High-resolution product quantization for Gaussian processes under sup-norm distortion
Harald Luschgy, Gilles Pagès

To cite this version:
Harald Luschgy, Gilles Pagès. High-resolution product quantization for Gaussian processes under sup-norm distortion. Bernoulli, 2007, 13 (3), 653-671; http://dx.doi.org/10.3150/07-BEJ6025. 10.3150/07-BEJ6025. hal-00013489v2

HAL Id: hal-00013489
https://hal.archivesouvertes.fr/hal-00013489v2
Submitted on 5 Sep 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
High-resolution product quantization for Gaussian processes under sup-norm distortion

HARALD LUSCHGY\textsuperscript{1} and GILLES PAGÈS\textsuperscript{2}

\textsuperscript{1}Universität Trier, FB IV-Mathematik, D-54286 Trier, Germany. E-mail: luschgy@uni-trier.de

\textsuperscript{2}Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5, France. E-mail: gpa@ccr.jussieu.fr

We derive high-resolution upper bounds for optimal product quantization of pathwise continuous Gaussian processes with respect to the supremum norm on $[0,T]^d$. Moreover, we describe a product quantization design which attains this bound. This is achieved under very general assumptions on random series expansions of the process. It turns out that product quantization is asymptotically only slightly worse than optimal functional quantization. The results are applied to fractional Brownian sheets and the Ornstein–Uhlenbeck process.

Keywords: Gaussian process; high-resolution quantization; product quantization; series expansion

1. Introduction

In this paper, we investigate the functional quantization problem for pathwise continuous Gaussian processes $X = (X_t)_{t \in I}, I = [0,T]^d$, where the path space $E = C(I)$ is endowed with the supremum norm. For any real separable space $(E, \| \cdot \|)$ and $r \in (0, \infty)$, optimal quantization means the best approximation in $L^r_r(E)$ of a random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ by random vectors $\hat{X} : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ taking finitely many values in $E$. If $N \in \mathbb{N}$, card($\hat{X}(\Omega)$) $\leq$ $N$, then $\hat{X}$ is called $N$-quantization. This leads to the minimal level $N$-quantization error defined by

$$e_{N,r}(X,E) := \inf \{ (E \| X - \hat{X} \|)^{1/r} : \hat{X} N\text{-quantization of } X \},$$

provided $X \in L^r_r(E)$. When $E = \mathbb{R}^d$, this problem is known as optimal vector quantization and has been extensively investigated since the early 1950s, with some applications to signal processing and transmission (see Gersho and Gray [11]) and to model-based clustering in statistics (see e.g., Tarpey [25]). Beyond these classical applications, optimal quantization has been used as a space discretization device to solve nonlinear problems, such as those arising in optimal stopping theory (American-style option pricing, reflected
BSDE, Bally and Pagès [2], nonlinear filtering (Pagès and Pham [22]), forward-backward SDE (see Delarue and Menozzi [5]) and SPDE (see Gobet et al. [12]). The mathematical foundations are treated in Graf and Luschgy [13]. Much attention has been paid to the infinite-dimensional case. This is the so-called functional quantization of stochastic processes: the aim is to quantize some processes viewed as random vectors taking values in their path spaces. Recently, a first application of functional quantization to statistical clustering of functional data has been investigated (see Tarpey and Kinateder [26] and Tarpey et al. [27]). The simplest application of functional quantization as a numerical method is to use it as an alternative to Monte Carlo simulation, using the quadrature formula

$$E(F(X)) \approx E(F(\hat{X})) = \sum_{a \in \alpha} F(a) P(\hat{X} = a),$$

where $\alpha = \hat{X}(\Omega)$, for sufficiently regular functionals $F : E \to \mathbb{R}$. If $\hat{X}$ is an $L^r$-optimal $N$-quantization and $F$ is Lipschitz continuous, then the induced error is bounded by $\|F\|_{\text{Lip}} N, r \leq 1$.

Some numerical applications are being developed for the pricing of path-dependent options (such as regular Asian options) in various models using $E = L^2([0, T], dt)$ (Black and Scholes, Heston, see Pagès and Printems [23], Wilbertz [29]). However, many important functionals of processes, like those related to barrier options or to options on maximum, are only continuous with respect to the sup-norm on $E = C([0, T])$.

Let us now describe what we will call the product quantization scheme. Let $X$ be a centered $E$-valued Gaussian random vector. Let $\xi_1, \xi_2, \ldots$ be i.i.d. $N(0, 1)$-distributed random variables and let $(f_j)_{j \geq 1}$ be a sequence in $E$ such that $\sum_{j=1}^{\infty} \xi_j f_j$ converges a.s. in $E$ and

$$X = \sum_{j=1}^{\infty} \xi_j f_j. \quad (1.2)$$

Let us call such a sequence admissible for $X$. For background on expansions for Gaussian random vectors, the reader is referred to Bogachev [4] and Ledoux and Talagrand [17].

One checks that $(f_j)_{j \geq 1}$ is admissible for $X$ if and only if $(f_j)_{j \geq 1}$ is a normalized tight frame in the reproducing kernel Hilbert space (Cameron–Martin space) $H = H_X$, that is, $\{f_j, j \geq 1\} \subset H$ and $\sum_{j \geq 1} (f_j, h)^2 = \|h\|^2_H$ for all $h \in H$ (see Luschgy and Pagès [21]). Then a sufficient (but not necessary) condition is that $(f_j)_{j \geq 1}$ is an orthonormal basis of $H_X$.

For $m, N_1, \ldots, N_m \in \mathbb{N}$ with $\prod_{j=1}^{m} N_j \leq N$, let $\hat{\xi}_j$ be an $L^r$-optimal $N_j$-quantization for $\xi_j$, that is, $(\mathbb{E}|\xi_j - \hat{\xi}_j|^r)^{1/r} = e_{N_j,r}(\xi_j, \mathbb{R})$. An $L^r$-product $N$-quantization of $X$ with respect to $(f_j)_{j \geq 1}$ is then defined by

$$\hat{X} := \hat{X}^{(N_1, \ldots, N_m)} := \sum_{j=1}^{m} \hat{\xi}_j f_j. \quad (1.3)$$
and the quantization error induced by \( \hat{X} \) is

\[
(\mathbb{E}\|X - \hat{X}\|^r)^{1/r}.
\]

Note that if \( \alpha_j = \hat{\xi}_j(\Omega) \), then the codebook \( \alpha = \hat{X}(\Omega) \) of \( \hat{X} \) satisfies \( \alpha = \{ \sum_{j=1}^m a_j f_j : a \in \prod_{j=1}^m \alpha_j \} \) and

\[
\left( \mathbb{E} \min_{a \in \alpha} \|X - a\|^r \right)^{1/r} \leq (\mathbb{E}\|X - \hat{X}\|^r)^{1/r}.
\]

The minimal \( N \)th product quantization error is then defined by

\[
e^{(\text{prod})}_{N,r}(X,E) := \inf \{ (\mathbb{E}\|X - \hat{X}\|^r)^{1/r} : (f_j)_{j \geq 1} \in E \text{ admitable for } X, \hat{X}L^r\text{-product } N\text{-quantization w.r.t. } (f_j) \}. \tag{1.4}
\]

Clearly, we have

\[
e_{N,r}(X,E) \leq e^{(\text{prod})}_{N,r}(X,E). \tag{1.5}
\]

We address the issue of high-resolution product quantization in \( E = \mathcal{C}(I) \) under the sup-norm, which concerns the performance of \( \hat{X} = \tilde{X}^{(N_1,\ldots,N_m)} \) under a suitable choice of the marginal quantization levels \( N_j \) and the behaviour of \( e^{(\text{prod})}_{N,r}(X,\mathcal{C}(I)) \) as \( N \to \infty \). For a broad class of Gaussian processes, we derive high-resolution upper estimates for \( e^{(\text{prod})}_{N,r}(X,\mathcal{C}(I)) \). Furthermore, we describe a product quantization design \( \hat{X} \) which attains this bound. Combining these estimates with precise high-resolution formulas for \( e_{N,r}(X,\mathcal{C}(I)) \) (see Dereich et al. [6], Dereich and Scheutzow [7], Graf et al. [14]), one may typically conclude that

\[
e^{(\text{prod})}_{N,r}(X,\mathcal{C}(I)) = O((\log \log N)^c e_{N,r}(X,\mathcal{C}(I))),
\]

for some suitable constant \( c > 0 \). This suggests that the asymptotic quality of product quantization, which is based on easy computations, is only slightly worse than optimal quantization. The optimality of this rate for product quantization rate remains open, although one may reasonably guess that it is optimal.

The paper is organized as follows. In Section 2, we derive high-resolution upper estimates for \( e^{(\text{prod})}_{N,r}(X,\mathcal{C}(I)) \) under very general assumptions on expansions. Section 3 contains a collection of examples, including fractional Brownian sheets, Riemann–Liouville processes and the Ornstein–Uhlenbeck process.

It is convenient to use the symbols \( \sim \) and \( \approx \), where \( a_n \sim b_n \) means \( a_n/b_n \to 1 \) and \( a_n \approx b_n \) means \( a_n = O(b_n) \) and \( a_n = \Omega(b_n) \). Throughout, all logarithms are natural logarithms and \([x]\) denotes the integer part of the real number \( x \).
We investigate high-resolution product functional quantization of centered continuous Gaussian processes \(X = (X_t)_{t \in I}\) on \(I = [0, T]^d\) in the space \(E = \mathcal{C}(I)\) equipped with the sup-norm \(\|x\| = \sup_{t \in I} |x(t)|\). Let
\[
e^{\text{prod}}_{N,r}(X) := e^{\text{prod}}_{N,r}(X, \mathcal{C}(I)).
\]

The subsequent setting comprises a broad class of processes.

Let \((f_j)_{j \geq 1} \in \mathcal{C}(I)^d\) satisfy the following assumptions:

(A1) \(\|f_j\| \leq C_1 j^{-\vartheta} \log(1 + j)^\gamma\) for every \(j \geq 1\) with \(\vartheta > 1/2, \gamma \geq 0\) and \(C_1 < \infty\);

(A2) \(f_j\) is \(a\)-Hölder-continuous and \([f_j]_a \leq C_2 j^b\) for every \(j \geq 1\) with \(a \in (0, 1], b \in \mathbb{R}\) and \(C_2 < \infty\), where
\[|f|_a = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^a}\]

(and \(|t|\) denotes the \(l_2\)-norm of \(t \in \mathbb{R}^d\)).

In the sequel, finite constants depending only on the parameters \(T, \vartheta, \gamma, a, b, C_1, C_2, d\) and \(r\) are denoted by \(C\) and may differ from one formula to another one. Other dependencies are explicitly indicated.

First, observe that by (A1),
\[
\sum_{j=1}^{\infty} [f_j]_a^2 < \infty
\]
for every \(\rho < 1\), and hence
\[
\sum_{j=1}^{\infty} [f_j]_a^2 < \infty \quad \text{for every } \rho < \frac{\vartheta - 1/2}{(b + \vartheta)_+}.
\]
This yields
\[
E |Y_s - Y_t|^2 = \sum_{j=1}^{\infty} |f_j(s) - f_j(t)|^2 \leq \left( \sum_{j=1}^{\infty} [f_j]_a^2 \right) |s - t|^{2\alpha \rho} \tag{2.1}
\]
and using the Gaussian feature of \(Y\), we obtain from the Kolmogorov criterion that \(Y\) has a continuous modification \(X\). Consequently, \((f_j)\) is admissible for \(X\) and
\[
X = \sum_{j=1}^{\infty} \xi_j f_j \quad \text{a.s.} \tag{2.2}
\]
Theorem 1. Assume that \( f_0 \) and \( f_1 \) are such that the upper estimate for \( E \|X - \hat{X}\|_{L_E^r(P)} \) satisfies

\[
(E \|X - \hat{X}\|^{1/r})^{1/r} = \|X - \hat{X}\|_{L_E^r(P)}
\]

so that

\[
(E \|X - \hat{X}\|^{r})^{1/r} \leq \sum_{j=1}^{m} \|f_j\| \epsilon_n, \gamma(N(0, 1)) + \left( E \left\| \sum_{j \geq m+1} \xi_j f_j \right\|^{r} \right)^{1/r}.
\]

(2.3)

For \( r \in (0, 1) \), we have

\[
(E \|X - \hat{X}\|^{r})^{1/r} \leq E \|X - \hat{X}\| \leq \sum_{j=1}^{m} \|f_j\| E|\xi_j - \hat{\xi}_j| + E \left\| \sum_{j \geq m+1} \xi_j f_j \right\|.
\]

(2.4)

Let us now consider the truncation error.

**Theorem 1.** Assume that \((f_j)_{j \geq 1} \in C(I)^N\) satisfies (A1)–(A2). Then, for every \( n \geq 2 \) and \( r \in (0, \infty) \),

\[
\left( E \left\| \sum_{j \geq n} \xi_j f_j \right\|^r \right)^{1/r} \leq \frac{C (\log n)^{\gamma + 1/2}}{n^{\vartheta - 1/2}}
\]

and

\[
\left( E \left\| \sum_{j \geq n} \xi_j f_j \right\|^r \right)^{1/r} \leq \frac{C (\log n)^{\gamma}}{n^{\vartheta - 1/2}}, \quad \text{if } b + \vartheta \leq 0.
\]

**Proof.** By equivalence of Gaussian moments,

\[
\left( E \left\| \sum_{j \geq n} \xi_j f_j \right\|^r \right)^{1/r} \leq DE \left\| \sum_{j \geq n} \xi_j f_j \right\|
\]

(2.5)

for some constant \( D \) depending on \( r \) (cf. Ledoux and Talagrand [17], Corollary 3.2). The upper estimate for \( E\| \sum_{j \geq n} \xi_j f_j \| \) is based on corresponding estimates for finite blocks.
of exponentially increasing length. For $m \geq 1$, set

$$Z = Z^{(m)} := \sum_{j=2^{m-1}+1}^{2^m} \xi_j f_j.$$ 

For a given $N \geq 1$, consider the grid $G_N = \{(\frac{(2i-1)T}{2N} : i = 1, \ldots, N)^d$. Then

$$\|Z\| \leq \sup_{t \in G_N} |Z_t| + \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t|.$$ 

It follows from the Gaussian maximal inequality that

$$E\sup_{t \in G_N} |Z_t| \leq C \sqrt{\log(1 + N^d)} \sup_{t \in G_N} \sqrt{EZ_t^2}.$$ 

Using (A1), we have, for every $t \in I$,

$$EZ_t^2 \leq \sum_{j=2^{m-1}+1}^{2^m} \|f_j\|^2 \leq C \sum_{j=2^{m-1}+1}^{2^m} j^{-2\vartheta} \log(1 + j)^{2\gamma} \leq C2^m(1 - 2\vartheta)_{m^{2\gamma}}$$

so that

$$E\sup_{t \in G_N} |Z_t| \leq C \sqrt{\log(1 + N)2^{-m(\vartheta-1/2)m^{2\gamma}}}.$$ 

Moreover, using (A2), we have, for $|s-t| \leq CN^{-1}$,

$$|Z_s - Z_t| \leq \sum_{j=2^{m-1}+1}^{2^m} |\xi_j||f_j(s) - f_j(t)|$$

$$\leq C|s-t|^a \sum_{j=2^{m-1}+1}^{2^m} |\xi_j||f_j|^a$$

$$\leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} |\xi_j|^b$$

and hence

$$E\sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t| \leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} j^b \leq CN^{-a}2^m(1+b).$$

Thus we have established the estimate

$$E\|Z^{(m)}\| \leq C(\sqrt{\log(1 + N)2^{-m(\vartheta-1/2)m^{2\gamma}}} + N^{-a}2^m(1+b)).$$

(2.6)
As concerns the choice of $N$, set $N := \lfloor 2^um \rfloor + 1$, with $u \in (0, \infty)$ satisfying $1 + b - au \leq \frac{1}{2} - \vartheta$. Equation (2.6) then becomes

$$E\|Z(m)\| \leq C2^{-m(\vartheta-1/2)}m^{\gamma+1/2}. \quad (2.7)$$

We note that in the case $b + \vartheta \leq -1/2$, we may choose $N = 1$ and thereby obtain a power reduction from $m^{\gamma+1/2}$ to $m^{\gamma}$. This can be improved. In fact, we have

$$E|Z_s - Z_t|^2 = \sum_{j=2^{m-1}+1}^{2^m} |f_j(s) - f_j(t)|^2 \leq C|s - t|^{2a} \sum_{j=2^{m-1}+1}^{2^m} j^{2b} \leq C|s - t|^{2a}2^{m(1+2b)}$$

so that

$$d_Z(s,t) := (E|Z_s - Z_t|^2)^{1/2} \leq C|s - t|^{a}2^{m(b+1/2)}.$$ 

If $N(\varepsilon, d_Z)$ denotes the covering numbers of $I$ with respect to the intrinsic semi-metric $d_Z$, then, by chaining,

$$E \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t| \leq E \sup_{d_Z(s,t) \leq \delta} |Z_s - Z_t| \leq C \int_0^\delta \sqrt{\log N(\varepsilon, d_Z)} \, d\varepsilon,$$

where $\delta := CN^{-a}2^{m(b+1/2)}$ (cf. Van der Waart and Wellner [28], page 101). Since

$$N(\varepsilon, d_Z) \leq C \left( \frac{2^{m(b+1/2)}}{\varepsilon} \right)^{d/a}, \quad 0 < \varepsilon \leq \varepsilon_0,$$

and $\int_0^1 \sqrt{\log(1/x)} \, dx < +\infty$, we obtain, for sufficiently large $N$,

$$\int_0^\delta \sqrt{\log N(\varepsilon, d_Z)} \, d\varepsilon \leq C2^{m(b+1/2)} \int_0^1 \sqrt{\log(1/x)} \, dx \leq C2^{m(b+1/2)}.$$ 

Consequently,

$$E\|Z^{(m)}\| \leq C(\sqrt{\log(1 + N)}2^{-m(\vartheta-1/2)}m^{\gamma} + 2^{m(b+1/2)}) \leq C2^{-m(\vartheta-1/2)}m^{\gamma} \quad \text{if } b + \vartheta \leq 0. \quad (2.8)$$

We now complete the proof. For $n \geq 2$, choose $m = m(n) \geq 1$ such that $2^{m-1} < n \leq 2^m$. Then

$$\left\| \sum_{j \geq n} \xi_j f_j \right\| \leq \sum_{j \geq m+1} \left\| Z^{(j)} \right\| + \left\| \sum_{j = n}^{2^m} \xi_j f_j \right\|.$$
Since \( E \| \sum_{n \leq j \leq 2m} \xi_j f_j \| \leq E \| Z^{(m)} \| \) by the Anderson inequality (cf. Bogachev \cite{4}, Corollary 3.3.7), we deduce from equation (2.7) that

\[
E \left\| \sum_{j \geq n} \xi_j f_j \right\| \leq C \sum_{j \geq m} \frac{j^{\gamma + 1/2}}{2^{j(\vartheta - 1/2)}} \leq \frac{C m^{\gamma + 1/2}}{2^{m(\vartheta - 1/2)}} \leq \frac{C (\log n)^{\gamma + 1/2}}{n^{\vartheta - 1/2}}.
\]

If \( b + \vartheta \leq 0 \), then it follows from (2.8) that

\[
E \left\| \sum_{j \geq n} \xi_j f_j \right\| \leq \frac{C (\log n)^{\gamma}}{n^{\vartheta - 1/2}}.
\]

Combining these estimates with (2.5) yields the assertion. \( \square \)

**Remarks.**

- The rate for the truncation error depends only on \( \vartheta \) and \( \gamma \), that is, on the decay of the size of functions \( f_j \) (provided \( b + \vartheta > 0 \)). The occurrence of expansions with \( b + \vartheta \leq 0 \) seems to be a rare event and otherwise \( b \) plays no role (see the subsequent example). The case \( \gamma = 0 \) typically corresponds to one-parameter processes with \( I = [0, T] \).
- The \( c_{X,r}^{(\text{prod})} \)-problem comprises the optimization of admissible sequences and, in view of (2.3) and (2.4), is thus related to the \( l \)-numbers of \( X \) defined by

\[
l_{n,r}(X) = l_{n,r}(X, \mathcal{C}(I)) := \inf \left\{ \left( E \left\| \sum_{j \geq n} \xi_j g_j \right\|_r^{1/r} \right) : (g_j) \text{ admissible for } X \text{ in } \mathcal{C}(I) \right\}.
\]

Rate-optimal solutions of the \( l_{n,r} \)-problem, in the sense of \( l_{n,r}(X) \approx (E \| \sum_{j \geq n} \xi_j g_j \|_r)^{1/r} \) as \( n \to \infty \), have recently been investigated (see Kühn and Linde \cite{16}, Dzhaparidze and van Zanten \cite{8, 9, 10}, Ayache and Taqqu \cite{1}). Admissible sequences of type (A1) and (A2) seem to be promising candidates.

**Example 1 (Weierstrass processes).** Let

\[
f_j(t) = j^{-\vartheta} \sin(j^{b+\vartheta} t), \quad j \geq 1, \vartheta > 1/2, b \in \mathbb{R}, t \in [0, T].
\]

Then \( \| f_j \| \leq j^{-\vartheta} \) and \([f_j]_1 = j^b\). Since \( f_j(0) = 0 \), we also have \( \| f_j \| \leq T j^b \), so (A1) and (A2) are satisfied, with \( \vartheta = \max\{\vartheta, -b\} \) and \( a = 1 \). The covariance function of \( X = \sum_{j=1}^\infty \xi_j f_j \) is given by

\[
\mathbb{E} X_s X_t = \sum_{j \geq 1} j^{-2\vartheta} \sin(j^{b+\vartheta} s) \sin(j^{b+\vartheta} t).
\]
Now, in the “Weierstrass case” \( b + \vartheta > 0 \), we obtain, from Theorem 1,
\[
\left( E \left\| \sum_{j \geq n} \xi_j f_j \right\|_r^\frac{1}{r} \right) \leq C \frac{\sqrt{\log n}}{n^{\vartheta^{-1}/2}},
\]
while in the “non-Weierstrass case,” \( b + \vartheta \leq 0 \) appears the better rate:
\[
\left( E \left\| \sum_{j \geq n} \xi_j f_j \right\|_r^\frac{1}{r} \right) \leq C \frac{n}{n^{b^{-1}/2}}.
\]

We pass to the minimal product quantization error \( e_{N,r}^{(\text{prod})} (X) \).

**Theorem 2.** Assume that \( X \) admits an admissible set \((f_j)_{j \geq 1}\) in \( C(I) \) satisfying (A1) and (A2). We then have, for every \( N \geq 3 \) and \( r \in (0, \infty) \),
\[
e_{N,r}^{(\text{prod})} (X) \leq C \frac{(\log \log N)^{\vartheta + \gamma}}{(\log N)^{\vartheta - 1/2}} \tag{2.10}
\]
and
\[
e_{N,r}^{(\text{prod})} (X) \leq C \frac{(\log \log N)^{\vartheta + \gamma - 1/2}}{(\log N)^{\vartheta - 1/2}} \quad \text{if } b + \vartheta \leq 0.
\]
Furthermore, the \( L^r \)-product \( N \)-quantization \( \hat{X} \) with respect to \((f_j)\), with tuning parameters defined in (2.11) and (2.15) below, achieves these rates.

**Proof.** Let \( r \in [1, \infty) \) and set \( \nu_j := j^{-\vartheta} \log(1 + j)^\gamma \) if \( j < j_0 := \lceil e^{\vartheta/\vartheta} \rceil \) and \( \nu_j := j^{-\vartheta} \log(1 + j)^\gamma \) if \( j \geq j_0 \). The sequence \((\nu_j)_j\) is then decreasing. Since
\[
\lim_{k \to \infty} k e_{k,r}(N(0, 1), \mathbb{R}) \text{ exists in } (0, \infty)
\]
(cf. Graf and Luschgy [13]), we deduce from (2.3), (A1) and Theorem 1 the estimate
\[
(E \|X - \hat{X}\|_r)^{1/r} \leq C \left( \sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1 + m)^{\gamma + 1/2}}{m^{\vartheta-1/2}} \right),
\]
for every \( m, N_1, \ldots, N_m \in \mathbb{N} \) with \( \prod_{j=1}^m N_j \leq N \). (The case \( b + \vartheta \leq 0 \) is treated analogously.) Consequently,
\[
e_{N,r}^{(\text{prod})} (X) \leq C \inf \left\{ \sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1 + m)^{\gamma + 1/2}}{m^{\vartheta-1/2}} : m, N_1, \ldots, N_m \in \mathbb{N}, \prod_{j=1}^m N_j \leq N \right\} \tag{2.11}
\]
For a given \( N \in \mathbb{N} \), we may first optimize the integer bit allocation given by the \( N_j \)'s for fixed \( m \) and then optimize \( m \). To this end, note that the continuous allocation problem reads

\[
\inf \left\{ \sum_{j=1}^{m} \nu_j y_j^{-1} : y_j > 0, \prod_{j=1}^{m} y_j \leq N \right\} = \sum_{j=1}^{m} \nu_j z_j^{-1} = N^{-1/m} \left( \prod_{j=1}^{m} \nu_j \right)^{1/m},
\]

where

\[
z_j = N^{1/m} \nu_j \left( \prod_{k=1}^{m} \nu_k \right)^{-1/m}
\]

and \( z_1 \geq \cdots \geq z_m \). One can produce an (approximate) integer solution by setting

\[
N_j = \lfloor z_j \rfloor = \left\lfloor N^{1/m} \nu_j \left( \prod_{k=1}^{m} \nu_k \right)^{-1/m} \right\rfloor, \quad j \in \{1, \ldots, m\}, \tag{2.12}
\]

provided \( z_m \geq 1 \). Then

\[
\sum_{j=1}^{m} \nu_j N_j^{-1} \leq 2m N^{-1/m} \left( \prod_{j=1}^{m} \nu_j \right)^{1/m} \leq C m N^{-1/m} \nu_m.
\]

Since the constraint on \( m \) reads \( m \in I(N) \) with

\[
I(N) := \left\{ m \in \mathbb{N} : N^{1/m} \nu_m \left( \prod_{j=1}^{m} \nu_j \right)^{-1/m} \geq 1 \right\}, \tag{2.13}
\]

we arrive at

\[
\epsilon_{N,r}^{(\text{prod})}(X) \leq C \inf_{m \in I(N)} \left( \frac{N^{-1/m} \log(1 + m)^\gamma}{m^{d-1}} + \frac{\log(1 + m)^{\gamma + 1/2}}{m^{d-1/2}} \right), \tag{2.14}
\]

for every \( N \in \mathbb{N} \). We check that \( I(N) \) is finite, \( I(N) = \{1, \ldots, m^*(N)\} \), \( m^*(N) \) increases to infinity and

\[
m^*(N) \sim \frac{\log N}{d} \quad \text{as} \quad N \to \infty. \tag{2.15}
\]

Finally, let

\[
m = m(N) \in I(N), \quad \text{with} \quad m(N) \leq \frac{2 \log N}{\log \log N} \quad \text{for} \quad N \geq 3 \tag{2.16}
\]

such that

\[
m(N) \sim \frac{2 \log N}{\log \log N} \quad \text{as} \quad N \to \infty.
\]
This is possible in view of (2.14). Using (2.4), the case \( r \in (0, 1) \) follows from \( r = 1 \) since the \( L^r \)-optimal \( N_j \)-quantizations \( \hat{\xi}_j \) satisfy \( \mathbb{E}|\xi_j - \hat{\xi}_j| \leq CN_j^{-1} \), \( j \geq 1 \); see Graf et al. [15].

We may reasonably conjecture that for many specific processes, the above rate is the true one. This would imply that product quantization achieves the optimal rate for quantization, namely the rate of convergence to zero of \( e_{N,r}(X) := e_{N,r}(X,\mathcal{C}(I)) \), only up to a \( \log \log N \) term in formula (2.16). This is in contrast to the Hilbert space setting, where the optimal rate is attained by product quantization (cf. Luschgy and Pagès [20]). To be precise, we summarize the results on \( e_{N,r}(X) \) in the present setting.

**Proposition 1.** (a) Assume that \( X \) admits an admissible sequence in \( \mathcal{C}(I) \) satisfying (A1) and (A2). Then

\[
e_{N,r}(X) = O\left( \frac{(\log \log N)^{\gamma+1/2}}{(\log N)^{\vartheta-1/2}} \right) \tag{2.17}
\]

and

\[
e_{N,r}(X) = O\left( \frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1/2}} \right), \quad \text{if } b + \vartheta \leq 0. \tag{2.18}
\]

(b) Assume that \( X \) admits an admissible sequence satisfying (A1). Let \( \mu \) be a finite Borel measure on \( I \) and let \( V: \mathcal{C}(I) \to L^2(I,\mu) \) denote the natural embedding. Then

\[
e_{N,r}(V(X),L^2(\mu)) = O\left( \frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1/2}} \right)
\]

and

\[
e_{N,2}^{(\text{prod})}(V(X),L^2(\mu)) = O\left( \frac{(\log \log N)^{\gamma}}{(\log N)^{\vartheta-1/2}} \right).
\]

**Proof.** (a) The proof is not constructive. We use Proposition 4.1 in Li and Linde [18], which relates \( l \)-numbers (see (2.9)) and small ball probabilities (but this relation is not always sharp). By combining this relation and Theorem 1, we obtain

\[
-\log(\mathbb{P}(\|X\| \leq \varepsilon)) = O\left( \varepsilon^{-1/(\vartheta-1/2)} \left( \log \left( \frac{1}{\varepsilon} \right) \right)^{(\gamma+1/2)/(\vartheta-1/2)} \right),
\]

\[
-\log(\mathbb{P}(\|X\| \leq \varepsilon)) = O\left( \varepsilon^{-1/(\vartheta-1/2)} \left( \log \left( \frac{1}{\varepsilon} \right) \right)^{\gamma/(\vartheta-1/2)} \right), \quad \text{if } b + \vartheta \leq 0
\]

as \( \varepsilon \to 0 \). We may then apply a known, precise relationship between these probabilities and \( e_{N,r}(X) \) (cf. Dereich et al. [6], Graf et al. [14]) and this leads to the desired estimate.
(b) Let \((f_j)_{j \geq 1}\) be an admissible sequence in \(C(I)\) for \(X\) satisfying (A1) and consider an \(L^2\)-product \(N\)-quantization of \(V(X)\) based on \((V f_j)_{j \geq 1}\),

\[
\hat{V}(X)^N = \sum_{j=1}^{m} \hat{\xi}_j V(f_j),
\]

where \(\hat{\xi}_j\) are \(L^2\)-optimal Voronoi \(N_j\)-quantizers; see Luschgy and Pagès [19]. Then, using the independence of \(\xi_j - \hat{\xi}_j\), \(j \geq 1\), and the stationarity property \(\hat{\xi}_j = E(\xi_j | \hat{\xi}_j)\) of the quantization \(\hat{\xi}_j\), we have

\[
E \left\| \sum_{j=1}^{\infty} \xi_j V(f_j) - \hat{V}(X)^N \right\|_{L^2(\mu)}^2 = \sum_{j=1}^{m} E|\xi_j - \hat{\xi}_j|^2 \|V f_j\|^2_{L^2(\mu)} + \sum_{j \geq m+1} \|V f_j\|^2_{L^2(\mu)}
\]

\[
\leq C \left( \sum_{j=1}^{m} N_j^{-2} j^{-2\theta} \log(1+j)^{2\gamma} + \sum_{j \geq m+1} j^{-2\theta} \log(1+j)^{2\gamma} \right).
\]

We then argue along the lines of Luschgy and Pagès [19] to conclude that

\[
e^{\text{prod}}_{N,r}(V(X), L^2(\mu)) = O\left( \frac{(\log \log N)^{\gamma}}{(\log N)^{d-1/2}} \right).
\]

Sometimes, (2.17) provides the true rate for \(e_{N,r}(X)\) (as for the two-parameter Brownian sheet), sometimes it yields the best known upper bound (as for the \(d\)-parameter Brownian sheet with \(d \geq 3\)) and sometimes (2.18) provides the true rate (as for Brownian motion). The latter typically occurs when the rate of \(e_{N,r}(X)\) and the “Hilbert rate” of \(e_{N,r}(V(X), L^2(dt))\) coincide (see Section 3). It remains an open question to find conditions for this to happen.

3. Examples

3.1. Fractional Brownian motions and fractional Brownian sheets

We consider the Dzaparidze–van Zanten expansion of the fractional Brownian motion \(X = (X_t)_{t \in [0,T]}\) with Hurst index \(\rho \in (0,1)\) and covariance function

\[
\mathbb{E}X_s X_t = \frac{1}{2}(s^{2\rho} + t^{2\rho} - |s-t|^{2\rho}).
\]
These authors discovered, in Dzhaparidze and van Zanten [9], that the sequence
\[
\begin{align*}
    f_j^1(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{1-\rho}(x_j)| x_j^{\rho+1}} \sin \left( \frac{x_j t}{T} \right), \quad j \geq 1, \\
    f_j^2(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{-\rho}(y_j)| y_j^{\rho+1}} \left( 1 - \cos \left( \frac{y_j t}{T} \right) \right), \quad j \geq 1,
\end{align*}
\]

in \( C([0,T]) \) is admissible for \( X \), where \( J_\nu \) denotes the Bessel function of the first kind of order \( \nu \), \( 0 < x_1 < x_2 < \cdots \) are the positive zeros of \( J_\rho \), \( 0 < y_1 < y_2 < \cdots \) the positive zeros of \( J_{1-\rho} \) and \( c_\rho = \Gamma(1 + 2\rho) \sin(\pi \rho) / \pi \).

Using the asymptotic properties
\[
x_j \sim y_j \sim \pi j, \quad J_{1-\rho}(x_j) \sim J_{-\rho}(y_j) \sim \frac{\sqrt{2}}{\pi} j^{-1/2} \quad \text{as} \quad j \to \infty
\]

(cf. Dzhaparidze and van Zanten [9]), one observes that a suitable arrangement of the functions (3.1) (like \( f_2 = f_1^1 \), \( f_{2j-1} = f_j^2 \)) satisfies (A1) and (A2) with parameters \( \vartheta = \rho + 1/2, \gamma = 0, a = 1 \) and \( b = 1/2 - \rho \). Consequently,
\[
e_{N,r}^{(\text{prod})}(FBM) = O \left( \frac{(\log \log N)^{\rho+1/2}}{(\log N)^{\rho}} \right),
\]

while (see Dereich and Scheutzow [7], Graf et al. [14])
\[
e_{N,r}(FBM) \approx (\log N)^{-\rho}.
\]

The tensor products of functions (3.1) are admissible for the fractional Brownian sheet \( X \) over \([0,T]^d\) with covariance function
\[
E X_s X_t = (\frac{1}{2})^d \prod_{i=1}^d (s_i^{2\rho_i} + t_i^{2\rho_i} - |s_i - t_i|^{2\rho_i}),
\]

\( \rho_i \in (0,1) \), and satisfy conditions (A1) and (A2) with \( \vartheta = \rho + 1/2, \rho = \min_{1 \leq i \leq d} \rho_i, \gamma = \vartheta(m-1) \), where \( m = \text{card}\{i \in \{1, \ldots, d\} : \rho_i = \rho\} \), \( a = 1 \) and \( b = \max_{1 \leq i \leq d} (1/2 - \rho_i) \). This is a consequence of the following lemma which ensures stability of conditions (A1) and (A2) under tensor products.

**Lemma 1.** For \( i \in \{1, \ldots, d\} \), let \( (f_i^j)_{j \geq 1} \in C([0,T])^{11} \) satisfy (A1) and (A2) with parameters \( \vartheta_i, \gamma_i, a_i, b_i \) such that \( \gamma_i = 0 \). Then a decreasing arrangement of \( (\bigotimes_{i=1}^d f_i^j)_{j \in \mathbb{N}^d} \) satisfies (A1) and (A2) with parameters \( \vartheta = \min_{1 \leq i \leq d} \vartheta_i, \gamma = \vartheta(m-1) \), where \( m = \text{card}\{i \in \{1, \ldots, d\} : \vartheta_i = \vartheta\} \), \( a = \min_{1 \leq i \leq d} a_i \) and \( b = (\max_{1 \leq i \leq d} b_i) \).
Proof. For \( j = (j_1, \ldots, j_d) \in \mathbb{N}^d \), set \( f_j = \bigotimes_{i=1}^{d} f_{j_i}^{i} \) so that \( f_j(t) = \prod_{j_i=1}^{d} f_{j_i}^{i}(t_i) \), \( t \in [0,T]^d \). We have

\[
\|f_j\| \leq \prod_{i=1}^{d} \|f_{j_i}^{i}\| \leq C \prod_{i=1}^{d} j_i^{-\theta_i} \quad \text{and} \quad |f_j(s) - f_j(t)| \leq C \max_{1 \leq i \leq d} j_i^{b_i} |s - t|^a.
\]

Let \( u_j := \prod_{i=1}^{d} j_i^{-\theta_i} \). Choose a bijective map \( \psi : \mathbb{N} \rightarrow \mathbb{N}^d \) such that \( u_k := u_{\psi(k)} \) is decreasing in \( k \geq 1 \). Set \( f_k := f_{\psi(k)} \). Then

\[
u_k \approx C k^{-\theta} (\log k)^{\theta(m-1)} \quad \text{as} \quad k \rightarrow \infty
\]

(cf. Papageorgiou and Wasilkowski [24], Theorem 2.1). Consequently,

\[
\|f_k\| \leq C k^{-\theta} (\log k)^{\theta(m-1)}
\]

and, for \( j = \psi(k) \),

\[
\prod_{i=1}^{d} j_i \leq \prod_{i=1}^{d} j_i^{\theta_i/\theta} \leq C k (\log k)^{-1} \leq C k,
\]

hence

\[
|f_k(s) - f_k(t)| \leq C k^{b} |s - t|^a.
\]

Therefore, by Theorem 2 and Proposition 1,

\[
\varepsilon_{N,r}^{(\text{prod})}(FBS) = O\left(\frac{(\log \log N)^{m(\rho+1/2)}}{(\log N)^\rho}\right)
\]

\[(3.4)\]

and

\[
\varepsilon_{N,r}(FBS) = O\left(\frac{(\log \log N)^{m(\rho+1/2)-\rho}}{(\log N)^\rho}\right).
\]

\[(3.5)\]

The Hilbert space setting \( E = L^2([0,T]^d, dt) \) provides the lower estimate

\[
\varepsilon_{N,r}(FBS) = \Omega\left(\frac{(\log \log N)^{(m-1)(\rho+1/2)}}{(\log N)^\rho}\right)
\]

\[(3.6)\]

(see Luschgy and Pagès [19, 20]). The true rate of \( \varepsilon_{N,r}(FBS) \) is known only for the case \( m = 1 \), where the true rate is the “Hilbert rate” \((2.6)\) (see Dereich et al. [6]), and for the case \( m = 2 \), where \((3.5)\) is the true rate (see Belinsky and Linde [3], Graf et al. [14]). A reasonable conjecture is that \((3.5)\) is also the true rate for \( m \geq 3 \).
3.2. Riemann–Liouville and other moving average processes

For $\psi \in L^2([0,T], dt)$ and a standard Brownian motion $W$, let

$$X_t = \int_0^t \psi(t-s) dW_s, \quad t \in [0,T],$$

and assume that $X$ has a pathwise continuous modification. Since

$$\mathbb{E}X_sX_t = \int_0^{s \wedge t} \psi(s-u)\psi(t-u) \, du,$$

$$f_j(t) = \sqrt{\frac{2}{T}} \int_0^t \psi(s) \cos \left( \frac{\pi(j-1/2)s}{T} \right) \, ds$$

$$= \sqrt{\frac{2}{T}} \int_0^t \psi(s) \cos \left( \frac{\pi(j-1/2)(t-s)}{T} \right) \, ds, \quad j \geq 1, \quad (3.7)$$

is an admissible sequence for $X$. Observe that (3.7) provides well-defined continuous functions, even for $\psi \in L^1([0,T], dt)$.

Lemma 2. Let $\psi \in L^1([0,1], dt)$.

(a) If $\varphi(t) = \int_0^t |\psi(s)| \, ds$ is $\beta$-Hölder continuous with $\beta \in (0,1]$, then the sequence $(f_j)$ from (3.7) satisfies $(A2)$ with $a = \beta$ and $b = 1$. In particular, if $\psi \in L^2([0,T], dt)$, then $(A2)$ is satisfied with $a = 1/2$ and $b = 1$.

(b) If $\psi$ has finite variation over $[0,T]$, then $(A1)$ is satisfied with $\vartheta = 1$ and $\gamma = 0$.

Proof. Let $\lambda_j = (\pi(j-1/2)/T)^{-2}$. (a) For $s < t$, we have

$$f_j(s) - f_j(t) = \sqrt{\frac{2}{T}} \left\{ \int_0^s \psi(u) \cos((s-u)/\sqrt{\lambda_j}) - \cos((t-u)/\sqrt{\lambda_j})) \right\} du$$

so that

$$|f_j(s) - f_j(t)| \leq \sqrt{\frac{2}{T}} \left( \frac{|s-t|}{\sqrt{\lambda_j}} \|\psi\|_{L^1(dt)} + \int_s^t |\psi(u)| \, du \right).$$

(b) We have

$$f_j(t) = -\sqrt{2\lambda_j/T} \int_0^t \psi(s) \, d(s/(\sqrt{\lambda_j}))$$

$$= \sqrt{2\lambda_j/T} \left( \psi(0) \sin(t/\sqrt{\lambda_j}) + \int_0^t \sin((t-s)/\sqrt{\lambda_j}) \, ds \right)$$
so that
\[ \|f_j\| \leq \sqrt{2\lambda_j/T(\|\psi(0)\| + \text{Var}(\psi, [0, T]))}. \]

This lemma yields a universal upper bound,
\[ c_{N,r}^{(\text{prod})}(X) = O\left( \frac{\log \log N}{(\log N)^{1/2}} \right), \]
for functions \( \psi \) having finite variation.

In the sequel, we do not concern ourselves with improvements of the parameter \( b \) in (A2) since the condition \( b + \vartheta \leq 0 \) cannot be achieved in this setting.

Lemma 3. Let \( \psi \in L^1([0, T], dt) \).

(a) If \( \psi \) is positive and decreasing on \((0, T]\) and \( \varphi(t) = \int_0^t \psi(s) \, ds \) is \( \beta \)-Hölder continuous with \( \beta \in (0, 1] \), then the sequence \((f_j)\) from (3.7) satisfies \( \|f_j\| \leq C j^{-\beta} \). If \( \beta > 1/2 \), then (A1) is satisfied with \( \vartheta = \beta \) and \( \gamma = 0 \).

(b) If \( \psi(0) = 0 \), \( \psi \) is \( \beta \)-Hölder continuous with \( \beta \in (0, 1] \) and \( \psi \) is differentiable on \((0, T]\) such that \( \psi' \) is positive and decreasing on \((0, T]\), then (A1) is satisfied with \( \vartheta = 1 + \beta \) and \( \gamma = 0 \).

Proof. Let \( \lambda_j = (\pi(j - 1/2)/T)^{-2} \). (a) For \( t \leq \sqrt{\lambda_j} \), we have
\[ |f_j(t)| \leq \sqrt{2/T} \varphi(\sqrt{\lambda_j}). \]

Using the second integral mean value formula, we obtain, for \( t \in [\sqrt{\lambda_j}, T] \) and some \( \delta_j \in [\sqrt{\lambda_j}, t] \),
\[ |f_j(t)| \leq \sqrt{2/T} \left( \int_0^{\sqrt{\lambda_j}} \psi(s) \cos((t-s)/\sqrt{\lambda_j}) \, ds \right) + \int_{\sqrt{\lambda_j}}^t \psi(s) \cos((t-s)/\sqrt{\lambda_j}) \, ds \)
\[ = \sqrt{2/T} \left( \int_0^{\sqrt{\lambda_j}} \psi(s) \cos((t-s)/\sqrt{\lambda_j}) \, ds \right) + \psi(\sqrt{\lambda_j}) \left( \int_{\sqrt{\lambda_j}}^{\delta_j} \psi((t-s)/\sqrt{\lambda_j}) \, ds \right) \]
\[ \leq \sqrt{2/T} \varphi(\sqrt{\lambda_j}) + 2 \sqrt{\lambda_j} \psi(\sqrt{\lambda_j}) \]
\[ \leq 3 \sqrt{2/T} \varphi(\sqrt{\lambda_j}). \]

Consequently,
\[ \|f_j\| \leq 3 \sqrt{2/T} \varphi(\sqrt{\lambda_j}) \leq C \lambda_j^{1/2}. \]

(b) The function \( \psi \) is absolutely continuous on \([0, T]\), so an integration by parts yields
\[ f_j(t) = \sqrt{2\lambda_j/T} \int_0^t \psi'(s) \sin((t-s)/\sqrt{\lambda_j}) \, ds. \]
Arguing as in (a) (with \( \psi \) replaced by \( \psi' \)), we deduce that
\[
\| f_j \| \leq 3 \sqrt{2 \lambda_j} / T \psi(\sqrt{\lambda_j}) \leq C \lambda_j^{(1+\beta)/2}.
\] □

Now, let \( \psi(t) = t^{\rho-1/2} \) with \( \rho \in (0, \infty) \). Then
\[
X_t = X_t^\rho = \int_0^t (t-s)^{\rho-1/2} dW_s, \quad t \in [0,T]
\] (3.8)
so that \( X^\rho \) is a Riemann–Liouville process of order \( \rho \). Using the \((\rho \wedge \frac{1}{2})\)-Hölder continuity of the application \( t \mapsto X_t^\rho \) from \([0,T]\) into \( L^2(\mathbb{P}) \) and the Kolmogorov criterion, we can check that \( X^\rho \) has a pathwise continuous modification.

**Lemma 4.** Let \( \psi(t) = t^{\rho-1/2}, \rho \in (0, \infty) \). Then the sequence \( (f_j) \) from (3.7) satisfies (A2) with \( a = \min \{1, \rho + 1/2\}, \ b = 1 \) and (A1) for \( \rho \in (0, 3/2] \) with \( \vartheta = \rho + 1/2 \) and \( \gamma = 0 \).

**Proof.** This is an immediate consequence of Lemmas 2 and 3. □

We deduce, for Riemann–Liouville processes of order \( \rho \in (0, 3/2]\), that
\[
e_{N,r}^{(\text{prod})}(RL) = O \left( \frac{(\log \log N)^{\rho+1/2}}{\log N} \right),
\] (3.9)
while for every \( \rho \in (0, \infty) \) (see [18], Graf et al. [14]),
\[
e_{N,r}(RL) \approx (\log N)^{-\rho}.
\] (3.10)

To go beyond \( \rho = 3/2 \), we must slightly change the way we quantize. Let \( \psi(t) = t^{\rho-1/2} \), with \( \rho > 3/2 \), and choose \( k \in \mathbb{N} \) such that \( k + 1/2 < \rho \leq k + 3/2 \). Set \( \lambda_j = (\pi(j-1/2)/T)^{-2} \). For \( k \in \{2n-1, 2n\} \ n \in \mathbb{N} \), integration by parts yields the expansion
\[
f_j(t) = \sum_{m=1}^n (-1)^{m-1} \lambda_j^m \sqrt{2/T} \psi(2m-1)(t) + (-1)^n \lambda_j^n \sqrt{2/T} \int_0^t \psi(2n)(s) \cos((t-s)/\sqrt{\lambda_j}) ds.
\]
\[= g_j(t) + h_j(t), \quad t \in [0,T].
\]
Since \( \psi(2n)(t) = Ct^{\beta-1} \) if \( k = 2n-1 \) and \( \psi(2n)(t) = Ct^\beta \) if \( k = 2n \) with \( \beta = \rho - k - 1/2 \in (0,1] \), we deduce from Lemma 2 and Lemma 3 that the sequence \( (h_j) \) in \( C([0,T]) \) satisfies (A1) with \( \vartheta = \rho + 1/2, \gamma = 0 \) and (A2) with \( a = \rho - k - 1/2, \ b = -k \) if \( k = 2n-1 \) and \( a = 1, \ b = -k + 1 \) if \( k = 2n \). Clearly, the sequence \( (g_j) \) also satisfies the conditions (A1) and (A2) (with \( \vartheta = 2, \gamma = 0, \ b = -2 \) and \( a = \rho - k - 1/2 \) if \( k = 2n-1 \) and \( a = 1 \) if \( k = 2n \)). Consequently, there exist centered continuous Gaussian processes \( U = (U_t)_{t \in [0,T]} \) and \( Z \) such that \( U = \sum_{j=1}^\infty \xi_j g_j \) a.s., \( Z = \sum_{j=1}^\infty \xi_j h_j \) a.s.,
\[
X = X^\rho \overset{d}{=} U + Z
\] (3.11)
and $U \in \text{span}\{\psi^{(2m-1)} : m = 1, \ldots, n\}$ a.s. Observe that
\[ U = \sum_{m=1}^{n} (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \eta_m, \]
where $\eta_m = \sum_{j=1}^{\infty} \lambda_j^m \xi_j$ is $\mathcal{N}(0, \sum_{j=1}^{\infty} \lambda_j^{2m})$-distributed.

Now use, for example, $[\sqrt{N}/2]$-quantizations of $\eta_m$ and a $[\sqrt{N}]$-product quantization of $Z$ for the quantization of $X$ (which is clearly not optimal in practise, but remains rate optimal). Let $\hat{\eta}_m$ be an $L^r$-optimal $[\sqrt{N}/2]$-quantization for $\eta_m$,
\[ \hat{U}^{\sqrt{N}} := \sum_{m=1}^{n} (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \hat{\eta}_m, \]
and let $\hat{Z}^{\sqrt{N}}$ be the $L^r$-product $[\sqrt{N}]$-quantization of $Z$ from Theorem 2. A (modified) $L^r$-product $N$-quantization of $X$ with respect to $(f_j)$ is then defined by
\[ \hat{X} := \hat{U}^{\sqrt{N}} + \hat{Z}^{\sqrt{N}}. \quad (3.12) \]

Using Theorem 2, we can show for the quantization error, that
\[ \|U + Z - \hat{X}\|_{L^E} \leq C\left(\|U - U^{\sqrt{N}}\|_{L^E_k} + \|Z - Z^{\sqrt{N}}\|_{L^E_k}\right) \]
\[ \leq C \left( \sum_{m=1}^{n} \sqrt{2/T} \|\psi^{(2m-1)}\| \|\eta_m - \hat{\eta}_m\|_{L^r} + \|Z - Z^{\sqrt{N}}\|_{L^E_k}\right) \]
\[ \leq C \left( \frac{\log \log \sqrt{N}}{\log \sqrt{N}} \right)^{\rho+1/2} \]
\[ \leq C \left( \frac{\log \log \sqrt{N}}{\log \sqrt{N}} \right)^{\rho} \]
so that, with the above modification, (3.9) remains true for $\rho > 3/2$.

Now, consider the stationary Ornstein–Uhlenbeck process as the solution of the Langevin equation
\[ dX_t = -\beta X_t \, dt + \sigma \, dW_t, \quad t \in [0, T], \]
with $X_0$ independent of $W$ and $\mathcal{N}(0, \sigma^2/2\beta)$-distributed, $\sigma > 0$, $\beta > 0$. It admits the explicit representation
\[ X_t = e^{-\beta t} X_0 + \sigma e^{-\beta t} \int_0^t e^{\beta s} \, dW_s \quad (3.13) \]
and
\[ \mathbb{E}X_t = \frac{\sigma}{2\beta} e^{-\beta|s-t|}. \]
By Lemma 2, the admissible sequence

\[ f_0(t) = \frac{\sigma}{\sqrt{2a}} e^{-\beta t}, \quad f_j(t) = \sigma \sqrt{\frac{2}{T}} \int_0^t e^{-\beta (t-s)} \cos \left( \frac{\pi(j-1/2)s}{T} \right) ds, \quad j \geq 1, \]

satisfies conditions (A1) and (A2) with \( \vartheta = 1, \gamma = 0, a = 1 \) and \( b = 1 \). Consequently,

\[ e_{N,r}^{\text{(prod)}}(OU) = O \left( \log \log N \big/ (\log N)^{1/2} \right), \quad (3.14) \]

while (see Graf et al. [14])

\[ e_{N,r}(OU) \approx (\log N)^{-1/2}. \quad (3.15) \]

References

[1] Ayache, A. and Taqqu, M.S. (2003). Rate optimality of Wavelet series approximations of fractional Brownian motion. *J. of Fourier Analysis and Applications* 9 451–471. MR2027888

[2] Bally, V. and Pagès, G. (2003). A quantization algorithm for solving discrete time multidimensional optimal stopping problems. *Bernoulli* 9 1003–1049. MR2046816

[3] Belinsky, E. and Linde, W. (2002). Small ball probabilities of fractional Brownian sheets via fractional integration operators. *J. Theoret. Probab.* 15 589–612. MR1922439

[4] Bogachev, V.I. (1998). *Gaussian Measures*. Providence, RI: AMS. MR1642391

[5] Delarue, F. and Menozzi, S. (2006). A forward-backward stochastic algorithm for quasilinear PDEs. *Ann. Appl. Probab.* 16 140–184. MR2209339

[6] Dereich, S., Fehringer, F., Matoussi, A. and Scheutzow, M. (2003). On the link between small ball probabilities and the quantization problem for Gaussian measures on Banach spaces. *J. Theoret. Probab.* 16 249–265. MR1956830

[7] Dereich, S. and Scheutzow, M. (2006). High resolution quantization and entropy coding for fractional Brownian motions. *Electron. J. Probab.* 11 700–722. MR2242661

[8] Dzhaparidze, K. and van Zanten, H. (2005). Optimality of an explicit series expansion of the fractional Brownian sheet. *Statist. Probab. Lett.* 71 295–301. MR2145497

[9] Dzhaparidze, K. and van Zanten, H. (2004). A series expansion of fractional Brownian motion. *Probab. Theory Related. Fields* 130 39–55. MR2092872

[10] Dzhaparidze, K. and van Zanten, H. (2005). Krein’s spectral theory and the Paley–Wiener expansion of fractional Brownian motion. *Ann. Probab.* 33 620–644. MR2123205

[11] Gersho, A. and Gray, R.M. (1992). *Vector Quantization and Signal Compression*. Boston: Kluwer.

[12] Gobet, E., Pagès, G., Pham, H. and Printems, J. (2005). Discretization and simulation for a class of SPDE’s with applications to Zakai and McKean–Vlasov equations. Preprint, LPMA-958, Univ. Paris 6 (France). *SIAM J. Control Optim.* To appear.

[13] Graf, S. and Luschgy, H. (2000). *Foundations of quantization for probability distributions*. *Lecture Notes in Math.* 1730. Berlin: Springer. MR1764176

[14] Graf, S., Luschgy, H. and Pagès, G. (2003). Functional quantization and small ball probabilities for Gaussian processes. *J. Theoret. Probab.* 16 1047–1062. MR2033197
[15] Graf, S., Luschgy, H. and Pagès, G. (2006). Distortion mismatch in the quantization of probability measures. Preprint, LPMA-1051, Univ. Paris 6 (France). ESAIM Probab. Statist. To appear.

[16] Kühn, T. and Linde, W. (2002). Optimal series representation of fractional Brownian sheets. Bernoulli 8 669–696. MR1935652

[17] Ledoux, M. and Talagrand, M. (1991). Probability in Banach Spaces. Berlin: Springer. MR1102015

[18] Li, W.V. and Linde, W. (1999). Approximation, metric entropy and small ball estimates for Gaussian measures. Ann. Probab. 27 1556–1578. MR1733160

[19] Luschgy, H. and Pagès, G. (2002). Functional quantization of Gaussian processes. J. Funct. Anal. 196 486–531. MR1943099

[20] Luschgy, H. and Pagès, G. (2004). Sharp asymptotics of the functional quantization problem for Gaussian processes. Ann. Probab. 32 1574–1599. MR2060310

[21] Luschgy, H. and Pagès, G. (2006). Expansion of Gaussian processes and Hilbert frames. Technical report.

[22] Pagès G. and Pham, H. (2005). Optimal quantization methods for nonlinear filtering with discrete-time observations. Bernoulli 11 893–932. MR2172846

[23] Pagès, G. and Printems, J. (2005). Functional quantization for numerics with an application to option pricing, Monte Carlo Methods Appl. 11 407–446. MR2186817

[24] Papageorgiou, A. and Wasilkowski, G.W. (1990). On the average complexity of multivariate problems. J. Complexity 6 1–23. MR1048027

[25] Tarpey, T. (1996). Self-consistency: a fundamental concept in statistics. Statist. Sci. 11 229–243. MR1436648

[26] Tarpey, T. and Kinateder, K.K.J. (2003). Clustering functional data. J. Classification 20 93–114. MR1983123

[27] Tarpey, T., Petkova, E. and Ogden, R.T. (2003). Profiling Placebo responders by self-consistent partitioning of functional data. J. Amer. Statist. Association 98 850–858. MR2055493

[28] Van der Waart, A.W. and Wellner, J.A. (1996). Weak Convergence and Empirical Processes. New York: Springer. MR1385671

[29] Wilbertz, B. (2005). Computational aspects of functional quantization for Gaussian measures and applications. Doctoral thesis, Univ. Trier.

Received January 2005 and revised January 2007