Symmetry enhancement interpolation, non-commutativity and Double Field Theory

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Abstract: We present a moduli dependent target space effective field theory action for (truncated) heterotic string toroidal compactifications. When moving continuously along moduli space, the stringy gauge symmetry enhancement-breaking effects, which occur at particular points of moduli space, are reproduced.

Besides the expected fields, originated in the ten dimensional low energy effective theory, a new vector and scalar fields are included. These fields depend on “double periodic coordinates” as usually introduced in Double Field Theory. Their mode expansion encodes information about string states, carrying winding and KK momenta, associated to gauge symmetry enhancements. Interestingly enough, it is found that a non-commutative product, which introduces an intrinsic non-commutativity on the compact target space, is required in order to make contact with string theory amplitude results.

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1 Introduction

In this article we propose a target space effective field theory description of string theory interactions. Clearly the subject is not new. Indeed, the conventional low energy effective action for given values of moduli fields can be found in string books [11, 2, 3]. However, recent analyses [4, 5, 6, 7] performed from the perspective of Double Field Theory (DFT)1 aiming at the inclusion of gauge symmetry enhancement aspects in the field theory description, as well as recent (and not so recent) proposals about non-commutativity of string zero modes [8, 9], point towards a richer structure with some intrinsic compact target space non-commutativity.

A key guide in our analysis is the field theory description of gauge symmetry enhancement on toroidal compactifications. Gauge symmetry enhancement is a very stringy phenomenon associated to the fact that the string is an extended object and, therefore, it can wind around non-contractible cycles. At certain moduli points (i.e., fixed points of T-duality transformations) vector boson states, associated to definite values of windings and compact momenta become massless. These vectors, combined with massless vectors inherited from the metric and antisymmetric tensor fields, give rise to an enhanced gauge symmetry group $G_1$ (see for instance [15, 16]). Further displacements on moduli space can lead to a different fixed point where, generically, other vectors associated to different values of winding and momenta will become massless leading to a different enhanced gauge group $G_2$, etc. At generic points only a $U(1)^r_R \times U(1)^{r+16}_L$ symmetry exists, where $r$ is the number of compactified dimensions associated to the KK zero modes of the metric and antisymmetric fields and 16 comes from Cartan generators of the ten dimensional gauge group, in the heterotic string case. The low energy effective theory, at a given moduli point, where massive states are neglected, can be described by a usual gauge field theory Lagrangian, coupled to gravity, with no explicit reference to any windings. By slightly moving away from this fixed moduli point, gauge symmetry gets broken. The symmetry breaking can be understood as a conventional higgsing mechanism and also, as found from a DFT approach [5, 7], as associated to a dependence of “would-be structure

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1 See [10, 11] for some original references on DFT and for instance [12, 13] for a reviews. DFT approaches to heterotic string can be found in [14].
constants” fluxes on moduli fields.

The aim of the present work is to write down a lower dimensional effective field theory action for a toroidally compactified heterotic string, dependent on moduli fields expectation values such that, by varying such values, different enhancement situations can be reached.

Very schematically, the idea is to incorporate a vector boson field $A_\mu(x, Y)$ and a scalar $M_I(x, Y)$ into the action, in addition to the fields inherited from the usual ten dimensional metric and Kalb-Ramond $B_2$. All fields must depend on both $d$ space-time $x^\mu$ coordinates as well as on internal compact toroidal $Y \equiv (y^I, y^m, \tilde{y}_m)$ coordinates. Besides the $y^I$ coordinates associated to the heterotic string degrees of freedom, $2r$ double coordinates $(y^m, \tilde{y}^m)$ conjugate to momenta and windings modes for each of the $r$ compact dimension are considered, in the spirit of DFT. A generalized mode expansion in periodic internal coordinates would produce $d$ dimensional fields $A^{(L)}_\nu(x)$ (and $M^{(L)}_I(x)$) with $L$ labeling modes, depending on windings and KK momenta. As mentioned before, for certain moduli values some of these modes become massless and, when combined with KK zero modes coming from metric and $B$ field (as well as heterotic Cartan fields) they enhance the gauge symmetry.

The resulting action, in terms of the “uplifted” $A_\mu(x, Y)$ and $M_I(x, Y)$ fields, appears to require a non-commutativity on fields introduced through a non-commutative $\star$-product in the compact space [8]. At the neighborhood of each specific moduli fixed point and when only the slightly massive modes that become massless at this point are kept, the usual, commutative, effective gauge theory action is recovered after integrating over the internal coordinates. The gauge symmetry gets enhanced exactly at the fixed point.

Therefore, the action provides an effective interpolation among theories at different points. It is worth mentioning that enhancement can be described in DFT constructions as an enlargement of the compactification tangent space [4, 5, 6, 7] at a fixed point. Here the compact manifold is an $r$ dimensional double torus and we find that this enlargement is effectively provided by Fourier modes associated to fields that “will-be massless at such point”. Interestingly enough, the mentioned non-commutativity can be traced back to cocycle factors in string vertices. These factors were first mentioned in [11] but did not
manifest in previous DFT constructions due to the considered level matching conditions.

We organize the article as follows: In Section 2 we introduce the proposed action in $D = d + 2r$ dimensions. In Section 3 we perform the mode expansion and analyze the different contributions. Section 4 deals with the physical content of the action, like vector and scalar masses, Goldstone bosons, enhancement-breaking of gauge symmetries, etc. An illustrative torus compactification ($r = 2$) example is briefly discussed. A summary and a discussion of the limitations and possible extensions of the present work are presented in Section 5. Notation and technical aspects are reserved to the Appendices as well as a more detailed description of the $\star$-product, extended to incorporate the heterotic string gauge modes.

2 The effective action

In this section we present the moduli dependent field theory effective action for the toroidal compactification of heterotic string. The basic ingredients and notation conventions are introduced here and the reader is referred to the appendices for details.

At a given fixed point on moduli space the heterotic gauge group is of the form $G_L \times U(1)_R^r$. The rank of $G_L$ is $r_L = r + 16 = 26 - d$ originated from the 16 Cartan generators of the ten dimensional heterotic gauge group plus the $r = 10 - d$ vector bosons coming from Left combinations of the KK reductions of the metric and the antisymmetric tensor. Therefore, the dimension of the gauge group is $\dim G_L = n_c + r_L$ where $n_c$ denotes the number of charged generators. These generators correspond to string vertex operators containing KK momenta and windings associated, generically, with massive fields that become massless at the fixed point. These fields will play a central role in our discussion. Let us stress that $n_c$ depends on the moduli point and that, at generic points, there is no enhancement at all ($n_c = 0$) and the generic gauge group is $U(1)_L^{r_L} \times U(1)_R$. The low energy effective action for the bosonic sector of heterotic string at a fixed point
\[ S = \int d^d x \sqrt{g} \left[ e^{-2\varphi} \left( R + 4 \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} \right) \right. \\
\left. - \frac{1}{4} \left( \delta_{AB} F^{A \mu \nu} F_{\mu \nu}^B + \delta_{IJ} \bar{F}_{\mu \nu}^I \bar{F}_{\mu \nu}^J - 2 g d \sqrt{\alpha'} M_{AI} F_{\mu \nu}^A \bar{F}_{\mu \nu}^I \right) \right. \\
\left. - \frac{1}{4} D_\mu M_{AI} D_\nu M_{A\bar{I}} g^{\mu \nu} + O(M^4) \right]. \tag{2.1} \]

Several indices are introduced: Right indices \( \bar{I} = 1, \ldots, r \) label the Abelian group \( U(1)_{\bar{I}} \) associated to Right vector bosons \( \bar{A}^\bar{I}_\mu \). Left indices \( A = (\hat{I}, a) \) where \( a = 1, \ldots, n_c \) label the Left gauge group charged generators with vector bosons \( A^a_\mu \) and \( \hat{I} \equiv (m, I) \) with \( m = 1 \ldots r \) and \( I = 1 \ldots 16 \) labeling the Left Cartan generators \( \hat{A}^\hat{I}_\mu \).

The scalar fields \( M_{AI} \) live in the \((\dim G_L)_{q=0}\) adjoint representation of \( G_L \) and carry zero vector charge \( \bar{q} = (\bar{q}_1, \ldots, \bar{q}_r) = 0 \) with respect to \( U(1)_R \) Abelian Right group. We have

\[
F^B = dA^B + \frac{g_d}{2} f_{CD}^B A^C \wedge A^D, \quad F^\bar{I} = dA^\bar{I} \tag{2.2}
\]

\[
D_\mu M_{AI} = \partial_\mu M_{AI} + g_d f^K_{SA} A^S_{L\mu} M_{K\bar{I}} \tag{2.3}
\]

\( H \) is the \( B \) field strength (with Chern-Simons interactions) defined as

\[
H = dB + F^B \wedge A_B, \tag{2.4}
\]

\( \phi \) is the dilaton and \( R \) the scalar curvature. We observe that the terms in this expression corresponding to “Cartan” indices \( \mathcal{T} = (\hat{I}, I) \) (which will play the role of \( O(r_1, r) \) indices), originated from reductions of the 10D fields will be always present, whereas terms associated with charged fields \( (a) \) index, will change when moving on moduli space. It proves convenient in our construction to separate these contributions and rewrite the
above action (2.1) as

\[
S = \int d^d x \sqrt{g} \left[ e^{-2\varphi} \left( R + 4 \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) ight.
\]

\[
- \frac{1}{4} \left( \delta_{ij} F^{i\mu\nu} F_{\mu\nu}^j + \delta_{ij} F^{i\mu\nu} F^{j\mu\nu} - 2 g_d \sqrt{\alpha'} M_{ij} F^i_{\mu\nu} F^{j\mu\nu} \right)
- \frac{1}{4} D_\mu M_{ij} D_\nu M^{ij} g_{\mu\nu}
- \frac{1}{4} \left( \delta_{ab} F^{a\mu\nu} F^b_{\mu\nu} + \delta_{ij} F^{i\mu\nu} F^{j\mu\nu} - 2 g_d \sqrt{\alpha'} M_{ai} F^a_{\mu\nu} F^{i\mu\nu} \right)
- \frac{1}{4} D_\mu M_{ai} D_\nu M^{ai} g_{\mu\nu} + \mathcal{O}(M^4) \right].
\]

(2.5)

Inspired by DFT constructions [7] we introduce a generalized $O(r_L, r)$ metric $H^{I\bar{J}}$ and we expand on fluctuations around a flat background as

\[
H^{I\bar{J}} = \delta^{I\bar{J}} + H^{(1)}_{I\bar{J}} + \frac{1}{2} H^{(2)}_{I\bar{J}} + \ldots
\]

(2.6)

where matrix elements vanish unless

\[
H^{(1)}_{i\bar{j}} = -M_{i\bar{j}}, \quad H^{(1)}_{j\bar{i}} = -M^{T}_{i\bar{j}},
\]

\[
H^{(2)}_{i\bar{j}} = (MM^{T})_{i\bar{j}}, \quad H^{(2)}_{i\bar{j}} = (M^{T} M)_{i\bar{j}}.
\]

(2.7)

In terms of this metric and keeping terms up to second order in fluctuations (assuming vector fields are first order) the action can be re-expressed as

\[
\int d^d x \sqrt{g} \left[ e^{-2\varphi} \left( R + 4 \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) ight.
\]

\[
- \frac{1}{4} D_\mu M_{ai} D_\nu M^{ai} g_{\mu\nu} + \frac{1}{8} D_\mu \mathcal{H}_{I\bar{J}} D_\nu \mathcal{H}^{I\bar{J}} g_{\mu\nu}
- \frac{1}{4} \left( \delta_{ab} F^{a\mu\nu} F^b_{\mu\nu} + \mathcal{H}_{I\bar{J}} F^{I\mu\nu} F^{J\mu\nu} \right)
- \frac{1}{2} g_d \sqrt{\alpha'} M_{ai} F^a_{\mu\nu} F^{i\mu\nu} \right],
\]

(2.8)

where the $I$ indices are contracted with the $O(r_L, r)$ invariant metric that we will generically express in the L-R basis (also called $C-$basis) as

\[
\eta^{I\bar{J}}_C = \begin{pmatrix} 1_{16+r} & 0 \\ 0 & -1_\tau \end{pmatrix}.
\]

(2.9)
Most of the time we are going to work in a Cartan-Weyl basis where charged generators are labeled by simple roots, denoted by Greek indices. We introduce a generalized vector \( A^I_\mu = (A^I_\mu, A^I_\mu) \) that incorporates the left and right Cartan fields, respectively. Then, the Left Cartan field strength reads

\[
F^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + ig\sum_\alpha f_{\alpha\dot{\beta}} A^\alpha_{[\mu}A^\beta_{\nu]} \quad F^J_{\mu\nu} = 2\partial_{[\mu}A^J_{\nu]} \tag{2.10}
\]

whereas

\[
F^\alpha_{\mu\nu} = 2\partial_{[\mu}A^\alpha_{\nu]} + igf^{\alpha\beta\gamma} A^\beta_{[\mu}A^\gamma_{\nu]} \tag{2.11}
\]

are the field strength for charged vectors\(^2\). Similarly for scalar fields we have

\[
D_{\mu}M_{i,j} = \partial_{\mu}M_{i,j} + igd_{\beta\dot{\alpha}} A^\beta_{\mu}M_{i,j} \tag{2.12}
\]

\[
D_{\mu}M_{\alpha,j} = \partial_{\mu}M_{\alpha,j} + igd_{\alpha\beta} A^\beta_{\mu}M_{\alpha,j} \tag{2.13}
\]

where a sum over repeated root indices is implicit. We are using a Cartan-Weyl basis such that

\[
f_{\alpha\beta\gamma} = f_{\alpha\beta\gamma} = 1 \quad \text{(with } \gamma = \alpha + \beta) \quad \text{and} \quad f_{\beta\alpha\dot{i}} = f_{-\alpha\dot{i}} = f_{\alpha\dot{i}} = \alpha^i \quad \text{(no sum on } \alpha) \text{ etc.}
\]

String vertex operators giving rise to vector bosons and scalars above generically contain an internal factor (see Appendix \(\Delta\) for notation)

\[
e^{iL^{(\tilde{F})}(\Phi), Y(z)} = e^{iL^{(\tilde{F})}, Y_L(z)+k_\mathcal{R}, Y_R(z)},
\]

where \(L^{(\tilde{F})}(\Phi) = (L^{(\tilde{F})}_L(\Phi), k^{(\tilde{F})}_{\mathcal{R}}(\Phi))\) is the generalized momentum (see \(\Delta.1\)) that depends on windings \(\tilde{p}^m\), KK momenta \(p_m\) and \(\Lambda_{16}\) weights \(P^I\) that we organize into the generalized Kaluza-Klein (GKK) momenta

\[
\tilde{\mathcal{P}} \equiv (P^I, p_m, \tilde{p}^m) \tag{2.15}
\]

and on moduli field values, denoted by \(\Phi \equiv (g, B, A)\), encoding the background metric \(g\), the \(B\) field and Wilson line values\(^3\)

\(^2\)Here we use the convention \(2A_{[\mu}B_{\nu]} = A_\mu B_\nu - B_\nu A_\mu\).

\(^3\)We will generally write \(L \equiv L^{(\tilde{F})}(\Phi)\) and omit the explicit writing of the dependence on \(\tilde{\mathcal{P}}\) and \(\Phi\) to lighten the notation.
The Cartan vectors \( A^I_\mu \) do originate in string vertex operators coming from KK reductions of the metric and antisymmetric field of the form \( V(\hat{I}_L) \propto \tilde{\psi}^\mu \partial_x y^I e^{i\hat{I}_L(\Phi).Y(z)} e^{iK.X(z)} \) where \( K^\mu \) is the space time momentum. The level matching condition (LMC) reads \( \mathbb{L}^2 = 0 \) and it is trivially satisfied by massless states that correspond to \( l_L = k_R = 0 \) (with null windings and KK momenta).

On the other hand, the left handed charged vector bosons arise from vertices

\[
V(l_L) \propto \epsilon(\tilde{P}, K)_\mu e^{i\mathbb{L}(\Phi).Y(z)} \tilde{\psi}^\mu(z) e^{iK.X(z)},
\]

where we have included a polarization vector \( \epsilon(\tilde{P}, K)_\mu \). Generalized momenta must satisfy the LMC

\[
\frac{1}{2} \mathbb{L}^2 = \frac{1}{2} l_L^2 - \frac{1}{2} k_R^2 = 1.
\]

At a fixed point \( \Phi_0 \) and for specific values of winding and momenta (i.e., for specific values of \( \tilde{P} \)),

\[
k_R^{(\tilde{P})}(\Phi_0) = 0, \quad l_L^{(\tilde{P})}(\Phi_0) = \alpha^{(\tilde{P})} \quad \text{with} \quad \frac{1}{2} \alpha^{(\tilde{P})^2} = 1
\]

the states become massless (see (A.6)) and \( l_L^{(\tilde{P})}(\Phi_0) \) become the roots \( \alpha^{(\tilde{P})} \) of the enhanced gauge group algebra charged generators such that \( e^{i\mathbb{L}(\Phi).Y(z)} \rightarrow J_{\alpha^{(\tilde{P})}} \) are the charged generators. Generically, at a different fixed point, other set of \( \tilde{P} \)'s will ensure (2.18), leading to a different enhanced gauge group. We will denote this set of \( n_c \) points by

\[
\tilde{G}(\Phi_0)_{nc} = \{ \tilde{P} \equiv (P^I, k_m, \tilde{k}^m) : k_R^{(\tilde{P})}(\Phi_0) = 0 \text{ (thus } l_L^{(\tilde{P})}(\Phi_0) = \alpha^{(\tilde{P})}, m^2 = 0) \}.
\]

Namely, \( \tilde{G}(\Phi_0)_{nc} \) encodes the \( n_c \) "will-be massless fields at fixed point \( \Phi_0 \)."

The main aim of our work is to provide a unified, moduli dependent, field theory description such that, at given fixed points the different gauge theories are reproduced.

Following the suggestions in Ref.[5] we propose to consider a sort of generalized Kaluza Klein expansion on generalized momenta \( \mathbb{L} \) of the different fields coming into play in the enhancement process. With this scope in mind we associate polarization vectors \( \epsilon(\mathbb{L}, K)_\mu \) or their space-time version \( A^{(L)}_\mu(x) \) as modes of a field

\[
A_\mu(x, Y) = \sum_{\tilde{P}} A^{(L)}_\mu(x) e^{i\mathbb{L}Y^\mathbb{L}} = \sum_{\mathbb{L}} A^{(L)}_\mu(x) e^{i\mathbb{L}Y_L + ik_R R} \delta\left(\frac{1}{2}\mathbb{L}^2, 1\right),
\]
where the prime in the sum indicates that LMC (2.17) must be imposed (with an abuse of notation we indicate the sum on mode index $\tilde{P}$ by $\mathbb{L}$). Recall that the LMC is a severe constraint. For the circle case, for instance, it reads $p\tilde{p} = 1$ and thus the sum contains only two terms corresponding to $(p, \tilde{p}) = \pm (1, 1)$.

Generically, if the mass of the GKK components $A^{(L)}_{\mu}(x)$ were given by the string mass formula (A.6), these modes would be massive. However, when moving continuously along the moduli space, for specific values $\tilde{P} \in \tilde{G}_{n_c}(\phi_0)$, $n_c$ vector fields $A^{(L)}_{\mu}(x) \equiv A^{(\ell)}_{\mu}(x)$ would become massless leading to the enhanced $G_L$ gauge group.  In a similar way we introduce the GKK expansion for scalar fields by associating the fields $M_a\bar{J}(x)$, coming from string vertex operators modes $M^a\bar{J}(x, Y) = \sum L M^{a}(L)\bar{J}(x)e^{iL_xY_x}.$ (2.21)

Notice that $A_{\mu}(x, Y), M_{a\bar{J}}(x, Y)$ could be interpreted as an uplifting of $d$ dimensional fields to $d + r + r_L$ dimensions with $r + r_L$ periodic.

With these expressions for charged vector and scalar fields we could now try to uplift the above $d$ dimensional action (2.5) to $d + r + r_L$ dimensions and thus, $\int dx^d \to \int dx^d dY = \int dx^d dy_L dY dy_R$. Notice that we should also consider expansions for fields originated from the $D = 10$ metric and $B$ fields (now with the level matching condition $L^2 = 0$). We will formally do it but for the moment only the zero modes, i.e, the massless ones, will be kept for these fields.

A crucial point is how to generate a non-Abelian structure out of these fields in order to give rise to enhancements at fixed points. We will see that a new so called “star product” [8], which we denote by $\star$, accounting for a non-commutativity of modes will do the job.

\footnote{A reality condition $A^{(L)*}_{\mu} = A^{(-L)}_{\mu}$ must be imposed.}
\footnote{Recall that Heterotic coordinates can be thought of as coordinates on a 16 dimensional torus.}
Let us present the action and discuss its particular features afterwards.

\[
S = \int dxdY \left[ e^{-2\varphi} (R + 4\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}) \right. \\
- \frac{1}{4} \left( F_{\mu\nu} \star F^{\mu\nu} + \mathcal{H}_{I,J} \star F_{I,\mu} \star F_{J,\nu} - \mathcal{D}_\mu M^I \star \mathcal{D}^\mu M_I - \frac{1}{2} \mathcal{D}_\mu \mathcal{H}^I_{J} \star \mathcal{D}^\mu \mathcal{H}_{I,J} \right) \\
- \frac{1}{2} M_I \star F^{\mu\nu} \star F_{I,\mu} + \frac{1}{4} \partial_I M^I \star \partial_K M_I (H^{J,K} - \eta^{J,K}) \\
+ \left. \frac{i}{2} \partial_I M^I \star M_J \star M^I + \mathcal{O}(M^4) \right] .
\]

(2.22)

All fields in the action depend on the compact coordinate $Y$ and can therefore be mode expanded. Integration over $Y$ will produce an effective action in $d$ space-time dimensions. Fields originated from $D = 10$ KK reductions $(G, B, \varphi, M_{I,J}, A_\mu^I)$ require a mode expansion with the constraint $\delta (L^2, 0)$.

If just the zero mode is kept we notice that the first two rows in (2.8) are formally reproduced with $g_{\mu\nu} = G_{\mu\nu}^{(0)}, M_{I,J} = M_{I,J}^{(0)}$, etc. However, a non trivial action of the $\star$-product arises whenever non zero modes come into play, as it happens in products of fields associated to enhancements (and thus requiring expansions with $\delta (\frac{1}{2}L^2, 1)$ constraint). We provide a more detailed discussion in the next section. Also, the different terms in the action are now defined as

\[
F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + igA_\mu \star A_\nu + 2gA_I^J \star \partial_J A_\nu \\
F_{\mu\nu}^I = 2\partial_{[\mu} A_{I,\nu]} + g\partial^I A_\mu \star A_\nu
\]

(2.23)

\[
\mathcal{D}_\mu M^I = \partial_\mu M^I + igA_\mu \star M^I + gA^J_\mu \star \partial_J M^I - g\partial_J A_\mu \star (H^{J,I} - \eta^{J,I}) \\
\mathcal{D}_\mu \mathcal{H}^I_{J} = \partial_\mu \mathcal{H}^I_{J} + g\partial^I A_\mu \star M^J,
\]

(2.24)

(where $g = \frac{1}{\sqrt{\alpha'}}$) by generalizing (2.11). The term $\frac{1}{2}(\eta^{J,I} - H^{J,I})$ acts as a covariant projector on Right sector indices. Actually, even if we have introduced $M^I = (M^J, M^\bar{J})$ fields, the left $\hat{I}$ components can be $O(r_L, r)$ covariantly set to zero, as in string vertex operators.

Finally, the three form $H$ is defined as

\[
H = dB + F^I \wedge A_I + F \star \wedge A.
\]

(2.25)
3 The action for GKK modes

In this section we perform the expansion of the fields in the above action in terms of GKK modes, compute the $\star$-products for these modes and finally integrate over the internal coordinates $\mathbb{Y}$ in order to obtain the, moduli dependent, $d$-dimensional effective action. In particular, we will show that, when only the massless GKK are taken into account and the background is at a self-dual point, Eq. (2.22) gives rise to the gauge enhanced action (2.8).

The particular $\star$-product we consider here is a generalization of the one proposed in [8] to the case of the heterotic string. It is described in Appendix B. For two mode expanded fields it reads

\[
(\phi \star \psi)(x, \mathbb{Y}) = \sum_{L_1, L_2} e^{i\pi(\tilde{k}_1 k_2 + \frac{1}{2} P_1 E P_2)} \cdot \phi^{(L_1)}(x)\psi^{(L_2)}(x)e^{i(L_1+L_2)\cdot \mathbb{Y}},
\]

where a phase dependence on the windings of the first $\tilde{k}_1^m$ field and KK momentum $k_{2m}$ of second mode, as well on Spin(32) weights, is generated. Recall that $L = L(\tilde{k}_m, k_m, P^I)$ and $E_{IJ} = G_{IJ} + B_{IJ}$ (see [B]). It is useful to notice that if $L_1 + L_2 = 0$, as we would find if we integrated on $\mathbb{Y}$, the phase becomes irrelevant due to LMC. Therefore $\star$ will be relevant only for terms in the action involving three charged fields. In this case, by using associativity, we have

\[
(\phi \star \psi \star \lambda)(x, \mathbb{Y}) = \sum_{L_1, L_2, L_3} \tilde{f}_{L_1 L_2 L_3} \phi^{(L_1)}(x)\psi^{(L_2)}(x)\lambda^{(L_3)}(x)e^{i(L_1+L_2+L_3)\cdot \mathbb{Y}},
\]

where

\[
\tilde{f}_{L_1 L_2 L_3} = e^{i\pi(\tilde{k}_1 k_2 + \frac{1}{2} P_1 E P_2)} e^{i\pi(\tilde{k}_1 k_2 + \tilde{k}_2 k_1 + P_1 P_2) E P_3}.
\]

If we integrate over internal coordinates, then momentum conservation $L_1 + L_2 + L_3 = 0$ is obtained, the second phase becomes $e^{-i\pi L_3^2} = e^{-i\pi(\tilde{k}_1 k_2 + \tilde{k}_2 k_1 + P_1 P_2)}$ and thus $\tilde{f}_{L_1 L_2 L_3} = e^{-i\pi(\tilde{k}_1 k_2 + \frac{1}{2} P_1 E P_2)}$. Even if $\frac{1}{2} L_1^2 = \frac{1}{2} L_2^2 = 1$, notice that $(L_1 + L_2)$ will not necessarily satisfy the same LMC condition. If this were indeed the case then $\frac{1}{2} L_3^2 = 1$ implies $L_1 L_2 = \tilde{k}_1 k_2 + \tilde{k}_2 k_1 + P_1 P_2 = -1$. Generically, if we allowed for different LMC for the $6$We will mainly refer to Spin(32) but results are valid for $E_8 \times E_8$ as well.
fields in the product, we would have
\[ \tilde{k}_1 k_2 + \tilde{k}_2 k_1 + P_1 P_2 = -1 + N \]  
(3.4)

(we will come back to this issue at the end of Section 4). Then,
\[ \tilde{f}_{L_1 L_2 L_3} = e^{-i\pi (\tilde{k}_2 k_1 + \frac{1}{2} P_1 EP_2 - \tilde{k}_2 k_1 - \frac{1}{2} P_1 EP_2)} \]
\[ = (1)^{1+N} e^{i\pi (\tilde{k}_2 k_1 + \frac{1}{2} P_1 EP_2)} = (1)^{1+N} e^{-i\pi (\tilde{k}_2 k_1 + \frac{1}{2} P_1 EP_2)} \]
\[ = (1)^{1+N} \tilde{f}_{L_2 L_1 L_3} = (1)^{1+N} \tilde{f}_{L_1 L_3 L_2} \]  
(3.5)

where we used again momentum conservation for the last equality. We conclude that
for \( N = 0 \), as it is the case for charged fields, \( \tilde{f}_{L_1 L_2 L_3} = \pm 1 \) and that it is completely
antisymmetric under index permutation. This result is valid for both massless and massive
states.

As mentioned, for modes originated in 10D massless modes we require
\( \delta(\mathbb{L}^2, 0) \). When
the zero mode sector is considered, for instance for \( \lambda^{(L_3)}(x) \rightarrow \lambda^{(L_3=0)}(x) \) above, then
\( \mathbb{L}_1 = -\mathbb{L}_2 \) and the phase becomes irrelevant so the \( \ast \)-product reduces to just the ordinary
product.

As a conclusion we see that, up to order three in fluctuations -as we are considering
here- only terms involving three charged fields will see a non trivial phase in the \( \ast \)-product.

We show below explicit expressions for mode expansions of some of the terms appearing
in (2.22).

In the following subsections we analyze the different contributions in the action (2.22).

### 3.1 The vectors kinetic term

Let us analyze first \( \int dxdY \ F_{\mu\nu} \ast F^{\mu\nu} \). In order to do it let us consider the Fourier
\( \mathbb{L} \)-component \( F_{\mu\nu}^{(L)} \) as follows
\[ F_{\mu\nu}^{(L)} = \int dY F_{\mu\nu} e^{-iL^\cdot Y} \]  
(3.6)

where:
\[ F_{\mu\nu}^{(L)} = 2\partial_{[\mu} A_{\nu]}^{(L)} + ig \sum_{L_2} \tilde{f}_{L_2 L_3 L_4} A_{\mu}^{(L_2)} A_{\nu}^{(L_3)} + 2ig \tilde{f}_{L_1 L_2 L_3} A_{\mu}^{(L_1)} A_{\nu}^{(L_2)} \]  
(3.7)

\( ^7\)We normalize the integration variables so as to have a unit volume factor. Also, we use that
\( \int d^{2n} Y e^{i(P_M + Q_M)Y} = \delta^{2n}(P_M + Q_M) \),

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Here, $L + L_2 + L_3 = 0$, and we have defined $\tilde{f}_{L-LI}$ as
\[
\tilde{f}_{L-LI} = l_I(\Phi), \quad \tilde{f}_{L-LI} = k_{RI}(\Phi).
\] (3.8)

In doing the Fourier mode decomposition we have assumed $A^T_{\mu}(x, Y) = A^T_{\mu}(x)$, i.e., we have kept only the zero mode of the generalized Cartan field. Also, recall that $F^{(L)}_{\mu\nu}$ depends on moduli point.

Now let's analyze what happens at self-dual points. If we restrict the sum to $\tilde{P} \in \tilde{G}_{nc}(\Phi_0)$ and move to $\Phi = \Phi_0$, then (3.7) will reduce to (2.11) as long as we identify
\[
A^{(L_i)}_\mu \leftrightarrow A^{(\alpha_{L_i})}_\mu, \quad -A^{(\alpha_{L_i})}_\mu \leftrightarrow A^{(-\alpha_{L_i})}_\mu,
\] (3.9)
where $L_i$ is in a one to one correspondence with the positive roots $\alpha_i \equiv \alpha^{(L_i)}$, $i = 1, ..., n$, (and $-L_i$ with $-\alpha_i \equiv \alpha^{(-L_i)}$) of the enhanced group. The reason for this identification is because $f_{L_2L_3L_4}$ is invariant under $L_i \rightarrow -L_i$. Thus, for each root $\alpha^{(\tilde{P})}$
\[
F^{(\tilde{P})}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}^{(\tilde{P})} + ig \sum_{\tilde{P}} \beta_{\alpha(-\tilde{P})} A^{(\tilde{P})}_{\alpha(-\tilde{P})} A_{\nu}^{(\tilde{P})} + 2igf_{I_\alpha(\tilde{P})} A^{(\tilde{P})}_{\nu}
\] (3.10)
\[
-F^{(-\tilde{P})}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}^{(-\tilde{P})} + ig \sum_{\tilde{P}} \beta_{\alpha(-\tilde{P})} A^{(-\tilde{P})}_{\alpha(-\tilde{P})} A_{\nu}^{(-\tilde{P})} + 2igf_{I_\alpha(-\tilde{P})} A^{(-\tilde{P})}_{\nu},
\] (3.11)
becomes the field strength, for the charged fields, of the corresponding gauge theory. Then, up to third order in fluctuations we can write
\[
\int dx dY \ F_{\mu\nu} \ast F^{\mu\nu} = \sum_{L} \int dx F^{(L)}_{\mu\nu} F^{(-L)\mu\nu}
\] (3.12)

We have thus matched the first term of the second row of action (2.8). The second term of the same row is reproduced by $H_{\mathcal{L}J} \ast F^{(\mathcal{L})}_{\mu\nu} \ast F^{(\mathcal{J})\mu\nu}$ in (2.22) since, when focusing only on massless GKK,
\[
F^{(\mathcal{L})}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}^{(\mathcal{L})} + 2ig \sum_{\tilde{P}} \beta_{\alpha(-\tilde{P})} A^{(\mathcal{P})}_{\alpha(-\tilde{P})} A_{\nu}^{(\mathcal{P})},
\] (3.13)
(remember $f_{\alpha(\tilde{P})\alpha(-\tilde{P})} = 0$ for massless GKK).
It is interesting to consider the $D = 10$ theory. In this case $\mathbb{L} \equiv l_L$ is always Left and its components $l^I = P^I$ are just the components of the $Spin(32)$ roots $P = (\pm 1, \pm 1, 0, \ldots)$ (underlining denoting permutation) and therefore (3.12) becomes the $Spin(32)$ gauge kinetic term for charged fields. Together with terms in the first row in (2.22) the low energy $D = 10$ heterotic effective action is recovered. Recall that all other terms are not present since there is no compactification at all.

### 3.2 Scalars kinetic term

Following similar steps as above we can write

$$\int dx dY \; \mathcal{D}_\mu M_J \star \mathcal{D}^\mu M^J = \int dx \sum_L \mathcal{D}_\mu M_J^{(L)} \mathcal{D}^\mu M^{(-L)J},$$

where

$$\mathcal{D}_\mu M_J^{(L)} = \partial_\mu M_J^{(L)} + ig \sum_{L2} f_{LL2L3} M_{(L2)J}^{(L3)} A_{\mu}^{(L3)} + ig \tilde{f}_{L-LI} M_{J}^{(L)} A_{\mu}^{(I)}$$

(3.14)

$$- 2igk_{RJ} A_{\mu}^{(L)} + ig \sum_{L2} \tilde{f}_{LL2L3} M_{IJ}^{(L)} \tilde{L}^I_{\mu} A_{\mu}^{(L3)}$$

is the Fourier transform of (2.13) and we have used that $(\bar{H}_{IJ} - \eta_{IJ})\mathbb{L}^J = (0, 2k_{RJ})$. The usual covariant derivative for charged vectors of $G$ (first term of the second row of (2.8)) is reproduced at enhancement point $\Phi_0$, for $\tilde{P} \in \tilde{G}_{\text{nc}}(\Phi_0)$ as long as we identify

$$M_J^{(L)} \leftrightarrow M_J^{a(l)} \leftrightarrow M_J^{(-l)}$$

(3.15)

Again, the second term in the second row of (2.8) is trivially obtained from $\frac{1}{8} \mathcal{D}_\mu \mathcal{H}_{IJ} \star \mathcal{D}^\mu \mathcal{H}_{IJ}$ because, besides having the same form,

$$\mathcal{D}_\mu \mathcal{H}^{(i)ij} = \partial_\mu M^{i} + ig f_{\alpha(\tilde{P}) \alpha(-\tilde{P})}^{i} A_{\mu}^{\alpha(\tilde{P})} M^{j} \alpha(-\tilde{P})$$

(3.16)

matches (2.12).

### 3.3 Scalar potential and other couplings

The scalar potential is such that it vanishes for massless states, it is $O(r_L, r)$ invariant and it reproduces the scalar potential away from the enhancing point for scalars that would
be massless at such point, as computed in [7]. It appears that the most general form is
\[ \frac{1}{4} \partial_J M^I \star \partial_K M_I (H^{JK} - \eta^{JK}) + i \frac{1}{2} \partial IM^J \star M^I + O(M^4). \] (3.17)

Finally our action (2.22) contains a last term, namely \( M^I \star F^\mu \star F_I^\mu \), which gives rise to the last term of (2.8) at the enhancement point (actually it is always present and outside of self-dual point gives rise to the adequate coupling between massive scalar and vector and massless \( U(1)_R \) vectors).

4 Breaking-enhancement of gauge symmetry along moduli space

We have shown the explicit mode expansions for some of the terms appearing in the action. Computation of other terms proceed by following similar steps.

Several interesting results like vector and scalar masses, presence of would-be Goldstone bosons, etc. can be straightforwardly read out from these expansions. We discuss some of these issues below.

4.1 Vector masses

Vector boson masses can be extracted from the fourth term in the covariant derivative-like term (3.14)
\[ \int dxdY \frac{1}{4} D_\mu M_J \star D^\mu M^J = \sum_L - \frac{1}{4} D_\mu M_\mu^{(L)} D^\mu M^{(-L)J} = \]
\[ = \cdots + \sum_L \frac{1}{4} g^2 2 [L \cdot (\bar{H} - \eta) \cdot L] A_\mu^{(L)} A_\mu^{(-L)} \] (4.1)
\[ = \cdots + \sum_L \frac{m_A^2}{2} A_\mu^{(L)} A_\mu^{(-L)}, \]
where we have used the string theory result \( m_A^2 = 2k_R^2 = L \cdot (\bar{H} - \eta) \cdot L \) for the masses of the charged vector fields. We also observe that there are no mass terms for Cartan vectors \( A_\mu^R \) so, generically, the gauge group is \( U(1)^L \times U(1)^R \). At \( \phi = \Phi_0 \) and for \( \bar{P} \in \bar{G}_{nc}(\Phi_0) \) vector bosons become massless leading to gauge symmetry enhancement.
Finally we check the normalizations. Since the kinetic term of the vectors reads:

\[ - \int dxdY \frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} = - \sum_{L} \frac{1}{4} \left( \partial_{\mu} A_{\nu}^{(L)} - \partial_{\nu} A_{\mu}^{(L)} \right) \left( \partial_{\mu} A_{\nu}^{(-L)} - \partial_{\nu} A_{\mu}^{(-L)} \right) + \ldots \] (4.2)

when adding (3.14) we find

\[ \sum_{L} \frac{1}{4} \left( \partial_{\mu} A_{\nu}^{(L)} - \partial_{\nu} A_{\mu}^{(L)} \right) \left( \partial_{\mu} A_{\nu}^{(-L)} - \partial_{\nu} A_{\mu}^{(-L)} \right) + \frac{m_{A}^{2}}{2} A_{\mu}^{(L)} A_{\mu}^{(-L)}, \] (4.3)

which is the Proca Lagrangian with signature \((-+++\ldots\)) (with a global different normalization) of a vector with mass \(m_{A}\).

4.2 Goldstone bosons

From the same scalars kinetic factors above we find the terms

\[ + \sum_{L} \frac{1}{4} D_{\mu} M_{j}^{(L)} D^{\mu} M^{(-L)j} = \ldots + \sum_{L} \frac{1}{4} 2 \bar{L} \tilde{J} \partial_{\mu} M_{j}^{(L)} A_{\mu}^{(-L)} \]

\[ = \ldots + \sum_{L} \frac{1}{2} k_{j}^{L} \partial_{\mu} M_{j}^{(L)} A_{\mu}^{(-L)}. \] (4.4)

As also discussed in [7] this coupling indicates that, for a given vector boson \(A_{\mu}^{(L)}\), there exists a combination of \(\bar{I} = 1, \ldots, r = 10 - d\), of would-be Goldstone bosons \(k_{R}^{j} \partial_{\mu} M_{j}^{(-L)}\) (this is exactly the Goldstone combination which can be read from the vertex operators in string theory [4]). In fact, at enhancement point \(\Phi_{0}\) and for \(\bar{P} \in \tilde{G}_{nc}(\Phi_{0})\) these terms are not present since \(k_{R} = 0\). However, when sliding away from \(\Phi_{0}, k_{R} \neq 0\), and these \(n_{c}\) combinations provide the longitudinal components of the \(n_{c}\) corresponding \(A_{\mu}^{(L)}\) massive vector bosons. Namely,

\[ A_{\mu}^{(L)\prime} = A_{\mu}^{(L)} + k_{j}^{L} \partial_{\mu} M_{j}^{(L)}. \] (4.5)

4.3 Scalar masses

The masses of the scalar fields can be read from the quadratic terms in the scalar potential (3.17)

\[ \frac{1}{4} \int dxdY (H^{IJ} - \eta^{IJ}) \partial_{I} M^{\ell} \partial_{J} M_{\ell} = \sum_{L} \frac{1}{4} [\bar{L} \cdot (\bar{\eta} - \eta) \cdot L] M^{(L)K}_{\ell} M^{(-L)K}_{\ell} \]

\[ = \sum_{L} \frac{1}{4} m_{M_{j}^{(L)}}^{2} M^{(L)K}_{\ell} M^{(-L)K}_{\ell}, \] (4.6)
with

\[ m_{M_j^{(L)}}^2 = 2k_R^2 = \mathbb{L} \cdot (\vec{H} - \eta) \cdot \mathbb{L}, \tag{4.7} \]
as expected from string theory. They coincide with the masses of the corresponding vector boson modes.

As in the vector case, it is still necessary to check for the relative coefficients, so we must compare the above terms with the scalar kinetic term

\[ \int dY \frac{1}{4} D_\mu M_J \ast D^\mu M^J = \sum_L \frac{1}{4} \partial_\mu M^{(L)J} \partial^\mu M^{(-L)J} + \ldots \tag{4.8} \]

Altogether we have

\[ \sum_L \frac{1}{4} \partial_\mu M^{(L)J} \partial^\mu M^{(-L)J} + \frac{1}{4} m_{M_j^{(L)}}^2 M^{(L)J} M^{(-L)J}, \tag{4.9} \]

which is the Lagrangian of a massive scalar field (with a global normalization) on the signature \((-++\ldots\)) with mass \(m_{M_j^{(L)}}\).

### 4.4 Duality and gauge invariance

Let us close this section by commenting on T-duality and gauge invariance.

We notice that, even if the different terms in the action (2.8) are written in an \(O(r_L, r)\) invariant way, by index contraction, the effect of the \(\ast\)-product on T-duality is not transparent. Consider for instance (3.1). Each term on the expansion contains a Fourier mode labeled by \(\vec{p}\), encoding momenta and winding modes (2.15), as well as an exponential term \(e^{i\bar{L} \cdot Y}\) where the exponent is explicitly \(O(r_L, r)\) invariant. On the other hand, a \(h \in O(r_L, r)\) rotation would send \(\vec{p} \rightarrow \vec{p}' = h\vec{p}\) but also, \(e^{i\bar{l}_1 l_2 \phi(\vec{p}_1)(x)\psi(\vec{p}_2)(x) \rightarrow e^{i\pi l_1 l_2 \phi(\vec{p}_1')(x)\psi(\vec{p}_2')(x)},\)

where the heterotic part is expressed in terms of 16 windings and momenta as discussed in (B.2). Therefore, the sum will be invariant if the transformed term is one of the terms we are summing over. This would actually be the case if \(\vec{p}'\) left the LMC invariant. Indeed, we can write the LMC as \(\bar{l}_1 l_2 + \bar{l}_2 l_1 = -1 + N\). The \(O(r_L, r)\) transformations are generated by exchanging momenta and winding (that when accompanied with Buscher rules transformations of moduli should become a symmetry) and by \(l_i \rightarrow l_i + N_{ij} \bar{B}\) where \(N_{ij}\) is an antisymmetric matrix of integers (that becomes a symmetry if \(B_{ij} \rightarrow B_{ij} + N_{ij}\)).
It is easy to see that both transformations maintain the LMC and therefore the sum is $O(r_L, r)$ invariant. Notice that, if we restricted to a set of GKK momenta in $\tilde{G}(\Phi_0)_{nc}$, the above transformations will take us out of this set. Namely, the $\tilde{p}'$ will not become massless at $\Phi_0$. However, $\tilde{p}' \in \tilde{G}(h\Phi_0)_{nc}$, namely, they will become massless at the transformed moduli point (note that the mass terms obtained, coinciding with string mode masses, are invariant). We will illustrate this fact in an example below. Let us stress that if any of the fields contained an $O(r_L, r)$ index, as $M^{(F)}_I$, it must appear contracted in an invariant way as it indeed happens in the action.

The action we are dealing with contains massive and massless states. At a $U(1)_r \times U(1)_{rL}$ generic points, besides the $2r + 16$ Abelian vectors and the gravity sector fields, all other vector and scalar fields will be massive. Recall that a field $\Phi(x)^{(L)}$ carries charge $(L)_I = \tilde{f}_{I-LI}$ with respect to the Abelian factor $A^{(Z)}_I$ and therefore, gauge invariance should be ensured by a covariant derivative containing $D_\mu \Phi(x)^{(L)} = \partial_\mu \Phi(x)^{(L)} - (L)_I A^{(Z)}_I \Phi(x)^{(L)} + \ldots$. In fact, it can be checked that this is indeed the case for the covariant derivative of scalars in Eq. (3.14) as well as for the derivative of the massive vectors given in Eq.(3.7).

On the other hand, at a given fixed point the Abelian gauge group is enhanced to some non Abelian gauge group $G$, and all fields in the theory, massless and massive, should organize into $G$ irreducible representations. However, this appears as a limitation of our construction since, generically, filling these multiplets requires a LMC with non zero oscillator number. Indeed, assume that $K$ with $k_R = 0$, $k^2_L = 2$ encode the charged gauge vector boson modes $A^{(K)}_I$ of the group $G$ and let us call the currents associated to these vectors as $J^{(K)}$. If we start with some massive field $\Phi(x)^{(L)}$ with mass $m^{(L)}$ and $L^2 = 2$, its OPE with the current should lead to another field $\Phi(x)^{(Q)}$ with $Q = L + K$ and the same mass, in order to be part of the same multiplet. By using (A.2) and (A.7) it is possible to show that $\tilde{N} = \tilde{N}_B + \tilde{N}_F + \tilde{E}_0 = 0$ and

$$\mathbb{P}_L K_L = -1 - N$$

or, equivalently, $\frac{1}{2} Q^2 = 1 - N$. Therefore, even if we started with an $N = 0$ case, we conclude that other values must be included. Presumably full consistency would be attained if $\delta(\frac{1}{2} Q^2, 1 - N)$ LMC is allowed. However, this would imply introducing
higher spin fields, as expected from string theory. Interestingly enough, it appears that, consistency (at tree level) can be achieved up to first mass level, with \( N = 0, 1 \) as we are indeed considering here.

We showed above (see [3.5]) that the constants \( \tilde{f}_{abc} \) are antisymmetric under index permutations for \( N = 0 \) (or even) as required to reproduce the gauge group structure constants for massless states but could also be symmetric for odd \( N \). Finally recall that several consistency conditions, are expected to be satisfied by physical states. For instance, physical massive vectors must satisfy \( \partial^\mu A_\mu^B = 0 \), etc.

In string theory such conditions arise from conformal invariance. Namely, physical field must satisfy the adequate OPE with the stress energy tensor. It was shown in Ref. [17], in the case of the bosonic string and for some specific fields, that these conditions can be understood from generalized diffeomorphism invariance. However, as mentioned above, when level matching conditions as \( \frac{1}{2}L^2 = 1 \) is considered our analysis points towards a modification of the generalized diffeomorphism algebra in order to incorporate the \( \star \)-product. Therefore consistency conditions expected from diffeomorphism invariance need further investigation in these cases.

In what follow we illustrate some of the issues discussed above in an explicit example for the torus case.

### 4.5 SU(3) example

Consider the 2-torus compactification case. For the sake of simplicity we turn off the Wilson lines and, therefore, the gauge group will be \( U(1)^2_R \times G \times SO(32) \) with \( G = U(1)^2_L \) at a generic point. The generalized momentum encoding KK and winding modes is \( \mathbf{P} = (P^I, p_1, p_2; \tilde{p}^1, \tilde{p}^2) \). We will not consider the \( P^I \) here since they do not play any role in the enhancement process when no WL are present. A given point in moduli space can be characterized by \( \Phi = (T, U) \) where \( U = U_1 + iU_2, T = T_1 + iT_2 \) are the complex and the Kahler structure of the torus defined in terms of the metric and \( B \) field as \( U_1 = \frac{g_{22}}{g_{22}}, U_2 = \frac{\sqrt{\text{det} g}}{g_{22}}, 2T_1 = B_{12}, 2T_2 = \sqrt{\text{det} g} \). At point \( \Phi_0 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}) \)

\( SU(3)_L \) enhancing occurs, whereas at \( \Phi_1 = (i, i) \) the enhanced group is \( (SU(2) \times SU(2) \times SU(2))_L \).
SU(2)_L. At the SU(3) point Left and Right momenta \([A.2]\) become

\[
\begin{align*}
    p^1_L &= \frac{1}{3} (2p_1 + p_2 + \bar{p}^1 + \bar{p}^2), \\
    p^2_L &= \frac{1}{3} (p_1 + 2p_2 - \bar{p}^1 + 2\bar{p}^2), \\
    p^1_R &= \frac{1}{3} (2p_1 + p_2 - 2\bar{p}^1 + \bar{p}^2), \\
    p^2_R &= \frac{1}{3} (p_1 + 2p_2 - \bar{p}^1 - \bar{p}^2),
\end{align*}
\]

where we have set \(\alpha' = 1\).

It is easy to check that the six weights \(\bar{P}_0 = \pm(-1, 1; 0, 1), \pm(1, 0; 1, 0), \pm(0, 1; 1, 1)\), lead to \(P^m_R = 0\) and therefore, \((p^1_L, p^2_L) = (1, 1), (1, 0), (0, 1)\), with \(p^2_L = 2\) correspond to the six massless charged vectors of \(SU(3)\). In particular \((1, 1)\) corresponds to the highest weight of the adjoint 8 representation. On the other hand, the generalized momenta \(\bar{P}_1 = \pm(1, 0, 1, 0), \pm(0, 1; 0, 1)\) provide the charged massless vectors of \(SU(2) \times SU(2)\) at \(\Phi_1\). We notice that states (vectors and scalars) associated to modes \(\pm(1, 0, 1, 0)\) are massless at both points whereas for \(\pm(0, 1; 0, 1)\) we have \(p_L = \pm(\frac{2}{3}, \frac{4}{3})\), \(p_R = \pm(\frac{2}{3}, \pm \frac{1}{3})\) that satisfy \(p_L^2 - p_R^2 = 2\) and correspond to massive states with \(\alpha' m_A^2 = \frac{4}{3}\). Interestingly enough, \(p_L = (\frac{2}{3}, \frac{4}{3}), (-\frac{1}{3}, \frac{1}{3})\) is the highest (lowest) weight of the symmetric 6 (\(\bar{6}\)) representation. Indeed, it is easy to see that, when combining with \(\bar{P}_0\) vectors we obtain the vectors \(\bar{L}_i \equiv (\frac{2}{3}, \frac{4}{3}), (\frac{3}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{1}{3})\), \((-\frac{2}{3}, \frac{2}{3}), (-\frac{1}{3}, -\frac{2}{3}), (-\frac{4}{3}, -\frac{2}{3})\), which fill the \(\bar{6}\) representation (and similarly \(\bar{6}\)). In order to have the same masses, fields with \(i = 1, 3, 6\) require \(N = 0\) whereas the other three \(i = 2, 4, 5\) have \(N = 1\). In terms of momenta and winding they correspond to \(\bar{L}_i \equiv (0, 1; 0, 1), (1, 0; 0, 0), (0, 0; -1, 0), (2, 0; 1, 0), (1, -1; -1, -1), (0, -1; -2, -1)\). These states are indeed present in our construction. As an illustration consider scalar states. The \(N = 0\) states correspond to modes in \(M_j(x, \bar{Y})\) expansion whereas states with \(N = 1\) are contained in \(M_{jI}(x, Y)\). It is enlightening to look at the “covariant” derivative (3.14) of a mode \(i = 1, 3, 6\). From (3.14) we will have

\[
D_\mu M^i_J = \partial_\mu M^i_J + \sum_{j=1,3,6} \bar{f}_{ij\alpha} M_j^a A^a_\mu + M_j \bar{f}_{i-\bar{\alpha}m} A^m_\mu
\]

where \(a = 1, 6\) spans charged vector indices and \(m = 1, 2\) label the Cartan generators.

\[\text{Notice that } (p^1_L, p^2_L) \text{ are the coordinates of the weight vectors of the representation in the simple root lattice, namely } \Lambda = p^i_L a_i \text{ with } a_i \text{ the simple roots.}\]
Generalized diffeomorphism invariance (see [17]) or equivalently, conformal invariance requires,
\[ L^I M_{IJ}^{(1)} = 0. \] \( (4.14) \)

In our case we are interested in \( L_i \) with \( i = 2, 4, 5 \). Moreover, we can express the scalar field modes in the simple root basis as \( M_{k,j} = (\lambda_{1,j}^{1} + \lambda_{2,j}^{2}\alpha^2)_k \) where \( (\alpha^i)_k \) are the components of the root \( \alpha^i \). We find that a solution of (4.14) leads to \( \lambda_{1(2)k, j} = \lambda_{1(2)k}^1 \lambda_J \)
with \( (\lambda_1, \lambda_2)_k = (1, 0)_k, (1, 1)_k, (0, 1)_k \) as from string theory computations.

Interestingly enough, when these results are introduced in (4.14), up to normalizations, the found values correspond to the elements of the group generators matrices \( T_a \) in the symmetric representation, as required by gauge invariance.

Even if we have mainly concentrated on the charged vectors sector, it is also worth noticing that the fields originated in \( D = 10 \) gravity sector, the first two rows in Eq. (2.5), will contain massive field modes that must also organize in multiplets of \( SU(3) \). This is the case for massive graviton and Kalb-Ramond \( g_{\mu\nu}^L, B_{\mu\nu}^L \) modes as well as for vectors and scalars. These modes must satisfy \( L^2 = p^2_l - p^2_R = 0 \) level matching condition. Actually, it is easy to see that the lowest massive level \( m^2 = \frac{4}{3} \) with \( N = 1 \), contains \( (p^1_L, p^2_L) = \pm (\frac{1}{3}, \frac{2}{3}), \pm (\frac{1}{3}, -\frac{1}{3}), \pm (\frac{2}{3}, -\frac{1}{3}) \) weights with plus (minus) sign corresponding to the fundamental (anti-fundamental) \( 3(\bar{3}) \) representation. Also, the \( U(1)_R \) charge is \( \pm (\frac{1}{3}, \frac{2}{3}) \). Therefore all fields organize into \( 3(\frac{1}{3}, \frac{2}{3}) \) (or conjugate) representations of \( U(1)_R^2 \times SU(3) \) at the enhancement point. Again, going beyond the first mass level would require the introduction of \( N > 1 \) and thus, higher spin fields. Also notice that, in order to ensure gauge invariance, a well defined gauge covariant derivative of massive modes would require the introduction of extra terms containing charged vectors. We have not analyzed such terms in detail but, similarly to (2.24) such gauge covariant derivative should look as \( D_\mu = \partial_\mu - A_\mu^I \partial_I + A_\mu^* + \ldots \)

5 Summary and Outlook

A striking and distinctive feature of string compactifications is that, at certain values of the compactification background -namely a point in moduli space- compact momenta
and winding modes, that we encoded here in a generalized vector $\vec{P}$, can combine to generate new (let’s say $n_c$) massless vector bosons leading to an enhancement of the gauge symmetry group. For other values of moduli and, generically, other values of winding and momenta a different enhancement can occur. In the notation presented above (2.19) we would say that, for a given number of compact dimensions $r$, several sets $\mathcal{G}(\Phi_i)_{n_i}$ of generalized momenta $\vec{P}$ could exist. These lead to enhancement at moduli point $\phi_i$ where $n_i^c$ vector bosons and scalars become massless. The structure gets richer for lower space-time dimensions. By restricting to fields associated to $\vec{P} \in \mathcal{G}(\Phi_i)_{n_i^c}$ it is straightforward to write down the effective low energy theory at each $\phi_i$. Slightly sliding away from it, can be interpreted as a Higgs mechanism. Actually, when moving along moduli space some (or all) of these fields become massive whereas other fields become lighter at a different point. Therefore, a moduli dependent description able to account for these different enhancements implies handling an infinite number of fields. In this work we have shown that this description, is indeed possible and that it can be encoded in a moduli dependent effective action.

The proposed action, written in $d$ space time dimensions contains a, generically, infinite number of fields labeled by allowed momenta and winding modes. In principle, it could have been obtained by carefully looking at string 3-point amplitudes of vertex operators associated with these modes. We have shown that these infinite fields in $d$ dimensions can be understood as modes of a GKK expansion in the internal double torus and heterotic coordinates $\mathcal{Y} \equiv (y^I, y^m_L, y^m_R)$, providing an uplifting to higher dimensions.

In this sense the action can be seen as a Kaluza-Klein inspired rewriting of a double field theory (see for instance Ref.[20]), where coordinates are split into space-time coordinates (that could be formally doubled) and internal double coordinates. However, once compact coordinates come into play we noticed that a $\star$-product that introduces a non-commutativity in the target compact space is called for.

It is this non-commutativity that leads to the adequate factors to reproduce the structure constants. As we have shown in an example, it also appears to have the right features to reproduce the generator matrix elements in higher order representations where massive states live, as required by gauge invariance. Let us stress that the $\star$-product is not needed,
at third order in fluctuations, if fields satisfy $P^2 = 0$ level matching condition. This is why it did not manifest in original DFT constructions \[11\].

It would be interesting to trace the origin of this product for the heterotic string case\[21\] back. In the context of bosonic string it was shown in \[8\] to be associated to non commutativity of string coordinate zero modes.

An interesting result of the construction is that, close to a given enhancement point $\Phi_0$, by keeping just the $n_c$ slightly massive fields, the Higgs mechanism can be cast in terms of $\tilde{f}$ moduli dependent “structure like constants” that become the enhanced group structure constants at $\Phi_0$. This description provides a field theory stringy version of the gauge symmetry breaking-enhancement mechanism. In fact, it was already addressed in the context of DFT in \[5, 7\] where it was shown that constants $\tilde{f}(\Phi)$ can be interpreted as DFT Scherk -Scherz\[12, 18, 19\] compactifications generalized fluxes. These fluxes can be read from the DFT generalized diffeomorphism algebra. Actually, it is worth noticing that these fluxes were explicitly constructed from a generalized frame only in the circle case where a $SU(2)$ enhancing at the self dual radio $R = \tilde{R} = \sqrt{\alpha'}$ occurs \[4, 5, 6, 7\].

Difficulties in going beyond this case were mentioned in \[5, 6\]. Interestingly enough, the $SU(2)$ case is the only situation where the $\star$-product is not needed (essentially due to the absence of a $B$ field). This suggests that this non commutative product could provide a solution. In Ref.\[6\] a connection among these difficulties and vertex operators cocycle factors was suggested. The $\star$ appears as a manifestation of the cocycle factors in the DFT context. Actually, in higher dimensional groups the problem arises when the generalized Lie algebra of three charged fields is considered, implying $\frac{1}{2}L^2 = 1$ level matching condition, which is just the situation where the $\star$-product phase is relevant. As mentioned above, a modified version of generalized diffeomorphisms is called for to handle these cases. The detailed construction is left for future investigation.

In our construction we started by proposing mode expansions restricted by the level matching constraint $\frac{1}{2}L^2 = 1$ (corresponding to $N = 0$ oscillators) necessary to contain massless vectors at the enhancement point. Even if it effectively interpolates among different enhancement points, we stressed that new ingredients must be incorporated. In particular, we noticed that, at first mass level, for massive states to organize into multiplets...
of the enhanced group $G$, $N = 1$ oscillator number is also required. Since we had already included the $N = 1$ case, to tackle the gravity sector, we showed in an example that indeed massive vector and scalar states nicely fill $G$ multiplets for first massive level. This happens to be the case also for gravity sector massive modes. However, for higher masses, other oscillator numbers are expected (this was noticed in [17]). Namely, if we consider next to first massive level, in order to complete a $G$ multiplet, a level matching condition with $N > 1$ is required. We see that a simple gauge symmetry consistency check points towards the necessity of including massive higher spin fields and higher derivative terms in the action (see [22], [23] for a discussion from another perspective), as is in fact expected from string theory. Let us stress that gauge invariance underscores the limitations of the construction but at the same time it is a guide for consistent extensions. Indeed, gauge invariance provides a tool to systematically include higher spin modes and $\alpha'$ corrections by looking for consistency all the way from the very first massive levels up to the highest ones.

Throughout our construction we have made intensive use of DFT tools. In particular, before mode expanding, all fields are expressed in terms of higher dimensional coordinates. However, a fully higher dimensional version is still lacking in the sense that fields are written in terms of space time $d$ dimensional indices. Formally it appears rather straightforward. On the one hand, the new fields we are introducing here associated to $N = 0$, can be cast in terms of a $D$ dimensional “charged vector” field $A^M(x, Y) \equiv (A^\mu(x, Y), M^I(x, Y))$ and, on the other hand, the sectors originated from the generalized metric in 10-dimensions were already addressed in [17] (up to a third order expansion) in terms of a generalized metric. However, it appears that the latter must be modified by the presence of the new fields, as required by gauge invariance. Moreover, the form of generalized diffeomorphism and $\star$-product should be understood.

Finally let us mention that even if we have restricted our analysis to the bosonic sector of the heterotic string, the inclusion of fermions could also be addressed by invoking supersymmetry, generalizing the discussion in [7] (see also [24], [25]) where “will-be massless fermions at fixed point”, namely, for modes in $\tilde{G}_{n_c}$, were considered. From a duality invariant field theory point of view, an uplift including fermions would require an analysis
from an Extended Field Theory (EFT)\textsuperscript{26} in order to include magnetic modes. The recent work in in Ref.\textsuperscript{27} might be helpful in this direction.
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A Some Heterotic string basics

We summarize here some string theory ingredients (that can be found in string books) needed in the body of the article. We mainly concentrate in the $SO(32)$ string.

For a heterotic string compactified to $d$ space-time dimensions, Left and Right momenta are encoded in momentum

$$\mathbb{L} = (l_L, l_R), \quad (A.1)$$

defined on a self-dual lattice $\Gamma_{26-d,10-d}$ of signature $(26-d,10-d)$. By writing

$$l_L^I = (K_L^I, k_L^m) \quad \text{with} \quad I = 1, \ldots, 16 \quad \text{and} \quad m = 1, \ldots 10 \quad - \quad d = r,$$

the moduli dependent momenta, read

$$K_L^I = P^I + RA_n^I \tilde{p}^n \quad (A.2)$$

$$k_L^m = \frac{\sqrt{\alpha'}}{2} \left[ \frac{\tilde{p}^m}{R} + 2 g^{mn} \left( \frac{p_n}{R} - \frac{1}{2} B_{nr} \tilde{p}^r \right) - P^I A^m_I - \frac{R}{2} A^m_I A^I_n \tilde{p}^n \right],$$

where $g_{mn}, B_{mn}$ are internal metric and antisymmetric tensor components, $A_m$ are Wilson lines and $p_n$ and $\tilde{p}^n$ are integers corresponding to KK momenta and windings, respectively. $P^I$ are $Spin(32)$ weight components. More schematically, by defining the vector $\tilde{\mathbb{P}} = (P^I, p_n, \tilde{p}^n)$ and $\mathbb{L} = (K_L^I, k_L^m, k_R^m)$ we can write

$$\mathbb{L} = \mathcal{R}(\Phi)\tilde{\mathbb{P}}, \quad (A.3)$$

where

$$\mathcal{R} = \begin{pmatrix}
1 & 0 & RA \\
-\frac{\sqrt{2}}{A} & \frac{\sqrt{2}}{R} g^{-1} & \frac{\sqrt{2}}{R} (1 - g^{-1}B - \frac{1}{2} A.A' ) \\
-\frac{\sqrt{2}}{A} & \frac{\sqrt{2}}{R} g^{-1} & \frac{\sqrt{2}}{R} (1 - g^{-1}B - \frac{1}{2} A.A' )
\end{pmatrix}, \quad (A.4)$$

26
performs the change of basis (we have set $\alpha' = 1$ here). It also rotates the coordinates $\tilde{Y} = (y^I, y_m, \tilde{y}^m)$ to $Y = (Y^I, y^m_L, y^m_R)$. In particular it transforms the $O(16 + r, r)$ metric $\eta_C$ defined in (2.9) to

$$\eta^{IJ} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix}. \quad (A.5)$$

Notice that $\mathcal{R}(\Phi)$ encodes the dependence on moduli.

The mass formulas for string states are (we mainly use the notation in [3])

$$\frac{\alpha'}{2}m^2_L = \frac{1}{2}l^2_L + (N - 1) = \frac{1}{2}K^2_L + \frac{1}{2}k^2_L + (N - 1)$$

$$\frac{\alpha'}{2}m^2_R = \frac{1}{2}k^2_R + \bar{N}, \quad (A.6)$$

where $N = N_B, \bar{N} = \bar{N}_B + \bar{N}_F + \bar{E}_0$ where $N_B, \bar{N}_B$ are the bosonic L and R-oscillator numbers, $\bar{N}_F$ is the R fermion oscillator number and $\bar{E}_0 = -\frac{1}{2}(0)$ for NS (R) sector. The level matching condition is $\frac{1}{2}m^2_L - \frac{1}{2}m^2_R = 0$ or, in terms of above notation

$$\frac{1}{2}l^2_L = \frac{1}{2}l^2_R - \frac{1}{2}k^2_R = \frac{1}{2}k^2_L + \frac{1}{2}K^2 - \frac{1}{2}k^2_R = \tilde{p}.p + \frac{1}{2}P^2 = (1 - N + \bar{N}). \quad (A.7)$$

In our discussion we restrict to $\bar{N}_B = 0, N_F = \frac{1}{2}$, namely $\bar{N} = 0$. The “charged vectors” sector, corresponds to $N = 0$, i.e. $L^2 = 2$. Massless vectors are a particular case with $\frac{1}{2}l^2_L = 1, k_r = 0$.

As is well known, there are $10 - d + 16$ Left gauge bosons corresponding to 16 Cartan generators $\partial_\gamma Y^I \tilde{\psi}^\mu$ of the original gauge algebra as well as $10 - d$ KK Left gauge bosons coming from a Left combination of the metric and antisymmetric field $\partial_\gamma Y^m \tilde{\psi}^\mu$. The $10 - d$ Right combinations $\partial_\gamma X^\mu \tilde{\psi}^m$ with $m = 1, \ldots, 10 - d$ generate the Right abelian group. These states have $k_R = 0$ and $l_L = 0$, with vanishing winding and KK momenta.

B The $\star$-product

A $\star$-product, was proposed in [8] in order to incorporate, in a “Double Field theory” description, information about bosonic string vertex cocycle factors. If $\hat{P} \equiv (p_m, \tilde{p}^m)$ is an

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\footnote{The normalizations are chosen such that, $\frac{p^2}{\sqrt{\alpha}}$, for an enhancement point, correspond to the coordinates of the weight vectors of a representation in the lattice span by simple roots $\alpha_m$ with $\alpha^2_m = 2$.}
\(O(n, n)\) vector encoding information about winding numbers \(\tilde{p}_m\) and Kaluza-Klein (KK) compact momenta \(p_m\), then for two fields depending on the compact double coordinate \(\tilde{\mathcal{Y}} \equiv (y^m, \tilde{y}_m)\) their \(\star\)-product proposed reads

\[
(\phi \star \psi)(x, \mathcal{Y}) = \sum_{P_1, P_2} e^{i\pi \tilde{p}_1 P_2} \cdot \phi^{(P_1)}(x)\psi^{(P_2)}(x)e^{i(P_1 + P_2)\cdot \mathcal{Y}}.
\]  

(B.1)

The product is associative.

Interestingly enough, the proposal can be understood as a non commutativity of the string compact coordinates zero modes (see also [9]).

Here we just extend this product to account for heterotic string degrees of freedom in an \(O(r_L, r)\) context. Actually, since it is possible to interpret the heterotic string momenta \(P^I\) as originated from a 16 dimensional torus with some winding and momenta \((\tilde{p}^I, p_I)\) (with \(I = 1, \ldots, 16\)) we can generalize above expression by including a phase that contains not only the compactified winding and momenta but also the gauge ones. More concretely, \(P^I_L, P^I_R\) can be computed using similar expressions as (A.2) above (no Wilson lines) but by imposing \(P^I_R = 0\). Then, \(P^I \equiv P^I_L\) root vectors are obtained with, \(G_{IJ}\) the Cartan Weyl metric of \(Spin(32)\) and \(B_{IJ} = G_{IJ} = -B_{IJ}\) for \(I > J\). It is possible to check then that for two vectors \(P_1, P_2\) we have \(\tilde{p}_1^I p_{2I} = \frac{1}{2} P^I_L E_{IJ} P_{2J}\) where \(E_{IJ} = G_{IJ} + B_{IJ}\). Therefore, for the heterotic string we would have (see (A.1) above) \(L = (l_L, k_R) \equiv (K^I_L, k^m_L, k^m_R)\) and \(\mathcal{Y} = (y_l, y_R) \equiv (y^I, y^m_L, y^m_R)\) and using that

\[
\tilde{l}_1 l_2 = \tilde{p}_1^m p_{2m} + \tilde{p}_1^I p_{2I} = \tilde{p}_1 \cdot p_2 + \frac{1}{2} P_1 E P_2,
\]  

(B.2)

we recover the expression in (B.1). Notice that the phase \(\epsilon(P_1, P_2) = e^{i\pi \frac{1}{2} P_1 E P_2}\) introduces a notion of ordering for \(Spin(32)\) roots. For two adjacent roots in the corresponding Dynkin diagram \(E_{IJ} = -1\) for \(I > J\) and vanishes otherwise. This provides an adequate representation of structure constants for \(Spin(32)\) charged operator algebra. Namely

\[
[E_{P_1}, E_{P_2}] = \epsilon(P_1, P_2) E_{P_3}\]  

(see e.g. the construction in [1]).

The same reasoning holds for the full enhanced group. At the enhancement point \(\Phi_0\) with \(\tilde{\Phi} \in \tilde{G}_{\Phi_0}(\phi_0), k_r = 0, l_i\) become the roots of the gauge group and from equations (A.2) above we can express windings and momenta in terms of the \(l_L\) such that

\[
\tilde{l}_1 l_2 = \tilde{p}_1 \cdot p_2 + \frac{1}{2} P_1 E P_2 = l_1 L \mathcal{E} l_2 L,
\]  

(B.3)
with $[16]$

$$E = \begin{pmatrix} (B + g + \frac{\alpha' g}{2} A_I A^I)_{nm} & g_{nm} A^n_{IJ} \\ 0 & (G + B)_{IJ} \end{pmatrix}. \quad (B.4)$$

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