EXACT SCALING IN THE EXPANSION-MODIFICATION SYSTEM

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Abstract. This work is devoted to the study of the scaling, and the consequent power-law behavior, of the correlation function in a mutation-replication model known as the expansion-modification system. The latter is a biology inspired random substitution model for the genome evolution, which is defined on a binary alphabet and depends on a parameter interpreted as a mutation probability. We prove that the time-evolution of this system is such that any initial measure converges towards a unique stationary one exhibiting decay of correlations not slower than a power-law. We then prove, for a significant range of mutation probabilities, that the decay of correlations indeed follows a power-law with scaling exponent smoothly depending on the mutation probability. Finally we put forward an argument which allows us to give a closed expression for the corresponding scaling exponent for all the values of the mutation probability. Such a scaling exponent turns out to be a piecewise smooth function of the parameter.

1. Introduction.

1.1. In recent years, several models have been introduced (such as n-step Markov chains or hidden Markov chains, among others [8, 9, 16, 17]) to describe the evolution of nucleotide sequences as well as the patterns and correlations occurring in the genome. In this paper we are concerned with one of those models, proposed by W. Li [5], which consists of a sequence (or chain) of symbols that evolve according to a given discrete-time stochastic dynamics. Such a dynamics captures the essential processes which are assumed to be responsible of the genome evolution: the random expansion and modification of symbols (hence the name of expansion-modification system). Originally introduced as a simple model exhibiting some spatial scaling properties, a behavior ubiquitous in natural phenomena [5], it was subsequently used to understand the scaling properties and the long-range correlations found in real DNA sequences [2, 6, 7, 8, 13]. Recently, the expansion-modification system has also been used to investigate the universality of the rank-ordering distributions [1, 11].

1.2. From the mathematical point of view, the expansion-modification system belongs to the class of random substitution dynamical systems, which attracted some attention in recent years for their possible applications in genome evolution studies (see [4] and references therein). A related class of stochastic processes, inspired by randomly generated grammars, were formalized and studied by Toom and coworkers (see [15]). Previously, Godrèche and Luck used random substitutions to study the robustness of quasiperiodic structures [8], in particular structures associated...
to random perturbations of Fibonacci sequences and Penrose tilings. They observed that the Fourier spectrum of the structures thus obtained are of mixed type: they contain both singular and continuous parts. Fourier spectra of mixed type appear in structures corresponding to random perturbation of quasicrystals. For instance, Zaks [18] observed mixed spectrum in structures generated by random-ized Thue-Morse sequences. In our case, the structure generated by the expansion-modification system cannot be seen as a random perturbation of a quasicrystal, as the corresponding Fourier spectrum turns out to be continuous. What the expansion-modification system shares with those randomly perturbed quasicrystals is the scaling property and the consequent power law behavior of the correlations and the Fourier spectrum. For those randomly perturbed quasicrystals, the scaling of the perturbed structure can be very easily deduced from the scaling already present in the underlying quasicrystal, by means of obvious recurrence relations derived from the inflation rules. As we will show, the scaling in the expansion-modification system derives from recurrence relations implied by the underlying dynamics, and contrary to the quasicrystal case, these recurrence relations grow in complexity in such a way that its treatment demands the implementation of non-trivial techniques. The rigorous study of this scaling behavior and the consequent power laws is the main contribution of this work.

1.3. The expansion-modification system can be described as follows. Consider the
random substitution

\[
\begin{align*}
0 & \mapsto \begin{cases} 1 \text{ with probability } p, \\ 00 \text{ with probability } 1 - p, \end{cases} \\
1 & \mapsto \begin{cases} 0 \text{ with probability } p, \\ 11 \text{ with probability } 1 - p, \end{cases}
\end{align*}
\]

in the binary set \{0, 1\}, and extend it coordinate-wise to the set \{0, 1\}^+ of finite binary strings. Starting at time zero with a seed in \(x^0 \in \{0, 1\}^+\), and iterating the above substitution, we obtain a sequence

\[
x^0 \mapsto x^1 \mapsto \cdots \mapsto x^n \mapsto \cdots
\]

of finite strings of non-decreasing length. Since the applied substitution is a random map, the sequence we obtain by successive iterations is a random sequence which is nevertheless supposed to converge, in a certain statistical sense, to a random string \(x^\infty\). It is easy to see that the probability of having a finite string after infinitely many iterations is zero, therefore it is more convenient to study the evolution of infinite strings under the infinite extension of the above substitution. This is precisely the point of view we will follow throughout this work.

1.4. The paper is organized as follows. In Section 2 we set up the mathematical framework where the expansion-modification system is defined. In Section 3 we state our main results, which we proof in Section 5. In Section 4 we compute a closed expression for the scaling exponent which presumably holds in the whole range of mutation probabilities. We finish the paper with some final remarks and comments.
2. The Expansion-Modification Dynamics.

2.1. In order to review the expansion-modification system, let us start fixing the relevant notation and terminology. Let \( X = \{0, 1\}^\mathbb{N} \), which we endow with the \( \sigma \)-algebra generated by the cylinder sets. Elements of \( X \) are called configurations, and will be denoted by boldface characters like \( \mathbf{x} = x_0x_1 \cdots \), with \( x_i \in \{0, 1\} \). Finite sequences of symbols, also called words, will be also denoted by boldfaced letters while their size will be denoted by \(| \cdot |\), i.e., for \( \mathbf{v} \in \{0, 1\}^k \) we have \(|\mathbf{v}| = k\). A word \( \mathbf{v} \in \{0, 1\}^k \) occurs as prefix of \( \mathbf{x} \in X \), which we denote by \( \mathbf{v} \sqsubset \mathbf{x} \), if \( \mathbf{v} = x_0x_1 \cdots x_{k-1} \). We will also use this notation when \( \mathbf{x} \in X \) is replaced by a finite word. Given a configuration \( \mathbf{x} = x_0x_1 \cdots \in X \), and integers \( 0 \leq i < j \), we denote with \( \mathbf{x}_j^i \) the word \( x_i x_{i+1} \cdots x_j \). Product of words will be understood as concatenation: given two words \( \mathbf{v} \in \{0, 1\}^k \) and \( \mathbf{w} \in \{0, 1\}^l \), we let \( \mathbf{vw} \) denote the word \( \mathbf{u} \) of size \( k + l \) satisfying \( \mathbf{u}_{k-1}^l = \mathbf{v} \) and \( \mathbf{u}_{k+l+1}^l = \mathbf{w} \). Consider \( S = \{e, m\}^\mathbb{N} \), where the symbols \( e \) and \( m \) stand for expansion and modification respectively. The space \( S \), which we will refer to as the space of substitutions, is endowed with the \( \sigma \)-algebra generated by the cylinder sets as well. We will use the same convention to denote the elements of \( S \), words and concatenation of words, as for the symbolic space \( X \).

2.2. Let us now define the local substitutions \( e, m : \{0, 1\} \to \{0, 1\}^+ := \bigcup_{n=1}^\infty \{0, 1\}^n \), which are given by

\[
e(x) = xx, \\
m(x) = 1 - x.
\]

A configuration \( \mathbf{s} \in S \) of local substitutions defines the global substitution \( \mathbf{s} : X \to X \) given by

\[
\mathbf{s}(\mathbf{x}) = \prod_{i \in \mathbb{N}_0} s_i(x_i).
\]

Here \( \prod \) stands for concatenation of words. Notice that \( \mathbf{s} \) replaces the \( i \)-th symbol of \( \mathbf{x} \) according to the \( i \)-th local substitution, i.e., if \( s_i = e \) then \( x_i \) is expanded, otherwise \( x_i \) is modified.

2.3. The expansion-modification dynamics is a random dynamical system whose orbits depend on an initial condition and a choice of global substitutions to be applied to that initial condition. To be more precise, an initial condition \( \mathbf{x} \in X \) and a sequence \( \mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \ldots \) of configurations in \( S \), define the orbit \( \mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots \) in \( X \) where \( \mathbf{x}^0 := \mathbf{x} \) and \( \mathbf{x}^{t+1} = s^t(\mathbf{x}^t) \) for each \( t > 0 \). At each time step \( t \), the global substitution \( s^t \) is randomly chosen according to the Bernoulli measure \( \mu_p \) such that \( \mu_p[\mathbf{m}] = p \) and \( \mu_p[\mathbf{e}] = 1 - p \). The parameter \( p \in (0, 1) \) is the mutation probability.

In terms of distributions, the expansion-modification system can be defined as follows. If \( \mu^t \) is the measure according to which the time-\( t \) configurations are distributed, then the distribution \( \mu^{t+1} \) of time-\( (t + 1) \) configurations is completely determined by \( \nu_p \) and \( \mu^t \) according to the following expression:

\[
\mu^{t+1}\left\{ (\mathbf{x}^{t+1})_0^\ell = \mathbf{a} \right\} = \sum_{\mathbf{c} \in \{e, m\}^{\ell+1}} \sum_{\mathbf{b} \in \{0, 1\}^{\ell+1}} \mu^t\left\{ (\mathbf{x}^t)_0^\ell = \mathbf{b} \right\} \nu_p\left\{ (\mathbf{s}^t)_0^\ell = \mathbf{c} \right\},
\]
for each \( \ell \in \mathbb{N}_0 \) and \( a \in \{0, 1\}^{\ell+1} \). As mentioned before, \( a \preceq b \) means that the word \( a \) occurs as a suffix of the word \( b \). Hence, the evolution of the \((\ell+1)\)-marginal is nothing but a Markov chain. Indeed, considering the \((\ell+1)\)-marginal \( \mu^t \) of a measure \( \mu^0 \) as a probability vector of dimension \( 2^{\ell+1} \), the \((\ell+1)\)-marginal \( \mu^t \) of the time-\( t \) distribution is given by matrix product
\[
\mu^t = \mu^0 M^t,
\]
where \( M^t : \{0, 1\}^{\ell+1} \times \{0, 1\}^{\ell+1} \rightarrow [0, 1] \) is the \( 2^{\ell+1} \times 2^{\ell+1} \)-stochastic matrix given by
\[
M^t(a, b) = \sum_{c \in \{e, m\}^{\ell+1}} \nu_p[c].
\]

3. Results

3.1. Our first result states the existence and uniqueness of the stationary distribution, which turns out to be a global attractor for the expansion-modification dynamics.

**Theorem 1 (Existence and Uniqueness).** For each \( p \in (0, 1) \) there exists a unique measure \( \mu_p \) on \( X \) which is invariant under the expansion-modification dynamics. Furthermore, starting from any measure \( \mu^0 \) determining the distribution of the initial conditions, the measure \( \mu^t \), corresponding to the distribution at time \( t \), converges in the \(*\)-weak sense to \( \mu_p \).

It is not hard to see that uniqueness does not hold for \( p \in \{0, 1\} \). For \( p = 0 \), when only expansion is possible, each convex combinations of the Dirac measures at all-zeros and all-ones, is an admissible invariant distribution. On the other hand, in the case \( p = 1 \), the dynamic of each initial distributions enters a period-two cycle, excepting for the measures that are flip-invariant (0 ↔ 1). In both cases, the asymptotic regime depends on the initial distribution.

In [15] Toom and coworkers prove the existence of invariant measures for substitution operators similar to, but not including, the expansion-modification dynamics. In their case, due to random deletion of words, the dynamics cannot be reduced to the action of stochastic matrices over finite length marginals.

3.2. The main result of this paper establishes the power-law decay of correlations exhibited by the unique stationary measure \( \mu_p \). Before stating this theorem, let us remind the main definitions.

The **two-sites correlation function**, \( C_p : \mathbb{N}_0 \rightarrow \mathbb{R} \), is given by
\[
C_p(n) := \int_X x_n x_0 d\mu_p(x) - \left( \int_X x_0 d\mu_p(x) \right) \left( \int_X x_n d\mu_p(x) \right),
\]
where \( x_n \) denotes, as usual, the projection of \( x \) on the \( n \)-th coordinate. Following the usual practice, we say that \( \mu_p \) has decay of correlations if \( \lim_{n \rightarrow \infty} |C_p(n)| = 0 \).

Let \( p^* := \sup\{p \in (0, 1) : C_p(n) > 0 \ \forall n \in \mathbb{N} \} \), and for each \( p \in (0, 1) \), let
\[
\beta_p := \frac{\log(2-p) - \log(1-2p) - \log(2-3p)}{\log(2-p)}.
\]
Theorem 2 (Power-law Decay of Correlations). For each \( p \in (0, p^*) \) there exist \( n_0 \in \mathbb{N} \) and constants \( A_p \leq 1 \leq B_p \) such that
\[
A_p n^{-\beta_p} \leq C_p(n) \leq B_p n^{-\beta_p}
\]
for all \( n \geq n_0 \).

3.3. The invariance of the expansion-modification dynamics under coordinate-wise negation, or flip invariance, \( 0 \leftrightarrow 1 \), implies that \( \mu_p\{x_0 = 1\} = \mu_p\{x_0 = 0\} = 1/2 \) for all \( n \in \mathbb{N}_0 \). Therefore
\[
C_p(n) := \int x_0 x_n d\mu_p(x) - 1/4 = \mu_p\{x_0 = x_n = 1\} - 1/4.
\]
Now, flip invariance also implies \( \mu_p\{x_0 = x_n = 1\} = \mu_p\{x_0 = x_n = 0\} \) and \( \mu_p\{x_0 = 0 \neq x_n = 1\} = \mu_p\{x_0 = 1 \neq x_n = 0\} \), for each \( n \in \mathbb{N}_0 \). With this, we obtain a very simple expression for the two-sites correlation
\[
C_p(n) := \frac{1}{2}(\mu_p\{x_0 = x_n\} - 1/2) = \frac{1}{4}(\mu_p\{x_0 = x_n\} - \mu_p\{x_0 \neq x_n\}).
\]
This expression and the invariance under the expansion-modification dynamics will allow to deduce a recurrence formula for the two-sites correlation function.

Let us denote by \( \ell(s_0^k) \) the length of the words obtained by applying the substitution \( s_0^k \), and for each \( k, n \in \mathbb{N} \) let
\[
\nu_p(k, n) := \nu_p(\ell(s_0^{k-2}) = n - 1) = \binom{k-1}{n-k}(1-p)^{n-k}p^{2k-n-1}.
\]
Since \( \mu_p \) is left invariant under the expansion-modification dynamics, and since \( \nu_p \) is a Bernoulli measure, then
\[
\mu_p\{x_0 = x_n\} = \sum_{k=\lceil n/2 \rceil}^n \mu_p\{x_0 = x_k\} (p^2 \nu_p(k, n) + (1-p)^2(\nu_p(k, n-1) + \nu_p(k, n-2))
\]
\[
+ \sum_{k=\lceil n/2 \rceil}^n \mu_p\{x_0 \neq x_k\} (2p(1-p)\nu_p(k, n-1) + p(1-p)\nu_p(k, n)),
\]
and similarly for \( \mu_p\{x_0 \neq x_n\} \). From the previous equation and its analogous for \( \mu_p\{x_0 \neq x_n\} \), it readily follows that
\[
C_p(n) = \sum_{k=\lceil n/2 \rceil}^n C_p(k) (f(p) \nu_p(k, n) + g(p) \nu_p(k, n-1) + h(p) \nu_p(k, n-2)),
\]
with \( f(p) := p(2p-1), g(p) := (1-p)(1-3p), \) and \( h(p) := (1-p)^2 \). This relation can be rewritten as
\[
C_p(n) = \frac{1}{1 + p^n(1-2p)} \sum_{k=\lceil n/2 \rceil}^{n-1} C_p(k) \mathcal{W}_p(k, n),
\]
with \( \mathcal{W}_p(k, n) \) defined as
\[
\mathcal{W}_p(k, n) := f(p) \nu_p(k, n) + g(p) \nu_p(k, n-1) + h(p) \nu_p(k, n-2).
\]
We will make extensive use of the previous relations in the proof of Theorem 2.
4. The Scaling Exponent

4.1. According to Theorem 2, the correlation function follows an asymptotic scaling law for mutation probabilities in the range $0 < p < p^*$. It also establishes an expression for the scaling exponent $\beta_p$. The following argument leads us to conjecture that the scaling property, and the expression for the corresponding scaling exponent, extend to the whole interval $0 < p < 1$.

Let us consider the recursive relation

$$ C_p(n) = \sum_{k=\lfloor n/2 \rfloor}^{n} C_p(k) \left( f(p) \nu_p(k, n) + g(p) \nu_p(k, n - 1) + h(p) \nu_p(k, n - 2) \right), $$

deducing above. The distribution $k \mapsto \nu_p(k, n)$, which is unimodal with maximum at $k \approx n/(2 - p)$, steepens around this maximum as $n$ goes to infinity in such a way that

$$ \sum_{k=\lfloor n/2 \rfloor}^{n} \nu_p(k, n) \approx \sum_{\ell(n) \leq k \leq u(n)} \nu_p(k, n), $$

where $\ell(n) < n/(2 - p) < u(n)$, which we define below in Subsection 5.2, are such that both $(2 - p)\ell(n)/n$ and $(2 - p)u(n)/n$ tend to 1 as $n$ goes to infinity. Hence, assuming a slow variation in $k \mapsto C_p(k)$, we have

$$ C_p(n) \approx \sum_{\ell(n) \leq k \leq u(n)} C_p(k) \left( f(p) \nu_p(k, n) + g(p) \nu_p(k, n - 1) + h(p) \nu_p(k, n - 2) \right) $$

$$ \approx C \left( \frac{n}{2 - p} \right) \sum_{\ell(n) \leq k \leq u(n)} \left( f(p) \nu_p(k, n) + g(p) \nu_p(k, n - 1) + h(p) \nu_p(k, n - 2) \right) $$

$$ \approx C \left( \frac{n}{2 - p} \right) \sum_{k=\lfloor u/2 \rfloor}^{n} \left( f(p) \nu_p(k, n) + g(p) \nu_p(k, n - 1) + h(p) \nu_p(k, n - 2) \right) $$

$$ = C \left( \frac{n}{2 - p} \right) \left( f(p) S_p(n) + g(p) S_p(n - 1) + h(p) S_p(n - 2) \right), $$

where $S_p(n) := \sum_{k=\lfloor u/2 \rfloor}^{n} \nu_p(k, n) = (1 - (p - 1)n)/(2 - p)$. From this we finally obtain the approximate scaling relation

$$ C_p((2 - p)^k n_0) \approx \left( \frac{1 - 2p(2 - 3p)}{2 - p} \right)^k C(n_0), $$

which traduces into the scaling law $C_p(n) \approx C_p(n_0)^{n/n_0} - \beta_p$, with

$$ \beta_p := \frac{\log(2 - p) - \log(1 - 2p) - \log(2 - 3p)}{\log(2 - p)}. $$

4.2. We have used the recurrence relation in Equation 4 to numerically compute the two-sites correlation function for different values of the mutation probability. As shown in Figure 1, the numerical computations confirm that the two-point correlation function approximatively follows a power-law behavior. Furthermore, according to Figure 2, the theoretically predicted exponents, $\{ -\beta_p : 0 < p < 1 \}$, fit very well the ones obtained by linear regression from the numerically computed correlation functions.
The argument developed above suggests that the stationary measure $\mu_p$ varies in a piecewise smooth way with $p$. This variation is reflected on the behavior of the two-sites correlation function $C_p$, which appears to follow a power law decay which prevails in the whole interval $0 < p < 1$, except for the two singularities located at $p = 1/2$ and $p = 2/3$. At precisely those values of $p$, the two-sites correlation function appears to decay faster than any power law.
5. Proofs

5.1. Our proof of Theorem 2 requires the following.

**Lemma 1.** For each $a, b \in \{0, 1\}^{\ell+1}$ there exists $\{s^0, s^1, \ldots, s^n\} \subset \{e, m\}^+$, such that $b \sqsubseteq s^n \circ \cdots \circ s^1 \circ s^0(a)$.

**Proof.** For each $\ell \in N_0$ and $a \in \{0, 1\}^{\ell+1}$, let $s = e^{\ell+1}$ whenever $a_0 = 0$, otherwise let $s = me^\ell$. Clearly, for each $n > \lceil \log(\ell + 1)/\log(2) \rceil$ we have

$$0^{\ell+1} \sqsubseteq t^n \circ \cdots \circ t^1 \circ s(a),$$

where $t^j \in \{e, m\}^+$ is such that $e^{\ell+1} \sqsubseteq t^j$ for each $1 \leq j \leq n$. We claim that for each $b \in \{0, 1\}^{\ell+1}$ there exists $\{s^0, s^1, \ldots, s^k\} \subset \{e, m\}^+$ such that $b \sqsubseteq s^k \circ \cdots \circ s^1 \circ s^0(0^{\ell+1})$.

The primitivity of $M^\ell$ in the $*$-weak sense.

On the other hand, if $k$ even and $c_0 = 1$ or $k$ odd and $c_0 = 0$, by taking $t^j := ms^j$ for $0 \leq j \leq k$, we have

$$c = c_0 c_0^n \sqsubseteq m^{k+1}(0) s^k \circ \cdots \circ s^1 \circ s^0(0^{\ell+1}) = t^k \circ \cdots \circ t^1 \circ t^0(0^{\ell+1}).$$

On the other hand, if $k$ even and $c_0 = 0$ or $k$ odd and $c_0 = 1$, by taking $t^{j+1} := ms^j$ for $0 \leq j \leq k$, and $t^0 := e^{\ell+1}$, we have

$$c = c_0 c_0^n \sqsubseteq m^{k+2}(0) s^k \circ \cdots \circ s^1 \circ s^0(0^{\ell+1}) = t^{k+1} \circ \cdots \circ t^1 \circ t^0 (0^{\ell+1}),$$

and the lemma follows. \hfill \Box

**Proof of Theorem 2**. Let us assume that for each $\ell \in N_0$, the stochastic matrix $M^\ell$ is primitive. This implies, by the Perron-Frobenius Theorem, that there is a unique probability vector $\mu^\ell : \{0, 1\}^{\ell+1} \to [0, 1]$ such that

$$\mu^\ell = \mu^\ell M^\ell \quad \text{and} \quad \lim_{\ell \to \infty} \mu_0^\ell M^\ell = \mu_\ell,$$

for every initial probability vector $\mu^0 : \{0, 1\}^{\ell+1} \to [0, 1]$. Hence, for any measure $\mu^0$ specifying the distribution of the initial conditions, for each $\ell \in N_0$, and for all $a \in \{0, 1\}^{\ell+1}$ we have

$$\lim_{\ell \to \infty} \mu^\ell[a] = \mu_\ell(a).$$

If in addition the probability vectors $\mu^\ell$ satisfy the compatibility condition

$$\sum_{x \in \{0, 1\}} \mu^\ell+1(ax) = \mu^\ell(a),$$

for each $\ell \in N_0$ and $a \in \{0, 1\}^{\ell+1}$, then Kolmogorov’s representation theorem implies the existence of a measure $\mu$ on $X$ such that $\mu[a] = \mu_\ell(a)$ for each $\ell \in N_0$ and $a \in \{0, 1\}^{\ell+1}$. Finally, Equation (6) ensures the convergence of $\mu^\ell$ towards $\mu$ in the $*$-weak sense.

The primitivity of $M^\ell$ follows straightforwardly from the following argument. As proved in Lemma 1, for each pair of words $a, b \in \{0, 1\}^{\ell+1}$, there exist a sequence of substitutions such that applied to $a$ produces a word having $b$ as prefix. Now, since all words in $\{e, m\}^{\ell+1}$ have positive probability, then the previous claim implies that
\[ M_t^n(a, b) > 0 \] for some \( n > 0 \), which proves that \( M_t^n \) is irreducible. Now, since the word \( 00 \cdots 0 \) occurs as the prefix of \( e(0)e(0) \cdots e(0) \), then \( M_t(00 \cdots 0, 00 \cdots 0) > 0 \), which implies that \( M_t \) is aperiodic, and so \( M_t \) is primitive.

To prove the compatibility condition \( (\mathbb{I}) \), we should notice that it is inherited from the analogous compatibility condition satisfied by all the marginals \( \mu_t^\ell \) at each time \( t \in \mathbb{N}_0 \). Indeed, for \( t = 0 \) we obviously have

\[
\sum_{x \in \{0, 1\}} \mu^{0}_t(ax) := \sum_{x \in \{0, 1\}} \mu^{0}(ax) = \mu^{0} \left( \bigsqcup_{x \in \{0, 1\}} [ax] \right) = \mu[a] =: \mu^{0}_0(a),
\]

for each \( \ell \in \mathbb{N}_0 \) and \( a \in \{0, 1\}^{\ell+1} \). Here \( \sqcup \) stands for the disjoint union. Now, from Equation \( (\mathbb{I}) \) it follows that

\[
\sum_{x \in \{0, 1\}} \mu^{\ell+1}_t(ax) = \sum_{x \in \{0, 1\}} \sum_{s \in \{e, m\}^{\ell+2}} \nu_p[s] \left( \sum_{b \in \{0, 1\}^{\ell+2}} \mu^\ell[b] \right) = \sum_{s \in \{e, m\}^{\ell+2}} \nu_p[s] \mu^\ell \left( \bigcup_{x \in \{0, 1\}} \bigcup_{a \in \{0, 1\}^{\ell+2}} [b] \right)
\]

Since \( |a| = \ell + 1 \), the statement \( a \sqsubseteq \prod_{i=0}^{\ell+1} s_i(b_i) \) is equivalent to \( a \sqsubseteq \prod_{i=0}^{\ell} s_i(b_i) \), and we have

\[
\sum_{x \in \{0, 1\}} \mu^{\ell+1}_t(ax) = \sum_{s \in \{e, m\}^{\ell+2}} \nu_p[s] \mu^\ell \left( \bigcup_{b \in \{0, 1\}^{\ell}} [b] \right)
\]

for each \( \ell \in \mathbb{N}_0 \) and \( a \in \{0, 1\}^{\ell+1} \). The compatibility condition \( (\mathbb{I}) \) follows by taking the limit \( t \to \infty \) on both sides of the equation, which completes the proof of the theorem. \( \square \)
5.2. This subsection is devoted to the proof of Theorem 2 which relies in Lemmas 2 and 3 and other auxiliary propositions which we state without proof. The proofs of these auxiliary results is referred to Subsections 5.3, 5.4 and 5.5.

Lemma 2 (Upper bound). For each \( p \in (0, 1) \), the stationary measure \( \mu_p \) has decay of correlations bounded above by a power law. Indeed, for each \( p \in (0, 1) \) there exist positive constants \( \alpha_p \) and \( K_p > 0 \) such that
\[
|C_p(n)| \leq K_p n^{-\alpha_p}
\]
for all \( n \geq 2 \).

Lemma 3 (Lower bound). There are constants \( 0 < b \) and \( n_p^* \in \mathbb{N} \), such that
\[
C_p(n) \geq n - b \quad \text{for all} \quad p \in (0, n_p^*) \quad \text{and all} \quad n \geq n_p^*.
\]
This lemma has the following straightforward consequences that we will use below.

Corollary 1. For each \( p \in (0, 1) \) and \( a < \alpha_p \), there exists \( n = n_a \) such that
\[
C_p(n) \leq n^{-a}
\]
for each \( n \geq n_a \).

Besides Lemmas 2 and 3, the proof of Theorem 2 requires some additional notation and preliminary results which we present now.

Let \( b \) be as in Theorem 3. Then for each \( \beta > b \) define the function \( d_\beta : [1, \infty) \to \mathbb{R} \) as follows
\[
d_\beta(x) := \sqrt{p(1 - p)(2 - p)(\beta + 1) \log(x)} / x
\]
With this, define the functions \( \ell_{\beta, p}, u_{\beta, p} : [1, \infty) \to [0, \infty) \) given by
\[
\ell_{\beta, p}(x) := \frac{x}{2 - p + d_\beta(x)}, \quad u_{\beta, p}(x) := \frac{x}{2 - p - d_\beta(x)}.
\]
Finally, to simplify the expressions that will appear, define
\[
\lambda_p := (2 - p), \quad \phi_{\beta, p}(x) := \ell_{\beta, p}(\lambda_p x) / \lambda_p \quad \text{and} \quad \psi_{\beta, p}(x) := u_{\beta, p}(\lambda_p x) / \lambda_p.
\]
Let us remind that, for \( p \in (0, 1) \) and each \( k, n \in \mathbb{N} \),
\[
W_p(k, n) := f(p)\nu_p(k, n) + g(p)\nu_p(k, n - 1) + h(p)\nu_p(k, n - 2),
\]
with \( f(p) := p(2p - 1), \quad g(p) := (1 - p)(1 - 3p), \) and \( h(p) := (1 - p)^2 \) and
\[
\nu_p(k, n) := \nu_p(\ell(\mathcal{S}_0^{k-2}) = n - 1) \equiv \binom{k - 1}{n - k} (1 - p)^{n-k} p^{2k-n-1}.
\]
We have the following.
Proposition 1. If \( p \in (0, p^*) \) and \( \lfloor n/2 \rfloor < k \leq 2(n-1)/3 \), then \( W_p(k, n) > 0 \).

Proof. A simple computation shows that
\[
W_p(k, n) = \frac{(1-p)^{n-k}p^{2k-n}(k-1)!}{(n-k)!(2k-n+1)!} Q_p(n, k), \quad \text{with}
Q_p(n, k) := (1 - 2p)(2n - 3k)(2k - n + 1) + p(2n - 3k - 2)(n - k).
\]
Since \( p^* < 1/2 \), then \( Q_p(k, n) > 0 \) for \( \lfloor n/2 \rfloor < k \leq 2(n-1)/3 \). The result follows from the fact that \( W_p(k, n) \) and \( Q_p(k, n) \) have the same sign. \( \square \)

Proposition 2. For each \( \beta > b \) there exists \( n_\beta \geq 5 \) such that
\[
\delta_n := n^b \left( \sum_{|n/k-(2-p)| > d_\beta(n)} \nu_p(k, n) \right) \leq n^{-(\beta-b)/2},
\]
for all \( n \geq n_\beta \).

Proposition 3. For each \( n \geq 5 \) and \( p \in (0, p^*) \) we have
\[
\left| \sum_{k > 2(n-1)/3} W_p(k, n) C_p(k) \right| \leq \frac{n \left(4p(1-p)\right)^{(n-2)/3}}{6}.
\]

Proposition 4. There exists \( x_0 \geq e \) such that, for each \( x \geq x_0 \) there are constants
\( 0 < Q(x) < 1 < R(x) \) such that for every \( 1 \leq j \leq k \in \mathbb{N} \) we have
\[
Q(x) \lambda_p^{k-j} x \leq \lambda_p^{d_j}(\lambda^{k-1} x) \leq u_{\beta,p}(\lambda^{k-1} x) \leq R(x) \lambda_p^{k-j} x.
\]
Furthermore, \( Q(x), R(x) \to 1 \) as \( x \to \infty \).

Proof of Theorem 2 Fix \( p \in (0, p^*) \) and \( \beta > b \), and let \( d = d_\beta \), \( u = u_{\beta,p} \) and \( \ell = \ell_{\beta,p} \) be as above. Let \( f, g, h : [0, 1] \to \mathbb{R} \) be as in Equation (3). Note that \( \ell(n) \leq k \leq u(n) \) implies \( \frac{n}{2} - (2 - p) \leq d(n) \). Note also that \( |C_p(k)| \leq 1/4 \) for all \( k \), and that \( |f(p)| + |g(p)| + |h(p)| \leq 2 \) for all \( p \in (0, 1) \). From this, using Equation (3) and Proposition 2 we obtain
\[
C_p(n) \leq \sum_{\ell(n) \leq k \leq u(n)} C_p(k) W_p(k, n) + n^{-b} \delta_n
\]
and
\[
C_p(n) \geq \sum_{\ell(n) \leq k \leq u(n)} C_p(k) W_p(k, n) - n^{-b} \delta_n
\]
for each \( n \geq n_\beta \).

A simple computation shows that for \( p < p^* \), there exists \( n_1 = n_1(p, \beta) \in \mathbb{N} \) such that \( u(n_1) < 2(n-1)/3 \) for all \( n \geq n_1 \). Hence, according to Proposition 1 \( W_p(k, n) > 0 \) for all \( n \geq n_1 \) and \( \ell(n) \leq k \leq u(n) \). In this case we can define a probability distribution \( k \mapsto P_p(k, n) \) proportional to \( k \mapsto C_p(k, n) \), in the interval \( \ell(n) \leq k \leq u(n) \).
For $n \geq \max(n_1, n_p, n_3)$ we can use the lower bound of Theorem 2 and rewrite
Inequalities (8) and (9) as

\[
C_p(n) \leq \left( \sum_{k=[n/2]}^{n} \mathcal{W}_p(n, k) + 2\delta_n \right) \mathbb{E}_{p,n}(C_p),
\]

\[
C_p(n) \geq \left( \sum_{k=[n/2]}^{n} \mathcal{W}_p(n, k) - 2\delta_n \right) \mathbb{E}_{p,n}(C_p),
\]

where $\mathbb{E}_{p,n}(C_p)$ denotes the mean value of $C_p$ with respect to $\mathbb{P}_p(k,n)$. Using the
fact that $\sum_{k=[n/2]}^{n} \mathcal{W}_p(n, k) = (1 - 2p)(2 - 3p)/(2 - p) - 2p(p - 1)^n$, we obtain

\[
C_p(n) \leq \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} + 3\delta_n \right) \mathbb{E}_{p,n}(C_p)
\]

\[
= \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} + 3\delta_n \right) \max_{\ell(n) \leq k \leq u(n)} C_p(k),
\]

\[
C_p(n) \geq \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} - 3\delta_n \right) \mathbb{E}_{p,n}(C_p)
\]

\[
\geq \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} + 3\delta_n \right) \min_{\ell(n) \leq k \leq u(n)} C_p(k).
\]

We can rewrite these inequalities as

\[
\lambda_p^{-\beta_p - \eta_{p,x}} \min_{\lambda_p \phi(x) \leq y \leq \lambda_p \psi(x)} C_p([y]) \leq C_p(\lambda_p x) \leq \lambda_p^{-\beta_p + \eta_{p,x}} \max_{\lambda_p \phi(x) \leq y \leq \lambda_p \psi(x)} C_p([y]),
\]

with $\eta_x := 3\lambda_p^{-\beta_p} (x - 1)^{-1/(\beta - b)/2}/\log(\lambda_p) \geq 3\lambda_p^{-\beta_p} \delta_x/\log(\lambda_p)$ and $\beta_p$ as defined
in Equation (2). It follows from here, by a straightforward induction, that

\[
C_p(\lambda_p^k x) \leq m_p \max_{\lambda_p \phi(k^{\frac{1}{\beta - b}} - 1)} C_p([z]),
\]

\[
C_p(\lambda_p^k x) \geq m_p \min_{\lambda_p \phi(k^{\frac{1}{\beta - b}} - 1)} C_p([z]),
\]

for all $k \in \mathbb{N}$ and $x \geq \max(n_1, n_p, n_3)$.

Using the above estimates and taking into account Corollary 1 and Theorem 3 we obtain, for all $p \in (0, p^*)$ and all $x \geq n_0 := \max(n_1, n_p, n_3, n_0)$, the inequalities

\[
C_p(\lambda_p^k x) \leq \lambda_p^{-k \beta_p + \sum_{j=0}^{k-1} \eta_{Q(x)} x^{\beta_p - b}} (Q(x) x)^{-a} \leq \lambda_p^{-k \beta_p + \epsilon_p(x)} (Q(x) x)^{-a},
\]

\[
C_p(\lambda_p^k x) \geq \lambda_p^{-k \beta_p - \sum_{j=0}^{k-1} \eta_{Q(x)} x^{\beta_p - b}} (R(x) x)^{-b} \geq \lambda_p^{-k \beta_p - \epsilon_p(x)} (R(x) x)^{-b},
\]

where

\[
\epsilon_p(x) := \frac{3 \times (2 \lambda_p)^{(\beta - b)/2}}{(Q(x) x)^{(\beta - b)/2} \lambda_p^{\beta_p} \log(\lambda_p) \left(\lambda_p^{(\beta - b)/2} - 1\right)} \to 0 \text{ as } x \to \infty.
\]

For $n \geq n_0$ let $k \in \mathbb{N}$ be such that $n_0 \lambda_p^k \leq n < n_0 \lambda_p^{k+1}$, and let $x_0 := n/\lambda_p^k \in [n_0, \lambda_p n_0)$. With this, the inequalities above become

\[
R(x_0)^{-b} x_0^{\beta_p - b} \lambda_p^{-\epsilon_p(x_0)} n^{-\beta_p} \leq C_p(n) \leq Q(x_0)^{-a} x_0^{\beta_p - a} \lambda_p^{-\epsilon_p(x_0)} n^{-\beta_p},
\]
and the result follows by taking
\[
A_p := \min_{n_0 \leq x_0 < \lambda_p n_0} R(x_0)^{-b} x_0^{\beta_p - b} \lambda_p^{-e_p(x_0)},
\]
\[
B_p := \max_{n_0 \leq x_0 < \lambda_p n_0} Q(x_0)^{-a} x_0^{\alpha_p - a} \lambda_p^{-e_p(x_0)}.
\]

5.3. **Proof of Lemma** [2] For each \(n \in \mathbb{N}\) let \(\bar{C}_p(n) := \max\{|C_p(m)| : m \geq n\}\). Obviously \(n \mapsto \bar{C}_p(n)\) is a non-increasing upper bound for \(n \mapsto |C_p(n)|\).

For each \(n \in \mathbb{N}\) let \(S_p(n) := \sum_{k=[n/2]}^n \nu_p(k,n)\). From Equation (3) we readily derive the inequality
\[
|C_p(n)| \leq \sum_{k=[n/2]}^n |C_p(k)| |g(p)| \nu_p(k,n) + |g(p)| \nu_p(k,n-1) + |h(p)| \nu_p(k,n-2)
\]
\[
\leq \bar{C}_p([n/2]) (|f(p)| S_p(n) + |g(p)| S_p(n-1) + |h(p)| S_p(n-2)),
\]
which holds for each \(n \geq 2\). Hence,
\[
\bar{C}_p(n) \leq \max_{m \geq n} \bar{C}_p([m/2]) (|f(p)| S_p(m) + |g(p)| S_p(m-1) + |h(p)| S_p(m-2))
\]
\[
\leq \bar{C}_p([n/2]) \max_{m \geq n} (|f(p)| S_p(m) + |g(p)| S_p(m-1) + |h(p)| S_p(m-2))
\]

It can be easily verified that \(S_p(n+1) = p S_p(n) + (1-p) S_p(n-1)\), and solving this recursion from \(S_p(0) = 0\) and \(S_p(1) = 1\), we obtain \(S_p(n) = (1 - (p-1)^n)/(2-p)\). Hence,
\[
\max_{m \geq n} S_p(m) \leq \bar{S}_p(n) := \frac{1 + (1-p)^n}{2-p}
\]
for each \(n \in \mathbb{N}\), and using the previous inequality for \(\bar{C}_p(n)\) it follows that
\[
(10) \quad \bar{C}_p(n) \leq \bar{C}_p([n/2]) (|f(p)| \bar{S}_p(m) + |g(p)| \bar{S}_p(m-1) + |h(p)| \bar{S}_p(m-2)).
\]

We have three cases:

a) For \(0 < p \leq 1/3\) we have \(f(p) \leq 0 \leq g(p), h(p)\). In this range (10) becomes
\[
\bar{C}_p(n) \leq \bar{C}_p([n/2]) \left(1 - 2p + \frac{2(1-p - p^2)}{2-p} (1-p)^n\right).
\]

b) For \(1/3 \leq p \leq 1/2\) we have \(f(p), g(p) \leq 0 \leq h(p)\), hence
\[
\bar{C}_p(n) \leq \bar{C}_p([n/2]) \left(\frac{p(3-4p)}{2-p} + 2p(1-p)^n\right).
\]

c) Finally, for \(1/2 \leq p < 1\) we have \(g(p) \leq 0 \leq f(p), h(p)\), and so
\[
\bar{C}_p(n) \leq \bar{C}_p([n/2]) \left(\frac{p}{2-p} + \frac{2p(p+1)}{2-p} (1-p)^n\right).
\]

In all three cases, the inequality has the form
\[
\bar{C}_p(n) \leq \bar{C}_p([n/2]) (\eta_p + A_p (1-p)^n),
\]
with \( \eta_p \in (0, 1) \) and \( A_p > 0 \). For \( n \in \mathbb{N} \), let \( k = k(n) \) be such that \( 2^k \leq n < 2^{k+1} \). Iterating the previous inequality, and taking into account that \( \bar{C}_p(1) \leq 1/4 \) we obtain

\[
\tilde{C}_p(n) \leq \bar{C}_p(2^k) \leq \frac{\eta_p}{4} \prod_{j=1}^{k} \left( 1 + A_p \eta_p^{-1} (1 - p)^{2^j} \right).
\]

Since \( 2^{k+1} > n \) then \( k > \log(n) / \log(2) - 1 \), it follows that

\[
|C_p(n)| \leq \tilde{C}_p(n) \leq \frac{\eta_p}{4} \prod_{j=1}^{k} \left( 1 + A_p \eta_p^{-1} (1 - p)^{2^j} \right),
\]

and the result follows with \( \alpha_p := -\log(\eta_p) / \log(2) \) and

\[
K_p := \frac{1}{4\eta_p} \prod_{j=1}^{\infty} \left( 1 + A_p \eta_p^{-1} (1 - p)^{2^j} \right) \leq \exp \left( \frac{A_p (p \eta_p)^{-1}}{4\eta_p} \right) < \infty.
\]

\[\square\]

5.4. This section is devoted to the proof of Propositions 2, 3 and 4.

**Proposition 5.** For each \( p \in (0, 1) \) the function

\[
q \mapsto I_p(q) := \frac{q}{q + 1} \log \left( \frac{q}{1 - p} \right) + \frac{1 - q}{q + 1} \log \left( \frac{1 - q}{p} \right)
\]

is non-negative, strictly convex, and satisfies

\[
\min \{ I_p(q) : q \in (0, 1) \} = I_p(1 - p) \equiv 0.
\]

**Proof.** Since \( x \mapsto -\log(x) \) is a concave function, then

\[
I_p(q) = \frac{1}{q + 1} \left( -q \log \left( \frac{1 - p}{q} \right) - (1 - q) \log \left( \frac{p}{1 - q} \right) \right)
\geq \frac{1}{q + 1} \left( - \log \left( \frac{1 - p}{q} + (1 - q) \frac{p}{1 - q} \right) \right) = 0.
\]

On the other hand,

\[
\frac{dI_p(q)}{dq} = \frac{1}{(q + 1)^2} \left( \log \left( \frac{q}{1 - p} \right) - 2 \log \left( \frac{1 - q}{p} \right) \right) = 0 \iff q = 1 - p.
\]

In this way we prove that \( I_p \) is non-negative with minimum at \( q = 1 - p \). Now,

\[
\frac{d^2I_p(q)}{dq^2} = \frac{1}{q(1 - q^2)} + \frac{2}{(q + 1)^3} \left( 2 \log \left( \frac{1 - q}{p} \right) + \frac{1 - p}{q} \right) > 0
\]

for all \( p, q \in (0, 1) \). For this, note that if \( 1 - q \geq p \) then \( 1 - p \geq q \), and in this case we have

\[
\frac{d^2I_p(q)}{dq^2} \geq \frac{1}{q(1 - q^2)} > 0,
\]
otherwise, for $1-q < p$ then $1-p < q$, and taking into account that $-\log(x) \geq 1-x$, we obtain
\[
\frac{d^2 I_p(q)}{dq^2} = \frac{1}{q(1-q^2)} - \frac{2}{(q+1)^3} \log \left( \frac{(1-q^2)^2}{p} \right) \left( \frac{1-p}{q} \right) \\
\geq \frac{1}{q(q+1)} + \frac{2}{(q+1)^3} \left( 1 - \frac{(1-q^2)^2}{p} \right) \left( \frac{1-p}{q} \right) \\
\geq \frac{1}{q(q+1)} \left( \frac{1}{1-q} - 2 \frac{(1-q)^2}{(q+1)^2} \right) = \frac{1+3q^2}{q(q+1)^3(1-q)} > 0.
\]
Therefore $I_p$ is strictly convex. \hfill \Box

**Proposition 6.** For $p \in (0, 1)$ and $k, n \in \mathbb{N}$ let
\[
\nu_p(k, n) := \left( \frac{k-1}{n-k} \right) (1-p)^{n-k} p^{2k-n-1}.
\]
Then, for each $n \geq 3$ there are constants $0 \leq A^- \leq A^+ \leq 1$ such that
\[
e^{-n I_p(n/k-1)} A^- \leq \nu_p(k+1, n+1) \leq e^{-n I_p(n/k-1)} A^+,
\]
for all $\lfloor n/2 \rfloor \leq k \leq n$.

**Proof.** A very useful refinement of Stirling’s approximation, first published in [14], states that
\[
\sqrt{2\pi} n^n \exp \left( -n + \frac{1}{12n+1} \right) \leq n! \leq \sqrt{2\pi} n^n \exp \left( -n + \frac{1}{12n} \right)
\]
for all $n \in \mathbb{N}$. Hence, for each $p \in (0, 1)$, $n \geq 3$, and $\lfloor n/2 \rfloor < k < n$, we have
\[
\exp (-\epsilon_{n,k}) \leq \sqrt{2\pi} k^n (k/n-1) (2-n/k) \leq \left( k/n-k \right)^{(2k-n)} \times \left( k/(n-k) \right)^{(n-k)} \leq \exp (+\epsilon_{n,k}),
\]
with $\epsilon_{n,k} = (4 \min(n-k, 2k-n))^{-1}$. A simple computation shows that
\[
k^k (n-k)^{(n-k)} (2k-n)^{(2k-n)} (1-p)^{n-k} p^{2k-n} = \exp (-n I_p(1-k/n)),
\]
with $I_p(q) := q/(q+1) \log (q/(1-p)) + (1-q)/(q+1) \log ((1-q)/p)$ for each $q \in (0, 1)$. Hence
\[
e^{-n I_p(n/k-1)-\epsilon_{n,k}} \leq \nu_p(k+1, n+1) \leq e^{-n I_p(n/k-1)+\epsilon_{n,k}} \leq \sqrt{2\pi} k^n (k/n-1)(2-n/k),
\]
for each $n \geq 3$ and $\lfloor n/2 \rfloor \leq k \leq n$. On the other hand,
\[
\nu_p(n/2+1, n+1) = (1-p)^{n/2} \equiv \lim_{q \to 1} \exp(-n I_p(q)), \\
\nu_p(n+1, n+1) = p^n \equiv \lim_{q \to 0} \exp(-n I_p(q)).
\]
Thus, using
\[
A^\pm := \begin{cases} e^{\pm(4 \min(n-k, 2k-n))^{-1}} (2\pi k (k/n-1)(2-n/k))^{-1/2} & \text{if } n/2 < k < n, \\
1 & \text{otherwise,}
\end{cases}
\]
we can extend \((11)\) to
\[
e^{-n I_p(n/k-1)} A^- \leq \nu_p(k+1, n+1) \leq e^{-n I_p(n/k-1)} A^+
\]
which holds for all \( n \geq 3 \) and \( |n/2| \leq k \leq n \). Finally, a simple computation shows that
\[
A^+ \leq \max \left( 1, \frac{\epsilon^{1/4} \sqrt{n-1}}{\sqrt{2\pi(n-2)}}, \frac{\epsilon^{1/4} \sqrt{n+2}}{\sqrt{4\pi(n-2)}} \right) = 1,
\]
for \( n \geq 3 \). \( \square \)

**Proof of Proposition 2**. We have already proved that the function \( q \mapsto I_p(q) \) is non-negative, vanishes only at \( q = 1 - p \), and is strictly convex. Hence, Taylor’s Theorem ensures that
\[
(q - (1 - p))^2 \times \min_{|q-(1-p)| \leq \epsilon} \frac{d^2 I_p}{dq^2} \leq I_p(q) \leq (q - (1 - p))^2 \times \max_{|q-(1-p)| \leq \epsilon} \frac{d^2 I_p}{dq^2}.
\]
Since \( \frac{d^2 I_p}{dq^2}(q=1-p) = (p(1-p)(2-p))^{-1} \) and \( q \mapsto \frac{d^2 I_p}{dq^2} \) is continuous, then, for all \( \alpha \in (0,1) \) there exists \( \epsilon_\alpha > 0 \) such that
\[
\frac{\alpha (q - (1 - p))^2}{p(1-p)(2-p)} \leq I_p(q) \leq \frac{\alpha^{-1} (q - (1 - p))^2}{p(1-p)(2-p)},
\]
for all \( q \in (1 - p - \epsilon_\alpha, 1 - p + \epsilon_\alpha) \). With this, and using Proposition 3 it follows that
\[
0 \leq \sum_{|k/n-(2-p)| \geq \epsilon} \nu_p(k+1,n+1) \leq n \times \exp(-n I_p(\epsilon)) \leq n \times \exp\left(\frac{-n \alpha \epsilon_\alpha^2}{p(1-p)(2-p)}\right)
\]
for all \( \epsilon \leq \epsilon_\alpha \) and \( n \geq 3 \). By taking \( n \) such that \( d_{\beta}(n) = \sqrt{p(1-p)(2-p)(\beta+1) \log(n)/n} \leq \epsilon_\alpha \), we obtain
\[
0 \leq \sum_{|k/n-(2-p)| \geq d_{\beta}(n)} \nu_p(k+1,n+1) \leq n^{1-\alpha(\beta+1)},
\]
and the claim follows if we fix \( \alpha = (b + \beta + 2)/(2\beta + 2) \) and \( n_{\beta} \) such that \( d_{\beta}(n) \leq \epsilon_\alpha \), for all \( n \geq n_{\beta} \). \( \square \)

**Proof of Proposition 3**. Fix \( n \geq 5 \) and \( p \in (0,1) \). Since \( C_p(k) \leq 1/4 \) for all \( k \in \mathbb{N} \) and \( \nu_p(n,k) > 0 \), then
\[
\left| \sum_{k > 2n/3} W_p(k,n) C_p(k) \right| \leq \frac{1}{4} \sum_{k > 2n/3} |W_p(k,n)| \leq \frac{n(|f(p)| + |g(p)| + |h(p)|)}{12} \left( \max_{2n/3 < k, n-2 \leq m \leq n} \nu_p(k,m) \right).
\]
Then, using Proposition 3 and the fact that \( |f(p)| + |g(p)| + |h(p)| \leq 2 \) for all \( p \in (0,1) \), we obtain,
\[
\left| \sum_{k > 2n/3} W_p(k,n) C_p(k) \right| \leq \frac{n}{6} \exp\left(-\left(n-2\right) \min_{2n/3 < k, n-2 \leq m \leq n} I_p((m-1)/(k-1) - 1) \right)
\]
for each \( n \geq 5 \). Now, according to Proposition 4 the function \( q \mapsto I_p(q) \) is monotonously decreasing in \([0,1-p]\) and monotonously decreasing in \([1-p,1]\).
Since for each \( p \in (0, p^*) \) and \( n \geq 5 \) we have \( 0 < (m-1)/(k-1) - 1 < 1/2 - 3/2n < 1/2 \leq 1 - p^* \), for each \( n - 2 \leq m \leq n \) and \( k \geq 2n/3 \). Hence,

\[
\min_{2n/3 < k, n-2 \leq m \leq n} I_p \left( (m-1)/(k-1) - 1 \right) > I_p(1/2) = \frac{1}{3} \log \left( \frac{1}{4p(1-p)} \right).
\]

which leads to the desired result, namely

\[
\left| \sum_{k > 2(n-1)/3} \mathcal{W}_p(k, n) C_p(k) \right| \leq \frac{n (4p(1-p))^{(n-2)/3}}{6}.
\]

\[ \square \]

**Proof of Proposition 4** A straightforward computation shows that

\[ \lambda_p \varphi_j(\lambda_p^{k-1} x) = \ell(\lambda_p^k x) \text{ and } \lambda_p \psi_j(\lambda_p^{k-1} x) = u(\lambda_p^k x), \]

for all \( x \geq e, k \in \mathbb{N} \) and \( 1 \leq j \leq k \). It is easily checked that, for each \( p \in (0, 1) \) and \( \beta > b \), there exists \( x_1 \geq e \) such that both \( \ell \) and \( u \) are increasing functions in \([x_0, \infty)\).

For each \( p \in (0, 1) \) and \( x \geq x_1 \) let \( Q(x) \) be the largest solution to

\[ Q(x) = \exp \left( -\frac{d(x)}{\sqrt{Q(x)}} \sum_{m=0}^{\infty} \frac{m+1}{\lambda_p^m} \right). \]

It is not difficult to check that \( Q(x) \in (0, 1) \) and since \( d(x) \to 0 \) as \( x \to \infty \), then \( Q(x) \to 1 \) as \( x \to \infty \).

Now, fix \( k \in \mathbb{N} \), and \( x_0 \geq x_1 \) so large that \( \lambda_p Q(x) \geq 1 \) for all \( x \geq x_0 \). Let \( Q_{k,0}(x) := 1 \), and define recursively

\[ Q_{k,j+1}(x) := \frac{Q_{k,j}(x)}{1 + \lambda_p^{-(k-j)/2-1} Q(x)^{-1/2} \sqrt{k-j+1} d(x)} \]

for \( 0 \leq j \leq k-1 \). Clearly

\[
1 \geq Q_{k,j}(x) = \prod_{i=0}^{j-1} \left( 1 + \lambda_p^{-(k-i)/2-1} Q(x)^{-1/2} \sqrt{k-i+1} d(x) \right)^{-1}
\]

\[
\geq \exp \left( -\frac{d(x)}{\sqrt{Q(x)}} \sum_{i=0}^{j-1} \sqrt{k-i+1} \lambda_p^{-(k-i)/2-1} \right)
\]

\[
\geq \exp \left( -\frac{d(x)}{\sqrt{Q(x)}} \sum_{m=0}^{\infty} \sqrt{m+1} \lambda_p^m \right) = Q(x).
\]

Since \( x \geq e > \lambda_p \), then \( x^{k+1} \geq \lambda_p^k x \), and therefore

\[
d(\lambda_p^k x) = \sqrt{\frac{p(1-p)(2-p)(\beta+1) \log(\lambda_p^k x)}{\lambda_p^k x}}
\]

\[
\leq \lambda_p^{-k/2} \sqrt{\frac{p(1-p)(2-p)(\beta+1)(k+1) \log(x)}{x}} = \sqrt{\frac{k+1}{\lambda_p^k d(x)}}.
\]
Hence, for $j = 1$ we have

$$\ell(\lambda_p^k x) = \frac{\lambda_p^{k-1} x}{1 + d(\lambda_p^k x)/\lambda_p} \geq \frac{\lambda_p^{k-1} x}{1 + \lambda_p^{-k/2} \sqrt{k + 1} d(x)} = Q_{k,1}^{(1)}(x) \lambda_p^{k-1} x.$$  

Suppose that $\ell^j(\lambda_p^k x) \geq Q_{k,j}(x) \lambda_p^{k-j} x$ for $j < k$. Since $Q_{k,j}(x) \lambda_p^{k-j} x \leq \lambda_p^{k-j} x \leq x^{k-j+1}$, then

$$d(Q_{k,j} \lambda_p^{k-j} x) = \sqrt{p(1-p)(2-p)(\beta + 1) \log(Q_{k,j}(x) \lambda_p^{k-j} x)}$$

$$\leq \lambda_p^{-(k-j)/2} Q_{k,j}(x)^{-1/2} \sqrt{k - j + 1} d(x)$$

$$\leq \lambda_p^{-(k-j)/2} Q(x)^{-1/2} \sqrt{k - j + 1} d(x).$$

Taking into account that $y \mapsto \ell(y)$ is an increasing function for $y \geq x_0$, and since $Q_{k,j}(x) \lambda_p^{k-j} x \geq Q(x) x \geq x_0$, then

$$\ell^{j+1}(\lambda_p^k x) \geq \frac{Q_{k,j}(x) \lambda_p^{k-j} x}{\lambda_p + d(Q_{k,j}(x) \lambda_p^{k-j} x)}$$

$$\geq \frac{Q_{j}(x) \lambda_p^{k-j-1} x}{1 + \lambda_p^{-(k-j)/2-1} Q(x)^{-1/2} \sqrt{k - j + 1} d(x)}$$

$$= Q_{k,j+1}(x) \lambda_p^{k-j-1} x.$$

In this way we have proved that

$$\ell^j(\lambda_p^k x) \geq Q_{k,j}(x) \lambda_p^{k-j} x \geq Q(x) \lambda_p^{k-j} x,$$

for all $k \in \mathbb{N}$ and $1 \leq j \leq k$, and $x \geq x_0$.

By taking $R = R(x)$ the smallest solution to the equation

$$R = \exp\left(\frac{d(x)}{\sqrt{R \lambda_p}} \sum_{m=0}^{\infty} \sqrt{\frac{m + 1}{\lambda_p^m}}\right),$$

the previous argument can be easily adapted to deduce

$$u^j(\lambda_p^k x) \leq R(x) \lambda_p^{k-j} x,$$

for all $k \in \mathbb{N}$ and $1 \leq j \leq k$, and $x \geq x_0$.}

5.5. **Proof of Lemma** Suppose that $n^* \geq 5$ and $b > 0$ are such that $\mathcal{C}_p(k) \geq n^{-b}$ for all $|n^*/2| \leq k < n^*$. Using (3), and using Proposition 1 and Proposition 3, we obtain

$$\mathcal{C}_p(n) \geq \left(\sum_{k<2(n-1)/3} \mathcal{W}_p(k, n)\right) \left(\min_{k<2(n-1)/3} \mathcal{C}_p(k)\right) - \frac{n}{6} \left(4p(1-p)\right)^{(n-2)/3}$$

$$\geq \left(\sum_{k\leq n} \mathcal{W}_p(k, n)\right) \left(\frac{2(n-1)}{3}\right)^{-b} - \frac{n}{6} \left(4p(1-p)\right)^{(n-2)/3},$$
for \( n^* \leq n < (3n^* + 1)/2 \). Since \( \sum_{k=0}^{n^*} W_p(k, n) = (1 - (p - 1)^n)/(2 - p) \), then \( \sum_{k=0}^{n} W_p(k, n) = (1 - 2p)(2 - 3p)/(2 - p) - 2p(p - 1)^n \), for all \( n \geq 2 \). From this it follows that

\[
C_p(n) \geq n^{-b} \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} - 2p(p - 1)^n \right) \left( \frac{3n}{2} \right)^b - \frac{n^{1+b} (4p(1-p))^{(n-2)/3}}{6},
\]

for \( n^* \leq n < (3n^* + 1)/2 \). Now, if \( n^* \) is large enough so that

\[
\left( \frac{(1 - 2p)(2 - 3p)}{2 - p} - 2p(p - 1)^n \right) \left( \frac{3}{2} \right)^b - \frac{n^{1+b} (4p(1-p))^{(n-2)/3}}{6} > 1
\]

for all \( n \geq n^* \), then we have \( C_p(n) \geq n^{-b} \) for \( |n^*/2| \leq n < (3n^* + 1)/2 \). In this way, the interval where the lower bound \( C_p(n) \geq n^{-b} \) holds, enlarges from \( |n^*/2| \leq n < n^* \) to \( |n^*/2| \leq n < (3n^* + 1)/2 \). This constitutes the induction step from which it follows that \( C_p(n) \geq n^{-b} \) for all \( n \geq |n^*/2| \) and \( p \in (0, p^*) \). The validity of this induction depends on the existence of \( n^* \) and \( b \) such that \( C_p(k) \geq n^{-b} \) for all \( |n^*/2| \leq k < n^* \) and all \( p \in (0, p^*) \), and

\[
\left( \frac{(1 - 2p)(2 - 3p)}{2 - p} - 2p(p - 1)^n \right) \left( \frac{3}{2} \right)^b - \frac{n^{1+b} (4p(1-p))^{(n-2)/3}}{6} > 1
\]

for all \( n \geq n^* \) and all \( p \in (0, p^*) \). We found such \( b \) and \( n^* \) by using [4], which allows us to explicitly compute \( C_p(n) \) as function of \( p \). By doing so, we verify that \( C_p(n) \geq n^{-3} \) for \( n \leq 402 \) and \( p \in (0, p^*) \). We also verified that the function

\[
p \mapsto \left( \frac{(1 - 2p)(2 - 3p)}{2 - p} - 2p(p - 1)^n \right) \left( \frac{3}{2} \right)^b - \frac{n^{1+b} (4p(1-p))^{(n-2)/3}}{6}
\]

increases monotonously with \( n \) for all \( n \geq 60 \). Since

\[
\min_{p \leq p^*} B_p(402) = B_{p^*}(402) \approx 1.0015,
\]

then we can choose \( n^* = 402 \) and \( b = 3 \). The theorem follows with \( b = 3 \) and \( n_{p^*} = \lfloor n^*/2 \rfloor = 201 \).

\[ \square \]

6. Conclusions.

6.1. Summary. We approach the expansion-modification system considering the action of the expansion-modification dynamics over the space of Borel measures on \( \{0, 1\}^{\mathbb{N}_0} \). Inside this framework, we have proved the existence and uniqueness of the stationary measure, which attracts all initial distributions as times goes to infinity (Theorem 1). We have also proved that the stationary measure has correlations decaying faster than a certain power-law (Theorem 2). Although the calculations leading to these results strongly rely on a recurrence relation very specific to this system, some degree of generality could be expected. For instance, existence and uniqueness of the stationary measure depends only on the primitivity of the stochastic matrices describing the dynamics of the finite-size marginals. Following a simple argument (see Lemma 2.4.4. in [4]), one can easily prove that the primitivity of the matrix associated to the 1-marginal implies the primitivity of the matrices associated to all marginals, from which it follows the *-weak convergence of the distributions towards a unique stationary measure. On the other hand, decay of correlations and asymptotic scaling of the correlation function could be proved for
cases where recurrence relations similar to (3) are satisfied. For this kind of increasingly complex recurrence relations, which cannot be solved in closed form, one cannot expect to obtain an exact scaling behavior.

The main contribution of this work is the rigorous proof of the asymptotic scaling of the correlation function, for a rather large interval of mutation probabilities (Theorem 2). In order to prove this it was necessary to find a power law bounding from below the correlation function (Theorem 3). The validity of is result, and consequently of our main theorem, could in principle be extended to the range $0 < p < 1/2$, where it seems that $C_p(n) > 0$ for all $n$ sufficiently large. However, our technique depends on the explicitly computation of $p \mapsto C_p(n)$ for values of $n$ in the range were $C_p(n) > 0$, and we observe that this range diverges as $p$ approaches $1/2$. Hence, the choice of the range $(0, p^*)$ is rather arbitrary and of practical nature.

6.2. Scaling. The argument developed in Section 4 suggests that the scaling property extends to the whole range of mutation probabilities and that the corresponding scaling exponent varies piece-wise smoothly as function of that parameter. The same argument allows us to furnish an explicit expression for the scaling exponent that fits very well the exponent calculated by numerically solving the recurrence relation (3).

The scaling behavior of a system is usually detected by observing a power law behavior in its power spectrum. It can be easily verified that a power law behavior in the correlation function implies a power law behavior for the power spectrum $f(\omega) := |\mathcal{F}(C_p)(\omega)|$ where $\mathcal{F}(C_p)$ denotes the discrete Fourier transform of the correlation function. In our case, a straightforward computation shows that $f(\omega) = O(\omega^{-\alpha_p})$ with

$$\alpha_p := 1 - \beta_p = \frac{\log(2 - p) - \log(1 - 2p) - \log(2 - 3p)}{\log(2 - p)}.$$

6.3. Related work. In [12], Messer, Arndt and Lässing study a model generalizing the expansion-modification systems for which they deduce an asymptotic scaling behavior and a closed expression for the scaling exponent. Although we have followed similar ideas, their model evolves in continuous time. Besides, ours are the first rigorous results concerning those kind of models. In [10], Mansilla and Cocho analyze the correlation function of the model we consider, and they claim the existence of several scaling exponents in the non-asymptotic regime.

In a more recent paper, the expansion-modification system was studied in relation to the universality of the rank-ordering distributions [1]. The authors numerically found an order-disorder transition which would manifest itself on the scaling behavior. According to them, there would be a critical mutation probability $p_c \approx 0.4$, such that for $p > p_c$, long-range correlations and consequently the scaling behavior of $C_p$ would disappear. As we have shown, this kind of order-disorder transition does not occur. The apparent drop of long-range correlations for large $p$ can be explained by the lack of statistics. Indeed, a huge amount of data is needed to empirically compute correlation functions with a fast power-law decay. In order to observe a power-law decay with exponent $-5$ up to two decades, one would need of the order of $10^{10}$ sample sequences in $\{0, 1\}^{100}$, obtained by the action of the
Some other models of random substitutions have been previously studied. In [3], Godrèche and Luck considered a “perturbed” Fibonacci substitution for which they show that the Fourier spectrum is of mixed type, i.e., it contains both singular and continuous parts. Zaks obtain the same for a substitution system which can be seen a perturbation of the Thue-Morse sequence [18]. In both cases the system can be seen as a random perturbation of a quasicrystal. These is not the case for the expansion-modification system for which we observe a continuous spectrum.

Random substitutions have been treated in some generality by Koslicki in [4], in the framework of countable Markov chains. We approach the expansion-modification system from another point of view, considering the action of the expansion-modification dynamics over the space of Borel measures on \( \{0,1\}^\mathbb{N} \). Nevertheless, we could in principle use Koslicki’s results to, for instance, derive our Theorem 1. More interestingly, its result ensuring the almost sure convergence of frequencies (Theorem 2.4.10 in [4]) can be used to prove that the stationary measure \( \mu_p \) is absolutely continuous with respect to a shift-invariant ergodic measures. More closely related to ours is the approach by Toom and coworkers (see [15] and references therein). They define a certain class of substitution operators acting on shift-invariant measures, nevertheless, the class of substitutions they consider do not include the expansion-modification system. None of the above mentioned approaches directly apply in our setting, therefore some technical work would be required to adapt their results to our case.

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