The imaginary time Path Integral and non-time-reversal-invariant- saddle points of the Euclidean Action

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Abstract

We discuss new bounce-like (but non-time-reversal-invariant-) solutions to Euclidean equations of motion, which we dub boomerons. In the Euclidean path integral approach to quantum theories, boomerons make an imaginary contribution to the vacuum energy. The fake vacuum instability can be removed by cancelling boomeron contributions against contributions from time reversed boomerons (anti-boomerons). The cancellation rests on a sign choice whose significance is not completely understood in the path integral method.

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1 Introduction

Formally the imaginary time path integral is related to the trace of the imaginary time evolution operator \( \exp(-Ht) \) by the relation:

\[
\lim_{t \to \infty} \text{Tr}[\exp(-Ht/\hbar)] = N \int \mathcal{D}\Phi \exp(-S_E[\Phi]/\hbar), \tag{1.1}
\]

where \( \Phi \) denotes the coordinates in the quantum theory, \( H \) is the Hamiltonian operator, \( S_E \) is the Euclidean action (the imaginary time action) and \( N \) is a normalization factor. This formula is often exploited to compute the decay rates of metastable false ground states of the quantum theory by evaluating the path integral on the right semiclassically. Briefly, the procedure [1] can be understood by considering particle mechanics in one dimension. The action in real time is \( S = \int dt [\frac{1}{2} \dot{q}^2 - V(q)] \). Consider a potential \( V(q) \) as shown in Fig. 1. The point \( q = a \) is a classical ground state. Quantum mechanically the particle penetrates the barrier to the right. The tunneling probability can be computed by the WKB method. The path integral (P.I.) method is an elegant alternative method, where one simply expands the right hand side of (1.1) about classical solutions of the Euclidean equations of motion that extremize the imaginary time action \( S_E \) and satisfy the boundary condition \( \lim_{t \to \pm \infty} q = a \). Since \( S_E = \int dt [\frac{1}{2} \dot{q}^2 + V(q)] \), the extrema of \( S_E \) are given by classical paths of a particle moving in the potential \(-V(q)\).

Fig. 1. The potential \( V \) has a local minimum at \( q = a \). The escape point is at \( q = b \).
In this case there is a nontrivial classical solution to the equations of motion known as the bounce. The bounce is given by the path \( \overline{q}(t) \) that begins at \( q = a \) when \( t \to -\infty \), “bounces” off the point \( q = b \) (with \( V(b) = V(a) = 0 \)) and ends at \( q = a \) as \( t \to \infty \). Expanding \( S_E \) about \( \overline{q} \) up-to second order in the fluctuations we get,
\[
S_E[\delta q] = S_E[\overline{q}] + \int dt \delta q \left[ -d_t^2 + \frac{d^2 V}{dq^2}(\overline{q}) \right] \delta q + \text{higher order, with } \delta q = q - \overline{q}.
\]
The functional integral over the quadratic fluctuations is a simple Gaussian one if the eigenvalues of the operator \( O[\overline{q}] = -d_t^2 + \frac{d^2 V}{dq^2}(\overline{q}) \) are all non-negative. It is well known however, that the bounce is a saddle point of the action \( S_E \) and \( O[\overline{q}] \) has a single negative eigenvalue. The ill defined integration over the negative eigenmode can be done by continuing the eigenmode \( \psi \) to complex values, i.e., the integral
\[
\int d\psi \exp(-\alpha \psi^2), \quad \alpha \text{ is the negative eigenvalue of } O[\overline{q}]
\]
can be performed through a contour along the imaginary \( \psi \) axis. The result is that the integral over the Gaussian fluctuations yields a formally imaginary quantity: \( [\det(O[\overline{q}])]^{-1/2} \). When contributions from multi-bounce solutions of the Euclidean equations of motion are taken into account, equation (1.1) reduces to \( [1] \):

\[
\lim_{t \to \infty} \text{Tr}[\exp(-Ht/\hbar)] = \lim_{t \to \infty} \left( \frac{\omega}{\pi \hbar} \right)^{1/2} e^{-\omega t/4} \exp \left[ \exp(-S_E(\overline{q})/\hbar) \times Kt \right] [1 + O(\hbar)],
\]
where \( K = (S_E[\overline{q}]/2\pi \hbar)^{1/2} \left( \frac{\det'O[\overline{q}]}{\det O[a]} \right)^{-1/2} \), \( \omega = \frac{d^2 V}{dq^2} \bigg|_{q=a} \), \( O[a] = -d_t^2 + \frac{d^2 V}{dq^2}(q = a) \) and the prime on one of the determinants indicates that the zero eigenvalue of \( O \) coming from the time translational invariance of the bounce is to be excluded in the computation of the determinant. Comparing this with (1.1) one sees that the ground state energy is given by \( E_0 = \frac{1}{2} \hbar \omega + \hbar K \exp(-S_E(\overline{q})/\hbar)[1 + O(\hbar)] \). The first term in this expression is just the ground state energy of a harmonic oscillator, but the second term is imaginary and may be interpreted as the decay width of the metastable state localized near \( q = a \). Although the imaginary term is formally smaller in magnitude than order \( \hbar^2 \) terms that are neglected in this approximation, it is still the dominant contribution to any imaginary part in the energy and, heuristically, should provide an estimate of the decay width.

The confidence in the above method comes from its ability to reproduce the results of the more rigorous WKB approximation where the tunneling is seen as happening through the path of least resistance \( \overline{q}(t) \), \( t \in (-\infty, 0) \) that extends between \( q = a \) and the “escape point” \( q = b \) on the “other side” of the barrier. The bounce solution of the P.I. approach is exactly the “sum” of the least resistance path of WKB and its time reversal conjugate. \textit{The bounce itself therefore, is time-reversal- (T) invariant.}

This formalism can be extended to multiple dimensions \([2]\). In a system with \( n \) co-ordinates \( q_i \), where \( i = 1, \ldots n \), the paths of least resistance connect the false vacuum \( q_i^f \)
to a surface $\sigma$ of zero potential that lies on the “other side” of the barrier. In the P.I. formalism, each such path extends to a $\mathcal{T}$ invariant bounce and makes an imaginary contribution to the vacuum energy. If there are several saddle points $\bar{q}_i(\alpha)$ ($\alpha = 1,2..m$), the imaginary part of the vacuum energy is given by

$$\delta E_0 = \hbar \sigma^m_{\alpha=1} K_{\alpha} \exp(-S_E[\bar{q}_i(\alpha)/\hbar]) [1 + O(\hbar)]. \quad (1.3)$$

In this approximation one need retain only the dominant bounces which are the bounces with the least (Euclidean) action.

While the saddle points in the P.I. formalism include all the WKB trajectories as bounces, it is not obvious whether the least action saddle point of $S_E$ is always a bounce. If in some system the dominant imaginary contribution to the ground state energy comes from saddle points that are not bounces, the corresponding “decay width” (if non-zero) has nothing to do with tunneling and will signal a limitation of the P.I. formalism. In a non-compact one dimensional system like the one shown in Fig. 1, all periodic solutions of the Euclidean equations of motion must retrace themselves backwards and be $\mathcal{T}$ invariant. This need not be true in systems with more degrees of freedom. Consider a two dimensional system with coordinates $q_1$ and $q_2$ (Fig. 2). The origin is a minimum of the potential and there may be a loop-like trajectory along $P$ that extremizes $S_E$ but is not invariant under time reversal.

Fig. 2. There is a trajectory along the path $P$ that is not invariant under time-reversal but extremizes $S_E$.

Because they lack $\mathcal{T}$ invariance, trajectories like this are not bounces and there is no corresponding WKB trajectory. Nevertheless, in the P.I. formalism, the determinant
factor in $K$ computed for them can in principle have a single negative eigenvalue and, like a bounce, they can make an imaginary contribution to the vacuum energy.

At this point the following questions arise:
(i) Do trajectories like this actually exist for which $K$ is imaginary?
(ii) Can such fake bounce solutions exist in quantum field theories?
(iii) Does the P.I. formalism for computing the decay width of the vacuum fail when such solutions exist?

We will find answers to these questions in the next sections. In particular we will find that such fake bounces can exist even in field theories and indicate how one can take their effect into account in the P.I. formalism. Because these solutions to the equations of motion resemble (allegedly possible) periodic trajectories of a boomerang, we will call them “boomerons” to facilitate distinction with the bounces.

2 Boomerons in Quantum Mechanics

Let us show that the answer to question (i) of the previous section is in the affirmative. Consider quantum mechanics for a particle of unit mass moving on a 2 dimensional sphere ($S^2$). Suppose the potential energy is minimized at the north pole where the particle attains its classical ground state. We are interested in closed trajectories on the sphere with one point fixed at the north pole. The space $\Omega(S^2)$ of these trajectories is not simply connected. In the language of algebraic geometry, $\Pi_1[\Omega(S^2)] \simeq \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers (see, for instance ref. [3]).

If $\Omega(S^2)$ were a compact and finite dimensional space, this fact alone would suffice to prove the existence of a boomeron. This is best illustrated by Fig. 3a where the torus is a compact and finite dimensional analogue of $\Omega(S^2)$. The height function on the torus plays the role of $S_E$. If we look at non-contractible curves of winding number 1 based at the global minimum of $S_E$ at $P$, the saddle point at $B$ emerges as the highest point of the curve with the lowest highest point (the mountain pass curve). The argument breaks down even for a simple non-compact space like the tapered cylinder of Fig. 3b where the saddle point $B$ “escapes” to infinity. In such cases, neither the existence nor the $T$-non-invariance of the saddle point can be guaranteed. Nonetheless, the non-triviality of $\Pi_1[\Omega(S^2)]$ is an encouraging signal for the boomeron hunter. In this particular example the boomeron can be explicitly constructed for simple potentials on $S^2$, such as a potential $U(\theta)$ that is invariant for rotations around the axis joining the north and south poles.
(θ is the latitude). The boomeron trajectory $B$ then traces out a large circle passing through the north pole (Fig. 4). It is simple to show, when $U(θ)$ increases monotonically toward the south pole, that the Hessian operator $O$ computed at the boomeron trajectory has a negative eigenvalue (roughly, this unstable direction corresponds to sliding off the trajectory to the left or right, so that it moves into the left or the right hemisphere defined by the boomeron). The action of the boomeron is $S_E = \int dθ \sqrt{\frac{R^2 U(θ)}{2}}$, where $R$ is the radius of the sphere. The corresponding boomeron contributions to the ground state energy are imaginary and of order $(\frac{S_E}{2\pi \hbar})^{1/2}\exp(-S_E/\hbar)$.

**Fig. 3a.** $P$ is the minimum of the height function on the torus. Of a set of non-contractible loops through $P$, the saddle point $B$ is the highest point on the loop with the lowest highest point.

**Fig. 3b.** Three non-contractible curves are drawn on an open cylinder. The non-contractible curve through $P$ with the lowest highest point “escapes to $\infty$” toward left and right.

**Fig. 4.** Attaching an infinite cylinder to the sphere creates a new minimum $P'$ of $S_E$, while the boomeron $B$ may continue to exist.
This example not only establishes the existence of boomerons in quantum mechanics but also sheds light on an interesting aspect of the boomerons: they are likely to exist whenever the classical vacuum is the global minimum of the potential and the space of closed trajectories passing through the classical vacuum is not a simply connected space. We do not preclude the existence boomerons that are not “topologically required” as above. For instance by attaching an infinite cylinder to the sphere (Fig. 4), the topology of the configuration space is changed from $S^2$ to $R^2$. Since $\Pi_1[\Omega(R^2)]$ is trivial, there are no topological requirements for the existence of the boomeron. Nevertheless, the boomeron continues to exist if the attached cylinder does not intersect the boomeron’s path. In this case however, a new minimum of the action must appear in the form of the trajectory $P'$. The boomeron $B$ is the saddle point between the two action minima at $P$ and $P'$.

3 Boomerons and Path Integrals

Before we move on to examples of boomerons in field theories let us attempt to answer the question posed in (iii) of section 1 at the level of quantum mechanics. Note that a boomeron is a nuisance in the P.I. formalism only if its action is lower than the action of all bounces. It is not hard to construct theories where this is true. Indeed, in the example constructed in section 2, bounces do not exist at all while boomerons do. In these theories, at first glance, the P.I. formalism seems to fail. However, there is a simple way of saving the method. Not surprisingly it rests on the single fact distinguishing boomerons from bounces, namely, the boomeron’s non-invariance under $T$. The time reversed boomeron is a different boomeron. One could give it the suggestive name anti-boomeron and ask if its contribution can cancel the contribution of the boomeron.

This does appear to be a possibility in the P.I. method. Recall that the imaginary term $K$ in (1.2) suffers from a sign ambiguity. When carrying out a formally divergent Gaussian integral of the kind $\int d\psi \exp(-\alpha\psi^2)$ with a negative $\alpha$ the analytic continuation of $\psi$ to imaginary values can be done in two ways. Integrating $\psi$ from $-i\infty$ to $i\infty$ and from $i\infty$ to $-i\infty$ give opposite signs for the integral. In a theory with no boomerons, the sign choice for the imaginary part of the vacuum energy does not change the physical value of the decay width. When both boomerons ($B$) and anti-boomerons ($\overline{B}$) are present and dominant, their imaginary contributions to the vacuum energy can be exactly cancelled.
against each other if the action is invariant under time reversal \( (S_E[B] = S_E[\bar{B}]) \). One does this simply by choosing opposite signs for \( K \) in the boomeron and the anti-boomeron in equation (1.3).

What is unsatisfactory about this otherwise simple and straightforward procedure is that the P.I. formalism does not require us to make this sign choice. This is not a disaster, since the P.I. method for computing vacuum decay widths is, despite its elegance, a heuristic method. In the presence case it is not so much the wrong result that one gets upon a “wrong” choice of signs, as the possibility of a perfect cancellation with the “correct” sign choice that we find intriguing. One would hope to understand the reason for this at a deeper level, perhaps in an extended version of the P.I. method where the orientation of the analytic continuation at one saddle point fixes the orientation of the analytic continuation at other saddle points in an un-ambiguous manner.

There is also the need to check the familiar regularizations of the P.I. method carefully to see if the boomeron and the anti-boomeron contributions can indeed have opposite signs. This is actually a tricky point which we would like to clarify with an example. Suppose a boomeron exists in a 4-dimensional bosonic field theory with real fields \( \phi_1, \ldots, \phi_n \) which are components of scalar or gauge fields. The boomeron \( \phi_i(x, y, z, t) \), is not invariant under time reversal \( (\phi_i(x, y, z, t) \neq \phi_i(x, y, z, -t)) \). The finiteness of the boomeron action demands that the fields should approach constant values as \( x, y, z, t \to \infty \). Applying an \( SO(4) \) transformation \( \Lambda \) to the above configuration preserves the boundary conditions and yields another field configuration \( \Lambda \phi_i \) which is also a boomeron with the same action. Integrating over the zero mode of \( SO(4) \) transformations therefore amounts to adding these boomeron contributions with the same sign. This is required in any regularization since the boomerons are connected by a continuous zero mode but the sign can not change continuously.

On the other hand, the anti-boomeron \( \overline{\phi_i} \) and its \( SO(4) \) transforms \( (\Lambda \overline{\phi_i}) \) are obtained from the original boomeron \( \phi_i \) by applying an \( O(4) \) transformation (namely, \( T \)) that is not connected to identity. Thus the boomerons and anti-boomerons remain separated in the space of field configurations, and there is no apparent obstruction to having opposite analytic continuation at boomerons and anti-boomerons. The fact that \( O(4) \) is not a connected group is vital. The same is true of the symmetry group \( O(d) \) in \( d \) dimensional field theories \( (d \geq 2) \). When \( d = 1 \), the appropriate symmetry group is \( Z_2 \) (\( T \)) which is also a discrete group.

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2 We discuss theories that are not invariant under time-reversal in a later section.
Although there is no obstruction to choosing opposite signs from continuous space-time symmetries relating boomerons and anti-boomerons, there may be continuous internal symmetries relating them. Once again an example will explain this situation best. In the 2-dimensional quantum mechanics described in the previous section, the boomeron and the antiboomeron trajectories are coincident large circles with opposite orientations. Because the potential $U(\theta)$ has a rotational symmetry in the $\phi$ (the longitude) direction, one can obtain the anti-boomeron from the boomeron by continuously changing $\phi$ to $\phi + \pi$ through a zero mode of boomerons.

The complication due to internal symmetries will also arise in field theories, especially in the form of gauge-symmetries. Here we suggest a regularization that may be used to preserve the boomeron-anti-boomeron cancellation. One simply adds a term $\delta U$ to the potential that breaks the discomforting continuous symmetry but preserves time-reversal-invariance of the action. Then the boomeron-anti-boomeron imaginary parts can cancel. The $\delta U \to 0$ limit is taken after the cancellation. Physically, if the cancellation gives the correct result in the regulated theory, it should be valid in the symmetric theory.

In higher than 2-dimensional gauge theories, regularizations that break gauge symmetries may spoil the renormalizability of the theory. However, the zero mode relating the boomeron and the anti-boomeron is an infra-red effect and its resolution should be independent of the high energy behaviour of the theory. In this spirit, one can define a cut-off theory by integrating out all the high energy modes. In the low energy effective theory, the term $\delta U$ may be added (it can be thought of as coming from a renormalizable Higgs interaction, with a massive Higgs field that has been integrated out). The subsequent regularization of boomeron contributions may proceed as sketched above. Having made our suggestion we remark that we will not demonstrate this procedure by working out an example in detail. This is not for lack of examples in gauge theories, but as we shall see shortly, all the gauge boomerons we could find, offer other (peculiar but easier) solutions to the symmetry problems described here. In the next section we consider in greater detail these peculiarities of boomerons in field theories.

4 Boomerons in Field Theories
4.1 Scalar Fields

It is simple to show that for purely scalar field theories in dimensions greater than 2, the existence of a boomeron implies the existence of a bounce with an equal or lower action. Consider a purely scalar field theory in \(d\) (\(d \geq 2\)) dimensions. The Euclidean action is

\[
S_E = T + V,
\]

\[
T = \int d^{d}x \frac{1}{2} \left[ \left( \frac{d\phi_i}{dx_{\mu}} \right)^2 \right],
\]

\[
V = \int d^{d}x \left[ U(\phi_1, ... \phi_n) \right],
\]

where \(\phi_i\) are \(n\) real scalar fields and \(U\) is a potential \(\text{which is normalized to be zero at the vacuum of interest}\). Under scale transformations \(x \to \lambda x\) with the positive parameter \(\lambda\) the terms \(T\) and \(V\) scale like: \(T \to \lambda^{d-2}T\); \(V \to \lambda^d V\). At a stationary point \(\bar{\phi}_i(x)\) of the action we must therefore have,

\[
\frac{dS_E[\bar{\phi}_i]}{d\lambda} \bigg|_{\lambda=1} = (d - 2)T[\bar{\phi}_i] + (d)V[\bar{\phi}_i] = 0 .
\]

(4.5)

This condition is satisfied by a non-trivial finite action field configuration only if \(V \leq 0\), which implies that the vacuum of interest is a false vacuum (\(i.e.\ U\ becomes negative somewhere) and bounce solutions also exist.

It is straightforward to show that there is always a bounce with an equal or lower action than any boomeron (should a boomeron exist). Suppose a boomeron \(\bar{\phi}_i\) exists with \(T\) and \(V\) obeying (4.3). We can slice the boomeron into two halves using a \(d-1\) dimensional plane. Let us call the two halves \(L\) and \(R\) (for left and right halves respectively). The integrals \(T\) and \(V\) appropriately split into integrals over \(L\) and \(R\) given by \(T(L), T(R), V(L)\) and \(V(R)\) respectively with \(T = T(L) + T(R)\) and \(V = V(L) + V(R)\). We slice the boomeron so that \(T(L) = T(R) = \frac{1}{2}T\). The different ways of slicing the boomeron while preserving this condition are in one to one correspondence with the points on the group space \(SO(d)\). We choose a slicing for which \(V(R)\) is least. Then \(V(R) \leq V(L)\). Let us call the axis perpendicular to the \(d-1\) dimensional plane separating \(L\) and \(R\) the time axis \(t\). The slicing plane intersects this axis at \(t = 0\) by convention, with \(t > 0\) in the right half.

Using equation (4.3), the action of the boomeron is \(T[\bar{\phi}_i] + V[\bar{\phi}_i] = \frac{2}{d}T[\bar{\phi}_i]\). A time-reversal- invariant- field configuration \(\hat{\phi}_i\) is obtained from the boomeron \(\bar{\phi}_i\) by replacing
the left half with a mirror image of the right half.

\[
\hat{\phi}_i(x, ...t) = \begin{cases} 
\overline{\phi}_i(x, ...-t) & \text{for } t \leq 0 \\
\overline{\phi}_i(x, ...t) & \text{for } t > 0 
\end{cases}
\]  

(4.6)

The action of the new field configuration is lower than or equal to the action of the
boomeron

\[
S_E[\hat{\phi}_i] = 2 \left[ T(R)[\overline{\phi}_i] + V(R)[\overline{\phi}_i] \right] \leq \frac{2}{d} T[\overline{\phi}_i].
\]  

(4.7)

Although \(\hat{\phi}_i\) is not likely to be smooth at \(t = 0\), there are smooth field configurations
in the neighbourhood of \(\hat{\phi}_i\) whose action is as close to that of \(\hat{\phi}_i\) as one wants. The
configuration \(\hat{\phi}_i\) does not satisfy (4.5) but one can scale transform the configuration by
the parameter \(\beta\) \((x, ...t \rightarrow \beta x, ...\beta t)\) so that condition (4.5) is satisfied. Then the following
statements are true for the scale transformed configuration \(\hat{\phi}_i^\beta\).

(i) \(\beta \leq 1\).

We show this as follows. Let us write \(T = T[\overline{\phi}_i] = T[\hat{\phi}_i]\) and \(V = V[\hat{\phi}_i]\). We scale
transform \(T\) and \(V\) so that \((d - 2)(\beta^{d-2} T) + (d)(\beta^d V) = (d - 2)(T[\hat{\phi}_i^\beta]) + (d)(V[\hat{\phi}_i^\beta]) = 0\). This
condition, along with the condition, \(V \leq \frac{2}{d} T\), implies that \(\beta \leq 1\).

(ii) \(S_E[\overline{\phi}_i] = \frac{2}{d} T \geq S_E[\hat{\phi}_i^\beta] = \beta^{d-2} T + \beta^d V\).

Define \(\alpha\) by the relation \(V = -\alpha T\). Then, \(\frac{2}{d} T \geq \beta^{d-2} T + \beta^d V = \beta^{d-2}[1 - \alpha \beta^2] T\), iff,
\(\beta^{d-2} \leq 1\) \((\alpha \geq \frac{d-2}{d})\). Therefore the assertion in (ii) follows from the assertion in (i).

\(\hat{\phi}_i^\beta\) is therefore a field configuration which satisfies (4.5) and is invariant under time-
reversal. It has been shown elsewhere \[4, 5\] that the action of such a field configuration
bounds the action of a bounce from above. By the result in (ii) a bounce exists whose
action is equal to or lower than the action of the boomeron.

The above results are actually valid for \(d = 2\) also. However one must remember that
the action in equation (4.4) does not include the most general renormalizable terms when
\(d = 2\). In particular, non-linear sigma models are excluded from the discussion. We will
come back to this point again in the next section.

4.2 Gauge and Topological Boomerons

When the field theory has gauge fields in addition to scalar fields, the arguments presented
in the preceding section do not hold. Let us show that in this case a boomeron may
exist even if no bounce exists in the theory. We begin with a simple gauge theory in \(d\)
dimensions. The action is

\[ S_E = T + V + F, \]
\[ T = \int d^d x \left[ |D_\mu \phi_1|^2 \right], \]
\[ V = \int d^d x \left[ U(\phi_1, \ldots \phi_n) \right], \]
\[ F = \int d^d x \frac{1}{4} \left[ F_{\mu \nu}^2 \right] \]  

(4.8)

where \( D_\mu \) is the covariant derivative and \( F_{\mu \nu} \) is the field strength. The potential \( U \) is gauge-invariant. One defines scale transformation by \( \beta \) as \((x, \ldots t) \rightarrow \beta(x, \ldots t), A_\mu \rightarrow \frac{1}{\beta} A_\mu. \)

Under this transformation \( T, V \) and \( F \) transform as

\[ T \rightarrow \beta^{d-2} T, \quad V \rightarrow \beta^d V, \quad F \rightarrow \beta^{d-4} F. \]

At a stationary point of the action, we have an equation analogous to (4.5)

\[ (d-2)T + (d)V + (d-4)F = 0. \]  

(4.9)

This equation may have non-trivial solutions for \( V \geq 0 \) provided \( d \leq 4 \). The dimension \( d = 4 \) is a critical dimension where equation (4.3) may have non-trivial and finite action solutions only for \( V = T = 0 \) (a possibility, for instance, in pure Yang-Mills theories).

Thus stationarity with respect to scale transformation produces no obstruction to boomerons in 2, 3 or 4 dimensional gauge theories even if there is no vacuum tunneling in the system.

Indeed, boomerons in gauge theories may exist in the literature! The well known sphaleron [6] of a 4-dimensional \( SU(2) \) gauge theory is likely to be a boomeron in a 3-dimensional \( SU(2) \) gauge theory. Recall that the 4-dimensional theory has the action

\[ S_E = \int d^4 x \left[ \frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a + |D_\mu \Phi|^2 + \lambda (\Phi^\dagger \Phi - \frac{1}{2} v^2)^2 \right] \]  

(4.10)

where the Higgs field \( \Phi \) is a doublet under \( SU(2) \), \( F_{\mu \nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon^{abc} W_\mu^b W_\nu^c \) is the \( SU(2) \) field strength, \( W_\mu = W_\mu^a \tau_a \) are the \( SU(2) \) gauge fields defined using the Pauli matrices \( \tau_a \) and the covariant derivative is \( D_\mu = \partial_\mu - \frac{1}{2} g W_\mu \). The sphaleron solution is an \( SO(3) \) invariant non-trivial time-independent solution of the equations of motion described by two real functions \( f(\xi) \) and \( h(\xi) \) (we follow the conventions of ref. [6]) of the dimensionless coordinate \( \xi = vgr \) where \( r^2 = x_1^2 + x_2^2 + x_3^2 \). The gauge and Higgs fields are given by:

\[ W_\mu = -\frac{2i}{g} f(\xi) (\partial_\mu U^\infty)(U^\infty)^{-1} \]
\[ \Phi = \frac{v}{\sqrt{2}} h(\xi) U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  

(4.11)
where the $SU(2)$ elements $U^\infty$ are defined as $U^\infty = i\vec{r} \cdot \hat{x}$.

The sphaleron is a saddle point of energy and has at least one unstable direction (a negative eigenvalue) associated with it. In ref. [8] it is shown that the sphaleron has in fact a single unstable direction when spherically symmetric perturbations are considered, provided $M_H < 12.03 \, M_W$, where $M_H$ is the Higgs mass and $M_W$ is the mass of the gauge field. One expects that non-spherically symmetric perturbations will increase the energy of these configurations. If this is true, then the sphaleron in these cases is the lowest energy saddle point in the theory and, from our point of view, a boomeron in a 3-dimensional theory where the Euclidean action is the same as the energy functional of the 4-dimensional theory. However there is no false vacuum in the 3-dimensional theory and the question of bounces does not arise.

The sphaleron is a topological boomeron in essentially the same way as our example from quantum mechanics in section 2 [7, 6]. One often says that the existence of the sphaleron is related to the presence of multiple topological vacua in the 4-dimensional theory. The vacua are actually physically indistinguishable, being equivalent up to a gauge transformation. However there are non-contractible trajectories in the space of field configurations connecting these vacua. Because the end points are gauge equivalent, they are very much like the loops in the space $\Omega(S^2)$ described in section 2.

The topological vacua correspond to distinct maps from the compactified three space $S^3$ to the gauge group $SU(2)$. Such maps are characterized by the integer winding number $n$. The sphaleron is a field configuration of winding number $1/2$ which is the “lowest” saddle point of energy between the vacua of winding numbers 0 and 1 [3]. In the 3-dimensional theory, where the sphaleron is a boomeron, a semiclassical estimate of the imaginary part of the vacuum energy will be zero only if the contributions of the sphaleron and the anti-sphaleron (whose winding number is $-1/2$) are made to cancel by design. This issue is complicated by the fact that the sphaleron and the anti-sphaleron are actually gauge equivalent [3]. It is instructive to see how the gauge equivalence is established. In the 3-dimensional theory, we identify the coordinate $x_3$ as Euclidean time. Then the sphaleron fields $(W, \Phi)$ and the anti-sphaleron fields $(\bar{W}, \bar{\Phi})$ are related by:

$$
\bar{W}_i(x_1, x_2, x_3) = \begin{cases} 
W_i(x_1, x_2, -x_3) & \text{for } i = 1, 2 \\
-W_i(x_1, x_2, -x_3) & \text{for } i = 3
\end{cases}
$$

$$
\bar{\Phi}(x_1, x_2, x_3) = \Phi(x_1, x_2, -x_3).
$$

(4.12)

\footnote{I am grateful to F. Klinkhamer for showing me this equivalence.}
Therefore the anti-sphaleron fields are given by similar expressions as (4.11) with $U^\infty$ replaced by $\bar{U}^\infty(x_1, x_2, x_3) = U^\infty(x_1, x_2, -x_3)$. However $-U^\infty = -i\tau^3U^\infty i\tau^3$. Therefore, a global $SU(2)$ transformation by $-i\tau^3$ takes $W$ to $\bar{W}$, while $\bar{\Phi}$ is obtained from $\Phi$ by the same transformation along with a $U(1)$ charge rotation by $\exp(i\pi/2)$ \textit{(i.e.} an extra $i$).

A peculiarity of the sphaleron as a boomeron can now be seen as follows. As we have said before, a saddle point of the Euclidean action can be interpreted as a boomeron only if it has trivial boundary conditions (all fields approach constant values at infinity). The sphaleron ansatz of (4.11) does not have this property. But there are no topological restrictions to choosing a different gauge in which $\Phi(x \to \infty) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; and $W(x \to \infty) = \frac{-2i}{g}(dU)(U)^{-1}$ where $U$ is the identity matrix ($U \equiv 1$). In such a gauge it is possible to define a fractional charge of the sphaleron by integrating the Chern-Simons action density: $Q = \frac{g^2}{32\pi^2} \int d^3x [F \wedge W - \frac{2}{3} W \wedge W \wedge W]$. The value of $Q$ for the sphaleron in this gauge is $1/2$. One is now interested in continuous symmetry transformations that preserve the boundary conditions. Global symmetry transformations that act non-trivially on $\Phi$ clearly do not preserve the boundary conditions. This leaves us with the small gauge transformations (those, that can be continuously obtained from the identity and do not change the boundary conditions). However, small gauge transformations do not change $Q$, yet the charge $Q$ for the anti-sphaleron must be $-1/2$ as is easily seen by applying time reversal to the form of $Q$. Because the sphaleron and the anti-sphaleron are gauge equivalent the gauge transformation relating them is a large gauge transformation which can not be continuously obtained from identity!

What is the significance of this result? In computing gauge invariant Green’s functions, it is redundant to consider multiple gauge copies in the path integral. However, if one is computing the imaginary part of the vacuum energy, the result depends crucially on the analytic continuation. The analytic continuation can be made gauge dependent provided opposite continuations are made only over gauge copies that are not connected by small gauge transformations.

This simple resolution of the symmetry problem can be seen in yet another gauge theory, namely a $U(1)$ gauge theory in $0 + 1$ dimensions \textit{(i.e.} quantum mechanics). A topological gauge boomeron exists much in the same way as sphalerons in $2+1$ dimensions. In this case the topological requirement is the existence of non-trivial maps from the compactified space-time $S^1$ (a circle) to the gauge group $U(1)$ ($\Pi_1[U(1)] = Z$). Let us
explicitly construct the boomeron in a model. The action is:

\[ S_E = \int dt \left[ |Dx|^2 + U(|x|) \right] \]

(4.13)

where \( x = \frac{1}{\sqrt{2}}(x_1 + ix_2) \) is a complex coordinate and the covariant derivative is given by \( Dx = d_t x - igAx \). The gauge field \( A \) has no kinetic term in the action. The potential \( U \) may be chosen to be of the form shown in Fig. 5a, with a minimum at \(|x| = a\).

![Potential U(|x|) graph](image)

Fig. 5a (Left). The potential \( U(|x|) \) has a minimum at \(|x| = a\).

Fig. 5b (Right). The circle in the \( x_1 - x_2 \) plane is defined by \(|x| = a\). Bold lines denote trajectories.

The theory can be quantized in the axial gauge \( A = 0 \), where Schrödinger's equations have no dependence on \( A \) and the wave functions are \( U(1) \) invariant wave functions of the corresponding theory with no gauge field. Where are the boomerons? There are only two classical solutions of the Euclidean equations of motion that begin from a classical ground state (say \( x = a \)) with zero initial velocity. The first is the trivial solution \( x \equiv a \). The second is the solution that starts at \( x = a \) and ends at \( x = -a \) and goes along the real line \( x_2 = 0 \). Consider now trajectories \( x(t) \) that begin \((t \to -\infty) \) at \( x = a \) and end \((t \to +\infty) \) at \( x = a_1 + ia_2 \) with \( a_1^2 + a_2^2 = a^2 \). On the space \( P(a, 0; a_1, a_2) \) of all such trajectories with fixed \( a_1, a_2 \), there is a trajectory \( x(a, 0; a_1, a_2) \) that has the least action. Fig. 5b shows three such trajectories for different values of \( a_1 \) and \( a_2 \). As one moves the end point \( x = a_1 + ia_2 \) of the trajectory \( x(a, 0; a_1, a_2) \) clockwise along the circle \(|x| = a\) through the points \( Q_0, Q_1, Q_2, Q_3, Q_0 \) the value of the action increases monotonically from its least value 0 for the trivial trajectory \( x(a, 0; a, 0) \) to a maximum at \( x(a, 0; -a, 0) \) and then decreases again to 0 at the trivial trajectory. Thus the solution \( x(a, 0; -a, 0) \) to the Euclidean equations of motion is a saddle point of the action. This trajectory still does not resemble a boomeron because the boundary conditions at \( t \to \pm \infty \) are different. However,
that shortcoming can now be removed by the gauge transformation $x \rightarrow \exp[i\alpha(t)]$, with any $\alpha(t)$ satisfying $\alpha(-\infty) = 0, \alpha(\infty) = \pi$ and $A(t) = \frac{1}{g} \int dt \alpha(t) = A(-t)$. This results in the trajectory ‘closing in’ to give periodic boundary conditions for both $x$ and $A$.

This is perhaps the simplest gauge-boomeron that possesses properties typical of the sphaleron. The charge $Q = \frac{2}{\pi} \int dt A(t)$ for the boomeron (anti-boomeron) is $1/2$ ($-1/2$). Zero modes of the boomeron from internal symmetries can involve only small gauge transformations, while the boomeron goes to the anti-boomeron only through a large gauge transformation. From a boomeron point of view, the cancellation of the imaginary part of the vacuum energy is identical to the case of the sphaleron in the $2 + 1$ dimensions.

The discussion above brings out the general features of gauge boomerons in arbitrary dimensions. However, there may be other physically relevant processes associated with boomerons which depend crucially on the dimension of space time. In the above $0 + 1$ dimensional case for instance, when one chooses the gauge $A = 0$, the boomeron has non-periodic boundary conditions that resemble an instanton. In fact it is easy to see that these instantons play a physical role: they mix the asymmetric classical ground-states of the theory to produce a $U(1)$ symmetric ground-state. Such an effect can not be expected to happen in $2 + 1$ dimensions, where the gauge fields can not be completely gauged away and spontaneous symmetry breaking does indeed take place ($SU(2) \rightarrow U(1)$). On the other hand, the boomerons in the $2 + 1$ dimensional theory may play a role in Fermion condensation and spontaneous breaking of global chiral symmetries (suitably defined for $2 + 1$ dimensions) that has no analog in the $0 + 1$ dimensional case. Suffice it to say here, that there are physical amplitudes that are affected by the boomerons. Typically, these effects are not related to the analytic continuations that yield imaginary integrals. They are obtained when one computes the amplitudes by making phenomenological estimates of the apparently divergent integrals. We propose to explore some of these aspects in a future publication [10] and restrict ourselves to the study of imaginary contributions to vacuum energy in this paper.

Finally we ask the question: Do boomerons exist in 4-dimensional gauge theories? A promising hunting ground would be topological sphalerons of 5-dimensional theories that arise if a nontrivial map exists from the compactified space $S^4$ to the gauge group space $G$, i.e. $\pi_4(G) \neq I$. The following table from ref. [10] is useful:

$$
\pi_4(G) \simeq \begin{cases} 
Z_2 \oplus Z_2 & \text{for } (G = SO(4), Spin(4)) \\
Z_2 & \text{for } (G = Sp(n), SU(2), SO(3), SO(5), Spin(3), Spin(5)) \\
I & \text{for } (G = SU(n)(n \geq 3), SO(n)(n \geq 6), G_2, F_4, E_6, E_7, E_8)
\end{cases}
$$

(4.14)
An $SU(2)$ or $SO(4)$ gauge theory in 4-dimensions therefore should have a boomeron-anti-boomeron pair purely on topological grounds. The boomeron for the $SU(2)$ gauge theory does in fact seem to exist provided the action contains some higher-than-dimension-4-operators to constrain the scale of the solution. This saddle point has been called the $I^*$ instanton in ref. [11]. Physically, the $I^*$ may make a noticeable impact on four Fermion scattering amplitudes, to which it contributes a term that grows exponentially with energy [11]. Once again there do not seem to be any small gauge transformations relating the $I^*$ to its time-reversal-conjugate. When $SU(2)$ or $SO(4)$ is embedded in a larger group, the boomerons continue to be extrema of $S_E$. The existence of non-topological boomerons such as these can not be ruled out although it is not clear if the number of negative eigenmodes can still remain 1.

Since topological boomerons are easier to find we will remark on another kind arising in 2-dimensional non-linear sigma models. When the target space is a manifold $M$ with local coordinates $X^\mu$, the action is:

$$S_E = \int d\tau d\sigma \left[ g_{\mu\nu}(X) \left[ (\partial_\tau X^\mu \partial_\tau X^\nu) + (\partial_\sigma X^\mu \partial_\sigma X^\nu) \right] + V(X) \right]$$

(4.15)

where $\sigma \in [0, 2\pi]$ is the compact space dimension, $\tau$ is time and $g_{\mu\nu}$ is a metric on $M$. The classical ground state is the trivial configuration $(X(\sigma) \equiv X_0)$ where $X_0$ minimizes the potential $V$. We are interested in field configurations $X(\sigma, \tau)$ which begin and end (as $\tau \to \pm \infty$) at the classical ground state $X(\sigma) \equiv X_0$. The space of these configurations is the space $[T^2, M]$ of based maps from a torus $T^2$ to $M$. It can be shown that the space of loops $[S^1, [T^2, M]]$ on $[T^2, M]$ is isomorphic to the space $[T^2, [S^1, M]]$ [3]. Therefore $\Pi_1([T^2, M])$ is non-trivial whenever there are non-trivial homotopy classes of the maps $[S^1, M]$ or $[T^2, M]$. In particular, topological boomerons should exist in this case if $M$ is a compact but not simply connected manifold.

# 5 Is Time Reversal Invariance Necessary?

We have shown that the boomeron and the anti-boomeron contributions cancel each other if the action is time-reversal-($T$) invariant. One can ask what happens if $S_E(B) \neq S_E(\overline{B})$? Although the sign of the contribution is opposite, the anti-boomeron is weighted by a different exponent than the boomeron and a perfect cancellation may not take place. Is it possible, then, to devise a situation where the P.I. formalism fails?

We will restrict ourselves to the purely bosonic case. Consider a 4-dimensional field
theory first. Let us denote a generic gauge field by $A$ and a generic scalar field by $\phi$. Both are multiplets of some representation of the internal symmetry groups. Then there is only one $\mathcal{T}$ non-invariant renormalizable term that can be added to the action. It is obtained by contracting the field strength tensor with its dual: $S = S^T + \theta \int d^4x [F \wedge F]$ where $S^T$ is the $\mathcal{T}$ invariant part of the action. In a Q.C.D. like theory the extra term may arise due to instanton effects and breaks $\mathcal{CP}$ and $\mathcal{T}$. An interesting property of this term is that in the Euclidean form of the action this term is pure imaginary, i.e., $S_E = S^T_E + i \theta \int d^4x [F \wedge F]$. Under $\mathcal{T}$ this term goes to negative of itself ($F \wedge F \rightarrow - F \wedge F$). Because the Euclidean action is complex, the equations of motion coming from the real and imaginary parts of the action must be individually satisfied. The new equation coming from the imaginary part of the action is

$$D \wedge F = 0 \quad (5.16)$$

which is an identity. Therefore the boomerons and anti-boomerons of the $\mathcal{T}$ invariant theory remain boomerons and anti-boomerons when the new term is added to the action. The contribution to the vacuum energy density from a boomeron anti-boomeron pair is (using the field theoretic generalization of (1.3))

$$\delta E_0 = J \times [(\text{det} O[B])^{-1/2} \exp(i n \theta) + (\text{det} O[B])^{-1/2} \exp(-i n \theta)] \quad (5.17)$$

where $J = \frac{(-S^T_E(B)/2\pi\hbar)^2 \exp[-S^T_E(B)/\hbar]}{(\text{det} \phi)^{1/2} \exp(-S^T_E(B)/\hbar)}$, $n = \int d^4x [F \wedge F]$ is the winding number of the gauge field in the boomeron background and $\phi$ is the trivial field configuration corresponding to the vacuum being considered. The determinant factors are imaginary and can be chosen to have opposite signs. Then the contribution of the boomeron is the complex conjugate of the contribution of the anti-boomeron and $\delta E_0$ is real!

We will give another example. In a 3-dimensional gauge theory the Chern-Simons term $k (A \wedge F + \frac{2}{3} A \wedge A \wedge A)$ is a $\mathcal{T}$ non-invariant term. In the presence of Higgs fields and if the gauge group is $SU(2)$, boomerons which are simply the dimensionally reduced sphalerons, are present in the $\mathcal{T}$ invariant theory ($k = 0$). With the inclusion of the Chern-Simons term ($k \neq 0$) the Euclidean action gets an imaginary contribution: $S_E = S^T_E + i (C.S)$, where $S^T_E$ is the $\mathcal{T}$ invariant action and $C.S$ is the Euclidean Chern Simons term. The extra equation of motion to be satisfied by a boomeron is:

$$F = 0 \quad (5.18)$$
Boomerons of the $\mathcal{T}$ invariant theory do not satisfy (5.18), and the problem has an easier resolution than the $\mathcal{T}$ invariant case! There are simply no boomerons now!

In general we would like to consider non-renormalizable terms as well. The Euclidean action may be written as: $S_E = S_T^E + iS_U^E$, where $S_T^E$ is the $\mathcal{T}$ non-invariant part. The boomeron satisfies the equations:

$$\frac{\delta S_T^E}{\delta \phi} = 0, \frac{\delta S_U^E}{\delta \phi} = 0 \quad (5.19)$$

where $\phi$ stands for all the fields in the theory. Under time reversal these equations are unchanged, therefore the boomeron $B$ has a corresponding anti-boomeron $\overline{B}$ which is obtained by time reversing the boomeron. Thus $S_E[B]$ is the complex conjugate of $S_E[\overline{B}]$. It also follows that the operator $O[B] = O[\overline{B}]^\dagger$, where $O[B]$ is defined as $\frac{\delta^2 \mathcal{L}_E}{\delta \phi^2}[B]$ with $\mathcal{L}_E$ given by $S_E = \int d^d x [\mathcal{L}_E]$.

Now consider the contribution of the boomeron-anti-boomeron pair to the vacuum energy density in a $d$ dimensional theory:

$$\delta E_0 = (\det O[\phi])^{1/2} \left[ (-S_E[B]/2\pi\hbar)^{d/2}\exp(-S_E[B]/\hbar)(\det O[B])^{-1/2} \right. \right. \quad (5.20)$$

$$\left. \left. + \quad (-S_E[\overline{B}]/2\pi\hbar)^{d/2}\exp(-S_E[\overline{B}]/\hbar)(\det' O[\overline{B}])^{-1/2} \right] ,$$

where $\phi$ denotes the trivial configuration at the vacuum of interest. In view of our discussion above, the second term on the R.H.S may be chosen to be the complex conjugate of the first term and $\delta E_0$ receives no imaginary contribution from boomerons and anti-boomerons.

The above result is surprising because the $\mathcal{T}$ invariance of the action, that seemed so vital for the cancellation between boomerons and anti-boomerons now seems redundant. On the other hand this is a satisfying result that establishes the independence of the robustness of the P.I. formalism from the symmetries of the action.

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