Signature of the chiral anomaly in ballistic Weyl junctions

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Abstract

Massless relativistic particles possess an extra quantum number: their chirality. However this chirality ceases to be a conserved quantity in the presence of colinear electric and magnetic fields, which is a manifestation of the chiral anomaly. Weyl materials possess low energy electrons whose dynamics mimics that of massless relativistic particles. Here we show that ballistic transport through such a material is the appropriate regime to unveil this chiral anomaly. We compute the magneto-conductance of a short junction made out of such a Weyl semi-metal. We show that it displays quantum oscillations at low magnetic field and low temperature, before reaching a universal high-field regime where it increases linearly with the field. At low fields, the algebraic-in-field magneto-conductance results from the interplay between both chiral and non-chiral conducting channels. By contrast, the linear conductance at higher fields, with a universal proportionality factor $e^2/h$ per unit area of the conductor’s cross-section, constitutes an unambiguous signature of the chiral anomaly in a Weyl conductor. Finally, we study the dependence of the ballistic magneto-conductance on the chemical potential, and discuss the cross-over towards the diffusive regime when elastic scattering is present.

1. Introduction

When relativistic particles in three dimensions become massless, they acquire an additional symmetry, the chirality (equivalent to their helicity). This chirality measures the correlation of the spin with the direction of propagation: for a right-handed (resp. left-handed) particle, the spin and momentum point in the same (resp. opposite) directions [1]. However, while this chirality is a conserved quantity of the Hamiltonian, it is no longer conserved by the associated field theory. This very intriguing and unanticipated property was called a chiral anomaly, or Adler–Bell–Jackiw anomaly [2, 3]. Much efforts have been devoted to identifying measurable consequences of this anomaly. In particular, it was realized that in the presence of a magnetic field, a chemical potential imbalance between particles with opposite chiralities induces a charge current: the chiral magnetic effect [4].

In recent years, the prospect of probing consequences of the chiral anomaly, which was initially discussed in the context of high energy physics, through the transport properties of solids has brought the study of this effect in the realm of condensed matter physics. Indeed, in several materials electrons behave as relativistic particles. Close to a linear crossing of two Bloch bands, the effective Hamiltonian takes the form initially introduced by Hermann Weyl to describe massless relativistic particles with a well-defined chirality [5]. Similarly, a Dirac Hamiltonian is obtained when two Weyl Hamiltonians of opposite chiralities coincide in the Brillouin zone, thus describing the crossing of four energy bands. The study of such crossings is not new [6]. However, it has been completely revived by the study of topological properties associated with these band crossings in the fields of Helium physics [7], strongly correlated iridates [8], and, more generally, the topological band theory of materials [9].

The discovery of materials in which electrons behave as massless relativistic particles triggered the search for a manifestation of the chiral anomaly, which is a specificity of the quantum relativistic physics in three
dimensions. This chiral anomaly was associated with the prediction of the positive magneto-conductance in materials with linear band crossing. Indeed, the presence of both electric \( E \) and magnetic \( B \) fields induces a chiral current proportional to \( E \cdot B \) and modifies the associated electromagnetic response of the massless relativistic particles. Recently, the anomalous linear response of Weyl quasi-particles to weak electromagnetic fields was determined in the presence of scattering processes between states of opposite chiralities. The equilibration between charge pumping and internode scattering leads to a steady state with a finite density imbalance between the two chiralities. According to the chiral magnetic effect, this leads to a positive magneto-conductance along the field \( B \), quadratic in \( |B| \) \([10–12]\). A first factor \( \propto |B| \) can be attributed to the degeneracy of Landau levels while a second factor originates from the chiral chemical potential proportional to \( E \cdot B \). Alternatively, recent proposals considered the so-called chiral magnetic effect induced by an oscillating magnetic field or chiral chemical potential in a junction \([13–15]\), its signature in non-local transport \([16]\), as well as in a quantized circular photogalvanic effect \([17]\).

The purpose of the present work is to study the longitudinal magneto-conductance of ballistic junctions made with Weyl semi-metals in the regime where equilibration takes place only in the leads. This corresponds to so-called cold electrons, by contrast with the hot-electron regime of \([10–12]\), where energy relaxation occurs within the conductor. The conceptual simplicity of the ballistic regime allows us to trace back unambiguously the relation between the chiral anomaly and the longitudinal magneto-conductance. We find that two regimes must be distinguished: (i) a low-field regime in which the magneto-conductance is still positive, behaves algebraically with \( |B| \), displays quantum oscillations at higher fields, but is not a unique manifestation of the chiral anomaly; (ii) a quantum regime reached for magnetic fields larger than \( B_c \approx (\hbar/e) \lambda_F^2 \) where \( \lambda_F \) is the Fermi wavelength, in which the conductance is linear in field with a universal slope \( e^2/\hbar^2 \) per unit cross-section area, identical to the proportionality factor of the quantized circular photogalvanic effect \([17]\). The predicted behavior in this regime constitutes an unambiguous signature of the chiral anomaly. Moreover this regime is found to be robust to weak elastic scattering. In this regime, the linear magneto-conductance is a direct measure of the chiral current that is related to the existence of an anomalous chiral Landau level, as it was discussed in relation with the chiral anomaly by Nielsen and Ninomiya \([18]\). This signature of the chiral anomaly in the ballistic high field regime is independent of specific details of the material. We confirm its generality by a combined analysis on a minimal continuum low-energy model with two Weyl cones, as well as on two lattice models with one and four pairs of Weyl cones.

On the experimental side, the magneto-conductance was studied in various materials that were identified as candidates for this relativistic physics, including \( \text{Cd}_3\text{As}_2 \) \([19]\), \( \text{Na}_3\text{Bi} \) \([20]\), \( \text{TaAs} \) \([21]\), and \( \text{NbP} \) \([22]\). In these compounds, the positive magneto-conductance for a parallel magnetic field, as well as its extreme sensitivity to the orientation of the magnetic field with respect to the electric field, were taken as manifestations of the underlying chiral anomaly of the relativistic massless equations of motion. Interestingly in narrow NbP wires in which the chemical potential is close to the band crossing, a longitudinal magnetic conductance linear at high fields was already observed \([22]\), in agreement with our results. Besides, a very large elastic mean free path of the order of 100 \( \mu m \) at 4K was achieved in the Weyl semi-metal \( \text{WP}_2 \) \([23]\). Although the transport above 4K was found to be described by a hydrodynamic regime dominated by the Coulomb interaction, a ballistic regime should be reached at lower temperatures. While scattering was claimed to be dominated by interactions between electrons, lowering even further the temperature in such a sample would drive the conductor into the so-called ballistic regime.

Our paper is organized as follows. In section 2, we determine analytically the ballistic magneto-conductance at zero temperature for the simplest model of two well-separated (in momentum space) Weyl cones with opposite chiralities. We find that the magneto-conductance displays quantum oscillations as the magnetic field increases, similar to Shubnikov-de Haas oscillations, but along the direction of the field. These quantum oscillations evolve into a robust linear magneto-conductance at large field. On the other hand, we discuss how the low-field oscillations are suppressed by the temperature, or by the broadening of the Landau levels due to their coupling with the leads. In section 3, we compare our predictions with the numerics for a tight-binding model with four pairs of Weyl cones. We obtain a good agreement with the analytics at low chemical potential, when Weyl cones are well separated in momentum space. On the other hand, we find that the slope of the linear magneto-conductance at large magnetic field decreases by a factor two as the chemical potential increases above the saddle-point energy corresponding to the merging of two Weyl points with opposite chirality. We explain this effect as the signature of the assymetry of the magneto-conductance with the magnetic field for a single pair of Weyl cones, as the chemical potential is tuned away from the band crossing. The symmetry is actually recovered in the specific model with eight cones, due to their distribution in momentum space. We confirm this interpretation with numerics for a tight-binding model with a single pair of Weyl cones. In section 4, we analyze the magneto-conductance within a ballistic semiclassical theory that allows recovering its linear-in-field dependence in the quantum regime. The quantum oscillations are beyond the semiclassical approximation, and we obtain a quadratic magneto-conductance at small magnetic field, which is distinct from the predictions of

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section 2 in the presence of level-broadening at low field. Furthermore, we study the stability of the linear magneto-conductance in the presence of elastic intra-cone and inter-cone scattering, we find that it is pushed toward very large fields in strongly disordered systems, in agreement with [24, 25]. In section 5, we discuss our results in the context of the chiral anomaly, and we argue that only the quantum regime is a clear signature of it, before concluding in section 6.

2. Ballistic magneto-transport

We consider the transport through a short junction made of a Weyl semi-metal in the presence of an external magnetic field $B$, represented in figure 1. We assume that $B$ is applied along the direction of the current, unless specified otherwise. In such a setup, the charge current is induced by the bias voltage between the leads on each side of the junction. For a short enough junction, scattering inside the conductor can be neglected. Hence, the energy and momentum relaxation only take place in the leads. This so-called ballistic regime is inherently out of equilibrium, in contrast with the linear response to an electric field, which has been considered so far. The current induced by a small difference of chemical potentials across the junction, $\mu_2/\mu_1 = \mu \mp \delta \mu/2$, takes the simple form $I = G(\delta \mu/e)$. The conductance of the junction, $G$, scales as its transverse area, $W \times W$, and does not depend on the junction’s length, $L$, in the ballistic regime.

In the presence of a magnetic field $B \parallel z$, the kinetic energy of electrons in the $x, y$-directions freezes into Landau levels, while the motion in the $z$-direction is unaffected. Hence each state in each Landau level provides a conduction channel in the ballistic junction. The number of such channels per Landau level is $W^2/(2\pi l_B^2)$ where $l_B = \sqrt{\hbar/eB}$ is the magnetic length. This yields a conductance

$$G(\mu, B) = \frac{e^2}{h} W^2 2\pi l_B^2 N(\mu, b),$$

(1)

where $N(\mu, b)$ is the number of Landau levels below the chemical potential $\mu$.

Let us now study the evolution of $N(\mu, b)$ for a ballistic junction built out of Weyl fermions. Weyl valleys necessarily come by pairs of chiral fermions with opposite chiralities. Around each valley, the local Bloch Hamiltonian can be written as

$$H = cp_x \sigma_x + cp_y \sigma_y + \chi c_p \sigma_z$$

(2)

with $\chi = \pm 1$ the right/left-handed chirality of the valley, $p_i = \hbar k_i$ are quasimomenta, while $c$ and $\chi$ are velocities in $x, y$- and $z$-directions respectively. (For simplicity, we assume same velocities along $x$ and $y$.) In the presence of a magnetic field $B \parallel z$, the spectrum of each valley consists of a series of bands $E_{n,\pm} = \pm \hbar c (\epsilon_2/k_2)^1/2 + \hbar \sqrt{\delta_\mu^2 / \lambda_B^2} n \geq 1$, which disperse along the $z$-direction, and which are separated from each other by gaps of the order $\hbar \omega_0 = \hbar c / \lambda_B = \mu \sqrt{\hbar eB} / B$ with $B/\lambda_B = \pi \hbar / e$ where $\lambda_B$ is the Fermi wavelength satisfying $\mu = \hbar 2\pi e / \lambda_B$ for $\mu > 0$ (see appendix A). Besides these bands, the energy spectrum admits an additional, linearly dispersing band, whose direction of propagation depends both on the chirality of the valley and the direction of the magnetic field, $E_0 = -\chi \text{ sgm}(B) \epsilon / \hbar k_2$, as shown in figure 2.

At large magnetic field, $|B| \gg B_s$, only the anomalous Landau level $n = 0$ contributes to transport, a situation reminiscent to that of [18] albeit considered here in the ballistic regime. We call this regime the quantum regime, in which $N(\mu, B) = 1$, and from equation (1) we find $G = (e^2/h^2)W^2|B|$. At smaller fields, $|B| \ll B_s$, other Landau bands intercept the Fermi level and contribute to the conductance, which thus oscillates.

![Figure 1. Schematic representation of a short junction of a Weyl semi-metal between two metallic leads at chemical potentials $\mu_1, \mu_2$, and submitted to a magnetic field $B$ parallel to the direction of the current $I$.](image)
as a function of the magnetic field $|B|$ or the chemical potential $\mu$. For an ideal ballistic junction, the number of filled Landau levels is deduced from the expression of Landau levels, $N(\mu, B) = 1 + \left[ (\mu / (\hbar \omega_0))^{2} \right] = 1 + 2 \left[ (B/T_B) \right]$, where $[x]$ stands for the integer part of $x$. By introducing the sawtooth function $sw(x) = \left[ x \right] - x + \frac{1}{2}$, we obtain the expression of the conductance of an ideal Weyl junction,

$$G(B) = G_{Sh} \left[ 1 + \frac{|B|}{B_c} \operatorname{sw} \left( \frac{B_c}{|B|} \right) \right],$$

where $G_{Sh}(\mu) = 2\pi (c^2 / \hbar)(W / \lambda_{c})^2$, which is proportional to the number of conduction channels at $B = 0$, is the Sharvin conductance of the junction. Let us note that quantum oscillations of the longitudinal conductance, similar to those we identified in the ballistic regime, were also observed in the diffusive regime for hot electrons [26].

The limit $B \to 0$ of the conductance (3) for an ideal junction is actually ill-defined: as usual, we expect the quantum oscillations described by equation (3) to be cut-off at low field by any broadening of the Landau bands [27]. In the ballistic junction that we consider, the finite dwell time of the electrons between the leads, $\tau_d = L / c_\parallel$ provides an inherent broadening of order $\eta \approx \hbar / \tau_d = \hbar c_\parallel / L$. We can phenomenologically incorporate this effect by a standard Lorentz broadening of the energy levels, leading to

$$G(\mu, B) = G(\mu, 0) + G_{Sh}(\mu) \frac{|B|}{B_c(\mu)} \int \frac{d\mu'}{\eta} f \left( \frac{\mu - \mu'}{\eta} \right) \operatorname{sw} \left( \frac{B_c(\mu')}{|B|} \right),$$

with $f(x) = 1 / [\pi (1 + x^2)]$. In the regime where a non-oscillating density of states is recovered, corresponding to $\hbar \omega_0 = \mu \sqrt{|B| / B_c} \ll \eta$, we find a non-analytic scaling of the positive magneto-conductance,

$$\frac{G(\mu, B)}{G_{Sh}(\mu)} = \frac{G(\mu, 0)}{G_{Sh}(\mu)} \approx \tilde{\alpha} \left( \frac{\mu}{\eta} \right) \left( \frac{|B|}{B_c(\mu)} \right)^{3/2},$$

with $\tilde{\alpha} \approx 0.59$, see appendix B. Noteworthy, this scaling is different from the typical $B^2$-behavior predicted in the diffusive regime [10]. Similarly if a thermal broadening of the levels supersedes the intrinsic broadening, we can still use equation (4), but now with $f(x) = -f'_T(x)$ with $f_T(x) = 1 / (1 + e^x)$. For Weyl fermions, it yields

3 Note that the Lorentz broadening of the Sharvin conductance, $G(\mu, B = 0)$, requires a high-energy cut-off below which the Hamiltonian (2) is defined, as opposed to the $B$ dependent correction described by equation (4).
In particular, the anomalous magneto-conductance is exponentially suppressed at $\mu \gg k_B T$.

Let us contrast this behavior with that for a standard non-relativistic parabolic dispersion relation $E = p^2 / (2m)$, whose Landau bands read $E_n = \hbar^2 \omega_0 (n + 1/2) + p_z^2 / (2m)$ with $n \geq 1$, $\hbar \omega_0 = \hbar |B| / m = \mu |B| / \hat{B}_z$. The number of filled Landau bands is now $N(\mu, B) = \left( \frac{\mu}{\hbar \omega_0} + \frac{1}{2} \right) \left( \frac{\hat{B}_z}{|B|} + \frac{1}{2} \right)$ corresponding to a conductance $G(\mu, B) = G_{\text{sh}}(\mu) \left( 1 + (|B|/\hat{B}_z) \right)$.

It yields strikingly different predictions compared with equation (3). In particular, the conductance vanishes in the quantum regime, which is reached for fields $|B| \geq 2 |\hat{B}_z|$, as opposed to the positive linear magneto-conductance at high fields. Furthermore, a Lorentz broadening of the Landau levels leads to an exponentially suppressed magneto-conductance at low fields, in contrast with equation (5) for the Weyl Hamiltonian. The sharp difference between the ballistic magneto-conductance of a Weyl junction with that of a standard material with a non-relativistic dispersion relation is highlighted in figure 3, which illustrates the presence versus absence of the positive magneto-conductance in the quantum regime. In particular, note that the linear positive regime for the magneto-conductance is robust in the presence of level broadening. The qualitative difference between a Weyl and a standard junction is still striking when the broadening is strong, even though the linear regime is pushed towards higher fields.

### 3. Numerical study

We now complement the previous arguments based on the simplified low-energy Bloch Hamiltonian (2) by a numerical study of transport for two lattice Hamiltonians displaying respectively four pairs and one pair of Weyl cones. First, we compute the conductance of a ballistic junction of size $W \times W \times L$ using the Kwant numerical software [28] applied to a tight-binding two-band model on a cubic lattice with nearest-neighbor couplings, such that the dispersion relation, $E_z^\pm(k) = t^2 (\sin^2 k_x a + \sin^2 k_y a) + t_z (1 - \cos k_z a) - \Delta_p^\pm$, possesses four pairs of Weyl cones with opposite chiralities at quasi-wavectors $k = (0/\pi, 0/\pi, \pm K_z)$ with $K_z = (2/a) \arcsin(\Delta / (2t_z))$, assuming $0 < \Delta / (2t_z) < 1$ (see appendix C for details). Throughout our study, we use $a = 1$ for the lattice spacing, $t = 1$ and $t_z = 1$ for the hopping matrix elements in $(xy)$-plane and along $z$-direction, respectively, and various values of the energy threshold $\Delta$. We study the transport along the $z$-direction as a function of the magnetic field $|B|$ aligned with the junction.

The numerical data are represented using a dimensionless and scale independent conductance, $g$, defined as $G = (e^2 / h)(W / a)^2 g$, as a function of the rescaled magnetic flux per unit cell (in units of the flux quantum), that is, $b = \phi / \phi_0 = a^2 B e / h = \text{sign}(B) a^2 / (2 \pi t^2_B)$. Results for $\Delta = 1$ are shown in figure 4 in which the conductance of the junction is plotted as a function the dimensionless magnetic flux $b$ threading a lattice unit.

![Figure 3. Magneto-conductance of ballistic junctions with Weyl (blue) and parabolic (yellow) dispersion relations. The finite dwell time of the electrons in the junction is included through the broadening amplitude $\eta$ of the spectrum and we chose $\mu = 0.7, \epsilon = 1$. For small magnetic fields, several Landau bands contribute to the conductance. Correspondingly the conductance oscillates as a function of the magnetic field, reflecting the discontinuous change of the number of filled Landau bands as the later varies. At high magnetic fields, in the quantum regime, the Weyl and parabolic magneto-conductances differ. In this regime a single, $n \neq 0$ Landau band contributes to the conductance of the Weyl junction, and it increases linearly with $|B|$. By contrast, for a junction with a parabolic dispersion relation, no Landau band crosses the chemical potential, and the conductance vanishes.](image-url)
For all chemical potential of the conductor below the energy threshold between Weyl valleys, $|\mu| < \Delta$, we find a quantitative agreement with the previous analysis. Namely, (i) at low magnetic field, the conductance $g(b)$ oscillates with $b$, with an amplitude of oscillations that increases linearly with $b$, (ii) at high magnetic field, the conductance $g(b)$ reaches a linear regime independent on energies, $g(b) \approx 4|b|$, where the factor 4 accounts for the presence of four pairs of Weyl points in our model. We have checked both that the conductance is independent of the length $L$ of the junction, and that the above behavior is a contribution of bulk states, while surface states provide a subdominant contribution $\propto W$ to the conductance (see appendix D). This allows us to rule out possible scenarios for the origin of the linear magnetoconductance involving surface states, such as Fermi arc surface states contributions [15] or surface spiral modes effects [29]. Note that the numerical Landauer technique that we use amounts to introducing semi-infinite systems on both sides of the conducting part, playing the role of leads with perfect contacts, thereby allowing to reach a ballistic regime. In doing so we effectively consider a long ballistic junction, corresponding to an artificially large dwell time for the electrons, which hampers the study of the algebraic low $b$ regime which is relevant experimentally. The thermal broadening of the magnetoconductance is illustrated in figure 5, in agreement with equation (6) for $\mu \gg kT$.

We now discuss the dependence of the ballistic magnetoconductance on the chemical potential, and show that the number of pairs of Weyl valleys contributing to this regime depends on this chemical potential. In particular, as shown in figure 6, we find that the linear regime at large fields survives even above the threshold energy separating the Weyl valleys, $\mu > \Delta$ (here taken as $\Delta = 0.7$), though its slope is reduced by a factor two at large $\mu$. To understand this reduction of the slope, we consider the simpler situation of a single pair of Weyl cones separated by an energy threshold $\Delta$. It can be realized with a tight-binding two-band model on a cubic lattice introduced in [30], where the Weyl points only occur at $k = \pm K/2 = (0, 0, \pm K_z)$ (see appendix D). The corresponding numerical results for the magnetoconductance are presented in figure 7. We observe that above the threshold energy $\Delta$ separating the Weyl valleys, the linear behavior at large field is lost for one direction of the magnetic field oriented along the current with $B \parallel \mathbf{I} \parallel \mathbf{K}$ (figures 7(A) and (B)). Moreover, the whole behavior at small magnetic field is now asymmetric in $b$, irrespective of $\mu$. Note that such a simple Weyl semi-metal with only two cones necessarily breaks time-reversal symmetry (TRS). This manifests itself in the breaking of Onsager relation $G(B) = G(−B)$, and henceforth an asymmetry of the curves $G(B)$ (or $g(b)$). The amplitude of this TRS breaking and this asymmetry originates from the separation $\mathbf{K}$ of the two cones in the Brillouin zone, or more precisely on its projection onto the direction of magnetic field. Indeed, when the vector $\mathbf{K}$ is aligned perpendicular to both the magnetic field and the junction, no sign of this TRS breaking is observed on magneto-transport and a magnetoconductance symmetric in $b$ is recovered as shown in figure 7(C). A related effect was discussed in the diffusive regime in [31].

The change of linear magnetoconductance in the quantum regime can be understood by considering the simple Bloch Hamiltonian that generalizes equation (2) and describes two Weyl valleys separated by an energy saddle point,
For $\Delta > 0$, two Weyl points at $\pm \mathbf{K}/2 = (0, 0, \pm \sqrt{2} m\Delta / \hbar)$, and having opposite chirality, are separated by an energy barrier $\Delta$. The chemical potential dependance in the large field regime can now be inferred from the behavior of the Landau levels for the model (7) (see appendix A). Depending on the sign of $b$, i.e., on the orientation of the field with respect to the ‘chirality vector’ $\mathbf{K}$ pointing from the left-handed Weyl cone with $\chi = -1$ to the right-handed Weyl cone with $\chi = +1$, the dispersion of the $n = 0$ Landau level changes dramatically. Indeed, it either exists for $\mu$ smaller than $\Delta$, or for $\mu$ larger than $-\Delta$ as illustrated in figure 2. We expect the larger energy barrier between the two Weyl valleys, neglected in the model (7), to cut-off the energy range of this $n = 0$ Landau band on the other side. Beyond this saddle point energy, the linear

$$H = cp_x\sigma_x + cp_y\sigma_y + \left(\frac{p_z^2}{2m} - \Delta\right)\sigma_z.$$

(7)
magneto-conductance for the pair of Weyl cones is lost. Depending on the sign of \( b \), this happens either above the positive or below the negative energy saddle point \( \pm \Delta \).

We are now ready to understand the reduction of the slope illustrated in figure 6 for the model with four pairs of Weyl cones. Indeed, due to the symmetrical distribution of the pairs of Weyl cones in momentum space in that model, above the saddle point energy, the linear magneto-conductance contribution vanishes for the half of Weyl pairs whose chirality vector is parallel with the magnetic field, while it persists for the other half having an anti-parallel chirality vector with respect to \( b \). Furthermore, the absence of an asymmetry of the magneto-conductance in that model can also be traced to the symmetrical distribution of Weyl cones in momentum space.

4. Semi-classical description of ballistic transport

Let us now describe within a semi-classical picture the above ballistic magneto-transport. The advantage of such an approach is that it allows describing the smooth cross-over from the ballistic to the diffusive regime, as well as compare the results in both regimes. In this approach, the evolution of a semi-classical wave-packet of Weyl eigenstates is described by the semi-classical equations of motion (for a vanishing electric field)\[32]:

\[
\partial_t \mathbf{x} = \frac{1}{\hbar} \partial_{\mathbf{k}} \varepsilon - \partial_{\mathbf{k}} \mathbf{\Omega} \times \mathbf{k}; \quad \hbar \partial_{\mathbf{k}} \mathbf{k} = -e \partial_{\mathbf{x}} \mathbf{x} \times \mathbf{B}.
\]

Here, \( \mathbf{x} \) and \( \mathbf{k} \) are the position and momentum of the wavepacket, \( \varepsilon_{\mathbf{k}} \) and \( \Omega_{\mathbf{k}} \) are the energy and Berry curvature for the conduction band, which, around a Weyl point described by a Hamiltonian \( H^L_{\mathbf{k}} = \hbar v_F \chi \cdot \mathbf{k} \), read \( \varepsilon_{\mathbf{k}} = \hbar v_F k \) and \( \Omega_{\mathbf{k}} = \chi |k|/2k^3 \). Solving for the semi-classical equations of motion (8), we obtain the anomalous velocity of the electrons, corrected by the Berry curvature: \( \mathbf{v}_{\mathbf{A}} = \partial_{\mathbf{k}} \varepsilon + (e/\hbar)(\partial_{\mathbf{k}} \varepsilon \cdot \mathbf{B}) \mathbf{B} \). The charge current density along the \( z \)-direction follows:

\[
J_z = -e \sum_{\chi, \mathbf{k}} (\mathbf{w}_{\chi} \cdot \hat{z}) f_{\chi} (\mathbf{x}, \varepsilon, \mathbf{k}),
\]

where the sum runs over constant energy contours and \( \hat{z} = \mathbf{k}/k \). The distribution function satisfies the stationary Boltzmann equation (dropping the energy dependence of \( f \))

\[
(\mathbf{w}_{\chi} \cdot \hat{z}) \partial_z f_{\chi} (z, \mathbf{k}) = \frac{1}{\tau} \sum_{\chi', \mathbf{k}'} [f_{\chi'} (z, \mathbf{k}') - f_{\chi'} (z, \mathbf{k})] + \frac{1}{\tau'} \sum_{\mathbf{k}''} [f_{\chi} (z, \mathbf{k}'') - f_{\chi} (z, \mathbf{k})].
\]

Here \( \tau \) and \( \tau' \) are the intra-cone and inter-cone elastic scattering times, respectively, which are assumed to be larger than the dwell time in the ballistic regime. We solve equation (10) with an ansatz for \( f \) that satisfies boundary conditions at the contacts [33],

\[
f = f_0 (\varepsilon_{\mathbf{k}} - \mu_L) \Theta (\mathbf{w}_{\chi} \cdot \hat{z}) + f_0 (\varepsilon_{\mathbf{k}} - \mu_R) \Theta (-\mathbf{w}_{\chi} \cdot \hat{z}).
\]
Solving these equations in the ballistic regime, $\tau, \tau' \rightarrow \infty$, we find
\[
G_d = \begin{cases} 
G_{SB} \left( 1 + \frac{B^2}{(4B_c)^2} \right) & \text{for } |B| < 4B_c, \\
\epsilon^4/\hbar^2W^2|B| & \text{for } |B| \geq 4B_c.
\end{cases}
\] (12)

Quite remarkably, incorporating the Berry curvature effect into the semi-classical description of the ballistic transport allows describing the anomalous linear magneto-conductance in the quantum regime at large field, as well as the cross-over towards an anomalous regime at low field. However, the semi-classical description is unable to accurately describe the $|B|^{1/2}$-dependence of the magneto-conductance, see equation (5), and predicts a $B^2$-behavior, similar to the diffusive regime [10]. Such a diffusive regime is indeed recovered in the present situation at finite scattering times $\tau, \tau'$. However it corresponds to a different situation from the one considered by Son et al. [10], as inelastic scattering occurs in the leads in our case, and not in the junction, thus yielding a quantitatively different result. Solving this problem, we find an expression for the conductance (see appendix F), which identifies with that previously derived via a topological nonlinear sigma field theory [24]. As expected, when intra-cone disorder is increased a diffusive regime is reached at small magnetic field when $\ell = c\tau \leq L$, with a conductance now scaling as $G \propto W^2/L$. The correction at small field in this regime remains quadratic in $B$. At high magnetic field, the previous ballistic quantum magneto-conductance regime $G(B) = (\epsilon^4/\hbar^2W^2|B|)$ is affected neither by intra-cone, nor by inter-cone disorder. Moreover as long as $L \ll \sqrt{\epsilon_c^2}$ with $\epsilon_c = c\ell'$, including the situation when only the intra-cone disorder is relevant, $\ell \leq L \leq \ell'$, this ballistic linear regime is reached at smaller magnetic fields, $|B| \geq 2B_c\ell/L$. On the other hand, when $L \geq \sqrt{\epsilon_c^2}$, or $L \geq \ell' \geq \ell$, inter-cone disorder pushes this ballistic regime to high magnetic field (possibly outside of the experimental regime) for $|B| \geq 2B_cL/\ell'$. The magneto-conductance corresponding to these different regimes is obtained by numerically solving the semi-classical diffusive equation (see appendix F), and the results are represented in figure 8.

5. Discussion
In this paper, we have shown that two regimes have to be distinguished when discussing the ballistic conductance of a Weyl junction in parallel magnetic field: (i) a linear regime at high fields with a universal slope; (ii) a weak magnetic field regime, in which the magneto-conductance behaves algebraically with the magnetic field, whose details are non-universal. Furthermore at higher fields the magneto-conductance displays quantum
oscillations when the broadening of Landau levels is weak (such as in long ballistic junctions) and at low temperature.

Let us now discuss the relation between these different regimes and the underlying chiral anomaly. In the ballistic regime addressed in this work, there exists an equilibrium chiral current density \( j = j_R - j_L \propto |B| \) for non-vanishing magnetic fields: the presence of the \( n = 0 \) Landau band implies that, irrespective of the chemical potential \( \mu \), there is an excess of Weyl electrons of, e.g., left chirality moving to the right of the junction, and electrons of right chirality moving to the left. This equilibrium chiral current is obviously uncorrelated with a charge density current \( j = j_R + j_L \), as the latter vanishes in equilibrium. On the other hand, a chemical potential bias, \( \mu_1 - \mu_2 = \delta \mu \), reveals this chiral current as a non-equilibrium charge current, as first discussed in [18]. At high enough field, i.e., when transport is fully taken over by the \( n = 0 \) state, charge current is fully chirally polarized and the charge current reflects the linear-in-field dependence of the chiral anomaly. At weak magnetic field only part of the conductance (1) is related to the \( n = 0 \) Landau-band contribution, and thus can be related to the chiral anomaly. Hence the anomalous positive ballistic magneto-conductance in the low field regime is not a unique signature of the chiral anomaly. This is in contrast with the situation of hot electrons, close to equilibrium, considered in [10]. There, the chiral current driven by the \( n = 0 \) Landau level leads to a chiral chemical potential \( \mu_R - \mu_L \) between left- and right-handed Weyl valleys, and a magneto-conductance via the chiral magnetic effect. Hence attributing the anomalous magneto-conductance at low fields to the chiral anomaly requires first identifying the relevant regime of transport, and the observation of a positive algebraic behavior is not sufficient.

In contrast to the low field regime, the quantum regime of a linear magneto-conductance \( G = (e^2/h^2)W^2|B| \) at high field is a unique contribution of the \( n = 0 \) Landau level. Thus it can be unambiguously associated with the chiral anomaly. This regime is reached for magnetic fields satisfying \( B\lambda_F^2 \supset \pi h/e \) where \( \lambda_F \) is the Fermi wavelength. Thus it can be reached experimentally for chemical potentials sufficiently close to the band crossing. Moreover this regime is robust, and persists even in the presence of disorder: only its domain of existence is affected by elastic scattering. Note that such linear magnetoconductance in presence of elastic scattering was derived in [24], our derivation allows to relate it to the contribution from the chiral \( n = 0 \) Landau level in presence of disorder and thus to a manifestation of the chiral anomaly. Furthermore, when energy relaxation occurs within the conductor the above ballistic conductance at high fields is replaced by \( G = (e^2/h^2)(e' / L)W^2|B| \), also linear in magnetic field [18, 26, 34]. In this regime the slope of the linear regime now depends explicitly on the amplitude of inter-cone scattering. Note that a different mechanism was proposed for the linear magnetoconductance in the diffusive regime with energy relaxation, leading to a different formula, but there again the slope depends on details of the scattering potential [35]. Hence the study of this linear regime should provide an unambiguous determination of the regime of transport. In the exact same regime of transport a linear magneto-resistance in transverse magnetic field was also predicted by Abrikosov in [36, 37] (see also [38]). We believe that this quantum regime of transport in Weyl materials is of high experimental interest. Quite remarkably a linear-in-field longitudinal magnetic conductance has already been observed in narrow wires of NbP, a Weyl semi-metal [22], hence validating that this regime is within experimental reach, although it has not been thoroughly studied yet.

6. Conclusion

In this paper, we have studied the conductance of a junction of Weyl material in the presence of a parallel magnetic field, and in the ballistic regime. We have shown that the low-field magneto-conductance displays low-temperature quantum oscillations, whose broadening results in an algebraic behavior at vanishing field. At high fields, the magneto-conductance becomes linear in the field. Besides its experimental relevance, this ballistic regime allows discussing in details the relation between this conductance and the chiral anomaly of Weyl fermions. This allows to unambiguously identify the large field regime as a signature of the chiral anomaly.

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Appendix A. Landau levels

We consider the model

\[ H = c p_x \sigma_x + c p_y \sigma_y + \left( \frac{p_z^2}{2m} - \Delta \right) \sigma_z, \]  

with \( p_i = \hbar k_i \). For \( \Delta > 0 \), it admits two Weyl points of opposite charge at \( k_x = k_y = 0 \) and \( \hbar k_z = \pm \sqrt{2m\Delta} \). By minimal coupling, with a charge of carriers \( q = -e \), we get \( p_i = \hbar k_i \rightarrow P_i = p_i - qA_i = p_i + eA_i \) in the presence of a magnetic field \( B = B \hat{z} = \text{rot } A \) along \( z \)-direction. We naturally introduce the ladder operators

\[ \hat{a} = \frac{\hbar}{\sqrt{2\hbar}} (\hat{\Pi}_x - i\hat{\Pi}_y); \quad \hat{a}^\dagger = \frac{\hbar}{\sqrt{2\hbar}} (\hat{\Pi}_x + i\hat{\Pi}_y) \]  

which satisfy \( [\hat{a}, \hat{a}^\dagger] = \eta_B [\,] \), with \( \eta_B = \pm 1 \) the sign of \( B \).

- Case \( B > 0 \): the Hamiltonian then reads

\[ H = H_{xy} + \left( \frac{p_x^2}{2m} - \Delta \right) \sigma_z, \]  

\[ H_{xy} = \hbar \omega_0 \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix}, \]  

with \( \hbar \omega_0 = \sqrt{2} \hbar v_c / l_B = \sqrt{4\pi \hbar^2 c^2 |b| / a^2} \). Let us consider the quanta \( |n\rangle \) defined by \( \sqrt{n!} |n\rangle = (\hat{a}^\dagger)^n |0\rangle \). The spectrum of relativistic Landau levels of \( H_{xy} \) is given by \( E_n = \eta \sqrt{n} \hbar \omega_0, \eta = \pm 1 , n \in \mathbb{N} \) with the corresponding eigenstates

\[ |\psi_0\rangle = \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}; \]  

\[ |\psi_{\eta,n}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |n\rangle - |n-1\rangle \\ \eta |n\rangle \end{pmatrix}, \quad n \geq 1. \]

- Case \( B < 0 \): there the algebra of ladder is reversed,

\[ H_{xy} = \hbar \omega_0 \begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{pmatrix}, \]  

and the eigenstates take the form

\[ |\psi_0\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}; \]  

\[ |\psi_{\eta,n}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |n\rangle \\ \eta |n-1\rangle \end{pmatrix}, \quad n \geq 1. \]
We readily get the full spectrum by writing a reduced Hamiltonian in the basis $|\psi_{+,n}\rangle$, $|\psi_{-,n}\rangle$,

$$H = \sqrt{n} \hbar \omega_0 \sigma_z + \left( \frac{k_z^2}{2m} - \Delta \right) \sigma_x,$$

(A10)

Namely, we find

$$E_{n,\pm} = \pm \sqrt{4\pi e^2 |b| n + \left( \frac{k_z^2}{2m} - \Delta \right)^2}$$

(A11)

with $n > 0$ and

$$E_0 = \begin{cases} -\frac{k_z^2}{2m} - \Delta, & \text{if } B > 0, \\ \frac{k_z^2}{2m} - \Delta, & \text{if } B < 0. \end{cases}$$

(A12)

**Appendix B. Lorentz broadening of the conductance**

We start from equation (4) in the main text:

$$G(\mu, B) = \frac{G_{01}(\mu)}{G_{01}(\mu)} \int \frac{d\mu'}{\eta'} f(\mu - \mu') \sw \left( \frac{B_0(\mu')}{|B|} \right)$$

$$= 1 + \frac{|B|}{B_0(\mu)} \int d\epsilon' f(\epsilon - \epsilon') \sw \left( \frac{\epsilon'^2}{N^2} \right)$$

(B1)

with $\epsilon = \mu/\eta f(x) = 1/[\pi (1 + x^2)]$, and we introduced $N^2 = (\hbar \omega_0/\eta)^2 = (\mu/\eta)^2 |B|/B_0(\mu)$. Then we use the Fourier expansion of the sawtooth function

$$sw(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} \sin(2\pi nx)$$

(B2)

to rewrite expression (B1) as

$$\frac{G(\mu, B)}{G_{01}(\mu)} = 1 = \frac{|B|}{B_0(\mu)} \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} d\epsilon' \frac{1}{1 + (\epsilon' - \epsilon)^2} \exp \left( i 2\pi n \frac{\epsilon'^2}{N^2} \right)$$

(B3)

Using the contour in the complex plane of figure B1 and the residue theorem gives

$$\int_{-\infty}^{\infty} d\epsilon' \frac{1}{1 + (\epsilon' - \epsilon)^2} \exp \left( i 2\pi n \frac{\epsilon'^2}{N^2} \right) = 2\pi\Theta(\epsilon - 1) \text{Res}_{\epsilon' = \epsilon} \left[ \frac{1}{1 + (\epsilon^2 - \epsilon)^2} \exp \left( i 2\pi n \frac{\epsilon^2}{N^2} \right) \right]$$

$$+ \int_{-\infty}^{\infty} du \frac{\epsilon^2}{1 + (\epsilon^2 u - \epsilon)^2} \exp \left( -2\pi n \frac{\epsilon^2 u^2}{N^2} \right).$$

(B4)

Summing over $n$

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -2\pi n \frac{\epsilon^2 u^2}{N^2} \right) = -\ln \left( 1 - \exp \left( \frac{2\pi \epsilon^2 u^2}{N^2} \right) \right)$$

(B5)

we obtain

$$\frac{G(\mu, B)}{G_{01}(\mu)} = 1 = \frac{|B|}{B_0(\mu)} \frac{\Theta(\epsilon - 1)}{\pi} \left[ \ln \left( 1 - \exp \left( \frac{2\pi (\epsilon + i)^2}{N^2} \right) \right) \right]$$

$$- \frac{|B|}{B_0(\mu)} \frac{1}{\pi^2} \int_{-\infty}^{\infty} du \frac{e^{i\epsilon^2 u^2}}{1 + (e^{i\epsilon^2 u - \epsilon})^2} \ln \left( 1 - \exp \left( \frac{2\pi \epsilon^2 u^2}{N^2} \right) \right).$$

(B6)
In the low-field limit, the first term of (B6) leads to exponentially small correction in $|B|$, while the second term can be expanded according to

$$\int_{-\infty}^{+\infty} du \frac{e^{iu}}{1 + (e^{iu} - \epsilon)^2} \ln \left(1 - \exp \left(\frac{2\pi}{\Lambda^2} e^{iu}\right)\right)$$

$$= \frac{|B|}{B_c(\mu)} \frac{1}{\pi^2} \Lambda^2 \int_{-\infty}^{+\infty} dy \frac{e^{iy}}{1 + (\Lambda^2 e^{iy} - \epsilon)^2} \ln \left(1 - e^{-2\pi y^2}\right)$$

$$= \frac{|B|}{B_c(\mu)} \frac{1}{\pi^2} \Lambda \int_{-\infty}^{+\infty} dy \left[ \frac{e^{y}}{1 + e^2} \frac{2\pi}{(1 + e^2)^2} \Lambda y + O((\Lambda y)^2) \right] \ln \left(1 - e^{-2\pi y^2}\right)$$

$$= \frac{|B|}{B_c(\mu)} \frac{1}{\pi^2} \Lambda \int_{-\infty}^{+\infty} dy \ln \left(1 - e^{-2\pi y^2}\right) + O((\Lambda y)^2)$$

$$(B7)$$

From parity arguments, the expansion will only contain terms with even powers of $\Lambda$, giving an expansion in half-integer powers of $B$ starting from $B^3$. At lowest order in $b$, we finally obtain

$$\frac{G(\mu, B)}{G_{Sh}(\mu)} - 1 = \tilde{\alpha} \left( \frac{|B|}{B_c(\mu)} \right)^2 \frac{\mu}{\eta} \left( \frac{\mu}{\eta} \right) + O(|B|^2)$$

$$(B8)$$

with

$$\tilde{\alpha} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dy \ln \left(1 - e^{-2\pi y^2}\right) \approx 0.59$$

$$(B9)$$

which is equation (5) in the main text.

**Appendix C. Thermal broadening of the conductance**

We use the Euler–Maclaurin formula,

$$\sum_{n=1}^{\infty} f(n) = \int_{1}^{+\infty} dx f(x) + \frac{1}{2} f(1) + \frac{1}{12} f'(1) - \frac{1}{30} \frac{1}{24} f''(1) + \ldots$$

$$(C1)$$

$\sum_{n=1}^{\infty} f(n) = \int_{1}^{+\infty} dx f(x) + \frac{1}{2} f(1) + \frac{1}{12} f'(1) - \frac{1}{30} \frac{1}{24} f''(1) + \ldots$$

$$(C1)$$

The conductance given by equation (4) in the main text with thermal broadening is (we assume $B, \mu > 0$)
\[ 2 \frac{G_T}{G_{Sh}} = \frac{B}{B_c} \left( 1 + 2 \sum_{n=1}^{\infty} f'_e \left( \frac{\epsilon_0 \sqrt{m_T}}{k_B T} - \frac{\mu}{k_B T} \right) \right), \]  
(C2)

where \( f_e(x) = 1/(1 + e^x) \). The development at low magnetic field, corresponding to \( z = \hbar \omega_0 / (k_B T) \ll 1 \), is written, using \( \epsilon = \mu / (k_B T) \), as

\[
2 \frac{G_T}{G_{Sh}} B = 1 + 2 \int_0^{\infty} dy \frac{y}{1 + e^{y - \epsilon}} = 1 + 4 \int_0^{\infty} dy \frac{y}{1 + e^{y - \epsilon}} - 4 \int_0^{\infty} dy \frac{y}{1 + e^{y - \epsilon}} \left( \frac{\epsilon_0 \sqrt{m_T}}{k_B T} - \frac{\mu}{k_B T} \right) + O(\epsilon^3)
\]

\[
= 1 + \frac{4}{\epsilon^2} \int_0^{\infty} dy \frac{y}{1 + e^{y - \epsilon}} - 4 \int_0^{\infty} dy \frac{y}{1 + e^{y - \epsilon}} \left( \frac{\epsilon_0 \sqrt{m_T}}{k_B T} - \frac{\mu}{k_B T} \right) + O(\epsilon^3)
\]

This expression can be rewritten

\[
\frac{G_T(B)}{G_{Sh}} - \frac{G_T(0)}{G_{Sh}} = \frac{B}{2 B_c} (1 - f_e(-\epsilon)) - \frac{1}{8} \left( \frac{B}{B_c} \right)^3 e^{f'_e(-\epsilon)}
\]

(C4)

which is equation (6) in the main text. We obtain in the limit \( k_B T \ll \mu (\epsilon \gg 1) \) that all algebraic corrections in \( B \) vanish, while for \( \mu \ll k_B T (\epsilon \ll 1) \)

\[
\frac{G_T(B)}{G_{Sh}} - \frac{G_T(0)}{G_{Sh}} \approx \frac{1}{4 B_c}
\]

(C5)

### Appendix D. Lattice models

Here we provide the explicit form of the two lattice Hamiltonians whose spectra were described—and which were studied numerically—in section 3 in the main text.

#### D.1. 8 Weyl cones model

We consider a two-band model on a cubic lattice with lattice spacing \( a = 1 \), whose sites are denoted by \( r \) below, with a nearest-neighbor Hamiltonian

\[
H = \sum_r \left[ \left( \frac{t}{2} \sum_{\vec{e}_r} \langle \vec{r} + \vec{e}_r | \hat{c}_r^\dagger \hat{c}_r \rangle + \left( \frac{t}{2} \sum_{\vec{e}_r} \langle \vec{r} + \vec{e}_r | h.c. \rangle \sigma_z \right) \sigma_z \right] + \sum_r \left[ -\frac{t_2}{2} \langle \vec{r} + \vec{e}_r | \hat{c}_r^\dagger \hat{c}_r \rangle + \langle t_x - \Delta \rangle \langle \vec{r} | \vec{r} \rangle \sigma_z \right] \sigma_z
\]

(D1)

The corresponding Bloch Hamiltonian reads

\[
H(k) = t \sin k_x \sigma_x + \mu \sin k_y \sigma_y + \frac{t_2}{2} (1 - \cos k_y) - \Delta \sigma_z
\]

(D2)

and its spectrum is

\[
E_{\pm} = \pm \left( t^2 \sin^2 k_x \sigma_x + \frac{t_2}{2} \sin^2 k_y + \frac{t_2}{2} (1 - \cos k_y) - \Delta \right)^{1/2}
\]

(D3)

Hence this lattice model possess four pair of Weyl points located at \( k_x = \pm \sqrt{2m\Delta} \) and \( k_y = 0, \pi \). In our numerical study we use \( t = m = 1 \) and study transport along the \( z \) direction for various values of the saddle-point energy \( \Delta \).
D.2. 2 Weyl cones model

We use the lattice model of \[30\], whose Bloch Hamiltonian is

\[
H = \left(2t \sin \frac{k_x + k_y}{2}\right) \sigma_x + \left(2t \sin \frac{k_x - k_y}{2}\right) \sigma_y + t_z \left(2m_t + 2 \cos k_z - m_2(\cos k_x + \cos k_y)\right) \sigma_z. \quad (D4)
\]

The lattice spacing is set to \(a = 1\) and we choose the parameters \(t = 1\), \(t_z = 0.2\), \(m_1 = 3/4\), \(m_2 = 1\) for which only two Weyl points exist and are located at \((0, 0, \pm K)\) with \(K = \arccos(m_2 - m_1)\). Around these Weyl cones the Hamiltonian reads (up to spin rotation)

\[
H(\pm K + q) = 2t(q_x + q_y) \sigma_x + 2t(q_x + q_y) \sigma_y + c_z q_z \sigma_z \quad (D5)
\]

with \(c_z = 2t_z \sqrt{1 - (m_2 - m_1)^2}\).

Appendix E. Numerical study

We compute the conductance using a numerical implementation of the recursive Green function technique with Kwant \[28\]. Samples are Weyl junctions sandwiched between semi-infinite leads made out of the same lattice model. This way, we ensure a perfect ballistic transport regime with a very long lifetime for the electronic wavepackets. The usual sizes of our samples are \(L/a = 4\) and \(W/a = 8, 16, 24, 32\), i.e., up to 4096 unit cells. Computing the conductance for a growing transverse size allows us to discriminate between bulk and surface contributions to the conductance. The bulk-modes contribution scales like the cross-section \((W/a)^2\), whereas the surface-modes contribution only scales with transverse size as \(W/a\). Hence, \((a/W)^2 G\) flows asymptotically to the bulk conductance for growing size. In the infinite-sample limit, we expect in the high field regime

![Figure E1. Left: dimensionless conductance \(g\) as a function of \(b\) for \(\mu = 0.7, \Delta = 1, L/a = 4\) and various \(W/a\). Inset shows surface corrections to the slope scaling like \(a/W\); in the limit \(W/a \to \infty\) the slope extrapolates to 4 which corresponds to \(g = 4\). 4 being the number of Weyl pairs. Right: conductance of a short junction with partially reflective contacts with the leads. The conductance is no longer ballistic and a \(b^2\)-behavior is recovered at small magnetic fields. Parameters are \(\Delta = 1, W/a = 24\).](image-url)
\[ g = G \left( \frac{a}{W} \right)^2 = N_{\text{pairs}} b + O \left( \frac{a}{W} \right), \]  

(E1)

in units of \( e^2/h \), where \( N_{\text{pairs}} \) counts the pairs of Weyl pairs and \( b = \phi / \phi_0 \). Indeed we show from linear fits of our results that the slope in the linear magneto-conductance regime flows toward \( N_{\text{pairs}} \) with additional surface corrections scaling like \( a/W \).

We have investigated the impact of reflective leads on transport, see figure E1. Reflection in the leads was implemented either by replacing the leads with semi-infinite 1D chains, as in \([39]\) or by changing the longitudinal velocity in the leads with respect to the junction. Both methods yielded a \( b^2 \)-regime for \( g \).

**Appendix F. Kinetic theory**

**F.1. Method**

We determine the magneto-conductance of a planar junction consisting a Weyl semimetallic layer of thickness \( L \) placed between two metallic leads.

The Hamiltonian around the two Weyl cones is written as \( H^\Sigma_k = \hbar c \chi \sigma \cdot k \) with energies \( \varepsilon^\Sigma_k(k) = \pm \hbar c k \) and eigenstates \( |u_{\chi,\pm,k} \rangle \). From now on we consider conduction band electrons, whose Berry curvature is

\[ \Omega^\Sigma_k = i \langle \partial_k u_{\chi,\pm,k} | \times | \partial_k u_{\chi,\pm,k} \rangle = \chi \frac{k}{2k^3}. \]  

(F1)

Within the semiclassical Boltzmann theory extended to account for topological properties of the Bloch band structure \([32, 10]\), the equations of motion are (neglecting orbital magnetization effects):

\[ \frac{\hbar}{c} \frac{\partial}{\partial \mathbf{r}} - \mathbf{v}_k \times \mathbf{B}, \]  

(F2)

\[ \hbar \dot{\mathbf{k}} = \frac{- \partial \varepsilon - \frac{c}{\hbar} \partial_k \varepsilon \times \mathbf{B} - e(\partial \varepsilon \mathbf{ \cdot B}) \Omega}{1 + \frac{c}{\hbar} (\mathbf{\mathbf{\Omega} \cdot B})}, \]  

(F3)

We solve these equations into

\[ \frac{\hbar}{c} \frac{\partial}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}} \times \mathbf{k}, \]  

(F4)

\[ \dot{\mathbf{k}} = \frac{1}{\hbar} \partial_k \varepsilon + \frac{e}{\hbar} \partial_k \varepsilon \times \mathbf{\Omega} + \frac{c}{\hbar} (\partial_k \varepsilon \mathbf{ \cdot B}) \frac{\mathbf{k}}{1 + \frac{c}{\hbar} (\mathbf{\Omega} \cdot B)}, \]  

(F5)

In the absence of electric field, corresponding to \( \partial \varepsilon = 0 \), and using the notation \( \mathbf{v}_k = \hbar^{-1} \partial_k \varepsilon = \mathbf{c} \hat{k} \), these equations simplify into

\[ \frac{\hbar}{c} \frac{\partial}{\partial \mathbf{r}} = \frac{- \varepsilon}{1 + \frac{c}{\hbar} (\mathbf{\mathbf{\Omega} \cdot B})} \mathbf{v}_k \times \mathbf{B}, \]  

(F6)

\[ \dot{\mathbf{r}} = \frac{1}{1 + \frac{c}{\hbar} (\mathbf{\mathbf{\Omega} \cdot B})} \left( \mathbf{v}_k + \frac{c}{\hbar} (\mathbf{v}_k \cdot \mathbf{B}) \mathbf{k} \right), \]  

(F7)

From equation (F7) the current density in the \( z \) direction is expressed as

\[ \mathbf{j}_z = -e \sum_{\chi,k} \mathbf{w}_{\chi,k} \cdot \hat{z} f_{\chi}(\mathbf{r}, \mathbf{k}), \]  

(F8)

where the anomalous velocity is

\[ \mathbf{w}_{\chi,k} = \left( \mathbf{v}_k + \frac{e}{\hbar} (\mathbf{v}_k \cdot \mathbf{\Omega_k}) \mathbf{B} \right) = \mathbf{c} \hat{k} + \frac{e c}{\hbar} \frac{\mathbf{B}}{2k^2}. \]  

(F9)

Introducing the rescaled magnetic flux we can express this anomalous velocity as

\[ \mathbf{w}_{\chi,k} = e \left( \hat{k} + \frac{\mathbf{B}}{\mathbf{B}} \right). \]  

(F10)

Furthermore, in the stationary regime, the occupations \( f_{\chi}(z, \mathbf{k}) \equiv f_{\chi}(z, \hat{k}, \phi_k) \) solve the kinetic equation

\[ w_{\chi,k}^z \partial_z f_{\chi}(z, \hat{k}) \equiv \frac{1}{\tau} \sum_{\chi'} \left[ f_{\chi}(z, \hat{k}') - f_{\chi}(z, \hat{k}) \right] + \frac{1}{\tau'} \sum_{\chi'} \left[ f_{\chi}(z, \hat{k}') - f_{\chi}(z, \hat{k}) \right]. \]  

(F11)

Here \( \tau \) and \( \tau' \) are the intra-node and inter-node elastic scattering times, respectively, and we dropped the energy dependence of \( f \). Furthermore, we neglected any inelastic process and assumed a vanishing electric field in the semimetal. The equilibration in the reservoirs is described by the boundary conditions.
\[ f_{\lambda}(z = 0, \kappa) = f_0(\varepsilon_k - \mu_1) \quad \text{if} \quad w_{\lambda, \kappa} \cdot \hat{z} > 0, \quad \text{(F12a)} \]
\[ f_{\lambda}(z = L, \kappa) = f_0(\varepsilon_k - \mu_2) \quad \text{if} \quad w_{\lambda, \kappa} \cdot \hat{z} < 0. \quad \text{ (F12b)} \]

Here, \( f_0 \) is the Fermi function and \( \mu_2 - \mu_1 \) is the electro-chemical potential bias.

Subsequently, we solve equation (F11) with
\[ f_{\lambda}(z, \kappa) = f_0(\varepsilon_k - \mu_1) T_\lambda(z, \theta) + f_0(\varepsilon_k - \mu_2)[1 - T_\lambda(z, \theta)], \quad \text{(F13)} \]
where \( \cos \theta = \hat{k} \cdot \hat{z} \), provided that \( T_\lambda(x, \theta) \) solve the equation
\[ \epsilon \left( \cos \theta + \frac{e}{h} \frac{B_z}{2k_F^2} \right) \partial_z T_\lambda(z, \theta) = \frac{1}{\tau}[\langle T_\lambda(z, \theta) \rangle_\theta - T_\lambda(z, \theta)] + \frac{1}{\tau}[\langle T_\lambda(z, \theta) \rangle_\theta - T_\lambda(z, \theta)], \quad \text{(F14)} \]
where \( B_z = B \cdot \hat{z} \) and we note \( \langle \ldots \rangle_\theta = \int_0^\pi d\theta \sin \theta \ldots \), together with the boundary conditions
\[ T_\lambda(z = 0, \theta) = 1 \quad \text{if} \quad \cos \theta + \frac{B_z}{B_c} > 0, \quad \text{(F15a)} \]
\[ T_\lambda(z = L, \theta) = 0 \quad \text{if} \quad \cos \theta + \frac{B_z}{B_c} < 0. \quad \text{(F15b)} \]

Using \( \sum_k ... = \nu_0 \int d\varepsilon \langle ... \rangle_\varepsilon \) in equation (F8), one can then straightforwardly perform the energy integration and find that the current \( I = W^2 \chi \) obeys the Ohm’s law, \( I = G V \), with the conductance
\[ G = e^2 W^2 c \nu_0 \sum_k \left( \left( \cos \theta + \frac{B_z}{B_c} \right) T_\lambda(x, \theta) \right)_\theta. \quad \text{(F16)} \]
Here \( W^2 \) is the wire’s cross-section and \( \nu_0 = \mu^2/(2\pi^2 \hbar^2) = k_F^2/(2\pi^2 \hbar) \) is the density of states at the Fermi level in a Weyl cone. Note that the kinetic equations guarantee the current conservation, so that equation (F16) does not depend on \( z \). By using the notation of the main text, \( G = (e^2/h)(W/a)^2 \chi \), we can rewrite equation (F16) as
\[ G = 2G_{Sh} \sum_\lambda \left( \left( \cos \theta + \frac{B_z}{B_c} \right) T_\lambda(x, \theta) \right)_\theta. \quad \text{(F17)} \]
with \( G_{Sh} \equiv g(B = 0) \) is the Sharvin (or contact) conductance associated with the two Weyl cones.

F.2. Clean limit

In the clean limit \( (\tau, \tau' \to \infty) \), we use the solution
\[ T_\lambda(z, \theta) = \begin{cases} 1 & \text{if} \quad \cos \theta + \frac{B_z}{B_c} > 0, \\ 0 & \text{if} \quad \cos \theta + \frac{B_z}{B_c} < 0. \end{cases} \quad \text{(F18)} \]
ton to obtain
\[ G = \begin{cases} G_{Sh} \left( 1 + \left( \frac{4B_c}{B_z} \right)^2 \right) & \text{if} \quad \left| B_z \right| < 4B_c, \\ \frac{4G_{Sh}}{B_c} = e^2 \frac{h}{W^2} \left| B_z \right| & \text{if} \quad 4B_c < \left| B_z \right|. \end{cases} \quad \text{(F19)} \]

F.3. Diffusive limit

When the mean-free path is short, \( \ell' \equiv \nu\tau \ll L \), the solution of equation (F14) is almost isotropic. Inserting the expansion \( T_\lambda(z, \theta) \approx T_\lambda(z) + \cos \theta \delta T_\lambda(z) \) (with \( \delta T_\lambda(z) \ll T_\lambda(z) \)) in equation (F14), and averaging it over angles, we find
\[ \frac{\ell}{3} \partial_z \delta T_\lambda(z) + \frac{e}{h} \frac{B_z}{2k_F^2} \partial_z T_\lambda(z) = \frac{1}{\tau}[T_\lambda(z) - T_\lambda(z)]. \quad \text{(F20)} \]
Multiplying equation (F14) by \( \cos \theta \), and then averaging it over angles, we get
\[ \epsilon \partial_z T_\lambda(z) \approx -\frac{1}{\tau} \delta T_\lambda(z), \quad \text{(F21)} \]
where we assumed \( 1/\tau', \epsilon/L, eB_z/(L\tau) \ll 1/\tau \). Thus the kinetic equation reduces to a diffusion equation including a magnetic-field induced ballistic contribution,
\[-D\partial_z^2 T_\chi(z) + \frac{B_z}{B_c} \partial_z T_\chi(z) - \frac{1}{\tau'} [T_\chi(z) - T_\chi(z)] = 0, \quad (F22)\]

where \(D = \frac{\tau}{3}\) is the diffusion constant, supplemented with the boundary conditions \(T_\chi(z = 0) = 1\) and \(T_\chi(z = L) = 0\), while the conductance is

\[G = W^2 \nu_0 \sum_x \left(-D\partial_z + \frac{B_z}{B_c}\right) \partial_z T_\chi(z). \quad (F23)\]

These equations match with those of [24] provided one identifies \(\tau' = 2\tau_n\).

An exact formula in the diffusive regime is provided in [24]. At \([B_z] \ll B_c \sqrt{\tau'/\tau}\), it reproduces the Drude conductance, \(G_D = 2\nu_0 DW^2/L\), which can be rewritten as \(G_D = 4G_{Sh} D/(cL)\), while at \([B_z] \gg B_c \sqrt{\tau'/\tau}\), it yields

\[G = \begin{cases} 
G_{Dr}, & [B]/[B_c] \ll \ell/L, \\
3(B/[B_c]^2 G_{Dr}, & [B]/[B_c] \ll \ell'/\ell', \\
\frac{1}{2} G_{Sh} [B_c]/[B], & \ell/L, L/\ell' \ll [B]/[B_c].
\end{cases} \quad (F24)\]

That is, a ballistic conductance again arises at sufficiently large magnetic fields.

**F.4. Ballistic-to-diffusive cross-over**

We follow the methods of [33] to obtain the conductance at arbitrary magnetic field and disorder strength. For this, we first observe that equation (F14) is a first-order differential equation on \(T_\chi\) (assuming that \(\langle T_\chi\rangle\) is known), which can be solved together with its boundary conditions as

\[T_\chi(z, \theta) = e^{-z^2/\ell^2} + \int_0^z dz' \langle T_\chi(z')\rangle e^{-z^2/\ell^2} / T_\chi(\theta), \quad (F25a)\]

for \(\ell_\chi(\theta) > 0\) and

\[T_\chi(z, \theta) = -\int_0^z dz' \langle T_\chi(z')\rangle e^{-z^2/\ell^2} / T_\chi(\theta), \quad (F25b)\]

for \(\ell_\chi(\theta) < 0\), where \(\ell_\chi(\theta) = \ell [\cos \theta + \chi B_z/B_c]\) and \(\langle T_\chi(z)\rangle = (\tau' T_\chi(z) + \tau \langle T_\chi(z)\rangle) / (\tau + \tau')\). Then averaging equation (F25) on angles, one gets an integral equation on \(\langle T_\chi\rangle\),

\[\langle T_\chi(z)\rangle = t_\chi(z) + \int_0^L dt g_\chi(z, z') \langle T_\chi(z')\rangle, \quad (F26)\]

with

\[t_\chi(z) = \langle e^{-z^2/\ell^2} \Theta(\ell_\chi(\theta))\rangle \quad (F27)\]

and

\[g_\chi(z, z') = \left\{ \frac{e^{-z^2/\ell^2} \Theta(\ell_\chi(\theta))}{\ell_\chi(\theta)} [\Theta(\ell_\chi(\theta)) \Theta(z - z') - \Theta(-\ell_\chi(\theta)) \Theta(z' - z)] \right\}, \quad (F28)\]

where \(\Theta\) is the Heaviside function.

Once the functions \(\langle T_\chi(z)\rangle\) solving equation (F26) have been obtained, one can then compute the conductance as

\[G = \frac{W^2 \nu_0}{\tau} \sum_x \left(s_\chi(z) + \frac{\tau + \tau'}{\tau'^2} \int_0^L dz' h_\chi(z, z') \langle T_\chi(z')\rangle \right), \quad (F29)\]

(it can be evaluated at an arbitrary position \(z\) with

\[s_\chi(z) = \langle \ell_\chi(\theta) e^{-z^2/\ell^2} \Theta(\ell_\chi(\theta))\rangle \quad (F30)\]

and

\[h_\chi(z, z') = \langle e^{-z^2/\ell^2} \Theta(\ell_\chi(\theta)) \Theta(z - z') - \Theta(-\ell_\chi(\theta)) \Theta(z' - z)\rangle \]. \quad (F31)\]

For the numerical solution, one should be careful with the log-divergency of \(g_\chi(z, z')\) at \(z \rightarrow z'\). As \(\langle T_\chi(z)\rangle\) is a smooth function of \(z\) on the scale of \(L\), we used the following regularization scheme: we transformed the integral equation (F26) into a matrix equation,
\langle T_{\chi}(z_n) \rangle \approx t_{\chi}(z_n) + \sum_{m=1}^{N} \int_{-\Delta z/2}^{\Delta z/2} dy \, g_{\chi}(z_m, z_n + y) \langle T_{\chi}(z_m) \rangle,
\]
\begin{align*}
= t_{\chi}(z_n) + \sum_{m=1}^{N} \tilde{g}_{\chi}(z_m, z_n) \langle T_{\chi}(z_m) \rangle
\end{align*}
\tag{F32}

with \( z_n = L(n - 1/2)/N \) (\( 1 \leq n \leq N \)), \( \Delta z = L/N \), and
\[
\tilde{g}_{\chi}(z_m, z_n) = \begin{cases} 
  t_{\chi}(z_n - z_m - \Delta z/2) - t_{\chi}(z_n - z_m + \Delta z/2), & n > m, \\
  1 - t_{\chi}(\Delta z/2) - t_{\chi}(\Delta z/2), & n = m, \\
  t_{\chi}(z_m - z_n - \Delta z/2) - t_{\chi}(z_m - z_n + \Delta z/2), & n < m,
\end{cases}
\tag{F33}
\]
where one can use Mathematica to get an explicit expression for \( t_{\chi}(z) \). Similarly, equation (F29) can be evaluated as
\[
G = \frac{A t_{\chi 0}}{\pi} \sum_{\chi} \left( s_{\chi}(z_n) + \frac{\pi + \pi'}{\pi t'} \sum_{m=1}^{N} \tilde{h}_{\chi}(z_m, z_n) \langle T_{\chi}(z_m) \rangle \right)
\tag{F34}
\]
with
\[
\tilde{h}_{\chi}(z_m, z_n) = \begin{cases} 
  s_{\chi}(z_n - z_m - \Delta z/2) - s_{\chi}(z_n - z_m + \Delta z/2), & n > m, \\
  \pm B/B_{z} - s_{\chi}(\Delta z/2) + s_{\chi}(\Delta z/2), & n = m, \\
  s_{\chi}(z_m - z_n - \Delta z/2) - s_{\chi}(z_m - z_n + \Delta z/2), & n < m,
\end{cases}
\tag{F35}
\]
where one can also use Mathematica to get an explicit expression for \( s_{\chi}(z) \).

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