The Thirring-Wess Model Revisited: A Functional Integral Approach

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Abstract

We consider the Wess-Zumino-Witten theory to obtain the functional integral bosonization of the Thirring-Wess model with an arbitrary regularization parameter. Proceeding a systematic of decomposing the Bose field algebra into gauge-invariant- and gauge-noninvariant field subalgebras, we obtain the local decoupled quantum action. The generalized operator solutions for the equations of motion are reconstructed from the functional integral formalism. The isomorphism between the QED$_2$ (QCD$_2$) with broken gauge symmetry by a regularization prescription and the Abelian (non-Abelian) Thirring-Wess model with a fixed bare mass for the meson field is established.

1 Introduction

The Thirring-Wess (TW) model \cite{1} was considered in Refs. \cite{2, 3, 4, 5, 6, 7} in order to investigate the way in which QED$_2$ can be understood as a limit of a vector meson theory when the bare mass ($m_o$) of the meson tends to zero. The standard TW model, in which the gauge invariant regularization prescription is adopted, corresponds to quantize the model absorbing all effects of the gauge symmetry breakdown in the bare mass of the meson theory. The computation of the vector current is performed using the gauge invariant regularization for the point-splitting limiting procedure, which corresponds to the regularization parameter $a = 2$. Within a formulation in a positive-definite metric Hilbert space, the limit $m_o \rightarrow 0$ does not exist for the Fermi field operator ($\psi$) and vector field operator ($A_\mu$) themselves \cite{7}. The zero mass limit is not well defined for the general Wightman functions that provide a representation of the gauge noninvariant field algebra.
of the TW model. However, the gauge invariant Wightman functions of the $QED_2$ are obtained as the zero mass limit of the Wightman functions of the gauge invariant field subalgebra of the standard TW model.

More recently, the $QED_2$ with broken gauge symmetry, by the use of an arbitrary regularization parameter $a$, has been considered in Refs.\cite{9} and entitled “anomalous” vector Schwinger model. The standard $QED_2$ should corresponds to the gauge invariant regularization $a = 2$. However, also in this case, the gauge invariant limit $a \to 2$ does not exist for the fields $\psi$ and $A_\mu$ themselves. The limit $a \to 2$ is well defined only for the gauge invariant subset of Wightman functions. The structural problem associated with the zero bare mass limit $m_0 \to 0$ of the Wightman functions for the standard TW model can be mapped into the corresponding problem of perform the limit $a \to 2$ in the $QED_2$ with gauge symmetry breakdown.

One of the purposes of the present work is to fill a gap in the existing literature by discussing structural aspects of the TW model from the functional integral formulation and by extending the analysis to the general TW model regularized with an arbitrary parameter $a$. To this end, we use the Abelian reduction of the Wess-Zumino-Witten theory to consider the functional integral bosonization of the generalized TW model. In order to obtain a better insight into the behaviour of gauge variant operators, in the present approach we adopt the systematic of decomposing step-by-step the Bose field algebra into gauge-invariant- (GI) and gauge-noninvariant (GNI) field subalgebras, such that the effective bosonized quantum action decouples into GI and GNI pieces. The gauge symmetry breakdown is characterized by the presence of a non-canonical free massless Bose field. The vector field with bare mass $m_0$ acquires a dynamical mass

$$\tilde{m}_0^2 = \frac{e^2}{4\pi} (a + 2) + m_0^2.$$  \hspace{1cm} (1.1)

Within the present approach we obtain a formulation in an indefinite-metric Hilbert space of states and the Proca equation is satisfied in the weak form. The generalized field operators are reconstructed from the functional integral formalism and are written as gauge transformed fields by an operator-valued gauge transformation. In the indefinite-metric formulation, the GI limit can be performed and we obtain the corresponding field operators of $QED_2$, as obtained by Lowenstein-Swieca \cite{6}. Performing a canonical transformation, the singular gauge part becomes the identity and we obtain the generalized operator solution for the coupled Dirac-Proca equations. For the gauge invariant regularization $a = 2$, we recover the operator solution of the standard TW model, as obtained by Lowenstein-Rothe-Swieca \cite{6,7}. Since in this case the longitudinal current does not carry any fermionic charge selection rule, commutes with itself and commutes with all operators belonging to the field algebra, it is reduced to the identity operator. This leads to a positive-metric Hilbert space of states and the coupled Dirac-Proca equations are satisfied in the strong form. Since in this case the Fermi field operator is given in terms of the charge-carrying fermion operator of the Thirring model, the zero mass limit exists only for the GI field subalgebra. This streamlines the presentation of Refs. \cite{6,7}.

Another purpose of the present paper is to discuss the isomorphism between the $QED_2$ with broken local gauge invariance (the “anomalous” vector Schwinger model considered in \cite{9}) and the TW model. Using the Wess-Zumino-Witten theory, we show that the effective quantum action of the $QED_2$ quantized with a gauge noninvariant regularization $b \neq 2$, is equivalent to the quantum action of the TW model with a vector field with bare mass
where \( a < b \) is the parameter used in the regularization of the TW model. In this way, the gauge invariant limit \( b \to 2 \) for the \( QED_2 \) with broken gauge symmetry, is mapped into the limit \( m_o \to 0 \) for the standard TW model \((a = 2)\). As is well known [7, 8], the confinement phenomenon in the standard \( QED_2 \) with broken gauge symmetry, is mapped into the limit \( m_o \to 0 \) for the standard TW model.

The introduction of the mass term for the Fermi field of the generalized TW model is also considered. For the GI regularization we recover the operator solution obtained by Rothe-Swieca [7] for the massive TW model.

In the last section we consider the functional integral bosonization of the non-Abelian TW model with an arbitrary regularization parameter. In this case, the quantum action of the non-Abelian TW model is mapped into the action of the \( QCD_2 \) with gauge symmetry breakdown.

## 2 Functional Integral Bosonization

The Thirring-Wess (TW) model is defined [1] from the classical Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu) \psi + \frac{1}{2} m_o^2 A_\mu A^\mu,
\]

where the field-strength tensor \( F_{\mu\nu} \) is given by

\[
F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.
\]

The local gauge invariance is broken by the mass term for the vector field. In the classical level, the local gauge invariance can be restored by performing in (2.1) the limit \( m_o \to 0 \). This limit corresponds to the classical Lagrangian of the two-dimensional electrodynamics. However, in the quantum level, the zero mass limit for the vector field, even for a gauge invariant regularization prescription, is well defined only for the gauge invariant subset of Wightman functions [7].

### 2.1 Decoupled Quantum Action

In order to obtain the bosonized action, we shall consider the Abelian reduction of the Wess-Zumino-Witten theory [11]. To this end, let us consider the generating functional (in Minkowski space),

\[
Z[\bar{\psi}, \psi, j^\mu] = \left\langle e^{i \int d^2z (\bar{\psi} \psi_0 + \bar{\psi} \psi_0 + j_\mu A^\mu)} \right\rangle,
\]

Our conventions are: \( g^{00} = 1, \epsilon^{01} = -\epsilon_{01} = -\epsilon^{10} = 1, \gamma^\mu \gamma^5 = \epsilon^{\mu\nu} \gamma_\nu, \partial = \gamma^\mu \partial_\mu; \bar{\psi}_0 = \epsilon_{\mu\nu} \phi^\nu, \gamma^5 = \gamma^0 \gamma^1; \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \partial_\pm = \partial_0 \pm \partial_1; A_{\pm} = A_0 \pm A_1; \partial = \gamma^\mu \partial_\mu; \) For a free massless scalar field, \( \bar{\Phi} = -\partial_\mu \Phi \). The Fermi field \( \Psi \) is the two-component spinor \( \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \).

\[
m_o^2 = \frac{e^2}{4\pi} (b - a),
\]
where the average is taken with respect to the functional integral measure,

\[ d\mu = \int D\mathcal{A}_\mu D\bar{\psi} D\psi e^{iS[\bar{\psi},\psi,\mathcal{A}_\mu]}, \tag{2.4} \]

and the Lagrangian density is written as,

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \psi_1^\dagger D_+(\mathcal{A}) \psi_1 + \psi_2^\dagger D_-(\mathcal{A}) \psi_2 + \frac{1}{2} m_o^2 \mathcal{A}_+ \mathcal{A}_-, \tag{2.5} \]

where

\[ D_{\pm}(\mathcal{A}) = (i \partial_{\pm} - e \mathcal{A}_{\pm}). \tag{2.6} \]

The vector field can be parametrized in terms of the \( U(1) \) group-valued Bose fields \((U,V)\) as follows \[13, 14, 15\]

\[ \mathcal{A}_+ = -\frac{1}{e} U^{-1} i \partial_+ U, \tag{2.7} \]
\[ \mathcal{A}_- = -\frac{1}{e} V i \partial_- V^{-1}, \tag{2.8} \]

with

\[ U = e^{2i \sqrt{\pi} u}, \tag{2.9} \]
\[ V = e^{2i \sqrt{\pi} v}, \tag{2.10} \]

such that,

\[ \bar{\psi} \mathcal{D}(\mathcal{A}) \psi = (U \psi_1)^\dagger (i \partial_+) (U \psi_1) + (V^{-1} \psi_2)^\dagger (i \partial_-) (V^{-1} \psi_2). \tag{2.11} \]

In order to decouple the Fermi- and vector fields in the Lagrangian \[2.1\], the spinor components \((\psi_1, \psi_2)\), are parametrized in terms of the Bose fields \((U,V)\) and the free Fermi field components \((\chi_1, \chi_2)\),

\[ \psi_1 = U^{-1} \chi_1, \tag{2.12} \]
\[ \psi_2 = V \chi_2. \tag{2.13} \]

In order to introduce the systematic of decomposing the Bose field algebra \( \mathcal{Z}^B \),

\[ \mathcal{Z}^B = \mathcal{Z}^B \{U,V\}, \tag{2.14} \]

into gauge invariant (GI) and gauge noninvariant (GNI) field subalgebras, let us consider a class of local gauge transformations \( g \) acting on the Bose fields \( U, V \), as follows,

\[ g : U \to gU = Ug, \tag{2.15} \]
\[ g : V \to gV = g^{-1}V, \tag{2.16} \]
with
\[ g(x) = e^{2i \sqrt{\pi} \Lambda(x)}. \] (2.17)

Under \( g \)-transformations the fields \((u, v)\) transform according with,

\[ g : u \to u + \Lambda, \] (2.18)
\[ g : v \to v - \Lambda. \] (2.19)

The field combination \((u + v)\) is gauge invariant, whereas the combination \((u - v)\) is gauge non-invariant. The vector field and the Fermi field transform under \( g \) according with,

\[ g A_\mu = A_\mu + \frac{1}{e} g i \partial_\mu g^{-1}, \] (2.20)
\[ g \psi_1 = (gU^{-1}) \chi_1 = g^{-1} \psi_1, \] (2.21)
\[ g \psi_2 = (gV) \chi_2 = g^{-1} \psi_2. \] (2.22)

Let us introduce the gauge invariant Bose field \( G \),

\[ G = UV = e^{2i \sqrt{\pi} (u + v)}, \] (2.23)
such that,

\[ g G = gU gV = UV = G. \] (2.24)

In terms of the field \( G \), the field-strenght tensor is given by,

\[ F_{01} = \frac{1}{2} (\partial_- A_+ - \partial_+ A_-) = - \frac{1}{2e} \partial_+ (G^{-1} i \partial_- G) = \frac{\sqrt{\pi}}{e} \Box (u + v). \] (2.25)

and the Maxwell action can be written as,

\[ S_M[UV] = \frac{1}{2 \tilde{\mu}_o^2} \int d^2 z (u + v) \Box^2 (u + v), \]
\[ S_M[G] = \frac{1}{8e^2} \int d^2 z [\partial_+(G i \partial_- G^{-1})]^2 \]
\[ = \frac{1}{8e^2} \int d^2 z [\partial_-(G^{-1} i \partial_+ G)]^2, \] (2.26)

where we have defined

\[ \tilde{\mu}_o^2 = \frac{e^2}{4 \pi}. \] (2.27)

Let us return to the functional integration in the generating functional. Introducing the identities,
\[ 1 = \int \mathcal{D}U \left[ \det \mathcal{D}_+(U) \right] \delta(e \mathcal{A}_+ - U^{-1} i \partial_+ U) , \quad (2.28) \]

\[ 1 = \int \mathcal{D}V \left[ \det \mathcal{D}_-(V) \right] \delta(e \mathcal{A}_- - V i \partial_- V^{-1}) , \quad (2.29) \]

the change of variables \( \{ \mathcal{A}_+, \mathcal{A}_- \} \to \{ U, V \} \) is performed by integrating over the vector field components \( \mathcal{A}_\pm \). Performing the Fermi field rotations (2.12)-(2.13), and taking due account of the Jacobians in the change of the integration measure in the generating functional, we obtain the effective integration measure \[13, 14, 15\],

\[ d\mu = \mathcal{D} \bar{\chi} \mathcal{D} \chi \mathcal{D} U \mathcal{D} V e^{i S[U,V,\bar{\chi},\chi]} e^{i S'[U,V,a,m_o]} , \quad (2.30) \]

where,

\[ S[U,V,\bar{\chi},\chi] = \int d^2 z \bar{\chi} i \partial_+ \chi + S_M[UV], \quad (2.31) \]

and \( S'[U,V,a,m_o] \) is the GNI contribution, which is given in terms of the Wess-Zumino-Witten (WZW) functionals \( \Gamma[U], \Gamma[V] \),

\[ S'[U,V,a,m_o] = -\Gamma[U] - \Gamma[V] - \frac{1}{2e^2} (a \bar{\mu}^2_o + m^2_o) \int d^2 z \left( U^{-1} \partial_+ U \right) \left( V \partial_- V^{-1} \right) . \quad (2.32) \]

The term carrying the regularization parameter \( a \) in the action (2.32) corresponds to the Jackiw-Rajaraman action,

\[ S_{JR} = \frac{a}{2} \bar{\mu}^2_o \int d^2 z \mathcal{A}_+ \mathcal{A}_- , \quad (2.33) \]

which in an anomalous model, such as chiral \( QED_2 \), characterizes the quantization ambiguity \[10\]. The value \( a = 2 \) corresponds to the GI regularization. The WZW functionals enter (2.32) with negative level \[11\]. In the Abelian case, the WZW functional reduces to the action of a free massless scalar field,

\[ \Gamma[h] = \frac{1}{8\pi} \int d^2 z \left( \partial_\mu h \right) \left( \partial^\mu h^{-1} \right) . \quad (2.34) \]

Using the Abelian reduction of the Polyakov-Wiegmann (PW) identity \[12\],

\[ \Gamma[gh] = \Gamma[g] + \Gamma[h] + \frac{1}{4\pi} \int d^2 z \left( g^{-1} \partial_+ g \right) \left( h \partial_- g^{-1} \right) , \quad (2.35) \]

the total effective action can be written as,

\[ S[U,V,\bar{\chi},\chi] = S[\bar{\chi},\chi,U,V] + S'[U,V] , \quad (2.36) \]

where

\[ S[\bar{\chi},\chi,U,V] = S_F^{(0)}[\bar{\chi},\chi] + S_M[UV] , \quad (2.37) \]
\[ S'[U,V] = -\frac{1}{2} \tilde{\mu}_o^2 \left\{ \tilde{\mu}_o^2 a + m_o^2 \right\} \Gamma[UV] + \frac{1}{2} \tilde{\mu}_o^2 \left\{ \tilde{\mu}_o^2 (a - 2) + m_o^2 \right\} \left( \Gamma[V] + \Gamma[U] \right). \]  

(2.38)

The standard TW model corresponds to \( a = 2 \) and in the limit \( m_o \to 0 \), the gauge invariance is restored,

\[ S'[U,V] \to S'[\mathbf{G}], \]

(2.39)

such that the action (2.36) reduces to the action of the standard \( QED_2 \),

\[ S[U,V,\bar{\chi},\chi] \to S_{QED}[\mathbf{G},\bar{\chi},\chi]. \]

(2.40)

For a gauge noninvariant regularization, in the limit \( m_o \to 0 \), we obtain the action for the \( QED_2 \) with broken gauge symmetry,

\[ S'[U,V,b,0]_{QED} = -\frac{b}{2} \Gamma[UV] + \frac{1}{2} (b - 2) \left( \Gamma[V] + \Gamma[U] \right). \]

(2.41)

where \( b \) is the corresponding regularization parameter. The actions (2.38) and (2.41) are equivalent,

\[ S'[U,V,b,0] \equiv S'[U,V,a,m_o], \]

(2.42)

provided that \( b > a \) and

\[ m_o^2 = \tilde{\mu}_o^2 (b - a). \]

(2.43)

This implies that the \( QED_2 \) quantized with a GNI regularization \( b \neq 2 \) (\( QED_2 \) with broken gauge symmetry) is isomorphic to the TW model with a regularization parameter \( a \) and with a fixed bare mass for the vector field given by (2.43).

In order to proceed further the decomposition of the Bose field algebra \( \mathcal{B}_B\{U,V\} \) into GI and GNI field subalgebras, let us introduce the gauge invariant pseudo-scalar field \( \tilde{\phi} \),

\[ \tilde{\phi} \equiv \frac{1}{2} (u + v), \]

(2.44)

and the gauge non-invariant scalar field \( \zeta \),

\[ \zeta \equiv \frac{1}{2} (u - v). \]

(2.45)

The Bose fields \( \{U,V\} \), defined by (2.9)-(2.10), can be decomposed in terms of the gauge invariant field \( g[\tilde{\phi}] \) and the gauge non-invariant field \( h[\zeta] \) as

\[ U = g \, h, \]

(2.46)

\[ V = g \, h^{-1}, \]

(2.47)

where

\[ g = e^{2i \sqrt{\pi} \, \tilde{\phi}}, \]

(2.48)
\[ h = e^{2i \sqrt{\pi} \zeta}. \]  
(2.49)

The gauge invariant field \( G \) can be rewritten as

\[ G = U \cdot V = g^2 = e^{4i \sqrt{\pi} \bar{\phi}}. \]  
(2.50)

The vector field can be decomposed in the standard form in terms of GI and GNI contributions,

\[ A_\mu = \sqrt{\pi} e^{\left\{ \bar{\partial}_\mu \bar{\phi} + \partial_\mu \zeta \right\}}, \]  
(2.51)

and the Maxwell action (2.26) is now given by

\[ S_M[g] = \frac{1}{2} \bar{\mu}^2 \int d^2z (\Box \bar{\phi})^2. \]  
(2.52)

The Bose field algebra can be decomposed into

\[ \mathcal{ℑ}_B \{ U, V \} = \mathcal{ℑ}_B \{ g \} \oplus \mathcal{ℑ}_{GNI} \{ h \}, \]  
(2.53)

and the effective quantum action \( S[U, V, \bar{\chi}, \chi] \) can be rewritten in terms of the fields \( g \) and \( h \),

\[ S[U, V, \bar{\chi}, \chi] = S[g, h, \bar{\chi}, \chi] = S[g, \bar{\chi}, \chi] + S'[g, h], \]  
(2.54)

where

\[ S[g, \bar{\chi}, \chi] = S_F^{(0)}[\bar{\chi}, \chi] + S_M[g], \]  
(2.55)

\[ S'[g, h] = -\frac{1}{2} \bar{\mu}^2 \left\{ \bar{\mu}^2 (a + m_o^2) \right\} \Gamma[g^2] + \frac{1}{2} \bar{\mu}^2 \left\{ \bar{\mu}^2 (a - 2) + m_o^2 \right\} \left\{ \Gamma[gh^{-1}] + \Gamma[gh] \right\}, \]  
(2.56)

with the Maxwell action given by

\[ S_M[g] = \frac{1}{8 \pi \bar{\mu}^2} \int d^2z \left[ \partial^+ (g_i \partial_- g^{-1}) \right]^2. \]  
(2.57)

Using the P-W identity, the GNI action \( S'[g, h] \) given by (2.56) decouples into,

\[ S'[g, h] = S[g] + S[h] = -\frac{\bar{m}^2}{\bar{\mu}^2} \Gamma[g] + \frac{\bar{m}_o^2}{\bar{\mu}_o^2} \Gamma[h], \]  
(2.58)

where

\[ \bar{m}^2 = \bar{\mu}^2 (a + 2) + m_o^2, \]  
(2.59)

\[ \bar{m}_o^2 = \bar{\mu}_o^2 (a - 2) + m_o^2. \]  
(2.60)
The fields $\bar{\phi}, h[\zeta]$, decouple in the action (2.58) and the GNI field $h[\zeta]$ is a free massless non-canonical scalar field. The total partition function factorizes as

$$
\mathcal{Z} = \left( \int \mathcal{D} h \ e^{iS[h]} \right) \left( \int \mathcal{D} g \mathcal{D} \bar{\chi} \mathcal{D} \chi \ e^{iS[g]+S[g,\bar{\chi},\chi]} \right) = \mathcal{Z}_h \times \mathcal{Z}_{\bar{\chi},\chi,g}.
$$

The partition function $\mathcal{Z}_h$ characterizes the local gauge symmetry breakdown. Although the partition function can be factorized, the generating functional, and thus the Hilbert space of states $\mathcal{H}$, cannot be factorized

$$
\mathcal{H} \neq \mathcal{H}_{\bar{\chi},\chi,g} \otimes \mathcal{H}_h.
$$

The bonafide gauge invariant vector model corresponds to $m_o = 0$, that is obtained with $m_o = 0$ and the gauge invariant regularization $a = 2$. In this case, all reference to the field $h[\zeta]$ has disappeared from the effective quantum action (2.58), which is given in terms of the gauge invariant field $g$. In this case, the field $\zeta$ is not a dynamical degree of freedom and corresponds to a pure $c$-number gauge excitation. The commutator (2.85) and the corresponding Hamiltonian of the effective bosonized quantum action are singular for $m_o = 0$. For these critical values of the parameters, the GNI action vanishes,

$$
S[h] = \frac{\bar{m}_o^2}{\bar{\mu}_o^2} \Gamma[h] \to 0,
$$

implying that for $m_o = 0$, the gauge invariance is formally restored and the field $h[\zeta]$ is not a dynamical degree of freedom. At this critical point the constraint structure of the model change. Since in the gauge invariant limit $\bar{m}_o \to 0$ the field $h$ becomes a pure gauge excitation, the corresponding partition function reduces to a gauge volume,

$$
\lim_{\bar{m}_o \to 0} \mathcal{Z}_h \to \int \mathcal{D} h = V_{\text{gauge}}.
$$

The “anomalous vector Schwinger model” considered in Ref. [9] is obtained considering $m_o = 0$ and $b \neq 2$. The GNI action is now given by

$$
S'[g,h] = -(b+2) \Gamma[g] + (b-2) \Gamma[h],
$$

and corresponds to the TW model with bare mass for the vector field given by (2.43).

### 2.2 Local Action

The Maxwell action (2.57) is non-local due to the quartic-self-interaction of the field $g$. In order to dequartize the action of the gauge invariant field $g$, let us consider the functional integral over the field $g$ in the partition function $\mathcal{Z}_{\bar{\chi},\chi,g}$. To begin with, let us introduce an auxiliary gauge invariant field $\Omega$, such that,

$$
\int \mathcal{D} g \exp i \int d^2z \left\{ \frac{1}{8\pi \bar{\mu}_o^2} \left[ \partial_+ \left( g i \partial_- g^{-1} \right) \right]^2 - \frac{1}{8\pi \bar{\mu}_o^2} \partial_+ g \partial_- g^{-1} \right\} \equiv 
\int \mathcal{D} \Omega \mathcal{D} g \exp i \int d^2z \left\{ -\frac{1}{2} \Omega^2 + \frac{1}{2\sqrt{\pi} \bar{\mu}_o} \Omega \left[ \partial_- \left( g^{-1} i \partial_+ g \right) \right] \right\}
$$
\[
- \frac{1}{8\pi} \frac{\bar{m}^2}{\bar{\mu}_o^2} \partial_+ g \partial_- g^{-1} \) \] (2.66)

Making the change of variables,

\[ \partial_+ \Omega = (\omega_+ \partial_+ \omega^{-1}) , \] (2.67)

we can write,

\[ \Omega = \partial_+ (\omega_+ \partial_+ \omega^{-1}) . \] (2.68)

Rescaling the exponential field \( \omega \),

\[ \frac{2\sqrt{\pi} \bar{\mu}_o}{\bar{m}^2} \omega^{-1} i \partial_+ \omega = \omega^{-1} i \partial_+ \bar{\omega} , \] (2.69)

we obtain from (2.66) after an integration by parts,

\[ \int D\bar{\omega} DG \exp i \int d^2z \left\{ - \frac{1}{8\pi} \frac{\bar{m}^4}{\bar{\mu}_o^2} [\partial_- (\omega_- \partial_- \omega^{-1})]^2 + \frac{1}{8\pi} \frac{\bar{m}^2}{\bar{\mu}_o^2} (g_-^{-1} i \partial_+ g - \omega_-^{-1} i \partial_+ \bar{\omega}) \left( g_i \partial_i g^{-1} - \omega_i \partial_i \omega^{-1} \right) \right. \]

\[ \left. - \frac{1}{8\pi} \frac{\bar{m}^2}{\bar{\mu}_o^2} (\omega_-^{-1} i \partial_+ \bar{\omega}) \left( \omega_i \partial_i \omega^{-1} \right) \right\} . \] (2.70)

Defining a new gauge invariant field \( \theta \),

\[ \theta^{-1} i \partial_+ \theta = g_-^{-1} i \partial_+ g - \omega_-^{-1} i \partial_+ \bar{\omega} , \] (2.71)

\[ \theta_+ i \partial_- \theta^{-1} = g_i \partial_i g^{-1} - \omega_i \partial_i \omega^{-1} , \] (2.72)

the field \( g \) can be factorized as,

\[ g = \theta \bar{\omega} . \] (2.73)

The total effective action is now written in terms of the gauge invariant fields \( \theta, \bar{\omega}, \) the gauge noninvariant field \( h \) and the free Fermi field \( \chi \). In order to obtain the complete bosonized action, we introduce the bosonized form for the action of the free Fermi field \[16],

\[ S_F^{(0)} = \Gamma[f_\varphi] = \frac{1}{2} \int d^2z \partial_\mu \varphi \partial^\mu \varphi , \] (2.74)

with

\[ f_\varphi = e^{2i \sqrt{\pi} \varphi} , \] (2.75)

and the Mandelstam representation for the bosonized free massive Fermi field,

\[ \chi(x) = \left( \frac{\kappa_o}{2\pi} \right)^{\frac{1}{2}} e^{-i \frac{\pi}{4} \gamma^5} : \exp i \sqrt{\pi} \{ \gamma^5 \bar{\varphi}(x) + \int_{x^1}^\infty dz^1 \partial_0 \bar{\varphi}(x^0, z^1) \} : , \] (2.76)
where $\kappa_o$ is an arbitrary finite mass scale. The total local action is given in the bosonized form by,

$$S[f, g, h] = S[f, \theta, \hat{\omega}, h] = \Gamma[f] - \Gamma[\theta] + \Gamma[\hat{\omega}] + \frac{\tilde{m}_o^2}{\mu_o^2} \Gamma[h] -$$

$$- \frac{1}{8\pi} \frac{\tilde{m}^4}{\mu_o^2} \int d^2z \left[ \partial_z^{-1} (\hat{\omega} i \partial_z \hat{\omega}) \right]^2,$$

(2.77)

where the field $\theta$ is quantized with negative metric. Parametrizing the fields $\theta$ and $\hat{\omega}$ as,

$$\theta(x) = e^{2i \sqrt{\pi} \tilde{\eta}(x)},$$

(2.78)

$$\hat{\omega}(x) = e^{2i \sqrt{\pi} \tilde{\mu} \tilde{m}_o \tilde{\Sigma}(x)},$$

(2.79)

performing the field scaling,

$$\tilde{\eta} \rightarrow \frac{\mu_o}{\tilde{m}} \tilde{\eta}',$$

(2.80)

$$\tilde{\Sigma} \rightarrow \tilde{m} \tilde{\Sigma}',$$

(2.81)

and streamlining the notation by dropping primes everywhere, we obtain for the gauge invariant field $\tilde{\phi}$,

$$\tilde{\phi} = \frac{\tilde{\mu}_o}{\tilde{m}} (\tilde{\eta} + \tilde{\Sigma}),$$

(2.82)

$$g = \theta \hat{\omega} = e^{2i \sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} (\tilde{\eta} + \tilde{\Sigma})}.$$  

(2.83)

The effective bosonized total Lagrangian density is then given by,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \tilde{\phi})^2 + \frac{1}{2} (\partial_\mu \tilde{\Sigma})^2 - \frac{\tilde{m}_o^2}{2} \tilde{\Sigma}^2 - \frac{1}{2} (\partial_\mu \tilde{\eta})^2 + \frac{1}{2} \frac{\tilde{m}_o^2}{\mu_o^2} (\partial_\mu \zeta)^2.$$  

(2.84)

The field $\tilde{\eta}$ is quantized with negative metric. The gauge non-invariant field $\zeta$ is a non-canonical free massless decoupled field,

$$[\zeta(x), \zeta(y)] = \frac{\tilde{\mu}_o^2}{\tilde{m}_o^2} \Delta(x - y; 0).$$  

(2.85)

For massless Fermi fields, although the total partition function factorizes into free field partition functions,

$$Z = Z_f \times Z_\theta \times Z_{\hat{\omega}} \times Z_h,$$  

(2.86)

the generating functional does not factorizes.
3 Field Operators and Hilbert Space

The Bose fields \((U, V)\) are given by,

\[
U = e^{2i\sqrt{\pi} \tilde{\phi} h[\zeta]} = e^{2i\sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} (\tilde{\eta} + \tilde{\Sigma})} h[\zeta],
\]

(3.1)

\[
V = e^{2i\sqrt{\pi} \tilde{\phi} h^{-1}[\zeta]} = e^{2i\sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} (\tilde{\eta} + \tilde{\Sigma})} h^{-1}[\zeta].
\]

(3.2)

The bosonized form of the Fermi field operator is given in terms of the free Fermi field as,

\[
\psi(x) = \left(\frac{\chi_0}{2\pi}\right) e^{-i\frac{\pi}{4} \gamma_5} e^{2i\sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} \gamma^5 \{\tilde{\eta}(x) + \tilde{\Sigma}(x)\}} \times
\]

\[
e^{i\sqrt{\pi} \gamma^5 \varphi(x) + \int_{x^0}^{x^1} \partial_0 \varphi(x^0, z^1) dz^1} h^{-1}[\zeta],
\]

(3.3)

and the vector field is given by,

\[
A_\mu = \frac{1}{\tilde{m}} \tilde{\partial}_\mu (\tilde{\Sigma} + \tilde{\eta}) + \frac{1}{e} h i \partial_\mu h^{-1}.
\]

(3.4)

The divergent part \(h[\zeta]\) has the form of a gauge term. The bosonized expressions for fields (3.3) and (3.4) correspond to those of the Schwinger model gauged by a divergent operator-gauge transformation as long as \(\tilde{m}_o \rightarrow 0\).

The vector current is computed with the standard point-splitting limit procedure,

\[
J_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x) = Z^{-1}(e) \left[\bar{\psi}(x + e) \gamma_\mu e^{ia\tilde{\mu}_o \int_{x^0}^{x^1} A_\mu(z) dz^1} \psi(x) - V.E.V.\right].
\]

(3.5)

In terms of the GI and GNI field combinations \((u \pm v)\), the vector current is given by,

\[
\frac{e}{2} J_\mu = \frac{e}{2} J_\mu - \frac{\tilde{\mu}_o}{2} (a + 2) \tilde{\partial}_\mu (u + v) - \frac{\tilde{\mu}_o}{2} (a - 2) \tilde{\partial}_\mu (u - v),
\]

(3.6)

where \(J_\mu\) is the conserved current associated with the free Fermi field \(\chi\),

\[
J_\mu = \bar{\chi} \gamma_\mu \chi = -\frac{2}{\sqrt{\pi}} \tilde{\partial}_\mu \tilde{\phi}.
\]

(3.7)

The bosonized form of the vector current is given by,

\[
\frac{e}{2} J_\mu = -2 \tilde{\mu}_o \tilde{\partial}_\mu \tilde{\phi} + \left(\frac{\tilde{m}^2 - m_o^2}{\tilde{m}}\right) \tilde{\partial}_\mu (\tilde{\Sigma} + \tilde{\eta}) - \tilde{\mu}_o (\tilde{m}_o^2 - m_o^2) \tilde{\partial}_\mu \zeta.
\]

(3.8)

The conserved current that acts as the source of \(F_{\mu\nu}\) is given by

\[
K_\mu = m_o^2 A_\mu - \frac{e}{2} J_\mu = -\frac{e}{2} J_\mu + \tilde{m}^2 \tilde{\partial}_\mu (u + v) + \tilde{m}_o^2 \tilde{\partial}_\mu (u - v),
\]

(3.9)

which can be written as,

\[
K_\mu = \tilde{m} \tilde{\partial}_\mu \tilde{\Sigma} + L_\mu,
\]

(3.10)
where $L_\mu$ is a longitudinal current of zero norm given by,

$$L_\mu = \partial_\mu \mathcal{L},$$

with the potential $L$ given by,

$$L = 2\mu_o \varphi + \tilde{m} \eta + \frac{\tilde{m}_o^2}{\mu_o} \zeta.$$  \hspace{1cm} (3.12)

The longitudinal current $L_\mu$ commutes with itself and thus generates zero norm states from the vacuum,

$$\langle \Psi_o | L_\mu(x) L_\nu(y) | \Psi_o \rangle = 0.$$  \hspace{1cm} (3.13)

Due to the presence of the longitudinal current $L_\mu$, the Proca equation is not satisfied as an operator identity,

$$\begin{align*}
\partial_\nu F^{\nu \mu} + m^2_o A_\mu &- e J_\mu = L_\mu.
\end{align*}$$

The fundamental fields $\{\bar{\psi}, \psi, A_\mu\}$ generate a local field algebra $\mathcal{Z}$. These field operators constitute the intrinsic mathematical description of the model and whose Wightman functions define the model. The field algebra $\mathcal{Z}$ is represented in the indefinite-metric Hilbert space of states $\mathcal{H}$,

$$\mathcal{H} = \mathcal{Z}|\Psi_o\rangle.$$  \hspace{1cm} (3.15)

In a gauge invariant model, the field $\zeta$ is not a dynamical degree of freedom. In the indefinite metric formulation, the Gauss’ law is satisfied in weak form,

$$\partial_\nu F^{\nu \mu} - e J_\mu = L_\mu,$$

where $L_\mu$ is the zero-norm longitudinal piece of the vector current that acts as the source in the Gauss’ law. The local gauge transformations of the intrinsic fields $\{\bar{\psi}, \psi, A_\mu\}$ are implemented by the longitudinal current $L_\mu$. The physical gauge invariant field algebra $\mathcal{Z}_{phys}$, is a subalgebra of the intrinsic field algebra $\mathcal{Z}$ that commutes with the longitudinal current,

$$\mathcal{Z}_{phys} \subset \mathcal{Z},$$

$$[\mathcal{Z}_{phys}, L_\mu] = 0.$$  \hspace{1cm} (3.17) \hspace{1cm} (3.18)

The physical Hilbert space

$$\mathcal{H}_{phys} = \mathcal{Z}_{phys}|\Psi_o\rangle,$$

is a subspace of the Hilbert space $\mathcal{H} = \mathcal{Z}|\Psi_o\rangle$,

$$\mathcal{H}_{phys} \subset \mathcal{H}.$$  \hspace{1cm} (3.19) \hspace{1cm} (3.20)

In a model with gauge symmetry breakdown, the field $\zeta$ is a dynamical degree of freedom. The intrinsic field algebra commutes with the longitudinal current,
This implies that the fields \( \{ \bar{\psi}, \psi, A_\mu \} \) are singlet under gauge transformations generated by the longitudinal current and thus are physical operators. The intrinsic field algebra is by itself the physical field algebra,

\[
\mathfrak{I} \equiv \mathfrak{I}_{\text{phys}} .
\]  

### 3.1 The \( QED_2 \) limit

It is very instructive to make some comments about the gauge invariant limit \( m_o \to 0 \). From the Lagrangian (2.84), written in terms of the non-canonical field \( \zeta \), we obtain in the zero mass limit,

\[
L_{m_o \to 0} \to L_{QED} .
\]  

In this indefinite-metric Hilbert space formulation, the GI limit can be performed in the operator field algebra written in terms of the non-canonical free field \( \zeta \). The operator solution of the \( QED_2 \), as given by Lowenstein-Swieca [6], can be formally obtained from (3.3), (3.4), (3.10) and (3.11). In the gauge invariant limit, the field \( \zeta \) decouples from the quantum action corresponding to the Lagrangian (2.84) and thus it is not a dynamical degree of freedom. The field \( \zeta \) becomes a \( c \)-number, \( \zeta(x) \to \Lambda(x) \), and we get,

\[
\psi(x) =: e^{i \sqrt{\pi} \gamma^5 \{ \bar{\eta}(x) + \bar{\zeta}(x) \}} : \chi(x) e^{-i \sqrt{\pi} \Lambda(x)} ,
\]  

\[
A_\mu = \frac{\sqrt{\pi}}{e} \bar{\eta} \delta_\mu (\bar{\Sigma} + \bar{\eta}) + \frac{\sqrt{\pi}}{e} \partial_\mu \Lambda(x) ,
\]  

\[
J_\mu = \frac{1}{\sqrt{\pi}} \delta_\mu \bar{\zeta} + L_\mu ,
\]  

where \( L_\mu \) is the longitudinal current of zero norm,

\[
L_\mu = -\frac{e}{\sqrt{\pi}} \partial_\mu (\varphi + \eta) .
\]

Taking this into account, we obtain for the generating functional,

\[
Z[\bar{\psi}, \psi, J_\mu]_{m_o \to 0} \to Z[\bar{\psi}, \psi, J_\mu]_{QED} .
\]  

It must be stressed that this gauge invariant limit can be formally performed only in this level. It cannot be performed in the general Wightman functions due to the singular commutation relation for the field \( \zeta \). This limit is well defined only for the subset of Wightman functions that are independent of the field \( \zeta \). By considering the quantum action and the field operators written in terms of the non-canonical field \( \zeta \), to take the limit \( m_o \to 0 \) is equivalent to start from the beginning with a zero bare mass for the vector meson. As a matter of fact, in this indefinite-metric Hilbert space formulation, the GI limit can be formally performed as above, since the Fermi field operator is written in terms of the free Fermi field and not in terms of the charge-carrying Fermi field operator of the Thirring model. The intrinsic field algebra \( \mathfrak{I} \) is the physical field algebra, which is singlet under gauge transformations generated by the longitudinal current.
4 The Positive-definite-metric Formulation

The bosonization in the indefinite-metric formulation introduces a larger Bose field algebra $\mathcal{Z}_B$ that contains more degrees of freedom than those needed for the description of the model, $\mathcal{Z} \subset \mathcal{Z}_B$. In order to extract the redundant degrees of freedom, as well as, to obtain the solution of the equations of motion as operator identities in a positive definite-metric Hilbert space of states, we introduce two free Bose fields $(\tilde{\Phi}, \tilde{\Xi})$, by the following canonical transformation

$$\frac{\beta}{2} \tilde{\Phi} = \sqrt{\pi} \tilde{\varphi} + 2 \sqrt{\pi} \frac{\tilde{\mu}_0}{m} \tilde{\eta}, \quad (4.1)$$

$$\frac{\beta}{2} \tilde{\Xi} = 2 \sqrt{\pi} \frac{\tilde{\mu}_0}{m} \tilde{\varphi} + \sqrt{\pi} \tilde{\eta}. \quad (4.2)$$

The negative metric quantization for the field $\tilde{\eta}$ ensures that the fields $\tilde{\Phi}$ and $\tilde{\Xi}$ are independent degrees of freedom,

$$[\tilde{\Phi}(x), \tilde{\Xi}(y)] = 0, \quad \forall (x, y). \quad (4.3)$$

Imposing canonical commutation relations for the fields $\Phi$ and $\Xi$, we get

$$\frac{\beta^2}{4\pi} = \left( \frac{\tilde{m}_o^2}{\mu_o^2} \right) > 0, \quad (4.4)$$

and the field $\tilde{\Xi}$ is quantized with negative metric. Defining the canonical free field,

$$\xi = \frac{\tilde{m}_o}{\mu_o} \zeta, \quad (4.5)$$

the Fermi field and the vector field operators (3.3), (3.4) can be re-written as “gauge transformed fields”,

$$\psi = \hat{\psi} \rho, \quad (4.6)$$

$$\mathcal{A}_\mu = \hat{\mathcal{A}}_\mu + \frac{1}{e} \rho i \partial_\mu \rho^{-1}. \quad (4.7)$$

Here, the operator $\rho$ is a pure gauge excitation given by,

$$\rho = : e^{2i \sqrt{\pi} \tilde{\mu}_m} (\Xi - \xi) :, \quad (4.8)$$

and

$$\hat{\psi} = : e^{-2i \gamma^5 \sqrt{\pi} \tilde{\mu}_m} \tilde{\Sigma} : \Psi \tilde{\Phi}, \quad (4.9)$$

$$\hat{\mathcal{A}}_\mu = - \frac{1}{m} \tilde{\partial}_\mu \left( \tilde{\Sigma} - 2 \frac{\tilde{\mu}_o}{\tilde{m}_o} \tilde{\Phi} \right), \quad (4.10)$$

Here $\Psi$ is the charge-carrying Fermi field operator of the Thirring model, which is given by the Mandelstam representation,
\[ \Psi(x) = \left( \frac{\mu_o}{2\pi} \right)^\frac{1}{2} e^{-i\frac{4}{3} x^5} : \exp \left\{ -i \gamma^5 \frac{\beta}{2} \bar{\Phi}(x) - i \frac{2\pi}{\beta} \int_{x_1}^{+\infty} \partial_0 \bar{\Phi}(x^0, z^1) dz^1 \right\} : . \] 

(4.11)

The vector current \( J_\mu \), given by (3.10), can be re-written as

\[ \frac{e}{2} J_\mu = \frac{\tilde{m}^2 - m_o^2}{\tilde{m}} \partial_\mu \tilde{\Sigma} + \frac{2\tilde{\mu}_o m_o^2}{\tilde{m} \tilde{m}_o} \partial_\mu \tilde{\Phi} . \] 

(4.12)

and the current \( K_\mu \) is given by,

\[ K_\mu = \tilde{m} \tilde{\partial}_\mu \tilde{\Sigma} + L_\mu , \] 

(4.13)

where the longitudinal current is now given by,

\[ \ell_\mu = \partial_\mu L = \tilde{m}_o \partial_\mu (\Xi - \xi) = \frac{\tilde{m}_o}{2\sqrt{\pi} \tilde{\mu}_o} \rho \partial_\mu \rho^{-1} . \] 

(4.14)

The longitudinal current carries no fermion selection rule since it is independent of the vector current of the Thirring model,

\[ J^{Th}_\mu = \frac{2\tilde{\mu}_o m_o^2}{\tilde{m} \tilde{m}_o} \tilde{\partial}_\mu \tilde{\Phi} . \] 

(4.15)

For the GI regularization \( a = 2 \), we obtain,

\[ \frac{\beta^2}{4\pi} = \frac{m_o^2}{\tilde{m}_o + m_o^2} , \] 

(4.16)

\[ J^{Th}_\mu = \frac{\beta}{\sqrt{\pi}} \tilde{\partial}_\mu \tilde{\Phi} , \] 

(4.17)

and the field operators given by (4.6)-(4.7) correspond to those operators obtained by Lowenstein-Rothe-Swieca [6, 7] gauged by a singular operator gauge transformation.

The operator \( \rho \) has zero scale dimension, commutes with itself and thus generates infinitely delocalized states that leads to constant Wightman functions

\[ \langle \Psi_o | \rho^\dagger(x_1) \cdots \rho^\dagger(x_n) \rho(y_1) \cdots \rho(y_n) | \Psi_o \rangle = 1 . \] 

(4.18)

The spurious operator \( \rho \) does not carry any fermionic charge selection rule, and since it commutes with all operators belonging to the field algebra \( \mathfrak{F} \), it is reduced to the identity operator in \( \mathcal{H} \).

The position independence of the state \( \rho(x) \Psi_o \) can be seen by computing the general Wightman functions involving the operator \( \rho \) and all operators belonging to the local field algebra \( \mathfrak{F} \). Thus, for any operator \( \mathcal{O}(f_z) = \int \mathcal{O}(z) f(z) d^2 z \in \mathfrak{F} \), of polynomials in the smeared fields \{\bar{\psi}, \psi, A_\mu\}, the position independence of the operator \( \rho \) can be expressed in the weak form as

\[ \langle \Psi_o , \rho^\dagger(x_1) \cdots \rho^\dagger(x_0) \rho(y_1) \cdots \rho(y_m) \mathcal{O}(f_{z_1}, \cdots, f_{z_n}) \Psi_o \rangle = \] 

\[ \mathcal{W}(z_1, \cdots, z_n) \equiv \langle \Psi_o , \mathcal{O}(f_{z_1}, \cdots, f_{z_n}) \Psi_o \rangle , \forall \mathcal{O}(f_z) \in \mathfrak{F} , \] 

(4.19)
where \( \mathcal{W}(z_1, \cdots, z_n) \) is a distribution independent of the space-time coordinates \((x_i, y_j)\).

Within the functional integral approach, the Hilbert space of states is constructed from the generating functional \((2.3)\), which can be rewritten as,

\[
\mathcal{Z}[\bar{\vartheta}, \vartheta, \mathbf{f}^\mu] = \left\langle \exp i \int d^2 z \left( \bar{\vartheta} \tilde{\psi} \varrho + \bar{\tilde{\psi}} \varrho^* \vartheta + f^\mu (\hat{A}_\mu + \frac{1}{e} \varrho^{-1} i \partial_\mu \varrho) \right) \right\rangle, \tag{4.20}
\]

where the average is taken with respect to the functional integral measure,

\[
d\mu = \int D\varXi D\xi e^{i S(0)[\varXi, \xi]} \int D\hat{\Phi} D\hat{\Sigma} e^{i \mathcal{S}[\hat{\Sigma}, \hat{\Phi}]}, \tag{4.21}
\]

and the actions appearing in \((4.21)\) are given in terms of the corresponding bosonized Lagrangian densities,

\[
\mathcal{L}^{(0)} = -\frac{1}{2} (\partial_\mu \varXi)^2 + \frac{1}{2} (\partial_\mu \xi)^2; \tag{4.22}
\]

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \hat{\Phi})^2 + \frac{1}{2} (\partial_\mu \hat{\Sigma})^2 - \frac{\tilde{m}^2}{2} \hat{\Sigma}^2. \tag{4.23}
\]

Since the free massless fields \( \varXi \) and \( \xi \) decouples in the quantum action, the partition function factorizes,

\[
\mathcal{Z} = \mathcal{Z}_\varXi^{(0)} \times \mathcal{Z}_\xi^{(0)} \times \mathcal{Z}_\varphi^{(0)} \times \mathcal{Z}_\sigma^{(0)}. \tag{4.24}
\]

In the computation of general correlation functions from the generating functional \((4.20)\), due to the opposite metric quantization, the space-time contributions of the functional integration over the field \( \xi \) cancel those contributions coming from the integration over the field \( \varXi \). In this way, the field \( \varrho \) becomes the identity with respect to the functional integration and we get the identities,

\[
\psi \equiv \tilde{\psi}, \quad A_\mu \equiv \hat{A}_\mu. \tag{4.25}
\]

The positive-definite metric Hilbert space \( \hat{\mathcal{H}} \) is builted from the generating functional,

\[
\mathcal{Z}[\bar{\vartheta}, \vartheta, \mathbf{f}^\mu] \equiv \hat{\mathcal{Z}}[\bar{\vartheta}, \vartheta, \mathbf{f}^\mu] = \left\langle \exp i \int d^2 z \left( \bar{\vartheta} \tilde{\psi} + \bar{\tilde{\psi}} \vartheta + f^\mu \hat{A}_\mu \right) \right\rangle, \tag{4.26}
\]

where the average is taken with respect to the functional integral measure

\[
d\hat{d}\mu = \int D\hat{\Phi} D\hat{\Sigma} e^{i \mathcal{S}[\hat{\Sigma}, \hat{\Phi}]}. \tag{4.27}
\]

The Hilbert space \( \hat{\mathcal{H}} \) corresponds to the quotient space

\[
\hat{\mathcal{H}} = \frac{\mathcal{H}}{\mathcal{H}_0}, \tag{4.28}
\]

where \( \mathcal{H}_0 \) is the zero-norm space. The fields \((4.6)-(4.7)\) provide the operator solution for the coupled Dirac-Proca equations

\[
i \gamma^\mu \partial_\mu \psi(x) + e \gamma^\mu : \mathcal{A}_\mu(x) \psi(x) : = 0, \tag{4.29}
\]
\[ \partial_\nu F^{\nu\mu} + m_o^2 A^\mu - \frac{e}{2} J^\mu = 0. \]  

(4.30)

For the gauge invariant regularization \( a = 2 \) and \( m_o \neq 0 \), we recover from (1.25) the operator solution for the TW model obtained by Lowenstein-Rothe-Swieca \[6, 7\]. For \( m_o = 0 \) and the gauge non-invariant regularization \( a \neq 2 \), we obtain the operator solution of \( QED_2 \) with broken gauge symmetry.

In the positive-metric Hilbert space formulation, the \( QED_2 \) limit does not exist for the Fermi field and vector field themselves. In this case, the equations of motion are satisfied as operator identities and the Fermi field is a charge-carrying operator,

\[ [J_{\mu}^0 (x), \Psi(z)]_{\rho=-\rho} = -e \frac{m_o^2}{m_o^2} \delta(x^1 - z^1) \Psi(z). \]  

(4.31)

As stressed in Ref. \[7\], if this limit were to exist, we would obtain for the \( QED_2 \) a local charge-carrying Fermi field operator, which is incompatible with the Maxwell’s equation being satisfied in the strong form. Nevertheless, this limit is well defined for the gauge invariant field subalgebra, as for instance,

\[ J_\mu \rightarrow \frac{1}{\sqrt{\pi}} \tilde{\partial}_\mu \tilde{\Sigma}, \]  

(4.32)

\[ F_{\mu\nu} \rightarrow \epsilon_{\mu\nu} \frac{e}{\sqrt{\pi}} \tilde{\Sigma}. \]  

(4.33)

### 4.1 Massive Fermions

The introduction of a mass term for the Fermi field gives a contribution to the action,

\[ \mathcal{M} = -M_o \bar{\psi} \psi = -M_o \left\{ \chi_1^\dagger \chi_2 (U V) + \chi_2^\dagger \chi_1 (U V)^{-1} \right\} = \]

\[ - M_o \left\{ \chi_1^\dagger \chi_2 g^2 + \chi_2^\dagger \chi_1 g^{-2} \right\}. \]  

(4.34)

Using the decomposition for the GI field \( g \) and the bosonized form for the free Fermi field and the bosonized chiral density of the free Fermi field,

\[ \chi_1^\dagger \chi_2 = \left( \frac{\kappa_o}{2\pi} \right) : e^{2i\sqrt{\pi} \varphi} : , \]  

(4.35)

we get,

\[ \mathcal{M} = \frac{M_o \kappa_o}{2\pi} \left\{ e^{2i\sqrt{\pi} \varphi} [\tilde{\omega} \theta]^{2} + e^{-2i\sqrt{\pi} \varphi} [\tilde{\omega} \theta]^{-2} \right\} = \]

\[ - \frac{M_o \kappa_o}{\pi} \cos\left\{ 2\sqrt{\pi} \varphi + 4\sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} (\tilde{\eta} + \tilde{\Sigma}) \right\}. \]  

(4.36)

Performing the canonical transformation \( (4.1)-(4.2) \), the mass term can be written as

\[ \mathcal{M} = - \frac{M_o \kappa_o}{\pi} \cos\{ 2 \beta \tilde{\Phi} + 4\sqrt{\pi} \frac{\tilde{\mu}_o}{\tilde{m}} \tilde{\Sigma} \}. \]  

(4.37)
Notice that, even for a massive fermion field, the field $\zeta (\xi)$ is a free field. The fields $\tilde{\Phi}$ and $\tilde{\Sigma}$ are coupled by the sine-Gordon interaction and the total Lagrangian density is now given by

$$L = \frac{1}{2} (\partial_{\mu} \Xi)^2 + \frac{1}{2} (\partial_{\mu} \xi)^2 + \frac{1}{2} (\partial_{\mu} \tilde{\Phi})^2 + \frac{1}{2} (\partial_{\mu} \tilde{\Sigma})^2 - \frac{\tilde{m}^2}{2} \tilde{\Sigma}^2 - \frac{M_o \kappa_o}{\pi} \cos \{ 2 \beta \tilde{\Phi} + 4 \sqrt{\pi} \beta \tilde{m} \tilde{\Sigma} \}. \quad (4.38)$$

For $a = 2$ we recover from (4.38) the bosonized Lagrangian density for the massive TW model, as obtained by Rothe-Swieca [7].

5 A Glance into the Non-Abelian Model

The classical Lagrangian density defining the non-Abelian TW model is given by

$$L = L_{YM} + \bar{\psi} \left( i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu \right) \psi + \frac{1}{2} m_o^2 \Box tr A_\mu A^\mu, \quad (5.39)$$

where the Yang-Mills Lagrangian is given by,

$$L_{YM} = -\frac{1}{4} \Box tr F_{\mu \nu} F^{\mu \nu}. \quad (5.40)$$

Let us apply the Wess-Zumino-Witten theory to obtain the effective bosonic action. To this end, we shall consider the change of variables (2.7), (2.8), (2.12), (2.13), in terms of the Lie-algebra-valued Bose fields ($U, V$). The partition function can be factorized as [13, 14],

$$Z = Z_F^{(0)} Z_{gh}^{(0)} \tilde{Z}, \quad (5.41)$$

where $Z_F^{(0)}$ is the partition function of free fermions,

$$Z_F^{(0)} = \int \mathcal{D} \chi \mathcal{D} \bar{\chi} e^{i \int d^2z \bar{\chi} \gamma^\mu \partial_\mu \chi}, \quad (5.42)$$

$Z_{gh}^{(0)}$ is the partition function of free ghosts associated with the change of variables (2.7)-(2.8),

$$Z_{gh}^{(0)} = \int \mathcal{D} b_+^{(0)} \mathcal{D} c_+^{(0)} e^{i \int d^2z \Box tr b_+^{(0)} i \partial_+ c_+^{(0)}} \int \mathcal{D} b_-^{(0)} \mathcal{D} c_-^{(0)} e^{i \int d^2z \Box tr b_-^{(0)} i \partial_- c_-^{(0)}}, \quad (5.43)$$

and

$$\tilde{Z} = \int \mathcal{D} U \mathcal{D} V e^{i S_{YM}[UV]} e^{-i \{ C_V \Gamma[UV] + \Gamma[U^{-1}] + \Gamma[V] \} - \frac{m_o^2}{2} \left( \partial_+ G \cdot \partial_- G \right) \int d^2z \Box tr \left( [U^{-1} \partial_+ U](V \partial_- U^{-1}) \right) \}, \quad (5.44)$$

with the Yang-Mills action given by,

$$S_{YM}[UV] = \frac{1}{4e^2} \int d^2z \Box tr \left( \frac{1}{2} \left( \partial_+ (G \cdot \partial_- G) \right)^2 \right). \quad (5.45)$$

In the non-Abelian case, the WZW functional is given by [11, 16],
\[ \Gamma[g] = S_{P\sigma M}[g] + S_{WZ}[g], \]  
\[ \text{(5.46)} \]

where \( S_{P\sigma M}[g] \) is the principal sigma model action,
\[ S_{P\sigma M}[g] = \frac{1}{8\pi} \int d^2x \square tr \left[ (\partial_{\mu} g) (\partial^{\mu} g^{-1}) \right], \]  
\[ \text{(5.47)} \]

and the functional \( S_{WZ}[g] \) is the Wess-Zumino action,
\[ S_{WZ}[g] = \frac{1}{12\pi} \int d^3x \varepsilon^{ijk} \square tr \left[ (\tilde{g}^{-1}\partial_i \tilde{g}) (\tilde{g}^{-1}\partial_j \tilde{g}) (\tilde{g}^{-1}\partial_k \tilde{g}) \right]. \]  
\[ \text{(5.48)} \]

Using the PW identity, the total partition function can be rewritten as,
\[ Z = Z_F^{(0)} Z_{gh}^{(0)} \int D U D V e^{iS_{YM}[UV]} \times \]
\[ e^{-i \frac{1}{2\rho_o} \{ \tilde{\rho}_o^2 (a+2C_V) + m_o^2 \} \Gamma[UV] + i \frac{1}{2\rho_o} \{ \tilde{\rho}_o^2 (a-2) + m_o^2 \} \Gamma[V] - i \Gamma[U^{-1}] + i \frac{1}{2\rho_o} \{ \tilde{\rho}_o^2 a + m_o^2 \} \Gamma[U]}. \]  
\[ \text{(5.49)} \]

From the partition function (5.49), we read off the equivalence of the \( QCD_2 \) provided with a GNI regularization \( b \neq 2 \) and the non-Abelian TW model presenting a fixed bare mass \( m_o^2 = \tilde{\rho}_o^2 (b-a) \) for the vector field. Similar to the Abelian case, this isomorphism also holds in the non-Abelian model with massive Fermi fields.

### 6 Conclusion

We have re-analyzed the TW model from the functional integral approach using the Wess-Zumino-Witten theory. The present approach gives us easiness to read off the equivalence of the \( QED_2 \) \( (QCD_2) \) with gauge symmetry breakdown and the TW model (non-Abelian TW model).

In the indefinite-metric formulation, the fields \( \{ \tilde{\psi}, \psi, A_{\mu} \} \) are singlet under gauge transformations generated by the longitudinal current. This implies that the field algebra is by itself the physical field algebra. The GI limit can be performed in the field operators written in terms of the non-canonical free field \( \zeta \), leading to the Lowenstein-Rothe-Swieca solution for the \( QED_2 \).

The positive-definite formulation is obtained by performing a canonical transformation that maps the redundant Bose degrees of freedom into a zero-norm gauge excitation. The fermion field operator is now written in terms of the charge-carrying fermion field of the Thirring model. In this case, the \( QED_2 \) limit is defined only for the GI field subalgebra.

The “anomalous” vector Schwinger model considered in Refs. [9] is nothing but the Thirring-Wess model. As a matter of fact, the structural physical aspect underneath the conclusion given in Ref. [9], according with the parameter \( a \) apparently controls the screening and confinement properties in the “anomalous” vector Schwinger model, is that the Hilbert space of states of the TW model exhibits charge sectors and thus there is no confinement at all.

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