Stability of Bounded Solutions for Degenerate Complex Monge-Ampère Equations

Sławomir Dinew
Jagiellonian University, Kraków
Zhou Zhang
Department of Mathematics, University of Michigan, at Ann Arbor

Abstract

We show a stability estimate for the degenerate complex Monge-Ampère operator that generalizes a result of Kołodziej [11]. In particular, we obtain the optimal stability exponent and also treat the case when the right hand side is a general Borel measure satisfying certain regularity conditions. Moreover our result holds for functions plurisubharmonic with respect to a big form generalizing thus the Kähler form setting in [11].

1 Introduction and the Main Theorem

In this work, we generalize and strengthen Kołodziej’s stability result concerning bounded solutions for complex Monge-Ampère equations, which is summarized in [11] (see also [12]). The solutions are understood in the sense of pluripotential theory, i.e. we do not impose any other regularity than upper semicontinuity and boundedness. It is, however, a classical fact that the image of the Monge-Ampère operator can be well defined as a Borel measure in this setting.

The equation we consider is over a closed Kähler manifold $X$ of complex dimension $n \geq 2$.

Suppose $\omega$ is a real smooth closed semi-positive $(1,1)$-form over $X$, $\Omega$ is a positive Borel measure on $X$ and $f \in L^p(X)$ for some $p > 1$ is non-negative, where the definition of the function space $L^p(X)$ is with respect to $\Omega$. The equation we consider is

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f \Omega.$$  

Using $d = \partial + \bar{\partial}$ and $d^c := \sqrt{-1}(\bar{\partial} - \partial)$ we have $dd^c = \sqrt{-1}\partial\bar{\partial}$ and this convention is also often used in the literature.

As mentioned above, we require regularity of $u$ much less than what is needed to make pointwise sense for the left hand side. More specifically, we look for

---

1When $n = 1$, the manifold is a Riemann surface and Monge-Ampère operator is just the Laplace operator.
solutions in the function class $PSH_\omega(X) \cap L^\infty(X)$, where $u \in PSH_\omega(X)$ means that $\omega + \sqrt{-1} \partial \bar{\partial} u$ is non-negative in the sense of distribution theory.

Of course, there is an obvious condition for the existence of such a solution coming from global integration over $X$, i.e. $\int_X \omega^n = \int_X f \Omega$. This condition follows from Stokes theorem in the smooth case, and hence (by smooth approximation) in our case either.

Kolodziej mainly studied the case when $\omega$ is a Kähler metric, or equivalently, $[\omega]$ is a Kähler class, and $\Omega$ is a smooth volume form. The existence of bounded solution in this case is proved. In fact, even more general $f$’s than $L^p$ functions are treated in [10], but for our main concern, we restrict to $L^p$ functions. Further, in this case, the bounded solution is always continuous as proved in [10]. So in the discussion of stability there, continuity of the solutions is naturally assumed.

The degeneration we want to consider in this note is in two places. First we allow $\omega$ to be just semi-positive instead of being Kähler; we are especially interested in the case when $\omega$ is the pullback of a Kähler metric under a holomorphic map preserving dimensions. The following theorem from [16] gives the precise picture of $\omega$ and the corresponding existence result. This result uses an argument very close to Kolodziej’s. Both of them have found the notion of relative capacity, introduced in [2], extremely useful.

**Theorem 1.1.** Let $X$ be a closed Kähler manifold with (complex) dimension $n \geq 2$. Suppose we have a holomorphic map $F : X \to \mathbb{CP}^N$ with the image $F(X)$ of the same dimension as $X$. Let $\omega_M$ be any Kähler form over some neighbourhood of $F(X)$ in $\mathbb{CP}^N$. For the following equation of Monge-Ampère type:

$$(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = f \Omega,$$

where $\omega = F^* \omega_M$, $\Omega$ is a fixed smooth (non-degenerate) volume form over $X$ and $f$ is a nonnegative function in $L^p(X)$ for some $p > 1$ with the correct total integral over $X$, i.e. $\int_X f \Omega = \int_X (F^* \omega_M)^n$, then we have the following:

1. (A priori estimate) If $u$ is a weak solution in $PSH_\omega(X) \cap L^\infty(X)$ of the equation with the normalization $\sup_X u = 0$, then there is a constant $C$ such that $\|u\|_{L^\infty} \leq C \|f\|_{L^p}^n$ where $C$ only depends on $F$, $\omega$ and $p$;
2. (Existence of a bounded solution) There exists a bounded (weak) solution for this equation;
3. (Continuity and uniqueness of bounded solution) If $F$ is locally birational, any bounded solution is actually the unique continuous solution.

The a priori estimate was obtained independently in [7] (even for more general big forms), and later generalized to more singular right hand side in [6]. As for the continuity of the solution, despite serious effort, the situation is still a little bit unclear. It is not known whether continuity holds when $\omega$ is a general semi-positive closed form with continuous (even smooth) potentials and positive total integral. This problem has attracted much interest recently, and for this reason we take the opportunity to present a detailed proof of the continuity in
the situation above. Indeed, the argument in [16] is a bit too sketchy therefore hard to follow. This will be done in Section 5.

Regardless of that in our discussion of stability we do not impose a priori continuity of the solutions. The methods we use are independent of that assumption. So, theoretically, solutions might be discontinuous in general, but uniformly close to each other if we perturb the data a little. Needless to say, this is quite an artificial situation. So our results strongly support (but in no way prove) the common belief that continuity holds in general.

Our second degeneration is that we allow Ω on the right hand side to be a Borel measure instead of smooth volume form. Then some restrictions must be imposed, since weak solutions for such an equation might not be bounded anymore (for example, if Ω is the Dirac delta measure at some point). Worse yet, there are measures for which existence of solutions (bounded or not) is not known so far. Therefore we impose some seemingly natural conditions on Ω that guarantee boundedness of the solutions.

**Definition 1.2.** We say that a Borel measure is well dominated by capacity for $L^p$ functions, if there exist constants $\alpha > 0$ and $\chi > 0$, such that for any compact $K \subset X$ and any non-negative $f \in L^p(\Omega)$, $p > 1$ one has for some constant $C$ independent of $K$, (but dependent on $f$)

$$\Omega(K) \leq C \text{cap}_\omega(K)^{1+\alpha}, \quad \int_K f \Omega \leq C \text{cap}_\omega(K)^{1+\chi}$$

A very similar notion (only the first condition is imposed) is discussed in [7]. Both are variations of the so-called condition (A), introduced by S. Kołodziej in [10]. These conditions (which actually are stronger than condition(A)) force boundedness for the solutions $u$ of

$$(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = f \Omega$$

(see [10] for the case $\omega$ is Kähler, and [7] for the case $\omega$ is merely semi-positive).

A few words on the second assumption. When $\Omega$ is a smooth volume form it is known (again see [10] and [7]) that the first condition is satisfied for every $\alpha > 0$. Hence by an elementary application of the Hölder inequality the second condition is also satisfied (for every $\chi > 0$). The same reasoning also shows that the second condition is a consequence of the first provided $p$ is big enough (if $\frac{(1+\alpha)(p-1)}{p} > 1$). Anyway, one has to impose some condition, since a priori $f \Omega$ is more singular than $\Omega$.

Note that, as in [10] or [12] the exponent $\chi > 0$ is used to construct an admissible function $Q$ with proper polynomial growth and afterwards function $\kappa$ and its inverse $\gamma$ (see below for a discussion). When the volume form is smooth one can take arbitrary $\chi > 0$ (of course the bigger $\chi$ we take, the better). Using this, in [11] it was shown that one can produce a function $\gamma(t)$ with growth like $t^\epsilon$, $\forall \epsilon > 0$ near 0. When $\chi$ is bounded from above (i.e. we assume it is a fixed constant dependent on the measure $\mu$), calculations as in [12] or [11] show that

---

2 See also in [14] where it is not so easily separated from the context.
one can take $\gamma(t) \approx t^n$. In order to avoid too much technicalities throughout the note we shall work with the assumption that $\chi$ can be taken arbitrarily large. At the end (see Remark 4.1) we will explain how to modify the argument in the case of fixed $\chi$ and obtain the stability exponent in this case either.

As mentioned in the thesis of the second named author [17], Kołodziej’s original argument is almost good enough for us except for two issues. One of them, about Comparison Principle, is doable using the regularizing result in [3]. The other one, an inequality for mixed Monge-Ampère measures, looks hard to justify for bounded functions. Recently, this has been treated by the first named author in [4] for even more general class of functions.

Now let’s state the main theorem.

**Theorem 1.3.** In the same set-up as in the theorem above (we assume that $\Omega$ is well dominated by capacity for $L^p$ functions), for any non-negative $L^p(\Omega)$-functions $f$ and $g$ with $p > 1$ which have the proper total integral over $X$, i.e., $\int_X f \Omega = \int_X g \Omega = \int_X \omega^n$, suppose that $\phi$ and $\psi$ in $PSH_\omega \cap L^\infty(X)$ satisfy $\omega^{\phi n} = f \omega^n$ and $\omega^{\psi n} = g \omega^n$ respectively and are normalized by the conditions $\max_X \{\phi - \psi\} = \max_X \{\psi - \phi\}$. Let also $\epsilon > 0$ be arbitrary.

If $\|f - g\|_{L^1} \leq \gamma(t) t^n + \epsilon$ for $\gamma(t) = C\kappa^{-1}(t)$ with some proper non-negative constant $C$ depending only on the $L^p$-norms of $f$ and $g$, where $\kappa^{-1}(t)$ the inverse function of the following $\kappa$ function,

$$\kappa(r) = C_n A^\frac{n}{2} \left( \int_r^\infty y^{-\frac{1}{n}} (Q(y))^{-\frac{1}{2}} dy + (Q(r^{-\frac{1}{n}}))^{-\frac{1}{2}} \right),$$

where $C_n$ is a positive constant only depending on the complex dimension $n$ and $Q$ is an increasing positive function with proper polynomial growth, then we can conclude that

$$\|\phi - \psi\|_{L^\infty} \leq C t$$

for $t < t_0$ where $t_0 > 0$ depends on $\gamma$ and $C$ depends on the $L^p$-norms of $f$ and $g$.

As a direct application, we have uniqueness of bounded solution from Theorem (1.1).

Another corollary is the following stability estimate.

**Corollary 1.4.** In the same setting as above there exists a constant $c = c(p, \epsilon, c_0)$ where $c_0$ is an upper bound for $\|f\|_p$ and $\|g\|_p$ such that

$$\|\phi - \psi\|_{L^\infty} \leq c \|f - g\|_1^{\frac{1}{1+p}}$$

**Remark 1.5.** The exponent in the last corollary is improved compared to [14]. As example 4.2 shows, the exponent we obtain is optimal.

**Remark 1.6.** The Monge-Ampère equation with $\omega$ big instead of Kähler has been studied extensively in the recent years (see [1], [2], [7]).

---

3The manifold $X$ and metric $\omega$ also affect $C$. 

---
The applications of the result above could go in two directions. The semi-positivity is particularly interesting in geometry, since the situation we have described above appears naturally in the study of algebraic manifolds of general type (or big line bundles in general) (see e.g. [15]). The degeneration of the measure on the right hand side, in turn, might be useful in complex dynamics and pluripotential theory. Complex dynamics often deals with such singular measures and it is an important question to obtain any regularity for the potential of such measures. The same question is crucial in pluripotential theory while studying extremal functions.

**Acknowledgment.** The authors would like to thank professor S. Kołodziej for all the generous help in the formation of this work and beyond. His suggestion for such a joint work is also very important for beginners like us. This work was initiated during the second named author’s visit at MRSI (Mathematical Sciences Research Institute) and he would like to thank the institute and the department of Mathematics at University of Michigan, at Ann Arbor, for the arrangement to provide such a wonderful opportunity.

### 2 Stability for Nondegenerate Monge-Ampère Equations

For readers’ convenience, Kołodziej’s stability argument will be included here. We are going to use global version of the notions, for example, capacity for the closed manifold $X$.

Specifically, in this part all the plurisubharmonic functions with respect to the Kähler metric $\omega$ ($\omega$-PSH for short) are continuous by definition. As explained before, this brings no difference in this case. So Comparison Principle between them can be justified by the Richberg’s approximation as in [11].

Basically, all the following argument is directly quoted from [11].

**Claim:** Let $\phi, \psi \in PSH_\omega(X)$ and satisfy $0 \leq \phi \leq C$, then for $s < C + 1$, we have

$$\text{Cap}_\omega(\{\psi + 2s < \phi\}) \leq \left(\frac{C + 1}{s}\right)^n \int_{\{\psi + s < \phi\}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n.$$  

**Proof.** Define $E(s) := \{\psi + s < \phi\}$. Take any $\rho \in PSH_\omega(X)$ valued in $[-1, 0]$. Set $V = \{\psi < \frac{s}{C+1}\rho + (1 - \frac{s}{C+1})\phi - s\}$. Since $-s \leq \frac{s}{C+1}\rho - \frac{s}{C+1}\phi \leq 0$, we can easily deduce the following chain relation of sets:

$$E(2s) \subset V \subset E(s).$$

Then we can have the following computation (with notation $\omega_\rho := \omega + \sqrt{-1} \partial \bar{\partial} \rho$):

$$\left(\frac{s}{C+1}\right)^n \int_{E(2s)} (\omega + \sqrt{-1} \partial \bar{\partial} \rho)^n \leq \int_V \left(\frac{s}{C+1}\omega_\rho + (1 - \frac{s}{C+1})\omega_\phi\right)^n \leq \int_V \omega_\phi^n \leq \int_{E(s)} \omega_\phi^n.$$
by the relation of sets above and applying comparison principle for the two functions appearing in the definition of the set \( V \).

Finally we can conclude the result from the definition of \( \text{Cap}_\omega \).

Now we state the following version of stability result, which is slightly weaker than the result in [12].

**Theorem 2.1.** In the same set-up as before, for any nonnegative \( L^p \)-functions \( f \) and \( g \) with \( p > 1 \) which have the proper total integral over \( X \), i.e., \( \int_X f \omega^n = \int_X g \omega^n \), suppose that \( \phi \) and \( \psi \) in \( \text{PSH}_\omega(X) \) satisfy \( \omega^n = f \omega^n \) and \( \omega^n = g \omega^n \) respectively and are normalized by the condition \( \max_X \{ \phi - \psi \} = \max_X \{ \psi - \phi \} \).

If \( \| f - g \|_{L^1} \leq \gamma(t) t^{n+3} \) for \( \gamma(t) = C \kappa^{-1}(t) \) with some proper nonnegative constant \( C \) depending only on the \( L^p \)-norms of \( f \) and \( g \), where \( \kappa^{-1}(t) \) the inverse function of the \( \kappa \) function in the main theorem, then we can conclude that

\[
\| \phi - \psi \|_{L^\infty} \leq C t
\]

for \( t < t_0 \) where \( t_0 > 0 \) depends on \( \gamma \) and \( C \) depends on the \( L^p \)-norms of \( f \) and \( g \).

**Proof.** Suppose \( \| f \|_{L^p}, \| g \|_{L^p} \leq A \). We will be careful about the fact that the constants in the argument will only depend on \( A \) and the function \( \gamma \).

For simplicity, let us normalize to have \( \int_X \omega^n = 1 \). And in fact, we can also assume \( \max_X \{ \phi - \psi \} = \max_X \{ \psi - \phi \} > 0 \) since the case for = 0 is trivial.

Without loss of generality, assume \( \int_{\{ \psi < \phi \}} (f + g) \omega^n \leq 1 \), since \( \int_X f \omega^n = \int_X g \omega^n = 1 \) and, if needed, one can interchange the roles of \( \psi \) and \( \phi \).

Then by adding the same constant to \( \phi \) and \( \psi \) which obviously affects nothing, we can assume \( 0 \leq \phi \leq a \) where ”a” is a positive constant only depending on \( A \) from the boundedness result before.

Of course we can take a larger ”a”, which we shall actually do below, as long as the dependence on \( A \) is clear, or say finally we can still fix it to be some positive constant only dependent on \( A \).

As \( \lim_{t \to 0} \gamma(t) = 0 \) by definition and the property of the function \( \kappa \), we can fix \( 0 < t_0 < 1 \) sufficiently small such that \( \gamma(t_0) t_0^{n+3} < \frac{1}{3} \), which will also hold for \( 0 < t < t_0 \) since \( \gamma \) is obviously decreasing.

Fix such a \( t \) for now and set \( E_k = \{ \psi < \phi - k a t \} \) where the ”a” is from above, but we still have not made the choice yet.

Clearly we have:

\[
\int_{E_0} g \omega^n = \frac{1}{2} \int_{E_0} ((f + g) + (g - f)) \omega^n \leq \frac{1}{2} (1 + \frac{1}{3}) = \frac{2}{3},
\]

\( ^4 \)The dependence on the manifold \( X \) and Kähler metric \( \omega \) should be clear.

\( ^5 \)In this case, we can have \( \phi - \psi \leq 0 \) and \( \psi - \phi \leq 0 \), which says \( \phi = \psi \). In other words, we have the compatible direction.
Now we construct a function $g_1$ which is equal to $\frac{3}{2}g$ over $E_0$ and some other nonnegative constant for the complement. By the above estimate, it is easy to see that one can choose a proper constant (in $[0,1]$) such that $g_1$ is still nonnegative with $L^p$-norm bounded by $\frac{3A}{2}$, and more importantly it has the proper total integral over $X$.

So we can find a continuous solution $\rho \in PSH_\omega(X)$ as before by the approximation method such that

$$\omega_\rho^n = g_1 \omega^n, \quad \max_X \rho = 0$$

with lower bound of $\rho$ only dependent on $A$. By enlarging "$a" if necessary which clearly won’t affect the set $E_0$, we can assume the lower bound of $\rho$ is $-a$. Now we can finally fix our constant "a", and it clearly depends only on $A$ in an explicit way.

By noticing that $-2at \leq -t\phi + t\rho \leq 0$, it is easy to see

$$E_2 \subset E := \{\psi < (1-t)\phi + t\rho\} \subset E_0.$$

Let’s denote the set $\{f < (1-t^2)g\}$ by $G$. Then over $E_0 \setminus G$, we have:

$$((1-t^2)^{-\frac{k}{n}}\omega_\phi)^n \geq g\omega^n, \quad ((\frac{3}{2})^{-\frac{k}{n}}\omega_\rho)^n = g\omega^n.$$

Hence we can conclude, using an inequality for mixed Monge-Ampère measures from [11], that over $E_0 \setminus G$,

$$\left((\frac{3}{2})^{-\frac{k}{n}}(1-t^2)^{-\frac{k}{n}}\omega_\phi^k \wedge \omega_\rho^{n-k}\right)^n \geq g\omega^n.$$

**Remark 2.2.** This is a rather trivial result in smooth case which is just a direct application of arithmetic-geometric mean value inequality. Then by approximation argument, it should also hold in our case here. For the conclusion above, there is no need to restrict ourselves to the set $E_0 \setminus G$. We can work globally on $X$ and use $g\chi_{E_0 \setminus G}\omega^n$ for the right hand side.

Actually the rigorous approximation argument is local and uses nontrivial results about Dirichlet problem for Monge-Ampère equation. The continuity of the functions is very involved in the proof which seems to be the main obstacle to carry over the whole argument in this part for merely bounded solutions.

This is the point where the recent result in [4] is applied.

Let’s set $q = (\frac{3}{2})^{-\frac{k}{n}} > 1$, and rewrite the above inequality as:

$$\omega_\phi^k \wedge \omega_\rho^{n-k} \geq q^{n-k}(1-t^2)^{\frac{k}{n}}g\omega^n$$

\[\text{Notice we’ve used the existence of continuous solution at this point for the solution } \rho.\]
over $E_0 \setminus G$. Now the following computation is quite obvious:

\[
\omega_{t \rho + (1-t)\phi}^n \geq (1-t((1-t^2) + qt))^n g^n \omega^n \\
\geq (1-t)((1-t^2) + qt))^n g^n \\
\geq (1 + t(q - 1) - t^2) g^n \\
\geq (1 + \frac{t}{2}(q - 1)) g^n.
\] (2.1)

From the definition of $G$ and assumption of the theorem, we also have:

\[
t^2 \int_G g^n \omega \leq \int_G \omega t^n \rho + (1-t)^\phi \gamma(t) t^{n+3}
\]

which is just:

\[
\int_G g^n \omega \leq \gamma(t) t^{n+1}.
\] (2.2)

Hence we can have the following inequalities:

\[
(1 + \frac{t}{2}(q - 1)) \int_{E \setminus G} g^n \omega \leq \int_E \omega_{t \rho + (1-t)\phi}^n \omega^n \leq \int_E \omega^n \leq \int_{E \setminus G} \omega^n + \gamma(t) t^{n+1} \ (the \ measure \ inequality \ (2.1))
\]

and arrive at:

\[
\frac{q-1}{2} \int_{E \setminus G} g^n \omega \leq \gamma(t) t^n.
\]

Therefore by noticing $E_2 \subset E$, we get:

\[
\frac{q-1}{2} (\int_{E_2} g^n \omega - \gamma(t) t^{n+1}) \leq \frac{q-1}{2} (\int_{E_2} g^n \omega - \int_{E_2} g^n \omega) \leq \frac{q-1}{2} \int_{E \setminus G} g^n \omega \leq \gamma(t) t^n,
\]

and so we have

\[
\int_{E_2} g^n \omega \leq (t + \frac{2}{q-1}) \gamma(t) t^n \leq \frac{3}{q-1} \gamma(t) t^n
\]

for $t$ small enough.

The claim proved before tells us:

\[
Cap_\omega(E_4) \leq \left(\frac{a + 1}{2at}\right)^n \int_{E_2} g^n \omega.
\]

Combining this with the previous inequality, we have:

\[
Cap_\omega(E_4) \leq \left(\frac{a + 1}{2at}\right)^n \frac{3}{q-1} \gamma(t).
\]
Thus if \( E' := \{ \psi < \phi - (4a + 2)t \} \) is nonempty, by the argument for boundedness result before, we should have:

\[
2t \leq \kappa(Cap_\omega(E_4)) \leq \kappa((\frac{a+1}{2a})^n \frac{3}{q-1}\gamma(t)) = t.
\]

Clearly this is a contradiction for \( t > 0 \).

Anyway, we have from above that \( \psi \geq \phi - (4a + 2)t \).

Hence \( max_X(\psi - \phi) = max_X(\phi - \psi) \leq (4a + 2)t \), which will give the desired conclusion.

Now from this stability result, it is easy to get uniqueness result for continuous plurisubharmonic solutions after normalization.

One can easily see the proof can be simplified a little if we only care about the uniqueness result. But this result above actually gives much better description of the variation of the solution under the perturbation of the right hand side of the equation (i.e., the measure).

Now in the same vein as in [11] one gets the following corollary:

**Corollary 2.3.** For any \( \epsilon > 0 \), there exists \( c = c(\epsilon, p, c_0) \), \( (c_0 \text{ is an upper bound for } L^p \text{ norms of } f \text{ and } g) \) such that

\[
||\phi - \psi||_\infty \leq c||f - g||_1^{\frac{1}{n+3} + \epsilon}
\]

provided \( \phi \) and \( \psi \) are normalised as before.

Before we proceed further we make a small improvement of the stability exponent in the last corollary.

Note that in the definition of set \( G = \{ f < (1 - t^2)g \} \) one can exchange \( t^2 \) with \( \frac{t^b}{b} \) for a sufficiently big independent constant \( b \), and the the same argument still goes through, so \( ||f - g||_1 \leq \gamma(t)^{n+2} \) implies \( ||\phi - \psi||_\infty \leq Ct. \) In particular the result in Corollary 2.3 holds with exponent \( \frac{1}{n+2+\epsilon} \).

## 3 Adjustment to Our Degenerate Case

Now we begin to adjust Kołodziej’s argument for the situation in our main theorem. All the places which need to be considered have been pointed out at the spot. Let us now treat them one by one.

### 3.1 Comparison Principle

In [3], authors constructed decreasing smooth approximation for bounded functions plurisubharmonic with respect to a Kähler metric. Using this, they got the following version of Comparison Principle,
Theorem 3.1. For \( \phi, \psi \in PSH_\omega(X) \cap L^\infty(X) \), where \((X, \omega)\) is a closed Kähler manifold, one has
\[
\int_{\{\phi < \psi\}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq \int_{\{\phi < \psi\}} (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n.
\]

Though the result we want would be for some background form \( \omega \geq 0 \), it would follow from the version above as we can perturb it by \( \varepsilon \omega_0 \) with \( \omega_0 > 0 \) and the constant \( \varepsilon > 0 \), since \( X \) is Kähler. Those functions plurisubharmonic with respect to \( \omega \) would still be plurisubharmonic with respect to \( \omega + \varepsilon \omega_0 \). Using the comparison principle above and letting \( \varepsilon \to 0 \), we get the following version

Theorem 3.2. For \( \phi, \psi \in PSH_\omega(X) \cap L^\infty(X) \), where \( X \) is a closed Kähler manifold and \( \omega \geq 0 \) is a real smooth \((1, 1)\)-form over \( X \), one has
\[
\int_{\{\phi < \psi\}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq \int_{\{\phi < \psi\}} (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n.
\]

This is Comparison Principle for the adjusted argument for stability.

3.2 Inequalities for Mixed Measures

Our first observation is that although we considered our equations of the form
\[
\omega^n \psi = f \omega^n, \quad \omega^n \phi = g \omega^n,
\]
the volume form \( \omega^n \) played no significant role in the proof. The only delicate point is the following inequality:

Suppose \( \phi \) and \( \psi \) are continuous \( \omega \)-PSH functions, and \( f, g \) are integrable functions on \( X \). Suppose we have (locally or globally) the inequalities
\[
\omega^n \psi \geq f \omega^n, \quad \omega^n \phi \geq g \omega^n,
\]
then (locally where we have those inequalities or globally)
\[
\forall k \in \{0, 1, \ldots, n\} \quad \omega^n_k \wedge \omega^{n-k}_\phi \geq f \frac{k}{n} g \frac{n-k}{n} \omega^n.
\]

In other words we want to generalize the above inequality, for more general measures and moreover for bounded (i.e. not necessarily continuous) functions \( \phi \) and \( \psi \). The following theorem is essentially taken from [4]:

Theorem 3.3. Suppose the nonnegative Borel measure \( \Omega \) is well dominated by capacity, and let \( \phi \) and \( \psi \) be two bounded \( \omega \)-psh functions on a Kähler manifold. Suppose the following inequalities hold
\[
\omega^n \psi \geq f \Omega, \quad \omega^n \phi \geq g \Omega,
\]
for some \( f, g \in L^p(\Omega), \quad p > 1 \). Then
\[
\forall k \in \{0, 1, \ldots, n\} \quad \omega^n_k \wedge \omega^{n-k}_\phi \geq f \frac{k}{n} g \frac{n-k}{n} \Omega.
\]
In [11] (Lemma 1.2) this inequality was proved under the assumption that both $\phi$ and $\psi$ are continuous and $\Omega = \omega^n$. The proof is local, it can be rephrased in a setting in a ball in $\mathbb{C}^n$. Then the argument goes via approximation for which a solution for the Dirichlet problem with boundary data is used. Since we deal with merely bounded functions (uppersemicontinuous by the plurisubharmonicity assumption), one cannot expect continuity on the boundary of the ball in general. But as observed in [4] we can line-by-line follow the approximation arguments from [11] whenever the measure on the right hand side is the Lebesgue measure. Indeed, approximants at the boundary will not converge uniformly towards discontinuous boundary data, but the sequence of approximate solutions is again decreasing. This implies convergence in capacity by [2], which is enough for the argument to go through. In the case when $\omega^n$ is exchanged with a general measure well dominated in capacity one cannot rely only on the argument from [11]. But domination by capacity forces the measure $\Omega$ to vanish on pluripolar sets, hence one can use the result form [4] to conclude. We refer to [4] for the details.

4 Improvement on the Stability Exponent

The exponent from Corollary 2.3 is quite important. In particular, since this inequality can be used to prove Hölder continuity for solutions of Monge-Ampère equations with right hand side in $L^p$ (see [13]), the bigger the exponent in the inequality, the better Hölder exponent one can get.

Trying to improve the exponent, one has to follow the main steps of the original proof and improve points where there is an exponent loss. Our strategy will be to iterate the original argument, defining at each step new function $\rho$ and use the previous step to get estimates for $||\rho - \psi||_{\infty}$, which in turn will be used to choose the new set $E$ in a "better" way.

The argument is divided into the following three parts.

The first part is the original argument quoted before with the improvement mentioned after Corollary 2.3 which is the starting point for us. In the sequel the original argument will be often denoted as Step 1.

The second part, (i.e. Step 2), is the description of the iteration procedure. Since Step 1 differs slightly from all the others, we outline Step 2 below and sketch how to proceed throughout the next iterations.

The mechanism is based on the fact that $||f - g||_1 \leq \gamma(t)t^\beta$ (in the improved original proof $\beta = n + 2$) yields $\int_{\{\psi + kt < \varphi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n \leq c_0 t^n$ for some constant $k$ and $c_0$ (in what follows $c_i$ denote constants independent of the relevant quantities). So we try to find $\beta$ as small as possible for which this implication holds true with uniform control on $c_0$ and enlarging $k$ if needed. Note that from now on instead of $\omega^n$ we use the measure $\Omega$. It follows from the discussion above that Step 1 is not affected by that.

So assume $||f - g||_1 \leq \gamma(t)t^\beta$, $t < 1$. Then if $l := \frac{n}{\beta - n - 2}$, $\beta < n + 2$, we obtain
\[ ||f - g||_1 \leq \gamma(t) t^{\alpha + 2}, \] so from Step 1 we know that
\[ \int_{E_2} g \partial_\Omega \leq \gamma(t) t^{\alpha}, \quad (4.1) \]

Where, as before \( E_k := \{ \psi < \phi - kat \} \). (Indeed, in Step 1 we have \( t = l \), but one can check that the proof can be repeated in this situation). Hence
\[ \int_{E_2} g \partial_\Omega \leq c_1 t^{\frac{\alpha}{\alpha + 2}}, \quad t \leq t_0 \quad (4.2) \]

(recall \( \gamma(t) \) decreases to 0, as \( t \searrow 0 \)).

Now fix a small positive constant \( \delta \) to be chosen later on.

Consider the "new" function
\[ g_1(z) = \begin{cases} (1 + \frac{\delta}{2}) g(z), & z \in E_2 \\ c_2 g(z), & z \in X \setminus E_2, \end{cases} \]

where \( 0 \leq c_2 \leq 1 \) is chosen such that \( \int_X g_1 \partial_\Omega = 1 \). (The constant \( \frac{1}{2} \) is taken to assure that the integral over \( E_2 \) is less than 1. Note that despite the fact that the case \( t \) being small is of main interest, when \( \delta \) is also small the quantity \( t^{\delta} \) cannot be controlled by a constant smaller than 1). As in Step 1 we find a solution \( \rho \) to the problem \((\partial_\rho)^n = g_1 \partial^n\). max \( X \rho = 0 \). Again \( \rho \geq -a \) and we renormalize \( \rho \) by adding a constant so that \( \max_X (\psi - \rho) = \max_X (\rho - \psi) \) (this can by done in an uniform way).

Now by Step 1
\[ ||\rho - \psi||_\infty \leq c_3 ||g - g_1||_1^{1 + \frac{\alpha}{\alpha + 2}} = c_3 (\int_{E_2} + \int_{X \setminus E_2} ||g - g_1||_1^{1 + \frac{\alpha}{\alpha + 2}}) = c_3 (2^{\delta} \int_{E_2} g \partial_\Omega)^{\frac{1}{1 + \frac{\alpha}{\alpha + 2}}} \leq c_4 t^{\frac{\delta + \frac{\alpha}{\alpha + 2}}{1 - \frac{\alpha}{\alpha + 2}}}. \]

If \( \delta \) is sufficiently small the last exponent is less than 1 and we define \( \alpha := 1 - \frac{\delta + \frac{\alpha}{\alpha + 2}}{1 - \frac{\alpha}{\alpha + 2}} \). Then by the above estimate
\[ E_s = \{ \psi + sat < \phi \} = \{ (1 - t^\alpha)(\psi + sat) < (1 - t^\alpha)\phi \} \subset \quad (4.3) \]
\[ \subset \{ \psi < (1 - t^\alpha)\phi + t^\alpha \rho + c_4 t - sat(1 - t^\alpha) \} = E \subset \]
\[ \subset \{ \psi < (1 - t^\alpha)\phi + t^\alpha \psi + 2c_4 t - sat(1 - t^\alpha) \} = \]
\[ = \{ \psi + (sa - \frac{2c_4}{1 - t^\alpha}) t < \phi \} \subset E_k, \]

provided \( s \geq 4c_4 + k \), (we take \( t < \frac{1}{2} \)).

Consider the "new" set
\[ G_1 := \{ f < (1 - \frac{t^\alpha + \delta}{8n^2 \frac{1}{2}}) g \}. \]
Using that $h(t) = (1 + \frac{t^\delta}{2})^{-\frac{1}{\delta}} - 1 - \frac{1}{4n^2(\frac{n}{n-1})}t^2\delta$ is increasing in $[0,1]$ and hence nonnegative there, we conclude as in Step 1 that on $E_k \setminus G$

$$(\omega_{\rho+\beta}(1-t^\alpha)g)^n \geq ((1-t^\alpha)(1 - \frac{t^{\alpha+3\delta}}{8n^2\frac{n}{n-1}}))^{\frac{1}{\alpha}} + (1 + \frac{t^\delta}{2})^\frac{1}{\alpha} t^\alpha)g \Omega \geq (4.4)$$

$$\geq ((1-t^\alpha)(1 - \frac{t^{\alpha+3\delta}}{8n^2\frac{n}{n-1}}) + (1 + \frac{1}{4n^2(\frac{n}{n-1})}t^2\delta)t^\alpha)g \Omega \geq (1 + \frac{t^{\alpha+2\delta}}{8n^2\frac{n}{n-1}})g \Omega.$$  

As in Step 1 on $G$ we have

$$\frac{t^{\alpha+3\delta}}{8n^2\frac{n}{n-1}} \int_G g\Omega \leq \int_G (g-f)\Omega \leq \gamma(t)t^\beta,$$  

so, using (4.4), (4.5) and the comparison principle we obtain

$$(1 + \frac{t^{\alpha+2\delta}}{8n^2\frac{n}{n-1}}) \int_{E_k \setminus G} g\Omega \leq \int_{E_k} (\omega_{1-t^\alpha})^n g\Omega \leq (4.6)$$

$$\leq \int_{E_k} g\Omega \leq \int_{E_k} g\Omega + c\gamma(t)t^{\beta-\alpha-3\delta}.$$  

Finally, as in Step 1, we obtain

$$\int_{E_k \setminus G} g\Omega \leq c\gamma(t)t^{\beta-2\alpha-5\delta}$$

and

$$\int_{E_s} g\Omega \leq c\gamma(t)t^{\beta-2\alpha-5\delta}.$$

If $\beta-2\alpha-5\delta = n$, we can proceed as in Step 1 to get $\max(\phi-\psi) = \max(\psi-\phi) \leq (2s+2)t$, and $||\phi-\psi||_{\infty} \leq C(\epsilon)||f-g||_1$, $\forall \epsilon > 0$. Now $\beta-2\alpha-5\delta = n$ yields

$$\beta(1 + \frac{2n}{n+2 + \epsilon}) = n + 2 + 5\delta = \frac{2\delta}{n+2 + \epsilon}.$$  

It is clear that if $\delta$ is sufficiently small $\beta$ is smaller than $n+2$, hence we get an improvement.  

Now in the last part we iterate the argument.

Consider $||f-g||_1 \leq \gamma(t)t^{\beta_k+1}$, then as before $l = t^{\frac{\alpha_{k+1}}{n+\epsilon}}$, $\int_{E_k} g\Omega \leq C t^{\frac{\alpha_{k+1}}{n+\epsilon}}$, (compare with (11), $r$ is now chosen so that we can use the estimate on appropriate sublevel set from the previous step).

Choosing $\delta_{k+1}$ small enough and proceeding in the same way as in the previous step one gets

$$\beta_{k+1} = n + 2\alpha_{k+1} + 5\delta_{k+1}.$$
\((\alpha_{k+1} = 1 - \frac{\delta_{k+1} n^{\beta_k + 1}}{n+2+\epsilon})\). This yields
\[
\beta_{k+1}(1 + \frac{2n}{\beta_k(\beta_k + \epsilon)}) = n + 2 + 5\delta_{k+1} - 2 \frac{\delta_{k+1}}{\beta_k + \epsilon}
\] (4.7)

If we choose \(\{\delta_k\}\) to be a sequence of sufficiently small numbers decreasing to 0, one can obtain that \(\{\beta_k\}\) is decreasing (recall \(n \geq 2\)). If \(A\) is the limit of the sequence \(\{\beta_k\}\) one gets
\[
A(1 + \frac{2n}{A(A + \epsilon)}) = n + 2 \Rightarrow A = \frac{n + 2 - \epsilon + \sqrt{(n - 2 - \epsilon)^2 + 8\epsilon}}{2}
\]

Now \(\epsilon \to 0^+ \Rightarrow A \to n\), so \(\beta_k\)'s can be arbitrarily close to \(n\) for \(k\) big enough if we take small enough \(\epsilon\).

Thus this argument yields in paritcular Corollary 1.3.

Remark 4.1. In the case when the measure \(\Omega\) is well dominated by capacity for \(L^p\) functions but the constant \(\chi\) is fixed one can construct \(Q(t)\) and afterwards \(\kappa(t), \gamma(t)\) in such a way that \(\gamma(t) \approx t^{\frac{\alpha}{2}}\). Then one can use the same iteration technique as above with the exception that inequality (4.2) should be improved to
\[
\int_{E_2} g \Omega \leq C t^{\frac{\alpha}{n+2}}
\]
(the factor \(t^{\frac{\alpha}{2}}\) comes from the estimate of \(\gamma\)). The recurrence (4.7) now reads
\[
\beta_{k+1}(1 + \frac{2n}{\beta_k(\beta_k + \frac{\alpha}{x})}) = n + 2 - \frac{n}{n + \frac{\alpha}{x}} + 5\delta_{k+1} - 2 \frac{\delta_{k+1}}{\beta_k + \frac{\alpha}{x}}
\] (4.8)

Again this is a convergent sequence and it can be computed that
\[
\lim_{k \to \infty} \beta_k = n.
\]

Hence the stability estimate in this case reads
\[
||\phi - \psi||_\infty \leq c(\epsilon, c_0, X, \mu)||f - g||_{L^1(\partial \mu)}^{\frac{\alpha}{n+2+\epsilon}}
\] (4.9)

The following example shows that the exponent we obtained is sharp:

Example 4.2. Fix appropriate positive constants \(B, D\) such that \(D < B\) and \(B^{2\alpha} < \log 2 + D\), for some fixed \(\alpha \in (0, 1)\) (such constants clearly exist). Then the function
\[
\tilde{\rho}(z) := \begin{cases} B||z||^{2\alpha}, & ||z|| \leq 1 \\ \max\{B||z||^{2\alpha}, \log(||z||) + D\}, & 1 \leq ||z|| \leq 2 \\ \log(||z||) + D, & ||z|| \geq 2 \end{cases}
\]
is well defined, plurisubharmonic in $\mathbb{C}^n$ and of logarithmic growth. One can smooth out $\hat{\rho}$, so that the new function $\rho$ is again of logarithmic growth, radial, smooth away from the origin and $\rho(z) = B\|z\|^{2\alpha}$ for $\|z\| \leq \frac{1}{2}$.

Via the standard inclusion

$$\mathbb{C}^n \ni z \mapsto [1 : z] \in \mathbb{P}^n$$

one identifies $\rho(z)$ with

$$\mathcal{P}([z_0 : z_1 : \cdots : z_n]) := \rho\left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right) - \frac{1}{2} \log(1 + \frac{||z||^2}{|z_0|^2}) \in PSH(\mathbb{P}^n, \omega_{FS})$$

(here $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{P}^n$, and the values of $\mathcal{P}$ on the hypersurface $\{z_0 = 0\}$ are understood as limits of values of $\mathcal{P}$ when $z_0$ approaches 0.)

It is clear that $\omega_{FS}^{\alpha} = (dd^c \rho)^n$ in the chart $z_0 \neq 0$ and in fact one can neglect what happens on the hypersurface at infinity.

Now for a vector $h \in \mathbb{C}^n$ one can define $\rho_h(z) := \rho(z + h)$ and analogously the corresponding $\mathcal{P}_h$. Note that when $||h|| \to 0$, $\mathcal{P}_h \to \mathcal{P}$.

One sees that

$$B||h||^{2\alpha} \leq ||\mathcal{P}_h - \mathcal{P}||_\infty \quad (4.10)$$

The Monge-Ampère measures of $\mathcal{P}$ and $\mathcal{P}_h$ are smooth functions except at the origin, and belong to $L^p(\omega_{FS}^p)$, for some $p > 1$ dependent on $\alpha$.

Now $\int_{\mathbb{P}^n} |\omega_{FS}^p - \omega_{FS}^p| = \int_{\mathbb{C}^n} |(dd^c \rho)^n - (dd^c \rho_h)^n|$. To estimate the last term we divide $\mathbb{C}^n$ into three pieces (we suppose $||h||$ is small):

$$\int_{\mathbb{C}^n} |(dd^c \rho)^n - (dd^c \rho_h)^n| = \int_{\{||z|| \leq 2||h||\}} + \int_{\{2||h|| < ||z|| \leq \frac{1}{2}||h||\}} + \int_{\{||z|| > \frac{1}{2}||h||\}}$$

Using the fact that $\rho$ and $\rho_h$ are smooth functions in a neighbourhood of $\{||z|| > \frac{1}{2}||h||\}$ one can easily estimate the last term by $||h||C_0$ for some constant independent of $h$. For the first two terms we observe that $(dd^c \rho)^n = B^n ||z||^{2n(\alpha - 1)}$, $(dd^c \rho_h)^n = B^n ||z + h||^{2n(\alpha - 1)}$.

Now we use a computation trick we found in [14].

$$\int_{\{||z|| \leq 2||h||\}} |(dd^c \rho)^n - (dd^c \rho_h)^n| =$$

$$= B^n \int_{\{||z|| \leq 2||h||\}} ||z||^{2n(\alpha - 1)} - ||z + h||^{2n(\alpha - 1)} \leq$$

$$\leq 2B^n \int_{\{||z|| \leq 3||h||\}} ||z||^{2n(\alpha - 1)} = C_1 ||h||^{2n\alpha}$$
For the second term

\[
\int_{\{2||h|| \leq ||z|| \leq \frac{1}{2}\}} |(dd^c \rho)^n - (dd^c \rho_h)^n| =
\]

\[
= B^n \int_{\{2||h|| \leq ||z|| \leq \frac{1}{2}\}} ||z||^{2n(\alpha - 1)} - ||z + h||^{2n(\alpha - 1)} |\leq
\]

\[
\leq B^n \int_{2||h|| < ||z||} \int_0^1 |\nabla ||z + th||^{2n(\alpha - 1)}, h > dt |\leq
\]

\[
\leq C_2 ||h|| \int_{2||h|| < ||z||} ||z||^{2n(\alpha - 1) - 1} \leq C_4 ||h||^{2n\alpha}
\]

provided \(\alpha < \frac{1}{2n}\), so that the integral is finite. Finally we obtain for small \(||h||\)

\[
\int_{\mathbb{P}^n} |\omega^n \alpha - \omega^n \alpha_{\rho_h}| \leq C_1 ||h||^{2n\alpha} + C_3 ||h|| \leq C_4 ||h||^{2n\alpha}
\](4.11)

Suppose finally that we have a stability estimate \(||\phi - \psi||_{\infty} \leq C_5 ||f - g||_1^{\frac{1}{s}}\). Then coupling 4.10 and 4.11 one gets

\[
||h||^{2\alpha} \leq C_6 (||h||^{2n\alpha})^{\frac{1}{m}}, \alpha \in (0, \frac{1}{2m})
\]

If we let \(||h|| \rightarrow 0\) this can hold only if \(m \geq n\).

**Remark 4.3.** In [7] Authors show a stability estimate of another type: In the setting as above (\(\Omega\) is now equal to \(\omega^n\))

\[
||\phi - \psi||_{\infty} \leq c(\epsilon, c_0, \omega)||\phi - \psi||_{L^2(\omega^n)}^{\frac{1}{s}} \leq C_7 ||f||_{L^2(\omega^n)}^{\frac{1}{s}}
\](4.12)

\((c_0\) is a constant that controls \(L^p\) norms of Monge-Ampère measures of \(\phi\) and \(\psi\)). Using the same reasoning as in [7] one can show more generally that

\[
||\phi - \psi||_{\infty} \leq c(\epsilon, c_0, \omega)||\phi - \psi||_{L^s(\omega^n)}, \forall s > 0.
\](4.13)

Using the same example and similar estimates one can show that this exponent is also sharp, provided that \(p < 2\) and \(s > \frac{2n}{p-2}\) (the reason for these obstructions is that the second integral we estimate as in the example would be divergent otherwise). It is, however, very likely that these exponents are sharp in general.

### 5 Continuity of Solutions in the Case of a Pullback Form via a Locally Birational Map

We give below a more detailed proof of the continuity statement in Theorem 1.1. Arguments used heavily rely on [10] and at some places we just follow it line by line. This section is unrelated with the other ones in the note. Recall once again, that this result is known already.
First of all we recall the geometrical background. Let $X$ be the base closed Kähler manifold we work on, and $F : X \to \mathbb{CP}^N$, is a map with the property that the image $F(X)$ has the same dimension and $F$ is itself locally birational i.e. for every small enough neighbourhood $U$ of any point on $F(X)$, each component of $F^{-1}(U)$ is birational to $U$. A typical global example of this situation is obtained as follows: if $X$ carries a big line bundle $L$, the linear series corresponding to $L^n$ generate (for sufficiently big $n \in \mathbb{N}$) a birational morphism into $\mathbb{CP}^N$ with the claimed properties. Note however that local and global birationality are different notions (see the example below) and if one has to deal with the global birationality one has to impose some additional assumptions for the argument to go through.

Consider now $Y := F(X)$. By the Proper Mapping Theorem $Y$ is a (singular in general) subvariety in $\mathbb{CP}^N$. It is also clear that $Y$ is irreducible and locally irreducible variety (the latter follows from the local birationality). Recall that an upper semicontinuous function $u$ on a singular variety $W$ is called weakly plurisubharmonic if for every holomorphic disc $f : \Delta \to W$ the function $u \circ f$ is a subharmonic function (see [8]). In that paper it is proved (in fact in a much more general situation of Stein spaces) that any such function $u$ can be extended locally to the ambient space to a classical plurisubharmonic function i.e. for every $x \in Y$ there exists a small Euclidean ball $B$ in $\mathbb{CP}^N$, centered at $x$ and a function $v \in PSH(B)$, such that $v|_{B \cap Y} = u$.

Now suppose $\phi$ is a positive discontinuous solution of the Monge-Ampere equation in question and let $d := \sup(\phi - \phi_*) > 0$, where $\phi_*$ denotes the lower semicontinuous regularization of $\phi$. Note that the supremum is attained, and if $E$ is the closed set $\{\phi - \phi_* = d\}$, there exists a point $x_0$ such that $\phi(x_0) = \min_E \phi$. Positivity is a technical assumption that can always be achieved by adding appropriate constant since we already know that $\phi$ is bounded.

By assumption there exist analytic sets $Z \subset X$ and $W \subset Y = F(X)$ such that $F|_{X \setminus Z} \to Y \setminus W$ is a biholomorphism and moreover $S := \{\omega^n = 0\} \subset Z$. Note that in the general case of a big form $S$ need not be contained in an analytic set- it may well happen that $S$ is open in $X$.

Two possibilities might take place

1. $x_0 \in X \setminus S$. In this case $\omega$ is strictly positive in a small ball centered at $x_0$ and repeating the argument from Section 2.4 in [10] we obtain a contradiction.

2. $x_0 \in S$. Then we shall produce a domain $V$ (not contained in a chart in general) and a potential $\theta$ of $\omega$ in $V$ with the property that $\inf_{\partial V} \theta > \theta(x_0) + b$, where $b$ is a positive constant.

Consider $F(x_0) = z$ and a neighbourhood $U$ of $z$ in $Y$, such that its preimages are birational to it. Choose the one $x_0$ sits in. For the rest of the argument we restrict ourselves to $F|_{F^{-1}(U) \ni x_0} \to U$. Consider the pushforward function

$$F_* \phi(z) := \begin{cases} \phi(w), & \text{if } z \in Y \setminus W, w \in X \setminus Z, F(w) = z \\ \limsup_{\chi \to z} F_*(\chi) & \text{if } z \in W \setminus Z \end{cases}$$
and a local potential $\eta$ for the Kähler form on $U \cap \mathbb{CP}^N$.

Claim: $\eta + F_\ast \phi$ is weakly subharmonic on $Y$.

Proof: Weak subharmonicity is a local property, hence it is enough to check it in a neighbourhood of any point on $Y$. For regular points of $Y$ this is evident. However at singularities of $Y$ one might a priori run into trouble as the example of a double point shows. Indeed, consider the following (classical) local example:

Let

$$F : \mathbb{C} \ni t \to (t^2 - 1, t(t^2 - 1)) \in \mathbb{C}^2$$

The image $F(\mathbb{C})$ sits in the variety $\{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 + z_2^3 = z_2^3\}$. Observe that $F$ is a bijection onto its image, except for the points 0 and $-1$ being mapped to $(0, 0)$. But then it is clear that the pushforward of a subharmonic function $w$ on $\mathbb{C}$ cannot be weakly subharmonic on the image if $w(1) \neq w(-1)$. Note that $F$ is not locally birational though.

Observe that local birationality forces the analytic set $Y$ to be locally irreducible. Then there is a classical theorem (see [5], Theorem 1.7) stating that on a locally irreducible variety $Y$ and a locally bounded plurisubharmonic function $w$ defined on $\text{Reg}_Y$ the regular part of $Y$ the extension via limsup technique

$$w(z) := \limsup_{\zeta \to z, \zeta \in \text{Reg}_Y} w(\zeta)$$

is a weak plurisubharmonic function. Moreover, it follows from the proof that for any $s \in Y$ and any birational modification $G : Y' \to Y$ of $Y$ the pulled-back function $G^\ast w$ is constant on the fiber $G^{-1}(s)$.

Now if $\omega_M$ is the Kähler metric which defines $\omega$ (i.e. $\omega = F^\ast \omega_M$), fix $\rho$ a local potential of $\omega_M$ near $z$ (in $\mathbb{CP}^n$). First we shall modify $\rho$ exactly as in [10]:

In local coordinates in a ball $B'$ centered at $z \rho$ is strictly plurisubharmonic smooth function and is expanded as

$$\rho(z + h) = \rho(z) + 2\Re \left( \sum_{j=1}^n a_j h_j + \sum_{j,k=1}^n b_{jk} h_j h_k + \sum_{j,k=1}^n c_{jk} h_j \bar{h}_k + o(|h|^2) \right)
= \Re P(h) + H(h) + o(|h|^2),$$

where $P$ is a complex polynomial in $h$ and $H$ is the complex Hessian at $z$.

Proceeding exactly as in [10] (Lemma 2.3.1) $\eta := \rho - \Re P(\cdot - z)$ is also a local potential for $\omega_M$, with the additional property that $\eta$ has a strict local minimum at $z$ (we use at this point that $H$ is strictly positive definite). This means that for a smaller ball (which after possible shrinking we again denote by $B'$) $\inf_{B'} \eta > \eta(z) + b'$ for some positive constant $b'$. Adding a constant if necessary one can further assume that $\eta(z) > 0$.

Now by Fornæss-Narasimhan theorem we find a small euclidean ball $B'$ in $\mathbb{CP}^n$ centered at $z$ and a function $\psi \in PSH(B')$, such that $\psi|_{Y \cap B'} = \eta + \phi$.

On a neighbourhood of a slightly smaller ball $B$ (everything is contained in $B'$ and $B'$) $\psi$ can be approximated by a sequence on smooth plurisubharmonic functions $\psi_j$ decreasing towards it. Again (decreasing a bit $b'$ if necessary) one obtains $\inf_{B} \eta > \eta(z) + b$ for some positive constant $b$. Now we pull back the ball and the regularizations: let $V := F^{-1}(B' \cap Y)$ and $u_j := \psi_j(F(w))$ ($u_j$ are assumed to be defined only on small neighbourhood of $V$). Of course these are continuous plurisubharmonic functions on $V$ which decrease towards
where $u := \eta \circ F + F^*(F_\phi) = \eta \circ F + \phi$ (the equality is due to the fact that $\phi$ has to be a constant on the fiber). Note that $V$ need not be an Euclidean domain anymore (i.e. it need not be contained in a coordinate chart), nevertheless $\eta \circ F$ is a global potential of $\omega$ on this set. This is the essential difference between this special situation and the general case. Next we state a lemma which is essentially contained in [10] (Section 2.4). We include the proof for the sake of completeness.

**Lemma 5.1.** There exist $a_0 > 0$, $t > 1$ such that the sets

$$W(j,c) := \{ tu + d - a_0 + c < u_j \}$$

are non-empty and relatively compact in $V$ for every constant $c$ belonging to an interval which does not depend on $j > j_0$.

**Proof.** Note that $E(0) := \{ u - u_* = d \} \cap \overline{V} = E \cap \overline{V}$, since the potential is continuous. Also the sets $E(a) := E := \{ u - u_* \geq d - a \} \cap \overline{V}$ are closed and decrease towards $E(0)$. Hence if $c(a) := \phi(x_0) - \min_{E(a) \phi}$ we have that $\limsup_{a \to 0^+} c(a) \leq 0$, for otherwise we would get a contradiction with the definition of $d$. Hence

$$c(a) < \frac{1}{3} b$$

for $0 < a < a_0 < \min(\frac{1}{3} b, d)$. Let $A := u(x_0)$. Note that $A > d$ since the potential is greater than 0 at $x_0$, and $\phi$ as a globally positive function has to be greater than $d$ at $x_0$. One can choose $t > 1$, such that it satisfies

$$(t - 1)(A - d) < a_0 < (t - 1)(A - d + \frac{2}{3} b)$$

Now if $y \in \partial V \cap E(a_0)$ one gets

$$u_*(y) \geq \eta(F(x_0)) + b + F^*F_\phi(x_0) \geq A - d + \frac{2}{3} b$$

Hence $u(y) \leq u_*(y) + d < tu_*(y) + d - a_0$. Note that this inequality extends to a neighbourhood of $\partial V \cap E(a_0)$. Taking another neighbourhood relatively compact in the first and applying Hartogs type argument one obtains

$$u_j < tu(y) + d - a_0, \quad \forall j > j_1$$

For the rest part of $\partial V$ the same inequality holds if we take big enough $j_1$ and the proof is even simpler, since $u - u_*$ is less than $d - a_0$ there. This proves the relative compactness on $W(j,c)$ in $V$.

Note that from the left inequality defining $t$ one gets $(t - 1)u_*(x_0) < a_0$, hence

$$tu_*(x_0) < u(x_0) - d - a_1 + a_0 < u_j(x_0) - d - a_1 + a_0$$

for some constant $a_1 > 0$. This implies that the sets $W(j,c)$, $c \in (0, a_1)$ contain some points near $x_0$, hence they are non empty. 

\[ \square \]
Now applying Lemma 2.3.1 from [10] (one can verify that despite the fact that $V$ can be a non-Euclidean set the argument still goes through) one can bound the capacity $\text{cap}(W(j,a_1),V)$ from below by an uniform positive constant. On the other hand $W(j,a_1) \subset \{u + (d - a_0 + a_1) < u_j\}$ and this contradicts the fact that the decreasing sequence $u_j$ has to converge towards $u$ in capacity. This proves that $d = 0$, hence $\phi$ is continuous.

Remark 5.2. As we have seen the argument cannot be applied in the case of a (globally) birational map. Then further assumption is needed to assure the pushforward to be plurisubharmonic. A satisfactory additional assumption is that the fibers in the preimage have to be connected. Then the function has to be constant on any nontrivial connected fiber and this is enough to push it forward onto the image.

6 Remarks

Complex Monge-Ampère equations are of great interest in geometry. In [17], the following version of the Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial \bar{\partial} u)^n = e^{u} \Omega$$

is the main focus. Of course anything new would be for a degenerate class $[\omega]$ as in the settings of Theorem 1.1. And using the argument in [11], we know that the main result in this work would also apply for it. To be precise, the following holds:

**Theorem 6.1.** Let $\omega$ be a big form and $u_1, u_2$ be $\omega$-psh solutions for the following Monge-Ampère equations:

$$(\omega_{u_1})^n = e^{u_1} \Omega_1, \quad (\omega_{u_2})^n = e^{u_2} \Omega_2,$$

where $\Omega_1$ and $\Omega_2$ are smooth volume forms such that

$$\int_X |\Omega_1 - \Omega_2| \leq \gamma(t) t^{n+3}.$$

Then

$$||u_1 - u_2||_{\infty} \leq Ct.$$

**Proof.** Since the comparison principle for big forms is available the proof is entirely the same as in Theorem 5.2 in [11]. \qed

The following problems are related to the results in [13] and [7], stating that when $\omega$ is Kähler form on a compact Kähler manifold, the solutions of

$$\omega^n = f \omega^n, \quad f \in L^p(\omega^n), \quad p > 1$$

are Hölder continuous. In general the Hölder exponent depends on the manifold $X$, and on $n$ and $p$ ([13]). Under the additional assumption that $X$ is homogeneous i.e. the automorphism group $\text{Aut}(X)$ acts transitively the exponent is independent of $X$ and is not less that $\frac{2}{m+2}, \quad q = \frac{m+2}{m+1}$ ([7]). One can ask the following questions:
1. Is the solution continuous when $\omega$ is semi-positive and big in general?

2. If so, does the Hölder continuity hold in the case $\omega$ is merely semi-positive and big?

3. Does the Hölder exponent on general manifold do really depend on the manifold? In the corresponding result in the flat case (12) the Hölder exponent is uniform and independent of the domain. Moreover the proof in (13) strongly depends on a regularization procedure for $\omega$-psh functions, which consists of patching local regularizations, and this is the point where the geometry of the manifold influences the exponent. In particular are there another regularization procedures of more global nature that are not so affected by the local geometry?

4. Is the exponent for the homogeneous case sharp? Note that in the flat case in (9) there is also a gap between the exponent given there $\frac{2}{qn+1}$ and the exponent $\frac{2}{qn}$, for which an example is shown.

5. It is interesting to compare the stability results we have proven and the one in (7). In particular, is the stability exponent in (7) sharp in general?

6. It would be very interesting to generalize Hölder continuity to more singular measures. One possible application of such a result would be a criterion for Hölder continuity of the Siciak extremal function of certain compact sets in $\mathbb{C}^n$ (see (12) for a definition). Such a property is very important from pluripotential point of view. So one has to study the equilibrium measure of the compact. The problem is that such measures are singular with respect to the Lebesgue measure, while (13) and (7) rely strongly on smoothness of $\omega^n$. However, as this note shows, some arguments can be adjusted to singular measures either.

We hope to address some of these questions in the future.

References

[1] Benelkourchi, Slimane; Guedj, Vincent; Zeriahi, Ahmed: A priori estimates for weak solutions of complex Monge-Ampère equations. Preprint. ArXiv 0704.0866.

[2] Bedford, Eric; Taylor, B. A.: A new capacity for plurisubharmonic functions. Acta Math. 149 (1982), no. 1-2, 1–40.

[3] Dinew, Slawomir: An Inequality for Mixed Monge-Ampère Measures. Preprint. ArXiv 0705.0974.
[5] Demailly, Jean-Pierre: Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. Mémoires Soc. Math. France 19 (1985) 1-125.

[6] Demailly, Jean-Pierre; Pali, Nefton: Degenerate complex Monge-Ampère equations over compact Kähler manifolds. Preprint. ArXiv 0710.5109.

[7] Eyssidieux, Phillipe; Guedj, Vincent; Zeriahi, Ahmed: Singular Kähler-Einstein metrics. Preprint. ArXiv math/0603431.

[8] Fornaess, John Erik; Narasimhan, Raghavan: The Levi problem on complex spaces with singularities. Math. Ann. 248 (1980), no. 1, 47–72.

[9] V. Guedj, S. Kolodziej, A. Zeriahi: Hölder continuous solutions to Monge-Ampère equations. preprint. ArXiv math/0607314.

[10] Kolodziej, Slawomir: The complex Monge-Ampere equation. Acta Math. 180 (1998), no. 1, 69–117.

[11] Kolodziej, Slawomir: The Monge-Ampere equation on Compact Kähler Manifold. Indiana Univ. Math. J. 52 (2003), 667-686.

[12] Kolodziej, Slawomir: The complex Monge-Ampere equation and pluripotential theory. Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.

[13] Kolodziej, Slawomir: Hölder continuity of solutions to the complex Monge-Ampere equation with the right hand side in $L^p$. Preprint. ArXiv math/0611051.

[14] H. Kozono, H. Wadade: Remarks on Gagliardo-Nirenberg type inequality with critical Sobolev space and BMO. to appear in Math Z.

[15] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, Number 2, 179–192.

[16] Zhang, Zhou: On Degenerate Monge-Ampere Equations over Closed Kähler Manifolds. Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

[17] Zhang, Zhou: Degenerate Monge-Ampere Equations over Projective Manifolds. PHD Thesis at MIT, 2006.

Jagiellonian University
Institute of Mathematics
Reymonta 4, 30-059 Kraków, POLAND.
E-mail slawomir.dinew@im.uj.edu.pl

Michigan University
Department of Mathematics
4835 East Hall, Ann Arbor, USA
E-mail zhangou@umich.edu
Key words and phrases: Kähler manifold, complex Monge-Ampère operator.
2000 Mathematics Subject Classification: Primary 32U05, 53C55; secondary: 32U40.