DYNAMICAL SOLUTIONS TO THE HORIZON AND FLATNESS PROBLEMS

Yue Hu, 1 Michael S. Turner, 2,3 and Erick J. Weinberg 1

1Department of Physics, Columbia University, New York, NY 10027
2Departments of Physics and of Astronomy & Astrophysics
Enrico Fermi Institute, The University of Chicago, Chicago, IL 60637-1433
3NASA/Fermilab Astrophysics Center, Fermi National Accelerator
Laboratory, Batavia, IL 60510-0500

ABSTRACT

We discuss in some detail the requirements on an early-Universe model that solves the horizon and flatness problems during the epoch of classical cosmology \(t \geq t_i \gg 10^{-43}\) sec. We show that a dynamical resolution of the horizon problem requires superluminal expansion (or very close to it) and that a truly satisfactory resolution of the flatness problem requires entropy production. This implies that a proposed class of adiabatic models in which the Planck mass varies by many orders of magnitude cannot fully resolve the flatness problem. Furthermore, we show that, subject to minimal assumptions, such models cannot solve the horizon problem either. Because superluminal expansion and entropy production are the two generic features of inflationary models, our results suggest that inflation, or something very similar, may be the only dynamical solution to the horizon and flatness problems.
1 Introduction

As successful as the standard cosmology is, it has two well known shortcomings: the horizon and flatness problems [1, 2]. These shortcomings do not indicate any logical inconsistency, but rather that in the standard cosmology the present state of the Universe depends strongly upon the initial state—a feature that many consider undesirable. It is possible that these shortcomings do not require an explanation. Penrose has suggested that there may be a law of physics that governs the initial state of the Universe [3]. Or perhaps the required initial state for the classical epoch of cosmology is dictated by the outcome of the quantum-gravity era. Here we focus on dynamical solutions to these problems during the era when gravity can be treated classically ($t \geq t_i \gg 10^{-43}$ sec).

Guth’s inflationary Universe paradigm provides an elegant solution involving the microphysics of the very early Universe ($t \sim 10^{-34}$ sec) [2]. While Guth’s original model based upon a first-order symmetry-breaking phase transition did not work, many viable implementations of inflation now exist [4]. All involve two key elements: a period of superluminal expansion (driven by scalar-field potential energy) and massive entropy production (conversion of that energy to radiation). We show that both features are essential for a true resolution of these problems: Superluminal expansion (or very close to it) is required to solve the horizon problem, while one cannot account for the flatness of the present Universe for arbitrary initial curvature without entropy production.

Of course, one might be less ambitious and try to solve only the horizon problem. Indeed, it has been proposed that an adiabatic scenario based on a time-varying Planck mass might provide a solution to the horizon problem [5, 6, 7]. We examine this proposal and show, with minimal assumptions (essentially positivity of energies), that in the context of scalar-tensor theories a varying Planck mass cannot lead to a solution of the horizon problem either.

In Sec. 2 we discuss the horizon and flatness problems. We show why superluminal expansion (which we define more precisely) and entropy production are essential ingredients of any solution to these problems. We describe how the horizon and flatness problems are solved in inflationary scenarios in Sec. 3. Scalar-tensor theories with a time-varying Planck mass are discussed in Sec. 4, and we show that they cannot solve the horizon problem without entropy production [8]. Section 5 contains our concluding remarks.
2 The Cosmological Problems

2.1 The horizon problem

The horizon problem involves the smoothness—that is, homogeneity and isotropy—of the presently observed Universe (size $\sim H_0^{-1} \sim 10^{28}$ cm). The uniformity of the cosmic background radiation indicates that this volume was smooth at $t \sim 300,000$ yr ($T \sim 0.3$ eV), when matter and radiation decoupled; further, the successful predictions of primordial nucleosynthesis are evidence that it was smooth at least as early as $t \sim 1$ sec ($T \sim 1$ MeV).

However, in the standard cosmology the distance that a light signal can travel by such early times, $d_{\text{HOR}}(t) = R(t) \int_t^{t_i} du/R(u)$, is much smaller than the size of the presently observed Universe at that time, $d_U(t) \sim R(t) H_0^{-1}/R_0$; this precludes causal physics operating at early times from accounting for the smoothness. (Here $R(t)$ is the cosmic-scale factor, $H \equiv \dot{R}/R$ is the expansion rate, and a subscript 0 denotes the present epoch. Throughout, we assume that recombination and last-scattering took place at a temperature $T \sim 0.3$ eV, as expected. None of our conclusions change if the Universe remained ionized so that these events occurred later.)

A necessary (though not sufficient) condition to solve the horizon problem is

$$d_{\text{HOR}}(t_S) = R(t_S) \int_{t_i}^{t_S} dt'/R(t') > R(t_S) H_0^{-1}/R_0; \quad (2.1)$$

where $t_S$, the time by which the horizon problem is solved, must be less than 1 sec, since the presently observed Universe was already smooth by then. Now note that if $R(t) \propto t^n$ with $n < 1$, then $d_{\text{HOR}}(t_S) \approx t_S/(1-n) = [n/(1-n)] H(t_S)^{-1} \approx H(t_S)^{-1}$. Thus, in the standard cosmology, where $n = 1/2$ in the radiation-dominated era ($t \lesssim 10^{11}$ sec) and $n = 2/3$ in the matter-dominated era ($t \gtrsim 10^{11}$ sec), the right-hand side of condition (2.1) is greater than the left by a factor of $(10^{15} \text{ sec}/t_S)^{1/2}$ for any $t_S$ in the radiation-dominated era and by a factor of $(3 \times 10^{17} \text{ sec}/t_S)^{1/3}$ in the matter-dominated era. Hence, condition (2.1) cannot be satisfied.

It is instructive to recast the horizon-problem-solving condition. Suppose first that at early times there was “standard” evolution (i.e., $R$ growing no faster than $t^n$ with $n$ less than, and not too close to, unity—say $n < .99$) up to some time $t_1$, and that the evolution was then nonstandard throughout the interval $t_1 < t < t_S$. Now write the integral in Eq. (2.1) as the sum
of the contribution from \( t_1 \) to \( t \) and that from \( t_1 \) to \( t_S \). The first integral is dominated by the upper end of the integration range and is equal, up to factors of order unity, to \((R_1H_1)^{-1}\), where subscript 1 refers to time \( t_1 \). If \( R \) grows faster than \( t^n \), with \( n \) greater than, and again not too close to, unity for \( t_1 < t < t_S \), then the second integral is dominated by the lower end of its integration range and is approximately equal to the first integral. Then, apart from factors of order unity, Eq. (2.1) becomes

\[
(R_1H_1)^{-1} > (R_0H_0)^{-1}. 
\]  
(2.2)

[If there were several alternating periods of standard and nonstandard evolution of the scale factor, we divide the integration range correspondingly, and find that the integral in Eq. (2.1) is given by a sum of terms of the form \([R(t_a)H(t_a)]^{-1}\), where the \( t_a \) are the times at which the periods of standard evolution terminate. In general, one of these terms will dominate; denoting as \( t_1 \) this time \( t_a \), we again obtain Eq. (2.2).]

Condition (2.2) says that the comoving Hubble volume at some very early time must grow large enough to contain the present Hubble volume. Since \( RH = \dot{R} \), it further implies that \( \dot{R}_0 > \dot{R}_1 \), which requires that \( \ddot{R} > 0 \) sometime between \( t_1 \) and \( t_S \lesssim 1 \text{ sec} \). Such expansion is often referred to as “superluminal.”

Let us clean up a several minor points before going on. First, suppose that there was no early period of “standard evolution” following the quantum-gravity era. If the horizon distance at the end of the quantum-gravity era, \( d_{\text{HOR}}(t_i) \), was of the order of \( H_i^{-1} \), we simply set \( t_1 = t_i \) in Eq. (2.2). If instead the size of causally connected regions at time \( t_i \) was actually significantly greater than \( H_i^{-1} \) and large enough to solve the horizon problem, then from our perspective the horizon problem was solved during the quantum-gravity era.

The second point involves the kinematical motivation for the term superluminal. Superluminal expansion might be most naturally defined as that where any two comoving points eventually lose causal contact; i.e., expansion so rapid that \( \int_{t_i}^{\infty} du/R(u) \) converges. For \( R \propto t^n \), this corresponds to \( n > 1 \), and hence \( \ddot{R} > 0 \), the definition adopted here.

Next, we note that there is a small loophole in the arguments going from Eq. (2.1) to Eq. (2.2). If \( R \) is very close to linear in time, the estimates we made for the integral in Eq. (2.1) are incorrect, and, in principle, the
horizon problem can be solved without superluminal expansion. However, we stress that achieving this requires a rather strictly constrained behavior for a long time. For example, condition (2.1) can be satisfied with $R \propto t^{1-\epsilon}$ (and hence $\ddot{R} < 0$) and $t_S \lesssim 1$ sec provided $\epsilon \lesssim 10^{-7}$. While not superluminal, such behavior might well be termed “almost superluminal”—or at least, nonstandard.

Finally, there is an apparent paradox in discussing the horizon problem in the context of a Friedmann-Robertson-Walker (FRW) model, which, by assumption, is isotropic and homogeneous. Consideration of the most general cosmological solutions, which are not homogeneous nor even fully classified, is very difficult though some attempts have been made [9]. A more modest approach to addressing this paradox is to consider a perturbed FRW model that is more inhomogeneous than ours (which is very easy to do since our Universe is so very smooth). In this case, the FRW equations apply to lowest order, and it makes sense to discuss whether the level of inhomogeneity can be reduced to a level consistent with that observed. Within this perturbative framework the horizon problem as usually discussed is correctly stated. One can also consider the isotropy issue alone by appealing to the homogeneous, but nonisotropic, Bianchi models which have tractable field equations [10].

2.2 The flatness problem

The flatness problem involves the observation that $\Omega$, the ratio of the energy density of the Universe to the critical density, is close to unity today in spite of the fact that $|\Omega - 1|$ grows as a power of the scale factor. In the FRW cosmology the expansion rate is governed by the Friedmann equation,

$$H^2 = \frac{8\pi G \rho}{3} - \frac{k}{R^2},$$

(2.3)

where $\rho$ is the total energy density, $\rho_{\text{crit}} = 3H^2/8\pi G$ is the critical density, and $R_{\text{curv}} = R(t)|k|^{-1/2}$ is the curvature radius. Eq. (2.3) can be rewritten as

$$\Omega - 1 = \left[ \left( \frac{8\pi G \rho}{3} \right) \left( \frac{R^2}{k} \right) - 1 \right]^{-1}.$$  

(2.4)

For $\Omega$ close to unity, $|\Omega - 1|$ grows as $R^m$ with $m = 2$ during the radiation-dominated era and $m = 1$ during the matter-dominated era. Since the cur-
vature radius is proportional to $|\Omega - 1|^{-1/2}$,

$$R_{\text{curv}} = R(t)|k|^{-1/2} = \frac{H^{-1}}{|\Omega - 1|^{1/2}}, \quad (2.5)$$

it decreases relative to the Hubble radius $H^{-1}$ and the Universe becomes “less flat” with time. This means that the Universe must have been very flat at the initial epoch in order that $R_{\text{curv}} \gtrsim H_0^{-1}$ today: $|\Omega_i - 1| = (R_{\text{curv}}H)^{-2} \ll 1$. (In alternative theories of gravity, we take the 3-curvature to be $6k/R(t)^2$ and use Eq. (2.5) as the definition of $\Omega$. This definition can be used for any metric theory of gravity.)

The resolution of the flatness problem is closely related to that of the horizon problem. Suppose that the horizon problem is solved in the post quantum-gravity era; i.e., that Eq. (2.2) is satisfied with

$$(R_1H_1)^{-1} = \beta(R_0H_0)^{-1},$$

and $\beta \geq 1$. It then follows by simple algebra that

$$|\Omega_0 - 1| = |\Omega_1 - 1|/\beta^2. \quad (2.6)$$

The deviation of $\Omega$ from unity is reduced by the square of the factor by which the horizon problem is solved, and thus $|\Omega - 1|$ need not be initially set to a very small value to insure that $|\Omega_0 - 1|$ is small today.

As we shall discuss later, it is possible that the sequence of events that solves the horizon problem (e.g., inflation) is prevented from occurring by large curvature (e.g., the Universe recollapses before it can inflate). The point here is simply that, if the horizon problem as formulated above is solved after the emergence from the quantum-gravity era, then one need not invoke an unnaturally flat initial state to account for the flatness of the present Universe.

### 2.3 Entropy considerations

It is instructive to reformulate the flatness problem in terms of entropy. By a very wide margin most of the entropy in the Universe exists in the form of radiation, today in the cosmic backgrounds of 3 K photons and 2 K neutrinos, and at very early times in a thermal bath of all particle species much less
massive than the temperature. If the Universe remains close to thermal equilibrium, the radiative entropy per comoving volume \( S \propto g_*(RT)^3 \) is constant and the expansion is said to be adiabatic. Here \( g_* \) counts the total number of effectively massless (mass \( m \ll T \)) degrees of freedom; although \( g_* \) varies during the evolution of the Universe we can neglect this variation in our considerations.

Our present Hubble volume contains an entropy \( S_0 \sim H_0^{-3}T_0^3 \sim 10^{88} \); for adiabatic expansion this is always the entropy within the comoving volume corresponding to the present Hubble volume since \( RT = \text{const} \) by adiabaticity. In the standard cosmology, the entropy within a horizon volume during the radiation-dominated era, \( S_{\text{HOR}}(t) \sim H^{-3}T^3 \sim (m_{\text{Pl}}/T)^3 \), was much smaller than this at early times, which is another way of stating the horizon problem.

The curvature of the Universe can also be characterized by the entropy contained within a curvature volume: \( S_{\text{curv}} \sim R_{\text{curv}}^3 T^3 = |k|^{-3/2}(RT)^3 \). Since the curvature radius at present is comparable to, or greater than, the Hubble radius, \( S_{\text{curv}} \) must be at least \( 10^{88} \). For adiabatic expansion, this is always the entropy within a curvature volume. This allows us to express the size of the curvature radius in terms of the temperature at any epoch:

\[
R_{\text{curv}}(t) = S_{\text{curv}}^{1/3} T^{-1} \gtrsim 10^{29} T^{-1} \tag{2.7}
\]

If furthermore \( H \gtrsim T^2/m_{\text{Pl}} \), as we usually expect, then

\[
R_{\text{curv}}(t) H(t) \gtrsim 10^{29} \left( \frac{T}{m_{\text{Pl}}} \right) . \tag{2.8}
\]

The upshot of Eqs. (2.7,2.8) is that in an adiabatic scenario it is not possible to evolve to a universe as flat as ours if the curvature radius at the initial epoch was smaller than \( 10^{29} T_i^{-1} \).

In the previous subsection, we described the flatness of the present-day Universe in terms of the small size of \( |\Omega - 1| = (R_{\text{curv}} H)^{-2} \). From that perspective, the flatness problem is that, in the standard cosmology, \( R_{\text{curv}} \) must have been very much larger than the Hubble radius at early times. From the entropy perspective, the puzzle is why the entropy within a curvature volume is so extraordinarily large. While one might envision an adiabatic

\footnote{A closer analysis shows that the variation in \( g_* \) does not affect any of our conclusions unless it changes by a factor of greater than \( 10^{180} \).}
scenario that resolved the first formulation of the flatness problem, such a
scenario cannot change the entropy within a curvature volume and hence
cannot explain the enormous size of \( S_{\text{curv}} \). Since the entropy within a cur-
vature volume is just another way of specifying the initial curvature of an
adiabatic universe, in an adiabatic model the present-state of the Universe
will always be sensitive to the initial curvature: curvature corresponding to
\( S_{\text{curv}} \lesssim 10^{88} \) can never lead to a universe that today is as flat as ours. It is
for this reason that we argue that a truly satisfactory solution of the flatness
problem cannot be obtained within an adiabatic scenario.

We showed previously that the horizon and flatness problems were linked:
Any scenario that solves the horizon problem automatically accounts for the
smallness of \( |\Omega - 1| \) today. This of course also applies to an adiabatic scenario,
and reveals a further problem. According to Eq. (2.8) the smallness of \( |\Omega_0 - 1| \)
requires either, (i) \( R(t_1)H(t_1) \) was very large, or (ii) \( T_1/m_{\text{Pl}} \) was very small.
The horizon-problem-solving condition, Eq. (2.2), does not allow the former.
Hence, \( \text{any} \) adiabatic solution to the horizon problem has the additional
drawback that it must take place at an unnaturally small temperature, many
orders of magnitude below the Planck mass. (In Section 4, we show that in
scalar-tensor theories the horizon problem cannot be solved adiabatically at
all.)

One final point; we have used adiabaticity to mean constant entropy per
comoving three volume. There have been attempts in higher-dimensional
theories of gravity to increase the entropy per comoving three volume while
maintaining adiabaticity in the higher dimensional theory. In such theories
adiabaticity refers to constant entropy per total comoving volume; the shrink-
ing of the volume of the extra dimensions causes the entropy per comoving
three volume to increase. Thus far this approach has not proven successful
[11]; further, in our terminology these scenarios are not adiabatic (though
this is largely a matter of semantics).

3 How Inflation Resolves the Problems

In this Section we illustrate how superluminal expansion and entropy pro-
duction solve the horizon and flatness problems in the context of inflationary
models. The Universe today is matter dominated with \( \rho_{\text{matter}} \sim 10^4 \rho_{\text{rad}} \sim
10^4 T_0^4 \), so that \( H_0 \sim 10^2 T_0^2/m_{\text{Pl}} \). If at time \( t_1 \) the Universe was radiation
dominated, then \( H_1 \sim T_1^2/m_{\text{Pl}} \); the horizon/flatness-problem-solving condition, Eq. (2.2), becomes

\[
\frac{m_{\text{Pl}}}{T_1 S_1^{1/3}} \gtrsim 10^{-2} \frac{m_{\text{Pl}}}{T_0 S_0^{1/3}}; \tag{3.1}
\]

where \( S \propto (RT)^3 \) is the entropy per comoving volume. Equation (3.1) can be satisfied if there is large-scale entropy production: \( S_0/S_1 \gtrsim (10^{-2}T_1/T_0)^3 \). This is the strategy underlying the inflationary solution: A period of superluminal expansion (nearly exponential) is followed by a reheating event which increases the entropy by a large factor.

It is quite natural that superluminal expansion should be followed by entropy production. In the FRW cosmology the energy density of a fluid with equation of state \( p = \gamma \rho \) evolves as \( \rho \propto R^{-3(1+\gamma)} \), and if it dominates the energy density the scale factor evolves as \( t^{2/(3(1+\gamma))} \). Superluminal expansion requires that \( \gamma < -1/3 \), which implies that \( \rho \) decreases more slowly than \( R^{-2} \). During the superluminal phase, the energy density of the “fluid” that drives inflation increases as \( R^2 \) (or faster) relative to the radiation energy density, and the Universe supercools to a very low temperature. In order that the Universe become radiation dominated once again the fluid driving inflation must “decay” into radiation, and the decay of this fluid into radiation “reheats” the Universe, thereby increasing the entropy by a large amount.

Let us now be more specific. Suppose that the vacuum energy that drives inflation is \( V_0 \equiv \mathcal{M}^4 \) and that the dynamics of the inflation are such that it takes a time \( NH_I^{-1} \) for the vacuum energy to decay, after which the Universe reheats to a temperature \( T \sim \mathcal{M} \) (perfect conversion of vacuum energy to radiation). Here \( \mathcal{M} \) is the energy scale of inflation and \( H_I \sim \mathcal{M}^2/m_{\text{Pl}} \) is the Hubble constant during inflation. It is convenient to write the Friedmann

\[\text{In Eq. (3.1) we need a lower bound to } H_1; \text{ the presence of additional forms of energy density (which we assume to be positive) or of negative curvature would only serve to increase } H_1, \text{ thus giving an even stronger condition. Positive curvature could decrease } H_1; \text{ however, a closer analysis shows that this decrease cannot be significant. In alternative theories of gravity there may be additional terms that might appear to be able to reduce } H_1; \text{ we will address this point, in the context of a theory with a variable Planck mass, below.} \]
equation in a slightly different form:
\[
H^2 = \frac{8\pi G}{3} \left( \frac{g_\ast \pi^2}{30} T^4 + V_0 \right) - \frac{k}{R^2} \sim \frac{T^4}{m_{\text{Pl}}^2} - \frac{\text{sign}(k) T^2}{S_{\text{curv}}^{2/3}} + \frac{\mathcal{M}^4}{m_{\text{Pl}}^2},
\] (3.2)
where for simplicity we ignore the energy density in nonrelativistic matter (which is negligibly small at early times) and all the purely numerical factors. As before, \( S_{\text{curv}} \) is the entropy within a curvature radius before inflation, which increases greatly after inflation due to the entropy production associated with reheating.

First, suppose that the curvature of the Universe is not “large;” specifically that, \( S_{\text{curv}} \gtrsim m_{\text{Pl}}^3/\mathcal{M}^3 \). In this case, when the vacuum-energy density starts to exceed the radiation-energy density, at a temperature \( T_1 \sim \mathcal{M} \), and the Universe begins to inflate, the curvature term is still small compared to these terms (at the beginning of inflation \( \Omega \) is still close to unity). The Universe then grows in size by a factor of \( e^N \); since the temperature after inflation is about the same as it was before inflation the entropy per comoving volume increases by a factor of \( e^{3N} \).

How large must \( N \) be to solve the horizon and flatness problems? A Hubble-radius-sized patch at the beginning of inflation, \( H_I^{-1} \sim m_{\text{Pl}}/\mathcal{M}^2 \), grows by a factor of \( e^N \) by the end of inflation. By today it has grown by another factor of \( \mathcal{M}/T_0 \), since we assume that the expansion is adiabatic after inflation. This patch will be larger than the present Hubble radius if
\[
N \gtrsim 68 + \ln(\mathcal{M}/m_{\text{Pl}}).
\] (3.3)

Next, the flatness problem; the entropy within a curvature radius after inflation increases by a factor of \( e^{3N} \) and is thus greater than \( e^{3N}(m_{\text{Pl}}/\mathcal{M})^3 \) (since by assumption \( S_{\text{curv}} \gtrsim m_{\text{Pl}}^3/\mathcal{M}^3 \)). Solving the flatness problem requires that this number be greater than about \( 10^{88} \), or
\[
N \gtrsim 68 + \ln(\mathcal{M}/m_{\text{Pl}}),
\] (3.4)
which is precisely the condition for solving the horizon problem. Thus, if there is enough inflation to solve the horizon problem the flatness problem is also be solved.

Now consider the opposite limit, large curvature, \( S_{\text{curv}} \lesssim m_{\text{Pl}}^3/\mathcal{M}^3 \). In this circumstance the curvature term becomes comparable to the radiation-energy density term at a time when they are both larger than the vacuum-energy term. If the Universe is positively curved this is very bad, as the
Universe will recollapse before can inflate, and thus the flatness and horizon problems cannot be solved. (There is a way out; in some models of inflation, e.g., chaotic inflation, the Universe begins vacuum dominated, i.e., $\rho \sim V(\phi) \sim m_{\text{Pl}}^4$, and inflating, and this problem never arises.)

For $k < 0$, the curvature term begins to dominate the right-hand side of the Friedmann equation at a temperature $T_{\text{curv}} \sim S_{\text{curv}}^{-1/3}m_{\text{Pl}}$; the Universe undergoes a period of “free expansion,” and $\Omega$ approaches zero. When the temperature reaches $T_{\text{infl}} \sim S_{\text{curv}}^{1/3}(\mathcal{M}/m_{\text{Pl}})^2m_{\text{Pl}}$, vacuum energy begins to dominate the right-hand side of the Friedmann equation and inflation begins. The requirement on $N$ to solve the horizon problem is precisely as before, cf. Eq. (3.3).

Now consider the flatness problem. After inflation, when the vacuum energy has been converted to radiation, the entropy per comoving volume has increased by a factor of $e^{3N} (\mathcal{M}/T_{\text{infl}})^3 \sim e^{3N} (m_{\text{Pl}}/\mathcal{M})^3/S_{\text{curv}}$. Thus the final entropy contained within a curvature radius is

$$S_{\text{curv}}(\text{final}) \sim e^{3N} (m_{\text{Pl}}/\mathcal{M})^3. \quad (3.5)$$

Note that the final entropy within a curvature radius does not depend upon the initial value. The condition that $S_{\text{curv}}(\text{final})$ be greater than about $10^{88}$ is the very same condition for solving the horizon problem,

$$N \gtrsim 68 + \ln (\mathcal{M}/m_{\text{Pl}}). \quad (3.6)$$

(One might worry that the number of e-foldings of inflation $N$, which is determined by the time it takes the inflaton to roll to the minimum of its potential, could depend upon the size of the curvature term since it influences the expansion rate; it does not in any significant way. For a given model of inflation, specified by the scalar potential, the amount of inflation is, to within a few e-foldings, independent of the size of the curvature term.)

To recapitulate, in the case of negative curvature sixty or so e-foldings of inflation serves to solve the horizon and flatness problems, regardless of the size of the initial curvature radius. In the case of positive curvature, for many models of inflation the Universe will only survive long enough to inflate if $S_{\text{curv}}$ is larger than $m_{\text{Pl}}^3/\mathcal{M}^3$, which is of the order of $10^{15}$ for the typical energy scale of inflation, $\mathcal{M} \sim 10^{15}$ GeV.

If one thought that the Universe began in an initial state as simple and special as an FRW model this would be a little disturbing. However, one
expects that a generic initial state is highly inhomogeneous, with regions of both negative and positive curvature. Many, though not all, of the positively curved regions would simple collapse to form black holes, while any sufficiently large region of negative curvature would undergo inflation and grow to a size large enough to encompass all that we see today. (Sufficiently large means much larger than the inverse of the local expansion rate; see Refs. [14].) In the context of a generic beginning for the classical epoch it is not very significant that only the negatively curved regions are certain to undergo inflation. In sharp contrast, in the case of an adiabatic scenario, only those negatively curved regions that contain an entropy greater than $10^{88}$ can ever be suitable to house our present Hubble volume.

4 Variable Planck Mass Cannot Solve the Horizon Problem

4.1 The horizon problem restated

We have seen in our previous discussion that an adiabatic scenario cannot address the flatness problem (unless $S_{\text{curv}} \gtrsim 10^{88}$ the universe will not be as flat as ours is today). However, one might well ask whether it is possible to solve just the horizon problem in the post-Planckian era without entropy production. In this Section we show that in a wide class of gravity theories this cannot be done, provided only that we assume that that the various contributions to the energy density are positive. In particular, we rule out proposed adiabatic solutions based on a time-varying Planck mass in the context of scalar-tensor theories of gravity [5, 6, 7].

For constant $m_{\text{Pl}}$, it is easy to see that entropy production is required to solve the horizon problem. For adiabatic expansion, $S_1 = S_0$ and Eq. (3.1) requires $T_1 < 10^2 T_0$ and thus $R_1 > 10^{-2} R_0$. Since time $t_1$ was taken to be during the radiation-dominated era, which began when $R \sim 10^{-4} R_0$, this inequality can only be satisfied if the scale factor was decreasing at early times, “bounced,” and began increasing. However, this cannot be, since in the absence of entropy production any FRW model that is now expanding must always have been expanding, provided only that all energy densities are positive.

From Eq. (3.1) it might seem that a decreasing Planck mass could dras-
tically alter this: A larger Planck mass at early times would imply a weaker effective gravitational constant. This would lead to slower expansion (at a given temperature), resulting in an older Universe and a larger horizon \[5, 6, 7\]. Specifically, Eq. \((3.1)\) would be satisfied without entropy production if

\[
m_{\text{Pl}}(t_1) \gtrsim (10^{-2} T_1/T_0) m_{\text{Pl},0} \sim 10^{30} T_1; \tag{4.1}
\]

where \(m_{\text{Pl},0} = m_{\text{Pl}}(t_0) = G_N^{-1/2} = 1.22 \times 10^{19} \text{ GeV}\) denotes the current value of the Planck mass, and \(m_{\text{Pl}}(t) = G_N(t)^{-1/2}\) its value as a function of time. We now show that it is not possible to decrease the Planck mass rapidly enough to satisfy Eq. \((4.1)\). Our strategy is to focus on \(T/m_{\text{Pl}}\). To reproduce the successful predictions of primordial nucleosynthesis, the Planck mass must have reached its present value by a temperature of 1 MeV. From this and Eq. \((4.1)\) it follows that at time \(t_1\) the value of \(T/m_{\text{Pl}}\) must have been smaller than its value at nucleosynthesis by at least a factor of \(10^8\); in fact, as we shall see, adiabatic expansion precludes \(T/m_{\text{Pl}}\) from increasing at all.

In describing a generic theory with a variable Planck mass we shall represent the Planck mass squared by a Brans-Dicke type field \(\Phi = m_{\text{Pl}}^2\). We write the action in the form

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{\Phi}{16\pi} R + \frac{\omega(\Phi)}{16\pi \Phi} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi) + \mathcal{L}_{\text{matter}} \right]. \tag{4.2}
\]

The unusual form of the \(\Phi\) kinetic energy term is not essential; it can be put in the standard form by transforming to a field \(\psi(\Phi)\) obeying \((d\psi/d\Phi)^2 = \omega(\Phi)/(8\pi \Phi)\). For constant \(\omega\) and vanishing \(V(\Phi)\) this reduces to the Brans-Dicke theory. We will assume that both the matter energy density \(\rho\) and \(V(\Phi)\) are non-negative (a negative potential would lead to a negative cosmological constant). It is also reasonable to require that \(\omega(\Phi)\) be positive to avoid the instabilities and quantum mechanical inconsistencies associated with negative kinetic and gradient energy terms (actually, only the weaker condition \(\omega \geq -3/2\) is needed). We do not consider the possibility of terms of second or higher order in the curvature; for the case of second-order terms, the theory can be reformulated as Einstein gravity with an additional field \[13\] and an analysis similar to ours can be applied.

This action leads to a Friedmann equation of the form

\[
H^2 = \frac{8\pi (\rho + V)}{3\Phi} - H \left( \frac{\dot{\Phi}}{\Phi} \right) + \frac{\omega}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - \frac{k}{R^2}. \tag{4.3}
\]
It is convenient to rewrite this as
\[
(H + \frac{1}{2}\dot{\Phi})^2 = \frac{8\pi(\rho + V)}{3\Phi} + \frac{1}{6}(\omega + \frac{3}{2})\left(\frac{\dot{\Phi}}{\Phi}\right)^2 - \frac{k}{R^2}. \tag{4.4}
\]

The quantity appearing on the left-hand side of this equation is
\[
H + \frac{1}{2}\dot{\Phi} = \frac{d}{dt}\ln(Rm_{Pl}) = -\frac{d}{dt}\ln(T/m_{Pl}); \tag{4.5}
\]
where the second equality follows from the assumption of adiabaticity. Hence, during epochs where the right-hand side of Eq. (4.4) is nonzero, the quantity \(T/m_{Pl}\) must evolve monotonically. For \(k \leq 0\), the right-hand side can never be negative, and so the variation of \(T/m_{Pl}\) is always monotonic. Since \(T/m_{Pl}\) has certainly been decreasing since the time of nucleosynthesis, it must always have been doing so; we thus have our result for an open or flat Universe.

The proof for a closed Universe requires more work. With \(k > 0\), the right-hand side of Eq. (4.4) has no definite sign, and so the Universe might have alternated between eras of increasing and decreasing \(T/m_{Pl}\). Suppose the current era of decreasing \(T/m_{Pl}\) began at some time \(t_\ast\), after the early era of rapid Planck mass variation and before nucleosynthesis. The vanishing of the right-hand side of Eq. (4.4) at \(t = t_\ast\) implies
\[
\frac{k}{R^2(t_\ast)} \geq \frac{8\pi}{3} \frac{\rho_{rad}(t_\ast)}{m_{Pl}^2(t_\ast)}; \tag{4.6}
\]
These quantities can be related to the corresponding quantities at the time of nucleosynthesis by using adiabaticity and the fact that \(T/m_{Pl}\) has been decreasing since \(t = t_\ast\) to obtain
\[
\frac{k}{R^2(t_{BBN})} \geq \frac{8\pi}{3} \frac{\rho_{rad}(t_{BBN})}{m_{Pl}^2(t_{BBN})}. \tag{4.7}
\]
This last inequality is false, since at the time of nucleosynthesis the curvature term in the Friedmann equation was in fact much smaller than the radiation energy density. Hence, the assumption of the existence of a time \(t_\ast\) must be abandoned, and we have proven our result.
4.2 Conformal transformation to Einstein gravity

It is instructive to use a conformal transformation to rewrite our action with a constant Planck mass, but time-varying particle masses. In the conformal frame, changing the sign of $d(T/m_p)/dt$ is equivalent to constructing a cosmological model that bounces. The conformal transformation is accomplished by defining a new metric

$$\tilde{g}_{\mu\nu} = \frac{\Phi}{m_{Pl,0}^2} g_{\mu\nu}. \quad (4.8)$$

When expressed in terms of this metric and the corresponding Ricci scalar $\tilde{R}$, the action of Eq. (4.2) becomes (after an integration by parts)

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[ -\frac{m_{Pl,0}^2}{16\pi} \tilde{R} + \left( \omega + \frac{3}{2} \right) \frac{m_{Pl,0}^2}{16\pi \Phi^2} \partial_{\mu} \Phi \partial^{\mu} \Phi 
- \frac{m_{Pl,0}^4}{\Phi^2} V(\Phi) + \frac{m_{Pl,0}^4}{\Phi^2} \tilde{L}_{\text{matter}} \right]. \quad (4.9)$$

($\tilde{L}_{\text{matter}}$ has further $\Phi$-dependence because of the metric factors which it contains.) In this frame, $\Phi$ is no longer the inverse of the effective gravitational constant, but simply another matter field whose contributions to the energy density and pressure (assuming spatial homogeneity) are

$$\rho_\Phi = \left( \omega + \frac{3}{2} \right) \frac{m_{Pl,0}^2}{16\pi \Phi^2} \left( \frac{d\Phi}{dt} \right)^2 + \frac{m_{Pl,0}^4}{\Phi^2} V(\Phi); \quad (4.10)$$

$$p_\Phi = \left( \omega + \frac{3}{2} \right) \frac{m_{Pl,0}^2}{16\pi \Phi^2} \left( \frac{d\Phi}{dt} \right)^2 - \frac{m_{Pl,0}^4}{\Phi^2} V(\Phi). \quad (4.11)$$

The time and scale factor for the transformed metric are related to those for the original metric by $d\tilde{t}/dt = \sqrt{\Phi/m_{Pl,0}^2}$ and $\tilde{R} = \sqrt{\Phi/m_{Pl,0}^2} R$, so the Hubble parameter for the transformed metric is

$$\tilde{H} \equiv \frac{1}{\tilde{R}} \frac{d\tilde{R}}{dt} = \frac{m_{Pl,0}}{\sqrt{\Phi}} \left( H + \frac{1}{2} \frac{\dot{\Phi}}{\Phi} \right). \quad (4.12)$$

We recognize Eq. (4.4) as the Friedmann equation, in standard form, in the new frame. Further, $\tilde{H}$ is, up to to a positive numerical factor, equal to
Thus, to have the present era of decreasing $T/m_{\text{Pl}}$ preceded by a period in which $T/m_{\text{Pl}}$ increased, $\dot{R}$ would have to first decrease and then increase. However, an adiabatically expanding FRW Universe with positive energy density cannot undergo such a bounce, as can be shown by a simple modification of the arguments we have given above.

The conformal transformation allows us to close a small loophole. In showing that an adiabatic solution to the horizon problem requires $T/m_{\text{Pl}}$ to increase, we used the bound $H_1 \gtrsim T_1^2/m_{\text{Pl}}$. One might worry about the potentially negative contribution of the $(\dot{\Phi}/\Phi)H$ term in Eq. (4.3), though this term is expected to be positive since $m_{\text{Pl}}$ is decreasing. In any case, it is simple to show that $H_1/\sqrt{8\pi p_{\text{rad}}/3m_{\text{Pl}}^2} \geq [1 + (3/2\omega)]^{-1/2}$ for $k \leq 0$. Thus, a problem arises only if $\omega$ is very small and $m_{\text{Pl}}$ is increasing. We could start with Eq. (2.1) and redo our analysis; it is simpler to work in the conformal frame where we have an ordinary FRW model, with an additional energy term, cf. Eq. (4.10). Here it is easy to see that there cannot be an adiabatic solution to the horizon problem. Since the existence of a horizon problem is independent of frame, we have our result.

5 Concluding Remarks

We have shown that a truly satisfactory dynamical resolution of the horizon and flatness problems associated with the standard cosmology requires both superluminal (or very close to it) expansion and massive entropy production. In general, we have shown that an adiabatic scenario cannot solve the horizon problem unless the Universe was very flat to begin with. Further, for a large class of adiabatic scenarios, those based on scalar-tensor theories, we have shown directly how adiabaticity by itself precludes a solution to the horizon problem regardless of how flat the Universe is.

Because both superluminal expansion and entropy production are necessary to solve in a satisfactory manner the horizon and flatness problems and because they are the two generic features of all inflationary models, it is suggestive to conclude that inflation, or something very similar, provides the only dynamical solution to these vexing cosmological problems.
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