Nonseparable CCR algebras

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Abstract

Extending a result of the first author and Katsura, we prove that for every UHF algebra \( A \) of infinite type, in every uncountable cardinality \( \kappa \) there are \( 2^{\kappa} \) nonisomorphic approximately matricial C*-algebras with the same \( K_0 \) group as \( A \). These algebras are group C*-algebras ‘twisted’ by prescribed canonical commutation relations (CCR), and they can also be considered as nonseparable generalizations of noncommutative tori.

Keywords: Nonseparable C*-algebras, canonical commutation relations, noncommutative tori.

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The inspiration for the research presented here comes from a question of Dixmier (\cite{Dixmier}), who asked whether Glimm’s result that every separable, unital, AM (also called matroid) C*-algebras is UHF (\cite{Glimm}) extends to the nonseparable case. (A C*-algebra is \textit{approximately matricial}, or AM, if it is an inductive limit of full matrix algebras. It is \textit{uniformly hyperfinite}, or UHF, if it is a tensor product of full matrix algebras.) The question stated in this form was answered in \cite{FarahKatsura}. The main result of Glimm’s paper was classification of separable UHF algebras using a smooth invariant that we now describe. For every separable, unital, AM algebra \( A \) there exists a sequence \( k(j) \in \mathbb{N} \cup \{\infty\} \), for \( j \in \mathbb{N} \), such that (with \( p(j) \), for \( j \in \mathbb{N} \) being an enumeration of the primes and \( M_{p^{\infty}}(\mathbb{C}) \) denoting \( \bigotimes_{\mathbb{N}} M_p(\mathbb{C}) \))

\[
A \cong \bigotimes_j M_{p(j)^{k(j)}}(\mathbb{C}).
\]
Thus the sequence \((k(j) | j \in \mathbb{N})\), identified with the generalized integer (also called ‘supernatural number’)

\[ n_A = \prod_j p(j)^{k(j)} \]

is a complete isomorphism invariant for separable, unital, AM algebras. A generalized integer \(n\) corresponds to the \(K_0\) group of \(A\), equal to (with the convention that \(l > 0\) divides \(n = \prod_j p(j)^{k(j)}\) if for every prime \(p(j)\), \(p(j)^k\) divides \(l\) implies that \(k \leq k(j)\))

\[ \mathbb{Z}[1/n] = \{ k/l | k \in \mathbb{Z}, l > 0, l \text{ divides } n \}. \]

In [16] it was shown that for every uncountable cardinal \(\kappa\) there are \(2^\kappa\) nonisomorphic AM algebras of density character \(\kappa\) with the same \(K_0\) as the CAR algebra \(M_{2^\infty}\). Since AM algebras are monotracial and have trivial \(K_1\) groups, these algebras have the same Elliott invariant as \(M_{2^\infty}\) (see e.g., [28]). These algebras were universal C*-algebras given by generators and relations. The relations were coded by a graph, in which the vertices corresponded to the generators (self-adjoint unitaries) while the edges determined which of the generators anticommute (i.e., satisfy the relation \(vw = -wv\)).

The question whether the analogous result can be proven for other UHF algebras in place of \(M_{2^\infty}\) remained open. Resolving it required a convenient coding of canonical commuting relations between non-self-adjoint unitaries.

**Theorem 1.** Suppose that \(A\) is a UHF algebra such that \(p^\infty\) divides \(n_A\) for some prime \(p\). Then for every uncountable cardinal \(\kappa\) there are \(2^\kappa\) nonisomorphic AM algebras of density character \(\kappa\) with the same \(K_0\) group, and even the same Elliott invariant \([\kappa]\) as \(A\).

In the proof of this theorem we use twisted group C*-algebras associated with canonical commutation relations (CCR). These algebras can also be considered as generalizations of noncommutative tori (see e.g., [3, §II.10.7.5], [27], or [25]).

### 1. Canonical commutation relations

In this section we introduce C*-algebras given by canonical commutation relations (CCR, not to be confused with completely continuous representations). Suppose that \(X\) is a set and

\[ \Gamma = \bigoplus_{\xi \in X} C_\xi \]

is a direct sum of cyclic groups. Such \(\Gamma\) can be coded by a function \(f_\Gamma : X \to \{2, 3, \ldots, n_0\}\) such that \(f_\Gamma(\xi) = |C_\xi|\) for all \(\xi\). Thus \(\Gamma\) can be presented as

\[ (g(\xi) : \xi \in X | g(\xi)f_\Gamma(\xi) = 1, g(\xi)g(\eta) = g(\eta)g(\xi) : \xi \in X, \eta \in X). \]

\(^a\)For the definitions of \(K_0\) and the Elliott invariant see e.g., [28]. It will be used only in Lemma 4.1 and even there only implicitly.
Throughout this paper, we adopt the convention that a group $\Gamma$ has a decomposition into a direct sum of cyclic groups as in \([11]\), and that $g(\xi)$ denotes a fixed generator of $C\xi$.

**Definition 1.1.** A CCR triple is a triple $(X, \Gamma, \Theta)$ where $X$ is a set with a fixed linear (i.e., total) ordering, $\Gamma = \bigoplus_{\xi \in X} C\xi$ is a direct sum of cyclic groups, and $\Theta : X^2 \to \mathbb{T}$ is such that for all $\xi$ and $\eta$ in $X$ the following requirements are met.

\[
\begin{align*}
(CCR.1) \quad \Theta(\xi, \xi) &= 1, \\
(CCR.2) \quad \Theta(\xi, \eta) &= \overline{\Theta(\eta, \xi)}, \\
(CCR.3) \quad \Theta(g(\xi)^m, g(\eta)^n) &= 1, \quad \text{where } \gcd(m, n) \text{ denotes the greatest common divisor of } m \text{ and } n.
\end{align*}
\]

The index-set $X$ is included in the CCR triple $(X, \Gamma, \Theta)$ as a reminder that $\Gamma$ is taken with a fixed decomposition as in \([11]\). The linear ordering on $X$ is used only in Lemma \([12]\) and it will be suppressed throughout. Typically, $X$ will be a cardinal, a subset of a cardinal, and in any case it will be equipped with a natural linear ordering.

The function $\Theta$ gives canonical commutation relations on $\Gamma$. In the literature (see \([3]\, \text{II.10.7.5}, \ [27], \ or \ [25]\), $\Theta$ is usually given indirectly, by a skew-symmetric matrix $\theta$, so that $\Theta(\xi, \eta) = \exp(\pi i \theta(\xi, \eta))$. In the existing literature, $X$ is finite and the order of each generator $g(\xi)$ is assumed to be infinite. This data gives a noncommutative torus. In this situation, condition \([CCR.3]\) is unnecessary and the following example is included here to justify it.

**Example 1.1.** Suppose that unitaries $u$ and $v$ in a C*-algebra $A$ satisfy $u^n = 1 = v^n$ and $uv = \lambda v$. Then $\lambda = 1$. To see this, note that by induction we have $u^nv = \lambda^nv$ and therefore $v = \lambda v$, hence $\lambda = 1$. A proof that $\lambda = 1$ is analogous. Since $\gcd(m, n)$ is an integral linear combination of $m$ and $n$, the conclusion follows.

In Proposition \([11]\) we will prove that conditions \([CCR.1], [CCR.3]\) are sufficient for the existence of a CCR algebra $A_{(X, \Gamma, \Theta)}$, in the sense that these relations are satisfied by some choice of unitaries on a Hilbert space of the appropriate order. In order to prove this, we will need the following (well-known) lemma.

**Lemma 1.1.** Suppose $\lambda \in \mathbb{T}$ and $n \geq 2$. If $\lambda$ is a primitive $n$th root of unity, then $M_n(\mathbb{C})$ is generated by unitaries $v_n$ and $w_n$ such that $v_nw_n = \lambda w_nv_n$ and each one of $v_n$ and $w_n$ has order $n$.

If $\lambda \neq 1$ for all $n \geq 1$, there are unitaries of infinite order $v_n$ and $w_n$ on a separable Hilbert space such that $v_nw_n = \lambda w_nv_n$.

**Proof.** Take $w_n = \text{diag}(1, \lambda, \ldots, \lambda^{n-1})$ and take $v_n$ to be the permutation unitary matrix such that $(v_n)_{i,i+1} = 1$ for $1 \leq i < n$ and $(v_n)_{n,1} = 1$, with all other entries equal to 0. Then (with $\xi_i$, for $1 \leq i \leq n$ denoting the standard basis of $\mathbb{C}^n$) we have $v_n\xi_i = \xi_{i+1}$ if $i < n$ and $v_n\xi_n = \xi_1$. Thus $v_nw_nv_n^*(\xi_i) = \lambda^i \xi_i$, and $v_nw_nv_n^* = \lambda w_n$. 
It remains to prove that $u_n$ and $v_n$ generate $M_n(\mathbb{C})$. We will prove that $u^k v^l$, for $0 \leq k, l < n$, are linearly independent. Otherwise, there are $k < n$ and $l < n$ such that $u^k v^l = \sum_{i<n,j<n} \lambda_{i,j} u^i v^j$, with $\lambda_{k,l} = 0$. By multiplying with $(u^*)^k$ on the left and $(v^*)^l$ on the right, we obtain $1 = \sum_{i<n,j<n} \lambda_{i,j} u^i v^j$, with $\lambda_{0,0} = 0$. With $\tau$ denoting the unique tracial state on $M_n(\mathbb{C})$, we have $\tau(u^i v^j) \neq 0$ if and only if $i = j = 0$; contradiction. Therefore $C^*(u,v)$ has dimension (at least) $n^2$, and it is therefore equal to $M_n(\mathbb{C})$.

Now suppose that $\lambda^* \neq 1$ for all $n$. In this case we use the generators of the irrational rotation algebra associated with $\lambda$ (see e.g., [3] §II.8.3.3 (i)). More precisely, let $v_0$ denote the standard generator of $C(\mathbb{T})$ (i.e., $v_0$ is a unitary whose spectrum is equal to $\mathbb{T}$). Let $\alpha$ be the rotation of $\mathbb{T}$ by angle $\theta$, chosen so that $\lambda = e^{i\theta}$. In the crossed product $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$, let $u_0$ be the unitary that implements $\alpha$. Then $u_0 v_0 u_0^* = \lambda v_0$, as required. Since the irrational rotation algebras are simple, any nondegenerate representation of $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$ on a Hilbert space yields $u_0$ and $v_0$ as required.

The proof of the following is based on the proof of [9] Proposition 10.1.3 (1)).

**Proposition 1.1.** For every CCR triple $(X, \Gamma, \Theta)$, there exists a universal $C^*$-algebra $\Lambda_{X,\Gamma,\Theta}$ given by generators $u_\xi$, for $\xi \in X$ and the following relations (the order function $f_\Gamma$ was defined after (1.1)):

$$
\mathcal{R}(\Gamma, \Theta) = \{ u_\xi u_\zeta^* = 1 = u_\zeta^* u_\xi, u_\xi u_\eta = \Theta(\xi, \eta) u_{\eta} u_\xi | \xi, \eta \in X \} \\
\cup \{ u_{\xi}^n = 1 | \xi \in X, n = f_\Gamma(\xi) \text{ is finite} \}
$$

In this algebra, the order of the unitary $u_\xi$ is equal to the order of the generator $g(\xi)$ in $\Gamma$.

**Proof.** Since all of the generators are unitaries (and therefore of norm 1), it suffices to prove that every $F \subseteq \mathcal{R}(\Gamma, \Theta)$ b is represented in some $C^*$-algebra such that for every $\xi$ such that $u_\xi$ occurs in $F$ the order of $u_\xi$ is equal to the order of $g(\xi)$ (this is well-known, see e.g., [9] Lemma 2.3.11)). Let $G = \{ \xi | g(\xi) \text{ occurs in } F \}$. We can re-enumerate $G$ and identify it with $n = \{ 0, \ldots, n - 1 \}$, with $n = |G|$. Let $\Gamma' = \prod_{i<n} C_i$.

For $i < j < n$, let $H_{i,j}$ be the Hilbert space of dimension (with $\aleph_0 = \min \emptyset$)

$$
d(i,j) = \begin{cases} 
\min \{ d(\Theta(g(i),g(j)), 1 \}, & \text{if } i < j. \\
\min \{ d(g(i)), 1 \}, & \text{if } i = j.
\end{cases}
$$

Let $H = \bigoplus_{0 \leq i \leq j < n} H_{i,j}$. Fix $k < n$. For $i < j < n$ we will define a unitary $u_{i,j,k}$ on

b By $F \subseteq X$ we abbreviate the assertion that $F$ is a finite subset of $X$. 
Claim 1.1. We have $u_k$ on $\Gamma$ and that this bicharacter (also denoted $\Theta$) controls the commutation on all $u_i,j,k$. Suppose that $(\xi, \eta, \rho) \in (\mathbb{R}^+)^3$. Let $u_k = \bigoplus_{i < j < k} u_{i,j,k}$. This is a unitary whose order is the least common multiple of the orders of $u_i,j,k$, for $i < j < k$. These orders range over the orders of $\Theta(g(k), g(l))$, for $l < n$, and the order of $g(k)$ (in the case when $i = j = k$). Therefore (CCR.3) implies that $u_k$ is a unitary of the same order as $g(k)$.

Proof. If $k = l$ then this is a consequence of (CCR.1) $\Theta(\Theta(g(k), g(l))) = 1$. Assume $k \neq l$. Fix for a moment $i < j < n$. If $\{i, j\} \neq \{k, l\}$, then a glance at the definitions of $u_{i,j,k}$ and $u_{i,j,l}$ confirms that $u_{i,j,k}$ and $u_{i,j,l}$ commute. Therefore for $k < l$ we have that $u_k u_l = \lambda_{i,l} u_k$ if and only if $u_{k,l,k} u_{k,l,l} = \lambda_{k,l,l} u_{k,l,k}$.

Fix $i < j$ and suppose that $\{i, j\} = \{k, l\}$. If $k < l$, then the choice of $u_{i,j,k}$ and $u_{i,j,l}$ implies that $u_{i,j,k} u_{i,j,l} = \Theta(g(k), g(l)) u_{i,j,l} u_{i,j,k}$. Similarly, if $k > l$, then the definition and (CCR.2) imply

$$u_{i,j,k} u_{i,j,l} = \Theta(g(k), g(l)) u_{i,j,l} u_{i,j,k} = \Theta(g(k), g(l)) u_{i,j,l} u_{i,j,k}.$$

By the previous paragraph, this completes the proof. $\square$

This completes the proof that every $F \in \mathcal{R}(\Gamma, \Theta)$ is represented in some C*-algebra. By [9, Lemma 2.3.11], there exists a universal C*-algebra given by generators $G(\Gamma)$ and relations $\mathcal{R}(\Gamma, \Theta)$. By construction, the order of $u_{\xi}$ is equal to the order of $g(\xi)$ for all $\xi$.

Suppose that $(X, \Gamma, \Theta)$ is a CCR triple. We associate elements of $\Gamma$ with finite products of the form

$$\prod_{\xi} g(\xi)^{m(\xi)}$$

where $0 \leq m(\xi) < f_{\Gamma}(\xi)$ for all $\xi$ and $m(\xi) = 0$ for all but finitely many $\xi$. For $g$ as in (1.2), in $A_{(X, \Gamma, \Theta)}$ consider the unitary

$$u_g = \prod_{\xi} u_{\xi}^{m(\xi)}$$

(in this product the generators $u_{\xi}$ are taken in the order increasing in the previously fixed linear ordering of $X$). In Lemma 1.2 we show how to extend $\Theta$ to a bicharacter on $\Gamma$ and that this bicharacter (also denoted $\Theta$) controls the commutation on all $u_g$, for $g \in \Gamma$.

Lemma 1.2. Suppose that $(X, \Gamma, \Theta)$ is a CCR triple. Extend $\Theta$ to $\Gamma$ by letting

$$\Theta(\prod_{\xi} g(\xi)^{m(\xi)}, \prod_{\eta} g(\eta)^{n(\eta)}) = \prod_{\xi, \eta} \Theta(\xi, \eta)^{m(\xi)n(\eta)}.$$
Then the unitaries in $\Lambda(X,\Gamma,\Theta)$ defined in (1.3) satisfy
\[ u_g u_h = \Theta(u_g, u_h) u_h u_g. \]

**Proof.** The proof is a straightforward induction on the sum of the lengths of the words $g$ and $h$. \hfill \square

**Remark 1.1.** ‘Twisted’ $C^*$-algebras associated to an abelian group and a bicharacter have been studied in [34] and [22]. Instead of CCR relations given by a bicharacter $\Theta$ and
\[ u_g u_h = \Theta(g, h) u_h u_g, \]
as in our setting, these authors started from an appropriate function $b: \Gamma^2 \to T$ and considered the relations
\[ u_g u_h = b(g, h) u_h u_g. \]
As pointed out e.g., in [25, Remark 1.2] or [26, Section 4], our $A(X,\Gamma,\Theta)$ can be recast as a $C^*$-algebra given by $\Gamma$ and $b$ and vice versa. In both [34, Theorem 3.6] and [23, §2.2.3] it was proven that if $b$ is nondegenerate (i.e., if $b(g, h) = 1$ for all $h \in \Gamma$ implies $g = e$) then the algebra associated to $\Gamma$ and $b$ is simple. Using this result would have shortened our arguments somewhat, but we chose to give a self-contained presentation.

## 2. Functoriality

Consider the category of CCR triples $(X,\Gamma,\Theta)$. The morphisms in this category, $\varphi: (X,\Gamma,\Theta) \to (X',\Gamma',\Theta')$, are injective homomorphisms from $\Gamma$ into $\Gamma'$ which preserve the CCR relations (as extended to the entire group in Lemma 1.2, i.e., $\Theta'(|\varphi(g), \varphi(h)|) = \Theta(g, h)$, for all $g$ and $h$ in $\Gamma$. We will prove that the universal construction $\Lambda(X,\Gamma,\Theta)$ is functorial in the subcategory of CCR triples associated with locally finite groups.

**Proposition 2.1.** If $\Gamma$ is a locally finite group and $(X,\Gamma,\Theta)$ and $(X',\Gamma',\Theta')$ are CCR triples, then every morphism
\[ \varphi: (X,\Gamma,\Theta) \to (X',\Gamma',\Theta') \]
gives a unique injective *-homomorphism $\Psi_\varphi: \Lambda(X',\Gamma',\Theta') \to \Lambda(X,\Gamma,\Theta)$ such that $\Psi_\varphi(u_g) = u_{\varphi(g)}$ for all $g \in \Gamma$.

The proof of Proposition 2.1 will be given after some preliminaries.

**Definition 2.1.** For a CCR triple $(X,\Gamma,\Theta)$, let $C_\Theta \Gamma$ denote the algebra of all finite linear combinations $\sum_g \lambda_g u_g$, with the multiplication ‘twisted’ by $\Theta$, as
\[ u_g u_h = \Theta(g, h) u_h u_g. \]
A group is called **locally finite** if every finitely generated subgroup is finite. For abelian groups this is equivalent to every element having finite order.
This is a complex algebra dense in $A(\mathcal{E}(X, \Gamma, \Theta))$.

The proof of Lemma 2.1 is based on the proofs of [25] Lemma 1.5 and Theorem 1.9, where its special case when $\Gamma$ is $\mathbb{Z}^n$ for some $n \geq 2$ was considered. An alternative proof of this lemma is analogous to the proof of [9 Lemma 10.1.3].

**Lemma 2.1.** Every CCR algebra $A(\mathcal{E}(X, \Gamma, \Theta))$ has a tracial state $\tau$ such that (c denotes the identity element in $\Gamma$)

$$\tau(\sum_g \lambda_g u_g) = \lambda_c$$

(2.1)

for all $\sum_g \lambda_g u_g$ in $\mathbb{C} \Theta \Gamma$. If $\Gamma$ is locally finite, then $\tau$ is faithful.

**Proof.** Fix $F \subseteq X$ and let $\Gamma' = \bigoplus_{i \in F} C_i$. By re-enumerating we may assume $F = \{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$. For $m < n$ let $A_m = C^*(\{u_i | i \leq m\})$ and let $\Gamma_m$ denote the subgroup of $\Gamma$ generated by $\{g(i) | i \leq m\}$. Then $A_0$ is naturally isomorphic to $C(\hat{\Gamma}_0)$, where $\hat{\Gamma}_0$ is the Pontryagin dual of $\Gamma_0$. Let $\tau_0$ be the tracial state on $A_0$ such that $\tau(1) = 1$ and $\tau_0(u_{g(0)}) = 0$. A simple Cauchy–Schwarz argument shows that this uniquely defines $\tau_0$, and that it agrees with (2.1). By induction we will prove that for every $1 \leq j \leq n$ there is a tracial state $\tau_j$ on $A_j$ such that $\tau_j | A_{j-1} = \tau_{j-1}$ and $\tau_j$ satisfies (2.1).

Suppose that $0 \leq j < n$ and $\tau_j$ as required has been constructed so that it satisfies (2.1). We will prove that it is a tracial state. The pertinent fact is that for every $g \in \Gamma$ we have $u_g u_{g^{-1}} = u_c = u_{g-1} u_g$. For $\sum_g \lambda_g u_g$ and $\sum_g \gamma_g u_g$ in $\mathbb{C} \Theta \Gamma_j$ we have

$$\tau_j(\sum_g \lambda_g u_g)(\sum_h \gamma_h u_h) = \sum_{gh = h} \lambda_g \gamma_h = \tau(\sum_h \gamma_h u_h)(\sum_g \lambda_g u_g)).$$

Since $\mathbb{C} \Theta \Gamma_j$ is dense in $A_j$, this shows that $\tau_j$ is tracial.

Conjugation by $u_{g(j+1)}$ induces an automorphism $\alpha = \alpha_{j+1}$ of $A_j$, since for every $h$ we have $u_g u_{g(j+1)} u_{g(j+1)} = \Theta(g(j+1), h) u_h$. The algebra $A_{j+1}$ is therefore isomorphic to the crossed product $A_j \rtimes_{\alpha_j} C_j$ (recall that $C_j$ is a, possibly finite, cyclic group). A glance at (2.1) reveals that $\alpha$ is $\tau_j$-preserving and therefore we can find $\tau_{j+1}$ as required.

We have proved that, for every $F \subseteq X$, with $\Gamma' = \bigoplus_{i \in F} C_i$, (2.1) defines a tracial state $\tau_{\Gamma'}$ on the $C^*$-subalgebra $C^*(u_g | g \in \Gamma')$ of $A(\mathcal{E}(X, \Gamma, \Theta))$. Since $\Gamma'$ is a direct limit of its finitely generated subgroups and the tracial states $\tau_{\Gamma'}$ are compatible (in the sense that they agree on $\mathbb{C} \Theta \Gamma$, when defined), this shows that (2.1) defines a tracial state $\tau$ on $A(\mathcal{E}(X, \Gamma, \Theta))$. It remains to verify that $\tau$ is faithful if $\Gamma$ is locally finite. Fix a finitely generated subgroup $\Gamma'$ of $\Gamma$. Since $\Gamma$ is locally finite, $\Gamma'$ is finite. Therefore every element of $A(\Gamma') = C^*(u_g | g \in \Gamma')$ is a finite linear combination of the canonical unitaries, and we can identify $A(\Gamma')$ with $\mathbb{C} \Theta \Gamma'$. To prove that $\tau$ is faithful on $A(\Gamma')$, fix a nonzero $\sum_g \lambda_g u_g$ in $\mathbb{C} \Theta \Gamma'$. Then (again using the fact that $u_g$ and $u_{g^{-1}}$ commute)

$$\tau_j((\sum_g \lambda_g u_g)^* (\sum_g \lambda_g u_g)) = \sum_g |\lambda_g|^2 > 0.$$
Therefore the restriction of $\tau$ to $A(\Gamma')$ is faithful for every finitely generated subgroup $\Gamma'$ of $\Gamma$. Let

$$J = \{a \in A_{(X, \Gamma, \Theta)} | \tau(a^* a) = 0\}.$$  

Since $\tau$ is a tracial state, $J$ is a two-sided, norm-closed, ideal ([9, Lemma 4.1.3]). We have proved that $A_{(X, \Gamma, \Theta)}$ is an inductive limit of $C^*$-subalgebras $A(\Gamma')$ for finite $\Gamma' \leq \Gamma$ such that the intersection of $J$ with each one of them is trivial. By a well-known property of $C^*$-algebras ([9, Proposition 2.5.3]), an ideal of an inductive limit is the inductive limit of its intersections with the algebras comprising the inductive system. Since $J \cap A(\Gamma')$ is trivial for all $\Gamma'$, this concludes the proof.

For a CCR triple $(X, \Gamma, \Theta)$ (with $\Gamma$ represented as a direct sum, as in (1.1)) and $F \subseteq X$, we consider the group

$$\Gamma_F = \bigoplus_{\xi \in F} C_\xi$$  

and the CCR triple $(F, \Gamma_F, \Theta_F)$.

**Lemma 2.2.** Suppose that $(X, \Gamma, \Theta)$ is a CCR triple and $\Gamma$ is locally finite.

1. If $\Gamma$ is finite then $A_{(X, \Gamma, \Theta)}$ is a finite-dimensional $C^*$-algebra, and its dimension is $|\Gamma|$.
2. The $C^*$-subalgebra $C^*(u_g | g \in \Gamma_F)$ of $A_{(X, \Gamma, \Theta)}$ is isomorphic to

$$A_F = A_{(F, \Gamma_F, \Theta | \Gamma_F)}$$  

for all $F \subseteq X$.
3. $A_{(X, \Gamma, \Theta)}$ is an AF algebra, and it is an inductive limit of the algebras $A_F$, for $F \subseteq X$.

**Proof.**

1. Let $u_g$, for $g \in \Gamma$, be the unitaries as defined in Lemma 1.2. Then $A_{(X, \Gamma, \Theta)}$ is the linear span of $u_g$, for $g \in \Gamma$, and therefore it is finite-dimensional of dimension at most $|\Gamma|$. For the converse inequality, we need to prove that $u_g$, for $g \in \Gamma$, are linearly independent. Assume otherwise, so that for some $g$ and scalars $\alpha_f$ we have $u_g = \sum_{f \neq g} \alpha_f u_f$. By multiplying the equation with $u_g^*$, we obtain $u_e = \sum_{f \neq e} \beta_f u_f$. However, the tracial state $\tau$ defined in Lemma 2.1 satisfies $\tau(u_e) = 1$ while $\tau(u_f) = 0$ for all $f \neq e$; contradiction.

2. In order to prove that $C^*(u_g | g \in \Gamma_F)$ isomorphic to $A_F$, it suffices to show that the $u_g$ are linearly independent. This follows by the argument using $\tau$ as in (1).

3. is an immediate consequence of (2).
The completion of the algebra with respect to this sesquilinear form is of course the GNS Hilbert space corresponding to $\tau$, and the associated norm

$$\|a\|_{2,\tau} = \langle a, a \rangle^{1/2}$$

is dominated by the operator norm (because $\|\tau\| = 1$).

**Lemma 2.3.** In every CCR algebra $A = A_{(X, \Gamma, \Theta)}$ such that $\Gamma$ is locally finite the unitaries $u_g$, for $g \in \Gamma$, form an orthonormal basis for the pre-Hilbert space $(A, \langle \cdot, \cdot \rangle_{\tau})$.

**Proof.** Let $\ell_2(\Gamma)$ denote the Hilbert space with the orthonormal basis $\delta_g$, for $g \in \Gamma$. The linear map that sends $\sum_j \lambda_j u_{g(j)}$ to $\sum_j \lambda_j \delta_{g(j)}$ is an isometry from $(A, \| \cdot \|_2)$ into $\ell_2(\Gamma)$. This is because (using the fact that $\tau(u_g) = 0$ if $g \neq e$ and $\tau(u_e) = 1$)

$$\| \sum_j \lambda_j u_{g(j)} \|_2^2 = \tau((\sum_j \lambda_j u_{g(j)})^* (\sum_j \lambda_j u_{g(j)})) = \sum_j |\lambda_j|^2.$$

Since these linear combinations are dense in $\ell_2(\Gamma)$, this linear map extends to an isometry from the $\| \cdot \|_2$-completion of $(A, \langle \cdot, \cdot \rangle_{\tau})$ onto $\ell_2(\Gamma)$, and $\delta_g$, for $g \in \Gamma$, is an orthonormal basis for $\ell_2(\Gamma)$. \qed

**Proof of Proposition 2.1** Fix locally finite groups $\Gamma$ and $\Gamma'$ and a morphism $\varphi: (X, \Gamma, \Theta) \to (X', \Gamma', \Theta')$. Let $u_g$, for $g \in \Gamma$, denote the canonical unitaries in $A_{(X, \Gamma, \Theta)}$ and let $v_g = u_{g(\varphi(\xi))}$, for $g \in \Gamma$, denote the corresponding unitaries in $A_{(X', \Gamma, \Theta')}$. Consider the map $u_g \mapsto v_g$ and let $\Psi_0$ be its extension to a linear map from $C_0 \Gamma$ into $C_0 \Gamma'$. We need to prove that $\Psi_0$ extends to a $^*$-homomorphism $\Psi_{\varphi}: A_{(X, \Gamma, \Theta)} \to A_{(X', \Gamma, \Theta')}$. Since $\Gamma$ is locally finite, $A_{(X, \Gamma, \Theta)}$ is by Lemma 2.2 an inductive limit of finite-dimensional subalgebras $A_F$, each of which is a linear span of $\{u_g | g \in \Gamma_F\}$, for $F \in X$. For $F \in X$ we have $A_F \subseteq C_0 \Gamma$, hence the restriction of $\Psi_1$ to $A_F$ is a $^*$-homomorphism between $C^*$-algebras, and therefore has norm 1 ([9] Lemma 1.2.10]). Since $g \neq g'$ implies $v_g \neq v_{g'}$, the restriction of $\Psi_1$ to $A_F$ is injective. By Lemma 2.2, $A_{(X, \Gamma, \Theta)}$ is the inductive limit of algebras $A_F$, and therefore $\Psi_1$ extends to a $^*$-homomorphism $\Psi: A_{(X, \Gamma, \Theta)} \to A_{(X', \Gamma, \Theta')}$. The intersection of $\ker(\Psi)$ with $A_F$ is trivial for every finite $F$, and by [9] Lemma 4.1.3], $\ker(\Psi)$ is trivial. This completes the proof. \qed

In the original version of this paper it was asserted that $\delta_g$ is a Schauder basis for $A_{(X, \Gamma, \Theta)}$. The referee pointed out that our proof of this lemma was incomplete, but that it can be replaced with the use of orthonormal basis $\delta_g$, for $g \in \Gamma$, in $\ell_2(\Gamma)$. Proposition 2.2 and Corollary 2.1 serve to formalize this argument. Analogous remarks apply to [9] Lemma 10.2.6 (1)].

If $B$ is a $C^*$-subalgebra of $A$, then a linear map $E: B \to A$ is a conditional expectation if $E(bac) = bE(a)c$ for all $a \in A$ and all $b$ and $c$ in $B$. By Tomiyama’s theorem, this is equivalent to having $E(b) = b$ for all $b \in B$ and $\|E\| = 1$ (see e.g., [4]).
The following proposition is related to \[9\] Lemma 10.2.7, where an analogous statement for graph CCR algebras, with an additional assumption that \(G\) is finite, was proved.

**Proposition 2.2.** Suppose that \(\Gamma\) is a locally finite group and \((X, \Gamma, \Theta)\) is a CCR triple. For \(G \leq \Gamma\) consider the \(C^*\)-subalgebra of \(A(X, \Gamma, \Theta)\) \[^4\]

\[
A(G) = C^*(u_g \mid g \in G).
\]

For subgroups \(G \leq H \leq \Gamma\) there exists a conditional expectation

\[
E_{HG} : A(H) \to A(G).
\]

This system of conditional expectations commutes: if \(G \leq H \leq K \leq \Gamma\) then \(E_{KG} = E_{HG} \circ E_{KH}\).

**Proof.** By Lemma \[24\] \(A = A(X, \Gamma, \Theta)\) has a faithful tracial state \(\tau\). We will use the notation \(\ell_2(G)\) and \(\langle \cdot, \cdot \rangle_\tau\) introduced in the proof of Lemma \[24\]. The GNS space associated to \(\tau\) is isomorphic to \(\ell_2(\Gamma)\) and we will identify \(A\) with its image \(\pi_\tau[A]\) under the GNS representation. For \(G \leq \Gamma\) let \(\tau_G = \tau \upharpoonright G\). Writing \(p_G\) for the projection from \(\ell_2(\Gamma)\) onto \(\ell_2(G)\), the representation

\[
\pi_G(a) = p_G \pi_\tau(a) p_G
\]

of \(A(G)\) is equivalent to the GNS representation \(\pi_{\tau_G}\) of \(A(G)\) associated with \(\tau_G\). To see this, note that for \(a \in C_\Theta[G]\) we have \(\tau_G(a) = \langle a \delta_c, \delta_c \rangle_\tau\) and that \(\delta_c\) is a cyclic vector for \(\pi_G[A(G)]\). Since \(C_\Theta[G]\) is dense in \(A(G)\), by the uniqueness of the GNS representation, \(\pi_{\tau_G}\) is equivalent to \(\pi_G\).

For \(G \leq \Gamma\) let \(\Xi : \pi_G[A(G)] \to A(G)\) be the \(*\)-isomorphism such that

\[
p_G \Xi_G(a) p_G = a
\]

for all \(a \in \pi_G[A(G)]\). This \(*\)-isomorphism will be needed after we verify the properties of the maps \(E_{HG}'\) defined in the following paragraph.

Fix \(G \leq H \leq \Gamma\) and define \(E_{HG}' : A(H) \to \mathcal{B}(\ell_2(\Gamma))\) by

\[
E_{HG}'(a) = p_G a p_G.
\]

This map is clearly linear and \(E_{HG}'(a) = \pi_G(a)\) for \(a \in H_G\) and the system of maps \(E_{HG}'\) is clearly commuting.

We will now verify that \(\|E_{HG}'(a)\| = 1\). By the density of \(C_\Theta[H]\) in \(A(H)\) it suffices to check (writing \(E' = E_{HG}'\)) \(\|E'(a)\| \leq \|a\|\) for \(a \in C_\Theta[H]\). Fix such \(a\) and \(\varepsilon > 0\).

Let \(\xi \in \ell_2(G)\) be a unit vector which satisfies \(\|E'(a)\xi\|^2 > \|E'(a)\|^2 - \varepsilon\). Since \(G\) is a subgroup and \(a \in C_\Theta[H]\), the vectors \((a - E'(a))\xi\) and \(\xi\) are orthogonal and

\[
\|a\|^2 \geq \|(a - E'(a))\xi\|^2 + \|E'(a)\xi\|^2 > \|E'(a)\|^2 - \varepsilon.
\]

\[^4\]If \(G = \Gamma_F\) as in \[22\], then \(A(G)\) is \(A_F\) as defined in the statement of Lemma \[22\].
Since $a \in A(H)$ and $\varepsilon > 0$ were arbitrary, we have $\|E'\| \leq 1$. The converse inequality follows from $E'(1) = 1$.

We prove that $E'_{HG}[A(H)] \subseteq \pi_G[A(G)]$. For $g \in G$ we have (with $\pi_G$ as in (2.3)) $p_gu_g\pi_G = \pi_G(u_g)$ and by linearity $E'_{HG}(a) = \pi_G(a)$ for all $a \in C_{\Theta|G}G$. For $h \in H \setminus G$ we have $u_h\delta_g \perp \ell_2(G)$ for all $g \in G$ and therefore $E'_{HG}(u_h) = 0$. If $G$ is a finite group, then $p_G$ has finite rank and $C_{\Theta|G}G = A(G)$, hence these computations show that $E'_{HG}(a) \in \pi_G[A(G)]$ for all $a \in A(H)$ and that $E'_{HG}(a) = \pi_G(a)$ for $a \in A(G)$. In this case, let (see (2.4))

$$E_{HG} = \Xi_G \circ E'_{EG}.$$ 

By the already verified $\|E_{HG}\| = 1$ and Tomiyama’s theorem, in the case when $G$ is finite, $E_{HG}$ is a conditional expectation.

Now consider the case when $G$ is not necessarily finite. Since $\Gamma$ is locally finite, so is $H$ and we can write it as an inductive limit of finite groups $H(\lambda)$, for $\lambda \in \Lambda$. With $G(\lambda) = G \cap H(\lambda)$, we have that $E_{HG(\lambda)}[A(H)] \subseteq A(G(\lambda))$ for every $\lambda$. For $a \in A(H)$, the net $(E_{HG(\lambda)}(a))_\lambda$ is Cauchy since it converges to $a$. By the contractivity and commutation, $(E_{HG(\lambda)}(a))_\lambda$ is a Cauchy net. Define

$$E_{HG}(a) = \lim_\lambda E_{HG(\lambda)}(a).$$

Therefore $E_{HG}(a) \in \lim_\lambda A(G(\lambda)) = A(G)$. Since $a \in A(H)$ was arbitrary, this proves $E_{HG}[A(H)] \subseteq A(G)$. It is clear that $E_{HG}(a) = a$ for $a \in A(G)$. Again, Tomiyama’s theorem implies that $E_{HG}$ is a conditional expectation as required.

Clearly, the system of the maps $E_{HG}$ is commuting, and this completes the proof.

\begin{corollary}
Suppose that $\Gamma$ is a locally finite group, $(X, \Gamma, \Theta)$ is a CCR triple, and $G \leq \Gamma$. If $a \in A_{(X, \Gamma, \Theta)}$ is such that the $\ell_2$-expansion $a = \sum \lambda_g u_g$ (as in Lemma 2.2) satisfies $\lambda_h = 0$ for all $h \in \Gamma \setminus G$, then $a \in C^*(u_g|g \in G)$.
\end{corollary}

\begin{proof}
For $a$ as in the assumptions we clearly have $E_{\Gamma G}(a) = a$.
\end{proof}

3. Properties of CCR algebras $A_{(X, \Gamma, \Theta)}$

In this section we analyze the structure of C*-algebras of the form $A_{(X, \Gamma, \Theta)}$ introduced in 3. In the following lemma and elsewhere in this paper, $\otimes$ stands for the minimal tensor product.

\begin{lemma}
Suppose that $(Z, \Gamma, \Theta)$ is a CCR triple, that $\Gamma$ is locally finite, that $Z = X \sqcup Y$ for some nonempty sets $X$ and $Y$, and that $\Theta(\xi, \eta) = 1$ if $\xi \in X$ and $\eta \in Y$. Then (using the notation of (2.2))

$$A_{(Z, \Gamma, \Theta)} \cong A_{(X, \Gamma_X, \Theta|\Gamma_X)} \otimes A_{(Y, \Gamma_Y, \Theta|\Gamma_Y)}.$$ 

\end{lemma}

\begin{proof}
The universal property and functoriality together imply that $A_{(Z, \Gamma, \Theta)} \cong A_{(X, \Gamma_X, \Theta|\Gamma_X)} \otimes A_{(Y, \Gamma_Y, \Theta|\Gamma_Y)}$ for some tensor product $\otimes_\alpha$. Lemma 2.2 implies that
these C*-algebras are AF, and therefore nuclear, hence $\otimes_\alpha$ is the minimal tensor product (see e.g., [3]).

In the following we follow von Neumann's convention and identify 2 with \{0,1\}.

Lemma 3.2. Suppose that $(\kappa \times 2, \Gamma, \Theta)$ is a CCR triple such that $\Gamma = \bigoplus_{\kappa \times 2} \mathbb{Z}/n\mathbb{Z}$ and $\Theta$ satisfies (with $\lambda = \exp(2\pi i/n)$)

$$\Theta(g(\alpha,i), g(\beta,j)) = \begin{cases} \lambda, & \text{if } \alpha = \beta, i = 0, \text{ and } j = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then $A_{(\kappa, \Gamma, \Theta)} \cong \bigotimes_\kappa M_n(\mathbb{C})$.

Moreover, if $\kappa$ is finite then every C*-algebra generated by unitaries $v_g, g \in G(\Gamma)$ that satisfy relations $R(\Gamma, \Theta)$ is isomorphic to $\bigotimes_\kappa M_p(\mathbb{C})$.

Proof. If $F \subseteq \kappa$, then $\Gamma_F = \bigoplus_{F \times 2} \mathbb{Z}/n\mathbb{Z}$ is a subgroup of $\Gamma$, and the restriction of $\Theta$ to $\Gamma_F$ entails that $A_{(F, \Gamma, \Theta|_{\Gamma_F})}$ is generated by the unitaries $v_\alpha = u_{\alpha,0}$ and $w_\alpha = u_{\alpha,1}$ such that $C^*(v_\alpha, w_\alpha)$ generate a copy of $M_n(\mathbb{C})$, and that these copies commute. Since $\Gamma$ is locally finite, by Lemma 3.1 $A_{(\Gamma_F, \Theta|_{\Gamma_F})} \cong \bigotimes_F M_n(\mathbb{C})$. Since $\Gamma$ is the direct limit of $\Gamma_F$, for $F \subseteq \kappa$, by Lemma 3.3 $A_{(\kappa, \Gamma, \Theta)}$ is isomorphic to $\bigotimes_\kappa M_p(\mathbb{C})$ as required.

We now prove that for a finite $\kappa$, every C*-algebra $A$ generated by unitaries $v_g, g \in G(\Gamma)$ that satisfy relations $R(\Gamma, \Theta)$ is isomorphic to $\bigotimes_\kappa M_p(\mathbb{C})$. The relations imply that $A$ is the closed linear span of 1 and finite products of generators taken in the lexicographic order such that each $g(\alpha,j)$ occurs fewer than $p$ times. Thus, if $\kappa$ is finite, the dimension of $A$ is $p^{2\kappa}$—i.e., equal to the dimension of $\bigotimes_\kappa M_p(\mathbb{C})$. By the universality of the latter, $A$ is isomorphic to it.

Lemma 3.3. Suppose $(\kappa, \Gamma, \Theta)$ is a CCR triple. For every $X \subseteq \Gamma$\footnote{The conclusion holds without the finiteness assumption on $\kappa$, but we will not need this fact.} $Z_{\Gamma, \Theta}(X) = \{g \in \Gamma | \Theta(x,g) = 1 \text{ for all } x \in X\}$ is a subgroup of $\Gamma$.

Proof. Suppose that $g$ and $h$ belong to $Z_{\Gamma, \Theta}(X)$. For $x \in X$ we have $\Theta(x,gh) = \Theta(x,g)\Theta(x,h) = 1$.

Similarly, $\Theta(x,g^{-1}) = \Theta(x,1)\Theta(x,g) = 1$. Since $x$ was arbitrary, this implies that $gh$ and $g^{-1}$ belong to $Z_{\Gamma, \Theta}(X)$. Since $g$ and $h$ were arbitrary, this proves that $Z_{\Gamma, \Theta}(X)$ is a subgroup of $\Gamma$.

If $A$ is a C*-subalgebra of $B$, then the relative commutant of $A$ in $B$ is $B \cap A' = \{b \in B | [a,b] = 0 \text{ for all } a \in A\}$.\footnote{In this definition we are using the extension of $\Theta$ to $\Gamma$ defined in Lemma 1.2}
The following definition is taken from [15] (see also [9, §7.4]).

**Definition 3.1.** A C*-subalgebra $B$ of a C*-algebra $A$ is *complemented* in $A$ if $A = C^*(B, A \cap B')$.

The following is an analog of [9, Lemma 10.2.10].

**Lemma 3.4.** Suppose that $(\kappa, \Gamma, \Theta)$ is a CCR triple, $\Gamma$ is locally finite, and $X \subseteq \kappa$ is nonempty. Then the following are true.

1. $A(\kappa, \Gamma, \Theta) \cap C^*(\{u_g | g \in \Gamma_X\})' = C^*(\{u_g | g \in Z_{\Gamma, \Theta}(X)\})$.
2. $C^*(\{u_g | g \in \Gamma_X\})$ is complemented in $A(\kappa, \Gamma, \Theta)$ if and only if $\Gamma$ is generated by $\Gamma_X$ and $Z_{\Gamma, \Theta}(X)$.

**Proof.** (1) Only the direct inclusion requires a proof.

Let $B = C^*(\{u_g | g \in \Gamma_X\})$. Fix $a \in A(\kappa, \Gamma, \Theta) \cap B'$; we will prove that $a \in C^*(\{u_g | g \in Z_{\Gamma, \Theta}(X)\})$. By Lemma 2.3, we can represent $a$ as a possibly infinite $\| \cdot \|_2, \tau$-convergent sum, $a = \sum_j \lambda_j u_{g(j)}$.

We will prove that $u_{g(j)} \in Z_{\Gamma, \Theta}(\Lambda)$ for all $j$ such that $\lambda_j \neq 0$. Towards contradiction, assume that there is $j$ such that $\lambda_j = 0$ but $g(j) \notin Z_{\Gamma, \Theta}(\Lambda)$. Fix $h \in \Gamma_X$ such that $\eta = \Theta(h, g(j))$ is not equal to 1. Then $u_h u_{g(j)} u_h^* = \eta u_{g(j)}$, and therefore in the $\| \cdot \|_2, \tau$-expansion of $u_h a u_h^*$, the $g(j)$-coefficient is equal to $\eta$. Therefore the $g(j)$-coefficient in the expansion of $a - u_h a u_h^*$ is $1 - \eta \neq 0$, contradiction.

By Corollary 2.1, this implies that $a \in C^*(\{u_g | g \in Z_{\Gamma, \Theta}(\Lambda)\})$, as required. Since $a$ was arbitrary, this proves the direct inclusion and completes the proof.

(2) follows immediately from (1).

The assumption that $\Gamma$ be locally finite used throughout §2 and §3 is probably unnecessary.

4. Non-uniqueness

Our main objective in this section is to prove Theorem 4.1 (a stronger result will be proven in §5). The following is [16, Proposition 3.2].

**Lemma 4.1.** If $A$ is a nonseparable AM algebra, then the following conditions hold.

1. $K_0(A) = K_0(B)$ for every separable elementary submodel $B$ of $A$ (in symbols, $B \prec A$).
2. $A$ has a unique tracial state.
3. $K_1(A)$ is trivial.

The proofs of Theorem 4.1 and Theorem 5.1 use the set-theoretic notion of a closed unbounded (club) set. We recall the definitions, and direct the reader to [9, §6.2–6.4] for additional information.

**Definition 4.1.** If $X$ is an uncountable set, then a family $C$ of countable subsets of $X$ is club if the following two requirements are met.
(1) For every increasing sequence \( Z_n \), for \( n \in \mathbb{N} \), in \( C \) we have \( \bigcup_n Z_n \in C \).
(2) For every countable \( Y \subseteq X \) there exists \( Z \in C \) such that \( Y \subseteq Z \).

**Definition 4.2.** If \( A \) is a nonseparable complete metric space, then a family \( C \) of separable closed subsets of \( A \) is a club if the following two requirements are met.

(3) For every increasing sequence \( C_n \), for \( n \in \mathbb{N} \), in \( C \) we have \( \bigcup_n C_n \in C \).
(4) For every countable \( B \subseteq A \) there exists \( C \in C \) such that \( B \subseteq C \).

We use the common jargon and say that ‘there are club many separable subspaces of \( A \) with property \( P \)’ if the set of separable subspaces of \( A \) with property \( P \) includes a club. By the Downwards Löwenheim–Skolem Theorem, club many separable subspaces of a \( C^* \)-algebras are \( C^* \)-subalgebras. By [9, Lemma 7.4.4], if two nonseparable \( C^* \)-algebras \( A \) and \( B \) are isomorphic then club many separable \( C^* \)-algebras of \( A \) are complemented if and only if club many separable \( C^* \)-subalgebras of \( B \) are complemented.

By using functoriality (Proposition 2.1) we can express the following lemma whose proof is, being straightforward, omitted.

**Lemma 4.2.** Suppose that \( (\kappa, \Gamma, \Theta) \) is a CCR triple such that \( \kappa \) is uncountable and that \( C \) is a club of countable subsets of \( \kappa \). Then the \( C^* \)-subalgebras of the form \( A(X, \Gamma X, \Theta | \Gamma X) \) for \( X \in C \) form a club of separable substructures of \( A(\kappa, \Gamma, \Theta) \). \( \square \)

**Theorem 4.1.** Suppose that \( A \) is a UHF algebra such that for some prime \( p \) every element of \( K_0(A) \) is divisible by \( p \). Then for every uncountable cardinal \( \kappa \) there exists an AM algebra of density character \( \kappa \) with the same Elliott invariant as \( A \) that is not UHF.

**Proof.** Let \( p(i) \), for \( i \in \mathbb{N} \), be an enumeration of all primes \( q \) such that \( M_q(\mathbb{C}) \) unitally embeds into \( A \) and let \( \prod_i p(i)^{k(i)} \) be the generalized integer of \( A \). Thus \( k(i) \geq 1 \) for all \( i \). We may assume that \( p = p(0) \) (the enumeration is not assumed to be in the increasing order).

Let \( J = \kappa \times 2 \sqcup \{ * \} \) and

(a) \( \Gamma = \bigoplus_{i \geq 0} \bigoplus_{1 < k(i)} (\mathbb{Z}/p(i)\mathbb{Z})^2 \times \bigoplus_j \mathbb{Z}/p\mathbb{Z} \).
(b) Let \( f_0(i, j), f_1(i, j) \) for \( i > 0 \) and \( j < k(i) \), \( g(\alpha, i) \), for \( \alpha < \kappa \), and \( i < 2 \), and \( g(*) \) be the generators of the corresponding direct summands of \( \Gamma \).
(c) Let \( G_- = \{ f_0(i, j), f_1(i, j) | i > 0, j < k(i) \} \) and \( G_+ = \{ g(\alpha, i) | \alpha < \kappa, i < 2 \} \).

Then \( \mathcal{G} = G_- \cup G_+ \cup \{ g(*) \} \) is a generating set for \( \Gamma \).

Let \( \lambda = \exp(2\pi i/n) \). Let \( \Theta: \mathcal{G}^2 \to T \) be such that the following conditions hold.

(1) \( \Theta(g(\alpha, 0), g(\alpha, 1)) = \lambda \) for \( \alpha < \kappa \).
(2) \( \Theta(g(\alpha, 1), g(\alpha, 0)) = \overline{\lambda} \), for \( \alpha < \kappa \).

\( ^8 \)A much easier proof of this theorem (which would not lead to a proof of Theorem 5.1) is described in the paragraphs following Question 6.1.
(3) $\Theta(g(\alpha, 0), g(\ast)) = \lambda$, for all $\alpha < \kappa$.
(4) $\Theta(g(\ast), g(\alpha, 0)) = \lambda$, for all $\alpha < \kappa$.
(5) $\Theta(f_0(l, j), f_1(l, j)) = \exp(2\pi/p(j))$, for $l > 0$ and $j < k(l)$.
(6) $\Theta(f_1(l, j), f_0(l, j)) = \exp(-2\pi/p(j))$, for $l > 0$ and $j < k(l)$.
(7) $\Theta(g, h) = 1$ for $g$ and $h$ in $\mathcal{G}$ for which $\Theta$ hasn’t been defined by (1)–(6).

We first prove that $B = A_{(\kappa, \Gamma, \Theta)}$ is AM. For $F \in \mathcal{G}$ let $\Gamma_F$ denote the subgroup of $\Gamma$ generated by $F$. By Proposition 2.1, $B_F = C^*(\Gamma_F)$ is naturally identified with $A_{(F, \Gamma_F, \Theta|\Gamma_F)}$.

Since $B$ is the inductive limit of $C^*$-subalgebras of the form $B_F$ for $F \in \mathcal{G}$, it will suffice to find a cofinal set $\mathcal{F}$ of $F \in \mathcal{G}$ such that $B_F$ is a full matrix algebra for every $F \in \mathcal{F}$.

Let $\mathcal{F}$ be the set of all $F \in \mathcal{G}$ such that the following conditions hold

1. For all $i > 0$ and $j < k(i)$, $f_0(i, j) \in F$ if and only if $f_1(i, j) \in F$.
2. For all $\alpha < \kappa$, $g(\alpha, 0) \in F$ implies $g(\alpha, 1) \in F$ and there exists a unique $\beta = \beta(F)$ such that $g(\beta, 1) \in F$ but $g(\beta, 0) \notin F$.
3. $g(\ast) \in F$.

Clearly for every $F_0 \in \mathcal{G}$ there is $F \in \mathcal{F}$ such that $F_0 \subseteq F$. We claim that $F \in \mathcal{F}$ implies $B_F$ is a full matrix algebra. The group $\Gamma$ is clearly locally finite, and this fact will be used in the remaining part of this proof. With $F(-) = F \cap \mathcal{G}_-$ and $F(+) = F \cap \mathcal{G}_+$, Lemma 3.1 implies that $B_F \cong B_{F(-)} \otimes B_{F(+)}$. The ‘moreover’ part of Lemma 3.2 and (1) together imply that $B_{F(-)}$ is isomorphic to a full matrix algebra. It remains to prove that $B_{F(+)}$ is isomorphic to a full matrix algebra. We will prove this by replacing the generator $g(\ast)$ with $h \in \Gamma$ such that $B_{F(+)} = C^*((F(+)) \setminus \{g(\ast)\}) \cup \{h\})$ and the $C^*$-algebra on the right-hand side is a full matrix algebra. Conditions (2) and (3) imply that there is a finite subset $\alpha(j)$, for $j \leq m$ of $\kappa$ such that

$$F \cap \mathcal{G}_+ = \{g(\alpha(j), 0), g(\alpha(j), 1) | j < m\} \cup \{g(\alpha(m), 0)\}.$$ 

Denoting the first set on the right-hand side by $G$, the ‘moreover’ part of Lemma 3.2 implies that $B_G$ is a full matrix algebra. Let

$$h = g(\ast) \prod_{j < m} g(\alpha(j), 0).$$ 

This is a product of generators, each of order $p$, and therefore it has order $p$. Moreover, for every $g \in \mathcal{G}$ we have

$$\Theta(g, h) = \prod_{j < m} \Theta(g, g(\alpha(j), 0)) \cdot \Theta(g, g(\ast)).$$ 

This implies that $\Theta(g(\alpha(m), 0), h) = \lambda$ and that $\Theta(g(\alpha(j), i), h) = 1$ for all $j < m$ and $i < 2$. Also, $g(\ast) = h \prod_{j < m} g(\alpha(j), 0)^{p-1}$ hence $B_F = C^*((F \setminus \{g(\ast)\}) \cup \{h\})$. Therefore Lemma 3.3 implies $B_{F(+)}$ is isomorphic to a full matrix algebra and concludes the proof that $B$ is AM.

Towards proving that $B$ is not a tensor product of full matrix algebras we first prove that $B$ does not have club many complemented separable $C^*$-subalgebras.
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Since $G_{-}$ is countable and $\kappa$ is uncountable, by using Lemma 4.2 it will suffice to prove that

Claim 4.2. If $G_{-}\cup\{g(*)\} \subseteq F \subseteq G$ and $F$ is countable, then $B_F$ is not complemented in $B$.

Proof. We claim that (writing an element of $\Gamma$ as a product of generators $\prod_{i<m} h(i)$ in the reduced form)

$$Z_{\Gamma,\Theta}(F) = \langle \prod_{i<m} h(i) | m \in \mathbb{N}, \quad h(i) \in G \setminus F, \quad \text{and} \quad \prod_{i<m} \Theta(h(i), g(*)) = 1 \rangle.$$ 

Only the direct inclusion requires a proof. Suppose that $\prod_{i<m} h(i)$ does not belong to the right-hand side. We consider two possible cases.

Suppose first that $h(r) \in F$ for some $r < m$. The word $\prod_{i<m} h(i)$ is in the reduced form and the generators in $F \setminus \{g(*)\}$ come in non-commuting pairs $(f_0(l, j))$ and $f_1(l, j), g(\alpha, 0)$ and $g(\alpha, 1))$. Therefore

$$\prod_{i<m} h(i) = h(r)^d h,$$

where $h$ is a product of generators distinct both from $h(r)$ and its pair, and $h(r)^d \neq 1$. Suppose in addition that $h(r) \neq g(*)$. This implies that $h(r)^d h$ does not commute with the generator paired with $h(r)$. Now assume that $h(r) = g(*)$. Since $F$ is infinite, there exists $g(\alpha, 1) \in F$ such that $g(\alpha, j)$ for $j < 2$ is not a factor of $h$. Then $\Theta(g(\alpha, 1), g(r)^d h) = \lambda^d \neq 1$.

It remains to analyze the remaining case, when $h(i) \notin F$ for all $i < m$ but $\prod_{i<m} \Theta(h(i), g(*)) \neq 1$. This clearly implies that $g(*)$ does not commute with $\prod_{i<m} h(i)$.

This proves that $Z_{\Gamma,\Theta}(F)$ is as claimed. Since $\Gamma$ is locally finite, by Lemma 3.4 $B_F$ is complemented in $A_{(\kappa, \Gamma, \Theta)}$ if and only if $\Gamma$ is generated by $F \cup Z_{\Gamma,\Theta}(F)$. Since $\kappa$ is uncountable, we can find $\alpha < \kappa$ such that $g(\alpha, 0) \notin F$. Then $\Theta(g(\alpha, 0), g(*)) = \lambda \neq 1$, and $g(\alpha, 0) \notin \Lambda$.

Since $\kappa$ is uncountable, the countable subsets of $G$ that include $G_{-}\cup\{g(*)\}$ form a club, and Claim implies that club many separable $C^*$-subalgebras of $A_{(\kappa, \Gamma, \Theta)}$ are not complemented. In a UHF algebra, club many separable $C^*$-subalgebras are complemented ( [9, Example 7.4.2]). By [9, Lemma 7.4.4], if two $C^*$-algebras $A$ and $B$ are isomorphic then club many separable $C^*$-algebras of $A$ are complemented if and only if club many separable $C^*$-subalgebras of $B$ are complemented, and therefore $B$ is not UHF.

5. Non-classification

In this section we prove our main non-classification result.

Theorem 5.1. Suppose that $A$ is a UHF algebra such that for some prime $p$ every element of $K_0(A)$ is divisible by $p$. Then for every uncountable cardinal $\kappa$ there
exists $2^\kappa$ nonisomorphic AM algebras of density character $\kappa$ with the same Elliott invariant as $A$.

The proof of this result uses basic continuous model theory and Shelah’s non-structure theory ([32], [33]), and we assume that the reader is familiar with the former (see e.g., [9, Appendix C] or [10]); the latter (incomparably more complex) ingredient will be treated as a blackbox. The following is an analog of Shelah’s order property (OP), first used in the context of $C^*$-algebras in [11].

**Definition 5.1.** Suppose that $\varphi(\bar{x}, \bar{y})$ is a formula of the language of $C^*$-algebras in $2n$ variables, in which $\bar{x}$ and $\bar{y}$ are $n$-tuples of the same sort. Define a binary relation $\varphi(\bar{a}, \bar{b})$ if and only if $\bar{a}$ and $\bar{b}$ are of the appropriate sort and the following holds

$$\varphi^A(\bar{a}, \bar{b}) = 1 \text{ and } \varphi^A(\bar{b}, \bar{a}) = 0.$$  

It is not required that $\varphi$ is transitive (but it is clearly antisymmetric). A $\varphi$-chain is an indexed set $\bar{a}_x$, for $x \in I$, where $(I, <)$ is a linear ordering and $x < y$ if and only if $\bar{a}_x \varphi \bar{a}_y$.

Lemma 5.1 below is an immediate consequence of Shelah’s non-structure results, and we sketch a proof for reader’s convenience. It should be emphasized that, in spite of being stated as a result about $C^*$-algebras, this result is about general metric structures.

**Lemma 5.1.** Suppose that $\kappa$ is an uncountable cardinal, $\mathcal{K}$ is a subcategory of $C^*$-algebras, and there exist a quantifier-free formula $\varphi$ and a functor $\mathbb{F}$ from the category of linear orderings into $\mathcal{K}$ such that for every $J$, $\mathbb{F}(J)$ is in $\mathcal{K}$ and it is generated by a $\varphi$-chain $\bar{a}_x$, for $x \in J$. Then for every uncountable cardinal $\kappa$ there are $2^\kappa$ nonisomorphic $C^*$-algebras of density character $\kappa$ in $\mathcal{K}$.

In addition, for every separable $C^*$-algebra $A$ there are $2^\kappa$ nonisomorphic $C^*$-algebras of the form $A \otimes B$ for $B \in \mathcal{K}$ of density character $\kappa$.

The assumption that $\varphi$ is quantifier-free can be replaced by the assumption that for $I \subseteq J$ the inclusion of $\mathbb{F}(I)$ into $\mathbb{F}(A)$ is sufficiently elementary (more precisely, that it preserves the values of $\varphi$).

The proof of Lemma 5.1 uses the the following definition and Lemma 5.2 stated below.

**Definition 5.2.** Suppose that $n \geq 1$ and $\varphi(x, y)$ is a $2n$-ary formula in which $\bar{x}$ and $\bar{y}$ are of the same sort. A $\varphi$-chain $\mathcal{C} = (\bar{a}_x | x \in I)$ in a $C^*$-algebra $A$ is weakly $(\mathbb{N}_1, \varphi)$-skeleton like inside $A$ if for every $\bar{a} \in A^n$ there is a countable $I_{\bar{a}} \subseteq I$ with the following property. If $x$ and $y$ are in $I$ and such that $\bar{a}_x \varphi \bar{a}_y$ and no $z \in I_{\bar{a}}$ satisfies $\bar{a}_z \varphi \bar{a}_x \varphi \bar{a}_y$, then $\varphi^A(\bar{a}_x, \bar{a}_y) = \varphi^A(\bar{a}_y, \bar{a}_x)$ and $\varphi^A(\bar{a}_x, \bar{a}_y) = \varphi^A(\bar{a}_y, \bar{a}_x)$.

The following is [16, Lemma 6.4], proved by a heavy use of the results of [17].

**Lemma 5.2.** Suppose that $\mathcal{K}$ is a subcategory of $C^*$-algebras, $\varphi(x, y)$ is a $2n$-ary formula in which $\bar{x}$ and $\bar{y}$ are of the same sort, and $\kappa$ is an uncountable cardinal. If
for every linear ordering \( \Lambda \) of cardinality \( \kappa \) there is \( B_\Lambda \in \mathcal{K} \) of density character \( \kappa \) such that \( B_\Lambda^n \) includes a \( \varphi \)-chain isomorphic to \( \Lambda \) which is weakly \( (\aleph_1, \varphi) \)-skeleton like, then \( \mathcal{K} \) contains \( 2^\kappa \) nonisomorphic C*-algebras of density character \( \kappa \).

**Proof of Lemma 5.1.** We first verify that the \( \varphi \)-chain \( \bar{a}_x \), for \( x \in J \), is weakly \( (\aleph_1, \varphi) \)-skeleton like in \( A = \mathbb{F}(J) \). To prove this, with \( n \) such that \( \varphi \) is \( 2n \)-ary fix \( \bar{b} \in A^n \) (of the appropriate sort). By the Downwards Löwenheim–Skolem Theorem, there is a countable \( J_\bar{b} \subseteq J \) such that \( \bar{b} \) belongs to \( \mathbb{F}(J_\bar{b}) \) (using functoriality to give meaning to this formula). Fix \( x \) and \( y \) such that \( \bar{a}_x \prec \bar{a}_y \) and no \( z \in J_\bar{b} \) satisfies \( \bar{a}_x \prec \varphi \) \( \bar{a}_z \prec \bar{a}_y \). By the functoriality of \( \mathbb{F} \), there is then an isomorphism

\[
\Phi: \mathbb{F}(J_\bar{b} \cup \{ x \}) \to \mathbb{F}(J_\bar{b} \cup \{ y \})
\]

that extends the identity on \( \mathbb{F}(J_\bar{b}) \) and sends \( \bar{a}_x \) to \( \bar{a}_y \).

Since \( \varphi \) is quantifier-free, this implies \( \varphi^A(\bar{a}_x, \bar{b}) = \varphi^A(\bar{a}_y, \bar{b}) \) and \( \varphi^A(\bar{b}, \bar{a}_x) = \varphi^A(\bar{b}, \bar{a}_y) \). If \( J \) is uncountable, then the density character of \( \mathbb{F}(J) \) is equal to \( |J| \), and Lemma 5.2 implies that there are \( 2^\kappa \) nonisomorphic algebras of the density character \( \kappa \) and the prescribed \( K \)-theory.

For the second part, we only need to prove that if a chain \( \bar{a}_x \), for \( x \in J \), is weakly \( (\aleph_1, \varphi) \)-skeleton like in \( \mathbb{F}(J) \) then it is weakly \( (\aleph_1, \varphi) \)-skeleton like in \( B \otimes \mathbb{F}(J) \). Since \( \varphi \) is quantifier-free, it is still a \( \varphi \)-chain. Let \( \bar{b} \) and \( J_\bar{a} \) be as in the first part of the proof. Mimicking this proof, while replacing separable algebras of the form \( \mathbb{F}(J') \) in this proof with \( B \otimes \mathbb{F}(J') \), one concludes the proof.

**Proof of Theorem 5.1.** In order to use Lemma 5.1 we define a functor that sends a linear ordering \( J \) to an AM C*-algebra \( A = \mathbb{F}(J) \). Fix \( J \). Suppose that the generalized integer associated with \( A \) is \( n_A = \prod_j p(j)^{k(j)} \), let \( \Gamma_A = \bigoplus_{j < k(j)} \mathbb{Z}/p(j)\mathbb{Z} \) and \( X_A = \bigcup_j k(j) \). With \( p \) as in the assumption of Theorem 5.1 let \( \Gamma_J = \bigoplus_{x \in J} \bigoplus_{i < j} \mathbb{Z}/p\mathbb{Z} \) and

\[
\Gamma = \Gamma_A \oplus \Gamma_J.
\]

Let \( g(x, j) \), for \( x \in J \) and \( j < 2 \), denote the generators of \( \Gamma_J \). Define \( \Theta \) on the generators of \( \Gamma \) as follows.

1. The restriction of \( \Theta \) to \( \Gamma_A \) is such that \( A_{(X_A \cup J, \Gamma, \Theta|_{\Gamma_A})} \cong A \) (as in the proof of Theorem 4.1).
2. \( \Theta(g(x, 0), g(y, 1)) = \lambda = \Theta(g(y, 1), g(x, 0)) \) if \( x \) and \( y \) are in \( J \) and \( x \leq y \).
3. \( \Theta(g, h) = 1 \) for all generators for which \( \Theta(g, h) \) is not determined by (1) or (2).

Let

\[
\mathbb{F}(J) = A_{(X_A \cup J, \Gamma, \Theta)}.
\]

The functoriality of \( \mathbb{F} \) follows by Proposition 2.1. Consider the following formula

\[
\varphi(a, b, c, d) = \frac{1}{2} \|ad - da\|.
\]
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Then by (2) we have \( \varphi(F)(u_{g(x,0)}, u_{g(x,1)}, u_{g(y,0)}, u_{g(y,1)}) = 1 \) if \( x \leq y \) and 0 otherwise. Therefore \((u_{g(x,0)}, u_{g(x,1)}), \) for \( x \in J \), form a \( \varphi \)-chain isomorphic to \( J \). Since these unitaries generate \( F(J) \), by Proposition 2.1 the assumptions of Lemma 5.1 are satisfied. Therefore for every uncountable \( \kappa \) there are \( 2^\kappa \) nonisomorphic algebras of the form \( A \otimes F(J) \) of density character \( \kappa \).

Claim 5.2. For every linear order \( J \), the algebra \( F(J) \) is AM.

**Proof.** As before, for \( F \in J \) let \( B_F = C^*(\{g(x,i) | x \in F, i < 2\}) \). Since \( F(J) \) is the inductive limit of such \( B_F \), it will suffice to prove that \( B_F \) is isomorphic to \( M_{p(|F|)}(C) \) for every \( F \in J \).

Fix \( F \in J \) and let \( x(j) \), for \( j < k \), be its increasing enumeration. For \( j < k \) and \( i < 2 \) define \( g'(j,i) \) as follows.

1. \( g'(x(j),0) = g(x(j),0) \) for all \( j < k \).
2. \( g'(x(0),1) = g(x(0),1) \).
3. \( g'(x(j),1) = g(x(j),1)\prod_{k<j} g(x(k),1) \), for \( 1 \leq j < k \).

Then \( g'(x(j),i) \), for \( j < k \) and \( i < 2 \), generate \( B_F \). Also, (2) implies that \( g'(x(j),i) \) and \( g'(x(j'),i') \) commute when \( j \neq j' \), while

\[
 g'(x(j),0)g'(x(j),1) = \lambda g'(x(j),1)g'(x(j),0)
\]

for all \( j < k \). Since \( \Gamma \) is locally finite, the ‘moreover’ part of Lemma 3.2 now implies that \( B_F \) is a full matrix algebra, as required.

Every AM algebra is, being a unital inductive limit of monotracial \( C^* \)-algebras, monotracial. Lemma 4.1 implies that each \( F(J) \) has the same \( K_0 \) and \( K_1 \) groups as \( A \), and therefore the same Elliott invariant as \( A \). This completes the proof.

6. Concluding remarks

In the present paper we explored constructions of nonseparable CCR algebras from bicharacters on uncountable direct sums of cyclic groups. Since the set-theoretic study of infinite abelian groups has a long and distinguished history (e.g., [31, 6, 21]), bicharacters on more interesting groups could lead to even more intriguing examples of nonseparable \( C^* \)-algebras. An alternative route towards our results worth exploring (suggested by Ilan Hirshberg) could proceed via uncountable spin systems (see [2]).

Another related source of nonseparable \( C^* \)-algebras are ‘twisted’ cocycle crossed products by an uncountable discrete group ( [24], already used (implicitly) in [15 §6] (see [15] Remark 6.9)).

For the definitions of classifiable \( C^* \)-algebras and Elliott invariant see e.g., [23].

**Question 6.1.** Suppose that \( A \) is a nuclear, simple, separable \( C^* \)-algebra and \( \kappa \) is an uncountable cardinal.
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(1) Is it true that there are $2^\kappa$ nonisomorphic simple nuclear C*-algebras of density character $\kappa$ with the Elliott invariant equal to $\text{Ell}(A)$?

(2) Is it true that there are at least two nonisomorphic simple nuclear C*-algebras of density character $\kappa$ with the Elliott invariant equal to $\text{Ell}(A)$?

(3) Is it true that there exists a simple nuclear C*-algebra of density character $\kappa$ with the Elliott invariant equal to $\text{Ell}(A)$?

If $A$ is as in Question 6.1, $D$ is a strongly self-absorbing C*-algebra (see [37], but all we need is that $\bigotimes_{\aleph_0} D \cong D$) and $A$ tensorially absorbs $D$, then Question 6.1 (3) has a positive answer. The algebra $A_1 = A \otimes \bigotimes_{\kappa} D$ has the same Elliott invariant as $A$ and density character $\kappa$. This is because by the assumption on $D$ club many separable subalgebras of $A_1$ are isomorphic to $A \otimes D \cong A$, and by the reflection argument as in Lemma 4.1, the Elliott invariant of $A_1$ is equal to that of $A$. In particular, Question 6.1 (3) has a positive answer for every classifiable C*-algebra which absorbs the Jiang–Su algebra $Z$. We don’t know the answer to Question 6.1 (3) for Rørdam’s simple C*-algebra with both finite and infinite projections ([29]) or for Villadsen’s algebras with perforated K-theory groups (see [28]).

If $A$ tensorially absorbs both $Z$ and some full matrix algebra $M_n(\mathbb{C})$ (e.g., if $A$ is a UHF algebra such that for some prime $p$ every element of $K_0(A)$ is divisible by $p$, as in the assumptions of Theorem 4.1), then Question 6.1 (2) has a positive answer, as witnessed by $A \otimes \bigotimes_{\kappa} Z$ and $A \otimes \bigotimes_{\kappa} M_n(\mathbb{C})$ (a proof of this is analogous to the proofs in [15, §3]).

An alternative approach to Question 6.1 (2) proceeds by finding a C*-algebra with the same invariant as a given $Z$-absorbing $A$ that is tensorially prime (i.e., not isomorphic to the tensor product of two infinite-dimensional C*-algebras). This was done for some AF and some purely infinite, simple C*-algebras in [36] Theorem B (1) and (4)]. In Theorem C of the same paper, a monotracial, tensorially prime, nuclear, simple C*-algebra with uncountable $K_1$-group was constructed. C*-algebras constructed in [36] are associated with iterated wreath products of groups and they even have no nontrivial central sequences. Every infinite-dimensional separable AM algebra has nontrivial central sequences (for much stronger results see [1] and [7]). The first examples of (necessarily nonseparable) AM algebras with this property were constructed in [12] using weakenings of the Continuum Hypothesis. These algebras have the same Elliott invariant as $M_{2^\infty}$.

Apparently the simplest instance of Question 6.1 (1) not resolved by our Theorem 1 is the case of $\bigotimes_{p \text{ prime}} M_p(\mathbb{C})$. Probably the most interesting case is when $A$ is the Jiang–Su algebra $Z$, whose Elliott invariant is equal to that of the complex numbers (see [20], also [18] and [30] for most recent treatments).

A programme analogous to one given by Question 6.1 was pursued in [38] and [35]. In these papers the authors studied the variety of the invariants of counterexamples to Naimark’s Problem (using Jensen’s diamond).

Graph CCR algebras provided the first example of a nuclear, simple C*-algebra with the property that its automorphism group did not act transitively on its pure
state space (§8, see §9.4). It is likely that the CCR algebras associated with bicharacters on uncountable abelian groups will find other uses. For example, every graph CCR algebra is a complexification of an operator algebra on a real Hilbert space, and therefore isomorphic to its opposite algebra. The only known examples of simple, nuclear, C*-algebras not isomorphic to their opposites were constructed using Jensen’s diamond in [13]. It is possible that, with an appropriately chosen group and a bicharacter, the CCR algebras considered in this paper may provide a ZFC example.

Our examples can also be used to construct $2^\kappa$ nonisomorphic hyperfinite II$_1$ factors with predual of density character $\kappa$, but this has already been proved in [16] (see also [35]). However, the II$_1$ factors obtained from our CCR algebras may have other interesting properties. For example, it is not known whether a hyperfinite II$_1$ factor not isomorphic to its opposite can be constructed in ZFC (see [14] for a construction using Jensen’s diamond).

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