Phase portrait of a matter bounce in Hořava–Lifshitz cosmology

E Czuchry

Instytut Problemów Jądrowych, ul. Hoża 69, 00-681 Warszawa, Poland

E-mail: eczuchry@fuw.edu.pl

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Abstract

The occurrence of a bounce in the FRW cosmology requires modifications of general relativity. An example of such a modification is the recently proposed Hořava–Lifshitz theory of gravity, which includes a ‘dark radiation’ term with a negative coefficient in the analog of the Friedmann equation. This paper describes a phase space analysis of models of this sort with the aim of determining to what extent bouncing solutions can occur. It is found that they are possible, but not generic in models under consideration. Apart from previously known bouncing solutions, some new ones are also described. Other interesting solutions found include ones that describe a novel sort of quasi stationary, oscillating universe.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The standard ΛCDM model has solved many issues in cosmology. However, in spite of all this success, it also leaves a number of issues unaddressed. Perhaps the most significant is the problem of initial singularity, where general relativity breaks down. There have been many attempts to modify Einstein’s theory to avoid this singularity. Some are made at the classical level, some involve quantum effects. Examples include the ekpyrotic/cyclic model [1–9] and loop quantum cosmology [10–15], which replace the Big Bang with the Big Bounce. Attempts to address these issues at the classical level include braneworld scenarios [16, 17], where the universe goes from an era of accelerated collapse to an expanding era without any divergences or singular behavior. There are also higher order gravitational theories and theories with scalar fields (see [18] for a review of bouncing cosmologies). It is, however, fair to say that the issue of the initial singularity remains one of the key questions of the early Universe cosmology, and the idea that it is avoided due to a bounce is an elusive (and promising) notion. As discussed
below, it is clear that close to the singularity the Friedmann equation has to be modified for a bounce to be possible.

In recent years, much effort has been devoted to studies of a proposal for a UV complete theory of gravity due to Hořava [19–21] and modifications of the theory [22–27] (for a recent review see [28, 29]). Because in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space, this theory is referred to as the Hořava–Lifshitz gravity. Soon after this theory was proposed, many specific solutions were found, including cosmological ones [22, 30–36]. It was also realized that the analog of the Friedmann equation in the HL gravity contains a term that scales in the same way as dark radiation in braneworld scenarios [30–32] and gives a negative contribution to the energy density. Thus, at least in principle it is possible to obtain non-singular cosmological evolution within the Hořava theory, as was pointed out in [30, 32, 37]. Propagation of linear cosmological perturbations through the bounce was studied in [38], and it was shown that their evolution remains non-singular throughout, despite a singularity in perturbations’ equation of motion at the bounce point. The scale invariance of the perturbation spectrum is preserved during the bounce—without the need for inflation. Thus, the HL gravity can provide a realization of the ‘matter bounce’ scenario.

The Hořava theory evolved in many aspects attempting to solve problems and inconsistencies appearing in the original formulation [25, 39–41]. Numerous sophisticated versions contain new terms added to the original Lagrangian with attempt to make the proposal more general [25] and to solve the so-called strong coupling problem [39, 42–44]. The latter one seems to be the key problem of Hořava theory. It concerns the extra scalar degree of freedom arising from the explicit breaking of the general covariance and appearing when expanding about flat spacetime [25, 39, 41]. While approaching the classical IR limit this mode becomes strongly coupled [39]. Blas et al proposed an extension of Hořava theory [44] (the so-called healthy extension), showing in [26, 45] that fluctuations about Minkowski spacetime can be made well behaved and this extra mode does not lead to any phenomenological inconsistencies. Another solution to this problem was proposed by Hořava and Melby-Thompson. In [27], they investigate the possibility of extending the gauge group by a local $U(1)$ symmetry, which eliminates scalar graviton. The presence of this new symmetry forces the coupling constant $\lambda$ to be equal to one (however, this result was questioned in [48], where an action with the extended gauge symmetry and $\lambda \neq 1$ is formulated). On the other hand, it was shown, e.g., in [26, 28] that to avoid instabilities of the scalar graviton, $\lambda$ has to be very close to 1 at low energies. The question on the value of $\lambda$—whether it has to be fixed at 1—is still a matter of debate.

Nevertheless, the influence of the dark radiation term on the existence and stability of a cosmological bounce remains an interesting issue. This contribution to the modified Friedmann equations appears both in the projectable formulation [25] and in the non-projectable healthy extension of the HL theory [29]. The dynamics of the version exhibiting $U(1)$ symmetry has not been adequately studied yet (though one expects that for cosmological solutions the terms suggested in [27] will not contribute to the equations of motion). In this paper, we are going to analyze how this specific modification of equations of motion affects the dynamics of the system. As a starting point, we will take the Friedmann equations of the projectable HL theory with detailed balance. This allows us to explore implications of the ‘dark radiation’ contribution and this formulation is convenient if one is interested in homogeneous and isotropic solutions. The dynamics of the system in the IR limit will be visualized with the help of the phase portrait techniques described in [59, 60] and compared to the standard GR cosmology. For the purpose of illustration we will assume that matter in the pre-bounce epoch is described by a scalar field $\varphi$ with a quadratic potential. In order to concentrate on
modifications created by the ‘dark radiation’ terms and to simplify the analysis, we will take the cosmological constant \( \Lambda \to 0 \) limit. Such a scenario is as an approximation to a general case with non-vanishing \( \Lambda \), valid in the regime of small scale factor \( a \), when standard curvature and \( \Lambda \) terms are not relevant. Thus, the present analysis can be regarded as an exploration of the cosmologies with modified equations of motion, where the modifications considered are inspired by Hořava cosmology.

Related analyses of Hořava–Lifshitz cosmology have also appeared in [46] and [47], which we became aware of while this work was being typed. Those papers address a somewhat different set of issues from what we have pursued. The analysis presented in [46] and [47] consider the full four-dimensional phase space of HL cosmology. The results presented here focus on the region close to where the scale factor vanishes, which admits a critical simplification: the number of dynamical equations under study can be reduced from 4 to 3 (as discussed in more detail in section 3). This makes it possible to visualize the possible phase space trajectories in a three-dimensional space.

The structure of this paper is as follows. In section 2, we briefly sketch the Hořava–Lifshitz gravity and cosmology. In section 3, the possibility of bounce is discussed. In section 4, we discuss phase portraits of the discussed system of equations and describe different families of phase trajectories.

2. Hořava–Lifshitz cosmology

The metric of Hořava–Lifshitz theory, due to anisotropy in UV, is written in the \((3 + 1)\)-dimensional ADM formalism,

\[
\text{d}s^2 = -N^2 \text{d}t^2 + g_{ij}(\text{d}x^i - N^i \text{d}t)(\text{d}x^j - N^j \text{d}t),
\]

where \( N, N^i \) and \( g_{ij} \) are dynamical variables. The general action of the Hořava–Lifshitz theory of gravity is [20]

\[
I_g = \frac{2}{\kappa^2} \int \text{d}^3x \sqrt{g} \{ (K_{ij}K^{ij} - \lambda K^2 - V_g(g_{ij}, N)) \},
\]

where \( K_{ij} = \frac{1}{N} \left[ \frac{1}{2} g_{ij} - \nabla_i N_j \right] \) is an extrinsic curvature of a spacelike hypersurface with a fixed time, covariant derivatives are defined with respect to the spatial metric \( g_{ij} \), \( \lambda \) is the dimensionless running coupling and \( \kappa \) is the gravitational coupling.

Potential \( V_g \) is a function of the metric \( g_{ij} \), lapse \( N \) and their spatial derivatives. It shall contain terms at least sixth order in spatial derivatives due to requirements set by power counting renormalizability. There are many possible forms of \( V_g \), different choices leading to a different version of the theory. In the original proposal [20], the potential \( V \) was written in the so-called detailed balance form, derivable from a superpotential \( W \),

\[
V_g = E^{ij} G^{ijkl} E_{kl},
\]

where

\[
E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W}{\delta g_{ij}},
\]

and \( G^{ijkl} \) is a generalized De Witt metric

\[
G^{ijkl} = \frac{1}{4}(g^{ik}g^{jl} + g^{il}g^{jk} - \lambda g^{ij}g^{kl}).
\]

This leads to the gravitational action in the form

\[
S_{\text{DB}} = \int \text{d}^4x N \sqrt{g} \left\{ \frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda R - 3\Lambda^2)}{8(1 - 3\lambda)} 
+ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2\omega^2} Z_{ij}Z^{ij} \right\},
\]

3
where

\[ Z_{ij} = C_{ij} - \frac{\mu \omega^2}{2} R_{ij}, \quad (7) \]

\( \mu, \omega \) and \( \Lambda \) are constant parameters and the Cotton tensor, \( C_{ij} \), is defined by

\[ C^{ij} = \epsilon^{ikl} \nabla_k (R^j_l - \frac{1}{4} R \delta^j_l) = \epsilon^{ikl} \nabla_k R^j_l - \frac{1}{4} \epsilon^{ikj} \partial_k R. \quad (8) \]

Imposing detailed balance condition reduces highly the number of terms in the potential part \( V \) of the action (2) and additionally introduces a superpotential \( W \), which may simplify quantization of the system. Although there is nothing fundamental in this formulation [25, 28, 29], it is convenient if one is interested in homogeneous and isotropic solutions.

In the HL gravity, opposite to GR, there is no direct prescription on how to couple matter to the gravitational field. In GR, there is the Lorentz invariance that provides arguments on how to couple matter, but in the Hořava theory there are no such guides for choosing a particular type from among the general family of couplings between the gravity and matter sectors. In some approaches, the minimal coupling is assumed [44, 49] to make more easy contact with GR, and there are also considered particular couplings [50–55]. This issue has been recently addressed in [56]. We follow the procedure described in [30–32], and for the scalar field take the same anisotropic scaling as the one chosen for the gravity sector. Thus, the action for matter is

\[ I_m = \int dt \, d^3x \sqrt{g} N L_m, \quad (9) \]

where

\[ L_m = \frac{3\lambda - 1}{2} \left[ \frac{1}{N^2} (\phi - N^i \partial_i \phi)^2 - F(\partial_i \phi, \phi) \right]. \quad (10) \]

The potential \( F \) can in principle contain arbitrary combinations of \( \phi \) and its spatial derivatives. The UV renormalizability indicates that \( F \) should contain up to six derivatives, but otherwise arbitrary powers of the scalar field.

In the IR, the terms with fewer spatial derivatives in the gravitational action (6) dominate. Thus, the leading terms would be the kinetic one \( I_k = \frac{\kappa^2}{2} \int dt \, d^3x \sqrt{g} N \left( (K_{ij} K^{ij} - \lambda K^2) \right) \), the cosmological constant \( \Lambda \) term and the Ricci scalar term, the fourth and six spatial derivative terms being irrelevant. Therefore, the following action describes the IR behavior of the theory,

\[ S_{IR} = \int dt \, d^3x \, N \sqrt{g} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\mu^2 \Lambda R}{8(1 - 3\lambda)} - \frac{3\mu^2 \Lambda^2}{8(1 - 3\lambda)} \right\}. \quad (11) \]

Comparing the action of the Hořava–Lifshitz theory near its IR limit to the Einstein–Hilbert action of general relativity, one can see that the speed of light, Newton’s constant and the cosmological constant are

\[ c = \frac{\kappa^2 \mu}{4 \sqrt{1 - 3\lambda}}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \Lambda_E = -\frac{3\mu^2 \Lambda^2}{3\lambda - 1}, \quad (12) \]

respectively. Setting dynamical constant \( \lambda = 1 \), reduces the terms in (11) to the usual ones of Einstein’s relativity.

The equations for Hořava–Lifshitz cosmology are obtained by imposing conditions of homogeneity and isotropy of the metric. The associated ansatz is \( N = N(t) \), \( N_i = 0 \), \( g_{ij} = a^2(t) \gamma_{ij} \) where \( a(t) \) is a scale factor and \( \gamma_{ij} \) is a maximally symmetric constant curvature metric, with a curvature \( k = \{-1, 0, 1\} \). On this background,
where \( H \equiv \dot{a}/a \) is the Hubble parameter.

The nonlinear function \( F \) in the scalar action (9) reduces effectively to a potential \( V(\phi) \) and matter Lagrangian \( \mathcal{L}_m \) takes the form

\[
\mathcal{L}_m = \frac{3\lambda - 1}{2} \left( \frac{\phi^2}{2N^2} - V(\phi) \right).
\]

The energy density and pressure of the scalar field may be defined in the following way,

\[
\rho = \frac{3\lambda - 1}{4} \phi^2 + V(\phi),
\]

\[
p = \frac{3\lambda - 1}{4} \phi^2 - V(\phi).
\]

In numerical calculations presented further on, a specific form of the scalar potential will be assumed (see equation (25)).

The total action becomes

\[
S_{\text{FRW}} = \int dt \int d^3x \; N a^3 \left\{ \frac{3(1 - 3\lambda)}{2k^2} \frac{H^2}{N^2} + \frac{3k^2\mu^2\Lambda}{4(1 - 3\lambda)} \left( \frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{k^2\mu^2}{8(1 - 3\lambda)} \frac{k^2}{a^2} \right\}
\]

\[
+ \int d^3x \; N a^3 \frac{3\lambda - 1}{2} \left( \frac{\phi^2}{2N^2} - V(\phi) \right).
\]

The equations of motion are obtained by varying action (17) with respect to \( N, a \) and \( \phi \), then setting \( N = 1 \) at the end of the calculations,

\[
H^2 = \frac{k^2 \rho}{6(3\lambda - 1)} + \frac{k^4 \mu^2 \Lambda}{8(3\lambda - 1)^2} - \frac{k^4 \mu^2}{16(3\lambda - 1)^2} \left( \frac{\Lambda^2}{3} + \frac{k^2}{a^2} \right),
\]

\[
\dot{H} = -\frac{k^2 (\rho + p)}{4(3\lambda - 1)} - \frac{k^4 \mu^2 \Lambda}{8(3\lambda - 1)^2} \frac{k^2}{a^2} + \frac{k^4 \mu^2}{32(3\lambda - 1)^2} \frac{k^2}{a^4},
\]

and also equation of motion for the scalar field,

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{2}{3\lambda - 1} V' = 0,
\]

where \( H = \dot{a}/a \), a prime denotes the derivative with respect to scalar field \( \phi \).

Equations (18) and (19) are the non-relativistic Friedmann equations. They are derived from the general HL action (not the one in the IR limit) and thus they differ from the GR Friedmann equations by the presence of the parameter \( \lambda \) and the \((1/a^4)\) terms. Taking the IR limit first would only reproduce the standard Friedmann equations, losing the additional structure. The \((1/a^4)\) terms, produced by the \( R^2 \) term in the HL action, are reminiscent of the dark radiation term in braneworld cosmology [57] and are present only if the spatial curvature of the metric is non-vanishing.

The coupling constant \( \lambda \) is dimensionless. In general, it runs (logarithmically in the UV) and may eventually reach on the three IR fixed points [19]: \( \lambda = 1/3, \lambda = 1 \) or \( \lambda = \infty \). The range \( 1 > \lambda > 1/3 \) leads to ghost instabilities in the IR limit of the theory [58]. However, this range of \( \lambda \) is exactly the flow-interval between the UV and IR regimes. The only physically interesting case that remains, allowing for a possible flow toward GR—at \( \lambda = 1 \)—is the regime \( \lambda \geq 1 \). Region \( \lambda \leq 1/3 \) is disconnected from \( \lambda = 1 \) and cannot be included in realistic considerations. Moreover, to avoid instabilities of the scalar graviton, \( \lambda \) has to be very close to 1 at low energies. Additionally, in one of the recent extensions of Hořava theory [27], \( \lambda \) is
only a coefficient in the action, forced by an imposed $U(1)$ symmetry to be equal to 1 (though this result was questioned in [48]). Nonetheless, the phenomenologically relevant range is $\infty > \lambda \geq 1$. In this range, the qualitative description of the above system does not depend on the specific value of $\lambda$: equations (18) and (19) show different behavior for $\lambda > 1/3$ and $\lambda < 1/3$. Therefore, if we want to stay within the IR limit, we may simplify further calculations and set the value $\lambda = 1$. This leads to

$$H^2 = \frac{k^2 \rho}{12} + \frac{k^4 \mu^2 \Lambda}{32} \frac{k}{a^2} \frac{\kappa^4 \mu^2}{64} \left( \frac{\Lambda^2 + k^2}{a^4} \right),$$

(21)

$$\dot{H} = -\frac{k^2 (\rho + p)}{8} - \frac{k^4 \mu^2 \Lambda}{32} \frac{k^2}{a^2} + \frac{k^4 \mu^2}{128} \frac{k^2}{a^4},$$

(22)

and an equation of motion for the scalar field,

$$\ddot{\phi} + 3H \dot{\phi} + V' = 0,$$

(23)

where in this limit $\rho = \phi^2/2 + V(\phi)$ and $p = \phi^2/2 - V(\phi)$.

3. Existence of bounce

New terms in the cosmological equations introduce the possibility of a bounce. The form of (21), with $k = \pm 1$, implies that it is possible that $H = 0$ at some moment in time. This is a necessary condition for the realization of the bounce. It was pointed out in [30], that it may happen in the presence of matter, at the critical time $t_*, a = a_*$, when the critical energy density is equal to

$$\rho = \rho_* = \frac{3k^2 \mu^2}{2} \left( -\frac{\Lambda}{4} \frac{a_*^2}{a^2} + \frac{\Lambda^2}{8} \frac{1}{a_*^4} \right),$$

(24)

which is determined by the couplings of the theory.

From the continuity equation, it follows that at the bounce point $\dot{H} > 0$. Therefore, a transition from a contracting to an expanding phase may be possible. It was shown in [37] that the necessary condition for a cosmological bounce is that the energy density of regular matter increases less fast than $a^{-4}$ as the scale factor decreases and $(\rho - p) > 0$.

We begin our considerations during a contracting phase. At the beginning, the scale factor is quite large and the contribution of dark radiation to the total energy density is quite small. As the universe contracts, the energy density increases and the scale factor decreases rapidly. When a critical density is achieved, a bounce is about to take place.

One would expect that near the bounce, when the scale factor $a$ is sufficiently small, the leading term in (21) and (22) would be the dark radiation one, with curvature and cosmological constant terms negligibly small. Specifically, assuming for a moment an equation of state of the form $p = w \rho$ with constant $w$, it is well known that $H^2, H$ and $\rho$ scale as $a^{-3(1+w)}$. Therefore, we may keep the density term and omit the curvature term $\sim 1/a^2$ if $w > -1/3$. In the case of a quadratic potential considered below (for which $w \neq \text{const}$, so the above argument does not directly apply), we have checked numerically that in all bounce scenarios discussed in this paper, this approximation is valid near the bounce point (up to $10^{-7}$).

We will model the matter sector in this pre-bounce epoch by assuming it is described by a scalar field $\phi$ with a potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2. $$

(25)

For calculational simplicity, we put $m = 1$. 

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In this way, we have the following equations modeling the bounce in the Hořava–Lifshitz cosmology,
\[ \dot{H} = -\frac{\kappa^2}{8} \dot{\phi}^2 + \frac{\kappa^4 \mu^2}{32} \frac{k^2}{a^4}, \]  
\[ H^2 = \frac{\kappa^2}{24} (\dot{\phi}^2 + \phi^2) - \frac{\kappa^4 \mu^2}{64} \frac{k^2}{a^4}, \]  
(26)  
(27)

The value of \( \kappa^2 \) may be expressed near the IR limit in terms of cosmological constants (12):
\[ \kappa^2 = 32\pi Gc. \]  
After the suitable time reparameterization, we can set \( 8\pi Gc = 1 \). Therefore, the Friedmann equations take the following form near the bounce,
\[ \dot{H} = -\frac{1}{2} \dot{\phi}^2 + \frac{\mu^2 k^2}{2a^4}, \]  
\[ H^2 = \frac{1}{6} (\dot{\phi}^2 + \phi^2) - \frac{\mu^2 k^2}{4a^4}. \]  
(28)  
(29)

Additionally, completing the dynamics of the system, there is the equation of motion for the scalar field and the definition of the Hubble parameter,
\[ \ddot{\phi} = -\phi - 3\dot{\phi} H, \]  
\[ \dot{a} = a H. \]  
(30)  
(31)

The value of the parameter \( \mu \) may be kept arbitrary. This parameter does not alter solutions of the system (28)–(31), but it specifies values of \( a \) on an obtained trajectory.

4. Phase portrait

4.1. Phase space

The local geometry of the phase portrait is characterized by the nature and position of its critical points. These points are locations where the derivatives of all of the dynamic variables, i.e. the r.h.s. of (42)–(44), vanish. Moreover, they are the only points where phase trajectories may start, end, or intersect. They can also begin or end in infinity, and then—after a suitable coordinate transformation projecting the complete phase space onto a compact region (the so-called Poincaré projection)—there may be well-defined infinite critical points. The set of finite and infinite critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, provides a qualitative description of the given dynamical system.

The dynamics of our system are described by the following set of first order ODEs,
\[ u = \dot{\phi}, \]  
\[ \dot{u} = -\phi - 3u H, \]  
\[ \dot{a} = a H, \]  
\[ \dot{H} = -\frac{1}{2} u^2 + \frac{\mu^2 k^2}{2a^4}, \]  
\[ H^2 = \frac{1}{6} (u^2 + \phi^2) - \frac{\mu^2 k^2}{4a^4}. \]  
(32)  
(33)  
(34)  
(35)  
(36)
If spatial curvature $k = 0$, one may consider a two-dimensional subsystem,

$$u = \dot{\phi},$$

$$\dot{u} = -\phi - 3uH,$$

$$H = -\frac{1}{2}u^2,$$  

with a constraint equation

$$H^2 = \frac{1}{3}(u^2 + \phi^2).$$

If $k \neq 0$, one may also consider a subsystem on variables $(\phi, u, H)$, obtained via reduction of the original system with respect to constraint (36). Namely, substituting

$$\mu^2 \frac{k^2}{4a^4} = \frac{1}{6}(u^2 + \phi^2) - H^2$$

into the equation for $\dot{H}$ and omitting equation on the dynamics of $a$ leads to the following set of equations,

$$u = \dot{\phi},$$

$$\dot{u} = -\phi - 3uH,$$

$$\dot{H} = \frac{1}{3} \left( \phi^2 - \frac{u^2}{2} \right) - 2H^2.$$}

This is a reduced three-dimensional (3D) subset of (32)–(36) on variables $(\phi, u, H)$. If you want to obtain also dynamics of $a$, you need to add to this system equation $\dot{a} = aH$ and also the constraint equation (36).

In subsequent considerations, we shall focus on a case $k \neq 0$ (when HL corrections play a significant role) and a phase portrait of solution of the system (42)–(44) in space of $(\phi, u, H)$, following a similar procedure to that described in [59, 60]. Reducing dimensionality of phase space enables 3D phase portrait visualizations. Moreover, we will discuss shortly also a case $k = 0$, which play a role of a limiting case for $k \neq 0$ dynamics.

We start by rewriting equations (42)–(44) in terms of the variables

$$x \equiv \phi, \quad y \equiv \dot{\phi}, \quad z \equiv \frac{\dot{a}}{a},$$

which give three ‘evolution’ equations

$$\dot{x} = y,$$

$$\dot{y} = -x - 3yz,$$

$$\dot{z} = \frac{1}{3} \left( x^2 - \frac{y^2}{2} \right) - 2z^2.$$}

The space of solution of the above dynamical system is a 3D region of the phase space $(x, y, z)$. This region is bounded by a 2D surface defined by a constraint equation (40)—space of trajectories of a flat universe $(k = 0)$. This limiting surface is a double cone $z^2 = \frac{1}{3}(x^2 + y^2)$, with the upper branch corresponding to expansion and lower one to contraction. These two branches connect at a point: $(0, 0, 0)$, which is a critical point (see below). Hence, there are no trajectories passing from one branch of the cone to the other. For $k = \pm 1$, all trajectories lie between the branches of this cone. Dynamical equations (32)–(36) contain only $k^2$, their solutions are the same for either non-zero value of $k$: $k = -1$ or $k = 1$. This cone is also a
limiting surface for trajectories with large $a$. The further a trajectory lies from this cone, the smaller the values of $a$ along it.

The bounce happens when a phase trajectory passes from the region $z < 0$ to the region $z > 0$, intersecting the plane $z = 0$. At the crossing point, $\dot{z} > 0$ must hold. Equation (48) implies that this happens if the crossing point lies between the lines $y = \sqrt{2}x$ and $y = -\sqrt{2}x$ on the plane $z = 0$. Those lines are the $z = 0$ section of an elliptic cone $\frac{1}{6}(x^2 - y^2) - z^2 = 0$, whose interior consists of trajectories with $\dot{z} > 0$ (equation (48)). The area outside this cone is filled with trajectories along which $\dot{z} < 0$.

To find the finite critical points, we set all right-hand sides of equations (46)–(48) to 0. This gives rise to the conditions

$$x = y = z = 0.$$  \hspace{1cm} (49)

Stability properties of this point are determined by the eigenvalues of the Jacobian of the system (46)–(48). More precisely, one has to linearize transformed equations (46)–(48) at each point. Inserting $\vec{x} = \vec{x}_0 + \delta \vec{x}$, where $\vec{x} = (x, y, z)$, and keeping terms up to first order in $\delta \vec{x}$, leads to an evolution equation of the form $\delta \dot{\vec{x}} = A \delta \vec{x}$. Eigenvalues of $A$ describe stability properties at the given point.

At the finite critical point $O = (0, 0, 0)$, the matrix $A$ has two purely imaginary eigenvalues, which implies there are closed orbits in the $xy$-plane encircling the $z$-axis, i.e. point $O$ lays on a center line surrounded by closed orbits.

To find critical points that occur at infinite values of the parameters, we rescale the infinite space $(x, y, z)$ into a finite Poincaré sphere by means of the coordinate change,

$$x = \frac{X}{1 - r},$$  \hspace{1cm} (50)

$$y = \frac{Y}{1 - r},$$  \hspace{1cm} (51)

$$z = \frac{Z}{1 - r},$$  \hspace{1cm} (52)

where

$$X = r \sin \theta \cos \varphi,$$  \hspace{1cm} (53)

$$Y = r \sin \theta \sin \varphi,$$  \hspace{1cm} (54)

$$Z = r \cos \theta,$$  \hspace{1cm} (55)

$$r^2 = X^2 + Y^2 + Z^2.$$  \hspace{1cm} (56)

We shall use both Cartesian coordinates $(X, Y, Z)$ and spherical ones: $(r, \theta, \varphi)$. We also rescale the time parameter $t$ by defining the new time parameter $T$ such that $dT = dt/(1 - r)$. In these coordinates, our phase space is contained within a sphere of radius one—infinity corresponds to $r = 1$.

This is a conformal transformation; hence, the limiting cone for phase trajectories is $Z^2 = \frac{1}{6}(X^2 - Y^2)$; all physical trajectories are contained within this cone. Bounce points are located on the plane $Z = 0$ within the region bounded by the lines $Y = \sqrt{2}X$ and $Y = -\sqrt{2}X$. The region containing trajectories with $\dot{z} > 0$ (i.e. with $H$ increasing) is bounded by the elliptic cone $\frac{1}{6}(X^2 - Y^2) - Z^2 = 0$. 


Table 1. The properties of the infinite critical points.

| Point | $\varphi$ | $\theta$ | Stability       |
|-------|-----------|----------|----------------|
| S1    | $\arcsin \sqrt{2/3}$ | $\pi/2$   | Saddle line    |
| S2    | $\pi - \arcsin \sqrt{2/3}$ | $\pi/2$   | Saddle line    |
| S3    | $\pi + \arcsin \sqrt{2/3}$ | $\pi/2$   | Saddle line    |
| S4    | $2\pi - \arcsin \sqrt{2/3}$ | $\pi/2$   | Saddle line    |
| A1    | 0         | $\arccos \sqrt{7/7}$ | Attracting line |
| R1    | 0         | $\pi - \arccos \sqrt{7/7}$ | Repelling line |
| R2    | $\pi/2$   | $\arccos \sqrt{7/7}$ | Repelling node |
| A2    | $\pi/2$   | $\pi - \arccos \sqrt{7/7}$ | Attracting node |
| A3    | $\pi$     | $\pi - \arccos \sqrt{7/7}$ | Attracting line |
| R3    | $3\pi/2$  | $\arccos \sqrt{7/7}$ | Repelling line |
| A4    | $3\pi/2$  | $\pi - \arccos \sqrt{7/7}$ | Attracting node |

After Poincaré transformation, equations (46)–(48) take the following form, written in the spherical coordinates $(r, \theta, \varphi)$,

$$r' = \frac{(r - 1)r^2}{48} \cos \theta [82 + 14 \cos 2\theta - 42 \cos 2\varphi + 21 \cos 2(\theta - \varphi) + 21 \cos 2(\theta + \varphi)],$$

$$\theta' = \frac{1}{24} r \sin \theta (5 + 7 \cos 2\theta)(1 + 3 \cos 2\varphi),$$

$$\varphi' = r - 1 - 3r \cos \theta \cos \varphi \sin \varphi.$$

The form of the above equations is similar to the ones obtained in [59, 60]. Taking the limit $r = 1$ and putting the r.h.s. of equations for $\theta'$ and $\varphi'$ to 0, we find 12 solutions for $\theta, \varphi$ at the Poincaré sphere, as shown in the table 1. As we can see, there are four saddle points (more precisely saddle lines with end points at $S_1, S_2, S_3, S_4$). In the contracting part of the phase portrait ($z < 0$), there are two attracting nodes $A_2$ and $A_4$ and two repulsing lines, starting at $R_1$ and $R_3$. Hyperbolic areas near the nodes are bounded by repulsing lines, which play the role of separatrices. The expanding part is a mirror (‘reversed in time’) of the contracting one.

Stability properties of infinite critical points are described in the table 1. Their position in 3D phase space, on a Poincaré sphere, is shown in figure 1.

4.2. Trajectories

When spatial curvature $k = 0$, then phase trajectories lie on the limiting cone $z^2 = \frac{1}{6}(X^2 + Y^2)$, as shown in figure 2. In the contracting part, all trajectories start winding around the $z$-axis, then some of them end at attracting node $A_2$, some at $A_4$. These two families are separated by repelling lines with end-points at $R_1$ or $R_3$, acting as separatrices. The expanding part is a mirror reflection with time reversed of the contracting part.

Trajectories of non-flat universes lie inside the limiting cone of the flat space. In the contracting part of the diagram ($Z < 0$), trajectories start spiraling outside from circles around the $Z$-axis. There are two families of such trajectories, separated by repelling lines ending at $R_1$ and $R_3$. In each family, there are two possible scenarios for subsequent evolution. The
Figure 1. Infinite critical points located on a Poincaré sphere.

Figure 2. Phase trajectories for a flat HL universe.

first one, shown in figure 3(a), is to end at an attractor node (A₂ or A₄). These also lie in the contracting part of the phase diagram. On the way between O and A₂ or A₄, a trajectory may go up through the Z = 0 surface, undergoing a bounce there, and then recollapse, crossing the Z = 0 plane again, or go straight to the attractor node, without bounce. In either case, the end is the Big Crunch.
Figure 3. Different types of phase trajectories for a non-flat Hořava–Lifshitz universe.

The second scenario is shown in figure 3(c). Here, after some oscillations and $H$ decreasing, the trajectories reach an attractor—a repelling line (that ends either at $R_1$ or $R_3$), along which they move until $\dot{H} = 0$. Then they rapidly go up, crossing the $Z = 0$ (i.e. $H = 0$) plane, undergoing a bounce. After that, and after a period of accelerated expansion, they reach another attractor—an attracting line lying in the expanding part, with the endpoint at either $A_1$ or $A_3$. Along this line, trajectories approach the $Z$-axis, winding around it. A subcase of this scenario is shown in figure 3(e), where trajectories do not go through accelerated contraction and expansion, but cross the $Z = 0$ surface during oscillations around the $Z$-axis. This is in fact the scenario described in [37].

Trajectories may also start at repelling nodes $R_2$ or $R_4$ in the expanding part of the diagram. At those points, $H = \infty$, which corresponds to the Bing Bang. After that and a period of extreme, but with decreasing rate, expansion, there are again two possible scenarios. One is
shown in figure 3(b), where trajectories reach an attracting line and end up winding around the Z-axis. Before that, some of them collapse, crossing the Z = 0 plane, then slow down and finally stop contraction, culminating in a bounce. Others show only slow expansion, without crossing Z = 0.

The last scenario is shown in figure 3(d). Trajectories start at Big Bang points R2 or R4, and after a period of slowing down expansion, reach a turning point and start accelerated contraction, ending at Big Crunch points A2 or A4.

For better visualization, we have gathered some described families of trajectories in figure 3(f).

Special attention has to be paid to circular motion around the z-axis. As stability properties of the point (0, 0, 0) and constraint equation (36) suggest, there may exist closed circular orbits lying on a Z = 0 plane (H = 0). But they cannot. Equation (48) does not allow for this, as \( \dot{Z} = 0 \) is fulfilled only on a class of curves lying on \( X^2 - \frac{Y^2}{2} - 6Z^2 = 0 \), i.e. on a surface of the elliptic cone mentioned before. Yet numerical simulations exhibit oscillating solutions, such as \( X^2 + Y^2 = \text{const.} \) and \( Z \) oscillated around 0, \( \dot{Z} > 0 \) between lines \( Y = \sqrt{2}X \) and \( Y = -\sqrt{2}X, \dot{Z} < 0 \) outside this region. Such a trajectory resembles the deformed circle. These solutions appear for sufficiently small X and Y, for larger values of X and Y, numerical simulations show slow decreasing of the radius of this 'circle'.

Finally, note that except for the special solution shown in figure 3(e), which is the bounce described by Brandenberger [37], there are also other types of bounces. One, probably the most interesting, is shown in figure 3(c). Here, a big existing universe slowly starts to contract, but later on the contraction becomes exponential, until a bounce is reached and an exponential expansion begins, which finally slows down. Another type of bounce, shown in figure 3(a), happens again when a big universe slowly starts contracting, stops and goes through an expanding phase for a while, then recollapses and ends at the Big Crunch. The last one, shown in figure 3(b), happens during a transition from the Big Bang to a quasi stationary final stage (with H slowly decreasing), however with a bounce on the way.

Trajectories shown in the figures discussed above are numerical solutions of the equations (57)–(59). To find a variety of bounce scenarios we investigated initial conditions: \( \theta = \pi/2 \pm 0.01, \phi = i \pi/20 \) (i = 1...20), each for \( r = j/10 \) (j = 1...9) and \( r = 0, 9 + j \cdot 0.01 \); time in range \([-20, 20]\). This procedure picked up the classes of trajectories discussed above. In general, it may not be exhaustive in the sense that qualitatively different behavior of solutions may be possible. However, it is sufficient for the purpose of understanding how bouncing scenarios emerge here due to the specific modification of general relativity which appears in Hořava’s theory.

5. Discussion and conclusions

In this paper, we have investigated the cosmological bounce in the Hořava–Lifshitz gravity. Using the 3D flow visualization technique, we have found that phase portraits in the considered theory have a different structure than in standard cosmology. Comparing with results from the paper [60], we can see that there are additional repellers (R1 and R2) in the contracting part of a phase space, and mirror attractors in the expanding part. Their presence allows for the existence of a bounce, because now there are possible new families of trajectories, starting at additional repellers in the contracting part, and possibly ending at new attractors in the expanding part, or surrounding the (0, 0, 0) point, which is now a center, compared to a saddle in standard cosmology. Those are realizations of the bounce. One of them is the solution with oscillatory behavior described in [37]; there are, however, additional possibilities. The most
interesting one contains a period of rapid contraction, and—after a bounce—a period of rapid expansion, which may fit the inflationary scenario.

Nevertheless, there are still initial conditions that lead to the Big Crunch, as shown in figures 3(a) and (d), or which start at initial singularity (figures 3(b) and (d)). Hence, the existence of a bounce is not generic for Hořava theory and depends on the initial conditions.

Another interesting class of solutions consists of quasi stationary universes. These solutions are described in phase space by closed orbits, winding around the critical point \((0,0,0)\)—a center. All trajectories in the neighborhood of this point end up as closed orbits, ‘deformed circles’. Equations of motion do not allow closed orbits lying on the \(Z = \text{const.}\) plane, resulting in slight deformation of the circular orbits. The values of \(H\) oscillate around the stationary stage, for sufficiently small values of \(\varphi\) and \(\dot{\varphi}\). Values of the scale parameter \(a\) during this evolution are much bigger than the regime for which our simplifications are valid. Therefore, this behavior is not a feature of the Hořava–Lifshitz theory, but of cosmologies with modified equations of motion, i.e. with the additional term \(\sim 1/a^4\) in the Friedmann equations. Still, presence of this term leads to a different solution than induced by a negative potential, as in [60], due to different stability properties of finite critical points there.

The visualizations presented in this paper describe the dynamics of the Hořava–Lifshitz universe in the regime of small scale factor \(a\), when standard curvature and \(\Lambda\) terms are not relevant. Even in such slightly limited framework, they answer the question of possible scenarios realizing a bounce, and whether it is generic for the theory or not. It appears not, as we have found solutions leading to infinite collapse, or starting at the initial singularity, both staying within the regime of small \(a\). There is also an interesting possibility of quasi stationary, oscillating universe, existence of which is clearly implied by the dark radiation term in the Friedmann equations.

The above portraits of the matter bounce in HL cosmology are attributed only to a homogenous and isotropic model. Possible deviations from isotropy may become dominant in the small volume limit, as it happens in GR [61–64]. Thus, the next step to be taken in the research on the realistic matter bounce is to analyze the effects of anisotropies in the HL gravity, extending the standard analysis of Belinskii, Khalatnikov and Lifshitz. The addition of shearing components, due to anisotropies, may make the bounce unstable leading possibly to the BKL chaotic behavior at the Big Crunch singularity. On the other hand, they may prevent the Universe from collapsing to the singularity and thus avoiding the Big Crunch, which is found in some solutions of the theory.

It was shown [65–67] that adding \((\mathcal{A} R)^2\) (and possibly other) curvature terms to the gravitational action in GR suppressed near the singularity chaotic behavior induced by the linear curvature term. One would expect a similar effect in the HL gravity, where the gravitational action potential consists of higher order curvature terms. Indeed, the recent study of the mixmaster universe in the Horava gravity [68] reveals that in certain cases, chaos is absent when the singularity is approached, there are also possible harmonic oscillations around the fully isotropic model. However different study [69] shows that in Hořava gravity with the zero cosmological constant the presence of the higher curvature terms in the HL action cannot suppress chaotic behaviors induced by the IR part of the action. Still, there is a possibility that extending the HL gravitational potential beyond the detailed balance form, i.e. adding additional higher curvature terms as in modified HL theories [20, 22–27] may have stabilizing effect. Clearly, further analysis is necessary, although it is beyond the scope of this paper. For this purpose the singularity theorems of general relativity have to be revisited and examined under which conditions they remain valid in theories with anisotroping scaling.
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References

[1] Khoury J, Ovrut B A, Steinhardt P J and Turok N 2001 Phys. Rev. D 64 123522 (arXiv:hep-th/0103239)
[2] Buchbinder E I, Khoury J and Ovrut B A 2007 Phys. Rev. D 76 123503 (arXiv:hep-th/0702154)
[3] Steinhardt P J and Turok N 2002 Science 296 1436 (arXiv:hep-th/0111030)
[4] Steinhardt P J and Turok N 2002 Phys. Rev. D 65 126003 (arXiv:hep-th/0111098)
[5] Steinhardt P J and Turok N 2002 Phys. Rev. D 66 101302 (arXiv:astro-ph/0112537)
[6] Tolley A J and Turok N 2002 Phys. Rev. D 66 106005 (arXiv:hep-th/0204091)
[7] Niz G and Turok N 2007 Phys. Rev. D 75 026001 (arXiv:hep-th/0601007)
[8] Copeland E J, Niz G and Turok N 2010 Phys. Rev. D 81 126006 (arXiv:1001.5291 [hep-th])
[9] Kallosh R, Kofman L and Linde A D 2001 Phys. Rev. D 64 123523 (arXiv:hep-th/0104073)
[10] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. Lett. 96 14130 (arXiv:gr-qc/0602086)
[11] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. D 73 124038 (arXiv:gr-qc/0604013)
[12] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. D 74 084003 (arXiv:gr-qc/0607039)
[13] Singh P, Vandersloot K and Vereshchagin G V 2006 Phys. Rev. D 74 043510 (arXiv:gr-qc/0606032)
[14] Corichi A and Singh P 2008 Phys. Rev. Lett. 100 161302 (arXiv:0710.4543 [gr-qc])
[15] Cailleteau T, Cardoso V, Vandersloot K and Wand D 2008 Phys. Rev. Lett. 101 251302 (arXiv:gr-qc/0808.0190)
[16] Maertens R and Koyama K 2010 Living Rev. Rel. 13 5 (arXiv:1004.3962) http://www.livingreviews.org/lrr-2010-5
[17] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4690
[18] Novello M and Perez S E 2008 Phys. Rep. 463 127–213 (arXiv:0802.1634 [astro-ph])
[19] Horava P 2009 J. High Energy Phys. JHEP03(2009)020 (arXiv:0812.4287 [hep-th])
[20] Horava P 2009 Phys. Rev. D 79 084008 (arXiv:0901.3775 [hep-th])
[21] Horava P 2009 Phys. Rev. Lett. 102 161301 (arXiv:0902.3567 [hep-th])
[22] Nastase H 2009 arXiv:0904.3604 [hep-th]
[23] Kehagias A and Stefanou K 2009 Phys. Lett. B 678 123 (arXiv:0905.0477 [hep-th])
[24] Sotiriou T P, Visser M and Weinhardt S 2009 Phys. Rev. Lett. 102 251601 (arXiv:0904.4641 [hep-th])
[25] Sotiriou T P, Visser M and Weinhardt S 2009 J. High Energy Phys. JHEP10(2009)033 (arXiv:0905.2798 [hep-th])
[26] Blas D, Pujolas O and Sibiryakov S 2010 Phys. Rev. Lett. 104 181302 (arXiv:0909.3525 [hep-th])
[27] Horava P and Melby-Thompson C M 2010 Phys. Rev. D 82 064027 (arXiv:0910.2410 [hep-th])
[28] Mukohyama S 2010 Class. Quantum Grav. 27 223101 (arXiv:1007.3199 [hep-th])
[29] Sotiriou T P 2010 based on a talk given at 14th Conference on Recent Developments in Gravity (NEBXIV) Ioannina, Greece, 8–11 June 2010 arXiv:1010.3218v1 [hep-th]
[30] Calcagni G 2009 J. High Energy Phys. JHEP09(2009)112 (arXiv:0904.0829 [hep-th])
[31] Kiritis E and Kofinas G 2009 Nucl. Phys. B 821 467 (arXiv:0904.1334 [hep-th])
[32] Saridakis E N 2010 Eur. Phys. J. C 67 225 (arXiv:0905.3532 [hep-th])
[33] Mukohyama S, Nakayama K, Takahashi F and Yokoyama S 2009 Phys. Lett. B 679 6 (arXiv:0905.0555 [hep-th])
[34] Minamitsuji M 2010 Phys. Lett. B 684 194 (arXiv:0905.3892 [astro-ph])
[35] Wang A and Wu Y 2009 J. Cosmol. Astropart. Phys. JCAP07(2009)012 (arXiv:0905.4117 [hep-th])
[36] Takahashi T and Soda J 2009 Phys. Rev. Lett. 102 231301 (arXiv:0904.0554 [hep-th])
[37] Brandenberger R 2009 Phys. Rev. D 80 043516 (arXiv:0904.2835 [hep-th])
[38] Gao X, Wang Y, Xue W and Brandenberger R 2010 J. Cosmol. Astropart. Phys. JCAP02(2010)020 (arXiv:0911.4161 [hep-th])
[39] Charmousis C, Niz G, Padilla A and Saffin P M 2009 J. High Energy Phys. JHEP08(2009)070 (arXiv:0905.2579 [hep-th])
[40] Li M and Pung Y 2009 J. High Energy Phys. JHEP08(2009)015 (arXiv:0905.2751 [hep-th])
[41] Bogdanos C and Saridakis E N 2010 Class. Quantum Grav. 27 075005 (arXiv:0907.1636 [hep-th])
[42] Iengo R, Russo I G and Serone M 2009 J. High Energy Phys. JHEP11(2009)020 (arXiv:0906.3477 [hep-th])
[43] Cai R G, Hu B and Zhang H B 2009 Phys. Rev. D 80 041501 (arXiv:0905.0255 [hep-th])
[44] Blas D, Pujolas O and Sibiryakov S 2009 J. High Energy Phys. JHEP01(2009)029 (arXiv:0906.3046 [hep-th])
[45] Blas D, Pujolas O and Sibiryakov S 2010 Phys. Lett. B 688 350 (arXiv:0912.0550 [hep-th])

[46] Carloni S, Elizalde E and Silva P J 2010 Class. Quantum Grav. 27 045004 (arXiv:0909.2219 [hep-th])

[47] Leon G and Saridakis E N 2009 J. Cosmol. Astropart. Phys. JCAP11(2009)006 (arXiv:0909.3571 [hep-th])

[48] da Silva A M 2011 Class. Quantum Grav. 28 055011 (arXiv:1009.4885 [hep-th])

[49] Germani C, Kehagias A and Sfetsos K 2009 J. High Energy Phys. JHEP09(2009)060 (arXiv:0906.1201 [hep-th])

[50] Suyama T 2010 J. High Energy Phys. JHEP01(2010)093 (arXiv:0909.4833 [hep-th]).

[51] Capasso D and Polychronakos A P 2010 J. High Energy Phys. JHEP02(2010)068 [arXiv:0909.5405 [hep-th])

[52] Rama S K 2009 arXiv:0910.0411 [hep-th]

[53] Kritsis E and Kofinas G 2010 J. High Energy Phys. JHEP01(2010)122 (arXiv:0910.5487 [hep-th])

[54] Sindoni L 2009 arXiv:0910.1329 [gr-qc]

[55] Kouretsis A P, Stathakopoulos M and Stavrinos P C 2010 Phys. Rev. D 82 064035 (arXiv:1003.5640 [gr-qc])

[56] Carloni S, Elizalde E and Silva P J 2010 arXiv:1009.5319v1 [hep-th]

[57] Binétruy P, Deffayet C, Ellwanger U and Langlois D 1989 Phys. Lett. B 285 (arXiv:hep-th/9910219)

[58] Bogdanos C and Saridakis E N 2010 Class. Quantum Grav. 27 075005 (arXiv:0907.1636)

[59] Felder G N, Frolov A and Kofman L 2002 Class. Quantum Grav. 19 2983 (arXiv:hep-th/0112165)

[60] Felder G N, Frolov A, Kofman L and Lande A 2002 Phys. Rev. D 66 023507 (arXiv:hep-th/0202017)

[61] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 Adv. Phys. 19 525

[62] Belinskii V A, Khalatnikov I M and Lifshitz E M 1972 Sov. Phys.—JETP 35(5) 838

[63] Belinskii V A, Khalatnikov I M and Lifshitz E M 1982 Adv. Phys. 31 639

[64] Damour T, Henneaux M and Nicolai H 2003 Class. Quantum Grav. 20 R145 (arXiv:hep-th/0212256)

[65] Barrow J D and Sirrouse-Zia H 1989 Phys. Rev. D 39 2187

[66] Barrow J D and Cotsakis S 1989 Phys. Lett. B 232 172

[67] Cotsakis S, Demaret J, de Rop Y and Querella L 1993 Phys. Rev. D 48 4595

[68] Bakas I, Bourliot F, Lust D and Petropoulos M 2010 Class. Quantum Grav. 27 045013 (arXiv:0911.2665 [hep-th])

[69] Myung Y S, Kim Y W, Son W S and Park Y J 2010 J. High Energy Phys. JHEP03(2010)085 (arXiv:1001.3921 [gr-qc])