ON THE APPROXIMATION OF DYNAMICAL INDICATORS IN SYSTEMS WITH NONUNIFORMLY HYPERBOLIC BEHAVIOR

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Abstract. Let $f$ be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold and $\mu$ an ergodic hyperbolic measure with positive entropy. We prove that for every continuous potential $\phi$ there exists a sequence of basic sets $\Omega_n$ such that the topological pressure $P(f|\Omega_n, \phi)$ converges to the free energy $P_\mu(\phi) = h(\mu) + \int \phi d\mu$. We also prove that for a suitable class of potentials $\phi$ there exists a sequence of basic sets $\Omega_n$ such that $P(f|\Omega_n, \phi) \to P(\phi)$.

1. Introduction

This is work is concerned with the approximation of dynamical indicators in systems with nonuniformly hyperbolic behavior.

(Uniformly) hyperbolic dynamics is characterized by a (continuous) decomposition of the tangent space $T_x M$ into invariant subspaces which are contracted (resp. expanded) by the derivative. The local instability of the orbits generated by this structure and the recurrence due to the compactness of the space gives rise to a complex and very rich orbit structure which is well understood. Among other things uniformly hyperbolic systems exhibit strong recurrence and mixing properties, many invariant measures, positive entropy and abundance of periodic points. Moreover, they are robust and structurally stable and can be modelled by Markov chains both topologically and from the measure-theoretical point of view. See [11] and [13] for a comprehensive presentation of the theory.

Nonuniformly hyperbolic dynamical systems were introduced by Pesin in the early seventies as a generalization the notion of uniformly hyperbolic dynamics. Invariant measures are at the heart of the theory of nonuniformly hyperbolic systems. We say that an $f$-invariant Borel probability $\mu$ is hyperbolic if all its Lyapunov exponents

$$\chi(x, v) = \lim_{n \to \pm \infty} \frac{\|Df^n(x)v\|}{n}, \quad \forall \ v \in T_x M - \{0\}$$

are nonzero $\mu$-a.e. By Oseledec’s theorem if $\mu$ is hyperbolic then there exists an $f$-invariant Borel subset $K \subset M$ and a splitting into stable $E^s_x$ and unstable $E^u_x$ Borel measurable fields of subspaces –in opposition to continuous– of the tangent space over $K$. Vectors in $E^s_x$ (resp. $E^u_x$) are asymptotically contracted (resp. expanded) by the derivative, that is, the time $N$ needed for every vector $v \in E^s_x$ (resp. $v \in E^u_x$) to be contracted (resp. expanded) depends on $x$ in a very irregular way –typically as a Borel function of the point– moreover the angle $\angle(E^s_x, E^u_x)$ is a Borel function of $x$ and decays to zero with subexponential rates along the orbits. This set of conditions define the notion of nonuniform hyperbolicity. See subsection 2.3 for details.
We refer to [4] for an up-to-date overview of the theory and to [18] for a survey on open problems on nonuniformly hyperbolic dynamical systems.

The following well known theorem due to A. Katok and L. Mendoza is a sample of the type of results that we are interested:

Let $f$ be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold $M$ and $\mu$ be a hyperbolic measure with positive metric entropy. Suppose in addition that $\mu$ is ergodic. Then there exists a sequence of hyperbolic horseshoes $\Omega_n$ and ergodic measures $\mu_n$ supported on $\Omega_n$ such that:

- $\mu_n \to \mu$, in the weak* topology and
- $h(\mu_n) \to h(\mu)$.

See [14, Theorem S.5.10].

A. Katok laid down the foundations to study this type of problems in his seminal paper [12] about relations between entropy, periodic orbits and Lyapunov exponents of systems with nonuniformly hyperbolic behavior. More recently these questions received attention in [7], [8], [9], [15], [16], [20], [21] and [24].

Katok-Mendoza’s theorem suggests to ask whether or not it is possible to approximate, along suitable sequences of hyperbolic sets, dynamical indicators such as topological pressure, fractal dimensions and Lyapunov exponents, in systems with nonuniformly hyperbolic behavior.

In this note we make the case for $P(\phi)$, the topological pressure of a continuous potential $\phi$. This quantity is a topological invariant of the dynamics which generalizes the notion of topological entropy and can be defined as a weighted rate of growing of the number of finite, dynamically non equivalent orbits, up to finite precision. A central result in the thermodynamic formalism is the following variational principle,

$$P(\phi) = \sup_{\mu \in \mathcal{M}_f} \left\{ h(\mu) + \int \phi d\mu \right\},$$

where $h(\mu)$ denotes the Kolmogorov-Sinai entropy, $P_\mu(\phi) := h(\mu) + \int \phi d\mu$ is the free energy or measure-theoretical pressure and $\mathcal{M}_f$ the set of $f$-invariant Borel probabilities. See [5] and [13].

This notion plays a central role in the ergodic theory of systems with some hyperbolicity as long as some valuable information about Lyapunov exponents, fractal dimensions, multifractal spectra and invariant measures which are extreme points of certain variational principles can be extracted from the topological pressure of suitable potentials. See [2] and [3].

Our first result is a generalization of Katok-Mendoza’s theorem for the free energy of a continuous potential $\phi$ with respect to a hyperbolic measure $\mu$ with positive entropy.

Let $\mu_n$ be the sequence of hyperbolic measures given by Katok-Mendoza’s theorem. Then, for every continuous $\phi$

$$h(\mu_n) + \int \phi d\mu_n \to h(\mu) + \int \phi d\mu,$$

and therefore, by the variational principle (2),

$$P(\phi) \geq \limsup_{n \to +\infty} P(f|\Omega_n, \phi) \geq \liminf_{n \to +\infty} P(f|\Omega_n, \phi) \geq h(\mu) + \int \phi d\mu.$$

We prove that $\Omega_n$ can be chosen carefully in such way that the limit exists and it is equal to the measure-theoretical pressure $P_\mu(\phi)$.

**Theorem A** Let $f$ be a $C^{1+\alpha}$ diffeomorphism of a compact manifold and $\mu$ an ergodic, hyperbolic measure with $h(\mu) > 0$. Then, for every continuous potential $\phi$ there exists a sequence
of basic sets $\Omega_n = \Omega_n(\mu, \phi)$ with rate of hyperbolicity $\chi(\Omega_n) \geq \beta$ for some constant $\beta > 0$ only depending on $\chi(\mu)$, the rate of hyperbolicity of $\mu$, such that

\begin{equation}
P(f|\Omega_n, \phi) \to h(\mu) + \int \phi d\mu.
\end{equation}

Furthermore $\Omega_n$ has the following strong approximation property: $\mu_n \to \mu$ for every sequence of ergodic measures $\mu_n \in M_f(\Omega_n)$.

Recently Gelfert announces a similar result in [10] for $C^{1+\alpha}$ diffeomorphisms or $C^1$ diffeomorphisms preserving a hyperbolic $f$-invariant probability having a dominated splitting on its support. We recall that the rate of hyperbolicity of an Oseledec’s regular point $x$ as

\[ \chi(x) := \min \{|\chi(x,v)| : v \in T_x M - \{0\}\}. \]

Given an $f$-invariant Borel probability $\mu$, we define the rate of hyperbolicity of $\mu$ as

\[ \chi(\mu) := \int \chi(x) d\mu(x) \]

Also, given a compact $f$-invariant subset $\Omega$ we define its rate of hyperbolicity as

\[ \chi(\Omega) := \inf \{ \chi(\mu) : \mu \in M_f(\Omega) \}, \]

where $M_f(\Omega)$ is the set of $f$-invariant probabilities in $\Omega$.

The probabilities $\mu_n$ provided by Katok-Mendoza’s theorem are measures of maximal entropy, that is, $h(\mu_n) = h_{top}(f|\Omega_n)$. Therefore if there would exists a sequence of ergodic hyperbolic measures $\mu_n$ with positive entropy such that $h(\mu_n) \to h_{top}(f)$ then, by an easy ‘diagonal’ argument we can find a sequence of hyperbolic horseshoes $\Omega_n$ such that

\[ h\text{top}(f|\Omega_n) \to h\text{top}(f). \]

Of course, a good amount of hyperbolicity in the phase space is necessary for this type of approximation results. Following this idea one may ask whether or not there exists in systems with sufficient hyperbolicity, a sequence of hyperbolic sets $\Omega_n$ such that

\[ P(f|\Omega_n, \phi) \to P(\phi), \]

for every continuous $\phi$. However, the following example shows that the answer to this question is, in general, negative, even if the system is nonuniformly hyperbolic.

**Example 1.1.** Let $\Omega_0 \subset \mathbb{D}^2$ be a horseshoe with internal tangencies defined inside a compact disc $\mathbb{D} \subset \mathbb{R}^2$. This is a nonuniformly hyperbolic set with positive topological entropy. See [6]. Let us plug $\Omega_0$ as a compact $f$-invariant subset of a $C^\infty$ diffeomorphism of the sphere in the usual way with a source at the north-pole $N$ and a sink at the south-pole $S$. Notice that $h_{top}(f) = h_{top}(f|\Omega_0)$ and that every $f$-invariant basic set $\Omega$ is contained in $\Omega_0$. Let $\phi$ be a continuous function such that $\phi(N) = 2h_{top}(f)$ and $\phi(x) = 0$, for every $x \notin U$, where $U$ is small neighborhood of $N$ contained in the connected component of $W^u(N)$ containing $N$. Then every point ergodic $f$-invariant Borel probability is either $\delta_N$, an ergodic measure supported on $\Omega$ or $\delta_S$, the Dirac measure concentrated at the south-pole. Then,

\[ P(\phi) = \sup_{\mu \in M_f} \{ h(\mu) + \int \phi d\mu \} = \phi(N) = 2h_{top}(f) \]

and, for every $f$-invariant basic set $\Omega \subset S^2$,

\[ P(\phi) > P(f|\Omega, \phi). \]
This happens since the support of $\phi$ is away from the part of phase space where basic sets are located. Therefore if $\phi$ captures the hyperbolicity of the phase space then it would be possible to approximate $P(\phi)$ by suitable sequences of hyperbolic sets.

**Definition 1.1.** We say that $\phi$ is a **hyperbolic potential** if

$$P(\phi) - \sup_{\mu \in \mathcal{M}} \int \phi d\mu > 0$$

and there exists a sequence of ergodic hyperbolic measures $\mu_n$ such that

$$h(\mu_n) + \int \phi d\mu_n \to P(\phi).$$

We denote by $C(\mathcal{H})$ the set of hyperbolic potentials.

**Theorem B** Let $f$ be a $C^{1+\alpha}$ diffeomorphism of a compact manifold with $h_{\text{top}}(f) > 0$. Then, for every $\phi \in C(\mathcal{H})$ there exists a sequence $\Omega_n$ of basic sets such that

$$P(f|\Omega_n, \phi) \to P(\phi).$$

In particular, it holds the following variational equation:

$$P(\phi) = \sup_{\Omega \in \mathcal{H}} P(f|\Omega, \phi),$$

where $\mathcal{H}$ is the family of $f$-invariant basic sets.

Hyperbolic potentials were introduced in [7] by K. Gelfert and C. Wolf. There they proved that the topological pressure of these potentials can be computed as a weighted rate of growing of hyperbolic periodic orbits filtrated according to the quality of its hyperbolicity.

**Proof of Theorem B:** let $\phi \in C(\mathcal{H})$ be a hyperbolic potential, the existence of a sequence of approximating basic sets $\Omega_n$ for $P(\phi)$ follows from Theorem A by the following straightforward 'diagonal' argument: let $\phi \in C(\mathcal{H})$ and $\mu_n$ be a sequence of hyperbolic $f$-invariant ergodic probabilities such that $h(\mu_n) + \int \phi d\mu_n \to P(\phi)$. Then $h(\mu_n) > 0$ for every sufficiently large $n$. Indeed, let $0 < \epsilon < P(\phi) - \sup_{\mu \in \mathcal{M}} \int \phi d\mu$. Then, $h(\mu_n) + \int \phi d\mu_n > P(\phi) - \epsilon$ for every large $n$. Therefore,

$$h(\mu_n) > P(\phi) - \int \phi d\mu_n - \epsilon \geq P(\phi) - \sup_{\mu \in \mathcal{M}} \int \phi d\mu - \epsilon > 0.$$

By Theorem A, for each $\mu_n$ there exists a sequence of basic sets $\Omega_n^m$ such that $P(f|\Omega_n^m, \phi)$ converges to the free energy $h(\mu_n) + \int \phi d\mu_n$. Passing to a suitable 'diagonal' sequence $\Omega_n = \Omega_n^m$ we get a sequence of basic sets such that $P(f|\Omega_n, \phi) \to P(\phi)$. The variational equation (4) holds since $P(\phi) \geq P(f|\Omega, \phi)$ for every compact $f$-invariant subset. QED

Theorem A is a consequence of the methods that we developed with S. Luzzatto in [16] and an idea of Mendoza in [17]. Of course hyperbolic horseshoes in Katok-Mendoza’s theorem [14, Theorem S.5.10] are basic sets. However our construction differ in several points from [14]. In particular, our approximating sets are $f$-invariant saturate of horseshoes with finitely many branches with variable return time (see [21]) with the strong approximation property mentioned at Theorem A.

We start observing that the measure-theoretical pressure $P_{\text{th}}(\phi) = h(\mu) + \int \phi d\mu$ is a weighted rate of growing of dynamically non equivalent finite typical orbits, up to finite precision (see Proposition 3.3 in section 3). Compare [17, Theorem 1.1]. Then we draw carefully finitely many finite orbits which are a good sample for this statistic with small precision. These orbits return to a suitable non invariant uniformly hyperbolic set or Pesin set giving rise to **hyperbolic**
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branches \( f^n_i : S_i \rightarrow U_i \) with variable return times \( n_i \), where by hyperbolic branches we mean diffeomorphisms mapping 'vertical' strips \( S_i \) onto 'horizontal' strips \( U_i \) inside a fixed rectangle crossing each other transversally according to Smale's horseshoe model, contracting in the vertical direction and expanding distances in the horizontal. See subsection 2.1 and definition 2.9. This is a consequence of the pseudo-Markov property of coverings of the hyperbolic Pesin sets by regular Lyapunov rectangles (see Proposition 2.11 in subsection 2.3). Let \( \Omega^* \) be the maximal invariant subset of the piecewise smooth map \( F \) defined by the hyperbolic branches so chosen. \( \Omega^* \) is endowed with a hyperbolic product structure according to [23, Definition 1], that is, two transversally intersecting continuous laminations \( F^s \) and \( F^u \) with an angle bounded from below which are contracted (resp. expanded) exponentially by iteration \( F \) and such \( \Omega^* = \bigcup F^s \cap \bigcup F^u \). See subsection 2.1.

The hyperbolic branches \( f^n_i : S_i \rightarrow U_i \) so chosen are quasi-generic meaning that the iterates of every point \( x \in S_i \) gives a good approximation of \( \mu \), up to a small precision. Then every ergodic measure supported on \( \Omega \), the \( f \)-invariant saturate of \( \Omega^* \), is near to \( \mu \) in the weak topology.

Then we prove that \( P(\mathbb{F}|\Omega, \phi) \) is good approximation of the measure-theoretical pressure \( P_{\mu}(\phi) \) by estimating the topological pressure of \( \Omega \) as a weighted rate of growing of hyperbolic periodic orbits in \( \Omega \),

\[
P(\mathbb{F}|\Omega, \phi) = \lim_{n \to +\infty} \frac{1}{n} \log \left( \sum_{x \in \text{Per}_n(\mathbb{F}|\Omega)} \exp \left( \sum_{j=0}^{n-1} \phi(f^j(x)) \right) \right).
\]

For this we use a shadowing argument to compare the weight of the periodic orbits of \( \Omega \) with the weight of the chosen sample of finitely many returning points generating \( \Omega^* \). See section 5. Here some care has to be taken to keep track of the combinatorics of periodic orbits, due to the variable return times defining \( \Omega^* \). This is done in section 4.

**Organization of the paper.** The paper is organized as follows: Section 2 contain preliminary material to the proof of Theorem A: in subsections 2.1 and 2.2 we recall the notion of an Alekseev set and the uniform approximation property. In subsection 2.3 we recall main definitions of nonuniformly hyperbolic dynamics, Pesin sets and pseudo-Markov property used in the construction of Alekseev sets. We give the arguments to choose \( \Omega \) at section 4. The estimation of the topological pressure is done at sections 4 and 5.

2. **Proof of Theorem A: preliminaries**

2.1. **The geometrical model: Alekseev sets.** Our geometric model will be defined by a finite collection \( \mathcal{S} \) of pairwise disjoint stable cylinders \( \{S_1, \ldots, S_N\} \) and corresponding pairwise disjoint collection \( \mathcal{U} \) of unstable cylinders \( \{U_1, \ldots, U_N\} \) contained in a rectangle \( R \) which are the domain (resp. co-domain) of suitable hyperbolic branches

\[
f^{R_i} : S_i \rightarrow U_i
\]
defined by finitely many return times \( R_i \). By hyperbolic we mean that \( f^{R_i} \) contracts (resp. expands) in the 'vertical' (resp. 'horizontal') directions; that is, they preserve suitable continuous families of cone. See Definition 2.2 in subsection 2.2 for details.

**Definition 2.1.** An Alekseev set \( \Omega^* \) is defined by an array of hyperbolic branches \( \{f^{R_i} : S_i \rightarrow U_i\} \) all whose stable cylinders \( S_i \) 'crosses' all \( U_i \)’s transversally and such that every \( U_i \) 'crosses' all \( S_i \)’s transversally. \( \Omega^* \) is the maximal invariant set in \( R \) under iterations of \( f^{R} \) and its inverse

\[
\Omega^* := \bigcap_{n \in \mathbb{Z}} (f^R)^n(R),
\]
where $f^R : \bigcup S_i \to \bigcup U_i$ is the piecewise smooth invertible map defined by

$$f^R|_{S_i} := f^R_i|_{S_i} \quad \text{and} \quad (f^R)^{-1}|_{U_i} := f^{-R_i}|_{U_i}.$$ 

**Remark 2.1.** This construction was originated in the work of M. V. Alekseev aiming at to describe topological analogues of Markov chains. See [1].

The next couple of technical results were proved in [16, Section 3].

**Lemma 2.2.** $\Omega^*$ is an $f^R$-invariant Cantor set endowed with a hyperbolic product structure by which we mean two continuous laminations of local $f^R$-invariant manifolds $F^S$ (resp. $F^U$) with angles uniformly bounded from below by a constant $>0$ which are exponentially contracted (resp. expanded) by $f^R$ and such that

$$\Omega^* = \left( \bigcup F^S \right) \cap \left( \bigcup F^U \right).$$

Remark 2.2. The sequences $\Omega_n$ in Theorem A approximate uniformly $\mu$ in that $\mu_n \to \mu$ for every sequence $\mu_n$ of ergodic measures such that $\text{supp} \mu \subset \Omega_n$. Actually, given an open neighborhood $\mathcal{N}$ of $\mu$ in the weak-* topology our methods allows to construct hyperbolic basic sets $\Omega = \Omega(\mathcal{N})$ such that $\nu \in \mathcal{N}$ for every ergodic Borel probability $\nu$ supported on $\Omega$. This is done as follows.

Let $\mu$ be a Borel probability satisfying our main hypotheses. First recall that a point $x$ is generic for $\mu$ if

$$\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) \to \int \phi d\mu \quad \text{as} \quad n \to \infty \quad \text{for all continuous functions} \phi \in C^0(M).$$

Given a countable dense subset $\{\psi_i\}$ of $C^0(M)$ we denote, given two constants $\rho, s > 0$, the weak-* $(\rho, s)$ neighborhood of $\mu$

$$\mathcal{O}(\mu, \rho, s) := \langle \nu : \left| \int \psi_i d\mu - \int \psi_i d\nu \right| < \rho, \quad i = 1, \cdots, s \rangle;$$

Clearly, $\mu_n \to \mu$ in the weak-* topology if and only if there are sequences $\rho_n \to 0^+$ and $s_n \to +\infty$ such that $\mu_n \in \mathcal{O}(\mu, \rho_n, s_n)$.

**Definition 2.4.** We say that a point $x$ is $(\rho, s, n)$ quasi-generic for the measure $\mu$ if

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(f^j(x)) \right| \leq \frac{1}{n} \int \phi_i d\mu \quad \forall i \leq s.$$

Furthermore, we say that a hyperbolic branch $f^n : S \to U$. 

is \((\rho, s)\)-quasi-generic for \(\mu\) if every \(x \in S\) is \((\rho, s, n)\) quasi-generic for \(\mu\).

We underline that to be \((\rho, s, n)\)-quasi-generic simply means that the empirical measure \(E_{x,n} = 1/n \sum_{k=0}^{n-1} \delta_{f^k(x)}\) belongs to \(O(\mu, \rho, s)\).

**Proposition 2.5.** Let \(\rho, s > 0\) and suppose there exists an Alekseev set \(\Omega^*(\rho, s)\) defined by \((\rho, s)\) quasi-generic branches. Then \(\mu_\Omega \in O(\mu, 3\rho, s)\) for every \(f\)-invariant ergodic probability measure \(\mu_\Omega\) supported on \(\Omega(\rho, s)\), the \(f\)-invariant saturate of \(\Omega^*(\rho, s)\). In particular,

\[
M_f(\Omega(\rho, s)) \subset O(\mu, 3\rho, s),
\]

where \(M_f(\Omega(\rho, s))\) denotes the set of \(f\)-invariant Borel probabilities supported on \(\Omega(\rho, s)\).

We refer the reader to [16] for details.

### 2.3. Nonuniform hyperbolicity and pseudo-Markov coverings

In this section we collect some facts on nonuniformly hyperbolic dynamics necessary to prove Theorem A.

Let \(f\) be a \(C^r\) \((r \geq 1)\) diffeomorphism of a compact manifold \(M\). We say that a point \(x\) is **Oseledec regular** if there exists numbers \(\chi_1(x) < \cdots < \chi_k(x)\) and a sum direct decomposition into subspaces \(T_xM = \bigoplus_{i=1}^k E_i(x)\) such that

\[
\lim_{n \to \pm \infty} \frac{\log \|DF^n(x)v\|}{n} = \chi_i(x) \quad \forall v \in E_i(x) - \{0\}.
\]

Notice that if \(x\) is regular then so is \(f^n(x)\) for every \(n \in \mathbb{Z}\) and therefore we can speak of a regular orbit.

Oseledec’s theorem (see [3] and [13]) says that the set of regular points \(\Sigma\) is a Borel subset of total probability i.e. it has \(\mu(\Sigma) = 1\), for every \(f\)-invariant Borel probability \(\mu\). Moreover, the functions \(\chi_i = \chi_i(x),\ k = k(x),\ E_i(x)\) are \(f\)-invariant and Borel measurable and the angle between the subspaces \(E_i(x)\) decays subexponentially along the orbits, that is,

\[
\lim_{n \to \pm \infty} \frac{\log \angle(E_S(f^n(x)), E_{N-S}(f^n(x)))}{|n|} = 0
\]

for every finite subset \(S \subset N := \{1, \cdots, k(x)\}\), where

\[
E_S(x) := \bigoplus_{i \in S} E_i(x).
\]

Given an \(f\)-invariant Borel probability hyperbolic measure \(\mu\) then the Lyapunov exponents \(\chi_i(x)\) are well defined for every \(x \in \Sigma\) and they are non-zero. Moreover, if \(\mu\) is ergodic there exists a constant \(\chi\) satisfying

\[
\inf\{|\chi_i(x)| : x \in \Sigma, i = 1, \cdots, k(x)\} > \chi > 0 \quad \text{for } \mu \text{-a.e.}
\]

Then, for all sufficiently small \(\epsilon > 0\) such that \(\chi > \epsilon\) by Oseledec’s theorem there exist measurable \(Df\)-invariant decompositions

\[
T_xM = E^*(x) \oplus E^u(x),
\]

and, for every \(\epsilon > 0\), tempered Borel measurable functions \(C_\epsilon, K_\epsilon : \Sigma \to (0, +\infty)\) with subexponential growth such that

\[
\begin{align*}
\|DF^n(x)v\| &\leq C_\epsilon(x) e^{n(-\chi+\epsilon)}\|v\| \quad \forall v \in E^*(x) \forall n \geq 0 \\
\|DF^{-n}(x)v\| &\leq C_\epsilon(x) e^{n(-\chi+\epsilon)}\|v\| \quad \forall v \in E^u(x) \forall n \geq 0
\end{align*}
\]

and \(\angle(E^*(x), E^u(x)) \geq K_\epsilon(x)\), where

\[
E^*(x) := \bigoplus_{\chi_i(x) < 0} E_i(x) \quad \text{and} \quad E^u(x) := \bigoplus_{\chi_i(x) > 0} E_i(x).
\]
By tempered we mean slowly growing and/or decay, that is,
\[(1 + \epsilon)^{-1} \leq \frac{C_r(f(x))}{C_r(x)} \frac{K_r(f(x))}{K_r(x)} \leq 1 + \epsilon, \quad \mu - a.e.\]

This follows from the tempering-kernel lemma \cite[Lemma S.2.12]{13}.

We remark that the properties given above as a consequence of the hyperbolicity of \(\mu\) can also be formulated without any reference to the measure \(\mu\) and are essentially nonuniform versions of standard uniformly hyperbolic conditions, see \cite[Theorem 6.6]{4}.

We refer the reader to \cite{4} and \cite{14} for an exposition of the ergodic theory of smooth dynamical systems with hyperbolic behavior.

We now introduce a standard “filtration” of \(\mu\) almost every point which gives a countable number of nested, uniformly hyperbolic (but not \(f\)-invariant) sets, often referred to as “Pesin sets”, whose points admit uniform hyperbolic bounds and uniform lower bounds on the sizes of the local stable and unstable manifolds.

For \(\chi > 0\) as in \cite{4} above, and every positive integer \(\ell > 0\), we define a (possibly empty) compact (not necessarily invariant) set \(A_{\chi, \ell} \subset M\) such that \(E^s|_{A_{\chi, \ell}}\) and \(E^u|_{A_{\chi, \ell}}\) vary continuously with the point \(x \in A_{\chi, \ell}\) and such that
\[
\begin{align*}
\|Df^n(x)v\| &\leq \ell e^{-n\chi}\|v\| &\|Df^{-n}(x)v\| &\geq \ell^{-1}e^{n\chi}\|v\| \forall \ v \in E^s(x) \forall \ n \geq 0 \\
\|Df^n(x)v\| &\geq \ell^{-1}e^{n\chi}\|v\| &\|Df^{-n}(x)v\| &\leq \ell e^{-n\chi}\|v\| \forall \ v \in E^u(x) \forall \ n \geq 0.
\end{align*}
\]

Moreover, the angles between the stable and unstable subspaces satisfy
\[\angle(E^s(x), E^u(x)) \geq \ell^{-1}\]
for every \(x \in A_{\chi, \ell}\). As the rate of hyperbolicity of \(\mu\) is bounded from below by \(\chi > 0\) we have
\[\mu(A_{\chi, \ell}) \to 1 \quad \text{as} \quad \ell \to +\infty.\]

**Definition 2.6.** We say that \(R(x)\) is a rectangle centered at \(x\) if it is the image of an embedding \(e_x : I^n \to M\) such that \(e_x(0) = x\), where \(I = [-1, 1]\).

By \cite[Theorem S.3.1]{14} for every \(\epsilon > 0\) and for \(\mu\)-a.e. \(x \in \Sigma\) there exists a local coordinate \(\phi_x : B(0, \rho(x)) \to M\), named Lyapunov charts, such that the representative \(f_x := \phi^{-1}_x \circ f \circ \phi_x\) of \(f\) in the new coordinates is a small perturbation of a hyperbolic linear isomorphism \(Df_x(0) : \mathbb{R}^n \to \mathbb{R}^n\) preserving the decomposition \(\mathbb{R}^n = \mathbb{R}^s \oplus \mathbb{R}^u\) such that:
\[
\begin{itemize}
\item \(D\phi_x(0)R^i = E^s_i\) for \(i = s, u\);
\item \(e^{\chi+\epsilon} \leq \|Df_x(0)R^s\|, \|Df_x(0)^{-1}R^u\| \leq e^{\chi-\epsilon};\)
\item \(f_x = Df_x(0) + h_x\) where \(\|h_x\|_{C^1} = \sup_{x \in I^n} \max\{\|h_x(x)\|, \|Dh_x(x)\|\} < \epsilon\)
\end{itemize}
\]

This is consequence of the \(C^{1+\alpha}\) hypotheses and the hyperbolicity of the orbit. Nonuniformity is captured by the slowly fluctuations of the radius \(\rho(x)\) along the orbit, i.e.
\[
\lim_{n \to \pm\infty} \frac{\log \rho(f^n(x))}{|n|} = 0 \quad \mu - a.e.
\]

Let \([-t(x), t(x)]^m \subset B(0, \rho(x))\) the largest \(m\)-cube contained in \(B(0, \rho(x))\) and let \(\sigma_x : [-1, 1]^m \to [-t(x), t(x)]^m\) a homothety. Then we introduce the modified Lyapunov chart
\[e_x := \sigma_x \circ \phi_x : [-1, 1]^m \to M.\]

**Definition 2.7.** We call \(R(x) := e_x([-1, 1]^m)\), the image of the modified Lyapunov chart, a rectangle.

Now we define admissible manifolds. They are good approximations to local stable and unstable invariant manifolds.
Definition 2.8. A admissible stable manifold is a graph $\gamma^s = \{e_z(s, z) : z \in I^s\}$, where $s \in C^1(I^s, I^u)$ is a smooth map with $\text{Lip}(s) := \sup_{z \in I^s} \|D\tilde{s}(z)\| \leq \gamma$. We define similarly admissible unstable manifolds: $\gamma^u = \{e_z(\tilde{u}(z), z) : z \in I^u\}$, where $\tilde{u} \in C^1(I^u, I^s)$ has $\text{Lip}(\tilde{u}) := \sup_{z \in I^u} \|D\tilde{u}(z)\| \leq \gamma$

Admissible manifolds endow $\mathbb{R}(x)$ with a product structure: any given pair of admissible manifolds $\gamma^s$ and $\gamma^u$ intersects transversally at a unique point with an angle bounded from below. Moreover, the map $(\gamma^s, \gamma^u) \to \gamma^s \cap \gamma^u$ so defined satisfies a Lipschitz condition $[14]$ §3.b] and $[4]$ §8. The transversal structure of the admissible stable and unstable manifolds inside a rectangle $R$ allows us to define the notion of admissible stable and unstable cylinders. An admissible stable cylinder $S \subseteq R$ is a non empty compact subset of $\mathbb{R}$ with piecewise smooth boundaries admitting a foliation by admissible stable manifolds which stretch fully across the rectangle $R$ and which is the closure of its interior points. Admissible unstable cylinder $U \subseteq \mathbb{R}$ are defined similarly with a foliation a foliation by admissible unstable manifolds stretching fully across the rectangle $R$.

The notion of admissible manifold is related to certain cone fields $K^s, K^u$. For every $z \in \mathbb{R}$ we define $K^s_z \subseteq T_zM$ as the image under $De_z(p)$ evaluated at $p(z) = e_z^{-1}(z) \in \mathbb{R}$, of the cone of width $\gamma$ centered at $\mathbb{R}^s \oplus \{0\}$, that is, the set of vectors in $\mathbb{R}^s$ making an angle bounded by $\gamma$ with $\mathbb{R}^u \oplus \{0\}$. We define $K^u_z \subseteq T_zM$ likewise considering a cone of width $\gamma$ centered at $\{0\} \oplus \mathbb{R}^u$. Notice that admissible manifolds are exactly those smooth graph-like submanifolds whose tangent spaces rest inside stable and unstable cones.

We say that a $C^1$ diffeomorphism $g : S \to U$ between admissible cylinders is hyperbolic if it preserves the cone fields $K^s, K^u$, that is,

$$Dg(z)K^s_z \subseteq \text{int} K^s_{g(z)} \forall z \in S \quad \text{and} \quad Dg^{-1}(z)K^u_z \subseteq \text{int} K^u_{g^{-1}(z)} \forall z \in U,$$

Definition 2.9. Let $R$ and $Q$ be regular rectangles. If some iterate $f^m$ maps an admissible stable cylinder $S \subseteq R$ diffeomorphically and hyperbolically to an admissible unstable cylinder $U \subseteq Q$, we shall say that

$$f^m : S \to U$$

is a hyperbolic branch.

The next pseudo-Markov property of coverings by Lyapunov rectangles is the key to the construction of hyperbolic $f$-invariant Cantor sets approximating the measure $\mu$ and satisfying $[3]$ in Theorem A.

Proposition 2.10. For every $\delta > 0$ and for every Pesin set $\Lambda$ there exists constants $\kappa > 0$, $\lambda = \lambda(\chi) > 1$ only depending on $\chi > 0$, the lower bound for the Lyapunov exponents introduced in $[9]$, subsection 2.3 and a finite covering by rectangles $\{R_i = R(p_i), p_i \in \Lambda; \; i = 1, \ldots, t\}$ such that:

1. $\Lambda \subseteq \bigcup_{i=1}^t B(p_i, \kappa)$, and $B(p_i, \kappa) \subseteq \text{int} R_i$;
2. $\text{diam}(R_i) \leq \delta$ for $i = 1, \ldots, t$;
3. if $x \in \Lambda \cap B(p_i, \kappa)$ and $f^m(x) \in \Lambda \cap B(p_j, \kappa)$ then there exists an admissible stable cylinder $S_z \subset R_i$ containing $x$ and an admissible unstable cylinder $U_{f^m(x)} \subset R_j$ containing $f^m(x)$ such that $f^m : S_z \to U_{f^m(x)}$ is a hyperbolic branch with nonlinear rate of expansion bounded from below by $\lambda > 1$, that is:

$$d_{W^s_k}(f^k(w), f^k(w')) \geq \lambda^k d_{W^s}(w, w') \quad \text{for} \; k = 1, \ldots, m, \; \forall \; w, w' \in W \cap S_z,$$

where $d_{W^s}$ and $d_{W^u}$ is the metric induced by the Lyapunov charts on $W \in \Gamma^u(R_i)$ and $W^s_k = f^k(W \cap S)$ and $\Gamma^u(R_i)$ is the set of admissible unstable manifolds in $R_i$ and $m > 0$ is the return time of $x$. Similarly

$$\text{dist}_{W^s_k}(f^{-k}(w), f^{-k}(w')) \geq \lambda^k \text{dist}_{W^s}(w, w') \quad \text{for} \; k = 1, \ldots, m, \; \forall \; w, w' \in W \cap U_{f^m(x)}.$$
for every $W \in \Gamma^s(R_j)$ and $W_k = f^{-k}(W \cap U_{f^k(x)})$, where $\Gamma^s(R_j)$ is the set of admissible stable manifolds in $R_j$;
(4) $\text{diam}(f^k(S_x)) \leq \delta$, for $0 \leq k \leq m$;

We emphasize that the formation of hyperbolic branches occurs for every return from $\Lambda \cap B(p_j, \kappa)$ to $\Lambda \cap B(p_j, \kappa)$ which are not necessarily first return times.

**Definition 2.11.** We call $\mathcal{R} = \{R_i\}$ a $(\delta, \kappa, \lambda)$-Markov covering of $\Lambda$.

See [14, Definition S.4.15] and [14, Theorem S.4.16].

3. Proof of Theorem A: first step, choosing $\Omega$

Theorem A’s (3) follows immediately from next lemma by taking $\Omega_n = \Omega(\rho_n, s_n, \phi)$, where $\rho_n \downarrow 0^+$ and $s_n \to +\infty$ are suitable sequences.

**Lemma 3.1.** Let $\rho, s > 0$, $\phi$ continuous and $\mu$ an ergodic non atomic hyperbolic Borel probability. Then, there exist a hyperbolic basic set $\Omega = \Omega(\rho, s, \phi)$ such that:

1. every ergodic measure $\nu$ supported on $\Omega$ belongs to the weak-* open neighborhood $\mathcal{O}(\mu, \rho, s)$;
2. the following estimate holds

\begin{equation}
\frac{\rho \inf \phi}{1 + \rho} + \frac{P_\nu(\phi) - 3\rho}{1 + \rho} \leq P(f|\Omega, \phi) \leq P_\nu(\phi) + 2\rho + \rho \sup \phi;
\end{equation}

3. and the rate of hyperbolicity of $\Omega$ is bounded from below by a constant $\log \lambda > 0$.

Here $\lambda = \Lambda(\chi) > 1$ is the rate of expansion along admissible unstable manifolds introduced in proposition 2.10, subsection 2.3.

We dedicate the rest of this paper to prove Lemma 3.1. In this section we prove item (1) and (3), while pressure estimates from (3) take up the final two sections.

Let us start recalling the following terminology:

1. $E \subset X$ is an $(\epsilon, n)$-spanning set in $X$ if for every $x \in X$ there exists $y \in E$ such that $d(f^k(x), f^k(y)) \leq \epsilon$, for every $0 \leq k \leq n - 1$;
2. $E \subset X$ is an $(\epsilon, n)$-separated set in $X$ if for every pair of different points $x \neq y$ in $E$ it holds $d(f^k(x), f^k(y)) > \epsilon$ for some $0 \leq k \leq n - 1$;
3. given an $f$-invariant Borel probability $\mu$ and a positive number $0 < \alpha < 1$, we say that $E$ is an $(\epsilon, n, \alpha)$-spanning set for $\mu$ if

\[ \mu \left( \bigcup_{x \in E} B(x, \epsilon, n) \right) \geq \alpha, \]

where $B(x, \epsilon, n) := \{ y \in X : \text{dist}(f^j(x), f^j(y)) < \epsilon, \; j = 0, \cdots, n - 1 \}$. $E$ is $(\epsilon, n)$-spanning in $X$ if and only if $M \subset \bigcup_{x \in E} B(x, \epsilon, n)$. Also notice that any maximal $(\epsilon, n)$-separated set in $X$ is $(\epsilon, n)$-spanning.

**Definition 3.2.** Let $f : X \to X$ continuous and $\mu$ and $f$-invariant Borel probability an $\phi$ continuous. We define the measure-theoretical pressure of $\phi$ w.r.t. $\mu$ as

\begin{equation}
P_\mu(\phi) := \lim_{\alpha \to 0^+} \lim_{\epsilon \to 0^+} \lim_{n \to +\infty} \frac{1}{n} \log \left( \inf_{E} \left\{ \sum_{x \in E} \exp S_n \phi(x) \right\} \right),
\end{equation}
where $\mu$ is an $f$-invariant Borel probability and the infimum taken over $(\epsilon,n,\alpha)$-spanning subsets $E \subset M$.

**Proposition 3.3.** Let $f : X \to X$ a continuous self map of a compact metric space $(X,d)$, $\phi$ continuous and $\mu \in M_f$ an ergodic $f$-invariant Borel probability. Then, for every $0 < \alpha < 1$,

$$P_\mu(\phi) = \lim_{\epsilon \to 0^+} \lim_{n \to +\infty} \frac{1}{n} \log \left( \inf_E \left\{ \sum_{x \in E} \exp S_n \phi(x) \right\} \right) = h(\mu) + \int \phi \, d\mu,$$

infimum taken over $(\epsilon,n,\alpha)$-spanning subsets $E \subset M$.

See [17, Theorem 1.1].

The construction of $\Omega$ will rely upon proposition 3.3. For this we need to fix $\alpha > 0$, $\delta > 0$, $n > 0$ and a finite $(\delta,n,\alpha)$-spanning subset $E_0$ such that each $x \in E_0$ is endowed with a hyperbolic branch $f^{R(x)} : S_x \to U^{f^s(x)}$ for a suitable return time to a hyperbolic Pesin set of quasi-generic points $\Lambda_0$. Then we choose a suitable subset of those hyperbolic branches to generate a horseshoe with finitely many branches and variable return times $\Omega^*$ and then we prove that $\Omega = \bigcup_{n \in \mathbb{Z}} f^n(\Omega^*)$, the $f$-invariant saturate of $\Omega^*$ satisfies the inequalities (12) in Lemma 3.1.

First we fix once for all $\rho > 0$ and $s > 0$ and $\{\psi_i\}$ a countable dense subset of continuous functions.

Then, we choose a hyperbolic Pesin set of quasi-generic points. For this we fix a hyperbolic Pesin set $\Lambda$ and define

$$\Lambda_N := \{ x \in \Lambda : \left| \sum_{k=0}^{n-1} \psi_i(f^k(x)) - \int \psi_i \, d\mu \right| < \rho/2 \ \forall i \leq s \ \forall n \geq N \}$$

Then we pick up a large integer $N_0 > 0$ such that $\Lambda_0 := \Lambda_{N_0}$ has

$$\mu(\Lambda_0) \geq \frac{\mu(\Lambda)}{2}$$

This is possible since,

$$G_{\rho,s,N} = \{ x \in M : \left| \sum_{k=0}^{n-1} \psi_i(f^k(x)) - \int \psi_i \, d\mu \right| < \rho/2 \ \forall i \leq s \ \forall n \geq N \}$$

is an increasing sequence and $\mu(G_{\rho,s,N}) \uparrow 1$ when $N \to +\infty$.

Then fix $\alpha > 0$ defining

$$\alpha := \frac{\mu(\Lambda)}{4}$$

Next step is to fix a small precision $\delta > 0$:

**Lemma 3.4.** There exists $\delta(\rho,s) > 0$ such that, for every $0 < \delta < \delta(\rho,s)$ it holds

$$\forall x,y \in M : \quad d(x,y) \leq \epsilon \implies |\psi_i(x) - \psi_i(y)| < \rho/2, \quad \forall i \leq s,$$

$$\forall x,y \in M : \quad d(x,y) \leq \epsilon \implies |\phi(x) - \phi(y)| < \rho,$$

and

$$\lim_{n \to +\infty} \frac{1}{n} \log \left( \inf_E \left\{ \sum_{x \in E} \exp S_n \phi \right\} \right) - P_\rho(\phi) < \rho/4.$$
infinum is taken over all the \((\delta, n, \alpha)\)-spanning subsets \(E\).

\((13)\) and \((19)\) follows from the continuity of \(\psi_i\) and \(\phi\); \((20)\) follows from the definition of the limit \((14)\).

Choosing a large time \(n \geq N_0 > 0\)

Now we fix a \((\delta/4, \kappa, \lambda)\)-Markov covering of \(\Lambda\) and choose \(N_0 > 0\) in the definition of \(\Lambda_0\) larger if necessary such that,

\[
\forall n \geq N_0 : \quad \left| \frac{1}{n} \log \left( \inf_{E} \left\{ \sum_{x \in E} \exp(S_n \phi(x)) \right\} \right) - P_\mu(\phi) \right| < \rho/2,
\]

and

\[
\forall n \geq N_0 : \quad \exp(n\rho) \geq \#R.
\]

Moreover, we choose \(N_0\) sufficiently large such that for every \(n \geq N_0\) a large portion of points in \(\Lambda_0\) return within a time \(R(x) \in [n, (1 + \rho)n]\) giving rise to quasi-generic branches. This is the content of the following

**Lemma 3.5.** There exists a large \(N_0 > 0\) satisfying \((16)\), \((21)\) and \((22)\) with the following property: for every \(n \geq N_0\) and for every open ball \(B(p_i, \kappa) \subset R_i\) of the \((\delta/4, \kappa, \lambda)\)-Markov covering of \(\Lambda\) there exists a subset \(\Lambda_{0,i} \subset B(p_i, \kappa) \cap \Lambda_0\) with

\[
\mu(\Lambda_{0,i}) \geq \mu(B(p_i, \kappa) \cap \Lambda_0)/2
\]

such that for every \(x \in \Lambda_{0,i}\) returns to \(B(p_i, \kappa) \cap \Lambda_0\) with a return time

\[
R(x) \in [n, (1 + \rho)n].
\]

**Remark 3.1.** We underline that \(R(x)\) is not necessarily the first return time of \(x\).

**Proof.** This follows from the ergodicity of \(\mu\). C.f. \([14]\). Let \(A \subset M\) be a Borel set with \(\mu(A) > 0\). Given \(\rho > 0\) and \(n > 0\) define

\[
A_{\rho,n} := \{ x \in A : x \text{ return to } A \text{ with return time } R(x) \in [n, (1 + \rho)n] \}
\]

Then given \(0 < \epsilon < 1\) there exists \(N > 0\) such that

\[
\mu(A_{\rho,n}) \geq (1 - \epsilon)\mu(A) \quad \text{for every } n \geq N.
\]

Then we apply this lemma to \(A = B(p_i, \kappa) \cap \Lambda_0\) and \(\epsilon = 1/2\). \(\square\)

We fix once for all some \(n \geq N_0\) satisfying \((16)\), \((21)\), \((22)\) and the return time property in Lemma 3.5.

**Choosing \(E_0\)**

Notice that,

\[
\mu(\bigcup_i \Lambda_{0,i}) \geq \alpha.
\]

Therefore we can choose a maximal \((\delta, n)\) separated subset \(E_0 \subset \bigcup_i \Lambda_{0,i}\) such that

\[
\left| \frac{1}{n} \log \left( \sum_{x \in E_0} \exp(S_n \phi(x)) \right) - P_\mu(\phi) \right| < \rho.
\]
By construction for each point \( x \in E_0 \) there exists a hyperbolic branch \( f^{R(x)} : S_x \to U_{f^{R(x)}(x)} \) contained in some \( R_i \) and such that

\[
\text{diam}(f^j(S_x)) < \delta/4 \quad \text{for every } j = 0, \cdots, R(x) - 1.
\]

This and the condition of separation of points in \( E_0 \) implies that any two different branches subordinated to the same rectangle are disjoint.

Then we choose \( \ell > 0 \) and a subset \( E_\ell := B(p_\ell, \kappa) \cap E_0 \) such that

\[
\sum_{x \in E_\ell} \exp S_n \phi(x) \geq \sum_{x \in E_{\ell'}} \exp S_n \phi(x) \quad \text{for every } \ell' \not= \ell,
\]

and define \( \Omega(\rho, s, \phi) \) as the \( f \)-invariant saturate of the horseshoe with finitely many branches defined by the collection of branches \( \{ f^{R(x)} : S_x \to U_x : x \in E_\ell \} \) chosen by condition (26):

\[
\Omega(\rho, s, \phi) = \bigcup_n f^n \left( \bigcap_{k \in \mathbb{Z}} \left( f^{R(x)} \bigcup_{x \in E_\ell} S_x \right) \right),
\]

where \( f^{R(x)}|_{S_x} = f^{R(x)} \).

**Lemma 3.6.** \( \nu \in O(\mu, \rho, s) \) for every ergodic \( f \)-invariant Borel probability \( \nu \) supported on \( \Omega \). Moreover, the rate of hyperbolicity of \( \Omega \) is bounded from below by \( \log \lambda > 0 \).

**Proof.** This follows from Proposition 2.5 since the branches \( \{ f^{R(x)} : S_x \to U_x : x \in E_\ell \} \) are \((\rho, s)\)-quasi-generics, that is,

\[
\forall y \in S_x : \left| \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(f^j(y)) - \int \psi_i \, d\mu \right| \leq \rho \quad \forall i \leq s.
\]

Indeed, for every \( i \) every return from \( x \in \Lambda_{0,i} \) to \( f^{R(x)}(x) \in \Lambda_{0,i} \) giving rise to a \((\rho, s)\)-quasi-generic branch (see Definition 2.4)

\[
f^{R(x)} : S \to U, \quad \text{with } S, U \subset \mathcal{R}_i.
\]

This follows from [14] in the definition of \( \delta \), since \( \text{diam}(f^j(S)) < \delta/4 \) for \( j = 0, \cdots, R(x) - 1 \) by the definition of a pseudo-Markov covering and \( R(x) \geq n \geq N_0 \), therefore for every \( y \in S \)

\[
\left| \sum_{k=0}^{n-1} \psi_i(f^k(y)) - \int \psi_i \, d\mu \right| \leq \left| \sum_{k=0}^{n-1} \psi_i(f^k(y)) - \sum_{k=0}^{n-1} \psi_i(f^k(x)) \right| + \left| \sum_{k=0}^{n-1} \psi_i(f^k(x)) - \int \psi_i \, d\mu \right| \\
\leq \rho/2 + \rho/2 \\
= \rho.
\]

The bound on the rate of hyperbolicity follows from nonlinear expansion property [10] and [11] in proposition 2.10. \( \square \)
4. Second step: counting periodic orbits

To estimate the topological pressure of $\Omega(\rho, s, \phi)$ we need to estimate the cardinality of periodic orbits. Let $\text{Per}(N)$ denote the set of periodic points with prime period $N$: $x \in \text{Per}(N)$ iff $f^N(x) = x$ and $f^k(x) \neq x$ for $0 < k < N$.

**Lemma 4.1.** Let $\Omega$ be a basic set for a $C^r$ ($r \geq 1$) diffeomorphism and $\phi$ continuous. Then:

\[
P(f|\Omega, \phi) = \limsup_{N \to +\infty} \frac{1}{N} \log \left( \sum_{x \in \text{Per}(N)} \exp S_N \phi \right).
\]

This was proved in [19, Section 7.19 (7.11)] for Smale spaces [19, Section 7.1] which includes basic sets of $C^r$ ($r \geq 1$) diffeomorphisms and bilateral full shifts of finitely many symbols.

To estimate the limsup in (29) one has to keep track of the combinatorics of periodic orbits. This is done as follows. Let $E^p = E \times \cdots \times E$ the cartesian product of $p$-copies of $E$, where we denote $E = E_\ell$ to simplify notation.

**Lemma 4.2.** $f^{R_\ell}|\Omega^*$ is topologically conjugated to the full-shift on $\#E$ symbols.

**Proof.** We first observe that, as the stable cylinders $S_x$, $x \in E$ are disjoint and

\[
\Omega^* = \bigcap_{n \in \mathbb{Z}} \left( f^{R_\ell(n)} \left( \bigcup_{x \in E} S_x \right) \right),
\]

where $x_n = (f^{R_\ell(n)}(x))$ is defined inductively as

\[
x_{n+1} = f^{R_\ell(n)}(x_n) \quad \text{for } n \geq 0 \quad \text{and} \quad x_{n-1} = f^{-R_\ell(n)}(x_n) \quad \text{for } n \leq 0.
\]

setting $x = x_0$. That is,

\[
\begin{cases}
(f^{R_\ell}(x)) = f^{\sum_{0 \leq i < n} R_\ell(x_i)}(x) & \text{for } n \geq 0 \\
(f^{R_\ell}(x)) = f^{-\sum_{n < 0} R_\ell(x)}(x) & \text{for } n \leq 0
\end{cases}
\]

In particular, for every $z \in \Omega^*$ there exists a unique $x \in E$ such that

\[
\text{dist}(f^j(x), f^j(z)) < \delta/4 \quad \text{for every } j = 0, \cdots, R_\ell(x) - 1.
\]

Unicity of $x \in E$ follows since $E$ is part of a $\delta/2$-separated set. Then we associate to every $x \in \Omega^*$ a unique bi-infinite sequence $\{x_n\} \in E^\mathbb{Z}$ defined by shadowing the orbit of $z$. This shows that $f^{R_\ell}|\Omega^*$ is topologically conjugated to the full-shift on $\#E$ symbols. \hfill $\Box$

**Corollary 4.1.** If $z \in \Omega$ is $f$-periodic, and assuming without loss of generality that $z \in \Omega^*$, then its successive returns to $\Omega^*$ define an $f^{R_\ell}$-periodic orbit given by a uniquely defined sequence $x_k \in E$, $0 \leq k < p$, $p > 1$, such that

\[
x_{k+1} = f^{R_\ell}(x_k), \quad \text{for } 0 \leq k < p \quad \text{and} \quad x_0 = f^{R_\ell}(x_{p-1}).
\]

Moreover,

\[
\text{dist}(f^{j + \sum_{i < k} R_\ell(x_i)}(z), f^{j + \sum_{i < k} R_\ell(x_k)}(x_k)) < \delta/4 \quad \text{for } j = 0, \cdots, R_\ell(x_k) - 1, \quad k = 0, \cdots p - 1.
\]

**Remark 4.1.** Let $z \in \text{Per}(f|\Omega)$ a periodic point. Then according to our previous discussion its prime period $N = N(z)$ is a linear combination of the basic periods $n_1, \cdots, n_{\#E}$ of the branches generating $\Omega^*$, namely, there exists integers $p_i \in \mathbb{Z}^+$, $i = 1, \cdots, \#E$ such that

\[
N = p_1 n_1 + \cdots + p_{\#E} n_{\#E}.
\]
Definition 4.3. We say that $N \in \mathbb{Z}^+$ is an admissible period if it satisfies (31) for some sequence of non-negative integers $p_i \in \mathbb{Z}^+$, $i = 1, \ldots, \#E$.

Lemma 4.4. Let $N > 0$ be an admissible period, $z \in \text{Per}(N)$ and $[x_0, \ldots, x_{p-1}] \in E^p$ the unique sequence of points in $E$ which successively $\delta/4$-shadows $\mathcal{O}(z)$ when the orbit cycles around $\Omega$ provided by Corollary 4.1. Then

$$\frac{N}{n(1 + \rho)} \leq p \leq \frac{N}{n},$$

where $n > 0$ were fixed after lemma 3.3.

Proof. Let $N = N(z)$ the prime period of $z$ then

$$N = \sum_{i=0}^{p-1} R(x_i).$$

Using (33) and that $n \leq R(x_k) \leq (1 + \rho)n$ for every $k = 0, \ldots, p - 1$, one conclude that $pn \leq N \leq (1 + \rho)n p$ and we get (32). \hfill $\Box$

Lemma 4.5. Let $N$ be an admissible period, $z \in \text{Per}(N)$ a periodic point for $f|\Omega$ of prime period $N$ and $[x_0, \ldots, x_{p-1}] \in E^p$ the encoding sequence defined in Corollary 4.1. Then,

$$\exp(S_N(\phi + \rho)(z)) \geq \prod_{k=0}^{p-1} \exp(S_{R(x_k)}\phi(x_k)) \quad (34)$$

and

$$\exp(S_N(\phi - \rho)(z)) \leq \prod_{i=0}^{p-1} \exp(S_{R(x_k)}\phi(x_k)). \quad (35)$$

Proof. Recall that the branches originating the Alekseev set $\Omega^*$ satisfy

$$\left| \sum_{j=0}^{R(x)-1} \phi(f^j(y)) - \sum_{j=0}^{R(x)-1} \phi(f^j(z)) \right| < R(x)\rho \quad \text{for every} \quad y, z \in S_x.$$

This is by our choice of $\delta > 0$ since $\text{diam}(f^j(S_x)) < \delta/4$ for $j = 0, \ldots, R(x) - 1$. Then by (33) and (38)

$$\left| \sum_{j=0}^{N-1} \phi(f^j(z)) - \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k)) \right| < N\rho.$$

Therefore

$$S_N(\phi + \rho)(z) \geq \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k)) \quad \text{and} \quad S_N(\phi - \rho)(z) \leq \sum_{k=0}^{p-1} \sum_{j=0}^{R(x_k)-1} \phi(f^j(x_k)).$$

Taking the exponential at both sides we get (34) and (35). \hfill $\Box$
5. Third step: pressure estimates and conclusion

To finish the proof we need to estimate the pressure of the $f$-invariant saturate of the Alekseev set chosen in previous step.

**Definition 5.1.** Let $p > 0$ be any positive integer. We denote $\Delta(p)$ the set of admissible periods of periodic orbits in $\Omega(\rho, s, \phi)$, encoded into $E^p$:

$$\Delta(p) := \{ N : \exists [x_0, \cdots, x_{p-1}] \in E^p \text{ such that } N = \sum_{k=0}^{p-1} R(x_k) \}.$$  

**Lemma 5.2.** For every positive integer $p > 0$ it holds:

$$\sum_{N \in \Delta(p)} \sum_{z \in \text{Per}(N)} \exp\left(S_N(\phi + \rho)(z)\right) \geq \left[ \sum_{x \in E} \exp\left(S_R(x)\phi(x)\right) \right]^p,$$

$$\sum_{N \in \Delta(p)} \sum_{z \in \text{Per}(N)} \exp\left(S_N(\phi - \rho)(z)\right) \leq \left[ \sum_{x \in E} \exp\left(S_R(x)\phi(x)\right) \right]^p.$$

**Proof.** Using inequality (34) in Lemma 4.5, (32) and the identity

$$(a_1 + \cdots + a_n)^m = \sum_{[i_1, \cdots, i_m]} a_{i_1} \cdots a_{i_m} \text{ where } [i_1, \cdots, i_m] \in \{1, \cdots, n\}^m$$

we get

$$\sum_{N \in \Delta(p)} \sum_{z \in \text{Per}(N)} \exp\left(S_N(\phi + \rho)(z)\right) \geq \sum_{[x_1, \cdots, x_p] \in E^p} \prod_{k=0}^{p-1} \exp\left(S_R(x_k)\phi(x_k)\right)$$

$$= \left[ \sum_{x \in E} \exp\left(S_R(x)\phi(x)\right) \right]^p,$$

thus proving (37). Inequality (38) follows similarly using inequality (35) in Lemma 4.5. □

**Proof of Lemma 3.1**

We proved already that $\nu \in \mathcal{O}(s, 3\rho)$ for every ergodic $f$-invariant Borel probability $\nu$ supported on $\Omega$ in Lemma 3.6. Now we want to prove the estimates (12) in Lemma 3.1. For this we use previous section results to estimate $P(f|\Omega, \phi)$.

We first notice that, as $R(x) \in [n, (1 + \rho)n]$ for every $x \in E$ we have

$$\sum_{x \in E} \exp(S_R(x)\phi(x)) \geq \sum_{x \in E} \exp(S_n\phi(x)) \times \exp(n\rho \inf \phi)$$

and

$$\sum_{x \in E} \exp(S_R(x)\phi(x)) \leq \sum_{x \in E} \exp(S_n\phi(x)) \times \exp(n\rho \sup \phi).$$

By (24) and the choice of $E = E_\ell$ as the set which maximizes the sums

$$\sum_{x \in E_\ell} \exp(S_n\phi(x))$$
in (26), we have
\[
\# \mathcal{R} \sum_{x \in E} \exp(S_n \phi(x)) \geq \sum_{x \in E_0} \exp(S_n \phi(x)) > \exp(n[P_{\mu}(\phi) - \rho])
\]
thus giving
\[
\sum_{x \in E} \exp(S_n \phi(x)) \geq \exp(n[P_{\mu}(\phi) - 2\rho]),
\]
since \(\exp(n\rho) > \# \mathcal{R}\), by (22). On the other hand, as \(E \subset E_0\)
\[
\sum_{x \in E} \exp(S_n \phi(x)) \leq \sum_{x \in E_0} \exp(S_n \phi(x)) < \exp(n[P_{\mu}(\phi) + \rho]).
\]
Therefore, substituting (41) into (39) and recalling (37) we get the lower bound
\[
\sum_{N \in \Delta(p)} \sum_{z \in \text{Per}(N)} \exp(S_N(\phi + \rho)(z)) \geq [\exp(n[P_{\mu}(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^P
\]
On the other hand, as \(R(x) \in [n, (1 + \rho)n]\) we have
\[
np \leq N \leq n(1 + \rho)p \quad \text{for every admissible period} \quad N \in \Delta(p).
\]
Therefore,
\[
\# \Delta(p) \leq n(1 + \rho)p - np = np
\]
Hence, maximizing the sums \(\sum_{z \in \text{Per}(N)} \exp(S_N(\phi + \rho)(z))\) over the set of admissible periods
\(N \in \Delta(p)\) we get
\[
\# \Delta(p) \sum_{z \in \text{Per}(N_p)} \exp(S_{N_p}(\phi + \rho)(z)) \geq [\exp(n[P_{\mu}(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^{\frac{N_p}{1 + \rho}}
\]
for a suitable admissible period \(N_p \in \Delta(p)\), where we have used (52) to bound from below \(p > 0\)
in terms of \(N_p\).
Therefore by (45) we get
\[
np \times \sum_{z \in \text{Per}(N_p)} \exp(S_{N_p}(\phi + \rho)(z)) \geq [\exp(n[P_{\mu}(\phi) - 2\rho]) \times \exp(n\rho \inf \phi)]^{\frac{N_p}{1 + \rho}}.
\]
Then, taking logarithms and dividing by \(N_p\) in (46) and letting \(p \to +\infty\), we get a sequence
\(N_p \to +\infty\) of admissible periods, such that,
\[
P(f|\Omega, \phi + \rho) = \lim_{N \to +\infty} \sup \frac{1}{N} \log \sum_{z \in \text{Per}(N)} \exp(S_N(\phi + \rho)(z))
\]
\[
\geq \lim_{p \to +\infty} \frac{\log(np)}{N_p} + \lim_{p \to +\infty} \frac{1}{N_p} \log \sum_{z \in \text{Per}(N_p)} \exp(S_{N_p}(\phi + \rho)(z))
\]
\[
\geq \frac{P_{\mu}(\phi) - 2\rho}{1 + \rho} + \frac{\rho \inf \phi}{1 + \rho}
\]
On the other hand, introducing (42) into (40) and recalling (38) we get
\[
\sum_{N \in \Delta(p)} \sum_{z \in \text{Per}(N)} \exp(S_N(\phi - \rho)(z)) \leq [\exp(n[P_{\mu}(\phi) + \rho]) \times \exp(n\rho \sup \phi)]^P
\]
Therefore, for every admissible period \(N \in \Delta(p)\) and for every \(p \geq 0\),
\[
\sum_{z \in \text{Per}(N)} \exp(S_N(\phi - \rho)(z)) \leq [\exp(n[P_{\mu}(\phi) + \rho]) \times \exp(n\rho \sup \phi)]^P
\]
and then, taking logarithms and dividing by \( N \), for every admissible period \( N \in \Delta(p) \), \( p \geq 0 \),

\[
\frac{1}{N} \log \left( \sum_{z \in \text{Per}(N)} \exp(S_N(\phi - \rho)(z)) \right) \leq \frac{Np}{N}(P_\mu(\phi) + \rho + \rho \sup \phi)
\]

\[
\leq P_\mu(\phi) + \rho + \rho \sup \phi,
\]

using (44). Hence,

\[
P(f|\Omega, \phi - \rho) = \limsup_{N \to +\infty} \frac{1}{N} \log \left( \sum_{z \in \text{Per}(N)} \exp(S_N(\phi - \rho)(z)) \right)
\]

\[
\leq P_\mu(\phi) + \rho + \rho \sup \phi.
\]

This proves estimative (12) at Lemma 3.1, after a straightforward calculation, using \( P(\phi + c) = P(\phi) \) ([Theorem 2.1, (vii)](22)). QED

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