Canonical metrics and ambiKähler structures on 4-manifolds with $U(2)$ symmetry

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Abstract

For $U(2)$-invariant 4-metrics, we show that the $B^t$-flat metrics are very different from the other canonical metrics (Bach-flat, Einstein, extremal Kähler, etc). We show every $U(2)$-invariant metric is conformal to two separate Kähler metrics, leading to ambiKähler structures. Using this observation we find new complete extremal Kähler metrics on the total spaces of $O(-1)$ and $O(+1)$ that are conformal to the Taub-bolt metric. In addition to its usual hyperKähler structure, the Taub-NUT’s conformal class contains two additional complete Kähler metrics that make up an ambi-Kähler pair, making five independent compatible complex structures for the Taub-NUT, each of which has a conformally Kähler $(1,1)$ form.

1 Introduction

Cohomogeneity-1 metrics with $U(2)$ symmetry have the form

$$g = A(r) dr^2 + B(r) (\eta^1)^2 + C(r) \left( (\eta^2)^2 + (\eta^3)^2 \right)$$

(1)

where $\eta^1$, $\eta^2$, $\eta^3$ are the usual left-invariant covector fields on $S^3$. Naively the topology is $\mathbb{R} \times S^3$, but topological changes occur at locations where $B$ or $C$ reach zero and there could be a quotient on the $S^3$ factor. We classify canonical metrics of this form particularly the $B^t$-flat metrics, and create some new explicit examples of canonical metrics using the ambiKähler techniques of [2]. This project began as a way to develop supporting examples for other work, and treads such familiar ground that we expected few surprises. But we did find some surprises, two of which we feel worth reporting to the wider community.

The first is how the $B^t$-flat metrics fit amongst the other canonical metrics. The space of extremal Kähler metrics is rather small—up to homothety the moduli space is 3-dimensional—and with one exception there are basically no other canonical metrics. Up to a choice of conformal factor, the Bach-flat metrics are a 2-parameter subspace of the extremal\footnote{We will use “extremal” to mean “extremal Kähler.”} metrics. The Einstein and harmonic-curvature metrics [11] are identical, and up to conformal factors are exactly the Bach-flat metrics. Half-conformally flat metrics are conformally extremal, and up to conformal factors the metrics with $W^+ = 0$ (or $W^- = 0$) form a 1-parameter subspace of the Bach-flat metrics. The Kähler-Einstein (KE) metrics and the Ricci-flat metrics are each a 1-parameter subclass of the Bach-flat metrics. Up to homothety there are exactly five Ricci-flat KE metrics: flat $\mathbb{R}^4$, the Eguchi-Hanson, the Taub-NUT, and two metrics with curvature singularities. The Taub-NUT is extraordinary; see Proposition 2.5 and Section 4.
The exception to this framework are the $B^t$-flat metrics (see [21]), which are generally not conformally extremal. A $B^t$-flat metric satisfies the Euler-Lagrange equations of the functional

$$B^t = \int |W|^2 + t \int s^2$$

for given $t \in (-\infty, \infty]$, where $B^\infty = \int s^2$. A metric that extremizes $B^\infty$ is either scalar-flat or Einstein [5] (the scalar-flat condition is not elliptic but the Einstein condition is second-order elliptic). The $B^t$ extremals are the Bach-flat metrics, and the Euler-Lagrange equations are $4^{th}$ order and underdetermined (although upon fixing a conformal gauge they become $4^{th}$ order elliptic). For $t \neq 0$, $\infty$ the $B^t$ Euler-Lagrange equations are an overdetermined $8^{th}$ order system; after an appropriate reduction we find a 5-dimensional moduli space of $B^t$-flat metrics up to homothety. If the constant scalar curvature (CSC) condition is imposed, the CSC $B^t$-flat metrics constitute a 4-parameter family up to homothety. Intuitively, as $t$ varies in $[0, \infty)$, the $B^t$-flat metrics would seem to interpolate between the Einstein metrics at $t = \infty$ and the Bach-flat metrics at $t = 0$. As we pointed out, up to conformal factors these are exactly the same class, so it would stand to reason that the $B^t$-flat metrics would stay within this class, perhaps up to conformal factors. But this is simply not the case, as we show in Theorem 3.11.

We characterize two other candidates for the status of “canonical” metrics: the CSC metrics and the “half-harmonic” (sometimes called “weakly self-dual”) metrics. The half-harmonic metrics are those with either $\delta W^+ = 0$ or $\delta W^- = 0$. Combining $\delta W^+ = 0$ with one other condition creates an elliptic system: this extra condition might be a scalar curvature condition such as $s = const$, a Kähler condition, or that $\delta W^- = 0$. The case of $\delta W^- = 0$ along with the Kähler condition was studied in [1].

The second surprise has to do with ambiKähler pairs. Any metric $\mathbb{I}$ is automatically compatible with two complex structures which give opposite orientations—in short, each Kähler metric $\mathbb{I}$ is a partner in an ambiKähler pair. From this observation we find four notable metrics: an ambiKähler pair that is conformal to the classic Taub-NUT on $\mathbb{C}^2$ and an ambiKähler pair that is conformal to the classic Taub-bolt $\mathbb{I}$. The ambiKähler pair conformal to the Taub-NUT are both complete extremal Kähler metrics, one of which has zero scalar curvature (ZSC) and is 2-ended, and the other of which is one-ended and strictly extremal. The two ambiKähler metrics conformal to the Taub-bolt are both complete extremal metrics, and exist on two different underlying complex surfaces, $\mathcal{O}(-1)$ and $\mathcal{O}(+1)$. The metric on $\mathcal{O}(+1)$ is the only complete extremal Kähler metric, known to the authors, on the total space of any $\mathcal{O}(k)$ where $k > 0$ (by contrast the Burns, Eguchi-Hanson, and LeBrun metrics are all Kähler metrics on $\mathcal{O}(k)$ with $k < 0$).

We first explain our computational framework. Solving $dz = 2\sqrt{AB-C} \, dr$ for $z$, the metric $\mathbb{I}$ is

$$g = C \left( \frac{1}{4F} dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 \right)$$

where we have abbreviated $F = \frac{\partial}{\partial z}$, now a function of $z$. If $f = f(z)$ is any function and $\{e_1, e_2, e_3\}$ is the $S^3$ frame dual to $\{\eta^1, \eta^2, \eta^3\}$, then

$$J_f = -2f \frac{\partial}{\partial z} \otimes \eta^1 + \frac{1}{2f} e_1 \otimes dz - e_2 \otimes \eta^3 + e_3 \otimes \eta^2$$

\footnote{Unlike the Taub-NUT, the Taub-bolt is non-Kähler so with the other two does not create any ambiKähler triple.}
is always a complex structure (where $f \neq 0$); see Lemma 2.1. Setting $f = \pm F$, the two complex structures $J^\pm = J_{\pm F}$ are also compatible with $g$, and produce opposite orientations. Their $(1,1)$ forms are

$$\omega^\pm = g(J^\pm \cdot, \cdot) = \pm \frac{1}{2} C dz \wedge \eta^3 + C \eta^1 \wedge \eta^3. \tag{5}$$

From $d\eta^i = -\epsilon_{ijk} \eta^j \wedge \eta^k$ we have $d\omega^\pm = (\pm C + C_\pm) dz \wedge \eta^j \wedge \eta^3$, so a $U(2)$-invariant metric $g$ is always conformally Kähler, and Kähler when the conformal factor is chosen to be $C = C_0 e^{\mp z}$, respectively.

The following linear operators appear frequently:

$$\mathcal{L}^+ = \left(-\frac{1}{2} \frac{d}{dz} + 1\right) \left(-\frac{d}{dz} + 1\right), \quad \mathcal{L}^- = \left(\frac{1}{2} \frac{d}{dz} + 1\right) \left(\frac{d}{dz} + 1\right) \tag{6}$$

as does the $4^\text{th}$ order linear operator $\mathcal{L}^+ \circ \mathcal{L}^- = \frac{1}{4} \frac{d^4}{dz^4} - \frac{5}{4} \frac{d^2}{dz^2} + 1$. The third-order nonlinear operator $\mathcal{B}$ is also important:

$$\mathcal{B}(F, F) = \left(-\frac{1}{2} F_x^3 + \frac{3}{2} F_x + F - 1\right) (\mathcal{L}^+(F) - 1) + F_x (\mathcal{L}^+(F))_x. \tag{7}$$

This operator appears messy, but a relationship exists between $\mathcal{B}$ and $\mathcal{L}^+ \circ \mathcal{L}^-$. Namely, $\mathcal{B}$ is a first integral of the inhomogeneous operator $F \mapsto \mathcal{L}^+ (\mathcal{L}^- (F)) - 1$, meaning that whenever $F$ solves $\mathcal{L}^+(\mathcal{L}^- (F)) - 1 = 0$ then $\mathcal{B}(F, F)$ is a constant. See equation (88).

We will often use $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$ where $\sigma^0 = \frac{1}{|\omega|^2} dz, \sigma^i = \frac{1}{|\omega|^2} \eta^i$ for the orthonormal frame corresponding to the orthogonal frame $\{dz, \eta^1, \eta^2, \eta^3\}$.

**Proposition 1.1.** The metric (3) has scalar curvature

$$s = -4 C^{-1} \left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2\right) - 24 C^{-\frac{3}{2}} \frac{\partial}{\partial z} \left(F \frac{\partial C^\frac{3}{2}}{\partial z}\right) \tag{8}$$

and trace-free Ricci tensor

$$\mathbf{R}^c = 4 FC^{-\frac{1}{2}} \left(\frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} - \frac{1}{4} C^{-\frac{1}{2}}\right) \cdot ((\sigma^0)^2 - (\sigma^1)^2) + 2 \left(C^{-\frac{1}{2}} \frac{\partial}{\partial z} \left(F \frac{\partial}{\partial z} C^{-\frac{1}{2}}\right) - C^{-1} \left(\frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{3}{4} F + 1\right)\right) \cdot ((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2). \tag{9}$$

The Weyl curvatures of $g$ are

$$W^\pm = -\frac{1}{C} (\mathcal{L}^\pm (F) - 1) \left(\omega^\pm \otimes \omega^\pm - \frac{2}{3} \text{Id}_{\Lambda^2}\right) \tag{10}$$

and the divergences of the Weyl tensors are

$$\delta W^\pm = W^\pm \left(\nabla \log \left|e^{\pm \frac{2}{3} (\mathcal{L}^\pm (F) - 1) \sqrt{C}\right| \cdot, \cdot, \cdot, \cdot\right). \tag{11}$$

The Bach tensor is

$$\text{Bach} = \frac{16}{3 C^2} \cdot F \cdot (\mathcal{L}^- (\mathcal{L}^+(F)) - 1) \cdot \left(-2 (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right) + \frac{8}{3 C^2} \cdot \mathcal{B}(F, F) \cdot \left(- (\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right). \tag{12}$$

If the metric is Kähler with respect to $J^+$, so $C = C_0 e^{-z}$, then the scalar curvature and Ricci form are

$$s = \frac{8}{C} (\mathcal{L}^+(F) - 1), \quad \text{and} \quad \rho = -\frac{2}{C} (\mathcal{L}^+(F) - 1) \omega^+ - \frac{2}{C} \left(-\frac{1}{2} \frac{\partial}{\partial z} + 1\right) \left(\frac{\partial}{\partial z} + 1\right) F - 1 \right) \omega^-. \tag{13}$$
We remark that the $U(2)$-ansatz linearizes the Bach-flat equations $Bach = 0$, reducing them to $L^+ \circ L^-(F) - 1 = 0$. The auxiliary equation $B(F, F) = 0$ is an algebraic restriction on initial conditions.

When studying metrics—rather than just solutions of ODEs—it is useful to reduce by homothetic equivalence. In our case this reduces the dimension of the solution space by two: one dimension for translation in $z$ and one for multiplication of $g$ by a positive constant.

**Proposition 1.2** (Extremal, Bach-flat, Kähler-Einstein metrics). The metric $\lbrack 3 \rbrack$ is extremal with complex structure $J^+$ if and only if $C = C_0 e^{-z}$ and $L^+ (L^-(F)) - 1 = 0$, meaning

$$F(z) = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}. \quad (14)$$

The metric $\lbrack 3 \rbrack$ is Bach-flat if and only if $F$ satisfies $\lbrack 14 \rbrack$ and $C_1 C_4 - C_2 C_3 = 0$. The metric $\lbrack 3 \rbrack$ is Kähler-Einstein with complex structure $J^+$ if and only if $C = C_0 e^{-z}$ and $F$ satisfies $\lbrack 14 \rbrack$ with $C_1 = C_3 = 0$. The metric has $W^\pm = 0$ if and only if $L^\pm (F) - 1 = 0$ meaning, respectively,

$$F(z) = 1 + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad \text{or} \quad F(z) = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z}. \quad (15)$$

Consequently, up to homothety, the extremal metrics constitute a 3-parameter family of metrics. Up to homothety and conformal transformation, the Bach-flat metrics constitute a 2-parameter family of metrics which, up to conformal factors, is a subspace of the extremal metrics. Up to homothety and conformal transformation, the metrics with $W^+ = 0$ (or $W^- = 0$) form a 1-parameter subspace of the Bach-flat metrics. The KE metrics also form a 1-parameter subspace of the Bach-flat metrics.

A metric is said to have harmonic curvature if $\delta Rm = 0$, which is equivalent to $\delta W = 0$ and $s = \text{const}$; see \[11\], \[6\]. In the $U(2)$-invariant case $\delta W = 0$ already implies scalar curvature is constant.

**Proposition 1.3** (Einstein and harmonic-curvature metrics). For the metric $\lbrack 3 \rbrack$ the following are equivalent: 1) $\delta W = 0$, 2) $\delta Rm = 0$, 3) the metric is Einstein: $Ric = 0$, and 4) $F$ and $C$ satisfy

$$F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \quad (16)$$

with the two relations $C_1 C_5 - C_2 C_6 = 0$ and $C_3 C_5 - C_4 C_6 = 0$. Given \[16\], scalar curvature is the constant $s = -24(C_2 C_5^2 - 2 C_5 C_6 + C_3 C_6^2)$.

Further, a metric $\lbrack 3 \rbrack$ is Bach-flat if and only if it is conformal to an Einstein metric. The metric \[16\] is KE with respect to $J^+$ if and only if $C_6 = 0$ (so also $C_1 = C_3 = 0$), and KE with respect to $J^-$ if and only if $C_5 = 0$ (so also $C_2 = C_4 = 0$).

Up to homothety, there is a 1-parameter family of Ricci-flat metrics. Up to homothety, there are exactly five Ricci-flat KE metrics: the flat metric, the Taub-NUT metric, the metric given by \[32\] below, the Eguchi-Hanson metric, and the metric given by \[33\] below.

**Proposition 1.4** (CSC and half-harmonic metrics). The metric $\lbrack 3 \rbrack$ has scalar curvature $s = s_0$ if and only if $F$, $C$ satisfy the second order relation

$$0 = s_0 C^2 + 4 C^2 \left( \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2 \right) + 24 \frac{\partial}{\partial z} \left( F \frac{\partial C^2}{\partial z} \right), \quad (17)$$

and has $\delta W^\pm = 0$ if and only if $e^\pm \frac{1}{2} \left( L^\pm (F) - 1 \right) \sqrt{C}$ is constant.
Suppose the metric is Kähler with respect to $J^+$, meaning $C = C_0 e^{-s^2}$. Then $\delta W^+ = 0$ if and only if $F = 1 + C_2 e^{-s^2} + C_3 e^s + \frac{1}{2} C_4 e^{2s}$, in which case scalar curvature is the constant $s = -24C_2/C_0$. Likewise $\delta W^- = 0$ if and only if $F = 1 + \frac{1}{2} C_1 e^{-s^2} + C_2 e^{-s} + \frac{1}{2} C_4 e^{2s}$, in which case the metric is extremal and $s = -\frac{24}{C_0} (C_1 e^{-s} + C_2)$.

See Section 3.6 for a table of the $U(2)$-invariant canonical metrics.

**Theorem 1.5.** In the $U(2)$-invariant case, the space of solutions to the $B^+$-flat equations is 7-dimensional. Up to homothety these constitute a 5-parameter family of metrics and the CSC $B^+$-flat metrics constitute a 4-parameter family of metrics. When $t \neq 0, \infty$, there exist CSC $B^+$-flat metrics that are not conformal to any extremal metric.

The 8th order system for the $B^+$-flat metrics is complicated, but appears explicitly in Lemma 2.1 below. In Section 4 we discuss the ambikähler transform and create extremal metrics on $C^2$, $C^2 \setminus \{(0,0)\}$ and $O(\pm 1)$ conformal to the classic Taub-NUT and Taub-bolt metrics.

## 2 Properties of the Ansatz

The metric $g$, complex structures $J^\pm$, and $(1,1)$ forms $\omega^\pm$ are

$$g = C \left( \frac{1}{4F} dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 \right)$$

$$J^\pm = \mp 2F \frac{\partial}{\partial z} \otimes \eta^1 \pm \frac{1}{2F} e_1 \otimes dz - e_2 \otimes \eta^3 + e_3 \otimes \eta^2$$

$$\omega^\pm = g(J^\pm, \cdot) = \pm \frac{1}{2} C dz \wedge \eta^1 + C \eta^2 \wedge \eta^3.$$  

We make three computations in this section. In Section 2.1 we show the left-invariant complex structures $J_f$ are always integrable, and establish the Kähler condition for some right-invariant complex structures as well. In Section 2.2 we compute the curvature tensors up through the Bach tensor. In Section 2.3 we examine the topology and asymptotics which the $U(2)$ ansatz may produce, determining when manifold ends might be ALE, ALF, cusp-like, Einstein-like, or have curvature singularities, and we characterize the nut-like and bolt-like topology changes.

### 2.1 The complex structures

Here we check the integrability of the left-invariant almost complex structures $J_f$, then study certain metric-compatible right-invariant structures.

**Lemma 2.1.** Given any $f = f(z) \neq 0$, the complex structure $J_f$ is integrable.

**Proof.** The splitting $\bigwedge^1 \mathbb{R} = \bigwedge^{0,0} \oplus \bigwedge^{0,1}$ into $\pm \sqrt{-1}$ eigenspaces of $J_f$ gives

$$\bigwedge^{0,1} = \text{span} \left\{ \frac{1}{2f} dz - \sqrt{-1} \eta^1, \eta^2 - \sqrt{-1} \eta^3 \right\}.$$  

On bases we compute

$$d \left( \frac{1}{2f} dz - \sqrt{-1} \eta^1 \right) = -2 \sqrt{-1} \eta^2 \wedge \eta^3 = 2\eta^2 \wedge (\eta^2 - \sqrt{-1} \eta^3),$$

$$d \left( \eta^2 - \sqrt{-1} \eta^3 \right) = 2\eta^1 \wedge \eta^3 + 2 \sqrt{-1} \eta^1 \wedge \eta^2 = 2 \sqrt{-1} \eta^1 \wedge (\eta^2 - \sqrt{-1} \eta^3).$$  

Therefore $d \bigwedge^{0,1} \subset \bigwedge^1 \wedge \bigwedge^{0,1} = \bigwedge^{1,1} \oplus \bigwedge^{0,2}$ and we conclude that $J_f$ is integrable. \qed
Lemma 2.2. The complex structures $J^\pm$ are metric compatible. Their $(1,1)$ forms $\omega^\pm = g(J^\pm \cdot, \cdot)$ are closed if and only if $C = C_0 e^{\mp z}$, respectively.

Proof. Checking compatibility with the metric is an elementary computation (which we omit). From \( [3] \), \( dw^\pm = 0 \) if and only if \( C = C_0 e^{z} \).

To create right-invariant complex structures and relate them to the metric (which is left-invariant) we require background coordinates. From the usual "Euler coordinates" on \( S^3 \) comes polar coordinates on \( \mathbb{R}^4 \approx \mathbb{C}^2 \) given by

\[
(r, \psi, \theta, \varphi) \mapsto \left( r \cos(\theta/2)e^{-i/2(\psi+\varphi)}, r \sin(\theta/2)e^{-i/2(\psi-\varphi)} \right).
\]

(21)

The coordinates \((\psi, \theta, \varphi)\), known historically as precession, nutation, and rotation, have ranges \(|\psi \pm \varphi| < 2\pi\) and \( \theta \in [0, \pi] \). The transitions between the coordinate coframe and left-invariant coframing \( dz, \eta^1, \eta^2, \eta^3 \) are

\[
\eta^0 = dz = \frac{\sqrt{F}}{2\sqrt{C}} dr \\
\eta^1 = \frac{1}{2} (d\psi + \cos \theta d\varphi) \quad e_0 = \frac{\partial}{\partial z} = \frac{\sqrt{F}}{2\sqrt{C}} \frac{\partial}{\partial r} \\
\eta^2 = \frac{1}{2} (\sin \psi d\theta - \cos \psi \sin \theta d\varphi) \quad e_2 = 2 \left( \cos \psi \cos \theta \frac{\partial}{\partial \psi} + \sin \psi \frac{\partial}{\partial \theta} + \cos \psi \frac{\partial}{\partial \varphi} \right) \\
\eta^3 = \frac{1}{2} (\cos \psi d\theta + \sin \psi \sin \theta d\varphi) \quad e_3 = 2 \left( \sin \psi \cos \theta \frac{\partial}{\partial \psi} + \cos \psi \frac{\partial}{\partial \theta} + \sin \psi \frac{\partial}{\partial \varphi} \right).
\]

(22)

To create the right-invariant frames we apply quaternionic conjugation \( T(z,w) = (\bar{z}, -\bar{w}) \) to \( \mathbb{C}^2 \), which changes the parameterization to

\[
(r, \psi, \theta, \varphi) \mapsto \left( r \cos(\theta/2)e^{i/2(\psi+\varphi)}, -r \sin(\theta/2)e^{i/2(\psi-\varphi)} \right).
\]

(23)

In coordinates \( T(r, \psi, \theta, \varphi) = (r, -\varphi, -\theta, -\psi) \). The left-invariant forms \( \eta^i \) pull back to right-invariant forms \( \bar{\eta}^i = T^*(\eta^i) \), and their dual vector fields are \( T_*(\bar{e}_i) = e_i \). Under this pullback,

\[
\bar{\eta}^0 = dz = \frac{\sqrt{F}}{2\sqrt{C}} dr \\
\bar{\eta}^1 = -\frac{1}{2} (d\varphi + \cos \theta d\psi) \quad \bar{e}_0 = -\frac{\partial}{\partial z} = \frac{\sqrt{F}}{2\sqrt{C}} \frac{\partial}{\partial r} \\
\bar{\eta}^2 = \frac{1}{2} (\sin \varphi d\theta - \cos \varphi \sin \theta d\psi) \quad \bar{e}_2 = 2 \left( \cos \varphi \sin \theta \frac{\partial}{\partial \varphi} + \sin \varphi \sin \theta \frac{\partial}{\partial \psi} + \cos \varphi \frac{\partial}{\partial \theta} \right) \\
\bar{\eta}^3 = -\frac{1}{2} (\cos \varphi d\theta + \sin \varphi \sin \theta d\psi) \quad \bar{e}_3 = 2 \left( \sin \varphi \sin \theta \frac{\partial}{\partial \varphi} + \cos \varphi \sin \theta \frac{\partial}{\partial \psi} + \sin \varphi \cos \varphi \frac{\partial}{\partial \theta} \right).
\]

(24)

In the \( \{\bar{\eta}^i\}, \{\bar{\eta}^i\} \) bases, the map \( T^* : \Lambda^1 \rightarrow \Lambda^1 \) giving \( \bar{\eta}^i = T^*(\eta^i) \) is the matrix

\[
T^* = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\cos \theta & \cos \psi \sin \theta & -\sin \psi \sin \theta \\
0 & -\sin \theta \cos \varphi & -\cos \psi \cos \varphi + \sin \psi \sin \varphi & \sin \psi \cos \theta \cos \varphi + \cos \psi \sin \varphi \\
0 & -\sin \theta \sin \varphi & -\cos \psi \cos \varphi - \sin \psi \sin \varphi & \sin \psi \cos \theta \sin \varphi - \cos \psi \cos \varphi
\end{pmatrix}.
\]

(25)

One may check directly that \( T^* \in SO(4) \). Let \( \sigma^i \) be the unit length forms

\[
\sigma^0 = \sqrt{\frac{C}{4r^2}} dz, \quad \sigma^1 = \sqrt{\sqrt{F}} \eta^1, \quad \sigma^2 = \sqrt{\sqrt{C}} \eta^2, \quad \sigma^3 = \sqrt{\sqrt{C}} \eta^3
\]

(26)

and let \( \{f_0, f_1, f_2, f_3\} \) be the corresponding left-invariant frame, so \( f_i = \frac{1}{\sqrt{|\sigma_i|}} e_i \). The left-invariant structures \( J^\pm \), from above, can be expressed

\[
J^\pm = \mp f_0 \otimes \sigma^1 \pm f_1 \otimes \sigma^0 - f_2 \otimes \sigma^3 + f_3 \otimes \sigma^2.
\]

(27)

Under \( T \) these are conjugate to the right-invariant complex structures we call \( I^- = T_\circ J^+ \circ T_\circ \) and \( I^+ = T_\circ J^- \circ T_\circ \). Because \( I^\pm \) are isomorphic to \( J^\pm \) under a diffeomorphism on \( M^4 \) (the antipodal map on the \( S^3 \) factor), \( I^+ \) and \( I^- \) are integrable. We summarize this in the following lemma.
Lemma 2.3. The structures $I^\pm$ are integrable, right-invariant, and $g$-compatible. The structures $J^+, I^+$ produce a common orientation, with corresponding $(1,1)$-forms $\omega^+, \omega^+_I \in \Lambda^+$. Similarly $J^-, I^-$ produce a common orientation, and $\omega^-, \omega^-_I \in \Lambda^-$. □

The complex structures $J^+, J^-$ produce a very flexible array of possible Kähler metrics, as $F$ may be chosen freely and only $C$ is constrained. By contrast, the Kähler condition on the $\omega^+_I$ is far more restrictive. This is because the left-action of $SU(2)$ fixes $g$ but permutes $I^\pm$ among an $S^2$ worth of complex structures, which in turn means that $d\omega^+_I = 0$ forces $\omega^+_I$ to be not just Kähler but a Kähler representative in a hyperKähler structure. In particular the Kähler condition forces $\text{Ric} = 0$.

Proposition 2.4. Letting $\omega^-_I = g(I^-, \cdot)$, then $d\omega^-_I = 0$ if and only if

$$F = (1 + C_1 e^z)^2 \quad \text{and} \quad C = \frac{C_0 e^z}{(1 + C_1 e^z)^2}.$$ (28)

In this case the metric $g$ is Ricci-flat. Replacing $z$ by $- z$ (28), the same is true for $I^+$.

Proof. We may compute $d\omega^-_I$ explicitly using the matrices for $T^*$ in (25) and its inverse-transpose $T$. The computation is tedious but completely elementary, and works out to be

$$\ast d\omega_1 = \frac{2}{\sqrt{C}} \left( \cos \theta \left( (-2 + F^2) + F^2 \frac{\partial}{\partial z} \log C \right) \eta^1 
- F^{-1} \sin \theta \cos \psi \left( 2 F^2 - 2 F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^2 
- F^{-1} \sin \theta \sin \psi \left( 2 F^2 - 2 F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^3 \right).$$ (29)

Setting this to zero gives the partially decoupled system

$$\frac{\partial}{\partial z} F^\pm = \left( -1 + F^2 \right), \quad \frac{\partial}{\partial z} \log C = \left( -1 + 2 F^{-1} \right)$$ (30)

which has general solution $F = (1 + C_1 e^z)^2, C = \frac{C_0 e^z}{(1 + C_1 e^z)^2}$. Ricci-flatness follows from the general fact (see [5]) that any hyperKähler metric is Ricci flat, or, more explicitly, from Proposition 3.2 below. □

Proposition 2.4 gives a two parameter family of solutions. Therefore up to homothety we have, not a 2-parameter family, but exactly two metrics.

Proposition 2.5. Up to homothety, there are exactly two metrics $g$ of the form (3) for which $I^-$ is a Kähler structure. The first is

$$F = (1 - e^z)^2 \quad \text{and} \quad C = \frac{e^z}{(1 - e^z)^2}$$ (31)

which on $z \in (0, \infty)$ is the classic Taub-NUT metric. It has an ALF end at $z = 0$ and a nut at $z = \infty$. The second is

$$F = (1 + e^z)^2 \quad \text{and} \quad C = \frac{e^z}{(1 + e^z)^2}.$$ (32)

This has a nut at $z = - \infty$ and curvature singularity at $z = \infty$.

For information about the Taub-NUT metric see Section 4. For an analysis of the nut-like topological change see 2.3.1 and for ALF ends see 2.3.2. To verify the claim that (32) has a curvature singularity as $z \to + \infty$, one may use (49) below to find $|W^+|^2 = 384 (-1 + e^z)^6$. 

7
2.2 Curvature quantities

A useful computational tool comes from placing the metric [18] into LeBrun ansatz form [27]. Referring to the polar coordinates of [21], from \((r, \varphi, \theta, \psi)\) we change to \((Z, \tau, x, y)\) where \(x = \log \tan \frac{\theta}{2}\), \(y = \varphi\), \(\tau = \psi\), and \(Z\) solves \(dz = \frac{1}{4}Cdz\). Then \((\eta^2)^2 + (\eta^3)^2 = \frac{1}{4}(d\theta^2 + \sin^2 \theta d\varphi^2) = \frac{1}{4 \cosh^2 \eta} \langle dx^2 + dy^2 \rangle\) and

\[
g = \frac{C}{4 \cosh^2 \eta} \langle dx^2 + dy^2 \rangle + \frac{FC}{4} \langle (dr - \tanh(x)dy)^2 + \frac{4}{FC} dz^2 \rangle. \tag{33}
\]

Written this way, the metric (33) is precisely in the form of Proposition 1 of [27]—the LeBrun ansatz—where \(w = \frac{4}{FC}\) and \(e^u = \frac{FC^2}{16 \cosh^2 \eta}\). The complex structures in these coordinates are

\[
J^\pm(dZ) = \mp 2FC \eta^1, \quad J^\pm(dx) = -dy \tag{34}
\]

and we record the useful fact that \(\eta^2 \wedge \eta^3 = \frac{1}{4 \cosh^2 \eta} dx \wedge dy\).

**Proposition 2.6** (Ricci Curvature in the Kähler case). If \(g\) is Kähler with respect to \(J^+\), its Ricci form \(\rho = \text{Ric}(J^+\cdot)\) and scalar curvature are

\[
\begin{align*}
\rho &= -\frac{2}{C} (L^+(F) - 1) \omega^+ - \frac{2}{C} \left( \frac{1}{2} \frac{\partial}{\partial z} + 1 \right) \left( \frac{\partial}{\partial z} + 1 \right) F - 1 \right) \omega^-, \tag{35} \\
s &= \frac{8}{C} (L^+(F) - 1). \tag{36}
\end{align*}
\]

**Proof.** Setting \(C = C_0 e^{-\tilde{z}}\) we follow the computation in [27]. From that paper, the Ricci form \(\rho = -i \partial \bar{\partial} u = \frac{1}{2} \langle d(Jdu) \rangle\) where in our case \(u = \log(FC^2) - \log(16 \cosh^2(x))\), as we found in (33). Using coordinates \((z, \tau, x, y)\) (specifically using \(z\), not \(Z\) from (33)), we have \(J(dz) = -2F \eta^1\) and \(J(dx) = -dy\) from (34). Using also \(dx \wedge dy = 4 \cosh^2(x) \eta^2 \wedge \eta^3\) and \(dy^2 = -2 \eta^2 \wedge \eta^3\),

\[
\begin{align*}
&u = \log F - 2z + 2 \log C_0 - 2 \log(4 \cosh x) \\
&du = (F_1, F_1 - 2) dz - 2 \tanh(x) dx \\
&Jdu = (-2 F^2 + 4F) \eta^1 + 2 \tanh(x) dy \\
&dJdu = (-2 F^2 + 4F) dz \wedge \eta^1 + (-4 F^2 - 8F + 8) \eta^2 \wedge \eta^3.
\end{align*}
\]

From (18), \(dz \wedge \eta^1 = C^{-1} \langle \omega^+ - \omega^- \rangle\) and \(\eta^2 \wedge \eta^3 = \frac{1}{2} C^{-1} \langle \omega^+ - \omega^- \rangle\). Therefore

\[
\rho = \frac{2}{C} \left( \frac{1}{2} F_{zz} + \frac{3}{2} F_{z} - F + 1 \right) \omega^+ + \frac{2}{C} \left( \frac{1}{2} F_{zz} - \frac{1}{2} F_{z} - F + 1 \right) \omega^- \tag{38}
\]

as claimed. Scalar curvature for any Kähler metric is \(s = 2*(\omega^+ \wedge \rho)\), so (36) along with the facts \(\omega^+ \wedge \omega^- = 0\) and \(* (\omega^+ \wedge \omega^-) = 2\) gives (30).

**Proposition 2.7** (Ricci curvature, general case). Scalar curvature is

\[
s = -4C^{-1} \left( \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2 \right) - 24C^{-\frac{3}{2}} \frac{\partial}{\partial z} \left( F \frac{\partial}{\partial z} C^2 \right). \tag{39}
\]

Using the unit frames \(\sigma^i\) of [20] the trace-free Ricci curvature is

\[
\text{Ric} = 4FC^{-\frac{1}{2}} \left( \frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} - \frac{1}{4} C^{-\frac{3}{2}} \right) \cdot ((\sigma^0)^2 - (\sigma^1)^2) + 2 \left( C^{-\frac{1}{2}} \frac{\partial}{\partial z} \left( F \frac{\partial}{\partial z} C^{-\frac{1}{2}} \right) - C^{-1} \left( \frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{3}{4} F + 1 \right) \right) \cdot ((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2). \tag{40}
\]

8
Proof. We use the conformal change formulas from [3]. The scalar curvature follows from [36] along with the formula $\tilde{s} = U^{-2} (s - 6 U^{-1} \Delta U)$ when $\tilde{g} = U^{-1} g$.

In the Kähler metric where $C = e^{-z}$, the Laplacian $\Delta$ acting on any $U = U(z)$ is $\Delta U = 4 e^{2z} \frac{\partial^2}{\partial z^2} \left( e^{-z} F \frac{dU}{dz} \right)$. To obtain $C$, use $U = e^{\frac{2}{3} C}$.

To compute $\tilde{R}$, again we start with the Kähler case; (30) gives

$$\tilde{R}_{\phi} = 2 e^z \left( \frac{1}{2} F_{\phi \phi} - \frac{1}{2} F_{\phi} - F + 1 \right) \left( - (\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right)$$

(41)

The conformal change formula for the trace-free Ricci is $\tilde{R}_{\phi} = \tilde{R}_{\phi} + 2 U \left( \nabla^2 U^{-1} - \frac{1}{4} (\Delta U^{-1}) g \right)$. Then

$$2 U \left( \nabla^2 U^{-1} - \frac{1}{4} (\Delta U^{-1}) g \right) = - 4 U F (e^z (U^{-1}) z \left( - (\sigma^0)^2 + (\sigma^1)^2 \right)$$

$$- 2 U (\epsilon F (U^{-1}) z \left( - (\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right)$$

(42)

so with $U = e^{\frac{2}{3} C}$ we add (42) to (41) to give (40).

Proposition 2.8. The metric (18) has Weyl curvatures

$$W^\pm = - C^{-1} (L^\pm (F) - 1) \left( \omega^\pm \otimes \omega^\pm - \frac{2}{3} Id_{A^\pm} \right)$$

(43)

Proof. We use Derdzinski’s Theorem ([12], section 3, proposition 2) to find $W^+$ in the Kähler case, then conformally change to the arbitrary case. By Derdzinski’s Theorem $W^+ = \frac{1}{12} \left( 2 \omega \otimes \omega - Id_{A^+} \right)$ where $\omega$ is a Kähler form. When $C = e^{-z}$, $\omega^+$ is Kähler and Proposition 2.6 gives

$$W^+ = - \frac{2}{3} e^z (L^+ (F) - 1) \left( \frac{3}{2} \omega^+ \otimes \omega^+ - Id_{A^+} \right).$$

(44)

Conformally changing from $C = e^{-z}$ to any $C = C(z)$ gives (43). Computing $W^-$ is the same, after setting $C = e^z$ so $\omega^-$ rather than $\omega^+$ is a Kähler form.

Proposition 2.9. The metric (18) has $\delta W^+$ and $\delta W^-$ given by

$$\delta W^\pm = W^\pm \left( \nabla \log |e^z (L^\pm (F) - 1) \sqrt{C^\pm}|, \cdot, \cdot, \cdot \right).$$

(45)

Proof. We begin again by conformally changing the metric so it is Kähler. By Lemma 2.4, the metric $\tilde{g} = e^{-z} C^{-1} g$ is Kähler and the form $\tilde{\omega} = \tilde{g} (J^+ , \cdot, \cdot)$ is closed. Then $\delta \tilde{\omega} = - * d \tilde{\omega} = 0$ so also $\delta (\tilde{\omega} \otimes \tilde{\omega}) = 0$, and $\delta (Id_{A^+}) = 0$ because $Id_{A^+}$ is covariant-constant. Therefore (43) gives

$$\delta \tilde{W}^+ (\cdot, \cdot, \cdot) = \tilde{\delta} \left( - e^z (L^+ (F) - 1) \left( \tilde{\omega} \otimes \tilde{\omega} - \frac{2}{3} Id_{A^+} \right) \right) (\cdot, \cdot, \cdot)$$

$$= \tilde{W}^+ \left( \nabla \log |e^z (L^+ (F) - 1)|, \cdot, \cdot, \cdot \right)$$

(46)

Derdzinski’s conformal change formula, equation (19) of [12], is

$$\delta \tilde{W}^+ = \delta W^+ - \frac{1}{2} W^+ (\nabla \log (e^z C), \cdot, \cdot, \cdot).$$

(47)
so changing the metric back with conformal factor $e^zC$, (46) and (47) give
\[ \delta W^+ = W^+ \left( \nabla \log \left[ e^{2z}(L^+(F) - 1)\sqrt{C} \right], \cdot, \cdot, \cdot \right). \] (48)

The argument for $\delta W^-$ is entirely the same, after conformally changing so that $\tilde{\omega}^-$ not $\tilde{\omega}$ is closed.

There are two basic ways to compute the Bach tensor. We could use direct computation using perhaps (24) of [12] or in the Kähler case that Bach $(J, \cdot)$ is a multiple of $(s\rho_+ + 2\sqrt{-1}\partial\bar{\partial}s)_0$; see (39) of [12], (21) of [1], or Lemma 6 of [9]. Alternatively we could compute the variation of $\int |W|^2 dVol_4$ directly. We choose to do the latter, because it helps elucidate the nature of the quadratic functional $B(F, F)$: it comes from varying the diffeomorphism gauge, and its constancy along solutions of $L^+ \circ L^-(F) - 1 = 0$ is an expression of the second Bianchi identity.

From Proposition 2.8 we easily compute $|W|^{\pm}|^2$ and $|W|^{\pm}|^2 dVol$. These are
\[ |W|^2 = \frac{32}{3C^2} (L^+(F) - 1)^2, \quad \text{and} \]
\[ |W|^2 dVol = \frac{16}{3} (L^-(F) - 1)^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \] (49)

With this in hand, we can compute the Bach tensor by carrying out variations of the metric and observing the changes in $\int |W|^2 dVol$. Only two independent variations are possible while maintaining $U(2)$ symmetry and the conformal gauge (maintaining the conformal gauge amounts to using only trace-free variations in the metric). Briefly, the reason this is true is that the metric as expressed in the $r$-coordinate (1) has three independent parameters $A$, $B$, and $C$ that may vary, but maintaining the conformal gauge creates a relation among these so there are really only two parameters.

The first of our two variations will be to vary $F = B/C$ via
\[ F \mapsto F + sf \] (50)

while keeping $C$ unchanged so the conformal gauge is preserved. Below, we’ll see that stabilizing this variation is equivalent to the linear equation $L^+ \circ L^-(F) - 1 = 0$.

For the second of our variations, we create a diffeomorphism flow that alters the coordinate $z$, while fixing both $F$ and the conformal gauge. Stabilizing this variation leads to the non-linear equation $B(F, F) = 0$. Such a diffeomorphism flow can be expressed
\[ L_t \frac{d}{dz} = \alpha dz, \quad L_t \frac{\partial}{\partial z} = -\alpha \frac{\partial}{\partial z} \] (51)

where time $t$ is the variation parameter, $\alpha = \alpha(z)$ is a function, and $L$ is the Lie derivative along the flow. For the variation in $z$ to occur in a compact region requires both that $\alpha$ have compact support and $\int_{-\infty}^{\infty} \alpha dz = 0$, in which case $\alpha$ has an antiderivative with compact support. We define $A = A(z)$ to be the unique function so that
\[ A'(z) = \alpha(z), \quad \text{and} \ A \text{ has compact support}. \] (52)

From (51) we have the variational field $\frac{d}{dt} = A \frac{d}{dz}$. The function $F$ changes with $z$, so to force $F$ to remain constant along the diffeomorphism flow we must vary $F$ explicitly with time. Its total derivative is
\[ 0 = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dz}{dt} \frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + A \frac{\partial F}{\partial z}. \] (53)
so we vary $F$ explicitly according to the transport equation $F_t - AF = 0$. We must also vary $C$ explicitly in order to maintain the conformal gauge. Letting \( \{ \sigma^0, \sigma^1, \sigma^2, \sigma^3 \} \) be the orthonormal frame of (54), then of course

\[
g = (\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2
\]

and \((\sigma^0)^2 = \frac{C}{t^2} (dz)^2\), \((\sigma^1)^2 = CF(\eta^1)^2\), \((\sigma^2)^2 = C(\eta^2)^2\), \((\sigma^3)^2 = C(\eta^3)^2\). Using \(\frac{dF}{dt} = 0\) and \(L_{\frac{dt}{dz}} (dz)^2 = 2 \alpha (dz)^2\), we obtain

\[
L_{\frac{dt}{dz}} g = \left( 2 \alpha + \frac{d}{dt} \log C \right) (\sigma^0)^2 + \left( \frac{d}{dt} \log C \right) (\sigma^1)^2
\]

\[
+ \left( \frac{d}{dt} \log C \right) (\sigma^2)^2 + \left( \frac{d}{dt} \log C \right) (\sigma^3)^2.
\]

Maintaining the conformal gauge is equivalent to requiring the right-hand side of (55) be trace-free. Thus we vary $C$ by requiring \(\frac{d}{dt} \log C = -\frac{1}{C} \alpha\) along the flow, which is the same as evolving $C$ explicitly by \((\log C)_t + A \cdot (\log C)_z = -\frac{1}{2} \alpha\). Thus $C$ obeys a transport equation with a source.

**Theorem 2.10 (The Bach Tensor).** The Bach tensor of (18) is

\[
Bach = \frac{16}{3} C^3 \cdot F \cdot (L^-(L^+(F)) - 1) \cdot \left( -2(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right)
\]

\[
+ \frac{8}{3} C^3 \cdot B(F, F) \cdot \left( -2(\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right).
\]

**Proof.** We compute the Bach tensor by directly computing the variation of \(\int |W^+|^2 \). First, we use the variation $F \mapsto F + sf$ of (50) to compute

\[
\frac{d}{ds} \int |W^+|^2 dV = \frac{32}{3} \int \left[ \int L^+(f) (L^+(F) - 1) dz \right] \eta^1 \wedge \eta^2 \wedge \eta^3.
\]

When integrating against $z$, $L^-$ is the $L^2$-adjoint of $L^+$ so therefore

\[
\frac{d}{ds} \int |W^+|^2 dV = \frac{32}{3} \int f \cdot (L^-(L^+(F)) - 1) C^{-2} dV.
\]

Next we use the variation from (51). Note that $L_\frac{dz}{dz} dz = \alpha dz$ implies $L_\frac{dz}{dz} \eta^1 \wedge \eta^2 \wedge \eta^3 = \alpha dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3$. Therefore

\[
\frac{d}{dt} \int |W^+|^2 dV = \frac{d}{dt} \left( \frac{16}{3} \int \left( (L^+(F)) - 1 \right)^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \right)
\]

\[
= \frac{16}{3} \int \left[ \left( -2 \frac{d}{dz} (L^+(F)) (L^+(F) - 1) + \alpha (L^+(F) - 1)^2 \right) dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \right.
\]

Computing $\frac{d}{dz} L^+(F)$ using $L_\frac{dz}{dz} \frac{dz}{dz} = -\alpha \frac{dz}{dz}$ and $\frac{dF}{dt} = 0$, we obtain

\[
2 \frac{d}{dt} L^+(F) = 2 \frac{d}{dt} \left[ \left( -1 \frac{dz}{dz} + 1 \right) \left( -2 \frac{dz}{dz} + 1 \right) F \right]
\]

\[
= \alpha \frac{dz}{dz} \left[ \left( -1 \frac{dz}{dz} + 1 \right) F \right] - 2 \left( -1 \frac{dz}{dz} + 1 \right) \left( \alpha \frac{dz}{dz} F \right)
\]

\[
= -\alpha (2F_{zz} - 3F_z) - \frac{dz}{dz} F_z.
\]

The variational integral $\frac{d}{dt} \int |W^+|^2 dV$ is therefore

\[
\frac{d}{dt} \left( \frac{16}{3} \int \left( (L^+(F)) - 1 \right)^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \right)
\]

\[
= \frac{16}{3} \int \left[ -\alpha (2F_{zz} - 3F_z) + \alpha (L^+(F) - 1) \right.
\]

\[
- F_z \frac{dz}{dz} \left( (L^+(F) - 1) dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \right]
\]
Applying integration by parts to remove the \( \frac{\partial g}{\partial z} \) factor gives

\[
\frac{d}{dt} \int |W^+|^2 \, dVol = \frac{16}{3} \int \alpha B(F, F) C^{-2} \, dVol
\]  
(62)

where \( B(F, F) \) is the quadratic third order operator of [7].

Next we compute the variation of the metric itself. Using \( (\sigma^0)^2 = \frac{df}{F^4}(dz)^2 \) and \( (\sigma^1)^2 = CF(\eta)^2 \) we have \( L \partial_z g = -\frac{f}{F} (\sigma^0)^2 + \frac{f}{F} (\sigma^1)^2 \). This and (55) give

\[
L \partial_z g = -\frac{f}{F} (\sigma^0)^2 + \frac{f}{F} (\sigma^1)^2, \\
L \partial_z g = \alpha^2 \frac{3}{2} (\sigma^0)^2 - \alpha^2 \frac{1}{2} (\sigma^1)^2 - \alpha^2 \frac{1}{2} (\sigma^2)^2 - \alpha^2 \frac{1}{2} (\sigma^3)^2. 
\]  
(63)

We have now computed \( \delta g \) and \( \delta \int |W^+|^2 \, dVol \). The Bach tensor is (implicitly) defined by

\[
\delta \int |W^+|^2 \, dVol = \int \langle \delta g, -2Bach \rangle \, dVol 
\]  
(64)

where we used the fact that the Bach tensor, as it is commonly expressed \( Bach_{ij} = W_{:ij}^{\alpha} + \frac{1}{2} W_{ij} \alpha \text{Ric}^{\alpha} \), is negative four times \( \alpha \) the \( L^2 \) gradient of \( g \mapsto \int |W|^2 \, dVol \), so negative two times the gradient of \( g \mapsto \int |W^+|^2 \, dVol \).

To compute \( Bach \), we choose an orthogonal basis for the subspace of the \( U(2) \)-invariant, trace-free \( (0, 2) \)-forms in which \( \delta g \) resides:

\[
I_1 = (\sigma^0)^2 - (\sigma^1)^2, \quad I_2 = (\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2. 
\]  
(65)

These are orthogonal, with \( |I_1|^2 = 2 \) and \( |I_2|^2 = 4 \). We write \( Bach = B_1 \cdot I_1 + B_2 \cdot I_2 \) where \( B_1, B_2 \) are functions of \( z \). The two metric variations (63) are

\[
\delta g = -\frac{f}{F} \cdot I_1 \quad \text{and} \quad \delta g = \alpha \cdot I_1 + \frac{\alpha}{2} \cdot I_2. 
\]  
(66)

Substituting each variation into (64) gives

\[
\int \frac{16}{3C^2} B(F, F) = \int \left( -\frac{f}{F} I_1, -B_1 I_1 - B_2 I_2 \right) = 2 \int \frac{f}{F} B_1 \\
\int \frac{16}{3C^2} B(F, F) = \int \left( \alpha I_1 + \frac{\alpha}{2} I_2, -B_1 I_1 - B_2 I_2 \right) = -2 \int \alpha (B_1 + B_2) 
\]  
(67)

which must hold for all \( f, \alpha \). Therefore \( B_1 = \frac{2f}{F} (\mathcal{L}^{-} (\mathcal{L}_{\alpha}^{+} (F)) - 1) \) and \( B_2 = -\frac{8}{3} B(F, F) - \frac{16}{3} F (\mathcal{L}^{-} (\mathcal{L}_{\alpha}^{+} (F)) - 1) \). We conclude, as claimed in (56), that

\[
Bach = \frac{16}{3C^2} F (\mathcal{L}^{-} (\mathcal{L}_{\alpha}^{+} (F)) - 1) (I_1 - I_2) + \frac{8}{3C^2} B(F, F) (-I_2). 
\]  
(68)

\[ \square \]

Compare (56) with (3.3) of [8]; after substituting \( C = 1, F = f^2 \) and \( dz = 2f \, dt \), these are the same. Recalling that \( B \) is a constant of the motion, our expression makes evident that Bach-flat metrics are in fact solutions of a linear equation.

### 2.3 Topology: “nuts,” “bolts,” and asymptotics

Much of our focus is on local analysis in this paper, but here we briefly discuss some global aspects of \( U(2) \)-invariant metrics. Ostensibly the metric (15) is well defined on \( \mathbb{R} \times S^3 \) but topology changes occur if \( F \) or \( C \) attain 0 somewhere. If \( F \) reaches zero, the metric most naturally lives on a quotient \( I \times \frac{S^3}{\Gamma} / \sim \) where \( \Gamma \) is some discrete subgroup of \( SU(2) \), and \( \sim \) identifies some \( 3 \)-sphere to a \( 2 \)-sphere, via the Hopf map. If \( F \) or \( C \) become infinite there might be a complete (or incomplete) manifold end. After reviewing “nut” and “bolt” topology changes, we discuss ALE, ALF, and asymptotically cusp-like ends.

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\[ \text{Compare (4.77) of [8] and (67') of [3].} \]
### 2.3.1 Bolts and Nuts

Figure 1: A compact manifold with two bolts, one of positive and one of negative self-intersection. At a bolt, the Hopf fiber pinches off to zero while the base $S^2$ remains finitely-sized.

The first kind of topology change occurs when the Hopf fiber collapses but the conformal factor remains non-zero, meaning $F$ but not $C$ reaches zero. When $F(z_0) = 0$, the locus $z = z_0$ is not a 3-sphere but a 2-sphere, colloquially known as a “bolt” \[13\] (see also \[29\], \[15\], \[26\]).

Recalling the coordinates of Section 2.1, the transversals to the bolt are 2-dimensional submanifolds locally given by $\theta = \text{const}$, $\varphi = \text{const}$, and the metric is smooth at the bolt provided it is smooth on such transversals. The inherited metric on the transversal is $\tilde{g}_2 = \frac{1}{4} r^2 dz^2 + \frac{r^2}{4} d\psi^2$, which we write $\tilde{g}_2 = dr^2 + (\sqrt{F} d(\frac{1}{2} \psi))^2$ by solving $dr = \frac{1}{\sqrt{F}} dz$ with $r = 0$ at $z = z_0$. If $\sqrt{F} = kr + O(r^2)$, where $k \neq 0$, then $\psi$ will only have range $(-\frac{\pi}{4}, \frac{\pi}{4})$ as we approach the bolt and the metric $\tilde{g}_2$ will be conical at $r = 0$ with cone angle $2\pi|k|$ (smooth if $k = \pm 1$). If $k \in \mathbb{Z} \setminus \{0\}$ however, we can obtain a smooth metric on the quotient $I \times S^2/\Gamma$ where $\Gamma$ is a cyclic subgroup of order $|k|$ of the Hopf action. From $\sqrt{F} = kr + O(r^2)$ we have $k = \frac{d\sqrt{F}}{dr}$, and because $\frac{d}{dz} = 2\sqrt{F} \frac{d}{dr}$, $k = \frac{dF}{dz}$. We summarize this in the following Proposition.

**Proposition 2.11** (The “bolting condition”). Let $z = z_0$ be a zero of $F(z)$ and not a zero of $C$. Assuming

\[
\left.\frac{dF}{dz}\right|_{z=z_0} = k
\]

where $k \in \mathbb{R} \setminus \{0\}$ then we may identify the locus $\{z = z_0\}$ with a 2-sphere (a “bolt”). Assuming $k \in \mathbb{Z} \setminus \{0\}$, then taking the $|k|$-to-1 quotient of the $S^2$ factor, the metric is smooth near $\{z = z_0\}$ and the “bolt” is a 2-sphere of self-intersection number $k$.

It is possible that two bolts occur, one at $z_0$ and one at $z_1$ where $z_0 < z_1$, as in Figure 1. We certainly must have $\frac{dF}{dz} \geq 0$ at $z = z_0$ and $\frac{dF}{dz} \leq 0$ at $z = z_1$ so the bolts, if they are both smooth after resolution, must have self-intersection numbers $k$ and $-k$ where $k \in \mathbb{Z} \setminus \{0\}$. With either complex structure $J^+$, $J^-$ this is the same “odd” Hirzebruch surface $\Sigma_{2k-1}$; see \[25\].

Figure 2: Depiction of a “nut.” With $F \approx 1$, as $z \to +\infty$ the 3-sphere is asymptotically round, not squashed. Then the conformal factor $e^{-z}$ collapses $z = \infty$ to a point at a finite distance away.

The second topological change, called a “nut”, occurs when the entire 3-sphere contracts to a point, and the nearby topology is that of a ball in $\mathbb{R}^4$. This occurs when $C$ becomes zero but $F$ remains finite. When $\omega$ is Kähler and $C = C_0 e^{-z}$, a nut may occur at $z = +\infty$; this is depicted in Figure 2. When $\omega^-$ is Kähler and $C = C_0 e^{-z}$ a nut may occur at $z = -\infty$. In the non-Kähler case, conceivably a nut could form if $C(z) = 0$ at some finite $z$, but this produces curvature singularities. By
considering the change of coordinates \( dr = \sqrt{C/4F} \, dz \) near \( z = \infty \) and examining the asymptotics of \( C \) and \( F \), the following proposition is straightforward.

**Proposition 2.12 (The Nut condition at \( z = \infty \)).** Assume \( C = O(e^{-z}) \) and \( F = 1 + O(e^{-z}) \) as \( z \to \infty \). Adding a point at \( z = \infty \), this point is a finite distance away and has a neighborhood with bounded curvature and the topology of a ball.

2.3.2 ALE, ALF, and cusp-like ends

Figure 3: Depiction of a Kähler ALE end. As \( z \to -\infty \), because \( F \approx 1 \) the \( S^2 \)-factor is approximately round. The conformal factor grows like \( e^z \), and the metric is asymptotically locally Euclidean (ALE).

If \( g \) is Kähler with respect to \( J^+ \) so \( C = C_0 e^{-z} \), an ALE end can occur as \( z \to -\infty \), as depicted in Figure 3. If instead \( g \) is Kähler with respect to \( J^- \) then replacing \( z \) by \(-z \) Figure 3 is flipped and an ALE end occurs as \( z \to \infty \).

**Proposition 2.13.** Assume \( g \) is Kähler with respect to \( J^+ \), so \( C = e^{-z} \). If \( F = 1 + O(z^{-2}) \) as \( z \to -\infty \), the metric is ALE with better-than-quadratically decaying curvature.

Proof. Letting \( r \) be the distance function that solves \( dr = \frac{1}{2} \sqrt{C/4F} \, dz = \frac{1}{2} e^{-\frac{1}{2}z} (1 + O(z^{-2})) \, dz \), by assumption we have \( r = e^{-\frac{1}{2}z} + O(z^{-1}) \). Then \( C = e^{-z} = r^2 + O(r^{-4}) \), so the metric is \( g \approx dr^2 + (r^2 + O(r^{-4})) d\sigma_{S^3} \) as \( r \to \infty \), so it is ALE. To check how quickly curvature decays, from Proposition 2.6

\[
\rho = -2C^{-1} \left( \frac{1}{2} F_{zz} - \frac{3}{2} F_z + F - 1 \right) \omega + 2C^{-1} \left( \frac{1}{2} F_{zz} - \frac{1}{2} F_z - F + 1 \right) \omega^{-}
\]

so asymptotically \( \rho \approx e^z O(z^{-2}) \omega + e^z O(z^{-2}) \omega^- = o(r^{-2}) \). The expressions for \( |W^+|, |W^-| \) in (49) give the same decay rates, so we conclude the Riemann tensor decays quadratically: \( |Rm| = o(r^{-2}) \).

Figure 4: An ALF end and a cusp-like end.

These are the two kinds of complete ends that can possibly occur at finite coordinate values in the Kähler setting. The ALF end is familiar from classical examples: these have cubic volume growth, cubic curvature decay, and R\(^3\) tangent cone at infinity. See for example [22], [14], [10], [13]. By a “cusp-like” end, we
mean an end that locally resembles a Riemannian product of a tractrix of revolution (sometimes called a pseudosphere) with a sphere. Toward infinity the scalar and Weyl curvatures decrease rapidly, whereas the Ricci curvature approaches a constant bilinear form of signature (−, −, +, +). These two kinds of ends are conformal to each other: we will have $C = \frac{e^z}{(1-e^z)^2}$ in the ALF case and $C = e^{-z}$ or $C = e^z$ in the cusp-like case. $F$ in both cases has a second-order zero at $z = 0$. See Figure 4.

Proposition 2.14. Assume $F = z^2 + O(z^3)$ near $z = 0$.

If $C$ remains finite then the manifold forms a complete, cusp-like end near $z = 0$. Asymptotically the Hopf fiber shrinks to zero and the metric has the local geometry of the product of a pseudosphere times a sphere.

If $C = O(z^{-2})$ then the the metric forms an ALF end near $z = 0$.

Proof. The distance function $r$ satisfies $dr = \frac{1}{2} \sqrt{F} dz$ so in the cusp-like case, where $C$ remains finite, then $\sqrt{F} = O(z)$ gives $r \approx \frac{1}{2} \log(-z)$ near 0 and indeed the distance to 0 is infinite so the metric is complete. From $\omega \wedge \omega = -C^2 dz \wedge \eta^j \wedge \eta^k \wedge \eta^l$, we see the volume is finite. Checking the tensors $W$, with $F = z^2 + O(z^3)$ we find that $\mathcal{L}^+(F) - 1 = O(z)$ and so $W^+ \searrow 0$ as $z \to 0$. In the Kähler case $\rho$ is a multiple of $\omega$ added to a multiple of $\omega_+$. The multiple on $\omega$ is also $O(z)$, but the multiple on $\omega_+$, by (36), approaches $4C^{-1}$. This justifies the assertion that, in the Kähler case, the local geometry approaches a +1 times a −1 curvature surface. In the non-Kähler case, the usual conformal change formulas for Ricci curvature shows this remains true.

Next we verify that when $C = z^{-2} + O(1)$ near $z = 0$, the metric has an ALF end. Then $dr = \frac{1}{2} \sqrt{F} dz = \left(\frac{1}{2} z^{-2} + O(1)\right) dz$ so $r = z^{-1} + O(z)$ near $z = 0$. To compute volume, we use $C^2 = O(r^3)$ and $F^\frac{1}{2} = O(z) = O(r^{-1})$, so we have

$$dV_{ol} = -C^2 F^{\frac{1}{2}} dr \wedge d\sigma_{3/2} \approx r^2 dr \wedge d\sigma_{3/2}.$$  

(71)

Integrating (71) and noting that $r$ is a distance function, indeed we observe cubed volume growth. Next we check curvature decay. From (19) we have $\mathcal{L}^+(F) - 1 = O(1)$ so that $|W^+| \approx \frac{32}{3} C^{-2} = O(z^2) = O(r^{-2})$ and similarly for $|W^-|$. Inserting $F$, $C$ into the Ricci form $\rho$ from (37), we see Ricci curvature decays quadratically.

We close by remarking that ALE ends and nuts are conformal to each other (by changing between two $C = e^{-z}$ and $C = e^z$). Similarly ALF ends and cusp-like ends are conformal to each other.

### 2.3.3 Asymptotically Einsteinian ends, and incomplete ends

![Figure 5: Left: an asymptotically Einsteinian end as $z \to +\infty$. Right: an end with a curvature singularity a finite distance away as $z \to +\infty$.](image-url)
Our special metrics all obey $F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2}C_4 e^{2z}$ and there is the possibility that $F = O(e^{-z})$ or $O(e^{-2z})$ as $z \to -\infty$. In the Kähler case $C = C_0 e^{-z}$ (or when $J^-$ is the complex structure, $C = C_0 e^z$), and $F$ might grow at about the same rate as $C$, or a faster rate. When $C = C_0 e^{-z}$ and $F = O(e^{-z})$ the end is asymptotically Einsteinian with negative Einstein constant, and when $F = O(e^{-2z})$ the end has a curvature singularity a finite distance away.

**Lemma 2.15.** Assume $g$ is Kähler with respect to $J^+$. Then as $z \to \infty$

- if $F = O(e^{-2z})$ the metric is asymptotically Einsteinian with negative Einstein constant.
- if $F = O(e^{-3z})$ the metric is incomplete and $s \searrow -\infty$ near this end.

The same holds if $g$ is Kähler with respect to $J^-$, after replacing $z$ by $-z$.

**Proof.** Proposition 2.6 states

$$\rho = -\frac{2}{C} \left( \frac{1}{2} F_{zz} - \frac{3}{2} F_z + F - 1 \right) \omega^+ + \frac{2}{C} \left( \frac{1}{2} F_{zz} - \frac{1}{2} F_z = F + 1 \right) \omega^-.$$  \hspace{1cm} (72)

If $F$ blows up like $C_3 e^{-z}$ as $z \searrow -\infty$, then (72) gives

$$\rho \approx -2C_0^{-1} e^z (3C_3 e^{-z} - 1) \omega^+ + 2C_0^{-1} e^z \omega^-.$$ \hspace{1cm} (73)

and asymptotically we obtain $\rho \approx -6C_3 C_0^{-1} \omega^+$, so $g$ is indeed asymptotically Einstein with negative Einstein constant.

Finally if $F$ blows up like $e^{2z}$ then the distance function satisfies $dr = \frac{1}{2} \sqrt{C/F} dz \approx e^{-z} dz$ and so the distance is finite as $z \searrow -\infty$. Using equation (72) again, we see that $\rho$ blows up like $e^{-z}$, so curvature also blows up at a finite distance. \hfill $\Box$

### 3 Special Metrics

We use the computations of Section 2.2 to determine what conditions are needed to make a $U(2)$-invariant metric special or canonical.

#### 3.1 Scalar Curvature

From (39) of Proposition 2.7 specifying scalar curvature is equivalent to

$$sC^2 + 4C \left( \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2 \right) + 24 \frac{\partial}{\partial z} \left( F \frac{\partial}{\partial z} C^2 \right) = 0.$$ \hspace{1cm} (74)

for given $s = s(z)$. This relation is linear in $F$, and degenerates only when $F$ or $C$ reaches zero. It is underdetermined. Imposing, for example, the Kähler condition creates a critically determined equation.

#### 3.2 Extremal Kähler metrics

A Kähler metric is extremal if the functional $g \mapsto \int s^2 dVol$ is stable under those perturbations of $g$ that preserve the Kähler class. From [8] the Euler-Lagrange equations are that the gradient $\nabla s$ is a holomorphic vector field, but there are several ways to assess whether (3) is extremal. We use the condition that a Kähler metric is extremal if and only if $J \nabla s$ is Killing.
Proposition 3.1 (The extremal condition). The metric \( \text{[18]} \) with complex structure \( J^+ \) is extremal Kähler if and only if \( C = C_0 e^{-z} \) and \( L^-(\mathcal{L}^+(F)) = 1 \), which is

\[
F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}. \tag{75}
\]

Scalar curvature is \( s = -\frac{24}{C_0^2} (C_1 e^{-z} + C_2) \), and \( g \) is CSC-K if also \( C_4 = 0 \).

Likewise, the metric with complex structure \( J^- \) is extremal Kähler if and only if \( C = C_0 e^z \) and again \( L^-(\mathcal{L}^+(F)) = 1 \). It has scalar curvature \( s = -\frac{24}{C_0} (C_3 + C_4 e^z) \), and the metric is CSC-K when \( C_4 = 0 \).

Proof. From \( \text{[18]} \) and \( \text{[22]} \), we have \( \nabla z = 4 F \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \). Because the coordinate field \( \frac{\partial}{\partial z} \) is itself a Killing field and because \( s = s(z) \) is a function of \( z \) alone, for \( J^\perp s \) to be Killing it must be proportional \( \frac{\partial}{\partial z} \). Thus the extremal condition is \( \nabla s = -4\alpha J \frac{\partial}{\partial z} = -\alpha e^z \nabla z = \nabla (ae^{-z}) \) for any constant \( \alpha \). Therefore \( s = \alpha e^{-z} + \beta \) where \( \beta \) is another constant. Using \( s = -8C_0^{-1} e^z (\mathcal{L}^+(F) - 1) \) from \( \text{[36]} \) we obtain

\[
-8C_0^{-1} e^z \left( \frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{3}{2} \frac{\partial F}{\partial z} + F - 1 \right) = \alpha e^{-z} + \beta. \tag{76}
\]

The general solution is \( \text{[75]} \), after setting \( C_1 = -\frac{1}{24} \alpha C_0 \) and \( C_2 = -\frac{1}{24} \beta C_0 \).

For \( J^- \) in place of \( J^\perp \), reverse the sign on \( z \) in all computations. \( \square \)

### 3.3 Ricci curvature and Einstein metrics

By \( \text{[40]} \), \( \text{Ric} = 0 \) and the metric is Einstein if and only if

\[
\frac{\partial^2}{\partial z^2} C^{-2} = \frac{1}{4} C^{-\frac{3}{2}} \quad \text{and} \quad C^{\frac{1}{2}} \frac{\partial}{\partial z} \left( F \frac{\partial}{\partial z} C^{-\frac{1}{2}} \right) = \left( \frac{1}{2} \frac{\partial^2 F}{\partial z^2} - \frac{3}{4} F + 1 \right). \tag{77}
\]

This is critically determined and partly decoupled. It is 4th order in total so we will have a 4-parameter solution space. The general solution is

\[
F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}. \tag{78}
\]

where \( C_1 C_5 - C_2 C_6 = 0 \), and \( C_3 C_5 - C_4 C_6 = 0 \).

With six constants and two algebraic relations we have the expected four-parameter family of solutions. (We remark that the Einstein metrics of Proposition 2.4 have this form.) The two algebraic relations can be expressed in vector form: the vectors \((C_1, C_2)\), \((C_3, C_4)\), and \((C_5, C_6)\) are proportional to each other. Because \((C_1, C_2)\) and \((C_3, C_4)\) are proportional, we have \( C_1 C_4 - C_2 C_3 = 0 \) so we recover the fact that Einstein metrics are Bach-flat; see \( \text{[87]} \) below. By Lemma 2.2 the metric is Kähler when \( C_6 = 0 \) (for \( J^+ \)) or \( C_5 = 0 \) (for \( J^- \)).

To be Ricci-flat, \( C \) and \( F \) require, in addition to \( \text{[77]} \), that \( s = 0 \). This third relation appears to make the overall system overdetermined, but it does not, for the reason that \( s \) is a first integral for the system \( \text{[77]} \) so only contributes an algebraic relation. From \( \text{[74]} \),

\[
s = -24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2). \tag{79}
\]

Proposition 3.2 (The Einstein conditions). The metric \( \text{[3]} \) is Einstein if and only if

\[
F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \tag{80}
\]

\( C_1 C_5 - C_2 C_6 = 0 \), and \( C_3 C_5 - C_4 C_6 = 0 \).
Its scalar curvature is the constant $s = -24(C_3 C_5^2 - 2C_5 C_6 + C_5 C_6^2)$.

Up to homothety, there is a 2-dimensional family of Einstein metrics. Up to homothety, there is a 1-dimensional family of Ricci-flat metrics, a 1-dimensional family of KE metrics with respect to $J^+$, and a 1-dimensional family of KE metrics with respect to $J^-$. Up to homothety and biholomorphism, there are exactly five Ricci-flat Kähler metrics.

Proof. We have proven everything except the final assertion, that exactly five metrics of the form \( [3] \) are Ricci-flat Kähler, up to homothety. We prove this regardless of the complex structure, whether one of the structures considered here or not. A $U(2)$-invariant metric is Einstein if and only if it has the form \( [30] \). By Derdzinski’s theorem \([12]\), if a scalar-flat metric is Kähler—regardless of the complex structure—then it is half-conformally flat. In particular either $C_1 = C_2 = 0$ or $C_3 = C_4 = 0$.

So assume $C_1 = C_4 = 0$; the case $C_1 = C_2 = 0$ is identical under the isomorphism $z \mapsto -z$. We have four remaining variables $C_1, C_2, C_3, C_6$ and two relations: $C_1 C_5 - C_2 C_6 = 0$ from \([78]\) and $C_2 C_5^2 - 2C_5 C_6 = 0$ from \([79]\). If in addition to $C_3 = C_4 = 0$ we have both $C_1 = C_2 = 0$ then either $C_5 = 0$ or else $C_6 = 0$ and in either case we have the flat metric: $F = 1$ and $C = C_6 e^{\pm z}$.

Suppose $C_1 = 0$ but $C_2 \neq 0$; then the two relations force $C_5 = C_6 = 0$, an impossibility. Suppose $C_1 \neq 0$ but $C_2 = 0$; then the relations force $C_6 = 0$ so

$$F = 1 + \frac{1}{2} C_1 e^{-2z}, \quad C = \frac{1}{C_5} e^{-z} \tag{81}$$

which is Kähler with respect to $J^+$. Up to homothety, there are exactly two such metrics: the first of these is given by $F = 1 - e^{-2z}, C = e^{-z}$, which is the Eguchi-Hanson metric (see the table in Appendix [A]), and the second is given by

$$F = 1 + e^{-2z}, \quad C = e^{-z} \tag{82}$$

which is incomplete and has a curvature singularity at $z = -\infty$.

Lastly it is possible that neither $C_1$ nor $C_2$ are zero. The two relations now give $\frac{C_5}{C_6} = \frac{C_2}{C_1}$ and $\frac{C_5}{C_5} = \frac{C_2}{C_2}$, so $C_1 = \frac{1}{2} C_2$. Therefore the metric is

$$F = 1 + \frac{1}{4} C_2^2 e^{-2z} + C_2 e^{-z} = \left(1 + \frac{1}{2} C_2 e^{-z} \right)^2, \quad C = \frac{C_2^2 e^{-z}}{(1 + \frac{1}{2} C_2 e^{-z})^2}. \tag{83}$$

Under the isomorphism $z \mapsto -z$ this is the Kähler metric of Proposition \([24]\) which is Kähler with respect to the complex structure $I^-$; therefore the metric \(83\) is Kähler with respect to the complex structure $I^+$. As in Proposition \([25]\) we obtain exactly two metrics: one where $C_2 < 0$ (which is isomorphic to the Taub-NUT) and one where $C_2 > 0$ (which has a curvature singularity).

\[\Box\]

### 3.4 Half-conformally flat, half-harmonic, and Bach-flat metrics

**Proposition 3.3** (Half-conformally flat metrics). The metric \([3] \) is half-conformally flat with $W^\pm = 0$ if and only if $L^z(F) = 1 = 0$, meaning

$$F = 1 + C_4 e^z + \frac{1}{2} C_4 e^{2z} \quad \text{or} \quad F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z}, \tag{84}$$

respectively. Up to homothety, each case constitutes a 1-parameter family of such metrics, each a subspace of the 2-parameter family of Bach-flat metrics.

In the case $g$ is Kähler with respect to $J^+$ so $C = C_6 e^{-z}$, then $W^+ = 0$ implies $s = 0$, and $W^- = 0$ implies $s = -\frac{24}{C_6} (C_1 e^{-z} + C_2)$. 

The half-harmonic condition $\delta W^+ = 0$ (or $\delta W^- = 0$) is underdetermined, and requires an additional condition to be critically determined. Three possibilities suggest themselves: $s = \text{const}$, the Kähler condition, and both $\delta W^\pm = 0$.

**Proposition 3.4** (Half-Harmonic metrics). The metric (18) has $\delta W^+ = 0$ if and only if a constant $k_1$ exists so $e^{\frac{3}{2}z} (\mathcal{L}^+(F) - 1) C = k_1$, and $\delta W^- = 0$ if and only if $e^{-\frac{3}{2}z} (\mathcal{L}^-(F) - 1) C = k_2$, some $k_2 \in \mathbb{R}$.

Assume (18) is Kähler with respect to $J^+$, meaning $C = C_0 e^{-z^2}$. Then

a) $\delta W^+ = 0$ if and only if $F = 1 + C_2 e^{-z^2} + C_3 e^z + \frac{1}{2} C_4 e^{2z}$. In particular scalar curvature $s = -24 \frac{C_2}{C_0}$ is constant.

b) $\delta W^- = 0$ if and only if $F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + \frac{1}{2} C_4 e^{2z}$. In particular the metric is extremal and $s = -24 \frac{1}{C_0} (C_1 e^{-z^2} + C_2)$.

**Proof.** The assertion for $\delta W^+ = 0$ follows from Proposition 2.9 using $C = C_0 e^{-z^2}$, $e^{\frac{3}{2}z} (\mathcal{L}^+(F) - 1) \sqrt{C} = k_1$ and finding the general solution. In the Kähler case, (a) and (b) follow from Proposition 3.1.

The harmonic-Weyl condition $\delta W = 0$ is the two conditions $\delta W^\pm = 0$. This is critically determined in the $U(2)$-invariant case, which may be surprising because it is underdetermined on a generic 4-manifold and requires an additional condition, usually on scalar curvature (11), to make it critically determined. In any case an Einstein metric has $\delta W = 0$, while in the $U(2)$-invariant case the converse is also true.

**Proposition 3.5** (Harmonic curvature). The metric (18) has $\delta W = 0$ if and only if $g$ is Einstein.

**Proof.** Because $\delta W^+ \in T^* M \otimes \Lambda^+$ and $\delta W^- \in T^* M \otimes \Lambda^-$, we have $\delta W = 0$ if and only if $\delta W^+$ and $\delta W^-$ are both zero. Then by Lemma 2.9 constants $k_1$, $k_2$ exist so

$$e^{\frac{3}{2}z} (\mathcal{L}^+(F) - 1) \sqrt{C} = k_1 \quad \text{and} \quad e^{-\frac{3}{2}z} (\mathcal{L}^-(F) - 1) \sqrt{C} = k_2. \quad (85)$$

Eliminating $C$, we obtain $k_2 e^{\frac{3}{2}z} (\mathcal{L}^+(F) - 1) = k_1 e^{-\frac{3}{2}z} (\mathcal{L}^-(F) - 1)$ which has general solution

$$F = 1 + k_1 \left( \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} \right) + k_2 \left( C_1 e^z + \frac{1}{2} C_4 e^{2z} \right). \quad (86)$$

Using either equation in (85), $C = \frac{C_0 e^{-z^2}}{(C_2 + C_1 e^{-z})^2}$. From Proposition 3.2 the metric is Einstein.

Next we consider the case of Bach-flat metrics. From Proposition 2.10 this requires $F$ solve the fourth order linear equation $\mathcal{L}^- (\mathcal{L}^+(F)) - 1 = 0$ and the third order non-linear equation $\mathcal{B}(F, F) = 0$. This seems to be overdetermined, but by (88) the two equations are not independent.

**Lemma 3.6.** If $F$ solves $\mathcal{L}^- (\mathcal{L}^+(F)) - 1 = 0$ then $\mathcal{B}(F, F) = \text{const}$. If $F$ solves $\mathcal{B}(F, F) = 0$, then $\mathcal{L}^- (\mathcal{L}^+(F)) - 1 = 0$. $F$ solves both equations if and only if

$$F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z} \quad \text{and} \quad C_1 C_4 - C_2 C_3 = 0. \quad (87)$$
Proof. A tedious but straightforward computation shows
\[
\frac{\partial}{\partial z} \mathcal{B}(F, F) = 2 \frac{\partial F}{\partial z} \cdot (\mathcal{L}^+(\mathcal{L}^- (F)) - 1).
\] (88)
Therefore \( \mathcal{B}(F, F) \) is indeed constant on solutions of \( \mathcal{L}^+(\mathcal{L}^- (F)) - 1 = 0 \), as claimed. Further, \( \mathcal{B}(F, F) = 0 \) implies that either \( F = \text{const} \) or \( \mathcal{L}^+(\mathcal{L}^- (F)) = 1 \). Direct computation shows the only constant that satisfies \( \mathcal{B}(F, F) = 0 \) is \( F = 1 \), which indeed solves \( \mathcal{L}^+(\mathcal{L}^- (F)) - 1 = 0 \). We conclude that \( \mathcal{B}(F, F) = 0 \) implies \( \mathcal{L}^+(\mathcal{L}^- (F)) - 1 = 0 \), as claimed.

The general solution of \( \mathcal{L}^+(\mathcal{L}^- (F)) = 1 \) is \( F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^{z} + \frac{1}{2}C_4e^{2z} \), and in this case direct computation shows that \( \mathcal{B}(F, F) = 3(C_2C_3 - C_1C_4) \).

Therefore the general solution of \( \mathcal{L}^+(\mathcal{L}^- (F)) = 1, \mathcal{B}(F, F) = 0 \) is the three parameter family of \( \mathcal{S}_7 \).

Proposition 3.7. The metric (18) is Bach-flat if and only if
\[
F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^{z} + \frac{1}{2}C_4e^{2z}, \quad \text{and} \quad C_1C_4 - C_2C_3 = 0.
\] (89)
In particular \( g \) is Bach-flat if and only if it is conformally Einstein. Up to choice of conformal factor and translation in \( z \), there is precisely a 2-parameter family of Bach-flat metrics.

Proof. The metric \( g \) is Bach-flat if and only if \( \mathcal{L}^+(\mathcal{L}^- (F)) - 1 = 0 \) and \( \mathcal{B}(F, F) = 0 \). From Lemma 3.6 this holds if and only if \( F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^{z} + \frac{1}{2}C_4e^{2z} \) and \( C_1C_4 - C_2C_3 = 0 \), giving a 2-parameter family of solutions. Factoring out by translation in \( z \), this is a 2-parameter family, as claimed. To see that any Bach-flat metric is conformal to an Einstein metric, simply let \( C \) be a conformal factor from Proposition 3.2.

3.5 \( B^3 \)-flat metrics

Lastly we discuss the \( B^3 \)-flat metrics of [21]. These are extremizers of the quadratic functional \( B^3(g) = \int |W|^2 + t \int s^2 \), and when \( t = \infty \) we take \( B^\infty = \int s^2 \). The Euler-Lagrange equations of this functional [21] are
\[-4Bach + tC = 0 \]
where \( C = 2(\nabla^2 s - (\triangle s)g - s\mathring{R}) \) and for the Bach tensor we use the classic expression \( Bach_{ij} = W_{ij,st} + \frac{1}{2}W_{ij}Ric_{st} \). Using \( Tr(-4Bach + tC) = 0 \), when \( t \neq 0 \) a \( B^3 \)-flat metric automatically has \( \triangle s = 0 \). This means the \( B^3 \)-flat equations are the two equations \( 2Bach + ts(\mathring{R} - \nabla^2 s) = 0 \) and \( \triangle s = 0 \). We re-express these as an ODE system.

Lemma 3.8 (The unreduced \( B^3 \)-flat equations). In the metric (18) the \( B^3 \)-flat equations \( \triangle s = 0, 2Bach + ts(\mathring{R} - \nabla^2 s) = 0 \) are equivalent to
\[
\frac{\partial}{\partial z} \left( CF \frac{\partial s}{\partial z} \right) = 0, \quad F_1(F, C) = 0, \quad F_2(F, C) = 0, \quad T(F, C) = 0 \] (90)
where $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{T}$ are the operators

$$
\mathcal{F}_1(F, C) = 24 \frac{\partial}{\partial z} \left( F \frac{\partial}{\partial z} C^2 \right) + 4C^2 \left( \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2 \right) + s C^2
$$

$$
\mathcal{F}_2(F, C) = \frac{8}{3} \left( \mathcal{L}^+(\mathcal{L}^-(F)) - 1 \right) + t s C^2 \left( \frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} - \frac{1}{4} C^{-\frac{1}{2}} \right) + t C \frac{\partial F}{\partial z} \frac{\partial s}{\partial z} + t C \frac{\partial C}{\partial z} \frac{\partial s}{\partial z}
$$

$$
\mathcal{T}(F, C) = 16 \mathcal{B}(F, F) - 18t F \frac{\partial C}{\partial z} \frac{\partial s}{\partial z} - 6t C \frac{\partial F}{\partial z} \frac{\partial s}{\partial z} - \frac{3}{4} ts C^{-1} \left( C^2 (-16 + 4F + Cs) + 12F \left( \frac{\partial C}{\partial z} \right)^2 + 8C \frac{\partial C}{\partial z} \frac{\partial F}{\partial z} \right)
$$

and $\mathcal{B}$ is the operator from even

Proof. In coordinates, $\Delta = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left( \sqrt{\det g} \frac{\partial}{\partial z} \right)$. Using $(Z, \tau, x, y)$-coordinates of $\frac{33}{3}$ we have $\det g = \frac{3}{4 \cosh^2(x)} C^2$ and $g^{11} = 4FC$. Because $s = s(Z)$ is a function of $Z$ alone, then $0 = \Delta s$ is

$$
0 = 4 \cosh^2(x) \frac{\partial}{\partial Z} \left( \frac{C}{4 \cosh^2(x)} 4FC \frac{\partial s}{\partial Z} \right) = 4 \frac{\partial}{\partial Z} \left( FC^2 \frac{\partial s}{\partial Z} \right)
$$

The coordinate transition from $z$ to $Z$ of $\frac{33}{3}$ is $C \frac{\partial}{\partial z} = \frac{\partial}{\partial z}$, so we obtain the first equation of $\frac{90}{10}$. The second equation $\mathcal{F}_1(F, C) = 0$ is precisely the scalar curvature equation $\frac{74}{4}$. With $\Delta s = 0$ the Hessian $\nabla^2 s$ is trace-free. One computes that

$$
\nabla^2 s = -2C^{-3} \frac{\partial s}{\partial z} \frac{\partial (FC^3)}{\partial z} \left( \sigma^2 \right)^2 + 2C^{-2} \frac{\partial s}{\partial z} \frac{\partial (FC)}{\partial z} \left( \sigma^2 \right)^2 + 2FC^{-2} \frac{\partial s}{\partial z} \frac{\partial C}{\partial z} \left( \sigma^2 \right)^2.
$$

(93)

For the third and fourth equations we use $\ref{90}$, $\ref{10}$. We expect precisely two additional relations, due to the fact that each of the tensors $Bach$, $Ric$, and $\nabla^2 s$ have four non-zero components, but also the two algebraic relations of being trace-free, and having identical $(3,3)$ and $(4,4)$ entries. We take one relation from $2 \left( Bach_{00} + Bach_{22} \right) + t(sRic_{00} + s Ric_{22} - s_{00} - s_{22}) = 0$. Using $\ref{10}$, $\ref{12}$, and $\ref{93}$, this is

$$
\frac{8}{3} \left( \mathcal{L}^+(\mathcal{L}^-(F)) - 1 \right) + t s C^2 \left( \frac{\partial^2 C^{-\frac{1}{2}}}{\partial z^2} - \frac{1}{4} C^{-\frac{1}{2}} \right) + t C \frac{\partial F}{\partial z} \frac{\partial s}{\partial z} + t C \frac{\partial C}{\partial z} \frac{\partial s}{\partial z} = 0
$$

(94)

which is $\mathcal{F}_2(C, F) = 0$. And we take another relation from $2 Bach_{00} + (s Ric_{00} - s_{00}) = 0$, which is

$$
0 = 16 \mathcal{B}(F, F) - 18t F \frac{\partial C}{\partial z} \frac{\partial s}{\partial z} - 6t C \frac{\partial F}{\partial z} \frac{\partial s}{\partial z} - \frac{3}{4} ts C^{-1} \left( C^2 (-16 + 4F + sC) + 12F \left( \frac{\partial C}{\partial z} \right)^2 + 8C \frac{\partial C}{\partial z} \frac{\partial F}{\partial z} \right)
$$

(95)

$$
+ \frac{3}{4} ts C^2 \left( 4C^2 \left( \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} F - 2 \right) + 24 \frac{\partial}{\partial z} \left( F \frac{\partial C}{\partial z} \right) + s C^2 \right).
$$

Using $\ref{74}$ to eliminate the last term, this is $\mathcal{F}_1(F, C) = 0$.

The equations $\ref{90}$ give four equations for the three unknowns $s$, $F$, $C$, so the system appears to be overdetermined. However the next lemma shows the equations of $\ref{90}$ are not independent.
Lemma 3.9. We have the following relation:

\[ \frac{\partial T}{\partial z} = \frac{-3t}{2\sqrt{C}} \frac{\partial (sC)}{\partial z} F_1 + 12 \frac{\partial F}{\partial z} F_2 - 6t \frac{\partial \log(C^2F)}{\partial z} \left( C^2 F \frac{\partial s}{\partial z} \right). \]  

(96)

Thus \( T(F,C) \) is a constant of the motion for the 8th order system \( F_1(F,C) = 0 \), \( F_2(F,C) = 0 \), \( \Delta s = 0 \).

Proof. \( \frac{\partial}{\partial z} \) follows from a tedious but elementary computation.

Lemma 3.10. At all points where \( C, F \neq 0 \), the 8th order system

\[ \frac{\partial}{\partial z} \left( C^2 F \frac{\partial s}{\partial z} \right) = 0, \quad F_1(F,C) = 0, \quad F_2(F,C) = 0 \]  

(97)

is critically determined. The operator \( T \) is a constant of the motion, and \( \frac{\partial}{\partial z} \) combined with \( T(F,C) = 0 \) admits a 7-parameter family of solutions.

Up to homothety, the \( U(2) \)-invariant \( B^2 \)-flat metrics is 5-dimensional, and the family of \( U(2) \)-invariant CSC \( B^2 \)-flat metrics is 4-dimensional.

Proof. To ascertain whether the system \( \frac{\partial}{\partial z} \) is critically determined, we examine the coefficients on the derivatives of \( s \), \( F \), and \( C \). We find coefficients of the form \( F^2, CF^{-1}, C^2 \), \( C^{-2} \), \( FC^{-2} \), and so on. Provided \( F \) and \( C \) remain bounded away from 0 and \( +\infty \), we never find degeneracy at highest order derivatives. We conclude that the system \( \frac{\partial}{\partial z} \) which has three unknowns and three equations, remains critically determined when \( F, C \) remain non-zero.

We count the degrees of freedom in the solution space. The equations \( \frac{\partial}{\partial z} \left( C^2 F \frac{\partial s}{\partial z} \right) = 0 \), \( F_1 = 0 \), and \( F_2 = 0 \) are fourth order in \( F \), second order in \( C \), and second order in \( s \), which makes eight derivatives in total, requiring eight initial conditions. Then we restrict to \( T = 0 \). From Lemma 3.9 \( T \) is constant along solutions so is completely determined by the system’s initial conditions. \( T(F,C) \) is third order in \( F \), second order in \( C \), and first order in \( s \), so \( T = 0 \) is a single algebraic relationship among the initial conditions, and reduces the solution space from eight dimensions to seven. Up to homothety the solution space is therefore 5-dimensional. Requiring \( s = \text{const} \) is the same as imposing an initial condition of \( s_0 = 0 \), so the solution space is reduced by one dimension. Thus the CSC \( B^2 \)-flat solution space is 4-dimensional up to homothety.

Theorem 3.11. The ZSC \( B^2 \)-flat solutions, \( t \neq \infty \), are precisely the ZSC Bach-flat solutions.

Assume \( g \) is \( B^2 \)-flat and conformally extremal, \( t \neq 0, \infty \). Then it is CSC if and only if it is ZSC or Einstein.

If \( t \neq 0, \infty \) there exist CSC \( B^2 \)-flat solutions that are not conformally extremal.

Proof. The CSC \( B^2 \)-flat equations are \( \frac{\partial}{\partial z} \) with initial condition \( s_0 = 0 \). As discussed above, this is a system with 6 degrees of freedom (4 up to homothety). First we examine the \( s = 0 \) case. In this case \( T = 16B \), so \( B(F,F) = 0 \) and so the metric is Bach-flat. Thus \( F \) lies in the 3-parameter family given by Lemma 3.6. Fixing \( F \), \( F_1 = 0 \) gives a 2-parameter family of solutions for \( C \) and we obtain the expected 5-parameter solution space of ZSC Bach-flat metrics (which has 3 free parameters up to homothety).

Next assume the metric is CSC \( B^2 \)-flat, \( s \neq 0 \), and \( g \) conformally extremal. By Proposition 3.1 \( F = \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{4}C_4e^{2z} \). Plugging in this, along with \( \frac{\partial s}{\partial z} = 0 \) into \( F_2 = 0 \), we obtain

\[ \left( \frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} - \frac{1}{4} C^{-\frac{3}{2}} \right) = 0. \]  

(98)
Therefore $C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})}$. Plugging this into $\mathcal{F}_1 = 0$ provides

$$0 = C_0 (C_1 C_5 - C_2 C_6) e^{-z} + \left( -\frac{s}{24} + C_2 C_5^2 - 2C_5 C_6 - C_5 C_6^2 \right) + C_6 (C_4 C_6 - C_3 C_5) e^{z}. \tag{99}$$

We have the seven unknowns $C_1, C_2, C_3, C_4, C_5, C_6$, and $s$, and $\eqref{99}$ contributes three relations so we have a 4-parameter solution space. We consider the possibilities. First, the expression for $C$ makes it impossible that $C_5$ and $C_6$ are both zero.

If $C_5 \neq 0, C_6 = 0$ then $C = C_5^{-2} e^{-z}$ so the metric is Kähler with respect to $J^+$, and $\eqref{99}$ forces $C_1 = 0, C_2 = \frac{s}{24 C_5^2}$. Then $0 = T$ is

$$0 = T = -\frac{1}{2} e^{2s} \left( 3st - 4e^{2s}(1 - 3t)C_5 e^{2z} \right), \tag{100}$$

and because $t \neq 0$, this forces $s = 0$, contradicting the assumption $s \neq 0$. (Similarly, assuming $C_5 = 0, C_6 \neq 0$ also gives $s = 0$, again contradicting $s \neq 0$.)

Therefore both $C_5, C_6 \neq 0$. Then $\eqref{99}$ forces $C_1 C_5 - C_2 C_6 = 0, C_4 C_6 - C_3 C_5 = 0$, and by Proposition $\ref{3.2}$ the metric is Einstein. We conclude that if a CSC $B^4$-flat metric is conformally extremal, it is ZSC or Einstein.

Finally we prove that some CSC $B^4$-flat metrics are not conformally extremal. The family of Einstein solutions (not modding by homothety) is 4-dimensional, and therefore, by what we just proved, the family of CSC $B^4$-flat that are conformally extremal is also 4-dimensional. But the space of CSC $B^4$-flat metrics is 6-dimensional, so we conclude that CSC $B^4$-flat metrics exist that are not conformally extremal.

\section*{3.6 Summary}

We summarize our findings in the following chart. All of the metrics we have discussed have $F$ of the form $F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}$, the exceptions being the $\delta W^\pm = 0$ and $B^4$-flat metrics, whose forms cannot be expressed easily.

| Condition on the Metric | Conditions on coefs. | Conf. Scalar | Factor | Curvature |
|------------------------|----------------------|--------------|--------|-----------|
| Extremal Kähler        | None                 | $C_0 e^{-z}$ | $-24 \frac{C_2 + C_5 e^{-z}}{C_0}$ |
| CSC-K                  | $C_1 = 0$            | $C_0 e^{-z}$ | $-24 C_0^{-1} C_2$ |
| Einstein               | $C_1 C_5 - C_2 C_6 = 0, C_3 C_5 - C_4 C_6 = 0$ | $\frac{e^{-z}}{(C_0 + C_0 e^{-z})^2}$ | $-24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2)$ |
| Kähler-Einstein        | $C_1 = C_3 = 0$      | $C_0 e^{-z}$ | $-24 C_0^{-1} C_2$ |
|                       | $C_1 C_5 - C_2 C_6 = 0, C_3 C_5 - C_4 C_6 = 0$ | $\frac{e^{-z}}{(C_0 + C_0 e^{-z})^2}$ | 0 |
| Ricci-Flat             | $C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2 = 0$ | 0 |
| $W^+ = 0$              | $C_1 = C_2 = 0$      | any          | 0, if K |
| $W^- = 0$              | $C_3 = C_4 = 0$      | any          | $-24 \frac{C_2 + C_5 e^{-z}}{C_0}$, if K |
| $\delta W = 0$         |                      | Identical to Einstein | any |
| $\delta W^\pm = 0$     | $e^{\mp \frac{1}{2} s} (L^\pm (F) - 1) \sqrt{C} = C_5$ |

\[23\]
| Condition on the Metric | Conditions on coefs. | Conf. Factor | Scalar Curvature |
|-------------------------|----------------------|--------------|------------------|
| Kähler, \( \delta W^+ = 0 \) | Identical to CSC-K | \( C_0 e^{-z} \) | \(-24C_0^{-1}C_2 \) |
| Kähler, \( \delta W^- = 0 \) | \( C_3 = 0 \) | \( C_0 e^{-z} \) | \(-24 \frac{C_1 e^{-z} + C_2}{C_0} \) |
| Kähler, \( \delta W = 0 \) | Identical to KE | \( C_0 e^{-z} \) | \(-24C_0^{-1}C_2 \) |
| Bach-flat | \( C_1 C_4 - C_2 C_3 = 0 \) | any | any |

Notes: “K” or “Kähler” means Kähler with respect to \( J^+ \); for the \( J^- \) case replace \( z \) with \(-z\) and exchange \( W^+ \) and \( W^- \).

4 AmbiKähler Pairs

AmbiKähler pairs are from [2], where they were studied in connection with toric manifolds. An **ambiKähler structure** on a manifold is a pair of Kähler manifolds \((M^n, J_1, g_1)\) and \((M^n, J_2, g_2)\) where the complex structures \( J_1 \) and \( J_2 \) produce opposite orientations and the Kähler metrics \( g_1 \) and \( g_2 \) are conformal. Either member of the pair the **ambiKähler transform** of the other. From Lemma 2.2, every \( U(2) \)-invariant metric on a 4-manifold has an ambiKähler structure using \( J^\pm \), conformally related by \( e^\pm 2z \).

Consequently the classic Kähler metrics—see the chart in Section A—all have ambiKähler transforms. Most of these ambiKähler transforms produce nothing interesting. The ambiKähler transform of the Burns metric is the Fubini-study metric, for example, and the transforms of the other LeBrun instanton metrics are extremal Kähler metrics on weighted projective spaces—we call these the “modified LeBrun metrics” on these weighted projective spaces (these metrics were found by Bryant in §2.2 of [7], although their ambiKähler relationship with the LeBrun instantons was unknown). The transform of an odd Hirzebruch surface is precisely itself. The transforms of the Taub-NUT-Λ and Eguchi-Hanson-Λ metrics have curvature singularities.

However two cases are more interesting: the Taub-NUT and Taub-bolt. The Taub-NUT is hyperKähler with complex structures \( I^- \) and its left-translates. By Proposition 2.4 we have

\[
F = (1 - e^{-z})^2, \quad C = \frac{C_0 e^{-z}}{(1 - e^{-z})^2}. \tag{101}
\]

(In the Appendix we compute \( F \) and \( C \) explicitly from the classic expression; see also Propositions 2.4 and 2.5). The coordinate range is \( z \in (0, \infty) \), the nut is located at \( z = \infty \), and the ALF end is located at \( z = 0 \) (Figure 6). Separate from this, we have an ambiKähler structure given by complex structures \( J^- \) and \( J^+ \) and respective conformal factors \( C_0 e^z \) and \( C_0 e^{-z} \) in place of the \( C \) of (101). These give the conformal orbit of the classic Taub-NUT three different canonical metrics: itself which is hyperKähler, a 2-ended complete scalar-flat Kähler metric, and a complete extremal Kähler metric. We call the latter two the **modified Taub-NUT metrics of the first and second kinds**.

The modified Taub-NUT of the first kind has complex structure \( J^- \) and conformal factor \( C = C_0 e^z \), which gives it the same orientation as the original Taub-NUT by Lemma 2.3. This metric is two-ended: the nut at \( z = -\infty \) becomes an ALE end, and the ALF end at \( z = 0 \) becomes a cusp-like end. This complete, 2-ended
metric is scalar flat: using Proposition 1.1 one computes $s = 0$. Second, letting $J^+$ be the complex structure and choosing conformal factor $C = C_0 e^{-z}$ produces the modified Taub-NUT of the second kind. This metric is one-ended: it still has a “nut” at $z = \infty$, but the conformal change turns the ALF end into a cusp-like end. By Theorem 3.1 it is an extremal Kähler metric. It has scalar curvature $s = 48(1 - e^{-z})$, which is positive and approaches 0 asymptotically along the cusp.

These modified Taub-NUT metrics have both been discovered already, although their conformal relationship with the classic Taub-NUT has not been uncovered until now. The modified Taub-NUT of the first kind on $\mathbb{C}^2 \setminus \{(0,0)\}$ is the ZSC-K metric of §16 for $n = 2$, and the modified Taub-NUT of the second kind is a complete Bochner-flat metric of the type considered in §2.2 of [7]; see also [35].

The Taub-bolt is Ricci-flat but not Kähler (and certainly not hyperKähler) with respect to any complex structure (this is possible in the non-compact but not the compact case). The metric is

$$C = \frac{C_0 e^{-z}}{(1 - e^{-z})^2}, \quad F = 1 - \frac{1}{8} e^{-2z} + \frac{1}{4} e^{-z} - \frac{9}{4} e^z + \frac{9}{8} e^{2z}$$

on $z \in [-\log(3), 0)$. This metric is complete, Ricci-flat, Bach-flat, but not half-conformally flat: both $W^+$ and $W^-$ are non-zero by Proposition 3.3. It has an ALF end at $z = 0$ and a bolt of self-intersection $-1$ at $z = -\log(3)$. The underlying manifold is the total space of $\mathcal{O}(-1)$. It is conformally Kähler with respect to either $J^-$ or $J^+$, creating an ambiKähler pair—the modified Taub-bolt metrics of the first and second kind, respectively. Changing between $J^-$ and $J^+$ reverses the orientation, so changes the self-intersection number of the bolt from $-1$ to $+1$.

With the complex structure $J^-$ and conformal factor $C = C_0 e^z$ we obtain an extremal Kähler metric we call the modified Taub-bolt of the first kind. This metric continues to have a bolt of self-intersection $-1$ at $z = -\log(3)$, but the ALF end at $z = 0$ has been transformed into a cusp-like end. The scalar curvature is $s = 54C_0^{-1}(1 - e^z)$, which is positive and approaches 0 along the cusp. Its underlying complex manifold is the total space of $\mathcal{O}(-1)$. Its ambiKähler transform

---

Figure 6: The Taub-NUT and the modified Taub-NUTs of the first and second kinds.

Figure 7: The first image is the Taub-bolt, and the second and third are the modified Taub-bolts of the first and second kinds.

4If it were Kähler with respect to any complex structure, whether a complex structure considered here or not, Derdzhinski’s theorem would imply it is half-conformally flat which it is not.
has complex structure $J^+$ and conformal factor $C = C_0 e^{-z}$; we call this extremal Kähler metric the modified Taub-bolt of the second kind. The orientation has been reversed and the bolt has self-intersection $+1$ at $z = -\log(3)$. The ALF end at $z = 0$ has been conformally transformed into a cusp-like end. The scalar curvature is $s = 6C_0^{-1}(-1 + e^{-z})$, which again is positive and approaches zero asymptotically along the cusp. Its underlying complex manifold is the total space of $O(1)$. The modified Taub-bolt of the second kind is the only complete extremal Kähler metric on any surface with a rational curve of positive self-intersection that is known to the authors.

A The Classic Metrics

Numerous $U(2)$-invariant 4-metrics have been developed, the basic method going back to (at least) 1916 [34]. In Euclidean signature, most of these metrics were developed in the late 1970’s with notable works by Plebanski-Demianski [33], Eguchi-Hanson [15], Gibbons-Hawking [17], Page [31] [32], Gibbons-Pope [19], and Gibbons-Perry [20]. The LeBrun metrics [26] appeared slightly later. See also [14] and [4].

A.1 Transcribing the classic metrics

Fitting the classic metrics into the present framework is straightforward. As an illustration we consider the Taub-NUT metric, classically

$$g = \frac{1}{4} r + m dr^2 + 4m^2 \frac{r - m}{r + m} (\eta^1)^2 + (r^2 - m^2) \left( (\eta^2)^2 + (\eta^3)^2 \right)$$

(103)

where $r \in [m, \infty)$. Solving $-dz = 2\sqrt{AB} dr$ for $z$, we obtain coordinate changes

$$z = -\log \frac{r - m}{r + m} \quad \text{and} \quad r = \frac{m(1 - e^{-z})}{1 + e^{-z}}$$

(104)

where $z \in (0, \infty]$. Then $C$ and $F = \frac{B}{C}$ are

$$C = (r^2 - m^2) = \frac{4m^2 e^{-z}}{1 - e^{-z}}^2 \quad \text{and} \quad F = \frac{4m^2}{(r + m)^2} = (1 - e^{-z})^2.$$  

(105)

Similarly one may transcribe any of the classic $U(2)$-invariant metrics in this way. We list below the classic metrics and their expressions via conformal factor $C$ and a function $F$ of the form $F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}$.

| Name               | Classic Coefs $A, B, C$ | Conf. Factor of $F$ | Special Metric | Underlying Manifold |
|--------------------|-------------------------|---------------------|----------------|---------------------|
| Taub-NUT           | $\frac{1}{4} r + m$     | $\frac{4m^2 e^{-z}}{1 - e^{-z}^2}$ | RF, HK         | $C^2$               |
|                   | $\frac{1}{4} r - m$     |                      | HCF            |                     |
|                   | $\frac{4m^2 r - m}{r + m}$ |                      | C-Extr         |                     |
| Modified           | $e^z$                   | $(1 - e^{-z})^2$     | Extr-K         | $C^2$               |
| Taub-NUT, First Kind | See Sec. [4]            |                     | HCF            | C-RF                |
|                    |                         |                     |                |                     |
| Name                      | Classic Coefs | Conf. Factor | F or coeffs. of F | Special Metric | Underlying Manifold |
|---------------------------|---------------|--------------|-------------------|----------------|---------------------|
| Modified Taub-NUT, Second Kind | \( e^{-z} \) | \((1 - e^{-z})^2\) | SFK             | C-Extr singular | \( C^2 \setminus \{0\} \) |
| Taub-NUT-\( \Lambda \)   | \( \frac{r^2 - m^2}{4m^2} \) | \( \frac{m - \frac{1}{2} m^3 \Lambda}{(1 - e^{-z})^2} \) | Einst-\( \Lambda \) sometimes, | BF but usually | \( O(-k) \) |
| \( \Delta = * \)         | \( \frac{e^{-z}}{r^2 - m^2} \) | \( \frac{m - \frac{1}{2} m^3 \Lambda}{(1 - e^{-z})^2} \) | BF             | \( O(1) \) |
| Taub-bolt                 | \( \frac{16m^2(r^2 - \frac{2}{5} mr + m^2)}{4(r^2 - m^2)^2} \) | \( \frac{16m^2 e^{-z}}{(1 - e^{-z})^2} - \frac{1}{4}, \frac{1}{4}, -\frac{9}{4}, \frac{9}{4} \) | BF             | \( O(-1) \) |
| Modified Taub-bolt, first kind | \( e^{z} \) | \(-\frac{1}{4}, \frac{1}{4}, -\frac{9}{4}, \frac{9}{4} \) | Extr-K          | BF             | \( O(-1) \) |
| Modified Taub-bolt, second kind | \( e^{-z} \) | \(-\frac{1}{4}, \frac{1}{4}, -\frac{9}{4}, \frac{9}{4} \) | Extr-K          | BF             | \( O(+1) \) |
| Burns                     | \( \frac{(1 - \frac{m^2}{r^2})^{-1}}{r^2} \) | \( e^{z} \) | 0, \(-m^2, 0, 0 \) | SFK             | \( O(-1) \) |
| Eguchi -Hanson            | \( \frac{(1 - \frac{m^4}{r^4})^{-1}}{r^2} \) | \( e^{z} \) | \(-2m^4, 0, 0, 0 \) | RF              | \( O(-2) \) |
| LeBrun                    | \( \frac{A^{-1}}{r^2} \) | \( e^{z} \) | \(-2m^4(k - 1), m^2(k - 2), 0, 0 \) | SFK             | \( O(-k) \) |
| Modified LeBrun           | \( \frac{e^{-z}}{m^2(k - 2)} \) | \( m^2(k - 2), 0, 0 \) | Extr-K          | One point       | \( O(+k) \) |
|                         | \( \frac{(1 - \frac{m^4}{r^4} - \frac{4}{5} r^2)^{-1}}{r^2} \) | \( e^{z} \) | \(-2m^4, 0, 0, 0 \) | Einst-\( \Lambda \) if \( m^4 = \frac{4(1 + k)}{5} \), \( \Lambda = 4 - 2k \), and \( k \geq 2 \) |
| Fubini-Study             | \( \frac{(1 + \frac{4}{5} r^2)^{-2}}{r^2} \) | \( 6e^{-z} \) | 0, \(-\Lambda, 0, 0 \) | Einst-\( \Lambda \) | \( CP^2 \) |
Notes: In the Taub-NUT-A metric, \( \Delta = (r^2 - L)^2 + \frac{1}{4} \left( -\frac{1}{4} r^2 + 2m^2 r^2 + m^4 \right) \). In the LeBrun metrics, \( A = \left( 1 - \frac{m^2}{r^2} \right)^2 \left( 1 + (k - 1) \frac{m^2}{r^2} \right) \) and \( k = 1 \) gives the Burns metric and \( k = 2 \) gives the Eguchi-Hanson metric. In the Page metric, the value of the coefficients is \( C_1 = C_2 = C_3 = C_4 = \frac{1}{(2 + \cosh(2z)) \sinh(z)} \) and \( z_0 \) is the unique real solution of \( e^{4z_0} - 4e^{z_0} - 3 = 0 \).

A.2 The Page Metric

The Page metric, which won’t fit on the chart, has classic coefficients

\[
\begin{align*}
A(r) &= \frac{3(1 + \nu^2)}{\Lambda} \left( \frac{1}{3 - \nu^2 - \nu^2 (1 + \nu^2) r^2} - \frac{1}{1 - r^2} \right), \\
B(r) &= \frac{3(1 + \nu^2)}{\Lambda} \left( \frac{1}{(1 - r^2) (3 - \nu^2 - \nu^2 (1 + \nu^2) r^2)} \right), \\
C(r) &= \frac{3(1 + \nu^2)}{\Lambda} \left( \frac{1}{\nu(3 + \nu^2)} \right),
\end{align*}
\]

where \( \nu \) is the positive root of \( \nu^4 + 4\nu^3 - 6\nu^2 + 12\nu - 3 = 0 \), about \( \nu \approx \pm 0.28 \). Any other choice both disrupts the bolting condition and also creates a non-canonical metric. In this paper’s formulation, \( C \) and \( F \) are

\[
\begin{align*}
C &= \frac{12(1 + \nu^2)}{\Lambda \nu(3 + \nu^2)} \frac{e^{-z}}{(1 + e^{-z})^2} \quad \text{and} \\
F &= \frac{-\nu^4 + 6\nu^2 + 3}{4\nu(3 + \nu^2)} \left( \frac{1 - \nu^2}{4\nu(3 + \nu^2)} \right) \left( \cosh(2z) + 2 \cosh(z) \right).
\end{align*}
\]

The stated choice of \( \nu \) makes the constant \( \frac{-\nu^4 + 6\nu^2 + 3}{4\nu(3 + \nu^2)} \) equal to 1.

The Page metric is not Kähler, but is conformal to one of the extremal Kähler metrics constructed by Calabi [8] on Hirzebruch surfaces. We mentioned the “odd” Hirzebruch surfaces \( \Sigma_{2k-1} \) in Section 2.3.1 using our techniques we can construct many extremal metrics on each \( \Sigma_{2k-1} \). To do so let \( C = C_0 e^{-z} \) so the metric is Kähler with respect to \( J^+ \). For a compact manifold, \( F \) must reach 0 at two places: after translating in \( z \) we may assume \( F(\pm z_0) = F(z_0) = 0 \), some \( z_0 > 0 \). We require a bolt of self-intersection \( +k \) at \( -z_0 \) and \( -k \) at \( +z_0 \), so by Lemma 2.11 we have the four conditions \( F(-z_0) = 0 \), \( F(z_0) = 0 \), \( F'(z_0) = k \), and \( F''(z_0) = -k \).

Using \( F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z} \), for every choice of \( z_0 > 0 \), \( k \in \mathbb{N} \), \( C_0 > 0 \) we obtain a unique smooth metric

\[
C = C_0 e^{-z} \quad \text{and} \quad F = 1 + C_1' \cosh(2z) + 2C_2' \cosh(z), \quad \text{where} \quad C_1' = \frac{\sinh(z_0) - k \cosh(z_0)}{(2 + \cosh(2z_0)) \sinh(z_0)}, \quad C_2' = \frac{-2 \sinh(2z_0) + k \cosh(2z_0)}{2(2 + \cosh(2z_0)) \sinh(z_0)}.
\]

By Lemma 2.2 these are each extremal Kähler metrics. For fixed \( k \) (which fixes the Hirzebruch surface \( \Sigma_{2k-1} \)), we have two choices—\( C_0 \) and \( z_0 \)—giving a 2-parameter family of extremal metrics on each surface. Since the Kähler cone is
two-dimensional—parameterized by the mass of the two bolts—it is easy to see that we have found an extremal Kähler metric in each Kähler class. (We remark that it is already known that every Kähler class on \( \Sigma_{2k-1} \) has an extremal representative; see [23] or [24].)

We remark that an ambiKähler transform of any Hirzebruch surface is itself. This is because \( F \) of (108) is invariant under \( z \mapsto -z \), whereas the conformal factor \( e^{-2z} \) and complex structure \( J^+ \) switch: \( e^z \) becomes \( e^{-z} \) and \( J^+ \) becomes \( J^- \). Therefore the map \( z \mapsto -z \) is not only a biholomorphism but an isometry between ambiKähler pairs.

**Corollary A.1.** Assume the \( U(2) \)-invariant Riemannian manifold \((M^4, g)\) is compact and Bach-flat. Then \( M^4 = \mathbb{CP}^2 \# \mathbb{CP}^2 \) and \( g \) is conformal to the Page metric.

**Proof.** By Proposition 1.1 the Bach-flat metrics are those with \( F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{-z} + \frac{1}{2} C_4 e^{2z} \) and \( C_1 C_4 = C_2 C_3 \). Ignoring the condition \( C_1 C_4 = C_2 C_3 \) for the moment, up to homothety the metric given by (108) encodes all possibilities for compact, extremal \( U(2) \)-invariant manifolds. In particular when \( k = 1 \) the surface \( \Sigma_{2k-1} = \Sigma_1 = \mathbb{CP}^2 \# \mathbb{CP}^2 \).

Next we impose the Bach-flat condition \( C_1 C_4 = C_2 C_3 \); for (108) this becomes \( C_1' = C_2' \), which is the same as

\[
k = \frac{2(1 + 2 \cosh(z_0)) \sinh(z_0)}{2 \cosh(z_0) + \cosh(2z_0)}. \tag{109}
\]

The expression on the right is increasing for \( z_0 \geq 0 \), is zero at \( z_0 = 0 \) and approaches 2 as \( z_0 \to \infty \). Therefore the only \( k \in \mathbb{N} \) for which we can solve (109) is \( k = 1 \). For this \( k = 1 \), a unique positive value of \( z_0 \) solves (109) — this is the positive root of \( e^{4z_0} - 4e^{2z_0} - 3 = 0 \) which works out to be about \( z_0 \approx 0.579 \).

We conclude that one unique function \( F \) gives a Bach flat metric on a compact \( U(2) \)-invariant manifold: the function \( F \) of (108) where \( k = 1 \) and \( z_0 \) is the unique positive solution of \( e^{4z_0} - 4e^{2z_0} - 3 = 0 \). Because the Page metric is \( U(2) \)-invariant and Bach-flat, certainly the \( F \) of (107) equals the \( F \) of (108) for these choices (as can be verified by direct computation).

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