Some problems for the processes with the compensation of the change-point event

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Abstract. In the paper we consider some problems for the processes with the compensation of the change-point event, namely for the adaptive changes of a system. We suppose that compensations could be continuous, discontinuous or combined. The approaches of the change-point description could differ. Depending on the observation method of a change-point stopping time, we consider three types of problems (similar to the methods of filtering, extrapolation or interpolation in partially observable schemes).

1. Introduction

Studies of processes with characteristics that change at random moments of time (for example, at moments of the change-point) are carried out in many papers (see, for example, [1-7]). The classical problem with change-point was formulated as early as the 60s by A. Wald [2] and A.N. Shiryaev [5]. Further this direction was developed in the works of A.L. Presman [8], G. Robbins [9], and today the development of this problem is reflected in the works of M.L. Nikolaev [10], V.V. Mazalova [11], G.I. Salova [12] and others. Technical areas of science are traditionally considered as an applied application. Also in the last decade, a number of articles have appeared on the application of problems of the change-point in Biology and Medicine.

It is customary to distinguish the following two classes of theoretical problems:
1. the analysis of the main and alternative hypotheses about the onset of the change-point time;
2. the numerical determination of the probability of occurrence of the change-point time to a specific time.

In technical applications, the moment of disorder can be interpreted as a breakdown.

In biological systems, significant disturbances can occur, which over time can be compensated by various adaptive reactions or external influences. In mathematical models, physiological impairments correspond to frustrations, and a single-step adaptive response can be called compensation. Compensation violations in living organisms increase the life expectancy. Disorder compensation is currently not well understood.

In this paper we consider the problem of the compensation of the change-point (some adaptive changes of a system).
It is supposed that compensations can be discontinuous, continuous or combined (mixed). The problem of the compensation depends also on the way of a description of the change-point.

In this paper the compensation has the piecewise linear form so it is discontinuous. We consider three types of the problem: similar to filtering, extrapolation or interpolation problems in partially observable schemes. We find the optimal values of the intensity and the values of jumps of the process of the compensation for the cases of the observable and unobservable values of the change-point time.

In the case of a given time of the change-point we propose to find the optimal parameters of the compensation with the method of solving the optimization problem using simulation methods. In the case of the unobservable change-point time we propose to find the optimal parameters of compensation similar to the case of the observable change-point time. The change-point time we find using interpolation and extrapolation.

We consider these three models corresponding to the method of observable process with the change-point.

2. The first model of the compensation of the change-point

Let the system with function $X^{(i)} = (X_{t}^{(i)})_{t \in \mathbb{R}}$ and process $Y^{(i)} = (Y_{t}^{(i)})_{t \in \mathbb{R}}$ be defined on the stochastic basis $B^{(i)} = (\Omega, F, \mathbb{P} = (F_{t})_{t \in \mathbb{R}})$ with usual C. Dellacherie conditions (see for example, [1, 13]):

$$
\begin{align*}
X_{t}^{(i)} &= \alpha^{(i)} \cdot I \{ \tau \leq t \} \\
Y_{t}^{(i)} &= \int_{0}^{t} X_{s}^{(i)} \, ds + \sigma^{(i)} W_{t}^{(i)}
\end{align*}
$$

(1)

where $\alpha^{(i)} > 0$, $\sigma^{(i)} > 0$ ($\alpha^{(i)}, \sigma^{(i)} \in \mathbb{R}^+$) are known. The change-point time $\tau$ is also supposed to be known, and $\tau \in [0; T]$, $0 < T < \infty$. Process $W^{(i)} = (W_{t}^{(i)})_{t \in \mathbb{R}}$ is a standard Wiener one.

We define the filtration $F_{t}^{(i)} = (F_{s}^{(i)})_{s \leq t}$ (imbedded in $F$) where $F_{0}^{(i)} = \{ \emptyset, \Omega \}$ and $F_{s}^{(i)} = \sigma \{ Y_{s}^{(i)} ; s \leq t \}$ is a $\sigma$-algebra, generated by observations $(Y_{s}^{(i)})_{0 \leq s \leq t}$ for every $t \geq 0$.

Function $X^{(i)} = (X_{t}^{(i)})_{t \geq 0}$ is the change-point «indicator» and $Y^{(i)} = (Y_{t}^{(i)})_{t \geq 0}$ is observable process with the change-point.

The accumulated compensation of the change-point $K_{t}^{(i)} = (K_{t}^{(i)})_{t \geq 0}$ is defined as:

$$
K_{t}^{(i)} = \int_{0}^{t} \left\{ I \{ X_{s}^{(i)} \ du - K_{s}^{(i)} \geq \beta^{(i)} \} \right\} I \{ \tau \leq t \} \cdot \beta^{(i)} \, d \pi_{s}^{(i)}
$$

(2)

where $\beta^{(i)} > 0$ is the level of the compensation, $\pi^{(i)} = (\pi_{s}^{(i)})_{0 \leq s \leq t}$ is a Poisson process with the intensity $\lambda^{(i)}$, which admits the decomposition:

$$
\pi_{s}^{(i)} = \lambda^{(i)} t + \int_{0}^{s} \mu_{t}^{(i)} \, dt
$$

(3)

with the intensity $\lambda^{(i)} > 0$ and the martingale $\mu_{t}^{(i)}$.

Thus, the process with the compensation of the change-point $Z_{t}^{(i)} = (Z_{t}^{(i)})_{t \geq 0}$ is of the following form:

$$
Z_{t}^{(i)} = Y_{t}^{(i)} - K_{t}^{(i)}
$$

(4)

We can formulate the problem of finding the optimal values of intensities and values of jumps of the compensation. This problem can be conceded as the problem of the sequential analysis (we make a decision as data are available) with the discontinuous compensation and the observable change-point time. In the model the parameter $\beta^{(i)}$ characterizes any effect and $\lambda^{(i)}$ characterizes the effect intensity. From the formula (2) we see that the compensations (in the first approach) are at times of jumps of the Poisson process $\pi^{(i)}$. 

3. The second model of the compensation of the change-point

Similarly to the first model, in the second model objects are considered: process with the change-point \( Y^{(2)} = (Y_{t}^{(2)})_{t>0} \) and the function of the «indicator» \( X^{(2)} = (X_{t}^{(2)})_{t>0} \) showing the changes in the main characteristics.

The function \( X^{(2)} = (X_{t}^{(2)})_{t>0} \) and the process are defined on the stochastic structure [14,15]:

\[
S = (\Omega, F, F, \mathbb{P}),
\]

where \( F^{Y^{(2)}} = (F_{t}^{Y^{(2)})}_{t>0} \) is the filtration with usual C. Dellacherie conditions ([13]), \( \mathbb{P} \) is a family of probability distributions (measures) and \( \mathbb{P} = \{ P_{\theta} : \theta \in \Theta \} \), where \( \Theta = [0,T] \), \( 0 < T < \infty \).

The function \( X^{(2)} = (X_{t}^{(2)})_{t>0} \) and the process \( Y^{(2)} = (Y_{t}^{(2)})_{t>0} \) are defined as:

\[
\begin{align*}
X_{t}^{(2)} &= \alpha^{(2)} \cdot I\{ \theta \leq t \} \\
Y_{t}^{(2)} &= \int_{0}^{t} X_{s}^{(2)} ds + \sigma^{(2)} W_{t}^{(2)},
\end{align*}
\]

where the parameters \( \alpha^{(2)} > 0 \), \( \sigma^{(2)} > 0 \) (\( \sigma^{(2)}, \alpha^{(2)} \in \mathbb{R}^{+} \)) are known and \( \theta \) is an unobservable time of the change-point with values from \( [0,T] \) \( (0 < T < \infty) \). The process \( W^{(2)} = (W_{t}^{(2)})_{t>0} \) is a standard Wiener one.

The system (6) has the form of a partially observable scheme and the filtration \( F_{t}^{Y^{(2)}} = \sigma\{ Y_{s}^{(2)} : s \leq t \} \) is a \( \sigma \) - algebra, generated by observations \( (Y_{s}^{(2)})_{0 \leq s \leq t} \) for every \( t \geq 0 \), with \( F_{0}^{Y^{(2)}} = \{ \varnothing, \Omega \} \).

The process of the accumulated compensation of the change-point \( K^{(2)} = (K_{t}^{(2)})_{t \geq 0} \) we define as:

\[
K_{t}^{(2)} = \int_{0}^{t} \left[ \alpha^{(2)} \cdot I\{ u - \tilde{\theta} \leq u \} - u - K_{s}^{(2)} \right] \cdot \tilde{\beta}^{(2)} d\tilde{\pi}_{s}^{(2)},
\]

where \( \tilde{\beta}^{(2)} > 0 \) is the level of the compensation, \( \pi^{(2)} = (\pi_{t}^{(2)})_{t \geq 0} \) is a Poisson process with the intensity \( \lambda^{(2)} \), which admits the decomposition:

\[
\pi_{t}^{(2)} = \lambda^{(2)} I + m_{t}^{(2)},
\]

with the intensity \( \lambda^{(2)} > 0 \) and the martingale \( m_{t}^{(2)} \).

The time \( \tilde{\theta} \) is defined as:

\[
\tilde{\theta} = \mathbb{E}\{ \theta \mid F_{t}^{Y^{(2)}} \}.
\]

As a result, the process with the compensation of the change-point \( Z^{(2)} = (Z_{t}^{(2)})_{t \geq 0} \) is the following:

\[
Z_{t}^{(2)} = Y_{t}^{(2)} - K_{t}^{(2)}.
\]

Similarly to the first model, the problem of finding the optimal values of intensities and the values of jumps compensation is formulated. The difficulty in analyzing of this model is the estimation of the change-point time, defined with the formula (9). Similar tasks were solved, for example, in [16-20]. Thus in [7] an estimation of the change-point time is for the case of the exponential distribution of the stopping time of the change-point.

In this paper, the time of the change-point is unobservable and we estimate it with the least squares method:

\[
\delta_{\theta} = \mathbb{E}\{ (\theta - \tilde{\theta})^{2} \mid F_{t}^{Y^{(2)}} \}, \theta \leq t \rightarrow m_{\text{min}}.
\]

While modeling the processes of the second problem the subtask (11) is solved with the method of a computer simulation.
4. The third model of the compensation of the change-point

Function $X^{(3)} = \begin{cases} \alpha \end{cases}$ (the indicator of change-point) and process $Y^{(3)} = \begin{cases} \beta \end{cases}$ (with the change-point) are defined on the stochastic structure [14-15]:

$$S = (\Omega, F, F, \mathbb{P}),$$

where $F^{Y^{(3)}} = \{F_t^{Y^{(3)}}\}_{t \in \mathbb{N}}$ is the filtration with usual C. Dellacherie conditions, $\mathbb{P}$ is a family of probability distributions (measures) and $\mathbb{P} = \{P_\xi, \xi \in \Sigma\}$, where $\Sigma = [0, T], 0 < T < \infty$.

The function $X^{(3)} = (X_t^{(3)})_{t \geq 0}$ and the process $Y^{(3)} = (Y_t^{(3)})_{t \geq 0}$ are defined as:

$$
\begin{align*}
X_t^{(3)} &= \alpha_t^{(3)} \cdot I\{\xi \leq t\}, \\
Y_t^{(3)} &= \int X_s^{(3)} ds + \sigma^{(3)} W_t^{(3)},
\end{align*}
$$

where the parameters $\alpha^{(3)} > 0, \sigma^{(3)} > 0 (\alpha^{(3)}, \sigma^{(3)} \in \mathbb{R}^+)$ are known and $\xi$ is the unobservable change-point time with values from $[0, T]$ ($0 < T < \infty$) and unknown. The process $W^{(3)} = (W_t^{(3)})_{t \geq 0}$ is a standard Wiener process.

The system (13) has the form of a partially observable scheme and the observation is possible only after the moment of time $N$ ($0 < N < T$). The filtration $F_t^{Y^{(3)}} = \sigma\{Y_s^{(3)}, s \leq t\}$ is a $\sigma$-algebra, generated by observations $(Y_s^{(3)})_{0 \leq s \leq t}$ for every $t \geq N$ with $F_0^{Y^{(3)}} = \{\emptyset, \Omega\}$.

The process of the accumulated compensation of a change-point $K^{(3)} = (K_t^{(3)})_{t \geq 0}$ is defined as:

$$K_t^{(3)} = \int \left\{ \alpha^{(3)} t \right\} I\{\xi \leq u\} du - \int K_s^{(3)} ds + \beta^{(3)} I\{\xi \leq t\},$$

where $\beta^{(3)} > 0$ is the level of compensation, $\pi^{(3)} = (\pi_i^{(3)})_{i \geq 0}$ is the Poisson process with the intensity which admits the $\lambda^{(3)}$, decomposition is:

$$
\pi^{(3)}_t = \lambda^{(3)} t + m_t^{(3)},
$$

with the intensity $\lambda^{(3)} > 0$ and the martingale $m_t^{(3)}$.

The time $\tilde{\xi}$ is defined as:

$$\tilde{\xi} = E\{\tilde{\xi} | F_t^{Y^{(3)}}\}.$$

The process with the compensation of the change-point $Z^{(3)} = (Z_t^{(3)})_{t \geq 0}$ is the following:

$$Z_t^{(3)} = \alpha^{(3)} t \int_0^\infty I\{\tilde{\xi} \leq u\} du + Y_t^{(3)} I\{N \leq t\} - K_t^{(3)}$$

Similarly to the second model, the change-point time is unobservable and we estimate it with the least squares method:

$$\delta^{(3)} = E\{(\xi - \tilde{\xi})^2 | F_t^{Y^{(3)}}, \xi \leq t, t \geq N\} \rightarrow \min$$

Problem (18) is solved with the method of imitation.

5. The optimization problems for three models of the compensation of the change-point

In the model we suppose that the «smaller» is the value of the processes with the compensation and the change-point at the time $T$, the «better» is the system.

The problem can be solved with a bigger frequency and with a bigger level of the compensation. But there is a payment for the value frequency and the level. Thus, the problem of finding the compromise arises.
To find the optimal parameters of the compensation (of the intensity and of the level) for three models of this paper, the objective function is considered:

$$\Phi_{z} = \beta \cdot (1 + \lambda) + \gamma \cdot E Z \cdot I \left[ Z_{z} \geq 0 \right].$$

(19)

In this paper we solve the optimization problem:

$$\Phi_{z}(\lambda, \beta) \rightarrow \min_{\lambda, \beta}.$$  

(20)

![Figure 1. Values of the objective function $\Phi_{z}(\lambda, \beta)$](image1)

![Figure 2. Values of the objective function $\Phi_{z}(\lambda, \beta)$ with parameters $\alpha$ from 0 to 30 with step 0.5 and $\beta = 33$.](image2)

![Figure 3. Values of the objective function $\Phi_{z}(\lambda, \beta)$ with parameters $\beta$ from 0 to 100 with step 0.5 and $\alpha = 3.5$.](image3)
In figures 1, 2, 3, the change of the objective function (19) is given depending on the parameters $\alpha$, $\beta$ for the first model of this paper (the system of equations (1) - (4)). Similar values are assumed by the objective function (19) for systems of the second and third models.

According to the obtained results, we can see that the solution of the problem (20) for the first model is achieved with $\alpha = 3.5$ and $\beta = 33$.

For the second and third models we get $\alpha = 3.5$ with $\beta = 34$ and $33.5$, respectively.

6. Conclusion
The models considered in this paper are of a practical importance. The application is possible for systems with violations of stable operations, for example, destruction or wear of devices. The change-point is the time of a system destruction, and the compensation is the restoration to the normal state.

The methods described in this article for detecting the change-point, determining the stopping time of the system (the stopping time for the process) show the need to use several methods in the simulation of real objects (operation of airplanes, ships, spacecraft, nuclear power plants, computer networks), and when setting system parameters based on real data, choose the most optimal method under the given constraints.

The obtained results can also be applied to other problems of the compensation of the change-point (in the frameworks of a chosen type), for example, the cross-boundary problem (the boundary in this situation can serve as a critical level of a system operation). Earlier, in [16], the problem of finding of an estimation of the time of the compensation for a given distribution function of the time of the change-point was solved.

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