Entropic bounds and continual measurements

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Abstract
Some bounds on the entropic informational quantities related to a quantum continual measurement are obtained and the time dependencies of these quantities are studied.

1 Introduction
In the problem of information transmission through quantum systems, various entropic quantities appear which characterize the performances of the encoding and decoding apparatuses. Due to the peculiar character of a quantum measurement, many bounds on the informational quantities involved have been proved to hold [1–8]. In the case of measurements continual in time, these bounds acquire new aspects (family of measurements are now involved) and new problems arise. A typical question is about which of the various entropic measures of information is monotonically increasing or decreasing in time. We already started the study of this subject in Refs. [9, 10]; here we apply to the case of continual measurements the new techniques developed [6–8] for the time independent case.

1.1 Notations and preliminaries
We denote by $L(A;B)$ the space of bounded linear operators from $A$ to $B$, where $A, B$ are Banach spaces; moreover we set $L(A) := L(A;A)$.

Let $\mathcal{H}$ be a separable complex Hilbert space; a normal state on $L(\mathcal{H})$ is identified with a statistical operator, $T(\mathcal{H})$ and $S(\mathcal{H}) \subset T(\mathcal{H})$ are the trace-class and the space of the statistical operators on $\mathcal{H}$, respectively, and $\|\rho\|_1 := \text{Tr} \sqrt{\rho^*\rho}$, $\langle \rho, a \rangle := \text{Tr}_\mathcal{H} \{\rho a\}$, $\rho \in T(\mathcal{H})$, $a \in L(\mathcal{H})$. 

More generally, if \( a \) belongs to a \( W^* \)-algebra and \( \rho \) to its dual \( \mathcal{M}^* \) or predual \( \mathcal{M}_* \), the functional \( \rho \) applied to \( a \) is denoted by \( \langle \rho, a \rangle \).

### 1.1.1 A quantum/classical algebra

Let \((\Omega, \mathcal{F}, Q)\) be a measure space, where \( Q \) is a \( \sigma \)-finite measure. By Theorem 1.22.13 of [11], the \( W^* \)-algebra \( L^\infty(\Omega, \mathcal{F}, Q) \otimes \mathcal{L}(\mathcal{H}) \) (\( W^* \)-tensor product) is naturally isomorphic to the \( W^* \)-algebra \( L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})) \) of all the \( \mathcal{L}(\mathcal{H}) \)-valued \( Q \)-essentially bounded weakly* measurable functions on \( \Omega \). Moreover ([11], Proposition 1.22.12), the predual of this \( W^* \)-algebra is \( L^1(\Omega, \mathcal{F}, Q; \mathcal{T}(\mathcal{H})) \), the Banach space of all the \( \mathcal{T}(\mathcal{H}) \)-valued Bochner \( Q \)-integrable functions on \( \Omega \), and this predual is naturally isomorphic to \( L^1(\Omega, \mathcal{F}, Q) \otimes \mathcal{T}(\mathcal{H}) \) (tensor product with respect to the greatest cross norm — [11], pp. 45, 58, 59, 67, 68).

Let us note that a normal state \( \sigma \) on \( L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})) \) is a measurable function \( \omega \mapsto \sigma(\omega) \in \mathcal{T}(\mathcal{H}) \), \( \sigma(\omega) \geq 0 \), such that \( \text{Tr}_\mathcal{H}\{\sigma(\omega)\} \) is a probability density with respect to \( Q \).

### 1.2 Quantum channels and entropies

#### 1.2.1 Relative and mutual entropies

The general definition of the relative entropy \( S(\Sigma|\Pi) \) for two states \( \Sigma \) and \( \Pi \) is given in [12]; here we give only some particular cases of the general definition.

Let us consider two quantum states \( \sigma, \tau \in \mathcal{S}(\mathcal{H}) \) and two classical states \( q_k \) on \( L^\infty(\Omega, \mathcal{F}, Q) \) (two probability densities with respect to \( Q \)). The quantum relative entropy and the classical one are

\[
S_q(\sigma|\tau) = \text{Tr}_\mathcal{H}\{\sigma(\log \sigma - \log \tau)\}, \quad (1a)
\]

\[
S_c(q_1|q_2) = \int_\Omega Q(d\omega) q_1(\omega) \log \frac{q_1(\omega)}{q_2(\omega)}. \quad (1b)
\]

We shall need also the von Neumann entropy of a state \( \tau \in \mathcal{S}(\mathcal{H}) \): \( S_q(\tau) := -\text{Tr}\{\tau \log \tau\} \).

Let us consider now two normal states \( \sigma_k \) on \( L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})) \) and set \( q_k(\omega) := \text{Tr}\{\sigma_k(\omega)\} \), \( q_k(\omega) := \sigma_k(\omega)/q_k(\omega) \) (these definitions hold where the denominators do not vanish and are completed arbitrarily where the denominators vanish). Then, the relative entropy is

\[
S(\sigma_1|\sigma_2) = \int_\Omega Q(d\omega) \text{Tr}_\mathcal{H} \{\sigma_1(\omega)(\log \sigma_1(\omega) - \log \sigma_2(\omega))\} \quad (2a)
\]

\[
= S_c(q_1|q_2) + \int_\Omega Q(d\omega) q_1(\omega) S_q(\sigma_1(\omega)|\sigma_2(\omega)). \quad (2b)
\]

We are using a subscript “c” for classical entropies, a subscript “q” for purely quantum ones and no subscript for general entropies, eventually of a mixed character.

Classically a mutual entropy is the relative entropy of a joint probability with respect to the product of its marginals and this key notion can be generalized immediately to states on von Neumann algebras, every times we have a state on a tensor product of algebras [6-8].
1.2.2 Channels

**Definition 1.** ([12], p. 137) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $W^*$-algebras. A linear map $\Lambda^*$ from $\mathcal{M}_2$ to $\mathcal{M}_1$ is said to be a *channel* if it is completely positive, unital (i.e. identity preserving) and normal (or, equivalently, weakly* continuous).

Due to the equivalence [13] of w*-continuity and existence of a preadjoint $\Lambda$, a channel is equivalently defined by: $\Lambda$ is a completely positive linear map from the predual $\mathcal{M}_1^*$ to the predual $\mathcal{M}_2^*$, normalized in the sense that $\langle \Lambda[\rho], 1 \rangle_2 = \langle \rho, 1 \rangle_1$, $\forall \rho \in \mathcal{M}_1^*$. Let us note also that $\Lambda$ maps normal states on $\mathcal{M}_1$ into normal states on $\mathcal{M}_2$.

A key result which follows from the convexity properties of the relative entropy is *Uhlmann monotonicity theorem* ([12], Theor. 1.5 p. 21), which implies that channels decrease the relative entropy.

**Theorem 1.** If $\Sigma$ and $\Pi$ are two normal states on $\mathcal{M}_1$ and $\Lambda^*$ is a channel from $\mathcal{M}_2 \rightarrow \mathcal{M}_1$, then $S(\Sigma|\Pi) \geq S(\Lambda[\Sigma]|\Lambda[\Pi])$.

1.3 Continual measurements

Let us axiomatize the properties of a probability space where an independent-increment process lives and that ones of the $\sigma$-algebras generated by its increments. The probability measure $Q_1$ we are introducing will play the role of a reference measure.

**Assumption 1.** Let $(X, \mathcal{X}, Q_1)$ be a probability space with $(X, \mathcal{X})$ standard Borel. Moreover:

1. $\{\mathcal{X}^t_s, 0 \leq s \leq t\}$ is a two-times filtration of sub-$\sigma$-algebras: $\mathcal{X}^s_t \subset \mathcal{X}^r_T \subset \mathcal{X}$ for $0 \leq r \leq s \leq t \leq T$;
2. $\forall t \geq 0, \mathcal{X}^t_t$ is trivial;
3. $\mathcal{X}^t_s = \bigwedge_{T:T > t} \mathcal{X}^r_T$ for $0 \leq s \leq t$;
4. $\mathcal{X}^t_s = \bigvee_{r:s < r < t} \mathcal{X}^r_r$ for $0 \leq s < t$;
5. $\mathcal{X} = \bigvee_{t:T > 0} \mathcal{X}^0_t$;
6. for $0 \leq r \leq s \leq t \leq T$, $\mathcal{X}^r_s$ and $\mathcal{X}^t_t$ are $Q_1$-independent.

Continual measurements are a quantum analog of classical processes with independent increments [10, 14]. As any kind of quantum measurement, a continual measurement is represented by *instruments* [15–17], but, as shown in [7], instruments are equivalent to particular types of channels. Here we introduce continual measurements directly as a family of channels satisfying a set of axioms (cf. also [10, 18]).

**Assumption 2.** Let $\mathcal{H}$ be a separable complex Hilbert space. For all $s, t$, $0 \leq s \leq t$, we have a channel

$$\tilde{\Lambda}_t^s : L^1(X, \mathcal{X}^0_s, Q_1; T(\mathcal{H})) \rightarrow L^1(X, \mathcal{X}^0_t, Q_1; T(\mathcal{H}))$$

such that
1. $\tilde{\Lambda}^t_1 = 1$, $t \geq 0$;
2. $\tilde{\Lambda}^s_r \circ \tilde{\Lambda}^t_s = \tilde{\Lambda}^t_r$, $0 \leq r \leq s \leq t$;
3. $\forall \eta \in \mathcal{T}(\mathcal{H})$, $\tilde{\Lambda}^s_t[\eta]$ is $\mathcal{X}^t_\eta$-measurable, $0 \leq s \leq t$;
4. $\forall \eta \in \mathcal{T}(\mathcal{H})$, $\forall q \in L^1(X, \mathcal{X}^t_\eta, Q_1)$, $\tilde{\Lambda}^s_t[q\eta] = q\tilde{\Lambda}^s_t[\eta]$, $0 \leq s \leq t$, (i.e. $\tilde{\Lambda}^s_t[q\eta](x) = q(x)\tilde{\Lambda}^s_t[\eta](x)$ a.s.).

By points (3), (4) of Assumption 2 and (6) of Assumption 1 one gets: $\forall \sigma_s \in L^1(X, \mathcal{X}^0_s, Q_1; \mathcal{T}(\mathcal{H})), 0 \leq s \leq t$,
\begin{equation}
\mathbb{E}_{Q_1} [\tilde{\Lambda}^s_t[\sigma_s]|\mathcal{X}^t_\eta] = \tilde{\Lambda}^s_t [\mathbb{E}_{Q_1}[\sigma_s]].
\end{equation}

Here $\mathbb{E}_{Q_1}$ and $\mathbb{E}_{Q_1}[\bullet|\mathcal{X}^t_\eta]$ are the classical expectation and conditional expectation extended to operator-valued random variable.

Let us also define the evolution
\begin{equation}
U(t, s)[\tau] := \mathbb{E}_{Q_1} [\tilde{\Lambda}^s_t[\tau]], \quad \tau \in \mathcal{T}(\mathcal{H}), \quad 0 \leq s \leq t;
\end{equation}
$U(t, s)$ is a channel from $\mathcal{T}(\mathcal{H})$ into $\mathcal{T}(\mathcal{H})$. By points (2), (3), (4) of Assumption 2 for $0 \leq r \leq s \leq t$, $\sigma_s \in L^1(X, \mathcal{X}^0_s, Q_1; \mathcal{T}(\mathcal{H}))$, we get
\begin{equation}
U(t, s) \circ U(s, r) = U(t, r), \quad \mathbb{E}_{Q_1} [\tilde{\Lambda}^s_t[\sigma_s]|\mathcal{X}^t_\eta] = U(t, s)[\sigma_s].
\end{equation}

The quantum continual measurements is represented by the operators $\tilde{\Lambda}^t_s$, in the sense that they give probabilities and state changes. If $\eta_0 \in \mathcal{S}(\mathcal{H})$ is the initial state at time 0 and $B \in \mathcal{X}^0_\eta$ is any event involving the output in the interval $(0, t)$, then $\int_B \text{Tr}\{\tilde{\Lambda}^0_\eta[\eta_0](x)\}Q_1(\text{d}x)$ is the probability of the event $B$ and $\frac{\tilde{\Lambda}^0_\eta[\eta_0](x)}{\text{Tr}(\tilde{\Lambda}^0_\eta[\eta_0])}$ is the state at time $t$, conditional on the result $x$ (the \textit{a posteriori} state). Instead, $U(t, 0)[\eta_0]$ represents the state of the system at time $t$, when the results of the measurement are not taken into account (the \textit{a priori} state).

2 The initial state and the measurement

2.1 Ensembles

In quantum information theory, not only single states are used, but also families of quantum states with a probability law on them, called ensembles. An ensemble $\{\mu, \rho\}$ is a probability measure $\mu(\text{d}y)$ on some measurable space $(Y, \mathcal{Y})$ together with a random variable $\rho : Y \to \mathcal{S}(\mathcal{H})$. Alternatively, an ensemble can be seen as a quantum/classical state of the type described in Section 1.1.1. Given an ensemble, one can introduce an average state $\overline{\rho} \in \mathcal{S}(\mathcal{H})$
\begin{equation}
\overline{\rho} := \mathbb{E}_\mu[\rho] = \int_Y \mu(\text{d}y)\rho(y);
\end{equation}
the integrals involving trace class operators are always understood as Bochner integrals. Finally, the average relative entropy of the states $\rho(y)$ with respect to $\overline{\rho}$ is called the “$\chi$-quantity” of the ensemble:
\begin{equation}
\chi(\mu, \rho) := \int_Y \mu(\text{d}y)S_\eta(\rho(y)|\overline{\rho}) = \mathbb{E}_\mu [S_\eta(\rho|\overline{\rho})].
\end{equation}
This new quantity plays an important role in the whole quantum information theory [3, 20] and can be thought as a measure of some kind of quantum information stored in the ensemble.

2.2 The letter states

Let us consider the typical setup of quantum communication theory. A message is transmitted by encoding the letters in some quantum states, which are possibly corrupted by a quantum noisy channel; at the end of the channel the receiver attempts to decode the message by performing measurements on the quantum system. So, one has an alphabet \( A \) and the letters \( \alpha \in A \) are transmitted with some a priori probabilities \( P_i \). Each letter \( \alpha \) is encoded in a quantum state and we denote by \( \rho_i(\alpha) \) the state associated to the letter \( \alpha \) as it arrives to the receiver, after the passage through the transmission channel. While it is usual to consider a finite alphabet, also general continuous parameter spaces are acquiring importance [19, 20].

Assumption 3. Let \((A, A, Q_0)\) be a probability space with \((A, A)\) standard Borel and let \( \sigma_i \) be a normal state on \( L^\infty(A, A, Q_0; L(H)) \).

Let us set

\[
q_i(\alpha) := \text{Tr} \{ \sigma_i(\alpha) \}, \quad \rho_i(\alpha) := \frac{\sigma_i(\alpha)}{q_i(\alpha)}, \quad P_i(\alpha) := q_i(\alpha) Q_0(\alpha); \quad (8)
\]

\( q_i \) is a probability density and \( \{ P_i, \rho_i \} \) is the initial ensemble. The average state and the \( \chi \)-quantity of the initial ensemble are

\[
\eta_0 := \mathbb{E}_{Q_0}[\sigma_i] = \int_A P_i(\alpha) \rho_i(\alpha), \quad (9)
\]

\[
\chi\{P_i, \rho_i\} := \int_A P_i(\alpha) S_q(\rho_i(\alpha)|\eta_0). \quad (10)
\]

The quantity \( \chi\{P_i, \rho_i\} \) is known also as Holevo capacity [3, 20].

2.3 Probabilities and states derived from \( \eta_0 \)

For \( 0 \leq r \leq s \leq t \) we define:

\[
\eta_t := U(t, 0)[\eta_0], \quad \tilde{\sigma}_t := \tilde{\Lambda}_t[\eta_t], \quad \tilde{q}_t := \| \tilde{\sigma}_t \|_1, \quad \tilde{q}_t := \frac{\tilde{\sigma}_t}{\tilde{q}_t}. \quad (11)
\]

Then, \( \eta_t \) and \( \tilde{q}_t(x) \) are states on \( L(H) \), \( \tilde{q}_t \) is a state on \( L^\infty(X, \mathcal{X}_t^0, Q_1) \) and \( \tilde{\sigma}_t \) a state on \( L^\infty(X, \mathcal{X}_t^0, Q_1; L(H)) \). We have also

\[
\mathbb{E}_{Q_1}[\tilde{q}_t^0|X_t^0] = \tilde{q}_t, \quad \mathbb{E}_{Q_1}[\tilde{q}_t^*|X_t^*] = \tilde{q}_t^*. \quad (12)
\]

Moreover, there exists a unique probability \( P_1 \) on \( (X, \mathcal{X}) \) such that \( P_1(dx)|_{X_t^0} = \tilde{q}_t^0(x) Q_1(dx) \) for all \( t \geq 0 \). Also \( P_1(dx)|_{X_t^*} = \tilde{q}_t^*(x) Q_1(dx) \) holds.
2.4 The general setup

It is useful to unify the initial distribution and the distribution of the measurement results in a unique filtered probability space. Let us set:

\[ \Omega := A \times X, \quad \omega := (\alpha, x), \quad \pi_0(\omega) := \alpha, \quad \pi_1(\omega) := x, \quad (13a) \]

\[ \sigma_0 := \sigma_1 \circ \pi_0, \quad q_0 := q_1 \circ \pi_0 = \|\sigma_0\|_1, \quad \rho_0 := \rho_1 \circ \pi_0 = \frac{\sigma_0}{\|\sigma_0\|_1}, \quad (13b) \]

\[ F := A \otimes X, \quad Q := Q_0 \otimes Q_1, \quad (13c) \]

\[ F_0 := \{ B \times X : B \in A \}, \quad F^t_0 := \{ A \times Y : Y \in X^t \}, \quad (13d) \]

\[ F_t := F_0 \vee F^t_0 = \sigma\{ B \times Y : B \in A, Y \in X^t \}. \quad (13e) \]

By defining \( \Lambda_t^\sigma := 1 \otimes \Lambda_t^\sigma \), we extend \( \Lambda_t^\sigma \) to \( L^1(\Omega, \mathcal{F}, Q; T(\mathcal{H})) \simeq L^1(A, \mathcal{A}, Q_0) \otimes L^1(X, \mathcal{A}^0, Q_1; T(\mathcal{H})) \). Similarly, we extend \( U(t, s) \) to \( L^1(\Omega, \mathcal{F}, Q) \otimes T(\mathcal{H}) \). Let us also set:

\[ \sigma_t := \Lambda_t^\sigma[\sigma_0], \quad \sigma_t^\sigma := \tilde{\sigma}_t \circ \pi_1 = \Lambda_t^\sigma[\eta_t], \quad q_t := \|\sigma_t\|_1, \quad (14a) \]

\[ q^t := \tilde{q}_t \circ \pi_1 = \|\sigma_t^\sigma\|_1, \quad \rho_t := \frac{\sigma_t}{\|\sigma_t\|_1}, \quad \sigma_t^\sigma := \tilde{\sigma}_t \circ \pi_1 = \frac{\sigma_t^\sigma}{\|\sigma_t^\sigma\|_1}. \quad (14b) \]

In the computations of the following sections we shall need various properties of the quantities we have just introduced; here we summarize such properties. Let \( r, s, t \) be three ordered times: \( 0 \leq r \leq s \leq t \). Then, \( \sigma_t \) and \( \sigma_t^\sigma \) are states on \( L^\infty(\Omega, \mathcal{F}, Q; L(\mathcal{H})) \) and

\[ \mathbb{E}_Q[q_t|F_s] = q_s, \quad \mathbb{E}_Q[q_t|F^t_0] = \mathbb{E}_Q[q_t^t|F^t_0] = q_t^t, \quad (15a) \]

\[ \mathbb{E}_Q[q_t|F_s] = q_t, \quad \mathbb{E}_Q[q_t|F^t_0] = \mathbb{E}_Q[q_t^t|F^t_0] = q_t^t, \quad (15b) \]

\[ \mathbb{E}_Q[\sigma_t|F_s] = U(t, s)[\sigma_s], \quad \mathbb{E}_Q[\sigma_t^\sigma|F_s] = U(t, s)[\sigma_s^\sigma], \quad (15c) \]

\[ \mathbb{E}_Q[\sigma_t^\sigma|F_s] = \eta_t, \quad \eta_t = \mathbb{E}_Q[\sigma_t], \quad \sigma_t = \Lambda_t^\sigma[\sigma_s], \quad (15d) \]

\[ \sigma_t^\sigma = \Lambda_t^\sigma[\sigma_s^\sigma], \quad \frac{\Lambda_t^\sigma[\rho_s]}{\|\Lambda_t^\sigma[\rho_s]\|_1} = \rho_t, \quad \frac{\Lambda_t^\sigma[\sigma_s^\sigma]}{\|\Lambda_t^\sigma[\sigma_s^\sigma]\|_1} = \sigma_t^\sigma. \quad (15e) \]

We have that \( \{ q_t, t \geq 0 \} \) is a non-negative, mean one, \( Q \)-martingale. Then, there exists a unique probability \( P \) on \( (\Omega, \mathcal{F}) \) such that \( \forall t \geq 0 \)

\[ P(d\omega)|_{F_t} = q_t(\omega)Q(d\omega). \quad (16) \]

Moreover,

\[ P(d\alpha \times X) = P_1(d\alpha), \quad P(A \times dx) = P_1(dx), \quad (17) \]

\[ P(d\omega)|_{F_t^t} = q_t^t(\omega)Q(d\omega), \quad \eta_t = \mathbb{E}_P[\rho_t] = U(t, s)[\eta_s]. \quad (18) \]

3 Mutual entropies and informational bounds

Here and in the following we shall have always \( 0 \leq u \leq r \leq s \leq t \).
3.1 The state $q_t$ and the classical information

Let us consider the state $q_t$ and its marginals $E_Q[q_t|F_r] = q_r$, $E_Q[q_t|F_s^r] = q_r^t$. Then, we can introduce the classical mutual entropy:

$$S_c(q_t|q_rq_r^t) = \int_\Omega P(d\omega) \log \frac{q_t(\omega)}{q_r(\omega)q_r^t(\omega)} =: I_c(r, t).$$ (19a)

Note that $I_c(t, t) = 0$. For $r = 0$ we have the input/output classical information gain:

$$I_c(0, t) = S_c(q_t|q_t^0) = \int_{A \times X} P(d\alpha \times dx) \log \frac{q_t(\alpha, x)}{q_t^0(\alpha)|q_t(\alpha)|}.$$ (19b)

By applying the monotonicity theorem and the channel $E_Q[\bullet|F_s]$ to the couple of states $q_t$ and $q_rq_r^t$, we get

$$S_c(q_t|q_rq_r^t) \geq S_c\left(E_Q[q_t|F_r]|E_Q[q_rq_r^t|F_s]\right) = S_c(q_s|q_rq_r^t),$$ (20)

which becomes

$$I_c(r, t) \geq I_c(r, s).$$ (21)

The function $t \mapsto I_c(s, t)$ is non decreasing.

3.2 The state $\sigma_s$ and the main bound

A useful quantity, with the meaning of a measure of the “quantum information” left in the a posteriori states, is the mean $\chi$-quantity

$$\overline{\chi}(s, t) := \int_\Omega P(d\omega) S_q(\rho_t(\omega)|q_t^s(\omega)) = E_P\left[S_q(\rho_t|q_t^s)\right].$$ (22)

The interpretation as a mean $\chi$-quantity is due to the fact that $\overline{\chi}(s, t) = E_P\left[S_q(\rho_t|q_t^s)|F_r^s\right]$. But by Eq. (19) and $E_P[\rho_t|F_s] = \chi_t^s$, $E_P[\chi_t|F_s]$ is a random $\chi$-quantity. Note that

$$\overline{\chi}(t, t) = \int_\Omega P(d\omega) S_q(\rho_t(\omega)|\eta_t) =: \chi\{P, \rho_t\}.$$ (23)

Let us consider the state $\sigma_s$ and its marginals $E_Q[\text{Tr}\{\sigma_s\}|F_r] = q_r$, $E_Q[\sigma_s|F_s^r] = \sigma_s^r$. Then, we have the mutual entropy

$$S(\sigma_s|q_r\sigma_s^r) = I_c(r, s) + \overline{\chi}(r, s).$$ (24)

For $r = s$ and for $r = 0$ this equation reduces to

$$S(\sigma_s|q_s\eta_s) = \chi\{P, \rho_s\}, \quad S(\sigma_0|q_0\eta_0) = \chi\{P, \rho_0\} = \chi\{P, \rho_t\}.$$ (25)

By applying the monotonicity theorem and the channel $\Lambda_t^s$ to the couple of states $\sigma_s$ and $q_r\sigma_s^r$, we get

$$S(\sigma_s|q_r\sigma_s^r) \geq S(\Lambda_t^s[\sigma_s]|\Lambda_t^s[q_r\sigma_s^r]) = S(\sigma_t|q_r\sigma_t^s),$$ (26)

which becomes

$$\overline{\chi}(r, s) - \overline{\chi}(r, t) \geq I_c(r, t) - I_c(r, s) \geq 0.$$ (27)
Therefore, the function \( t \mapsto \chi(s, t) \) is non increasing.

For \( r = s \) we get
\[
S(\sigma_s | q_s \eta_s) \geq S(\sigma_t | q_s \sigma_t^*),
\]
which gives the upper bound for \( I_c \):
\[
0 \leq I_c(s, t) \leq \chi\{P, \rho_s\} - \chi(s, t).
\] (29)

For \( s = r = 0 \), it reduces to
\[
0 \leq I_c(0, t) \leq \chi\{P_i, \rho_i\} - \int_{A \times X} P(d\alpha \times dx) S_q(\rho_t(\alpha, x) | \tilde{\rho}_t^0(x)).
\] (30)

The bound (30) is the translation in terms of continual measurements of the bound of Section 3.3.4 of [7], which in turn is a generalization of a bound by Schumacher, Westmoreland and Wootters [5]. Equation (30) is a strengthening of the Holevo bound [3] \( I_c(0, t) \leq \chi\{P_i, \rho_i\} \).

3.3 Quantum information gain

Let us consider now the quantum information gain defined by the quantum entropy of the pre-measurement state minus the mean entropy of the a posteriori states [1,2,4]. It is a measure of the gain in purity (or loss, if negative) in passing from the pre-measurement state to the post-measurement a posteriori states.

In the continual case, we can consider the quantum information gain in the time interval \((s, t)\) when the system is prepared in the ensemble \(\{P_i, \rho_i\}\) at time 0 or when it is prepared in the state \(\eta_r\) at time \(r\):
\[
I_q(s, t) := \int_{\Omega} P(d\omega) \left[ S_q(\rho_s(\omega)) - S_q(\rho_t(\omega)) \right], \quad (31a)
\]
\[
I_q(r; s, t) := \int_{\Omega} P(d\omega) \left[ S_q(\tilde{\rho}_s^r(\omega)) - S_q(\tilde{\rho}_t^r(\omega)) \right]. \quad (31b)
\]

By this definition we have immediately
\[
I_q(r, t) = I_q(r, s) + I_q(s, t), \quad I_q(u; r, t) = I_q(u; r, s) + I_q(u; s, t). \quad (32)
\]

It has been proved [4] that the quantum information gain is positive for all initial states if and only if the measurement sends pure initial states into pure a posteriori states.

As in the single time case [6–8], inequality (27) can be easily transformed into an inequality involving \(I_q\):
\[
I_q(r; s, t) - I_q(s, t) \geq I_c(r, t) - I_c(r, s) \geq 0. \quad (33)
\]

Let us take an initial ensemble made up of pure states: \(\rho_i(\alpha)^2 = \rho_i(\alpha), \forall \alpha \in A\). Let us assume that the continual measurement preserve pure states: the states \(\rho_i(\alpha, x)\) are pure for all choices of \(t, \alpha, x\). Then, the von Neumann entropy of \(\rho_t(\omega)\) vanishes and we have \(I_q(s, t) = 0\) for all choices of \(s\) and \(t\).

From the second of Eqs. (32) and Eq. (33) we get
\[
I_q(u; r, t) - I_q(u; r, s) = I_q(u; s, t) \geq I_c(u, t) - I_c(u, s) \geq 0, \quad (34)
\]

8
i.e. the function $t \mapsto I_q(u; r; t)$ is non decreasing for “pure” continual measurements.

In particular, by taking $u = r = 0$ we have

$$I_q(0; 0, t) = S_q(\eta_0) - \int_X P_1(dx) S_q(\tilde{g}^0(x)). \quad (35)$$

For a continual measurement sending every pure initial state into pure a posteriori states, $\forall \eta_0 \in \mathcal{S}(\mathcal{H})$ the quantum information gain $I_q(0; 0, t)$ is non negative, non decreasing in time and with $I_q(0; 0, 0) = 0$.

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