Birman-Wenzl-Murakami Algebra and the Topological Basis

Zhou Chengcheng,1∗ Xue Kang,1† Wang Gangcheng,1 Sun Chunfang,1 and Du Guijiao1

1School of Physics, Northeast Normal University, Changchun 130024, People’s Republic of China

In this paper, we use entangled states to construct $9 \times 9$-matrix representations of Temperley-Lieb algebra (TLA), then a family of $9 \times 9$-matrix representations of Birman-Wenzl-Murakami algebra (BWMA) have been presented. Based on which, three topological basis states have been found. And we apply topological basis states to recast nine-dimensional BWMA into its three-dimensional counterpart. Finally, we find the topological basis states are spin singlet states in special case.

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∗Zhoucc237@nenu.edu.cn
†Xuekang@nenu.edu.cn
I. INTRODUCTION

Quantum entanglement (QE) is the most surprising nonclassical property of quantum systems which plays a key role in quantum information and quantum computation processing\cite{1–4}. Because of these applications, QE has become one of the most fascinating topics in quantum information and quantum computation. To the best of our knowledge, the Yang-Baxter equation (YBE) plays an important role in quantum integrable problem, which was originated in solving the one-dimensional δ-interacting models\cite{5} and the statistical models\cite{6}. Braid group representations (BGRs) can be obtained from YBE by giving a particular spectral parameter. BGRs of two and three eigenvalues have direct relationship with Temperley-Lieb algebra (TLA)\cite{7} and Birman-Wenzl-Murakami algebra (BWMA)\cite{8} respectively. TLA and BWMA have been widely used to construct the solutions of YBE\cite{9–12}.

The TLA first appeared in statistical mechanics as a tool to analyze various interrelated lattice models\cite{7} and was related to link and knot invariants\cite{13}. In the subsequent developments TLA is related to knot theory, topological quantum field theory, statistical physics, quantum teleportation, entangle swapping and universal quantum computation\cite{14, 15}. On the other hand, the BWMA\cite{8} including braid algebra and TLA was first defined and independently studied by Birman, Wenzl and Murakami. It was designed partially help to understand Kauffman’s polynomial in knot theory. Recently, Ref.\cite{16} applied topological basis states for spin-1/2 system to recast 4-dimensional YBE into its 2-dimensional counterpart. As we know, few studies have reported topological basis states for spin-1 system. The motivation for our works is to find topological basis states for spin-1 system and study the topological basis states.

The purpose of this paper is twofold: one is that we construct a family of $9 \times 9$-matrix representations of BWMA; the other concerns topological basis states for spin-1 system. This paper is organized as follows. In Sec. 2, we use entangled states to construct the $9 \times 9$ matrix representations of TLA, then we present a family of $9 \times 9$-matrix representations of BWMA, and study the entangled states. In Sec. 3, we obtain three topological basis states of BWMA, and we recast nine-dimensional BWMA into its three-dimensional counterpart. We end with a summary.
II. 9 × 9-MATRIX REPRESENTATIONS OF BWMA

The 4 × 4 Hermitian matrix $E$, which satisfies TLA and can construct the well-known six-vertex model \[17\], takes the representation

$$
E = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & q & \eta & 0 \\
0 & \eta^{-1} & q^{-1} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

(1)

where $\eta = e^{i\varphi}$ with $\varphi$ being any flux. We can rewrite $E$ as

$$
\begin{cases}
E = d |\Psi\rangle\langle\Psi|, \\
|\Psi\rangle = d^{-1/2}(q^{1/2}|\uparrow\downarrow\rangle + q^{-1/2}e^{-i\varphi}|\downarrow\uparrow\rangle),
\end{cases}
$$

(2)

where $d = q + q^{-1}$.

So like this symmetrical method, we found the 9 × 9 Hermitian matrices $E$’s, which satisfies TLA, take the representations as follows

$$
\begin{cases}
E = d |\Psi\rangle\langle\Psi|, \\
|\Psi\rangle = d^{-1/2}(q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}e^{i\phi_{\mu\lambda}}|\mu\lambda\rangle),
\end{cases}
$$

(3)

where $d = q + 1 + q^{-1}$, $\lambda \neq \mu \neq \nu \in (1, 0, -1)$ and $(d, q, \phi_\nu, \phi_{\mu\lambda}) \in \text{real}$. Recently, a 9 × 9–matrix representation of BWMA has been presented \[18, 19\]. We notice that $E$ is the same as Gou et al. \[18, 19\] presented, when $\phi_\nu = \varphi_2 - \varphi_1 + \pi$, $\phi_{\mu\lambda} = -2\varphi_1$, $\lambda = 1$, $\mu = -1$ and $\nu = 0$.

As we know the BWMA relations \[8, 9, 20, 21\] including braid relations and TLA relations satisfy the following
relations,

\[
\begin{align*}
S_i - S_i^{-1} &= \omega(I - E_i), \\
S_i S_{i\pm 1} S_i &= S_i S_{i\pm 1} S_i, \quad S_i S_j = S_j S_i, |i - j| \geq 2, \\
E_i E_{i\pm 1} E_i &= E_i, \quad E_i E_j = E_j E_i, \quad |i - j| \geq 2, \\
E_i S_i &= S_i E_i = \sigma E_i, \\
S_{i\pm 1} E_i S_{i\pm 1} &= E_i S_{i\pm 1} S_i = E_i E_{i\pm 1}, \\
S_{i\pm 1} E_i S_{i\pm 1} &= S_{i\pm 1}^{-1} E_i S_{i\pm 1}^{-1}, \\
E_{i\pm 1} E_i S_{i\pm 1} &= E_{i\pm 1} S_{i\pm 1}^{-1}, \quad S_{i\pm 1} E_i E_{i\pm 1} = S_{i\pm 1}^{-1} E_{i\pm 1}, \\
E_i S_{i\pm 1} E_i &= \sigma^{-1} E_i, \\
E_i^2 &= \left(1 - \frac{\sigma - \sigma^{-1}}{\omega}\right) E_i,
\end{align*}
\]

where \(S_i, S_{i\pm 1}\) satisfy the braid relations, \(E_i, E_{i\pm 1}\) satisfy the TLA relations [7]

\[
\begin{align*}
E_i E_{i\pm 1} E_i &= E_i, \quad E_i E_j = E_j E_i, \quad |i - j| \geq 2, \\
E_i^2 &= dE_i,
\end{align*}
\]

where \(0 \neq d \in \mathbb{C}\) is a topological parameter in the knot theory which does not depend on the sites of lattices. We denote \(\sigma = q^{-2}\) and \(\omega = q - q^{-1}\) throughout the text. The notations \(E_i \equiv E_{i,i+1}\) and \(S_i \equiv S_{i,i+1}\) are used, \(E_{i,i+1}\) and \(S_{i,i+1}\) are abbreviation of \(I_1 \otimes \ldots \otimes I_{i-1} \otimes E_{i,i+1} \otimes I_{i+2} \otimes \ldots \otimes I_N\) and \(I_1 \otimes \ldots \otimes I_{i-1} \otimes S_{i,i+1} \otimes I_{i+2} \otimes \ldots \otimes I_N\) respectively, and \(I_j\) represents the unit matrix of the \(j\)-th particle.

Following the matrix representation of TLA we obtain a family of 9 \times 9-matrix representations of BWMA as follows

\[
\begin{align*}
E &= d|\Psi\rangle\langle\Psi|, \\
|\Psi\rangle &= d^{-1/2}(|\lambda\mu\rangle|\lambda\mu\rangle + e^{i\phi_{\nu\mu}}|\nu\nu\rangle + q^{-1/2}e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle),
\end{align*}
\]

\[
S = q(|\lambda\lambda\rangle\langle\lambda\lambda| + |\mu\mu\rangle\langle\mu\mu|) + |\nu\nu\rangle\langle\nu\nu| \\
+ (q - q^{-1})(|\nu\lambda\rangle\langle\nu\lambda| + |\mu\nu\rangle\langle\mu\nu|) + (q - 1)^2(q + 1)q^{-2}|\mu\lambda\rangle\langle\mu\lambda| \\
+ e^{-i\varphi_{\mu\lambda}/2}(|\lambda\nu\rangle\langle\nu\nu| + |\nu\mu\rangle\langle\mu\nu|) + e^{i\varphi_{\nu\mu}/2}(|\nu\lambda\rangle\langle\nu\lambda| + |\mu\nu\rangle\langle\mu\nu|) \\
+ q^{-1}e^{-i\varphi_{\lambda\mu}}|\lambda\mu\rangle\langle\lambda\mu| + q^{-1}e^{i\varphi_{\lambda\mu}}|\mu\lambda\rangle\langle\mu\lambda| \\
- q^{-3/2}(q - 1)(e^{i\phi_{\nu\mu}}|\nu\nu\rangle\langle\mu\lambda| + e^{-i\phi_{\nu\mu}}|\mu\lambda\rangle\langle\nu\nu|),
\]
\[ S^{-1} = q^{-1}(\langle \lambda \lambda | \langle \lambda \lambda | + | \mu \mu \rangle \langle \mu \mu |) + | \nu \nu \rangle \langle \nu \nu | \]
\[ + (q^{-1} - q)(\langle \nu \nu | \langle \lambda \lambda | + | \mu \mu \rangle \langle \nu \mu |) + (q - 1)^2(q + 1)q^{-1}| \lambda \mu \rangle \langle \lambda \mu | + e^{-i\phi_{\mu \lambda}}| \lambda \mu \rangle \langle \lambda \mu | \]
\[ + e^{i\phi_{\nu \lambda}}| \mu \lambda \rangle \langle \mu \lambda | + e^{i\phi_{\nu \mu}}| \nu \mu \rangle \langle \nu \mu |) + q^{-1/2}(q^2 - 1)(e^{-i\phi_{\lambda \nu}}| \nu \nu \rangle \langle \lambda \mu | + e^{i\phi_{\nu \mu}}| \mu \nu \rangle \langle \nu \mu | \]
\[ + \) \]
\[ (8) \]

where \( d = q + 1 + q^{-1}, \lambda \neq \mu \neq \nu \in (1, 0, -1) \) and \( (d, q, \phi_{\nu \lambda}, \phi_{\mu \lambda}) \in \text{real}. \)

It is worth noticing that the states \( | \Psi \rangle \)‘s are entangled states. By means of negativity, we study these entangled states. The negativity for two qutrits is given by,

\[ N(\rho) \equiv \frac{\| \rho^{T_A} \| - 1}{2}, \]

(9)

where \( \| \rho^{T_A} \| \) denotes the trace norm of \( \rho^{T_A} \), which denotes the partial transpose of the bipartite state \( \rho \). In fact, \( N(\rho) \) corresponds to the absolute value of the sum of negative eigenvalues of \( \rho^{T_A} \), and negativity vanishes for unentangled states. By calculation, we can obtain the negativity of states \( | \Psi \rangle \)’s as

\[ N(q) = \frac{q^{1/2} + 1 + q^{-1/2}}{d}, \]

(10)

where \( d = q + 1 + q^{-1} \). The Fig corresponds to the negativity \( N(q) \). One demonstrates that the states \( | \Psi \rangle \)’s become maximally entangled states of two qutrits as \( | \Psi \rangle = (| \lambda \mu \rangle + e^{i\phi_{\nu \lambda}}| \nu \nu \rangle + e^{i\phi_{\mu \lambda}}| \mu \lambda \rangle) / \sqrt{3} \) when \( q = 1 \).

### III. Topological Basis States

In the topological quantum computation theory, the two-dimensional (2D) braid behavior under the exchange of anyons has been investigated based on the \( \nu = 5/2 \) fractional quantum Hall effect (FQHE). The orthogonal topological basis states read

\[ | e_1 \rangle = \frac{1}{d} \bigcup \bigcup \]
\[ | e_2 \rangle = \frac{1}{\sqrt{d^2 - 1}} \bigcup \bigcup - \frac{1}{d} \bigcup \bigcup \]

(11)

where the parameter \( d \) represents the values of a unknotted loop. In Eq. there are two topological graphics \( \bigcup \bigcup \) and \( \bigcup \bigcup \). For four lattices, we can easy find four graphics \( \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \bigcup \). If we use Skein relations \( \bigtimes = q^{1/2} \bigcup + q^{-1/2} \bigcup (S = q^{1/2}I + q^{-1/2}E) \) and \( \bigtimes = q^{-1/2} \bigcup + q^{1/2} \bigcup (S^{-1} = q^{-1/2}I + q^{1/2}E) \),
where the unknotted loop \( d = \bigcirc = -q - q^{-1} \), the third and the fourth graphics recast to \( \bigcup \bigcup \) and \( \bigcup \bigcup \) (\( \bigcup \bigcup = q^{1/2} \bigcup + q^{-1/2} \bigcup \bigcup; \bigcup \bigcup = q^{-1/2} \bigcup + q^{1/2} \bigcup \bigcup \)). So the topological basis states[11] are self-consistent. But in this paper, we focus on BWMA, the braid group representations \((S)\) is independent of TLA representations \((E)\), and in BWMA \( S - S^{-1} = \omega(I - E) \). So we know the graphics \( \bigcup \bigcup \) and \( \bigcup \bigcup \) have one independent graphic. We choose three independent graphics as \( \bigcup \bigcup \), \( \bigcup \bigcup \) and \( \bigcup \bigcup \).

We define

\[
\begin{align*}
\bigcup = d^{1/2} |\Phi_{ij}\rangle = q^{1/2} |\lambda\mu\rangle + e^{i\phi}\nu\nu\rangle + q^{-1/2} |\mu\lambda\rangle, \\
\bigcap = d^{1/2} \langle \Phi_{ij} | = q^{1/2} \langle \lambda\mu | + e^{-i\phi}\nu\nu\langle | + q^{-1/2} \langle \mu\lambda |, \\
[ \bigcup \bigcup ]^\dagger = \bigcap \bigcap, \\
[ \bigcup \bigcup ]^\dagger = \bigcap \bigcap, \\
[ \bigcup \bigcup ]^\dagger = \bigcap \bigcap.
\end{align*}
\]

(12)

So \( E \) recasts to \( E_{ij} = \bigcup \bigcup \). Following the BWMA, we define the graphic rules

\[
\begin{align*}
\otimes = \sigma \bigcup \bigcirc = \sigma^{-1} \bigcup \\
\times - \times = \omega(\bigcup \bigcup), \\
\bigcirc = d \text{ (the unknotted loop).}
\end{align*}
\]

(13)
The orthogonal basis states read
\[
\begin{align*}
|e_1\rangle &= \frac{q}{(1 + q^2)\sqrt{d^2 - d - 1}} (1 + q) - \frac{q(q + 1)}{d} |\rangle,
|e_2\rangle &= \frac{1}{d} |\rangle,
|e_3\rangle &= \frac{q}{(1 + q^2)\sqrt{d}} (1 + q^{-1}) - \frac{q^2 - q^{-1}}{d} |\rangle.
\end{align*}
\] (14)

Let’s introduce the reduced operators $E_A, E_B, A$ and $B$
\[
\begin{align*}
(E_A)_{ij} &= \langle e_i | E_{12} | e_j \rangle, \\
(E_B)_{ij} &= \langle e_i | E_{23} | e_j \rangle, \\
A_{ij} &= \langle e_i | S_{12} | e_j \rangle, \\
B_{ij} &= \langle e_i | S_{23} | e_j \rangle.
\end{align*}
\] (15)

Due to the limited length, we only show how $S_{23}$ acts on $|e_3\rangle$ in detail as follows
\[
S_{23}|e_3\rangle = \frac{q}{(1 + q^2)\sqrt{d}} (1 + q^{-1}) - \frac{q^2 - q^{-1}}{d} |\rangle
= \frac{q}{(1 + q^2)\sqrt{d}} (\omega + q^{-1}) - \frac{q^2 - q^{-1}}{d} |\rangle
= \frac{q}{(1 + q^2)\sqrt{d}} (\omega - q^2 - q^{-1}) - \frac{q^2 - q^{-1}}{d} |\rangle
= -\frac{\sqrt{d^2 - d - 1}}{q^2\sqrt{d^2 - d - 1}} |e_1\rangle + \frac{q}{\sqrt{d}} |e_2\rangle + \frac{d - 2}{d - 1} |e_3\rangle.
\] (16)

Thus their matrix representations in the basis states $(|e_1\rangle, |e_2\rangle, |e_3\rangle)$ are given by
\[
E_A = \text{diag}\{0, d, 0\},
\] (17)
\[
E_B = \begin{pmatrix}
\frac{d^2 - d - 1}{d} & \frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} & -\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} \\
\frac{\sqrt{d^2 - d - 1}}{d} & \frac{1}{\sqrt{d}} & -\frac{1}{\sqrt{d}} \\
-\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} & -\frac{1}{\sqrt{d}} & 1
\end{pmatrix},
\] (18)
\[
A = \text{diag}\{q, q^{-2}, -q^{-1}\}.
\] (19)
where $E_A$, $E_B$, $A$ and $B$ are Hermitian matrices. It is worth noting that $E_B = U E_A U^{-1}$, $B = U A U^{-1}$,

$$ U = \begin{pmatrix} \frac{1}{\sqrt{d^2 - d - 1}} & \frac{\sqrt{d^2 - d - 1}}{d} & -\frac{\sqrt{d^2 - d - 1}}{d} \\ \frac{(d-1)d}{d} & \frac{d}{d} & \frac{1}{d} \\ \frac{\sqrt{d^2 - d - 1}}{d} & \frac{1}{d} & -\frac{d-2}{d-1} \end{pmatrix}, $$

(21)

and they satisfy the reduced BWMA relations

$$ \begin{align*}
A - A^{-1} &= \omega(I - E_A), \quad B - B^{-1} = \omega(I - E_B), \\
ABA &= BAB, \\
E_A E_B E_A &= E_A, \quad E_B E_A E_B = E_B, \\
E_A A &= \sigma E_A, \quad E_B B = BE_B = \sigma E_B, \\
ABE_A &= E_B AB = E_B E_A, \quad BAE_B = E_A BA = E_A E_B, \\
AE_B A &= B^{-1} E_A B^{-1}, \quad BE_A B = A^{-1} E_B A^{-1}, \\
E_A E_B A &= E_A B^{-1}, \quad E_B E_A B = E_B A^{-1}, \\
AE_B E_A &= B^{-1} E_A, \quad BE_A E_B = A^{-1} E_B, \\
E_A B E_A &= \sigma^{-1} E_A, \quad E_B A E_B = \sigma^{-1} E_B, \\
E_A^2 &= (1 - \sigma^{-1}/\omega) E_A, \quad E_B^2 = (1 - \sigma^{-1}/\omega) E_B.
\end{align*} $$

(22)

We emphasize that (22) acts on the basis $(|e_1\rangle, |e_2\rangle, |e_3\rangle)$.

It is worth noting that the topological basis states are singlet states, when $\phi_\nu = \pi$, $\lambda = 1$, $\mu = -1$, $\nu = 0$ and $q = 1$. In other words, $S^z|e_i\rangle = 0$ and $S_z|e_i\rangle = 0$, where $S = \sum_{j=1}^3 S_j$, $S_j$ are the operators of spin-1 angular momentum for the $j$-th particle, $i = 1, 2, 3$.

**IV. SUMMARY**

In this paper we construct $9 \times 9$-matrix representations of TLA, where we used the entangled states $|\Psi\rangle = d^{-1/2}(q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}e^{i\phi_\lambda}|\mu\lambda\rangle)$. Then we get a family of $9 \times 9$ representations of BWMA. We study
the entangled states $|\Psi\rangle$'s, and find the negativity related parameter $q$. The negativity became the maximum value if $q = 1$. In Sec. 3, we defined the third topological graphic $\bigotimes \bigoplus$ and find three orthogonal topological basis states of BWMA, based on the former researchers. It was mentioned that the Hermitian matrices $E_A$, $E_B$, $A$ and $B$ have an interesting similar transformation matrix $U$ which satisfies $B = UAU^{-1}$ and $E_B = UE_AU^{-1}$. Based on them, we obtain a three-dimensional representation of BWMA. Finally we find the topological basis states are the spin singlet states, if $\phi_\nu = \pi, \lambda = 1, \mu = -1, \nu = 0$ and $q = 1$. Our next work will study how the topological basis states play a role in quantum theory.

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