Continuum Hartree-Fock-Bogoliubov theory for weakly bound deformed nuclei using coordinate-space Green’s function method

Hiroshi Oba and Masayuki Matsuo

1Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
2Department of Physics, Faculty of Science, Niigata University, Niigata 950-2181, Japan

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We formulate a new scheme of the Hartree-Fock-Bogoliubov mean-field theory applicable to weakly bound and pair correlated deformed nuclei using the coordinate-space Green’s function technique. On the basis of a coupled-channel representation of the quasiparticle wave function expanded in terms of the partial waves, we impose the correct boundary condition of the asymptotically outgoing waves on the continuum quasiparticle states. We perform numerical analysis for $^{38}$Mg to illustrate properties of the continuum quasiparticle states and the pair correlation in deformed nuclei near the neutron drip-line.

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I. INTRODUCTION

The RI-beam facilities in the new generation will enlarge significantly the experimentally accessible region in the nuclear chart, in particular in medium and heavy mass domains. An interesting area may be the $10 \lesssim Z < 20$ and $N \gtrsim 20$ region, where the selfconsistent mean-field theories predict that the shape deformation systematically occurs even if nuclei are close to the drip-line. It is the presence of weakly bound neutrons that makes these nuclei interesting, and furthermore the possible shape deformation will bring about additional mechanism influencing the single-particle motion and the many-body correlations such as the pairing and the collective excitations.

A promising theoretical framework to describe this situation may be the selfconsistent mean-field approach. We consider more specifically the Hartree-Fock-Bogoliubov (HFB) method to describe the pair correlated and deformed ground state, and the quasiparticle random phase approximation (QRPA), to describe excitation modes built on the ground state. Note here that, since the nucleons are bound only weakly and the threshold energy for the nucleon separation is low, one has to formulate the HFB and QRPA methods on the basis of correct description of the asymptotic forms of the wave functions of weakly bound and unbound continuum quasiparticle states. The formalisms that fulfill this requirement, which we shall call the continuum HFB or the continuum QRPA, are limited mostly to spherical nuclei. Therefore we need to extend, as the first step, the continuum HFB to deformed nuclei, i.e., we need to formulate the deformed continuum HFB theory. Hamamoto has analyzed in detail the quasiparticle motion in deformed Woods-Saxon potential by solving the HFB equation in the coupled-channel representation imposing the boundary condition of the correct asymptotics. Recently Stoitsov et al. have introduced a formulation of the deformed continuum HFB utilizing the Pöschel-Teller-Ginochio basis. In the present paper, we intend to give another new formulation of the deformed continuum HFB method by extending the Green’s function approach. Our eventual target is not just to formulate the deformed continuum HFB, but also to formulate a continuum QRPA for deformed nuclei. Although the latter is not pursued here, the deformed continuum QRPA can be easily formulated once the quasiparticle Green’s function for deformed nuclei is constructed in the coordinate representation.

The kernel of the present formulation is the quasiparticle Green’s function (called also the HFB Green’s function) which satisfies the correct boundary conditions of the continuum quasiparticle states. This can be achieved by utilizing the coupled-channel representation of the quasiparticle Schrödinger equation (the HFB equation) based on the partial wave expansion. The exact form of the HFB Green’s function is known for spherical nuclei where the channels - the partial waves - decouple, but what we need is the one in deformed nuclei for which there exists the coupling among the partial waves. We mention here that the Green’s function for deformed potentials has been utilized in describing other physical systems, e.g. the electronic response in molecules and electrons in matter scattering on deformed ion potentials, for which the coupled-channel representation based on the partial wave expansion is also employed. Indeed the exact form of the Green’s function in the general coupled-channel system can be extended to our problem, i.e. to describe the quasiparticle wave functions in pair correlated deformed nuclei. In the HFB theory, one also needs to calculate the density and the pair density, or the generalized density matrix in general, by summing up the wave functions of all the quasiparticle states including those in the continuum. The Green’s function formalism can be utilized
also to efficiently perform this summation. In this way we obtain a complete scheme to calculate the HFB ground state and the single-particle properties influenced by the pair correlation. We explain the details of this formalism in Section II.

As a demonstration of the deformed continuum HFB method, we perform in Section III numerical analysis for a neutron-rich nucleus $^{38}$Mg, which is situated near the neutron drip-line and is predicted to be prolately deformed in many selfconsistent mean-field calculations [1, 2, 3, 4, 5, 6, 7]. The purpose of this analysis is two fold. The first is to investigate how the neutron pair correlation changes as the binding of neutrons becomes weaker and stronger. Secondly we would like to reveal peculiar properties of the single-particle motion in the pair correlated deformed nuclei near the neutron drip-line, where the coupling of the quasiparticle states to the continuum orbits may introduce new features in the single-particle motion. We shall compare our results with those of Hamamoto [20, 21], who performed a pioneering analysis of the continuum quasiparticle states in a formalism satisfying the correct asymptotic forms, but not on the basis of the selfconsistent treatment of the pair correlation. Section IV is devoted to the conclusions.

II. DEFORMED CONTINUUM HFB THEORY IN THE GREEN’S FUNCTION FORMALISM

A. Coordinate-space HFB equation

The Bogoliubov’s quasiparticle plays a central role in the HFB theory [10]. The wave function of the quasiparticle state has two components, and is written in the coordinate-space representation as

$$\phi(r\sigma) = \begin{pmatrix} \phi^{(1)}(r\sigma) \\ \phi^{(2)}(r\sigma) \end{pmatrix}. \quad (1)$$

It obeys the HFB equation

$$\begin{pmatrix} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{2m} \frac{\partial}{\partial r} + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} - \lambda - E \end{pmatrix} \phi^{(1)}(r, E) + \sum_{L'} \left( u^{(0)}_{LL'}(r) + u^{(1)}_{LL'}(r) \frac{\partial}{\partial r} \right) \phi^{(1)}(r, E) + \sum_{L'} \Delta_{LL'}(r) \phi^{(2)}(r, E) = 0, \quad (5)$$

$$\begin{pmatrix} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{2m} \frac{\partial}{\partial r} + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} - \lambda + E \end{pmatrix} \phi^{(2)}(r, E) - \sum_{L'} \left( u^{(0)}_{LL'}(r) + u^{(1)}_{LL'}(r) \frac{\partial}{\partial r} \right) \phi^{(2)}(r, E) + \sum_{L'} \Delta_{LL'}(r) \phi^{(1)}(r, E) = 0. \quad (6)$$

Here the “channels” are labeled by the quantum number $jlm (= L)$ and the index $i = 1, 2$ specifying the upper and lower components of the quasiparticle wave function. The coupling among the channels is governed by

$$u^{(0)}_{LL'}(r), u^{(1)}_{LL'}(r) and \Delta_{LL'}(r), which are defined by$$

$$u^{(0)}_{LL'}(r) + u^{(1)}_{LL'}(r) \frac{\partial}{\partial r} = \int d\tilde{r} \sum_{\sigma\sigma'} Y^*(\tilde{r}\sigma') V(r\sigma) Y_{L'}(\tilde{r}\sigma'), \quad (7)$$

as is realized in the cases of some Skyrme effective interactions, e.g. SkP [12] with the effective mass $m^* = 1$, or of the phenomenological Woods-Saxon potential. The pair Hamiltonian becomes a local pair potential $\Delta(r)$ when the contact force is adopted for the pairing interaction. The position dependent effective mass $m^*(r)$ is often encountered in the Skyrme Hartree-Fock(-Bogoliubov) theories. An extension to this case is straightforward, but we do not deal with here.

We expand the quasiparticle wave function with respect to the partial waves specified by the angular quantum numbers $jlm$, abbreviated by $L$ hereafter:

$$\begin{pmatrix} \phi^{(1)}(r\sigma, E) \\ \phi^{(2)}(r\sigma, E) \end{pmatrix} = \sum_L \begin{pmatrix} \phi^{(1)}_L(r, E) Y_L(\hat{r}\sigma) \\ \phi^{(2)}_L(r, E) Y_L(\hat{r}\sigma) \end{pmatrix}, \quad (4)$$

where $Y_L(\hat{r}\sigma)$ is the spin spherical harmonics. The HFB equation is then transformed to a coupled-channel form [20, 21] for the radial wave functions $\phi^{(i)}_L(r, E)$:
\[ \Delta_{LL'}(r) = \int d\hat{r} \sum_{\sigma} Y_L^*(\hat{r}\sigma) \Delta(\hat{r}) Y_{L'}(\hat{r}\sigma). \] (8)

Note that the coupled-channel representation is often employed to describe scattering states in the non-spherical potential problems [22, 23, 24, 25, 26]. We have to truncate the partial wave expansion in practical calculations, and we denote \( N \) to represent the number of partial waves to be included.

Using the two component radial wave function
\[
\phi_{L}(r, E) = \begin{pmatrix} \phi_{L}^{(1)}(r, E) \\ \phi_{L}^{(2)}(r, E) \end{pmatrix},
\] (9)
the coupled-channel equation can be written as

\[
\partial_r^2 \phi_{L}(r, E) + \frac{2}{r} \partial_r \phi_{L}(r, E) + \sum_{L'} v_{LL'}^{(1)}(r) \partial_r \phi_{L'}(r, E) + \sum_{L'} v_{LL'}^{(0)}(r, E) \phi_{L'}(r, E) = 0,
\] (10)

where we introduced \( 2 \times 2 \) matrices

\[
v_{LL'}^{(0)}(r, E) = \begin{pmatrix} -\alpha u_{LL'}^{(0)}(r) + \alpha E & \alpha \Delta_{LL'}(r) \\ \alpha \Delta_{LL'}(r) & -\alpha u_{LL'}^{(0)}(r) + \alpha E \end{pmatrix},
\] (11)

and a constant \( \alpha = 2m/h^2 \). It is also possible to write this equation in a grand matrix form

\[
\begin{pmatrix} \partial_r^2 + \frac{2}{r} \partial_r + v^{(1)}(r) \partial_r + v^{(0)}(r, E) \end{pmatrix} \phi(r, E) = 0,
\] (13)

where the radial wave functions with different \( L \)'s are combined to form a \( 2N \) dimensional vector

\[
\phi(r, E) = \begin{pmatrix} \phi_{L_1}(r, E) \\ \phi_{L_2}(r, E) \\ \vdots \\ \phi_{L_N}(r, E) \end{pmatrix},
\] (14)

and \( 2N \times 2N \) matrices \( v^{(0)} \) and \( v^{(1)} \) are defined by

\[
[v^{(n)}]_{LL'} = v_{LL'}^{(n)}. \quad (n = 1, 2)
\] (15)

### B. Boundary conditions for the quasiparticle wave function

The quasiparticle states with the energy higher than the nucleon separation energy form a continuum spectrum [11, 12, 13]. They are the states with \( |E| > |\lambda| \) while the discrete quasiparticle states lie in the energy range \( |E| < |\lambda| \). In order to describe the continuum quasiparticle states, it is an essential condition that the quasiparticle wave functions must have correct asymptotic forms at far outside \( r \to \infty \) where the potentials vanish.

We therefore impose the boundary condition at \( r \to \infty \) that the radial wave functions should be connected to the following asymptotic form. There are \( 2N \) independent solutions where the upper or lower component of a given partial wave \( L' \) is dominant:

\[
\phi_{L}(r, E) \xrightarrow{r \to \infty} \begin{pmatrix} H^{+}_{L}(k, r) \delta_{LL'} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ H^{+}_{L}(k, r) \delta_{LL'} \end{pmatrix}.
\] (16)

where \( H^{+}_{L}(k, r) = \{ j_{l}(k, r) + in_{l}(k, r) \} \) (for neutron) \( \{ F_{l,0}(k, r) + iG_{l,0}(k, r) \} \) (for proton) is the Hankel/Coulomb function. The wave numbers in Eq. (16) are given by \( k_{L}(E) = \sqrt{2 \hbar |\lambda \pm E|} \) and their branch cuts are chosen so that \( \text{Im}k_{\pm}(E) > 0 \) is satisfied [13]. This boundary condition is equivalent to imposing that the wave functions are connected to the asymptotic outgoing or exponentially decaying waves. Combining these solutions in columns, we introduce a \( 2 \times 2 \) form \( \varphi^{(out)}_{LL'}(r, E) \) satisfying

\[
\varphi^{(out)}_{LL'}(r, E) \xrightarrow{r \to \infty} \begin{pmatrix} H^{+}_{L}(k, r) \delta_{LL'} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ H^{+}_{L}(k, r) \delta_{LL'} \end{pmatrix},
\] (18)

and similarly a \( 2N \times 2N \) matrix form \( \Phi^{(out)}_{LL'}(r, E) \) defined by

\[
[\Phi^{(out)}_{LL'}(r, E)]_{LL'} = \varphi^{(out)}_{LL'}(r, E).
\] (19)

At the center of the nucleus, we consider the radial wave functions which are regular at \( r = 0 \):

\[
\varphi^{(in)}_{LL'}(r, E) \xrightarrow{r \to 0} \begin{pmatrix} r^{|\lambda|} \delta_{LL'} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ r^{|\lambda|} \delta_{LL'} \end{pmatrix}.
\] (20)

We denote it also in the \( 2N \times 2N \) matrix form as

\[
[\Phi^{(in)}_{LL'}(r, E)]_{LL'} = \varphi^{(in)}_{LL'}(r, E).
\] (21)
C. HFB Green’s function in the coupled-channel representation

Now we introduce the Green’s function defined for the coordinate-space HFB equation \([24]\). It is expressed formally by

\[
G(E) = \left[ E - \begin{pmatrix} h - \lambda & \tilde{h} \\ \tilde{h} & -h + \lambda \end{pmatrix} \right]^{-1},
\]

and may be denoted

\[
G(r\sigma, r'\sigma', E) = \begin{pmatrix} G^{(11)}(r\sigma, r'\sigma', E) & G^{(12)}(r\sigma, r'\sigma', E) \\ G^{(21)}(r\sigma, r'\sigma', E) & G^{(22)}(r\sigma, r'\sigma', E) \end{pmatrix},
\]

in the coordinate-space representation. Note that the HFB Green’s function \(G(E)\) has the 2 \(\times\) 2 matrix form, whose diagonal and off-diagonal components are often referred to as the normal and abnormal Green’s function, respectively. Using the partial wave expansion it may be expanded as

\[
G(r\sigma, r'\sigma', E) = \sum_{LL'} Y_L(\hat{r}\sigma)g_{LL'}(r, r', E)Y_{L'}^*(\hat{r}'\sigma'),
\]

\[
g_{LL'}(r, r', E) = \begin{pmatrix} g^{(11)}_{LL'}(r, r', E) & g^{(12)}_{LL'}(r, r', E) \\ g^{(21)}_{LL'}(r, r', E) & g^{(22)}_{LL'}(r, r', E) \end{pmatrix},
\]

We can also introduce the 2\(N\) \(\times\) 2\(N\) matrix form \(g(r, r', E)\) of \(g_{LL'}(r, r', E)\):

\[
[g(r, r', E)]_{LL'} = g_{LL'}(r, r', E).
\]

It is easy to derive from the definition of the Green’s function that \(g(r, r', E)\) must satisfy the equation

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + v^{(1)}(r) \frac{\partial}{\partial r} + v^{(0)}(r, E) \right) g(r, r', E) = \frac{\delta(r-r')}{rr'} \alpha J,
\]

where \(J\) is defined by

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_N
\]

with \(I_N\) being the \(N\) dimensional unit matrix.

We now seek the Green’s function \(g(r, r', E)\) which satisfies both Eq. (24) and the boundary conditions, \([13]\) and \([20]\), imposed at \(r \rightarrow \infty\) and \(r = 0\), respectively. According to Ref. \([24]\), we write it in a form

\[
g_{LL'}(r, r', E) = \sum_{L''} \left( C^{(in)}_{L'L''}(r, r', E)\varphi^{(in)}_{LL'}(r, E)\theta(r' - r) + C^{(out)}_{L'L''}(r, r', E)\varphi^{(out)}_{LL'}(r, E)\theta(r - r') \right)
\]

or equivalently

\[
g(r, r', E) = \Phi^{(in)}(r, E)C^{(in)}(r', E)\theta(r' - r) + \Phi^{(out)}(r, E)C^{(out)}(r', E)\theta(r - r'),
\]

where \(C^{(in)}(r, E)\) and \(C^{(out)}(r, E)\) are matrix functions

\[
[C^{(in/out)}(r, E)]_{LL'} = C^{(in/out)}_{LL'}(r, E).
\]

Substitution of Eq. (30) into Eq. (27) leads to the equations for \(C^{(in)}(r, E)\) and \(C^{(out)}(r, E)\):

\[
\begin{pmatrix} \Phi^{(in)}(r, E) & -\Phi^{(out)}(r, E) \\ -\frac{d}{dr}\Phi^{(in)}(r, E) & \frac{d}{dr}\Phi^{(out)}(r, E) \end{pmatrix} \begin{pmatrix} C^{(in)}(r, E) \\ C^{(out)}(r, E) \end{pmatrix} = \frac{\alpha}{r^2} \begin{pmatrix} 0 \\ J \end{pmatrix}
\]

which corresponds to Eq. (23) of Ref. \([24]\). If the Hamiltonian does not contain the coupling term with the first derivative, i.e. \(v^{(1)}(r) = 0\), the solution has a simple form

\[
\begin{align*}
C^{(in)}(r, E) & = \alpha (M^T)^{-1} \Phi^{(out)}(r, E), \\
C^{(out)}(r, E) & = \alpha M^{-1} \Phi^{(in)}(r, E),
\end{align*}
\]

with the Wronskian matrix defined by

\[
M(E) = r^2 \begin{pmatrix} \Phi^{(in)}(r, E)J \left( \frac{d}{dr}\Phi^{(out)}(r, E) \right) \\ - \left( \frac{d}{dr}\Phi^{(in)}(r, E) \right) J\Phi^{(out)}(r, E) \end{pmatrix},
\]

and consequently the Green’s function is written as

\[
g(r, r', E) = \alpha \left( \Phi^{(in)}(r, E) (M^T)^{-1} \Phi^{(out)}(r', E) \theta(r' - r) + \Phi^{(out)}(r, E)M^{-1} \Phi^{(in)}(r', E)\theta(r - r') \right),
\]

in parallel to Eq. (36) of Ref. \([24]\).

In the following numerical application, however, we do not use Eq. (35), but instead we solve directly Eq. (32) to evaluate the HFB Green’s function. This procedure is applicable to the case where the coupling with the first derivative is present. We found also that the procedure provides us better numerical accuracy than to use Eq. (35) even when there is no first derivative term.

D. Generalized density matrix

Key quantities in the HFB theory are the normal density matrix

\[
\rho(r\sigma, r'\sigma') = \langle \Psi | \psi^{(1)}(r'\sigma')\psi(r\sigma) | \Psi \rangle,
\]

and the abnormal density (pair density) matrix

\[
\tilde{\rho}(r\sigma, r'\sigma') = \langle \Psi | \psi^{(1)}(r'\sigma')\psi(r\sigma) | \Psi \rangle,
\]

which can be combined as the generalized density matrix

\[
R(r\sigma, r'\sigma') = \begin{pmatrix} \rho(r\sigma, r'\sigma') & \tilde{\rho}(r\sigma, r'\sigma') \\ \tilde{\rho}(r'\sigma, r\sigma') & \rho(r'\sigma, r\sigma') \end{pmatrix} \delta_{rr'}\delta_{\sigma\sigma'} - \rho(r\sigma, r'\sigma') \delta_{rr'}\delta_{\sigma\sigma'}.
\]
Let $i$ be the index to specify the quasiparticle states, and $\phi_i(r\sigma)$ be the eigen solutions of the HFB equation \(^{[2]}\). Then the generalized density matrix is obtained by summing up products of the quasiparticle wave functions over all the quasiparticle states \(^{[14]}\):

$$R(r\sigma, r'\sigma') = \sum_i \tilde{\phi}_i(r\sigma)\phi_i^\dagger(r'\sigma') \hspace{1cm} (39)$$

where $\tilde{\phi}_i$ is a conjugate wave function of $\phi_i$, defined by

$$\tilde{\phi}_i(r\sigma) \equiv \begin{pmatrix} -\phi_i^{(2)*}(r\tilde{\sigma}) \\ \phi_i^{(1)*}(r\tilde{\sigma}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \phi_i(r\sigma).$$

Here $\tilde{\sigma}$ implies $\phi_i^{(i)}(r\tilde{\sigma}) \equiv (-2\sigma)\phi_i^{(i)}(r-\sigma)$, and $\phi_i^{(i)}(r\sigma)$ is the time-reversal $\phi_i^{(i)}(r\sigma) \equiv T\phi_i^{(i)}(r\sigma) = \phi_i^{(i)*}(r\tilde{\sigma})$ of $\phi_i^{(i)}(r\sigma)$.

We perform this summation by using a contour integral of the HFB Green’s function in the complex plane of the quasiparticle energy $E$ \(^{[13]}\). Then the generalized density matrix is given by

$$R(r\sigma, r'\sigma') = \frac{1}{2\pi i} \int_C G(r\sigma, r'\sigma', E)dE. \hspace{1cm} (40)$$

We use the partial wave expansion also for the generalized density matrix:

$$R(r\sigma, r'\sigma') = \sum_{L,L'} Y_{LL'}(\hat{r}\hat{\sigma}) R_{LL'}(r, r'\sigma'), \hspace{1cm} (41)$$

$$R_{LL'}(r, r') = \frac{1}{2\pi i} \int_C g_{LL'}(r, r', E)dE. \hspace{1cm} (42)$$

The contour $C$ should enclose the negative energy side of the real $E$ axis. The use of the contour integral enables us to impose the proper boundary condition for the continuum quasiparticle states using the HFB Green’s function $g_{LL'}$ described in the previous section.

In order to make the numerical integral efficiently, we choose $C$ as a circular path shown in Fig.1 although a rectangular contour is adopted in Ref.\(^{[13]}\). Then the integral reads

$$R_{LL'}(r, r') = \frac{E_{\text{cut}}}{4\pi i} \int_0^{2\pi} g_{LL'}(r, r', E(\zeta))i e^{i(\zeta - \pi)}d\zeta, \hspace{1cm} (43)$$

where $E_{\text{cut}}$ is the maximal quasiparticle energy which define the cut-off of the sum $|E| < E_{\text{cut}}$ of the quasiparticle states. The circular path makes the integrand to behave smoothly. In actual numerical calculations, we split $C$ into two semicircles $0 < \zeta < \pi$ and $\pi < \zeta < 2\pi$, and then apply the higher-order Gauss-Legendre quadrature with $M_\zeta/2$ points to perform the numerical integration in each semicircle ($M_\zeta$ points in total). The local density $\rho(r) = \sum_\sigma \rho(r\sigma, r\sigma)$ and the local pair density $\tilde{\rho}(r) = \sum_\sigma \tilde{\rho}(r\sigma, r\sigma)$ are calculated accordingly.

Given the methods to calculate the quasiparticle wave functions and the generalized density matrices, an usual iteration procedure can be applied to obtain a converged HFB ground state $|\Psi\rangle$, the associated quasiparticle states and the densities. The Fermi energy $\lambda$ is also determined selfconsistently so that the expectation value of the neutron/proton number is constrained to $N/Z$ of the nucleus under consideration.

III. APPLICATION TO A DEFORMED DRIP-LINE NUCLEUS

In the following we shall illustrate the deformed continuum HFB method with numerical examples. We would like to discuss some features of the quasiparticle spectra and the pair correlation which are characteristic to deformed nuclei near the neutron drip-line. As an example, we choose $^{38}\text{Mg}$ which is predicted to be prolate deformed and close to the neutron drip-line in the mean-field calculations \(^{[11,24,34,35,27]}\). Experimentally, $^{39}\text{Mg}$ is unbound and $^{40}\text{Mg}$ is the most neutron-rich bound isotope identified so far \(^{[28]}\). We do not deal with $^{40}\text{Mg}$ for which the prediction of the deformation is more subtle \(^{[11,24,34,35,27]}\).

A. Model and numerical procedure

In the following analysis we intend to clarify qualitative features rather than to make precise quantitative predictions of the specific nucleus. We simplify the Hartree-Fock potential by replacing it with an axially-symmetric
deformed Woods-Saxon potential

\[ V_{ws}(r) = V_{ws}^0 f(r, \theta), \]  

(44)

where \( f(r, \theta) = (1 + e^{-(r-R(\theta))/a_0})^{-1} \) and \( R(\theta) = R_0(1 + \beta Y_{20}(\theta)) \), together with the spin-orbit potential

\[ V_{ls}(r, s) = V_{ls}^0 \frac{\partial}{\partial \beta} \ell \cdot s. \]  

(45)

We choose the deformation parameter \( \beta = 0.3 \), being a typical value of the mean-field predictions \([1, 2, 3, 4, 5, 6, 7, 27]\). The other Woods-Saxon parameters are taken from Bohr and Mottelson\([29]\). We neglect the deformation in the spin-orbit potential. The neutron Nilsson diagram of the adopted Woods-Saxon potential is shown in Fig.2. If there is no pair correlation, the highest occupation of the deformed Woods-Saxon potential is shown in Fig.2. If there is no pair correlation, the highest occupied orbit is the one labeled with the Nilsson asymptotic quantum number \([310]_2^1\]

![Fig. 2: The neutron Nilsson diagram of the adopted Woods-Saxon potential for $^{38}$Mg. The neutron Fermi energy is located near the orbits labeled with the Nilsson asymptotic quantum numbers $[310]_4^1$ and $[312]_4^1$. The other orbits with $\Omega = 1/2$ are also marked with the arrows.](image)

Concerning the pairing properties, we derive it in a selfconsistent way from an effective pairing interaction, for which we adopt the density-dependent delta interaction (DDD),

\[ v_\tau(r-r') = v_\tau - \frac{P_\tau}{2} \left( 1 - \eta \left( \frac{\rho_\tau(r)}{\rho_c} \right)^\alpha \right) \delta(r-r'). \]  

\( (\tau = n, p) \)  

(46)

The selfconsistent pair potential is then given by

\[ \Delta(r) = \frac{1}{2} v_0 \left( 1 - \eta \left( \frac{\rho_\tau(r)}{\rho_c} \right)^\alpha \right) \hat{\rho}(r). \]  

(47)

Here we adopt the parameters \( v_0 = -458.4 \text{ MeVfm}^{-3} \), \( \eta = 0.76, \alpha = 0.59, \rho_c = 0.08 \text{ fm}^{-3} \) given in Ref.\([31, 31]\), where \( v_0 \) is determined so that the DDDI describes the \(^1S\) scattering length \( a = -18.5 \text{ fm} \) under the given cut-off energy \( E_{cut} \).

The selfconsistent pair potential depends on the pair density \( \hat{\rho}(r) \), and it varies in different physical situations. To provide a contrast to this case, we perform another calculation where the pair potential is replaced by a fixed phenomenological one having a Woods-Saxon shape

\[ \Delta(r) = V_p^0 f(r, \theta), \]  

(48)

as adopted in Refs.\([20, 21]\). According to Refs.\([20, 21]\) we constrain the depth \( V_p^0 \) of the pair potential by the value of an average gap \( \Delta \) defined by \( \Delta = \int dr \Delta(r) f(r, \theta) / \int dr f(r, \theta) \).

![Fig. 3: Dependence of the neutron pair interaction energy \( E_{pair}^{\Omega_{max}} \) on the maximal value \( \Omega_{max} \) of the magnetic quantum number of the quasiparticle states.](image)

The numerical details are as follows. We consider only neutrons. Since we assume the axial symmetry, the projection of the total angular momentum along the z-axis is conserved. The quasiparticle states have the magnetic quantum number \( \Omega \), the eigenvalue of \( j_z \), and we solve the HFB equation separately for each \( \Omega \). The coupling potentials \( a_{LL}^{(0)}(r) \) and \( \Delta_{LL}^{(0)}(r) \) are evaluated according to Eqs.\((47) \) and \((48) \), where the integral over the azimuthal angle \( \theta \) is performed numerically by means of the higher-order Gauss-Legendre quadrature with \( N_\theta \) points. The coupled-channel HFB equation \((13) \) is solved by means of the Runge-Kutta-Nystöm method \([32]\) in the region \( r = 0 - R_{max} \) with \( R_{max} = 15 \text{ fm} \) and an equidistant interval \( \Delta r = 0.2 \text{ fm} \). We impose the boundary condition that the wave function is connected to the asymptotic form, Eq.\((18) \), at \( r = R_{max} \). We use the code \texttt{cufcomplex}\([33]\) to calculate the Hankel and Coulomb functions at complex energies. The cut-off quasiparticle energy is \( E_{cut} = 60 \text{ MeV} \). We also truncate the quasiparticle states in terms of the magnetic number by including those with \(-\Omega_{max} \leq \Omega \leq \Omega_{max} \). Figure\([34]\) shows the dependence of the neutron pair interaction energy \( E_{pair} = \frac{1}{2} \int dr \Delta(r) \hat{\rho}(r) \) on the choice of \( \Omega_{max} \). A reasonable convergence is obtained with \( \Omega_{max} \gtrsim 21/2 \), and
hence we adopt $\Omega_{\text{max}} = 21/2$ in the following analysis. The parameters for the Gauss-Legendre quadrature are $N_\theta = 60$ and $M_\zeta = 100$.

**B. Density profiles**

The deformed continuum HFB method can describe the exponentially decaying tails of the densities. Figure 4 shows the density asymptotics of the HFB ground state. Here we make the multipole expansion of the densities as

$$\rho(r) = \sum_\ell \rho_\ell(r) Y_\ell(\hat{r}), \quad (49)$$

$$\tilde{\rho}(r) = \sum_\ell \tilde{\rho}_\ell(r) Y_\ell(\hat{r}), \quad (50)$$

and plot the monopole and quadrupole parts $\ell = 0, 2$. For comparison, we also show the results of another calculation where we impose the box boundary condition $\phi_L(R_{\text{max}}) = 0$ at the boundary $r = R_{\text{max}}$. In the latter case the whole quasiparticle spectrum including the continuum part is discretized. It is clear that the calculation with the correct boundary condition can describe the exponentially decaying asymptotics both for the density and the pair densities while the calculation with the box boundary condition fails.

We remark here that both the monopole and quadrupole parts exhibit the same slope in the exponentially decaying tail, i.e., the ratios $\rho_2(r)/\rho_0(r)$ and $\tilde{\rho}_2(r)/\tilde{\rho}_0(r)$ are approximately independent of $r$ in the tail region. This means that we can define the “deformation” of the exponentially decaying tails in terms of the shape of the equi-density contour curves. Note however that the size of the deformation in the tail differs from that of the nuclear surface (which may be defined by the deformation of the equi-density contour at the half central density). It is also seen that the tail deformation in the density $\rho(r)$ and in the pair density $\tilde{\rho}(r)$ are different. It is interesting to quantify the deformations in the tail region, but we leave it for future works since in the present calculation we do not make the selfconsistent treatment of the deformation of the Hartree-Fock potential. The calculation in Ref.19 indicates also that the exponentially decaying tail has the same slope in different directions.

**C. Weak-binding effect on pair correlation**

We can regard the calculated $^{35}$Mg as a quite weakly bound system since the neutron Fermi energy in the present calculation is just $\lambda = -0.889$ MeV. It is interesting to see what happens if the system is weakly bound further more, or if the system is more strongly bound. To see the effects of the weak binding, we investigate the pairing correlation by varying artificially the depth parameter $V_{\text{ws}}^0$ of the Woods-Saxon potential as

$$V_{\text{ws}}^0 \rightarrow V_{\text{ws}}^0 + \alpha \quad (51)$$

where we choose $\alpha = +1, 0, -1, -2, -3$ and $-4$ MeV. The neutron Fermi energy is $\lambda = -0.471, -0.889, -1.335, -1.802, -2.289$ and $-2.793$ MeV for $\alpha = +1, 0, -1, -2, -3$ and $-4$ MeV, respectively.

Figure 5 shows the monopole parts $\rho_0(r)$ and $\tilde{\rho}_0(r)$ of the density and pair density, respectively, of neutrons for different values of $\alpha$. It is seen that the exponentially decaying tails of the density and the pair density extends (shrinks) as the Fermi energy becomes shallower (deeper). This is naturally expected as the exponentially decaying behavior kinematically as $\rho(r) \propto e^{-2\kappa r}$ and $\tilde{\rho}(r) \propto e^{-\kappa r}$ with the exponent $\kappa = \sqrt{2m|\lambda|}/\hbar$ related to the Fermi energy $\lambda$.34. We also see another noticeable feature that as $\lambda$ becomes shallower the magnitude of the pair density increases not only in the tail region but also in the whole region of $r$, including the interior and the surface areas. Apparently the pair correlation is enhanced by the weak binding. To characterize the overall magnitude of the pair correlation we evaluate the
average pairing gaps $\Delta_{vv}$ and $\Delta_{uv}$ defined by

$$\Delta_{vv} = \frac{\int dr \Delta(r) \rho(r)}{\int dr \rho(r)}, \quad (52)$$

$$\Delta_{uv} = \frac{\int dr \Delta(r) \tilde{\rho}(r)}{\int dr \tilde{\rho}(r)}, \quad (53)$$

and the pair interaction energy

$$E_{\text{pair}} = \frac{1}{2} \int dr \Delta(r) \tilde{\rho}(r). \quad (54)$$

They are plotted in Fig. 6, where the enhancement due to the weak binding is seen directly in these quantities.

In order to investigate the origin of the enhanced pairing in the weak binding cases, we compare with another calculation where the pair potential is replaced by a phenomenological one, Eq. (48), with fixed strength $\bar{\Delta} = 1.5$ MeV. The calculated average pairing gaps $\Delta_{vv}$ and $\Delta_{uv}$ and the pair interaction energy $E_{\text{pair}}$ are shown also in Fig. 6, where, in contrast to the selfconsistent case, $\Delta_{vv}$ and $\Delta_{uv}$ stay almost constant or even decreases slightly with lifting the Fermi energy. The monopole density $\rho_0(r)$ and the monopole pair density $\tilde{\rho}_0(r)$ are shown in Fig. 7. It is seen that the pair density inside the nucleus decreases as the Fermi energy becomes shallower. This is opposite to the trend seen in the case of the selfconsistent calculation. Consequently there is very little influence of the weak binding on the pair correlation provided that we fix the pair potential. In other words, the enhancement of the pair correlation caused by the weak binding is described only when we calculate the selfconsistent pair potential. In the selfconsistent case, the increase of the pair density in the tail region induces the increase of the pair correlation not only in the tail region but also in the entire region.
D. Structure of continuum quasiparticle states

In the HFB theory, both the neutron density $\rho(r)$ and the expectation value $\langle N \rangle = \int d\rho(r)$ of the neutron number can be expressed as summations of contributions of the individual quasiparticle states. Since the quasiparticle states are specified by the energy eigenvalue $E$, we can define the contributions per unit energy (the density per energy) to the neutron number $\langle N \rangle$. We denote it by $n(E)$, and call the occupation number density. It satisfies $\langle N \rangle = \int_0^{E_{cut}} dE n(E)$ by definition, and can be calculated in terms of the HFB Green’s function by

$$n(E) = \frac{1}{\pi} \text{Im} \sum_\sigma \int dr G^{(11)}(r\sigma, r\sigma, -E - i\epsilon). \quad (55)$$

This is the level density of the quasiparticle states, but it is weighted with the occupation number (corresponding to the $\nu^2$ factor in the case of the BCS approximation). Similarly we can define the pair number density $\tilde{n}(E)$ by

$$\tilde{n}(E) = \frac{1}{\pi} \text{Im} \sum_\sigma \int dr G^{(12)}(r\sigma, r\sigma, -E - i\epsilon) \quad (56)$$

for which the level density is weighted with the pairing factor (the $uv$-factor in the BCS). $\tilde{n}(E)$ represents the contribution of the quasiparticle state at energy $E$ to the pair number $\langle \tilde{N} \rangle = \int dr \tilde{\rho}(r)$. Note that the constant $\epsilon$ plays a role of the smoothing parameter. If the quasiparticle spectrum contains discrete states, there emerge delta functions in $n(E)$ and $\tilde{n}(E)$ at the energies of the discrete quasiparticle states. Having the imaginary part $i\epsilon$ in Eqs. (55) and (56), the delta function becomes a Lorentzian function with FWHM of $\gamma = 2\epsilon$. In the following, we take $\epsilon = 25$ keV ($\gamma = 50$ keV), and we shall investigate the structure of the quasiparticle spectrum in terms of $n(E)$ and $\tilde{n}(E)$.

Figure 8 shows $n(E)$ and $\tilde{n}(E)$ for neutron states with $\Omega = 1/2$. Note that the threshold energy of the continuum spectrum is $E_{th} = |\lambda| = 889$ keV. It is seen that the continuum spectrum above $E_{th}$ consists of several resonances forming narrow peaks, and a broad continuum...
that the single-particle energies of the orbits \([310\frac{1}{2}]\) of the deformed Woods-Saxon potential (cf. Fig.2). Narrow resonances located at \(E = 3.36, 4.61, 7.36\) and \(11.68\) MeV correspond to the other \(\Omega = 1/2\) orbits marked with the arrows in Fig.2. The widths of these resonances are smaller than the smoothing width \(\gamma = 50\) keV except the lowest energy resonance at \(E = 1.24\) MeV, which has a width apparently larger than those of the others. Note that the \([310\frac{1}{2}]\) orbit is a bound state if we neglect the pair potential, and the finite and relatively large width is brought by the pair potential.

Another important feature seen in Fig.8 is that the non-resonant continuum states have significant contribution to the pair number density \(\tilde{n}(E)\) in an extent comparable to that of the narrow resonance states. This should be contrasted to that the contributions of the non-resonant continuum states to the occupation number density \(n(E)\) is very small. The small contribution of the non-resonant continuum states to the occupation number density \(n(E)\) is pointed out in Ref. [21], but we emphasize here that this is not the case for the pair number density \(\tilde{n}(E)\).

In Fig. 8 we also show the results obtained with the box boundary condition \(\phi(R_{max}) = 0\). In this case the quasiparticle states are all discretized. It is obvious that the discretized quasiparticle states obtained with the box size \(R_{max} = 15\) fm fail to describe the width of the resonances, for instance, that of the \([310\frac{1}{2}]\). Obviously the discretization is too crude to describe the continuous behavior of the non-resonant continuum states.

In Figs. 9 and 10 we show \(n(E)\) and \(\tilde{n}(E)\) combining several values of \(\Omega\). We mention here two features. (i) Firstly, the lowest energy resonance (with \(\Omega = 1/2\)) and the second lowest one (with \(\Omega = 5/2\)) have significantly different widths even though the centroid energies of both resonances are comparable. We note here that the single-particle energies of the orbits \([310\frac{1}{2}]\) and \([312\frac{5}{2}]\), which correspond to the lowest and the second lowest resonances, respectively, are also very close (cf. Fig.2). The difference in the width may be attributed to the difference in the magnetic quantum number \(\Omega\) in a way discussed in Ref. [20]: the centrifugal barrier for the \([310\frac{1}{2}]\) state is lower than that of the \([312\frac{5}{2}]\) state since the former contains the partial wave with \(l = 1\) while the latter does not. (ii) Secondly, it is seen in the pair number density \(\tilde{n}(E)\) that the contribution of the non-resonant continuum states is larger for states with smaller \(\Omega\). Note, however, that the relative importance of high/low-\(\Omega\) non-resonant states depends on the quasiparticle energy. The low-\(\Omega\) states such as \(\Omega = 1/2\) and 3/2 are dominant as far as the non-resonant quasiparticle states at low energies \(E \leq 10\) MeV are concerned (cf. Fig.9). Concerning the non-resonant continuum states at higher energies \(E \gtrsim 30\) MeV, however, we see in Fig.10 that contributions from various \(\Omega\)'s do not depend very strongly on \(\Omega\): there is only a small difference by a factor of \(\sim 2\) in \(\tilde{n}(E)\) between \(\Omega = 1/2\) and 11/2 states at \(E \gtrsim 30\) MeV. In other words, the high-\(\Omega\) states also have sizable contributions to the pair correlation.

The decomposition of the quasiparticle states into the partial waves is another useful tool to analyze the structure of the quasiparticle states. In fact we can decompose the occupation number density \(n(E)\) and the pair number density \(\tilde{n}(E)\) into different partial waves:

\[
n(E) = \sum_L n_L(E), \quad \tilde{n}(E) = \sum_L \tilde{n}_L(E)
\]

(57)

where the contribution from each partial wave \(L\) is given by

\[
n_L(E) = \frac{1}{\pi} \Im \int drr^2 g^{(11)}_{L L}(r, r, -E - i\epsilon)
\]

(58)

and similarly for \(\tilde{n}_L(E)\). The partial wave decomposition of the \(\Omega = 1/2\) states is shown in Fig.11. It is seen that each of the resonances has specific partial wave contents reflecting the character of the resonances. For instance, the lowest energy resonance \([310\frac{1}{2}]\) consists of dominant

\[
\begin{align*}
\text{FIG. 9: (Top)} & \quad \text{The occupation number density } n(E) \text{ of neutrons shown separately for different values of } \Omega = 1/2, 3/2, \ldots, 11/2. \text{ The arrow indicates the threshold energy } |\lambda| \text{ for the continuum quasiparticle states.} \\
\text{(Bottom)} & \quad \text{The same but for the pair number density } \tilde{n}(E).
\end{align*}
\]
$p_1/2 - p_3/2$ components and sub-dominant $f_7/2$ component. The second resonance consists of dominant $f_7/2$ and sub-dominant $p_3/2$.

We can also argue the structure of the non-resonant quasiparticle states in terms of the partial wave contents in $\tilde{n}(E)$. For instance, the continuum states in the interval $1.5 \lesssim E \lesssim 3.0$ MeV is characterized by the relatively largest component of $p_1/2 - p_3/2$ waves and sub-dominant $f_7/2 - f_5/2$ components. We see also some structures in the intervals $2.5 \lesssim E \lesssim 5.5$ MeV and $5.5 \lesssim E \lesssim 8$, which are characterized the relatively largest components $f_5/2$ and $g_9/2$, respectively. These structures presumably originate from resonance states in the deformed Woods-Saxon potential lying high above the Fermi energy. They coexist in the same energy region with smooth (almost flat in $E$) components containing several partial waves with $l = 0 - 4$. At higher energies, say $E \gtrsim 30$ MeV, there remain only smooth non-resonant components including all the partial waves up to very high orbital angular momentum $l \lesssim 6 - 10$. The contributions of high-$l$ orbits is consistent with the high-$\Omega$ components observed in Fig.10.

FIG. 10: The same as Fig.9 but in the log scale and in a wider energy range.

FIG. 11: The partial wave decomposition of the occupation number density $n(E)$ (top panel) and the pair number density $\tilde{n}(E)$ (bottom) of neutron $\Omega = 1/2$ states. Each curve labeled with $l_j$ represents the partial sum $\sum_{L<l_j} n_L(E)$ ($\tilde{n}_L(E)$) containing from $s_1/2, p_1/2, p_3/2, \cdots$ to $l_j$.

E. Weak-binding effects on quasiparticle motion

In this subsection we shall elucidate how the weak binding influences the neutron quasiparticle motion and its coupling to the pair correlation. For this purpose we first analyze how the neutron pair number density $\tilde{n}(E)$ evolves when we vary the Fermi energy by shifting the depth $V_{ws}^0$ of the Woods-Saxon potential as $V_{ws}^0 \rightarrow V_{ws}^0 + \alpha$.

Figure [12] shows the pair number density $\tilde{n}(E)$ of $\Omega = 1/2$ and $5/2$ states calculated for various Woods-Saxon potential depths. It is seen first of all that the influence of the weak binding is stronger on the $\Omega = 1/2$ states than on the $\Omega = 5/2$ states. (This is along with the observation in Refs. [20, 21, 26].) The influence is especially strong on the low-lying quasiparticle states. Here we concentrate on $\Omega = 1/2$, and focus on the lowest energy quasiparticle state (or the quasiparticle resonance)
The same but for the $\Omega = \frac{5}{2}$ case. The steep change in the width is quite contrasting to that of the $\Omega = \frac{1}{2}$ case, i.e. the $[312]\frac{5}{2}$ state, for which there is only a small change in the width. Note also that a similar steep increase of the width can be seen in the second lowest peak, which corresponds to the Woods-Saxon orbit marked with the asterisk in Fig. 13. The second lowest peak is a bound discrete state in the case of $\alpha = -4$, -3, -2, -1, 0 and +1 MeV. The positive energy states are discretized with the box boundary condition at $r = R_{\text{max}} = 15 \text{fm}$, and shown within the parenthesis. The calculated neutron Fermi energy $\lambda$ is plotted also with the symbol +. $\lambda$ calculated for the fixed phenomenological pair potential is also plotted with the symbol $\times$.

Let us look into the $[310]\frac{1}{2}$ state. The binding energy of the $[310]\frac{1}{2}$ orbit of the deformed Woods-Saxon potential becomes very small in the cases of $\alpha = 0$ and +1 MeV (cf. Fig. 13). In these cases the quasiparticle peak corresponding to the $[310]\frac{1}{2}$ orbit is located in the continuum region $E + \lambda > 0$ ($E > |\lambda| = E_{\Delta^2}$). It has a finite width due to the coupling to the continuum. It is seen that the width increases quite steeply for the variation of the potential depth $\alpha = -1 \rightarrow 0 \rightarrow +1$ MeV. The steep change in the width is quite contrasting to that of the $\Omega = 5/2$ case, i.e. the $[312]\frac{5}{2}$ state, for which there is only a small change in the width. Note also that a similar steep increase of the width can be seen in the second lowest peak, which corresponds to the Woods-Saxon orbit marked with the asterisk in Fig. 13. The second lowest peak is a bound discrete state in the case of $\alpha = -4$, -3, -2, -1, 0 and +1 MeV. The positive energy states are discretized with the box boundary condition at $r = R_{\text{max}} = 15 \text{fm}$, and shown within the parenthesis. The calculated neutron Fermi energy $\lambda$ is plotted also with the symbol +. $\lambda$ calculated for the fixed phenomenological pair potential is also plotted with the symbol $\times$.

The most important feature which we observe in Fig. 12 is that the contribution of the low-lying non-resonant continuum quasiparticle states to $\tilde{n}(E)$ increases significantly as the neutron Fermi energy becomes close to zero, especially in the cases $\alpha = 0$ and +1 MeV (corresponding to $\lambda = -0.89$ and -0.47 MeV, respectively). If we look closely at the second lowest resonance discussed above, it keeps a sizable contribution to $\tilde{n}(E)$ in the cases $\alpha = -2$ and -1 MeV where the width becomes very large. The contribution is still sizable even in the cases $\alpha = 0$ and +1 MeV where only a non-resonant continuum structure is visible as a remnant in the expected energy region. It is argued in Refs. [20, 21] that weakly bound or resonant $\Omega = 1/2$ quasiparticle states contribute little to the pair correlation compared with that of the strongly bound states. Our calculation does not exhibit this behavior. Our analysis rather suggests that all of the weakly bound, resonant and non-resonant quasiparticle states contribute

![Fig. 12: (Bottom) The pair number density $\tilde{n}(E)$ of the neutron $\Omega = 1/2$ states is shown for various shifts $\alpha = -4, -3, -2, -1, 0$ and +1 MeV of the Woods-Saxon potential depth $V^{\alpha}_{ws} \rightarrow V^{\alpha}_{ws} + \alpha$. The horizontal axis represents $E + \lambda$ whose origin corresponds to the threshold energy $E_{\Delta^2} = |\lambda|$ for the continuum spectrum. (Top) The same but for the $\Omega = 5/2$ states.](image_url)

![Fig. 13: The single-particle energy of the neutron $\Omega = 1/2$ orbits in the deformed Woods-Saxon potential without the pair correlation for various shifts $\alpha = -4, -3, -2, -1, 0$ and +1 MeV of the Woods-Saxon potential depth $V^{\alpha}_{ws} \rightarrow V^{\alpha}_{ws} + \alpha$. The positive energy states are discretized with the box boundary condition at $r = R_{\text{max}} = 15 \text{fm}$, and shown within the parenthesis. The calculated neutron Fermi energy $\lambda$ is plotted also with the symbol +. $\lambda$ calculated for the fixed phenomenological pair potential is also plotted with the symbol $\times$.](image_url)
significantly to the pair correlation in the case when the neutron Fermi energy is small. This is also supported by
the observation in subsection III C that the weak binding enhances the net pair correlation.

Let us then investigate how the weakly bound and resonant quasiparticle states couple with the pair correlation. For this purpose we shall analyze the pairing gap associated with these orbits. As far as the $[310]_{1/2}$ state is concerned, we can evaluate an effective pairing gap of this state in a simple way. This is because, as seen in Fig. 13, the neutron Fermi energy is small. This is also supported by the selfconsistent pair potential derived from the DDDI while the lower half is for those obtained with the fixed phenomenological pair potential, Eq (59).

![Figure 14](image)

**FIG. 14:** (Top) The pair number density $\tilde{n}(E)$ of the neutron $\Omega = 1/2$ states for the shifted Woods-Saxon potential depths $V_{ws} + \alpha$ with $\alpha = +1, 0, -4$ MeV. The arrow indicates the positions of the Fermi energy. This is the result of the calculation using the selfconsistent pair potential derived from the DDDI. (Bottom) The same but for the calculation assuming the fixed phenomenological pair potential. **Selfconsistent** "Fixed pair potential" $g_{OgBo} = 0, -4gB$

valid in the BCS approximation leads to $E_i \approx \Delta_i^{(eff)}$ for $|e_i - \lambda| \ll \Delta_i^{(eff)}$. We read from Fig. 13 and Table I that the peak position of the $[310]_{1/2}$ resonance stays at $E_i \approx 1.3$ MeV, and hence $\Delta_i^{(eff)} \approx 1.3$ MeV in all the cases. Namely we see that the effective pairing gap is almost unchanged even when $e_i \to 0$.

We can evaluate more directly effective pairing gap associated with the quasiparticle states. Note here that the density $\rho(r, E)$ and the pair density $\hat{\rho}(r, E)$ associated with the quasiparticle state at $E$ is given by

$$\rho(r, E) = \frac{1}{\pi} \Im \sum_{\sigma} \int dr G^{(11)}(r \sigma, r \sigma, -E - i\epsilon),$$

$$\hat{\rho}(r, E) = \frac{1}{\pi} \Im \sum_{\sigma} \int dr G^{(12)}(r \sigma, r \sigma, -E - i\epsilon).$$

Utilizing these quantities, we can evaluate the effective pairing gaps of this state by

$$\Delta_{uv}(E) = \frac{\int dr \Delta(r) \rho(r, E)}{\int dr \rho(r, E)},$$

$$\Delta_{uv}(E) = \frac{\int dr \Delta(r) \hat{\rho}(r, E)}{\int dr \hat{\rho}(r, E)}.$$
In order to clarify the behavior of the effective pairing gap of the weakly bound or resonant $[310]_{1}^{2}$ state, we compare again with the calculations where the pair potential is replaced by the fixed phenomenological one (cf. Eq. (18)). The result is shown in the bottom panel of Fig. 14. The peak energy of the lowest energy quasiparticle state $[310]_{1}^{2}$ varies as the Woods-Saxon potential depth $V_{\text{WS}}$ is shifted, and we see that the effective pairing gap $\Delta_{s}(\epsilon_{f})$ decreases as the Woods-Saxon potential depth becomes shallower. The reduction of the pairing gap of the $[310]_{1}^{2}$ state caused by the weak binding is seen more explicitly in $\Delta_{uv}(E)$ and $\Delta_{vv}(E)$ listed in Table I. This reduction of the effective pairing gap due to the weak binding effect has been pointed out in Refs. [20, 21], where the mechanism of the reduction is ascribed to a sort of decoupling, i.e., the mechanism that the spatially extended wave function of the weakly bound or resonant quasiparticle state has less overlap with the pair potential. But our analysis indicates that the reduction is a consequence of the use of the fixed phenomenological pair potential. The reduction of the effective pairing gap does not show up if we describe the pair correlation using the selfconsistent pair potential. This is because the weak binding has two effects, one causing the enhancement of the pair correlation and the other causing the decoupling of the wave function, which have a tendency to compensate each other.

IV. CONCLUSIONS

We have given a new formulation of the deformed continuum HFB method, which can be applied to deformed nuclei near the drip-lines. The kernel of this formulation is the use of the exact quasiparticle Green’s function constructed on the basis of the coupled-channel representation in the partial wave expansion. This enables us to impose the correct asymptotics on the quasiparticle wave functions of the weakly bound and continuum quasiparticle states. Consequently we can describe the quasiparticle states above the neutron separation energy without energy discretization.

We have shown several numerical examples to illustrate effects of the weak binding and the continuum coupling on the pair correlation and the quasiparticle spectrum of neutrons in deformed nuclei near the neutron drip-line. The calculations are performed for $^{38}$Mg which is chosen as an example of prolately deformed nuclei. It is found that there arises a significant contribution to the pair correlation from the non-resonant part of the continuum quasiparticle states, and the contribution grows as the neutron Fermi energy approaches zero. This trend is most significant for the quasiparticle states with $\Omega = 1/2$. We confirm the strong effects of the continuum coupling on the $\Omega = 1/2$ states [21, 26] causing a steep increase of the width of the resonances, but we found that the contribution to the pair correlation never reduces even with the large continuum coupling effects. We also found that the effective pairing gap of the weakly bound or resonant $\Omega = 1/2$ orbits stay at sizable values. The effect of the decoupling from the pair correlation [20, 21] is not large, and it may be compensated by the enhancement of the pair correlation due to the weak binding. The use of the selfconsistent pair potential is crucial to describe this feature.

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