A Novel Parametric Bound for Information Retrieval from Black Hole Radiation

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Abstract

Hawking’s argument about non-unitary evolution of black holes is often questioned on the ground that it doesn’t acknowledge the quantum correlations in radiation process. However, recently it has been shown that adding ‘small’ correction to leading order Hawking analysis, accounting for the correlations, doesn’t help to restore unitarity. This paper generalizes the bound on entanglement entropy by relaxing the ‘smallness’ condition and configures the parameters for possible recovery of information from an evaporating black hole. The new bound effectively puts an upper limit on increase in entanglement entropy. It also facilitates to relate the change in entanglement entropy to the amount of correction to Hawking state.

Index Terms

Information paradox, Hawking radiation, Black hole evaporation.
I. INTRODUCTION

Black hole information paradox is the most exciting testing ground for unitarity, locality and Poincaré symmetry, as these three pillars of physics are in essential conflict here [1]. Quantum information, as well as quantum mechanics, is built upon unitarity. However, semiclassical analysis shows that unitarity is violated when a black hole forms and eventually evaporates. There are some arguments demanding that the semiclassical analysis is incomplete and ‘small’ corrections to it would restore unitarity. It has been shown recently that these arguments are not precise; ‘small’ corrections cannot reduce entanglement entropy [2]. This paper quantifies the ‘amount’ of correction that is necessarily required to restore unitarity. A rigorous bound for change in entanglement entropy is established and the parameters are configured for possible information retrieval from black hole radiation.

The paper is organized as follows: Section II introduces the essential features of the paradox, section III describes the pair production process in leading order, section IV explores Mathur’s bound on entanglement entropy based on ‘small’ correction, section V presents a generalization of Mathur’s bound and finally section VI describes the necessary conditions for information retrieval from black hole radiation.

II. BLACK HOLE INFORMATION PARADOX - AN OVERVIEW

Bekenstein first introduced the idea that black holes have entropy directly related to their area and the amount of entropy can be associated to the information contained in them [3]. Since black holes have entropy and energy, thermodynamically they should have temperature and therefore should radiate. Hawking calculated the thermal emission from a classical black hole [4]. Event horizon of a black hole acts like a one way membrane allowing nothing to emerge locally out of its interior to its exterior. Hawking, therefore, considered a quantum process of radiation in which particle pairs are created in the vacuum due to stretching of space-time near the horizon [4]. Of the pair, one particle, created slightly outside the horizon, can escape to infinity, whereas the other one, created slightly inside the horizon, must remain inside and eventually hit the singularity. This description of particle creation is validated by local quantum field theory.

The nature of the pair production process is such that outside quanta are entangled with their inside counterparts. If only the radiation subsystem is considered, it is found to be in a mixed state and can be purified by the subsystem consisting of the inside particles and matter shell. An asymptotic observer sees that positive mass quanta are being radiated from the black hole. Conservation of energy then dictates that the inside quanta must have negative
energy. Therefore, black hole loses mass in the process of evaporation. Hawking showed that entanglement entropy of the hole increase in every step of pair creation [4]. At the end of evaporation, the outside quanta will be in a mixed state; but there will be nothing inside that can purify it. Therefore, Hawking’s calculation implies that a pure state has evolved into a permanently mixed state and theory doesn’t prevent it! Unitarity is lost in principle [5]!

In standard quantum mechanics, mixed states occur when information is deliberately discarded. For any mixed state, there is a purifier state known as ‘ancilla’, which can readily make the state pure. Any loss of information due to creation of a mixed state is thus ‘apparent’. In black hole evaporation process, the radiation subsystem is mixed and the internal subsystem acts as its purifier. However, at the end of evaporation the internal subsystem vanishes and the radiation subsystem becomes ‘permanently’ mixed. Thus the information that has been discarded can no longer be recovered. This is the essence of black hole information paradox.

III. Pair creation - Leading order picture

Particle creation in black hole horizon is described by quantum field theory in curved spacetime. A formal description cannot be accommodated here (see [6] for review); but for our purpose it suffices to consider that particles are created in a state of the form [7]

\[ |\Psi_{\text{pair}}\rangle = Ce^{\frac{\beta c\hat{c}\hat{b}\hat{b}^\dagger}}|0\rangle_c|0\rangle_b \]  

where \( \beta \) is number of order unity, \( \hat{c}\dagger \) and \( \hat{b}\dagger \) are creation operators and \(|0\rangle\) is the vacuum state. \( c \) and \( b \) denote the inside and outside quanta respectively. In leading order Hawking calculation, this state is simplified in the form [2]

\[ |\Psi_{\text{pair}}\rangle = \frac{1}{\sqrt{2}}|0\rangle_c|0\rangle_b + \frac{1}{\sqrt{2}}|1\rangle_c|1\rangle_b. \]  

We will refer to this state as ‘Hawking state’ in this paper. If we foliate the black hole geometry by spacelike slices, as done in [1], the matter shell that created the black hole resides very far away from horizon on the spacelike slice. We can approximate it as residing on a different Hilbert space. If the matter shell is denoted by the state \(|\Psi_M\rangle\), the complete state of the black hole system after first step of pair production takes the form

\[ |\Psi\rangle \approx |\Psi_M\rangle \otimes |\Psi_{\text{pair}}\rangle \]

\[ = |\Psi_M\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_c|0\rangle_b + \frac{1}{\sqrt{2}}|1\rangle_c|1\rangle_b\right). \]

Entanglement entropy of \( b_i \) quanta with \( \{M, c_1\} \) is

\[ S_{\text{ent}} = \log 2. \]  

\[ (3) \]
At further time-steps, new pairs are created in Hawking state and entanglement entropy increases in each step by $\log 2$. Thus, after $N$ step, $S_{\text{ent}} = N \log 2$ [2].

This monotonic increase in entanglement entropy in black hole radiation is at the heart of the information paradox. It was widely speculated that Hawking’s calculation is not very precise in the sense that it approximates the pair creation process in the leading order. However, there is no sharp reason why Hawking’s semiclassical analysis should be invalid until the black hole comes down to Planck scale, where our known physics ceases to provide any consistent picture. Some advocated that pair production in Hawking state at every step should be modified when ‘back-reaction’ is considered. Under such conditions, emitted pair at any stage is slightly correlated to the earlier emitted pair. It has been suggested that such ‘small’ correlation can eventually lead to substantial decrease in entanglement entropy and thus resolve the paradox [8]. However, a bound on entanglement entropy proposed by Mathur states that ‘small’ corrections cannot avoid the consequences of Hawking’s argument [2]. An analytical model incorporating ‘small’ corrections accounting for the correlations as outlined in [8] has been demonstrated in [9], reestablishing the results of [2]. In the following section, we take a brief review of the bound on entanglement entropy proposed by Mathur, closely following [2].

**IV. Small correction to Hawking state - Mathur’s bound**

Let us assume that the created pair at each time-step of evolution is not invariably in the Hawking state, rather it can be in any state of the space spanned by the basis states

$$S^{(1)} = \frac{1}{\sqrt{2}} |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} + \frac{1}{\sqrt{2}} |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}$$

(5)

and

$$S^{(2)} = \frac{1}{\sqrt{2}} |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} - \frac{1}{\sqrt{2}} |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}.$$  

(6)

Here we deliberately choose to avoid the subspace spanned by the states $|0\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}$ and $|1\rangle_{c_{n+1}} |0\rangle_{b_{n+1}}$, because there is not much physical explanation for pair creation in such states. Moreover, a four dimensional space considering all these four states as basis states has been considered in [9], and it shows no result essentially different from that obtained from a two dimensional analysis.

The complete system consists of the matter $M$, inside quanta $c_i$ and outside quanta $b_i$. Let us choose a basis $|\psi_i\rangle$ for the subsystem comprising matter $M$ and inside quanta $c_i$ and another basis $|\chi_i\rangle$ for the radiation subsystem comprising of the $b_i$ quanta. Then the state of
the complete system can be expressed as

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_{m,n} C_{m,n} \psi_m \chi_n.$$  \hspace{1cm} (7)

We can always perform Schmidt decomposition to express this state as

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_i C_i \psi_i \chi_i.$$ \hspace{1cm} (8)

At next time-step of evolution, the $b_i$ quanta move farther apart from the vicinity of the hole. Since the hole can no longer influence their evolution, we consider that no further evolution takes place for the outgoing quanta. The created pair can be in a superposition of the states $S^{(1)}$ and $S^{(2)}$. Hence the state $\psi_i$ can evolve into

$$\psi_i \rightarrow \psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)}$$

where the state $\psi_i$ has been expressed as the tensor product of the state $\psi_i^{(i)}$ representing $\{M_i, c_i\}$ subsystem and $S^{(i)}$ representing the newly created pair. Since $S^{(1)}$ and $S^{(2)}$ are orthonormal states, unitarity requires that

$$\|\psi_i^{(1)}\|^2 + \|\psi_i^{(2)}\|^2 = 1.$$ \hspace{1cm} (9)

In leading order case, newly created pair is invariably in the state $S^{(1)}$; hence $\psi_i^{(1)} = \psi_i$ and $\psi_i^{(2)} = 0$.

Now,

$$|\Psi_{M,c}, \psi_b(t_{n+1})\rangle = \sum_i C_i [\psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)}] \chi_i$$

$$= \left[\sum_i C_i \psi_i^{(1)} \chi_i\right] S^{(1)} + \left[\sum_i C_i \psi_i^{(2)} \chi_i\right] S^{(2)}$$

$$= \Lambda^{(1)} S^{(1)} + \Lambda^{(2)} S^{(2)}$$ \hspace{1cm} (10)

where $\Lambda^{(1)} = \sum_i C_i \psi_i^{(1)} \chi_i$, $\Lambda^{(2)} = \sum_i C_i \psi_i^{(2)} \chi_i$.

Entanglement entropy of the $\{b\}$ quanta

$$S_{b_n} = -\text{tr} \rho_{b_n} \log \rho_{b_n}$$

$$= \sum_i |C_i|^2 \log |C_i|^2 = S_0.$$ \hspace{1cm} (11)

Since earlier emitted outside quanta can no longer be influenced, we have the same entanglement entropy of $\{b\}$ quanta at time-step $t_{n+1}$.

Now, entanglement entropy of the pair $(b_{n+1}, c_{n+1})$ with the rest of the system is given by

$$S(b_{n+1}, c_{n+1}) = -\text{tr} \rho_{b_{n+1}, c_{n+1}} \log \rho_{b_{n+1}, c_{n+1}}.$$ \hspace{1cm} (12)
Density matrix for the system \((b_{n+1}, c_{n+1})\) is

\[
\rho_{b_{n+1}, c_{n+1}} = \begin{pmatrix}
\langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\
\langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle
\end{pmatrix}.
\tag{15}
\]

Again, normalization of \(|\Psi_{\text{d}, \psi_{b}(t_{n+1})}\rangle\) requires

\[\|\Lambda^{(1)}\|^2 + \|\Lambda^{(2)}\|^2 = 1.\]

Mathur defined the correction to leading order Hawking state to be ‘small’ in the sense that \(\|\Lambda^{(2)}\|^2 < \epsilon\), where \(\epsilon \ll 1\) [2]. This definition implies that there is very small admixture of the \(S^{(2)}\) state with the \(S^{(1)}\) state when new particle pairs are generated. Under such ‘small’ departure from leading order semi-classical Hawking analysis, Mathur showed that entanglement entropy at each time-step increases by at least \(\log 2 - 2\epsilon\) [2]. Since \(\epsilon\) is a very small number, by definition, there is still order unity increase in entanglement entropy at each time-step, when ‘small’ corrections are allowed. This result has been exemplified by a simple model incorporating small correlations between quanta created in consecutive steps [9]. Some other toy models of evaporation have been studied in [10], which also conform to this result.

V. Generalization of Mathur’s bound

In this section, we generalize Mathur’s bound on entanglement entropy by relaxing the ‘smallness’ condition. We establish a more rigorous lower bound as well as an upper bound on change in entanglement entropy at each time-step. To facilitate the derivation, we first derive two lemmas leading to the derivation of the final result as a theorem.

Let us define some quantities for convenience:

\[
\langle \Lambda^{(2)} | \Lambda^{(2)} \rangle = \epsilon^2, \tag{16}
\]

\[
\langle \Lambda^{(1)} | \Lambda^{(1)} \rangle = 1 - \epsilon^2, \tag{17}
\]

\[
\langle \Lambda^{(1)} | \Lambda^{(2)} \rangle = \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle = \epsilon_2, \tag{18}
\]

\[
\gamma^2 = 1 - 4[\epsilon^2(1 - \epsilon^2) - \epsilon_2^2]. \tag{19}
\]

Lemma 1: Entanglement entropy of the newly created pair is given by

\[
S(p) \leq \sqrt{1 - \gamma^2} \log 2.
\]

Proof: Reduced density matrix for the pair

\[
\rho_p = \begin{pmatrix}
1 - \epsilon^2 & \epsilon_2 \\
\epsilon_2 & \epsilon^2
\end{pmatrix}.
\tag{20}
\]
Eigenvalues of this matrix are: $\lambda_1 = \frac{1+\gamma}{2}$ and $\lambda_2 = \frac{1-\gamma}{2}$. Hence entanglement entropy of the pair is

$$S(p) = -tr\rho_p \log \rho_p = -\sum_{i=1}^{2} \lambda_i \log \lambda_i$$

$$= \log 2 - \frac{1}{2}[(1 + \gamma) \log(1 + \gamma) + (1 - \gamma) \log(1 - \gamma)]. \quad (21)$$

It can be shown easily for $0 \leq x \leq 1$ that

$$(1 - x^2) \log 2 \leq \log 2 - \frac{1}{2}[(1 + x) \log(1 + x) + (1 - x) \log(1 - x)] \leq \sqrt{1 - x^2} \log 2. \quad (22)$$

Now, the result follows from (22).

**Lemma 2:**

$$S(b_{n+1}) = S(c_{n+1}) \leq \sqrt{1 - 4\epsilon_2^2} \log 2 \quad (23)$$

*Proof:* The complete state of the system after creation of the first pair is

$$|\Psi_{M,ct}; \psi_b(t_{n+1})\rangle = \left|0\rangle_{cn+1}|0\rangle_{bn+1} \frac{1}{\sqrt{2}}((\Lambda^{(1)} + \Lambda^{(2)})

+ |1\rangle_{cn+1}|1\rangle_{bn+1} \frac{1}{\sqrt{2}}((\Lambda^{(1)} - \Lambda^{(2)})]. \quad (24)$$

Now, the reduced density matrix describing $c_{n+1}$ or $b_{n+1}$ quanta is

$$\rho_{b_{n+1}} = \rho_{c_{n+1}}$$

$$= \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} (\Lambda^{(1)} + \Lambda^{(2)}) & (\Lambda^{(1)} - \Lambda^{(2)}) \\
(\Lambda^{(1)} - \Lambda^{(2)}) & (\Lambda^{(1)} - \Lambda^{(2)}) \end{bmatrix}$$

$$= \begin{bmatrix}
\frac{1 + 2\epsilon_2}{2} & 0 \\
0 & \frac{1 - 2\epsilon_2}{2} \\
\end{bmatrix}. \quad (25)$$

Then, entanglement entropy of the $c_{n+1}$ or $b_{n+1}$ quanta is

$$S(b_{n+1}) = S(c_{n+1}) \leq \log 2 - \frac{1 + 2\epsilon_2}{2} \log(1 + 2\epsilon_2) - \frac{1 - 2\epsilon_2}{2} \log(1 - 2\epsilon_2). \quad (26)$$

Now the result follows directly from (22).

With the use of these lemmas, we now prove our desired result as a theorem.
Theorem 1: Change of entanglement entropy from time-step $t_n$ to $t_{n+1}$ is restricted by the following bound:

$$1 - 4\epsilon^2 - \sqrt{1 - \gamma^2} \leq \frac{\Delta S}{\log 2} \leq \sqrt{1 - 4\epsilon^2}$$

(28)

where $\Delta S = S(b_{n+1}, \{b\}) - S(\{b\})$.

Proof: Let us assume $A = \{b\}, B = b_{n+1}, C = c_{n+1}$.

Using strong subadditivity inequality [11] we have,

$$S(A) + S(C) \leq S(A, B) + S(B, C)$$

$$\Rightarrow S(\{b\}) + S(c_{n+1}) \leq S(\{b\}, b_{n+1}) + S(b_{n+1}, c_{n+1})$$

$$\Rightarrow \Delta S \geq (1 - 4\epsilon^2) \log 2 - \sqrt{1 - \gamma^2} \log 2.$$  

(29)

Inequality (29) follows from using Lemma 1 and Lemma 2.

Now using subadditivity inequality [12] we have,

$$S(A) + S(B) \geq S(A, B)$$

$$\Rightarrow S(\{b\}) + S(b_{n+1}) \geq S(\{b\}, b_{n+1})$$

$$\Rightarrow \Delta S \leq \sqrt{1 - 4\epsilon^2} \log 2$$

(30)

Inequality (30) follows from Lemma 2.

The result follows from combining (29) and (30).

Theorem 1 provides an upper as well as lower bound on change in entanglement entropy at each time-step. Mathur’s bound on the same is only a weaker lower bound and is based on ‘smallness’ of correction to leading order Hawking state. On the other hand, (28) has been derived without any ‘smallness’ constraint on the parameters. Thus this inequality serves two very important purposes that Mathur’s inequality fails to do – (a) it puts an upper bound on the change in entanglement entropy, i.e. how far entanglement entropy can non-trivially increase at each time-step in presence of some correction factor, and (b) it provides us the opportunity to decide the ‘amount’ of correction to pair production in leading order Hawking state, and quantify the change in entanglement entropy accordingly.

Mathur’s inequality tells us that entanglement entropy must increase by at least $\log 2 - 2\epsilon$ when ‘small’ corrections to leading order Hawking state is allowed. However, it doesn’t tell us anything about how far it can increase. If we want to have any hope of avoiding the informataion paradox, a necessary condition is to limit the increase of entanglement entropy.
at each stage. Feature (a) of (28) serves this purpose. The utility of feature (b) will be more evident in the next section.

The lower bound on change in entanglement entropy is nontrivial and stronger than it may seem. To illustrate this, let us find another bound by straightforward use of the Araki-Lieb inequality [13]. Taking \( A = \{b\}, B = b_{n+1} \), we employ the Araki-Lieb inequality

\[
S(A, B) \geq |S(A) - S(B)| \geq S_0 - \log 2\sqrt{1 - 4\epsilon^2}
\]

\[
\Rightarrow S(\{b\}, b_{n+1}) - S_0 \geq -\log 2\sqrt{1 - 4\epsilon^2}
\]

\[
\Rightarrow \Delta S \geq -\log 2\sqrt{1 - 4\epsilon^2}.
\]

The proposed bound in (28) is stronger than that trivially emerges here. This can be seen by letting \( \epsilon \to 0 \), when the inequality in (28) approaches unity from both ends but the trivial lower bound obtained from Araki-Lieb inequality lies at a wasted boundary of \( -\log 2 \).

VI. NECESSARY CONDITION FOR INFORMATION RETRIEVAL

In this section, we will derive the necessary condition for retrieving fallen information from black hole radiation. We will find the optimum range of values for the parameters that can facilitate information transfer from black hole.

Don Page had shown that if information paradox is to be avoided, entanglement entropy must start to decrease at some step of evaporation and finally fall to zero at the end [14]. This condition requires that the lower bound of (28) goes negative at some time-step. In that case, we have

\[
4\epsilon^2(1 - 4\epsilon^2) > 1 - 4\epsilon^2(1 - \epsilon^2)
\]

after replacing \( \gamma^2 = 1 - 4[\epsilon^2(1 - \epsilon^2) - \epsilon^2] \) and some manipulation.

Now, LHS of (33) has maximum value of \( \frac{1}{4} \); therefore RHS must have a value lower than this.

\[
1 - 4\epsilon^2(1 - \epsilon^2) < \frac{1}{4}
\]

\[
\Rightarrow \frac{1}{2} < \epsilon < \frac{\sqrt{3}}{2}.
\]

This gives us bounds for the parameter \( \epsilon \). Recall that the quantity \( \epsilon \) represents the ‘amount’ of admixture of the \( S^{(2)} \) state with the Hawking state \( S^{(1)} \) in pair creation. We have derived the range of allowed admixture that can actually decrease the entanglement entropy. This result is in conformity with Mathur’s theorem, which states that ‘small’ amount of admixture cannot reduce entanglement entropy. We have shown that it is, however, in principle, possible
to reduce entanglement entropy by allowing ‘not-so-small’ to ‘large’ correction to Hawking state. It is important to remember that this result is a necessary condition for decreasing entanglement entropy. One must check for the sufficiency of the condition before modeling black hole evaporation that respects conservation of information.

Now, let us find the bounds for the remaining parameter $\epsilon_2$. It follows from (33):

$$16\epsilon_2^4 - 4\epsilon_2^2 + 1 - 4\epsilon^2(1 - \epsilon^2) < 0. \quad (35)$$

Solving (35) we get the desired bound for $\epsilon_2$ in terms of $\epsilon$:

$$1 - \sqrt{16\epsilon^2(1 - \epsilon^2) - 3} < \epsilon_2 < 1 + \sqrt{16\epsilon^2(1 - \epsilon^2) - 3}. \quad (36)$$

However, it can be shown using Cauchy-Schwarz inequality that

$$\epsilon_2 \leq \epsilon \sqrt{1 - \epsilon^2}. \quad (37)$$

This leaves the possibility open that $\epsilon_2$ can be arbitrarily small. However, (36) restricts the value of $\epsilon_2$ with a lower bound. Recall that $\epsilon_2$ represents the overlap between $\Lambda^{(1)}$ and $\Lambda^{(2)}$. The value of $\epsilon_2$ depends on the value of $\epsilon$. It is possible for $\epsilon_2$ to become zero for certain value of $\epsilon$, but for other allowed values of $\epsilon$ it must be a positive number. This implies that $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are not necessarily orthogonal in all cases and the overlap between them cannot be arbitrarily small.

**VII. Conclusion**

A necessary condition for maintaining unitarity and thus recovering information from Hawking radiation is established. The condition shows that although ‘small’ correction cannot help to decrease entanglement entropy, ‘not-so-small’ to somewhat ‘large’ correction might be able to do it. The precise limit of such correction is found by the bound on the parameter $\epsilon$. This result helps us to relate the change in entanglement entropy to different amount of corrections to leading order Hawking state. Based upon this bound and upon verification of its sufficiency, black hole evaporation models can be designed that respect unitarity and conserve information.

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