Exact eigenfunctions of FQHE systems at fractional filling factors $\nu = \frac{1}{q}$. I. Formal results.

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Eigenstates of the FQHE hamiltonian problem after to be projected on the $LLL$ are determined for filling factors $\nu = 1/q$, with $q$ an odd number. The solutions are found for an infinite class of finite samples in which the Coulomb potential is periodically extended. Therefore, a thermodynamic limit solution is also identified. The results suggest the presence of integrability properties in FQHE systems. The many particle states are simple Slater determinants constructed with special single particle states. These orbitals are defined as powers of order $q$ of composite fermion like wavefunctions associated to a reduced magnetic field $B/q$. At the same time, those composite fermion states were obtained by factorizing and canceling fixed position (quasi-momentum independent) zeros in previously derived exact Hartree-Fock orbitals. A formula for the energy per particle of the FQHE states is given for finite samples as well as for the thermodynamic limit state. As a side result, the same composite fermions like orbitals are employed to construct variational wavefunctions of the system, showing zeros of order $q$ as two electrons approach each other, as Laughlin states do. The long range spatial correlation associated to the starting HF solutions may further reduce the energy of these states.

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I. INTRODUCTION

The Fractional Quantum Hall effect has been a central theme of investigation in condensed matter physics for decades. Up to now its understanding rests on Laughlin’s variational approach for Landau level filling fractions of the form $1/q$, for $q$ odd, and its generalization to general fractional fillings developed by Jain. The wave functions proposed by these authors are not eigenstates of the system hamiltonian although their energy is quite close to the true ground state, as numerical calculations have shown. An alternative approach to FQHE originated in the special Hartree-Fock (HF) solutions proposed by F. Claro many years ago, which although lacking correlations, capture the essential odd-denominator rule observed in the fractional quantum Hall regime. Later, further work (Refs. ) has shown promising properties of such solutions, raising the hope that by adding electron correlations one could approach the true ground state of the system, possibly leading to the long-standing idea of a weak dynamical breaking of the translation symmetry.

In this work, in first place, we construct exact eigenstates of the restricted to the lowest Landau level ($LLL$) FQHE problem when the Coulomb potential is periodically extended. These wavefunctions corresponds to filling factors of the form $\nu = 1/q$, with $q$ an odd integer. They satisfy periodic boundary conditions, and the Coulomb potential is defined to have the same periodicity. The possibility for the determination of these exact wavefunctions is offered by

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II. EXACT SOLUTIONS OF THE FQHE PROBLEM

Let us consider a 2-dimensional electron gas (2DEG) in the presence of a perpendicular magnetic field $\mathbf{B}$ such that a fraction $\nu = 1/q$ of the lowest Landau level ($LLL$) is filled, where $q$ is an odd integer. The electronic systems will be periodically extended in what follows to satisfy magnetic periodic von Karman conditions in the lattice generated by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\frac{\mathbf{L}_1}{q} = \frac{N}{q} \mathbf{a}_1, \quad \frac{\mathbf{L}_2}{q} = \frac{N}{q} \mathbf{a}_2.$$  

Here $N$ is an even multiple of $q$, such that precisely one flux quanta pierces the cell of sides $\mathbf{L}_1/(qN)$ and $\mathbf{L}_2/N$. The unit cell vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ define the smaller spatial magnetic translations under which the HF orbitals motivating this work, were eigenfunctions (see Appendix A for further definitions and conventions).
As it was mentioned in the Introduction, the Coulomb potential between particles will be periodically extended to be invariant under translations in the same lattice defining the von Karman periodicity of the electron wavefunctions

\[ R_q = R_{q,1} \frac{Na_1}{q} + R_{q,2} \frac{Na_2}{q} \]
\[ = R_{q,1} \frac{L_1}{q} + R_{q,2} \frac{L_2}{q} \]

in which \( R_{q,1} \) and \( R_{q,2} \) are arbitrary integers. Then, the extended Coulomb potential has the expression

\[ V_C(x - x') = \sum_{n_q} \frac{e^2}{|x - x' + R_q|} \]

which is periodic in the lattice \( R_q \). After projection on the LLL, the Coulomb Hamiltonian can be written in the Landau gauge, in the form (See Appendix A and Ref. [5])

\[ H = \frac{1}{A_c} \sum_1^{2\pi} \sum_{i<j} 2\pi e^2 r_i r_j \exp(-\frac{l l^*}{2r_i r_j}) T(1, z_i, z^*, -1, z_j), \]

\[ A_c = \frac{n \cdot L_1 \times L_2}{q^2}, \]

where the magnetic translation operators in an arbitrary vector \( v \) are given by (See Appendix A)

\[ T(v, z^*) = \exp(-\frac{(v^2_2 - v^2_1)}{2} + i z^*) \exp(v^* \partial / \partial z^*), \]

and the vectors \( l \) are defined in terms of the normal vector \( n \) and the discrete reciprocal lattice momenta (associated to the von Karman lattice) \( k \) as

\[ l = r_o^2 n \times k, \]
\[ k = q_1 \frac{1}{q} s_1 + q_2 \frac{1}{q} s_2, \]
\[ q_1, q_2 = -\infty, ..., -2, -1, 0, 1, 2, ..., \infty, \]

in which the vectors \( s_1 \) and \( s_2 \), are defined in Appendix A. Technical definitions and a derivation of the expression of the above periodically extended Hamiltonian can be also found in Appendix A.

Let us now consider the following Slater wavefunction for a system of \( N_e = (\frac{N}{a})^2 \) electrons,

\[ \Psi(z_1, z_2, ..., z_{N_e}) = \text{Det}[\Theta_k(z^*_j)], \]
\[ \text{Det}[\Theta_k(z^*_j)] = \sum_p (-1)^p \Theta_{1,1'}(z^*_1) \Theta_{1,2'}(z^*_2) ... \Theta_{1,N_e'}(z^*_N), \]

where \( p \) is the order of a given permutations \( P \) of the set of numbers \( \{1, 2, 3, ..., N_e\} \), defined by \( P\{1, 2, 3, ..., N_e\} = \{1^p, 2^p, 3^p, ..., N_e^p\} \). The single particle orbitals employed to construct the determinant have the expressions

\[ \Theta_k(z^*) = \exp(-i \kappa z^*)(\chi_k(z^*))^q, \]
\[ l = r_o^2 n \times k = -(\frac{q_2}{N}) a_1 + (\frac{q_1}{N}) a_2, \]
\[ q_1 = -\frac{N}{2}, -\frac{N}{2} + 1, ..., 0, ..., \frac{N}{2} - 1, \]
\[ q_2 = -\frac{N}{2}, -\frac{N}{2} + 1, ..., 0, ..., \frac{N}{2} - 1, \]

and \( \kappa = -\pi q(q-1)/a \). Note that the quantum numbers \( I \) (or their equivalent \( k \)) of the set of one particle functions \( \Theta_1 \) take \( N_e = N^2 / q^2 \) values. That is, the subspace formed by this set of functions is filled in the proposed may particle
state. Since the total number of flux quanta traversing the unit cell of the periodicity region is \( \Phi = q \frac{\pi^2}{2} \), the filling fraction of the ansatz state is just \( \nu = 1/q \). The functions \( \chi \) entering the above definition are defined as follows,

\[
\chi_q(z^*) = \exp(i \frac{q a_1}{a} z^*) \times \prod_{R} \vartheta_1(\frac{\pi}{L} (z^* - R^* - C_q)) - \tau^*),
\]

\[
C_q = \frac{a}{2\pi} (q a_1 \tau^* - q a_2) + \frac{q \tau^* a}{2},
\]

where the vectors \( R \) (note that the vectors will be designed either by their boldface symbols of by usual symbols associated to their complex representations: See Appendix A) defining the product, and \( q \) are given by

\[
R = r_1 a + r_2 a_2, \quad R^* = r_1 a + r_2 a^*_2, \quad a = a \exp(i \sqrt{3}),
\]

\[
r_1, r_2 = -\frac{N}{2}, ..., 0, ..., \frac{N}{2} - 1,
\]

\[
q = \frac{q_1}{N} s_1 + \frac{q_2}{N} s_2,
\]

\[
q_1, q_2 = -\frac{N}{2}, ..., -\frac{N}{2} + 1, ..., 0, ..., \frac{N}{2} - 1,
\]

\[
\tau^* = \frac{a_2^*}{a},
\]

and \( \vartheta_1 \) is the elliptic Theta function

\[
\vartheta_1(u|\alpha) = 2 \sum_{n=1}^{\infty} (-1)^n \exp(i \pi \alpha (n - \frac{1}{2})^2) \sin((2n - 1)u), \quad \text{Im}(\alpha) > 0.
\]

The above functions \( \chi \) are representations in terms of the Elliptic Theta functions of the composite fermion like states derived from the exact HF single particle solutions discussed in the Ref.\textsuperscript{11,12,13,14,15,16}, after factorizing and canceling zeros situated at momentum independent positions. Note that the momenta argument of the functions \( \chi \) in he definition of the orbitals in (11) has been divided by \( q \). This selection was heuristically motivated by considering the power \( q \) of the composite fermion should be expected to determine a momentum of the single particle wavefunction being \( q \) times greater than the one associated to each of the \( q \) composite fermion constituent blocks.

It should be noted, that the case of HF states associated to a crystal periodicity showing one electron per unit cell (\( \gamma = 1 \)) and \( \nu = 1/q \) was considered here. In this sense, it can be underlined that a similar discussion to the one given here seems possible to be done for the case (\( \gamma = \frac{1}{4} \)) and \( \nu = 1/q \).

In the next subsection we will show that the functions \( \Theta_l \) satisfy the following closure properties under a magnetic translation in an arbitrary vector \( \text{of the form} \ l' \):

\[
T(l', z^*) \Theta_l(z^*) = \exp(\Phi_l(l')) \Theta_{|l-l'|_{\text{red}}}(z^*),
\]

where \( |l-l'|_{\text{red}} \) means the vector equivalent to \( l - l' \) modulo an element of the lattice generated by the unit cell vectors \( a_1 \) and \( a_2 \). The satisfaction of the periodic boundary conditions in the von Karman periodicity region of the functions \( \Theta_l \) will be shown in Appendix B.

### A. Action of the magnetic translations on the functions \( \Theta_l(z^*) \)

Let us now argue that the space generated by the set of functions \( \Theta_l(z^*) \) for all \( l \) values of the quantum numbers is closed under the action of arbitrary magnetic translations in any vector of the form \( l' \) which are the spacial shift defining the Hamiltonian in (3). This action can be expressed as follows

\[
T(l', z^*) \Theta_l(z^*) = \exp(-\frac{i(l')_2}{r_o} (l'_o^2 + z^*) - i\varpi(z + l'^*)) \times (\chi_q(z^* + l'^*))^q.
\]

But, the appearing power \( q \) of the functions \( \chi \) can be explicitly written in the form
\[
(\chi_\frac{1}{4}(z^* + l^*))^q = \exp\left(\frac{i(1)_2}{r_o^2} l^* + \frac{i(1)_1}{r_o^2} z^*\right) \left(\prod_{R} \theta_1(\frac{\pi}{L}(z^* - R^* - \frac{q}{2}a_2^* - (l^* - l'^*)) - \tau)\right)^q.
\] 
(25)

After substituting this expression in (24), it follows

\[
T(\textbf{l}', z^*) \Theta_1(z^*) = \exp\left(-\frac{i(1)_2}{r_o^2} l'^* + \frac{i(1)_1}{r_o^2} l^* - i \varpi l'^*\right) \times
\]
\[
\exp\left(-i \varpi z^*\right) \left(\prod_{R} \theta_1(\frac{\pi}{L}(z^* - R^* - \frac{q}{2}a_2^* - q(l^* - l'^*)) - \tau)\right)^q,
\]
(26)

\[
= \exp\left(-\frac{i(1)_2}{r_o^2} l'^* + \frac{i(1)_1}{r_o^2} l^* - i \varpi l'^*\right) \times \Theta_{1-1}(z^*).
\]
(27)

As it can be noted, the last line in the above equation is the same analytic expression defining the function \(\Theta_1\), but in which the shifted quantum number value \(1 - \textbf{l}'\), may lay outside the set defining the allowed values of such quantum numbers. However, the difference \(1 - \textbf{l}'\) can be expressed in terms of a proper value of a quantum number modulo a linear combination of some basic quasi periods of the functions \(\Theta\). On another hand the values \(l^* - l'^*\) can always be written as

\[
l^* - l'^* = [l^* - l'^*]_{\text{red}} - t_1 a - t_2 a_2^*.
\]
(28)

where \([l^* - l'^*]_{\text{red}}\) has the quantum number form

\[
[l^* - l'^*]_{\text{red}} = \frac{r_1}{q} a + \frac{r_2}{q} a_2^*,
\]
(29)

\[
r_1, r_2 = -\frac{N}{2q} - \frac{N}{2q} + 1, ..., 0, ..., \frac{N}{2q} - 1,
\]

and \(t_1, t_2\) are defined as functions of \(l^*\) and \(l'^*\) as the unique pair of integer numbers required to express \([l^* - l'^*]_{\text{red}}\) in the above form. The definitions of \(1\) and \(\textbf{l}'\) permits to write for them

\[
t_1 = -\text{Floor}\left[\frac{r_1 - r_1' + \frac{N}{2q}}{\frac{N}{q}}\right],
\]
(30)

where the operation \(\text{Floor}[n]\) as usual, is defined as the nearest integer number being lower or equal that the real number \(n\). Note that the indices \(r_1', r_2'\) take arbitrarily large values because they define the translations entering in the periodic expansion of the Coulomb potential. On another hand the values \(r_1, r_2\) only define the set quantum numbers.

Then, we can write for the main quantity entering in the arguments of the Theta functions as follows

\[
(l^* - l'^*) + \frac{q}{2} a_2^* + R^* = [l^* - l'^*]_{\text{red}} + \frac{q}{2} a_2^* + R^* - t_1 a - t_2 a_2^*.
\]
(31)

Therefore, the shifts in the vectors \(l^*\) produce a new argument which coincides with an argument of another function in the considered set, plus a shift in an integer number of the unit cell vectors \(a_1\) and \(a_2\). However, the functions \(\Theta_1\) under consideration satisfy exact recurrence relations under such displacements. They are direct consequences of the usual transformation properties of the Theta functions under shifts in their quasiperiods. The derivation of these relations is presented in Appendix B. In order to start making use of this property, the expression (23) for the action of the translations on the functions \(\Theta_1\) can be written in the following form

\[
T(\textbf{l}', z^*) \Theta_1(z^*) = \exp\left(-\frac{i(1)_2}{r_o^2} l'^* + \frac{i(1)_1}{r_o^2} l^* - i \varpi l'^*\right) \times
\]
\[
\exp\left(-i \varpi z^*\right) \exp\left(\frac{i(1)_2}{r_o^2} l'^* - \frac{i(1)_1}{r_o^2} I\right) \times \left(\prod_{R} \theta_1(\frac{\pi}{L}(w^* - R^*)) - \tau^*\right),
\]
(32)

\[
\Omega(w^*) = \prod_{R} \theta_1\left(\frac{\pi}{L}(w^* - R^*)\right) - \tau^*,
\]
(33)

\[
v^*(l', l, z^*) = z^* - \frac{q}{2} a_2^* - [l^* - l'^*]_{\text{red}}.
\]
(34)
Then, we can make use of the relation \((31)\) in Appendix B, for the above defined functions \(\Omega\)

\[
\Omega(v^* + m^*) = \exp(i\pi m_2 + 2i\pi m_2 \frac{v^*}{a} + i\pi \tau^* m_2(m + 1)) \Omega(v^*),
\]

(35)
in which the vector \(\mathbf{m}\) for our situation should be chosen as \(\mathbf{m} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2\) being equal to \(t_1 \mathbf{a}_1 + t_2 \mathbf{a}_2\). Applying this formula for transforming the product \(P_1\) of \(\Omega\) functions appearing in the previous expression:

\[
P_1 = (\Omega(v^*(l', l, z^*)) + t_1 a + t_2 a_2) \right)^q
\]

(36)

we can write the shifted \(\Omega\) functions as multiplicative factors of the original ones as follows

\[
\Omega(v^*(l', l, z^*) + t_1 a + t_2 a_2) = \exp(i\pi t_2 + 2i\pi t_2 \frac{v^*(l', l, z^*)}{a} + i\pi t_2(1 + t_2))\Omega(v^*(l', l, z^*)). \tag{37}
\]

Then, \(P_1\) have the expression

\[
P_1 = \exp(i\pi q t_2 + 2i\pi q t_2 \frac{v^*(l', l, z^*)}{a} + i\pi q t_2(1 + t_2)) \times
\]

(38)

\[
((\Omega(v^*(l', l, z^*)) \right)^q).
\]

Therefore, the action of the translations on the function \(\Theta_1\) can be expressed as

\[
T(l', z^*) \Theta_1(z^*) = \exp(-i(\frac{1}{2})_2 l'^* + i(\frac{1}{2})_2 l'^* - i\pi d'^* - i\pi z'^*)
\]

\[
\exp\left(\frac{i}{r_2}(q - (1 - l')_{\text{red}})z^* - \frac{i}{r_2}(t_1 \frac{a_1}{q} + t_2 \frac{a_2}{q})_2 z^*ight)
\]

\[
\exp(i\pi t_2 + 2\pi q t_2 \frac{a}{a_2} (z^* - + \pi t^* \pi t_2(1 + t_2)) \times
\]

\[
\left(\prod_R \theta_1(\frac{\pi}{L}(z^* - R^* - \frac{a_2}{a} - [l^* - l']_{\text{red}}) - \tau^*)\right)^q. \tag{39}
\]

After employing the following identity

\[
(a_2)_2 a = q \times 2\pi r_2^2,
\]

which reflects that exactly \(q\) flux quanta pass through the unit cell formed by the vectors \(a_1\) and \(a_2\), by also noticing that \((l_{\text{red}} a_2) = 0\), because the vector \(a_1 = a(1, 0, 0)\), the following closure relation arises

\[
T(l', z^*) \Theta_1(z^*) = \exp(-i(\frac{1}{2})_2 l'^* + i(\frac{1}{2})_2 l'^* - i\pi d'^* - i\pi z'^*)
\]

\[
\exp(i\pi t_2 + 2\pi q t_2 \frac{a}{a_2} (q - (1 - l')_{\text{red}}) + \pi t^* \pi t_2(1 + t_2)) \times
\]

\[
\exp(-i\pi z'^*) \exp\left(\frac{i}{r_2}(q - (1 - l')_{\text{red}})z^* \prod_R \theta_1(\frac{\pi}{L}(z^* - R^* - \frac{a_2}{a} - q[l^* - l']_{\text{red}}) - \tau^*)\right)^q
\]

(40)

The space independent multiplicative factor has the formula

\[
\exp(\Phi_1(l')) = \exp(-i(\frac{1}{2})_2 l'^* + i(\frac{1}{2})_2 l'^* - i\pi d'^*)
\]

\[
\exp(i\pi q t_2 + 2\pi q t_2 \frac{a}{a_2} (q - (1 - l')_{\text{red}}) + \pi t^* \pi t_2(1 + t_2)). \tag{41}
\]

The integer numbers \(t_1\) and \(t_2\) were fixed before in \((30)\) as functions of \(l, l'\) by the condition of enforcing the vector \([l - l']_{\text{red}}\) to be an allowed quantum number.
B. The eigenstates of the FQHE in the LLL for projected Coulomb interaction

Let us consider now the central point in this work. Note that the action of an arbitrary translation entering in the definition \( \Theta \) of the periodic potential, on any of the functions \( \Theta \), reduces to a permutation of the set of quantum numbers times a space independent exponential factor. Thus, the defined subspace of functions on the LLL remains invariant under the class of translations appearing in the expansion of the interaction potential.

Let us recall here that the number of inequivalent translations \( \mathbf{v}' \) in vectors \( \mathbf{v}' \) is \( N^2/q^2 \), which is \( q \) times smaller than the number of flux quanta \( q(N^2/q^2) \) passing though one von Karman periodicity cell of area \( \mathbf{n.L}_1 \times \mathbf{n.L}_2/q^2 \). By inequivalent translations we mean the whole set of translations obtained for arbitrary values of the quantum number \( l \). Note that due to the closure relation, the translations differing from the inequivalent ones in an arbitrary linear combination with a integral coefficients of the vectors \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \), reduce to inequivalent translation times a multiplicative factor. For \( q = 1 \) the set of translations of the form \( \mathbf{v}' \) transforms the LLL level in itself since the number of translations coincides with the number of independent function in this level and any translation implements a permutation of the functions. However, for \( q > 1 \), the number of translations \( \mathbf{v}' \) is again equal to the number of functions \( \Theta \), but this common number is \( q \) times smaller than number of independent functions in the LLL (equal to the number of flux quanta passing the region of area \( \mathbf{n.L}_1 \times \mathbf{n.L}_2/q^2 \)).

Let us now inspect how the Hamiltonian of the system acts over the wavefunction \( \Psi \) defined in (9). Through the help of relation (42) the result of this action can be written in the form

\[
H \Psi(z_1^*, z_2^*, ..., z_{N_c}^*) = \frac{1}{A_c} \sum_{l<j} \sum_{1}^{2\pi e^2 r_0^2} \exp\left(-\frac{ll^*}{2r_0^2}\right) T(l, z_1^*) T(-l, z_2^*) \times \\
\sum_{\mathbf{p}} (-1)^p \Theta_{1,\mathbf{p}}(z_1^*) \Theta_{1,\mathbf{p}}(z_2^*) ... \Theta_{N_c,\mathbf{p}}(z_{N_c}^*) \\
= \frac{1}{A_c} \sum_{l<j} \sum_{1}^{2\pi e^2 r_0^2} \exp\left(-\frac{ll^*}{2r_0^2}\right) \times \\
\sum_{\mathbf{p}} (-1)^p \Theta_{1,\mathbf{p}}(z_1^*) \Theta_{1,\mathbf{p}}(z_2^*) ... \exp(\Phi_{1,\mathbf{p}}(1)) \Theta_{1,\mathbf{p}}(1^*) ... \\
\exp(\Phi_{1,\mathbf{p}}(-1)) \Theta_{1,\mathbf{p}}(-1^*) ... \Theta_{N_c,\mathbf{p}}(z_{N_c}^*). \tag{42}
\]

Let us now recall that the action of the Hamiltonian on an antisymmetric function of the coordinates gives again an antisymmetric function of the coordinates. Therefore, it should be invariant under the operation of adding its expression for all permutations of the arguments \( z_1^*, z_2^*, ..., z_{N_c}^* \), after multiplication by the order of the permutations, and a division by \( N_c! \). However, it can be noted that the action of the two translations involved produce other functions of the same set, showing modified momenta quantum numbers and being multiplied by constant exponential factors. Therefore, when the increment \( \mathbf{l} \) is unable to transform both quantum numbers \( l_1^\mathbf{p} \) and \( l_2^\mathbf{p} \) between themselves modulo a vector of the form \( m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 \) \( (m_1 \) and \( m_2 \) integers), the corresponding antisymmetrized term in the sum over permutations in equation (42) will vanish. This is a result of the fact that, in the mentioned situation, there will be two coincident states in the Slater determinant resulting from the above described auxiliary symmetrization process. Therefore, these properties imply the following eigenvalue equation to hold

\[
H \Psi = E_p \Psi, \tag{43}
\]

\[
E_p = -\frac{1}{A_c} \sum_{l<j} \sum_{1}^{2\pi e^2 r_0^2} \exp\left(-\frac{ll^*}{2r_0^2}\right) \times \\
\exp(\Phi_{1,\mathbf{p}}(1)) + \Phi_{1,\mathbf{p}}(-1)) \times \delta^{(K,P)}(1 - (l_1^\mathbf{p} - l_2^\mathbf{p})), \tag{44}
\]

where \( \delta^{(K,P)}(1) = 1 \) if \( 1 = 0 \) modulo a lattice vector of the form \( m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 \), and vanishes otherwise. Equation (43) is the main result of the present work. It states that \( \Psi \) defined above is an exact eigenfunction of the many electron system in FQHE regime for a periodically extended Coulomb interaction. Note that the solution is obtained for any finite particle systems satisfying the conditions assumed here. Thus, it can be employed to investigate the thermodynamic limit. The evaluation of the energy per particle associated to the obtained solution will be pursued in paper II.

Let us briefly comment here about how the large number of translations entering the definition of the periodic potential in the growing number of particles limit, is however able produce the closure relation which was central in obtaining the argued result.
It can be noted that for \( q = 1 \), the set of translations of the form \( \mathbf{l} \) transforms the LLL level set of functions satisfying the assumed periodic conditions, in itself. That is, the number of translations coincides with the number of independent functions in this level which satisfy the boundary conditions, and any translation implements a permutation of the elements of a special complete basis of eigenfunctions \( \phi_q \) defined in Refs.\textsuperscript{10,11,31}. These orbitals are eigenfunctions of the magnetic translations on a lattice showing exactly one unit of flux per unit cell. The form of this unit cell can be arbitrary, the only requirement on it, is to being pierced by one flux quanta. An important property of those translations which is argued by example in Refs.\textsuperscript{11,31}, is that any magnetic translation in the LLL is equivalent to a shift in the momentum quantum numbers of the functions in the mentioned basis. Let us consider such a one flux quantum unit cell as formed by the vectors \( \mathbf{a}_1/q \) and \( \mathbf{a}_2/q \). The number of such cells in the area of the sample we are considering is exactly \( q(N^2/q^2) \), that is, the number of flux quanta passing through the sample area. In this case, the momenta associated to the complete set of functions \( \phi_q \) in the chosen sample area are\textsuperscript{10,11,31}.

\[
\mathbf{q} = \frac{q_1}{N} \mathbf{s}_1 + \frac{q_2}{q} \mathbf{s}_2,
\]

\[
s_1 = -\frac{\mathbf{n} \times \mathbf{a}_2}{r_0^q}, \quad s_2 = -\frac{\mathbf{n} \times \mathbf{a}_2}{r_0^q},
\]

\[
q_1 = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, 0, \ldots, \frac{N}{2} - 1,
\]

\[
q_2 = -\frac{N}{2q}, -\frac{N}{2q} + 1, \ldots, 0, \ldots, \frac{N}{2q} - 1.
\]

Let us consider now the \( (\frac{N}{q})^2 \) translations generated by the vectors \( \mathbf{l}' \) when \( \mathbf{l}' \) is restricted to be a quantum number of the functions \( \Theta_{\mathbf{l}}' \). Then, the shifts in the momenta \( \mathbf{q} \) produced on the functions \( \phi_q \) after acting on them with a translation \( \mathbf{l}' \) is given by\textsuperscript{11}.

\[
\delta \mathbf{q} = -\frac{2e}{\hbar c} A(\mathbf{l}')
= -\frac{2e}{\hbar c} B\mathbf{n} \times (\mathbf{l}')
= \frac{1}{r_0^q} \mathbf{n} \times \mathbf{l}'
= \frac{1}{r_0^q} \mathbf{n} \times \left( \frac{r_1}{N} \mathbf{a}_1 + \frac{r_2}{N} \mathbf{a}_2 \right)
= (-q_2 s_2 + \frac{r_1}{N} s_1 + \frac{r_1}{q} s_2).
\]

This expression indicates that when \( q > 1 \) there exist a proper subset of the functions \( \phi_q \) that are closed under all the magnetic translations defining the specially periodic Coulomb operator. This conclusion follows form the fact that the shifts \( \delta \mathbf{q} \) are closed under addition, but they only have \( (\frac{N}{q})^2 \) values within the first Brillouin zone, of the total number \( q(N^2/q^2)^2 \) of momenta forming this zone. Therefore, the existence of a close proper subset formed by the functions \( \Theta_{\mathbf{l}} \) within the LLL subject to the considered periodic conditions is allowed.

### C. Formula for the energy per particle of the exact eigenstates

Let us employ in what follows the periodicity conditions for finding a formula for the energy per particle in terms of the \( \Theta_{\mathbf{l}} \) functions, for the FQHE eigenstates. Since the state \( \Psi \) is a Slater determinant the total energy in the von Karman periodicity zone can be expressed in the form

\[
E = \langle \Psi \mid V_c \mid \Psi \rangle
= \int_{A_0} \int_{A_0} d\mu(x)d\mu(x') V_c(x-x') g(x, x'),
\]

\[
g_{NS}(x, x') = \rho(x, x)\rho(x', x') - \rho(x, x')\rho(x', x),
\]

\[
\rho(x, x') = \sum_{i} \Theta_i(x)\Theta_i^*(x'),
\]

\[
\rho(x, x') = \sum_{i} \Theta_i(x)\Theta_i^*(x'),
\]

\[
\Theta_i(x) = \langle x \mid \Theta_i \rangle,
\]

\[\Theta_i(x) = \langle x \mid \Theta_i \rangle.
\]
where $\Theta_l(x)$ as a function of the coordinate vector $x$ is naturally defined by

$$
\Theta_l(x) = \Theta_l(((x)_1, (x)_2)) \equiv \Theta_l(z^*) = \Theta_l((x)_1 - i(x)_2),
$$

and as usual, in the Landau gauge the integration measure is given by

$$
d\mu(x) = d\mathbf{x} \exp\left(-\frac{((x)_2)^2}{r_o^2}\right). \quad (50)
$$

We also assumed here that the functions $\Theta_l(z^*)$ were already normalized in the von Karman periodicity zone.

Let us consider in what follows the following type of integrals

$$
I = \int_{A_c} d\mu(x)V_c(x - x')\Theta_l(x)\Theta_l(x'), \quad (51)
$$

which are building elements of the expression for the energy \textcolor{red}{[17]}. To start, note that the region of integration is limited to a cell of the lattice $R_q$ which was named above as $A_q$. For this purpose, we can rewrite \textcolor{red}{[135]}, after substituting the explicit form of the magnetic translation invariance in the Landau gauge \textcolor{red}{[130]}, as follows

$$
\Theta_l(z^*) = T(R_q, z^*)\Theta_l(z^*) = \exp\left(-i\frac{(R_q)_2R_q^*}{2r_o^2} - i\frac{(R_q)_2(z^* - z)}{r_o^2}\right)\Theta_l(z^* + R_q). \quad (52)
$$

Employing this relation for the two functions $\Theta_l$ and $\Theta_l$ in \textcolor{red}{[51]} and taken into account the periodicity of the potential in the lattice $R_q$, it follows

$$
I = \int_{A_c} d\mu(x)V_c(x - x')\Theta_l(x + R_q)\Theta_l(x + R_q) \times \exp\left(-i\frac{(R_q)_2(R_q^* - R_q)}{2r_o^2} - i\frac{(R_q)_2(z^* - z)}{r_o^2}\right) = \int_{A_c} d\mu(x + R_q)V_c(x - x')\Theta_l(x + R_q)\Theta_l(x + R_q). \quad (53)
$$

This expression explicitly shows that the integral $I$ performed over an arbitrary cell of the lattice $R_q$, give the same result when performed over any of such cells. This property directly check that total energy can be evaluated as an integer limited to the first cell of the lattice $R_q$ as the von Karman conditions implies. Then, the total energy has the formula

$$
E_{N_c} = \langle \Psi | V_c | \Psi \rangle = \int_{A_c} \int_{A_c} d\mu(x)d\mu(x')V_c(x - x')g_{N_c}(x, x'). \quad (54)
$$

The evaluation of the energy per particles of the infinite systems, their thermodynamic limits and the pair correlation functions will be considered in paper II, which is now in preparation.

D. Remarks on the thermodynamic limit

Let us briefly comment in this subsection on the thermodynamic limit of the determined exact wavefunctions. As known, in the large magnetic field regime in which the system is projected in $LLL$ level, the only term in the Hamiltonian is the Coulomb interaction. Here we assumed von Karman periodic conditions in a magnetic field (See \textcolor{red}{[1]}), with spacial periods $L_1/q$ and $L_2/q$. Let us assume as usual, that the systems has a homogeneously distributed jellium which compensates the net charge of all the particles contained in it. In this situation, when the $|L_1/q| = |L_2/q| = L/q = aN/q$ tends to be large in the thermodynamic limit, the interaction between any pair of particles sitting at arbitrary but finite distances from the origin, should approximate the usual Coulomb potential, as $\frac{L}{q}$ becomes very much greater than the distance between the mentioned two particles. This property is expected to be valid because when $L/q$ is very much larger than the distance between the particles, their interaction potential becomes negligible assumed the compensating fields of the jellium homogeneous charges are considered (or
equivalently, imposing that the inter particle potential is assumed as short ranged, by substituting the Coulomb one by a Yukawa screened potential at large distances). In other words, the jellium charge, should reduce the fields of the far away laying image charges to be dipolar or higher multipole short ranged contributions, which should produce vanishing forces in the thermodynamic limit. Therefore, it can be expected the solution found in last section, when considered in the limit of a large number of particles, should describe the FQHE systems when the limit \( L/q \to \infty \) is taken. In this case of large samples the following formula for the energy per particle arises from expression (54) for the total energy of arbitrary finite systems

\[
\epsilon_{FQHE} = \lim_{N_e \to \infty} \left( \frac{1}{N_e} \langle \Psi | V_c | \Psi \rangle \right) = \lim_{N_e \to \infty} \left( \frac{1}{N_e} \int_{A_e} \int_{A_e} d\mu(x)d\mu(x') V_c(x - x') g_{N_e}(x, x') \right).
\] (55)

### III. CONCLUSIONS

Exact eigenstates of a two dimensional electron gas (2DEG) in a magnetic field are constructed for finite samples of arbitrary sizes, and periodic extensions of the Coulomb potential after restricted to the LLL level. Magnetic von Karman periodic boundary conditions are imposed in the Landau gauge. Formulae for the energy per particle and the two particle density matrix of the realistic FQHE state in the thermodynamic limit are also determined by considering the large limit of finite systems. The found states are associated to filling factors \( \nu = 1/q \), for odd \( q \). The results suggest the presence of integrability properties in FQHE problems. The energy per particle and pair correlation functions of the states for the finite system and the thermodynamic limit ones, will be evaluated in a second work (paper II) which is now in preparation. In the case that the energies per particle associated to the wavefunctions result to be lower than the estimated values for the Laughlin variational states, they could surprisingly furnish the ground states of the FQHE problem. However, in the opposite outcome, the determined wavefunctions could be also helpful in describing experimentally detected phases shown by FQHE samples at variable temperature experiences. In any situation, the obtained kind of solution constitutes a valuable primer of exactly solvable model in the context of the FQHE theory. The solutions could also be exact realizations of the early proposed cooperative rings of exchange states. The evaluations of the energy per particle and pair correlation functions of the identified eigenstates will be presented in a related work (paper II) which preparation is in progress.

The special one particle orbitals constructed here, were also employed (See Appendix C) to construct ansatz wave functions for a class of low energy states having a similar structure as the Laughlin’s ones. That is, showing zeros of the form \((z_i - z_j)^q\) when the coordinates of the particles coincide. Then, these states exhibit a similar sort of short range behavior as the Laughlin states. However, since the basic elements in their definition are the HF one particle orbitals, we expect that long range correlations incorporated in the original HF solution can remain in the proposed ansatz states, helping in this way to slightly lower the energy per particle.

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### APPENDIX A: BASIC DEFINITIONS

#### 1. General definitions

In this work, bold symbols represent vectors with components \( \mathbf{v} = (v_1, v_2, v_3) \) and 2-dimensional vectors are equivalently defined by the two complex numbers

\[
v = (v_1 + i v_2), \quad v^* = (v_1 - i v_2).
\]
Let us consider the plane containing the 2DEG. One kind of useful periodicity planar box employed in previous works consists in a parallelogram of sides \( L_1 = N a_1 \) and \( L_2 = N a_2 \), where the unit cell vectors \( a_1 \) and \( a_2 \) are defined by

\[
\begin{align*}
a_1 &= a (1, 0, 0), \quad a_2 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right), \\
n &= (0, 0, 1), \quad a = \frac{4\pi q}{\sqrt{3} r_o},
\end{align*}
\]

where \( q \) is an odd integer, \( n \) is the unit vector normal to the 2DEG plane, and \( r_o = \sqrt{\frac{\hbar c}{|e| B}} \) is the magnetic length.

The Landau gauge \( A = B(-x_2, 0, 0) \) is employed for the vector potential in accordance with Ref. [4]. Note that the cell with sides \( a_1 \) and \( a_2 \) intercepts exactly \( q \) flux quanta of the magnetic field \( B = B(0, 0, 1) \). Also, \( N^2 \) particles laying inside the above defined particular planar box correspond to a filling factor \( \nu = 1/q \). Note that this box will not be the von Karman periodicity box considered for determining the exact solutions, which will have sizes given by \( L_1/q \) and \( L_2/q \). The electric charge \( e \) is assumed to be negative. For convenience, \( N \) will be assumed to be an even positive integer number and also an exact multiple of \( q \). Is helpful to define a spatial lattice given by the vectors

\[
R = R_1 a_1 + R_2 a_2,
\]

in which \( R_1 \) and \( R_2 \) are arbitrary positive or negative integers. The reciprocal lattice cell vectors associated to this lattice has unit vectors

\[
\begin{align*}
s_1 &= -\frac{n \times a_2}{q r_o^2}, \quad s_2 = \frac{n \times a_1}{q r_o^2}, \\
s_i, a_j &= 2\pi \delta_{ij}, \quad i, j = 1, 2.
\end{align*}
\]

The free Hamiltonian of the single planar electron in the Landau gauge will be

\[
H = \frac{1}{2m} (p - \frac{e}{c} A)^2,
\]

where the generators of the magnetic translation symmetry operations are

\[
\mathbf{G} = \nabla - i\frac{e}{\hbar c} A + \frac{i e B}{\hbar c} n \times x,
\]

\[
[G, H] = 0,
\]

\[
[(G)_i, (G)_j] = \frac{i e B}{\hbar c} \epsilon_{ijk} (n)_k.
\]

The structure of the free electron wavefunctions of the LLL in the Landau gauge is

\[
\varphi(x) = \phi(z^*) \exp\left(-\frac{y^2}{2r_0^2}\right), \quad y = (x)_2 \\
z = (x)_1 + (x)_2 i, \quad z^* = (x)_1 - (x)_2 i.
\]

where \( \phi(z^*) \) is an analytic function of \( z^* \). In the axial gauge, the electron states and the vector potential are

\[
\varphi_A(x) = \phi_A(z^*) \exp\left(-\frac{x^2}{4r_0^2}\right),
\]

\[
A_A = \frac{1}{2} B n \times x,
\]

with \( \phi_A(z^*) \) an analytic function of \( z^* \) different from \( \phi_A(z^*) \). The general connection between wavefunctions in both gauges, and between the analytic forms associated with the LLL are

\[
\begin{align*}
\varphi_A(x) &= \exp\left(\frac{i e B}{2\hbar c} (x)_1 (x)_2 \right) \varphi(x), \\
\phi_A(z^*) &= \exp\left(\frac{(z^*)^2}{4r_0^2}\right) \phi(z^*).
\end{align*}
\]
Then, the Landau gauge versions of the Hamiltonian $H$, and the symmetry translations generators $G$, can be obtained from the similarity transformations

$$H_A = \exp(i e B/2\hbar c)(x)_1(x)_2)H \exp(-i e B/2\hbar c)(x)_1(x)_2)$$
$$= \frac{1}{2m}(p - eA)^2 = \frac{\hbar^2}{2m}\Pi^2_A$$
$$G_A = \exp(i e B/2\hbar c)(x)_1(x)_2)G \exp(-i e B/2\hbar c)(x)_1(x)_2)$$
$$= \nabla + \frac{ie}{\hbar c}A_A,$$
$$\Pi_A = \nabla - \frac{ie}{\hbar c}A_A.$$

These quantities have the following properties

$$[(G_A)_i, (\Pi_A)_j] = 0$$
$$[(G_A)_i, (G_A)_j] = \frac{i}{r^2_o}c_{ij}(n)_l,$$
$$[(\Pi_A)_i, (\Pi_A)_j] = -\frac{i}{r^2_o}c_{ij}(n)_l$$
$$[H, G_A] = 0$$
$$T_A(L, x) = \exp(i e B/2\hbar c)(x)_1(x)_2)T(L, x) \exp(-i e B/2\hbar c)(x)_1(x)_2).$$

Helpful quantities are also the lowering and rising operators $\Pi_\pm$ and $G_\pm$ defined by and satisfying the following relations

$$\Pi_\pm = (\Pi_A)_1 \pm (\Pi_A)_2 i,$$
$$G_\pm = (G_A)_1 \pm (G_A)_2 i,$$
$$[H, \Pi_\pm] = \pm \frac{\hbar^2}{mr^2_o}\Pi_\pm, [H, G_\pm] = 0,$$
$$[\Pi_+, \Pi_-] = \frac{2}{r^2_o}, [G_+, G_-] = -\frac{2}{r^2_o}.$$}

They have the explicit forms in terms of the complex representation of the coordinates

$$\Pi_+ = 2\partial_2 - \frac{z}{2r^2_o}, \quad \Pi_- = 2\partial_2 + \frac{z^*}{2r^2_o},$$
$$G_+ = 2\partial_2 + \frac{z}{2r^2_o}, \quad G_- = 2\partial_2 - \frac{z^*}{2r^2_o}.$$}

Now let us consider a property that becomes useful for projecting the Coulomb interaction in the LLL, since it is convenient for studying the high magnetic field limit. Firstly, let us express the scalar product of a 2D coordinate vector $x$ with a vector $Q$, as

$$iQ \cdot x = -2l \frac{ie}{\hbar c}A_A(x),$$
$$l = r^2_o n \times Q.$$}

But the gradient and the axial gauge vector potential can be expressed in terms of the magnetic translation (MT) symmetry $G_A$ and the operator $\Pi_A$ as

$$\nabla = \frac{1}{2}(\Pi_A + G_A),$$
$$\frac{ie}{\hbar c}A_A(x) = \frac{1}{2}(G_A - \Pi_A).$$
Then, the exponential \( \exp(iQ \cdot x) \) can be represented as follows,

\[
\exp(iQ \cdot x) = \exp(-2i \frac{\hbar}{\epsilon c} A_A(x)) = \exp(i \mu_A) \exp(-iG_A) = \exp\left(\frac{1}{2}(l^* \Pi_+ + l \Pi_-) \exp\left(-\frac{1}{2}(l^* G_+ + lG_-)\right)\right) = \exp\left(-\frac{l^*}{4r_o^2}\right) \exp\left(\frac{l^*}{2} \Pi_+ \right) \exp\left(\frac{l}{2} \Pi_- \right) \times \exp\left(-\frac{1}{2}(l^* G_+ + lG_-)\right),
\]

where the well known identity

\[
\exp(A + B) = \exp(A) \exp(B) \exp\left(-\frac{1}{2}[A, B]\right),
\]

valid when \([A, B]\) commutes with \(A\) and with \(B\), has been used. Applying the one particle projection operator in the LLL:

\[
P_o = \sum_n |n\rangle \langle n| \quad \text{being any complete set in the LLL},
\]

it follows that

\[
P_o(i \mu_A) P_o^{(i)} = \exp\left(-\frac{l^*}{4r_o^2}\right) \exp\left(-iG_A\right) = \exp\left(-\frac{l^*}{4r_o^2}\right) \exp\left(-\frac{1}{2}(l^* G_+ + lG_-)\right) = \exp\left(-\frac{l^*}{4r_o^2}\right) \exp\left(-\frac{1}{2}(l^* (2\partial z_i + \frac{z_i}{2r_o^2}) + l(2\partial z_i - \frac{z_i}{2r_o^2}))\right).
\]

2. **LLL projection of the Coulomb interaction operator**

Let us next consider the Fourier expansion of the 2D function \(f(x)\) which is periodic in a parallelepiped with sides \(L_1^g\) and \(L_2^g\).

\[
f(x) = \frac{1}{A_{cell}} \sum_Q f(Q) \exp(iQ \cdot x),
\]

\[
f(Q) = \int d\mathbf{x} \ f(x) \exp(-iQ \cdot x),
\]

\[
\delta^P(x - x') = \frac{1}{A_{cell}} \sum_Q \exp(iQ \cdot (x - x')),
\]

\[
A_{cell} = \mathbf{n} \cdot L_1^g \times L_2^g.
\]

It should noted that here in this subsection \(A_{cell}, L_1^g, \text{and } L_2^g\) are the area and the unit cell vectors of a general periodic lattice. Employing this expansion, the Coulomb interaction Hamiltonian can be expressed as follows,

\[
V_C = \sum_{i < j} \sum_{R_e} e^2 \frac{\epsilon^2}{|x_i - x_j + R_e|} = \frac{1}{A_{cell}} \sum_Q \sum_{i < j} 2\pi e^2 \frac{1}{|Q|} \exp(iQ \cdot (x_i - x'_j)).
\]
the projected Coulomb interaction \( \hat{P}_o(x_1, x_2, \ldots, x_{N_i}) = \prod_{i=1}^{N_i} P_o^{(i)} \),

the projected Coulomb interaction in the axial gauge acquires the form

\[
\hat{V}_{C,A} \equiv \hat{P}_o V_C \hat{P}_o = \frac{1}{A_{\text{cell}}} \sum_{i,j} \frac{2 \pi e^2}{|Q|} P_o(x_i) \exp(iQ \cdot x_i) P_o(x_j) \exp(-iQ \cdot x_j') P_o(x_j),
\]

This operator may be written in the representation given by the analytic factors by considering the following rules expressing how the translation operators transform when extracting the exponential factors from the wavefunctions:

\[
2 \partial z_j - \frac{z_j^*}{2r_o^2} \rightarrow \frac{z_j^*}{r_o^2}, \quad 2 \partial z_j^* + \frac{z_j^*}{2r_o^2} \rightarrow 2 \partial z_j^*.
\]

Then, the projected Coulomb interaction in the axial gauge acquires the form

\[
\hat{V}^\text{Anal}_{C,A} = \frac{1}{A_{\text{cell}}} \sum_{i,j} \frac{2 \pi r_o^2 e^2}{|l|} \exp(-\frac{4z_j^*}{2r_o}) \exp\left(l^* \partial z_j^* - \frac{z_j^*}{2r_o^2}\right) \times
\]

\[
\exp\left(-\frac{4z_j^*}{2r_o} \frac{l^*}{2} \left(\frac{T_A(1,z_j^*)}{T_A(1,z_j^*)}\right) \right)
\]

where \( T_A(1,z^*) \) is the translation operator in the axial gauge acting in the space of analytic functions. After performing the many-particle similarity transformation which transforms axial gauge operators into the Landau gauge, the Coulomb interaction reduced to the \( LLL \) in this latter gauge takes the form

\[
\hat{V}_C^\text{Landau} = \left( \prod_{i=1}^{N_i} \exp(-\frac{4z_i^*}{2r_o^2}) \right) \hat{V}^\text{Anal}_{C,A} \left( \prod_{i=1}^{N_i} \exp\left(\frac{4z_i^*}{2r_o^2}\right) \right)
\]

\[
= \frac{1}{A_{\text{cell}}} \sum_{i,j} \frac{2 \pi r_o^2 e^2}{|l|} \exp\left(\frac{4z_i^*}{2r_o^2}\right) T(-1,z_i^*) T(1,z_i^*)
\]

where now the Landau gauge translation operators acting on the analytic parts of the single particle wavefunctions have the form

\[
T(1,z^*) = \exp\left(-\frac{i}{2r_o^2} l^* \frac{z^*}{r_o^2}\right) \exp\left(l^* \frac{\partial}{\partial z^*}\right).
\]
APPENDIX B: VON KARMAN BOUNDARY CONDITIONS

Let us argue below that the functions $\Theta_l(z^*)$ defined above are invariant under the translations.

$$T\left(\frac{L_1}{q}, z^*\right) = T\left(\frac{Na_1}{q}, z^*\right),$$

$$T\left(\frac{L_2}{q}, z^*\right) = T\left(\frac{Na_2}{q}, z^*\right).$$

This property, implies that those functions satisfy von Karman periodicity condition under translations in the periods $\frac{L_1}{q}, \frac{L_2}{q}$. This result can be directly derived from the relation (40).

Consider first the shift in $\frac{L_1}{q}$. Then, let us fix the parameters in (40) in the form

$$l' = \frac{L_1}{q},$$

$$1 - l' = 1 - \frac{L_1}{q},$$

$$[1 - l']_{\text{red}} = 1,$$

$$t_1 = \frac{N}{q}, t_2 = 0,$$

where it has been employed that, since $\frac{L_1}{q} = Na_1/q$ is a linear combination with integer coefficients of the vectors $a_1$ and $a_2$ thanks to the fact that $N$ was supposed as even and also as a multiple of $q$. Substituting these values in (40) directly leads to the first von Karman boundary condition

$$T\left(\frac{L_1}{q}, z^*\right) \Theta_l(z^*) = \Theta_l(z^*).$$

Similarly for a shift in $\frac{L_2}{q}$, the required parameters in (40) become

$$l' = \frac{L_2}{q},$$

$$1 - l' = 1 - \frac{L_2}{q},$$

$$[1 - l']_{\text{red}} = 1,$$

$$t_1 = 0, t_2 = \frac{N}{q},$$

which after performing a little more complicated algebra implies the second von Karman boundary condition

$$T\left(\frac{L_2}{q}, z^*\right) \Theta_l(z^*) = \Theta_l(z^*).$$

Therefore, the functions $\Theta_l$ are invariant under magnetic translations in the lattice vectors pertaining to $R_q$ as

$$T(R_q, z^*) \Theta_l(z^*) = \Theta_l(z^*).$$

1. An auxiliary relation

Let us consider the product of Theta functions

$$\Omega(v^* + m^*) = \prod_R \theta_1\left(\frac{\pi}{L}(v^* - (R^* + m^*)) - \tau^*, \frac{\pi}{L}(v^*) - \frac{\pi}{L}(R^* + m^*) - \tau^*\right),$$

where $m^*$ is the complex representation of the vector $m = m_1 a_1 + m_2 a_2$, and the product over $R$ runs as before over the vectors
\[ R = R_1 a_1 + R_2 a_2, \]  
\[ R_1, R_2 = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots \frac{N}{2} - 1. \]

The simple properties of the Theta functions under shifts in vectors of the form \( R_1 a_1 \), and the even character of \( N \), make \( \Omega \) independent of \( R_1 \). Therefore

\[
\Omega(v^* + m^*) = \Omega(v^* + m_2 a_2^*) \\
= \prod_R \theta_1\left( \frac{\pi}{L} (v^* - (R_1 a + (R_2 - m_2) a_2^*)) \right) \tau^*. 
\]

Now, it can be noted that \( m_2 \) can be interpreted as a shift in the arguments \( R \) of the factors, defining a modified set of arguments \( R' \). Then, let us write the product \( P \) of all the factors in which the modified arguments \( R' \) are not appearing in the original set \( R \). Assuming \( m_2 > 0 \), this product can be given the form

\[
P = \prod_{R_1}^{m_2} \prod_{j=1}^{m_2} \theta_1\left( \frac{\pi}{L} (v^* - (R_1 a + (\frac{N}{2} - j) a_2^*)) \right) - \tau^*. 
\]

After iterating the usual formula of the \( \theta_1 \) Theta function under a single shift in \( \pi \tau^* \), the transformation rule when the argument is shifted an integer \( k \) times \( \pi \tau^* \), can be written in the form

\[
\theta_1 (u^* + k \pi \tau^* \mid - \tau^*) = (-1)^k \frac{q^{k^2} (\tau^*)}{q^{k^2} (\tau^*)} \exp(2 i k u^* \theta_1 (u^*) \theta_1 (u^* - \tau^*)) \\
q(\tau^*) = \exp(-i \pi \tau^*), \quad k = -\infty, \ldots, -1, 0, 1, \ldots \infty. 
\]

The above expression allows to write \( P \) in the form

\[
P = \prod_{R_1}^{m_2} \prod_{j=1}^{m_2} \left( -1 \right) \frac{2i \pi m_2}{N} \sum_{R_1=-N}^{N} R_1 a + \\
+ 2i \pi \tau^* \sum_{j=1}^{m_2} j \times \prod_{R_1}^{m_2} \prod_{j=1}^{m_2} \theta_1\left( \frac{\pi}{L} (v^* - (R_1 a + (\frac{N}{2} - j) a_2^*)) \right) \tau^*. 
\]

Then, the use of the relations

\[
\sum_{R_1=-N}^{N} R_1 = -\frac{N}{2}, \\
\sum_{j=1}^{m_2} j = \frac{m_2(m_2 + 1)}{2},
\]

reduces the expression for \( P \) to the same product in which the \( m_2 \) shift is absent, after multiplication by an exponential factor, as follows
\[ \mathcal{P} = \prod_{R_1} \prod_{j=1}^{m_2} \theta_1(\pi\left( v^* - \left( R_1 a - \frac{N}{2} a_2^* - ja_2^* \right) \right) - \tau^*) \]

\[ = \exp(i\pi m_2 + 2i\pi m_2 \frac{v^*}{a} + i\pi \tau^*) m_2(m_2 + 1) \times \prod_{R_1} \prod_{j=1}^{m_2} \theta_1(\pi\left( v^* - \left( R_1 a + \frac{N}{2} - ja_2^* \right) \right) - \tau^*). \]

Then, the following resulting relation arises

\[ \Omega(v^* + m^*) = \exp(i\pi m_2 + 2i\pi m_2 \frac{v^*}{a} + i\pi \tau^*) m_2(m_2 + 1) \Omega(v^*), \]  

checking that shifts in the vectors \( m^* \) of the functions \( \Omega \) reproduce the same function times an exponential factor.

**APPENDIX C: ANSATZ STATES FOR THE FQHE PROBLEM SHOWING TRANSITION SYMMETRY BREAKING AT \( \nu = \frac{1}{q} \)**

Let us employ in this section the composite fermion-like basis defined in Refs.\textsuperscript{14,15,16} and in [16], to construct a special class of many-particle state associated with the fractional filling factor \( 1/q \). These wavefunctions show a two particle pair correlation function having the known optimized behavior \( g(x_1 - x_2) \sim c(z_1^* - z_2^*)^q \) which is helpful in reducing the correlation energy. However, the original source of the definitions for the basis functions \( \chi_k \) (which come from HF single particle solutions after extracting the fixed position zeros) leads to expecting the presence of long range correlations further diminishing the energy per particle\textsuperscript{14,15,16}. The ansatz states will be simply defined by

\[ \Psi_G(z_1^*, z_2^*, ...z_{N_e}) = \text{Det}[\Theta_1(z_j^*)]^q, \]

\[ \text{Det}[\Theta_1(z_j^*)] = \sum_{\nu} (-1)^{\nu} \Theta_{1,\nu}(z_1^*) \Theta_{1,\nu}(z_2^*) ... \Theta_{1,\nu}(z_{N_e}^*). \]

These wavefunctions satisfy the boundary conditions

\[ T(L_1/q, z_i^*) \Psi_G(z_1^*, z_2^*, ...z_{N_e}^*) = \Psi_G(z_1^*, z_2^*, ...z_{N_e}^*), \]

\[ T(L_2/q, z_i^*) \Psi_G(z_1^*, z_2^*, ...z_{N_e}^*) = \Psi_G(z_1^*, z_2^*, ...z_{N_e}^*), \]

which directly follow from the boundary conditions for the functions \( \Theta_1 \).

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