Component Coloring of Proper Interval and Split Graphs

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Abstract

We introduce a generalization of the well known graph (vertex) coloring problem, which we call the problem of component coloring of graphs. Given a graph, the problem is to color the vertices using the minimum number of colors so that the size of each connected component of the subgraph induced by the vertices of the same color does not exceed $C$. We give a linear time algorithm for the problem on proper interval graphs. We extend this algorithm to solve two weighted versions of the problem in which vertices have integer weights. In the splittable version the weights of vertices can be split into differently colored parts, however, the total weight of a monochromatic component cannot exceed $C$. For this problem on proper interval graphs we give a polynomial time algorithm. In the non-splittable version the vertices cannot be split. Using the algorithm for the splittable version we give a 2-approximation algorithm for the non-splittable problem on proper interval graphs which is NP-hard. We also prove that even the unweighted version of the problem is NP-hard for split graphs.

Keywords: Graph, Chordal, Proper, Interval, Split, Component, WDM, Light-trail, Reconfigurable Bus Architecture, Weighted, Splittable, Coloring, Partition, Scheduling, Routing, Algorithm, Hardness, Complexity, NP-Complete, Approximation

1. Introduction

The vertex coloring problem is to color the vertices of a graph using the minimum number of colors so that no two adjacent vertices are assigned the same color. In this
paper, we introduce and study a generalization of the vertex coloring problem. In this
generalized problem, called the problem of \textit{component coloring of graphs}, we allow two
adjacent vertices to be assigned the same color. It is customary to consider two variations:
unweighted and weighted. In the unweighted version of the problem, given an graph
\( G = (V, E) \), the objective is to color the vertices using the minimum number of colors
such that the size of any monochromatic component, \textit{i.e.}, the connected component of
the subgraph induced by the vertices of the same color, does not exceed \( C \). The vertex
coloring problem is a special case of the unweighted component coloring problem where
\( C = 1 \), and each monochromatic component consists of a single vertex.

In the weighted version of the problem, given an graph \( G = (V, E) \) and for each \( v \in V \)
a rational weight \( W(v) \in (0, 1] \), the objective is to color the vertices using the minimum
number of colors such that the total weight of any monochromatic component, does not
exceed 1.

Since the vertex coloring problem is NP-hard on general graphs \cite{1}, the unweighted
(and hence weighted) component coloring problem is also NP-hard on general graphs.

Our formulation of the component coloring problem is motivated by a problem on
scheduling transmission requests on \textit{light-trails}, a hardware solution for bandwidth pro-
visioning in optical WDM (Wavelength Division Multiplexing) networks \cite{2}. In a path
network of processors using light-trails, each processor has an optical shutter for each
wavelength which can be configured to be switched ON/OFF for allowing/blocking the
light signal pass through it. For each wavelength, by suitably configuring the optical shut-
ter at each processor, the logical path network can be partitioned into subpath networks
in which multiple transmissions can happen in parallel, provided the total bandwidth
requirement of the transmissions assigned to a subpath does not exceed the capacity of a
wavelength. Such subpaths, in which only the end processors have their optical shutters
blocked, are called \textit{light-trails}. A light-trail can serve only the transmissions having both
source and destination within the light-trail. If a transmission is assigned to a light-trail,
it uses the complete physical span of the light-trail. Given a set of transmission requests,
each with a bandwidth requirement, the scheduling problem is to configure the optical
shutters at the processors so that the minimum number of wavelengths is required by
the light-trails to serve all transmission requests.
A graph $G$ is an interval graph if there exists a family $\mathcal{I}$ of intervals in a linearly ordered set (like the real line), and there exists a one-to-one correspondence between the vertices of $G$ and the intervals in $\mathcal{I}$ such that two vertices are adjacent if and only if the corresponding intervals intersect. If no interval of $\mathcal{I}$ properly contains another, set theoretically, then $G$ is called a proper interval graph.

The light-trail scheduling problem on path networks can be posed as a component coloring problem on interval graphs as follows. For each transmission request, create a vertex with weight equal to the bandwidth requirement, expressed as a fraction of the wavelength capacity. Two vertices are adjacent if the corresponding transmissions overlap, i.e., they use at least one common link. Given a solution to the component coloring problem, a solution to the light-trail scheduling problem can be constructed as follows. For each of the used colors, use a separate wavelength. For each wavelength, construct a separate light-trail for each monochromatic component of the corresponding color. Note that the light-trails on a wavelength do not intersect with each other. All transmission requests corresponding to the vertices of a monochromatic component are served by the corresponding light-trail. The physical span of the light-trail is the union of the physical spans of all requests in it. For each wavelength, the optical shutters in only the processors at the endpoints of all light-trails on the corresponding wavelength are configured to be OFF; optical shutters of other processors are configured to be ON.

As mentioned in [3], the light-trail scheduling problem is similar to the problem of scheduling in reconfigurable bus architectures [4, 5], and hence component coloring applies there too.

The unweighted component coloring problem for $C = 1$, i.e., the vertex coloring problem, has a polynomial time algorithm on interval graphs [6]. However, the complexity of the problem on interval graphs for general $C$ is not known. In this paper we give a polynomial time algorithm for the problem on proper interval graphs for general $C$. Since the problem arises in scheduling light-trails on path networks, we assume that an interval representation of the graph is also available. Our first result is the following.

**Theorem 1.** Given a proper interval graph $G = (V, E)$ with an interval representation, there exists an algorithm that solves the unweighted component coloring problem on $G$ in $O(|V|)$ time.
We also consider a splittable weighted version of the component coloring problem in which each vertex of the input graph has an integer weight which can be divided among multiple copies of the vertex and these copies can be colored separately. However, the total weight of a monochromatic component in the resultant graph should not exceed $C$. Again, this is motivated by a variation of the light-trail scheduling problem in which the bandwidth requirement of a transmission can be divided into multiple transmissions between the same source-destination pair. We extend the algorithm for the unweighted problem to solve this splittable weighted problem on proper interval graphs. So our second result is the following.

**Theorem 2.** Given a proper interval graph $G = (V, E)$ with an interval representation, there exists an algorithm that solves the splittable weighted component coloring problem on $G$ in $O(|V|^2)$ time.

However, the (non-splittable) weighted version of the problem is NP-hard even on proper interval graphs. This comes from the fact that the complete graph $K_n$ is a proper interval graph and the weighted component coloring problem on $K_n$ is an instance of NP-hard Bin Packing problem [1]. We use the algorithm for the splittable weighted problem to get a 2-approximation algorithm for the non-splittable weighted problem on proper interval graphs.

**Theorem 3.** Given a proper interval graph $G$ with an interval representation, there exists a 2-approximation algorithm for the non-splittable weighted component coloring problem on $G$.

The vertex coloring problem also has a polynomial time algorithm for split graphs, i.e., when the vertex set can be partitioned into an independent set and a clique [7]. However, for general $C$, we prove that the unweighted component coloring is NP-hard on split graphs. So our final result is the following.

**Theorem 4.** The component coloring problem is NP-hard for split graphs.

The rest of the paper is organized as follows. We begin in Section 2 by comparing our work with previous related work. In Section 3 we present some pertinent definitions and known results. In Section 4 we show that for the class of chordal graphs, the component
The coloring problem is equivalent to a vertex partitioning problem. Note that the interval graphs and the split graphs are chordal. We show in Section 5 that for the class of proper interval graphs, it is enough to solve a simpler version of the partitioning problem which we call the block-partitioning problem. We give an LP based algorithm for the block-partitioning problem in Section 6. We give a combinatorial algorithm for the same problem in Section 7. In Section 8 we extend this algorithm to solve the splittable weighted problem. Based on the algorithm for the splittable weighted problem we give a 2-approximation algorithm for the non-splittable weighted problem in Section 9. We prove the NP-hardness of the problem on split graphs in Section 10.

2. Previous Work

In the graph coloring literature, there are papers [8, 9] to solve a problem that is a kind of dual to the unweighted component coloring problem. Here, the objective is to minimize the size of the largest monochromatic component in a coloring using a fixed number of colors. The paper [8] shows that for a \( n \)-vertex graph of maximum degree 4, there exists an algorithm that uses 2 colors and produces a coloring in which the size of the largest monochromatic component is \( O(\log n) \). For a family of minor-closed graphs, the paper [9] shows that if \( \lambda \) colors are used, the size of the largest monochromatic component is in between \( \Omega(n^{2/(2\lambda-1)}) \) and \( O(n^{2/(\lambda+1)}) \) for every fixed \( \lambda \). However, in our knowledge, there is no work in the graph coloring literature for the versions of the problem we formulated.

The NP-hard light-trail scheduling problem with arbitrary bandwidth requirements on ring networks and general networks has generally been solved using heuristics and evaluated experimentally [10-15] without any bound on the performance. For path/ring networks, the paper [3] gives an approximation algorithm that uses \( O(\omega + \log p) \) wavelengths where \( p \) is the number of processors in the network and \( \omega \) is the congestion, i.e., the maximum total traffic required to pass through any link. For the corresponding component coloring problem, \( p \) is the number of distinct end points of the intervals in the given interval representation, and \( \omega \) is the weight of a maximum clique and hence a lower bound on the number of colors used. Thus the algorithm in [3] is a constant factor approximation algorithm with an additive term \( \log p \) for the component
coloring problem on interval/circular-arc graphs. Note that in general $p \ll 2n$.

3. Preliminaries

Throughout this paper, let $G = (V, E) = (V(G), E(G))$ be a simple, undirected graph and let $n = |V|$ and $m = |E|$. We also assume that $G$ is connected. If $G$ is not connected, the results in this paper can be applied separately to each of its connected components.

The set of vertices adjacent to a vertex $v \in V$ is represented as $N(v)$. For a set $S \subseteq V$, the sub-graph of $G$ induced by $S$ is $G[S] = (S, E(S))$ where $E(S) = \{ (u, v) \in E \mid u, v \in S \}$.

A clique of $G$ is a set of pairwise adjacent vertices of $G$. The size of a clique is the number of vertices in it. A maximal clique is a clique of $G$ that is not properly contained in any clique of $G$. A maximum clique is a clique of maximum size. The clique number of $G$, denoted by $\omega(G)$ or simply $\omega$, is the size of a maximum clique of $G$. An independent set of $G$ is a set of pairwise non-adjacent vertices in it.

A weighted graph $G = (V, E, W)$ has a weight $W(v) \in \mathbb{Z}_{\geq 0}$ associated with each vertex $v \in V$. The weight-split graph of a weighted graph $G = (V, E, W)$, in short $WSP(G)$, is the weighted graph $G' = (V', E', W')$ such that the weight of each $v \in V$ is divided among a separate set of vertices $v_1, \ldots, v_{n_v}$ in $V'$, i.e., $\sum_{j=1}^{n_v} W'(v_j) = W(v)$ and for each edge $(u, v) \in E$ there is an edge $(u_i, v_j) \in E'$ for all $i = 1, \ldots, n_u$ and $j = 1, \ldots, n_v$.

The weight-expanded graph of a weighted graph $G = (V, E, W)$, in short $WXP(G)$, is the unweighted graph $G' = (V', E')$ such that if we put a weight 1 to each vertex in $V'$ then the resulting weighted graph is a weight-split graph of $G$. We will use the following notations: $n' = |V'|$ and $m' = |E'|$.

Coloring of a graph is an assignment of colors to its vertices. A $\lambda$-assignment of a graph $G = (V, E)$ is a map from $V$ to some set of $\lambda$ colors such as $\{1, \ldots, \lambda\}$; this assignment may not be ‘proper’ in the standard notion of graph (vertex) coloring that two adjacent vertices must be assigned different colors. A color class $i$ is the set of vertices assigned color $i$ under the $\lambda$-assignment. A monochromatic component of $G$ under a $\lambda$-assignment is a component of the sub-graph induced by a single color class, or in other words, a maximal connected monochromatic sub-graph. Following the terminology of [8], we call a monochromatic component a chromon. The size of a chromon is the number
of vertices in it. For a weighted graph, the weight of a chromon is the sum of weights of vertices in it.

An unweighted (weighted) graph is \([\lambda, C]-colorable\) if it has a \(\lambda\)-assignment in which every chromon has size (weight) at most \(C\) and such an assignment is called \([\lambda, C]\)-coloring. A \(C\)-component coloring of graph \(G\) is a \([\lambda, C]\)-coloring with the minimum \(\lambda\). Sometimes we will simply refer to the problem of finding a \(C\)-component coloring of a graph as the coloring problem.

A weighted graph \(G\) is \([\lambda, C]\)-split colorable if it has a weight-split graph \(G'\) which is \([\lambda, C]\)-colorable and such a coloring is called a \([\lambda, C]\)-split coloring of \(G\). A \(C\)-split component coloring of graph \(G\) is a \([\lambda, C]\)-split coloring with the minimum \(\lambda\). Sometimes we will simply refer to the problem of finding a \(C\)-split component coloring of a graph as the split coloring problem.

The component coloring problem can be seen as solving two problems simultaneously, (i) partitioning the vertex set into chromons and (ii) assigning colors to the chromons. The partitioning should be such that if each part is contracted to a single vertex, the resulting graph can be colored using as few colors as possible. Since the size of a maximum clique in the contracted graph plays a major role in determining the number of colors used, at least for some graph classes such as perfect graphs, we have to ensure that the cliques in the original graph does not intersect too many parts. We formally define the partitioning problem as follows:

**Definition** A graph \(G = (V, E)\) is said to have a \([\lambda, C]\)-partition if and only if there is a partition \(\Pi = \{P_1, P_2, \ldots, P_t\}\) of \(V\), \(P_i \subseteq V\), \(P_i \cap P_j = \emptyset\) for all \(i \neq j\) such that the following constraints are satisfied:

- **connectedness** – the subgraph induced by each part \(P_i\), i.e., \(G[P_i]\) is connected,
- **size** – each part \(P_i\) has at most \(C\) vertices, and
- **clique intersection** – any clique in \(G\) intersects at most \(\lambda\) parts (\(\lambda\) will subsequently be called the clique intersection of the partition).

A \(C\)-component partition of a graph is a \([\lambda, C]\)-partition with the minimum \(\lambda\). We will refer to the problem of finding a \(C\)-component partition as the partition problem.
We study the coloring problem on interval graphs and split graphs. Each of these classes of graphs is a subclass of the class of chordal graphs. A graph is chordal if each of its cycles of four or more vertices has a chord, which is an edge joining two vertices that are not adjacent in the cycle. There are many characterizations of chordal graphs (see [7] for more details). We will use the characterization of a chordal graph based on perfect elimination ordering or, in short, PEO. A vertex \( v \) of \( G \) is called simplicial if its neighbors \( N(v) \) form a clique. An ordering \( \sigma = [v_1, v_2, \ldots, v_n] \) of vertices is a PEO if each vertex \( v_i \) is a simplicial vertex of the induced subgraph \( G[v_1, \ldots, v_n] \).

**Proposition 5** ([7]). *Let \( G = (V, E) \) be an undirected graph. Then \( G \) is a chordal graph if and only if \( G \) has a PEO. Moreover, any simplicial vertex can start a PEO.*

A graph \( G = (V, E) \) is a split graph if there is a partition \( V = S + Q \) of its vertex set into an independent set \( S \) and a clique \( Q \). There is no restriction on edges between vertices of \( S \) and \( Q \). A graph \( G = (V, E) \) is an interval graph if there exists a family \( \mathcal{I} = \{I_v \mid v \in V\} \) of intervals on a real line such that for distinct vertices \( u, v \) in \( G \), \( (u, v) \in E \) if and only if \( I_u \cap I_v \neq \emptyset \). Such a family \( \mathcal{I} \) of intervals is commonly referred to as the interval representation of \( G \). Given an interval representation of \( G \), consider a cycle of more than 3 vertices, and the corresponding intervals in ascending left endpoints. Since the rightmost interval intersects the leftmost interval, it also intersects the intervals in between them. Hence \( G \) is also chordal. It will be convenient to let \( \text{Left}(I_v) \) and \( \text{Right}(I_v) \) stand for the left and right endpoint of the interval \( I_v \), respectively. The family \( \mathcal{I} \) is the interval representation of a proper interval graph (PIG) if and only if no interval is properly contained in another. Interval graphs and split graphs are easily seen to be chordal.

**Proposition 6** ([6]). *There exists an \( O(m + n) \) time algorithm to get an interval representation of a given interval graph.*

However, since the component coloring problem is motivated by the light-trail scheduling problem, in this paper we will assume that an interval representation \( \mathcal{I} = \{I_v \mid v \in V\} \) is given for the input PIG \( G = (V, E) \).

Now consider the linear order \( < \) on \( V \) defined as follows. For \( u, v \in V \), \( u < v \) if and only if \( \text{Left}(I_u) < \text{Left}(I_v) \) or \( (\text{Left}(I_u) = \text{Left}(I_v)) \) and \( \text{Right}(I_u) \leq \text{Right}(I_v)) \).
We call this ordering $v_1 < v_2 < \cdots < v_n$ the canonical ordering. In the rest of the paper, we use numbers 1 to $n$ to represent the vertices where $i$ represents the vertex that appears $i$th in the canonical ordering. Hence, $v$ will be interchangeably used to represent a vertex $v \in V$ as well as its position in the canonical ordering. If $u < v$ then $u$ is said to be on the left of $v$ and $v$ is said to be on the right of $u$.

**Proposition 7** ($\square$). A graph $G = (V, E)$ is an interval graph if and only if there exists a linear order $<$ on $V$ such that for every choice of vertices $u, v, w$ with $u < v < w$, $(u, w) \in E$ implies $(u, v) \in E$.

For PIGs the canonical ordering not only satisfies the conditions in Proposition 7 but, in fact, satisfies a stronger property:

**Proposition 8** (“The Umbrella Property” $\square$). A graph $G = (V, E)$ is a PIG, if and only if, there exists a linear order $<$ on $V$ such that for every choice of vertices $u, v, w$, with $u < v < w$, $(u, w) \in E$ implies both $(u, v) \in E$ and $(v, w) \in E$.

A block in a PIG is a set of vertices which are consecutive in the canonical ordering. We will represent a block starting at a vertex $u$ and ending at a vertex $v$ as the interval $[u, v]$. An immediate corollary of Proposition 8 is that every edge $(u, v) \in E$ induces a clique $[u, v]$. Also, any maximal clique of a PIG can be represented by a single edge between the two end vertices, say $(u, v)$, or by the block $[u, v]$.

**Corollary 9.** Let $S$ be a connected subgraph of a PIG and $v_1, v_2, \ldots, v_t$ be the vertices of $S$ arranged in the canonical ordering. Then there must be an edge $(v_i, v_{i+1})$ for all $i = 1, \ldots, t - 1$.

**Proof.** Consider the two vertices $v_i$ and $v_{i+1}$. Since $S$ is connected there must be an edge $(v_j, v_k)$ where $j \leq i$ and $i + 1 \leq k$. Then $[v_j, v_k]$ is a clique. Thus there is an edge $(v_i, v_{i+1})$.

**Proposition 10.** If an interval representation is given for a PIG $G$, then the vertices of $G$ can be arranged in canonical ordering in $O(n)$ time.

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1Some authors use the term block to represent what we call a clique.
Proof. Since there are at most \( t \leq 2n \) endpoints of all intervals in the representation, the intervals can be sorted in canonical ordering using bucket sort in \( O(n) \) time.

**Proposition 11.** If an interval representation is given for a PIG \( G \), then the maximal cliques of \( G \) can be found in \( O(n^2) \) time.

*Proof.* Let \( I = \{ I_v \mid v \in V \} \) be an interval representation of \( G \). Without loss of generality we assume that the endpoints of all intervals are unique. Otherwise we can suitably extend some of the intervals on either side so that all endpoints become distinct without altering the maximal cliques. We construct the sorted array \( A(1, \ldots, 2n) \) of all endpoints in \( O(n) \) time using bucket sort. The maximal cliques are identified as follows. Traverse \( A \) left to right and whenever \( A(p) \) is \( \text{Left}(I_v) \) and \( A(p+1) \) is \( \text{Right}(I_u) \) for some \( u, v \in V \) then output \([u,v]\). Clearly \( u \) and \( v \) are adjacent and hence \([u,v]\) is a clique. Since \( u \) is the leftmost possible and \( v \) is the rightmost possible for such a clique, \([u,v]\) is a maximal clique. The traversal takes \( O(n) \) time.

4. Equivalence of Coloring and Partition on Chordal Graphs

**Lemma 12.** If a graph \( G \) has a \([\lambda,C]\)-coloring then it has a \([\lambda,C]\)-partition.

*Proof.* Suppose \( G \) has a \([\lambda,C]\)-coloring \( C \). Consider the partition \( \Pi \) induced by \( C \) where each part is exactly a chromon. The connectedness constraint is immediately satisfied. Since a chromon has size at most \( C \), the size constraint is also satisfied. Since any pair of vertices in a clique is directly connected by an edge, the chromons in \( C \) intersected by a clique are all of different colors. Hence, a clique intersects at most \( \lambda \) parts in \( \Pi \). Thus the clique intersection constraint is also satisfied. Hence, \( \Pi \) is a \([\lambda,C]\)-partition.

Next we will show that for chordal graphs the converse is also true.

**Lemma 13.** If a chordal graph \( G \) has a \([\lambda,C]\)-partition then it has a \([\lambda,C]\)-coloring.

*Proof.* Suppose \( G \) has a \([\lambda,C]\)-partition \( \Pi = \{P_1, P_2, \ldots, P_t\} \). We prove that there exists a \([\lambda,C]\)-coloring of \( G \) in which each \( P_i \) is a chromon. Let the colors be numbered \( 1, 2, \ldots \). We prove by induction on number of vertices \( n \). For \( n = 1 \), assigning color 1 to the single vertex gives a \([\lambda,C]\)-coloring for any \( \lambda, C \geq 1 \).
For \( n \geq 1 \), let \( u \) be a simplicial vertex of \( G \). Without loss of generality assume \( u \in P_1 \). Consider the graph \( G' \) obtained by removing \( u \) from \( G \). Then \( \Pi' = \{ P_1 \setminus \{ u \}, P_2, \ldots, P_t \} \) is a \([\lambda, C] \)-partition for \( G' \). By induction, there is a \([\lambda, C] \)-coloring \( C' \) of \( G' \) in which each part of \( \Pi' \) is a chromon. We obtain a coloring \( C \) of \( G \) as follows. If \( |P_1| > 1 \) we assign the color of other vertices in \( P_1 \) to \( u \) too. Otherwise we assign \( u \) the lowest numbered color that is not assigned to any of the neighbors of \( u \) in \( C \). To show that \( C \) is a \([\lambda, C] \)-coloring, it is enough to show that at most \( \lambda \) colors are used in \( C \). For \( |P_1| > 1 \) it is obvious as no new color is used. For \( |P_1| = 1 \) if it requires \( \lambda + 1 \) colors then it implies that the clique \( u \cap N(u) \) intersects \( \lambda + 1 \) parts which is not possible.

Thus, on chordal graphs, solving the coloring problem is equivalent to solving the partition problem. In the rest of the paper we solve the partition problem only because the solution can be converted to a solution to the coloring problem using the procedure described in the proof of Lemma 13.

5. Equivalence of Coloring and Block-partition on PIGs

For PIGs, we introduce a more restricted way of partitioning the vertex set.

**Definition** A PIG is said to have a \([\lambda, C] \)-block partition if it has a \([\lambda, C] \)-partition in which each part also satisfy consecutiveness constraint, i.e., each part is also a block.

**Lemma 14.** A PIG \( G \) has a \([\lambda, C] \)-partition if and only if \( G \) has a \([\lambda, C] \)-block partition.

**Proof.** A \([\lambda, C] \)-block partition is also a \([\lambda, C] \)-partition. Now suppose \( G \) has a \([\lambda, C] \)-partition \( \Pi \). If the parts in \( \Pi \) also satisfy the consecutiveness constraint, we are done. So assume not. We convert \( \Pi \) to a new partition \( \Pi' \) that also satisfies the consecutiveness constraint. The conversion is done by exchanging vertices among the parts in \( \Pi \), step-by-step, as follows.

We call a vertex \( u \) to be terminal if \( u \) and some \( v > u + 1 \) belong to one part but \( u + 1 \) belongs to a different part, non-terminal otherwise. Let \( P_1 \) be the leftmost part whose vertices are not consecutive. Let \( i \in P_1 \) be smallest terminal vertex such that \( i + 1 \) is in some \( P_2 \neq P_1 \), and there exists \( i + k \in P_1 \) for some \( k > 1 \). We will show how to repartition \( P = P_1 \cup P_2 \) into parts \( P_1' \) and \( P_2' \) such that in the new partition, each vertex
in the range $[1,i]$ is a non-terminal vertex. Then by repeating this process all vertices
can be made non-terminal and hence consecutiveness constraint will be satisfied. Note
that $P$ is connected as both $P_1, P_2$ are connected and $P_2$ has a vertex in between two
vertices of $P_1$. There are two cases.

Case 1: There are at most $C$ vertices in $P$ to the right of $i$. In this case we set $P'_2$
to be the vertices in $P$ to the right of $i$, and the $P'_1$ to be the vertices in $P$ to the left
of and including $i$. Clearly, $i$ is no more a terminal vertex. Since $P$ is connected, the
vertices of $P$ considered in the canonical ordering form a path. $P'_1, P'_2$ are formed by
breaking this path in the middle, so $P'_1, P'_2$ are both connected. Let $Q$ be any maximal
clique which intersects $P'_1, P'_2$. Since we know that the vertices of $Q$ are consecutive, and
$i$ is the rightmost vertex in $P'_1$ and $i+1$ the leftmost vertex in $P'_2$, the vertices $i, i+1$
must be in $Q$. Thus $Q$ intersects $P_1, P_2$ as well. All other parts intersecting $Q$ remain
unchanged, so the number of parts intersected by $Q$ is the same in the new partition as
the old.

Case 2: There are more than $C$ vertices in $P$ to the right of $i$. In this case we set $P'_2$
to be the $C$ rightmost vertices in $P$, and the remaining go to $P'_1$. As before we see that $i$
is no more a terminal vertex and $P'_1, P'_2$ satisfy the connectedness property. Consider a
maximal clique $Q$ that intersects $P'_1, P'_2$. We show that it must intersect the same number
of parts in the new partition as the old. $Q$ must contain the rightmost vertex $u$ of $P'_1$
and leftmost vertex $v$ of $P'_2$.

Note first that $P'_1$ contains both $i, i+1$, i.e., it has at least one vertex from $P_1$ and
one vertex from $P_2$. But $P'_2$ has $C$ vertices, so they cannot all be from $P_1$, or all from
$P_2$ because both $P_1, P_2$ had at most $C$ vertices each. Thus $P'_2$ also contains at least one
vertex $j$ from $P_1$ and one vertex $k$ from $P_2$. Since $i, j \in P_1$, there must be a path in $P_1$
from $i$ to $j$. There must exist an edge $(u', v')$ in this path such that $u' \leq u$, and $v \leq v'$
(see Fig. [1]). Since $Q$ is maximal, it must contain $u', v'$. Thus $Q$ intersects $P_1$. In a
similar manner, we see that it must intersect $P_2$. Thus it follows that $Q$ intersects the
same number of parts in the old and new partitions.

There is a simple example of a general (non-proper) interval graph where Lemma[14]
does not work. Consider the example graph given by the intervals in canonical order-
ing: $a = [1, 9], b = [2, 5], c = [3, 6], d = [4, 12], e = [7, 10], f = [8, 11]$. It has two
maximal cliques $Q_1 = \{a, b, c, d\}, Q_2 = \{a, d, e, f\}$. For $C = 2$, the optimal partition
$\{\{a, d\},\{b, c\}, \{e, f\}\}$ has clique intersection 2 but the part $\{a,d\}$ is not a block as $a,d$
are not consecutive according to canonical ordering. All block partitions have clique
intersection 3 or more. The reason is as follows. If $a$ is the only vertex in a part then to
cover the remaining 3 vertices of $Q_1$ we need at least 2 more parts. On the other hand
if $a$ is paired with $b$ then to cover the remaining 3 vertices of $Q_2$ we need at least 2 more
parts.

Since for a PIG, the notions of partition and block partition are equivalent, in the rest
of the paper we will abuse the notation $[\lambda, C]$-partition to actually mean a $[\lambda, C]$-block
partition in the context of PIGs.

**Lemma 15.** Given a PIG $G$ with an interval representation, if there is an $O(f(n))$
algorithm to solve the (block) partition problem on $G$, then there is an $O(n + f(n))$
algorithm to solve the coloring problem on $G$.

**Proof.** We first get the canonical ordering of the vertices using the procedure given in
Proposition 10. Suppose the partition algorithm returns a partition with clique intersec-
tion $\lambda$ and the parts sorted in canonical ordering are $P_1, P_2, \ldots, P_t$. For each $1 \leq i \leq t$,
we assign color $(i - 1) \mod \lambda + 1$ to $P_i$. This is a valid coloring because otherwise, there
is an edge $(u, v)$ between two parts of same color implying the clique $[u, v]$ in $G$ intersects
more that $\lambda$ parts which is not possible in a $[\lambda, C]$-partition. \hfill $\Box$

6. An LP Based Algorithm for Block-partition on PIGs

Let $G = (V, E)$ be a PIG with vertices in $V$ already sorted in canonical ordering and
$Q$ be the set of maximal cliques. Let $Left(Q)$ denote the leftmost vertex of $Q$. Then
partition problem on $G$ can be formulated as the following integer linear program:

\[
\text{ILPPart: } \min \lambda \\
\text{s.t. } \begin{align*}
    & x_n = 1 & (1) \\
    & \sum_{j=i}^{i+C-1} x_j \geq 1 & 1 \leq i \leq n - C + 1 & (2) \\
    & \sum_{j=\text{Left}(Q)+1}^{\text{Left}(Q)+|Q|-1} x_j \leq \lambda - 1 & \forall Q \in Q & (3) \\
    & x_j \in \{0, 1\} & 1 \leq j \leq n & (4) \\
    & \lambda \text{ integer} & (5)
\end{align*}
\]

where $x_j$ is a binary variable to denote if vertex $j$ is the rightmost vertex of a block and $\lambda$ denotes maximum clique intersection by any clique. Constraint (1) ensures that some block must end at $n$. Constraints (2) ensure that among $C$ consecutive vertices there must be at least one vertex which is the rightmost vertex of a block because a block has size at most $C$. Since a clique $Q$ intersects at most $\lambda$ blocks, constraints (3) ensure that the vertices in $Q$, except the rightmost, can include the rightmost vertices of at most $\lambda - 1$ blocks. The objective is to minimize the maximum clique intersection $\lambda$.

Let LPPart be the LP relaxation of ILPPart obtained by making $x_j$ a real variable in $[0, 1]$ and making $\lambda$ unconstrained.

**Lemma 16.** If $x, \lambda$ is a fractional solution to LPPart then it can be rounded to an integer feasible solution $\bar{x}, \bar{\lambda}$ in polynomial time.

**Proof.** Consider the following rounding scheme which takes $O(n)$ time. We use a set of intermediate variables $y_0, y_1, \ldots, y_n$. We set

\[
     y_0 = 0, \quad y_j = \sum_{i=1}^{j} x_i, \quad \bar{\lambda} = \lfloor \lambda \rfloor \quad \text{and} \quad \bar{x}_j = \begin{cases} 
    1 & \text{if } [y_{j-1}] \neq [y_j] \\
    0 & \text{otherwise} 
\end{cases} \text{ for all } 1 \leq j \leq n.
\]

Note that each $\bar{x}_j$ is a 0-1 variable and $\bar{\lambda}$ is an integer. Since $x_1 = 1$, by construction $\bar{x}_1 = 1$. Hence $\bar{x}$ satisfies constraint (1).

Now we prove that $\bar{x}$ satisfies the constraints in (2). Since $x$ satisfies $j$th of such constraints, $x_j + x_{j+1} + \ldots + x_{j+C-1} \geq 1$, i.e., $y_{j+C-1} - y_{j-1} \geq 1$. So there must be at
least one index $k$ in $[j, j + C - 1]$ such that $[y_{k-1}] \neq [y_k]$ implying that $\bar{x}_k = 1$. Thus $\bar{x}$ also satisfies the $j$th constraint in (2).

Finally we prove that $\bar{x}, \bar{\lambda}$ satisfy constraints in (3) too. Consider the constraint for clique $Q$ and let $j = Left(Q)$. Since $x$ satisfies this constraint, $x_{j+1} + x_{j+2} + \ldots + x_{j+|Q|-1} \leq \lambda - 1$, i.e., $y_{j+|Q|-1} - y_j \leq \lambda - 1$. So there can be at most $\lambda - 1$ indices $k$ in $[j + 1, j + |Q| - 1]$ such that $[y_{k-1}] \neq [y_k]$ implying that at most $\lfloor \lambda \rfloor - 1$ of the corresponding $\bar{x}_k$s are set to 1. Thus $\bar{x}, \bar{\lambda}$ also satisfy the constraint for clique $Q$.

Clearly $\lambda \leq \bar{\lambda}$. Since the integer objective value $\bar{\lambda}$ cannot be strictly less than the fractional value $\lambda$, $\bar{\lambda} = \lambda$ and hence $\bar{x}, \bar{\lambda}$ optimally solve ILP$\text{PART}$. Thus by solving LP$\text{PART}$ and rounding the solution using the procedure given in the proof of Lemma 16 gives a polynomial time algorithm for the partition problem.

7. A Combinatorial Algorithm for Block-partition on PIGs

We now give a combinatorial algorithm for the partition problem on PIGs. The algorithm does not use LP scaffolding and hence is more efficient.

7.1. Lower Bound

**Lemma 17.** If a PIG $G$ has a $[\lambda, C]$-partition then $\lambda \geq \lfloor (\omega(G) + C - 1)/C \rfloor$.

**Proof.** Let $Q$ be a maximum clique of $G$, i.e., $|Q| = \omega(G)$. To cover all vertices of $Q$ by parts of size at most $C$, we need at least $\lfloor \omega(G)/C \rfloor$ parts. Hence, clique intersection $\lambda \geq \lfloor \omega(G)/C \rfloor = \lfloor (\omega(G) + C - 1)/C \rfloor$.

There is a simple example where $\lambda = \lfloor \omega/C \rfloor$ is not enough to have a $[\lambda, C]$-partition. Consider the graph given by the intervals $[1, 3], [2, 5], [4, 6]$. Here $\omega = 2$. For $C = 2$, the number of parts given by lower bound $\lfloor \omega/C \rfloor = 1$ is not enough as the single part would contain 3 > $C$ connected vertices.

7.2. Upper Bound

**Lemma 18.** If an interval representation is given for a connected PIG $G$ then there exists an algorithm that produces a $\lceil (\omega(G) + C - 1)/C \rceil, C$-partition.
Proof. Consider the following algorithm which we call SimplePart. We first arrange the vertices of $G$ in the canonical ordering time using Proposition 10. Then we assign the block of vertices $[(i - 1)C + 1, \min\{n, iC\}]$ to part $P_i$ for each $1 \leq i \leq \lfloor n/C \rfloor$.

Each part $P_i$ produced by SimplePart clearly has consecutive vertices and has size at most $C$. Since $G$ is connected, by Corollary 9 there is an edge between $v_j$ and $v_{j+1}$ for all $j = 1, 2, \ldots, n - 1$. Hence $P_i$ is also connected. Thus it will be enough to show that any clique intersects at most $\lambda = \lfloor (\omega + C - 1)/C \rfloor$ parts.

If a clique $Q$ intersects $\lambda$ parts $P_1, \ldots, P_{\lambda - 1}$, then $Q$ must contain at least one vertex of each of $P_i$ and $P_{\lambda - 1}$ and all vertices of remaining parts $P_2, \ldots, P_{\lambda - 2}$. So the minimum size of $Q$ is $1 + (\lambda - 2)C + 1 = \lambda C - 2C + 2$. Thus $\omega \geq \lambda C - 2C + 2$. Hence $\lambda \leq (\omega + 2C - 2)/C$. This implies $\lambda \leq \lfloor (\omega + 2C - 2)/C \rfloor = \lfloor (\omega + C - 1)/C \rfloor$.

There is a simple example where SimplePart does not give the optimal partition. Consider the graph given by the intervals $a = [1, 6], b = [2, 7], c = [3, 10], d = [4, 11], e = [5, 12], f = [8, 13], g = [9, 14]$. It has two maximal cliques $\{a, b, c, d, e\}, \{c, d, e, f, g\}$ and $\omega = 5$. For $C = 3$, SimplePart produces the partition $\{\{a, b, c\}, \{d, e, f\}, \{g\}\}$ which has clique intersection 3. But there exists a better partition $\{\{a, b\}, \{c, d, e\}, \{f, g\}\}$ with clique intersection 2.

However, a close analysis reveals that SimplePart is not that bad. In fact, when $\omega(G) = kC + 1$ for some integer $k$, the two bounds match and hence SimplePart gives the optimal solution. Again for other values of $\omega(G)$, which can be represented as $kC + r$ for integer $k, r$ such that $2 \leq r \leq C$, the two bounds are $k + 1$ and $k + 2$ respectively, hence differ by 1 and one of the two bounds is optimum. Thus it will be enough to solve the following special case of the problem.

**Definition** Partition subproblem: given a PIG $G$ with $\omega(G) = kC + r$, $k$ integer and $2 \leq r \leq C$, check if there is a $[k + 1, C]$-partition and if so, generate the partition.

If there is an algorithm ALG for the partition subproblem then we apply ALG to $G$ to check if there is a $[k + 1, C]$-partition. If yes, ALG also gives the required partition. Otherwise, SimplePart gives an optimal solution.

In the rest of the paper we will let $k(G)$, or in short $k$, denote $\lfloor (\omega(G) - 1)/C \rfloor$. 

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7.3. Forbidden vertices

The key idea in our algorithm is to first identify those vertices that cannot be right endpoints of a block in a possible \([k + 1, C]\)-partition.

**Definition** A vertex \(i\) in a PIG is said to be primarily forbidden if the block \([i - kC, i + 1]\) is a clique.

**Lemma 19.** If the vertex \(i\) in a PIG is primarily forbidden then no block in a \([k + 1, C]\)-partition can end at \(i\).

**Proof.** Suppose a block in a \([k + 1, C]\)-partition ends at the vertex \(i\). Consider the clique \([i - kC, i + 1]\) which must be covered by at most \(k + 1\) blocks. To cover the vertex \(i + 1\) we need one block. Then the remaining \(kC + 1\) vertices \([i - kC, i]\) must be covered by at most \(k\) blocks. This is not possible as the size of a block is at most \(C\). \(\square\)

**Definition** A vertex in a PIG is forbidden if it is primarily forbidden or secondarily forbidden, where secondarily forbidden vertices are defined recursively as follows. If (a) there exists a block of forbidden vertices \([v - s + 1, v]\) where \(1 \leq s \leq C - 1\), and (b) the block \([v - kC, v - s + 1]\) is a clique then the set of vertices \(P(v) = \{v - qC | 1 \leq q \leq k\}\) is secondarily forbidden. Further, we will say that \(v\) is the leader of all secondarily forbidden vertices in \(P(v)\) and \(v\) itself. Similarly each secondarily forbidden vertex in \(P(v)\) is a follower of \(v\).

Note that a primarily forbidden vertex is the leader of itself. Furthermore, any forbidden vertex \(i\) has a leader \(i + qC\) where \(q\) is an integer and \(0 \leq q \leq k\).

**Lemma 20.** If the vertex \(i\) in a PIG is secondarily forbidden then no block in a \([k + 1, C]\)-partition can end at \(i\).

**Proof.** The leader of \(i\) is \(i + qC\) where \(1 \leq q \leq k\). Let \(Q\) be the clique \([i + qC - kC, i + qC - s + 1]\). Since the vertices \([i + qC - s + 1, i + qC]\) are forbidden, they must be covered by a single block, say \(B\). The block \(B\) must end at a vertex on the right of \(i + qC\). In the best case \(B\) ends at \(i + qC + 1\) and covers \(C - s\) vertices of \(Q\), i.e., \([i + (q - 1)C + 2, i + qC - s + 1]\). The remaining \((k - 1)C + 2\) vertices \([i - (k - q)C, i + (q - 1)C + 1]\) of \(Q\) must be covered by at most \(k\) blocks.
Now suppose a block of a \([k+1,C]\)-partition ends at the vertex \(i\). To cover the \((k-q)C+1\) vertices \([i-(k-q)C,i]\) we need at least \((k-q)C+1\) blocks. The remaining \((q-1)C+1\) vertices \([i+1,i+(q-1)C+1]\) must be covered by at most \(q-1\) blocks. This is not possible as the size of a block is at most \(C\).

Our algorithm is as follows. We mark all forbidden vertices, and then try to form blocks by a greedy left to right strategy.

### 7.4. Marking forbidden vertices

The algorithm for marking forbidden vertices is given in Algorithm 1. The algorithm assumes that we are given an array \(Lmn\), where \(Lmn(i)\) denotes the leftmost neighbor of \(i\). It is easily seen that \(Lmn\) can be computed in \(O(n)\) time given an interval representation of the input graph. The algorithm constructs the array \(F\), where \(F(i) = 1\) if and only if \(i\) is forbidden.

**Algorithm 1: MarkForbidden**

**Input**: \(Lmn(1,\ldots,n)\) for a PIG \(G = (V,E)\)

**Output**: \(F(1,\ldots,n)\)

1. **foreach** \(i = 1\) to \(n\) do \(F(i) = \text{Ldist}(i) = 0;\)

2. **foreach** \(i = n\) downto 1 do /* phase 1 */

3. **if** \(Lmn(i) \leq i - kC - 1\) then

4. \(F(i-1) = 1;\)

5. \(Rnf(n) = n;\) /* \(Rnf(i)\) is rightmost non-forbidden vertex \(j\) where \(j \leq i\) */

6. **foreach** \(i = n-1\) downto 1 do /* phase 2 */

7. \(Rnf(i) = \min\{i,Rnf(i+1)\};\)

8. **while** \(F(Rnf(i)) = 1\) do \(Rnf(i) = Rnf(i) - 1;\) /* Extend \(Rnf(i)\) */

9. **if** \((F(i) = 1)\) and \((\text{Ldist}(i) \leq (k-1)C)\) then /* \(i\) is a follower */

10. \(F(i-C) = 1;\) \(\text{Ldist}(i-C) = \text{Ldist}(i) + C;\)

11. **if** \(Lmn(Rnf(i)+1) \leq i - kC\) then /* \(i\) is a leader */

12. \(F(i-C) = 1;\) \(\text{Ldist}(i-C) = C;\)
In phase 1, the primarily forbidden vertices are marked. We check at each vertex \(i\) if there is a clique \([i - kC - 1, i]\) of size \(kC + 2\), that is, if \(\text{Lmn}(i) \leq i - kC - 1\), then we mark the primarily forbidden vertex \(i - 1\) by setting \(F(i - 1) = 1\).

In phase 2, we mark the secondarily forbidden vertices. It is enough to identify all the leaders because their followers are all the secondarily forbidden vertices. We identify the leaders and mark their followers in an interleaved manner in a single traversal through the vertices. If \(i\) is identified as a leader, we mark its rightmost follower \(i - C\) immediately, we mark the second rightmost follower \(i - 2C\) when we visit \(i - C\), and so on.

Note that the rightmost vertex \(n\) is never forbidden and hence is not a leader. So in phase 2, we visit each vertex \(i\) starting with the second rightmost. We check if the vertex is a follower of a already discovered leader. This is easy to do, we merely check if \(i\) is a follower of a vertex that is not far, i.e., at most a distance \((k - 1)C\) from \(i\). Then we mark \(i - C\). For this we maintain the auxiliary array \(\text{Ldist}(1,\ldots,n)\) where \(\text{Ldist}(i)\) is the distance of \(i\) from its leader if \(i\) is secondarily forbidden, unspecified otherwise. Then we also check if \(i\) is itself a leader. The condition for \(i\) being a leader is that there should exist a block of forbidden vertices \([j, i]\), and a clique \([i - kC, j]\) where \(i - j + 1 < C\). However, as following Lemma shows, we can assume without loss of generality that the block of forbidden vertices is left maximal.

**Lemma 21.** In a PIG the vertex \(i\) is a leader if and only if the left maximal forbidden block at \(i\) is \([j, i]\) and the block \([i - kC, j]\) is a clique.

**Proof.** \(\iff\) If the block \([i - kC, j]\) is a clique and \([j, i]\) is a forbidden block then clearly \(i\) is a leader.

\(\implies\) If \(i\) is a leader then there exists \(j'\) such that \(0 \leq i - j' < C - 1\), the block \([j', i]\) is forbidden and the block \([i - kC, j']\) is a clique. If \([j', i]\) is not left maximal then let \([j, i]\) be the left maximal forbidden block at \(i\). Then \(j < j'\). Hence, \([i - kC, j]\) is a subclique of \([i - kC, j']\). \(\square\)

Now, checking if a vertex is a leader is easy. We need to know the leftmost endpoint \(j\) of a block of forbidden vertices ending at \(i\), and whether a clique starting at \(i - kC\) ends at \(j\). The leftmost endpoint of the forbidden block need not be calculated afresh for every \(i\). If the leftmost endpoint \(j\) of the forbidden block ending at \(i\) is already calculated,
then we only need to check if the forbidden block extends further on the left of \( j \), when considering the vertex \( i \). For this we maintain the auxiliary array \( Rnf(1, \ldots, n) \) where \( Rnf(i) \) is the rightmost non-forbidden vertex such that \( Rnf(i) \leq i \). The elements of \( Rnf \) can be recursively computed as follows: \( Rnf(i) \) is the rightmost non-forbidden vertex \( x \) such that the block \([x + 1, \min\{i, Rnf(i + 1)\}]\) is forbidden. To make sure the existence of \( Rnf(i) \) for all \( 1 \leq i \leq n \), we assume without loss of generality an imaginary vertex numbered 0 that is not forbidden.

**Lemma 22.** If the array \( Lmn \) for a PIG \( G \) is given, the algorithm \textsc{MarkForbidden} correctly marks the forbidden vertices of \( G \) in \( O(n) \) time.

**Proof.** It is easy to see that phase 1 of \textsc{MarkForbidden} correctly computes correct values of \( F \) for primarily forbidden vertices. Let us refer each iteration of the loop in phase 2 by the corresponding value of \( i \). We now claim that, at the beginning of iteration \( i \) in phase 2, the array \( F \) contains correct values for all vertices in the range \([\max\{1, i - C + 1\}, n]\). This in turn proves the correctness of \textsc{MarkForbidden}. We show by induction on \( i \).

For \( i = n \) there cannot be any secondarily forbidden vertex in the range \([i - C + 1, n]\). Hence the claim is trivially true. For \( i \leq n \) assume at the beginning of iteration \( i \), the array \( F \) contains correct values in the range \([i - C + 1, n]\) (for simplicity we assume \( i \geq C + 1 \), the cases \( i \leq C \) can be shown similarly). The vertex \( i - C \) is a secondarily forbidden vertex if and only if it is either the right most follower of \( i \) or it is a follower of some vertex on the right of \( i \). In \textsc{MarkForbidden} we handle the second case first and update \( F \) accordingly. After adjusting \( Rnf(i) \) suitably, \([Rnf(i + 1), i]\) correctly denotes the left maximal forbidden block at \( i \) because by induction hypothesis the forbidden vertices in the range \([i - C + 1, i]\) are already marked. If there is a clique \([i - kC, j + 1]\) then \( i \) is a leader and we update \( F \) for the rightmost follower \( i - C \) of \( i \). Hence \textsc{MarkForbidden} correctly computes if \( i - C \) is a forbidden vertex in iteration \( i \). Thus at the end of iteration \( i \), i.e., at the beginning of iteration \( i - 1 \), the array \( F \) contains correct values in the range \([(i - 1) - C + 1, n]\).

The pseudocode of Algorithm 1 clearly shows that \textsc{MarkForbidden} takes overall \( O(n) \) time. \( \square \)
7.5. Algorithm CombPart

We now give our algorithm to solve the partition subproblem. The algorithm first marks all forbidden vertices and then forms blocks greedily such that no block ends at a forbidden vertex. We call this algorithm CombPart, which is shown in Algorithm 2.

Algorithm 2: CombPart

Input: A PIG $G = (V, E)$
Output: If $G$ has a $[k + 1, C]$-partition; if Yes also output such a partition

1. $F(1, \ldots, n) =$ array returned by MarkForbidden on $G$; $u = 1$;
2. while $u \leq n$ do
3.   $v = \min\{u + C - 1, n\};$
4.   while $(v \geq u)$ and $(F(v) == 1)$ do $v = v - 1;$
5.   if $v < u$ then return No else create part $[u, v]; u = v + 1;$
6. return Yes;

Lemma 23. Let $B$ be any block, except the rightmost, created by CombPart. Let $u$ be the leftmost vertex of $B$. Then the block of vertices $[u + jC + |B|, u + jC + (C - 1)]$ are forbidden for $0 \leq j \leq k - 1$.

Proof. If $|B| = C$ then the block $[u + jC + |B|, u + jC + (C - 1)]$ is empty and hence the lemma is vacuously true. So we assume $|B| < C$. Note that $u + |B|, \ldots, u + C - 1$ are all forbidden because otherwise CombPart would have created the block $B$ of bigger size. Also note that either $B$ is the leftmost block or $u - 1$ is the rightmost vertex of a block. So without loss of generality we assume that $u - 1$ is not forbidden.

It will be enough if we prove that for each $|B| \leq t \leq C - 1$ the vertex $u + kC + t$ is a leader because then its followers $P(u + kC + t)$, i.e., $u + t, u + C + t, \ldots, u + (k - 1)C + t$ are all forbidden. We prove by induction on $t$.

Base case: $t = C - 1$. Since $u + C - 1$ is forbidden, its leader is the vertex $v = u + C - 1 + qC$ for some $0 \leq q \leq k$ and the followers of $v$, the vertices in $P(v)$, are forbidden. But $u - 1$ is not forbidden, i.e., $u - 1 \notin P(v)$. Hence $u - 1 < v - kC$, implying $q > k - 1$. Thus $q = k$, which implies our claim.
Induction case: suppose the claim is true for $t = t'$ where $|B| < t' \leq C - 1$, i.e., the vertex $u + kC + t'$ is a leader. We need to prove that $u + kC + t' - 1$ is also a leader.

Since $u + t' - 1$ is forbidden, its leader is the vertex $v = u + t' - 1 + qC$ for some $0 \leq q \leq k$. Thus, if $q = k$ then we are done. So assume that $q < k$, i.e., $q = k - z, 1 \leq z \leq k$. We will show that this leads to a contradiction.

Since $v$ is a leader, by Lemma 21 for some vertex $x$, the block $F_1 = [x, v]$ is the left maximal forbidden block at $v$ and the block $Q = [v - kC, x]$ is a clique. Again, by induction hypothesis, each of the vertices $[u + kC + t', u + (k + 1)C - 1]$ is a leader. Thus, the set of vertices $F_2 = [u + qC + t', u + (q + 1)C - 1]$ is forbidden. But $F_2$ can be rewritten as $[v + 1, u + (q + 1)C - 1]$. Thus $F = F_1 \cup F_2 = [x, u + (q + 1)C - 1]$ is a left maximal forbidden block at $u + (q + 1)C - 1$. By applying Lemma 21 to $Q$ and $F$, the vertex $u + (q + 1)C - 1$ is a leader. Among its followers, $P(u + (q + 1)C - 1)$, the $z$th from the left is $u + (q - k + z)C - 1 = u - 1$. This is a contradiction because $u - 1$ is not forbidden.

**Lemma 24.** Suppose COMBPart creates consecutive blocks $B_1, \ldots, B_{k+1}$. Let $u$ be the leftmost vertex of $B_1$. Let $s_i = C - |B_i|$ and $S_i = \sum_{j=1}^i s_j$. Then the $S_k$ consecutive vertices $[u + kC - S_k, u + kC - 1]$ are all forbidden.

**Proof.** For $1 \leq i \leq k$, let $u_i$ be the leftmost vertex of block $B_i$. Note $u_i = u + (i - 1)C - S_i$ where $S_0 = 0$.

Applying Lemma 23 to $B_i$ for all $1 \leq i \leq k$ and considering the set of consecutive forbidden vertices $F_i$ corresponding to $j = k - i + 1$ we get $F_i = [u_i + (k - i + 1)C - s_i, u_i + (k - i + 1)C - 1] = [u + kC - S_i, u + kC + S_{i-1} - 1]$.

The set $\cup_{i=1}^k F_i$ is indeed the required set of consecutive forbidden vertices.

**Lemma 25.** If an interval representation for a PIG $G$ is given then COMBPart correctly solves the partition subproblem on $G$ in $O(n)$ time.

**Proof.** If COMBPart outputs NO, then there is a set of $C$ consecutive forbidden vertices. To cover these vertices we need a block of size at least $C + 1$. So there cannot be any valid partition. Hence COMBPart is correct.

Now we prove that if COMBPart outputs YES then the partition generated is a valid partition. Since the algorithm generates blocks of size at most $C$, the size constraint
is satisfied. We only need to prove that no clique intersects more than \( k + 1 \) blocks generated by \textsc{CombPart}. We prove this by contradiction.

Suppose there is a clique \( Q \) that intersects \( k + 2 \) blocks \( B_0, B_1, \ldots, B_{k+1} \). Without loss of generality, we assume that only the leftmost vertex of \( Q \) is covered by \( B_0 \) and only the rightmost vertex of \( Q \) is covered by \( B_{k+1} \). Because, otherwise we can take a sub-clique \( Q' \subset Q \) with this property. Also let \( v \) be the leftmost vertex of \( Q \), i.e., rightmost vertex of \( B_0 \) and hence not forbidden. Let \( s_i = C - |B_i| \) and \( S_i = \sum_{j=1}^{i} s_j \) for \( 1 \leq i \leq k \). Let \( u \) be the leftmost vertex of \( B_1 \).

Note \( |Q| \leq kC + 1 \) because otherwise \( v \) would be forbidden. By Lemma 24 there are \( S_k \) consecutive forbidden vertices. Since the algorithm outputs \textsc{Yes}, \( S_k < C \). Hence \( |Q| = 2 + \sum_{i=1}^{k} |B_i| = 2 + \sum_{i=1}^{k} (C - s_i) = 2 + kC - S_k > (k - 1)C + 2 \).

So \( Q \) has size \( kC + 2 - S_k \) where \( 1 \leq S_k \leq C - 1 \) and by Lemma 24 there are \( S_k \) consecutive forbidden vertices \([u + kC - S_k, u + kC - 1] = [v + kC + 1 - S_k, v + kC]\).

By Lemma 20 \( v \) is forbidden. It is a contradiction.

If an interval representation is given then we can easily find the maximal cliques and hence can compute the array \( Lmn \) in \( O(pnq) \) time. By Lemma 22 the marking of forbidden vertices takes time \( O(pnq) \). The greedy procedure for generating the parts also takes \( O(pnq) \) time. Overall time taken is \( O(pnq) \).

Combining Lemma 15, Lemma 26 and the discussions at the end of the subsection 7.2 we get a proof of Theorem 1.

8. Algorithm for Splittable Weighted Problem on PIGs

**Lemma 26.** A weighted graph \( G = (V, E, W) \) is \([\lambda, C]\)-split colorable if and only if \( WX\text{P}(G) \) is \([\lambda, C]\)-colorable.

**Proof.** Let \( G' = (V', E') = WX\text{P}(G) \).

\([\Rightarrow] \) The weighted graph \( G'' \), obtained by putting weight 1 to every vertex of \( G' \), is also a weight-split graph of \( G \). So if \( G' \) is \([\lambda, C]\)-colorable then \( G'' \) and \( G \) both are \([\lambda, C]\)-split colorable.

\([\Rightarrow] \) Let the weighted graph \( G''(V'', E'', W'') \) be the weight-split graph corresponding to the \([\lambda, C]\)-split coloring of \( G \). Then \( G' \) is the weight-expanded graph of \( G'' \) too. A
[\lambda, C\text{-coloring of } G'\text{ can be obtained by assigning the vertices in } G'\text{ corresponding to a vertex } v'' \text{ in } G'' \text{ the same color of } v''.]$

Thus solving the split coloring problem on a weighted PIG $G = (V, E, W)$ is equivalent to solving the unweighted coloring problem on $\text{WXP}(G)$. In the rest of the section we will use $G' = (V', E')$ to represent $\text{WXP}(G)$. Applying the algorithm described in Section 7 on $G'$ gives correct result but it makes the algorithm pseudo-polynomial as it takes $O(n')$ time, proportional to the sum of weights. This is mainly because the algorithm iterates over each vertex in $G'$.

However, it turns out that iterating over each vertex in $G'$ is not necessary. The forbidden vertices in $G'$ can be divided into blocks such that if the vertices $u$ and $v$ are in the same block $b$ then leader of $u$ and leader of $v$ are in the same block $l$. We call such forbidden blocks FBs. Parallel to the vertices, we say that FB $l$ is the leader of FB $b$ and $b$ is the follower of $l$. It can be seen that all vertices in an FB can be marked together. Hence it is enough to iterate through the FBs instead of iterating through the vertices of $G'$.

8.1. Marking forbidden blocks

We now modify the algorithm presented in Section 7 to let it work with FBs instead of forbidden vertices. The modified algorithm to mark all the FBs, which we call \text{SplitMark}, is shown in Algorithm 3.

We use the following correspondence between a vertex $v \in V$ and a vertex $v' \in V'$. The vertex $v' = h(v, q)$ if $v'$ is the $q$th copy of $v$ where $1 \leq q \leq W(v)$ and $v = h(v')$ if $v'$ is a copy of $v$. The set $\{h(v, 1), \ldots, h(v, W(v))\}$ of copies of $v$ is represented by $H(v)$. As usual, we will interchangeably use $1 \leq v' \leq n'$ ($1 \leq v \leq n$) to denote a vertex $v' \in V'$ ($v \in V$) as well as its position in the canonical ordering of vertices in $G' (G)$. We also use an auxiliary array $Z(0, \ldots, n)$ such that $Z(0) = 0$ and for all $v > 0$, the entry $Z(v)$ denotes the rightmost copy of $v$ in $G'$, i.e., $h(v, W(v))$. Since all the copies $h(v, t) \in V'$ of $v \in V$ appear consecutively in the canonical ordering of $G'$, we have $Z(v) = \sum_{i=1}^{v} W(i)$. Note that given $Z$, the values of the function $h(u)$ for all $u$ belonging to a subset of vertices $S \subseteq V'$, can be computed in right to left order, in overall $O(|S| + n)$ time.
F.

if $(u-1) + (kC + 2) 

\textbf{Algorithm 3: SplitMark}

\begin{algorithm}
\textbf{Input} : Maximal cliques of a PIG $G$, $\text{Lmn}(1, \ldots, n)$, $Z(0, \ldots, n)$

\textbf{Output} : Doubly linked list of FBs $F$ in $G' = WXP(G)$

\begin{algorithmic}[1]
\STATE \textbf{foreach} maximal clique $[u, v]$ in $G$ \textbf{do} \hfill /* phase 1 */
\STATE \quad \textbf{if} $Z(v) - Z(u-1) - (kC + 2) \geq 0$ \textbf{then} \hfill /* $[Z(u-1) + 1, Z(v)] \in G' \equiv [u, v] \in G$ */
\STATE \quad \quad $F$.Inlay($Z(v) - 1, Z(v) - Z(u-1) - (kC + 2) + 1, 0$);
\STATE \quad $i = F$.end$\rightarrow$prev; \hfill /* $F$.end$\rightarrow$prev is the rightmost FB */
\STATE \quad \textbf{while} $i \neq F$.begin \textbf{do} \hfill /* phase 2 */
\STATE \quad \quad \quad $v = i \rightarrow$right;
\STATE \quad \quad \quad $j = i \rightarrow$rnf = $i \rightarrow$next$\rightarrow$rnf; \quad \textbf{if} $v < i \rightarrow$rnf$\rightarrow$right \textbf{then} $j = i \rightarrow$rnf = $i$;
\STATE \quad \quad \quad \textbf{while} $j \rightarrow$prev$\rightarrow$right $= j \rightarrow$right $- j \rightarrow$size \textbf{do} \hfill /* $i$ is a follower block */
\STATE \quad \quad \quad \quad $j \rightarrow$rnf = $j$; $j = j \rightarrow$prev;
\STATE \quad \quad \quad \textbf{if} $i \rightarrow$ldist $\leq (k-1)C$ \textbf{then} \hfill /* $i$ is a follower block */
\STATE \quad \quad \quad \quad $F$.Inlay($v - C, i \rightarrow$size, $i \rightarrow$ldist $+ C$);
\STATE \quad \quad \quad $u = i \rightarrow$rnf$\rightarrow$right $- i \rightarrow$rnf$\rightarrow$size $+ 1$; \hfill /* new leader block ending at $v$ */
\STATE \quad \quad \quad $f = v - u + 1$;
\STATE \quad \quad \quad $s = (v - \text{Lmn}(\bar{h}(u)) + 1) - (kC + 2)$; \quad \textbf{if} $f + s \geq 0$ \hfill /* function $\bar{h}$ is computed using $Z$ */
\STATE \quad \quad \quad \textbf{then} \hfill /* new leader block ending at $v$ */
\STATE \quad \quad \quad \quad $F$.Inlay($v - C, f + s + 1, C$);
\STATE \quad \quad \quad $i = i \rightarrow$prev;
\end{algorithmic}

We store the information about the FBs in a linked list. Thus $F$ is now a doubly linked list of non-intersecting FBs sorted according to canonical ordering. We also keep the information stored in auxiliary arrays $\text{Ldist}$ and $\text{Rnf}$ earlier, in the list $F$ itself. Thus each entry $b$ of $F$ has the following fields: (i) $\text{right}$ denotes the rightmost vertex of the FB $b$, (ii) $\text{size}$ denotes the size of $b$, and (iii) $\text{ldist}$ denotes the distance of $b$.right from its leader, (iv) $\text{rnf}$ points to the leftmost FB such that all FBs between $b$.rnf and $b$ are consecutive, i.e., all vertices in $[b$.$\text{rnf}$.$\rightarrow$right $- b$.rnf.$\rightarrow$size $+ 1, b$.right$]$ are forbidden, (v) $\text{prev}$ points to the FB on the left of $b$, and (vi) $\text{next}$ points to the FB on the right of $b$. Note that we use the notation $p$.$\rightarrow$q to represent the field $q$ of the FB pointed by the pointer $p$. We keep two sentinel FBs in $F$ always, (i) the leftmost FB $[-2, -1]$ and (ii) the rightmost FB $[Z(n) + 2, Z(n) + 3]$ each having $\text{ldist} = 0$ and $\text{rnf}$ pointing to itself.
Two pointers $F.\text{begin}$ and $F.\text{end}$ point to these two FBs, respectively.

In addition to the standard operations of insert, delete and both way traversals though the list, we define a new operation on $F$ which we call $F.\text{Inlay}(rt, sz, ld)$. This operation inserts a new FB $b$ with $b.\text{right} = rt, b.\text{size} = sz, b.\text{ldist} = ld$ into $F$ but makes sure that the FBs in $F$ remain non-intersecting and sorted. Let $b_1, \ldots, b_s$ be the FBs in $F$ which intersect $b$. The operation $\text{Inlay}$ does the following: (i) deletes all the FBs in $F$ which are subsets of $b$, (ii) if $b$ partly intersects $b_1$, i.e., if $t = (b.\text{right} - b.\text{size} - b_1.\text{right}) < b_1.\text{size}$ then updates $b_1.\text{size} = b_1.\text{size} - t, b_1.\text{right} = b_1.\text{right} - t$, (iii) if $b$ partly intersects $b_s$, i.e., if $b_s.\text{right} > b.\text{right}$ then updates $b_s.\text{size} = b_s.\text{right} - b.\text{right}$, and (iv) inserts $b$ at its proper position in $F$.

We slightly modify the definition of the array $Lmn(p,\ldots,n)$. Now $Lmn(v)$ denotes the leftmost neighbor of $Z(v)$ in $G'$. If the maximal cliques of $G$ are given, then the elements of $Lmn$ can be computed in $O(pn)$ time.

In phase 1 we mark the FBs due to the primarily forbidden vertices given by the following lemma:

**Lemma 27.** Let a PIG have a maximal clique $Q$ with rightmost vertex $v$ and let $s = |Q| - (kC + 2) \geq 0$. Then the set of all primarily forbidden vertices in $Q$ is exactly the FB $[v - s - 1, v - 1]$.

**Proof.** Let $u$ be the leftmost vertex of $Q$. The only subcliques of $Q$ which satisfy the conditions of Lemma 19 are $Q_p = [u + p, u + kC + 1 + p]$ for all $0 \leq p \leq s$. The primarily forbidden vertex for $Q_p$ is $u + kC + p$. Thus the set of all phase 1 forbidden vertices for $Q$ is $\{u + kC + p, 0 \leq p \leq s\}$, i.e., the FB $[v - s - 1, v - 1]$. \hfill $\square$

So in phase 1 we go through the maximal cliques of $G'$ which have one-to-one correspondence with the maximal cliques in $G$ and mark the FBs for the primarily forbidden vertices. Note that the maximal clique $[u, v] \in G$ corresponds to the clique $[Z(u - 1) + 1, Z(v)] \in G'$.

It is clear that in phase 2 we need not check for leaders at the vertices which are not forbidden; checking only at the rightmost vertex in each FB suffices. The following lemma determines the size of the leader block ending at the rightmost vertex in a FB.
Lemma 28. Let a PIG have a left maximal block $B$ of forbidden vertices with the rightmost vertex $v$ and size $f$ where $1 \leq f < C$. Let $Q$ be the largest clique with the rightmost vertex $v' - f + 1$. Let $s = |Q| - (kC + 2)$. Then the leaders in $B$ are exactly the FB $[v - f - s, v]$.

Proof. Let $u$ be the leftmost vertex of $Q$. Thus $u = v - f + 1 - |Q| + 1 = v - kC - (f + s)$. Consider any vertex $v' \in B$. If $f + s < 0$ then $u > v - kC \geq v' - kC$. Since $Q$ is the largest possible there cannot be a clique $[v' - kC, v - f + 1]$. Hence by Lemma 21 the vertex $v'$ is not a leader. So we assume $f + s \geq 0$.

Note that $s < 0$ because otherwise $v - f$ would be a primarily forbidden vertex on the immediate left of $B$ which is not possible as $B$ is left maximal. Thus $[v - f - s, v]$ is a subset of $B$. Now for all $0 \leq p \leq f + s$, consider the vertex $v_p = v - p$. Since $v_p - kC = v - kC - p \geq u$, the block $[v_p - kC, v - f + 1]$ is a subclique of $Q$. Thus by Lemma 21 the vertex $v_p$ is a leader. Since $Q$ is the largest possible, $[v - f - s, v]$ is exactly the set of leaders in $B$. 

Thus in phase 2 we visit each FB $i$ starting with the rightmost FB created in phase 1. We check if $i$ is a follower of some previously discovered FB $l$ at a distance at most $(k - 1)C$ and mark the next follower of $l$ on the left of $i$. We also check if $i$ is a leader itself using Lemma 28. If a leader FB $l$ is identified then we insert the rightmost follower of $l$, and so on, similar to secondarily forbidden vertices in Section 7.

Lemma 29. If the arrays $Z, Lmn$ and the maximal cliques of a PIG $G$ are given, then SplitMark correctly marks the FBs of $G' = WX(P(G))$ in $\Theta(n^2)$ time.

Proof. It is easy to see that phase 1 of SplitMark correctly inserts into $F$ the FBs due to the primarily forbidden vertices while ensuring that the FBs in $F$ are non-intersecting and sorted in canonical ordering. Let us refer each iteration of the loop in phase 2 by the corresponding value of $i$. We now claim that, at the beginning of iteration $i$ in phase 2, the list $F$ correctly contains all FBs in the range $[\max\{1,i\rightarrow right - C + 1\}, n']$. This in turn proves the correctness of SplitMark. We show by induction on $i$.

For $i$ pointing to the rightmost FB there cannot be any secondarily forbidden vertex in the range $[i\rightarrow right - C + 1, n]$. Hence the claim is trivially true. For other values of $i$, assume at the beginning of iteration $i$, the list $F$ correctly contains FBs in the range
\[i \mapsto \text{right} - C + 1, n]\) (for simplicity we assume \(i \mapsto \text{right} \geq C + 1\), the cases \(i \mapsto \text{right} \leq C\) can be shown similarly). In iteration \(i\) \text{SPLITMARK} correctly inserts a new FB \([i \mapsto \text{right} - i \mapsto \text{size} - C + 1, i \mapsto \text{right} - C]\) or \([i \mapsto \text{right} - (f + s + 1) - C + 1, i \mapsto \text{right} - C]\) depending upon whether (i) FB \(i\) is a follower of a previously discovered FB or (ii) there is a leader FB with rightmost vertex \(i \mapsto \text{right}\), given by Lemma 28. In case (i) there are non forbidden vertex in \([b \mapsto \text{right} + 1, i \mapsto \text{right} - i \mapsto \text{size}]\). Hence there can not be any secondarily forbidden vertex in the \([b \mapsto \text{right} - C + 1, i \mapsto \text{right} - i \mapsto \text{size} - C]\). Thus at the end of iteration \(i\), the list \(F\) correctly contains FBs in the range \([b \mapsto \text{right} - C + 1, n']\).

Similarly in case (ii) there can not be any new secondarily forbidden vertex \([b \mapsto \text{right} - C + 1, i \mapsto \text{right} - (f + s + 1) - C]\) and hence at the end of iteration \(i\), the list \(F\) correctly contains FBs in the range \([b \mapsto \text{right} - C + 1, n']\). At the end of iteration new value of \(i\) is \(b\). Hence our claim is true at the beginning of the next iteration too.

Note that for each FB the algorithm takes \(O(1)\) time except the operation \text{Inlay}. Note that since the operation \text{Inlay} is invoked with FBs in right to left order, it can be implemented by maintaining an extra pointer that traverses through the FBs in right to left order, in overall \(O(|F|)\) time. If the weight of each vertex is at most \(O(2)\) then the set of vertices \(H(v)\) for \(v\) in \(G\) can contain at most one follower of each leader FB. There can be as many leader FBs as the number of maximal cliques in \(G'\), i.e., at most \(n\). Hence \(|F| = O(n^2)\). Thus, time complexity is \(O(n^2)\).

Now we show that there is a class of PIGs for which \text{SPLITMARK} takes \(\Omega(n^2)\) time. The class of PIGs is obtained by varying some parameter \(t\). A PIG \(G\) in this class has \(n = 3t\) vertices given by the intervals \(I_1, \ldots, I_{3t}\) where for \(1 \leq j \leq t + 1\) the interval \(I_j = [j, 2t + 2t - 1]\) has weight 2, for \(2 \leq j \leq t\) the interval \(I_{t+j} = [t + 1, 4t + j]\) has weight \(2t\), and again for \(1 \leq j \leq t\) the interval \(I_{2t+j} = [2t + 2t, 5t + j]\) has weight 2. Clearly \(G' = WX\ P(G)\) has \((t + 1) \ast 2 + (t - 1) \ast 2t + t \ast 2 = 2(t^2 + t + 1)\) vertices and each of the \(t + 1\) maximal cliques \([2j - 1, 2j + 2t^2], 1 \leq j \leq t + 1\), has size \(2t^2 + 2\).

For \(C = 2t\), we have \(k(G') = t\). In phase 1 \text{SPLITMARK} creates \(t + 1\) FBs each having a single vertex \(2j + 2t^2 - 1\) for all \(1 \leq j \leq t + 1\). In phase 2 \text{SPLITMARK} creates a

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2If the weights are unrestricted, then we can still solve the decision version of the problem in \(O(n^2)\) time by considering only the interesting FBs that fall within the set \(H(v)\) for a vertex \(v\) in \(G\). We omit the details here.

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FB from each of the remaining odd numbered vertices in \( G' \). Thus the total number of FBs created by \text{SplitMark} on \( G \) is equal to the number of odd vertices in \( G' \). Hence \(|F| = t^2 + t + 1 = (n/3)^2 + n/3 + 1 = \Omega(n^2)\). Thus \text{SplitMark} takes \( \Omega(n^2) \) time on \( G \).

### 8.2. Algorithm SplitPart

We now give the modifications to \text{CombPart} to use the FBs. We call this modified algorithm \text{SplitPart}, which is shown in Algorithm 4.

**Algorithm 4: SplitPart**

**Input** : A PIG \( G = (V, E, W) \)

**Output**: If \( G' = WXP(G) \) has a \([k+1, C]\)-partition; if Yes also output the partition

1. \( F = \) list of FBs returned by \text{SplitMark} on \( G \); \( u = 1; \) \( i = F.beg\rightarrow next; \)

2. while \( u \leq n' \) do
3. \hphantom{2.} \( v = \min\{u + C - 1, n'\}; \)
4. \hphantom{2.} while \( i.right < v \) do \( i = i.beg\rightarrow next; \)
5. \hphantom{2.} \( v = i.right - i.size; \)
6. \hphantom{2.} if \( v < u \) then return No else create part \([u, v]; u = v + 1; \)

7. return Yes;

**Lemma 30.** If an interval representation for a weighted PIG \( G \) is given then \text{SplitPart} correctly solves the partition subproblem on \( WXP(G) \) in \( O(n^2) \) time.

**Proof.** Given that \text{SplitMark} correctly marks the forbidden blocks of \( G' = WXP(G) \), it is easy to see that \text{SplitPart} generates the same partition that \text{CombPart} would have generated on \( G' \). Given an interval representation of \( G, Z, \text{Lmn} \) and maximal cliques of \( G \) can be computed in \( O(n) \) time. Thus by Lemma 29, computing \( F \) takes \( O(n^2) \) time. The block generation step also takes \( O(|F|) = O(n^2) \) time.

Combining Lemmas 15, 26, 30 and the discussions at the end of Subsection 7.2, we get a proof of Theorem 2. Note that \text{SimplePart} can be slightly modified to use vertices in \( WXP(G) \) but still taking \( O(n) \) time.

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9. A 2-approximation Algorithm for Weighted Problem on PIGs

Lemma 31. There exists a polynomial time algorithm for the non-splittable weighted partition problem that generates a $[\lambda, C]$-partition on a PIG $G$ such that $\lambda$ is at most 2 times the clique intersection of the partition generated by an optimal algorithm on $G$.

Proof. We first solve the corresponding splittable weighted problem on $G$ in $O(n^2)$ time using the algorithm described in Section 8. Let the blocks in the $[\lambda', C]$-partition created by the algorithm be $\mathcal{P}' = \{P_1, P_2, \ldots, P_t\}$. Note that $\lambda'$ is a lower bound on the clique intersection $\lambda^*$ of the partition generated by any optimal algorithm on $G$.

Since the weight of a vertex in a non-splittable problem is at most $C$, a vertex of $G$ is split into at most two consecutive blocks $P_i$ and $P_{i+1}$. We convert the splittable partition $\mathcal{P}'$ into a non-splittable partition $\mathcal{P}$ in $O(n)$ time as follows. Consider each vertex $v$ left to right. If $v$ is split into blocks $P_i, P_{i+1}$ and $v$ cannot be put completely in $P_i$ then create a copy $P_i'$ of $P_i$, insert $P_i'$ in between $P_i$ and $P_{i+1}$, put $v$ completely in $P_i'$ and repeat with the rest of the vertices. Note that $\mathcal{P}$ contains at most 2 copies of each block $P_i$ and hence $\mathcal{P}$ is a $[\lambda, C]$-partition with clique intersection $\lambda \leq 2\lambda' \leq 2\lambda^*$. Overall it takes $O(n^2)$ time.

Combining Lemma 15 and Lemma 31 we get a proof of Theorem 3.

10. Partition Problem on Split Graphs

Since split graphs are also chordal, solving the partition (not block-partition) problem is enough. It can be noted that the same lower bound of Lemma 17 applies here too.

10.1. Upper Bound

Lemma 32. Let $\omega$ be the clique number of a split graph $G$. There exists a polynomial time algorithm that gives a $[[\omega/C] + 1, C]$-partition for $G$.

Proof. Let the vertex set of $G$ be split into clique $Q$ and independent set $S$. Without loss of generality, we assume that $Q$ is a maximum clique. Because otherwise $|Q| = \omega - 1$ and we can move a vertex in $S$ that is adjacent to all vertices in $Q$ to $Q$. Now consider the following partition of vertices: $\Pi = \{P_1, P_2, \ldots, P_t\} \cup \{\{v\}|v \in S\}$ where $t = [\omega/C]$,
and \( \{P_i\}_{i=1}^t \) is an arbitrary partition of \( Q \) such that \( |P_i| = C \) for all \( i = 1, \ldots, (t-1) \). Note that this partition can be created in polynomial time. Each part is connected and has at most \( C \) vertices. Moreover, any maximal clique in \( G \) intersects at most \( t + 1 \) parts. Thus, \( \mathcal{II} \) is a \( [[\omega/C] + 1, C] \)-partition. 

10.2. NP-hardness

Since the upper bound and the lower bound differ by 1, it is enough to decide if \( G \) has a \( [[\omega/C], C] \)-partition or not. If the answer is \( \text{Yes} \) then we have an optimal solution to the partition problem with clique intersection \( \lambda = [\omega/C] \). Otherwise the partition given in the proof of Lemma 32 gives an optimal solution with clique intersection \( \lambda = [\omega/C] + 1 \). Thus Lemma 33 directly gives a proof of Theorem 4.

Lemma 33. The problem of deciding if a split graph \( G \) has a \( [[\omega(G)/C], C] \)-partition for \( C \geq 2 \) is NP-complete.

Proof. We show that the decision problem is NP-complete even for \( C = 2 \). We call the problem in this special case as CP. First we show that CP is in NP. A maximal clique in \( G \) is either \( Q \) or the closed neighborhood of a vertex in \( S \). So the maximal cliques in \( G \) can be found in polynomial time. Suppose a partition of the vertices is given. Size constraints can be easily checked. Each part contains a single vertex or a pair of vertices. A single vertex is trivially connected. Connectedness of a part of size 2 can be checked by just checking if there is an edge between the two vertices. Clique intersection constraint can also be checked in polynomial time.

We now introduce a set partitioning problem (SP) is defined as follows. Given a set of \( 2n \) elements \( e_1, e_2, \ldots, e_{2n} \) and a collection of \( m \) subsets \( S_1, S_2, \ldots, S_m \), can the elements be partitioned into \( n \) groups of size 2 such that each subset has both elements of at least one group?

We complete the proof by first showing a polynomial time reduction from SP to CP (Lemma 34) and then a polynomial time reduction from the well known NP-complete problem SAT to SP (Lemma 35). 

Lemma 34. \( SP \leq_p CP \).
Proof. Given an instance of SP, we construct an instance of CP as follows. The complete set \( Q \) has a vertex \( v_i \) corresponding to each element \( e_i \) and the independent set \( S \) has a vertex \( w_j \) corresponding to each subset \( S_j \). There is an edge between \( v_i \) and \( w_j \) if and only if \( e_i \not\in S_j \). Clearly \( \omega = 2n \) and hence \( \lfloor \omega/C \rfloor = n \).

Suppose there is a Yes solution to the SP instance where the groups are \( G_1, G_2, \ldots, G_n \). Then create a partition \( \Pi = \{ P_1, P_2, \ldots, P_n \} \cup \{ w_j \}_{j=1}^m \) for the CP instance where \( v_k \in P_i \) if and only if \( e_k \in G_i \). Clearly each part is connected and has at most 2 vertices. Clique intersection constraint is satisfied for \( Q \). Since \( S_j \) contains both elements of at least one \( G_i \), the maximal clique \( Q' \) containing \( w_j \) does not intersect at least one part \( P_i \). Including the part \( \{ w_j \} \), \( Q' \) intersects at most \( (n - 1) + 1 = n \) parts. Hence \( \Pi \) is a \([n, 2]\)-partition.

On the other hand, suppose there is a Yes solution to the CP instance. Since the clique intersection constraint is satisfied for \( Q \), the vertices of \( Q \) are divided into parts of size exactly 2. These parts give the required groups of SP because, for a maximal clique \( Q' \) containing \( w_j \) has clique intersection at most \( n \), and hence it must not intersect with at least one part \( P_i \) which implies that \( S_j \) contains both elements of \( G_i \).

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**Lemma 35.** \( SAT \leq_p SP \).

Proof. Suppose an instance of SAT has \( p \) Boolean variables \( x_1, x_2, \ldots, x_p \) and \( q \) clauses \( C_1, C_2, \ldots, C_q \). Without loss of generality, we assume that there is at most one literal for each variable in each clause. Now we construct an instance of SP as follows. There are \( 4p \) elements \( x_1, x'_1, T_1, F_1, x_2, x'_2, T_2, F_2, \ldots, x_p, x'_p, T_p, F_p \) and the subsets are of two types as follows: (1) the subsets \( \{ x_i, x'_i, T_i \}, \{ x_i, x'_i, F_i \}, \{ x_i, T_i, F_i \}, \{ x'_i, T_i, F_i \} \) for all \( 1 \leq i \leq p \), and (2) the subset \( \bigcup_{i \in C_j} \{ l_i, T_i \} \) for all clause \( C_j \), where \( l_i \) is either \( x_i \) or \( x'_i \) (e.g., for \( C_j = (x_1 + x'_2 + x_3) \) the subset \( \{ x_1, T_1, x'_2, T_2, x_3, T_3 \} \)).

Suppose there is a satisfying assignment for the SAT instance. Then construct a grouping for the SP instance as follows. For all \( i \), if \( x_i \) is true then construct two groups \( \{ x_i, T_i \} \) and \( \{ x'_i, F_i \} \); otherwise (i.e., \( x_i \) is false) construct two groups \( \{ x_i, F_i \} \) and \( \{ x'_i, T_i \} \). Clearly each subset of type (1) has two elements belonging to the same group. Since each clause is satisfied there must be a variable \( x_i \) such that one of \( x_i \) and \( x'_i \) is true. Hence the corresponding subset of type (2) must have two elements belonging to the same group.

On the other hand, suppose there is a Yes solution for the SP instance. The subsets of type (1) force the elements \( x_i, x'_i, T_i, F_i \) to form 2 groups amongst themselves. The
subsets of type (2) ensure that one of the literals \( l_i \) in the clause \( C_j \) must group with \( T_i \) and hence \( C_j \) must be true. This implies that the SAT instance has an satisfying assignment implied by the groups. For some \( x_i \) the grouping may contain \( \{x_i, x'_i\}, \{T_i, F_i\} \), in which case the value for the variable \( x_i \) can be chosen arbitrarily.

11. Conclusions and Future Work

We gave polynomial time algorithms for unweighted and splittable weighted versions of the component coloring problem for proper interval graphs and showed that it is NP-hard for split graphs. However the complexity of both the versions are not known for general interval graphs. We would like to get polynomial time algorithms for general interval graphs using similar ideas. This may lead to a constant factor approximation algorithm for the weighted version of the problem for general interval graphs which is known to be NP-hard, using ideas from Bin-packing.

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