Quantum Zeno effect: A possible resolution to the leakage problem in superconducting quantum computing architectures

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We propose to use the continuous version of the quantum Zeno effect to eliminate leakage to higher energy states in superconducting quantum computing architectures based on Josephson phase and flux qubits. We are particularly interested in the application of this approach to the single-step Greenberger-Horne-Zeilinger (GHZ) state protocol described in [A. Galiautdinov and J. M. Martinis, Phys. Rev. A 78, 010305(R) (2008)]. While being conceptually appealing, the protocol was found to be plagued with a number of spectral crowding and leakage problems. Here we argue that by coupling the qubits to a measuring device which continuously monitors leakage to higher energy states (say, to a very lossy resonator of frequency $\hbar \omega_{\text{dump}} = E_3^{\text{qubit}} - E_1^{\text{qubit}}$, with 1 labeling the ground state of the qubit), we could potentially restrict the multi-qubit system’s evolution to its computational subspace, thus circumventing the above mentioned problems.

1. INTRODUCTION

The spectral crowding problem and the closely related problem of leakage to higher-energy states have pestered the field of superconducting quantum computing since its very inception [1]. For almost twenty years a tremendous amount of effort has been made to resolve both of these problems (see., e. g., [2][12]). The effort was certainly worth making, because every time an operation lifting the system off its ground state is performed (such as, e.g., a strong Rabi pulse), leakage out of the computational subspace develops, which results in the degradation of the gate fidelity. One may argue that even if the other problems facing quantum computing, such as scalability and decoherence, are ever successfully resolved, the crowding and the leakage problems would still be with us, simply as a matter of principle. As long as one insists on using Josephson junctions, one will have to deal with the matrix elements that kick the system all over the spectra of the corresponding anharmonic potentials.

As an example, consider the Greenberger-Horne-Zeilinger (GHZ) state protocol proposed in Ref. [13] for a fully connected network of three identical Josephson phase qubits. In the rotating wave approximation (RWA) the system is described by the Hamiltonian,

$$
H = \hat{\Omega}_1 \cdot \vec{\sigma}_1 + \hat{\Omega}_2 \cdot \vec{\sigma}_2 + \hat{\Omega}_3 \cdot \vec{\sigma}_3
+ \frac{1}{2} \left[ g \left( \sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2 \right) + \tilde{g} \sigma_z^1 \sigma_z^2
+ g \left( \sigma_x^2 \sigma_x^3 + \sigma_y^2 \sigma_y^3 \right) + \tilde{g} \sigma_z^2 \sigma_z^3
+ g \left( \sigma_x^3 \sigma_x^1 + \sigma_y^3 \sigma_y^1 \right) + \tilde{g} \sigma_z^3 \sigma_z^1 \right],
$$

(1)

where $\Omega$’s are the Rabi frequencies, $g$ and $\tilde{g}$ are the coupling constants, and $\sigma$’s are the Pauli matrices. The protocol consists of the following sequence of symmetric pulses:

$$
X_{\pi/2} U_{\text{int}} Y_{\pi/2} |000\rangle = e^{-i\alpha} e^{i(\pi/4)} |\text{GHZ}\rangle,
$$

(2)

with the entangling time set to $t_{\text{GHZ}} = \pi/(2|g - \tilde{g}|)$, and

$$
X_\theta = X_\theta^{(3)} X_\theta^{(2)} X_\theta^{(1)},
Y_\theta = Y_\theta^{(3)} Y_\theta^{(2)} Y_\theta^{(1)},
$$

(3)

being the simultaneous single-qubit rotations (the unimportant overall phase $\alpha$ depends on the values of $g$ and $\tilde{g}$). The crucial step in the protocol is the initial $Y$-rotation by $\pi/2$ performed on all the qubits in the circuit, which is supposed to result in the fully uniform superposition of the computational states,

$$
|\psi\rangle_{\text{unif}} = (1/\sqrt{8}) (|000\rangle + |001\rangle + \cdots + |110\rangle + |111\rangle).
$$

(5)

The protocol is very fast, but, unfortunately, is impossible to implement due to severe problems with spectral crowding [14]. The energy levels, such as $|200\rangle$, etc., quickly get populated during the initial rotations.

Since the problem of leakage out of the computational subspace is not going away any time soon, one may try to tackle it by using the proverbial “If you can’t beat them, lead them” principle. Since the higher energy levels are here to stay, we should look for ways to minimize their influence or make them irrelevant altogether. It seems that Nature herself, by way of quantum mechanical trickery, offers a curious possibility of doing just that, in the form of the so-called quantum Zeno effect [15][22]. Before analyzing that effect in superconducting qubits, let us take a quick look at how it works in the simplest possible scenario.

2. BRIEF REVIEW OF THE QUANTUM ZENO EFFECT

Consider a two-level system, with basis states $|1\rangle$ and $|2\rangle$, whose dynamics is described by the Hamiltonian

$$
H = V (|1\rangle \langle 2| + |1\rangle \langle 2|) = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix},
$$

(6)
where the off-diagonal matrix elements are chosen to be real, for convenience. Denote the general time-dependent state of the system by

$$|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle,$$  \hspace{1cm} (7)

subject to the normalization condition,

$$|a_1(t)|^2 + |a_2(t)|^2 = 1.$$  \hspace{1cm} (8)

The evolution of the corresponding amplitudes, \(a_1\) and \(a_2\), is given by (here we use \(\hbar = 1\))

$$i\frac{da_1}{dt} = Va_2, \quad i\frac{da_2}{dt} = Va_1.$$  \hspace{1cm} (9)

Assume,

$$a_1(0) = 1, \quad a_2(0) = 0.$$  \hspace{1cm} (10)

Then, in linear order, for small times, \(t = \Delta t\), such that \(|V\Delta t| \ll 1\), the amplitudes are given by

$$a_1(\Delta t) \approx 1 + o(\Delta t^2), \quad a_2(\Delta t) \approx -iV\Delta t,$$  \hspace{1cm} (11)

with the probability of finding the system in state \(|2\rangle\) being

$$w_2(\Delta t) = |a_1(\Delta t)|^2 \approx V^2\Delta t^2.$$  \hspace{1cm} (12)

Normalization condition then gives the probability of finding the system in state \(|1\rangle\),

$$w_1(\Delta t) = 1 - w_2(\Delta t) = 1 - V^2\Delta t^2.$$  \hspace{1cm} (13)

Assume now that the total time, \(T\), of system’s evolution is divided into \(n\) small equal time intervals, \(\Delta t\), such that \(T = n\Delta t\), with \(|V\Delta t| \ll 1\). Assume additionally that at the end of the first time interval an ideal instantaneous measurement represented by the projection operator \(P_2 \equiv |2\rangle\langle 2|\) has been made – typically implemented by some kind of tunneling process out of state \(|2\rangle\), – that gave a negative result. Then the state of the system immediately after the measurement is described by the same ket \(|1\rangle\) in which the system was initially prepared at \(t = 0\).

We now calculate the probability \(W_1^{(n)}(T)\) of the complex event, \(\mathcal{E}\), consisting of a series of \(n\) \(P_2\)-measurements, resulting in the system remaining in the \(|1\rangle\) state at time \(T\). For this to be possible, each of the intermediate measurements has to produce a negative result. Since all intermediate “negative” events are independent and each occurs with probability \(w_1\), the overall probability of the complex event \(\mathcal{E}\) is given by

$$W_1^{(n)}(T) = [w_1(\Delta t)]^n \approx (1 - V^2\Delta t^2)^n = \left(1 - \frac{V^2T^2}{n^2}\right)^n \equiv \left(1 - \frac{q}{n^2}\right)^n,$$  \hspace{1cm} (14)

where \(q \equiv V^2T^2\). In the limit \(n \to \infty\), we get

$$W_1^{(\infty)}(T) = \lim_{n \to \infty} \left[\left(1 - \frac{q}{n^2}\right)^{q/n}\right] = \lim_{n \to \infty} e^{-q/n} = 1,$$  \hspace{1cm} (15)

which constitutes the so-called quantum Zeno effect. Essentially, and counter-intuitively, by continually “watching” the system’s \(|2\rangle\) state, we “freeze” the system in state \(|1\rangle\).

3. APPLICATION TO A RABI-DRIVEN SUPERCONDUCTING QUBIT

It is now clear how to use the quantum Zeno effect to restrict qubit’s dynamics to the computational subspace. By continually measuring the qubit’s higher state(s), we will automatically confine the qubit to its computational subspace.

By expressing the leakage to states \(|E_1, E_2, E_3\rangle\), whose Rabi oscillation dynamics is implemented by a microwave drive of frequency, \(\omega = \omega_{12}\), which is resonant with the \(|1\rangle \leftrightarrow |2\rangle\) transition. This is described by the Hamiltonian, which, in the rotating wave approximation, is given by \([2]\)

$$H = \begin{pmatrix} 0 & \Omega e^{i\phi} & 0 \\ \Omega e^{-i\phi} & 0 & \sqrt{2}\Omega e^{i\phi} \\ 0 & \sqrt{2}\Omega e^{-i\phi} & \eta \end{pmatrix},$$  \hspace{1cm} (16)

where \(\eta \equiv E_3 - 2E_2 < 0\) is the qubit anharmonicity (assuming \(E_1 = 0\)), \(\phi\) is the phase of the drive, and \(\Omega\) is the time-dependent (pulse-shaped) Rabi frequency, which we take to be constant, for simplicity.

For future use, we will be interested in the case \(\phi = -\pi/2\), which implements qubit rotations around the \(Y\)-axis of the Bloch sphere, with the corresponding Hamiltonian being

$$H = \begin{pmatrix} 0 & -i\Omega & 0 \\ i\Omega & 0 & -i\sqrt{2}\Omega \\ 0 & i\sqrt{2}\Omega & \eta \end{pmatrix}.$$  \hspace{1cm} (17)

Writing the unitary evolution operator in the form

$$U(\Delta t) = 1 + (-i)H\Delta t + \frac{(-i)^2}{2!}H^2\Delta t^2 + \ldots,$$  \hspace{1cm} (18)

and assuming that the initial state of the system is

$$|\psi(0)\rangle = a_1(0)|1\rangle + a_2(0)|2\rangle,$$  \hspace{1cm} (19)

$$|a_1(0)|^2 + |a_2(0)|^2 = 1,$$  \hspace{1cm} (20)

$$a_3(0) = 0,$$  \hspace{1cm} (21)
we find, in second order, the state at a later time $\Delta t$ to be
\begin{align*}
a_1(\Delta t) &= a_1(0) \left(1 - \frac{1}{2}\Omega^2 \Delta t^2\right) - a_2(0) \Omega \Delta t, \quad (22) \\
a_2(\Delta t) &= a_2(0) \left(1 - \frac{3}{2}\Omega^2 \Delta t^2\right) + a_1(0) \Omega \Delta t, \quad (23) \\
a_3(\Delta t) &= \sqrt{2}a_2(0)\Omega \Delta t + a_2(\Delta t^2). \quad (24)
\end{align*}
Correspondingly, the probabilities for finding the system in state $|3\rangle$ and in the computational subspace are, respectively,
\begin{align*}
w_3(\Delta t) &= |a_3(\Delta t)|^2 = 2|a_2(0)|^2 \Omega^2 \Delta t^2, \quad (25) \\
w_{\text{comp.}}(\Delta t) &\equiv w_1(\Delta t) + w_2(\Delta t) = 1 - 2|a_2(0)|^2 \Omega^2 \Delta t^2. \quad (26)
\end{align*}

By analogy with the discussion around Eq. (14), in the case of a long sequence of $n$ $P_3$-measurements with negative outcomes, the probability $W^{(n)}_{\text{comp.}}(T)$ of the complex event $E_{\text{comp.}}$, in which the qubit remains in its computational subspace at time $T$, can be estimated as,

\begin{equation}
W^{(n)}_{\text{comp.}}(T) \approx \left(1 - 2|a_2|^2\right)^n \approx 1 - 2|a_2|^2 + \left(2\Omega^2 T^2 / n^2\right)^n, \quad (27)
\end{equation}

\begin{equation}
geq (1 - 2\Omega^2 \Delta t^2)^n = \left(1 - \frac{2\Omega^2 T^2}{n^2}\right)^n \rightarrow 1, \quad n \rightarrow \infty, \quad (28)
\end{equation}

since $|a_2|^2 \leq 1$ at every step of system’s evolution.

As far as the evolution of the amplitudes is concerned, we return to Eqs. (22), (23), (24), and notice that if at time $\Delta t$, $|V\Delta t| \ll 1$, an ideal $P_3$-measurement is made with the negative outcome (for the system to be found in its $|3\rangle$-state), the updated renormalized state of the system will be, in second order,
\begin{align*}
a_1(\Delta t) &= a_1 - a_2 \Omega \Delta t - a_1 \left(\frac{1}{2} - |a_2|^2\right) \Omega^2 \Delta t^2, \quad (29) \\
a_2(\Delta t) &= a_2 + a_1 \Omega \Delta t - a_2 \left(\frac{1}{2} + |a_1|^2\right) \Omega^2 \Delta t^2, \quad (30) \\
a_3(\Delta t) &= 0, \quad (31)
\end{align*}

where on the right hand side of each equation the label $(0)$ has been dropped for notational simplicity. For $a_1 = 1$, $a_2 = 0$, we get
\begin{align*}
a_1(\Delta t) &= 1 - \frac{\Omega^2 \Delta t^2}{2} \approx \cos(V \Delta t), \quad (32) \\
a_2(\Delta t) &= \Omega \Delta t \approx \sin(V \Delta t), \quad (33) \\
a_3(t) &= 0, \quad (34)
\end{align*}

and for $a_1 = 0$, $a_2 = 1$, we get
\begin{align*}
a_1(\Delta t) &= -\Omega \Delta t \approx -\sin(\Omega \Delta t), \quad (35) \\
a_2(\Delta t) &= 1 - \frac{\Omega^2 \Delta t^2}{2} \approx \cos(\Omega \Delta t), \quad (36) \\
a_3(\Delta t) &= 0. \quad (37)
\end{align*}

The above discussion indicates that in the presence of continuous higher-energy state measurement the qubit will keep evolving as if the higher-energy state did not exist.

Figures 1, 2, and 3 show the time dependence of various state populations,
\begin{equation}
p_i(t) \equiv |a_i(t)|^2, \quad i = 1, 2, 3, \quad (38)
\end{equation}
during the execution of the Rabi pulse applied to the system initially prepared in its ground state, $|\psi(0)\rangle = (1, 0, 0)$, and interrupted by $n = 25, 50$, and 100 ideal $P_3$-measurements, with the assumption that the detector never clicked. We distinguish the following three cases:

The ideal case, shown just for comparison, corresponds to the evolution under the action of the model Hamiltonian,
\begin{equation}
H_{\text{ideal}} = \begin{pmatrix} 0 & -i\Omega & 0 \\ i\Omega & 0 & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (39)
\end{equation}
without any leakage out of the computational subspace.

The Zeno case corresponds to the evolution under the action of $H$ given in (17) and subjected to $n$ projective $P_3$-measurements. The evolution was numerically calculated in accordance with the iterative scheme,
\begin{align*}
|\psi_1\rangle &= |1\rangle, \\
|\psi_k\rangle &= \frac{P_{\text{comp.}}|\psi_{k-1}\rangle}{\sqrt{||P_{\text{comp.}}|\psi_{k-1}\rangle||}}, \\
|\psi_{k+1}\rangle &= \exp\{-iH\Delta t\}|\psi_k\rangle, \quad (40)
\end{align*}
for $k = 2, 3, \ldots, n$, where $P_{\text{comp.}}$ is the projection operator onto the computational subspace,
\begin{equation}
P_{\text{comp.}} \equiv 1 - P_3 = 1 - |3\rangle \langle 3|. \quad (41)
\end{equation}
At the final time $T = n \Delta t$ the qubit remains in its computational subspace (that is, $p_3(T) = |a_3(T)|^2 = 0$),
with the probability (let us call it the survival probability) given by the product (compare to (27)),

\[
W^{(n)}_{\text{comp.}}(T) = \prod_{k=1}^{n} (1 - p_3(t_k)).
\]

Finally, the no-Zeno case corresponds to the exact evolution,

\[
U(t) = \exp\{-iHt\},
\]

with the leakage to \(|3⟩⟩\) taken into account. The probability for the system to be found in the computational subspace in a single measurement performed at the end of the evolution is

\[
W_{\text{comp.}}(T) = 1 - p_3(T).
\]

A quick glance at the figures shows the presence of the Zeno effect in our system. There is a critical number of the measuring steps, \(n_{\text{crit}}\), depending on the choice of the system parameters, above which the survival probability \(W^{(n)}_{\text{comp.}}(T)\) exceeds \(W_{\text{comp.}}(T)\). In the limit \(n \to \infty\), \(W^{(n)}_{\text{comp.}}(T)\) approaches 1, as expected.

Assume now that the \(|3⟩⟩\) level is allowed to tunnel out with the rate \(\Gamma\). The third diagonal matrix element of the Hamiltonian (17) would then get modified as

\[
\eta \to \eta - i\Gamma/2,
\]

4. TAKING A CLOSER LOOK AT EQ. (24) AND DESIGNING A MEASUREMENT MODEL

We now take a closer look at Eq. (24), in which the second order term was ignored. Restoring that \(o(\Delta t^2)\) term, we find

\[
a_3(\Delta t) = \sqrt{2}a_2(0)\Omega\Delta t + \frac{\sqrt{2}}{2} [a_1(0)\Omega^2 - a_2(0)\Omega\eta] \Delta t^2.
\]

(45)
The first order term, rate is chosen to be canceled, giving

\[ a_3(\Delta t) = \frac{\sqrt{2}}{2} \Omega a_2(0) \Delta t \]

\[ + \frac{\sqrt{3}}{2} \left[ a_1(0) \Omega^2 - a_2(0) \Omega \left( \frac{\Gamma}{2} + \eta i \right) \right] \Delta t^2. \]  

(47)

We now make the following observation: if the tunneling rate is chosen to be

\[ \Gamma = \frac{4}{\Delta t}, \]  

(48)

the first order term, \( \sqrt{2} a_2(0) \Omega \Delta t \), in Eq. (47) would get canceled, giving

\[ a_3(\Delta t) = \frac{\sqrt{3}}{2} \left[ a_1(0) \Omega^2 - a_2(0) \Omega \eta i \right] \Delta t^2, \]  

(49)

which would result in the fourth order population of the unwanted state [3].

Figure 4 shows the result of exact simulation for the system initially prepared in the ground state, using the Hamiltonian

\[ H = \begin{pmatrix} 0 & -i\Omega & 0 \\ i\Omega & 0 & -i\sqrt{2}\Omega \\ 0 & i\sqrt{2}\Omega & -\eta - i\Gamma/2 \end{pmatrix}. \]  

(50)

with \( \Gamma = 40 \text{ [ns}^{-1}] \) and \( \Omega = 0.05 \text{ [GHz]} \) (this effectively corresponds to \( n = 50 \) with \( \Delta t = 0.1 \text{ [ns]} \) of the Zeno case). The survival probability is defined by

\[ W_{\text{comp.}}^{(\text{tunnel})}(t) = |a_1(t)|^2 + |a_2(t)|^2. \]  

(51)

The corresponding dynamics can be viewed as a reasonable measurement model for state [3]. We see that for the chosen system’s parameters leakage out of the computational subspace is virtually nonexistent.

5. CONCLUSIONS

Working with the RWA Hamiltonian of a three-level qubit driven by a Rabi pulse, we showed that, by allowing the system to continuously tunnel out of its higher-energy state (the greater, \( \Gamma \) the better), we effectively confine the system’s dynamics to the computational subspace. Such quantum Zeno behavior can potentially be used to resolve the long standing spectral crowding problem that plagued the field of superconducting quantum computing for almost two decades.

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