Clustering of Monopoles in the Instanton Vacuum

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Abstract

We generate a random instanton vacuum with various densities and size distributions. We perform numerically the maximally abelian gauge fixing of these configurations in order to find monopole trajectories induced by instantons. We find that instanton-induced monopole loops form enormous clusters occupying the whole physical volume, provided instantons are sufficiently dense. It indicates that confinement might be caused by instantons.

1. Introduction

It is widely discussed that confinement, characterized by the area behavior of the Wilson loop, is related to monopoles \cite{1, 2, 3}. There is also growing evidence that instantons play a key role in the nonperturbative QCD. Instantons should be present in the Yang-Mills vacuum if only because a non-zero topological susceptibility is needed to resolve the so-called $U_A(1)$ paradox \cite{4, 5}. Also, instantons are playing an important role in all phenomena related to chiral symmetry breaking \cite{6}. Until now, however, there has been no evidence that instantons have anything to do with confinement. Rather on the contrary: the string tension is known to decrease with lattice cooling, whereas instantons survive many cooling steps \cite{7, 8, 9, 10, 11, 12, 13, 14, 15, 16}.

It may seem thus that instantons and monopoles are representing completely different sectors of physics, and are hardly related to one another. Recently, however, there has been evidence indicating that there might be a relation between the two objects \cite{10, 11, 12, 13, 14, 15, 16}. The lattice QCD simulations show strong correlations between the instantons and the monopole currents \cite{10, 12, 14}. Analytic studies of one and two instanton systems have
been performed by making the abelian gauge fixing as the Polyakov gauge [11, 14]. It has been shown that monopole trajectories tend to pass through instanton centers, while those far from instantons seem to be unstable under small variations. Further in the recent studies, another type solution of the monopole trajectories are found to surround the instanton configurations in the maximally abelian gauge.

Theoretically, the relation between instantons and monopoles is suggested by the following consideration. The presence of a monopole can be signaled by a test Wilson loop surrounding it: its value is $-1$ if the monopole is inside the contour (we imply the $SU(2)$ color for simplicity). As for instantons, if one takes a Wilson loop whose one side is going through the center of an (anti)instanton and the other three sides are closing the loop at infinity, its value is also $-1$ [17]. This is true for any plane cutting through the center of an instanton. From this point of view, any instanton contains spherically-distributed loops of monopole currents winding around the 4-d center of the instanton. The length of the loop is typically given by the instanton size, $\rho$. When one makes an abelian gauge fixing of the instanton field to pinpoint the monopole loop inside the instanton, its concrete form and orientation in space depends on the particular abelian gauge fixing used and on the orientation of the instanton in color space. However, the presence of monopole loops inside instantons is, by itself, a gauge-invariant statement.

What happens with monopole trajectories when one takes an ensemble of instantons? The answer to this question depends on the ratio of the typical instanton size to the average separation between instantons. Let us introduce the dimensionless packing fraction of instantons, $f = (\pi^2/2)\rho_0^4N/V$, where $\rho_0$ is the most probable instanton size, $N$ is the total number of pseudoparticles (instantons and anti-instantons) and $V$ is the 4-d volume. If this packing fraction is small, the pseudoparticles are well separated. Hence, the monopole loops associated with instantons do not mix. As the packing fraction increases, the instantons start to contact each other, and their monopole loops can well form large clusters going through many instantons. This has been observed in an idealized case [18] and in lattice investigations [15, 19]. A quantitative question arises, at what packing fraction does the “clusterization” happen, and what is the size of a typical cluster. If clusters of the size of the physical volume appear, it may witness a condensation of monopoles [3, 20].

In principle, different intermediate situations may take place. For example, let the most probable size of instantons in the ensemble, $\rho_0$, be such that the packing fraction is, on the average, small. However, if the size distribution has a long power-like tail at large sizes, certain rare but large instantons may overlap or just touch each other. As a result, the monopole loops inside them may hop from one pseudoparticle to another, so that a “percolation”-type phenomenon takes place, with monopole trajectories forming
large clusters.

The present paper reports on the results of a numerical investigation on the relation between the clustering of monopoles and the instanton densities.

2. Instantons

The field of one instanton with the size $\rho$ and the center $z_\mu$ in the singular gauge is

$$A^I_\mu(x; z_k, \rho_k, O_k) = \frac{i \rho^2 \tau^a O^i a}{(x - z)^2 [(x - z)^2 + \rho^2]},$$

(1)

where $O^a_i$ is the instanton orientation matrix and $\bar{\eta}^{i}_{\mu
u}$ is the 't Hooft symbol [4]. For anti-instantons $A^\bar{I}_\mu(x; z_k, \rho_k, O_k)$ one has to replace the $\bar{\eta}^{i}_{\mu
u}$ symbol by $\eta^{i}_{\mu
u}$. We shall work with the color $SU(2)$ group, hence the orientation matrix $O^a_i$ is an orthogonal $3 \times 3$ matrix which can be parametrized by 3 real parameters. We use random orientation matrices for the ensemble of instantons and anti-instantons ($I$’s and $\bar{I}$’s for short).

For simplicity we take a sum ansatz [5] for the system of $I$’s and $\bar{I}$’s ,

$$A_\mu(x) = \sum_k A^I_\mu(x; z_k, \rho_k, O_k) + \sum_k A^\bar{I}_\mu(x; z_k, \rho_k, O_k).$$

(2)

An important point is the distribution in the sizes of instantons in the ensemble. It has been recently shown analytically [17] that if the instanton size distribution falls off as $1/\rho^3$ at large $\rho$, one gets a linear confining potential, with the string tension proportional to the coefficient in the “one over cube” law. Previously certain arguments have been presented that the large-size tail of the instanton size distribution should behave as $1/\rho^5$ [21, 22]. Therefore, in our investigation we take the power of the tail $1/\rho^\nu$ of the instanton size distribution at large $\rho$ to be $\nu = 3$ or $\nu = 5$. As for small sizes, the distribution has to follow the one-loop result [4]: $d(\rho) \sim \rho^{b-5}$ where $b = 11N_c/3$ is the first Gell-Mann–Low coefficient.

We thus adopt the following size distribution,

$$d(\rho) = \frac{1}{(\rho_1^\nu + (\rho_2^b)^{b-5})}.$$  

(3)

Here, $\rho_1$ and $\rho_2$ are certain parameters so that the distribution (3) is normalized to unity, while the maximum of the distribution is fixed to a given value $\rho_0$, which is the most probable size of the pseudoparticles in the ensemble.

3. Maximally Abelian Gauge Fixing

We generate ensembles of $N$ pseudoparticles with random orientations and centers; the size distribution is given by Eq.(3). We take equal numbers of instantons and anti-instantons; $N_I = N_\bar{I} = N/2$. This is performed in the continuum theory. We then
introduce a lattice and express the classical field in terms of the unitary matrices \( U_\mu = \exp(i a A_\mu) \) living on the links. The continuum field is thus discretized. From now on, we do exactly the same as it is done in lattice calculations: we apply the maximally abelian gauge fixing [23] to extract the monopole trajectories for each instanton ensemble. The only difference with the usual lattice calculations is that we start not from the “hot” vacuum dominated by normal and probably innocent zero-point fluctuations, but from the ensemble of \( I \)'s and \( \bar{I} \)'s. It should be stressed, however, that our ensemble does not necessarily coincide with the ensemble of instantons obtained by cooling down the true or “hot” Yang–Mills vacuum: the lattice cooling procedure does not only kill the zero-point oscillations but may also distort considerably the original ensemble of instantons, especially if \( I \)'s and \( \bar{I} \)'s happen to overlap.

We take a \( 16^4 \) lattice with the lattice spacing of \( a = 0.15 \text{fm} \) and the most probable instanton size, \( \rho_0 = 0.4 \text{fm} \). The volume is thus fixed and equal to \( V = (2.4 \text{fm})^4 \). We consider four cases with the number of pseudoparticles equal to \( N = 20, 40, 60 \) and 80, corresponding to the density \( (N/V)^{1/4} = 174, 206, 228 \) and 245 MeV, respectively. Note that these values are to be compared with 200 MeV suggested by the study of instanton vacuum [25, 26]. The corresponding packing fractions are \( f = (\pi^2/2)\rho_0^4 N/V = 0.076, 0.152, 0.228 \) and 0.305. In all four cases we consider two regimes of the falloff of the size distributions: \( 1/\rho^3 \) and \( 1/\rho^5 \). The abelian gauge fixing is done by maximizing \( R = \sum_{\mu,s} \text{Tr}[U_\mu(s)\tau^3U_\mu^{-1}(s)\tau^3] \) in the maximally abelian gauge. We plot the distribution in the lengths of the monopole loops from 240 initial instanton ensembles.

4. Numerical Results

We show in Figs.1 and 2 the distributions in the lengths of monopole loops. We see that for low densities (Figs.1a and 2a) one has only relatively short monopole loops, whose distribution follows the distribution in the sizes of instantons [1]. It means that at low density there is no “hopping” of monopole trajectories from one instanton to another. At larger densities the length distribution starts to broaden. Fig. 1b demonstrates the first appearance of very large monopole clusters; in Fig. 1c their typical length is already around 4000 lattice units, whereas the total number of sites in our \( 16^4 \) lattice is 65,536. It means that about 1/16 of the total number of sites is connected by monopole currents. Since \( 1/16 \ll 1 \) we are not having problems in identifying monopole currents. At the same time a 4000-link cluster actually covers the whole physical volume. We conclude

\[\text{It is worth mentioning that the distribution of relatively short monopole loops in true lattice QCD has been recently measured by Hart and Teper who find a power behavior: } P(L) \sim 1/L^{2.87} \text{ [24]. It indicates that in reality the size distribution of instantons is rather } 1/\rho^3 \text{ than } 1/\rho^5.\]
that somewhere between Figs. 1b and 1c a percolation-type phase transition occurs. Note
that it occurs when the bulk of instantons are still relatively dilute (the average packing
fraction is between 1/6 and 1/4), and the instanton density is of the order of \((200\text{MeV})^4\)
which is a very reasonable value from the point of view of the instanton vacuum \[25, 26\].
The same phenomenon happens in the \(1/\rho^5\) case, see Fig. 2d, but at instanton densities
about twice larger than in the \(1/\rho^3\) case, which is probably less realistic.

It is instructive to compare our results with those of the true \(SU(2)\) lattice QCD with
\(16^3 \times 4\) at different temperatures \[19, 20\] as shown Fig. 3. We see that the distribution
of the monopole lengths in the confinement phase (\(\beta = 2.2\)) resembles the distribution
we get from the instanton ensemble at the density around \((228 \text{ MeV})^4\). The distribution
obtained in the deconfinement phase is very similar to that which we get from a low-density
instanton ensemble. Since the density of instantons is known to be suppressed at high
temperatures, the deconfinement phase transition can be thus qualitatively understood as
due to the inevitable dilution of instantons at high temperatures.

To conclude, we have demonstrated that a random ensemble of instantons and anti-
instantons produces, after the standard maximally abelian gauge fixing, enormous clusters
of monopole loops. This is quite similar to what is obtained in the true lattice QCD, where
the appearance of enormous clusters is believed to have relevance to the confining behavior
of the Wilson loop. The percolation-like phase transition to monopole clusters covering
the whole physical volume happens when the bulk of instantons is still relatively dilute
(their packing fraction is from 1/6 to 1/4). It should be stressed that our ensemble of
instantons has much entropy due to random positions, sizes and orientations; cooling
such an ensemble would probably freeze out most of its degrees of freedom. That might
explain why the well-cooled lattice where instantons are usually detected, looses most of
its original string tension.

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**Figure Captions**

Fig.1: Histograms of monopole loop lengths for the number of pseudoparticles; \( N = 20, 40, 60 \) and 80, which correspond to the instanton densities of \( (N/V)^{1/4} = 174, 206, 228 \) and 245 MeV, respectively. This is the case of the instanton size distribution with the fall off as \( 1/\rho^3, \nu = 3 \) in Eq.(3).

Fig.2: Histograms of monopole loop lengths for the number of pseudoparticles; \( N = 20, 40, 60 \) and 80. The numbers indicated are the corresponding instanton densities; \( (N/V)^{1/4} \). The instanton size distribution falls off as \( 1/\rho^5, \nu = 5 \) in Eq.(3).

Fig.3: The comparison of the distributions of the monopole loop length. Shown in the upper part are the \( SU(2) \) lattice QCD results at \( \beta = 2.35 \) and \( \beta = 2.2 \), which correspond to the deconfinement (high temperature) and the confinement (low temperature) phases, respectively [19]. The lower part denotes the results for the instanton ensemble with the instanton size distribution, \( \nu = 3 \), at the instanton densities \( (N/V)^{1/4} = 174 \) MeV (low instanton density) and \( (N/V)^{1/4} = 228 \) MeV (high instanton density).