Convex Hulls of Dragon Curves

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Abstract

Dragon curves $K_\eta$, $\eta \in (0, \pi/3)$, are a family of self-similar sets in the plane. We prove that every dragon curve has a polygonal convex hull. Moreover, the vertices of the convex hull $\text{co}(K_\eta)$ are given in the case $8\cos^4 \eta \geq 1$.

Keywords Heighway Dragon, Convex Hull, Fractal, Iterated Function System

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1 Introduction

Let $A$ be a $d \times d$ matrix and $d_i \in \mathbb{R}^d$. We assume that $A$ is contractive. The convex hull of the attractor of iterated function system (IFS for short) $\{f_i \mid i = 1, 2, \ldots, m\}$ with $f_i = Az + d_i$ is studied by Strichartz-Wang [9]. They observed an important property of extreme points of the convex hull and deduced that the attractor has a polygonal convex hull if and only if there exists a positive integer $s$ such that $A^s$ is a scalar matrix. Kirat-Kocyigit [7] considered the case that the linear part of $f_i$ may not be identical and proved that, if the attractor has a polygonal convex hull, the vertices must have eventually periodic codings. In contrast with this result, we further get the following theorem.

Theorem 1.1. Let $K$ be the attractor of an IFS $\{f_i \mid i = 1, 2, \ldots, m\}$ on the complex plane $\mathbb{C}$ with

$$f_i(z) = a_i z + b_i, \ a_i, b_i \in \mathbb{C}, \ 0 < |a_i| < 1.$$ 

Suppose $K$ is not a singleton. If an eventually periodic word $i_1i_2\cdots i_l(j_1\cdots j_k)^\infty$ in $\{1, 2, \ldots, m\}^\mathbb{N}$ is a coding of an extreme point of $\text{co}(K)$ then $a_{j_1}a_{j_2}\cdots a_{j_k} > 0$. 

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For an infinite word $i_1i_2\cdots$ in $\{1, 2, \ldots, m\}^\mathbb{N}$ and an integer $k \geq 1$ denote by $f_{i_1\cdots i_k}$ the composition $f_{i_1} \circ \cdots \circ f_{i_k}$. If $\cap_{k=1}^\infty f_{i_1\cdots i_k}(K) = z$, then $i_1i_2\cdots$ is called a coding of the point $z$ in $K$.

Kirat-Kocyigit [7] also gave a sufficient and necessary condition such that the attractor of a given IFS has a polygonal convex hull. Moreover, they found an algorithm to check their condition, but the termination of the algorithm is not discussed.

The present paper is devoted to studying the convex hulls of dragon curves. Let $C$ be the complex plane. For $z \in C$ denote by $\text{arg } z$ the argument of $z$ in $[0, 2\pi)$, by $\text{Re } z$ and $\text{Im } z$ the real and imaginary part of $z$, and by $\bar{z}$ the conjugate of $z$. Let $\eta \in (0, \pi/3)$ and let

$$a := a(\eta) = \frac{e^{-i\eta}}{2\cos \eta}. \quad (1)$$

The $\eta$-Dragon curve $K_\eta$ is the attractor of the IFS

$$f_1(z) = az, f_2(z) = 1 - \bar{a}z, z \in C.$$ 

In other words, $K_\eta$ is an unique nonempty compact subset of $C$ satisfying

$$K_\eta = f_1(K_\eta) \cup f_2(K_\eta). \quad (2)$$

The $\eta$-dragon curve has also been obtained as the limit of the renormalized paperfolding curves in the Hausdorff metric as well; see [1, 10]. By using their algorithm, Kirat-Kocyigit [7] verified that the dragon curve $K_{\pi/4}$ has a polygonal convex hull. We will prove that every dragon curve has this property.

**Theorem 1.2.** For each $\eta \in (0, \pi/3)$ the convex hull $\text{co}(K_\eta)$ is a polygon.

Actually, we find out a countable subset $V$ of $K_\eta$ and prove that its convex hull $\text{co}(V)$ is a polygon with $\text{co}(V) = \text{co}(K_\eta)$.

Given $\eta \in (0, \pi/3)$, let $a, f_1, f_2$, and $K_\eta$ be defined as above. One has

$$\frac{1}{2} < |a| < 1, a + \bar{a} = 1, \text{ and } 2|a| \cos \eta = 1. \quad (3)$$

Let $z_0$ be the fixed point of the composition $f_{2211}$. Then one has $z_0 = ca \in K_\eta$ by a simple computation, where

$$c = \frac{1}{1 - |a|^4}. \quad (4)$$

For every integer $k \geq 0$ let

$$z_k = f_1^k(z_0), \quad w_k = f_2(z_k), \quad \text{and } \quad b_k = f_2(w_{k+1}). \quad (5)$$

Then

$$z_k = ca^{k+1}, \quad w_k = 1 - c|a|^2a^k, \quad \text{and } \quad b_k = a + c|a|^3a^k. \quad (6)$$
We define a countable subset $V$ of $K_\eta$ by

$$V = \{b_0\} \cup \{z_k : k \geq 0\} \cup \{w_k : k \geq 1\}. \quad (7)$$

Since $z_0 \in K_\eta$, one has $V \subset K_\eta$ by (5). For every integer $k \geq 1$ let

$$V_k = \{b_0, z_0, z_1, \ldots, z_k, w_1, \ldots, w_k\}. \quad (8)$$

We shall show

$$\text{co}(V) = \text{co}(V_k) \quad (9)$$

for sufficiently large integer $k$ depending on $\eta$. Therefore $V$ has a polygonal convex hull. By the construction of the attractor $K_\eta$ we may further prove that $\text{co}(K_\eta) = \text{co}(V)$, which gives Theorem 1.2. Detailed proof will be given in Section 3.

We shall see that, in the proof of Theorem 1.2, the vertices of $\text{co}(K_\eta)$ are not determined completely. To answer this question, the first work is to find the smallest integer with the property (9). Let

$$\Phi_k(\eta) = (1 - |a|^4) \sin(k - 1)\eta - |a|^3 \sin(k - 2)\eta + |a|^k \sin \eta. \quad (10)$$

We will show that for each integer $k \geq 4$ the function $\Phi_k$ has a unique null in the interval $(\pi/k, \pi/(k - 1))$. We denote this zero point of $\Phi_k$ by $\eta_k$. Then the interval $(0, \pi/3)$ has a partition as

$$(0, \pi/3) = \left[\eta_4, \pi/3\right) \cup \bigcup_{k=4}^{\infty} [\eta_{k+1}, \eta_k). \quad (11)$$

For $\eta \in (0, \eta_4)$ we get the following result.

**Theorem 1.3.** Let $k \geq 4$ be an integer and let $\eta \in [\eta_{k+1}, \eta_k)$. Then the vertices of the polygon $\text{co}(K_\eta)$ are $b_0, z_0, z_1, \ldots, z_k, w_1, \ldots, w_k$ in clockwise.

We shall prove Theorem 1.3 in Section 2 and Theorem 1.2 in Section 3. For our purpose, some properties of functions $\Phi_k$ will be given in Section 4. Some properties of dragon curves will be given in Section 5. Theorem 1.3 will be proved in Section 6. We give an outline here for the convenience of readers. For $u, v, w \in \mathbb{C}$ denote by $\angle uvw$ the counterclockwise angle of $u$ to $v$ to $w$. That is,

$$\angle uvw = \arg \frac{w - v}{u - v}.$$  

Then $\angle uvw \in (0, \pi)$ means that $v$ is in the left-hand side of the straight line passing through $u$ and $w$ of direction $(w - u)/|w - u|$. We have the implications:

$$\angle uvw \in (0, \pi) \iff \text{Im} \frac{w - v}{u - v} > 0 \iff \text{Im}(\bar{u} - \bar{v})(w - v) > 0.$$

It is not difficult to get a generic result for all dragon curves as follows:

$$\angle b_0 z_0 z_1 = \angle z_k z_{k+1} z_{k+2} = \angle w_k w_{k+1} w_{k+2} = \pi - \eta.$$
for each $\eta \in (0, \pi/3)$ and each integer $k \geq 0$. Moreover, given $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$, we may prove

$$\angle z_{k-1}z_kw_1, \angle z_kw_1w_2, \angle w_{k-1}w_kb_0, \angle w_kb_0z_0 \in (0, \pi)$$

and

$$\text{co}(V_k) = \text{co}(V).$$

After that, we infer that for each $\eta \in [\eta_{k+1}, \eta_k)$ the points

$$b_0, z_0, z_1, \ldots, z_k, w_1, w_2, \ldots, w_k$$

are in turn the vertices of the polygon $\text{co}(V_k)$ in clockwise. Once these results are proved, Theorem 1.3 will follow from the proof of Theorem 1.2.

We remark that dragon curves are a class of path-connected self-similar sets in the plane [5], for which some basic geometric questions are subtle. For example, we know very little about when a dragon curve satisfies the open set condition; see [3, 4, 8]. Motivated by a question of Tabachnikov [10], Albers [1], Allouche et al [2], and Kamiya [6] studied self-intersecting and non-intersecting dragon curves, but the study on the question when a dragon curve is an arc is far from conclusive. As for the convex hull of $K_\eta$, we shall see that, in the case of $\eta \in [\eta_4, \pi/3)$, the point $z_4$ is no longer any vertex of $\text{co}(K_\eta)$. Moreover, we shall prove that, if $\eta$ is near to $\pi/3$, it is not true that $b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3$ are the vertices of the polygon $\text{co}(K_\eta)$ in clockwise. See Remark 6.2 at the end of Section 5. The vertex question of $\text{co}(K_\eta)$ is still open for $\eta \in [\eta_4, \pi/3)$.

2 The proof of Theorem 1.1

Let $K$ be the attractor of IFS $\{f_n| n = 1, 2, \ldots, m\}$ on the plane with

$$f_n(z) = a_nz + b_n, \ a_n, b_n \in \mathbb{C}, \ 0 < |a_n| < 1.$$  

Let $i_1i_2\cdots i_l(j_1\cdots j_k)\infty$ be a coding of an extreme point of $\text{co}(K)$. We are going to show $a_{j_1}a_{j_2}\cdots a_{j_k} > 0$.

We may write $a_{j_1}\cdots a_{j_k} = re^{i\alpha}$, where $r \in (0,1)$ and $\alpha \in [0,2\pi)$ are the modulus and argument of $a_{j_1}\cdots a_{j_k}$. Then

$$a_{j_1}\cdots a_{j_k} > 0 \iff \alpha = 0.$$  

Let $w$ be the unique fixed point of $f_{j_1}\cdots j_k$. Then $w \in K$, with coding $(j_1\cdots j_k)\infty$.

Since $K$ is not a singleton, a point $v \in K$ exists with $v \neq w$. Denote $v_p = f_{(j_1\cdots j_k)p}(v)$ for every positive integer $p$, then $v_p \in K$, $v_p \neq w$ and

$$v_p = (a_{j_1}\cdots a_{j_k})^p(v-w) + w = r^pe^{ip\alpha}(v-w) + w.$$  

(12)
If $\alpha \neq 0$, in view of (12), there is an integer $p \geq 2$ such that

$$w \in \text{co}\{v_1, v_2, \ldots, v_p\}$$

and $w$ is not a vertex of $\text{co}\{v_1, v_2, \ldots, v_p\}$.

We know that $f_{j_1\cdots j_k}(w)$ is the point of coding $i_1i_2\cdots i_l(j_1\cdots j_k)\infty$. Then

$$f_{j_1\cdots j_k}(w) \in \text{co}\{f_{j_1\cdots j_k}(v_1), f_{j_1\cdots j_k}(v_2), \ldots, f_{j_1\cdots j_k}(v_p)\}$$

and $f_{j_1\cdots j_k}(w)$ is not a vertex of $\text{co}\{f_{j_1\cdots j_k}(v_1), f_{j_1\cdots j_k}(v_2), \ldots, f_{j_1\cdots j_k}(v_p)\}$.

Since $v_p \in K$, then $f_{j_1\cdots j_k}(v_p) \in K$ for every positive integer $p$. The previous discussion implies that $f_{j_1\cdots j_k}(w)$ is not an extreme point of $\text{co}(K)$, contradicting the assumption of Theorem 1.1. This proves $\alpha = 0$ and thus completes the proof.

3 The proof of Theorem 1.2

Given $\eta \in (0, \pi/3)$, let $f_1, f_2, K_\eta, c, z_k, w_k, b_k, \Phi, V$ and $V_k$ be defined as in Section 1. To prove Theorem 1.2, we only need to show that $\text{co}(V)$ is a polygon and that $\text{co}(K_\eta) = \text{co}(V)$. The definitions of the above parameters and their relationships will be used frequently without mentioning.

Lemma 3.1. $\text{co}(V)$ is a polygon for each $\eta \in (0, \pi/3)$.

Proof. Let $\eta \in (0, \pi/3)$. By the definition of $z_k$, we see that there are points of $\{z_k: k \geq 0\}$ in the inner part of every quadrant of the plane. Therefore, 0 is an inner point of the convex hull $\text{co}(V)$. By the definition of $w_k$, we easily check that 1 is also an inner point of $\text{co}(V)$. Since by (6)

$$\lim_{k \to \infty} z_k = 0 \quad \text{and} \quad \lim_{k \to \infty} w_k = 1,$$

there exists an integer $n$ such that $\{z_k: k > n\} \cup \{w_k: k > n\}$ is in the inner part of $\text{co}(V)$. It then follows that

$$\text{co}(V) = \text{co}(V_n),$$

so $\text{co}(V)$ is a polygon.

Remark 3.1. By the above proof, $\text{co}(V)$ is closed and 0, 1 $\in \text{co}(V)$. Moreover, since $a^k = c^{-1}z_{k-1} = (1 - |a|^2)z_{k-1}$, one has $a^k \in \text{co}(V)$ for all integers $k \geq 1$.

For $z \in \mathbb{C}$ recall that $\arg z$ is the argument of $z$ in $[0, 2\pi)$. For $u, v, w \in \mathbb{C}$ denote by $uv$ the segment of endpoints $u$ and $v$, and by $\triangle(u, v, w)$ the closed solid triangle of vertices $u, v$ and $w$. By the definition, $w_0 = 1 - c|a|^2$ is a real number. It can be nonnegative or negative, depending on the choice of $\eta$. The next lemma is useful.

Lemma 3.2. We have $w_0 \in \triangle(0, z_2, z_3) \cap \triangle(0, z_2, z_3)$, if $w_0 < 0$. 

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Proof. Let \( \eta \in (0, \pi/3) \) be given such that \( w_0 < 0 \). Then \( 1 - |a|^2 - |a|^4 < 0 \), which occurs only if \( \eta \in (\pi/4, \pi/3) \). Thus,

\[
\arg z_2 = 2\pi - 3\eta > \pi \quad \text{and} \quad \arg z_3 = 2\pi - 4\eta < \pi,
\]
giving \( \arg z_3 < \arg w_0 < \arg z_2 \). On the other hand, let

\[
t = \frac{1}{|a|^2} - 1.
\]

Then \( t \in (0, 1) \) by \( |a| < 1 \) and \( 1 - |a|^2 - |a|^4 < 0 \). And we have

\[
(1-t)z_2 + tz_3 = a^2(1-t + ta) = a^2(1-t\bar{a})
\]

\[
= a^2(1 - \frac{\bar{a}}{|a|^2} + \bar{a}) = a^2(a - 1 + |a|^2)
\]

\[
= a^2(-\bar{a} + |a|^2) = c|a|^2a(1 + a) = -c|a|^4.
\]

Thus the segment \( z_2z_3 \) intersects the real axis at \(-c|a|^4\). Since

\[
1 - c|a|^2 + c|a|^4 = c(1 - |a|^2) > 0,
\]

one has \(-c|a|^4 < w_0\). The above facts imply \( w_0 \in \Delta(0, z_2, z_3) \).

Acting on \( w_0 \in \Delta(0, z_2, z_3) \) with \( f(z) = \bar{z} \), we get \( w_0 \in \Delta(0, \bar{z}_2, \bar{z}_3) \), as desired. \( \square \)

**Lemma 3.3.** \( f_1(\co(V)) \cup f_2(\co(V)) \subset \co(V) \) for each \( \eta \in (0, \pi/3) \).

**Proof.** It suffices to show \( f_1(V) \cup f_2(V) \subset \co(V) \). Clearly,

\[
f_1(V) = \{ f_1(b_0) \} \cup \{ z_k : k \geq 1 \} \cup \{ f_1(w_k) : k \geq 1 \}
\]

and

\[
f_2(V) = \{ f_2(b_0) \} \cup \{ w_k : k \geq 0 \} \cup \{ b_k : k \geq 0 \}.
\]

Since

\[
f_1(b_0) = a(a + c|a|^4) = a^2 + c|a|^4a = (1 - |a|^4)z_1 + |a|^4z_0,
\]

we get \( f_1(b_0) \in \co(V) \). On the other hand, since

\[
b_k = a + c|a|^4a^k = (1 - |a|^4)z_0 + |a|^4z_{k-1},
\]

we have \( b_k \in \co(V) \) for all integers \( k \geq 1 \). Thus, to complete the proof, we only need to prove \( f_1(w_k), f_2(b_0), w_0 \in \co(V) \) for all integers \( k \geq 1 \).

**The proof of** \( w_0 \in \co(V) \). \( \) If \( w_0 \geq 0 \), since \( w_0 = 1 - c|a|^2 \) and \( 0, 1 \in \co(V) \), we have \( w_0 \in \co(V) \). In the other case, we have \( w_0 \in \Delta(0, z_2, z_3) \) by Lemma 3.2, so \( w_0 \in \co(V) \).

**The proof of** \( f_2(b_0) \in \co(V) \). \( \) First, we have

\[
f_2(b_0) = 1 - \bar{a}(a + c|a|^4) = 1 - |a|^2 - c|a|^4\bar{a} = 1 - |a|^2 - c|a|^4 + c|a|^4a.
\]
By a long but elementary computation, we get

\[ f_2(b_0) = \frac{|a|^4 h + (1 - 2|a|^4 + |a|^6)(1 - |a|^2)}{1 - |a|^4 + |a|^6}, \]

where

\[ h = (1 - |a|^2 + |a|^4)z_1 + (|a|^2 - |a|^4)z_2. \]

Clearly, \( h \in \co(V). \) As was known, \( 0, 1 \in \co(V), \) which gives \( 1 - |a|^2 \in \co(V). \)

Additionally,

\[ 1 - 2|a|^4 + |a|^6 = (1 - |a|^2)(1 + |a|^2 - |a|^4) > 0. \]

It then follows that \( f_2(b_0) \in \co(V). \)

The proof of \( f_1(w_k) \in \co(V) \) for all integers \( k \geq 1. \) It will be done by induction.

We first show \( f_1(w_1) \in \co(V). \) Since \( a + \bar{a} = 1 \) and \( f_1(z) = az \), we have

\[ |a|^2(1 - \bar{a}w_0) + (1 - |a|^2)aw_0 \]
\[ = |a|^2 - |a|^2 \bar{a}w_0 + aw_0 - |a|^2aw_0 = |a|^2 + aw_0 - |a|^2w_0 \]
\[ = |a|^2 + a - c|a|^2a - |a|^2 + c|a|^4 = a(1 - c|a|^2 + c|a|^2\bar{a}) \]
\[ = a(1 - c|a|^2a) = aw_1 = f_1(w_1). \]

Thus the proof of \( f_1(w_1) \in \co(V) \) can be reduced to showing

\[ aw_0, 1 - \bar{a}w_0 \in \co(V). \quad (14) \]

If \( w_0 \geq 0, \) one has by \( 0, 1, a \in \co(V) \)

\[ aw_0 \in \co(V) \text{ and } 1 - \bar{a}w_0 = (1 - w_0) + w_0a \in \co(V). \]

If \( w_0 < 0, \) one has \( w_0 \in \triangle(0, z_2, z_3) \) by Lemma 2.2. Acting on it with \( f_1 \) and \( f_2 \) respectively, we get

\[ aw_0 = f_1(w_0) \in \triangle(f_1(0), f_1(z_2), f_1(z_3)) = \triangle(0, z_3, z_4) \]

and

\[ 1 - \bar{a}w_0 = f_2(w_0) \in \triangle(f_2(0), f_2(z_2), f_2(z_3)) = \triangle(1, w_2, w_3). \]

Therefore \( aw_0, 1 - \bar{a}w_0 \in \co(V). \) This proves \( f_1(w_1) \in \co(V). \)

Secondly, we show for every integer \( k \geq 1 \)

\[ f_1(w_{k+1}) = (1 - |a|^2)f_1(w_k) + |a|^2b_{k-1}. \quad (15) \]

In fact, one has by \( a + \bar{a} = 1 \)

\[ -a^{k+1} + a^{k+2} + |a|^2a^{k+1} + |a|^4a^{k-1} = 0. \]
Using this equality, we get
\[ f_1(w_{k+1}) = a - c|a|^2 a^{k+2} \]
\[ = a - c|a|^2 a^{k+2} + c|a|^3 (-a^{k+1} + a^{k+2} + |a|^2 a^{k+1} + |a|^4 a^{k-1}) \]
\[ = a - c|a|^2 a^{k+1} + c|a|^4 a^{k+1} + c|a|^6 a^{k-1} \]
\[ = a - c|a|^2 a^{k+1} - |a|^2 a + c|a|^4 a^{k+1} + |a|^2 a + c|a|^6 a^{k-1} \]
\[ = (1 - |a|^2)(a - c|a|^2 a^{k+1}) + |a|^2(a + c|a|^4 a^{k-1}) \]
\[ = (1 - |a|^2)f_1(w_k) + |a|^2b_k. \]

Finally, since \( \{b_k : k \geq 0\} \subseteq \text{co}(V) \) and \( f_1(w_1) \in \text{co}(V) \) have been proved, by the formula (15) we get \( f_1(w_k) \in \text{co}(V) \) for all integers \( k \geq 1 \) by induction. \( \Box \)

**Proof of Theorem 1.2.** Let \( \eta \in (0, \pi/3) \). Since \( V \subseteq K_\eta \), one has \( \text{co}(V) \subseteq \text{co}(K_\eta) \). On the other hand, since \( \text{co}(V) \) is closed, the self-similarity construction of the dragon curve \( K_\eta \) together with Lemma 3.3 implies \( K_\eta \subseteq \text{co}(V) \), which yields \( \text{co}(K_\eta) \subseteq \text{co}(V) \). Thus, \( \text{co}(K_\eta) = \text{co}(V) \). It then follows from Lemma 3.4 that \( \text{co}(K_\eta) \) is a polygon.

### 4 The properties of functions \( \Phi_k, \Psi_k, \) and \( \Theta_k \)

Let
\[ \Phi_k(\eta) = (1 - |a|^4)\sin(k-1)\eta - |a|^3\sin(k-2)\eta + |a|^k \sin \eta, \]
\[ \Theta_k(\eta) = (1 - |a|^4)\sin \eta + |a|^k \sin k\eta \]
and
\[ \Psi_k(\eta) = \sin \eta + |a|^k \sin(k+1)\eta. \]

These functions are closely related to the geometry of dragon curves. In what follows we write \( A < B \) if \( A = CB \) for some \( C > 0 \).

**Lemma 4.1.** For each \( \eta \in (0, \pi/3) \) and for each integer \( k \geq 1 \) we have
\[ \text{Im}((\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) \asymp \Phi_k(\eta) \quad \text{and} \quad \text{Im}((\bar{z}_k - \bar{w}_1)(w_2 - w_1)) \asymp \Psi_k(\eta). \]

**Proof.** Let \( \eta \in (0, \pi/3) \) and let \( k \geq 1 \) be an integer. Observing that
\[ z_{k-1} - z_k \asymp a^k - a^{k+1} = a^k(1 - a) = a^k\bar{a} \asymp a^{k-1}, \]
we have
\[ \text{Im}((\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) \]
\[ \asymp \text{Im}(a^{k-1}(1 - c|a|^2 a - ca^{k+1})) \]
\[ \asymp \text{Im}(a^{k-1}(1 - |a|^4 - |a|^2 a - a^{k+1})) \]
\[ \asymp (1 - |a|^4)\sin(k-1)\eta - |a|^3\sin(k-2)\eta + |a|^{k+1} \sin 2\eta \]
\[ = (1 - |a|^4)\sin(k-1)\eta - |a|^3\sin(k-2)\eta + |a|^k \sin \eta. \]
On the other hand, since \( w_2 - w_1 = c|a|^2(a - a^2) = c|a|^4 \), one has
\[
\text{Im}((\bar{z}_k - \bar{w}_1)(w_2 - w_1)) = \text{Im}((\bar{z}_k - \bar{w}_1)
= \text{Im}(ca^{k+1} - 1 + c|a|^2\bar{a}) = \text{Im}(ca^{k+1} + c|a|^2\bar{a})
\geq |a|^{k-2}\sin(k + 1)\eta + \sin\eta,
\]
This completes the proof. \(\square\)

**Lemma 4.2.** For each integer \( k \geq 4 \) the function \( \Phi_k(\eta) \) has a unique null in the interval \((\pi/k, \pi/(k - 1))\).

**Proof.** We first show that \( \Phi_4(\eta) \) has a unique null in \((\pi/4, \pi/3)\). From the definition
\[
\Phi_4(\eta) = (1 - |a|^4)\sin 3\eta - |a|^3\sin 2\eta + |a|^4\sin \eta
\geq (1 - |a|^4)(3 - 4\sin^2\eta) - 2|a|^3\cos\eta + |a|^4
= (1 - |a|^4)(-1 + 4\cos^2\eta) - |a|^2 + |a|^4
= (1 - |a|^4)(\frac{1}{|a|^2} - 1) - |a|^2 + |a|^4
\geq (1 - 2|a|^4)(1 - |a|^2) \geq 1 - 2|a|^4.
\]
Since \(|a|\) is a strictly increasing function of \( \eta \), we see from the last relationship that \( \Phi_4(\eta) \) has a unique null in \((\pi/4, \pi/3)\). Denote this null of \( \Phi_4(\eta) \) by \( \eta_4 \). Then we have for each \( \eta \in (0, \eta_4) \)
\[
1 - 2|a|^4 > 0. \tag{16}
\]

Given an integer \( k \geq 5 \), we are going to show that \( \Phi_k(\eta) \) has a unique null in \((\pi/k, \pi/(k - 1))\). By the definition of \( \Phi_k \) and a simple triangular formula, we have
\[
\Phi_k(\eta) = (1 - |a|^2 - |a|^4)\sin(k - 1)\eta + |a|^3\sin k\eta + |a|^k\sin \eta.
\]
Thus
\[
\Phi_k\left(\frac{\pi}{k}\right) = (1 - |a|^2 - |a|^4 + |a|^k)\sin \frac{\pi}{k} > 0
\]
and
\[
\Phi_k\left(\frac{\pi}{k-1}\right) = (-|a|^3 + |a|^k)\sin \frac{\pi}{k-1} < 0.
\]
We only need to show the derivative \( \Phi'_k(\eta) < 0 \) for each \( \eta \in (\pi/k, \pi/(k - 1)) \).
Since the derivative \( d|a|/d\eta = 2|a|^2\sin \eta \), we have
\[
\Phi'_k(\eta) = -8|a|^3\sin \eta \sin(k + 1)\eta + (1 - |a|^4)(k - 1)\cos(k - 1)\eta
-6|a|^4\sin \eta \sin(k - 2)\eta - |a|^3(k - 2)\cos(k - 2)\eta
+2k|a|^{k+1}\sin^2 \eta + |a|^k\cos \eta.
\]
Given \( \eta \in (\pi/k, \pi/(k - 1)) \), since the cosine is decreasing in \((0, \pi)\), we have
\[
\cos(k - 1)\eta < \cos \frac{(k - 1)\pi}{k} = -\cos \frac{\pi}{k} < -\cos \eta
\]
and
\[ \cos(k-2)\eta > \cos \left( \frac{(k-2)\pi}{k-1} \right) = -\cos \frac{\pi}{k-1} > -\cos \eta. \]

In addition, since \( k \geq 5 \), we easily get \((k-1)\sin^2 \eta \leq 2\) for the given \( \eta \), in fact, for \( k = 5 \) we have \((k-1)\sin^2 \eta < 4\sin^2(\pi/4) = 2\), and for \( k \geq 6 \) we have \((k-1)\sin^2 \eta \leq (k-1)(\pi/(k-1))^2 \leq \pi^2/5 < 2\). Therefore
\[
2k|a|^{k+1} \sin^2 \eta + |a|^k \cos \eta \\
= 2k|a|^{k+1} \sin^2 \eta + 2|a|^{k+1} \cos^2 \eta \\
= 2|a|^{k+1} (1 + (k-1) \sin^2 \eta) \\
\leq 6|a|^{k+1} = 12|a|^{k+2} \cos \eta.
\]

Now, using the above inequalities, we get
\[
\frac{\Phi_k'(\eta)}{\cos \eta} < -(k-1)(1 - |a|^4) + (k-2)|a|^3 + 12|a|^7.
\]

Then, since \(|a| < 1/\sqrt{2}\) for the given \( \eta \), we get
\[
\frac{\Phi_k'(\eta)}{\cos \eta} < -\frac{3(k-1)}{4} + \frac{k-2}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} \leq -3 + \frac{3}{\sqrt{2}} < 0.
\]

This proves that \( \Phi_k \) has a unique null in \((\pi/k, \pi/(k-1))\).

For \( k \geq 4 \) denote by \( \eta_k \) the null of \( \Phi_k \) in \((\pi/k, \pi/(k-1))\). Clearly,
\[
\frac{1}{2} < |a| < \frac{1}{\sqrt{2}} \text{ for each } \eta \in (0, \frac{\pi}{4})
\]
and
\[
\frac{1}{\sqrt{2}} \leq |a| < \frac{1}{\sqrt{2}} \text{ for each } \eta \in \left[\frac{\pi}{4}, \eta_k\right).
\]

The latter is due to \([16]\). We shall use these two estimates without mentioning them. The properties of functions \( \Phi_k, \Theta_k, \text{ and } \Psi_k \), which will be used in the proof of Theorem \([13]\) are formulated in the following three lemmas.

**Lemma 4.3.** \( \Theta_j(\eta) > 0 \) for \( k \geq 4, \eta \in [\eta_{k+1}, \eta_k], \text{ and } j \geq k. \)

**Proof.** If \( k = 4 \) and \( \eta \in [\eta_5, \eta_4] \), one has
\[
\Theta_4(\eta) = (1 - |a|^4) + |a|^5 4 \cos \eta \cos 2\eta \\
= (1 - |a|^4) + |a|^4 \cos 2\eta \\
> 1 - 2|a|^4 > 0.
\]

If \( k = 4, j \geq 5, \text{ and } \eta \in [\pi/4, \eta_4] \), one has
\[
\Theta_j(\eta) = (1 - |a|^4) \sin \eta + |a|^{j+1} \sin j\eta \\
\geq (1 - |a|^4) \sin \frac{\pi}{4} - |a|^6 \\
\geq 1 - |a|^4 - \sqrt{2}|a|^6 \\
> 1 - \frac{1}{2} - \frac{1}{2} = 0.
\]
If \( k = 4, j \geq 5 \), and \( \eta \in [\eta_5, \pi/4) \), one has
\[
\Theta_j(\eta) = (1 - |a|^4) \sin \eta + |a|^{j+1} \sin j\eta \\
\geq (1 - |a|^4) \sin \frac{\pi}{5} - |a|^6 \\
\geq (1 - |a|^4) \frac{2}{5} - |a|^6 \\
> \frac{3 \cdot 2}{4 \cdot 5} - \frac{1}{8} > 0.
\]

If \( j \geq k \geq 5 \) and \( \eta \in [\eta_{k+1}, \eta_k) \), one has
\[
\Theta_j(\eta) = (1 - |a|^4) \sin \eta + |a|^{j+1} \sin j\eta \\
> (1 - |a|^4) \frac{2\eta}{\pi} - |a|^{k+1} \\
> (1 - |a|^4) \frac{2}{k+1} - |a|^{k+1} \\
> \frac{3}{2(k+1)} - \left( \frac{1}{\sqrt{2}} \right)^{k+1} > 0.
\]

This completes the proof.

**Lemma 4.4.** \( \Psi_j(\eta) > 0 \) for \( k \geq 4, \eta \in [\eta_{k+1}, \eta_k), \) and \( j \geq k. \)

**Proof.** If \( k = 4, \eta \in [\pi/4, \eta_4) \) and \( j \geq 4, \) one has
\[
\Psi_j(\eta) = \sin \eta + |a|^{j-2} \sin(j+1)\eta \geq \sin \frac{\pi}{4} - |a|^2 > \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.
\]

If \( k = 4 \) and \( \eta \in [\eta_5, \pi/4) \), one has
\[
\Psi_4(\eta) > \sin \frac{\pi}{5} + |a|^2 \sin \frac{5\pi}{4} = \sin \frac{\pi}{5} - |a|^2 \sin \frac{\pi}{4} > \frac{2}{5} - \frac{1}{2\sqrt{2}} > 0
\]
and for \( j \geq 5 \)
\[
\Psi_j(\eta) = \sin \eta + |a|^{j-2} \sin(j+1)\eta > \frac{2\eta}{\pi} - |a|^3 > \frac{2}{5} - \frac{1}{2\sqrt{2}} > 0.
\]

If \( k = 5 \) and \( \eta \in [\eta_6, \eta_5) \), one has
\[
\Psi_5(\eta) = \sin \eta + |a|^3 \sin 6\eta = \sin \frac{\pi}{6} - |a|^3 > \frac{1}{2} - \frac{1}{2\sqrt{2}} > 0
\]
and for \( j \geq 6 \)
\[
\Psi_j(\eta) > \frac{2\eta}{\pi} - |a|^4 > \frac{2}{6} - \frac{1}{4} > 0.
\]

If \( j \geq k \geq 6 \) and \( \eta \in [\eta_{k+1}, \eta_k) \), one has
\[
\Psi_j(\eta) > \frac{2\eta}{\pi} - |a|^{k-2} > \frac{2}{k+1} - \frac{1}{(\sqrt{2})^{k-2}} > 0.
\]

This completes the proof.
Lemma 4.5. Let $k \geq 4$. We have the following statements.

1. $\Phi_k(\eta) > 0$ for $\eta \in [\eta_{k+1}, \eta_k]$.
2. $\Phi_j(\eta) < 0$ in both the cases C1 and C2, where
   
   C1: $\eta \in [\pi/k, \eta_k)$ and $j \in \{k+1, k+2, \ldots, 2k-3\}$;
   C2: $\eta \in [\eta_{k+1}, \pi/k)$ and $j \in \{k+1, k+2, \ldots, 2k-1\}$.

Proof. (1) We have $\Phi_k(\eta) > 0$ for $\eta \in [\pi/k, \eta_k)$ by the proof of Lemma 4.2. In the case $\eta \in [\eta_{k+1}, \pi/k)$, we have
   
   $$1 - |a|^2 - |a|^4 > 0, \sin(k-1)\eta > 0, \text{ and } \sin k\eta > 0,$$

   which imply
   
   $$\Phi_k(\eta) = (1 - |a|^2 - |a|^4) \sin(k-1)\eta + |a|^3 \sin k\eta + |a|^k \sin \eta > 0.$$  

   (2) Let $k \geq 4$. For case C1 one has
   
   $$\pi + \eta \leq (k+1)\eta < (k+2)\eta < \cdots < (2k-3)\eta < 2\pi - \eta.$$

   Therefore
   
   $$\sin(j-1)\eta \leq 0 \text{ and } \sin j\eta < -\sin \eta.$$  

   Thus, in the case $1 - |a|^2 - |a|^4 \geq 0$, we immediately get
   
   $$\Phi_j(\eta) = (1 - |a|^2 - |a|^4) \sin(j-1)\eta + |a|^3 \sin j\eta + |a|^j \sin \eta < 0.$$ 

   The other case $1 - |a|^2 - |a|^4 < 0$ occurs only if $k = 4$ and $\eta \in [\pi/4, \eta_4)$, for which we get by (16) and (17)
   
   $$\Phi_5(\eta) < (|a|^4 - |a|^2) \sin 4\eta - |a|^3 \sin \eta + |a|^5 \sin \eta$$
   
   $$\times - \sin 4\eta - |a|^3 \sin \eta = -\sin 4\eta - |a|^2 \sin 2\eta$$
   
   $$\times (-2 \cos 2\eta - |a|^2 = 2 - \frac{1}{|a|^2} - |a|^2 < 0.$$ 

   For the case C2 one has $1 - |a|^2 - |a|^4 > 0$ and
   
   $$\pi + \eta < (k+2)\eta < \cdots < (2k-1)\eta < 2\pi - \eta.$$ 

   Then $\Phi_j(\eta) < 0$ for $j \in \{k+2, k+3, \ldots, 2k-1\}$ by an easier argument than what we just did. In addition, we have $\Phi_{k+1}(\eta) < 0$ for $\eta \in [\eta_{k+1}, \pi/k)$ by the proof of Lemma 4.2 and thus finish the proof.

5 The properties of dragon curves

The following properties on dragon curves are useful.
Lemma 5.1. For each \( \eta \in (0, \eta_k) \) we have \( \text{Re}z_0, \text{Rew}_1, \text{Im}w_1 \in (0, 1) \).

Proof. Given \( \eta \in (0, \eta_1) \), since \( 1 - 2|a|^4 > 0 \) by (16), one has
\[
|a|^4 < 1 \quad \text{and} \quad c < 2. \tag{18}
\]
Then \( \text{Re}z_0 = c|a| \cos \eta = c/2 \in (0, 1) \). As for \( w_1 \) we have
\[
\text{Im}w_1 = c|a|^3 \sin \eta < |a|^{-1} \sin \eta = \sin 2\eta \leq 1
\]
and
\[
\text{Rew}_1 = 1 - c|a|^3 \cos \eta = 1 - \frac{c|a|^2}{2} > 1 - |a|^2 > 0.
\]
Then we easily get \( \text{Rew}_1, \text{Im}w_1 \in (0, 1) \). This completes the proof. \( \Box \)

Lemma 5.2. \( \angle 1z_jw_1 \in (0, \pi) \) for \( k \geq 4 \), \( \eta \in [\eta_{k+1}, \eta_k) \), and \( j \geq k \).

Proof. For \( \eta \in (0, \pi/3) \) one has
\[
\text{Im}((1 - \bar{z}_j)(w_1 - z_j)) = \text{Im}((1 - \bar{z}_j)(w_1 - 1)) \\
\times \text{Im}((ca^j+1 - 1)a \sin \eta + c|a|^j+1 \sin j\eta) \\
\times (1 - |a|^4) \sin \eta + |a|^{j+1} \sin j\eta = \Theta_j(\eta),
\]
which together with Lemma 4.3 implies the desired result. \( \Box \)

Lemma 5.3. Let \( k \geq 4 \). We have in both the cases \( C1 \) and \( C2 \)
\[
z_j \in \text{co}\{0, z_{j-1}, w_1, 1\} \quad \text{and} \quad w_j \in \text{co}\{1, w_{j-1}, b_0, a\}.
\]

Proof. Remember that \( \text{arg} z \) denotes the argument of \( z \) in \( [0, 2\pi) \). Let \( k \geq 4 \).

In the case \( C1 \), i.e. \( \eta \in [\pi/k, \eta_k) \) and \( j \in \{k + 1, k + 2, \cdots, 2k - 3\} \), one has
\[
\pi + \eta \leq (k + 1)\eta < \cdots < (2k - 3)\eta < 2\pi - \eta,
\]
\[
\text{arg} z_k = 2\pi - (k + 1)\eta \in (\pi - 2\eta, \pi - \eta],
\]
\[
\text{arg} z_{2k-3} = 2\pi - (2k - 2)\eta \in (0, 2\eta], \quad \text{and}
\]
\[
0 < \text{arg} z_{2k-3} \leq \text{arg} z_j < \text{arg} z_{j-1} \leq \text{arg} z_k \leq \pi - \eta.
\]

By Lemma 5.2 we have \( \angle 1z_jw_1 \in (0, \pi) \). On the other hand, by Lemma 4.1 and Lemma 5.2, we have
\[
\text{Im}((\bar{w}_1 - \bar{z}_j)(z_{j-1} - z_j)) = -\text{Im}((\bar{z}_{j-1} - \bar{z}_j)(w_1 - z_j)) \approx -\Phi_j(\eta) > 0.
\]
which implies \( \angle w_1z_jz_{j-1} \in (0, \pi) \). In summary, the point \( z_j \) is located in the sector \( 0 < \text{arg} z < \text{arg} z_{j-1} \), with \( \angle 1z_jw_1, \angle w_1z_jz_{j-1} \in (0, \pi) \), by which we get
\[
z_j \in \text{co}\{0, z_{j-1}, w_1, 1\}.
\]
Now, acting on this relationship with \( f_2 \), we get
\[
w_j \in \text{co}\{1, w_{j-1}, b_0, a\}.
\]
In case \( C2 \) the argument is the same as that for case \( C1 \). \( \Box \)
Denote by $D(0, r)$ the closed disk of radius $r$ centered at the origin and by $v_0 v_1 \cdots v_j$ the broken segment formed by segments $v_k v_{k+1}$, $k = 1, \cdots, j$. All $\eta$-dragon curves with $\eta \in (0, \eta_k)$ have the following disk property.

**Lemma 5.4.** For each $\eta \in (0, \eta_k)$ and each integer $j \geq 1$ we have

$$\{ z_k : k \geq j \} \subset D(0, |z_j|)$$

and $z_0 z_1 \cdots z_{j-1} \subset \mathbb{C} \setminus D(0, |z_j|)$.

**Proof.** Let $\eta \in (0, \eta_k)$ and $j \geq 1$ be given. For every integer $k \geq 0$ we have

$$|z_{k+1}| < |z_k|,$$

which implies

$$\{ z_k : k \geq j \} \subset D(0, |z_j|).$$

On the other hand, one has for each $t \in [0, 1]$ 

$$|1 - t + t^2 a| = (1 - t + t^2(|a| \cos \eta)^2 + t^2|a|^2 \sin^2 \eta - |a|^4$$

$$= (1 - t)^2 + 2(1 - t)t|a| \cos \eta + t^2|a|^2 - |a|^4$$

$$= 1 - t + t^2|a|^2 - |a|^4.$$ 

If $2|a|^2 < 1$, we have 

$$1 - t + t^2|a|^2 - |a|^4 > |a|^2 - |a|^4 > 0.$$ 

If $2|a|^2 \geq 1$, we have by (16)

$$1 - t + t^2|a|^2 - |a|^4 = 1 - |a|^4 + |a|^2(t - \frac{1}{2|a|^2})^2 - \frac{1}{4|a|^2}$$

$$\geq 1 - |a|^4 - \frac{1}{4|a|^2} \geq 4|a|^2 - 4|a|^6 - 1$$

$$\geq 2|a|^2 - 4|a|^6 = 2|a|^2(1 - 2|a|^4) > 0.$$ 

Thus $|1 - t + ta| > |a|^2$ for each $t \in [0, 1]$. Then, given $j \geq 1$, we have 

$$|(1 - t)z_{j-1} + t z_j| > |z_{j+1}|$$

for any $t \in [0, 1]$. It follows that the broken segment $z_0 z_1 \cdots z_{j-1} \subset \mathbb{C} \setminus D(0, |z_j|)$.

\[\square\]

6 The Proof of Theorem 1.3

Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. To prove Theorem 1.3 we first prove $\text{co}(V_k) = \text{co}(V)$, which implies $\text{co}(V_k) = \text{co}(K_\eta)$ by the proof of Theorem 1.2. Secondly, we show that the points $b_0, z_0, z_1, \cdots, z_k, w_1, w_2, \cdots, w_k$ are in turn the vertices of the polygon $\text{co}(V_k)$ in clockwise.

The next lemma is generic for all $\eta$-dragon curves with $\eta \in (0, \pi/3)$.
**Lemma 6.1.** For each $\eta \in (0, \pi/3)$ and each integer $n \geq 0$ we have

$$\angle b_0 z_0 z_1 = \angle z_n z_{n+1} z_{n+2} = \angle w_n w_{n+1} w_{n+2} = \pi - \eta.$$ 

**Proof.** One has

$$\frac{z_1 - z_0}{b_1 - z_0} = \frac{c(a^2 - a)}{a + c|a|^4 - ca} = -\frac{a}{|a|^4}$$

and

$$\frac{z_{n+2} - z_{n+1}}{z_n - z_{n+1}} = \frac{w_{n+2} - w_{n+1}}{w_n - w_{n+1}} = -a.$$

Thus

$$\angle b_0 z_0 z_1 = \angle z_n z_{n+1} z_{n+2} = \angle w_n w_{n+1} w_{n+2} = \arg(-a) = \pi - \eta,$$

as desired. \qed

**Lemma 6.2.** For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have

$$\angle z_{k-1} z_k w_1, \angle z_k w_1 w_2, \angle w_{k-1} w_k b_0, \angle w_k b_0 z_0 \in (0, \pi).$$

**Proof.** Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. By Lemma 4.5(1), we have $\Phi_k(\eta) > 0$. By Lemma 4.4, we have $\Psi_k(\eta) > 0$. Then by Lemma 4.1 we get

$$\text{Im}(\bar{z}_{k-1} - \bar{z}_k)(w_1 - z_k)) > 0 \text{ and } \text{Im}((\bar{z}_k - w_1)(w_2 - w_1)) > 0,$$

which implies $\angle z_{k-1} z_k w_1, \angle z_k w_1 w_2 \in (0, \pi)$.

Since $f_2$ preserves angles, one has by the action of $f_2$

$$\angle z_{k-1} z_k w_1 = \angle w_{k-1} w_k b_0 \text{ and } \angle z_k w_1 w_2 = \angle w_k b_0 f_2(w_2).$$

As

$$f_2(w_2) = 1 - \bar{a}(1 - c|a|^2a^2) = a + c|a|^4a = ca = z_0,$$

we then get $\angle w_{k-1} w_k b_0, \angle w_k b_0 z_0 \in (0, \pi)$. This completes the proof. \qed

**Lemma 6.3.** For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have $0, 1 \in \co(V_k)$.

**Proof.** Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. We consider two cases.

Case 1. $\eta \in [\pi/k, \eta_k)$. We shall prove

$$0 \in \triangle(z_1, z_{k-1}, w_1) \text{ and } 1 \in \triangle(w_1, w_{k-1}, b_0).$$

In this case, one has

$$\arg z_{k-1} = 2\pi - k\eta \in (\pi - \eta, \pi],$$

so $z_{k-1}$ is in the second quadrant of the plane. Since

$$\text{Im}(\bar{z}_1 w_1) = \text{Im}(ca^2(1 - c|a|^2a)) \propto \sin 2\eta - c|a|^3 \sin \eta \propto 2 \cos \eta - c|a|^3 \propto 1 - c|a|^4 \propto 1 - 2|a|^4 > 0,$$
one has $\angle z_1 0 w_1 \in (0, \pi)$. As for the angle $\angle z_{k-1} 0 z_1$, it is obvious that 

$$\angle z_{k-1} 0 z_1 = (k - 2) \eta \in (0, \pi).$$

Since $w_1$ is in the first quadrant by Lemma 5.1, the above facts imply 

$$0 \in \triangle(z_1, z_{k-1}, w_1),$$

which in turn implies $1 \in \triangle(w_1, w_{k-1}, b_0)$ by the action of $f_2$.

Case 2. $\eta \in [\eta_{k+1}, \pi/k)$. In this case, $z_1$ is in the fourth quadrant; $z_k$ is in the second quadrant due to $\arg z_k \in (\pi - \eta, \pi)$; $\angle z_k 0 z_1 = (k - 1) \eta \in (0, \pi)$; and $w_1$ is in the first quadrant. These facts imply $0 \in \triangle(z_1, z_k, w_1)$, which in turn implies $1 \in \triangle(w_1, w_k, b_0)$ by the action of $f_2$. This completes the proof. \hfill $\square$

Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$. One has $\arg z_{2k-3} = 2\pi - (2k - 2) \eta \in (0, 2\eta]$.

**Lemma 6.4.** Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$. In the case $\arg z_{2k-3} \in (0, \eta)$ we have

$$z_{2k-2} \in \triangle(0, z_0, 1), \quad z_{2k-1} \in \triangle(0, z_1, z_0),$$

$$w_{2k-2} \in \triangle(1, w_0, a), \quad \text{and} \quad w_{2k-1} \in \triangle(1, w_1, w_0).$$

In the case $\arg z_{2k-3} \in (\eta, 2\eta]$ we have

$$z_{2k-2} \in \triangle(0, 1, z_{2k-3}), \quad z_{2k-1} \in \triangle(0, 0, 1),$$

$$w_{2k-2} \in \triangle(1, a, w_{2k-3}), \quad \text{and} \quad w_{2k-1} \in \triangle(1, w_0, a).$$

**Proof.** We only prove the latter. The proof of the former is similar.

Let $k \geq 4$ and $\eta \in [\pi/k, \eta_k)$ and assume $\arg z_{2k-3} \in (\eta, 2\eta]$. Then

$$\angle z_0 0 z_{2k-1} \leq \eta = \angle z_0 0 1. \tag{19}$$

On the other hand, using the inequality $1 - 2|a|^4 > 0$, we easily get $c < 2$ and

$$|z_{2k-1}| = c |a|^{2k} < c |a|^4 < c/2 < 1.$$

As $\Re z_0 = c |a| \cos \eta = c/2$, we then get

$$\Re z_{2k-1} < \Re z_0 < 1. \tag{20}$$

By (19) and (20) we obtain $z_{2k-1} \in \triangle(0, z_0, 1)$, which yields $w_{2k-1} \in \triangle(1, w_0, a)$ by the action with $f_2$.

Next we prove $z_{2k-2} \in \triangle(0, 1, z_{2k-3})$. As $\arg z_{2k-3} \in (\eta, 2\eta]$ is assumed, we have

$$0 < \arg z_{2k-2} < \arg z_{2k-3} < \pi.$$

It suffices to show $\angle 1 z_{2k-2} z_{2k-3} \in (0, \pi)$, which can be reduced to showing

$$\Im((1 - \bar{z}_{2k-2})(z_{2k-3} - z_{2k-2})) > 0. \tag{21}$$
In fact, if $k = 4$, by using $a + \bar{a} = 1$, $1 - 2|a|^4 > 0$, $c|a|^4 < 1$, $2|a| \cos \eta = 1$, and $|a| > 1/\sqrt{2}$, one has

$$\text{Im}((1 - \bar{z}_5)(z_5 - z_6)) = \text{Im}((1 - \bar{z}_5)(z_5 - z_6))$$

$$> \text{Im}(a^5 - c|a^{10}\bar{a}) \asymp - \sin 5\eta - c|a|^6 \sin \eta$$

$$> -4 \cos^2 \eta \cos 2\eta - \cos 4\eta - c|a|^6$$

$$= -16 \cos^4 \eta + 12 \cos^2 \eta - 1 - |a|^2$$

$$> -1 + 3|a|^2 - |a|^4 - |a|^6$$

$$= -(1 - |a|^2)^2 + |a|^2(1 - |a|^2)(1 + |a|^2)$$

$$> -1 + 2|a|^2 + |a|^4 > -1 + 2|a|^2 > 0.$$ 

If $k > 4$, one has $2\pi - (2k - 3)\eta \in (\eta, 3\eta) \subset (\eta, \pi - \eta)$, so

$$\text{Im}((1 - \bar{z}_{2k-2})(z_{2k-3} - z_{2k-2}))$$

$$> - \sin(2k - 3)\eta - c|a|^{2k-2} \sin \eta$$

$$= \sin(2\pi - (2k - 3)\eta) - c|a|^{2k-2} \sin \eta$$

$$> \sin \eta - c|a|^{2k-2} \sin \eta > 0.$$

This proves the inequality (21), so we have $z_{2k-2} \in \Delta(0, 1, z_{2k-3})$, which in turn gives $w_{2k-2} \in \Delta(1, a, w_{2k-3})$ by the action of $f_2$. 

Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \pi/k)$. One has arg $z_{2k-1} = 2\pi - 2k\eta \in (0, 2\eta]$.

**Lemma 6.5.** Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \pi/k)$. For the case arg $z_{2k-1} \in (0, \eta]$ we have

$$z_{2k} \in \Delta(0, z_0, 1), \quad z_{2k+1} \in \Delta(0, z_1, z_0),$$

$$w_{2k} \in \Delta(1, w_0, a), \quad w_{2k+1} \in \Delta(1, w_1, w_0).$$

For the case arg $z_{2k-1} \in (\eta, 2\eta]$ we have

$$z_{2k} \in \Delta(0, 1, z_{2k-1}), \quad z_{2k+1} \in \Delta(0, z_0, 1),$$

$$w_{2k} \in \Delta(1, a, w_{2k-1}), \quad w_{2k+1} \in \Delta(1, w_0, a).$$

**Proof.** The proof is the same as that of Lemma 6.4. 

**Lemma 6.6.** For $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ we have $\text{co}(V) = \text{co}(V_k)$.

**Proof.** Let $k \geq 4$ and $\eta \in [\eta_{k+1}, \eta_k)$ be given. One has

$$a, 0, 1, w_0 \in \text{co}(V_k)$$

by $a = c^{-1}z_0$, Lemma 6.3 and Lemma 3.2. It suffices to prove

$$\{z_j : j > k\} \cup \{w_j : j > k\} \subset \text{co}(V_k).$$  

(23)
We consider two cases.

Case 1. \( \eta \in [\pi/k, \eta_k) \). By using (22) and Lemma 5.3 we get

\[ z_{k+1}, z_{k+2}, \ldots, z_{2k-3}, w_{k+1}, w_{k+2}, \ldots, w_{2k-3} \in \text{co}(V_k) \]

by induction. Moreover, we have by Lemma 6.4

\[ z_{2k-2}, z_{2k-1}, w_{2k-2}, w_{2k-1} \in \text{co}(V_k) \]

and

\[ z_0 z_{2k-1} \subset \mathbb{C} \setminus D(0, |z_{2k}|). \] (24)

Then, by (24) and Lemma 5.3 we get

\[ \{z_j : j \geq 2k\} \subset D(0, |z_{2k}|) \subset \text{co}(\{z_0, z_1, \ldots, z_{2k-1}\}) \]

which in turn implies

\[ \{w_j : j \geq 2k\} \subset \text{co}(\{w_0, w_1, \ldots, w_{2k-2}, w_{2k-1}\}) \]

by the action of \( f_2 \). This proves (23).

Case 2. \( \eta \in [\eta_k+1, \pi/k) \). As we just did, by using (22), Lemma 5.3, 5.4 and 6.5 we may prove (23) by showing step in step

\[ z_{k+1}, \ldots, z_{2k-1}, w_{k+1}, \ldots, w_{2k-1} \in \text{co}(V_k), \]

\[ z_{2k}, z_{2k+1}, w_{2k}, w_{2k+1} \in \text{co}(V_k), \]

and

\[ \{z_j : j > 2k + 1\} \cup \{w_j : j > 2k + 1\} \subset \text{co}(V_k). \]

This completes the proof.

The proof of Theorem 1.3. Let \( k \geq 4 \) and \( \eta \in [\eta_k+1, \eta_k) \) be given. By Lemma 6.6 and the proof of Theorem 1.2 we have

\[ \text{co}(V_k) = \text{co}(V) = \text{co}(K_\eta). \]

Since Lemma 6.1 and Lemma 6.2 have been proved, to show that

\[ b_0, z_0, z_1, \ldots, z_k, w_1, \ldots, w_k \]

are the vertices of the polygon \( \text{co}(K_\eta) \) in clockwise, it suffices to show that the broken segment \( b_0 z_0 z_1 \ldots z_k w_1 \ldots w_k b_0 \) is a loop.

Clearly, \( z_0, b_0 \) are in the lower half-plane and \( w_1 \) is in the upper half-plane.

Denote by \( l_1 \) the directed straight line passing through 0 of direction \(-a/|a|\) and by \( l_2 \) the directed straight line passing through 1 of direction \(-a/|a|\). Then

\[ \text{Im}(\tilde{z}_0 - \tilde{b}_0)(0 - b_0)) \times -\sin \eta < 0 \]
we see that \( w_0 \) is on the right side of \( l_1 \) and the left side of \( l_2 \).

Case 1. \( \eta \in [\pi/k, \eta_k) \). In this case,

\[
\pi \leq k\eta < \pi + \eta.
\]

Thus \( \text{Im} w_k \approx \sin k\eta < 0 \), implying that \( w_k \) is in the lower half-plane.

Since \( \text{Im}((\bar{z}_0 - \bar{z}_j)(0 - z_j)) \approx \sin j\eta \)

and

\[
\text{Im}((1 - \bar{w}_j)(w_1 - w_j)) \approx -\sin(j - 1)\eta,
\]

we easily check that the broken segment \( z_0 z_1 \cdots z_{k-1} \) is a simple arc on the left side of \( l_1 \) and that \( w_1 w_2 \cdots w_k \) is a simple arc on the right side of \( l_2 \).

Since

\[
\pi - 2\eta < \arg z_k = 2\pi - (k + 1)\eta \leq \pi - \eta,
\]

\( z_k \) is in the upper half-plane. Moreover, since

\[
\text{Im}((\bar{z}_0 - \bar{z}_k)(0 - z_k)) \approx \sin k\eta \leq 0
\]

and

\[
\text{Im}((1 - \bar{z}_k)(w_1 - z_k)) \approx (1 - |a|^4) \sin \eta - |a|^{k+1} \sin k\eta > (1 - |a|^4) \sin \eta > 0,
\]

\( z_k \) is on the right side of \( l_1 \) and the left side of \( l_2 \).

The above facts together imply that the broken segment \( b_0 z_0 \cdots z_k w_1 \cdots w_k b_0 \)

is a loop.

Case 2. \( \eta \in [\eta_{k+1}, \pi/k) \). The proof is similar.

**Remark 6.1.** The vertices of the polygon \( co(K_\eta) \) are \( b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3 \) in clockwise when \( \eta = \eta_4 \). The argument of this case is the same as the proof of Theorem 4.1.

**Remark 6.2.** In the case \( \eta \in (\eta_4, \pi/3) \), we have \( \Phi_4(\eta) < 0 \) by Lemma 4.2. Then, by Lemma 4.4, we have \( \angle z_3 z_4 w_1 \in (\pi, 2\pi) \). Moreover, we easily see that \( z_4 \) is in the inner part of \( co(K_\eta) \). One may ask: Is it true that \( co(K_\eta) \) is a polygon of vertices \( b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3 \) in clockwise?

The answer to this question is no. In fact, by simple computation we get

\[
\text{Im}((\bar{b}_0 - \bar{z}_6)(w_3 - z_6)) \approx 6 - 9|a|^2 - 6|a|^4 + 16|a|^6 - 7|a|^8.
\]

Let \( h(x) = 6 - 9x - 6x^2 + 16x^3 - 7x^4 \). Then \( h(1) = 0 \) and \( h'(1) > 0 \), which implies \( \text{Im}((\bar{b}_0 - \bar{z}_6)(w_3 - z_6)) < 0 \), provided that \( \eta \in [\eta_4, \pi/3) \) is sufficiently near to \( \pi/3 \). Moreover, for such \( \eta \), one has \( \angle b_0 z_6 w_3 \in (\pi, 2\pi) \), so \( z_6 \) is not in the polygon of vertices \( b_0, z_0, z_1, z_2, z_3, w_1, w_2, w_3 \) in clockwise.
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