NONNEGATIVE HERMITIAN VECTOR BUNDLES AND CHERN NUMBERS

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ABSTRACT. We show in this article that if a holomorphic vector bundle has a nonnegative Hermitian metric in the sense of Bott and Chern, which always exists on globally generated holomorphic vector bundles, then some special linear combinations of Chern forms are strongly nonnegative. This particularly implies that all the Chern numbers of such a holomorphic vector bundle are nonnegative and can be bounded below and above respectively by two special Chern numbers. As applications, we obtain a family of new results on compact connected complex manifolds which are homogeneous or can be holomorphically immersed into complex tori, some of which improve several classical results.

1. INTRODUCTION

Throughout this article, all the vector bundles considered are holomorphic and over compact complex manifold.

The concept of positivity/nonnegativity has played a central role in complex differential geometry and algebraic geometry. Bott and Chern ([BC65]) introduced a notion of nonnegativity on Hermitian vector bundles and applied it to initiate the study of high-dimensional value distribution theory. As showed in [BC65], the existence of such a metric implies that all the Chern forms are strongly nonnegative and on globally generated vector bundles such metrics always exist. Later, globally generated vector bundles were investigated in detail by Matsushima and Stoll in [MS73] and they, among other things, related the non-vanishing of the Chern classes and Chern numbers to the transcendence degree of meromorphic functions on the base manifolds.

Soon after the appearance of [BC65], the notions of ampleness and Griffiths-positivity were introduced respectively by Hartshorne ([Ha66]) and Griffiths ([Gr69]) and it turns out that Bott-Chern nonnegativity implies Griffiths-nonnegativity while Griffiths-positivity implies ampleness. Griffiths raised in [Gr69] the question of characterizing the polynomials in the Chern classes/forms which are positive as cohomology classes/differential forms for Griffiths-positive or ample vector bundles. On the class level this was answered completely by Fulton and Lazarsfeld ([FL83]), extending an earlier result of Bloch and Gieseker ([BG71]). Indeed Fulton and Lazarsfeld showed that the set of such polynomials in the Chern classes for ample vector bundles is exactly the cone generated by Schur polynomials of the Chern classes. These inequalities of Fulton-Lazarsfeld type were showed by Demailly, Peternell and...
Schneider ([DPS94]) to remain true for numerically effective ("nef" for short) vector bundles over compact Kähler manifolds.

On the other hand, Griffiths’ question on the form level is still largely unknown. Griffiths himself showed in [Gr69, p. 249] that the second Chern form is positive on a Griffiths-positive vector bundle. Note that its proof is purely algebraic, which seems to be difficult to be generalized to higher dimensions. Recently Guler ([Gu12]) showed that the dual Segre forms, which are formal inverse of the total Chern forms, are positive on Griffiths-positive vector bundles using some geometric arguments.

The main purposes of this article are two-folded. Our first main purpose is to show a useful technical result, Proposition 3.1, which says that on a Bott-Chern nonnegative Hermitian vector bundles, the cone generated by the Schur polynomials in the Chern forms are strongly nonnegative, thus providing some positive evidence to the aforementioned Griffiths’ question on the form level. Our proof of Prop. 3.1 is built on a relationship established in [FL83] that the cone generated by the Schur polynomials coincides to the Griffiths cone defined in [Gr69]. When taking some special Schur polynomials in Prop. 3.1, we shall see in Theorem 3.2 that the self-products of Chern forms and thus all the Chern numbers of such vector bundles can be bounded below and above by two of them respectively. As is well-known globally generated vector bundles admit Bott-Chern nonnegative metrics (see Example 4.1), this implies that Theorem 3.2 imposes strong constraints on the Chern numbers of such vector bundles.

Our second main purpose is to apply Theorem 3.2 to such vector bundles and especially to compact complex manifolds whose holomorphic tangent or cotangent bundles are globally generated to yield a family of new results on them, some of which improve several related classical results. To be more precise, our first application (Thm 5.1, Coro. 5.2) is to give lower and upper bounds for Chern numbers of various nonnegative-type holomorphic vector bundles on compact complex manifolds, whose nonnegativity has been well-known for several decades. Our second application (Thms 5.3, 5.4 and 5.5) is concerned with the projectivity of compact complex manifolds equipped with globally generated vector bundles and homogeneous compact complex manifolds, and its relationship with the non-vanishing of Chern classes/numbers. Our third application (Thm 5.9) is to give an in-depth investigation on the structure of compact complex manifolds holomorphically immersed into complex tori in detail.

The rest of this article is organized as follows. We introduce some necessary notation and symbols in Section 2 and then state in Section 3 our main technical results in this article, Prop. 3.1 and Thm. 3.2. Then we give in Section 4 some examples where the Bott-Chern nonnegativity is satisfied. In Section 5 we apply our established technical results to obtain various related consequences, some of which improve several classical results, and the proof of Prop. 3.1 will be given in the last section, Section 6.

Acknowledgements

This paper was initiated during the author’s visit to the Max-Planck Institut für Mathematik in Bonn from September 2016 to February 2017. The author thanks the Institute for hospitality and financial support. The author also thanks Xiaokui Yang for informing him the related results in [Zha97] and Vamsi Pingali for some crucial comments and pointing out the reference [BP13].
2. Preliminaries

Suppose that \((E^r, h) \rightarrow M^n\) is a rank \(r\) Hermitian vector bundle over a compact complex manifold \(M\) with \(\dim \mathbb{C} M = n\). This means that \(E^r\) is a rank \(r\) holomorphic vector bundle over \(M\) and equipped with a Hermitian metric \(h\) on each fiber varying smoothly. We denote by \(\nabla\) the Chern connection of \((E^r, h) \rightarrow M^n\), i.e., \(\nabla\) is the unique connection compatible with both the complex structure and the Hermitian metric \(h\). Then the curvature tensor \(R\) of \(\nabla\) is given by

\[
R := \nabla^2 \in A^{1,1}(M; \text{Hom}(E, E)) = A^{1,1}(M; E^* \otimes E),
\]

where \(A^{1,1}(M; \cdot)\) is the set of complex-valued smooth \((1, 1)\)-forms with values in some bundle. If \(\{e_1, \ldots, e_r\}\) is a locally defined frame field of \(E\), then

\[
R(e_1, \ldots, e_r) = (e_1, \ldots, e_r)(\Omega^i_j),
\]

where \(\Omega := (\Omega^i_j)\) \((i\text{ row, } j\text{ column})\) is the curvature matrix with respect to \(\{e_i\}\) whose entries are \((1, 1)\)-forms. If \(\{\tilde{e}_i\}\) is another frame with

\[
(\tilde{e}_1, \ldots, \tilde{e}_r) = (e_1, \ldots, e_r)P,
\]

then the curvature matrix \(\tilde{\Omega}\) with respect to \(\{\tilde{e}_i\}\) is related to \(\Omega\) by

\[
\tilde{\Omega} = P^{-1}\Omega P.
\]

It is well-known that the following \(c_i(E, h)\) \((0 \leq i \leq n)\):

\[
\det(tI_r + \frac{\sqrt{-1}}{2\pi}(\Omega^i_j)) = \sum_{i=0}^{n} c_i(E, h) \cdot t^{n-i}, \quad I_r : \text{ } r \times r \text{ identity matrix},
\]

are globally well-defined, real and closed \((i, i)\)-forms and called the \(i\)-th Chern forms of \((E, h)\), which represent the Chern classes \(c_i(E)\) of \(E\).

The following definition was introduced by Bott and Chern ([BC65, p. 90]).

**Definition 2.1** (Bott-Chern). A Hermitian vector bundle \((E^r, h) \rightarrow M^n\) is called Bott-Chern nonnegative, denoted by \(h \geq_{BC} 0\), if for any point of \(M\), there exist a unitary frame field around it and a matrix \(A\) with \(r\) rows and whose entries are \((1, 0)\)-forms, such that the curvature matrix \(\Omega\) under this unitary frame field satisfies

\[
\Omega = A \wedge A^t.
\]

Here “\(t\)” denotes the transpose of a matrix.

**Remark 2.2.**

1. This definition is independent of the unitary frame we choose as the transformation matrix \(P\) is a unitary matrix between them and thus from (2.1) we have

\[
\tilde{\Omega} = P^{-1}\Omega P = P^{-1}(A \wedge A^t)P = (P^t A \wedge A^t)P = (\overline{P^t A}) \wedge (\overline{P^t A})^t.
\]

2. Here we don’t have any requirement on the number of columns of the matrix \(A\), which may vary along the choice of the points on \(M\).

3. We shall see in Example 4.1 that globally generated vector bundles over compact complex manifolds form an important subclass of those admitting Bott-Chern nonnegative Hermitian metrics.
Before stating our first main result, we need some more notation. Following [FL83], we denote by \( \Gamma(i, r) \) the set of partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i) \) of weight \( i \) by nonnegative integers \( \lambda_j \leq r \):

\[
r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq 0, \quad \sum_{j=1}^{i} \lambda_j = i.
\]

For each partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i) \in \Gamma(i, r) \), a Schur polynomial

\[
S_{\lambda}(c_1, \ldots, c_r) \in \mathbb{Z}[c_1, \ldots, c_r]
\]

is attached to as follows:

\[
S_{\lambda}(c_1, \ldots, c_r) := \det(c_{\lambda_j-k+i})_{1 \leq j,k \leq i} \quad (j : \text{row}, \ k : \text{column})
\]

\[
= \begin{vmatrix}
    c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+i-1} \\
    c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+i-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{\lambda_i-i+1} & c_{\lambda_i-i+2} & \cdots & c_{\lambda_i}
\end{vmatrix}
\]

Here we make the convention that \( c_0 = 1 \) and \( c_j = 0 \) if \( j < 0 \) or \( j > r \). In particular, we have

\[
S_{(i,0,\ldots,0)}(c_1, \ldots, c_r) = c_i
\]

and

\[
S_{(i-j,j,0,\ldots,0)}(c_1, \ldots, c_r) = c_{i-j}c_j - c_{i-j+1}c_{j-1}. \quad \text{(2.5)}
\]

The expression (2.5) shall play a decisive role in establishing the lower and upper bounds of the Chern classes/numbers in our Theorem 3.2, which was partly inspired by [DPS94] as the special case

\[
S_{(i-1,1,0,\ldots,0)}(c_1, \ldots, c_r) = c_{i-1}c_1 - c_i
\]

has been observed and applied in [DPS94, Coro. 2.6].

Now we recall the notions of nonnegativity and strong nonnegativity for real \((p,p)\)-forms ([De12, Ch. 3, §1.A]). Recall that a real \((p,p)\)-form \( \varphi \) \((1 \leq p \leq n) \) on a compact complex manifold \( M^n \) is called nonnegative, denoted by \( \varphi \geq 0 \), if for any point \( x \in M \) and any \((1,0)\)-type tangent vectors \( X_1, \ldots, X_p \) at \( x \), we have

\[
(- \sqrt{-1})^p \varphi(X_1, \ldots, X_p, \overline{X_1}, \ldots, \overline{X_p}) \geq 0.
\]

A real \((p,p)\)-form \( \varphi \) is called strongly nonnegative, denoted by \( \varphi \geq_s 0 \), if it can be written as

\[
\varphi = (\sqrt{-1})^p \sum \psi_i \wedge \overline{\psi_i},
\]

where these \( \psi_i \) are \((p,0)\)-forms. A strongly nonnegative form is clearly nonnegative.

Two \((p,p)\)-forms \( \varphi_1 \) and \( \varphi_2 \) are said to be \( \varphi_1 \leq \varphi_2 \) (resp. \( \varphi_1 \leq_s \varphi_2 \)) if \( \varphi_2 - \varphi_1 \geq 0 \) (resp. \( \geq_s 0 \)). It is clear that the sum of two (strongly) nonnegative forms of the same degree is still (strongly) nonnegative. In general the product of two nonnegative forms may not
be nonnegative (cf. [BP13]). Nevertheless, it is obvious that the product of two strongly nonnegative forms is still strongly nonnegative.

3. Main technical results

Bott and Chern noticed that all the Chern forms \( c_i(E, h) \) \((1 \leq i \leq n)\) of a Bott-Chern nonnegative Hermitian vector bundle \((E^r, h) \to M^n\) are strongly nonnegative ([BC65, p. 91]). The following proposition, which is our first main technical observation in this article, states that indeed many more real forms involving in the Chern forms, including \( c_i(E, h) \) themselves, are strongly nonnegative.

**Proposition 3.1.** Suppose that \((E^r, h) \to M^n\) is a Hermitian vector bundles with \( h \geq_{BC} 0 \). Then the following closed real \((i, i)\)-forms

\[
S_\lambda(c_1(E, h), \ldots, c_r(E, h)), \quad (\forall \lambda \in \Gamma(i, r), \forall 1 \leq i \leq n)
\]

are strongly nonnegative. In particular, by (2.4) and (2.5) the closed real \((i, i)\)-forms

\[
\begin{cases}
  c_i(E, h) & (1 \leq i \leq n) \\
  c_{i-j}(E, h)c_j(E, h) - c_{i-j+1}(E, h)c_{j-1}(E, h) & (1 \leq i \leq n, 1 \leq j \leq \left[ \frac{i}{2} \right])
\end{cases}
\]

are strongly nonnegative.

For later simplicity we denote by

\[
c_\lambda(E, h) := \prod_{j=1}^{i} c_{\lambda_j}(E, h), \quad \forall \lambda = (\lambda_1, \ldots, \lambda_i) \in \Gamma(i, r),
\]

\[
c_\lambda[E] := \int_M \prod_{j=1}^{n} c_{\lambda_j}(E, h) \in \mathbb{Z}, \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma(n, r),
\]

and

\[
c_\lambda(E) := \prod_{j=1}^{i} c_{\lambda_j}(E) \in H^{2i}(M; \mathbb{Z}), \quad \forall \lambda = (\lambda_1, \ldots, \lambda_i) \in \Gamma(i, r).
\]

We are now able to deduce our second technical result, which says that the strongly nonnegative forms in (3.1) in fact implies that the self-products of Chern forms and thus the Chern numbers can be bounded below and above as follows.

**Theorem 3.2.** Suppose that \((E^r, h) \to M^n\) is a Hermitian vector bundle with \( h \geq_{BC} 0 \). Then as real \((i, i)\)-forms, \( c_\lambda(E, h) \geq_{s} 0 \), and bounded below and above respectively by \( c_i(E, h) \) and \( c_1^\lambda(E) \):

\[
0 \leq_{s} c_i(E, h) \leq_{s} c_\lambda(E, h) \leq_{s} c_1(E, h)^i, \quad \forall \lambda \in \Gamma(i, r), \forall 1 \leq i \leq n.
\]

Consequently the Chern numbers satisfy

\[
0 \leq c_n[E] \leq c_\lambda[E] \leq c_1^n[E], \quad \forall \lambda \in \Gamma(n, r).
\]
Proof. For any $\lambda = (\lambda_1, \ldots, \lambda_i) \in \Gamma(i, r)$, repeated use of (3.1) leads to

$$0 \leq s c_i(E, h) \leq s c_{i-1}(E, h)c_1(E, h) \leq s c_{i-2}(E, h)c_2(E, h) \leq s \cdots \leq s c_{i-\lambda_1}(E, h)c_{\lambda_1}(E, h), \quad \text{if } \lambda_1 \leq \frac{i}{2},$$

or

$$0 \leq s c_i(E, h) \leq s c_{i-1}(E, h)c_1(E, h) \leq s \cdots \leq s c_{\lambda_1}(E, h)c_{i-\lambda_1}(E, h), \quad \text{if } \lambda_1 > \frac{i}{2}.$$  

This means that in any case we have

$$(3.2) \quad 0 \leq s c_i(E, h) \leq s c_{\lambda_1}(E, h)c_{i-\lambda_1}(E, h)$$

Similarly,

$$c_{i-\lambda_1}(E, h) \leq s c_{\lambda_2}(E, h)c_{i-\lambda_1-\lambda_2}(E, h),$$

$$c_{i-\lambda_1-\lambda_2}(E, h) \leq s c_{\lambda_3}(E, h)c_{i-\lambda_1-\lambda_2-\lambda_3}(E, h),$$

$$\cdots.$$  

Note that all these forms discussed here are strongly nonnegative by Prop. 3.1 and thus their products are still strongly nonnegative:

$$(3.4) \quad 0 \leq s c_i(E, h) \leq s \prod_{j=1}^{i} c_{\lambda_j}(E, h) = c_{\lambda}(E, h).$$

On the other hand, similar arguments yield

$$c_{\lambda_j}(E, h) \leq s c_{\lambda_j-1}(E, h)c_1(E, h) \leq s [c_{\lambda_j-2}(E, h)c_1(E, h)]c_1(E, h) \leq s \cdots \leq s c_1(E, h)^{\lambda_j}. \quad (3.5)$$

Therefore,

$$(3.6) \quad c_{\lambda}(E, h) = \prod_{j=1}^{i} c_{\lambda_j}(E, h) \leq s \prod_{j=1}^{i} c_1(E, h)^{\lambda_j} = c_1(E, h)^i, \quad \forall \lambda \in \Gamma(i, r).$$

Remark 3.3. This theorem is indeed partly inspired by an observation in [DPS94, Coro. 2.6], where they noticed that $S_{(i-1,1,0,\ldots,0)} = c_1c_{i-1} - c_i$ and thus gave the upper bound $c_i^1$ for $c_{\lambda}$ as we have done in (3.5) and (3.6) in the context of nef vector bundles on compact Kähler manifolds. The reason why the deductions in (3.5) and (3.6) are true in their situation is due to the facts that nonnegative real $(1, 1)$-forms (in particular for $c_1(E, h)$) are strongly nonnegative and the product of a strongly nonnegative form with a nonnegative form is still nonnegative ([De12, p. 132]). Nevertheless, in their context the inequality (3.4) is no longer true and so their is no respectively lower bound in their situation. Indeed we shall see in Section 5 that it is this lower bound that plays key roles in many related applications.
In this section we shall illustrate by some examples that in many important situations the Bott-Chern nonnegativity can be satisfied. Although these examples and facts are well-known to experts, for the reader’s convenience, we still outline some necessary details and/or point out some references.

Recall that a holomorphic vector bundle $E \rightarrow M$ is called globally generated if the global holomorphic sections of $E$ span the fiber over each point of $M$. If $M$ is compact then $H^0(M, E)$, the complex vector space consisting of holomorphic sections of $M$, is finite-dimensional. Then the property of being globally generated implies that the following bundle sequence

$$0 \rightarrow \ker(\phi) \rightarrow M \times H^0(M, E) \xrightarrow{\varphi} E \rightarrow 0$$

$$(x, s) \mapsto (x, s(x)).$$

is exact. This means that a globally generated vector bundle over a compact complex manifold can be realized as a quotient bundle of a trivial vector bundle. Then the fact that

any globally generated vector bundle over a compact complex manifold

admits a Bott-Chern nonnegative Hermitian metric

(4.2)

follows from the following general result.

**Example 4.1.** If $(E', h) \rightarrow M$ is a Hermitian vector bundle over a compact complex manifold $M$ and $S$ is an $l$-dimensional holomorphic subbundle of $E$ ($l < r$) and thus $Q := E/S$ is a $(r - l)$-dimensional holomorphic quotient bundle of $E$, i.e., we have the following short exact sequence:

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0.$$

The Hermitian metric $h$ on $E$ naturally induces a metric on $S$ and on $Q$. Bott and Chern noticed that ([BC65, p. 91]), under any unitary frame $\{e_1, \ldots, e_r\}$ of $E$ such that $\{e_1, \ldots, e_l\}$ and $\{e_{l+1}, \ldots, e_r\}$ are unitary frames of $S$ and $Q$ respectively, the curvature matrices of $E$ and $Q$, denoted by $\Omega_E$ and $\Omega_Q$, are related by (cf. [GH78, p. 79], [De12, p. 275])

$$\Omega_E|_Q = \Omega_Q - A \wedge \overline{A'},$$

where $(\cdot)|_Q$ denotes the restriction to $Q$ and $A$ is a matrix whose entries are $(1, 0)$-forms representing the second fundamental form of the subbundle $S$ in $E$. This property is usually called “curvature increases in holomorphic quotient bundles”. This means that if $E$ can be equipped with a flat Hermitian metric as in (4.1), which implies that $\Omega_E|_Q = 0$, then the quotient bundle $Q$ admits a Bott-Chern nonnegative Hermitian metric. Thus the fact (4.2) follows.

Our principal concerns in this article are compact complex manifolds whose holomorphic tangent or cotangent bundles, denoted for simplicity in the sequel by $T$ and $T^*$ respectively, are globally generated. To each kind of them there is a well-known result, which we record in the following example for our later purpose.

**Example 4.2.** (1) Recall that a compact complex manifold $M$ is called homogeneous if the holomorphic automorphism group of $M$, denoted by $\text{Aut}(M)$, acts transitively on it. It is well-known that a compact connected complex manifold $M$ has globally
generated $T_M$ if and only if it is homogeneous. This particularly implies that the holomorphic tangent bundles of Hermitian symmetric spaces, projective-rational manifolds and tori are globally generated.

(2) A compact connected Kähler manifold has globally generated $T^*$ if and only if it can be holomorphically immersed into some complex torus. This result should be due to Matsushima and Stoll ([MS73, p. 100] or [Ma92, p. 606]), at least to the author's best knowledge. The basic idea is to consider the Albanese map related to this Kähler manifold. A more compact proof can be found in [Sm76, p. 271]. Note that the term "globally generated" was called "ample" and "weakly ample" respectively in [MS73] and [Sm76]. Also note that a compact complex manifold holomorphically immersed into some torus is automatically Kähler as the restriction of the obvious Kähler metric on the complex torus is still Kähler. This implies that any compact connected complex manifold holomorphically immersed into some complex torus has globally generated $T^*$.

The notion of nefness of line/vector bundles over projective algebraic manifolds is well-known and has been extended to general compact complex manifolds by Demailly, which can be viewed as a limiting case of ampleness. For its definition and basic properties we refer to [De92, §1] or [DPS94, §1]. Before ending this section, we need to point out an implication, which will be used in the next section, that the Bott-Chern nonnegativity implies nefness. Indeed it turns out that Griffiths positivity implies ampleness ([Gr69, Theorem B], [Zhe00, p. 206]). Note that ampleness was called cohomological positivity in [Gr69]. Taking the limiting case we know that Griffiths nonnegativity implies nefness. Then the above-mentioned implication follows from the fact that the Bott-Chern nonnegativity in fact implies the Griffiths nonnegativity. The latter fact is well-known to experts, but we are not able to find a reference. So we record it in the following example and include a proof for the reader’s convenience as well as for completeness.

**Example 4.3.** We have the following implications:

(4.3) Bott-Chern nonnegativity $\Rightarrow$ Griffiths nonnegativity $\Rightarrow$ Nefness.

**Proof.** We show the first implication in (4.3). Under a local local coordinate $(z^1, \ldots, z^n)$, the entries $\Omega^i_j$ in the curvature matrix $\Omega$ with respect to some unitary frame field can be written as

$$\Omega^i_j = R^i_{jpq} dz^p \wedge \overline{dz^q}.$$ 

The Hermitian metric $h$ on $E$ is called Griffiths nonnegative ([Gr69, p. 181]) if, at any point of $M$, we have

$$\sum_{i,j,p,q} R^i_{jpq} \xi^j \eta^p \overline{\xi^q \eta^q} \geq 0$$

for any $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{C}^n$, and any $\eta = (\eta^1, \ldots, \eta^n) \in \mathbb{C}^n$. Now we assume that $h \geq_{BC} 0$, i.e.,

$$(\Omega^i_j) = A \wedge \overline{A}, \quad A = (\sum_p T^{(p)}_{ij} dz^p).$$

This implies that

$$R^i_{jpq} = \sum_k T^{(p)}_{ik} \overline{T^{(q)}_{jk}}.$$
and thus
\[
\sum_{i,j,p,q} R^i_{jpq} \xi^j \eta^p \xi^q \eta^q = \sum_{k,i,j,p,q} T^{(p)}_{ik} T^{(q)}_{jk} \xi^j \eta^p \xi^q \eta^q = \sum_{k,i,j,p,q} (T^{(p)}_{ik} \xi^j \eta^p) (T^{(q)}_{jk} \xi^q \eta^q)
\]
\[
= \sum_k \left| \sum_{i,p} T^{(p)}_{ik} \xi^i \eta^p \right|^2 \geq 0.
\]

5. Applications

In this section we give three kinds of related applications, whose main contents have been briefly described in the Introduction.

5.1. The first application. Bott and Chern observed that ([BC65, p. 91]) the Chern forms of Bott-Chern nonnegative Hermitian vector bundles are all strongly nonnegative and thus it is immediate that their Chern numbers are all nonnegative.

Our first direct application of our Theorem 3.2 yields the following improvements of the above-mentioned nonnegativity by giving lower and upper bounds.

Theorem 5.1. Suppose \((E^r, h) \rightarrow M^n\) is a Bott-Chern nonnegative Hermitian vector bundle. Then all their Chern numbers are nonnegative and bounded below and above by \(c_n[E]\) and \(c^1_1[E]\):

\[
(5.1) \quad 0 \leq c_n[E] \leq c_\lambda[E] \leq c^1_1[E], \quad \forall \lambda \in \Gamma(n,r).
\]

This particularly holds for globally generated vector bundles. In particular, if the Chern number \(c^1_1[E] = 0\), then all the Chern numbers vanish.

Compact connected homogeneous complex manifolds have been well-studied and it is also a classical result that their Chern numbers are nonnegative ([Go54],[BR62],[GR62],[MS73]). The nonnegativity of the signed Chern numbers of compact connected complex manifolds holomorphically immersed into complex tori has also been obtained by Matsushima and Stoll ([MS73, Thm 6.5], [Ma92, p. 607]). So the following corollary improves the nonnegativity results for Chern numbers/signed Chern numbers of these manifolds by giving lower and upper bounds respectively.

Corollary 5.2. (1) The Chern numbers of any compact connected homogeneous complex manifold \(M^n\) are nonnegative and bounded below and above by \(c_n[M]\) and \(c^1_1[M]\):

\[
(5.2) \quad 0 \leq c_n[M] \leq c_\lambda[M] \leq c^1_1[M], \quad \forall \lambda \in \Gamma(n,n).
\]

(2) The signed Chern numbers of any compact connected complex manifold \(M^n\) whose holomorphic cotangent bundle is globally generated are nonnegative and bounded below and above by \((-1)^n c_n[M]\) and \((-1)^n c^1_1[M]\):

\[
(5.3) \quad 0 \leq (-1)^n c_n[M] \leq (-1)^n c_\lambda[M] \leq (-1)^n c^1_1[M], \quad \forall \lambda \in \Gamma(n,n).
\]

This particularly holds for compact connected complex manifolds holomorphically immersed into complex tori.

In particular, in these two cases, if the Chern number \(c^1_1[M] = 0\), then all the Chern numbers of \(M\) vanish.
5.2. The second application. Now we come to our second application. First let us recall some more notation. For a compact connected complex manifold $M^n$, the \textit{algebraic dimension} of $M$, denoted by $a(M)$, is defined to be the maximal number of meromorphic functions on $M$ that can be algebraically independent. In other words, $a(M)$ is the transcendental degree of function field of $M$ over $\mathbb{C}$. $a(M)$ satisfies $0 \leq a(M^n) \leq n$. A compact complex manifold $M^n$ is called \textit{Moishezon} if the algebraic dimension $a(M^n) = n$, in which case it was investigated in detail by Moishezon ([Mo66]). We refer the reader to [MM07, Ch. 2] for a detailed presentation of the materials related to Moishezon manifolds. The starting point of our second application is to recall several basic results due to Matsushima-Stoll, Siu-Demailly and Demailly respectively.

A major result of Matsushima-Stoll in [MS73] says that ([MS73, Thms 5.5, 5.6] or [Ma92, p. 600]) if a globally generated vector bundle $E^r \rightarrow M^n$ satisfies $0 \neq c_\lambda(E) \in H^2(M, \mathbb{Z})$ for some $\lambda \in \Gamma(i, r)$, then the algebraic dimension $a(M) \geq i$. In particular, if some Chern number of a globally generated vector bundle over a compact Kähler manifold is nonzero, then $M$ is Moishezon. This result, together with Moishezon’s result that (cf. [MM07, p. 95])

\begin{equation}
\text{(5.4) a Moishezon manifold is Kähler if and only if it is projective algebraic,}
\end{equation}

leads to another main theorem of [MS73] ([MS73, p. 94, Main Thm] or [Ma92, p. 600]) saying that if some Chern number of a globally generated vector bundle over a compact Kähler manifold is nonzero, then this manifold is projective algebraic.

A basic result due to Siu-Demailly (cf. [MM07, p. 96]) solving the Grauert-Riemenschneider conjecture gives another effective sufficient condition for a compact connected complex manifold to be Moishezon, which says

\begin{equation}
\text{(5.5) a compact connected complex manifold $M$ is Moishezon provided that there exists a quasi-positive element in $H^{1,1}(M; \mathbb{Z}) := H^2(M; \mathbb{Z}) \cap H^{1,1}(M; \mathbb{R})$.}
\end{equation}

Here an element in $H^{1,1}(M; \mathbb{Z})$ is called \textit{quasi-positive} if there exists a real $(1, 1)$-form representing it which is nonnegative everywhere and positive somewhere.

Another basic result due to Demailly ([De92, Coro. 1.6] or [DPS94, Thm 4.1]) tells us that

\begin{equation}
\text{(5.6) a Moishezon manifold with nef tangent bundle is projective algebraic.}
\end{equation}

Our starting observation in this second application is to give a new proof of the above-mentioned result due to Matsushima-Stoll by combining our Prop. 3.1 and Thm. 5.1 with Siu-Demailly’s result (5.5).

\textbf{Theorem 5.3} (Matsushima-Stoll). Suppose $E^r \rightarrow M^n$ is a globally generated vector bundle over a compact complex (resp. Kähler) manifold. If some Chern number of $E \rightarrow M$ is nonzero, then $M$ is a Moishezon (resp. projective algebraic) manifold.

\textit{Proof.} By (4.2) there exists a Hermitian metric $h$ on $E$ such that $h \geq_{BC} 0$. Therefore Prop. 3.1 says that the first Chern form $c_1(E, h)$ is nonnegative as a real $(1, 1)$-form. If by the assumption some Chern number of $M$ is nonzero, then the Chern number

\[ c_1^n[E] = \int_M c_1(E, h)^n > 0 \]

by 5.1. This means that $c_1(E, h)$ must be positive somewhere and hence the first Chern class $[c_1(E, h)] \in H^{1,1}(M; \mathbb{Z})$ is quasi-positive. So the desired conclusion follows from the Siu-Demailly criterion (5.5). \qed
Compact connected homogeneous complex manifolds have been well-studied and a well-known related fact, which should be due to H.C. Wang, states (cf. [MS73, Thm 6.2] or [Ma92, p. 601]) they are projective algebraic provided that their Euler characteristic are nonzero. Our following result is a refinement of this classical fact, and, moreover its proof is built on the aforementioned two basic results due to Matsushima-Stoll and Demailly and thus different from the original one.

**Theorem 5.4.** Suppose $M^n$ is a compact connected homogeneous complex manifold. Then $M$ is a projective algebraic manifold if some Chern number of $M$ is nonzero.

**Proof.** If some Chern number is nonzero then $M$ is Moishezon due to Theorem 5.3. Now the property of being globally generated on $T$ implies the existence of a Bott-Chern nonnegative Hermitian metric on $T$ and thus implies nefness by (4.3). Then the fact of $M$ being projective algebraic follows from Demailly’s result (5.6). □

Note that in Theorem 5.4 the condition of some Chern number be nonzero is only sufficient and its converse part is not true in general, i.e., even if a compact connected homogeneous complex manifold is projective algebraic, it may happen that its all Chern numbers vanish. For instance, the product of an abelian variety over $\mathbb{C}$ and a projective-rational manifold is projective algebraic but its Chern numbers are all zero. Indeed, a classical result of Borel and Remmert tells us that ([BR62])

\begin{equation}
\text{any compact connected homogeneous Kähler manifold is the product of a complex torus and a projective-rational manifold.}
\end{equation}

(5.7)

This result (5.7) implies that the condition of some Chern number being nonzero is also necessary in Theorem 5.4 if we further impose simple-connectedness on the manifolds in question and thus we have the following result.

**Theorem 5.5.** Suppose $M^n$ is a simply-connected compact connected homogeneous complex manifold. Then the following four conditions are equivalent:

1. The Chern number $c_1^n[M] \neq 0$.
2. Some Chern number of $M$ is nonzero.
3. $M$ is a projective algebraic manifold.
4. $M$ is a Kähler manifold.

**Remark 5.6.**

1. The manifolds considered in Theorem 5.5 were called C-spaces by Wang ([Wa54]) and investigated in detail by many authors ([BR62], [Go54], [GR62], [Wa54] etc.). However, the contents in our Theorem 5.5 should be completely new, at least to the author’s best knowledge.

2. There are homogeneous complex structures on the product of odd-dimensional spheres: $M_{p,q} := S^{2p+1} \times S^{2q+1}$ $(p,q) \neq (0,0)$, which are called Calabi-Eckmann manifolds ([CE53]) and whose Chern numbers are all zero. $M_{p,q}$ are indeed non-Kähler as its second Betti number is zero and so these examples match Theorem 5.5 very well.

5.3. The third application. **Our third application** is to give an in-depth investigation on the structure of compact connected complex manifolds holomorphically immersed into complex tori in detail. As we have seen in Example 4.2, these manifolds are precisely those compact connected Kähler manifolds whose holomorphic cotangent bundles are globally generated. The structure of these manifolds was first investigated by Matsushima and his coauthors
(\[HM75\], [Ma74], [MS73]), Yau ([Ya74, Chapter 3]) and Smyth ([Sm76]) etc around the same time. Later in [Zha97] Zhang related the ampleness of the canonical line bundles of these manifolds to their Todd genera.

Before continuing, let us digress to recall the notions of Kodaira dimension $\kappa(M)$ for compact complex manifolds $M$ ([Zhe00, p. 132]), which has several equivalent definitions. Here for our later purpose we adopt the following one. Denote by $K_M$ the canonical line bundle of $M$. It turns out that $\dim_{\mathbb{C}} H^0(M, K_M^{\otimes m})$, the complex dimension of the holomorphic sections of $K_M^{\otimes m}$, has the following property: either $H^0(M, K_M^{\otimes m}) = 0$ for all $m \geq 1$ or there exists an integer $0 \leq \kappa(M) \leq \dim_{\mathbb{C}}(M)$ and constants $0 \leq C_1 < C_2$ such that

\[
C_1 m^{\kappa(M)} \leq \dim_{\mathbb{C}} H^0(M, K_M^{\otimes m}) \leq C_2 m^{\kappa(M)}, \quad m \gg 0,
\]

which means $\dim_{\mathbb{C}} H^0(M, K_M^{\otimes m})$ grows at a rate of $m^{\kappa(M)}$. In the former case $\kappa(M) := -\infty$. $M$ is called of general type if $\kappa(M)$ attains its maximum: $\kappa(M) = \dim_{\mathbb{C}}(M)$. It is an elementary fact between the Kodaira dimension $\kappa(M)$ and the algebraic dimension $a(M)$ that ([Zhe00, p. 135]):

\[
\kappa(M) \leq a(M) \leq \dim_{\mathbb{C}}(M).
\]

Denote by $\text{Aut}_0(M)$ the identity component of the holomorphic automorphism group of a compact complex manifold $M$, which is a connected complex Lie group. The most fundamental structure of compact connected complex manifolds holomorphically immersed into complex tori is the following result.

**Theorem 5.7.** ([Ma74, Prop. 1], [Ya74, Th. 5], [Sm76, Th. 1]) Suppose $M^n$ is a compact connected complex manifold holomorphically immersed into some complex torus with $\text{Aut}_0(M) \neq \{0\}$. Then $\text{Aut}_0(M)$ is a complex torus and acts freely on $M$. Moreover, the quotient manifold $N := M/\text{Aut}_0(M)$ is also a compact connected complex manifold and can be holomorphically immersed into some complex torus and $\text{Aut}_0(N) = \{0\}$. Consequently, $M$ is a holomorphic principal complex torus bundle over some compact connected complex manifold $N$ also holomorphically immersed into some complex torus with $\text{Aut}_0(N) = \{0\}$.

This structure theorem has several applications in [Ma74], [Ya74] and [Sm76]. Let us record several of them and some results in [Zha97] related to our next result in the following theorem.

**Theorem 5.8.** Suppose $M^n$ is a compact connected complex manifold holomorphically immersed into some complex torus. Then

1. ([Ya74, p. 238]) $M$ is a (possibly trivial) torus bundle over a compact Kähler manifold of general type.
2. ([Sm76, p. 278]) The following three conditions are equivalent:

\[
\text{Aut}_0(M) = \{0\} \iff c^n_1[M] \neq 0 \iff c_n[M] \neq 0.
\]

3. ([Zha97, Thm 3, Coro. 3]) The following three conditions are equivalent:

\[
K_M \text{ is ample} \iff (-1)^n \text{td}(M) > 0 \iff (-1)^n c_n[M] > 0,
\]

where $\text{td}(M)$ is the Todd genus of $M$, which by definition is the alternating sum of the Hodge numbers

\[
\text{td}(M) = \sum_q (-1)^q h^{0,q}(M)
\]

and the Hirzebruch-Riemann-Roch theorem says that it is a rationally linear combination of Chern numbers.
We can now give our third application, which relates $\text{Aut}_0(M)$, all the Chern numbers of $M$, the Kodaira dimension $\kappa(M)$, the canonical line bundle $K_M$, and its torus bundle structure and particularly improves various results summarized in Theorem 5.8.

**Theorem 5.9.** Suppose $M^n$ is a compact connected complex manifold holomorphically immersed into some complex torus, or equivalently, $M^n$ is a compact connected Kähler manifold with globally generated holomorphic cotangent bundle. Then the following six conditions are equivalent:

1. $\text{Aut}_0(M) = \{0\}$,
2. all the signed Chern numbers $(-1)^n c_\lambda[M]$ are strictly positive,
3. $M$ is of general type,
4. the canonical line bundle $K_M$ is ample,
5. the signed Todd genus is positive: $(-1)^n \text{td}(M) > 0$,
6. $M$ cannot be realized as a total space of some nontrivial torus bundle;

and the following six conditions are equivalent:

7. $\text{Aut}_0(M) \neq \{0\}$,
8. all the Chern numbers of $M$ vanish,
9. the Kodaira dimension $\kappa(M) < n$,
10. the canonical line bundle $K_M$ is not ample,
11. the Todd genus $\text{td}(M)$ vanishes,
12. $M$ is a nontrivial holomorphic principal complex torus bundle over a compact connected Kähler manifold with ample canonical line bundle.

Moreover, under (any one of) the first six equivalent conditions, $M$ is projective algebraic.

A direct corollary of Theorem 5.9 is the following result, which tells us that whether or not the Chern numbers and Todd genus vanish is simultaneous.

**Corollary 5.10.** Suppose $M^n$ is a compact connected complex manifold holomorphically immersed into some complex torus. Then

1. if the Todd genus $\text{td}(M) = 0$, then all the Chern numbers vanish;
2. if the Todd genus $\text{td}(M) \neq 0$, then it has sign $(-1)^n$ and all the Chern numbers are nonzero and have the same sign $(-1)^n$.

**Proof of Theorem 5.9.**

*Proof.* First we show that the condition (2) implies that $M$ is projective algebraic and thus prove the last conclusion in Theorem 5.9. Indeed, Theorem 5.3 says that $M$ is Moishezon. Then $M$ being projective algebraic follows from (5.4) and the fact that $M$ be Kähler.

“(1) $\iff$ (2)” follows from (5.10) and (5.3).

“(2) $\iff$ (6)”: If $M$ can be realized as a total space of some nontrivial torus bundle, then $c_0[M] = 0$ as the Euler characteristic is multiplicative for fiber bundles, which implies that “(2) $\implies$ (6)”. “(6) $\implies$ (1)” follows from Theorem 5.7.

“(2) $\Rightarrow$ (3)” The proof is a combination of a Kodaira-type vanishing theorem and the Hirzebruch-Riemann-Roch theorem. By definition we can choose a Hermitian metric $h$ on $T_M^*$.
such that $(h, T_M^*) \geq_{BC} 0$. Then (3.1) tells us that $c_1(T_M^*, h) \geq 0$. This means that the Chern class
\begin{equation}
(5.12) \quad c_1(K_M^{\otimes m}) = mc_1(K_M) = m[c_1(T_M^*, h)] \geq 0, \quad m \geq 1.
\end{equation}

Condition (2) implies that
\begin{equation}
(5.13) \quad \int_M (c_1(K_M^{\otimes m}))^n = m^n(-1)^nc_n[M] > 0, \quad m \geq 1.
\end{equation}

We have shown above that $M$ is projective algebraic under the condition (2). Then a combination of (5.12), (5.13), $M$ being projective algebraic and the Kodaira-Kawamata-Viehweg vanishing theorem for projective algebraic manifolds ([Ko87, p. 74, (3.10)]) yields
\begin{equation}
(5.14) \quad H^q(M, K_M^{\otimes m}) = 0, \quad q \geq 1, \quad m \geq 2.
\end{equation}

Therefore the Hirzebruch-Riemann-Roch theorem tells us that ([Hi66])
\begin{align*}
\dim_{\mathbb{C}} H^0(M, K_M^{\otimes m}) &= \sum_{q=0}^n (-1)^q \dim_{\mathbb{C}} H^q(M, K_M^{\otimes m}) \quad (5.14) \\
&= \int_M [\text{Td}(M) \cdot \text{Ch}(K_M^{\otimes m})] \quad (\text{Td: Todd class, Ch: Chern character}) \\
&= \int_M \left\{ \left[ 1 + \frac{1}{2} c_1(M) + \cdots \right] \cdot \exp \left[ -mc_1(M) \right] \right\} \\
&= \frac{(-1)^n c_1^n[M]}{n!} m^n + O(m^{n-1}), \quad (m >> 0)
\end{align*}

which, together with $(-1)^n c_1^n[M] > 0$ under the condition (2), implies (5.8) with the Kodaira dimension $\kappa(M) = n$, i.e., $M$ is of general type.

“(3) \Rightarrow (4)”: This follows from a result of Ran ([Ra84, p. 176, Coro. 3]), which says that a compact connected complex manifold of general type holomorphically immersed into some complex torus must have ample canonical line bundle.

“(4) \Leftrightarrow (5)”: this follows from (5.11).

“(4) \Rightarrow (2)”: First note that the ampleness of $K_M$ implies that $M$ is projective algebraic. If on the contrary the Euler characteristic $c_n[M] = 0$, then a result of Howard and Matsushima ([HM75, Th. 6] or [Ma92, p. 655]) says that $M$ admits a holomorphic one-form whose zero-locus is empty. However, another major result of Zhang in [Zha97, Thm 1] tells us that the zero-locus of any holomorphic one-form on a projective algebraic manifold with ample canonical line bundle is non-empty, which leads to a contradiction. This means that $c_n[M] \neq 0$ and thus proves condition (2) by (5.3).

“(7) \Leftrightarrow (8)” follows from (5.3) and (5.10).

“(8) \Leftrightarrow (11)”: Note that the Todd genus $\text{td}(M)$ is a rationally linear combination of Chern numbers and thus “(8) \Rightarrow (11)”. “(11) \Rightarrow (8)” follows from “(2) \Leftrightarrow (5)”.

“(7) \Rightarrow (12)”: Theorem 5.7 tells us that, under the condition $\text{Aut}_0(M) \neq \{0\}$, $M$ is a nontrivial holomorphic principal complex torus bundle over a compact connected complex manifold $N$, where $N$ can also be holomorphically immersed some complex torus with $\text{Aut}_0(N) = \{0\}$. Then “(1) \Leftrightarrow (4)” implies that $K_N$ is ample.
“(12) ⇒ (7)”: Condition (12) implies that the Euler characteristic \(c_n[M] = 0\) and thus \(\text{Aut}_0(M) \neq \{0\}\) follows from “(1) ⇔ (2)”.

Other equivalent relations among (7) – (12) are direct via those among (1) – (6).

**Remark 5.11.** A recent result of Popa and Schnell ([PS14]) refined [Zha97, Thm 1] by showing that the zero-locus of any holomorphic one-form on a projective algebraic manifold of general type is non-empty. Thus we can also apply it to deduce “(3) ⇒ (2)”.

**6. Proof of Proposition 3.1**

In this last section we shall prove Proposition 3.1 and complete this article.

Let us begin by recalling some well-known facts about \(GL_r(\mathbb{C})\)-invariant polynomial functions. In what follows \(GL_r(\mathbb{C})\) and \(M_r(\mathbb{C})\) denote respectively the general linear group of order \(r\) and the \(r \times r\) matrix group, both over \(\mathbb{C}\). A map \(f : M_r(\mathbb{C}) \to \mathbb{C}\) is called a \(GL_r(\mathbb{C})\)-invariant polynomial function of homogeneous degree \(i\) if \(f\) can be written as

\[(6.1) \quad f(A = (T_{\alpha\beta})) = \sum_{1 \leq \alpha_j, \beta_j \leq r} p_{\alpha_1 \cdots \alpha_i, \beta_1 \cdots \beta_i} T_{\alpha_1 \beta_1} \cdots T_{\alpha_i \beta_i}\]

for \(p_{\alpha_1 \cdots \alpha_i, \beta_1 \cdots \beta_i} \in \mathbb{C}\) and satisfies

\[(6.2) \quad f(BAB^{-1}) = f(A), \quad \forall A \in M_r(\mathbb{C}), \forall B \in GL_r(\mathbb{C}).\]

Denote by \(I_i\) the complex linear space consisting of all \(GL_r(\mathbb{C})\)-invariant polynomial function of homogeneous degree \(i\):

\[I_i := \{ f : M_r(\mathbb{C}) \to \mathbb{C} \mid f \text{ satisfy (6.1) and (6.2)} \}\]

and the graded ring

\[I := \bigoplus_{i \geq 0} I_i.\]

If we set

\[\det(tI_r + A) =: \sum_{i=0}^r c_i(A) \cdot t^{r-i},\]

then these \(c_i(\cdot)\) are \(GL_r(\mathbb{C})\)-invariant polynomial functions of homogeneous degree \(i\). In particular, \(c_0(A) = 1\), \(c_1(A) = \text{trace}(A)\) and \(c_n(A) = \det(A)\). It is well-known that the graded ring \(I\) is multiplicatively generated by \(c_1, \ldots, c_r\), i.e.,

\[I = \mathbb{C}[c_1, \cdots, c_r].\]

Now if \((E^r, h) \longrightarrow M\) is a Hermitian holomorphic vector bundle and \((\Omega^i_j)\) is the curvature matrix of its Chern connection, then (recall (2.1) and (2.2))

\[c_i\left(\frac{\sqrt{-1}}{2\pi}(\Omega^i_j)\right) =: c_i(E, h), \quad (0 \leq i \leq r)\]

are globally defined, closed, real-valued \((i, i)\)-forms over \(M\) and represent the \(i\)-th Chern classes of \(E \longrightarrow M\). Thus we have a natural graded ring homomorphism \(\varphi\) sending \(c_i\) to \(c_i(E, h)\):

\[I = \bigoplus_{i \geq 0} I_i = \mathbb{C}[c_1, \cdots, c_r] \overset{\varphi}{\longrightarrow} \mathbb{C}[c_1(E, h), \cdots, c_r(E, h)]\]

\[(6.3) \quad f \longmapsto f\left(\frac{\sqrt{-1}}{2\pi}(\Omega^i_j)\right).\]
Griffiths first showed that ([Gr69, p. 242, (5.6)]) any \( f \in I_i \) can be written, which may not be unique, in the form

\[
f(A = (T_{ij})) = \sum_{\pi, \tau \in S_i, \rho = (\rho_1, \ldots, \rho_i) \in [1, r]^i} p_{\rho, \pi, \tau} T_{\rho^{(1)} \rho^{(1)}} \cdots T_{\rho^{(i)} \rho^{(i)}}, \quad p_{\rho, \pi, \tau} \in \mathbb{C},
\]

where \( S_i \) denotes the permutation group on \( i \) objects.

The following definition was introduced by Griffiths ([Gr69, p. 242, (5.9)]).

**Definition 6.1** (Griffiths). A polynomial function \( f \in I_i \) is called Griffiths positive if it can be expressed in the form (6.4) with

\[
p_{\rho, \pi, \tau} = \sum_{j \in I} \lambda_{\rho, j} q_{\rho, j, \pi} q_{\rho, j, \tau}, \quad \forall \, \rho, \pi, \tau,
\]

for some real numbers \( \lambda_{\rho, j} \geq 0 \), some complex numbers \( q_{\rho, j, \pi} \) and some finite set \( J \). Denote by

\[
\Pi_i := \{ f \in I_i \mid f \text{ are Griffiths positive} \}.
\]

The following key fact relating Griffiths positive polynomials and Schur polynomials was observed in [FL83, p. 54, Prop. A.3], whose proof is built on the representation theory and a previously related result in [UT77].

**Proposition 6.2** (Fulton-Lazarsfeld).

\[
\left\{ \sum_{\lambda \in \Gamma(i, r)} a_{\lambda} S_{\lambda}(c_1, \ldots, c_r) \mid \text{all } a_{\lambda} \geq 0 \right\} = \Pi_i.
\]

Now we are in the position to prove Proposition 3.1.

**Proof.** Assume that the Hermitian holomorphic vector bundle \((E^r, h) \rightarrow M^n\) satisfies \( h \geq_{BC} 0 \), i.e., under the local coordinates \((z^1, \ldots, z^n)\), the curvature matrix \((\Omega^j_i)\) with respect to some unitary frame field can be written in the following form

\[
(\Omega^j_i) = A \wedge \overline{A}, \quad A = \left( \sum_p T_{ij}^{(p)} dz^p \right).
\]

This implies that

\[
(6.7) \quad \Omega^j_i = \sum_{k, p, q} T_{ik}^{(p)} T_{jk}^{(q)} dz^p \wedge d\overline{z^q}.
\]
Therefore, for each $\lambda \in \Gamma(i, r)$ and $1 \leq i \leq n$, we have

$$S_{\lambda}(c_1(E, h), \ldots, c_r(E, h))$$

$$= \varphi(S_{\lambda}(c_1, \ldots, c_r)) \quad (6.3)$$

$$= S_{\lambda}(\sqrt{-1} \frac{1}{2\pi} (\Omega^1_j)) \ldots, c_r(\sqrt{-1} \frac{1}{2\pi} (\Omega^1_j)) \quad (6.3)$$

$$= (\frac{-1}{2\pi})^i \sum \lambda_{\rho, j} q_{\rho, j, \pi} \bar{q}_{\rho, j, \pi} T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(r_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(s_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i$$

$$\cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i$$

$$\cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i$$

$$= (\frac{-1}{2\pi})^i (-1)^{i(\frac{n-1}{2})} \sum \lambda_{\rho, j} \left\{ \begin{bmatrix} [ \sum q_{\rho, j, \pi} T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i ] \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \cdot \cdot \cdot T^{(r_1)}_{\rho}(1) k_1 \cdot \cdot \cdot T^{(s_i)}_{\rho}(1) k_i \end{bmatrix} \right\}$$

$$= (\frac{-1}{2\pi})^i \sum \lambda_{\rho, j} \psi_{\rho, j} \wedge \psi_{\rho, j}.$$

Now $\psi_{\rho, j}$ are $(i, 0)$-forms and by the definition $\lambda_{\rho, j} \geq 0$ (recall (6.5)), which means that the last expression is a strongly nonnegative $(i, i)$-form. This gives the desired proof and thus completes this article.
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