WITH COLUMBUS: "BUSCAR EL LEVANTE POR EL PONENTE" AND WITH ALICE’S BACKWARD WALK: VERIFYING THE RIEMANN HYPOTHESIS BY FALSIFYING ITS COUNTER-HYPOTHESIS

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Abstract. The RIEMANN Hypothesis (R.H.) to the effect that all the complex conjugate (c.c.) pairs of zeroes of the Zeta Function should lie on the Critical Line (C.L.) \( \text{Re}(s) = \frac{1}{2} \) is analyzed by a 'reductio ad absurdum' argument. Assuming that the R.H. is false, so that a c.c. ‘outlying’ pair of zero-points has been found, the necessary implications of this assumption are worked out by purely algebraic developments on the GRAM-BACKLUND extension of the Zeta Function. It is inferred that the assumed outlying pair must fall onto the C.L., thus falsifying the assumption. It follows that the R.H. is verified.

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1. Introduction

Christopher COLUMBUS is said to have synthetically expressed his discovery goal as ”To the West lies the way to the East”. In LEWIS CARROLL’s masterpiece, one of Alice’s initial forays into Wonderland aims at reaching a distant castle. But going straight for it along a bee-line does not seem to work: the destination looms ever farther and keeps shifting perversely now to the left, now to the right side.
of the line of sight. Taking a hint from an enigmatic remark of the Red Queen, Alice decides to change her strategy and begins walking resolutely in the opposite direction. And, lo and behold! after a short time, she finds herself at the castle drawbridge.

As in many an episode of LEWIS CARROLL’s masterpiece, this apparent flight of fancy can be interpreted on a higher semantic plane, e.g. to the effect that the proof of a mathematical hypothesis can sometimes be more easily pursued by searching for a falsification, rather than for a direct verification, of the conjecture. If through a chain of unassailable inferences the falsification is found to fail, then thanks to the principle of the third excluded it can be safely said that the original statement is verified.

The sentence attributed to Christopher COLUMBUS and quoted in the title of the present essay is not as tightly fitting an analogy as LEWIS CARROLL’s tale, but still evokes an effective intuition of a conceptual plan of action. In the present essay, the Authors propose to adopt a similar indirect strategy to reach a verification of the celebrated RIEMANN Hypothesis (1859). The latter, as well-known, states that all c.c. pair of zeroes of the Zeta Function, $s_H$ and $\bar{s}_H$, should belong to the critical line $Re(s_H) = \frac{1}{2}$. Postulating that an outlying zero-point, i.e. one in which $Re(s_H) \neq \frac{1}{2}$, has actually been found, the verification procedure of this counter-hypothesis is first presented in a broad outline, highlighting the connections of the successive logical steps. Then the internal structure of each step is analyzed in detail to bring out the necessary implications of the conditions to be imposed to proceed to the next step. This phase of the Authors’ study allows to evidence an internal contradiction, or an incompatibility, of two essential conditions: the zero-condition $Z(s_H) = 0$ and the requirement that the postulated outlying zero-point belongs to a c.c. pair. These conditions are seen to become compatible if, and only if, the two components of the pair fall back onto the C.L. The final conclusion is that the hypothesis of existence of outlying c.c. zero-pairs is falsified, and consequently the opposite conjecture, i.e. the RIEMANN Hypothesis, is verified.

2. OUTLINE OF THE SUCCESSION OF THE DISCUSSION STEPS

A chain of straightforward algebraic developments are carried out as described in the following five preliminary PROPOSITIONS paving the way to the final conclusion.

To start with, a synthetic expression for the GRAM-BACKLUND extension of the Zeta function is derived, and the general zero-condition is accordingly formulated, evidencing the presence of a main term containing the product $s.(s - 1)$, which comes from the EULER-McLAURIN procedure of ‘summation by integration’, and of a further term $Q(s)$ which is made up by the sum of several functions of $s$ (PROPOSITIONS 1 and 2).

Then, contrary to the RIEMANN Hypothesis, the existence of an outlying c.c. pair of zero-points is assumed (PROPOSITION 3) and the conditions for this existence to be allowed are analyzed in detail, underscoring the implications of the need for a factorization of the general zero-condition in the form of a binomial product whose factors reflect the assumed locations of the two c.c. zero-points split into their real and imaginary parts, the real part explicitly containing the distance $\xi$ of the pair from the C.L., i.e. $s_H = \frac{1}{2} + \xi + i.Y_H, \bar{s}_H = \frac{1}{2} + \xi - i.Y_H$. 
Another factorization is implied necessarily by the assumption that the two outlying locations form a c.c. pair (PROPOSITION 4). Those two factorizations are expressed by Eqs. (20) and (24), which cannot be fulfilled simultaneously unless 
\[ \frac{1}{2} - \xi - iY_H = \frac{1}{2} + \xi - iY_H, \]
and unless \( \xi = 0 \). This result entails the further consequence (PROPOSITION 4, Eq.(22) and PROPOSITION 5, Eq.(24)):
\[ Q(s_H) = s_H.(1 - s_H) = s_H.s_H \text{ or } s_H = 1 - s_H, \]
also identically fulfilled only if \( \xi = 0 \).

These conclusions mean that the assumedly outlying c.c. pair is constrained to fall back onto the C.L.: a result which amounts to a falsification of the postulated negation of the RIEMANN Hypothesis. Thus, with a procedure that falls into the general category of "proof by reductio ad absurdum", the RIEMANN Hypothesis is verified.

3. Details of the successive argumentation steps

3.1. Proposition 1 - Recap of fundamental properties of the Zeta function.

The basic definition of the Zeta function:

\[ Z(s) = \sum_{n=1}^{\infty} n^{-s} \]

does not lend itself to a direct determination of its non-trivial zero-points, because the convergence domain of the series figuring on the right hand side of (1) is limited to the half-plane \( \text{Re}(s) > 1 \), while the looked-for non-trivial zeroes are known to lie, see [1] and [2], inside the critical strip:

\[ 0 < \text{Re}(s) < 1 \]

Therefore it is necessary to work on an extension of the Zeta Function, i.e. a different representation which gives the same results in the convergence domain of (1) but is convergent on all of the complex plane. One such extension, which was chosen by the Authors of the present essay, is the GRAM-BACKLUND formulation:

\[ Z(s)_{GB} = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{B_2}{2} s.N^{-s-1} + \]

\[ \sum_{\mu=1}^{\nu} \frac{B_{2\mu}}{(2\mu)!} (s+1).(s+2)\ldots(s+2\mu-2).N^{-s-2\mu+1} + R_{2\nu} \]

where the remainder \( R_{2\nu} \) can be made as small as desired by choosing large values of \( N \) and \( \nu \), see [3] (in particular, page 114), and the coefficients \( B_{2\mu} \) are BERNOULLI numbers.

This extension derives from applying to definition (1) the EULER-McLaurin 'summation by integration' procedure and can be put into the more expedient,
abbreviated form:

\[ Z(s)_{GB} = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + s.r(N,s) \]

(see op. cit) where

\[ r(N,s) = \frac{N^{-s}}{2.s} + \frac{B_2}{2} N^{-s-1} \]

\[ + \sum_{\mu=1}^{\nu} \frac{B_{2\mu}}{(2\mu)!} (s+1)(s+2)\ldots(s+2.\mu-2) N^{-s-2.\mu+1} + \frac{R_{2\nu}}{s} \]

In each of the postulated c.c. zero-points, \( s = s_H = \frac{1}{2} + \xi \pm i.Y_H \) and \( s = \bar{s}_H = \frac{1}{2} + \xi - i.Y_H \), the condition \( Z_{GB}(s_H) = 0 \) or \( Z_{GB}(\bar{s}_H) = 0 \) is to be imposed, meaning that in the c.c. points \( s = s_H \) and \( s = \bar{s}_H \) the Zeta Function is zero, irrespective of the actual location of the zero (on the C.L. or elsewhere).

3.2. Proposition 2 - Formulation of the Riemann Hypothesis.

Accordingly, the condition \( Z_{GB}(s) = 0 \) in either of the pair of c.c. zero-points defined by \( s = s_H = \frac{1}{2} + \xi \pm i.Y_H \) can be put in the equivalent form:

\[ \frac{1}{s_H.N^{1-s_H}} \sum_{n=1}^{N-1} n^{-s_H} + \frac{1}{s_H.(s_H-1)} + \frac{r(N,s_H)}{N^{1-s_H}} = 0 \]

where \( \xi \in \mathbb{R}, Y_H \in \mathbb{R} \) and \(-\frac{1}{2} < \xi < \frac{1}{2}\), because all zero-points must lie inside the critical strip, see [1] and [2].

Defining a function \( Q_{GB}(s) \) of \( s \) by

\[ \frac{1}{Q_{GB}(s)} = \frac{1}{s.N^{1-s}} \sum_{n=1}^{N-1} n^{-s} + \frac{r(N,s)}{N^{1-s}} \]

from this definition Eq. (5) becomes:

\[ \frac{1}{s(s-1)} + \frac{1}{Q_{GB}(s)} = 0 \]

and successively:

\[ \frac{1}{s(s-1)} = -\frac{1}{Q_{GB}(s)} \].

Since it is \( s \neq 0, s \neq 1, Q_{BG}(s) \neq 0 \) it comes:

\[ s(s-1) = -Q_{GB}(s) \]

and at last:

\[ s(s-1) + Q_{GB}(s) = 0 \].

This is the form chosen for the zero condition in the present essay, where \( Q_{GB}(s) \) is treated as a given (generally complex unless differently inferred by the results progressively obtained) constant and \( s \) as an ‘unknown’ complex variable.

Assume now that in \( s = s_H \) and in \( s = \bar{s}_H \), see definitions after Eq. (4), there are two zero-points forming a c.c. outlying pair. The following equation can then be written:

\[ s^2 - s + Q_{GB}(s) = 0 \]
i.e. the zero-condition, which after our assumption is fulfilled by either \( s = s_H \) or \( s = \bar{s}_H \), so that also the following Eq.\( 12 \) is to be fulfilled by the ‘unknown’ \( s \):

\[
(s - s_H)(s - \bar{s}_H) = 0 \text{ or } s^2 - s(s_H + \bar{s}_H) + s_H \bar{s}_H = 0
\]

where from definitions following Eq.\( 4 \) it comes:

\[
s_H + \bar{s}_H = 1 + 2 \xi \text{ and } Q_{GB}(s) = s_H \bar{s}_H = \frac{1}{4} + \xi + \xi^2 + Y_H^2.
\]

Then Eq.\( 12 \) becomes:

\[
s^2 - s(1 + 2 \xi) + \frac{1}{4} + \xi + \xi^2 + Y_H^2 = 0
\]

so that comparing \( 14 \) with \( 11 \) it comes:

\[
\xi = 0, \quad Q_{GB}(s_H) = Q_{GB}(\bar{s}_H) = s_H \bar{s}_H = \frac{1}{4} + Y_H^2 \in \mathbb{R}
\]

**NOTE** - Two conditions are simultaneously to be fulfilled in either \( s = s_H \) or \( s = \bar{s}_H \):

\[
s^2 - s + Q_{GB}(s) = 0, \quad \text{see Eq.} \( 10 \), \text{ or:}
\]

\[
Q_{GB}(s) = s(1 - s), \quad \text{and } Q_{GB}(s) = s \bar{s}, \quad \text{see} \ (13)
\]

From Eq.\( 16 \) it is inferred that:

\[
\bar{s} = (1 - s)
\]

or, substituting in the *in extenso* expressions of \( s = s_H \) or \( s = \bar{s}_H \), Eqs. from \( 13 \) to \( 15 \) immediately follow as necessary consequences of the initial assumptions. These heuristic findings are confirmed by the lengthy algebraic developments carried out under paragraph 3.4

The RIEMANN Hypothesis states that in all non-trivial c.c. pairs of zero-points it must be \( \xi = 0 \). If a c.c. pair should be found with \( \xi \neq 0 \), the RIEMANN Hypothesis would be falsified.

### 3.3. Proposition 3 - The counter-hypothesis: assuming that the Riemann Hypothesis is false.

An hypothetical pair of c.c. zero-points \( s = s_H \) and \( s = \bar{s}_H \) lying externally from the C.L. is henceforth termed an "outlying zero-pair" (or simply an "outlier"). Assuming now that contrary to the RIEMANN Hypothesis such a non-trivial outlying zero-pair has actually been found, i.e. that:

\[
s_H^2 - s_H + Q_{GB}(s_H) = 0 \quad \text{and} \quad s_H^2 - \bar{s}_H + Q_{GB}(\bar{s}_H) = 0
\]

which come from Eq.\( 10 \), it is then proposed to analyze the necessary consequences of the conditions imposed on the pair location parameters (in particular on \( \xi \)) by the structure of the zero-condition equations \( 18 \) where the term \( Q_{GB}(s_H) \) is to be considered as a known complex quantity as a consequence of the hypothesis of the existence of the outlier.
3.4. Proposition 4 - Imposing that in two outlying c.c. points the zero-condition is fulfilled: the compatibility condition. Let us look more closely to the results exposed in the preceding PROPOSITION 3. The basic zero-condition, (10), is assumedly fulfilled by two c.c. values of \( s \), i.e. \( s = s_H = \frac{1}{2} + \xi + i.Y \) and \( s = \overline{s}_H = \frac{1}{2} + \xi - i.Y \) (the index \( H \) is henceforth dropped from \( Y_H \), and the index \( GB \) is dropped from \( Q_{GB} \), where no confusion is possible, in favor of simplicity). The zero-condition equation \( s^2 - s + Q(s) = 0 \) is assumedly fulfilled by the two c.c. values:

\[
(19) \quad s = s_H = \frac{1}{2} + \xi + i.Y_H, \quad s = \overline{s}_H = \frac{1}{2} + \xi - i.Y_H
\]

These two values of \( s \) must be identical with the two c.c. solutions of the quadratic algebraic equation \((s - s_H)(s - \overline{s}_H) = 0\):

\[
(20) \quad [s - (\frac{1}{2} + \xi + i.Y)](s - (\frac{1}{2} + \xi - i.Y)) = 0, \quad \text{or}
\]

\[
(21) \quad s^2 - s(1 + 2\xi) + (\frac{1}{2} + \xi)^2 + Y_H^2 = 0.
\]

From Eq. (21), compared with \( s^2 - s + Q(s) = 0 \), it comes:

\[
\xi = 0 \quad \text{and} \quad Q(s) = (\frac{1}{2} + \xi)^2 + Y_H^2 = \frac{1}{4} + Y_H^2 \in \mathbb{R}
\]

\[
(22) \quad \text{therefore} \quad Q(s_H) = Q(\overline{s}_H) ; \quad Q(s_H) = Q(\overline{s}_H) = \frac{1}{4} + Y_H^2 = s_H \overline{s}_H.
\]

In order to approach the question from a different angle, it is now proposed to factorize the left-hand side of the zero-condition (10), i.e. \( s^2 - s + Q(s) \), in such a way as to bring forth the necessarily ensuing constraints on the distance \( \xi \) of the assumed outlying zero-points from the C.L.

The factorization is carried out in the following way: considering the zero-point assumedly found on the outlier \( s = s_H = \frac{1}{2} + \xi + i.Y_H \), the polynomial division \( \frac{s(s-1)+Q(s_H)}{s-s_H} \) is performed on the left-hand side of Eq. (10) with the divisor \( s-s_H \), where \( s_H = \frac{1}{2} + \xi + i.Y \), as well as \( Q(s_H) \), are to be treated as known complex quantities as a consequence of the hypothesis of the existence of the outlier, while \( s \) in the numerator and denominator is to be formally considered as the unknown location (alternative to \( s_H \)) implied by the zero-condition (10). The polynomial division \( \frac{s(s-1)+Q(s_H)}{s-s_H} \), performed according to the usual rules of Algebra, yields the result (details omitted as devoid of interest, but as a check it can be easily verified that the product \((s-s_H)(s-(1-s_H))\) yields \(s(s-1)+Q(s_H)\) if the rest \(R(s_H)=0\):

\[
(23) \quad \frac{s(s-1)+Q(s_H)}{s-s_H} = s-(1-s_H), \quad \text{with the rest} \quad R(s_H) = Q(s_H) - s_H(1-s_H)
\]

which is zero as a consequence of Eq. (18). Thus \( s-s_H \) divides exactly \( s(s-1)+Q(s_H) \) and for \( s \neq s_H \) Eq. (10) is factorized as:

\[
(24) \quad s(s-1)+Q(s_H) = (s-s_H)(s-(1-s_H)) = 0 \quad (\equiv 0 \text{ for } s = s_H)
\]

Likewise, \( s - \overline{s}_H \) divides exactly \( s(s-1)+Q(\overline{s}_H) \), so that changing the divisor to \( s - \overline{s}_H \) we get:
\[
\frac{s(s-1)+Q(\overline{s_H})}{s-\overline{s_H}} = s - (1 - \overline{s_H}), \text{ with the rest } R(\overline{s_H}) = Q(\overline{s_H}) - \overline{s_H}.(1 - \overline{s_H}) \text{ which is zero as a consequence of Eq.}\,(15), \text{ and therefore:}
\]
(25) \[s.(s-1) + Q(\overline{s_H}) = (s - \overline{s_H}).|s - (1 - \overline{s_H})| = 0 \text{ (\equiv 0 for } s = \overline{s_H})\]
(recall, Eq.\,(22), that \(Q(\overline{s_H}) = Q(s_H)\)).

Thus for \(s = \overline{s_H}\), while Eq.\,(25) becomes an identity, in order that Eq.\,(24) be fulfilled as well it must be:
(\(\overline{s_H} - s_H\)).[\(\overline{s_H} - (1 - s_H)\]] = 0, and since \(\overline{s_H} - s_H = -2.i.Y_H \neq 0\) it comes
(26) \[\overline{s_H} - (1 - s_H) = 0 \text{ or } s_H = 1 - \overline{s_H}\]
Likewise, for \(s = s_H\), while Eq.\,(24) becomes an identity, in order that Eq.\,(25) be fulfilled as well it must be:
(27) \[s_H - (1 - \overline{s_H}) = 0 \text{ or } s_H = 1 - \overline{s_H}.
\]

3.5. Proposition 5 - The compatibility conditions required for the existence of the assumed c.c. pair of outlying zero-points.

Eqs.\,(22) in the preceding PROPOSITION 4 are confirmed independently by both Eqs.\,(25) and \,(26), which are incompatible with the assumption \(\xi \neq 0\) in \(s_H = \frac{1}{2} + \xi + i.Y_H\); indeed, from \,(26): \(\overline{s_H} = \frac{1}{2} + \xi - i.Y_H = 1 - s_H = \frac{1}{2} - \xi - i.Y_H\) or:
(28) \[\xi = 0\]
The same result is obtained from Eq.\,(25).

The condition \,(28) implied by Eqs.\,(25) and \,(26), i.e. \(\xi = 0\), means that the assumedly "outlying" pair is in fact constrained to lie on the C.L. The assumption that outlying pairs of c.c. zeroes can exist is thus seen to have been falsified. Besides, it comes:
(29) \[s_H = \frac{1}{2} + i.Y_H \text{ and } \overline{s_H} = \frac{1}{2} - i.Y_H\]

4. Final remarks

Let us now look back on the substance and interconnections of the steps through which our proof was built up. The details, which were illustrated in the five PROPOSITIONS, are here omitted in favor of a "panoramic" view of our procedure.

I The RIEMANN HYPOTHESIS.: All c.c. zero-pairs \(s, \overline{s}\) of the Zeta Function, fulfilling \(Z(s) = 0\) with \(\text{Im}(s) \neq 0\), fall onto the C.L. \(\text{Re}(s) = \frac{1}{2}\).

II The zero-condition \(Z(s) = 0\) can be put under the form \(s.(s-1) + Q(s) = 0\) (GRAM-BACKLUND and EULER-McLAURIN 'summation by integration')

III We adopt the counter hypothesis: the RIEMANN HYPOTHESIS is false, so that we can assume that it was found a c.c. pair \(s_H, \overline{s_H}\) fulfilling the above mentioned zero-condition with \(s_H, \overline{s_H} = \frac{1}{2} + \xi \pm i.Y_H\) and \(\xi \neq 0\).

IV Then it should be possible to factorize the zero-condition \(s.(s-1) + Q(s) = 0\) in either of two forms:

V \(s.(s-1) + Q(s) = (s - s_H).(s - \overline{s_H}) = 0\) [from which \(Q(s) = s_H.\overline{s_H} \in \mathbb{R}\) and \(Q(s_H) = Q(s_H) = Q]\).

VI ... and by performing the division \((s.(s-1)+Q)/(s - s_H)\) or \((s.(s-1)+Q)/(s - \overline{s_H})\). In this way we get for \(s \neq s_H\) the factorization:

VII \(s.(s-1) + Q = (s - s_H).[s - (1 - s_H)] = 0\) and a similar factorization valid for \(s \neq \overline{s_H}\) by replacing \(s_H\) with \(\overline{s_H}\).
VIII Then comparing V with VI or VII we see that it should be $\pi_H = 1 - s_H$, which is possible only if $\xi = 0$, contrary to III.

The consequence of the preceding elaborations purports to the invalidation of the assumed existence of outlying pairs of c.c. zero-points, i.e. to the falsification of the negation of the RIEMANN Hypothesis. The logical implication is that the latter Hypothesis is verified.

It is easily seen that this conclusion is strictly tied to the structure of the GRAM-BACKLUND extension of the Zeta Function, see (3), and within this expression, more specifically, to the terms originating from the EULER-McLAURIN summation by integration procedure. Since this form of summation is indispensable to obtain an extension of the Zeta function convergent within the critical strip, the result is deemed to hold in general.

Another relevant feature of the zero-condition (10) is, see [4] and [5], the invariance of the product $s(s - 1)$ with respect to the exchange $s \leftrightarrow 1 - s$.

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