Estimates for lower bounds of eigenvalues of the poly-Laplacian and the quadratic polynomial operator of the Laplacian

Qing-Ming Cheng  
Department of Applied Mathematics, Faculty of Sciences,  
Fukuoka University, Fukuoka 814-0180, Japan

He-Jun Sun  
Department of Applied Mathematics,  
Nanjing University of Science and Technology, Nanjing 210094,  
People’s Republic of China (hejunsun@163.com)

Guoxin Wei  
School of Mathematical Sciences, South China Normal University,  
Guangzhou 510631, People’s Republic of China

Lingzhong Zeng  
Department of Mathematics, Graduate School of Science and Engineering, Saga University, Saga 840-8502, Japan

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In this paper, we investigate the Dirichlet eigenvalue problems of the poly-Laplacian with any order and the quadratic polynomial operator of the Laplacian. We give some estimates for lower bounds of the sums of their first $k$ eigenvalues.

1. Introduction

Let $\Omega$ be a bounded domain in an $n$-dimensional Euclidean space $\mathbb{R}^n$, where $n \geq 2$. The Dirichlet eigenvalue problem of the poly-Laplacian is described by

$$
\begin{aligned}
(-\Delta)^{l} u &= \lambda u \quad \text{on } \Omega, \\
\left. u \right|_{\partial \Omega} &= \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega} = \cdots = \left. \frac{\partial^{l-1} u}{\partial \nu^{l-1}} \right|_{\partial \Omega} = 0,
\end{aligned}
$$

(1.1)

where $\Delta$ is the Laplacian and $\nu$ denotes the outward unit normal vector field of $\partial \Omega$. As we know, this problem has a real and discrete spectrum (see [8, 20, 21]): $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$, where each eigenvalue repeats with its multiplicity. It is useful to make some estimates for eigenvalues of this problem (see, for example, [5, 8, 22] and the references therein).

When $l = 1$, (1.1) is called the Dirichlet Laplacian problem or the fixed membrane problem. The asymptotic behaviour of its $k$th eigenvalue $\lambda_k$ relates to geometric
properties of $\Omega$ when $k \to \infty$. In fact, Weyl’s asymptotic formula,

$$
\lambda_k \sim \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} \quad \text{as } k \to \infty,
$$

(1.2)

holds, where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $V(\Omega)$ denotes the volume of $\Omega$. In 1961, Pólya [17] proved that (1.2) is a one-sided inequality on tiling domains in $\mathbb{R}^2$. His proof also works on tiling domains in $\mathbb{R}^n$. Moreover, he conjectured that the inequality

$$
\lambda_k \geq \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n}
$$

(1.3)

holds for any bounded domain in $\mathbb{R}^n$. Berezin [3] and Lieb [14] made some contributions to the partial solution of this conjecture. In 1983, Li and Yau [13] proved the following so-called Li–Yau inequality:

$$
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n}.
$$

(1.4)

In 1997, Laptev [10] justified Pólya’s conjecture for certain product domains. In 2000, Laptev and Weidl [11] pointed out that (1.4) can be derived by the Legendre transform of a result derived by Berezin [3]. Hence, (1.4) is also called the Berezin–Li–Yau inequality. In 2003, adding an additional positive term to the right-hand side of (1.4), Melas [15] improved (1.4) to

$$
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \frac{1}{24(n+2)} \frac{V(\Omega)}{I(\Omega)},
$$

(1.5)

where

$$
I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 \, dx
$$

is the moment of inertia of $\Omega$. In 2009, Kovařík et al. [9] improved (1.4) by adding a positive correction term to its right-hand side when $n = 2$ and assuming some geometric properties of the boundary of $\Omega$. Recently, Ilyin [7] obtained the following asymptotic lower bound for eigenvalues of (1.1):

$$
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \frac{n}{48} \frac{V(\Omega)}{I(\Omega)} (1 - \varepsilon_n(k)),
$$

(1.6)

where $0 \leq \varepsilon_n(k) = O(k^{-2/n})$ is an infinitesimal of $k^{-2/n}$. Moreover, he derived some explicit inequalities for the particular cases of $n = 2, 3, 4$:

$$
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \frac{n}{48} \frac{V(\Omega)}{I(\Omega)} \beta_n,
$$

(1.7)

where $\beta_2 = \frac{119}{120}$, $\beta_3 = 0.986$ and $\beta_4 = 0.983$. 

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When $l = 2$, (1.1) is called the clamped plate problem. Agmon [1] and Pleijel [16] obtained that
\[
\lambda_k \sim \frac{(2\pi)^4}{(\omega_n V(\Omega))^{4/n}} k^{4/n} \text{ as } k \to +\infty.
\] (1.8)

In 1985, Levine and Protter [12] proved that
\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n + 4} \frac{(2\pi)^4}{(\omega_n V(\Omega))^{4/n}} k^{4/n}.
\] (1.9)

For the special case of $n = 2$, Ilyin [7] proved that
\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{16\pi^2}{3(V(\Omega))^2} k^2 + \frac{12095\pi}{3 \cdot 12096 I(\Omega)} k.
\] (1.10)

In 2011, Cheng and Wei [4] strengthened (1.9) to
\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n + 4} \frac{(2\pi)^4}{(\omega_n V(\Omega))^{4/n}} k^{4/n} + \frac{n}{n + 2} \left[ \frac{n + 2}{12n(n + 4)} - \frac{1}{1152 n^2(n + 4)} \right] \frac{(2\pi)^2 V(\Omega) I(\Omega)}{(\omega_n V(\Omega))^{2/n}}
\]
\[
+ \left[ \frac{1}{576 n(n + 4)} - \frac{1}{27648 n^2(n + 2)(n + 4)} \right] \frac{(V(\Omega))^2}{I(\Omega)}.
\] (1.11)

When $l \geq 3$, Levine and Protter [12] proved that
\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n + 2l} \frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{2l/n}} k^{2l/n}.
\] (1.12)

Recently, adding $l$ terms of lower order of $k^{2l/n}$ to the right-hand side of (1.12), Cheng et al. [6] derived that
\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n + 2l} \frac{(2\pi)^{2l}}{(\omega_n V(\Omega))^{2l/n}} k^{2l/n}
\]
\[
+ \frac{n}{(n + 2l)} \sum_{p=1}^{l} \frac{l + 1 - p}{(24)^{p(n)}} \frac{(2\pi)^{2l-p}}{(\omega_n V(\Omega))^{2(l-p)/n}} \left( \frac{V(\Omega) I(\Omega)}{I(\Omega)} \right)^p k^{2(l-p)/n}.
\] (1.13)

When $l = 1$, (1.13) becomes (1.5).

In this paper, we obtain the following result for (1.1).
Theorem 1.1. Let \( \Omega \) be a bounded domain in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Denote by \( \lambda_j \) the \( j \)th eigenvalue of (1.1). We then have that

\[
\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2l} \left( \frac{2\pi}{\omega_n V(\Omega)} \right)^{2l/n} k^{2l/n} + \frac{n l}{48} \frac{(2\pi)^{2l-2}}{(\omega_n V(\Omega))^{(2l-2)/n}} \frac{V(\Omega)}{I(\Omega)} k^{(2l-2)/n} (1 - \varepsilon_n(k)),
\]

(1.14)

where \( 0 \leq \varepsilon_n(k) = O(k^{-2/n}) \) is an infinitesimal of \( k^{-2/n} \).

Taking \( l = 1 \) in (1.14), we obtain (1.6). Moreover, the second term on the right-hand side of (1.13) is

\[
\frac{l}{24(n+2l)} \frac{(2\pi)^{2l-2}}{(\omega_n V(\Omega))^{(2l-2)/n}} \frac{V(\Omega)}{I(\Omega)} k^{(2l-2)/n}.
\]

Hence, the second term on the right-hand side of (1.14) is \( n(n+2l)/2 \) times larger than that of (1.13). Thus, for large \( k \), (1.14) is sharper than (1.13).

Furthermore, we investigate the following Dirichlet eigenvalue problem of the quadratic polynomial operator of the Laplacian:

\[
\begin{aligned}
\Delta^2 u - a \Delta u &= \Gamma u & \text{on} & \Omega, \\
u|_{\partial\Omega} &= \frac{\partial u}{\partial \nu} \bigg|_{\partial\Omega} &= 0,
\end{aligned}
\]

(1.15)

where \( a \) is a non-negative constant. It is an ideal model, which is abstracted from the problems of physics and mechanics (see [19]). Levine and Protter [12] proved that the eigenvalues of this problem satisfy

\[
\Gamma_k \geq \frac{n}{n+4} \left( \frac{2\pi}{\omega_n V(\Omega)} \right)^{4/n} k^{4/n} + \frac{na}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n}.
\]

(1.16)

In this paper, we derive the following results for (1.15).

Theorem 1.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Denote by \( \Gamma_j \) the \( j \)th eigenvalue of (1.15). We then have that

\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \left( \frac{2\pi}{\omega_n V(\Omega)} \right)^{4/n} k^{4/n} + \left( \frac{n}{24} \frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2} \right) \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \left[ \frac{n(n^2 - 4)}{3840} \frac{V(\Omega)}{I(\Omega)} + \frac{na}{48} \right] \frac{V(\Omega)}{I(\Omega)} (1 - \varepsilon_n(k)),
\]

(1.17)

where \( 0 \leq \varepsilon_n(k) = O(k^{-2/n}) \) is an infinitesimal of \( k^{-2/n} \).

For the special cases of \( n = 2, 3, 4 \), we prove the following sharper result.
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**Theorem 1.3.** Denote by $\Gamma_j$ the $j$th eigenvalue of (1.15) on a bounded domain $\Omega$ in $\mathbb{R}^n$, where $n = 2, 3, 4$. We then have that

$$
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \left( \frac{(2\pi)^4}{(\omega_n V(\Omega))^{4/n}} k^{4/n} + \left( \frac{n}{24} \alpha_n \frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2} \right) \right) \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \frac{na \beta_n}{48} \frac{V(\Omega)}{I(\Omega)},
$$

(1.18)

where $\alpha_2 = \frac{12095}{12096}$, $\beta_2 = \frac{119}{120}$, $\alpha_3 = 0.991$, $\beta_3 = 0.986$, $\alpha_4 = 0.985$ and $\beta_4 = 0.983$.

Making a modification in the proof of theorem 1.3, we get the following result.

**Theorem 1.4.** Denote by $\Gamma_j$ the $j$th eigenvalue of (1.15) on a bounded domain $\Omega$ in $\mathbb{R}^n$, where $n = 3, 4$. We then have that

$$
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \left( \frac{(2\pi)^4}{(\omega_n V(\Omega))^{4/n}} k^{4/n} + \left( \frac{n}{24} \alpha_n \frac{V(\Omega)}{I(\Omega)} + \frac{na}{n+2} \right) \right) \frac{(2\pi)^2}{(\omega_n V(\Omega))^{2/n}} k^{2/n} + \frac{n(a^2 - 4)}{3840} \frac{V(\Omega)}{I(\Omega)} + \frac{na \beta_n}{48} \frac{V(\Omega)}{I(\Omega)},
$$

(1.19)

Taking $a = 0$ in (1.17)–(1.19), we can obtain some results for the clamped plate problem.

**2. Proofs of the main results**

In order to prove theorem 1.1, we need the following lemma, derived by Ilyin [7] (see [9]).

**Lemma 2.1.** Let

$$
\Psi_s(r) = \begin{cases} 
M & \text{for } 0 \leq r \leq s, \\
M - L(r - s) & \text{for } s \leq r \leq s + \frac{M}{L}, \\
0 & \text{for } r \geq s + \frac{M}{L}.
\end{cases}
$$

Suppose that

$$
m^* = \int_{0}^{+\infty} r^{b} \Psi_s(r) \, dr
$$

and $d \geq b$. Then, for any decreasing and absolutely continuous function $F$ satisfying the conditions

$$
0 \leq F \leq M, \quad \int_{0}^{+\infty} r^{b} F(r) \, dr = m^*, \quad 0 \leq -F' \leq L,
$$

(2.1)

the following inequality holds:

$$
\int_{0}^{+\infty} r^{d} F(r) \, dr \geq \int_{0}^{+\infty} r^{d} \Psi_s(r) \, dr.
$$

(2.2)
We now give the proof of theorem 1.1.

**Proof of theorem 1.1.** Let \( u_j \) be an orthonormal eigenfunction corresponding to the \( j \)th eigenvalue \( \lambda_j \) of (1.1). Denote by \( \hat{u}_j(\xi) \) the Fourier transform of \( u_j(x) \), which is defined by

\[
\hat{u}_j(\xi) = (2\pi)^{-n/2} \int_{\Omega} u_j(x)e^{ix\cdot\xi} \, dx. \tag{2.3}
\]

It follows from Plancherel’s theorem that

\[
\int_{\Omega} \hat{u}_j(\xi)\hat{u}_q(\xi) \, d\xi = \delta_{jq}. \tag{2.4}
\]

Set \( h(\xi) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 \). From (2.4) and Bessel’s inequality, one can get that

\[
h(\xi) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} |e^{ix\cdot\xi}|^2 \, dx = (2\pi)^{-n} V(\Omega). \tag{2.5}
\]

Moreover, Parseval’s identity implies that

\[
\int_{\mathbb{R}^n} h(\xi) \, d\xi = \sum_{j=1}^{k} \int_{\Omega} |u_j(x)|^2 \, dx = k. \tag{2.6}
\]

Since

\[
\nabla \hat{u}_j(\xi) = (2\pi)^{-n/2} \int_{\Omega} ixe^{-ix\cdot\xi} \, dx,
\]

we have that

\[
\sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} |ixe^{ix\cdot\xi}|^2 \, dx = (2\pi)^{-n} I(\Omega). \tag{2.7}
\]

It follows from (2.5) and (2.7) that

\[
|\nabla h(\xi)| \leq 2 \left( \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2 \right)^{1/2} \left( \sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)}. \tag{2.8}
\]

Denote by \( h^*(\xi) = \psi(|\xi|) \) the symmetric decreasing rearrangement (see [2,18]) of \( h \). From

\[
k = \sum_{j=1}^{k} \int_{\Omega} |u_j(x)|^2 \, dx = \int_{\mathbb{R}^n} h(\xi) \, d\xi = \int_{\mathbb{R}^n} h^*(\xi) \, d\xi = n\omega_n \int_0^{+\infty} r^{n-1}\psi(r) \, dr,
\]

we get that

\[
\int_0^{+\infty} r^{n-1}\psi(r) \, dr = \frac{k}{n\omega_n}. \tag{2.9}
\]
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At the same time, using integration by parts and Parseval’s identity, we have that

\[
\int_{\mathbb{R}^n} |\xi|^{2l} h(\xi) \, d\xi = \sum_{j=1}^{k} \sum_{p_1,\ldots,p_l=1}^{n} \left| (2\pi)^{-n/2} \int_{\Omega} \xi_{p_1} \cdots \xi_{p_l} u_j(x) e^{ix \cdot \xi} \, dx \right|^2 \, d\xi
\]

\[
= \sum_{j=1}^{k} \sum_{p_1,\ldots,p_l=1}^{n} \left| (2\pi)^{-n/2} \int_{\Omega} \frac{\partial^l u_j(x)}{\partial x_{p_1} \cdots \partial x_{p_l}} e^{ix \cdot \xi} \, dx \right|^2 \, d\xi
\]

\[
= \sum_{j=1}^{k} \sum_{p_1,\ldots,p_l=1}^{n} \int_{\mathbb{R}^n} \left| \frac{\partial^l u_j(\xi)}{\partial x_{p_1} \cdots \partial x_{p_l}} \right|^2 \, d\xi
\]

\[
= \sum_{j=1}^{k} \int_{\Omega} u_j(x)(-\Delta)^l u_j(x) \, dx. \tag{2.10}
\]

Thus, it yields that

\[
\sum_{j=1}^{k} \lambda_j = \int_{\mathbb{R}^n} |\xi|^{2l} h(\xi) \, d\xi. \tag{2.11}
\]

Making use of (2.11) and the properties of symmetric decreasing rearrangement, we obtain that

\[
\sum_{j=1}^{k} \lambda_j = \int_{\mathbb{R}^n} |\xi|^{2l} h(\xi) \, d\xi \geq \int_{\mathbb{R}^n} |\xi|^{2l} h^+(\xi) \, d\xi = n\omega_n \int_0^{+\infty} r^{n+2l-1} \psi(r) \, dr. \tag{2.12}
\]

Noting (2.5), (2.8) and (2.9), we can apply lemma 2.1 to \( \psi \) with \( b = n - 1 \) and \( d = n + 2l - 1 \). Therefore, using (2.12), we have that

\[
\sum_{j=1}^{k} \lambda_j \geq n\omega_n \int_0^{+\infty} r^{n+2l-1} \psi(r) \, dr \geq n\omega_n \int_0^{+\infty} r^{n+2l-1} \Psi_s(r) \, dr, \tag{2.13}
\]

with \( M = (2\pi)^{-n} V(\Omega) \), \( m_* = k/n\omega_n \) and \( L = 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} \). Set \( t = Ls/M \).

Combining (2.9) and

\[
\int_0^{+\infty} r^{n-1} \psi(r) \, dr = \int_0^{+\infty} r^{n-1} \Psi_s(r) \, dr = \frac{M^{n+1}}{n(n+1)L^n} [(t+1)^{n+1} - t^{n+1}]
\]

yields that

\[
(t+1)^{n+1} - t^{n+1} = k_*, \tag{2.14}
\]

where

\[
k_* = k \frac{(n+1)L^n}{\omega_n M^{n+1}}.
\]
Set $\eta = t - \frac{1}{2}$. Then, (2.14) becomes

$$(\eta + \frac{1}{2})^{n+1} - (\eta - \frac{1}{2})^{n+1} = k_*.$$  \hfill (2.15)

The asymptotic expansion for the unique positive root of (2.15) is

$$\eta(k_*) = \zeta - \frac{n-1}{24} \zeta^{-1} + \frac{(n-1)(n-3)(2n+1)}{5760} \zeta^{-3} + \cdots,$$  \hfill (2.16)

where $\zeta = (k_*/(n+1))^{1/n}$. We can then deduce that

$$(t(k^*)+1)^{n+2l+1} - t(k^*)^{n+2l+1}$$

$$= \left(\begin{array}{c} n+2l+1 \\ 1 \end{array}\right) \zeta^{n+2l} + \left[ \frac{1}{4} \left(\begin{array}{c} n+2l+1 \\ 3 \end{array}\right) - \frac{n-1}{12} \left(\begin{array}{c} n+2l+1 \\ 2 \end{array}\right) \right] \zeta^{n+2l-2}$$

$$+ \left[ \frac{1}{16} \left(\begin{array}{c} n+2l+1 \\ 5 \end{array}\right) - \frac{n-1}{24} \left(\begin{array}{c} n+2l+1 \\ 4 \end{array}\right) + \frac{(n-1)^2}{192} \left(\begin{array}{c} n+2l+1 \\ 3 \end{array}\right) \right] \zeta^{n+2l-4} + \cdots$$

$$= (n+2l+1) \left[ \zeta^{n+2l} + \frac{l(n+2l)}{12} \zeta^{n+2l-2} + \frac{(n+2l)C(n,l)}{5760} \zeta^{n+2l-4} + \cdots \right],$$  \hfill (2.17)

with the binomial coefficient

$$\binom{q}{t} = \frac{q!}{t!(q-t)!}$$

and the constant

$$C(n,l) = (n+2l-1)[(n+2l-2)(6l-7n+1) + 5(n-1)^2]$$

$$+ (n-1)(n-3)(2n+1).$$

Using (2.17), we get

$$n\omega_n \int_0^{+\infty} r^{n+2l-1} \overline{\Psi}_*(r) \, dr$$

$$= \frac{n\omega_n M^{n+2l+1}}{(n+2l)(n+2l+1)L^{n+2l}} [ (t(k_*)+1)^{n+2l+1} - t(k_*)^{n+2l+1} ]$$

$$= \frac{n\omega_n M^{n+2l+1}}{(n+2l)L^{n+2l}} \left[ \left(\frac{k_*}{n+1}\right)^{(n+2l)/n} + \frac{l(n+2l)}{12} \left(\frac{k_*}{n+1}\right)^{(n+2l-2)/n} \right.$$

$$+ \left. \frac{(n+2l)C(n,l)}{5760} \left(\frac{k_*}{n+1}\right)^{(n+2l-4)/n} + \cdots \right].$$  \hfill (2.18)
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Substituting the expressions \(k_4 = k(n + 1)L_n/\omega_n M^{n+1}, M = (2\pi)^{-n}V(\Omega)\) and \(L = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}\) into (2.18), we have that

\[
n\omega_n \int_0^{+\infty} r^{n+2l-1} \psi_s(r) \, dr = \frac{n}{n + 2l} \omega_n^{-2l/n} M^{-2l/n} \sum_{j=1}^{k_4} 
\]

\[
= \frac{n}{n + 2l} \omega_n^{-2l/n} M^{-2l/n} k^{1+2l/n} + \frac{nl}{12} \omega_n^{-2l/n} \frac{M^{2-(2l-2)/n}}{L^2} k^{1+(2l-2)/n} 
\]

\[
+ \frac{nC(n, l)}{5760} \omega_n^{-2l/n} M^{-2l/n} \frac{M^{4-(2l-4)/n}}{L^4} k^{1+(2l-4)/n} + O(k^{1+(2l-6)/n}) 
\]

\[
= \frac{n}{n + 2l} \omega_n^{-2l/n} M^{-2l/n} k^{1+2l/n} + \frac{nl}{48} \omega_n V(\Omega) V(\Omega) \frac{V(\Omega)^{2l-2/n}}{I(\Omega)} k^{1+(2l-2)/n} 
\]

\[
+ \frac{nC(n, l)}{92160} \omega_n V(\Omega) V(\Omega) \frac{V(\Omega)^{2l-4/n}}{I(\Omega)} k^{1+(2l-4)/n} + O(k^{1+(2l-6)/n}). \quad (2.19) 
\]

Inserting (2.19) into (2.13), we show that (1.14) is true. This completes the proof of theorem 1.1.

\[\square\]

**Proof of theorem 1.2.** It follows from (2.10) that

\[
\sum_{j=1}^{k} \Gamma_j = \sum_{j=1}^{k} \int_{\Omega} u_j(x)(\Delta^2 u_j(x) - a\Delta u_j(x)) \, dx 
\]

\[
= \int_{\mathbb{R}^n} |\xi|^4 h(\xi) \, d\xi + a \int_{\mathbb{R}^n} |\xi|^2 h(\xi) \, d\xi 
\]

\[
\geq \int_{\mathbb{R}^n} |\xi|^4 h^*(\xi) \, d\xi + a \int_{\mathbb{R}^n} |\xi|^2 h^*(\xi) \, d\xi 
\]

\[
= n\omega_n \left( \int_0^{+\infty} r^{n+3}\psi_s(r) \, dr + a \int_0^{+\infty} r^{n+1}\psi_s(r) \, dr \right). \quad (2.20) 
\]

Then, applying lemma 2.1 to \(\psi\) and using (2.20), we obtain that

\[
\sum_{j=1}^{k} \Gamma_j \geq n\omega_n \left( \int_0^{+\infty} r^{n+3}\psi_s(r) \, dr + a \int_0^{+\infty} r^{n+1}\psi_s(r) \, dr \right). \quad (2.21) 
\]

Observe that \(C(n, l) = -24n^2 + 96\) when \(l = 2\), and \(C(n, l) = -4(3n + 2)(n - 1)\) when \(l = 1\). Therefore, from (2.19), we have that

\[
n\omega_n \left( \int_0^{+\infty} r^{n+3}\psi_s(r) \, dr + a \int_0^{+\infty} r^{n+1}\psi_s(r) \, dr \right) 
\]

\[
= \frac{n}{n + 4} \left( \frac{2\pi}{\omega_n V(\Omega)} \right)^{4/n} k^{1+4/n} + \left( \frac{n V(\Omega)}{24 I(\Omega)} + a \frac{n}{n + 2} \right) \left( \frac{2\pi}{\omega_n V(\Omega)} \right)^{2/n} k^{1+2/n} 
\]

\[
+ \left[ - \frac{n(n^2 - 4) V(\Omega)}{3840 I(\Omega)} + a \frac{n}{48} \right] \frac{V(\Omega)^2}{I(\Omega)} k + O(k^{2/n-2}). \quad (2.22) 
\]

It is then easy to find that (1.17) holds. This completes the proof of theorem 1.2. \[\square\]
Proof of theorem 1.3. When \( n = 2 \), making use of (1.7) and (1.10), we have that

\[
\sum_{j=1}^{k} T_j = \int_{\mathbb{R}^n} |\xi|^4 h(\xi) \, d\xi + a \int_{\mathbb{R}^n} |\xi|^2 h(\xi) \, d\xi
\]

\[
\geq 2\omega_2 \int_0^{+\infty} r^5 \psi_s(r) \, dr + 2a\omega_2 \int_0^{+\infty} r^3 \psi_s(r) \, dr
\]

\[
\geq \frac{1}{3} \left( \frac{2\pi}{\omega_2 V(\Omega)} \right)^3 \cdot k^3 + \left( \frac{\alpha_2}{12V(\Omega)} + \frac{a}{2V(\Omega)} \right) \frac{(2\pi)^2}{\omega_2} k^2 + \frac{a}{24} \beta_2 \frac{1}{V(\Omega)},\quad (2.23)
\]

where \( \alpha_2 = \frac{12.095}{12.096} \) and \( \beta_2 = \frac{119}{20} \).

When \( n = 3 \), it follows from (2.21) that

\[
\sum_{j=1}^{k} T_j \geq 3\omega_3 \int_0^{+\infty} r^6 \psi_s(r) \, dr + 3a\omega_3 \int_0^{+\infty} r^4 \psi_s(r) \, dr.\quad (2.24)
\]

We now make an estimate for the lower bound of \( \int_0^{+\infty} r^6 \psi_s(r) \, dr \). Since

\[
\int_0^{+\infty} r^6 \psi_s(r) \, dr = \frac{M^8}{56L^7}[(t(k_*) + 1)^8 - t(k_*)^8],
\]

we need to estimate \( (t(k_*) + 1)^8 - t(k_*)^8 \). Equation (2.14) becomes \( t + 1 = k_* \) when \( n = 3 \). Its positive root \( t(k_*) \) is

\[
t(k_*) = \frac{1}{2}(\rho(k_*) - \varrho(k_*)) - \frac{1}{2},
\]

where

\[
\rho(k_*) = \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \quad \text{and} \quad \varrho(k_*) = \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3}.
\]

Set \( \vartheta(k_*) = \frac{1}{2}(\rho(k_*) - \varrho(k_*)) \). We then have that

\[
(t(k_*) + 1)^8 - t(k_*)^8 = 8\vartheta(k_*)^7 + 14\vartheta(k_*)^5 + \frac{7}{2}\vartheta(k_*)^3 + \frac{1}{2}\vartheta(k_*)
\]

\[
= \frac{1}{16}[(\rho(k_*) - \varrho(k_*))^7 + 7\rho(k_*)^3 - \varrho(k_*)^5]
\]

\[
+ 7(\rho(k_*) - \varrho(k_*))^3 + (\rho(k_*) - \varrho(k_*))].\quad (2.25)
\]

Observe that

\[
\rho(k_*) \cdot \varrho(k_*) = \frac{1}{3}.\quad (2.26)
\]

Then, using (2.26), we have that

\[
(\rho(k_*) - \varrho(k_*))^7
\]

\[
= \rho(k_*)(\rho(k_*)^6 + 7\varrho(k_*)^6) + 21\rho(k_*)^2 \varrho(k_*)^2 (\rho(k_*)^3 - \varrho(k_*)^3)
\]

\[
- 35\rho(k_*)^3 \varrho(k_*)^3 (\rho(k_*) - \varrho(k_*)) - \varrho(k_*)(7\rho(k_*)^6 + \varrho(k_*)^6)
\]

\[
= \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \left( 16k_*^2 - 12k_* \sqrt{k_*^2 + \frac{1}{27}} - 1 + \frac{14}{3} k_* \right)
\]

\[
- \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \left( 16k_*^2 + 12k_* \sqrt{k_*^2 + \frac{1}{27}} - 1 \right),\quad (2.27)
\]
Estimates for eigenvalues of the poly-Laplacian

\((\rho(k_*) - \varrho(k_*)^5) \leq \rho(k_*)^5 - \varrho(k_*)^5 - 5\rho(k_*)\varrho(k_*)(\rho(k_*)^3 - \varrho(k_*)^3) + 10\rho(k_*)^2\varrho(k_*)^2(\rho(k_*) - \varrho(k_*)) \leq \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{5/3} \leq \frac{7}{10} \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{\frac{5}{3}} + \frac{10}{9} \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} - \frac{10}{3} k_* - \frac{10}{9} \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \right\). \quad (2.28)

And

\((\rho(k_*) - \varrho(k_*)^3) + (\rho(k_*) - \varrho(k_*)^3) \leq 7[\rho(k_*)^3 - \varrho(k_*)^3 - 3\rho(k_*)\varrho(k_*)(\rho(k_*) - \varrho(k_*)] + (\rho(k_*) - \varrho(k_*)) \leq 14k_* - 6 \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} + 6 \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \right\}. \quad (2.29)

Substituting (2.27)–(2.29) into (2.25), we obtain that

\((t(k_* + 1)^5 - t(k_*)^8) \leq \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \left( k_*^2 - \frac{3}{4} k_*^1 \sqrt{k_*^2 + \frac{1}{27}} \right) \geq \frac{2^{1/3}}{4} k_*^{7/3} - \frac{2^{1/3}}{72} k_*^{1/3}. \quad (2.31)

Here we use the inequality \( \sqrt{k_*^2 + \frac{1}{27}} \leq k_* + 1/54k_* \), since \( k_* \) is large. The second term is

\[ -\frac{7}{16} \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{5/3} \geq -\frac{7}{16} \cdot \frac{2^{1/3}}{16} \cdot 18 k_*^{-5/3}. \quad (2.32)\]

The third term is

\[ \frac{7}{16} \left( k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{5/3} \geq -\frac{7}{12} \cdot \frac{2^{2/3}}{432} k_*^{-1/3}. \quad (2.33)\]

The fourth term is

\[ \frac{7}{144} \left( -k_* + \sqrt{k_*^2 + \frac{1}{27}} \right)^{1/3} \geq \frac{7}{144} \cdot k_*^{1/3} - \frac{7}{16} \cdot \frac{2^{2/3}}{16} k_*^{-1/3}. \quad (2.34)\]
Therefore, using (2.31)–(2.34) in (2.30), we have that
\[
(t(k_\ast) + 1)^8 - t(k_\ast)^8 \geq \frac{21^3}{4} k_\ast^{7/3} + \frac{7 \cdot 2^{2/3}}{12} k_\ast^{5/3} - \frac{7}{24} k_\ast
+ \frac{5 \cdot 2^{1/3}}{144} k_\ast^{1/3} - \frac{22/3}{96} k_\ast^{-1/3} - \frac{7 \cdot 2^{1/3}}{16 \cdot 54 \cdot 18} k_\ast^{-5/3}
\geq \frac{21^3}{4} k_\ast^{7/3} + \frac{7 \cdot 2^{2/3}}{12} k_\ast^{5/3} - \frac{7}{24} k_\ast.
\] (2.35)

Here, we used the fact that \( k_\ast \geq 1 \). In fact, noting that
\[
k_\ast \geq \frac{(n + 1)(4\pi)^n}{\omega_n^2} \left( \frac{n}{n + 2} \right)^{n/2},
\]
it is not difficult to observe that
\[
k_\ast \geq 4kL^3(\omega_3)^{-1}M^{-4} \geq \tau := \frac{432\sqrt{15}\pi}{25} \approx 210.25
\]
when \( n = 3 \). Hence, when \( \alpha \geq \frac{7}{24} \tau^{-2/3} \), the inequality \( \alpha k_\ast^{5/3} \geq \frac{7}{24} \tau \) holds for \( k_\ast \in [\tau, +\infty) \). Since
\[
1 - \frac{6}{7} \cdot 2^{1/3} \alpha \leq 1 - \frac{1}{4} \cdot 2^{1/3} \tau^{-2/3} \approx 0.991,
\]
we can conclude that
\[
(t(k_\ast) + 1)^8 - t(k_\ast)^8 \geq \frac{21^3}{4} k_\ast^{7/3} + \frac{7 \cdot 2^{2/3}}{12} \alpha_3 k_\ast^{5/3},
\] (2.36)
where \( \alpha_3 = 0.991 \). Therefore, using (2.36), we derive that
\[
3\omega_3 \int_0^{+\infty} r^6 \Psi_\ast(r) \, dr = \frac{3\omega_3 M^8}{56L^7} [(t(k_\ast) + 1)^8 - t(k_\ast)^8]
\geq \frac{3 \cdot 2^{1/3} \omega_3 M^8}{224L^7} k_\ast^{7/3} + \frac{2^{2/3} \omega_3 M^8}{32L^7} \alpha_3 k_\ast^{5/3}
= \frac{3}{7} (\omega_3 V(\Omega))^{4/3} k_\ast^{7/3} + \frac{10}{8} \alpha_3 (\omega_3 V(\Omega))^{2/3} \frac{V(\Omega)}{T(\Omega)} k_\ast^{5/3}. \tag{2.37}
\]
At the same time, it follows from (1.7) that
\[
3\omega_3 \int_0^{+\infty} r^4 \Psi_\ast(r) \, dr \geq \frac{3}{5} \frac{(2\pi)^2}{\omega_3 V(\Omega)^2/3} k_\ast^{5/3} + \frac{1}{16} \beta_3 \frac{V(\Omega)}{T(\Omega)} k,
\] (2.38)
where \( \beta_3 = 0.986 \). Substituting (2.37) and (2.38) into (2.24), we obtain that
\[
\sum_{j=1}^{k} \Gamma_j \geq \frac{3}{7} \frac{(2\pi)^4}{(\omega_3 V(\Omega))^{4/3}} k^{7/3}
+ \left( \frac{1}{8} \alpha_3 \frac{V(\Omega)}{T(\Omega)} + \frac{3a}{5} \right) \frac{(2\pi)^2}{\omega_3 V(\Omega)^{2/3}} k^{5/3} + \frac{a}{10} \beta_3 \frac{V(\Omega)}{T(\Omega)} k. \tag{2.39}
\]
When \( n = 4 \), it follows from (2.21) that
\[
\sum_{j=1}^{k} \Gamma_j \geq 4\omega_4 \int_{0}^{+\infty} r^7 \Psi_s(r) \, dr + 4a\omega_4 \int_{0}^{+\infty} r^5 \Psi_s(r) \, dr. \tag{2.40}
\]

We now make an estimate for the lower bound of \( \int_{0}^{+\infty} r^7 \Psi_s(r) \, dr \). Since
\[
\int_{0}^{+\infty} r^7 \Psi_s(r) \, dr = \frac{M^9}{27L^8} \left[ (t(k_*) + 1)^9 - t(k_*)^9 \right],
\]
we need to estimate \( (t(k_*) + 1)^9 - t(k_*)^9 \). Equation (2.14) becomes \( (t + 1)^5 - t^5 = k_* \) when \( n = 4 \). Its positive root \( t(k_*) \) is \( t(k_*) = \theta(k_*) - \frac{1}{2} \), where
\[
\theta(k_*) = \sqrt{\frac{20k_* + 5}{10}} - \frac{1}{4}.
\]

We then have that
\[
(t(k_*) + 1)^9 - t(k_*)^9 = 9\theta(k_*)^8 + 21\theta(k_*)^6 + \frac{63}{8}\theta(k_*)^4 + \frac{9}{16}\theta(k_*)^2 + \frac{1}{256}
\]
\[
= \frac{9}{25}k_*^2 + \frac{6}{25}k_*\sqrt{20k_* + 5} - \frac{18}{25}k_* + \frac{1}{50}\sqrt{20k_* + 5} - \frac{7}{50}
\]
\[
\geq \frac{9}{25}k_*^2 + \frac{12}{25}\sqrt{5k_*^{3/2}} - \frac{18}{25}k_*.
\tag{2.41}
\]

Here, we used the fact that \( k_* \geq 1 \). In fact, noting that
\[
k_* \geq \frac{(n + 1)(4\pi)^n}{\omega_n^2} \left( \frac{n}{n + 2} \right)^{n/2},
\]
it is not difficult to observe that
\[
k_* = 5kL^4(\omega_4)^{-1}M^{-5} \geq \sigma := \frac{5 \cdot 2^{12}}{9} \approx 2275.56
\]
when \( n = 4 \). Hence, when \( \alpha \geq \frac{18}{25}\sigma^{-1/2}, \) the inequality \( \alpha k_*^{3/2} \geq \frac{18}{25}k_* \) holds for \( k_* \in [\sigma, +\infty) \). Since
\[
1 - \frac{5\sqrt{5}}{12} \alpha \leq 1 - \frac{3\sqrt{5}}{10} \sigma^{-1/2} \approx 0.9859,
\]
we can conclude that
\[
(t(k_*) + 1)^9 - t(k_*)^9 \geq \frac{9}{25}k_*^2 + \frac{12\sqrt{5}}{25}k_*^{3/2}, \tag{2.42}
\]
where \( \alpha_4 = 0.985 \). Therefore, making use of (2.42), we deduce that
\[
4\omega_4 \int_{0}^{+\infty} r^7 \Psi_s(r) \, dr = \frac{\omega_4 M^9}{18L^8} \left[ (t(k_*) + 1)^9 - t(k_*)^9 \right]
\]
\[
\geq \frac{\omega_4 M^9}{50L^8}k_*^2 + \frac{2\sqrt{5}\omega_4 M^9}{75L^8}\alpha_4 k_*^{3/2}
\]
\[
= \frac{1}{2} \frac{(2\pi)^4}{\omega_4 V(\Omega)} k_*^2 + \frac{1}{6} \alpha_4 (\omega_4 V(\Omega))^{1/2} V(\Omega) k_*^{3/2}. \tag{2.43}
\]
Meanwhile, from (1.7), we have that
\[ 4\omega_4 \int_0^{\infty} r^5 \psi_s(r) \, dr \geq \frac{2}{3} \left( \frac{(2\pi)^2}{\omega_4 V(\Omega)} \right)^{1/2} k^{3/2} + \frac{1}{12} \beta_4 \frac{V(\Omega)}{I(\Omega)} k, \tag{2.44} \]
where \( \beta_4 = 0.983 \). Substituting (2.43) and (2.44) into (2.40), we obtain that
\[
\sum_{j=1}^{k} \Gamma_j \geq \frac{1}{2} \omega_4 V(\Omega) k^2 + \left( \frac{1}{6} \alpha_4 \frac{V(\Omega)}{I(\Omega)} + \frac{2a}{3} \right) \left( \frac{(2\pi)^2}{\omega_4 V(\Omega)} \right)^{1/2} k^{3/2}
+ \frac{a}{12} \beta_4 \frac{V(\Omega)}{I(\Omega)} k. \tag{2.45} \]
Therefore, synthesizing (2.23), (2.39) and (2.45), we conclude that (1.18) is true. This completes the proof of theorem 1.3. \( \square \)

**Proof of theorem 1.4.** When \( n = 3 \), using (2.35), we derive that
\[
3\omega_3 \int_0^{\infty} r^6 \psi_s(r) \, dr \geq \frac{3}{224L^7} \frac{(2\pi)^4}{\omega_3 M^8} k^{7/3} + \frac{2^2 \beta_3 M^8}{32L^7} k^{5/3} - \frac{\omega_3 M^8}{64L^7} k^3
= \frac{3}{7} \left( \frac{(2\pi)^4}{\omega_3 V(\Omega)} \right)^{1/3} k^{7/3} + \left( \frac{1}{8} \alpha_3 \frac{V(\Omega)}{I(\Omega)} + \frac{3a}{5} \right) \left( \frac{(2\pi)^2}{\omega_3 V(\Omega)} \right)^{2/3} k^{5/3}
+ \left( - \frac{1}{256} \frac{V(\Omega)}{I(\Omega)} + \frac{a \beta_3}{16} \right) \frac{V(\Omega)}{I(\Omega)} k. \tag{2.46} \]
Substituting (2.38) and (2.46) into (2.24), we have that
\[
\sum_{j=1}^{k} \Gamma_j \geq \frac{3}{7} \left( \frac{(2\pi)^4}{\omega_3 V(\Omega)} \right)^{1/3} k^{7/3} + \left( \frac{1}{8} \alpha_3 \frac{V(\Omega)}{I(\Omega)} + \frac{3a}{5} \right) \left( \frac{(2\pi)^2}{\omega_3 V(\Omega)} \right)^{2/3} k^{5/3}
+ \left( - \frac{1}{256} \frac{V(\Omega)}{I(\Omega)} + \frac{a \beta_3}{16} \right) \frac{V(\Omega)}{I(\Omega)} k. \tag{2.47} \]
When \( n = 4 \), it follows from (2.41) that
\[
4\omega_4 \int_0^{\infty} r^7 \psi_s(r) \, dr \geq \frac{\omega_4 M^9}{50L^8} k^2 + \frac{2\sqrt{5}\omega_4 M^9}{75L^8} k^{3/2} - \frac{\omega_4 M^9}{25L^8} k^3
= \frac{1}{2} \frac{(2\pi)^4}{\omega_4 V(\Omega)} k^2 + \frac{1}{6} \left( \frac{(2\pi)^2}{\omega_4 V(\Omega)} \right)^{1/2} k^{3/2} - \frac{1}{80} \left( \frac{V(\Omega)}{I(\Omega)} \right)^2 k. \tag{2.48} \]
Substituting (2.44) and (2.48) into (2.40), we obtain that
\[
\sum_{j=1}^{k} \Gamma_j \geq \frac{1}{2} \omega_4 V(\Omega) k^2 + \left( \frac{1}{6} \alpha_4 \frac{V(\Omega)}{I(\Omega)} + \frac{2a}{3} \right) \left( \frac{(2\pi)^2}{\omega_4 V(\Omega)} \right)^{1/2} k^{3/2}
+ \left( - \frac{1}{80} \frac{V(\Omega)}{I(\Omega)} + \frac{a \beta_4}{12} \right) \frac{V(\Omega)}{I(\Omega)} k. \tag{2.49} \]
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Therefore, combining (2.47) and (2.49), we conclude that (1.19) is true. This completes the proof of theorem 1.4.

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