Compactification of the moduli space of minimal instantons on the Fano threefold $V_4$

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Abstract
We study semistable sheaves of rank 2 with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on the Fano threefold $V_4$ of Picard number 1, degree 4 and index 2. We show that the moduli space of such sheaves is isomorphic to the moduli space of semistable rank 2, degree 0 vector bundles on a genus 2 curve. This also provides a natural smooth compactification of the moduli space of Ulrich bundles of rank 2 on $V_4$.

Keywords Moduli spaces · Instanton bundles · Fano threefolds · Semiorthogonal decomposition

Mathematics Subject Classification 14J10 · 14J30 · 14F05 · 14H60

1 Introduction

Instanton bundles first appeared in [3] as a way to describe Yang–Mills instantons on a 4-sphere $S^4$. They provide extremely useful links between mathematical physics and algebraic geometry. The notion of mathematical instanton bundle was first introduced on $\mathbb{P}^3$. By definition a mathematical instanton of charge $n$ is a stable rank 2 vector bundle with on $\mathbb{P}^3$ with Chern classes $c_1(E) = 0$, $c_2(E) = n$, satisfying the instantonic vanishing condition

$$h^1(E(-2)) = 0.$$

Since then the irreducibility [34] and smoothness [19] of their moduli space were heavily investigated. Faenzi [11] and Kuznetsov [21] generalized this notion to Fano threefolds, we recall

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Definition 1.1 (21) Let $Y$ be a Fano threefold of index 2. An instanton bundle of charge $n$ on $Y$ is a stable vector bundle $E$ of rank 2 with $c_1(E) = 0, c_2(E) = n$, enjoying the instantonic condition

$$H^1(Y, E(-1)) = 0.$$  

We mention that $c_2(E) \geq 2$ [21, Corollary 3.2]. The instanton bundles of charge 2 are called the minimal instantons.

In this paper, we will be concerned with minimal instantons and natural compactification of their moduli on the Fano threefold of Picard rank 1, index 2 and degree 4, which we denote by $V_4$. Such a threefold is an intersection of two quadrics in $\mathbb{P}^5$. The moduli space of minimal instanton bundles on $V_4$ was discussed in [21] and it was shown to be an open subscheme of the moduli space of rank 2 even degree bundles on a genus 2 curve $C$ which is naturally associated to $V_4$ (see [21, Theorem 5.10]).

On the other hand, Ulrich bundles are defined as vector bundles on a smooth projective variety $X$ of dimension $d$ such that

$$H^b(X, E(-t)) = 0$$

for all $t = 1, \ldots, d$. They first appeared in commutative algebra and entered the world of algebraic geometry via [10]. The existence and the moduli space of Ulrich bundles provide a great amount of information about the original variety. For example, in the case when $X$ is a smooth hypersurface, the existence of Ulrich bundles is equivalent to the fact that $X$ can be defined set-theoretically by a linear determinant [4]. Inspired by [21], Lahoz et al. [22,23] studied moduli spaces of Ulrich bundles on cubic threefolds and fourfolds using derived categories. In a recent paper [7], the moduli space of stable Ulrich bundles of rank $r$ was shown to be an open subscheme of the moduli space of rank $r$ degree $2r$ vector bundles on $C$. We will see that on $V_4$, the minimal instanton bundles and the Ulrich bundles of rank 2 differ only by a twist by $\mathcal{O}_{V_4}(1)$. Thus they share the same moduli space and compactifications.

Our first result states that the stable rank 2 vector bundles on $V_4$ with Chern classes $c_1 = 0, c_2 = 2$ automatically satisfy the instantonic vanishing condition, and hence are minimal instantons.

Theorem 1.2 Let $E$ be a stable rank 2 vector bundle on $V_4$, with $c_1(E) = 0$ and $c_2(E) = 2$, then $H^1(V_4, E(n)) = 0$ for all $n \in \mathbb{Z}$. In particular, $E$ is a minimal instanton bundle.

In light of this result, we consider the moduli space of semistable rank 2 sheaves with Chern classes $c_1 = 0, c_2 = 2$ and $c_3 = 0$ on $V_4$ as a natural compactification of the moduli space of minimal instanton bundles.

We also mention that similar phenomenon was observed on cubic threefolds (see [8, Theorem 2.4]), but the proof used properties of the cubic.

Our next result is the classification of semistable rank 2 sheaves with Chern classes $c_1 = 0, c_2 = 2$ and $c_3 = 0$ on $V_4$. [8] classified sheaves with same numerical conditions on cubic threefolds and proved that their moduli space is isomorphic to
the blow-up of the intermediate Jacobian in the Fano surface of lines. We follow his method and prove that a parallel classification happens on $V_4$.

**Theorem 1.3** Let $E$ be a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$. If $E$ is stable, then either $E$ is locally free or $E$ is associated to a smooth conic $Y \subset V_4$ such that we have the exact sequence

$$0 \to E \to \mathcal{H}^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1) \to 0,$$

where $\theta$ is the theta-characteristic of $Y$.

If $E$ is strictly semistable, then $E$ is the extension of two ideal sheaves of lines.

Recall that a theta-characteristic of a non-singular curve $Y$ is a line bundle $L$ such that $L \otimes^2$ is the canonical bundle. In the case when $Y$ is a smooth conic, $Y \cong \mathbb{P}^1$ and a theta-characteristic is just the negative of the ample generator in the Picard group of $Y$.

Unfortunately, the method to study the moduli space in [8] does not transfer well to $V_4$. However, we note that $\mathcal{D}^b(V_4)$ has a semi-orthogonal decomposition

$$\mathcal{D}^b(V_4) = (\mathcal{B}_{V_4}, \mathcal{O}_{V_4}, \mathcal{O}_{V_4}(1)).$$

There is a genus 2 smooth curve $C$ naturally associated to $V_4$ such that there is a natural choice of equivalence of triangulated categories $\Phi: \mathcal{D}^b(C) \simeq \mathcal{B}_{V_4}$ (see Sect. 2.2 for the precise definition). This functor is Fourier–Mukai by Orlov’s result. Our second result connects semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ with rank 2 bundles on $C$.

**Theorem 1.4** Let $E$ be a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$, then $E \in \mathcal{B}_{V_4}$ and $\Phi^*(E)[-1]$ is a rank 2 semistable vector bundle of degree 0 on $C$. Moreover, if $E$ is stable (strictly semistable), then the vector bundle $\Phi^*(E)[-1]$ is stable (strictly semistable).

Using this relation, we construct a morphism between the moduli space $M_{V_4}$ of semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$ and the moduli space $M_C$ of semistable vector bundles on $C$ of rank 2 and degree 0 and prove it is an isomorphism.

**Theorem 1.5** There exists a morphism $\psi: M_{V_4} \to M_C$ which is an isomorphism. As a result, the moduli space of semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$ is a $\mathbb{P}^3$-bundle over the Jacobian of $C$, thus a smooth projective variety of dimension 5.

We mention some related work. On the projective space $\mathbb{P}^3$, the question of how instanton bundles degenerate has been intensely studied. Maruyama and Trautmann [27] were the first to consider limits of instantons. Instanton sheaves of charge $n$ on $\mathbb{P}^3$ were first defined by Jardim [17] as torsion free sheaves $E$ with $c_1(E) = 0, c_2(E) = n, c_3(E) = 0$ satisfying

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$
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(See also [17] for a study of instanton sheaves on general \( \mathbb{P}^n \).) Gargate and Jardim [13] showed (among other properties) the singular locus of \( E \), i.e., the quotient of the double dual of \( E \) by \( E \), has pure dimension 1. Jardim, Markushevich and Tikhomirov [18] considered the boundary of moduli space of instanton bundles of charge \( n \) on \( \mathbb{P}^3 \) in the moduli space of semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = n \) and \( c_3 = 0 \). They showed that the boundary contains \( n \) irreducible divisors \( \mathcal{D}(m, n) \), where \( m = 1, 2, \ldots, n \), whose generic points represent instanton sheaves singular along normal rational curves of degree \( m \). Moreover, these divisors only suffice to compactify the moduli space when \( n = 1 \) is the minimal charge. We note that the boundary sheaves in our compactification do not satisfy instantonic vanishing conditions (translate to \( h^0(E) = h^1(E(-1)) = h^2(E(-1)) = h^3(E(-2)) = 0 \) on Fano threefolds of index 2). But they do satisfy the property that they are singular on rational curve of degree \( \leq 2 \) and suffice to compactify the moduli space of minimal instantons.

Faenzi [11] and Kuznetsov [21] generalized the notion of instanton bundles to Fano threefolds and studied instanton bundles on Fano threefolds of index 2 and Picard rank 1 using techniques in derived categories. Properties of instanton bundles and their moduli spaces on degree 4, 5 cases were investigated in detail in [11,21]. The author [32] studied compactification of the moduli space of minimal instantons on the degree 5 Fano threefold of index 2 by associating such sheaves with quiver representations via the Kuznetsov component.

Casnati, Coskun, Genc and Malaspina [6] proposed a general definition of instanton bundles with given charge on any Fano threefold, unifying all previous notions. They studied instanton bundles on the blow-up \( \tilde{\mathbb{P}}^3 \) of \( \mathbb{P}^3 \) at a point in the same paper. Henni [16] examined instanton sheaves and their moduli on \( \mathbb{P}^3 \).

This paper is organized as follows. In the second section the reader can find some preliminary definitions and results that are used throughout the paper. In the third section we classify the semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \), showing the parallel result to that holding for cubic threefolds. In the fourth section we connect such sheaves to vector bundles on \( C \) using derived category. In the last section we describe the compactification of the moduli space of instantons on \( V_4 \).

\textbf{Notation and conventions}

- We work over the complex numbers \( \mathbb{C} \).
- Let \( E \) be a sheaf on \( V_4 \). We use \( H^i(E) \) to denote \( H^i(V_4, E) \) for simplicity. Also we use \( h^i(E) \) to denote the dimension of \( H^i(V_4, E) \) as a complex vector space.
- \( M_{V_4} \) denotes the moduli space of semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \) on \( V_4 \).
- \( M_C \) denotes the moduli space of semistable vector bundles of rank 2 and degree 0 on the associated curve \( C \).
- Let \( F \) be a sheaf with certain characterization, we will use \([F]\) to denote the point it corresponds to in the moduli space.
2 Preliminaries

2.1 Derived categories

Let $X$ be an algebraic variety, we use $\mathcal{D}^b(X)$ to denote the derived categories of coherent sheaves on $X$. For objects $F, G \in \mathcal{D}^b(X)$, we denote $\text{Ext}^p(F, G) = \text{Hom}(F, G[p])$ and $\text{Ext}^\bullet(F, G) = \bigoplus_{p \in \mathbb{Z}} \text{Ext}^p(F, G)[-p]$. Recall that a sequence of full admissible triangulated subcategories of a triangulated category $\mathcal{T}$,

$$D_1, \ldots, D_n \subset \mathcal{T}$$

is semi-orthogonal if for all $j > i$

$$D_i \subset D_j^\perp,$$

where $D_j^\perp$ is the full subcategory of objects $C \in \mathcal{T}$ such that $\text{Hom}(B, C) = 0$ for all objects $B \in D_j$. Such a sequence defines a semi-orthogonal decomposition of $\mathcal{T}$ if the smallest full subcategory of $\mathcal{T}$ containing $D_1, \ldots, D_n$ is itself, in this case we use the notion $\mathcal{T} = \langle D_1, \ldots, D_n \rangle$. An easy way to produce a semi-orthogonal decomposition is by using exceptional objects or collections.

**Definition 2.1** An object $F \in \mathcal{T}$ is called exceptional if $\text{Ext}^\bullet(F, F) = \mathbb{C}$. A collection of exceptional objects $F_1, \ldots, F_n$ is called an exceptional collection if $\text{Ext}^p(F_j, F_i) = 0$ for all $j > i$ and all $p \in \mathbb{Z}$.

On a smooth Fano threefold $V$ of index 2, Kodaira vanishing theorem implies

$$H^i(V, \mathcal{O}_V) = 0$$

for all $i > 0$. Thus all line bundles on $V$ are exceptional objects. Moreover, we can check that $\{\mathcal{O}_V, \mathcal{O}_V(1)\}$ is an exceptional collection using the fact that $V$ has index 2. We denote their left orthogonal complement by $\mathcal{B}_V$ and obtain the following semi-orthogonal decomposition:

$$\mathcal{D}^b(V) = \langle \mathcal{B}_V, \mathcal{O}_V, \mathcal{O}_V(1) \rangle.$$

Note an object $F \in \mathcal{D}^b(V)$ is in $\mathcal{B}_V$ if and only if

$$\text{Hom}(\mathcal{O}_V, F[i]) = 0, \quad \text{Hom}(\mathcal{O}_V(1), F[i]) = 0$$

for all $i \in \mathbb{Z}$.

2.2 Vector bundles

In this section we recall several useful results about vector bundles on smooth projective varieties.
Proposition 2.2 ([15, Section 1]) Let $X$ be a smooth projective variety of dimension at least 2 and $E$ a vector bundle of rank 2 on $X$. Suppose there exists a global section of $E$ whose zero locus $Y$ is of pure codimension 2, then we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to I_Y \otimes \det(E) \to 0.$$ 

Theorem 2.3 (Serre Construction, [2, Section 1]) Suppose $X$ is a smooth projective variety of dimension at least 3. Let $L$ be an invertible sheaf such that $h^1(L^{-1}) = 0$ and $h^2(L^{-1}) = 0$, and $Y \subset X$ a closed subscheme of pure codimension 2. We have an isomorphism

$$\text{Ext}^1(I_Y \otimes L, \mathcal{O}_X) \cong H^0(\mathcal{O}_Y).$$

The subscheme $Y$ is the zero locus of a section of a vector bundle of rank 2 with determinant $L$ if and only if $Y$ is locally complete intersection and $\omega_Y = (\omega_X \otimes L)|_Y$.

We recall the following useful result:

Proposition 2.4 (Mumford–Castelnuovo Criterion, [12, Lemma 5.1]) Let $F$ be a coherent sheaf on a projective variety $X$. Suppose $h^i(X, F(-i)) = 0$ for all $i \geq 1$, then $h^i(X, F(k)) = 0$ for all $i \geq 1$ and $k \geq -i$. Moreover $F$ is generated by global sections.

2.3 Fano threefold $V_4$

(See also [21, Section 5.1].) A Fano threefold of Picard rank 1, index 2 and degree 4 is a smooth intersection of two quadrics in $\mathbb{P}^5$. We let $V_4$ be such a threefold. There is a smooth curve $C$ of genus 2 associated to $V_4$. We briefly recall its construction. Let $V$ be a complex vector space of dimension 6 and $A$ a vector space of dimension 2. A pair of quadrics gives a map $A \to S^2 V^*$, so $\mathbb{P}(A)$ parametrizes a family of quadrics in $\mathbb{P}(V)$. There are six degenerate quadrics in this family, giving six points on $\mathbb{P}(A) \simeq \mathbb{P}^1$. $C$ is defined to by the double cover of $\mathbb{P}(A)$ ramified at the six points. Clearly $C$ is a curve of genus 2. We use $\tau : C \to C$ to denote its hyperelliptic involution.

By looking at the spinor bundles on quadrics in $\mathbb{P}(A)$, one can show there is a vector bundle $S$ of rank 2 on $C \times V_4$. The Fourier–Mukai functor with kernel $S$ connects $C$ with $V_4$ in the following way:

Theorem 2.5 ([5, Theorem 2.7]) The Fourier–Mukai functor $\Phi_S : \mathcal{D}^b(C) \to \mathcal{D}^b(V_4)$ provides an equivalence of $\mathcal{D}^b(C)$ onto $\mathcal{B}_{V_4}$.

From now on, we use $\Phi$ to denote $\Phi_S$ for simplicity.

Another way to understand the relation between $C$ and $V_4$ and the vector bundle $S$ is due to Mukai. It is shown that $V_4$ is the moduli space of rank 2 bundles on $C$ with a fixed determinant $\xi$ of odd degree. Then $S$ is the universal family. We follow [21]’s convention and assume $\deg \xi = 1$. Then

$$\det(S) = \xi \boxtimes \mathcal{O}_{V_4}(-1).$$

We can also describe the Fano variety of lines $F(V_4)$ using this functor. First note it is straightforward to check that the ideal sheaf $I_l$ of a line is an object in $\mathcal{B}_{V_4}$. Then
Theorem 2.6 ([21, Lemma 5.5]) There is an isomorphism $G : F(V_4) \sim \text{Pic}^0(C)$ given by

$$G(l) = \Phi^{-1}(I_l[-1]).$$

In particular $\Phi^{-1}(I_l[-1])$ is a degree 0 line bundle on $C$.

This result will be crucial in our analysis of strictly semistable sheaves.

Finally we provide some topological information that will be useful later on. Let $[h], [l], [p]$ be the class of a hyperplane section, a line and a point respectively. Then

$$H^2(V_4, \mathbb{Z}) \simeq \mathbb{Z}[h], \quad H^4(V_4, \mathbb{Z}) \simeq \mathbb{Z}[l], \quad H^6(V_4, \mathbb{Z}) \simeq \mathbb{Z}[p]$$

with $h \cdot l = p$, $h^2 = 4l$, $h^3 = 4p$.

The natural embedding $V_4 \subset \mathbb{P}^5$ provides a very ample divisor $O_{V_4}(1)$. We will always use this polarization when talking about stability. A general section of $|O_{V_4}(1)|$ is a del Pezzo surface of degree 4, hence isomorphic to $\mathbb{P}^2$ blown up at five points in general position.

We compute the Todd class of $V_4$

$$\text{td}(\mathcal{O}_{V_4}) = 1 + h + \frac{7}{3} l + p.$$ 

We also recall the Grothendieck–Riemann–Roch for the functor $\Phi$.

Lemma 2.7 ([21, Lemma 5.2]) For any $F \in \mathcal{D}^b(C)$ we have

$$\text{ch}(\Phi(F)) = (2 \text{deg}(F) - \text{rk}(F)) - \text{deg}(F) h + \text{rk}(F) l + \frac{\text{deg}(F)}{3} p.$$ 

2.4 Stability of sheaves

Let $X$ be a smooth projective variety of dimension $n$ and $O_X(1)$ a fixed ample line bundle. Let $E$ be a coherent sheaf of rank $r$, then the slope of $E$ is defined as

$$\mu(E) = \frac{c_1(E) \cdot c_1(O_X(1))^{n-1}}{r \cdot c_1(O_X(1))^n}.$$ 

The sheaf $E$ is called (semi)stable if it is torsion free and for any torsion free subsheaf $F \subset E$, we have

$$\frac{\chi(F(n))}{\text{rk}(F)} \leq \frac{\chi(E(n))}{\text{rk}(E)}$$

for $n \gg 0$. 

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The sheaf $E$ is called $\mu$-(semi)stable if it is torsion free and for any torsion free subsheaf $F \subset E$, we have

$$\mu(F) (\leq) < \mu(E).$$

We have the following implications:

$$\mu\text{-stable} \implies \text{stable} \implies \text{semistable} \implies \mu\text{-semistable}.$$ 

When $X$ is a smooth projective curve, any torsion free sheaf is locally free. We have the following criterion for semistability which is due to Faltings.

**Lemma 2.8** ([31, Exercise 2.8]) Let $F, G$ be vector bundles on a curve $X$ such that $H^i(F \otimes G) = 0$ for $i = 0, 1$, then both $F$ and $G$ are semistable.

Let $C$ be the associated curve of $V_4$. Then $C$ has genus 2. We use $M_C$ to denote the moduli space of semistable vector bundles of rank 2 and degree 0 on $C$. $M_C$ was studied in detail in [29]. Suppose $J^1$ is the moduli space of line bundles of degree 1. Then $C$ is naturally embedded in $J^1$ as a divisor. We denote the corresponding divisor by $\Theta$. To understand $M_C$, it suffices to understand the 3-dimensional subvariety $S \subset M_C$ consisting of bundles with trivial determinant. [29] showed $S$ is naturally isomorphic to the projective space associated to $H^0(J^1, 2\Theta)$. From this it is not hard to conclude:

**Theorem 2.9** ([29, Theorem 7.3]) $M_C$ is canonically isomorphic to the space of positive divisors on $J^1$ algebraically equivalent to $2\Theta$. In particular, $M_C$ is a projective bundle over the Jacobian of $C$.

As a consequence, we have

**Corollary 2.10** $M_C$ is a non-singular projective algebraic variety of dimension 5.

### 2.5 Instanton bundles and Ulrich bundles

Let $Y$ be a Fano threefold of index 2. By definition a minimal instanton bundle is a stable vector bundle $E$ of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$, enjoying the instantonic condition

$$H^1(Y, E(-1)) = 0.$$ 

We will see in Theorem 3.2 that the instantonic condition is automatically satisfied on $V_4$. We use $M_{V_4}$ to denote the moduli space of semistable sheaves of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$. It is clear that the moduli space of minimal instanton bundles $M_{\text{inst}}(V_4)$ is an open subscheme of $M_{V_4}$.

We also recall two equivalent definitions of an Ulrich bundle.

**Definition 2.11** Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n$ and degree $d$. An Ulrich bundle $E$ is a vector bundle on $X$ satisfying

$$H^s(X, E(-t)) = 0$$

for all $s > 0$ and $t > 0$. If $s = 0$, we require $E$ to be a line bundle on $X$. 

This definition will be useful for our purposes.
for all $t = 1, \ldots, n$. Equivalently, it is a vector bundle of rank $r$ satisfying

$$H^i(X, E(t)) = 0$$

for all $t \in \mathbb{Z}$ and $0 < i < n$ and having Hilbert polynomial $P_E(t) = dr \binom{n+t}{r}$. 

We list some well-known facts about Ulrich bundles that will be useful to us (see [4, Sections 3.4]):

- There are no Ulrich line bundles on a variety $X$ with $\text{Pic}(X) = \mathcal{O}_X(1)$ and degree $d > 1$.
- An Ulrich bundle is semistable. If it is not stable, it is an extension of Ulrich bundles of smaller ranks.

To see the relation between minimal instanton bundles and Ulrich bundles of rank 2 on $V_4$, we first recall the following result.

**Proposition 2.12** ([7, Proposition 3.4]) Let $E$ be an Ulrich bundle of rank $r$ on $V_4$, then $\mu(E) = 1$.

As a result, if $E$ is a rank 2 Ulrich bundle, $E(-1)$ is a semistable rank 2 bundle with Chern class $c_1 = 0$. Moreover, we have the Hilbert polynomial

$$P_{E(-1)}(t) = 8 \binom{t + 2}{3}.$$ 

Using Riemann–Roch, it is not hard to see that $c_2(E(-1)) = 2$. Combine this with the following result.

**Lemma 2.13** ([21]) Let $E$ be a minimal instanton bundle on a Fano threefold of index 2. Then

$$H^k(E(t)) = 0$$

for $t = 0, -1, -2$.

We obtain

**Proposition 2.14** A vector bundle $E$ on $V_4$ is a minimal instanton bundle if and only if $E(1)$ is a stable Ulrich bundle of rank 2.

If we use $M^S_{2U}$ to denote the moduli space of stable Ulrich bundles of rank 2. It follows immediately from this result that

$$M^S_{2U} \cong M^{\text{inst}}(V_4).$$

To describe $M^{\text{inst}}(V_4)$, we first recall that by definition a second Raynaud bundle on $C$ is the (shift of the) Fourier–Mukai transform of the bundle $\mathcal{O}_{\text{Pic}(C)}(-2\Theta)$ with the kernel given by the Poincaré bundle, where $\Theta$ is the theta divisor of $\text{Pic}(C)$ (see [21,31]). It is a semistable rank 4 vector bundle of degree 4 on $C$. Kuznetsov gave the following description of the moduli space $M^{\text{inst}}_n(V_4)$ of instanton bundles of charge $n$:
Theorem 2.15 ([21, Theorem 5.10]) Let $R$ be a second Raynaud bundle. The moduli space $M_{n}^{\text{inst}}(V_4)$ of instantons of charge $n$ is isomorphic to the moduli space of simple vector bundles $F$ on $C$ of rank $n$ and degree $0$ such that

$$F^* \simeq \tau^* F,$$
$$H^0(C, F \otimes S_y) = 0 \text{ for all } y \in V_4,$$
$$\dim \text{Hom}(F, R) = \dim \text{Ext}^1(F, R) = n - 2.$$

Remark 2.16 Apply this theorem to $n = 2$ and note that in this case the last equation shows that $F$ is semistable by Lemma 2.8, so we see that

$$M_2^{\text{SU}} \simeq M^{\text{inst}}(V_4) \subset M_C.$$

See also [7, Theorem 4.14] for a similar result from the perspective of Ulrich bundles.

2.6 Curves and surfaces of low degrees

In this section we recall some results about varieties of (almost) minimal degrees in projective spaces. Recall a variety $V \subset \mathbb{P}^n$ is said to be non-degenerate if $V$ is not contained in any hyperplane in $\mathbb{P}^n$.

Regarding the degree of a curve we have first the classical Castelnuovo bound:

Theorem 2.17 (Castelnuovo Bound) Let $C \subset \mathbb{P}^n$ be an irreducible non-degenerate smooth curve of degree $d$ and genus $g$. Let $m, \epsilon$ be the quotient and remainder when dividing $d - 1$ by $n - 1$, and $\pi(d, n) = (n - 1)m(m - 1)/2 + m\epsilon$, then

$$g \leq \pi(d, n).$$

In fact this bound is sharp, and extremal curves (curves with $g = \pi(d, n)$) can be explicitly described [1, Chapter 3]. This result was improved by Eisenbud and Harris.

Theorem 2.18 ([9, Theorem 3.15]) With the same notation as above, for any $d$ and $n \geq 4$, set $m_1, \epsilon_1$ to be the quotient and remainder when dividing $d - 1$ by $n$. Let $\mu_1 = 1$ if $\epsilon_1 = n - 1$ and $0$ otherwise. Let

$$\pi_1(d, n) = \left(\frac{m_1}{2}\right)n + m_1(\epsilon_1 + 1) + \mu_1,$$

then

- if $g > \pi_1(d, n)$ and $d \geq 2n + 1$, then $C$ lies on a surface of degree $n - 1$; and
- if $g = \pi_1(d, n)$ and $d \geq 2n + 3$, then $C$ lies on a surface of degree $n$ or $n - 1$.

We now look at surfaces of small degrees. It is well known that for an irreducible, non-degenerate variety $M \subset \mathbb{P}^n$ of dimension $m$, the minimal degree is $n - m + 1$. (See [Section 1.3] [14].) Minimal surfaces are well understood.
Theorem 2.19 (\cite{14}, Section 4.3) Every non-degenerate irreducible surface of degree \( n - 1 \) in \( \mathbb{P}^n \) is either a rational normal scroll or the Veronese surface in \( \mathbb{P}^5 \).

The case of degree \( n \) surface in \( \mathbb{P}^n \) is a little more involved.

Theorem 2.20 (\cite{28}, Theorem 8) Every non-degenerate irreducible surface \( S \) of degree \( m \neq 8 \) in \( \mathbb{P}^m \) is one of the following:

- Projection of a minimal surface in \( \mathbb{P}^{m+1} \).
- A del Pezzo surface with isolated double points.
- A cone with a smooth elliptic base curve.

Remark 2.21 When \( m = 8 \) we have two more kinds of surfaces, we will not need this case. The reader can find more details in \cite{28}, Theorem 8.

3 Classification of semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \) on \( V_4 \)

We first show that for stable rank 2 vector bundles with Chern classes \( c_1 = 0, c_2 = 2 \) on \( V_4 \), the instantonic vanishing condition is satisfied.

Lemma 3.1 (\cite{8}, Lemma 2.2) Let \( S \subset \mathbb{P}^4 \) be a del Pezzo surface of degree 4 and \( E \) a \( \mu \)-semistable vector bundle of rank 2 with Chern classes \( c_1(E) = 0 \) and \( c_2(E) = 2 \). If \( h^0(E) = 0 \), then \( h^1(E(n)) = 0 \) for \( n \in \mathbb{Z} \) and \( h^2(E(n)) = 0 \) for \( n \geq -1 \). If \( h^0(E) \neq 0 \), then \( h^0(E) = 1 \), \( h^1(E(n)) = 0 \) for \( n \leq -2 \) and \( n \geq 1 \), \( h^1(E(-1)) = h^1(E) = 1 \) and \( h^2(E(n)) = 0 \) for \( n \geq 0 \).

Theorem 3.2 Let \( E \) be a stable rank 2 vector bundle on \( V_4 \), with \( c_1(E) = 0 \) and \( c_2(E) = 2 \). Then \( H^1(V_4, E(n)) = 0 \) for all \( n \in \mathbb{Z} \). In particular, \( E \) is a minimal instanton bundle.

Proof Let \( S \in |\mathcal{O}_{V_4}(1)| \) be a general hyperplane section of \( V_4 \). Then \( E_S \) is \( \mu \)-semistable with respect to the polarization \( \mathcal{O}_S(1) \) \cite{26}, Theorem 3.1.

Suppose \( h^0(E_S) = 0 \). Consider the short exact sequence

\[
0 \to E(n-1) \to E(n) \to E_S(n) \to 0.
\]

Since \( h^1(E_S(n)) = 0 \) for \( n \in \mathbb{Z} \), we have \( h^1(E(n)) \leq h^1(E(n-1)) \). Thus \( h^1(E(n)) = 0 \) for all \( n \in \mathbb{Z} \) since \( h^1(E(n)) = 0 \) for \( n \ll 0 \).

Suppose \( h^0(E_S) \neq 0 \), we will try to get a contradiction. We claim \( E(2) \) is generated by global sections. Using the above exact sequence, we obtain \( h^1(E(-n)) = 0 \) for \( n \geq 2 \). Note \( h^2(E) = h^1(E(-2)) = 0 \) and \( h^3(E) = h^0(E(-2)) = 0 \). Since \( \chi(E) = 0 \), we have \( h^1(E) = 0 \) and the exact sequence

\[
0 \to E \to E(1) \to E_S(1) \to 0
\]

gives \( h^1(E(1)) = h^1(E_S(1)) = 0 \). We have then \( h^3(E(-1)) = h^0(E(-1)) = 0 \). Thus \( E(2) \) is generated by global sections by the Mumford–Castelnuovo criterion (Proposition 2.4).

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If there exists a nowhere vanishing section of $E(2)$, then $E$ is isomorphic to $\mathcal{O}_{V_4}(2) \oplus \mathcal{O}_{V_4}(-2)$, which is absurd. We have then an exact sequence

$$0 \to \mathcal{O}_{V_4}(-4) \to E(-2) \to I_Y \to 0$$  \hspace{1cm} (2)

where $Y \subset V_4$ is a smooth curve of degree $c_2(E(2)) = 18$. We have $h^1(I_Y) = 0$, so the curve $Y$ is connected. We have $\omega_Y = \mathcal{O}_Y(2)$ and $g(Y) = 19$. Finally, the curve $Y$ is non-degenerate since $E$ is stable. Using (1), (2), it is not hard to find that $h^0(\mathcal{O}_Y(1)) = 7$. Thus the curve $Y$ is the projection to $\mathbb{P}^5$ of a non-degenerate curve in $\mathbb{P}^6$ with degree 18 and genus 19. The next lemma shows that this leads to a contradiction. \hfill $\square$

Lemma 3.3 Let $\tilde{Y} \subset \mathbb{P}^6$ be a non-degenerate curve of degree 18 and genus 19. Let $O \notin \tilde{Y}$ be a point such that the projection from $O$ induces an embedding of $\tilde{Y}$ into $\mathbb{P}^5$. Then the image $Y$ of $\tilde{Y}$ cannot lie in the intersection $V_4$ of two quadrics in $\mathbb{P}^5$.

Proof We first apply Theorem 2.18 to $\tilde{Y}$. In this case, we have $d = 18$ and $n = 6$, then $m_1 = 2$ and $\epsilon_1 = 5 = 6 - 1$. So $\mu_1 = 1$. We compute $\pi_1(18, 6) = 6 + 2 \times 6 + 1 = 19 = g$. Thus by the second part of Theorem 2.18, $\tilde{Y}$ lies on a surface $S$ of 5 or 6 in $\mathbb{P}^6$.

Degree 5 surfaces in $\mathbb{P}^6$ are the images of $\mathbb{F}_{1+2k}$, $k = 0, 1, 2$, of the morphisms $\tau_k$ induced by complete linear systems $|C_0 + (3 + k)f|$, where $C_0$ is the unique section with $C_0^2 = -1 - 2k$ and $f$ is a general fibre. We have $\tilde{Y} \in |3C_0 + (12 + 3k)f|$. Note when $k = 0, 1$, $\tau_k$ is an closed embedding while $\tau_2$ contracts the section $C_0$ and the image is a cone.

Degree 6 surfaces in $\mathbb{P}^6$ are:

- A cone over smooth elliptic curve of degree 6 in $\mathbb{P}^5$.
- A del Pezzo surface of degree 6 with possibly isolated double points.
- A projection of $\mathbb{F}_{2k} \subset \mathbb{P}^7$, $k = 0, 1, 2, 3$, embedded via the complete linear system $|C_0 + (3 + k)f|$, from a point outside of the surface.

Let $\pi$ be the projection from $O \in \mathbb{P}^6$. Let $\pi(S)$ be the image of $S$ under the rational map $\pi$. If $\pi(S)$ is one-dimensional, then $S$ is a cone with base $\tilde{Y}$, which is absurd, since $S$ can only be a cone over a rational or elliptic curve by the classification above. Thus $\pi(S)$ is two-dimensional. If $S$ is a cone, then its vertex is different from the point $O$.

Now suppose $Y \subset \mathbb{P}^5$ is contained in $V_4$ and use $\overline{V}_4 \subset \mathbb{P}^6$ to denote the cone with vertex $O$ and base $V_4$.

Suppose $\overline{V}_4$ does not contain the surface $S$. Recall $V_4$ is the complete intersection of two smooth quadrics $Q_0, Q_1$. Use $\overline{Q}_t$ to denote the cone with vertex $O$ and base $Q_t$. Then $\overline{V}_4$ is the intersection of $\overline{Q}_t$’s. Then $S$ is not contained in at least one of the $\overline{Q}_t$’s, say $\overline{Q}_0$. Then $\overline{Q}_0$ cuts the surface $S$ at a curve of degree 10 or 12, which cannot contain $\tilde{Y}$, contradiction. So $S \subset \overline{V}_4$. We thus also have $\pi(S) \subset V_4$.

Suppose $O \notin S$. Denote the degree of $\pi$ by $d$, then $\pi(S)$ is a surface of degree $5/d$ or $6/d$. On the other hand, $\pi(S)$ corresponds to a Cartier divisor $\mathcal{O}_{V_4}(l)$ where $l$ is an integer, thus has degree $4l$. We have now a contradiction since $4dl = 5$ or $4dl = 6$ have no integral solutions.
Suppose \( O \in S \). If \( S \) has degree 5, then \( S \) is one of the surfaces \( \tau_k(\mathbb{P}^1_{1+2k}) \) for \( k = 0, 1, 2 \). The fibre \( f \) passing through \( O \) is contracted by \( \pi \). But we have \( \widetilde{Y}.f = 3 \) and \( \pi \) cannot induce a closed embedding on \( \widetilde{Y} \). If \( S \) is a cone over a smooth elliptic curve \( E \) of degree 6 in \( \mathbb{P}^5 \), then \( S \) is the image of a ruled surface over \( E \) by the morphism associated to a linear system numerically equivalent to \( |C_0 + 6f| \), where \( C_0 \) is the section we obtained by blowing up the vertex \( (C_0^2 = −6) \). Then \( \widetilde{Y} \equiv mC_0 + 18f \), where \( m = 3 \) or 4. Then again if \( f \) is the general fibre passing through \( O, \widetilde{Y}.f = m > 1 \) and this contradicts the fact that \( \pi \) induced a closed embedding on \( \widetilde{Y} \). If \( S \) is a del Pezzo surface of degree 6 with possible isolated double points, then \( S \) is the image of \( \mathbb{P}^2 \) blowing up three points, i.e., \( S \) is the image of the linear system \( |3H − E_1 − E_2 − E_3| \) where \( H \) is the pull-back of a hyperplane section in \( \mathbb{P}^2 \) and \( E_i \) are the exceptional divisors. Note that the singularities occur when the points are not in general position. Then (compactification) of \( \pi(S) \subset \mathbb{P}^5 \) is the image of the blow-up of \( S \) at \( O \) by the morphism associated to \( |3H − E_1 − E_2 − E_3 − G| \), where \( G \) is the exceptional divisor of the blow-up at \( O \). Then \( \pi(S) \subset \mathbb{P}^5 \) is one of the following:

- The Veronese surface \( \mathbb{P}^2 \subset \mathbb{P}^5 \).
- \( \mathbb{F}_0 \) or \( \mathbb{F}_2 \) embedded into \( \mathbb{P}^5 \) as a minimal surface.
- A del Pezzo surface of degree 5 with possibly isolated double points.

On the other hand, \( \pi(S) \subset V_4 \), and hence corresponds to a Cartier divisor of the form \( \mathcal{O}_{V_4}(l) \). Thus \( \pi(S) \) has degree \( 4l \). This will immediately lead to a contradiction in the third case. In the first and second case, we will have \( l = 1 \). But remember that a smooth hyperplane section of \( V_4 \) can only be a del Pezzo surfaces of degree 4, which is impossible in the first two cases.

The case when \( S \) is of the third type is more complicated. Note when \( k \neq 3 \), the secant variety of \( \mathbb{F}_{2k} \subset \mathbb{P}^7 \) is a proper subvariety of \( \mathbb{P}^7 \) and projection from a point off the secant variety induces an isomorphism. Since a rational normal scroll does not have a tri-secant line (not contained in the surface), any secant line can only meet the scroll at two points transversely. Similarly, a tangent line (not contained in the surface) can be tangent at one point and does not meet the scroll again. Moreover, two distinct secant lines, a tangent line and a secant line or two tangent lines cannot meet at any points other than a point of the scroll unless the scroll intersects the plane spanned by these two lines in a conic. In conclusion, \( S \) can either be smooth, or have only one double point or only one double line (which comes from a conic on \( \mathbb{F}_{2k} \)). Now suppose \( O \in S \) is a smooth point. Then there exists a unique corresponding point, which we also call \( O \) on \( \mathbb{F}_{2k} \). Then \( \pi(S) \) is the image of a codimension one base-point-free linear system in \( |C_0 + (3 + k)f − E| \), where \( E \) is the exceptional divisor obtained by blowing up \( O \). Thus \( \pi(S) \subset V_4 \) is a Cartier divisor of degree 5, which leads to a contradiction as before.

When \( O \) is the double point, then there are either two distinct points or a point and a tangent direction at it on \( \mathbb{F}_{2k} \) corresponding to \( O \). For simplicity of argument, we will only discuss the case of two distinct points, the other case can be handled similarly. Denote the two points by \( p_1, p_2 \), then \( \pi(S) \) is the image of \( \mathbb{F}_{2k} \) via the complete linear system \( |C_0 + (3 + k)f − E_1 − E_2| \) where \( E_1, E_2 \) are the exceptional divisors obtained by blowing up \( p_1, p_2 \). Note by our choice, \( p_1, p_2 \) cannot lie on a line contained in \( S \) nor a conic in \( S \). When \( k = 0 \), there are no conics on \( S \). When \( k = 1 \), \( C_0 \) is the
Table 1 Description of \( \pi(S) \) based on the position of \( p_1, p_2 \)

| \( k \) | Position of \( p_1, p_2 \) | \( \pi(S) \) |
|---|---|---|
| 0  | \( p_1, p_2 \) not on a \( C_0 \) | (A) |
| 0  | \( p_1, p_2 \) on a \( C_0 \) | (B) |
| 1  | \( p_1 \in C_0 \) and \( p_2 \notin C_0 \) | (B) |
| 1  | \( p_1, p_2 \notin C_0 \) | (A) |
| 2  | \( p_1, p_2 \notin C_0 \) | (B) |

only conic in \( S \). When \( k = 2 \), \( C_0 + f \) will be a conic for any general fibre \( f \). When \( k = 3 \), any two points not on the same line will be on a conic (which is the union of the lines containing each point). It is not hard to check that \( \pi(S) \) is one of the following surfaces:

(A) Image of \( \mathbb{F}_0 \) associated to complete linear system \( |C_0 + 2f| \).
(B) Image of \( \mathbb{F}_2 \) associated to complete linear system \( |C_0 + 3f| \).
(C) Image of \( \mathbb{F}_4 \) associated to complete linear system \( |C_0 + 4f| \).

We present in Table 1 what \( \pi(S) \) is depending on the position of \( p_1, p_2 \). Note all \( \pi(S) \) are smooth surfaces in \( \mathbb{P}^5 \) of degree 4. On the other hand, \( \pi(S) \subset V_4 \) corresponds to a Cartier divisor \( \mathcal{O}_{V_4}(l) \). By looking at the degree, we see that \( l = 1 \). But then \( \pi(S) \) has to be a del Pezzo surface of degree 4, which leads to a contradiction.

When \( O \) is a point on the double line, then \( k = 1, 2 \) or 3. In each case, the strict transform of the conic under the blow-up of \( E_1, E_2 \) will be contracted by the system \( |C_0 + (3 + k)f - E_1 - E_2| \), and it is not hard to check that \( \pi(S) \) is of type (C) by computing the degree and the self-intersection of the strict transform for each \( k \). Since \( \pi(S) = (C) \) has degree 4, it corresponds to the Cartier divisor \( \mathcal{O}_{V_4}(1) \), which means \( \pi(S) \subset H \), where \( H \) is a hypersurface in \( \mathbb{P}^5 \). This is absurd since (C) is non-degenerate.

The paper [8] classified semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \) on cubic threefolds. Now we classify semistable rank 2 sheaves with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \) on \( V_4 \), closely following the argument of [8]. When the proof transfers almost verbatim, we will only point out the changes in our situation and refer the readers to [8].

**Proposition 3.4** Let \( E \) be a semistable rank 2 sheaf with Chern classes \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0 \) on \( V_4 \). Let \( F \) be the double dual of \( E \). Then either \( E \) is locally free or \( F \) is locally free with the second Chern class \( c_2(F) = 1 \) and \( h^0(F) = 1 \) or \( F = H^0(F) \otimes \mathcal{O}_{V_4} \).

**Proof** See [8, Proposition 3.1]. The proof of [8, Proposition 3.1] used mainly general arguments about semistable sheaves on projective varieties and can be directly applied here. We highlight a few similarities between \( V_4 \) and a cubic threefold \( V_3 \) which allow us to mimic the argument:

- Both \( V_4 \) and \( V_3 \) are Fano threefolds of index 2.
- General hyperplane sections of both \( V_4 \) and \( V_3 \) are del Pezzo surfaces in their anticanonical embedding.
The following inequalities for Hilbert polynomials on $V_3$:

$$\chi(E(n)) < 2\chi(\mathcal{O}_{V_3}(n)), \quad \chi(E(n)) < 2\chi(I_p(n))$$

remain true on $V_4$, in fact

$$\chi(E(n)) = \frac{4}{3}n^3 + 4n^2 + \frac{8}{3}n,$$
$$\chi(\mathcal{O}_{V_4}(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1,$$
$$\chi(I_p(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1$$

where $p$ is a point.

**Lemma 3.5** Suppose $\theta$ is the theta-characteristic of a smooth conic $C \subset V_4$. We consider the sheaf $E$ which is the kernel of the surjection $H^0(\mathcal{O}(1)) \otimes \mathcal{O}_{V_4} \rightarrow \mathcal{O}(1)$. Then $E$ is stable with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$.

**Proof** See [8, Lemma 3.4]. Again the arguments in [8] can be applied because of the similarities highlighted in the previous proof. The only new fact we need is $2\chi(I_C(n)) < \chi(E(n))$. This is true since

$$\chi(I_C(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{1}{3}n.$$

**Theorem 3.6** Let $E$ be a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$. If $E$ is stable, then either $E$ is locally free or $E$ is associated to a smooth conic $Y \subset V_4$ such that we have the exact sequence

$$0 \rightarrow E \rightarrow H^0(\mathcal{O}(1)) \otimes \mathcal{O}_{V_4} \rightarrow \mathcal{O}(1) \rightarrow 0,$$

where $\theta$ is the theta-characteristic of $Y$.

If $E$ is strictly semistable, then $E$ is the extension of two ideal sheaves of lines.

**Proof** See [8, Theorem 3.5]. Again the arguments in [8] can be applied due to the highlights in the previous two proofs. The only new fact we need in this proof is

$$\chi(I_Z(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1 - l(Z),$$

when $Z$ is a zero-dimensional subscheme.

**4 Relation to semistable rank 2 bundles on $C$**

We now use the classification of semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ to understand their relation with semistable rank 2 bundles on $C$. 
Lemma 4.1 Let $E$ be a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$. Then $E \in \mathcal{B}_{V_4}$.

Proof It suffices to show that $H^*(E(-1)) = H^*(E) = 0$. If $E$ is a stable vector bundle, the result follows from Theorem 3.2 and [20, Lemma B.2].

If $E$ is associated to a smooth conic, we have the short exact sequence

$$0 \rightarrow E \rightarrow H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \rightarrow \theta(1) \rightarrow 0.$$  

Since $H^*(\mathcal{O}_{V_4}(-1)) = H^*(\theta) = 0$, we immediately obtain $H^*(E(-1)) = 0$. On the other hand, we have $H^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_4}) = 2$ and $H^0(\theta(1)) = 2$. It is clear that the map $H^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_4}) \rightarrow H^0(\theta(1))$ is surjective. Moreover, $H^i(\mathcal{O}_{V_4}) = H^i(\theta(1)) = 0$ for all $i > 0$. Thus $H^*(E) = 0$.

If $E$ is the extension of the ideal sheaves of lines in $V_4$, the result follows from the fact that $\mathcal{I}_l \in \mathcal{B}_{V_4}$.  

Remark 4.2 It is worth noting that when $E$ is associated to a smooth conic, the short exact sequence

$$0 \rightarrow E \rightarrow H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \rightarrow \theta(1) \rightarrow 0$$

expresses the fact that $E$ is the left mutation $L_{\mathcal{O}_{V_4}}(\theta(1))[-1]$. This will imply $E \in \mathcal{O}_{V_4}^\perp$ (hence $H^*(E) = 0$) immediately.

[21, Theorem 5.10] proved that for any minimal instanton bundle $E$ on $V_4$, $\Phi^*(E)[-1]$ is a simple rank 2 degree 0 vector bundles on $C$. We generalized this result in the following theorem.

Theorem 4.3 Let $E$ be a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$, then $\mathcal{F} := \Phi^*(E)[-1]$ is a semistable vector bundle of rank 2 and degree 0 on $C$.

Proof If $E$ is a vector bundle, by [21, Theorem 5.10], $\mathcal{F} = \Phi^*(E)[-1]$ is a rank 2 degree 0 vector bundle on $C$ such that

$$\text{Hom}_C(\mathcal{F}, \mathcal{R}) = \text{Ext}_C^1(\mathcal{F}, \mathcal{R}) = 0,$$

where $\mathcal{R}$ is a second Raynaud bundle. By Lemma 2.8, $\mathcal{F}$ is semistable.

If $E$ is associated to a smooth conic $Y$, we have the exact triangle in $\mathcal{D}^b(V_4)$:

$$\theta(1)[-1] \rightarrow E \rightarrow H^0(\theta(1)) \otimes \mathcal{O}_{V_4}.$$  

Apply the functor $\Phi^*(\cdot)[-1]$ and noting that $\Phi^*(\mathcal{O}_{V_4}) = 0$, we obtain

$$\mathcal{F} = \Phi^*(\theta(1))[-2].$$
Recall, as pointed out in the proof of [21, Theorem 5.10], that $\Phi^*$ is a Fourier–Mukai transform with the kernel $S^* \otimes p_{V_4}^* \mathcal{O}_{V_4}(-2)[3]$. Thus the fiber of the object $\mathcal{F}$ at a point $x \in C$ is given by

$$\mathcal{F}_x = H^{*+1}(V_4, S^* \otimes \theta(-1))$$

$$= H^{*+1}(Y, S^*_x|_Y \otimes \theta(-1)) = H^{-*}(\mathbb{P}^1, S_x|_Y \otimes \mathcal{O}_{\mathbb{P}^1}(1))^*.$$

Now $S_x|_Y$ is a rank 2 bundle on $Y \simeq \mathbb{P}^1$ with degree $-2$. Moreover, we note $H^0(V_4, S^*_x) = \mathbb{C}^4$ and the induced map $\mathcal{O}_{V_4} \to S^*_x$ is surjective (see [21, Proposition 5.7]). Hence $S^*_x|_Y$ as a sheaf on $Y$ is generated by global sections. Thus

$$\mathcal{F}_x = \mathbb{C}^2[0]$$

for all $x \in C$. Hence $\mathcal{F}$ is a vector bundle of rank 2. By Lemma 2.7, we see that $\mathcal{F}$ has degree 0.

It remains to see that $\mathcal{F}$ is semistable. Note that a vector bundle $\mathcal{F}$ of rank 2 and degree 0 is not stable if and only if there is a non-trivial morphism $\mathcal{F} \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle of degree 0. By adjunction

$$\text{Hom}(\mathcal{F}, \mathcal{L}) = \text{Hom}(\Phi^*(\theta(1))[2], \mathcal{L})$$

$$= \text{Hom}(\theta(1), \Phi(\mathcal{L}))[2] = \text{Hom}(\theta(1), I_l[1]) = \text{Ext}^1(\theta(1), I_l)$$

where $l$ is a line on $V_4$ by Theorem 2.6. Apply $\text{Ext}^1(\theta(1), -)$ to the short exact sequence $0 \to I_l \to \mathcal{O}_{V_4} \to \mathcal{O}_l \to 0$, we have

$$\text{Hom}(\theta(1), \mathcal{O}_l) \to \text{Ext}^1(\theta(1), I_l) \to \text{Ext}^1(\theta(1), \mathcal{O}_{V_4}).$$

Now the first space is zero since $\theta$ is supported on a smooth conic, while the last space is Serre dual to $\text{Ext}^2(\mathcal{O}_{V_4}(2), \theta(1)) = H^2(\theta(-1)) = 0$. Thus $\text{Ext}^1(\theta(1), I_l) = 0$ and $\mathcal{F}$ is in fact stable in this case.

If $E$ is the extension of ideal sheaves of lines in $V_4$, then we have the short exact sequence

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0.$$

Applying the functor $\Phi^*(\cdot)[-1]$, we have the exact triangle

$$\Phi^*(I_{l_1})[-1] \to \mathcal{F} \to \Phi^*(I_{l_2})[-1].$$

By [21, Lemma 5.5], $\Phi^*(I_{l_1})[-1]$ are line bundles of degree 0 on $C$, thus $\mathcal{F}$ is a strictly semistable rank 2 bundle of degree 0.

**Proposition 4.4** Let $E$ be a stable (strictly semistable) rank 2 sheaf with Chern classes $c_1 = 0, c_2 = 2$ and $c_3 = 0$ on $V_4$, then $\Phi^*(E)[-1]$ is a stable (strictly semistable) vector bundle on $C$.
Proof By the above theorem, $\Phi^*(E)[-1]$ is always semistable. By the proof of the above theorem, we see that for a strictly semistable instanton $E$, $\Phi^*(E)[-1]$ is a strictly semistable vector bundle on $C$. Suppose for a semistable rank 2 sheaf with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$, $\mathcal{F} = \Phi^*(E)[-1]$ is a strictly semistable vector bundle on $C$. Let

$$0 \to \mathcal{L}_1 \to \mathcal{F} \to \mathcal{L}_2 \to 0$$

be a Jordan–Hölder filtration. Then $\mathcal{L}_1, \mathcal{L}_2$ have to be degree 0 line bundles. Apply the functor $\Phi(\cdot)[1]$, by [21, Lemma 5.5], there exist two lines $l_1, l_2$ in $V_4$ such that

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0$$

is exact. Hence $E$ is a strictly semistable sheaf.

By now we have established a well-behaved correspondence between (semi)stable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on $V_4$ and (semi)stable rank 2 degree 0 vector bundles on $C$. Next is to use this correspondence to analyze the two moduli spaces.

5 Moduli space of instantons

We start by showing the smoothness of $M_{V_4}$. To do this we first compute some related invariants.

**Lemma 5.1** Let $\theta$ be the theta-characteristic of a smooth conic $Y$ in $V_4$. Let $E$ be the kernel of the natural surjection $H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1)$. Then $\text{Ext}^2(E, E) = 0$ and $\text{Ext}^1(E, E)$ has dimension 5.

**Proof** By Theorem 4.3, $E = \Phi(\mathcal{F})[1]$, where $\mathcal{F}$ is a rank 2 bundle on $C$. Thus

$$\text{Ext}^2(E, E) = \text{Ext}^2(\Phi(\mathcal{F})[1], \Phi(\mathcal{F})[1]) = \text{Ext}^2_C(\mathcal{F}, \mathcal{F}).$$

The last space is 0 since $C$ is a curve.

Now $\text{Ext}^3(E, E) \simeq \text{Hom}(E, E(-2))^* = 0$ and $\text{Hom}(E, E) = \mathbb{C}$. By Riemann–Roch, $\chi(E, E) = -4$. Thus $\text{Ext}^1(E, E)$ is 5-dimensional.

**Lemma 5.2** Let $l_1, l_2 \subset V_4$ be two lines. Then $\text{Ext}^2(I_{l_1}, I_{l_2}) = 0$ and

$$\dim \text{Ext}^1(I_{l_1}, I_{l_2}) = \begin{cases} 1 & \text{if } l_1 \neq l_2, \\ 2 & \text{if } l_1 = l_2. \end{cases}$$

**Proof** Since $I_{l_1}, I_{l_2} \in \mathcal{B}_{V_4}$, we have

$$\text{Ext}^2(I_{l_1}, I_{l_2}) = \text{Ext}^2_{\text{Db}(C)}(\Phi^{-1}(I_{l_1}[-1]), \Phi^{-1}(I_{l_2}[-1])) = \text{Ext}^2_C(\mathcal{L}_1, \mathcal{L}_2)$$
where \( L_i \) is the line bundle corresponding to \( l_i \) as in Theorem 2.6. Since \( C \) is a curve, we see that the above extension group is 0.

Moreover, \( \text{Ext}^3(I_{l_1}, I_{l_2}) \cong \text{Hom}(I_{l_2}, I_{l_1}(-2))^\ast = 0 \). By Riemann–Roch, \( \chi(I_{l_1}, I_{l_2}) = -1 \). \( \square \)

Let \( N \geq 1 \) be an integer and \( V \) a complex vector space. Let \( Q \) be the Quot scheme parametrizing quotients \( V \otimes \mathcal{O}_{V_4}(-N) \to E \) of rank 2 on \( V_4 \) and Chern classes \( c_1(E) = 0, c_2(E) = 2 \) and \( c_3(E) = 0 \). Let \( L \) denote the natural polarization on \( Q \) (see [33, Section 1]). The group \( G = \text{PGL}(V) \) acts on \( Q \) and a suitable power of \( L \) is \( G \)-linearized. Let \( Q^\text{ss}_c \) be the \( \text{PGL}(V) \)-semistable points corresponding to quotients without torsion and \( Q_c \) the closure of \( Q^\text{ss}_c \) in \( Q \). When the integer \( N \) and the vector space \( V \) are suitably chosen the following properties are satisfied. The map \( V \otimes \mathcal{O}_{V_4} \to E(N) \) induces an isomorphism \( V \cong H^0(E(N)) \) and \( h^i(E(k)) = 0 \) for \( k \geq N \) and \( i \geq 1 \) and for all \( E \in Q_c \). The point \([E] \in Q_c \) is semistable if and only if the sheaf \( E \) is semistable if and only if \( E \in Q^\text{ss}_c \). The stabilizer of \([E] \) in \( \text{GL}(V) \) is identified with the group of automorphisms of the sheaf \( E \) and the moduli space is then the GIT quotient \( Q^\text{ss}_c // G \).

**Lemma 5.3** With the above hypotheses, the scheme \( Q^\text{ss}_c \) is smooth.

**Proof** The tangent space of \( Q^\text{ss}_c \) at a point \([E] \) is isomorphic to \( \text{Hom}(F, E) \) where \( F \) is the kernel of the map \( V \otimes \mathcal{O}_{V_4}(-N) \to E \). The scheme \( Q^\text{ss}_c \) is smooth at the point if \( \text{Ext}^1(F, E) = 0 \). Consider the exact sequence

\[
\text{Ext}^1(V \otimes \mathcal{O}_{V_4}(-N), E) \to \text{Ext}^1(F, E) \to \text{Ext}^2(E, E).
\]

We then obtain an inclusion \( \text{Ext}^1(F, E) \to \text{Ext}^2(E, E) \) since \( h^1(E(N)) = 0 \). It suffices then to prove that \( \text{Ext}^2(E, E) = 0 \). But this follows from the fact \( E = \Phi(\mathcal{F})[1] \) where \( \mathcal{F} \) is a sheaf on \( C \), as we have seen in the proof of Lemmas 5.1 and 5.2. \( \square \)

**Theorem 5.4** The moduli space \( M_{V_4} \) of semistable sheaves of rank 2 with Chern classes \( c_1(E) = 0, c_2(E) = 2, c_3(E) = 0 \) on \( V_4 \) is smooth of dimension 5.

**Proof** See [8, Theorem 4.6]. Let \( x \in Q^\text{ss}_c \) and \( E \) be the corresponding sheaf. Let \( Q_c \subseteq Q_e \) be the set of stable sheaves and \( M^\text{ss} \) be the moduli space of stable sheaves. The scheme \( M^\text{ss} \) is a principal \( G \)-space over \( M^s \) and hence \( M^s \) is smooth by the above lemma. It remains to study the case when \( E = I_{l_1} \oplus I_{l_2} \), where \( l_1, l_2 \) are two lines in \( V_4 \). The orbit \( O(x) \) of \( x \) under \( G \) is closed. The stabilizer \( G_x \) of \( x \) is a reductive group and there exists an affine subscheme \( W \subseteq Q^\text{ss}_c \) containing \( x \) which is locally closed and stable under \( G_x \) so that the morphism \( W // G_x \to Q^\text{ss}_c // G \) is étale [25]. Let \( (W, x) \) be the germ from \( W \) to \( x \) and let \( F \) be the restriction to \( X \times (W, x) \) of the tautological quotient on \( X \times Q \). Then \( ((W, x), F) \) is a space of versal deformation for the sheaf \( E \), [30, Proposition 1.2.3]. The germ \( W \) is then smooth at \( x \) by Lemma 5.2. Since the morphism \( W // G_x \to Q^\text{ss}_c // G \) is étale, it suffices to show the quotient \( W // G_x \) is smooth at \( x \). Now there exists a \( G_x \) linear morphism \( (W, x) \to (T_x W, 0) \) étale at \( x \) so that the induced morphism \( W // G_x \to T_x W // G_x \) is étale at \( [x] \) [25]. It is therefore sufficient to prove that the quotient \( T_x W // G_x \) is smooth at 0.
Suppose \( l_1 \) and \( l_2 \) are distinct. The tangent space \( T_x W = \text{Ext}^1(E, E) \) is of dimension 6 and \( G_x = G_m \times G_m \) acts on the space by the formula [30, Lemma 1.4.16]

\[
(t, t') \cdot \left( \sum_{i,j} e_{i,j} \right) = e_{1,1} + t/t'e_{1,2} + t'/te_{2,1} + e_{2,2}.
\]

It is easy to verify the quotient \( T_x W / G_x \) is the affine space \( \mathbb{A}^5 \) and in particular smooth at 0.

Suppose \( l_1 \) and \( l_2 \) are the same and use \( l \) to denote the line. The tangent space \( \text{Ext}^1(E, E) \) is then of dimension 8 and \( G_x = \text{PGL}(2) \). Let \( T = \text{Ext}^1(I_l, I_l) \) and let \( U \) be a vector space of dimension 2. The group \( G_x \) acts on \( T_x W = T \otimes \text{End}(U) \) by conjugation on \( \text{End}(U) \), [30, Lemma 1.4.6]. The quotient \( T_x W / G_x \) is then isomorphic to \( \mathbb{A}^5 \) [24, III, case 2] and in particular smooth at 0.

We now construct a morphism from \( M_{V_4} \) to \( M_C \), the moduli space of semistable vector bundles of rank 2 and degree 0 on \( C \). \( M_{V_4} \) is the GIT quotient \( Q^s_c / G \). Let \( \mathcal{E} \) be a universal family on \( Q^s_c \times V_4 \). For any \( t \in Q^s_c \), by Lemmas 4.3 and 4.4, the map

\[
\Psi : Q^s_c \rightarrow M_C
\]

\[
t \mapsto [\Phi^*([\mathcal{E}_t] - 1)]
\]

is well defined. Since \( \Phi^*([-1]) \) is Fourier–Mukai, \( \Psi \) is algebraic. To see that this morphism is invariant under \( G \), it suffices to check that \( \Psi(t) = \Psi(t_0) \) for any \( t \) representing a strictly semistable sheaf \( \mathcal{E}_t \), an extension of \( I_{l_1} \) and \( I_{l_2} \), with \( t_0 \) representing \( I_{l_1} \oplus I_{l_2} \). Applying the functor \( \Phi^*([-1]) \) to the short exact sequence

\[
0 \rightarrow I_{l_1} \rightarrow E \rightarrow I_{l_2} \rightarrow 0,
\]

we obtain

\[
0 \rightarrow \Phi^*([I_{l_1}] - 1) \rightarrow \Phi^*([E] - 1) \rightarrow \Phi^*([I_{l_2}] - 1) \rightarrow 0.
\]

Recall that \( \Phi^*([I_{l_1}] - 1) \) are line bundles of degree 0 thus \( \Phi^*([E] - 1) \) lies in the \( S \)-equivalence class of \( \Phi^*([I_{l_1}] - 1) \oplus \Phi^*([I_{l_2}] - 1) \). As a result \( \Psi \) descends to a morphism \( \psi : M_{V_4} \rightarrow M_C \).

**Theorem 5.5** The morphism \( \psi : M_{V_4} \rightarrow M_C \) is an isomorphism. As a result, the moduli space of semistable rank 2 sheaves with Chern classes \( c_1 = 0 \), \( c_2 = 2 \) and \( c_3 = 0 \) on \( V_4 \) is a projective bundle over the Jacobian of \( C \).

**Proof** Since both \( M_{V_4} \) and \( M_C \) are projective, \( \psi \) is proper. We claim \( \psi \) is injective. Let \( \psi([E_1]) = \psi([E_2]) \). By Proposition 4.4, either both \( E_i \) are stable or both \( E_i \) are strictly semistable. If \( E_i \) are stable, then \( \psi([E_1]) = \psi([E_2]) \) implies \( E_1 \simeq E_2 \), i.e., \( [E_1] = [E_2] \). If \( E_i \) are strictly semistable, the injectivity follows from Theorem 2.6.

Any proper quasi-finite morphism is finite, so \( \psi \) is a finite morphism. Thus the image \( \psi(M_{V_4}) \) has dimension 5, which must be all of \( M_C \). So \( \psi \) is surjective. Since
\(\psi\) is injective and \(M_C\) is integral, we see that \(M_{V_4}\) must be connected. Along with Theorem 5.4, we know that \(M_{V_4}\) is a smooth variety.

Let \([E] \in M_{V_4}\) be a stable point, then the tangent space at \([E]\) is given by \(\text{Ext}^1(E, E)\). The tangent space at \(\psi(E)\) (which is also stable) is given by \(\text{Ext}^1(\Phi^*(E)[-1], \Phi^*(E)[-1])\). Since \(\Phi^*: B_{V_4} \to D^b(C)\) is an equivalence, \(\psi\) induces isomorphism between tangent spaces, so \(\psi\) is étale when restricted to the open stable locus. Since \(\psi\) is also injective and we are working over \(\mathbb{C}\), \(\psi\) is an open immersion over the stable locus (see for example Stack Project 40.14). Now \(\psi\) is a bijective birational proper morphism, it has to be an isomorphism. \(\square\)

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