GRBs as standard candles: There is no “circularity problem”
(and there never was)

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Abstract

Beginning with the 2002 discovery of the “Amati Relation” of GRB spectra, there has been much interest in the possibility that this and other correlations of GRB phenomenology might be used to make GRBs into standard candles. One recurring apparent difficulty with this program has been that some of the primary observational quantities to be fit as “data” — to wit, the isotropic-equivalent prompt energy \(E_{\text{iso}}\) and the collimation-corrected “total” prompt energy energy \(E_{\text{pk}}\) — depend for their construction on the very cosmological models that they are supposed to help constrain. This is the so-called “circularity problem” of standard candle GRBs. This paper is intended to point out that the circularity problem is not in fact a problem at all, except to the extent that it amounts to a self-inflicted wound. It arises essentially because of an unfortunate choice of data variables — “source-frame” variables such as \(E_{\text{iso}}\), which are unnecessarily encumbered by cosmological considerations. If, instead, the empirical correlations of GRB phenomenology which are formulated in source-variables are mapped to the primitive observational variables (such as fluence) and compared to the observations in that space, then all taint of circularity disappears. I also indicate here a set of procedures for encoding high-dimensional empirical correlations (such as between \(E_{\text{iso}}, E_{\text{pk}}(\text{src}), t_{\text{jet}}(\text{src}), T_{45}(\text{src})\)) in a “Gaussian Tube” smeared model that includes both the correlation and its intrinsic scatter, and how that source-variable model may easily be mapped to the space of primitive observables, to be convolved with the measurement errors and fashioned into a likelihood. I discuss the projections of such Gaussian tubes into sub-spaces, which may be used to incorporate data from GRB events that may lack some element of the data (for example, GRBs without ascertained jet-break times). In this way, a large set of inhomogeneously observed GRBs may be assimilated into a single analysis, so long as each possesses at least two correlated data attributes.

Keywords: Gamma Rays: Bursts, Cosmology: Cosmological Parameters, Methods: Data Analysis, Methods: Statistical

1. Introduction

Since the earliest published evidence of tight correlations in gamma-ray burst (GRB) spectral properties (Amati et al., 2002), there has been sustained interest in pressing those correlations into service to make GRBs into standard candles, which is the same office that the Phillips correlation performs for SN Ia (Phillips, 1993; Riess et al., 1998; Goldhaber and Perlmutter, 1998). The intriguing possibility is that GRBs may open a window in redshift space \((z \sim [1 – 8])\) beyond what is provided by SN Ia studies, for the purpose of constraining the parameters that characterize Dark Energy (Dai et al., 2004; Ghirlanda et al., 2004; Friedman and Bloom, 2005; Liang and Zhang, 2005; Firmani et al., 2005).

The earliest correlation, the “Amati Relation”, discovered by Amati et al. (2002), was between the isotropic-equivalent prompt emission energy \(E_{\text{iso}}\) and the peak energy \(E_{\text{pk}}\) of the Band-function spectrum fit to the time-integrated prompt emission from the burst, boosted to the source frame by the expansion factor \(1 + z\). Other correlations were discovered in short order, ostensibly exhibiting tighter scatter that could make them more suitable for standardizing candles. Examples are the “Ghirlanda Relation” (Ghirlanda et al., 2004), connecting the collimation-corrected prompt energy \(E_{\gamma}\) and \(E_{\text{pk}}(\text{src})\); the “Liang-Zhang Relation” (Liang and Zhang, 2005), connecting \(E_{\text{iso}}\) with a fit-determined function constructed from \(E_{\text{pk}}(\text{src})\) and the source-frame jet-break time \(t_{\text{jet}}(\text{src})\); and the “Firmani Relation” (Firmani et al., 2006), analogous to the Liang-Zhang relation, but replacing the dependence on \(t_{\text{jet}}(\text{src})\) with one on \(T_{45}(\text{src})\), the source-frame “emission time,” which is a duration measure that robustly stands up to the diversity of duty cycles observed in prompt GRB emission (Reichart et al., 2001).

The later correlations of Liang and Zhang (2005) and

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Firmani et al. (2006) constitute considerable advances over the earlier work on constructing GRB distance indicators. By passing from \((E_{\gamma}, E_{pk}^{(\text{src})})\) to a 2-D projection of the space \((E_{iso}, E_{pk}^{(\text{src})}, T_{45}^{(\text{src})})\), Liang and Zhang (2005) eliminated all reference to highly uncertain theoretical factors — the density of the ISM in the burst source neighborhood, and the conversion efficiency of kinetic energy to radiation in the afterglow — required to convert \(t_{\text{jet}}^{(\text{src})}\) to a jet opening angle. This purged an important source of systematic error from the problem. Firmani et al. (2006) went a step further, passing to 2-D projections of the space \((E_{iso}, E_{pk}^{(\text{src})}, T_{45}^{(\text{src})})\), which, by replacing the difficult-to-obtain jet-break time with the more easily measured prompt duration made many more GRBs available as potential standard candles.

A fly in the ointment was noticed early on by several authors (Friedman and Bloom, 2005; Liang and Zhang, 2005; Firmani et al., 2005): The Amati and Ghirlanda relations were calibrated assuming a standard concordance \(\Lambda\text{CDM}\) cosmology. That is to say, it is not possible to construct quantities such as \(E_{iso}\) or \(E_{\gamma}\) from the observed GRB prompt fluence \(S\) without reference to a specific cosmological model to supply the luminosity distance. As the cosmological model is precisely what is to be constrained from the data, an inconsistency would appear to have been introduced into the problem. This is the (apparent) “Circularity Problem” of GRB standard candles.

Much effort and ingenuity has gone into the abatement of the circularity problem. Friedman and Bloom (2005) performed fits of the Ghirlanda Relation assuming a wide range \((0 \leq \Omega_M, \Omega_L \leq 2)\) of “fiducial” cosmologies, and used each such fit to infer confidence regions on the true parameters, reporting regions that bracket all the results. Aside from being rather conservative, it is difficult to understand what sort of confidence probability is to be ascribed to such intervals. This is problematic if confidence intervals from GRB studies are to be combined with those from other types of Dark Energy studies, such as SN Ia.

The procedures adopted by Liang and Zhang (2005) and by Firmani et al. (2005) are, from a conceptual point of view, even more problematic. Liang and Zhang (2005) re-fit the correlation for each “fiducial” cosmology, to obtain an \(\chi^2_{\text{corr}}\). For each such fit to the correlation, they fit to the cosmological parameters, and re-weight the probability of the cosmological parameter fit by \(\exp(-\chi^2_{\text{corr}}/2)\) — in effect, an ad hoc, tacit, and oddly data-dependent choice of prior.

Firmani et al. (2005) explicitly embrace Bayesian logic, by interpreting the likelihood obtained for cosmology \(\Omega\) using Ghirlanda-relation fits obtained assuming “fiducial” cosmology \(\hat{\Omega}\) as a conditional probability \(P(\hat{\Omega} | \Omega)\) — an interpretation that is both mathematically inconsistent (such an expression should be proportional to the Dirac delta function \(\delta(\hat{\Omega} - \Omega)\)), and logically dubious (what information could cosmology \(\hat{\Omega}\) possibly supply about cosmology \(\Omega\)?) They then eliminate \(\hat{\Omega}\) from their results by marginalizing this probability with some prior on \(\hat{\Omega}\). This removes the nuisance parameter \(\hat{\Omega}\) from the final expressions, but does not correct the logical inconsistency that underlies the calculation.

More recently, an “astronomical” fix has been proposed for the circularity problem. Liang et al. (2008) interpolate distance moduli from SN Ia at the same redshift \((z < 1.4)\) to “train” the correlations at low redshift. This is not terribly different from using nearby SN Ia to calibrate the Phillips relation for all SN Ia, and is not conceptually as problematic as some of the above approaches. However it is a rather weak solution, since the relation must be calibrated using a small subset of GRBs, and since it means that GRB distance moduli can never even in principle be determined more accurately than SN Ia distance moduli. Moreover, if confidence regions on Dark Energy parameters obtained using such a calibration are to be combined with confidence regions obtained from SN Ia, a new hidden statistical dependence will have been introduced that will be difficult to characterize.

It is unfortunate that so much effort has been thus addressed to solving this problem. As I show below, there is no real circularity problem, and there never was. To the extent that a “problem” exists, it is, in effect, a self-inflicted wound, arising from an unfortunate choice of data variables — “source-frame” variables such as \(E_{iso}\) and \(E_{\gamma}\), which are, by their construction, unnecessarily encumbered by cosmological considerations. If, instead, the empirical correlations of GRB phenomenology which are formulated in source-variables are mapped to the primitive observational variables such as fluence (so that the model may discharge its duty of predicting the data, without at the same time being obliged to assist in constructing it), then the circularity disappears, and the analysis may be carried out without fear of inconsistency or paradox.

A recent paper by Basilakos and Perivolaropoulos (2008) addresses the circularity issue by making explicit the dependence of “data” such as \(E_{iso}\) or \(E_{\gamma}\) on the cosmological parameter \(\Omega_M\) in the expression for the log-likelihood, and allows both the “data” and the model to vary with the model parameters in the fit. As such, this work does not make a clean separation between model and data, in the way that is in my opinion desirable. Nonetheless, for reasons that will be discussed in §2.2, the resulting formalism has some features that are similar to the one presented here.

Concomitantly with the necessary disentangling of data from cosmological modeling, I show below how multi-dimensional correlations of the sort projected down to two dimensions by Liang and Zhang (2005) and Firmani et al. (2006) can be fully, and more informatively, modeled in the higher-dimensional space in which they reside, by a Gaussian Tube model, which represents the correlation together with its intrinsic scatter. The Gaussian nature assumed for the scatter yields the benefit of easy convolution with measurement errors to furnish a likelihood function that may be put to the usual inferential work. The Gaussian Tube
will be illustrated in this work by formulating it in the 4-D space of source-frame variables \((t_{\text{jet}}^{\text{src}}, r_{45}^{\text{src}}, E_{pk}^{\text{src}}, E_{\text{iso}})\), and mapping it to the space of observables \((t_{\text{jet}}^{\text{obs}}, T_{45}^{\text{obs}}, E_{pk}^{\text{obs}}, S)\). Generalization to higher-dimensional spaces or to other observables is obvious and immediate.

For those GRBs that are endowed with all four observations, the full Gaussian Tube model is used to produce the event likelihood \(\mathcal{L}_i\). For GRBs that are missing some measured observables, we may still calculate an event likelihood by using the projection of the Tube model into the space of available observables, whether that be 2-D or 3-D (the projection onto 1-D is a uniform distribution, which is uninformative). Thus it is possible to fit simultaneously to all GRBs for which at least two correlated observables are measured. This is a substantial technical advance, in that it was previously necessary to use separately samples of GRBs with different available measurements.

Projection of the tube model has additional uses beyond extending the data set. A mysterious multi-dimensional tube correlation model, however technically satisfying, is not persuasive unless one can verify that the data in fact justify the model. Fortunately, this is not hard to do. Once a best-fit cosmology \(\Omega\) has been obtained (or once we have fixed \(\Omega\) at the concordance model), we may project the best-fit Gaussian Tube model into various 2-D planes — exhibiting both its orientation and its Gaussian width — and superpose the applicable data in that plane, including measurement errors. We are thus able to exhibit the various existing 2-D correlations as different perspectives on the full, multi-dimensional correlation in a series of 2-D plots, and visually inspect the agreement with the data.

The organization of the remainder of this paper is as follows: in §2 I introduce the variables in play, define some notation, and exhibit the Gaussian Tube model in technical detail. In §3, I discuss the mathematical details associated with projection operations of the model into lower-dimensional spaces. In §4 I discuss the procedures required to compare the model to data — formulation of the event likelihoods for the cases of full and partial data, and how the event likelihoods are (trivially) strung together into a full likelihood function for the ensemble of GRBs. In §5 I discuss the use of model projections to verify that the data in fact has a nodding acquaintance with the difficult-to-visualize, multi-dimensional model. A discussion of likely data requirements of the method presented here is in §6. Final discussion and conclusions appear in §7.

2. The Gaussian Tube Correlation Model

The Gaussian Tube is defined as a density which is Gaussian about a symmetry axis along the direction of the correlation, and invariant along that axis. The finite-width density is intended to represent the intrinsic scatter of the correlation. The model is a somewhat crude empirical approximation, since it does not allow for the nature of the intrinsic scatter in the correlation to change as one moves up or down the symmetry axis. The benefit of the simplification is that the likelihood function of data endowed with Gaussian measurement errors may be computed analytically, as I will show in §4. Some possibilities for moving beyond this simplification are indicated at the end of §5.

It is convenient to work with the logs of the observables as primary quantities. Accordingly, we define

\[
X_{t_{\text{jet}}}^{\text{src}} \equiv \log t_{\text{jet}}^{\text{src}}; \quad X_{45}^{\text{src}} \equiv \log T_{45}^{\text{src}}; \quad X_{\text{iso}} \equiv \log E_{\text{iso}}; \quad X_{pk}^{\text{src}} \equiv \log E_{pk}^{\text{src}};
\]

and introduce the vectors

\[
x^{\text{src}} = \begin{bmatrix} X_{t_{\text{jet}}}^{\text{src}} \\ X_{45}^{\text{src}} \\ X_{\text{iso}} \\ X_{pk}^{\text{src}} \end{bmatrix}
\]

\[
x^{\text{obs}} = \begin{bmatrix} X_{t_{\text{jet}}}^{\text{obs}} \\ X_{45}^{\text{obs}} \\ X_{S} \\ X_{pk}^{\text{obs}} \end{bmatrix}
\]

\[
f(z, \Omega) \equiv \begin{bmatrix} \log(1 + z) \\ \log(1 + z) \\ -\log[4\pi(1 + z)^{-2}d_L(z, \Omega)^2] \\ -\log(1 + z) \end{bmatrix},
\]

where \(z\) is the redshift of a particular GRB. In terms of this notation, the transformation from source variables to primitive observables of a particular GRB is simply

\[
x^{\text{obs}} = x^{\text{src}} + f(z, \Omega).
\]

The transformation is thus an elementary shift, albeit one that is different for each GRB (since each is at a different redshift \(z\)).

We will define the model in the space of \(x^{\text{src}}\), and use this relation to move it to the observable space when the time comes to compare the model to data.

2.1. The Axis Of The Tube

The symmetry axis of the event density distribution is easily defined in terms of elementary analytical geometry. The direction of of the tube axis is along a vector \(\mathbf{n}\), and the axis passes through a point \(x_0\), so that points on the axis are defined by the parametric relation \(\mathbf{x} = x_0 + t\mathbf{n}\), for all real \(t\).

This parametrization is not unique, since \(\mathbf{n}\) may be multiplicatively rescaled, and \(x_0\) may be shifted by a multiple of \(\mathbf{n}\). In order to fix a non-degenerate parametrization it is necessary to choose a definite scale for \(\mathbf{n}\) and a definite
intercept for \( \mathbf{x}_0 \). We will choose \( \mathbf{n}^T = [n_1, n_2, n_3, 1]^T \), and \( \mathbf{x}_0 = [x_{0,1}, x_{0,2}, x_{0,3}, 0] \). Thus 6 parameters are required to specify the axis.

2.2. The Gaussian Density

The Gaussian Tube density is denoted by \( \rho(\mathbf{x}^{(\text{src})}) d^2 \mathbf{x}^{(\text{src})} \), where

\[
\rho(\mathbf{x}^{(\text{src})}) = N \times \exp \left[ -\frac{1}{2} (\mathbf{x}^{(\text{src})} - \mathbf{x}_0)^T \mathbf{B} (\mathbf{x}^{(\text{src})} - \mathbf{x}_0) \right].
\]

and where \( \mathbf{B} \) is a non-negative-definite matrix.

The eigenvectors of \( \mathbf{B} \) are the principal directions of the ellipsoids of constant density. If the eigenvalue corresponding to a certain eigenvector should become very small, the ellipsoids will become very elongated along the corresponding direction. In the limit of an eigenvalue going to zero, the distribution will be infinitely elongated, becoming, in effect, a tube. The condition that the tube should be oriented along the direction \( \mathbf{n} \) is therefore \( \mathbf{B} \cdot \mathbf{n} = 0 \).

We require a useful parametrization of \( \mathbf{B} \) that will satisfy this condition. Such a parametrization may be exhibited by introducing dual vectors (linear maps from vectors to numbers) \( \mathbf{w}_i \), \( i = 1, 2, 3 \), satisfying \( \mathbf{w}_i(\mathbf{n}) = 0 \). Then, we may choose

\[
\mathbf{B} = \sum_{i,j=1}^{3} b^{ij} \mathbf{w}_i \mathbf{w}_j, \tag{7}
\]

where the \( b^{ij} \) are components of a positive-definite matrix. By construction, this \( \mathbf{B} \) evidently satisfies \( \mathbf{B} \cdot \mathbf{n} = 0 \).

A convenient choice of the \( \mathbf{w}_i \) may be specified in terms of the dual basis \( \mathbf{g}_\nu \), \( \nu = 1, \ldots, 4 \), which is dual to the coordinate direction vectors \( \mathbf{e}_\mu \), \( \mu = 1, \ldots, 4 \), in the sense that \( \mathbf{g}_\nu(\mathbf{e}_\mu) = \delta_\nu^\mu \). Then we may choose

\[
\mathbf{w}_i = \mathbf{g}_i - n_i \mathbf{g}_4. \tag{8}
\]

It is straightforward to verify that \( \mathbf{w}_i(\mathbf{n}) = 0 \) (recall that \( n_4 = 1 \) by convention). The \( \mathbf{w}_i \) may be written in component form \( \mathbf{w}_i = \sum_{k=1}^{4} w_{ik} \mathbf{g}_k \), where from Eq. (8)

\[
w_{ik} = \delta_{ik} - n_4 \delta_{ik}. \tag{9}
\]

By substituting the components of the \( \mathbf{w}_i \) from Eq. (9) into Eq. (7), we may obtain the matrix components of \( \mathbf{B} \) along the coordinate dual basis \( \mathbf{g}_\nu \) (which is what we mean by the “matrix” \( \mathbf{B} \)):

\[
[B]_{km}^{ij} = \sum_{i,j=1}^{3} b^{ij} w_{ik} w_{jm}. \tag{10}
\]

It remains to guarantee that the components \( b^{ij} \) produce a matrix \( \mathbf{B} \) satisfying \( \mathbf{x}^T \cdot \mathbf{B} \cdot \mathbf{x} \geq 0 \) for all vectors \( \mathbf{x} \), and \( \mathbf{x}^T \cdot \mathbf{B} \cdot \mathbf{x} = 0 \) only when \( \mathbf{x} \propto \mathbf{n} \). From Eq. (7), this is clearly equivalent to the condition that the matrix \( \mathbf{b} \) with components \( b^{ij} \) should be a positive-definite matrix. A parametrization that guarantees this is the Cholesky Decomposition \( \mathbf{L} \) of \( \mathbf{b} \) (see, e.g. Golub and Loan, 1989, p. 141). This is the unique lower-triangular matrix with components satisfying \( L_{ii} > 0 \) and \( L^{ij} = 0 \), \( j > i \), in terms of which \( \mathbf{b} = \mathbf{L} \mathbf{L}^T \), that is,

\[
b^{ij} = \sum_{n=1}^{3} L_{in} L_{jn}. \tag{11}
\]

We therefore adopt the \((3 \times 4)/2 = 6\) components \( L^{ij} \) of \( \mathbf{L} \) as the parameters which, together with \( \mathbf{n} \), control the quadratic form \( \mathbf{B} \).

It is necessary at this point to be more definite about the normalization “constant” \( N \) that figures in Eq. (6). This normalization is of course constant with respect to the variables \( \mathbf{x} \). It is not constant with respect to the parameters \( \mathbf{L} \), however. This is because we must require that the model be somehow normalized, so that we can vary the shape of the tube (the \( \mathbf{L} \)) without varying the predicted overall rate of GRB events. This is an essential feature of the model, without which the task of inferring the \( \mathbf{L} \) from the data will certainly fail.\(^2\)

The normalization must have the following property: the integral of \( \rho(\mathbf{x}) \) on any 3-D hyperplane must be independent of \( \mathbf{L} \). Loosely speaking, this guarantees that changing the width and “cross-sectional shape” of the Gaussian Tube does not change the overall event rate of predicted GRBs. This allows us to decouple the aspects of the model that predict the correlation shape (which we care about) from the aspects that predict GRB event rates (which we do not).

This normalization is easily exhibited: it is

\[
N = \prod_{i=1}^{3} L^{ii}, \tag{12}
\]

which is just the square root of the determinant of the matrix \( \mathbf{b} \). This is roughly speaking \( 1/(\text{product of } \sigma \text{ over...}) \)

\(^1\) The more familiar scale choice of \( \mathbf{n} \cdot \mathbf{n} = 1 \) (i.e. choosing a unit vector for \( \mathbf{n} \)) is not particularly natural in this context. The reason is that there is no natural Euclidean metric defined in the space of observables, and we have no particular reason to import one. The cost of the additional complexity introduced by a quadratic normalization convention is not offset by any benefit (such as, for example, a normalization that is invariant under a relevant class of reparametrizations).

\(^2\) The failure would take the form of an inability of the likelihood function to prefer tight correlations to dispersed ones. Since the term in the exponential of the Gaussian is negative quadratic, and hence bounded above by zero, the fit to the data could simply proceed by making \( \mathbf{B} \to 0 \) (which makes the model a uniform density), reaching the maximum attainable likelihood irrespective of how bad the correlation really is. It is the job of the normalization “constant” to prevent this catastrophe. The normalization of Eq. (12) guarantees that the likelihood will decline to zero if \( \mathbf{B} \) attempts to go to zero.
non-degenerate principal directions), which is the standard normalizing factor of Gaussian distributions (except for inessential π-related factors, which fortunately are constant).

At this point, enough of the model is in view to allow a comparison with the work of Basilakos and Perivolaropoulos (2008), who consider separately various 2-D correlations. As pointed out in §1, there was no clean separation made between model and data in that paper. Nonetheless, Basilakos and Perivolaropoulos (2008) also work with log-space observables, so that the relation between their source- and observer-frame variables is still given by an offset, as in Eq. (5). Since they use χ², which is a function of model-data difference, it is immaterial from the point of view of the formula whether the offset is applied to the model, as here, or the negative offset is applied to the data, as in their work. Note, however, that while χ² is effectively the argument of the exponential of the likelihood in Eq. (6), it does not represent the dependence of the likelihood on the parameters through the normalization N, which, as argued above, is an important omission. The effect of the omission may be seen from Eq. (5) of Basilakos and Perivolaropoulos (2008), where the limit of the slope parameter α → ∞, the expression for χ² saturates at a constant value, leading to open confidence contours. It is precisely this circumstance that the normalization of Eq. (12) avoids.

2.3. Sampling From A Gaussian Tube

Methods for sampling from the distribution defined by a Gaussian Tube are obviously of some interest if one intends to simulate events from such a distribution. Sampling from the tube is not a straightforward exercise in multidimensional Gaussian sampling, as one might imagine upon contemplating Eq. (6), since the degenerate direction n complicates matters somewhat.

Nonetheless there are no insurmountable difficulties or disputing complications here. The main idea is that one samples a vector x₁ from a 3-D multivariate Gaussian in the space of vectors dual to the dual vectors wᵢ — that is, in the 3-D vector space of equivalence classes of vectors differing only by a multiple of n (this is the so-called “Quotient Space” V/n of the full vector space V by the subspace spanned by n). One then samples a real number λ from a uniform distribution in some chosen range. The full sample vector is x = x₁ + λn + x₀. The functional formula is straightforward. From Eq. (8), it is apparent that we can choose as representative vectors for an orthogonal basis of the quotient space the vectors eᵢ, i = 1, 2, 3. The reduced matrix bᵢ is in Eq. (10) may be thought to operate on components of vectors expressed in this basis. That is to say, we may sample from a 3-D multivariate Gaussian with inverse covariance components given by bᵢ, ascribing the components of the sampled vectors to the first three components of x₁ (whose fourth component is zero). One then proceeds from x₁ to x as described above.

Note that the choices n₄ = 1, x₀,₄ = 0 imply that a vector x sampled in this way satisfies x₄ = λ. Thus the chosen range of λ is also the chosen range of x₄.

The fact that we choose the Cholesky decomposition L⁻¹ to parametrize bᵢ, so that b = LLᵀ, is of some assistance here. If we sample three numbers s = (s₁, s₂, s₃) independently from a 1-D standard normal distribution, then the vector (L⁻¹)ᵀs is easily seen to be sampled from a Gaussian distribution with inverse covariance b, as required.

2.4. Summary Of The Model

The 4-D Gaussian Tube model is therefore characterized by 12 parameters: 6 parameters required to establish the location and orientation of the tube through the vectors n and x₀, and another 6 parameters required to set up the actual Gaussian distribution about that axis, through the lower-diagonal matrix L, which is used to obtain the quadratic form B using Eqs. (7), (8), and (11). In a more general N-dimensional space of observables, the parameter count would be (N - 1)(N + 4)/2.

Including the normalization, the expression for the model density is

\[ p(x^{SRC}) = \left( \prod_{i=1}^{3} L^{ii} \right) \exp \left( -\frac{1}{2} (x^{SRC} - x_0)^T B (x^{SRC} - x_0) \right). \]

3. Projection

As discussed in the Introduction, we require the ability to project the Tube onto lower-dimensional subspaces. This may be for the sake of visualizing the correlation in 2-D, or it may be in order to compare the model to the data from a GRB that is not supplied with all four possible observations.

The process of “projecting” a 4-D correlation down to a subspace (such as a visualizable 2-D plane) is, in effect, marginalization over the remaining dimensions. This is a standard operation in Gaussian probability theory, which will now be briefly reviewed.

Suppose, that we wish to project onto a subspace, by marginalizing the Gaussian Tube over the complement of the subspace. We partition all vectors and matrices into the two subspaces:

\[ n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}; \quad x_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \]

We will project onto subspace “1”, by marginalizing over subspace “2”.

5
The projected density is

\[
\eta(x_1) = \left( \prod_{i=1}^{3} L^i \right) \times \int dx_2 \exp \left\{ -\frac{1}{2} \left[ (x_1 - x_{0,1})^T B_{11} (x_1 - x_{0,1}) + 2 (x_1 - x_{0,1})^T B_{12} (x_2 - x_{0,2}) + (x_2 - x_{0,2})^T B_{22} (x_2 - x_{0,2}) \right] \right\},
\]

(15)

The integral may be performed by completing the square. The result is

\[
\eta(x_1) = \left( \prod_{i=1}^{3} L^i \right) \det |B_{22}|^{-1/2} \times \exp \left\{ -\frac{1}{2} (x_1 - x_{0,1})^T A_{11} (x_1 - x_{0,1}) \right\},
\]

(16)

where

\[
A_{11} \equiv B_{11} - B_{12} B_{22}^{-1} B_{12}^T.
\]

(17)

The Gaussian quadratic form in the projected space is therefore \(A_{11}\). Note that the term \(\det |B_{22}|^{-1/2}\) may not be dropped from the normalization of \(\eta(x_1)\), for the same reasons that motivate respect for the parameter-dependence of the normalization of the full 4-D model density \(\rho(x)\).

It is not hard to show that \(A_{11} \cdot n_1 = 0\), as expected. This is because the partitioned version of \(B \cdot n = 0\) is

\[
\begin{align*}
B_{11} \cdot n_1 + B_{12} n_2 &= 0 \\
B_{12}^T \cdot n_1 + B_{22} n_2 &= 0,
\end{align*}
\]

(18, 19)

so that

\[
A_{11} \cdot n_1 = B_{11} \cdot n_1 - B_{12} B_{22}^{-1} B_{12}^T \cdot n_1 = B_{11} \cdot n_1 + B_{12} B_{22}^{-1} B_{22} \cdot n_2 = B_{11} \cdot n_1 + B_{12} \cdot n_2 = B_{11} \cdot n_1 - B_{11} \cdot n_1 = 0.
\]

(20)

In summary, all there really is to know about projection is the partitioning trick: the projected Gaussian Tube has a direction \(n_1\) and an offset \(x_{0,1}\) that are merely the appropriate partitions of their higher-dimensional counterparts, and a quadratic form given by Eq. (17).

4. Model-Data Comparison

As was mentioned in the introduction, the comparison of model and data is necessarily to be carried out in the observable space, and not, as is unfortunately customary, in the source variable space. The reason is that this is the only sensible way to disentangle the cosmology from the data, and permit well-defined estimation of cosmological parameters.

Suppose that the \(i\)th GRB (with precisely-determined redshift \(z_i\)) resulted in a measurement \(y_i\) of the event’s true observables \(x^{(obs)}\). We will not assume that all four observables are available to encode in \(y_i\). Instead, we will assume that \(y_i\) is a \(n\)-dimensional vector, with \(2 \leq n \leq 4\) (so, for example, if all that is available is \(P_{pk}\) and \(S\) then \(n = 2\)). We will also encode the measurement errors of the components of \(y_i\), as the matrix elements of an \(n \times n\) diagonal matrix \(D_i\), defined as

\[
[D_i]_{kl} = \delta_{kl} \sigma_{il}^{-2},
\]

(21)

where \(\sigma_{il}\) is the measurement error on the \(l\)th component of \(y_i\).

The strategy for calculating the likelihood function of all the data is to calculate the event likelihood \(P(y_i | z_i, n, x_0, L, \Omega)\). Since the data for different GRBs is statistically independent, the total likelihood is the product of all the individual event likelihoods:

\[
L(n, x_0, L, \Omega) = \prod_{i=1}^{N} P(y_i | z_i, n, x_0, L, \Omega).
\]

(22)

The problem is therefore reduced to the calculation of the event likelihood for each GRB.

We first require the transformed model density in the full observable space, \(\xi(x^{(obs)}) \; d^4x^{(obs)} = \rho(x^{(src)}) \; d^4x^{(src)}\). This is easily obtained, given the redshift \(z_i\), using the transformation of Eq. (5). As this transformation is a pure constant offset, its Jacobian is 1, and we have

\[
\xi(x^{(obs)}) = \rho(x^{(obs)} - f(z_i, \Omega)) = \left( \prod_{i=1}^{3} L^i \right) \; \exp \left[ -\frac{1}{2} \Delta x^T B \Delta x \right],
\]

(23)

where

\[
\Delta x \equiv x^{(obs)} - x_0 - f(z_i, \Omega).
\]

(24)

If the \(i\)th GRB is endowed with all 4 observations, this is sufficient. If, on the other hand, \(n < 4\), we must project \(\xi(x^{(obs)})\) down to the appropriate space. We adopt the partitioning \(x^{(obs)} T = [u^T, v]^T\), and project out \(v\) to obtain \(\eta(u)\) by the technique of §3, obtaining

\[
\eta(u) = \left( \prod_{i=1}^{3} L^i \right) \det |B_{uvu}|^{-1/2} \times \exp \left[ -\frac{1}{2} \Delta x_u^T A_{uu} \Delta x_u \right],
\]

(25)

where

\[
A_{uu} \equiv B_{uu} - B_{uv} B_{vv}^{-1} B_{uv}^T,
\]

\[
\Delta x_u \equiv u - x_{0,u} - f_u(z_i, \Omega),
\]

(26, 27)

and where the meaning of the partitioned vectors and matrices should be clear from context.
We may now convolve this distribution with the measurement error distribution on $y_i$. This is tantamount to integrating over the entire space $u$ the probability that the true value of the observables should have been $u$ and that the actual measured values should have been $y$. By a routine Gaussian integration, we obtain

$$P(y_i | z_i, n, x_0, L, \Omega) = \int d^n u \eta(u) \times \exp \left[ -\frac{1}{2} (u - y_i)^T D_i (u - y_i) \right] \prod_{i=1}^3 L^3 \\det |B_{vv}|^{-1/2} \\det |A_{uu} + D_i|^{-1/2} \times \exp \left[ -\frac{1}{2} \Delta y_i^T Q_i \Delta y_i \right],$$

where

$$Q_i \equiv D_i - D_i (D_i + A_{uu})^{-1} D_i,$$

$$\Delta y_i \equiv y_i - x_{0,u} - f_u(z_i, \Omega).$$

Eqs. (28), (29), and (30) are the final result for the event likelihood, in the case where data is incomplete and projection is necessary. Obviously, if the full set of observations are available, projection is not necessary, and these formulas are to be applied by replacing $A_{uu}$ by $B$, $\det |B_{vv}|$ by 1, $x_{0,u}$ by $x_0$, and $f_u(z_i, \Omega)$ by $f(z_i, \Omega)$.

As complicated as these formulas may appear, they do not represent much of a computational challenge, as the determinants and inverses that are required are of symmetric, positive-definite matrices with dimensionality less than or equal to 4. Perhaps a slightly greater computational challenge is the code organization required to arrange for the capability of carrying out the projection/partition of relevant matrices and vectors along arbitrary subsets of coordinates. This is nonetheless a manageable programming task.

With the event likelihood in hand, we may proceed to the calculation of the total likelihood $L$, as given by Eq. (22).

And now, we’re in business. For example, we may simultaneously optimize $L(G, \Omega)$ (where $G$ represents the Gaussian Tube parameters) with respect to $G$ and $\Omega$, obtaining fully internally-calibrated point estimates of both sets of parameters, and perhaps even frequentist confidence intervals.

We may also play Bayesian games, using some choice of prior over the parameters to trade $L$ in for a posterior density distribution $P(G, \Omega | O)$, where $O$ represents the observations. We may then marginalize some of the parameters to produce Bayesian confidence regions on others. This may require a Markov Chain Monte Carlo approach, given the large number of parameters. Or, we may make the approximation that marginalization is equivalent to extremization (which is true for Gaussian distributions)\(^3\), and calculate an approximate posterior density on the cosmology parameters $P(\Omega | O)$ by maximizing the full posterior $P(G, \Omega | O)$ with respect to $G$ at every value of $\Omega$.

The point is, $L$ is a genuine likelihood — the probability of some data given a model — and may be pressed into service in exactly the sort of ways that likelihoods are normally used. The circularity concerns that derive from the use of “fiducial cosmologies” to create the source-variable “data” have been short-circuited by the simple expedient of calculating the probability of the data that is directly observed.

5. Sanity Checking

The program of data analysis outlined so far relies on some rather abstract and difficult-to-visualize constructions. It is crucial that there should be some way to visualize the relationship between the model and the data, both to spot possible problems and to get an intuitive feeling for the predictive content of the model.

Once a best-fit Gaussian Tube and a best-fit cosmology (or the concordance cosmology) $\Omega$ have been fixed, this is a straightforward thing to do. There are six 2-D planes that may be formed from the 4 available source variables. The best-fit Gaussian Tube model may be projected according to the method of §3 onto each of these planes. The projected best-fit straight line and the $1 - \sigma$ confidence region from the projected distribution may be plotted on each plane. Each GRB endowed with observations that are representable on some of those planes may have those observations mapped to the appropriate plane (for example, a GRB with measured $S$, $\epsilon_{\text{pk}}^{\text{obs}}$, and $T_{45}^{\text{obs}}$ may be mapped to the $X_{\text{iso}} - X_{\text{pk}}^{\text{obs}}$, $X_{\text{iso}} - X_{45}^{\text{obs}}$, and $X_{\text{iso}} - X_{45}^{\text{obs}}$ planes).

We finally end up with a series of six plots, each one displaying the projected model and the mapped data. From these plots, it should be possible to visualize directly the properties of the various projected aspects of the correlation model, and the extent to which the best-fit model really respects the data.

Besides this sort of visual verification of the various 2-D correlations against the data, there is another model verification issue that merits consideration. The observed redshift distribution of GRBs drops dramatically below $z \leq 1$, where most SN Ia redshifts are found, and extends out past $z = 6$. This is an opportunity, of course, since it means that GRB-derived confidence regions in, say, the $\Omega_M - \Omega_A$ plane cut across those derived from SN Ia (Ghirlanda et al., 2006), furnishing tighter constraints on those parameters. However, the much broader range of GRB redshifts raises a troubling question: Even if we find

\(^3\) Note, however, that the posterior probability density over model parameters can at best be only approximately Gaussian, despite the Gaussian nature of the GRB density model.
a reasonable-seeming fit of the Gaussian Tube model’s projections to the data, how do we know whether the properties of the tube should be considered to have evolved with redshift? That is to say, is it reasonable to assume, as the model does, that the correlations of GRB energetics follow the same distributions irrespective of redshift? If so, how do we know? If not, how would this affect the inferred values of the cosmological parameters $\Omega$?

This question cannot be addressed merely by inspecting the projection plots described above, since the redshifts are all intermingled in those plots. Instead, it seems advisable to adopt a model-comparison strategy, wherein the “vanilla” Gaussian Tube model described above is compared to more complicated models (via a frequentist likelihood ratio, or a Bayesian test based on posterior odds ratios) that allow the correlation parameters to vary with redshift. That is, we may introduce another hierarchical level in the model by allowing some of the parameters $G$ to be some parametrized empirical function of $z$ (a linear function is an obvious thing to try), and calculate the amount by which the log-likelihood (say) is improved in this model over a model in which the $G$ are the same for all redshifts. Significant improvements would be evidence for evolution of the distributions. Additionally, significant shifts in the confidence regions in the $\Omega_M - \Omega_A$ plane in the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as an indication of trouble, whereas stability of those contours as the more complicated model could be interpreted as a reassuring sign of robustness of the results.

Clearly this approach entails some considerable expansion of the parameter space. A somewhat more modest approach, similar to the calibration approach of Basilakos and Perivolaropoulos (2008), is to partition the GRBs into a small number of bins, fit them separately, and determine whether the sum of the log likelihoods is significantly better than the log-likelihood for the full sample. Again, any significant shifts in $\Omega_M - \Omega_A$ contours, or lack of such shifts, would be telling of the robustness of the inferences drawn from the model.

6. Data Requirements

This paper is a “methods” paper, and I have not yet attempted to collect a carefully-calibrated sample of GRB data to subject to this analysis. I can therefore not suggest precise guidelines as to how many GRBs, bearing what kind of information, may be necessary to obtain interesting constraints on cosmological parameters using the present methodology. Nonetheless the question is worth addressing, so I offer a few tentative thoughts on the matter.

The “vanilla” (i.e. not redshift-dependent) Gaussian Tube model presented here has 12 free parameters, and operates on 2 to 4 observable quantities per GRB. In addition, a minimally interesting cosmological model offers two additional parameters for constraining ($\Omega_M$ and $\Omega_A$), so that a total of 14 parameters must be managed in the fit.

Consider the Tube parameters $G$ first. The role of these 12 parameters is to ensure that the 6 2-D projections of the model adequately fit the projections of available data into those planes. The model is more compact than a model composed of 6 2-D Tube models (which would require 18 parameters). Therefore, a conservative estimate of the amount of data required to constrain the full Gaussian Tube model is the amount required to constrain the 6 independent 2-D Tube models. In each plane, this number would depend on the tightness of the correlation, the size of the measurement errors, and the dynamic range of the data. In the case of the original Amati relation (Amati et al., 2002), with 10 constrained data points, measurement errors in the 10-30% range, and a dynamic range in $E_{iso}$ of nearly 3 orders of magnitude, the fit parameters that resulted had a statistical error of about 10%.

Suppose, then, for the sake of making a conservative estimate, that we require 15 points in each plane for adequate constraints on $G$. The number of events required to furnish this information is bounded below by 15 (if all events bear all information, so that 60 numbers are used), and above by 90 (if all points on all planes are due to different GRBs, so that 180 numbers are used).

Turning to the cosmological parameters, one may observe that initially, the inflation of the Tube parameter count from 3 (for a single 2-D correlation such as the Ghirlanda relation) to 12 adds uncertainty to the contours in the $\Omega_M - \Omega_A$ plane, uncertainty which must be made up by adding data that constrains those additional Tube parameters. If the data is in fact constraining on those additional parameters, then one may expect the statistical errors on cosmological parameters to shrink roughly as $N_{pair}^{-1/2}$, where $N_{pair}$ is the number of independent pairs of event data (i.e. the total number of points in the 6 projected planes). This is the point of the exercise: by passing to the Gaussian Tube model, one pays a price in parameter count inflation, in the expectation that one will reap a dividend through the larger and more informative dataset that thereby becomes accessible.

In other words, the design of this framework requires a cost/benefit analysis. It is not necessarily the case that the particular choice of observables made in this work for the sake of illustration is optimal. Possibly a different set, or a smaller or larger set, might be preferable. Much depends on visual inspection of correlations. If one or more of the 2-D projections of the data appear not to show evidence for a strong correlation, it may be the case that the pa-
rameters controlling the correlation in that projected plane may be adding more noise than signal, and it might be a good idea to change observables, or to freeze the responsible parameters at some harmless value. On the other hand, reasonably clear correlations of the data in all projected planes would constitute evidence for a good choice of observables, one which is likely to reward the analysis with statistical errors on cosmological parameters that are smaller in consequence of more abundant data, and that shrink more rapidly with increasing data than would errors inferred from a lower-dimensional model.

7. Discussion

It is my hope that readers are persuaded that the circularity problem of GRB standard candle enterprise was merely a diversion, an own-goal brought about by an unfortunate choice of space for model-data comparison, and readily corrected by making a better choice. The various fix-ups for the “problem” that have been proposed in the literature, and which were discussed in §1, are not merely unnecessary: by taking an excessively conservative attitude towards parameter constraints, or actually introducing incoherent features to their statistical model, they almost certainly do more harm than good.

The view advocated here is that the various correlations that are discussed in the literature must necessarily be projected aspects of a higher-dimensional “super-correlation”. At its root, this is really no more than the observation that if A is correlated with B, and B is correlated with C, then A is necessarily correlated with C, and a correlation structure must therefore exist in the joint space of A, B, and C.

I cannot say at this stage what the various correlations look like in all six 2-D planes that may be constructed from the present variables. However, it would be difficult to understand if they weren’t about as tight as the Amati relation, unless there is something wrong with the Ghirlanda/Firmani/Liang-Zhang/etc. correlations, which, as I explain below, I do not believe. Turning this around, however, there is a very interesting possibility: exhibiting correlations in alternative planes — including some built from burst durations and afterglow break times — strengthens the case for the reality of all these correlations, in the sense that it is difficult to imagine a selection effect of such perversity that it can produce both $E_{\text{pk}}^{\text{src}}$–$E_{\text{iso}}$ and $t_{\text{jet}}^{\text{src}}$–$t_{\text{45}}^{\text{src}}$ correlations (for example).

An additional remark concerning projections seems apposite. It is possible to search for 2-D projections that are not necessarily along the coordinate axes, which in some sense minimize scatter in the data. The Ghirlanda, Liang-Zhang, and and Firmani relations are of this character. All that is required is to effect linear transformations of the coordinate axes, together with the corresponding similarity transformations on all matrices. One could imagine searching for the linear transformation that makes a correlation in a 2-D projection look maximally tight. However, it seems to me that the motivation for doing so is not as strong in the current picture as it once was, since all the content of these relations is already embodied in the best-fit Gaussian Tube model. Certainly, the construction of such a transformation would have no effect whatever on the likelihood function computed above, or on any of the cosmological inferences drawn therefrom.

This remark underlines the essential fact that the most suitable space for visualizing the relationship between model and data is not necessarily the most suitable space for analyzing that relationship. It was the failure to understand this truism of data analysis that gave rise to the circularity problem in the first place.

While the reality of these correlations has been harshly questioned (Nakar and Piran, 2005; Band and Preece, 2005; Butler et al., 2007), in my opinion the assuredness of these critiques is out of all proportion to the questionable cogency of the evidence upon which they rest. It should be kept in mind that in order to even observe the correlations, the essential requirements are (a) rapid, accurate astrometry (to furnish afterglow redshifts), and (b) accurate broadband spectroscopy (to obtain the essential spectral fit parameters).

The critiques of Nakar and Piran (2005) and Band and Preece (2005) rely upon the fits of BATSE spectral data reported in Band et al. (1993), despite the fact that BATSE had no afterglows. Furthermore, as attested by columns 5, 6, and 7 of Table 4 of Band et al. (1993), many of these spectral fits were of questionable quality.

The critique of Butler et al. (2007), relies purely on Swift spectral fits, but as Swift’s bandpass is essentially 20–120 keV, there is no possibility of securing actual spectral fit parameters. BATSE-informed priors must therefore do some extremely heavy lifting. Nonetheless, Butler et al. (2007) find an Amati Relation correlation in Swift data, with the correct slope, but with the wrong normalization. Curiously, rather than conclude that their priors might be exerting some uncontrolled influence, they infer instead that the inconsistency exposes the Amati relation as being due to a somewhat vaguely-specified instrumental selection effect.

Meanwhile, every analysis based on data from instrument complements capable of both prompt, accurate astrometry and accurate broad-band spectroscopy, such as from BeppoSAX (Amati et al., 2002) or from HETE (Sakamoto et al., 2005) has found that with the exception of a small number of conspicuous outliers (such as the under-luminous GRB980425), new data invariably drapes itself across the old, known correlations. In addition, analysis of time-resolved spectroscopy of selected BATSE bursts by Liang et al. (2004) showed that flux and $E_{\text{pk}}$ are Amati-correlated within the time history of each GRB. Finally, Ghirlanda et al. (2009), using 12 long GRBs jointly observed by Swift and by Fermi/GBM (with GBM supplying the spectroscopic coverage), not only confirm the time-resolved GRB-personalized mini-Amati relations of long GRBs discovered by Liang et al. (2004), but also show that the
normalizations of those mini-relations actually place them on the BeppoSAX/HETE Amati relation, with the BeppoSAX/HETE parameters (and, of course, the time-integrated spectral parameters of all 12 events also fall on the BeppoSAX/HETE Amati relation).

This debate would appear to be over: the various long GRB phenomenological correlations, are (except for a small fraction of conspicuous outliers) convincingly confirmed, and appear to be manifestations of “internal” features of GRB emission. They must certainly be taken seriously. Given that Swift and Fermi/GBM appear capable of producing about a dozen joint events with the required spectral data per year (Ghirlanda et al., 2009), and given that one may expect that a sample of GRBs of about 150 events may make an impact on Dark Energy studies comparable to that of SN Ia (Ghirlanda et al., 2006), it is possible that GRBs may be put to useful cosmological work sooner rather than later.

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