Transverse Structure of Nucleon Parton Distributions from Lattice QCD

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This work presents the first calculation in lattice QCD of the three moments of spin-averaged and
spin-polarized generalized parton distributions in the proton. It is shown that the slope of the
associated generalized form factors decreases significantly as the momentum increases, indicating that
the transverse size of the light-cone quark distribution decreases as the momentum fraction of the
struck parton increases.

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INTRODUCTION

High energy lepton scattering reveals the quark and
and gluon structure of the nucleon by measuring matrix elements of the light-cone operator

\[ O_q(x) = \int \frac{d\lambda}{4\pi} \epsilon(\lambda, n) \lambda \bar{q} P e^{-i \lambda/2} \alpha n A(0) q(\lambda/2 n). \]  

(1)

The familiar quark distribution \( q(x) \) specifying the
probability of finding a quark carrying a fraction \( x \) of the
nucleon’s momentum in the light-cone frame is measured by
the diagonal nucleon matrix element, \( \langle P | O(x) | P \rangle = q(x) \). Expanding \( O(x) \) in local operators via the
operator product expansion generates the tower of twist-two
operators,

\[ O_q(\mu_1, \mu_2, ..., \mu_n) = \bar{q} \gamma_\mu_1 \gamma_\mu_2 \ldots \gamma_\mu_n q, \]  

(2)

and the diagonal matrix element \( \langle P | O(\mu_1, \mu_2, ..., \mu_n) | P \rangle \) spec-
ifies the \((n-1)\)th moment of the quark distribution

\[ \int dx x^{n-1} q(x). \]

The generalized parton distributions \( H(x, \xi, t) \) and
\( E(x, \xi, t) \) are measured by off-diagonal matrix elements of the light-cone operator

\[ \langle P' | O(x) | P \rangle = \langle \hat{\Gamma} \rangle H(x, \xi, t) + \frac{i \Delta^4}{2m} \langle \sigma_{\mu \nu} n_\mu \rangle E(x, \xi, t), \]  

(3)

where \( \Delta^4 = P'^4 - P^4, \xi = -n \cdot \Delta/2, n \) is
a light-cone vector, and \( \langle \hat{\Gamma} \rangle = \hat{U}(P') \hat{U}(P) \). Off-
diagonal matrix elements of the tower of twist-two
operators \( \langle P' | O(\mu_1, \mu_2, ..., \mu_n) | P \rangle \) yield moments of the generalized
parton distributions, which for \( \xi = 0 \), are

\[ \int dx x^{n-1} H(x, 0, t) = A_{n0}(t) \]

\[ \int dx x^{n-1} E(x, 0, t) = B_{n0}(t), \]  

(4)

where \( A_{n0}(t) \) and \( B_{n0}(t) \) are referred to as generalized
form factors (GFFs).

Analogous expressions in which the light-cone operator
\( O_q(x) \) and twist-two operators contain an additional \( \gamma_5 \)
measure the longitudinal spin density, \( \Delta q(x) \) and
spin-dependent generalized parton distributions \( H(x, \xi, t) \) and
\( E(x, \xi, t) \) with moments \( A_{ni}(t) \) and \( B_{ni}(t) \). In this work,
we present calculations of the generalized form factors
\( A(n=1,2,3),0(t) \) and \( A(n=1,2,3),0(t) \) in full QCD and discuss
their physical significance.

TRANSVERSE STRUCTURE OF PARTON DISTRIBUTION

In general, \( H(x, \xi, t) \) is complicated to interpret physically because it combines features of both parton distributions
and form factors, and depends on three kinematical variables: the momentum fraction, \( x \), the longitudi-
dinal component of the momentum transfer, \( \xi \), and the
total momentum transfer squared, \( t \). In the particular
case in which \( \xi = 0 \), however, Burkardt \[4\] has shown
that \( H(x, 0, t) \), as well as its spin-dependent counterpart
\( \tilde{H}(x, 0, t) \), has a simple and revealing physical interpretation.

It is useful to consider a mixed representation in which
transverse coordinates are specified in coordinate space,
the longitudinal coordinate is specified in momentum space, and one uses light-cone coordinates for the lon-
gitudinal and time directions: \( x^\pm = (x^0 \pm x^3)/\sqrt{2}, \)
\( p^\pm = (p^0 \pm p^3)/\sqrt{2} \). Using these variables, letting \( x \) denote
the momentum fraction and \( b_\perp \) denote the transverse
displacement (or impact parameter) of the light-cone operator relative to the proton state, one may de-
fine an impact parameter dependent parton distribution in
light-cone gauge

\[ q(x, b_\perp) \equiv \langle P^+, \bar{R}_\perp = 0, \lambda | O_q(x, b_\perp) | P^+, \bar{R}_\perp = 0, \lambda \rangle, \]  

(5)
where

\[ O_q(x, \vec{b}_\perp) = \int \frac{dx^-}{4\pi} e^{ix_-^+ x_-} q\left(-\frac{x_-}{2}, \vec{b}_\perp\right) \gamma^+ q\left(-\frac{x_-}{2}, \vec{b}_\perp\right). \]  

(6)

Burkardt shows that the generalized parton distribution \( H(x, 0, t) \) is the Fourier transform of the impact parameter dependent parton distribution, so that

\[ H(x, 0, -\Delta^2) = \int d^2 b_\perp q(x, \vec{b}_\perp) e^{i \vec{b}_\perp \cdot \vec{\Delta}}, \]

\[ A_{n0}(-\Delta^2) = \int d^2 b_\perp \int dxx^{n-1} q(x, \vec{b}_\perp) e^{i \vec{b}_\perp \cdot \vec{\Delta}}. \]  

(7)

where the second form follows from Eq. (6). Although one normally only expects a form factor to reduce to a Fourier transform of a density in the non-relativistic limit, Ref. [4] shows that special features of the light-cone field theory. Thus, \( H(x, 0, t) \) specifies how the transverse distribution of quarks varies with the longitudinal momentum fraction \( x \).

Physically, we expect the transverse size of the nucleon to depend significantly on \( x \). Averaging \( q(x, \vec{b}_\perp) \) over all \( x \) produces \( A_{10}(t) \) and thus corresponds to calculating the form factor. In this case the active parton represents the (transverse) center of momentum, and the distribution in impact parameter reduces to a delta function \( \delta^2(\vec{b}_\perp) \) with zero spatial extent. Indeed, explicit light-cone wave functions \[ \Psi_{n,c}(x_1, \ldots; \vec{r}_{1}, \ldots) \] bear out this expectation, with the result

\[
q(x, \vec{b}_\perp) = (4\pi)^{n-1} \sum_{n,c} \int \left[ \prod_{j=1}^{n} dx_j d^2 r_{j,\perp} \right] \times \delta \left( 1 - \sum_{j=1}^{n} x_j \right) \delta^2 \left( \sum_{j=1}^{n} x_j \vec{r}_{j,\perp} \right) \delta (x - x_a) \\
\times \delta^2 \left( \vec{b}_\perp + (1 - x) \vec{r}_{a,\perp} - \sum_{j\neq a}^{n} x_j \vec{r}_{j,\perp} \right) \\
\times \Psi_{n,c}^* (x_1, \ldots; \vec{r}_{1}, \ldots) \Psi_{n,c} (x_1, \ldots; \vec{r}_{1}, \ldots),
\]

where \( a \) denotes the index of the active parton, \( n \) is the number of partons in the Fock state and the sum over \( c \) represents the sum over all additional quantum numbers characterizing the Fock state. Here, one explicitly observes \( \lim_{x^+ \to 1} q(x, \vec{b}_\perp) \propto \delta^2(\vec{b}_\perp) \). Since \( H(x, 0, t) \) is the Fourier transform of the transverse distribution, the slope in \(-t = \Delta^2\) at the origin measures the rms transverse radius. As a result, we expect the substantial change in transverse size with \( x \) to be reflected in an equally significant change in slope with \( x \). In particular, as \( x \to 1 \) the slope should approach zero. Hence, when we calculate moments of \( H(x, 0, t) \), the higher the power of \( x \), the more strongly large \( x \) is weighted, and the smaller the slope should become. Therefore, this argument makes the qualitative prediction that the slope of the generalized form factors \( A_{n0}(t) \) and \( \tilde{A}_{n0}(t) \) should decrease with increasing \( n \), and we expect that this effect should be strong enough to be clearly visible in lattice calculations of these form factors.

**LATTICE MEASUREMENT**

We consider the three spin independent moments

\[
\langle P' | O^{\mu_1} | P \rangle = \langle \gamma^{\mu_1} \rangle A_{10}(t) \\
+ \frac{i}{2m} \langle \sigma^{\mu_1} \rangle \Delta_0 B_{10}(t),
\]

\[
\langle P' | O^{\mu_1\mu_2} | P \rangle = \langle \gamma^{\mu_1} \gamma^{\mu_2} \rangle A_{20}(t) \\
+ \frac{i}{2m} \tilde{P}^{(\mu_1 \mu_2)} \langle \sigma^{\mu_1} \sigma^{\mu_2} \rangle \Delta_0 B_{20}(t),
\]

\[
\langle P' | O^{\mu_1\mu_2\mu_3} | P \rangle = \langle \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \rangle A_{30}(t) \\
+ \frac{i}{2m} \tilde{P}^{(\mu_1 \mu_2 \mu_3)} \langle \sigma^{\mu_1} \sigma^{\mu_2} \sigma^{\mu_3} \rangle \Delta_0 B_{30}(t),
\]

where \( \tilde{P}^{(\mu_1 \mu_2)} = \langle P''^{\mu_1} + P''^{\mu_2} \rangle / 2 \), as well as the analogous spin dependent momenta.

Generalized form factors \( A_{(n=1,2,3),0}(t) \) and \( \tilde{A}_{(n=1,2,3),0}(t) \) were calculated using the new method introduced in Ref. [8]. We considered all the combinations of \( \tilde{P} \) and \( \tilde{P}' \) that produce the same four-momentum transfer \( t = (P' - P)^2 \), subject to the conditions that \( \tilde{P} = \frac{2\kappa}{\sqrt{3}} (n_x, n_y, n_z) \) and \( \tilde{P}' = (0, 0, 0) \) or \( \frac{2\kappa}{\sqrt{3}} (-1, 1, 0) \). Using all these momentum combinations for a given \( t \) below 3.5 GeV, we calculated all the hypercubic lattice operators and index combinations that produce the same continuum GFFs, obtaining an overdetermined set of equations from which we extracted a statistically accurate measurement. The errors are substantially smaller than obtained by the common practice of measuring a single operator with a single momentum combination. Our calculations are based on unquenched SESAM configurations on \( 16^3 \times 32 \) lattices with \( \kappa = 0.1560 \) and \( \kappa = 0.1570 \), corresponding to pion masses of \( m_\pi = 897 \) and 744 MeV respectively.
Figure 1 presents our principal results, showing the generalized form factors $A_{n0}(t)$ and $\tilde{A}_{n0}(t)$ for the lowest three moments: $n = 1$, 2, and 3. The form factors have been normalized to unity at $t = 0$ to make the dependence of the shape on $n$ more obvious. Note that $A_{1,0}$, $A_{3,0}$, and $\tilde{A}_{2,0}$ depend on the difference between the quark and antiquark distributions whereas $\tilde{A}_{1,0}$, $\tilde{A}_{3,0}$, and $A_{2,0}$ depend on the sum. Hence only moments differing by two compare the same physical quantity with different weighting in $x$. To facilitate determination of the slope of the form factors and to guide the eye, the data have been fit using a dipole form factor

$$A_{n0}^{\text{dipole}} = \frac{A}{(1 - \frac{t}{m_q^2})^2}.$$  \hspace{1cm} (9)

The solid line denotes the least-squares fit and the shaded error band shows the error arising from the statistical error in the fit mass, $\Delta m_d$. Although the dipole fit is purely phenomenological, we note that it is consistent with the lattice data. For reference, the normalization factors $A_{n0}$ and dipole masses are tabulated in Table I.

The top panel in Fig. 1 shows the flavor non-singlet case $A^u - A^d$, for which the connected diagrams we have calculated yield the complete answer. It is calculated at the heaviest quark mass we have considered, corresponding to $m_\pi = 897$ MeV. Note that the form factors are statistically very well separated, and differ dramatically for the three moments. Indeed, the slope at the origin decreases by more than a factor of 2 between $n = 1$ and $n = 3$, indicating that the transverse size decreases by more than a factor of 2. The second panel shows analogous results for lighter quarks, $m_\pi = 744$ MeV, where we observe the same qualitative behavior but slightly weaker dependence on the moment. The third panel shows the flavor singlet combination $A^u + A^d$, for which we have had to omit the disconnected diagram because of its significantly greater computational cost. Comparing this figure with the top panel calculated at the same quark mass, we observe that while the connected contributions to $A^u + A^d$ are qualitatively similar, there is significant quark flavor dependence that can be used to explore the nucleon wave function. It is useful to note our results for the $u$ and $d$ GFFs are consistent with the $n=2$ moments calculated in Ref. [11]. The bottom panel shows the spin-dependent flavor non-singlet form factors $\tilde{A}^u - \tilde{A}^d$ at the heaviest quark mass. Thus, comparing the top and bottom figures displays the difference between the spin averaged and spin dependent densities. We observe a striking difference, in that the change between the $n = 1$ and $n = 3$ form factors for $q(x, \vec{b}_\perp)|_t - \bar{q}(x, \vec{b}_\perp)|_t$ is roughly 6 times smaller than for $\frac{1}{2} \left( q(x, \vec{b}_\perp)|_t + \bar{q}(x, \vec{b}_\perp)|_t \right)$.

Finally, it is useful to use the slope of the form factors at $t = 0$ to determine the transverse rms radius,

$$\langle r^2_{\perp} \rangle^{(n)} = \frac{\int d^2b_\perp b_\perp^2 \int dx x^{n-1} q(x, \vec{b}_\perp)}{\int d^2b_\perp \int dx x^{n-1} q(x, \vec{b}_\perp)}.$$  \hspace{1cm} (10)

Transverse rms radii calculated in this way are tabulated in Table I. To set the scale, the transverse charge radius at this mass is $\langle r^2_{\perp} \rangle_{\text{charge}} = 0.48$ fm, which is two-thirds the experimental transverse size 0.72 fm, reflecting the absence of a significant pion cloud. For the heaviest
In the “heavy pion world” presently accessible to full lattice QCD, we have calculated the lowest 3 generalized form factors $A_{n0}$ and $\bar{A}_{n0}$ up to $|t|=3$ GeV as shown in Fig. 1. We obtain excellent precision for $n=1$ and sufficient precision for $n=2$ and 3 to clearly distinguish each form factor and observe striking differences in slope and hence transverse size. Whereas there are other calculations of isolated moments, three moments are crucial for the present investigation since $n=1$ and 3 are necessary to measure the same combination of quark and antiquark distributions. The dependence of the transverse size on $x$ is most dramatic for the heaviest $u-d$ combination, for which $(r_{T}^{2})_{u-d}$ decreases by 62% between the first and third moment. We also observed clear dependence of the transverse distribution on flavor and spin. Our results show that the commonly used factorization Ansatz $H(x,0,t) = Q(x) F(t)$ is fundamentally wrong in the “heavy pion world” and we are aware of no arguments as to why it should be restored for lighter quarks.

The most immediate challenges are to extend these calculations to the chiral regime of realistic quark masses, which is being explored using a hybrid calculation of dynamical staggered sea quarks and domain wall valence quarks [12], and to extend techniques for evaluating disconnected diagrams [13] to GFFs. When precise, controlled extrapolations to the physical pion mass are finally achieved, moments calculated from first principles will play an essential role in complementing experimental results because of the impracticality of measuring the full $x$, $\xi$, and $t$ dependence of $H(x,\xi,t)$ and $\bar{H}(x,\xi,t)$ experimentally. In addition, they will provide rich insight into the flavor and spin dependence of the transverse wave function.

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**CONCLUSIONS AND OUTLOOK**

In the “heavy pion world” presently accessible to full lattice QCD, we have calculated the lowest 3 generalized form factors $A_{n0}$ and $\bar{A}_{n0}$ up to $|t|=3$ GeV as shown in Fig. 1. We obtain excellent precision for $n=1$ and sufficient precision for $n=2$ and 3 to clearly distinguish each form factor and observe striking differences in slope and hence transverse size. Whereas there are other calculations of isolated moments, three moments are crucial for the present investigation since $n=1$ and 3 are necessary to measure the same combination of quark and antiquark distributions. The dependence of the transverse size on $x$ is most dramatic for the heaviest $u-d$ combination, for which $(r_{T}^{2})_{u-d}$ decreases by 62% between the first and third moment. We also observed clear dependence of the transverse distribution on flavor and spin. Our results show that the commonly used factorization Ansatz $H(x,0,t) = Q(x) F(t)$ is fundamentally wrong in the “heavy pion world” and we are aware of no arguments as to why it should be restored for lighter quarks.

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