Improving the bound for maximum degree on Murty-Simon Conjecture

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Abstract

A graph is said to be diameter-$k$-critical if its diameter is $k$ and removal of any of its edges increases its diameter. A beautiful conjecture by Murty and Simon, says that every diameter-2-critical graph of order $n$ has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges and equality holds only for $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. Haynes et al. proved that the conjecture is true for $\Delta \geq 0.7n$. They also proved that for $n > 2000$, if $\Delta \geq 0.6789n$ then the conjecture is true. We will improve this bound by showing that the conjecture is true for every $n$ if $\Delta \geq 0.6755n$.

1 Introduction

Throughout this paper we assume that $G$ is a simple graph. Our notation is the same as [3], let $G = (V, E)$ be a graph with vertex set $V$ of order $n$ and edge set $E$ of size $m$. For a vertex $v \in G$ we denote the set of its neighbors in $G$ by $N_G(u)$. Also we denote $N_G(u) \cup u$ by $N_G(u)$. The maximum and minimum degrees of $G$ will be denoted by $\Delta$ and $\delta$, respectively. The distance $d_G(u, v)$ between two vertices $u$ and $v$ of $G$, is the length of the shortest path between

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them. The diameter of $G$, $(diam(G))$, is the maximum distance among all pairs of vertices in $G$.

We say graph $G$ is diameter-$k$-critical if its diameter is $k$ and removal of any of its edges increases its diameter. Based on a conjecture proposed by Murty and Simon [5], there is an upper bound on the number of edges in a diameter-2-critical graph.

**Conjecture 1.1.** Let $G$ be a diameter-2-critical graph. Then $m \leq \lfloor n^2/4 \rfloor$ and equality holds only if $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$.

Several authors have conducted some studies on the conjecture proving acceptable results nearly close to the original one, however, no complete proof has been provided yet. Plesnik [6] showed that $m < \frac{3n(n-1)}{8}$. Moreover, Caccetta and Haggkvist [5] proved $m < \frac{n^2}{27}$. Fan [7] also proved the fact that for $n \leq 24$ and for $n = 26$ we have $m \leq \lfloor n^2/4 \rfloor$. For $n = 25$, he achieved $m < \frac{n^2}{4} + \frac{(n^2-16,2n+56)}{320} < 0.2532n^2$. Another proof was presented by Xu [8] in 1984, which was found out to have a small error. Afterwards, Furedi [9] provided a considerable result showing that the original conjecture is true for large $n$, that is, for $n > n_0$ where $n_0$ is a tower of 2s of height about $10^{14}$. This result is highly significant though not applicable to those graphs we are currently working with.

## 2 Total Domination

Domination number and Total domination number are parameters of graphs which are studied, respectively, in [2, 14] and [15]. Assume $G = (V, E)$ is a simple graph. Let $X$ and $Y$ be subsets of $V$; We say that $X$ dominates $Y$, written $X \succ Y$, if and only if every element of $Y - X$ has a neighbor in $X$. Similarly, we say that $X$ totally dominates $Y$, written $X \succ_t Y$ if and only if every element of $Y$ has a neighbor in $X$. If $X$ dominates or totally dominates $Y$, we might write, $X \succ G$ or $X \succ_t G$ instead of $X \succ V$ and $X \succ_t V$, respectively. Domination number and total domination number of $G = (V, E)$ are the size of smallest subset of $V$ that, dominates and totally dominates $V$, respectively. A graph $G$ with total domination number of $k$ is called $k_t$-critical, if every graph constructed by adding an edge between any nonadjacent vertices of $G$ has total domination number less than $k$. It is obvious that adding any edge to $k_t$-critical graph $G$ would result a graph which has total domination number of $k - 1$ or $k - 2$. Assume $G$ is $k_t$-critical graph. If for every pair of non adjacent vertices $\{u, v\}$ of $G$, the total domination number of $G + uv$ is $k - 2$, then $G$ is called $k_t$-supercritical. As shown in [4] there is a great connection between diameter-2-critical graphs and total domination critical graphs: **Theorem 2.1.** ([4]) A graph is diameter-2-critical if and only if its complement is $3_t$-critical or $4_t$-supercritical.

By this theorem in order to prove Murty-Simon conjecture, it suffices to prove that every graph which is $3_t$-critical, or $4_t$-critical , has at least $\lfloor n(n -
2)/4 | edges where \( n \) is order of graph. This problem is solved in some cases in [10] [11] [12]:

**Theorem 2.2.** ([10]) A graph \( G \) is 4\(_t\)-supercritical if and only if \( G \) is disjoint union of two nontrivial complete graphs.

**Theorem 2.3.** ([11]) If \( G \) is a 3\(_t\)-critical graph, then \( 2 \leq \text{diam}(G) \leq 3 \).

**Theorem 2.4.** ([12]) Every 3\(_t\)-critical graph of diameter 3 and order \( n \) has size \( m \geq n(n - 2)/4 \).

By this theorems a proof for following conjecture will show that Murty-Simon conjecture is true.

**Conjecture 2.5.** A 3\(_t\)-critical graph of order \( n \) and of diameter 2 has size \( m \geq n(n - 2)/4 \).

More recently Haynes et al. proved the following:

**Theorem 2.6.** ([13]) Let \( G \) be a 3\(_t\)-critical graph of order \( n \) and size \( m \). Let \( \delta = \delta(G) \). Then the following holds:

a) If \( \delta \geq 0.3n \), then \( m > \lceil n(n - 2)/4 \rceil \).

b) If \( n \geq 2000 \) and \( \delta \geq 0.321n \), then \( m > \lceil n(n - 2)/4 \rceil \).

Also G. Fan et al. proved that:

**Theorem 2.7.** ([7]) The Murty-Simon conjecture is true for every graph with less than 25 vertices.

In next section, in order to improve this bound, we will prove that, every simple diameter-2-critical graph of order \( n \) and size \( m \) satisfies \( m < \lfloor n^2/4 \rfloor \) if \( \Delta \geq 0.6756n \).

### 3 Main Result

In this section we will prove Murty-Simon conjecture for graphs which their complement are 3\(_t\)-critical and have less restriction on their minimum degree and improve the result proposed by Haynes et al in [13]. First we recall the following lemma, which was proposed in that paper.

**Lemma 3.1.** Let \( u \) and \( v \) are nonadjacent vertices in 3\(_t\)-critical graph \( G \), clearly \( \{u, v\} \not\in G \). Then there exists a vertex \( w \), such that \( w \) is adjacent to exactly one of \( u, v \), say \( u \), and \( \{u, w\} \not\in G - v \). We will call \( uw \) quasi-edge associated with \( uv \). Further \( v \) is the unique vertex not dominated by \( \{u, w\} \) in \( G \); In this case we call \( v \) supplement of \( \{u, w\} \).

**Definition 3.1.** Let \( G = (V, E) \) be a 3\(_t\)-critical graph. If \( S \subseteq V \) then we say that \( S \) is a quasi-clique if for each nonadjacent pair of vertices of \( S \) there exists a quasi-edge associated with that pair, and each quasi-edge associated with that pair at contains at least on vertex outside \( S \). Edges associated with quasi-clique \( S \) are the union of the edges with both ends in \( S \) and the quasi-edges associated with some pair of nonadjacent vertices of \( S \).
**Definition 3.2.** Let $G = (V, E)$ be a $3_t$-critical graph. Let $A$ and $B$ be two disjoint subsets of $V$. We define $E(G; A, B)$ as set of all edges $\{a, b\}$ where $a \in A$ and $b \in B$, and $\{a, b\}$ is associated with a non adjacent pair $\{a, c\}$, where $c$ is in $A$. By lemma 3.1, we know that every two members of $E(G; A, B)$ are associated with different non adjacent pairs.

**Lemma 3.2.** Let $G$ be a $3_t$-critical graph. Let $S \subset V(G)$, if $S^* = \cap_{s \in S} N(s)$, then the following holds:

$$|E(G[S^*])| + |E(G; S^*, V(G) - (S^* \cup S))| \geq \frac{|S^*|^2 - 2|S^*|}{c}$$

Where $c$ is the greatest root of $x^2 - 4x - 4 = 0$, which is equal to $2 + 2\sqrt{2} \approx 4.83$.

**Proof.** We apply induction on size of $S^*$ to prove the theorem. Note that for every pair of non-adjacent vertices in $S^*$ such as $\{u, v\}$, If $\{u, w\}$ is the quasi-edge associated to it, then, since $v$ is adjacent $u$, we can conclude that $w \notin S$. Note that when $|S^*| \leq 2$, since $\frac{|S^*|^2 - 2|S^*|}{c} \leq 0$, then the inequality is obviously true. Let $v$ be the vertex having minimum degree in $G[S^*]$. We denote the set of neighbors of $v$ in $S^*$ by $A$. Since every vertex in $S^* - (A \cup \{v\})$ is not adjacent to $v$, so $S^* - (A \cup \{v\})$ is a quasi-clique. Also $A$ is $\cap_{s \in S \cup \{v\}} N(s)$, so $|E(G[A])| + |E(G; A, V(G) - (A \cup S \cup \{v\}))| \geq \frac{|A|^2 - 2|A|}{c}$. For every pair of non-adjacent vertices $\{x, y\}$, one of them is the supplement of quasi-edge associated to this pair, so quasi-edges associated to non-adjacent pairs in $A$ and $S^* - (A \cup \{v\})$ are disjoint. With statements mentioned above we can conclude that:

$$|E(G[S^*])| + |E(G; S^*, V(G) - (S^* \cup S))| \geq \frac{|A|^2 - 2|A|}{c} + \left(\frac{|S^*| - |A| - 1}{2}\right) + |A|.$$

The right side of the inequality is a function of $|A|$, that we call it $f(|A|)$. One can find out that:

$$f'(|A|) = \frac{(c + 2)|A|}{c} + \left(\frac{5}{2} - \frac{2}{c}\right) - |S^*|$$

So $f'(|A|)$ has negative value whenever $0 \leq |A| \leq \frac{2|S^*| - 4}{c}$ and $|S^*| \geq 3$. So it suffices to prove that $f\left(\frac{2|S^*| - 4}{c}\right) \geq \frac{|S^*|^2 - 2|S^*|}{c}$, which is done by Lemma A.2. On the other hand when $|A| \geq \frac{2|S^*| - 4}{c}$ by definition of $A$, we can easily conclude that:

$$|E(G[S^*])| \geq \frac{|A||S^*|}{2} \geq \frac{|S^*|^2 - 2|S^*|}{c}.$$

\[\square\]

**Lemma 3.3.** Let $G = (V, E)$ be a $3_t$-critical graph. If $v \in V$, then $V - N_G[v]$ is a quasi-clique.
Proof. This lemma is generalized from a lemma in (13), in which $v$ was assumed as a vertex with minimum degree in $G$. Since the proof was independent of such assumption, the same proof is correct.

Now, we present the main result of this paper:

**Theorem 3.4.** Suppose that $c = 2 + \sqrt{2}$, and $a$ is the smallest root of the equation $(2c + 4)x^2 - 4cx + c = 0$, which is equal to $\frac{\sqrt{2} - \sqrt{2 - \sqrt{2}}}{2} \approx 0.32442$. Let $G(V, E)$ be a $3_t$-critical graph of order $n$, size $m$ and minimum degree $\delta$. If $n \geq 3$ and $\delta \leq an - 1$ then,

$$m > \left\lceil \frac{n(n - 2)}{4} \right\rceil$$

Proof. First, note that for every positive integer $n$:
- if $n$ is even $n(n - 2)$ is divisible by 4.
- if $n$ is odd $n(n - 2) + 1$ is divisible by 4.

So it suffices to prove that:

$$m > \frac{n(n - 2) + 1}{4}$$

Let $v \in V(G)$ be a vertex with $\delta$ neighbors and $A = N_G[v]$. Also let $B = V - N_G[v]$; then by Lemma 3.3, $B$ is a quasi-clique. Also by Lemma 3.2,

$$|E(G[A])| + |E(G; A, B)| \geq \frac{\delta^2 - 2\delta}{c}.$$  

$A$ and $B$ are disjoint, so the quasi-edges associated to non-adjacent pairs in $A$ are disjoint from the quasi-edges associated to non-adjacent pairs in $B$, because every quasi-edge has unique supplement. Therefore, we have:

$$m \geq \delta + \frac{\delta^2 - 2\delta}{c} + \left( \frac{n - 1 - \delta}{2} \right)$$

So by Lemma A.1 we have:

$$m > \frac{n(n - 2) + 1}{4}$$

Theorem 3.5. For every diameter-2-critical graph $G$ of order $n$ and size $m$, if $\Delta(G) \geq 0.6756n$, then $m < \left\lfloor \frac{n^2}{4} \right\rfloor$

Proof. Since $diam(G) = 2$, so $n \geq 3$. Let $\bar{G}$ be complement of $G$. Assume that size of $\bar{G}$ is $m'$. Since $m + m' = \binom{n}{2}$, so it suffices to prove that:

$$m' > \left\lfloor \frac{n(n - 2)}{4} \right\rfloor.$$
We have:
\[ \delta(\overline{G}) = n - 1 - \Delta(G) \leq 0.3244n - 1 \]

Note that by Theorem 2.1, \( \overline{G} \) is either 3\( t \)-critical or 4\( t \)-supercritical. If \( \overline{G} \) is 4\( t \)-supercritical, then by Theorem 2.2, \( \overline{G} \) is disjoint union of two non-trivial graphs and size of the smaller one is less than \( 0.3244n - 1 \), which means

\[ m' \geq \left( \frac{0.3244n - 1}{2} \right) + \left( \frac{0.6756n + 1}{2} \right) > \frac{n(n - 2)}{4}. \]

So we may consider that \( \overline{G} \) is 3\( t \)-critical, which is shown in Theorem 3.4.

\[ \square \]

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Proof of Inequalities

**Lemma A.1.** Suppose that $c = 2 + \sqrt{2}$, and $a$ is smaller root of the equation $(2c + 4)x^2 - 4cx + c = 0$, which is equal to $\sqrt{2} - \sqrt{2 - \sqrt{2}} \approx 0.3244$.

If $an - 1 \geq y \geq 0$ and $n \geq 3$ then:

$$y + \frac{y^2 - 2y}{c} + \left(\frac{n - 1 - y}{2}\right) > \frac{n(n - 2) + 1}{4}$$

**Proof.** Let $f(y) = y + \frac{y^2 - 2y}{c} + \left(\frac{n - 1 - y}{2}\right)$. We have:

$$f'(y) = 1 + \frac{2y - 2}{c} - n + y + \frac{3}{2}$$

$$= -n + \frac{5}{2} + \sqrt{2}y - \frac{2}{c}$$

$$< -n + \frac{5}{2} + \sqrt{2}(an - 1) - \frac{2}{c} < 0$$

Which means $f(y) \geq f(an - 1)$. Let $g(n) = f(an - 1) - \frac{n(n - 2) + 1}{4}$. Now it suffices to prove $g(n)$ has positive value for every $n \geq 3$.

$$g(n) = \frac{1}{4}((-8 + 7\sqrt{2} + 4\sqrt{4 - 2\sqrt{2} - 7\sqrt{2 - 2\sqrt{2}}})n + 6\sqrt{2} - 11)$$

So the coefficient of $n$ is positive and $g(3) \approx 0.025 > 0$, so we can conclude that $g(n)$ is positive when $n \geq 3$. 

**Lemma A.2.** Let $n \geq 3$ be a positive integer and $c = 2 + 2\sqrt{2}$, then

$$\frac{(2n - 4)^2 - 2(2n - 4)}{c} + \left(n - \frac{(2n - 4)}{2}\right) + \frac{(2n - 4)}{c} \geq \frac{n^2 - 2n}{c}.$$ 

**Proof.** We prove that $f(n) = \frac{(2n - 4)^2 - 2(2n - 4)}{c} + \left(n - \frac{(2n - 4)}{2}\right) + \frac{(2n - 4)}{c} - \frac{n^2 - 2n}{c}$ has positive value.

$$f(n) = \frac{1}{2}((3\sqrt{2} - 4)n + (8 - 6\sqrt{2})$$
\[
\frac{(3\sqrt{2} - 4)}{2}(n - 2) > 0
\]

So \( f(n) \) is positive for \( n \geq 3 \). \( \square \)