LIOUVILLE’S THEOREM FOR A FRACTIONAL ELLIPTIC SYSTEM

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Abstract. In this paper, we investigate the following fractional elliptic system

\[
\begin{align*}
(-\Delta)^{\alpha/2} u(x) &= f(x)v^q(x), & x \in \mathbb{R}^n, \\
(-\Delta)^{\beta/2} v(x) &= h(x)u^p(x), & x \in \mathbb{R}^n,
\end{align*}
\]

where \(1 \leq p, q < \infty\), \(0 < \alpha, \beta < 2\), \(f(x)\) and \(h(x)\) satisfy suitable conditions. Applying the method of moving planes, we prove monotonicity without any decay assumption at infinity. Furthermore, if \(\alpha = \beta\), a Liouville theorem is established.

1. Introduction. In recent years, the fractional Laplace equations have been frequently used to describe many science phenomena, such as the turbulence, water waves, anomalous diffusion and quasi-geostrophic flows (see [7], [16], [17], [33] and the references therein). It also has various applications in probability and finance (see [3], [5]).

The fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\) in \(\mathbb{R}^n\) is a nonlocal pseudo-differential operator with the form

\[
(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+\alpha}} \, dz
\]

\[
= C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x-z|^{n+\alpha}} \, dz,
\]

where \(0 < \alpha < 2\) and \(\text{PV}\) stands for the Cauchy principal value.

Let

\[
L_\alpha(\mathbb{R}^n) = \{ u \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} \, dx < \infty \}.
\]

It is easy to verify that for \(u(x) \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)\), the integral in (1) is well defined. In this paper, we consider the fractional Laplacian in this space \(C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)\).

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In the past few decades, the famous Lane-Emden equation

\[
\begin{cases}
-\Delta u(x) = u^p, & x \in \mathbb{R}^n, \\
\quad u(x) > 0, & x \in \mathbb{R}^n
\end{cases}
\]  

(2)

has been widely studied by many authors (see [8], [20], [27]). To the Lane-Emden system

\[
\begin{cases}
-\Delta u(x) = v^q(x), & x \in \mathbb{R}^n, \\
-\Delta v(x) = u^p(x), & x \in \mathbb{R}^n
\end{cases}
\]  

(3)

where \( p, q \geq 0 \), Serrin and Zou [29] derived existence result and confirmed the Lane-Emden conjecture in \( \mathbb{R}^3 \) on the assumptions that \((u(x), v(x))\) is bounded by polynomials at infinity.

Brandle [4] studied the properties of the positive solutions to fractional Lane-Emden equation

\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = u^p(x), \quad x \in \mathbb{R}^n
\]  

(4)

by using the \textit{extension method} introduced by Caffarelli and Silvestre [6], which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions, and then applied the \textit{method of moving planes} to show that nonexistence of solutions in the subcritical case.

**Theorem A.** ([4]) Assume that \( 1 \leq \alpha < 2 \) and \( 1 < p < \frac{n+\alpha}{n-\alpha} \). Then equation (4) possesses no bounded positive solution.

Existence and uniqueness of positive viscosity solutions to the fractional Lane-Emden system

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = v^q(x), & x \in \Omega, \\
(-\Delta)^{\frac{\beta}{2}} v(x) = u^p(x), & x \in \Omega, \\
\quad u(x) = v(x) = 0, & x \notin \Omega,
\end{cases}
\]  

(5)

with \( p, q > 0 \), \( pq \neq 1 \) and \( \frac{1}{p+1} + \frac{1}{q+1} > \frac{n+\alpha}{n} \), were established by Leite and Montenegro [22], where \( \Omega \subseteq \mathbb{R}^n \) is a smooth bounded domain. Quass and Xia [28] also considered (5) with \( \Omega = \mathbb{R}^n_+ \) and proved nonexistence of positive viscosity solutions under the condition \( 1 < p, q < \frac{n+2\alpha}{n-2\alpha} \).

Liouville theorems are very useful in studying semi-linear elliptic equations and systems. For example, they played an essential role in deriving a priori bounds for solutions in [2], [19] and [20], and were used to obtain uniqueness of solutions in [15], [21] and [24]. Recently, equations involving the fractional Laplacian have aroused general concern (see [1], [7], [14], [23], [36] and the references therein). The nonlocal nature of fractional operators brings many new difficulties comparing with the usual Laplacian. 2005, Chen, Li and Ou [12] introduced the method of moving planes in integral forms to the fractional Laplacian equations. Subsequently, Chen, Li and Li [10] developed a direct method of moving planes which can be used directly to the nonlocal operators. These methods have been applied to semi-linear differential equations, free boundary problems and other problems, and a series of fruitful results have been obtained (see [9], [18], [26], [34], [35], and the references therein).

In this paper, we consider the following fractional elliptic system

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = f(x)v^q(x), & x \in \mathbb{R}^n, \\
(-\Delta)^{\frac{\beta}{2}} v(x) = h(x)u^p(x), & x \in \mathbb{R}^n, \\
\quad u(x), v(x) > 0, & x \in \mathbb{R}^n
\end{cases}
\]  

(6)

where \( 1 \leq p, q < \infty, 0 < \alpha, \beta < 2 \). Since we do not assume any decay assumption on the solutions at infinity, we construct a function and combine the method of
moving planes to derive monotonicity. If $\alpha = \beta$, we obtain nonexistence of positive solutions to system (6).

The following is our main result:

**Theorem 1.1.** Assume that $0 < \alpha, \beta < 2$, and $1 \leq p, q < \infty$. Then the positive bounded solutions of system (6) in $(C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)) \times (C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\beta(\mathbb{R}^n))$ are monotone increasing along the $x_1$ direction under the following conditions

(i) $f(x) \sim o(\frac{1}{|x_1|^p})$, $h(x) \sim o(\frac{1}{|x_1|^p})$ as $x_1 \to -\infty$; $f(x) \to \infty$, $h(x) \to \infty$ as $x_1 \to +\infty$.

(ii) $\frac{\partial f(x)}{\partial x_1} > 0$, $\frac{\partial h(x)}{\partial x_1} > 0$, $f(x)$ and $h(x)$ are continuous in other variables;

(iii) $f(x) > 0$, $h(x) > 0$, $f(x) \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and $h(x) \in L^\infty_{\text{loc}}(\mathbb{R}^n)$.

**Theorem 1.2.** Assume that $f(x)$ and $h(x)$ satisfy the conditions in Theorem 1.1. If $\alpha = \beta$, then system (6) possesses no positive bounded solution in $(C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n)) \times (C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n))$.

The paper is organized as follows. In Section 2 we construct the function $g(x)$ and use the method of moving planes to show that every positive bounded solution to system (6) must be monotone increasing along the $x_1$ direction. The proof of Theorem 1.2 is given in Section 3.

Throughout the paper, we use $C$ to denote positive constants whose values may vary from line to line.

2. Monotonicity of positive solutions. In this section, we prove Theorem 1.1.

**Proof of Theorem 1.1.** We will show that every positive bounded solution to system (6) must be monotone increasing along the $x_1$ direction.

For $\lambda \in \mathbb{R}$, denote by

$$T_\lambda = \{x \in \mathbb{R}^n | x_1 = \lambda\}$$

the moving plane, by

$$\Sigma_\lambda = \{x \in \mathbb{R}^n | x_1 < \lambda\}$$

the left region of the plane $T_\lambda$, by

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$$

the reflection of $x$ about $T_\lambda$, and let

$$U_\lambda(x) = u_\lambda(x) - u(x)$$

and

$$V_\lambda(x) = v_\lambda(x) - v(x).$$

**Step 1.** Move the plane $T_\lambda$ from $-\infty$ to the right along the $x_1$-direction.

We claim that for $\lambda$ sufficiently negative,

$$U_\lambda(x) \geq 0, \ V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda. \quad (7)$$

In fact, it follows by the mean value theorem that

$$(-\Delta)^{\frac{1}{2}} U_\lambda(x) = f(x)\nu^\lambda_1(x) - f(x)\nu^\beta_1(x)$$

$$= (f(x^\lambda) - f(x))\nu^\beta_1(x) + f(x)[\nu^\beta_1(x) - \nu^{\beta_1}(x)] \quad (8)$$

$$\geq f(x)\eta_\lambda(x)1^{-1}(x)V_\lambda(x),$$

where $\eta_\lambda(x)$ is between $v_\lambda(x)$ and $v(x)$.

To prove (7), we apply a contradiction argument. Without loss of generality, suppose that $U_\lambda(x)$ is negative at some point in $\Sigma_\lambda$.

**Claim 1.** If either $U_\lambda(x) < 0$ or $V_\lambda(x) < 0$ at some points, then $U_\lambda(x) < 0$ and $V_\lambda(x) < 0$ at these points.
Proof. We only consider that $U_{\lambda}(x) < 0$ at some points and another situation is similarly treated. Notice that $u(x)$ and $v(x)$ do not have the decay condition near the infinity and the usual approaches for the case are to use the Kelvin transform or build an auxiliary function. Because system (6) contains $f(x)$ and $h(x)$, it is going to be complicated to use the Kelvin transform. So we construct a function

$$g(x) = |x - (\lambda + 1)e_1|^\sigma > 0,$$

where

$$\lambda \in R, \ x \in \Sigma_{\lambda}, \ e_1 = (1, 0, \cdots, 0),$$

and $\sigma$ is a small positive number to be chosen later.

Let

$$\bar{U}_\lambda(x) = \frac{U_\lambda(x)}{g(x)}.$$

Due to $g(x) = \infty$, it follows $\lim_{|x| \to \infty} \bar{U}_\lambda(x) = 0$. Obviously, $\bar{U}_\lambda(x)$ and $U_\lambda(x)$ have the same sign; also $U_\lambda(x) = 0, V_\lambda(x) = 0$ for $x \in T_\lambda$. There exists some $\bar{x} \in \Sigma_{\lambda}$ such that $\bar{U}_\lambda(x)$ achieves its negative minimum, i.e. $\bar{U}_\lambda(\bar{x}) = \min_{\Sigma_{\lambda}} \bar{U}_\lambda(x) < 0$. Therefore,

$$(-\Delta)^{\frac{\sigma}{2}} U_\lambda(\bar{x})$$

$$= CPV \int_{R^n} \frac{U_\lambda(x) - U_\lambda(y)}{|x - y|^{n+\alpha}} dy$$

$$= CPV \int_{\Sigma_{\lambda}} \frac{U_\lambda(x) - U_\lambda(y)}{|x - y|^{n+\alpha}} dy + CPV \int_{\Sigma_{\lambda}} \frac{U_\lambda(x) - U_\lambda(y^\lambda)}{|x - y^\lambda|^{n+\alpha}} dy$$

$$= CPV \int_{\Sigma_{\lambda}} \frac{(\bar{U}_\lambda g)(\bar{x}) - (\bar{U}_\lambda g)(y)}{|\bar{x} - y|^{n+\alpha}} dy + CPV \int_{\Sigma_{\lambda}} \frac{(\bar{U}_\lambda g)(\bar{x}) + (\bar{U}_\lambda g)(y)}{|\bar{x} - y|^{n+\alpha}} dy$$

$$= CPV \int_{\Sigma_{\lambda}} [(\bar{U}_\lambda(\bar{x}) - \bar{U}_\lambda(y))g(y) \left( \frac{1}{|\bar{x} - y|^{n+\alpha}} - \frac{1}{|\bar{x} - y^\lambda|^{n+\alpha}} \right) + 2\bar{U}_\lambda(\bar{x})g(\bar{x})] dy$$

$$+ CPV \int_{\Sigma_{\lambda}} \bar{U}_\lambda(\bar{x})(g(\bar{x}) - g(y)) \left( \frac{1}{|\bar{x} - y|^{n+\alpha}} - \frac{1}{|\bar{x} - y^\lambda|^{n+\alpha}} \right) dy$$

$$\leq C\bar{U}_\lambda(\bar{x}) \left[ \int_{\Sigma_{\lambda}} \frac{g(\bar{x})}{|\bar{x} - y^\lambda|^{n+\alpha}} dy + \int_{\Sigma_{\lambda}} \{ \frac{g(\bar{x})}{|\bar{x} - y^\lambda|^{n+\alpha}} + (g(\bar{x}) - g(y))(\frac{1}{|\bar{x} - y|^{n+\alpha}} - \frac{1}{|\bar{x} - y^\lambda|^{n+\alpha}}) \} dy \right]$$

$$:= C\bar{U}_\lambda(\bar{x})[I_1 + I_2].$$

An elementary calculus gives

$$I_1 \geq g(\bar{x}) \frac{1}{4} \int_{\sqrt{|\bar{x}| - |\lambda|}}^{\infty} \frac{|S_r|}{r^{1+\alpha}} dr = \frac{C_1 g(\bar{x})}{|\bar{x} - \lambda|^\alpha}$$

with a positive constant $C_1$ and $|S_r|$ being the $n - 1$ dimensional sphere with radius $r$. 

(9)
We split $I_2$ into three parts:

$$I_2 = \left(-\triangle\right)^{\frac{\sigma}{2}}g(\tilde{x}) - \int_{\Sigma_\lambda} \frac{g(\tilde{x}) - g(y)}{|\tilde{x} - y|^{n+\alpha}} dy + \int_{\Sigma_\lambda} \frac{g(y^\lambda)}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy$$

$$:= I_{21} - I_{22} + \int_{\Sigma_\lambda} \frac{g(y^\lambda)}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy. \quad (11)$$

To calculate $I_{21}$, we need a known lemma.

**Lemma 2.1.** ([13]) We have

$$\left(-\triangle\right)^{\frac{\sigma}{2}}(|x - a|^t) = C_t|x - a|^{t - \sigma},$$

where $C_t$ depends continuously on $t$,

$$C_t : \begin{cases} 
> 0, & \text{if } \alpha - n < t < 0; \\
= 0, & \text{if } t = 0 \text{ or } \alpha - n; \\
< 0, & \text{if } 0 < t < \alpha.
\end{cases}$$

Lemma 2.1 implies

$$I_{21} = \left(-\triangle\right)^{\frac{\sigma}{2}}(|\tilde{x} - (\lambda + 1)e_1|^\sigma)$$

$$= \text{CPV} \int_{R^n} \frac{|\tilde{x} - (\lambda + 1)e_1|^\sigma - |y - (\lambda + 1)e_1|^\sigma}{|\tilde{x} - y|^{n+\alpha}} dy$$

$$= C \int_{R^n} \frac{|\tilde{x} - (\lambda + 1)e_1|^\sigma - |y - (\lambda + 1)e_1|^\sigma}{|\tilde{x} - (\lambda + 1)e_1 - z|^n |\tilde{x} - (\lambda + 1)e_1|^{n+\alpha}} dz$$

$$= C|\tilde{x} - (\lambda + 1)e_1|^{\sigma - \alpha} \int_{R^n} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz$$

$$= C|\tilde{x} - (\lambda + 1)e_1|^{\sigma - \alpha} \int_{R^n} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz + \int_{B_{e_1}(1)} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz$$

$$= C|\tilde{x} - (\lambda + 1)e_1|^{\sigma - \alpha} \int_{B_{e_1}(1)} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz + \int_{B_{e_1}(1)} (1 - |z|^{-\sigma}) |z|^{\alpha - n} dz$$

$$= C|\tilde{x} - (\lambda + 1)e_1|^{\sigma - \alpha} \int_{B_{e_1}(1)} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz$$

$$:= -C_\sigma|\tilde{x} - (\lambda + 1)e_1|^{\sigma - \alpha}$$

$$= \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz = \int_{B_{e_1}(1)} \frac{1 - |z|^{-\sigma}}{|e_1 - z|^{n+\alpha}} dz$$

where we have $\int_{B_{e_1}(1)} \frac{1 - |z|^\sigma}{|e_1 - z|^{n+\alpha}} dz = \int_{B_{e_1}(1)} \frac{1 - |z|^{-\sigma}}{|e_1 - z|^{n+\alpha}} dz$ by the coordinate transformation. $C_\sigma > 0$ is as small as we wish for sufficiently small $\sigma$.

Next we evaluate $I_{22}$. Consider two regions

$$D_1 = \Sigma_{\lambda} \cap (|y| \leq K|\tilde{x}|) \text{ and } D_2 = \Sigma_{\lambda} \cap (|y| > K|\tilde{x}|), \text{ for some } K > 0.$$
In $D_1$, we have
\[ |g(\tilde{x}) - g(y)| \leq |\nabla g(\xi)||\tilde{x} - y| \]
where $\xi \in \Sigma_\lambda$ is some point on the line segment from $\tilde{x}$ to $y$ and $C_2$ is some positive constant depending only on $\tilde{x}$. Consequently,
\[
\left| \int_{D_1} \frac{g(\tilde{x}) - g(y)}{|\tilde{x} - y|^\alpha} dy \right| \\
\leq C_2 g(\tilde{x}) \left( \int_{D_1} \frac{1}{|\tilde{x} - y|^\alpha} dy \right) \\
\leq C_2 g(\tilde{x}) \left( \int_{B_{|\tilde{x} - \lambda|}(\tilde{x})} \frac{1}{r^\alpha} dr \right) \\
\leq C_2 g(\tilde{x}) \left( \frac{1}{|\tilde{x} - \lambda|^{\alpha+1}} \right) \\
\leq \frac{C_2 g(\tilde{x})}{|\tilde{x} - \lambda|^{\alpha}}.
\]

On $D_2$, since
\[ |g(\tilde{x}) - g(y)| \leq |g(\tilde{x})| + |g(y)| \leq C_3 |y|^\alpha \quad \text{and} \quad |\tilde{x} - y|^\alpha \geq |\tilde{x} - y| \sim |y|,
\]
we derive
\[
\left| \int_{D_2} \frac{g(\tilde{x}) - g(y)}{|\tilde{x} - y|^\alpha} dy \right| \leq C_3 \int_{D_2} \frac{|y|^\alpha}{|y|^\alpha} dy \leq \frac{C_3}{|K(\tilde{x})|^{\alpha-\sigma}} \leq \frac{C_3 g(\tilde{x})}{|\tilde{x} - \lambda|^{\alpha}},
\]
where $C_3$ is some positive constant depending only on $\tilde{x}$. The proof of the final inequality in (14) is as follows.

We only need to prove $|\tilde{x}|^{-\alpha} \geq |\tilde{x} - \lambda|^{\alpha}$. First, we always have $|\tilde{x} - (\lambda + 1)e_1| \geq |\tilde{x} - (\lambda + 1)| > |\tilde{x} - \lambda|$.

Next, let's consider two possibilities.

**Possibility 1.** $\lambda \leq 0$. So $|\tilde{x}| \geq |\tilde{x}_1| \geq |\tilde{x}_1 - \lambda|$.

**Possibility 2.** $\lambda > 0$. Since $0 < \lambda < \infty$ in the proof of the whole paper, we have $|\tilde{x}| \geq C_0|\tilde{x}_1 - \lambda|$, i.e. $|\tilde{x}|^{-\alpha} |\tilde{x} - (\lambda + 1)e_1|^{\alpha} \geq C_0|\tilde{x} - \lambda|^{\alpha}$, where $C_0$ is some positive constant depending only on $\tilde{x}$.

Combining (13) and (14),
\[
|I_{22}| \leq \left( C_2 + \frac{C_3}{|K^{\alpha-\sigma}|} \right) \frac{g(\tilde{x})}{|\tilde{x} - \lambda|^{\alpha}}.
\]

Hence
\[
I_2 \geq -C_2 g(\tilde{x}) \frac{1}{|\tilde{x} - \lambda|^{\alpha}} + \int_{\Sigma_\lambda} \frac{g(y)}{|x - y|^\alpha} dy - \left( C_2 + \frac{C_3}{|K^{\alpha-\sigma}|} \right) \frac{g(\tilde{x})}{|\tilde{x} - \lambda|^{\alpha}}.
\]

Taking into account of (10) and (16), we obtain from (9) that
\[
(-\Delta)^{\alpha/2} U_\lambda(\tilde{x}) \leq C U_\lambda(\tilde{x}) (I_1 + I_2) \\
\leq C U_\lambda(\tilde{x}) \left( C_1 - C_\sigma - C_2 - \frac{C_3}{|K^{\alpha-\sigma}|} \right) \frac{g(\tilde{x})}{|\tilde{x} - \lambda|^{\alpha}} + \int_{\Sigma_\lambda} C U_\lambda(\tilde{x}) g(y^\lambda) \frac{1}{|x - y|^\alpha} dy.
\]
\[
\leq C \bar{U}_\lambda(\bar{x}) \left( C_1 - C_\sigma - C_2\sigma - \frac{C_3}{K^{\alpha-\sigma}} \right) \frac{g(\bar{x})}{|\bar{x}_1 - \lambda|^\alpha}
\leq \frac{C U_\lambda(\bar{x})}{|\bar{x}_1 - \lambda|^\alpha} < 0,
\]
where we choose a large \( K \) and then a sufficiently small \( \sigma \) (hence \( C_\sigma \) becomes sufficiently small), such that \( C_1 - C_\sigma - C_2\sigma - \frac{C_3}{K^{\alpha-\sigma}} \) is a positive constant, which guarantees \( I_1 + I_2 > 0 \).

From (8) and (17), one has
\[
V_\lambda(\bar{x}) < 0
\]
and ends the proof of Claim 1.

Now we continue to prove (7). Let
\[
\bar{V}_\lambda(x) = \frac{V_\lambda(x)}{g(x)}.
\]
Obviously, \( \bar{V}_\lambda(x) \) and \( V_\lambda(x) \) have the same sign and \( \lim_{|x| \to \infty} \bar{V}_\lambda(x) = 0 \). So there exists \( \bar{x} \) such that
\[
\bar{V}_\lambda(\bar{x}) = \min_{\Sigma_\lambda} \bar{V}_\lambda(x) < 0.
\]
Similarly to (9), we conclude
\[
(\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}) \leq CV_\lambda(\bar{x})(J_1 + J_2),
\]
where
\[
J_1 = \int_{\Sigma_\lambda} \frac{g_\lambda(x)}{|x - y|^\alpha} dy,
J_2 = (\Delta)^{\frac{\beta}{2}} g(\bar{x}) - \int_{\Sigma_\lambda} \frac{g_\lambda(x) - g(y)}{|x - y|^\alpha} dy + \int_{\Sigma_\lambda} \frac{g(y)}{|x - y|^\alpha} dy.
\]
Since the estimates to \( J_1, J_2 \) are similar to \( I_1, I_2 \) respectively (just replace \( \alpha \) with \( \beta \)), it follows
\[
(\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}) \leq \frac{CV_\lambda(\bar{x})}{|x_1 - \lambda|^\beta}.
\]
By the mean value theorem, we derive
\[
(\Delta)^{\frac{\beta}{2}} V_\lambda(x) = (h(x^\lambda)u_\lambda^p(x) - h(x)u^p(x))
= (h(x^\lambda) - h(x))u_\lambda^p(x) + h(x)[u_\lambda^p(x) - u^p(x)]
\geq h(x)p\xi_\lambda^{p-1}(x)U_\lambda(x),
\]
where \( \xi_\lambda(x) \) is between \( u_\lambda(x) \) and \( u(x) \). From (8) and (17), it shows
\[
U_\lambda(x) \geq Cf(\bar{x})q\eta_\lambda^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha V_\lambda(\bar{x}),
\]
i.e.
\[
\bar{U}_\lambda(\bar{x}) \geq Cf(\bar{x})q\eta_\lambda^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha \bar{V}_\lambda(\bar{x}).
\]
Similarly, combining (21), (22) and (23), we have
\[
CV_\lambda(\bar{x}) \frac{g(\bar{x})}{|\bar{x}_1 - \lambda|^\beta} \geq h(\bar{x})p\xi_\lambda^{p-1}(\bar{x})U_\lambda(\bar{x})
\geq h(\bar{x})p\xi_\lambda^{p-1}(\bar{x})\bar{U}_\lambda(\bar{x})
\geq h(\bar{x})p\xi_\lambda^{p-1}(\bar{x})\bar{U}_\lambda(\bar{x})
\geq Ch(\bar{x})p\xi_\lambda^{p-1}(\bar{x})f(\bar{x})q\eta_\lambda^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha \bar{V}_\lambda(\bar{x})
\geq Ch(\bar{x})p\eta_\lambda^{p-1}(\bar{x})f(\bar{x})q\eta_\lambda^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha \bar{V}_\lambda(\bar{x}),
\]
then
\[
\bar{V}_\lambda(\bar{x}) \geq Ch(\bar{x})p\eta_\lambda^{p-1}(\bar{x})f(\bar{x})q\eta_\lambda^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha \bar{V}_\lambda(\bar{x}),
\]
which gives

\[ 1 \leq Ch(\bar{x})u^{p-1}(\bar{x})f(\bar{x})v^{q-1}(\bar{x})|\bar{x}_1 - \lambda|^\alpha|\bar{x}_1 - \lambda|^\beta. \]  

(25)

But this contradicts with the boundedness of \( u(x), v(x) \) and the condition (i). So we obtain

\[ U_\lambda(x) \geq 0, \ x \in \Sigma_\lambda. \]  

(26)

Let us prove \( V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda \). If \( V_\lambda(x) < 0 \) at some point, it implies from (21) and (22) that

\[ 0 \leq h(\bar{x})p\xi_\lambda^{p-1}(\bar{x})U_\lambda(\bar{x}) \leq (-\triangle)^{\frac{\beta}{2}}V_\lambda(\bar{x}) \leq \frac{CV_\lambda(\bar{x})}{|\bar{x}_1 - \lambda|^\beta} < 0, \]

which is a contradiction. So \( V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda \).

Now (7) is correct.

**Step 2.** Move the plane \( T_\lambda \) to the limiting position as long as (7) holds.

Letting

\[ \lambda_0 = \sup\{ \lambda \mid U_\mu(x), V_\mu(x) \geq 0, \ x \in \Sigma_\mu, \ \mu \leq \lambda \}, \]

we conclude

\[ \lambda_0 = \infty. \]  

(27)

If \( \lambda_0 < \infty \), then by the definition of \( \lambda_0 \), there exist two sequences \( \{\lambda_k\}_{k=1}^{\infty} \) and \( \{x^k\}_{k=1}^{\infty} \) satisfying

\[ \lambda_0 < \lambda_{k+1} < \lambda_k < +\infty, \ k = 1, 2, \ldots ; \ \lim_{k \to \infty} \lambda_k = \lambda_0; \]

\( x^k \in \Sigma_{\lambda_k} \) and either \( U_{\lambda_k}(x^k) < 0 \) or \( V_{\lambda_k}(x^k) < 0 \). Due to Claim 1, \( U_{\lambda_k}(x^k) \) and \( V_{\lambda_k}(x^k) \) are negative. We let \( \tilde{x}^k \) and \( \bar{x}^k \) be the negative minimum points of \( U_{\lambda_k} \) and \( V_{\lambda_k} \) in \( \Sigma_{\lambda_k} \) respectively, i.e.

\[ \tilde{U}_{\lambda_k}(\tilde{x}^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k}(x) < 0, \ \tilde{V}_{\lambda_k}(\tilde{x}^k) = \min_{\Sigma_{\lambda_k}} V_{\lambda_k}(x) < 0. \]

It implies by (24) that

\[ CV_{\lambda_k}(\tilde{x}^k) \frac{1}{|\tilde{x}^k_1 - \lambda_k|^\beta} \geq h(\tilde{x}^k)pu^{p-1}(\tilde{x}^k)f(\tilde{x}^k)qv^{q-1}(\tilde{x}^k)|\tilde{x}^k_1 - \lambda_k|^\alpha V_{\lambda_k}(\tilde{x}^k), \]

i.e.

\[ \frac{1}{|\tilde{x}^k_1 - \lambda_k|^\beta} \leq Ch(\tilde{x}^k)pu^{p-1}(\tilde{x}^k)f(\tilde{x}^k)qv^{q-1}(\tilde{x}^k)|\tilde{x}^k_1 - \lambda_k|^\alpha. \]  

(28)

Denote

\[ x = (x_1, x'), \ x^k = (x^k_1, x'^k), \ u_k(x) = u(x_1, x' - x'^k), \ v_k(x) = v(x_1, x' - x'^k), \]

\[ f_k(x) = f(x_1, x' - x'^k) \text{ and } h_k(x) = h(x_1, x' - x'^k), \]

then \((u_k(x), v_k(x))\) also satisfies (6). So we also have

\[ \frac{1}{|x^k_1 - \lambda_k|^\beta} \leq Ch_k(\tilde{x}^k)pu_k^{p-1}(\tilde{x}^k)f_k(\tilde{x}^k)qv_k^{q-1}(\tilde{x}^k)|\tilde{x}^k_1 - \lambda_k|^\alpha. \]  

(29)

Noticing that \( u_k(x), v_k(x), (-\triangle)^{\frac{\beta}{2}} u_k(x) \) and \( (-\triangle)^{\frac{\beta}{2}} v_k(x) \) are bounded, by the fractional Sobolev embedding [25] and regularity of solutions to fractional Laplace equations [11], one can derive that \( \{u_k(x)\} \) and \( \{v_k(x)\} \) are convergent. We will prove that there exist nonnegative functions \( \tilde{u}(x) \) and \( \tilde{v}(x) (\neq 0) \) such that as \( k \to \infty, \)

\[ u_k(x) \to \tilde{u}(x) \text{ and } (-\triangle)^{\frac{\beta}{2}} \tilde{u}(x) = f(x)\tilde{v}(x), \ x \in \mathbb{R}^n, \]  

(30)

and

\[ v_k(x) \to \tilde{v}(x) \text{ and } (-\triangle)^{\frac{\beta}{2}} \tilde{v}(x) = h(x)\tilde{u}(x), \ x \in \mathbb{R}^n. \]  

(31)
To verify (30) and (31), we need to establish a uniform $C^{0,\alpha+\theta}$ estimate for $u_k$ in a neighborhood of any point $x \in R^n$, which is independent of $k$ and $x$. This is done in two steps. We first obtain a $C^{\theta}$ estimate ( $0 < \theta < 1$ ), and then boost $C^{\theta}$ up to $C^{0,\alpha+\theta}$ by using the equation satisfied by $u_k(x)$. For convenience, we denote

$$C^{\theta,\gamma} := \begin{cases} C^{\gamma}, & \text{if } 0 < \gamma < 1, \\ C^{1,\gamma-1}, & \text{if } 1 < \gamma \leq 2. \end{cases}$$

We recall three known propositions.

**Proposition 1.** ([32]) Let $v_k \in C^{m,\theta}$ and suppose that $m + \theta - \alpha$ is not an integer. Then $(-\Delta)^{\alpha/2} v_k \in C^{l,\gamma}$, where $l$ is the integer part of $m + \theta - \alpha$ and $\gamma = m + \theta - \alpha - 1$.

**Proposition 2.** ([11]) Assume that $\alpha + \gamma$ ($0 < \gamma < 1$) is not an integer. If $f(x) \in C^{\gamma}(B_3)$, $u(x) \in L^{\infty}(R^n)$ solves

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), \quad x \in B_3,$$

then

$$\|u\|_{C^{\alpha,\alpha+\gamma}(B_1)} \leq C(\|f\|_{C^{\gamma}(B_3)} + \|u\|_{L^{\infty}(R^n)}).$$

**Proposition 3.** (30) For $f(x) \in L^{\infty}(B_3)$, let $u(x) \in H^{\frac{\alpha}{2}}(B_3) \cap L^{\infty}(R^n)$ be a weak solution to

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), \quad x \in B_3.$$

Then $u \in C^{\gamma}(B_1)$ for any $0 < \gamma < \min\{\alpha, 1\}$ and

$$\|u\|_{C^{\gamma}(B_1)} \leq C(\|f\|_{L^{\infty}(B_3)} + \|u\|_{L^{\infty}(R^n)}),$$

for a suitable $C > 0$ depending on $n$, $\alpha$ and $\gamma$.

Since $u(x)$ and $v(x)$ are positive bounded solutions to system (6), $f_k(x)$ and $h_k(x)$ are bounded, we have $|u^k(x)| \leq C, |v^k(x)| \leq C$, and

$$\|(-\Delta)^{\frac{\alpha}{2}} u_k(x)\| = \|f_k(x) v_k(x)\| \leq C,$$

$$\|(-\Delta)^{\frac{\alpha}{2}} v_k(x)\| = \|h_k(x) u_k(x)\| \leq C.$$

Hence for any $x^0 \in R^n$, $B_3(x^0) \subset R^n$, it implies by proposition 3 that

$$\|u_k(x)\|_{C^{\gamma}(B_3(x^0))} \leq C$$

and

$$\|v_k(x)\|_{C^{\gamma}(B_3(x^0))} \leq C,$$

where $C$ is independent of $k$ and $x^0$.

Let us first consider $\{v_k\}$. Due to the above uniform estimate (33), it follows from the Arzelà-Ascoli theorem that there exists a converging subsequence of $\{v_k\}$ in $B_1(0)$, denoted by $\{v_{1m}\}$. Then one can find a subsequence of $\{v_{1m}\}$, denoted by $\{v_{2m}\}$, that converges in $B_2(0)$, and then a subsequence of $\{v_{2m}\}$, denoted as $\{v_{3m}\}$, converging in $B_3(0)$. By induction, we get a chain of sub-sequences

$$\{v_{1m}\} \supset \{v_{2m}\} \supset \{v_{3m}\} \supset \ldots$$

such that $\{v_{jm}\}$ converges in $B_j(0)$ as $m \rightarrow \infty$. Taking the diagonal sequence $\{v_{jj}\}$, it seems that $\{v_{jj}\}$ converges at all points in any $B_R(0)$. Thus we have constructed a subsequence (still denoted by $\{v_k\}$) of solutions that converges point-wisely in $R^n$ to a function $\tilde{v}(x)$. From the condition (ii), it follows $f_k(x) \rightarrow f(x)$.

Similarly, we also have $u_k(x) \rightarrow \tilde{u}(x)$ and $h_k(x) \rightarrow h(x)$. 
Next we show that \((-\triangle)^{\beta/2}v_k(x)\) also converges point-wisely to \((-\triangle)^{\beta/2}v(x)\). To do so, we use the equation
\[
g_k = (-\triangle)^{\beta/2}v_k(x) = h_k(x)u_k^p(x) \tag{34}
\]
and derive from (32) that for any \(x^0 \in R^n\) and some \(0 < \theta < 1\),
\[
\|g_k\|_{C^{0,\theta}(B_1)} < C,
\]
where \(C\) is independent of \(k\) and \(x^0\). Now applying proposition 2 to the equation (34), we show that there exists a positive constant \(C\) independent of \(k\) and \(x^0\), such that
\[
\|v_k\|_{C^{0,\theta}(B_1)} \leq C. \tag{35}
\]
By the definition of \((-\triangle)^{\beta/2}\), we have
\[
(-\triangle)^{\beta/2}v_k(x) = \frac{C_{n,\beta}}{2} \int_{R^n} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \, dy
\]
\[
= \frac{C_{n,\beta}}{2} \left[ \int_{R^n \setminus B_1(0)} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \, dy + \int_{B_1(0)} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \, dy \right]
\]
\[
:= \frac{C_{n,\beta}}{2} (I_3 + I_4).
\]
It follows from (35) that for \(y \in B_1(0)\),
\[
\frac{|2v_k(x) - v_k(x+y) - v_k(x-y)|}{|y|^{n+\beta}} \leq C|y|^{\beta+\theta} = \frac{C}{|y|^{n-\theta}}
\]
and for \(y \in B_1^c(0)\),
\[
\frac{|2v_k(x) - v_k(x+y) - v_k(x-y)|}{|y|^{n+\beta}} \leq \frac{4C}{1 + |y|^{n+\beta}}.
\]
Therefore, for \(y \in R^n\),
\[
\frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \leq \frac{C}{1 + |y|^{n+\beta}} \left( 1 + \frac{1}{|y|^{n-\theta}} \right).
\]
Since
\[
|I_3 + I_4| \leq \int_{R^n} \frac{C}{1 + |y|^{n+\beta}} \left( 1 + \frac{1}{|y|^{n-\theta}} \right) \, dy < \infty,
\]
it yields from Lebesgue’s dominated convergence theorem that
\[
\lim_{k \to \infty} (-\triangle)^{\beta/2}v_k(x)
\]
\[
= \lim_{k \to \infty} \frac{C_{n,\beta}}{2} \int_{R^n} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \, dy
\]
\[
= \frac{C_{n,\beta}}{2} \int_{R^n} \lim_{k \to \infty} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\beta}} \, dy
\]
\[
= (-\triangle)^{\beta/2}v(x).
\]
Similarly, we obtain
\[
\lim_{k \to \infty} (-\triangle)^{\alpha/2}u_k(x) = (-\triangle)^{\alpha/2}u(x).
\]
This proves (30) and (31).

Let
\[ U^k(x) = u_k(x^\lambda) - u_k(x), \quad \tilde{U}_\lambda(x) = \tilde{u}(x^\lambda) - \tilde{u}(x), \]
and
\[ V^k(x) = v_k(x^\lambda) - v_k(x), \quad \tilde{V}_\lambda(x) = \tilde{v}(x^\lambda) - \tilde{v}(x), \]
then
\[ U^k(x) \to \tilde{U}_\lambda(x), \quad \tilde{V}_\lambda(x) \to \tilde{V}_\lambda(x), \quad \text{as} \ k \to \infty. \]

Due to boundedness of \( \{\tilde{x}^k_1\} \) and \( \{\tilde{x}^k_1\} \), there exists subsequences (still denoted by \( \{\tilde{x}^k_1\} \) and \( \{\tilde{x}^k_1\} \)) which converges to \( \tilde{x}^\infty_1, \tilde{x}^\infty_1 \) respectively. Hence
\[ \tilde{U}_{\lambda_0}(x^\infty_1, 0) = \lim_{k \to \infty} U^k_{\lambda_0}(\tilde{x}^k_1) \leq 0, \quad \tilde{V}_{\lambda_0}(x^\infty_1, 0) = \lim_{k \to \infty} V^k_{\lambda_k}(\tilde{x}^k_1) \leq 0. \quad (36) \]

On the other hand, we arrive at for each \( x \in \Sigma_{\lambda_0}, \)
\[ 0 \leq U^k_{\lambda_0}(x) \to \tilde{U}_{\lambda_0}(x), \quad 0 \leq V^k_{\lambda_0}(x) \to \tilde{V}_{\lambda_0}(x). \]

Furthermore, recall the equations
\[ (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) = f(x^\lambda)v^q(x) - f(x)v^q(x) \]
and
\[ (-\Delta)^{\frac{\alpha}{2}} V_\lambda(x) = h(x^\lambda)u^p(x) - h(x)u^p(x). \]

By the strong maximum principle, we obtain
\[ \tilde{U}_{\lambda_0}(x) = 0, \tilde{V}_{\lambda_0}(x) = 0, \quad x \in \Sigma_{\lambda_0} \quad (37) \]
or
\[ \tilde{U}_{\lambda_0}(x) > 0, \tilde{V}_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \quad (38) \]

If (37) holds, then \( f(x^\lambda) = f(x), h(x^\lambda) = h(x) \), which contradicts the assumption (ii). Therefore, (38) must be true. So \( \tilde{x}_1^\infty = \lambda_0, \tilde{x}_1^\infty = \lambda_0 \). It follows that
\[ |\tilde{x}^k_1 - \lambda_0| \to 0, \quad |\tilde{x}^k_1 - \lambda_k| \to 0, \quad \text{as} \ k \to \infty. \]

This contradicts (29) and it proves (27).

Now we have shown that every positive solution must be monotone increasing along the \( x_1 \) direction. So Theorem 1.1 is proved. \( \square \)

3. Liouville’s theorem. In this section, we prove Theorem 1.2. Its proof is based on the following Maximum Principle:

**Lemma 3.1.** ([10]) Let \( \Omega \) be a bounded open domain. Assume that \( u(x) \in C^{1,1}_{loc}(\Omega) \cap L_\alpha \) is a lower semi-continuous function in \( \Omega \) and satisfies
\[ \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) \geq 0, & x \in \Omega, \\ u(x) \geq 0, & x \in R^n \setminus \Omega. \end{cases} \]

Then
\[ u(x) \geq 0, \quad x \in R^n. \]

If \( u(x) = 0 \) at some point in \( \Omega \), then \( u(x) = 0 \) almost everywhere in \( R^n \).

**Proof of Theorem 1.2.** If \( \alpha = \beta \), the fractional elliptic system (6) becomes
\[ \begin{cases} (-\Delta)^{\alpha/2} u(x) = f(x)v^q(x), & x \in R^n, \\ (-\Delta)^{\alpha/2} v(x) = h(x)u^p(x), & x \in R^n, \\ u(x), v(x) > 0, & x \in R^n, \end{cases} \quad (39) \]

where \( 1 \leq p, q < \infty, \ 0 < \alpha < 2 \).
Denote by \( \varphi(x) \in H^2_0(B_1(x)) \) the first eigenfunction to \((-\triangle)^{\frac{2}{3}} \) in \( B_1(x) \), \( \lambda_1 \) is the first nonzero eigenvalue to the problem ([30])

\[
\begin{align*}
(-\triangle)^{\frac{2}{3}} \varphi(x) &= \lambda_1 \varphi(x), & x &\in B_1(x), \\
\varphi(x) &> 0, & x &\in B_1(x), \\
\varphi(x) &= 0, & x &\in B^c_1(x).
\end{align*}
\] (40)

By Proposition 4 in [31] we know that \( \varphi(x) \in L^\infty(B_1(x)) \). From Theorem 2 in [30] it is easily seen that \( \varphi(x) \in C(R^n) \). And then we know from the Theorem 1 in [30] that \( \varphi(x) \) is a viscosity solution of (40). Hence, by Theorem 2.5 and Theorem 2.6 in [28] we have that \( \varphi(x) \in L_\alpha \cap C^{1,1}_{lo} (B_1(x)) \).

Denoting \( x_0 = (R, 0, \cdots, 0) \in R^n \), let \( B_1(x_0) := \{ x \in R^n \mid ||x - x_0|| < 1 \} \) be the unit ball centered at \( x_0 \). Due to (i), \( u(x) > 0, v(x) > 0, f(x) > 0 \) and \( h(x) > 0 \) are monotone increasing along the \( x_1 \) direction, it follows

\[
f(x)v^{q-1}(x) \geq M_1, \quad h(x)w^{p-1}(x) \geq M_2, \quad x \in B_1(x_0).
\]

Taking \( R \) sufficiently large, \( M = \min \{ M_1, M_2 \} \) and \( M \geq \lambda_1 \), then

\[
(-\triangle)^{\frac{2}{3}} u(x) = f(x)v^{q-1}(x)v(x) \geq M v(x) \geq \lambda_1 v(x), \quad x \in B_1(x_0),
\]

and

\[
(-\triangle)^{\frac{2}{3}} v(x) = h(x)w^{p-1}(x)u(x) \geq M u(x) \geq \lambda_1 u(x), \quad x \in B_1(x_0).
\]

Denote \( A_1 = \max_{B_1(x_0)} \frac{\varphi(x)}{v(x)}, \ A_2 = \max_{B_1(x_0)} \frac{\varphi(x)}{u(x)}, \ D_1(x) = A_1 u(x) \in L_\alpha \cap C^{1,1}_{lo} (B_1(x_0)) \) and \( D_2(x) = A_2 v(x) \in L_\alpha \cap C^{1,1}_{lo} (B_1(x_0)) \). We have that in \( B_1(x_0) \),

\[
\frac{(-\triangle)^{\frac{2}{3}} D_1(x)}{\varphi(x)} = A_1 (-\triangle)^{\frac{2}{3}} u(x) = A_1 f(x)v^{q-1}(x)v(x) \geq \lambda_1 \max_{B_1(x_0)} \frac{\varphi(x)}{v(x)} v(x) \geq \lambda_1 \varphi(x),
\]

\[
\frac{(-\triangle)^{\frac{2}{3}} D_2(x)}{\varphi(x)} = A_2 (-\triangle)^{\frac{2}{3}} v(x) = A_2 h(x)w^{p-1}(x)u(x) \geq \lambda_1 \max_{B_1(x_0)} \frac{\varphi(x)}{u(x)} u(x) \geq \lambda_1 \varphi(x).
\]

Noting that \( D_1(x) \) satisfies

\[
\begin{align*}
(-\triangle)^{\frac{2}{3}} (D_1(x) - \varphi(x)) &\geq 0, & x &\in B_1(x_0), \\
D_1(x) - \varphi(x) &\geq 0, & x &\in B^c_1(x_0)
\end{align*}
\] (41)

and applying Lemma 3.1, it holds

\[
D_1(x) \geq \varphi(x), \quad x \in B_1(x_0).
\] (42)

Similarly, we obtain

\[
D_2(x) \geq \varphi(x), \quad x \in B_1(x_0).
\] (43)

Assume that \( x_1 \) is a maximum point of \( \frac{\varphi(x)}{v(x)} \), i.e. \( A_1 = \frac{\varphi(x_1)}{v(x_1)} \), and \( x_2 \) is a maximum point of \( \frac{\varphi(x)}{u(x)} \), i.e. \( A_2 = \frac{\varphi(x_2)}{u(x_2)} \). We can deduce from (42) that

\[
D_1(x) = A_1 u(x) = \max_{B_1(x_0)} \frac{\varphi(x)}{v(x)} v(x) = \frac{\varphi(x_1)}{v(x_1)} v(x) \geq \varphi(x).
\]

Thus

\[
\frac{\varphi(x_1)}{v(x_1)} \geq \frac{\varphi(x)}{u(x)}, \quad x \in B_1(x_0).
\]

Similarly, we have from (43) that

\[
D_2(x) = A_2 v(x) = \max_{B_1(x_0)} \frac{\varphi(x)}{u(x)} u(x) = \frac{\varphi(x_2)}{u(x_2)} u(x) \geq \varphi(x)
\]
and
\[
\frac{\varphi(x_2)}{u(x_2)} \geq \frac{\varphi(x)}{v(x)}, \quad x \in B_1(x_0).
\]
Hence
\[
A_1 = \frac{\varphi(x_1)}{v(x_1)} = \frac{\varphi(x_2)}{u(x_2)} = A_2.
\]

Using Lemma 3.1 to (42), we consider two possibilities: \(D_1(x) - \varphi(x) \equiv 0\) and \(D_1(x) > \varphi(x), \quad x \in B_1(x_0)\), and will show that they will not happen.

**Case 1.** If \(D_1(x) - \varphi(x) \equiv 0\) in \(B_1(x_0)\), then
\[
\max_{B_1(x_0)} \frac{\varphi(x)}{v(x)} u(x) \equiv \varphi(x)
\]
or
\[
A_1 u(x) = \varphi(x).
\]
Since \(\varphi(x) = 0\) on \(\partial B_1(x_0)\), \(u(x) > 0\) on \(\partial B_1(x_0)\), it is impossible.

**Case 2.** If \(D_1(x) > \varphi(x)\) in \(B_1(x_0)\), since \(A_1 = A_2\) at the maximum point \(x_2\) of \(\frac{\varphi(x_2)}{u(x_2)}\), we have
\[
D_1(x_2) = A_1 u(x_2) = A_2 u(x_2) = \varphi(x_2),
\]
which contradicts \(D_1(x) > \varphi(x)\).

So far, we claim that (42) is not valid. Similarly, one can prove that (43) is not also valid.

Therefore, system (6) possesses no bounded positive solution with \(\alpha = \beta\). This completes the proof of Theorem 1.2.

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