FIBERS OF PENCILS OF CURVES ON SMOOTH SURFACES

FRANCISCO MONSERRAT

Abstract. Let $X$ be a smooth projective surface such that linear and numerical equivalence of divisors on $X$ coincide and let $\sigma \subseteq |D|$ be a linear pencil on $X$ with integral general fibers. A fiber of $\sigma$ will be called special if either it is not integral or it has non-generic multiplicity at some of the base points (including the infinitely near ones) of the pencil. In this note we provide an algorithm to compute the integral components of the special fibers of $\sigma$.

1. Introduction

In this note we consider a (linear) pencil of curves without fixed components on a smooth projective surface $X$ over an algebraically closed field $k$ of arbitrary characteristic. Our goal is to provide an algorithm, that uses infinitely near points, for computing the integral (reduced and irreducible) components of the special fibers of the pencil, assuming that it has integral general fibers and that linear and numerical equivalence of divisors on $X$ coincide (condition which is satisfied, for instance, by projective smooth rational surfaces). The concept of special fiber we shall use will be defined later (in particular, the reducible fibers are special).

There are several results in the literature related with special fibers of a pencil. In [9, 1, 8, 10], bounds either on the number of reducible fibers or on the number of reducible fibers of a particular type are provided. The articles [6] and [4] are the starting point of a new motivation for the interest of completely reducible fibers of a pencil of projective plane curves (that is, unions of lines not necessarily reduced) due to their relationship with the theory of line arrangements. In [2], certain special fibers of pencils appear in the study of characteristic varieties of local systems on a plane curve arrangement complement.

We consider an effective divisor $D$ on $X$ and a linear system $\sigma \subseteq |D|$ without fixed components and with projective dimension 1 (a pencil on $X$, in the sequel). It corresponds to the projectivization of the sub-vector space $V_\sigma$ of $H^0(X, \mathcal{O}_X(D))$ given by \( \{ s \in H^0(X, \mathcal{O}_X(D)) \mid (s)_0 \in \sigma \} \cup \{ 0 \} \), (s)_0 denoting the divisor of zeros of the section $s$ (see [5 II.7]). A basis of $V_\sigma$ provides a rational map $f_\sigma : X \cdots \rightarrow \mathbb{P}_k^1$, which is independent from the basis up to composition with an automorphism of

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The closures of the fibers of \( f_\sigma \) are exactly the curves of the pencil \( \sigma \). Moreover, there exists a minimal composition of point blowing-ups \( \pi_\sigma : Z_\sigma \rightarrow X \) eliminating the indeterminacies of the rational map \( f_\sigma \), that is, the map \( h_\sigma := f_\sigma \circ \pi_\sigma : Z_\sigma \rightarrow \mathbb{P}^1_k \) is a morphism. We denote by \( BP(\sigma) \) to the set of centers of the blowing-ups used to get it; \( Z_\sigma \) is the so-called sky of \( BP(\sigma) \). The morphism \( \pi_\sigma \) and the set \( BP(\sigma) \) are essentially unique, since we identify those corresponding to \( X \)-isomorphic skies.

For each point \( p \in BP(\sigma) \), set \( E_p^* \) the total transform on \( Z_\sigma \) of the exceptional divisor created by the blowing-up at \( p \). Now, assume that \( C \) is a curve on \( X \). Then,

\[
\pi_\sigma^* C = \tilde{C} + \sum_{p \in BP(\sigma)} m_p(C)E_p^*,
\]

where \( \tilde{C} \) denotes the strict transform of \( C \) on \( Z_\sigma \) and \( m_p(C) \) is the multiplicity at \( p \) of the strict transform of \( C \) on the surface to which \( p \) belongs. The successive strict transforms of the fibers of the pencil \( \sigma \), except a finite number, have the same multiplicity at every point \( p \in BP(\sigma) \), that we shall denote by \( m_p(\sigma) \). The strict transforms on \( Z_\sigma \) of these (infinitely many) fibers belong to the complete linear system \( |G_\sigma| \), where \( G_\sigma := \pi_\sigma^* D - \sum_{p \in BP(\sigma)} m_p(\sigma)E_p^* \). If the pencil \( \sigma \) is irreducible (that is, with reduced and irreducible general fibers), we shall say that a fiber \( C \) of \( \sigma \) is special if either it is not integral or there exists \( p \in BP(\sigma) \) such that \( m_p(C) \neq m_p(\sigma) \).

2. Computation of the components of the special fibers

Consider, as above, a smooth projective surface \( X \) over \( k \) such that linear and numerical equivalence of divisors coincide. Fix a closed immersion \( i : X \rightarrow \mathbb{P}^1_k \). For each curve \( C \) on \( X \) (resp., its linear equivalence class \( \mathcal{L} \)), \( \deg C \) (resp., \( \deg \mathcal{L} \)) will denote the degree of \( C \) (resp., \( \mathcal{L} \)) with respect to the immersion \( i \) (that is, the intersection product \( i^*\mathcal{O}_{\mathbb{P}^1_k}(1) \cdot C \)). For each divisor \( D \) on a surface \( Z \), \([D]\) will denote its class in \( \text{Pic}(Z) \) (and also in \( \text{Pic}(Z) \otimes \mathbb{R} \) via the natural inclusion map). Next, we shall prove a statement that will support our method to compute the integral components of the special fibers of an irreducible pencil.

**Proposition 1.** Let \( \sigma \subseteq |D| \) be a pencil and \( C \) an integral curve, both on \( X \). Then, \( C \) is a component of a fiber of \( \sigma \) if, and only if, \( G_\sigma \cdot \tilde{C} = 0 \). Moreover, in this case, \( \tilde{C}^2 \leq 0 \).

*Proof.* If \( C \) is a component of a fiber of \( \sigma \), then \( \tilde{C} \) does not meet a general fiber of \( h_\sigma \) and so \( G_\sigma \cdot \tilde{C} = 0 \). Conversely, if \( C \) is not included in a fiber of \( \sigma \), then \( h_\sigma(\tilde{C}) = \mathbb{P}^1 \) and \( \tilde{C} \) meets properly each fiber; therefore \( G_\sigma \cdot \tilde{C} > 0 \). For the last assertion observe that, as a consequence of the Hodge Index Theorem \[ IV.1.9 \], the set of elements \( x \) of the real vector space \( \text{Pic}(Z_\sigma) \otimes \mathbb{R} \) such that \( x \cdot x \geq 0 \) and \([H] \cdot x \geq 0 \), where \( H \) is an ample divisor, (considering the bilinear pairing induced by the intersection product of divisors) is the half-cone over an Euclidean ball of dimension \( \rho(Z_\sigma) - 1 \), \( \rho(Z_\sigma) \)
being the Picard number of \( Z_\sigma \); the orthogonal hyperplane to \([G_\sigma]\) is tangent to that half-cone, since \( G_\sigma^2 = 0 \). Taking these facts into account, the second assertion follows from the first one and the strict convexity of the Euclidean ball. \( \square \)

Now, consider an irreducible pencil \( \sigma \subseteq |D| \) on \( X \). For each integer \( e \) such that \( 0 < e \leq \deg D \) let \( \Lambda(\sigma, e) \) be the set of pairs \( W = (\mathcal{L}, (v_p)_{p \in BP(\sigma)}) \), where \( \mathcal{L} \) is an effective class of Pic(\( X \)) of degree \( e \) and \( (v_p)_{p \in BP(\sigma)} \) is a sequence of non-negative integers, satisfying the following properties:

(a) \( v_p \leq e \) for all \( p \in BP(\sigma) \),
(b) \( v_p \geq \sum_q v_q \) for all \( p \in BP(\sigma) \), where the sum is taken over the set of points \( q \in BP(\sigma) \) which belong to the strict transform of the prime exceptional divisor associated with the blowing-up centered at \( p \),
(c) \( \mathcal{L}^2 \leq \sum_{p \in BP(\sigma)} v_p \),
(d) either \( K_X \cdot \mathcal{L} + \sum_{p \in BP(\sigma)} v_p \geq 0 \) and \( \mathcal{L}^2 + K_X \cdot \mathcal{L} + 2 \geq \sum_{p \in BP(\sigma)} v_p (v_p - 1) \), or \( \mathcal{L}^2 = \sum_{p \in BP(\sigma)} v_p^2 \) and \( K_X \cdot \mathcal{L} + \sum_{p \in BP(\sigma)} v_p = -2 \), or \( K_X \cdot \mathcal{L} + \sum_{p \in BP(\sigma)} v_p = \mathcal{L}^2 - \sum_{p \in BP(\sigma)} v_p^2 = -1 \), where \( K_X \) stands for the canonical class of \( X \),
(e) \( \mathcal{L} \cdot D = \sum_{p \in BP(\sigma)} v_p m_p(\sigma) \).

Taking into account the imposed conditions on the surface \( X \) and \[7, \text{Lecture 16}\] one has that there exist finitely many effective classes in Pic(\( X \)) of degree \( e \) and, therefore, \( \Lambda(\sigma, e) \) is finite.

**Algorithm 1.**

**Input:** \( BP(\sigma) \).

**Output:** The set of integral components of the special fibers of \( \sigma \).

**Begin**

Let \( \Xi_0 := \emptyset \)

Let \( e := 1 \)

While \( e \leq d \) do:

\( \Xi_e := \emptyset \)

For each \( W = (\mathcal{L}, (v_p)_{p \in BP(\sigma)}) \in \Lambda(\sigma, e) \) do:

If the complete linear system \( |\pi_\sigma^* \mathcal{L} - \sum_{p \in BP(\sigma)} v_p [E_p]| \) satisfies the following properties:

(1) it has projective dimension 0,
(2) it has no exceptional part,
(3) no curve in \( \bigcup_{0 \leq j < e} \Xi_j \) is a component of the curve \( C_W \) of \( X \) given by the direct image by \( \pi_\sigma \) of the unique element of the linear system,

then let \( \Xi_e := \Xi_e \cup \{C_W\} \)

Let \( e := e + 1 \)

Return the set \( \bigcup_{0 < j \leq e} \Xi_j \)

**End**
Observe that, to check if a pair \(W = (\mathcal{L}, (v_p)_{p \in BP(\sigma)})\) belonging to some set \(\Lambda(\sigma, e)\) satisfies the above properties (1) and (2) involves the knowledge of a basis of the space of global sections of \(\mathcal{L}\) and the resolution of a system of linear equations. Also, a possibility for checking the property (3) is to verify that, for all \(Q \in \bigcup_{0 \leq j < e} \Xi_j\), the linear system \(|\tilde{C}_W - \tilde{Q}|\) is empty (and, again, this can be done solving a system of linear equations if one knows a basis of \(H^0(X, \mathcal{O}_X(C_W - Q))\).

The next result justifies the correctness of the algorithm.

**Lemma 1.** For each integer \(e\) such that \(1 \leq e \leq \deg D\), \(\Xi_e\) is the set of integral components \(C\) of special fibers of \(\sigma\) such that \(\deg C = e\).

**Proof.** Assume that \(C\) is an integral component of degree \(e\) of a special fiber of \(\sigma\). Define \(\mathcal{L} = [C]\) and \(v_p := m_p(C)\) for all \(p \in BP(\sigma)\), and take \(W = (\mathcal{L}, (v_p)_{p \in BP(\sigma)})\). Let us see that \(W \in \Lambda(\sigma, e)\). The first property defining this set, (a), must be obviously satisfied by \(W\), since the integers \(v_p\) coincide with the multiplicities of the strict transforms of the curve \(C\). (b) means that the proximity inequalities [3 Chap. II, book 4] are satisfied for \(C\). (c) means that the self-intersection of \(\tilde{C}\) is non-positive, which is true by Proposition [1] (d) follows from the Adjunction Formula and (e) follows also from Proposition [1] We conclude, then, that \(W \in \Lambda(\sigma, e)\).

Taking into account again Proposition [1] each element of the linear system \(|\tilde{C}| = |\pi_* \mathcal{L} - \sum_{p \in BP(\sigma)} v_p [E_p]|\) is sum of strict transforms of integral components of special fibers of the pencil \(\sigma\) and (possibly) strict transforms of exceptional divisors. Hence, since the set of special fibers is finite, Property (1) in Algorithm [1] is satisfied for \(W\). (2) holds clearly, since \(\tilde{C}\) belongs to the linear system. Therefore \(C\) will be obtained as \(C_W\) in Algorithm [1].

Conversely, set \(C \in \Xi_e\) and let \(W \in \Lambda(\sigma, e)\) be such that \(C = C_W\). Due to (e), Proposition [1] and Properties (2) and (3) in Algorithm [1] it follows that \(C\) is an integral component of a fiber of the pencil \(\sigma\). But, taking into account (1), it is clear that this fiber must be special. \(\square\)

**Remark.** Notice that, from a basis of \(\sigma\) and the output of Algorithm [1] one can determine which obtained curves \(C_W\) are components of the same special fiber and, therefore, compute all the special fibers of the pencil.

**Example.** Consider the complex projective plane \(\mathbb{P}^2_\mathbb{C}\), with projective coordinates \((X : Y : Z)\). Let \(H\) be a hyperplane section and consider the pencil \(\sigma \subseteq [3H]\) spanned by the divisors given by the projective curves with equations \(F := 27X^3 - 27X^2Y + 9XY^2 - Y^3 - 8XZ^2 + 5YZ^2 = 0\) and \(G := X^3 + 6X^2Y + 12XY^2 + 8Y^3 - 7YZ^2 = 0\). \(BP(\sigma)\) consists of 9 points in \(\mathbb{P}^2_\mathbb{C}\) and \(m_p(\sigma) = 1\) for all \(p \in BP(\sigma)\). Therefore, \(G_\sigma = 3H* - \sum_{p \in BP(\sigma)} E_p^*\) and \(\sigma\) is irreducible (in other case, \(\sigma\) would be composite with an irreducible pencil, which contradicts the fact that \(m_p(\sigma) = 1\) for any point \(p \in BP(\sigma)\)). The conditions (a)-(e) given in the above definition
of the elements \( W = (\mathcal{L}, (v_p)_{p \in BP(\sigma)}) \) of \( \Lambda(\sigma, 1) \) (resp., \( \Lambda(\sigma, 2) \)) force \( C_W \) to be a line (resp., conic) passing through three (resp., six) points of \( BP(\sigma) \), and \( \Lambda(\sigma, 3) \) is empty. Using Algorithm 1 we can deduce that the integral components of the special fibers of \( \sigma \) are given by the following equations:

\[
\begin{align*}
L_1 &= 4X + Y = 0; \quad L_2 := 2X - 3Y = 0; \\
L_3 &= 2X - (7\sqrt{5} + 17)Y = 0; \quad L_4 := 2X + (7\sqrt{5} - 17)Y = 0; \\
C_1 &= (11\sqrt{5} - 15)Y^2 + 2(2\sqrt{5} - 25)XY + 4(10 + 9\sqrt{5})X^2 - 16\sqrt{5}Z^2 = 0; \\
C_2 &= (11\sqrt{5} + 15)Y^2 + 2(2\sqrt{5} + 25)XY + 4(9\sqrt{5} - 10)X^2 - 16\sqrt{5}Z^2 = 0; \\
C_3 &= 3Y^2 + 3XY + 13X^2 - 4Z^2 = 0; \quad C_4 := 7Y^2 - 7XY + 7X^2 - 2Z^2 = 0.
\end{align*}
\]

The special fibers of the pencil are those corresponding with the equations:

\[
\begin{align*}
F + G &= L_1C_4 = 0; \quad F - G = L_2C_3 = 0; \\
4\sqrt{5}F - 4(9\sqrt{5} + 20)G &= L_3C_2 = 0; \quad 4\sqrt{5}F - 4(9\sqrt{5} - 20)G = L_4C_1 = 0.
\end{align*}
\]

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Current address: Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Valencia (Spain)

E-mail address: framonde@mat.upv.es