MATHEMATICAL ANALYSIS OF AN AGE STRUCTURED HEROIN-COCAIN EPIDEMIC MODEL

ABDENNASSER CHEKROUN AND MOHAMMED NOR FRIOUI
Laboratoire d’Analyse Nonlinéaire et Mathématiques Appliquées
University of Tlemcen
Tlemcen 13000, Algeria

TOSHIKAZU KUNIYA*
Graduate School of System Informatics
Kobe University
1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan

TARIK MOHAMED TOUAOULA
Laboratoire d’Analyse Nonlinéaire et Mathématiques Appliquées
University of Tlemcen
Tlemcen 13000, Algeria

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Abstract. This paper is devoted to studying the dynamics of a certain age structured heroin-cocaine epidemic model. More precisely, this model takes into account the following unknown variables: susceptible individuals, heroin users, cocaine users and recovered individuals. Each one of these classes can change or interact with others. In this paper, firstly, we give some results on the existence, uniqueness and positivity of solutions. Next, we obtain a threshold value \( r(\Psi'[0]) \) such that an endemic equilibrium exists if \( r(\Psi'[0]) > 1 \). We then show that if \( r(\Psi'[0]) < 1 \), then the disease-free equilibrium is globally asymptotically stable, whereas if \( r(\Psi'[0]) > 1 \), then the system is uniformly persistent. Moreover, for \( r(\Psi'[0]) > 1 \), we show that the endemic equilibrium is globally asymptotically stable under an additional assumption that epidemic parameters for heroin users and cocaine users are same. Finally, some numerical simulations are presented to illustrate our theoretical results.

1. Introduction. The mathematical and statistical techniques applied to the modeling of infectious diseases had an extensive development in last decades. In 1927, Kermack and McKendrick \([15]\) applied the mathematical modeling to study the dynamics of human infectious disease transmission. Their model, as well as several modifications of it, has been widely investigated in \([1, 5, 6, 14, 27]\) and references therein by introducing the age dependence and in \([2, 8, 12, 17, 29–31, 33, 35]\) by considering the space and diffusion effects.

Recently, several mathematical models have been developed to describe the drug epidemics. In \([4]\), Burattini \textit{et al}. constructed an ODEs (ordinary differential equations) system to study the impact of the crack-cocaine use on the final prevalence

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* Corresponding author.
of HIV/AIDS in a community of drug users. In [32], White and Comiskey constructed an ODEs system to consider the heroin epidemics. They obtained the basic reproduction number $R_0$ for their model and performed the sensitivity analysis of it. Moreover, they showed that the backward bifurcation of the endemic equilibrium can occur at $R_0 = 1$. In [24], Mulone and Straughan showed the stability of the equilibria in the White-Comiskey model. In [25], Samanta modified the White-Comiskey model into a nonautonomous model with distributed time delay and obtained sufficient conditions for the permanence and the extinction of the epidemic. In [19], Liu and Zhang studied the global behavior of a heroin epidemic model with distributed time delay: the drug-free equilibrium is globally asymptotically stable if the basic reproduction number $R_0$ is less than 1, whereas a disease endemic equilibrium is locally asymptotically stable and the system is uniformly persistent if $R_0 > 1$. For other studies on the drug epidemic models with time delay, see, e.g., [9, 13, 20, 21].

The above models are systems of ODEs or RDEs (retarded differential equations). Recently, some authors have focused on the effect of the age that implies the time elapsed since the entering into a particular class. In [10] and [11], Fang et al. studied the global stability of heroin epidemic models with age-dependent susceptibility and treat-age, respectively. In [34], Yang et al. studied a heroin epidemic model with treat-age and nonlinear incidence rate. In [7], Djilali et al. considered a heroin epidemic model with treat-age and general nonlinear incidence rate. In [18], Liu and Liu studied the global behavior of a heroin epidemic model with age-dependent susceptibility and treat-age with relapse.

Most of the above studies focused on the epidemics of just one kind of drugs such as heroin. In this paper, we formulate a new mathematical model describing the dynamics of heroine and cocaine epidemics. The model enables us to consider the effect of the interaction between the epidemics of these two drugs. In our model, a susceptible to the use of drug is defined as an individual who is exposed to both of heroin and cocaine. The affected vector is defined as an individual using either one of these two drugs. We consider the dynamics in each class given by a system of age structured partial differential equations where the age structure means here the time spent by consuming heroin and cocaine. We obtain a threshold value allowing to indicate the global stability of each equilibrium: the disease-free equilibrium and the endemic equilibrium. More precisely, each situation correspond either for the eradication of the consummation of drug from the population or the persistence and spreading of it in the population. It is therefore of interest to explore the problem mathematically.

In this paper, we denote by $S$ the density of individuals in the susceptible class, $i_1$ the density of individuals using heroin, $i_2$ the density of individuals using cocaine and $R$ the density of recovered individuals. We assume that all newborns, given and noted by $A$ per unit time, are susceptible. Let $\mu$ be the per capita natural death rate per unit time. We model the contact or the rate of becoming a drug user by the law of mass action given by a transmission rate $\beta$. Let $\theta_1$ and $\theta_2$ be the per capita age-specific recovery rates from the consumption of heroin and cocaine, respectively. A part of them becomes new user of the other drug by a rate $k_1$ for the conversion from heroin to cocaine users and by a rate $k_2$ for the converse situation. The part $(1 - k_1)$ of heroin and $1 - k_2$ of cocaine users becomes totally cured. We design also by the rates $\delta_1$ and $\delta_2$ to describe the individuals who have relapsed into one of the two infected classes (see Table 1 and Figure 1). The model is given by,
for $t > 0$,

\[
\begin{align*}
\frac{dS(t)}{dt} &= A - \mu S(t) - S(t) \int_{0}^{\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 - S(t) \int_{0}^{\infty} \beta(\xi_2) i_2(t, \xi_2) d\xi_2, \\
\frac{\partial i_1(t, \xi_1)}{\partial t} + \frac{\partial i_1(t, \xi_1)}{\partial \xi_1} &= -(\mu + \theta_1(\xi_1)) i_1(t, \xi_1), \quad \xi_1 > 0, \\
\frac{\partial i_2(t, \xi_2)}{\partial t} + \frac{\partial i_2(t, \xi_2)}{\partial \xi_2} &= -(\mu + \theta_2(\xi_2)) i_2(t, \xi_2), \quad \xi_2 > 0, \\
\frac{dR(t)}{dt} &= (1 - k_1) \int_{0}^{\infty} \theta_1(\xi_1) i_1(t, \xi_1) d\xi_1 + (1 - k_2) \int_{0}^{\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 - (\mu + \delta_1 + \delta_2) R(t),
\end{align*}
\]

where $\xi_1$ is the time spent consuming heroin and $\xi_2$ is the time spent consuming cocaine. The boundary and initial conditions are given by, for $t > 0$,

\[
\begin{align*}
i_1(t, 0) &= S(t) \int_{0}^{\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 + k_2 \int_{0}^{\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 + \delta_1 R(t), \\
i_2(t, 0) &= S(t) \int_{0}^{\infty} \beta(\xi_2) i_2(t, \xi_2) d\xi_2 + k_1 \int_{0}^{\infty} \theta_1(\xi_1) i_1(t, \xi_1) d\xi_1 + \delta_2 R(t),
\end{align*}
\]

**Table 1.** Description of each symbol in model (1).

| Symbol | Description |
|--------|-------------|
| $S(t)$ | Density of susceptible individuals at time $t$ |
| $i_1(t, \xi_1)$ | Density of heroin users at time $t$ and age $\xi_1$ |
| $i_2(t, \xi_2)$ | Density of cocaine users at time $t$ and age $\xi_2$ |
| $R(t)$ | Density of recovered individuals at time $t$ |
| $A$ | Number of all newborns per unit time |
| $\mu$ | Natural death rate per capita and unit time |
| $\beta(\xi_i) (i = 1, 2)$ | Transmission rate for drug users with age $\xi_i$ $(i = 1, 2)$ |
| $\theta_1(\xi_1)$ | Recovery rate from the consumption of heroin at age $\xi_1$ |
| $\theta_2(\xi_2)$ | Recovery rate from the consumption of cocaine at age $\xi_2$ |
| $k_1$ | Rate at which an individual recovered from the consumption of heroin becomes a cocaine user |
| $k_2$ | Rate at which an individual recovered from the consumption of cocaine becomes a heroin user |
| $\delta_1$ | Rate at which a recovered individual becomes a heroin user |
| $\delta_2$ | Rate at which a recovered individual becomes a cocaine user |

**Figure 1.** Transfer diagram for model (1).
and
\[
\begin{align*}
& i_1(0, \cdot) = i_{1,0}(\cdot) \in L^1(\mathbb{R}^+, \mathbb{R}^+), \quad i_2(0, \cdot) = i_{2,0}(\cdot) \in L^1(\mathbb{R}^+, \mathbb{R}^+), \\
& S(0) = S_0 \in \mathbb{R}^+ \quad \text{and} \quad R(0) = R_0 \in \mathbb{R}^+.
\end{align*}
\]

(3)

For the above system, we make the following assumptions,

1. \( A > 0, \mu > 0, \delta_1 > 0 \) and \( \delta_2 > 0 \).
2. \( 0 < k_1 < 1 \) and \( 0 < k_2 < 1 \).
3. \( \beta(\cdot) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+), \theta_1(\cdot) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+) \) and \( \theta_2(\cdot) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+) \).
4. \( \beta(\cdot), \theta_1(\cdot) \) and \( \theta_2(\cdot) \) are Lipschitz continuous on \( \mathbb{R}^+ \), that is, there exist positive Lipschitz constants \( L_\beta, L_{\theta_1} \) and \( L_{\theta_2} \) such that

\[
|\beta(t + h) - \beta(t)| \leq L_\beta h, \quad |\theta_j(t + h) - \theta_j(t)| \leq L_{\theta_j} h, \quad j = 1, 2,
\]

hold for all \( t \geq 0 \) and \( h > 0 \).
5. There exist nonempty intervals \( I_i \subset \mathbb{R}^+ \), \( i = 1, 2 \), such that \( \theta_i(\cdot) > 0 \) on \( I_i \), \( i = 1, 2 \).

In this paper, we will be interested in analyzing the dynamics of model. In the next section 2, we formulate the main model as an abstract equation and show the existence and uniqueness of solutions. Section 3 is devoted to the study of equilibria. Our main result in this section shows that there exists a threshold value that guarantees the existence of an endemic equilibrium. In Section 4, we prove the existence of a compact attractor. In Section 5, we prove the global stability of the disease-free equilibrium when the threshold value is less than 1. In Section 6, we prove the uniform persistence of the system when the threshold value is greater than 1. In Section 7, in a special case that \( \theta_1(\cdot) = \theta_2(\cdot) \) and \( k_1 = k_2 \), we prove the global stability of the endemic equilibrium when the threshold value is greater than 1. Finally, in Section 8, some numerical simulations are presented to illustrate our theoretical results.

2. Existence and uniqueness of the solution. To prove the existence and uniqueness of the solution of system (1), we use the approach in [22, 28]. Let \( X := L^1(\mathbb{R}^+, \mathbb{R}) \) and let \( X^+ := L^1(\mathbb{R}^+, \mathbb{R}^+) \) be the positive cone of \( X \). Let \( Y := \mathbb{R} \times X \times X \times \mathbb{R} 

Let \( (\lambda I_d - \mathcal{A})^{-1} \) denote the resolvent set of an operator, and \( \mathbb{R} \) denotes the identity operator.

**Lemma 2.1.** \( \mathcal{A} \) is a Hille-Yosida operator, that is, there exist real constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( (\omega, +\infty) \subset \rho(\mathcal{A}) \), and

\[
\left\| (\lambda I_d - \mathcal{A})^{-n} \right\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N}^* = \{1, 2, \cdots\}, \quad \lambda > \omega,
\]

where \( \rho(\cdot) \) denotes the resolvent set of an operator, and \( I_d \) denotes the identity operator.

**Proof.** Let \( (\lambda I_d - \mathcal{A})^{-1} (r_1, r_2, \ell_1(\cdot), r_3, \ell_2(\cdot), r_4)^T = (\varphi, 0, \phi_1(\cdot), 0, \phi_2(\cdot), \psi)^T \). We then have

\[
\varphi = \frac{r_1}{\lambda + \mu}, \quad \phi_1(\xi_1) = r_2 e^{-\int_0^{\xi_1} (\lambda + \mu + \theta_1(\rho))d\rho} + \int_0^{\xi_1} \ell_1(\sigma) e^{-\int_1^{\xi_1} (\lambda + \mu + \theta_1(\rho))d\rho} d\sigma,
\]

\[
\psi = \frac{r_4}{\lambda + \mu + \delta_1 + \delta_2}, \quad \phi_2(\xi_2) = r_3 e^{-\int_0^{\xi_2} (\lambda + \mu + \theta_2(\rho))d\rho} + \int_0^{\xi_2} \ell_2(\sigma) e^{-\int_2^{\xi_2} (\lambda + \mu + \theta_2(\rho))d\rho} d\sigma.
\]

Hence, we have, for \( \lambda > -\mu \),

\[
\left\| (\lambda I_d - \mathcal{A})^{-1} (r_1, r_2, \ell_1(\cdot), r_3, \ell_2(\cdot), r_4)^T \right\|_W
\]

\[
= \left| r_1 \right| + \left| \psi \right| + \int_0^{+\infty} \left| \phi_1(\xi_1) \right| d\xi_1 + \int_0^{+\infty} \left| \phi_2(\xi_2) \right| d\xi_2,
\]

\[
\leq \frac{\left| r_1 \right|}{\lambda + \mu} + \frac{\left| r_2 \right|}{\lambda + \mu} + \left| r_2 \right| \int_0^{+\infty} e^{-(\lambda + \mu)(\xi_1 - 0)} d\xi_1 + \left| \ell_1(\sigma) \right| \int_0^{+\infty} e^{-(\lambda + \mu)(\xi_1 - 0)} d\xi_1 d\sigma
\]

\[
+ \left| r_3 \right| \int_0^{+\infty} e^{-(\lambda + \mu)(\xi_2 - 0)} d\xi_2 + \int_0^{+\infty} \left| \ell_2(\sigma) \right| \int_0^{+\infty} e^{-(\lambda + \mu)(\xi_2 - 0)} d\xi_2 d\sigma,
\]

\[
\leq \frac{\left| r_1, r_2, \ell_1(\cdot), r_3, \ell_2(\cdot), r_4 \right|}{\lambda + \mu}.
\]

This implies that \( (-\mu, +\infty) \subset \rho(\mathcal{A}) \) and, for \( \lambda > -\mu \),

\[
\left\| (\lambda I_d - \mathcal{A})^{-1} \right\| \leq \frac{1}{\lambda + \mu} \quad \text{and hence},
\]

\[
\left\| (\lambda I_d - \mathcal{A})^{-n} \right\| \leq \left\| (\lambda I_d - \mathcal{A})^{-1} \right\|^n \leq \frac{1}{(\lambda + \mu)^n}, \quad n \in \mathbb{N}^*.
\]

This completes the proof with \( M = 1 \) and \( \omega = -\mu \).
Lemma 2.2. For any $C > 0$, $\mathcal{F}$ is Lipschitz continuous on bounded set $\overline{B}_C(0) := \{ y \in Y_0 : \|y\|_W \leq C \}$. That is, there exists a positive constant $K(C) > 0$ such that $\| \mathcal{F}(y) - \mathcal{F}(\tilde{y}) \|_W \leq K(C) \| y - \tilde{y} \|_W$ holds for all $y, \tilde{y} \in \overline{B}_C(0)$.

Proof. Let $y = (\varphi, 0, \phi_1(\cdot), 0, \phi_2(\cdot), \psi)^T \in \overline{B}_C(0)$ and $\tilde{y} = (\tilde{\varphi}, 0, \tilde{\phi}_1(\cdot), 0, \tilde{\phi}_2(\cdot), \tilde{\psi})^T \in \overline{B}_C(0)$. We then have

$$
\| \mathcal{F}(y) - \mathcal{F}(\tilde{y}) \|_W = \left| -\varphi \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 - \varphi \int_0^{+\infty} \beta(\xi_2)\phi_2(\xi_2)d\xi_2 
+ \tilde{\varphi} \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 + \tilde{\varphi} \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 
+ \varphi \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + k_2 \int_0^{+\infty} \theta_2(\xi_2)\phi_2(\xi_2)d\xi_2 + \delta_1 \psi 
- \tilde{\varphi} \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 - k_2 \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 - \delta_1 \tilde{\psi} 
+ \varphi \int_0^{+\infty} \beta(\xi_2)\phi_2(\xi_2)d\xi_2 + k_1 \int_0^{+\infty} \theta_1(\xi_1)\phi_1(\xi_1)d\xi_1 + \delta_2 \psi 
- \tilde{\varphi} \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 - k_1 \int_0^{+\infty} \theta_1(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 - \delta_2 \tilde{\psi} 
+ (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1)\phi_1(\xi_1)d\xi_1 + (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2)\phi_2(\xi_2)d\xi_2 
- (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 - (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 \right|. \tag{6}
$$

Let

$$
\beta^\infty := \operatorname{esssup}_{\sigma \geq 0} \beta(\sigma), \quad \theta_1^\infty := \operatorname{esssup}_{\sigma \geq 0} \theta_1(\sigma), \quad \theta_2^\infty := \operatorname{esssup}_{\sigma \geq 0} \theta_2(\sigma).
$$

We then have from (6) that

$$
\| \mathcal{F}(y) - \mathcal{F}(\tilde{y}) \|_W \leq \beta^\infty C \left( 2 \| \varphi - \tilde{\varphi} \| + \int_0^{+\infty} |\phi_1(\xi_1) - \tilde{\phi}_1(\xi_1)|d\xi_1 + \int_0^{+\infty} |\phi_2(\xi_2) - \tilde{\phi}_2(\xi_2)|d\xi_2 \right) 
+ \beta^\infty C \left( |\varphi - \tilde{\varphi}| + \int_0^{+\infty} |\phi_1(\xi_1) - \tilde{\phi}_1(\xi_1)|d\xi_1 \right) + k_2 \theta_2^\infty \int_0^{+\infty} |\phi_2(\xi_2) - \tilde{\phi}_2(\xi_2)|d\xi_2 
+ \delta_1 |\psi - \tilde{\psi}| 
+ \beta^\infty C \left( |\varphi - \tilde{\varphi}| + \int_0^{+\infty} |\phi_2(\xi_2) - \tilde{\phi}_2(\xi_2)|d\xi_2 \right) + k_1 \theta_1^\infty \int_0^{+\infty} |\phi_1(\xi_1) - \tilde{\phi}_1(\xi_1)|d\xi_1 
+ \delta_2 |\psi - \tilde{\psi}| 
+ (1 - k_1) \theta_1^\infty \int_0^{+\infty} |\phi_1(\xi_1) - \tilde{\phi}_1(\xi_1)|d\xi_1 + (1 - k_2) \theta_2^\infty \int_0^{+\infty} |\phi_2(\xi_2) - \tilde{\phi}_2(\xi_2)|d\xi_2,
$$

and hence,

$$
\| \mathcal{F}(y) - \mathcal{F}(\tilde{y}) \|_W \leq K(C) \| y - \tilde{y} \|_W,
$$

where

$$
K(C) := \max \{ 4\beta^\infty C, \ 2\beta^\infty C + \theta_1^\infty, \ 2\beta^\infty C + \theta_2^\infty, \ \delta_1 + \delta_2 \} > 0.
$$

This completes the proof. \qed
Using Lemmas 2.1 and 2.2, we prove the following proposition on the existence and uniqueness of a local integral solution of system (1).

**Proposition 1.** For any $C > 0$, there exists a $T_C > 0$ such that system (1) has the unique integral solution $u \in C([0, T_C], Y_0^+)$ satisfying

$$u(t) = u_0 + A \int_0^t u(\sigma)d\sigma + \int_0^t F(u(\sigma))d\sigma, \quad t \in [0, T_C],$$

provided $u_0 \in B_C(0) \cap Y_0^+$.

**Proof.** By Lemmas 2.1 and 2.2, we can apply [22, Lemma 3.1 and Proposition 3.2] to obtain that system (1) has the unique integral solution $u \in C([0, T_C], Y_0)$. To show its positivity, it is sufficient from [22, Proposition 3.6] to show the following two properties.

(i): $(\lambda I - A)^{-1}W^+ \subset W^+$ for any $\lambda > -\mu$;

(ii): there exists a $\gamma_C > 0$ such that $F(u_0) - \mu u_0 + \gamma_C u_0 \in W^+$ for any $u_0 \in B_C(0) \cap Y_0^+$,

where $W := \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. (i) obviously holds from (5). We can easily see that (ii) holds by choosing $\gamma_C := \mu + 2\beta^\infty C$. This completes the proof.

To prove the existence and uniqueness of the global classical solution of system (1), we need the following lemma.

**Lemma 2.3.** $\mathcal{F} : Y_0 \to W$ is continuously differentiable. That is, for any $y \in Y_0$, there exists a bounded linear operator $\mathcal{F}'[y] : Y_0 \to W$ such that

$$\lim_{\|y\|_W \to 0} \frac{\|\mathcal{F}(y + \tilde{y}) - \mathcal{F}(y) - \mathcal{F}'[y]\tilde{y}\|_W}{\|\tilde{y}\|_W} = 0.$$

**Proof.** For any $y := (\varphi, 0, \phi_1(\cdot), 0, \phi_2(\cdot), \psi)^T \in Y_0$, let us define the linear operator,

$$\mathcal{F}'[y] := \begin{pmatrix} \mathcal{F}_1[y]\tilde{y} \\ \mathcal{F}_2[y]\tilde{y} \\ 0 \\ \mathcal{F}_3[y]\tilde{y} \\ \mathcal{F}_4[y]\tilde{y} \end{pmatrix}, \quad \tilde{y} := \begin{pmatrix} \tilde{\varphi} \\ 0 \\ \tilde{\phi}_1(\cdot) \\ 0 \\ \tilde{\phi}_2(\cdot) \\ \psi \end{pmatrix} \in Y_0,$

where

$$\mathcal{F}_1[y]\tilde{y} := -\varphi \left( \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + \int_0^{+\infty} \beta(\xi_2)\phi_2(\xi_2)d\xi_2 \right)$$

$$-\varphi \left( \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 + \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 \right),$$

$$\mathcal{F}_2[y]\tilde{y} := \tilde{\varphi} \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + \varphi \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1$$

$$+ k_2 \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 + \delta_1 \tilde{\psi},$$

$$\mathcal{F}_3[y]\tilde{y} := -\varphi \left( \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + \int_0^{+\infty} \beta(\xi_2)\phi_2(\xi_2)d\xi_2 \right)$$

$$-\varphi \left( \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1 + \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 \right),$$

$$\mathcal{F}_4[y]\tilde{y} := \tilde{\varphi} \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + \varphi \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1$$

$$+ k_2 \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 + \delta_1 \tilde{\psi},$$

$$\mathcal{F}_5[y]\tilde{y} := \tilde{\varphi} \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 + \varphi \int_0^{+\infty} \beta(\xi_1)\tilde{\phi}_1(\xi_1)d\xi_1$$

$$+ k_2 \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 + \delta_1 \tilde{\psi}.$$
\[ F'_t[\hat{y}] := \hat{\phi} \int_0^{+\infty} \beta(\xi_2)\phi_2(\xi_2)d\xi_2 + \varphi \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 + k_1 \int_0^{+\infty} \theta_1(\xi_1)\hat{\phi}_1(\xi_1)d\xi_1 + \delta_2\tilde{\psi}, \]

\[ F'_t[\hat{y}] := (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1)\hat{\phi}_1(\xi_1)d\xi_1 + (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2. \]

We then have

\[
\lim_{\|\hat{y}\|_W \to 0} \frac{\|F(y + \hat{y}) - F(y) - F'_t[\hat{y}]\|_W}{\|\hat{y}\|_W} = \lim_{\|\hat{y}\|_W \to 0} \frac{1}{\|\hat{y}\|_W} \left( \left\| \begin{pmatrix} -\hat{\phi} \int_0^{+\infty} \beta(\xi_1)\hat{\phi}_1(\xi_1)d\xi_1 - \hat{\phi} \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 \\ \hat{\phi} \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 \\ 0 \\ \hat{\phi} \int_0^{+\infty} \beta(\xi_2)\tilde{\phi}_2(\xi_2)d\xi_2 \\ \hat{\phi} \int_0^{+\infty} \beta(\xi_1)\phi_1(\xi_1)d\xi_1 \end{pmatrix} \right\|_W \right) + \left\| \begin{pmatrix} 0 \\ 0 \\ 2\beta^2|\hat{\phi}| \left( \int_0^{+\infty} |\hat{\phi}_1(\xi_1)|d\xi_1 + \int_0^{+\infty} |\tilde{\phi}_2(\xi_2)|d\xi_2 \right) \right\|_W.
\]

It is easy to see that \( F'_t[y] \) is a bounded operator for any fixed \( y \). Thus, \( F \) is continuously differentiable. This completes the proof.

**Proof.** Let \( \hat{C} := \|u_0\|_W \). Then, by Proposition 1 and Lemma 2.3, we can apply [22, Theorem 4.4] (see also [28, Theorem 3.7]) to obtain that there exists a \( T_0^+ > 0 \) such that system (1) has the unique classical solution \( u \in C^1([0, +\infty), Y^+_0) \cap C([0, +\infty), D(A)) \) satisfying (4).

**Theorem 2.4.** Assume that \( u_0 \in D(A) \cap Y^+_0 \) and \( Au_0 + F(u_0) \in \overline{D(A)} = Y_0 \). Then, system (1) has the unique classical solution \( u \in C^1([0, +\infty), Y^+_0) \cap C([0, +\infty), D(A)) \) satisfying (4).

**Proof.** Using Proposition 1 and Lemma 2.3, we establish the following theorem on the existence and uniqueness of the global classical solution of system (1).

By (1), we have

\[ \frac{dN(t)}{dt} = A - \mu N(t), \quad t > 0, \]

and hence, \( N(t) = A/\mu + (N(0) - A/\mu)e^{-\mu t}, t \geq 0 \). Hence, we have

\[ \|u(t)\|_W = N(t) \leq \max \left\{ \frac{A}{\mu}, N(0) \right\} = \max \left\{ \frac{A}{\mu}, \|u_0\|_W \right\} < +\infty, \quad t \geq 0. \]

This completes the proof.
3. Equilibria. It is easy to see that system (1) always has the disease-free equilibrium \( E^0 := (S, i_1, i_2, R) = (S^0, 0, 0) \in Y^+ \), where \( S^0 := A/\mu \). Let \( E^* := (S, i_1, i_2, R) = (S^*, i_1^*, i_2^*, R^*) \in Y^+ \) denote the endemic equilibrium of system (1). Then, the following equations hold.

\[
\begin{align*}
0 &= A - \mu S^* - S^* \int_0^{+\infty} \beta(\xi_1) i_1^*(\xi_1) d\xi_1 - S^* \int_0^{+\infty} \beta(\xi_2) i_2^*(\xi_2) d\xi_2, \\
\frac{di_1^*(\xi_1)}{d\xi_1} &= -(\mu + \theta_1(\xi_1)) i_1^*(\xi_1), \quad \xi_1 > 0, \\
\frac{di_2^*(\xi_2)}{d\xi_2} &= -(\mu + \theta_2(\xi_2)) i_2^*(\xi_2), \quad \xi_2 > 0, \\
0 &= (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1) i_1^*(\xi_1) d\xi_1 \\
&\quad + (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2) i_2^*(\xi_2) d\xi_2 - (\mu + \delta_1 + \delta_2) R^*, \\
i_1^*(0) &= S^* \int_0^{+\infty} \beta(\xi_1) i_1^*(\xi_1) d\xi_1 + k_2 \int_0^{+\infty} \theta_2(\xi_2) i_2^*(\xi_2) d\xi_2 + \delta_1 R^*, \\
i_2^*(0) &= S^* \int_0^{+\infty} \beta(\xi_2) i_2^*(\xi_2) d\xi_2 + k_1 \int_0^{+\infty} \theta_1(\xi_1) i_1^*(\xi_1) d\xi_1 + \delta_2 R^*,
\end{align*}
\]

Let \( \Theta_1(\xi_1) := e^{-\int_0^{\xi_1}(\mu + \theta_1(\rho)) d\rho} \) and \( \Theta_2(\xi_2) := e^{-\int_0^{\xi_2}(\mu + \theta_2(\rho)) d\rho} \). Then, from the second and third equations in (7), we have

\[
i_1^*(\xi_1) = i_1^*(0) \Theta_1(\xi_1), \quad \xi_1 > 0, \quad i_2^*(\xi_2) = i_2^*(0) \Theta_2(\xi_2), \quad \xi_2 > 0.
\]

Let

\[
B_1 := \int_0^{+\infty} \beta(\xi_1) \Theta_1(\xi_1) d\xi_1, \quad B_2 := \int_0^{+\infty} \beta(\xi_2) \Theta_2(\xi_2) d\xi_2, \\
\Gamma_1 := \int_0^{+\infty} \theta_1(\xi_1) \Theta_1(\xi_1) d\xi_1, \quad \Gamma_2 := \int_0^{+\infty} \theta_2(\xi_2) \Theta_2(\xi_2) d\xi_2.
\]

Note that

\[
0 < \Gamma_1 = \int_0^{+\infty} \theta_1(\xi_1) e^{-\int_0^{\xi_1}(\mu + \theta_1(\rho)) d\rho} d\xi_1, \\
< \int_0^{+\infty} \theta_1(\xi_1) e^{-\int_0^{\xi_1}(\mu + \theta_1(\rho)) d\rho} d\xi_1 = 1 - e^{-\int_0^{+\infty} \theta_1(\rho) d\rho} \leq 1, \quad i = 1, 2.
\]

From the first and fourth equations in (7), we have

\[
S^* = \frac{A}{\mu + B_1 i_1^*(0) + B_2 i_2^*(0)}, \quad R^* = \frac{(1 - k_1) \Gamma_1 i_1^*(0) + (1 - k_2) \Gamma_2 i_2^*(0)}{\mu + \delta_1 + \delta_2}.
\]

Substituting them into the last two equations in (7), we have

\[
\begin{align*}
i_1^*(0) &= \frac{AB_1 i_1^*(0)}{\mu + B_1 i_1^*(0) + B_2 i_2^*(0)} + k_2 \Gamma_2 i_2^*(0) + \frac{\delta_1 (1 - k_1) \Gamma_1 i_1^*(0) + \delta_1 (1 - k_2) \Gamma_2 i_2^*(0)}{\mu + \delta_1 + \delta_2}, \\
i_2^*(0) &= \frac{AB_2 i_2^*(0)}{\mu + B_1 i_1^*(0) + B_2 i_2^*(0)} + k_1 \Gamma_1 i_1^*(0) + \frac{\delta_2 (1 - k_1) \Gamma_1 i_1^*(0) + \delta_2 (1 - k_2) \Gamma_2 i_2^*(0)}{\mu + \delta_1 + \delta_2}.
\end{align*}
\]
Rewriting them, we have
\[
(\mu + \delta_1 + \delta_2) i_1^*(0) = \frac{(\mu + \delta_1 + \delta_2) \mu B_{1i_1^*(0)}}{\mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + (\mu + \delta_1 + \delta_2) k_2 \Gamma_{2i_2^*(0)} + \delta_1 (1 - k_1) \Gamma_{1i_1^*(0)} + \delta_1 (1 - k_2) \Gamma_{2i_2^*(0)},
\]
\[
(\mu + \delta_1 + \delta_2) i_2^*(0) = \frac{(\mu + \delta_1 + \delta_2) \mu B_{2i_2^*(0)}}{\mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + (\mu + \delta_1 + \delta_2) k_1 \Gamma_{1i_2^*(0)} + \delta_2 (1 - k_1) \Gamma_{1i_1^*(0)} + \delta_2 (1 - k_2) \Gamma_{2i_2^*(0)},
\]
and hence,
\[
\begin{align*}
\frac{\Psi_1(x)}{\Psi_2(x)} &= \frac{\mu + \delta_1 + \delta_2}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2 \mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + \frac{A B_{1i_1^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2}, \\
\frac{\Psi_2(x)}{\Psi_2(x)} &= \frac{\mu + \delta_1 + \delta_2}{\mu + \delta_1 [1 - (1 - k_2) \Gamma_2] + \delta_2 \mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + \frac{A B_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2}.
\end{align*}
\]
Let us define the following vector-valued operator \( \Psi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \times \mathbb{R}^+ \).
\[
\Psi(x) := \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}, \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^+ \times \mathbb{R}^+,
\]
\[
\Psi_1(x) := \mu + \delta_1 + \delta_2 \frac{A B_{1i_1^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2 \mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + \frac{A B_{1i_1^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2},
\]
\[
\Psi_2(x) := \mu + \delta_1 + \delta_2 \frac{A B_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2 \mu + B_{1i_1^*(0)} + B_{2i_2^*(0)}} + \frac{A B_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2}.
\]
Then, positive solution \((i_1^*(0), i_2^*(0))\) of \((10)\) exists if and only if operator \( \Psi \) has a nontrivial fixed point. We now prove the following lemma.

**Lemma 3.1.** Operator \( \Psi \) has a strong Fréchet derivative \( \Psi'[0] \) and a strong asymptotic derivative \( \Psi'[\infty] \). The spectrum of \( \Psi'[\infty] \) lies in the circle \(|\lambda| < 1\).

**Proof.** For any \( x \in \mathbb{R}^+ \times \mathbb{R}^+ \) such that \( x \neq 0 \), we have
\[
\lim_{h \to 0} \frac{\Psi(h \mathbf{x}) - \Psi(\mathbf{0})}{h} = \begin{pmatrix} \frac{(\mu + \delta_1 + \delta_2) A B_{1i_1^*(0)} + [(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2} \\ \frac{(\mu + \delta_1 + \delta_2) A B_{2i_2^*(0)} + [(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2} \end{pmatrix},
\]
and
\[
\lim_{h \to +\infty} \frac{\Psi(h \mathbf{x})}{h} = \begin{pmatrix} \frac{[(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2} \\ \frac{[(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_2) \Gamma_2] + \delta_2} \end{pmatrix}.
\]
They imply that
\[
\Psi'[0] = \begin{pmatrix} \frac{(\mu + \delta_1 + \delta_2) A B_{1i_1^*(0)} + [(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2} \\ \frac{[(\mu + \delta_2) B_{1i_1^*(0)} + \delta_2 \Gamma_{2i_2^*(0)}}{\mu + \delta_1 [1 - (1 - k_1) \Gamma_1] + \delta_2} \end{pmatrix}.
\]
and

\[ \Psi'[\infty] = \begin{pmatrix} 0 & \frac{[(\mu + \delta_2)k_2 + \delta_1] \Gamma_2}{\mu + \delta_1[1 - (1 - k_1)\Gamma_1] + \delta_2} \\ \frac{[(\mu + \delta_1)k_1 + \delta_2] \Gamma_1}{\mu + \delta_1[1 - (1 - k_2)\Gamma_2] + \delta_2} & 0 \end{pmatrix}, \]

respectively. The eigenvalues \( \lambda \in \mathbb{C} \) of matrix \( \Psi'[\infty] \) are

\[ \lambda = \pm \sqrt{\frac{(\mu + \delta_1)k_1 + \delta_2}{\mu + \delta_1[1 - (1 - k_1)\Gamma_1] + \delta_2} \frac{(\mu + \delta_2)k_2 + \delta_1}{\mu + \delta_1[1 - (1 - k_2)\Gamma_2] + \delta_2} \Gamma_1 \Gamma_2}. \]

Since \( 0 < k_i < 1 \) and \( 0 < \Gamma_i < 1, i = 1, 2 \), we have

\[ |\lambda| < \sqrt{\Gamma_1 \Gamma_2} < 1, \]

which completes the proof. \( \square \)

Since matrix \( \Psi'[0] \) is positive, it follows from the Perron-Frobenius theorem [3] that its spectral radius \( r(\Psi'[0]) \) is a simple eigenvalue, greater than the magnitude of any other eigenvalue. Moreover, \( \Psi'[0] \) has a positive eigenvector \( v \) corresponding to \( r(\Psi'[0]) \), and any nonnegative eigenvector of \( \Psi'[0] \) is a multiple of \( v \). That is, there exists no nonnegative eigenvector of \( \Psi'[0] \) corresponding to eigenvalue 1 if \( r(\Psi'[0]) > 1 \). Hence, by Lemma 3.1 and [16, Theorem 4.11], we establish the following theorem on the existence of the endemic equilibrium \( E^* \).

**Theorem 3.2.** Assume that \( r(\Psi'[0]) > 1 \). Then, system (1) has at least one endemic equilibrium \( E^* \).

**Proof.** By the above argument and Lemma 3.1, we can apply [16, Theorem 4.11] to see that operator \( \Psi \) has a positive fixed point \( x^* = \Psi(x^*) \) in \((\mathbb{R}^+ \setminus \{0\}) \times (\mathbb{R}^+ \setminus \{0\})\). Thus, as stated above, positive solution \((i^*_1(t), i^*_2(t))\) of (10) exists. Substituting it into (8) and (9), we obtain an endemic equilibrium \( E^* = (S^*, i^*_1, i^*_2, R^*) \in Y^+ \).

This completes the proof. \( \square \)

4. Existence of a compact attractor. By Proposition 1 and Theorem 2.4, we can define a semiflow \( \{\Phi(t)\}_{t \geq 0} \), which is generated for system (1), by

\[ \Phi(t)u_0 := u(t) = (S(t), 0, i_1(t, \cdot), 0, i_2(t, \cdot), R(t))^T \in Y^+_0, \quad t \geq 0. \]

The following lemma immediately follows.

**Lemma 4.1.** \( \Phi(\cdot) \) is point dissipative and eventually bounded (see, for instance, [26, Definition 2.25]) on every bounded set in \( Y^+_0 \).

**Proof.** As in the proof of Theorem 2.4, we see that for any \( u_0 \in Y^+_0 \),

\[ \|\Phi(t)u_0\|_W = N(t) = \frac{A}{\mu} + \left( N(0) - \frac{A}{\mu} \right) e^{-\mu t} \to \frac{A}{\mu} \text{ as } t \to +\infty, \quad (11) \]

which implies that \( \Phi(\cdot) \) is point dissipative. Let \( C \subset Y^+_0 \) be an arbitrary bounded set such that for some \( K > 0, \|y\|_W \leq K \) holds for any \( y \in C \). As in the proof of Theorem 2.4, we have, for any \( u_0 \in C \),

\[ \|\Phi(t)u_0\|_W = N(t) \leq \max \left\{ \frac{A}{\mu}, K \right\}, \quad t \geq 0, \]

which implies that \( \Phi(\cdot) \) is eventually bounded on \( C \). This completes the proof. \( \square \)
To show the existence of a compact attractor, we have to prove the asymptotic smoothness of semiflow $\Phi(\cdot)$. Let $J_1(t) := i_1(t, 0)$ and $J_2(t) := i_2(t, 0)$ for all $t \geq 0$. Now we have the following lemma.

**Lemma 4.2.** $S(\cdot)$, $J_1(\cdot)$, $J_2(\cdot)$ and $R(\cdot)$ are Lipschitz continuous on $\mathbb{R}^+$, that is, there exist positive Lipschitz constants $L_S, L_{J_1}, L_{J_2}$ and $L_R$ such that

$$
|S(t+h) - S(t)| \leq L_S h, \quad |R(t+h) - R(t)| \leq L_R h,
$$

$$
|J_1(t+h) - J_1(t)| \leq L_{J_1} h, \quad |J_2(t+h) - J_2(t)| \leq L_{J_2} h,
$$

hold for all $t \geq 0$ and $h > 0$.

**Proof.** Let $K := \max \{ A/\mu, \| u_0 \|_W \}$. Then, as in the proof of Lemma 4.1, we have $\| \Phi(t) u_0 \|_W \leq K$ for all $t \geq 0$. From the first equation in (1), we have

$$
\left| \frac{dS(t)}{dt} \right| \leq A + \mu K + 2\beta^\infty K^2, \quad t > 0,
$$

which implies that $S(\cdot)$ is Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz constant $L_S := A + \mu K + 2\beta^\infty K^2$. Similarly, we can show that $R(\cdot)$ is Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz constant $L_R := (1 - k_1)\theta^\infty K + (1 - k_2)\theta^\infty K + (\mu + \delta_1 + \delta_2)K$. For any $t \geq 0$ and $h > 0$, we have

$$
|J_1(t+h) - J_1(t)| = \left| S(t+h) \int_0^{+\infty} \beta(\xi_1) i_1(t+h, \xi_1) d\xi_1 
+ k_2 \int_0^{+\infty} \theta_2(\xi_2) i_2(t+h, \xi_2) d\xi_2 + \delta_1 R(t+h) 
- S(t) \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 - k_2 \int_0^{+\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 - \delta_1 R(t) \right|,
$$

$$
\leq \beta^\infty K |S(t+h) - S(t)| + \delta_1 |R(t+h) - R(t)|
+ K \left| \int_0^{+\infty} \beta(\xi_1) i_1(t+h, \xi_1) d\xi_1 - \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 \right|
+ k_2 \left| \int_0^{+\infty} \theta_2(\xi_2) i_2(t+h, \xi_2) d\xi_2 - \int_0^{+\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 \right|. \quad (12)
$$

Note that we have, for $j = 1, 2$,

$$
i_j(t, \xi_j) = \begin{cases}
J_j(t - \xi_j) \Theta_j(\xi_j), & t - \xi_j > 0, \\
\frac{J_j(\xi_j) - \xi_j}{\Theta_j(\xi_j)}, & \xi_j - t \geq 0,
\end{cases}
(13)
$$

and hence,

$$
i_j(t+h, \xi_j + h) = i_j(t, \xi_j) \frac{\Theta_j(\xi_j + h)}{\Theta_j(\xi_j)}.
$$

Hence, we have

$$
\left| \int_0^{+\infty} \beta(\xi_1) i_1(t+h, \xi_1) d\xi_1 - \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 \right|
= \left| \int_0^h \beta(\xi_1) i_1(t+h, \xi_1) d\xi_1 + \int_h^{+\infty} \beta(\xi_1) i_1(t+h, \xi_1) d\xi_1 - \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 \right|,
$$

respectively.
\[ \leq \beta^\infty \int_0^h J_1(t+h-\xi_1)\Theta_1(\xi_1)d\xi_1 + \int_0^{+\infty} |\beta(\xi_1+h)i_1(t+h,\xi_1+h) - \beta(\xi_1)i_1(t,\xi_1)|d\xi_1, \]
\[ \leq \beta^\infty J_1^\infty h + \int_0^{+\infty} |\beta(\xi_1+h)\frac{\Theta_1(\xi_1+h)}{\Theta_1(\xi_1)} - \beta(\xi_1)|i_1(t,\xi_1)d\xi_1, \]

where \( J_1^\infty := \beta^\infty K^2 + k_2 \theta_2^\infty K + \delta_1 K \). Since \( \beta(\cdot) \) is Lipschitz continuous, we have
\[
\left| \int_0^{+\infty} \beta(\xi_1)i_1(t+h,\xi_1)d\xi_1 - \int_0^{+\infty} \beta(\xi_1)i_1(t,\xi_1)d\xi_1 \right| \\
\leq \beta^\infty J_1^\infty h + \int_0^{+\infty} \beta(\xi_1 + h) \left| \frac{\Theta_1(\xi_1 + h)}{\Theta_1(\xi_1)} - 1 \right| i_1(t,\xi_1)d\xi_1 \\
+ \int_0^{+\infty} |\beta(\xi_1 + h) - \beta(\xi_1)| i_1(t,\xi_1)d\xi_1,
\]
\[
\leq (\beta^\infty J_1^\infty + L_\beta K) h + \beta^\infty \int_0^{+\infty} \left( 1 - e^{-\left(\mu + \theta_1^\infty\right) h} \right) i_1(t,\xi_1)d\xi_1,
\]
\[
\leq [\beta^\infty J_1^\infty + L_\beta K + \beta^\infty (\mu + \theta_1^\infty) K] h =: L_1 h.
\]

In a similar way, we have
\[
\left| \int_0^{+\infty} \theta_2(\xi_2)i_2(t+h,\xi_2)d\xi_2 - \int_0^{+\infty} \theta_2(\xi_2)i_2(t,\xi_2)d\xi_2 \right| \\
\leq [\theta_2^\infty J_2^\infty + L_\theta_2 K + \theta_2^\infty (\mu + \theta_2^\infty) K] h =: L_2 h,
\]

where \( J_2^\infty := \beta^\infty K^2 + k_1 \theta_1^\infty K + \delta_2 K \). Consequently, from (12), we have
\[
|J_1(t+h) - J_1(t)| \leq (\beta^\infty KL_S + \delta_1 L_R + KL_1 + k_2 L_2) h,
\]
which implies that \( J_1(\cdot) \) is Lipschitz continuous on \( \mathbb{R}^+ \). In a similar way, we can show that \( J_2(\cdot) \) is Lipschitz continuous on \( \mathbb{R}^+ \). This completes the proof. \( \square \)

Using Lemma 4.2, we prove the following proposition on the asymptotic smoothness of semiflow \( \Phi(\cdot) \).

**Proposition 2.** \( \Phi(\cdot) \) is asymptotically smooth.

**Proof.** Let \( C_0 \) be an arbitrary bounded subset of \( Y_0^+ \) with bound \( K_0 \geq A/\mu \), that is, \( \|u\|_{W} \leq K_0 \) holds for any \( u \in C_0 \). Note that \( C_0 \) is forward invariant under \( \Phi(\cdot) \) as shown in the proof of Lemma 4.1. By [26, Theorem 2.46] (see also [23, Theorem 5.1]), \( \Phi(\cdot) \) is asymptotically smooth if there exist two maps \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) such that \( \Phi(\cdot) = \Phi_1(\cdot) + \Phi_2(\cdot) \) and the following two properties hold,

(P1): \( \lim_{t \to +\infty} \text{diam} \Phi_1(t)(C_0) = 0 \),
(P2): there exists \( t_0 \geq 0 \) such that \( \Phi_2(t)(C_0) \) has compact closure for any \( t > t_0 \),

where \( \text{diam} A := \sup\{\|u - v\|_{W} : u, v \in A\} \) for any set \( A \) in \( Y_0^+ \). To use the method as in [23, Proof of Theorem 5.3], we define \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) by, for \( u_0 \in C_0 \) and \( t \geq 0 \),

\[
\Phi_1(t)u_0 := (0, 0, \tilde{i}_1(t, \cdot), 0, \tilde{i}_2(t, \cdot), 0)^T \quad \text{and} \\
\Phi_2(t)u_0 := (S(t), 0, \tilde{i}_1(t, \cdot), 0, \tilde{i}_2(t, \cdot), R(t))^T,
\]
where, for $j = 1, 2$,

\[
\hat{i}_j(t, \xi_j) := \begin{cases} 
0, & t - \xi_j > 0, \\
\tilde{i}_{j,0}(\xi_j - t) \frac{\Theta_j(\xi_j)}{\Theta_j(\xi_j - t)}, & \xi_j - t \geq 0,
\end{cases}
\]

\[
\hat{i}_j(t, \xi_j) := \begin{cases} 
J_j(t - \xi_j) \Theta_j(\xi_j), & t - \xi_j > 0, \\
0, & \xi_j - t \geq 0.
\end{cases}
\]

Then, we see from (13) that $\Phi(\cdot) = \Phi_1(\cdot) + \Phi_2(\cdot)$. For $j = 1, 2$, we have

\[
0 \leq \int_0^{+\infty} \dot{\hat{i}}_j(t, \xi_j) d\xi_j = \int_t^{+\infty} \tilde{i}_{j,0}(\xi_j - t) \frac{\Theta_j(\xi_j)}{\Theta_j(\xi_j - t)} d\xi_j,
\]

\[
\leq e^{-\mu t} \int_0^{+\infty} \tilde{i}_{j,0}(\xi_j) d\xi_j \leq K_0 e^{-\mu t}.
\]

This implies that $\Phi_1(t)u_0$ converges uniformly for $u_0 \in C_0$ to $0 \in Y_0^+$ as $t \to +\infty$. Hence, $\lim_{t \to +\infty} \text{diam} \Phi_1(t)(C_0) = 0$, and thus, (P1) holds.

To show (P2), as in [23, Proof of Theorem 5.3], it suffices to show that, for $j = 1, 2$,

\[
\lim_{h \to +0} \int_0^{+\infty} |\dot{\hat{i}}_j(t, \xi_j + h) - \dot{\hat{i}}_j(t, \xi_j)| d\xi_j = 0 \quad \text{uniformly for } t > 0 \text{ and } u_0 \in C_0. \quad (14)
\]

In fact, for $t > 0$, $h \in (0, t)$ and $j = 1, 2$, we have

\[
\int_0^{+\infty} |\dot{\hat{i}}_j(t, \xi_j + h) - \dot{\hat{i}}_j(t, \xi_j)| d\xi_j
\]

\[
= \int_0^{t-h} |J_j(t - \xi_j - h) \Theta_j(\xi_j + h) - J_j(t - \xi_j) \Theta_j(\xi_j)| d\xi_j + \int_{t-h}^t J_j(t - \xi_j) \Theta_j(\xi_j) d\xi_j,
\]

\[
\leq \int_0^{t-h} J_j(t - \xi_j - h) |\Theta_j(\xi_j + h) - \Theta_j(\xi_j)| d\xi_j
\]

\[
+ \int_{t-h}^t |J_j(t - \xi_j - h) - J_j(t - \xi_j)| |\Theta_j(\xi_j)| d\xi_j + J_j^\infty h,
\]

where $J_j^\infty$, $j = 1, 2$ are positive upper bounds for $J_j(\cdot)$, $j = 1, 2$ defined as in the proof of Lemma 4.2. By Lemma 4.2, we have, for $t > 0$, $h \in (0, t)$ and $j = 1, 2$,

\[
\int_0^{+\infty} |\dot{\hat{i}}_j(t, \xi_j + h) - \dot{\hat{i}}_j(t, \xi_j)| d\xi_j
\]

\[
\leq J_j^\infty \int_0^{t-h} |\Theta_j(\xi_j + h)| d\xi_j + \int_0^{t-h} L_j h \Theta_j(\xi_j) d\xi_j + J_j^\infty h,
\]

\[
\leq J_j^\infty \left( \int_0^{t-h} \Theta_j(\xi_j) d\xi_j - \int_{t-h}^t \Theta_j(\xi_j) d\xi_j \right) + L_j h \int_0^{t-h} e^{-\mu \xi_j} d\xi_j + J_j^\infty h,
\]

\[
\leq \left( 2J_j^\infty + \frac{L_j h}{\mu} \right) h,
\]

which converges to 0 as $h \to +0$ uniformly for $t > 0$ and $u_0 \in C_0$. Hence, we obtain (14). As in [23, Proof of Theorem 5.3], we see that $\dot{\hat{i}}_j(t, \cdot)$, $j = 1, 2$ remains in a pre-compact subset of $X^+$ for $t > 0$. Since $S(t)$ and $R(t)$ remain in the compact set $[0, K_0]$ for $t > 0$, we see that $\Phi_2(t)(C_0)$ has compact closure for any $t > 0$. Thus, (P2) holds for $t_0 = 0$. This completes the proof. \qed
From Lemma 4.1 and Proposition 2, we can apply [26, Theorem 2.33] to establish the following proposition on the existence of a compact attractor.

**Proposition 3.** There exists a compact attractor $\mathcal{A}_0 \subset Y_0^+$, which attracts any bounded subset of $Y_0^+$.

**Proof.** The assertion directly follows from Lemma 4.1, Proposition 2 and [26, Theorem 2.33]. This completes the proof. □

We end this section by describing the total trajectories of the system (1)-(2)-(3). Let $\mathcal{U} : \mathbb{R} \to Y_0^+$ such that $\mathcal{U}(t) = (S(t), 0, i_1(t, \cdot), 0, i_2(t, \cdot), R(t))^T$ be a total trajectory of semiflow $\Phi(\cdot)$, that is, $\mathcal{U}(\cdot)$ is a function satisfying $\mathcal{U}(r + t) = \Phi(t)\mathcal{U}(r)$ for all $r \in \mathbb{R}$ and $t \geq 0$. For $\mathcal{U}(\cdot)$, by a simple computation see also [26] and for all $t \in \mathbb{R}$, we have

$$
\begin{align*}
S'(t) &= A - \mu S(t) - S(t) \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 - S(t) \int_0^{+\infty} \beta(\xi_2) i_2(t, \xi_2) d\xi_2, \\
i_j(t, \xi_j) &= i_j(t - \xi_j, 0) \Theta_j(\xi_j), \quad j = 1, 2, \\
R'(t) &= (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1) i_1(t, \xi_1) d\xi_1 + (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 \\
&\quad - (\mu + \delta_1 + \delta_2) R(t),
\end{align*}
$$

and

$$
\begin{align*}
i_1(t, 0) &= S(t) \int_0^{+\infty} \beta(\xi_1) i_1(t, \xi_1) d\xi_1 + k_2 \int_0^{+\infty} \theta_2(\xi_2) i_2(t, \xi_2) d\xi_2 + \delta_1 R(t), \\
i_2(t, 0) &= S(t) \int_0^{+\infty} \beta(\xi_2) i_2(t, \xi_2) d\xi_2 + k_1 \int_0^{+\infty} \theta_1(\xi_1) i_1(t, \xi_1) d\xi_1 + \delta_2 R(t).
\end{align*}
$$

5. **Global stability of the disease-free equilibrium.** From a well-known fact and Proposition 3, the compact attractor $\mathcal{A}_0$ is consisting of total trajectories (see Proposition 2-34 [26]), that is, for any $\mathcal{U}_0 \in \mathcal{A}_0$, there exists a total trajectory such that $\mathcal{U}(0) = \mathcal{U}_0$ and $\mathcal{U}(t) \in \mathcal{A}_0$ for all $t \in \mathbb{R}$. Let $\bar{J}_j := \sup_{t \in \mathbb{R}} J_j(t), j = 1, 2$, where $J_j(t) = i_j(t, 0)$ for all $t \in \mathbb{R}$ and $j = 1, 2$. We now prove the following lemma.

**Lemma 5.1.** For $\mathcal{U}(\cdot) \in \mathcal{A}_0$, we have

$$
0 \leq S(t) \leq \frac{A}{\mu}, \quad 0 \leq R(t) \leq \frac{(1 - k_1) \Gamma_1 \bar{J}_1 + (1 - k_2) \Gamma_2 \bar{J}_2}{\mu + \delta_1 + \delta_2}, \quad t \in \mathbb{R}.
$$

**Proof.** The nonnegativity of $S(t)$ and $R(t)$ for all $t \in \mathbb{R}$ is obvious since $\mathcal{A}_0 \subset Y_0^+$. From the first equation of (1), we have $dS(t)/dt \leq A - \mu S(t), \quad t \in \mathbb{R}$. This implies that $S(t) \leq A/\mu, \quad t \in \mathbb{R}$. Similarly, from the fourth equation of (1), we have

$$
\frac{dR(t)}{dt} = (1 - k_1) \int_0^{+\infty} \theta_1(\xi_1) J_1(t - \xi_1) \Theta_1(\xi_1) d\xi_1 \\
+ (1 - k_2) \int_0^{+\infty} \theta_2(\xi_2) J_2(t - \xi_2) \Theta_2(\xi_2) d\xi_2 - (\mu + \delta_1 + \delta_2) R(t),
$$

$$
\leq (1 - k_1) \Gamma_1 \bar{J}_1 + (1 - k_2) \Gamma_2 \bar{J}_2 - (\mu + \delta_1 + \delta_2) R(t), \quad t \in \mathbb{R}.
$$
This implies that \( R(t) \leq \left( (1 - k_1) \Gamma_1 J_1 + (1 - k_2) \Gamma_2 J_2 \right) / (\mu + \delta_1 + \delta_2), \) \( t \in \mathbb{R}. \) This completes the proof. \( \square \)

By using Lemma 5.1, we prove the global asymptotic stability of the disease-free equilibrium for \( r(\Psi'[0]) < 1. \)

**Theorem 5.2.** Assume that \( r(\Psi'[0]) < 1. \) Then, the disease-free equilibrium \( E^0 = (A/\mu, 0, 0, 0) \in Y^+ \) is globally asymptotically stable.

**Proof.** First recall that the compact attractor of bounded set is the union of bounded total orbits. Let \( \mathcal{U}(t) = (S(t), 0, i_1(t, .), 0, i_2(t, .), R(t))^T \) be a total trajectory of semiflow \( \Phi(t). \) By (15)-(16) and Lemma 5.1, we have, for \( t \in \mathbb{R}, \)

\[
J_1(t) \leq \frac{A}{\mu} \int_0^{+\infty} \beta(\xi_1) J_1(t - \xi_1) \Theta_1(\xi_1) d\xi_1 + k_2 \int_0^{+\infty} \theta_2(\xi_2) J_2(t - \xi_2) \Theta_2(\xi_2) d\xi_2
+ \frac{(1 - k_1) \Gamma_1 J_1 + (1 - k_2) \Gamma_2 J_2}{\mu + \delta_1 + \delta_2},
\]

\[
J_2(t) \leq \frac{A}{\mu} \int_0^{+\infty} \beta(\xi_2) J_2(t - \xi_2) \Theta_2(\xi_2) d\xi_2 + k_1 \int_0^{+\infty} \theta_1(\xi_1) J_1(t - \xi_1) \Theta_1(\xi_1) d\xi_1
+ \frac{(1 - k_1) \Gamma_1 J_1 + (1 - k_2) \Gamma_2 J_2}{\mu + \delta_1 + \delta_2},
\]

Taking the supremum of the right-hand sides of these inequalities, we have

\[
J_1 \leq \frac{A}{\mu} B_1 J_1 + k_2 \Gamma_2 J_2 + \frac{(1 - k_1) \Gamma_1 J_1 + (1 - k_2) \Gamma_2 J_2}{\mu + \delta_1 + \delta_2},
\]

\[
J_2 \leq \frac{A}{\mu} B_2 J_2 + k_1 \Gamma_1 J_1 + \frac{(1 - k_1) \Gamma_1 J_1 + (1 - k_2) \Gamma_2 J_2}{\mu + \delta_1 + \delta_2}.
\]

Rearranging as in the calculation in Section 3, we obtain

\[
\left( \begin{array}{c} J_1 \\ J_2 \end{array} \right) \leq \Psi'[0] \left( \begin{array}{c} J_1 \\ J_2 \end{array} \right). \quad (17)
\]

Since matrix \( \Psi'[0] \) is positive, it follows from the Perron-Frobenius theorem [3] that \( \Psi'[0] \) has a positive left eigenvector \( \ell := (\ell_1, \ell_2), \ell_j > 0, \) \( j = 1, 2 \) corresponding to eigenvalue \( r(\Psi'[0]) < 1. \) Multiplying \( \ell \) by both sides of (17), we obtain

\[
(\ell_1 J_1 + \ell_2 J_2) = \ell \left( \begin{array}{c} J_1 \\ J_2 \end{array} \right)
\]

\[
\leq \ell \Psi'[0] \left( \begin{array}{c} J_1 \\ J_2 \end{array} \right) = r(\Psi'[0]) \ell \left( \begin{array}{c} J_1 \\ J_2 \end{array} \right) = r(\Psi'[0]) (\ell_1 J_1 + \ell_2 J_2).
\]

This inequality holds only if \( J_1 = J_2 = 0 \) since \( r(\Psi'[0]) < 1 \) and \( \ell \) is positive. Hence, \( J_1(t) = J_2(t) = 0 \) for all \( t \in \mathbb{R}. \) Moreover, from the inequality in Lemma 5.1, we see that \( R(t) = 0 \) for all \( t \in \mathbb{R}. \) We then have that \( dS(t)/dt = A - \mu S(t), \) \( t \in \mathbb{R} \) and thus, \( S(t) = A/\mu, t \in \mathbb{R}. \) Consequently, the compact attractor \( A_0 \) is the singleton \( \{(A/\mu, 0, 0, 0, 0)^T\} \subset Y^+_0. \) Moreover the disease-free equilibrium is locally stable.
by [26, Theorem 2.39]. This implies that the $E^0$ is globally asymptotically stable. This completes the proof. \hfill $\Box$

6. Uniform persistence of the system. Let $\varrho : Y^+_0 \to \mathbb{R}$ be a function defined by
\[
\varrho(\Phi(t)u_0) := i_1(t,0) + i_2(t,0) = J_1(t) + J_2(t), \quad t \geq 0, \quad u_0 \in Y^+_0.
\]
Let
\[
\Omega := \{u_0 = (S_0,0,i_{1,0}(\cdot),0,i_{2,0}(\cdot),R_0)^T \in Y^+_0: \varrho(u_0) > 0\}.
\]
We first prove the following proposition.

**Proposition 4.** Assume that $r(\Psi[0]) > 1$. Then, system (1) is uniformly weakly $\varrho$-persistent for nontrivial initial condition, that is, there exists an $\epsilon > 0$ such that $\limsup_{t \to +\infty} \varrho(\Phi(t)u_0) > \epsilon$, provided $u_0 \in \Omega_0$.

**Proof.** Suppose on the contrary that $\limsup_{t \to +\infty} \varrho(\Phi(t)u_0) \leq \epsilon$ for an arbitrary small $\epsilon > 0$. Then, there exists a $T_0 > 0$ such that $\varrho(\Phi(t)u_0) \leq \epsilon$ for all $t \geq T_0$. Without loss of generality, we can assume that $\varrho(\Phi(t)u_0) \leq \epsilon$ for all $t \geq 0$ by regarding $\Phi(T_0)u_0$ as a new initial condition. We then have from the first equation in (1) that
\[
\frac{dS(t)}{dt} \geq A - \epsilon - \mu S(t), \quad t \geq 0,
\]
and hence,
\[
S(t) \geq (A - \epsilon) \int_0^t e^{-\mu(t-\sigma)}d\sigma = \frac{A - \epsilon}{\mu} (1 - e^{-\mu t}), \quad t \geq 0.
\]
Then, there exists a sufficiently large $T > 0$ such that
\[
S(t) \geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}), \quad t \geq T. \tag{18}
\]
Without loss of generality, we can assume that inequality (18) holds for all $t \geq 0$ by regarding $\Phi(T)u_0$ as a new initial condition. We then have, for all $t \geq 0$,
\[
i_1(t,0) \geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) \int_0^{+\infty} \beta(\xi_1)i_1(t,\xi_1)d\xi_1
\]
\[
+ k_2 \int_0^{+\infty} \theta_2(\xi_2)i_2(t,\xi_2)d\xi_2 + \delta_1 R(t),
\]
\[
i_2(t,0) \geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) \int_0^{+\infty} \beta(\xi_2)i_2(t,\xi_2)d\xi_2
\]
\[
+ k_1 \int_0^{+\infty} \theta_1(\xi_1)i_1(t,\xi_1)d\xi_1 + \delta_2 R(t).
\]
Hence, recalling that $i_j(t,\xi_j) = J_j(t-\xi_j)\Theta_j(\xi_j), \quad t - \xi_j > 0, \quad j = 1,2$, we have, for all $t \geq 0$,
\[
J_1(t) \geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) \int_0^t \beta(\xi_1)\Theta_1(\xi_1)J_1(t-\xi_1)d\xi_1
\]
\[
J_2(t) \geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) \int_0^t \beta(\xi_2)\Theta_2(\xi_2)J_2(t-\xi_2)d\xi_2.
\]
For $\lambda > 0$, let $\tilde{f}(\lambda) := \int_{0}^{+\infty} e^{-\lambda t} f(t) dt$ be Laplace transform of function $f(\cdot)$. We then have from (19) and (20) that
\[
\begin{align*}
\dot{J}_1[\lambda] &\geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) B_{1,\lambda} \dot{J}_1[\lambda] + k_2 \Gamma_{2,\lambda} \dot{J}_2[\lambda] + \delta_1 \tilde{R}[\lambda], \\
\dot{J}_2[\lambda] &\geq \frac{A - \epsilon}{\mu} (1 - e^{-\mu T}) B_{2,\lambda} \dot{J}_2[\lambda] + k_1 \Gamma_{1,\lambda} \dot{J}_1[\lambda] + \delta_2 \tilde{R}[\lambda], \\
\tilde{R}[\lambda] &\geq \frac{(1 - k_1)\Gamma_{1,\lambda}}{\lambda + \mu + \delta_1 + \delta_2} \dot{J}_1[\lambda] + \frac{(1 - k_2)\Gamma_{2,\lambda}}{\lambda + \mu + \delta_1 + \delta_2} \dot{J}_2[\lambda],
\end{align*}
\]
where $B_{j,\lambda} := \int_{0}^{+\infty} e^{-\lambda \xi_1} \beta(\xi_1) \Theta_j(\xi_1) d\xi_1$ and $\Gamma_{j,\lambda} := \int_{0}^{+\infty} e^{-\lambda \xi_2} \theta_j(\xi_2) \Theta_j(\xi_2) d\xi_2$, $j = 1, 2$. Rearranging these inequalities as in the calculation in Section 3, we obtain
\[
\begin{pmatrix}
\dot{J}_1[\lambda] \\
\dot{J}_2[\lambda]
\end{pmatrix} \geq \Psi_{\epsilon, T, \lambda} \begin{pmatrix}
J_1[\lambda] \\
J_2[\lambda]
\end{pmatrix},
\]
(21)
where
\[
\Psi_{\epsilon, T, \lambda} := \begin{pmatrix}
\frac{(\lambda + \mu + \delta_1 + \delta_2)(A - \epsilon)(1 - e^{-\mu T}) B_{1,\lambda}}{\lambda + \mu + \delta_1 [1 - (1 - k_1) \Gamma_{1,\lambda}]} + \delta_2 \\
\frac{[(\lambda + \mu + \delta_1 + \delta_2)(A - \epsilon)(1 - e^{-\mu T}) B_{2,\lambda}] + \delta_1}{\lambda + \mu + \delta_1 [1 - (1 - k_2) \Gamma_{2,\lambda}]} + \delta_2
\end{pmatrix}.
\]
Note that $\Psi_{\epsilon, T, \lambda} \to \Psi'(0)$ as $\epsilon \to 0$, $T \to +\infty$ and $\lambda \to 0$. Since $r(\Psi'(0)) > 1$, from the continuity, without loss of generality, we can assume that $\epsilon$ and $\lambda$ are sufficiently small and $T$ is sufficiently large such that $r(\Psi_{\epsilon, T, \lambda}) > 1$. By (21), as in the proof of Theorem 5.2, we can prove that $\dot{J}_1[\lambda] = \dot{J}_2[\lambda] = 0$. This contradicts to the positivity of the solution for $u_0 \in \Omega_0$. This completes the proof. \qed

To prove the uniform strong $\varphi$-persistence from Proposition 4, we shall use [26, Theorem 5.2]. To this end, we need to prove that there exists no total trajectory $\mathcal{U} : \mathbb{R} \to Y_0^+$ such that $\varphi(\mathcal{U}(0)) = 0$ and $\varphi(\mathcal{U}(-r)) > 0$ and $\varphi(\mathcal{U}(t)) > 0$ for some
Moreover, from the fourth equation of (1), we have, for $t > 0$,

$$
\frac{dR(t)}{dt} \leq (1-k_1)\theta_1^\psi \int_0^t J_1(\xi_1)d\xi_1 + (1-k_2)\theta_2^\psi \int_0^t J_2(\xi_2)d\xi_2 - (\mu + \delta_1 + \delta_2)R(t),
$$

and thus, we have, for $t > 0$,

$$
R(t) \leq R(0)e^{-(\mu + \delta_1 + \delta_2)t} + (1-k_1)\theta_1^\psi \int_0^t e^{-(\mu + \delta_1 + \delta_2)(t-s)} \int_s^t J_1(\xi_1)d\xi_1 ds
$$

$$
+ (1-k_2)\theta_2^\psi \int_0^t e^{-(\mu + \delta_1 + \delta_2)(t-s)} \int_s^t J_2(\xi_2)d\xi_2 ds,
$$

$$
\leq \frac{(1-k_1)\theta_1^\psi}{\mu + \delta_1 + \delta_2} \int_0^t J_1(\xi_1)d\xi_1 + \frac{(1-k_2)\theta_2^\psi}{\mu + \delta_1 + \delta_2} \int_0^t J_2(\xi_2)d\xi_2.
$$

(23)

Note that $R(0) = 0$ since $J_1(0) = J_2(0) = 0$. Combining (22) and (23), we obtain the following Gronwall’s inequalities of vector type, for $t > 0$,

$$
\begin{pmatrix}
J_1(t) \\
J_2(t)
\end{pmatrix} \leq \int_0^t \begin{pmatrix}
\frac{\lambda \beta_1^\psi}{k_1 \theta_1^\psi} + \frac{\delta_1(1-k_1)\theta_1^\psi}{\mu + \delta_1 + \delta_2} & \frac{k_2 \theta_2^\psi}{\mu + \delta_1 + \delta_2} + \frac{\delta_1(1-k_2)\theta_2^\psi}{\mu + \delta_1 + \delta_2} \\
\frac{\lambda \beta_2^\psi}{k_2 \theta_2^\psi} & \frac{\lambda \beta_2^\psi}{k_2 \theta_2^\psi}
\end{pmatrix}\begin{pmatrix}
J_1(\xi) \\
J_2(\xi)
\end{pmatrix}d\xi.
$$

(24)

This implies that $J_1(t) = J_2(t) = 0$ for all $t > 0$, and thus, $g(U(t)) = 0$ for all $t > 0$. This completes the proof.

By adding the equations in (2), we have, for all $t \in \mathbb{R}$,

$$
g(U(t)) = S(t) \int_0^t \beta(\xi) [i_1(t, \xi) + i_2(t, \xi)] d\xi + \hat{I}(t),
$$

(25)

where, for all $t \in \mathbb{R}$,

$$
\hat{I}(t) = S(t) \int_t^{+\infty} \beta(\xi) [i_1(t, \xi) + i_2(t, \xi)] d\xi + k_1 \int_0^{+\infty} \theta_1(\xi_1) i(t, \xi_1) d\xi_1
$$

$$
+ k_2 \int_0^{+\infty} \theta_2(\xi_2) i(t, \xi_2) d\xi_2 + (\delta_1 + \delta_2)R(t).
$$

Note that $\hat{I}(0) = g(U(0))$. We next prove the following lemma.

**Lemma 6.2.** For total trajectory $U(\cdot) \in A_0$, we have, for all $t \in \mathbb{R}$,

$$
S(t) \geq \frac{A}{\mu + 2\beta^\infty A/\mu} =: S_L > 0.
$$
Proof. By (11) in the proof of Lemma 4.1, we see that \( \|U(t)\|_W \leq A/\mu \) for all \( t \in \mathbb{R} \). This implies that \( \int_0^{+\infty} \omega_j(t, \xi) d\xi \leq A/\mu \) for all \( t \in \mathbb{R} \) and \( j = 1, 2 \). We then have from the first equation in (1) that, for all \( t \in \mathbb{R} \),

\[
\frac{dS(t)}{dt} \geq A - \left( \mu + 2\beta^\infty \frac{A}{\mu} \right) S(t).
\]

For any \( t \in \mathbb{R} \) and \( r < t \), we then have

\[
S(t) \geq S(r) e^{-\left( \mu + 2\beta^\infty \frac{A}{\mu} \right)(t-r)} + \int_r^t A e^{-\left( \mu + 2\beta^\infty \frac{A}{\mu} \right)(t-\sigma)} d\sigma
\]

\[
\geq \frac{A}{\mu + 2\beta^\infty \frac{A}{\mu}} \left( 1 - e^{-\left( \mu + 2\beta^\infty \frac{A}{\mu} \right)(t-r)} \right).
\]

Letting \( r \to -\infty \), we obtain \( S(t) \geq S_L \). This completes the proof. \( \square \)

By using Lemmas 6.1 and 6.2, we prove the following proposition.

**Proposition 5.** Assume that \( \beta(\cdot) \) is not zero almost everywhere. Either \( g(U(\cdot)) \) is identically zero or strictly positive on \( \mathbb{R} \).

**Proof.** By (24) and Lemma 6.2, we have, for all \( t \in \mathbb{R} \),

\[
g(U(t)) \geq S_L \int_0^t \beta(\xi) \left[ J_1(t-\xi)e^{-\left( \mu + \theta_{j^*}^\infty \right) \xi} + J_2(t-\xi)e^{-\left( \mu + \theta_{j^*}^\infty \right) \xi} \right] d\xi + \tilde{I}(t),
\]

\[
\geq S_L \int_0^t \beta(\xi)e^{-\left( \mu + \theta_{j^*}^\infty + \theta_{j^*}^\infty \right) \xi} g(U(t-\xi)) d\xi + \tilde{I}(t). \tag{25}
\]

By Lemma 6.1 and appropriate shifts, for any \( r \in \mathbb{R} \), we see that if \( g(U(t)) = 0 \) for all \( t \leq r \), then \( U(\cdot) \) is identically zero. Suppose that there exists a decreasing sequence \( \{ r_j \}_{j=1}^{+\infty} \) such that \( r_1 < r \) and \( r_j \to -\infty \) as \( j \to +\infty \) and \( g(U(r_j)) > 0 \) for all \( j \in \mathbb{N} \). For any \( t \in \mathbb{R} \) and \( j \in \mathbb{N} \), let \( U_j(t) := U(t + r_j) \) and \( \tilde{I}_j(t) := \tilde{I}(t + r_j) \). We then have from (25) that, for all \( t \in \mathbb{R} \),

\[
g(U_j(t)) \geq S_L \int_0^t \beta(\xi)e^{-\left( \mu + \theta_{j^*}^\infty + \theta_{j^*}^\infty \right) \xi} g(U_j(t-\xi)) d\xi + \tilde{I}_j(t).
\]

Note that \( \tilde{I}_j(\cdot) \) is continuous at 0 and \( \tilde{I}_j(0) = \tilde{I}(r_j) = g(U(r_j)) > 0 \). By [26, Corollary B.6], we see that there exists a \( b > 0 \) such that \( g(U_j(t)) > 0 \) for all \( t > b \). This implies that \( g(U(t)) > 0 \) for all \( t > b + r_j \). Since \( r_j \to -\infty \) as \( j \to +\infty \), we obtain that \( g(U(t)) > 0 \) for all \( t \in \mathbb{R} \). This implies that \( g(U(\cdot)) \) is strictly positive on \( \mathbb{R} \). This completes the proof. \( \square \)

By using Propositions 4 and 5, we establish the following theorem.

**Theorem 6.3.** Assume that \( r(\Phi'[0]) > 1 \) and \( \beta(\cdot) \) is not zero almost everywhere. Then, system (1) is uniformly strongly \( g \)-persistent for nontrivial initial condition, that is, there exists an \( \epsilon > 0 \) such that \( \lim_{t \to +\infty} g(U(t)u_0) > \epsilon \), provided \( u_0 \in \Omega_0 \).

**Proof.** By Propositions 4 and 5, we can apply [26, Theorem 5.2] to conclude that the uniform weak \( g \)-persistence implies the uniform strong \( g \)-persistence. This completes the proof. \( \square \)

From [26, Theorem 5.7], we have the following result.
Theorem 6.4. There exists a compact attractor $\mathcal{A}_1$ that attracts all solutions with initial condition belonging to $\Omega_0$. Moreover there exists some $\varepsilon > 0$ such that, for any total trajectory $\mathcal{U}(\cdot)$ in $\mathcal{A}_1$,

$$g(\mathcal{U}(t)) \geq \varepsilon \quad \text{for all } t \in \mathbb{R}. \quad (26)$$

$\mathcal{A}_1$ is called the persistence attractor [23, Section 8]. Note that (26) states that the sum of $i_1(t,0)$ and $i_2(t,0)$ is bounded below by a positive constant. To show that both of $i_1(t,0)$ and $i_2(t,0)$ are bounded below by a positive constant is important for constructing a Lyapunov function. The main purpose of the following proposition is to prove that our solution is bounded away from 0 in the persistence attractor $\mathcal{A}_1$.

Proposition 6. For any total trajectory $\mathcal{U}(\cdot) \in \mathcal{A}_1$, the following estimates hold for all $t \in \mathbb{R}$:

$$S(t) \geq \frac{A}{\mu + 2\beta \infty A/\mu},$$

$$i_1(t,0) \geq \varepsilon, \quad i_2(t,0) \geq \varepsilon k_2 \Gamma_1,$$

$$i_1(t,\xi_1) \geq \varepsilon \frac{i_1^*(\xi_1)}{i_1^*(0)}, \quad i_2(t,\xi_2) \geq \varepsilon \frac{i_2^*(\xi_2)}{i_2^*(0)},$$

$$R(t) \geq \frac{\varepsilon}{\mu + \delta} \left((1 - k_1) \Gamma_1 + (1 - k_2) \Gamma_2\right).$$

Proof. The first inequality follows directly from Lemma 6.2. By (26) we have $i_1(t,0) + i_2(t,0) \geq \varepsilon$ for all $t \in \mathbb{R}$. Without loss of generality, we suppose that $i_1(t,0) \geq \varepsilon$ for all $t \in \mathbb{R}$. Hence from the equation of $i_1$ in (15), we get

$$i_1(t,\xi_1) \geq \varepsilon \Theta_1(\xi_1), \quad t \in \mathbb{R}, \quad \xi_1 \in \mathbb{R}^+.$$

Since $i_1^*(\xi_1) = \Theta_1(\xi_1)i_1^*(0)$ for all $\xi_1 \in \mathbb{R}^+$, we have

$$i_1(t,\xi_1) \geq \varepsilon \frac{i_1^*(\xi_1)}{i_1^*(0)}, \quad t \in \mathbb{R}, \quad \xi_1 \in \mathbb{R}^+.$$

Similarly, in view of the equation of $i_2(t,0)$ in (16) and combining with the above result,

$$i_2(t,0) \geq k_1 \int_0^{+\infty} \theta_1(\xi_1)i_1(t,\xi_1)d\xi_1,$$

$$i_2(t,0) \geq \varepsilon k_1 \Gamma_1, \quad t \in \mathbb{R},$$

and hence

$$i_2(t,\xi_2) \geq \varepsilon k_1 \Gamma_1 \Theta_2(\xi_2),$$

$$i_2(t,\xi_2) \geq \varepsilon k_1 \frac{i_2^*(\xi_2)}{i_2^*(0)}, \quad t \in \mathbb{R}, \quad \xi_2 \in \mathbb{R}^+.$$

From the equation of $R$ in (15) and combining with the above results, we have

$$R'(t) \geq (1 - k_1) \varepsilon \Gamma_1 + (1 - k_2) \varepsilon \Gamma_2 - (\mu + \delta)R(t), \quad t \in \mathbb{R},$$

so by a simple computation we conclude that

$$R(t) \geq \frac{\varepsilon}{\mu + \delta} \left((1 - k_1) \Gamma_1 + (1 - k_2) \Gamma_2\right), \quad t \in \mathbb{R}.$$

This completes the proof. \qed
7. Global stability and uniqueness of the endemic equilibrium. In this section, we restrict our attention to a special case where
\[ \theta_1(\cdot) = \theta_2(\cdot) =: \theta(\cdot) \quad \text{and} \quad k_1 = k_2 =: k. \]

In this case, by adding the equations of \( i_1 \) and \( i_2 \) in (1), we obtain the following simplified system, for \( t > 0 \),

\[
\begin{cases}
\frac{dS(t)}{dt} = A - \mu S(t) - S(t) \int_0^{+\infty} \beta(\xi)i(t, \xi)d\xi, \\
\frac{d(i(t, \xi))}{dt} + \frac{\partial i(t, \xi)}{\partial \xi} = -(\mu + \theta(\xi))i(t, \xi), \quad \xi > 0, \\
i(t, 0) = S(t) \int_0^{+\infty} \beta(\xi)i(t, \xi)d\xi + k \int_0^{+\infty} \theta(\xi)i(t, \xi)d\xi + \delta R(t), \\
\frac{dR(t)}{dt} = (1 - k) \int_0^{+\infty} \theta(\xi)i(t, \xi)d\xi - (\mu + \delta)R(t),
\end{cases}
\]

where \( i = i_1 + i_2 \) and \( \delta = \delta_1 + \delta_2 \). By Theorem 3.2, if \( r(\Psi([0])) > 1 \), then system (27) has an endemic equilibrium \( E^* = (S^*, i^*(\cdot), R^*) \in \mathbb{R}^+ \times X^+ \times \mathbb{R}^+ \), which satisfies

\[
\begin{cases}
0 = A - \mu S^* - S^* \int_0^{+\infty} \beta(\xi)i^*(\xi)d\xi, \\
\frac{di^*(\xi)}{d\xi} = -(\mu + \theta(\xi))i^*(\xi), \quad \xi > 0, \\
i^*(0) = S^* \int_0^{+\infty} \beta(\xi)i^*(\xi)d\xi + k \int_0^{+\infty} \theta(\xi)i^*(\xi)d\xi + \delta R^*, \\
0 = (1 - k) \int_0^{+\infty} \theta(\xi)i^*(\xi)d\xi - (\mu + \delta)R^*.
\end{cases}
\]

Now, we rewrite the functions already defined in the last section for this particular case. Let \( Y_0^+ := \mathbb{R}^+ \times \{0\} \times X^+ \times \mathbb{R}^+ \) and \( \varrho : Y_0^+ \to \mathbb{R} \) becomes
\[
\varrho(\Phi(t)u_0) := i(t, 0) = J(t) \quad t \geq 0, \quad u_0 \in Y_0^+.
\]

The set \( \Omega_0 \) is rewritten as
\[
\Omega_0 := \left\{ u_0 = (S_0, 0, i_0(\cdot), R_0)^T \in Y_0^+ : \varrho(u_0) > 0 \right\}.
\]

In addition, recall that \( U : \mathbb{R} \to A_1 \) is a total trajectory, then for all \( t \in \mathbb{R} \), \( U(t) = (S(t), 0, i(t, \cdot), R(t))^T \) is a solution of the following problem

\[
\begin{cases}
\frac{dS(t)}{dt} = A - \mu S(t) - S(t) \int_0^{+\infty} \beta(\xi)i(t, \xi)d\xi, \\
i(t, \xi) = i(t - \xi, 0)\Theta(\xi), \\
i(t, 0) = S(t) \int_0^{+\infty} \beta(\xi)i(t, \xi)d\xi + k \int_0^{+\infty} \theta(\xi)i(t, \xi)d\xi + \delta R(t), \\
\frac{dR(t)}{dt} = (1 - k) \int_0^{+\infty} \theta(\xi)i(t, \xi)d\xi - (\mu + \delta)R(t).
\end{cases}
\]

We now prove the following lemma.
Lemma 7.1. The following equality holds,
\[
0 = \int_0^{+\infty} S^* \beta(\xi) i^*(\xi) \left[ \frac{S(t, \xi) i^*(\xi)}{S^* i^*(\xi)} - \frac{i(t, 0)}{i^*(0)} + 1 - \frac{S(t, \xi) i^*(0)}{S^* i^*(\xi) i(t, 0)} \right] d\xi
\]
\[+
\int_0^{+\infty} k\theta(\xi) i^*(\xi) \left[ \frac{i(t, \xi)}{i^*(\xi)} - \frac{i(t, 0)}{i^*(0)} + 1 - \frac{i(t, \xi) i^*(0)}{i^*(\xi) i(t, 0)} \right] d\xi
\]
\[+
\delta R^* \left[ \frac{R}{R^*} - \frac{i(t, 0)}{i^*(0)} + 1 - \frac{R i^*(0)}{R^* i(t, 0)} \right].
\]

where, for simplicity, we write \( S = S(t) \) and \( R = R(t) \).

Proof. We have from the boundary conditions in (27) and (28) that
\[
0 = i(t, 0) - i^*(0) \frac{i(t, 0)}{i^*(0)},
\]
\[
= S(t) \int_0^{+\infty} \beta(\xi) i(t, \xi) d\xi + k \int_0^{+\infty} \theta(\xi) i(t, \xi) d\xi + \delta R(t)
\]
\[-
\left[ S^* \int_0^{+\infty} \beta(\xi) i^*(\xi) d\xi + k \int_0^{+\infty} \theta(\xi) i^*(\xi) d\xi + \delta R^* \right] \frac{i(t, 0)}{i^*(0)},
\]
\[
= \int_0^{+\infty} S^* \beta(\xi) i^*(\xi) \left[ \frac{S(t, \xi)}{S^* i^*(\xi)} - \frac{i(t, 0)}{i^*(0)} \right] d\xi
\]
\[+
\int_0^{+\infty} k\theta(\xi) i^*(\xi) \left[ \frac{i(t, \xi)}{i^*(\xi)} - \frac{i(t, 0)}{i^*(0)} \right] d\xi + \delta R^* \left[ \frac{R}{R^*} - \frac{i(t, 0)}{i^*(0)} \right].
\]

Multiplying \( 1 - i^*(0)/i(t, 0) \) by both sides of (31), we obtain (30). This completes the proof.

Let \( g(x) = x - 1 - \ln x, \ x > 0 \). We now prove the following theorem.

Theorem 7.2. Assume that \( r(\Psi'[0]) > 1 \). Then, the endemic equilibrium \( E^* = (S^*, i^*, R^*) \in \mathbb{R}^+ \times X^+ \times \mathbb{R}^+ \) is globally asymptotically stable in \( \Omega_0 \).

Proof. Let \( \mathcal{U} : \mathbb{R} \to A_1 \) be a total trajectory such that \( \mathcal{U}(t) = (S(t), 0, i(t, \cdot), R(t))^T \), \( S(0) = S_0 \), \( i(0, \cdot) = i_0(\cdot) \) and \( R(0) = R_0 \). We set \( V_1(S) := S^* g(S/S^*) \). Using the first equation in (28), the derivative of \( V_1(S) \) is calculated as follows.
\[
V_1'(S) = \left( 1 - \frac{S^*}{S} \right) \frac{dS}{dt} = \left( 1 - \frac{S^*}{S} \right) \left[ A - \mu S - S \int_0^{+\infty} \beta(\xi) i(t, \xi) d\xi \right],
\]
\[
= \left( 1 - \frac{S^*}{S} \right) \left[ \mu S^* + S \int_0^{+\infty} \beta(\xi) i^*(\xi) d\xi - \mu S^{\prime} - S \int_0^{+\infty} \beta(\xi) i(t, \xi) d\xi \right],
\]
\[
= -\frac{\mu(S - S^*)}{S} \int_0^{+\infty} S^* \beta(\xi) i^*(\xi) \left[ 1 - \frac{S^*}{S} \right] - \frac{S(t, \xi) i(t, \xi)}{S^* i^*(\xi) + i^*(\xi)} d\xi.
\]

Let \( V_2(i) := \int_0^{+\infty} p(\xi) g(i(\cdot, \xi)/i^*(\xi)) d\xi \), where \( p(\xi) := \int_0^{+\infty} q(\sigma) d\sigma, \ \xi > 0 \) and \( q(\cdot) \in X^+ \) is defined below. By [23, Lemma 9.4], the derivative of \( V_2(i) \) is calculated as follows.
\[
V_2'(i) = \int_0^{+\infty} q(\xi) \left[ g \left( \frac{i(t, \xi)}{i^*(\xi)} \right) - g \left( \frac{i(t, \xi)}{i^*(\xi)} \right) \right] d\xi,
\]
\[
= \int_0^{+\infty} q(\xi) \left[ i(t, 0) - i^*(0) \right] d\xi + \ln i^*(0) i(t, \xi) d\xi.
\]
Let $V_3(R) := R^*g(R/R^*)$. Using the fourth equation in (28), the derivative of $V_3(R)$ is calculated as follows.

$$\frac{dV_3}{dR} = \left(1 - \frac{R^*}{R}\right) \left(1 - \frac{R^*}{R}\right) \left(1 - k\right) \int_0^{+\infty} \theta(\xi) i(t, \xi) d\xi - (\mu + \delta) R,$$

$$= \left(1 - \frac{R^*}{R}\right) \left(1 - k\right) \int_0^{+\infty} \theta(\xi) i(t, \xi) d\xi + (\mu + \delta) R^* \left(1 - \frac{R^*}{R}\right).$$

$$= \int_0^{+\infty} (1 - k) \theta(\xi)i^*(\xi) \left[1 - \frac{R}{R^*} - \frac{R^* i(t, \xi)}{R^* i^*(\xi)} + \frac{i(t, \xi)}{i^*(\xi)}\right] d\xi. \quad (34)$$

Let $V(\mathcal{U}) := V_1(S) + V_2(i) + CV_3(R)$, where $C > 0$ is a constant defined below. By (32)-(34), the derivative of $V(\mathcal{U})$ is calculated as follows.

$$V'(\mathcal{U}) = -\frac{\mu(S - S^*)^2}{S} + \int_0^{+\infty} S^* \beta(\xi)i^*(\xi) \left[1 - \frac{S^*}{S} - \frac{Si(t, \xi)}{S^* i^*(\xi)} + \frac{i(t, \xi)}{S^* i^*(\xi)}\right] d\xi$$

$$+ \int_0^{+\infty} q(\xi) \left[i(t, 0) - \frac{i(t, \xi)}{i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi$$

$$+ \int_0^{+\infty} C(1 - k) \theta(\xi)i^*(\xi) \left[1 - \frac{R}{R^*} - \frac{R^* i(t, \xi)}{R^* i^*(\xi)} + \frac{i(t, \xi)}{i^*(\xi)}\right] d\xi. \quad (35)$$

Let

$$q(\xi) := S^* \beta(\xi)i^*(\xi) + k \theta(\xi)i^*(\xi) + C(1 - k) \theta(\xi)i^*(\xi), \quad \xi > 0.$$ 

We then have from (35) that

$$V'(\mathcal{U}) = -\frac{\mu(S - S^*)^2}{S}$$

$$+ \int_0^{+\infty} S^* \beta(\xi)i^*(\xi) \left[1 - \frac{S^*}{S} - \frac{Si(t, \xi)}{S^* i^*(\xi)} + \frac{i(t, \xi)}{S^* i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi$$

$$+ \int_0^{+\infty} k \theta(\xi)i^*(\xi) \left[i(t, 0) - \frac{i(t, \xi)}{i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi$$

$$+ \int_0^{+\infty} C(1 - k) \theta(\xi)i^*(\xi) \left[1 - \frac{R}{R^*} - \frac{R^* i(t, \xi)}{R^* i^*(\xi)} + \frac{i(t, \xi)}{i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi. \quad (36)$$

By Lemma 7.1, adding (30) to (36), we have

$$V'(\mathcal{U}) = -\frac{\mu(S - S^*)^2}{S}$$

$$+ \int_0^{+\infty} S^* \beta(\xi)i^*(\xi) \left[1 - \frac{S^*}{S} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)} + 1 - \frac{Si(t, \xi)i^*(0)}{S^* i^*(\xi) i(t, \xi)}\right] d\xi$$

$$+ \int_0^{+\infty} k \theta(\xi)i^*(\xi) \left[1 - \frac{i(t, \xi)}{i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi$$

$$+ \int_0^{+\infty} C(1 - k) \theta(\xi)i^*(\xi) \left[1 - \frac{R}{R^*} - \frac{R^* i(t, \xi)}{R^* i^*(\xi)} + \frac{i(t, \xi)}{i^*(\xi)} + \ln \frac{i^*(0) i(t, \xi)}{i(t, \xi)}\right] d\xi$$

$$+ \delta R^* \left[\frac{R}{R^*} - \frac{i(t, 0)}{i^*(0)} + 1 - \frac{R^* i(0)}{R^* i(t, 0)}\right]. \quad (37)$$
Let $C = \delta R^* / \int_0^{+\infty} (1 - k)\theta(\xi)i^*(\xi)d\xi$. We then obtain from (37) that

\[
V'(\mathcal{U}) = -\frac{\mu(S - S^*)^2}{S} + \int_0^{+\infty} S^* \beta(\xi)i^*(\xi) \left[-g \left(\frac{S^*}{S} \right) - \ln \frac{i^*(0)i(t, \xi)}{i(t, 0)i^*(\xi)} + 1 - \frac{Si(t, \xi)i^*(0)}{S^*i^*(\xi)i(t, 0)} \right] d\xi
+ \int_0^{+\infty} \mu(\xi)i^*(\xi) \left[-g \left(\frac{i(t, \xi)i^*(0)}{i^*(\xi)i(t, 0)} \right) \right] d\xi
+ \int_0^{+\infty} C(1 - k)\theta(\xi)i^*(\xi) \left[1 - \frac{R^*i(t, \xi)}{R^*i(\xi)} + \ln \frac{i^*(0)i(t, \xi)}{i(t, 0)i^*(\xi)} + 1 - \frac{R^*i(t, 0)}{R^*i(\xi)} \right] d\xi
= -\frac{\mu(S - S^*)^2}{S} + \int_0^{+\infty} S^* \beta(\xi)i^*(\xi) \left[-g \left(\frac{S^*}{S} \right) - g \left(\frac{Si(t, \xi)i^*(0)}{S^*i^*(\xi)i(t, 0)} \right) \right] d\xi
+ \int_0^{+\infty} \mu(\xi)i^*(\xi) \left[-g \left(\frac{i(t, \xi)i^*(0)}{i^*(\xi)i(t, 0)} \right) \right] d\xi
+ \int_0^{+\infty} C(1 - k)\theta(\xi)i^*(\xi) \left[-g \left(\frac{R^*i(t, \xi)}{R^*i(\xi)} \right) - g \left(\frac{R^*i(t, 0)}{R^*i(\xi)} \right) \right] d\xi \leq 0. \quad (38)
\]

Thus, we obtain $V'(\mathcal{U}) \leq 0$. This implies that $V(\mathcal{U}(t))$ is nonincreasing for all $t \in \mathbb{R}$. Since $V$ is bounded for all $\mathcal{U} \in \mathcal{A}_1$, the alpha limit set of $\mathcal{U}(\cdot)$ must be contained in the largest invariant subset $M$ of the set $\{(S, 0, i(\cdot), R) \in \mathcal{A}_1 \subset Y^+_0 : V' = 0\}$. Notice that $V' = 0$ implies that

\[
S(t) = S^*, \quad \frac{i(t, \xi)i^*(0)}{i^*(\xi)i(t, 0)} = 1 \quad \text{and} \quad \frac{R(t)i^*(0)}{R^*i(t, 0)} = 1, \quad t \in \mathbb{R}. \quad (39)
\]

Using the fact that $i^*(\xi) = \Theta(\xi)i^*(0)$, the second equality in (39) is equivalent to $i(t, \xi) = \Theta(\xi)i(t, 0)$. Now since $S' = 0$ then from (29) we obtain

\[
A = \mu S^* - S^* \int_0^{+\infty} \beta(\xi)i(t, \xi)d\xi,
= \mu S^* - S^* \int_0^{+\infty} \beta(\xi)\Theta(\xi)d\xi.
\]

Combining this with the fact that $A = \mu S^* - S^* \int_0^{+\infty} \beta(\xi)\Theta(\xi)d\xi$, we get

\[
i(t, 0) = i^*(0), \quad t \in \mathbb{R}.
\]

Finally, from the third equality in (39), we conclude that $M$ consists of only the endemic equilibrium. Since $V$ attains its minimum at the endemic equilibrium and it is nonincreasing, $\mathcal{U}(t) = (S^*, 0, i^*(\cdot), R^*)$ holds for all $t \in \mathbb{R}$. Therefore, the compact attractor $\mathcal{A}_1$ is reduced to the endemic equilibrium. In addition by [26, Theorem 2.39] the endemic equilibrium is also locally stable. Uniqueness of this one is a direct consequence of the fact that $V'(\mathcal{U}(t)) = 0$ holds only on the line $S = S^*$. This completes the proof. \qed

8. **Numerical simulations.** In this section, we present some numerical simulations for model (1)-(2) to illustrate the different cases obtained in the previous sections.

We consider the following values of parameters

\[
A = 10^{-1}, \quad \mu = 7.5 \times 10^{-2} \quad \text{and} \quad \delta_1 = \delta_2 = 0.01
\]
with the initial conditions
\[ S_0 = 5 \times 10^{-1}, \quad R_0 = 10^{-1} \quad \text{and} \quad i_{1,0}(a) = i_{2,0}(a) = e^{-0.08a} \times 10^{-2}. \]

The transmission function \( \beta \) is given by
\[
\beta(a) = \begin{cases} 
0, & \text{if } a \leq 10, \\
(a - 10)^2 e^{-0.1(a - 10)} \times 10^{-2}, & \text{if } a > 10.
\end{cases}
\]

In the first time, set \( k_1 = k_2 = 0.1 \) and the functions \( \theta_1 \) and \( \theta_2 \) are chosen as (see Figure 2)
\[
\theta_1(a) \equiv \theta_2(a) = \begin{cases} 
0, & \text{if } a \leq 12, \\
5(a - 12)^2 e^{-0.12(a - 12)} \times 10^{-3}, & \text{if } a > 12.
\end{cases}
\]

For these parameters \( r(\Psi'[0]) = 0.9414 < 1. \) According to the result stated, the disease-free equilibrium is globally asymptotically stable (Figures 3 and 4).
Figure 4. The evolution of solutions $i_1$ and $i_2$ with respect to time $t$ and age $a$. The case of disease-free equilibrium with $r(\Psi'[0]) < 1$.

In the second time, we take $k_1 = 0.6$, $k_2 = 0.4$ and the functions $\theta_1$ and $\theta_2$ are chosen as

$$
\theta_1(a) = \begin{cases} 0, & \text{if } a \leq 12, \\ 6(a-12)^2 e^{-0.12(a-12)} \times 10^{-3}, & \text{if } a > 12, \end{cases}
$$

and

$$
\theta_2(a) = \begin{cases} 0, & \text{if } a \leq 12, \\ 2(a-12)^2 e^{-0.12(a-12)} \times 10^{-3}, & \text{if } a > 12. \end{cases}
$$

In this case the recovery rate $\theta_1$ is more important than the recovery rate $\theta_2$ (see Figure 2), then $r(\Psi'[0]) = 1.4163 > 1$. From Figures 5 and 6, it is seen that the solution converge to the endemic equilibrium and the density of individuals $i_1$ is significantly lower than the density of individuals $i_2$ at equilibrium.

Figure 5. The evolution of solution $S$ and $R$ with respect to time $t$ are drawn. The case of endemic equilibrium with $r(\Psi'[0]) > 1$.

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Figure 6. The evolution of solutions $i_1$ and $i_2$ with respect to time $t$ and age $a$. The case of endemic equilibrium with $r(\Psi'[0]) > 1$.

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E-mail address: chekrour@math.univ-lyon1.fr
E-mail address: fr_med13@yahoo.fr
E-mail address: tkuniya@port.kobe-u.ac.jp
E-mail address: tarik.touaoula@mail.univ-tlemcen.dz