THE \textsc{t}-\textsc{analogs of string functions for} $A^{(1)}_1$ AND 
\textsc{hecke indefinite modular forms}.

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\textbf{Abstract.} We study generating functions for Lusztig’s $t$-analog of weight multiplicities associated to integrable highest weight representations of the simplest affine Lie algebra $A^{(1)}_1$. At $t = 1$, these reduce to the string functions of $A^{(1)}_1$, which were shown by Kac and Peterson to be related to certain Hecke indefinite modular forms. Using their methods, we obtain a description of the general $t$-string function; we show that its values can be realized as radial averages of a certain extension of the Hecke indefinite modular form.

1. \textsc{Introduction}

1.1. Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let $\delta$ denote its null root. Let $\Lambda$ be a dominant integral weight of $\mathfrak{g}$, and $L(\Lambda)$ the corresponding irreducible highest weight representation. A weight $\lambda$ of $L(\Lambda)$ is maximal if $\lambda + \delta$ is not a weight of this module. For a maximal dominant weight $\lambda$ of $L(\Lambda)$, the \textit{string function} $c_{\Lambda}^{\lambda}(\tau)$ is (up to multiplication by a power of $q = e^{2\pi i \tau}$) the generating function of weight multiplicities along the $\delta$-string $\{\lambda - k\delta : k \geq 0\}$. String functions are known to be modular forms of weight $-1/2$ for certain congruence subgroups of $\text{SL}_2(\mathbb{Z})$ [1].

We now consider the simplest affine algebra $\mathfrak{g} = A^{(1)}_1$. This is the only case for which an explicit description of all string functions is known.

\textbf{Theorem 1.} (\cite{2}) Let $\mathfrak{g} = A^{(1)}_1$. Let $\Lambda$ be a dominant integral weight of $\mathfrak{g}$, and $\lambda$ be a maximal dominant weight of $L(\Lambda)$. Then

$$c_{\Lambda}^{\lambda}(\tau) = \theta_{L}(\tau) \eta(\tau)^{-3}.$$ 

Here $\theta_{L}(\tau)$ is a Hecke indefinite modular form and $\eta(\tau)$ is the Dedekind eta function.

We explain these notions further in the next subsection. In this paper, we consider Lusztig’s $t$-analog of weight multiplicities (Kostka-Foulkes polynomials) and the corresponding $t$-string functions (see section 2.1). The level 1 $t$-string functions are explicitly known for all simply-laced untwisted affines and for the twisted affines $[3,4]$. Our present goal is to obtain a description of all $t$-string functions for $\mathfrak{g} = A^{(1)}_1$, thereby generalizing theorem [1].

\textbf{2000 Mathematics Subject Classification.} 33D67, 17B67.
1.2. In order to state our main theorem, we recall some background from [2]. Let \( g = A_1^{(1)} \). Fix a dominant integral weight \( \Lambda \) of \( g \) of level \( m \geq 1 \), and let \( \lambda \) be a maximal dominant weight of \( L(\Lambda) \). Let \( N \) denote the quadratic form defined on \( \mathbb{R}^2 \) by:

\[
N(x, y) := 2(m + 2)x^2 - 2my^2 \quad (x, y \in \mathbb{R})
\]

and let \( (\cdot | \cdot) \) denote the corresponding symmetric bilinear form. Let \( M := \mathbb{Z}^2 \) and let \( M^* \) denote the lattice dual to \( M \) with respect to this form.

Let \( O(N) \) denote the group of invertible linear operators on \( \mathbb{R}^2 \) preserving \( N \), and \( SO_0(N) \) be the connected component of \( O(N) \) containing the identity. We then have the groups \( G := \{ g \in SO_0(N) : gM = M \} \) and \( G_0 := \{ g \in G : g \text{ fixes } M^*/M \text{ pointwise} \} \). The set \( U^+ := \{(x, y) \in \mathbb{R}^2 : N(x, y) > 0 \} \) is preserved under the action of \( O(N) \) on \( \mathbb{R}^2 \). We let \( A := \left( \frac{\lambda + \rho_0}{2m+2} \right) \) and \( B := \left( \frac{\lambda}{2m} \right) \) where \( \rho_0 \) is the coroot corresponding to the underlying finite type diagram (\( A_2 \) in this case), and \( \rho \) is the Weyl vector. Then, \( (A, B) \in M^* \), and we set \( L := (A, B) + M \). The Hecke indefinite modular form that occurs in theorem \( \boxed{} \) is the following sum:

\[
\theta_L(\tau) := \sum_{(x, y) \in L \cap U^+} \text{sign}(x, y)e^{\pi i\tau N(x, y)},
\]

where \( \text{sign}(x, y) = 1 \) for \( x \geq 0 \) and \( -1 \) for \( x < 0 \). This is an absolutely convergent sum for \( \tau \) in the upper half plane \( \mathbb{H} \), and defines a cusp form of weight 1. We set \( \mathbb{D} := \{ \omega \in \mathbb{C} : |\omega| < 1 \} \), and \( \mathbb{D} \) its closure in the metric topology.

We now consider the group \( \tilde{G} := \langle \zeta \rangle \ltimes G \) where \( \zeta \in O(N) \) is defined by \( \zeta(x, y) := (-x, y) \). We have

\[
\tilde{F} := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \text{ or } 0 > y > x \}
\]

is a fundamental domain for the action of \( \tilde{G} \) on \( U^+ \). Given \( (x, y) \in \mathbb{R}^2 \), we let \( (x^\dagger, y^\dagger) \) denote the unique element of \( \tilde{F} \) which is in the \( \tilde{G} \)-orbit of \( (x, y) \). We now extend \( \theta_L(\tau) \) to a function \( \theta_L(\omega; \tau) \) on \( \mathbb{D} \setminus \{0\} \times \mathbb{H} \) as follows:

\[
\theta_L(te^{2\pi i u}; \tau) := \sum_{(x, y) \in L \cap U^+} \text{sign}(x, y)e^{\pi i\tau N(x, y)}e^{2(\tau^\dagger - B)}e^{2\pi i u((m+2)x^\dagger - my^\dagger - \frac{1}{2})}
\]

(1.1)

where \( 0 < t \leq 1 \) and \( -1/2 \leq u < 1/2 \). This turns out to be a well-defined function, continuous in \( \omega = te^{2\pi i u} \) and holomorphic in \( \tau \). It can be viewed as a (specialization of a) kind of theta function associated to the indefinite lattice \( L \).

We also extend \( \eta(\tau)^{-3} \) to the function \( \eta^{(-3)}(\omega; \tau) : \mathbb{D} \times \mathbb{H} \to \mathbb{C} \) defined by

\[
\eta^{(-3)}(\omega; \tau) := e^{-\pi i \tau/4} \prod_{n=1}^{\infty} \frac{1}{1 - e^{2\pi i n \tau}(1 - \omega e^{2\pi i n \tau})(1 - \overline{\omega} e^{2\pi i n \tau})}.
\]
Clearly, \( \vartheta_L(1; \tau) = \theta_L(\tau) \) and \( \eta^{(-3)}(1; \tau) = \eta(\tau)^{-3} \).

Finally, recall that the Poisson kernel \( P(\omega) : \mathbb{D} \to \mathbb{C} \) is the positive harmonic function defined by:

\[
P(\omega) = \frac{1 - |\omega|^2}{(1 - \omega)(1 - \overline{\omega})}.
\]

We set \( P_t(u) := P(te^{2\pi i u}) \).

The following, which is our main theorem, states that the values of the \( t \)-string function \( c^\Lambda(\lambda; t, \tau) \) for \( 0 < t < 1 \) are the radial averages of \( \vartheta_L(\omega; \tau) \eta^{(-3)}(\omega; \tau) \) with respect to the measure defined by \( P_t(u) \).

**Theorem 2.** Let \( F(\omega; \tau) := \vartheta_L(\omega; \tau) \eta^{(-3)}(\omega; \tau) \). Then

\[
c^\Lambda(\lambda; t, \tau) = \int_0^1 F(te^{2\pi i u}; \tau) P_t(u) \, du.
\]

We recall that the collection \( \{P_t(u)\} \) is an approximate identity on the unit circle; as \( t \to 1 \), the measure defined by \( P_t(u) \) approaches the Dirac delta measure supported on the single point 1. In this limit, we have

\[
\int_0^1 F(te^{2\pi i u}; \tau) P_t(u) \, du \to \vartheta_L(1; \tau) \eta^{(-3)}(1; \tau) = \theta_L(\tau) \eta(\tau)^{-3},
\]

giving us back theorem 1.

The proof of theorem 2 occupies the rest of the paper. Our proof closely follows that of theorem 1 by Kac and Peterson [2].

2.

2.1. We assume throughout that \( \mathfrak{g} = A_1^{(1)} \). Let \( \mathfrak{h} \) denote the Cartan subalgebra of \( \mathfrak{g} \), \( K \in \mathfrak{h} \) the canonical central element, \( \Delta_+ \) the set of positive roots and \( \delta \) the null root. Let \( Q, P \) denote the root and weight lattices of \( \mathfrak{g} \). The standard basis [1, chapter 6] of \( \mathfrak{h}^* \) is \( \{\alpha_1, \delta, \Lambda_0\} \), where \( \alpha_1 \) is the simple root corresponding to the underlying \( \mathfrak{sl}_2 \) diagram, and \( \Lambda_0 \) is a fundamental weight corresponding to the extended node. Given \( \lambda \in \mathfrak{h}^* \) of level \( m = \langle \lambda, K \rangle \), we have \( \lambda = b(\lambda)\alpha_1 + d(\lambda)\delta + m\Lambda_0 \) for unique scalars \( b(\lambda) \) and \( d(\lambda) \).

The \( t \)-Kostant partition function is given by:

\[
K(\beta; t) := |e(-\beta)| \prod_{\alpha \in \Delta_+} \frac{1}{(1 - te(-\alpha))^{m_\alpha}}.
\]

i.e., the coefficient of \( e(-\beta) \) in the product. For \( \mathfrak{g} = A_1^{(1)} \), \( m_\alpha = 1 \) for all \( \alpha \in \Delta_+ \). Lusztig’s \( t \)-analog of weight multiplicity or (affine) Kostka-Foulkes polynomial \( m^\Lambda_\mu(t) \) is defined by

\[
m^\lambda_\mu(t) := \sum_{w \in W} \epsilon(w) K(w(\lambda + \rho) - (\mu + \rho); t)
\]

where \( \lambda, \mu \) are dominant integral weights, \( W \) is the Weyl group, and \( \epsilon \) is its sign character.
For \( \beta \in Q \), define the function \( K'(\beta; t) \) as follows:
\[
K'(\beta; t) := K(\beta; t) + t K(r_1 \cdot \beta; t).
\]
We have the following simple observation: 
\( K(\beta; t) = 0 \) iff \( \beta \notin Q_+ \), and 
\( K'(\beta; t) = 0 \) iff \( \beta \notin Q_+ \cup r_1 \cdot Q_+ \) iff \( d(\beta) \leq -1 \).

We consider the corresponding generating functions for values along \( \delta \)-strings, defined by (for \( \beta \in Q \), \( \lambda, \mu \in P^+ \)):
\[
K_\beta := \sum_{n \geq 0} K(\beta + n\delta; t) q^n
\]
\[
K'_\beta := \sum_{n \geq 0} K'(\beta + n\delta; t) q^n
\]
\[
a^\lambda(t, q) := \sum_{k \geq 0} m^\lambda_{\mu - k\delta}(t) q^k
\]
where \( q := e(-\delta) \). We will also think of these as functions of \( \tau \in \mathbb{H} \) by setting \( q := e^{2\pi i \tau} \). Now, let
\[
\Gamma_t := \prod_{\alpha \in \Delta_+} \frac{1}{1 - te(-\alpha)} = \sum_{\beta \in Q_+} K(\beta; t)e(-\beta)
\]
\[
\xi_t := \frac{1}{\prod_{n \geq 1}(1 - tq^n)(1 - tq^n e(-\alpha_1))(1 - tq^n e(\alpha_1))}
\]
Then, we observe that \( \Gamma_t = \frac{\xi_t}{1 - te(-\alpha_1)} \).

2.2. We recall the constant term map \( \text{ct}(\cdot) \), defined on formal sums \( \sum_{\alpha \in Q} c_\lambda e(\lambda) \) by \( \text{ct}(\sum_{\alpha \in Q} c_\lambda e(\lambda)) := \sum_{n \in \mathbb{Z}} c_n e(n\delta) \). Let \( \mathcal{L} := \{ \beta \in Q : d(\beta) \leq 0 \} \).

**Lemma 1.** If \( \beta \in \mathcal{L} \), then
\[
K'_\beta = \text{ct}(e(\beta) \xi_t P_t)
\]
where \( P_t := \sum_{n \in \mathbb{Z}} t^{\lvert n \rvert} e(n\alpha_1) \) is the (formal) Poisson kernel of the unit disc.

**Proof.** Let \( \beta \in \mathcal{L} \). Observe that in this case \( r_1 \cdot \beta \in \mathcal{L} \), and the sums on the right hand sides of equations (2.1) and (2.2) can be replaced by \( \sum_{n \in \mathbb{Z}} \). It then follows from definitions that (i) \( K'_\beta = K_\beta + t K_{r_1 \cdot \beta} \), (ii) \( K_\beta = \text{ct}(\Gamma_t e(\beta)) \) and (iii) \( K_{r_1 \cdot \beta} = \text{ct}(\Gamma_t e(r_1 \cdot \beta)) \).

For \( \xi = \sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t) e(\lambda) \), define \( \overline{\xi} := \sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t) e(r_1(\lambda)) \). Note that \( \text{ct}(\xi) = \text{ct}(\overline{\xi}) \). For \( \xi = \Gamma_t e(r_1 \cdot \beta) \), we have \( \overline{\xi} = \overline{\Gamma_t} e(\beta + \alpha_1) \). Further, it is easy to see that \( \Gamma_t + t e(\alpha_1) \overline{\Gamma_t} = P_t \xi_t \). Putting all these facts together completes the proof.

Let \( W \) denote the Weyl group of \( \mathfrak{g} \); it can be written as \( W = T \times \hat{W} \) where \( T \) is the group of translations by elements of the finite root lattice \( \hat{Q} = \mathbb{Z} \alpha_1 \), and \( \hat{W} = \{ 1, r_1 \} \) is the Weyl group of the underlying \( \mathfrak{sl}_2 \). The extended affine Weyl group \( \hat{W} := \hat{T} \times \hat{W} \) where \( \hat{T} \) is the set of translations
by elements of the finite weight lattice. Let \( t_\alpha \) denote the translation by the element \( \alpha \) of the finite weight lattice. Define \( \tau := t_{\alpha_1/2} \) and \( \sigma := \tau r_1 \).

Then \( T = \{ \tau^{2n} : n \in \mathbb{Z} \} \), and the element \( \sigma \) permutes the simple roots of \( g \), \( \sigma(\alpha_0) = \alpha_1 \) and \( \sigma(\alpha_1) = \alpha_0 \), and fixes the Weyl vector \( \rho \). We also have the following formula for the action of \( \tau \):

\[
\tau(\lambda) = \lambda + \frac{1}{2} (\langle \lambda, \delta \rangle \alpha_1 - \langle \lambda, \alpha_1 \rangle \delta - \langle \lambda, \delta \rangle \frac{\delta}{2}).
\] (2.5)

We note that \( \langle \lambda, \delta \rangle \) is the level of \( \lambda \). Define a function \( I : Q \times \mathbb{Z} \to \{0, \pm 1\} \) by

\[
I(\beta, j) := \begin{cases} 
1 & \text{if } b(\beta) \geq 0, j \geq 0, \\
-1 & \text{if } b(\beta) < 0, j < 0, \\
0 & \text{otherwise}.
\end{cases}
\] (2.6)

**Lemma 2.** Let \( \beta \in Q \). Then

\[
K_\beta = \sum_{j \in \mathbb{Z}} (-1)^j I(\beta, j) t^j K'_{\tau^j \cdot \beta}.
\]

**Proof.** Since \( \sigma \) interchanges the simple roots \( \alpha_0, \alpha_1 \) and fixes \( \rho \), it is clear that \( K(\beta; t) = K(\sigma \beta; t) = K(\sigma \cdot \beta; t) \) for all \( \beta \in Q \). Now, this implies that \( K(\beta; t) = K'(\beta; t) - t K'(r_1 \cdot \beta; t) = K'(\beta; t) - t K(\tau \cdot \beta; t) \). Iterating the last expression gives

\[
K(\beta; t) = \sum_{j \geq 0} (-1)^j t^j K'(\tau^j \cdot \beta; t).
\] (2.7)

Similarly, replacing \( \beta \) by \( \sigma \beta \), one obtains the relations \( K(\beta; t) = t^{-1} K'(\beta; t) - t^{-1} K(\tau^{-1} \cdot \beta; t) \) and hence

\[
K(\beta; t) = -\sum_{j < 0} (-1)^j t^j K'(\tau^j \cdot \beta; t).
\] (2.8)

The sums in equations (2.7) and (2.8) are in fact finite (as can be seen from equation (2.10) below) and either expression can be used for a given \( \beta \in Q \). But choosing the expression (2.7) (resp. (2.8)) when \( b(\beta) \geq 0 \) (resp. \( b(\beta) < 0 \)), we obtain

\[
K(\beta; t) = \sum_{j \in \mathbb{Z}} (-1)^j I(\beta, j) t^j K'(\tau^j \cdot \beta; t).
\] (2.9)

To complete the proof, it only remains to replace \( \beta \) by \( \beta + n\delta \) \((n \geq 0)\) in (2.9) and observe that (i) \( I(\beta + n\delta, j) = I(\beta, j) \) and (ii) \( \tau^j \cdot (\beta + n\delta) = (\tau^j \cdot \beta) + n\delta \).

**Lemma 3.** Let \( \beta \in Q \).

1. \( \{ j \in \mathbb{Z} : I(\beta, j) \neq 0 \} \subset \{ j \in \mathbb{Z} : d(\tau^j \cdot \beta) \leq d(\beta) \} \).
2. If \( I(\beta, j) \neq 0 \) and \( \beta \in \mathcal{L} \), then \( \tau^j \cdot \beta \in \mathcal{L} \).
Proof. The second assertion clearly follows from the first. To prove (1), we use equation (2.5) to obtain:

$$\tau^j \cdot \beta - \beta = \tau^i(\beta + \rho) - (\beta + \rho) = j \alpha_1 - \left( j b(\beta) + \frac{j(j+1)}{2} \right) \delta. \quad (2.10)$$

Thus \(d(\tau^j \cdot \beta - \beta) = -j b(\beta) - \frac{j(j+1)}{2} \). It is clear from equation (2.6) that this is non-positive for all pairs \((\beta, j)\) for which \(I(\beta, j) \neq 0\).

2.3. We will henceforth fix \(\Lambda\), a dominant integral weight of \(g\), and \(\lambda\) a maximal dominant weight of \(L(\Lambda)\). We may assume assume without loss of generality that \(\Lambda - \lambda \in \mathbb{Z} \alpha_1\). For \(w \in W\), define

\[s(w) := w(\Lambda + \rho) - (\Lambda + \rho) = w \cdot \Lambda - \lambda \in \mathcal{Q}.
\]

**Lemma 4.** \(s(w) \in \mathcal{L}\) for all \(w \in W\).

**Proof.** We have \(d(\beta) = \langle \beta, \Lambda_0 \rangle\) for all \(\beta \in \mathcal{Q}\). Thus \(d(s(w)) = \langle w(\Lambda + \rho) - (\Lambda + \rho), \Lambda_0 \rangle + \langle \Lambda - \lambda, \Lambda_0 \rangle\). The second term is zero since \(\Lambda - \lambda \in \mathbb{Z} \alpha_1\), while the first term equals \((\Lambda + \rho, w^{-1} \Lambda_0 - \Lambda_0)\) which is non-positive since \(\Lambda_0\) is a dominant weight. \(\square\)

**Lemma 5.** \(a^\lambda(t, q) = \sum_{w \in W} (-1)^{\ell(w)} K_{s(w)}\)

**Proof.** This follows from the definitions. \(\square\)

By lemma 2 we get

\[a^\lambda(t, q) = \sum_{w \in W} \sum_{j \in \mathbb{Z}} (-1)^{\ell(w)+j} I(s(w), j) t^j K^\prime_{\tau^j \cdot s(w)}.\]

By lemmas 3 and 4 it is clear that \(\tau^j \cdot s(w) \in \mathcal{L}\) for all pairs \((w, j)\) for which \(I(s(w), j) \neq 0\). Thus by lemma 4 \(K^\prime_{\tau^j \cdot s(w)} = ct(P_t \xi_t e(\tau^j \cdot s(w))).\)

Now, define a function \(\tilde{e} : W \times \mathbb{Z} \to \{0, \pm 1\}\) by \(\tilde{e}(w, j) := (-1)^{\ell(w)+j} I(s(w), j)\), and let:

\[\mathcal{H} := \sum_{(w, j) \in W \times \mathbb{Z}} \tilde{e}(w, j) t^j e(\tau^j \cdot s(w)).\]

Then \(a^\lambda(t, q) = ct(P_t \xi_t \mathcal{H}).\)

3.

3.1. Let \(U := \mathbb{R} \alpha_1 \oplus \mathbb{R} \alpha_1\) and \(M := \mathbb{Z} \alpha_1 \oplus \mathbb{Z} \alpha_1\). We identify \(U\) with \(\mathbb{R}^2\) and \(M\) with \(\mathbb{Z}^2\). Define a quadratic form \(N\) on \(U\) by:

\[N(x, y) := 2(m+2)x^2 - 2my^2, x, y \in \mathbb{R}.
\]

We observe that \(N(\nu)\) is a non-zero even integer for \(\nu \in M \setminus \{0\}\).

The dual lattice \(M^* = \frac{1}{2(m+2)} \mathbb{Z} \oplus \frac{1}{2m} \mathbb{Z}\). Given elements \(\mu_1, \mu_2 \in P\) of levels \(m + 2\) and \(m\) respectively, observe that \(\left( \frac{b(\mu_1)}{m+2}, \frac{b(\mu_2)}{m} \right) \in M^*, \) since \(b(\mu_i) = \frac{\left( \mu_i, \alpha_1 \right)}{2} \in \frac{1}{2} \mathbb{Z}\).
Lemma 6. For \((w,j) \in W \times \mathbb{Z}\), we have

\[
\tau^j \cdot s(w) = \left( (m + 2)x - my - \frac{1}{2} \right) \alpha_1 - \frac{1}{2} N(x,y) \delta + \left( s_\Lambda(A) + \frac{1}{8} \right) \delta
\]

where \(x := \frac{j}{2} + \frac{b(w(\Lambda + \rho))}{m+2}\) and \(y := \frac{j}{2} + \frac{b(\Lambda)}{m}\).

Proof. This is an easy calculation. The coefficient of \(\delta\) was computed in [2, (5.13)] for \(w \in T\).

Corollary 1. \(c^A_\chi(t,q) := q^{s_\Lambda(A)} a^A_\chi(t,q) = ct(P_t(q^{\frac{1}{2}} \xi_t) \vartheta)\) where

\[
\vartheta := \sum_{(w,j) \in W \times \mathbb{Z}} \tilde{e}(w,j) j^{\frac{1}{2} N(x,y)} z^{(m+2)x-my-1/2}. \quad (3.1)
\]

Here \(z := e(\alpha_1)\), and \(x, y \in M^*\) are the functions of \((w,j)\) defined in lemma 6.

We remark that \(q^{-\frac{1}{2}} \xi_t\) reduces to \(\eta(q)^{-3}\) at \(t = 1\), where \(\eta(q)\) is the Dedekind eta function.

We now turn to the map \(\phi : W \times \mathbb{Z} \to M^*\) given by \((w,j) \mapsto (x,y)\) where

\[
x := \frac{j}{2} + \frac{b(w(\Lambda + \rho))}{m+2} \quad \text{and} \quad y := \frac{j}{2} + \frac{b(\Lambda)}{m}.
\]

as in lemma 4. Let \(\phi(e,0) = (A,B)\) where \(e\) is the identity element of \(W\). Thus \(A = \frac{b(\Lambda + \rho)}{m+2}\) and \(B = \frac{b(\Lambda)}{m}\). As in [2], we also assume (without loss of generality) that \(0 \leq B < A < \frac{1}{2}\).

Lemma 7. (1) \(\phi\) is injective.
(2) \(\text{Im } \phi = \bigcup_{i=1}^4 L_i\), where

\[
L_1 := (A,B) + M
\]

\[
L_2 := (A + \frac{1}{2}, B + \frac{1}{2}) + M \quad (3.3)
\]

\[
L_3 := (-A,B) + M \quad (3.4)
\]

\[
L_4 := (-A + \frac{1}{2}, B + \frac{1}{2}) + M \quad (3.5)
\]

i.e., a union of translates of \(M\).

Proof. Using lemma 5, it is clear that \(\phi(w_1,j_1) = \phi(w_2,j_2)\) implies \(j_1 = j_2\) and \(\tau^{j_1} \cdot s(w_1) = \tau^{j_2} \cdot s(w_2)\). In turn, this means \(s(w_1) = s(w_2)\), and hence \(w_1 = w_2\), since \(\Lambda + \rho\) is regular dominant. This proves (1).

Next, let \((w,j) \in W \times \mathbb{Z}\). Recall that since \(W = T \times W\), \(w\) can be uniquely written as \(\tau^{2n} \omega\) for some \(n \in \mathbb{Z}, \omega \in W = \{1, r_1\}\). Now, equation (2.3) implies that

\[
x = \frac{j}{2} + n + (\text{sgn } \omega)A, \quad y = \frac{j}{2} + B \quad (3.6)
\]
where sgn is the sign character of $\tilde{W}$. It is now clear that if $\tilde{W} \times \bar{Z}$ is written as the disjoint union of the four subsets $S_1 := T \times 2Z, S_2 := T \times (2Z + 1), S_3 := Tr_1 \times 2Z, S_4 := Tr_1 \times (2Z + 1)$, then $\phi(S_i) = L_i$ for $1 \leq i \leq 4$. □

We remark that our assumption on $A, B$ ensures that the $L_i$ are pairwise disjoint. From lemma 7, we see that $\vartheta$ has the following equivalent expression:

$$\vartheta = \sum_{(x, y) \in \bigcup_{i=1}^{4} L_i} \epsilon(x, y) t^{2(y-B)} q^{\frac{1}{2}N(x,y)} z^{(m+2)x-my-1/2} \tag{3.7}$$

where $\epsilon(x, y) := \bar{\epsilon}(\phi^{-1}(x, y))$ for $(x, y) \in \bigcup_{i=1}^{4} L_i$.

Next, we analyze the set of pairs $(x, y)$ for which $\epsilon(x, y) \neq 0$.

**Lemma 8.** For $1 \leq i \leq 4$, we have

$$\{ (x, y) \in L_i : \epsilon(x, y) \neq 0 \} = L_i \cap \tilde{F}$$

where $\tilde{F} := \{(u, v) \in U : 0 \leq v \leq u \text{ or } 0 > v > u\}$.

**Proof.** We prove this only for $i = 1$, the rest of the cases being similar. Fix $(x, y) \in L_1$; by lemma 7 we have $(x, y) = \phi(w, j)$ where $w = \tau^{2n}, n \in \mathbb{Z}, j \in 2Z$. Now $\epsilon(x, y) \neq 0$ iff $I(s(w), j) \neq 0$ iff either (i) $n, j \geq 0$ or (ii) $n, j < 0$. From equation (3.6) and our assumption that $0 \leq B \leq A < \frac{1}{2}$, it follows that (i) is equivalent to $0 \leq y \leq x$ and (ii) is equivalent to $0 > y > x$. □

3.2. Let $O(U, N)$ denote the group of invertible linear operators on $U$ preserving the quadratic form $N$, and let $SO_0(U, N)$ be the connected component of $O(U, N)$ containing the identity. Let $a \in GL(U)$ be defined by

$$a(u, v) := ((m + 1)u + mv, (m + 2)u + (m + 1)v).$$

Let $G$ be the subgroup of $GL(U)$ generated by $a$, and $G_0$ be the subgroup of $G$ generated by $a^2$. It is known that

$$G = \{ g \in SO_0(U, N) : gM = M \}$$

We note that elements of $G$ also leave $M^*/M$ invariant, and thus $G$ has a natural action on $M^*/M$. It is known that

$$G_0 = \{ g \in G : g \text{ fixes } M^*/M \text{ pointwise} \}.$$

Define $\zeta \in O(U, N)$ by $\zeta(u, v) := (-u, v)$, and let

$$\tilde{G} := \langle \zeta \rangle \times G \text{ and } \tilde{G}_0 := \langle \zeta \rangle \times G_0.$$

We have the following easy properties: (i) $\zeta^2$ is the identity, (ii) $\zeta a \zeta^{-1} = a^{-1}$, (iii) $\tilde{G}$ is an infinite dihedral group. We have the following diagram of inclusions between the four groups. Each inclusion is as an index 2 subgroup.
Proof. The first statement follows from the fact that \( L \)-wise. To show Lemma 9.

Let \( U \) where \( z \)

\[ L_1 = aL_4 = \zeta L_3 = a\zeta L_2. \]

We now show that the \( \tilde{G} \)-orbit of \( L_1 \) is \( \{ L_i : i = 1 \cdots 4 \} \).

\[ \text{Lemma 9.} \]

(1) If \( g \in G_0 \) then \( gL_i = L_i \) for \( i = 1 \cdots 4 \).

(2) \( L_1 = aL_4 = \zeta L_3 = a\zeta L_2 \).

Proof. The first statement follows from the fact that \( G_0 \) fixes \( M^*/M \) point-wise. To show \( L_1 = aL_4 \), observe using lemma \( \mathfrak{A} \) that \( a^{-1}(A,B) = (-A + \frac{1}{2}, B + \frac{1}{2}) + b(s(e))(1,-1) \). Since \( s(e) = \lambda - \lambda \in Q \), \( b(s(e)) \in \mathbb{Z} \) and we are done. The remaining two equalities are obvious.

Let \( U^+ := \{ (u,v) \in U : N(u,v) > 0 \} \).

\[ \text{Lemma 10.} \]

(1) \( U^+ \) is \( \tilde{G} \)-invariant.

(2) \( \tilde{F} \) is a fundamental domain for the action of \( \tilde{G} \) on \( U^+ \).

(3) \( F_0 := \tilde{F} \cup a\tilde{F} \cup \zeta \tilde{F} \cup a\zeta \tilde{F} \) is a fundamental domain for the action of \( G_0 \) on \( U^+ \).

Proof. (1) is clear since \( \tilde{G} \subset O(U,N) \). Now, \( F := \tilde{F} \cup \zeta \tilde{F} \) and \( F_0 = F \cup a\tilde{F} \) are known to be fundamental domains for the actions of \( G \) and \( G_0 \) (respectively) on \( U^+ \) \( \mathfrak{A} \). It follows that \( \tilde{F} \) is a fundamental domain for the action of \( \tilde{G} \) on \( U^+ \).

Lemmas \( \mathfrak{A} \) and \( \mathfrak{B} \) allow us to identify the sets \( \bigsqcup_{i=1}^{4} L_i \cap \tilde{F} \) and \( L_1 \cap F_0 \).

More precisely, define the map \( \psi : \bigsqcup_{i=1}^{4} L_i \cap \tilde{F} \to L_1 \cap F_0 \) by

\[ \psi(x,y) := \begin{cases} (x,y) & \text{if } (x,y) \in L_1 \cap \tilde{F} \\ a\zeta(x,y) & \text{if } (x,y) \in L_2 \cap \tilde{F} \\ \zeta(x,y) & \text{if } (x,y) \in L_3 \cap \tilde{F} \\ a(x,y) & \text{if } (x,y) \in L_4 \cap \tilde{F} \end{cases} \quad (3.8) \]

By lemmas \( \mathfrak{B} \) and \( \mathfrak{C} \) it is clear that \( \psi \) is well-defined, and is a bijection. In fact, the inverse map \( \psi^{-1} \) is easy to describe. Given \( (x,y) \in L_1 \cap F_0 \), \( \psi^{-1}(x,y) \) is the unique element in the \( \tilde{G} \)-orbit of \( (x,y) \) which lies in \( \tilde{F} \). We will denote \( \psi^{-1}(x,y) =: (x^\dagger, y^\dagger) \).

3.3. We now return to \( \vartheta \) in equation \( (3.7) \):

\[ \vartheta = \sum_{(x,y) \in \bigsqcup_{i=1}^{4} L_i} \epsilon(x,y) t^{2(y-B)} q^{\frac{1}{2}N(x,y)} z^{(m+2)x-my-1/2} \]

where \( z := e(\alpha_1) \). Using lemma \( \mathfrak{D} \) we can split this into four sums, one over each \( L_i \cap \tilde{F} \). We then perform a change of variables, replacing \( (x,y) \in \)
\[ \bigoplus_{i=1}^{L_i} L_i \] by \( \psi(x, y) \in L_1 \cap F_0 \). Since \( N(x, y) = N(x^\dagger, y^\dagger) \), the resulting sum becomes:
\[
\vartheta = \sum_{(x, y) \in L_1 \cap F_0} \epsilon(x^\dagger, y^\dagger) t^2(y^\dagger - B) q^{1/2} N(x, y) z^{(m+2)x^\dagger - my^\dagger - 1/2}.
\]

For \((x, y) \in U^+\), define \( \text{sign}(x, y) := 1 \) if \( x > 0 \) and \(-1 \) if \( x < 0 \). We then have:

**Lemma 11.** For \((x, y) \in L_1 \cap F_0\), \( \epsilon(x^\dagger, y^\dagger) = \text{sign}(x, y) \).

**Proof.** As in the above discussion, we split this into the four cases \((x, y) \in L_1 \cap g \tilde{F} \) for (i) \( g = e \), (ii) \( g = a \zeta \), (iii) \( g = \zeta \) and (iv) \( g = a \). We only consider case (ii), which is representative of the calculation needed for the other cases. For \((x, y) \in L_1 \cap a \zeta \tilde{F}\), we have \((x^\dagger, y^\dagger) = (a \zeta)^{-1} (x, y) \in L_2 \cap \tilde{F}\).

Let \((x^\dagger, y^\dagger) = \phi(w, j)\) where \( w = \tau^{2n} \), \( n \in \mathbb{Z} \), \( j \in 2\mathbb{Z} + 1 \). Now \( \epsilon(x^\dagger, y^\dagger) = \epsilon(w, j) = -I(s(w), j) \). Now, \( \epsilon(x^\dagger, y^\dagger) \) equals \(-1 \) if \( n, j \geq 0 \) and \( 1 \) if \( n, j < 0 \). In other words \( \epsilon(x^\dagger, y^\dagger) = -\text{sign}(x^\dagger, y^\dagger) = \text{sign}(x, y) \). The last equality follows from the fact that \( a \) leaves \( \text{sign} \) invariant, while \( \zeta \) reverses it. \( \square \)

Since \( \text{sign}(x, y) \) and \( N(x, y) \) are constant on \( G_0 \)-orbits, we have:
\[
\vartheta = t^{-2B} z^{-1/2} \sum_{(x, y) \in L_1 \cap U^+ \mod G_0} \text{sign}(x, y) q^{1/2} N(x, y) t^2 y^\dagger \ z^{(m+2)x^\dagger - my^\dagger} \tag{3.9}
\]

where for \((x, y) \in U^+\), we let \((x^\dagger, y^\dagger)\) denote the unique element in \( \tilde{F} \cap (\tilde{G} \text{-orbit of } (x, y)) \).

Finally, we observe that corollary 1 and equation (3.9) imply theorem 2.

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