Efficient Certification of Spatial Robustness

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Abstract

Recent work has exposed the vulnerability of computer vision models to spatial transformations. Due to the widespread usage of such models in safety-critical applications, it is crucial to quantify their robustness against spatial transformations. However, existing work only provides empirical quantification of spatial robustness via adversarial attacks, which lack provable guarantees. In this work, we propose novel convex relaxations, which enable us, for the first time, to provide a certificate of robustness against spatial transformations. Our convex relaxations are model-agnostic and can be leveraged by a wide range of neural network verifiers. Experiments on several network architectures and different datasets demonstrate the effectiveness and scalability of our method.

1 Introduction

It was recently shown that neural networks are susceptible not only to standard noise-based adversarial perturbations (Szegedy et al., 2013; Goodfellow et al., 2015; Carlini, Wagner, 2017; Madry et al., 2018), but also to spatially transformed images that are visually indistinguishable from the original (Kanbak et al., 2017; Alaifari et al., 2019; Xiao et al., 2018; Engstrom et al., 2019). Such spatial attacks can be modeled by smooth vector fields that describe the displacement of every pixel. Common geometric transformations, e.g., rotation and translation, are particular instances of these smooth vector fields, which indicates that they capture a wide range of naturally occurring image transformations.

Since the vulnerability of neural networks to spatially transformed adversarial examples can pose a security threat to computer vision systems relying on such models, it is critical to quantify their robustness against spatial transformations. A common approach to estimate neural network robustness is to measure the success rate of strong attacks (Carlini, Wagner, 2017; Madry et al., 2018). However, many networks which are indeed robust against these attacks were later broken using even more sophisticated attacks (Athalye, Carlini, 2018; Athalye et al., 2018; Engstrom et al., 2018; Tramer et al., 2020). The key issue is that such attacks do not provide provable robustness guarantees.

![Image instances and corresponding deforming vector fields](a) Original, (b) \( \delta = 1, \gamma = \infty \), (c) \( \delta = 3, \gamma = 0.25 \).

Figure 1: Image instances and corresponding deforming vector fields (\( \gamma \) and \( \delta \) explained below): (a) original, (b) adversarially deformed (label 5) by non-smooth vector field, and (c) adversarially deformed (label 3) by smooth vector field.
To address this, our goal is to provide a provable certificate that a neural network is robust against all possible deforming vector fields within an attack budget for a given dataset of images. While various certification methods exist, they are limited to noise-based perturbations (Katz et al., 2017; Gehr et al., 2018; Wong, Kolter, 2018; Singh et al., 2018; Zhang et al., 2018) or compositions of common geometric transformations (e.g., rotations and translations) (Pei et al., 2017; Singh et al., 2019b; Balunovic et al., 2019; Mohapatra et al., 2020) and thus cannot be applied to our setting.

A common approach in robustness certification is to compute, for a given image, pixel bounds that contain all possible perturbed images within some attack budget, and to then propagate these bounds through the network to obtain bounds on the output neurons. Then, if all images within the output bounds classify to the correct label, the network is provably robust against all attacks limited to the same attack budget. In our work, we intuitively parametrize the attack budget by the magnitude of pixel displacement, denoted by $\delta$, and the smoothness of the vector field, denoted by $\gamma$. This allows us to efficiently compute the tightest-possible pixel interval bounds on vector field deformations limited to a given displacement magnitude $\delta$ by using a mathematical analysis of the transformation. However, even small but non-smooth vector fields (i.e., small $\delta$ but large $\gamma$) can generate large pixel differences resulting in recognizably perturbed images (Figure 1b) and leading to large pixel bounds, which limit certification performance. Thus, a key challenge is to define smoothness constraints that can be efficiently incorporated with neural network verifiers to enable certification of smooth vector fields with large displacement magnitude (Figure 1c).

Hence, we tighten our convex relaxation for smooth vector fields by introducing smoothness constraints that can be efficiently incorporated into state-of-the-art verifiers (Tjeng et al., 2017; Singh et al., 2019a,c). To that end, we leverage the idea of computing linear constraints on the pixel values in terms of the transformation parameters (Balunovic et al., 2019; Mohapatra et al., 2020). We show that our mathematical analysis of the transformation induces an optimization problem for computing the linear constraints, which can be efficiently solved by linear programming. Finally, we show that the idea by Balunovic et al. (2019; Mohapatra et al., 2020) alone is insufficient for the setting of smooth vector fields and only the combination with our smoothness constraints yields superior certification performance. We implement our method in an open-source system and show that our convex relaxations can be leveraged to, for the first time, certify neural network robustness against vector field attacks.

**Key contributions** We make the following contributions:

- A novel method to compute the tight interval bounds for norm-constrained vector field attacks, enabling the first certification of neural networks against vector field attacks.
- A tightening of our relaxation for smooth vector fields and integration with state-of-the-art robustness certifiers.
- An open-source implementation together with extensive experimental evaluation on the MNIST and CIFAR-10 datasets, with convolutional and large residual networks. We make our code publicly available as part of the ERAN framework for neural network verification (https://github.com/eth-sri/eran).

**2 Related Work**

Here we discuss the most relevant related work on spatial robustness and certification of neural networks.

**Empirical spatial robustness** In addition to previously known adversarial examples based on $\ell_p$-norm perturbations, it has recently been demonstrated that adversarial examples can also be constructed via geometric transformations (Kanbak et al., 2017), rotations and translations (Engstrom et al., 2019), Wasserstein distance (Wong et al., 2018; Levine, Feizi, 2019; Hu et al., 2020), or via smooth vector field deformations (Alaifari et al., 2019; Xiao et al., 2018). Here, we use a threat model based on vector field deformations for which both prior works have proposed attacks. Alaifari et al. (2019) perform first-order approximations to find minimum-norm adversarial vector fields, and Xiao et al. (2018) relax vector field smoothness constraints with a continuous loss to find perceptually realistic adversarial examples.
Robustness certification There is a long line of work on certifying the robustness of neural networks to noise-based perturbations. These approaches employ SMT solvers (Katz et al., 2017), mixed-integer linear programming (Tjeng et al., 2017; Singh et al., 2019c), semidefinite programming (Raghunathan et al., 2018), or linear relaxation methods (Gehr et al., 2018; Wong, Kolter, 2018; Singh et al., 2018; Zhang et al., 2018; Singh et al., 2019b; Wang et al., 2018; Weng et al., 2018; Wang et al., 2018b; Salman et al., 2019b; Lin et al., 2019). Another line of work considers randomized smoothing (Lecuyer et al., 2018; Cohen et al., 2019; Salman et al., 2019a) which provides probabilistic robustness guarantees.

Certified spatial robustness Prior work introduced certification methods for some special cases of spatial transformations: a finite number of transformations (Pei et al., 2017), rotations (Singh et al., 2019b) and compositions of common geometric transformations (Balunovic et al., 2019; Mohapatra et al., 2020). Some randomized smoothing approaches exist, but they only handle single parameter transformations (Li et al., 2020) or transformations without compositions (Fischer et al., 2020). In another setting, Wu, Kwiatkowska (2020) compute the maximum safe radius on optical flow video perturbations but only for a finite set of neighboring grid points. Overall, previous approaches are limited by the fact that they only certify transformations in specific templates which can be characterized as special cases of smooth vector fields deformations. Certifying these vector field deformations is precisely the goal of our work.

3 Background

Here we introduce our notation, define the similarity metric for spatially transformed images, and provide the necessary background for certification of neural networks.

Spatial transformations We represent an image as a function \( I : P \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^C \), where \( C \) denotes the number of color channels, and \( P = \{1, 2, \ldots, W\}^2 \) corresponds to the set of pixel coordinates of the image of dimension \( W \times W \).

Spatial transformations are parameterized by a vector field \( \tau : P \rightarrow \mathbb{R}^2 \) assigning a displacement vector to every pixel. For an image \( I \), we define the transformed image \( I + \tau : P \rightarrow \mathbb{R}^2 \), mapping the pixel coordinates \((i, j)\) to \((I + \tau)(i, j) = (i, j) + \tau(i, j)\). Since these deformed coordinates may not lie on the integer grid, we use bilinear interpolation \( I_I : \mathbb{R}^2 \rightarrow \mathbb{R}^C \) evaluated at \((i, j) + \tau(i, j)\) to get the new pixel values:

\[
I_I(i, j) := \begin{cases} 
I_{\text{mn}}^{in}(i, j) & \text{if } (i, j) \in A_{mn}, \\
\sum_{p \in \{m, m+1\}} \sum_{q \in \{n, n+1\}} I(p, q) (1 - |p - i|) (1 - |q - j|), 
\end{cases}
\]

where \( A_{mn} := [m, m+1] \times [n, n+1] \) is an interpolation region. Like Alaifari et al. (2019), we only study vector fields that do not move pixels outside the image.

Estimating perceptual similarity For noise-based adversarial attacks, an adversarial image \( I_{\text{adv}} \) is typically considered to be perceptually similar to the original image \( I \) if the \( \ell_p \)-norm of the perturbation \( \|I - I_{\text{adv}}\|_p \) is small (Szegedy et al., 2013; Goodfellow et al., 2015; Carlini, Wagner, 2017; Madry et al., 2018). However, prior work (Alaifari et al., 2019, Xiao et al., 2018) has demonstrated that this is not necessarily a good measure for spatially transformed images. For example, translating an image by a small amount will typically produce an image that looks very similar to the original, but results in a large perturbation with respect to the \( \ell_p \)-norm. For this reason, the similarity of spatially transformed images is typically estimated with a norm on the deforming vector field and not on the pixel value perturbation. Here, we consider the \( T_p \)-norm (Alaifari et al., 2019), defined as

\[
\|\tau\|_{T_p} := \max_{(i, j) \in P} \|\tau(i, j)\|_p.
\]
We consider an instance of restricted polyhedra, introduced in (Singh et al., 2019b), as it provides state-of-the-art precision.

We consider \( p \in \{1, 2, \infty\} \) and note that the case \( p = 2 \) corresponds to the norm used by Alaifari et al. (2019). Intuitively, a vector field with \( T_2\)-norm at most 1 will displace any pixel by at most one grid length on the image grid. In general, for a vector field \( \mathbf{\tau} \) with \( \|\mathbf{\tau}\|_p \leq \delta \), the set of reachable coordinates from a single pixel \((i, j) \in P\) is

\[
B_{\delta}^p (i, j) := \left\{ x \in \mathbb{R}^2 \mid \| (i, j) - x \|_p \leq \delta \right\}.
\]

However, there may be vector fields with small \( T_p \)-norm that produce unrealistic images. For example, moving every pixel independently by at most 1 grid length can already result in images that are very pixelated and that can be easily recognized as unnatural when comparing with the original (Figure 1b). To address this, Xiao et al. (2018b) introduced a flow loss that penalizes the lack of smoothness of a vector field. Following this approach, we say that vector field \( \mathbf{\tau} \) has flow \( \gamma \) if it satisfies, for each pixel \((i, j)\), the flow-constraints

\[
\| \mathbf{\tau}(i, j) - \mathbf{\tau}(i', j') \|_{\infty} \leq \gamma, \forall (i', j') \in N(i, j),
\]

where \( N(i, j) \subseteq P \) represents the set of neighboring pixels in the 4 cardinal directions of pixel \((i, j)\). For instance, translation is parametrized by a vector field that has flow 0 (each pixel has the same displacement vector). These constraints enforce smoothness of the vector field \( \mathbf{\tau} \), which in turn ensures that transformed images look realistic and better preserve image semantics— even for large values of \( \delta \) (Figure 1c). We provide a more thorough visual investigation of the norms and constraints considered in Appendix D.

Robustness certification Robustness of neural networks is typically certified by (i) computing a convex shape around the input we want to certify (an over-approximation) and (ii) propagating this shape through all operations in the network to obtain a final output shape. Robustness is then proven if all concrete outputs inside the output shape classify to the correct class. For smaller networks, an input shape can be propagated exactly using mixed-integer linear programming (Tjeng et al., 2017). To scale to larger networks, standard approaches over-approximate the shape using various convex relaxations: intervals (Gowal et al., 2019), zonotopes (Gehr et al., 2018; Singh et al., 2018; Weng et al., 2018), and restricted polyhedra (Zhang et al., 2013; Singh et al., 2019b), just to name a few. In this work, we build on the convex relaxation DeepPoly, an instance of restricted polyhedra, introduced in (Singh et al., 2019b), as it provides state-of-the-art precision while scaling to larger networks. For every pixel, DeepPoly receives a lower and upper bound on the pixel
We now provide an end-to-end example of how to compute our convex relaxation of spatial transformations (i.e., potential extrema of bilinear interpolation, see Section 5) shown as dots in Figures 3b and 3c are bounded by linear planes to enforce flow-constraints.

To address this, we tighten the convex relaxation for smooth vector fields by incorporating flow-constraints.

Figure 3: Image with bilinear interpolation and dots showing pixel positions. Interval bound candidates (i.e., potential extrema of bilinear interpolation, see Section 5) shown as dots in Figures 3b and 3c are bounded by linear planes to enforce flow-constraints.

value, i.e., an input shape in the form of a box. This shape is then propagated by maintaining one lower and upper linear constraint for each neuron. Here, we first show how to construct a tight box around all spatially transformed images. However, since this box does not capture the relationship between neighboring pixels induced by flow-constraints, it contains spurious images that cannot be produced by smooth vector fields. To address this, we tighten the convex relaxation for smooth vector fields by incorporating flow-constraints.

4 Overview

We now provide an end-to-end example of how to compute our convex relaxation of spatial transformations and use it to certify the robustness of the toy network in Figure 2. This network propagates inputs $x_0$ and $x_1$ according to the weights annotated on the edges. Neurons $x_2$ ($x_3$) and $x_4$ ($x_5$) denote the pre- and post-activation values, and $x_6$ and $x_7$ are the network logits. We augment this network by vector field components $v_x, v_y, w_x,$ and $w_y,$ to introduce the flow-constraints.

The concrete inputs to the neural network are the pixels marked with blue ($x_0$) and green ($x_1$), shown in Figure 3a. The image is perturbed by a vector field of $T_{\infty}$-norm $\delta = 0.5$ and flow $\gamma = 0.25,$ thus the blue and green pixels are allowed to move in the respective rectangles shown in Figure 3a. However, in order to satisfy the flow-constraints, their deformation vectors can differ by at most $\gamma = 0.25$ in each coordinate.

Our objective is to certify that the neural network classifies the input to the correct label regardless of its deformed pixel positions. A simple forward pass of the pixel values $x_0 = 0$ and $x_1 = 0.5$ yields logit values $x_6 = 0$ and $x_7 = 0.625.$ Assuming that the image is correctly classified, we thus need to prove that the value of neuron $x_7$ is greater than the value of neuron $x_6$ for all admissible smooth vector field transformations. To that end, we will first compute interval bounds for the pixels, without using the relationship between vector field components $v_x, v_y, w_x,$ and $w_y,$ and then, in a second step, we tighten the relaxation for smooth vector fields by introducing linear constraints on $x_0$ and $x_1$ in terms of $v_x, v_y, w_x,$ and $w_y$ to exploit the flow-constraint relationship. Propagation of our convex relaxation through the network closely follows Singh et al. (2019b), and we provide a full formalization of our methods in Section 5.

Calculating interval bounds The first part of our convex relaxation is computing upper and lower interval bounds for the values that the blue and green pixel can attain on their $\ell_{\infty}$-neighborhood of radius $\delta = 0.5.$ For both pixels, the minimum and maximum are attained on the left and right border of the $\ell_{\infty}$-ball respectively. Using bilinear interpolation from Equation 1, we thus obtain the interval bounds $[l_0, u_0] = [0, 0.25]$ for $x_0$ and $[l_1, u_1] = [0.25, 0.75]$ for $x_1.$

Interval bound propagation The intervals $[l_0, u_0]$ and $[l_1, u_1]$ can be utilized directly for verification using standard interval propagation to estimate the output of the network (Gehr et al., 2018; Gowal et al.)
While this method is fast, it is also imprecise. The interval bounds for \( x_2 \) and \( x_4 \) are

\[
\begin{align*}
[l_2, u_2] &= 2 \cdot [ l_0, u_0] - [ l_1, u_1] + 0.25 \\
&= [2l_0 - u_1 + 0.25, 2u_0 - l_1 + 0.25] = [-0.5, 0.5], \\
[l_4, u_4] &= \max(l_2, 0), \max(u_2, 0)] = [0, 0.5].
\end{align*}
\]

All lower and upper interval bounds are given in Figure 2. The output for \( x_6 \) is thus between \( l_0 = -1 \) and \( u_0 = 0 \), while the one for \( x_7 \) is between \( l_7 = -0.375 \) and \( u_7 = 0.875 \). As this is insufficient to prove \( x_7 > x_6 \), certification fails.

**Backsubstitution** To gain precision one can keep track of the relationship between the neurons by storing linear constraints (Zhang et al., 2018; Singh et al., 2019b). In addition to \([l_2, u_2]\) we store upper and lower bounding linear constraints

\[
2x_0 - x_1 + 0.25 \leq x_2 \leq 2x_0 - x_1 + 0.25.
\]

Similarly, for \( x_4 \) we store, in addition to \([l_4, u_4]\),

\[
0 \leq x_4 \leq 0.5x_2 + 0.25,
\]

where we used the rules given in Singh et al. (2019b) to calculate the upper and lower linear constraints. All linear constraints are shown in Figure 2 next to the corresponding neurons. Certification succeeds, if we can show that \( x_7 - x_6 > 0 \). Using the linear constraints, we thus obtain

\[
\begin{align*}
x_7 - x_6 &\geq (x_2 - x_1) - (2x_4) = x_4 + x_5 \\
&\geq x_3 \geq x_1 - x_0 + 0.125 \geq 0.125.
\end{align*}
\]

This proves that \( x_7 - x_6 > 0 \), implying that the network classifies to the correct class under all considered deformations.

**Spatial constraints** Although the above method can certify robustness, certification fails for a more challenging network where the bias of \( x_3 \) is equal to \(-0.125\) instead of \(0.125\). With the previous approach we can only prove \( x_7 - x_6 \geq x_1 - x_0 - 0.125 \geq -0.125 \), which is insufficient for certification. However, we can leverage our vector field smoothness condition (Equation 4), namely that deformation vectors of \( x_0 \) and \( x_1 \) can differ by at most \( \gamma = 0.25 \) in the \( \ell_\infty \)-norm. Unfortunately, these constraints cannot be directly applied since they are defined on the vector field components and not on the pixel values. To amend this, we build on the idea from Balunovic et al. (2019), Mohapatra et al. (2020) and add upper and lower bounding linear constraints on the pixel values \( x_0 \) and \( x_1 \). That is, we compute upper and lower planes in terms of vector field components \( v_x \) and \( v_y \) for \( x_0 \) and \( w_x \) and \( w_y \) for \( x_1 \) as shown in Figures 3b and 3c. The plane equations are shown in Figure 3 and details on the computation are provided in Section 5. By substituting these plane equations into our expression and considering that all vector field components \( v_x \), \( v_y \), \( w_x \), and \( w_y \) are bounded from above and below by \([\delta, \delta] = [-0.5, 0.5] \), we thus obtain

\[
\begin{align*}
x_7 - x_6 \geq x_1 - x_0 - 0.125 \\
&\geq (0.5 + 0.5w_x) - (0.125 + 0.25v_x) - 0.125 \\
&= 0.25 + 0.5w_x - 0.25v_x \geq -0.125.
\end{align*}
\]

showing that a simple instantiation of the idea by Balunovic et al. (2019), Mohapatra et al. (2020) is insufficient for certification in our setting. Only with our flow-constraints \(-\gamma \leq v_x - w_x \leq \gamma \) with \( \gamma = 0.25 \) can we finally certify:

\[
\begin{align*}
x_7 - x_6 \geq x_1 - x_0 - 0.125 \\
&\geq 0.25 + 0.5w_x - 0.25v_x \\
&= 0.25 + 0.25w_x + 0.25(w_x - v_x) \geq 0.0625.
\end{align*}
\]

In practice, the resulting expression may have more than two pixels, and we use a linear program (described in Section 5) to perform the substitution of flow-constraints.
5 Convex Relaxation

Here, we introduce a novel convex relaxation tailored to spatial transformations. First, we compute the tightest interval bounds for each pixel in the transformed image. As intervals do not capture dependencies between variables, we then propose a method for introducing linear constraints on the pixel values in terms of vector field components and show how to use flow-constraints to further tighten our convex relaxation.

5.1 Computing Tight Interval Constraints

Consider an image $I$ and a maximum pixel displacement $\delta$. Our goal is to compute, for pixel $(i, j)$, interval bounds $l_{i,j}$ and $u_{i,j}$ such that $l_{i,j} \leq I_t \circ (I + \tau)(i,j) \leq u_{i,j}$, for any vector field $\tau$ of $T_\delta$-norm of at most $\delta$. We now show how to compute tight interval pixel bounds $[l_{i,j}, u_{i,j}]$.

Within a given interpolation region $A_{mn}$, the pixel $(i, j)$ can move to positions in $B_\delta^\circ (i,j) \cap A_{mn}$. Thus, for every pixel, we construct a set of candidates containing the possible maxima and minima of that pixel in $B_\delta^\circ (i,j) \cap A_{mn}$. However, this could potentially yield an infinite set of candidate points, and we thus make the key observation that the minimum and maximum pixel values of $I_I^{mn}$ in $B_\delta^\circ (i,j) \cap A_{mn}$ are always obtained at the boundary (see Lemma 5.1 below, with proof provided in Appendix A). Hence, for any reachable interpolation region $A_{mn}$, it suffices to consider the boundary of $B_\delta^\circ (i,j) \cap A_{mn}$ to analytically derive the candidate points. Finally, we set the lower and upper bound of the pixel value to the minimum and maximum of the candidate set, respectively.

Lemma 5.1. The minimum and maximum pixel values of $I_I^{mn}$ in $B_\delta^\circ (i,j) \cap A_{mn}$ are always obtained at the boundary.

We note that it is important to calculate the candidates analytically in order to guarantee soundness of the interval bounds. Furthermore, for a single-channel image, our interval bounds are exact, i.e., for every deformed image within our pixel bounds there exists a corresponding vector field $\tau$ with $||\tau||_\delta \leq \delta$. For the derivation of candidates we make use of the following auxiliary lemma (with proof in Appendix A):

Lemma 5.2. The bilinear interpolation $I_I^{mn}(v,w)$ on the region $A_{mn}$ can be rewritten as

$$I_I^{mn}(v,w) = A + Bv + Cw + Dwv.$$  \hspace{1cm} (3)

Candidates for $T_\infty$  The boundary of $B_\delta^\infty (i,j) \cap A_{mn}$ consists of line segments parallel to the coordinate axes. This means that for all pixels $(v,w)$ on such a line segment either $v$ or $w$ is constant. Thus, according to Lemma 5.2 if $v$ is fixed, then $I_I^{mn}$ is linear in $w$ and vice-versa. Hence, we only need to consider the line segment endpoints to obtain the candidate set of all possible extremal values in $B_\delta^\infty (i,j) \cap A_{mn}$.

Candidates for $T_1$  The boundary of $B_\delta^1 (i,j) \cap A_{mn}$ consists of line segments parallel to the coordinate axes, or lines defined by $w = \pm v + a$. In the first case, we add the pixel values of the line segment endpoints to the set of candidates. In the second case, the bilinear interpolation $I_I^{mn}$ restricted to $w = \pm v + a$ is a polynomial of degree two given by

$$I_I^{mn}(v, \pm v + a) = A + Bv + C(\pm v + a) + Dw(\pm v + a)\hspace{1cm} = (A + Ca) + (B \pm C)v \pm Dw^2.$$

Thus, the extremum can be attained on the interior of that line, unlike the $T_\infty$-case. Hence, we add both the endpoint values and the extremum of the polynomial to the candidate set if the corresponding extremal point lies on the line segment.
Candidates for $T_2$ The boundary of $B^2_2(i, j) \cap A_{mn}$ consists of line segments parallel to the coordinate axes or an arc with circle center $(i, j)$. We handle the line segments by adding the pixel values at the endpoints to the candidate set. However, the interpolation can also have minima and maxima on the interior of the arc. To find those, we extend the interpolation $T^mn$ from $A_{mn}$ to $\mathbb{R}^2$ and use Lagrange multipliers to find the extrema on the circle $v^2 + w^2 = \delta^2$ (assuming $i = j = 0$ for notational convenience). The Lagrangian is

$$\Lambda(v, w, \lambda) := T^mn(v, w) - \lambda(v^2 + w^2 - \delta^2),$$

which yields

$$\nabla_{v,w,\lambda} \Lambda(v, w, \lambda) = \begin{pmatrix}
B + Dw - 2\lambda v \\
C + Dv - 2\lambda w \\
\delta^2 - v^2 - w^2
\end{pmatrix} = 0.$$

Solving the first two equations for $\lambda$ we obtain

$$\lambda = \frac{B + Dw}{2v} \quad \text{and} \quad \lambda = \frac{C + Dv}{2w},$$

assuming $v \neq 0 \neq w$ (else $\delta = 0$). Eliminating $\lambda$, we have

$$w(B + Dw) - v(C + Dv) = 0.$$

We solve this quadratic equation and substitute the solutions

$$w = -\frac{B \pm \sqrt{B^2 + 4Dv(C + Dv)}}{2D}$$

into $\delta^2 - v^2 - w^2 = 0$ to obtain

$$\delta^2 = v^2 + \left(\frac{-B \pm \sqrt{B^2 + 4Dv(C + Dv)}}{2D}\right)^2.$$

Setting $E := \frac{B}{2D}$, $F := \frac{B^2}{4D^2}$, and $G := \frac{C}{D}$, we have

$$\delta^2 = v^2 + \left(E \pm \sqrt{F + Gv + v^2}\right)^2 = v^2 + E^2 \pm 2E\sqrt{F + Gv + v^2} + F + Gv + v^2 = 2v^2 + Gv \pm 2E\sqrt{F + Gv + v^2} + H,$$

for $H := (E^2 + F)$. Solving for $v$ requires squaring both sides to resolve the square root, thus yielding

$$\left(\delta^2 - H - Gv - 2v^2\right)^2 = \left(\pm 2E\sqrt{F + Gv + v^2}\right)^2 = 4E^2(F + Gv + v^2).$$

We are thus interested in finding the roots of

$$J + Kv + Lv^2 + Mv^3 + Nv^4$$

with

$$J := (\delta^2 - H)^2 - 4FE^2,$$
$$K := -2G \left( (\delta^2 - H) + 2E^2 \right),$$
$$L := G^2 - 4 \left( (\delta^2 - H) + E^2 \right),$$
$$M := 4G,$$
$$N := 4.$$
which is a polynomial of degree four. The roots of a polynomial of degree four are known in closed-
form (Shmakov, 2011) and we use Durand-Kerner’s root finding method (Kerner, 1966) to compute them analytically. We recall that computing the roots approximately, e.g., via gradient descent or Newton’s method, does not guarantee soundness of the interval bounds. If the coordinates obtained from the roots of Equation 4 lie within $A_{\text{fin}}$, we add the corresponding pixel values to the set of candidates. Finally, we add the pixel values of the endpoints of the arc to the set of candidates.

5.2 Computing Spatial Constraints

While our interval bounds are tight, they can contain spurious images, which cannot be produced by smooth vector fields of flow $\gamma$. To address this, we build upon the idea of Balunovic et al. (2019), Mohapatra et al. (2020) and introduce linear constraints on the pixel values in terms of the vector field, which can then be paired with flow-constraints to yield a tighter convex relaxation. We demonstrate how our method can be applied to tighten the DeepPoly relaxation (Singh et al., 2019b) and robustness certification via mixed integer linear programming (MILP) (Tjeng et al., 2017).

Upper and lower bounding planes We seek to compute sound linear constraints on the spatially transformed pixel value in terms of the deforming vector field $\tau$. Since every pixel is displaced independently by its corresponding vector, the linear constraints induce bounding planes of the form

$$\lambda_0 + \lambda \cdot \tau(i, j) \leq I(x(i, j)) \leq v_0 + v \cdot \tau(i, j),$$

where $\lambda^T = (\lambda_1, \lambda_2)$ and $v^T = (v_1, v_2)$. To compute a sound lower bounding plane, we apply our method from Section 5.1 to compute the set of candidate coordinates $C$ of potential minima and maxima in $B_{\text{fin}}^n(i, j)$ and then solve

$$\arg \min_{\lambda_0, \lambda_1, \lambda_2} \sum_{(p, q) \in C} I(p, q) - (\lambda_0 + \lambda_1(p - i) + \lambda_2(q - j))$$

$$I(p, q) \geq (\lambda_0 + \lambda_1(p - i) + \lambda_2(q - j)), \quad \forall (p, q) \in C.$$

Since both, the objective and the constraints are linear, we can compute this plane in polynomial time using linear programming. The upper bounding plane is obtained analogously.

Given these linear bounding planes, one could be tempted to simply instantiate the framework from Balunovic et al. (2019), Mohapatra et al. (2020). Unfortunately, their approach only works in the setting where multiple pixels are transformed by the same spatial parameters (e.g., rotation angle). However, we make the key observation that these linear constraints can be leveraged to enforce the flow-constraints, thus tightening our convex relaxation. We now describe how this can be achieved for the DeepPoly relaxation and MILP.

Tightening DeepPoly relaxation To compute precise bounds, DeepPoly performs backsubstitution for each neuron in the network (recall the example in Section 4). That is, every backsubstitution step computes a linear expression $e = a_1 x_1 + \ldots + a_n x_n$ in terms of the input pixels. A naive way to obtain the upper and lower bounds of $e$ is to substitute the interval bounds for each pixel $x_i$ which is equivalent to

$$\min a_1 x_1 + \ldots + a_n x_n$$

$$x_i \in [l_i, u_i].$$

However, this can be imprecise as intervals do not capture flow-constraints. Thus, we extend the above linear program with variables $v^{(i)}_x$ and $v^{(i)}_y$ denoting the vector field components of every pixel $x_i$, thus allowing us to add the constraints

$$\lambda^{(i)}_0 + (v_x, v_y)^{(i)} \lambda^{(i)} \leq x_i \leq v^{(i)}_0 + (v_x, v_y)^{(i)} v^{(i)},$$

$$-\gamma \leq v^{(i)}_x - v^{(i)}_y \leq \gamma, \quad \text{and} \quad -\gamma \leq v^{(i)}_y - v^{(i)}_y \leq \gamma,$$
| $\delta$ | $\gamma$ | MNIST | CIFAR-10 |
|---|---|---|---|
| | | CONVSMALL | PGD | DIFFAI | CONVSMALL | DIFFAI | CONVSMALL | DIFFAI | CONVSMALL | DIFFAI | CONVSMALL | DIFFAI |
| $\infty$ | | 51 | 97 | 90 | 91 | 40 | 47 | 51 | 56 | 0.3 | 6 | 89 | 74 | 76 | 78 | 84 | 31 | 38 | 36 | 42 |
| 0.1 | | 78 | 98 | 90 | 92 | 90 | 94 | 45 | 56 | 53 | 60 | 0.1 | 40 | 91 | 75 | 84 | 78 | 90 | 32 | 46 | 37 | 51 |
| 0.001 | | 91 | 99 | 91 | 95 | 91 | 95 | 50 | 70 | 57 | 70 | 0.001 | 77 | 99 | 77 | 92 | 78 | 92 | 37 | 69 | 43 | 58 |
| $\infty$ | | 6 | 89 | 74 | 76 | 78 | 84 | 31 | 38 | 36 | 42 | 0.1 | 40 | 91 | 75 | 84 | 78 | 90 | 32 | 46 | 37 | 51 |
| 0.01 | | 75 | 98 | 77 | 92 | 78 | 90 | 35 | 67 | 43 | 57 | 0.001 | 77 | 99 | 77 | 92 | 78 | 92 | 37 | 69 | 43 | 58 |
| 0.001 | | 32 | 88 | 44 | 91 | 37 | 89 | 27 | 53 | 31 | 47 |
| $\infty$ | | 0 | 76 | 40 | 50 | 36 | 62 | 20 | 32 | 29 | 34 |
| 0.1 | | 7 | 85 | 44 | 69 | 37 | 79 | 23 | 47 | 30 | 42 |
| 0.001 | 32 | 88 | 44 | 91 | 37 | 89 | 27 | 53 | 31 | 47 |
| 0.001 | 35 | 89 | 44 | 92 | 37 | 90 | 27 | 53 | 33 | 48 |

Table 1: $T_\infty$-norm certification rates (%) for vector fields $\tau$ with displacement magnitude $\|\tau\|_{T_\infty} = \delta$ and flow $\gamma$.

where $\lambda^T = (\lambda_1^{(i)}, \lambda_2^{(i)})$ and $\upsilon^T = (\upsilon_1^{(i)}, \upsilon_2^{(i)})$. Here, Equation (5) corresponds to the upper and lower bounding planes of pixel $x_i$, and Equation (6) enforces the flow-constraints for neighboring pixels $i$ and $j$. Minimization of this linear program then directly yields the tightest lower bound on the expression and can be performed in polynomial time. The upper bounding plane can be obtained analogously.

**Tightening MILP certification**  To encode a neural network as MILP we employ the method from [Tjeng et al. (2017)] which is exact for models with ReLU activations. Our approach of leveraging linear planes on pixel values to enforce flow-constraints can then be directly applied to the resulting MILP by adding the same variables and linear constraints (Equations (5) and (6)) as in the DeepPoly case.

### 6 Experiments

We now investigate the precision and scalability of our certification method by evaluating it on a rich combination of datasets and network architectures. We make our networks and code publicly available as part of the ERAN framework for neural network verification (https://github.com/eth-sri/eran) to ensure reproducibility.

**Experiment setup**  We select a random subset of 100 images from the MNIST [LeCun et al. (2010)] and CIFAR-10 [Krizhevsky (2009)] test datasets on which we run all experiments. We consider adversarially trained variants of the CONVSMALL, CONVMED, and CONVBIG architectures proposed by [Mirman et al. (2018)], using PGD [Madry et al. (2018)] and DiffAI [Mirman et al. (2018)] for adversarial training. Furthermore, for CIFAR-10, we also consider a ResNet with 4 residual blocks with 16, 16, 32, and 64 filters from [Wong et al. (2018)]. We present the model accuracies and training hyperparameters in Appendix B. We use a desktop PC with a single GeForce RTX 2080 Ti GPU and a 16-core Intel(R) Core(TM) i9-9900K CPU @ 3.60GHz, and we report all running times in Appendix C.

**Robustness certification**  We demonstrate the precision of our convex relaxations via robustness certification against spatial transformations. To that end, we run DeepPoly and MILP with our interval and spatial constraints and compute the percentage of certified MNIST and CIFAR-10 images for different networks and values of $\delta$ and $\gamma$. Note that $\gamma = \infty$ corresponds to Section 5.1 and $\gamma < \infty$ corresponds to Section 5.2. We limit MILP to 5 minutes and display the results in Table 1 (showing only $T_\infty$-norm results for brevity). We observe that our interval bounds successfully enable certification of vector field attacks and that our tightened convex relaxation for smooth vector fields offers (at times substantial) improvements across all datasets, verifiers, and networks. For example, for CONVSMALL PGD on MNIST, DeepPoly certification
Table 2: Certification rates (%) of CONVSMALL DIFFAI on MNIST for different $T_p$-norms with $\gamma = \infty$.

| $\delta$ | $T_1$-NORM | $T_2$-NORM | $T_\infty$-NORM |
|----------|-------------|-------------|-----------------|
| 0.3      | 96          | 97          | 95              |
| 0.5      | 75          | 79          | 70              |
| 0.7      | 23          | 43          | 13              |
| 0.9      | 4           | 21          | 2               |

...increases from 6% ($\gamma = \infty$) to 77% ($\gamma = 0.001$) for $\delta = 0.4$. Similarly, for CONVSMALL DIFFAI on CIFAR-10, MILP certification increases from 38% ($\gamma = \infty$) to 69% ($\gamma = 0.001$) for $\delta = 0.4$. In fact, our convex relaxation can also be applied with the k-ReLU verifier [Singh et al., 2019a] where it increases certification from 24% ($\gamma = \infty$) to 51% ($\gamma = 0.1$) for $\delta = 0.4$ for CONVSMALL PGD on MNIST.

We also compare the certification rates for the different $T_p$-norms on a CONVSMALL network trained to be provably robust with DiffAI on MNIST. For brevity, we only consider the case where $\gamma = \infty$, and we display the percentage of certified images in Table 2 (with MILP timeout of 5 minutes).

### 7 Conclusion

We introduced a novel convex relaxation for images obtained from spatial transformations and showed that this relaxation enables the first certification method for vector field attacks. Our evaluation across different datasets and architectures demonstrates the practical effectiveness of our methods.

### Ethics Statement

It is well known that neural networks can be successfully employed in settings with positive (e.g., personalized healthcare) and negative (e.g., autonomous weapons systems) social impacts. Moreover, even for settings with potentially beneficial impacts, such as personalized healthcare, the concrete implementation of these models remains challenging, e.g., concerning privacy. In that regard, robustness certification is more removed from the practical application, as it provides guarantees for a class of models (e.g., image classification networks), irrespective of the particular task at hand. For example, our method could be applied to certify spatial robustness for self-driving cars in the same way that it could be employed to prove robustness for weaponized drones. Since the spatial transformations considered in our work present a natural way of describing distortions arising from the fact that cameras map a 3D world to 2D images, our certification method can be used to comply with regulations or quality assurance criteria for all applications that require robustness against these types of transformations.

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A Calculation of Candidates

Here, we prove Lemmas 5.1 and 5.2 which we employed to efficiently compute the interval bounds \(l_{i,j}\) and \(u_{i,j}\) such that \(l_{i,j} \leq I_f \circ (I + \tau)(i, j) \leq u_{i,j}\), for a pixel \((i, j)\) of an image \(I\) and any vector field \(\tau\) of \(T_p\)-norm of at most \(\delta\). To ease notation, we view images as a collection of pixels on a regular grid. In this case, bilinear interpolation is given by

\[
I_f(i, j) := \left\{ \begin{array}{ll}
I_{mn}^{mn}(i, j) & \text{if } (i, j) \in A_{mn}, \\
\sum_{p \in \{m, m+1\}} \sum_{q \in \{n, n+1\}} I(p, q) (1 - |p - i|) (1 - |q - j|), & \text{otherwise}
\end{array} \right.
\]

where \(A_{mn} := [m, m+1] \times [n, n+1]\) is an interpolation region.

**Lemma 5.2.** The bilinear interpolation \(I_{mn}^{mn}(v, w)\) on the region \(A_{mn}\) can be rewritten as

\[
I_{mn}^{mn}(v, w) = A + Bv + Cw + Dwv.
\]

**Proof.**

\[
I_{mn}^{mn}(v, w) = \sum_{p \in \{m, m+1\}} \sum_{q \in \{n, n+1\}} I(p, q) (1 - |p - v|) (1 - |q - w|)
\]

\[
= I(m, n)(1 + m - v)(1 + n - w)
+ I(m + 1, n)(v - m)(1 + n - w)
+ I(m, n + 1)(1 + m - v)(w - n)
+ I(m + 1, n + 1)(v - m)(w - n)
= A + Bv + Cw + Dwv,
\]

where the last equality follows from expanding the parentheses and grouping the terms.

**Lemma 5.1.** The minimum and maximum pixel values of \(I_{mn}^{mn}\) in \(B_2^p(i, j) \cap A_{mn}\) are always obtained at the boundary.

**Proof.** Let \((v, w) \in B_\infty^p(i, j) \cap A_{mn}\) be an interior point such that \(I_{mn}^{mn}\) is extremal at \((v, w)\). Then, assuming \(v\) is fixed, it can be seen from Equation (5) that \(I_{mn}^{mn}\) is linear in \(w\). Hence, the pixel value is non-decreasing in one direction and non-increasing in the opposite direction of \(w\), implying that both the minimum and the maximum are obtained at the boundary.

Hence, for any reachable interpolation region \(A_{mn}\) it suffices to consider the boundary of \(B_\infty^p(i, j) \cap A_{mn}\) to construct the set of candidates containing the possible minima and maxima of the pixel in \(B_2^p(i, j) \cap A_{mn}\). The bounds \(l_{i,j}\) and \(u_{i,j}\) are then the minimum and maximum of the set of candidates, respectively.

B Models

Here, we describe the model accuracies and training hyperparameters. We recall that, for MNIST and CIFAR-10, we consider defended variants of the CONVSMALL, CONVMED, and CONVBIG architectures proposed by Mirman et al. (2018), using PGD (Madry et al. 2018) and DiffAI (Mirman et al. 2018) for adversarial training. Moreover, for CIFAR-10, we also evaluate our certification method on a ResNet with 4 residual blocks with 16, 16, 32, and 64 filters (108k neurons) trained with the provable defense from Wong et al. (2018).

To differentiate between the training techniques, we append suffixes to the model names: DiffAI for DiffAI and PGD for PGD. All DiffAI networks were pre-trained against \(\ell_\infty\)-noise perturbations by Mirman et al. (2018) with \(\epsilon = 0.3\) on MNIST and \(\epsilon = 8/255\) on CIFAR-10. For CONVSMALL PGD we used 20 epochs of PGD training against vector field attacks with \(\delta = 0.5\), using cyclic learning rate scheduling from \(1e-8\) to \(1e-2\), and \(\ell_1\)-norm weight decay with trade-off parameter \(1e-3\). We provide the model accuracies on the respective test sets in Table 3.
Table 3: Model test accuracies and training parameters.

| Dataset | Model          | Accuracy | Training   |
|---------|----------------|----------|------------|
| MNIST   | ConvSmall PGD  | 97.53%   | Spatial PGD|
|         | ConvSmall DiffAI| 94.52%   | DiffAI     |
|         | ConvBig DiffAI | 97.03%   | DiffAI     |
| CIFAR-10| ConvSmall DiffAI| 42.60%   | DiffAI     |
|         | ConvMed DiffAI | 43.57%   | DiffAI     |
|         | ResNet         | 27.70%   | WONG ET AL. (2018) |

Table 4: Average running times (seconds) for certification of vector fields $\tau$ with displacement magnitude $\|\tau\|_{T_p} = \delta$ and flow $\gamma$.

| $\delta$ | $\gamma$ | MNIST  | CIFAR-10 |
|----------|----------|--------|----------|
|          |          | ConvSmall PGD | MILP | ConvSmall DiffAI | MILP | ConvBig DiffAI | MILP | ConvSmall DiffAI | MILP | ConvMed DiffAI | MILP | ResNet |
| $\infty$ | $\infty$ | 0.1    | 1.4    | 3.8    | 5.2    | 0.1    | 1.7    | 0.1    | 1.4    | 3.8    | 5.2    | 159.9  | 404.1  | 109.4  | 373.7  |
| 0.3      | 0.1      | 9.8    | 76.6   | 34.6   | 62.3   | 225.5  | 0.5    | 128.8  | 0.7    | 214.6  |
| 0.4      | 0.1      | 9.4    | 71.3   | 3.1    | 15.9   | 65.1   | 261.6  | 0.2    | 59.1   | 0.6    | 153.4  |
|          | 0.01     | 11.9   | 61.6   | 3.1    | 8.4    | 77.1   | 319.8  | 104.9  | 268.8  | 107.0  | 368.3  |
| 0.001    | 12.0     | 9.8    | 76.6   | 7.2    | 34.6   | 62.3   | 225.5  | 103.0  | 320.7  | 93.8   | 353.3  |
| 0.01     | 11.2     | 87.0   | 7.3    | 19.5   | 72.4   | 231.6  | 159.9  | 404.1  | 109.4  | 373.7  |
|          | 0.001    | 85.6   | 12.1   | 7.1    | 28.6   | 65.9   | 288.1  | 139.8  | 370.1  | 125.3  | 410.4  |

C Running Times

Here, we provide the running times averaged over our random subsets of 100 test images. As mentioned in Section 6, we run all experiments on a desktop PC with a single GeForce RTX 2080 Ti GPU and a 16-core Intel(R) Core(TM) i9-9900K CPU @ 3.60GHz.

In Table 4, we display the average certification times corresponding to the experiment from Table 1 in Section 6. Likewise, we display the running times corresponding to the $T_p$-norm comparison experiment from Table 2 in Table 5. We recall that every instance of the MILP is limited to 5 minutes.

The average running times for the k-ReLU verifier (Singh et al., 2019a) using our convex relaxation with $k = 15$, an LP-timeout of 5 seconds, and a MILP-timeout of 10 seconds are: 176.4s ($\gamma = \infty$) and 285.3s ($\gamma = 0.1$) for $\delta = 0.4$.

Table 5: Average running times (in seconds) for certification of ConvSmall DiffAI on MNIST for different $T_p$-norms with $\gamma = \infty$. 

| $\delta$ | $T_1$-NORM | $T_2$-NORM | $T_{\infty}$-NORM |
|----------|-------------|-------------|------------------|
|          | DeepPoly    | MILP        | DeepPoly         | MILP | DeepPoly    | MILP     | DeepPoly    | MILP |
| 0.3      | 0.1         | 1.4         | 3.8              | 5.2  | 0.1         | 1.7      |
| 0.5      | 0.1         | 1.5         | 4.1              | 6.4  | 0.1         | 1.9      |
| 0.7      | 0.1         | 2.4         | 2.6              | 3.6  | 0.1         | 4.0      |
| 0.9      | 0.1         | 5.6         | 4.3              | 11.5 | 0.1         | 40.8     |
\[ \gamma = \infty \]
\[ \gamma = 0.1 \]
\[ \gamma = 0.01 \]
\[ \gamma = 0.001 \]

\[ \delta = 0.3 \]

| Image       | Label |
|-------------|-------|
| \(8\)\(8\)  | (a) label 8 |
| \(8\)\(8\)  | (b) label 8 |
| \(8\)\(8\)  | (c) label 8 (certified) |
| \(8\)\(8\)  | (d) label 8 (certified) |

\[ \delta = 0.4 \]

| Image       | Label |
|-------------|-------|
| \(4\)\(4\)  | (e) label 5 (adversarial) |
| \(4\)\(4\)  | (f) label 4 |
| \(4\)\(4\)  | (g) label 4 (certified) |
| \(4\)\(4\)  | (h) label 4 (certified) |

\[ \delta = 0.5 \]

| Image       | Label |
|-------------|-------|
| \(3\)\(3\)  | (i) label 5 (adversarial) |
| \(3\)\(3\)  | (j) label 9 (adversarial) |
| \(3\)\(3\)  | (k) label 3 (certified) |
| \(3\)\(3\)  | (l) label 3 (certified) |

\[ \delta = 0.6 \]

| Image       | Label |
|-------------|-------|
| \(0\)\(0\)  | (m) label 4 (adversarial) |
| \(0\)\(0\)  | (n) label 0 (certified) |
| \(0\)\(0\)  | (o) label 0 (certified) |
| \(0\)\(0\)  | (p) label 0 (certified) |

\[ \delta = 0.7 \]

| Image       | Label |
|-------------|-------|
| \(6\)\(6\)  | (q) label 6 (certified) |
| \(6\)\(6\)  | (r) label 6 (certified) |
| \(6\)\(6\)  | (s) label 6 (certified) |
| \(6\)\(6\)  | (t) label 6 (certified) |

Figure 4: Image instances and corresponding deforming vector fields \(\tau\) with displacement magnitude \(\|\tau\|_{T_\infty} = \delta\) and flow \(\gamma\).


D Visual Investigation

In this section, we compare original and deformed MNIST images for different $T_\infty$-norm values $\delta$ and flow-constraints $\gamma$. To that end, we employ the CONVSMALL DIFFAI network together with our convex relaxation with MILP to evaluate whether the deformed images are adversarial or certified (or neither) and we display the images in Figure 4. Note that since both MILP and our interval bounds are exact for single-channel images, the fact that Figure 4a is not adversarial is due to failure of the attack, as an adversarial vector field with $\delta = 0.3$ and $\gamma = \infty$ must exist (else MILP would certify this image). On the other hand, for Figure 4b we cannot say whether the attacker fails to find an adversarial image or if our convex relaxation is not precise enough to certify the image (since we are using an over-approximation to enforce the flow-constraints).