BOUNDNESS AND EXPONENTIAL STABILIZATION IN A PARABOLIC-ELLIPTIC KELLER–SEGEL MODEL WITH SIGNAL-DEPENDENT MOTILITIES FOR LOCAL SENSING CHEMOTAXIS

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Abstract  In this paper we consider the initial Neumann boundary value problem for a degenerate Keller–Segel model which features a signal-dependent non-increasing motility function. The main obstacle of analysis comes from the possible degeneracy when the signal concentration becomes unbounded. In the current work, we are interested in the boundedness and exponential stability of the classical solution in higher dimensions. With the aid of a Lyapunov functional and a delicate Alikakos–Moser type iteration, we are able to establish a time-independent upper bound of the concentration provided that the motility function decreases algebraically. Then we further prove the uniform-in-time boundedness of the solution by constructing an estimation involving a weighted energy. Finally, thanks to the Lyapunov functional again, we prove the exponential stabilization toward the spatially homogeneous steady states. Our boundedness result improves those in [1] and the exponential stabilization is obtained for the first time.

Key words  Classical solution; boundedness; exponential stabilization; degeneracy; Keller–Segel models

2010 MR Subject Classification  35B60; 35K20; 35K65; 35M33; 35Q92

1 Introduction

Chemotaxis is a biased movement of cells due to a chemical gradient, and it plays a significant role in diverse biological phenomena. In the 1970s, Keller and Segel proposed in their seminal work [2] the following model for chemotaxis:

\[
\begin{align*}
    u_t &= \nabla \cdot (\gamma(v)\nabla u - u\chi(v)\nabla v), \\
    \varepsilon v_t &= \Delta v - v + u.
\end{align*}
\]  

(1.1)
Here, \( u \) and \( v \) denote the density of cells and the concentration of signals, respectively. The signal-dependent cell diffusion rate \( \gamma \) and chemo-sensitivity \( \chi \) are linked via
\[
\chi(v) = (\sigma - 1)\gamma'(v), \tag{1.2}
\]
where the parameter \( \sigma \geq 0 \) is a constant proportional to the distance between the signal receptors in the cells. In the case \( \sigma > 0 \), a cell determines its moving direction due to a gradient sensing mechanism by calculating the difference of concentrations at different spots, while in the case \( \sigma = 0 \), the distance between receptors is zero and thus chemotactic movement occurs because of an undirected effect on the activity due to the presence of a chemical signal sensed by a single receptor (local sensing). One notices that in the latter case, the first equation of (1.1) has the concise form
\[
 u_t = \Delta(\gamma(v)u), \tag{1.3}
\]
where \( \gamma \) stands for a signal-dependent motility. More recently, in [3, 4], by adding a logistic source term on the right hand side of (1.3), this model was also applied to describe the process of pattern formations via the so-called “self-trapping” mechanism, where the cellular motility \( \gamma(\cdot) \) was assumed to be suppressed by the concentration of signals. In other words, \( \gamma(\cdot) \) is a signal-dependent decreasing function, i.e., \( \gamma'(v) < 0 \). We remark that \( \gamma'(v) < 0 \) indicates that cells are attracted by a high concentration of signals.

In this paper, we are interested in the boundedness and stability of classical solutions to the following initial boundary value problem for the parabolic-elliptic simplification of the original Keller–Segel model with signal-dependent motility for local sensing chemotaxis (i.e., \( \varepsilon = \sigma = 0 \) in (1.1)–(1.2)):
\[
\begin{cases}
  u_t = \Delta(\gamma(v)u) & x \in \Omega, \ t > 0, \\
  -\Delta v + v = u & x \in \Omega, \ t > 0, \\
  \partial_\nu u = \partial_\nu v = 0 & x \in \partial\Omega, \ t > 0, \\
  u(x, 0) = u_0(x) & x \in \Omega.
\end{cases} \tag{1.4}
\]
Here \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \) being a smooth bounded domain and
\[
u_0 \in C^3(\overline{\Omega}), \quad u_0 \geq 0, \quad u_0 \not\equiv 0. \tag{1.5}
\]
In general, we require that
\[
(A0): \quad \gamma(\cdot) \in C^3[0, +\infty), \quad \gamma(\cdot) > 0, \quad \gamma'(\cdot) \leq 0 \text{ on } (0, +\infty), \quad \lim_{s \to +\infty} \gamma(s) = 0. \tag{1.6}
\]
In view of the asymptotically vanishing property of \( \gamma \), an apparent difficulty in analysis lies in the possible degeneracy when \( v \) becomes unbounded. Theoretical analysis for the above Keller–Segel model with signal-dependent motility has attracted a lot of interest in recent years; see e.g. [1, 5–18], and see [19, 20] for some recent works on the gradient sensing model (1.1). A common strategy used in most literature is to derive the \( L^\infty_tL^p_x \)-boundedness of \( u \) with some \( p > \frac{n}{2} \) by an energy method. Then the \( L^\infty_tL^\infty_x \)-boundedness of \( v \) follows via an application of standard elliptic/parabolic regularity theory to the equation for \( v \). However, this method seems only to work under restrictive conditions, such as specific choices of \( \gamma \) [1, 8, 15], or with the presence of logistic source terms [5–7]. Recently, a new comparison approach was proposed in [11–14]. By introducing an auxiliary elliptic problem that enjoys a comparison principle, the
explicit point-wise upper bound estimates of \( v \) were established directly for generic motility functions satisfying (A0) in any spatial dimensions. In fact, it was proved that \( v(x,t) \) grows at most exponentially in time and hence degeneracy cannot happen in finite time. In addition, a delicate Alikakos–Moser type iteration was further developed in [13] to deduce the uniform-in-time boundedness of \( v \) directly in higher dimensions without the help of any integrability of \( u \).

Previous studies on (1.4) strongly indicate that the dynamics of solutions are closely related to the decay rate of \( \gamma \). When \( n = 2 \), it was proved in [11] that a classical solution always exists globally with any large initial datum and a generic \( \gamma \) satisfying (A0). Furthermore, if \( \gamma \) additionally satisfies that

\[
\lim_{s \to +\infty} e^{\alpha s} \gamma(s) = +\infty, \quad \text{for all } \alpha > 0,
\]

then for any large initial datum, the global classical solution is uniformly-in-time bounded [13]. Note that assumption (1.7) allows \( \gamma \) to take any decreasing form within a finite region and moreover that any motility function decreases more slowly than a standard exponentially decreasing function at high concentrations can guarantee the boundedness; for example, \( \gamma(s) = e^{-s^\beta} \) with any \( \beta \in (0,1) \). If \( \gamma \) decays even faster so that there is \( \chi > 0 \) such that

\[
\lim_{s \to +\infty} e^{\chi s} \gamma(s) = +\infty,
\]

then the solution of (1.4) is uniformly-in-time bounded provided that \( \|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi} \) [13]. In particular, if \( \gamma(s) = e^{-s^\beta} \), a critical-mass phenomenon was observed in [11, 12, 15] such that with any sub-critical mass, the classical solution is uniformly-in-time bounded. Moreover, it was pointed out that no finite-time blowup occurs and that the global solution may blow up at time infinity with certain super-critical mass [10–12].

For higher dimensions \( n \geq 3 \), the boundedness was studied in several works provided that \( \gamma \) satisfies some algebraically decreasing assumptions [1, 13, 19]. In particular, if \( \gamma(s) = s^{-k} \) with some \( k > 0 \), one notices that a variant form of (1.4) reads as

\[
\begin{cases}
    u_t = \nabla \cdot \left[ v^{-k} (\nabla u - ku \nabla \log v) \right], \\
    -\Delta v + v = u,
\end{cases}
\]

which resembles the logarithmic Keller–Segel model

\[
\begin{cases}
    u_t = \nabla \cdot (\nabla u - ku \nabla \log v), \\
    -\Delta v + v = u.
\end{cases}
\]

The above two systems share the same set of equilibria. In addition, they have similar scaling structures. Supposing that \((u,v)\) is a solution of (1.10), one checks that \( u_\lambda(t,x) = \lambda u(t,\lambda^{-k} x) \) and \( v_\lambda(t,x) = \lambda v(t,\lambda^{-k} x) \) is also a solution with any \( \lambda > 0 \), while for (1.9), the scaling is given by \( u_\lambda(t,x) = \lambda u(\lambda^{-k} t, x) \) and \( v_\lambda(t,x) = \lambda v(\lambda^{-k} t, x) \). Such a scaling invariance indicates that existence results are usually independent of the size of initial datum.

There is very limited theoretical research on both (1.9) and (1.10). Roughly speaking, the dynamics of solutions seem to be determined by the size of \( k \). For the logarithmic Keller–Segel system (1.10) there are several studies on the admissible range of \( k \) for global existence/boundedness, and on the other hand, blowup solutions were constructed only in the
radial symmetric case \( n \geq 3 \) and \( k > \frac{2n}{n-2} \) [21], though the threshold number is still unclear. In comparison, the existence of blowup solutions remains unknown for the fully parabolic version of (1.10). In [22], it was proved that a classical solution exists globally provided that \( k < \frac{2}{n} \). For the radially symmetric case, the existence of weak solutions for \( k < \infty, n = 2 \) and \( k < \frac{2}{n-2} \), \( n \geq 3 \) was shown in [23]. More recently, the asymptotic stabilization toward the constant equilibria was obtained in [24] for \( n \geq 2 \) with \( k < \frac{1}{n} \) under the smallness of the size of \( |\Omega| \). We refer the reader to [25–27] for a complete description of topics related to the logarithmic Keller-Segel model.

For the degenerate system (1.9), the boundedness of global solutions with any \( 0 < k < \frac{2}{n-2} \) was first shown in [1], and later in [13, 19] via different methods. Moreover, the existence of a global (but likely growing up) classical solution was obtained in [13] within a larger range \( 0 < k < \frac{2}{n-2} + \frac{n}{n+2} \). We also mention that, recently, in [32], some doubly nonlinear diffusion operators involving porous medium type degeneracies were considered in similar frameworks, which indicates that the simultaneous density-determined enhancement of diffusion and cross-diffusion can foster boundedness in a system of this form. It would be interesting to consider whether the comparison method developed in [12] and the idea in the present contribution can be generalized to that case.

In the present work, we aim to improve the uniform-in-time boundedness results for (1.4) in higher dimensions. The key observation of this contribution is that under an assumption

(A1) : \( \gamma(s) + s\gamma'(s) \geq 0, \quad \forall s > 0 \),

system (1.4) possesses a Lyapunov functional (see also [1]). Then we can perform an Alikakos–Moser iteration to derive a time-independent upper bound of \( v \) under weakened conditions compared with [1, 13, 19]. Also, with the aid of the Lyapunov functional, exponential stabilization of the global solution toward the spatially homogeneous steady states \((\overline{u}_0, \overline{u}_0)\) is obtained for the first time. A direct consequence of our result to the specific case \( \gamma(s) = s^{-k} \) is that the boundedness of solutions can be improved to any \( k \leq 1 \) when \( n = 4, 5 \), or \( k < \frac{4}{n-2} \) when \( n \geq 6 \). Furthermore, the solution will converge to \((\overline{u}_0, \overline{u}_0)\) exponentially as time tends to infinity.

In order to formulate our result in a more general framework, we introduce the following condition:

(A2) : there is \( k > 0 \) such that \( \lim_{s \to +\infty} s^k\gamma(s) = +\infty \).

Note that (A2) allows \( \gamma \) to take other algebraically decreasing functions. For example, if \( \gamma(s) = \frac{1}{s^{k_0} \log(1+s)} \) with any given \( k_0 > 0 \), we may take \( k = k_0 + \varepsilon \) with any \( \varepsilon > 0 \) in (A2).

Now we are in a position to state the main results of the current work.

**Theorem 1.1** Assume that \( n \geq 3 \). Suppose that \( \gamma \) satisfies (A0), (A1) and

(A3) : \( l_0|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s) \), with some \( l_0 > \frac{n+2}{4} \) for all \( s > 0 \).

Then, for any initial datum satisfying (1.5), problem (1.4) possesses a unique global classical solution that is uniformly-in-time bounded.

Moreover, there exist \( \alpha > 0 \) and \( C > 0 \) depending on \( u_0, \gamma, n \) and \( \Omega \) such that, for all \( t \geq 1 \),

\[
\|u(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{u}_0\|_{W^{1, \infty}(\Omega)} \leq Ce^{-\alpha t},
\]

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where \( \overline{u_0} = \frac{1}{|\Omega|} \int_{\Omega} u_0 dx \).

**Remark 1.2** Thanks to the strictly positive time-independent lower bound \( v_* \) of \( v \) for \((x,t) \in \overline{\Omega} \times [0,\infty)\) given in Lemma 2.2 in the next section, assumptions (A1) and (A3) can be weakened so as to hold for all \( s \geq v_* \). Thus, our existence and boundedness results also hold true if \( \gamma(s) \) has singularities at \( s = 0 \); for example \( \gamma(s) = s^{-k} \) with \( k > 0 \). In such cases, we can simply replace \( \gamma(s) \) by a new motility function \( \tilde{\gamma}(s) \) which satisfies (A0) and coincides with \( \gamma(s) \) for \( s \geq \frac{4}{n-2} \).

In particular, for the typical case \( \gamma(v) = v^{-k} \), we have

**Theorem 1.3** Suppose that \( \gamma(v) = v^{-k} \) and \( n \geq 4 \). Then for any initial datum satisfying (1.5), problem (1.4) has a unique global classical solution which is uniformly-in-time bounded provided that \( 0 \leq k \leq 1 \) when \( n = 4, 5 \), or \( 0 \leq k < \frac{4}{n-2} \) when \( n \geq 6 \). Moreover, there exist \( C > 0 \) depending on \( u_0, k, n \) and \( \Omega \) such that, for all \( t \geq 1 \),

\[
\|u(\cdot, t) - \overline{u_0}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{u_0}\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\alpha t}. \quad (1.15)
\]

**Remark 1.4** Combined with the results in [1, 13, 19], uniform-in-time boundedness for the case \( \gamma(s) = s^{-k} \) is now available if

\[
0 < k \begin{cases} < \infty, & n = 2, \\ < 2, & n = 3, \\ \leq 1, & n = 4, 5, \\ < \frac{4}{n-2}, & n \geq 6. \end{cases} \quad (1.16)
\]

The exponential decay (1.15) also holds when \( n \leq 3 \) if \( 0 < k \leq 1 \). We remark that the convergence of \((u,v)\) toward the constant solution was found in [1] when \( \gamma(s) = s^{-k} \), supposing that \( k \in (0, \frac{2}{m-2}) \cap (0, 1] \), however, no convergence rate was given.

Now, let us sketch the main idea of our proof. First, we would like to recall the following identity, which unveils the key mechanism of our system:

\[
v_t + u\gamma(v) = (I - \Delta)^{-1}[u\gamma(v)].
\]

Here \( \Delta \) denotes the usual Neumann Laplacian operator. The above key identity was first observed in [11, 12], and along with a new comparison approach, gives rise to a point-wise upper bound of \( v \) with generic functions satisfying (A0). Furthermore, one notices that a substitution of the second equation of (1.4) gives a variant form of this key identity:

\[
v_t - \gamma(v)\Delta v + v\gamma(v) = (I - \Delta)^{-1}[u\gamma(v)]. \quad (1.17)
\]

Thanks to the comparison principle of elliptic equations and the decreasing property of \( \gamma \), one has that

\[
(I - \Delta)^{-1}[u\gamma(v)] \leq \gamma(v_*) (I - \Delta)^{-1}[u] = \gamma(v_*) v,
\]

with \( v_* \) being the strictly positive lower bound for \( v \) given by Lemma 2.2 below. Then under the assumption (A2), based on a delicate Alikakos–Moser type iteration, we can show that the uniform-in-time upper bound of \( v \) is obtainable if we have time-independent estimates for \( \sup_{t \geq 0} \|v\|_{L^q} \) with any \( q > \frac{nk}{2} \) beforehand.
On the other hand, system (1.4) possesses a Lyapunov functional (see also [1]) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \|v\|^2 \right) + \int_\Omega (v) |\nabla v|^2 dx + \int_\Omega (\gamma(v) + v\gamma'(v)) |\nabla v|^2 dx = 0, \tag{1.18}
\]
which implies a time-independent estimate of \(\sup_{t \geq 0} \|v\|_{H^q(\Omega)}\) under the assumption (A1). Then the Sobolev embedding \(H^1 \hookrightarrow L^{q_*}\) with \(q_* = \frac{2n}{n-2}\) yields a time-independent estimate for \(\sup_{t \geq 0} \|v\|_{L^{q_*}}\), which, together with the Alikakos–Moser iteration, indicates that \(v\) is uniform-in-time bounded, provided that \(q_* > \frac{n}{2}\), i.e., \(k < \frac{1}{n-2}\).

Next, in order to prove the boundedness of solutions, it suffices to establish the \(L^\infty_t L^p_x\)-boundedness of \(u\) with some \(p > \frac{n}{2}\), since higher-order estimates can then be proven by standard iterations and a bootstrap argument. Recalling that \(v\) is now bounded from above, \(\gamma(v)\) is bounded from below by a strictly positive time-independent constant due to its decreasing property. With the aid of the key identity again, we construct an estimation involving a weighted energy \(\int_\Omega u^p \gamma^q(v)\), which, with a proper choice of \(p > \frac{n}{2}\) and \(q > 0\), will finally imply the boundedness.

Last, the Lyapunov functional also plays a crucial role in the study of exponential stabilization. Since \(v(t) = u_0\), we have \(\int_0^t v_0 dx = 0\) for all \(t > 0\), and hence the energy-dissipation relation (1.18) can be rewritten as
\[
\frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \|v - u_0\|^2 \right) + \int_\Omega (v) |\nabla v|^2 dx + \int_\Omega (\gamma(v) + v\gamma'(v)) |\nabla v|^2 dx = 0. \tag{1.19}
\]
With the boundedness of \(v\) in hand, one can deduce from the above by Poincaré’s inequality that \(\|v - u_0\|_{H^1}\) decay exponentially. Then, by a bootstrapping strategy, the exponential stabilization of \((u, v)\) can be further acquired in \(L^\infty \times W^{1,\infty}\).

We remark that if the second equation of (1.4) is of parabolic type, it is still unknown whether the system possesses a Lyapunov functional like (1.18). Thus, at the present stage, we cannot improve the results in [13] for the fully parabolic case using the same idea.

The rest of the paper is organized as follows: in Section 2, we provide some preliminary results and recall some useful lemmas. In Section 3 we first construct a Lyapunov functional which satisfies a certain dissipation property. Then, using a delicate Alikakos–Moser iteration, we derive the uniform-in-time upper bounds of \(v\). In Section 4, we first establish the boundedness of the weighted energy which gives rise to the boundedness of the global classical solutions. Then using the Lyapunov functional again we prove the exponential stabilization toward the constant steady states.

## 2 Preliminaries

In this section we recall some useful lemmas. First, the local existence and uniqueness of classical solutions to system (1.4) can be established by the standard fixed point argument and regularity theory for elliptic/parabolic equations. A similar proof can be found in [1, Lemma 3.1].

**Theorem 2.1** Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^n\). Suppose that \(\gamma(\cdot)\) satisfies (A0) and \(u_0\) satisfies (1.5). Then there exists \(T_{\text{max}} \in (0, \infty]\) such that problem (1.4) permits a unique non-negative classical solution \((u, v) \in (C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})))^2\). Moreover,
the following mass conservation holds:

$$\int_{\Omega} u(\cdot, t) dx = \int_{\Omega} v(\cdot, t) dx = \int_{\Omega} u_0 dx \quad \text{for all } t \in (0, T_{\text{max}}).$$

If $T_{\text{max}} < \infty$, then

$$\lim_{t \to T_{\text{max}}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

A strictly positive uniform-in-time lower bound for $v = (I - \Delta)^{-1}[u](x, t)$ is given in [1, Lemma 2.2]; see also [31, Lemma 3.3].

**Lemma 2.2** Suppose that $(u, v)$ is the classical solution of (1.4) up to the maximal time of existence $T_{\text{max}} \in (0, \infty]$. Then, there exists a strictly positive constant $v_* = v_*(n, \Omega, \|u_0\|_{L^1(\Omega)})$ such that, for all $t \in (0, T_{\text{max}})$, it holds that

$$\inf_{x \in \Omega} v(x, t) \geq v_*.$$

Next, we recall a key identity and an explicit point-wise upper bound estimate for $v [11, Lemma 3.1].$

**Lemma 2.3** Assume that $n \geq 1$ and suppose that $\gamma$ satisfies (A0). For any $0 < t < T_{\text{max}}$, it holds that

$$v_t + \gamma(v)u = (I - \Delta)^{-1}[\gamma(v)u].$$

Moreover, for any $x \in \Omega$ and $t \in [0, T_{\text{max}})$, we have

$$v(x, t) \leq v_0(x)e^{\gamma(v_*)t},$$

with $v_0 \triangleq (I - \Delta)^{-1}[u_0].$

Last, we recall the following $L^p - L^q$ estimates for the Neumann heat semigroup on bounded domains (see e.g., [28, 29]):

**Lemma 2.4** Suppose that $\{e^{t\Delta}\}_{t \geq 0}$ is the Neumann heat semigroup in $\Omega$, and that $\mu_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary conditions. Then there exist $k_1 > 0$ and $k_2 > 0$ which only depend on $\Omega$ such that the following properties hold:

(i) if $1 \leq q \leq p \leq \infty$, then

$$\|e^{t\Delta}w\|_{L^p(\Omega)} \leq k_1(1 + t^{-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)})e^{-\mu_1 t}\|w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

for all $w \in L^q_0(\Omega) \triangleq \{w \in L^q(\Omega), \int_{\Omega} wx dx = 0\}$;

(ii) if $1 < q \leq p \leq \infty$, then

$$\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \leq k_2(1 + t^{-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)})e^{-\mu_1 t}\|w\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

for any $w \in (W^{1,p}(\Omega))^n$.

## 3 Time-Independent Upper Bounds of $v$

In this part, we aim to establish the uniform-in-time upper bound for $v$ in higher dimensions when $\gamma$ decreases algebraically at large concentrations. The proof consists of several steps. To begin with, we introduce a Lyapunov functional.
Thus, we obtain that
\[ \frac{1}{2} \frac{d}{dt} (\|\nabla v\|^2 + \|v\|^2) + \int_{\Omega} \gamma(v) |\Delta v|^2 dx + \int_{\Omega} (\gamma(v) + v\gamma'(v))|\nabla v|^2 dx = 0 . \tag{3.1} \]
In particular, under the assumption (A1), there is \( C > 0 \) depending only on \( u_0 \) such that
\[ \sup_{0 \leq t < T_{max}} (\|\nabla v\|^2 + \|v\|^2) \leq C . \tag{3.2} \]

**Proof** Multiplying the first equation of (1.4) by \( v \), integrating over \( \Omega \) and substituting the second equation of (1.4) yields that
\[ \frac{1}{2} \frac{d}{dt} (\|\nabla v\|^2 + \|v\|^2) = \int_{\Omega} u\gamma(v) \Delta v dx = \int_{\Omega} \gamma(v) \Delta v (v - \Delta v) dx . \]
By integration by parts, it holds that
\[ \int_{\Omega} \gamma(v) v \Delta v dx = - \int_{\Omega} \nabla v \cdot \nabla (v\gamma(v)) dx = - \int_{\Omega} (\gamma(v) + v\gamma'(v)) |\nabla v|^2 dx . \]
Thus, we obtain that
\[ \frac{1}{2} \frac{d}{dt} (\|\nabla v\|^2 + \|v\|^2) + \int_{\Omega} \gamma(v) |\nabla v|^2 dx + \int_{\Omega} (\gamma(v) + v\gamma'(v)) |\nabla v|^2 dx = 0 . \]
Then the uniform-in-time estimate (3.2) follows by integration of the above identity with respect to time. Finally, since \( -\Delta v_0 + v_0 = u_0 \) in \( \Omega \) and \( \partial_{\nu} v_0 = 0 \) on \( \partial\Omega \), we infer, by Young’s inequality, that
\[ \|\nabla v_0\|^2 + \|v_0\|^2 = \int_{\Omega} u_0 v_0 dx \leq \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \|v_0\|^2 . \tag{3.3} \]
Hence,
\[ \|\nabla v_0\|^2 + \|v_0\|^2 \leq 2 \|\nabla v_0\|^2 + \|v_0\|^2 \leq \|u_0\|^2 . \]
This completes the proof. \( \square \)

**Remark 3.2** In view of the time-independent lower bound \( 0 < v_* \leq v(x,t) \), one can slightly weaken assumption (A1) as follows:
\[ \gamma(s) + s\gamma'(s) \geq 0 , \quad \forall \ s \geq v_* . \tag{3.4} \]
On the other hand, a direct calculation indicates that the above assumption yields that
\[ s\gamma(s) \geq v_* \gamma(v_*), \quad \forall \ s \geq v_* , \tag{3.5} \]
and hence \( \gamma \) fulfills (A2) with any \( k > 1 \). In particular, if \( \gamma(s) = s^{-k} \), assumption (A1) is satisfied when \( k \leq 1 \).

With the above result, we can proceed to derive the uniform-in-time upper bounds of \( v \) based on a delicate Alikakos–Moser iteration [30]. First, we demonstrate

**Lemma 3.3** Assume that \( n \geq 3 \). Suppose that \( \gamma \) satisfies (A0) and (A2) with some \( k > 0 \). Then there exist \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) independent of time such that, for any \( p > 1 + k \),
\[ \frac{d}{dt} \int_{\Omega} v^p dx + \frac{\lambda_1 p(p - k - 1)}{(p - k)^2} \int_{\Omega} |\nabla v|^{2(p-k)} dx + \lambda_1 p \int_{\Omega} v^{p-k} dx \leq 2\lambda_2 p \int_{\Omega} v^p dx . \tag{3.6} \]
Thanks to (3.7), one has

\[ 1/\gamma(s) \leq bs^k, \]

and on the other hand, since \( \gamma(\cdot) \) is non-increasing,

\[ 1/\gamma(s) \leq 1/\gamma(s_b) \]

for all \( 0 \leq s < s_b \). Therefore, for all \( s \geq 0 \), it holds that

\[ 1/\gamma(s) \leq bs^k + 1/\gamma(s_b). \] (3.7)

Now, multiplying the key identity (2.1) by \( v^{p-1} \) with some \( p > 1 + k \), we obtain that

\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p dx + \int_{\Omega} u_\gamma(v)v^{p-1} dx = \int_{\Omega} (I - \Delta)^{-1}[u_\gamma(v)]v^{p-1} dx. \] (3.8)

Since \( \gamma(v) \leq \gamma(v_*) \), we deduce, by the comparison principle of elliptic equations, that

\[ (I - \Delta)^{-1}[u_\gamma(v)] \leq \gamma(v_*) v, \]

and hence,

\[ \int_{\Omega} (I - \Delta)^{-1}[u_\gamma(v)]v^{p-1} dx \leq \gamma(v_*) \int_{\Omega} v^p dx. \]

Thanks to (3.7), one has

\[ \int_{\Omega} u_\gamma(v)v^{p-1} dx \geq \int_{\Omega} u(bv^k + 1/\gamma(s_b))^{-1} v^{p-1} dx \]

\[ \geq C_1 \int_{\Omega} (u^k + 1)^{-1} v^{p-1} dx, \]

with \( 1/C_1 = \frac{\max\{b\gamma(s_b), 1\}}{\gamma(s_b)} > 0 \) independent of \( p \) and time, in view of the fact that

\[ bv^k + 1/\gamma(s_b) = \frac{1}{\gamma(s_b)} (b\gamma(s_b)v^k + 1) \leq \frac{\max\{b\gamma(s_b), 1\}}{\gamma(s_b)} (v^k + 1). \]

Since \( v^k \geq v_*^k \) by Lemma 2.2, it holds that

\[ (u^k + 1)^{-1} v^{p-1} \geq (v^k + v_*^{-k} v^k)^{-1} v^{p-1} = \frac{v^{p-k-1}}{1 + v_*^{-k}}, \] (3.9)

from which we deduce that

\[ \int_{\Omega} u_\gamma(v)v^{p-1} dx \geq C_2 \int_{\Omega} v^{p-k-1} v dx, \] (3.10)

where \( C_2 > 0 \) may depend on the initial datum, \( n, \Omega \) and \( \gamma \), but is independent of \( p \) and time.

Next, recalling that \( v - \Delta v = u \), we observe that

\[ \int_{\Omega} v^{p-k-1} v dx = \int_{\Omega} v^{p-k-1}(v - \Delta v) dx \]

\[ = \int_{\Omega} v^{p-k} dx + (p - k - 1) \int_{\Omega} |\nabla v|^2 v^{p-k-2} dx \]

\[ = \int_{\Omega} v^{p-k} dx + \frac{4(p - k - 1)}{(p - k)^2} \int_{\Omega} |\nabla v|^{2\gamma} dx. \]

Finally, we arrive at

\[ \frac{d}{dt} \int_{\Omega} v^p dx + \frac{\lambda_1 p(p - k - 1)}{(p - k)^2} \int_{\Omega} |\nabla v|^{\frac{p-k}{2}} dx + \lambda_2 p \int_{\Omega} v^{p-k} dx \leq \lambda_2 p \int_{\Omega} v^p dx, \]
with some $\lambda_1, \lambda_2 > 0$ independent of $p$ and time. The proof is completed by adding $\lambda_2 p \int_{\Omega} v^p$ to both sides of the above inequality. \hfill \square

**Lemma 3.4** Assume that $n \geq 3$. Suppose that $\gamma$ satisfies (A0) and (A2) with some $0 < k < \frac{4}{n-2}$. Let $L > 1$ be a generic constant. There exists $C_0 > 0$ depending only on the initial datum, $\Omega, k, \lambda_1, \lambda_2$ and $n$ such that, for any $p > q \geq q_*$, we infer that

$$\frac{q}{p} < 2 - \frac{nk}{2},$$

it holds that

$$\frac{d}{dt} \int_{\Omega} v^p dx + \lambda_2 p \int_{\Omega} v^p dx \leq C_0 L^\frac{n+2}{p} \left( \int_{\Omega} v^q dx \right)^2. \tag{3.11}$$

**Proof** First, one notices that $q_* > 1 + k$ and $q_* > \frac{nk}{2}$, provided that $n \geq 3$ and $0 < k < \frac{4}{n-2}$. Let

$$q_* \leq q < p = 2q - \frac{nk}{2} = 2q - \frac{kq_*}{q_* - 2}.$$  

Denote $\eta = v^{\frac{p-k}{2}}$ and define

$$\alpha = \frac{(p-k)(p-q)}{p(p-k - 2q/q_*)}. \tag{3.12}$$

One easily checks that $\alpha \in (0, 1)$. Indeed,

$$p - k - \frac{2q}{q_*} > q - \frac{2q}{q_*} = \frac{q_* - 2}{q_*} q - k$$

and on the other hand, solving $\alpha < 1$ yields that $p > \frac{kq_*}{q_* - 2} = \frac{nk}{2}$. Moreover, since $q > \frac{nk}{2}$ as well, one checks that $\frac{2\alpha q}{p-k} < 2$. Then an application of Hölder’s inequality yields that

$$\int_{\Omega} v^p dx = \int_{\Omega} \eta^{\frac{2p}{p-k}} dx = \int_{\Omega} \eta^{\frac{2p}{p-k}} \eta^{\frac{2(p-k)-2}{p-k}} dx$$

$$\leq \left( \int_{\Omega} \eta^{2p} dx \right)^{\frac{2p}{2p+(p-k)q_*}} \left( \int_{\Omega} \eta^{\frac{2(p-k)-2}{p-k}(p-n)} dx \right)^{\frac{p-k-q_*-2\alpha}{(p-k)q_*}} \quad \text{since} \quad \frac{2\alpha q}{p-k} < 2.$$  

Therefore, the Sobolev embedding inequality

$$\|\eta\|_{L^{p^*} (\Omega)} \leq \lambda_* \|\eta\|_{H^1 (\Omega)},$$

where $\lambda_* > 0$ depends only on $n$ and $\Omega$. In view of the fact that $\frac{2\alpha q}{p-k} < 2$, invoking Young’s inequality, we infer that

$$\lambda_2 p \int_{\Omega} v^p dx \leq \lambda_2 p \|\eta\|_{L^{p^*} (\Omega)} \left( \int_{\Omega} v^q dx \right)^{\frac{2\alpha q}{p-k(q_* - 2\alpha)}} \left( \int_{\Omega} v^q dx \right)^{\frac{(p-k)q_* - 2\alpha}{(p-k)q_*}}$$

$$\leq \lambda_2 p \left( \lambda_* \|\eta\|_{H^1 (\Omega)} \right)^{\frac{2\alpha q}{p-k} \left( \frac{p-k(q_* - 2\alpha)}{(p-k)q_*} \right)} \left( \int_{\Omega} v^q dx \right)^{\frac{(p-k)q_* - 2\alpha}{(p-k)q_*}}$$

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It follows from the above and from (3.12) that
\[
\delta > 0 \text{ such that}
\]
\[
\frac{p \alpha \delta}{p - k} \|v\|_{H^{1}(\Omega)} + \frac{p - k - p \alpha}{p - k} \lambda_{2}^{p - k - p \alpha} \delta - 1 \lambda_{2} p \frac{p - k}{p - k - p \alpha} \left( \int_{\Omega} v^{q} \, dx \right) \frac{(p - k)q - 2p}{q(p - k - 2q)}.
\]
where \( \delta > 0 \) such that
\[
\frac{p \alpha \delta}{p - k} = \frac{\lambda_{1}(p - k - 1)}{2L(p - k)^{2}}.
\]
(3.13)

It follows from the above and from (3.12) that
\[
\frac{p - k - p \alpha}{p - k} \lambda_{2}^{p - k - p \alpha} (\delta - 1) \lambda_{2} p \frac{p - k}{p - k - p \alpha} \left( \int_{\Omega} v^{q} \, dx \right) \frac{(p - k)q - 2p}{q(p - k - 2q)}
\]
\[
= (q_{*} - 2)q - kq_{*} \left( \frac{2L(p - k)^{2}(p - q)\lambda_{*}^{2}}{q_{*}(p - k) - 2q} \left( \frac{\lambda_{1}(p - k - 1)(p - k - 2q/q_{*})}{n} \right) \frac{(p - q_{*})q_{*}}{(p - k)^{2}q_{*} - 2q} \left( \lambda_{2} p \right) \frac{q_{*}(p - k - 2q)}{q_{*}(p - k - 2q)} \left( q_{*}(p - k) - 2q \right) \right)
\]
\[
\times \left( \int_{\Omega} v^{q} \, dx \right) \frac{(p - k)q - 2p}{q_{*}(p - k) - 2q}.
\]
(3.14)

Noticing that
\[
\frac{n}{2} = \frac{q_{*}}{q_{*} - 2},
\]
and recalling that
\[
\frac{nk}{2} < q < p = 2q - \frac{nk}{2} = 2q - \frac{kq_{*}}{q_{*} - 2},
\]
(3.15)

one easily checks that
\[
q_{*}(p - k) - 2p = q_{*}(2q - \frac{nk}{2} - k) - 2(2q - \frac{nk}{2})
\]
\[
= q_{*}(2q - 2k - \frac{n - 2}{2}k) - 4q + nk
\]
\[
= 2q_{*}(q - k) - 4q - \frac{q_{*}(n - 2)}{2}k + nk
\]
\[
= 2q_{*}(q - k) - 4q.
\]
(3.16)

It follows that
\[
\frac{q_{*}(p - k) - 2p}{q_{*}(p - k) - 2q} = 2.
\]
(3.17)

Moreover, we note by (3.16) again that
\[
(q_{*} - 2)q - kq_{*} = q_{*}(q - k) - 2q
\]
\[
= \frac{1}{2} \left( q_{*}(p - k) - 2p \right)
\]
\[
= \frac{1}{2} \left( q_{*}(p - k) - 2q \right) \left( 1 + \frac{2(q - p)}{q_{*}(p - k) - 2q} \right)
\]
(3.18)

and
\[
1 + \frac{2(q - p)}{q_{*}(p - k) - 2q} = \frac{q_{*}(p - k) - 2q + 2(q - p)}{q_{*}(p - k) - 2q}
\]
\[
= \frac{q_{*}(p - k) - 2p}{q_{*}(p - k) - 2q}.
\]
Hence, one can find that
\[
\frac{q_*(2q - nk/2 - k) - 4q + nk}{q_*(2q - nk/2 - k) - 2q} = \frac{(2q_* - 4q + k(n - \frac{\mu^2}{2}q_*))}{(2q_* - 2q - \frac{\mu^2}{2}q_*k)}
\]
\[
= \frac{\frac{8}{n^2} q - \frac{4nk}{n-2}}{\frac{2n+4}{n-2} q - \frac{4nk}{n^2} (n+2)k} = \frac{4}{n + 2}.
\]
(3.19)

Thus, we obtain that
\[
\frac{(q_* - 2)q - kq_*}{q_*(p - k) - 2q} = \frac{2}{n + 2}
\]
(3.20)
and
\[
\frac{q_*(p - k) - 2q}{(q_* - 2)q - kq_*} = \frac{n + 2}{2}.
\]
(3.21)

In addition, by substituting (3.15), we infer that
\[
\frac{(p - q)q_*}{q(q_* - 2) - kq_*} = \frac{(q - nk/2)q_*}{q(q_* - 2) - kq_*} = \frac{2ng}{n^2} - \frac{n^2k}{n^2} = \frac{n}{2},
\]
(3.22)
and similarly, by (3.19),
\[
\frac{p - q}{p - k - 2q/q_*} = \frac{(p-q)q_*}{q_*(p-k) - 2q} = (1 - \frac{4}{n+2}) \cdot \frac{q_*}{2} = \frac{n}{n+2}.
\]

Moreover, since \( p > q_* > 1 + k \), we deduce that
\[
\frac{(p-k)^2}{p(p-k-1)} = \frac{(p-k-1)^2 + 2(p-k-1) + 1}{p(p-k-1)}
\]
\[
= \frac{p-k-1}{p} + \frac{2}{p} + \frac{1}{p(p-k-1)}
\]
\[
< 3 + \frac{1}{p-k-1}
\]
\[
< 3 + \frac{1}{q_* - k - 1}.
\]

Since \( p > \frac{nk}{2} \), we also observe that
\[
\frac{(p-k)^2}{p(p-k-1)} > \frac{p-k}{p} = 1 - \frac{k}{p} > 1 - \frac{2}{n} = \frac{n-2}{n} > 0.
\]
(3.23)

Now, based on the above calculations, we obtain that
\[
\frac{2L(p-k)^2(p-q)\lambda_*^2}{\lambda_1 p(p-k-1)(p-k-2q/q_*)} = \frac{2L\lambda_*^2}{\lambda_1} \cdot \frac{n+2}{n} \cdot \frac{(p-k)^2}{p(p-k-1)}
\]
\[
< \frac{2Ln\lambda_*^2}{\lambda_1 (n+2)} \left( 3 + \frac{1}{q_* - k - 1} \right).
\]

Hence, one can find that \( C_0 > 0 \) is a constant depending only on the initial datum, \( \Omega, k, \lambda_1, \lambda_2, \lambda_* \) and \( n \) such that
\[
\frac{(q_* - 2)q - kq_*}{q_*(p-k) - 2q} \left( \frac{2L(p-k)^2(p-q)\lambda_*^2}{\lambda_1 p(p-k-1)(p-k-2q/q_*)} \right) < \frac{(p-q)q_*}{q_*(p-k) - 2q} \left( \frac{q_*(p-k) - 2q}{q_* (p-k) - 2q} \right)
\]
\[
\left( \lambda_2 p \right)^\frac{(p-q)q_*}{q_*(p-k) - 2q} \left( \frac{q_*(p-k) - 2q}{q_* (p-k) - 2q} \right)
\]

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\[
< \frac{2}{n + 2} \left\{ \frac{2Lm\lambda_k^2}{\lambda_k(n + 2)} \left( 3 + \frac{1}{q_s - k - 1} \right) \right\} \frac{2}{n + 2} (\lambda_2 p)^{n+2} \\
\leq \frac{C_0}{2} L^{\frac{n}{2}} p^{\frac{n}{2+2}}.
\]
Therefore, by the above and by (3.13), we have that
\[
2\lambda_2 p \int_\Omega v^p dx \leq \frac{\lambda_1 p(p - k - 1)}{L(p - k)^2} \|v^{\frac{p}{p-k}}\|^2_{L^1(\Omega)} + C_0 L^{\frac{n}{2}} p^{\frac{n}{2+2}} \left( \int_\Omega v^q dx \right)^{2}.
\]
Combining Lemma 3.3 and recalling that \(L > 1\), we finally arrive at the following inequality:
\[
\frac{d}{dt} \int_\Omega v^p dx + \lambda_2 p \int_\Omega v^p dx \leq C_0 L^{\frac{n}{2}} p^{\frac{n}{2+2}} \left( \int_\Omega v^q dx \right)^{2}.
\]
This completes the proof. \(\square\)

**Proposition 3.5** Assume that \(n \geq 3\). Suppose that \(\gamma\) satisfies (A0), (A1) and (A2) with some \(k \in (0, \frac{1}{n-2})\). Then there is \(v^* > 0\) depending only on the initial datum, \(\gamma, n\) and \(\Omega\) such that
\[
\sup_{0 \leq t < T_{\max}} \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq v^*.
\]

**Proof** For all \(r \in \mathbb{N}\) we define
\[
p_r \triangleq 2^r \left( q_s - \frac{nk}{2} \right) + \frac{nk}{2}, \quad p_0 = q_s.
\]
Then \(p_r > q_s > \frac{nk}{2}\) and \(p_r = 2p_{r-1} - \frac{nk}{2}\). We apply Lemma 3.4 with \((p, q) = (p_r, p_{r-1})\) to get
\[
\frac{d}{dt} \int_\Omega v^{p_r} dx + \lambda_2 p_r \int_\Omega v^{p_r} dx \leq \lambda_2 p_r A_r (M_{r-1})^2,
\]
where \(\mathcal{M}_r \triangleq \sup_{0 \leq t < T_{\max}} \int_\Omega v^{p_r} dx\) and \(A_r \triangleq \frac{C_0 L^{\frac{n}{2}} p_r^{\frac{n}{2+2}}}{\lambda_2} \).

Note that \(\mathcal{M}_r\) is finite for all \(r\) in view of (2.2). Now, letting \(y(t) = \int_\Omega v^{p_r} dx\), we infer from the above that
\[
[e^{\lambda_2 p_r t} y(t)]' \leq \lambda_2 p_r e^{\lambda_2 p_r t} A_r \mathcal{M}_{r-1}^2,
\]
which implies that
\[
y(t) \leq A_r \mathcal{M}_{r-1}^2 (1 - e^{-\lambda_2 p_r t}) + y(0) e^{-\lambda_2 p_r t} \\
\leq A_r \mathcal{M}_{r-1}^2 (1 - e^{-\lambda_2 p_r t}) + \|v_0\|_{L^\infty(\Omega)} e^{-\lambda_2 p_r t} \\
\leq A_r \mathcal{M}_{r-1}^2 + (\|v_0\|_{L^\infty(\Omega)} e^{-\lambda_2 p_r t} \\
\leq A_r \mathcal{M}_{r-1}^2,
\]
provided that \(A_r \mathcal{M}_{r-1}^2 \geq \|v_0\|_{L^\infty(\Omega)}^{p_r}\). Otherwise, it holds that
\[
A_r \mathcal{M}_{r-1}^2 (1 - e^{-\lambda_2 p_r t}) + \|v_0\|_{L^\infty(\Omega)}^{p_r} e^{-\lambda_2 p_r t} \\
\leq \|v_0\|_{L^\infty(\Omega)}^{p_r} (1 - e^{-\lambda_2 p_r t}) + \|v_0\|_{L^\infty(\Omega)}^{p_r} e^{-\lambda_2 p_r t} \\
\leq \|v_0\|_{L^\infty(\Omega)}^{p_r}.
\]
As a result, we obtain that, for all \(r \in \mathbb{N}\),
\[
\mathcal{M}_r = \sup_{0 \leq t < T_{\max}} \int_\Omega v^{p_r} dx \leq \max\{A_r \mathcal{M}_{r-1}^2, \|v_0\|_{L^\infty(\Omega)}^{p_r}\}.
\]
Since $p_r \geq q_*$ for all $r \geq 1$, one can choose $L > 1$ sufficiently large depending only on the initial datum, $\Omega$, $n$ and $k$ such that $A_r > 1$ for all $r \geq 1$. Moreover, adjusting $C_0$ by a proper larger number, we have that

$$A_r \leq C_0a^r,$$

with some $a > 0$ depending only on the initial datum, $\Omega$, $k$ and $n$. In addition, since $\gamma$ satisfies (A0) and (A1), due to Lemma 3.1 and the Sobolev embedding $H^1 \hookrightarrow L^{2^*}$, we may find some large constant $K_0 > 1$ that dominates $\|v\|_{L^\infty}$ and $\int_\Omega v^{p_r} \, dx$ for all time (note that $\|v\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$, by the definition of $v_0$).

Iteratively, we deduce that

$$\int_\Omega v^{p_r} \, dx \leq \max\{A_rA_r^{-1}M_r^{4r-2}, A_rK_0^{2p_r-1}, K_0^{p_r}\}$$

$$= \max\{A_rA_r^{-1}M_r^{4r-2}, A_rK_0^{2p_r-1}\}$$

$$\leq \ldots$$

$$\leq \max\{A_rA_r^{-1}A_r^{-2} \cdots A_r^{2r-1} M_r^{2r}, A_rA_r^{-1} \cdots A_r^{2r-2} K_0^{2r-1-p_r}\}$$

$$\leq \max\{A_rA_r^{-1}A_r^{-2} \cdots A_r^{2r-1} K_0^{2r}, A_rA_r^{-1} \cdots A_r^{2r-2} K_0^{2r-1-p_r}\}$$

$$\leq C_0^{2^{n+2} + \cdots + 2^{r-1}} \times a^{1-r+2(r-1)+2^2(r-2)+\cdots+2^{r-1}(r-(r-1)) \times K_0^{2r}}$$

$$= C_0^{2^{n-1} - a^{2^{r-1} - r+2} K_0^{2r}},$$

where $\tilde{K} = \max\{K_0, K_0^{2r}\}$. Finally, recalling that $p_r = 2^r(q_* - \frac{nk}{2}) + \frac{nk}{2}$, we deduce that

$$\|v\|_{L^\infty(\Omega)} \leq \lim_{r^2 + \infty} \left(C_0^{2^{n-1} - a^{2^{r-1} - r+2} K_0^{2r}}\right)^{1/p_r} = \left(C_0a^2\tilde{K}\right)^{\frac{1}{2^{n-1} - r+2}},$$

which concludes the proof. \hfill \Box

**Corollary 3.6** If $\gamma(v) = v^{-k}$, then $v$ has a uniform-in-time upper bound provided that $k \leq 1$ when $n = 4, 5$, or $k < \frac{4}{n-2}$ when $n \geq 6$.

Next, we recall the following lemma, established in [13]:

**Lemma 3.7** A function satisfying (A0), (A1) and

$$(A3') : \quad l|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \forall \ s > 0$$

with some $l > 1$ must fulfill assumption (A2) with any $k > \frac{1}{l-1}$.

**Lemma 3.8** Assume that $n \geq 4$. Suppose that $\gamma(\cdot)$ satisfies (A0), (A1) and (A3). Then $v$ has a uniform-in-time upper bound in $\overline{\Omega} \times [0, T_{\text{max}})$.

**Proof** Note that $\frac{1}{l_0-1} < \frac{4}{n-2}$, since $l_0 > \frac{n+2}{4}$. Thus $\gamma$ satisfies (A2) with some $k < \frac{4}{n-2}$ and, due to Proposition 3.5, $v$ has a uniform-in-time upper bound such that $v \leq v^*$ on $\overline{\Omega} \times [0, T_{\text{max}})$. \hfill \Box

### 4 Proof of Theorem 1.1

#### 4.1 Uniform-in-time boundedness

This part is devoted to the proof of Theorem 1.1. With the time-independent upper bound of $v$ in hand, it suffices to establish an estimation involving a weighted energy $\int_\Omega u^{1+p}\gamma(q)(v)$.
for some $1 + p > \frac{3}{2}$ and $q > 0$. Higher-order estimates can be then proven via a standard bootstrapping argument. To begin with, we provide

**Lemma 4.1** For any $0 \leq t < T_{\text{max}}$ and $p, q > 0$, it holds that

$$
\frac{d}{dt} \int_{\Omega} u^{p+1}q^{q}(v)dx + (p + 1) \int_{\Omega} u^{p-1}q^{q+1}e^{2u}dx \\
+ q \int_{\Omega} \left((p + q + 1) |\gamma'(v)|^2 + \gamma''\right)u^{p+1}q^{q-1}e^{2u}dx \\
- q \int_{\Omega} (I - \Delta)^{-1}[u\gamma(v)]u^{p+1}q^{q-1}(v)\gamma'(v)dx
$$

$$
= -(p + 1)(p + 2q) \int_{\Omega} u^{p-1}q^{q}(v)u \cdot \nabla v dx - q \int_{\Omega} u^{p+1}q^{q}(v)\gamma'(v)v dx. \tag{4.1}
$$

**Proof** For any $p, q > 0$, by direct computations, we infer by the key identity that

$$
\frac{d}{dt} \int_{\Omega} u^{p+1}q^{q}(v)dx \\
= (1 + p) \int_{\Omega} u^{p}u\gamma'(v)dx + q \int_{\Omega} u^{p+1}q^{q-1}(v)\gamma'(v)u dx \\
= (1 + p) \int_{\Omega} u^{p}u\gamma'(v)dx + q \int_{\Omega} u^{p+1}q^{q-1}(v)\gamma'(v) \left((I - \Delta)^{-1}[u\gamma(v)] - u\gamma(v)\right)dx \\
= (1 + p) \int_{\Omega} u^{p}u\gamma'(v)dx + q \int_{\Omega} u^{p+1}q^{q-1}(v)\gamma'(v)(I - \Delta)^{-1}[u\gamma(v)]dx \\
- q \int_{\Omega} u^{p+1}q^{q}(v)\gamma'(v)u dx. \tag{4.2}
$$

Then a substitution of $v - \Delta v = u$ to the last term on the right hand side of (4.2) yields that

$$
\frac{d}{dt} \int_{\Omega} u^{p+1}q^{q}(v)dx - (p + 1) \int_{\Omega} \gamma'(v)u dx - q \int_{\Omega} u^{p+1}q^{q}(v)\gamma'(v)\Delta v dx \\
- q \int_{\Omega} (I - \Delta)^{-1}[u\gamma(v)]u^{p+1}q^{q-1}(v)\gamma'(v)dx \\
= - q \int_{\Omega} u^{p+1}q^{q}(v)\gamma'(v)\Delta v dx. \tag{4.3}
$$

By integration by parts and the first equation of (1.4), we infer that

$$
- (p + 1) \int_{\Omega} \gamma'(v)u dx \\
= - (p + 1) \int_{\Omega} \gamma'(v)u \Delta \gamma(v)u dx \\
= (p + 1) \int_{\Omega} \gamma'(v)u \Delta \gamma(v)u dx \\
= p(p + 1) \int_{\Omega} u^{p-1}q^{q+1}(v)\nabla u^2 dx + q(p + 1) \int_{\Omega} u^{p+1}q^{q-1}(v)\gamma'(v)\nabla v^2 dx \\
+ (p + 1)(p + q) \int_{\Omega} u^{p}q^{q}(v)\gamma'(v)u \cdot \nabla v dx. \tag{4.4}
$$

Last, by integration by parts again, it holds that

$$
- q \int_{\Omega} u^{p+1}q^{q}(v)\gamma'(v)\Delta v dx \\
= q^2 \int_{\Omega} u^{p+1}q^{q-1}(v)\gamma'(v)^2\nabla v^2 dx + q \int_{\Omega} u^{p+1}q^{q}(v)\gamma''(v)\nabla v^2 dx.
$$
Collecting the above equalities completes the proof. □

**Lemma 4.2** Assume that \( \gamma(\cdot) \) satisfies (A0), (A1) and (A3). For any \( 1 + p \in (0, l_0^2) \), there exist time-independent constants \( q = \frac{p_0}{2} > 0 \) and \( \delta_0 = \delta_0(p, q) \in (0, 1) \) such that

\[
\frac{(p + 1)(p + 2q)^2}{4p(1 - \delta_0)} \int_\Omega u^{1 + p} \gamma^{-1} |\gamma'|^2 |\nabla v|^2 \text{d}x \leq q \int_\Omega \left( (p + q + 1) |\gamma'(v)|^2 + \gamma'' \right) u^{p + 1} \gamma^{-1} |\nabla v|^2 \text{d}x. \tag{4.6}
\]

**Proof** Define

\[
f(\lambda) = 4\lambda l_0 - 4\lambda^2
\]

for all \( \lambda > 0 \). We observe that \( f(\lambda) \) attains its maximum value \( l_0^2 \) at \( \lambda_0 = l_0/2 \). Thus, for any \( 1 + p \in (1, l_0^2) \), it holds that

\[
1 + p < l_0^2 = f(\lambda_0).
\]

In other words,

\[
\frac{1 + p + 4\lambda^2}{4\lambda_0} < l_0.
\]

In addition, we can further find a time-independent constant \( \delta_0 = \delta_0(p, \lambda_0) \in (0, 1) \) such that

\[
\frac{1 + p + 4\lambda_0^2 + 4\lambda_0 \delta_0(1 + p + \lambda_0 p)}{4\lambda_0(1 - \delta_0)} = \frac{1}{2} \left( \frac{1 + p + 4\lambda_0^2}{4\lambda_0} + l_0 \right) \in \left( \frac{1 + p + 4\lambda_0^2}{4\lambda_0}, l_0 \right).
\]

It follows that

\[
\frac{1 + p + 4\lambda_0^2 + 4\lambda_0 \delta_0(1 + p + \lambda_0 p)}{4\lambda_0(1 - \delta_0)} |\gamma'|^2 < l_0 |\gamma'|^2 \leq \gamma'', \quad \forall \ s > 0
\]

for any \( 1 + p \in (1, l_0^2) \). On the other hand, according to Lemma 2.2 and Lemma 3.8, there exist the following time-independent lower and upper bounds for \( v \):

\[
v_* \leq v(x, t) \leq v^* \quad \text{on } \Omega \times [0, T_{\text{max}}).
\]

Thus we infer that

\[
\frac{1 + p + 4\lambda_0^2 + 4\lambda_0 \delta_0(1 + p + \lambda_0 p)}{4\lambda_0(1 - \delta_0)} |\gamma'(v(x, t))|^2 < \gamma(v(x, t)) \gamma''(v(x, t)), \quad \text{on } \Omega \times [0, T_{\text{max}})
\]

for any \( 1 + p \in (1, l_0^2) \). As a result, assertion (4.6) holds with \( q = \lambda_0 p \) and \( \delta_0 \) chosen above. □

**Lemma 4.3** Assume that \( n \geq 3 \) and that \( \gamma(\cdot) \) satisfies (A0), (A1) and (A3). Then there exist \( p > \frac{n}{2} - 1 \) and \( C > 0 \) independent of time such that

\[
\sup_{0 \leq t < T_{\text{max}}} \int_\Omega u^{1 + p} \text{d}x \leq C.
\]

**Proof** Invoking Young’s inequality, it holds that

\[
- (p + 1)(p + 2q) \int_\Omega u^p \gamma(q) \gamma'(v) \nabla u \cdot \nabla v \text{d}x
\]

\[
\leq (p + 1)p(1 - \delta_0) \int_\Omega u^{p - 1} \gamma' |\nabla u|^2 \text{d}x + \frac{(p + 1)(p + 2q)^2}{4p(1 - \delta_0)} \int_\Omega u^{1 + p} \gamma^{-1} |\gamma'|^2 |\nabla v|^2 \text{d}x, \tag{4.7}
\]

with any \( 1 + p \in (1, l_0^2) \) and \( q \) and \( \delta_0 \) chosen in Lemma 4.2.

\( \square \) Springer
Then it follows from (4.1) and Lemma 4.2 that
\[
\frac{d}{dt} \int_{\Omega} u^{p+1} \gamma(v) dx + \delta_0 (p+1) \int_{\Omega} u^{p-1} \gamma^{p+1} |\nabla u|^2 dx \\
- q \int_{\Omega} (I - \Delta)^{-1} [u \gamma(v)] u^{p+1} \gamma^{p-1}(v) \gamma'(v) dx
\leq - q \int_{\Omega} u^{p+1} \gamma^{3}(v) \gamma'(v) dx
\]  
(4.8)

with any \(1 + p \in (1, l_0^p)\) and \(q = \frac{\mu}{2}\).

Since \(v_* \leq v \leq v^*\), there exist \(C_3\) and \(C_3' > 0\) depending on \(p, \delta_0, \gamma\) and the initial datum only such that, for any \(1 + p \in (1, l_0^p)\) and \(q = \frac{n}{n+2}\),
\[
\frac{d}{dt} \int_{\Omega} u^{p+1} \gamma^q(v) dx + C_3 \int_{\Omega} u^{p-1} |\nabla u|^2 dx \leq C_3' \int_{\Omega} u^{p+1} dx.
\]  
(4.9)

Recall the Gagliardo-Nirenberg inequality
\[
\|\xi\|_{L^2(\Omega)} \leq C_{GN} \|\nabla \xi\|^\theta_{L^2(\Omega)} \|\xi\|^{1-\theta}_{L^1(\Omega)} + C_{GN} \|\xi\|_{L^1(\Omega)}
\]  
(4.10)

with \(\theta = \frac{n}{n+2}\). Denote that \(\xi = u^{\frac{p+1}{2}}\). Then, in view of the uniform-in-time boundedness of \(\gamma^q(v)\), we infer by Young’s inequality that
\[
\int_{\Omega} u^{1+p} \gamma^q(v) dx \leq C_4 \int_{\Omega} u^{1+p} dx = C_4' \|\xi\|_{L^2(\Omega)}^2 \leq \varepsilon \|\nabla \xi\|^2 + C_5(\varepsilon) \|\xi\|_{L^1(\Omega)}^2,
\]
with any \(\varepsilon > 0\) and some \(C_4, C_4', C_5(\varepsilon) > 0\) independent of time. Thus, by choosing proper small \(\varepsilon > 0\), we infer from (4.9) that, for any \(1 + p \in (1, l_0^p)\) and \(q = \frac{n}{n+2}\),
\[
\frac{d}{dt} \int_{\Omega} u^{1+p} \gamma^q(v) dx + C_6 \int_{\Omega} u^{1+p} \gamma^q(v) dx \leq C_7 \left( \int_{\Omega} u^{\frac{p+1}{2}} dx \right)^2,
\]  
(4.11)

with \(C_6\) and \(C_7 > 0\) independent of time. Then, in view of the fact that \(\|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}\), together with the uniform-in-time lower and upper boundedness of \(\gamma^q(v)\), one can deduce iteratively from (4.11) that, for any \(1 + p \in (1, l_0^p)\) and \(q = \frac{n}{n+2}\),
\[
\sup_{t \geq 0} \int_{\Omega} u^{1+p} dx \leq C_8.
\]  
(4.12)

Finally, we note that, for any \(n \geq 3\),
\[
\frac{n}{2} \leq \left( \frac{n+2}{4} \right)^2 < l_0^2.
\]

Thus we can always find \(p > 0\) satisfying \(1 + p > \frac{n}{2}\) such that (4.12) holds. This completes the proof. \(\square\)

**Proof of Theorem 1.1** Boundedness: Once we obtain Lemma 4.3 with \(1 + p > \frac{n}{2}\), we can proceed to deduce the uniform-in-time boundedness of the solutions in the same manner as was done in [1, Lemma 4.3]. We omit the details here. \(\square\)

**Corollary 4.4** Assume that \(\gamma(v) = v^{-k}\) and that \(n \geq 4\). Then there exists a unique globally bounded classical solution, provided that \(k \leq 1\) when \(n = 4, 5\), or \(k < \frac{4}{n+2}\) when \(n \geq 6\). \(\square\) Springer
4.2 Exponential stabilization toward constant steady states

In this part, we derive the exponential stabilization of the global solutions relying on a slightly modified version of the Lyapunov functional (3.1).

**Lemma 4.5** There exist constants \( \alpha > 0 \) and \( C > 0 \) depending on \( u_0, \gamma, n \) and \( \Omega \) such that

\[
\|v(\cdot,t) - \overline{u_0}\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\alpha t}, \quad \forall \, t > 0.
\]  

**Proof** Observing that \( \overline{v}(t) = \overline{v}(t) = \overline{u_0} \) for all \( t \geq 0 \), it follows that \( \frac{d}{dt} \int_{\Omega} v(t) dx = 0 \), and hence,

\[
\frac{d}{dt}\|v(\cdot,t) - \overline{u_0}\|^2 = \frac{d}{dt} \int_{\Omega} (v^2 - 2\overline{u_0}v + \overline{u_0}^2) \, dx
\]

\[
= \frac{d}{dt} \int_{\Omega} v^2 \, dx - 2\overline{u_0} \frac{d}{dt} \int_{\Omega} v \, dx
\]

\[
= \frac{d}{dt}\|v(\cdot,t)\|^2.
\]

Therefore, we deduce from (3.1) that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla v\|^2 + \|v - \overline{u_0}\|^2 \right) + \int_{\Omega} \gamma(v) |\Delta v|^2 \, dx + \int_{\Omega} (\gamma(v) + \nu\gamma'(v)) |\nabla v|^2 \, dx = 0.
\]  

(4.14)

Since \( v \leq v^* \) and \( \gamma \) is non-increasing, we have that

\[
\int_{\Omega} \gamma(v) |\Delta v|^2 \, dx \geq \gamma(v^*) \int_{\Omega} |\Delta v|^2 \, dx,
\]

which together with Poincaré’s inequality \( \mu_1 \|v - \overline{u_0}\|^2 \leq \|\nabla v\|^2 \), yields that

\[
\int_{\Omega} \gamma(v) |\Delta v|^2 \, dx \geq \mu_1 \gamma(v^*) \int_{\Omega} |\nabla v|^2 \, dx \geq \frac{\mu_1 \gamma(v^*)}{1 + \mu_1} \left( \|\nabla v\|^2 + \|v - \overline{u_0}\|^2 \right).
\]  

(4.15)

Here \( \mu_1 > 0 \) denotes the first positive eigenvalue of the Neumann Laplacian operator, and we also use the fact that \( \mu_1 \|\nabla v\|^2 \leq \|\Delta v\|^2 \) if \( \partial_{\nu} v = 0 \) on \( \partial\Omega \).

Thus, we infer from (4.14) that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla v\|^2 + \|v - \overline{u_0}\|^2 \right) + \frac{\mu_1 \gamma(v^*)}{1 + \mu_1} \left( \|\nabla v\|^2 + \|v - \overline{u_0}\|^2 \right) \leq 0,
\]

which by standard ODI analysis yields that, for all \( t \geq 0 \),

\[
\|\nabla v\|^2 + \|v - \overline{u_0}\|^2 \leq e^{-\frac{2\mu_1 \gamma(v^*)}{1 + \mu_1}} \left( \|\nabla v_0\|^2 + \|v_0 - \overline{u_0}\|^2 \right).
\]  

(4.16)

where \( v_0 = (I - \Delta)^{-1} [u_0] \).

Next, from the first equation of (1.4), we have that

\[
\partial_t (u - \overline{u_0}) = \Delta (uv (v)).
\]  

(4.17)

Multiplying (4.17) by \( u - \overline{u_0} \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u - \overline{u_0})^2 \, dx + \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx = \int_{\Omega} u\gamma'(v) \nabla u \cdot \nabla v \, dx
\]

\[
\leq \frac{1}{2} \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} u^2 |\gamma'(v)|^2 |\nabla v|^2 \, dx.
\]

Since now \( u \) and \( v \) are both uniformly-in-time bounded, there is a time-independent constant \( C_0 > 0 \) such that

\[
\frac{d}{dt} \int_{\Omega} |u - \overline{u_0}|^2 \, dx + \gamma(v^*) \int_{\Omega} |\nabla u|^2 \, dx \leq C_0 \int_{\Omega} |\nabla v|^2 \, dx.
\]

\( \odot \) Springer
Applying Poincaré’s inequality, one can find a constant \(0 < \alpha_1 < \frac{2\mu_1 \gamma(v^*)}{1 + \mu_1}\) and \(C_{10} > 0\) depending only on initial datum, \(\gamma, n\) and \(\Omega\) such that

\[
\frac{d}{dt} \int_{\Omega} |u - \overline{u_0}|^2 \, dx + \alpha_1 \int_{\Omega} |u - \overline{u_0}|^2 \, dx \leq C_{10} \int_{\Omega} |\nabla v|^2 \, dx.
\]

In view of (4.16), solving the above differential inequality yields that

\[
||u - \overline{u_0}||^2 \leq C_{11} e^{-\alpha_1 t}, \tag{4.18}
\]

with \(C_{11} > 0\) depending on initial datum, \(n, \gamma\) and \(\Omega\) only.

Next, for any \(p > 2\), we multiply (4.17) by \(|u - \overline{u_0}|^{p-2}(u - \overline{u_0})\) to get that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} |u - \overline{u_0}|^p \, dx + (p - 1) \int_{\Omega} \gamma(v) |u - \overline{u_0}|^{p-2} |\nabla u|^2 \, dx \leq \frac{p-1}{2} \int_{\Omega} \gamma(v) |u - \overline{u_0}|^{p-2} |\nabla v|^2 \, dx.
\]

Similarly, there are time-independent constants \(C_{12} = C_{12}(p) > 0\) and \(\alpha_2 < \alpha_1\) such that

\[
\frac{d}{dt} \int_{\Omega} |u - \overline{u_0}|^p \, dx + \alpha_2 \int_{\Omega} |u - \overline{u_0}|^p \, dx \leq C_{12} \int_{\Omega} |\nabla v|^2 \, dx + \alpha_2 \int_{\Omega} |u - \overline{u_0}|^p \, dx.
\]

Observing that

\[
\int_{\Omega} |u - \overline{u_0}|^p \, dx \leq ||u - \overline{u_0}||_{W^{1,\infty}(\Omega)}^p \int_{\Omega} |u - \overline{u_0}|^2 \, dx \leq C_{13} \int_{\Omega} |u - \overline{u_0}|^2 \, dx,
\]

we arrive at

\[
\frac{d}{dt} \int_{\Omega} |u - \overline{u_0}|^p \, dx + \alpha_2 \int_{\Omega} |u - \overline{u_0}|^p \, dx \leq C_{14} \left( \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |u - \overline{u_0}|^2 \, dx \right), \tag{4.19}
\]

which yields that

\[
\int_{\Omega} |u - \overline{u_0}|^p \, dx \leq C_{15} e^{-\alpha_2 t}. \tag{4.20}
\]

Last, note that from the second equation of (1.4),

\[
(v - \overline{u_0}) - \Delta (v - \overline{u_0}) = u - \overline{u_0}.
\]

Choosing some \(p_0 > n\) in (4.20), one may deduce, by elliptic regularity and Sobolev embeddings, that

\[
||v - \overline{u_0}||_{W^{1,\infty}(\Omega)} \leq C_{16} ||v - \overline{u_0}||_{W^{2,p_0}(\Omega)} \leq C_{16}' ||u - \overline{u_0}||_{L^{p_0}(\Omega)} \leq C_{16}'' e^{-\alpha t},
\]

with \(\alpha = \frac{\omega_2}{p_0}\). This completes the proof.

Next, we assert

**Lemma 4.6**  There are positive constants \(C = C(n, \Omega, \gamma, \overline{u_0})\) and \(\theta \in (0, 1)\) such that

\[
||u||_{C^{2+\theta,1+\theta}([0,\infty) \times [t,t+1])} \leq C, \quad \forall \, t \geq 1. \tag{4.21}
\]

**Proof**  Since \(||u||_{L^\infty(\Omega)}\) and \(||v||_{W^{1,\infty}(\Omega)}\) are now uniform-in-time bounded, one infers from the key identity (2.1) that

\[
\sup_{t > 0} ||u_\ell||_{L^\infty(\Omega)} \leq \sup_{t > 0} \left( \| (I - \Delta)^{-1} [u\gamma(v)] \|_{L^\infty(\Omega)} + \| u\gamma(v) \|_{L^\infty(\Omega)} \right)
\]

\(\square\) Springer
As a result, we deduce by Lemma 2.4 that
\[
\sup_{t>0} \|u\|_{L^\infty(\Omega)} < \infty,
\]
and from the second equation of (1.4),
\[
\sup_{t>0} \|v\|_{W^{2,p}(\Omega)} \leq C_{17}' \sup_{t>0} \|u\|_{L^\infty(\Omega)} < \infty,
\]
with any \(1 < p < \infty\) with \(C_{17}, C_{17}' > 0\) depending on \(n, p\) and \(\Omega\) at most. Thus, we have \(v \in W^{2,1}_{p} (\Omega \times [t, t+1])\) with any \(p > \frac{n+2}{n} \) for any \(t > 0\). Then, by the Sobolev embedding theorem, there exist \(\theta_1 \in (0, 2 - \frac{n+2}{p}]\) and time-independent constants \(C_{18}, C_{18}' > 0\) such that
\[
\|v\|_{C^{\theta_1, \frac{2}{2}}(\overline{\Omega} \times [t,t+1])} \leq C_{18}' \|v\|_{W^{2,1}_{p}(\Omega \times [t,t+1])} \leq C_{18}, \quad \forall \ t > 0.
\]
(4.22)

On the other hand, in the same manner as in [1, Lemma 5.1], there exists a time-independent constant \(C_{19} > 0\) such that
\[
\|u\|_{C^{\theta_2, \frac{2}{2}}(\overline{\Omega} \times [t,t+1])} \leq C_{19}, \quad \forall \ t \geq 1,
\]
with some \(\theta_2 \in (0, 1)\).

Then we recall the following variant form of the key identity:
\[
v_t - \gamma(v) \Delta v = (I - \Delta)^{-1}[\gamma(v)] - v\gamma(v).
\]
Since \(\gamma(v(x,t))\) is uniformly bounded from above and below, \(\gamma(v(x,t))\Delta\) is a uniform elliptic operator. Also, in view of our assumption (A0), as well as the regularizing effect of the operator \((I - \Delta)^{-1}\), both \(v\gamma(v)\) and 
\((I - \Delta)^{-1}[\gamma(v)]\) are now bounded in \(C^{\theta_3, \frac{2}{2}}(\overline{\Omega} \times [t+1])\). Thus we can further deduce, by a standard version of Schauder’s theory for parabolic equations, that with some \(\theta_3 \in (0, 1)\),
\[
\|v\|_{C^{2+\theta_3, \frac{2}{2}}(\overline{\Omega} \times [t,t+1])} \leq C_{18}', \quad \forall \ t \geq 1.
\]
In turn, we may finally deduce from the equation for \(u\), by Schauder’s theory, that
\[
\|u\|_{C^{2+\theta_1, \frac{2}{2}}(\overline{\Omega} \times [t,t+1])} \leq C_{19}', \quad \forall \ t \geq 1.
\]
\[
\square
\]

With the above preparations, we are now ready to prove the exponential decay of \(\|u - \overline{u_0}\|_{L^\infty}\). Denoting \(w = u - \overline{u_0}\) and \(\gamma_0 = \gamma(\overline{u_0})\), by the semigroup theory, we infer from (4.17) that, for any \(t > \tau_0 \geq 1\),
\[
w(t) = e^{\gamma_0 \Delta t} w(\tau_0) + \int_{\tau_0}^{t} e^{\gamma_0 \Delta (t-s)} \Delta ((\gamma(v(s) - \gamma_0)) u(s)) ds.
\]
(4.23)

As a result, we deduce by Lemma 2.4 that
\[
\|w(t)\|_{L^\infty(\Omega)} \leq \|e^{\gamma_0 \Delta t} w(\tau_0)\|_{L^\infty(\Omega)} + \int_{\tau_0}^{t} \|e^{\gamma_0 \Delta (t-s)} \Delta ((\gamma(v(s) - \gamma_0)) u(s))\|_{L^\infty(\Omega)} ds
\]
\[
\leq \|e^{\gamma \Delta t} w(\tau_0)\|_{L^\infty(\Omega)}
\]
\[
+ C_{20} \int_{\tau_0}^{t} e^{-\gamma_0 \Delta_1 (t-s)} (1 + (t-s)^{-\frac{1}{2}}) \|\nabla ((\gamma(v(s) - \gamma_0)) u(s))\|_{L^\infty(\Omega)} ds.
\]
Since $\| \nabla u \|_{L^\infty(\Omega)} \leq C'_9$ with some $C'_9 > 0$ for all $t \geq 1$, due to Lemma 4.6, we obtain that, for $t \geq 1$,
\[
\| \nabla ((\gamma(v(t) - \gamma_0)u(t))) \|_{L^\infty(\Omega)} \leq \| \gamma'(v(t))v(t)\nabla v(t) \|_{L^\infty(\Omega)} + \| (\gamma(v(t)) - \gamma(\overline{w}_0)) \nabla u(t) \|_{L^\infty(\Omega)} \\
\leq C_{21} \| \nabla v \|_{L^\infty(\Omega)} + C_{21} \| \gamma(v(t)) - \gamma(\overline{w}_0) \|_{L^\infty(\Omega)} \\
\leq C'_{21} (\| \nabla v \|_{L^\infty(\Omega)} + \| v(t) - \overline{w}_0 \|_{L^\infty(\Omega)}),
\]
where we use the fact that
\[
|\gamma(v) - \gamma(\overline{w}_0)| = |(v(t) - \overline{w}_0)\int_0^1 \gamma'(sv + (1-s)\overline{w}_0)ds| \leq C_{22}|v(t) - \overline{w}_0|,
\]
(4.24)
since $sv(t,x) + (1-s)\overline{w}_0$ is uniformly bounded from above and below on $[0, +\infty) \times \overline{\Omega}$ for all $s \in [0, 1]$.

As a result, recalling Lemma 4.5 and Lemma 2.4, we may infer that
\[
\| u(t) \|_{L^\infty(\Omega)} \leq \| e^{\gamma_0\Delta t}w(\tau_0) \|_{L^\infty(\Omega)} + \int_{\tau_0}^t \| e^{\gamma_0\Delta(t-s)} \Delta((\gamma(v(s) - \gamma_0)u(s))) \|_{L^\infty(\Omega)}ds \\
\leq C_{23} e^{-\gamma_0\mu_1t} \| w(\tau_0) \|_{L^\infty(\Omega)} + C_{23} \int_0^t e^{-\gamma_0\mu_1(t-s)}(1 + (t-s)^{-\frac{1}{2}})e^{-\alpha s}ds \\
\leq C'_3 e^{-\alpha' t},
\]
(4.25)
with any $\alpha' < \min\{\gamma_0\mu_1, \alpha\}$. Here, we use the fact that, for any $\beta \geq \kappa > 0$,
\[
\int_0^t e^{-\beta(t-s)}(1 + (t-s)^{-\frac{1}{2}})e^{-\kappa s}ds = e^{-\beta t} \int_0^t e^{(\beta - \kappa)s}(1 + (t-s)^{-\frac{1}{2}})ds \\
\leq e^{-\kappa t}(t + 2t^{\frac{1}{2}}) \leq C_{24} e^{-\kappa' t},
\]
with any $\kappa' < \kappa$. Here, $C_{24} > 0$ depends only on $\kappa$ and $\kappa'$. On the other hand, for $0 < \beta < \kappa$,
\[
\int_0^t e^{-\beta(t-s)}(1 + (t-s)^{-\frac{1}{2}})e^{-\kappa s}ds = e^{-\beta t} \int_0^t e^{(\beta - \kappa)s}(1 + (t-s)^{-\frac{1}{2}})ds \\
\leq e^{-\beta t}(t + 2t^{\frac{1}{2}}) \leq C_{25} e^{-\beta' t},
\]
with any $\beta' < \beta$ and $C_{25} > 0$ depending on $\beta, \beta'$.

**Proof of Theorem 1.1** Convergence: By Lemma 4.5 and (4.25), we conclude that
\[
\| u(\cdot, t) - \overline{w}_0 \|_{L^\infty(\Omega)} + \| u(\cdot, t) - \overline{w}_0 \|_{W^{1, \infty}(\Omega)} \leq C_{26} e^{-\alpha' t}, \quad \forall t \geq 1,
\]
with some $\alpha' > 0$ and $C_{26} > 0$ depending on $u_0, \gamma, n$ and $\Omega$. \qed

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