STANDARD CONJECTURES FOR ABELIAN FOURFOLDS

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Abstract. Let $A$ be an abelian fourfold. We prove the standard conjecture of Hodge type for $A$, namely that the intersection product

$$Z_{\text{num}}^2(A)_{\mathbb{Q}} \times Z_{\text{num}}^2(A)_{\mathbb{Q}} \rightarrow \mathbb{Q}$$

is of signature $(\rho_2 - \rho_1 + 1; \rho_1 - 1)$, with $\rho_n = \dim Z_{\text{num}}^n(A)_{\mathbb{Q}}$. The approach consists in reformulating this question into a $p$-adic problem and then using $p$-adic Hodge theory to solve it.

By combining this result with a theorem of Clozel we deduce that numerical equivalence on $A$ coincides with $\ell$-adic homological equivalence on $A$ for infinitely many prime numbers $\ell$. Hence, what is missing among the standard conjectures for abelian fourfolds is $\ell$-independency of $\ell$-adic homological equivalence.

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Introduction

In this paper we prove that the standard conjecture of Hodge type holds for abelian fourfolds. This is the first unconditional result on the conjecture

\[1\] Milne showed that the Hodge conjecture for complex abelian varieties implies the standard conjecture of Hodge type for abelian varieties over any field [Mil02].
since its formulation. Before giving the precise statement and a sketch of
the proof, we briefly recall the history of the problem.

In this introduction $X$ will be a smooth, projective and geometrically
connected variety over a base field $k$ of characteristic $p \geq 0$. We denote
by $Z^n_{\text{num}}(X)_\mathbb{Q}$ the space of algebraic cycles of codimension $n$ with rational
coefficients modulo numerical equivalence. This is a finite dimensional
$\mathbb{Q}$-vector space and its dimension will be denoted by

\[ (*) \quad \rho_n = \dim Z^n_{\text{num}}(X)_\mathbb{Q}. \]

The history of the problem starts with the so called Hodge index theorem.

**Theorem 0.1.** Suppose that $X$ has dimension two. Then the intersection
product

\[ Z^1_{\text{num}}(X)_\mathbb{Q} \times Z^1_{\text{num}}(X)_\mathbb{Q} \longrightarrow \mathbb{Q} \]

is of signature $(s_+; s_-) = (1; \rho_1 - 1)$.

When the characteristic $p$ is zero, the above theorem was proved by Hodge
by relating the intersection product to the cup product in singular cohomol-
ogy through the cycle class map [Hod33]. An algebraic proof, valid in any
characteristic, was found by Segre [Seg37] and Bronowski [Bro38].

Elaborating on an argument of Mattuck–Tate [MT58], Grothendieck re-
alized that the Lang–Weil estimate for the number of rational points on a
smooth and projective curve $C$ over a finite field follows from the Hodge
index theorem applied to the surface $X = C \times C$ [Gro58]. He then proposed
a program to show the Weil conjectures for varieties of higher dimension
based on a conjectural generalisation of the Hodge index theorem, known
as standard conjecture of Hodge type. Together with the three other stan-
dard conjectures (and the resolution of singularities), it was considered by
Grothendieck as the most urgent task in algebraic geometry [Gro69].

This conjecture is connected with other arithmetic contexts such as the
conjectural description of the rational points on a Shimura variety over a
finite field [LR87] and the weight-monodromy conjecture [Sai].

The four standard conjectures imply that the category of numerical mo-
tives is semisimple and polarizable just as the category of Hodge structures of
smooth projective complex varieties [SR72]. Surprisingly enough, Jannsen
proved semisimplicity unconditionally and independently of the standard
conjectures [Jan92]. Polarizability is still open and it is intimately related
to the standard conjecture of Hodge type.

As a conclusion of this historical panorama, let us mention that Gillet
and Soulé have proposed an arithmetic version of the standard conjectures
(i.e. over a ring of integers) [GS94]. Some results on the arithmetic standard
conjecture of Hodge type can be found in [Künn95, Künn98, KM00, KT01].
Let us now recall the statement of the conjecture. Let $d$ be the dimension of $X$ and fix a hyperplane section $L$ of $X$. For $n \leq d/2$ define the space of primitive cycles $\mathcal{Z}_{\text{num}}^{\text{prim}}(X)_{\mathbb{Q}}$ as

$$\mathcal{Z}_{\text{num}}^{\text{prim}}(X)_{\mathbb{Q}} = \left\{ \alpha \in \mathcal{Z}_{\text{num}}^n(X)_{\mathbb{Q}}, \; \alpha \cdot L^{d-2n+1} = 0 \text{ in } \mathcal{Z}_{\text{num}}^{d-n+1}(X)_{\mathbb{Q}} \right\}$$

and define the pairing

$$\langle \cdot, \cdot \rangle_n : \mathcal{Z}_{\text{num}}^{\text{prim}}(X)_{\mathbb{Q}} \times \mathcal{Z}_{\text{num}}^{\text{prim}}(X)_{\mathbb{Q}} \to \mathbb{Q}$$

via the intersection product

$$\alpha, \beta \mapsto (-1)^n \alpha \cdot \beta \cdot L^{d-2n}.$$ 

The standard conjecture of Hodge type predicts that this pairing is positive definite.

The evidences for Grothendieck were the case $n = 1$ (which can be shown by reducing it to Theorem 0.1) and the case where $p = 0$. Indeed, in characteristic zero, one can use the cycle class map to relate the quadratic form $\langle \cdot, \cdot \rangle_n$ to the quadratic form given by the cup product on singular cohomology. Then this kind of positivity statements can be deduced from positivity statements in cohomology, such as the Hodge–Riemann relations in Hodge theory.

The following is our main theorem.

**Theorem 0.2.** The standard conjecture of Hodge type holds for abelian fourfolds.

It turns out that this statement is equivalent to the following, which is maybe a more direct formulation (see also Proposition 1.15).

**Theorem 0.3.** Let $X$ be an abelian fourfold. Then the intersection product

$$\mathcal{Z}_{\text{num}}^2(X)_{\mathbb{Q}} \times \mathcal{Z}_{\text{num}}^2(X)_{\mathbb{Q}} \to \mathbb{Q}$$

is of signature $(s_+; s_-) = (\rho_2 - \rho_1 + 1; \rho_1 - 1)$, with $\rho_n$ as in (5).

This formulation should be reminiscent of the Hodge index theorem.

**Remark 0.4.** As we explained above, the standard conjecture of Hodge type is known in characteristic zero. Hence Theorem 0.2 is new only in positive characteristic and for those algebraic classes that cannot be lifted to characteristic zero. We discuss the existence of such classes in Appendix A.

By combining Theorem 0.2 with a theorem of Clozel [Clo99] we deduce the following.

**Theorem 0.5.** Let $X$ be an abelian fourfold. Then numerical equivalence on $X$ coincides with $\ell$-adic homological equivalence on $X$ for infinitely many prime numbers $\ell$. 

The fact that homological and numerical equivalence should always coincide is also one of the four standard conjectures. The two others (namely Künneth and Lefschetz) being known for abelian varieties, Theorems 0.2 and 0.5 imply that in order to fully understand standard conjectures for abelian fourfolds what is missing is \( \ell \)-independency of \( \ell \)-adic homological equivalence.

**Idea of the proof.** The starting point of the proof\(^2\) of Theorem 0.2 is a classical fact from the theory of quadratic forms: let \( q \) be a \( \mathbb{Q} \)-quadratic form, if we know \( q \otimes \mathbb{Q}_\ell \) for all prime numbers \( \ell \) then we have information on the signature of \( q \), more precisely we know the difference \( s_+ - s_- \) modulo 8. When \( q \) is the quadratic form \( \langle \cdot, \cdot \rangle \) as above, then one can hope to compute \( q \otimes \mathbb{Q}_\ell \) through the cycle class map.

In characteristic \( p = 0 \) one computes \( q \otimes \mathbb{R} \) by directly embedding it into singular cohomology. Our strategy should be thought as a way of circumventing the impossibility (when \( p > 0 \)) of embedding \( q \otimes \mathbb{R} \) in some Weil cohomology by instead embedding all the other completions of \( q \) in Weil cohomologies.

In order to deduce the full signature from the information modulo 8 one needs to be sure that the rank of \( q \) is small. Hence, one cannot apply this strategy to the whole space of algebraic cycles but rather one first decomposes it into smaller quadratic spaces and then computes \( q \otimes \mathbb{Q}_\ell \) for each of the quadratic subspaces.

To do so, one first reduces the question to varieties defined over a finite field (this is a classical specialization argument) where abelian varieties are known to always admit complex multiplication [Tat66]. Then one uses complex multiplication to decompose the space of algebraic cycles into smaller quadratic spaces. Finally, it turns out that when the abelian variety has dimension four then the subspaces constructed are of rank two (at least those where the problem is not trivial).

Technically speaking, we do not compute \( q \otimes \mathbb{Q}_\ell \) directly, but rather construct another \( \mathbb{Q} \)-quadratic form \( \tilde{q} \) and try to compare these two. This quadratic form \( \tilde{q} \) is constructed as follows. First, the abelian fourfold \( X \) admits a lifting \( \tilde{X} \) to characteristic zero on which complex multiplication still acts (Honda–Tate). The action of complex multiplication decomposes the singular cohomology \( H^4_{\text{sing}}(\tilde{X}, \mathbb{Q}) \) into subspaces. Those are quadratic subspaces endowed with the cup product. To a given factor \( q \) of \( Z^2_{\text{num}}(X)_\mathbb{Q} \) one associates the factor \( \tilde{q} \) of \( H^4_{\text{sing}}(\tilde{X}, \mathbb{Q}) \) which is the same irreducible representation for the action of complex multiplication.

\(^2\)The reader could also read the report [Anc16] to have an idea of the strategy.
The comparison \( q \otimes \mathbb{Q}_\ell \cong \tilde{q} \otimes \mathbb{Q}_\ell \) holds for all \( \ell \neq p \) by smooth proper base change in \( \ell \)-adic cohomology. On the other hand one computes \( \tilde{q} \otimes \mathbb{R} \) using the Hodge–Riemann relations; whether it is positive or negative definite depends on the Hodge types appearing in the Hodge structure \( \tilde{q} \).

Now the theory of quadratic forms (in particular the product formula on Hilbert symbols) tells us that the positivity of \( q \otimes \mathbb{R} \) is equivalent to the following statement. The quadratic forms \( q \otimes \mathbb{Q}_p \) and \( \tilde{q} \otimes \mathbb{Q}_p \) are not isomorphic precisely if the Hodge types of \( \tilde{q} \) force \( \tilde{q} \otimes \mathbb{R} \) to be negative definite. (Note that this equivalence holds because our quadratic spaces have rank two).

This formulation translates the problem into a question in \( p \)-adic Hodge theory. To solve it we use the \( p \)-adic comparison theorem which gives a canonical isomorphism \( (q \otimes \mathbb{Q}_p) \otimes B_{\text{crys}} \cong (\tilde{q} \otimes \mathbb{Q}_p) \otimes B_{\text{crys}} \) over a large \( \mathbb{Q}_p \)-algebra \( B_{\text{crys}} \). The strategy then consists in writing explicitly the matrix of the isomorphism with respect to well chosen bases of the two \( \mathbb{Q}_p \)-structures (i.e. computing the \( p \)-adic periods) and then exploit this explicit isomorphism to determine whether the two quadratic forms are isomorphic also over \( \mathbb{Q}_p \).

The reason for which a \( p \)-adic period is in general more computable than a complex one is that it comes equipped with some extra structures (in particular the action of an absolute Frobenius and the de Rham filtration) which sometimes characterize it\(^3\). Moreover, in our particular situation, these extra structures are particularly simple: the absolute Frobenius must act trivially (because \( q \otimes \mathbb{Q}_p \) is spanned by algebraic cycles) and the only non-trivial subspace appearing in the de Rham filtration is a line (as these quadratic spaces have rank two).

Once the matrix \( f \in M_{2 \times 2}(B_{\text{crys}}) \) of \( p \)-adic periods is explicitly computed we exploit the relation \( (q \otimes \mathbb{Q}_p) = f^\top \cdot (\tilde{q} \otimes \mathbb{Q}_p) \cdot f \). Even though in this equality we only know \( f \), this information is enough to determine whether the two quadratic forms \( q \otimes \mathbb{Q}_p \) and \( \tilde{q} \otimes \mathbb{Q}_p \) have the same discriminant and represent the same elements of \( \mathbb{Q}_p \). Again because we are in rank two these properties control whether \( q \otimes \mathbb{Q}_p \) and \( \tilde{q} \otimes \mathbb{Q}_p \) are isomorphic.

**Organisation of the paper.** In Section 1 we recall the standard conjecture of Hodge type and present some first reduction steps, among which the fact that it is enough to work over a finite field. We state Theorem 0.2 (Theorem 1.18 in the text) and deduce from it Theorem 0.5 (Theorem 1.20 in the text). In Section 2 we recall classical results on the motive of an abelian variety. In Section 3 we show that Theorem 1.18 holds true for Lefschetz classes on abelian varieties, i.e. those algebraic classes that are linear combinations of intersections of divisors. In Section 4 we recall that an abelian variety \( X \) over a finite field admits complex multiplication and

\(^3\)For example, an element of \( B_{\text{crys}} \) which is invariant under the Frobenius and sits in the zeroth step of the filtration must be an element of \( \mathbb{Q}_p \) [Fon94, Theorem 5.3.7].
use this to decompose its cohomology (or its motive). We also recall that $X$ together with its complex multiplication lifts to characteristic zero, hence its motivic decomposition lifts too. In Section 5 we study the interesting subfactors of this decomposition and we call them exotic. By definition, they are those containing algebraic classes which are not Lefschetz classes. The main result of the section says that if $X$ has dimension four then exotic motives have rank two (in the sense that their realizations are cohomology groups of rank two).

From there a somehow independent text starts, in which abelian varieties are not involved anymore: we study motives of rank two, living in mixed characteristic and having algebraic classes in positive characteristic. The main result (Theorem 6.1) predicts the signature of the intersection product on those algebraic classes. In Section 6 we put all pieces together and explain how Theorem 6.1 implies Theorem 1.18. In Section 7 we recall classical facts from the theory of quadratic forms and use them to reduce Theorem 6.1 to a $p$-adic question. The tool to attack this $p$-adic question is $p$-adic Hodge theory which we recall in Section 8. Finally in Section 9 we solve the problem through an explicit computation of $p$-adic periods.

Appendix A contains examples of exotic motives and in particular of those having algebraic classes that cannot be lifted to characteristic zero.

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Conventions

Throughout the paper we will use the following conventions.

(1) A **variety** will mean a smooth, projective and geometrically connected scheme over a base field $k$.

(2) Let $k$ be a base field and $F$ be a field of coefficients of characteristic zero, we will denote by

$$
\text{CHM}(k)_F
$$

the category of **Chow motives** over $k$ with coefficients in $F$. (The subscript $F$ may sometimes be omitted.) For generalities, we refer to [And04] in particular to [And04, Definition 3.3.1.1] for the notion of Weil cohomology and to [And04, Proposition 4.2.5.1] for the associated realization functor.
We will denote by
\[ h : \text{Var}_k \to \text{CHM}(k)^{\text{op}}_F \]
the functor associating to each variety its motive.

We will work also with **homological and numerical motives**. The motive of a variety \( X \) in these categories will still be denoted by \( h(X) \). We will denote by
\[ \text{NUM}(k)_F \]
the category of numerical motives over \( k \) with coefficients in \( F \).

Finally, we will need also to work with **relative motives** as defined for example in [O’S11 §5.1]. The relative situation that will appear in this paper will always be over the ring of integers \( W \) of a \( p \)-adic field. We will use analogous notations as in the absolute setting, for example Chow motives over \( W \) will be denoted by
\[ \text{CHM}(W). \]

(3) By **classical realization** we will mean Betti realization
\[ R_B : \text{CHM}(\mathbb{C})_\mathbb{Q} \to \text{GrVec}_\mathbb{Q}, \]
\( \ell \)-adic realization (when \( \ell \) is invertible in \( k \))
\[ R_\ell : \text{CHM}(k)_\mathbb{Q} \to \text{GrVec}_{\mathbb{Q}_\ell}, \]
and crystalline realization (when \( k \) is perfect and of positive characteristic)
\[ R_{\text{crys}} : \text{CHM}(k)_\mathbb{Q} \to \text{GrVec}_{\text{Frac}(W(k))}. \]

We tried to distinguish the properties which hold true for any realization and those which are known only for classical realizations. In any case, the main results of the paper only make use of classical realizations and the reader can safely think only about them.

Classical realizations are endowed with extra-structures (Galois action, Hodge decomposition, absolute Frobenius, . . .) which will appear in the paper.

(4) The **unit object** in one of the above categories of motives (Chow, homological or numerical motives) over a base \( S \) is the motive \( h(S) \) and it will be denote by
\[ 1 := h(S). \]

When \( S = \text{Spec}(k) \) we have the identification
\[ \text{End}(1) = \mathbb{Q}. \]

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4When \( k \) is of characteristic zero, algebraic de Rham cohomology is a classical realization, but we will not make an explicit use of it.
(5) An **algebraic class** of a motive $T$ is a map
$$\alpha \in \text{Hom}(\mathbb{1}, T).$$

The realization of an algebraic class gives a map
$$R(\alpha) : F \rightarrow R(T),$$
where $F$ is the field of coefficients of the realization. The map $R(\alpha)$ is characterized by its value at $1 \in F$. An element of the cohomology group $R(T)$ of this form
$$R(\alpha)(1) \in R(T)$$
will be called an algebraic class of $R(T)$. By abuse of language, an algebraic class of the cohomology group $R(T)$ may also be called an algebraic class of the motive $T$.

Finally, we may sometimes ignore Tate twists and say that $T$ has algebraic classes when $T(n)$ does. For example, for a variety $X$, the cohomology group $H^{2n}(X)$ may have algebraic classes, although, strictly speaking, these classes belong to $H^{2n}(X)(n)$. Hopefully this will not bring confusion as we will work with motives whose realization is concentrated in one single cohomological degree.

(6) An algebraic class $\alpha$ of a motive $T$ is **numerically trivial** if, for all $\beta \in \text{Hom}(T, \mathbb{1})$, the composition $\beta \circ \alpha$ is zero. Two algebraic classes are numerically equivalent if their difference is numerically trivial.

(7) Let $X$ be a variety. The space of algebraic cycles on $X$ with coefficients in $F$ modulo numerical equivalence will be denoted by
$$\mathcal{Z}_{\text{num}}^n(X)_F.$$ Similarly, for a fixed Weil cohomology, $\mathcal{Z}_{\text{hom}}^n(X)_F$ will denote the space of algebraic cycles on $X$ with coefficients in $F$ modulo homological equivalence.

1. **Standard conjecture of Hodge type**

In this section we recall some classical conjectures (due to Grothendieck) and give some reformulations of those. We state our main result (Theorem 1.18) and prove a consequence (Theorem 1.20).

Throughout the section, $k$ is a base field, $H^*$ is a fixed classical Weil cohomology and $R$ the associated realization functor (see Conventions). We fix a variety $X$ over $k$ of dimension $d$ and an ample divisor $L$ on $X$. We will write $h(X)$ for its homological motive (with respect to $H^*$).

**Conjecture 1.1.** (Standard conjecture of Künneth type)
There exists a (unique) decomposition
$$h(X) = \bigoplus_{n=0}^{2d} h^n(X)$$
such that $R(h^n(X)) = H^n(X)$ and $h^n(X) = h^{2d-n}(X)^\vee(-d)$. 

Conjecture 1.2. (Standard conjecture of Lefschetz type)
For all $n \leq d$, there is an isomorphism
\[ h^n(X) \sim \to h^{2d-n}(X)(d-n) \]
induced by the cup product with $L^{d-n}$. Moreover, the Künneth decomposition can be refined to a (unique) decomposition
\[ h^n(X) = h^n,\text{prim}(X) \oplus h^{n-2}(X)(-1) \]
which realizes to the primitive decomposition of $H^n(X)$.

Remark 1.3. By the work of Lieberman and Kleiman the two conjectures above are known for abelian varieties [Kle68, Lie68], see also Theorem 2.1.

Assumption 1.4. From now on we will assume that $X$ verifies the Conjectures 1.1 and 1.2.

Remark 1.5. Conjectures 1.1 and 1.2 induce an isomorphism
\[ h^n(X) \sim \to h^n(X)^\vee(-n). \]
By adjunction this gives a pairing
\[ q_n : h^n(X) \otimes h^n(X) \to \mathbb{1}(-n). \]
By construction the Lefschetz decomposition
\[ h^n(X) = h^n,\text{prim}(X) \oplus h^{n-2}(X)(-1) \]
is orthogonal with respect to this pairing.

Definition 1.6. For $n \leq d$, we define the pairing
\[ \langle \cdot, \cdot \rangle_{n,\text{mot}} : h^n(X) \otimes h^n(X) \to \mathbb{1}(-n) \]
recursively on $n$, by slightly modifying the pairing $q_n$ of the above remark. We impose that the Lefschetz decomposition
\[ h^n(X) = h^n,\text{prim}(X) \oplus h^{n-2}(X)(-1) \]
is still orthogonal, we impose the equality $\langle \cdot, \cdot \rangle_{n,\text{mot}} = \langle \cdot, \cdot \rangle_{n-2,\text{mot}}(-2)$ on $h^{n-2}(X)(-1)$ and finally on $h^n,\text{prim}(X)$ we define
\[ \langle \cdot, \cdot \rangle_{n,\text{mot}} = (-1)^{(n+1)/2}q_n. \]

Remark 1.7. If $k$ is embedded in $\mathbb{C}$, the Betti realization of $\langle \cdot, \cdot \rangle_{n,\text{mot}}$ is a polarization of the Hodge structure $H^n(X)$.

Definition 1.8. We define the space $Z^n_{\text{num}}(X)\mathbb{Q}$ and the pairing $\langle \cdot, \cdot \rangle_n$.

1. For $2n \leq d$, we define the pairing
\[ \langle \cdot, \cdot \rangle_n : Z^n_{\text{hom}}(X)\mathbb{Q} \times Z^n_{\text{hom}}(X)\mathbb{Q} \to \mathbb{Q} = \text{End}(\mathbb{1}) \]
as follows. For $\alpha, \beta \in Z^n_{\text{hom}}(X)\mathbb{Q} = \text{Hom}(\mathbb{1}, h^{2n}(X)(n))$ consider
\[ (\alpha \otimes \beta) \in \text{Hom}(\mathbb{1}, h^{2n}(X) \otimes h^{2n}(X)(2n)) \]
and define
\[ \langle \alpha, \beta \rangle_n = \langle \cdot, \cdot \rangle_{2n,\text{mot}}(2n) \circ (\alpha \otimes \beta). \]
(2) For $2n \leq d$, define $Z^{n,\text{prim}}_{\text{num}}(X)_Q \subset Z^{n}_{\text{num}}(X)_Q$ as 

$$Z^{n,\text{prim}}_{\text{num}}(X)_Q = \text{Hom}_{\text{NUM}(k)}(\mathbb{1}, h^{2n,\text{prim}}(X)(n)).$$

We will keep the same notation as in (1) for the induced pairing 

$$\langle \cdot, \cdot \rangle_n : Z^{n,\text{prim}}_{\text{num}}(X)_Q \times Z^{n,\text{prim}}_{\text{num}}(X)_Q \to \mathbb{Q}.$$ 

**Remark 1.9.** The above definition is equivalent to the one given in the introduction (where we did not make the Assumption 1.4). Instead Definition 1.6 cannot be formulated without the Assumption 1.4.

**Conjecture 1.10.** (Standard conjecture of Hodge type) For all $n \leq d$, the pairing 

$$\langle \cdot, \cdot \rangle_n : Z^{n,\text{prim}}_{\text{num}}(X)_Q \times Z^{n,\text{prim}}_{\text{num}}(X)_Q \to \mathbb{Q}$$

of Definition 1.8 is positive definite.

**Proposition 1.11.** The kernel of the pairing $\langle \cdot, \cdot \rangle_n$ on the vector space $Z^{n}_{\text{hom}}(X)_Q$ is the space of algebraic classes which are numerically trivial. The analogous statement holds true for $Z^{n,\text{prim}}_{\text{num}}(X)_Q$.

**Proof.** This is a direct consequence of a general fact. Let $T$ be a homological motive and suppose that it is endowed with an isomorphism $f : T \sim \to T^\vee$. Call $q : T \otimes T \to \mathbb{1}$ the pairing induced by adjunction. In analogy to Definition 1.8, $q$ induces a pairing $\langle \cdot, \cdot \rangle$ on the space $\text{Hom}(\mathbb{1}, T)$. First, notice that this pairing can also be described in the following way

$$\langle \alpha, \beta \rangle = \alpha^\vee \circ f \circ \beta.$$ 

This equality holds for formal reasons in any rigid category (alternatively, for homological motives, one can check it after realization). Using this equality one has that $\beta$ is in the kernel of the pairing if and only if $\langle \alpha^\vee \circ f \circ \beta \rangle = 0$ for all $\alpha$. As $f$ is an isomorphism, this is equivalent to the fact that $\gamma \circ \beta = 0$ for all $\gamma \in \text{Hom}(T, \mathbb{1})$. This means precisely that $\beta$ is numerically trivial.

To conclude, note that, by construction, one can apply this general fact to $T = h^{2n}(X)(n)$ or $T = h^{2,n,\text{prim}}(X)(n)$. \qed

**Corollary 1.12.** The pairing $\langle \cdot, \cdot \rangle_n$ is perfect on $Z^{n}_{\text{num}}(X)_Q$. Moreover the pairing is positive definite on $Z^{n}_{\text{num}}(X)_Q$ if and only if it is positive semidefinite on $Z^{n}_{\text{hom}}(X)_Q$. The analogous statements hold true for $Z^{n,\text{prim}}_{\text{num}}(X)_Q$.

**Remark 1.13.** If $k$ is embedded in $\mathbb{C}$ and if we work with Betti cohomology (cf. Remark 1.7), the Hodge–Riemann relations imply that $\langle \cdot, \cdot \rangle_n$ is positive definite on $(n, n)$-classes, hence on $Z^{n}_{\text{hom}}(X)_Q$. In particular the standard conjecture of Hodge type holds true.

Moreover, homological and numerical equivalence coincide (recall that we work under the Assumption 1.4). The argument is due to Lieberman \[\text{Lic68}\] (see also \[And04\, Corollary 5.4.2.2\]) and we recall it here.

If $\alpha \in Z^{n}_{\text{hom}}(X)_Q$ is a nonzero class in codimension $n \leq d/2$ the inequality $\langle \alpha, \alpha \rangle_n > 0$ implies that $\alpha$ is not numerically trivial. On the other
hand, the iterated intersection product with the hyperplane section \( L \) induces an isomorphism \( \mathcal{Z}^n_{\text{hom}}(X)_\mathbb{Q} \cong \mathcal{Z}^{d-n}_{\text{hom}}(X)_\mathbb{Q} \). Hence the intersection product \( \mathcal{Z}^n_{\text{hom}}(X)_\mathbb{Q} \times \mathcal{Z}^{d-n}_{\text{hom}}(X)_\mathbb{Q} \to \mathbb{Q} \) is non-degenerate in one variable if and only if it is non-degenerate in the other, which implies that homological and numerical equivalence must coincide also in codimension \( n \geq d/2 \).

Together with the case of characteristic zero (see the above remark), the following is the only known result on the standard conjecture of Hodge type [And04, 5.3.2.3].

**Theorem 1.14.** The pairing \( \langle \cdot, \cdot \rangle_1 \) on \( \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \) is positive definite. Equivalently, the pairing

\[
\mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \times \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \to \mathbb{Q} \quad D, D' \mapsto D \cdot D' \cdot L^{d-2}
\]

is of signature \( (s_+, s_-) = (1; \rho_1 - 1) \) with \( \rho_1 = \dim \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \).

**Proposition 1.15.** The standard conjecture of Hodge type for a fourfold \( X \) is equivalent to the statement that the intersection product

\[
\mathcal{Z}^2_{\text{num}}(X)_\mathbb{Q} \times \mathcal{Z}^2_{\text{num}}(X)_\mathbb{Q} \to \mathbb{Q}
\]

is of signature \( (s_+, s_-) = (\rho_2 - \rho_1 + 1; \rho_1 - 1) \) with \( \rho_n = \dim \mathcal{Z}^n_{\text{num}}(X)_\mathbb{Q} \).

In particular the conjecture does not depend on the ample divisor \( L \) chosen nor on the classical realization.

**Proof.** By Theorem 1.14 the standard conjecture of Hodge type holds true for divisors. Hence we have to study codimension 2 cycles. Consider the decomposition

\[
\mathcal{Z}^2_{\text{num}}(X)_\mathbb{Q} = \mathcal{Z}^2_{\text{prim}}(X)_\mathbb{Q} \oplus L \cdot \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q}
\]

induced by the Lefschetz decomposition

\[
\mathfrak{h}^4(X) = \mathfrak{h}^{4,\text{prim}}(X) \oplus \mathfrak{h}^2(X)(-1).
\]

It is orthogonal with respect to the intersection product as the Lefschetz decomposition is orthogonal with respect to the cup product (already at homological level).

We claim that the intersection product on \( L \cdot \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \) is of signature \( (1; \rho_1 - 1) \). Assuming the claim we notice that, the intersection product is positive definite on \( \mathcal{Z}^2_{\text{prim}}(X)_\mathbb{Q} \) if and only if the intersection product of the total space \( \mathcal{Z}^2_{\text{num}}(X)_\mathbb{Q} \) has the predicted signature.

For the claim, consider \( \alpha = L \cdot D \) an element of \( L \cdot \mathcal{Z}^1_{\text{num}}(X)_\mathbb{Q} \). Then we have \( \alpha \cdot \alpha = D \cdot D \cdot L^2 \) hence the claim follows from Theorem 1.14. \( \square \)

**Proposition 1.16.** Suppose that \( X \) is of dimension four and let \( X_0 \) be a specialization of \( X \). Suppose that \( X_0 \) as well satisfies the Assumption 1.4 (with respect to the specialization of \( L \)). Then the standard conjecture of Hodge type for \( X_0 \) implies the standard conjecture of Hodge type for \( X \).
Proof. By Proposition 1.15 we can work with one suitable realization. We will work with $\ell$-adic cohomology. Then, smooth proper base change induces a canonical inclusion $Z_{n,\text{prim}}(X)\subset Z_{n,\text{prim}}(X_0)\mathbb{Q}$. Then, if the pairing $\langle \cdot, \cdot \rangle_n$ is positive semidefinite on $Z_{n,\text{prim}}(X_0)\mathbb{Q}$ it must be so for $Z_{n,\text{prim}}(X)\mathbb{Q}$. On the other hand, by Corollary 1.12 this semipositivity property is a reformulation of the standard conjecture of Hodge type. □

Remark 1.17. The above proposition reduces the standard conjecture of Hodge type to varieties defined over a finite field. Such a reduction is classical, see for example [And04, Remark 5.3.2.2(2)].

The following is our main result. It is shown at the end of Section 6.

Theorem 1.18. The standard conjecture of Hodge type holds for abelian fourfolds.

Corollary 1.19. Let $A$ and $A_0$ be two abelian fourfolds and suppose that $A_0$ is a specialization of $A$. Let us fix a prime number $\ell$. If $\ell$-adic homological equivalence on $A_0$ coincide with numerical equivalence then the same holds true on $A$.

Proof. First, notice that the question whether homological and numerical equivalence coincide only matters for $Z_{2,\text{prim}}(A)\mathbb{Q}$. Indeed, homological and numerical equivalence coincide for divisors [Mat57] (see also [And04, Proposition 3.4.6.1]). This implies that they coincide also on dimension one cycles, as the standard conjecture of Lefschetz type holds true for abelian varieties. Finally, consider the decomposition

$$Z_{2,\text{prim}}(A)\mathbb{Q} = Z_{2,\text{prim}}^1(A)\mathbb{Q} \oplus L \cdot Z_{1,\text{hom}}(A)\mathbb{Q},$$

and notice that on the complement of $Z_{2,\text{prim}}^1(A)\mathbb{Q}$ the equivalences again coincide, as a consequence of the case of divisors.

Now, by smooth proper base change we have $Z_{2,\text{prim}}^1(A)\mathbb{Q} \subset Z_{2,\text{prim}}(A_0)\mathbb{Q}$. If homological and numerical equivalence coincide on $A_0$ then the pairing $\langle \cdot, \cdot \rangle_2$ on $Z_{2,\text{prim}}(A_0)\mathbb{Q}$ is positive definite by Theorem 1.18. Hence it is also positive definite on $Z_{2,\text{prim}}^1(A)\mathbb{Q}$. By Proposition 1.11 this means that there are no non-zero algebraic classes in $Z_{2,\text{prim}}^1(A)\mathbb{Q}$ which are numerically trivial. □

Theorem 1.20. Let $A$ be an abelian fourfold. Then numerical equivalence on $A$ coincides with $\ell$-adic homological equivalence on $A$ for infinitely many prime numbers $\ell$.

Proof. When $A$ is defined over a finite field, this result is due to Clozel [Clo99]. (Clozel’s result actually holds true without the dimensional restriction.) We can reduce to the finite field case by Corollary 1.19. □

Remark 1.21. The fact that homological and numerical equivalence should always coincide is also one of the four standard conjectures. The two others,
namely Künneth and Lefschetz, being known for abelian varieties, Theorems 1.18 and 1.20 imply that in order to fully understand the standard conjectures for abelian fourfolds what is missing is $\ell$-independency of $\ell$-adic homological equivalence.

2. THE MOTIVE OF AN ABELIAN VARIETY

In this section we recall classical results on motives of abelian type. We will work with the category of Chow motives $\text{CHM}(k)_F$, we fix a Weil cohomology $H^*$ together with its realization functor $R$. Generalities on this category can be found in the Conventions.

**Theorem 2.1.** Let $A$ be an abelian variety of dimension $g$. Let $\text{End}(A)$ be its ring of endomorphisms (as group scheme) and $h(A) \in \text{CHM}(k)_F$ be its motive. Then the following holds:

1. [DM91] The motive $h(A)$ admits a Chow–Künneth decomposition

$$h(A) = \bigoplus_{n=0}^{2g} h^n(A)$$

natural in $\text{End}(A)$ and such that

$$R(h^n(A)) = H^n(A).$$

2. [Kün94] There is a canonical isomorphism of graded algebras

$$h^*(A) = \text{Sym}^* h^1(A).$$

3. [Kin98, Proposition 2.2.1] The action of $\text{End}(A)$ on $h^1(A)$ (coming from naturality in 1) induces a morphism of algebras

$$\text{End}(A) \otimes \mathbb{Z} F \to \text{End}_{\text{CHM}(k)_F}(h^1(A))$$

and if $A$ is isogenous to $B \times C$ then $h^1(A) = h^1(B) \oplus h^1(C)$.

4. [Kün93] The Chow–Lefschetz conjecture holds true for $A$. In particular, the classical isomorphism in $\ell$-adic cohomology induced by a polarization $H^1_\ell(A) \cong H^1_\ell(A)^\vee(-1)$ lifts to an isomorphism

$$h^1(A) \cong h^1(A)^\vee(-1).$$

**Remark 2.2.** We will need the above results also in a slightly more general context, namely over the ring of integers of a $p$-adic field. Nowadays these results are known over very general bases, see [O'S11, Theorem 5.1.6] or [AHPL16, Theorem 3.3].

**Definition 2.3.** A motive is called of abelian type if it is a direct factor of the motive of an abelian variety (up to Tate twist).

We say that a motive of abelian type $T$ is of rank $d$ if the cohomology groups of $R(T)$ are all zero except in one degree and in that degree the cohomology group is of dimension $d$. In this case we will write $\dim T = d$. 
Remark 2.4. For motives of abelian type this definition is known to be independent of \( R \), see for example [Jan07, Corollary 3.5] and [Anc18, Corollary 1.6].

Proposition 2.5. Let \( T \) be a motive of abelian type of dimension \( d \). Consider its space of algebraic classes modulo numerical equivalence

\[
V_Z = V_Z(T) = \text{Hom}_{\text{NUM}}(\mathbb{Q}, T).
\]

Then the inequality \( \dim_{\mathbb{Q}} V_Z \leq d \) holds.

Moreover, if the equality \( \dim_{\mathbb{Q}} V_Z = d \) holds then we have the following facts:

1. All realizations of \( T \) are spanned by algebraic classes.
2. Numerical equivalence on \( \text{Hom}_{\text{CHM}}(\mathbb{Q}, T) \) coincides with homological equivalence (for all cohomologies).
3. Call \( L \) the field of coefficients of the realization \( R \), then the equality \( V_Z \otimes_{\mathbb{Q}} L = R(T) \) holds.

Proof. Let us consider \( n \) elements \( \bar{v}_1, \ldots, \bar{v}_n \) which are linearly independent in \( V_Z \) and fix \( v_1, \ldots, v_n \in \text{Hom}_{\text{CHM}}(\mathbb{Q}, T) \) which are liftings of those. By definition of numerical equivalence, there exist \( n \) elements \( f_1, \ldots, f_n \) in \( \text{Hom}_{\text{CHM}}(T, \mathbb{Q}) \) such that \( f_i(v_j) = \delta_{ij} \). By applying a realization we get

\[
R(f_i)(R(v_j)) = \delta_{ij}
\]

which implies the inequality \( n \leq \dim R(T) = d \).

When \( n = d \), the argument just above shows that \( R(v_1), \ldots, R(v_d) \) form a basis of \( \text{Hom}(R(\mathbb{Q}), R(T)) \), which means that \( R(T) \) is spanned by the algebraic classes \( R(v_1)(1), \ldots, R(v_d)(1) \), hence we have (1).

Fix an element \( v \in \text{Hom}_{\text{CHM}}(\mathbb{Q}, T) \) and write its realization in the previous basis \( R(v) = \lambda_1 R(v_1) + \ldots + \lambda_d R(v_d) \). Consider now the composition \( f_i(v) \). On the one hand, it is a rational number, on the other hand it equals \( \lambda_i \). This means that any algebraic class is a rational combination of the basis \( R(v_1)(1), \ldots, R(v_d)(1) \). This implies the points (2) and (3). \( \square \)

3. Lefschetz classes

The standard conjecture of Hodge type is known for divisors (Theorem 1.14). In this section we explain how this implies the standard conjecture of Hodge type for algebraic classes on abelian varieties that are linear combinations of intersections of divisors. This has been already pointed out by Milne [Mil02, Remark 3.7].

Throughout the section \( A \) is a polarized abelian variety of dimension \( g \). We will work with the motive \( h^n(A) \) from Theorem 2.1 and the pairing \( \langle \cdot, \cdot \rangle_{n, \text{mot}} \) from Definition 1.6.

Definition 3.1. Let \( \mathbb{Q}[Z^n_{\text{num}}(A)_{\mathbb{Q}}] \) be the subalgebra of \( Z^n_{\text{num}}(A)_{\mathbb{Q}} \) generated by \( Z^n_{\text{num}}(A)_{\mathbb{Q}} \). An element of \( \mathbb{Q}[Z^n_{\text{num}}(A)_{\mathbb{Q}}] \) is called a Lefschetz class. The subspace of Lefschetz classes in \( Z^n_{\text{num}}(A)_{\mathbb{Q}} \) is denoted by \( \mathcal{L}^n(A) \).
Remark 3.2. Given two positive integers $a, b$ such that $a \cdot b \leq g$ we have the canonical inclusions

$$\mathfrak{h}^{a-b}(A) = \text{Sym}^{a-b}\mathfrak{h}^1(A) \subset (\text{Sym}^a\mathfrak{h}^1(A))^{\otimes b} = \mathfrak{h}^a(A)^{\otimes b}.$$  

induced by Theorem 2.1(2). In particular any pairing on the right hand side induces a pairing on the left hand side.

Lemma 3.3. Fix an integer $n$ such that $2 \leq 2n \leq g$. Then the three motivic pairings $\langle \cdot, \cdot \rangle_{1, \text{mot}}, \langle \cdot, \cdot \rangle_{2, \text{mot}}$ and $\langle \cdot, \cdot \rangle_{2n, \text{mot}}$ on $\mathfrak{h}^{2n, \text{prim}}(A) \in \text{NUM}(k)_\mathbb{Q}$ coincide up to a positive rational scalar. Moreover, for each of these pairings, the Lefschetz decomposition $\mathfrak{h}^{2n}(A) = \mathfrak{h}^{2n, \text{prim}}(A) \oplus \mathfrak{h}^{2n-2, \text{prim}}(A)(-1)$ is orthogonal.

Proof. Our statement holds true even at homological level and not just numerically. For this, it is enough to check the statement after realization, for instance on the cohomology group $H^{2n}(A)$.

To do this, take the moduli space of polarized abelian varieties (in mixed characteristic, with some level structure fixed). These pairings are defined on the relative cohomology of the abelian scheme. Recall that the Zariski closure of the monodromy group associated to this relative cohomology is $\text{GSp}_{2n} = \text{GSp}(H^1_1(A))$ and that $H^{2n, \text{prim}}(A) \subset H^{2n}(A) = \Lambda^gH^1(A)$ is an irreducible representation. This implies that these pairings coincide up to a scalar, by Schur’s lemma. As these pairings are also defined on Betti cohomology this scalar must be rational. Moreover, they are polarizations of the underlined Hodge structure, so this scalar must be positive.

For the orthogonality part, the argument is the same. If, for a fixed pairing, the decomposition was not orthogonal, we would have a non-zero map between $H^{2n, \text{prim}}_\ell(A)$ and $H^{2n-2}_\ell(A)(-1)^\vee$ which would be $\text{GSp}_{2g}$-equivariant. This is impossible again by Schur’s lemma. \hfill \Box

Proposition 3.4. Fix an integer $n$ such that $2 \leq 2n \leq g$. The pairings $\langle \cdot, \cdot \rangle_{1, \text{mot}}, \langle \cdot, \cdot \rangle_{2, \text{mot}}$ and $\langle \cdot, \cdot \rangle_{2n, \text{mot}}$ on $\mathcal{Z}^n_{\text{num}}(A)_\mathbb{Q}$ are positive definite if and only if anyone of them is so. Moreover, they are positive definite on $\mathcal{L}^n(A)$.

Proof. By Lemma 3.3 the Lefschetz decomposition

$$\mathfrak{h}^{2n}(A) = \mathfrak{h}^{2n, \text{prim}}(A) \oplus \mathfrak{h}^{2n-2, \text{prim}}(A)(-1)$$

is orthogonal with respect to any of these pairings, so, arguing by induction on $n$, it is enough to check positivity on algebraic classes of $\mathfrak{h}^{2n, \text{prim}}(A)$. Again by Lemma 3.3 the positivity on the primitive part does not depend on the pairing.

The argument just above works also for Lefschetz classes, indeed each component in the Lefschetz decomposition of a Lefschetz class is again a Lefschetz class [Mil99, p. 640]. Hence we can check positivity for one of the pairings, we will do it for $\langle \cdot, \cdot \rangle^{\otimes n}_{2, \text{mot}}$. Note that, by construction, the restriction of $\langle \cdot, \cdot \rangle^{\otimes n}_{2, \text{mot}}$ to algebraic classes is $\langle \cdot, \cdot \rangle^{\otimes n}_{1, \text{mot}}$, see Definition 1.8.
Now, the pairing $\langle \cdot , \cdot \rangle_1$ on 
\[ Z^1_{\text{num}}(A)_{\mathbb{Q}} = \text{Hom}_{\text{NUM}(k)}(\mathbb{F}, \mathfrak{h}^2(A)(1)) \]
is positive definite by Theorem 1.14. Hence the pairing $\langle \cdot , \cdot \rangle_1^{\otimes n}$ on 
\[ \text{Hom}_{\text{NUM}(k)}(\mathbb{F}, \mathfrak{h}^2(A)(1))^{\otimes n} \]
is positive definite as well. By (3.1), we have the inclusion 
\[ Z^n_{\text{num}}(A)_{\mathbb{Q}} = \text{Hom}_{\text{NUM}(k)}(\mathbb{F}, \mathfrak{h}^{2n}(A)(n)) \subset \text{Hom}_{\text{NUM}(k)}(\mathbb{F}, (\mathfrak{h}^2(A)(1))^{\otimes n}). \]
Finally notice that Theorem 2.1(2) implies the equality 
\[ L^n(A)_{\mathbb{Q}} = Z^n_{\text{num}}(A)_{\mathbb{Q}} \cap Z^1_{\text{num}}(A)_{\mathbb{Q}}^{\otimes n} \]
where the intersection is taken inside $\text{Hom}_{\text{NUM}(k)}(\mathbb{F}, (\mathfrak{h}^2(A)(1))^{\otimes n})$. As the pairing $\langle \cdot , \cdot \rangle_1^{\otimes n}$ on $Z^1_{\text{num}}(A)_{\mathbb{Q}}^{\otimes n}$ is positive definite then its restriction to $L^n(A)_{\mathbb{Q}}$ will also be positive definite. □

4. ABELIAN VARIETIES OVER FINITE FIELDS

We start by recalling classical results on abelian varieties over finite fields and afterwards we draw some consequences on motives and algebraic cycles.

Throughout the section, we fix a polarized abelian variety $A$ of dimension $g$ over a finite field $k$. We denote by $\text{End}(A)$ the ring of endomorphisms of $A$, we write $\text{End}_0(A) = \text{End}(A) \otimes \mathbb{Z} \otimes \mathbb{Q}$ and $\ast$ for the Rosati involution on it (induced by the fixed polarization).

**Theorem 4.1** ([Tat66]). There exists a commutative $\mathbb{Q}$-subalgebra $B$ of $\text{End}_0(A)$ which has dimension $2g$, which contains the Frobenius $\text{Frob}_A$ of $A$ and which is a product of CM number fields $B = L_1 \times \cdots \times L_t$.

**Proposition 4.2.** Keeping notation from the above theorem the following holds:

1. [Mum08, pp. 211-212] If an algebra $B$ as above is $\ast$-stable then $\ast$ acts as the complex conjugation on each factor $B = L_1 \times \cdots \times L_t$.

2. [Shi71, Proposition 5.12] The compositum of CM number fields is itself a CM field. The Galois closure of a CM number field is a CM number field as well.

**Definition 4.3.** The choice of a $\mathbb{Q}$-subalgebra $B$ as in Theorem 4.1 is called a CM-structure for $A$. If $B$ is $\ast$-stable we will say that the CM-structure is $\ast$-stable.

**Notation 4.4.** Given a CM structure for $A$, we write $L$ for the CM number field which is the Galois closure of the compositum of the fields $L_i$, see Proposition 4.2.

Let $\Sigma_i$ be the set of morphisms from $L_i$ to $L$ and $\Sigma$ the disjoint union of the $\Sigma_i$ (with $i$ varying).
Write $\bar{\sigma}$ for the action on $\Sigma$ induced by composition with the complex conjugation. We will use the same notation for the induced action on subsets of $\Sigma$.

**Corollary 4.5.** Let $L$ and $\Sigma$ be as in Notation 4.4. Then, in $\text{CHM}(k)_L$, the motive $h^1(A)$ decomposes into a sum of $2g$ motives

$$h^1(A) = \bigoplus_{\sigma \in \Sigma} M_\sigma,$$

where the action of $b \in L_i$ on $M_\sigma$ induced by Theorem 2.1(3) is given by multiplication by $\sigma(b)$ if $\sigma \in \Sigma_i$ and by multiplication by zero otherwise. Moreover, each motive $M_\sigma$ is of rank one (in the sense of Definition 2.3). Finally, if the CM-structure is $\ast$-stable, the isomorphism $p : h^1(A) \cong h^1(A)^\vee(-1)$ of Theorem 2.1(4) restricts to an isomorphism

$$M_\sigma \cong M_\sigma^\vee(-1)$$

for all $\sigma \in \Sigma$, and to the zero map

$$M_\sigma \xrightarrow{0} M_{\bar{\sigma}}^\vee(-1)$$

for all $\sigma' \neq \bar{\sigma}$.

**Proof.** See [Anc18, Corollary 3.2].

**Proposition 4.6.** In the setting of Notation 4.4 and Corollary 4.5 the following holds:

1. In $\text{CHM}(k)_L$ the motive $h^n(A)$ decomposes into a sum

$$h^n(A) = \bigoplus_{I \subset \Sigma, |I|=n} M_I,$$

with $M_I = \otimes_{\sigma \in I} M_\sigma$. Each motive $M_I$ is of rank one (in the sense of Definition 2.3).

Moreover, if the CM-structure is $\ast$-stable, then the motives $M_I$ and $M_J$ are mutually orthogonal in $\text{NUM}(k)_L$ with respect to $\langle \cdot, \cdot \rangle_{1,\text{mot}} \otimes n$ (Definition 1.6) except if $I = J$.

2. In $\text{CHM}(k)_Q$ the motive $h^n(A)$ decomposes into a sum of motives

$$h^n(A) = \bigoplus \mathcal{M}_I,$$

with $\mathcal{M}_I = \oplus_{g \in \text{Gal}(L/Q)} \eta_g(I)$.

Moreover, if the CM-structure is $\ast$-stable, this decomposition in $\text{NUM}(k)_Q$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{1,\text{mot}} \otimes n$ (Definition 1.6).

**Proof.** Part (2) follows from part (1). For the latter, Corollary 4.3 gives the case $n = 1$. Using Theorem 2.1(2) we deduce the result for higher $n$.

**Proposition 4.7.** Let $\mathcal{M}_I \in \text{CHM}(k)_Q$ be a direct factor of $h^{2n}(A)$ as constructed in Proposition 4.6(2). Then the following holds:
(1) If the vector space $\text{Hom}_{\text{NUM}(k)}^\mathbb{Q}(\mathbb{1}, M_I(n))$ is not zero then
\[
\dim_{\mathbb{Q}} \text{Hom}_{\text{NUM}(k)}^\mathbb{Q}(\mathbb{1}, M_I(n)) = \dim M_I.
\]
In this case homological and numerical equivalence coincide on $M_I$ and the realizations of $M_I$ are spanned by algebraic classes.

(2) If the vector space $\text{Hom}_{\text{NUM}(k)}^\mathbb{Q}(\mathbb{1}, M_I(n))$ contains a nonzero Lefschetz class (Definition 3.1), then all classes in it are Lefschetz.

Proof. As numerical equivalence commutes with extension of scalars [And04, Proposition 3.2.7.1], we have
\[
\dim_{\mathbb{Q}} \text{Hom}_{\text{NUM}(k)}^\mathbb{Q}(\mathbb{1}, M_I(n)) = \dim L \text{Hom}_{\text{NUM}(k)}^L(\mathbb{1}, M_I(n)).
\]
On the other hand, by construction of $M_I$, we have
\[
\text{Hom}_{\text{NUM}(k)}^L(\mathbb{1}, M_I(n)) = \bigoplus_{g \in \text{Gal}(L/\mathbb{Q})} \text{Hom}_{\text{NUM}(k)}^L(\mathbb{1}, M_{g(I)}(n)).
\]
Moreover, as $\text{Gal}(L/\mathbb{Q})$ acts on the space of algebraic cycles modulo numerical equivalence, we have
\[
\dim L \text{Hom}_{\text{NUM}(k)}^L(\mathbb{1}, M_I(n)) = \dim L \text{Hom}_{\text{NUM}(k)}^L(\mathbb{1}, M_{g(I)}(n)),
\]
for all $g \in \text{Gal}(L/\mathbb{Q})$.

Now, by Proposition 4.6(1), the motives $M_I \in \text{CHM}(k)_L$ are of rank one hence, by Proposition 2.5, the above dimension is either zero or one. This implies the equality in part (1). The rest of part (1) follows from Proposition 2.5 as well.

For part (2), the proof just goes as part (1) as all the properties of motives and algebraic classes that we used hold true for Lefschetz motives and Lefschetz classes by [Mil99, p. 640].

Remark 4.8. The previous proposition is not accessible if one replaces numerical with homological equivalence. One finds the same issues in [Clo99]. This is the crucial reason why the main results in this paper are in the setting of numerical equivalence.

Theorem 4.9 ([Tat71, Theorem 2]). For any CM-structure $B$ for $A$ (Definition 4.3), the pair $(A, B)$ lifts to characteristic zero. More precisely there exists a $p$-adic field $K$ with ring of integers $W$ whose residue field $k'$ is a finite extension of $k$, and there is an abelian scheme $A$ over $W$ such that $B \hookrightarrow \text{End}^0(A)$ and $A \times_k k'$ is isogenous to $A \times_k k'$.

Corollary 4.10. The decompositions of $h(A \times_k k')$ in Proposition 4.6 lift to decompositions of $h(A) \in \text{CHM}(W)$. Moreover, if the CM-structure of $A$ is $\ast$-stable (and if the polarisation lifts as well) then the orthogonality statement in Proposition 4.6 holds true in $\text{CHM}(W)$ as well.

Proof. The proof of Proposition 4.6 is a formal combination of Theorem 2.1 together with the CM-structure. It works also over $W$ because of Remark 2.2 and Theorem 4.9.
5. Exotic classes

In this section we fix an abelian variety $A$ of dimension four over a finite field $k = \mathbb{F}_q$ of cardinality $q$ (and we fix an algebraic closure $\bar{k}$ of $k$). After choosing a CM-structure for $A$ (Definition 4.3), Proposition 4.6(2) gives us motives $M_I \in \text{CHM}(k)_\mathbb{Q}$. Some of them, that we will call exotic, are essential in the proof of the standard conjecture of Hodge type. The main result of the section (Proposition 5.3) tells us that they are of rank two (in the sense of Definition 2.3).

**Definition 5.1.** Let $M_I \in \text{CHM}(k)_\mathbb{Q}$ be a direct factor of $h^4(A)$ as constructed in Proposition 4.6(2). It is called exotic if
\[ \dim_\mathbb{Q} \text{Hom}_{\text{NUM}(k)_\mathbb{Q}}(1, M_I(2)) = \dim_\mathbb{Q} M_I \]
and the space $\text{Hom}_{\text{NUM}(k)_\mathbb{Q}}(1, M_I(2))$ does not contain any nonzero Lefschetz class (Definition 3.1).

An element in the vector space $\text{Hom}_{\text{NUM}(k)_\mathbb{Q}}(1, M_I(2))$ will be called an exotic class.

**Remark 5.2.** Examples of exotic classes on abelian fourfolds (and especially of those that cannot be lifted to characteristic zero) will be discussed in Appendix A.

**Proposition 5.3.** Suppose that $A$ has dimension four, then, after possibly a finite extension of $k$ and after a good choice of the CM-structure, any exotic motive $M_I$ has dimension two.

The proof is decomposed in a series of lemmas and will take the rest of the section. We first fix notations and assumptions that will be used for the lemmas below.

**Notation 5.4.** Let $\Sigma$ be as in Notation 4.4. Consider the decomposition $h^1(A) = \bigoplus_{\sigma \in \Sigma} M_\sigma$ from Corollary 4.5 (Recall that Frobenius acts on each of the eight $M_\sigma$ by Theorem 4.1). Let us denote $\alpha_\sigma \in \overline{\mathbb{Q}}$ the eigenvalues for the action of Frobenius on each $M_\sigma$. We will denote by $\overline{\cdot}$ the action of complex conjugation on $\Sigma$ (or on the set of parts of $\Sigma$, or on $\overline{\mathbb{Q}}$).

**Assumption 5.5.** We will suppose that all the algebraic classes $Z^*_{\text{num}}(A_{\bar{k}})_\mathbb{Q}$ are defined over $k$.

**Remark 5.6.** Note that the assumption above always holds after a finite extension of $k$. Note also that, under this assumption, a class which becomes Lefschetz after a finite extension of the base field must be already Lefschetz. Hence, a motive which is exotic over the base field will still be exotic after a finite extension.

**Lemma 5.7.** Let $q$ be the cardinality of $k$ and let $M_I$ be an exotic motive. Then we have the relation
\[ \prod_{\sigma \in I} \alpha_\sigma = q^2. \]
Moreover, we have the property
(5.2) \[ \alpha \in \{ \alpha_{\sigma} \}_{\sigma \in I} \Rightarrow \bar{\alpha} \notin \{ \alpha_{\sigma} \}_{\sigma \in I} \]
and its slight generalization
(5.3) \[ \alpha^{n} \in \{ \alpha_{\sigma}^{n} \}_{\sigma \in I} \Rightarrow \bar{\alpha}^{n} \notin \{ \alpha_{\sigma}^{n} \}_{\sigma \in I} \]
(for all positive integer \( n \)).

Proof. By Proposition 4.7(1) the \( \ell \)-adic realization of \( M_{I} \) is spanned by algebraic cycles hence Frobenius acts by multiplication by \( q^{2} \). On the other hand, by construction (see Proposition 4.6(1)), Frobenius acts on the line \( M_{I} \subset M_{I} \) via the multiplication by \( \prod_{\sigma \in I} \alpha_{\sigma} \). This gives (5.1).

Suppose now that (5.2) is not satisfied. This means that among the four eigenvalues \( \{ \alpha_{\sigma} \}_{\sigma \in I} \) we would find a pair of the form \( (\alpha, \bar{\alpha}) \). Recall that the Weil conjectures imply \( \bar{\alpha} = q/\alpha \). Then (5.1) would force the set \( \{ \alpha_{\sigma} \}_{\sigma \in I} \) to be of the form \( \alpha, q/\alpha, \beta, q/\beta \). Each of the pairs \( (\alpha, q/\alpha) \) and \( (\beta, q/\beta) \) would correspond to a Frobenius invariant class in \( H^{2}_{\ell}(A)(1) \). As the Tate conjecture for divisors on abelian varieties is known [Tat66, Theorem 4], each of these pairs corresponds to the class of a divisor, hence \( M_{I} \) would contain a Lefschetz class.

If we now extend the scalars to \( \mathbb{F}_{q^{n}} \) the motive \( M_{I} \) will still be exotic (Remark 5.6). The eigenvalues of Frobenius become \( \{ \alpha_{\sigma}^{n} \} \). Hence, by applying (5.2) over this new base field we deduce (5.3).

Definition 5.8. A subset \( I \) of \( \Sigma \) verifying (5.1) and (5.2) is called an exotic subset.

Remark 5.9. Lemma 5.7 implies that the dimension of the space of exotic classes is at most the number of exotic subsets. The Tate conjecture for \( A \) predicts that this inequality should actually be an equality.

Lemma 5.10. An exotic motive \( M_{I} \) is even dimensional.

Proof. The property (5.2) tells us in particular that complex conjugation cannot fix \( I \). This property holds for each of the \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-conjugates of \( I \) as well, hence the cardinality of the orbit of \( I \) under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) is even. On the other hand, this cardinality is the dimension of \( M_{I} \) (see Proposition 4.6). \( \square \)

Lemma 5.11. Under the Assumption 5.5, two exotic subsets cannot intersect in exactly two elements.

Proof. Suppose that there are two exotic subsets \( I \) and \( J \) whose intersection has cardinality two. Let us call \( \alpha, \beta \) the two Frobenius eigenvalues corresponding to \( I - I \cap J \) and \( \gamma, \delta \) those corresponding to \( J - I \cap J \). Properties (5.1) and (5.2) give
\[ \alpha \cdot \beta = \gamma \cdot \delta \]
and (after reordering if necessary)
\[ \alpha = q/\gamma, \beta = q/\delta. \]
Putting these relations together we have $\alpha^2 = q^2/\beta^2 = \alpha^2$. This equality gives a contradiction by applying (5.3) to $I$ for $n = 2$.

**Lemma 5.12.** Under the Assumption 5.5, the dimension of the space of exotic classes is zero, two or four. More precisely, the exotic subsets are either zero, two or four. In case they are two, they are complex conjugate to each other. In case they are four, they are of the form $I, \bar{I}, J, \bar{J}$, with $I \cap J$ of cardinality three.

*Proof.* The statement on exotic classes is implied by the statement on exotic subsets. Indeed, the dimension of the space of exotic classes is even (Lemma 5.10) and it is at most the number of exotic subsets (Remark 5.9).

Let us now show the statement on exotic subsets. We have already pointed out in the proof of Lemma 5.10 that complex conjugation acts without fixed points on the exotic subsets. Suppose now that there are at least four such subsets, call them $I, \bar{I}, J, \bar{J}$. Then one subset among $J$ and $\bar{J}$ intersects $I$ in at least two elements. Without loss of generality we suppose it is $J$. Then by Lemma 5.11 $I \cap J$ must be of cardinality three.

If there were more than four exotic subsets, then, by the same arguments there would be an exotic subset $K$ intersecting $I$ in exactly three elements. Then the intersection of $J$ and $K$ would have exactly two elements. This is impossible by Lemma 5.11.

**Lemma 5.13.** Suppose that there are four exotic subsets and that the Assumption 5.5 is satisfied. Then, after extending $k = \mathbb{F}_q$ to its quadratic extension $\mathbb{F}_{q^2}$, the abelian fourfold $A$ becomes isogenous to $E \times X$ where $X$ is an abelian threefold and $E$ is a supersingular elliptic curve on which Frobenius acts as $q \cdot \text{id}$.

*Proof.* If the element of $I - I \cap J$ is $\sigma$ then the element of $J - I \cap J$ must be $\sigma$ because of (5.2). Let $\alpha$ be the Frobenius eigenvalue for the action on $M_{\sigma}$, then (5.11) applied to $I$ and $J$ implies $\alpha = \bar{\alpha}$, hence $\alpha^2 = q$.

Let us now extend the field of definition to $\mathbb{F}_{q^2}$. Among the eight eigenvalues of Frobenius we will find $q$, which means that there is a nonzero Frobenius-equivariant map between the Tate module of the supersingular elliptic curve $E$ and the Tate module of $A$. This implies the statement by [Tat66, Theorem 4].

**Lemma 5.14.** We keep Notation 5.4 and Assumption 5.5. Consider on the base field $k = \mathbb{F}_{q^2}$ a supersingular elliptic curve $E$ on which Frobenius acts as $q \cdot \text{id}$. Let $A$ be an abelian fourfold of the form $A = E \times X$, and consider the induced decomposition $\Sigma = \Sigma_X \cup \Sigma_E$. Then, any exotic subset $I$ verifies that $I \cap \Sigma_X$ has cardinality three. Moreover, the motive $\mathcal{M}_{I \cap \Sigma_X} \in \text{CHM}(k)_{\mathbb{Q}}$, direct factor of $h^3(X)$ as constructed in Proposition 4.6, is of dimension two.

Finally, if the space of exotic classes on $A$ is four dimensional, it is contained in the four dimensional motive $\mathcal{M}_{I \cap \Sigma_X} \otimes h^1(E)$. 

Proof. By property \((5.4)\), the sets \(I\) and \(I\) have empty intersection. As \(\Sigma_X\) has cardinality six and is stable by the action of complex conjugation one must have that \(I \cap \Sigma_X\) has cardinality three.

Consider the subsets \(K \subset \Sigma_X\) verifying
\[
\prod_{\sigma \in K} \alpha_{\sigma} = q^3
\]
and
\[
\alpha \in \{\alpha_{\sigma}\}_{\sigma \in K} \Rightarrow \bar{\alpha} \notin \{\alpha_{\sigma}\}_{\sigma \in K}.
\]
Clearly the relations \((5.1)\) and \((5.2)\) for \(I\) imply that \(K \cap \Sigma_B\) verifies \((5.4)\) and \((5.5)\). Conversely, the relations \((5.4)\) and \((5.5)\) for \(K\) imply that \(K = I \cup \{\sigma\}\) verifies \((5.1)\) and \((5.2)\) for any \(\sigma \in \Sigma_E\). As there are at most four exotic subsets for \(A\), there must be at most two subsets of \(\Sigma_X\) verifying \((5.4)\) and \((5.5)\). On the other hand \(I \cap \Sigma_X\) and \(I \cap \Sigma_X\) are two of those, which implies that there are exactly two sets verifying \((5.4)\) and \((5.5)\).

Finally, as the Galois group \(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\) acts on the subsets of \(\Sigma_X\) verifying \((5.4)\) and \((5.5)\) the motive \(M_{I \cap \Sigma_X}\) has dimension two. The rest follows from the construction of \(M_{I \cap \Sigma_X}\). \(\square\)

Lemma 5.15. Consider an abelian fourfold of the form \(A = E \times X\), where \(E\) is a supersingular elliptic curve such that \(\dim \text{End}(E) \otimes \mathbb{Q} = 4\). Consider a direct factor of \(h^3(X)\) of the form \(M_I \subset \text{CHM}(k)\mathbb{Q}\), as constructed in Proposition 4.6. Suppose that \(M_I\) has rank two and that the motive \(M_I \otimes h^1(E)\) is spanned by algebraic classes. Then, given any CM structure of \(X\), there exists a CM structure \(\Sigma_E = \{\sigma, \bar{\sigma}\}\) of \(E\) such that the motive \(M_I \otimes h^1(E)\) becomes the sum of two motives of rank two
\[
M_I \otimes h^1(E) = M_{I,\sigma} \oplus M_{I,\bar{\sigma}}
\]
via Proposition 4.6(2).

Proof. By construction of \(M_I\) we can find a quadratic number field \(F\) such that the motive \(M_I\) decomposes in the category \(\text{CHM}(k)F\) into the sum
\[
M_I = M_I \oplus M_I
\]
of two motives of rank one. In order to conclude, it is enough to show that also \(h^1(E)\) decomposes in the category \(\text{CHM}(k)F\) into the sum of two motives of dimension one.

This fact is equivalent to saying that \(F\) can be embedded in the division algebra \(\text{End}(E) \otimes \mathbb{Z} \mathbb{Q}\). As this algebra splits at all primes except infinity and the characteristic of the base field (call it \(p\)), we have to show that \(F\) does not split at infinity and \(p\) as well, i.e. that the embeddings \(F \subset \mathbb{R}\) and \(F \subset \mathbb{Q}_p\) cannot exist. We will give two proofs of this.

A first proof uses that these motives lift to characteristic zero (Proposition 4.10). Now if \(F\) was contained in \(\mathbb{R}\) then the Betti realization of the lifting of \(M_I\) would respect the Hodge symmetry. This is impossible as it is one dimensional and of weight three. Similarly, if \(F\) was contained in \(\mathbb{Q}_p\) then the
crystalline realization of $M_I$ and $M_{\bar{I}}$ would be two isocrystals of dimension one with different filtrations (again because the weight of $M_I$ is three). This implies that $\det(M_I \otimes h^1(E))$ and $\det(M_{\bar{I}} \otimes h^1(E))$ would realize in two isocrystals of dimension one and different filtrations. On the other hand absolute Frobenius acts on both in the same way (namely by multiplication by $p^4$) because they are spanned by algebraic classes. This implies that at least one of the two isocrystals is not admissible and concludes the proof as all isocrystals coming from geometry (in particular isocrystals that are realizations of motives) must be admissible.

We give an alternative proof, which does not use the lifting to characteristic zero. We write it for $\mathbb{Q}_p$, it works in the same way for $\mathbb{R}$. If $F$ was contained in $\mathbb{Q}_p$ then the motive $M_I \otimes h^1(E)$ would live in $\text{CHM}(k)_{\mathbb{Q}_p}$. On such a motive the division algebra $\text{End}(E) \otimes \mathbb{Z} \mathbb{Q}_p$ acts hence it acts also on the $\mathbb{Q}_p$ vector space spanned by the algebraic classes of the motive. On the other hand, as this motive is spanned by algebraic classes, the division algebra $\text{End}(E) \otimes \mathbb{Z} \mathbb{Q}_p$ would act on a $\mathbb{Q}_p$-vector space of dimension two. This gives a contradiction as such an action cannot exist. □

Proof of Proposition 5.3. This is a combination of the last four lemmas. □

6. Orthogonal motives of rank 2

We start by stating our main technical result, Theorem 6.1. The remark and proposition right after will hopefully give some intuitions on the hypothesis of the statement. We conclude the section by showing that Theorem 6.1 implies Theorem 1.18. The proof of Theorem 6.1 will take the next three sections.

Theorem 6.1. Let $K$ be a $p$-adic field, $W$ its ring of integers and $k$ its residue field. Let us fix an embedding $\sigma : K \hookrightarrow \mathbb{C}$. Let

$$M \in \text{CHM}(W)_{\mathbb{Q}}$$

be a motive in mixed characteristic and consider the motives induced by pullbacks $M_{/C} \in \text{CHM}(\mathbb{C})_{\mathbb{Q}}$ and $M_{/k} \in \text{CHM}(k)_{\mathbb{Q}}$. Consider $V_B$ and $V_Z$ the two $\mathbb{Q}$-vector spaces defined as

$$V_B = R_B(M_{/C}) \quad \text{and} \quad V_Z = \text{Hom}_{\text{NUM}(k)_{\mathbb{Q}}}(1, M_{/k}).$$

Let $q$ be a quadratic form on $M$, by which we mean a morphism in $\text{CHM}(W)_{\mathbb{Q}}$ of the form $q : \text{Sym}^2 M \to 1$. Consider the two $\mathbb{Q}$-quadratic forms induced by $q$ on $V_B$ and $V_Z$ respectively,

$$q_B = R_B(q) \quad \text{and} \quad q_Z(\cdot) = (q_{/k} \circ \text{Sym}^2(\cdot)).$$

Suppose that the following holds:

(1) The two $\mathbb{Q}$-vector spaces $V_B$ and $V_Z$ are of dimension 2,

(2) The pairing $q_B$ on $V_B$ is a polarization of Hodge structures.

Then the quadratic form $q_Z$ is positive definite.
Remark 6.2. (1) The main example of such a motive $M$ we have in mind is an exotic motive of an abelian fourfold (see Section 5). Another example coming from geometry is given in Proposition A.9.
(2) One can actually make the hypothesis a little bit more flexible and work with homological motives instead of Chow motives. This will not matter for our application to abelian varieties.

Proposition 6.3. Under the hypothesis of Theorem 6.1 the following holds:
(1) The quadratic form $q_Z$ on $V_Z$ is non-degenerate.
(2) Given a classical realization $R$, the vector space $R(M_{/k})$ is of dimension two.
(3) Given a classical realization $R$, the vector space $R(M_{/k})$ is spanned by algebraic classes.
(4) Numerical equivalence on $\text{Hom}_{\text{CHM}(k)\text{Q}}(1, M_{/k})$ coincides with the homological equivalence for any classical cohomology.
(5) Fix a classical realization $R$ and call $L$ the field of coefficients of $R$. Then the equality $V_Z \otimes \mathbb{Q} L = R(X)$ holds.

Proof. Let us start with (2). By the comparison theorem between singular and $\ell$-adic cohomology, we have $\dim_{\mathbb{Q}} V_B = \dim_{\mathbb{Q}_\ell} R_\ell(M_{/C})$, for all primes $\ell$, including $\ell = p$. Then, by smooth proper base change we have $\dim_{\mathbb{Q}_\ell} R_\ell(M_{/C}) = \dim_{\mathbb{Q}_\ell} R_\ell(M_{/k})$ for all $\ell \neq p$ and finally by the $p$-adic comparison theorem we have $\dim_{\mathbb{Q}_p} R_p(M_{/C}) = \dim_{\text{Frac}(W(k))} R_{\text{crys}}(M_{/k})$. This concludes (2) as, by hypothesis, $\dim_{\mathbb{Q}} V_B = 2$.

The proof of (3)-(5) goes as in Proposition 2.5. There, the hypothesis that the motive was of abelian type was used to ensure that all realizations have the same dimension (see Remark 2.4). Here, this is replaced by part (2).

Let us now show part (1). First notice that it is enough to show that $R(q_{/k})$ is non-degenerate (for some classical realization) because of parts (2)-(5). Then, again by using the comparison theorems, this is equivalent to the fact that $R_B(q)$ is non-degenerate. On the other hand this is the case as $R_B(q)$ is a polarization of Hodge structures.

Proof of Theorem 1.18. Let $A$ be an abelian fourfold and $L$ be a hyperplane section. By Proposition 1.16 we can suppose that $A$ and $L$ are defined over a finite field $k = \mathbb{F}_q$. Note that, in order to prove Theorem 1.18 we can (and will) replace $k$ by a finite extension.

Consider now the decomposition from Proposition 4.6(2) and, among the factors of this decomposition, consider those that are exotic (Definition 3.1). After choosing a good CM structure (and after a finite extension of the base field), these exotic motives are of dimension two (Proposition 5.3). By Theorem 4.9 the CM-structure can be lifted to characteristic zero. Moreover, by Proposition 4.13 it is enough to work with a single $L$ for a given abelian

\footnote{For our main application the motive $M$ will be anyway of abelian type.}
Now, by Lemma 3.3 we can work with the pairing \( \langle \cdot, \cdot \rangle_{1, \text{mot}} \otimes^4 \) instead of \( \langle \cdot, \cdot \rangle_{4, \text{mot}} \). For this pairing, the decomposition from Proposition 4.6(2) is orthogonal, hence we can work with a single motive of the decomposition. By Propositions 3.4 and 4.7(2) we are reduced to motives that are exotic.

Finally, those exotic motives \( M_f \) are settled by Theorem 6.1 by setting \( M = M_f(2) \). Note that all the hypothesis of Theorem 6.1 are satisfied. Indeed, the motive lives in mixed characteristic (Corollary 4.10), together with its quadratic form (because \( L \) lifts to characteristic zero). The space \( V_B \) is clearly of dimension two; so it is \( V_Z \) by Proposition 4.7(1). Last, the quadratic form \( q_B = R_B(\langle \cdot, \cdot \rangle_{1, \text{mot}}) \) is a polarization as \( R_B(\langle \cdot, \cdot \rangle_{1, \text{mot}}) \) is so.

\[ \Box \]

7. Quadratic forms

We recall here some classical facts on quadratic forms. They will allow us to reduce Theorem 6.1 to a \( p \)-adic question (Proposition 7.6). For simplicity, we will work only in the context we will need later, namely with non-degenerate \( \mathbb{Q} \)-quadratic forms of rank 2. In what follows \( \mathbb{Q}_\nu \) denotes the completion of \( \mathbb{Q} \) at the place \( \nu \).

**Definition 7.1.** Let \( q \) be a \( \mathbb{Q} \)-quadratic form of rank 2. Define \( \varepsilon_\nu(q) \), the Hilbert symbol of \( q \) at \( \nu \), as +1 if

\[ x^2 - q(y, z) = 0 \]

has a nonzero solution in \( x, y, z \in \mathbb{Q}_\nu \), and as −1 otherwise. Depending on the context we may write \( \varepsilon_p(q) \) or \( \varepsilon_\mathbb{R}(q) \).

**Remark 7.2.** Let \( q \) be a \( \mathbb{Q} \)-quadratic form of rank 2. It is positive definite if and only if its discriminant is positive and \( \varepsilon_\mathbb{R}(q) = +1 \) and it is negative definite if and only if its discriminant is positive and \( \varepsilon_\mathbb{R}(q) = -1 \).

**Proposition 7.3 ([Ser77, §2.3]).** Let \( p \) be a prime number and \( q_1 \) and \( q_2 \) two non-degenerate \( \mathbb{Q} \)-quadratic forms of rank 2. Then

\[ q_1 \otimes \mathbb{Q}_p \cong q_2 \otimes \mathbb{Q}_p \]

if and only if the discriminants of \( q_1 \) and \( q_2 \) coincide in \( \mathbb{Q}_p^*/(\mathbb{Q}_p)^2 \) and

\[ \varepsilon_p(q_1) = \varepsilon_p(q_2). \]

**Theorem 7.4 ([Ser77, §3.1]).** Let \( q \) be non-degenerate \( \mathbb{Q} \)-quadratic form of rank 2 and \( \varepsilon_\nu(q) \) as in Definition 7.1. Then for all but finite places \( \nu \) the equality \( \varepsilon_\nu(q) = +1 \) holds. Moreover, the following product formula running on all places holds

\[ \prod \varepsilon_\nu(q) = +1. \]
Corollary 7.5. Let $q_1$ and $q_2$ be two non-degenerate $\mathbb{Q}$-quadratic forms of rank 2 and let $p$ be a prime number. Suppose that, for all primes $\ell$ different from $p$, we have

$$q_1 \otimes \mathbb{Q}_\ell \cong q_2 \otimes \mathbb{Q}_\ell.$$ 

Then $q_1$ is positive definite if and only if one of the following two cases happens:

1. The quadratic forms $q_1 \otimes \mathbb{Q}_p$ and $q_2 \otimes \mathbb{Q}_p$ are isomorphic and $q_2$ is positive definite.
2. The quadratic forms $q_1 \otimes \mathbb{Q}_p$ and $q_2 \otimes \mathbb{Q}_p$ are not isomorphic and $q_2$ is negative definite.

Proof. The $\ell$-adic hypothesis implies in particular that the discriminants of $q_1$ and $q_2$ coincide in $\mathbb{Q}_\ell^*/(\mathbb{Q}_\ell^*)^2$ for all $\ell \neq p$. This implies that they coincide in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ by [IR82, Theorem 3 in 5.2].

If the discriminants are negative, none of the conditions in the statement holds and the equivalence is clear. From now on we suppose that the discriminants are positive. By Remark 7.2, $q_1$ is positive definite if and only if $\varepsilon_{\mathbb{R}}(q_1) = +1$.

There are then two cases, namely $q_2$ is positive definite, respectively $q_2$ is negative definite. Again by Remark 7.2 they are equivalent to $\varepsilon_{\mathbb{R}}(q_2) = +1$, respectively $\varepsilon_{\mathbb{R}}(q_2) = -1$.

Now, Theorem 7.4 implies that $\prod_\nu \varepsilon_\nu(q_1) = \prod_\nu \varepsilon_\nu(q_2)$. Combining this with the $\ell$-adic isomorphisms we deduce

$$\varepsilon_{\mathbb{R}}(q_1)^{\varepsilon_p(q_1)} = \varepsilon_{\mathbb{R}}(q_2)^{\varepsilon_p(q_2)}.$$ 

This means that $q_1$ is positive definite if and only if

$$\varepsilon_p(q_1) = \varepsilon_p(q_2).$$ 

This relation is equivalent to the fact that one of the following two situations hold:

1. $q_2$ is positive definite and $\varepsilon_p(q_1) = \varepsilon_p(q_2)$.
2. $q_2$ is negative definite and $\varepsilon_p(q_1) \neq \varepsilon_p(q_2)$.

As the discriminant of $q_1$ and $q_2$ coincide, the equality $\varepsilon_p(q_1) = \varepsilon_p(q_2)$ is equivalent to the fact that $q_1 \otimes \mathbb{Q}_p$ and $q_2 \otimes \mathbb{Q}_p$ are isomorphic (Proposition 7.3).

Proposition 7.6. Let us keep notation from Theorem 6.1. Let $p$ be the characteristic of $k$ and $i$ be the unique non-negative integer such that the Hodge structure $V_B$ is of type $(i, -i)$ and $(-i, i)$. Then, the quadratic form $q_Z$ is positive definite if and only if the following holds:

1. When $i$ is even: the quadratic forms $q_B \otimes \mathbb{Q}_p$ and $q_Z \otimes \mathbb{Q}_p$ are isomorphic.
2. When $i$ is odd: the quadratic forms $q_B \otimes \mathbb{Q}_p$ and $q_Z \otimes \mathbb{Q}_p$ are not isomorphic.
Proof. By Proposition 6.3 we have that $q_Z \otimes \mathbb{Q}_\ell = R_\ell(q_{/k})$. By the comparison theorem, we have $q_B \otimes \mathbb{Q}_\ell = R_\ell(q_{/k})$. Combining these equalities with smooth proper base change in $\ell$-adic cohomology we deduce that

$$q_B \otimes \mathbb{Q}_\ell = q_Z \otimes \mathbb{Q}_\ell.$$ 

On the other hand, by hypothesis, $q_B$ is a polarization for the Hodge structure $V_B$ hence it is positive definite if $i$ is even and negative definite if $i$ is odd. We can now conclude by applying Corollary 7.5 to $q_1 = q_Z$ and $q_2 = q_B$. □

8. The $p$-adic comparison theorem

We keep notation from Theorem 6.1 and Proposition 7.6. The aim of this section (and the next one) is to compare the two $\mathbb{Q}_p$-quadratic spaces of rank two

$$(V_{B,p}, q_{B,p}) := (V_B, q_B) \otimes_{\mathbb{Q}} \mathbb{Q}_p \quad \text{and} \quad (V_{Z,p}, q_{Z,p}) := (V_Z, q_Z) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

and deduce from this study Theorem 6.1 (via Proposition 7.6). The key ingredient is the $p$-adic comparison theorem which we recall in Theorem 8.2. The core of the proof is in the next section, we give here some preliminary results.

**Theorem 8.1.** [Fon82] There are two integral $\mathbb{Q}_p$-algebras

$$B_{\text{cris}} \subset B_{\text{dR}}$$

the first endowed with actions of the Galois group $\text{Gal}_K$ and of the absolute Frobenius $\varphi$ and the second endowed with a decreasing filtration $\text{Fil}^i$ such that the multiplication in $B_{\text{dR}}$ verifies

$$\text{Fil}^i \cdot \text{Fil}^j \subset \text{Fil}^{i+j}.$$ 

The $\mathbb{Q}_p$-algebras $B_{\text{dR}}$ contains $\overline{\mathbb{Q}_p}$, an algebraic closure of $\mathbb{Q}_p$, and the intersection $B_{\text{dR}} \cap \overline{\mathbb{Q}_p}$ is the biggest non-ramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$.

Finally, the following equality holds [Fon94, Theorem 5.3.7]

$$B_{\text{crys}}^{\varphi = \text{id}} \cap \text{Fil}^0 B_{\text{dR}} = \mathbb{Q}_p.$$ 

**Theorem 8.2** (FM87, Fal90, CF00). There is an equivalence of $\mathbb{Q}_p$-linear rigid categories

$$D : \text{REP} \rightarrow \text{MOD}$$

between the category $\text{REP}$ of crystalline $\text{Gal}_K$-representations and the category $\text{MOD}$ of admissible filtered $\varphi$-modules. This equivalence verifies the following properties:

1. There is a canonical identification

$$V \otimes B_{\text{crys}} = D(V) \otimes B_{\text{crys}}$$

which is $\text{Gal}_K$-equivariant and $\varphi$-equivariant. Moreover, the induced isomorphism

$$V \otimes B_{\text{dR}} = D(V) \otimes B_{\text{dR}}$$
respects the filtrations.

(2) The functor $D$ is given by
\[ V \mapsto D(V) = (V \otimes B_{\text{crys}})^{\text{Gal}_K} \]
and its inverse is given by
\[ [W \otimes B_{\text{crys}}]^{\varphi = \text{id}} \cap \text{Fil}^0[W \otimes B_{\text{dR}}] \leftrightarrow W. \]

(3) [DN18, 4.15] The functor $D$ commutes with the realization functors $R_p((\cdot)/K) : \text{CHM}(W) \to \text{REP}$ and $R_{\text{crys}}((\cdot)/k) : \text{CHM}(W) \to \text{MOD}$.

**Corollary 8.3.** There are canonical identifications (commuting with the extra structures):

(8.2) $(V_{B,p}, q_{B,p}) = [(V_{Z,p}, q_{Z,p}) \otimes B_{\text{crys}}]^{\varphi = \text{id}} \cap \text{Fil}^0[(V_{Z,p}, q_{Z,p}) \otimes B_{\text{dR}}]$.

(8.3) $(V_{B,p}, q_{B,p}) \otimes B_{\text{crys}} = (V_{Z,p}, q_{Z,p}) \otimes B_{\text{crys}}$.

(8.4) $(V_{B,p}, q_{B,p}) \otimes B_{\text{dR}} = (V_{Z,p}, q_{Z,p}) \otimes B_{\text{dR}}$.

Moreover, under these identifications, the equality of $\mathbb{Q}_p$-algebras

(8.5) $\text{End}_{\text{Gal}_K}(V_{B,p}) = \text{End}_{\varphi, \text{Fil}^*}(V_{Z,p} \otimes \text{Frac}(W(k)))$

holds.

**Proof.** Proposition [6,3] gives the identification
\[ V_{Z,p} \otimes \text{Frac}(W(k)) = R_{\text{crys}}(M/k). \]

On the other hand, we have the equality
\[ V_{B,p} = R_p(M/K) \]
because of the comparison theorem between singular and $p$-adic cohomology. Finally, Theorem [8.2,3] implies
\[ D(R_p(M/K)) = R_{\text{crys}}(M/k). \]

Altogether we have the equality
\[ D(V_{B,p}) = V_{Z,p} \otimes \text{Frac}(W(k)). \]

Notice that these identifications are compatible with the quadratic forms as realization functors and the equivalence $D$ are tensor functors. Hence Theorem [8.2,2] gives (8.2) and Theorem [8.2,1] gives (8.3) and (8.4). The identification (8.5) follows from the fact that $D$ is an equivalence of $\mathbb{Q}_p$-linear categories.

**Proposition 8.4.** Suppose that the Hodge structure $V_B$ is of type $(0,0)$, then $q_{B,p}$ and $q_{Z,p}$ are isomorphic, hence Theorem [6,1] holds true in this case.
Proof. First note that Frobenius acts trivially on $V_{Z,p}$ because the latter is spanned by algebraic classes. Second, the hypothesis on the Hodge types gives $\text{Fil}^0 V_{Z,p} = V_{Z,p}$. Hence the relation (8.2) implies $(V_{B,p}, q_{B,p}) = (V_{Z,p}, q_{Z,p}) \otimes (B_{\text{crys}}^{\varphi = \text{id}} \cap \text{Fil}^0 B_{\text{dR}})$. Using (8.1) we deduce the equality $q_{B,p} = q_{Z,p}$. Theorem 6.1 then follows using Proposition 7.6.

Remark 8.5. (1) This proposition (together with the arguments of the previous sections) gives a full proof of Theorem 1.18 for ordinary abelian fourfolds.

(2) The hypothesis that $V_B$ is of type $(0,0)$ corresponds to the only case where one can hope that the algebraic classes in $V_Z$ might be lifted to characteristic zero, in which case Theorem 6.1 would follow from the Hodge–Riemann relations. We find remarkable that this conjecturally easier case corresponds to an easier $p$-adic analysis.

(3) The proof of the above proposition shows that, under the comparison isomorphisms of Corollary 8.3, one has the equality $V_{B,p} = V_{Z,p}$. The Hodge conjecture predicts that actually also the two $\mathbb{Q}$-structures $V_B$ and $V_Z$ should coincide as well. Is it possible to show the equality $V_B = V_Z$ without assuming the Hodge conjecture? We do not know.

An analogous question can be formulated in the $\ell$-adic setting. Consider an ordinary abelian variety $A$ together with its canonical lifting $\tilde{A}$. Fix an algebraic class on $A$. Does it corresponds to a Hodge class on $\tilde{A}$? This is a priori weaker than the Hodge conjecture: we do not ask that the algebraic cycle does lift to an algebraic cycle. Note that if the answer to the question was affirmative then one would have a proof of the standard conjecture of Hodge type for $A$.

Assumption 8.6. From now on we will suppose that the Hodge structure $V_B$ is not of type $(0,0)$. Equivalently there is a well-defined positive integer $i$ such that $\text{Fil}^i (V_Z \otimes K)$ is a $K$-line.

Proposition 8.7. Under Assumption 8.6 the $\mathbb{Q}_p$-algebra $\text{End}_{\text{Gal}_K}(V_{B,p})$ is a field $F$ such that $[F : \mathbb{Q}_p] = 2$. Moreover, for all $v \in V_{B,p}$ and all $f \in F$, we have the equality

$$q_{B,p}(f \cdot v) = N_{F/\mathbb{Q}_p}(f) q_{B,p}(v).$$

Proof. Consider the ($\mathbb{Q}_p$-points of the) orthogonal group $G = O(q_{B,p})$. We claim that the $\mathbb{Q}_p$-algebra $F := \mathbb{Q}_p[G] \subset \text{End}(V_{B,p})$ satisfies all the properties of the statement.

By construction, this algebra has dimension two and (once we will exclude the possibility that $F$ is isomorphic to $\mathbb{Q}_p \times \mathbb{Q}_p$) the relation (8.6) will follow.
As the quadratic form is induced by an algebraic cycle, the Galois group \( \text{Gal}_K \) must act on \( V_{B,p} \) through \( G \). Hence we have the inclusions

\[ \mathbb{Q}_p[\text{Gal}_K] \subset F \subset \text{End}(V_{B,p}). \]

Let us show that \( \mathbb{Q}_p[\text{Gal}_K] \) is not of dimension one. If it were so, the algebra \( \text{End}_{\text{Gal}_K}(V_{B,p}) \) would be isomorphic to \( M_{2 \times 2}(\mathbb{Q}_p) \). By (8.5) the algebra \( \text{End}_{\varphi,\text{Fil}^\ast}(V_{Z,p} \otimes \text{Frac}(W(k))) \) would also be \( M_{2 \times 2}(\mathbb{Q}_p) \), which would imply that \( V_{Z,p} \otimes \text{Frac}(W(k)) \) would be decomposed into the sum of two isomorphic filtered \( \varphi \)-modules. This is impossible as it would in particular imply that the filtration on \( V_{Z,p} \otimes \text{Frac}(W(k)) \) would be a one step filtration, hence contradicting Assumption 8.6.

As the \( \mathbb{Q}_p \)-algebra \( \mathbb{Q}_p[\text{Gal}_K] \) is not of dimension one, we deduce that the inclusion \( \mathbb{Q}_p[\text{Gal}_K] \subset F \) is an equality. The commutator of \( F \) being \( F \) itself, we also have \( F = \text{End}_{\text{Gal}_K}(V_{B,p}) \).

To conclude we need to show that \( F \) is not isomorphic to \( \mathbb{Q}_p \times \mathbb{Q}_p \). If it were so, arguing as before, we would have a decomposition of filtered \( \varphi \)-modules \( (V_{Z,p} \otimes \text{Frac}(W(k)) = W \oplus W' \) and each of the lines \( W, W' \) would be isotropic. On the other hand the line \( \text{Fil}^i(V_Z \otimes K) \) is also isotropic, hence, we would have the equality

\[ W \otimes K = \text{Fil}^i(V_Z \otimes K) \]

after possibly replacing \( W \) with \( W' \).

Now, as \( W \) must be admissible, for any non-zero vector \( w \) of \( W \), the scalar \( \alpha \) such that \( \varphi(w) = \alpha w \) has \( p \)-adic valuation equal to \( i \). On the other hand, as \( V_{Z,p} \) is spanned by algebraic classes, there is a non-zero vector of \( W \) which is fixed by \( \varphi \). As \( i \neq 0 \) by Assumption 8.6, we deduce a contradiction. \( \square \)

**Remark 8.8.** For our main application, i.e. for exotic motives (Section 5), the action of \( F \) is induced by algebraic correspondences. The Hodge conjecture predicts that this should always be the case.

**Corollary 8.9.** Keep Assumption 8.6 and let \( F \) be the field of Proposition 8.7. Then \( F \) acts on \( V_{Z,p} \) and the equality (8.2) is \( F \)-equivariant. Moreover, for all \( v \in V_{Z,p} \) and all \( f \in F \) we have the equality

\[ q_{Z,p}(f \cdot v) = N_{F/\mathbb{Q}_p}(f)q_{Z,p}(v). \]

**Proof.** If one replaces \( V_{Z,p} \) by \( V_{Z,p} \otimes \text{Frac}(W(k)) \) the statement is a combination of Corollary 8.3 and Proposition 8.7. As \( F \) commutes with the action of \( \varphi \) and \( V_{Z,p} \subset V_{Z,p} \otimes \text{Frac}(W(k)) \) is precisely the space of vectors which are fixed by \( \varphi \), we deduce that \( V_{Z,p} \) is stable by \( F \) and the statement follows. \( \square \)

**Corollary 8.10.** Keep Assumption 8.6 and let \( F \) be the field of Proposition 8.7. The following statements are equivalent:

1. The quadratic forms \( q_{B,p} \) and \( q_{Z,p} \) are isomorphic.
(2) There exists a pair of non-zero vectors \( v_B \in V_{B,p} \) and \( v_Z \in V_{Z,p} \) such that \( q_{B,p}(v_B) \) and \( q_{Z,p}(v_Z) \) are equal in \( \mathbb{Q}_p^*/N_{F/\mathbb{Q}_p}(F^*) \).

(3) For any pair of non-zero vectors \( v_B \in V_{B,p} \) and \( v_Z \in V_{Z,p} \) we have that \( q_{B,p}(v_B) \) and \( q_{Z,p}(v_Z) \) are equal in \( \mathbb{Q}_p^*/N_{F/\mathbb{Q}_p}(F^*) \).

**Proof.** This is a formal consequence of formulas (8.6) and (8.7). □

**Remark 8.11.** By the very construction of \( F \), there are two actions of \( F \) on \( V_{Z,p} \otimes F \) and this allows us to write the decomposition

\[ V_{Z,p} \otimes F = V_{Z,+} \oplus V_{Z,-} \]

where \( V_{Z,+} \) is the line where the two actions coincide and \( V_{Z,-} \) is the line where the two actions are permuted by \( \text{Gal}(F/\mathbb{Q}_p) \). Using (8.7), we see that \( V_{Z,+} \) and \( V_{Z,-} \) are also the two isotropic lines of the hyperbolic plane \( V_{Z,p} \otimes F \). Note that there is also an analogous decomposition

\[ V_{B,p} \otimes F = V_{B,+} \oplus V_{B,-} \]

with analogous properties and that these decompositions are respected by Corollary 8.3.

Finally, as \( \text{Fil}^iV_{Z,p} \otimes K \) is an isotropic line it must coincide with \( V_{Z,+} \) or \( V_{Z,-} \) (after extension of scalars to a field containing \( F \) and \( K \)). We decide that the identification of \( F \) with a subfield of \( \mathbb{Q}_p \) is made to have the equality

\[ \text{Fil}^iV_{Z,p} \otimes \overline{\mathbb{Q}_p} = V_{Z,+} \otimes \overline{\mathbb{Q}_p}. \]

9. \( p \)-ADIC PERIODS

This section continues the comparison between the two quadratic spaces \((V_{B,p}, q_{B,p})\) and \((V_{Z,p}, q_{Z,p})\) which we initiated in the previous section. We keep notation from there, in particular we work under the Assumption 8.6 which gives a well defined positive integer \( i \) and we will make constant use of the field \( F \) constructed in Proposition 8.7.

The goal is to show Theorem 6.1 (via Proposition 7.6). This proof will appear at the very end of the section. The point is to describe explicitly the period matrix (relating the two quadratic spaces \( V_{B,p} \) and \( V_{Z,p} \)) given by Corollary 8.3. This description depends on \( i \) and especially on \( F \). This will oblige us to consider different cases, depending whether \( F \) is ramified or not.

**Unramified case.** In this subsection we work under Assumption 8.6 and we assume that the field \( F \) constructed in Proposition 8.7 is unramified over \( \mathbb{Q}_p \). This means that we have the inclusion \( F \subset B_{\text{crys}} \) and that the absolute Frobenius \( \varphi \) of \( B_{\text{crys}} \) restricted to \( F \) is the non-trivial element of \( \text{Gal}(F/\mathbb{Q}_p) \).

**Definition 9.1.** We define the \( F \)-vector subspace \( P_i \) of \( B_{\text{crys}} \) as the set of \( \lambda \in B_{\text{crys}} \) verifying the following properties:

1. \( \varphi^2(\lambda) = \lambda \).
2. \( \lambda \in \text{Fil}^iB_{\text{DR}} \).
$\varphi(\lambda) \in \text{Fil}^{-i}B_{\text{dR}}$.

**Definition 9.2.** Consider $V_{Z,+}$ and $V_{B,+}$ as constructed in Remark 8.11 and consider them as $F$-vector subspaces of $V_{Z,p} \otimes B_{\text{crys}}$. We define the $F$-vector subspace $Q_i$ of $B_{\text{crys}}$ as the set of $\lambda \in B_{\text{crys}}$ such that

$$\lambda \cdot V_{B,+} \subset V_{Z,+}.$$ 

**Proposition 9.3.** The $F$-vector subspaces $P_i, Q_i \subset B_{\text{crys}}$ introduced in Definitions 9.1 and 9.2 coincide and they have dimension one.

**Proof.** Following Remark 8.11 we have

$$(9.1) \quad V_{B,+} \cdot B_{\text{crys}} = V_{Z,+} \cdot B_{\text{crys}}.$$ 

As the $F$-vector spaces $V_{B,+}$ and $V_{Z,+}$ have dimension one then $Q_i \subset B_{\text{crys}}$ must have dimension one.

Fix a non-zero element $\lambda_i \in Q_i$. By definition of $Q_i$ there are non-zero vectors $v_B \in V_{B,+}$ and $v_Z \in V_{Z,+}$ such that

$$(9.2) \quad \lambda_i \cdot v_B = v_Z$$

which implies

$$(9.3) \quad \varphi(\lambda_i) \cdot \varphi(v_B) = \varphi(v_Z).$$

Now $\varphi$ acts trivially on $V_{B,p}$ and, because of the presence of algebraic classes, also on $V_{Z,p}$. Hence $\varphi$ acts as the non-trivial element of $\text{Gal}(F/\mathbb{Q}_p)$ on $V_{B,p} \otimes F$ and $V_{Z,p} \otimes F$. This implies that

$$(9.4) \quad \varphi(v_B) \in V_{B,-} \quad \text{and} \quad \varphi(v_Z) \in V_{Z,-}$$

as well as

$$(9.5) \quad \varphi^2(v_B) = v_B \quad \text{and} \quad \varphi^2(v_Z) = v_Z.$$ 

We claim that the above relations (through Corollary 8.3) imply that $\lambda_i$ verifies conditions (1), (2) and (3) from Definition 9.1, hence the inclusion $Q_i \subset P_i$. Indeed (9.2) gives (2), (9.3) and (9.4) give (3) and finally (9.5) gives (1).

We now show that the inclusion $Q_i \subset P_i$ is actually an equality by showing that $P_i$ is an $F$-vector space of dimension one as well. Keep $\lambda_i \in Q_i$ as before. Relations (9.1) and (9.2) imply that $\lambda_i$ is invertible in $B_{\text{crys}}$. Let $\mu \in P_i$ and consider $f = \mu \cdot \lambda_i^{-1} \in B_{\text{crys}}$. As both $\lambda_i$ and $\mu$ belong to $P_i$, $f$ verifies the following properties:

1. $\varphi^2(f) = f$.
2. $f \in \text{Fil}^0 B_{\text{dR}}$.
3. $\varphi(f) \in \text{Fil}^0 B_{\text{dR}}$.

This implies that $f \in F$ by [Col02, Proposition 9.2], hence concludes the proof. \qed
Proposition 9.4. Consider a non-zero element $\lambda_i \in P_i$, then

$$\lambda_i \cdot \varphi(\lambda_i) \in \mathbb{Q}_p^*.$$  

Moreover the quadratic forms $q_{B,p}$ and $q_{Z,p}$ are isomorphic if and only if

$$\lambda_i \cdot \varphi(\lambda_i) \in N_{F/\mathbb{Q}_p}(F^*).$$

Proof. By construction of $P_i$ we have $\lambda_i \cdot \varphi(\lambda_i) \in (B^\varphi_{\text{crys}} \cap \text{Fil}^0 B_{\text{dR}}) = \mathbb{Q}_p$ (see Theorem 8.1). Now let $v_B \in V_{B,+}$ and $v_Z \in V_{Z,+}$ as in the proof of Proposition 9.3. By (9.5) we have

$$v_B + \varphi(v_B) \in V_{B,p} \quad \text{and} \quad v_Z + \varphi(v_Z) \in V_{Z,p}.$$  

Write $\langle \cdot, \cdot \rangle$ for the bilinear form associated to the quadratic form $q_{\cdot}$. Then we have

$$q_{B,p}(v_B + \varphi(v_B)) = 2\langle v_B, \varphi(v_B) \rangle_{B,p} \quad \text{and} \quad q_{Z,p}(v_Z + \varphi(v_Z)) = 2\langle v_Z, \varphi(v_Z) \rangle_{Z,p}$$

because $v_B, v_Z, \varphi(v_B)$ and $\varphi(v_Z)$ are isotropic vectors (see Remark 8.11 and (9.4)). Hence, using the relations (9.2), (9.3), together with (8.4), we have

$$\lambda_i \cdot \varphi(\lambda_i) \cdot q_{B,p}(v_B + \varphi(v_B)) = q_{Z,p}(v_Z + \varphi(v_Z)).$$

We conclude by Corollary 8.10. \qed

Proposition 9.5. (Lubin–Tate) There is an element $t_2 \in B_{\text{crys}}$ which verifies the following properties:

1. $t_2 \in \text{Fil}^1 B_{\text{dR}} - \text{Fil}^2 B_{\text{dR}}$.
2. $\varphi(t_2) \in \text{Fil}^0 B_{\text{dR}} - \text{Fil}^1 B_{\text{dR}}$.
3. $\varphi^2(t_2) = p \cdot t_2$.

Proof. This is [Col02, §9.3] applied to $E = \mathbb{Q}_p^2$, the quadratic unramified extension of $\mathbb{Q}_p$. \qed

Corollary 9.6. Fix a positive integer $i$ and define $\lambda_i \in B_{\text{crys}}$ as

$$\lambda_i = (t_2/\varphi(t_2))^i.$$  

Then $\lambda_i$ is a non-zero element of $P_i$ and moreover it verifies

$$\lambda_i \cdot \varphi(\lambda_i) = 1/p^i.$$  

Lemma 9.7. The element $1/p \in \mathbb{Q}_p^*$ does not belong to the group of norms $N_{F/\mathbb{Q}_p}(F^*)$.

Proof. For $f \in F$, the $p$-adic valuation of $f$ and $\varphi(f)$ coincide, hence a norm must have even $p$-adic valuation. \qed
**Ramified case.** In this subsection we work under the Assumption 8.6 and we assume that the field $F$ constructed in Proposition 8.7 is ramified over $\mathbb{Q}_p$. This subsection is written in analogy to the unramified case, although some extra computations are needed to deal with wild ramification.

Define $\sigma$ to be the non-trivial element of $\text{Gal}(F/\mathbb{Q}_p)$ and $B_{\text{crys},F} \subset B_{\text{dR}}$ as the smallest subring containing $F$ and $B_{\text{crys}}$. The inclusions $B_{\text{crys}} \subset B_{\text{crys},F}$ and $F \subset B_{\text{crys},F}$ induce an identification

$$F \otimes_{\mathbb{Q}_p} B_{\text{crys}} = B_{\text{crys},F}.$$

This allows us to extend $\sigma$ and $\varphi$ to two automorphisms of $\mathbb{Q}_p$-algebras

$$\sigma, \varphi : B_{\text{crys},F} \longrightarrow B_{\text{crys},F}$$

which commute.

Finally, we will denote by $\mathbb{Q}_p^n \subset B_{\text{crys}}$ the unique unramified field extension of $\mathbb{Q}_p$ of degree $n$.

**Definition 9.8.** We define the $F$-vector subspace $P_i$ of $B_{\text{crys},F}$ as the set of $\lambda \in B_{\text{crys}}$ verifying the following properties.

1. $\lambda \in \text{Fil}^i B_{\text{dR}}$.
2. $\sigma(\lambda) \in \text{Fil}^{-i} B_{\text{dR}}$.
3. $\varphi(\lambda) = \lambda$.

**Definition 9.9.** Consider $V_{Z,+}$ and $V_{B,+}$ as constructed in Remark 8.11 and consider them as $F$-vector subspaces of $V_{Z,p} \otimes B_{\text{crys},F}$ via Corollary 8.3. We define the $F$-vector subspace $Q_i$ of $B_{\text{crys},F}$ as the set of $\lambda \in B_{\text{crys},F}$ such that

$$\lambda \cdot V_{B,+} \subset V_{Z,+}.$$ 

**Proposition 9.10.** The $F$-vector subspaces $P_i, Q_i \subset B_{\text{crys},F}$ introduced in Definitions 9.8 and 9.9 coincide and they have dimension one.

**Proof.** Analogous to the proof of Proposition 9.3. $\square$

**Proposition 9.11.** Consider a non-zero element $\lambda_i \in P_i$, then

$$\lambda_i \cdot \varphi(\lambda_i) \in \mathbb{Q}_p^*.$$

Moreover the quadratic forms $q_{B,p}$ and $q_{Z,p}$ are isomorphic if and only if

$$\lambda_i \cdot \sigma(\lambda_i) \in N_{F/\mathbb{Q}_p}(F^*).$$

**Proof.** Analogous to the proof of Proposition 9.4. $\square$

**Proposition 9.12.** (Colmez) For each uniformizer $\pi \in F$ there exists an element $t_\pi \in B_{\text{crys},F}$ which verifies the following properties.

1. $t_\pi \in \text{Fil}^1 B_{\text{dR}} - \text{Fil}^2 B_{\text{dR}}$.
2. $\sigma(t_\pi) \in \text{Fil}^0 B_{\text{dR}} - \text{Fil}^1 B_{\text{dR}}$.
3. $\varphi(t_\pi) = \pi \cdot t_\pi$.

**Proof.** This is [Col02, §9.3] applied to $E = F$. $\square$
Corollary 9.13. Suppose that there is a uniformizer \( \pi \in F \) such that 
\[
\sigma(\pi) = -\pi.
\]
Let \( a \in \mathbb{Q}_p^2 \) be an element of the quadratic unramified extension of \( \mathbb{Q}_p \) such that 
\[
\varphi(a) = -a.
\]
Define \( \lambda_i \in B_{\text{crys},F} \) as 
\[
\lambda_i = (a \cdot t_\pi/\sigma(t_\pi))^i.
\]
Then \( \lambda_i \) is a non-zero element of \( P_i \) and moreover it verifies 
\[
\lambda_i \cdot \sigma(\lambda_i) = (a^2)^i.
\]

Lemma 9.14. Let \( a \in \mathbb{Q}_p^2 \) be as in Corollary 9.13. Then the element \( a^2 \in \mathbb{Q}_p^2 \) does not belong to the group of norms \( N_{\mathbb{F}/\mathbb{Q}_p}(F^*) \).

Proof. This is about finding non-zero solutions \( x, y \in \mathbb{Q}_p \) of the equation 
\[
x^2 - \pi^2 y^2 = a^2.
\]
We can suppose that \( a^2 \) has \( p \)-adic valuation zero. Then \( x \) has valuation zero and \( y \) has non-negative valuation. By reducing modulo \( p \) (respectively modulo \( 8 \) if \( p = 2 \)) we see that the existence of a solution \( (x, y) \) would imply that \( a^2 \) would be a square in \( \mathbb{F}_p \) (respectively \( 3, 5, \) or \( 7 \) would be a square in \( \mathbb{Z}/8\mathbb{Z} \)) which gives a contradiction. \( \square \)

Remark 9.15. If \( F \) is tamely ramified then one can find a uniformizer \( \pi \in F \) such that \( \sigma(\pi) = -\pi \) as in Corollary 9.13. If \( p \neq 2 \) then \( F \) is automatically tamely ramified. For \( p = 2 \) there are two quadratic extensions which are not tamely ramified namely \( F = \mathbb{Q}_2(\sqrt{3}) \) and \( F = \mathbb{Q}_2(\sqrt{-1}) \). What follows treats these two exceptions.

Corollary 9.16. We keep notation from Proposition 9.12 and we work with \( p = 2 \) and the field \( F = \mathbb{Q}_2(\sqrt{-1}) \). Let \( \alpha \in \mathbb{Q}_2(\sqrt{-1}) \) be an element such that 
\[
\varphi(\alpha) = \sqrt{-1} \cdot \alpha.
\]
Define \( \lambda_i \in B_{\text{crys},F} \) as 
\[
\lambda_i = (\alpha \cdot t_{1-\sqrt{-1}}/\sigma(t_{1-\sqrt{-1}}))^i.
\]
Then \( \lambda_i \) is a non-zero element of \( P_i \) and moreover it verifies 
\[
\lambda_i \cdot \sigma(\lambda_i) = (\alpha \cdot \sigma(\alpha))^i.
\]

Lemma 9.17. The element \( \alpha \cdot \sigma(\alpha) \in \mathbb{Q}_2^* \) does not belong to the group of norms \( N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^*) \).

Proof. Write \( \alpha = \alpha_1 + \sqrt{-1} \cdot \alpha_2 \) with \( \alpha_1, \alpha_2 \in \mathbb{Q}_2 \), then we have 
\[
\varphi(\alpha_1) = \alpha_2, \quad \varphi(\alpha_2) = -\alpha_1, \quad \alpha \cdot \sigma(\alpha) = \alpha_1^2 + \varphi(\alpha_1^2).
\]
Let us make explicit computations. Fix the presentations 
\[
\mathbb{F}_{21} = \mathbb{F}_2[x]/(x^2 + x + 1) \quad \mathbb{F}_{21} = \mathbb{F}_2[y]/(y^2 + xy + 1)
\]
This is a norm if and only if one can find a nonzero solution \( u, v \). Then \( \Delta = \sqrt{(-1/2)^2 - 4} = \sqrt{-1 - \sqrt{-3}} - 4 \) can be chosen as \( \alpha_1 \). Thus, 
\[
\alpha \cdot \sigma(\alpha) = \Delta^2 + \varphi(\Delta^2) = -9.
\]
This is a norm if and only if one can find a nonzero solution \( u, v \in \mathbb{Q}_2 \) of the equation \( u^2 + v^2 = -9 \). This is impossible by looking at the equation modulo 8. \( \square \)

**Proposition 9.18.** (Colmez) Consider the ring \( B_{\text{crys}, F} \) for the prime \( p = 2 \) and \( F = \mathbb{Q}_2(\sqrt{3}) \). There exists an element \( t_{F(\sqrt{-1})} \in B_{\text{crys}, F} \) which verifies the following properties.

1. \( t_{F(\sqrt{-1})} \in \text{Fil}^1 B_{\text{dR}} - \text{Fil}^2 B_{\text{dR}} \).
2. \( \sigma(t_{F(\sqrt{-1})}), \varphi(t_{F(\sqrt{-1})}), \varphi \circ \sigma(t_{F(\sqrt{-1})}) \in \text{Fil}^0 B_{\text{dR}} - \text{Fil}^1 B_{\text{dR}} \).
3. \( \varphi^2(t_{F(\sqrt{-1})}) = (1 - \sqrt{-1}) \cdot t_{F(\sqrt{-1})} \).

**Proof.** This is [Col02 §9.3] applied to \( E = F(\sqrt{-1}) = \mathbb{Q}_2(\sqrt{-1}) \). \( \square \)

**Corollary 9.19.** Let \( \alpha \in \mathbb{Q}_2^2(\sqrt{3}) = \mathbb{Q}_2^2(\sqrt{-1}) \) be an element such that \( \varphi(\alpha) = \sqrt{-1} \cdot \alpha \).

Define \( \lambda_i \in B_{\text{crys}, F} \) as
\[
\lambda_i = \left( \frac{\alpha \cdot t_{F(\sqrt{-1})} \cdot \varphi(t_{F(\sqrt{-1})})}{\sigma(t_{F(\sqrt{-1})}) \cdot (\varphi \circ \sigma(t_{F(\sqrt{-1})}))} \right)^i.
\]
Then \( \lambda_i \) is a non-zero element of \( P_i \) and moreover it verifies
\[
\lambda_i \cdot \sigma(\lambda_i) = (\alpha \cdot \sigma(\alpha))^i.
\]

**Lemma 9.20.** The element \( \alpha \cdot \sigma(\alpha) \in \mathbb{Q}_2^* \) does not belong to the group of norms \( N_{\mathbb{Q}_2(\sqrt{3})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{3}))^* \).

**Proof.** We argue as in the proof of Lemma 9.17. Write \( \alpha = \alpha_1 + \sqrt{-1} \cdot \alpha_2 \) with \( \alpha_1, \alpha_2 \in \mathbb{Q}_{24} \), then we have
\[
\varphi(\alpha_1) = \alpha_2, \quad \varphi(\alpha_2) = -\alpha_1.
\]
Moreover, as the elements fixed by \( \sigma \) are exactly those in \( B_{\text{crys}} \), we have
\[
\sigma(\sqrt{-1}) = -\sqrt{-1}, \quad \sigma(\alpha_1) = \alpha_1, \quad \sigma(\alpha_2) = \alpha_1
\]
and we deduce that \( \alpha \cdot \sigma(\alpha) = \alpha_1^2 + \varphi(\alpha_1^2) \). This means that we can use the same computations as in the proof of Lemma 9.17 and for the same choice of \( \alpha \) in there we have
\[
\alpha \cdot \sigma(\alpha) = -9.
\]
This is a norm if and only if one can find a nonzero solution \( u, v \in \mathbb{Q}_2 \) of the equation \( u^2 - 3v^2 = -9 \). This is impossible by looking at the equation modulo 8. \( \square \)
Proof of Theorem 6.1. Thank to Proposition 8.4 we can work under Assumption 8.6 and fix the positive integer \(i\) as in the assumption as well as the field \(F\) from Proposition 8.7.

Let us first suppose that \(F\) is unramified. We can then combine Proposition 9.4 and Corollary 9.6 to conclude that the quadratic forms are \(q_{B,p}\) and \(q_{Z,p}\) are isomorphic if and only if \(1/p^i\) is a norm. Because of Lemma 9.7 this is the case if and only if \(i\) is even.

Suppose now that \(F\) is ramified but it is not \(\mathbb{Q}_2(\sqrt{3})\) or \(\mathbb{Q}_2(\sqrt{-1})\). We can then combine Proposition 9.11 and Corollary 9.13 to conclude that the quadratic forms are \(q_{B,p}\) and \(q_{Z,p}\) are isomorphic if and only if \((a^2)^i\) is a norm. Because of Lemma 9.14 this is the case if and only if \(i\) is even.

If \(F\) is \(\mathbb{Q}_2(\sqrt{-1})\) we can combine Proposition 9.11 and Corollary 9.16 to conclude that the quadratic forms are \(q_{B,p}\) and \(q_{Z,p}\) are isomorphic if and only if \((\alpha \cdot \sigma(\alpha))^i\) is a norm. Because of Lemma 9.17 this the case if and only if \(i\) is even.

Finally, if \(F\) is \(\mathbb{Q}_2(\sqrt{3})\) we can combine Proposition 9.11 and Corollary 9.19 to conclude that the quadratic forms are \(q_{B,p}\) and \(q_{Z,p}\) are isomorphic if and only if \((\alpha \cdot \sigma(\alpha))^i\) is a norm. Because of Lemma 9.20 this is the case if and only if \(i\) is even.

In conclusion, we showed for that, for any possible \(F\), the quadratic forms \(q_{B,p}\) and \(q_{Z,p}\) are isomorphic if and only if \(i\) is even. This shows Theorem 6.1 via Proposition 7.6. \(\square\)

Remark 9.21. The formulation of Proposition 7.6 does not involve the ring \(B_{\text{crys}}\). We do not know if there is a way to show this proposition without computing explicitly the \(p\)-adic periods.

A. Geometric examples

In this section we discuss several examples to which Theorems 1.18 and 6.1 apply non-trivially. We are particularly interested in exotic classes on abelian fourfolds (Definition 5.1). The main result is Proposition A.1 where we discuss the existence of exotic classes that cannot be lifted to algebraic classes in characteristic zero. The techniques of construction there are inspired by [LO74] and [Zar15].

Other examples of exotic classes will be found in Remark A.8. We end the section with an example (other than abelian fourfolds) for which the standard conjecture of Hodge type holds true via Theorem 6.1.

Proposition A.1. Let \(p\) be a prime number and let \(K\) be an imaginary quadratic number field where \(p\) does not split. Then there exists an abelian fourfold \(A\) over \(\mathbb{F}_p\) verifying the following properties.

1. The endomorphism algebra can be written as the compositum

   \[\text{End}(A)_{\mathbb{Q}} = K \cdot R\]

   where \(R\) is a totally real number field such that \([R : \mathbb{Q}] = 4\). (In particular \(A\) is simple and has a unique CM-structure).
Consider the motivic decomposition from Proposition 4.6(2). Among the factors $M_I$ of $h^4(A)$ there exists a (unique) factor $M$ such that $M(2)$ does not contain any Lefschetz class (Definition 3.1) but it is Frobenius invariant (for a model of $A$ over a finite field).

For any CM-lifting of $A$ to characteristic zero (see Theorem 4.9 and its Corollary) the Hodge structure $R_B(M)$ is of type $(3,1), (1,3)$.

For any CM-lifting of $A$ to characteristic zero we have the equality $\text{End}_{\text{NUM}(\mathbb{C})}(M) = K$.

**Proof.** The proof is decomposed in a series of lemmas. The final step is Lemma A.7.

**Remark A.2.** Let us make some comments on the above proposition.

1. If (a model of) the fourfold $A$ verifies the Tate conjecture, then $M$ is an exotic motive in the sense of Section 5. In particular, in each characteristic, there should be infinitely many non-isogenous abelian fourfolds having exotic motives.

   If the Tate conjecture was highly false and no such exotic classes existed then our main result (Theorem 1.18) would follow directly from the arguments in Section 3. The Tate conjecture and the standard conjecture of Hodge type should be thought as two independent and different problems. The first is about the construction of algebraic classes, the second is about how they intersect (independently whether there are a lot of algebraic classes or not). It seems likely that a solution of one problem does not imply a solution for the other. For example the proof of the Tate conjecture for divisors on abelian variety [Tat66, Theorem 4] does not imply the standard conjecture of Hodge type for divisor on abelian variety (which is known by a different argument).

2. Because of the Hodge types in part (3), the (expected) algebraic classes in positive characteristic cannot be lifted to algebraic classes in characteristic zero.

3. Note that the field $F = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ can be any quadratic extension of $\mathbb{Q}_p$. This field $F$ coincides with the one in Proposition 8.7. This shows that the different cases studied in Section 9 were needed.

4. The hypothesis that $K$ is totally imaginary is necessary. If $K$ were a real quadratic number field, conditions (3) and (4) in Proposition A.1 would not be compatible (see the proof of Lemma 5.15).

**Lemma A.3.** Let $p$ and $K$ be as in Proposition A.1. There exists a totally real number field $R$ such that the following holds:

1. The prime $p$ does not split in $R$.
2. The degree $[R : \mathbb{Q}]$ is four.
3. The field $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is embeddable in the field $R \otimes_{\mathbb{Q}} \mathbb{Q}_p$.
4. If $\bar{R}$ is the normal closure of $R$ over $\mathbb{Q}$, then $\text{Gal}(R/\mathbb{Q}) = S_3$. In particular we have $\text{Gal}(\bar{R}/\mathbb{Q}) = S_4$. 


Proof. Same as in [LO74, §3]. □

**Lemma A.4.** Let \( R \) be as in the above lemma. The following holds:

1. The subfields of the compositum \( R \cdot K \) are \( \mathbb{Q}, K, R \) and \( R \cdot K \).
2. The prime \( p \) factorises in \( R \cdot K \) as
   \[
   p = (p \cdot \bar{p})^e,
   \]
   where \( p \) and \( \bar{p} \) are two prime ideals which are exchanged by complex conjugation and \( e \) is the ramification index.

Proof. Let \( \tilde{R} \) be as in Lemma A.3. The equality \( \text{Gal}(\tilde{R}/\mathbb{Q}) = S_4 \) implies
\[
\text{Gal}(\tilde{R} \cdot K/\mathbb{Q}) = S_4 \times \mathbb{Z}/2\mathbb{Z},
\]
and similarly \( \text{Gal}(\tilde{R}/R) = S_3 \) implies \( \text{Gal}(\tilde{R} \cdot K/R \cdot K) = S_3 \). We deduce that the subfields of \( R \cdot K \) are in bijection with the subgroups of \( S_4 \times \mathbb{Z}/2\mathbb{Z} \) containing \( S_3 \). Those are precisely \( S_3, S_4, S_3 \times \mathbb{Z}/2\mathbb{Z} \) and \( S_4 \times \mathbb{Z}/2\mathbb{Z} \), which implies (1).

Note that the equality \( R \cdot K \cong R \otimes_{\mathbb{Q}} K \) holds. Hence we have
\[
(R \cdot K) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong (K \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} (R \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cong (R \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\text{Gal}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p/\mathbb{Q}_p)},
\]
where the last equality comes from Lemma A.3(3). As \( R \otimes_{\mathbb{Q}} \mathbb{Q}_p \) is a field (by Lemma A.3(1)), the prime \( p \) factorizes in \( R-K \) as the product of two different primes. Moreover, those two primes are exchanged by \( \text{Gal}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p/\mathbb{Q}_p) \). As complex conjugation generates this Galois group it exchanges these two prime ideals. In particular they must have the same ramification index. This concludes part (2). □

**Lemma A.5.** Consider the prime ideals \( p \) and \( \bar{p} \) of \( R \cdot K \) from the above lemma. Then there exist an integer \( n \), a \( p \)-power \( q \) and a \( q \)-Weil number \( \alpha \in R \cdot K \) verifying the following properties:

1. The ideal generated by \( \alpha \) factorizes as
   \[
   (\alpha) = p^n \cdot \bar{p}^{3n}.
   \]
2. For any positive integer \( s \), we have the equality
   \[
   \mathbb{Q}(\alpha^s) = R \cdot K.
   \]
3. The norm \( N_{R \cdot K/K}(\alpha) \in K \) equals \( q^2 \).

Proof. Consider the ideal \( I = (p \cdot \bar{p}^3)^e \). We have that \( I \cdot \bar{I} = p^4 \) by Lemma A.3(2). Actually, for any \( g \in \text{Gal}(\tilde{R} \cdot K/\mathbb{Q}) \), we have the equality \( g(I) \cdot g(\bar{I}) = p^4 \) of ideals in \( \tilde{R} \cdot K \). This follows from the case \( g = \text{id} \) together with the explicit description of the Galois group in the proof of the lemma above (which implies that the complex conjugation is in the center of the Galois group). Hence, we can apply [Hon88, Lemma 1] and deduce that there exists a \( q \)-Weil number \( \alpha \in R \cdot K \) verifying (1).

Let us show (2). Because of Lemma A.4(1), this amounts to showing that \( \alpha^s \) does not belong to \( K \) nor to \( R \). Now, if \( \alpha^s \) belongs to \( K \) (or to \( R \)) then its
factorization in $R \cdot K$ would have the same exponent in $p$ and in $\overline{p}$ because there is only one prime above $p$ in $K$ (or in $R$); see the hypothesis on $K$ in Proposition A.1 (respectively by Lemma A.3(1)).

Let us now show (3). If we compute the norm of the ideal $(\alpha)$ we obtain

$$(N_{R,K/K}(\alpha)) = N_{R,K/K}(p^n \cdot \overline{p}^3)^n = N_{R,K/K}(p)^n \cdot N_{R,K/K}(\overline{p})^3 = \tilde{p}^m,$$

where $m$ is an integer and $\tilde{p}$ is the only prime ideal above $p$ in $K$. Hence, after possibly replacing $\alpha$ by a power, we obtain that the ideal $N_{R,K/K}(\alpha)$ is generated by a power of $p$. For weight reasons we have

$$(N_{R,K/K}(\alpha)) = (q^2).$$

This is equivalent to the relation $N_{R,K/K}(\alpha) = \xi \cdot q^2$, where $\xi$ is an invertible element of the ring of integers of $K$. As the group of invertible elements of the ring of integers of an imaginary quadratic field is finite, after replacing $\alpha$ by a power we get (3). Note that such a power of $\alpha$ will still have properties (1) and (2).

\[\Box\]

Lemma A.6. Let $\alpha$ be a $q$-Weil number verifying the properties as in the above lemma. Let $A$ be an abelian variety over $\mathbb{F}_q$ whose isogeny class corresponds to $\alpha$ under the Honda–Tate correspondence [Tat71]. Then the following holds:

1. The dimension of $A$ is four.
2. $A$ is geometrically simple.
3. $\text{End}(A)_{\mathbb{Q}} = \text{End}(A_{\mathbb{F}_q}) = K \cdot R$.
4. The slopes of $A$ are $(1/4, 3/4)$.
5. There are Frobenius invariant classes in $H^4_\ell(A)$ which are not of Lefschetz type.

Proof. By [Tat66] the division algebra $\text{End}(A)_{\mathbb{Q}}$ has center equal to $\mathbb{Q}(\alpha)$ which is $R \cdot K$ by Lemma A.5. Moreover, by [Tat71, Theorem 1], this division algebra splits at every place except possibly at the places $p$ and $\overline{p}$ above $p$. The local invariants there are computed by the formula in [Tat71, Theorem 1], which gives

$$\text{inv}_p(\text{End}(A)_{\mathbb{Q}}) = \frac{v_p(\alpha)}{v_p(q)} \cdot [(R \cdot K)_p : \mathbb{Q}_p] \mod \mathbb{Z},$$

and similarly for $\overline{p}$.

We claim that these local invariants are trivial as well. Indeed, using the factorisation in Lemma A.5(1), we deduce that $\frac{v_p(\alpha)}{v_p(q)} = \frac{1}{4}$. On the other hand, the degree $[(R \cdot K)_p : \mathbb{Q}_p]$ equals four, because $[R \cdot K : \mathbb{Q}] = 8$ and the two primes above $p$ are exchanged by complex conjugation (Lemma A.4(2)). Altogether we have that $\text{inv}_p(\text{End}(A)_{\mathbb{Q}}) = 0$ and similarly one shows $\text{inv}_{\overline{p}}(\text{End}(A)_{\mathbb{Q}}) = 0$.

Because all the invariants of the $(R \cdot K)$-central algebra $\text{End}(A)_{\mathbb{Q}}$ are trivial, we have $R \cdot K = \text{End}(A)_{\mathbb{Q}}$. As $[R \cdot K : \mathbb{Q}] = 8$ we deduce (1).
Consider now the abelian variety \( A_s \) over \( \mathbb{F}_{q^s} \) whose isogeny class corresponds to \( \alpha^s \). Following the Honda–Tate correspondence, \( A_s \) is a simple factor of \( A \times_{\mathbb{F}_q} \mathbb{F}_{q^s} \). On the other hand, all the arguments above work by replacing \( \alpha \) by \( \alpha^s \), because of Lemma A.5(2). In particular, \( A_s \) has also dimension four and \( R \cdot K = \text{End}(A_s) \mathbb{Q} \). This implies (2) and (3).

One slope has already been computed, namely \( v_p(\alpha) = 1/4 \). Duality implies that there is also the slope \( 3/4 \). As \( A \) has dimension four there are no more slopes.

Let us now show (5). The existence of a class such as the ones claimed is equivalent to the existence of a set \( I \) consisting of four Galois conjugates of \( \alpha \) whose product equals \( q^2 \) and such that \( I \) is not stable under the action of complex conjugation (see the proof of Lemma 5.7 or [Zar15, §2]). We claim that the relation

\[
N_{R \cdot K/K}(\alpha) = q^2
\]

(Lemma A.5(3)) gives precisely the existence of those four Galois conjugates. Indeed, let \( \tilde{R} \) be the normal closure of \( R \) over \( \mathbb{Q} \), by definition we have

\[
N_{R \cdot K/K}(\alpha) = \prod_{g \in \text{Hom}_K(R \cdot K, \tilde{R} \cdot K)} g(\alpha).
\]

Hence it is enough to show that complex conjugation does not stabilize the set \( J = \{g(\alpha)\}_{g \in \text{Hom}_K(R \cdot K, \tilde{R} \cdot K)} \). As the set \( J \) is of size four and the total Galois orbit of \( \alpha \) is of size 8 there is an element of \( \text{Gal}(\tilde{R} \cdot K/\mathbb{Q}) \) which does not stabilize \( J \). On the other hand, thanks to the equality

\[
\text{Gal}(\tilde{R} \cdot K/\mathbb{Q}) = \text{Gal}(\tilde{R} \cdot K/K) \times \text{Gal}(\tilde{R} \cdot K/\tilde{R})
\]

we have that the total Galois group \( \text{Gal}(\tilde{R} \cdot K/\mathbb{Q}) \) is generated by its subgroup \( \text{Gal}(\tilde{R} \cdot K/K) \) and complex conjugation. As \( \text{Gal}(\tilde{R} \cdot K/K) \) stabilizes \( J \), complex conjugation cannot stabilize it.

\[\blacklozenge\]

**Lemma A.7.** Let \( A \) be an abelian fourfold which satisfies the properties of the lemma above. Then it also satisfies all the conditions of Proposition A.1.

**Proof.** Part (1) has already been showed. Part (2) follows from Lemma A.6(5). (Unicity comes from Lemma 5.13)

Let us now show part (3). Write \( \alpha, \beta, \gamma, \delta, q/\alpha, q/\beta, q/\gamma, q/\delta \) for the eight (distinct) Frobenius eigenvalues and consider the decomposition in eigenlines

\[
h^1(A) = V_\alpha \oplus V_\beta \oplus V_\gamma \oplus V_\delta \oplus V_{q/\alpha} \oplus V_{q/\beta} \oplus V_{q/\gamma} \oplus V_{q/\delta}
\]

as in Corollary 5.5. Among these eight eigenvalues, four have slope \( 1/4 \) and four have slope \( 3/4 \).

Fix a CM-lifting (Theorem A.9). The above decomposition in eigenlines will lift as well (Corollary 5.10). Among the eight lines, four will belong to \( H^{1,0} \) and four will belong to \( H^{0,1} \). The Shimura–Taniyama formula [Lat71, Lemma 5] implies that there is exactly one eigenvalue, call it \( \alpha \), whose slope is \( 1/4 \) and such that \( V_\alpha \subset H^{1,0} \).
Now decompose $M = M_{\alpha,\beta,\gamma,\delta} \oplus M_{q/\alpha,q/\beta,q/\gamma,q/\delta}$ via Proposition 4.6. (After possibly renaming the eigenvalues.) With this notation we have the relation

$$\alpha \cdot \beta \cdot \gamma \cdot \delta = q^2.$$  

By looking at the $p$-adic valuation we deduce that, among $\beta, \gamma, \delta$ there is exactly one eigenvalue of slope $1/4$, say $\beta$. Hence we have $V_\beta \subset H^{0,1}$. Finally we have $V_\gamma, V_\delta \subset H^{1,0}$ because there is exactly one eigenvalue, namely $q/\alpha$, whose slope is $3/4$ and such that $V_{q/\alpha} \subset H^{0,1}$. Altogether we deduce 

$$V_\alpha \otimes V_\beta \otimes V_\gamma \otimes V_\delta \subset H^{3,1}$$  

which gives (3).

Let us now show part (4). By construction we can find a quadratic number field $F \subset \tilde{R} \cdot K$ such that the motive $M$ decomposes in the category $\text{CHM}(k)_F$ into a sum

$$M_I = M_I \oplus \tilde{M}_I$$  

of two motives of rank one (see Proposition 4.9). We first claim that such a field $F$ must be imaginary. If $F$ were contained in $\mathbb{R}$ then the Betti realization of the lifting of $M_I$ would respect the Hodge symmetry. As it is one dimensional for weight reasons it would be of type $(2,2)$. This contradicts part (3).

By [Jan92], $D = \text{End}_{\text{NUM}(\mathbb{C})_Q}(M)$ is a division algebra. By construction, $F$ splits $D$. We claim that $D = F$. Otherwise we would have $D \otimes F \cong M_{2 \times 2}(F)$ which would imply that $M_I$ and $\tilde{M}_I$ are isomorphic as numerical motives. As homological and numerical equivalence is known to coincide for complex abelian varieties [Lie68], this would imply that their Betti realization are isomorphic, which is impossible because of the different Hodge types.

In conclusion, $\text{End}_{\text{NUM}(\mathbb{C})_Q}(M)$ is an imaginary quadratic field contained in $\tilde{R} \cdot K$. On the other hand, there is only one such field (namely $K$) because of the description of $\text{Gal}(\tilde{R} \cdot K/Q)$ in the proof of Lemma A.4. □

Remark A.8. Let us comment on other examples of exotic motives coming from abelian fourfolds.

(1) One can construct an abelian fourfold $A$ over a finite field having an exotic motive whose lifting to $\mathbb{C}$ has Betti realization of type $(2,2)$. Such a condition means that the CM-lifting of $A$ over $\mathbb{C}$ has Hodge classes which are not Lefschetz. This situation (over $\mathbb{C}$) has been classified in [MZ95]. So, any reduction modulo $p$ of their examples will give an abelian fourfold over a finite field of the desired type. (To avoid that the reduction modulo $p$ creates more Lefschetz classes, one can take an ordinary prime).

As already pointed out, these examples are less interesting for the standard conjecture of Hodge type, see Remark 8.5(2).
(2) There are no exotic motives over $\mathbb{F}_p$ (coming from abelian fourfolds) whose lifting to $\mathbb{C}$ have Betti realization of type $(4, 0), (0, 4)$. To show this, consider a model of the abelian fourfold over a finite field $\mathbb{F}_q$. Let $I$ be the set of Frobenius eigenvalues such that the corresponding eigenspaces are lifted into $H^{1,0}$ and $\bar{I}$ be the set of Frobenius eigenvalues such that the corresponding eigenspaces are lifted into $H^{0,1}$. If the cohomology group $H^{4,0} \oplus H^{0,4}$ becomes Frobenius invariant over $\mathbb{F}_q$, then $\prod_{\alpha \in I} \alpha = \prod_{\beta \in \bar{I}} \beta\left(= q^2\right)$. On the other hand, using the Shimura–Taniyama formula [Tat71, Lemma 5], we have that the $p$-adic valuation of $\prod_{\alpha \in I} \alpha$ is greater than the one of $\prod_{\beta \in \bar{I}} \beta$, except if all Frobenius eigenvalues have the same slopes. In this case the abelian variety would be isogenous to the forth power of the supersingular elliptic curve and hence all algebraic classes would be Lefschetz. 

(3) Having the results of Section 5 in mind, the last example that needs to be discussed is that of an abelian fourfold with a four dimensional space of exotic classes. By Lemma 5.13 this reduces to an abelian fourfold over $\mathbb{F}_q^2$ of the form $X \times E$, where $X$ is an abelian threefold and $E$ is a supersingular elliptic curve on which Frobenius acts as $q \cdot \text{id}$. Now the equations (5.4) and (5.5) imply that the existence of an exotic class on $X \times E$ is equivalent to the existence of an exotic class on $X^2$. There are infinitely many such threefolds $X$, they have been classified in [Zar15].

Proposition A.9. The standard conjecture of Hodge type holds true for Fermat’s cubic fourfold $X = \{x_0^3 + \cdots + x_5^3 = 0\} \subset \mathbb{P}^5$.

Proof. Let us first consider $X$ as variety over $\mathbb{C}$. By [Bea14, Proposition 11] its Hodge structure decomposes as $H^*_B(X, \mathbb{Q}) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \oplus \mathbb{Q}(-2)^{\oplus 21} \oplus V_B \oplus \mathbb{Q}(-3) \oplus \mathbb{Q}(-4)$ where $V_B$ is a $\mathbb{Q}$-Hodge structure of rank 2 and of type $(3, 1), (1, 3)$. As the Hodge conjecture is known for $X$ and its powers [Shi79], this decomposition holds true at the level of homological motives $M(X) = 1 \oplus 1(-1) \oplus 1(-2)^{\oplus 21} \oplus V \oplus 1(-3) \oplus 1(-4)$. This implies that the primitive part of the motive is of the form $h^{4,\text{prim}}(X) = 1(-2)^{\oplus 20} \oplus V$. Note that the decomposition is orthogonal with respect to the cup product as the types of the Hodge structures are different. Finally, as the motive of $X$ is finite dimensional [BLPT18, Lemma 5.2], this decomposition lifts to the level of Chow motives. (Alternatively, see Remark 5.2 [2].)

Let us now work over $\overline{\mathbb{F}}_p$. (This is enough for our purpose, thanks to Proposition 1.16.) The positivity of the cup product on algebraic classes on the factor $1(2)^{\oplus 20}$ is clear as all these classes come from characteristic zero, see Remark 1.13. We are reduced to the study of algebraic classes on the
two dimensional motive $V$. As the characteristic polynomial of Frobenius acting on $V$ is a rational polynomial of degree two, there are either zero or two rational solutions. In the first case the space of algebraic classes on $V$ is reduced to zero hence the standard conjecture of Hodge type holds trivially. In the second case the Fermat variety is supersingular and $V$ is spanned by algebraic cycles [SK79]. In this case the standard conjecture of Hodge type holds true via Theorem 6.1 (Note that there are infinitely many primes for which the non-trivial case occurs [SK79, Theorem 2.10]).

Remark A.10. Let us comment on applications and limits of Theorem 6.1.

(1) Theorem 6.1 cannot be applied to show the standard conjecture of Hodge type for abelian varieties of dimension at least five. Indeed, let $A$ be a simple abelian variety of dimension $g$ and let $M_I \subset H^g(A)$ be a factor as constructed in Proposition 4.6. By its very construction, the dimension of $M_I$ is at least $g/i$ as $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ acts transitively on $\Sigma$, see Notation 4.4. Hence the rank of $M_I$ will never be two (except possibly in middle degree).

(2) It seems likely that using Theorem 6.1 one can show the standard conjecture of Hodge type for some special varieties as we did in Proposition A.9 for Fermat’s cubic fourfold. On the other hand we do not know examples (other than abelian fourfolds) where this strategy applies for a whole family of varieties and we expect such examples to be rare. It is rather a miracle, based on the computations of Section 5, that for all abelian fourfolds only motives of rank two turn out to be significant.

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\[6\] For example the proof of Proposition A.9 cannot apply to all cubic fourfolds as the smallest $\mathbb{Q}$-Hodge structure containing the $H^{1,3}$ of a cubic fourfold has in general dimension greater than two.
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