Continuous Variable Entropic Uncertainty Relations in the Presence of Quantum Memory

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We generalize entropic uncertainty relations in the presence of quantum memory in terms of conditional von Neumann entropy [Nat. Phys. 6 (659), 2010] as well as conditional min- and max-entropy [Phys. Rev. Lett. 106 (110506), 2011] in two aspects. Namely, we consider measurements with continuous outcomes and allow for infinite-dimensional quantum memory. To achieve this, we introduce differential conditional von Neumann entropy and differential conditional min- and max-entropy for classical-quantum states on von Neumann algebras, and show that these entropies can be approximated by their discrete counterparts. As an illustration, we evaluate the uncertainty relations for position and momentum measurements, which has applications in continuous variable quantum cryptography and quantum information theory.

1. INTRODUCTION

The uncertainty principle expresses the fundamental quantum feature that measurements of two non-commuting observables necessarily lead to statistical ignorance of at least one of the outcomes [22, 25]. Entropic uncertainty relations establish a quantitative formulation of this principle. They were first studied for position and momentum operators by Hirschman [23], and subsequently improved by Beckner [4] as well as Bia lynicki-Birula and Mycielski [9]. Deutsch stated in [16] an entropic uncertainty relation for finite-dimensional observables which was tightened by Maassen and Uffink [31] to the inequality

\[ H(X) + H(Y) \geq \log \frac{1}{c}, \]

(1)

where \( H(X) \) and \( H(Y) \) are the Shannon entropies of the outcome distributions of non-degenerate measurements \( X \) and \( Y \) and \( c = \max_{i,j} |\langle x_i | y_j \rangle|^2 \) with \( |x_i \rangle \) and \( |y_j \rangle \) the eigenvectors of \( X \) and \( Y \), respectively. The inequality (1) was further generalized to measurements described by positive operator valued measures and different entropies (see [10, 60] and references therein).

Interestingly, there exists a deep connection between uncertainty relations and another fundamental quantum feature, entanglement. This is because uncertainty should not be treated as absolute, but with respect to the knowledge of an observer. In particular, if the observer has a quantum memory at hand, one obtains a subtle interplay between the observed uncertainty, and the entanglement between the measured system and the quantum memory. Quantitatively, this can be expressed by uncertainty relations in terms of quantum conditional entropies. In particular, for a bipartite quantum state \( \rho_{AB} \) and measurements as above, we have [5]

\[ H(X | B) + H(Y | B) \geq \log \frac{1}{c} + H(A | B). \]

(2)

Here, \( H(X | B) \) and \( H(Y | B) \) are the conditional von Neumann entropy of the measurement outcomes of \( X \) and \( Y \) given the information of the quantum memory \( B \), and \( H(A | B) \) is the conditional von Neumann entropy of the state \( \rho_{AB} \). The latter quantity measures the initial correlations between the measured system \( A \) and the quantum memory \( B \), and can be negative if \( A \) is entangled with \( B \). Furthermore, the monogamy property of entanglement also allows to elegantly state this extended uncertainty principle in the tripartite scenario: it holds for any tripartite quantum state as well.

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More precisely, \( H(X | B) = H(XB) - H(B) \) is the conditional von Neumann entropy of the post-measurement state \( \rho_{XB} = \sum_i \langle x_i | x_i \rangle_A \otimes I_B \rho_{AB} \langle x_i | x_i \rangle_A \otimes 1_B \).
It is remarkable that the constant $c$ is the same as in the relation (1) and most importantly independent of the state. Beside its fundamental interest, entropic uncertainty relations in the presence of quantum memory found various applications in quantum information theory, as for instance in quantum key distribution [54] or entanglement witnessing [20, 38]. For that purpose, the inequality (4) was generalized to other entropies [13], and in particular, to conditional min- and max-entropy [55], which are quantum generalizations of Rényi entropies of order $\infty$ and $1/2$, respectively [32]. The conditional min-entropy can for instance be used to quantize the extractable key rate in quantum key distribution protocols [10].

However, all of the previously mentioned results involving quantum memory assume quantum systems with finitely many degrees of freedom. In contrast to this, the first papers about the uncertainty principle discuss position and momentum measurements [22, 25], and thus, infinite-dimensional quantum systems. This problem was recently addressed by Frank and Lieb [18]. They discuss entropic uncertainty relations in terms of conditional von Neumann entropy, which also apply to continuous position-momentum distributions. Their analysis, however, is limited to a restricted definition of conditional von Neumann entropy that does not apply to all quantum states and quantum memories. The difficulties for defining conditional entropies in an infinite-dimensional setup were already noted by Kuznetsova [28].

In this work, we derive entropic uncertainty relations of the form (3) for infinite-dimensional quantum systems without restrictions on the measurements or the quantum memory. In previous work [6], we have shown such a relation in terms of conditional min- and max-entropy for measurements with a finite number of outcomes. Under certain conditions this can also be extended to conditional von Neumann entropy by means of the asymptotic quantum equipartition property [20, 53]. Here, we follow a different approach and introduce differential conditional von Neumann entropy, $h(X|B)$, and differential conditional min- and max-entropy, $h_{\min}(X|B)$ and $h_{\max}(X|B)$. The classical system $X$ is described by a measure space, and the quantum memory $B$ modeled by an observable algebra.

From a physics point of view, continuous classical systems may be thought of as being approximated by discrete systems in the limit of infinite precision. Thus, we might expect that operationally meaningful quantities are continuous in the transition to infinite precision. We make such a statement precise by proving that differential conditional entropies can be approximated by their discretized counterparts. Let $X$ be a classical system with outcome range being the real line, and $X_\alpha$ its restriction to a covering of $\mathbb{R}$ by half open intervals of length $\alpha$: \(^2\) then

$$h(X|B) = \lim_{\alpha \to 0} \left( H(X_\alpha|B) + \log \alpha \right) .$$

(4)

The uncertainty relation (3) for measurements with a continuous outcome range is then obtained by means of these approximation results from the finite case.

Finally, we apply our uncertainty relations to position and momentum measurements for continuous and discretized outcome ranges. The aforementioned approximation property can be seen as a coarse-graining of the outcome range with finer and finer intervals. The uncertainty relation for continuous outcomes reads for the differential conditional von Neumann entropy as

$$h(Q|B) + h(P|C) \geq \log 2\pi .$$

(5)

We further show that the uncertainty relation (5) is indeed sharp. Remarkably, the constant is different to the case without quantum memory where the optimal constant on the right hand side is given by $\log e\pi$ [4, 9]. \(^3\) We show a similar uncertainty relation for differential conditional min- and max-entropy, which generalizes the uncertainty relation for the differential $\alpha$-Rényi entropy pair $\alpha = 1/2$ and $\infty$ to quantum memory [8]. This relation was already proven to be sharp without quantum memory.

We consider further an uncertainty relation for finite precision position and momentum measurements, and thus, discrete outcomes. Such an inequality for conditional min- and max-entropy was recently applied to prove security of a continuous variable quantum key distribution protocol against coherent attacks including finite-size effects [21]. We then show that for conditional min- and max-entropy the uncertainty relations are sharp for any precision even without quantum memory.

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\(^2\) We mean that the random variables on $X_\alpha$ are given by functions which are constant on the elements of a given covering of $\mathbb{R}$ by half open intervals of length $\alpha$ (which do not overlap).

\(^3\) This bound was originally derived for pure states, but also holds for mixed states (see e.g. [5]).
The paper is structured as follows. In Section 2 we review the algebraic formalism to describe quantum and classical systems. In Section 3 we discuss differential conditional entropies. The entropic uncertainty relations in the presence of quantum memory are proven in Section 4. We then discuss the special case of position and momentum measurements (Section 5), and end with a discussion and an outlook (Section 6).

2. PRELIMINARIES

The aim is to derive entropic uncertainty relations in the presence of quantum memory for measurements with a continuum of outcomes. Such measurements are usually modeled on an infinite-dimensional Hilbert space $\mathcal{H}$, as for example the position and momentum observables which are defined on the Hilbert space of square integrable functions. However, more general settings are certainly of interest and may be modeled using the algebraic language of operator algebras, or more precisely von Neumann algebras. It is natural to allow the same generality for the system modeling the quantum memory. For an introduction to von Neumann algebras we refer to [11], and a more detailed discussion of the following description of quantum mechanics by means of von Neumann algebras can be found in [6]. A reader not familiar with the framework of von Neumann algebras can think of a von Neumann algebra $\mathcal{M}$ throughout the paper as a full set of bounded linear operators on a possibly infinite-dimensional Hilbert space $\mathcal{B}(\mathcal{H})$.

2.1. Algebraic Approach to Physical Systems

Opposite to the standard description of quantum mechanics where the structure of the system is related to a Hilbert space, the basic objects in the algebraic approach are the observables or, respectively, the algebra generated by the quantum mechanical expectation values, that is, the algebra of bounded operators on some Hilbert space $\mathcal{H}$. The algebraic approach has for instance the benefit that one can treat classical and quantum systems on the same footing. We start with specifying general quantum systems.

Quantum Systems. We associate to every quantum system a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. The set of linear, normal (i.e. $\sigma$-weakly continuous), and positive functionals on $\mathcal{M}$ is denoted by $\mathcal{P}(\mathcal{M})$. The set of sub-normalized states $S_\leq(\mathcal{M})$ is defined as the elements in $\mathcal{P}(\mathcal{M})$ satisfying $\omega(\mathbf{1}) \leq 1$, where $\mathbf{1}$ denotes the identity element in $\mathcal{M}$. Elements $\omega \in S_\leq(\mathcal{M})$ with $\omega(\mathbf{1}) = 1$ are called (normalized) states, and the corresponding set is denoted by $S(\mathcal{M})$. If $\mathcal{M} \cong \mathcal{B}(\mathcal{H})$, then there exists a one to one correspondence between states on $\mathcal{M}$ and density matrices on $\mathcal{H}$. We then have for every $\omega \in S(\mathcal{M})$ a unique positive trace-one operator $\rho$ on $\mathcal{H}$, such that for all $E \in \mathcal{M}$, $\omega(E) = \text{tr}[\rho E]$. We denote the set of density operators on $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$.

A multipartite system is a composite of different local subsystems $A, B, C$ associated with mutually commuting von Neumann algebras $\mathcal{M}_A, \mathcal{M}_B, \mathcal{M}_C$ acting on the same Hilbert space $\mathcal{H}$. The combined system is denoted by $\mathcal{M}_{ABC}$ and is given by the von Neumann algebra generated by the individual subsystems, that is, $\mathcal{M}_{ABC} = \mathcal{M}_A \vee \mathcal{M}_B \vee \mathcal{M}_C$ is the $\sigma$-weak closure of the algebra $\{abc : a \in \mathcal{M}_A, b \in \mathcal{M}_B, c \in \mathcal{M}_C\}$. If it is not clear from context, we label the correspondence of states, operators and algebras to different subsystems by lower indexes.

By the Gelfand-Naimark-Segal (GNS) construction every $\mathcal{M}$ admits a purification [11 Chapter 2.3.3], that is a triple $(\mathcal{K}, \pi, \xi_\omega)$, $\mathcal{K}$ being a Hilbert space, $\pi$ a representation of $\mathcal{M}$ on $\mathcal{K}$, and a sub-normalized vector $\xi_\omega \in \mathcal{H}$ such that $\omega(x) = \langle \xi_\omega | \pi(x) \xi_\omega \rangle$ for all $x \in \mathcal{M}$. We often speak of the commutant $\pi(\mathcal{M})'$ of $\pi$ on $\mathcal{K}$ as the purifying system.

The space $\mathcal{P}(\mathcal{M})$ can be equipped with two different, albeit equivalent notions of distance [12] [56]. The first one is the usual norm induced from $\mathcal{M}$, that is defined for $\omega \in \mathcal{P}(\mathcal{M})$ as

$$||\omega|| = \sup_{E \in \mathcal{M}, ||E|| \leq 1} |\omega(E)|^2.$$  

(6)

For $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and density matrices this corresponds to the usual trace-distance. The second one is called the fidelity

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4 This is for instance necessary if one considers a state with an infinite number of Bosonic modes.

5 The $\sigma$-weak topology on $\mathcal{B}(\mathcal{H})$ is the locally convex topology induced by the semi-norms $A \mapsto |\text{tr}(\pi A)|$ for trace-class operators $\pi \in \mathcal{B}(\mathcal{H})$, see [11 Chapter 2.4.1].

6 If they act on different Hilbert spaces, we just consider their action on the tensor product of the Hilbert spaces.
and was introduced by Uhlmann [56]. The fidelity for $\omega, \sigma \in S_\infty(\mathcal{M})$ is defined as
\begin{equation}
F(\omega, \sigma) = \sup \left| \langle \xi_\omega | \xi_\sigma \rangle \right|^2 ,
\end{equation}
where the supremum runs over all purifications of $\omega$ and $\sigma$ being defined with respect to the same Hilbert space.\(^7\)

**Classical Systems.** A classical system is specified by the property that all possible observables commute, and can thus be described by an abelian von Neumann algebra. This perspective allows to use the same definitions for states on classical systems as defined for quantum systems in the previous paragraph. Since classical systems will play a major role in the sequel, we discuss them in more detail.

For the sake of illustration, we start with countable classical systems denoted by $X$.\(^8\) The corresponding von Neumann algebra is $\ell^\infty(X)$, that is, the set of functions from $X$ to $\mathbb{C}$ equipped with the supremum norm. Here, one can think of $\varepsilon_x = (\delta_{x,k})_k$ as the measurement operator corresponding to the outcome $x \in X$. A classical state is then a normalized positive functional $\omega_X$ on $\ell^\infty(X)$, which can be identified with a probability distribution on $X$, that is, $\omega_X \in \ell^1(X)$. It is often convenient to embed the classical system $\ell^\infty(X)$ into the quantum system with Hilbert space dimension $X$ as the algebra of diagonal matrices with respect to a fixed basis $\{|x\rangle\}_{x \in X}$. A classical state $\omega_X$ can then be represented by a density operator
\begin{equation}
\rho_{\omega_X} = \sum_x \omega_X(x) |x\rangle \langle x| ,
\end{equation}
such that the probability distribution can be identified with the eigenvalues of the corresponding density operator.

Let us now go a step further and consider classical systems with continuous degrees of freedom. In order to define such systems properly, we start with $(X, \Sigma, \mu)$ a measure space with $\sigma$-algebra $\Sigma$, and measure $\mu$. In the following, we will always assume that the measure space is $\sigma$-finite. The von Neumann algebra of the system is given by the essentially bounded functions denoted by $L^\infty(X)$. A classical state on $X$ is defined as a normalized positive and normal functional on $L^\infty(X)$, and may be identified with an element of the integrable complex functions $L^1(X)$, which is almost everywhere non-negative and satisfies
\begin{equation}
\int_X \omega_X(x) d\mu(x) = 1 .
\end{equation}
Such functions in $L^1(X)$ are also called probability distributions on $X$. The most prominent example of a continuous classical system is $X = \mathbb{R}$ with the usual Lebesgue measure. This describes for instance the outcome of a position or momentum measurement (as discussed in Section 5).

Note that the case of a discrete classical system is obtained if the measure space $X$ is discrete, and equipped with the equally weighted discrete measure $\mu(I) = \sum_{x \in I} 1$ for $I \subset X$. In the discrete case, (8) defines a representation of a classical state as a diagonal matrix of trace-one on the Hilbert space with dimension equal to the classical degrees of freedom. However, in the case of continuous variables this representation is not possible if we demand that the image is a valid density operator. This is easily seen from the fact that every density operator is by definition of trace class, and hence, has discrete spectrum.

**Classical-Quantum Systems.** Let us take a closer look at bipartite systems consisting of a classical part $X$ and a quantum part $B$. For a countable classical part $X$, the combined system is described by the von Neumann algebra $\mathcal{M}_X$ [Chapter 6.3]\(^9\)
\begin{equation}
\mathcal{M}_{XB} = \ell^\infty(X) \lor \mathcal{M}_B \cong \ell^\infty(X) \otimes \mathcal{M}_B
\end{equation}
\begin{equation}
\cong \ell^\infty(X, \mathcal{M}_B) = \{ f : X \to \mathcal{M}_B : \sup_x \| f(x) \| \leq \infty \} .
\end{equation}
A state on $\mathcal{M}_{XB}$ is called a classical-quantum state and can be written as $\omega_{XB} = (\omega_B^x)$ with $\omega_B^x \in S_\infty(\mathcal{M}_B)$ and $\sum_x \omega_B^x(1) = 1$. If the quantum system $B$ is given by the set of all bounded linear operators on a Hilbert space $\mathcal{H}_B$, we can represent $\omega_{XB}$ uniquely by the density operator
\begin{equation}
\rho_{\omega_{XB}} = \sum_x |x\rangle \langle x| \otimes \rho_{\omega_B^x} .
\end{equation}

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\(^7\) This is a non-empty set since there exists a Hilbert space $\mathcal{K}$ and a representation $\pi$ of $\mathcal{M}$ on $\mathcal{K}$, called standard form, such that every state on $\mathcal{M}$ has a purification on $\mathcal{K}$ [50] Chapter 9).

\(^8\) In the following, we denote quantum systems by indexes $A, B, C$ and classical systems by indexes $X, Y, Z$. In the classical case we use $X, Y, Z$ to specify the subsystem as well as the range of the classical variable.

\(^9\) From now on, $\otimes$ denotes either the von Neumann tensor product or the usual Hilbert space tensor product (depending on the context).
It is now straightforward to generalize the above introduced classical-quantum systems from countable to continuous classical systems. The combined system is then described by the von Neumann algebra \[ \mathcal{M}_{XB} = L^\infty(X) \vee \mathcal{M}_B \cong L^\infty(X) \otimes \mathcal{M}_B \]

(13)  

\[ \cong L^\infty(X, \mathcal{M}_B) , \]

(14)

where \( L^\infty(X, \mathcal{M}_B) \) denotes the space of essentially bounded functions with values in \( \mathcal{M}_B \). The normal, positive functionals on \( \mathcal{M}_{XB} \) are given by elements in \( L^1(X, \mathcal{P}(\mathcal{M}_B)) \), and states can be identified with integrable functions \( \omega_{XB} \) on \( X \) with values in \( \mathcal{P}(\mathcal{M}_B) \) satisfying the normalization condition

\[ \int_X \omega^\circ_{XB}(1) d\mu(x) = 1 . \]

(15)

In analogy to the discrete case, we write the argument of the map \( \omega_{XB} \) as an upper index. The evaluation of \( \omega_{XB} \) on an element \( E_{XB} \in L^\infty(X, \mathcal{M}_B) \) is computed by \( \omega_{XB}(E_{XB}) = \int_X \omega^\circ_{XB}(E_B(x)) d\mu(x) \). For further details we refer to [40] Chapter 4.6.4.7.

### 2.2. Channels, Measurements, and Post-Measurement States

We call an evolution of a system a channel. As we work with von Neumann algebras it is convenient to define channels as maps on the observable algebra, which is also called the Heisenberg picture. A channel from system \( A \) to system \( B \) described by von Neumann algebras \( \mathcal{M}_A \) and \( \mathcal{M}_B \), respectively, is given by a linear, normal, completely positive, and unital map \( \mathcal{E} : \mathcal{M}_B \to \mathcal{M}_A \).\(^{10}\) Note that \( \mathcal{M}_A \) and \( \mathcal{M}_B \) can be either a classical or a quantum system. If both systems are quantum (classical), we call the channel a quantum (classical) channel.

A measurement with outcome range \( X \) is a channel which maps \( L^\infty(X) \) to a von Neumann algebra \( \mathcal{M}_A \). Its predual then maps states of the quantum system \( A \) to states of the classical system \( X \). We denote the set of all measurements \( \mathcal{E} : L^\infty(X) \to \mathcal{M}_A \) by \( \text{Meas}(X, \mathcal{M}_A) \). If \( X \) is countable, we can identify a measurement \( \mathcal{E} : L^\infty(X) \to \mathcal{M}_A \) by a collection of positive operators \( E_x = \mathcal{E}(e_x) \) \((x \in X)\) satisfying \( \sum_x E_x = 1 \) (we denote by \( e_x \) the sequence with 1 at position \( x \) and 0 elsewhere). More generally, given a \( \sigma \)-finite measure space \((X, \Sigma, \mu)\) and the associated algebra \( L^\infty(X) \), the mapping \( \mathcal{O} \to \chi_{\mathcal{O}} \to \mathcal{E}(\chi_{\mathcal{O}}) \), for \( \mathcal{O} \in \Sigma \), \( \chi_{\mathcal{O}} \) being its indicator function, defines a measure on \( X \) with values in the positive operators of \( \mathcal{M}_A \). Note that therefore our definition coincides with usual definition of a measurement as a positive operator valued measure [15] Chapter 3.1. We define the post-measurement state obtained when measuring the state \( \omega_A \in \mathcal{S}(\mathcal{M}_A) \) with some \( E_X \in \text{Meas}(X, \mathcal{M}_A) \) by the concatenation \( \omega_X = \omega_A \circ E_X \), that is, \( \omega_X(f) = \omega_A(E_X(f)) \) for \( f \in L^\infty(X) \). Since \( \omega_A \) and \( E_X \) are normal, so is \( \omega_X \), such that \( \omega_X \) is an element of the predual of \( L^\infty(X) \), which is \( L^1(X) \). Hence, the obtained post measurement state is a proper classical state and can be represented by a probability distribution on \( X \).

In the following, we are particularly interested in the situation where we start with a bipartite quantum system \( \mathcal{M}_{AB} \), and measure the \( A \)-system with some \( E_X \in \text{Meas}(X, \mathcal{M}_A) \). The post-measurement state is then given by \( \omega_{XB} = \omega_{AB} \circ E_X \). Similarly as in the case of a trivial \( B \)-system, one can show that the state \( \omega_{XB} \) is a proper classical-quantum state on \( L^\infty(X) \otimes \mathcal{M}_B \) as introduced in the previous paragraph.

### 2.3. Discretization of Continuous Classical Systems

Let us consider a classical system \( L^\infty(X) \) with \((X, \Sigma, \mu)\) a \( \sigma \)-finite measure space, where \( X \) is also equipped with a topology. The aim is to introduce a discretization of \( X \) into countable measurable sets along which we can show convergence of differential conditional von Neumann as well as differential conditional min- and max-entropy (see Section 3).

We call a countable set \( \mathcal{P} = \{I_k\}_{k \in A} \) (\( A \) any countable index set) of measurable subsets \( I_k \in \Sigma \) a partition of \( X \) if \( X = \bigcup_k I_k \), \( \mu(I_k \cap I_l) = \delta_{kl} \cdot \mu(I_k) \), \( \mu(I_k) < \infty \), and the closure \( \bar{I}_k \) is compact for all \( k \in \Lambda \). If \( \mu(I_k) = \mu(I_l) \) for all \( k, l \in A \), we call \( \mathcal{P} \) a balanced partition, and denote \( \mu(\mathcal{P}) = \mu(I_k) \). Note that the property \( \mu(I_k \cap I_l) = \delta_{kl} \cdot \mu(I_k) \) implies that the step functions associated to a fixed partition form a subalgebra of all essentially bounded functions

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\(^{10}\) A linear map \( \Phi : \mathcal{N} \to \mathcal{M} \) between two von Neumann algebras is called completely positive, if the map \( (d_n \otimes \Phi) : \mathcal{M}_n \otimes \mathcal{N} \to \mathcal{M}_n \otimes \mathcal{M} \) is positive for all \( n \in \mathbb{N} \). The map is called unital, if \( \Phi(\mathbb{I}_n) = \mathbb{I}_M \).
on $X$. If for two partitions $P_1$, $P_2$ all sets of $P_1$ are subsets of elements in $P_2$, we say that $P_2$ is finer than $P_1$ and write $P_2 \leq P_1$. A family of partitions $\{P_\alpha\}_{\alpha \in \Delta}$ with $\Delta$ a discrete index set approaching zero such that each $P_\alpha$ is balanced, $P_\alpha \leq P_{\alpha'}$ for $\alpha \leq \alpha'$, $\mu(P_\alpha) = \alpha$, and $\bigcup_\alpha P_\alpha$ generates $\Sigma$, is called an ordered dense sequence of balanced partitions. For simplicity, we usually omit the index set $\Delta$.

Definition 1. We call an ordered dense sequence of balanced partitions $\{P_\alpha\}$ of a measure space $(X, \Sigma, \mu)$ a coarse graining of $X$. A quadruple $(X, \Sigma, \mu, \{P_\alpha\})$ is called a coarse grained measure space if the measure space is $\sigma$-finite and $\{P_\alpha\}$ is a coarse graining of $X$.

Note that not every $\sigma$-finite measure space admits a coarse graining in the sense of the above definition. As a simple example, we can take a discrete measure space with the counting measure where each partition consists of sets with measure at least one. In the case that $X = \mathbb{R}$, $\Sigma$ the Borel $\sigma$-algebra, and $\mu$ the Lebesgue measure, a coarse graining can be easily constructed. For a positive real number $\alpha$, let us take a partition $P_\alpha$ of $\mathbb{R}$ into intervals $I_k = [k\alpha, (k+1)\alpha]$, $k \in \mathbb{Z}$, with $\mu(P_\alpha) = \alpha$. Choosing for $\alpha$ the sequence $\frac{1}{n}$, $n \in \mathbb{N}$ then gives rise to a coarse graining.

Remark 2. Every Lebesgue measurable subset $X \subset \mathbb{R}$ equipped with the Lebesgue measure restricted to $X$ admits a coarse graining.

For a classical-quantum system $M_{XB} = L^\infty(X) \otimes M_B$, and a partition $P = \{I_k\}_{k \in \Lambda}$ of $X$, we can define the discretized state $\omega_{XB} \in S(\ell^\infty(\Lambda) \otimes M_B)$ of $\omega_X \in S(M_X)$ by

$$\omega_{XB}\left(\langle b_k \rangle\right) = \sum_{k \in \Lambda} \int_{I_k} \omega_B^\tau(b_k) d\mu(x) = \sum_{k \in \Lambda} \omega_B^{P,k}(b_k),$$

where $\langle b_k \rangle \in \ell^\infty(\Lambda) \otimes M_B$. The new classical system induced by the partition is denoted by $X_P$ and thus, $X_P \cong \Lambda$. In a similar way we define the discretization of a measurement $E \in \text{Meas}(X, M_\Lambda)$ with respect to a partition $P = \{I_k\}_{k \in \Lambda}$ as the element $E^P \in \text{Meas}(X_P, M_\Lambda)$ determined by the collection of positive operators

$$E_k^P = E(\chi_{I_k}),$$

where $\chi_{I_k}$ denotes the indicator function of $I_k$. Note that the concept of a discretized measurement and a discretized state are compatible in the sense that the post-measurement state obtained from the discretized measurement $E^P$ is equal to the one which is obtained where one first measures $E$ and then discretizes the state. Hence, we have that $\omega_{XB} = \omega_{AB} \circ E^P$ if $\omega_{XB} = \omega_{AB} \circ E$.

3. CONDITIONAL ENTROPY MEASURES

Conditional entropy measures are used to quantify the uncertainty of a system when having partial information about it, as for instance in the form of access to an additional correlated system. Interested in uncertainties of measurement outcomes in the presence of quantum memory, we focus on the particular situation where the system is classical, possibly continuous, and the correlated system is modeled by a von Neumann algebra. For that, we define and discuss (differential) conditional min- and max-entropy as well as (differential) conditional von Neumann entropy for classical quantum states (see Section 2.1). To the best of our knowledge, the definitions given in the following have not appeared in the literature in this generality before.

Von Neumann entropy based measures are important in physics as well as asymptotic information theory where they are also referred to as Shannon entropy in the purely classical case. The conditional min- and max-entropy appear in the context of non-asymptotic information theory (see, e.g. [14, 52] and references therein). The most important application of min- and max-entropy is in quantum key distribution, where they can be used to quantify the information-theoretic secure key rate [10]. In this context, entropic uncertainty relations in the presence of quantum memory provide a powerful tool for the security analysis (see, e.g. [21, 54]).

The main results of the following sections are discrete approximation theorems for the differential conditional von Neumann entropy and the differential conditional min- and max-entropy along finer and finer coarse grainings of the continuous outcome range (Proposition 4 and Proposition 5).

3.1. Conditional Min- and Max-Entropy

Conditional min- and max-entropy have already been investigated on infinite-dimensional Hilbert spaces [20] and von Neumann algebras [5, 19]. For finite classical systems $X$, the conditional min- and max-entropy for a state $\omega_{XB}$
on the bipartite system $\mathcal{M}_{\mathcal{X} \mathcal{B}} = L^\infty(\mathcal{X}) \otimes \mathcal{M}_B$ with $\mathcal{M}_B$ a von Neumann algebra are given by \cite{3}

$$
H_{\min}(X|B)_\omega = - \log \sup \left\{ \sum_x \omega_B^x(E_B^x) : E_B^x \in \mathcal{M}_B, E_B^x \geq 0, \sum_x E_B^x = 1_B \right\} \quad (18)
$$

$$
H_{\max}(X|B)_\omega = 2 \log \sup \left\{ \sum_x \sqrt{F(\omega_B^x, \sigma_B)} : \sigma_B \in \mathcal{S}(\mathcal{M}_B) \right\}, \quad (19)
$$

where $F(\cdot, \cdot)$ denotes the fidelity \cite{7}. These quantities admit natural extensions to classical-quantum systems involving continuous classical variables.

**Definition 3.** Let $\mathcal{M}_{\mathcal{X} \mathcal{B}} = L^\infty(\mathcal{X}) \otimes \mathcal{M}_B$ with $(\mathcal{X}, \Sigma, \mu)$ a $\sigma$-finite measure space, $\mathcal{M}_B$ a von Neumann algebra, and $\omega_{\mathcal{X} \mathcal{B}} \in \mathcal{S}_\mathcal{L}(\mathcal{M}_{\mathcal{X} \mathcal{B}})$. Then, the conditional min-entropy of $X$ given $B$ is defined as

$$
h_{\min}(X|B)_\omega = - \log \sup \left\{ \int \omega_B^x(E_B^x) \, d\mu(x) : E \in L^\infty(\mathcal{X}) \otimes \mathcal{M}_B, E \geq 0, \int E_B^x \, d\mu(x) \leq 1_B \right\} . \quad (20)
$$

Furthermore, the conditional max-entropy of $X$ given $B$ is defined as

$$
h_{\max}(X|B)_\omega = 2 \log \sup \left\{ \int \sqrt{F(\omega_B^x, \sigma_B)} \, d\mu(x) : \sigma_B \in \mathcal{S}(\mathcal{M}_B) \right\}. \quad (21)
$$

These quantities are well defined since the integrands are measurable and positive. In the following we are interested in two different cases, namely, the one where the measure space is discrete and the one where the measure space is coarse grained. In the latter case we refer to $h_{\min}(X|B)_\omega$ and $h_{\max}(X|B)_\omega$ as the differential conditional min- and max-entropy since the variable $X$ takes values in a continuous range. An important example of this situation is given for $X = \mathbb{R}$. Then, for trivial quantum memory $\mathcal{M}_B = \mathbb{C}$, the differential min- and max-entropy correspond to the differential Rényi entropy of order $\infty$ and $1/2$, respectively,

$$
h_{\min}(X)_\omega = - \log \|\omega_X\|_\infty \quad (22)
$$

$$
h_{\max}(X)_\omega = 2 \log \int \sqrt{\omega^x} \, dx = \log \|\omega_X\|_{\frac{1}{2}}, \quad (23)
$$

where $\| \cdot \|_p$ denotes the usual $p$-norm on $L^p(\mathbb{R})$. We note that like any differential entropy, the differential conditional min- and max-entropy can get negative. In particular,

$$
-\infty \leq h_{\min}(X)_\omega < \infty, \quad -\infty < h_{\max}(X)_\omega \leq \infty . \quad (24)
$$

and the same bounds also hold for the conditional versions in \cite{18} and \cite{19}.

In the case where the measure space $X$ is discrete and equipped with the counting measure, we retrieve the usual definitions as in \cite{18} and \cite{19}, with sums now involving infinitely many terms. We therefore use uppercase letters for the entropies, $H_{\min}(X|B)_\omega$ and $H_{\max}(X|B)_\omega$, if $X$ is discrete. For $X$ having infinite cardinality, we can assume that $X \cong \mathbb{N}$ and since all the terms inside the sums are positive, the conditional min- and max-entropy can be obtained from finite sum approximations

$$
H_{\min}(X|B)_\omega = - \log \sup \sup \left\{ \sum_{x=1}^m \omega_B^x(E_B^x) : E_B^x \in \mathcal{M}_B, E_B^x \geq 0, \sum_{x=1}^m E_B^x \leq 1_B \right\} \quad (25)
$$

$$
H_{\max}(X|B)_\omega = 2 \log \sup \sup \left\{ \sum_{x=1}^m \sqrt{F(\omega_B^x, \sigma_B)} : \sigma_B \in \mathcal{S}(\mathcal{M}_B) \right\} . \quad (26)
$$

The following proposition shows that the differential conditional min- and max-entropy for coarse grained measure spaces can be interpreted as regularized quantities.

**Proposition 4.** Let $\mathcal{M}_{\mathcal{X} \mathcal{B}} = L^\infty(\mathcal{X}) \otimes \mathcal{M}_B$ with $\mathcal{M}_B$ a von Neumann algebra and $(\mathcal{X}, \Sigma, \mu, \{P_\alpha\})$ a coarse grained measure space. Then, we have that for $\omega_{\mathcal{X} \mathcal{B}} \in \mathcal{S}(\mathcal{M}_{\mathcal{X} \mathcal{B}})$,

$$
h_{\min}(X|B)_\omega = \lim_{\alpha \to 0} \left( H_{\min}(X_{P_\alpha}|B)_\omega + \log \alpha \right), \quad (27)
$$
There exists \( \omega_{X|P_n} \) defined as in [16]. Furthermore, if there exists an \( \alpha_0 > 0 \) such that \( H_{\text{max}}(X_{P_{\alpha_0}}) < \infty \), then we have that
\[
\lim_{\alpha \to 0} \left( H_{\text{max}}(X_{P_{\alpha}}) + \log \alpha \right) = \frac{h_{\text{max}}(X|B) + \alpha}{\alpha}.
\]  
(28)

A similar result under additional conditions and with different techniques is derived in the thesis of one of the authors [19]. From the following proof of Proposition [4] it is further evident that the limits for \( \alpha \to 0 \) in [27] and [28] can be replaced by an infimum over \( \alpha \).

Proof. We start with the differential conditional min-entropy. Let us fix an \( \alpha \) and consider \( P_{\alpha} = \{ P^\alpha_l \}_{l \in \Lambda} \) where we can assume that \( \Lambda = \{ 1, 2, 3, \ldots \} \subset \mathbb{N} \). For \( k \in \Lambda \), we then define \( C_k = \bigcup_{l=1}^k I^\alpha_l \) which is compact according to the definition of a coarse graining. We then write the differential min-entropy as
\[
h_{\text{min}}(X|B) = \log \sup_k \left\{ \omega_{XB}(E) : E \in L^\infty(C_k) \otimes \mathcal{M}_B, E \geq 0, \int E_B d\mu(x) \leq 1_B \right\}.
\]  
(29)

Since \( C_k \) is compact, the set of step functions \( T^k = \bigcup_{k \in \alpha} T^k_\alpha \) with \( T^k_\alpha \) the step functions corresponding to partitions \( P_\alpha \) defined as the restriction of \( P \) to \( C_k \) is \( \sigma \)-weakly dense in \( L^\infty(C_k) \). Because \( \omega_{XB} \) is \( \sigma \)-weakly continuous we get that
\[
h_{\text{min}}(X|B) = \log \sup_k \left\{ \omega_{XB}(E) : E \in T^k_\alpha \otimes \mathcal{M}_B, E \geq 0, \int E_B d\mu(x) \leq 1_B \right\},
\]  
(30)

where we used that a \( \{ P_\alpha \} \) is an ordered family of partitions. By exchanging the two suprema, we find that the right hand side of (29) reduces to the supremum of \( H_{\text{min}}(X_{P_{\alpha}}, B) + \log \alpha \) over \( \alpha \), with \( \omega_{X|P_{\alpha}} \) defined as in [16]. Finally, we note that since the expression on the right hand side of (30) is monotonic in \( \alpha \), the supremum over \( \alpha \) in (30) can be exchanged by the limit \( \alpha \to 0 \).

To show the approximation of the differential conditional max-entropy, we start by using an expression for the fidelity [11]
\[
h_{\text{max}}(X|B) = 2 \log \sup_{\sigma_B \in S(\mathcal{M}_B)} \left\{ \int \sup_{\pi \in (\mathcal{M}_B)'} \frac{\|\pi \|}{\|\pi \|} \text{tr}(\sigma_B \otimes \rho|U(x)) dx : \pi_U \in S(\mathcal{M}_B') \right\},
\]  
(31)

where \( \pi \) is some fixed representation of \( \mathcal{M}_B \) in which all \( \omega_{\beta} \) and \( \sigma_B \) admit vector states \( |\xi^\pi_\beta \rangle \), \( |\xi_\sigma \rangle \), respectively, and the supremum is taken over all elements \( U(x) \in \pi(\mathcal{M}_B)' \) with \( \|U(x)\| \leq 1 \). We note that we can always choose \( U(x) \) such that \( \langle \xi^\pi_\beta|U(x)|\xi_\sigma \rangle \) is positive. It follows by the \( \sigma \)-finiteness of the measure space together with the theorem of monotone convergence, that we can find a sequence of sets \( X^n \) all having finite measure, and
\[
h_{\text{max}}(X|B) = 2 \log \sup_{\sigma_B \in S(\mathcal{M}_B)} \lim_{n \to \infty} \sup_{U(x) \in \pi(\mathcal{M}_B)} \int_{X^n} \langle \xi^\pi_\beta|U(x)|\xi_\sigma \rangle dx.
\]  
(32)

For later reasons we note that the sequence \( X^n \) can be chosen such that it is compatible with the partitions in the sense that for every \( n \) the restriction of \( P_\alpha \) onto \( X^n \) forms again a balanced partition with measure \( \alpha \). It then follows from disintegration theory of von Neumann algebras [49, Chapter IV.7] that the expression involving the second supremum and the integral can again be recognized as a fidelity, more precisely, as the square root of \( F(\omega_{X^n|B}, \mu_{X^n} \otimes \sigma_B) \), where \( \omega_{X^n|B} \) is the state restricted to the subalgebra \( L^\infty(X^n) \otimes \mathcal{M}_B \subseteq L^\infty(X) \otimes \mathcal{M}_B \), and \( \mu_{X^n} \) is the Lebesgue measure restricted to the set \( X^n \). Because \( X^n \) of finite measure, \( \mu_{X^n} \) can be identified as a positive functional on \( L^\infty(X^n) \otimes \mathcal{M}_B \). We now employ similar ideas to the min-entropy case. The fidelity between the two positive forms \( \omega_{X^n|B} \) and \( \mu_{X^n} \otimes \sigma_B \) can be approximated by evaluating [2]
\[
F(\omega_{X^n|B}, \mu_{X^n} \otimes \sigma_B) = \inf \left\{ \sum_j \omega_{X^n|B}(e_j) \mu_{X^n} \otimes \sigma_B(e_j) : \right\},
\]  
(33)

11 There exists \( \omega_X \in L^1(X) \) with \( h_{\text{max}}(X) < \infty \) but \( h_{\text{max}}(X_{P_{\alpha}}) = \infty \) for all \( \alpha > 0 \). As an example, take \( X = \mathbb{R} \) and \( \omega_X \) to be the normalization of the function which is equal to 1 for \( x \in [k, k+1-k^2] \), \( k \in \mathbb{N} \), and 0 else. But conversely, \( h_{\text{max}}(X_{P_{\alpha}}) < \infty \) implies that \( h_{\text{max}}(X) < \infty \) since the relation \( h_{\text{max}}(X|B) < \infty \) holds for all \( \alpha > 0 \) and \( \omega_{XB} \in S(\mathcal{M}_B) \).

12 One can take for instance \( X^n \) to be generated by finite increasing unions of the sets in a partition \( P_{\alpha_0} \) for a fixed \( \alpha_0 \). Then for all \( \alpha \leq \alpha_0 \) the condition is satisfied.
where \( \{ e_j \} \subset L^\infty(X^n) \otimes \mathcal{M}_B \) are finitely many orthogonal projections summing up to the identity. Using the same reasoning as for the conditional min-entropy, we can restrict this infimum to finite sets of orthogonal projections in \( \mathcal{T} \otimes \mathcal{M}_B \). Since any such projection is of the form \( \chi_{I^j} \otimes P^j_B \), for a projection \( P^j_B \in \mathcal{M}_B \), we find

\[
F(\omega_{X^nB}, \mu_{X^n} \otimes \sigma_B) = \inf_{\alpha > 0} \int F(\omega_{X^n_{P^j_B}B}, \mu_{X^n_{P^j_B}} \otimes \sigma_B) = \lim_{\alpha \to 0} F(\omega_{X^n_{P^j_B}B}, \mu_{X^n_{P^j_B}} \otimes \sigma_B),
\]

where the restricted states are defined as in \((16)\). Note that the infimum can be replaced by the limit since the family \( \{ P^j_B \} \) is ordered and the fidelity is monotonic under restrictions \[1\]. This leads to

\[
h_{\max}(X|B)_\omega = 2 \log \sup_{\sigma_B \in \mathcal{S}(M_B)} \lim_{n \to \infty} \lim_{\alpha \to 0} \sqrt{F(\omega_{X^n_{P^j_B}B}, \mu_{X^n_{P^j_B}} \otimes \sigma_B)},
\]

and in order to proceed, we have to interchange the limits. For this, we define

\[
f_n(\sigma, \alpha) = \sqrt{F(\omega_{X^n_{P^j_B}B}, \mu_{X^n_{P^j_B}} \otimes \sigma_B)} = \sum_{k \in \Lambda(\alpha,n)} \sqrt{\alpha_0 \cdot F(\omega_{\alpha_0^k}^n \otimes \sigma_B)},
\]

where we have used that \( \mu_{X^n} \) restricted to \( \mathcal{N}_\alpha \) is just the counting measure multiplied by \( \alpha \). Since \( f_n(\sigma, \alpha) \) is monotonously increasing in \( \alpha \), there exists by assumption an \( \alpha_0 \) such that

\[
f_n(\sigma, \alpha) \leq \sum_{k \in \Lambda(\alpha_0,n)} \sqrt{\alpha_0 \cdot F(\omega_{\alpha_0^k}^n \otimes \sigma_B)} \leq \sum_{k \in \Lambda(\alpha_0,n)} \sqrt{\alpha_0 \cdot \omega_{\alpha_0^k}^n (\mathbb{I})}
\]

is finite in the limit \( n \to \infty \). It follows by the Weierstrass’ uniform convergence theorem that the sequence \( f_n(\sigma, \alpha) \) converges uniformly in \( \sigma \) and \( \alpha \) to the limiting function \( f(\sigma, \alpha) = \lim_{n \to \infty} f_n(\sigma, \alpha) \). Hence, the limits in \((35)\) can be interchanged, and we arrive at

\[
h_{\max}(X|B)_\omega = 2 \log \sup_{\sigma_B \in \mathcal{S}(M_B)} \lim_{\alpha \to 0} f(\sigma, \alpha) = 2 \log \sup_{\sigma_B \in \mathcal{S}(M_B)} \inf_{\alpha > 0} f(\sigma, \alpha).
\]

As the last step, we need to interchange the supremum with the infimum. For this, we extend \( f(\sigma, 0) = \int \sqrt{F(\omega_B^\sigma, \sigma_B)} d\mu(x) \) and get

\[
h_{\max}(X|B)_\omega = 2 \log \sup_{\sigma_B \in \mathcal{S}(M_B)} \inf_{\alpha \in [0, \alpha_0]} f(\sigma, \alpha). \tag{39}
\]

Since \( f_n(\sigma, \alpha) \) converges uniformly in \( \sigma \) and \( \alpha \), and \( f(\sigma, \alpha) \) is monotonically increasing in \( \alpha \), we have

\[
\inf_{\alpha > 0} f(\sigma, \alpha) = f(\sigma, 0). \tag{40}
\]

Hence, we find that

\[
\sup_{\sigma_B \in \mathcal{S}(M_B)} \inf_{\alpha \in [0, \alpha_0]} f(\sigma, \alpha) = \inf_{\alpha \in [0, \alpha_0]} \sup_{\sigma_B \in \mathcal{S}(M_B)} f(\sigma, \alpha), \tag{41}
\]

and by using that \( f(\sigma, \alpha) \) is monotonically increasing in \( \alpha \) we obtain with \((39)\) that

\[
h_{\max}(X|B)_\omega = \lim_{\alpha \to 0} \left( H_{\max}(X_{P^j_B})_\omega + \log \alpha \right). \tag{42}
\]

The approximation result for the differential conditional max-entropy (Proposition 4) is only proven under the assumption that there exists a partition \( \mathcal{P}_\alpha \) into intervals of length \( \alpha \) such that \( H_{\max}(X_{P^j_B})_\omega \) is finite. But in the important case where \( X = \mathbb{R} \), this condition is satisfied under the assumption that the second moment of the distribution \( \omega_X \) is finite (which is often a valid assumption in physical applications).

**Lemma 5.** Let \( X = \mathbb{R} \) and \( \omega \in \mathcal{S}(L^\infty(X)) \) such that \( \int \omega(x)x^2 dx < \infty \). Then, there exists a partition \( \mathcal{P}_\alpha \) of \( X \) into intervals of length \( \alpha > 0 \) such that \( H_{\max}(X_{P^j_B})_\omega \) is finite.
Proof. Let us fix $\alpha$ and take the partition $\mathcal{P}_\alpha$ of $X$ into intervals $I_k = [k\alpha, (k+1)\alpha]$ for $k \in \mathbb{Z}$. The max-entropy $H_{\max}(X_{\mathcal{P}_\alpha})_\omega$ is finite if and only if $\sum_k \sqrt{\omega_k}$ with $\omega_k = \int_{I_k} \omega(x) dx$ is finite. By means of the monotone convergence theorem, we can write
\[
\int x^2 \omega(x) dx = \sum_{k \geq 0} \int_{I_k} \omega(x)x^2 dx + \sum_{k < 0} \int_{I_k} \omega(x)x^2 dx .
\] (43)

For the following we only consider the sum over $k \geq 0$, but the same argument can also be applied to the sum over $k < 0$. From the monotonicity of the square and the definition of $I_k$ follows that $\int_{I_k} \omega(x)x^2 dx \geq (\alpha k)^2 \int_{I_k} \omega(x) dx = (\alpha k)^2 \omega_k$. Hence, we find that
\[
\alpha^2 \sum_k k^2 \omega_k \leq \sum_k \int_{I_k} \omega(x)x^2 dx < \infty ,
\] (44)
and since all terms are positive we conclude that the sum $\sum k^2 \omega_k$ converges absolutely. We set $\Delta = \{ k \in \mathbb{N} : k^2 \sqrt{\omega_k} \geq 1 \}$ and write
\[
\sum_{k \in \mathbb{N}} k^2 \omega_k = \sum_{k \in \Delta} k^2 \omega_k + \sum_{k \in \mathbb{N}\setminus\Delta} k^2 \omega_k \geq \sum_{k \in \Delta} \sqrt{\omega_k} ,
\] (45)
where we used that absolute converging series can be reordered and that $\sum_{k \in \Delta} k^2 \omega_k = \sqrt{\omega_k}(k^2 \sqrt{\omega_k}) \geq \sqrt{\omega_k}$ for all $k \in \Delta$. Hence, we find that $\sum_{k \in \Delta} \sqrt{\omega_k}$ converges absolutely. Moreover, by definition of $\Delta$, it holds that $\sqrt{\omega_k} < 1/k^2$ for all $k \in \mathbb{N}\setminus\Delta$ such that $\sum_{k \in \mathbb{N}\setminus\Delta} \sqrt{\omega_k} \leq \sum_{k \in \mathbb{N}\setminus\Delta} 1/k^2 < \infty$. Using again that absolutely converging series can be reordered, we finally obtain
\[
\sum_{k \in \mathbb{N}} \sqrt{\omega_k} = \sum_{k \in \Delta} \sqrt{\omega_k} + \sum_{k \in \mathbb{N}\setminus\Delta} \sqrt{\omega_k} < \infty .
\] (46)

Note that the discretized entropies are regularized by the logarithm of the measure of the partition. This is in accordance with the fact that discretized entropies have no units while the power of the differential entropies $2^{-h_{\min}(X|B)_\omega}$ and $2^{-h_{\max}(X|B)_\omega}$ are in units of $X$.

Apparent from (18), we have that the operational interpretation of $2^{-h_{\min}(X_{\alpha}|B)_\omega}$ is given by the optimal success probability to guess the variable $X_{\alpha}$ given the quantum memory $B$ (see [24]). The approximation proposition leads then to a clarification of the operational interpretation of the differential conditional min-entropy. Denoting the guessing probability $2^{-h_{\min}(X_{\alpha}|B)_\omega}$ by $p_{\text{guess}}(X_{\alpha}|B)_\omega$, the approximation result, Proposition 4, directly yields
\[
2^{-h_{\min}(X|B)_\omega} = \lim_{\alpha \to 0} \frac{p_{\text{guess}}(X_{\alpha}|B)_\omega}{\alpha} .
\] (47)

Hence, the differential conditional min-entropy can be regarded as the derivative of the guessing probability at $\alpha = 0$. For $\alpha \approx 0$ we obtain the linear approximation
\[
p_{\text{guess}}(X_{\alpha}|B)_\omega = 2^{-h_{\min}(X|B)_\omega} \cdot \alpha - \Theta(\alpha^2) .
\] (48)

### 3.2. Conditional von Neumann Entropy

Shannon already introduced measures for the information content of discrete and continuous random variables. If $p$ is the corresponding probability distribution on $X \subseteq \mathbb{R}$, then its differential Shannon entropy is given by [27]
\[
h(X) = - \int p(x) \log p(x) dx ,
\] (49)
where $dx$ denotes the Lebesgue measure. We note that the differential Shannon entropy can take values in $[-\infty, \infty]$. We now wish to define its quantum conditional version.

In order to motivate our definition of the differential conditional von Neumann entropy, let us first recall the situation for discrete finite classical systems and finite-dimensional Hilbert spaces. For a classical-quantum density operator
The sesquilinear form \( \rho_{XB} = \sum_x p_x |x\rangle \langle x| \otimes \rho_B^x \), the conditional von Neumann entropy is defined as \( H(X|B)_\rho = H(XB)_\rho - H(B)_\rho \), where \( H(XB)_\rho = -\text{tr}[\rho_{XB} \log \rho_{XB}] \) denotes the von Neumann entropy. In the following, we use that the conditional von Neumann entropy can also be rewritten as

\[
H(X|B)_\rho = -\sum_x \text{tr}[p_x \rho_B^x (\log p_x \rho_B^x - \log \rho_B)] = -\sum_x D(p_x \rho_B^x \| \rho_B),
\]

where the quantum relative entropy of two density matrices \( \rho \) and \( \sigma \) acting on a finite-dimensional Hilbert space \( \mathcal{H} \) is defined as (see e.g. [37])

\[
D(\rho\|\sigma) = \text{tr}[\rho \log \rho] - \text{tr}[\rho \log \sigma],
\]

in the case where the support of \( \rho \) is contained in the support of \( \sigma \), and \( \infty \) else. Writing the conditional von Neumann entropy in terms of the quantum relative entropy is motivated by the fact that the latter has a well behaved extension to states on von Neumann algebras which was introduced by Araki [3] and further studied by Petz and various authors (see [34] and references therein). This generalization can be understood in the finite-dimensional case by writing

\[
D(\rho\|\sigma) = \text{tr}\left[\rho^{1/2} \log (\Delta(\rho/\sigma)) \rho^{1/2}\right],
\]

where the so-called spatial derivative is defined as \( \Delta(\rho/\sigma) = L(\sigma^{-1}) R(\rho) \), where \( L(a) \) and \( R(a) \) denote the left and right multiplication by an element \( a \in \mathcal{B}(\mathcal{H}) \), respectively. Here, \( \sigma^{-1} \) denotes the pseudo inverse on the support of \( \sigma \).

Note that \( \Delta(\rho/\sigma) \) defines a linear positive operator acting on the Hilbert space \( HS(\mathcal{H}) \) of Hilbert-Schmidt operators on \( \mathcal{H} \). We emphasize that the mapping \( \pi : a \mapsto L(a), a \in \mathcal{B}(\mathcal{H}) \) defines a representation of the algebra \( \mathcal{B}(\mathcal{H}) \) on the Hilbert space \( HS(\mathcal{H}) \). Before discussing the spatial derivative on von Neumann algebras we first consider its properties in the case of density operators (see also [37] Chapter 3.4). The spatial derivative may then be defined by the quadratic form\(^\text{13}\)

\[
q : a \mapsto \text{tr}\left[\rho R(\sigma^{-\frac{1}{2}}a)R(\sigma^{-\frac{1}{2}}a)^*\right] = \text{tr}\left[\rho a^* \sigma^{-1} a\right],
\]

where again \( R(\sigma^{-\frac{1}{2}} a) \) is the right multiplication by \( \sigma^{-\frac{1}{2}} a \). The operator \( R(\sigma^{-\frac{1}{2}} a)R(\sigma^{-\frac{1}{2}} a)^* \) commutes with all operators acting by left multiplication, and hence is an element of the commutant of \( \pi(\mathcal{B}(\mathcal{H})) \). This characterization of the spatial derivative can be generalized to states on von Neumann algebras.

For a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \), let \( (\xi_\sigma, \pi_\sigma, H_\sigma) \) be the GNS-triple associated with \( \sigma \in \mathcal{P}(\mathcal{M}) \). For vectors \( \eta \) in the set \( \{ \eta \in H : ||\eta|| \leq c_\eta \sigma(a^*a), a \in \mathcal{M}, c_\eta > 0 \} \) with closure equal to the support of \( \sigma \), we may define a linear bounded operator from \( H_\sigma \) to \( H \) by

\[
r_\sigma(\eta) : x \xi_\sigma \mapsto x \eta.
\]

Note that the GNS construction ensures that the linear span of vectors of the form \( x \xi_\sigma, x \in \mathcal{M} \) are dense in \( H_\sigma \). If the Hilbert spaces \( H_\sigma \) and \( H \) are isomorphic and \( \eta = cx \xi_\sigma \) for \( c \in \mathcal{M}' \), then \( r_\sigma(\eta) = c \). Moreover, the operator \( r_\sigma(\eta)r_\sigma(\eta)^* \) is always an element of \( \mathcal{M}' \). Let now \( \omega \) be a state on \( \mathcal{M} \), which is implemented by a vector \( \xi \in H \), that is, \( \xi \) is a purifying vector of \( \omega \). The vector \( \xi \) also defines a state \( \omega_\xi \) on the commutant \( \mathcal{M}' \) by \( \omega_\xi(y) = \langle \xi | y \xi \rangle \) for \( y \in \mathcal{M}' \). We define the spatial derivative \( \Delta(\omega_\xi/\sigma) \) as the self-adjoint operator associated with the quadratic form on \( H \) given by

\[
q : \eta \mapsto \omega_\xi(r_\sigma(\eta)r_\sigma(\eta)^*).
\]

For a detailed derivation of its properties, see [34] Chapter 9.3, and [33] Chapter 4] and references therein. In analogy with the finite-dimensional case, we can now define the quantum relative entropy in terms of this operator (following Araki [3]).

**Definition 6.** Let \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \), \( \omega \in \mathcal{S}(\mathcal{M}) \) implemented by a vector \( \xi \in \mathcal{H} \), and \( \sigma \in \mathcal{P}(\mathcal{M}) \). If \( \xi \) is in the support of \( \sigma \), then the quantum relative entropy of \( \omega \) with respect to \( \sigma \) is defined as

\[
D(\omega\|\sigma) = \langle \xi | \log (\Delta(\omega_\xi/\sigma)) \rangle \xi.
\]

The logarithm of the possibly unbounded operator \( \Delta(\omega_\xi/\sigma) \) is defined via the functional calculus. If \( \xi \) is not in the support of \( \sigma \), we set \( D(\omega\|\sigma) = \infty \).

\(^{13}\) The sesquilinear form \( s : (a, b) \mapsto \text{tr}[a^* \Delta(\rho/\sigma)(b)] \) defining the positive linear operator \( \Delta(\rho/\sigma) \) is derived from \( q \) by the polarization identity \( s(a, b) = \frac{1}{4}(q(a+b) - q(a-b) + iq(a-ib) - iq(a+ib)) \).
It can be shown that the quantum relative entropy is independent of the particular choice of some purifying vector $\xi$ of $\omega$, see the discussion in [28] Chapter 5] together with [57]. The definition of the differential conditional von Neumann entropy of a classical-quantum states is now motivated by (50). For later purposes, we also include an additional finite-dimensional quantum system.

**Definition 7.** Let $M_{XAB} = L^\infty(X) \otimes B(H_A) \otimes M_B$ with $(X, \Sigma, \mu, \{\mathcal{P}_\alpha\})$ a $\sigma$-finite measure space, $H_A$ a finite-dimensional Hilbert space, $M_B$ a von Neumann algebra, and $\omega_{XAB} \in \mathcal{S}_\infty(M_{XAB})$. Then, the conditional von Neumann entropy of $XA$ given $B$ is defined as

$$h(XA|B)_\omega = - \int D(\omega_{XAB}^x \| \omega_B) \, d\mu(x) \, .$$

(57)

where $\text{tr}_A$ is the trace on $H_A$.

Similar as for the conditional min- and max-entropy, we call $h(X|B)_\omega$ differential conditional von Neumann entropy if $X$ is a coarse grained measure space and use uppercase letter, $H(X|B)_\omega$, if $X$ is discrete. In the latter case of a discrete measure space, we recover the formula in (50) with now a possible infinite sum

$$H(X|B)_\omega = - \sum_{x \in X} D(p_x \omega_B^x \| \omega_B) \, .$$

(58)

We derive an approximation result similar to the ones for the differential conditional min- and max-entropy.

**Proposition 8.** Let $\mathcal{M}_{XB} = L^\infty(X) \otimes M_B$ with $M_B$ a von Neumann algebra and $(X, \Sigma, \mu, \{\mathcal{P}_\alpha\})$ a coarse grained measure space. Consider $\omega_{XB} \in \mathcal{S}(\mathcal{M}_{XB})$ such that $-\infty < h(X|B)_\omega$, and assume that there exists $\alpha_0 > 0$ for which $H(X_{\mathcal{P}_\alpha}|B)_\omega < \infty$. Then, we have that

$$h(X|B)_\omega = \lim_{n \to \infty} (H(X_{\mathcal{P}_n}|B)_\omega + \log \alpha) \, .$$

(59)

where $\omega_{X_{\mathcal{P}_n},B}$ is defined as in (16). Furthermore, if $h(X|B)_\omega < \infty$, then it follows that

$$h(X|B)_\omega = h(X)_\omega - D(\omega_{XB} \| \omega_X \otimes \omega_B) \, .$$

(60)

**Proof.** Following an analogue reasoning as for the differential conditional max-entropy, we write the integral as a series of integrals over a covering $\{X^k\}_{k=0}^\infty$ of $X$ by compact measurable sets with $\mu(X^k \cap X^l) = 0$ for $k \neq l$. Using that the Lebesgue integral can be split over positive and negative parts of the integrand, we can use the monotone convergence theorem to obtain

$$-h(X|B)_\omega = \lim_{n \to \infty} \sum_{k=1}^n \int_{X^k} D(\omega_B^x \| \omega_B) \, d\mu(x) \, .$$

(61)

For a fixed $k$, it follows from disintegration theory [49] Chapter IV.7] that

$$\int_{X^k} D(\omega_B^x \| \omega_B) \, dx = D(\omega_{XB} \| \mu_{X^k} \otimes \omega_B) \, ,$$

(62)

where $\mu_{X^k}$ denotes the restriction of the Lebesgue measure on $X^k$. Note that $\mu_{X^k}$ is now a positive finite functional such that we can apply the approximation result for the quantum relative entropy of states on a von Neumann algebra along a net of increasing subalgebras in $L^\infty(X^k)$ (Lemma [28]). In particular, we take the net of subalgebras given by the step functions corresponding to the fixed partitions $\mathcal{P}_\alpha$ obtained by restricting $\mathcal{P}_\alpha$ to $X^k$. We assume here that the covering $\{X^k\}$ is taken such that it is compatible with the partitions $\mathcal{P}_\alpha$ for a small enough $\alpha$ such that $\mathcal{P}_\alpha^k$ is balanced as well. Let us denote the corresponding alphabet of the induced discrete and finite abelian algebra by $X^k_{\alpha}$. Hence, we obtain that

$$-h(X|B)_\omega = \lim_{n \to \infty} \lim_{\alpha \to 0} \sum_{k=1}^n D(\omega_{X^k_{\alpha}B} \| \mu_{X^k_{\alpha}} \otimes \omega_B) \, .$$

(63)

14 From $-\infty < h(X|B)$, it follows by the monotonicity of the von Neumann relative entropy under channels (Lemma [24]) that $-\infty < h(X)$.

15 In [28], conditional von Neumann entropy was defined as in (60) for $\omega_{AB} \in \mathcal{S}(B(H_A \otimes H_B))$ with $H(A)_\omega < \infty$, and separable Hilbert spaces $H_A$ and $H_B$.

16 Note that such a covering exists since one can for instance take the the sets of a fixed partition $\mathcal{P}_\alpha$ for a large enough $\alpha$. 
where \( \omega_{X_kB} \) and \( \mu_{X_k} \otimes \omega_B \) are states in \( \ell^\infty(X^k) \otimes \mathcal{M}_B \) and defined as in (10). We therefore have that \( \mu_{X_k} = \alpha \mathbf{1} \), where the identity is the one in \( \ell^\infty(X^k) \), and it follows by an elementary property of the quantum relative entropy (Lemma 27) that

\[
D(\omega_{X_kB} \otimes \omega_B) = D(\omega_{X_kB} \| \mathbf{1} \otimes \omega_B) - p_k \log \alpha,
\]

where \( p_k = \int_{X_k} \omega_B^\mu(\mathbf{1}) \, dx \) is the probability that an event in the interval \( X^k \) occurs. Hence, in order to obtain the approximation result in the proposition, we have to show that the limits on the right hand side of (63) can be interchanged. Note that it is sufficient to show that the sum \( \sum f_k(\alpha) \) converges uniformly. By assumption, \( H(X_{P,\alpha}|B)_\omega < \infty \), and due to the monotonicity of the quantum relative entropy under restrictions (Lemma 26), we then get that \( f_k(\alpha) \leq f_k(0) \) for all \( k \). Together with (64) it follows that

\[
h(X|B)_\omega = - \sum_k f_k(0) \leq - \sum_k f_k(0) = H(X_{P,\alpha}|B)_\omega + \log \alpha_0 < \infty,
\]

and since by assumption \( h(X|B)_\omega > -\infty \), we conclude that \( h(X|B)_\omega \) is finite. Further, we have that \( |f_k(\alpha)| \leq |f_k(0)| + |f_k(0)| = M_k \). Using the Weierstrass uniform convergence criteria, it remains to show that \( \sum_k M_k \) is finite. The series \( \sum_k |f_k(0)| \) is finite since it is upper bounded by \( \int |D(\omega_{X_kB} \| \omega_B)| \, dx \), which is finite since \( h(X|B)_\omega \) is finite. Using (64) and the fact that \( D(\omega_{X_kB} \| \omega_B) \geq 0 \) for all \( k \), it is easy to see that the series \( \sum_k |f_k(0)| \) is bounded by \( H(X_{P,\alpha}|B) + \log(\alpha_0) \) (which is finite by assumption). This concludes the first statement of the proposition.

The second statement follows from the first together with the chain rule for the quantum relative entropy (Lemma 29).

4. ENTROPIE UNCERTAINTY RELATIONS IN THE PRESENCE OF QUANTUM MEMORY

Let us consider a tripartite system modeled by the von Neumann algebra \( \mathcal{M}_{ABC} \) and measurements \( E \in \text{Meas}(X, \mathcal{M}_A) \) and \( F \in \text{Meas}(Y, \mathcal{M}_A) \) on the \( A \)-system. Given a state on \( \mathcal{M}_{ABC} \), we are interested in quantifying the uncertainty of the measurement statistics of \( E \) with respect to \( B \) and of the outcome statistics of \( F \) with respect to \( C \). While it is clear that the uncertainty of each individual part can be zero, its sum is bounded due to the complementarity of the measurements and the monogamy of entanglement. In the following, we use differential conditional min- and max-entropy as well as differential conditional von Neumann entropy to obtain a quantitative bound on the sum of these uncertainties. Our starting point is a recent proof technique developed by Coles et al. (see also [55]) for finite-dimensional Hilbert spaces, which we generalize to von Neumann algebras and continuous measure spaces by means of the approximation results derived in Section 3. The advantage of the applied proof strategy is that it only requires basic properties of the involved entropies.

4.1. Uncertainty Relations in Terms of Conditional Min- and Max-Entropy

An uncertainty relation for conditional min- and max-entropy was first derived in the finite-dimensional setting in [55], and then generalized to measurements with a finite number of outcomes on von Neumann algebras in [9]. Before stating the extension to measurements with continuous outcomes, we first prove the uncertainty relation for the case of measurements with an infinite, but countable number of outcomes.

**Proposition 9.** Let \( \mathcal{M}_{ABC} \) be a tripartite von Neumann algebra, \( \omega_{ABC} \in \mathcal{S}(\mathcal{M}_{ABC}) \), \( X \) and \( Y \) countable, and \( E_X = \{E_x\}_{x \in X} \in \text{Meas}(X, \mathcal{M}_A) \) and \( F_Y = \{F_y\}_{y \in Y} \in \text{Meas}(Y, \mathcal{M}_A) \). Then, we have that

\[
H_{\text{max}}(X|B)_\omega + H_{\text{min}}(Y|C)_\omega \geq -\log c(E_X, F_Y),
\]

where the overlap of the measurements is given by

\[
c(E_X, F_Y) = \sup_{x,y} \|E_x^{1/2}F_y^{1/2}\|^2.
\]

\[13\] Note that the terms \( f_k(\alpha) \) in the sum can be negative or positive and we need lower and upper bounds in order to bound the absolute value of \( f_k(\alpha) \).

\[18\] Note that if the integral of a function is finite so is the integral of the absolute value of the function.
Proof. The main difference to the proofs of the uncertainty relations in [13, 55] is that we have to take an infinite number of outcomes into account. We achieve this by first showing an inequality for sub-normalized measurements with a finite number of outcomes, and then use a limit argument to obtain the uncertainty relation for measurements with an infinite number of outcomes. We describe sub-normalized measurements with a finite number of outcomes, and then use a limit argument to obtain the uncertainty relation for measurements with an infinite number of outcomes.

Let $\mathcal{H}$ be a Hilbert space such that $\mathcal{M}_{ABC} \subseteq \mathcal{B}(\mathcal{H})$ is faithfully embedded, and there exist a purifying vector $|\psi\rangle \in \mathcal{H}$ for $\omega_{ABC}$, that is, $\omega_{ABC}(\cdot) = \langle \psi | \cdot | \psi \rangle$. We choose a Stinespring dilation (see [36, Theorem 4.1]) for $E_X$ of the form

$$V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathbb{C}^X \otimes \mathbb{C}^{X'}, \quad V|\psi\rangle = \sum_x E_x^{1/2}|\psi\rangle \otimes |x, x\rangle,$$

where $\mathbb{C}^X$ denotes a quantum system with dimension $X$ in which the classical output of the measurement $E_X$ is embedded, and $X \cong X'$. Since $|\psi\rangle \in \mathcal{H}$ is a purifying vector of $\omega_{ABC}$, we have that $V|\psi\rangle \in \mathcal{H} \otimes \mathbb{C}^X \otimes \mathbb{C}^{X'}$, is a purifying vector of $\omega_{XB} = \omega_{AB} \circ E_X$. Denoting the commutant of $\mathcal{M}_{ABC}$ in $\mathcal{B}(\mathcal{H})$ by $\mathcal{M}_D$, we find that the purifying system of $\omega_{XB}$ is $B(\mathbb{C}^{X'}) \otimes \mathcal{M}_{ACD}$. It then follows from the duality of the conditional min- and max-entropy (Lemma [23]) that

$$H_{\text{max}}(X|B)_{\omega} = -H_{\text{min}}(X|X'ACD)_{\psi \circ V},$$

where $\psi_{ACD} \circ V(\cdot) = \langle \psi | V^{\ast}(\cdot) V | \psi \rangle$. Since the conditional min-entropy can be written as a max-relative entropy (Definition [18]), we have that

$$-H_{\text{min}}(X|X'ACD)_{\psi \circ V} = \inf_{\sigma} D_{\text{max}}(\psi_{ACD} \circ V \circ \tau_X \otimes \sigma_{X'ACD}),$$

where the max-relative entropy is given by (Definition [17])

$$D_{\text{max}}(\psi_{ACD} \circ V \circ \tau_X \otimes \sigma_{X'ACD}) = \inf \{ \epsilon \in \mathbb{R} : \psi_{ACD} \circ V \leq 2^\epsilon \cdot \tau_X \otimes \sigma_{X'ACD} \},$$

and the infimum is over $\sigma_{X'ACD} \in S(\mathcal{M}_{X'ACD})$, and $\tau_X$ denotes the trace on $B(\mathbb{C}^X)$. Let us now define the completely positive map $\mathcal{E} : B(\mathcal{H}) \rightarrow B(\mathcal{H} \otimes \mathbb{C}^X \otimes \mathbb{C}^{X'})$ given by $\mathcal{E}(a) = V a V^{\ast}$. The map is sub-unital since $\mathcal{E}(I) = V V^{\ast}$ and $\| V V^{\ast} \| = \| V^{\ast} V \| = \| \sum_x E_x \| = \| M \| \leq 1$. Due to the monotonicity of the max-relative entropy under applications of sub-unital, completely positive maps (Lemma [19]), we obtain for fixed $\sigma_{X'ACD} \in S(\mathcal{M}_{X'ACD})$,

$$D_{\text{max}}(\psi_{ACD} \circ V \circ \tau_X \otimes \sigma_{X'ACD}) \geq D_{\text{max}}((\psi_{ACD} \circ V) \circ \mathcal{E}(\tau_X \otimes \sigma_{X'ACD}) \circ \mathcal{E})$$

$$= D_{\text{max}}(\omega_{ACD}^{\mathcal{E}} \circ \gamma_{\mathcal{E}ACD}^{\mathcal{E}}),$$

where we denoted $\omega_{ACD}^{\mathcal{E}} = (\psi_{ACD} \circ V) \circ \mathcal{E}$ and $\gamma_{\mathcal{E}ACD}^{\mathcal{E}} = (\tau_X \otimes \sigma_{X'ACD}) \circ \mathcal{E}$. Due to the fact that $V^{\ast} V = M$, we have that $\omega_{ACD}^{\mathcal{E}}(\cdot) = \psi_{ACD} \circ V \circ V^{\ast}(\cdot) = \langle \psi | V^{\ast} V(\cdot) V^{\ast} V | \psi \rangle = \omega_{ACD}(M \cdot M)$ with $\omega_{ACD}$ the state on $\mathcal{M}_{ACD}$ corresponding to $|\psi\rangle$. Using once more the monotonicity of the max-relative entropy under application of channels (Lemma [19]), we obtain by first restricting onto the subalgebra $\mathcal{M}_{AC}$ and then measuring the $A$ system with $F$, and $\gamma_{\mathcal{E}ACD}^{\mathcal{E}}$ we set $\omega_{ACD}^{\mathcal{E}} = \omega_{ACD} \circ F_Y$ and $\gamma_{\mathcal{E}ACD}^{\mathcal{E}} = \gamma_{\mathcal{E}ACD} \circ F_Y$. By definition, we have that $\gamma_{\mathcal{E}ACD}(a) = \sum_y \gamma_{\mathcal{E}ACD}(F_y a y)$ for $a = (a_y) \in \mathcal{M}_{YC}$. Hence, it holds for all positive $a_y \in \mathcal{M}_C$ with $y \in Y$ that

$$\gamma_{\mathcal{E}ACD}^{\mathcal{E}}(F_y a y) = \tau_X \otimes \sigma_{X'AC}(F_y a_y V^{\ast}) = \sum_x \sigma_{AC}^{x, x}(\sqrt{E_x} F_y (E_x a_y) \leq \sup_{x,y} \| E_x^{1/2} F_y^{1/2} \|^2 \sigma_C(a_y),$$

where we denoted $\sigma_{X'ACD} = (\sigma_{X'AC})^{x, x}$, and used that $a_y$ commutes with $E_x$ and $F_y$. Thus, we conclude that

$$\gamma_{\mathcal{E}ACD}^{\mathcal{E}} \leq \sup_{x,y} \| E_x^{1/2} F_y^{1/2} \|^2 \cdot \tau_Y \otimes \sigma_C.$$

By some elementary properties of the max-relative entropy (Lemma [20] and Lemma [21]), it then follows for any $\sigma_{X'ACD} \in S(\mathcal{M}_{X'ACD})$ that

$$D_{\text{max}}(\omega_{ACD}^{\mathcal{E}} \circ \gamma_{\mathcal{E}ACD}^{\mathcal{E}}) \geq D_{\text{max}}(\omega_{ACD}^{\mathcal{E}} \circ \gamma_{\mathcal{E}ACD}^{\mathcal{E}}) - \log \sup_{x,y} \| E_x^{1/2} F_y^{1/2} \|^2$$

$$\geq \inf_{\eta} D_{\text{max}}(\omega_{ACD}^{\mathcal{E}} \circ \gamma_{\mathcal{E}ACD}^{\mathcal{E}}) - \log \sup_{x,y} \| E_x^{1/2} F_y^{1/2} \|^2$$

$$= -H_{\text{min}}(Y|C)_{\omega} - \log \sup_{x,y} \| E_x^{1/2} F_y^{1/2} \|^2,$$
where the infimum is over $\eta_C \in S(\mathcal{M}_C)$, Combining this with all the steps going back to (69), we therefore obtain
\[
H_{\text{max}}(X|B)_\omega \geq -H_{\text{min}}(Y|C)_\omega \nu - \log \sup_{x,y} \left\| E_x^{1/2} F_y^{1/2} \right\|^2.
\] (80)

Recall that $\omega_{Y|C}^V = (\omega_{C|Y}^V)_y$ with $\omega_{C|Y}^V(\cdot) = \omega(MF_yM^\perp)$, and thus, if $E$ is normalized we obtain the uncertainty relation for measurements with a finite number of outcomes.

Let us now lift the relation to the case of discrete $X$ and $Y$ with infinite cardinality. We take sequences of increasing finite subsets $X_1 \subset X_2 \subset \ldots \subset X$ and $Y_1 \subset Y_2 \subset \ldots \subset Y$ such that $\bigcup_n X_n = X$ and $\bigcup_n Y_n = Y$. The strategy is to apply the inequality (80) derived for sub-normalized measurements to $E_{X_n} = \{E_x\}_{x \in X_n}$ and $F_{Y_m} = \{F_y\}_{y \in Y_m}$. It is straightforward to see that (80) for fixed $n$ and $m$ reads as
\[
H_{\text{max}}(X_n|B)_\omega \geq -H_{\text{min}}(Y_m|C)_\omega - \log \sup_{x \in X_n, y \in Y_m} \left\| E_x^{1/2} F_y^{1/2} \right\|^2,
\] (81)
where we denoted $\omega_{X_n,B} = \omega_{AB} \circ E_{X_n}$ and $\omega_{Y_m,C}^0 = (\omega_{C|Y}^0)_y \in Y_m$ with $\omega_{C|Y}^0(\cdot) = \omega_{AB}(M_n F_y M_n^\perp)$, and $M_n = \sum_{x \in X_n} E_x$.

Note that we already used that taking the supremum over $X$ and $Y$ instead of $X_n$ and $Y_m$ only decreases the right hand side of the above inequality. We now take the limit for $n \to \infty$ on both sides. By using the definition of the conditional max-entropy in (20), it is straightforward to see that $H_{\text{max}}(X_n|B)_\omega$ converges to $H_{\text{max}}(X|B)_\omega$ for $n \to \infty$. The only term on the right hand side depending on $n$ is the conditional min-entropy of the state $\omega_{Y_m,C}^V$, which is given by (see [18])
\[
H_{\text{min}}(Y_m|C)_\omega = -\log \sup_G \sum_{y \in Y_m} \omega_{AC}(M_n F_y M_n G_y),
\] (82)
where the supremum is taken over all $G = \{G_y\}_{y \in Y_m}$ in $\text{Meas}(Y_m, \mathcal{M}_C)$. It holds for every $y \in Y_m$ and $0 \leq G_y \leq 1$ that
\[
|\omega_{AC}(F_y G_y) - \omega_{AC}(M_n F_y M_n G_y)| \leq |\omega_{AC}(F_y G_y (\mathbb{I} - M_n))| + |\omega_{AC}((\mathbb{I} - M_n) F_y G_y M_n)| \leq 2\sqrt{\omega_{AC}((\mathbb{I} - M_n)^2)},
\] (83)
where we used the Cauchy-Schwarz inequality for states, that is, $\omega(ab)^2 \leq \omega(a^*a)\omega(b^*b)$ for any operators $a, b$. Hence, we have that the functionals $\omega_{C|Y}^0(\cdot) = \omega_{AC}(M_n F_y M_n^\perp)$ converge uniformly to $\omega_{C|Y}^V(\cdot) = \omega_{AC}(F_y^\perp)$ on the unit ball of $\mathcal{M}_C$ for any $y \in Y_m$ (since $M_n$ converges in the $\sigma$-weak topology to $\mathbb{I}$). Because the set $Y_m$ is finite, this also implies that $\omega_{Y_m,C}^V = (\omega_{C|Y}^V)_y \in Y_m$ converges uniformly to $\omega_{Y_m,C}^0 = (\omega_{C|Y}^0)_y \in Y_m$ on the unit ball of $\mathcal{M}_Y$. Hence, we can interchange the limit for $n \to \infty$ with the supremum over $\text{Meas}(Y_m, \mathcal{M}_C)$ and obtain
\[
H_{\text{max}}(X|B)_\omega \geq -H_{\text{min}}(Y_m|C)_\omega - \log \left( \sup_{x \in X_n, y \in Y_m} \left\| E_x^{1/2} F_y^{1/2} \right\|^2 \right).
\] (85)

In a final step we take the infimum over all $m \in \mathbb{N}$ which gives the desired inequality due to the definition of the conditional min-entropy in (23).

Using the discrete approximation of the differential conditional min- and max-entropy from Proposition\[1\] we obtain the uncertainty relation for continuous outcome measurements.

**Theorem 10.** Let $\mathcal{M}_{ABC}$ be a tripartite von Neumann algebra, $\omega_{ABC} \in S(\mathcal{M}_{ABC})$ and $E_X \in \text{Meas}(X, \mathcal{M}_A)$ and $F_Y \in \text{Meas}(Y, \mathcal{M}_A)$ with coarse grained measure spaces $(X, \Sigma_X, \mu_X, \{P_x\})$ and $(Y, \Sigma_Y, \mu_Y, \{Q_y\})$. If for the post-measurement states $\omega_{X|BC} = \omega_{ABC} \circ E_X$ and $\omega_{Y|BC} = \omega_{ABC} \circ F_Y$, there exists an $\alpha_0 > 0$ such that $H_{\text{max}}(X_{P_{\alpha_0}}) < \infty$, then we have that
\[
h_{\text{max}}(X|B)_\omega + h_{\text{min}}(Y|C)_\omega \geq -\log c(E_X, F_Y),
\] (86)
where the overlap of the measurements is given by
\[
c(E_X, F_Y) = \liminf_{\alpha, \beta \to 0} \sup_{P_{\alpha}, Q_{\beta}} \frac{\left\| (E_X(\chi_I))^{1/2} \cdot (F_Y(\chi_J))^{1/2} \right\|^2}{\alpha \cdot \beta},
\] (87)
where $E_X(\chi_I)$ and $F_Y(\chi_J)$ are defined as in [17].
A similar relation derived under stronger conditions and with different techniques can be found in the thesis of one of the authors \[19\]. Note that in the case \( X = \mathbb{R} \) the assumption that there exists an \( a_0 > 0 \) such that \( H_{\max}(X_{F_{\beta}}) < \infty \) is satisfied if the second moment of the distribution of \( X \) is finite, or equivalently, the expectation value of the observable \( \int x^2 E_X(x) dx \) with respect to \( \omega_A \) is finite (see Lemma \[5\]).

\[ \text{Proof.} \] We first apply the uncertainty relation for measurements with a discrete number of outcomes (Proposition \[9\]) to obtain for any partitions \( P_\alpha \) and \( Q_\beta \) the inequality

\[
\left( H_{\max}(X_{P_\alpha}|B) + \log \alpha \right) + \left( H_{\min}(Y_{Q_\beta}|C) + \log \beta \right) \geq - \log \sup_{k,l} \frac{\|(E_{k}^{P_\alpha})^{1/2} \cdot (F_{l}^{Q_\beta})^{1/2}\|_2^2}{\alpha \beta},
\]

where the supremum in the logarithm is taken over all possible measurements \( E_k^{P_\alpha} = E_X(\chi_{I_k}) \) and \( F_l^{Q_\beta} = E_Y(\chi_{J_l}) \) with \( P_\alpha = \{I_k\} \) and \( Q_\beta = \{J_l\} \). Taking the limit superior for \( \alpha, \beta \to 0 \) on both sides, we obtain the desired uncertainty relation by means of the approximation of the differential conditional min- and max-entropy (Proposition \[4\]).

### 4.2. Uncertainty Relations in Terms of Conditional von Neumann Entropy

We follow the same strategy as in the case of conditional min- and max-entropy and start with countably many outcomes. The following uncertainty relation for conditional von Neumann entropy was first derived in the finite-dimensional setting in \[5\].

**Proposition 11.** Let \( M_{ABC} \) be a tripartite von Neumann algebra, \( \omega_{ABC} \in S(M_{ABC}) \), \( X \) and \( Y \) countable, and \( E_X = \{E_x\}_{x \in X} \in \text{Meas}(X, M_A) \) and \( F_Y = \{F_y\}_{y \in Y} \in \text{Meas}(Y, M_A) \). Then, we have that

\[
H(X|B)_\omega + H(Y|C)_\omega \geq - \log c(E_X, F_Y),
\]

where the overlap is given in \( \[67\] \).

\[ \text{Proof.} \] The result is obtained by following the same steps as in the proof of the statement for the conditional min- and max-entropy (Proposition \[9\]). Doing so, one has to replace the conditional min- and max-entropy by the conditional von Neumann entropy and the max-relative entropy by the quantum relative entropy, respectively. Again, we first assume that \( E_X \) and \( F_Y \) are sub-normalized measurements with a finite number of outcomes \( X \) and \( Y \).

In the following, we use the same notation as in the proof of Proposition \[9\]. By a similar argument as in the case of the conditional min- and max-entropy, the self duality of the conditional von Neumann entropy (Lemma \[30\]) leads to

\[
H(X|B)_\omega = D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V)
\]

with \( \omega_{X',ACD}^V \) the restriction of \( \psi_{ACD} \circ V \) onto \( \mathbb{C}^{X'} \otimes M_{ABC} \). Let us define the projector \( \Pi = \sum_x |x,x\rangle\langle x,x| \) on \( \mathbb{C}^X \otimes \mathbb{C}^{X'} \). Denoting by \( T_a \) the transformation \( a \mapsto O^*aO \) with a suitable operator \( O \), we can use the monotonicity of the quantum relative entropy under application of channels (Lemma \[25\]) to get

\[
D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi) \geq D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi, T_{\Pi} || \tau_X \otimes \omega_{X',ACD}^V, \Pi, T_{I_{1-II}}).
\]

Note now that the channel \( T_{\Pi} + T_{I_{1-II}} \) projects \( \mathbb{C}^X \otimes \mathbb{C}^{X'} \) onto two orthogonal subspaces such that the term on the right hand side can be written as

\[
D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi) = D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi, T_{I_{1-II}}).
\]

Since \( \psi_{ACD} \circ V \circ T_{I_{1-II}} = 0 \), we have that the right term in the above sum is zero, and thus,

\[
D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi) \geq D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi).
\]

In a next step, we apply the sub-unital map \( E(a) = V a V^* \) and get according to Lemma \[25\]

\[
D(\psi_{ACD} \circ V || \tau_X \otimes \omega_{X',ACD}^V, \Pi) \geq D(\omega_{ABC} || \tau_X \otimes \omega_{X',ACD}^V, \Pi \circ E) + \omega_A(M - M^2) \log \omega_A(M - M^2),
\]

where we used that \( \tau_X \otimes \omega_{X',ACD}^V \circ \Pi \leq 1 \). Using again the monotonicity of the quantum relative entropy under sub-unital maps (Lemma \[25\]), the restriction onto the systems \( AC \) followed by the application of the measurement \( F_Y \) leads to the bound

\[
D(\omega_{ABC} || \tau_X \otimes \omega_{X',ACD}^V, \Pi \circ E) \geq D(\omega_{AC} || \tau_X \otimes \omega_{X',ACD}^V, \Pi \circ E \circ F_Y) + \omega_A(N) \log \omega_A(N),
\]

\[ \square \]
with $N = \sum_y F_y$. A straightforward computation similar to (75) shows that
\[
\tau_X \otimes \hat{\omega}^Y_{X,ACD} \circ T_\Pi \circ \mathcal{E} \circ F_Y \leq c(E_X, F_Y) \tau_Y \otimes \hat{\omega}^V_C,
\]
from which via basic properties of the quantum relative entropy (Lemma 26 and Lemma 27) the bound
\[
D(\omega^V_Y \| \tau_X \otimes \hat{\omega}^Y_{X,ACD} \circ T_\Pi \circ \mathcal{E} \circ F_Y) \geq D(\omega^V_Y \| \tau_Y \otimes \hat{\omega}^V_C) - \omega^V_Y(\mathbb{I} \otimes N) \log c(E_X, F_Y),
\]
follows. Plugging all the steps together we finally arrive at
\[
H(X|B)_{\omega} \geq D(\omega^V_Y \| \tau_Y \otimes \hat{\omega}^V_C) - \omega^V_Y(\mathbb{I} \otimes N) \log c(E_X, F_Y) + \omega_A(M - M^2) \log \omega_A(M - M^2) + \omega^\sigma(1 I - N_m) \log \omega_A^\sigma(1 I - N_m).
\]
Note that the above inequality reduces to (89) if both measurements $E_X$ and $F_Y$ are normalized. This can easily be seen by using that in such a case $M = N = 1$, and thus, $\omega^V_Y = \omega_Y$ and $\hat{\omega}^V_C = \omega_C$.

We now use the inequality (98)-(99) in a similar way as in the proof of Proposition 10 to obtain (89) for an infinite number of outcomes. For that we let $X_n$ and $Y_n$, $n \in \mathbb{N}$, as well as $E_{X_n}$ and $F_{Y_n}$ be as in the proof of Proposition 10.

For fixed sub-normalized measurements $E_{X_n}$ and $F_{Y_n}$, the inequality (98)-(99) then reads
\[
H(X_n|B)_{\omega} \geq D(\omega^V_{Y_n} \| \tau_{Y_n} \otimes \hat{\omega}^V_{X_n}) - \omega^V_{Y_n}(\mathbb{I} \otimes N_n) \log c(E_X, F_Y) + \omega_A(M_n - M_n^2) \log \omega_A(M_n - M_n^2) + \omega^\sigma_n(1 I - N_m) \log \omega^\sigma_n(1 I - N_m).
\]

Here, we used the same notation as in the proof of Proposition 9 and set further $\hat{\omega}^\sigma_n(a) = \omega(M_n a)$ as well as $N_m = \sum_{y \in Y_m} F_y$. Let us take the limit inferior for $n, m \to \infty$. According to the definition of the conditional von Neumann entropy (Definition 7) we have that $H(X_n|B)_{\omega^n}$ converges to $H(X|B)_{\omega}$. Furthermore, we use the lower semi-continuity of the quantum relative entropy [34, Corollary 5.12] and that $\omega^\sigma_n$ converge to $\omega_Y$ and $\omega_C$, respectively, to get that $\liminf_{n,m} D(\omega^\sigma_n \| \tau_{Y_n} \otimes \hat{\omega}^\sigma_n) \geq H(Y|C)_{\omega}$. Using that $M_n$ and $N_m$ converge $\sigma$-weakly to $1_A$, it is straightforward to see that $\omega^\sigma_n(1 I - N_m) \to 0$, as well as $\omega^\sigma_n(1 I - N_n) \to 0$. Using that $x \log x \to 0$ for $x \to 0$, we finally obtain (89).

\[\square\]

Theorem 12. Let $M_{ABC}$ be a tripartite von Neumann algebra, $\omega_{ABC} \in S(M_{ABC})$ and $E_X \in \text{Meas}(X, M_A)$ and $F_Y \in \text{Meas}(Y, M_A)$ with coarse grained measure spaces $(X, \Sigma_X, \mu_X, \{P_\alpha\})$ and $(Y, \Sigma_Y, \mu_Y, \{Q_\beta\})$. If the post-measurement states $\omega_{Y|BC} = \omega_{ABC} \circ E_X$ and $\omega_{Y|BC} = \omega_{ABC} \circ F_Y$ satisfy $-\infty < h(X|B)_{\omega}$, $-\infty < h(Y|C)_{\omega}$, and if there exists $c_0 > 0$ for which $h(X|B)_{\omega} < c_0$ as well as $\beta_0 > 0$ for which $h(Y|C)_{\omega} < c_0$, then it holds that
\[
h(X|B)_{\omega} + h(Y|C)_{\omega} \geq -\log c(E_X, F_Y),
\]
where $c(E_X, F_Y)$ is as in (87).

The theorem is obtained via the approximation result for the differential conditional von Neumann entropy (Proposition 8) using the exactly same steps as in the proof of the corresponding result for the differential conditional min- and max-entropy (Theorem 10).

5. **ENTROPIC UNCERTAINTY RELATIONS FOR POSITION AND MOMENTUM OPERATORS**

Entropic uncertainty relations for position-momentum measurements have been intensively studied in the history of quantum mechanics. For an overview and further references we refer to the recent review article [10] and references therein. There are basically two type of relations, such for finite spacing position-momentum measurements, and such for continuous position-momentum distributions. Whereas relations for continuous distributions in terms of differential entropies are useful to express the general uncertainty principle in quantum mechanics, finite spacing relations are needed in the sense that any position-momentum measurement in the laboratory always has a finite resolution. These discrete relations then allow to actually test the uncertainty principle, and also allow for information theoretic and cryptographic applications [21]. In the following, we apply the uncertainty relations derived in the previous section to finite spacing and continuous position-momentum measurements.

Let $Q$ and $P$ be a pair of position and momentum operators defined via the canonical commutation relation $[Q, P] = i$ where we set $\hbar = 1$. The unique representation space is $\mathcal{H} = L^2(\mathbb{R})$, with $Q$ the multiplication operator and $P$ the first order differential operator. Both operators possess a spectral decomposition with a positive operator
valued measure $E_Q$ and $E_P$ in $\text{Meas}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$.

Let us start with finite spacing measurements and assume that the precision of the position and momentum measurement are given by intervals of length $\delta q$ and $\delta p$ for the entire range of the spectrum. As we will see, the overlap in the uncertainty relation only depends on the spacings $\delta q$ or $\delta p$ but not the explicit coarse grainings $Q_{\delta q} = \left\{ I_{\delta q}^{k} \right\}_{k=1}^{\infty}$ and $P_{\delta p} = \left\{ J_{\delta p}^{k} \right\}_{k=1}^{\infty}$. For the following, we thus fix arbitrary coarse grainings $Q_{\delta q}$ and $P_{\delta p}$ for given spacings $\delta q$ and $\delta p$. The corresponding measurements are then formed by the positive operators $Q^k = E_Q(I_{\delta q}^k)$ and $P^k = E_P(J_{\delta p}^k)$. We denote the discrete classical systems induced by these position and momentum measurements by $Q(\delta q)$ and $P(\delta p)$. For the sake of notation, we omit here the dependence of the distributions on the particular coarse grainings. According to Proposition [9], the quantity which enters the entropic uncertainty relation is the overlap of the measurement operators

$$
\sup_{k,l} \| \sqrt{Q^k} \sqrt{P^l} \|^2 = \sup_{k,l} \| Q^k P^l Q^k \| .
$$

Note first that $\| Q^k P^l Q^k \|$ can only depend on the length of the intervals and is similar for any $k$ and $l$, which are taken to be $k = l = 1$ in the following. The reason is that the translation in position and momentum space are given by the unitary transformations $\exp[-ix_0 P]$ and $\exp[-i\tau_0 Q]$, which leave the norm invariant. Furthermore, since dilation operators are unitary, the constant only depends on the invariant product $\delta q \delta p$. The operator $H(\delta q, \delta p) = Q^1 P^1$ is important for time-limiting and lowpassing signals, and its largest eigenvalue, and thus, its norm can be expressed by [18] (see also [20] and references therein)

$$
c(\delta q, \delta p) = \frac{1}{2\pi} \delta q \delta p \cdot S_0^{(1)} \left( 1, \frac{\delta q \delta p}{4} \right)^2 ,
$$

where $S_0^{(1)}(1, \cdot)$ denotes the 0th radial prolate spherical wave function of the first kind. For $\delta q \delta p \to 0$, it follows that $S_0^{(1)} \left( 1, \frac{\delta q \delta p}{4} \right) \to 1$, such that the overlap behaves as $c(\delta q, \delta p) \approx \frac{1}{2\pi} \cdot \delta q \delta p$ for small spacing. A plot of the overlap $c(\delta q, \delta p)$ as well as the complementarity constant $-\log c(\delta q, \delta p)$ are shown for $\delta q = \delta p$ in Fig. 1.

FIG. 1: The plot on the left hand side shows the overlap $c(\delta q, \delta p)$ depending on $\delta$ where $\delta^2 = \delta q \delta p$. One can see that for $\delta \geq 4$ the value is approximately 1 and the uncertainty relation gets trivial. The plot on the right hand side shows the behavior of the complementarity constant $-\log c(\delta q, \delta p)$ for small $\delta = \sqrt{\delta q \delta p}$. Note that we have set $\hbar = 1$ such that the minimal uncertainty product of the standard deviations is $\sqrt{\text{Var}(P)} \sqrt{\text{Var}(Q)} = 1/2$ and the vacuum has a variance of 1/2.

**Corollary 13.** Let $\mathcal{M}_{ABC} = \mathcal{B}(L^2(\mathbb{R})) \otimes \mathcal{M}_{BC}$ with $\mathcal{M}_{BC}$ a von Neumann algebra, and consider position and momentum measurements with spacing $\delta q > 0$ and $\delta p > 0$ on system $A$. Then, we have that

$$
H_{\text{max}}(Q(\delta q)|B)_{\omega} + H_{\text{min}}(P(\delta p)|C)_{\omega} \geq -\log c(\delta q, \delta p) ,
$$

and

$$
H(Q(\delta q)|B)_{\omega} + H(P(\delta p)|C)_{\omega} \geq -\log c(\delta q, \delta p) ,
$$

where $c(\delta q, \delta p)$ is given in (104).
The corollary follows directly from Proposition [9]. Since the statement is invariant under exchanging $Q$ and $P$, the uncertainty relation in [105] also holds for the conditional min-entropy of $\omega_Q(\delta q)\omega_P(\delta p)$ and the conditional max-entropy of $\omega_P(\delta p)\omega_Q(\delta q)$. Corollary [13] generalizes known results for the Shannon entropy [11, 12, 13] and for the Rényi entropy (for the order pair $\infty - 1/2$, cf. [22] and [23]) to the case of quantum memory.

Let us now address the sharpness of the uncertainty relations. More precisely, we say that an uncertainty relation is sharp if there exists a state for which equality is attained.

**Proposition 14.** The entropic uncertainty relation in terms of the conditional min- and max-entropy in [105] is sharp for any spacing $\delta q$ and $\delta p$.

**Proof.** By the data-processing inequalities for the conditional min- and max-entropy (Proposition 22), it is enough to show sharpness for

$$H_{\text{max}}(Q(\delta q))_\omega + H_{\text{min}}(P(\delta p))_\omega \geq \log c(\delta q, \delta p) .$$

Let us assume that the partitions are centralized around 0, such that they contain the intervals $I = [-\delta q/2, \delta q/2]$ and $J = [-\delta p/2, \delta p/2]$, respectively. We take a normalized state given by $\psi(q) \in L^2(\mathbb{R})$ with support on $I$. It then follows that the distribution of the discretized position measurement is peaked and thus, $H_{\text{max}}(Q(\delta q))_\omega = 0$. The continuous probability distribution of the momentum measurement is given by $|F(\psi(q))|^2$, where $F$ denotes the Fourier transform. Therefore, we find for the min-entropy that

$$2^{H_{\text{min}}(P)_\omega} \leq \int \chi_I(p)|F(\psi)|^2 dp = \frac{1}{2\pi} \int \chi_I(q_1)\chi_J(q_2)\bar{\psi}(r)\psi(q)e^{-i(q-r)p} dq_1 dp d q_2 = \langle \psi|Q[I]P[J]Q[I]\psi \rangle ,$$

where $\chi_I$ denotes the indicator function on $I$. But also note that by [103] the overlap is given by

$$\|Q[I]P[J]Q[I]\| = \sup_{\phi \in L^2(X)} \langle \phi|Q[I]P[J]Q[I]\phi \rangle .$$

Since the supremum can be restricted to functions with support on $I$, we find the claim by choosing for $\psi$ the element for which the optimum is attained. We further note that this state $\psi(q)$ is given by the normalized projection of the radial prolate spheroidal wave function of the first kind onto the interval $I$ (see [20] and references therein).

We are unable to show sharpness for the uncertainty relation in terms of the conditional von Neumann entropy [106]. By the data-processing inequality for the conditional von Neumann entropy (Lemma 24) it would be sufficient to find sharpness for

$$H(Q(\delta q))_\omega + H(\delta p)_\omega \geq \log \left( \frac{2\pi}{\delta q \delta p \cdot S_0^{(1)}(1, \frac{\delta q \delta p}{4})^2} \right) .$$

However, it also holds that [7]

$$H(Q(\delta q))_\omega + H(\delta p)_\omega \geq \log \left( \frac{e\pi}{\delta q \delta p} \right) ,$$

which becomes a better bound for small enough spacing.\(^{19}\)

Let us now consider continuous position-momentum distributions. In order to compute the measurement overlap given in (57), we can simply take the limit of $c(\delta, \delta)$ for $\delta \to 0$ yielding

$$c(E_Q, E_P) = \lim_{\delta \to 0} \frac{1}{2\pi} \frac{1}{2} S_0^{(1)}(1, \frac{\delta^2}{4})^2 = \frac{1}{2\pi} ,$$

where we used (104), and that $S_0^{(1)}(1, \frac{\delta^2}{4}) \to 1$ for $\delta \to 0$. Hence, we immediately obtain the following corollary which generalizes known results for the differential Shannon entropy [4, 22], and for the differential Rényi entropy [8] (for the order pair $\infty - 1/2$, cf. [22] and [23]) to the case of quantum memory.

\(^{19}\) This bound was originally derived for pure states [7], but also holds for mixed states (see e.g. [8]).
Corollary 15. Let \( \mathcal{M}_{ABC} = \mathcal{B}(L^2(\mathbb{R})) \otimes \mathcal{M}_B \) with \( \mathcal{M}_B \) a von Neumann algebra, \( \omega_{ABC} \in \mathcal{S}(\mathcal{M}_{ABC}) \), and denote the post-measurement states obtained by continuous position and momentum measurements on system \( A \) by \( \omega_{QBC} \) and \( \omega_{PBC} \). If there exists a finite spacing \( \delta q \) such that \( H_{\text{max}}(Q|\delta q)_{\omega} < \infty \), then we have that
\[
h_{\text{min}}(Q|B)_{\omega} + h_{\text{max}}(P|C)_{\omega} \geq \log 2 \pi . \tag{113}
\]
Furthermore, if \( -\infty < h(Q|B)_{\omega} < \infty \), \( -\infty < h(P|C)_{\omega} \), and if there exists finite spacings \( \delta q, \delta p \) for which \( H(Q|\delta q|B)_{\omega} < \infty \) and \( H(P|\delta p|C)_{\omega} < \infty \), then we have that
\[
h(Q|B)_{\omega} + h(P|C)_{\omega} \geq \log 2 \pi . \tag{114}
\]

Let us first note that for states with finite expectation for the operator \( Q^2 + P^2 \) we can always find a spacing for which \( H_{\text{max}}(Q|\delta q)_{\omega} \), \( H(Q|\delta q|B)_{\omega} \) and \( H(P|\delta p|C)_{\omega} \) are less than \( \infty \). This follows from Lemma 5 which says that the condition is satisfied whenever \( \omega_A(P^2) \) and \( \omega_A(Q^2) \) are finite.\(^{20}\) Hence, if considering modes of an electromagnetic field, the uncertainty relation for the conditional min-and max-entropy holds for any state with finite mean energy, while for the conditional von Neumann entropy the only further assumption is that \( h(Q|B)_{\omega} \) and \( h(P|C)_{\omega} \) are not \( -\infty \).

The uncertainty relation for the differential min- and max-entropy (113) is already sharp without quantum memory.\(^{20}\) The minimal uncertainty states are pure Gaussian states, where the product of the variances of the position and momentum measurements are minimal, that is, \( \text{Var}(Q) \text{Var}(P) = \frac{1}{4} \). This follows from the fact that for a Gaussian distribution \( X \) with variance \( \sigma \), \( H_{\text{min}}(X) = \log \sqrt{2\pi \sigma} \) and \( H_{\text{max}}(X) = \log 2 \sqrt{2\pi \sigma} \). However, the von Neumann entropy version (114) is not sharp in the case of no quantum memory, since the sharp inequality
\[
H(Q)_{\omega} + H(P)_{\omega} \geq \log 2 \pi \tag{115}
\]
holds.\(^{11,9}\) Another sharp uncertainty relation without quantum memory has recently been shown by Lieb and Frank \(^{17}\): \( H(Q)_{\omega} + H(P)_{\omega} \geq \log 2 \pi + H(A)_{\omega} \) (see also \( \text{13, 15} \)). But strikingly, the uncertainty relation (114) is sharp if we include quantum memory. In particular, take \( \mathcal{M}_B = \mathcal{B}(L^2(\mathbb{R})) \) and \( \mathcal{M}_C \) trivial, then the EPR state is\(^{20}\) on \( AB \) for infinite squeezing saturates inequality (114). Note that the EPR state is a Gaussian state with covariance matrix
\[
\Gamma^{AB}(\nu) = \frac{1}{2} \left( \nu I_2 - \sqrt{\nu^2 - 1} Z \right), \tag{116}
\]
where \( \nu = \cosh(2r) \) with \( r \) the squeezing strength and \( Z = \text{diag}(1, -1) \). The covariance matrix is written with respect to a phase space parametrization given by \( (q_A, p_A, q_B, p_B) \). In the following we denote by \( \omega_{AB}^{\nu} \) the Gaussian state corresponding to \( \Gamma^{AB}(\nu) \). The variance of the outcome distribution of the \( P \) measurement on the \( A \) system (which is Gaussian as well) is given by \( \Gamma^{AB,2}_{2,2}(\nu) = \nu/2 \), and thus,
\[
\text{var}(P)_{\omega^{\nu}} = \log(\nu)/2 + \log \sqrt{\nu} . \tag{117}
\]
In order to compute \( h(Q|B)_{\omega^{\nu}} \), we first note that since \( h(Q)_{\omega^{\nu}} < \infty \) we can use Proposition 8 to write
\[
h(Q|B)_{\omega^{\nu}} = h(Q)_{\omega^{\nu}} - D(\omega^{\nu}_{Q|B} || \omega_Q \otimes \omega_B) . \tag{118}
\]
Due to \( \Gamma^{AB,1}_{1,1}(\nu) = \Gamma^{AB,2}_{2,2}(\nu) \), we get \( h(Q)_{\omega^{\nu}} = h(P)_{\omega^{\nu}} \). By using disintegration theory (\textit{Chapter IV.7}) we can further compute
\[
D(\omega^{\nu}_{Q|B} || \omega_Q \otimes \omega_B) = \int \omega_Q(x) \text{tr}(\omega^{\nu}_{Q|B} \log \omega^{\nu}_{Q|B}) dx + h(Q)_{\omega^{\nu}} - h(Q)_{\omega^{\nu}} + H(B)_{\omega^{\nu}} , \tag{119}
\]
where \( \omega^{\nu}_{Q|B} \) is the normalized post-measurement state on \( B \) conditioned on the outcome \( x \) of the \( Q \) measurement on \( A \). Note that for every \( x \) the state \( \omega^{\nu}_{Q|B} \) is pure\(^{21}\) and thus, we end up with \( D(\omega^{\nu}_{Q|B} || \omega_Q \otimes \omega_B) = H(B)_{\omega^{\nu}} \). Note that in the case of a Gaussian state the von Neumann entropy only depends on the symplectic eigenvalues of the covariance

\(^{20}\) Note that due to the data processing inequality and the fact that max-entropy is larger as the von Neumann entropy, the latter is as well bounded whenever the assumption of Lemma 5 are satisfied.

\(^{21}\) The states \( \omega^{\nu}_{Q|B} \) are Gaussian states with a covariance matrix independent on \( x \) and a displacement of \( x \)\(^{30}\).
matrix \[21\] [59], and in our case it is given by \( H(B)_{\nu} = t \log t - (t-1) \log(t-1) \) with \( t = (\nu + 1)/2 \). Hence, we finally get a closed expression for the left hand side of (114)

\[
f(\nu) = \log(e \pi \nu) - \frac{\nu + 1}{2} \log \frac{\nu + 1}{2} + \frac{\nu - 1}{2} \log \frac{\nu - 1}{2}.
\]

(120)

Note that \( f(\nu) \to \log(2 \pi) \) for \( \nu \to \infty \) and the uncertainty relation (114) gets sharp. Note further that the speed of convergence is exponentially (which is illustrated in Figure 2). Hence, we obtain the following sharpness results for our position and momentum uncertainty relations.

**Proposition 16.** The entropic uncertainty relations for continuous position and momentum measurements stated in (113) and (114) are sharp.

![Graph](image)

**FIG. 2:** The plots show the gap \( f(\nu) - \log(2 \pi) \) in dependence on the squeezing \( r (\nu = \cosh 2r) \). The plot on the right hand side illustrates the exponentially fast convergence in \( r \). Note that the mean energy of an EPR state with squeezing \( r \) with respect to the harmonic oscillator Hamiltonian \( Q_A^2 + P_A^2 + Q_B^2 + P_B^2 \) is given by \( 1 + 2 \sinh^2(r/2) \). Considering two-mode squeezed vacuum states of light, an experimentally achievable squeezing of 10dB [51] corresponds to a squeezing of \( r \approx 1.5 \) for which the gap is already negligible.

6. DISCUSSION AND OUTLOOK

We have shown entropic uncertainty relations in the presence of quantum memory for states on von Neumann algebras. Our relations are expressed in terms of differential conditional von Neumann entropy and differential conditional min- and max-entropy. Whereas von Neumann entropy based measures are the most studied in quantum physics and asymptotic quantum information theory, the conditional min- and max-entropy have applications in non-asymptotic quantum information theory and quantum cryptography. For example, our finite spacing position and momentum relation for conditional min- and max-entropy (105) was the key ingredient for the first quantitative security analysis against coherent attacks in continuous variable quantum key distribution [21]. We have further shown that the uncertainty relations in the continuous case are sharp in the sense that there exists a state for which equality holds. Hence, they provide the best possible state-independent bounds.

An interesting open question concerns the derivation of bipartite continuous variable entropic uncertainty relations. Recently, Frank and Lieb [18] extended the relation (2), and proved that for any finite-dimensional bipartite quantum state \( \rho_{AB} \) as well as finite measurements \( \{ E_x \}_{x \in X} \) and \( \{ F_y \}_{y \in Y} \) on \( A \),

\[
H(X|B) + H(Y|B) \geq \log \frac{1}{c_1} + H(A|B),
\]

(121)

where \( c_1 = \max_{x,y} \text{tr}[E_x F_y] \). The constant \( c_1 \) agrees with the constant \( c \) if at least one of the measurements is rank-one projective [14], and is otherwise an upper bound. Frank and Lieb then go on and extend this to continuous position and momentum measurements, for which they get \( c_1 = \frac{1}{2 \pi} \). An alternative generalization of (2) (that is

\[22\] Yet, Lieb and Frank only consider a somewhat restricted form of conditional von Neumann entropy.
tighter than the relation \[121\] for some natural applications) was presented by one of the authors in his thesis \[52\].

This approach is motivated by the following gedankenexperiment. Consider a quantum system \(A\) comprised of two qubits, \(A_1\) and \(A_2\), where \(A_1\) is maximally entangled with a second system, \(B\), and \(A_2\) is in a fully mixed state, product with \(A_1\) and \(B\). We employ projective measurements \(E^1\) and \(F^1\) which measure \(A_1\) in two mutually unbiased bases and leave \(A_2\) intact.\(^{23}\) Analogously, \(E^2\) and \(F^2\) measure \(A_2\) in mutually unbiased bases and leave \(A_1\) intact. Evaluating the terms of interest for this setup yields \(c(E^1, F^1) = c(E^2, F^2) = \frac{1}{2}\) and \(c_1(E^1, F^1) = c_1(E^2, F^2) = 1\) as well as \(H(A|B) = H(A_1|B) + H(A_2) = -1 + 1 = 0\). Indeed, if the maximally entangled system \(A_1\) is measured, we find that \(H(X|B) + H(Y|B) = 0\), and the bound by Frank and Lieb \[121\] is sharp. On the other hand, if \(A_2\) is measured instead, we expect that \(H(X|B) + H(Y|B) = 2\) and the bound is far from sharp. Examining the above example, it is clear that the expected uncertainty depends strongly on which of the two systems is measured and thus, how much entanglement is consumed. However, this information is not taken into account by the overlaps \(c\) or \(c_1\), nor by the entanglement of the initial state, \(H(A|B)\). In the above example, it is straightforward to see that if \(A_1\) (\(A_2\)) is measured, the average entanglement left in the post-measurement state measured by the von Neumann entropy is given by \(H(A_2|B)\) (\(H(A_1|B)\)). Hence, \[2\] would turn into

\[
H(X|B) + H(Y|B) \geq \log \frac{1}{c} + \left( H(A|B) - H(A'|B) \right),
\]

where \(A'\) corresponds to \(A_2\) if \(A_1\) is measured and \(A_1\) if \(A_2\) is measured instead. It is easy to verify that the above inequality is sharp for both examples considered so far. This suggests that \[2\] can be generalized by considering the difference in entanglement of the state before and after measurement. The entanglement of the post-measurement state vanishes for rank-one projective measurements, which is why this contribution was initially overlooked — however, it must be accounted for when considering general measurements. An alternative bipartite entropic uncertainty relation can now be stated in the following way. Let \(\{E_x\}_{x \in X}\) and \(\{F_y\}_{y \in Y}\) be finite measurements on a finite-dimensional quantum system \(A\), and denote the Stinespring dilations of \(\{E_x\}_{x \in X}\) and \(\{F_y\}_{y \in Y}\) by \(U\) and \(V\), respectively. Then, it holds for any finite-dimensional bipartite quantum state \(\rho_{AB}\) that

\[
H(X|B)_{\omega \circ E} + H(Y|B)_{\omega \circ F} \geq -\log c(E_X, F_Y) + H(A|B)_{\omega} - \min \left\{ H(A|XB)_{\omega \circ V}, H(A|YB)_{\omega \circ U} \right\}.
\]

It is possible to reformulate the relation \[123\] in terms of von Neumann mutual information or von Neumann conditional mutual information, and it would be interesting to find formulations of \[123\] that also hold for continuous measurements and infinite-dimensional quantum memory.

Finally, it was already suggested in \[8\] to use bipartite entropic uncertainty relations in the presence of quantum memory for entanglement witnessing. This would be certainly interesting for continuous variable systems since it provides a simple criterion. Ideas in this direction have been developed in \[39\] \[46\] \[58\].

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\(^{23}\) These are thus rank-two projective measurements of the form, e.g. \(E_0^n = |0\rangle|0\rangle \otimes \text{id}, \ F_0^n = |+\rangle|+\rangle \otimes \text{id}, \text{where } |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\).
Appendix A: Properties of Entropies

1. Min- and Max-Entropy

**Definition 17.** Let $\mathcal{M}$ be a von Neumann algebra, $\omega \in \mathcal{P}(\mathcal{M})$, and $\sigma \in \mathcal{P}(\mathcal{M})$. Then, the max-relative entropy of $\omega$ with respect to $\sigma$ is defined as

$$D_{\max}(\omega \| \sigma) = \inf \{ c \in \mathbb{R} : \omega \leq 2^c \cdot \sigma \}.$$  \hspace{1cm} (A1)

**Definition 18.** Let $\mathcal{M}_{AB} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{M}_B$ with $\mathcal{H}_A$ a finite-dimensional Hilbert space, $\mathcal{M}_B$ a von Neumann algebra, and $\omega_{AB} \in \mathcal{S}_\omega(\mathcal{M}_{AB})$. Then, the conditional min-entropy of $A$ given $B$ is defined as

$$H_{\min}(A|B)_\omega = - \inf_{\sigma_B \in \mathcal{S}(\mathcal{M}_B)} D_{\max}(\omega_{AB} \| \tau_A \otimes \sigma_B),$$  \hspace{1cm} (A2)

where $\tau_A$ denotes the trace on $\mathcal{B}(\mathcal{H}_A)$. Furthermore, the conditional max-entropy of $A$ given $B$ is defined as

$$H_{\max}(A|B)_\omega = \sup_{\sigma_B \in \mathcal{S}(\mathcal{M}_B)} F(\omega_{AB}, \tau_A \otimes \sigma_B).$$  \hspace{1cm} (A3)

**Lemma 19.** Let $\mathcal{M}_A, \mathcal{M}_B$ be von Neumann algebras, $\omega_A, \sigma_A \in \mathcal{P}(\mathcal{M}_A)$, and let $\mathcal{E} : \mathcal{M}_B \rightarrow \mathcal{M}_A$ be a normal, completely positive, and sub-unital map. Then, we have that

$$D_{\max}(\omega_A \| \sigma_A) \geq D_{\max}(\omega_A \circ \mathcal{E} \| \sigma_A \circ \mathcal{E}).$$  \hspace{1cm} (A4)

**Lemma 20.** Let $\mathcal{M}$ be a von Neumann algebra, and $\omega, \sigma \in \mathcal{P}(\mathcal{M})$ with $\sigma \geq \gamma$. Then, we have that

$$D_{\max}(\omega \| \gamma) \geq D_{\max}(\omega \| \sigma).$$  \hspace{1cm} (A5)

**Lemma 21.** Let $\mathcal{M}$ be a von Neumann algebra, $\omega, \sigma \in \mathcal{P}(\mathcal{M})$, and $c > 0$. Then, we have that

$$D_{\max}(\omega \| c \cdot \sigma) = D_{\max}(\omega \| \sigma) + \log 1/c.$$  \hspace{1cm} (A6)

**Proposition 22.** Let $\mathcal{M}_{XBC} = L^\infty(X) \otimes \mathcal{M}_{BC}$ with $(X, \Sigma, \mu)$ a $\sigma$-finite measure space, $\mathcal{M}_{BC}$ a bipartite von Neumann algebra, and $\omega_{XBC} \in \mathcal{S}_\omega(\mathcal{M}_{XBC})$. Then, we have that

$$h_{\min}(X|BC)_\omega \leq h_{\min}(X|B)_\omega \leq h_{\min}(X|BC)_\omega \leq h_{\max}(X|BC)_\omega \leq h_{\max}(X|B)_\omega.$$  \hspace{1cm} (A7)

**Proof.** The inequality for the differential conditional min-entropy is obtained by using that any $E \in \text{Meas}(X, \mathcal{M}_B)$ can be embedded into $\text{Meas}(X, \mathcal{M}_{BC})$ such that $\omega_E^B(E_a) = \omega^B_{\mathcal{E}}(E_a)$. For the differential conditional max-entropy, one exploits the fact that the fidelity can only increase under restrictions to a subsystem, that is, $F(\omega_{BC}, \sigma_{BC}) \leq F(\omega_B, \sigma_B)$ (as shown in [1]). \hfill $\square$

**Lemma 23.** [13] Proposition 4.14 Let $\mathcal{M}_{AB} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{M}_B$ with $\mathcal{H}_A$ a finite-dimensional Hilbert space, $\mathcal{M}_B$ a von Neumann algebra, and $\omega_{AB} \in \mathcal{S}_\omega(\mathcal{M}_{AB})$. Then, we have that

$$H_{\min}(A'|C)_\omega = -H_{\max}(A|B)_\omega,$$  \hspace{1cm} (A9)

where $\omega_{A'B'C}$ is a purification $(\pi, \mathcal{K}, |\xi\rangle)$ of $\omega_{AB}$ with $\mathcal{M}_{A'B'} = \pi(\mathcal{M}_{AB})$ the principal system, and $\mathcal{M}_C = \pi(\mathcal{M}_{A'B'})'$ the purifying system.

2. Von Neumann Entropy

**Lemma 24.** [54] Corollary 5.12 (iii) Let $\mathcal{M}_A, \mathcal{M}_B$ be von Neumann algebras, $\omega_A, \sigma_A \in \mathcal{P}(\mathcal{M}_A)$, and let $\mathcal{E} : \mathcal{M}_B \rightarrow \mathcal{M}_A$ be a normal, completely positive, and unital map. Then, we have that

$$D(\omega_A \| \sigma_A) \geq D(\omega_A \circ \mathcal{E} \| \sigma_A \circ \mathcal{E}).$$  \hspace{1cm} (A10)

We further need a slight extension of the above lemma to sub-unital maps.
Lemma 25. Let $M_A, M_B$ be von Neumann algebras, $\omega_A, \sigma_A \in \mathcal{P}(M_A)$, and let $E : M_B \to M_A$ be a normal, completely positive, and sub-unital map. Then, we have that

$$D(\omega_A \| \sigma_A) \geq D(\omega_A \circ E \| \sigma_A \circ E) + D(\sigma_A(1 - E(1)) \| \sigma_A(1 - E(1))) \quad (A11).$$

Furthermore, if $\sigma(1) \leq 1$, we have that

$$D(\omega_A \| \sigma_A) \geq D(\omega_A \circ E \| \sigma_A \circ E) + \omega_A(1 - E(1)) \log \omega_A(1 - E(1)) \quad (A12).$$

Proof. Let $M_A \subset B(H_A)$. According to Stinespring’s dilation theorem [34, Theorem 4.1], there exists a Hilbert space $K$ together with a representation $π : M_B \to B(K)$ and a bounded operator $V : H_A \to K$ such that $E(a) = V^* π(a) V$ for all $a \in M_B$. Without loss of generality, we can assume that $K \cong H_A$ by taking $H_A$ large enough. We define now the isometry $V : H_A \to K \otimes \mathbb{C}^2$ by $V = V \otimes |0\rangle + \sqrt{1 - V^* V} \otimes |1\rangle$, where $|0\rangle, |1\rangle$ denotes an orthonormal basis of $\mathbb{C}^2$. Let us further define the projectors $P = I \otimes |0\rangle\langle 0|$ and $P_⊥ = I \otimes |1\rangle\langle 1|$ in $K \otimes \mathbb{C}^2$. It is then easy to see that $E = T_V \circ T_P \circ π_A$, where $π_A = π_A \otimes I$ and $T_M$ denotes the map $a \mapsto M^* a M$ for every $M$ with suitable range. Applying the monotonicity of the quantum relative entropy under unital completely positive maps, Lemma 24, we find that

$$D(\omega_A \| \sigma_A) \geq D(\omega_A \circ T_V \circ (T_P + T_{P⊥}) \| \sigma_A \circ T_V \circ (T_P + T_{P⊥})) \quad (A13)$$

$$= D(\omega_A \circ T_V \circ T_P \| \sigma_A \circ T_V \circ T_P) + D(\omega_A \circ T_V \circ T_{P⊥} \| \sigma_A \circ T_V \circ T_{P⊥}) \quad (A14),$$

where we used in the second step that the map $T_P + T_{P⊥}$ divides the range into two orthogonal subspaces for which the spatial derivative decays in a direct sum with respect to these orthogonal subspaces. Applying $π_A$ to the first term, we obtain $D(\omega_A \circ T_V \circ T_P \| \sigma_A \circ T_V \circ T_P) \geq D(\omega_A \circ E \| \sigma_A \circ E)$ due to $E = T_V \circ T_P \circ π_A$ and Lemma 24.

To the second term we apply the restriction onto the subalgebra generated by $P_⊥$, which is isomorphic to $\mathbb{C}$, to find $D(\omega_A \circ T_V \circ T_{P⊥} \| \sigma_A \circ T_V \circ T_{P⊥}) \geq D(\omega_A(1 - E(1)) \| \sigma_A(1 - E(1)))$ with Lemma 24. This proves the first assertion. The second one follows from the first by using that the term $-\omega_A(1 - E(1)) \log \sigma_A(1 - E(1))$ is positive whenever $\sigma(1) \leq 1$. 

Lemma 26. [34, Corollary 5.12 (ii)] Let $M$ be a von Neumann algebra, and $\omega, σ \in \mathcal{P}(M)$ with $σ \geq γ$. Then, we have that

$$D(\omega \| γ) \geq D(ω \| σ) \quad (A15).$$

Lemma 27. [34, Proposition 5.1] Let $M$ be a von Neumann algebra, $ω, σ \in \mathcal{P}(M)$, and $c > 0$. Then, we have that

$$D(ω \| c \cdot σ) = D(ω \| σ) + ω(1) \log \frac{1}{c} \quad (A16).$$

Lemma 28. [34, Corollary 5.12 (iv)] Let $M$ be a von Neumann algebra, and let $\{M_i\}_{i \in \mathbb{N}}$ be a sequence of von Neumann subalgebras of $M$ such that their union is weakly dense in $M$. If $ω, σ \in \mathcal{P}(M)$, then the increasing sequence $D(ω_{M_i} \| σ_{M_i})$ converges to $D(ω \| σ)$, where $ω_{M_i}$ denotes the restriction of $ω$ onto the subalgebra $M_i$.

Lemma 29. [34, Corollary 5.20] Let $M_{AB} = M_A \otimes M_B$ be a tensor product of von Neumann algebras, and let $ω_{AB} \in \mathcal{S}(M_{AB})$, $σ_A \in \mathcal{S}(M_A)$, as well as $σ_B \in \mathcal{S}(M_B)$. Then, we have that

$$D(ω_{AB} \| σ_A \otimes σ_B) = D(ω_A \| σ_A) + D(ω_B \| σ_B) \quad (A17).$$

Lemma 30. Let $M_{AB} = B(H_A) \otimes M_B$ with $H_A$ a finite-dimensional Hilbert space, $M_B$ a von Neumann algebra, and $ω_{AB} \in \mathcal{S}_≤(M_{AB})$. Then, we have that

$$H(A'|C)_ω = -H(A|B)_ω \quad (A18),$$

where $ω_A'B'C$ is a purification $(π, K, |ξ⟩)$ of $ω_{AB}$ with $M_{A'B'} = π(M_{AB})$ the principal system, and $M_C = π(M_{A'B'})'$ the purifying system.

Proof. Let $V$ be the Stinespring dilation of the trace map, i.e., $tr_A(x) = V^* x \otimes I V$ for $x \in M_A$. By the definition of the quantum relative entropy (Definition 6), the assertion would follow from the identity

$$Δ(ω_{AB}^* / V ω_B V^*) = Δ(V ω_B V^*/ω_{AB}^*)^{-1} = Δ(V ω_C V^*/ω_{AC}^*) \quad (A19),$$
where the first equality can be found in [34, Chapter 4]. However, the commutant of $\mathcal{M}_{AC}$ is exactly $\mathcal{M}_B$, hence $\omega'_{AC} = \omega_B$, and similarly $\omega'_{AB} = \omega_C$. We are left to show that for $c \in \mathcal{M}_C$

$$\omega_C(r_{\omega_{B}V^*} (cV(\xi^*)) r_{\omega_{B}V^*} (cV(\xi^*)^*)) = \omega_C(\text{tr}_A[r_{\omega_{B}}(cV(\xi^*)) r_{\omega_{B}}(cV(\xi^*)^*)]) ,$$

(A20)

for $\xi'$ the GNS vector associated with $\omega_B$. Indeed, the state $\text{tr}_A \otimes \omega_B$ may assumed to be faithful for $\mathcal{M}_{AB}$ (otherwise restrict everything to the support of $\omega_{AB}$) from which it follows that both $\mathcal{M}_{AB}$ as well as $\mathcal{M}_C = \mathcal{M}'_{AB}$ are faithfully represented on the associated GNS Hilbert space $\mathcal{H}$. It also follows that $\mathcal{H} = HS(\mathcal{H}_A) \otimes \mathcal{H}_B$, where $|A|$ denotes the dimension of $A$ and $\mathcal{H}_B$ is the GNS Hilbert space associated to $\mathcal{M}_B$ with respect to $\omega_B$. The discussion so far implies that the linear span of vectors of the form $cV(\xi')$ is dense in $\mathcal{H}$ and $r_{\omega_{B}V^*} (cV(\xi^*)) = c$ for $c \in \mathcal{M}_C$, and with this, the left hand side of (A20) becomes

$$\omega_C(r_{\omega_{B}V^*} (cV(\xi^*)) r_{\omega_{B}V^*} (cV(\xi^*)^*)) = \omega_C(cc^*) .$$

(A21)

Moreover, the linear span of vectors $b\xi'$, $b \in \mathcal{M}_B$ is dense in $\mathcal{H}_B$, and we have

$$r_{\omega_{B}}(cV(\xi'))(b\xi') = bcV(\xi') = cV(b\xi')$$

(A22)

since the isometry $V$ just acts as tensoring with the identity in $\mathcal{M}_A$ (in the Hilbert-Schmidt-picture), from which it follows that $r_{\omega_{B}}(cV(\xi'))^* = V^*c$. It may be checked that the action of $V^*$ on $HS(\mathcal{H}_A)$ is given by

$$V^* = \sum_{|A|} |i\rangle \langle i| \pi_L(\langle i|) \cdot \pi_R(|i\rangle) ,$$

(A23)

where $\pi_L$ respectively $\pi_R$ again denote the left respectively right action of some matrix on $HS_{|A|}$. The operator $\pi_L(\langle i|)$ is by definition an element of $\mathcal{M}_C$, whereas $\pi_R(|i\rangle)$ is an element of $\mathcal{M}_A$. The partial trace taken on $\mathcal{M}_A$ then reduces the operator $r_{\omega_{B}}(cV(\xi')) r_{\omega_{B}}(cV(\xi')^*)$ to $cc^*$, which proves the assertion. \hfill \Box

[1] P. M. Alberti. A note on the transition probability over C*-algebras. Letters in Mathematical Physics, 7:25–32, 1983.
[2] P. M. Alberti and A. Uhlmann. On Bures distance and *-algebraic transition probability between inner derived positive linear forms over W*-algebras. Acta Applicandae Mathematicae, 60:1–37, 2000.
[3] H. Araki. Relative entropy of state of von Neumann algebras,. Publications Research Institute for Mathematical Science Kyoto University, 9:809–833, 1976.
[4] W. Becker. Inequalities in Fourier analysis. Annals of Mathematics, 102:159, 1975.
[5] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner. The uncertainty principle in the presence of quantum memory. Nature Physics, 6:659, 2010.
[6] M. Berta, F. Furrer, and V. B. Scholz. The smooth entropy formalism on von Neumann algebras. 2011. arXiv:1107.5460v1.
[7] I. Bialynicki-Birula. Entropic uncertainty relations. Physics Letters, 103:253, 1984.
[8] I. Bialynicki-Birula. Formulation of the uncertainty relations in terms of the Rényi entropies. Physical Review A, 74:052101, 2006.
[9] I. Bialynicki-Birula and J. Mycielski. Uncertainty relations for information entropy in wave mechanics. Communications in Mathematical Physics, 44:129, 1975.
[10] I. Bialynicki-Birula and L. Rudnicki. Entropic uncertainty relations in quantum physics. In Statistical Complexity, pages 1–34. Springer Netherlands, 2011.
[11] O. Bratteli and D. W. Robinson. Operator Algebras and Quantum Statistical Mechanics 1. Springer, 1979.
[12] D. Bures. An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite W*-algebras. Transactions of the American Mathematical Society, 135:199–212, 1969.
[13] P. J. Coles, R. Colbeck, L. Yu, and M. Zwolak. Uncertainty relations from simple entropic properties. Physical Review Letters, 108:210405, 2012.
[14] P. J. Coles, L. Yu, V. Gheorghiu, and R. B. Griffiths. Information theoretic treatment of tripartite systems and quantum channels. Physics Review A, 83:062338, 2011.
[15] E. B. Davies and J. T. Lewis. An operational approach to quantum probability. Communications in Mathematical Physics, 17:239–260, 1970.
[16] D. Deutsch. Uncertainty in quantum measurements. Physical Review Letters, 50:631–633, 1983.
[17] R. L. Frank and E. H. Lieb. Entropy and the uncertainty principle. Annales Henri Poincaré, 13:1711–1717, 2012.
[18] R. L. Frank and E. H. Lieb. Extended quantum conditional entropy and quantum uncertainty inequalities. 2012. arXiv:1204.0825v1.
[19] F. Furrer. Security of Continuous-Variable Quantum Key Distribution and Aspects of Device-Independent Security. PhD thesis, Leibniz University Hannover, 2012.
[20] F. Furrer, J. Aberg, and R. Renner. Min- and max-entropy in infinite dimensions. *Communications in Mathematical Physics*, 306:165–186, 2011.
[21] F. Furrer, T. Franz, M. Berta, A. Leverrier, V. B. Scholz, M. Tomamichel, and R. F. Werner. Continuous variable quantum key distribution: Finite-key analysis of composable security against coherent attacks. *Physical Review Letters*, 109:100502, 2012.
[22] W. Heisenberg. Ueber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 34:376, 1927.
[23] I. I. Hirschman. A note on entropy. *American Journal of Mathematics*, 79:152, 1957.
[24] A. S. Holevo, M. Sohma, and O. Hirota. Capacity of quantum Gaussian channels. *Physical Review A*, 59:1820–1828, 1999.
[25] E. H. Kennard. Zur Quantenmechanik einfacher Bewegungstypen. *Zeitschrift für Physik A*, 44:326–352, 1927.
[26] J. Kiukas and R. F. Werner. Maximal violation of Bell inequalities by position measurements. *Journal of Mathematical Physics*, 51:072105, 2010.
[27] R. König, R. Renner, and C. Schaffner. The operational meaning of min- and max-entropy. *IEEE Transactions on Information Theory*, 55:4674, 2009.
[28] A. A. Kuznetsova. Quantum conditional entropy for infinite-dimensional systems. *Theory of Probability and its Applications*, 55:782–790, 2010.
[29] C.-F. Li, J.-S. Xu, X.-Y. Xu, K. Li, and G.-C. Guo. Experimental investigation of the entanglement-assisted entropic uncertainty principle. *Nature Physics*, 7:752–756, 2011.
[30] J. Lodewyck, M. Bloch, R. García-Patrón, S. Fossier, E. Karpov, E. Diamanti, T. Debuisschert, N. J. Cerf, R. Tualle-Brouri, S. W. McLaughlin, and F. Grangier. Quantum key distribution over 25 km with an all-fiber continuous-variable system. *Physical Review A*, 76:042305, 2007.
[31] H. Maassen and J. Uffink. Generalized entropic uncertainty relations. *Physical Review Letters*, 60:1103, 1988.
[32] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel. On quantum Rényi entropies: a new definition and some properties. 2013. arXiv:1306.3142v1.
[33] G. Murphy. *C*-algebras and Operator Theory*. Academic Press, Inc., Boston, 1990.
[34] M. Ohya and D. Petz. *Quantum Entropy and Its Use*. Springer Verlag Berlin, Heidelberg, New York, 1993.
[35] M. H. Partovi. Entropic formulation of uncertainty for quantum measurements. *Physical Review Letters*, 50:1883, 1983.
[36] D. Petz. *Quantum Information Theory and Quantum Statistics*. Springer Verlag Berlin, Heidelberg, New York, 2008.
[37] R. Prevedel, D. R. Hamel, R. Colbeck, K. Fisher, and K. J. Resch. Experimental investigation of the uncertainty principle in the presence of quantum memory and its application to witnessing entanglement. *Nature Physics*, 7:757–761, 2011.
[38] M. R. Ray and S. J. van Enk. Missing data outside the detector range: application to continuous variable entanglement verification and quantum cryptography. 2013. arXiv:1302.5087v1.
[39] R. Renner. Security of quantum key distribution. *International Journal of Quantum Information*, 6:1, 2008.
[40] L. Rudnicki. Uncertainty related to position and momentum localization of a quantum state. 2010. arXiv:1010.3269v1.
[41] L. Rudnicki. Shannon entropy as a measure of uncertainty in positions and momenta. *Journal of Russian Laser Research*, 32:393, 2011.
[42] L. Rudnicki, S. P. Walborn, and F. Toscano. Optimal uncertainty relations for extremely coarse-grained measurements. *Physical Review A*, 85:042115, 2012.
[43] M. Rumin. Balanced distribution-energy inequalities and related entropy bounds. *Duke Mathematical Journal*, 160:567–597, 2011.
[44] M. Rumin. An entropic uncertainty principle for positive operator valued measures. *Letters in Mathematical Physics*, 100:291–308, 2012.
[45] J. Schneeloch, P. B. Dixon, G. A. Howland, C. J. Broadbent, and J. C. Howell. Witnessing continuous variable entanglement with discrete measurements. 2012. arXiv:1210.4234v1.
[46] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656, 1948.
[47] D. Slepián and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty-I. *The Bell System Technical Journal*, 40:43, 1964.
[48] M. Takesaki. *Theory of Operator Algebras 1*. Springer, 2001.
[49] M. Takesaki. *Theory of Operator Algebras 2*. Springer, 2002.
[50] E. Tobias, V. Händchen, and R. Schnabel. Stable control of 10 db two-mode squeezed vacuum states of light. *Optical Express*, 21:11546–11553, 2013.
[51] M. Tomamichel. A Framework for Non-Asymptotic Quantum Information Theory. PhD thesis, ETH Zürich, 2013.
[52] M. Tomamichel, R. Colbeck, and R. Renner. A fully quantum asymptotic equipartition property. *IEEE Transactions on Information Theory*, 55:5840–5847, 2009.
[53] M. Tomamichel, C. C. W. Lim, N. Gisin, and R. Renner. Tight finite-key analysis for quantum cryptography. *Nature Communications*, 3:634, 2012.
[54] M. Tomamichel and R. Renner. The uncertainty relation for smooth entropies. *Physical Review Letters*, 106:110506, 2011.
[55] A. Uhlmann. The transition probability in the state space of a *-algebra. *Report on Mathematical Physics*, 9:273, 1976.
[56] A Uhlmann. Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. *Journal of Mathematical Physics*, 51:072105, 2010.
[57] A. Uhlmann. The transition probability in the state space of a *-algebra. *Report on Mathematical Physics*, 9:273, 1976.
[58] M. Tomamichel and R. Renner. The uncertainty relation for smooth entropies. *Physical Review Letters*, 109:100502, 2012.
information. Reviews of Modern Physics, 84:621–669, 2012.

[60] S. Wehner and A. Winter. Entropic uncertainty relations - a survey. New Journal of Physics, 12:025009, 2010.