Some Notes on the Similarity of Priority Vectors Derived by the Eigenvalue Method and the Geometric Mean Method

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ABSTRACT
This paper examines the differences in ordinal rankings obtained from a pairwise comparison matrix using the eigenvalue method and the geometric mean method. First, we introduce several propositions on the (dis)similarity of both rankings concerning the matrix size and its inconsistency expressed by the Koczkodaj’s inconsistency index. Further on, we examine the relationship between differences in both rankings and Kendall’s rank correlation coefficient $\tau$ and Spearman’s rank coefficient $\rho$. Apart from theoretical results, intuitive numerical examples and Monte Carlo simulations are also provided.

1. Introduction

In many cases, comparing entities in pairs is easier and more intuitive for experts than having to provide a complete ranking of a larger set of items. As such, it had been used since the thirteenth century [1].

However, until the early twentieth century, comparisons were only qualitative (e.g. “is $a$ better/more preferred than $b$?”). The first attempts to use quantitative comparisons date back to the 1920s, when they were used for applications such as comparing physical stimuli [18, 19] and calculating the results of chess tournaments [21]. Subsequent developments led to applications in other fields, including consumer research, healthcare and economics.

A major breakthrough occurred in the 1970s with the development of the Analytic Hierarchy Process (AHP) by Thomas L. Saaty [16]. Besides the theoretical background, it provided an end-to-end solution supporting multi-criteria decision-making, and was soon followed by a professional-grade software package, used e.g. in strategic planning [17], environmental studies [20], risk management [14], agriculture [15] and manufacturing [3].

Having a set of comparisons made by an expert, usually in the form of a matrix, one may use various algorithms to transform that into a vector of weights. Several algorithms have been defined for that purpose [16, 4]. These weights, in turn, can be used to...
order the entities to form a ranking. For the most commonly-used algorithms, for a given set of comparisons, these rankings will be identical, provided the comparisons are consistent. However, as one can rarely expect full consistency, the resulting order may differ. Inconsistency in pairwise comparisons is determined by the indices [2, 12]. In addition to inconsistency, the ranking can be affected by incompleteness [11]. In this work, however, we will limit our considerations to complete pairwise comparisons. We discuss the issues of inconsistency in 2.

This paper elaborates the issue of ranking similarity by providing several propositions which specify the conditions for the rankings to be identical and to estimate the difference of the rankings (by the means of Kendall’s and Spearman’s correlation coefficients) based on the inconsistency of the comparison matrices.

The rest of the paper is organised as follows. 2 describes the most important concepts of pairwise comparisons. It is followed by 3, which provides the details of ranking comparison methods and the aforementioned propositions, which are the primary contribution of this work. It is followed by 4, which gives a practical context to these contributions by means of Monte Carlo analysis. The work is summarised in 5.

2. Preliminaries

2.1. Multiplicative pairwise comparisons – concepts and notation

This chapter provides brief preliminaries into the multiplicative pairwise comparisons framework.

Let $\mathcal{C} = \{c_1, c_2, ..., c_n\}, n \in N, n > 1$ be a non-empty and finite set of objects (alternatives, criteria, sub-criteria, etc.) being compared. Let $S$ be a pairwise comparison scale\footnote{The most popular is the fundamental scale [16] where $S = \{1/9, 1/8, ..., 1/2, 1, 2, ..., 9\}$}. Let $a_{ij} \in S, a_{ij} > 0$ denote the relative importance/preference of an object $i$ over object $j$, where $i, j \in \{1, \ldots, n\}$. For instance, $a_{ij} = 3$ means that an object $i$ is $3$ times more important, or more preferred, than an object $j$.

Pairwise comparisons of $n$ objects form a $n \times n$ square matrix $A = [a_{ij}]$, called a pairwise comparison matrix (PC matrix, PCM):

$$
A = \begin{bmatrix}
1 & a_{12} & \ldots & a_{1n} \\
{a_{21}} & 1 & \ldots & \ldots \\
\ldots & \ldots & 1 & \ldots \\
{a_{n1}} & \ldots & \ldots & 1 \\
\end{bmatrix},
$$

A pairwise comparisons matrix $A = [a_{ij}]$ is reciprocal if

$$
a_{ij} = 1/a_{ji} \ \forall i, j \in \{1, \ldots, n\}.
$$

Hereinafter, it is assumed that all pairwise comparison matrices are reciprocal.

Let’s call a pairwise comparison method any decision-making method that involves pairwise comparisons, and let a prioritisation method (a priority generating method) be
any procedure that derives a priority vector \( \mathbf{w} = (w_1, \ldots, w_n) \) (vector of weights of all \( n \) compared objects) from a PC matrix.

Based on the values \( w_1, \ldots, w_n \), the compared objects can be ranked from the best to the worst (with possible ties), which is the goal of the majority of pairwise comparison methods.

We say that priority vector \( \mathbf{w} \) is associated with a PC matrix \( \mathbf{A} \), or, that the priority vector \( \mathbf{w} \) is derived by a priority generating method based on the PC matrix \( \mathbf{A} \).

In addition, the priority vector \( \mathbf{w} \) is usually *normalized*, i.e.

\[
\sum_{i=1}^{n} w_i = 1.
\]

*Koczkodaj’s inconsistency index*, \( KI(\mathbf{A}) \), of an \( n \times n \) PC matrix \( \mathbf{A} = [a_{ij}] \) is defined as follows [8]:

\[
KI(\mathbf{A}) = \max \left\{ 1 - \min \left\{ \frac{a_{ij}}{a_{ik}a_{kj}} : k \in \{1, \ldots, n\} \right\} \right\} \quad (1)
\]

It is obvious from (1) that \( 0 \leq KI(\mathbf{A}) < 1 \).

2.2. The eigenvalue method and the geometric mean method for the derivation of a priority vector

The geometric mean method and the eigenvalue (eigenvector) method are two main procedures for derivation of a priority vector from a PC matrix \( \mathbf{A} = [a_{ij}] \). The eigenvalue (EV) method was proposed by Saaty [16], and the geometric mean (GM) method was introduced by Crawford [4].

In the EV method, the priority vector is an eigenvector corresponding to the largest eigenvalue of \( \mathbf{A} = [a_{ij}] \), i.e.

\[
\mathbf{A} \mathbf{w} = \lambda_{\max} \mathbf{w},
\]

where \( \lambda_{\max} \geq n \) is a positive eigenvalue (the existence of \( \lambda_{\max} \) is guaranteed by the Perron-Frobenius theorem), and \( \mathbf{w} \) is the corresponding (right) eigenvector of \( \mathbf{A} \).

Usually, it is assumed that \( \mathbf{w} \) is normalised: \( \|\mathbf{w}\| = 1 \).

In the GM method (the *least logarithmic squares* method), the priority vector \( \mathbf{w} \) is derived as the geometric mean of all rows of \( \mathbf{A} \):

\[
w_i = \left( \prod_{k=1}^{n} a_{ik} \right)^{1/n} / \left( \sum_{j=1}^{n} \left( \prod_{k=1}^{n} a_{ik} \right)^{1/n} \right), \forall i.
\]
This formula is equivalent to finding a solution of the following non-linear programming problem:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} (\ln a_{ij} - \ln \frac{w_i}{w_j})^2
\]

s.t. \( \sum_{i=1}^{n} w_i = 1, w_i \geq 0, \forall i \).

Again, the priority vector \( w \) is normalized.

2.3. Rank correlation coefficients

Rank correlation measures an ordinal relationship between rankings of ordinal variables, where a ranking is the assignment of ordering labels “first”, “second”, “third”, etc. to different observations of a particular variable. A rank correlation coefficient measures the degree of similarity (relation) between two rankings, and can be used to assess the significance of this relation. The best-known rank correlation coefficients include Kendall’s rank correlation coefficient (Kendall’s \( \tau \)) and Spearman’s rank correlation coefficient (Spearman’s \( \rho \)).

Thereinafter, it is assumed that rankings do not contain ties, hence they provide a total order.

Below, definitions of Kendall’s \( \tau \) and Spearman’s \( \rho \) are provided.

**Definition 1.** Let \( X \) and \( Y \) be two rankings (without ties) of \( n \) objects. Pairs \((x_i, x_j)\) and \((y_i, y_j)\), \( i < j \) are called concordant, if either \((x_i > x_j \wedge y_i > y_j)\) or \((x_i < x_j \wedge y_i < y_j)\) holds (in other words, when both objects are ranked in the same order in both rankings. Otherwise, the pairs are called discordant. Let \( n_c \) be the number of concordant pairs and let \( n_d \) denote the number of discordant pairs. Then Kendall’s \( \tau \) is defined as follows:

\[
\tau = \frac{n_c - n_d}{\binom{n}{2}} = \frac{2}{n(n-1)} \sum_{i<j} \text{sgn}(x_i - x_j) \text{sgn}(y_i - y_j)
\]

From the definition it follows that \(-1 \leq \tau \leq 1\). If two rankings are identical, then \( \tau = 1 \); if they are reversed, then \( \tau = -1 \).

**Definition 2.** Let \( X \) and \( Y \) be two rankings (without ties) of \( n \) objects, and let \( d_i \) be the difference between the rank of object \( i \) in \( X \) and in \( Y \). Then, Spearman’s \( \rho \) is defined as follows:

\[
\rho = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}
\]
Again, $-1 \leq \rho \leq 1$, $\rho = 1$ for identical rankings and $\rho = -1$ for reversed rankings.

Both rank coefficients are applied in the next section to measure the association between rankings of objects obtained by the eigenvalue method and the geometric mean method with respect to the inconsistency of the original pairwise comparison matrix.

### 3. Ordinal rankings’ similarity of EVM and GMM

From the previous example, it can be seen that priority vectors obtained by different methods are different.

In general, priority vectors derived using EV and GM methods are identical in the case of a consistent pairwise comparison matrix [5]. Also, in the case of $n = 3$, priority vectors for both methods coincide even when the PC matrix is inconsistent, see e.g. [5] or [9]. The study [6] established that for not-so-inconsistent matrices, both EVM and GMM produce very similar results, with differences ‘small beyond human perception’.

For numerical comparisons of prioritisation methods see e.g. [7] or [13]. Ishizaka and Lusti [7] concluded their study by saying: “There is a high level of agreement between the different (…) techniques (…). We do not think that one method is superior to another. We advice decision makers also to consider other criteria like ‘easy to use’ in selecting of their derivation method”.

From a theoretical point of view, Kułakowski et al. [10] proved the following theorem about the distance between priority vectors obtained via EV and GM methods.

**Theorem 1.** For every $n \times n$ PC matrix $A$ and two rankings $w_{EV}$ and $w_{GM}$, $\|w_{EV}\|$, $\|w_{GM}\|$, obtained by the eigenvalue and geometric mean method respectively, it holds that:

$$\kappa^2 - 1 \leq MD(w_{EV}, w_{GM}) \leq \frac{1}{\kappa^2} - 1,$$

where $\kappa = 1 - KI(A)$, $KI$ is the Koczkodaj’s inconsistency index and $MD$ denotes the Manhattan distance.

**Proof.** See [10].

This result provides a lower and upper bound on how “far apart” both priority vectors can be. Of course, when a PC matrix is consistent ($KI = 0$), then both priority vectors are identical and their distance is zero.

However, in many situations the ordinal ranking of compared objects is more important than the absolute values of their weights. Certainly, a situation when a decision maker obtains a different ranking from the EVM method than from the GMM method constitutes a serious inconvenience.

The proposition below, which directly follows from Theorem 1, addresses this issue.

**Proposition 1.** Let $A$ be a pairwise comparison matrix with Koczkodaj’s inconsistency index $KI(A)$. Let $w^{EM}$ and $w^{GM}$ be the vectors of weights (priority vectors of compared objects) obtained via the EV method and GM method, respectively. Let $O_1, \ldots, O_n$ be the
ordinal ranking of all objects \(O_i\) from the best \((O_1)\) to the worst \((O_n)\) obtained by the EV method, hence \(w_1^{EV} \geq w_2^{EV} \geq \ldots \geq w_n^{EV}\). Further on, let \(d = \min|w_{i+1}^{EV} - w_i^{EV}|\), \(i \in \{1,\ldots,n-1\}\). Then, ordinal rankings of all objects obtained by the EV method and GM method are identical if

\[
d > \frac{1}{\kappa^2} - 1 \geq MD(w_{EV}, w_{GM}).
\]

**Proof.** Let \(d = \min|w_{i+1}^{EV} - w_i^{EV}|\). To obtain a different priority vector \(w^{GM}\) from the GM method, it is necessary that the (Manhattan) distance between \(w^{EV}\) and \(w^{GM}\) is greater than \(d\) (this distance is necessary to "bridge the smallest gap" between adjacent weights \(w_{i+1}^{EV}\) and \(w_i^{GM}\)). However, this contradicts the assumption that \(d > \frac{1}{\kappa^2} - 1 \geq MD(w_{EV}, w_{GM})\). (The gap \(d\) is larger than the distance between \(w^{EV}\) and \(w^{GM}\), hence it is impossible to be “bridged”.)

In real-world situations, only objects ranked at the top of the list are important, and differences at the bottom of a ranking might be irrelevant. An analogous proposition can be formulated for the change of the best object.

**Proposition 2.** Let \(A\) be a pairwise comparison matrix with Koczkodaj’s inconsistency index \(KI(A)\). Let \(w^{EV}\) and \(w^{GM}\) be the vectors of weights (priority vectors of compared objects) obtained via the EV method and GM method, respectively. Let \(O_1,\ldots,O_n\) be the ordinal ranking of all compared objects, from the best to the worst, obtained by the EV method, hence \(w_1^{EV} \geq w_2^{EV} \geq \ldots \geq w_n^{EV}\). Further on, let \(d^* = w_1^{EV} - w_2^{EV}\). Then, the object ranked first by the EV method is also ranked first by the GM method, if \(d^* > \frac{1}{\kappa^2} - 1 \geq MD(w_{EV}, w_{GM})\).

**Proof.** It is analogous the the proof of Proposition 1.

It is obvious that Proposition 2 is a special case of Proposition 1.

To demonstrate the use of Propositions 1 and 2 we provide a following simple, yet illustrative Example 1.

**Example 1.** Let’s consider a PC matrix \(A\) of the order \(n = 3\), where weights derived by the EV method are: \(w = (w_1, w_2, w_3) = (0.60, 0.30, 0.10)\), and \(KI(A) = 0.03\). We will show (prove) that the rank of the best object does not change in the GM method, and also the entire rankings obtained by the EV and GM methods are identical. First, let’s evaluate the best object. We have \(d^* = 0.30\). Now, we evaluate \(\frac{1}{\kappa^2} - 1 = \frac{1}{0.97^2} - 1 = 0.062\). Since \(d^* = 0.30 > 0.062\), then according to Proposition 2, the best object obtained by the GM method is the same. As for the entire ordinal ranking, we get: \(d = \min|w_{i+1}^{EV} - w_i^{EV}| = 0.20 > 0.062\), hence, according to Proposition 1, the ranking by GM method must be identical.

**Remark 1.** Provided that the weights of all alternatives sum up to 1, it is natural that the distance \(d\) is also limited by 1. Indeed, for two alternatives \(a_1\) and \(a_2\), such that \(a_1 \prec a_2\), the greatest possible \(d = w(a_2) - w(a_1)\) is \(1 - \epsilon\), where \(\epsilon\) is a small number greater than 0. However, for three alternatives, the highest possible \(d\) can be achieved
if the priorities of different alternatives are evenly spaced from each other. Indeed, a simple optimisation exercise

\[
\begin{align*}
\max d \\
\text{s.t.} \\
    d &= \min \{w(a_2) - w(a_1), w(a_3) - w(a_2)\} \\
    w(a_1) + w(a_2) + w(a_3) &= 1 \\
    w(a_1) > w(a_2) > w(a_3) &\geq 0,
\end{align*}
\]

indicates that \( d = 1/3 \) providing that \( w(a_1) = 2/3, w(a_2) = 1/3, \) and \( w(a_3) = 0. \) Since all of the weights have to be positive, then in practice we have to assume \( w(a_3) = \epsilon \) and \( d = 1/3 - \epsilon. \) Thus, the maximal \( d \) is upper-bounded by \( 1/3. \) Similar reasoning can be repeated for more alternatives. For example, it is easy to observe that for 4 alternatives \( d = 1/6 - \epsilon, \) and similarly for 5 alternatives, \( d = 1/10 - \epsilon, \) etc. In general, for \( n \) alternatives,

\[
d = \frac{1}{1 + 2 + \ldots + n - 1} - \epsilon.
\]

Thus, the more alternatives, the smaller the allowed distance \( d, \) and hence the lower the required value of the inconsistency index \( KI. \)

Next, we will provide a consequence of Proposition 1 on the values of Kendall’s rank correlation coefficient \( \tau \) and Spearman’s rank correlation coefficient \( \rho. \) Both coefficients measure the similarity of two ordinal rankings, which is applied to the rankings obtained by the EV and GM methods.

**Proposition 3.** Let \( A \) be a pairwise comparison matrix with Koczkodaj’s inconsistency index \( KI(A). \) Let \( w^{EV} \) and \( w^{GM} \) be the vectors of weights (priority vectors of compared objects), obtained via the EV method and GM method, respectively. Let \( O^{EV} = (O_1, ..., O_n) \) be the ordinal ranking of all objects, from the best to the worst, obtained by the EV method and let \( O^{GM} \) be the ordinal ranking of all objects, from the best to the worst, obtained by the GM method. Further on, let \( d = \min \{d_i\} \) be the smallest difference between adjacent EVM weights and let \( \frac{1}{k^2} - 1 = K. \) Let \( k \) be an integer number such that \( 0 \leq k \leq \binom{n}{2}, k \cdot d < K \) and \( (k + 1) \cdot d > K. \) Then \( \tau(O^{EV}, O^{GM}) \geq \frac{(\binom{n}{2}) - 2k}{(\binom{n}{2})}. \)

**Proof.** For \( k = 0, \) we get \( 0 < K \) and \( d > K, \) hence no change in the ranking can occur (see the proof of Proposition 1). Therefore, \( \tau(O^{EV}, O^{GM}) \geq \frac{(\binom{n}{2}) - 0}{(\binom{n}{2})} = 1, \) and the EVM and GMM rankings are identical. For \( k = 1 \) we get \( d < K \) and \( 2d > K; \) hence, at most one difference (one discordant pair) in the rankings can occur, which means that one originally concordant pair changes into a discordant pair and the numerator decreases by two, therefore \( \tau(O^{EV}, O^{GM}) \geq \frac{(\binom{n}{2}) - 2}{(\binom{n}{2})}. \) The number \( k \) denotes the maximum number of differences in the ranking that can occur, which is equal to the maximum number of discordant pairs that can appear (and replace concordant pairs). Therefore, in general, \( \tau(O^{EV}, O^{GM}) \geq \frac{(\binom{n}{2}) - 2k}{(\binom{n}{2})}. \)
Proposition 3 postulates how much the EV and GM rankings can differ in terms of Kendall’s \( \tau \) (in other words, it postulates its lower bound on \( \tau \)). The lower is the upper boundary \( (K) \) given by Theorem 1 on the difference between \( w^{EV} \) and \( w^{GM} \) (in other words, the lower the inconsistency of the original PC matrix), and the larger the differences between adjacent weights in \( w^{EV} \), the higher is the value of Kendall’s \( \tau \).

**Example 2.** Estimate Kendall’s \( \tau \) between EVM and GMM rankings if a PC matrix \( A \) is of the order \( n = 5 \), \( KI(A) = 0.11 \) and \( d = 0.08 \).

First, we evaluate \( K = \frac{1}{k^2} - 1 = \frac{1}{0.89^2} - 1 = 0.262 \). Next, we find the value of \( k \) such that \( 0 \leq k \leq \binom{n}{2} \), \( k \cdot d < K \) and \( (k + 1) \cdot d > K \), which is \( k = 3 \). Therefore, according to Proposition 3, \( \tau \geq \frac{6}{\binom{n}{2}} = 0.40 \).

**Proposition 4.** Let \( A \) be a pairwise comparison matrix with Koczkodaj’s inconsistency index \( KI(A) \). Let \( w^{EV} \) and \( w^{GM} \) be the vectors of weights (priority vectors of compared objects) obtained via the EV method and GM method, respectively. Let \( O^{EV} = (O_1, \ldots, O_n) \) be the ordinal ranking of all objects, from the best to the worst, obtained by the EV method, and let \( O^{GM} \) be the ordinal ranking of all objects, from the best to the worst, obtained by the GM method. Further on, let \( d = \min \{d_i\} \) be the smallest difference between adjacent EVM weights and let \( \frac{1}{k^2} - 1 = K \). Let \( k \) be an integer number such that \( 0 \leq k \leq \binom{n}{2} \), \( k \cdot d < K \) and \( (k + 1) \cdot d > K \). Then, \( \rho(O^{EV}, O^{GM}) \geq 1 - \frac{6(k^2+k)}{n(n^2-1)} \).

**Proof.** Let’s start with the value of \( \sum_{i=1}^{n} d_i^2 \). This value is the sum of quadratic differences in both rankings. For a given, \( k \) – the maximal feasible number of adjacent changes (swaps) between both rankings – is the maximal value of \( D \) obtained when one object changes its rank by not more than \( k \) positions (since this change is squared in \( D \)), while \( k \) other objects change their rank by 1. This gives \( D_{\text{max}} = k^2 + k \). Since actual \( D \) is smaller than or equal to \( D_{\text{max}} \), we obtain the desired formula – a lower bound for Spearman’s \( \rho \).

In the next example, the use of Proposition 4 is demonstrated.

**Example 3.** Estimate Spearman’s \( \rho \) between EVM and GMM rankings, if a PC matrix \( A \) is of the order \( n = 5 \), \( KI(A) = 0.07 \) and \( d = 0.07 \). First, we evaluate \( K = \frac{1}{k^2} - 1 = \frac{1}{0.93^2} - 1 = 0.1562 \). Next, we find the value of \( k \) such that \( 0 \leq k \leq \binom{n}{2} \), \( k \cdot d < K \) and \( (k + 1) \cdot d > K \), which is \( k = 2 \). Therefore, according to Proposition 4, \( \rho(O^{EV}, O^{GM}) \geq 1 - \frac{6(k^2+k)}{n(n^2-1)} = 1 - \frac{6(2^2+2)}{5(5^2-1)} = 0.70 \).

4. Monte Carlo analysis

The first proposition (3) formulates sufficient conditions for the EVM and GMM rankings to be identical in an ordinal sense. Indeed, if \( d \) (the smallest distance between subsequent priorities in \( w^{EV} \)) is greater than \( 1/k^2 - 1 \) then the orders of alternatives determined by \( w^{EV} \) and \( w^{GM} \) are identical. In practice, however, this guarantee is difficult
to obtain. This is due to two reasons. The first of them was mentioned in Remark 1, according to which the more alternatives, the smaller the possible $d$. Similarly, the more alternatives, the greater the right side of 2 is. It suggests that the effect described by Proposition 1 can be observed with a small number of alternatives. The second reason stems from the local nature of Koczkodaj’s index $KI$. For the value of this index to be high, it is enough for an expert to make a mistake in only one comparison. Hence, a matrix with a relatively low Saaty’s $CI$ but a relatively high Koczkodaj’s $KI$ can be found quite often in practice.

![Figure 1](image1.png)

**Figure 1.** Distribution of randomly generated PC matrices $3 \times 3$ for which condition (2) is met with respect to the value of $KI$.

In the presented Monte Carlo analysis, we generate 362,750 matrices with varying degrees of disturbance$^2$. We generate them in such a way that each comparison in an initially connected matrix is multiplied by some scalar $\alpha \in [1, \beta]$, where the value $\beta$ is gradually increased.

![Figure 2](image2.png)

**Figure 2.** Distribution of randomly generated PC matrices $3 \times 3$ for which condition (2) is met with respect to the value of $CI$.

$^2$We prepare 250 consistent random matrices and then disturb them using the disturbance coefficient from 1, 1.02, 1.04, up to 30. Hence, the total number of matrices in the study is $250 \times 1451 = 362,750$. 

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For these matrices, we check for how many of them the condition in 2 is met, depending on the value of the inconsistency index $KI$. In 1, we can see that in the case of a $3 \times 3$ matrix, the higher the inconsistency index $KI$, the smaller the number of cases in which the postulate from Proposition 1 is satisfied. Moreover, out of the total of 362,750 $3 \times 3$ matrices, only for 14,262, i.e. for 3.93% of all matrices, this condition was met. These values significantly decrease for matrices of $4 \times 4$ size. With the same number of analyzed matrices, i.e. 362,750, the sufficient criterion (2) is met only in 636 cases (0.175% of analyzed cases).

Figure 3. Distribution of randomly generated PC matrices $3 \times 3$ for which condition (2) is not met with respect to the value of $KI$.

Figure 4. Distribution of randomly generated PC matrices $3 \times 3$ for which condition (2) is not met with respect to the value of $CI$.

The values of the Saaty index for which the condition (2) is met are low. Here, we can also see the regularity according to which the higher the inconsistency, the fewer cases in which the condition is met (2).

Since the proposed condition is sufficient but not necessary, there are often matrices for which this condition is not met, and yet the order of alternatives induced by both vectors $w_{EV}$ and $w_{GM}$ is identical. Among these cases, the most numerous group are matrices with a high value of the Koczkodaj’s inconsistency index (3).
This result is somewhat surprising, since it is pretty straightforward to generate a matrix that is highly inconsistent in terms of $KI$ – it is enough to disturb only one comparison in the matrix significantly. As expected, the number of PC matrices for which the order induced by both vectors $w_{EV}$ and $w_{GM}$ is identical decreases as the inconsistency measured by $CI$ rises.

The conducted experiments show that the criterion based on Koczkodaj’s index is quite restrictive. As a result, its practical usefulness is moderate. However, thanks to this restrictiveness, it was possible to formulate an analytical condition for the ordinal identity of vectors calculated using two different ranking methods. The large number of cases not covered by the criterion (2) for which the order induced by both ranking methods is identical may suggest that there is a better criterion for the ordinal agreement of the vectors. Unfortunately, at the moment, such a criterion is unknown to the authors of this study.

5. Conclusions

In the AHP method, the order of the alternatives is often more critical than their numerical priorities. Therefore, in our study, we focused on the ordinal aspect of the ranking calculated using the two methods, EVM and GMM. We proposed a sufficient criterion for the ordinal compliance of these two methods. We also estimated the Kendall’s $\tau$ coefficient under certain assumptions. A Monte Carlo analysis accompanies the theoretical considerations. It shows that the nature of the theoretically formulated criteria is relatively restrictive. i.e. the set of pairwise comparisons must have a really high consistency in order to have guaranteed stability of the result under criterion (2). For example it is easy to see (Fig. 1) that most $3 \times 3$ matrices that meet the criterion have a CI smaller than $10^{-2}$. It is not much. On the other hand, a small (acceptable) inconsistency also does not guarantee that the criterion will be met (Fig. 4). This restrictiveness limits the practical applicability of the obtained academic results; on the other, it suggests the that better criteria, applicable in more cases, may be determined. The search for such criteria will be the subject of further research by the authors.

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