Asymptotics of the number of standard Young tableaux of skew shape

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\textbf{A B S T R A C T}

We give new bounds and asymptotic estimates on the number of standard Young tableaux of skew shape in a variety of special cases. Our approach is based on Naruse’s hook-length formula. We also compare our bounds with the existing bounds on the numbers of linear extensions of the corresponding posets.

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1. Introduction

The classical hook-length formula (HLF) allows one to compute the number $f^\lambda = |\text{SYT}(\lambda)|$ of standard Young tableaux of a given shape \cite{Macdonald}. This formula had profound applications in Enumerative and Algebraic Combinatorics, Discrete Probability, Representation Theory and other fields (see e.g. \cite{Stembridge, Stanley, Stembridge2}). Specifically, the HLF allows to derive asymptotics for $f^\lambda$ for various families of “large” partitions $\lambda$. This was famously used to compute the diagram of a random representation of $S_n$ with respect to the Plancherel measure $(f^\lambda)^2/n!$, see \cite{Macdonald3, Stanley3} (see also \cite{Arias-Castro, Stembridge2}).

For skew shapes $\lambda/\mu$, little is known about the asymptotics, since there is no multiplicative formula for $f^{\lambda/\mu} = |\text{SYT}(\lambda/\mu)|$. For large $n = |\lambda/\mu|$, the asymptotics are known for a few special families of skew shapes (see \cite{Arias-Castro}), and for fixed $\mu$ (see \cite{Stembridge2, Stembridge3, Stembridge4}). In this paper we show that the “naive HLF” gives good approximations for $f^{\lambda/\mu}$ in many special cases of interest.

Formally, let $\lambda$ be a partition of $n$. Denote by $f^\lambda = |\text{SYT}(\lambda)|$ the number of standard Young tableaux of shape $\lambda$. We have:

\[
f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)} ,
\]

(\textit{HLF})

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where \( h(u) = \lambda_i - i + \lambda'_j - j + 1 \) is the hook-length of the square \( u = (i, j) \). Now, let \( \lambda/\mu \) be a skew shape, \( n = |\lambda/\mu| \). By analogy with the HLF, define

\[
F(\lambda/\mu) := \frac{n!}{\prod_{u \in \lambda/\mu} h(u)}, \quad \text{(naive HLF)}
\]

where \( h(u) \) are the usual hook-lengths in \( \lambda \).

Our main technical tool is Theorem 3.3, which gives

\[
(*) \quad F(\lambda/\mu) \leq f^{\lambda/\mu} \leq \xi(\lambda/\mu)F(\lambda/\mu),
\]

where \( \xi(\lambda/\mu) \) is defined in Section 3. These bounds turn out to give surprisingly sharp estimates for \( f^{\lambda/\mu} \), compared to standard bounds on the number \( e(\mathcal{P}) \) of linear extensions for general posets.\(^2\) We also give several examples when the lower bound is sharp but not the upper bound, and vice versa (see e.g. Sections 9.1 and 10.2).

Let us emphasize an important special case of thick ribbons \( \delta_{k+r}/\delta_k \), where \( \delta_k = (k - 1, k - 2, \ldots, 2, 1) \) denotes the staircase shape. The following result illustrates the strength of our bounds (cf. Section 3.3).

**Theorem 1.1.** Let \( v_k = (\delta_{2k}/\delta_k) \), where \( \delta_k = (k - 1, k - 2, \ldots, 2, 1) \). Then

\[
\frac{1}{6} - \frac{3}{2} \log 2 + \frac{1}{2} \log 3 + o(1) \leq \frac{1}{n} \left( \log f^{v_k} - \frac{1}{2} n \log n \right) \leq \frac{1}{6} - \frac{7}{2} \log 2 + 2 \log 3 + o(1),
\]

where \( n = |v_k| = k(3k - 1)/2 \).

Here the LHS \( \approx -0.3237 \), and the RHS \( \approx -0.0621 \). Note that the numbers \( f^{\delta_{k+r}/\delta_k} \) of standard Young tableaux for thick ribbons (see [44, A278289]) have been previously considered in [2], but the tools in that paper apply only for \( r \to \infty \).

We should mention that in the theorem and in many other special cases, the leading terms of the asymptotics are easy to find. Thus most of our effort is over the lower order terms which are much harder to determine (see Section 12). In fact, it is the lower order terms that are useful for applications (see Sections 8.3 and 12.6).

The rest of the paper is structured as follows. We start with general results on linear extensions (Section 2), standard Young tableaux of skew shape (Section 3), and excited diagrams (Section 4). We then proceed to our main results concerning the asymptotics for \( f^{\lambda/\mu} \) in the following cases:

1. when both \( \lambda, \mu \) have the Thoma–Vershik–Kerov limit (Section 5). Here the Frobenius coordinates scale linearly and \( f^{\lambda/\mu} \) grow exponentially.
2. when both \( \lambda, \mu \) have the stable shape limit (Section 6). Here the row and column lengths scale as \( \sqrt{n} \), and \( f^{\lambda/\mu} \approx n^{O(1)} \) up to an exponential factor.
3. when \( \lambda/\mu \) have subpolynomial depth (Section 7). Here both the row and column lengths of \( \lambda/\mu \) grow as \( n^{O(1)} \), and \( f^{\lambda/\mu} \approx n! \) up to a factor of intermediate growth (i.e. super-exponential and subfactorial), which can be determined by the depth growth function.
4. when \( \lambda/\mu \) are a large ribbon hook (Section 9). Here \( \lambda/\mu \) scale linearly along a fixed curve, and \( f^{\lambda/\mu} \approx n! \) up to an exponential factor.
5. when \( \lambda/\mu \) is a slim shape (Section 10). Here \( \mu \) is fixed and \( \ell, \lambda/\ell \to \infty \), where \( \ell = \ell(\lambda) \). Here \( f^{\lambda/\mu} \sim f^2 f^{\mu/|\mu|} \).

We illustrate these cases with various examples. Further examples and more specialized applications are given in Sections 8 and 11. We conclude with final remarks in Section 12.

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1 Note that \( F(\lambda/\mu) \) is not necessarily an integer.
2 Both \( e(\mathcal{P}) \) and \( f^{\lambda/\mu} \) are standard notation in respective areas. For the sake of clarify and to streamline the notation, we use \( e(\lambda/\mu) = |\text{SYT}(\lambda/\mu)| \) throughout the paper (except for the Introduction and Final Remarks Sections 1 and 12).
2. Linear extensions of posets

2.1. Notation

We assume the reader is familiar with standard definitions and notation of Young diagrams, Young tableaux, ranked posets, linear extensions, chains, antichains, etc. In case of confusion, we refer the reader to [49] and the clarifications of notation throughout the paper.

To further simplify the notation, we use the same letter to denote the partition, the corresponding Young diagram, as well as the poset corresponding to the Young diagram. To avoid the ambiguity, unless explicitly stated otherwise, we always assume that skew partitions are connected. To describe disconnected shapes, we use \( \lambda / \mu \circ \pi / \tau \) notation.

We make heavy use of Stirling’s formula \( \log n! = n \log n - n + O(\log n) \). Here and everywhere below \( \log \) denotes the natural logarithm. There is a similar formula for the double factorial \((2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1)\), the superfactorial \( \phi(n) = 1! \cdot 2! \cdots n! \), the double superfactorial \( \psi(n) = 1! \cdot 3! \cdots (2n - 1)! \), and the super doublefactorial \( \Lambda(n) = 1!! \cdots 3!! \cdots 5!! \cdots (2n - 1)!! \):

\[
\log(2n - 1)!! = n \log n + (\log 2 - 1)n + O(1),
\log \phi(n) = \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + 2n \log n + O(n),
\log \psi(n) = n^2 \log n + \left(\log 2 - \frac{3}{2}\right) n^2 + \frac{5}{2} n \log n + O(n),
\log \Lambda(n) = \frac{1}{2} n^2 \log n + \left(\log 2 - \frac{3}{4}\right) n^2 + \frac{1}{2} n \log n + O(n)
\]

(see [44, A001147], [44, A008793], [44, A168467]), and [44, A057863]).

Finally, we use the standard asymptotics notations \( f \sim g, f = o(g) \), \( f = \Omega(g) \), see e.g. [15, Section A.2]. For functions, we use \( f \approx g \) to denote \( \log f \sim \log g \), see the introduction. For constants, we use \( c \approx c' \) to approximate their numerical value with the usual rounding rules, e.g. \( \pi \approx 3.14 \) and \( \pi \approx 3.1416 \).

2.2. Ranked posets

Let \( \mathcal{P} \) be a ranked poset on a finite set \( X \) with linear ordering denoted by \( \prec \). Unless stated otherwise, we assume that \( |X| = n \). Let \( e(\mathcal{P}) \) be the number of linear extensions of \( \mathcal{P} \).

Denote by \( \ell = L(\mathcal{P}) \) and \( m = M(\mathcal{P}) \) the length of the longest chain and the longest antichain, respectively. Let \( r_1, \ldots, r_k \) denote the number of elements in \( X \) of each rank, so \( k \geq \ell \) and \( n = r_1 + \cdots + r_k \). Similarly, let \( \mathcal{P} \) have a decomposition into chains \( C_1, \ldots, C_m \), and denote \( \ell_i = |C_i| \), so \( \ell_1 + \cdots + \ell_m = n \). Recall that such decompositions exist by Dilworth’s theorem (see e.g. [50]).

**Theorem 2.1.** For every ranked poset \( \mathcal{P} \) as above, we have:

\[
|C_1|! \cdots |C_m|! \leq e(\mathcal{P}) \leq \frac{n!}{\ell_1! \cdots \ell_m!}.
\]

These bounds are probably folklore; for the lower bound see e.g. [4]. They are easy to derive but surprisingly powerful. We include a quick proof for completeness.

**Proof.** For the lower bound, label elements of rank 1 with numbers \( 1, \ldots, r_1 \) in any order, elements of rank 2 with numbers \( r_1 + 1, \ldots, r_1 + r_2 \) in any order, etc. All these labelings are clearly linear extensions and the bound follows. For the upper bound, observe that every linear extension of \( \mathcal{P} \) when restricted to chains \( C_1, \ldots, C_m \) defines an ordered set-partition of \( \{1, \ldots, n\} \) into \( m \) subsets of sizes \( |C_1|, \ldots, |C_m| \). Since this map is an injection, this implies the upper bound. \( \square \)

**Remark 2.2.** Note that the lower bound in the theorem extends to more general antichain decompositions which respect the ordering of \( \mathcal{P} \). Although in some cases this can lead to small improvements in
the lower bounds, for applications we consider the version in the theorem suffices. It is also worth noting that the upper bound in Theorem 2.1 is always better than the easy to use upper bound $e(\mathcal{P}) \leq m^n$ (cf. [4,5]).

Let us also mention the following unusual bound for the number of linear extensions of general posets. Denote also by $\mathfrak{b}(x) = \# \{ y \in \mathcal{P}, y \succ x \}$ the size of the upper ideal in $\mathcal{P}$ spanned by $x$.

**Theorem 2.3 ([20]).** For every poset $\mathcal{P}$, in the notation above, we have:

$$e(\mathcal{P}) \geq \frac{n!}{\prod_{x \in \mathcal{P}} \mathfrak{b}(x)}.$$ 

The lower bound was proposed by Stanley [49, Exc. 3.57] and proved by Hammett and Pittel [20]. It is tight for forests, i.e. disjoint unions of tree posets (see e.g. [42]). Note that this bound can be different for the poset $\mathcal{P}$ and the dual poset $\mathcal{P}^*$. We refer to Section 12.1 for further references on the number of linear extensions.

2.3. Square shape

The following is the motivating example for this paper. Let $\lambda = (k^k), \mu = \emptyset, n = k^2$. Clearly, $e(k^k) = |\text{SYT}(\lambda)|$ (see [44, A039622]).

Observe that $(r_1, r_2, \ldots) = (1, 2, \ldots, k - 1, k, k - 1, \ldots, 1)$ in this case and $m = M(k^k) = k$.

**Theorem 2.1** gives $e(k^k) \geq \Phi(k)\Phi(k - 1)$, which implies

$$(\oplus) \log e(k^k) \leq \log \Phi(k) + \log \Phi(k - 1) = \frac{1}{2} n \log n - \frac{3}{2} n + O(\sqrt{n \log n}).$$

For the upper bound, observe that $Q_n = (k^k)$ can be decomposed into chains of lengths $2k-1, \ldots, 3, 1$, each involving two adjacent diagonals. We have then:

$$(\ominus) \log e(k^k) \leq \log \left( \begin{array}{c} n \\ 2k - 1, 2k - 3, \ldots, 3, 1 \end{array} \right) = \log n! - \log \Psi(k)$$

$$\leq \frac{1}{2} n \log n + \left( \frac{1}{2} - \log 2 \right) n + O(\sqrt{n \log n}).$$

In other words, the lower and upper bounds agree in the leading term of the asymptotics but not in the second term. Let us compare this with an exact value of $e(k^k)$. The HLF gives:

$$e(k^k) = \frac{n!\Phi(k - 1)^2}{\Phi(2k - 1)},$$

which implies

$$\log e(k^k) = \frac{1}{2} n \log n + \left( \frac{1}{2} - 2 \log 2 \right) n + O(\sqrt{n \log n}).$$

Since $\left( \frac{1}{2} - 2 \log 2 \right) \approx -0.8863$ and $\left( \frac{1}{2} - \log 2 \right) \approx -0.1931$, we conclude that the true constant $-0.8863$ of the second asymptotic term lies roughly halfway between the lower bound $-1.5$ in $(\ominus)$ and the upper bound $-0.1931$ in $(\oplus)$. Let us mention also that the lower bound $e(k^k) \geq n!/(k!)^{2k}$ in Theorem 2.3 is much too weak.

2.4. Skew shapes

Let $\lambda/\mu$ be a skew shape Young diagram (see Fig. 1). To simplify the notation, we use $\lambda/\mu$ to also denote the corresponding posets of squares increasing downward and to right. The main object of this paper is the asymptotic analysis of

$$e(\lambda/\mu) = f^{\lambda/\mu} = |\text{SYT}(\lambda/\mu)|,$$
the number of standard Young tableaux of shape $\lambda/\mu$. Note that both are standard notation in different areas; we use them interchangeably throughout the paper.

The following determinant formula due to Feit [13] is a standard result in the area, often referred to as the Jacobi–Trudi identity (see e.g. [42,49]):

$$f_{\lambda/\mu} = n! \det \frac{1}{(\lambda_i - \mu_j - i + j)!}^{(\ell(\lambda))}_{i,j=1}.$$ 

Unfortunately, due to the alternating sign nature of the determinant, this formula is difficult to use in the asymptotic context. Here is the only two (rather weak) general bounds that easily follow from the existing literature.

**Proposition 2.4.** For every skew shape $\lambda/\mu$, we have:

$$f_{\lambda/\mu} \leq f_\lambda.$$ 

For the proof, simply observe that $f_\mu \cdot f_{\lambda/\mu} \leq f_\lambda$ follows immediately from the combinatorial interpretation of SYT($\lambda/\mu$).

**Proposition 2.5.** For every skew shape $\lambda/\mu$, we have:

$$f_{\lambda/\mu} \leq |\lambda|! f_{\mu}.$$ 

**Proof.** Recall the standard equalities for the Littlewood–Richardson (LR–) coefficients $c_{\mu,\nu}^\lambda$, where $|\lambda| = |\mu| + |\nu|$. We have:

$$f_\mu f_\nu \left( \frac{|\mu| + |\nu|}{|\mu|} \right) = \sum_{\lambda \vdash |\mu| + |\nu|} c_{\mu,\nu}^\lambda f_\lambda \quad \text{and} \quad f_{\lambda/\mu} = \sum_{v \vdash |\lambda| - |\mu|} c_{\mu,\nu}^\lambda f^v.$$ 

From here and the Burnside identity [42,49], we have:

$$f_{\lambda/\mu} = \sum_v c_{\mu,v}^\lambda f^v \leq \sum_v \left( \frac{\lambda}{|\mu|} \right) f_\mu f_\nu f^v = \frac{f_\mu}{f_\mu} \left( \frac{|\lambda|}{|\mu|} \right) \sum_v (f^v)^2 = \frac{f_\mu |\lambda|!}{|\mu|! |\nu|!} |v|! = \frac{|\lambda|! f_\mu}{|\mu|! f_\nu},$$

proving the first inequality. The equality follows from the HLF. □

**Example 2.6.** Let $\lambda/\mu = (4^232/21)$ be a skew shape of size $n = 10$. The Jacobi–Trudi formula (\ref{Jacobi-Trudi}) gives $f_{\lambda/\mu} = e(\lambda/\mu) = 3060$.

The poset of $\lambda/\mu$, the length of the longest antichain is $m = 4$, and the number of elements of each rank are 3, 4, 3. Similarly, the poset can be decomposed into four chains of sizes 3, 3, 3, 1 (see Fig. 1). Theorem 2.1 gives:

$$864 = 3!4!3! \leq e(\lambda/\mu) \leq \frac{10!}{3!3!3!3!} = 16800.$$
The sizes $b(x)$ are given in Fig. 1. Then Theorem 2.3 gives a slightly weaker bound:
\[ e(\lambda/\mu) \geq \frac{10!}{2^2 3^2 5^2 6} = 672. \]

Proposition 2.4 and the HLF gives a very reasonable upper bound:
\[ e(\lambda/\mu) \leq \frac{e(4^2 32)}{e(21)} = \frac{8580}{2} = 4290 \]

(this bound gets exponentially worse for larger shapes). Finally, Proposition 2.5 gives the following very weak upper bound:
\[ e(\lambda/\mu) \leq \frac{13! e(21)}{3! e(4^2 32)} = \frac{13! \cdot 2}{3! \cdot 8580} = 241920. \]

Remark 2.7. Calculations similar to the square shape can be done for various other geometric shapes with known nice product formulas. Beside the usual Young diagrams these include shifted diagrams and various ad hoc shapes as in Fig. 2 (see [1,25,37] and the last example in [32]). In all these cases,
\[ \log e(\mathcal{P}) = \frac{1}{2} n \log n + O(n). \]

While the bounds in Theorem 2.1 again give the leading term correctly, they are all off in the second asymptotic term. This observation is the key starting point for this work. Roughly speaking, in many cases, the inequalities $\ast$ in the introduction make the gap between the second asymptotic term smaller.

3. Hook formulas for skew shapes

3.1. Definition of excited diagrams

Let $\lambda/\mu$ be a skew partition and $D$ a subset of the Young diagram of $\lambda$. A cell $u = (i, j) \in D$ is called active if $(i + 1, j), (i, j + 1)$ and $(i + 1, j + 1)$ are all in $\lambda \setminus D$. Let $u$ be an active cell of $D$, define $\alpha_u(D)$ to be the set obtained by replacing $(i, j)$ in $D$ by $(i + 1, j + 1)$. We call this replacement an excited move. An excited diagram of $\lambda/\mu$ is a subset of squares in $\lambda$ obtained from the Young diagram of $\mu$ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of $\lambda/\mu$, and let $\xi(\lambda/\mu) = |\mathcal{E}(\lambda/\mu)|$.

The following explicit characterization in [31] is also very helpful. Let $D$ be a subset of squares in $\lambda$ with the same number of squares in each diagonal as $\mu$. Define an order relation on squares $(i, j) \preceq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. Then $D$ is an excited diagram if and only if the relation $\preceq$ on squares of $\mu$ holds for $D$.

We conclude with an explicit formula for $\xi(\lambda/\mu)$. For the diagonal $\Delta$ that passes through the cell $(i, \mu_i)$, denote by $\vartheta_i$ the row in which $\Delta$ intersects the boundary of $\lambda$.

**Theorem 3.1 ([31]).** Let $\lambda/\mu$ be skew partition and let $\ell = \ell(\mu)$. In the notation above, we have:
\[ \xi(\lambda/\mu) = \det \left[ \begin{array}{c} \vartheta_i + \mu_i - i + j - 1 \\ \vartheta_i - 1 \end{array} \right]_{i,j=1}^\ell \]
This formula follows from a characterization of the excited diagrams as certain flagged tableaux of shape $\mu$ with entries in row $i$ at most $\vartheta_i$, a different border strip decomposition, and the Lindström–Gessel–Viennot lemma. We refer to [30,31] for details and references.

3.2. NHLF and its implications

The following recent result is the crucial advance which led to our study (cf. [30,31]).

**Theorem 3.2** (Naruse [34]). Let $\lambda$, $\mu$ be partitions, such that $\mu \subset \lambda$. We have:

$$e(\lambda/\mu) = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in \lambda \setminus D} \frac{1}{h(u)}.$$  \hspace{1cm} (NHLF)

We can now present a corollary of the NHLF, which is the main technical tool of this paper. For a general bound of this type, it is quite powerful in applications (see below). It is also surprisingly easy to prove.

**Theorem 3.3.** For every skew shape $\lambda/\mu$, $|\lambda/\mu| = n$, we have:

$$F(\lambda/\mu) \leq e(\lambda/\mu) \leq \xi(\lambda/\mu) F(\lambda/\mu),$$

where

$$F(\lambda/\mu) = n! \prod_{u \in \lambda \setminus \mu} \frac{1}{h(u)}$$

is defined as in the introduction.

Note that when restricted to skew shapes, the lower bound in Theorem 2.3 is clearly weaker than the lower bound ($\ast$), and coincides with it exactly for the ribbon hooks (shapes with at most one square in every diagonal).

**Proof.** The lower bound follows from the NHLF, since $\mu$ is an excited diagram in $\mathcal{E}(\lambda/\mu)$. For the upper bound, note that under the excited move the product $\prod_{u \in \lambda \setminus \mu} h(u)$ increases. Thus this product is minimal for $D = \mu$, and the upper bound follows from the NHLF. □

**Remark 3.4.** Note also that both bounds in Theorem 2.1 are symmetric with respect to taking a dual poset. On the other hand, one can apply Theorem 3.3 to either $(\lambda/\mu)$ or the dual shape $(\lambda/\mu)^*$ obtained by a 180° rotation.

**Example 3.5.** As in Example 2.6, let $\lambda/\mu = (4^2 32/21)$ and $e(\lambda/\mu) = 3060$. The hook-lengths in this case are given in Fig. 1. We have

$$F(\lambda/\mu) = \frac{10!}{54^2 3^2 2^2} = 1260.$$  

In this case we have $\vartheta_1 = 2$, $\vartheta_2 = 3$, and by Theorem 3.1

$$\xi(\lambda/\mu) = \det \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} = 5.$$  

In this case, Theorem 3.3 gives:

$$1260 \leq e(\lambda/\mu) \leq 1260 \cdot 5 = 6300.$$  

Both bounds are better than the bounds in Example 2.6.
3.3. Comparison of bounds

To continue the theme of this section, we make many comparisons between the bounds on $e(\lambda/\mu)$ throughout the paper, both asymptotically and in special cases. Here is the cleanest comparison, albeit under certain restrictions.

Fix a skew shape $\nu = (\lambda/\mu)$ with $\ell = L(\lambda)$. Let $A_1, \ldots, A_{\ell}$ be an antichain decomposition with elements in $A_k$ lying in the antidiagonal $A_k = \{(i, j) \in \nu \mid i + j = k + s - 1\}$, where $s = \min(i + j \mid (i, j) \in \nu)$. Denote $r_k = |A_k|$. 

**Theorem 3.6.** In the notation above, suppose $r_1 \leq r_2 \leq \ldots \leq r_k$. Then we have:

$$r_1! \cdot \ldots \cdot r_k! \leq F(\lambda/\mu).$$

For example, the theorem applies to the thick ribbon shapes (see Section 8).

**Proof.** Consider the hooks of the squares on a given antidiagonal $A_i$. Let the number of squares of rank $\geq i$ (i.e. on the antidiagonal or below/ right of it) be $N_i$. Every such square belongs to at most two hooks $h_u$ and $h_v$ with $u, v \in A_i$, and that happens only when it is contained in the "subtriangle" with antidiagonal $A_i$ on the outside (everything left after erasing rows and columns which have no boxes in $A_i$, and also excluding the antidiagonal itself). There are at most $r_i!$ boxes in such a triangle, and cannot be more than all boxes below diagonal $i$, which is $N_i - r_i$. In fact, since $r_i + 1 \geq r_i$, there is at least one box below the diagonal which is not counted twice, so the bound is $N_i - r_i - 1$. Hence by counting the squares covered by a hook, we have:

$$\sum_{u \in A_i} h_u \leq N_i + \min\left\{N_i - r_i - 1, \left(\frac{r_i}{2}\right)\right\}.$$ 

Noting that for the last diagonal we have $h_u = 1$ and $N_k = r_k$, we obtain the following

$$\prod_{i=1}^{k} r_i! \prod_{u \in A_i} h_u \leq \prod_{i=1}^{k} r_i! \prod_{i=1}^{k-1} \left(\frac{\sum_{u \in A_i} h_u}{r_i}\right)^{r_i} \leq r_k! \prod_{i=1}^{k-1} r_i! \left\{N_i + \min\left\{N_i - r_i - 1, \left(\frac{r_i}{2}\right)\right\}\right\}^{r_i}. $$

For the first inequality is the AM–GM inequality for the product of hooks on a given diagonal, and for the second—the estimate for their sum.

Next we will need the following inequality for binomial coefficients, the proof is included at the end.

**Lemma 3.7.** Let $t \geq r \geq 3$. Then

$$\left(\frac{t + r}{r}\right) \geq \left(\frac{2t + r - 1}{r}\right)^{r}. $$

We use the lemma to estimate the RHS of inequality $\heartsuit$. For $r_i \geq 3$, take $t = N_i - r_i \geq r_i$ for $i < k$. We have:

$$\frac{(2N_i - 1 - r_i)^{r_i}}{r_i^{r_i}} \leq \left(\frac{N_i}{r_i}\right).$$

For $r_i = 2$, we have $\min(N_i - 3, 1) \leq 1$ and the inequality becomes

$$\frac{(N_i + \min(N_i - 3, 1))^2}{2^2} \leq \frac{(N_i + 1)^2}{4} \leq \left(\frac{N_i}{2}\right),$$

which holds trivially whenever $N_i \geq 5$. When $N_i \leq 4$ and $r_i = 2$, under the assumptions, we must have at most 4 boxes on or after the ith diagonal, with the previous diagonals having at most 2 boxes.
Such skew shapes would necessarily be of the form $(k + 1, k)/(j)$ when the desired inequality becomes $2^{k-j+1} \leq \frac{(2k+1+j)/2}{(k-j+2)\pi}$, which is easy to verify. For partitions of size at most 5, we check the main inequality in the statement by direct computation. Finally, for $r_i = 1$, we have $\min(N_i - 1, 0) = 0$, and we have the equality in (3.1). Putting all these together in the RHS of (3.1) we have

$$\prod_{i=1}^{k} r_i! \prod_u h_u \leq \prod_{i=1}^{k} r_i! \prod_{i=1}^{k} \left( \frac{N_i}{r_i} \right) = N!,$$

since $N_i = r_i + N_{i+1}$ and $N_1 = N$. Dividing both sides by the product of hooks gives the desired inequality. □

Proof sketch of Lemma 3.7. Note that the inequality is trivial when $2t$ is replaced by $t$ in the RHS for all $t$, and the complication comes from the coefficients 2, then the inequality does not hold for all values. First, we show that the inequality holds for $t = r \geq 3$:

$$\left( \frac{2r}{r} \right) \geq (3 - 1/r)^r,$$

(3.2)

which holds for $r = 3, 4, 5$ (and not for $r = 2$) and for larger $r$ it follows from the lower and upper bounds from Stirling’s approximation of the logarithm on the RHS:

$$\log(2r)! - 2 \log r! \geq 1 + (2r + 1/2) \log 2 - \frac{1}{2} \log r - 2 \log \sqrt{2\pi} \geq r \log 3,$$

where the last inequality holds for all $r \geq 6$.

Next, one sees that the ratio is an increasing function of $t$ when $t \geq r$. This follows by taking the logarithms, and estimating the derivative of $t$ via the arithmetic mean–harmonic mean inequality. We omit the details. □

Remark 3.8. The antichain decomposition in the theorem can be generalized from rank antichains to all ordered antichain decompositions $(A_1, \ldots, A_k)$, such that if $x \in A_i$, $y \in A_j$, $i < j$, then $x < y$. The proof extends verbatim; we omit the details. Note also that the theorem cannot be extended to general skew shapes. The examples include $(2^2/1)$ and $(3^3/21)$.

4. Bounds on the number of excited diagrams

4.1. Non-intersecting paths

We recall that the excited diagrams of $\lambda/\mu$ are in bijection with families of certain non-intersecting grid paths $\gamma_1, \ldots, \gamma_k$ with a fixed set of start and end points, which depend only on $\lambda/\mu$. This was proved in [30,31], and was based on the earlier works by Kreiman [26], Lascoux and Schützenberger [28], and Wachs [52], on flagged tableaux.

Formally, given a connected skew shape $\lambda/\mu$, there is a unique family of border-strips (i.e. non-intersecting paths) $\gamma_1^*, \ldots, \gamma_k^*$ in $\lambda$ with support $\lambda/\mu$ (set of path squares), where each border strip $\gamma_i^*$ begins at the southern box $(a_i, b_i)$ of a column and ends at the eastern box $(c_i, d_i)$ of a row [26, Lemma 5.3]. Moreover, all non-intersecting paths $(\gamma_1, \ldots, \gamma_k)$ contained in $\lambda$ with $\gamma_i : (a_i, b_i) \rightarrow (c_i, d_i)$ are in correspondence with excited diagrams of the shape $\lambda/\mu$ [26, Section 5.5].

Proposition 4.1 (Kreiman [26], see also [30]). The non-intersecting paths $(\gamma_1, \ldots, \gamma_k)$ in $\lambda$ where $\gamma_i : (a_i, b_i) \rightarrow (c_i, d_i)$ are uniquely determined by their support and moreover these supports are in bijection with complements of excited diagrams of $\lambda/\mu$.

Corollary 4.2. In the notation above,

$$\xi(\lambda/\mu) = \# \{ \text{non-intersecting paths } (\gamma_1, \ldots, \gamma_k) \mid \gamma_1 \subseteq \lambda, \gamma_i : (a_i, b_i) \rightarrow (c_i, d_i) \}.$$

See Fig. 3 for an example of the proposition and the corollary.
Fig. 3. The eight non-intersecting paths \((\gamma_1, \gamma_2)\) whose support are the complement of the excited diagrams of the shape \((54^2/121)\).

4.2. Bounds on \(\xi(\lambda/\mu)\)

We prove the following two general bounds. They are elementary but surprisingly powerful in applications.

**Lemma 4.3.** Let \(\lambda/\mu\) be a skew shape, \(n = |\lambda/\mu|\). Then \(\xi(\lambda/\mu) \leq 2^n\).

**Proof.** Each path of fixed size is determined by its vertical and horizontal steps between its fixed endpoints. We conclude:

\[
\xi(\lambda/\mu) \leq \prod_{i=1}^{k} 2^{n_i - 1} = 2^{n-k} \leq 2^n,
\]

where \(k\) is the number of paths \(\gamma_i\), as above. □

The Durfee square in Young diagram \(\lambda\) is the maximal square which fits inside \(\lambda\). Let \(d(\lambda)\) be the size of the Durfee square.

**Lemma 4.4.** Let \(\lambda/\mu\) be a skew shape and let \(d = d(\lambda)\). Then \(\xi(\lambda/\mu) \leq n^{2d^2}\).

**Proof.** Since \(\lambda/\mu\) is connected, the path traced by the border of \(\lambda\) is a ribbon of length at most \(n\) and height at least \(d\), so we must have that \(\lambda_1, \ell(\lambda) \leq n - d\). Then \(\lambda \subset \nu\), where \(\nu = ((n - d)^d, d^{n-2d})\). This implies \(\xi(\lambda/\mu) \leq \xi(\nu/\mu)\).

As before, we estimate \(\xi(\nu/\mu)\) by the number of non-intersecting lattice paths inside that shape. Consider the lattice paths above the diagonal \(i = j\). Since the shape \(\nu/\mu\) within the top \(d\) rows is the complement of a straight shape, the lattice paths there all start at the bottom row and follow the shape \(\mu\), ending at the rightmost column. So there are at most \(d\) paths above the diagonal and their endpoints are no longer fixed (see Fig. 4). Clearly, each such path has length at most \(n\). Those above the diagonal have at most \((d - 1)\) vertical steps, and those below the diagonal have at most \((d - 1)\) horizontal steps. Therefore,

\[
\xi(\lambda/\mu) \leq \xi(\nu/\mu) \leq \left(\frac{n-1}{d-1}\right)^{2d} \leq n^{2d^2},
\]

as desired. □

5. The Thoma–Vershik–Kerov limit shape

Denote by \(a(\lambda) = (a_1, a_2, \ldots)\), \(b(\lambda) = (b_1, b_2, \ldots)\) the Frobenius coordinates of \(\lambda\), defined as \(a_i = \lambda_i - i + 1\), \(b_j = \lambda_j' - j\), for all \(i, j \geq 1\) for which these coordinates are non-negative. Fix an integer
$k \geq 1$. Let $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}^k_+$. We say that a sequence of partitions $(\lambda^{(n)})$ has a Thoma–Vershik–Kerov (TVK) limit $(\alpha, \beta)$, write $\lambda^{(n)} \rightarrow (\alpha, \beta)$, if $a_i/n \rightarrow \alpha_i$ and $b_i/n \rightarrow \beta_i$ as $n \rightarrow \infty$, for all $1 \leq i \leq k$, and $d(\lambda^{(n)}) \leq k$ for all $n \geq 1$.

**Theorem 5.1.** Let $\{\upsilon_n = \lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with a TVK limit. Formally, in the notation above, suppose $\lambda^{(n)} \rightarrow (\alpha, \beta)$, where $\alpha_1, \beta_1 > 0$, and $\mu^{(n)} \rightarrow (\pi, \tau)$ for some $\alpha, \beta, \pi, \tau \in \mathbb{R}^k_+$. Then

$$\log e(\upsilon_n) = cn + o(n) \quad \text{as} \quad n \rightarrow \infty,$$

where

$$c = c(\alpha, \beta, \pi, \tau) = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i).$$

**Proof.** First, in the notation of TVK limit, we have $d(\lambda^{(n)}) \leq k$, so $\xi(\upsilon_n) < n^{2k^2}$. Thus, by Theorem 3.3, it suffices to compute the leading term in the asymptotics of $F(\lambda^{(n)}/\mu^{(n)})$. Observe that $|\upsilon_n| \sim \gamma n$. Next, observe that in the $i$th row, up to an additive constant $\leq d(\lambda) = k$, all the hooks are $1, \ldots, (\alpha_i - \pi_i + o(1))n$, and in the $i$th column all the hooks are $1, \ldots, (\beta_i - \tau_i + o(1))n$. Therefore, by the Stirling formula, we have:

$$\log F(\upsilon_n) \sim \log(\gamma n)! - \sum_i \log[(\alpha_i - \pi_i)n]! - \sum_i \log[(\beta_i - \tau_i)n]!$$

$$\sim n \left( \gamma \log \gamma - \sum_i (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_i (\beta_i - \tau_i) \log(\beta_i - \tau_i) \right),$$

as $n \rightarrow \infty$. We omit the easy details. $\square$

**Remark 5.2.** When $\mu = \emptyset$, the exponential nature of $f^\lambda$ was recently studied in [17]; one can view our results as an asymptotic version in the skew shape case. For a fixed partition $\mu$, Okounkov and Olshanski [35] give an explicit formula for the ratio $f^{\lambda/\mu}/f^\lambda$ which can be computed explicitly in this case (see also [9,46]). Note also that the notion of TVK limit is rather weak, as it is oblivious to adding $s = o(n)$ to the rows (columns), which can affect $f^{\lambda/\mu}$ by a factor of $2^{o(s)}$. Thus the error term in the theorem cannot be sharpened.
Note finally that the property \( d(\lambda^{(n)}) = O(1) \) is essential in the theorem, as otherwise \( \xi(\lambda/\mu) \) can be superexponential (see Section 10).

6. The stable shape

6.1. The usual stable shape

Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) be a non-increasing continuous function, s.t. \( \lim_{n \to \infty} \omega(x) = 0. \) Consider a sequence of partitions \( \{\lambda^{(n)}\} \), such that the rescaled diagrams \( \frac{1}{\sqrt{n}}[\lambda^{(n)}] \) converge uniformly to \( \omega \), where \( [\lambda] \) denotes the curve giving the boundary of Young diagram \( \lambda \), which we contract by a factor \( 1/\sqrt{n} \) in both directions. In this case we say that a sequence of partitions \( \{\lambda^{(n)}\} \) has a stable shape \( \omega \), and write \( \lambda^{(n)} \to \omega \).

Suppose we are given two stable shapes \( \omega, \pi : \mathbb{R}_+ \to \mathbb{R}_+ \), such that \( \pi(x) \leq \omega(x) \) for all \( x \geq 0. \) To simplify the notation, denote by \( C = C(\omega/\pi) \subset \mathbb{R}_+^2 \) the region between the curves. One can view \( C \) as the stable shape of skew diagrams, and denote by \( \text{area}(\omega/\pi) \) the area of \( C \).

**Theorem 6.1.** Let \( \omega, \pi : [0, a] \to [0, b] \) be continuous non-increasing functions, and suppose that \( \text{area}(\omega/\pi) = 1. \) Let \( \{v_n = \lambda^{(n)}/\mu^{(n)}\} \) be a sequence of skew shapes with the stable shape \( \omega/\pi \), i.e. \( \lambda^{(n)} \to \omega, \mu^{(n)} \to \pi \). Then
\[
\log e(v_n) \sim \frac{1}{2} n \log n \quad \text{as} \quad n \to \infty.
\]

We prove the theorem below. Let us first state a stronger result for a more restrictive notion of stable shape limit, which shows how the stable shape \( \omega/\pi \) appears in the second term of the asymptotic expansion.

**Remark 6.2.** Note that as stated, the theorem applies to nice geometric shapes with \( \sqrt{n} \) scaling and the Plancherel shape \( \omega \), see [51] (see also [41,47]), but not the Erdős–Szekeres limit shape (see e.g. [10]), since random partitions have \( \Theta(\sqrt{n} \log n) \) parts. The proof, however, can be adapted to work in this case (see also Section 12.8).

6.2. Strongly stable shape

Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) be a non-increasing continuous function. Suppose sequence of partitions \( \{\lambda^{(n)}\} \) satisfies the following property
\[
(\sqrt{n} - L)\omega < [\lambda^{(n)}] < (\sqrt{n} + L)\omega, \quad \text{for some} \quad L > 0,
\]
where we write \( [\lambda] \) to denote a function giving the boundary of Young diagram \( \lambda \). In this case we say that a sequence of partitions \( \{\lambda^{(n)}\} \) has a strongly stable shape \( \omega \), and write \( \lambda^{(n)} \to \omega \).

In the notation above, define the hook function \( h : C \to \mathbb{R}_+ \) to be the scaled function of the hooks: \( h(x, y) := h([x, \sqrt{n}], [y, \sqrt{n}]) / \sqrt{n} \).

**Theorem 6.3.** Let \( \omega, \pi : [0, a] \to [0, b] \) be continuous non-increasing functions, and suppose that \( \text{area}(\omega/\pi) = 1. \) Let \( \{v_n = \lambda^{(n)}/\mu^{(n)}\} \) be a sequence of skew shapes with the strongly stable shape \( \omega/\pi \), i.e. \( \lambda^{(n)} \to \omega \) and \( \mu^{(n)} \to \pi \). Then
\[
-(1 + c(\omega/\pi))n + o(n) \leq \log e(v_n) - \frac{1}{2} n \log n \leq -(1 + c(\omega/\pi))n + \log \xi(v_n) + o(n),
\]
as \( n \to \infty \), where
\[
c(\omega/\pi) = \iint_C log h(x, y) \text{d}x \text{d}y.
\]

Note that by Lemma 4.3, we always have \( \log \xi(v_n) \leq (\log 2)n \).
\textbf{Proof.} Theorem 3.3 gives:

\[ \log F(\nu_n) \leq \log e(\nu_n) \leq \log F(\nu_n) + \log \xi(\nu_n). \]

Now, observe that \(|\nu_n| = n + O(\sqrt{n}) as n \to \infty. Using the Stirling formula, the definition and compactness of the stable shape \( C \subset [a \times b]. \) \footnote{In [14], such shapes are called balanced.} we have:

\[ \log F(\nu_n) = \log |\nu_n|! - \sum_{u \in \nu_n} \log h(u) = n \log n - n + O(\sqrt{n} \log n) \]

\[ - n \cdot \int_{\mu} \log(\sqrt{\nu h(x, y)}) \, dx\,dy + o(n) = \frac{1}{2} n \log n - n - c(\omega/\pi) n + o(n), \]

where the \( o(n) \) error term comes from approximation of the sum with the scaled integral. This implies both parts of the theorem. \( \square \)

\textbf{Proof of Theorem 6.1.} The proof is similar, but the error terms are different. First, by uniform convergence, we have \(|\nu_n| = n + o(n)\), which only implies

\[ \log |\nu_n|! = \frac{1}{2} n \log n + o(n \log n) \quad \text{as} \quad n \to \infty. \]

The same error term \( o(n \log n) \) also appears from the scaled integral bound. The details are straightforward. \( \square \)

7. The subpolynomial depth shape

Let \( g(n) : \mathbb{N} \to \mathbb{N} \) be an integer function which satisfies \( 1 \leq g(n) \leq n, g(n) \to \infty \) and \( \log g(n) = o(\log n) \) as \( n \to \infty. \) The last condition is equivalent to \( g(n) = n^{o(1)} \), hence the name. Function \( g(n) \) is then said to have subpolynomial growth (cf. [18]). Examples include \( e^{\log^2 n}, (\log n)^n, n^{1/\log \log n} \), etc.

For a skew shape \( \lambda/\mu \), define \( \text{width}(\lambda/\mu) := \min(\lambda_1, \lambda_1', \lambda_1'') \) and \( \text{depth}(\lambda/\mu) := \max_{u \in \lambda/\mu} h(u). \) For example, for thick ribbons \( \nu_k = \delta_{3k}/\delta_{2k} \), we have \( \text{width}(\nu_k) = 3k - 1 \) and \( \text{depth}(\nu_k) = 2k - 1. \)

We say that a sequence of skew partitions \( \{ \nu_n = \lambda^{(n)}/\mu^{(n)} \} \) has subpolynomial depth shape if

\[ (\circ) \quad \text{width}(\nu_n) = \Theta \left( \frac{n}{g(n)} \right) \quad \text{and} \quad \text{depth}(\nu_n) = \Theta(g(n)), \]

for some subpolynomial growth function \( g(n) \).

\textbf{Theorem 7.1.} Let \( \{ \nu_n = \lambda^{(n)}/\mu^{(n)} \} \) be a sequence of skew partitions with a subpolynomial depth shape associated with the function \( g(n) \). Then

\[ \log e(\nu_n) = n \log n - \Theta(n \log g(n)) \quad \text{as} \quad n \to \infty. \]

\textbf{Proof.} First, by Lemma 4.3, we have \( \xi(\nu_n) \leq 2^n \). Thus, by Theorem 3.3, we have:

\[ \log e(\nu_n) = \log F(\nu_n) + O(n) = n \log n + O(n) - \sum_{u \in \nu_n} \log h(u). \]

First, observe:

\[ \sum_{u \in \nu_n} \log h(u) \leq n \log \left[ \text{depth}(\nu_n) \right] = O(n \log g(n)). \]

In the other direction, suppose for simplicity that \( \text{width}(\nu_n) = \lambda_1. \) By \( (\circ) \) and total area count, we have: of the \( \text{width}(\nu_n) \) columns, at least \( \alpha \) proportion of them have length \( > \beta \text{depth}(\nu_n) \), for some
\( \alpha, \beta > 0 \). Here the constants \( \alpha, \beta \) depend only on the constants implied by the \( \Theta(\cdot) \) notation and independent on \( n \). We conclude:

\[
\sum_{u \in \nu_n} \log h(u) \geq \alpha \text{width}(\nu_n) \cdot \log \left[ \beta \text{depth}(\nu_n) ! \right] \\
\geq \alpha \Theta \left( \frac{n}{g(n)} \right) \cdot \beta \Theta (g(n)) \log \left[ \beta \Theta (g(n)) \right] = \Omega \left( n \log g(n) \right),
\]

as desired. □

Remark 7.2. The subpolynomial depth shape introduced in this section is a generalization of certain thick ribbon shapes (see below). It is of interest due largely to the type of asymptotics for \( e(\lambda/\mu) \). Here the leading term of \( \log e(\lambda/\mu) \) is \( n \log n \), which implies that the lower order terms are of interest. Furthermore, we have \( \log \xi(\lambda/\mu) = o \left( \log n! - \log F(\lambda/\mu) \right) \) in this case, which implies that NHLF is tight for the first two terms in the asymptotics.

8. Thick ribbons

8.1. Proof of Theorem 1.1

As in the introduction, let \( \delta_k = (k - 1, k - 2, \ldots, 2, 1) \) and \( \nu_k = \delta_{2k}/\delta_k \) (see [30] and for the sequence \( \{e(\nu_k)\} \) see [44, A278289]). Note that thick ribbons \( \nu_k \) have strongly stable trapezoid shape. We have \( n = |\nu_k| = 3k(k - 1)/2 \). For simplicity, we assume that \( k \) is even (for odd \( k \) the formulas are slightly different). In the notation of Section 2.1, we have:

\[
F(\nu_k) = \frac{n!}{(2k - 1)!! \Lambda(k - 1)}. 
\]

This gives the lower bound in Theorem 1.1:

\[
\log e(\nu_k) \geq \log F(\nu_k) = \frac{1}{2} n \log n + \left( \frac{1}{6} - \frac{3 \log 2}{2} + \frac{\log 3}{2} \right) n + o(n) \\
\geq \frac{1}{2} n \log n - 0.3238n + o(n). 
\]

For the upper bound, we obtain a closed formula for the number of excited diagrams.

Lemma 8.1. Let \( \nu_k = (\delta_{2k}/\delta_k) \) be a thick ribbon, and let \( k \) be even. We have:

\[
\xi(\nu_k) = \prod_{1 \leq i < j \leq k} \frac{k + i + j - 1}{i + j - 1} = \left[ \frac{\Phi(3k - 1)\Phi(k - 1)^3(2k - 1)!!(k - 1)!!}{\Phi(2k - 1)^3(3k - 1)!!} \right]^{1/2}.
\]

Proof. The product in the RHS is the Proctor’s formula for the number of (reverse) plane partitions of shape \( \delta_k \) with entries \( \leq k/2 \) [39, Cases CG]. Let us show that the number also counts excited diagrams of the shape \( \delta_{2k}/\delta_k \). Indeed, by the proof of Theorem 3.1 in [31], the excited diagrams in this case are in bijection with flagged tableaux of shape \( \delta_k \) with entries in row \( i \) at most \( \vartheta_i = k/2 + i \). By subtracting \( i \) from the entries in row \( i \) of such tableaux we obtain reverse plane partitions of shape \( \delta_k \) with entries at most \( k/2 \), as desired. □

Remark 8.2. Proctor’s formula also counts the number of plane partitions inside a \([k \times k \times k] \) cube whose (matrix) transpose is the same as its complement (see also [24, Class 6], [45, Case 6] and [44, A181119]). This is because the anti-diagonal consists of the values \( k/2 \), and the plane partition below of the diagonal is of shape \( \delta_k \) with entries at most \( k/2 \). Clearly, this determines the rest of the partition.
From the lemma we obtain:
\[
\log \xi(\nu_k) \sim \left( \frac{3 \log 3}{2} - 2 \log 2 \right) n + o(n) \approx 0.2616n + o(n),
\]
and therefore
\[
\log e(\nu_k) \leq \log F(\nu_k) + \log \xi(\nu_k) = \frac{1}{2} n \log n + \left( \frac{7}{6} - \frac{2 \log 2}{2} + 2 \log 3 \right) n + o(n)
\]
\[
\leq \frac{1}{2} n \log n - 0.0621n + o(n).
\]
This proves the upper bound in Theorem 1.1.

8.2. Comparison with general bounds

Denote by \( \mathcal{P}_k \) the poset corresponding to \( \nu_k \). We have \( M(\mathcal{P}_k) = 2k - 1 \), \( L(\mathcal{P}_k) = k \), and \( r_1 = k \), ..., \( r_k = 2k - 1 \). Consider partition of the poset \( \mathcal{P}_k \) corresponding to \( \nu_k \) into chains of lengths 1, 2, ..., \( k - 1 \) and \( k \) chains of length \( k \). Now Theorem 2.1 gives
\[
r_1! \cdots r_k! = \frac{\Phi(2k - 1)}{\Phi(k - 1)} \leq e(\nu_k) \leq \left( \frac{n}{1, 2, \ldots, k - 1, k, \ldots, k} \right) = \frac{n!}{(k!)^{k-1} \Phi(k)}.
\]
In other words,
\[
\log e(\nu_k) \geq \log \frac{\Phi(2k - 1)}{\Phi(k - 1)} = \frac{1}{2} n \log n + \left( \frac{11 \log 2}{6} - \frac{\log 3}{2} - \frac{3}{2} \right) n + o(n)
\]
\[
\geq \frac{1}{2} n \log n - 0.7786n + o(n), \quad \text{and}
\]
\[
\log e(\nu_k) \leq \log \frac{n!}{(k!)^{k-1} \Phi(k)} = \frac{1}{2} n \log n + \left( \frac{1}{6} - \frac{\log 2}{2} + \frac{\log 3}{2} \right) n + o(n)
\]
\[
\leq \frac{1}{2} n \log n + 0.3695n + o(n).
\]
In notation of conjecture in Section 12.7, denote by \( c \) the (conjectural) constant in the asymptotics
\[
\log e(\nu_k) = \frac{1}{2} n \log n + cn + o(n).
\]
Our bounds imply that \( c \in [-0.3238, -0.0622] \). This is much sharper than the bounds \( c \in [-0.7786, 0.3695] \) which follows from Theorem 2.1.

Finally, by the HLF for \( e(\delta_k) \) (see \cite{A405118}), we have:
\[
\log e(\delta_k) = \log \frac{\binom{k}{2}!}{\lambda(k)} = \frac{1}{2} n \log n + \left( \frac{1}{2} - \frac{3}{2} \log 2 \right) n + o(n)
\]
\[
\approx \frac{1}{2} n \log n - 0.5397n + o(n),
\]
where \( n = |\delta_k| = \binom{k}{2} \). Proposition 2.5 then gives:
\[
\log e(\nu_k) \leq \log \frac{|\delta_{2k}| e(\delta_k)}{|\delta_k| e(\delta_{2k})} = \frac{1}{2} n \log n + \left( \frac{3}{2} - \frac{1}{2} \log 2 \right) n + o(n)
\]
\[
\leq \frac{1}{2} n \log n + 1.1535n + o(n).
\]
Note that this bound has correct first asymptotic term once again, but is weak in the \( O(n) \) term.
8.3. Application to Littlewood–Richardson coefficients

Let us show how the upper bounds in Theorem 1.1 imply new bounds on the LR-coefficients (cf. the proof of Proposition 2.5). Denote by $\nabla$ the (triangular) strongly stable shape of staircase shapes $\delta_k$.

**Corollary 8.3.** Let $\lambda^{(k)} = \delta_{2k}$, $\mu^{(k)} = \delta_k$, and $n = |\lambda/\mu| = 3k(k + 1)/2$ as above. Suppose $\nu^{(k)} \vdash n$ has strongly stable shape $\nabla$. Then

$$\log c_{\mu^{(k)}, \nu^{(k)}}^{\lambda^{(k)}} \leq \left( 2 \log 3 - \frac{1}{3} - 2 \log 2 \right) n + o(n).$$

The constant in the RHS is approximately 0.4776. It is unlikely to be sharp, but is a rare explicit result currently available in the literature (see [33] and Section 12.6).

**Proof.** Recall that $c_{\mu/\nu}^{\lambda} \leq f_{\lambda/\mu} / f^{\nu}$, see the proof of Proposition 2.5. Now apply the upper bound for $e(\nu_k)$ in Theorem 1.1 and the asymptotics for $e(\delta_r)$ given above, with $r = \sqrt{2n/3}$. □

**Remark 8.4.** Note that the corollary likely extends to all three shapes having strongly stable shape $\nabla$, if one could obtain an asymptotic version of Lemma 8.1.

8.4. Thick ribbons of subpolynomial depth

Consider now thick ribbons $\nu_k = \delta_{k+g(k)/\delta_k}$, where $g(k)$ is a subpolynomial growth function. Note that $n = |\nu_k| = kg(k) + O(g(k)^2)$.

**Theorem 8.5.** Let $g(k)$ be a subpolynomial growth function, and let $\{\nu_k\}$ be thick ribbons defined as above. Then

$$- \log 2 + o(1) \leq \frac{1}{n} \left( \log e(\nu_k) - n \log n + n \log g(k) \right) \leq o(1) \quad \text{as} \quad k \to \infty,$$

where $n = kg(k)$.

**Proof.** By a direct computation:

$$\log F(\nu_k) = n \log n - \log((2g(k) - 1)!)^k - O(g(k)^2 \log g(k)) = n \log n - n \log g(k) - n \log 2 - O(g(k)^2 \log g(k)).$$

The result follows from Theorem 3.3 and Lemma 4.3. □

**Remark 8.6.** We conjecture that the lower bound is tight in the theorem. This is supported by the zigzag ribbon shapes calculations (see below). Note also that Proctor’s formula applies to all thick ribbons $\delta_{k+2r}/\delta_k$. It gives $\xi(\nu_k) = 2^{n(1+o(1))}$ for $r = g(k)$ of subpolynomial growth.

For comparison, in notation of Theorem 8.5, the lower bound in Theorem 2.1 gives a weaker lower bound of $-1 + o(1)$. Curiously, the upper bound

$$e(\nu_k) \leq \frac{n!}{(g(k)!)^{k-1} \Phi(g(k))}$$

given by Theorem 2.1 matches the upper bound in Theorem 8.5.

9. Ribbon shapes

9.1. Zigzag shapes

Let $\text{Alt}(n) = \{\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots \} \subset S_n$ be the set of alternating permutations. The number $E_n = |\text{Alt}(n)|$ is the $n$th Euler number (see [48] and [44, A000111]), with the asymptotics
\[ E_n \sim n! \left( \frac{2}{\pi} \right)^n \frac{4}{\pi} (1 + o(1)) \quad \text{as} \quad n \to \infty \]

(see e.g. [15,48]). Consider the zigzag ribbon hook \( \rho_k = \delta_{k+2}/\delta_k \), \( n = |\rho_k| = 2k + 1 \) (see Fig. 5). Clearly, \( e(\rho_k) = E_{2k+1} \). Observe that \( \xi(\rho_k) = C_{k+1} \), the \((k+1)\)-st Catalan number.

\[ C_m = \frac{1}{m+1} \left( \frac{2m}{m} \right) \sim 4^m m^{-3/2} \pi^{-1/2} \]

(see [30]). Thus \( \xi(\rho_k) = \Theta\left(\frac{2^n}{n^{3/2}}\right) \), so Lemma 4.3 is asymptotically tight for the staircase shape.

Writing \( \xi(\rho_k) \sim cn^n n^\alpha \), let us compare the estimates from different bounds. We have \( F(\rho_k) = n! / 3^k \), so Theorem 3.3 for \( \rho_k \) gives

\[ \frac{n!}{3^k} \leq E_n \leq \frac{n! \cdot C_k}{3^k}, \]

implying that \( 1/\sqrt{3} \leq \eta \leq 2/\sqrt{3} \). Note that \( 1/\sqrt{3} \approx 0.577, 2/\pi \approx 0.636 \) and \( 2/\sqrt{3} \approx 1.155 \). This example shows that the lower bound in Theorem 3.3 is nearly tight, while the upper bound is vacuous for large \( n \) (we always have \( e(P) \leq n! \), of course).

Similarly, Theorem 2.1 for \( \rho_k \) gives

\[ k!(k+1)! \leq E_n \leq \frac{n!}{2^k}, \]

implying \( 0.5 \leq \eta \leq 1/\sqrt{2} \approx 0.707 \). Thus, in this case the general poset lower bound is weaker, while the upper bound is much sharper than (*) in Theorem 3.3.

**Remark 9.1.** The asymptotics of \( e(\lambda/\mu) \) is known for other ribbons with a periodic pattern (see Fig. 5). For example, for two up, two right ribbons \( \rho_n \) we have \( e(\rho_n) = \Theta(n! n^\alpha) \), where \( \alpha \approx 0.533 \) is the smallest positive solution of \( \cos \left( \frac{1}{\alpha} \right) \cosh \left( \frac{1}{\alpha} \right) = -1 \), see [8] and [44, A131454]. It would be interesting to find the asymptotics for more general ribbons, e.g. ribbons along a curve as in the figure.

### 9.2. Ribbons with subpolynomial depth

Let \( n = km \), and let \( \rho(k, m) \) be the unique ribbon hook where all columns have length \( m \) (see Fig. 5). We have \( e(\rho(k, m)) \) is the number of \((m-1)\) up, 1 down" permutations in \( S_n \).

Consider the case \( m = g(k) \), where \( g(k) \) is a subpolynomial function. The ribbon \( \rho(k, m) \) has a subpolynomial depth, so Theorem 7.1 applies. The following result gives a sharper estimate (cf. Section 8.4):

**Theorem 9.2.** Let \( n = kg(k) \), where \( g(k) \to \infty, g(k) = k^{o(1)} \) as \( k \to \infty \). Then:

\[ \log e(\rho(k, g(k))) = n \log n - n \log g(k) - \frac{n \log g(k)}{2g(k)} - \frac{n}{g(k)} + O\left( \frac{n}{g(k)^2} \right) \quad \text{as} \quad n \to \infty. \]
Proof. Denote $a(k, m) = e(\rho(k, m))$. By the lower bound in Theorem 3.3, we have:

$$a(k, g(k)) \geq F(\tau_n) \sim \frac{n!}{(m - 1)^k(m + 1)^k} \quad \text{as} \quad k \to \infty,$$

where $m = g(k)$. On the other hand, by the upper bound in Theorem 2.1, we have:

$$a(k, g(k)) \leq \left(\frac{n}{m, m, \ldots, m}\right) = \frac{n!}{m^k}.$$

We conclude:

$$\log a(k, g(k)) = \log F(\tau_n) + kO(\log(1 + 1/m)) = n \log n - k \log m! + O(k/m)$$

$$+ O(k/m)$$

$$= n \log n - n \log g(k) - \frac{n \log g(k)}{2g(k)} - \frac{n}{g(k)} + O\left(\frac{n}{g(k)^2}\right),$$

as desired. □

10. Slim shapes

10.1. Number of excited diagrams

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be such that $\lambda_\ell \geq \mu_1 + \ell$. Such partitions are called slim. The following result improves the bound in Lemma 4.4 in this case.

Proposition 10.1. In the notation above, let $\mu \vdash m$ be fixed and let $\{\lambda^{(k)}\}$ be a family of slim partitions, such that $\ell(k) := \ell(\lambda^{(k)}) \to \infty$ as $k \to \infty$. We have:

$$\xi(\lambda^{(k)}/\mu) \sim \ell(k)^m \prod_{x \in \mu} h(x) \quad \text{as} \quad k \to \infty.$$

Note that there no direct dependence on $n = |\lambda^{(k)}/\mu|$ in this case.

Proof. We think of every finite $S \subset \mathbb{N}^n$ as a poset with order relations given by $(x, y) \preceq (x', y')$ if $x \preceq y$ and $y \preceq y'$. Consider a random subset $S$ of squares of $\mu$ on the corresponding diagonals inside $\lambda$, which all have lengths between $\ell - O(1)$ and $\ell$. The subset $S \subseteq \mu$ is an excited diagram if and only if its poset is a refinement of the poset $\mu$. In one direction this follows by induction and the definition of excited diagrams (order relations in $\mu$ do not disappear under moves). In the other direction, this is more delicate and follows from [31, Section 3.3].

Now, observe that for the points sampled from the unit interval, the probability the order relations are satisfied is exactly

$$\frac{|\text{SYT}(\mu)|}{m!} = \prod_{x \in \mu} \frac{1}{h(x)}.$$

Indeed, this probability is equal to the volume of the poset polytope (see e.g. [2,49]). From above, the number of such diagrams is $\left(\ell(k) - O(1)\right)^m$, which implies the result. □

10.2. Complementary shapes in large slim rectangles

The case of a fixed $\mu$ and $\lambda$ a rectangle is especially interesting (cf. Section 11.1).
Theorem 10.2 ([40]). Let $\mu$ be fixed and $\lambda = (k^\ell)$ with $\ell, k/\ell \to \infty$. We have:

$$\frac{e(\lambda/\mu)}{e(\lambda)} \sim \prod_{x \in \mu} \frac{1}{h(x)}, \quad \text{as } \ell, k/\ell \to \infty.$$ 

This theorem is obtained by Regev and Vershik in [40] by a calculation based on [35]. Let us show that the upper bound in the theorem follows from the upper bound in Theorem 3.3.

Let $m = |\mu|$. We have in this case:

$$F(\lambda/\mu) e(\lambda) = \frac{(k\ell - m)!}{(k\ell)!} \left( \prod_{x \in \mu} h_i(x) \right),$$

where $h_i(x)$ is a hook-length in $\lambda$. Letting $k/\ell \to \infty$, we get

$$F(\lambda/\mu) e(\lambda) \sim \frac{(k\ell - m)!}{(k\ell)!} \sim \frac{1}{\ell^m}.$$ 

Now the upper bound in Theorem 3.3 and Proposition 10.1 give:

$$\frac{e(\lambda/\mu)}{e(\lambda)} \lesssim \frac{\xi(\lambda/\mu) F(\lambda/\mu)}{e(\lambda)} \sim \prod_{x \in \mu} \frac{1}{h(x)}, \quad \text{as } \ell, k/\ell \to \infty,$$

where we write $f(z) \lesssim g(z)$ for $f(z) \leq g(z)(1+o(1))$. This implies that the upper bound in Theorem 3.3 is asymptotically tight in this case, while the lower bound is off by a $\Theta(\ell^{(|\mu|)})$ factor.

10.3. Slim stripes

Let $\nu = \lambda/\mu$ be slim skew shape defined above. It follows from the proof of Proposition 10.1, that the number of excited diagrams $\xi(\lambda/\mu)$ in this case depends only on $\mu$ and $\ell$, but not on $\lambda$. The following result considers a special case of a staircase shape $\mu = \delta_\ell$.

Proposition 10.3. Let $\lambda/\mu$ be a skew shape s.t. $\mu = \delta_\ell$, $\ell(\lambda) = \ell$ and $\lambda_\ell \geq \mu_1 + \ell$. Then $\xi(\lambda/\mu) = 2^{\ell_1}$.

For example, for $\lambda/\mu = (765/21)$, we have $\ell = 3$ and $\xi(\lambda/\mu) = 8$.

Proof. By the proof of Theorem 3.1 in [31], the set of corresponding flagged tableaux in this case is in bijection with SSYT of shape $\delta_\ell$ with all entries $\leq \ell$. In general these constraints vary from row to row, but for $\mu = \delta_\ell$ and slim $\lambda$ as above, we have $f_i = \ell$ for all $1 \leq i \leq \ell$.

Now, the number of SSYT of shape $\mu$ with all entries $\leq \ell$ is equal to the value of Schur function $s_{\mu}(1, \ldots, 1, \ell)$ times, which can be computed by the hook-content formula [49, Cor. 7.21.4]. A direct calculation gives $s_{\delta_\ell}(1, \ldots, 1) = 2^{\ell_1}$. □

Remark 10.4. There is a curious bijection between excited diagrams in this case and domino tilings of an Aztec diamond $AD_\ell$, since SSYT of shape $\delta_\ell$ with all entries $\leq \ell$ have Gelfand–Tsetlin patterns given by complete monotone triangles of size $\ell$. In turn, the latter are in bijection with domino tilings of $AD_\ell$, see [11].

11. Further examples and applications

11.1. Dual shapes

Let $\lambda = (k^\ell)$ be a rectangle and $\mu \subset \lambda$. Denote by $\nu = (\lambda/\mu)^*$ the partition obtained by the 180° rotation of $\lambda/\mu$. Thus we can apply the lower bound in Theorem 3.3 to obtain the following unusual result.
Proposition 11.1. Let $\nu$ be a partition of $n$. Denote by $h^*(x), x = (i, j) \in \nu$, the dual hooks defined by $h^*(i, j) = i + j - 1$, and let

$$H(\nu) = \prod_{x \in \nu} h(x), \quad H^*(\nu) = \prod_{x \in \nu} h^*(x).$$

Then $H(\nu) \leq H^*(\nu)$, and the equality holds only when $\nu$ is a rectangle.

Note that this result cannot be proved by a simple monotonicity argument, since

$$\sum_{x \in \nu} h(x) = \sum_{x \in \nu} h^*(x) = n + \sum_i \binom{\nu_i}{2} + \sum_j \binom{\nu'_j}{2}.$$

Proof. Observe that $e(\lambda/\mu) = e(\nu) = n!/H(\nu)$ and $F(\lambda/\mu) = n!/H^*(\nu)$, where $n = |\lambda/\mu| = |\nu|$. Now the inequality $e(\lambda/\mu) \geq F(\lambda/\mu)$ implies the first part. For the second part, recall from the proof of Theorem 3.3 that $e(\lambda/\mu) = F(\lambda/\mu)$ only if $\xi(\lambda/\mu) = 1$, i.e. when no excited moves are allowed. It is easy to see that this can happen only when $\nu$ is a rectangle. □

Remark 11.2. After the proposition was obtained, in response to second author’s MathOverflow question, F. Petrov found a generalization of this result to all concave functions. In particular, his proof implies that the variance of hooks is larger than that of complementary hooks in all Young diagrams.

11.2. Regev–Vershik shapes

Let $\sigma \subset \tau$, where $\tau = (\ell^k)$ is a rectangle. As above, denote by $\sigma^*$ the $180^\circ$ rotation of $\sigma$. Consider a skew shape $\lambda/\mu$ obtained by attaching two copies of $\sigma^*$ above and to the left of $\tau$, and removing $\sigma^*$ from $\tau$ (see Fig. 6). The theorem by Regev and Vershik in [40] states that

$$\left(\diamondright\right) \prod_{x \in \lambda/\mu} h(x) = \left[\prod_{x \in \sigma} h(x)\right] \cdot \left[\prod_{x \in \tau} h(x)\right].$$

Proposition 11.3. In the notation above, let $s := |\sigma|$, $t := |\tau| = k\ell$. Then:

$$e(\lambda/\mu) \geq \binom{s + t}{s} e(\sigma)e(\tau).$$

The inequality is trivially tight for $\sigma = \emptyset$ or $\sigma = \tau$, when the skew shapes coincide: $\lambda/\mu = \sigma \circ \tau$.

Proof. Note that $|\lambda/\mu| = |\sigma| + |\tau| = s + t$. By the Regev–Vershik theorem $\left(\diamondright\right)$, we have

$$\frac{F(\lambda/\mu)}{(s + t)!} = \frac{e(\sigma)}{s!} \cdot \frac{e(\tau)}{t!}.$$

By the lower bound in Theorem 3.3, we have:

$$e(\lambda/\mu) \geq F(\lambda/\mu) = \binom{s + t}{t} e(\sigma)e(\tau),$$

as desired. □

Remark 11.4. Regev and Vershik conjectured that $\lambda/\mu$ and $\sigma \circ \tau$ have the same multiset of hooks. This was proved bijectively and generalized in a number of directions by Janson, Regev and Zeilberger, Bessenrodt, Krattenthaler, Goulden and Yong, and others. We refer to [23] for the “master bijection” and to [1, Section 12] for further references.

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4 See http://mathoverflow.net/q/243846.
12. Final remarks and open problems

12.1. Computing $e(\mathcal{P})$ is known to be #P-complete [6], which partly explains relatively few good general bounds (see [12,50]). For a nice counterpart of the upper bound in Theorem 2.1, relating $\log e(\mathcal{P})$ and the entropy, defined in terms of the $M(\mathcal{P})$, see [7,22]. Unfortunately this bound is not sharp enough for bounds on $\log e(\lambda/\mu)$ as it is off by a multiplicative constant.

Note that this approach was used to improve the bounds on the second term in the asymptotic expansion of $e(B_k)$, the number linear extensions of the Boolean lattice [5] (see also [4,43]). Namely, for even $k$, Theorem 2.1 gives:

$$\log_2\left(\frac{k}{k/2}\right) - \frac{3}{2} \log_2 e + o(1) \leq \log_2 e(B_k) \leq \log_2\left(\frac{k}{k/2}\right).$$

Brightwell and Tetali in [5] show that the lower bound is tight.

12.2. We show that in many special cases of interest our bounds give the first or the first two terms in the asymptotic expansion. Although the number $\xi(\lambda/\mu)$ of excited diagrams can be large, even exponential in many cases, this is dwarfed by the number of standard Young tableaux, implying that the “naive HLF” $F(\lambda/\mu)$ is indeed a good estimate in these cases.

12.3. The arXiv version of [31] contains an extended survey and comparison between NHLF and other known formulas for $f^{\lambda/\mu}$. Of course, the Jacobi–Trudy and other determinant formulas (see e.g. [19,27,49]) are computationally efficient, but difficult to use for asymptotic estimates. While the Littlewood–Richardson rule is positive, the LR-coefficients have a complicated combinatorial structure and are hard to compute [33] and estimate (cf. Section 12.6). Finally, the Okounkov–Olshanski formula [35] mentioned earlier seems to be weak for large $\mu$ (cf. [32]).

12.4. The notion of stable limit shape goes back to Erdős and Szekeres in the context of random partitions, and to Vershik and Kerov in the context of Young tableaux and asymptotic representation theory (see e.g. [3,41]). Our notion of strong stable shapes is more restrictive as we need faster convergence for the proof of Theorem 6.3.

12.5. Note that the shifted skew diagrams greatly increase the number of limit shapes of area 1, for which we have

$$\log e(\nu_n) \sim \frac{1}{2} n \log n \text{ as } n \to \infty.$$
As we mentioned in Remark 2.7, this holds also for some “truncated shapes” (see [1,37]). We generalize this result in a forthcoming [36] to all piecewise linear shapes in the plane.

12.6.

Let us emphasize once again that it is the lower order terms in the asymptotics of $f^{\lambda/\mu}$ that turn out to be most relevant for applications (cf. [3,38]). For example, for the LR-coefficients we have:

$$c_{\mu,\nu}^\lambda \leq f^{\lambda/\mu} / f^\nu$$

(see Section 8.3). Following the proof of Corollary 8.3, in the stable shape case the leading terms for the RHS coincide since $|\lambda/\mu| = |\nu|$, while the second order terms give an exponential upper bound (see Theorem 6.3). In other words, any improvement in the second order terms for $f^{\lambda/\mu}$ in every particular stable shape case improves the upper bound (○) for the LR-coefficients.

Note that computing or even approximating the LR-coefficients is a major problem in the area (see e.g. [21,33]). In fact, the LR-coefficients $c_{\mu,\nu}^\lambda$ are always at most exponential in $n = |\lambda|$, as recently announced by Stanley in [49, Supp. Exc. 7.79]. In the case $\lambda_1, \ell(\lambda) = O(\sqrt{n})$ which includes the stable shapes in Theorem 6.1, this easily follows from the Knutson–Tao puzzle interpretation of the LR-coefficients.

12.7.

For the thick ribbons $\nu_k = \delta_{2k}/\delta_k$, we conjecture that

$$\log f^{\nu_k} = \frac{1}{2} n \log n + cn + o(n), \quad \text{for some } c < 0.$$ 

There seem to be no available tools to even approach this problem. If true, the lower and upper bounds for $e(\nu_k)$ in Theorem 1.1 imply that $-0.3238 < c < -0.0621$ (cf. Section 8.2).

Recently, Jay Pantone used his implementation of the method of differential approximants on 150+ terms of the sequence $\{e(\nu_k)\}$ [44, A278289] to approximate the constant above as $c \approx -0.1842$. We plan to refine our techniques to improve our bounds towards this value.

12.8.

There is a shifted version of the NHLF also obtained by Naruse [34], which should give similar asymptotic results for the number of SYT of shifted shapes (cf. [1, Section 5.3]). In fact, the results should follow verbatim for the stable shape limit case. Let us also mention that our calculations for thick ribbons (see Section 8) can be translated to this case; the excited diagrams in the shifted case correspond to type B and have been extensively studied. We refer to [30,32] for details and further references.

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