Refined Factorizations of Solvable Potentials

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Abstract

A generalization of the factorization technique is shown to be a powerful algebraic tool to discover further properties of a class of integrable systems in Quantum Mechanics. The method is applied in the study of radial oscillator, Morse and Coulomb potentials to obtain a wide set of raising and lowering operators, and to show clearly the connection that link these systems.

I. INTRODUCTION

We shall begin this section by recalling some basic facts of the standard factorization method, as can be found for instance in [1,2], mainly to fix the notation. Afterwards, we will set up the general lines to define more general factorizations, and the way they depart from the conventional ones previously characterized.

Let us consider a sequence of stationary one dimensional Schrödinger equations, labeled by an integer number $\ell$, written in the form

$$H_\ell^0 \psi_\ell^n \equiv \left\{ -\frac{d^2}{dx^2} + V_\ell^0(x) \right\} \psi_\ell^n(x) = E_\ell^n \psi_\ell^n(x), \quad (1)$$

where the constants $\hbar$ and $m$ have been conveniently reabsorbed. If such a set (or 'hierarchy') of Hamiltonians can be expressed as

$$H_\ell = X_\ell^+ X_\ell^- - q(\ell) = X_{\ell-1}^- X_{\ell-1}^+ - q(\ell-1), \quad (2)$$

where

$$X_\ell^\pm = \mp \frac{d}{dx} + w_\ell(x), \quad (3)$$

being $w_\ell(x)$ functions and $q(\ell)$ constants, then we will say that they admit a factorization. From (3) we have that $X_\ell^\pm$ are hermitian conjugated of each other, $\left(X_\ell^->\right)^\dagger = X_\ell^+$, with

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respect to the usual inner product of the Schrödinger equation. This is consistent with the factorization (2) and the hermiticity of $H^\ell$.

We shall focus our interest in studying the discrete spectrum of each Hamiltonian, so we further impose that the equation

$$X^-_\ell \psi^\ell_\ell(x) = 0$$

will determine the ground state of $H^\ell$ if it exists. Of course many other properties related with the continuous spectrum can also be derived with the help of factorizations, but they are out of our present scope.

Some consequences that can immediately be derived from the previous conditions are enumerated below.

**i) Spectrum.** Let $\psi^\ell_\ell$ be the ground state of $H^\ell$ as stated in (4), then its energy is precisely $E^\ell_\ell = -q(\ell)$. When there are excited bounded states $\psi^\ell_n$, with $n = \ell, \ell + 1, \ldots$, their energy is given by $E^\ell_n = -q(n)$. Therefore, in these circumstances, $-q(\ell)$ should be an increasing function on $\ell$.

**ii) Eigenfunctions.** It is straightforward to check, for each $\ell$, the intertwining relations

$$H^\ell X^+_{\ell} = X^+_{\ell} H^{\ell+1}, \quad H^{\ell+1} X^-_{\ell} = X^-_{\ell} H^\ell.$$  

Let us designate by $H^\ell = \langle \{ \psi^\ell_n \}_{n \geq \ell} \rangle$ the Hilbert space spanned by the bounded states of $H^\ell$, for $\ell \in \mathbb{Z}$. Then, due to (3) the operators $X^\pm_{\ell}$ link these spaces as

$$X^-_{\ell} : \mathcal{H}^\ell \to \mathcal{H}^{\ell+1}, \quad X^+_{\ell} : \mathcal{H}^{\ell+1} \to \mathcal{H}^\ell,$$

$$X^-_{\ell} \psi^\ell_n(x) \propto \psi^{\ell+1}_n(x), \quad X^+_{\ell} \psi^{\ell+1}_n(x) \propto \psi^\ell_n(x).$$

Remark that the action of $X^\pm_{\ell}$ preserve the label $n$, that is, they connect eigenfunctions with the same energy $E^\ell_n$. If the eigenfunctions are normalized we can be more explicit: up to an arbitrary phase factor,

$$X^-_{\ell} \psi^\ell_n(x) = \sqrt{q(\ell) - q(n)} \psi^{\ell+1}_n(x), \quad n \geq \ell$$

$$X^+_{\ell} \psi^{\ell+1}_n(x) = \sqrt{q(\ell) - q(n)} \psi^\ell_n(x), \quad n > \ell.$$  

Similar considerations would also apply if the ground states were defined through $X^+$. Depending on each particular problem we will use one of the following notations

$$X^-_{\ell-1}(r) \psi^\ell_\ell = 0, \quad \text{if } \ell \leq 0,$$

$$X^+_{\ell-1}(r) \psi^\ell_\ell = 0, \quad \text{if } \ell \geq 0.$$  

For such a case $-q(\ell)$ must be a decreasing function of $\ell$. We shall also have the opportunity to illustrate this situation in some examples along the next sections.

Now, it is natural to define a set of free-index linear operators $\{X^\pm, L\}$ acting on the direct sum of the Hilbert spaces $\mathcal{H} \equiv \oplus_\ell \mathcal{H}^\ell$ by means of
where one must have in mind (3) and (7). That is, the operators $X^\pm$ act on each function $\psi_n(x)$ by means of the differential operators (3) changing $\ell$ into $\ell \mp 1$. The action on any other vector of $\mathcal{H}$ can be obtained from (10) by linearization, but we shall never need it. At this moment we are not in conditions to guarantee that the space $\mathcal{H}$ is invariant under this action (it might happen that the action of $X^\pm$ on $\mathcal{H}$ could lead us to the continuous spectrum, or even to an unphysical eigenfunction), but we postpone this problem to the examples of Section III.

Taking into account our definitions (10), it is straightforward to arrive at the following commutators,

$$[L, X^\pm] = \mp X^\pm, \quad [X^+, X^-] = q(L) - q(L-1).$$

It is clear that the set of operators $\{X^\pm, L\}$ in general does not close a Lie algebra; relations (11) only allow us to speak formally of an associative algebra.

There are many aspects of the conventional factorizations above characterized which can be modified, mainly with the objective of being applicable to a wider class of systems (see for example [3]). However, in this paper we are interested in going deeply into the possibilities of this method on a class of systems where the usual factorization can already be applied, so that it could supply us with additional information. With this aim, we shall stress here on two points that will be useful in the next sections.

First, we shall assume that the operators $X^\pm_\ell$ do not have to take necessarily the form given in (3). In particular, if we have a family of invertible operators $D_\ell$ and define $Y^+_\ell = X^+_\ell D^{-1}_\ell$, $Y^-_\ell = D_\ell X^-_\ell$, we will also have

$$H^\ell = X^+_\ell X^-_\ell - q(\ell) = Y^+_\ell Y^-_\ell - q(\ell).$$

The new factor $D_\ell$ may be a function (which would add nothing specially new) but also a local operator, i.e., an operator acting on wavefunctions in the form

$$D_\ell \psi(x) = \psi(g_\ell(x)),$$

where $g_\ell$ is a bijective real function. An example of such an operator, which was already used in [3], is given by the dilation,

$$D(\mu) \psi(x) = \psi(\mu x), \quad \mu > 0.$$  

Second, an eigenvalue equation can be characterized by more than one label; this consideration has also been explored by Barut et al [3], but in another context. In the next section we shall deal with two real labels; this will enable us to have more possible ways to factorize the Hamiltonian hierarchy, and the sequence of labels will not be limited (essentially) to the integers, but it will be constituted by a lattice of points in $\mathbb{R}^2$. This increasing of factorizations will reflect itself in a larger algebra of free-index operators. In particular, among them, there can be lowering and raising operators for each Hamiltonian, which can never be obtained by the conventional factorization method. Section III will illustrate how our general method works when it is applied to three well known potentials: radial oscillator, Morse, and radial Coulomb. For each of these potentials we shall see that the results so obtained can be used to recover, as special cases, those corresponding to the standard factorizations. Finally some comments and remarks will end this paper.
II. REFINED FACTORIZATIONS

Once the spectrum $E_n^\ell$ of the hierarchy $H^\ell$ is known, we propose a somewhat more general factorization of the eigenvalue equations than that one already displayed in (3), as follows:

$$h_{n,\ell}(x) \left[ H^\ell - E_n^\ell \right] = B_{n,\ell} A_{n,\ell} - \phi(n, \ell) = A_{\tilde{n},\tilde{\ell}} \tilde{B}_{\tilde{n},\tilde{\ell}} - \phi(\tilde{n}, \tilde{\ell}).$$

(15)

This must be understood as a series of relationships valid for a class of allowed values of the parameters $(n, \ell) \in \mathbb{R}^2$. Here $B_{n,\ell}$ and $A_{n,\ell}$ are first order differential operators in the wider sense specified in the previous section, $h_{n,\ell}(x)$ denote functions, and $\phi(n, \ell)$ are constants. The $(\tilde{n}, \tilde{\ell})$ values depend on $(n, \ell)$, i.e., $(\tilde{n}, \tilde{\ell}) = F(n, \ell)$, being $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ an invertible map defined on a certain domain. The iterated action of $F$ or $F^{-1}$ on a fixed initial point $(n_0, \ell_0) \in \mathbb{R}^2$ originates a sequence of points in $\mathbb{R}^2$ that will play a role similar to the integer sequence $\ell$ in $\mathbb{R}$ for the usual factorizations. In principle the points $(n, \ell)$ obtained by this new approach can take integer values for both arguments, but we do not discard a priori other possibilities.

The problem of finding solutions to this kind of factorizations becomes more involved because we have additional functions $h_{n,\ell}(x)$ to be determined. Nevertheless, an important and immediate consequence of (15) is that the operators $B_{n,\ell}, A_{n,\ell}$ share properties similar to (3) with respect to their analogs $\{X_\ell^\pm\}$:

$$\left[ h_{\tilde{n},\tilde{\ell}}(x) \left( H^\ell - E_{\tilde{n}}^\ell \right) \right] A_{n,\ell} = A_{n,\ell} \left[ h_{n,\ell}(x) \left( H^\ell - E_n^\ell \right) \right],$$

$$B_{n,\ell} \left[ h_{\tilde{n},\tilde{\ell}}(x) \left( H^\ell - E_{\tilde{n}}^\ell \right) \right] = \left[ h_{n,\ell}(x) \left( H^\ell - E_n^\ell \right) \right] B_{n,\ell},$$

(16)

where $F(\tilde{n}, \tilde{\ell}) = (n, \ell)$. Therefore, using the same notation as in (3),

$$A_{n,\ell} : \mathcal{H}_n^\ell \rightarrow \mathcal{H}_n^\ell,$$

$$A_{n,\ell} \psi_n^\ell(x) \propto \psi_n^\ell(x),$$

$$B_{n,\ell} : \mathcal{H}_n^\ell \rightarrow \mathcal{H}_n^\ell,$$

$$B_{n,\ell} \psi_n^\ell(x) \propto \psi_n^\ell(x).$$

(17)

In this case, the most relevant differences with respect to the usual factorizations are:

i) $B_{n,\ell}, A_{n,\ell}$ in general do not preserve the energy eigenvalue, they may change both labels $n$ and $\ell$.

ii) $A_{n,\ell}$ does not act on the whole space $\mathcal{H}_n^\ell$, it acts just on the eigenfunction $\psi_n^\ell(x) \in \mathcal{H}_n^\ell$ (the same can be said of $B_{n,\ell}$ with respect to $\psi_n^\ell(x) \in \mathcal{H}_n^\ell$).

When $n = \tilde{n}$ and $h_{n,\ell}(x) = 1$, we recover the conventional case with $B_{n,\ell}, A_{n,\ell}$ playing the role of $X_\ell^+$, $X_\ell^-$, respectively. However, the hermiticity properties for the general case are lost because the product $B_{n,\ell} A_{n,\ell}$ gives not the Hamiltonian operator alone, but it includes also a non constant multiplicative factor.

We can define the free-index operators $\{A, B, L, N\}$ as we did in (10), where the latter is defined by $N \psi_n^\ell = n \psi_n^\ell$. They satisfy the following commutation rules

$$[L, B] = B(\tilde{L} - L), \quad [N, B] = B(\tilde{N} - N), \quad [B, A] = \phi(N, L) - \phi(\tilde{N}, \tilde{L})$$

$$[L, A] = (L - \tilde{L})A, \quad [N, A] = (N - \tilde{N})A, \quad [N, L] = 0,$$

(18)
where ($\tilde{N}, \tilde{L}) = F(N, L)$. As the operators $L, N$ commute, their eigenvalues are used to label the common eigenfunctions $\psi_n^{\ell}(x)$. We must also notice that the equation $A_n, \ell \psi_n^{\ell}(x) = 0$ (or $B_n, \ell \psi_n^{\ell}(x) = 0$) does not necessarily give an eigenfunction of $H^{\ell}$ (or $H^{\ell \dagger}$); this happens to be the case only when $\phi(n, \ell) = 0$.

### III. APPLICATIONS

#### A. Radial Oscillator Potential

As usual the Hamiltonian of the two dimensional harmonic oscillator includes the effective radial potential $V^{\ell}(r) = r^2 + \frac{(2\ell+1)(2\ell-1)}{4r^2}$, where $\ell = 0, 1, \ldots$ is for the angular momentum. The related stationary Schrödinger equation has discrete eigenvalues denoted according to the following convention,

$$E_n^{\ell} = 2n + 2, \quad n = 2\nu + \ell; \quad \nu = 0, 1, \ldots$$

It can be factorized in two ways according to our general scheme:

$$-\frac{1}{4} \left[ H^{\ell} - E_n^{\ell} \right] = \frac{1}{4} \left[ \frac{d^2}{dr^2} - \frac{2\ell + 1}{4r^2} + 2n + 2 \right] = B_{n, \ell}^i A_{n, \ell}^i - \phi^i(n, \ell), \quad i = 1, 2 \quad (19)$$

with $\phi^i(n, \ell)$ given by

$$\phi^1(n, \ell) = -\frac{1}{2}(n + \ell + 2), \quad \phi^2(n, \ell) = -\frac{1}{2}(n - \ell + 2), \quad (20)$$

and where the action on the parameters associated to each factorization is given respectively by the functions

$$(n, \ell) = F^1(n + 1, \ell + 1), \quad (n, \ell) = F^2(n + 1, \ell - 1). \quad (21)$$

This can also be written in an easier notation,

$$A^1: (n, \ell) \rightarrow (n + 1, \ell + 1) \quad A^2: (n, \ell) \rightarrow (n + 1, \ell - 1)$$

$$B^1: (n + 1, \ell + 1) \rightarrow (n, \ell) \quad B^2: (n + 1, \ell - 1) \rightarrow (n, \ell). \quad (22)$$

The explicit form of these intertwining operators is

$$\begin{align*}
A_{n, \ell}^i(r) &= \frac{1}{2} \left[ \frac{d}{dr} - r - (\ell+1/2) \frac{1}{r} \right] \\
B_{n, \ell}^i(r) &= \frac{1}{2} \left[ \frac{d}{dr} + r + (\ell+1/2) \frac{1}{r} \right] \\
A_{n, \ell}^2(r) &= \frac{1}{2} \left[ \frac{d}{dr} - r + (\ell-1/2) \frac{1}{r} \right] \\
B_{n, \ell}^2(r) &= \frac{1}{2} \left[ \frac{d}{dr} + r - (\ell-1/2) \frac{1}{r} \right]
\end{align*} \quad (23)$$

Observe that in this case, as $h_{n, \ell}(r)$ is a constant, we are able to implement also the hermiticity properties $(A^i)^\dagger = -B^i$. The nonvanishing commutation rules for the free-index operators $\{N, L, A^i, B^i; i = 1, 2\}$ are shown to be, in agreement with [IS],

$$[L, B^i] = (-1)^i B^i, \quad [N, B^i] = -B^i, \quad [A^i, B^i] = 1,$$

$$[L, A^i] = -(-1)^i A^i, \quad [N, A^i] = A^i, \quad i = 1, 2. \quad (24)$$

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These commutators correspond to two independent boson algebras with $N, L$ being a linear combination of their number operators. Formally we can extend the values of $\ell$ so to include the negative integers. This is physically appealing because in two space dimensions (only!) $\ell$ represents the $L_z$-component of angular momentum, so that it could take negative integer values. Of course the extension $\psi_{-\ell}(r) := \psi_\ell(r)$ above proposed is consistent with such an interpretation: (i) The radial components for opposite $L_z$-values have to coincide, and (ii) The potential $V^\ell$ is invariant under the interchange $\ell \to -\ell$. With this convention, the Hilbert space $H$ of bounded states is invariant under the action of the operators $\{N, L, A_i, B_i; i = 1, 2\}$, so that it constitutes the support for a lowest weight irreducible representation for the algebra (24) based on the fundamental state $\psi_{\ell=0}^0$.

It is worth to notice that, taking into account (22), the composition $\{A^1 A^2, B^1 B^2\}$ constitutes the lowering and raising operators for each Hamiltonian $H^\ell$, while the pair $\{A^1 B^2, A^2 B^1\}$ connects states of different Hamiltonians $H^\ell$ with the same energy, changing only the label $\ell$.

We shall compare briefly the above results with the conventional factorizations of the two-dimensional radial oscillator potential [6]. It is well known that there are two such factorizations which we will write in the form:

\begin{align}
(a) \quad X^+ X^- - q_x(\ell) &= H_x^\ell = H^\ell - 2\ell \\
(b) \quad Z^+ Z^- - q_z(\ell) &= H_z^\ell = H^\ell + 2\ell,
\end{align}

with $H^\ell = -\frac{d^2}{dr^2} + r^2 + \frac{(2\ell + 1)(2\ell - 1)}{4r^2}$. Then we have the following identification:

Case (a)

1. Operators: $X^+ = -2B_{n,\ell}^1, X^- = 2A_{n,\ell}^1, q_x(\ell) = 4\ell - 2$.

2. Ground states: $X^+_{\ell-1} \psi_\ell = 0, \ell \leq 0$.

3. Energy eigenvalues: $E_n^\ell = 4n + 2$, with $n \in \mathbb{Z}^+$ and $n \geq -\ell$.

In this case we have used a notation in agreement with (8).

Case (b)

1. Operators: $Z^+ = -2A_{n-1,\ell+1}^2, Z^- = 2B_{n-1,\ell+1}^2, q_z(\ell) = -4\ell - 2$.

2. Ground states: $Z^- \psi_\ell = 0, \ell \geq 0$.

3. Energy eigenvalues: $E_n^\ell = 4n + 2$, with $n \in \mathbb{Z}^+$ and $n \geq \ell$.

Therefore, as there is a correspondence between the results of the conventional and our factorizations, one might conclude the total equivalence of both treatments. However, we make a remark worth to take into account: the conventional factorizations make use of two Hamiltonian hierarchies, $H_x^\ell$ and $H_z^\ell$, whose terms differ in a constant $4\ell$, while the new factorizations use only one $H^\ell$. If we want that both factorizations (a) and (b) be valid inside the same hierarchy it is necessary to adopt the properties of our approach in the following sense: either the operators $X^\pm$ or $Z^\pm$ (or both pairs) must change not only the quantum number $\ell$ but also $n$. In this way we have shown, by means of this simple example, that the factorizations presented here prove to be quite useful providing directly a more natural viewpoint.
B. Morse Potential

In this case we have eigenvalue Schrödinger equations for the whole real line \( x \in \mathbb{R} \) with the potentials

\[
V^\ell(x) = \left( \frac{\alpha}{2} \right)^2 \left( e^{2ax} - 2(\ell + 1) e^{ax} \right), \quad \alpha > 0, \ell \geq 0.
\]

(27)

Often in the literature \cite{7} the Morse potentials are written \( V(y) = A (e^{-2\alpha y} - 2 e^{-\alpha y}) \). This form can be reached from (27) by a simple change of the variable \( x = -y + k \), with \( e^{\alpha k} = \ell + 1 \).

The energy eigenvalues can be expressed as

\[
E_n^\ell = -\frac{\alpha^2}{4} n^2, \quad n = \ell - 2\nu > 0; \quad \nu = 0, 1, 2 \ldots
\]

(28)

In order to have bounded states it is necessary the restriction \( \ell > 0 \); the critical value \( \ell = 0 \) has in this respect an special limiting character, and it is convenient to take it into account as we shall see later. According to (28), the eigenfunctions \( \psi^\ell_n \) are characterized by labels satisfying \( n \leq \ell \); this means that the ground states will be defined through (9).

There are two new factorizations

\[
-\frac{e^{-\alpha x}}{\alpha^2} \left[ H^\ell - E_n^\ell \right] = B^i_{n,\ell}(x)A^i_{n,\ell}(x) - \phi^i(n, \ell), \quad i = 1, 2,
\]

(29)

with \( \phi^i(n, \ell) \) given by

\[
\phi^1(n, \ell) = -\frac{1}{2}(\ell + n + 2), \quad \phi^2(n, \ell) = -\frac{1}{2}(\ell - n + 2),
\]

(30)

and the action on the parameters \( (n, \ell) \) for each factorization by the functions

\[
(n, \ell) = F^1(n + 1, \ell + 1), \quad (n, \ell) = F^2(n - 1, \ell + 1).
\]

(31)

The explicit form of the intertwining operators (29) is

\[
\begin{aligned}
B^1_{n,\ell}(x) &= \frac{e^{-\alpha x/2}}{\alpha} \frac{d}{dx} + \frac{1}{2} e^{\alpha x/2} + \frac{n+1}{2} e^{-\alpha x/2} \\
A^1_{n,\ell}(x) &= \frac{e^{-\alpha x/2}}{\alpha} \frac{d}{dx} - \frac{1}{2} e^{\alpha x/2} - \frac{n-1}{2} e^{-\alpha x/2}
\end{aligned}
\]

(32)

\[
\begin{aligned}
B^2_{n,\ell}(x) &= \frac{e^{-\alpha x/2}}{\alpha} \frac{d}{dx} + \frac{1}{2} e^{\alpha x/2} - \frac{n-1}{2} e^{-\alpha x/2} \\
A^2_{n,\ell}(x) &= \frac{e^{-\alpha x/2}}{\alpha} \frac{d}{dx} - \frac{1}{2} e^{\alpha x/2} + \frac{n}{2} e^{-\alpha x/2}
\end{aligned}
\]

(33)

As in the oscillator case we have two pairs of operators that change simultaneously two types of labels: one, \( \ell \), is related to the intensity of the potential, although here it can not be interpreted as due to a centrifugal term. The second one, \( n \), is directly related to the energy through formula (28). The (nonvanishing) commutators of the free-index operators are

\[
[L, B^i] = -B^i, \quad [N, B^i] = (-1)^i B^i, \quad [A^i, B^i] = 1
\]

\[
[L, A^i] = A^i, \quad [N, A^i] = -(-1)^i A^i, \quad i = 1, 2.
\]

(34)
Observe that in this case the function \( h_{n,\ell}(x) = -e^{-ax}/a^2 \) is not a constant, so the hermiticity relations among the operators \( \{A^i, B^i; i = 1, 2\} \) are spoiled. Let us take \( \ell \in \mathbb{Z}^+ \), and formally allow for negative \( n \)-values in (28), i.e., \( \pm n = \ell - 2\nu \); this is admissible because in the operators of (22)-(23) we have a symmetry under the change \( n \rightarrow -n \). Then the Hilbert space \( \mathcal{H} \) of bounded states enlarged with the (not square integrable) states \( \psi_{n=0}^\ell \), \( \ell = 0, 1, 2 \ldots \), will be invariant under the action of all the operators defined in this section. The lowest weight state is played in this case by a not square-integrable wavefunction, \( \psi_{n=0}^\ell \).

We can of course build other operators out of the previous ones, changing exclusively one of the labels: the pair \( \{A^1A^2, B^1B^2\} \) change \( \ell \) (in +2 or −2 units, respectively), while \( \{A^1B^2, A^2B^1\} \) change \( n \) (also in +2 or −2 units, respectively). It is interesting to show explicitly the form taken by the former couple:

\[
\begin{cases}
(B^1B^2)_{n,\ell} = -\frac{1}{\alpha} \frac{d}{dx} + \frac{1}{2} (e^{\alpha x} - (\ell + 2)) \\
(A^1A^2)_{n,\ell} = -\frac{1}{\alpha} \frac{d}{dx} + \frac{1}{2} (e^{\alpha x} - (\ell + 2))
\end{cases}
\]

where \((A^1A^2)_{n,\ell} = A_{n-1,\ell+1}^1A_{n,\ell}^2\) and \((B^1B^2)_{n,\ell} = B_{n,\ell}^1B_{n+1,\ell+1}^2\) according to the rules of the action of free index operators (38), (39). They can be identified with the usual factorization operators for the Morse Hamiltonians \( H^\ell \) described in the first section in the following way:

1. Factorization: \( X^\ell_+ X^\ell_- - q(\ell') = H^{2\ell'} \), \( \ell' \in \mathbb{Z}^+ \).
2. Operators: \( X^\ell_+ = -\alpha (B^1B^2)_{n,2\ell} \), \( X^\ell_- = -\alpha (A^1A^2)_{n,2\ell} \), \( q(\ell') = \alpha^2 (\ell' + 1)^2 \).
3. Ground states: \( X^\ell_{\ell'-1} \psi^\ell_{2n'} = 0 \), \( \ell' > 0 \).
4. Energy eigenvalues: \( E^\ell_{2n'} = -\alpha^2 (n')^2 \), with \( n = 2n' \), \( n' \in \mathbb{Z}^+ \), and \( 0 \leq n' \leq \ell' \).

This time the notation, as it was mentioned above, is in agreement with (9).

C. Radial Coulomb Potential

After the separation of the angular variables, the stationary radial Schrödinger equation for the Coulomb potential in two dimensions takes the form

\[
H^\ell \psi^\ell_n(r) = \left\{ -\frac{d^2}{dr^2} + \frac{(2\ell + 1)(2\ell - 1)}{4r^2} - \frac{2}{r} \right\} \psi^\ell_n(r) = E^\ell_n \psi^\ell_n(r),
\]

where the values of the orbital angular momentum are positive integers \( \ell = 0, 1, 2 \ldots \)

The computation of the discrete spectrum associated to the bounded states of \( H^\ell \) can be easily obtained by means of the conventional factorizations (2) with

\[
X^\ell_+ = + \frac{d}{dr} - \frac{2\ell + 1}{2} + \frac{2}{2\ell + 1}, \quad q(\ell) = \frac{-1}{(\ell + 1/2)^2}.
\]

Therefore, according to the results quoted in Section I, we have

\[
E^\ell_n = -\frac{1}{(n + 1/2)^2}, \quad n = \ell + \nu, \quad \nu = 0, 1, \ldots
\]
When our method is applied to the hydrogen Hamiltonians $H^\ell$ of equation (30) with the eigenvalues $E^\ell_n$ (38), we obtain two independent solutions that read as follows

$$B^1_{n,\ell}A^1_{n,\ell} + \ell + n + 1 = -\frac{(2n+1)r}{4} \left[ H^\ell - E^\ell_n \right],$$

$$B^2_{n,\ell}A^2_{n,\ell} - \ell + n + 1 = -\frac{(2n+1)r}{4} \left[ H^\ell - E^\ell_n \right].$$

The explicit form of the operators $\{A^i, B^j\}_{i=1,2}$, is displayed below:

$$B^1_{n,\ell}(r) = (2n+1)^{1/2} \left( \frac{r^{1/2}}{2} \frac{d}{dr} + \frac{r^{1/2}}{2n+1} + \frac{\ell}{2r^{1/2}} \right) c_n^{-1/2} D(c_n)$$

$$A^1_{n,\ell}(r) = D(c_n^{-1}) \frac{1}{2} (2n+1)^{1/2} \left( \frac{r^{1/2}}{2} \frac{d}{dr} - \frac{r^{1/2}}{2n+1} - \frac{2\ell+1}{4r} \right)$$

$$B^2_{n,\ell}(r) = (2n+1)^{1/2} \left( \frac{r^{1/2}}{2} \frac{d}{dr} + \frac{r^{1/2}}{2n+1} - \frac{\ell}{2r^{1/2}} \right) c_n^{-1/2} D(c_n)$$

$$A^2_{n,\ell}(r) = D(c_n^{-1}) \frac{1}{2} (2n+1)^{1/2} \left( \frac{r^{1/2}}{2} \frac{d}{dr} - \frac{r^{1/2}}{2n+1} + \frac{2\ell+1}{4r} \right)$$

The symbol $D(\mu)$ in (11)-(12) is for the dilation operator (14), and $c_n = \frac{2n+2}{2n+1}$. Thus, in this example we are dealing with general first order differential operators as explained in Section I. For the first couple $\{A^1, B^1\}$ we have $(\hat{n}, \hat{\ell}) = (n + 1/2, \ell + 1/2)$, while for the second pair $\{A^2, B^2\}$, $(\hat{n}, \hat{\ell}) = (n + 1/2, \ell - 1/2)$.

The nonvanishing commutators among the free-index operators are

$$[N, B^i] = \frac{1}{2} B^i, \quad [L, B^i] = (-1)^i \frac{1}{2} B^i, \quad [A^i, B^j] = I,$$

$$[N, A^i] = \frac{1}{2} A^i, \quad [L, A^i] = -(-1)^i \frac{1}{2} A^i, \quad i = 1, 2.$$  (43)

In other words, as in the previous examples, we have a set of two independent boson operator algebras. The problem with these operators is that they change the quantum numbers $(n, \ell)$ in half-units, so that they do not keep inside the sector of physical wavefunctions. To avoid this problem we can build quadratic operators $\{A^iA^j, B^iA^j, B^iB^j\}_{i,j=1,2}$ satisfying this requirement; such second-order operators close the Lie algebra $sp(4, \mathbb{R})$ (10), which includes the subalgebra $su(2)$ (whose generators connect eigenstates with the same energy but different $\ell$s). It is worth to write down these quadratic operators:

$$\left\{\begin{array}{l}
(B^2A^1)_{n,\ell} = \frac{(2\ell+1)(2n+1)}{2} \left( \frac{1}{2} \frac{d}{dr} + \frac{2\ell+1}{4r} - \frac{1}{2\ell+1} \right) \\
(B^1A^2)_{n,\ell} = \frac{(2\ell+1)(2n+1)}{2} \left( -\frac{1}{2} \frac{d}{dr} + \frac{2\ell+1}{4r} - \frac{1}{2\ell+1} \right)
\end{array}\right.$$  (44)

They constitute, up to global constants, the usual factorization operators given in (17): $X^+ = (A^1B^2)_{n,\ell}$, $X^- = (A^2B^1)_{n,\ell}$. Another subalgebra is $su(1,1)$ (relating states with the same $\ell$ but different energies or $n$ values). Once included the negative $\ell$ values, as we did for the radial oscillator potential, the space $\mathcal{H}$ is the support for what it is called a ‘singleton representation’ (11) of $so(3,2) \approx sp(4, \mathbb{R})$. There is one lowest weight eigenvector $\psi_{n=0}^{\ell=0} \in \mathcal{H}$, from which all the representation space is generated by applying raising operators.
IV. CONCLUSIONS AND REMARKS

We have shown that a refinement of the factorization method allows us to study the maximum of relations among the Hamiltonian hierarchies that the conventional factorizations are not able to appreciate. The operators involved obey commutation rules which show the connection existing among the three examples dealt with in this paper: they have the same underlying Lie algebra associated with confluent hypergeometric functions. In other occasions the conventional factorizations have been used in this respect, but we have seen that such an approach is partial and not complete at all.

Usually the Hamiltonian hierarchies are obtained from higher dimensional systems after separation of variables (or by any other way of reduction). Such systems have symmetries that are responsible for their analytical treatment. These symmetries are reflected in the many factorizations that the hierarchies can give rise to by means of the technique we have developed. We have limited our study to \( N = 2 \) space dimensions for the radial oscillator and Coulomb potentials because they are the simplest cases to deal with. For other dimensions there appear certain subtleties, in the sense that the Hilbert space \( \mathcal{H} \) of bounded states is no longer invariant under the involved operators \([1]\).

Finally, let us mention that we have limited ourselves to some examples (all of them inside the class of shape invariant potentials \([2]\)), but it is clear that the whole treatment is applicable to the remaining Hamiltonians in the classification of Infeld and Hull \([2]\).

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REFERENCES

[1] W. Miller Jr., Lie Theory and Special Functions, Mathematics in Science and Engineering 43 (Academic Press, N. Y., 1968)
[2] I. Infeld and T.E. Hull, Rev. Mod. Phys. 23, 21 (1951)
[3] B. Mielnik, J. Math. Phys. 25, 3387 (1984); D.J. Fernández C., Lett. Math. Phys. 8, 337 (1984)
[4] V. Spiridonov, Phys. Rev. Lett. 69, 398 (1992)
[5] A.O. Barut, A. Inomata and R Wilson, J. Phys A 20, 4075 (1987); J. Phys A 20, 4083 (1987)
[6] D.J. Fernández C., J. Negro and M.A. del Olmo, Ann. Phys. 252, 386 (1996)
[7] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Pergamon (1965).
[8] Y.F. Liu, Y.A. Lei and J.Y. Zeng, Phys. Lett A 231, 9 (1997)
[9] Y. Alhassid, F. Gürsey and F. Iachello, Ann. Phys. 148, 346 (1983).
[10] M. Flato and C. Fronsdal, Phys. Lett. 97B, 236 (1980); P.A.M. Dirac, J. Math. Phys. 4, 901 (1963)
[11] J. Negro, L M Nieto, O Rosas-Ortiz, Preprint UVA (1999)
[12] A.B. Balantekin, Phys. Rev. A 57, 4188 (1998)