SPIN-STRUCTURES ON REAL BOTT MANIFOLDS WITH KÄHLER STRUCTURE

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Abstract. Let $M$ be a real Bott manifold with Kähler structure. Using Ishida characterization [6] we give necessary and sufficient condition for the existence of the spin-structure on $M$. In proof we use the technic developed in [10] and characteristic classes.

1. Introduction

Let $\Gamma$ be a fundamental group of a real Bott manifold $M$. From [7] we know that $\Gamma$ defines a short exact sequence

\begin{equation}
0 \to \mathbb{Z}^n \to \Gamma \to \mathbb{C}_2^k \to 1,
\end{equation}

where $\mathbb{C}_2$ is the cyclic group of order 2. Conjugation in $\Gamma$ defines the integral holonomy representation $\theta: \mathbb{C}_2^k \to \text{GL}(n, \mathbb{Z})$ of $\Gamma$, i.e.

$$\theta_\gamma(z) = \gamma^{-1}(\gamma z \gamma^{-1}),$$

where $\gamma \in \mathbb{C}_2^k$, $\gamma \in \Gamma$ is such that $\pi(\gamma) = g$ and $z \in \mathbb{Z}^n$. Moreover, the image of $\theta$ is a group of diagonal matrices, i.e. $\Gamma$ is a diagonal Bieberbach group and $M$ a diagonal flat manifold.

Up to diffeomorphism, $M$ is determined by a certain square matrix $A$ with coefficients in $\mathbb{F}_2$. We call $A$ a Bott matrix and we denote the manifold $M$ by $M(A)$. In [6, Theorem 3.1] Ishida gives the necessary and sufficient condition for existence of the Kähler structure on $M(A)$: $A \in \mathbb{F}_2^{2n \times 2n}$ and one can split columns of $A$ into $n$ pairs of equal ones.

Let $M$ be a real Bott and Kähler manifold, an RBK-manifold for short. In this note we examine the existence of spin structures on $M$. We would like to mention that the general condition for existence of spin structures on diagonal flat manifolds is considered in [4, 5]. However, in the specific case of RBK-manifolds, this condition can be formulated in much simpler – purely combinatorial – form. Namely, if $\tilde{A}$ is obtained from $A$ by a removal of one column from each pair of equal ones and $S_i$ is the sum of elements from the $i$-th row of $\tilde{A}$, then:

**Main Theorem.** RBK-manifold $M(A)$ admits a spin structure if and only if

$$\forall_{1 \leq i \leq 2n} S_i = 1 \implies A^{(i)} = 0,$$

where $A^{(i)}$ denotes the $i$-th column of the matrix $A$.

We prove the above theorem in Section 3. We take advantage of the description of spin diagonal flat manifolds presented in [9]. For the convenience of readers it is recalled in Section 2.

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2. New definition of real Bott manifold

In this section we recall methods introduced in [10] and developed in [9]. Let $S^1$ be a unit circle in $\mathbb{C}$ and we consider automorphisms $g_i : S^1 \to S^1$ given by
\begin{equation}
(2) \quad g_0(z) = z, \quad g_1(z) = -z, \quad g_2(z) = \bar{z}, \quad g_3(z) = -\bar{z},
\end{equation}
for all $z \in S^1$. If we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$, then for each $[t] \in \mathbb{R}/\mathbb{Z}$ we get
\begin{equation}
(3) \quad g_0([t]) = [t], \quad g_1([t]) = \left[ t + \frac{1}{2} \right], \quad g_2([t]) = [-t], \quad g_3([t]) = \left[ -t + \frac{1}{2} \right].
\end{equation}
Let $D = \langle g_i : i = 0, 1, 2, 3 \rangle$. Then $D \cong C_2 \times C_2$ and $g_3 = g_1 g_2$. We define an action $D^n$ on $T^n$ by
\begin{equation}
(4) \quad (t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)
\end{equation}
for $(t_1, \ldots, t_n) \in D^n$ and $(z_1, \ldots, z_n) \in T^n = S^1 \times \cdots \times S^1$.

By taking any $d$ generators of the group $C_d^d \subseteq D^n$, we define a $(d \times n)$-matrix with entries in $D$. This in turn defines a matrix with entries in the set $P = \{0, 1, 2, 3\}$, under the identification $i \leftrightarrow g_i$ for $i = 0, 1, 2, 3$. We call it a $P$-matrix of $C_d^d$. Note that under the above identification, $P$ has the natural structure of vector space over $\mathbb{F}_2$.

Although a group $C_d^d \subseteq D^n$ can have many $P$-matrices in general, every such a matrix $E$ encodes some important properties of its action the torus $T^n$. Namely, $C_d^d$ acts freely on $T^n$ if and only if there is 1 in the sum of any distinct collection of rows of $E$. In this case $C_d^d$ is the holonomy group of the flat manifold $T^n/C_d^d$ if and only if there is either 2 or 3 in each row of $E$ (see [9] Lemma 2.4]).

Let us consider the linear forms $\alpha, \beta : P \to \mathbb{F}_2$ given by the following table
\[
\begin{array}{c|cccc}
\alpha & 0 & 1 & 2 & 3 \\
\hline
\beta & 0 & 1 & 1 & 0 \\
\end{array}
\]
Let $C_d^d \subseteq D^n$ and $j \in \{1, 2, \ldots, n\}$. We define epimorphisms
\begin{equation}
(5) \quad \alpha_j : C_d^d \subseteq D^n \xrightarrow{pr'_j} P \xrightarrow{\alpha} \mathbb{F}_2, \quad \beta_j : C_d^d \subseteq D^n \xrightarrow{pr'_j} P \xrightarrow{\beta} \mathbb{F}_2,
\end{equation}
where
\[pr'_j(g_1, \ldots, g_n) = i_n\]
for $g_1, g_2, \ldots, g_n \in D$. Since $H^1(C^d_2, \mathbb{F}_2) = \text{Hom}(C^d_2, \mathbb{F}_2)$, we can view $\alpha_j$ and $\beta_j$ as 1-cocycles and define
\begin{equation}
(6) \quad \theta_j = \alpha_j \cup \beta_j \in H^2(C^d_2, \mathbb{F}_2),
\end{equation}
where $\cup$ denotes the cup product. It is well known that $H^*(C^d_2, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_d]$ where $\{x_1, \ldots, x_d\}$ is a basis of $H^1(C^d_2, \mathbb{F}_2)$. Hence, elements $\alpha_j$ and $\beta_j$ correspond to
\begin{equation}
(7) \quad \alpha_j = \sum_{i=1}^{d} \alpha(pr_j(b_i)) x_i, \quad \beta_j = \sum_{i=1}^{d} \beta(pr_j(b_i)) x_i \in C^d_2[x_1, \ldots, x_d],
\end{equation}
where $b_1, \ldots, b_d$ are generators of $C^d_2$ (see [1] Proposition 1.3]). Moreover, if $[p_{ij}]$ is a $P$-matrix of $C_d^d$, which corresponds to those generators, we can write equations (5) and (7) as follows
\begin{equation}
(8) \quad \alpha_j = \sum_{i=1}^{d} \alpha(p_{ij}) x_i, \quad \beta_j = \sum_{i=1}^{d} \beta(p_{ij}) x_i, \quad \theta_j = \alpha_j \beta_j.
\end{equation}
There is an exact sequence

\[ 0 \to H^1(C^d_2, \mathbb{F}_2) \xrightarrow{\pi} H^1(\Gamma, \mathbb{F}_2) \xrightarrow{\iota} H^1(\mathbb{Z}^n, \mathbb{F}_2) \xrightarrow{d_2} H^2(C^d_2, \mathbb{F}_2) \xrightarrow{\pi^*} H^2(\Gamma, \mathbb{F}_2) \]

where \( d_2 \) is the transgression and \( \pi^* \) is induced by the quotient map \( \pi : \Gamma \to C^d_2 \), see [2].

**Proposition 2.1** ([3 Proposition 3.2]). Suppose that free and diagonal action of \( C^d_2 \) on \( T^n \) corresponds to a \( P \)-matrix \( E \), which defines elements \( \alpha_j, \beta_j \) and \( \theta_j \) as in [3], for \( 1 \leq j \leq n \). Let \( M = T^n/C^d_2 \) and \( \Gamma = \pi_1(M) \) be associated to the group extension ([1]). Then

1. \( \forall 1 \leq i \leq n \), \( \theta_i = d_2(\varepsilon_i) \), where \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) is the basis of \( H^1(\mathbb{Z}^n, \mathbb{F}_2) \) dual to the standard basis of \( \mathbb{Z}^n \otimes \mathbb{F}_2 \).
2. The total Stiefel-Whitney class of \( M \) is given by

\[ w(M) = \pi^*(w) \in H^*(\Gamma, \mathbb{F}_2) = H^*(M, \mathbb{F}_2), \]

where

\[ w = \prod_{j=1}^{n} (1 + \alpha_j + \beta_j) \in \mathbb{F}_2[x_1, \ldots, x_d]. \]

We call the ideal

\[ I_E = (\text{Im}(d_2)) = \langle \theta_1, \ldots, \theta_n \rangle \subseteq \mathbb{F}_2[x_1, x_2, \ldots, x_n] \]

the characteristic ideal of \( E \) and the quotient \( C_E = \mathbb{F}_2[x_1, \ldots, x_d]/I_E \) – the characteristic algebra of \( E \).

**Corollary 2.1** ([3 Corollary 3.3]). Suppose that free and diagonal action of \( C^d_2 \) on \( T^n \) corresponds to a \( P \)-matrix \( E \). There is a canonical homomorphism of graded algebras \( \Phi : C_E \to H^*(T^n/C^d_2, \mathbb{F}_2) \) such that \( \Phi([w]) = w(T^n/C^d_2) \). Moreover, \( \Phi \) is a monomorphism in degree less than or equal to two.

**Definition 2.1.** Given a \( P \)-matrix \( E \in P^{d \times n} \), we define the Stiefel-Whitney class of \( E \), to be the class \([w] \in C_E \) defined by ([3]).

**Corollary 2.2.** Using notation of Corollary 2.1, let \( w_2 \) be the sum of degree 2 summands of \( w_2 \). Then, by [3 Proposition on page 40], \( T^n/C^d_2 \) admits a spin structure if and only if \( w_2 \in I_E \).

Now, we describe a real Bott manifold \( M(A) \). Let \( A = [a_{ij}] \) be an strictly upper triangular matrix with entries 0 or 1 and let \( s_i, i = 1, \ldots, n \) be Euclidean motions on \( \mathbb{R}^n \) defined by

\[ s_i = \left( \text{diag}[1, \ldots, 1, (-1)^{a_{i,i+1}}, \ldots, (-1)^{a_{i,n}}], \left( 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \right)^T \right) \]

where \((-1)^{a_{i,i+1}} \) is at the \((i+1)\)-th position and \( \frac{1}{2} \) is the \( i \)-th coordinate of the column, for \( i = 1, 2, \ldots, n-1 \) and \( s_n = (I, (0, 0, \ldots, 0, \frac{1}{2})) \). The group \( \Gamma(A) \) generated by \( s_1, \ldots, s_n \) is a crystallographic group. The subgroup generated by \( s_1^2, s_2^2, \ldots, s_n^2 \) consists of all translations of \( \mathbb{Z}^n \). The action of \( \Gamma(A) \) on \( \mathbb{R}^n \) is free and the orbit space \( \mathbb{R}^n/\Gamma(A) \) is compact.
3. Main results

We keep the notation of the previous section. Using the same methods as in \[10\] and \[8\], for each strictly upper triangular matrix \( A = [a_{ij}] \) which generates the fundamental group of real Bott manifold \( M(A) \) we get that the corresponding \( P \)-matrix \( P_A = [p_{ij}] \) is of the form

\[
P_A = \begin{bmatrix}
1 & 2a_{12} & \ldots & 2a_{1,n-1} & 2a_{1n} \\
0 & 1 & \ldots & 2a_{2,n-1} & 2a_{2n} \\
& \ddots && \\
0 & 0 & \ldots & 1 & 2a_{n-1,n} \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}.
\]

(11)

Note that in the above notation \( 2a_{ij} \) is multiplication in integers. To be more specific, we have

\[
p_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j \text{ and } a_{ij} = 0, \\
2 & \text{if } i \neq j \text{ and } a_{ij} = 1,
\end{cases}
\]

for \( 1 \leq i, j \leq n \). Using the form (11) of \( P_A \) and definition of forms \( \alpha \) and \( \beta \) we get that

\[
\alpha_j = \sum_{i=1}^{j} \alpha(p_{ij})x_i = \sum_{i=1}^{j-1} \alpha(p_{ij})x_i + \alpha(p_{jj})x_j = \sum_{i=1}^{j-1} a_{ij}x_i + x_j,
\]

\[
\beta_j = \sum_{i=1}^{j} \beta(p_{ij})x_i = \beta(p_{jj})x_j = x_j
\]

and hence

\[
\alpha_j + \beta_j = \sum_{i=1}^{j-1} a_{ij}x_i,
\]

(12)

\[
\theta_j = \sum_{i=1}^{j-1} a_{ij}x_ix_j + x_j^2 = \sum_{i \neq j} a_{ij}x_ix_j + x_j^2
\]

(13)

for all \( 1 \leq j \leq n \). Note that in the last formula we take advantage of the definition of a Bott matrix.

We go back to the matrix \( A \) of the real Bott manifold \( M(A) \). From \[6\] we have the following necessary and sufficient condition for the existence of a Kähler structure on \( M(A) \):

**Theorem 3.1 (\[6\] Theorem 3.1).** Let \( A \) be \( 2n \)-dimensional matrix of real Bott manifold \( M(A) \). Then the following conditions are equivalent:

1. there exist \( n \) subsets \( \{j_1,j_{n+1}\} \ldots \{j_n,j_{2n}\} \) of the set \( \{1,2,\ldots,2n\} \) such that
   
   (a) \( \bigcup_{k=1}^{n} \{j_k,j_{k+n}\} = \{1,2,\ldots,2n\} \),
   
   (b) \( A^{(ij)} = A^{(j_{k}i)} \) for all \( 1 \leq i < j_k \), where \( A^{(k)} \) is the \( k \)-th column of the matrix \( A \).

2. there exist a Kähler structure on \( M(A) \).

Let \( A \) be a Bott matrix of an RBK-manifold. Using the notation from Theorem 3.1 let \( \tilde{A} = [A^{(i_1)}A^{(j_2)} \ldots A^{(j_n)}] \) be a matrix obtained from \( A \) by removing duplicated columns. Then \( \tilde{A} \in \mathbb{F}_2^{2n \times 2n} \). Let

\[
S_i = \sum_{k=1}^{n} a_{ij_k} \in \mathbb{F}_2
\]
denote the sum of elements in the \(i\)-th row of the matrix \(\tilde{A}\), for \(1 \leq i \leq 2n\). Let us recall our main theorem.

**Main Theorem.** Let \(A\) be a matrix of of \(2n\)-dimensional RBK manifold \(M(A)\). Then \(M(A)\) admits a spin structure if and only if

\[
\forall 1 \leq i \leq 2n \ S_i = 1 \implies A^{(i)} = 0.
\]

Proof. \(M(A)\) is \(2n\)-dimensional RBK manifold, so from (12) we get

\[
\alpha_{jk+n} + \beta_{jk+n} = \alpha_{jk} + \beta_{jk} = \sum_{i=1}^{j_k-1} a_{ijk} x_i,
\]

and

\[
w(M(A)) = \prod_{k=1}^{2n} \left(1 + \alpha_{jk} + \beta_{jk}\right) = \prod_{k=1}^{n} \left(1 + \alpha_{jk} + \beta_{jk}\right)^2
\]

\[
= \prod_{k=1}^{n} \left(1 + \sum_{i=1}^{j_k-1} a_{ijk} x_i\right) = \prod_{k=1}^{n} \left(1 + \sum_{i=1}^{j_k-1} a_{ijk} x_i^2\right).
\]

From the above considerations and from the definition of a Bott matrix we have

\[
w_2 = w_2(M(A)) = \sum_{k=1}^{n} \sum_{i=1}^{2n} a_{ijk} x_i^2 = \sum_{k=1}^{n} \sum_{i=1}^{2n} a_{ijk} x_i^2
\]

\[
= \sum_{i=1}^{2n} \sum_{k=1}^{n} a_{ijk} x_i^2 = \sum_{i=1}^{2n} S_i x_i^2.
\]

Let \(J = \{j : S_j = 1\} \subset \{1, \ldots, 2n\}\). Then

(14)

\[
w_2 = \sum_{j \in J} x_j^2.
\]

By Corollary 2.2, the existence of a spin structure on \(M(A)\) is equivalent to \(w_2 \in \text{IP}_A\), which – by (13) – occurs if and only if

(15)

\[
\sum_{j \in J} x_j^2 = \sum_{j \in J} \theta_j.
\]

The above holds if \(A^{(j)} = 0\), since then, using formula (13) again, we get \(\theta_j = x_j^2\), for \(j \in J\).

Now, let

\[
K = \{(i, j) : a_{ij} = 1, j \in J, 1 \leq i \leq 2n\}.
\]

Then

(16)

\[
\sum_{j \in J} \theta_j = \sum_{j \in J} \left(\sum_{i \neq j} a_{ijk} x_i x_j + x_j^2\right) = \sum_{j \in J} x_j^2 + \sum_{(i, j) \in K} x_i x_j.
\]

From (14), (15) and (16) we have

\[
\sum_{j \in J} x_j^2 = \sum_{j \in J} x_j^2 + \sum_{(i, j) \in K} x_i x_j,
\]

hence \(K = \emptyset\), which means that the \(j\)-th column of the Bott matrix \(A\) has only zero entries.

\[\square\]
Example 3.1. Let

\[ A = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

be a matrix of a manifold \( M(A) \). Then

\[ \bar{A} = \begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \]

\[ S_1 = S_2 = S_5 = S_6 = 0, \quad S_3 = S_4 = 1 \]

and there are entries equal to 1 in columns \( A^{(3)} \) and \( A^{(4)} \), so \( M(A) \) has no spin-structure.

At the end let us note an easy corollary of our main theorem.

Corollary 3.1. Let \( A \in \mathbb{F}_2^{2n \times 2n} \) be a Bott matrix such that \( M(A) \) is a RBK-manifold. Let \( T = \{ j : A^{(j)} \neq 0 \} \). If

\[ T = T_1 \sqcup \ldots \sqcup T_l \]

such that for every \( 1 \leq k \leq l \) \( T_k \) is a four-elements set and

\[ \forall i, j \in T_k A^{(i)} = A^{(j)}, \]

then \( M(A) \) admits a spin structure.

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