ON THE COFINALITY OF THE SPLITTING NUMBER

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Abstract. The splitting number $s$ can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and the second author [Ultrafilters with small generating sets, Israel J. Math., 65, (1989)]

1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [2 van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subset \omega$ is unsplit by a family $\mathcal{Y} \subset [\omega]^\omega$ if $S$ is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $s$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (equivalently every $S \in [\omega]^\omega$ is split by some member of $\mathcal{Y}$). It is mentioned in [2] that it is currently unknown if $s$ can be a singular cardinal.

Proposition 1.1. The cofinality of the splitting number is not countable.

Proof. Assume that $\theta$ is the supremum of $\{\kappa_n : n \in \omega\}$ and that there is no splitting family of cardinality less than $\theta$. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \theta\}$ be a family of subsets of $\omega$. Let $S_0 = \omega$ and by induction on $n$, choose an infinite subset $S_{n+1}$ of $S_n$ so that $S_{n+1}$ is not split by the family $\{Y_\alpha : \alpha < \kappa_n\}$. If $S$ is any pseudointersection of $\{S_n : n \in \omega\}$, then $S$ is not split by any member of $\mathcal{Y}$. $\square$

Date: October 15, 2018.

1991 Mathematics Subject Classification. 03E15.

Key words and phrases. splitting number, cardinal invariants of the continuum, matrix forcing.

The research of the first author was supported by the NSF grant No. nsf-dms 1501506. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant no. 2010405), and by the NSF grant No. nsf-dms 1101597. Paper no. 1127 on Shelah’s publication list.
One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least \( t \). In this paper we prove the following.

**Theorem 1.2.** If \( \kappa \) is any uncountable regular cardinal, then there is a \( \lambda > \kappa \) with \( \text{cf}(\lambda) = \kappa \) and a ccc forcing \( \mathbb{P} \) satisfying that \( s = \lambda \) in the forcing extension.

To prove the theorem, we construct \( \mathbb{P} \) using matrix iterations.

2. A special splitting family

**Definition 2.1.** Let us say that a family \( \{x_i : i \in I\} \subset [\omega]^{\omega} \) is \( \theta \)-Luzin (for an uncountable cardinal \( \theta \)) if for each \( J \in [I]^{\theta} \), \( \bigcap \{x_i : i \in J\} \) is finite and \( \bigcup \{x_i : i \in J\} \) is cofinite.

Clearly a family is \( \theta \)-Luzin if every \( \theta \)-sized subfamily is \( \theta \)-Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal \( \theta \), each \( \theta \)-Luzin family is a splitting family. A poset being \( \theta \)-Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal \( \theta \) is \( \theta \)-Luzin preserving.

**Lemma 2.2.** If \( \theta \) is a regular uncountable cardinal then any ccc finite support iteration of \( \theta \)-Luzin preserving posets is again \( \theta \)-Luzin preserving.

**Proof.** We prove this by induction on the length of the iteration. Fix any \( \theta \)-Luzin family \( \{x_i : i \in I\} \) and let \( \langle \langle P_\alpha : \alpha \leq \gamma \rangle, \langle \dot{Q}_\alpha : \alpha < \gamma \rangle \rangle \) be a finite support iteration of ccc posets satisfying that \( P_\alpha \) forces that \( \dot{Q}_\alpha \) is ccc and \( \theta \)-Luzin preserving, for all \( \alpha < \gamma \).

If \( \gamma \) is a successor ordinal \( \beta + 1 \), then for any \( P_\beta \)-generic filter \( G_\beta \), the family \( \{x_i : i \in I\} \) is a \( \theta \)-Luzin family in \( V[G_\beta] \). By the hypothesis on \( \dot{Q}_\beta \), this family remains \( \theta \)-Luzin after further forcing by \( \dot{Q}_\beta \).

Now we assume that \( \alpha \) is a limit. Let \( \dot{J}_0 \) be any \( P_\gamma \)-name of a subset of \( I \) and assume that \( p \in P_\gamma \) forces that \( |\dot{J}_0| = \theta \). We must produce a \( q < p \) that forces that \( \dot{J}_0 \) is as in the definition of \( \theta \)-Luzin. There is a set \( J_1 \subset I \) of cardinality \( \theta \) satisfying that, for each \( i \in J_1 \), there is a \( p_i < p \) with \( p_i \vDash i \in \dot{J}_0 \). The case when the cofinality of \( \alpha \) not equal to \( \theta \) is almost immediate. There is a \( \beta < \alpha \) such that \( J_2 = \{i \in J_1 : p_i \in P_\beta \} \) has cardinality \( \theta \). There is a \( P_\beta \)-generic filter \( G_\beta \) such that \( J_3 = \{i \in J_2 : p_i \in G_\beta \} \) has cardinality \( \theta \). By the induction hypothesis, the family \( \{x_i : i \in I\} \) is \( \theta \)-Luzin in \( V[G_\beta] \) and so we have that \( \bigcap \{x_i : i \in J_3\} \) is finite and \( \bigcup \{x_i : i \in J_3\} \) is co-finite. Choose any \( q < p \) in \( G_\beta \) and a name \( \dot{J}_3 \) for \( J_3 \) so that \( q \) forces this
property for $\dot{J}_3$. Since $q$ forces that $\dot{J}_3 \subseteq \dot{J}_0$, we have that $q$ forces the same property for $\dot{J}_0$.

Finally we assume that $\alpha$ has cofinality $\theta$. Naturally we may assume that the collection $\{\text{dom}(p_i) : i \in J_1\}$ forms a $\Delta$-system with root contained in some $\beta < \alpha$. Again, we may choose a $\mathbb{P}_\beta$-generic filter $G_\beta$ satisfying that $J_2 = \{i \in J_1 : p_i \upharpoonright \beta \in G_\beta\}$ has cardinality $\theta$. In $V[G_\beta]$, let $\{J_{2,\xi} : \xi \in \omega_1\}$ be a partition of $J_2$ into pieces of size $\theta$. For each $\xi \in \omega_1$, apply the induction hypothesis in the model $V[G_\beta]$, and so we have that $\bigcap\{x_i : i \in J_{2,\xi}\}$ is finite and $\bigcup\{x_i : i \in J_{2,\xi}\}$ is co-finite. For each $\xi \in \omega_1$ let $m_\xi$ be an integer large enough so that $\bigcap\{x_i : i \in J_{2,\xi}\} \subseteq m_\xi$ and $\bigcup\{x_i : i \in J_{2,\xi}\} \supset \omega \setminus m_\xi$. Let $m$ be any integer such that $m_\xi = m$ for uncountably many $\xi$. Choose any condition $\overline{p} \in \mathbb{P}_\alpha$ so that $\overline{p} \upharpoonright \beta \in G_\beta$. We prove that for each $n > m$ there is a $\overline{p}_n < \overline{p}$ so that $\overline{p}_n \Vdash n \notin \bigcap\{x_i : i \in I\}$ and $\overline{p}_n \Vdash n \in \bigcup\{x_i : i \in I\}$. Choose any $\xi \in \omega_1$ so that $m_\xi = m$ and $\text{dom}(p_i) \cap \text{dom}(\overline{p}) \subset \beta$ for all $i \in J_{2,\xi}$. Now choose any $i_0 \in J_{2,\xi}$ so that $n \notin x_{i_0}$. Next choose a distinct $\xi'$ with $m_{\xi'} = m$ so that $\text{dom}(p_i) \cap (\text{dom}(\overline{p}) \cup \text{dom}(p_{i_0})) \subset \beta$ for all $i \in J_{2,\xi'}$. Now choose $i_1 \in J_{2,\xi'}$ so that $n \in x_{i_1}$. We now have that $\overline{p} \cup p_{i_0} \cup p_{i_1}$ is a condition that forces $\{i_0, i_1\} \subseteq I$. \hfill \Box

Next we introduce a $\sigma$-centered poset that will render a given family non-splitting.

\textbf{Definition 2.3.} For a filter $\mathcal{D}$ on $\omega$, we define the Laver style poset $\mathbb{L}(\mathcal{D})$ to be the set of trees $T \subseteq \omega^{<\omega}$ with the property that $T$ has a minimal branching node $\text{stem}(T)$ and for all $\text{stem}(T) \subseteq t \in T$, the branching set $\{k : t \upharpoonright k \in T\}$ is an element of $\mathcal{D}$. If $\mathcal{D}$ is a filter base for a filter $\mathcal{D}'$, then $\mathbb{L}(\mathcal{D})$ will also denote $\mathbb{L}(\mathcal{D}')$.

The name $\check{\mathcal{L}} = \{\langle k, T \rangle : (\exists t) t \upharpoonright k \subset \text{stem}(T)\}$ will be referred to as the canonical name for the real added by $\mathbb{L}(\mathcal{D})$.

If $\mathcal{D}$ is a principal (fixed) ultrafilter on $\omega$, then $\mathbb{L}(\mathcal{D})$ has a minimum element and so is forcing isomorphic to the trivial poset. If $\mathcal{D}$ is principal but not an ultrafilter, then $\mathbb{L}(\mathcal{D})$ is isomorphic to Cohen forcing. If $\mathcal{D}$ is a free filter, then $\mathbb{L}(\mathcal{D})$ adds a dominating real and has similarities to Hechler forcing. As usual, for a filter (or filter base) $\mathcal{D}$ of subsets of $\omega$, we use $\mathcal{D}^+$ to denote the set of all subsets of $\omega$ that meet every member of $\mathcal{D}$.

\textbf{Definition 2.4.} If $E$ is a dense subset of $\mathbb{L}(\mathcal{D})$, then a function $\rho_E$ from $\omega^{<\omega}$ into $\omega_1$ is a rank function for $E$ if $\rho_E(t) = 0$ if and only if $t = \text{stem}(T)$ for some $T \in E$, and for all $t \in \omega^{<\omega}$ and $0 < \alpha \in \omega_1$, $\rho_E(t) \leq \alpha$ providing the set $\{k \in \omega : \rho_E(t \upharpoonright k) < \alpha\}$ is in $\mathcal{D}^+$. 
When $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ has cardinality $\mathfrak{c}$, but nevertheless, if $\mathfrak{D}$ has a base of cardinality less than a regular cardinal $\theta$, then $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.

**Lemma 2.5.** If $\mathfrak{D}$ is a free filter on $\omega$ and if $\mathfrak{D}$ has a base of cardinality less than a regular uncountable cardinal $\theta$, then $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.

**Proof.** Let $\{x_i : i \in \theta\}$ be a $\theta$-Luzin family with $\theta$ as in the Lemma. Let $\mathcal{J}$ be a $\mathbb{L}(\mathfrak{D})$-name of a subset of $\theta$. We prove that if $\mathcal{J} = \mathcal{J}_0 \in \theta$ is not finite, then $\mathcal{J}$ is bounded in $\theta$. By symmetry, it will also prove that if $\mathcal{J} = \mathcal{J}_0 \in \theta$ is not cofinite, then $\mathcal{J}$ is bounded in $\theta$. Let $\mathcal{J}_1$ be the $\mathbb{L}(\mathfrak{D})$-name of the intersection, and let $\mathcal{T}_0$ be any member of $\mathbb{L}(\mathfrak{D})$ that forces that $\mathcal{J}_1$ is infinite. Let $\mathcal{M}$ be any $<\theta$-sized elementary submodel of $H((2^\theta)^+)$. Since $\mathcal{T}_0, \mathcal{D}, \mathcal{J}_0$ such that $\mathcal{T}_0 < \mathcal{D}, \mathcal{J}_0$, and $\{x_i : i \in \theta\}$ are all members of $\mathcal{M}$ and such that $\mathcal{M} \cap \mathfrak{D}$ contains a base for $\mathfrak{D}$. Let $\mathcal{M}_M = \sup(M \cap \theta)$. If $x \in \mathcal{M} \cap [\omega]^{<\theta}$, then $I_x = \{i \in \theta : x \subset x_i\}$ is an element of $\mathcal{M}$ and has cardinality less than $\theta$. Therefore, if $i \in \mathcal{M} \setminus \mathcal{M}_M$, then $x_i$ does not contain any infinite subset of $\omega$ that is an element of $M$. We prove that $x_i$ is not finite, then $\mathcal{J}_1$ is finite. This proves that $\mathcal{J}$ is bounded by $\mathcal{M}_M$. Let $\mathcal{T}_0 < \mathcal{T}_0$ be any condition in $\mathbb{L}(\mathfrak{D})$ and let $t_1 = \text{stem}(\mathcal{T}_0)$. We show that $\mathcal{T}_0$ does not force that $x_i \supset \mathcal{J}_1$. We define the relation $\mathcal{L}_w$ on $\mathcal{T}_0 \times \omega$ to be the set

$$\{(t, n) \in \mathcal{T}_0 \times \omega : \text{there is no } T \leq \mathcal{T}_0, \text{stem}(T) = t, \text{s.t. } T \models n \notin \mathcal{J}_1\}.$$

For convenience we may write, for $T \leq \mathcal{T}_0$, $T \models n \in \mathcal{J}_1$ providing (stem$(T)$, $n$) is in $\mathcal{L}_w$, and this is equivalent to the relation that $T$ has no stem preserving extension forcing that $n$ not in $\mathcal{J}_1$. Let $\mathcal{T}_2 \in \mathcal{M}$ be any extension of $\mathcal{T}_0$ with stem $t_0$. Let $\mathcal{L}$ denote the set of $\ell \in \omega$ such that $\mathcal{T}_2 \models \ell \in \mathcal{J}_1$. If $L$ is infinite, then, since $\mathcal{L} \in \mathcal{M}$, there is in $\mathcal{L} \setminus x_i$. This implies that $\mathcal{T}_0$ does not force $x_i \supset \mathcal{J}_1$, since $\mathcal{T}_2 \models \ell \in \mathcal{J}_1$ implies that $\mathcal{T}_0$ fails to force that $\ell \notin \mathcal{J}_1$.

Therefore we may assume that $L$ is finite and let $\ell$ be the maximum of $\mathcal{L}$. Define the set $E \subset \mathbb{L}(\mathfrak{D})$ according to $T \in E$ providing that either $t_1 \notin T$ or there is a $\ell > \ell$ such that $T \models \ell \in \mathcal{J}_1$. Again this set $E$ is in $\mathcal{M}$ and is easily seen to be a dense subset of $\mathbb{L}(\mathfrak{D})$. By the choice of $\ell$, we note that $\rho_E(t_1) > 0$. If $\rho_E(t_1) > 1$, then the set $\{k \in \omega : 0 < \rho_E(t_1, k) < \rho_E(t_1)\}$ is in $\mathfrak{D}^+$ and so there is a $k_1$ in this set such that $t_1^{+}k_1 \in \mathcal{T}_1 \cap \mathcal{T}_2$. By a finite induction, we can choose an extension $\mathcal{T}_1 \supset \mathcal{J}_1$ so that $\mathcal{T}_2 \models \ell \in \mathcal{T}_1 \cap \mathcal{T}_2$ and $\rho_E(t_1) = 1$. Now, there is a set $D \in \mathfrak{D} \cap M$ contained in $\{k : t_2^{+}k \in \mathcal{T}_1 \cap \mathcal{T}_2\}$ since $M$ contains a base for $\mathfrak{D}$. Also, $D_E = \{k \in D : \rho_E(t_1^{+}k) = 0\}$ is in $\mathfrak{D}^+$. For each $k \in D_E$, choose the minimal $j_k$ so that $\mathcal{T}_2 \models j_k \notin \mathcal{J}_1$. The set
\{j_k : k \in D_E\} is an element of M. This set is not finite because if it were then there would be a single \(j\) such that \(\{k \in D_E : j_k = j\} \in D^+\), which would contradict that \(\rho_E(t_2) > 0\). This means that there is a \(k \in D_E^+\) with \(j_k \notin x_i\), and again we have shown that \(T_1\) fails to force that \(x_i\) contains \(\dot{y}\).

\[\square\]

3. Matrix Iterations

The terminology “matrix iterations” is used in [3], see also forthcoming preprint (F1222) from the second author. The paper [3] nicely expands on the method of matrix iterated forcing first introduced in [1].

Let us recall that a poset \((P, \prec_P)\) is a complete suborder of a poset \((Q, \prec_Q)\) providing \(P \subset Q, \prec_P \subset \prec_Q\), and each maximal antichain of \((P, \prec_P)\) is also a maximal antichain of \((Q, \prec_Q)\). Note that it follows that incomparable members of \((P, \prec_P)\) are still incomparable in \((Q, \prec_Q)\), i.e. \(p_1 \perp_P q_2\) implies \(p_1 \perp_Q q_2\). We use the notation \((P, \prec_P)^{\prec_Q}\) to abbreviate the complete suborder relation, and similarly use \(P^{\prec_Q}\) if \(\prec_P\) and \(\prec_Q\) are clear from the context. An element \(p\) of \(P\) is a reduction of \(q \in Q\) if \(r \not\prec_Q q\) for each \(r \prec_P p\). If \(P \subset Q, \prec_P \subset \prec_Q, \perp_P \subset \perp_Q\), and each element of \(Q\) has a reduction in \(P\), then \(P^{\prec_Q}\). The reason is that if \(A \subset P\) is a maximal antichain and \(p \in P\) is a reduction of \(q \in Q\), then there is an \(a \in A\) and an \(r \prec_P a\) such that \(r \not\prec_Q q\).

Definition 3.1. We will say that an object \(\mathbf{P}\) is a matrix iteration if there is an infinite cardinal \(\kappa\) and an ordinal \(\gamma\) (thence a \((\kappa, \gamma)\)-matrix iteration) such that \(\mathbf{P} = \langle \{P_{i,\alpha} : i \leq \kappa, \alpha \leq \gamma\}, \langle \dot{Q}_{i,\alpha} : i \leq \kappa, \alpha < \gamma\rangle \rangle\) where, for each \((i, \alpha)\) \in \(\kappa + 1 \times \gamma\) and each \(j < i\),

1. \(P_{i,\alpha}\) is a complete suborder of the poset \(P_{i,\alpha}\) (i.e. \(P_{j,\alpha}^{\prec_P} P_{i,\alpha}\)),
2. \(\dot{Q}_{i,\alpha}\) is a \(P_{i,\alpha}\)-name of a ccc poset, \(P_{i,\alpha + 1}\) is equal to \(P_{i,\alpha} * \dot{Q}_{i,\alpha}\),
3. for limit \(\delta \leq \gamma\), \(P_{i,\delta}\) is equal to the union of the family \(\{P_{i,\beta} : \beta < \delta\}\),
4. \(P_{\kappa,\alpha}\) is the union of the chain \(\{P_{j,\alpha} : j < \kappa\}\).

When the context makes it clear, we omit the superscript \(\mathbf{P}\) when discussing a matrix iteration. Throughout the paper, \(\kappa\) will be a fixed uncountable regular cardinal.

Definition 3.2. A sequence \(\vec{\lambda}\) is \(\kappa\)-tall if \(\vec{\lambda} = \langle \mu_\xi, \lambda_\xi : \xi < \kappa \rangle\) is a sequence of pairs of regular cardinals satisfying that \(\mu_0 = \omega < \kappa < \lambda_0\) and, for \(0 < \eta < \kappa\), \(\mu_\eta < \lambda_\eta\) where \(\mu_\eta = (2^{\sup(\lambda_\xi : \xi < \eta)})^+\).
Also for the remainder of the paper, we fix a \( \kappa \)-tall sequence \( \vec{\lambda} \) and \( \lambda \) will denote the supremum of the set \( \{ \lambda_{\xi} : \xi \in \kappa \} \). For simpler notation, whenever we discuss a matrix iteration \( P \) we shall henceforth assume that it is a \((\kappa, \gamma)\)-matrix iteration for some ordinal \( \gamma \). We may refer to a forcing extension by \( P \) as an abbreviation for the forcing extension by \( P_{\kappa, \gamma} \).

For any poset \( P \), any \( P \)-name \( \dot{D} \), and \( P \)-generic filter \( G \), \( \dot{D}[G] \) will denote the valuation of \( \dot{D} \) by \( G \). For any ground model \( x \), \( \dot{x}[G] = x \). When \( x \) is an ordinal (or an integer) we will suppress the accent in \( \dot{x} \). A \( P \)-name \( \dot{D} \) of a subset of \( \omega \) will be said to be nice or canonical if for each integer \( j \in \omega \), there is an antichain \( A_j \) such that \( \dot{D} = \bigcup \{ \{ j \} \times A_j : j \in \omega \} \). We will say that \( \mathcal{D} \) is a nice \( P \)-name of a family of subsets of \( \omega \) just to mean that \( \mathcal{D} \) is a collection of nice \( P \)-names of subsets of \( \omega \). We will use \( (\mathcal{D})_P \) if we need to emphasize that we mean the \( P \)-name. Similarly if we say that \( \mathcal{D} \) is a nice \( P \)-name of a filter (base) we mean that \( \mathcal{D} \) is a nice \( P \)-name such that, for each \( P \)-generic filter, the collection \( \{ \dot{D}[G] : \dot{D} \in \mathcal{D} \} \) is a filter (base) of infinite subsets of \( \omega \).

Following these conventions, the following notation will be helpful.

**Definition 3.3.** For a \((\kappa, \gamma)\)-matrix \( P \) and \( i < \kappa \), we let \( B_{i, \gamma} \) denote the set of all nice \( P \)-names of subsets of \( \omega \). We note that this then is the nice \( P \)-name for the power set of \( \omega \). As usual, when possible we suppress the \( P \) superscript.

For a nice \( P \)-name \( \mathcal{D} \) of a filter (or filter base) of subsets of \( \omega \), we let \( (\mathcal{D})^+ \) denote the set of all nice \( P \)-names that are forced to meet every member of \( \mathcal{D} \). It follows that \( (\mathcal{D})^+ \) is the nice \( P \)-name for the usual defined notion \( (\mathcal{D})^+ \) in the forcing extension by \( P \). We let \( \langle \mathcal{D} \rangle \) denote the nice \( P \)-name of the filter generated by \( \mathcal{D} \). We use the same notational conventions if, for some poset \( P \), \( \mathcal{D} \) is a nice \( P \)-name of a filter (or filter base) of subsets of \( \omega \).

The main idea for controlling the splitting number in the extension by \( P \) will involve having many of the subposets being \( \theta \)-Luzin preserving for \( \theta \in \{ \lambda_{\xi} : \xi \in \kappa \} \). Motivated by the fact that posets of the form \( L(\mathcal{D}) \) (our proposed iterands) are \( \theta \)-Luzin preserving when \( \mathcal{D} \) is sufficiently small we adopt the name \( \vec{\lambda} \)-thin for this next notion.

**Definition 3.4.** For a \( \kappa \)-tall sequence \( \vec{\lambda} \), we will say that a \((\kappa, \gamma)\)-matrix-iteration \( P \) is \( \vec{\lambda} \)-thin providing that for each \( \xi < \kappa \) and \( \alpha \leq \gamma \), \( P_{\xi, \alpha} \) is \( \lambda_{\xi} \)-Luzin preserving.
ON THE COFINALITY OF THE SPLITTING NUMBER

Now we combine the notion of $\vec{\lambda}$-thin matrix-iteration with Lemma 2.2. We adopt Kunen’s notation that for a set $I$, $\text{Fn}(I, 2)$ denotes the usual poset for adding Cohen reals (finite partial functions from $I$ into 2 ordered by superset).

**Lemma 3.5.** Suppose that $\mathbb{P}$ is a $\vec{\lambda}$-thin $(\kappa, \gamma)$-matrix iteration for some $\kappa$-tall sequence $\vec{\lambda}$. Further suppose that $\mathbb{Q}_{i,0}$ is the $\mathbb{P}_{i,0}$-name of the poset $\text{Fn}(\lambda_{\xi}, 2)$ for each $\xi \in \kappa$, and therefore $\mathbb{P}_{\kappa,1}$ is isomorphic to $\text{Fn}(\lambda, 2)$. Let $\check{g}$ denote the generic function from $\lambda$ onto 2 added by $\mathbb{P}_{\kappa,1}$ and, for $i < \lambda$, let $\check{x}_i$ be the canonical name of the set $\{n \in \omega : \check{g}(i + n) = 1\}$. Then the family $\{\check{x}_i : i < \lambda\}$ is forced by $\mathbb{P}$ to be a splitting family.

**Proof.** Let $G_{\kappa, \gamma}$ be a $\mathbb{P}_{\kappa, \gamma}$-generic filter. For each $\xi \in \kappa$ and $\alpha \leq \gamma$, let $G_{\xi, \alpha} = G_{\kappa, \gamma} \cap \mathbb{P}_{\xi, \alpha}$. Let $\check{y}$ be any nice $\mathbb{P}_{\kappa, \gamma}$-name for a subset of $\omega$. Since $\check{y}$ is a countable name, we may choose a $\xi < \kappa$ so that $\check{y}$ is a $\mathbb{P}_{\xi, \gamma}$-name. It is easily shown, and very well-known, that the family $\{\check{x}_i : i < \lambda_{\xi}\}$ is forced by $\mathbb{P}_{\xi, 1}$ (i.e. $\text{Fn}(\lambda_{\xi}, 2)$) to be a $\lambda_{\xi}$-Luzin family. By the hypothesis that $\mathbb{P}$ is $\vec{\lambda}$-thin, we have, by Lemma 2.2 that $\{\check{x}_i : i < \lambda_{\xi}\}$ is still $\lambda_{\xi}$-Luzin in $V[G \cap \mathbb{P}_{\xi, \gamma}]$. Since $\check{y}$ is a $\mathbb{P}_{\xi, \gamma}$-name, there is an $i < \lambda_{\xi}$ such that $\check{y}[G_{\xi, \gamma}] \cap \check{x}_i[G_{\xi, \gamma}]$ and $\check{y}[G_{\xi, \gamma}] \setminus \check{x}_i[G_{\xi, \gamma}]$ are infinite. \qed

4. THE CONSTRUCTION OF $\mathbb{P}$

When constructing a matrix-iteration by recursion, we will need notation and language for extension. We will use, for an ordinal $\gamma$, $\mathbb{P}^\gamma$ to indicate that $\mathbb{P}^\gamma$ is a $(\kappa, \gamma)$-matrix iteration.

**Definition 4.1.** (1) A matrix iteration $\mathbb{P}^\gamma$ is an extension of $\mathbb{P}^\delta$ providing $\delta \leq \gamma$, and, for each $\alpha \leq \delta$ and $i \leq \kappa$, $\mathbb{P}_{i, \alpha} = \mathbb{P}_{i, \alpha}^\delta$.

We can use $\mathbb{P}^\gamma \upharpoonright \delta$ to denote the unique $(\kappa, \delta)$-matrix iteration extended by $\mathbb{P}^\gamma$.

(2) If, for each $i < \kappa$, $\hat{Q}_{i, \gamma}$ is a $\mathbb{P}_{i, \gamma}$-name of a ccc poset satisfying that, for each $i < j < \kappa$, $\mathbb{P}_{i, \gamma} \ast \hat{Q}_{i, \gamma}$ is a complete subposet of $\mathbb{P}_{j, \gamma} \ast \hat{Q}_{j, \gamma}$, then we let $\mathbb{P} \ast (\hat{Q}_{i, \gamma} : i < \kappa)$ denote the $(\kappa, \gamma + 1)$-matrix $\langle \langle \hat{Q}_{i, \alpha} : i \leq \kappa, \alpha \leq \gamma + 1 \rangle, \hat{Q}_{i, \alpha} : i \leq \kappa, \alpha < \gamma + 1 \rangle \rangle$, where $\hat{Q}_{\kappa, \gamma}$ is the $\mathbb{P}$-name of the union of $\{\hat{Q}_{i, \gamma} : i < \kappa\}$ and, for $i \leq \kappa$, $\mathbb{P}_{i, \gamma} = \mathbb{P}_{i, \gamma}^\kappa$, $\mathbb{P}_{i, \gamma + 1} = \mathbb{P}_{i, \gamma} \ast \hat{Q}_{i, \gamma}$, and for $\alpha < \gamma$, $(\mathbb{P}_{i, \alpha}, \hat{Q}_{i, \alpha}) = (\mathbb{P}_{i, \alpha}^\kappa, \hat{Q}_{i, \alpha}^\kappa)$.

The following, from [3, Lemma 3.10], shows that extension at limit steps is canonical.
Lemma 4.2. If $\gamma$ is a limit and if $\{P_\delta : \delta < \gamma\}$ is a sequence of matrix iterations satisfying that for $\beta < \delta < \gamma$, $P_\delta \upharpoonright \beta = P_\beta$, then there is a unique matrix iteration $P_\gamma$ such that $P_\gamma \upharpoonright \delta = P_\delta$ for all $\delta < \gamma$.

Proof. For each $\delta < \gamma$ and $i < \kappa$, we define $P_{i,\delta}$ to be $P_{i,\delta}^\gamma$ and $Q_{i,\delta}$ to be $Q_{i,\delta}^\gamma$. It follows that $Q_{i,\delta}^\gamma$ is a $P_{i,\delta}^\gamma$-name. Since $\gamma$ is a limit, the definition of $P_{i,\gamma}^\gamma$ is required to be $\bigcup\{P_{i,\delta}^\gamma : \delta < \gamma\}$ for $i < \kappa$. Similarly, the definition of $P_{i,\gamma}^\gamma$ is required to be $\bigcup\{P_{i,\delta}^\gamma : i < \kappa\}$. Let us note that $P_{i,\gamma}^\gamma$ is also required to be the union of the chain $\bigcup\{P_{i,\delta}^\gamma : \delta < \gamma\}$, and this holds by assumption on the sequence $\{P_\delta : \delta < \gamma\}$.

To prove that $P_\gamma^\gamma$ is a $(\kappa, \gamma)$-matrix it remains to prove that for $j < i \leq \kappa$, and each $q \in P_{i,\gamma}^\gamma$, there is a reduction $p$ in $P_{j,\gamma}^\gamma$. Since $\gamma$ is a limit, there is an $\alpha < \gamma$ such that $q \in P_{i,\alpha}^\gamma$ and, by assumption, there is a reduction, $p$, of $q$ in $P_{j,\alpha}^\gamma$. By induction on $\beta$ ($\alpha \leq \beta \leq \gamma$) we note that $q \in P_{i,\beta}^\gamma$ and that $p$ is a reduction of $q$ in $P_{j,\beta}^\gamma$. For limit $\beta$ it is trivial, and for successor $\beta$ it follows from condition (1) in the definition of matrix iteration. □

We also will need the next result taken from [3, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

Lemma 4.3. Let $P, Q$ be partial orders such that $P$ is a complete suborder of $Q$. Let $\hat{A}$ be a $P$-name for a forcing notion and let $\hat{B}$ be a $Q$-name for a forcing notion such that $\Vdash_{Q} \hat{A} \subseteq \hat{B}$, and every $P$-name of a maximal antichain of $\hat{A}$ is also forced by $Q$ to be a maximal antichain of $\hat{B}$. Then $P \ast \hat{A} \prec Q \ast \hat{B}$.

Let us also note if $\hat{B}$ is equal to $\hat{A}$ in Lemma 4.3, then the hypothesis and the conclusion of the Lemma are immediate. On the other hand, if $\hat{A}$ is the $P$-name of $L(\mathcal{D})$ for some $P$-name of a filter $\mathcal{D}$, then the $Q$-name of $L(\mathcal{D})$ is not necessarily equal to $\hat{A}$.

Lemma 4.4 ([2, 1.9]). Suppose that $P, Q$ are posets with $P \prec Q$. Suppose also that $\mathcal{D}_0$ is a $P$-name of a filter on $\omega$ and $\mathcal{D}_1$ is a $Q$-name of a filter on $\omega$. If $\Vdash_{Q} \mathcal{D}_0 \subseteq \mathcal{D}_1$ then $P \ast L(\mathcal{D}_0)$ is a complete subposet of $Q \ast L(\mathcal{D}_1)$ if either of the two equivalent conditions hold:

1. $\Vdash_{Q} ((\mathcal{D}_0)^+) \subseteq \mathcal{D}_1^+$,
2. $\Vdash_{Q} \mathcal{D}_1 \cap V^P \subseteq \langle \mathcal{D}_0 \rangle$ (where $V^P$ is the class of $P$-names).

Proof. Let $\hat{E}$ be any $P$-name of a maximal antichain of $L(\mathcal{D}_0)$. By Lemma 4.3 it suffices to show that $Q$ forces that every member of
\( \mathbb{L}(\dot{\mathcal{D}}_i) \) is compatible with some member of \( \dot{E} \). Let \( G \) be any \( \mathbb{Q} \)-generic filter and let \( E \) denote the valuation of \( \dot{E} \) by \( G \cap \mathbb{P} \). Working in the model \( V[G \cap \mathbb{P}] \), we have the function \( \rho_E \) as in Lemma 2.4. Choose \( \delta \in \omega_1 \) satisfying that \( \rho_E(t) < \delta \) for all \( t \in \omega^{<\omega} \). Now, working in \( V[G] \), we consider any \( T \in \mathbb{L}(\dot{\mathcal{D}}_1) \) and we find an element of \( E \) that is compatible with \( T \). In fact, by induction on \( \alpha < \delta \), one easily proves that for each \( T \in \mathbb{L}(\dot{\mathcal{D}}_1) \) with \( \rho_E(\text{stem}(T)) \leq \alpha \), \( T \) is compatible with some member of \( E \).

**Definition 4.5.** For a \((\kappa, \gamma)\)-matrix-iteration \( \mathbb{P} \), and ordinal \( i_\gamma < \kappa \), we say that an increasing sequence \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases, if for each \( i < j < \kappa \):

1. \( \mathcal{D}_i \) is a subset of \( \mathbb{B}_{i, \gamma} \) (hence a nice \( \mathbb{P}_{i, \gamma} \)-name)
2. \( \Vdash_{\mathbb{P}_{i, \gamma}} \mathcal{D}_i \) is a filter with a base of cardinality at most \( \mu_{i, \gamma} \),
3. \( \Vdash_{\mathbb{P}_{j, \gamma}} (\mathcal{D}_j) \cap \mathbb{B}_{i, \gamma} \subseteq \mathcal{D}_i \).

Notice that a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases can be (essentially) eventually constant. Thus we will say that a sequence \( \langle \mathcal{D}_i : i \leq j \rangle \) (for some \( j < \kappa \)) is a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases if the sequence \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases where \( \mathcal{D}_i \) is the \( \mathbb{P}_{i, \gamma} \)-name for \( \mathbb{B}_{i, \gamma} \cap \langle \mathcal{D}_j \rangle \) for \( j < i \leq \kappa \). When \( \mathbb{P} \) is clear from the context, we will use \( \bar{\lambda}(i_\gamma) \)-thin as an abbreviation for \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin.

**Corollary 4.6.** For a \((\kappa, \gamma)\)-matrix-iteration \( \mathbb{P} \), ordinal \( i_\gamma < \kappa \), and a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases \( \langle \mathcal{D}_i : i < \kappa \rangle \), \( \mathbb{P} \ast (\mathbb{Q}_{i, \gamma} : i \leq \kappa) \) is a \( \gamma + 1 \)-extension of \( \mathbb{P} \), where, for each \( i \leq i_\gamma \), \( \mathbb{Q}_{i, \gamma} \) is the trivial poset, and for \( i_\gamma \leq i < \kappa \), \( \mathbb{Q}_{i, \gamma} \) is \( \mathbb{L}(\dot{\mathcal{D}}_i) \).

**Definition 4.7.** Whenever \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases, let \( \mathbb{P} \ast \mathbb{L}(\langle \mathcal{D}_i : i_\gamma \leq i < \kappa \rangle) \) denote the \( \gamma + 1 \)-extension described in Corollary 4.6.

This next corollary is immediate.

**Corollary 4.8.** If \( \mathbb{P} \) is a \( \bar{\lambda} \)-thin \((\kappa, \gamma)\)-matrix and if \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \( (\mathbb{P}, \bar{\lambda}(i_\gamma)) \)-thin sequence of filter bases, then \( \mathbb{P} \ast \mathbb{L}(\langle \mathcal{D}_i : i_\gamma \leq i < \kappa \rangle) \) is a \( \bar{\lambda} \)-thin \((\kappa, \gamma + 1)\)-matrix.

We now describe a first approximation of the scheme, \( \mathcal{K}(\bar{\lambda}) \), of posets that we will be using to produce the model.

**Definition 4.9.** For an ordinal \( \gamma > 0 \) and a \((\kappa, \gamma)\)-matrix iteration \( \mathbb{P} \), we will say that \( \mathbb{P} \in \mathcal{K}(\bar{\lambda}) \) providing for each \( 0 < \alpha < \gamma \),
(1) for each $i \leq \kappa$, $\mathbb{P}^i_{P_i, \lambda}$ is $\text{Fn}(\lambda_i, 2)$, and

(2) there is an $i_\alpha = i_\alpha P_{\alpha, \lambda}(i_\alpha)$ such that $\mathbb{P}^i_{\alpha, \lambda}$ is a $\mathbb{P}^i_{\alpha, \lambda}$-thin sequence of filter bases, such that $\mathbb{P}^i_{\alpha, \lambda}$ is equal to $\mathbb{P}^i_{\alpha, \lambda}$. For each $0 < \alpha < \gamma$, we let $\dot{D}^i_{\alpha, \lambda}$ denote the $\mathbb{P}^i_{\alpha, \lambda}$-name of the union $\bigcup \{ \dot{D}^i_{\alpha, \lambda} : i \leq \iota < \kappa \}$, and we let $\dot{L}^i_{\alpha, \lambda}$ denote the canonical $\mathbb{P}^i_{\alpha, \lambda}$-name of the subset of $\omega$ added by $\mathbb{P}^i_{\alpha, \lambda}$. Let us note that each $\mathbb{P}^i_{\alpha, \lambda}$ is $\mathbb{P}^i_{\alpha, \lambda}$-thin. Furthermore, by Lemma 3.5, this means that each $\mathbb{P}^i_{\alpha, \lambda}$ forces that $s \leq \lambda$. We begin a new section for the task of proving that there is a $\mathbb{P}^i_{\alpha, \lambda}$ that forces that $s \geq \lambda$.

It will be important to be able to construct $(\mathbb{P}^i_{\alpha, \lambda}(\iota_\gamma))$-thin sequences of filter bases, and it seems we will need some help.

**Definition 4.10.** For an ordinal $\gamma > 0$ and a $(\kappa, \gamma)$-matrix iteration $\mathbb{P}$ we will say that $\mathbb{P}^i_{\gamma, \lambda}$ is $\mathbb{P}^i_{\gamma, \lambda}$-thin if $\mathbb{P}^i_{\gamma, \lambda}$ is in $\mathbb{K}^i_{\lambda}$ and for each $0 < \alpha < \gamma$, if $i_\alpha = i_\alpha > 0$ then $\omega_1 \leq \text{cf}(\alpha) \leq \mu_i$ and there is a $\beta_\alpha < \alpha$ such that

1. for $\beta_\alpha \leq \xi < \alpha$, $i_\xi \in \{0, i_\alpha\}$,
2. if $\beta_\alpha < \eta < \alpha$, $i_\eta > 0$ and $\xi = \eta + \omega_1 \leq \alpha$, then $\dot{L}_\eta \in \dot{D}^i_{\xi, \alpha}$, and $\mathbb{P}^i_{\eta, \xi} \models \dot{D}^i_{\xi, \alpha}$ has a descending mod finite base of cardinality $\omega_1$,
3. if $\beta_\alpha < \xi < \alpha$, $i_\xi > 0$, and $\eta + \omega_1 < \xi$ for $\eta < \xi$, then $\{ \dot{L}_\eta : \beta_\alpha \leq \eta < \alpha, \text{cf}(\eta) \geq \omega_1 \}$ is a base for $\dot{D}^i_{\xi, \alpha}$.

5. PRODUCING $\tilde{\lambda}$-THIN FILTER SEQUENCES

In this section we prove this main lemma.

**Lemma 5.1.** Suppose that $\mathbb{P}^\gamma_{\lambda} \in \mathbb{H}(\tilde{\lambda})$ and that $\mathcal{Y}$ is a set of fewer than $\lambda$ nice $\mathbb{P}^\gamma_{\lambda}$-names of subsets of $\omega$, then there is a $\delta < \gamma + \lambda$ and an extension $\mathbb{P}^\delta_{\gamma}$ of $\mathbb{P}^\gamma_{\lambda}$ in $\mathbb{H}(\tilde{\lambda})$ that forces that the family $\mathcal{Y}$ is not a splitting family.

The main theorem follows easily.

**Proof of Theorem 1.2.** Let $\theta$ be any regular cardinal so that $\theta^{<\lambda} = \theta$ (for example, $\theta = (2^\lambda)^+$). Construct $\mathbb{P}^\theta_{\lambda} \in \mathbb{H}(\tilde{\lambda})$ so that for all $\mathcal{Y} \subset \mathbb{B}_{\kappa, \theta}$ with $|\mathcal{Y}| < \lambda$, there is a $\gamma < \delta < \theta$ so that $\mathcal{Y} \subset \mathbb{B}_{\kappa, \gamma}$ and, by applying Lemma 5.1 such that $\mathbb{P}^\theta_{\lambda} \models \delta$ forces that $\mathcal{Y}$ is not a splitting family. We begin by reducing our job to simply finding a $(\mathbb{P}, \tilde{\lambda}(i_\gamma))$-thin sequence.
Lemma 5.3. For any $\mathbb{P}^\gamma \in \mathcal{H}(\check{\lambda})$, and any $(\mathbb{P}^\gamma, \check{\lambda}(i_\gamma))$-thin filter base $\mathcal{E}$, there is an $\alpha \leq \gamma + \mu_{i_\gamma} + 1$ and extensions $\mathbb{P}^\alpha, \mathbb{P}^{\alpha+1}$ of $\mathbb{P}^\gamma$ in $\mathcal{H}(\check{\lambda})$, such that, $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha \ast \mathbb{L}(\langle \check{\lambda}^\alpha : i_\alpha \leq i < \kappa \rangle)$ and $\mathbb{P}^\alpha$ forces that $\mathcal{E} \cap \mathbb{B}_{i_\gamma}$ is a subset of $\check{\mathcal{D}}_i^\alpha$ for all $i < \kappa$.

Proof. The case $i_\gamma = 0$ is trivial, so we assume $i_\gamma > 0$. There is no loss of generality to assume that $\mathcal{E} \cap \mathbb{B}_{i_\gamma}$ has character $\mu_{i_\gamma}$. Let $\{\check{E}_\xi : \xi < \mu_{i_\gamma}\} \subset \mathcal{E} \cap \mathbb{B}_{i_\gamma}$ enumerate a filter base for $\langle \mathcal{E} \rangle \cap \mathbb{B}_{i_\gamma}$. We can assume that this enumeration satisfies that $\check{E}_\xi \setminus \check{E}_{\xi+1}$ is forced to be infinite for all $\xi < \mu_{i_\gamma}$. Let $\mathcal{A}$ be any countably generated free filter on $\omega$ that is not principal mod finite. By induction on $\xi < \mu_{i_\gamma}$ we define $\mathbb{P}^{\gamma+\xi}$ by simply defining $i_\gamma + \xi$ and the sequence $(\check{\mathcal{D}}^{\gamma+\xi}_i : i_{\gamma+\xi} \leq i \leq \kappa)$. We will also recursively define, for each $\xi < \mu_{i_\gamma}$, a $\mathbb{P}^{\gamma+\xi}$-name $\check{D}_\xi$ such that $\mathbb{P}^{\gamma+\xi}$ forces that $\check{D}_\xi \subset \check{E}_\xi$. An important induction hypothesis is that $\{\check{D}_\eta : \eta < \xi\} \cup \{\check{E}_\zeta : \zeta < \mu_{i_\gamma}\} \cup \mathcal{E}$ is forced to have the finite intersection property.

For each $\xi < \gamma + \omega_1$, let $i_\zeta = 0$ and $\check{\mathcal{D}}^\zeta_i$ be the $\mathcal{P}^\zeta$-name $\langle \mathcal{A} \rangle \cap \mathbb{B}_{i_\zeta}$ for all $i \leq \kappa$. The definition of $\check{D}_0$ is simply $\check{E}_0$. By recursion, for each $\eta < \omega_1$ and $\xi = \eta + 1$, we define $\check{D}_\xi$ to be the intersection of $\check{D}_\eta$ and $\check{E}_\xi$. For limit $\xi < \omega_1$, we note that $\mathbb{P}_{i_\zeta}^{\xi}$ forces that $\mathbb{L}(\langle \mathcal{A} \rangle)$ is isomorphic to $\mathbb{L}(\{\{\check{D}_\eta \cap \check{E}_\xi : \eta < \xi\}\})$. Therefore, we can let $\check{D}_\xi$ be a $\mathbb{P}^{\xi+1}$-name for the generic real added by $\mathbb{L}(\{\{\check{D}_\eta \cap \check{E}_\xi : \eta < \xi\}\})$. A routine density argument shows that this definition satisfies the induction hypothesis.

The definition of $i_{\gamma+\omega_1}$ is $i_\gamma$ and the definition of $\check{\mathcal{D}}^{\gamma+\omega_1}_i$ is the filter generated by $\{\check{D}_\xi : \xi < \omega_1\}$. The definition of $\check{D}_{\omega_1}$ is $\check{L}_{\gamma+\omega_1}$.

Let $S$ denote the set of $\eta < \mu_{i_\gamma}$ with uncountable cofinality. We now add additional induction hypotheses:

1. if $\zeta = \text{sup}(S \cap \xi) < \xi$ and $\xi = \nu + 1$, then $\check{D}_\xi = \check{D}_\nu \cap \check{E}_\xi$, and $i_\zeta = 0$ and $\check{\mathcal{D}}^{\zeta+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$.

2. if $\zeta = \text{sup}(S \cap \xi) < \xi$ and $\xi$ is a limit of countable cofinality, then $i_\zeta = 0$ and $\check{\mathcal{D}}^{\zeta+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$, and $\check{D}_\xi$ is forced by $\mathbb{P}^{\zeta+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\check{D}_\eta \cap \check{E}_\xi : \eta < \xi\})$.

3. if $\zeta = \text{sup}(S \cap \xi)$ and $\xi = \zeta + \omega_1$, then $i_\zeta = i_\gamma$, $\check{\mathcal{D}}^{\zeta+\xi}_i$ is the filter generated by $\{\check{E}_\xi \cap \check{D}_\eta : \zeta \leq \eta < \xi\}$ and $\check{D}_\xi$ is $\check{L}_{\gamma+\zeta}$.
\(4\) if \(S \cap \xi\) is cofinal in \(\xi\) and \(\text{cf}(\xi) > \omega\), then \(i_\xi = i_\gamma\) and \(\hat{D}_{i_\xi}^{\gamma+\xi}\) is the filter generated by \(\{\hat{D}_{\gamma+\eta} : \eta \in S \cap \xi\}\) and \(\hat{D}_{i_\xi} = \hat{L}_{\gamma+\xi}\).

\(5\) if \(S \cap \xi\) is cofinal in \(\xi\) and \(\text{cf}(\xi) = \omega\), then \(i_\xi = 0\) and \(\hat{D}_{i_\xi}^{\gamma+\xi} = \langle A \rangle\) for all \(i \leq \kappa\), and \(\hat{D}_{i_\xi}\) is forced by \(P^{\gamma+\xi+1}\) to be the generic real added by \(L(\{\hat{D}_{\eta_n} \cap \hat{E}_\xi : n \in \omega\})\), where \(\{\eta_n : n \in \omega\}\) is some increasing cofinal subset of \(S \cap (\gamma, \xi)\).

It should be clear that the induction continues to stage \(\mu_{i_\gamma}\) and that \(P^{\gamma+\xi} \in \mathcal{H}(\bar{\lambda}(i_\gamma))\) for all \(\xi \leq \mu_{i_\gamma}\), with \(\beta_{\gamma_\xi} = \gamma\) being the witness to Definition 4.10 for all \(\xi\) with \(\text{cf}(\xi) > \omega\).

The final definition of the sequence \(\langle \hat{D}_{i_\delta} : i_\delta = i_\gamma \leq i \leq \kappa \rangle\), where \(\delta = \gamma + \mu_{i_\gamma}\) is that \(\hat{D}_{i_\delta}\) is the filter generated by \(\{\hat{L}_{\gamma+\xi} : \text{cf}(\xi) > \omega\}\), and for \(i_\gamma < i \leq \kappa\), \(\hat{D}_{i_\delta}\) is the filter generated by \(\hat{D}_{i_\gamma} \cup (E \cap B_{i_\gamma})\). \(\square\)

**Lemma 5.4.** Suppose that \(\mathcal{E}\) is a \((P^\gamma, \bar{\lambda}(i_\gamma))\)-thin filter base. Also assume that \(i < \kappa\) and \(\alpha \leq \gamma\) and \(\mathcal{E}_1 \subset B_{i_\alpha}\) is a \((P^\alpha, \bar{\lambda}(i_\alpha))\)-thin filter base satisfying that \(\langle \mathcal{E} \rangle \cap B_{i_\alpha} \subset \langle \mathcal{E}_1 \rangle\), then \(\mathcal{E} \cup \mathcal{E}_1\) is a \((P^\gamma, \bar{\lambda}(i_\gamma))\)-thin filter subbase.

**Proof.** Let \(\mathcal{E}_2\) be equal to \(\mathcal{E} \cup \mathcal{E}_1\). The fact that each member of the sequence \(\langle \hat{D}_j : E_j = \langle \mathcal{E}_2 \cap B_{i_\gamma} \rangle : j < \kappa \rangle\) is a name of a filter base with character at most \(\mu_{i_\gamma}\) is immediate. Now we verify that if \(j_1 < j_2 < \kappa\), then \(\mathcal{E}_1 \cup \mathcal{E}_2\) is regular and \(\mathcal{D}_{j_2} \cap B_{j_1,\gamma} \subset \mathcal{D}_{j_1}\). Let \(b \in B_{j_2,\gamma}\) and suppose there are \(p \in P_{j_2,\gamma}\), \(E_0 \in E \cap B_{j_2,\gamma}\), and \(E_1 \in \mathcal{E}_1\) such that \(p \Vdash b \cap E_0 \cap E_1\). It suffices to produce an \(\hat{E} \in \langle \mathcal{E}_2 \rangle \cap B_{j_1,\gamma}\) satisfying that \(p \Vdash \hat{b} \cap \hat{E} = \emptyset\).

First, using that \(\mathcal{E}\) is \((P^\gamma, \bar{\lambda}(i_\gamma))\)-thin, choose \(\hat{E}_2 \in \langle \mathcal{E} \rangle \cap B_{j_1,\gamma}\) such that \(p \Vdash (b \setminus E_0) \cap \hat{E}_2 = \emptyset\). Equivalently, we have that \(p \Vdash (b \cap \hat{E}_2) \subset E_0\), and therefore \(p \Vdash (b \cap \hat{E}_2) \cap E_1 = \emptyset\). Since \(\hat{E}_1\) is a \(P_{j_1,\alpha}\)-name, there is a \(P_{j_1,\alpha}\)-name (which we can denote as) \((b \cap \hat{E}_2) \cup \alpha\) satisfying that \(p \Vdash E_2 \cap (b \cap \hat{E}_2) \cup \alpha\) is empty and that \(p \Vdash (b \cap \hat{E}_2) \subset (b \cap \hat{E}_2) \cup \alpha\). Now using that \(\mathcal{E}_1\) is \((P^\alpha, \bar{\lambda}(i_\alpha))\)-thin, choose \(\hat{E}_3 \in \langle \mathcal{E}_1 \rangle \cap B_{j_1,\alpha}\) so that \(p \Vdash \hat{E}_3 \cap (b \cap \hat{E}_2) \cup \alpha\) is empty. Naturally we have that \(p \Vdash \hat{E}_3 \cap (b \cap \hat{E}_2)\) is also empty. This completes the proof since \(\hat{E}_2 \cap \hat{E}_3\) is in \(\langle \mathcal{E}_2 \rangle \cap B_{j_1,\gamma}\). \(\square\)

Let \(P^\gamma \in \mathcal{H}(\bar{\lambda})\) and let \(\hat{y} \in B_{\kappa,\gamma}\). For a family \(\mathcal{E} \subset B_{\kappa,\gamma}\) and condition \(p \in P^\gamma\) say that \(p\) forces that \(\mathcal{E}\) measures \(\hat{y}\) if \(p \Vdash_p \{\hat{y}, \omega \setminus \hat{y}\} \cap \langle \mathcal{E} \rangle \neq \emptyset\). Naturally we will just say that \(\mathcal{E}\) measures \(\hat{y}\) if \(1\) forces that \(\mathcal{E}\) measures \(\hat{y}\).

Given Lemma 5.3, it will now suffice to prove.
Lemma 5.5. If \( Y \subset B_{\gamma,\gamma} \) for some \( P^\gamma \in H(\bar{\lambda}) \) and \( |Y| \leq \mu_i, \) for some \( i_\gamma < \kappa, \) then there is a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter \( E \subset B_{\kappa,\gamma} \) that measures every element of \( Y. \)

In fact, to prove Lemma 5.5 it is evidently sufficient to prove:

**Lemma 5.6.** If \( P^\gamma \in H(\bar{\lambda}), \dot{y} \in B_{\kappa,\gamma}, \) and if \( E \) is a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter, then there is a family \( E_1 \supset E \) measuring \( \dot{y} \) that is also a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter.

**Proof.** Throughout the proof we suppress mention of \( P^\gamma \) and refer instead to component member posets \( P_{i,\alpha}, Q_{i,\alpha} \) of \( P^\gamma. \) Let \( i_\dot{y} \) be minimal such that \( \dot{y} \in B_{i_\dot{y},\gamma}. \) Proceeding by induction, we can assume that the lemma holds for all \( \dot{y}, \) \( \dot{x} \in B_{i_\dot{y},\gamma} \) and all \( j < i_\dot{y}. \)

We can replace \( \dot{y} \) by any \( \dot{x} \in B_{i_\dot{y},\gamma} \) that has the property that \( 1 \models \dot{x} \in \{ \dot{y}, \omega \setminus \dot{y} \} \) since if we measure \( \dot{x} \) then we also measure \( \dot{y}. \) With this reduction then we can assume that no condition forces that \( \omega \setminus \dot{y} \) is in the filter generated by \( E. \)

**Fact 1.** If \( i_\dot{y} \leq i_\gamma, \) then there is a \( \dot{E} \in B_{i_\dot{y},\gamma} \) such that \( E \cup \{ \dot{E} \} \) is contained a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter that measures \( \dot{y}. \)

**Proof of Fact 1.** It is immediate that \( \langle \{ \dot{y} \} \cup (B_{i_\dot{y},\gamma} \cap E) \rangle \) is a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter. Therefore, by Lemma 5.4, \( E \cup \{ \dot{y} \} \) is a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter subbase. \( \square \)

We may thus assume that \( 0 < i_\dot{y} \) and that the Lemma has been proven for all members of \( B_{i,\gamma} \) for all \( i < i_\dot{y}. \) Similarly, let \( \alpha_\dot{y} \) be minimal so that \( \dot{y} \in B_{i_\dot{y},\alpha_\dot{y}} \) and assume that the Lemma has been proven for all members of \( B_{i_\dot{y},\beta} \) for all \( \beta < \alpha_\dot{y}. \) We skip proving the easy case when \( \alpha_\dot{y} = 1 \) and henceforth assume that \( 1 < \alpha_\dot{y}. \) Notice also that \( \alpha_\dot{y} \) has countable cofinality since \( P_{i_\dot{y},\gamma} \) is ccc.

Now choose an elementary submodel \( M \) of \( H((2^{\lambda_\gamma})^+) \) containing \( \bar{\lambda}, P^\gamma, E, \dot{y} \) and so that \( M \) has cardinality equal to \( \mu_i, \) and, by our cardinal assumptions, \( M^{\lambda_\gamma} \subset M \) for all \( j < i_\gamma. \) Naturally this implies that \( M^\omega \subset M. \)

By the inductive assumption we may assume that there is an \( E_1 \supset E \) that is \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin and measures every element of \( M \cap B_{j,\gamma} \) for \( j < i_\gamma \) as well as every element of \( M \cap B_{i_\dot{y},\beta} \) for all \( \beta \in M \cap \alpha_\dot{y}. \) Moreover, it is easily checked that we can assume that \( E_1 \) is a subset of \( M. \) Furthermore, we may assume that \( E_1 \) contains a maximal family of subsets of \( M \cap B_{i_\dot{y},\alpha_\dot{y}} \) that forms a \((P^\gamma,\bar{\lambda}(i_\gamma))\)-thin filter subbase.

**Fact 2.** There is a maximal antichain \( A \subset P_{i_\dot{y},\gamma} \) and a subset \( A_1 \subset A \) such that...
such that

(1) each \( p \in A_1 \) forces that \( \mathcal{E}_1 \) measures \( \dot{y} \),
(2) for each \( p \in A \setminus A_1 \), \( p \) forces that there is an \( i_p < i_\dot{y} \) such that \( \mathbb{B}_{i_p, \gamma} \cap \langle \mathcal{E}_1 \cup \{ \dot{y} \} \rangle \) is not generated by the elements in \( M \),
(3) for each \( p \in A \setminus A_1 \), \( p \) forces that there is a \( j_p < i_\dot{y} \) such that \( i_p \leq j_p \) and \( \mathbb{B}_{j_p, \gamma} \cap \langle \mathcal{E}_1 \cup \{ \omega \setminus \dot{y} \} \rangle \) is not generated by the elements in \( M \).

Proof of Fact 2. Suppose that \( p \in \mathbb{P}_{i_\dot{y}, \gamma} \) forces that the conclusion (2) fails. We have already arranged that \( p \forces_{p_{i_\dot{y}, \gamma}} \dot{y} \in \langle \mathcal{E}_1 \cap \mathbb{B}_{i_\dot{y}, \gamma} \rangle^+ \). Define \( \dot{E} \in \mathbb{B}_{i_\dot{y}, \gamma} \) so that \( p \) forces \( \dot{E} = \dot{y} \) and each \( q \in \mathbb{P}_{i_\dot{y}, \gamma} \cap p^+ \) forces that \( \dot{E} = \omega \). It is easily checked that \( \mathbb{B}_{i_\dot{y}, \gamma} \cap \langle \mathcal{E}_1 \cup \{ \dot{E} \} \rangle \) is then \( (\mathbb{P}^\gamma, \dot{\lambda}(i_\gamma))-\)thin and that \( p \) forces that it measures \( \dot{y} \). This condition ensures that \( p \) is compatible with an element of \( A_1 \).

If (2) holds but (3) fails, then by a symmetric argument as in the previous paragraph we can again define \( \dot{E} \) so that \( \mathbb{B}_{i_\dot{y}, \gamma} \cap \langle \mathcal{E}_1 \cup \{ \dot{E} \} \rangle \) is then \( (\mathbb{P}^\gamma, \dot{\lambda}(i_\gamma))-\)thin and that \( p \) forces that it measures \( \omega \setminus \dot{y} \). \( \square \)

If by increasing \( M \) we can enlarge \( A_1 \) we simply do so. Since \( \mathbb{P}^\gamma \) is ccc we may assume that this is no longer possible, and therefore we may also assume that \( A \) is a subset of \( M \). Now we choose any \( p \in A \setminus A_1 \). It suffices to produce an \( \dot{E}_p \in \mathbb{B}_{i_\dot{y}, \gamma} \) that can be added to \( \mathcal{E}_1 \) that measures \( \dot{y} \) and satisfies that \( q \forces \dot{E}_p = \omega \) for all \( q \in p^+ \). This is because we then have that \( \mathcal{E}_1 \cup \{ \dot{E}_p : p \in A \setminus A_1 \} \) is contained in a \( \dot{\lambda}(i_\gamma) \)-thin filter that measures \( \dot{y} \).

Fact 3. There is an \( \alpha \) such that \( \alpha_\dot{y} = \alpha + 1 \).

Proof of Fact 3. Otherwise, let \( j = i_p \) and for each \( r \prec p \in \mathbb{P}_{i_\dot{y}, \alpha_\dot{y}} \), choose \( \beta \in M \cap \alpha_\dot{y} \) such that \( r \in \mathbb{P}_{i_\dot{y}, \beta} \), and define a name \( \dot{y}[r] \) in \( M \cap \mathbb{B}_{j, \gamma} \) according to \( (\ell, q) \in \dot{y}[r] \) providing there is a pair \( (\ell, p_\ell) \in \dot{y} \) such that \( q <_j p_\ell \) and \( q \forces \beta \) is in the set \( M \cap \mathbb{P}_{j, \beta} \setminus (r \setminus p_\ell \setminus \beta)^+ \). This set, namely \( \dot{y}[r] \), is in \( M \) because \( \mathbb{P}_{j, \beta} \) is ccc and \( M^x \subseteq M \).

We prove that \( r \) forces that \( \dot{y}[r] \) contains \( \dot{y} \). Suppose that \( r_1 < r \) and there is a pair \( (\ell, p_\ell) \in \dot{y} \) with \( r_1 < p_\ell \). Choose an \( r_2 \in \mathbb{P}_{j, \gamma} \) so that \( r_2 <_j r_1 \). It suffices to show \( r_2 \forces \ell \in \dot{y}[r] \). Let \( q <_j p_\ell \) with \( q \in M \). Then \( r_2 \not\forces p_\ell \) implies \( r_2 \not\forces q \). Since \( r_2 \) was any \( <_j \)-projection of \( r_1 \) we can assume that \( r_2 < q \). Since \( r_2 \forces \beta \) is in \( (\mathbb{P}_{j, \beta} \cap (r \setminus p_\ell \setminus \beta)^+)^+ \), it follows that \( q \forces \beta \not\in (r \setminus p_\ell \setminus \beta)^+ \). This implies that \( (\ell, q) \in \dot{y}[r] \) and completes the proof that \( r_2 \forces \ell \in \dot{y}[r] \).

Now assume that \( \beta < \alpha_\dot{y} \) and \( r \forces \dot{b} \cap \dot{E} \cap \dot{y} \) is empty for some \( r < p \) in \( \mathbb{P}_{i_\dot{y}, \beta} \), \( \dot{b} \in \mathbb{B}_{j, \gamma} \), and \( \dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_\dot{y}, \gamma} \). Let \( \dot{x} = (\dot{E} \cap \dot{y})[r] \) (defined as above for \( \dot{y}[r] \)). We complete the proof of Fact 3 by proving that \( r \forces \dot{b} \cap \dot{x} \) is...
Now assume that $i_3$ to prove that $i_3$.

Proof of Fact 4. Since $r$ and $r \not< p$ (in $P_{i,y+1}$ and $r \not< p$), it follows that $r \not< p$. Now it follows that $r_1$ has an extension forcing that $\ell \in b \cap (\dot{E} \cap \dot{y})$ which is a contradiction. \qed

Fact 4. $i_y = i_\alpha$ and so also $i_p < i_\alpha$.

Proof of Fact 4. Since $P_{i,a+1} = P_{i,a}$ for $i < i_\alpha$, we have that $i_\alpha \leq i_y$.

Now assume that $i_\alpha < i_y$ and we proceed much as we did in Fact 3 to prove that $i_p$ does not exist. Assume that $r < p$ (in $P_{i,y+1}$ and $r \not< p$) is empty for some $\dot{E} \in M \cap (E_1) \cap B_{i,y+1}$ and $\dot{b} \in B_{i,y+1}$. It follows from Lemma 3.4 that we can simply assume that $\dot{E} = E_1 \cap B_{i,a+1}$, and similarly that $\dot{b} \in B_{i,a+1}$.

Let $T_\alpha$ be the $P_{i,a}$-name such that $r \upharpoonright \alpha \models r(\alpha) = T_\alpha \in L(D_{i,y})$. We may assume that there is a $t_\alpha \in \omega^{<\omega}$ such that $r \upharpoonright \alpha \models t_\alpha = \text{stem}(T_\alpha)$.

Choose any $M \cap P_{i,a}$-generic filter $\dot{G}$ such that $r \upharpoonright \alpha \in \dot{G}$. Since $P_{i,a}$ is ccc and $M^\omega \subseteq M$, it follows that $M[\dot{G}]$ is closed under $\omega$-sequences in the model $V[\dot{G}]$.

In this model, define an $L(D_{i,y})$-name $\dot{x}$. A pair $(\ell, T_\ell) \in \dot{x}$ if $t_\alpha \leq \text{stem}(T_\ell) \subseteq L(D_{i,y})$ and for each $\text{stem}(T_\ell) \leq t \in T_\ell$, there is a pair $(\ell, q_{\ell,t}) \in M$ in the name $(\dot{y} \cap \dot{E})$ such that $q_{\ell,t} \upharpoonright \alpha \in \dot{G}$, $q_{\ell,t} \upharpoonright \alpha \models t = \text{stem}(q_{\ell,t}(\alpha))$, and $(q_{\ell,t} \upharpoonright \alpha \wedge r \upharpoonright \alpha)$ does not force (over the poset $\dot{G}$) that $t \notin T_\alpha$. We will show that $r$ forces over the poset $\dot{G}$ that $\dot{x}$ contains $\dot{E} \cap \dot{y}$ and that $\dot{x} \cap \dot{b}$ is empty. This proves that $p$ forces that $(E_1) \cap B_{i,a+1}$ generates $(E_1 \cap \{\dot{y}\}) \cap B_{i,a+1}$ since $\dot{x}$ must be forced to be in $\langle E_1 \rangle$. It then follows from Lemma 3.4 that $E_1 \cap B_{i,y+1}$ generates $(E_1 \cap \{\dot{y}\}) \cap B_{i,y+1}$, contradicting the assumption on $i_p$.

To prove that $r$ forces that $\dot{x}$ contains $\dot{y} \cap \dot{E}$, we consider any $r < r$ that forces over $\dot{G}$ that $\ell \in \dot{y} \cap \dot{E}$. We may choose $(\ell, p_\ell) \in M$ in the name $(\dot{E} \cap \dot{y})$ such that (wlog) $r_\ell < p_\ell$. We may assume that $r_\ell \upharpoonright \alpha$ forces a value $t$ on $\text{stem}(r_\ell(\alpha))$ and that this equals $\text{stem}(p_\ell(\alpha))$. Now show there is a $T_\ell \in L(D_{i,y})$. In fact, assume $t \in T_\ell$ with $q_{\ell,t}$ as the witness. Let $L^- = \{k : t \cap k \notin T_\ell\}$; it suffices to show that $L^- \notin (D_{i,y})^+$. By assumption that $q_{\ell,t}$ is the witness, there is an $r_\ell < (q_{\ell,t} \upharpoonright \alpha \wedge r \upharpoonright \alpha)$ such that $r_\ell \upharpoonright t \in T_\alpha$ and $r_\ell \upharpoonright t = \text{stem}(q_{\ell,t}(\alpha))$. By strengthening $r_\ell$ we can assume that $r_\ell$ forces a value $\dot{D} \in D_{i,y}$ on $\{k : t \cap k \in T_\alpha \cap q_{\ell,t}(\alpha)\}$. But now, it follows that $r_\ell$ forces that $\dot{D}$ is disjoint from $L^-$ since if $r_\ell,k \upharpoonright k \notin \dot{D}$ for some $r_\ell,k < r_\ell$, $r_\ell,k$ is the witness to $(\ell, q_{\ell,t-k})$ is in $(\dot{y} \cap \dot{E})$ etc., where $q_{\ell,t-k} \upharpoonright \alpha = q_{\ell,t} \upharpoonright \alpha$ and $q_{\ell,t}(\alpha) = (q_{\ell,t}(\alpha))_{t-k}$.\[\]
Since some condition forces that \(L^-\) is not in \((\hat{D}_x)^\alpha\) it follows that \(L^-\) is not in \((\hat{D}_i)^\alpha\)

Finally we must show that \(r\) forces over \(G^+\) that \(\hat{b}\) is disjoint from \(\hat{x}\).

Since each are \(\mathbb{P}_{\lambda_\alpha+1}\)-names, it suffices to assume that \(\bar{r} \in \bar{G}^+\) is some \(\mathbb{P}_{\lambda_\alpha+1}\)-reduct of \(r\) that forces some \(\ell\) in \(\hat{b} \cap \hat{x}\), and to then show that \(r\) fails to force that \(\ell \notin \hat{b} \cap (\hat{E} \cap \hat{y})\). Choose \((\ell, q_{\ell,\alpha}) \in (\hat{y} \cap \hat{E})\) witnessing that \(\bar{r} \Vdash \ell \in \hat{x}\). That is, we may assume that \(\bar{r} \upharpoonright \alpha \Vdash t = \text{stem}(r(\alpha))\), that \(q_{\ell,\alpha} \upharpoonright \alpha \in G^+\), and \((q_{\ell,\alpha} \land r \upharpoonright \alpha)\) does not force over \(G^+\) that \(t \notin T_\alpha\).

Of course this means that the condition \(\bar{r} \land r \land [t \in T_\alpha] \land q_{\ell,\alpha}\) is not 0. This condition forces that \(\ell\) is in \(\hat{b} \cap (\hat{E} \cap \hat{y})\) as required. \(\square\)

**Fact 5.** The character of \(\hat{D}_i\) is greater than \(\mu_i\).

**Proof of Fact 5.** We know that \(\hat{D}_i\) is forced to have an \(\omega\)-closed base (in fact, descending mod finite with uncountable cofinality). Even more, \(\mathbb{P}_{\lambda_\alpha}\) forces that for all \(T \in \mathbb{L}(\hat{D}_i)\), there is a \(D \in \hat{D}_i\) such that the condition \((|D|^{< \omega})_{\text{stem}(T)}\) is below \(T\). Let \(\chi_\alpha\) be the cofinality of \(\alpha\) and fix a list \(\{\hat{D}_\beta : \beta < \chi_\alpha\} \subseteq M\) (closed under mod finite changes) of \(\mathbb{P}_{\lambda_\alpha}\)-names of elements of \(\hat{D}_i\) that is forced to be a base.

Now, suppose that \(\hat{b} \in \mathbb{B}_{\lambda_\alpha+1} = \mathbb{B}_{\lambda_\alpha}\) and there is an \(\hat{E} \in E_1\) and an \(r < p\) forcing that \(\hat{b} \cap (\hat{E} \cap \hat{y})\) is empty. We prove there is an \(\hat{x} \in E_1\) and an \(r_2 < r \upharpoonright \alpha\) in \(\mathbb{P}_{\lambda_\alpha}\) such that \(r_2 \Vdash \hat{b} \cap \hat{x}\) is empty. We may assume that \(r_2\) forces a value \(t\) on \(\text{stem}(r(\alpha))\) and that, for some \(\beta < \chi_\alpha\), \(r_2 \Vdash (\hat{D}_\beta^{< \omega})_t < r(\alpha)\).

Let

\[
\hat{x} = \{(\ell, q_{\ell} \upharpoonright \alpha) : (\ell, q_{\ell}) \in (\hat{E} \cap \hat{y})\} \quad \text{and} \quad q_{\ell} \upharpoonright \alpha \Vdash q_{\ell}(\alpha) \leq (\hat{D}_\beta^{< \omega})_t.
\]

It is immediate that \(\hat{x} \in M\) and that \((r_2 \land r) \Vdash_{\mathbb{P}_{\lambda_\alpha+1}} \hat{x} \supseteq (\hat{E} \cap \hat{y})\). Since \(\hat{E} \cap \hat{y}\) is forced to be in \(E_1^+\), it follows that \(\hat{x}\) is forced by \(r_2\) to be in \(E_1\). Now we verify that \(r_2 \Vdash \hat{b} \cap \hat{x}\) is empty. Assume that \(r_3 < r_2\) in \(\mathbb{P}_{\lambda_\alpha}\) and that \(r_3 \Vdash \ell \in \hat{b} \cap \hat{x}\). We may assume there is \((\ell, q_{\ell} \upharpoonright \alpha) \in \hat{x}\) such that \(r_3 < q_{\ell} \upharpoonright \alpha\). But now \(r_2 \Vdash q_{\ell}(\alpha) \leq r(\alpha)\) and so \(r_2 \land r \Vdash \ell \in \hat{b} \cap (\hat{E} \cap \hat{y})\) – a contradiction.

The conclusion now follows from Lemma 5.4. \(\square\)

**Definition 5.7.** For each \(t \in \omega^{< \omega}\), define that \(\mathbb{P}_{\lambda_\alpha}\)-name \(\hat{E}_t\) according to the rule that \(r \Vdash \ell \in \hat{E}_t\) providing \(r \in \mathbb{P}_{\lambda_\alpha}\) forces that there is a \(\hat{T}\) with \(r \Vdash \hat{T} \in \mathbb{L}(\hat{D}_i), r \Vdash t = \text{stem}(\hat{T})\), and \(r \cup \{(\alpha, \hat{T})\} \Vdash \ell \notin \hat{y}\).

**Fact 6.** There is a \(\hat{T} \in \mathbb{L}(\hat{D}_i) \cap M\) such that \(p \upharpoonright \alpha\) forces the statement: \(\hat{E}_t \in E_1\) for all \(t\) such that \(\text{stem}(\hat{T}) \leq t \in \hat{T}\).
Proof of Fact 6. By elementarity, there is a maximal antichain of $\mathbb{P}_{i_{\alpha}}$, each element of which decides if there is a $\hat{T}$ with $\hat{E}_t \in \mathcal{E}_1$ for all $t \in \hat{T}$ above stem$(\hat{T})$. Since $p \in A \setminus A_1$ it follows that there is an $i_p < i_\alpha$ as in condition (2) of Fact 2. Let $t_0 \in \omega^{<\omega}$ so that $p \upharpoonright \alpha \Vdash t_0 = \text{stem}(p(\alpha))$.

By the maximum principle, there is a $\hat{b} \in \mathbb{B}_{\beta,\gamma}$ and a $\hat{E}_0 \in \mathcal{E}_1$ satisfying that $p \Vdash \hat{b} \cap \hat{E}_0 \cap \hat{y}$ is empty, while $p \Vdash \hat{b} \cap \hat{E}$ is infinite for all $\hat{E} \in \langle \mathcal{E}_1 \rangle$. This means that $p$ forces that $\hat{b} \cap \hat{E}_0$ is an element of $\langle \mathcal{E}_1 \rangle$ that is contained in $\omega \setminus \hat{y}$. As in the proof of Lemma 5.4, there is an $\hat{E}_2 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{\beta,\gamma}$ such that $p$ forces that $\hat{b} \cap \hat{E}_2$ is contained in $\hat{E}_0$. We also have that $(\hat{b} \cap \hat{E}_2) \upharpoonright \alpha$ is forced to be contained in $\omega \setminus \hat{y}$. It now follows that $p \upharpoonright \alpha$ forces that for all $t_0 \leq t \in p(\alpha)$, $p \upharpoonright \alpha$ forces that $\hat{E}_t$ contains $(\hat{b} \cap \hat{E}_2) \upharpoonright \alpha$ and so is in $\langle \mathcal{E}_1 \rangle$. Since $\hat{E}_t$ is also measured by $\mathcal{E}_1$, we have that $p \upharpoonright \alpha$ forces that such $\hat{E}_t$ are in $\mathcal{E}_1$. This completes the proof. $\square$

Now we show how to extend $\mathcal{E}_1 \cap \mathbb{B}_{i_{\alpha}}$ so as to measure $\hat{y}$. Let $\beta = \sup(M \cap \alpha)$. By Fact 3, $\beta < \alpha$ and by the definition of $\mathcal{H}(\hat{\lambda})$, $M \cap \mathcal{D}_{i_{\alpha}}^\beta$ is a subset of $\langle \mathcal{D}_{i_{\alpha}}^\beta \rangle$, $\hat{\lambda}_\beta \in \mathcal{D}_{i_{\alpha}}^\alpha$, and $i_\beta = i_{\alpha}$. We also have that the family

$$\{ \hat{\lambda}_\xi : \text{cf}(\xi) \geq \omega_1 \text{ and } \beta_\alpha \leq \xi \in M \cap \beta \}$$

is a base for $\mathcal{D}_{i_{\alpha}}^\beta$. For convenience let $q_M \in M$ denote the relation that $q$ is an $M \cap \mathbb{P}_{i_{\alpha \alpha}}$-reduct of $p$.

Let $\hat{p}$ be any condition in $\mathbb{P}_{i_{\beta},i_{\alpha+1}}$ satisfying that $\hat{p} \upharpoonright \beta = p \upharpoonright \alpha$ and $\hat{p} \upharpoonright \beta \Vdash \text{stem}(\hat{p}(\beta)) = t_\alpha$ (recall that $p \upharpoonright \alpha \Vdash t_\alpha = \text{stem}(p(\alpha))$).

Let us note that for each $q \in M \cap \mathbb{P}_{i_{\alpha \alpha+1}}$, $q \upharpoonright \alpha = q \upharpoonright \beta$ and $q \upharpoonright \beta \Vdash q(\alpha)$ is also a $\mathbb{P}_{i_{\beta},i_{\alpha}}$-name of an element of $\mathcal{L}(\mathcal{D}_{i_{\beta}}^\beta)$. Let $\hat{x}$ be the following $\mathbb{P}_{i_{\beta},i_{\alpha+1}}$-name

$$\hat{x} = \{(\ell, q \upharpoonright \beta \cup \{(\beta, q(\beta))\}) : (\ell, q) \in \hat{y} \cap M \text{ and } q <_M p\}.$$

We will complete the proof by showing that there is an extension of $p$ that forces that $\mathcal{E}_1 \cup \{ \omega \setminus (\hat{x}[\hat{\lambda}_\beta]) \}$ measures $\hat{y}$ and that 1 forces that $\langle \mathcal{E}_1 \cup \{ \omega \setminus (\hat{x}[\hat{\lambda}_\beta]) \} \rangle \cap \mathbb{B}_{i_{\beta},i_{\alpha+1}}$ is $\lambda(i_{\alpha})$-thin. Here $\hat{x}[\hat{\lambda}_\beta]$ abbreviates the $\mathbb{P}_{i_{\beta},i_{\alpha+1}}$-name

$$\{(\ell, r) : (\exists q)(\ell, q) \in \hat{x}, q \upharpoonright \beta = r \upharpoonright \beta, \text{ and } r \Vdash \text{stem}(q(\beta)) \in \hat{L}_\beta^\omega\}.$$ 

The way to think of $\hat{x}[\hat{\lambda}_\beta]$ is that if $\hat{p}$ is in some $\mathbb{P}_{i_{\alpha \alpha}}$-generic filter $G$, then $\hat{y}[G]$ is now an $\mathcal{L}(\mathcal{D}_{i_{\alpha}}^\alpha)$-name, $L_\beta^\omega = \hat{L}_\beta[G]^{<\omega}$ is in $\mathcal{L}(\mathcal{D}_{i_{\alpha}}^\alpha)$, and $(\hat{x}[\hat{\lambda}_\beta])[G]$ is equal to $\{ \ell : L_\beta^\omega \not\models \ell \notin \hat{y} \}$. We will use the properties of $\hat{x}$ to help show that $\mathcal{E}_1 \cup \{ \omega \setminus (\hat{x}[\hat{\lambda}_\beta]) \}$ is $\lambda(i_{\alpha})$-thin. This semantic description of $\hat{x}[\hat{\lambda}_\beta]$ makes clear that $\hat{p} \cup \{(\alpha, (L_\beta^\omega))\} \in \mathbb{P}_{i_{\alpha \alpha}}$ forces that $\hat{x}[\hat{\lambda}_\beta]$ contains $\hat{y}$. This implies that $\mathcal{E}_1 \cup \{ \omega \setminus (\hat{x}[\hat{\lambda}_\beta]) \}$ measures $\hat{y}$.

Claim: It is forced by $\hat{p}$ that $\omega \setminus \hat{x}$ is not measured by $\mathcal{E}_1$. 

Each element of $\mathcal{E}_1$ is in $M$ and simple elementarity will show that for any condition in $q$ in $M$ that forces $\hat{E} \cap (\omega \setminus \hat{y})$ is infinite, the corresponding $\bar{q} = q \upharpoonright \alpha \cup \{(\beta, q(\alpha))\}$ will also force that $\hat{E} \cap (\omega \setminus \hat{x})$ is infinite.

It follows from Fact \ref{fact:thin} with $\omega \setminus \hat{x}$ playing the role of $\hat{y}$, that $\mathcal{E}_1 \cup \{\omega \setminus \hat{x}\}$ is $\bar{\lambda}(i_r)$-thin. Recall that $q \Vdash \hat{x} = \emptyset$ for all $q \perp \bar{p}$. Now to prove that $\mathcal{E}_1 \cup \{\omega \setminus (\hat{x}[\dot{L}_\beta])\}$ is also $\bar{\lambda}(i_r)$-thin, we prove that

\[
\langle \mathcal{E}_1 \cup \{\omega \setminus \hat{x}\} \rangle \cap \mathbb{B}_{i,\alpha} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\hat{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\alpha}
\]

for all $i < i_\alpha$. In fact, first we prove

\[
\langle \mathcal{E}_1 \cup \{\omega \setminus \hat{x}\} \rangle \cap \mathbb{B}_{i,\beta} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\hat{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\beta}
\]

for all $i < i_\alpha$.

We begin with this main Claim.

**Claim 1.** If $\dot{b} \in \mathbb{B}_{i,\beta}$ ($i < i_\beta$) and there is an $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\beta}$ and a $\bar{p} \geq q \in \mathbb{P}_{i_\beta,\alpha+1}$ such that $q \Vdash \dot{b} \cap (\dot{E} \setminus \hat{x}) = \emptyset$ then $q \upharpoonright \beta \Vdash (\exists \dot{E} \in \mathcal{E}_1) \dot{b} \cap \dot{E} = \emptyset$.

**Proof of Claim:** We may assume that $q \upharpoonright \beta$ forces a value $t$ on $\text{stem}(q(\beta))$. Recall that $q \upharpoonright \beta$ forces the statement: there is a $\dot{D} \in M \cap \mathcal{D}_{i_\alpha}^\alpha$ such that $(\dot{D}^{<\omega})_t \leq q(\beta)$. The definition of $\hat{x}$ ensures that $q \upharpoonright \beta \cup \{(\alpha, (\dot{D}^{<\omega})_t)\} \Vdash \dot{b} \cap (\dot{E} \setminus \hat{y})$ is empty. There is a $\mathbb{P}_{i_\alpha,\alpha}$-name $\dot{E}_1 \in M$ such that $q \upharpoonright \alpha \Vdash \dot{E}_1 = \{\ell : (\dot{D}^{<\omega})_t \not\Vdash \ell \notin (\dot{E} \setminus \hat{y})\}$. By assumption $q \upharpoonright \alpha \Vdash \dot{E}_1 \in \mathcal{E}_1$. Since $\dot{b}$ is also a $\mathbb{P}_{i_\alpha,\alpha}$-name, we have that $q \upharpoonright \alpha \Vdash \dot{b} \cap \dot{E}_1 = \emptyset$. \hfill \Box

Now assume that $\dot{b} \in \mathbb{B}_{i_\beta,\alpha}$ and $q \Vdash \dot{b} \cap (\dot{E} \setminus (\hat{x}[\dot{L}_\beta]))$ is empty for some $q < \bar{p}$ in $\mathbb{P}_{i_\beta,\alpha+1}$. By Lemma \ref{lemma:extension} it suffices to assume that $\dot{E} \in \mathbb{B}_{i_\beta,\alpha}$. To prove that $q$ forces $\dot{b} \notin \langle \mathcal{E}_1 \rangle^+$, it suffices to prove that there is some $\dot{E}_1 \in \mathcal{E}_1$ such that $q \Vdash \dot{b} \cap (\dot{E}_1 \cap (\omega \setminus \hat{x}))$ is finite. We proceed by contradiction.

We may again assume that $q \upharpoonright \beta$ forces that $q(\beta)$ is $(\dot{D}^{<\omega})_t$ for some $t \supset t_\alpha$ and some $\dot{D} \in \mathcal{D}_{i_\alpha}^\alpha \cap M$. Let $H$ be the range of $t$. Let, for the moment, $G$ be a $\mathbb{P}_{i_\alpha,\alpha}$-generic filter with $q \in G$. Now in $M[G]$ we have the value $L_\beta$ of $\dot{L}_\beta$ and $H \subset L_\beta$. We can also let $E$ denote the value of $\dot{E}[G]$. Recall that for each $s \in H^{<\omega}$, $E_s$ denotes the set of $\ell \in E$ such that there is some $T \in \mathcal{L}(\mathcal{D}_{i_\beta}^\alpha)$ with $s = \text{stem}(T)$ and $T \Vdash \ell \notin \dot{y}$. We have shown in Fact \ref{fact:extension} that there is a $T \in \mathcal{L}(\mathcal{D}_{i_\beta}^\alpha) \cap M$ such that $E_s \in \mathcal{E}_1$ for all $s \in T$ above $\text{stem}(T)$. This means that there is an $\ell \in \dot{b} \cap E$ such that $\ell \in E_s$ for each of the finitely many suitable $s$. For each $s$, choose $T_s \subset T$ witnessing $\ell \in E_s$. As before, and since there are only finitely many $s$ involved, we can assume that $\dot{T}_s = (\dot{D}^{<\omega})_s$ for
some $H \subset \dot{D} \in \dot{\mathcal{D}}_\alpha \cap M$ and we then define an extension $q$ of $q$ so that $q'(\beta) = (\dot{D}^{<\omega})_{t_\alpha}$ ensures that $(\dot{L}_\beta^{<\omega})_s < T_s$ for each $s$. Note that such a condition $q'$ we have that $q' \cup \{ (\alpha, (\dot{L}_\beta)^{<\omega}) \}$ forces that $\ell \notin \dot{y}$. But then it should be clear that $q'$ that forces $\ell \notin \dot{x}[\dot{L}_\beta]$. This contradicts that $q$ forces $\ell \notin b \cap (E \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$.

□

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