ABSTRACT. We show the the existence of noncontractible periodic orbits for every compactly supported time-dependent Hamiltonian on the open unit disk cotangent bundle of a Finsler manifold provided that the Hamiltonian is sufficiently large over the zero section. This result solves a conjecture of Irie [19] and generalizes the previous results [8, 40, 42] etc.

We then obtain a number of applications including: (1) preservation of Finsler lengths of closed geodesics by symplectomorphisms, (2) existence of periodic orbits for Hamiltonian systems separating two Lagrangian submanifolds, (3) existence of periodic orbits for Hamiltonians on noncompact domains, (4) existence of periodic orbits for Lorentzian Hamiltonian in higher dimensional case, (5) partial solution to a conjecture of Kawasaki in [22], (6) results on squeezing/nonsqueezing theorem on torus cotangent bundles.

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1. INTRODUCTION

The main theme of this article is the existence of noncontractible periodic orbits for compactly supported time-dependent Hamiltonian systems on the unit disk cotangent bundle of a closed Finsler manifold $M$. Closely related earlier results on finding noncontractible Hamiltonian periodic orbits on cotangent bundles go back to the papers by Gatien and Lalonde [16] and by Biran, Polterovich and Salamon [8] etc. Weber [40] extended the result in [8], relaxing the condition that $M$ is either the Euclidean torus $\mathbb{T}^n$ or of negative sectional curvature, to the case that $M$ is just a closed connected Riemannian manifold. Applications of the results include the existence of periodic orbits for certain magnetic Hamiltonian systems on cotangent bundles [18, 33].

On the other hand, there are some other results [42, 22] on the existence of periodic orbits on cotangent bundle which does not fit into the general Riemannian framework of [40]. In particular, these results have important applications including the existence of noncontractible periodic orbits for Hamiltonian Lorentzian systems, etc.

The main result in this paper is a generalization of [40] to the Finsler setting, which provides a unified framework hence enables us to recover many results in the literature including [8, 40, 42] etc. Such a Finslerian generalization is by no means obvious. There are many Riemannian results that do not admit a Finslerian version, for instance, Katok’s example [21] of a Finsler metric with only two closed geodesics for $S^2$.

Our result allows us to find periodic orbits for Hamiltonian systems supported in fiberwise convex domains on the cotangent bundle (Theorem 1.1) and compute the BPS capacity on such domains (Theorem 1.4). The setting is so flexible that it leads to a number of applications. These include:

1. preservation of minimal Finsler lengths of closed geodesics by symplectomorphisms (Corollary 1.5),
2. existence of periodic orbits for Hamiltonian systems separating two Lagrangian submanifolds (Theorem 1.7),
3. existence of periodic orbits for Hamiltonians on noncompact domains (Theorem 1.8),
4. existence of periodic orbits for Lorentzian Hamiltonian in higher dimensional case (Theorem 1.9),
5. partial solution to a conjecture of Kawasaki in [22] (Theorem 1.11),
6. results on squeezing/nonsqueezing theorem on torus cotangent bundles (Theorem 1.12, 1.13, 1.14).
We expect that the list is not exhaustive and there are a lot more applications to come. Moreover, we expect that the method of this paper may have broader interests in further studying geometry, topology and dynamics on Finsler manifold using Floer theory.

To state our result, let us first give a brief introduction to the Finsler geometry. For more comprehensive materials on Finsler geometry, we refer to the books [30, 36].

1.1. Finsler geometry. Let $M$ be a smooth manifold of dimension $n$. A Finsler metric on $M$ is a continuous function $F : TM \to [0, \infty)$ satisfying the following properties:
   
   (i) $F$ is $C^\infty$ on $TM \setminus M \times \{0\}$.
   
   (ii) $F(x, \lambda v) = \lambda F(x, v)$ for every $\lambda > 0$ and $(x, v) \in TM$.
   
   (iii) For any $(x, y) \in TM \setminus \{0\}$, the symmetric bilinear form
       
       $$g_F(x, y) : T_x M \times T_x M \to \mathbb{R}, \quad (u, v) \mapsto \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)_{s=t=0}$$
   
   is positive definite.

A smooth manifold $M$ endowed with a Finsler metric $F$ is called a Finsler manifold. Let $\gamma : (a, b) \to M$ be a smooth curve. We define the $F$-length of $\gamma$ as

$$\text{len}_F(\gamma) = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt.$$  

Define the co-Finsler metric $F^*$ on the cotangent bundle $T^*M$ as

$$F^*(x, \cdot) : T^*_x M \to \mathbb{R}, \quad F^*(x, p) := \max_{F(x, v) \leq 1} \langle p, v \rangle, \quad \forall \ x \in M.$$  

Denote by $D^F T^*M$ the unit open disk cotangent bundle, and $S^F T^*M$ denotes its boundary:

$$D^F T^*M := \{ (x, p) \in T^* M \mid F^*(x, p) < 1 \},$$

$$S^F T^*M := \{ (x, p) \in T^* M \mid F^*(x, p) = 1 \}.$$  

It is obvious from (iii) that for every $x \in M$, the disk cotangent fiber

$$(D^F T^*M)_x := \{ p \in T^*_x M \mid F^*(x, p) < 1 \}$$

is a strictly convex set. Note that, on all $TM$, $L_0 := F^2$ is of class $C^{1,1}$, and is of class $C^2$ if and only if $F$ is the square of the norm of a Riemannian metric (cf. [29]). In the latter case, every $(D^F T^*M)_x$ is an ellipsoid in $T^*_x T^*M$.

Conversely, given a fiberwise strictly convex domain $U \subset T^* M$ with smooth boundary and containing $M$ in its interior, a Finsler metric $F$ can be associated to it to realize $U$ as the unit disc cotangent bundle as follows

$$F(x, v) := \sup_{p \in T^*_x U} \langle p, v \rangle, \quad \forall \ v \in TM.$$  

1.2. Main result. Let $(M, F)$ be a closed connected Finsler manifold, and let $\pi : T^* M \to M$ denote the natural projection. Define $\lambda_0 \in \Omega^1(T^* M)$ as

$$\lambda_0(\xi) := \langle p, d\pi(\xi) \rangle \quad \forall \ x \in M, \ p \in T^*_x M, \ \xi \in T_{(x, p)} T^* M.$$  

The cotangent bundle $T^* M$ admits a canonical symplectic form $\omega_0 := d\lambda_0$. For any $H \in C^\infty(S^1 \times T^* M)$, denote $H_t := H(t, \cdot)$, the time-dependent Hamiltonian vector field $X_{H_t}$ is defined as $\omega_0(X_{H_t}, \cdot) = -dH_t$. Set $S^1 := \mathbb{R}/\mathbb{Z}$. Let $[S^1, M]$ denote the set of homotopy classes of free loops on $M$. Since $M$ is compact, for any non-trivial homotopy class $\alpha \in [S^1, M]$, we
have
\[
I^F_\alpha := \inf \{ \text{len}_F(\gamma) \mid \gamma \in C^\infty(S^1, M), \, [\gamma] = \alpha \} > 0. \tag{1.1}
\]

The set of Finslerian lengths of all periodic Finslerian geodesics in a Finsler manifold \((M, F)\) representing \(\alpha\) is said to be the marked length spectrum \(\Lambda_\alpha\) of the Finsler manifold. It is closed and nowhere dense by Lemma 3.2, and hence
\[
I^F_\alpha = \inf \Lambda_\alpha \in \Lambda_\alpha.
\]

The main result of this paper is the following. It was conjectured by Kei Irie [19].

**Theorem 1.1.** Let \((M, F)\) be a closed connected smooth Finsler manifold. Let \(\alpha \in [S^1, M]\) be a nontrivial free homotopy class. Then, for any \(H \in C^\infty(S^1 \times T^* M)\) which is compactly supported in \(DF^{\bullet} T^* M\) and satisfies
\[
H(t, x, 0) \geq I^F_\alpha \quad \forall \, t \in S^1, \, \forall \, x \in M
\]
there exists \(z : S^1 \to T^* M\) with \(\dot{z}(t) = X_H(t, z(t))\) and \([z] = -\alpha\).

**Remark 1.2.** When \((M, g)\) is a Riemannian manifold, Theorem 1.1 was proved by Weber (Theorem A in [40]).

Our proof of the main theorem is based on the isomorphism between Floer homology and singular homology of the free loop space. Such an isomorphism was firstly shown by Viterbo [39] using the generating function method. After that Salamon and Weber [32] and Abbondandolo and Schwarz [3, 4] gave different constructions of this kind of isomorphisms. Instead of using the main result in [32] which is the key ingredient in the proof of [40], we apply the Abbondandolo-Schwarz isomorphism to compute BPS capacities (see Section 7). This is because in the Finsler case we can not rewrite naturally the Floer equation as a heat flow equation and estimate as in [40], while for a Finsler manifold \((M, F)\) the Hamiltonian \(H_{F^*} = F^{*2}/2\) is naturally uniformly convex and grows quadratically in the fibers. Although the non-smoothness of \(H_{F^*}\) on all \(T^* M\) in general may cause the obstruction to utilize the Abbondandolo-Schwarz isomorphism, the perturbing method used in [25] can help us to overcome this problem.

The main technical result that we obtain in this paper is the following generalization of the results of [40, Theorem 2.9] to the Finsler setting.

**Theorem 1.3** (Floer homology of convex radial Hamiltonians). Let \((M, F)\) be a closed connected smooth Finsler manifold, and let \(\alpha \in [S^1, M]\). Let \(f : [0, \infty) \to \mathbb{R}\) be a smooth function such that \(f' \geq 0, f'' \geq 0, \) and \(f = f(0)\) on \([0, \epsilon_f)\) for some constant \(\epsilon_f > 0\). Assume that \(\lambda \in (0, \infty) \setminus \Lambda_\alpha\) and \(f'(r) = \lambda\) for some \(r > \epsilon_f\). Let \(c_{f, \lambda} := r f'(r) - f(r)\). Then it holds the following
(i) There exists a natural isomorphism
\[
\Psi^\lambda_f : HF^{(-\infty, \epsilon_f, \lambda)}(f \circ F^*; \alpha) \to H^*_c(\Lambda^{\lambda^2/2}_\alpha M)
\]
where \(\Lambda^{\lambda^2/2}_\alpha M := \{ x \in C^\infty(S^1, M) \mid \int_{S^1} F^2(x(t), \dot{x}(t)) dt < \lambda^2, \, [x] = \alpha \}\).
(ii) If \(\mu \in (0, \lambda) \setminus \Lambda_\alpha\) is another slope of \(f\), then the following diagram commutes:
(iii) Let \( g \) be another such function, then there exists an isomorphism \( \Psi^\lambda_{gf} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{HF}^*\left(-\infty, c_f, \lambda \right) & \xrightarrow{\text{[I]}} & \text{HF}^*\left(-\infty, c_g, \lambda \right) \\
\Psi^\lambda_f & \cong & \Psi^\lambda_g \\
H_*(\Lambda_{\alpha}^{\lambda^2/2} M) & \xrightarrow{\sim} & H_*(\Lambda_{\alpha}^{\lambda^2/2} M)
\end{array}
\] (1.3)

The proof of the above theorem is based on the Abbondandolo-Schwarz isomorphism, which is given in Section 6.

Following [8], we define the BPS capacity. For \( c > 0 \), an open set \( W \subset T^*M \) and a compact set \( A \subset W \), denote

\[
\mathcal{K}_c(W, A) := \{ H \in C_0^\infty(S^1 \times W) \mid \sup_{S^1 \times A} H \leq -c \}.
\]

Define the BPS capacity for the pair \((W, A)\) and a nontrivial free homotopy class \( \alpha \)

\[
C_{\text{BPS}}(W, A; \alpha) := \inf \left\{ c > 0 \mid \forall H \in \mathcal{K}_c(W, A), \mathcal{P}_\alpha(H) \neq 0 \right\}.
\]

Our main theorem computes the BPS capacity for fiberwise convex sets, see also Theorem 7.3.

**Theorem 1.4.** Let \( M \) be a closed connected manifold. Let \( U \) be a fiberwise strictly convex and compact sets on \( T^*M \) with smooth boundary and containing \( M \) in its interior, and let \( F \) be the associated Finsler metric to \( U \). Then for all nontrivial free homotopy class \( \alpha \), we have

\[
C_{\text{BPS}}(U, M; \alpha) = l^F_\alpha.
\]

We remark that when \( U \) is noncompact, for properly chosen \( \alpha \), the capacity may also be estimated as in the proof of Theorem 1.8.

Our next result shows that a symplectomorphism in the identity component \( \text{Symp}_0(T^*M) \) of the symplectomorphic group \( \text{Symp}(T^*M) \) does not change the minimal length of the Finsler closed geodesic though the Finsler metric is deformed.

**Theorem 1.5.** Let \( F_1, F_2 \) be two Finsler metrics on a closed connected manifold \( M \) with nontrivial free homotopy group. If \( \psi \in \text{Symp}_0(T^*M) \) is a symplectomorphism from \( D^{F_1}T^*M \) to \( D^{F_2}T^*M \) such that \( \psi(M \times \{0\}) \) is the graph of an exact one-form on \( M \), then for every non-trivial \( \alpha \in [S^1, M] \), we have \( l^F_\alpha = l^F_\alpha \).

In the proof of Theorem 1.5, the assumption that \( \psi(M \times \{0\}) \) is a graph of an exact one-form on \( M \) is technical. It is likely that this assumption can be dropped. After this paper finished, Zhang told us that whenever \( \psi \) is a Hamiltonian symplectomorphism then the assertion in Theorem 1.5 can also be proved by combining Theorem 7.4 with the tool of persistent homology as developed in [38].
Problem 1: Let $\psi \in \text{Symp}(T^*M)$ be a symplectomorphism between the unit co-disc bundles of two Finsler metrics $F_1$ and $F_2$ on a closed connected manifold $M$ with nontrivial free homotopy group. Is it true that the manifolds $(M, F_1)$ and $(M, F_2)$ are isometric?

Here we remark that if $F_1, F_2$ are any two flat Finsler metrics on $M = T^n$, a positive answer of the above question is implied by a rigidity theorem of Benci and Sikorav [37].

The paper is organized as follows.

In Subsection 1.3, we give a number of applications of our main result. In Section 2, we introduce Floer theory for the Liouville domain. In Section 3, we introduce radial Hamiltonian systems associated to a Finsler metric and compute its action of periodic orbits. In Section 4, we introduce filtered Floer homology group and study its properties such as homotopical invariance, direct and inverse limits, etc following [8]. In Section 5, we introduce Abbondandolo-Schwarz isomorphism and adapt it to our Finsler setting by introduce a quadratic modification. In Section 6, we give the proof of Theorem 1.3. In Section 7, we compute the BPS capacities. In Section 8, we prove Theorem 1.1 and its applications in Section 1.3. In Section 9, we prove the proof of all the main technical results.

1.3. Applications. In this section, we give a number of applications of our main result.

1.3.1. Recover known results. By taking an appropriate Finsler (non-Riemannian) metric on $T^n$, Theorem 1.1 recovers the following result in [42]. Its proof (see Section 8.2), assuming Theorem 1.1, is due to Irie [19].

Theorem 1.6 ([42, Theorem 2]). Let $C$ be a closed cone in $\mathbb{R}^n$, and $C^*$ denotes its dual cone, that is, $C^* = \{v \cdot w \geq 0 \quad (\forall v \in C)\}$. Let $0 \neq \alpha \in [S^1, T^*T^n] \cong \mathbb{Z}^n \subseteq \mathbb{R}^n$. Suppose that $p^*$ belongs to the interior of the cone $C^*$ (denoted by $\text{int} C^*$), and that $c > 0$ satisfies

$$(p^*, \alpha) \leq c, \quad \forall \alpha \in C.$$  

Then, for any $H \in C_0^\infty(S^1 \times T^*T^n)$ which is supported in $S^1 \times T^n \times \text{int} C^*$ and satisfies

$$H(t, x, p^*) \geq c \quad \forall t \in S^1, \forall x \in T^n,$$

there exists $z : S^1 \to T^*T^n$ with $\dot{z}(t) = X_H(t, z(t))$ and $[z] = \alpha$.

1.3.2. Periodic orbits for Hamiltonians separating two Lagrangian submanifolds. The problem of existence of periodic orbits for Hamiltonian systems separating two Lagrangian submanifolds were studied in [16, 24, 42], etc. As an application of our main theorem, we have the following result.

Theorem 1.7. Let $M$ be a closed connected smooth manifold. Let $\sigma$ be a smooth closed one-form on $M$ whose graph does not intersect the zero section. Suppose there exists a compact set $U \subset T^*M$ with $C^\infty$-boundary, containing graph in its interior, not containing the zero section and satisfies that $U \cap T^*_xM$ is strictly convex for all $x \in M$. Then for any nontrivial free homotopy class $\alpha$, there exists a number $c(U, \alpha) > 0$ such that for any Hamiltonian $H : S^1 \times T^*M \to \mathbb{R}$ compactly supported in $U$ and satisfying $\min_{t,x} H(t, x, \sigma(x)) \geq c(U, \alpha)$, there exists a 1-periodic orbit with homotopy class $-\alpha$. 
We introduce the symplectic map $\Phi: (x, p) \mapsto (x, p - \sigma(x))$. Then apply the main theorem to the Hamiltonian $H \circ \Phi^{-1}$ and the domain $\Phi(U)$. The constant $c(U, \alpha)$ equals to $l_{\alpha}^F$ where $F$ is the Finsler metric associated to the fiberwise convex set $\Phi U$.

In case when $\sigma$ is non closed, it represents a special class of magnetic Hamiltonian systems in the following way. Given a manifold $M$, we endow its cotangent bundle with the twisted symplectic form $\omega_{\sigma} = \omega_0 + d\pi^*\sigma$, where $\omega_0$ is the standard symplectic form and $\sigma$ is a $C^\infty$ non closed 1-form. Given a Hamiltonian $H: T^*M \to \mathbb{R}$, the Hamiltonian flow determined by the twisted symplectic form is the same as the Hamiltonian flow of the Hamiltonian $H(x, p - \sigma(x)) : T^*M \to \mathbb{R}$ with the standard symplectic form. Existences of periodic orbits for this kind of magnetic systems are studied in [15].

Theorem 1.7 partially generalizes [42, Theorem 2]. However, there is a special feature in Theorem 2 of [42] which is not generalized by Theorem 1.7. Indeed, the domain $C^*$ in Theorem 1.6 is noncompact while the BPS capacity $(\alpha, p)^*$ is bounded. To recover this feature, we have the following result, which is inspired by Irie [19].

**Theorem 1.8.** Let $M$ be a closed connected $C^\infty$-manifold. Let $K_0 \subset K_1 \subset \ldots$ be a sequence of compact fibrewise strictly convex sets with $C^\infty$-boundaries in $T^*M$ and $F_0, F_1, \ldots$ be the associated Finsler metrics. Let $\alpha \in [S^1, M]$ be a nontrivial free homotopy class. Suppose that $\{K_i\}, \{F_i\}$ and $\alpha$ satisfy the following property:

1. $K_0$ contains a neighborhood of the zero section;
2. There exists a compact set $A \subset T^*M$ such that the Legendre transform of the lift of length minimizing closed $F_i$-geodesics in class $\alpha$ to $TM$ lies in $A$ for all $i = 0, 1, \ldots$

Then there exists a constant $c > 0$ such that for any $C^\infty$-Hamiltonian $H : S^1 \times T^*M \to \mathbb{R}$ with compact support in $K = \lim K_i$, satisfying $\min_{t,x} H(t, x, 0) \geq c$, there exists a 1-periodic orbit representing $-\alpha$.

**1.3.3. Periodic orbits for higher dimensional Minkowskian systems.** We next generalize Theorem 5 of [42] to higher dimensional cases.

Consider the Lorentzian Hamiltonian system $H: (T^*\mathbb{R}^n, \omega_0) \to \mathbb{R}$, $n \geq 2$, via

$$H(q, p) = \frac{1}{2}(p_1^2 - (p_2^2 + \ldots + p_n^2)) + V(q),$$

where $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$. We normalize $V$ such that $\max V = 0$.

We introduce the cone $C^* := \{-p_1^2 + p_2^2 + \ldots + p_n^2 < 0, \ p_1 > 0\} \subset \mathbb{R}^n$ and its dual cone

$$C := \{v \in \mathbb{R}^n | \langle v, p \rangle > 0, \ \forall \ p \in C^*\}.$$

**Theorem 1.9.** Let $H$ and $C$ be as above. Then for any homology class $\alpha \in C \cap H_1(\mathbb{T}^n, \mathbb{Z})$, there exists a dense subset $S_\alpha \subset (0, \infty)$ such that on each energy level in $S_\alpha$, the Hamiltonian system $H$ admits a periodic orbit in the homology class $\alpha$.

**Remark 1.10.** We consider here only Minkowski type kinetic energy with signature $(1, -1, \ldots, -1)$. The same result holds for the signature $(-1, 1, \ldots, 1)$ by setting $H \mapsto -H$. The main reason is that the cone $C^*$ is convex. For other signatures, we do not have this property.

**1.3.4. A conjecture of Kawasaki.** The following result partially confirms Conjecture 1.3 of [22].

**Theorem 1.11.** Given a homology class $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_1(\mathbb{T}^n, \mathbb{Z}) \setminus \{0\}$, for any Hamiltonian function $H : S^1 \times T^*\mathbb{T}^n \to \mathbb{R}$ with support in $\mathbb{T}^n \times (\bigcup_{i=1}^{n}(-R_i, R_i))$ and satisfying
that
\[ \min_{q,t} H(t, q, 0) \geq \sum_{i=1}^{n} R_i |\alpha_i|, \]
there exists a 1-periodic orbit in the homology class \( \alpha. \)

1.3.5. **Symplectic squeezing vs. nonsqueezing on \( T^*\mathbb{T}^n.** Symplectic nonsqueezing problems on \( T^*\mathbb{T}^n \) were studied by many authors, c.f. [37, 26] etc. In this section, we show that BPS capacity provides obstructions for symplectic embeddings on \( T^*\mathbb{T}^n \). We have the flexibility to choose the fiber convex domain as well as the free homotopy type to compute the BPS capacity. We first state two nonsqueezing type results explaining the way of applying BPS capacity to the nonsqueezing problem. We do not claim the originality of the results since they can be also proved using Sikarov’s rigidity theorem. After that, we show that the BPS capacity may be infinity simultaneously and provide no obstruction for the symplectic embedding problem. We do not claim the originality of the results since they can be also proved using Sikarov’s rigidity theorem. After that, we show that the BPS capacity may be infinity simultaneously and provide no obstruction for the symplectic embedding problem.

Let \( \Delta^n(r) \) denote the interior of the \( n \)-dimensional simplex with the \( n + 1 \) vertices \((0,\ldots,0), (r,0,\ldots),\ldots,(0,\ldots,0,r)\). Denote open subsets \( B^n, Z^n \subset \mathbb{R}^n \) and \( P^{2n}, Y^{2n} \subset (T^*\mathbb{T}^n, \omega_0) \):
\[
\begin{align*}
B^n(r) &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq r\}, \\
Z^n(r) &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 \leq r\}, \\
P^{2n}(r) &:= \mathbb{T}^n \times \Delta^n(r), \\
Y^{2n}(r) &:= \mathbb{T}^n \times (0,r) \times (\mathbb{R}^+)^{n-1},
\end{align*}
\]
where \( r \in \mathbb{R}^+ := (0, \infty) \) and \( T^*\mathbb{T}^n \) is naturally symplectically identified to \( \mathbb{T}^n \times \mathbb{R}^n \).

Let \( U, V \) be two open subset of \( T^*M \). Recall that a symplectic embedding \( \psi : U \to V \) is called \( \tilde{\pi}_1(M) \)-trivial if \( \psi_* \alpha = \alpha \) for any \( \alpha \in \pi_1([S^1, M]) \).

**Theorem 1.12.** There is a \( \tilde{\pi}_1(M) \)-trivial symplectic embedding \( \phi : P^{2n}(s) \to Y^{2n}(r) \) such that for every \( u \in \Delta^n(r), \phi(\mathbb{T}^n \times \{u\}) \) is a \( C^\infty \) section in \( T^*\mathbb{T}^n \) if and only if \( s \leq r \).

Under a weaker topological condition, namely, the induced map \( \phi_* : H_*(P^{2n}(s); \mathbb{Z}) \to H_*(Y^{2n}(r); \mathbb{Z}) \) is an isomorphism, Maley, Mastrangeli and Traynor [26] proved the above nonsqueezing theorem for any symplectic embedding \( \phi : P^{2n}(s) \to Y^{2n}(r) \) and used it to study symplectic packing problems.

**Theorem 1.13.** There is a \( \tilde{\pi}_1(M) \)-trivial symplectic embedding \( \phi : \mathbb{T}^n \times B^n(s) \to \mathbb{T}^n \times Z^n(r) \) such that for a sequence \( \{u_i\} \subseteq B^n(s) \) with \( \lim u_i = (s,0,\ldots,0) \) every \( \phi(\mathbb{T}^n \times \{u_i\}) \) is a \( C^\infty \) section in \( T^*\mathbb{T}^n \) if and only if \( s \leq r \).

In general we have infinitely many choices for the BPS capacity by varying the homotopy type. However, they may be infinity simultaneously and provide no obstruction for the symplectic embedding.

Let \( v \) be a unit vector in \( \mathbb{R}^n \), we define the following tilted cylinder
\[
Y^{2n}(r,v) := \mathbb{T}^n \times (-r,r) v \times v^\perp.
\]

We have the following theorem.

**Theorem 1.14.** Let \( v \) be a unit vector in \( \mathbb{R}^n \). Then it holds the following.
(1) If \( v \) is the scalar multiple of an integer vector \( \alpha \in \mathbb{Z}^n \setminus \{0\} \). Then
\[
C_{\text{BPS}}(Y^{2n}(r,v), \mathbb{T}^n, \pm \alpha) = r \|\alpha\|.
\]
And for all \( \beta \in \mathbb{Z}^n \setminus \text{span}\{\alpha\} \) and all \( r > 0 \), we have
\[
C_{\text{BPS}}(Y^{2n}(r,v), \mathbb{T}^n, \beta) = \infty.
\]
(2) If \( v \) is not a scalar multiple of any integer vector. Then for all \( \alpha \in H_1(\mathbb{T}^n, \mathbb{Z}) \setminus \{0\} \) and all \( r > 0 \), we have
\[
C_{\text{BPS}}(Y^{2n}(r,v), \mathbb{T}^n, \alpha) = \infty.
\]
We remark that similar statements can be formulated for \( Y \) of the form \( \mathbb{T}^n \times D^k \times (D^k)^\perp \), where \( D^k \) is a disk of dimension \( k \) lying in a \( k \)-dimensional plane in \( \mathbb{R}^n \). This theorem also explains the necessity of assumption 2 of Theorem 1.8.

In the case of infinite capacity for all classes, squeezing may occur. We illustrate the squeezing by the following example.

**Example:**

Let \( A := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \) and the symplectic map \( \Phi_A : T^*\mathbb{T}^2 \to T^*\mathbb{T}^2 \) via \((x, y) \mapsto (Ax, A^{-1}y)\).

We next denote by \( v \) the eigenvector associated to the smaller eigenvalue of \( A \). Then introduce the cylinder \( Y^4(r,v) \). Let \( U \) be any bounded domain in \( \mathbb{R}^2 \), then the image of \( U \) under \( A^{-n} \) is expanded along \( v \) and contracted along \( v^\perp \). So for any \( r > 0 \), there exists \( N \) such that \( \Phi^n(U) \subset Y^4(r,v) \) for all \( n > N \).

Note that embeddings produced in this way can only be induced by a matrix \( A \in \text{PSL}_2(\mathbb{Z}) \) with \( \text{tr}(A) > 2 \). Its eigenvalue has to be an algebraic number solving \( x^2 - \text{tr}(A)x + 1 = 0 \).

The vector \( v \) has to be a scalar multiple of a vector in \( \mathbb{Z}^2[\lambda] \) where \( \lambda \) solves the above quadratic equation.

We are naturally led to the following question.

**Problem 2:** Let \( v \) be an irrational vector but not a scalar multiple of a vector in \( \mathbb{Z}^2[\lambda] \) where \( \lambda \) solves the above quadratic equation. Is it true that for all \( r > 0 \), there exists a symplectic map \( \Phi \) on \( T^*\mathbb{T}^2 \) such that \( \Phi(P^4(1)) \subset Y^4(r,v) \)?

Since we have the capacity \( C_{\text{BPS}}(Y^4(r,v), \mathbb{T}^2, \alpha) = \infty \) for all \( \alpha \in \mathbb{Z}^2 \setminus \{0\} \), there is no obstruction provided by the BPS capacity.

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2. FLOER HOMOLOGY ON A LIOUVILLE DOMAIN \((X, \lambda)\)
2.1. Basic definitions and convexity results. Let \((X, \lambda)\) be a Liouville domain, meaning that \(X\) is a 2n-dimensional compact manifold with boundary \(\partial X\). \(\lambda\) is a 1-form on \(X\) such that \(d\lambda\) is a symplectic form on \(X\) and \(\lambda \wedge (d\lambda)^{n-1} > 0\) on \(\partial X\). There exists a vector field on \(X\), called the Liouville vector field \(Z\) of \((X, \lambda)\), which points transversely outward at \(\partial X\) and satisfies
\[
\mathcal{L}_Z d\lambda = d\lambda.
\]
Then \((\partial X, \theta := \lambda|_{\partial X})\) is a contact manifold. The Reeb vector field \(R\) is defined by \(d\theta(R, \cdot) = 0\) and \(\theta(R) = 1\). The action spectrum
\[
\text{Spec}(X, \lambda) := \left\{ \int_{\gamma} \lambda \mid \gamma \text{ is a periodic Reeb orbit of } R \right\},
\]
is closed and nowhere dense in \(\mathbb{R}\). Moreover, \(\lambda \wedge (d\lambda)^{n-1} > 0\) implies \(\text{Spec}(X, \lambda) \subset (0, \infty)\).

The vector field \(Z\) gives rise to an embedding \(\phi : [0, 1] \times \partial X \to X\) satisfying
\[
\phi(1, z) = z \quad \text{and} \quad \rho \partial_{\rho} \phi(\rho, z) = Z(\phi(\rho, z)).
\]
It is easy to verify that \(\phi^* \lambda = \rho \phi\), and thus \(\phi^* d\lambda = d(\rho \phi)\). So a neighborhood of \(\partial X\) in \(X\) can be symplectically identified with the symplectic manifold \(((0, 1] \times \partial X, d(\rho \phi))\). By attaching a cylindrical end to \([0, 1] \times \partial X\) we obtain a completion of \((X, \lambda)\) defined by
\[
(X, \hat{\lambda}) := (X, \lambda) \cup_{\partial X} ([1, \infty) \times \partial X, \rho \phi).
\]
Obviously, \((\hat{X}, \hat{\lambda})\) is an open symplectic manifold. Let \(H_t\) be a smooth time-dependent Hamiltonian on \(\hat{X}\). The Hamiltonian vector field \(X_H\) associated to \(H\) is defined by \(-dH_t = d\hat{\lambda}(X_H, \cdot)\). Denote by \(\phi^t_{H_t}\) the flow of \(X_{H_t}\). Let \(\tilde{Z}\) be the extended Liouville vector field of \(Z\) satisfying \(\tilde{Z} = Z\) on \(X\) and \(\tilde{Z} = \rho \partial \rho\) on \([0, \infty] \times \partial X\). \(\tilde{Z}\) is complete since its flow exists for all times.

Let \(J_t\) be the \(t\)-dependent 1-periodic smooth \(d\hat{\lambda}\)-compatible almost complex structure on \(\hat{X}\), that is, \(\langle \cdot, \cdot \rangle := d\hat{\lambda}(J_t \cdot, \cdot)\) is a loop of Riemannian metrics on \(\hat{X}\) and \(J^2 = -I\). The corresponding norm is defined as \(| \cdot |_{J_t} := \sqrt{\langle \cdot, \cdot \rangle}\), and the set consisting of all such \(J_t\) is denoted by \(\mathcal{J}\). We call an almost complex structure \(J_t \in \mathcal{J}\) of contact type on \([\rho_0, \infty) \times \partial X\) for some \(\rho_0 > 0\) if
\[
d\rho \circ J_t = \hat{\lambda} \quad \text{on } [\rho_0, \infty) \times \partial X.
\]
Equivalently, \(J_t\) preserves the symplectic splitting
\[
T_{(\rho, z)} \hat{X} = \ker \lambda(z) \oplus \mathbb{R} R(z) \oplus \mathbb{R} \tilde{Z}(\rho, z) \quad \forall (\rho, z) \in [\rho_0, \infty) \times \partial X
\]
and
\[
J_t R = \tilde{Z}, \quad J_t \tilde{Z} = -R. \tag{2.1}
\]
Denote by \(\mathcal{J}(\hat{X}, \hat{\lambda})\) the subset of \(\mathcal{J}\) which consists of almost complex structures of contact type at infinity.

Disk cotangent bundles of a closed Finsler manifold are examples of Liouville domains which we are mainly interested in throughout this paper. For more examples of Liouville domains, we refer the reader to the survey article [35] by Seidel.

Example 2.1. Let \(M\) be a closed \(C^\infty\) manifold with a Finsler metric \(F\), which induces a co-Finsler metric \(F^*\) defined on the cotangent bundle \(T^* M\) (see Subsection 1.1). Denote by \(W := DF^* T^* M\) the unit cotangent disk bundle. The restriction \(\theta = \lambda_0|_{\partial W}\) of the standard Liouville form \(\lambda_0 := pdz\) to the boundary \(\partial W\) is a contact form on \(\partial W\). So \((W, \lambda_0)\) is a Liouville domain. The standard Liouville vector field \(Z = p \frac{\partial}{\partial p}\) is transverse to \(\partial W\). The flow \(\varphi\) of \(Z\) induces the
diffeomorphism
\[(0, \infty) \times \partial W \to T^* M \setminus \{0\}, \quad (\rho, z) \to \varphi_{\log \rho}(z)\]
which has the inverse
\[T^* M \setminus \{0\} \to (0, \infty) \times \partial W, \quad z \to \left( F^*(z), \frac{z}{F^*(z)} \right).\]
Then \(T^* M\) is naturally identified with its completion
\[\hat{W} := W \cup_{\partial W} [1, \infty) \times \partial W.\]
Under this identification, on \(T^* M \setminus D^*_F T^* M \cong (r, \infty) \times \partial W\) (see (3.1) for the definition of \(D^*_F T^* M\)), a function \(f \in C^\infty(T^* M, \mathbb{R})\) can be written as \(f(\rho, z)\) with respect to the variables \(\rho\) and \(z\).

To define Floer homology on \((\hat{X}, d\hat{\lambda})\), we will need the \(C^0\)-bounds for solutions \(u : \mathbb{R} \times S^1 \to \hat{X}\) to the \(s\)-dependent Floer equation
\[\partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{s,t}}(u)) = 0. \quad (2.2)\]
Based on the method of the maximum principle, various \(C^0\)-bounds for the solutions of Floer equations given by the almost complex structures of contact type are established in the literature, see, for instance, [4, 6, 20, 35, 39]. For our purpose we will show the following convexity results for those Hamiltonians which are constant, linear and/or superlinear (with respect to the \(\rho\)-variable) at infinity in a unified way.

**Lemma 2.2.** Let \((X, \lambda)\) be a Liouville domain. Let \(\{H_{s,t}\}_{s,t \in \mathbb{R} \times S^1}\) be a smooth family of Hamiltonians on \(\hat{X}\), and \(\{J_{s,t}\}_{s,t \in \mathbb{R} \times S^1}\) be a smooth family of elements in \(\mathcal{J}\). Assume that there exists a constant \(\rho_0 > 0\) and two functions \(f, g \in C^\infty(\mathbb{R})\) such that for every \((s, t) \in \mathbb{R} \times S^1\),
- \(H_{s,t}(\rho, z) = f(s)\rho^\mu + g(s)\) on \([\rho_0, \infty) \times \partial X\), where \(\mu \geq 1\) and \(f'(s) \geq 0, \) or \(\mu = 0\) without any restriction on \(f\) and \(g\).
- \(J_{s,t}\) is of contact type on \([\rho_0, \infty) \times \partial X\).
If \(u : \mathbb{R} \times S^1 \to \hat{X}\) satisfies (2.2) and \(u^{-1}([\rho_1, \infty) \times \partial X]\) is bounded for some \(\rho_1 > \rho_0\), then \(u(\mathbb{R} \times S^1) \subset \hat{X} \setminus ((\rho_1, \infty) \times \partial X)\).

This lemma is proved in Section 9.1.

**Remark 2.3.** Suppose that the Hamiltonians \(H\) are radial outside a compact set of \((\hat{X}, d\hat{\lambda})\), that is,
\[H(\rho, z) = h(\rho), \quad \forall (\rho, z) \in [\rho_0, \infty) \times \partial X\]
for some \(\rho_0 > 0\), where \(h : [\rho_0, \infty) \to \mathbb{R}\) is a smooth function. The Hamiltonian vector field \(X_H\) has the form
\[X_H(\rho, z) = h'(\rho)R(z) \quad \text{on} \quad [\rho_0, \infty) \times \partial X.\]
Thus a \(T\)-periodic Reeb orbit of \(R\) gives rise to a 1-periodic orbit of \(X_H\) in \([r_0] \times \partial X\) if and only if \(T = |h'(r_0)|\), and conversely every 1-periodic orbit in \([r_0] \times \partial X\) for \(r_0 \geq \rho_0\) is given by a \(T\)-periodic orbit of \(R\) with \(T = |h'(r_0)|\).

### 2.2. Floer homology of admissible Hamiltonians
For \(H \in C^\infty(S^1 \times \hat{X})\), we denote \(\mathcal{P}(H) \subseteq \hat{X}\) as the set of all 1-periodic orbits of \(X_{H}\):
\[\mathcal{P}(H) = \{ z \in C^\infty(S^1, \hat{X}) | \dot{z}(t) = X_{H}(z(t)) \}.\]
Define the set of *admissible Hamiltonians* \( \mathcal{H}_{\text{ad}} \) to consist of all smooth \( H : S^1 \times \hat{X} \) with the following properties:

- \( H \) is linear at infinity, meaning that there exist \( \rho_0 > 0 \), \( \mu_H > 0 \) and \( a_H \in \mathbb{R} \) such that
  \[
  H_t(\rho, z) = \mu_H \rho + a_H \quad \text{on } [\rho_0, \infty) \times \partial X
  \]
  for all \( t \in S^1 \). Here \( \mu_H \) is required to satisfy \( \mu_H \notin \text{Spec}(X, \lambda) \).
- All elements \( z \in \mathcal{P}(H) \) are non-degenerate, that is, the linear map
  \[
  d\phi^1_{H}(z(0)) : T_{z(0)} \hat{X} \to T_{z(0)} \hat{X}
  \]
  does not have 1 as an eigenvalue.

Observe that since \( \mu_H \notin \text{Spec}(X, \lambda) \), the linear behavior of \( H_t := H(t, \cdot) \) means that there are no 1-periodic orbits in \([\rho_0, \infty) \times \partial X\), hence there are finitely many in total.

For a free homotopy class \( \alpha \in [S^1, \hat{X}] \), denote \( \Lambda_{\alpha} \hat{X} := \{ z \in C^\infty(S^1, \hat{X}) | [z] = \alpha \} \) and \( \mathcal{P}_\alpha(H) := \mathcal{P}(H) \cap \Lambda_{\alpha} \hat{X} \). Note that if \( \alpha \neq 0 \), there is no canonical way to assign \( \mathbb{Z} \)-valued Conley-Zehnder index to elements \( z \in \mathcal{P}_\alpha(H) \). However, for \((\hat{X}, d\lambda) = (T^*M, dp \wedge dx)\), one can follow Abbondandolo and Schwarz’s line to define \( \mathbb{Z} \)-valued Conley-Zehnder index, cf. [3]. In this paper we only work on \((X, \lambda) = (DFT^*M, pdx)\), and so let us assume that the \( \mathbb{Z} \)-valued Conley-Zehnder index of noncontractible Hamiltonian periodic orbits is well-defined.

The *Floer action functional* \( \mathcal{A}_H : \Lambda_{\alpha} \hat{X} \to \mathbb{R} \) is defined by

\[
\mathcal{A}_H(z) = \int_{S^1} z^* \lambda - \int_0^1 H(t, z) \, dt.
\]

It is easy to check that a loop \( z \in \mathcal{P}_\alpha(H) \) if and only if \( z \) is a critical point of \( \mathcal{A}_H \) on \( \Lambda_{\alpha} \hat{X} \). The set of values of \( \mathcal{A}_H \) on \( \mathcal{P}_\alpha(H) \) is called the *action spectrum* with respect to \( \alpha \), and we denote it by

\[
\text{Spec}(H; \alpha) := \{ \mathcal{A}_H(x) | x \in \mathcal{P}_\alpha(H) \}.
\]

For every \( k \in \mathbb{Z} \), we define the *Floer chain group* \( \text{CF}_k(H; \alpha) \) to be the free \( \mathbb{Z}_2 \)-module generated by the elements \( z \) of \( \mathcal{P}_\alpha(H) \) with Conley-Zehnder index \( \mu_{\text{CZ}}(z) = k \).

Assume that \( H \in \mathcal{H}_{\text{ad}} \) and \( J \in \mathcal{J}(\hat{X}, \hat{\lambda}) \). Given critical points \( z_\pm \in \mathcal{P}_\alpha(H) \), denote by

\[
\hat{\mathcal{M}}_\alpha(z_-, z_+, H, J) \subseteq C^\infty(\mathbb{R} \times S^1, \hat{X})
\]

the set of smooth maps \( u : \mathbb{R} \times S^1 \to \hat{X} \) that satisfy the Floer equation

\[
\partial_s u + J(u)(\partial_t u - X_H(u)) = 0
\]

and the asymptotic conditions

\[
\lim_{s \to -\infty} u(s, t) = z_- \quad \text{and} \quad \lim_{s \to +\infty} u(s, t) = z_+
\]

uniformly in \( t \in S^1 \). Note that each moduli space \( \hat{\mathcal{M}}_\alpha(z_-, z_+, H, J) \) carries a free \( \mathbb{R} \)-action given by \((\tau \cdot u)(s, t) := u(s - \tau, t)\), denote by \( \mathcal{M}_\alpha(z_-, z_+, H, J) \) the quotient space under this action. For generic \( J \in \mathcal{J}(\hat{X}, \hat{\lambda}) \), \( \mathcal{M}_\alpha(z_-, z_+, H, J) \) carries a smooth manifold structure with dimension \( \mu_{\text{CZ}}(z_-) - \mu_{\text{CZ}}(z_+) - 1 \). Such \( J \) is called a regular almost complex structure for \( H \).

If \( \mu_{\text{CZ}}(z_-) = \mu_{\text{CZ}}(z_+) + 1 \), \( \mathcal{M}_\alpha(z_-, z_+, H, J) \) is compact, and hence consists of finitely many points. The boundary operator

\[
\partial_k = \partial_k(H, J) : \text{CF}_k(H; \alpha) \to \text{CF}_{k-1}(H; \alpha)
\]
is defined by
\[
\partial_k(z_-) = \sum_{z_+ \in \mathcal{P}_\alpha(H)_{k-1}} n(z_-, z_+) z_+, \quad x \in \mathcal{P}_\alpha(H)_k
\]
with \(n(z_-, z_+) := \sharp_2(\mathcal{M}_\alpha(z_-, z_+, H, J))\). The standard transversality and gluing arguments (see e.g. [31]), combining with a \(C^0\)-bounds for solutions of the Floer equation (see Lemma 2.2) show that
\[
\partial_{k-1} \circ \partial_k = 0.
\]
So \(\{\text{CF}_*(H; \alpha), \partial_*(H, J)\}\) is a chain complex, called the Floer chain complex. The Floer homology \(\text{HF}_*(H, J; \alpha)\) is defined to be the homology of such a chain complex. Moreover, the homology group of \(\{\text{CF}_*(H; \alpha), \partial_*(H, J)\}\) does not depend on \(J\), and we denote it by \(\text{HF}_*(H; \alpha)\).

3. RADIAL HAMILTONIAN SYSTEMS RELATED TO FINSLER STRUCTURES

Let \((M, F)\) be a smooth \(n\)-dimensional Finsler manifold. The Euler theorem implies \(F^2(x, y) = g^F(x, y)[y, y]\). Note that \(F^2\) is only \(C^1\) on the zero section of \(TM\). In a local coordinate centered at \(x \in M\), the Finsler metric can be represented by
\[
g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y) \quad \forall \ y \in T_x M \setminus \{0\}.
\]
For \(r \in (0, \infty)\), denote the Finsler cotangent disk \(D^F_rT^*M\) with radius \(r\) by
\[
D^F_rT^*M := \{(x, p) \in T^*M | F^*(x, p) \leq r\}.
\]
Let \((g^{ij})\) be the inverse matrix of the matrix \((g_{ij})\). \(\nabla^F\) denotes the Chern connection associated to the Finsler metric on \(M\), see [34].

**Definition 3.1.** A smooth curve \(c : (a, b) \to M\) in a Finsler manifold \(M\) is called a \(F\)-geodesic if \(\text{len}_F\) is stationary at \(c\), that is, for any variation \(c_s(t)\) of \(c\) it holds that
\[
\frac{d}{ds} \text{len}_F(c_s) |_{s=0} = 0.
\]
Here \(c_s(\cdot) = \tau(s, t)\), and \(\tau : (-\varepsilon, \varepsilon) \times [a, b] \to M\) is a smooth function such that \(\tau_0(t) = c(t)\) for all \(t \in (a, b)\), and \(c_s(a), c_s(b)\) are constants not depending on \(s \in (-\varepsilon, \varepsilon)\).

**Lemma 3.2.** Let \(M\) be a closed connected Finsler manifold, and let \(\alpha \in [S^1, M]\) be a free homotopy class in \(M\). Then the marked length spectrum \(\Lambda_\alpha\) is a closed and nowhere dense subset of \(\mathbb{R}\).

The proof of the above lemma is analogous to the Riemannian case, see, e.g. [40, Lemma 3.3], and thus we omit it here.

The geometries \((T^*M, F^*)\) and \((TM, F)\) are related by the Legendre transforms \(\ell\).

**Definition 3.3.** The Legendre transform \(\ell_x : T_x M \to T^*_x M\) is defined as
\[
(\ell_x(v))_i = g_{ij}(x, v) v^j \quad \forall \ v \in T_x M \setminus \{0\} \quad \text{and} \quad \ell_x(0) = 0;
\]
and the Legendre transform \(\ell_x^* : T^*_x M \to T_x M\) is given by
\[
(\ell_x^*(p))^i = g^{ij}(x, p) p_j \quad \forall \ p \in T^*_x M \setminus \{0\} \quad \text{and} \quad \ell_x^*(0) = 0.
\]

**Lemma 3.4.** Let \(h : [0, \infty) \to \mathbb{R}\) be a smooth function being constant near 0. Let \(H\) be a \(C^\infty\)-radial function on the cotangent bundle \(T^*M\) which is given by
\[
H(x, y) = f(F^*(x, y)) = h(F^{*2}(x, y)),
\]
where \( f(x) = h(x^2) \) is a smooth function on \([0, \infty)\). Assume that \( z(t) = (x(t), y(t)) \) is a smooth loop on \( T^*M \). Then \( z \) is a critical point of the action functional \( \mathcal{A}_H \) if and only if \( z(t) \) is a \( F \)-geodesic loop with constant speed satisfying

\[
f'(r_z)y(t) = r_z \xi_x(\dot{x})(t), \quad F(x(t), \dot{x}(t)) \equiv \pm f'(r_z) \quad \forall t \in S^1
\]

for some constant \( r_z = F^*(z(t)) \). Then, in this case the action of \( \mathcal{A}_H \) at \( z(t) \) is given by

\[
\mathcal{A}_H(z(t)) = f'(F^*(z(t)))F^*(z(t)) - f(F^*(z(t))).
\]

The proof is postponed to Section 9.2.

4. Filtered Floer homology of cotangent bundles

Let \((M, F)\) be a closed connected Finsler manifold. In this section we introduce a filtration on the Floer chain complex via the action for a class of time-dependent 1-periodic Hamiltonians defined on the cotangent bundle \( T^*M \), which can be viewed as the completion of the Liouville domain \((D^F T^*M, \lambda)\) (see Example 2.1).

4.1. Radial and linear Hamiltonians at infinity. Fix a free homology class \( \alpha \in [S^1, T^*M] \). Let \( a, b \in \mathbb{R} \cup \{\pm \infty\} \) with \( a < b \). For \( \eta > 0 \), define

\[
\mathcal{H}_{\eta;\alpha}^{a,b} := \{ H \in C^\infty(S^1 \times T^*M) \mid \exists \tau \geq 0 \exists c \in \mathbb{R} \text{ such that } H_t(x, p) = \tau F^*(x, p) - c \text{ for } F^*(x, p) \geq \eta, \{a, b\} \cap \text{Spec}(H; \alpha) = \emptyset, \text{ and } \tau \notin \Lambda_\alpha \text{ or } c \notin [a, b] \}.
\]

Let us give some explanations about the above definition:

1. the condition that \( a, b \) are not in the action spectrum allows for small perturbations of the Hamiltonian without changing Floer homology;
2. if \( \tau \in \Lambda_\alpha \), by Lemma 3.4, such a Hamiltonian admits degenerate 1-periodic orbits of action \( c \) on all hypersurfaces \( \{(x, p) \in T^*M | F^*(x, p) = \tau\} \) with \( \tau \geq \eta \); if \( \tau = 0 \), every point in the complement of \( D_\eta^F T^*M \) is a degenerate 1-periodic orbit with action \( c \); the condition \( c \notin [a, b] \) excludes all these degenerate orbits;
3. if \( \tau \notin \Lambda_\alpha \) and \( \tau > 0 \), by Lemma 3.4 again, there are no Hamiltonian 1-periodic orbits on \( T^*M \setminus D_\eta^F T^*M \), so from Remark 2.3 one can see that \( \tau \notin \text{Spec}(D^F T^*M, \lambda) \).

For each \( H \in \mathcal{H}_{\eta;\alpha}^{a,b} \), denote

\[
\mathcal{P}_\alpha^{(a,b)}(H) := \{ z \in \mathcal{P}_\alpha(H) \mid \mathcal{A}_H(z) \in (a, b) \}.
\]

Observe that one can perturb \( H \) along the periodic orbits \( z \in \mathcal{P}_\alpha^{(a,b)}(H) \) by a smooth function \( h \in C^\infty_c(S^1 \times D_\eta^F T^*M) \) with sufficiently small \( \| \cdot \|_{C^2} \)-norm such that \( H + h \in \mathcal{H}_{\eta;\alpha}^{a,b} \) and all elements in \( \mathcal{P}_\alpha^{(a,b)}(H + h) \) are nondegenerate, see, [31, Section 9] or [40].

4.2. The definition of filtered Floer homology. Assume that \( H \in \mathcal{H}_{\eta;\alpha}^{a,b} \) is nondegenerate. Consider the \( \mathbb{Z}_2 \)-vector space

\[
CF^{(a,b)}(H, \alpha) := \bigoplus_{x \in \mathcal{P}_\alpha^{(a,b)}(H)} \mathbb{Z}_2 x
\]

which is graded by the index \( \mu_{CZ} \). Here we use the convention that the complex generated by the empty set is zero.
Given $z_\pm \in \text{CF}^{(a,b)}(H; \alpha)$, every solution $u$ of $(2.4)$ connecting $z_-$ to $z_+$ satisfies
\[
\mathcal{A}_H(z_-) - \mathcal{A}_H(z_+) = \int_{\mathbb{R} \times S^1} |\partial_s u(s,t)|^2 ds dt \geq 0.
\]
Similar to Section 2.2, if $\mu C^*Z(z_-) = \mu C^*Z(z_+) + 1$, then for generic $J \in \mathcal{J}(T^*M, \lambda_0)$ the moduli space $\mathcal{M}_\alpha(z_-, z_+, H, J)$, in consideration of the fact that $\mathcal{A}_H(z_-) \geq \mathcal{A}_H(z_+)$ if it is not empty, gives rise to the boundary operator
\[
\partial_* : \text{CF}^{(a,b)}(H; \alpha) \to \text{CF}^{(a,b)}_{s-1}(H; \alpha)
\]
satisfying $\partial_{s-1} \partial_* = 0$. So $(\text{CF}^{(a,b)}(H; \alpha), \partial_*)$ is a chain complex. The above claims are proved by the usual transversality and gluing arguments, together with the $C^0$-estimate given by Lemma 2.2. Its homology group $\text{HF}^{(a,b)}_*(H; \alpha)$ is called the filtered Floer homology of $H \in \mathcal{H}^{a,b}_{\eta; \alpha}$.

4.3. Homotopy invariance. Suppose that Hamiltonians $H^\pm \in \mathcal{H}^{a,b}_{\eta; \alpha}$ are nondegenerate. Then $H^\pm = \tau \pm F^*(x, p) - c_\pm$ on $T^*M \setminus D^F_T T^*M$ for some constants $\tau_+ \geq \tau_- \geq 0$ and $c_\pm \in \mathbb{R}$. Let $\beta : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that $\beta = 0$ for $s \leq 0$, $\beta(s) = 1$ for $s \geq 1$ and $0 \leq \beta(s) \leq 1$. Let $H_s = \{H_{s,t}\}$ be a smooth homotopy from $\mathbb{R}$ to $\mathcal{H}^{a,b}_{\eta; \alpha}$ defined by
\[
H_{s,t} := (1 - \beta(s))H^-_t + \beta(s)H^+_t.
\]
Identifying $T^*M \setminus D^F_T T^*M$ with $(\eta, \infty) \times DT^*M$ yields for any $(\rho, z) \in (\eta, \infty) \times DT^*M$, $H_{s,t}(\rho, z) := H_{s,t}(x, p) = \beta(s)(\tau_+ - \tau_-)\rho + \tau_+ \rho + \beta(s)(c_- - c_+) - c_-$ with $\rho = F^*(x, p)$ and $z = (x, p/F^*(x, p)) \in SF^*T^*M := \partial DF^*T^*M$.

Given $z_\pm \in \mathcal{P}_\alpha(H^\pm)$, consider the parameter-dependent Floer equation
\[
\partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{s,t}}(u)) = 0 \quad (4.1)
\]
which satisfies uniformly in $t \in S^1$ the asymptotic boundary conditions
\[
\lim_{s \to \pm \infty} u(s,t) = z_\pm \quad \text{and} \quad \lim_{s \to \pm \infty} \partial_s u(s,t) = 0 \quad (4.2)
\]
Here $J_{s,t} : \mathbb{R} \to \mathcal{J}$ is a regular homotopy of smooth families of $d\lambda_0$-compactible almost complex structure on $T^*M$ satisfying

- $J_{s,t}$ is of contact type on $(\eta, \infty) \times DT^*M$.
- $J_{s,t} = J^-_t$ is regular for $H^-_t$ for $s \leq 0$.
- $J_{s,t} = J^+_t$ is regular for $H^+_t$ for $s \geq 1$.
- The linearized operator for equation (4.1) is surjective for each finite-energy solution of (4.1) in the homotopy class $\alpha$.

By our assumption,
\[
\frac{\partial^2 H_{s,t}}{\partial s \partial \rho} = \beta(s)(\tau_+ - \tau_-) \geq 0
\]
thus, combining Lemma 2.2 with the exactness of the canonical symplectic form $d\lambda_0$ implies that the moduli spaces $\mathcal{M}_\alpha(z_-, z_+, H_{s,t}, J_{s,t})$ of smooth solutions of (4.1) satisfying the boundary conditions (4.2) are $C^\infty_{loc}$-compact. Observe that every solution $u$ of (4.1) and (4.2) satisfies the
energy identity

\[
E(u) : = \int_{-\infty}^{\infty} \int_{0}^{1} |\partial_s u|_{s,t} \, ds \, dt
= \mathcal{A}_{H^{-}}(z_{-}) - \mathcal{A}_{H^{+}}(z_{+}) - \int_{-\infty}^{\infty} \int_{0}^{1} (\partial_s H_{s,t})(u(s, t)) \, ds \, dt.
\]

Therefore, for \(|H^{+} - H^{-}|\) sufficiently small we can define a chain map from \(\mathcal{CF}_{a,b}^{(a,b)}(H^{-}; \alpha)\) to \(\mathcal{CF}_{a,b}^{(a,b)}(H^{+}; \alpha)\), (see [8, Section 4.4]). This defines an homomorphism

\[
\sigma_{H^{+}H^{-}} : \mathcal{HP}_{a,b}^{(a,b)}(H^{-}; \alpha) \to \mathcal{HP}_{a,b}^{(a,b)}(H^{+}; \alpha)
\]

which is independent of the choice of homotopy by standard arguments, see, e.g., [31, 34]. Actually, we have the following.

![Figure 4.1. The functions \(H^{\pm}, H'\) and homotopies](image)

**Lemma 4.1** (Local isomorphisms). If \(H^{\pm}\) are sufficiently close to \(H \in \mathcal{K}_{\eta, \alpha}^{a,b}\), then the homomorphism \(\sigma_{H^{+}H^{-}}\) is an isomorphism.

The proof of the above lemma, essentially due to Weber (see [40, Lemma 2.5]), is based on the convexity result in Section 2.1 and the properties of the marked length spectrum (see Lemma 3.2).

**Sketch of the proof.** As in [40], consider an appropriate intermediate Hamiltonian \(H'\) (see Figure 4.1), and two homotopies \(H_{a,t}^{0} := (1 - \beta(s))H_{t}^{+} + \beta(s)H_{t}^{-}\) and \(H_{a,t}^{1} := (1 - \beta(s))H_{t}^{+} + \beta(s)H_{t}^{-}\). Here the \(C^{\infty}\)-Hamiltonian \(H'\) is required to satisfy: \(H' = H^{-}\) on \(D_{\eta}^{F_{\eta}T^{*}M}\), \(H' = \tau_{+}F^{*} - c_{0}\) for some constant \(c_{0} \in \mathbb{R}\) outside \(D_{2\eta}^{F_{\eta}T^{*}M}\) and \(\frac{\partial H'}{\partial \rho} \geq 0\) for \(\rho \in (\eta, 2\eta)\). The standard arguments (see [31, Section 3.4]) implies that the homomorphisms \(\sigma_{H^{+}H^{-}}\) and \(\sigma_{H^{+}H'} \circ \sigma_{H'H^{-}}\) are equal. Hence it suffices to prove that both \(\sigma_{H^{+}H'}\) and \(\sigma_{H'H^{-}}\) are isomorphisms whenever \(H^{\pm}\) are sufficiently close to \(H \in \mathcal{K}_{\eta, \alpha}^{a,b}\).
To prove this, let us first note that since $\mathbb{R} \setminus \Lambda_\alpha$ is open by Lemma 3.2, $(\tau_-, \tau_+ ) \cap \Lambda_\alpha = \emptyset$ provided that $H^+$ and $H^-$ are sufficiently close to $H \in \mathcal{H}_{a,b}$. From Lemma 3.4 we know that no 1-periodic Hamiltonian orbits of $H^+$ and $H^-$ appear outside $D^F_\eta T^* M$. Then, by lemma 2.2, the solution $u$ of Floer equation (4.1) connecting $z^\in \mathcal{P}_a(H^-)$ to $z \in \mathcal{P}_a(H^+)$ can not escape from $D^F_\eta T^* M$. Moreover, we have $H_0^s = H^-$ on $D^F_\eta T^* M$ for all $s \in \mathbb{R}$. Therefore, the homomorphism $\sigma'_{H^+ H^-}$ is the identity map.

Secondly, observe that outside $D^F_\eta T^* M$ it holds that $\frac{\partial^2 H_{s,t}}{\partial s \partial \rho} = 0$. By lemma 2.2 again all solutions of Floer equation (4.1) corresponding to the homotopy $H_{s,t}^1$ remains in $D^F_\eta T^* M$, and this also holds for its inverse homotopy $G_{s,t} := (1 - \beta(s))H^+_t + \beta(s)H^-_t$. So the usual argument of reversing the homotopy achieves $\sigma_{H^+ H'} = \sigma'_{H^+ H'}$, and hence $\sigma_{H^+ H'}$ is an isomorphism. This completes the proof of Lemma 4.1. □

A direct consequence of the above lemma is that one can still define $\text{HF}^{(a,b)}(H^-; \alpha)$ for every $H \in \mathcal{H}_{a,b}$ when $H$ is degenerate, by simply setting

$$\text{HF}^{(a,b)}(H; \alpha) := \text{HF}^{(a,b)}(\tilde{H}; \alpha)$$

for any nondenerate Hamiltonian $\tilde{H} \in \mathcal{H}_{a,b}$ which is sufficiently close to $H$. Moreover, by composing the above local isomorphisms we deduce that in each component of $\mathcal{H}_{a,b}$, the functions have identical Floer homology groups.

4.4. Monotone homotopies. Let $H, K \in \mathcal{H}_{a,b}$ be two functions with $H(t, z) \leq K(t, z)$ for all $(t, z) \in S^1 \times T^* M$. Choose a smooth homotopy $s \to H_s = \{H_{s,t}\}$ from $H$ to $K$ such that $\partial_t H_s \geq 0$ everywhere and $\frac{\partial^2 H_{s,t}}{\partial s \partial \rho} \geq 0$ for every $\rho \geq \eta$. Here we do not require $H_{s,t}$ to be in $\mathcal{H}_{a,b}$ for every $s \in [0, 1]$. From the energy identity (4.3) we deduce that such a homotopy induces a natural homomorphism, which is called monotone homomorphism

$$\sigma_{KH} : \text{HF}^{(a,b)}(H; \alpha) \to \text{HF}^{(a,b)}(K; \alpha)$$

(see, e.g., [8, 13, 10, 34, 39]):

**Lemma 4.2.** These monotone homomorphisms are independent of the choice of the monotone homotopy of Hamiltonians and satisfy the following properties

$$\sigma_{HH} = \text{id} \quad \forall \ H \in \mathcal{H}_{a,b}$$

$$\sigma_{KH} \circ \sigma_{HG} = \sigma_{KG}$$

whenever $G, H, K \in \mathcal{H}_{a,b}$ satisfy $G \leq H \leq K$.

As a corollary we have

**Lemma 4.3** (see [39] or [8, Section 4.5]). If $K_s$ is a monotone homotopy from $H$ to $K$ such that $K_s \in \mathcal{H}_{a,b}$ for every $s \in [0, 1]$, then $\sigma_{KH}$ is an isomorphism.

4.5. Symplectic homology. Following closely the discussion in [8, Sections 4.6 and 4.7], we consider the direct and inverse limits of Floer homology groups, see also [40, Section 3.1].

4.5.1. Partially ordered set $(\mathcal{H}_{a,b}, \leq)$. Fix a free homotopy class $\alpha \in [S^1, M]$. For $-\infty \leq a < b \leq +\infty$, we define

$$\mathcal{H}_{a,b}^\alpha := \{ H \in C^\infty_0(S^1 \times D^F T^* M) \mid a, b \notin \text{Spec}(H; \alpha)\}$$
of all Hamiltonians which are compactly supported in $D^FT^* M$ and do not contain $a$ and $b$ in their action spectrum. We introduce the partial-order relation $\leq$ on the set $\mathcal{H}_\alpha^{a,b}$ by

$$H_0 \leq H_1 \iff H_0(t, z) \leq H_1(t, z) \quad \text{for all } (t, z) \in S^1 \times D^FT^* M.$$ 

4.5.2. Inverse limits. Note that $\alpha \neq 0$ implies $\mathcal{H}_\alpha^{a,b} \subseteq \mathcal{H}_1^{a,b}$. When $\alpha$ is the homotopy class of constant loops, we ask the intervals $[a, b]$ to not contain 0. In this case, we still have $\mathcal{H}_\alpha^{a,b} \subseteq \mathcal{H}_1^{a,b}$. So the monotone homomorphisms $\sigma_{H_1, H_0}$ in Section 4.4 yields the partially ordered system $(HF, \sigma)$ of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}_\alpha^{a,b}$, that is, $HF$ assigns to every $H \in \mathcal{H}_\alpha^{a,b}$ the $\mathbb{Z}_2$-vector space $HF_*(a,b)(H; \alpha)$, and $\sigma$ assigns to any two elements $H_0 \leq H_1$ the monotone homomorphism $\sigma_{H_1, H_0}$ satisfying (4.2). The partial order system $(\mathcal{H}_\alpha^{a,b}, \leq)$ is downward directed, meaning that for any $H_1, H_2 \in \mathcal{H}_\alpha^{a,b}$, there exists $H_0 \in \mathcal{H}_\alpha^{a,b}$ such that $H_0 \leq H_1$ and $H_0 \leq H_2$. The functor $(HF, \sigma)$ is an inverse system of $\mathbb{Z}_2$ vector spaces over $\mathcal{H}_\alpha^{a,b}$, which has an inverse limit, called symplectic homology of $D^FT^* M$ in the free homotopy class $\alpha$ for the action interval $(a, b)$ and defined by

$$SH_*(a,b)(D^FT^* M; \alpha) := \lim_{H \in \mathcal{H}_\alpha^{a,b}} HF_*(a,b)(H; \alpha)$$

$$:= \left\{ (e_H)_{H \in \mathcal{H}_\alpha^{a,b}} \in \prod_{H \in \mathcal{H}_\alpha^{a,b}} HF_*(a,b)(H; \alpha) \mid H_1 \leq H_2 \Rightarrow \sigma_{H_2 H_1}(e_{H_1}) = e_{H_2} \right\}.$$  

For $H \in \mathcal{H}_\alpha^{a,b}$, one can define the natural projection

$$\pi_H : SH_*(a,b)(D^FT^* M; \alpha) \to HF_*(a,b)(H; \alpha)$$

which satisfies $\pi_{H_1} = \sigma_{H_1, H_0} \circ \pi_{H_0}$ whenever $H_0 \leq H_1$.

4.5.3. Direct limits. Fix $c > 0$. Next we introduce relative symplectic homology of the pair $(D^FT^* M, M)$ at the level $c$ in the homotopy class $\alpha$ for the action interval $(a, b)$. Consider the subset

$$\mathcal{H}_\alpha^{a,b,c} := \{ H \in \mathcal{H}_\alpha^{a,b} \mid \sup_{S^1 \times M} H < -c \}.$$  

This set is upward directed. Namely, for any $H_0, H_1 \in \mathcal{H}_\alpha^{a,b,c}$, there exists $H_2 \in \mathcal{H}_\alpha^{a,b,c}$ such that $H_0 \leq H_2$ and $H_1 \leq H_2$. The functor $(HF, \sigma)$ can be viewed as a direct system of $\mathbb{Z}_2$ vector spaces over $\mathcal{H}_\alpha^{a,b,c}$, whose direct limit is defined as

$$SH_*(a,b,c)(D^FT^* M, M; \alpha) := \lim_{H \in \mathcal{H}_\alpha^{a,b,c}} HF_*(a,b)(H; \alpha)$$

$$:= \left\{ (H, e_H) \mid H \in \mathcal{H}_\alpha^{a,b,c}, e_H \in HF_*(a,b)(H; \alpha) \right\}.$$  

Here the equivalence relation is defined as follows: $(H_0, e_{H_0}) \sim (H_1, e_{H_1})$ if and only if there exists $H_2 \in \mathcal{H}_\alpha^{a,b,c}$ such that $H_0 \leq H_2$, $H_1 \leq H_2$ and $\sigma_{H_2 H_0}(e_{H_0}) = \sigma_{H_2 H_1}(e_{H_1})$. The direct limit is a $\mathbb{Z}_2$-vector space with the operations

$$k[H_0, e_{H_0}] := [H_0, ke_{H_0}], \quad [H_0, e_{H_0}] + [H_1, e_{H_1}] := [H_2, \sigma_{H_2 H_0}(e_{H_0}) + \sigma_{H_2 H_1}(e_{H_1})],$$

for any $k \in \mathbb{Z}_2$ and $H_2 \in \mathcal{H}_\alpha^{a,b,c}$ with $H_0 \leq H_2$ and $H_1 \leq H_2$. For $H \in \mathcal{H}_\alpha^{a,b,c}$, let

$$\iota_H : HF_*(a,b)(H; \alpha) \to SH_*(a,b,c)(D^FT^* M, M; \alpha), \quad e_H \mapsto [H, e_H]$$
be the natural homomorphism which satisfies \( \iota_{H_0} = \iota_{H_1} \circ \sigma_{H_1, H_0} \).

4.5.4. Exhausting sequences. A sequence \( \{ H_i \in \mathcal{H}_{\alpha_i}^{a,b} \mid i \in \mathbb{N} \} \) is called downward exhausting for \((HF, \sigma)\) if it holds the following two properties

- for every \( i \in \mathbb{N} \) we have \( H_{i+1} \leq H_i \) and \( \sigma_{H_i, H_{i+1}} : HF^*(a,b)(H_{i+1}; \alpha) \rightarrow HF^*(a,b)(H_i; \alpha) \) is an isomorphism
- for every \( H \in \mathcal{H}_{\alpha_i}^{a,b} \) there exists a \( i \in \mathbb{N} \) such that \( H_i \leq H \).

Correspondingly, a sequence \( \{ H_i \in \mathcal{H}_{\alpha_i}^{a,b;c} \mid i \in \mathbb{N} \} \) is called upward exhausting for \((HF, \sigma)\) if and only if the following holds:

- for every \( i \in \mathbb{N} \) we have \( H_i \leq H_{i+1} \) and \( \sigma_{H_{i+1}, H_i} : HF^*(a,b)(H_i; \alpha) \rightarrow HF^*(a,b)(H_{i+1}; \alpha) \) is an isomorphism
- for every \( H \in \mathcal{H}_{\alpha_i}^{a,b;c} \) there exists a \( i \in \mathbb{N} \) such that \( H \leq H_i \).

Given an exhausting downward (upward) sequence it is possible to actually compute a inverse (direct) limit.

**Lemma 4.4.** Let \((HF, \sigma)\) be the partially ordered system of \( \mathbb{Z}_2 \)-vector spaces over \((\mathcal{H}_{\alpha_i}^{a,b} ; \leq )\).

(i) If \( \{ H_i \in \mathcal{H}_{\alpha_i}^{a,b} \mid i \in \mathbb{N} \} \) is a downward exhausting sequence for \((HF, \sigma)\), then the homomorphism \( \pi_{H_i} : SH^*(a,b)(DF^*T^* M ; \alpha) \rightarrow HF^*(a,b)(H_i; \alpha) \) is an isomorphism for every \( i \in \mathbb{N} \).

(ii) If \( \{ H_i \in \mathcal{H}_{\alpha_i}^{a,b;c} \mid i \in \mathbb{N} \} \) is an upward exhausting sequence for \((HF, \sigma)\), then the homomorphism \( \iota_{H_i} : SH^*(a,b;c)(DF^*T^* M, M ; \alpha) \rightarrow HF^*(a,b)(H_i; \alpha) \) is an isomorphism for every \( i \in \mathbb{N} \).

Before the end of this section, let us state the following proposition, which is an adaptation of [8, Proposition 4.8.2].

**Proposition 4.5** (4.8.2). Let \( \alpha \in [S^1, M] \) be a free homotopy class and suppose that \( -\infty \leq a < b \leq +\infty \). Then for any \( c \in \mathbb{R} \) there exists a unique homomorphism

\[
T_{\alpha}^{(a,b);c} : SH^*(a,b)(DF^*T^* M ; \alpha) \longrightarrow SH^*(a,b;c)(DF^*T^* M, M ; \alpha)
\]

such that for every \( H \in \mathcal{H}_{\alpha_i}^{a,b;c} \), the following diagram commutes:

\[
\begin{array}{ccccc}
SH^*(a,b)(DF^*T^* M ; \alpha) & \xrightarrow{T_{\alpha}^{(a,b);c}} & SH^*(a,b;c)(DF^*T^* M, M ; \alpha) \\
\pi_H & & & & \iota_H \\
& HF_{\alpha}^{(a,b)}(H; \alpha) & & &
\end{array}
\tag{4.6}
\]

5. **The Abbondandolo-Schwarz isomorphism between the Morse and the Floer complex**

5.1. **The Abbondandolo-Schwarz isomorphism.** In order to compute the groups \( SH^*(a,b)(DF^*T^* M ; \alpha) \) and \( SH^*(a,b;c)(DF^*T^* M, M ; \alpha) \), we use Abbondandolo-Schwarz isomorphism between the Morse and the Floer complex, which we introduce next.
Let $M$ be a smooth compact connected $n$-dimensional manifold without boundary. Recall that a smooth Lagrangian

$$L : S^1 \times TM \to \mathbb{R}$$

is fiberwise convex and quadratic at infinity if it satisfies:

- **(L1)** there exists $l_1 > 0$ such that
  $$\partial_{uv}L(t, x, v) \geq l_1 I \quad \forall \ (t, x, v) \in S^1 \times TM;$$

- **(L2)** there exists $l_2 > 0$ such that
  $$\|\partial_{uv}L(t, x, v)\| \leq l_2, \quad \|\partial_{xv}L(t, x, v)\| \leq l_2(1 + |v|),$$
  $$\|\partial_{xx}L(t, x, v)\| \leq l_2(1 + |v|^2) \quad \forall \ (t, x, v) \in S^1 \times TM$$

with respect to some Riemannian metric $g$ on $M$ with $|v|^2 := g_x(v, v)$.

Equivalently, there exists a finite atlas on $M$ and two constants $0 < c_0 < c_1$ such that in every chart of this atlas the following conditions hold:

- **(L1)** \(\sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j} L(t, x, v)u_i u_j \geq c_0 |u|^2 \quad \forall \ t \in S^1, \ \forall \ u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n\).

- **(L2)** \(\left| \frac{\partial^2}{\partial v_i \partial v_j} L(t, x, v) \right| \leq c_1, \quad \left| \frac{\partial^2}{\partial x_i \partial v_j} L(t, x, v) \right| \leq c_1(1 + |v|) \quad \text{and} \quad \left| \frac{\partial^2}{\partial x_i \partial x_j} L(t, x, v) \right| \leq c_1(1 + |v|^2), \quad \forall \ (t, x, v) \in S^1 \times TM\).

Denote by $\mathcal{L}(M)$ the space of absolutely continuous curves $\gamma : S^1 \to M$ such that

$$\int_{S^1} g(\dot{\gamma}(t), \dot{\gamma}(t)) dt < \infty,$$

and denote by $\mathcal{L}_\alpha(M)$ the connected component whose elements represent the free homotopy class $\alpha$. It is well known that $\mathcal{L}_\alpha(M)$ has Hilbert manifold structures, and its tangent space at $\gamma T\gamma \mathcal{L}_\alpha(M)$ can be identified with the Hilbert space $W^{1,2}(\gamma^* TM)$ equipped with the inner product

$$\langle \xi, \eta \rangle_{W^{1,2}} := \int_0^1 g(\xi(t), \eta(t)) dt + \int_0^1 g(\nabla g^\gamma \xi(t), \nabla g^\gamma \eta(t)) dt.$$ 

Here $W^{1,2}(\gamma^* TM)$ denotes the space of all 1-periodic vector fields along $\gamma$ with finite $W^{1,2}$-norm.

**Remark 5.1.** Since $\mathcal{L}_\alpha(M)$ is homotopy equivalent to $\Lambda_\alpha(M)$, if no ambiguity is possible, we sometimes just work on $\mathcal{L}_\alpha(M)$ instead of the complete space $\mathcal{L}_\alpha(M)$ (see, e.g., using Lemma 6.1 to show Theorem 1.3).

The Lagrangian action functional $\mathcal{L} : \mathcal{L}_\alpha(M) \to \mathbb{R}$ is defined by

$$\mathcal{L}(x) := \int_0^1 L(t, x(t), \dot{x}(t)) dt \quad \forall \ x \in \mathcal{L}_\alpha(M).$$

Here we need to point out that in general the functional $\mathcal{L}$ is not of class $C^2$. Indeed, $\mathcal{L}$ is of class $C^2$ if and only if the function $v \to L(t, x, v)$ is a polynomial of degree at most 2 for all $(t, x) \in S^1 \times M$ (see [5, Proposition 3.2] for the detailed proof). To do Morse theory one may need at least $C^2$-regularity since the Morse lemma requires $C^2$-regularity (see [9]). The lack of $C^2$-regularity of $\mathcal{L}$ seems to prevent us from doing infinite dimensional Morse theory for $\mathcal{L}$. However, under the following non-degeneracy hypothesis (see (L0) below), one still construct a Morse complex for $\mathcal{L}$ by using the so-called pseudo-gradient for $\mathcal{L}$ (cf. [5]).
(L0) Every critical point $x$ of $\mathcal{L}$ is non-degenerate, that is, the symmetric bilinear form $d^2\mathcal{L}(x)$ on $T_x\mathcal{L}_\alpha(M)$ is non-degenerate.

We next cite a result given by Abbondandolo and Schwarz [4].

**Theorem 5.2** (Abbondandolo-Schwarz). Let $M$ be a closed manifold. Assume that $H \in C^\infty(S^1 \times T^*M, \mathbb{R})$ is the Legendre transform of the Lagrangian $L \in C^\infty(S^1 \times T^*M, \mathbb{R})$ satisfying (L0) – (L2). Then there exists a chain isomorphism $\Psi$ from the Floer complex of $H$ to the Morse complex of the Lagrangian action functional $\mathcal{L}$ associated to the Lagrangian $L$ with coefficients in $\mathbb{Z}_2$. Moreover, such an isomorphism preserves the action filtrations, that is, $\Psi$ induces an isomorphism $\Psi_* \mid _{H\mathcal{F}^\le \alpha(H; \alpha)}$ and $H\mathcal{M}^\le \alpha(\mathcal{L}; \alpha)$ for every $\alpha \in \mathbb{R} \cup \{\pm \infty\}$.

5.2. **Convex quadratic modifications.** The main problem of applying Abbondandolo and Schwarz to our setting is that $F^2$ is not smooth at the zero section. To solve this problem, we introduce the following quadratic modification which smoothens $F^2$ near the zero section maintaining the fiberwise convexity.

**Lemma 5.3.** Let $(M, F)$ be a closed connected Finsler manifold and $L_0 = F^2 : TM \to \mathbb{R}$. Then there exists a family of convex quadratic modifications $L_\eta$ of $L_0$ which satisfies (L1), (L2) and the following on $TM$:

(a) $L_0(x, v) - \eta \le L_\eta(x, v) \le L_0(x, v)$ for all $(x, v) \in TM$,

(b) $L_\eta(x, v) = L_0(x, v)$ for $L_0(x, v) \ge \eta$.

**Remark 5.4.** Actually, we can construct a family of convex quadratic Lagrangians $\{L_\eta\}_{0 < \eta < \eta_0}$ which satisfy (a) and (b), and in addition satisfy $L_{\eta_1} \ge L_{\eta_2}$ for $0 < \eta_1 < \eta_2$. This can be done by picking a Lagrangian $L_{\eta_0}$ as in Lemma 5.3, and then rescaling the parameters in the auxiliary functions $\chi_{\epsilon, \delta}^\mu$ and $\chi_{\delta, \rho}^\kappa$ (see Section 9.3) by $\Delta \in (0, 1]$, for instance, $\kappa \to \kappa/\Delta$, $\mu \to \mu/\Delta$, $\delta \to \Delta \delta$, $\epsilon \to \Delta \epsilon$ and $\eta_0 \to \Delta \eta_0$ without changing $\sigma$ and $\rho$.

The strong convexity condition (L1) can be used to define a Hamiltonian on $T^*M$ by means of the Legendre transform

$$\mathcal{L} : S^1 \times T^* M \to S^1 \times T^* M, \quad (t, x, v) \to (t, x, d_v L(t, x, v)), \quad (5.1)$$

which is is a fiber-preserving diffeomorphism, cf. [27]. The Hamiltonian $H : S^1 \times T^* M \to \mathbb{R}$ takes the form

$$H(t, x, p) := \max_{v \in T_x M} (p(v) - L(t, x, v)),$$

called the Fenchel dual Hamiltonian of $L$.

For each $L_\eta$ in Lemma 5.3, its Legendre transform $H_\eta$ coincides with $\frac{1}{2}F^{*2}$ on $T^*M \setminus D_{\sqrt{\eta}}T^*M$. Since $L_\eta \le L_0$, by definition it holds that $H_\eta \ge \frac{1}{2}F^{*2}$ on $T^*M$. Moreover, since we have $\partial_x H_\eta(x, p) \partial_{x^i} L_\eta(x, v) = I$, we get that $H_\eta$ is also fiberwise uniformly convex since $L_\eta$ satisfies (L1) by our construction.

For $V \in C^\infty_0(S^1 \times T^*M, \mathbb{R})$, we define the Lagrangian action functional $\mathcal{L}_V^0$ on $\Lambda_\alpha M$ by

$$\mathcal{L}_V^0(x) := \int_0^1 \frac{1}{2} L_\eta(x(t), \dot{x}(t)) - V(t, x(t), \dot{x}(t)) dt.$$

Now following directly from Theorem 5.2 we have the corollary.

**Corollary 5.5.** Let $(M, F)$ be a connected closed Finsler manifold, and let $L_\eta (0 < \eta \le \eta_0)$ be a Lagrangian constructed in Section 5.2. Fix $r \in (0, \infty) \setminus \Lambda_\alpha$ with $r^2 > \eta_0$. Suppose that...
\[ L_{\eta}/2 + V \text{ for } V \in C_0^\infty(S^1 \times D_r^s TM) \text{ satisfies } (L1) \text{ and } (L2), \text{ and that every critical point } x \text{ of } \mathcal{L}_{V}^\eta(x) \leq r^2/2 \text{ is non-degenerate. Let } H_\eta + U \text{ be its Fenchel dual Hamiltonian with } U \in C_0^\infty(S^1 \times D_r^s T^*M).

(1) Then there is an isomorphism

\[ \Psi_* : HF_*^{<r^2/2}(H_\eta + U; \alpha) \longrightarrow H_*(\{ x \in \Lambda_\alpha M | \mathcal{L}_{V}^\eta(x) < r^2/2 \}) \tag{5.2} \]

which preserves the action filtrations, i.e., for every \( a \in (0, r^2/2) \), \( \Psi_* \) is an isomorphism between \( HF_*^{<a}(H_\eta + U; \alpha) \) and \( H_* (\{ x \in \Lambda_\alpha M | \mathcal{L}_{V}^\eta(x) < a \}) \).

(2) For every \( a \in (0, r^2/2) \) and \( 0 < \eta_1 < \eta_2 < \eta_0 \), the following diagram commutes:

\[ \begin{array}{ccc}
\Psi_* (\mathcal{L}_{V}^{\eta_1}) & \longrightarrow & \Psi_* (\mathcal{L}_{V}^{\eta_2}) \\
H_* (\{ x \in \Lambda_\alpha M | \mathcal{L}_{V}^{\eta_1}(x) < a \}) & \longrightarrow & H_* (\{ x \in \Lambda_\alpha M | \mathcal{L}_{V}^{\eta_2}(x) < a \})
\end{array} \tag{5.3} \]

Here \( U_1, U_2 \in C_0^\infty(S^1 \times T^*M, \mathbb{R}) \) are chosen so that \( H_{\eta_1} + U_1 \) and \( H_{\eta_2} + U_2 \) satisfy \( H_{\eta_2} + U_2 \leq H_{\eta_1} + U_1 \) (and hence \( \mathcal{L}_{V}^{\eta_1} \geq \mathcal{L}_{V}^{\eta_2} \)) and the non-degeneracy condition

\((H0)\) All the elements \( z \in \mathcal{D}_\alpha^{r^2/2}(H) \) with \( H = H_\eta + U_i, i = 1, 2 \) are non-degenerate, that is to say, the linear map \( d\phi_H^1(z(0)) \in \text{Sp}(T_{z(0)}T^*M) \) does not have 1 as an eigenvalue.

6. The proof of Theorem 1.3

In this section we use Abbondandolo-Schwarz isomorphism to compute Floer homology of a class of convex radial Hamiltonians and prove Theorem 1.3. Let \((M, F)\) be a smooth connected closed Finsler manifold. Let \( \{L_\eta\}_{0 < \eta < \eta_0} \) be a family of Lagrangians constructed in Section 5.2, and \( H_\eta \) the Fenchel dual Hamiltonians of \( L_\eta/2 \). Denote \( Q_\eta := \sqrt{2H_\eta} \). Obviously, it holds that

- \( Q_\eta = F^* \text{ on } T^*M \setminus D_\sqrt{2H_\eta}T^*M \),
- \( F^* \leq Q_\eta \leq \sqrt{F^* + \eta} \text{ on } T^*M \text{ and} \)
- \( Q_\eta \geq Q_{\eta_1} \text{ for } 0 < \eta_1 \leq \eta_2 < \eta_0 \).

For \( c \in \mathbb{R} \), denote \( H_\alpha(\Lambda^{0,c}_\alpha M) \) the singular homology with \( \mathbb{Z}_2 \)-coefficients of the sublevel set

\[ \Lambda^{0,c}_\alpha M := \{ x \in \Lambda_\alpha M | \mathcal{L}_V^\eta(x) \leq c \}. \]

Hereafter, we abbreviate \( \Lambda^{0,c}_\alpha M \) by \( \Lambda^c_\alpha M \), and for \( a \leq b \) denote by \( I_{a,b}^0 \) the natural inclusion from \( \Lambda^a_\alpha M \) to \( \Lambda^b_\alpha M \). For the sake of simplicity, we also denote \( [I] : H_\alpha(\Lambda^a_\alpha M) \to H_\alpha(\Lambda^b_\alpha M) \) the homomorphism induced by \( I_{a,b}^0 \).

In the proof of Theorem 1.3, we use the following two lemmata.

**Lemma 6.1.** ([11, Proposition 3.1]) Let \( \{f_t\}_{t \in [0,1]} \) be a family of \( C^1 \)-functions from a Banach space \( X \) to \( \mathbb{R} \). Let \( a \in \mathbb{R} \) and \( \varepsilon > 0 \). For any \( t \in [0,1] \), define

\[ \Sigma_t := \{ u \in X | f_t(u) \leq a \}. \]

Suppose that

(a) \( \inf_u \{ \|f'(u)\|_{X^*} : a - \varepsilon \leq f(u) \leq a \} > 0 \), \( \forall t \in [0,1] \);

(b) if \( t_k \to t \) in \([0,1] \), then \( f_{t_k} \to f_t \) uniformly on \( \bigcup_{0 \leq \tau \leq 1} \Sigma_\tau \).
Then $H_*(\Sigma_0) \cong H_*(\Sigma_1)$.

**Lemma 6.2.** Let $\{L_\eta\}_{0 < \eta < \eta_0}$ be a family of Lagrangians constructed in Section 5.2. Suppose that $\alpha$ is a nontrivial homotopy class of free loops in $M$ and $\alpha \in (0, +\infty) \setminus \Lambda_\alpha$. Then there exists a natural isomorphism

$$\lim_{\eta \to (0, \eta_0]} H_*(\{x \in \Lambda_\alpha M \mid \mathcal{L}_0^\eta(x) \leq \alpha^2 / 2\}) \to H_*(\Lambda^{a^2/2}_\alpha M).$$

**Remark 6.3.** Actually, in the proof of Lemma 6.2 we have shown that there exists a sufficiently small constant $\bar{\eta} = \bar{\eta}(\alpha) > 0$ such that for any $0 \leq \eta \leq \bar{\eta}$, the natural inclusion

$$\Lambda^{a^2/2}_\alpha M \hookrightarrow \{x \in \Lambda_\alpha M \mid \mathcal{L}_0^\eta(x) \leq \alpha^2 / 2\}$$

induces an isomorphism between the $\mathbb{Z}_2$-singular homologies $H_*(\{x \in \Lambda_\alpha M \mid \mathcal{L}_0^\eta(x) \leq \alpha^2 / 2\})$ and $H_*(\Lambda^{a^2/2}_\alpha M)$. Besides, one can also see that for $\eta$ sufficiently small, the Lagrangian functional $\mathcal{L}_0^\eta$ has no critical points $x \in \Lambda_\alpha M$ such that $\dot{x}(\tau) \in D_{\sqrt{\eta}}^F TM$ for some $\tau \in S^1$. Hence, by the property of Legendre transform, we find that whenever $\eta$ is sufficiently small, say, not larger than $\bar{\eta} > 0$, the functional $\omega_{H_\eta}$ does not admit critical points on $\Lambda_\alpha T^*M$ intersecting $D_{\sqrt{\eta}}^F T^*M$.

We now complete the proof of Theorem 1.3 assuming Lemma 6.2.

**The proof of Theorem 1.3.** We introduce a smooth function $f^{(\lambda)}$ with slope $\lambda$ at infinity associated to $f$ as following: firstly follow the graph of $f$ until it takes on slope $\lambda$ for the first time, say, at a point $r = r(f, \lambda)$, then continue linearly with slope $\lambda$, finally smoothing near $r$ yields a $C^\infty$-function $f^{(\lambda)}$ which coincides with $f$ outside a small neighborhood of $r$. Define the function $f_0 \in C^\infty([0, \infty), \mathbb{R})$ by $f_0(s) := s^2 / 2$. Pick $\bar{\eta} > 0$ as in Remark 6.3. Denote $\tilde{\eta} := \min\{\bar{\eta}, \lambda^2, \epsilon_f^2 / 2\}$. Obviously, $f \circ Q_\eta = f \circ F^*, r(f_0, \lambda) = \lambda$, and $H_{\tilde{\eta}} = f_0 \circ Q_\eta$ for all $\eta \in (0, \bar{\eta}]$.

To prove Theorem 1.3 (i) and (ii), we show that the vertical homomorphisms in diagram (6.1) are isomorphisms, and prove that the following diagram commutes:

$$\begin{array}{cccc}
\mathrm{HF}_*^{(-\infty,\epsilon_f,\mu)}(f \circ Q_\eta + K_1; \alpha) & [i^F] & \mathrm{HF}_*^{(-\infty,\epsilon_f,\lambda)}(f \circ Q_\eta + K_1; \alpha) \\
\downarrow & & \downarrow \\
\mathrm{HF}_*^{(-\infty,\infty)}(f(\mu) \circ Q_\eta + K_2; \alpha) & [i^F]_{[\sigma]} & \mathrm{HF}_*^{(-\infty,\infty)}(f^{(\lambda)} \circ Q_\eta + K_1; \alpha) \\
\downarrow & & \downarrow \\
\mathrm{HF}_*^{(-\infty,\infty)}(f_0^{(\mu)} \circ Q_\eta + K_3; \alpha) & [i^F]_{[\sigma]} & \mathrm{HF}_*^{(-\infty,\infty)}(f_0^{(\lambda)} \circ Q_\eta + K_4; \alpha) \\
\downarrow & & \downarrow [\mathrm{id}] \\
\mathrm{HF}_*^{(-\infty,\mu^2/2)}(f_0 \circ Q_\eta + K_4; \alpha) & [i^F]_{[\sigma]} & \mathrm{HF}_*^{(-\infty,\lambda^2/2)}(f_0 \circ Q_\eta + K_4; \alpha) \\
\downarrow & & \downarrow [\mathrm{id}] \\
H_*(\Lambda^{a^2/2}_\alpha M) & [\mathrm{id}] & H_*(\Lambda^{a^2/2}_\alpha M) \\
\end{array}$$

(6.1)

where $K_i \in C^\infty_0(S^1 \times T^*M), i = 1, 2, 3, 4$, are compactly supported perturbations such that the corresponding Hamiltonian 1-periodic orbits within the action windows are non-degeneracy. Once
this is completed, composing of the vertical maps on the right hand side of diagram (6.1) yields the isomorphism
\[ \Psi_f^\eta,\lambda : \text{HF}^{(-\infty,\epsilon_f,\lambda)}(f \circ Q_\eta + K_1; \alpha) \rightarrow \text{H}_\ast(\Lambda^{\alpha^2/2}_\alpha) \]
Then combining \( \Psi_f^\eta,\lambda \) with the natural isomorphism \( \text{H}_\ast(\Lambda^{\alpha^2/2}_\alpha) \rightarrow \text{H}_\ast(\Lambda^{\alpha^2/2}_\alpha) \) by our choice of \( \eta \) we obtain the desired isomorphism \( \Psi_f^\lambda \).

The proof of (i). Observe that for every \( \eta \in (0, \hat{\eta}] \), \( f \circ Q_\eta \) is constant on \( D_{\sqrt{\pi}}T^*M \) (since \( Q_\eta \leq \sqrt{F} + \eta \leq 2\sqrt{\eta} \leq \epsilon_f \) and \( f \equiv f(0) \) on \( [0, \epsilon_f) \)), and is radial outside of \( D_{\sqrt{\pi}}T^*M \). Then by Lemma 3.4, for sufficiently small perturbations \( K_1 \), every 1-periodic orbit of \( f \circ Q_\eta + K_1 \) with action less than \( c_f, \lambda \), takes value in \( D_fT^*M \) with \( r = r(f, \lambda) \), and the connecting trajectories between two such 1-periodic orbits are located in \( D_fT^*M \). The same is true for \( f^{(\lambda)} \circ Q_\eta + K_1 \) without any restriction on action. Since in \( D_fT^*M \), Hamiltonians \( f \circ Q_\eta + K_1 \) and \( f^{(\lambda)} \circ Q_\eta + K_1 \) are the same, so both chain complexes are identical.

The second vertical homomorphism is induced by Floer’s continuation map \( \sigma(H_s) \) associated to a homotopy \( H_s \) from \( f^{(\lambda)} \circ Q_\eta + K_1 \) to \( f_0^{(\lambda)} \circ Q_\eta + K_4 \). By the homotopy invariance (see subsection 4.3), \( [\sigma(H_s)] \) is indeed an isomorphism because outside some compact set, say, \( D_{\sqrt{\pi}+1}T^*M \) with \( \hat{r} := \max\{r(f_0, \lambda), r(f, \lambda)\} \), \( f^{(\lambda)} \) and \( f_0^{(\lambda)} \) are of slope \( \lambda \) and hence belong to the same component in \( \mathcal{H}_{r=1/\alpha}^{\infty} \).

The third vertical homomorphism is induced by the identity map between two identical chain complexes, and hence is an isomorphism. In fact, although \( f_0 \) is not constant near the origin, by our choice of \( \eta \) we know that \( H_\eta = f_0 \circ Q_\eta \) has no Hamiltonian periodic orbits \( z \in \Lambda_\alpha T^*M \) intersecting \( D_{\sqrt{\eta}}T^*M \). Thus \( z \in \Lambda_\alpha T^*M \) is a 1-periodic orbit of the Hamiltonian \( f_0^{(\lambda)} \circ Q_\eta \) if and only if \( z \) is a 1-periodic orbit of the Hamiltonian \( H_\eta \) with action \( \mathcal{A}_{\mathcal{H}_\eta}(z) < \lambda^2/2 \). Hence, for sufficiently small perturbations \( K_4 \), all 1-periodic orbits and their connecting trajectories are located in \( D_{\lambda}T^*M \). Moreover, in \( D_{\lambda}T^*M \) both Hamiltonians are the same. So both chain complexes are identical.

The fourth vertical isomorphism is given by the Abbondandolo and Schwarz’s isomorphism. Indeed, for \( K_4 \in C_0^{\infty}(S^1 \times D_\lambda T^*M) \) with sufficiently small \( \|K_4\|_{C^2} \), the Hamiltonian \( H_\eta + K_4 \) are still fiber-wise uniformly convex and quadratic at infinity. Note that the Legendre transform (see (5.1)) is involutive, namely,
\[ \mathcal{L}^{-1} : S^1 \times T^*M \rightarrow S^1 \times TM, \quad (t, x, p) \rightarrow (t, x, d_pH(t, x, v)). \]
For any fiber-wise uniformly convex Lagrangian \( L \) and its Fenchel dual Hamiltonian \( H \), we have
\[ v = \partial_p H(t, q, \partial_v L(t, q, v)), \quad \partial_{pp} H(t, q, \partial_v L(t, q, v)) \partial_{vv} L(t, q, v) = Id. \] (6.2)
Denote by \( L' \) the Fenchel dual Lagrangian of the Hamiltonian \( H_\eta + K_4 \). Set
\[ V(t, x, v) := L'(t, x, v) - \frac{1}{2} L_\eta(t, x, v) \quad \forall (t, x, v) \in S^1 \times TM. \]
By (6.2) and the implicit function theorem, we have \( \|V\|_{C^2} \) is also small enough. Since \( K_4 \) has a compactly supported set in \( D_\lambda T^*M \), by the Legendre transform we find that \( V \) is compactly supported in the disk tangent bundle \( D_\lambda TM := \{(x, v) \in TM | F(x, v) \leq \lambda\} \).
For \( \lambda \in (0, +\infty) \setminus \Lambda_\alpha \), if \( \|V\|_{C^2} \) is sufficiently small, \( \lambda^2/2 \) is a regular value of \( \mathcal{L}_V^\eta \) on \( \Lambda_\alpha M \). Therefore, by Lemma 6.1, we have the isomorphism
\[
H_*(\{ x \in \Lambda_\alpha M | \mathcal{L}_V^\eta(x) < \lambda^2/2 \}) \cong H_*(\{ x \in \Lambda_\alpha M | \mathcal{L}_V^\eta(x) < \lambda^2/2 \}).
\] (6.3)
Furthermore, whenever the perturbation \( K_4 \) is chosen such that all \( z \in \mathcal{P}_\alpha(\mathcal{H}_\eta + K_4) \) are non-degenerate, the Abbondandolo and Schwarz’s isomorphism (see Corollary 5.5) implies
\[
HF_*^{(-\infty, \lambda^2/2)}(\mathcal{H}_\eta + K_4; \alpha) \cong H_*(\{ x \in \Lambda_\alpha M | \mathcal{L}_V^\eta(x) < \lambda^2/2 \}).
\] (6.4)
Combining (6.3) and (6.4) we get the fourth vertical isomorphism.

The proof of (ii). It suffices to show that for \( 0 < \eta \leq \tilde{\eta} \), the following diagram commutes:
\[
\begin{array}{ccc}
HF_*^{(-\infty, c_{f,\mu})}(f \circ Q_\eta; \alpha) & \xrightarrow{\lambda^F} & HF_*^{(-\infty, c_{f,\lambda})}(f \circ Q_\eta; \alpha) \\
\Phi_{\eta, \mu}^\lambda \downarrow & & \downarrow \Phi_{\eta, \lambda}^\mu \\
H_*(\Lambda_\alpha^\eta \lambda^{2/2} M) & \xrightarrow{[\iota]} & H_*(\Lambda_\alpha^\eta \lambda^{2/2} M)
\end{array}
\] (6.5)
This reduces to show the commutativity of the diagram 6.1. Let us note that the vertical maps on the left hand side of (6.1) are similar to the ones on the right hand side. The perturbations \( K_2 \) and \( K_3 \) are the restrictions of \( K_1 \) and \( K_4 \) respectively. The first four horizontal maps are induced by inclusion of subcomplexes. The first and third blocks in diagram (6.1) commute on the chain level. The second, third and fourth horizontal maps can also be viewed as being induced by continuation map associated to monotone homotopies. Due to the convexity of the unperturbed Hamiltonians, no orbits could enter from infinity during the homotopy, and hence continuation maps satisfy the usual composition rule for concatenations of homotopies, see, e.g., [31, 34]. So the second block in (6.1) commute. (5.3) implies the commutativity of block four which is already described at the end of (i).

The proof of (iii). Let \( \tilde{\eta} := \min\{ \tilde{\eta}, \lambda^2, \epsilon_f^2/2, \epsilon_g^2/2 \} \). It is obvious that \( f \circ Q_{\eta_1} = f \circ F^* \) and \( g \circ Q_{\eta_2} = g \circ F^* \). By Remark 6.3, we only need to prove that for \( 0 < \eta_1, \eta_2 < \tilde{\eta} \) with \( \eta_1 \leq \eta_2 \), there exists an isomorphism \( \Psi_{\eta, f}^\lambda \) such that the following diagram commutes:
\[
\begin{array}{ccc}
HF_*^{(-\infty, c_{f,\lambda})}(f \circ Q_{\eta_1}; \alpha) & \xrightarrow{\Psi_{\eta_1}^\lambda} & HF_*^{(-\infty, c_{g,\lambda})}(g \circ Q_{\eta_2}; \alpha) \\
\Psi_{\eta_1}^\lambda \downarrow & & \downarrow \Psi_{\eta_2}^\lambda \\
H_*(\Lambda_\alpha^\eta_1 \lambda^{2/2} M) & \xrightarrow{[\iota]} & H_*(\Lambda_\alpha^\eta_2 \lambda^{2/2} M)
\end{array}
\] (6.6)
Consider the following diagram:
Since the Lagrangian functional \( L \) 

topology. If the compactly supported smooth perturbations \( T \) 

All homomorphisms in (6.7) induced by Floer’s continuation maps \( \sigma \) are isomorphisms which commute on homology because no orbits could enter from \( T^*M \setminus D^F \eta T^*M \) during the homotopy. If the compactly supported smooth perturbations \( K_1, K_4, K_5 \) and \( K_6 \) are chosen such that \( \| K_1 \|_{C^2}, \| K_4 \|_{C^2}, \| K_5 \|_{C^2} \) and \( \| K_6 \|_{C^2} \) are sufficiently small, \( H_{\eta_2} + K_6 \leq H_{\eta_1} + K_4 \), and the associated 1-periodic Hamiltonian orbits (after perturbing) with action in the given action windows are non-degeneracy, then different perturbations achieve the identical Floer homology groups, and by a homotopy-of-homotopies argument two blocks in the diagram (6.7) commute. Using (5.3) and (6.7) we can define an isomorphism \( \Psi_\lambda^f \) such that diagram (6.6) commutes.

We now give the proof of Lemma 6.2.

**Proof of Lemma 6.2.** We first claim that there exists \( \hat{\eta} > 0 \) such that for all \( \eta < \hat{\eta} \), there exists \( \delta > 0 \) such that we have \( \| d\mathcal{L}_0^\eta \| > \delta \) on \( \{ x \in \Lambda_\alpha M \mid \mathcal{L}_0^\eta(x) = a^2/2 \} \). Arguing by contradiction, since the Lagrangian functional \( \mathcal{L}_0^\eta \) satisfies the Palais-Smale condition (cf. [1]) we may assume that for any \( \hat{\eta} > 0 \), there exists \( \eta \in (0, \hat{\eta}) \) and a \( x_\eta \in \Lambda_\alpha M \) such that \( |d\mathcal{L}_0^\eta(x_\eta)| = 0 \) and \( \mathcal{L}_0^\eta(x_\eta) = a^2/2 \).

Suppose \( \alpha < l_\alpha \), then for \( \eta \) sufficiently small,

\[
\int_0^1 F^2(x(t), \dot{x}(t)) dt \leq \int_0^1 L_\eta(x(t), \dot{x}(t)) dt + \eta \leq a^2 + \eta < l_\alpha^2.
\]

This contradicts the fact that \( \int_0^1 F^2(x(t), \dot{x}(t)) dt \geq l_\alpha^2 \) on \( \Lambda_\alpha M \). So \( \alpha \geq l_\alpha \). From (9.7) and the property of Legendre transform we find that if for some \( \tau \in S^1 \), \( \dot{x}_\eta(\tau) \notin D^F\sqrt{\eta} TM \) then \( \dot{x}_\eta(t) \notin D^F\sqrt{\eta} TM \) for all \( t \in S^1 \), and in this case, \( x_\eta \) is also a critical point of the energy functional

\[
E_F(x) = \frac{1}{2} \int_0^1 F^2(x(t), \dot{x}(t)) dt \quad \forall x \in \Lambda_\alpha M,
\]

and \( E_F(x_\eta) = \mathcal{L}_0^\eta(x_\eta) = a^2/2 \). This contradicts \( \alpha \in (0, +\infty) \setminus \Lambda_\alpha \). Therefore, \( x_\eta(t) \in D^F\sqrt{\eta} T^* M \) for any \( t \in S^1 \). But in this case,

\[
\frac{l_\alpha^2}{2} \leq \frac{a^2}{2} = \mathcal{L}_0^\eta(x_\eta) \leq \mathcal{L}(x_\eta) < \frac{\eta}{2}.
\]
which is obviously a contradiction for \( \eta \) sufficiently small. So, for all \( \eta \), there exists \( \delta \) such that on \( \{ x \in \Lambda_\alpha M \mid L^n_0(x) = a^2/2 \} \), we have \( \| dL^n_0 \| > \delta \). Moreover, since \( \Lambda_\alpha \) is a closed and nowhere dense subset of \( \mathbb{R} \), there exist \( \varepsilon \) and \( \eta_0 \) such that for \( \eta < \eta_0 \) there exists \( \delta > 0 \) such that \( \| dL^n_0 \| > \delta \) on \( \{ x \in \Lambda_\alpha M \mid a^2/2 - \varepsilon \leq L^n_0(x) \leq a^2/2 \} \).

Recall that \( L_0 - \eta \leq L_\eta \leq L_0 \), and for any \( \eta_1, \eta_2 \in (0, \eta_0] \) satisfying \( \eta_1 \leq \eta_2 \), we have
\[
\{ x \in \Lambda_\alpha M \mid L^n_0(x) \leq a^2/2 \} \subseteq \{ x \in \Lambda_\alpha M \mid L^n_2(x) \leq a^2/2 \}.
\]
Thus \( L^n_0 \) converges to \( L \) uniformly on
\[
\bigcup_{\eta \in (0, \eta_0]} \{ x \in \Lambda_\alpha M \mid L^n_0(x) \leq a^2/2 \} = \{ x \in \Lambda_\alpha M \mid L^n_0(x) \leq a^2/2 \}.
\]
Therefore, by Lemma 6.1, for any sufficiently small \( \eta_1 \) and \( \eta_2 \) with \( 0 \leq \eta_1 \leq \eta_2 < \eta_0 \), the inclusions
\[
\{ x \in \Lambda_\alpha M \mid L^n_0(x) \leq a^2/2 \} \hookrightarrow \{ x \in \Lambda_\alpha M \mid L^n_2(x) \leq a^2/2 \}
\]
are homotopy equivalences, and hence
\[
H_* \left( \{ x \in \Lambda_\alpha M \mid L^n_0(x) \leq a^2/2 \} \right) = \tilde{H}_* \left( \{ x \in \Lambda_\alpha M \mid L^n_2(x) \leq a^2/2 \} \right).
\]
The proof of the lemma completes.

\[
\square
\]

7. Computing the BPS Capacity

Following closely \([40, 8]\), we will now define certain symplectic capacities. The finiteness of these capacities in various cases will be shown in this section. For \( c > 0 \), denote
\[
\mathcal{K}_c := \{ H \in C_0^\infty(S^1 \times DT^*M) \mid \sup_{S^1 \times M} H \leq -c \}.
\]
In the following, we use the conventions \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \).

**Definition 7.1.** Let \( \alpha \in [S^1, M] \) be a free homotopy class. For \( a \in \mathbb{R} \), define the Biran-Polterovich-Salamon (BPS) capacities of \( DT^*M \) relative to \( M \) as
\[
C_{BPS}(DT^*M, M; \alpha, a) := \inf \left\{ c > 0 \mid \forall H \in \mathcal{K}_c, \exists z \in \mathcal{P}_\alpha(H) \text{ such that } \mathcal{A}_H(z) \geq a \right\},
\]
\[
C_{BPS}(DT^*M, M; \alpha) := \inf \left\{ c > 0 \mid \forall H \in \mathcal{K}_c, \mathcal{P}_\alpha(H) \neq \emptyset \right\}.
\]

Indeed, for general open subsets \( W \subset T^*M \) containing \( M \), one can also define BPS-capacities \( C_{BPS}(W, M; \alpha) \) of \( W \) relative to \( M \). Moreover, the BPS capacity has the following property.

**Proposition 7.2** (Monotonicity \([8, Proposition \ 3.3.1]\)). If \( W_1 \subset W_2 \subset T^*M \) are open subsets containing \( M \) and \( \alpha \in [S^1, M] \), then \( C_{BPS}(W_1, M; \alpha) \leq C_{BPS}(W_2, M; \alpha) \).

Associated to the homomorphism \( T^{(a,b)}_\alpha(c) \) given in Proposition 4.5, for \( c > 0 \), we set
\[
\Theta_c(DT^*M, M; \alpha) := \{ a \in \mathbb{R} \mid (a > 0 \text{ if } \alpha = 0) \mid T^{(a,\infty)}_\alpha(c) \neq 0 \}.
\]
To compute BPS-capacities, we introduce the homological relative capacity
\[
\hat{C}_{BPS}(DT^*M, M; \alpha, a) := \inf \left\{ c > 0 \mid \sup \Theta_c(DT^*M, M; \alpha) > a \right\}
\]
which bounds BPS-capacity \( C_{BPS} \) from above \((8, \text{Proposition 4.9.1})\).

The main result about calculating the BPS-capacity is as follows.
Theorem 7.3. Let $M$ be a closed connected Finsler manifold. Then for every non-trivial free homotopy class $\alpha \in [S^1, M]$, and every $a \in \mathbb{R}$, the BPS capacities are finite and given by

$$C_{\text{BPS}}(D^F T^* M, M; \alpha, a) = \max \{ l_\alpha, a \}, \quad C_{\text{BPS}}(D^F T^* M, M; \alpha) = l_\alpha.$$  

The proof of this theorem follows that of Theorem 3.2.1 and Theorem 3.3.4 in [8] respectively.

The main ingredient in the proof is to use the following Theorem 7.4 to compute $\hat{C}_{\text{BPS}}(D^F T^* M, M; \alpha, a)$.

Theorem 7.4. Assume that $\alpha$ is a homotopy class of free loops in $M$. Then it holds that

(i) if $a \in \mathbb{R} \setminus \Lambda_\alpha$, we have a natural isomorphism $\mathrm{SH}^{(a, +\infty)}_*(D^F T^* M; \alpha) \cong H_*(\Lambda^{a^2/2} M)$;

(ii) for $a, c > 0$, there exists a natural isomorphism

$$\mathrm{SH}^{(a, +\infty); c}_*(D^F T^* M, M; \alpha) \cong \begin{cases} H_*(\Lambda_\alpha M) & \text{if } a \in (0, c], \\ 0 & \text{if } a > c; \end{cases}$$

(iii) for any $a \in (0, c) \setminus \Lambda_\alpha$, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{SH}^{(a, +\infty)}_*(D^F T^* M; \alpha) & \cong & H_*(\Lambda^{a^2/2} M) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{SH}^{(a, +\infty); c}_*(D^F T^* M, M; \alpha) & \cong & H_*(\Lambda_\alpha M)
\end{array}$$

where $I_{a^2/2}$ denotes the natural inclusion $\Lambda^{a^2/2} M \hookrightarrow \Lambda_\alpha M$. 

Figure 7.2. Two sequences of functions

\[ f_k \]

\[ h_k \]
The proof is similar to that of [8] and [40]. We will construct two sequences of profile functions \( \{ f_k \}_{k \in \mathbb{N}} \) and \( \{ h_k \}_{k \in \mathbb{N}} \), one of which is upward exhausting and the other one is downward exhausting. The shape of the graphs of these functions are shown in Figure 7.2. However, there is a main difference between our profile functions and those used in [40]. Indeed, to apply our Theorem 1.3 to compute the Floer homology groups, we require all these functions to be constant near \( \rho = 0 \), which is dictated by the fact that the Hamiltonian \( H_F = F^{*2}/2 \) is not smooth on \( T^*M \) in general. Therefore, we use a different construction of the upward exhausting profile functions from [40] and use some new arguments to prove Theorem 7.4.

The proof of this theorem is postponed to Section 9.4.

8. PROOF OF THE MAIN THEOREM AND ITS APPLICATIONS

8.1. Proofs of the main Theorem 1.1 and Theorem 1.5.

Proof of Theorem 1.1. Consider the Hamiltonian function defined by
\[
\mathcal{T}(t,z) := -H(-t,z) \quad \forall \ (t,z) \in S^1 \times DFT^*M.
\]
Obviously, \( x(t) \) is a periodic orbit of \( H \) representing \(-\alpha\) if and only if \( x(-t) \) is a periodic orbit of \( \mathcal{T} \) representing \( \alpha \), and it holds that
\[
\sup_{S^1 \times M} \mathcal{T} \leq -l_\alpha.
\]
By Theorem 7.3, we have \( C_{\text{BPS}}(DFT^*M,M;\alpha,l_\alpha) = l_\alpha \). This implies that the set
\[
\{ c > 0 | \text{For any } H \in \mathcal{K}_c, \text{ there exists } z \in \mathcal{P}_\alpha(H) \text{ such that } \mathcal{A}_H(z) \geq c \}
\]
is nonempty. From [8, Proposition 3.3.4] we know that this set is empty or has a minimum. Therefore, \( \mathcal{T} \) has a Hamiltonian periodic orbits whose projection on \( M \) belongs to \( \alpha \). This completes the proof.

Proof of Theorem 1.5. Since \( \psi(M \times \{0\}) \) is the graph of an exact one-form on \( M \) in \( D^{F_2}T^*M \), there is a \( C^\infty \) function \( S \) on \( M \) such that
\[
\psi(M \times \{0\}) = \text{graph}(dS) =: \Sigma_\psi.
\]
Note that \( D^{F_2}T^*M - \Sigma_\psi \) is a fiberwise strictly convex subset of \( T^*M \) containing \( M \times \{0\} \). There exists a Finsler metric on \( M \), denoted by \( F_\psi \), such that the unit open disk bundle \( D^{F_\psi}T^*M \) equals \( D^{F_2}T^*M - \Sigma_\psi \). Now we define a vertical diffeomorphism \( \nu_S \) associated to the exact 1-form \( dS \)
\[
\nu_S : T^*M \to T^*M, \quad \nu_S(x,p) = (x, p - dS(x)). \quad (8.1)
\]
It follows from the definition that
\[
\nu_S^*\lambda_0 - \lambda_0 = \pi^*dS.
\]
Denote by \( \alpha_2 \) and \( \alpha_\psi \) the restriction of the canonical 1-form \( \lambda_0 \) to the unit co-sphere bundles \( S^{F_2}T^*M \) and \( S^{F_\psi}T^*M \) respectively. Then (8.1) shows that \( \nu_S(S^{F_2}T^*M) = S^{F_\psi}T^*M \), and that the contact forms \( \alpha_2 \) and \( \alpha_\psi \) satisfy
\[
\tilde{\nu}_S^*\alpha_\psi - \alpha_2 = t^*\pi^*dS = df, \quad (8.2)
\]
where \( \tilde{\nu}_S : S^{F_2}T^*M \to S^{F_\psi}T^*M \) is induced by \( \nu_S \), \( t : S^{F_2}T^*M \to T^*M \) is the natural inclusion map, and \( f := S \circ \pi \circ t \) is a \( C^\infty \) function on \( S^{F_2}T^*M \). Observe that if \( \gamma \) is any smooth
curve on a Finsler manifold \((M, F)\), then we have
\[
\text{len}_F(\gamma) = \int_{\ell_F \circ \gamma} F(q, \dot{q}) \, dt = \int_{\ell_F \circ \gamma} \langle p, \dot{q} \rangle \, dt = \int_{\ell_F \circ \gamma} \alpha_F
\]
where \(\ell_F\) is the Legendre transform associated to \(F\), and \(\alpha_F\) is the corresponding contact form on \(S^2 T^* M\). This, together with (8.2), implies that \(\tilde{\nu}_S\) maps closed orbits of the geodesic flow on \(S^2 T^* M\) to closed orbits of the geodesic flow on \(S^2 F^* T^* M\) with the same length. Therefore, the length spectra (the set of the lengths of closed geodesics) with respect to \(F_1\) and \(F_2\) on \(M\) are the same. Moreover, since \(\nu_S\) is isotopic to the identity map by the isotopy \(t \mapsto \nu_{tS}\), we have \(l_{\alpha}^{F_2} = l_{\alpha}^{F_1}\) for any nontrivial free homotopy class \(\alpha \in [S^1, M]\).

On the other hand, since BPS capacities are invariant under symplectomorphisms \(\psi\) and \(\nu_S\), we have
\[
C_{\text{BPS}}(D^F T^* M, M; \alpha) = C_{\text{BPS}}(D^F S^2 T^* M, \Sigma_\psi; \psi_* \alpha) = C_{\text{BPS}}(D^F S^2 T^* M, M; \nu_{S*} \psi_* \alpha).
\]
Note that \(\psi\) and \(\nu_S\) are isotopic to \(\text{Id}\), we have \(\nu_{S*} \psi_* \alpha = \psi_* \alpha = \alpha\). This, together with Theorem 7.3 and \(l_{\alpha}^{F_2} = l_{\alpha}^{F_1}\), shows that \(l_{\alpha}^{F_1} = l_{\alpha}^{F_2}\).

\[\square\]

8.2. Recovering the main theorem of [42], proof of Theorem 1.6. This proof is communicated to us by Irie [19].

Proof of Theorem 1.6. Since \(\text{supp} \, H\) is compact and contained in \(S^1 \times \mathbb{T}^n \times \text{int} \, C^*\), there exists a compact strictly convex domain with \(C^\infty\)-boundary which is denoted by \(K^*\) such that
\[
p^* \in \text{int} \, K^*, \quad K^* \subseteq C^*, \quad \text{supp} \, H \subseteq S^1 \times \mathbb{T}^n \times \text{int} \, K^*.
\]
This implies
\[
c \geq \max \{\langle -\alpha, v \rangle | v \in K^* - p^*\} \tag{8.3}
\]
because \(\alpha \in C\), and for every \(v \in K^* - p\), one has \(v + p^* \in C^*\) and
\[
\langle -\alpha, v \rangle = -\langle \alpha, v + p^* \rangle = \langle \alpha, p^* \rangle \leq \langle \alpha, p^* \rangle \leq c.
\]

Now let us take a Minkowskian metric \(F\) on \(\mathbb{R}^n\) which induces constant Finsler metric on \(\mathbb{T}^n\) (for brevity, we simply denote it by \(F\)) such that for every \(x \in \mathbb{T}^n\)
\[
(D^F S^2 T^* T^n)_x = x \times (K^* - p^*).
\]
Define the Hamiltonian \(H_{p^*} : S^1 \times T\mathbb{T}^n \to \mathbb{R}\) as
\[
H_{p^*}(t, x, p) := H(t, q, p^* + p) \quad \forall \, t \in S^1, \forall \, x \in \mathbb{T}^n, \forall \, p \in T_x^* \mathbb{T}^n.
\]
Observe that (8.3) shows
\[
c \geq F(-\alpha) \geq \inf \{\text{len}_F(\gamma) | [\gamma] = -\alpha\}.
\]
Then by Theorem 1.1, there exists \(S^1 \to T\mathbb{T}^n\) such that \(\dot{z}(t) = X_H(t, z(t))\) and \([z] = \alpha\). Since the map
\[
T\mathbb{T}^n \to T\mathbb{T}^n, \quad (x, p) \mapsto (x, p + p^*)
\]
preserves the canonical symplectic form \(\omega_0, \gamma + p^*\) is a \(1\)-periodic orbit of \(X_H\), representing \(\alpha\).

\[\square\]

8.3. Noncompact domains, proof of Theorem 1.8.
Proof of Theorem 1.8. By our assumption, for every \( K_i, \ i = 0, 1, \ldots \) the Finsler metric \( F_i : TM \to \mathbb{R} \) is given by
\[
F_i(x,v) := \sup_{p \in K_i \cap T^*_x M} \langle p, v \rangle.
\]
Since \( K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \), it is obvious that \( F_0 \leq F_1 \leq F_2, \ldots \) Let \( \gamma_i \in C^\infty(S^1, M) \) be the length minimizing \( F_i \)-geodesic loop representing \( \alpha \). Denote by \( D^{F_i}T^*M \) the unit open Finslerian disk cotangent bundle. Then we find \( D^{F_i}T^*M = \text{int} K_i \). Set
\[
\ell_i^i := \text{len}_{F_i}(\gamma_i).
\]
By Theorem 7.3 we have
\[
C_{\text{BPS}}(D^{F_i}T^*M, M; \alpha) = \ell_i^i.
\]
We claim that the sequence of numbers \( \{\ell_i^i\}_{i=0}^\infty \) are bounded. In fact, by our assumption, we have
\[
\ell^{F_i}(\gamma_i) \subseteq A
\]
where \( \ell^{F_i} \) denotes the Legendre transform associated to \( F_i \) (see Definition 3.3). This implies that for every \( t \in S^1 \),
\[
F_i(\gamma_i(t)) = F_i^* \circ \ell^{F_i}(\gamma_i(t)) \leq \sup_{(x,p) \in A} F_i^*(x,p).
\]
Note that \( A \) is a compact set in \( T^*M \), and \( F_0^* \geq F_1^* \geq F_2^* \ldots \). Thus integrating (8.4) over \( S^1 \) yields
\[
c := \sup_i \ell_i^i \leq \sup_{(x,p) \in A} F_0^*(x,p) < \infty.
\]
Since the Hamiltonian \( H \in C^\infty_0(S^1 \times T^*M) \) is compactly supported in \( K = \lim_i K_i \), for \( i \) large enough, we have \( \text{supp} \ H \subset S^1 \times \text{int} K_i = S^1 \times D^{F_i}T^*M \). If \( \min_{t,q} H(t,q,0) \geq c \), then the finiteness of the BPS capacity \( C_{\text{BPS}}(D^{F_i}T^*M, M; \alpha) \) shows that there there is a \( 1 \)-periodic orbit that represents \( -\alpha \).

8.4. Lorentzian Hamiltonian, proof of Theorem 1.9. To prove this theorem, we first formulate the following result.

Proposition 8.1. Let \( H : S^1 \times T^*\mathbb{T}^n \to \mathbb{R} \) be a \( C^\infty \) Hamiltonian compactly supported in the interior of \( C^* \). Given \( p^* \in C^*, \ \alpha \in C \cap H_1(\mathbb{T}^n, \mathbb{Z}) \) and \( c > 0 \) satisfying
\[
c \geq \langle p^*, \alpha \rangle,
\]
we assume
\[
\min_{q,t} H(t,q,p^*) \geq c.
\]
Then \( H \) admits a \( 1 \)-periodic orbit in the homology class \( \alpha \).

This proposition is proved in the same way as the above Theorem 1.6, so we skip the proof.

Proof of Theorem 1.9. We introduce a nondecreasing function \( \phi : \mathbb{R} \to \mathbb{R}_+ \) such that \( \phi(x) = 0 \) for \( x \leq 0 \) and \( \phi(x) = 1 \) for \( x \geq 1 \), and a function \( \varphi : \mathbb{R}_+ \to [0, 1] \) which is nonincreasing and satisfies \( \varphi(x) = 1 \) for \( x \in [0, 0.9] \) and \( \varphi(x) = 0 \) for \( x \geq 1 \). We next introduce an auxiliary Hamiltonian for any given \( 0 < a < b \)
\[
G(q,p) = \begin{cases} c\phi(\frac{H(q,p)}{b-a})\varphi(\frac{\|p\|}{R}), & p \in C, \\ 0, & p \notin C^*, \end{cases}
\]
where $c$ and $R$ are to be determined later. We fix choose $p^* \in C^*$ such that $\min_q H(q, p^*) > b$. For a given homology class $\alpha \in C$, we choose $c \geq \langle p^*, \alpha \rangle$. Finally, we choose $R \gg \|p^*\|$ to be further determined later.

Since we have normalized $\max V = 0$, we get that $\{H > a\} \subset \mathbb{T}^n \times C^*$, hence $\text{supp} \ G \subset C^*$ and $G \in C^{\infty}$. Applying Proposition 8.1, we get that $G$ admits a periodic orbit in the homology class $\alpha$. We assume for a moment that the periodic orbit is not created by $\varphi$, i.e., the periodic orbit does not intersect $\text{supp} \varphi' \left( \frac{||p||}{R} \right) \frac{p}{R\|p\|}$. So the periodic orbit is also a periodic orbit of the Hamiltonian $c\phi(\frac{H(q, p)}{b-a})$. Then by the energy conservation and the injectivity of $\phi$, we get that the periodic orbit is a periodic orbit of $H$ on some energy level in $(a, b)$.

It remains to prove that the periodic orbit is not created by $\varphi$. Suppose there exists such a 1-periodic orbit $\gamma$. We write down the Hamiltonian equation

\[
\begin{cases}
\dot{q} = \frac{c}{b-a} \phi' \left( \frac{H(q, p)}{b-a} \right) \varphi \left( \left| \frac{p}{R} \right| \right) \frac{\partial H}{\partial p} + c\phi \left( \frac{H(q, p)}{b-a} \right) \varphi' \left( \left| \frac{p}{R} \right| \right) \frac{p}{R\|p\|}, \\
\dot{p} = -\frac{c}{b-a} \phi' \left( \frac{H(q, p)}{b-a} \right) \varphi \left( \left| \frac{p}{R} \right| \right) \frac{\partial V}{\partial q}.
\end{cases}
\]

We consider two cases depending on whether $\gamma$ intersects the region $D := C^* \cap \{p_1^2 - (p_2^2 + \ldots + p_n^2) \leq - \min V + b\}$ or not.

Case 1, suppose that $\gamma \cap D = \emptyset$. This implies $H > b$ hence $\phi' \left( \frac{H(q, p)}{b-a} \right) = 0$. We get that

\[
\|\dot{q}\| = \left\| c\phi' \frac{p}{R\|p\|} \right\| \leq \frac{c}{R} |\varphi'|, \quad \dot{p} = 0.
\]

For large $R$, the 1-periodic orbit $\gamma$ cannot have homology class $\alpha \neq 0$.

We remark that in this case, once $\gamma(t) \cap D = \emptyset$ for some $t \in S^1$, due to the openness of $D$ and the fact $\dot{p} = 0$, we get that $\gamma(t) \cap D = \emptyset$ for all $t \in S^1$. This implies that if $\gamma \cap D \neq \emptyset$, then $\gamma \subset D$.

Case 2, suppose $\gamma \subset D$. Since we have assumed that $\gamma \cap \text{supp} \varphi' \left( \frac{||p||}{R} \right) \neq \emptyset$, we get that for $0.9R \leq \|p(t^*)\| \leq R$ some $t^* \in S^1$. Applying the Hamiltonian equation again we have

\[
\begin{cases}
\dot{q} = \frac{c\phi' \varphi}{b-a} (p_1, -q_2, \ldots, -q_n) + O(1/R) \\
\dot{p} = \frac{c\phi' \varphi}{b-a} O(1)
\end{cases}
\]
as $R \to \infty$. Since $\dot{p}$ is uniformly bounded, within time 1, we have $0.8R \leq \|p\| \leq 1.1R$ along the periodic orbit $\gamma$. Since we also know $\gamma \subset D$, denoting $r(p) = \sqrt{p_1^2 + \ldots + p_n^2}$, we get that $|p_1 - r(p)| < \frac{C}{R}$ and $|p_1 + r(p)| \geq R/C$ for some constant $C$ independent of $R$. For a fixed homology class $\alpha \in C$, we have $\alpha_1^2 - r(\alpha)^2 > 0$, hence $0 < \frac{1}{C_\alpha} < |\alpha_1 \pm r(\alpha)| \leq C_\alpha$ for some constant $C_\alpha$. Choosing $R$ large such that $C/R \ll 1/C_\alpha \ll C_\alpha \ll R/C$, we see that the 1-periodic orbit $\gamma$ cannot have homology class $\alpha$. Indeed, since we have $|p_1 + r(p)| \geq R/C$, to attain homology class $|\alpha_1 \pm r(\alpha)| \leq C_\alpha$, we must have $\phi' \varphi = O(1/R)$, hence $\dot{p} = O(1/R)$. Combined with $|p_1 - r(p)| < \frac{C}{R}$, this will imply that $|\alpha_1 - r(\alpha)| < C'/R$ contradicting $\frac{1}{C_\alpha} < |\alpha_1 - r(\alpha)|$. □

8.5. Kawasaki’s conjecture, proof of Theorem 1.11.
Proof of Theorem 1.11. We take a sequence of Finsler metrics \( \{ F_n \} \) to approximate the degenerate Finsler metric \( F(q, p) = \sum_{i=1}^{n} R_i |p_i| : T^*\mathbb{T}^n \to \mathbb{R} \) in the \( C^0 \) norm (we say that it is degenerate since the disk unit tangent bundle is not strictly convex). After Legendre transform, the disk unit cotangent bundles associated to \( F_n \) approximate in the \( C^0 \) norm the following set
\[
D^F T^*\mathbb{T}^n = \mathbb{T}^n \times \left\{ \max_i \frac{1}{R_i} |p_i| \leq 1 \right\} = \mathbb{T}^n \times \left( \prod_{i=1}^{n} [-R_i, R_i] \right).
\]
We next pick a closed geodesic \( \gamma_0 \) on \( \mathbb{T}^n \) in the homology class \(-\alpha\) in the Euclidean metric. The length of the geodesic in the metric \( F \) is \( \text{len}_F(\gamma_0) = \sum_i R_i |\alpha_i| \). Then, for any \( \epsilon > 0 \), there exists \( N \) such that for all \( n > N \), it holds that \( \text{len}_{F_n}(\gamma_0) \leq \sum_i R_i |\alpha_i| + \epsilon \). So we get for all \( n > N \)
\[
\sum_i R_i |\alpha_i| + \epsilon \geq \inf \{ \text{len}_{F_n}(\gamma) : [\gamma] = \alpha \}.
\]
For any Hamiltonian \( H \) compactly supported in the interior of \( D^F T^*\mathbb{T}^n \), there exists \( N' \) such that for all \( n > N' \) we have that \( H \) is also compactly supported in the interior of \( D^F_n T^*\mathbb{T}^n \). Then the theorem follows directly from the main theorem.\( \square \)

8.6. Symplectic nonsqueezing, proof of Theorem 1.12 and 1.13.

Proof of Theorem 1.12. If \( s \leq r \), the inclusion \( P^{2n}(s) \to Y^{2n}(r) \) is a symplectic embedding. Conversely, suppose that there exists a symplectic embedding \( \phi : P^{2n}(s) \to Y^{2n}(r) \). For sufficiently small \( \epsilon > 0 \), let \( p^* \in \Delta^n(s) \) so that \( |p^*| < \epsilon \). By our assumption, the image \( \phi(T^n \times \{ p^* \}) \) is a smooth section in \( T^*\mathbb{T}^n \), equivalently,
\[
\Sigma_{p^*} := \phi(T^n \times \{ p^* \}) = \{(x, \sigma(x)) | x \in T^n \}
\]
where a \( C^\infty \) closed 1-form \( \sigma \) on \( T^n \) (since \( \phi(T^n \times \{ p^* \}) \) is a Lagrangian submanifold in \( T^*\mathbb{T}^n \)). Obviously, \( \Sigma_{p^*} \subseteq Y^{2n}(r) \). Now we choose a strictly convex (closed) subset \( K \subseteq \Delta^n \) with \( C^\infty \)-boundary sufficiently close to \( \partial \Delta^n \) such that \( p^* \in \text{int} K \). Since \( \phi \) is an embedding from \( P^{2n}(s) \) into \( Y^{2n}(r) \), \( \phi(T^n \times K) \) is a compact set in \( Y^{2n}(r) \) containing \( \Sigma_{p^*} \). Hence, one can find a strictly fiberwise convex compact subset \( T \subseteq Y^{2n}(r) \) such that \( \phi(T^n \times K) \subseteq T \). Let \( F_K \) and \( F_T \) be the Finsler metrics on \( T^n \) associated to \( T^n \times (K - p^*) \) and \( T - \Sigma_{p^*} \) respectively, namely,
\[
F_K(x, v) := \sup_{p \in (K - p^*)} \langle p, v \rangle,
F_T(x, v) := \sup_{p \in (T - \Sigma_{p^*}) \cap T^n} \langle p, v \rangle.
\]
It is obvious from the above definitions that \( D^F K T^n = T^n \times (K - p^*) \) and \( D^F T T^n = T - \Sigma_{p^*} \). Observe that for any closed 1-form \( \sigma \) on \( T^n \), the map
\[
\tau_{\sigma} : TT^n \to TT^n, \quad (x, p) \mapsto (x, p - \sigma(x))
\]
preserves the canonical symplectic form \( \omega_0 \). The map
\[
\Phi : T^n \times (K - p^*) \to T - \Sigma_{p^*}, \quad \Phi = \tau_{\sigma} \circ \phi \circ \tau_{-p^*}
\]
is a symplectic embedding satisfying \( \Phi(T^n \times \{ 0 \}) = T^n \times \{ 0 \} \). Since the BPS-capacity is invariant under symplectic diffeomorphism, Proposition 7.2 shows that for every non-trivial \( \alpha \in [S^1, T^n] \cong \mathbb{Z}^n \),
\[
C_{BPS}(D^F K T^n, T^n; \alpha) = C_{BPS}(\Phi(D^F K T^n), \Phi(T^n); \alpha) \leq C_{BPS}(D^F T T^n, T^n; \alpha) \tag{8.5}
\]

Set $e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}^n$ and let $\gamma(t) = [e_1 t]$ be a closed curve in $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ representing $e_1$. To calculate the length of $\gamma$ with respect to the Finsler metrics $F_K$ and $F_T$ respectively, we find

\[
F_K(x, e_1) = \sup_{p \in (K - p^*)} \langle p, e_1 \rangle = \sup_{p \in K} \langle p, e_1 \rangle - \langle p^*, e_1 \rangle \geq s - 2\epsilon,
\]

\[
F_T(x, e_1) = \sup_{p \in (T - \Sigma^*) \cap T^* \mathbb{T}^n} \langle p, e_1 \rangle \leq r.
\]

So we have $\text{len}_{F_K} (\gamma) \geq s - 2\epsilon$ and $\text{len}_{F_T} (\gamma) \leq r$. Note that $F_K$ is invariant under translations $\tau_a : \mathbb{T}^n \to \mathbb{T}^n$, $\tau_a(y) = y + a$, $\gamma(t)$ is the length minimizing closed $F_K$-geodesics in class $e_1$. Theorem 7.3, together with (8.5), implies

\[
s - 2\epsilon \leq \text{len}_{F_K} (\gamma) = l_{e_1} \leq \text{len}_{F_T} (\gamma) \leq r.
\]

Since $\epsilon$ is arbitrary, the proof of the theorem completes.

The proof of Theorem 1.13 is essentially analogous to that of Theorem 1.12.

**Proof of Theorem 1.13.** It is obvious that for $s \leq r$ the inclusion $\mathbb{T}^n \times B^n(s) \to \mathbb{T}^n \times \mathbb{Z}^n(r)$ is a symplectic embedding. Conversely, if there is a symplectic embedding $\phi : \mathbb{T}^n \times B^n(s) \to \mathbb{T}^n \times \mathbb{Z}^n(r)$. By the assumption, every image $\phi(\mathbb{T}^n \times \{u_i\})$ is a smooth section in $T^* \mathbb{T}^n$, meaning that

\[
\Sigma_i := \phi(\mathbb{T}^n \times \{u_i\}) = \{(x, \sigma_i(x)) | x \in \mathbb{T}^n \}
\]

with closed 1-forms $\sigma_i \in \Omega^1(\mathbb{T}^n)$. One can choose strictly convex compact subsets $K_i \subseteq B^n(s)$ containing $u_i$ with $C^\infty$ boundaries approximating $\partial B^n(s)$ and fibewise strictly convex compact subsets $\bar{T}_i \subseteq \mathbb{T}^n \times \mathbb{Z}^n(r)$ containing $\phi(\mathbb{T}^n \times \bar{K}_i)(\geq \Sigma_i)$. Pick the homotopy class $-e_1 = (-1, 0, \ldots, 0) \in \mathbb{Z}^n$. Then $\gamma(t) = [-e_1 t]$ is a closed curve in $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ representing $-e_1$. Let $F_K$ and $F_T$ be the Finsler metrics on $T^* \mathbb{T}^n$ associated to $\mathbb{T}^n \times (K_i - u_i)$ and $\bar{T}_i - \Sigma_i$ respectively. Finally, arguing as in the proof of Theorem 1.12 by making use of the monotonicity property of BPS-capacity and the fact that $|u_i - (s, 0, \ldots, 0)|$ is sufficiently small for large $i > N$ leads to $s \leq r$.

8.7. Symplectic squeezing, proof of Theorem 1.14.

**Proof of Theorem 1.14.** Let $U$ be an open convex set in $\mathbb{R}^n$, not necessarily strictly convex or compact, and $\{U_i\}$ be a sequence of strictly convex sets with $C^\infty$ boundaries and satisfying $U_i \subseteq U_{i+1}$ and $\lim U_i = U$. Let $F_i$ be the sequence of Finsler metrics associated to $U_i$ via $F_i(q, \dot{q}) = \sup_{p \in U_i} \langle p, \dot{q} \rangle$ for any $(q, \dot{q}) \in T^* U_i$.

For each $H$ compactly supported in $\mathbb{T}^n \times U$, we have that there exists some $I$ such that $\text{supp}(H) \subseteq \mathbb{T}^n \times U_i$ for all $i > I$. This shows that $C_{\text{BPS}}(\mathbb{T}^n \times U, \mathbb{T}^n, \alpha) = \lim C_{\text{BPS}}(\mathbb{T}^n \times U_i, \mathbb{T}^n, \alpha)$.

Suppose now $v$ is proportional to an integer vector $\alpha$.

We define $U_i$ to be a sequence of ellipsoids centered at the origin with minor axis being the vector $rv$ so that they are all tangent to $Y^{2n}(r, v)$ at the two points $\pm rv$.

A closed geodesic with homology class $\alpha$ has constant velocity $\dot{q} = \alpha$. For each $i$, the sup $r|\alpha|$ in the definition of $F_i$ is always attained at the point $rv$ or $-rv$. In fact, for all $p$ on one boundary of
\( Y^{2n}(r, v) \), the inner product \( \langle p, q \rangle \) is constant \( \pm r|\alpha| \). So in this case, \( C_{\text{BPS}}(Y^{2n}(r, v), \mathbb{T}^n, \pm \alpha) = r|\alpha| \).

Suppose \( v \) is not proportional to an integer vector. In this case, for any \( \alpha \in H_1(\mathbb{T}^n, \mathbb{Z}) \setminus \{0\} \), we have that \( \alpha \) is not perpendicular to \( v^\perp \). So we can find \( p \) on the boundary of \( Y^{2n}(r, v) \) such that \( \langle \alpha, p \rangle \) is as large as we wish. For any \( N > 0 \), we can find \( i \) and \( p_i \in \partial U_i \) such that \( \langle \alpha, p_i \rangle > N \). This shows that \( C_{\text{BPS}}(Y^{2n}(r, v), \mathbb{T}^n, \pm \alpha) = \infty \).

\[ \square \]

9. PROOF OF THE MAIN TECHNICAL RESULTS

9.1. Convexity results, proof of Lemma 2.2.

**Proof of Lemma 2.2.** Consider the function \( r : \mathbb{R} \times S^1 \to \mathbb{R} \) defined by

\[ r(s, t) := \rho(u(s, t)). \]

Arguing by contradiction, we assume the open subset of \( \mathbb{R} \times S^1 \)

\[ \Sigma := r^{-1}((\rho_1, \infty)) = u^{-1}((\rho_1, \infty) \times \partial X) \]

is not empty. By our assumption, \( \Sigma \) is bounded, so \( r \) achieves its maximum on \( \Sigma \), and

\[ r_0 = \max_{(s, t) \in \Sigma} r(s, t) > \rho_1. \] (9.1)

It is easy to verify that

\[ X_{H_{s,t}} = \frac{\partial H_{s,t}}{\partial \rho} R \quad \text{on } [\rho_0, \infty) \times \partial X. \]

Then, by (2.1) and our assumptions concerning \( H_{s,t} \) and \( J_{s,t} \),

\[ \frac{\partial r}{\partial s} = d\rho(\partial_s u) = d\rho \circ J_{s,t}(u)(-\partial_t u + X_{H_{s,t}}(u)) = \hat{\lambda}(-\partial_t u + \partial_\rho H_{s,t}(u) R) = -\hat{\lambda}(\partial_t u) + \mu f(s) r^\mu(s, r). \] (9.2)

Similarly,

\[ \frac{\partial r}{\partial t} = d\rho(\partial_t u) = d\rho \circ J_{s,t}(\partial_s u - J_{s,t} X_{H_{s,t}}) = \hat{\lambda}(\partial_s u - \partial_\rho H_{s,t}(u) Z(u)) = \hat{\lambda}(\partial_s u). \] (9.3)

Write (9.2) and (9.3) in a less coordinate-bound way as

\[ d^c r = \frac{\partial r}{\partial t} ds - \frac{\partial r}{\partial s} dt = u^* \hat{\lambda} - \mu f(s) r^\mu dt. \] (9.4)

Note that \( dd^c r = -\Delta r ds \wedge dt \) and

\[ |\partial_s u|_{J_{s,t}}^2 = d\hat{\lambda}(J_{s,t}\partial_s u, \partial_s u) = d\hat{\lambda}(\partial_t u - X_{H_{s,t}}(u), \partial_s u) = -d\hat{\lambda}(\partial_s u, \partial_t u) + dH_{s,t}(\partial_s u). \]

Then by differentiating (9.4) we arrive at

\[ \Delta r - \mu(\mu - 1) f(s) r^{\mu-1} \partial_s r(s, t) = |\partial_s u|_{J_{s,t}}^2 + \mu f(s) r^\mu. \] (9.5)

Keeping in mind that \( r > 0 \) on \( \Sigma \) by definition, the left hand side of (9.5) is nonnegative by our assumptions, then the maximum principle implies that \( r \) achieves its maximum on the boundary
\( \partial \Sigma \) of \( \Sigma \). But \( r_{|\partial \Sigma} = \rho_1 \), this contradicts with (9.1). So \( \Sigma \) must be empty. This completes the proof of Lemma 2.2. \( \square \)

9.2. The radial Hamiltonian and its action, proof of Lemma 3.4.

Proof of Lemma 3.4. For each \((x, y) \in T^* M \setminus \{0\} \), locally, we write
\[
F^{*2}(x, y) = g^{ij}(x, y)y_iy_j.
\]
Then,
\[
dH(x, y) = h'(F^{*2}(x, y)) \left[ \frac{\partial g^{ij}}{\partial x_k} y_iy_j dx_k + \frac{\partial g^{ij}}{\partial y_l} y_iy_j dy_l + 2g^{ij}y_i dy_j \right].
\]
Using \(-dH = \omega_0(X_H, \cdot)\), we get
\[
X_H = h'(F^{*2}(x, y)) \left[ -\frac{\partial g^{ij}}{\partial x_k} y_iy_j \frac{\partial}{\partial y_k} + \left( \frac{\partial g^{ij}}{\partial y_l} y_iy_j + 2g^{il}y_i \right) \frac{\partial}{\partial x_l} \right]
\]
which gives the Hamiltonian equation
\[
\begin{cases}
\dot{x}_l = h'(F^{*2}(x, y)) \left( \frac{\partial g^{ij}}{\partial y_l} y_iy_j + 2g^{il}y_i \right) = 2h'(F^{*2}(x, y))g^{il}y_i, \\
\dot{y}_k = -h'(F^{*2}(x, y)) \frac{\partial g^{ij}}{\partial x_k} y_iy_j
\end{cases}
\tag{9.6}
\]
Here we have use the homogeneity property of the Finsler metric \( F \):
\[
\frac{\partial g^{ij}}{\partial y_l} y_iy_j = \frac{\partial^3 (F^2)}{2\partial y_i \partial y_j \partial y_l} y_i = 0
\]
For a solution \( z(t) \) of the Hamiltonian equation (9.6) sitting in \( T^* M \setminus \{0\} \), we compute
\[
\frac{d}{dt} [F^{*2}(x(t), y(t))] = \frac{\partial g^{ij}}{\partial x_k} \dot{x}_k y_iy_j + \frac{\partial g^{ij}}{\partial y_l} y_iy_j \dot{y}_l + 2g^{ij}(x, y)y_i \dot{y}_j = \frac{\partial g^{ij}}{\partial x_k} h'(F^{*2}(x, y)) \left( 2g^{ik}y_iy_j - 2g^{ij}(x, y)h'(F^{*2}(x, y)) \right) \frac{\partial g^{kl}}{\partial x_l} y_ky_ly_j = 0
\tag{9.7}
\]
which implies
\[
F^{*2}(x(t), y(t)) \equiv \text{constant} \quad \forall \ t \in \mathbb{R}.
\]
Denote \( C = h'(F^{*2}(x(t), y(t))) \). The Legendre transform,
\[
\tau_x : T^*_x M \rightarrow T_x M \quad (x, y) \mapsto (x, v),
\]
where \( v = \sum v^i \frac{\partial}{\partial x^i} \) and \( v^l = \sum g^{il} y_i \), implies
\[
\dot{v}^l = \frac{\partial g^{il}}{\partial x^k} \dot{x}^k y_i + \frac{\partial g^{il}}{\partial y_k} \dot{y}_k y_i + g^{il}(x, y) \dot{y}_i
\]
\[
= \frac{\partial g^{il}}{\partial x^k} \dot{x}^k y_i + g^{il}(x, y) \dot{y}_i
\]
\[
= \frac{\partial g^{il}}{\partial x^k} 2C g^{jk} y_j y_i - C g^{il}(x, y) g^{iu} \frac{\partial g^{jk}}{\partial x^i} y_j y_k
\]
\[
= \frac{\partial g^{il}}{\partial x^k} 2C v^k g^{vt} - C g^{il}(x, y) g^{iu} \frac{\partial g^{jk}}{\partial x^i} y_j y_k
\]
\[
= -2C g^{il} \frac{\partial g^{ji}}{\partial x^k} v^k v^t + C g^{il} \frac{\partial g^{jt}}{\partial x^i} g^{sk} g^{st} v^s
\]
\[
= -C g^{il} \left( 2 \frac{\partial g^{jl}}{\partial x^k} - \frac{\partial g^{kt}}{\partial x^i} \right) v^k v^t.
\] (9.8)

Hence,
\[
\nabla^F_x v = [\dot{v}^l(t) + v^i(t) \dot{x}^k(t) \Gamma^l_{jk}(x(t), v(t))] \left. \frac{\partial}{\partial x^i} \right|_{x(t)}
\]
\[
= [- C g^{il} \left( 2 \frac{\partial g^{jl}}{\partial x^k} - \frac{\partial g^{kt}}{\partial x^i} \right) v^k v^t + v^i(t) 2C v^k(t) \Gamma^l_{jk}(x(t), v(t))] \left. \frac{\partial}{\partial x^i} \right|_{x(t)}.
\]

Here
\[
\Gamma^l_{jk} = \gamma^l_{jk} \frac{g^{il}}{F} \left( A_{ij} N^s_k - A_{jks} N^s_i + A_{kl} N^s_j \right)
\]
are the components of the Chern connection with the Christoffel symbols
\[
\gamma^l_{jk} := \frac{1}{2} g^{ls} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right),
\]
the Cartan tensor
\[
A_{ij} (x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y_k} = \frac{F}{4} \frac{\partial^3 (F^2)}{\partial y_i \partial y_j \partial y_k}
\]
and the coefficients of the nonlinear connection on \( TM \setminus \{0\} \)
\[
N^l_{jk} (x, y) := \gamma^l_{jk} y_k - \frac{1}{F} A_{j}^{l} \gamma^{lk}_{rs} y_r y_s
\]
Using the homogeneity property of Finsler metric again, we have
\[
\Gamma^l_{jk}(x, v) v^j v^k = \frac{1}{2} g^{ls} \left( 2 \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} \right) v^j v^k
\]
Therefore,
\[
\nabla^F_x v = 0.
\]
So, under the Legendre transform \( \ell : T^* M \rightarrow TM \) induced by the Finsler metric, the Hamiltonian equation associated to \( H \) can be written as
\[
\left\{ \begin{array}{l}
\dot{x} = 2C v, \\
\nabla^F_x v = 0
\end{array} \right. \quad (9.9)
\]
Then
\[ \nabla^F_x \hat{x} = 2C \nabla^F_x v = 0 \]
which means that \( x(t) \) is a Finsler geodesic loop on \( M \). By (9.9), we have
\[
F^2(x(t), \dot{x}(t)) = g_{ij} \dot{x}^i \dot{x}^j
= 4C^2 g_{ij} v^i v^j
= 4C^2 g_{ij} g^{ik} y_k y^l y_l
= 4C^2 F^{*2}(x(t), y(t))
= 4h'(F^{*2}(x(t), y(t)))^2 F^{*2}(x(t), y(t))
= \left( f'(F^*(x(t), y(t))) \right)^2
\]
which implies that the Finsler length of the geodesic loop \( x(t) \) equals to \( \pm f'(r_z) \) for some constant \( r_z = F^*(x(t), y(t)) \geq 0 \). Then
\[
r_z \ell(\dot{x}) = 2r_z C \ell(v) = 2r_z C y = 2r_z h'(r_z^2) y = f'(r_z) y.
\]
Moreover, the action of \( \mathcal{A}_H \) at \( z(t) \) can be computed as follow:
\[
\mathcal{A}_H(z(t)) = \int_{S^1} \langle y_t(t), \dot{x}_t(t) \rangle - \int_{S^1} h(F^{*2}(x, y))
= \int_{S^1} 2C g^{il} y_i y_l - \int_{S^1} h(F^{*2}(x, y))
= 2h'(F^{*2}(z(t))) F^{*2}(z(t)) - h(F^{*2}(z(t)))
= f'(F^*(z(t))) F^*(z(t)) - f(F^*(z(t))).
\]
Conversely, if \( x \) is a \( F \)-geodesic loop satisfying (3.2), it is easy to verify that the loop \( z = (x, y) \) is a critical point of the action functional \( \mathcal{A}_H \), we omit it here. \( \square \)

9.3. Quadratic modifications, proof of Lemma 5.3.

Proof of Lemma 5.3. Set
\[
a_0 := \inf_{x \in M, |v|_x = 1} \inf_{u \neq 0} \frac{g^F(x, v)[u, u]}{g_x(u, u)} \quad \text{and} \quad a_1 := \sup_{x \in M, |v|_x = 1} \sup_{u \neq 0} \frac{g^F(x, v)[u, u]}{g_x(u, u)}.
\]
Due to the compactness of \( M \), we have \( 0 < a_0 \leq a_1 \). Since \( g^F \) is positively homogeneous of degree 0, it holds that for any \( (x, v) \in TM \setminus \{0\} \) and \( (x, u) \in TM \),
\[
a_0 g_x(u, u) \leq g^F(x, v)[u, u] \leq a_1 g_x(u, u).
\]
By rescaling the metric \( g \) (for instance, by choosing \( \tilde{g} := a_0 g \)), we may assume that
\[
g_x(u, u) \leq g^F(x, v)[u, u] \leq A g_x(u, u) \quad \forall (x, v) \in TM \setminus \{0\}, \forall (x, u) \in TM
\]
for some constant \( A > 1 \). In particular, we have
\[
|v|_x \leq L_0(x, v) \leq A|v|_x \quad \forall (x, v) \in TM.
\]
In order to construct convex quadratic Lagrangians, we follow closely the line of [25, Section 2] by modifying \( L_0 \) near the zero section of \( TM \). In the following we first construct two auxiliary functions, see Figure 9.3.
Choose positive parameters $0 < \epsilon < \delta < \frac{\eta}{2}$. Let $\lambda_{\epsilon,\delta}^\mu : (0, \infty) \to \mathbb{R}$ be a smooth function such that $\lambda_{\epsilon,\delta}^\mu(s) = 0$ for $s \in [0, \epsilon)$, $\lambda_{\epsilon,\delta}^\mu(s) = \mu s + \sigma$ for $s \in (\delta, \infty)$, $\lambda_{\epsilon,\delta}^\mu$ is convex and $(\lambda_{\epsilon,\delta}^\mu)'(s) > 0$ on $(\epsilon, \infty)$, where $\mu > 0$ and $\sigma < 0$ are suitable constants. Let $\chi_{\delta,\rho}^\kappa$ be another smooth function such that $\chi_{\delta,\rho}^\kappa(s) = \kappa(s - \delta)$ for $s \in [0, \delta]$, $\chi_{\delta,\rho}^\kappa(s) = \rho$ for $s \in [\eta/A, \infty)$, and $\chi_{\delta,\rho}^\kappa$ is concave and nondecreasing, here $\kappa > 0$ and $\rho > 0$ are suitable constants.

![Diagram of auxiliary functions](image)

**Figure 9.3.** The auxiliary functions

Define the new Lagrangians $L_\eta$ by

$$L_\eta(x, v) := \frac{1}{\mu} \left\{ \lambda_{\epsilon,\delta}^\mu(L_0(x, v)) + \chi_{\delta,\rho}^\kappa(|v|^2_x) - \sigma - \rho \right\}.$$

We check properties (a) and (b) in two steps.

(i) Obviously, for every $\eta > 0$, $L_\eta$ is smooth on the whole $TM$. Suppose that $\mu \geq \kappa$ and $\kappa$ is sufficiently large such that

$$\delta \kappa + \sigma + \rho > 0 \quad \text{and} \quad \kappa(\delta - \epsilon) + \sigma > 0. \quad (9.14)$$

If $0 \leq L_0(x, v) \leq \epsilon$ (hence $|v|^2_x \leq \epsilon$ by (9.13)), by the first inequality in (9.14),

$$L_\eta(x, v) = \frac{1}{\mu} \left\{ \kappa(\epsilon - \delta) \right\} - \sigma - \rho \}
= \frac{\kappa}{\mu} |v|^2_x - \frac{\kappa \delta + \sigma + \rho}{\mu} \leq L_0(x, v). \quad (9.15)$$
If \( \epsilon \leq L_0(x, v) \leq \delta \), observe that
\[
\mathcal{L}_{\epsilon, \delta}^\mu(s) \leq \frac{\mu \delta + \sigma}{\delta - \epsilon}(s - \epsilon)
\]
for every \( s \in (\epsilon, \delta) \), we have
\[
L_\eta(x, v) \leq \frac{1}{\mu} \left\{ \frac{\mu \delta + \sigma}{\delta - \epsilon}(L_0(x, v) - \epsilon) + \kappa(|v|^2_x - \delta) - \sigma - \rho \right\}
\leq \frac{1}{\mu} \left\{ \frac{\mu \delta + \sigma}{\delta - \epsilon}(L_0(x, v) - \epsilon) + \kappa(L_0(x, v) - \delta) - \sigma - \rho \right\}
\leq \frac{1}{\mu} \left\{ \left( \frac{\mu \delta + \sigma}{\delta - \epsilon} + \kappa \right)L_0(x, v) - \frac{\mu \delta + \sigma}{\delta - \epsilon} \epsilon - \kappa \delta - \sigma - \rho \right\}. 
\] (9.16)

Then
\[
L_\eta(x, v) - L_0 \leq \frac{1}{\mu} \left\{ \left( \frac{\mu \epsilon + \sigma}{\delta - \epsilon} + \kappa \right)L_0(x, v) - \frac{\mu \delta + \sigma}{\delta - \epsilon} \epsilon - \kappa \delta - \sigma - \rho \right\}
\leq \frac{1}{\mu} \left\{ \left( \frac{\mu \epsilon + \sigma}{\delta - \epsilon} + \kappa \right)\delta - \frac{\mu \delta + \sigma}{\delta - \epsilon} \epsilon - \kappa \delta - \sigma - \rho \right\}
= -\frac{\rho}{\mu} < 0. 
\] (9.17)

By the second inequality in (9.14),
\[
\frac{\mu \epsilon}{\delta - \epsilon} + \kappa > \frac{\mu \epsilon}{\delta - \epsilon} > 0.
\]

So, by (9.16), when \( \epsilon \leq L_0(x, v) \leq \delta \),
\[
L_\eta(x, v) - L_0 \leq \frac{1}{\mu} \left\{ \left( \frac{\mu \epsilon + \sigma}{\delta - \epsilon} + \kappa \right)\delta - \frac{\mu \delta + \sigma}{\delta - \epsilon} \epsilon - \kappa \delta - \sigma - \rho \right\}
\leq \frac{1}{\mu} \left\{ \left( \frac{\mu \epsilon}{\delta - \epsilon} + \kappa \right)\delta - \frac{\mu \delta + \sigma}{\delta - \epsilon} \epsilon - \kappa \delta - \sigma - \rho \right\}
= -\frac{\rho}{\mu} < 0.
\] (9.18)

Therefore, if \( \epsilon \leq L_0(x, v) \leq \delta \), \( L_\eta(x, v) \leq L_0 \).

If \( L_0(x, v) \geq \delta \),
\[
L_\eta(x, v) = \frac{1}{\mu} \left\{ \mu L_0(x, v) + \sigma + \chi_{\delta, \rho}^\kappa(|v|^2_x) - \sigma - \rho \right\} \leq L_0(x, v), 
\] (9.19)

and in this case, if \( L_0(x, v) \geq \eta \) (thus \(|v|^2_x > \eta/A \) by (9.13)), we have
\[
L_\eta(x, v) = L_0(x, v). 
\] (9.20)

(ii) By our assumption, \( \mathcal{L}_{\epsilon, \delta}^\mu \) is convex, so for any \( s \in [0, \infty) \) we have
\[
\mathcal{L}_{\epsilon, \delta}^\mu(s) \geq \frac{d\mathcal{L}_{\epsilon, \delta}^\mu}{ds} \bigg|_{s=\delta} (s - \delta) + \mathcal{L}_{\epsilon, \delta}^\mu(\delta)
\geq \mu(s - \delta) + \mu \delta + \sigma = \mu s + \sigma.
\] (9.21)

\[
\chi_{\delta, \rho}^\kappa(s) \geq \chi_{\delta, \rho}^\kappa(0) = -\kappa \delta \text{ because } \chi_{\delta, \rho}^\kappa \text{ is nondecreasing on } [0, \infty). 
\]

Combining this with (9.21) shows
\[
L_\eta \geq \frac{1}{\mu} \left\{ \mu L_0 + \sigma - \kappa \delta - \sigma - \rho \right\} = L_0 - \frac{\kappa \delta + \rho}{\mu}. 
\]

Since \( 0 < \delta < \eta \) and \( 0 < \kappa \leq \mu \) by our assumption, taking \( \mu > 0 \) sufficiently large yields
\[
L_\eta(x, v) \geq L_0 - \eta.
\]

To complete the proof, it suffices to prove that \( L_\eta \) is fiberwise convex and quadratic at infinity. Due to the compactness of \( M \) and \( L_0(x, v) = g^F(x, v)[v, v] \) for any \( (x, v) \in TM \setminus \{0\} \), the condition (L2) holds obviously. To show (L1), for every \( v \in TM \setminus \{0\} \) and \( u \in T_x M \) we
compute
\[
\partial_{vv}L_\eta(x, v)[u, u] = \frac{\partial^2}{\partial s \partial t} L_\eta(x, v + su + tu) \bigg|_{s=t=0}
\]

\[
= \frac{1}{\mu} \left\{ (\lambda_{\epsilon, \delta}^\mu)''(L_0(x, v)) (\partial_v L_0(x, v)[u])^2 + 4(\chi_{\delta, \rho}^\kappa)(|v|_x^2)g_x(v, u)^2 
+ (\lambda_{\epsilon, \delta}^\mu)'(L_0(x, v))\partial_{vv} L_0(x, v)[u, u] + 2(\chi_{\delta, \rho}^\kappa)'(|v|_x^2)|u|_x^2 \right\}. \tag{9.22}
\]

We prove \((L_1)\) in two cases.

Case 1. If \(L_0(x, v) < \delta, |v|_x < \delta\) by (9.13), and thus
\[
\partial_{vv}L_\eta(x, v)[u, u] \geq \frac{2\kappa}{\mu} |u|_x^2. \tag{9.23}
\]

Here we have use the properties that \(\lambda_{\epsilon, \delta}^\mu\) is convex, \((\lambda_{\epsilon, \delta}^\mu)' \geq 0\) and \((\chi_{\delta, \rho}^\kappa)'' = 0\) on \([0, \delta]\), and the fact that \(g^x = \frac{1}{2} \partial_{vv} L_0\) is positive definite.

Case 2. If \(L_0(x, v) \geq \delta, |v|_x^2 \geq \delta/A\) by (9.13). \(\lambda_{\epsilon, \delta}^\mu\) equals to the affine function \(\mu s + \sigma\) and \(\chi_{\delta, \rho}^\kappa\) is non-decreasing on \([0, \infty)\), thus
\[
\partial_{vv}L_\eta(x, v)[u, u] = \mu \partial_{vv} L_0(x, v)[u, u] + 4(\chi_{\delta, \rho}^\kappa)'(|v|_x^2)|u|_x^2 
\geq \mu \partial_{vv} L_0(x, v)[u, u] + 4(\chi_{\delta, \rho}^\kappa)'(|v|_x^2)|v|_x^2 |u|_x^2 
\geq \mu |u|_x^2 + \frac{4\delta}{A} (\chi_{\delta, \rho}^\kappa)'(|v|_x^2)|u|_x^2. \tag{9.24}
\]

Since \((\chi_{\delta, \rho}^\kappa)''(s) = 0\) for \(s \in [\eta/A, \infty)\), and \((\chi_{\delta, \rho}^\kappa)'(|v|_x^2)\) is bounded for \(|v|_x^2 \in [\delta/A, \eta/A]\), we may choose \(\mu\) so large that
\[
\mu + \frac{4\delta}{A} (\chi_{\delta, \rho}^\kappa)'(|v|_x^2) > 0.
\]

This and (9.24) imply that \(L_\eta\) satisfy \((L_1)\) for \(L_0(x, v) \geq \delta\).
\[
\square\]
9.4. Construction of the profile functions, proof of Theorem 7.4.

Proof of Theorem 7.4. We prove Theorem 7.4 in three steps.

Step 1. The idea of proving (i) is to construct a downward exhausting sequence of profile functions \( \{h_k\}_{k \in \mathbb{N}} \) and use Proposition 4.5 to compute symplectic homology.

Fix \( \alpha \in \mathbb{R} \setminus \Lambda_\alpha \). The functions \( h_k : [0, \infty) \to +\infty \) are smooth functions such that \( h_k|_{[0, \rho_k]} = h_k(0) \) and \( h_k|_{[1, +\infty)} = 0 \) with \( h_k(0) \to -\infty \) and \( \rho_k \to 1 \) as \( k \to \infty \). Moreover, the derivatives are required to satisfy \( h'_k \geq 0 \) everywhere, \( h''_k \geq 0 \) near \( \rho_k \), \( h'''_k \leq 0 \) near \( \rho_k \), and \( h'''_k = 0 \) elsewhere, see Figure 9.4.

We prove that for each \( k \in \mathbb{N} \), there is a natural isomorphism

\[
\Psi_{h_k}^\alpha : HF_*^{(a, +\infty)}(h_k \circ F^*; \alpha) \to H_*(\Lambda_\alpha^{a^2/2} M).
\] (9.25)

We first consider a monotone homotopy, indicated in Figure 9.4, from \( h_k \circ F^* \) to \( \tilde{h}_k \circ F^* \), where \( \tilde{h}_k \) is obtained by making the graph of \( h_k \) linear with slope \( a \) near \( \rho_k \). By Lemma 3.4, no Hamiltonian 1-periodic orbit with action in the action window \( [a, \infty) \) appears during the homotopy, this gives the isomorphism

\[
\sigma_{\tilde{h}_k h_k} : HF_*^{(a, +\infty)}(h_k \circ F^*; \alpha) \to HF_*^{(a, +\infty)}(\tilde{h}_k \circ F^*; \alpha).
\] (9.26)
Next, we consider another monotone homotopy from $h_k^a \circ F^* \circ F^*$ to $\tilde{h}_k \circ F^*$ as shown in Figure 9.5. Here $h_k^a$ is obtained by making the graph of $h_k$ linear with slope $a$ near $\rho_{k1}$. This homotopy induces the monotone homomorphism

$$\sigma_{\tilde{h}_k h_k^a} : \text{HF}_s^{(a, +\infty)}(h_k^a \circ F^*; \alpha) \rightarrow \text{HF}_s^{(a, +\infty)}(\tilde{h}_k \circ F^*; \alpha),$$

(9.27)

which is actually an isomorphism because no Hamiltonian 1-periodic orbit has action $a$ during the homotopy. Observe that the minimal action of the Hamiltonian 1-periodic orbit of $h_k^a \circ F^*$ is larger than $a$, and the maximal action of the Hamiltonian 1-periodic orbit of $h_k \circ F^*$ is less than $C_{h_k, a} := c_{h_k, a}$ (here $c_{f, \lambda}$ is defined as in Theorem 1.3). So we obtain the following isomorphisms

$$\text{HF}_s^{(a, +\infty)}(h_k^a \circ F^*; \alpha) \xrightarrow{\frac{\alpha}{[F]^{-1}}} \text{HF}_s^{(-\infty, +\infty)}(h_k^a \circ F^*; \alpha) \xrightarrow{\frac{\alpha}{[F]^{-1}}} \text{HF}_s^{(-\infty, C_{h_k, a})}(h_k^a \circ F^*; \alpha).$$

(9.28)

For $a \notin \Lambda_{\alpha}$, Theorem 1.3 yields the isomorphism

$$\Psi_{h_k^a}^\alpha : \text{HF}_s^{(-\infty, C_{h_k, a})}(h_k^a \circ F^*; \alpha) \rightarrow H_s(A_{\alpha}^{2/2} M).$$

(9.29)

By (9.26) – (9.29), we arrive at the isomorphism (9.25).

To conclude the proof of (i) in Theorem 7.4, we check that the sequence $\{h_k \circ F^*\}_{k \in \mathbb{N}}$ is downward exhausting for the inverse limiting defining symplectic homology. Firstly, for any $H \in M^{a, +\infty}$, by choosing $k \in \mathbb{R}$ sufficiently large, one has $H(t, z) \geq h_k \circ F^*(z)$ for all $(t, z) \in S^1 \times DFT^* M$. Secondly, for every $k \in \mathbb{R}$, the monotone homomorphism

$$\sigma_{h_k h_k+1} : \text{HF}_s^{(a, +\infty)}(h_k+1 \circ F^*; \alpha) \rightarrow \text{HF}_s^{(a, +\infty)}(h_k \circ F^*; \alpha)$$

(9.30)
induced by a monotone homotopy between $h_{k+1} \circ F^*$ and $h_k \circ F^*$ is an isomorphism, since no Hamiltonian 1-periodic orbit has action $a$ during the homotopy, for more details about this fact see [40, Section 3.2].Taking the inverse limit in (9.25), by Lemma 4.4, we obtain the isomorphism $SH_*^{(a, +\infty)}(D^*T^*M; \alpha) \cong H_*(\Lambda_{a^2/2} M)$.

**Step 2.** To prove (ii), we construct an upward exhausting sequence of functions $\{f_k\}_{k \in \mathbb{N}}$ to compute relative symplectic homology. Here let us emphasize that the functions $f_k$ are required to be constant near $\rho = 0$, and thus are different from those functions $h_\delta$ constructed in [40]. So we need to define $f_k$ and compute the filtered Floer homology of $f_k \circ F^*$ carefully. Fix $a, c > 0$.

Now we discuss in two cases.

**Case 1.** If $a > c > 0$, we show

$$SH_*^{(a, +\infty); c}(D^*T^*M; \alpha) = 0.$$  

Let $\{m_k\}_{k \in \mathbb{N}}$ be a monotone decreasing sequence of numbers such that $c < m_k < \frac{a+c}{2}$ and $m_k \to c$ as $k \to \infty$. Let $\{\delta_k\}_{k \in \mathbb{N}}$ be another monotone decreasing sequence of numbers such that $0 < \delta_k < 1/4$ and $\delta_k \to 0$ as $k \to \infty$. For every $k \in \mathbb{N}$, set

$$\nu_k := \frac{a - c}{2\delta_k}, \quad S_k := \max \left\{ \frac{a - m_k}{2\sqrt{\delta_k}} - m_k, 0 \right\}.$$  

Here we require in addition that the sequence $\{S_k\}_{k \in \mathbb{N}}$ is increasing and tends to $\infty$ as $k \to \infty$, which is possible by choosing a rapidly decreasing sequence $\{\delta_k\}_{k \in \mathbb{N}}$.

Next, we consider the piecewise linear curve in $\mathbb{R}^2$ obtained by beginning with a line segment with the starting point $(0, -m_k)$. Upon reaching the point $(\delta_k, -m_k)$, follow the line with slope
\[ f_k \] until meeting the horizontal line through \((0, S_k)\), then follow this horizontal line to the right until it has a point of intersection with the vertical line through \((1, 0)\), then go straight down to the point \((1, 0)\), finally follow the horizontal coordinate axis to \(+\infty\). Smooth out this piecewise linear curve near its corners so that it becomes the graph of a smooth function \(f_k\), see Figure 9.6.

We claim that

\[
\text{HF}^{(a, +\infty)}(f_k \circ F^*; \alpha) = 0. \tag{9.31}
\]

In fact, since \(f_k \circ F^*\) is a radial function with respect to \(\rho\), and all tangents to the graph of \(f_k\) intersects the vertical coordinate axis strictly above \(-a\), the action of 1-periodic orbits is strictly less than \(a\). By (9.31), the monotone homomorphism

\[
\sigma_{f_{k+1} f_k} : \text{HF}^{(a, +\infty)}(f_k \circ F^*; \alpha) \to \text{HF}^{(a, +\infty)}(f_{k+1} \circ F^*; \alpha).
\]

is obviously an isomorphism for every \(k \in \mathbb{N}\). To show that \(\{f_k \circ F^*\}\) is an upward exhausting sequence in \(\mathcal{H}^{a,b,c}_\alpha\), it suffices to check that for any \(H \in \mathcal{H}^{a,b,c}_\alpha\), there exists \(f_k\) such that \(f_k \circ F^*(z) \geq H(t,z)\) for all \((t,z) \in S^1 \times T^*M\). But this is clearly true by our construction. This, combining with (9.31), completes the proof of Case 1.

**Figure 9.7.** The function \(f_k\) for \(c \geq a\)

**Case 2.** If \(c \geq a > 0\), we show

\[
\text{SH}^{(a, +\infty)}(D^F T^* M; \alpha) \cong H_*(\Lambda_\alpha M). \tag{9.32}
\]

In the similar fashion to the proof of case 1, we construct an upward directed sequence to compute the relative symplectic homology. Let \(\{\delta_k\}_{k \in \mathbb{N}}\) be a decreasing sequence of numbers such that \(0 < \delta_k < a/c\),

\[
\mu_k := \frac{a}{\delta_k} - c \notin \Lambda_\alpha \quad \text{for all} \quad k \in \mathbb{N}
\]
and $\delta_k \to 0$ as $k \to \infty$. Set

$$
\mu_k^+ := \inf ((\mu_k, \infty) \cap \Lambda_\alpha), \quad \mu_k^- := \sup ((0, \mu_k) \cap \Lambda_\alpha), \quad \mu_k' := \frac{\mu_k^- + \mu_k^+}{2}.
$$

Let $T_k := \min\{\mu_k' - a, 0\}$ for every $k \in \mathbb{N}$, and $\{m_k\}_{k \in \mathbb{N}}$ be a decreasing sequence of numbers such that $m_k > c$ and $m_k \to c$ as $k \to \infty$. In the case of $c \geq a$, the shape of these functions $f_k$ are similar to those in case 1. Namely, $f_k$ are obtained by smoothing out a piecewise linear curve in $\mathbb{R}^2$. The difference here is to replace $S_k$ and $\nu_k$ by $T_k$ and $a/\delta_k$ respectively, see Figure 9.7.

Now we prove that there is a natural isomorphism

$$
\Psi_{f_k} : HF_*^{(a, +\infty)}(f_k \circ F^*; \alpha) \to H_\alpha(\Lambda_{\alpha}^{\mu_k^2/2}; M).
$$

(9.33)

To prove (9.33), we initially deform $f_k$ by the monotone homotopy as showed in Figure 9.8 to a smooth function $\tilde{f}_k$, which is obtained by following the graph of $f_k$ until its first right turn. Then keep going linearly with constant slope $a/\delta_k$. Upon meeting the line $\mu_k \rho - a$, turn right and follow that line closely and linearly as shown in Figure 9.8. Note that all points on the graph of the function during the homotopy, at which tangential lines pass through the point $(0, -a)$, lie strictly between the lines $\mu_k \rho - a$ and $\mu_k' \rho - a$. Since $(\mu_k^-, \mu_k^+) \cap \Lambda_\alpha = \emptyset$, there are no 1-periodic orbits of action $a$ during the monotone homotopy, and therefore, we obtain the monotone isomorphism

$$
\sigma_{f_k, \tilde{f}_k} : HF_*^{(a, +\infty)}(f_k \circ F^*; \alpha) \to HF_*^{(a, +\infty)}(\tilde{f}_k \circ F^*; \alpha).
$$

(9.34)
Next, we deform \( \tilde{f}_k \) by the monotone homotopy as indicated in Figure 9.9 to a convex function \( f^{(\mu_k)} \). The graph of this function is obtained by following \( \tilde{f}_k \) until it takes on slope \( \mu_k \) for the first time (near the point \((\delta_k, -m_k)\)), and then making a smooth turn and continuing linearly with slope \( \mu_k \). Since the vertical axis intercepts of the tangential lines of the graphs during the homotopy as showed in Figure 9.9 are less than \(-a\), we obtain the monotone isomorphism with inverse
\[
\sigma_{f^{(\mu_k)}}^{-1} : \text{HF}_*(a, +\infty) (j_{f^{(\mu_k)}_k} \circ F^*; \alpha) \rightarrow \text{HF}_*^{(a, +\infty)} (j^{(\mu_k)}_k \circ F^*; \alpha).
\]
(9.35)

The fact that \( f^{(\mu_k)} \circ Q_{f_k} \) has neither 1-periodic orbits of action less than \( a \) nor 1-periodic orbits of action larger than \( C_{f_k, \mu_k} := c_{f^{(\mu_k)}_k, \mu_k} \) implies the following isomorphisms
\[
\text{HF}_*^{(a, +\infty)} (j^{(\mu_k)}_k \circ F^*; \alpha) \xrightarrow{\sigma_{f^{(\mu_k)}}} \text{HF}_*^{(-\infty, +\infty)} (j^{(\mu_k)}_k \circ F^*; \alpha) \xrightarrow{\sigma_{f^{(\mu_k)}}} \text{HF}_*^{(-\infty, C_{f_k, \mu_k})} (j^{(\mu_k)}_k \circ F^*; \alpha).
\]
(9.36)

Since \( j^{(\mu_k)}_k \circ F^* \) is convex and radial with respect to \( \rho \), Theorem 1.3 implies the isomorphism
\[
\Psi_{f^{(\mu_k)}} : \text{HF}_*^{(-\infty, C_{f_k, \mu_k})} (j^{(\mu_k)}_k \circ F^*; \alpha) \rightarrow H_*(\Lambda_{\alpha}^{\mu_k^2/2} M).
\]
(9.37)

Composing (9.34)–(9.37) yields the desired isomorphism (9.33).

By our construction, for any \( H \in \mathcal{K}_\alpha^{a,b,c} \), we can find some \( f_k \circ F^* \in \mathcal{K}_\alpha^{a,b,c} \) such that \( H \preceq f_k \circ F^* \), and for any \( k \in \mathbb{N} \) it holds that \( f_k \circ F^* \preceq f_{k+1} \circ F^* \). Thus, \( f_k \circ F^* \) is an upward directed sequence in \( \mathcal{K}_\alpha^{a,b,c} \). Taking the direct limit of both side of (9.33) yields the desired isomorphism (9.32).
Step 3. Given \( a \in (0, c) \setminus A_\alpha \), we adopt the notation used in Step 1 and Step 2, and choose \( k \in \mathbb{N} \) sufficiently large so that \( \mu_k > a, \ h_k \in \mathcal{H}^{a,b,c}_\alpha \) and \( f_k \geq h_k \). Following closely [40, Section 3.2], the proof of the commutativity of the diagram

\[
\begin{align*}
\text{SH}_{\ast}^{(a, +\infty)}(DFT^*M; \alpha) \xrightarrow{\cong} H_\ast(\Lambda^{a^2/2}_\alpha M) \\
\text{SH}_{\ast}^{(a, +\infty);C}(DFT^*M, M; \alpha) \xrightarrow{\cong} H_\ast(\Lambda^{2}_\alpha M)
\end{align*}
\]

reduces to show the commutativity of the diagram

\[
\begin{align*}
\text{HF}_{\ast}^{(a, \infty)}(h_k \circ F^*; \alpha) & \xrightarrow{\sigma f_k h_k} \text{HF}_{\ast}^{(a, \infty)}(f_k \circ F^*; \alpha) \\
\Psi_{h_k}^a \cong & \Psi_{f_k}^\mu \cong \\
H_\ast(\Lambda^{a^2/2}_\alpha M) & \xrightarrow{[f]} H_\ast(\Lambda^{2}_\alpha M)
\end{align*}
\]

This can be deduced from the following commutative diagram

\[
\begin{align*}
\text{HF}_{\ast}^{(a, \infty)}(h_k \circ F^*; \alpha) & \xrightarrow{\sigma} \text{HF}_{\ast}^{(a, \infty)}(f_k \circ F^*; \alpha) \\
\sigma \left( 9.26 \right) & \left( 9.34 \right) \sigma \\
\text{HF}_{\ast}^{(a, \infty)}(\tilde{h}_k \circ F^*; \alpha) & \xrightarrow{\sigma} \text{HF}_{\ast}^{(a, \infty)}(\tilde{f}_k \circ F^*; \alpha) \\
\sigma \left( 9.27 \right) & \left( 9.35 \right) \sigma \\
\text{HF}_{\ast}^{(a, \infty)}(h_k^{(a)} \circ F^*; \alpha) & \xrightarrow{\sigma} \text{HF}_{\ast}^{(a, \infty)}(f_k^{(\mu_k)} \circ F^*; \alpha) \\
\left[ j^F \right] \left( 9.28 \right) & \left( 9.36 \right) \left[ j^F \right] \\
\text{HF}_{\ast}^{(-\infty, \infty)}(h_k^{(a)} \circ F^*; \alpha) & \xrightarrow{\sigma} \text{HF}_{\ast}^{(-\infty, \infty)}(f_k^{(\mu_k)} \circ F^*; \alpha) \\
\left[ j^F \right] \left( 9.28 \right) & \left[ j^F \right] \\
\text{HF}_{\ast}^{(-\infty, C_{h_k}, \alpha)}(h_k^{(a)} \circ F^*; \alpha) & \xrightarrow{\Psi_{f_k}^\mu} \text{HF}_{\ast}^{(-\infty, C_{f_k}, \alpha)}(f_k^{(\mu_k)} \circ F^*; \alpha) \\
\left( 6.6 \right) & \left[ j^F \right] \\
\Psi_{h_k}^\alpha \left( 9.29 \right) & \Psi_{f_k}^\mu \left( 9.37 \right) \\
H_\ast(\Lambda^{a^2/2}_\alpha M) & \xrightarrow{[f]} H_\ast(\Lambda^{2}_\alpha M)
\end{align*}
\]

where \( \Psi_{f_k}^\mu \) is an isomorphism by Theorem 1.3 (iii). In fact, Lemma 4.2 implies the commutativity of the first two rectangular blocks. The third and fourth block commutes by the naturality of long exact sequences concerning Floer homology. The lowest rectangular block commutes by (1.2), and the triangle to its left commutes by (1.3). The remaining part of the proof proceeds exactly like [40, Section 3.2] by taking direct and inverse limits in the diagram (9.39). The proof of Theorem 7.4 (iii) completes.
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