Extremes of $\alpha(t)$-locally Stationary Gaussian Random Fields

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Abstract: The main result of this contribution is the derivation of the exact asymptotic behaviour of the supremum of $\alpha(t)$-locally stationary Gaussian random field over a finite hypercube. We present two applications of our results; the first one deals with the extremes of aggregate multifractional Brownian motions, whereas the second one establishes the exact asymptotics of the supremum of $\chi$-processes generated by multifractional Brownian motions.

1 Introduction and Main Result

The classical Central Limit Theorem and its ramifications show that the Gaussian model is a natural and correct paradigm for building an approximate solution to many otherwise unsolvable problems encountered in various research fields. While the theory of Gaussian processes and Gaussian random fields (GRF’s) is well-developed and mature, the range of their applications is constantly growing. Recently, applications in brain mapping, cosmology, quantum chaos and some other fields have been added to its palmarès, see e.g., Adler (2000), Adler and Taylor (2007), Anderes and Chatterjee (2009), Azaïs and Wschebor (2009) and Adler et al. (2012a,b). In applications related to extremes of Gaussian processes the fractional Brownian motion (fBm) appears inevitably in the definition of the Pickands constant, see e.g., Pickands (1969), Berman (1992) and Piterbarg (1996). Numerous research articles have shown the importance of fBm in both theoretical models and applications. For certain applications, the stationarity of increments, which together with the self-similarity property characterises fBm in the class of Gaussian processes can be a severe restriction. A natural way to avoid the stationarity of increments property is to introduce the multifractional Brownian motion (mfBm), see e.g., Stoev and Taqqu (2006) and Ayachea et al. (2011). In order to make the problem tractable, we discuss in this paper a simple class of mfBm. By definition, a mean-zero Gaussian process \( \{ B_{\alpha(t)}(t), t \geq 0 \} \) is called a mfBm with parameter \( \alpha(t), t \geq 0 \), if

\[
\mathbb{E} \left( B_{\alpha(t)}(t) B_{\alpha(s)}(s) \right) = \frac{1}{2} D(\alpha(s,t)) \left( s^{\alpha(s,t)} + t^{\alpha(s,t)} - |t - s|^{\alpha(s,t)} \right), \quad \alpha(s,t) := \frac{s^{\alpha}}{2} + \frac{t^{\alpha}}{2}, s, t \geq 0, \tag{1.1}
\]

where \( D(x) = \frac{2\pi}{(x+1)\sin(\pi x/2)} \) and \( \alpha(\cdot) \) is a H"older function of exponent \( \gamma > 0 \) such that \( 0 < \alpha(t) < 2 \min(1, \gamma), t \geq 0, \) see e.g., Ayache et al. (2000). For \( \alpha(t) = \alpha \in (0, 2), t \geq 0, \) the \( B_{\alpha} \) reduces to a fBm (not necessarily standard).

Inspired by the structure of the mfBm, the recent paper Dębicki and Kisowski (2008) introduces the class of \( \alpha(t) \)-locally stationary Gaussian processes. Therein the exact asymptotic of the tail behaviour of the supremum of \( \alpha(t) \)-locally stationary Gaussian process is derived which can be applied, for instance, to analyse the extremes of standardized mfBm.

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It is worth noting that this new class includes locally stationary ones, see Berman (1974), Hüsler (1990) and Piterbarg (1996) for results concerning the asymptotic behaviour of their extremes. If \( \{X_i(t), t \in [0, T]\}, i \leq k, \) are independent real-valued Gaussian processes a natural GRF associated with these processes is the aggregate random field

\[
X(t) = \sum_{i=1}^{k} X_i(t_i), \quad t = (t_1, \ldots, t_k) \in [0, T]^k.
\]

Extremes of GRF’s can not be analysed by aggregating the corresponding results for processes. Moreover, the analysis of the extremes of GRF’s leads to technical difficulties, see e.g., the excellent monographs Piterbarg (1996) and Adler and Taylor (2007). Recently, Abramowicz and Seleznjev (2011) deal with multivariate piecewise linear interpolation of locally stationary random fields, whereas Hashorva et al. (2012) investigates the piece-wise approximation of \( \alpha(t) \)-locally stationary processes. With motivation from the aforementioned papers and Dębicki and Kisowski (2008), we consider, in this paper, extremes of \( \alpha(t) \)-locally stationary GRF \( \{X(t), t \in [0, T]^k\} \) (to be defined below). Specifically, we are interested in the exact asymptotic behaviour of

\[
P \left( \sup_{t \in [0, T]^k} X(t) > u \right), \quad u \to \infty,
\]

with \( T > 0 \) a given constant and \( k \in \mathbb{N} \) a positive integer.

Let \( C(D) \) denote the set of all continuous functions on \( D \subset \mathbb{R}^k \). Next, we give a formal definition of the GRF’s of interest.

**Definition.** A real-valued separable GRF \( \{X(t), t \in [0, T]^k\} \) is said to be \( \alpha(t) \)-locally stationary if

1. \( E(X(t)) = 0 \) and \( Var(X(t)) = 1 \) for all \( t \in [0, T]^k \);
2. \( \alpha_i(t_i) \in C([0, T]) \) and \( \alpha_i(t_i) \in (0, 2] \) for all \( t_i \in [0, T], i = 1, \ldots, k; \)
3. \( C_i(t) \in C([0, T]^k) \) and \( 0 < \inf \{C_i(t) : t \in [0, T]^k\} \leq \sup \{C_i(t) : t \in [0, T]^k\} = C^*_i < \infty, i = 1, \ldots, k; \)
4. uniformly with respect to \( t \in [0, T]^k \)

\[
1 - Cov(X(t), X(t + s)) = \sum_{i=1}^{k} C_i(t)|s_i|^\alpha_i(t_i) + o \left( \sum_{i=1}^{k} C_i(t)|s_i|^\alpha_i(t_i) \right)
\]

as \( s \to 0 \) with \( 0 := (0, \ldots, 0) \in \mathbb{R}^k \).

A canonical example of \( \alpha(t) \)-locally stationary GRF’s is the aggregate mfBm defined by aggregating independent standardized mfBm’s, see Section 2.

In this paper we consider the case that there exists some integer \( k_1 \leq k \) such that:

1. each of \( \alpha_i(t_i), i = 1, \ldots, k_1, \) attains its global minimum on \( [0, T] \) at a unique point \( t^0_i \), and further for any \( i = k_1 + 1, \ldots, k, \) there is some \( [a_i, b_i] \subset (0, T) \) such that \( \alpha_i(t_i) \equiv \alpha_i \) in \( [a_i, b_i] \) which is the global minimum of \( \alpha_i(t_i) \) on \( [0, T] \);
2. there exist \( M_i, \beta_i > 0, \) and \( \delta_i > 1, i = 1, \ldots, k_1, \) such that

\[
\alpha_i(t_i + t^0_i) = \alpha_i(t^0_i) + M_i|t_i|^\beta_i + o(|t_i|^\beta_i|\ln|t_i||^{-\delta_i}), \quad as \ t \to 0,
\]

and there exist \( M_i, \beta_i, \tilde{M}_i, \tilde{\beta}_i > 0, \) and \( \tilde{\delta}_i, \delta_i > 1, i = k_1 + 1, \ldots, k, \) such that

\[
\alpha_i(b_i + t_i) = \alpha_i(b_i) + M_i t_i^\beta_i + o(t_i^\beta_i|\ln t_i|^{-\delta_i}), \quad as \ t_i \downarrow 0,
\]
\[ \alpha_i(a_i - t_i) = \alpha_i(a_i) + M_i t_i^{\delta_i} + o(t_i^{\delta_i} |\ln t_i|^{-\delta_i}), \quad \text{as } t_i \downarrow 0. \] (1.6)

The assumption A1 is initially suggested in Dębicki and Kisowski (2008), whereas assumption A2 is a weaker version of a similar condition given therein which assumes (1.4-1.6) with \(|t_i|^{\delta_i} (t_i^k)\) instead of \(|\ln t_i|^{-\delta_i} (|\ln t_i|^{-\delta_i})\).

For notational simplicity, set
\[ \alpha_i := \alpha_i(t_i^0), \quad i = 1, \ldots, k_1, \]
and
\[ \int_{x \in \{ x_0 \} \times D_1} C(x) dx := \int_{x \in D_1} C(x_0, x) dx \]
for all integrable function \( C(\cdot) \). Further, denote by \( \Psi(\cdot) \) the survival function a standard normally distributed random variable, and by \( \Gamma(\cdot) \) the Euler’s Gamma function.

The crucial step of the proof of our main result Theorem 1.1 is an application of the double-sum method that was developed by Pickands (1969). As expected, the Pickands constant defined by
\[ \mathcal{H}_\alpha = \lim_{T \to \infty} T^{-1} \mathbb{E} \left\{ \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2B_\alpha(t) - t^\alpha} \right) \right) \right\} \in (0, \infty), \quad \alpha \in (0, 2], \]
appears in the asymptotic expansion, where \( \{ B_\alpha(t), t \geq 0 \} \) is a fBm with Hurst index \( \alpha/2 \). See Pickands (1969), Piterbarg (1996) or Dębicki (2002) for the basic properties of Pickands constant and generalisations.

**Theorem 1.1.** Let \( \{ X(t), t \in [0, T]^k \} \) be an \( \alpha(t) \)-locally stationary GRF that satisfies
\[ \text{Cov}(X(t), X(s)) < 1, \quad \forall t, s \in [0, T]^k, \quad t \neq s. \] (1.7)

If both conditions A1 and A2 are satisfied, then we have (set \( q_k := \# \{ i \in \mathbb{N} : 1 \leq i \leq k_1, t_i^0 \in (0, T) \} \})
\[ \mathbb{P} \left( \sup_{t \in [0, T]^k} X(t) > u \right) = K_\mathcal{O} u^{\alpha}(\ln u)^{\beta} \Psi(u)(1 + o(1)), \quad u \to \infty, \] (1.8)

where \( \alpha = 2 \sum_{i=1}^{k} 1/\alpha_i, \quad \beta = -\sum_{i=1}^{k_1} 1/\beta_i \) and
\[ K_\mathcal{O} = 2^{q_k} \left( \prod_{i=1}^{k_1} \left( \frac{\alpha_i^2}{2M_i} \right)^{1/\beta_i} \Gamma(1/\beta_i + 1) \right) \left( \prod_{i=1}^{k} \mathcal{H}_\alpha \right) \int_{x \in \mathcal{O}} \int_{i=1}^{k} (C_i(x))^{1/\alpha_i} dx \in (0, \infty), \] (1.9)

with \( \mathcal{O} = \prod_{i=1}^{k_1} \{ t_i^0 \} \times \prod_{i=k_1+1}^{k} [a_i, b_i] \).

**Remarks:**

a) Under the conditions of Theorem 1.1 if, for the chosen \( k_1 < k \), \( \alpha_i(t_i) \equiv \alpha_i \), \( i = k_1 + 1, \ldots, k \), on some compact set \( \mathcal{O}_2 \subset \mathbb{R}^{k-k_1} \), with positive Lebesgue measure, then (1.8) holds for \( \{ X(t), t \in [0, T]^{k_1} \times \mathcal{O}_2 \} \) with \( \mathcal{O} = \prod_{i=1}^{k_1} \{ t_i^0 \} \times \mathcal{O}_2 \). In addition, Theorem 1.1 coincides with Theorem 7.1 in Piterbarg (1996) when \( k_1 = 0 \).

b) In the proof of Theorem 1.1 an extension of Pickands theorem (see Lemma 3.3 below) plays an important role. We remark that Pickands theorem (see Pickands (1969)) has been rigorously proved in Piterbarg (1972).

**Brief outline of the paper:** We give two applications of our main result in Section 2. In Section 3 we present some preliminary results. All the proofs are relegated in Section 4 and Appendix.
2 Applications

In this section we apply our results to two interesting cases of \( \alpha(t) \)-locally stationary GRF’s, namely, the aggregate mfBm’s and the \( \chi \)-processes generated by mfBm’s defined below. Let \( \{ B_{\alpha(t)}(t), t \geq 0 \} \) be a mfBm with parameter \( \alpha(t) \in (0, 2], t \geq 0 \). We define the standardized/normalized mfBm by

\[
\overline{B}_{\alpha}(t) = \frac{B_{\alpha(t)}(t)}{\sqrt{\text{Var}(B_{\alpha(t)}(t))}}, t \in [T_1, T_2], \quad \text{with } 0 < T_1 < T_2 < \infty.
\]

As shown in Dębicki and Kisowski (2008)

\[
1 - \text{Cov}(\overline{B}_{\alpha}(t), \overline{B}_{\alpha}(s + t)) = \frac{1}{2} t^{-\alpha(t)}|s|^\alpha(t) + o(|s|^\alpha(t))
\]

uniformly with respect to \( t \in [T_1, T_2] \), as \( s \to 0 \).

**Aggregate multifractional Brownian motions:** Let \( \{ \overline{B}_{\alpha_i}(t_i), t_i \in [T_1, T_2] \}, i = 1, \ldots, k \), be independent standardized mfBm’s, with parameters \( \alpha_i(t_i), t \geq 0, i = 1, \ldots, k \), respectively. Assume, for any fixed \( i = 1, \ldots, k \), that \( \alpha_i(t_i) \) attains its minimum at the unique point \( t_i^0 \in (T_1, T_2) \), and that there exist some positive \( M_i, \beta_i, \) and \( \delta_i > 1, i = 1, \ldots, k \), such that A2 is satisfied. Set \( X(t) = \frac{1}{\sqrt{k}} (\overline{B}_{\alpha_1}(t_1) + \cdots + \overline{B}_{\alpha_k}(t_k)), t \in [T_1, T_2]^k \). It follows that, as \( s \to 0 \),

\[
1 - \text{Cov}(X(t), X(t + s)) = 1 - \frac{1}{k} \sum_{i=1}^{k} \text{Cov}(\overline{B}_{\alpha_i}(t_i), \overline{B}_{\alpha_i}(t_i + s_i))
\]

\[
= \frac{1}{2k} \sum_{i=1}^{k} \left( (t_i)^{-\alpha_i(t_i)}|s_i|^\alpha_i(t_i) \right) (1 + o(1))
\]

uniformly with respect to \( t \in [T_1, T_2]^k \). Therefore, conditions D1 – D4 are satisfied and we have from Theorem 1.1 (recall \( \alpha_i := \alpha_i(t_i^0) \))

\[
\mathbb{P} \left( \sup_{t \in [T_1, T_2]^k} X(t) > u \right) = 2^k(2k)^{-\sum_{i=1}^{k} \frac{1}{\beta_i}} \prod_{i=1}^{k} \frac{H_{\alpha_i} \Gamma(1/\beta_i + 1)}{\beta_i^{\alpha_i(t_i^0) + 1}} \left( \frac{\alpha_i^2}{2M_i} \right)^{1/\beta_i} \frac{\sum_{i=1}^{k} \frac{\alpha_i}{\beta_i}}{(\ln u)^{\sum_{i=1}^{k} 1/\beta_i}} \Psi(u)(1 + o(1))
\]

(2.10)

as \( u \to \infty \).

**\( \chi \)-processes:** Let \( \{ \overline{B}_{i,\alpha}(t), t \in [T_1, T_2] \}, i = 1, \ldots, k \), be independent copies of \( \{ \overline{B}_{\alpha}(t), t \in [T_1, T_2] \} \). Assume that \( \alpha(t) \) attains its minimum at the unique point \( t^0 \in (T_1, T_2) \), and that there exist some positive \( M, \beta, \) and \( \delta > 1 \), such that again A2 holds. Consider the \( \chi \)-process defined by

\[
\chi_k(t) = \sqrt{\overline{B}_{1,\alpha}^2(t) + \cdots + \overline{B}_{k,\alpha}^2(t)}, \quad t \in [T_1, T_2].
\]

Further, we introduce a GRF

\[
Y(t, u) = \overline{B}_{1,\alpha}(t)u_1 + \cdots + \overline{B}_{k,\alpha}(t)u_k, \quad u = (u_1, \ldots, u_k)
\]

defined on the cylinder \( \mathcal{G}_T = [T_1, T_2] \times S_{k-1} \), with \( S_{k-1} \) being the unit sphere in \( \mathbb{R}^k \) (with respect to \( L_2 \)-norm). In the light of Piterbarg (1996)

\[
\sup_{t \in [T_1, T_2]} \chi_k(t) = \sup_{(t, u) \in \mathcal{G}_T} Y(t, u).
\]
Further we have as \((s, v) \to (0, 0)\)

\[
1 - \text{Cov}(Y(t, u), Y(t + s, u + v)) = \frac{1}{2} t^{-\alpha(t)}|s|^{\alpha(t)} + \frac{1}{2} \sum_{i=1}^{k-1} |v_i|^2 + o\left(|s|^{\alpha(t)} + \sum_{i=1}^{k-1} |v_i|^2\right)
\]

uniformly with respect to \((t, u) \in \mathcal{G}_T\). Therefore, the conditions \(D1 - D4\) are satisfied and we have that (recall Remark a) above)

\[
\mathbb{P}\left(\sup_{t \in [T_1, T_2]} \chi_k(t) > u\right) = 2^{\frac{k}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}} \frac{H_\alpha(t^0)(\alpha(t^0))}{M_{1/\beta}^{1/\beta} \Gamma(k/2)} \frac{u^{k-1 + \frac{1}{2} - \frac{1}{2}}}{(\ln u)^{1/\beta}} \Psi(u)(1 + o(1)), \ u \to \infty. \tag{2.11}
\]

### 3 Preliminary Lemmas

This section is concerned with some preliminary lemmas used for the proof of Theorem 1.1. We assume, without loss, that \(1 \leq k_1 < k\) and \(M_i = 1, i = 1, \ldots, k_1\). As pointed out in Dębicki and Kisowski (2008), for the asymptotics of the original process, we have to replace \(C_i(t)\) with \((M_i)^{-\alpha_i/\beta_i} C_i(t)\), \(i = 1, \ldots, k_1\). We may further assume that \(t_i^0 = 0, i = 1, \ldots, k_1\), and thus the final general result should be multiplied by \(2^{q_1}\). Hereafter, consider \(\{X(t), t \in [0, T]^{k}\}\) to be an \(\alpha(t)\)-locally stationary GRF with the above simplification (called simplified \(\alpha(t)\)-locally stationary GRF). Set next

\[
t_i^u = \left(\frac{\alpha_i}{\beta_i} \ln \ln u \ln u\right)^{\frac{1}{\beta_i}}, \ i = 1, \ldots, k.
\]

Clearly

\[
\mathbb{P}\left(\sup_{t \in [0, T]^{k}} X(t) > u\right) \leq \mathbb{P}\left(\sup_{t \in \prod_{i=1}^{k-1} [a_i, b_i]} X(t) > u\right) \leq \mathbb{P}\left(\sup_{t \in \prod_{i=1}^{k-1} [a_i, b_i]} X(t) > u\right) + \mathbb{P}\left(\sup_{t \in (0, T) / \prod_{i=1}^{k-1} [a_i, b_i]} X(t) > u\right). \tag{3.12}
\]

There are two steps in the proof of Theorem 1.1. In step 1, we focus on the asymptotics of

\[
\Pi(u) := \mathbb{P}\left(\sup_{t \in \prod_{i=1}^{k-1} [a_i, b_i]} X(t) > u\right), \ u \to \infty, \tag{3.13}
\]

which is the main part of our proof. In step 2, we shall show that (see Lemma 3.3 below)

\[
\mathbb{P}\left(\sup_{t \in (0, T) / \prod_{i=1}^{k-1} [a_i, b_i]} X(t) > u\right) = o(\Pi(u)), \ u \to \infty. \tag{3.14}
\]

The idea of finding the asymptotics of (3.13) is based on the so-called double-sum method; see e.g., Pickands (1969) or Piterbarg (1996). Before going to the detail of the proof, let us recall the brief outline of the double-sum method. First of all, we need to find a suitable partition, say cubes \(\{W_i\}\), of the set \(\prod_{i=1}^{k_1} [0, t_i^u] \times \prod_{i=k_1 + 1}^{k} [a_i, b_i]\). Then using the well-known Bonferroni’s inequality we find upper and lower bounds for \(\Pi(u)\), i.e.,

\[
\sum_{i} \mathbb{P}\left(\sup_{t \in W_i} X(t) > u\right) \geq \mathbb{P}\left(\sup_{t \in \prod_{i=1}^{k_1} [0, t_i^u] \times \prod_{i=k_1 + 1}^{k} [a_i, b_i]} X(t) > u\right) \geq \mathbb{P}\left(\sup_{t \in \prod_{i=1}^{k_1} [0, t_i^u] \times \prod_{i=k_1 + 1}^{k} [a_i, b_i]} X(t) > u\right).
\]
Moreover, let $k$ for $\triangle$ and let $m_i$ and double-sum terms in the last formula. For $u$, we denote $g_i(u) := \frac{(\alpha_i)^2}{\beta_i} (\ln \ln u)^{1+1/\beta_i}$, where $\lfloor x \rfloor$ denotes the integer part of $x$. Further, let $S > 1$ be a fixed constant; by dividing each $A^i_{p_i}$ into subintervals of length $S/u^{2/(\alpha_i(c_{p_i}+1))}$ (recall function $\alpha_i(\cdot)$ in (13)), we define

$$B^i_{r_j,p_i}(u) := \left[ c_{p_i} + \frac{r_j S}{u^{2/(\alpha_i(c_{p_i}+1))}}, c_{p_i} + \frac{(r_j+1)S}{u^{2/(\alpha_i(c_{p_i}+1))}} \right]$$

for $j_i = 0, 1, \cdots, n_i p_i = n_i p_i(u) := \left\lfloor \frac{c_{p_i} + 1 - c_{p_i+1}}{S} u^{2/(\alpha_i(c_{p_i}+1))} \right\rfloor$.

Moreover, let $k_2 := k-k_1$, $a = (a_{k_1+1}, \cdots, a_k)$, and let $k = (K_1, \cdots, K_{k_2}) \in \mathbb{Z}^{k_2}$ be a vector with integer coordinates. For $\delta > 0$, we denote

$$\delta_k = (a + \delta k + [0, \delta]^k) \cap \prod_{i = k_1+1}^k [a_i, b_i],$$

where $k \in \mathcal{B}$ with

$$\mathcal{B} = \{ k \in \mathbb{Z}^{k_2} : \delta_k \neq \emptyset \}.$$

Define an operator $g_u$ on $\mathbb{R}^{k_2}$ as in Piterbarg (1996), i.e., for $t = (t_{k_1+1}, \cdots, t_k) \in \mathbb{R}^{k_2}$

$$g_u t = \left( u^{-\frac{1}{\alpha_k+1}} t_{k_1+1}, u^{-\frac{1}{\alpha_k+1}} t_k \right).$$

(3.15)

Denote $\Delta_0 = g_u[0,1]^{k_2}$, and, for fixed $k \in \mathcal{B}$, $\Delta_{\delta_k} = \Delta_{\delta_k} : g_u S \mathbb{I}_k + \Delta_0 S$ with $\mathbb{I}_k = (I_k^1, \cdots, I_k^{k_2}) \in \mathbb{Z}^{k_2}$ being a vector with integer coordinates. Further, let $V_{\delta_k,k} := a + \delta_k + \Delta_{\delta_k}$, where $\mathbb{I}_k \in \mathcal{A}_k$ with

$$\mathcal{A}_k = \{ \mathbb{I}_k \in \mathbb{Z}^{k_2} : V_{\delta_k,k} \cap \delta_k \neq \emptyset \}.$$

Denote

$$N_k^+ = \# \{ \mathbb{I}_k \in \mathbb{Z}^{k_2} : V_{\delta_k,k} \cap \delta_k \neq \emptyset \} \quad \text{and} \quad N_i = \left\lfloor \frac{\delta}{S} u^{2/\alpha_{k_1+1}} \right\rfloor, \quad \text{for} \quad i = 1, \cdots, k_2.$$

Moreover, let, for $i = 1, \cdots, k_1$,

$$\mathcal{L}_i^1 = \{ (j_i, p_i) : j_i, p_i \in \mathbb{Z}, 0 \leq p_i \leq m_i - 1, 0 \leq j_i \leq n_i p_i - 1 \},$$

and

$$\mathcal{L}_2 = \{ (\mathbb{I}_k, k) : k \in \mathcal{B}, V_{\delta_k,k} \in \delta_k \}, \quad \mathcal{U}_2 = \{ (\mathbb{I}_k, k) : k \in \mathcal{B}, \mathbb{I}_k \in \mathcal{A}_k \}.$$
We have
\[
\bigcup_{(j,p) \in \mathcal{L}_1, i=1, \ldots, k_1} \prod_{i=1}^{k_1} B_{j_i, p_i} \times V_{j_i, k_i} \subset \bigcup_{(j,p) \in \mathcal{L}_1, i=1, \ldots, k_1} \prod_{i=1}^{k_1} [0, t_i^p] \times \prod_{i=k_1+1}^k [a_i, b_i] \subset \bigcup_{(j,p) \in \mathcal{L}_2} \prod_{i=1}^{k_1} B_{j_i, p_i} \times V_{j_i, k_i}.
\]

In order to specify the 'distance' between segments of the type \(\prod_{i=1}^{k_1} B_{j_i, p_i} \times V_{j_i, k_i}\), we introduce the following order relation: for any \((j, p), (j', p') \in \mathbb{Z}^2\), we write
\[
(j, p) \prec (j', p') \text{ iff } (p < p' \text{ or } p = p' \text{ and } j < j').
\]

Further, for \(j, p, j', p' \in \mathbb{Z}^{k_1}\) with \((j, p_i), (j', p'_i) \in \mathcal{L}_1, i = 1, \ldots, k_1, \)
\[
(j, p) \prec (j', p') \text{ iff } (j, p_i) \prec (j', p'_i) \text{ for some } i = 1, \ldots, k_1, \text{ and } (j_i, p_i) = (j'_i, p'_i) \text{ for } l = 1, \ldots, i - 1,
\]
and, for \((\mathbb{I}_k, k), (\mathbb{I}'_k, k') \in \mathcal{L}_2, \)
\[
(\mathbb{I}_k, k) \prec (\mathbb{I}'_k, k') \text{ iff } (I_i^{\mathbb{I}_k}, k_i) \prec (I_i^{\mathbb{I}'_k}, k'_i) \text{ for some } i = 1, \ldots, k_2, \text{ and } (I_i^{\mathbb{I}_k}, k_i) = (I_i^{\mathbb{I}'_k}, k'_i) \text{ for } l = 1, \ldots, i - 1.
\]

Moreover, define, for \(j, p, j', p' \in \mathbb{Z}, \)
\[
N_{j, p}^{j', p'} := \# \{(j'', p'') \in \mathbb{Z}^2 : (j, p) \prec (j'', p'') \prec (j', p')\}.
\]

In the sequel, for fixed \(j_i, p_i, \mathbb{I}_k, k\) such that \((j, p_i) \in \mathcal{U}_1, i = 1, 2, \ldots, k_1\) and \((\mathbb{I}_k, k) \in \mathcal{U}_2\), we consider the GRF
\[
X(\nu) := X(v_1, \ldots, v_k) \text{ on } A_{j_i, p}^{\mathbb{I}_k, k} := \prod_{i=1}^{k_1} B_{j_i, p_i} \times V_{j_i, k_i}.
\]

In order to obtain the estimates of the tail probabilities of the supremum of \(X\) on \(A_{j_i, p}^{\mathbb{I}_k, k}\) (see Lemmas 3.11 and 3.2 below), we introduce the following stationary GRF’s, for a fixed (marked) point \(v^0 = (v_1^0, \ldots, v_k^0) := v^0_{j_i, p, \mathbb{I}_k, k}\) in \(A_{j_i, p}^{\mathbb{I}_k, k}\) :

\(\{Y_{v^0, \alpha}^{\nu}(\nu), \nu \in [0, S]^k\}\) is a family of centered stationary GRF’s with
\[
\text{Cov}(Y_{v^0, \alpha}^{\nu}(\nu), Y_{v^0, \alpha}^{\nu}(\nu + \mathbf{x})) = e^{-(1-\varepsilon)(\sum_{i=1}^{k_1} C_i(v^0)u^{-\beta_i} - \sum_{i=k_1+1}^{k} C_i(v^0)u^{-\beta_i} + \sum_{i=k_1+1}^{k} C_i(v^0)u^{-\beta_i})}
\]
for \(\varepsilon \in (0, 1), u > 0\) such that \(\alpha_i + 2(t_i^p)^{\beta_i} \leq 2, i = 1, \ldots, k_1, \) and \(\nu, \nu + \mathbf{x} \in [0, S]^k.\)

\(\{Z_{v^0, \alpha}^{\nu}(\nu), \nu \in [0, S]^k\}\) is a family of centered stationary GRF’s with
\[
\text{Cov}(Z_{v^0, \alpha}^{\nu}(\nu), Z_{v^0, \alpha}^{\nu}(\nu + \mathbf{x})) = e^{-(1+\varepsilon)(\sum_{i=1}^{k_1} C_i(v^0)u^{-\beta_i} - \sum_{i=k_1+1}^{k} C_i(v^0)u^{-\beta_i})},
\]
for \(\varepsilon > 0, u > 0\) and \(\nu, \nu + \mathbf{x} \in [0, S]^k.\)

**Lemma 3.1.** For any \(\varepsilon \in (0, 1),\) there exists \(u_\varepsilon > 0\) such that for \(u > u_\varepsilon,\)

\[
(i) \quad \mathbb{P}\left(\sup_{\nu \in A_{j_i, p}^{\mathbb{I}_k, k}} X(\nu) > u\right) \geq \mathbb{P}\left(\sup_{\nu \in [0, S]^k} Y_{v^0, \alpha}^{\nu}(\nu) > u\right),
\]

\[
(ii) \quad \mathbb{P}\left(\sup_{\nu \in A_{j_i, p}^{\mathbb{I}_k, k}} X(\nu) > u\right) \leq \mathbb{P}\left(\sup_{\nu \in [0, S]^k} Z_{v^0, \alpha}^{\nu}(\nu) > u\right).
\]
Remark 3.2. Due to continuity of the functions $C_i(\cdot), i = 1, \cdots, k$, the point $v^0$ can also be chosen as a fixed (marked) point in $\prod_{i=1}^{k} A_{\mu_i} \times \delta_k$ when $\delta$ is sufficiently small and $u$ is sufficiently large. In the sequel, we chose $v^0$ in this way. Actually $v^0$ depends on $p, k$, but, if no confusion is caused, for notational simplicity we still write $v^0$.

Next we introduce a structural modulus on $\mathbb{R}^k$ by

$$|s|_\alpha = \sum_{i=1}^{k} |s_i|^{\alpha_i}, \ s \in \mathbb{R}^k.$$  

The following result inspired by Lemma 7 of Hüsler and Piterbarg (2004) is crucial for our investigation; its proof is relegated to Appendix.

Lemma 3.3. For any compact set $D \subset \mathbb{R}^k$, let $\{X_u(t), t \in D\}, u > 0$, be a family of a.s. continuous GRF’s, with $\mathbb{E}(X_u(t)) \equiv 0$, $\mathbb{E}((X_u(t))^2) \equiv 1$ for all $u$, and with correlation function $r_u(t, s) = \mathbb{E}(X_u(t)X_u(s))$. If

$$\lim_{u \to \infty} u^2(1 - r_u(t, s)) = |t - s|_\alpha$$  

(3.18)

uniformly with respect to $t, s \in D$, then

$$\mathbb{P} \left( \sup_{t \in D} X_u(t) > u \right) = \mathcal{H}_{(k, \alpha)}[D] \Psi(u)(1 + o(1))$$

as $u \to \infty$, where

$$\mathcal{H}_{(k, \alpha)}[D] = \mathbb{E} \left( \exp \left( \sup_{t \in D} (\tilde{B}_\alpha(t) - |t|_\alpha) \right) \right) \in (0, \infty)$$  

(3.19)

as defined in Piterbarg (1996), with

$$\tilde{B}_\alpha(t) = \sqrt{2} \sum_{i=1}^{k} \tilde{B}_\alpha^{(i)}(t_i)$$

and $\tilde{B}_\alpha^{(i)}$, $1 \leq i \leq k$, being independent fBm’s with Hurst indexes $\alpha_i/2 \in (0, 2]$, respectively.

Lemma 3.4. For any $S > 1$ and $\varepsilon \in (0, 1)$, we have, as $u \to \infty$,

(i) $\mathbb{P} \left( \sup_{\nu \in [0, S]^k} Y_{\varepsilon, u}^{(\nu)}(\nu) > u \right) = \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \left[ 0, (C_1(u^0)(1 - \varepsilon))^{1/\alpha_i} S \right] \Psi(u)(1 + o(1))$,

(ii) $\mathbb{P} \left( \sup_{\nu \in [0, S]^k} Z_{\varepsilon, u}^{(\nu)}(\nu) > u \right) = \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \left[ 0, (C_1(u^0)(1 + \varepsilon))^{1/\alpha_i} S \right] \Psi(u)(1 + o(1))$,

where (recall (3.1)) we set $\mathcal{H}_{\alpha_i}[0, S] := \mathcal{H}_{(1, \alpha_i)}[(0, S)]$, $i = 1, 2, \cdots, k$.

In order to estimate the double-sum term in the derivation of (3.13), we need the following two lemmas.

Lemma 3.5. Let GRF $\tilde{Z}_{\varepsilon, u}^{(\nu)}; \nu \in [0, S]^k$, having covariance structure (3.16) with $u^0$ replaced by $w^0$, be independent of $\{Z_{\varepsilon, u}^{(\nu)}(\nu) \in [0, S]^k\}$, with $\varepsilon > 0$. Then there exists some positive constant $F_\varepsilon$, for $u$ large enough, we have

$$\mathbb{P} \left( \sup_{\nu, \mu \in [0, S]^k} \frac{1}{\sqrt{2}} \left( \tilde{Z}_{\varepsilon, u}^{(\nu)}(\nu) + \tilde{Z}_{\varepsilon, u}^{(\mu)}(\mu) \right) > u \right) \leq F_\varepsilon S^{2k} \Psi(u).$$
Next, we introduce a distance of two sets $D_1, D_2 \subset \mathbb{R}^k_+$ by

$$
\text{dist}(D_1, D_2) = \inf_{t \in D_1, s \in D_2} |t - s|_{\alpha}.
$$

Further, we fix some sufficiently small $\gamma_0 > 0$ in the following way: uniformly with respect to $t \in [0, T]^k$,

$$
1 - \text{Cov}(X(t), X(t + s)) < \gamma_0 \in [0, 1/2)
$$

for $|s|_{\alpha} < \gamma_0$ (recall (1.3)).

**Lemma 3.6.** There exist some universal positive constants $C, C_1$ such that, for sufficiently large $u$, the following statements are established.

1. For $(j_i, p_i), (j'_i, p'_i) \in \mathcal{L}_1, i = 1, \cdots, k_1, (\mathcal{I}_k, \mathcal{K}), (\mathcal{I}'_k, \mathcal{K}') \in \mathcal{L}_2$ satisfying

   $$
   \text{dist} \left( A_{i,j,\mathcal{I}, \mathcal{K}}, A_{i',j',\mathcal{I}', \mathcal{K}'} \right) < \gamma_0
   $$

   and

   $$
   N_{j_i, p_i} > 0 \text{ for some } i = 1, \cdots, k_1, \text{ or } N_{l_k, K_i} > 0 \text{ for some } i = 1, \cdots, k_2,
   $$

   we have

   $$
   \mathbb{P} \left( \sup_{v \in A_{i,j,\mathcal{I}, \mathcal{K}}} X(v) > u, \sup_{v' \in A_{i',j',\mathcal{I}', \mathcal{K}'}} X(v') > u \right) 
   \leq CS^{2k} \exp \left( -C_1 \left( \sum_{i=1}^{k_1} \sqrt{N_{j_i, p_i}} S_{\alpha_i} + \sum_{i=1}^{k_2} \left( N_{l_k, K_i} S_{\alpha_{k+1}} \right) \right) \right) \Psi(u).
   $$

2. Let $(j_i, p_i), (j'_i, p'_i) \in \mathcal{L}_1, i = 1, \cdots, k_1, (\mathcal{I}_k, \mathcal{K}), (\mathcal{I}'_k, \mathcal{K}') \in \mathcal{L}_2$ satisfy

   $$
   N_{j_i, p_i} = 0 \text{ for all } i = 1, \cdots, k_1, \text{ and } N_{l_k, K_i} = 0 \text{ for all } i = 1, \cdots, k_2.
   $$

If $(j, p) \prec (j', p')$, then the following number $\kappa$ can be defined:

$$
\kappa = \begin{cases} 
\| \| : = \inf \{ 1 \leq i \leq k_1 : p_i = p'_i, j_i = j_i + 1 \}, & \text{if } \| \| \exists, \\
\| \| : = \inf \{ 1 \leq i \leq k_1 : p_i = p_i + 1, j_i = n_{i, p_i}, j'_i = 0 \}, & \text{if } \| \| \not\exists.
\end{cases}
$$

Similarly, if $(j, p) = (j', p')$ and $(\mathcal{I}_k, \mathcal{K}) \prec (\mathcal{I}'_k, \mathcal{K}')$, then we can define $\kappa$ as

$$
\kappa = \begin{cases} 
\| \| : = k_1 + \inf \{ 1 \leq i \leq k_2 : K_i = K'_i, I_k = I_k + 1 \}, & \text{if } \| \| \exists, \\
\| \| : = k_1 + \inf \{ 1 \leq i \leq k_2 : K_i = K_i + 1, I_k = N_i, I'_k = 0 \}, & \text{if } \| \| \not\exists.
\end{cases}
$$

Assume, without loss of generality, that $\kappa = \| \| \exists$ exists. We have

$$
\mathbb{P} \left( \sup_{v \in A_{i,j,\mathcal{I}, \mathcal{K}}} X(v) > u, \sup_{v' \in A_{i',j',\mathcal{I}', \mathcal{K}'}} X(v') > u \right) 
\leq CS^{2k} \exp \left( -C_1 S^{\alpha_{\kappa}/2} \right) \Psi(u),
$$

where

$$
A_{\kappa} = \prod_{i=1}^{\kappa-1} B_{j_i, p_i} \times \left[ c_{p_\infty}^\kappa + \frac{(j_\infty + 1)S + \sqrt{S}}{u^{2/(\alpha_{\infty}(c_{p_\infty}^\kappa + 1))}}, c_{p_\infty}^\kappa + \frac{(j_\infty + 2)S}{u^{2/(\alpha_{\infty}(c_{p_\infty}^\kappa + 1))}} \right] \times \prod_{i=\kappa+1}^{k_1} B_{j_i, p_i} \times V_{l_k, \mathcal{K}'}.
$$
Lemma 3.7. The next lemma gives the asymptotics of (3.13), which is the main part of the proof of Theorem 1.1.

Proof of Lemma 3.7. Let \( \{X(t), t \in [0, T]\} \) be the simplified \( \alpha(t) \)-locally stationary GRF. We have

\[
\Pi(u) = \left( \prod_{i=1}^{k_1} \frac{a_i}{2} \right)^{1/\beta_i} \Gamma(1/\beta_i + 1) \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \int_{x \in \prod_{i=k+1}^{k} (\alpha_i, \beta_i)} \prod_{i=1}^{k} (C_i(x))^{1/\alpha_i} \, dx 
\times u^\alpha (\ln u)^\beta \Psi(u)(1 + o(1)), \quad u \to \infty,
\]

where \( \alpha, \beta \) are the same as in Theorem 1.1.

The next lemma gives the asymptotics of (3.13), which is the main part of the proof of Theorem 1.1.

Lemma 3.8. Let \( \{X(t), t \in [0, T]\} \) be the simplified \( \alpha(t) \)-locally stationary GRF. Then

\[
\mathbb{P} \left( \sup_{t \in [0, T]} X(t) > u \right) = o(\Pi(u)), \quad u \to \infty.
\]

4 Proofs

Proof of Theorem 1.1. Taking into account of the (simplification) statement in the beginning of Section 3, we conclude that the claim follows directly from (3.12) and Lemmas 3.7 and 3.8.

Proof of Lemma 3.7. Set

\[
X_{j,p,u}^{k,0}(u) = X \left( \left( \begin{array}{c} c_{p_1} \frac{j_1 S + \nu_1}{u^2(\alpha_1(c_{p_1}+1))} \\
\cdots \\
c_{p_k} \frac{j_k S + \nu_k}{u^2(\alpha_k(c_{p_k}+1))} \end{array} \right) + \delta k + g_u S \mathbb{I} + \Delta_0 u \right)
\]

with \( \Delta_0 = g_u \prod_{i=k+1}^{k} [\alpha_i, \beta_i] \). It follows that

\[
\sup_{v \in A_{j,p}^{k,0}} X(v) = \sup_{v \in [0, S]} X_{j,p,u}^{k,0}(v).
\]

Furthermore, we derive, for the fixed point \( v^0 \) in \( A_{j,p}^{k,0}, \) and \( u \) sufficiently large,

\[
1 - \text{Cov} \left( X_{j,p,u}^{k,0}(v), X_{j,p,u}^{k,0}(v + \mathbf{x}) \right)
\geq (1 - \varepsilon/4)^{1/3} \left( \sum_{i=1}^{k_1} C_i(v) u^{-2/(\alpha_1(c_{p_1}+1))} x_i \right)^{\alpha_1 \left( \frac{c_{p_1}}{u} + \frac{j_1 S + \nu_1}{u^2(\alpha_1(c_{p_1}+1))} \right)} + \sum_{i=k+1}^{k} C_i(v) u^{-2|x_i|^{\alpha_i}}
\geq (1 - \varepsilon/2)^{1/3} \left( \sum_{i=1}^{k_1} C_i(v^0) u^{-2/(\alpha_1(c_{p_1}+1))} x_i \right)^{\alpha_1 \left( \frac{c_{p_1}}{u} + \frac{j_1 S + \nu_1}{u^2(\alpha_1(c_{p_1}+1))} \right)} + \sum_{i=k+1}^{k} C_i(v^0) u^{-2|x_i|^{\alpha_i}}
\]
uniformly with respect to $\nu, \nu + x \in [0, S]^k$, where we used the fact that $C_i(\cdot), i = 1, \ldots, k$, are continuous functions. In view of the proof of Lemma 4.1 of Debicki and Kisowski (2008) for sufficiently large $u$ we obtain

\[
1 - \text{Cov} \left( X_{j,p_i,u}^i(\nu), X_{j,p_i,u}^i(\nu + x) \right) \geq (1 - \varepsilon/2) \left( \sum_{i=1}^{k_1} C_i(v_0^0)u^{-2}|x_i|^\alpha_i + \sum_{i=k_1+1}^k C_i(v_0^0)u^{-2}|x_i|^\alpha_i \right) \quad (4.27)
\]

uniformly with respect to $\nu, \nu + x \in [0, S]^k$. Similarly, for sufficiently large $u$

\[
1 - \text{Cov} \left( \tilde{X}_{j,p_i,u}^i(\nu), \tilde{X}_{j,p_i,u}^i(\nu + x) \right) \leq (1 + \varepsilon/2) \left( \sum_{i=1}^{k_1} C_i(v_0^0)u^{-2}|x_i|^\alpha_i \right), \quad (4.28)
\]

uniformly with respect to $\nu, \nu + x \in [0, S]^k$. The claim follows now by the Slepian’s inequality. \hfill \Box

**Proof of Lemma 3.3** The proofs of (i) and (ii) are similar, therefore we present below only the proof of (i). Note that

\[
\lim_{u \to \infty} u^2(1 - \text{Cov}(Y_{\varepsilon,u}^0(t), Y_{\varepsilon,u}^0(s))) = (1 - \varepsilon) \sum_{i=1}^k C_i(v_0^0)|t_i - s_i|^\alpha_i
\]

uniformly with respect to $s, t \in [0, S]^k$. Hence (i) follows from Lemma 3.3. \hfill \Box

**Proof of Lemma 3.5** Let

\[
W_{\varepsilon,u}(\nu, \nu') := \frac{1}{\sqrt{2}} \left( Z_{\varepsilon,u}^0(\nu) + \tilde{Z}_{\varepsilon,u}^0(\nu') \right), \quad \nu, \nu' \in [0, S]^k.
\]

Since $\mathbb{E}(W_{\varepsilon,u}(\nu, \nu')) = 0$, $\mathbb{E}((W_{\varepsilon,u}(\nu, \nu'))^2) = 1$, and

\[
\lim_{u \to \infty} u^2(1 - \text{Cov}(W_{\varepsilon,u}(\nu, \nu'), W_{\varepsilon,u}(\mu, \mu'))) = (1 + \varepsilon) \left( \sum_{i=1}^k C_i(v_0^0)|\nu_i - \mu_i|^\alpha_i + \sum_{i=1}^k C_i(w_0^0)|\nu'_i - \mu'_i|^\alpha_i \right)
\]

uniformly with respect to $\nu, \mu, \nu', \mu' \in [0, S]^k$, it follows immediately from Lemma 3.3 that, as $u \to \infty$,

\[
\mathbb{P} \left( \sup_{\nu, \mu \in [0, S]^k} \frac{1}{\sqrt{2}} \left( Z_{\varepsilon,u}^0(\nu) + \tilde{Z}_{\varepsilon,u}^0(\mu) \right) > u \right) = \left( \prod_{i=1}^k \mathcal{H}_{\alpha_i} \left( 0, (C_i(0^0) + \varepsilon^{1/\alpha_i})^{1/\alpha_i} S_1 \right) \right)^2 \Psi(u)(1 + o(1))
\]

where in the last inequality we used the fact that $\mathcal{H}_{\alpha_i}[0, R] \leq \mathcal{H}_{\alpha_i}[0, 1] R$, for any $R > 1$ (cf. Piterbarg (1996)), hence the proof is complete. \hfill \Box

**Proof of Lemma 3.7** Since the proof of (1) and (2) are similar, we present next only the proof of (1). Let

\[
Y_u(\nu, \nu') = X_{1,u}(\nu) + X_{2,u}(\nu'),
\]

where

\[
X_{1,u}(\nu) = X \left( c_{p_1}^1 + \frac{j_1 S + \nu_1}{u^2(\alpha_1(c_{p_1}^1 + \nu_1))}, \ldots, c_{p_k}^k + \frac{j_k S + \nu_k}{u^2(\alpha_k(c_{p_k}^k + \nu_k))}, \alpha + \delta k + g_u SD k + \Delta_u^2 \right)
\]
and
\[ X_{2,u}(\nu') = X \left( c_{i_1}^{j_1} + \frac{j_1^2 S + \nu_1'}{u^{2/(\alpha_1(c_{i_1}^{j_1} + 1))}}, \ldots, c_{i_k}^{j_k} + \frac{j_k^2 S + \nu_k'}{u^{2/(\alpha_k(c_{i_k}^{j_k} + 1))}}, \alpha + \delta k' + g_u S \delta k' + \Delta_0' \right), \]

with \( \Delta_0' = g_u \prod_{i=k+1}^{k} [1 + \nu_i'] \). For any \( u > 0 \), we have
\[
\mathbb{P} \left( \sup_{\nu' \in A_{j_1,p',k}'} X(\nu') > u \right) \leq \mathbb{P} \left( \sup_{\nu, \nu' \in [0, S]^k} Y_u(\nu, \nu') > 2u \right).
\]

We see from \([3.20],[3.21]\) that, for sufficiently large \( u \),
\[
\text{Var}(Y_u(\nu, \nu')) = 4 - 2(1 - \text{Cov}(X_{1,u}(\nu), X_{2,u}(\nu'))).
\]

It follows, for fixed \( i = 1, \ldots, k_1 \), and \( v_i \in B_{j_i,p_i} \), \( v_i' \in B_{j_i,p_i}' \), that \( |v_i - v_i'| \geq N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \). Further, we have, for fixed \( i = 1, \ldots, k_2 \), \( v_{k_1+i} \in [K_1 \delta + \frac{\nu_{k_1+i}^{j_i} S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}}, K_1 \delta + \frac{(\nu_{k_1+i}^{j_i} + 1) S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}}] \) and \( v_{k_1+i}' \in [K_1 \delta + \frac{\nu_{k_1+i}^{j_i'} S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}}, K_1 \delta + \frac{(\nu_{k_1+i}^{j_i'} + 1) S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}}] \) that \( |v_{k_1+i} - v_{k_1+i}'| \geq N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \). Therefore, there exists some \( C_2 > 0 \) such that for sufficiently large \( u \)
\[
\text{Var}(Y_u(\nu, \nu')) \leq 4 - C_2 \left( \sum_{i=1}^{k_1} \left( N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \right)^{\alpha_i(c_{i_i}^{j_i} + 1)} + \sum_{i=1}^{k_2} \left( N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \right)^{\alpha_{k_1+i}} \right). 
\]

With the help of Lemma 4.4 of Dębicki and Kosiowski (2008), we have, for some \( C_3 > 0 \),
\[
\text{Var}(Y_u(\nu, \nu')) \leq 4 - C_3 \left( \sum_{i=1}^{k_1} \left( \sqrt{N_{j_i,p_i}^{j_i',p_i'}} S \right)^{\alpha_i} + \sum_{i=1}^{k_2} \left( N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \right)^{\alpha_{k_1+i}} \right) u^{-2} =: H(S, u). 
\]

Consequently,
\[
\mathbb{P} \left( \sup_{\nu, \nu' \in [0, S]^k} Y_u(\nu, \nu') > 2u \right) \leq \mathbb{P} \left( \sup_{\nu, \nu' \in [0, S]^k} \sqrt{\varphi(U(\nu, \nu'))} > \frac{2u}{\sqrt{H(S, u)}} \right), 
\]

where \( \varphi(U(\nu, \nu')) = Y_u(\nu, \nu') / \sqrt{\text{Var}(Y_u(\nu, \nu'))} \). Furthermore, following the argumentation analogous to that given in the proof of Lemma 6.3 in Piterbarg (1996) (see alternatively the proof of Lemma 4.5 in Dębicki and Kosiowski (2008)), for \( \nu, \nu', \mu, \mu' \in [0, S]^k \),
\[
\mathbb{E} \left( (\sqrt{\varphi(U(\nu, \nu'))} - \varphi(U(\mu, \mu')))^2 \right) \leq 4 \left( \mathbb{E} \left( (X_{1,u}(\nu) - X_{1,u}(\mu))^2 \right) + \mathbb{E} \left( (X_{2,u}(\nu') - X_{2,u}(\mu'))^2 \right) \right) 
\]
\[
\leq \frac{1}{8} \left( \mathbb{E} \left( (Z_{S,u}^{\nu_o}(\nu) - Z_{S,u}^{\nu_o}(\mu))^2 \right) + \mathbb{E} \left( (Z_{S,u}^{\nu_o}(\nu') - Z_{S,u}^{\nu_o}(\mu'))^2 \right) \right),
\]

where the GRF \( Z_{S,u}^{\nu_o} \) is independent of \( Z_{S,u}^{\nu_o} \), and has covariance structure \([3.10]\) with \( \nu_o \) replaced by \( \nu^{o} \) (chosen similarly as \( \nu^{o} \)). Next, by Slepian’s inequality (see e.g., Theorem C.1 of Piterbarg (1996)) and \([3.20],[3.21]\) we obtain
\[
\mathbb{P} \left( \sup_{\nu, \nu' \in [0, S]^k} \sqrt{\varphi(U(\nu, \nu'))} > \frac{2u}{\sqrt{H(S, u)}} \right) \leq \mathbb{P} \left( \sup_{\nu, \nu' \in [0, S]^k} \sqrt{\varphi(U(\nu, \nu'))} > \frac{2u}{\sqrt{H(S, u)}} \right) 
\]
\[
\leq F_{S,u}^{2k} \Psi \left( \frac{2u}{\sqrt{H(S, u)}} \right) 
\]
\[
\leq CS^{2k} \exp \left( -C_1 \left( \sum_{i=1}^{k_1} \left( \sqrt{N_{j_i,p_i}^{j_i',p_i'}} S \right)^{\alpha_i} + \sum_{i=1}^{k_2} \left( N_{j_i,p_i}^{j_i',p_i'} \frac{S}{u^{2/\alpha_i(c_{i_i}^{j_i} + 1)}} \right)^{\alpha_{k_1+i}} \right) \right) \Psi(u) 
\]
for \( u \) sufficiently large. Next, in order to prove (3) we apply the Borell theorem (e.g., Piterbarg (1996)). By [1.4] and \( 3.24 \), we see that

\[
\sup_{v \in A^{\tilde{p}, k}_{j, p}, \tilde{v} \in A^{\tilde{p}', k}'} \text{Var}(X(v) + X(v')) = 4 - 2 \inf_{v \in A^{\tilde{p}, k}_{j, p}, \tilde{v} \in A^{\tilde{p}', k}'} (1 - \text{Cov}(X(v), X(v'))) < 4 - 2\lambda,
\]

with some \( \lambda \in (0, 1) \). Further, there exists some \( h > 0 \), such that

\[
\mathbb{P}\left( \sup_{v \in A^{\tilde{p}, k}_{j, p}, \tilde{v} \in A^{\tilde{p}', k}'} X(v) + X(v') > h \right) \leq 2\mathbb{P}\left( \sup_{v \in [0, T]} X(v) > h/2 \right) < \frac{1}{2}
\]

Consequently, utilising Borell theorem, we obtain, for \( u \) sufficiently large

\[
\mathbb{P}\left( \sup_{v \in A^{\tilde{p}, k}_{j, p}, \tilde{v} \in A^{\tilde{p}', k}'} X(v) + X(v') > 2u \right) \leq 2\mathbb{P}\left( \frac{u - h/2}{\sqrt{1 - \lambda/2}} \right)
\]

establishing thus the claim. \( \square \)

**Proof of Lemma 3.7** Let \( \varepsilon \in (0, 1) \) be an arbitrarily chosen constant, and set \( \varepsilon := 1 + \varepsilon \). We first give the upper bound. Noting that \( n_{i, p_i} = \left\lfloor \frac{c_{p_i} - c_{p_i}}{s} u^{2/\alpha_i} (c_{p_i + 1}) \right\rfloor \), we derive that, as \( u \to \infty \),

\[
\Pi(u) \leq \sum_{(j, p_j) \in U^1, 1 \leq i \leq k_1, (k, j \in \mathbb{N})} \mathbb{P}\left( \sup_{v \in A^{\tilde{p}, k}_{j, p}, \tilde{v} \in A^{\tilde{p}', k}'} X(v) > u \right) \leq \sum_{(j, p_j) \in U^1, 1 \leq i \leq k_1} \sum_{k \in \mathbb{N}} \sum_{\nu \in [0, S]^k} \mathbb{P}\left( \sup_{v \in [0, S]^k} Z^0\varepsilon(v, \nu) > u \right)
\]

\[
= \sum_{p_i \leq m_1 \leq \ldots \leq k_1} \left( \prod_{i=1}^{k_1} \mathbb{H}_{\alpha_i} \left[ 0, C_i(v)^{\varepsilon} \right]^{1/\alpha_i} S \right) \prod_{i=1}^{k} \left( \mathbb{C}_i \left( 0, C_i(v)^{\varepsilon} \right)^{1/\alpha_i} S \right) \frac{1}{S^{k_1}} \prod_{i=1}^{k_1} \left( \frac{u^{2/\alpha_i}}{(\ln u)^{1/\beta_i}} \right) \prod_{i=1}^{k_1} \left( \ln u \right)^{1/\beta_i} \left( c_{p_i}^2 - c_{p_i}^2 \right)^{\varepsilon} \left( \ln u \right)^{1/\beta_i} \left( c_{p_i}^2 - c_{p_i}^2 \right)^{\varepsilon} \left( \ln u \right)^{1/\beta_i} \left( c_{p_i}^2 - c_{p_i}^2 \right)^{\varepsilon}
\]

where

\[
\eta(u, k, \alpha, \beta) := \frac{\prod_{i=1}^{k} u^{2/\alpha_i}}{\prod_{i=1}^{k} (\ln u)^{1/\beta_i}},
\]

with \( \prod_{i=m+1}^{k} \cdot := 1, m \in \mathbb{N} \). It follows that (see also Debicki and Kisowski (2008))

\[
\lim_{S \to \infty} \frac{\mathbb{H}_{\alpha_i} \left[ 0, C_i(v)^{\varepsilon} \right]^{1/\alpha_i} S}{\prod_{i=1}^{k} (C_i(v)^{\varepsilon})^{1/\alpha_i} S} = \prod_{i=1}^{k} \mathbb{H}_{\alpha_i},
\]

\[
\lim_{u \to \infty} e^{\frac{2(1 - 2\alpha_i)}{s} (\ln u)^{1/\beta_i} (c_{p_i}^2 - c_{p_i}^2) \left( \ln (c_{p_i}^2 - c_{p_i}^2) \right)^{1/\beta_i}} = 1.
\]
\[
\lim_{\delta \to 0} \sum_{k=1}^{\infty} \prod_{i=1}^{k} (C_i(x)^{1/\alpha_i}) \left( N^+_{k_i} \prod_{i=k+1}^{k} (S_{u^{-2/\alpha_i}}) \right) = \int_{\mathbb{R}^+} \prod_{i=1}^{k_1} (e^{i}) \times \prod_{i=k+1}^{k} (C_i(x)^{1/\alpha_i}) \, dx
\]

and
\[
\lim_{u \to \infty} \sum_{p_1 \leq m_1} \prod_{i=1}^{k_1} \left( \ln(u)^{1/\beta_i} (c_{p_1} - c_{p_1+1}) e^{-\frac{2(1-\varepsilon)}{\alpha_i^2}} (\ln(u)^{1/\beta_i} c_{p_1+1})^{\beta_i} \right) = \int_{\mathbb{R}^+} e^{-\sum_{i=1}^{k_1} \frac{2(1-\varepsilon)}{\alpha_i^2} x_{i}^{\beta_i}} \, dx
\]

since \( \prod_{i=1}^{k_1} (\ln(u)^{1/\beta_i} (c_{p_1} - c_{p_1+1}) \to 0 \) and \((\ln(u)^{1/\beta_i} c_{m_1+1} \to \infty, \text{ as } u \to \infty). \) Consequently, the upper bound is given as
\[
\Pi(u) \leq \sum_{(j,p_i) \in \mathcal{L}^1, 1 \leq i \leq k_1} \sum_{(k_1, k_2)} \mathbb{P} \left( \sup_{v \in A_{j,k_1}^{k_2,k}} X(v) > u \right)
\]
\[
-2 \sum_{(j,p_i), (j',p'_i) \in \mathcal{L}^1, 1 \leq i \leq k_1, (k_1, k_2) \in \mathcal{L}_2} \mathbb{P} \left( \sup_{v \in A_{j,k_1}^{k_2,k}} X(v) > u, \sup_{v' \in A_{j',k_1}^{k_2,k}} X(v') > u \right)
\]

Similar arguments as in the derivation of the upper bound yield, as \( u \to \infty, \)
\[
\lim_{\delta \to 0, S \to \infty} \sum_{(j,p_i) \in \mathcal{L}^1, 1 \leq i \leq k_1} \mathbb{P} \left( \sup_{v \in A_{j,k_1}^{k_2,k}} X(v) > u \right)
\]
\[
\geq \sum_{(j,p_i) \in \mathcal{L}^1, 1 \leq i \leq k_1} \sum_{(k_1, k_2) \in \mathcal{L}_2} \mathbb{P} \left( \sup_{v \in [0,S]^k} Y_{x,u}^0(\nu) > u \right)
\]
\[
\geq (1-\varepsilon) \sum_{i=1}^{k_1} \frac{1}{\beta_i} \left( \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \right) \left( \int \prod_{i=1}^{k_1} (e^{i}) \times \prod_{i=k+1}^{k} (C_i(x)^{1/\alpha_i}) \, dx \right)
\]
\[
\times \prod_{i=1}^{k_1} \left( \frac{\alpha_i^2}{2(1-\varepsilon)} \right) \frac{\Gamma(1/\beta_i)}{\beta_i} \eta(u, k_1, \alpha, \beta) \Psi(u)(1 + o(1)).
\]

Therefore, by letting \( \varepsilon \to 0, \) in order to complete the proof, it is sufficient to show that
\[
\lim_{\delta \to 0, S \to \infty} \lim_{u \to \infty} \frac{\sum_{i=1}^{3} \mathbb{P} \left( \sup_{v \in A_{j,k_1}^{k_2,k}} X(v) > u, \sup_{v' \in A_{j',k_1}^{k_2,k}} X(v') > u \right)}{\eta(u, k_1, \alpha, \beta) \Psi(u)} = 0,
\]
where
\[
\Sigma_u^i := \sum_{(j,p),(j',p') \in A_{j,p},(j',p') \in E_i} \mathbb{P} \left( \sup_{v \in \mathbb{A}_{j,p}} X(v) > u, \sup_{v' \in \mathbb{A}_{j',p'}} X(v') > u \right), \quad i = 1, 2, 3,
\]
with
\[
E_i = \left\{ (j,p),(j',p'),(l_k,k),(l_k',k') : \text{conditions of (i) in Lemma 3.6 are satisfied, and} \right. \\
\left. (j,p) \not\prec (j',p'), \text{or } (j,p) = (j',p') \text{ and } (l_k,k) \not\prec (l_k',k') \right\}, \quad i = 1, 2, 3.
\]

Eq. (1.29) follows from Lemma 3.6 and the details are given in Appendix. \[\square\]

**Proof of Lemma 3.8** It is easy to see that the set \( [0,T]^k / \prod_{i=1}^{k_1} [0,t_i^u] \times \prod_{i=k_1+1}^k [a_i, b_i] \) is the union of \( 2^{k_1} - 1 \) sets of the form \( \prod_{i=1}^{k_1} A_{i,u} \times \prod_{i=k_1+1}^k \Theta_i \), with
\[
A_{i,u} = [0,t_i^u] \text{ or } [t_i^u,T], \quad i = 1, \ldots, k_1, \quad \text{and} \quad \Theta_i = [0,a_i] \text{ or } [a_i, b_i] \text{ or } [b_i,T], \quad i = k_1 + 1, \ldots, k,
\]
where at least one of \( \{[t_i^u,T], i = 1, \ldots, k_1, [0,a_i], [b_i,T], i = k_1 + 1, \ldots, k \} \) appears. Since the other cases are similar, without loss of generality, it suffices to prove that
\[
\mathbb{P} \left( \sup_{t \in \prod_{i=1}^{k_1-1} [0,t^u_i] \times [t^u_i,T] \times \prod_{i=k_1+1}^{k_1-1} [a_i, b_i] \times [b_k,t_k]} X(t) > u \right) = o(\Pi(u)).
\]
We see that
\[
\mathbb{P} \left( \sup_{t \in \prod_{i=1}^{k_1-1} [0,t^u_i] \times [t^u_i,T] \times \prod_{i=k_1+1}^{k_1-1} [a_i, b_i] \times [b_k,t_k]} X(t) > u \right) \leq \mathbb{P} \left( \sup_{t \in \prod_{i=1}^{k_1-1} [0,t^u_i] \times [t^u_i,T] \times \prod_{i=k_1+1}^{k_1-1} [a_i, b_i] \times [b_k,b_k+t_k]} X(t) > u \right) + \mathbb{P} \left( \sup_{t \in \prod_{i=1}^{k_1-1} [0,t^u_i] \times [t^u_i,T] \times \prod_{i=k_1+1}^{k_1-1} [a_i, b_i] \times [b_k+b_k+t_k,T]} X(t) > u \right)
\]
It is sufficient to analyze the first probability on the right-hand side of the last inequality since the analysis of the second one is similar. It is derived that
\[
\theta(u) := \mathbb{P} \left( \sup_{t \in \prod_{i=1}^{k_1-1} [0,t^u_i] \times [t^u_i,T] \times \prod_{i=k_1+1}^{k_1} [a_i, b_i] \times [b_k,b_k+t_k]} X(t) > u \right) \leq \sum_{(j,p) \in \mathbb{U}_1, i = 1, \ldots, k_1-1, k, (l_k,k) \in \mathbb{U}_2} \mathbb{P} \left( \sup_{v \in \prod_{i=1}^{k_1-1} B_{j,p}^i \times \prod_{i=k_1}^k [t^u_i,T] \times W_{l_k,k} \times (b_k+B_{l_k,k}^i)} X(v) > u \right), \quad (4.30)
\]
where \( k = (K_1, \ldots, K_{k_2-1}) \in \mathbb{Z}^{k_2-1}, \mathbb{U}_k = (l_1, \ldots, l_{k_2-1}) \in \mathbb{Z}^{k_2-1}, \) and \( B_{j,p}^i, \mathbb{U}_2 \) and \( W_{l_k,k} \) are defined similarly as \( B_{j_1,p_1}^1, \mathbb{U}_2 \) and \( W_{l_1,k} \), respectively.

For any fixed \( j_i, p_i, i = 1, \ldots, k_2-1, k, l_k,k \) such that \( (j_i,p_i) \in \mathbb{U}_1, i = 1, 2, \ldots, k_1-1, k \) and \( (l_k,k) \in \mathbb{U}_2, \) consider the GRF \( X(v) := X(v_1, \ldots, v_k) \) on the set
\[
\mathbb{A}_{j,p,l,k} := \prod_{i=1}^{k_1-1} B_{j_i,p_i}^i \times [t^u_i,T] \times W_{l_k,k} \times (b_k+B_{l_k,k}^i). \]
For notational simplicity write next $X_{k_1,u}(\nu)$ instead of

$$X \left( c_{p_1} + \frac{j_1 S + \nu_1}{u^2/((\alpha_1(c_{p_1+1}))}, \ldots, c_{p_{k_1-1}} + \frac{j_{k_1-1} S + \nu_{k_1-1}}{u^2/((\alpha_1(c_{p_{k_1-1}})))}, \nu_{k_1}, \alpha' + \delta_k + g'_u S l k + \Delta_{\alpha}' b_k + c k + \frac{j_k S + \nu_k}{u^2/((\alpha_1(c_{p_{k_1+1}})))}, \right) ,$$

where $\Delta_{\alpha}' = g'_u \prod_{i=k_1+1}^{k-1}[0, \nu_i]$, $\alpha' = (a_{k_1+1}, \ldots, a_{k-1})$ and $g'_u$ is defined in a similar way as $g_u$ (see \textbf{B.14}). It follows that

$$\sup_{\nu \in [0, S]^{k_1-1} \times [t_{u_1}^{k_1}, T] \times [0, S]^{k_2}} X_{k_1,u}(\nu) = \sup_{\nu \in [0, S]^{k_1-1} \times [t_{u_1}^{k_1}, T] \times [0, S]^{k_2}} X_{k_1,u}(\nu).$$

(4.31)

Let $b_{k_1,u} = u^{-1}(\alpha_k + \frac{3}{4}(l_k^{(u)})^\beta_k)$, and fix $v^0 \in \prod_{i=1}^{k_1-1} A_{p_i} \times [0, T] \times \delta_k \times (b_k + A_{p_i})$ with $A_{p_i}$, $\delta_k$ defined similarly as before (the only difference is the dimension). In view of the proof of \textbf{3.17}, there exists a constant $C_0$ such that, for sufficiently large $u$

$$1 - Cov(X_{k_1,u}(\nu), X_{k_1,u}(\nu + x)) \leq 1 - e^{-\frac{1}{2} \sum_{i=1}^{k_1} C_i(v^0) u^{-2} |x_i|^{\alpha_i} - C_0 |x_k|^{\alpha_k + \frac{3}{4} (l_k^{(u)})^\beta_k}}$$

uniformly with respect to $\nu, \nu + x \in [0, S]^{k_1-1} \times [t_{u_1}^{k_1}, T] \times [0, S]^{k_2}$ such that $|x_k| \leq b_{k_1,u}$. Let $\{Z_{u}^{v_0}(t), t \in [0, S]^{k_1-1} \times [t_{u_1}^{k_1}, T] \times [0, S]^{k_2}, u > 0$, be a family of centered stationary GRF's such that

$$Cov(Z_{u}^{v_0}(\nu), Z_{u}^{v_0}(\nu + x)) = e^{-\frac{1}{2} \sum_{i=1}^{k_1} C_i(v^0) u^{-2} |x_i|^{\alpha_i} - C_0 |x_k|^{\alpha_k + \frac{3}{4} (l_k^{(u)})^\beta_k}}$$

for $u$ such that $\alpha_k + \frac{3}{4} (l_k^{(u)})^\beta_k \leq 2$, and $\nu, \nu + x \in [0, S]^{k_1-1} \times [t_{u_1}^{k_1}, T] \times [0, S]^{k_2}$. In view of the Slepian's inequality, continuing (4.30) we get, as $u \to \infty$

$$\theta(u) \leq \sum_{(j, p_i) \in U_{k_1-1} \times \cdots, k_1-1, k, (\mathcal{D}_{k, \nu}) \in U_{k_1-1}^2} P \left( \sup_{\nu \in A_{j}^{p_1} \cdot A_{j}^{p_2}, \cdots} X_{k_1,u}(\nu) > u \right)$$

$$\leq \sum_{(j, p_i) \in U_{k_1-1} \times \cdots, k_1-1, k, (\mathcal{D}_{k, \nu}) \in U_{k_1-1}^2} \sum_{l=0}^{[T(b_{k_1,u})^{-1}]+1} P \left( \sup_{\nu \in [0, S]^{k_1-1} \times [b_{k_1,u} t(l+1), b_{k_1,u} t]} X_{k_1,u}(\nu) > u \right)$$

$$\leq \left( [T(b_{k_1,u})^{-1}] + 2 \right) \sum_{(j, p_i) \in U_{k_1-1} \times \cdots, k_1-1, k, (\mathcal{D}_{k, \nu}) \in U_{k_1-1}^2} \sum_{l=0}^{[T(b_{k_1,u})^{-1}]+1} P \left( \sup_{\nu \in [0, S]^{k_1-1} \times [0, b_{k_1,u} t]} Z_{u}^{v_0}(\nu) > u \right)$$

$$\leq u^{1/(\alpha_k)} \left( \sum_{(j, p_i) \in U_{k_1-1} \times \cdots, k_1-1, k, (\mathcal{D}_{k, \nu}) \in U_{k_1-1}^2} \sum_{l=0}^{[T(b_{k_1,u})^{-1}]+1} P \left( \sup_{\nu \in [0, S]^{k_1-1} \times [0, b_{k_1,u} t]} Z_{u}^{v_0}(\nu) > u \right) \right) ,$$

where in the last inequality we used that $(b_{k_1,u})^{-1} \leq u^{1/(\alpha_k)} \left( \sum_{(j, p_i) \in U_{k_1-1} \times \cdots, k_1-1, k, (\mathcal{D}_{k, \nu}) \in U_{k_1-1}^2} \right)$ given in Dębicki and Kisowski (2008).

Furthermore, it follows from Lemma \textbf{5.3} that, as $u \to \infty$,

$$\sup_{\nu \in [0, S]^{k_1-1} \times [0, b_{k_1,u} t]} Z_{u}^{v_0}(\nu) > u$$

$$= \left( \prod_{i=1}^{k_1-1} \mathcal{H}_{\alpha_i} \left[ 0, \frac{3}{2} C_i(v^0)^{1/\alpha_i} S \right] \right) \times \mathcal{H}_{\alpha_{k_1}} \left[ 0, \frac{3}{2} C_i(v^0)^{1/\alpha_i} S \right] \right) \times \mathcal{H}_{\alpha_{k+1}} \left[ 0, \frac{3}{2} C_i(v^0)^{1/\alpha_i} S \right] \right) \times \mathcal{H}_{\alpha_{k+1}} \left[ 0, \frac{3}{2} C_i(v^0)^{1/\alpha_i} S \right] \right)$$

$$\leq C_3 \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \left[ 0, 1 \right] s^{k-1} \Psi(u)(1 + o(1))$$

$$\leq C_3 \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \left[ 0, 1 \right] s^{k-1} \Psi(u)(1 + o(1))$$

$$\leq C_3 \prod_{i=1}^{k} \mathcal{H}_{\alpha_i} \left[ 0, 1 \right] s^{k-1} \Psi(u)(1 + o(1))$$
for some positive constant $C_3$. Consequently, similar arguments as in the proof of the upper bound in Theorem 3.1 implies

$$
\theta(u) \leq C_3 T \left( \prod_{i=k_i+1}^{k_i-1} (b_i - a_i) \right) \left( \prod_{i=1}^{k_i-1} \frac{(\alpha_i)^{2/\beta_i} \Gamma(1/\beta_i) (\alpha_k)^{2/\beta_k} \Gamma(1/\beta_k)}{\beta_i} \right) \left( \prod_{i=1}^{k_i-1} u^{2/\alpha_i} \right) \left( \frac{1}{\ln u} \right)^{k_i-1} \Psi(u)
$$

as $u \to \infty$, and thus the proof is complete.

\[ \square \]

5 Appendix

PROOF OF LEMMA 5.3 Using the classical approach (see, e.g., Piterbarg (1996)) we have for $u > 0$

$$
P \left( \sup_{t \in D} X_u(t) > u \right) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{z}{u}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2u}} P \left( \sup_{t \in D} X_u(t) > u \right) d\xi. \tag{5.32}
$$

It follows that, for any $u > 0$

$$
\left\{ X_u(t) \mid X_u(0) = u - \frac{z}{u}, t \in D \right\} \text{ and } \left\{ X_u(t) - r_u(t, 0)X_u(0) + r_u(t, 0) \left( u - \frac{z}{u} \right), t \in D \right\}
$$

have the same distribution (cf. Alder and Taylor (2007)) from which we see that

$$
P \left( \sup_{t \in D} X_u(t) > u \mid X_u(0) = u - \frac{z}{u} \right) = P \left( \sup_{t \in D} \left( \zeta_u(t) - u^2(1 - r_u(t, 0)) + z(1 - r_u(t, 0)) \right) > z \right),
$$

with $\{ \zeta_u(t) = u(X_u(t) - r_u(t, 0)X_u(0)), t \in D \}$. By 5.18

$$
\lim_{u \to \infty} \left( u^2(1 - r_u(t, 0)) - z(1 - r_u(t, 0)) \right) = |t|_\alpha
$$

uniformly with respect to $t \in D$ for any $z \in R$.

Next we show that $\zeta_u, u > 0$ converges weakly to $\bar{B}_\alpha$ in $\mathcal{C}(D)$ as $u \to \infty$. To this end, we need to show (e.g., Wichura (1969) or Neuhaus (1971)):

i) finite-dimensional distributions of $\zeta_u$ converge in distribution to those of $\bar{B}_\alpha$ as $u \to \infty$

ii) tightness, i.e., for any $\eta > 0$

$$
\lim_{u \to \infty} \sup_{t \in D} \left( \max_{1 \leq i \leq k} |s_i - t_i| < \delta \right) \sup_{s, t \in D} |\zeta_u(t) - \zeta_u(s)| > \eta = 0.
$$

First note that the increments of the centered GRF $\{ \zeta_u(t), t \in D \}$ have the following property

$$
\lim_{u \to \infty} Var(\zeta_u(t) - \zeta_u(s)) = \lim_{u \to \infty} \mathbb{E} \left( (\zeta_u(t) - \zeta_u(s))^2 \right)
$$

$$
= \lim_{u \to \infty} \left( 2u^2(1 - r_u(t, s)) - u^2(r_u(t, 0) - r_u(s, 0))^2 \right)
$$

$$
= 2|t - s|_\alpha
$$

$$
= Var(\bar{B}_\alpha(t) - \bar{B}_\alpha(s)). \tag{5.33}
$$
Furthermore, the above holds uniformly with respect to $t, s \in D$, implying $i)$. In order to prove the tightness, we use a similar approach as in Dieker (2005) and Dębicki et al. (2012). We start by defining, for fixed $u > 0$, a metric $d_u$ on $\mathbb{R}_k^+$ as

$$d_u(s, t) = \sqrt{E((\zeta_u(t) - \zeta_u(s))^2)}.$$ 

Further write

$$B_{d_u}(t, u, \vartheta) := \{s \in \mathbb{R}_k^+ : d_u(s, t) \leq \vartheta\}$$

for the $d_u$-ball centered at $t \in \mathbb{R}_k^+$ and of radius $\vartheta$, and let

$$H_{d_u}(D', u, \vartheta) := \ln(N'(D', u, \vartheta)),$$

with $N'(D', u, \vartheta)$ being the smallest number of such balls that cover $D'$, a compact set in $\mathbb{R}_k^+$. Here $H_{d_u}(D', u, \vartheta)$ is called (metric) entropy for $D'$ induced by $d_u$. See Adler and Taylor (2007) for more detail on metric entropy.

We see from (5.33) that, for sufficiently large, there exists some constant $C_0$ such that

$$d_u(s, t) \leq C_0 \sqrt{|s - t|_\alpha} \leq k C_0 \delta^\frac{1}{2}, \quad (5.34)$$

if $\max_{1 \leq i \leq k} |s_i - t_i| < \delta < 1$, where $\alpha := \min_{1 \leq i \leq k} \alpha_i$. By utilising Corollary 1.3.4 of Adler and Taylor (2007), it follows that there exists some universal constant $Q_0 > 0$ such that, for any $0 < \delta < 1$,

$$P \left( \sup_{s, t \in D} \left( \max_{1 \leq i \leq k} |s_i - t_i| < \delta \right) | \zeta_u(t) - \zeta_u(s) | > \eta \right) \leq P \left( \sup_{s, t \in D} \left( \max_{1 \leq i \leq k} |s_i - t_i| < \delta \right) | \zeta_u(t) - \zeta_u(s) | > \eta \right) \leq \frac{Q_0}{\eta} \int_0^{k C_0 \delta^\frac{1}{2}} H_{d_u}([0, R]_k, u, \vartheta) d\vartheta,$$

with $R < \infty$ being a sufficiently large constant. Define, for $t, s \in \mathbb{R}_k^+$, a semimetric

$$d(t, s) = C_0 \sqrt{|s - t|_\alpha}.$$ 

Thanks to (5.34) it follows that, for sufficiently large $u$ and small $\vartheta$,

$$H_{d_u}([0, R]_k, u, \vartheta) \leq H_d([0, R]_k, u, \vartheta) \leq k \ln \left( \frac{R}{\delta^{\frac{1}{2}}} + 1 \right) \leq C_1 \ln \left( \frac{1}{\vartheta} \right),$$

for some positive constant $C_1$, with $H_d([0, R]_k, u, \vartheta)$ being the entropy induced by $d$.

Consequently, we have that

$$\lim_{\delta \to 0} \lim_{u \to \infty} \sup \left( \sup_{s, t \in D} \left( \max_{1 \leq i \leq k} |s_i - t_i| < \delta \right) | \zeta_u(t) - \zeta_u(s) | > \eta \right) \leq \lim_{\delta \to 0} \frac{Q_0 \sqrt{C_1}}{\eta} \int_0^{\infty} \frac{\sqrt{\ln \vartheta}}{\vartheta^2} d\vartheta = 0,$$

establishing the claim $i)$. Moreover, since the functional $\sup_{t \in D} f(t)$ is continuous on $C(D)$, we conclude, for any $z \in R$, that

$$\lim_{u \to \infty} P \left( \sup_{t \in D} X_u(t) > u | X_u(0) = u - \frac{z}{u} \right) = P \left( \sup_{t \in D} \tilde{B}_\alpha(t) - |t|_\alpha > z \right).$$
In order to use dominate convergence theorem to the integral in (5.32) when taking limit in \( u \), we need a uniform (in \( u \) large enough) upper bound of

\[
P_u(z) := \mathbb{P} \left( \sup_{t \in D} (\zeta_u(t) - u^2(1 - r_u(t, 0)) + z(1 - r_u(t, 0))) > z \right)
\]

for \( z > 0 \) sufficiently large. It follows that, for \( u \) sufficiently large,

\[
P_u(z) \leq \mathbb{P} \left( \sup_{t \in D} \zeta_u(t) + \sup_{t \in D} (1 - r_u(t, 0))z > z \right) \leq \mathbb{P} \left( \sup_{t \in D} \zeta_u(t) > (1 - \varrho_0)z \right)
\]

(5.35)

for some \( \varrho_0 \in (0, 1) \). Further, we see from (5.33) that, for sufficiently large \( u \), there exists some positive constant \( C_2 \) such that

\[
Var(\zeta_u(t) - \zeta_u(s)) \leq C_2 Var(\widetilde{B}_\alpha(t) - \widetilde{B}_\alpha(s))
\]

for all \( s, t \in D \), implying, by Sudakov-Fernique inequality (e.g., Adler and Taylor (2007))

\[
\mathbb{E} \left( \sup_{t \in D} \zeta_u(t) \right) \leq \sqrt{C_2} \mathbb{E} \left( \sup_{t \in D} \widetilde{B}_\alpha(t) \right) := U_0.
\]

(5.36)

The constant \( U_0 \) is finite, which follows thanks to Theorem 2.1.1 of Adler and Taylor (2007). Moreover,

\[
\sup_{t \in D} Var(\zeta_u(t)) \leq \sigma^2_D := C_2 \sup_{t \in D} Var(\widetilde{B}_\alpha(t)) = 2C_2 \sup_{t \in D} |t|_\alpha < \infty.
\]

(5.37)

With the help of (5.35), (5.36) and (5.37), Borell-TIS inequality (Theorem 2.1.1 of Adler and Taylor (2007)) gives, for any \( z > \frac{U_0}{1 - \varrho_0} \) and \( u \) sufficiently large,

\[
P_u(z) \leq \mathbb{P} \left( \sup_{t \in D} \zeta_u(t) > (1 - \varrho_0)z \right) \leq \exp \left( -\frac{(1 - \varrho_0)z - U_0}{2\sigma^2_D} \right).
\]

Applying dominate convergence theorem to the integral in (5.32), we conclude that

\[
\lim_{u \to \infty} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} P_u(z)dz = \mathbb{E} \left( \exp \left( \sup_{t \in D} (\widetilde{B}_\alpha(t) - |t|_\alpha) \right) \right),
\]

thus the proof is completed. \( \square \)

**Proof of Eq. (4.29)** According to Lemma 3.6, the three parts of the double-sum in (4.29) can be estimated in different ways. It follows from (4.29) that

\[
\lim_{u \to \infty} \frac{\Sigma^1_u}{\eta(u, k_1, \alpha, \beta)\Psi(u)} \leq \lim_{u \to \infty} \frac{2\Psi \left( \frac{u - b/2}{\sqrt{1 - h/2}} \right)}{\eta(u, k_1, \alpha, \beta)\Psi(u)} \sum_{(j_1, k_1) \in \mathcal{L}_2, (j_1, p_1) \in \mathcal{L}_1, 1 \leq i \leq k_1} 1^2 = 0,
\]

where the sum in the middle term can be estimated using the same arguments as the upper bound in Theorem 1.1

Next, for sake of simplicity, we only give the estimates of the first two sums for \( k_1 = k_2 = 1 \), since the general cases \( (k_1, k_2) \) are arbitrary integers) follow from similar arguments. For the first sum, we derive, using (4.22) that, for \( u \) sufficiently large

\[
\Sigma^1_u \leq \sum_{(t_1^1, k_1) \in \mathcal{L}_2, (j_1, p_1) \in \mathcal{L}_1} 4 \sum_{(j_1', p_1') \in \mathcal{L}_1} \sum_{(t_1^1, k_1') \in \mathcal{L}_2} \sum_{(j_1', p_1') \in \mathcal{L}_1} \sum_{N_{j_1, p_1}^N, N_{j_1', p_1'}^N > 0} \mathbb{C} S^4 \exp \left( - \mathbb{C}_1 \left( (N_{j_1, p_1}^N)^{\alpha_1 / 2} S_1 \right) \right).
\]


for suitably chosen constant $\tilde{\eta}$ and the project RARE -318984, a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme.

Lastly we estimate the second sum. According to (2) of Lemma 3.6, the sum $\Sigma_2^2$ can be divided into four parts, denoted by $\Sigma_1^1,u, \Sigma_1^2,u, \Sigma_2^1,u$ and $\Sigma_2^2,i^2,u$, respectively. Applying (5.38) Lemma 3.4 and Lemma 3.5, we find that, for $u$ large enough,

$$\Sigma_1^1,u \leq (3^2 - 1) \sum_{(p^1, K_1) \in \mathcal{L}_2, (j_1, p_1) \in \mathcal{L}_1^1} \left[ \mathbb{P} \left( \sup_{c_{p_1} + \frac{1}{n^{1/2}}(c_{j_1} + \frac{1}{n^{1/2}} c_{p_1} + 1)} X(u) > u \right) + \mathbb{P} \left( \sup_{c_{p_1} + \frac{1}{n^{1/2}}(c_{j_1} + \frac{1}{n^{1/2}} c_{p_1} + 1)} X(u) > u \right) \right]$$

for suitably chosen constant $\tilde{C}$. Note that in the last formula $V_{t^1,K_1}$ is one of the adjacent sets of $V_{t^1,K_1}$, and the number of it is at most $3^2 - 1$. Using the same arguments we can obtain similar upper bounds for $\Sigma_1^2,i^1,u, \Sigma_1^2,i^2,u$ and $\Sigma_2^2,i^2,u$. Consequently, the same reasoning as (5.38) yields

$$\lim_{S \to \infty} \lim_{u \to \infty} \frac{\Sigma_2^u}{\tilde{\eta}(u, k_1, \alpha, \beta)} = 0,$$

for the claim follows. \hfill $\Box$

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