GLOBAL EXISTENCE AND CONVERGENCE RATES OF SOLUTIONS FOR THE COMPRESSIBLE EULER EQUATIONS WITH DAMPING

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Abstract. The Cauchy problem for the 3D compressible Euler equations with damping is considered. Existence of global-in-time smooth solutions is established under the condition that the initial data is small perturbations of some given constant state in the framework of Sobolev space $H^3(\mathbb{R}^3)$ only, but we don’t need the bound of $L^1$ norm. Moreover, the optimal $L^2-L^2$ convergence rates are also obtained for the solution. Our proof is based on the benefit of the low frequency and high frequency decomposition, here, we just need spectral analysis of the low frequency part of the Green function to the linearized system, so that we succeed to avoid some complicate analysis.

1. Introduction. The compressible Euler equations with damping are used to model and simulate the compressible flow through a porous medium. In the present paper, we consider the compressible Euler equations with damping

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \otimes u) + \nabla p(\rho) + \alpha \rho u &= 0, \\
(\rho, u)(x, t)|_{t=0} &= (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3,
\end{aligned}
$$

(1)

where $\rho = \rho(t, x), u = u(t, x)$ represent the density and velocity functions respectively, $x \in \mathbb{R}^3$ is the space variable, $t > 0$ is the time variable. Furthermore, the pressure $p$ is a suitably smooth function of $\rho$. Assuming the flow is a polytropic perfect gas, then $p(\rho) = A\rho^r$ with constants $A > 0$ and $r > 1$ the adiabatic exponent; the constant $\alpha > 0$ models friction and $1/\alpha$ may be regarded as the relaxation time for some physical flows.

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There are many important progresses on the investigation of the global existence and large time behavior of solutions to (1). For one dimension, the global existence of a smooth solution with small data was proved by Nishida [13, 14], and the behavior of the smooth solution was studied in many papers; see the excellent survey paper by Dafermos [3], the book by Hsiao [5], the papers [6, 7, 9, 10, 15, 21, 25, 26], and their references.

We note that the multi-dimensional compressible Euler equations with damping describe more realistic phenomena and carry some unique features, such as the effect of vorticity, which make the problem more mathematically challenging. Due to its physical importance and significant mathematical challenge, a deep research on the multi-dimensional model (1) is of great importance. In three dimensions, Wang and Yang [22] proved the global existence of the small smooth solutions and obtained the pointwise estimates of the solutions, then [17] used a different energy method to obtain the similar results. The authors in [17] proved that the $L^2$-norm of the solution decays at the rate $(1 + t)^{-3/4}$ provided that the initial data is small in $L^1$, while the authors in [20] proved that the $L^2$-norm of the solution decays at the rate $(1 + t)^{-3/4 - s/2}$ provided that the initial data is small in $B^{-s}_{1,\infty}$. [18] studied the global existence and time-asymptotic behavior of small smooth solutions to the system (1) by a refined pure energy method. [2] studied in more details the effect of the damping on the decay rate of the solutions, along with this end, they removed the smallness of those low frequency assumption of the initial data and showed the optimal decay rates of the higher-order spatial derivatives. For initial boundary value problem, refer for instance to [8, 16] and the references therein.

In this paper, we prove the global existence and optimal convergence rates of the solution when $H^3(\mathbb{R}^3)$-norm of the initial perturbation around a constant state is sufficiently small only. There are two main differences between the analysis of this paper and the known results for Navier-Stokes(-Poisson) equations [4, 19]. For Navier-Stokes equations, there are strong dissipative terms (i.e. viscous of fluid) in the momentum equations, which is helpful for energy estimates. However, there is only a weak dissipative term (i.e. damping term) in the momentum equations of Euler equations, which causes troubles for energy estimates. Another difference is that we introduce the low frequency and high frequency decomposition $u = u^l + u^h$ to consider the decay rates, where the two terminologies (i.e. the low frequency part $u^l$ and the high frequency part $u^h$) have been used in [23]. In particular, we do not require that the $L^p$ norm of initial data is sufficiently small.

We reformulate the Cauchy problem of the compressible Euler system (1) as in [17]. The main point is to obtain a symmetric system. Introduce the sound speed

$$\mu(\rho) = \sqrt{\rho'(\rho)},$$

and set $\bar{\mu} = \mu \bar{\rho}$ corresponding to the sound speed at a background density $\bar{\rho} > 0$.

Define

$$n = \frac{2}{\gamma - 1} (\mu(\rho) - \bar{\mu}).$$

Then the Euler equations (1) are transformed into the following system:

$$\begin{cases}
\partial_t n + \bar{\mu} \text{div} u = F_1,
\partial_t u + \bar{\mu} \nabla n + a u = F_2,
(n, u)(x, t)|_{t=0} = (n_0, u_0)(x), \quad x \in \mathbb{R}^3,
\end{cases}$$

(2)
where
\[ F_1 := -u \cdot \nabla n - vn, \]
\[ F_2 := -u \cdot \nabla u - vn \nabla n, \]
\[ \nu := \frac{\gamma - 1}{2}, \quad n_0 = \frac{2}{\gamma - 1}(\mu(\rho_0) - \bar{\mu}). \]

The main result of the present paper is stated in the following theorem:

**Theorem 1.1.** Let the initial data \((n_0, u_0)\) be such that \(\|(n_0, u_0)\|_{H^3(\mathbb{R}^3)}\) is sufficiently small. Then the initial value problem (2) admits a unique global solution \((n, u)\) satisfying that for all \(t \geq 0\),
\[
\|(n, u)(t)\|_{H^3}^2 + \int_0^t \|\nabla n(\tau)\|_{H^2}^2 + \|u(\tau)\|_{H^3}^2 d\tau \leq C\|(n_0, u_0)\|_{H^3}^2.
\]

Moreover, there is a constant \(C\) such that for any \(t \geq 0\), the solution \((n, u)\) enjoys the decay properties
\[
\|\nabla^k n(t)\|_{L^2} \leq C(1 + t)^{-\frac{k}{2}}, \quad \text{for} \quad k = 0, 1, 2, \tag{4}
\]
\[
\|\nabla^3 n(t)\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}}, \tag{5}
\]
\[
\|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{k+1}{4}}, \quad \text{for} \quad k = 0, 1, \tag{6}
\]
\[
\|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}}, \quad \text{for} \quad k = 2, 3, \tag{7}
\]
\[
\|\nabla^k n(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{1}{2} - \frac{k+1}{4}}, \quad \text{for} \quad k = 0, 1, \tag{8}
\]
\[
\|\nabla^k u(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{1}{2} - \frac{k+1}{4}}, \quad \text{for} \quad k = 0, 1. \tag{9}
\]

**Notations.** Throughout this paper, \(C\) denotes a generic positive constant which may vary in different estimates. The norms in the usual \(L^p\) and Sobolev Space \(H^m\) on \(\mathbb{R}^3\) are denoted by \(\|\cdot\|_{L^p}\) and \(\|\cdot\|_{H^m}\) respectively. Moreover, we use \(\langle \cdot \rangle\) to denote the inner product in \(L^2(\mathbb{R}^3)\). \(\partial_j\) stands for \(\partial_{x_j}\), \(\nabla^l\) with an integer \(l \geq 0\) stands for the usual any spatial derivatives of order \(l\), when \(l < 0\) or \(l\) is not a positive integer, \(\nabla^l\) stands for \(A^l\) defined by \(A^l u = \mathcal{F}^{-1}(\|\xi|^l \hat{u}(\xi))\), where \(\hat{u}\) is the Fourier transform of \(u\) and \(\mathcal{F}^{-1}\) its inverse. We will employ the notation \(A \lesssim B\) to mean that \(A \leq CB\) for a universal constant \(C > 0\) that only depends on the parameters coming from the problem. For the sake of conciseness, we write \(\|(A, B)\|_X := \|A\|_X + \|B\|_X\).

**2. Reformulations.** In this subsection, we will consider the global existence and time decay rates of the solutions to the reformulated system (2). By the standard continuity argument, the global existence of solutions to (2) will be obtained by combining the local existence result with global a priori estimates. The following local existence result can be established using the arguments in [11, 12].

**Proposition 1.** Suppose that the initial data satisfy \((n_0(x), u_0(x)) \in H^3(\mathbb{R}^3)\), then there exists a unique local solution \((n(x, t), u(x, t))\) of the Cauchy problems (2) in \(C([0, T), H^3) \cap C^1([0, T), H^2)\) for some finite \(T > 0\).

To prove global existence of a smooth solution with small initial data, we also need to establish the following a priori energy estimates.
Proposition 2. Let \((n_0, u_0) \in H^3(\mathbb{R}^3)\), suppose that the initial value problem (2) has a solution \((n, u)(x, t)\) on \(\mathbb{R}^3 \times [0, T]\) for some \(T > 0\). Then there exist a small constant \(\delta > 0\) and a constant \(C\), which are independent of \(T\), such that if

\[
\sup_{0 \leq t \leq T} \| (n, u)(t) \|_{H^3} \leq \delta,
\]

then for any \(t \in [0, T]\), it holds that

\[
\| (n, u)(t) \|_{H^3}^2 + C \int_0^t (\| \nabla n(\tau) \|_{H^2} + \| u(\tau) \|_{H^3}^2) d\tau \leq \| (n_0, u_0) \|_{H^3}^2. \tag{10}
\]

3. Some a priori estimates. In this section, we will establish some a priori estimates of the solution \((n, u)\). We first make the a priori assumption that

\[
\| (n, u)(t) \|_{H^3} \leq \delta \tag{11}
\]

for sufficiently small \(\delta > 0\).

3.1. Spectral analysis and linear decay structure. Let us consider the initial value problem for the linearized system:

\[
\begin{cases}
   n_t + \mu \text{div} u = 0, \\
   u_t + \mu \nabla n + au = 0, \\
   (n, u)|_{t=0} = (p_0, u_0).
\end{cases}
\tag{12}
\]

which gives rise to

\[
U(t) = S(t)U_0 = e^{tB}U_0, \quad t \geq 0. \tag{13}
\]

Applying the Fourier transform to system, we have

\[
\hat{U}_t = A(\xi)\hat{U}, \quad \hat{U}(0) = \hat{U}_0, \quad t \geq 0, \tag{15}
\]

and \(A(\xi)\) is defined as

\[
A(\xi) = \begin{pmatrix} 0 & -i\mu \xi^t \\ -i\mu \xi & -aI_{3 \times 3} \end{pmatrix}. \tag{16}
\]

The eigenvalues of the matrix \(A\) are computed from the determinant

\[
\det(A(\xi) - \lambda I) = (\lambda + a)(\lambda^2 + a\lambda + \mu^2|\xi|^2) = 0. \tag{17}
\]

The semigroup \(e^{tA}\) is expressed as

\[
e^{tA} = e^{\lambda_0 t}P_0 + e^{\lambda_1 t}P_1 + e^{\lambda_2 t}P_2, \tag{18}
\]

where the project operators \(P_i\) can be computed as

\[
P_i = \prod_{i \neq j} \frac{A(\xi) - \lambda_j I}{\lambda_i - \lambda_j}. \tag{19}
\]

Denote

\[
\hat{G}(t, \xi) = e^{tA}
\]

\[
= \begin{pmatrix} \lambda_1 e^{\lambda_2 t - \lambda_2 \xi t} & -i\xi(\xi^t e^{\lambda_1 t - \lambda_2 \xi t} - e^{\lambda_2 t} I_{3 \times 3}) \\
-\xi(\xi^t e^{\lambda_1 t - \lambda_2 \xi t} - e^{\lambda_2 t} I_{3 \times 3}) & -i\xi^t e^{\lambda_1 t - \lambda_2 \xi t} \\
\lambda_1 - \lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix} e^{\lambda_0 t} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) + \frac{\xi \otimes \xi}{|\xi|^2} \left( \frac{\xi \otimes \xi}{|\xi|^2} e^{\lambda_1 t - \lambda_2 \xi t} - \frac{\xi \otimes \xi}{|\xi|^2} e^{\lambda_2 t} I_{3 \times 3} \right)
\]

\[
= \begin{pmatrix} \hat{P}_1(t) & \hat{P}_2(t) \\
\hat{U}_1(t) & \hat{U}_2(t) \end{pmatrix}, \tag{20}
\]
Lemma 3.1. Let \( (n, u) = G \ast U_0 \) as
\[
\hat{n} = \hat{P}_1 \cdot \hat{n}_0 + \hat{P}_2 \cdot \hat{u}_0, \quad \hat{u} = \hat{U}_1 \cdot \hat{n}_0 + \hat{U}_2 \cdot \hat{u}_0.
\]
We are able to obtain that it holds for \(|\xi| \ll 1\),
\[
\lambda_0 = -a \text{(double)}, \quad \lambda_1 = -\frac{a}{2} + \frac{\sqrt{a^2 - 4\mu^2|\xi|^2}}{2} \sim \frac{\mu^2 a}{\lambda^2} |\xi|^2 + O(|\xi|^3), \quad \lambda_2 = -\frac{a}{2} - \frac{\sqrt{a^2 - 4\mu^2|\xi|^2}}{2} \sim -\frac{a}{2} + O(|\xi|).
\]
We are also able to obtain that it holds for \(|\xi| \gg 1\),
\[
\lambda_0 = -a \text{(double)}, \quad \lambda_1 = -\frac{a}{2} + i\frac{\sqrt{4\mu^2|\xi|^2 - a^2}}{2} \sim -\frac{a}{2} + i(\mu|\xi| + O(|\xi|^{-1})), \quad \lambda_2 = -\frac{a}{2} - i\frac{\sqrt{4\mu^2|\xi|^2 - a^2}}{2} \sim -\frac{a}{2} - i(\mu|\xi| + O(|\xi|^{-1})).
\]
Then we obtain that
\[
\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \sim \begin{cases} O(1) e^{-\frac{a^2}{4}|\xi|^2 t}, & \text{for } |\xi| \ll 1, \\ O(1) e^{-\frac{a}{2} t}, & \text{for } |\xi| \gg 1, \end{cases}
\]
\[
\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \sim \begin{cases} O(1)|\xi|^2 e^{-\frac{a^2}{4}|\xi|^2 t}, & \text{for } |\xi| \ll 1, \\ O(1) e^{-\frac{a}{2} t}, & \text{for } |\xi| \gg 1, \end{cases}
\]
\[
\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \sim \begin{cases} O(1)|\xi|^2 e^{-\frac{a^2}{4}|\xi|^2 t}, & \text{for } |\xi| \ll 1, \\ O(1) e^{-\frac{a}{2} t}, & \text{for } |\xi| \gg 1. \end{cases}
\]
We can now derive the decay rates for the solution of the linear system (12).

Lemma 3.1. Let \( U_0 = (n_0, u_0) \in W^{l,q}(\mathbb{R}^3) \) with \( l \geq 1 \), where \( p, q \in [1, \infty) \) and \( p \geq q \), then \( U = (n, u) \) satisfies for \( 0 \leq k \leq l - 1 \) that
\[
\|\partial_x^k n\|_{L^p(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{k}{2} - \frac{3}{4} - \frac{3}{2}} \|n_0, u_0\|_{W^{l,q}(\mathbb{R}^3)},
\]
\[
\|\partial_x^k u\|_{L^p(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{k}{2} - \frac{3}{4} - \frac{3}{2} - \frac{1}{2}} \|n_0, u_0\|_{W^{l,q}(\mathbb{R}^3)},
\]
where \( C > 0 \) is a positive constant independent of time.

Proof. With the help of the formula for Green’s function in Fourier space and the asymptotical analysis on its elements, we have
\[
\hat{n} = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \hat{n}_0 - \frac{i\xi \cdot \hat{u}_0(e^{\lambda_1 t} - e^{\lambda_2 t})}{\lambda_1 - \lambda_2}
\]
\[
\sim \begin{cases} O(1)e^{-\frac{a^2}{4}|\xi|^2 t}(|\hat{n}_0| + |\hat{u}_0|), & \text{for } |\xi| \ll 1, \\ O(1)|\xi|^2 e^{-\frac{a^2}{4}|\xi|^2 t}(|\hat{n}_0| + |\hat{u}_0|), & \text{for } |\xi| \gg 1, \end{cases}
\]

Therefore, we have the $L^2$-decay rate on the derivatives of $n$ as

$$
\|\partial_x^k n\|_{L^p(\mathbb{R}^3)} = \|\partial_x^k n\|_{L^q(\mathbb{R}^3)} \leq C \left[ \int_{|\xi| \leq q} e^{-\frac{\mu^2}{4}|\xi|^2} |\xi|^{kq} |\hat{\eta}_0|^q + |\hat{u}_0|^q d\xi \right]^{\frac{1}{q}}
$$

$$
+ C \left[ \int_{|\xi| \geq q} e^{-\frac{\mu t}{4}|\xi|} |\xi|^{(k+1)q} |\hat{\eta}_0|^q + |\hat{u}_0|^q d\xi \right]^{\frac{1}{q}}
$$

$$
\leq C \left[ \int_{|\xi| \leq q} e^{-\frac{\mu^2}{4}|\xi|^2} |\xi|^{kq} d\xi \right] \|\hat{\eta}_0, \hat{u}_0\|_{L^p}
$$

$$
+ C \left[ \int_{|\xi| \geq q} e^{-\frac{\mu t}{4}|\xi|} |\xi|^{(k+1)q} d\xi \right] \|\xi|^{k+1} \hat{\eta}_0, |\xi|^{k+1} \hat{u}_0\|_{L^p}
$$

$$
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \|n_0, u_0\|_{W^{1,q}(\mathbb{R}^3)},
$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $\eta$ are some positive constants. As for $u$, in the similar fashion, we have

$$
\hat{u} \sim \begin{cases} 
O(1) |\xi|^{-\frac{\mu}{4}|\xi|^2} (|\hat{\eta}_0| + |\hat{u}_0|), & \text{for } |\xi| \ll 1, \\
O(1) |\xi|^{-\frac{\mu}{4}|\xi|^2} (|\hat{\eta}_0| + |\hat{u}_0|), & \text{for } |\xi| \gg 1,
\end{cases}
$$

(28)

then we also obtain that

$$
\|\partial_x^k u\|_{L^p(\mathbb{R}^3)} = \|\partial_x^k \hat{u}\|_{L^p(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \|n_0, u_0\|_{W^{1,q}(\mathbb{R}^3)}.
$$

(29)

The proof of the Lemma is completed. \hfill \Box

3.2. Low frequency and high frequency decomposition. Based on the Fourier transform, we can define a low frequency and high frequency decomposition $(f^l(x), f^h(x))$ for a function $f(x) \in L^2(\mathbb{R}^3)$ as follows

$$
f^l(x) = \lambda_0(D_x)f(x), f^h(x) = \chi_1(D_x)f(x),
$$

(30)

where $\lambda_0(D_x)$ and $\chi_1(D_x), D_x = \sqrt{-1} \nabla = \sqrt{-1}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}),$ are the pseudo-differential operators with symbols $\lambda(\xi)$ and $1 - \chi(\xi)$, respectively. Here, $\chi(\xi)$ is a smooth cut-off function satisfying $0 \leq \chi(\xi) \leq 1$ and

$$
\chi(\xi) = \begin{cases} 
1, & |\xi| < \varepsilon \\
0, & |\xi| > 2\varepsilon,
\end{cases}
$$

(31)

for some chosen constant $\varepsilon$. Thus, it is obvious that

$$
f(x) = f^l(x) + f^h(x).
$$

(32)

Similarly, we can also define a low frequency and high frequency decomposition $(f^l(x), f^{h'}(x))$ for $f(x)$ as follows

$$
f^l(x) = \lambda'_0(D_x)f(x), f^{h'}(x) = \chi'_1(D_x)f(x),
$$

(33)

where $\lambda'_0(D_x)$ and $\chi'_1(D_x),$, are the pseudo-differential operators with symbols $1 - (1 - \chi(\xi))^2$ and $(1 - \chi(\xi))^2$, respectively.

Noticing the definition of $f^h(x)$ and using the Plancherel theorem, it is obvious that there exist two positive constants $C_3$ and $C_4$ such that if $f(x) \in H^3(\mathbb{R}^3)$,

$$
C_3\|\nabla^2 f^h\|_{L^2} \leq \|\nabla^3 f^h\|_{L^2}, \quad C_4\|\nabla^2 f^h\|_{H^1} \leq \|\nabla^2 f\|_{H^1}.
$$

(34)
3.3. **Estimates of the low-frequency part.** To begin with, we first rewrite the solution of (2) as

\[ U(t) = S(t)U_0 + \int_0^t S(t - \tau)(F_1, F_2)(s) ds, \quad t \geq 0. \]  

(35)

Taking \( \chi_0(D_x) \) on both sides of (35), we have

\[ U'(t) = S'(t)U_0 + \int_0^t S'(t - \tau)(F_1, F_2)(s) ds, \quad t \geq 0, \]  

(36)

where \( S'(t) = \chi_0(D_x)S(t) \) and \( U'(t) = (n'(t), u'(t)) \). Making use of Lemma 3.1, we have the \( L^p - L^q \) type of the time decay estimates of \( S(t) \) as follows.

**Lemma 3.2.** Let \( k \geq 0 \) be integers and \( p, q \in [1, \infty) \) and \( p \geq q \), then for any \( t > 0 \), it holds that

\[
\| \partial_x^k n'(t) \|_{L^p(R^3)} \leq C(1 + t)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| (n_0, u_0) \|_{L^q(R^3)}, \\
\| \partial_x^k u'(t) \|_{L^p(R^3)} \leq C(1 + t)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| (n_0, u_0) \|_{L^q(R^3)},
\]

(37)

where \( C > 0 \) is a positive constant independent of time.

In the following, we show the estimates on \( U'(t) = (n'(t), u'(t)) \). By the way, one can get the same estimates for \( U''(t) \), where \( U''(t) = (n''(t), u''(t)) \).

**Lemma 3.3.** Under the assumptions of Proposition 2, it holds that for \( 2 \leq p \leq +\infty \) and any integer \( k \geq 0 \)

\[
\| \nabla^k n'(t) \|_{L^p} \leq C(1 + t)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| U_0 \|_{L^2} + C \delta \int_0^t (1 + t - \tau)^{-\frac{2}{3}(1 - \frac{1}{p}) - \frac{k}{2}} \| \nabla U(\tau) \|_{L^2} d\tau \\
+ C \delta \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| U(\tau) \|_{L^\infty} d\tau.
\]

(38)

\[
\| \nabla^k u'(t) \|_{L^p} \leq C(1 + t)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k+1}{2}} \| U_0 \|_{L^2} + C \delta \int_0^t (1 + t - \tau)^{-\frac{2}{3}(1 - \frac{1}{p}) - \frac{k+1}{2}} \| \nabla U(\tau) \|_{L^2} d\tau \\
+ C \delta \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k+1}{2}} \| U(\tau) \|_{L^\infty} d\tau.
\]

(39)

**Proof.** From (36) and Lemma 3.2, we obtain

\[
\| \nabla^k n'(t) \|_{L^p} \leq C(1 + t)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| U_0 \|_{L^2} + C \int_0^t (1 + t - \tau)^{-\frac{2}{3}(1 - \frac{1}{p}) - \frac{k}{2}} \| (F_1, F_2)(\tau) \|_{L^1} d\tau \\
+ C \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{2}{3}(\frac{1}{q} - \frac{1}{p}) - \frac{k}{2}} \| (F_1, F_2)(\tau) \|_{L^2} d\tau.
\]

(40)

We shall estimate the second and third terms on the right hand side of (40), by Hölder’s inequality and the a priori assumption (11), we have

\[
\| (F_1, F_2) \|_{L^1} \lesssim \| (n, u) \|_{L^3} \| \nabla (n, u) \|_{L^2} \lesssim \delta \| \nabla (n, u) \|_{L^2},
\]

(41)

\[
\| (F_1, F_2) \|_{L^2} \lesssim \| (n, u) \|_{L^3} \| \nabla (n, u) \|_{L^\infty} \lesssim \delta \| \nabla (n, u) \|_{L^\infty}.
\]

(42)
Putting (41)-(42) into (40), we get (38). Doing the similar argument to (39), we thus complete the proof of Lemma 3.3.

3.4. Estimates of the high-frequency part. In this section, we will carry out the energy estimates on the high frequency component of the solution. First of all, taking the operator $\chi_1(D_x)$ on both sides of (2)$_{1,2}$, we have a problem of $(p^h(t), u^h(t))$ as follows

$$
\begin{cases}
\partial_t n^h + \mu \text{div} u^h = F_1^h, \\
\partial_t u^h + \mu \nabla n^h + au^h = F_2^h, \\
(n^h, u^h)(x, 0) = (n_0^h, u_0^h)(x),
\end{cases}
$$

where $(F_1^h, F_2^h) := (\chi_1(D_x)F_1, \chi_1(D_x)F_2)$. In what follows, we show the energy estimate for $(n^h, u^h)$.

**Lemma 3.4.** For $k=2$, it holds that

$$
\frac{1}{2} \frac{d}{dt} \| \nabla^k (n^h, u^h)(t) \|_{L^2}^2 + \frac{a}{2} \| \nabla^k u^h \|_{L^2}^2 
\lesssim \delta \{ \| \nabla^k (n^h, u^h) \|_{L^2}^2 + \| \nabla^2 (n^h, u^h) \|_{H^1}^2, \\
+ \| \nabla (n', u') \|_{L^\infty} + \| \nabla (n', u') \|_{L^2} + \| \nabla^{k+1} (n', u') \|_{L^2} \},
$$

for any $0 \leq t \leq T$ and some small constant $\delta > 0$.

**Proof.** First we take $\nabla^k$ to (43), and multiply the equations by $\nabla^k n^h$ and $\nabla^k u^h$ respectively. After adding them together, we integrate the resulting over $\mathbb{R}^3$ by parts

$$
\frac{1}{2} \frac{d}{dt} \| \nabla^k (n^h, u^h)(t) \|_{L^2}^2 + a \| \nabla^k u^h \|_{L^2}^2 
= -\langle \nabla^k n^h, \chi_1(D_x) \nabla^k (u \cdot \nabla n) \rangle - \nu \langle \nabla^k n^h, \chi_1(D_x) \nabla^k (n \text{div} u) \rangle \\
- \langle \nabla^k u^h, \chi_1(D_x) \nabla^k (u \cdot \nabla u) \rangle - \nu \langle \nabla^k u^h, \chi_1(D_x) \nabla^k (n \nabla n) \rangle.
$$

We shall estimate each term on the right hand side of (45). The second and fourth terms on the right hand side of (45) can be rewritten as follows

$$
- \nu \langle \nabla^k n^h, \chi_1(D_x) \nabla^k (n \text{div} u) \rangle - \nu \langle \nabla^k u^h, \chi_1(D_x) \nabla^k (n \nabla n) \rangle \\
= - \nu \langle \nabla^k n^h, \nabla^k (\text{div} u) \rangle - \nu \langle \nabla^k u^h, \nabla^k (n \nabla n) \rangle \\
= - \nu \langle \nabla^k n^h, [\nabla^k, n] \text{div} u \rangle - \nu \langle \nabla^k u^h, [\nabla^k, n] \nabla n \rangle \\
- \nu \langle \nabla^k n^h, \nabla^k \text{div} u \rangle - \nu \langle \nabla^k u^h, \nabla^k \nabla n \rangle.
$$

Using Lemma 6.1 and (32), the first two terms on the right hand side of (46) can be estimated as follows

$$
\lesssim \| \nabla n \|_{L^\infty} \| \nabla^k \text{div} u \|_{L^2} \| \nabla^k n^h \|_{L^2} + \| \nabla^k n \|_{L^2} \| \text{div} u^h \|_{L^\infty} \| \nabla^k n^h \|_{L^2} \\
+ \| \nabla u \|_{L^\infty} \| \nabla^k \text{div} u \|_{L^2} \| \nabla^k n^h \|_{L^2} + \| \nabla^k n \|_{L^2} \| \text{div} u^h \|_{L^\infty} \| \nabla^k n^h \|_{L^2} \\
+ \| \nabla n \|_{L^\infty} \| \nabla^k u^h \|_{L^2} + \| \nabla^k n \|_{L^2} \| \nabla^k u^h \|_{L^\infty} \| \nabla^k u^h \|_{L^2} \\
+ \| \nabla u \|_{L^\infty} \| \nabla^k u^h \|_{L^2} + \| \nabla^k n \|_{L^2} \| \nabla^k u^h \|_{L^\infty} \| \nabla^k u^h \|_{L^2} \\
\lesssim \delta \{ \| \nabla^k (n^h, u^h) \|_{L^2}^2 + \| \nabla^k (n', u') \|_{L^2}^2 + \| \nabla (n^h, u^h) \|_{L^\infty}^2 + \| \nabla (n^h, u^h) \|_{L^2} \}.
$$
The last two terms on the right hand side of (46) can be estimated as follows
\[ -\nu(\nabla^k h', n \nabla^k \text{div} u) - \nu(\nabla^k u', n \nabla^k \nabla n) \]
\[ = -\nu(\nabla^k h', n \nabla^k \text{div} u) - \nu(\nabla^k u', n \nabla^k \nabla n) \]
\[ = -\nu(n, \text{div}(\nabla^k u h' \nabla^k n')) - \nu(\nabla^k h', n \nabla^k \text{div} u') - \nu(\nabla^k h', n \nabla^k \nabla n') \]
\[ = \nu(\nabla n, \nabla^k u' \nabla^k n') - \nu(\nabla^k h', n \nabla^k \text{div} u') - \nu(\nabla^k h', n \nabla^k \nabla n') \]
\[ \lesssim \|\nabla n\|_{L^\infty} \|\nabla^k h'\|_{L^2} \|\nabla^k n'\|_{L^2} + \|n\|_{L^2} \|\nabla^{k+1} u'\|_{L^\infty} \|\nabla^k h'\|_{L^2} \]
\[ + \|n\|_{L^2} \|\nabla^{k+1} n'\|_{L^\infty} \|\nabla^k u'\|_{L^2} \]
\[ \lesssim \delta \{\|\nabla^k (n, u)\|_{H^2}^2 + \|\nabla^{k+1} (n', u')\|_{L^\infty}^2\} . \]

For the first term on the right-hand side of (45), using the Paserval theorem and the decomposition (32), we obtain
\[ \langle \nabla^k h', \chi_1(D_x) \nabla^k (u \cdot \nabla n) \rangle \]
\[ = \langle \nabla^k \chi_1(D_x) n h', \nabla^k (u \cdot \nabla n) \rangle \]
\[ = \langle \nabla^k h', \nabla^k (u \cdot \nabla n) \rangle + \langle \nabla^k h', \nabla^k (u \cdot \nabla n') \rangle \]
\[ := I_1 + I_2 . \]

In the following, we will estimate the term $I_1$. Using Lemma 6.1, it is obvious that $I_1$ can be rewritten as
\[ I_1 = \langle \nabla^k h', \nabla (u \cdot \nabla n) \rangle + \langle \nabla^k h', [\nabla^k, n] \nabla h' \rangle := I_1^1 + I_1^2 . \] (50)

For $I_1^1$, using integration by parts, Lemmas 6.1-6.2, the assumption (11) and the Plancherel theorem, we obtain
\[ I_1^1 = -\langle |\nabla^k n'|^2, \text{div} u \rangle \lesssim \|\nabla^k n'\|_{L^2}^2 \|\text{div} u\|_{L^\infty} \lesssim \delta \|\nabla^k n'\|_{L^2}^2 , \] (51)

where we have used the fact that $n'(x, t) = \chi_1(D_x)n(x, t)$. By a similar argument, we have
\[ I_1^2 \lesssim \|\nabla^k n'\|_{L^2} \{\|\nabla u\|_{L^2} \|\nabla^k h'\|_{L^2} + \|\nabla^{k} u\|_{L^2} \|\nabla n'\|_{L^\infty}\} \]
\[ \lesssim \delta \{\|\nabla^{k} n'\|_{L^2}^2 + \|\nabla^{k+1} n'\|_{L^\infty}^2\} . \] (52)

With the above two inequalities at hands, we obtain
\[ I_1 \lesssim \delta \{\|\nabla^k n'\|_{L^2}^2 + \|\nabla^{k+1} n'\|_{H^1}^2\} . \] (53)

For $I_2$, by Lemma 6.1, the assumption (11) and the Young inequality, we get
\[ I_2 \lesssim \|\nabla^k n'\|_{L^2} \|\nabla^k (u \cdot \nabla n')\|_{L^2} \]
\[ \lesssim \|\nabla^k n'\|_{L^2} \{\|u\|_{L^2} \|\nabla^{k+1} n'\|_{L^\infty} + \|\nabla^k u\|_{L^2} \|\nabla n'\|_{L^\infty}\} \]
\[ \lesssim \delta \{\|\nabla^{k} n'\|_{L^2}^2 + \|\nabla^{k+1} n'\|_{L^\infty}^2 + \|\nabla n'\|_{L^\infty}^2\} . \] (54)

Putting (53) and (54) into (49), we get
\[ \langle \nabla^k h', \chi_1(D_x) \nabla^k (u \cdot \nabla n) \rangle \]
\[ \lesssim \delta \{\|\nabla^k n'\|_{L^2}^2 + \|\nabla^{2} n'\|_{H^1}^2 + \|\nabla^{k+1} n'\|_{L^\infty}^2 + \|\nabla n'\|_{L^\infty}^2\} . \] (55)
Lemma 3.5. It holds that
\[ \delta (\| \nabla \nabla \nabla (n^h, u^h) \|_L^2 + \| \nabla (n^h, u^h) \|_L^2 + \| \nabla (u^h, n^h) \|_L^\infty \| \nabla \bar{n}^h \|_L^\infty \} \times \bar{n}^h \}
\]

Combining (46)-(48) with (55)-(56) and (45) and taking \( \delta \) sufficiently small, we get (44).

Proof. From (43)2, we have
\[ \bar{\mu} \nabla n^h = - \partial_t u^h - a u^h + F_2^h. \]

Multiplying \( \nabla^2 (58) \) by \( \nabla^2 \nabla n^h \), and then integrating it over \( \mathbb{R}^3 \), we get that
\[ \bar{\mu} \| \nabla^3 n^h \|_L^2 = - \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle - a \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle + \langle \nabla^2 \nabla n^h, \nabla^2 F_2^h \rangle \]
\[ = - \frac{d}{dt} \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle + \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle - a \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle + \langle \nabla^2 \nabla n^h, \nabla^2 F_2^h \rangle. \]

Using (43)1, we obtain
\[ \bar{\mu} \| \nabla^3 n^h \|_L^2 + \frac{d}{dt} \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle \]
\[ = -a \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle + \langle \nabla^2 \nabla n^h, \nabla^2 F_2^h \rangle \]
\[ - \bar{\mu} \langle \nabla^2 u^h, \nabla^2 \nabla u^h \rangle + \langle \nabla^2 u^h, \nabla^2 \nabla F_2^h \rangle. \]

Then, it follows from integration by parts and the Young inequality that
\[ \bar{\mu} \| \nabla^3 n^h \|_L^2 + \frac{d}{dt} \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle \]
\[ \leq \frac{\bar{\mu}}{4} \| \nabla^3 n^h \|_L^2 + \frac{a}{\bar{\mu}} \| \nabla^2 u^h \|_L^2 + \bar{\mu} \| \nabla^3 u^h \|_L^2 \]
\[ + \langle \nabla^2 \nabla n^h, \nabla^2 F_2^h \rangle + \langle \nabla^2 u^h, \nabla^2 \nabla F_2^h \rangle. \]

Similar to the estimate on the terms on the right hand side of (45) in the proof of Lemma 3.3, we have
\[ \langle \nabla^2 \nabla u^h, \nabla^2 F_2^h \rangle \]
\[ \lesssim \delta \{ \| \nabla^3 (n^h, u^h) \|_L^2 + \| \nabla (n^h, u^h) \|_L^2 + \| \nabla (u^h, n^h) \|_L^2 \} \times \| \nabla \bar{n}^h \|_L^\infty \times \bar{n}^h \}
\]
\[ \langle \nabla^2 \nabla n^h, \nabla^2 F_2^h \rangle \]
\[ \lesssim \delta \{ \| \nabla^3 (n^h, u^h) \|_L^2 + \| \nabla (n^h, u^h) \|_L^2 + \| \nabla (u^h, n^h) \|_L^2 \} \times \| \nabla \bar{n}^h \|_L^\infty \times \bar{n}^h \}
\]

Thus, putting (62) and (63) into (61) and noticing the smallness of \( \delta \), we get (57).
Choosing some positive constant $D_1 > \max\{8, \frac{4\mu}{a}\}$ independent of $\delta$, let $\delta > 0$ be sufficiently small and satisfying $D_1 \delta \leq 1$, then $D_1 \times \sum_{2 \leq k \leq 3}(44) + (57)$ gives

$$
\frac{d}{dt}\left\{ \frac{D_1}{2} \sum_{2 \leq k \leq 3} \|\nabla^k (n^h, u^h)(t)\|_{L^2}^2 + \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle \right\}
+ \frac{\mu}{8} \|\nabla^3 n^h\|_{L^2}^2 + aD_1 \frac{\mu}{8} \|\nabla^2 u^h\|_{H^1}^2
\lesssim \delta\{\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2 + \|\nabla (n^l, n^h)\|_{L^\infty}^2 + \|\nabla (u^l, u^h)\|_{L^\infty}^2
+ \|\nabla^3 (n^l, u^l)\|_{L^\infty}^2 \} + \delta \sum_{1 \leq k \leq 4} \|\nabla^k (n^l, u^l)\|_{L^\infty}^2.
$$

(64)

Noticing that $\delta$ is small, we may then use Sobolev inequality in Lemma 6.2 and (64) to bound

$$
\frac{d}{dt}\left\{ \frac{D_1}{2} \sum_{2 \leq k \leq 3} \|\nabla^k (n^h, u^h)(t)\|_{L^2}^2 + \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle \right\}
+ \frac{\mu}{8} \|\nabla^3 n^h\|_{L^2}^2 + aD_1 \frac{\mu}{8} \|\nabla^2 u^h\|_{H^1}^2
\lesssim \delta\{\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2 + \|\nabla u^l\|_{L^\infty}^2 + \|\nabla u^h\|_{L^\infty}^2 + \|\nabla^3 (n^l, u^l)\|_{L^\infty}^2 \} + \delta \sum_{1 \leq k \leq 4} \|\nabla^k (n^l, u^l)\|_{L^\infty}^2.
$$

(65)

3.5. Estimates on $(n, u)(x, t)$. In this section, we show the energy estimate on $(n, u)(x, t)$.

**Lemma 3.6.** There exists a suitably large constant $D_2 > 0$ such that

$$
\frac{d}{dt}\left\{ \frac{D_2}{2} \|\nabla \nabla (n^h, u^h)\|_{L^2}^2 + \langle \nabla \nabla n^h, \nabla \nabla u^h \rangle \right\} + \frac{\mu}{4} \|\nabla n^h\|_{L^2}^2 + \frac{D_1 a}{2} \|u^h\|_{L^2}^2 \lesssim \|\nabla u^h\|_{L^2}^2,
$$

(66)

for any $0 \leq t \leq T$.

**Proof.** Multiplying (2)$_1$, (2)$_2$ by $n$, $u$ respectively, adding them together and then integrating the resulting over $\mathbb{R}^3$ by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt}\|\nabla (n, u)\|_{L^2}^2 + a\|u\|_{L^2}^2 = \langle n, F_1 \rangle + \langle u, F_2 \rangle.
$$

(67)

The two terms on the right hand side of (67) can be estimated as follows. Firstly, for the first term, by Hölder’s inequality, the Lemma 6.1, the Young inequality and the a priori assumption (11), we have

$$
\langle n, F_1 \rangle \lesssim \langle n, n, \text{div} u \rangle + \langle n, n \cdot \nabla n \rangle
\lesssim \|n\|_{L^3}\|n\|_{L^6}\|\text{div} u\|_{L^2} + \|n\|_{L^3}\|u\|_{L^6}\|\nabla n\|_{L^2}
\lesssim \delta\|\nabla (n, u)\|_{L^2}^2.
$$

(68)

For the second term, it follows from Lemma 6.1, the Hölder’s inequality and (11) that

$$
\langle u, F_2 \rangle \lesssim \langle u, u \cdot \nabla u \rangle + \langle u, n \nabla n \rangle
\lesssim \|u\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^2} + \|u\|_{L^6}\|n\|_{L^3}\|\nabla n\|_{L^2}
\lesssim \delta\|\nabla (n, u)\|_{L^2}^2.
$$

(69)
Then, combining (67) with (68) and (69) yields
\[
\frac{1}{2} \frac{d}{dt} \|(n, u)\|^2_{L^2} + a \|(u)\|^2_{L^2} \lesssim \delta \|\nabla (n, u)\|^2_{L^2}.
\] (70)

From (10)\textsubscript{2}, we have
\[
\bar{\mu} \nabla n = -\partial_t u - au + F_2.
\] (71)

Multiplying it by \(\nabla n\), and then integrating it over \(\mathbb{R}^3\), we have that
\[
\bar{\mu} \|\nabla n\|_{L^2}^2 = -\langle \nabla n, \partial_t u \rangle - a \langle \nabla n, u \rangle + \langle \nabla n, F_2 \rangle.
\] (72)

Using integration by parts and (2)\textsubscript{1}, the first term on the right hand side of (72) can be rewritten as
\[
-\langle \nabla n, \partial_t u \rangle = -\frac{d}{dt} \langle \nabla n, u \rangle - \langle \partial_t n, \text{div} u \rangle
\] (73)
\[
= -\frac{d}{dt} \langle \nabla n, u \rangle + \langle \bar{\mu} \text{div} u - F_1, \text{div} u \rangle.
\]

Adding (72) and (73) and using the Young inequality, we obtain
\[
\bar{\mu} \|\nabla n\|_{L^2}^2 + \frac{d}{dt} \langle \nabla n, u \rangle \leq \frac{\bar{\mu}}{4} \|\nabla n\|_{L^2}^2 + \frac{a^2}{\bar{\mu}} \|u\|_{L^2}^2 + \bar{\mu} \|\text{div} u\|_{L^2}^2 - \langle \text{div} u, F_1 \rangle + \langle \nabla n, F_2 \rangle.
\] (74)

In a similar way, the fourth term on the right hand side of (68) can be estimated as follows
\[
-\langle \text{div} u, F_1 \rangle \lesssim \langle \text{div} u, n \text{div} u \rangle + \langle \text{div} u, u \cdot \nabla n \rangle \] (75)
\[
\lesssim \|\text{div} u\|_{L^2} \|n\|_{L^\infty} \|\text{div} u\|_{L^2} + \|\text{div} u\|_{L^2} \|u\|_{L^\infty} \|\nabla n\|_{L^2}
\]
\[
\lesssim \delta \|\nabla (n, u)\|_{L^2}^2,
\]

and
\[
\langle \nabla n, F_2 \rangle \lesssim \delta \|\nabla (n, u)\|_{L^2}^2.
\] (76)

Since \(\delta > 0\) is sufficiently small, combining (74) and (75) with (76) yields
\[
\frac{\bar{\mu}}{2} \|\nabla n\|_{L^2}^2 + \frac{d}{dt} \langle \nabla n, u \rangle \leq C \delta \|\nabla u\|_{L^2}^2 + \frac{a^2}{\bar{\mu}} \|u\|_{L^2}^2 + \bar{\mu} \|\text{div} u\|_{L^2}^2.
\] (77)

Finally, multiplying (70) by \(D_2\) suitably large and adding it to (77), one has (66) since \(\delta > 0\) is sufficiently small. \(\square\)

Our next goal is to deal with the higher order estimate of \((n, u)\).

Lemma 3.7. It holds that
\[
\frac{d}{dt} \left( \frac{D_3}{2} \|(n, u)\|^2_{H^1} + \sum_{0 \leq k \leq 2} \langle \nabla^k \nabla n, \nabla^k u \rangle \right) + \frac{D_3 a}{2} \|u\|^2_{H^1} + \frac{\bar{\mu}}{4} \|\nabla n\|^2_{H^2} \leq 0,
\] (78)

for any \(0 \leq t \leq T\).
Proof. Applying $\nabla^k(0 \leq k \leq 3)$ to (2)$_{1,2}$, then multiplying them by $\nabla^k n$ and $\nabla^k u$ respectively, summing up and integrating the resulting over $\mathbb{R}^3$ by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^k (n, u)\|_{L^2}^2 + a \|\nabla^k u\|_{L^2}^2 = \langle \nabla^k n, \nabla^k F_1 \rangle + \langle \nabla^k u, \nabla^k F_2 \rangle. \tag{79}
\]

For $k = 0$, it follows from (68) and (69) that
\[
\langle n, F_1 \rangle + \langle u, F_2 \rangle \leq \delta \|\nabla (n, u)\|_{L^2}^2. \tag{80}
\]

For $1 \leq k \leq 3$, it follows from Lemma 6.1, the Hölder’s inequality and (11) that
\[
\langle \nabla^k n, \nabla^k F_1 \rangle + \langle \nabla^k u, \nabla^k F_2 \rangle \lesssim \delta \|\nabla^k (n, u)\|_{L^2}^2. \tag{81}
\]

Then, combining (79) with (80) and (81) yields
\[
\frac{d}{dt} \|\nabla (n, u)\|_{H^3}^2 + a \|u\|_{H^3}^2 \lesssim \delta \|\nabla (n, u)\|_{H^3}^2. \tag{82}
\]

Applying $\nabla^k(0 \leq k \leq 2)$ to (71), multiplying it by $\nabla^k \nabla n$, and then integrating it over $\mathbb{R}^3$, we have that
\[
\bar{\mu} \|\nabla^k \nabla n\|_{L^2}^2 = -\langle \nabla^k \nabla n, \nabla^k \partial_t u \rangle - a \langle \nabla^k \nabla n, \nabla^k u \rangle + \langle \nabla^k \nabla n, \nabla^k F_2 \rangle. \tag{83}
\]

Similar to the estimate (72) in Lemma 3.6, we have
\[
\frac{\bar{\mu}}{2} \|\nabla n\|_{H^3}^2 + \frac{d}{dt} \sum_{0 \leq k \leq 2} \langle \nabla^k \nabla n, \nabla^k u \rangle \tag{84}
\]
\[
\leq C\delta \|\nabla u\|_{H^3}^2 + \frac{a^2}{\bar{\mu}} \|u\|_{H^3}^2 + \bar{\mu} \sum_{0 \leq k \leq 2} \|\nabla^k \text{div} u\|_{L^2}^2.
\]

Finally, multiplying (82) by $D_3$ suitably large and adding it to (84), one has (78) since $\delta > 0$ is sufficiently small.

4. Global existence. In this section, we will prove Proposition 2. We are aiming to close the a priori assumption about $(n, u)$. Up to now, we only can get a priori decay-in-time estimates of the low frequency part $(n^l, u^l)$. Under a priori decay-in-time assumptions blow, we continue to obtain a decay-in-time estimates on $(n, u)$. Precisely, we have the following lemma.

Lemma 4.1. Under the assumptions of Proposition 2, the solution $(n, u)$ to the problem satisfies
\[
\|(n^l, u^l)(t)\|_{L^{\infty}} \leq C(1 + t)^{-\frac{3}{2}} \|(n_0, u_0)\|_{H^3},
\]
\[
\|\nabla^k (n^l, u^l)(t)\|_{L^{\infty}} \leq C(1 + t)^{-\frac{3}{2}} \|(n_0, u_0)(t)\|_{H^3}, \quad k \geq 1,
\]
\[
\|\nabla^k (n^l, u^l)(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{2}} \|(n_0, u_0)\|_{H^3}, \quad 0 \leq k \leq 2,
\]
\[
\|\nabla^k (n^l, u^l)(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{2}} \|(n_0, u_0)\|_{H^3}, \quad k \geq 3,
\]
\[
\|(n^h, u^h)(t)\|_{H^3} \leq C(1 + t)^{-\frac{3}{2}} \|(n_0, u_0)\|_{H^3}.
\]

Proof. Assume that the classical solution to the problem exists for $t \in [0, T]$ and denote that
\[
M_0(t) := \sup_{0 \leq \tau \leq t} \{(1 + \tau)^{\frac{3}{2}} \|\nabla (n^l, u^l)(\tau)\|_{L^\infty} + (1 + \tau)^{\frac{3}{2}} \|(n^l, u^l)(\tau)\|_{L^\infty}
\]
\[
+ (1 + \tau)^{\frac{3}{2}} \|\nabla (n^l, u^l)(\tau)\|_{L^2} \}.
\]
\( M_1(t) := \sup_{0 \leq \tau \leq t} \{(1 + \tau)^{\frac{3}{5}} \|(n^h, u^h)(\tau)\|_{H^3}\} \).

Define the temporal energy functional
\[
\mathcal{F}(t) = \frac{D_1}{2} \|\nabla^2(n^h, u^h)(t)\|_{L^2}^2 + \langle \nabla^2 \nabla n^h, \nabla^2 u^h \rangle.
\]

Then, there exist two positive constants \(C_5\) and \(C_6\) satisfying
\[
C_5 \|\nabla^2(n^h, u^h)(t)\|_{H^1}^2 \leq \mathcal{F}(t) \leq C_6 \|\nabla^2(n^h, u^h)(t)\|_{H^1}^2.
\]

From (65), we have
\[
\frac{d\mathcal{F}(t)}{dt} + \frac{H}{8} \|\nabla^3(n^h, u^h)(t)\|_{L^2}^2 \leq \delta \{ \|\nabla^2 n^h\|_{L^2}^2 + \|\nabla n^l\|_{W^{2,\infty}}^2 + \|\nabla u^l\|_{W^{2,\infty}}^2 + \|\nabla n^\prime\|_{W^{3,\infty}}^2 + \|\nabla u^\prime\|_{W^{3,\infty}}^2 \}. \tag{86}
\]

Using (34), we have
\[
C_3 \|\nabla^2(n^h, u^h)(t)\|_{L^2} \leq \|\nabla^3(n^h, u^h)(t)\|_{L^2}. \tag{87}
\]

Then, using (87), it follows from (86) that
\[
\frac{d\mathcal{F}(t)}{dt} + \frac{C_3H}{8} \|\nabla^2(n^h, u^h)(t)\|_{L^2}^2 + \frac{\mu_2}{8} \|\nabla^3(n^h, u^h)(t)\|_{L^2}^2 \leq \delta \{ \|\nabla^2 n^h\|_{L^2}^2 + \|\nabla n^l\|_{W^{2,\infty}}^2 + \|\nabla u^l\|_{W^{2,\infty}}^2 + \|\nabla n^\prime\|_{W^{3,\infty}}^2 + \|\nabla u^\prime\|_{W^{3,\infty}}^2 \}. \tag{88}
\]

Noticing the smallness of \(\delta\) and the definition of \(\mathcal{F}(t)\), (88) gives
\[
\frac{d\mathcal{F}(t)}{dt} + \mathcal{F}(t) \leq \delta \{ \|\nabla n^l\|_{W^{2,\infty}}^2 + \|\nabla u^l\|_{W^{2,\infty}}^2 + \|\nabla n^\prime\|_{W^{3,\infty}}^2 + \|\nabla u^\prime\|_{W^{3,\infty}}^2 \}. \tag{89}
\]

Using Lemma 3.3 with \(p = +\infty\), (32) and Lemma 6.2, for \(k \geq 0\) we have
\[
\|\nabla^k(n^l, u^l)(t)\|_{L^\infty} \leq (1 + t)^{-\frac{k}{2} - \frac{3}{2}} \|U_0\|_{L^2} + \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} \|\nabla U(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} \|\nabla U(\tau)\|_{L^\infty} d\tau
\]
\[
\leq (1 + t)^{-\frac{k}{2} - \frac{5}{2}} \|U_0\|_{L^2} + \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} \|\nabla U^l(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} \|\nabla U^h(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} (\|\nabla U^h(\tau)\|_{L^\infty} + \|\nabla U^l(\tau)\|_{L^\infty}) d\tau
\]
\[
\leq (1 + t)^{-\frac{k}{2} - \frac{5}{2}} \|U_0\|_{L^2} + \delta M_0(t) \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau
\]
\[
+ \delta M_1(t) \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau
\]
\[
+ \delta (M_0 + M_1) \int_0^t (1 + t - \tau)^{-\frac{k}{2} - \frac{5}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau.
\]
For $k = 0$, we have
\[
\|(n^t, u^t)(t)\|_{L^\infty} \lesssim (1 + t)^{-\frac{3}{2}} \{(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\},
\]
and for $k \geq 1$,
\[
\|\nabla^k(n^t, u^t)(t)\|_{L^\infty} \lesssim (1 + t)^{-\frac{k}{2}} \{(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\},
\]
which implies that
\[
\|\nabla n^t\|_{W^{2,\infty}} + \|\nabla u^t\|_{W^{2,\infty}} \lesssim (1 + t)^{-\frac{3}{2}} \{(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\}. \tag{92}
\]
In a similar way, we get
\[
\|\nabla n^t\|_{W^{3,\infty}} + \|\nabla u^t\|_{W^{3,\infty}} \lesssim (1 + t)^{-\frac{2}{3}} \{(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\}. \tag{93}
\]
Multiplying (89) by $e^t$ and integrating respect to $t$ in $[0, t]$ yields
\[
\mathcal{F}(t) \lesssim \mathcal{F}(0)e^{-t} + \delta \int_0^t e^{-(t-\tau)} \{(\|\nabla n^\tau\|^2_{W^{2,\infty}} + \|\nabla u^\tau\|^2_{W^{2,\infty}} + \|\nabla n^\tau\|^2_{W^{3,\infty}} + \|\nabla u^\tau\|^2_{W^{3,\infty}}\}d\tau,
\]
which combining with (92) and (93) gives
\[
\mathcal{F}(t) \lesssim \mathcal{F}(0)e^{-t} + \delta \int_0^t e^{-(t-\tau)} (1 + \tau)^{-\frac{3}{2}} \{(n_0, u_0)\|^2_{L^2} + \delta^2 M_0^2 + \delta^2 M_1^2\}d\tau
\lesssim (1 + t)^{-\frac{1}{2}} \{\mathcal{F}(0) + \|(n_0, u_0)\|^2_{L^2} + \delta^2 M_0^2 + \delta^2 M_1^2\}.
\]
From (85) and (34), we obtain
\[
\|(n^h, u^h)(t)\|^2_{H^1} \lesssim \|\nabla^2 (n^h, u^h)(t)\|^2_{H^1} \lesssim \mathcal{F}(t).
\]
Noticing the definitions of $M_1(t)$ and using the smallness of $\delta$, we get
\[
M_1(t)^2 \lesssim \mathcal{F}(0) + \|(n_0, u_0)\|^2_{L^2} + \delta^2 M_0(t)^2. \tag{94}
\]
In a similar way, using Lemma 3.3 and taking $p = 2$, for $k \geq 0$ we obtain
\[
\|\nabla^k(n^t, u^t)(t)\|_{L^2}
\lesssim (1 + t)^{-\frac{k}{2}} \|U_0\|_{L^2} + \delta \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} \|\nabla U(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^t (1 + t - \tau)^{-\frac{k}{2}} \|\nabla U(\tau)\|_{L^\infty} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{k}{2}} \|U_0\|_{L^2} + \delta \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} \|\nabla U^t(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} \|\nabla U^h(\tau)\|_{L^2} d\tau
\]
\[
+ \delta \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} (\|\nabla U^h(\tau)\|_{L^\infty} + \|\nabla U^t(\tau)\|_{L^\infty}) d\tau
\]
\[
\lesssim (1 + t)^{-\frac{k}{2}} \|U_0\|_{L^2} + \delta M_0(t) \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} (1 + \tau)^{-\frac{k}{2}} d\tau
\]
\[
+ \delta M_1(t) \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{k}{2}} (1 + \tau)^{-\frac{k}{2}} d\tau
\]
+ \delta(M_0 + M_1) \int_{\frac{1}{2}}^{t} (1 + t - \tau)^{-\frac{7}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau.

Then, it follows that for $0 \leq k \leq 2$, we have
\begin{equation}
\|\nabla^k(n', u')(t)\|_{L^2} \lesssim (1 + t)^{-\frac{7}{4}} \{\|(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\}, \tag{96}
\end{equation}
and for $k \geq 3$,
\begin{equation}
\|\nabla^k(n', u')(t)\|_{L^2} \lesssim (1 + t)^{-\frac{7}{4}} \{\|(n_0, u_0)\|_{L^2} + \delta(M_0 + M_1)\}. \tag{97}
\end{equation}
Noticing the definition of $M_0(t)$ and using (90), (91), (96) and the smallness of $\delta$, it is easy to get
\begin{equation}
M_0(t) \lesssim \|(n_0, u_0)(t)\|_{L^2} + \delta M_1(t). \tag{98}
\end{equation}
Since $\delta$ is small, from (94) and (98), we obtain
\begin{equation}
M_0^2 + M_1^2 \lesssim \mathcal{F}(0) + \|(n_0, u_0)\|_{L^2}^2 \lesssim \|(n_0, u_0)\|_{H^3}^2, \tag{99}
\end{equation}
where by (34), we have used the fact that
\begin{equation}
\mathcal{F}(0) \lesssim \|\nabla^2(n_0^h, u_0^h)\|_{H^1}^2 \lesssim \|\nabla^2(n_0, u_0)\|_{H^1}^2.
\end{equation}
Thus, from (90), (91) and (94)-(99), we complete the proof of Lemma 4.1.

**Proof of Proposition 2.** Since $D_3 > 0$ is sufficiently large, integrating (78) over $[0, t]$, it follows
\begin{equation}
\|\nabla^k(n, u)\|_{H^3}^2 + C \int_0^t \|\nabla n\|_{L^2}^2 + \|u\|_{H^3}^2 d\tau \leq \|(n_0, u_0)\|_{H^3}^2.
\end{equation}
Thus, we get (10), then, the proof of Proposition 2 is completed.

5. **Convergence rate of the solution.**

**Lemma 5.1.** Under the assumptions of Proposition 2, there is a constant $C$ such that for any $t \in [0, T]$, the global solution $(n, u)(x, t)$ we obtain from Propositions 1 and 2 has the decay properties
\begin{equation}
\|\nabla^k(n, u)\|_{L^2} \leq C(1 + t)^{-\frac{7}{4}}, \quad k = 0, 1, 2, \tag{100}
\end{equation}
\begin{equation}
\|\nabla^3(n, u)\|_{L^2} \leq C(1 + t)^{-\frac{7}{4}}. \tag{101}
\end{equation}

**Proof.** Using the decomposition (32) and Lemma 4.1, we have
\begin{equation}
\|\nabla^k(n, u)\|_{L^2} \leq C(1 + t)^{-\frac{7}{4}} \|(n_0, u_0)\|_{H^3}, \quad k = 0, 1, 2,
\end{equation}
\begin{equation}
\|\nabla^3(n, u)\|_{L^2} \leq C(1 + t)^{-\frac{7}{4}} \|(n_0, u_0)\|_{H^3}.
\end{equation}
Thus, the proof of Lemma 5.1 is completed.

Next, we will establish the optimal time decay rates (6) and (7) for velocity of the flow.

**Lemma 5.2.** Under the assumptions of Theorem 1.1, the global solution $(n, u)$ of problem (2) satisfies
\begin{equation}
\|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{k+1}{4}}, \quad \text{for } k = 0, 1, \tag{102}
\end{equation}
\begin{equation}
\|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{k}{2}}, \quad \text{for } k = 2, 3. \tag{103}
\end{equation}
Proof. Similar to the estimate (95) in Lemma 4.1, for $k \geq 0$, we obtain
\[
\|\nabla^k u^l(t)\|_{L^2} \leq (1 + t)^{-k/2} \|U_0\|_{L^2} + \delta \int_0^t (1 + t - \tau)^{-k/2} \|\nabla U(\tau)\|_{L^2} d\tau + \delta \int_0^t (1 + t - \tau)^{-k/2} \|\nabla U^h(\tau)\|_{L^2} d\tau + \delta \left( \| \nabla U^h(\tau) \|_{L^\infty} + \| \nabla U^l(\tau) \|_{L^\infty} \right) d\tau
\]
which together with (99), (32) and Lemma 4.1 leads to the estimate (102) and (103).

Proof of Theorem 1.1. With the help of Lemma 5.1 and Lemma 5.2, it is easy to obtain the conclusion (4) – (7). Using the Sobolev imbedding inequality and Lemma 5.1,Lemma 5.2, we have
\[
\|\nabla^k u\|_{L^\infty} \leq C(1 + t)^{-3/2 - k/2}, \quad k = 0, 1,
\]
\[
\|\nabla^k u\|_{L^\infty} \leq C(1 + t)^{-3/2 - \frac{k}{2} - k/2}, \quad k = 0, 1.
\]
Thus, the proof of Theorem 1.1 is completed.

6. Appendix A. Analytic tools. We recall the following commutator estimate:

Lemma 6.1. [24] Let $m \geq 1$ be an integer and define the commutator
\[
[\nabla^m, f] g = \nabla^m (fg) - f \nabla^m g.
\]
then we have
\[
\| [\nabla^m, f] g \|_{L^p} \lesssim \| \nabla f \|_{L^p_1} \| \nabla^{m-1} g \|_{L^p_2} + \| \nabla^m f \|_{L^p_3} \| g \|_{L^p_4}.
\]
and for $m \geq 0$
\[
\| \nabla^m (fg) \|_{L^p} \lesssim \| f \|_{L^p_1} \| \nabla^m g \|_{L^p_2} + \| \nabla^m f \|_{L^p_3} \| g \|_{L^p_4}.
\]
where \(1 \leq p_i \leq \infty\) (\(i = 1, \cdot \cdot \cdot, 4\)) and \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}\).

We should now recall the following elementary but useful inequality.

**Lemma 6.2.** [1] Let \(f \in H^2(\mathbb{R}^3)\). Then we have

\[
\|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{H^1}^{\frac{1}{2}} \lesssim \|\nabla f\|_{H^1},
\]

\[
\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2},
\]

\[
\|f\|_{L^q} \lesssim \|f\|_{H^1}, \quad 2 \leq q \leq 6.
\]

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