Towards a Natural Representation of Quantum Theory:
I. The Dirac Equation Revisited

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ABSTRACT

An alternative (yet equivalent) formulation of the Dirac equation and corresponding 4-spinor is investigated by considering a representation of the Lorentz Group over a simple non-division algebra. The (commutative) algebra admits a natural operation of ‘conjugation’, and ‘unitarity’ can easily be defined. In contrast to the usual complex representation of the Lorentz group, the one introduced here turns out to be unitary (with respect to the algebra).
1 Introduction and Motivation

This article is motivated by the observation that the Dirac equation can be concisely formulated without any explicit reference to complex-valued quantities. To see this, consider the following:

1.1 Realising the Dirac Equation

If

\[ \Psi_C = \begin{pmatrix} u_1 + iv_1 \\ u_2 + iv_2 \\ u_3 + iv_3 \\ u_4 + iv_4 \end{pmatrix} \]  

(1)

is a Dirac 4-spinor satisfying the Dirac equation

\[(i\gamma^\mu \partial_\mu - m)\Psi_C = 0,\]  

(2)

then the real functions \(u_i, v_i(i = 1, \ldots, 4)\) satisfy the following eight differential equations:

\[
\begin{align*}
\partial_t u_1 &= -\partial_x u_2 - \partial_y v_2 - \partial_z u_1 + m v_3, \\
\partial_t v_1 &= -\partial_x v_2 + \partial_y u_2 - \partial_z v_1 - m u_3, \\
\partial_t u_2 &= -\partial_x u_1 + \partial_y v_1 + \partial_z u_2 + m v_4, \\
\partial_t v_2 &= -\partial_x v_1 - \partial_y u_1 + \partial_z v_2 - m u_4, \\
\partial_t u_3 &= \partial_x u_4 + \partial_y v_4 + \partial_z u_3 + m v_1, \\
\partial_t v_3 &= \partial_x v_4 - \partial_y u_4 + \partial_z v_3 - m u_1, \\
\partial_t u_4 &= \partial_x u_3 - \partial_y v_3 - \partial_z u_4 + m v_2, \\
\partial_t v_4 &= \partial_x v_3 + \partial_y u_3 - \partial_z v_4 - m u_2.
\end{align*}\]  

(3)

Curiously, there is a commutative algebra closely related to the real numbers which admits a similarly concise formulation of the equations listed above. Some basic definitions concerning this algebra are presented in the next section.

1.2 The Semi-Complex Number System

In this article, we will be considering ‘numbers’ of the form

\[ w = t + jx, \]  

(4)

where \(t\) and \(x\) are real numbers, and \(j\) is a commuting variable satisfying the relation

\[ j^2 = 1. \]  

(5)

We call \(t\) and \(x\) the real and imaginary part of \(w = t + jx\) (respectively). Addition, subtraction, and multiplication\(^2\) are defined in the obvious way:

\[
\begin{align*}
(t_1 + jx_1) \pm (t_2 + jx_2) &= (t_1 \pm t_2) + j(x_1 \pm x_2), \\
(t_1 + jx_1) \cdot (t_2 + jx_2) &= (t_1 t_2 + x_1 x_2) + j(t_1 x_2 + x_1 t_2).
\end{align*}\]  

(6)  

(7)

The zero element is \(0 = 0 + j0\), and two numbers \(t_1 + jx_1, t_2 + jx_2\) are equal if and only if \(t_1 = t_2\) and \(x_1 = x_2\).

The set of all such numbers will be denoted by the symbol \(\mathbb{D}\), and we shall refer to this set as the semi-complex number system\(^3\). For a more detailed account of this number system, the reader is referred to [1], [2].

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1. The chiral representation for the \(\gamma^\mu\) matrices is assumed here. See [3] for details.
2. One can also easily define division; see section [3].
3. The more familiar term hyperbolic quasi-real numbers is avoided for the sake of brevity.
1.3 Dirac’s Equation Revisited

Define four (real) $4 \times 4$ matrices, $\xi^\mu$, $(\mu = 0, \ldots, 3)$ by writing

$$
\xi^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix},
$$

(8)

where $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $1$ is the $2 \times 2$ identity matrix. We choose this notation for later convenience. The anti-commutation relations for the $\xi^\mu$ matrices take the form

$$
\{ \xi^\mu, \xi^{\nu} \} = -2g^{\mu\nu}, \quad \mu, \nu = 0, \ldots, 3.
$$

(9)

We now introduce the semi-complex analogue of the Dirac 4-spinor: Let

$$
\Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix},
$$

(10)

where $a_i, b_i (i = 1, \ldots, 4)$ are real, and suppose $\Psi_D$ satisfies the equation

$$
(j\xi^\mu \partial_\mu - m)\Psi_D = 0.
$$

(11)

Separating the real and imaginary parts of this last equation yields a set of eight real partial differential equations connecting the real functions $a_i, b_i$. Interestingly, if we make the substitutions

$$
\begin{align*}
a_1 &\to u_2 & b_1 &\to v_3 \\
a_2 &\to v_1 & b_2 &\to -u_4 \\
a_3 &\to u_1 & b_3 &\to -v_4 \\
a_4 &\to v_2 & b_4 &\to u_3,
\end{align*}
$$

(12)

then these eight differential equations become identical to the set of equations listed in (3). In other words, if we make the identification

$$
\Psi_C = \begin{pmatrix} u_1 + iv_1 \\ u_2 + iv_2 \\ u_3 + iv_3 \\ u_4 + iv_4 \end{pmatrix} \leftrightarrow \Psi_D = \begin{pmatrix} u_2 + jv_3 \\ v_1 - ju_4 \\ u_1 - jv_4 \\ v_2 + ju_3 \end{pmatrix},
$$

(13)

then equation (2) (Dirac’s equation), and equation (11), are entirely equivalent. The suggestion here is that the Dirac equation is not a fundamentally ‘complex’ equation.

Fortunately, this observation turns out to be more than just a curious accident, and a better understanding can be obtained by investigating a particular representation of the Lorentz Group. This topic will be taken up next.
2 The Lorentz Algebra

2.1 A Complex Representation

Under Lorentz transformations, the Dirac 4-spinor $\Psi_C$ transforms as follows:

$$\Psi_C \rightarrow \left( \begin{array}{cc} e^{i\theta (\sigma - i\phi)} & 0 \\ 0 & e^{i\theta (\sigma + i\phi)} \end{array} \right) \Psi_C.$$  \hspace{1cm} (14)

The six real parameters $\theta = (\theta_1, \theta_2, \theta_3)$ and $\phi = (\phi_1, \phi_2, \phi_3)$ correspond to three generators for spatial rotations, and three for Lorentz boosts respectively. The matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the well known Pauli matrices.

Let us introduce six matrices $E_i, F_i$ ($i = 1, 2, 3$) by writing

$$E_1 = \frac{i}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad E_2 = -\frac{i}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad E_3 = \frac{i}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$$

$$F_1 = \frac{i}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \quad F_2 = \frac{i}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \quad F_3 = \frac{i}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}.$$  \hspace{1cm} (15)

Then transformation (14) may be written as follows:

$$\Psi_C \rightarrow \exp(\phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3) \Psi_C.$$  \hspace{1cm} (16)

The algebra of commutation relations for the matrices $E_i, F_i$ is given below:

$$[E_1, E_2] = E_3 \quad [F_1, F_2] = -E_3 \quad [E_1, F_2] = F_3 \quad [F_1, E_2] = F_3$$

$$[E_2, E_3] = E_1 \quad [F_2, F_3] = -E_1 \quad [E_2, F_3] = F_1 \quad [F_2, E_3] = F_1$$

$$[E_3, E_1] = -E_2 \quad [F_3, F_1] = -E_2 \quad [E_3, F_1] = -F_2 \quad [F_3, E_1] = -F_2.$$  \hspace{1cm} (17)

All other commutators vanish. Abstractly, these commutation relations define the Lie algebra of the Lorentz Group $O(1,3)$, and the complex matrices $E_i, F_i$ defined by (17) correspond to a complex representation of this algebra.

2.2 A Semi-Complex Representation

It turns out that there exists a semi-complex representation of the Lorentz algebra (17). In order to obtain an explicit presentation, we proceed as follows:

Define three $2 \times 2$ matrices $\tau = (\tau_1, \tau_2, \tau_3)$ by setting

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (18)

The reader may like to verify the following commutation relations for the matrices $\tau_i$:

$$[\tau_1, \tau_2] = 2j\tau_3, \quad [\tau_2, \tau_3] = 2j\tau_1, \quad [\tau_3, \tau_1] = -2j\tau_2.$$  \hspace{1cm} (19)

Now redefine the matrices $E_i, F_i$ ($i = 1, 2, 3$) by setting

$$E_i = \frac{i}{2} \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}, \quad F_i = \frac{1}{2} \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$  \hspace{1cm} (20)

4
A straightforward calculation shows that these matrices do indeed satisfy the Lorentz algebra defined by the commutation relations (17). Consequently, if \( \Psi_D \) is a semi-complex 4-spinor (as in expression (10)), then under Lorentz transformations, \( \Psi_D \) transforms as follows:

\[
\Psi_D \rightarrow \exp(\phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3) \Psi_D, \tag{21}
\]

where this time, the matrices \( E_i, F_i \) are semi-complex.

We shall discover in the next section that the semi-complex representation has the distinct advantage of being unitary. This means that the exponential in transformation (21) is a (semi-complex) unitary matrix, which we shall define next. In fact, since the semi-complex algebra admits a very natural operation of 'conjugation', the definition of unitarity presented in the next section should look very familiar, and very natural.

## 3 Unitarity

We shall endeavour to present an elementary (i.e. brief) introduction to the semi-complex unitary groups. We begin by defining 'conjugation' for semi-complex numbers.

### 3.1 Conjugation

Given any semi-complex number \( w = t + jx \), we define the conjugate of \( w \), written \( \overline{w} \), to be

\[ \overline{w} = t - jx. \]

Two simple consequences can be immediately deduced; for any \( w_1, w_2 \in \mathbb{D} \), we have

\[
\overline{w_1 + w_2} = \overline{w_1} + \overline{w_2} \quad \text{and} \quad \overline{w_1 \cdot w_2} = \overline{w_1} \cdot \overline{w_2}. \tag{22}
\]

We also have the identity

\[ \overline{w} \cdot w = t^2 - x^2. \tag{23} \]

Hence \( \overline{w} \cdot w \) is real for any semi-complex number \( w \), although unlike the complex case, it may take on negative values. In order to strengthen the analogy between the semi-complex and complex numbers, we often write

\[ |w|^2 = \overline{w} \cdot w \]

where \( |w|^2 \) is referred to as the 'modulus squared' of \( w \). A nice consequence of these definitions can now be stated: For any semi-complex numbers \( w_1, w_2 \in \mathbb{D} \),

\[ |w_1 \cdot w_2|^2 = |w_1|^2 \cdot |w_2|^2. \]

Now observe that if \( |w|^2 \) does not vanish, the quantity

\[ w^{-1} = \frac{\overline{w}}{|w|^2} \tag{25} \]

is a well defined inverse for \( w \). So \( w \) fails to have an inverse if (and only if) \( |w|^2 = t^2 - x^2 = 0 \).
3.2 The Semi-Complex Unitary Groups

**Hermiticity:** Suppose $H$ is a matrix of arbitrary dimensions with semi-complex entries. The adjoint of $H$, written $H^\dagger$, is obtained by transposing $H$, and then conjugating each of the entries. We say $H$ is Hermitian if $H^\dagger = H$, and anti-Hermitian if $H^\dagger = -H$. For example, the $\tau_i$ matrices defined in (18) are Hermitian.

**Unitarity:** The semi-complex unitary group $U(n,D)$ is the set of all $n \times n$ semi-complex matrices $U$ satisfying the identity

$$U^\dagger U = 1.$$  \hfill (26)

The special unitary group $SU(n,D)$ is defined to be the set of all elements $U \in U(n,D)$ with unit determinant:

$$\det U = 1.$$  \hfill (27)

**Unitarity and Hermiticity:** If $H$ is an $n \times n$ Hermitian matrix (over $D$), then $e^{jH}$ is an element of the unitary group $U(n,D)$. Equivalently, $e^H$ is unitary if $H$ is anti-Hermitian. If the trace of $H$ vanishes, then $e^{jH}$ (or $e^H$ in the anti-Hermitian case) is contained in the special unitary group $SU(n,D)$.

Remark: According to these definitions, the exponential appearing in the Lorentz transformation (21) is an element of the special unitary group $SU(4,D)$, since the traceless generators $E_i, F_i$ defined in (20) are anti-Hermitian. So the Lorentz group is just a six-dimensional (Lie) subgroup of the fifteen dimensional Lie group $SU(4,D)$.

3.3 An Isomorphism: The Spin Group

We claim that the semi-complex group $SU(2,D)$ is just the familiar complex group $SU(1,1)$. In fact, if we restrict our attention to real numbers $a_1, a_2, b_1, b_2$ satisfying the constraint $a_1^2 + a_2^2 - b_1^2 - b_2^2 = 1$, then the identification

$$\begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ b_1 - ib_2 & a_1 - ia_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 + jb_1 & -a_2 + jb_2 \\ a_2 + jb_2 & a_1 - jb_1 \end{pmatrix}$$  \hfill (28)

establishes a (group) isomorphism between $SU(1,1)$ and $SU(2,D)$, as claimed. The terminology ‘spin group’ for $SU(2,D)$ anticipates a forthcoming article investigating the intimate relationship between this group, spin, and Lorentz invariance in $2 + 1$ space-time.

The identification (28) above suggests that the conformal group $SU(2,2)$ might be just the group $SU(4,D)$; at any rate, they are both fifteen dimensional. Initial investigations suggest that the differences between these groups might be very subtle. A definitive proof demonstrating the (non)existence of an isomorphism — at least between the associated Lie algebras — would be an important initial step before investigating the physics of $SU(4,D)$.
4 The Dirac Equation Revisited II

4.1 Quantisation: A New Perspective

The Dirac equation (11) with zero mass \( (m = 0) \) may be written in the form

\[
\frac{j}{\partial t} \Psi_D = H \Psi_D, \tag{29}
\]

where the operator \( H \) turns out to be Hermitian\(^4\) with respect to the semi-complex inner product

\[
< \Psi_D, \Phi_D > := \int \Psi_D^\dagger \Phi_D d^3x. \tag{30}
\]

So the scalar quantity \( |\Psi_D|^2 := \Psi_D^\dagger \Psi_D \) may be viewed as some kind of Lorentz invariant density function\(^5\).

So how do we quantise field equations in general such as the massless Dirac equation given by expression (29)? Associated with any equation of this form is a propagator \( K \), which, in the path integral formalism, necessitates the evaluation of a path integral

\[
K = \int e^{iS} D\psi^\dagger D\psi. \tag{31}
\]

The functional \( S = S[\psi^\dagger, \psi] \) is an appropriately chosen action, which we may assume to be real-valued\(^6\). Note that the integrand of this path integral is a semi-complex phase \( e^{iS} \), contrasting the usual prescription of integrating over a complex phase \( e^{iS} \).

Now any semi-complex number may be uniquely decomposed into a real linear combination of the following two projections:

\[
p_+ = \frac{1}{2}(1 + j) \quad p_- = \frac{1}{2}(1 - j) \tag{32}
\]

\((p_+ \text{ and } p_- \text{ are projections since } p_+ + p_- = 1, \quad p_+^2 = p_+, \quad p_-^2 = p_-, \text{ and } p_+ p_- = p_- p_+ = 0).\)

In particular, \( e^{iS} = \cosh S + j \sinh S \) admits the following decomposition:

\[
e^{iS} = e^S \cdot p_+ + e^{-S} \cdot p_. \tag{33}
\]

So the path integral (31) takes the form \( K = K_+ \cdot p_+ + K_- \cdot p_- \), where the projection coefficients \( K_+, K_- \) are given by

\[
K_+ = \int e^S D\psi^\dagger D\psi \tag{34}
\]

\[
K_- = \int e^{-S} D\psi^\dagger D\psi \tag{35}
\]

If \( S \) were positive-definite (e.g. if we replace \( S \) with \(|S|\)), one would expect (34) to be divergent, although (35) might be well defined. So the original integral (31) may be undefined as a whole, but it may nevertheless possess a well defined projection. Mathematically, this is an elegant way of separating unwanted infinities.

Moreover, the semi-complex formalism appears to have removed the troublesome complex phase in the path integral without resorting to an analytic continuation to Euclidean space-time. The integrity of Minkowski space is thus preserved.

\(^4\) i.e. \( < H\Psi_D, \Phi_D > = < \Psi_D, H\Phi_D >\).

\(^5\) Non-positive definiteness suggests it might be charge density.

\(^6\) Say, a real Grassmann variable, for the fermionic case.
4.2 Gauge Invariance

Finally, we indulge in some fanciful speculation: First, observe that transformations of the type

$$\Psi_D \rightarrow e^{i\theta} \cdot \Psi_D \quad (\theta \in \mathbb{R})$$  \hspace{1cm} (36)$$

leave $|\Psi_D|^2$ invariant. So the Lagrangian

$$L = \Psi_D^\dagger (j\xi^\mu \partial_\mu - m) \Psi_D,$$  \hspace{1cm} (37)$$

which gives rise to the Dirac equation (11) under variation, is also invariant under transformations of this type. Gauging this global transformation (i.e. allowing $\theta$ to vary at different space-time points) augments the Lagrangian into the following expression:

$$L = \Psi_D^\dagger (j\xi^\mu D_\mu - m) \Psi_D - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \hspace{1cm} (38)$$

Here, $D_\mu = \partial_\mu - jG_\mu$ is the covariant derivative operator, $G_\mu$ is the associated gauge field, and $F_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu$ is the (gauge invariant) field tensor. The Lagrangian (38) is now invariant under the gauge transformations

$$\Psi_D \rightarrow e^{i\theta(x)} \cdot \Psi_D$$  \hspace{1cm} (39)$$

$$G_\mu \rightarrow G_\mu + \partial_\mu \theta. \hspace{1cm} (40)$$

The transformation (39) is not a Lorentz transformation, so what can it be? The similarities between the electromagnetic field and the gauge field $G_\mu$ are rather striking. For example, in the source free case, the field tensors are exactly equivalent. In the physical world, there are two forces which, classically, appear very similar; namely, the Coulomb attraction between two stationary charges, and the gravitational force between two masses, each of which are described by an inverse square law. It is tempting, then, to view the $U(1,D)$ gauge theory just presented as a simple gauge theory of gravity. We leave such speculations to the interested reader!

5 Concluding Remarks

Our observations suggest that it is unnecessary (and potentially restrictive) to view complex-valued quantities as fundamental to a relativistic quantum theory.

Quantum physics depends heavily on the concept of Hermitian operators, since it is precisely these operators which guarantee a real (i.e. measurable) spectrum of eigenvalues. Consequently, if we are seeking a quantum theory which is consistent with relativity—an essential requirement for quantum gravity—then it is appropriate that a unitary-like representation of the Lorentz group be considered. Such a representation can be obtained, but it requires the somewhat unorthodox step of embracing an unfamiliar number algebra.

Hopefully, the effort of venturing beyond the familiar complex number system will be more than compensated by a new and fruitful perspective on the old quantum world.

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References

[1] A.Kwasniewski & R.Czech *Reports on Mathematical Physics* Vol 31 No. 3; June 1992.

[2] Antonuccio F.A. *Semi-Complex Analysis and Mathematical Physics* gr-qc 9311032

[3] Ryder, Lewis H 1992 *Quantum Field Theory* Cambridge University Press.

[4] D.H. Sattinger & O.L. Weaver 1986 *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*. Springer-Verlag New york Inc.