ALMOST CONTACT CURVES IN TRANS-SASAKIAN 3-MANIFOLDS

S. K. Srivastava
Department of Mathematics, Central University of Himachal Pradesh, Dharamshala - 176215, Himachal Pradesh, India

Abstract
This paper is devoted to the study of curvature and torsion of almost contact curves in trans-Sasakian 3-Manifolds. The conditions for the frenet curves to be almost contact curves in trans-Sasakian 3-manifolds have been obtained.

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Key words: Almost contact curve, Frenet curve, trans- Sasakian manifold.

1 Introduction

Almost contact curves play an important role in geometry and topology of contact metric manifolds, e.g. a diffeomorphism of a contact manifold is a contact transformation if and only if any almost contact curves in a domain of it go to almost contact curves [5]. Several authors have studied almost contact curves in contact geometry such as [1, 5, 6, 9, 11, 12, 16, 18]. In [5] Baikoussis and Blair have studied almost contact curves in contact metric 3-manifold and gave the Frenet 3-frame in this class of manifold.

Belkhelfa et al. have extended some of the results of [5] from the Riemannian to the Lorentzian case, and classified all biharmonic almost contac curves in Sasaki-Heisenberg spaces[18]. A result of Blair [see theorem 8.2, p.134 [1]] had been generalized by Welyczko [11] to the case of 3-dimensional Quasi-Sasakian manifolds. Moreover the author had also obtained some interesting properties of non-Frenet almost contact curves in normal almost paracontact metric 3-manifolds [12]. Özgür and Tripathi established necessary and sufficient conditions for almost contact curves having parallel mean curvature vector, proper mean curvature vector, being harmonic and being of type $AW(k), k = 1, 2, 3$; in $\alpha$- Sasakian manifolds[6]. In [9] Lee characterized almost contact curves in a Sasakian manifold having the following properties: (i) a pseudo-Hermitian parallel mean curvature vector field (ii) a pseudo-Hermitian proper mean curvature vector field in the normal bundle. Recently, Inoguchi and Lee have studied almost contact curves in normal almost contact metric 3-manifold satisfying $\nabla H = \lambda H$ or $\nabla^\perp H = \lambda H$ and gave natural equations for planar biminimal curves [16].

The purpose of this paper is to investigate the properties of almost contact curves in trans-Sasakian 3-manifolds. This paper is organized as follows: In §2 we recall some basic definitions and facts about almost contact metric (in brief a.c.m.) manifolds, trans-Sasakian manifolds and Frenet curves. The curvature, torsion of almost contact curves and the conditions for the frenet curves to be almost contact curves in trans-Sasakian 3-manifolds have been obtained, and finally we construct the examples in §3.

1Email: sachink.ddumath@gmail.com
2 Preliminaries

2.1 Contact metric manifolds

A $(2n+1)$-dimensional differentiable manifold $M$ is said to be an almost contact manifold if its structural group $GL_{2n+1} \mathbb{R}$ of linear frame bundle is reducible to $U(n) \times \{1\}$ (Gray [10]). This is equivalent to existence of a tensor field of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2.1}$$

From these conditions one can easily obtain

$$\phi \xi = 0, \quad \eta \phi = 0. \tag{2.2}$$

Moreover, since $U(n) \times \{1\} \subset SO(2n+1)$, $M$ admits a Riemannian metric $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$. Such a metric is called an associated metric (Sasaki [19]) of the almost contact manifold $M$. With respect to $g$, $\eta$ is metrically dual to $\xi$, that is

$$g(X, \xi) = \eta(X) \tag{2.4}$$

A structure $(\phi, \xi, \eta, g)$ on $M$ is called an almost contact metric structure and a manifold $M$ equipped with an almost contact metric structure is said to be an almost contact metric manifold. The fundamental 2-form $\Phi$ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$. An almost contact metric manifold $M$ is said to be a contact metric manifold if $\Phi = d\eta$. Here the exterior derivative $d\eta$ is defined by

$$d\eta(X, Y) = \frac{1}{2} (X\eta(Y) - Y\eta(X) - \eta([X,Y])). \tag{2.6}$$

On a contact metric manifold, $\eta$ is contact form, i.e., $\eta \land (d\eta)^n \neq 0$ everywhere on $M$. In particular, $\eta \land (d\eta)^n \neq 0$ is a volume element on $M$ so that a contact manifold is orientable. Define a $(1, 1)$ type tensor field $h$ and $l$ by $h = \frac{1}{2}L_\xi \phi$, $lX = R(\cdot, \xi)\xi$, where $L$ denotes the Lie differentiation and $R$ the curvature tensor respectively. The operators $h$ and $l$ are self-adjoint and satisfy: $h\xi = l\xi = 0$ and $h\phi = -\phi h$. Also we have $Tr.h = Tr.\phi h = 0$. Moreover, if $\nabla$ denotes the Levi-Civita connection on $M$, then following formulas holds on a contact metric manifold.

$$\nabla_X \xi = -\phi X - \phi hX. \tag{2.7}$$

$$l = \phi l \phi - 2(h^2 + \phi^2). \tag{2.8}$$
On the direct product manifold $M \times \mathbb{R}$ of an almost contact metric manifold $M$ and the real line $\mathbb{R}$, any tangent vector field can be represented as the form $(X, f \frac{d}{dt})$, where $X \in \Gamma(TM)$ and $f$ is a function on $M \times \mathbb{R}$ and $t$ is the cartesian coordinate on the real line $\mathbb{R}$.

Define an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$J \left( X, \lambda \frac{d}{dt} \right) = \left( \phi X - \lambda \xi, \eta(X) \frac{d}{dt} \right).$$

(2.9)

If $J$ is integrable then $M$ is said to be normal. Equivalently, $M$ is normal if and only if

$$[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0,$$

(2.10)

where $[\phi, \phi]$ is the Nijenhuis torsion tensor of $\phi$ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

(2.11)

for all $X, Y \in \Gamma(TM)$.

For an arbitrary almost contact metric 3-manifold $M$, we have (20):

$$(\nabla X \phi)Y = g(\phi \nabla_X \xi, Y)\xi - \eta(Y)\phi \nabla_X \xi, X \in \Gamma(TM)$$

(2.12)

where $\nabla$ is the Levi-Civita connection on $M$.

### 2.2 trans-Sasakian manifolds

This class of manifolds arose in a natural way from the classification of almost contact metric structures and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. In [4] Gray Harvella classification of almost Hermite manifolds appear as a class $W_4$ of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_6 \oplus C_5$ [13] coincides with the class of trans-Sasakian structure of type $(\alpha, \beta)$.

An almost contact metric structure $(\phi, \xi, \eta)$ on a connected manifold $M$ is called trans- Sasakian structure [14] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [4], where $J$ is the almost complex structure defined by (2.9) and $G$ is the product metric on $M \times \mathbb{R}$. This may be expresses by the condition [3]

$$(\nabla X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

(2.13)

for the smooth functions $\alpha$ and $\beta$ on $M$. Hence we say that the trans - Sasakian structure is of type $(\alpha, \beta)$. From (2.13) it follows that

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)$$

(2.14)

We note that trans-Sasakian structure of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are the cosymplectic, $\alpha$–Sasakian and $\beta$–Kenmotsu manifold respectively.
2.3 Frenet Curves

Let \((M, g)\) be a Riemannian \(n\)-manifold with Levi-Civita connection \(\nabla\). A unit speed curve \(\gamma : I \rightarrow M\) is said to be an \(r\)-Frenet curve if there exists an orthonormal \(r\)-frame field \((E_1 = \gamma', E_2, ..., E_r)\) along \(\gamma\) such that there exist positive smooth functions \(k_1, k_2, ..., k_{r-1}\) satisfying

\[
\nabla_{\gamma'} E_1 = k_1 E_2, \quad \nabla_{\gamma'} E_2 = -k_1 E_1 + k_2 E_3, \ldots, \nabla_{\gamma'} E_r = -k_{r-1} E_{r-1}.
\]

The function \(k_r\) is called the \(r\)-th curvature of \(\gamma\). A Frenet curve is said to be

- a geodesic if \(r = 1\), i.e., \(\nabla_{\gamma'} \gamma' = 0\).
- a Riemannian circle if \(r = 2\), and \(k_1\) is non-zero constant.
- a helix of order \(r\) if \(k_1, k_2, ..., k_{r-1}\) are constants.

In case \(n = 3\), we denote by \((E_1, E_2, E_3) = (T, N, B)\). Then we have the Serret-Frenet equation:

\[
\nabla_T T = kN, \quad \nabla_T N = -kT + \tau B \quad \text{and} \quad \nabla_T B = -\tau N \tag{2.15}
\]

where \(T = \gamma'\).

The first curvature \(k = k_1\) and the second curvature \(\tau = k_2\) are called the geodesic curvature and geodesic torsion of \(\gamma\), respectively. The vector field \(N\) and \(B\) are called the unit normal vector field and binormal vector field of \(\gamma\), respectively.

3 Almost contact Curves

Let \(\gamma : I \rightarrow M\) be a curve parameterized by arc-length (the natural parametrization) in an almost contact metric 3-manifold \(M\) with Frenet frame \((T, N, B)\).

**Definition 3.1** A Frenet curve \(\gamma\) in an almost contact 3-manifold \(M\) is said to be an almost contact curve if it is an integral curve of the contact distribution \(D = \ker \eta\), equivalently, \(\eta(\gamma') = 0\).

In particular, when \(\eta \wedge (d\eta)^n \neq 0\), almost contact curves are traditionally called Legendre curves (cf.\([5]\)).

We begin with a proposition that will motivate the main result:

**Proposition 3.2** Let \(M\) be a trans-Sasakian 3-manifold. Then for non-geodesic almost contact curve \(\gamma : I \rightarrow M\), curvature (\(\kappa\)) and torsion (\(\tau\)) are given by

\[
\kappa = \sqrt{\beta^2 + \vartheta^2} \tag{3.1}
\]

\[
\tau = |\alpha + \frac{\beta \vartheta' - \beta' \vartheta}{\kappa^2}| \tag{3.2}
\]

**Proof:** Let \(\gamma\) be an almost contact curve on \(M\). Then

\[
\nabla_{\gamma'} T = \nabla_{\gamma'} \gamma' = -\beta \xi + \vartheta \phi \gamma'
\]

for some function \(\vartheta\). The unit normal vector field \(N\) is given by

\[
N = \frac{1}{\kappa} \nabla_{\gamma'} T = -\frac{\beta}{\kappa} \xi + \frac{\vartheta}{\kappa} \phi \gamma'
\]

\[
(3.4)
\]
Differentiating (3.4) along \( \gamma' \), we get

\[
\nabla_{\gamma'} N = -\kappa'\gamma' + p\xi + q\phi' \tag{3.5}
\]

where

\[
p = \frac{\vartheta}{\kappa} \alpha - \frac{\beta'\kappa - \beta\kappa'}{\kappa^2}, \quad q = \frac{\alpha\beta}{\kappa} + \frac{\vartheta'\kappa - \vartheta\kappa'}{\kappa^2}.
\]

Here \( \beta' \), \( \delta' \) and \( \kappa' \) are

\[
\beta'(s) = \frac{d}{ds}\beta(\gamma(s)), \quad \delta'(s) = \frac{d}{ds}\delta(\gamma(s)) \quad \text{and} \quad \kappa'(s) = \frac{d}{ds}\kappa(\gamma(s)).
\]

From (3.4) and \( \tau B = \nabla_{\gamma'} N + \kappa T = p\xi + q\phi' \), we have (3.1) and (3.2).

**MAIN RESULT**

**Theorem 3.3** For a Frenet curve \( \gamma : I \to M \) in a trans-Sasakian 3-manifold \( M \) with \( \alpha \neq 0 \) and \( \beta \neq 0 \). Set \( \sigma = \eta(\gamma') \). If \( \tau = \l_1\alpha + \l_2\beta + \l_3 \) and at one point of \( I \), \( \sigma = \sigma' = \sigma'' = 0 \), then \( \gamma \) is an almost contact curve.

Where

\[
l_1 = \frac{1}{\sqrt{1 - \sigma^2}}, \quad l_2 = -\frac{pq\sigma}{\sqrt{1 - \sigma^2}(p^2 + q^2)}, \quad l_3 = -\frac{p^2}{(p^2 + q^2)}\gamma \left( \frac{\beta}{p} \right),
\]

\( p \) is non-zero constant on \( I \) and \( q \) is certain function on \( I \).

**Proof:** Suppose that \( \gamma' \) is not collinear with \( \xi \) and describe curvature (\( \kappa \)) and torsion (\( \tau \)) of \( \gamma \) on \( I \). We may decompose \( \nabla_{\gamma'} \gamma' \) as

\[
\nabla_{\gamma'} \gamma' = \nabla_{\gamma'} T = \frac{p}{\sqrt{1 - \sigma^2}} \phi' + \frac{q}{\sqrt{1 - \sigma^2}}(\xi - \sigma\gamma') \tag{3.6}
\]

Therefore

\[
\kappa = \sqrt{p^2 + q^2} \tag{3.7}
\]

is curvature of \( \gamma \).

Using (2.12) and (3.6), we have

\[
\sigma' = \gamma' (g(\xi, \gamma'))
\]

\[
= g(\nabla_{\gamma'} \xi, \gamma') + g(\xi, \nabla_{\gamma'} \gamma')
\]

\[
= \beta (1 - \sigma^2) + q\sqrt{1 - \sigma^2}. \tag{3.8}
\]

From (3.6), we find

\[
N = \frac{1}{k} \nabla_{\gamma'} T
\]

\[
= \frac{p}{k\sqrt{1 - \sigma^2}} \phi' + \frac{q}{k\sqrt{1 - \sigma^2}}(\xi - \sigma\gamma'). \tag{3.9}
\]
Let us write,

\[ p_1 = \frac{p}{k\sqrt{1 - \sigma^2}}, \quad q_1 = \frac{q}{k\sqrt{1 - \sigma^2}}. \]  

(3.10)

Then (3.9) becomes

\[ N = p_1 \phi' + q_1 (\xi - \sigma' \gamma'). \]  

(3.11)

From (3.7) and (3.8), we compute

\[ p_1' = \frac{q'(pq - pq')}{k^3\sqrt{1 - \sigma^2}} + \frac{pq\sigma}{k(1 - \sigma^2)} + \frac{p\sigma\beta}{k\sqrt{1 - \sigma^2}}, \]

\[ q_1' = \frac{p(pq' - pq)}{k^3\sqrt{1 - \sigma^2}} + \frac{q^2\sigma}{k(1 - \sigma^2)} + \frac{q\sigma\beta}{k\sqrt{1 - \sigma^2}}. \]  

(3.12)

Differentiating (3.11) along \( \gamma' \), we have

\[ \nabla_{\gamma'} N = p_1' \phi' + p_1 \left( (\nabla_{\gamma'} \gamma' + \phi \nabla_{\gamma'} \gamma') + q_1 (\xi - \sigma' \gamma') + q_1 (\nabla_{\gamma'} \xi - \sigma' \gamma' - \sigma \nabla_{\gamma'} \gamma') \right). \]  

(3.13)

Using (2.13), (2.14), (3.6), (3.8) and (3.5); we get

\[ \nabla_{\gamma'} N = \frac{q}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} - \frac{p\sigma}{\sqrt{1 - \sigma^2}} - \alpha \right] \phi' + \frac{p}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right] (\xi - \sigma' \gamma') - k\gamma' \]

or,

\[ \nabla_{\gamma'} N = -\frac{q}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right] \phi' + \frac{p}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right] (\xi - \sigma' \gamma') - k\gamma' \]

or,

\[ \nabla_{\gamma'} N + k\gamma' = \tau B = p_2 \phi' + q_2 (\xi - \sigma' \gamma') \]  

(3.14)

where

\[ p_2 = -\frac{q}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right], \]

\[ q_2 = \frac{p}{k\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - pq)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right] \]  

(3.15)
\[ p_2^2 + q_2^2 = \frac{1}{(1 - \sigma^2)} \left[ \frac{(pq' - p'q)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right]^2 \]
\[ = \tau^2. \]  
(3.16)

Since, we have assumed that \( \tau = |l_1\alpha + l_2\beta + l_3| \). Therefore, we have
\[ |l_1\alpha + l_2\beta + l_3| = \left| \frac{1}{\sqrt{1 - \sigma^2}} \left[ \frac{(pq' - p'q)}{k^2} + \frac{p\sigma}{\sqrt{1 - \sigma^2}} + \alpha \right] \right| \]
that is,
\[ \sigma'' + \frac{2\sigma'^2}{1 - \sigma^2} + p^2\sigma = 0. \]  
(3.17)

For \( \sigma \) not constant, write \( \mu = \frac{\sigma'}{p} \). Equation (3.17) yields
\[ \frac{\mu}{\sigma} \frac{d\mu}{d\sigma} + \frac{2\sigma\mu^2}{1 - \sigma^2} + \sigma = 0, \]  
(3.18)
where \( p \) is non-zero constant. Integrating (3.18), we have
\[ \mu^2 = p^2 \left( C \sqrt{1 - \sigma^2} - 1 \right) \left( 1 - \sigma^2 \right) \]  
(3.19)
where \( C \) is constant of integration. Using at one point of \( I \), \( \sigma = \sigma' = 0 \) and \( p \neq 0 \) we have \( C = 1 \).

Therefore
\[ \sigma'^2 = -p^2\sigma^2(1 - \sigma^2). \]

Recalling \( \sigma^2 \leq 1 \), we have \( \sigma = 0 \), which is a contradiction. \( \square \)

Let us suppose that \( M = \mathbb{R}^2 \times \mathbb{R}_+ \), \( \omega : M \to \mathbb{R}_+ \) and \((x, y, z)\) be cartesian coordinates in \( M \), we define a trans-Sasakian structure on \( M \) by
\[ \xi = \frac{\partial}{\partial z}, \quad \eta = dz - ydx \]
\[ \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \]
\[ g = \begin{pmatrix} \omega + y^2 & 0 & -y \\ 0 & \omega & 0 \\ -y & 0 & 1 \end{pmatrix}. \]

Certain almost contact curves in the above class of manifolds are given below:

**Example 3.4.** Let us suppose that \( \omega = \exp(z) \), then the structure \((\phi, \xi, \eta, g)\) is trans-Sasakian structure of type \( \left( \frac{z-1}{2\exp(z)}, \frac{1}{2} \right) \).

A curve \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \) in \( M \) is almost contact curve if and only if

(i) \( \dot{\gamma}^3 = \gamma^2 \dot{\gamma}^1 \)
\[
\left( \dot{\gamma}^1 \right)^2 + \left( \dot{\gamma}^2 \right)^2 = \exp (-\gamma^3) \cdot
\]

The concrete examples of almost contact curves in \( M \) are

(3.4.1) \[ \gamma(t) = (1, t, 0), t > 0 \text{ -- a helix with } \kappa = \tau = 1/2. \]

(3.4.2) \[ \gamma(t) = (\ln t, 2\ln t, t), t > 0 \text{ -- a curve with } \kappa = 1/2 \text{ and } \tau = 1/2t^2. \]

**Example 3.5.** Let us suppose that \( \omega = \imath \), then the structure \((\phi, \xi, \eta, g)\) is trans-Sasakian structure of type \((\frac{1}{2\pi}, \frac{1}{2\pi})\).

A curve \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \) in \( M \) is almost contact curve if and only if

(i) \[ \dot{\gamma}^3 = \gamma^2 \dot{\gamma}^1 \]

(ii) \[ \left( \dot{\gamma}^1 \right)^2 + \left( \dot{\gamma}^2 \right)^2 = \left( \gamma^3 \right)^{-1}. \]

The concrete examples of almost contact curves in \( M \) are

(3.5.1) \[ \gamma(t) = (1, t, 0), t > 0 \text{ -- a helix with } \kappa = \tau = 1/2. \]

(3.5.2) \[ \gamma(t) = (\sqrt{2t}, \sqrt{2t}, t), t > 0 \text{ -- a generalized helix with } \kappa = \tau = 1/2t. \]

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