STRONG KÄHLER WITH TORSION STRUCTURES FROM ALMOST CONTACT MANIFOLDS

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Abstract. For an almost contact metric manifold $N$, we find conditions for which either the total space of an $S^1$-bundle over $N$ or the Riemannian cone over $N$ admits a strong Kähler with torsion (SKT) structure. In this way we construct new 6-dimensional SKT manifolds. Moreover, we study the geometric structure induced on a hypersurface of an SKT manifold, and use such structures to construct new SKT manifolds via appropriate evolution equations. Hyper-Kähler with torsion (HKT) structures on the total space of an $S^1$-bundle over manifolds with three almost contact structures are also studied.

1. Introduction

On any Hermitian manifold $(M^{2n}, J, h)$ there exists a unique Hermitian connection $\nabla^B$ with totally skew-symmetric torsion, called in the literature as Bismut connection [4]. The torsion 3-form $h(X, T^B(Y, Z))$ of $\nabla^B$ can be identified with the 3-form

$$-JdF(\cdot, \cdot, \cdot) = -dF(J\cdot, J\cdot, J\cdot),$$

where $F(\cdot, \cdot) = h(\cdot, J\cdot)$ is the fundamental 2-form associated to the Hermitian structure $(J, h)$.

Hermitian structures with closed $JdF$ are called strong Kähler with torsion (shortly SKT) or also pluriclosed [9]. Since $\bar{\partial}\partial$ acts as $\frac{1}{2}dJd$ on forms of bidegree $(1, 1)$, the latter condition is equivalent to $\bar{\partial}\partial F = 0$. SKT structures have been recently studied by many authors and they have also applications in type II string theory and in 2-dimensional supersymmetric $\sigma$-models [15, 25, 22].

The class of SKT metrics includes of course the Kähler metrics, but as in [12] we are interested on non-Kähler geometry, so for SKT metrics we will mean Hermitian metrics $h$ such that its fundamental 2-form $F$ is $\partial\bar{\partial}$-closed but not $d$-closed.

Gauduchon in [19] showed that on a compact complex surface an SKT metric can be found in the conformal class of any given Hermitian metric, but in higher dimensions the situation is more complicated.

SKT structures on 6-dimensional nilmanifolds, i.e. on compact quotients of nilpotent Lie groups by discrete subgroups, were classified in [12, 28]. Simply-connected examples of 6-dimensional SKT manifolds have been found in [17] by using torus bundles and recently Swann in [27] has reproduced them via the twist construction, by extending them to higher dimensions, and finding new other compact simply-connected SKT manifolds. Moreover, in [14] it has been showed that the SKT condition is preserved by the blow-up construction.

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The odd dimensional analog of Hermitian structures are given by normal almost contact metric structures. Indeed, on the product $N^{2n+1} \times \mathbb{R}$ of a $(2n+1)$-dimensional almost contact metric manifold $N^{2n+1}$ by the real line $\mathbb{R}$ it is possible to define a natural almost complex structure, which is integrable if and only if the almost contact metric structure on $N^{2n+1}$ is normal [25]. More in general, it is possible to construct Hermitian manifolds starting from an almost contact metric manifold $N^{2n+1}$ by considering a principal fibre bundle $P$ with base space $N^{2n+1}$ and structural group $S^1$, i.e. an $S^1$-bundle over $N^{2n+1}$ (see [24]). Indeed, in [24] by using the almost contact metric structure on $N^{2n+1}$ and the connection 1-form $\theta$, Ogawa constructed an almost Hermitian structure $(J, h)$ on $P$ and found conditions for which $J$ is integrable and $(J, h)$ is Kähler.

In Section 2 we determine conditions for which in general an $S^1$-bundle over an almost contact metric $(2n+1)$-dimensional manifold $N^{2n+1}$ is SKT (Theorem 2.3). We study the particular case when $N^{2n+1}$ is quasi-Sasakian, i.e. it has an almost contact metric structure for which the fundamental form is closed (Corollary 2.4). In this way we are able to construct some new 6-dimensional SKT examples, starting from 5-dimensional quasi-Sasakian Lie algebras and also from Sasakian ones.

A Sasakian structure can be also seen as the analog in odd dimensions of a Kähler structure. Indeed, by [7] a Riemannian manifold $(N^{2n+1}, g)$ of odd dimension $2n+1$ admits a compatible Sasakian structure if and only if the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ is Kähler. In Section 3 we study which conditions has to satisfy the compatible almost contact metric structure on a Riemannian manifold $(N^{2n+1}, g)$ in order to the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ to be SKT (Theorem 3.1). An example of an SKT manifold constructed as Riemannian cone is provided and the particular case that the Riemannian cone is 6-dimensional is considered in Section 4. This case is interesting since one can impose that the SKT structure is in addition an SKT SU(3)-structure and one can find relations with the SU(2)-structures studied by Conti and Salamon in [8].

In Section 5 we study the geometric structure induced naturally on any oriented hypersurface $N^{2n+1}$ of a $(2n+2)$-dimensional manifold $M^{2n+2}$ carrying an SKT structure and in Section 6 we use such structures to construct new SKT manifolds via appropriate evolution equations [20, 8], starting from a 5-dimensional manifold endowed with an SU(2)-structure (Theorem 6.4).

A good quaternionic analog of Kähler geometry is given by hyper-Kähler with torsion (shortly HKT) geometry. An HKT manifold is a hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ admitting a hyper-Hermitian connection with totally skew-symmetric torsion, i.e. for which the three Bismut connections associated to the three Hermitian structures $(J_r, h)$, $r = 1, 2, 3$, coincide. This geometry was introduced by Howe and Papadopoulos [21] and later studied for instance in [16, 11, 2, 3, 27].

A particular interesting case is when the torsion 3-form of such hyper-Hermitian connection is closed. In this case the HKT manifold is called strong.

In the last section we find conditions for which an $S^1$-bundle over a $(4n+3)$-dimensional manifold endowed with three almost contact metric structures is HKT and in particular when it is strong HKT (Theorem 7.1).
2. SKT structures arising from $S^1$-bundles

Consider a $(2n + 1)$-manifold $N^{2n+1}$ with an almost contact metric structure $(I, \xi, \eta, g)$, that is, $I$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $N^{2n+1}$ satisfying the following conditions:

$$I^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(IU, IV) = g(U, V) - \eta(U)\eta(V),$$

for any vector fields $U, V$ on $N^{2n+1}$. Denote by $\omega$ the fundamental 2-form of $(I, \xi, \eta, g)$, i.e. $\omega$ is the 2-form on $N^{2n+1}$ given by

$$\omega(\ldots) = g(\ldots, I\ldots).$$

Given the tensor field $I$ consider its Nijenhuis torsion $[I, I]$ defined by

$$[I, I](X, Y) = I^2[X, Y] + [IX, IY] - I[IX, Y] - I[X, IY].$$

On the product $N^{2n+1} \times \mathbb{R}$ it is possible to define a natural almost complex structure $J$ on $N^{2n+1} \times \mathbb{R}$ by

$$J(X, f \, dt) = (IX + f\xi, -\eta(X) \frac{d}{dt}),$$

where $f$ is a $C^\infty$-function on $N^{2n+1} \times \mathbb{R}$ and $t$ is the coordinate on $\mathbb{R}$.

We recall the following:

**Definition 2.1.** [25] An almost contact metric structure $(I, \xi, \eta, g)$ on $N^{2n+1}$ is called normal if the almost complex structure $J$ on $N^{2n+1} \times \mathbb{R}$ is integrable, or equivalently if

$$[I, I](X, Y) + 2d\eta(X, Y)\xi = 0,$$

for any vector fields $X, Y$ on $N^{2n+1}$.

By [3] Lemma 2.1] for a normal almost contact metric structure $(I, \xi, \eta, g)$, one has that $i_\xi d\eta = 0$.

**Remark 2.2.** The normality of the almost contact structure implies also that $I d\eta = d\eta$. Indeed, we have that $d(\eta - idt) = d\eta$ has no $(0, 2)$-part and therefore it has also no $(2, 0)$-part since $d\eta$ is real. Thus $J d\eta = d\eta$, but we have also that $J d\eta = I d\eta$ since $i_\xi d\eta = 0$.

We recall that a Hermitian manifold $(M, J, h)$ is SKT if and only if the 3-form $J dF$ is closed, where $F$ is the fundamental 2-form of $(J, h)$. In the paper we will use the convention that $J$ acts on $r$-forms $\beta$ as

$$(J\beta)(X_1, \ldots, X_r) = \beta(JX_1, \ldots, JX_r),$$

for any vector fields $X_1, \ldots, X_r$.

We now show conditions for which in general an $S^1$-bundle over an almost contact metric $(2n + 1)$-dimensional manifold is SKT.

Let $(N^{2n+1}, I, \xi, \eta)$ be a $(2n + 1)$-dimensional almost contact manifold, and let $\Omega$ be a closed 2-form on $N^{2n+1}$ which represents an integral cohomology class on $N^{2n+1}$. From the well-known result of Kobayashi [23], we can consider the circle bundle $S^1 \to P \to N^{2n+1}$, with connection 1-form $\theta$ on $P$ whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N^{2n+1}$ is the projection.
By using the almost contact structure \((I, \xi, \eta)\) and the connection 1-form \(\theta\), one can define an almost complex structure \(J\) on \(P\) as follows (see [24]). For any right-invariant vector field \(X\) on \(P\), \(JX\) is given by

\[
\begin{align*}
\theta(JX) &= -\pi^*(\eta(\pi_*X)), \\
\pi_*(JX) &= I(\pi_*X) + \theta(X)\xi,
\end{align*}
\]  
(2)

where \(\tilde{\theta}(X)\) is the unique function on \(N^{2n+1}\) such that

\[
\pi^*\tilde{\theta}(X) = \theta(X).
\]

The above definition can be extended to arbitrary vector fields \(X\) on \(P\), since \(X\) can be written in the form

\[X = \sum_j f_j X_j,\]

with \(f_j\) smooth functions on \(P\) and \(X_j\) right-invariant vector fields. Then \(JX = \sum_j f_j JX_j\).

In [24] it has been showed that if \((N^{2n+1}, I, \xi, \eta)\) is normal, then the almost complex structure \(J\) on \(P\) defined by (2) is integrable if and only if \(d\theta\) is \(J\)-invariant, that is,

\[J(d\theta) = d\theta,
\]

or equivalently

\[d\theta(JX, Y) + d\theta(X, JY) = 0,
\]

for any vector fields \(X, Y\) on \(P\), i.e. \(d\theta\) is a complex 2-form on \(P\) having bidegree (1, 1) with respect to \(J\).

In terms of the 2-form \(\Omega\) whose lifting to \(P\) is the curvature of the circle bundle \(S^1 \hookrightarrow P \rightarrow N^{2n+1}\), the previous condition means that \(\Omega\) is \(I\)-invariant, i.e. \(I(\Omega) = \Omega\), and therefore \(i_\xi \Omega = 0\).

If \(\{e^1, \ldots, e^{2n}, \eta\}\) is an adapted coframe on a neighborhood \(U\) on \(N^{2n+1}\), i.e. such that

\[I e^{2j-1} = -e^{2j}, \quad I e^{2j} = e^{2j-1}, \quad 1 \leq j \leq n,
\]

then we can take \(\{\pi^* e^1, \ldots, \pi^* e^{2n}, \pi^* \eta, \theta\}\) as a coframe in \(\pi^{-1}(U)\). Using the coframe \(\{\pi^* e^1, \ldots, \pi^* e^{2n}\}\), we may write

\[d\theta = \pi^* \alpha + \pi^* \beta \wedge \pi^* \eta,
\]

where \(\alpha\) is a 2-form in \(\wedge^2 < e^1, \ldots, e^{2n} >\) and \(\beta \in \wedge^1 < e^1, \ldots, e^{2n} >\).

Next, suppose that \(N^{2n+1}\) has a normal almost contact metric structure \((I, \xi, \eta, g)\). We consider a principal \(S^1\)-bundle \(P\) with base space \(N^{2n+1}\) and connection 1-form \(\theta\), and endow \(P\) with the almost complex structure \(J\) (associated to \(\theta\)) defined by (2). Since \(N^{2n+1}\) has a Riemannian metric \(g\), a Riemannian metric \(h\) on \(P\) compatible with \(J\) (see [24]) is given by

\[h(X, Y) = \pi^* g(\pi_* X, \pi_* Y) + \theta(X)\theta(Y),
\]

for any right-invariant vector fields \(X, Y\). The above definition can be extended to any vector field on \(P\).

**Theorem 2.3.** Let \((N^{2n+1}, I, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional almost contact metric manifold and let \(\Omega\) be a closed 2-form on \(N^{2n+1}\) which represents an integral cohomology class. Consider the circle bundle \(S^1 \hookrightarrow P \rightarrow N^{2n+1}\) with connection 1-form \(\theta\) whose curvature form is \(d\theta = \pi^*(\Omega)\), where \(\pi : P \rightarrow N^{2n+1}\) is the projection.
Then, the almost Hermitian structure \((J, h)\) on \(P\), defined by (2) and (3), is SKT if and only if \((I, \xi, \eta, g)\) is normal, \(d\theta\) is \(J\)-invariant and such that

\[
\begin{align*}
    d(\pi^*(I(i_\xi d\omega))) &= 0, \\
    d(\pi^*(I(d\omega) - d\eta \wedge \eta)) &= (-\pi^*(I(i_\xi d\omega)) + \pi^*\Omega) \wedge \pi^*\Omega,
\end{align*}
\]

where \(\omega\) denotes the fundamental form of the almost contact metric structure \((I, \xi, \eta, g)\).

**Proof.** As we mentioned previously, a result of Ogawa [24] asserts that the almost complex structure \(J\) is integrable if and only if \((g, I, \xi, \eta)\) is normal and \(J(d\theta) = d\theta\). Thus \((J, h)\) is SKT if and only if the 3-form \(JdF\) is closed. By using the first equality of (2), we have that the fundamental 2-form \(F\) is

\[
F(X, Y) = h(X, JY) = \pi^*g(\pi_\ast X, \pi_\ast JY) + \theta(X)\theta(JY)
\]

Thus (7)

\[
\begin{align*}
    J(\pi^*(d\omega)) &= \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta.
\end{align*}
\]

Indeed, locally and in terms of the adapted basis \(\{e^1, \ldots, e^{2n+1}\}\) such that

\[
Ie^{2j-1} = -e^{2j}, \quad 1 \leq j \leq n, \quad Ie^{2n+1} = 0, \quad \eta = e^{2n+1},
\]

we can write

\[
d\omega = \alpha + \beta \wedge \eta,
\]

where the local forms \(\alpha \in \Lambda^3 < e^1, \ldots, e^{2n}\) and \(\beta \in \Lambda^2 < e^1, \ldots, e^{2n}\) are generated only by \(e^1, \ldots, e^{2n}\). Furthermore, we have

\[
I\alpha = I(d\omega), \quad \beta = i_\xi d\omega.
\]

Thus,

\[
J(\pi^*(d\omega)) = J(\pi^*(\alpha)) + J(\pi^*(i_\xi d\omega)) \wedge \theta.
\]

Now, by using (2) and (3), we see that \(J(\pi^*(\alpha)) = \pi^*(I\alpha)\) and \(J(\pi^*(i_\xi d\omega)) = \pi^*(I(i_\xi d\omega))\), which proves (7). As a consequence of Remark 2.2 we have

\[
J(\pi^*(d\eta)) = \pi^*(I(d\eta)) - \pi^*(I(i_\xi d\eta)) \wedge \theta = \pi^*(d\eta),
\]

since \(i_\xi d\eta = 0\) and \(I d\eta = d\eta\).

By using (7) and (8) we get

\[
JdF = \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta - \pi^*(d\eta) \wedge \pi^*\eta - \theta \wedge d\theta.
\]

Therefore

\[
d(JdF) = d(\pi^*(I(d\omega))) + d(\pi^*(I(i_\xi d\omega))) \wedge \theta + \pi^*(I(i_\xi d\omega)) \wedge d\theta
\]

\[-d(\pi^*(d\eta)) \wedge \pi^*\eta - \pi^*(d\eta) \wedge d\pi^*\eta - d\theta \wedge d\theta.
\]
Consequently, \(d(JdF) = 0\) if and only if
\[d(\pi^*(I(i_\xi d\omega))) = 0,\]
and
\[d(\pi^*(I(d\omega) - d\eta \wedge \eta)) = (\pi^*(-I(i_\xi d\omega)) + d\theta) \wedge d\theta,\]
which completes the proof. \(\square\)

We recall that an almost contact metric manifold \((N^{2n+1}, I, \xi, \eta, g)\) is **quasi-Sasakian** if it is normal and its fundamental form \(\omega\) is closed. If, in particular, \(d\eta = \alpha \omega\), then the almost contact metric structure is called **\(\alpha\)**-Sasakian. When \(\alpha = -2\), the structure is said to be **Sasakian**.

By [15, Theorem 8.2] an almost contact metric manifold \((N^{2n+1}, I, \xi, \eta, g)\) admits a connection \(\nabla^c\) preserving the almost contact metric structure and with totally skew-symmetric torsion tensor if and only if the Nijenhuis tensor of \(I\), given by (1), is skew-symmetric and \(\xi\) is a Killing vector field. Moreover, this connection is unique.

Then, in particular on any quasi-Sasakian manifold \((N^{2n+1}, I, \xi, \eta, g)\) there exists a unique connection \(\nabla^c\) with totally skew-symmetric torsion such that
\[\nabla^c I = 0, \quad \nabla^c g = 0, \quad \nabla^c \eta = 0.\]
Such connection \(\nabla^c\) is uniquely determined by
\[g(\nabla^c_X Y, Z) = g(\nabla^h_X Y, Z) + \frac{1}{2}(d\eta \wedge \eta)(X, Y, Z),\]
where \(\nabla^h\) denotes the Levi-Civita connection and \(\frac{1}{2}(d\eta \wedge \eta)\) is the torsion 3-form of \(\nabla^c\).

**Corollary 2.4.** Let \((N^{2n+1}, I, \xi, \eta, g)\) be a quasi-Sasakian \((2n + 1)\)-manifold and let \(\Omega\) be a closed 2-form on \(N^{2n+1}\) which represents an integral cohomology class. Consider the circle bundle \(S^1 \rightarrow P \rightarrow N^{2n+1}\) with connection 1-form \(\theta\) whose curvature form is \(d\theta = \pi^*(\Omega)\), where \(\pi : P \rightarrow N^{2n+1}\) is the projection. Then, the almost Hermitian structure \((J, h)\) on \(P\), defined by (2) and (4), is SKT if and only if \(\Omega\) is \(I\)-invariant, \(i_\xi \Omega = 0\) and
\[d\eta \wedge d\eta = -\Omega \wedge \Omega.\]
Moreover, the Bismut connection \(\nabla^B\) of \((J, h)\) on \(P\) and the connection \(\nabla^c\) on \(N\) given by (10) are related by
\[h(\nabla^B_X Y, Z) = \pi^* g(\nabla_{\pi_*X} \pi_*Y, \pi_*Z),\]
for any vector fields \(X, Y, Z\) on \(P\), where \(\nabla^h\) is the Levi-Civita connection associated to \(h\). Then, for any \(X, Y, Z\) in the kernel of \(\theta\) we have
\[h(\nabla^B_X Y, Z) = \pi^* g(\nabla^h_X Y, Z) + \frac{1}{2}(\pi^*(d\eta) \wedge \pi^*\eta)(X, Y, Z).\]
By [24, Lemma 3] and the definition of $\nabla^c$ we get
\[ h(\nabla^B_X Y, Z) = \pi^* g(\nabla^c_{\pi_* X} \pi_* Y, \pi_* Z) + \frac{1}{2} (\pi^*(d\eta) \wedge \pi^* \eta)(X, Y, Z) = \pi^* g(\nabla^c_{\pi_* X} \pi_* Y, \pi_* Z), \]
for any $X, Y, Z$ in the kernel of $\theta$. \qed

Remark 2.5. If the structure $(I, \xi, \eta, g)$ is $\alpha$-Sasakian, equation (11) reads as
\[ \Omega \wedge \Omega = -\alpha^2 \omega \wedge \omega. \]
In the case of a trivial $S^1$-bundle, i.e. by considering the natural almost Hermitian structure on the product $N^{2n+1} \times \mathbb{R}$, we get the following

Corollary 2.6. Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold. Consider on the product $N^{2n+1} \times \mathbb{R}$ the almost complex structure $J$ given by
\[ JX = IX, \quad X \in \text{Ker} \eta, \quad J\xi = -\frac{d}{dt}, \]
and the product metric $h = g + (dt)^2$. The Hermitian structure $(J, h)$ is SKT if and only if $(I, \xi, \eta, g)$ is normal and such that
\[ d(I(d\omega)) = d(d\eta \wedge \eta), \quad d(I(\xi d\omega)) = 0, \]
where $\omega$ denotes the fundamental 2-form of the almost contact metric structure $(g, I, \xi, \eta)$.

As a consequence of previous results we get

Corollary 2.7. Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n + 1)$-dimensional quasi-Sasakian manifold such that $d\eta \wedge d\eta = 0$. Then, the Hermitian structure $(J, h)$ on $N^{2n+1} \times \mathbb{R}$ is SKT. Moreover, its Bismut connection $\nabla^B$ coincides with the unique connection $\nabla^c$ on $N^{2n+1}$ given by (10).

Proof. In this case, since $d\omega = 0$ we get
\[ d(JdF) = -d(d\eta \wedge \eta). \]
Moreover, by using (12)
\[ h(\nabla^B_X Y, Z) = g(\nabla^c_X Y, Z), \]
for any vector fields $X, Y, Z$ on $N^{2n+1}$. \qed

2.1. Examples. We will start presenting three examples of quasi-Sasakian Lie algebras satisfying the condition $d\eta \wedge d\eta = 0$. By applying Corollary 2.7, one gets an SKT structure on the product of the corresponding simply-connected Lie group by $\mathbb{R}$.

Example 2.8. Let $\mathfrak{s}$ be the 5-dimensional Lie algebra with structure equations
\[
\begin{cases}
  de^1 = e^{13} + e^{23} + e^{25} - e^{34} + e^{35}, \\
  de^2 = 2e^{12} - 2e^{13} + e^{14} - e^{15} - e^{24} + e^{34} + e^{45}, \\
  de^3 = -e^{12} + e^{13} + e^{14} - e^{15} + 2e^{24} - 2e^{34} + e^{45}, \\
  de^4 = -e^{12} - e^{23} - e^{24} - e^{25} - e^{35}, \\
  de^5 = e^{12} - e^{13} - e^{24} + e^{34},
\end{cases}
\]
where by $e^{ij}$ we denote $e^i \wedge e^j$. 
Consider on $\mathfrak{s}$ the quasi-Sasakian structure $(I, \xi, \eta, g)$ given by

\begin{equation}
\eta = e^5, \quad Ie^1 = -e^2, \quad Ie^3 = -e^4, \quad \omega = -e^{12} - e^{34}, \quad g = \sum_{j=1}^{5} (e^j)^2.
\end{equation}

We have that the above quasi-Sasakian structure satisfies the condition $d(\eta \wedge \eta) = 0$.

The Lie algebra $\mathfrak{s}$ is 2-step solvable since the commutator

$$\mathfrak{s}^1 = [\mathfrak{s}, \mathfrak{s}] = \mathbb{R} < e_1 - e_4, e_2 + e_3, e_1 - e_2 + 2e_3 - e_5 >$$

is abelian, where $\{e_1, \ldots, e_5\}$ denotes the dual basis of $\{e^1, \ldots, e^5\}$. Moreover $\mathfrak{s}$ has trivial center, it is irreducible and non unimodular, since we have that the trace of $ad_{e_i}$ is equal to $-3$.

**Example 2.9.** Consider the family of 2-step solvable Lie algebras $\mathfrak{s}_a$, $a \in \mathbb{R} - \{0\}$, given by

\[
\begin{align*}
de^1 &= a e^{33} + 3 e^{25}, \\
de^2 &= -a e^{13} - 3 e^{15}, \\
de^3 &= a e^{34}, \\
de^4 &= 0, \\
de^5 &= -\frac{a^2}{3} e^{34}.
\end{align*}
\]

The almost contact metric structure $(I, \xi, \eta, g)$ given by (14) is quasi-Sasakian and satisfies the condition $d\eta \wedge d\eta = 0$. Moreover, the second cohomology group of $\mathfrak{s}_a$ is generated by $e^{12}$ and $e^{45}$.

**Example 2.10.** Another example of family of quasi-Sasakian Lie algebras satisfying the condition $d\eta \wedge d\eta = 0$ is $\mathfrak{g}_b$, $b \in \mathbb{R} - \{0\}$, with structure equations

\[
\begin{align*}
de^1 &= b(e^{13} + e^{14} - e^{23} + e^{24}) + e^{25}, \\
de^2 &= b(-e^{13} + e^{14} - e^{23} - e^{24}) - e^{15}, \\
de^3 &= 2 e^{45}, \\
de^4 &= -2 e^{35}, \\
de^5 &= -4b^2 e^{34},
\end{align*}
\]

and endowed with the quasi-Sasakian structure given by (14). The second cohomology group of $\mathfrak{g}_b$ is generated by $e^{12}$. The Lie algebras $\mathfrak{g}_b$ are not solvable since for the commutator we have $[\mathfrak{g}_b, \mathfrak{g}_b] = \mathfrak{g}_b$.

The Lie groups underlying examples 2.9 and 2.10 satisfy also the conditions of Corollary 2.4 with $\Omega \wedge \Omega = 0$ just by considering as connection 1-form the 1-form $e^6$ such that $de^6 = \lambda e^{12}$ and then $\Omega = \lambda e^{12}$. With this expression of $de^6$ we have that: $d^2e^6 = 0, J(de^6) = de^6$ and $de^6 \wedge de^6 = 0$, and therefore equation (14) is satisfied.

Observe that $\lambda = 0$ provides examples of trivial $S^1$-bundles.

We can recover also one of the 6-dimensional nilmanifolds found in [12].

**Example 2.11.** Consider the 5-dimensional nilpotent Lie algebra with structure equations

\[
\begin{align*}
de^j &= 0, \quad j = 1, \ldots, 4, \\
de^5 &= e^{12} + e^{34},
\end{align*}
\]

and endowed with the quasi-Sasakian structure given by (14). If we consider the closed 2-form $\Omega = e^{13} + e^{24}$ and we apply Corollary 2.4 we have that there exists a
non trivial $S^1$-bundle over the corresponding 5-dimensional nilmanifold. Moreover, since $de^5 \wedge de^5 = -\Omega \wedge \Omega \neq 0$, the total space of this $S^1$-bundle is an SKT nilmanifold. More precisely, according to the classification given in [12] (see also [28]), the nilmanifold is the one with underlying Lie algebra isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, where by $\mathfrak{h}_3$ we denote the real 3-dimensional Heisenberg Lie algebra.

Since the starting Lie algebra in Example 2.11 is Sasakian, it is natural to start with other 5-dimensional Sasakian Lie algebras to construct new SKT structures in dimension 6. A classification of 5-dimensional Sasakian Lie algebras was obtained in [1].

**Example 2.12.** Consider the 5-dimensional Lie algebra $\mathfrak{e}_3$ with structure equations

\[
\begin{cases}
  de^3 = 0, & j = 1, 4, \\
  de^2 = -e^{13}, \\
  de^3 = e^{12}, \\
  de^5 = \lambda e^{14} + \mu e^{23},
\end{cases}
\]

where $\lambda, \mu < 0$. By [1] $\mathfrak{e}_3$ admits the Sasakian structure given by

\[
Ie^1 = e^4, \quad Ie^2 = e^3, \quad \eta = e^5,
\]

\[
g = -\frac{\lambda}{2} e_1 \otimes e_1 - \frac{\lambda}{2} e_2 \otimes e_2 - \frac{\mu}{2} e_3 \otimes e_3 - \frac{\mu}{2} e_4 \otimes e_4 + e_5 \otimes e_5,
\]

and it is isomorphic to $\mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R})$. Moreover, by [1] the corresponding solvable simply-connected Lie group admits a compact quotient by a discrete subgroup.

Consider on $\mathfrak{e}_3$ the closed 2-form $\Omega = \lambda e^{14} - \mu e^{23}$. $\Omega$ is $I$-invariant and satisfies $\Omega \wedge \Omega = -2\lambda e^{1234}$. Since $e^5$ is the contact form and $de^5 \wedge de^5 = 2\lambda e^{1234}$, again we get by Corollary 2.12 an SKT structure on a non trivial $S^1$-bundle over the 5-dimensional solvmanifold. We will denote by $e^6$ the connection 1-form.

The orthonormal basis $\{\alpha^1 = e^1, \alpha^2 = e^4, \alpha^3 = e^2, \alpha^4 = e^3, \alpha^5 = e^5, \alpha^6 = \theta\}$ for the SKT metric satisfies the equations

\[
da^1 = da^2 = 0, \quad da^3 = -a^{14}, \quad da^4 = a^{13},
\]

\[
da^5 = \lambda a^{12} + \mu a^{34}, \quad da^6 = \lambda a^{12} - \mu a^{34},
\]

and the complex structure is given by $J(X_1) = X_2, J(X_3) = X_4, J(X_5) = X_6$, where $\{X_i\}_{i=1}^6$ denotes the basis dual to $\{\alpha^i\}_{i=1}^6$. Since the fundamental 2-form is $F = \alpha^{12} + \alpha^{34} + \alpha^{56}$, one has that the 3-form torsion $T$ of the SKT structure is

\[
T = \lambda \alpha^{12}(\alpha^5 + \alpha^6) + \mu \alpha^{34}(\alpha^5 - \alpha^6).
\]

Moreover, $*T = \lambda \alpha^{12}(\alpha^5 + \alpha^6) - \mu \alpha^{34}(\alpha^5 - \alpha^6)$, where $*$ denotes the Hodge operator of the metric, which implies that the torsion form is also coclosed.

The only nonzero curvature forms $(\Omega^B)_i$ of the Bismut connection $\nabla^B$ are

\[
(\Omega^B)_1^1 = -2 \lambda^2 \alpha^{12}, \quad (\Omega^B)_1^2 = -2 \mu^2 \alpha^{34}.
\]

A direct calculation shows that the 1-forms $\alpha^5, \alpha^6$ and the 2-forms $\alpha^{12}, \alpha^{34}$ are parallel with respect to the Bismut connection, which implies that $\nabla^B T = 0$.

Finally, since $\nabla^B \alpha^i \neq 0$ for $i = 1, 2, 3, 4$, we conclude that $\text{Hol}(\nabla^B) = U(1) \times U(1) \subset U(3)$. 
3. SKT structures arising from Riemannian cones

Let $N^{2n+1}$ be a $(2n + 1)$-dimensional manifold endowed with an almost contact metric structure $(I, \xi, \eta, g)$ and denote by $\omega$ its fundamental 2-form.

The Riemannian cone of $N^{2n+1}$ is defined as the manifold $N^{2n+1} \times \mathbb{R}^+$ equipped with the cone metric:

$$h = t^2 g + (dt)^2.$$  \hfill (15)

The cone $N^{2n+1} \times \mathbb{R}^+$ has a natural almost Hermitian structure defined by

$$F = t^2 \omega + t \eta \wedge dt.$$ \hfill (16)

The almost complex structure $J$ on $N^{2n+1} \times \mathbb{R}^+$ defined by $(F, h)$ is given by

$$JX = IX, \quad X \in \text{Ker} \eta, \quad J\xi = -t \frac{d}{dt}.$$ \hfill (17)

In terms of a local orthonormal adapted coframe $\{e^1, \ldots, e^{2n}\}$ for $g$ such that

$$\omega = -\sum_{j=1}^n e^{2j-1} \wedge e^{2j},$$ \hfill (18)

we have

$$Je^{2j-1} = -e^{2j}, \quad Je^{2j} = e^{2j-1}, \quad j = 1, \ldots, n,$$

$$J(te^{2n+1}) = dt, \quad J(dt) = -te^{2n+1}.$$ \hfill (19)

The almost Hermitian structure $(J, h)$ on $N^{2n+1} \times \mathbb{R}^+$ is Kähler if and only if the almost contact metric structure $(I, \xi, \eta, g)$ on $N^{2n+1}$ is Sasakian, i.e. a normal contact metric structure.

If we impose that the almost Hermitian structure $(J, h)$ on $N^{2n+1} \times \mathbb{R}^+$ is SKT, we can prove the following

**Theorem 3.1.** Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold. The almost Hermitian structure $(J, h)$ on the Riemannian cone $(N^{2n+1} \times \mathbb{R}^+, h)$, given by (15) and (16), is SKT if and only if $(I, \xi, \eta, g)$ is normal and

$$-4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta = d(I(\xi d\omega)),$$ \hfill (19)

where $\omega$ denotes the fundamental 2-form of the almost contact metric structure $(I, \xi, \eta, g)$.

**Proof.** $J$ is integrable if and only if the almost contact metric structure is normal. Now we compute $JdF$. We have that

$$dF = 2tdt \wedge \omega + t^2 d\omega + td\eta \wedge dt,$$

and

$$JdF = -2t^2 \eta \wedge \omega + t^2 J(d\omega) - t^2 d\eta \wedge \eta,$$

since

$$J\omega = \omega, \quad J(dt) = -t\eta, \quad Jd\eta = d\eta.$$ \hfill (20)

Moreover, with respect to an adapted basis $\{e^1, \ldots, e^{2n+1}\}$ we may prove, in a similar way as in the proof of Theorem 2.3, that

$$Jd\omega = I(d\omega) + I(\xi d\omega) \wedge J\eta.$$ \hfill (19)

As a consequence we get

$$JdF = -2t^2 \eta \wedge \omega + t^2 I(d\omega) + tdt \wedge I(\xi d\omega) - t^2 d\eta \wedge \eta.$$
Therefore, by imposing $d(JdF) = 0$ we obtain the two equations
\[
\begin{align*}
-4\eta \wedge \omega + 2I(d\omega) - d\eta \wedge d(I(\xi d\omega)) &= 0, \\
-2d(\eta \wedge \omega) + d(I(d\omega)) - d(d\eta \wedge \eta) &= 0.
\end{align*}
\]
Since the second equation is consequence of the first one, we have that the Hermitian structure $(F, h)$ on the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ is SKT if and only if the almost contact metric structure $(I, \eta, \xi, g, \omega)$ on $N^{2n+1}$ satisfies the equation (19). □

Remark 3.2. As a consequence of previous theorem we have that, if $n = 1$, equation (19) is satisfied if and only if the 3-dimensional manifold $N$ is Sasakian. On the other hand, if $n > 1$ and the almost contact metric structure on $N^{2n+1}$ is quasi-Sasakian (i.e. $d\omega = 0$), then the structure has to be Sasakian, i.e. $d\eta = -2\omega$.

Example 3.3. Consider the 5-dimensional Lie algebras $g_{a,b,c}$ with structure equations
\[
\begin{align*}
de e^1 &= a e^{23} + 2 e^{25} + \left(-\frac{1}{2} ab + \frac{b^2}{2a} + \frac{2b}{a}\right) e^{34} + b e^{45}, \\
de e^2 &= -a e^{13} - 2 e^{15} - \frac{1}{2} bc e^{34} - b e^{35}, \\
de e^3 &= \left(-\frac{4}{a} - \frac{b^2}{a}\right) e^{34}, \\
de e^4 &= c e^{34}, \\
de e^5 &= 2 e^{12} + b e^{14} - b e^{23} + (2 + b^2) e^{34},
\end{align*}
\]
where $a, b, c \in \mathbb{R}$ and $a \neq 0$, endowed with the normal almost contact metric structure $(I, \xi, \eta, g, \omega)$ with
\[
I e^1 = -e^2, \quad I e^3 = -e^4, \quad \eta = e^5, \quad \omega = -e^{12} - e^{34}.
\]
This structure satisfies (19) and therefore, the Riemannian cones over the corresponding simply-connected Lie groups are SKT.

4. SKT SU(3)-structures

Let $(M^6, J, h)$ be a 6-dimensional almost Hermitian manifold. An $SU(3)$-structure on $M^6$ is determined by the choice of a $(3,0)$-form $\Psi = \Psi_+ + i\Psi_-$ of unit norm. If $\Psi$ is closed, then the underlying almost complex structure $J$ is integrable and the manifold is Hermitian. We will denote the $SU(3)$-structure $(J, h, \Psi)$ simply by $(F, \Psi)$, where $F$ is the fundamental 2-form, since from $F$ and $\Psi$ we can reconstruct the almost Hermitian structure.

We can give the following

Definition 4.1. We say that an $SU(3)$-structure $(F, \Psi)$ on $M^6$ is SKT if
\[
d\Psi = 0, \quad d(JdF) = 0,
\]
where $J$ is the associated complex structure.

We will see the relation between SKT $SU(3)$-structures in dimension 6 and $SU(2)$-structures in dimension 5.

First we recall some facts about $SU(2)$-structures on a 5-dimensional manifold. An $SU(2)$-structure on a 5-dimensional manifold $N^5$ is an $SU(2)$-reduction of the principal bundle of linear frames on $N^5$. By [8] Proposition 1, these structures are
in 1 : 1 correspondence with quadruplets \((\eta, \omega_1, \omega_2, \omega_3)\), where \(\eta\) is a 1-form and \(\omega_i\) are 2-forms on \(N^5\) satisfying
\[
\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,
\]
for some 4-form \(v\), and
\[
i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X,Y) \geq 0,
\]
where \(i_X\) denotes the contraction by \(X\). Equivalently, an SU(2)-structure on \(N^5\) can be viewed as the datum of \((\eta, \omega_1, \Phi)\), where \(\eta\) is a 1-form, \(\omega_1\) is a 2-form and \(\Phi = \omega_2 + i \omega_3\) is a complex 2-form such that
\[
\eta \wedge \omega_1 \wedge \omega_1 \neq 0, \quad \Phi \wedge \Phi = 0, \quad \omega_1 \wedge \Phi = 0, \quad \Phi \wedge \Phi = 2 \omega_1 \wedge \omega_1,
\]
and \(\Phi\) is of type \((2,0)\) with respect to \(\omega_1\).

SU(2)-structures are locally characterized as follows (see [8]): If \((\eta, \omega_1, \omega_2, \omega_3)\) is an SU(2)-structure on a 5-manifold \(N^5\), then locally, there exists an orthonormal basis of 1-forms \(\{e_1, \ldots, e_5\}\) such that
\[
\omega_1 = e_1^2 + e_3^4, \quad \omega_2 = e_1^3 - e_2^4, \quad \omega_3 = e_1^4 + e_2^3, \quad \eta = e_5.
\]
We can also consider the local tensor field \(I\) given by
\[
Ie_1 = -e_2, \quad Ie_2 = e_1, \quad Ie_3 = -e_4, \quad Ie_4 = e_3, \quad Ie_5 = 0.
\]
This tensor gives rise to a global tensor field of type \((1,1)\) on the manifold \(N^5\) defined by \(\omega_1(X,Y) = g(X, IY)\), for any vector fields \(X, Y\) on \(N^5\), where \(g\) is the Riemannian metric on \(N^5\) underlying the SU(2)-structure. The tensor field \(I\) satisfies
\[
I^2 = -Id + \eta \otimes \xi,
\]
where \(\xi\) is the vector field on \(N^5\) dual to the 1-form \(\eta\).

Therefore, given an SU(2)-structure \((\eta, \omega_1, \omega_2, \omega_3)\) we also have an almost contact metric structure \((I, \xi, \eta, g)\) on the manifold, where \(\omega_1\) is the fundamental form.

**Remark 4.2.** Notice that we have two more almost contact metric structures when one considers \(\omega_2\) and \(\omega_3\) as fundamental forms.

If \(N^5\) has an SU(2)-structure \((\eta, \omega_1, \omega_2, \omega_3)\), the product \(N^5 \times \mathbb{R}\) has a natural SU(3)-structure given by
\[
F = \omega_1 + \eta \wedge dt, \quad \Psi = (\omega_2 + i \omega_3) \wedge (\eta - idt).
\]
Moreover, by Corollary [2.6] the previous SU(3)-structure is SKT if and only if
\[
d(I(d\omega_1)) = d(\eta \wedge \eta), \quad d(I(\xi d\omega_1)) = 0, \quad d\omega_2 = -3 \omega_3 \wedge \eta, \quad d\omega_3 = 3 \omega_2 \wedge \eta.
\]
Then we have proved the following

**Theorem 4.3.** Let \(N^5\) be a 5-dimensional manifold endowed with an SU(2)-structure \((\eta, \omega_1, \omega_2, \omega_3)\). The SU(3)-structure \((F, \Psi)\), given by (22), on the product \(N^5 \times \mathbb{R}\) is SKT if and only if the equations (23) are satisfied.
Example 4.4. Consider on the 5-dimensional Lie algebras, introduced in Examples 2.8, 2.9 and 2.10 the $SU(2)$-structure given by
\[
\omega = \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}.
\]
For the example 2.8 we have:
\[
d\omega_2 = -2 \omega_3 \wedge \eta - 4(e^{124} - e^{134}),
\]
\[
d\omega_3 = 2 \omega_2 \wedge \eta + 4(e^{123} + e^{234}).
\]
For the examples 2.9 and 2.10 we get
\[
d\omega_2 = -3 \omega_3 \wedge \eta \quad \text{and} \quad d\omega_3 = 3 \omega_2 \wedge \eta,
\]
therefore on the product of the corresponding simply-connected Lie groups by $\mathbb{R}$ one gets an SKT $SU(3)$-structure.

We will study the existence of SKT $SU(3)$-structures on a Riemannian cone over a 5-dimensional manifold $N^5$ endowed with an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then $N^5$ has an induced almost contact metric structure $(I, \xi, \eta, g)$ and $\omega_1$ is its fundamental form.

The Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ of $(N^5, g)$ has a natural $SU(3)$-structure defined by
\[
F = t^2 \omega_1 + t \eta \wedge dt,
\]
\[
\Psi = t^2(\omega_2 + i \omega_3) \wedge (t \eta - idt).
\]
In terms of a local orthonormal coframe \{e_1, \ldots, e_5\} for $g$ such that
\[
\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}, \quad \eta = e^5,
\]
we have that
\[
Je_1 = -e^2, \quad Je_2 = e^1, \quad Je_3 = -e^4, \quad Je_4 = e^3, \quad J(te^5) = dt, \quad J(dt) = -te^5.
\]
We recall that the $SU(3)$-structure $(F, \Psi)$ on $N^5 \times \mathbb{R}^+$ is integrable if and only if the $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on $N^5$ is Sasaki-Einstein, or equivalently if and only if
\[
d\eta = -2 \omega_1, \quad d\omega_2 = -3 \omega_3 \wedge \eta, \quad d\omega_3 = 3 \omega_2 \wedge \eta.
\]

For the Riemannian cones we can prove the following

Corollary 4.5. Let $N^5$ be a 5-dimensional manifold endowed with an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$. The $SU(3)$-structure $(F, \Psi)$ on the Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ is SKT if and only if
\[
\begin{cases}
-4\eta \wedge \omega_1 + 2I(d\omega_1) - 2d\eta \wedge \eta = d(I(\xi d\omega_1)), \\
d\omega_2 = 3 \omega_3 \wedge \eta, \\
d\omega_3 = -3 \omega_2 \wedge \eta.
\end{cases}
\]

Proof. By imposing that $d\Psi = 0$ we get the conditions
\[
d\omega_2 = -3 \omega_3 \wedge \eta, \quad d\omega_3 = 3 \omega_2 \wedge \eta.
\]

By imposing $d(JdF) = 0$, we obtain, as in the proof of Theorem 3.1 the equation \[(19)\] for $\omega = \omega_1$.\hfill $\square$
5. Almost contact metric structure induced on a hypersurface

Here we study the almost contact metric structure induced naturally on any oriented hypersurface $N^{2n+1}$ of a $(2n+2)$-manifold $M^{2n+2}$ equipped with an SKT structure.

Let $f: N^{2n+1} \to M^{2n+2}$ be an oriented hypersurface of a $(2n+2)$-dimensional manifold $M^{2n+2}$ endowed with an SKT structure $(J,h,F)$ and denote by $\mathcal{U}$ the unitary normal vector field. It is well known that $N^{2n+1}$ inherits an almost contact metric structure $(I,\xi,\eta,g)$ such that $\eta$ and the fundamental 2-form $\omega$ are given by

$$\eta = -f^*(i_\mathcal{U}F), \quad \omega = f^*F,$$

where $F$ is the fundamental 2-form of the almost Hermitian structure (see for instance [6]).

**Proposition 5.1.** Let $f: N^{2n+1} \to M^{2n+2}$ be an immersion of an oriented $(2n+1)$-dimensional manifold into a $(2n+2)$-dimensional Hermitian manifold $(M^{2n+2},J,h)$. If the Hermitian structure $(J,h)$ is SKT, then the induced almost contact metric structure $(I,\xi,\eta,g)$ on $N^{2n+1}$, with $\eta$ and $\omega$ given by (25), satisfies

$$d(\text{Id} \omega - I(f^*(i_\mathcal{U}F)) \wedge \eta) = 0.$$

**Proof.** We can choose locally an adapted coframe $\{e^1,\ldots,e^{2n+2}\}$ for the Hermitian structure such that the unitary normal vector field $\mathcal{U}$ is dual to $e^{2n+2}$. Since the almost complex structure $J$ is given in this adapted basis by

$$Je^{2j-1} = -e^{2j}, \quad Je^{2j} = e^{2j-1}, \quad j = 1,\ldots,n,$$

$$Je^{2n+1} = e^{2n+2}, \quad Je^{2n+2} = -e^{2n+1},$$

the tensor field $I$ on $N^{2n+1}$ satisfies that $If^*e^i = f^*Je^i$, $i = 1,\ldots,2n+1$, that is,

$$If^*e^{2j-1} = -f^*e^{2j}, \quad If^*e^{2j} = f^*e^{2j-1}, \quad j = 1,\ldots,n, \quad If^*e^{2n+1} = 0.$$

However, $If^*e^{2n+2} = 0 \neq f^*e^{2n+1} = -f^*Je^{2n+2}$.

Now we compute $f^*JdF$. First we decompose (locally and in terms of the adapted basis) the differential of $F$ as follows:

$$dF = \alpha + \beta \wedge e^{2n+1} + \gamma \wedge e^{2n+2} + \mu \wedge e^{2n+1} \wedge e^{2n+2},$$

where the local forms $\alpha \in \Lambda^1 < e^1,\ldots,e^{2n}>$, $\beta,\gamma \in \Lambda^2 < e^1,\ldots,e^{2n}>$ and $\mu \in \Lambda^1 < e^1,\ldots,e^{2n}>$ are generated only by $e^1,\ldots,e^{2n}$. Then,

$$JdF = J\alpha + J\beta \wedge e^{2n+2} - J\gamma \wedge e^{2n+1} + J\mu \wedge e^{2n+1} \wedge e^{2n+2}.$$

Since $f^*e^{2n+2} = 0$ and using that $f^*e^{2n+1} = \eta$, we get

$$f^*JdF = f^*J\alpha - (f^*J\gamma) \wedge \eta.$$

But $f^*(i_\mathcal{U}dF) = f^*\gamma + f^*\mu \wedge \eta$, which implies that

$$I(f^*(i_\mathcal{U}dF)) = I(f^*\gamma) = f^*J\gamma.$$

On the other hand,

$$\text{Id} \omega = Id f^*\omega = f^*d\omega = f^*\alpha = f^*J\alpha.$$

We conclude that

$$f^*JdF = f^*J\alpha - (f^*J\gamma) \wedge \eta = Id \omega - I(f^*(i_\mathcal{U}dF)) \wedge \eta.$$

Now, if the Hermitian structure is SKT, then $JdF$ is closed and the induced structure satisfies (26).
Remark 5.2. Notice that using that $i_{\nu}dF = \mathcal{L}_{\nu}F - di_{\nu}F$ we can write (20) as
\[
d(I d\omega - I(f^*(\mathcal{L}_{\nu}F) + d\eta) \wedge \eta) = 0.
\]
Therefore, if $f^*(\mathcal{L}_{\nu}F) = 0$, the induced almost contact metric structure has to satisfy the equation
\[
d(I d\omega - (d\eta) \wedge \eta) = 0.
\]
In the case of the product $N^{2n+1} \times \mathbb{R}$ the condition $f^*(\mathcal{L}_{\nu}F) = 0$ is satisfied.

In the case of the Riemannian cone we have that
\[
\mathcal{L}_{\nu}^*F = 2t \omega + dt \wedge \eta,
\]
and therefore we get $f^*(\mathcal{L}_{\nu}^*F) = 2\omega$.

In this way we recover some of the equations obtained in Corollary 2.6 and in Theorem 5.1.

Now we study the structure induced naturally on any oriented hypersurface $N^5$ of a 6-manifold $M^6$ equipped with an SKT SU(3)-structure. Let $f: N^5 \rightarrow M^6$ be an oriented hypersurface of a 6-manifold $M^6$ endowed with an SU(3)-structure ($F, \Psi = \Psi_+ + i \Psi_-$) and denote by $U$ the unitary normal vector field. Then $N^5$ inherits an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ given by
\[
(27) \quad \eta = -f^*(i_{\nu}F), \quad \omega_1 = f^*F, \quad \omega_2 = -f^*(i_{\nu}\Psi_-), \quad \omega_3 = f^*(i_{\nu}\Psi_+).
\]
As a consequence of Proposition 5.1 we have the following

Corollary 5.3. Let $f: N^5 \rightarrow M^6$ be an immersion of an oriented 5-dimensional manifold into a 6-dimensional manifold with an SU(3)-structure. If the SU(3)-structure is SKT, then the induced SU(2)-structure on $N^5$ given by (27) satisfies
\[
d(I d\omega_1 - If^*(i_{\nu}dF) \wedge \eta) = 0,
\]
and
\[
d(\omega_2 \wedge \eta) = 0, \quad d(\omega_3 \wedge \eta) = 0.
\]

Proof. The equation (28) follows by Proposition 5.1 taking $\omega = \omega_1$. We can choose locally an adapted coframe $\{e^1, \ldots, e^5, e^6\}$ for the SU(3)-structure such that the unitary normal vector field $U$ is dual to $e^6$. From (27) it follows that $\omega_2 \wedge \eta = f^*\Psi_+$ and $\omega_3 \wedge \eta = f^*\Psi_-$. Now, if $\Psi = \Psi_+ + i \Psi_-$ is closed then the induced structure satisfies (29).

5.1. A simple example. Consider the 6-dimensional nilmanifold $M^6$ whose underlying nilpotent Lie algebra has structure equations
\[
\begin{cases}
de e^j = 0, & j = 1, 2, 3, 6, \\
de e^4 = e^{12}, \\
de e^5 = e^{14},
\end{cases}
\]
and it is endowed with the SU(3)-structure given by
\[
F = -e^{14} - e^{26} - e^{53}, \quad \Psi = (e^4 - ie^4) \wedge (e^2 - ie^6) \wedge (e^5 - ie^3).
\]
The oriented hypersurface with normal vector field dual to $e^2$ is a 5-dimensional nilmanifold $N^5$, which has by [8] no invariant hypo structures, but the SU(2)-structure on $N^5$
\[
(30) \quad \eta = e^2, \quad \omega_1 = -e^{14} - e^{53}, \quad \omega_2 = -e^{15} - e^{34}, \quad \omega_3 = -e^{13} - e^{45},
\]
satisfies (28) and (29). In section 6 we will show that by using this SU(2)-structure and appropriate evolution equations we can construct an SKT SU(3)-structure on the product of $N^5$ with an open interval.

6. SKT Evolution Equations

The goal here is to construct SKT SU(3)-structures by means of appropriate evolution equations starting from a suitable SU(2)-structure on a 5-dimensional manifold, following ideas of [20] and [8].

Lemma 6.1. Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of SU(2)-structures on a 5-dimensional manifold $N^5$, for $t \in (a, b)$. Then, the SU(3)-structure on $M^6 = N^5 \times (a, b)$ given by

$$F = \omega_1(t) + \eta(t) \wedge dt, \quad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - i dt),$$

satisfies the condition $d\Psi = 0$ if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is an SU(2)-structure such that

$$\begin{align*}
\hat{d}(\omega_2(t) \wedge \eta(t)) &= 0, \quad \hat{d}(\omega_3(t) \wedge \eta(t)) = 0, \\
d_t(\omega_1(t) \wedge \eta(t)) &= -\hat{d}\omega_2(t), \quad d_t(\omega_3(t) \wedge \eta(t)) = \hat{d}\omega_2(t),
\end{align*}$$

(31)

hold, for any $t$ in the open interval $(a, b)$.

Here $\hat{d}$ denotes the exterior differential on $N^5$ and $d$ the exterior differential on $M^6$. Now we show which are the additional evolution equations to add to the last two equations of (31) to ensure that $dJdF = 0$.

Proposition 6.2. Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of SU(2)-structures on $N^5$, for $t \in (a, b)$. Then, the SU(3)-structure on $M^6 = N^5 \times (a, b)$ given by

$$F = \omega_1(t) + \eta(t) \wedge dt, \quad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - i dt),$$

satisfies that $JdF$ is closed if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ satisfies the following evolution equations

$$\begin{align*}
\hat{d} \left( I_t \hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t) \right) &= 0, \\
\partial_t \left( I_t \hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t)) \wedge \eta(t) \right) &= \\
-\hat{d} \left( I_t(i\xi\hat{d}\omega_1(t)) - I_t(i\xi(\partial_t\omega_1(t) + \hat{d}\eta(t))) \wedge \eta(t) \right),
\end{align*}$$

(33)

where, for each $t \in (a, b)$, $\xi(t)$ denotes the vector field on $N^5$ dual to $\eta(t)$.

Proof. Since $F = \omega_1(t) + \eta(t) \wedge dt$, we have that

$$dF = \hat{d}\omega_1 + (\partial_t\omega_1 + \hat{d}\eta) \wedge dt.$$
Now, given $\tau(t) \in \Omega^k(N^5)$, $t \in (a, b)$, we can decompose it locally as

$$\tau(t) = \alpha(t) + \beta(t) \wedge \eta(t),$$

where $\alpha(t) \in \Lambda^k < e^1(t), \ldots, e^4(t) >$ and $\beta(t) \in \Lambda^{k-1} < e^1(t), \ldots, e^4(t) >$. Therefore

$$J \tau(t) = J\alpha(t) + J\beta(t) \wedge J\eta(t) = \hat{I}_t \alpha(t) + \hat{I}_t \beta(t) \wedge dt = \hat{I}_t \tau(t) - (-1)^k \hat{I}_t (i\xi(t) \tau(t)) \wedge dt.$$

Applying this to $JdF$ we get

$$JdF = \hat{J} d\omega_1 - j(\partial \omega_1 + \hat{d} \eta) \wedge \eta(t)$$

$$= \hat{I}_t d\omega_1 - \hat{I}_t (\partial \omega_1 + \hat{d} \eta) \wedge \eta(t) + I_t (i\xi \hat{d} \omega_1) \wedge dt - I_t (i\xi (\partial \omega_1 + \hat{d} \eta)) \wedge \eta(t) \wedge dt.$$

Finally, taking the differential of $JdF$ we get

$$dJdF = \hat{d} \left( \hat{I}_t d\omega_1 - \hat{I}_t (\partial \omega_1 + \hat{d} \eta) \wedge \eta(t) \right) + \partial_t \left( \hat{I}_t d\omega_1 - \hat{I}_t (\partial \omega_1 + \hat{d} \eta) \wedge \eta(t) \right) \wedge dt$$

$$+ \hat{d} \left( I_t (i\xi \hat{d} \omega_1) - I_t (i\xi (\partial \omega_1 + \hat{d} \eta)) \wedge \eta(t) \right) \wedge dt.$$

$$\square$$

Remark 6.3. Observe that the first equation in (33) is exactly condition (28) for $F = \omega_1(t) + \eta(t) \wedge dt$ (see Remark 5.2).

As a consequence of Lemma 6.1 and Proposition 6.2, we get

Theorem 6.4. Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$, $t \in (a, b)$, be a family of SU(2)-structures on a 5-dimensional manifold $N^5$, such that

$$\hat{d} (\omega_2(t) \wedge \eta(t)) = 0, \quad \hat{d} (\omega_3(t) \wedge \eta(t)) = 0,$$

for any $t$. If the following evolution equations

$$\begin{align*}
\hat{d} (I_t \hat{d} \omega_1(t) - I_t (\partial \omega_1(t) + \hat{d} \eta(t)) \wedge \eta(t)) &= 0, \\
\partial_t (I_t \hat{d} \omega_1(t) - I_t (\partial \omega_1(t) + \hat{d} \eta(t)) \wedge \eta(t)) &= \\
-\hat{d} (I_t (i\xi \hat{d} \omega_1(t)) - I_t (i\xi (\partial \omega_1(t) + \hat{d} \eta(t))) \wedge \eta(t)), \\
\partial_t (\omega_2(t) \wedge \eta(t)) &= -\hat{d} \omega_3(t), \\
\partial_t (\omega_3(t) \wedge \eta(t)) &= \hat{d} \omega_2(t),
\end{align*}$$

are satisfied, then the SU(3)-structure on $M = N \times (a, b)$ given by

$$F = \omega_1(t) + \eta(t) \wedge dt, \quad \Psi = (\omega_2(t) + i\omega_3(t)) \wedge (\eta(t) - i dt),$$

is SKT.

Example 6.5. Let us consider the Lie algebra with structure equations

$$\begin{align*}
\{ &d e^j = 0, j = 1, 2, 3, \\
d e^4 = e^{12}, \\
d e^5 = e^{14},
\end{align*}$$

underlying the 5-dimensional nilmanifold $N^5$ considered in Example 5.1 and endowed with the $SU(2)$-structure given by (30). It is straightforward to verify that

$$d (\omega_2 \wedge \eta) = d (\omega_3 \wedge \eta) = d (\omega_1 \wedge \eta) = 0.$$
Let us evolve the previous SU(2)-structure in the following way:

\[
\begin{align*}
\omega_1(t) &= -e^{14} - e^{53}, \\
\omega_2(t) &= -(1 + \frac{3}{2}t)^{1/3} e^{15} - (1 + \frac{3}{2}t)^{-1/3} e^{34}, \\
\omega_3(t) &= -(1 + \frac{3}{2}t)^{1/3} e^{13} - (1 + \frac{3}{2}t)^{-1/3} e^{45}, \\
\eta(t) &= (1 + \frac{3}{2}t)^{1/3} e^2,
\end{align*}
\]

where \( t \in (-2/3, \infty) \).

It is immediate to observe that the family \((\omega_1(t), \omega_2(t), \omega_3(t), \eta(t))\) verifies equations (33) and the two last equations in (35) for any \( t \in (-2/3, \infty) \). Moreover, it verifies the following conditions:

\[
\partial_t \omega_1(t) = 0, \quad d(\eta(t)) = 0, \quad i_\xi \left( d(\omega_1(t)) \right) = 0, \quad \partial_t \left( I_t(\hat{d} \omega_1(t)) \right) = 0,
\]

which implies that the evolution equations (33) are also satisfied.

On the product \( N^5 \times \mathbb{R} \) let us consider the local basis of 1-forms given by

\[
\begin{align*}
\beta^1 &= (1 + \frac{3}{2}t)^{1/3} e^1, & \beta^2 &= (1 + \frac{3}{2}t)^{-1/3} e^4, & \beta^3 &= e^5, & \beta^4 &= e^3, \\
\beta^5 &= (1 + \frac{3}{2}t)^{1/3} e^2, & \beta^6 &= dt.
\end{align*}
\]

The structure equations are:

\[
\begin{align*}
d\beta^1 &= -\frac{1}{2} \left( 1 + \frac{3}{2}t \right)^{-1} \beta^{16}, \\
d\beta^2 &= \left( 1 + \frac{3}{2}t \right)^{-1} \left( \beta^{15} + \frac{1}{2} \beta^{26} \right), \\
d\beta^3 &= \beta^{12}, \\
d\beta^4 &= 0, \\
d\beta^5 &= -\frac{1}{2} \left( 1 + \frac{3}{2}t \right)^{-1} \beta^{56}, \\
d\beta^6 &= 0.
\end{align*}
\]

\( J \) is given locally by \( J \beta^1 = -\beta^2, \quad J \beta^3 = -\beta^4, \quad J \beta^5 = \beta^6 \). The fundamental form \( F = -\beta^{12} - \beta^{34} + \beta^{56} \) verifies that \( d(JdF) = 0 \) and the \((3,0)\)-form \( \Psi = (\beta^1 + i \beta^2) \wedge (\beta^3 + i \beta^4) \wedge (\beta^5 - i \beta^6) \) is closed. Therefore, \((F, \Psi)\) is a local SKT SU(3)-structure on \( N^5 \times \mathbb{R} \).

Remark 6.6. A Hermitian structure \((J, h)\) on a 6-dimensional manifold \( M^6 \) is called balanced if \( F \wedge F \) is closed, \( F \) being the associated fundamental 2-form. In [10] it was introduced the notion of balanced SU(2)-structures on 5-dimensional manifolds, together with appropriate evolution equations whose solution gives rise to a balanced SU(3)-structure in six dimensions.

If \( M^6 \) is compact, then a balanced structure cannot be SKT (see for instance [12]).

The SU(2)-structure (3) on the previous example is also balanced and it gives rise to a balanced metric on the product of \( N^5 \) with an open interval (see (11) in [10]). However one can check directly that this solution is not SKT.

Notice that if \( G \) is the nilpotent Lie group underlying \( N^5 \), the product \( G \times \mathbb{R} \) has no left-invariant SKT structures and it does not admit any left-invariant complex structures; however we find a local SKT SU(3)-structure on it.
7. HKT structures

In this section we will find conditions for which an $S^1$-bundle over a $(4n + 3)$-dimensional manifold endowed with three almost contact metric structures is hyper-Kähler with torsion (HKT for short). We recall that a $4n$-dimensional hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is a hypercomplex manifold $(M^{4n}, J_1, J_2, J_3)$ endowed with a Riemannian metric $h$ which is compatible with the complex structures $J_r$, $r = 1, 2, 3$, i.e. such that
\[ h(J_r X, J_r Y) = h(X, Y), \]
for any $r = 1, 2, 3$ and any vector fields $X, Y$ on $M^{4n}$.

A hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is called HKT if and only if
\[ J_1 dF_1 = J_2 dF_2 = J_3 dF_3, \]
where $F_r$ denotes the fundamental 2-form associated to the Hermitian structure $(J_r, h)$ (see [16]).

Let us consider a $(4n + 3)$-dimensional manifold $N^{4n+3}$ endowed with three almost contact metric structures $(I_r, \xi_r, \eta_r, g)$, $r = 1, 2, 3$, such that
\[ I_k = I_1 I_j - \eta_j \otimes \xi_i = -I_j I_i + \eta_i \otimes \xi_j, \]
\[ \xi_k = I_1 \xi_j = -I_1 \xi_i, \quad \eta_k = -\eta_i J_j, \]
By applying Theorem 2.3 we can construct hyper-Hermitian structures on $S^1$-bundles over $N^{4n+3}$ and study when they are strong HKT.

**Theorem 7.1.** Let $N^{4n+3}$ be a $(4n + 3)$-dimensional manifold with three normal almost contact metric structures $(I_r, \xi_r, \eta_r, g)$, $r = 1, 2, 3$, satisfying (38), and let $\Omega$ be a closed 2-form on $N^{4n+3}$ which represents an integral cohomology class which is $I_r$-invariant for every $r = 1, 2, 3$. Consider the circle bundle $S^1 \hookrightarrow P \rightarrow N^{4n+3}$ with connection 1-form $\theta$ whose curvature form is $d\theta = \pi^* (\Omega)$, where $\pi : P \rightarrow N$ is the projection. Then, the hyper-Hermitian structure $(J_1, J_2, J_3, h)$ on $P$, defined by (2) and (4), is HKT if and only if
\[ \pi^*(I_1 (d\omega_1)) = \pi^*(I_2 (d\omega_2)) = \pi^*(I_3 (d\omega_3)), \]
\[ \pi^*(I_1 (i(\xi_1, d\omega_1)) - \pi^*(d\eta_1) \land \pi^* \eta_1 = \pi^*(I_2 (i(\xi_2, d\omega_2)) - \pi^*(d\eta_2) \land \pi^* \eta_2 \]
\[ = \pi^*(I_3 (i(\xi_3, d\omega_3))) - \pi^*(d\eta_3) \land \pi^* \eta_3, \]
where $\omega_r$ denotes the fundamental form of the almost contact structure $(I_r, \xi_r, \eta_r, g)$. Moreover, the HKT structure is strong if and only if
\[ d(\pi^*(I_r (i(\xi_r, d\omega_r)))) = 0, \]
\[ d(\pi^*(I_r (d\omega_r - d\eta_r \land \eta_r))) = (\pi^*(-I_r (i(\xi_r, d\omega_r))) + \pi^* \Omega) \land \pi^* \Omega, \]
for every $r = 1, 2, 3$.

**Proof.** The almost hyper-Hermitian structure $(J_1, J_2, J_3, h)$ on $P$, defined by (2) and (4), is hyper-Hermitian if and only $(I_r, \xi_r, \eta_r, g)$ is normal and $d\theta$ is $I_r$-invariant for every $r = 1, 2, 3$. The HKT condition is equivalent to (57). By (59) we have
\[ J_r dF_r = \pi^*(I_r (d\omega_r)) + \pi^*(I(i(\xi_r, d\omega_r)) \land \theta - \pi^*(d\eta_r) \land \pi^* \eta_r - \theta \land d\theta, \]
where $F_r$ is the fundamental 2-form of $(J_r, h)$. Therefore, the condition (57) is satisfied if and only if (59) holds. Finally, $J_r dF_r$ are closed forms if and only if (40) holds.

\[ \Box \]
Consider on $N^{4n+3} \times \mathbb{R}$ the almost Hermitian structures $(J_r, F_r, h)$ defined by
\begin{equation}
    h = g + (dt)^2, \quad F_r = \omega_r + \eta_r \wedge dt,
\end{equation}
and
\begin{equation}
    J_r(\eta_r) = dt, \quad J_r(X) = I_r(X), \quad X \in \text{Ker} \eta_r.
\end{equation}
Moreover, by (38) we have:
\begin{align*}
    J_1J_2 = J_3 &= -J_2J_1, \\
    J_1\eta_2 = I_1\eta_2 &= -\eta_3, \\
    J_2\eta_3 = I_2\eta_3 &= -\eta_1, \\
    J_3\eta_1 = I_3\eta_1 &= -\eta_2.
\end{align*}
Therefore $(J_r, F_r, h)$, $r = 1, 2, 3$, is a hyper-Hermitian structure on $N^{4n+3} \times \mathbb{R}$ if and only if the structures $(I_r, \xi_r, \eta_r)$ for $r = 1, 2, 3$ are normal.

**Corollary 7.2.** Let $N^{4n+3}$ be a $(4n + 3)$-dimensional manifold endowed with three normal almost contact metric structures $(I_r, \xi_r, \eta_r)$, $r = 1, 2, 3$. Consider on the product $N^{4n+3} \times \mathbb{R}$ the hyper-Hermitian structure $(J_1, J_2, J_3, h)$ defined by (41) and (42). Then, $(J_1, J_2, J_3, h)$ is HKT if and only if
\begin{equation}
    d\eta_1 \wedge \eta_1 = d\eta_2 \wedge \eta_2 = d\eta_3 \wedge \eta_3,
\end{equation}
and one of the following conditions is satisfied:
\begin{enumerate}
    \item[(a)] $d\omega_r = 0$ for any $r = 1, 2, 3$, i.e. $(I_r, \xi_r, \eta_r)$ is quasi-Sasakian for any $r = 1, 2, 3$ or
    \item[(b)] $d\omega_i \wedge \eta_j \wedge \eta_k \neq 0$, where $(i, j, k)$ is a permutation of $(1, 2, 3)$, and
\end{enumerate}
\begin{align*}
    I_1(d\omega_1) &= I_2(d\omega_2) = I_3(d\omega_3), \\
    I_1(i_{\xi_1}d\omega_1) &= I_2(i_{\xi_2}d\omega_2) = I_3(i_{\xi_3}d\omega_3).
\end{align*}
In the case (a) the HKT structure is strong. In the case (b) the HKT structure is strong if and only if
\begin{equation}
    d(I_1(d\omega_1)) = d(I_1(i_{\xi_1}d\omega_1)) = 0.
\end{equation}

**Proof.** By Theorem 7.1 the hyper-Hermitian structure $(J_r, F_r, h)$, $r = 1, 2, 3$, is HKT if and only if
\begin{equation}
    I_1(d\omega_1) = I_2(d\omega_2) = I_3(d\omega_3),
\end{equation}
\begin{align*}
    I_1(i_{\xi_1}d\omega_1) &= d\eta_1 \wedge \eta_1 = I_2(i_{\xi_2}d\omega_2) = d\eta_2 \wedge \eta_2 = I_3(i_{\xi_3}d\omega_3) = d\eta_3 \wedge \eta_3.
\end{align*}
Let us express locally
\begin{equation}
    d\omega_r = \alpha_r + \sum_{i=1}^{3} \beta_i^r \wedge \eta_i + \sum_{i<j=1}^{3} \gamma_{ij}^r \wedge \eta_i \wedge \eta_j + \rho_r \eta_1 \wedge \eta_2 \wedge \eta_3,
\end{equation}
where $\alpha_r$, $\beta_i^r$ and $\gamma_{ij}^r$ are 3-forms, 2-forms and 1-forms respectively in $\bigcap_{i=1}^{3} \text{Ker} \eta_i$ and $\rho_r$ are smooth functions.

By using the normality of the three almost contact metric structures, and then that $i_{\xi_r}d\eta_r = 0$ and $I_r(d\eta_r) = d\eta_r$, we can write locally:
\begin{align*}
    d\eta_1 &= A_1 + B_1 \wedge \eta_2 - I_1B_1 \wedge \eta_3 + C_1 \eta_2 \wedge \eta_3, \\
    d\eta_2 &= A_2 + B_2 \wedge \eta_1 + I_2B_2 \wedge \eta_3 + C_2 \eta_1 \wedge \eta_3, \\
    d\eta_3 &= A_3 + B_3 \wedge \eta_1 - I_3B_3 \wedge \eta_2 + C_3 \eta_1 \wedge \eta_2,
\end{align*}
where $I_rA_r = A_r$, $A_r$ and $B_r$ are 2-forms and 1-forms respectively in $\bigcap_{i=1}^{3} \text{Ker} \eta_i$ and $C_r$ are smooth functions.
We have
\[ J_r(\omega) = J_r(\omega) + J_r(d\xi \wedge dt) = J_r(\omega) - d\eta \wedge \eta. \]
Therefore, by using (44) and (45), we obtain
\[
J_1(\omega) = I_1 \alpha_1 + I_1 \beta_1 \wedge dt - A_1 \wedge \eta_1 - I_1 \beta_2 \wedge \eta_2 - I_1 \beta_3 \wedge \eta_3
- I_1 \gamma_1 \wedge \eta_2 \wedge dt + I_1 \gamma_1 \wedge \eta_3 \wedge dt + B_1 \wedge \eta_1 \wedge \eta_2 - I_3 \gamma_1 \wedge \eta_1 \wedge \eta_3 + I_1 \gamma_2 \wedge \eta_3 \wedge dt - C_1 \eta_1 \wedge \eta_2 \wedge \eta_3,
\]
\[
J_2(\omega) = I_2 \alpha_2 + I_2 \beta_2 \wedge dt - I_2 \beta_3 \wedge \eta_1 - A_2 \wedge \eta_2 + I_2 \beta_2 \wedge \eta_3 + I_2 \gamma_2 \wedge \eta_1 \wedge \eta_3 - B_2 \wedge \eta_1 \wedge \eta_2 + I_2 \gamma_2 \wedge \eta_1 \wedge \eta_3 + I_2 \gamma_2 \wedge \eta_3 \wedge dt + C_2 \eta_1 \wedge \eta_2 \wedge \eta_3,
\]
\[
J_3(\omega) = I_3 \alpha_3 + I_3 \beta_3 \wedge dt - I_3 \beta_3 \wedge \eta_1 - I_3 \beta_3 \wedge \eta_2 - A_3 \wedge \eta_3 + I_3 \gamma_3 \wedge \eta_1 \wedge \eta_2 + I_3 \gamma_3 \wedge \eta_2 \wedge dt + C_3 \eta_1 \wedge \eta_2 \wedge \eta_3 + I_3 \gamma_3 \wedge \eta_3 \wedge dt - C_3 \eta_1 \wedge \eta_2 \wedge \eta_3.
\]

The conditions (45) are satisfied if and only if
\begin{align*}
\gamma_1 & = \gamma_2 = \gamma_3 = 0, & \rho_r & = 0, & C_1 & = -C_2 = C_3, \\
I_1 \alpha_1 & = I_2 \alpha_2 = I_3 \alpha_3, & I_1 \beta_1 & = I_2 \beta_2 = I_3 \beta_3, \\
A_1 & = I_2 \beta_3 = -I_3 \beta_2, & A_2 & = I_1 \beta_1 = I_3 \beta_1, & A_3 & = I_1 \beta_2 = -I_2 \beta_2, \\
B_1 & = -B_2 = I_3 \gamma_1, & -I_1 B_1 & = -B_3 = I_2 \gamma_1, & I_2 B_2 & = I_3 B_3 = I_1 \gamma_1.
\end{align*}

(46)

Since \( I_r A_r = A_r \) we obtain that \( \beta_r = 0 \) for \( r, i = 1, 2, 3 \), with \( r \neq i \). The last three equations in (46) are satisfied if and only if \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \).

Thus, finally, we obtain:
\[
d\omega_r = \alpha_r + \beta_r \wedge \eta_r, \quad d\eta_i = \lambda \eta_j \wedge \eta_k,
\]
for any even permutation of \( (1, 2, 3) \).

Now, the expression for \( d(J_1 \omega) \) is the following:
\[
d(J_1 \omega) = d(I_1(\omega_1) + I_1(i \xi_1 \omega_1) \wedge dt) - d((d\eta) \wedge \eta)
= d(I_1(\omega_1)) + d(I_1(i \xi_1 \omega_1)) \wedge dt - d\eta \wedge d\eta
= d(I_1(\omega_1)) + d(I_1(i \xi_1 \omega_1)) \wedge dt - \lambda^2 \eta_2 \wedge \eta_3 \wedge \eta_2 \wedge \eta_3
= d(I_1(\omega_1)) + d(I_1(i \xi_1 \omega_1)) \wedge dt,
\]
and thus the HKT structure is strong if and only if \( d(I_1(\omega_1)) = 0 \) and \( d(I_1(i \xi_1 \omega_1)) = 0 \).

In particular, in the case (a) we have that \( d(J_r \omega) = 0 \), for any \( r \) and then the HKT structure is strong. \( \square \)
Example 7.3. Consider the 7-dimensional Lie group $G = SU(2) \ltimes \mathbb{R}^4$ with structure equations

$$
\begin{align*}
&d\mathbb{e}^1 = - \frac{1}{2} \mathbb{e}^{25} - \frac{1}{2} \mathbb{e}^{36} - \frac{1}{2} \mathbb{e}^{47}, \\
&d\mathbb{e}^2 = \frac{1}{2} \mathbb{e}^{15} + \frac{1}{2} \mathbb{e}^{37} - \frac{1}{2} \mathbb{e}^{46}, \\
&d\mathbb{e}^3 = \frac{1}{2} \mathbb{e}^{16} - \frac{1}{2} \mathbb{e}^{27} + \frac{1}{2} \mathbb{e}^{45}, \\
&d\mathbb{e}^4 = \frac{1}{2} \mathbb{e}^{17} + \frac{1}{2} \mathbb{e}^{26} - \frac{1}{2} \mathbb{e}^{35}, \\
&d\mathbb{e}^5 = \mathbb{e}^{67}, \\
&d\mathbb{e}^6 = - \mathbb{e}^{57}, \\
&d\mathbb{e}^7 = \mathbb{e}^{56}.
\end{align*}
$$

By $[\mathbf{13}]$ $G$ admits a compact quotient $M^7 = \Gamma \backslash G$ by a uniform discrete subgroup $\Gamma$ and it is endowed with a weakly generalized $G_2$-structure. Moreover, by $[\mathbf{3}]$ $M^7 \times S^1$ admits a strong HKT structure. We can show that $M^7$ has three normal almost contact metric structures $(I_r, \xi_r, \eta_r, g)$ for $r = 1, 2, 3$ given by

$$
\begin{align*}
&I_1 \mathbb{e}^1 = \mathbb{e}^2, \quad I_1 \mathbb{e}^3 = \mathbb{e}^4, \quad I_1 \mathbb{e}^5 = \mathbb{e}^6, \quad \eta_1 = \mathbb{e}^7, \\
&I_2 \mathbb{e}^1 = \mathbb{e}^3, \quad I_2 \mathbb{e}^2 = - \mathbb{e}^4, \quad I_2 \mathbb{e}^5 = - \mathbb{e}^7, \quad \eta_2 = \mathbb{e}^6, \\
&I_3 \mathbb{e}^1 = \mathbb{e}^4, \quad I_3 \mathbb{e}^2 = \mathbb{e}^3, \quad I_3 \mathbb{e}^6 = \mathbb{e}^7, \quad \eta_3 = \mathbb{e}^5,
\end{align*}
$$

satisfying the conditions (a) of Corollary 7.2.

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