Twisted Parafermions

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A new type of nonlocal currents (quasi-particles), which we call twisted parafermions, and its corresponding twisted Z-algebra are found. The system consists of one spin-1 bosonic field and six nonlocal fields of fractional spins. Jacobi-type identities for the twisted parafermions are derived, and a new conformal field theory is constructed from these currents. As an application, a parafermionic representation of the twisted affine current algebra $A_2^{(2)}$ is given.

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The introduction of the $Z_k$ parafermions [20,21] in the context of statistical models and conformal field theory [22] is perhaps one of the most significant conceptual advances in modern theoretical physics. From field theory point of view, parafermions generalize the Majorana fermions and have found important applications in superstring theory [23], fractional supersymmetry and fractional superstring [24]. In a very recent work by Maldacena, Moore and Seiberg, $D$-branes were constructed with the help of the $Z_k$ parafermions [25]. From statistical physics point of view, parafermions are related to the exclusion statistics introduced by Haldane [26]. In particular, the $Z_k$ parafermion models offer various extensions of the Ising model which corresponds to the $k = 2$ case [1]. Other examples include the 3-state Potts model ($k = 3$) [23] and the Ashkin-Teller model ($k = 4$) [1]. Parafermions also have applications in Quantum Hall Effects [27], Bose-Einstein Condensates [28] and Quantum Computation [14].

The category for parafermions (nonlocal operators) is the generalized vertex operator algebra $V$ [29]. The $Z_k$ parafermion algebra was referred to as $Z$-algebra in [20,21], and the $Z_k$ parafermions are canonically modified $Z$-algebras acting on certain quotient spaces $A_1^{(1)}$-modules defined by the action of an infinite cyclic group. It was proved that the $Z$-algebra is identical with the $A_1^{(1)}$ parafermion. The $Z_k$ parafermion characters and their singular vectors were studied in [17].

The importance of parafermions inspired many researchers to study various extensions of the $Z_k$ parafermions which are basically related to the simplest $A_1^{(1)}$ algebra. Gepner proposed a parafermion algebra associated with any given untwisted affine Lie algebra $G^{(1)}$ [30,31], which has been subsequently used in the study of $D$-branes. The operator product expansions (OPEs) and the corresponding $Z$-algebra of the untwisted parafermions were studied in [32,33].

In this paper, we find a new type of nonlocal currents (quasi-particles), which will be referred to as twisted parafermions. The system contains a bosonic spin-1 field and six nonlocal fields with fractional spins. Some of the fields are in the Ramond sector and some are in the Neveu-Schwarz (NS) sector. They correspond to a new type of parafermions. The system contains a bosonic spin-1 field and six nonlocal fields with fractional spins. Jacobi-type identities for the twisted parafermions are derived, and a new conformal field theory is constructed from these currents. As an application, a parafermionic representation of the twisted affine current algebra $A_2^{(2)}$ is given.

It is well-known that Euclidian correlation functions are defined only if operators in the correlators are time-ordered [30]. In the radial picture, $|z| > |w|$ means that $z$ is later than $w$. In the Euclidian functional integral definition of correlation functions, the time ordering is automatic. So, in Euclidian field theory the operator products $A(z)B(w)$ are only defined for $|z| > |w|$. Therefore the radial ordering is implied throughout this paper.

Now we propose the twisted parafermion current algebra:

$$\psi_l(z)\psi_l(w)(z-w)^{l'/2k} = \frac{\delta_{l+l'/0}}{(z-w)^2} + \frac{\epsilon_{l,l'}}{z-w}\psi_{l+l'}(w) + \cdots,$$

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\[ \psi_I(z) \psi_J(w)(z - w)^{l''/2k} = \frac{\delta_{I+\hat{I},0}}{(z-w)^2} + \frac{\epsilon_{I+\hat{I}}}{z-w} \psi_I(w) + \cdots, \]
\[ \psi_I(z) \psi_J(w)(z - w)^{l''/2k} = \frac{\epsilon_{I+\hat{I}}}{z-w} \psi_I(w) + \cdots, \]

where \( l, l' = \pm 1 \) and \( \hat{I}, \hat{I}' = 0, \pm 1, \pm 2; \epsilon_{I,I'}, \epsilon_{I,\hat{I}} \) and \( \epsilon_{I,\hat{I}'} \) are structure constants. If we denote \( \psi_I \) or \( \psi_J \) by \( \Psi_a \), then we can rewrite the above relations as:

\[ \Psi_a(z) \Psi_b(w)(z - w)^{ab/2k} = \sum_{n=-\infty}^{\infty} (z-w)^n [\Psi_a \Psi_b]_{-n}, \]

where \( a, b = 0, \pm 1, \pm 1, \pm 2 \). So we have \([\Psi_a \Psi_b]_1 = 0 \) (if \( g \geq 3 \)), \([\Psi_a \Psi_b]_2 = \delta_{a+b,0} \) and \([\Psi_a \Psi_b]_3 = \epsilon_{a,b} \Psi_{a+b} \). For consistency, \( \epsilon_{a,b} \) must have the properties: \( \epsilon_{a,b} = -\epsilon_{b,a} = -\epsilon_{a+b} = \epsilon_{-a,a+b} \) and \( \epsilon_{a,-a} = 0 \). Due to the mutually semilocal property between two parafermions, the radial ordering products are multivalued functions. So we define the radial ordering product of (generating) twisted parafermions (TPFs) as

\[ \Psi_a(z) \Psi_b(w)(z - w)^{ab/2k} = \Psi_b(w)\Psi_a(z)(w - z)^{ab/2k}, \]

which, like the untwisted case, is an extension of the fermion (i.e. \( ab = 1, k = 1 \)) and boson (i.e. \( k \to \infty \)) theories.

For every field in the parafermion theory there are a pair of charges (\( \lambda, \bar{\lambda} \)), which take values in the weight lattice. We denote such a field by \( \phi_{\lambda,\bar{\lambda}}(z, \bar{z}) \). The OPE of \( \Psi_a \) with \( \phi_{\lambda,\bar{\lambda}}(z, \bar{z}) \) is given by

\[ \Psi_a(z) \phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \sum_{m=-\infty}^{\infty} (z-w)^{-m-1-a\lambda/2k} A^a_{m,\lambda} \phi_{\lambda,\bar{\lambda}}(w, \bar{w}), \]

where \( m \in \mathbb{Z} \) (Ramond sector) for \( a = l \) and \( m \in \mathbb{Z} + \frac{1}{2} \) (Neveu-Schwarz sector) for \( a = \tilde{l} \). The action of \( A^a_{m,\lambda} \) on \( \phi_{\lambda,\bar{\lambda}}(z) \) is defined by the integration

\[ A^a_{m,\lambda} \phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \oint_{c_w} dz (z-w)^{m+a\lambda/2k} \Psi_a(z) \phi_{\lambda,\bar{\lambda}}(w, \bar{w}), \]

where \( c_w \) is a contour around \( w \). Consider the difference of integrals

\[ \oint_{c_w} du \oint_{c_w} dz \Psi_a(u) \Psi_b(z) \phi_{\lambda,\bar{\lambda}}(w, \bar{w}) (u-w)^{-p+1+ab/2k} \times (u-w)^{m+q+1+a\lambda/2k} (z-w)^{n-p+q+b\lambda/2k}, \]

along two contours satisfying \( |u-w| > |z-w| \) and \( |u-w| < |z-w| \), respectively. The difference of the two contour integrals can be expressed as a two-fold integration of a \( u \)-contour enclosing \( z \) once followed by a \( z \)-contour enclosing \( w \) once. Properly carrying out the Taylor expansion of \((u-w)^z\), we then obtain the so-called twisted \( Z \)-algebra relations,

\[ \sum_{l=0}^{\infty} C^{(l)}_{p-1+ab/2k} A^a_{m-l-p+q} A^b_{n+l+p-q} + (1)^p A^a_{n-l-q-1} A^b_{m+l+q+1} \]

\[ = C_{m+q+1+a\lambda/2k} \delta_{a,b} \delta_{m,-n} + C_{m+q+1+p+2k} \varepsilon_{a,b} A^a_{m+n} \]

\[ + \sum_{r=0}^{\infty} C_{m+q+1+ab/2k} [\Psi_a \Psi_b]_{-r,m+n}, \]

where \( p = 2q \) or \( 2q + 1 \) and \([\Psi_a \Psi_b]_{-m,n} \) is defined by

\[ [\Psi_a \Psi_b]_{-m,n} \phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \oint_{c_w} dz (z-w)^{m+n+1+(a-b)\lambda/2k} [\Psi_a \Psi_b]_{-m}(z) \phi_{\lambda,\bar{\lambda}}(w, \bar{w}). \]

Let \( A_a \) and \( B_b \) be two arbitrary functions of the twisted parafermions with charges \( a \) and \( b \), respectively. These fields are local \((a \text{ or } b = 0)\) or semilocal \((a \text{ and } b \neq 0)\). The OPEs can be written as
\[A_a(z)B_b(w)(z - w)^{ab/2k} = \sum_{n=-[h_A+h_B]}^\infty [A_aB_b]_{-n}(w)(z - w)^n, \] (9)

where \([h_A]\) stands for the integral part of the dimension of \(A\). Hence we have \([A_aB_b]_{-n}(w) = \oint w \, d \zeta \, A_a(z)B_b(w)(z - w)^{n-1+ab/2k}\). It is easy to find the following relation between the three-fold radial ordering products

\[
\left\{ \oint w \, d \zeta \oint w \, d \zeta \oint w \, d \zeta \right\} \left( R(A(u)R(B(z)C(w)))
- \oint w \, d \zeta \oint w \, d \zeta \oint w \, d \zeta \right) \left( (-)^{ab/2k} R(B(z)R(A(u)C(w)))
- \oint w \, d \zeta \oint w \, d \zeta \oint w \, d \zeta \right) \left( R(A(u)B(z)C(w)) \right) \right\} \\
(z - w)^{p-1+bc/2k}(u - w)^{q-1+ac/2k}(u - z)^{r-1+ab/2k} = 0,
\] (10)

where integers \(p, q, r\) are in the regions: \(-\infty < p \leq [h_B + h_C], -\infty < q \leq [h_C + h_A], -\infty < r \leq [h_A + h_B] \); and \(a, b, c\) are parafermionic charges of the fields \(A, B\) and \(C\) respectively. The above equation is an extension of \(A(BC) - B(AC) - [A, B]C = 0\). Performing the binomial expansion, we obtain the following twisted Jacobi-like identities:

\[
\left[ \begin{array}{c} [h_B + h_C] \\ i \\ p \end{array} \right] \sum_{i=p}^{[h_B + h_C]} C_r^{(i-p)} [A^{(BC)}]_r Q-i(w) + (-)^r \left[ \begin{array}{c} [h_C + h_A] \\ j \\ q \end{array} \right] \sum_{j=q}^{[h_C + h_A]} C_r^{(j-q)} [B^{(AC)}]_r Q-j(w)
= \sum_{i=r}^{[h_B + h_C]} (-)^{i-r} C_r^{(i-r)} Q-r(w),
\] (11)

where \(Q = p + q + r - 1\), \(C_r^{(i)} = (-)^{i} \frac{(x-1)(x-2)\ldots(x-i+1)}{i!}\) and \(C_r^{(0)} = C_r^{(1)} = 1, C_r^{(p)} = 0\), for \(l > p > 0\). This identity will be used extensively for our purpose. Performing analytic continuation one obtains

\[
[B^A]_r(w) = \sum_{i=r}^{[h_B + h_C]} \frac{(-)^i}{(t-r)!} \partial^{t-r}[AB]_r(w).
\] (12)

We remark that \(A, B, C\) can be any composite operators and we can calculate any coefficient in the OPEs from the fundamental equation (4).

For the twisted parafermion theory to be a conformal field theory, we require that the spin-2 terms in the OPEs contain an energy-momentum tensor. It is obvious that the spin-2 terms in the OPE are \([\Psi_a\Psi_b]_0\). Since the parafermionic charge for the energy-momentum tensor should be zero, so the relevant terms are \([\Psi_a\Psi_b]_0\). Note that \([\Psi_a\Psi_{- \alpha}]_0(z) = [\Psi_{- \alpha}\Psi_a]_0(z)\), we calculate the OPE of \([\Psi_a\Psi_{- \alpha}]_0\) with \(\Psi_a\) and \(\Psi_b\Psi_{- \alpha}\). Setting \(Q = p = 2, q = 1\) and \(r = 0\) in (11), we have

\[
[[\Psi_a\Psi_{- \alpha}]_0\Psi_b]_2 = \delta_{a,b}\Psi_a + \varepsilon_{a,b}\varepsilon_{-a,-a+b}\Psi_b + (1 + a^2/2k)\delta_{a,-b}\Psi_{- \alpha} + \frac{ab}{4k}(1 - \frac{ab}{2k})\Psi_b.
\] (13)

From the general theory of conformal field theory [3], the conformal dimension of \(\Psi_a\) should be \(1 - \frac{a^2}{4k}\). So we impose the constraints:

\[
\sum_a \varepsilon_{a,b}\varepsilon_{-a,-a+b} = \frac{6-b^2}{k}, \quad \sum_a ab = 0, \quad \sum_a (ab)^2 = 12b^2.
\] (14)

One solution to these constraints is given by \(\varepsilon_{1,2} = \varepsilon_{1,3} = \varepsilon_{-1,2} = \sqrt{3}k\) and \(\varepsilon_{1,-1} = \varepsilon_{0,1} = \sqrt{3/2k}\). Then we have

\[
\sum_a [[\Psi_a\Psi_{- \alpha}]_0\Psi_b]_2 = \frac{2k+6}{k}(1 - \frac{k^2}{4k})\Psi_b.
\] Choose a proper normalization and write

\[
T_\Psi = \frac{k}{2k+6} \sum_a [[\Psi_a\Psi_{- \alpha}]_0, \ (15)
\]


Then $[T \psi \psi_b]_2 = \left(1 - \frac{b^2}{4k}\right) \psi_b$. Repeating the above process, we obtain

\[
[[\psi_a \psi_{-a}, \psi_b]]_1 = \frac{1}{2} \delta_{a,b} \psi_{-a} + \frac{1}{2} \delta_{a,b} \partial \psi_b + (1 + a^2/2k) \delta_{a,b} \partial \psi_a + (1 + a^2/2k) \delta_{-a,b} \partial \psi_{-a},
\]

(16)
or equivalently $[T \psi \psi_b]_1 = \partial \psi_b$. These results can be written in the form of OPEs

\[
T \psi(z) \psi_b(w) = \frac{1 - b^2/4k}{(z-w)^4} \psi_b(w) + \frac{1}{z-w} \partial \psi_b(w) + \ldots.
\]

(17)

It follows that the conformal dimension of the twisted parafermion is 1, $1 - \frac{1}{4k}$ or $1 - \frac{1}{k}$, for a given level $k$. Carrying out a similar program for $T$, we obtain the OPE:

\[
T \psi(z) T \psi(w) = \frac{c_{tpf}/2}{(z-w)^4} + \frac{2T \psi(w)}{(z-w)^2} + \frac{\partial T \psi(w)}{z-w} + \ldots,
\]

(18)

where $c_{tpf} = 7 - \frac{24}{k+3} = \frac{8k}{k+3} - 1$ is the central charge of the twisted parafermion theory.

One of the applications of the twisted parafermionic currents is that they give a representation of the twisted affine current algebra $A^{(2)}$. Introduce eight currents:

\[
\begin{align*}
J^+(z) &= 2\sqrt{k} \psi_1(z) e^{\frac{i}{\sqrt{8k}} \phi_0(z)}, \\
J^-(z) &= 2\sqrt{k} \psi_{-1}(z) e^{-\frac{i}{\sqrt{8k}} \phi_0(z)}, \\
J^0(z) &= 2\sqrt{2k} i \delta \phi_0(z), \\
J^+(-z) &= 2\sqrt{k} \psi_{-2}(z) e^{i\sqrt{\frac{k}{2}} \phi_0(z)}, \\
J^-(z) &= 2\sqrt{k} \psi_2(z) e^{-i\sqrt{\frac{k}{2}} \phi_0(z)}, \\
J^0(z) &= 2\sqrt{2k} i \delta \phi_0(z),
\end{align*}
\]

where $\phi_0$ is an $U(1)$ current obeying $\phi_0(z) \phi_0(w) = -\ln(z-w)$. Then it can be checked that the above currents satisfy the OPEs of the twisted affine currents algebra $A^{(2)}$.

In summary, we have found a new type of nonlocal currents (quasi-particles) and corresponding twisted $Z$-algebra. We derive the Jacobi-type identities for the twisted parafermion currents. These Jacobi-type identities are expected to be useful in proving some of the Ramanujan identities, which play an important role in statistical physics. From the twisted parafermions, we construct a new conformal field theory, and give a parafermionic representation of twisted current algebra $A^{(2)}$. This representation is expected to be useful in the description of entropy of the $AdS_3$ black hole.

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**Note Added:** P. Mathieu pointed to us the reference [25], where graded parafermions associated with the $osp(1|2)$ algebra were studied and fields with conformal dimensions of $1 - \frac{1}{4k}$ and $1 - \frac{1}{k}$ also appeared. However, our twisted parafermion algebra is quite different from the graded parafermion algebra in [25]. Our theory contains more fields, is unitary and has a different central charge.

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