DEFORMATIONS OF SINGULAR MINIMAL HYPERSURFACES I, ISOLATED SINGULARITIES

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Abstract. Locally stable minimal hypersurface could have singularities in dimension $\geq 7$ in general, locally modeled on stable and area-minimizing cones in the Euclidean spaces. In this paper, we present different aspects of how these singularities may affect the local behavior of minimal hypersurfaces.

First, given a non-degenerate minimal hypersurface with strictly stable and strictly minimizing tangent cone at each singular point, under any small perturbation of the metric, we show the existence of a nearby minimal hypersurface under new metric. For a generic choice of perturbation direction, we show the entire smoothness of the resulting minimal hypersurface.

Second, given a strictly stable minimal hypersurface $\Sigma$ with strictly minimizing tangent cone at each singular point, we show that $\Sigma$ is uniquely homologically minimizing in its neighborhood.

Lastly, given a family of stable minimal hypersurfaces multiplicity 1 converges to a stable minimal hypersurface $\Sigma$ in an eight manifold, we show that there exists a non-trivial Jacobi field on $\Sigma$ induced by these converging hypersurfaces, which generalizes a result by Ben Sharp in smooth case; Moreover, positivity of this Jacobi field implies smoothness of the converging sequence.

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1. Introduction

Minimal hypersurface has been frequently studied in recent years in dimension between 2 and 6 through Almgren-Pitts min-max theory, see for example [MN14, MN16, MN17, MNS19, IMN18, Son18].
In higher dimensions, the minimal hypersurfaces given by min-max constructions are known to have codimension \( \geq 7 \) singular set, locally modeled on stable minimal hypercones in the Euclidean spaces, see [SS81]. Two natural questions are then to study the structures of singular set, as well as how they affect the "behavior" of minimal hypersurfaces. In a measure theoretic sense, the former question was addressed in [AST00, Sim93a, Sim93b, Whi97, CN13, NV20], providing a natural stratification of singular set together with a quantitative volume estimate, curvature estimate and rectifiability for each stratum.

In this article, we study the latter question above in different aspect.

**Local Moduli.** [Whi91, Whi17] studied the space \( \Gamma \) of Riemannian metrics and smooth minimal surfaces pairs \((g, \Sigma)\), which admits a natural Banach manifold structure and makes the projection onto the first variable a Fredholm mapping with index 0.

When \( \Sigma \) is allowed to have singularities, no local structure result of \( \Gamma \) is known in the literature. In fact, when \( g \) is Euclidean, possible singular structures for minimal surfaces in a neighborhood of \( \Sigma \) near singular set of \( \Sigma \) was studied in [CHS84, SS86, CES17, ES19], for certain classes of singularity models. And it’s shown that nearby minimal surface can have different singularity type from \( \Sigma \), making it hard to study the local structure of \( \Gamma \) using parametric approach.

In this paper, we shall address the following local existence problem,

**Question:** Given a closed singular minimal hypersurfaces \( \Sigma_0 \) in a Riemannian manifold \((M, g_0)\) and a smooth perturbed metric \( g \) of \( g_0 \), is there always at least one, possibly singular, hypersurface \( \Sigma \subset M \) minimal with respect to \( g \) and staying close to \( \Sigma_0 \)?

Our first result provide partial answers to the question above when the singular set are strongly isolated.

**Definition 1.1.** Call \( p \in \text{Sing}(\Sigma) \) strongly isolated, if the tangent cone of \( \Sigma \) at \( p \) is of multiplicity 1 and singular only at \( \emptyset \).

**Theorem 1.2.** Suppose \( \Sigma \subset (M, g) \) be an optimal regular, locally stable, two-sided minimal hypersurface with only strongly isolated singularities. Also suppose that

1. \( \Sigma \) is nondegenerate
2. the tangent cone at each singularity is strictly minimizing.
3. the tangent cone at each singularity is strictly stable

Then for every \( \epsilon > 0 \), there’s some \( \delta > 0 \) depending on \( \Sigma, M, g, \epsilon \), such that if \( g' \) is a smooth metric with \( \|g' - g\|_{C^0} \leq \delta \), then there exists a hypersurface \( \Sigma' \subset M \) minimal, locally stable under \( g' \) with optimal regularity lying in the \( \epsilon \)-neighborhood of \( \Sigma \) and satisfying

\[
P(\Sigma', \Sigma) \leq \epsilon
\]

The notions of nondegeneracy, index, strict stability and strictly minimizing property are discussed in section 2 and 3. Note that the nondegeneracy condition is a generic condition, see proposition 3.5 (6). Also note that theorem 1.2 fails if we drop the assumption (2), see proposition 8.1.

Examples of minimal hypercones have been constructed in [HLJ71, FKM81, FK85], many of which has been verified to be area-minimizing by [Law72, Sim74, Law91, WS94]. It’s also worth mentioning that all examples of area-minimizing hypercones in the literature above are proved to be strictly stable and strictly minimizing, see [WS94, TZ20].

Unlike its proof in case \( \text{Sing}(\Sigma) = \emptyset \), theorem 1.2 is not derived by fix point method when singularities exist. In fact, unlike in the smooth setting, the uniqueness of such nearby minimal hypersurfaces under a fixed perturbed metric is not known, even when tangent cones are Simons cones. See the problem (P4.1) in section 3.

**Generic Regularity.** When \( \Sigma \) is the unique homologically area-minimizer in some homology class of \( M \), theorem 1.2 is automatic by a compactness argument. By appropriately choosing
the perturbed metric \( g' \), [Sma93] was able to establish the smoothness of homologically area-minimizer for \( C^{\infty} \)-generic metric in dimension 8; Combined with a similar perturbation argument locally, [CLS20] used min-max approach to show that in generic Riemannian 8-manifold with positive Ricci curvature, there exists a closed embedded smooth minimal hypersurface. We prove a local analogue of these regularity theorems in section \( 8 \).

**Theorem 1.3.** Let \( \Sigma \subset (M, g) \) be as in theorem \( 1.2 \) satisfying (1)-(3).

Then for \( C^k \)-generic \( f \in C^\infty(M) \) on \( M \) \((k \geq 4)\), if \( \{g_t = g(1 + tf) + o_4(t)\}_{t \in (-1, 1)} \) is a family of smooth Riemannian metric near \( g \), then for each sufficiently small \( t \), there’s a closed embedded smooth minimal hypersurface \( \Sigma_t \subset (M, g_t) \), and \( \Sigma_t \to \Sigma \) in varifold sense when \( t \to 0 \).

Here \( o_4(t) \) means some symmetric 2-tensor converges to 0 in \( C^4 \) after dividing by \( t \).

- **Induced Jacobi Fields.** The proof of regularity in theorem \( 1.3 \) is based on a general result deriving the smoothness of \( \Sigma_t \) from the behavior of Jacobi field they induce on \( \Sigma \).

Recall that by [Sha17], if \( \Sigma_j \) are a family of distinct minimal hypersurfaces in \((M, g)\) multiplicity 1 converges to a smooth minimal hypersurface \( \Sigma_\infty \), then the convergence is in \( C^\infty \) graphical sense and the normalization of graphical functions sub-converges to a smooth Jacobi field on \( \Sigma \), called an induced Jacobi field on \( \Sigma \). As an analogue we have,

**Theorem 1.4.** Let \( \{\Sigma_j, \nu_j\}_{1 \leq j \leq \infty} \subset (M, g) \) be a family of two-sided stable minimal hypersurfaces. Suppose \( \Sigma_j \) multiplicity 1 converges to \( \Sigma_\infty \) and \( \Sigma_\infty \) has only strongly isolated singularities.

Then after passing to a subsequence, \( \{\Sigma_j\}_{j \geq 1} \) induces a Jacobi field \( u \in C^\infty(\Sigma) \), i.e. \( (\Delta_\Sigma + |A_\Sigma|^2 + \text{Ric}_M(\nu_\infty, \nu_\infty))u = 0 \) on \( \Sigma \).

Moreover, if \( u > 0 \) near some singularity \( p \in \text{Sing}(\Sigma_\infty) \), then for infinitely many \( j \gg 1 \), \( \Sigma_j \) are smooth near \( p \).

We emphasis that here no further assumption is required on singularities of \( \{\Sigma_j\} \). In particular, the theorem is always true in 8 manifolds. In a upcoming paper joint with Yangyang Li, a slightly generalized version of this theorem is used to get the existence of closed embedded smooth minimal hypersurface in arbitrary 8-manifolds for generic metric, which generalizes the results in [CLS20].

The index of a singular minimal hypersurface is introduced in [Dey19], based on which, theorem \( 1.4 \) also holds when replacing the stability assumption on \( \{\Sigma_j\}_{1 \leq j \leq \infty} \) by that they all have the same index. Certain metric variations are also allowed. See theorem \( 1.2 \) for the precise statement.

Along the way to theorem \( 1.3 \), the first step is to look at minimal hypersurfaces asymptotic to a cone near infinity. This object was studied in [BDGG69, HSS83, SSS80, Wli89, WS94, Cha97, Maz13, ES19], most focusing on minimizing hypersurfaces and hypercones. In section \( 3.1 \) we prove the following generalization of [HSS5] in stable case, which is itself interesting.

**Theorem 1.5.** Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a stable minimal hypersurface with optimal regularity asymptotic to a regular minimal hypercone \( C \subset \mathbb{R}^{n+1} \) near infinity. If write \( \Sigma \) as the graph over \( C \) of some function \( u \) outside a large ball, then we have

\[
\liminf_{R \to +\infty} R^{-2\gamma^- - n} \int_{A_{R, 2R}} u^2 > 0
\]

where \( A_{R, 2R} \) is the annulus in \( C \) and \(-1 < \gamma^- \leq -(n-2)/2\) are some geometric constant depending on \( C \), defined in section \( 2.2 \).

Moreover, if \( \Sigma \) lies on one side of \( C \) near infinity, then \( \text{Sing}(\Sigma) = \emptyset \) and it’s contained entirely in that side and foliate the component of \( \mathbb{R}^{n+1} \setminus C \) containing it by rescaling.

- **Local Minimizing Property.** A natural question by Lawlor [Law91] is that given a (strictly) minimizing hypercone \( C \subset \mathbb{R}^{n+1} \), is every minimal hypersurface \( \Sigma \) with \( C \) to be the tangent cone at some singularity \( p \) minimizing in some neighborhood of \( p \)? When \( C \) is strictly stable or the asymptotic rate of \( \Sigma \) to \( C \) near \( p \) is strictly greater than 1, [HSS5, theorem 4.4] confirmed this question.
We get a positive answer in this article to a more general version of the question under stability assumption.

**Theorem 1.6.** Suppose $\Sigma \subset (M, g)$ be a stable minimal hypersurface with only strongly isolated singularities satisfying (1) and (2) in theorem 1.2.

Then there's some small neighborhood $U$ of $\text{Clos}(\Sigma)$ which admits a mean convex foliation. More precisely, there's a family of piecewise smooth mean convex neighborhood $\{U_t\}_{t \in (0,1)}$ of $\text{Clos}(\Sigma)$ such that

1. $U_{t_1} \subset U_{t_2}$ if $t_1 \leq t_2$
2. $U = \bigcup_{t \in (0,1)} U_t$, $\text{Clos}(\Sigma) = \bigcap_{t \in (0,1)} U_t$; $U_s = \bigcup_{t < s} U_t$; $\text{Clos}(U_s) = \bigcap_{t > s} U_t$, $\forall s \in (0,1)$.

In particular, $\Sigma$ is homologically area-minimizing in $U$ and is the unique stationary varifold in $U$ up to multiplicity.

For a precise description of mean convexity of piecewise smooth domain, see theorem 5.1.

Similar local minimizing property was proved by a contradiction argument in [Sma99, section 3, lemma 4] assuming that each tangent cone are strictly stable and strictly minimizing, and that $\Sigma$ is a minimal cone in the Euclidean space near each singularity; By a calibration argument, [Zha18] obtained similar local minimizing result for singular minimal submanifolds of higher codimension with Lawlor’s cones in [Law91] to be the tangent cone, and with specifically chosen metrics away from singular points.

**Organization of the Article.** Section 2 contains the basic notations and assumption we use in this article. We also present a brief review of analysis on singular minimal surfaces, geometric features of minimal cones and asymptotic conical minimal hypersurfaces. They will be frequently used throughout this article.

In section 3, we study the analytic behavior of the linearized minimal surfaces operator, namely, the Jacobi operator $L_{\Sigma}$, on a singular minimal hypersurface $\Sigma$. The key property is the $L^2$-compactness of a family of functions with uniformly bounded $L^2$-norm and uniformly bounded quadratic form associated to $L_{\Sigma}$. Similar compactness has been proved by Schoen-Yau using a uniform $L^2$-nonconcentration property and coercivity in [SY17] for the operators associated to the specific quadratic form of minimal slicings and by Lohkamp using the associated skin structure in [Loh19] for the $S$-adapted Schrödinger operators. In fact, their argument is sufficient to obtain the desired compactness if each tangent cone is strictly stable. A brief review of Schoen-Yau’s argument in [SY17] is contained in section 2.4.

For a general $\Sigma$, the coercivity of $L_{\Sigma}$ might fail. Hence, we establish another $L^2$-compactness in section 3.1 and 3.2 when $\Sigma$ has strongly isolated singularities, which allows us to still get spectral decomposition of $L_{\Sigma}$ and define the Green’s function. We also derive an asymptotic estimate of Green’s function near singularities in section 3.3 which is used in section 5 to deduce certain mean convexity of one-sided perturbations of minimal hypersurfaces.

Section 4 studies the induced Jacobi fields of a family of converging stable minimal hypersurfaces. In particular, we prove some more general version of theorem 1.4 and 1.5, relating the asymptotic rate of induced Jacobi fields with specific blow-up limit of the converging sequence near each singularity. In particular, we show that the induced Jacobi fields are actually fall in some specific function space described in section 3.

In section 5, we prove theorem 1.6. The key is an existence of minimal graph over compact subset of regular part with weighted estimate, see theorem 5.3. Note that this estimate doesn’t require the tangent cone to be minimizing.

In section 6, inspired by [Whi94], we prove an analogue of strong min-max property for $\Sigma$ in theorem 1.2 in sufficiently small neighborhood $U$. The proof closely follows the argument in [Whi94], except that we have to estimate error terms near singularities using similar argument as [Sma99]. This allows us to construct an admissible family of sweepouts under any perturbed metric, with width close to the volume of $\Sigma$. Then by applying the min-max theory for obstacle
problem in [Wan20], for any slightly perturbed metric $g'$, there’s a constrained embedded minimal hypersurface $\Sigma' \subset (\text{Clos}(U), g')$.

To conclude that $\Sigma'$ doesn’t intersect $\partial U$ for certain small $U$ and $g$ sufficiently close to $g_0$, we prove in section 7 a local rigidity result for constrained embedded minimal hypersurfaces, as an analogue of [Wan20, Theorem 5.1].

In section 8 we complete the proof of theorem 1.2 and theorem 1.3. In the end, we list some open problems related with singular minimal hypersurfaces.

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2. Preliminaries

Throughout this article, let $n \geq 3$, $(M, g)$ will be an $n+1$ dimensional closed oriented Riemannian manifold. We may assume that $(M, g)$ isometrically embedded in some $\mathbb{R}^L$ if necessary. Write

- $\text{Int}(A)$ the interior of a subset $A \subset M$
- $\text{Clos}(A)$ the closure of a subset $A \subset M$
- $\text{dist}_M$ the intrinsic distant function on $M$
- $B^M_r(p)$ the open geodesic ball in $M$ of radius $r$ centered at $p$
- $B^M_r(E) := \bigcup_{p \in E} B^M_r(p)$, the $r$ neighborhood of $E \subset M$
- $A^M_{s,M}(p) := B^M_r(p) \setminus \text{Clos}(B^M_s(p))$ the open geodesic annuli in $M$
- $\mathcal{H}^k$ $k$-dimensional Hausdorff measure
- $\nabla_M$ the Levi-Civita connection on $M$
- $\exp^M$ the exponential map of $M$
- $\text{inj}(M)$ the intrinsic injectivity radius of $M$
- $dvol_M$ the unit volume form on $M$
- $\text{Ric}_M$ the Ricci curvature tensor of $M$

For an open subset $U \subset M$, let

- $\mathcal{X}^c(U)$ the space of compactly supported vector fields in $U$
- $e^{tV}$ the one-parameter family of diffeomorphism generated by $V \in \mathcal{X}^c(U)$
- $\mathcal{D}^k(U)$ the space of compactly supported smooth $k$-forms in $U$
For fixed $p \in M$, we usually identify the tangent space $T_pM$ with $\mathbb{R}^{n+1}$ and view $B_{\text{inj}(M)}(p) \hookrightarrow \mathbb{R}^{n+1}$ by $\exp^M$. In an $n + 1$ dimensional Euclidean space $\mathbb{R}^{n+1}$, write

- $\mathbb{B}_r^{n+1}(p)$ the open ball of radius $r$ centered at $p$
- $\overline{B}_r(p)$ the sphere of radius $r$ centered at $p$
- $\mathcal{A}_{s,r}^{n+1}(p)$ the open annuli $\mathcal{B}_s(p) \setminus \text{Clos}(\mathcal{B}_r(p))$ centered at $p$
- $\eta_{p,r}$ the map between $\mathbb{R}^{n+1}$, maps $x$ to $(x - p)/r$
- $G(V,k)$ the Grassmannian of $k$ dimensional unoriented linear subspace in a real vector space $V$
- $\omega_k$ the volume of $k$-dimensional unit ball

$\theta^k(x,\mu)$ the $k$-th density (if exists) of a Radon measure $\mu$ at $x$, i.e. $\lim_{r \to 0} \frac{1}{\omega_k r^k} \mu(\mathbb{B}_r^{n+1}(x))$

For $E \subset \mathbb{R}^{n+1}$, we sometimes write for simplicity $\frac{1}{r}(E - p) := \eta_{p,r}(E)$

Let $\Sigma \subset U$ be a two-sided smooth hypersurface with normal field $\nu$ (extended to its neighborhood by $\nabla^M \nu = 0$). Throughout this article, unless stated otherwise, every hypersurface will be two-sided and have optimal regularity, i.e. $\mathcal{H}^{n-2}(U \cap \text{Clos}(\Sigma) \setminus \Sigma) = 0$ and $\mathcal{H}^n \Sigma$ is locally finite.

Let

- $\text{Reg}(\Sigma) := \{ x \in \text{Clos}(\Sigma) \cap U : \text{Clos}(\Sigma) \text{ is smooth, embedded hypersurface near } x \}$
- $\text{Sing}(\Sigma) := U \cap \text{Clos}(\Sigma) \setminus \text{Reg}(\Sigma)$

By adding points to $\Sigma$ if necessary, we may identify $\Sigma = \text{Reg}(\Sigma)$ for simplicity. Call $\Sigma$ closed in $U$, if $\text{Clos}(\Sigma) \subset \subset U$; Call $\Sigma$ regular in $U$, if $\text{Sing}(\Sigma) = \emptyset$.

The Fermi coordinate of $M$ near $\Sigma$ is a parametrization of some neighborhood of $\Sigma$ in $M$ by a sub-domain $\Omega \subset \Sigma \times \mathbb{R}$, $\Omega \ni (x,z) \mapsto \exp^M_x(z \cdot \nu(x)) \in M$

The Riemannian metric on $\Sigma$ is induced from $(M,g)$. We usually omit the subscript for differential operators $\nabla^\Sigma$, $\text{div}^\Sigma$, $\nabla^\Sigma_k$ and $\Delta^\Sigma_k$ etc. of $\Sigma$. Also write

- $\text{dist}_\Sigma$ the intrinsic distant on $\text{Clos}(\Sigma)$
- $B_r(p) := \{ y \in \Sigma : \text{dist}_\Sigma(p,y) < r \}$ the intrinsic open ball in $\Sigma$, $p \in \text{Clos}(\Sigma)$
- $H^\Sigma := -\text{div}^\Sigma(\nu) \cdot \nu = H^M \cdot \nu$, the mean curvature vector of $\Sigma$
- $A^\Sigma := -\nabla \nu$, the 2nd fundamental form of $\Sigma$

Recall that $\Sigma$ is called minimal if $H^\Sigma = 0$.

A minimal hypersurface $(\Sigma,\nu) \subset M$ is called stable in $U$ if

$$\frac{d^2}{ds^2} |_{s=0} \mathcal{H}^n(e^{sX}(\Sigma)) \geq 0 \quad \forall X \in \mathcal{X}_c(U)$$

By [SSS1], this is equivalent to $\mathcal{H}^{n-7+\epsilon}(\text{Sing}(\Sigma)) = 0, \forall \epsilon > 0$ and that

$$Q_\Sigma(\phi,\phi) := \int_{\Sigma} |\nabla \phi|^2 - (|A^\Sigma|^2 + \text{Ric}_M(\nu,\nu))\phi^2 \geq 0 \quad \forall \phi \in C^2(\Sigma \cap U)$$

Let

$$L_\Sigma := \Delta + |A^\Sigma|^2 + \text{Ric}_M(\nu,\nu)$$

be the operator associated to $Q_\Sigma$, called Jacobi operator. Any solution $u \in C^2(\Sigma)$ to $L_\Sigma u = 0$ is called a Jacobi field on $\Sigma$. 
For \( E \subset \Sigma \) and \( u : E \to \mathbb{R} \), let
\[
\text{graph}_{\Sigma,g}(u) := \{ e^{x^M_g (u(x) \cdot \nu(x))} : x \in E \}
\]
be the graph of \( u \) over \( \Sigma \) under metric \( g \), and we shall omit \( g \) if there’s no confusion. If \( E \) is a smooth domain, for sufficiently small \( u \), \( \text{graph}_{\Sigma,g}(u) \) is a smooth hypersurface, oriented by taking normal field having positive inner product with \( \nu \). Let \( \mathcal{A}_\Sigma \) be the Euler-Lagrangian operator associated to the area function of \( \text{graph}_{\Sigma,g}(u) \), see section 2.3 for precise definition and discussion.

2.1. Currents, varifolds and convergence. We recall the basic notions of currents and varifolds and refer the readers to [Sim83b] and [Fed69] for details. For \( 1 \leq k \leq n+1 \), let
\[
I_k(M) \quad \text{the space of integral } k \text{ currents on } M
\]
\[
\mathcal{Z}_k(M) := \{ T \in I_k(M) : \partial T = 0 \}
\]
\[
\mathcal{V}_k(M) \quad \text{the closure of } \mathcal{Z}_k(M) \text{ under varifold weak topology}
\]
\[
\mathcal{IV}_k(M) \quad \text{the space of integral } k\text{-varifolds on } M
\]
If \( \Sigma \hookrightarrow M \) is an immersed \( k \)-dimensional submanifold with finite volume, then let \( |\Sigma| \in \mathcal{IV}_k(M) \) be the associated integral varifolds. If further \( \Sigma \) is oriented, let \( |\Sigma| \in I_k(M) \) be the associated integral current. For \( T \in I_k(M) \), let \( |T| \) and \( \|T\| \) be its associated integral varifold and Radon measure correspondingly.

For \( U \subset M \) open, let \( \mathcal{F}_{U,g} \) and \( \mathcal{M}_{U,g} \) be the flat metric and mass norm on \( I_k(U) \) with respect to metric \( g \). We may omit the subscript \( g \) if there’s no confusion and the subscript \( U \) if \( U = M \). Let \( \mathcal{F}_g \) be the varifolds metric on \( \mathcal{V}_k(M) \), i.e. the metric compatible with weak converges. The \( \mathcal{F}_g \)-metric on integral currents is defined to be \( \mathcal{F}_g(S,T) := \mathcal{F}_g(|S|,|T|), S,T \in I_k(M) \).

Let \( \text{spt}(V) \) be the support of \( V \in \mathcal{V}_k(M) \); \( \text{spt}(T) := \text{spt}(|T|) \) for \( T \in I_k(M) \).

For a proper Lipschitz map \( f \), \( f_* \) be the push forward of varifolds or currents. For \( E \subset M \) measurable, \( T \in I_k(M) \) and \( V \in \mathcal{V}_k(M) \), \( T \circ E \) and \( V \circ E \) be the restrictions onto \( E \).

Let \( T \in \mathcal{Z}_k(M) \), \( E \supset \text{spt}T \). Call \( T \) \textbf{homologically area minimizing} in \( E \) if for all \( W \subset M \),
\[
\mathcal{M}_W(T + \partial P) \geq \mathcal{M}_W(T) \quad \forall P \in I_{k+1}(M) \text{ with } \text{spt}(P) \subset E \cap W
\]

For \( U \subset M \) open, \( V \in \mathcal{V}_k(U) \) and \( \mathcal{X}_c(U) \), let
\[
\delta V(X) = \frac{d}{dt} \bigg|_{t=0} \mathcal{M}(t^2 \mathcal{I}_k) = \int \text{div}^\pi X \, dV(x, \pi)
\]
be the first variation of \( V \), where \( \text{div}^\pi X = \sum \langle \nabla^M_{e_i} X, e_i \rangle \), \( \{e_i\} \) be an orthonormal basis of \( \pi \in \mathcal{G}(T_x M, k) \). Call \( V \) stationary in \( U \) if \( \delta V(X) = 0 \) for every \( X \in \mathcal{X}_c(U) \).

2.2. Geometry of hypercones. Let \( C \) be a hypercone in \( \mathbb{R}^{n+1} \) with normal field \( \nu := \partial \mathbb{B}^{n+1} \cap C \) be the cross section of \( C \). Call \( C \) a \textbf{regular cone}, if \( S \) is a closed smooth hypersurface in \( \mathbb{S}^n \), in which case \( \text{Sing}(C) = \{0\} \). Let \( A_S \) be the second fundamental form of \( (S, \nu|_S) \) in \( \mathbb{S}^n \). Parametrize \( C \) by
\[
(0, +\infty) \times S \to C, \quad (r, \omega) \mapsto x := r\omega
\]

Now suppose \( C \subset \mathbb{R}^{n+1} \) be a regular stable minimal hypercone. Then by [Sim68], the Jacobi operator of \( C \) is decomposed w.r.t. this parametrization,
\[
L_C = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} (\Delta_S + |A_S|^2)
\]
Let \( L_S := \Delta_S + |A_S|^2 \), \( \mu_1 < \mu_2 \leq \ldots \), \( +\infty \) be the eigenvalues of \( -L_S \) on \( L^2(S) \) and \( w_1, w_2, \ldots \) be a corresponding family of \( L^2 \)-orthonormal eigenfunctions, \( w_1 > 0 \). By the stability of \( C \),
\[
\mu_1 \geq -\left( \frac{n-2}{2} \right)^2
\]
Following [CHS84], call \( C \) \textbf{strictly stable}, if \( \mu_1 > \left( \frac{n-2}{2} \right)^2 \).
By [CHSS4] section 1, for every $0 \leq R_1 < R_2 \leq +\infty$, general solution of $L_C u = f$ on $C \cap A_{R_1, R_2}(0)$ is given by

\begin{equation}
(2.4) \quad u(r, \omega) = \sum_{k \geq 1} (u_k(r) + v_k(r))w_k(\omega)
\end{equation}

where

\begin{equation}
(2.5) \quad u_k(r, f) := \begin{cases}
    \frac{1}{2\pi k} \int_0^1 s - \gamma^+ f_k(s) \, ds - r^{\gamma^+} \int_0^1 s - \gamma^- f_k(s) \, ds, & b_k \neq 0; \\
    r^{-\frac{n-2}{2}} \left( \log r \int_0^r s^\frac{n-2}{2} f_k(s) \, ds - \int_0^r s^\frac{n-2}{2} \log s f_k(s) \, ds \right), & b_k = 0.
\end{cases}
\end{equation}

be a special solution,

\begin{equation}
fk(r) := \int_S f(r, \omega)w_k(\omega) \, d\omega
\end{equation}

be the Fourier coefficients of $f$; and

\begin{equation}
v_k(r) = v_k(r; \gamma^+_k, \gamma^-_k) := \begin{cases}
    c^+_k r^{\gamma^+_k} + c^-_k r^{\gamma^-_k} = r^{-\frac{n-2}{2}} (c^+_k r^{b_k} + c^-_k r^{-b_k}), & b_k \neq 0; \\
    c^+_k + c^-_k \log r, & b_k = 0.
\end{cases}
\end{equation}

be the general homogeneous solutions, $c^+_k \in \mathbb{R}$; $\gamma^+_k = -\frac{n-2}{2} \pm b_k$ be the two solutions of

\begin{equation}
(2.7) \quad \gamma^2 + (n-2)\gamma - \mu_k = 0
\end{equation}

and $b_k = \sqrt{(\frac{n-2}{2})^2 + \mu_k}$. For later reference, write $\Gamma \subset \{ \gamma_j^\pm : j \geq 1 \}$ and $\forall \gamma \in \Gamma$, write $W_\gamma := \text{span}_k \{ w_j : \gamma^\pm_j = \gamma \}$ be the eigenspace of $L_S$.

We list some corollary of this expansion formula. In what follows, $B_r := C \cap B^{n+1}_r$, $A_{r,s} = C \cap A^{n+1}_r$.

**Lemma 2.1.** For each $\phi \in C^\infty_0(C)$ we have,

\[ \int_C |\nabla_C \phi|^2 - |A_C|^2 \phi^2 \geq (\mu_1 + (\frac{n-2}{2})) \int_C \frac{1}{r^{n-2}} \phi^2 \]

**Proof.** Write $\phi(r, \omega) = \sum_{k \geq 1} \phi_k(r)w_k(\omega)$, plug in the integration and change variables. The lemma follows from the Hardy inequality

\[ \int_0^{+\infty} 4a'(r)^2 - a(r)^2/r^2 \, dr \geq 0 \quad \forall a \in C^1_0(0, +\infty) \]

\[ \square \]

**Lemma 2.2.** If $\gamma \in R \setminus \Gamma$, $u$ is a Jacobi field on $B_R$ (resp. $C \setminus B_R$). Suppose

\[ I^2_\gamma(r) := \int_{A_{r,2r}} u^2(t, \omega) t^{-n} \, dx \]

satisfies that for some $r_0 < R$ (resp. $r_0 > R$) and $C > 0$,

\begin{equation}
(2.8) \quad I^2_\gamma(r) \leq Cr^{2\gamma}, \quad \forall \ r < r_0 \quad (\text{resp. } \forall \ r > r_0)
\end{equation}

Then

\[ u(r, \omega) = \sum_{k \geq 1} v_k(r; c^+_k, c^-_k)w_k(\omega) \]

where $c^+_k = 0$ if $\gamma^+_k < \gamma$ (resp. $\gamma^+_k > \gamma$); $v_k(r; c^+_k, c^-_k)$ is defined in (2.7).
Proof. We only prove for Jacobi fields on $B_R$ since the proof for Jacobi fields on $C \setminus B_R$ is similar. By (2.4),
\[
v_k(r; c_+^k, c_-^k) = \int_S u(r, \omega) w_k(\omega) \, d\omega \leq (\int_S u(r, \omega)^2 \, d\omega)^{1/2}
\]
Hence, for all $r \in (0, r_0)$,
\[
\int_r^{2r} t^{-1} v_k(t; c_+^k, c_-^k)^2 \, dt \leq \int_r^{2r} t^{-1} dt \int_S u(t, \omega)^2 \, d\omega \leq C(n) I_2^r = C r^{2\gamma + 1}
\]
And then by the expression (2.6) of $v_k(r; c_+^k, c_-^k)$, $c_+^k = 0$ if $\gamma_+^k < \gamma$.

Combined with Harnack inequality and that $w_k$ changes signs when $k \geq 2$, the following lemma is a corollary of lemma 2.2 and the proof is left to readers. See also [Whi80, Lemma 4.2].

**Lemma 2.3.** If $u$ is a positive Jacobi field on $B_R$ (resp. $C \setminus B_R$), then
\[
u(r, \omega) = v_1(r) w_1(\omega) + \sum_{k \geq 2} c_k r^{\gamma_+^k} w_k(\omega)
\]
( resp. $u(r, \omega) = v_1(r) w_1(\omega) + \sum_{k \geq 2} c_k r^{\gamma_-^k} w_k(\omega)$)
for some $c_k \in \mathbb{R}$, where $v_1$ is defined in (2.6) for some $c_1^\pm$ such that $v_1 > 0$ on $B_R$ (resp. on $C \setminus B_R$).

In particular, if $u$ is a positive Jacobi field on $C$, then
\[
u = \begin{cases} (c^+ r^{\gamma_+^1} + c^- r^{\gamma_-^1}) w_1, & \text{if } \gamma_+^1 > \gamma_-^1 \\
(1 - c^+ r^{\gamma_+^1} + c^- r^{\gamma_-^1}) w_1, & \text{if } \gamma_+^1 = \gamma_-^1
\end{cases}
\]
for some $c^\pm, c \geq 0$.

(2.4) also guarantees the existence of solutions with prescribed growth or decaying bound, which is analogous to [CHSS4, Theorem 1.1]. We start with some notations.

For $\sigma \in \mathbb{R} \setminus \Gamma_C$, define $\Pi_\sigma^\pm$ (resp. $\Pi_\sigma^\pm$) : $L^2(S) \to L^2(S)$ to be the orthogonal projection onto $\bigoplus_{\gamma \geq \sigma} W_{\gamma_+}^+$ (resp. $\bigoplus_{\gamma < \sigma} W_{\gamma_-}^-$). Note that $1 - \Pi_\sigma^\pm$ are the projection onto their $L^2$-orthogonal complement; when $\sigma < \gamma_1^+$ (resp. $\sigma > \gamma_1^-$), $\Pi_\sigma^+$ = id (resp. $\Pi_\sigma^- = id$).

For $E \subset C$ and $f \in L^2_{loc}(E)$, define
\[
\|f\|_{L^2(E)}^2 := \int_E |f(x)|^2 \cdot |x|^{-2\sigma - n} \, dx
\]
Also define for $k \geq 1$, $\|f\|_{W^{2,k}(E)}^2 := \sum_{j=0}^k \|\nabla^j f\|_{L^2(E)}^2$. $L^2(S)$ and $W^{2,k}(E)$ will be the space of functions with finite $\| \cdot \|_{L^2(S)}$ and $\| \cdot \|_{W^{2,k}(E)}$ norm respectively.

**Lemma 2.4.** Suppose $\sigma \in \mathbb{R} \setminus \Gamma_C$; $\varphi, \psi \in L^2(S)$; $f \in L^2_{-2}(B_1)$ (resp. $f \in L^2_{-2}(C \setminus B_1)$). Then there’s a unique $u \in L^2_{loc}(E)$ (resp. $u \in L^2_{loc}(C \setminus B_1)$) such that
\[
\begin{cases}
L^\sigma_C u = f & \text{on } B_1 \\
\Pi_\sigma^+ \varphi & \text{on } \partial B_1 \\
(1 - \Pi_\sigma^-) \partial_t u = (1 - \Pi_\sigma^-) \psi & \text{on } \partial B_1
\end{cases}
\]
Moreover, $u$ satisfies the estimate
\[
\sup_{t \in (0,1)} \|u(t, \cdot)\|_{L^2(S)} \cdot t^{-\sigma} + \|u\|_{L^2(S)} \leq C(C, \sigma)(\|f\|_{L^2_{-2}(B_1)} + \|\varphi\|_{L^2(S)} + \|\psi\|_{L^2(S)})
\]
( resp. $\sup_{t \in (1, +\infty)} \|u(t, \cdot)\|_{L^2(S)} \cdot t^{-\sigma} + \|u\|_{L^2(S)} \leq C(C, \sigma)(\|f\|_{L^2_{-2}(C\setminus B_1)} + \|\varphi\|_{L^2(S)} + \|\psi\|_{L^2(S)})$ )
The proof is almost the same as \cite{HSS1} Theorem 1.1 and is left to readers.

The following perturbed version of lemma 2.4 is used in section 5.

**Corollary 2.5.** Suppose \( \sigma \in \mathbb{R} \setminus \Gamma_C \). Then there exists \( \varepsilon = \varepsilon(C, \sigma) > 0 \) such that, if \( f \in L^2_{\sigma-2}(B_1) \); \( \mathcal{R} \) is a second order differential operator satisfying

\[
\| \mathcal{R}u \|_{L^2_{\sigma-2}} \leq \varepsilon \| u \|_{W^{2,2}} \quad \forall u \in W^2_{\sigma}(u)
\]

Then there exists a unique solution \( u \in W^2_{\sigma}(B_1) \) to the equation

\[
\begin{cases}
L_C u + \mathcal{R}u = f & \text{on } B_1 \\
\Pi^+ u = 0 & \text{on } \partial B_1 \\
(1 - \Pi^-) \partial_t u = 0 & \text{on } \partial B_1
\end{cases}
\]

Moreover, \( u \) satisfies the estimate

\[
\sup_{t \in (0, 1)} \| u(t, \cdot) \|_{L^2(S)} \cdot t^{-\sigma} + \| u \|_{L^2_I} \leq C(C, \sigma) \| f \|_{L^2_{\sigma-2}(B_1)}
\]

**Proof.** Apply Banach fixed point theorem on \( T : W^2_{\sigma}(B_1) \to W^2_{\sigma}(B_1) \), \( v = Tu \) solving

\[
\begin{cases}
L_C v = f - \mathcal{R}u & \text{on } B_1 \\
\Pi^+ v = 0 & \text{on } \partial B_1 \\
(1 - \Pi^-) \partial_t v = 0 & \text{on } \partial B_1
\end{cases}
\]

Combine lemma 2.4 and \( L^2 \) estimate for elliptic PDE to show that \( T \) is a contraction map provided \( \varepsilon \) is small enough.

To prove the finiteness of associated Jacobi field in section 4, we need the following growth rate monotonicity.

**Lemma 2.6.** Let \( \sigma \in \mathbb{R} \setminus \Gamma_C \). Then there exists \( K_0 = K_0(\sigma, C) > 2 \) such that if \( 0 \neq u \in L^2_{\sigma}(C \setminus B_1) \) is a Jacobi field, then

\[
J^u_{\sigma}(r, s) := \int_{A_{r,s}} u^2(t, \omega) t^{-\alpha - 2\sigma} \, dt
\]

satisfies \( J^u_{\sigma}(Kr, K^2r) < J^u_{\sigma}(r, Kr) \), \( \forall r > 1, \forall K \geq K_0 \).

**Proof.** By the same argument as lemma 2.2, \( u = \sum_{k \geq 1} v_k(r; c_k^+, c_k^-) w_k(\omega) \), where \( c_k^\pm = 0 \) if \( \gamma_k^\pm > \sigma \).

And then,

\[
J^u_{\sigma}(r, Kr) = \int_r^{Kr} \int_S t^{-\alpha - 2\sigma} v_k(t; c_k^+, c_k^-)^2 \, dt \, d\omega = \sum_{k \geq 1} \int_r^{Kr} t^{-\alpha - 2\sigma} v_k(t; c_k^+, c_k^-)^2 \, dt
\]

Hence, to show the lemma, it suffices to verify that for each \( k \geq 1 \),

\[
\int_r^{Kr} t^{-\alpha - 2\sigma} v_k(t; c_k^+, c_k^-)^2 \, dt < \int_r^{Kr} t^{-\alpha - 2\sigma} v_k(t; c_k^+, c_k^-)^2 \, dt
\]

provided \( c_k^\pm \) do not vanish simultaneously. If one of \( \gamma_k^\pm \) is greater than \( \sigma \), then the corresponding \( c_k^\pm \) vanishes and (2.12) immediately follows from (2.6). If not, (2.12) would be derived from (2.6) and the following:

**Claim 1:** suppose \( \alpha < 0, K \geq 2 \); \( I_K(r; c, c') := \int_r^{Kr} (cs^\alpha + c's^\alpha \log s)^2 s^{-1} \, ds \). Then \( \exists K_1 = K_1(\alpha) \geq 2 \) such that if \( K \geq K_1 \), then

\[
I_K(Kr; c, c') < I_K(r; c, c') \quad \forall r > 0 \text{ and } c, c' \in \mathbb{R}
\]

doesn’t vanish simultaneously

**Claim 2:** suppose \( \alpha < \beta < 0, K \geq 2 \); \( \tilde{I}_K(r; c, c') := \int_r^{Kr} (cs^\alpha + c's^\beta)^2 s^{-1} \, ds \). Then \( \exists K_2 = K_2(\alpha, \beta) \geq 2 \) such that if \( K \geq K_2 \), then

\[
\tilde{I}_K(Kr; c, c') < \tilde{I}_K(r; c, c') \quad \forall r > 0 \text{ and } c, c' \in \mathbb{R}
\]

doesn’t vanish simultaneously
We shall prove claim 1, the proof of claim 2 is similar.

**Proof of claim 1:** Claim is trivially true when \( c' = 0 \); if \( c' \neq 0 \), by the arbitrariness of \( r \), WLOG \((c,c') = (0,1)\). Then,

\[
I_K(r) = \frac{1}{2\alpha} (s^{2\alpha} (\log s)^2)^{\frac{r}{K}} - \frac{1}{2\alpha^r} (s^{2\alpha} \log s)^{\frac{r}{K}} + \frac{1}{4\alpha^r} (s^{2\alpha})^{\frac{r}{K}}
\]

and

\[
I_K(Kr) - I_K(r) = \frac{1}{2\alpha} \left( (K^{2\alpha} - 1)^2 \cdot r^{2\alpha} (\log r)^2 \right)
+ \left( - \frac{1}{\alpha} (K^{2\alpha} - 1)^2 + 4(K^{2\alpha} - 1)K^{2\alpha} \log K \right) \cdot r^{2\alpha} \log r
+ \left( \frac{1}{2\alpha^2} (2K^{2\alpha} - 1)^2 - \frac{2}{\alpha} (K^{2\alpha} - 1)K^{2\alpha} \log K + 2(2K^{2\alpha} - 1)K^{2\alpha} (\log K)^2 \right) \cdot r^{2\alpha}
\]

Hence, to see \( I_K(Kr) < I_K(r) \) for all \( r > 0 \), it suffices to show that the discriminant \( \Delta(K) \) is negative. Since for \( K > r > 1, K^{2\alpha}(\log K)^2 < r^{2\alpha} \), and the coefficient for the constant term of \( \Delta(K) \) is \( 1/\alpha^2 - 4/(2\alpha^2) < 0 \), we have \( \Delta(K) < 0 \) for \( K \geq K_1(\alpha) > 1 \). This proves the claim 1. \( \square \)

The following corollary is a quantitative version of lemma 2.6, and is proved by contradiction.

**Corollary 2.7.** Let \( \sigma \in \mathbb{R} \setminus \Gamma_C, K_0(\sigma,C) > 2 \) be as in lemma 2.6, \( K \geq K_0 \). Then \( \exists \theta_0 = \delta_0(\sigma,C,K) > 0, N = N(\sigma,C,K) \geq 2 \) such that if \( 0 \neq u \in W^{1,2}_{loc} \cap L^2(A_{1,K^{N+1}}) \) is a weak solution of

\[
\text{div}_C(\nabla_C u + b_0(x,u,\nabla_C u)) + |A_C|^2 u + b_1(x,u,\nabla_C u) = 0
\]

on \( A_{1,K^{N+1}} \), where

\[
|b_0(x,z,p)| + |b_1(x,z,p)| \leq \delta_0(|z|/|x| + |p| + J_0^\sigma(K,K^2)) \quad \forall x \in C, z \in \mathbb{R}, p \in T_z C
\]

And if further \( \sup_{2 \leq j \leq N} J_0^\sigma(K^j,K^{j+1}) \leq J_0^\sigma(K,K^2) \), then \( J_0^\sigma(K,K^2) < J_0^\sigma(1,K) \).

Now suppose \((C,\nu) \subset \mathbb{R}^{n+1} \) be a regular area-minimizing hypercone, i.e. \([C] \) is an area-minimizing current in \( \mathbb{R}^{n+1} \). Then \( \mathbb{R}^{n+1} \setminus C \) has exactly 2 connected component, denoted by \( E_{\pm} \), with \( \nu \) pointing in \( E_+ \). Recall the following existence and uniqueness theorem by Hardt and Simon.

**Theorem 2.8.** \([HS53] \) Let \( E \) be either one of the component \( E_{\pm} \). Then there exists a smooth area-minimizing hypersurface \( \Sigma \subset E \) with \( dist_{\mathbb{R}^{n+1}}(0,\Sigma) = 1, [\Sigma] = \partial P \) for some \( P \in I_{n+1}(\mathbb{R}^{n+1}) \) with \( spt(P) \subset E \).

Furthermore, \( \exists h \in C^\infty(C) \) such that \( \Sigma = \text{graph}_C(h) \) outside some large ball, and when \( R \to +\infty \), either

\[
h(R,\omega) = \begin{cases} 
 cR^\gamma w_1(\omega) + O(R^{\gamma'-\epsilon}) & \text{if } \mu > \left( \frac{n-2}{2} \right)^2 \\
 cR^\gamma \log R \ w_1(\omega) + O(R^{\gamma'-\epsilon}) & \text{if } \mu = \left( \frac{n-2}{2} \right)^2 
\end{cases}
\]

or

\[
h(R,\omega) = cR^\gamma w_1(\omega) + O(R^{\gamma'-\epsilon})
\]

where \( \gamma_{\pm} \) is defined in (2.7).

Moreover, \( \Sigma \) foliates \( E \) by rescaling. And any other area-minimizing integral n-cycle \( \partial [Q] \) with \( Q \subset E \) is a rescaling of \([E]\).

We shall prove an analogue of this regularity theorem in lemma 4.7 using a different method.

Let \( \Sigma_{\pm} \subset E_{\pm} \) be the smooth hypersurface in Theorem 2.8. Pick the normal field \( \nu_{\pm} \) on \( \Sigma_{\pm} \) by setting

\[
\pm \nabla_{\mathbb{R}^{n+1}} \left( \frac{|x|^2}{2} \right) \cdot \nu_{\pm} > 0
\]
And let \( h_{\pm} \) be the graphical functions of \( \Sigma_{\pm} \) outside a large ball on \( C \). Following [HS85], call \( C \) strictly minimizing if (2.11) holds for \( h_{\pm} \). Equivalent definitions of strict minimizing is discussed in [HS85] Theorem 3.2.

### 2.3. Minimal hypersurface near singularity

Let \( \Sigma \subset (M, g) \) to be a locally stable minimal hypersurface with normal field \( \nu, p \in \text{Sing}(\Sigma) \). Suppose that \( V \) is a tangent varifold of \( |\Sigma| \) at \( p \).

By [SS81] and [Ilm96, Theorem B], \( V = |C| \) for some stable minimal hypercone \( C \subset \mathbb{R}^{n+1} \). Call \( p \) a strongly isolated singularity if \( C \) is a regular cone.

Suppose now that \( C \) is regular. Identify \( C \subset T_{p}M = \mathbb{R}^{n+1} \) and \( \Sigma \cap B_{M}^{\tau}(p) \rightarrow (T_{p}M = \mathbb{R}^{n+1}, \exp_{p}g) \) by the exponential map of \( M \) at \( p \). We shall abuse the notations and still write \( \exp_{p}g \) as \( g \).

Since \( C \) is regular, by [Sim83a], \( |C| \) is the unique tangent cone of \( |\Sigma| \) at \( p \) and there’s a smooth function \( v \in C^{\infty}(C) \) such that for each integer \( j \geq 0 \)

\[
r^{j-1}|\nabla^{j}v|(r, \cdot) \to 0 \quad \text{as} \quad r \to 0
\]

and

\[
\text{graph}_{C}(v) \cap B_{M}^{\tau} = \Sigma \cap B_{M}^{\tau}
\]

For some \( r_{0} > 0 \). Parametrize \( \Sigma \cap B_{M}^{\tau} \) by

\[
\Sigma \cap B_{M}^{\tau} \rightarrow (0, +\infty) \times S \quad x \mapsto (r, \omega) = (\text{dist}_{\Sigma}(p, x), \Pi(x))
\]

and call it the conic coordinates near \( p \), where \( \Pi \) maps each \( x \) to the closest point on \( S \) from \( x/\text{dist}_{\Sigma}(p, x) \).

The proof of the following differential geometric lemma is left to readers.

**Lemma 2.9.** There exist an increasing function \( \tau = \tau_{p} : (0, 1] \rightarrow (0, +\infty) \) sufficiently small, depending on \( \Sigma, M, g, p \) such that,

1. the map in (2.16) is a diffeomorphism from \( \Sigma \cap B_{M}^{\tau}(p) \) onto its image;
2. There’s a smooth family of metrics \( \{h_{r}\}_{\tau > 0} \) on \( S \) such that the Riemannian metric on \( \Sigma \cap B_{M}^{\tau}(p) \) is \( g = dr^{2} + r^{2}h_{r} \), and \( \forall \epsilon \in (0, 1] \),

\[
\|r(h_{r} - gs)\|_{C^{2}(0, (\tau(\epsilon)) \times S)} \leq \epsilon
\]

where for a tensor \( \phi \) on \( (0, \tau) \times S \),

\[
\|\phi\|_{C^{2}(0, \tau) \times S} := \sum_{0 \leq l + j \leq k} r^{l+j-1}||\partial_{r}^{l}\nabla_{S}^{j}\phi||_{C^{0}}
\]

3. The second fundamental form of \( \Sigma \) also satisfies the estimate

\[
||r \cdot A_{\Sigma}\big|_{(r, \cdot)} - \Pi^{*}A_{S}||_{C^{0}(0, (\tau(\epsilon)) \times S)} \leq \epsilon
\]

4. The Jacobi operator of \( \Sigma \) has decomposition

\[
L_{\Sigma} = \partial^{2}_{r} + \frac{H_{r}}{r}\partial_{r} + \frac{1}{r^{2}}\mathcal{L}_{r}
\]

where \( H_{r} \) is a smooth family of functions on \( S \); \( \mathcal{L}_{r} = \partial_{j}(a^{ij}_{r}\partial_{i}) + V_{r} \) is a smooth family of self-adjoint elliptic operators on \( S \) under normal coordinate of \( S \), with

\[
||H_{r} - (n-1)||_{C^{0}(0, (\tau(\epsilon)) \times S)} + ||a^{ij}_{r} - \delta^{ij}||_{C^{0}(0, (\tau(\epsilon)) \times S)} + ||V_{r} - |A_{S}|^{2}||_{C^{0}(0, (\tau(\epsilon)) \times S)} \leq \epsilon
\]

If \( \Sigma \) only has strongly isolated singularities, let

\[
\tau_{\Sigma}(\epsilon) := \min\{\tau_{p}(\epsilon) : p \in \text{Sing}(\Sigma)\} \cup \{\text{dist}_{M}(p, q)/2 : p \neq q \in \text{Sing}(\Sigma)\}, \quad \epsilon \in (0, 1]
\]

We are interested in the behavior of nearby hypersurfaces of \( \Sigma \) given by graphs over \( \Sigma \). More precisely, let \( (\Sigma, \nu) \) be a locally stable minimal hypersurface as above with only strongly isolated singularities; \( \tau \in (0, \tau_{\Sigma}(1)/2) \) will be chosen differently from section to section, \( \rho(x) \) will be of
form $\rho(x) := \min\{\text{dist}_{\Sigma}(\text{Sing}(\Sigma), x), 2\tau\}$. For $\Omega \subset \Sigma$ and $u \in C^k(\Omega)$, we shall frequently use the following norm,

$$\|u\|_{k,\Omega}^2 := \|u\|_{C^k(\Omega)}^2 := \sum_{j=0}^{k} \sup_{\Omega} \rho^{j-1} \cdot |\nabla^j u|$$

Lemma 2.10. \exists \delta_1 = \delta(\Sigma, M, g, \tau) \in (0, 1) such that if $u \in C^1(\Sigma)$ with

$$\rho^{-1} \cdot |u| + |\nabla u| \leq \delta_1 \quad \forall x \in \Sigma$$

Then $\text{graph}(u)$ is an embedded $C^1$ hypersurface.

Moreover, if $g' = g + \beta$ is another $C^3$ Riemannian metric on $M$, let $F^{g'}(x, z, p)$ be the area integrand for graphs over $\Sigma$, i.e.

$$\text{Area}_{g'}(\text{graph}_{\Sigma,g}(\phi)) = \int_{\Sigma} F^{g'}(x, \phi(x), \nabla_{\Sigma} \phi(x)) \text{dvol}_{g^\Sigma}(x)$$

and let $\mathcal{M}^{g'}$ be the Euler-Lagrange operator, i.e.

$$\mathcal{M}^{g'} u := -\text{div}_{\Sigma}(\partial_p F^{g'}(x, u, \nabla u)) + \partial_z F^{g'}(x, u, \nabla u)$$

where $\partial_p F^{g'}$ could be written as

$$\partial_p F^{g'}(x, z, p) := (a^{ij}(x, z, p)p_i + b^{i}(x, z, p))\partial_j$$

$$\partial_z F^{g'}(x, z, p) := -(|A_\Sigma|^2 + \text{Ric}_M(\nu, \nu))z + \frac{1}{2}tr_{\Sigma} \nabla_{\Sigma}^M \beta + c(x, z, p)$$

under normal coordinates of $\Sigma$, where $a^{ij}, b^i, c \in C^1(\Sigma \times \mathbb{R} \times T, \Sigma)$. Here the derivatives and trace and all taken with respect to $g$. Then $\exists \delta_2 = \delta_2(\Sigma, M, g, \tau) \in (0, \delta_1)$, $C = C(\Sigma, M, g, \tau) > 1$ such that,

$$|a^{ij}(x, z, p) - \delta^{ij}| \leq C(||\beta||_{C^1(\Sigma)} + |p| + |z|/\rho(x))$$

$$|b^{i}(x, z, p)\partial_j - \delta_{ij}| \leq C(||\beta||_{C^1(\Sigma)} + |p| + |z|/\rho(x))^2$$

$$|c(x, z, p)| \leq C(||\beta||_{C^1(\Sigma)} + |p| + |z|/\rho(x))^2/\rho(x)$$

for $\forall (x, z, p) \in \{x \in \Sigma, |z| \leq \delta_2(\rho(x), |p| \leq \delta_2\} \subset \Sigma \times \mathbb{R} \times \Sigma$ and $\forall ||\beta||_{C^2(\Sigma)} \leq \delta_2$.

When $g' = g$, we also have estimate for higher derivatives,

Lemma 2.11. $\exists \delta_2' = \delta_2'(\Sigma, M, g, \tau) \in (0, \delta_1), C' = C'(\Sigma, M, g, \tau) > 1$ such that, if denote for simplicity $F := F^{g'}$, $\mathcal{M} := \mathcal{M}^{g'}$ and $V_2 := (|A_\Sigma|^2 + \text{Ric}_M(\nu, \nu))$ in lemma 2.10, then we have under normal coordinate of $\Sigma$,

$$|\partial_p F(x, z, p) - p| + \rho(\partial_z F(x, z, p) + V_2 z) + \rho(\partial^2_{pz} F(x, z, p)) \leq C'(|z|/\rho + |p|)^2$$

$$\rho(\partial_z F) + |\partial^2_{pzz} F - \delta_{ij}| + \rho(\partial^2_{zz} F) + \rho^2(|\partial^3_{zzz} F + V_2 z| + |\partial^3_{zz} F|)$$

$$+ |\partial^3_{ppz} F + \rho(\partial^3_{pzz} F) + \rho^2(|\partial^3_{ppzz} F| + |\partial^3_{pzz} F|)) \leq C'(|z|/\rho + |p|)$$

$$\rho(\partial^2_{ppz} F) + \rho^2(\partial^3_{pzz} F) \leq C'$$

for $\forall (x, z, p) \in \{x \in \Sigma, |z| \leq \delta_2'(\rho(x), |p| \leq \delta_2'\} \subset \Sigma \times \mathbb{R} \times T, \Sigma$.

In particular, if denote $\mathcal{M} u := -L_\Sigma u + N(x, u, \nabla u) \cdot \nabla^2 u + P(x, u, \nabla u)$, then

$$\rho(x)|P(x, z, p)| \leq C'(|p| + |z|/\rho(x))^2$$

$$|N| + |\partial_z N| + \rho(x)|\partial_z P| + \rho(x)^2|\partial_z P| \leq C'(|p| + |z|/\rho(x))$$

$$\rho(x)|\partial_z N| \leq C'$$

for $\forall (x, z, p) \in \{x \in \Sigma, |z| \leq \delta_2'(\rho(x), |p| \leq \delta_2'\} \subset \Sigma \times \mathbb{R} \times T, \Sigma$. 

Here, as a function defined on $T \Sigma \times \mathbb{R}$, $\partial_x F$ denote that derivatives in horizontal directions and $\partial_y F$ be the derivatives in vertical directions.

The calculation of both lemma are straightforward, and are left to readers.

2.4. Linear analysis of Jacobi operator. Let $(\Sigma, \nu) \subset (M, g)$ be a minimal hypersurface, $U \subset M$ be a smooth domain whose boundary transversely intersect with $\Sigma$. Let $\mathcal{U} := U \cap \Sigma$ and $\mathcal{U} := \text{Clos}(U) \cap \Sigma$. Suppose that $\Sigma$ is stable in $U$. The following result is a direct corollary of [CN13, (6.18)] by a cut-off argument,

**Lemma 2.12.** Suppose $\phi \in W^{1,2}_0(\Sigma)$ such that $\phi|_{\Sigma \cap \partial \mathcal{U}} = 0$ and that

$$\int_{\mathcal{U}} \phi^2 + |\nabla \phi|^2 < +\infty$$

Then $\exists \{\phi_j\}_{j \geq 1} \subset C_c^\infty(\mathcal{U})$ such that

$$\int_{\mathcal{U}} |\nabla (\phi - \phi_j)|^2 \to 0$$

From this lemma, we see that

$$W^{1,2}_0(\mathcal{U}) := \overline{C_c^\infty(\mathcal{U})}^{W^{1,2}} = \{ \phi \in W^{1,2}_0(\mathcal{U}) : \|\phi\|_{W^{1,2}}^2 := \int_{\mathcal{U}} \phi^2 + |\nabla \phi|^2 < +\infty, \phi|_{\partial \mathcal{U} \cap \Sigma} = 0 \}$$

In [SY17, Proposition 3.1], a uniform $L^2$ nonconcentration property is proved for minimal slicings, which is used to establish the compactness for minimal slicings. The proof also works for stable minima hypersurfaces, which gives

**Lemma 2.13.** Suppose $S \subset M$ be a subset with $\mathcal{H}^{n-1}(S) = 0$. Then for every $\epsilon > 0$, there exists a neighborhood $V_\epsilon \subset S$ in $M$, depending only on $M, g, U, S, \epsilon$, such that for every stable minimal hypersurfaces $\Sigma$ transversely intersect with $\partial \mathcal{U}$, we have

$$\int_{\Sigma \cap V_\epsilon} \phi^2 \leq \epsilon \int_{\Sigma} |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(\Sigma \cap U)$$

With this lemma, $W^{1,2}_0(\Sigma \cap U) \hookrightarrow L^2(\Sigma \cap U)$ is compact, and the first eigenvalue and eigen-function are well-defined for Schrödinger operators $L = \Delta_{\Sigma} + V$ satisfying the coercivity bound for some $\delta > 0$:

$$\int_{\Sigma} |\nabla \phi|^2 \leq \delta^{-1}(\int_{\Sigma} -\phi \cdot L \phi + \int_{\Sigma} \phi^2) \quad \forall \phi \in C_c^2(\Sigma \cap U)$$

In particular, for each $s > 0$, the perturbed Jacobi operator $L^s_{\Sigma} := L_{\Sigma} - s|A_{\Sigma}|^2$ satisfies (2.19), since

$$Q^s_{\Sigma}(\phi, \phi) := \int_{\Sigma} -\phi \cdot L^s_{\Sigma} \phi = \int_{\Sigma} |\nabla \phi|^2 - (1 - s)|A_{\Sigma}|^2 \phi^2 - Ric_{\Sigma}(\nu, \nu)\phi^2$$

$$\geq (1 - s)Q_{\Sigma}(\phi, \phi) + s \int_{\Sigma} |\nabla \phi|^2 - s\|Ric_{\Sigma}\| \int_{\Sigma} \phi^2$$

$$\geq s \int_{\Sigma} |\nabla \phi|^2 - s\|Ric_{\Sigma}\| \int_{\Sigma} \phi^2$$

where $\|Ric_{\Sigma}\|$ is the $L^\infty$ norm of $Ric_{\Sigma}$ on $M$. However, it’s not known whether (2.19) is true for Jacobi operators of stable minimal hypersurfaces in general.

It is also mentioned in [SY17] that the positive weighted functions of minimal slicings, which is the first eigenfunctions of the certain perturbed quadratic form associated to the minimal slicings, will tend to infinity near singular set. We give an alternative proof of an analogue of this fact for $L^s_{\Sigma}$ in a special case,
**Lemma 2.14.** Suppose \((M,g) = (\mathbb{R}^{n+1},|dx|^2)\) be the Euclidean space, \(\Sigma\) be a stable minimal hypersurface in a domain \(U \subset \mathbb{R}^{n+1}\) with \(\text{Sing}(\Sigma) \subset U\) being a compact subset; \(s \in [0,\frac{2}{n}]\) is fixed. Let \(0 \neq v \in C^\infty_{\text{loc}}(U)\) be a non-negative supersolution of \(L_\Sigma^q v = 0\) on \(U \cap \Sigma\), i.e.

\[
Q^e_\Sigma(v,\phi) \geq 0 \quad \forall \phi \in C^0_\infty (\Sigma \cap U, \mathbb{R}_+)
\]

Then

\[
v(x) \to +\infty \quad \text{as} \quad x \to \text{Sing}(\Sigma)
\]

**Proof.** The idea is to find an auxiliary function bounded from above by \(v\) and tending to infinity near \(\text{Sing}(\Sigma)\).

First set \(\mathcal{U} \subset \subset \text{Clos}(\Sigma) \cap U\) be a smooth domain containing \(\text{Sing}(\Sigma)\). We can assume WLOG that \(0 < v \in W^{1,2}(\mathcal{U})\) is a solution of \(L_\Sigma^{2/n} v = 0\). In fact, consider the minimizer \(\bar{v}\) of \(Q^{2/n}_\Sigma(\phi,\phi)\) among

\[
\{ \phi \in W^{1,2}(\mathcal{U}) : \phi = v \text{ on } \partial \mathcal{U}, \phi \leq v \text{ on } \mathcal{U} \}
\]

By weak and strong maximum principle, \(\bar{v} \in C^\infty_{\text{loc}}(\mathcal{U}'; \mathbb{R}_+)\) is a solution of \(L_\Sigma^{2/n} \bar{v} = 0\) with \(0 < \bar{v} \leq v\). Hence it suffices to prove the lemma for \(\bar{v}\). From now on denote \(s := 2/n\).

Recall that by [CN13], if \(\Sigma\) is area minimizing, then for every \(\epsilon \in (0,1)\), \(|A\Sigma| \in L^{7-\epsilon}\) and \(|\nabla A\Sigma| \in L^{7/2-\epsilon}\); This is also true for stable minimal hypersurface using the same proof. Hence, \(B(x) := (1 + |A\Sigma|^2)\) is a solution of \(W^{1,2}(\mathcal{U}')\).

Also, recall the following identity proposed by [Sim68],

\[
\frac{1}{2} \Delta |A\Sigma|^2 = |\nabla A\Sigma|^2 - |A\Sigma|^4
\]

and the refined Kato’s inequality for minimal hypersurfaces proposed in [SSY75],

\[
|\nabla |A\Sigma|^2| \leq \frac{n}{n+2} |\nabla A\Sigma|^2
\]

Denote for simplicity, \(A = A\Sigma\), we have

\[
L_\Sigma^q B = \Delta (1 + |A|^2) + (1 - s)|A|^2(1 + |A|^2) - (1 - s)(1 + |A|^2)(\nabla A)^2 - |A|^4 + (-s - 1)|A|^2|\nabla A|^2
\]

\[
 \geq 0
\]

In other words, \(B\) is a subsolution of \(L_\Sigma^q\).

Now fixed \(\epsilon > 0\) be sufficiently small such that \(\epsilon B < v\) near \(\partial \mathcal{U}'\). Then \(\phi := (\epsilon B - v)^+ \in W^{1,2}_0(\mathcal{U}')\) and

\[
Q^e_\Sigma(\phi,\phi) = \int_{\Sigma} \nabla \phi \cdot \nabla (\epsilon B - v) - (1 - s)|A\Sigma|^2 \phi (\epsilon B - v) = \int_{\Sigma} -\phi \cdot L_\Sigma^q (\epsilon B - v) \leq 0
\]

But since \(\Sigma\) is stable in \(U\), \(L_\Sigma^q\) is strictly positive on \(W^{1,2}_0(\mathcal{U}')\), hence \((\epsilon B - v)^+ = 0\), i.e. \(v \geq \epsilon B\).

Now we can complete the proof of lemma 2.14. For any sequence of points \(\Sigma \ni x_j \to p \in \text{Sing}(\Sigma)\), let \(r_j := \text{dist}_{\mathbb{R}^{n+1}}(x_j, p)\) and write for simplicity \(B_r(x) := \mathbb{B}_{\mathbb{R}^{n+1}}^n(x) \cap \Sigma\). Since \(\Delta v \leq L_\Sigma^q v = 0\), by applying the Harnack inequality in [BC72] to \(v\) and combining the fact that \(v \geq \epsilon B \geq \epsilon |A\Sigma|^{1-s}\), we have

\[
v(x_j) \geq C(\Sigma, n) \int_{B_{2r_j}(x_j)} v \geq C(\Sigma, n) \epsilon \int_{B_{r_j}(p)} |A\Sigma|^{1-s} \to +\infty
\]

when \(j \to \infty\). This finish the proof.

This lemma is used in section 3 to give a new proof of Hardt-Simon typed regularity.
3. Linear analysis for strongly isolated singularities

Throughout this section, let $\Sigma \subset (M, g)$ be a locally stable minimal hypersurface in a Riemannian manifold with normal field $\nu$; $U \subset M$ will always be a smooth domain with $\partial U$ transversely intersect with $\text{Reg}(\Sigma)$; $\mathcal{U} := U \cap \Sigma$ and $\bar{\mathcal{U}} := \text{Clos}(\mathcal{U}) \cap \Sigma$. Let $L_\Sigma$, $Q_\Sigma$ be the Jacobi operator and its associated quadratic form as in (2.2), (2.1). For our applications below, we may also deal with the weighted quadratic forms

$$Q_\Sigma^h(\phi, \phi) := Q_\Sigma(\phi, \phi) + \int_\Sigma h\phi^2$$

for a given $h \in L^\infty(\Sigma)$, and its associated operator $L_\Sigma - h$.

3.1. Linear analysis in large. We start with a basic observation

**Lemma 3.1.** There’s a constant $C_0 = C_0(\Sigma, M, g, U) > 0$ such that

$$Q_\Sigma(\phi, \phi) + C_0\|\phi\|_{L^2(\Sigma)}^2 \geq 0 \quad \forall \phi \in C^1_c(\mathcal{U})$$

**Proof.** Let $\{U_j \subset M\}$ be a fine finite open cover of $\text{Clos}(\mathcal{U})$ such that $\Sigma$ is stable in each $U_j$; Let $\{\eta_j^2\}$ be a partition of unity on $\{U_j\}$ such that $\eta_j$ is also smooth. Then $\forall \phi \in C^1_c(\mathcal{U} \cap \Sigma)$, since $\eta_j \cdot \phi$ is support in $U_j$,

$$Q_\Sigma(\phi, \phi) = \int_\Sigma \sum_j |\nabla \phi_j|^2 \eta_j^2 - \sum_j (|A_\Sigma|)^2 + \text{Ric}_M(\nu, \nu))\phi^2 \eta_j^2$$

$$\geq \sum_j Q_\Sigma(\phi \eta_j, \phi \eta_j) + \int_\Sigma (\nabla (\phi^2) \cdot \eta_j \nabla \eta_j + \phi^2 |\nabla \eta_j|^2)$$

(3.1)

$$\geq \int_\Sigma (\sum_j \eta_j \cdot \Delta \eta_j)\phi^2$$

Since $\Sigma$ is minimal,

$$|\Delta \eta_j| = |\Delta_M \eta_j - \nabla_M^2 \eta_j(\nu, \nu)| \leq n|\nabla_M^2 \eta_j|$$

Hence, $\sum_j \eta_j \cdot \Delta \eta_j$ is bounded from below by some constant $-C_0 = -C_0(M, g, \Sigma, U)$. This proves the lemma. $\square$

Such $C_0$ in the Lemma will be fixed throughout this section. Consider the Hilbert space $\mathcal{B}(\mathcal{U}) := C^1_c(\mathcal{U})/\|\phi\|_a$ and $\mathcal{B}_0(\mathcal{U}) := C^1_c(\bar{\mathcal{U}})^2/\|\phi\|_a$ where

$$\|\phi\|_{\mathcal{B}(\mathcal{U})}^2 := Q_\Sigma(\phi, \phi) + (C_0 + 1)\|\phi\|_{L^2(\mathcal{U})}^2$$

(3.2)

Note that by definition and lemma 3.1

$$-C_0\|\phi\|_{\mathcal{B}(\mathcal{U})}^2 \leq Q_\Sigma(\phi, \phi) \leq \|\phi\|_{\mathcal{B}(\mathcal{U})}^2 \quad \forall \phi \in C^1_c(\bar{\mathcal{U}})$$

Hence, the quadratic form $Q_\Sigma$ extends to a bilinear form on $\mathcal{B}(\mathcal{U})$.

$\mathcal{B}(\mathcal{U})$ (resp. $\mathcal{B}_0(\mathcal{U})$) will play the role as $W^{1,2}(\mathcal{U})$ (resp. $W_0^{1,2}(\mathcal{U})$) for inverting $L_\Sigma$ or finding eigenfunctions. In fact, $\mathcal{B}_0(\mathcal{U}) = W_0^{1,2}(\mathcal{U})$ provided $L = L_\Sigma$ satisfies the coercivity condition (2.19) for some $\delta > 0$. In particular, this is true when every singularity is strongly isolated and strictly stable. However, when $\Sigma$ has non-strictly stable tangent cones, this will always fail, and $W_0^{1,2} \subset \mathcal{B}_0$.

On the other hand, lemma 3.1 asserts that $\|\phi\|_a \geq \|\phi\|_{L^2}$, hence there’s a natural continuous embedding $\mathcal{B}(\mathcal{U}) \hookrightarrow L^2(\mathcal{U})$. It’s convenient to have the following characterization for $\phi \in L^2(\mathcal{U})$ to be in $\mathcal{B}(\mathcal{U})$.

**Lemma 3.2.** Suppose $\phi \in L^2(\mathcal{U})$. Then the following are equivalent,
(1) $\phi \in \mathcal{B}(U)$ (resp. $\mathcal{B}_0(U)$);
(2) $\phi \in W_{loc}^{1,2}(U)$; And for any decreasing family $V_j \supset \supset V_{j+1}$ of neighborhood of $\text{Sing}(\Sigma)$ with $\bigcap_j V_j = \text{Sing}(\Sigma)$, there are $\phi_j \in W^{1,2}(U)$ (resp. $W_0^{1,2}(U)$) such that $\text{spt}(\phi_j - \phi) \subset V_j$, $\text{spt}(\phi_j) \cap V_{N_j} = \emptyset$ for some $N_j \gg j$, and $\sup_j \|\phi_j\|_\infty < +\infty$.
(3) $\exists \phi_j \in \mathcal{B}(U) \cap L^2$ (resp. $\mathcal{B}_0(U) \cap L^2$) such that $\phi_j \to \phi$ a.e. and $\limsup \|\phi_j\|_\infty < +\infty$.

Proof. Clearly (2) implies (3); To see (3) implies (1), first observe that by weak compactness of Hilbert space and Banach-Saks theorem, if $\phi_j$ is uniformly bounded in $\mathcal{B}$, then there’s a subsequence $\phi_{k_j}$ such that $(\sum_{j=1}^m \phi_{k_j})/m \to \phi'$ in $\mathcal{B}(U)$ (resp. in $\mathcal{B}_0(U)$). Since $B \to L^2$ and that $\phi_j \to \phi$ a.e., we have $\phi = \phi' \in \mathcal{B}(U)$ (resp. $\phi \in \mathcal{B}_0(U)$).

Now we shall prove (1) implies (2) beginning with $\phi \in \mathcal{B}_0(\Sigma)$; the proof when $\phi \in \mathcal{B}(\Sigma)$ is similar.

Claim: If $\Omega \subset \subset U$, then for every $\varphi \in C^1_0(U)$,
$$
\|\varphi\|_{W^{1,2}(\Omega)} \leq C(\Omega, U, \Sigma, M, g)\|\varphi\|_{\mathcal{B}(U)}
$$

Proof of the claim: Take a finite open cover \{W_i\}_i of $\text{Clos}(U)$ in $M$ such that $\text{Clos}(\Omega) \subset W_0 \subset \subset M \setminus \text{Sing}(\Sigma)$, and that $W_i \cap \Omega = \emptyset$ and $\Sigma$ is stable in $W_i$ for $l \geq 1$. Let \{\eta_i\}_i be a partition of unity for \{W_i\} such that $\eta_l$ are also smooth. By the same computation in [3.1] we have,

$$
Q_{\Sigma}(\varphi, \varphi) = \sum_{I \geq 1} Q_{\Sigma}(\eta_l \varphi, \eta_l \varphi) + \int_{l \geq 0} \varphi^2(\eta_l \Delta_\Sigma \eta_l) \\
\geq \sum_{I \geq 0} Q_{\Sigma}(\eta_l \varphi, \eta_l \varphi) - C(\Omega, \{\eta_l\})\|\varphi\|_{L^2}^2 \geq \int_{\Omega} |\nabla \varphi|^2 - C(\Omega, \{\eta_l\})\|\varphi\|_{L^2}^2
$$

Together with the fact that $\|\varphi\|_{L^2} \leq \|\varphi\|_{\infty}$, this completes the proof of claim.

Now to prove (1) implies (2), suppose $\varphi_k \in C^1_0(U)$ be a Cauchy sequence in $\mathcal{B}_0(U)$ which converges to $\phi$. Then by the claim, $\phi \in W_{loc}^{1,2}(U)$.

For every nested neighborhood \{V_j\}_j in (2), let $\xi_j \in C^\infty_0(V_j, [0, 1])$ be a cut-off which equals 1 on $V_{j+1}$. For each fixed $j$, let $\psi_k := \psi_k^{(j)} := (1 - \xi_j)\phi + J_j \phi_k$. Then by the similar computation as in [3.1] we have,

$$
Q_{\Sigma}(\psi_k - \psi_k', \psi_k - \psi_k') = \int_{U \setminus V_{j+1}} -\xi_j \Delta_\Sigma \xi_j (\varphi_k - \varphi_k')^2 + \xi_j^2 |\nabla (\varphi_k - \varphi_k')|^2 - |\Delta_\Sigma|^2 |(\varphi_k - \nabla_\Sigma (\varphi_k - \varphi_k'))^2 | \\
\leq C(\xi_j)\|\varphi_k - \varphi_k'\|_{L^2}^2 + Q_{\Sigma}(\varphi_k - \varphi_k', \varphi_k - \varphi_k') \\
+ \int_{U \setminus V_j} |\nabla (\varphi_k - \varphi_k')|^2 + |\Delta_\Sigma|^2 |(\varphi_k - \varphi_k')^2 | \\
\leq C(\xi_j, V_j, V_{j+1})\|\varphi_k - \varphi_k'\|_{\infty}^2
$$

where the last inequality follows form the claim. Hence $\{\psi_k\}_{k \geq 1}$ is also a Cauchy sequence in $\mathcal{B}_0(U)$, which definitely converges to $\phi$. Thus we can choose $k_j \gg j$ such that $\|\psi_k^{(j)} - \phi\|_{\infty} \leq 2^{-j}$, and $N_j \gg j$ such that $\text{spt}(\psi_{k_j}^{(j)}) \cap V_{N_j} = \emptyset$. Thus $\phi_j := \psi_{k_j}^{(j)}$ is what we want in (2). \hfill $\Box$

Example 3.3. Let $\Sigma = C \subset \mathbb{R}^{n+1}$ be a regular stable minimal hypercone which is not strictly stable, i.e. $\mu_1 = -\langle \frac{\partial c}{\partial u} \rangle^2$; Let $U = \mathbb{R}^{n+1}_1$ and $U = B_1 \subset C$. Use the notations and parametrization as in [3.4].

Let $u(r, \omega) := r^{-(n-2)/2}w_1(\omega)$. Then

$$
\|\nabla u\|_{L^2(B_1(0))}^2 = \int_0^1 r^{n-1} dr \int_0^1 |\partial_r u|^2 + \frac{1}{r^2} |\nabla_\Sigma u|^2 \, d\omega = +\infty
$$
But we can construct
\[ u_j(r, \omega) := \max \{ r, \frac{1}{j} \}^{-(n-2)/2} \cdot w_1(\omega) \]
Clearly \( u_j \in W^{1,2}(U \cap C) \subset \mathcal{B}(U) \) and \( u_j \to u \) pointwisely. We verify now that \( \sup_j \| u_j \|_B < +\infty \).
In fact, notice that \( L_C u = 0 \) on \( B_1 \setminus B_{1/j} \), thus
\[
Q_C(u_j, u_j) = \int_{B_1 \setminus B_{1/j}} |\nabla u_j|^2 - |AC|^2 u_j^2 + \int_{B_{1/j}} |\nabla u_j|^2 - |AC|^2 u_j^2 \\
\leq \int_{\partial B_{1/j}} u \partial_r u - \int_{\partial B_{1/j}} u \partial_r u - \int_{B_1 \setminus B_{1/j}} u \cdot L_C u \\
+ \int_0^{1/j} r^{n-1} \int_{S} r^{-2} (|\nabla u_j|^2 - |AC|^2 u_j^2) \, d\omega \\
+ \int_{\partial B_{1/j}} \frac{n-2}{2} r^{-n/2} - (n-2)/2 w_1^2 - \int_{\partial B_{1/j}} (-\frac{n-2}{2}) r^{-n/2} -(n-2)/2 w_1^2 = 0 \\
+ \int_0^{1/j} r^{n-3} \int_{S} j^{n-2} (|\nabla u_1|^2 - |AC|^2 w_1^2) \, d\omega \\
\leq C(n)
\]
And \( \| u_j \|_{L^2}^2 \leq \| u \|_{L^2}^2 \leq C(n) \). Therefore, by lemma \ref{lemma:}, \( u \in \mathcal{B}(U) \).

On the other hand, let \( v = -r \log r \cdot w_1 \) on \( B_1 \), then \( v \in W^{1,2}_0(C) \) and \( v \) solves the equation \( L_C v = 0 \) on \( B_1 \). However, we shall see that \( v \notin \mathcal{B}(U) \). In fact, by Lemma \ref{lemma:}, consider \( \forall 0 < s < t < 1 \), the function \( v_{s,t} \) that achieves
\[
\inf \{ Q_C(\phi, \phi) : \phi \in C^1_0(C), \phi = v \text{ on } C \setminus B_t, \text{spt}(\phi) \cap B_s = \emptyset \}
\]
By writing down the E-L equation and solving an ODE,
\[
v_{s,t}(r, \omega) = \begin{cases} \frac{v(r, \omega)}{\log t - \log s} - \frac{\log t}{\log r} \cdot \frac{\log r - \log s}{\log r} w_1(\omega), & \text{if } t \leq r \leq 1 \\
\frac{\log t}{\log r} \cdot \frac{\log r - \log s}{\log r} w_1(\omega), & \text{if } s \leq r \leq t \\
0, & \text{if } 0 < r < s
\end{cases}
\]
Therefore,
\[
Q_C(v_{s,t}, v_{s,t}) = (-\log t) + \frac{(-\log t)^2}{\log t - \log s} \geq (-\log t)
\]
And then \( \| v_{s,t} \|_B \to +\infty \) when \( t \to 0 \). Hence by lemma \ref{lemma:}, \( v \notin \mathcal{B}(U) \).

To study the spectral theory of \( L_C \), we may wish that \( \mathcal{B} \hookrightarrow L^2 \) is also compact.

**Definition 3.4.** Call \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property in \( U \), if for any \( \epsilon > 0 \), there’s a sufficiently small neighborhood \( V_\epsilon \supset \text{Sing}(\Sigma) \cap U \) such that
\[
(3.3) \quad \int_{\Sigma \setminus V_\epsilon} \phi^2 \leq \epsilon \cdot \| \phi \|_B^2 \quad \forall \phi \in C^1_0(U)
\]

**Proposition 3.5.** Suppose \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property in \( U \); \( h \in L^\infty(U) \). Then,

(1) \( \mathcal{B}_0(U) \hookrightarrow L^2(U) \) is compact.

(2) \( \exists \lambda_1 < \lambda_2 < \lambda_3 < \ldots \nearrow +\infty \) and finite dimensional pairwise \( L^2 \)-orthogonal linear subspaces \( \{E_j\}_{j \geq 1} \) of \( \mathcal{B}_0(U) \cap C^\infty(U) \) such that
\[
(-L_\Sigma + h) \phi = \lambda_j \phi \quad \forall \phi \in E_j
\]
and that
\[
L^2(U) = \bigoplus_{j \geq 1} L^2(E_j) \quad ; \quad \mathcal{B}_0(U) = \bigoplus_{j \geq 1} E_j
\]
Moreover, dim $E_1 = 1$ and any $0 \neq \phi_1 \in E_1$ is nowhere vanishing in $\mathcal{U}$.

(3) If $(L_{\Sigma} - h)|_{\mathcal{B}_0}$ is nondegenerate, i.e. $\lambda_j \neq 0$ for all $j \geq 1$ in (2). Then for each $f \in L^2(\Sigma)$,

$$(-L_{\Sigma} + h)\phi = f \quad \text{on } \mathcal{U}$$

has a unique solution $\phi \in \mathcal{B}_0(\mathcal{U})$. Moreover, we have energy-type estimate

$$\|\phi\|_{H(\mathcal{U})} \leq C(h, \Sigma, U, M, g)\|f\|_{L^2}$$

(4) Suppose $-L_{\Sigma} + h$ is strictly positive, i.e. $\lambda_1 > 0$ in (2). Suppose $u \in \mathcal{B}(\mathcal{U})$ satisfies

$$(-L_{\Sigma} + h)u \geq 0 \text{ in the distribution sense, i.e.}$$

$$\int_{\mathcal{U}} \nabla u \cdot \nabla \phi - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu) - h)u \cdot \phi \geq 0 \quad \forall \phi \in C_0^\infty(\mathcal{U}; \mathbb{R}_+)$$

Also assume that either $\mathcal{U} \cap \text{Sing}(\Sigma) = \emptyset$ and $u|_{\partial \mathcal{U}} \geq 0$, or $u \not\in \mathcal{B}_0(\mathcal{U})$. Then $u \geq 0$ on $\mathcal{U}$.

(5) There exists $\vartheta \in C_0^\infty(\mathcal{U})$ such that $-L_{\Sigma} + h + \vartheta$ is strictly positive on $\mathcal{B}_0(\mathcal{U})$.

(6) $\{f \in C^\infty(\mathcal{U}) : (-L_{\Sigma} + h + f)|_{\mathcal{B}_0} \text{ is nondegenerate}\}$ is open and dense in $C^\infty(\mathcal{U})$.

Proof. (1) Suppose $\{\phi_k\}$ be a bounded sequence in $\mathcal{B}_0(\mathcal{U})$, WLOG $\|\phi_k\|_{H} \leq 1$. Then by the proof of lemma 3.2 for every subdomain $\Omega \subset \subset \Sigma$, $\|\phi_k\|_{W^{1,2}} \leq C(\Sigma, M, g, \Omega)$. By Reilly theorem [G-T] and lemma 3.2 up to a subsequence, $\phi_k \rightarrow \varphi$ for some $\varphi \in \mathcal{B}_0(\mathcal{U})$ in $L^2(\Omega)$, $\forall \Omega \subset \subset \Sigma$. We shall see that $\|\phi_k - \varphi\|_{L^2} \rightarrow 0$, which proves the compactness. In fact, $\forall \epsilon > 0$, let $V_\epsilon$ be the neighborhood of Sing($\Sigma$) in definition 3.3, then

$$\limsup_{k \rightarrow \infty} \int_{\Sigma} (\phi_k - \varphi)^2 \leq \limsup_{k \rightarrow \infty} \int_{V_\epsilon} (\phi_k - \varphi)^2 + \lim_{k \rightarrow \infty} \int_{V_\epsilon^c} (\phi_k - \varphi)^2$$

$$\leq 4\epsilon \|\phi_k\|^2_{\mathcal{B}_0} + 0 \leq 4\epsilon$$

Let $\epsilon \rightarrow 0$, we see that $\phi_k \rightarrow \varphi$ in $L^2(\Sigma)$.

(2) and (3) follows directly from (1) and the standard method in spectral theory. We omit the proof here.

To prove (4), the basic strategy is to multiply the differential inequality by $u^-$ and integration by part. However, it’s unclear whether $u^-$ still lies in $\mathcal{B}(\mathcal{U})$. Hence we do this by approximation. Consider for each $s \in (0, 1)$ the approximated operators $L^s := L_{\Sigma} - h - s|A_{\Sigma}|^2$. Denote for simplicity $P := |A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu) - h$. Let

$$\delta^s(v) := \frac{1}{2} \int_{\mathcal{U}} |\nabla v|^2 + (-P + s|A_{\Sigma}|^2)v^2 \, dx - \int_{\mathcal{U}} \nabla u \cdot \nabla v - Pu \cdot v \quad \forall v \in C^1_c(\mathcal{U})$$

Note that the second integration above is clearly bounded from above by $(1 + \|h\|_{L^\infty} + \|\text{Ric}_M\|_{L^\infty}) \cdot \|u\|_{H(\mathcal{U})} \cdot \|v\|_{H(\mathcal{U})}$, $\delta^s$ extends to be defined on $W^{1,2}(\mathcal{U})$.

Let $v^{(s)}$ be the minimizer of $\delta^s$ among $\{v \in W^{1,2}(\mathcal{U}) : v = u$ on $\partial \mathcal{U}\}$. Then by lemma 2.13 and 2.20, $v^{(s)} \in W^{1,2}(\mathcal{U})$, satisfies the equation $-L^s v^{(s)} = (-L_{\Sigma} + h)u$ in the distribution sense on $\mathcal{U}$ and $v^{(s)} \geq 0$ on $\partial \mathcal{U}$. Multiply the equation by $v^{(s)} := -\min\{0, v^{(s)}\} \in W^{1,2}_0(\mathcal{U})$ and take integration by part to get

$$\int_{\mathcal{U}} |\nabla v^{(s)}|^2 + (-P + s|A_{\Sigma}|^2)(v^{(s)})^2 \leq 0$$

Since $-L_{\Sigma} + h$ is strictly positive on $\mathcal{B}_0(\mathcal{U})$, so is $-L^s$. Thus we conclude that $v^{(s)} = 0$, i.e. $v^{(s)} \geq 0$.

On the other hand, by the equation satisfied by $v^{(s)}$, we see $\|v^{(s)}\|_{H(\mathcal{U})} \leq C(\Sigma, h)\|u\|_{H(\mathcal{U})}$ for every $s \in (0, 1)$. Hence, by (1), up to a subsequence of $s \rightarrow +$, $v^{(s)} \rightarrow v^{(0)}$ in $L^2(\mathcal{U})$ for some $0 \leq v^{(0)} \in \mathcal{B}(\mathcal{U})$ equal to $u$ on $\partial \mathcal{U}$ and satisfying the equation $(-L_{\Sigma} + h)v^{(0)} - u = 0$. By the nondegeneracy of $-L_{\Sigma} + h$ we see that $u = v^{(0)} \geq 0$, which completes the proof of (4).
To prove (5), first observe that since \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property in \( U \), there exists a small neighborhood \( \mathcal{V}_0 \supset Sing(\Sigma) \) such that

\[
Q^k_{\Sigma}(\phi,\phi) := Q^k_{\Sigma}(\phi,\phi) + \int_{\Sigma} h\phi^2 \geq 0 \quad \forall \phi \in C_c^1(U \cap \mathcal{V}_0)
\]

Let \( \mathcal{V}_1 \subset \subset M \setminus Sing(\Sigma) \) be an open subset such that \( \mathcal{V}_0 \cup \mathcal{V}_1 \supset Clos(U) \). Let \( \{\eta^0_j, \eta^1_j\} \) be the partition of unity corresponding to \( \mathcal{V}_0, \mathcal{V}_1 \). Then by the virtual of \( (3.1) \), for each \( \phi \in C_c^\infty(U) \),

\[
Q^k_{\Sigma}(\phi,\phi) = Q^k_{\Sigma}(\phi\eta_0,\phi\eta_0) + Q^k_{\Sigma}(\phi\eta_1,\phi\eta_1) + \int_{\Sigma} \phi^2(\eta_0\Delta\eta_0 + \eta_1\Delta\eta_1)
\]

\[\geq \int_{\mathcal{V}_1} \phi^2(\beta\eta_0^2 + \eta_0\Delta\eta_0 + \eta_1\Delta\eta_1)\]

Hence take \( \vartheta \in C_c^\infty(U;\mathbb{R}) \) such that \( \vartheta \geq 1 + |\beta + \eta_0\Delta\eta_0 + \eta_1\Delta\eta_1| \) on \( \mathcal{V}_1 \), we see from (3.4) that \( -L_\Sigma + \vartheta \) is non-degenerate on \( \mathcal{B}_0(U) \) for sufficiently small \( |t| \).

Choose \( f \) by the following approach. Let \( \{U_t \subset \subset U\}_{t \geq 1} \) be an increasing exhaustion of \( U \), and \( f_t \in C_c^\infty(U;[0,1]) \) be a family of cut-off which restricts to 1 on \( U_t \). Clearly, \( f_t \to 1 \) on \( U \) weakly in \( L^\infty \), hence for sufficiently large \( t \), the bilinear form on \( K \times K, (\phi_1,\phi_2) \mapsto \int_{U_t} f_t\phi_1\phi_2 \, dx \) is non-degenerate. Choose \( f = f_1 \) for such \( t \).

To show \( -L^h + tf \) is non-degenerate for \( |t| < \delta \), suppose for contradiction that \( \exists t_i \to 0 \) and \( u_i \in Ker(-L^h + t_if) \cap \mathcal{B}_0(U) \) with \( \|u_i\|_{L^2} = 1 \). Let \( v_i \) be the \( L^2 \)-orthogonal projection of \( u_i \) onto \( K \); And let \( \{E_j\}_{j \geq 1} \) be the \( L^2 \)-orthogonal eigensubspaces of \( -L^h \) given in (2), with corresponding eigenvalues \( \lambda_1 < \lambda_2 < \ldots \to +\infty \). Multiply the equation \( L^h(u_i - v_i) = t_i f u_i \) with \( \phi \in E_j \), integrate by parts and take sum over \( j \) we see that

\[
\|u_i - v_i\|_{L^2} \leq \lambda^{-1} |t_i| \|f\|_{L^\infty}
\]

where \( \lambda := \inf\{\lambda_j : \lambda_j \neq 0\} > 0 \). Let \( \bar{u}_i := (u_i - v_i)/t_i \), since \( \bar{u}_i \) satisfies the equation \( L^h\bar{u}_i = f u_i \) and the \( L^2 \) bound above, by integration by part we have \( \limsup_i \|\bar{u}_i\|_{L^2(\mathcal{U})} \leq C < +\infty \); Also, since \( v_i \in K \) and \( \|v_i\|_{L^2} \leq 1 \), we have \( \limsup_i \|v_i\|_{L^2} < +\infty \).

By compactness assumption for \( \mathcal{B}_0(U) \hookrightarrow L^2 \) we conclude that up to a subsequence, \( \bar{u}_i \to \bar{u}_\infty \), \( v_i \to v_\infty \) both in \( L^2(U) \); And they satisfy \( \|v_\infty\|_{L^2} = 1 \), \( L^h v_\infty = f v_\infty \). Multiply by \( \phi \in K \) and take integration by part, we see that

\[
\int_{U_t} f v_\infty \cdot \phi = 0 \quad \text{for every} \ \phi \in K
\]

This contradicts to the choice of \( f \).

\[\square\]

Remark 3.6. Proposition 3.5 (4) is usually called weak maximum principle for \( -L_\Sigma + h \). The assumption on boundary value of \( u \), which is used in the proof to guarantee the solutions to perturbed equations fall in \( W^{1,2}_\Sigma \), is technical and subtle and is conjectured to be dropped.

Proposition 3.5 (6) shows that \( -L_\Sigma + h \) being non-degenerate is a generic property.

We finish this subsection by discussing a basic corollary of the \( L^2 \)-nonconcentration property. Following [Dey19], the index of a singular minimal hypersurface \( \Sigma \subset M \) is defined by

\[
ind(\Sigma,U) := \sup \left\{ \dim \mathcal{V} : \mathcal{V} \subset \mathcal{D}_c(\Sigma) \text{ is a linear subspace s.t.} \right\}
\]

\[
\frac{d^2}{ds^2} \mathcal{H}^n(e^{sX}(\Sigma)) < 0 \quad \forall X \in \mathcal{V}
\]
Corollary 3.7. Suppose \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property in \( U \). Let \( \lambda_1 < \lambda_2 < \cdots \) be the spectrum of \( L_\Sigma \) on \( \mathcal{B}_r \), where \( h = 0 \); and \( E_j \) be the eigenspaces. Then

\[
\text{ind}(\Sigma, U) = \sum_{\lambda_j < 0} \dim E_j
\]

In particular, \( \Sigma \) has finite index.

Combined with lemma 3.9 in the subsection below, this partially answers a question proposed by A. Neves.

Corollary 3.8. Let \( \Sigma \) be a closed, locally stable minimal hypersurface in an 8 dimensional manifold, then \( \Sigma \) has finite index.

Proof of corollary 3.7. First recall that by lemma 2.12 if \( X \in \mathcal{X}_c(U) \), then \( \phi := X \cdot \nu \in W^{1,2}_0(\mathcal{U}) \), and

\[
\frac{d^2}{ds^2}\big|_{s=0} \mathscr{H}^n(e^{\phi X}(\Sigma)) < 0 \quad \text{iff} \quad Q_\Sigma(\phi, \phi) < 0
\]

Hence, define

\[
\text{ind}_c(\Sigma, U) := \sup \{ \dim \mathcal{V} : \mathcal{V} \subset C_c^\infty(\Sigma \cap U) \text{ and } Q_\Sigma(\phi, \phi) < 0 \quad \forall \phi \in \mathcal{V} \}
\]

\[
\text{ind}_{\mathcal{B}_r}(\Sigma, U) := \sup \{ \dim \mathcal{V}_r : \mathcal{V}_r \subset \mathcal{B}_0(\mathcal{U}) \text{ and } Q_\Sigma(\phi, \phi) < 0 \quad \forall \phi \in \mathcal{V}_r \}
\]

Since for each \( \phi \in C_c^\infty(\mathcal{U}) \), \( \phi \nu \) could be extended to a smooth vector field in \( M \) supported in \( U \), by (3.5) and proposition 3.5

\[
\text{ind}_c(\Sigma, U) \leq \text{ind}(\Sigma, U) \leq \text{ind}_{\mathcal{B}_r}(\Sigma, U) = \sum_{\lambda_j < 0} \dim E_j
\]

On the other hand, since \( C_c^\infty(\mathcal{U}) \) is dense in \( \mathcal{B}_0(\mathcal{U}) \), we conclude that \( \text{ind}_c(\Sigma, U) = \text{ind}_{\mathcal{B}_r}(\Sigma, U) \). \( \square \)

3.2. \( L^2 \)-nonconcentration for strongly isolated singularities. In this and next subsection, we keep assuming that \( \Sigma \) has only strongly isolated singularities. The goal of this subsection is the following

Lemma 3.9. Suppose \( \Sigma \) has only strongly isolated singularities and \( \partial U \cap \text{Sing}(\Sigma) = \emptyset \). Then, \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property in \( U \).

The proof is based on a refined analysis near singularities. Since \( \Sigma \) has only strongly isolated singularities, by a cut-off argument similar to the proof of lemma 3.11 it suffices to work near each singularity \( p \). In the proof below, we keep using the parametrization and notations of \( \Sigma \cap B^{M}_{\tau_1}(p) \) as in section 2.3. Recall that \( B_r(p) \) denotes the ball in \( \Sigma \) with intrinsic distance, and since we have parametrized \( \Sigma \cap B^{M}_{\tau_1}(p) \) by its tangent cone \( C_p \subset T_p M = \mathbb{R}^{n+1} \) as in lemma 2.9, \( p = 0 \) and \( B_r(p) \) is abbreviated as \( B_r \). Let \( r_2 = r_2(\Sigma, M, g) \in (0, \{ \tau(1/10n)/4 \}) \) be fixed such that \( \Sigma \) is stable in \( B^{M}_{\tau_1}(p) \).

We shall prove lemma 3.9 when \( U \cap \Sigma = B_{r_2} \). To do so, first observe that if \( C_p \) is strictly stable, then by lemma 2.11 and 2.9 \( \exists r_3 = r_3(\Sigma, p) \in (0, r_2) \) and \( \delta \Sigma > 0 \) such that

\[
\delta \Sigma \int_{\Sigma} |\nabla \phi|^2 \leq Q_\Sigma(\phi, \phi) \quad \forall \phi \in C^1_c(B_r)
\]

Hence lemma 3.9 holds directly by lemma 2.13 and a cut-off argument. Thus, it suffices to deal with non-strictly stable tangent cones. From now on, assume \( \mu_1(C_p) = -(n-2)^2/4 \) (see the notations in section 2.3).

Consider for each \( s \in (0, 1) \), the perturbed Jacobi operator

\[
L^s_\Sigma := L_\Sigma - s|A_\Sigma|^2 = \Delta_\Sigma + (1-s)|A_\Sigma| + \text{Ric}(\nu, \nu)
\]
and its associated quadratic form
\[ Q_\Sigma^s(\phi, \phi) = Q_\Sigma(\phi, \phi) + s \int_\Sigma |A_\Sigma|^2 \phi^2 = \int_\Sigma |\nabla^\Sigma \phi|^2 - ((1 - s)|A_\Sigma|^2 + \text{Ric}(\nu, \nu)) \phi^2 \]

By Lemma 2.13 and (2.20), there’s a continuous function \( 1 \leq \alpha(r, \omega) \leq 1 + \log(r_1/r) \) on \( B_{2r_2} \) (independent of choice of \( s \)) such that

(3.6) \[ \alpha(x) = \infty \quad \text{as} \quad x \to \text{Sing}(\Sigma) = \{ p \} \]

and that for each \( \epsilon > 0 \), there’s a neighborhood \( V_{\epsilon, s} \ni p \) such that

\[ \int_{\Sigma \setminus V_{\epsilon, s}} \phi^2 \cdot \alpha \leq \epsilon \cdot Q_\Sigma^s(\phi, \phi) \quad \forall \phi \in C^1_c(B_{r_2}) \]

Hence, by the similar argument of proposition 3.5, there’s a unique \( u^{(s)} \in C^0_{\text{loc}}(B_{2r_2}) \) with

(3.7) \[ u^{(s)}(r, \cdot) \begin{cases} = 0 & \text{if} \ r \geq r_2 \\ > 0 & \text{if} \ r < r_2 \end{cases} \quad \int_\Sigma (u^{(s)})^2 \cdot \alpha = 1 \]

and

(3.8) \[ \lambda_s := Q_\Sigma^s(u^{(s)}, u^{(s)}) = \inf \{ Q_\Sigma^s(\phi, \phi) : \phi \in C^1_c(B_{r_2}), \int_\Sigma \phi^2 \cdot \alpha = 1 \} \]

In particular, \( u^{(s)} \) satisfies the equation

(3.9) \[ -L^s_{\Sigma} u^{(s)} = \lambda_s \alpha \cdot u^{(s)} \quad \text{on} \ B_{r_2} \]

Also by definition and stability of \( \Sigma \) in \( B_{2r_2} \), for any \( 0 < s_1 < s_2 < 1 \) and any \( \phi \in C^1_c(B_{r_2}) \),

\[ 0 \leq Q_\Sigma^s(\phi, \phi) \leq Q_\Sigma^{s_2}(\phi, \phi) \quad \text{and} \]

(3.10) \[ \lambda_s := Q_\Sigma^{s_2}(u^{(s_2)}, u^{(s_2)}) = \min \{ Q_\Sigma^{s_2}(\phi, \phi) : \phi \in C^1_c(B_{r_2}) \} \]

We need a uniform lower bound on \( u^{(s)} \) away from singularities. To derive it, we need the following Lemma 3.10. In fact, this is the only place where structures of strongly isolated singularity is essentially made use of.

**Lemma 3.10.** \( \exists s_0 = s_0(\Sigma, M, g) \in (0, 1), \delta = \delta(\Sigma, M, g) > 0 \) s.t. for any \( 0 < s < s_0 \),

\[ \sup \{ u^{(s)}(r, \omega)^2 : \frac{r_2}{2} < r < r_2 \} \geq \delta \]

**Proof.** For each \( 0 < r < r_2 \), \( s \in (0, 1) \), let \( m^{(s)}(r) := \inf \{ u^{(s)}(r, \omega) : \omega \in S \} \). By 3.7 and Harnack inequality, there’s a \( C_1 = C_1(\Sigma, M, g) > 0 \) (independent of \( s \) and \( r \)) such that

(3.10) \[ u^{(s)}(r, \omega) \leq C_1 m^{(s)}(r) \quad \forall 0 < r < r_2/2, \forall s \in (0, 1), \forall \omega \in S \]

On the other hand, let \( 0 < s < 1 < \) such that

\[ r^2|A_\Sigma|^2 \frac{w_1}{r} \leq \frac{1}{10n \cdot s_0} \quad \forall \ 0 < r < r_1 \]

Hence, let \( U(r) := (\frac{r}{r_2})^{-(n-1)/2} - (\frac{r_2}{r})^{-n-3/2} \). For any \( s \in (0, s_0), \) any constant \( D > 0, \)

\[ L^s_{\Sigma}(B \cdot U(r))w_1(\omega) \]

\[ = D \cdot (U''(r))w_1(\omega) + U'(r)H_\nu(\omega)w_1(\omega) + \frac{U(r)}{r^2} (L_{\nu}w_1 - sr^2|A_\Sigma|^2 w_1) \]

\[ \geq D \cdot (U''(r)) + \left( n - 1 + \frac{2n}{10n} \right) U'(r) + (\mu_1 - \frac{2}{10n} \frac{U(r)}{r^2}) w_1(\omega) \]

\[ \geq 0 \quad \geq -\lambda_s \alpha \cdot u^{(s)} = L^s_{\Sigma} u^{(s)} \]

on \( B_{r_2} \), where the first equality and the first inequality follow from Lemma 2.9 and the choice of \( r_2 \), the second inequality is a direct calculation since \( \mu_1 = -(n - 2)^2/4 \) as assumed, and the third is by 3.7. Hence, by weak maximum principle, \( \forall 0 < r_3 < r_2, \) if

\[ D \cdot U(r_3) \sup_{\Theta} w_1 \leq m^{(s)}(r_3) \]
Also, by lemma 3.10, 

\[(3.14)\]

and the equation holds increasing in \( s \), \( \epsilon > 0 \), then by classical elliptic estimates \([GT01]\), up to a subsequence, 

\[(3.7)\]

Choose \( \delta \) from \((3.10)\), the third inequality is by \((3.12)\) and the last inequality is by \((3.11)\).

Thus, by \((3.7)\),

\[
1 = \int_\Sigma (u(s))^2 \cdot \alpha = \int_{B_{r_2} \setminus B_{r_2/2}} (u(s))^2 \cdot \alpha + \int_{B_{r_2/2}} (u(s))^2 \cdot \alpha
\]

\[
\leq C(\Sigma, M, g) \left( \sup_{B_{r_2/2}} (u(s))^2 + \int_0^{r_2/2} r^{-1} \log r \int_B (u(s))^2(r, \omega) \ d\omega \ dr \right)
\]

\[
\leq C(\Sigma, M, g) \left( \sup_{B_{r_2/2}} (u(s))^2 + \int_0^{r_2/2} r^{-1} \log r \cdot m(s)(r)^2 \ dr \right)
\]

\[
\leq C(\Sigma, M, g, r_2) \left( \sup_{B_{r_2/2}} (u(s))^2 + \int_0^{r_2/2} r^{-1} \log r \cdot m(s)(r_2/2)^2 \ U(r)^2 \ dr \right)
\]

\[
\leq C_2(\Sigma, M, g, r_2) \left( \sup_{B_{r_2/2}} (u(s))^2 \right)
\]

Where the first inequality follows by definition of \( \alpha \) and lemma 2.9, the second inequality follows from \((3.10)\), the third inequality is by \((3.12)\) and the last inequality is by 

\[
\int_0^1 r^{-1} \log r \cdot U(r)^2 < +\infty \ dr
\]

Hence, the lemma is proved by choosing \( \delta = C_2(\Sigma, M, g, r_2) \).

**Proof of lemma 3.11** Recall that \( u(s) \) satisfies the equation \((3.9)\) and \((3.7)\), where \( \lambda_s > 0 \) is decreasing in \( s \). Let \( s \to 0 \). Then by classical elliptic estimates \([GT01]\), up to a subsequence,

\[(3.13)\]

Clearly, 

\[
\lambda_0 \geq \inf \{ Q_\Sigma(\phi, \phi) : \phi \in C^1_{\text{loc}}(\Sigma \cap B_{r_2}), \int_\Sigma \phi^2 \cdot \alpha = 1 \} \geq 0
\]

and the equation holds

\[(3.14)\]

Also, by lemma 3.10 \( u_0 \neq 0 \).

**Claim:** \( \lambda_0 > 0 \).

We first complete the proof assuming this claim to be true: For each \( \epsilon > 0 \), let \( V_\epsilon := \{ x \in B_{r_2} : \alpha(x) > 1/(\lambda_0 \cdot \epsilon) \} \). By \((3.8)\), \( V_\epsilon \neq \emptyset \) is a neighborhood of \( \text{Sing}(\Sigma) \cap B_{r_2}(p) = \{ p \} \). For any \( \phi \in C^1_{\text{loc}}(B_{r_2}) \) fixed,

\[
Q_\Sigma(\phi, \phi) = \lim_{s \to 0} Q^s_\Sigma(\phi, \phi) \geq \lim_{s \to 0} \lambda_s \cdot \int_\Sigma \phi^2 \cdot \alpha = \lambda_0 \cdot \int_\Sigma \phi^2 \cdot \alpha
\]

\[
\geq \frac{\lambda_0}{\lambda_0 \epsilon} \cdot \int_{V_\epsilon} \phi^2 = \frac{1}{\epsilon} \int_{V_\epsilon} \phi^2
\]
Hence, $\Sigma$ satisfies the $L^2$-nonconcentration property in $B_{r_2}$.

**Proof of the claim:** We shall argue by contradiction. Suppose $\lambda_0 = 0$. By (3.7), (3.13) and (3.14),

$$L_\Sigma u_0 = 0 \quad \text{on } B_{r_2}; \quad u_0(r, \cdot) \begin{cases} = 0 & \text{if } r > r_2 \\ \geq 0 & \text{if } 0 < r < r_2 \end{cases}$$

Also notice that for $s \in (0,1/2)$,

$$Q_\Sigma(u^{(s)}, u^{(s)}) \leq Q^k_\Sigma(u^{(s)}, u^{(s)}) = \lambda_s \leq \lambda_{1/2}$$

Hence by (3.15) and lemma 3.2, $u_0 \in \mathcal{B}(B_{2r_2})$ and $Q_\Sigma(u_0, u_0) = 0$.

On the other hand, recall that $\Sigma$ is stable in $B_{2r_2}$, i.e.

$$Q_\Sigma(\phi, \phi) \geq 0 \quad \forall \phi \in C^1_c(B_{2r_2})$$

Therefore, $u_0 \neq 0$ achieves the infimum of $Q_\Sigma$ in $\mathcal{B}(B_{2r_2})$. The Euler-Lagrange equation gives

$$L_\Sigma u_0 = 0 \quad \text{on } B_{2r_2}$$

However, (3.15) contradict to the strong maximum principle of $u_0$ in $B_{2r_2}$. This completes the proof of the claim. \qed

**Remark 3.11.** The proof of the Claim above also tells that $-L_\Sigma$ is strictly positive on $\mathcal{B}_0(B_{r_2})$.

3.3. **Asymptotic analysis near singularities.** The aim of this subsection is to study the asymptotic behavior of special functions near strongly isolated singularities. They will be used in section 4 to describe associated Jacobi fields and in section 5 to construct desired one-sided perturbations. We keep the assumptions on $\Sigma$, $(M, g)$, $U$, $p \in \text{Sing}(\Sigma) \cap U$ and $r_1 > r_2 > 0$ as in section 3.2. Also keep parametrizing $\Sigma$ by $\mathcal{B}(\Sigma)$ and using the notation that $B_r := B_r(p)$. Recall that $-L_\Sigma$ is strictly positive on $\mathcal{B}_0(B_{r_2})$. For later applications, we shall consider operators of a more general form: $L^h := L_\Sigma - h$ and $Q^h(\phi, \phi) := Q_\Sigma(\phi, \phi) + \int_\Sigma h \cdot \phi^2$, where $h \in L^\infty(\Sigma)$ be fixed throughout this subsection. Assume that $-L^h$ is also strictly positive on $\mathcal{B}_0(B_{r_2})$.

We first slightly extend the function space we are dealing with.

**Lemma 3.12.** Up to a normalization, there exists a unique $G \in C^\infty_{loc}(\text{Clos}(B_{r_2}))$ such that $L^h G = 0$ on $B_{r_2}(p)$, $G > 0$ on $B_{r_2}$, $G = 0$ on $\partial B_{r_2}$.

**Proof.** We first prove the existence. Consider $(r_2/2, t_2) \to 0_+$ and the solution $u_j$ of $L^h u_j = 0$ on $A_{t_j, r_2}$, $u_j = 0$ on $\partial B_{r_2}$, $u_j = 1$ on $\partial B_{t_j}$. By weak maximum principle, $u_j \to 0$ on $A_{t_j, r_2}$. By Harnack inequality and standard elliptic estimate, $u_j/\sup_{B_{r_2}} u_j \to G$ in $C^\infty_{loc}(\text{Clos}(B_{r_2}))$ and $G > 0$ on $B_{r_2}$, $G = 0$ on $\partial B_{r_2}$.

To see the uniqueness, if $G'$ is another positive smooth function vanishing on $\partial B_{r_2}$ and satisfies the equation above, first note that by weak maximum principle, $\forall t \in (0, r_2)$,

$$\sup_{A_{t, r_2}} G/G' = \sup_{A_{t, r_2}} G/G' \quad \text{and} \quad \inf_{A_{t, r_2}} G/G' = \inf_{A_{t, r_2}} G/G'$$

and by Harnack inequality, $\sup_{\partial B_t} G/G' \leq C(\Sigma) \inf_{\partial B_t} G/G'$, $\forall t \in (0, r_2/2)$. Hence,

$$0 < a_+ := \inf_{B_{r_2}} G/G' \leq a_- := \sup_{B_{r_2}} G/G' < +\infty$$

We now show that $a_- = a_+$, with which the lemma is proved.

By (3.16), $a_- = \liminf_{t \to 0_+} \inf_{\partial B_t} G/G'$ and $a_+ = \liminf_{t \to 0_+} \inf_{\partial B_t} G/G'$. Thus, up to a subsequence of $t \to 0_+$, $G_t(x) := G(tx)/\inf_{\partial B_t} G \to G_0$, $G_t(x) := G'(tx)/\inf_{\partial B_t} G \to G'_0$ and $G_0, G'_0$ are positive Jacobi fields on $C_p$ with $a_- = \inf_{\partial B_t} G_0/G'_0$, $a_+ = \sup_{\partial B_t} G_0/G'_0$. By lemma 2.8, $a_- = a_+$. \qed
Call such $G$ in lemma 3.12 a Green’s function of $L^h$ at $p$ in $B_{r_2}$. Since $-L^h$ is strictly positive on $\mathcal{R}(B_{r_2})$, $G$ doesn’t belong to $\mathcal{R}(B_{r_2})$. Just like Green’s functions in a smooth domain, $G$ dominates certain functions in $\mathcal{R}(B_{r_2})$.  

**Lemma 3.13.** Let $\varphi_1 \in \mathcal{R}(B_{r_2})$ be the first eigenfunction of $-L^h$ in $\mathcal{R}(B_{r_2})$ with $\|\varphi_1\|_{L^2} = 1$; $\lambda_1 > 0$ to be the corresponding eigenvalue, i.e. \[
\lambda_1 = \inf\{Q^h(\varphi, \cdot) : \varphi \in \mathcal{R}(B_{r_2}), \|\varphi\|_{L^2} = 1\} = Q^h(\varphi_1, \varphi_1) > 0 \]
Then $\lim_{r \to 0} \sup_{\partial B_r}(\varphi_1/G) = 0$.
In particular, if $u \in \mathcal{R}(B_{r_2})$ satisfies $L^h u \in L^\infty(B_{r_2})$, then $\lim_{r \to 0} \sup_{\partial B_r}(\|u\|/G) = 0$

**Proof.** By weak maximum principle and Harnack inequality, for each $r \in (0, r_2)$,
\[\sup\varphi_1/G \leq C(\Sigma) \inf\varphi_1/G = C(\Sigma) \inf_{\partial B_{r_2}} \varphi_1/G \]
Hence, if assume for contradiction that $\lim_{r \to 0} \sup_{\partial B_r}(\varphi_1/G) > 0$, then by a renormalization in $G$, WLOG $G \leq \varphi_1$ on $B_{r_2}$.

For each $r < r_2$, consider the minimizer $v_r$ of $Q^h(\varphi, \cdot)$ among
\[\{\varphi \in \mathcal{R}(B_{r_2}) : \varphi = \varphi_1 \text{ in } B_r\} \]

The existence of such minimizer follows from proposition 3.3 and lemma 3.9. Moreover, $L^h v_r = 0$ onystem. $G \leq v_r \leq \varphi_1$ on $B_{r_2}$ by weak maximum principle; $\|v_r\|_{L^\infty(B_{r_2})} \leq (1 + \|h\|_{L^\infty}) \|\varphi_1\|_{L^\infty(B_{r_2})}$ by definition. Let $r \to 0_+$, up to a subsequence, $v_r \to v_0$ in $C^\infty_0(\text{Clos}(B_{r_2}))$ for some $v_0 \in \mathcal{R}(B_{r_2})$ satisfying $L^h v_0 = 0$ on $B_{r_2}$ and $G \leq v_0$. But since $L^h$ is nondegenerate in $\mathcal{R}(B_{r_2})$, $v_0 \equiv 0$, contradicts to that $G > 0$ on $B_{r_2}$.

To see that near $p$ $G$ also dominates $u \in \mathcal{R}(B_{r_2})$ with $L^h u \in L^\infty(B_{r_2})$, first observe that by Harnack inequality, lemma 2.11 and a blow up argument, $\varphi_1(x) \to +\infty$ as $x \to p$. Hence there exists $C > 1$ depending on $u$ and $\varphi_1$ such that $|L^h u| \leq -C \cdot L^h \varphi_1$ in $B_{r_2/2}$ and $|u| \leq C \varphi_1$ on $\partial B_{r_2/2}$. Then by proposition 3.3 (4), $|u| \leq C \varphi_1$ on $B_{r_2/2}$ and the lemma is proved.  

Now we introduce the notion of asymptotic rate at $p$. Recall for $\sigma \in \mathbb{R} \setminus \Gamma_{C_p}$,
\[J^\sigma_u(r, s) := \int_{A_{r,s}} u^2(x)|x|^{-n-2\sigma} \]
is introduced in (2.11). Here, the integration is with respect to volume density on $C_p \subset T_p M = \mathbb{R}^{n+1}$. Define the asymptotic rate of $u \in L^1_0(B_{r_2})$ at $p$ to be
\[(3.17) \quad AR_p(u) := \sup\{\sigma : \lim_{t \to +} J^\sigma_u(t, 2t) = 0\} \]
where we use the convention that $\sup\emptyset = -\infty$ and $AR_p(0) = +\infty$.

**Lemma 3.14.** Suppose $0 \neq u \in W^{2,2}_{loc}(B_{r_2})$ such that $L^h u \in L^\infty$. Also suppose that $\gamma := AR_p(u) \in (-\infty, 1)$, then

1. $\forall \sigma \in (\gamma, 1) \setminus \Gamma_{C_p}$, let $K_0(\sigma, C_p) > 2$ be determined in lemma 2.6. For every $K \geq K_0(\sigma, C_p)$, there exists some $t_0 = t_0(K, u, \sigma, C_p, L^h, \Sigma, M, g) \in (0, r_2)$ such that $t \mapsto J^\sigma_u(t_0K^{-\ell-1}, t_0K^{-\ell})$ is strictly increasing in $\ell$;  
2. $\exists \ell_0 \to 0_+$ and $c_\ell \to 0_+$ such that $u_j(x) := u(t_jx)/c_\ell \to u_\infty$ in $W^{1,2}_{loc}(C_p)$ for some $u_\infty(r, \omega) = r^{\gamma}(c + c_\ell \log r) \cdot \omega$, where $0 \neq \omega \in W^{1,2}_{loc}(C_p)$ is the corresponding eigenfunction of $\gamma$ defined in section 2.3 (c, c') \neq (0, 0)$ and $c' = 0$ unless $\gamma = -(n-2)/2$.

In particular, $AR_p(u) \in \Gamma_{C_p}$.

**Proof.** To prove (1), first notice that for every $\sigma' < \sigma$ we have $J^\sigma_u(r, s) \geq J^{\sigma'}_u(r, s) \cdot s^{2(\sigma' - \sigma)}$. Hence by definition of $AR_p(u), \forall \sigma \in (AR_p(u), 1) \setminus \Gamma_{C_p}, \forall K \geq K_0(\sigma, C_p) > 2$ as in lemma 2.6 fixed, $\limsup_{t \to 0_+} J^\sigma_u(t, Kt) = +\infty$. Thus, there exists $s_j \to 0_+$ such that for every $j$, 
\[J^\sigma_u(Ks_j, K^2s_j) \geq J^\sigma_u(t, Kt) \quad \forall t \in (Ks_j, r_2/K) \]
Also note that \( u \) satisfies the equation \( L^h u = f \in L^\infty(\Sigma) \). For each \( r \in (0, r_2) \), let \( u^{(r)}(x) := u(rx)/r \) defined on \( B_{r_2/r} \subset C_p \), then by lemma 3.9, \( u^{(r)} \) satisfies an equation of form

\[
(3.18) \quad \text{div}_C (\nabla_C u^{(r)} + b_0^{(r)}(x, u^{(r)}, \nabla_C u^{(r)})) + |A_C|^2 u^{(r)} + b_1^{(r)}(x, u^{(r)}, \nabla_C u^{(r)}) = 0
\]

where \(|b_0^{(r)}(x, z, p)| + |b_1^{(r)}(x, z, p)| \leq \epsilon(r)(|z|/|x| + |p|) + r \cdot \|f\|_{L^\infty(\Sigma)}\) on \( A_1, A_2/r \) and \( \epsilon(r) \rightarrow 0_+ \) as \( r \rightarrow 0_+ \). Also by changing variable, \( J_0^{(r)}(s, t) = J_0^{(s)}(r, s, r t) \cdot r^2(\sigma - 1) \). Therefore, we conclude that there exists \( J > > 1 \), \( u^{(x, y)} \) satisfies the assumptions in corollary 2.7. Then by applying corollary 2.7 inductively to \( u^{(s, t \cdot K^{-1})} \), we have \( l \rightarrow J_0^{(s, t \cdot K^{-l-1}, s, t \cdot K^{-l})} \) is strictly increasing. Take \( t_0 := s, t \).

To prove (2), consider \( \sigma' \in (-\infty, \mathcal{AR}_p(u)) \) and \( \sigma \in \mathcal{AR}_p(u, 1) \backslash C_p \) TBD; Let \( K \geq K_0(C_p, \sigma) > 2 \) be fixed as in (1). By definition of \( \mathcal{AR}_p(u) \), \( \lim_{t \rightarrow 0_+} J_0^{(t)}(t, K t) = 0 \). Thus, there exists \( t_0 \rightarrow 0_+ \) such that

\[
J_0^{(t_0)}(t_0, K t_0) \geq J_0^{(t_0)}(t_0, K t) \quad \forall t \in (0, t_0)
\]

Therefore, together with (1) we see that if we denote \( u_j := u(t_j)/\|u(t_j)\|_{L^2(A_1, \xi)} \), then \( u_j \) satisfies an elliptic equation of form (3.18) and a local-bound-global estimate for \( j > > 1 \):

\[
\|u_j\|_{L^2(A_{R-1, R})} \leq C(R, \sigma, \sigma') \|u_j\|_{L^2(A_1, \xi)} \quad \forall 1 < R < t_0/t_0
\]

By standard elliptic estimate \( \text{GT01} \), up to a subsequence, \( u_j \rightarrow u_\infty \) in \( W_{1,loc}^{1,2}(C_p) \) and \( u_\infty \) satisfies the Jacobi field equation \( L_{C_p} u_\infty = 0 \) and the growth estimate near 0 and \( \infty \):

\[
(3.19) \quad \limsup_{r \rightarrow 0_+} J_0^{(u_\infty)}(r, K r) + \lim_{r \rightarrow +\infty} J_0^{(u_\infty)}(r, K r) < +\infty
\]

Now choose \( \sigma, \sigma' \) such that \( \Gamma_{C_p} \cap \{\sigma', \sigma\} \subset \{\mathcal{AR}_p(u)\} = \emptyset \). Then by lemma 3.2 and 3.19, \( \mathcal{AR}_p(u) \in \Gamma_{C_p} \) and \( u_\infty(r, \omega) = \gamma^{-1}((c + c')^\log r) \cdot u(\omega) \) where \( w \in W_{1, loc}^{1,2}(C_p) \) and \( u_\infty \) satisfies an elliptic field \( L_{C_p} u_\infty = 0 \) on \( B_{r_2} \) and \( \xi = 1 \) on \( \partial B_{r_2} \). This completes the proof.

**Corollary 3.15.** Let \( G \) be a Green’s function of \( L^h \) in \( B_{r_2} \) at \( p \); \( \varphi_1 \in \mathcal{B}_0(B_{r_2}) \) be the first eigenfunction of \( -L^h \) in \( \mathcal{B}_0(B_{r_2}) \); \( \xi \in \mathcal{B}(B_{r_2}) \) be the solution of \( -L^h \xi = 0 \) on \( B_{r_2} \) and \( \xi = 1 \) on \( \partial B_{r_2} \).

Then \( \mathcal{AR}_p(G) = \gamma \), \( \mathcal{AR}_p(\varphi_1) = \mathcal{AR}_p(\xi) = \gamma^+ \).

**Proof.** By weak maximum principle, \( \varphi_1, \xi > 0 \); Also by definition, \( G > 0 \). By lemma 3.14 (2), lemma 2.3 and Harnack inequality, \( \mathcal{AR}_p(G), \mathcal{AR}_p(\varphi_1), \mathcal{AR}_p(\xi) \subset \{\gamma^+\} \). And to distinguish them, it suffices to assume in addition that \( C_p \) is strictly stable, in which case \( \gamma^+ > -(n - 2)/2 > \gamma^+ \).

With this assumption, by lemma 2.3 and 3.9 \( \|\cdot\|_{\mathcal{B}} \sim \|\cdot\|_{W_1,2} \) and in particular \( \mathcal{B}_0(B_{r_2}) = W_{0,1,2}(B_{r_2}) \). Therefore, combined with lemma 3.14 (1) and the standard elliptic estimate, we have for every \( u \in \{\varphi_1, G, \xi\}, \mathcal{AR}_p(u) = \gamma^+ \) if and only if \( u \in W_{0,1,2}(B_{r_2}) \). Thus proves the corollary.

**Corollary 3.16.** Let \( G \) be a Green’s function of \( L^h \) in \( B_{r_2} \) at \( p \). Then

\[
\lim_{r \rightarrow 0_+} G(rx)/\inf_{\partial B_r} G = G_0(x)
\]

where \( G_0(r, \omega) = r^{\gamma^+} w_1(\omega)/\inf_{\partial B_{r_2}} w_1 \) is a homogeneous positive Jacobi field on \( C_p \) with fast asymptotic rate. Moreover, the convergence is in \( C^{\infty}_{\text{loc}}(C_p) \).

**Proof.** First note that by Harnack inequality, when \( r \rightarrow 0_+ \), \( G(r^x)/\inf_{\partial B_{r}} \) always subconverges in \( C^{\infty}_{\text{loc}}(C_p) \) to some positive Jacobi field on \( C_p \) with infimum 1 on \( \partial B_{r_2} \). The goal now is to show that the limit is unique.

By lemma 2.3 if \( C_p \) is not strictly stable, then there’s only one such positive Jacobi field on \( C_p \). Hence, it suffices to prove in case when \( C_p \) is strictly stable, i.e. \( \gamma^+ < \gamma^+ \). By corollary 3.15, \( \mathcal{AR}_p(G) = \gamma^+ \); By applying lemma 3.14 (1) to \( \sigma \in (\gamma^+, \gamma^+) \), it’s easy to see that for every sequence \( \gamma_j \rightarrow 0_+ \) such that \( G(r^x) \rightarrow G^0 = (c^+ r^+ + c^- r^-) w_1 \) we have, \( l \rightarrow J^{\sigma}_{G^0}(l_0 K^{-l-1}, l_0 K^{-l}) \)

\[
\lim_{r \rightarrow 0_+} G(rx)/\inf_{\partial B_r} G = G_0(x)
\]
is increasing in \( l \in \mathbb{Z} \) for some \( t_0 \in [1, K) \). Hence \( c^+ = 0 \) and thus \( G_0 \) is of the form in corollary \( 3.16 \).

**Corollary 3.17.** Suppose \( u \in W^{2,2}_0(B_{r_2}) \) with \( L^h u \in L^\infty(B_{r_2}) \). \( G \) be a Green’s function of \( L^h \) in \( B_{r_2} \) at \( p \). Then,

(1) If \( u \in \mathcal{D}(B_{r_2}) \), then \( \mathcal{A}R_p(u) \geq \gamma^+_1 \).

(2) If \( \mathcal{A}R_p(u) \geq \gamma^-_1 \), then there exists a unique \( c \in \mathbb{R} \) such that \( u - c \cdot G \in \mathcal{D}(B_{r_2/2}) \).

**Proof.** (1) follows directly from weak maximum principle and lemma \( 3.15 \) which assert that \( \mathcal{A}R_p(\varphi_1) = \gamma^+_1 \).

To prove (2), consider \( v \in \mathcal{D}(B_{r_2/2}) \) be the solution of \( L^h v = L^h u \) on \( B_{r_2/2}, v = u \) on \( A_{r_2/2, r_2} \). Let \( w := u - v \). By (1) and definition we see that \( \mathcal{A}R_p(w) \geq \gamma^-_1 \) and \( L^h(u - v) = 0 \) on \( B_{r_2/2} \), \( w = 0 \) outside \( B_{r_2/2} \).

If \( \mathcal{A}R_p(w) > \gamma^-_1 \), then by corollary \( 3.15 \) \( \limsup_{t \to 0} \sup_{B_{r_2/2}} |w|/G' = 0 \), where \( G' \) denotes the Green’s function of \( L^h \) at \( p \) in \( B_{r_2/2} \). Then by weak maximum principle, \( w = 0 \).

If \( \mathcal{A}R_p(w) = \gamma^-_1 \), then by lemma \( 3.14 \) and lemma \( 2.2 \) WLOG there exists \( t_1 \to 0_+ \) such that \( w > 0 \) on \( \partial B_{r_1} \). Then by classical weak and strong maximum principle for elliptic PDE, \( w > 0 \) on \( B_{r_2/2} \). Hence by lemma \( 3.12 \), \( w = c'G' \) for some \( c' \in \mathbb{R} \setminus \{0\} \). To go back to a Green’s function on \( B_{r_2} \), consider \( \eta \in C_0^\infty(B_{r_2}((p)) \) which equals to 1 near \( p \) and the solution \( \psi \in \mathcal{D}(B_{r_2}) \) of \( L^h \psi = L^h(\eta \cdot w) \) on \( B_{r_2} \). By lemma \( 3.13 \) and weak maximum principle, \( -\psi + \eta \cdot w \) is a Green’s function at \( p \) in \( B_{r_2} \), combined with lemma \( 3.12 \) \( u - cG \in \mathcal{D}(B_{r_2/2}) \) for some \( c \in \mathbb{R} \), which completes the proof.

Now we turn to the globally defined functions on \( \Sigma \). Recall that \( U := U \cap \Sigma \). Denote \( G_{r,p} \) to be a Green’s function of \( L^h \) in \( B_r(p) \), where \( p \in \text{Sing}(\Sigma), r \in (0, r_2] \).

**Definition 3.18.** Define \( \mathcal{B}_0(U) \bigoplus \mathbb{R}_{L^h}(\text{Sing}(\Sigma) \cap U) \) to be the function space linearly generated by \( \mathcal{B}_0(U) \) and \( G_{r,p}, p \in \text{Sing}(\Sigma) \cap U \).

The proof of corollary \( 3.17 \) (2) actually guarantees that this space depends only on \( \Sigma, U \) and \( p \), but not on the choice of \( r \).

**Corollary 3.19.** Suppose \( L^h \) be nondegenerate on \( \mathcal{B}_0(U) \). Then, for each \( p \in \text{Sing}(\Sigma) \cap U \), there is a unique (up to normalization) \( 0 \neq G_p \in \mathcal{B}_0(U) \bigoplus \mathbb{R}_{L^h}(\text{Sing}(\Sigma) \cap U) \) such that \( L^h G_p = 0 \) on \( U \), \( G_p \in \mathcal{B}_0(U) \cap \text{Clos}(B_{r_2}((p))) \). Moreover, \( \mathcal{B}_0(U) \bigoplus \mathbb{R}_{L^h}(\text{Sing}(\Sigma) \cap U) \) is linearly generated by \( \mathcal{B}_0(U) \) and \( \{G_p : p \in \text{Sing}(\Sigma) \cap U\} \).

If further \( -L^h \) is strictly positive on \( \mathcal{B}_0(U) \), then \( G_p \) can be chosen positive on \( U \).

Call such \( G_p \) a **Green’s function of \( L^h \) on \( U \) at \( p \).**

**Proof.** The existence of \( G_p \) follows the same construction in the proof of corollary \( 3.17 \) (2) from \( G' \) to \( G \); The uniqueness of such \( G_p \) up to normalization follows directly from lemma \( 3.12 \) and corollary \( 3.15 \). The positivity of \( G_p \) when \( -L^h \) is strictly positive follows from weak maximum principle.

**Corollary 3.20.** Suppose \( u \in W^{2,2}(U) \), \( L^h u \in L^\infty(\Sigma) \) and \( u|_{\partial U} = 0 \). If \( \mathcal{A}R_p(u) \geq \gamma^-_1 \) for every \( p \in \text{Sing}(\Sigma) \cap U \), then \( u \in \mathcal{B}_0(U) \bigoplus \mathbb{R}_{L^h}(\text{Sing}(\Sigma) \cap U) \).

**Proof.** This directly follows from corollary \( 3.17 \).

The following lemma shows that having top asymptotic growth rate near each singularity is a generic property for solutions of \( L^h u = f \).

**Lemma 3.21.** Suppose \( L^h \) be nondegenerate on \( \mathcal{B}_0(U), p \in \text{Sing}(\Sigma) \cap U \). Then there exists an open subset \( \mathcal{V}_p \subset L^\infty(U) \) satisfying

(1) \( f \in \mathcal{V}_p \) implies \( cf \in \mathcal{V}_p, \forall c \in \mathbb{R} \setminus \{0\} \);
(2) If \( f \in \mathcal{Y}_p \) and \( g \in L^\infty(U) \setminus \mathcal{Y}_p \), then \( f + g \in \mathcal{Y}_p \);

(3) \( \mathcal{Y}_p \cap C_c^\infty(U) \) is dense in \( C_c^\infty(U) \);

(4) If \( u \in \mathcal{R}_0(U) \) is the solution of \( L^h u = f \) for some \( f \in \mathcal{Y}_p \), then \( \mathcal{AR}_p(u) = \gamma^+(C_p) \).

In particular, for every \( k \in \mathbb{N} \cup \{0\} \) and \( \alpha \in [0,1] \), \( \{ f \in C^{k,\alpha}(\text{Clos}(U)) : f|_\Sigma \in \cap_{\gamma \in \Sigma} \mathcal{R}_p \} \) is open and dense in \( C^{k,\alpha}(\text{Clos}(U)) \).

Proof. For each \( p \in \text{Sing}(\Sigma) \cap U \), define \( \xi_p \in \mathcal{R}(\mathcal{B}_{r_2}(p)) \) to be the solution of \( L^h \xi_p = 0 \) with \( \xi_p|_{\partial \mathcal{B}_{r_2}(p)} = 1 \); and let \( \hat{\xi}_p \in \mathcal{B}_{r_2}(p) \) be the solution of \( (-L^h + 1)\hat{\xi}_p = 0 \) with \( \hat{\xi}_p|_{\partial \mathcal{B}_{r_2}(p)} = 1 \). Then by weak maximum principle, \( \xi_p \geq \hat{\xi}_p > 0 \) on \( \mathcal{B}_{r_2}(p) \) for every \( p \in \text{Sing}(\Sigma) \cap U \), and by corollary 3.15 \( \hat{\xi}_p(x) \to +\infty \) as \( x \to p \).

Now fix \( p \), define

\[
\mathcal{Y}_p := \{ f \in L^\infty(U) : u := (L^h)^{-1}f \in \mathcal{R}_0(U) \text{ satisfies } \limsup_{x \to p} |u|(x)/\xi_p(x) > 0 \}
\]

(1), (2) directly follows from this definition; (4) follows from lemma 3.14 and corollary 3.15. If (1)-(4) and openness of \( \mathcal{Y}_p \) is verified, then by Baire category theorem,

\[
\bigcap_{p \in \text{Sing}(\Sigma) \cap U} \{ f \in C^{k,\alpha}(\text{Clos}(U)) : f|_\Sigma \in \mathcal{Y}_p \} \subset C^{k,\alpha}(\text{Clos}(U)) \text{ is open and dense}
\]

Now we proceed to prove (3) and openness of \( \mathcal{Y}_p \) in \( L^\infty \).

Proof of openness: Clearly, it suffices to show that there exists a \( C = C(h, \Sigma, U, M, g) > 0 \) such that for every \( \| f \|_{L^\infty(U)} \leq 1 \) we have

\[
(3.20) \limsup_{x \to p} |(L^h)^{-1}f|(x)/\xi_p(x) \leq C
\]

To prove this bound, denote \( (L^h)^{-1}f \) by \( v \). First recall that by proposition 3.5 (3) and interior \( C^0 \)-estimate, \( |v| \leq C_1 = C_1(h, \Sigma, U, M, g, r_2) \) on \( \mathcal{A}_{r_2/2r_2} \). Let \( \kappa := \inf_{\mathcal{B}_{r_2}(p)} \hat{\xi}_p(0) > 0 \), \( C := \max\{1, C_1\}/\kappa \). Then by \( |L^h v| \leq 1 \leq -C \cdot (2\xi_p - \hat{\xi}_p) \) on \( \mathcal{B}_{r_2}(p) \), \( |v| \leq C \cdot \kappa \leq C(2\xi_p - \hat{\xi}_p) \) on \( \partial \mathcal{B}_{r_2}(p) \) and weak maximum principle, we see that \( |v| \leq C(2\xi_p - \hat{\xi}_p) \) on \( \mathcal{B}_{r_2}(p) \). Hence (3.20) is proved.

Proof of (3): By (1) and (2), it suffices to show that \( \mathcal{Y}_p \cap C_c^\infty(U) \neq \emptyset \). To see this, first recall that by proposition 3.5 (5), there exists \( \vartheta \in C_c^\infty(U; \mathbb{R}_+) \) such that \( -L^h \vartheta \) is strictly positive.

Let \( w := (-(L^{h+\vartheta})^{-1}(\vartheta) \in \mathcal{R}_0(U) \), then \( w > 0 \) on \( U \) by weak and strong maximum principle; Let \( r_3 \in (0, r_2) \) such that \( \mathcal{B}_{r_3}(p) \cap \text{spt}(\vartheta) = \emptyset \). Then \( w \) satisfies the equation \( -L^h w = \vartheta - \vartheta w = 0 \) on \( \mathcal{B}_{r_3}(p) \), and by weak maximum principle, \( w \geq \varepsilon \xi_p \) for some \( \varepsilon > 0 \). Therefore, \( \vartheta - \vartheta w \in \mathcal{Y}_p \cap C_c^\infty(U) \). \( \square \)

4. Asymptotics and associated Jacobi fields

Let \( (\Sigma, \nu) \subset (M, g) \) be a minimal hypersurface with normal field \( \nu; U \subset \subset M \) be a smooth domain as in section 2. Suppose \( \{ g_j \} \) be a family of metric smoothly converges to \( g_0 \). \( V_j \) be a family of stationary integral varifolds in \( (U, g_j) \) different from \( \Sigma \) and converging to \( |\Sigma| \). Then by Allard regularity theorem \( \text{All72, Sim83b} \), the convergent is in \( C_c^\infty(U \cap \Sigma) \).

Definition 4.1. Call \( \exists \neq \phi \in C^\infty(\Sigma \cap U) \) a generalized Jacobi field associated to \( \{ (V_j, g_j) \}_{j \geq 1} \), if \( \exists t_j \to 0_+ \) and \( v_j \in C^\infty(\Sigma \cap U) \) such that \( v_j/t_j \to \phi \) in \( C^\infty_\text{loct}(U \cap \Sigma) \) and that \( \forall U' \subset \subset U \setminus \text{Sing}(\Sigma) \),

\[
|\text{graph}_{\Sigma}(v_j)|_{U'} = V_j \cup U' \text{ for sufficiently large } j
\]

Clearly, if \( \phi \) is an associated Jacobi operator to \( \{ (V_j, g_j) \} \), then so is \( c\phi \) for \( c > 0 \). Also, if \( g_j = g \) and \( \phi \) is a Jacobi field associated to some \( (V_j, g_j) \), then \( \phi \) satisfies the Jacobi field equation \( L_{\Sigma} \phi = 0 \).

When \( g \equiv g_j, U = M \) and \( \Sigma \) is regular, after passing to subsequences, the existence of associated Jacobi field directly follows from Allard regularity and elliptic estimates, see \( \text{Sim17} \). However, if \( \text{Sing}(\Sigma) \neq \emptyset \), even when \( g_j = g \), it’s unclear whether for any such family of stationary integral
varifold \{V_j\}, there exists an associated Jacobi field.

From now on, suppose that \(\Sigma\) is closed, locally stable and has only strongly isolated singularities, and \(U = M\). Let \(\beta\) be a symmetric 2-tensor on \(M\) such that

\[
q(\beta) := \operatorname{div}_\Sigma(\beta(\nu, \cdot)) - \frac{1}{2} \operatorname{tr}_\Sigma(\nabla^M\beta) \quad \text{is not identically 0 on } \Sigma
\]

The goal of this section is to prove the following

**Theorem 4.2.** Let \(\Sigma, \beta\) be as above, \(\{g_j\}\) be a family of Riemannian metric. Suppose either \(g_j \equiv g\) for all \(j\), or \(\exists c_j \geq 0\) and \(\beta_j \to \beta\) in \(C^4\) such that \(g_j = g + c_j\beta_j\).

Let \(V_j \neq |\Sigma|\) be a stationary integral varifold in \((M, g_j)\), \(V_j \to |\Sigma|\). Suppose for every \(p \in \text{Sing}(\Sigma)\), one of the following holds

(A) the tangent cone \(C_p\) of \(\Sigma\) at \(p\) is area-minimizing at least in one side, see [Lin87, Liu19].

(B) there exists some neighborhood \(U_p \subset M\) of \(p\) such that \(V_j \cap U_p\) are stable minimal hypersurfaces (possibly with multiplicity) in \((U_p, g_j)\), \(j > 1\).

Then after passing to a subsequence, there exists a (generalized) associated Jacobi field \(\phi \in \mathscr{B}(\Sigma) \bigoplus \mathbb{R}_L\{\text{Sing}(\Sigma)\}\) to \(\{(V_j, g_j)\}_{j \geq 1}\); Moreover, \(\phi\) satisfies \(L_\Sigma \phi = 0\) provided \(g_j \equiv g\); Or \(L_\Sigma \phi = cq(\beta)\) for some \(c \geq 0\) provided \(g_j = g + c_j\beta_j\).

**Remark 4.3.**
1. The assumption (B) is always satisfied when \(n = 7\) and \(V_j\) are locally stable minimal hypersurfaces with the same index as \(\Sigma\). Hence, the theorem applies in this case.
2. In section 4.2, we study in more details the possible asymptotic rate of \(\phi\) above near each \(p \in \text{Sing}(\Sigma)\). Roughly speaking, they are modeled on the asymptotic rate of stable minimal hypersurface in \(\mathbb{R}^{n+1}\) near infinity to the tangent cone \(C_p\). See corollary 4.12.

4.1. **Asymptotics of minimal hypersurfaces near infinity.** Let \(C \subset \mathbb{R}^{n+1}\) be a regular hypercone (not necessarily stable) throughout this subsection, with cross section \(S\) and normal field \(\nu\). The notions of geometric quantities on \(C\) and \(S\) will be the same as section 4.2. Recall that \(E_\pm\) are the two connected components of \(\mathbb{R}^{n+1} \setminus C\), with \(\nu\) pointing into \(E_+\).

Suppose that \(V\) is an integral varifold in \(\mathbb{R}^{n+1} \setminus B_1^{n+1}\) (not necessarily closed), with finite density at infinity, i.e.

\[
\limsup_{R \to +\infty} \frac{1}{R^n} \|V|((\mathbb{B}_1^{n+1} \setminus B_R^{n+1})) < +\infty
\]

**Definition 4.4.** Call \(V\) asymptotic to \(C\) near \(\infty\), if there’s a function \(h \in C^2(C)\) and \(R_0 > 1\) such that

\[
\frac{1}{R} |h|((R, \cdot)) + |\nabla h|((R, \cdot)) + R|m^2 h|((R, \cdot)) \to 0 \quad \text{in } C^0(S) \quad \text{as } R \to +\infty
\]

and that

\[
|\text{graph}_C(h)|((\mathbb{R}^{n+1} \setminus B_R^{n+1})) = V|((\mathbb{R}^{n+1} \setminus B_R^{n+1}))
\]

For such \(V\), define

\[
\mathcal{AR}_{\infty}(V) = \mathcal{AR}_{\infty}(V; C) := \inf \{\sigma : \limsup_{R \to +\infty} R^{-\sigma} h(R, \cdot) = 0 \quad \text{in } C^0(S)\}
\]

Called the asymptotic rate of \(V\) to \(C\) at infinity. We use the convention that \(\inf \emptyset := +\infty\) and \(\mathcal{AR}_{\infty}(C; C) = -\infty\).

If \(\Sigma \subset \mathbb{R}^{n+1} \setminus B_1^{n+1}\) be a hypersurface, denote \(\mathcal{AR}_{\infty}(\Sigma) := \mathcal{AR}_{\infty}(|\Sigma|)\).

For stationary integral varifolds with finite density at infinity, by Allard regularity [All72, Sim83b], being asymptotic to \(C\) is equivalent to having \(|C|\) to be the tangent varifold at \(\infty\).

The goal of this subsection is to prove the following asymptotic rate estimate of minimal hypersurface towards it’s tangent cone at infinity.
Theorem 4.5. Suppose $V \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ be a stationary integral varifold asymptotic to a regular minimal hypercone $C$ near infinity, $V \not\subset [C]$. Then

1. If $C$ is one-sided minimizing (resp. strictly minimizing), then $\mathcal{AR}_\infty(V) \geq \gamma_1^+(\text{resp. } \gamma_1^+)$. Moreover, if $C$ is area minimizing and $\mathcal{AR}_\infty(\Sigma) = \gamma_1^+$, then $\Sigma$ has no singularity and lies on one side of $C$.
2. If $V = [\Sigma]$ for some stable minimal hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, then $\mathcal{AR}_\infty(\Sigma) \geq \gamma_1^-$. If further, $\mathcal{AR}_\infty(\Sigma) = \gamma_1^-$, then $\Sigma$ has no singularity and lies on one side of $C$, inside which $\Sigma$ and $C$ are minimizing.

We start with some preparation.

Lemma 4.6. Suppose $\Sigma \subset \mathbb{R}^{n+1} \setminus \mathbb{B}_1^{n+1}$ be a minimal hypersurface asymptotic to $C$ near infinity. Then

1. $\mathcal{AR}_\infty(\Sigma) \in \Gamma_C \cup \{-\infty\}$, where $\Gamma_C$ is defined below (2.7).
2. If $\gamma := \mathcal{AR}_\infty(\Sigma) \in (-\infty, 1)$, $h$ be the graphical function as in (4.2), then

$$h(r, \omega) = r^\gamma (c + c' \log r) w_j(\omega) + O_2(r^{\gamma - \epsilon'})$$

for some $(c, c') \neq (0, 0), c' = 0$ unless $\gamma = \gamma_1^+ = \gamma_1^-$ (in which case $C$ is not strictly stable); $\epsilon = \epsilon(\Sigma) > 0; w_j(\omega)$ is a unit eigenfunction of $\mathcal{L}_S$ as in section 2.2, and $\gamma \in \{\gamma_1^\pm\}$.

This characterization is well-known, at least when $\gamma = \gamma_1^+$, in [HS85, 1.9]. For sake of completeness, we sketch the proof here. In what follows, every constant depends on the cone $C$.

Proof. Since $1 \in \Gamma_C$, (1) follows from (2). To prove (2), since $\Sigma$ is minimal, $h$ satisfies the minimal surface equation

$$\mathcal{M}_C h = -L_C h + \mathcal{R}_C h = 0$$

where by [CHS84, (2.2)],

$$\mathcal{R}_C h (r, \omega) := N(r, h, r, \nabla h) \cdot \nabla^2 h + r^{-1} P(r, h, r, \nabla h)$$

and

$$|P(x, z, p)| \leq C(|z| + |p|)^2$$

$$(N(x, z, p)| + |P_z(x, z, p)| + |P_p(x, z, p)| \leq C(|z| + |p|)$$

By elliptic estimate, for $R >> 2R_0$,

$$\|\nabla h\|_{C^0(\mathbb{R}^{n+1})} + R\|\nabla^2 h\|_{C^0(\mathbb{R}^{n+1})} \leq \frac{C}{R} \|h\|_{C^0(\mathbb{R}^{n+1})}$$

Hence by definition, for any $\delta > 0$, since $R^{-\gamma - \delta} h(R, \cdot) \to 0$ as $R \to +\infty$, we have

$$\|\mathcal{R}_C h(R, \cdot)\|_{C^0} \leq C_0 R^{\gamma - 2 + (\gamma + 1 + 2\delta)} \forall R >> 2R_0$$

Fix $\delta > 0$ such that $\gamma - 1 + 3\delta < 0$ and $\sigma := \gamma + (\gamma - 1 + 3\delta) \notin \Gamma_C$. By lemma 2.3, $\exists u \in C^\infty(C \setminus B_1)$ solving $L_C u = \mathcal{R}_C h$ and

$$\|u(R, \cdot)\|_{L^2(S)} \leq CR^\sigma$$

But since $L_C (h - u) = 0$ and $\forall \delta' > 0, \limsup_{R \to \infty} R^{-\gamma + \delta'} \|h(R, \cdot)\|_{C^0} < 0$, by (2.6), $h + u = R^{\gamma + \epsilon} (c + c' \log r) + o.t.$ and so does $h$.

The following lemma is an analogue of [HS85, Theorem 2.1] for stable minimal cones.

Lemma 4.7. Suppose $\Sigma \subset \mathbb{R}^{n+1}$ be a connected stable minimal hypersurface with finite density at infinity; Also suppose that $\Sigma$ is contained in $E_+$ outside a large ball, i.e. $\exists R_0 > 1$ such that $\Sigma \subset \mathbb{B}_0^{n+1} \cup E_+$. Then

1. Sing$(\Sigma) = 0$, $\Sigma$ is contained entirely in $E_+$ and area-minimizing in $E_+$.
2. $C$ is area-minimizing in Clos$(E_+)$.
3. $\{r \cdot \Sigma\}_{r > 0}$ foliates $E_+$. 

(4) Any stationary integral $n$-varifold $V$ with $\text{spt}V \subset E_+$ and having the same density at $\infty$ as $C$ is a rescaling of $|\Sigma|$.

**Remark 4.8.**

1. When $C$ is a priori assumed to be area-minimizing, (1), (3), (4) of lemma 4.4 is a corollary of [HSS85, Theorem 2.1] by a barrier argument. However, in the proof we present here, (1)(3) are established first based on the linear analysis on $\Sigma$ in section 2.4 and the one-sided minimizing property of $C$ is part of the conclusion. With (1) and (2), the proof (4) is similar to [HSS85]. For sake of completeness, we also sketch the idea of (4).

2. By solving Plateau problem with one-sided perturbed boundary, [H-S] shows that such $\Sigma$ in lemma 4.4 do exist if $C$ is area-minimizing. The argument also works for one-sided area-minimizers.

3. We assert here that the stability assumption on $\Sigma$ can’t be dropped. See (P1.2) for further discussion in section 8.

**Proof.** Recall that we use the notion $B_R := B_R^{n+1} \cap \Sigma$. Since $\Sigma$ is stable with finite density at infinity, by [SS81], any tangent varifold $V_{\infty}$ of $|\Sigma|$ at infinity is a stable minimal cone. And since $\Sigma$ is contained in $E_+$ outside a large ball, $\text{spt}(V_{\infty}) \subset \text{Clos}(E_+)$.

Then by $0 \in \text{spt}V_{\infty} \cap C$ and strong maximum principle [SW89][Mm96], $\text{spt}(V_{\infty}) = C$: Since $C$ is regular, by [SS81] and [CS97], $V = |\Sigma|$, i.e. $\Sigma$ asymptotic to $C$ near infinity. Take $R_0 > 1$ and $h \in C^2(C)$ such that (4.1) and (4.2) holds.

Since $\Sigma$ is contained in $E_+$ near $\infty$, by a blow down argument and combining lemma 4.6 and 4.7 see also [HSS85, 1.9], when $R \to + \infty$

\begin{align}
(4.3) \quad \begin{cases}
\text{either } h(R, \omega) = (c_1 + c_2 \log R) R^{\gamma_1} w_1(\omega) + O(R^{\gamma_1 - \epsilon}) \\
\text{or } h(R, \omega) = c_1 R^{\gamma_1} w_1(\omega) + O(R^{\gamma_1 - \epsilon})
\end{cases}
\end{align}

for some $\epsilon > 0$; $c_1 + c_2 \log R > 0$ for $R > R_0$; $c_2 = 0$ unless $\gamma_1 = \gamma_1 = -(n - 2)/2$.

Let $\nu_\Sigma$ be the normal field of $\Sigma$, pointing away from $C$. Consider the function $\psi = (\nabla R^{n+1})|x|^2 \cdot \nu_\Sigma$ on $\Sigma$. Note that $|\nabla \psi|(x) \leq |x| \cdot |A_\Sigma|_x$ and $L_\Sigma \psi = 0$; And by [CN13], $|A_\Sigma| \in L^{7/2}_{\text{weak}}(B_R)$, $|\nabla A_\Sigma| \in L^{7/2}_{\text{weak}}(B_R)$, thus $\psi \in W^{1,2}(B_R)$ for each $R > 0$. By (4.3), $\psi > 0$ outside some large ball $B_{R_1}$, $R_1 > R_0$; Hence by stability of $\Sigma$ and weak maximum principle, $\psi > 0$ on the whole $\Sigma$. Hence by lemma 2.14, $\psi(x) \to \infty$ as $x \to \text{Sing}(\Sigma)$. But notice that by its definition, $\psi$ is bounded on every $B_R$. This means $\text{Sing}(\Sigma) = \emptyset$.

To show that $\Sigma \subset E_+$, let $\Omega_+ := E_+ \cap S^n$ and consider the smooth map

$$p : \Sigma \setminus \{0^{n+1}\} \to S^n \quad x \mapsto \frac{x}{|x|}$$

By $\psi > 0$, $p$ is a diffeomorphism between $\Sigma \setminus B_{R_1}$ and a collar neighborhood of $S = \partial \Omega_+$ in $\Omega_+$. By the following topological lemma 4.9 $p$ is a global diffeomorphism onto $\Omega_+$. Hence, $\Sigma \subset E_+$ and $\{r \cdot \Sigma\}_{r > 0}$ foliates $E_+$.

To show that $\Sigma$ and $C$ are area-minimizing in $\text{Clos}(E_+)$, it suffices to observe that $\{r \cdot \Sigma\}_{r > 0}$ determines a calibration $\Theta := \nu_\Sigma dVol_{R^{n+1}}$ on $\text{Clos}(E_+) \setminus \{0\}$, where $\nu_\Sigma$ be the normal vector field of $\{r \cdot \Sigma\}_{r > 0}$, $C^1$ extended to $\text{Clos}(E_+) \setminus \{0\}$. Hence, for any $R > 0$, and any integral cycle $T \in \mathcal{Z}_n(B^{n+1}_R)$,

$$M_{\mathbb{R}^{n+1}}(T) \geq \langle T, \Theta \rangle = \langle |C|, \Theta \rangle = M_{\mathbb{R}^{n+1}}(|C|) \quad \text{if } \text{spt}(|C| - T) \subset \mathbb{B}^{n+1}_R$$

$$M_{\mathbb{R}^{n+1}}(T) \geq \langle T, \Theta \rangle = \langle |\Sigma|, \Theta \rangle = M_{\mathbb{R}^{n+1}}(|\Sigma|) \quad \text{if } \text{spt}(|\Sigma| - T) \subset \mathbb{B}^{n+1}_R$$

We finally sketch the idea of (4). If $V$ is a stationary integral $n$-varifold with the same density at $\infty$ as $C$ and $\text{spt}V \subset E_+$, then by a similar argument as the first part of this proof, $|C|$ is the
tangent varifold of $V$ at infinity, and hence $V$ could be parametrized as a graph over $C$ near infinity as in (4.2) and (4.3). Let

$$r_+ := \inf \{ r : \text{for any } R > r, \text{spt } V \cap R \cdot \Sigma = \emptyset \}$$
$$r_- := \sup \{ r : \text{for any } R < r, \text{spt } V \cap R \cdot \Sigma = \emptyset \}$$

By (4.3) and strong maximum principle, $0 < r_- = r_+ < \infty$ and $V = |r_+ \cdot \Sigma|$. \hfill \Box

**Lemma 4.9.** Let $N$ be a connected compact $n$ manifold with nonempty boundary, $M$ be a closed simply connected $n$ manifold, $p : N \to M$ be a local diffeomorphism onto its image, and restricted to a bijection near $\partial N$. If $M \setminus p(\partial N) = M_+ \cup M_-$ has 2 connected component, with $p(N) \cap M_+ \neq \emptyset$, then $p$ is a diffeomorphism onto $M_+$. \hfill \Box

**Proof.** Notice that the glued map $p_\partial : N \cap \partial N \to M$ is a local homeomorphism between closed manifolds, hence a covering map. Since $N$ is connected and $M$ is simply connected, this is a bijection, so is $p$. \hfill \Box

**Lemma 4.10.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stable minimal hypersurface. Then the space of positive Jacobi field on $\Sigma$ is nonempty, i.e.

$$G(\Sigma) := \{ u \in C^\infty(\Sigma) : L_\Sigma u = 0, \ u > 0 \} \neq \emptyset$$

And if $\text{Sing}(\Sigma)$ is bounded, then $\forall u \in G(\Sigma)$, we have $u(x) \to +\infty$ if $x \to x_\infty \in \text{Sing}(\Sigma)$.

Moreover, if $\Sigma$ is asymptotic to $C$ near infinity for some regular minimal hypercone $C$, assume WLOG (by a rescaling) $\text{Sing}(\Sigma) \subset B_1$, then for every $\epsilon > 0$,

$$u(x) \geq C(\Sigma, \epsilon, u)|x|^n - \epsilon > 0 \quad \forall x \in \Sigma \setminus B_2$$

**Proof.** To see $G(\Sigma) \neq \emptyset$, consider an exhaustion $\{ U_i \}_{i \geq 1}$ of $\Sigma$ and $B_i := B_i(p_i) \subset U_i$, where $p_i \to \infty$, $r_i \in (0, 1)$. Let $u_i \in W^{1,2}_0(U_i)$ be the solution of $L_\Sigma u_i = 0$ on $U_i \setminus B_i$ and $u_i \equiv 1$ on $B_i$. By the stability of $\Sigma$ and maximum principle, $u_i > 0$ on $U_i$; Also let $q \in \Sigma$ be fixed. Then by Harnack inequality, up to a subsequence,

$$u_i / u_\infty(q) \to u_\infty \quad \text{in } C^\infty_\text{loc}(\Sigma)$$

for some positive Jacobi field $u_\infty \in G(\Sigma)$.

The unboundedness of any $u \in G(\Sigma)$ towards singular set follows from lemma 2.11

To deduce the lower bound of $u$ near infinity, suppose WLOG that $h \in C^\infty(C)$ and $R_0 = 1$ such that (1.12) and (1.13) holds for $V = |\Sigma|$. Parametrize $\Sigma \setminus B_1$ by

$$C \to \Sigma \quad x \mapsto x + h(x)\nu(x)$$

By (1.12), under this parametrization and the normal coordinates of $C$, $L_\Sigma = a_{ij} \partial_i(a^{ij} \partial_j u) + |A_\Sigma|^2 u$, where

$$\sup_{\omega \in S} |a_{ij} - \delta^{ij}|(R, \omega) + R^3 |A_\Sigma|^2 - |A_\Sigma|^2 |R, \omega| \to 0 \quad \text{as } R \to \infty$$

Consider for each $R > 2$, $c_R := \inf_{\partial B_R} u$ and $v_R \in C^\infty(C \setminus \mathbb{B}_{2/R})$ defined by

$$v_R(r, \omega) := u(rR, \omega) / c_R$$

By elliptic estimate, up to subsequences, $v_R \to v_\infty$ in $C^0_\text{loc}(C)$ for some positive Jacobi fields $v_\infty$ on $C$ with $\inf_{\partial \mathbb{B}_1/C} v_\infty = 1$. By lemma 2.3

$$v_\infty(r, \omega) \geq C(S)r^{n/2}$$

(4.4)

The lower bound of decaying rate for $u$ then follows from (4.4) and an approximation argument. \hfill \Box
Proof of theorem 4.5 (1). Let \( \Sigma_+ = \partial P_+ \) be the leaf of Hardt-Simon foliation in theorem 2.8 lying on the side \( E \) where \( C \) is minimizing (resp. strictly minimizing), \( P_+ \subset E \); Let \( h_+, h \) be the graphical functions of \( \Sigma_+ \) and \( V \) correspondingly over \( C \). Suppose otherwise that \( \mathcal{A}_{R_{\infty}}(V) < \gamma_1^+ \) (resp. \( \gamma_1^- \)). Then by lemma 4.6, (2.14) and (2.15), \( h_+ > h \) near infinity. Hence for \( R >> 1 \), \( R \cdot P_+ \cap \text{spt}(V) = \emptyset \). By considering

\[
\inf \{ R : s \cdot P_+ \cap \text{spt}(V) = \emptyset, \forall s > R \}
\]

and using the strong maximum principle [SW89], we see that \( \text{spt}(V) \subset \mathbb{R}^{n+1} \setminus \text{Clos}(E) \), i.e. lying on the other side of \( C \). Then by lemma 4.7 and theorem 2.8, \( V \) is a rescaling of Hardt-Simon foliation on the compliment of \( \text{Clos}(E) \) and thus \( \mathcal{A}_{R_{\infty}}(V) \geq \gamma_1^+ \) (resp. \( \gamma_1^- \)), from which we get a contradiction.

If \( \mathcal{A}_{R_{\infty}}(V) \in \{ \gamma_1^\pm \} \), then by lemma 4.6 \( \text{spt}(V) \) lies on one side of \( C \) near infinity. By using the Hardt-Simon foliation on the other side and strong maximum principle as above, one see that \( \text{spt}(V) \) lies entirely on one side of \( C \), hence becomes a leaf of Hardt-Simon foliation by theorem 2.8.

Proof of theorem 4.5 (2) Consider \( \psi := \nabla_{R^{n+1}} |x|^2 \cdot \nu_\Sigma \). By lemma 4.6 if \( \mathcal{A}_{R_{\infty}}(\Sigma) < \gamma_1^- \), then for each \( \epsilon > 0 \), \( \exists R(\Sigma, \epsilon) > 1 \) such that

\[
|\psi(x)| \leq |x|^{\gamma_1^- + \epsilon} \quad \forall |x| > R(\Sigma, \epsilon)
\]

On the other hand, let \( v \in G(\Sigma) \) be a positive Jacobi field on \( \Sigma \). By lemma 2.14 and weak maximum principle, \( \exists \, c > 0 \) such that \( cv > \psi \) on \( \Sigma \). Consider the smallest one among all such \( c \) and using strong maximum principle, we conclude that \( \psi = 0 \) and thus \( \Sigma = C \), which contradicts to our assumption. This proves the lower bound on \( \mathcal{A}_{R_{\infty}}(\Sigma) \).

When \( \mathcal{A}_{R_{\infty}}(\Sigma) \in \{ \gamma_1^\pm \} \), the conclusion follows from lemma 4.6 and 4.7. \( \square \)

4.2. Finiteness of associated Jacobi fields. The first goal of this subsection is to prove theorem 4.7. The key is a growth rate estimate near each singularity. We begin with some notations. Recall \( (\Sigma, \nu) \subset (M, g) \) is a two sided, locally stable minimal hypersurfaces with strongly isolated singularities. The geometric quantities of \( \Sigma \) are defined in section 2.3.

Let \( \tau_\Sigma(1) \) be defined below (2.17): \( 0 < \tau < \tau_\Sigma(1)/2 \). When working in \( B^M_{\tau}(p) \) for some singular point \( p \), we identify \( \Sigma \cap B^M_{\tau}(p) \leftrightarrow T_p M \) as in section 2.3 and parametrize it by the tangent cone \( C_p \) of \( \Sigma \) at \( p \) as in (2.16). Denote \( \Gamma_{C_p} = \{ \chi_k(C_p) \}_{k \geq 1} \) to be the asymptotic spectrum defined in (2.7) for \( C = C_p \). We may simply write \( \gamma_k^\pm(C_p) \) as \( \gamma_k^\pm \) if there’s no confusion.

We shall first prove a general local dichotomy near every fixed \( p \in \text{Sing}(\Sigma) \). Recall for a domain \( \Omega \subset \Sigma \) and \( k > 0 \), \( \| \cdot \|_{k, \Omega} \) norm is introduced in (2.18) and will be used frequently in this subsection.

Consider in general \( \{ g_j \}_{j \geq 1} \) a family of smooth metrics \( C^k \) converges to \( g \) on \( M \); \( \{ V_j \}_{j \geq 1} \) are stationary integral varifolds in \( (B^M_{\tau}(p), g_j) \) which converges to \( (\Sigma)_0, B^M_{2\tau}(p) \) in \( B^M_{\tau}(p) \) in varifold sense. Suppose either of the following holds,

(A') \( C_p \) is area-minimizing at least in one side.
(B') \( \text{spt}(V_j) \) are stable minimal hypersurfaces in \( B^M_{\tau}(p) \).

For any \( \delta \in (0, 1) \), call \( \{ (\eta_{p, \tau_j})_j V_j \} \) \( \delta \)-minimal blow up sequence of \( \{ V_j \}_{j \geq 1} \) at \( p \) if

\[
\tau_j := \inf \{ t > 0 : V_j \cap A^M_{t, \tau}(p) = \text{graph}_{\Sigma}(v_j) \cup A^M_{1, \tau} \text{ for some } \| v_j \|_{2, A_1, \tau} \leq \delta \} > 0
\]

Note that by Allard compactness [All72, Sim83b] and the minimality of \( \tau_j \), for every \( 0 < \delta < \tau_j \), the \( \delta \)-minimal blow up sequence sub-converges to some stationary integral varifold \( |C_p| \neq V_{\infty} \subset \mathbb{R}^{n+1} \) which is a exterior graph over \( C_p \). Hence by (A'), (B') and theorem 4.3

\[
\mathcal{A}_{R_{\infty}}(V_{\infty} ; C_p) \geq \gamma_1^-(C_p)
\]

Call such \( V_{\infty} \) a \( \delta \)-minimal blow-up limit of \( \{ V_j \}_{j \geq 1} \).
For every $\sigma \in (-\infty,1) \setminus \Gamma_{C_p}$ fixed, let $K = K_0(C_p,\sigma)$ be determined in lemma 2.6. Let $\delta_0, N$ be determined in corollary 2.7 both depend on $C_p,\sigma, K$. For each $Q \in \mathbb{R}_+, \phi \in L^2(C_p)$ and $r > 0$, define

\begin{equation}
\hat{J}_{p,v}(r;Q) := \sup \left\{ Q, \int_{A_{r,v}(p)} \phi^2(x) \cdot |x|^{-n-2\sigma} \, d\nu_{C_p}(x) \right\}
\end{equation}

Lemma 4.11. Let $\Sigma, M, g, \tau, \sigma, K, N$ be described above; $\kappa > 0$. Then there exists $\delta_1 > 0$ and $\delta_0 \in (0,\tau)$ such that, if $\{ (V_j, g_j) \}_{j \geq 1}$ satisfying either (A') or (B') are given as above; then one of the following holds,

(i) $\exists \delta_1$-minimal blow-up limit $\hat{V}_\infty$ of $\{ V_j \}_{j \geq 1}$ such that $\mathcal{AR}_\infty(\hat{V}_\infty; C_p) \leq \sigma$;

(ii) If $\tau_j \to 0_+$ and $v_j \in C^2(\mathbb{B}_{2r}(p))$ be the $\delta_3$-graphical function of $V_j$ over $A_{r_j,\tau_j}$, $i.e v_j$ satisfies $\| v_j \|^2_{L^2(A_{r_j/2r,p}(p))} \leq \delta_3$ and $V_j, A_{r_j,\tau_j} = \text{graph}_\Sigma(v_j) \subseteq A_{r_j,\tau_j}$, then for $j >> 1$,

\begin{equation}
\hat{J}_j(l) \leq \sup_{1 \leq i \leq t-1} \hat{J}_j(i) \quad \text{for all integers } N + 1 \leq l \leq (\log s_0 - \log \tau_j)/\log K
\end{equation}

where $\hat{J}_j(l) := \hat{J}_{p,v}(s_0K^{-l}; \kappa \| g_j - g \|_{C^2(\Sigma)}).$

Proof. First observe that by lemma 2.9 and ??, $\exists s_1 \in (0,\tau)$ and $\delta_3 > 0$ such that if $U \subset B_{s_1}(p)$ is an open subset, $\| g_j - g \|_{C^2(\Sigma)} \leq \delta_3$ and $\| v \|^2_{L^2(U)} \leq \delta_3$ such that $\text{graph}_\Sigma(v)$ is minimal in $(M,g')$ in its interior, then $v$ satisfies the equation

\begin{equation}
-\text{div}_C(\nabla_Cv + b_0) + |A_C|^2v + b_1 = 0
\end{equation}

on $U$, where

\begin{equation}
|b_0(x,z,p) + |x| \cdot |b_1(x,z,p)| / \delta_0 \leq 0 \|z\|/|x| + |p| + \| g_j - g \|_{C^2(\Sigma)} / 2 \quad \text{on } U \times \mathbb{R} \times T_\Sigma
\end{equation}

and $\delta_0$ is specified in corollary 2.7. Moreover, $v^{(r)}(y) := v(ry)/r$ also satisfies the equation of the same form with the same estimate on its domain. Fix the choice of $\delta_3$ from now on. Also notice that

\begin{equation}
\hat{J}_{p,v}(s;Q) = v^{2(1-\sigma)}(s/r; \rho^{2(\sigma-1)}Q)
\end{equation}

Take $s_0 \in (0,s_1)$ such that $s_0^{2(\sigma-1)} \kappa \geq 1$.

Suppose (ii) fails, then after passing to a subsequence, $\exists n \geq N + 1, \tau_j := s_0 \cdot K^{-l_j} \in (\tau_j, s_0)$ such that

\begin{equation}
\hat{J}_{p,v}(\tau_j; \kappa \| g_j - g \|_{C^2(\Sigma)} > \sup_{1 \leq i \leq l-1} \hat{J}_{p,v}(\tau_j K; \kappa \| g_j - g \|_{C^2(\Sigma)}
\end{equation}

Then by inductively considering $v_j(\tau_j K^{-d})$, $d \geq 1$, by the choice of $s_0$, 4.14 and corollary 2.7

\begin{equation}
\hat{J}_{p,v}(s_0 K^{-d};0) > \sup_{1 \leq i \leq l-1} \hat{J}_{p,v}(s_0 K^{i}; \kappa \| g_j - g \|_{C^2(\Sigma)}
\end{equation}

WLOG $\tau_j \to 0_+$ are such that $(\eta_{p,\tau_j})_j V_j$ is the $\delta_3$-minimal blow up sequence for $\{ V_j \}_{j \geq 1}$, note that it also follows from 4.14 and 4.13 that $\tau_j > 0$. Then by 4.14, the varifold limit $V_\infty$ of this blow up sequence satisfies $\mathcal{AR}_\infty(V_\infty) \leq \sigma$. This completes the proof.

Proof of theorem 4.2. Recall that $V_j$ is stationary in $(G, g_j)$ and $V_j \to [\Sigma], V_j \neq [\Sigma]$. We consider first the simpler setting where $g_j \equiv g$. By assumption (A) and (B) in theorem 4.2 suppose $\tau \in (0,\gamma_1(1)/2)$ such that for each $p \in \text{Sing}(\Sigma)$, either $C_p$ is minimizing in one side, or $V_j$ is a stable minimal hypersurface in $B^\mathbb{B}_{2r}(p)$. Let $\sigma_p \in (\gamma_2(C_p), \gamma_1(C_p)) \setminus \Gamma_{C_p}$; $K_p \geq 2$; $N_p \in \mathbb{N}$ be specified in corollary 2.7 as above; $s_0(p) \in (0,\tau)$ and $\delta_3(p)$ be specified in lemma 4.11 for each $p \in \text{Sing}(\Sigma)$, where $\kappa = 1$ and $g_j = g$; And let $\delta_4 := \inf_{p \in \text{Sing}(\Sigma)} \delta_3(p)$. $s_0 := \inf_{p \in \text{Sing}(\Sigma)} s_0(p) \cdot K_p^{-N_p-1}$.

Note that by 4.15 and the choice of $\sigma_p$, only (ii) in lemma 4.11 could happen for every singularity $p$. 
Since $V_j \to |\Sigma|$, by Allard regularity, $\exists s_j \to 0$ and $v_j \in C^2(\Sigma)$ such that $|v_j|_{C^2(\Sigma)}^2 \leq \delta_4$ and that

$$V_j \setminus M \setminus B^M_{s_j}(\text{Sing}(\Sigma)) = |\text{graph}_{\Sigma} (v_j)\cup M \setminus B^M_{s_j}(\text{Sing}(\Sigma))$$

Then by lemma 4.11 (ii), for all $p \in \text{Sing}(\Sigma)$,

$$\int_{A_{r,\kappa_p}(p)} v_j^2 \rho^{-n-2\sigma_p} \text{dvol}_\Sigma(x) \leq C(\Sigma, M, g) \int_{A_{r,\kappa_p}(p)} v_j^2 \rho^{-n-2\sigma_p} \text{dvol}_\Sigma(x) \quad \forall r \in (s_j, \bar s_0)$$

In particular, $\forall j > 1$ and $\forall \Omega \subset \setminus B_{s_j}(\text{Sing}(\Sigma))$, we have

$$\int_{\Omega} v_j^2 \text{d}x \leq C(\Sigma, M, g, \{\sigma_p\}_{p \in \text{Sing}(\Sigma)}, \Omega) \int_{\setminus B_{s_j}(\text{Sing}(\Sigma))} v_j^2 \text{d}x$$

The key point of this estimate is that the constant on RHS does NOT depend on $j$. Take $a_j := \|v_j\|_{L^2(\Sigma \setminus B_{s_j})} > 0$ for $j > 1$. Since $V_j \neq |\Sigma|$, by Allard regularity, unique continuation property of minimal surface equation and assumption (A) (B) on $\delta_4 \tau < \text{dist} \nu$, every blow up limit holds immediately by elliptic estimate. We close this section by pointing out the following improved characterization of nearby minimal surfaces.

The proof for the case where $g_j = g + c_j \beta_j$ as in the statement of theorem is similar but more delicate in choosing the renormalizing constant. Since $q(\beta)$ is not identically 0, let $\Omega_0 \subset \subset \Sigma$ such that $\text{spt}(q(\beta)) \cap \Omega_0 \neq \emptyset$. Let $v_j \in C^2(\Omega_0)$ be the graphical function of $V_j$ over $\Omega_0$, $\|v_j\|_{C^2} \to 0$.

**Claim:** $\text{lim inf}_{j \to \infty} \|v_j\|_{L^2(\Omega_0)/c_j} > 0$.

**Proof of the claim:** Otherwise, up to a subsequence, $v_j/c_j \to 0$ in $L^2(\Omega_0)$. Since $V_j$ is stationary in $(M, g_j)$, by lemma 2.10 $\forall \phi \in C^2_\delta(\Omega_0)$,

$$0 = \int_{\Omega_0} \mathcal{H}^{\mathcal{M}} v_j : \phi$$

$$= \int_{\Omega_0} a_{g_j}^{ik}(x, v_j, \nabla v_j) \partial_i (v_j) \partial_k \phi + b_{g_j}^{ik}(x, v_j, \nabla v_j) \partial_k \phi + \partial_x F^{g_j}(x, v_j, \nabla v_j)\phi$$

$$= \int_{\Omega_0} -v_j \cdot \partial_i (a_{g_j}^{ik}(x, v_j, \nabla v_j) \partial_k \phi) + b_{g_j}^{ik}(x, v_j, \nabla v_j) \partial_k \phi + \partial_x F^{g_j}(x, v_j, \nabla v_j)\phi$$

multiply both side of (4.12) by $1/c_j$ and let $j \to \infty$, by lemma 2.10 and the assumption that $\langle g_j \rangle/c_j \to \beta$ in $C^4(M)$ and $v_j/c_j \to 0$ in $L^2(\Omega_0)$, we get

$$\int_{\Omega_0} \beta(\nu, \nabla \phi) + \frac{1}{2} \text{tr}_2(\mathcal{M}) \cdot \phi = 0 \quad \forall \phi \in C^2_\delta(\Omega_0)$$

This contradicts to that $\text{spt}(q(\beta)) \cap \Omega_0 \neq \emptyset$. Thus the claim is proved.

With this claim, we see that $\exists \kappa > 0$ such that $\kappa\|g_j - g\|_{C^4} \leq \|v_j\|_{L^2(\Omega)}$. Now repeat the previous proof by applying lemma 4.11 with also taking $\tau < \text{dist}(\text{Sing}(\Sigma), \Omega_0)$ and $\kappa > 0$ specified above. We still get a local-control-global typed $L^2$-estimate as 4.11, and then the conclusion holds immediately by elliptic estimate.

We close this section by pointing out the following improved characterization of nearby minimal hypersurfaces using their associated Jacobi fields.

**Corollary 4.12.** Suppose $\sigma < 1$; $\{\{V_j, g_j\}_{j \geq 1}\}$ is a sequence of varifold-metric pair described in theorem 4.3. Suppose $\tilde v \in C^2_{\text{loc}}(\Sigma)$ is an associated generalized Jacobi field of $\{\{V_j, g_j\}_{j \geq 1}\}; p \in \text{Sing}(\Sigma)$ such that $\mathcal{A}_{\text{R}_p} (\tilde v) \leq \sigma$. Then up to a subsequence, there exists $\delta \bar \kappa(p)$-minimal blow up sequence of $\{V_j\}_{j \geq 1}$ near $p$, every blow up limit $\tilde V_\infty$ of which is asymptotic to $C_p$ near infinity with $\mathcal{A}_{\text{R}_\infty} (\tilde V_\infty) \leq \sigma$. 

In particular, suppose $\sigma \leq \gamma_2^+$, and either (B) of theorem 4.2 hold or $C_p$ is area-minimizing in both side. Then $\text{Sing}(V_j) \cap B_{r_\mathrm{max}}^M(p) = \emptyset$ for $j \gg 1$.

**Proof.** Directly apply lemma 4.11. By lemma 3.14 and lemma 4.6, take WLOG $\sigma \in (-\infty, 1) \setminus \Gamma_{C_p}$.

By lemma 5.14 since $\mathcal{A} \mathcal{R}_p(v) \leq \sigma$, (ii) of lemma 4.11 cannot happen. Hence there exists a blow up limit with asymptotic rate to $C_p$ near infinity $\leq \sigma$.

When $\sigma \leq \gamma_2^+(C_p) < \gamma_1^+(C_p)$, by theorem 3.3 $\hat{V}_\infty$ is a closed embedded smooth minimal hypersurface in $\mathbb{R}^{n+1}$ of multiplicity 1, hence by Allard regularity, for those $V_j$ whose blow-up sequence converges to $\hat{V}_\infty$, $V_j$ has no singularity near $p$.

**Corollary 4.13.** Suppose $p \in \text{Sing}(\Sigma)$ such that $C_p$ is strictly minimizing. Let $\{V_j\}_{j \geq 1}$ be a family of stationary integral varifold in $B_{r_{\max}}^M(p)$ which converges in varifold sense to $[\Sigma]_\ast B_{r_{\max}}^M(p)$.

Let $\sigma \in (-\infty, \gamma_1^+ \setminus \Gamma_{C_p}$ be fixed; $N, s_0, \delta_3$ be the same as in lemma 4.11; $s_0' := s_0 \cdot K^{-N-1}$.

Suppose $r_j \to 0_+$ and $\|v_j\|_{L^2_{A_{r_j}, 2r_j}} \leq 3\delta_3$ be the graphical function of $V_j$ over $\Sigma$, i.e. $V_j A_{r_j, 2r_j} = \{\text{graph}_\Sigma(v_j)\} \setminus A_{r_j, 2r_j}$. Then for $j > 1$, the following pointwise bounds holds on $A_{K r_j, \tau/K}$

$$|v_j|/\rho + |\nabla v_j| \leq C(\sigma, \Sigma, M, g)\rho^{\sigma-1} \cdot \left(\int_{A_{r_j, \tau}} v_j^2\right)^{1/2}$$

**Proof.** Directly apply lemma 4.11 one get an $L^2$ bound of $v_j$ on each $A_{r_j, 2r_j}$, then since $v_j$ solves the minimal surface equation, classical interior Schauder estimate yields the corollary.

**Proof of theorem 4.2** The existence of associated Jacobi field $u$ follows from theorem 4.2. If assuming $u > 0$ near some $p \in \text{Sing}(\Sigma)$, then by a blow up argument and lemma 2.3, $\mathcal{A} \mathcal{R}_p(u) \in \{\gamma_1^+(C_p)\}$.

The regularity of $\Sigma_j$ near $p$ then follows from corollary 4.12.

**5. One-sided deformations of minimal hypersurfaces**

Let $(\Sigma, \nu) \subset (M, g)$ be a strictly stable, closed minimal hypersurface with strongly isolated singularities, i.e. $-L_\Sigma$ is strictly positive on $\mathcal{B}_0(\Sigma)$ as in proposition 3.3 (4). Recall "closed" means $\text{Clos}(\Sigma)$ is compact. Suppose $M \setminus \text{Clos}(\Sigma)$ has 2 connected component $M_\pm$. This is always possible if we replace $M$ by a small neighborhood of $\Sigma$. Suppose $M_\pm$ be the one which $\nu$ points in.

Let $\lambda_1 > 0$ be the first $L^2$-eigenvalue of $-L_\Sigma$ on $\mathcal{B}_0(\Sigma)$; $\Lambda \in (0, \lambda_1/2]$ be fixed throughout this section. Note that by definition we have

$$Q_\Sigma(\phi, \phi) \geq 2\Lambda \int_{\Sigma} \phi^2 \quad \forall \phi \in C_0^1(\Sigma)$$

The aim of this section is to prove theorem 4.6. More precisely, we shall prove the following

**Theorem 5.1.** Let $\Sigma, M$ be as above; $\tau \in (0, \tau_\Sigma(1)/4)$, where $\tau_\Sigma$ is defined in (2.17). Suppose each tangent cone of $\Sigma$ is strictly minimizing. Then there exists a family of piecewise smooth neighborhood $\{U_t\}_{t \in (0,1)}$ of $\text{Clos}(\Sigma)$ such that

1. $U_{t_1} \subset U_{t_2}$ for any $t_1 < t_2$;
2. $\bigcap_{t \in (0, 1)} U_t = \text{Clos}(\Sigma)$; $\bigcap_{t \in (0, 1)} U_x = \text{Clos}(U_t)$ and $\bigcup_{t \in (0, 1)} U_x = U_t, \forall t \in (0, 1)$;
3. smooth domains $\{\Omega_t\}_{t \in (0,1)} = \Sigma \setminus B_t(\text{Sing}(\Sigma)) \subset U_t \subset \subset C_\Sigma$ and decomposition $\partial U_t \cap M = \Sigma_t^{\pm, \text{ext}} \cup \bigcup_{p \in \text{Sing}(\Sigma)} \Sigma_t^{\pm, \text{int}}$

where $\Sigma_t^{\pm, \text{ext}}$ are smooth minimal graphs over $\Omega_t$ with smooth boundaries; $\Sigma_t^{\pm, \text{int}}$ are smooth minimal hypersurfaces in $B_r^M(p)$ with boundary $\partial \Sigma_t^{\pm, \text{int}} = \partial \Sigma_t^{\pm, \text{ext}} \cap B_r^M(p)$.

4. Along their intersection, $\Sigma_t^{\pm, \text{ext}}$ and $\Sigma_t^{\pm, \text{int}}$ forms a convex wedge.

**Remark 5.2.** Note that $\partial U_t$ are uniform local Lipschitz hypersurfaces. Hence by [EH07], by running mean curvature flow starting from $\partial U_t$, we see from (3) and (4) above that $U_t$ can be approximated by smooth mean convex domains.
We first sketch the construction of $U_t$. Let

$$\rho(x) := \min \{ \text{dist}_{\Sigma}(x, p) : p \in \text{Sing}(\Sigma) \} \cup \{2\tau\}$$

When working near a certain singularity $p$, we shall parametrize $B_{2\tau}(p)$ by the tangent cone $C_p$ of $\Sigma$ near $p$. We may abuse the notations and simply write any point $x \in B_{2\tau}(p)$ as $x = (r, \omega)$ as in (2.16). Then near each $p \in \text{Sing}(\Sigma)$, $\rho(r, \omega) = r$ if $r \leq 2\tau$.

Denote for simplicity $L = L_\Sigma$ and $L_\Lambda := L + \Lambda$. By (5.1), $-L_\Lambda$ is strictly positive, with the first $L^2$-eigenvalue $\geq \Lambda > 0$. Fixed $p_0 \in S \setminus B_{4\tau}(\text{Sing}(\Sigma))$. For each $p \in \text{Sing}(\Sigma)$, let $G_p$ be the Green’s function of $L_\Lambda$ at $p$ constructed in corollary 3.19 with $G_p(p_0) = 1$. Let $G := \sum_{p \in \text{Sing}(\Sigma)} G_p$. By corollary 3.19 and 3.15, $G > 0$ satisfies the equation $LG = -\Lambda \cdot G$ on $\Sigma$ and growth estimate

$$G(x)/\rho(x)^{\gamma_1}(C_p) \to +\infty \quad \text{as } x \to p$$

for every $\epsilon > 0$, where recall $\gamma_1(C_p)$ is the spectrum of $C_p$ defined in (2.7). Therefore, there’s a sufficiently large $T_0 > 0$ such that

$$\{ x \in \Sigma : G(x)/\rho(x)^{-1} > T_0 \} \subset B_r(\text{Sing}(\Sigma))$$

From now on in this section, every constant will depend on $M, g, \Sigma, \Lambda, \tau, p_0$.

For each $T > T_0$ and $p \in \Sigma$, let

$$r_p(T) := \sup\{ R : G(x)/\rho(x)^{-1} > T \text{ on } B_R(p) \}$$

and

$$\Sigma_T := \Sigma \setminus \bigcup_{p \in \text{Sing}(\Sigma)} B_{r_p(T)}(p)$$

By corollary 3.16, there’s a $T_1 > T_0$ such that on $[T_1, +\infty)$, $r_p(T)$ varies continuously in $T$, $\Sigma_T$ are smooth manifolds with boundary. Also by Harnack inequality, for every $p \in \text{Sing}(\Sigma)$,

$$T \cdot r_p(T) = \inf_{\partial \Sigma_T \cap B_r(p)} G \leq \sup_{\partial \Sigma_T \cap B_r(p)} G \leq C(\Sigma, \Lambda, \tau) T \cdot r_p(T)$$

$\Sigma_T$ will be the spaces on which we shall build graph.

For each $T > T_1$, we shall choose specific small positive function $\varphi^+ \in C^\infty(\partial \Sigma_T)$ and construct a graph of some positive function $u^+ \in C^\infty(\Sigma_T)$ with mean curvature $h$ and boundary graph$\Sigma_T \varphi^+$. Such a graph will be called an exterior hypersurface with boundary $\varphi^+$;

Then near each singularity $p$, solve Plateau problem with boundary graph$\Sigma_T \varphi^+$ in $B_r(p)$ to find a minimizing hypersurface, called an interior hypersurface with boundary $\varphi^+$ near $p$. The union of these exterior and interior hypersurfaces, denoted by $\Sigma_T$, is piecewise smooth and will lie in $M_+$ for sufficiently large $T$. Repeat the same process to construct $\Sigma_T$ lying in $M_+$. Then for each $t \in (0, 1)$, a reparametrization $T = T(t) \geq T_1$ will be specified below such that regions $U_t$ bounded by $\Sigma_T$ are what we want.

The mean convexity of $\partial U_t$ will be derived by a blow up argument letting $T \to +\infty$ and using the Hardt-Simon foliation as barriers. To make it precise, we need an existence result of exterior graphs over $\Sigma_T$ with uniform estimate near each singularity. We assert here that this existence result does NOT require the tangent cones of $\Sigma$ to be strictly stable or minimizing.

**Theorem 5.3.** Let $\alpha \in (0, 1)$. Then there exists $C = C(\alpha) > 1$, $T_2(\alpha) > T_1$ and $\delta_5 = \delta_5(\alpha) > 0$ sufficiently small such that, for each $T \in (T_2, +\infty)$, each $0 < \delta \leq \delta_5$ and each $\varphi \in C^{2, \alpha}(\partial \Sigma_T)$ with estimate near each singularity $p$

$$\frac{1}{r_p(T)} |\varphi| + |\nabla \varphi| + r_p(T)|\nabla^2 \varphi| \leq \delta$$

there’s a unique $u \in C^{2, \alpha}(\Sigma_T)$ solving

$$\begin{cases}
-\Delta u(x) = 0 & \text{on } \Sigma_T \\
u = \varphi & \text{on } \partial \Sigma_T
\end{cases}$$
Claim 2 in $\phi_u$. By continuous dependence of $B_\phi$ with $u$.

Proof of claim 1: By the uniqueness of solution above and the estimate (5.6), we know further that $u$ depends continuously with respect to $\phi$ in $C^2$-norm and in $T$.

We first finish the proof of theorem 5.1 assuming theorem 5.3 to be true.

Proof of theorem 5.1: Let $\delta_0$ be determined in theorem 5.3. For each singularity $p$ with strictly minimizing tangent cone $(C_p, \nu_p)$ at $p$, let $S_p$ be the cross section of $C_p$. For sufficiently small $\delta_0 < \delta_0/2$, there’s a leaf of Hardt-Simon foliation $\Sigma_p^+ \subset \mathbb{R}^{n+1}$ and a positive function $u_p^+ \in C^\infty(C_p)$ satisfying

\[
\Sigma_p^+ \setminus B_{1/4}^{n+1} = \text{graph}_{\mathcal{M}}(u_p^+) \setminus B_{1/4}^{n+1}
\]

\[
\|u_p^+(1,\cdot)\|_{C^2(S_p)} = \delta_0 \leq C \inf_{S_p} u_p^+(1,\cdot)
\]

We fix such a $\delta_0$ from now on so that (5.7) is true with certain $u_p^+$ for every $p \in \text{Sing}(\Sigma)$.

Now for each $T > T_2$, define $\varphi_p^+ \in C^2(\partial \Sigma_T)$ to be

\[
\varphi_p^+(r_p(T),\omega) := r_p(T) u_p^+(1,\omega) \quad \forall \omega \in S_p
\]

By a standard compactness argument, for sufficiently large $T$, $\varphi_p^+$ satisfies (5.4) near each singularity $p$ with $\delta = 2\delta_0$, and

\[
\inf_{\partial \Sigma_T} \varphi_p^+ / r_p(T) \geq \delta_0 / C
\]

Let $u_T^+$ be the solution to (5.6) with $\varphi = \varphi_T^+$ satisfying the pointwise estimate (5.6). Then $\Sigma_T^+,\text{ext} := \text{graph}_{\Sigma_T}(u_T^+)$ is a smooth minimal graph over $\Sigma_T$ with boundary $\text{graph}_{\Sigma_T}(\varphi_T^+)$. 

Claim 1: $u_T^+ > 0$ for every $T \in (T_2, +\infty)$.

Proof of claim 1: For fixed $T > T_1$ and every $s \in (0, 1]$, consider the solution $u_{T,s}^+$ of (5.6) on $\Sigma_T$ with $\varphi = s \varphi_T^+$. Then by theorem 5.3, $u_{T,s}^+$ is unique and varies continuously in $s$. Moreover, by (5.6) and the definition of $\varphi_T^+$ we have, $u_{T,s}^+ > 0$ in some neighborhood of $\partial \Sigma_T$ independent of $s$.

When $s \to 0_+$, by (5.6) and classical elliptic estimate [GT01], $u_{T,s}^+ / s \to u_T^+ 0$ in $C^2(\Sigma_T)$ for some $u_T^+$ satisfying $Lu_T^+ = 0$ on $\Sigma_T$ and $u_T^+ = \varphi_T^+$ on $\partial \Sigma_T$. Then by the strict positivity of $-L$, $\hat{u}_{T,0}^+ > 0$ on $\Sigma_T$. Hence $u_{T,s}^+ > 0$ on $\Sigma_T$ for $s < 1$.

Suppose for contradiction that $\{u_T^+ \leq 0\} \neq \emptyset$. Consider $\inf \{s \in (0, 1] : \{u_{T,s}^+ \leq 0\} \neq \emptyset\} =: s_0 > 0$. By continuous dependence of $u_{T,s}^+$ in $s$, $u_{T,s_0}^+ \geq 0$, nonzero on $\partial \Sigma_T$ and vanishes somewhere in $\text{Int}(\Sigma_T)$. This contradicts to strong maximum principle since 0 is also a solution to (5.6) with $\varphi = 0$.

Now for each $p \in \text{Sing}(\Sigma)$, let $\Sigma_{T,p}^+,\text{int}$ be the minimizing hypersurface with boundary $\partial \Sigma_{T,p}^+,\text{ext} \cap B_r(p)$. Such hypersurface is unique and smooth for $T \in (T_3, +\infty)$ for some $T_3 > T_2$ by the uniqueness of Hardt-Simon’s foliation $\Sigma_p^+$, nondegeneracy of $L_{\Sigma_p^+}$ on bounded subsets, Allard regularity theorem [All72, Sim83] and implicit function theorem. Let

\[
\Sigma_T^+ := \Sigma_T^+,\text{ext} \cup \bigcup_{p \in \text{Sing}(\Sigma)} \Sigma_{T,p}^+,\text{int}
\]

with normal field $\nu_T^+$ in $M$ pointing away from $\Sigma$, defined in the interior of each smooth component.

Claim 2: For each $p \in \text{Sing}(\Sigma)$, let $\nu_T^+,\text{ext} := \nu_T^+|_{\Sigma_T^+,\text{ext}}$ be the normal field of $\Sigma_T^+,\text{ext}$ in $M$ (pointing
away from $\Sigma$), and $\xi_p^{+, \text{int}}$ be the inward normal field of $\partial \Sigma_T^{+, \text{int}}$ in $\Sigma_T^{+, \text{int}}$. Then there exists $T_4 > T_3$ such that for every $T \in (T_4, +\infty)$,

$$
\nu_T^{+, \text{ext}} \cdot \xi_T^{+, \text{int}} < 0 \quad \text{on } \partial \Sigma_T^{+, \text{int}}
$$

**Proof of claim 2:** Identify $\Sigma_T^{+, \text{ext}}$ as a hypersurface with boundary in $T_p M$ using $\exp^M_p$. Suppose up to a subsequence,

$$
\frac{1}{r_p(T)} \Sigma_T^{+, \text{ext}} \to \Sigma_T^{+, \text{ext}} \quad \text{as } T \to +\infty
$$

By (5.6), the convergence is in $C^1$ sense as an exterior graph over $C_p$, and $\Sigma_T^{+, \text{ext}}$ is a smooth graph with vanishing mean curvature of some $u_T^{+, \text{ext}} > 0$ over $C_p \setminus B_{1}^{n+1}$. Moreover, by the boundary condition (5.7) & (5.8) in (5.5),

$$
\left| u_T^{+, \text{ext}}(1, \omega) - u_T^{+}(1, \omega) \right| < \text{every } \omega \in S_p
$$

and by (5.9), corollary 8.16 and the definition of strictly minimizing, we have $u_T^{+, \text{ext}} < u_T^{+}$ near $\Sigma$. Hence, by considering $\inf\{E > 0 : R \cdot \Sigma_T^{+, \text{ext}} \text{ lies above } \Sigma_T^{+, \text{ext}}\}$ and using strong maximum principle for minimal hypersurfaces,

$$
(5.9)
$$

Since both $u_T^{+, \text{ext}}$ and $u_T^{+}$ satisfies the minimal surface equation on $C_p \setminus \text{Clos}(B_{1}^{n+1})$, by (5.9) and Hopf boundary lemma,

$$
\partial_n u_T^{+, \text{ext}} (1, \omega) < \partial_n u_T^{+}(1, \omega) \quad \text{for every } \omega \in S_p
$$

And since the convergence of $\frac{1}{r_p(T)} \Sigma_T^{+, \text{ext}}$ is in $C^1$ sense, the strict inequality in claim 2 is true for sufficiently large $T$.

With claim 2, $(\Sigma_T^{+}, \nu_T^{+})$ is mean convex in $B_{2}^{M}(p)$ for each $p \in \text{Sing}(\Sigma)$ and all $T > T_4$. Let $U_T^{+}$ be the open domain bounded by $\Sigma_T^{+}$ and $\text{Clos}(\Sigma)$. Using the same blow up argument as claim 2, one can show that there’s a $T_5 > T_4$ such that for every $T \in (T_5, +\infty)$,

$$
\partial \Sigma_T^{+, s} \subset U_T^{+} \quad \forall s > 0
$$

Hence by the uniqueness of solutions to (5.5), the uniqueness of $\Sigma_T^{+, \text{int}}$ and a barrier argument similar to claim 1, we see that for every $T > T'$, $\Sigma_T^{+} \subset U_T^{+}$. In particular, combined with the uniqueness of solutions to (5.5) and $\Sigma_T^{+, \text{int}}$, we have $\{\Sigma_T^{+}\}_{T>T'}$ forms a foliation of $U_T^{+}$.

Similarly, one can construct $\Sigma_T^{+, s}$ on the other side of $\Sigma$ satisfying the similar properties for $T > T_6$.

Let $T = \max\{T_3, T_6\}$ and $U_T$ be the domain bounded by $\Sigma_T^{+}$ and $\Sigma_T^{-}$. In particular, combined with the uniqueness of solutions to (5.5), $\Sigma_T^{+, \text{ext}}$ and $\Sigma_T^{+, \text{int}}$, we have $\{\Sigma_T^{+}\}_{T>T'}$ forms a foliation of $U_T^{+}$.

We start to prove Theorem 5.3. We first introduce the following weighted norms for functions $u$ defined on $E \subset \Sigma$. Let $\alpha \in (0,1)$, $q \geq 1$, $k \in \mathbb{N}$.

\begin{align}
\|u\|_{C^{0,\alpha}_G(E)} := \sum_{j=0}^{k} \left\| \nabla^j u \cdot \rho^j \right\|_{C^0(E)} + \sup_{x \neq y \in E} \frac{\left| \nabla^k u(x) - \nabla^k u(y) \right|}{\text{dist}(x,y)^\alpha} \cdot \frac{\rho(x)^{k+\alpha}}{G(x)} : x \neq y \in E, \quad \frac{1}{10} < \frac{\rho(x)}{\rho(y)} < 10 \\
\|u\|_{L^q_G(E)} := \sup_{t > 0} \left( t^{-n} \int_{E \cap \{t \leq \rho \leq 100t\}} (u/G)^q \, d\mathcal{H}^n(x) \right)^{1/q} \\
\|u\|_{W^{k,q}_G(E)} := \sum_{j=0}^{k} \left\| \nabla^j u \rho^j \right\|_{L^q_G(E)}
\end{align}
By applying Morrey’s inequality in every annuli $A_{t,10r}(p) \subset \Sigma_T$, for $q \geq n/(1 - \alpha)$,
\begin{equation}
\|u\|_{C^{2,\alpha}(\Sigma_T)} \leq C(\alpha, q)\|u\|_{W^{2,q}(\Sigma_T)}
\end{equation}
for every $u$ and $T > T_1$. We emphasis that by Harnack inequality for $G$, the constant here doesn’t depend on $T$.

Fix $q = n/(1 - \alpha)$ from now on, and define for each $T > T_1$, $\|u\|_{X_T} := T \cdot \|u\|_{W^{2,q}(\Sigma_T)}$. Note that (5.4) hold for all $p \in \text{Sing}(\Sigma)$ is equivalent to that $T \cdot \|\varphi\|_{C^2_{\alpha}(\partial\Sigma_T)} \leq \delta$, and (5.6) is equivalent to $T \cdot \|u\|_{C^{2,\alpha}} \leq C\delta$, hence is implied by that $\|u\|_{X_T} \leq C\delta$ by (5.11).

Let $\delta_2'$ be determined in lemma 2.11. Let $\mathcal{R}$ be the error term operator given by
$$\mathcal{R}u(x) := -Lu + \mathcal{R}u$$
By lemma 2.11 if $\|u_1\|_{X_T}, \|u_2\|_{X_T} \leq \theta \leq \delta_2'$, then the following pointwise inequality holds,
\begin{equation}
|\mathcal{R}u_1 - \mathcal{R}u_2| \leq C\theta \cdot \frac{G}{\rho^2T} \left(\left|\{u_1 - u_2\} + \rho|\nabla(u_1 - u_2)|\right| \left(1 + \frac{\rho^2|\nabla^2 u_2|}{G \rho} \cdot T + \rho^2|\nabla^2(u_1 - u_2)|\right)\right)
\end{equation}
Note that if write $f := \left(\left|\{u_1 - u_2\} + \rho|\nabla(u_1 - u_2)|\right| \left(1 + \frac{\rho^2|\nabla^2 u_2|}{G \rho} \cdot T + \rho^2|\nabla^2(u_1 - u_2)|\right)\right)$, then by definition and (5.11),
\begin{equation}
\|f\|_{L_0^2(\Sigma_T)} \leq C \left(\left|\{u_1 - u_2\} \right|_{C^0} + \|\nabla(u_1 - u_2)\|_{C^0} \cdot \left(1 + \|u_2\|_{W^{2,q}(\Sigma_T) \cdot \frac{T}{\rho}} + \|u_1 - u_2\|_{W^{2,q}}\right)\right)
\end{equation}
\leq C\|u_1 - u_2\|_{X_T} \cdot T^{-1}

The following linear estimate is the key part of the proof.

**Lemma 5.5.** $\exists C_2 = C_2(q) > 1$, $T_2(q) > T_1$ such that, if $T > T_2$, $v \in W^{2,q}(\Sigma_T)$ is the solution to
\begin{equation}
\begin{cases}
Lv = \frac{G}{\rho^2T} \cdot f & \text{on } \Sigma_T \\
v = \phi & \text{on } \partial\Sigma_T
\end{cases}
\end{equation}
Then
$$\|v\|_{X_T} \leq C_2T(\|f\|_{L_0^2(\Sigma_T)} + \|\phi\|_{C^2_{\alpha}(\partial\Sigma_T)})$$
**Proof.** The key is that $C_2$ doesn’t depend on $T$. Note that $G/\rho^2T \sim \rho^{-2}$ near $\partial\Sigma_T$. Hence, we need to carefully deal with the estimate near $\partial\Sigma_T$.

**Step 1 (Local estimate)** We shall work near any fixed singularity $p$ of $\Sigma$ and parametrize $\Sigma \cap B_{t_r}(p)$ by the tangent cone $C_p$ or by $(0, +\infty) \times S_p$ as in section 2.2. By corollary 5.16 $\rho(x) \cdot \partial_t G/G(x) \rightarrow \gamma^{-1}_1(C_p)$ as $x \rightarrow p$. Hence by Harnack inequality and the definition of $r_p(T)$ (5.2), for every $\sigma < \gamma^{-1}_1(C_p) < \sigma'$ and every $T > T_1$,
\begin{equation}
C(\Sigma, \sigma)(\rho / r_{p}(T))^{\sigma} \leq \frac{G}{r_{p}(T)T} \leq C(\Sigma, \sigma')((\rho / r_{p}(T))^{\sigma'})
\end{equation}
on $\Sigma_T \cap B_{t_r}(p)$. We now fix a choice of $\sigma_p \in (-\infty, \gamma^{-1}_1(C_p)) \setminus \Gamma_{C_p}$ and $\sigma'_p \in (\gamma^{-1}_1(C_p), (1 + \sigma_p)/2)$.

Let $t_p = t_p(\sigma_p, \Sigma, M) \in (0, \tau)$ such that $L = L_{C_p} + \mathcal{R}_p$ satisfies the condition in corollary 2.3. The existence of such $t_p$ is guaranteed by lemma 2.9. Also let $T_{2,p} > T_1$ such that $r_{p}(T_{2,p}) < t_{p}/10$. By corollary 2.6 for every $T > T_{2,p}$, there exists a solution $v_p \in W^{2,2}_{\sigma} (B_{t_p}(p))$ to the equation $Lv_p = \chi_{\Sigma_T} f G/\rho^2T$ on $B_{t_p}(p)$ satisfying the weighted estimate
\begin{equation}
\sup_{t \in (0,t_p)} \|v_p(t, \cdot)\|_{L^2(S_{t_p})} \cdot t^{-\sigma_p} \leq C(\sigma_p, t_p) \|G/\rho^2T\|_{L^2_{\sigma_p - 2}(B_{t_p}(p))}
\end{equation}
And if denote for simplicity $A_j := A_{100^{-j}r,100^{-j+1}r}(p)$, then by definition,
\[
\|G\|_{\rho^T} \cdot \chi_{p,\rho}^j \cdot \|f\|^2 \cdot \|G\|_{\rho^T} \cdot f^2 \rho^{-2\sigma_2+4-n} \, dx
\]
\[
= T^2 \cdot \int A_j \cdot \chi_{p,\rho}(G) \cdot f^2 \rho^{-2\sigma_2-2n+2n/q} \, dx
\]
\[
\leq C(q)T^2 \|f\|_{L^2_S(\Sigma_T)}^2 \cdot \left( \int A_j \chi_{p,\rho}(G) \cdot f^2 \rho^{-2\sigma_2-2n+2n/q} \, dx \right)^{1/q}
\]
\[
\leq C(q, \sigma_p, \sigma_p', \tau_p)T^2 \|f\|_{L^2_S(\Sigma_T)}^2 \cdot \left( \int A_j \chi_{p,\rho}(G) \cdot f^2 \rho^{-2\sigma_2-2n+2n/q} \, dx \right)^{1/q}
\]
where the first inequality is by Holder inequality with $q' = q/(q-2)$; the second inequality is by RHS of (5.15) and the last inequality is true since we choose $L$ with (5.16), the interior
\[
|\nabla f| \leq C\rho^{-2\sigma_2+4-n} \, dx
\]
whence the constant
\[
C(5.19)
\]
is independent of $C$. Hence the desired estimate for $\|v\|_{L^2(\Sigma_T)}$ follows from LHS of (5.15). Since we have fixed the choice of $C$, we have $\|v\|_{L^2(\Sigma_T)}$ independent of $\rho$. Combined with (5.13), we get a pointwise estimate for $v$ on $B_{\epsilon}\rho^T/p(\rho)$.

(5.17) $|v| + \rho|\nabla v| \leq C(q, \sigma_p, \sigma_p', \tau_p)\rho \cdot \tau_p(T)^2 \cdot \|f\|_{L^2_S(\Sigma_T)} \cdot G$

where the last inequality above follow from LHS of (5.15). Since we have fixed the choice of $\sigma_p, \sigma_p', \tau_p$, $\rho$ on from now on every constant might depend on them in this proof.

**Step 2 (Global $C^0$ estimate)** Now let $T > T_2 := \sup_{p \in \Sigma_T} \eta_p v_p$, where $v_p$ is given by step 1 satisfying the estimate (5.16) in $B_{\epsilon\rho^T/p}(\rho)$; $\eta_p \in C_\infty(\Sigma_T)$ is a cut-off function which equals to 1 on $B_{\epsilon\rho^T/4}(\rho)$ and has $|\nabla \eta_p| + |\nabla^2 \eta_p| \leq C(\tau_p(\rho))$. Let $v := v - \sum_{p \in \Sigma_T} \eta_p v_p$. By definition of $v_p$ we have
\[
L\bar{v} = \frac{G}{\rho^T} \cdot f(1 - \sum_{p \in \Sigma_T} \eta_p) - \sum_{p \in \Sigma_T} (2\nabla \eta_p \cdot \nabla v_p + v_p \Delta \eta_p)
\]
In particular, $spt(L\bar{v}) \subset \Sigma \setminus \bigcup_{p \in \Sigma_T} B_{\epsilon\rho^T/4}(\rho)$ and by (5.17), $\|L\bar{v}\|_{L^q(\Sigma_T)} \leq C\|f\|_{L^2_S(\Sigma_T)}$, where $C$ is independent of $T$.

Let $\mathcal{U}_0 := \Sigma \setminus \bigcup_{p \in \Sigma_T} B_{\epsilon\rho^T/8}(\rho)$ and $\bar{v}_0 \in W^{1,2}(\mathcal{U}_0)$ be the solution of $L\bar{v}_0 = L\bar{v}$ on $\mathcal{U}_0$. Then by global $L^q$ estimate and Morrey’s inequality,
\[
\|\bar{v}_0\|_{C^{1,2}(\mathcal{U}_0)} \leq C\|f\|_{L^2_S(\Sigma_T)}
\]
Let $\eta_0 \in C_\infty(\mathcal{U}_0)$ be a cut-off function with value 1 on $\Sigma \setminus \bigcup_{p \in \Sigma_T} B_{\epsilon\rho^T/4}(\rho)$ and $|\nabla \eta_0| + |\nabla^2 \eta_0| \leq C(\mathcal{U}_0)$. Then
\[
|L(v - \eta_0\bar{v}_0)| = |2\nabla \eta_0 \cdot \nabla \bar{v}_0 + \eta_0 \Delta \eta_0| \leq C\|f\|_{L^2_S(\Sigma_T)} \cdot G \quad \text{on} \Sigma_T
\]
\[
|v - \eta_0\bar{v}_0| = |v - \sum_{p \in \Sigma_T} \eta_p v_p| \leq C\|f\|_{L^2_S(\Sigma_T)} + \|\phi\|_{C^{1,2}(\partial\Sigma_T)} \cdot G \quad \text{on} \partial\Sigma_T
\]
But recall that by definition, $-LG = A \cdot G$ and $-L$ is strictly positive on $\Sigma$. Hence by weak maximum principle for $v - \eta_0\bar{v}_0$ and the estimate (5.17) (5.18), we have weighted $C^0$ estimate for $v$
\[
|v| \leq C\|f\|_{L^2_S(\Sigma_T)} + \|\phi\|_{C^{1,2}(\partial\Sigma_T)} \cdot G \quad \text{on} \Sigma_T
\]
and the constant $C$ is independent of $f, \phi, T$.

**Step 3 (Global $W^{2,q}$ estimate)** Note that since $0 < G/\rho^T \leq C(\Sigma)$ on $\Sigma_T$, we have $|Lv| \leq C(\Sigma)f/\rho^2$. Hence the desired estimate for $\|v\|_{X_T}$ follows from (5.19) and interior and boundary
\[ L^q \] estimate on every domain \( \{ t < r < 100t \} \cap \Sigma_T, \forall t > 0. \] This completes the proof of lemma 5.4.

**Proof of theorem 5.3** Recall that \( \delta_3^* \) is determined in lemma 2.11. Let \( \bar{C} \geq 1 \) and \( \delta_5 \in (0, \delta_2^*/2\bar{C}) \) TBD subsequently. For every \( T > T_1, \delta \in (0, \delta_5] \) and \( \varphi \in C^2(\partial \Sigma_T) \) with \( T \cdot \| \varphi \|_{C^2_0(\partial \Sigma_T)} \leq \delta \), define \( X_{T,C} := \{ u \in W^{2,q}(\Sigma_T): \| u \|_{X_T} \leq \bar{C} \delta \}. \)

Consider \( \mathcal{T}: X_{T,C} \to W^{2,q}(\Sigma_T), \mathcal{T} u =: v \) solves

\[
\begin{cases}
Lv = \mathcal{R} u & \text{on } \Sigma_T \\
v = \varphi & \text{on } \partial \Sigma_T
\end{cases}
\]

Note that by non-degeneracy of \( L \), the equation above has a unique solution. We shall appropriately choose \( \bar{C} \) and \( \delta_5 \) to make \( \mathcal{T} \) a contraction map from \( X_{T,C} \) to itself.

For every \( u \in X_{T,C} \), let \( v := \mathcal{T} u \). By taking \( u_1 = u \) and \( u_2 = 0 \) in (5.12) and applying (5.13) and lemma 5.5

\[ \| u \|_{X_T} \leq C_2 (C_1 \bar{C} \delta \cdot \| u \|_{X_T} + \delta) \leq C_2 (C_1 \bar{C} \delta^2 + \delta) \]

For every \( u_1, u_2 \in X_{T,C} \), let \( v_i : = \mathcal{T} u_i \). Also by (5.12), (5.13) and lemma 5.5

\[ \| v_1 - v_2 \|_{X_T} \leq C_2 (C_1 \bar{C} \delta \cdot \| u_1 - u_2 \|_{X_T} + 0) \]

Hence, take \( \bar{C} := 4C_2 \) and \( \delta_5 := 1/((C_1 \bar{C}^2)^2) \), we see from (5.20) and (5.21) that \( \mathcal{T} \) is a contraction map from \( X_{T,C} \) to itself. Therefore, the theorem follows from Banach fixed point theorem.

**Remark 5.6.** The uniqueness of solution \( u \) in the statement of theorem 5.3 seems slightly stronger since we only assume \( \| u \|_{C^1_{\bar{C}}} \) bound. However, by similar argument as step 3 in the proof of lemma 5.3, for sufficiently small \( \delta' \) (not depending on \( T \)), (5.6) hold for \( u \) with \( \delta = \delta' \) implies \( \| u \|_{X_T} \leq C \delta' \). Thus the uniqueness of solutions with (5.6) estimate to the uniqueness of solutions in \( X_{T,C} \) above, which comes from Banach fixed point theorem.

**Remark 5.7.** By theorem 5.1 and strong maximum principle [SW89], \( \Sigma \) is homologically area minimizing in \( \text{Clos}(U_1) \) and is the unique stationary integral varifold supported in \( \text{Clos}(U_1) \). Hence theorem 1.2 is proved.

Also notice that, directly by a compactness argument, theorem 1.2 is proved in case when \( \Sigma \) is nondegenerate stable, even without assuming (3).

**Remark 5.8.** Theorem 5.1 also guarantees that a locally stable minimal hypersurface \( \Sigma \) must be uniquely area-minimizing near the singular point \( p \) at which the tangent cone is regular and strictly minimizing. In fact, by proposition 5.5 (5), one can change the metric outside a small neighborhood of \( p \) to make \( \Sigma \) strictly stable. Then theorem 5.1 provides a mean convex foliation of \( \text{Clos}(\Sigma) \), hence contains a small neighborhood of \( p \), inside which \( \Sigma \) is uniquely area minimizing.

6. **Strong min-max property for singular minimal hypersurfaces**

The goal of this section is to prove a strong min-max property for certain non-degenerate singular minimal hypersurface. The proof closely follows the argument in [Whi94]. The only difference is the treatment of singularities, which is based on the results in section 4 and 5.

Let’s begin with the basic setup:

- **S1.** \( (\Sigma, \nu) \subset (M^{n+1}, g) \) be a two-sided closed minimal hypersurface with only strongly isolated singularities; \( M \setminus \text{Clos}(\Sigma) = M_+ \cup M_- \) where \( \nu \) points into \( M_+ \).
- **S2.** Assume the Jacobi operator \( L_\Sigma \) is nondegenerate in \( \mathcal{R}_0(\Sigma) \) with \( I := \text{ind}(\Sigma; M) \geq 1 \);
- **S3.** Assume that \( \forall p \in \text{Sing}(\Sigma) \), the tangent cone \( C_p \) of \( \Sigma \) at \( p \) is strictly minimizing and strictly stable.
- **S4.** Let \( \tau \in (0, \tau_\Sigma(1)/4), K > 1 \); Let \( \{ U_j \}_{j \geq 1} \) be a decreasing family of smooth domains satisfying...
\[ \bigcap_{j \geq 1} U_j = \text{Clos}(\Sigma). \]
\[ \partial U_j \text{ are mean convex in } B^M_{\mathcal{U}}(\text{Sing}(\Sigma)) \]
\[ \partial U_j \cap M_{\pm} \text{ are graphs of smooth functions } u_j^\pm \in C^2(\Sigma) \text{ in } M \setminus B^M_{\mathcal{U}}(\text{Sing}(\Sigma)) \text{ such that} \]
\[ |\nabla u_j^\pm| + |\nabla^2 u_j^\pm| \leq K|u_j^\pm| \text{ on } \Sigma \setminus B_r(\text{Sing}(\Sigma)) \]
\[ \sup_{\Sigma \setminus B_r(\text{Sing}(\Sigma))} |u_j^\pm| \leq K \cdot \inf_{\Sigma \setminus B_r(\text{Sing}(\Sigma))} |u_j^\pm| \rightarrow 0_+ \text{ as } j \rightarrow \infty \]

The existence of such \( K, r \) and \( \{U_j\}_{j \geq 1} \) can be seen from the following argument. By Remark 5.8 for each \( p \in \text{Sing}(\Sigma) \), let \( O_p \subset M \) be a smooth convex neighborhood of \( p \) in which \( \Sigma \) is uniquely area-minimizing; \( \Sigma_{p,j}^\pm \subset M_{\pm} \cap O_p \) is a family of area-minimizing hypersurfaces in \( O_p \) which converges to \( \Sigma \cap O_p \) from both sides; Such \( \Sigma_{p,j}^\pm \) exists by solving Plateau problem with boundary to be the one-sided perturbation of \( \partial O_p \cap \Sigma \); For every compact subset \( Z \subset O_p \), \( \text{Sing}(\Sigma_{p,j}) \cap Z = \emptyset \) for \( j \gg 1 \) by Allard regularity \cite{All72, Sim83b}, theorem 28 and a blow-up argument similar to \cite{Sim93}. Hence \( U_j \) in (S4) can be constructed by (passing to a subsequence if necessary) gluing these \( \Sigma_{p,j}^\pm \) with a constant function over a neighborhood of \( \Sigma \setminus \bigcup_{p \in \text{Sing}(\Sigma)} O_p \) by partition of unity.

Throughout this and next section every constant will depend on the quantities in (S1)-(S4).

**Theorem 6.1.** There exists \( k_1 \geq 1 \), \( \Phi_0 : \text{Clos}(B^1) \rightarrow Z_n(\text{Clos}(U_{k_1})) \) \( \mathcal{F} \)-continuous and \( \epsilon : (0,1] \rightarrow (0,1) \) non-decreasing such that

(i) \( \Phi_0(0^1) = \Sigma \);

(ii) \( \mathcal{M}(\Phi_0(z)) \leq \mathcal{M}(\Sigma) - \epsilon(r) \) for every \( z \in \partial B^1 \);

(iii) If \( r \in (0,1] \) and \( \{\Phi_i\}_{i \geq 1} : \text{Clos}(B^1) \rightarrow Z_n(\text{Clos}(U_{k_1})) \) is a family of \( \mathcal{F} \)-continuous map such that

\[
\lim_{i \rightarrow \infty} \sup_{\mathcal{F}} \{\Phi_i(z): z \in \partial B^1\} = 0
\]

Then \( \lim_{i \rightarrow \infty} \inf_{\mathcal{F}} \sup_{\mathcal{F}} \{\Phi_i(z): z \in \partial B^1\} \geq \mathcal{M}(\Sigma) \).

We start with some analytic preparation.

**Lemma 6.2.** There exists \( \phi_1, \phi_2, ..., \phi_I \in C^\infty_c(\Sigma) \) and \( \lambda, \mu > 0 \) such that

(i) For every \( u \in \text{span}_{1 \leq i \leq I} \{\phi_i\} \),
\[ Q_\Sigma(u, u) \leq -\lambda \|u\|_{L^2(\Sigma)}^2 \]

(ii) For every \( v \in C^1_c(\Sigma) \),
\[ Q_\Sigma(v, v) + \mu^2 \sum_{1 \leq i \leq I} \left( \int_{\Sigma} v \cdot \phi_i \right)^2 \geq \lambda \int_{\Sigma} |\nabla v|^2 + v^2 / \rho^2 \]

(iii) The matrix \( [(\phi_i, \phi_j)_{L^2(\Sigma)}]_{(i,j) \in I \times I} \) is nondegenerate.

**Proof.** Let \( \psi_1, ..., \psi_I \in \mathcal{B}_0(\Sigma) \) be the first \( I \) pairwise orthogonal unit \( L^2 \)-eigenfunctions of \(-L_\Sigma\). Since \( C^\infty_c(\Sigma) \) is dense in \( \mathcal{B}_0(\Sigma) \), take \( \phi_i \in C^\infty_c(\Sigma) \) such that \( \sup_{1 \leq i \leq I} \|\phi_i - \psi_i\|_{\mathcal{B}(\Sigma)} \leq \epsilon \), where \( \epsilon \in (0,1) \) TBD.

To prove (i), notice that for every \( a \in \mathbb{R}^I \),
\[ Q_\Sigma(\sum_i a_i \phi_i, \sum_i a_i \phi_i) \leq -|\lambda'| \sum_i a_i^2 \]
where \( \lambda' \) be the largest negative eigenvalue of \(-L_\Sigma\); Also notice that
\[ |Q_\Sigma(\sum_i a_i \phi_i, \sum_i a_i \phi_i) - Q_\Sigma(\sum_i a_i \phi_i, \sum_i a_i \phi_i)| \]
\[ = | \sum_{1 \leq i,j \leq I} a_i a_j (Q_\Sigma(\phi_i, \phi_j) - Q_\Sigma(\psi_i, \psi_j)) | \leq C(\Sigma, I) \epsilon \cdot \sum_i a_i^2 \]
Hence by taking \( \epsilon < \lambda'|2C(\Sigma, I) \), we have (i) for \( \lambda \leq |\lambda'|/2 \).
Lemma 6.4. A of $A$ of $\Sigma$ for $T$ of convergence of minimizers for $H$. Hence if take $\varepsilon < \lambda''/2\mu'$ and $\lambda \leq \lambda''/2$, then we have

$$Q_{\Sigma}(v, v) + \frac{1}{2} \sum_{i=1}^{27} \left( \int_{\Sigma} v \cdot \psi_i \right)^2 \geq \lambda'' \|v\|_{L^2(\Sigma)}^2$$

where $\lambda''$ be the smallest positive eigenvalue of $-L_{\Sigma}$ and $\mu'$ be the difference between $\lambda''$ and the first eigenvalue of $-L_{\Sigma}$. Also note that

$$\left( \int_{\Sigma} v \cdot \psi_i \right)^2 - \left( \int_{\Sigma} v \cdot \phi_i \right)^2 \leq C'(\Sigma) \varepsilon \cdot \|v\|_{L^2}^2$$

Hence if $\varepsilon < \lambda''/(2\mu')(\Sigma)$, $\mu' \geq |\mu'|$ and $\lambda \leq \lambda''/2$, then we have

$$Q_{\Sigma}(v, v) + \frac{1}{2} \sum_{i=1}^{27} \left( \int_{\Sigma} v \cdot \phi_i \right)^2 \geq \lambda'' \|v\|_{L^2(\Sigma)}^2$$

(6.1)

Also note that since each tangent cone of $\Sigma$ is strictly stable, by lemma 2.1 and lemma 2.9, there is some small neighborhood $\mathcal{V}$ of $Sing(\Sigma)$ and $\vartheta(\Sigma) > 0$ such that

$$Q_{\Sigma}(v, v) - \vartheta \int_{\mathcal{V}} \frac{v^2}{\rho^2} \geq 0 \quad \forall v \in C^1_\varepsilon(\mathcal{V})$$

Hence by a similar argument as lemma 3.1, we have, for some $C''(\Sigma) > 0$,

$$Q_{\Sigma}(v, v) - \vartheta \int_{\mathcal{V}} \frac{v^2}{\rho^2} \geq C'' \|v\|_{L^2(\Sigma)}^2 \geq 0 \quad \forall v \in C^1_\varepsilon(\mathcal{V})$$

(6.2)

Also by lemma 3.1, $\|A\Sigma\|^2 + \text{Ric}_M(v, v) \leq C''''(\Sigma)/\rho^2$ on $\Sigma$, hence $\forall v \in C^1(\Sigma)$,

$$\left( 1 + a \right) \left( Q_{\Sigma}(v, v) + C'' \|v\|_{L^2(\Sigma)}^2 \right) \geq a \int_{\Sigma} |\nabla v|^2 + \left( Q_{\Sigma}(v, v) - \vartheta \int_{\mathcal{V}} \frac{v^2}{\rho^2} + C'' \|v\|_{L^2(\Sigma)}^2 \right) \geq a \int_{\Sigma} |\nabla v|^2$$

(6.3)

where $0 < a \leq \vartheta/C''''$. Hence lemma 5.2 (ii) follows by combining (6.1), (6.2) and (6.3) and take $0 < \lambda \leq \min\{1, \lambda''/(2C''') \cdot \min \{\vartheta, a/(1+a)\} \}/2$.

To prove (iii), it suffices to notice that $|\phi_i \cdot \phi_j|_{L^2(\Sigma)} - \delta_{ij} \leq 10\varepsilon$ and take $\varepsilon < 1/100H^2$. $\square$

Remark 6.3. We see from its proof that lemma 6.2 (i) and (6.1) holds for any $\Sigma$ satisfying $L^2$ nonconcentration property; while the proof of (ii) essentially use strictly stability of tangent cones of $\Sigma$.

Now take $\tilde{f} = (f_1, ..., f_l) \in C^\infty(M \setminus Sing(\Sigma); \mathbb{R}^l)$ such that $\tilde{f} = 0$ on $\Sigma$ and $\nu(f_i) = \mu \phi_i$ on $\Sigma$; Let $\tau \in (0, \tau)$ such that $B^\Sigma_\tau(Sing(\Sigma)) \cap \text{sp}(\tilde{f}) = \emptyset$.

Let $\zeta \in C^2(\mathbb{R}^l; [0, +\infty))$ such that $\zeta(0) = 0$, $\nabla^2 \zeta(0) = id$ and $\sup_{\mathcal{R}^l} |\nabla \zeta| \cdot \sup_{\mathcal{R}^l} |\tilde{f}| < 1$. Define for $T \in Z_n(M)$,

$$A^*(T) := M(T) + \zeta(\int_M \tilde{f}(x) \, d\|T\|_M(x))$$

Lemma 6.4. $A^*$ is lower semi continuous in flat topology.

The proof is the same as Section 3, claim 2 in [Whi94].

For a general parametric elliptic integrand $F$, the regularity problem of $(F, c_{\alpha}, \delta)$-almost minimizing surfaces is studied in [Alm75, SSS82, Rom82], and is used in [Whi94] to gain better regularity of convergence of minimizers for $F^*$ in a smaller and smaller neighborhood of the given surface stationary with respect to $F$. $F \equiv 1$ corresponds to the mass functional.

Observe that since $U_j$ are mean convex near $Sing(\Sigma)$ and $\tilde{f}$ vanishes near $Sing(\Sigma)$, minimizers of $A^*$ in $\text{Clos}(U_j)$ are minimal near each singularity. Hence the same argument in Section 3, claim 1 in [Whi94] yields the following,
**Lemma 6.5.** For every $\alpha \in (0, 1)$, there exists some constant $c, k_2 > 0$ such that for $j \geq k_2$, every minimizer $T \in \mathcal{Z}_n(Clos(U_j))$ of $A^*$ among cycles homologous to $[\Sigma]$ in $U_j$ is $(ct^n, \delta_7)$-almost minimizing in $M$.

In particular, if $T_j$ is a minimizer of $A^*$ in $Clos(U_j)$ homologous to $[\Sigma]$, then $T_j \to [\Sigma]$ in flat topology as $j \to \infty$ and

1. $T_j$ are homologically area minimizing in $B^2_{\infty}(Sing(\Sigma)) \cap Clos(U_j)$.
2. There are $u_j \in C^1(\Sigma)$ such that for every $U \subset M \setminus Sing(\Sigma)$,

$$T_j \cap U = [\text{graph} u_j]\cap U \quad \text{for} \quad j \gg 1$$

**Lemma 6.6.** There exists $k_1 \geq k_2$ such that $[\Sigma]$ is the only $A^*$-minimizer in its homology class in $\mathcal{Z}_n(Clos(U)_{k_1})$.

**Proof.** The proof closely follows [Whi94, Theorem 2]. The major extra effort is made to estimate error terms near singularities.

Suppose for contradiction that after passing to a subsequence, there are $[\Sigma] \neq T_j \in \mathcal{Z}_n(Clos(U_j))$ homologous to $[\Sigma]$, i.e. $T_j - [\Sigma] = \partial P_j$ for some $P_j \in I_{n+1}(M)$ supported in $Clos(U_j)$, which minimize $A^*$ among its homology class in $\mathcal{Z}_n(Clos(U_j))$.

Let $0 \neq u_j$ be the graphical functions of $T_j$ over $\Sigma$ as described in lemma 6.5 (ii). Moreover, for each $p \in Sing(\Sigma)$, let

$$r_j(p) := \inf \{t > 0 : T_j \cup A^M_{t, 2^2} = [\text{graph} u_j]\cup A^M_{t, 2\tau}(p) \text{ and } \|u_j\|_{2, A^*_{t, 2\tau}(p)} \leq \delta_3\}$$

where $\delta_3$ is specified in lemma 4.11. Clearly, $r_j(p) \to 0_+$ as $j \to \infty$; WLOG $r_j(p) < \tau$.

Since each $C_p$ is strictly stable, one can take $\sigma \in \left(-\frac{n-2}{2}, \inf_{p \in Sing(\Sigma)} \gamma_1^1(C_p)\right)$. By corollary 4.13 there exists $\Omega_0 \subset \subset \Sigma$ such that for $j \gg 1$,

$$|u_j|/p + |\nabla u_j| \leq C(\sigma)p^{\sigma-1} \cdot \left(\int_{\Omega_0} u_j^2\right)^{1/2} \quad \text{on} \quad A_{2r_j(p), \tau}(p)$$

**Claim:** $\liminf_{j \to \infty} r_j(p)^{2\sigma-2} \int_{\Omega_0} u_j^2 > 0$.

**Proof of claim:** Otherwise, consider $V_j := (t_{p, r_j(p)})_2 V_j$. By (6.4), up to a subsequence, $V_j \to V_\infty$ for some stationary integral varifold $V_\infty$ equal to $|C_p|$ outside the ball of radius 2. Since $C_p$ is area minimizing, by a barrier argument using Hardt-Simon foliations, $V_\infty = |C_p|$ in $\mathbb{R}^{n+1}$. By Allard regularity, this contradict to the minimality of $r_j(p)$.

Let $\tilde{U}_j := \Sigma \setminus \bigcup_{p \in Sing(\Sigma)} B_{2r_j(p)}(p)$; $U_j := \Sigma \setminus \bigcup_{p \in Sing(\Sigma)} B_{4r_j(p)}(p)$. Take $j \gg 1$ such that $U_j \supset \Omega_0 \cup (\Sigma \cap \text{spt}(\bar{f}))$.

Let $\xi_j \in C^\infty_c(U_j)$ be a cut-off function which equal to 1 on $U_j$ and $|\nabla \xi_j| \leq 10/r_j(p)$ on $B_\rho(p)$; Let $\tilde{u}_j := u_j \cdot \xi_j$. Note that by (6.4),

$$\int_{B_{4r_j(p)}(p)} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/p^2 \leq C(\sigma)r_j(p)^{2\sigma-2+n} \cdot \int_{\Omega_0} u_j^2$$
Now we are ready to estimate $A^*(T_j) - A^*(\Sigma)$. Work under Fermi coordinate near $\Sigma$, let $F(x, z, p)$ be the volume element for graphs over $\Sigma$ as in lemma 2.11 we have

$$0 \geq A^*(T_j) - A^*(\Sigma)$$

$$\geq \int_{U_j} (F(x, u_j, \nabla u_j) - 1) - \mathcal{H}^n(\Sigma \setminus U_j) + \zeta \left( \int_{\Sigma} \tilde{f}(x, u_j(x))F(x, u_j, \nabla u_j) \, d\mathcal{H}^n(x) \right)$$

$$\geq \left[ \int_{U_j} dx \int_0^1 \partial_p F(x, su_j, s\nabla u_j) \cdot \nabla u_j + \partial_x F(x, su_j, s\nabla u_j) \cdot u_j \, ds \right] - \mathcal{H}^n(\Sigma \setminus U_j)$$

$$+ \zeta \left( \int_{\Sigma} \tilde{f}(x, u_j(x))F(x, u_j, \nabla u_j) \, d\mathcal{H}^n(x) \right)$$

To estimate (I), notice that by lemma 2.11

$$|\partial_p F(x, z, p) - p| + |p| \partial_x F(x, z, p) - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)) |z| \leq C(|z|/\rho + |p|)^2$$

Hence, combined with (6.4) and (6.5) we have,

$$(I) \geq \frac{1}{2} \int_{U_j} |\nabla u_j|^2 + (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)) u_j^2 - C(|\nabla u_j| + |u_j|/\rho)^3$$

$$\geq \frac{1}{2} \int_{\Sigma} |\tilde{u}_j|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)) \tilde{u}_j^2 \, d\mathcal{H}^n$$

$$- C(\sigma, \Omega_0) \int_{\Sigma} (|\nabla \tilde{u}_j| + |\tilde{u}_j/\rho| + r_j(p)^2)^2 (|\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)$$

$$= \frac{1}{2} \int_{\Sigma} |\nabla \tilde{u}_j|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)) \tilde{u}_j^2 \, d\mathcal{H}^n + o(\int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)$$

To estimate (II), it suffices to see that by claim 1, $r_j(p)^n = o(\int_{\Omega_0} u_j^2)$ as $j \to \infty$. Hence,

$$(II) \leq \sum_{p \in \text{Sing}(\Sigma)} \mathcal{H}^n(B_{4r_j(p)}(p)) \leq C(\Sigma) \cdot \sum_p r_j(p)^n = o(\int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)$$

To estimate (III), note that by definition, $\zeta(\nu) \geq |\nu|^2/2 - C_\zeta|\nu|^3$ for some $C_\zeta > 0$; and by the choice of $f$,

$$|\int_{\Sigma} \tilde{f}(x, u_j(x))F(x, u_j, \nabla u_j) - \mu \tilde{\phi} \cdot \tilde{u}_j \, d\mathcal{H}^n| \leq C(f) \int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2$$

Hence, we have

$$(III) \geq \frac{\mu^2}{2} |\int_{\Sigma} \tilde{u}_j \cdot \tilde{\phi} \, d\mathcal{H}^n|^2 - o(\int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)$$

Now, plug (6.7), (6.8) and (6.9) into (6.6) and apply lemma 6.2 (ii), we get,

$$0 \geq \int_{U_j} (F(x, u_j, \nabla u_j) - 1) - \mathcal{H}^n(\Sigma \setminus U_j) + \zeta \left( \int_{\Sigma} \tilde{f}(x, u_j(x))F(x, u_j, \nabla u_j) \, d\mathcal{H}^n(x) \right)$$

$$\geq \lambda \int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2 \, d\mathcal{H}^n - o(\int_{\Sigma} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)$$

This contradicts to that $\lambda > 0$ in lemma 6.2 and that $u_j \neq 0$. Thus the proof is completed. \(\square\)

**Proof of theorem 6.7** Let $k_1$ be specified in lemma 6.6 Define $\Phi_0 : Clos(B_1^+) \to \mathbb{Z}_n(\text{Clos}(U_{k_1}))$ by

$$\Phi_0(z) := \left[ \text{graph}_\Sigma (\tilde{t} \cdot (z_1 \phi_1 + z_2 \phi + \ldots + z_t \phi_t)) \right]$$
where \( \bar{t} \in (0,1) \) small enough such that \( \text{Im}(\Phi_i) \subset \mathbb{Z}_n(\text{Clos}(U_j)) \) and that (ii) in theorem 6.1 hold for some \( \epsilon(r) > 0 \). Such \( \bar{t} \) and \( \epsilon(r) \) exists by lemma 6.2 (i).

Also by lemma 6.2 (iii), we can take \( \bar{t} > 0 \) even smaller such that

\[
\tilde{f}_0 : \text{Clos}(B_1^0) \rightarrow \mathbb{R}^{2} \quad z \mapsto \int_M f d\|\Phi_0(z)\|
\]

has non-zero degree at \( 0' \in \mathbb{R}^{2} \). Fix the choice of \( \bar{t} \) and then \( \Phi_0 \) from now on.

Now if \( \{\Phi_i\}_{i \geq 1} \) be a family of \( F \)-continuous sweepouts described in theorem 6.1 (iii), then

\[
\tilde{f}_i : \text{Clos}(B_1^i) \rightarrow \mathbb{R}^{2} \quad z \mapsto \int_M f d\|\Phi_i(z)\|
\]

\( C^0 \) converges to \( \Phi_0 \) restricted on \( \partial B_1^0 \), and hence also has non-zero degree at \( 0' \) for \( i >> 1 \). In particular, \( \exists z_i \in B_1^i \) such that \( \tilde{f}_i(z_i) = 0' \). Therefore, by lemma 6.6

\[
\sup\{M(\Phi_i(z)) : z \in B_1^i \} \geq M(\Phi(z_i)) = A^+_{\delta}(\Phi(z_i)) \geq A^+_{\delta}(|\Sigma|) = M(\Sigma)
\]

for \( i >> 1 \), thus proves (iii) of theorem 6.1. \( \square \)

7. A rigidity result for constrained minimal hypersurfaces

Recall as introduced in [Wan20], for a smooth domain \( U \subset (M,g) \), an integral varifold \( V \in \mathcal{D}_v(U) \) is called constrained embedded minimal hypersurface in \( \text{Clos}(U) \) if it’s a locally stable minimal hypersurface with optimal regularity inside \( U \), locally sum of \( C^{1,1} \) graphs with multiplicity near \( \partial U \), possibly with self-touching on \( \partial U \), and satisfies the variational inequality

\[
\delta V(X) \geq 0
\]

for every vector field \( X \in \mathcal{X}(M) \) which pointing inward \( U \) along \( \partial U \). Note that by [Wan20] Remark 2.9, (1)], the mean curvature vectors of these \( C^{1,1} \) graph comprising \( V \) all points outward \( U \) along \( \partial U \) and equals to the mean curvature of \( \partial U \) a.e. on \( \text{spt}(V) \cap \partial U \). This is the only additional fact we shall use in the proof of lemma 7.1.

[Wan20] discusses the existence of such constrained embedded minimal hypersurface through min-max construction, as well as a local rigidity result [Wan20] Theorem 5.1 near a smooth nondegenerate minimal hypersurface. The goal of this section is to prove an analogue of this local rigidity in the singular setting. The prove is almost the same, but more effort is made to deal with error terms near singularities.

We keep the same set up as section 6 i.e. suppose (S1)-(S4) holds. Note that by strong maximum principle and mean convexity of \( \partial U_j \) in (S4) inside \( B^M_{\bar{r}}(\text{Sing}(\Sigma)) \), any constrained embedded minimal hypersurface \( V \in \text{Clos}(U_j) \) must be stationary (i.e have vanishing mean curvature) in \( B^M_{\bar{r}}(\text{Sing}(\Sigma)) \).

**Lemma 7.1.** Let \( \epsilon \in (0,1) \). There exists \( k_2 = k_2(\epsilon) \geq 1 \) such that for every \( j \geq k_2(\epsilon) \), if \( V \) is a constrained embedded minimal hypersurface in \( \text{Clos}(U_j) \) with

\[
\mathcal{H}^n(\Sigma) \leq ||V||(M) \leq (2 - \epsilon)\mathcal{H}^n(\Sigma)
\]

then \( V = |\Sigma| \).

**Proof.** We shall argue by contradiction as is done in [Wan20] Theorem 5.1.

Suppose for sake of contradiction that after passing to a subsequence, there exist constrained embedded minimal hypersurfaces \( |\Sigma| \neq V_j \in \mathcal{D}_v(U_j) \), \( j \rightarrow \infty \). Since by the choice of \( \{U_j\}_{j \geq 1} \) in (S4), the mean curvature of \( \partial U_j \) on \( \text{spt}(h) \) tends to 0 as \( j \rightarrow \infty \), we see that the mean curvature of \( V_j \) converges to 0 in \( L^\infty \). Hence by Allard compactness theorem [All72][Sim83], up to a subsequence, \( V_j \rightarrow V \) for some stationary integral varifold \( V \in \mathcal{D}_v(M) \) supported in \( \text{Clos}(\Sigma) \) and has \( \mathcal{H}^n(\Sigma) \leq ||V||(M) \leq (2 - \epsilon)\mathcal{H}^n(\Sigma) \). This implies \( V = |\Sigma| \). Thus by Allard regularity theorem [All72][Sim83], this convergence is in \( C^{1,0} \)-graphical sense on each compact subset of \( \Sigma \), \( \forall \alpha \in (0,1) \).
Let $\alpha \in (0,1)$ be fixed from now on: Let $\delta_3 > 0$ be specified in lemma 4.11, $v_j \in C^{1,\alpha}(\Sigma)$ be the graphical function of $V_j$ over $\Sigma$, $r_j(p) \to 0_+$ for each singularity $p$ such that

$$r_j(p) = \inf \{ t > 0 : V_j \cup A^M_{1,2r}(p) = \text{graph} \Sigma(v_j)|_{A^M_{1,2r}(p)}, \|v_j\|_{L^2(A^M_{1,2r}(p))} \leq \delta_3 \}$$

Take $j > > 1$ such that $r_j(p) < \tau/2$. Let $\sigma \in (- (n - 2)/2, \inf_{p \in \text{Sing}(\Sigma)} \gamma_1^- (C_p))$ be fixed. By corollary 4.13 there exists $\Sigma \setminus B_{\tau}(\text{Sing}(\Sigma)) \subset \Omega_0 \subset \subset \Sigma$ such that for $j > > 1$, we have that up to a subsequence,

$$|v_j|/\rho + |\nabla v_j| \leq C(\sigma)\rho^{\sigma - 1} \cdot \left( \int_{\Omega_0} v_j^2 \right)^{1/2} \quad \text{on } A_{2\tau, \tau}(p)$$

Hence by the same argument as in the proof of lemma 6.6, we have

$$\lim \inf_{j \to \infty} r_j(p)^{2\sigma - 2} \int_{\Omega_0} v_j^2 > 0$$

Let $F = F^\sigma(x, z, p)$ be the area integrand of graph over $\Sigma$ defined in lemma 2.11. $M := \mathcal{M}^\sigma = -\text{div}_\Sigma(\partial_0 F(x, u, \nabla u)) + \partial_1 F(x, u, \nabla u)$ be the minimal surface operator. Also let $U_j := \Sigma \setminus \bigcup_{p \in \text{Sing}(\Sigma)} B_{\tau,j}(p)$; $u_j^\pm$ be $C^2$ functions over $\Omega_0$ such that under Fermi coordinates of $M$ near $\Sigma$, i.e.

$$\{(x, t) : x \in \Omega_0, u_j^+ (x) < t < u_j^- (x)\} \setminus B_{2\tau}(\text{Sing}(\Sigma)) = U_j \setminus B_{2\tau}^M(\text{Sing}(\Sigma))$$

Since $V_j$ are constrained embedded minimal hypersurfaces in $\text{Clos}(U_j)$, we have

$$\begin{cases}
v_j \cdot M v_j \leq 0 \quad \text{weakly on } \Omega_0; \\
M v_j = 0 \quad \text{on } \{ u_j^- < v_j < u_j^+ \} \cup (U_j \setminus \Omega_0); \\
|M v_j| \leq C|v_j| \quad \text{weakly on } U_j.
\end{cases}$$

Let $\varepsilon_j^2 := \int_{\Omega_0} |\nabla v_j|^2 + v_j^2/\rho^2$, then by (7.3), lemma 2.11 and standard elliptic estimate [GT01], we have that up to a subsequence, $v_j/\varepsilon_j \to \hat{v}_\infty$ in $W^{1,2} \cap C^0_{\text{loc}}(\Sigma)$ for some $0 \neq \hat{v}_\infty \in W^{1,2}(\Sigma)$ satisfying

$$\begin{cases}
-\hat{v}_\infty \cdot L_\Sigma \hat{v}_\infty \leq 0 \quad \text{weakly on } \Sigma; \\
-L_\Sigma \hat{v}_\infty = 0 \quad \text{on } \{ \hat{v}_\infty < \hat{v}_\infty < \hat{u}_\infty^+ \} \cap \Omega_0 \cup \Sigma \setminus \Omega_0.
\end{cases}$$

where $\hat{u}_\infty^\pm$ are the subsequential limit of $u_j^\pm/\varepsilon_j$, which is either either everywhere $\pm \infty$ or everywhere finite and nonzero by the assumption in (S4) and that $|v_j| \leq \max\{|u_j^+|, |u_j^-|\}$ on $\Omega_0$ for $j \geq 1$. Moreover by (7.1), $\mathcal{A}_p(\hat{v}_\infty) \geq \sigma > \gamma_1^- (C_p)$ for every singular point $p$. Hence by corollary 5.17 (2) and corollary 6.15, $\hat{v}_\infty \in \mathcal{B}_0(\Sigma)$.

Claim:

$$\int_{\Sigma} |\nabla \hat{v}_\infty|^2 - (|A_\Sigma|^2 + \text{Ric}_M (\nu, \nu)) \hat{v}_\infty^2 \, dx = \int_{\Sigma} -\hat{v}_\infty \cdot L_\Sigma \hat{v}_\infty \geq 0$$

Note that combine this claim with (7.3), we see that $\hat{v}_\infty$ actually satisfies the Jacobi field equation $L_\Sigma \hat{v}_\infty = 0$ on $\Sigma$. Together with that $\hat{v}_\infty \in \mathcal{B}_0(\Sigma)$, this contradicts to the non-degeneracy of $L_\Sigma$ and completes the proof of lemma 6.14.
Proof of the claim: By lemma 2.11

\[ 0 \leq \|V_j\| (M) - \mathcal{H}^n (\Sigma) \leq \int_{U^j} F(x, v_j, \nabla v_j) - 1 \; d\mathcal{H}^n (x) + \sum_{p \in \text{Sing}(\Sigma)} \|V_j\| (B^M_{\text{sr},r}(p)) \]

\[ \leq \int_{U^j} dx \int_0^1 \partial_p F(x, sv_j, s\nabla v_j) \cdot \nabla v_j + \partial_x F(x, sv_j, s\nabla v_j) \cdot v_j \; ds + \sum_{p \in \text{Sing}(\Sigma)} \|V_j\| (B^M_{\text{sr},r}(p)) \]

\[ = \frac{1}{2} \int_{U^j} \partial_p F(x, v_j, \nabla v_j) \cdot \nabla v_j + \partial_x F(x, v_j, \nabla v_j) \cdot v_j \; dx + \mathcal{E} + \sum_{p \in \text{Sing}(\Sigma)} \|V_j\| (B^M_{\text{sr},r}(p)) \]

(7.5)

where by the fact that \( \int_0^1 f(s) \; ds - (f(0) + f(1))/2 = (\int_0^1 f''(s)(s^2 - s) \; ds)/2 \) and lemma 2.11, we have

\[ |\mathcal{E}| \leq C \int_{U^j, s \in [0,1]} \left[ |\partial^3_{ppp} F(x, sv_j, s\nabla v_j)||\nabla v_j|^3 + |\partial^3_{ppz} F(x, sv_j, s\nabla v_j)||\nabla v_j|^2 v_j \right. \]

\[ + \left. |\partial^3_{pzv} F(x, sv_j, s\nabla v_j)||\nabla v_j|^2 v_j^2 + |\partial^3_{zzv} F(x, sv_j, s\nabla v_j)||\nabla v_j|^3 \right] dx \]

\[ \leq C \int_{U^j} (|\nabla v_j| + |v_j|/|z|) \; dx = o(c_j^2) \]

(7.6)

And by (7.2) and volume monotonicity for stationary varifolds in \( M \),

\[ (II) \leq \sum_{p \in \text{Sing}(\Sigma)} C(\tau) r_j(p) \cdot \|V_j\| (B^M_{\tau}(p)) = o(c_j^2) \]

(7.7)

Now to estimate (I), first observe by

\[ |\partial_p F(x, z, p) - p| + \rho |\partial_x F(x, z, p) + (|A_S|^2 + Ric_M(\nu, \nu))| \leq C(\tau)(|p| + |z|/\rho(x))^2 \]

Hence,

\[ (I) \leq \frac{1}{2} \int_{U^j} |\nabla v_j|^2 - (|A_S|^2 + Ric_M(\nu, \nu))v_j^2 \; dx + C(\tau) \int_{U^j} (|\nabla v_j| + |v_j|/\rho)^3 \]

(7.8)

\[ \leq \frac{1}{2} \int_{U^j} |\nabla v_j|^2 - (|A_S|^2 + Ric_M(\nu, \nu))v_j^2 \; dx + o(c_j^2) \]

Note that by (7.1), as \( j \to \infty \),

\[ \int_{U^j} |\nabla (v_j/c_j)|^2 - (|A_S|^2 + Ric_M(\nu, \nu))(v_j/c_j)^2 \; dx \to \int_{\Sigma} |\nabla \tilde{v}_\infty|^2 - (|A_S|^2 + Ric_M(\nu, \nu))\tilde{v}_\infty \; dx \]

Therefore, combined with (7.3), (7.7) and (7.8), by dividing (7.5) by \( c_j^2 \) and taking \( j \to \infty \), the proof of claim is completed.

8. Applications and further discussions

We first finish the proof of theorem 1.2 and 1.3

Proof of theorem 1.2 First recall that by remark 5.7, theorem 1.2 is already proved when \( \Sigma \) is non-degenerate stable. We now assume ind(\( \Sigma \)) \geq 1.

Since \( \Sigma \subset (M, g) \) in theorem 1.2 satisfies (S1)-(S3) in section 6 in this case, we can choose \( K, \tau \) and \( \{U_j\}_{j \geq 1} \) as in (S4).

Let \( k_1 \geq 1 \) be given in theorem 6.1, \( k_2 \geq 1 \) be given in lemma 7.11. Let \( \tilde{k} \geq \max\{k_1, k_2(1/2)\} \) be fixed; \( R_0 \subset (0, 1) \) is also fixed such that sp(\( \Phi_0(x) \)) \subset U_{\tilde{k}+1} for all \( x \in \text{Clos}(B^3_{R_0}) \), where \( \Phi_0 \) is specified in theorem 6.1. Also let \( \eta \in C^\infty_c (U_{\tilde{k}+1}, [0, 1]) \) be a fixed cut off function with \( \eta|_{U_{\tilde{k}+2}} = 1 \).
Let $\Pi$ be the family of sequences of $F$-continuous maps $S = \{ \Phi_i : \operatorname{Clos}(\mathbb{B}_z) \to \mathbb{Z}_n(\operatorname{Clos}(U)) \}, i \geq 1$ such that

$$\sup \{ \Phi_i(z), \Phi_0(z) : z \in \partial \mathbb{B}_z \} \to 0 \quad \text{as} \quad i \to \infty.$$

Hence by theorem [6.1] we have

$$L(\Pi; g) := \inf_{S = \{ \Phi_i \}_{i \geq 1} \in \Pi} \limsup_{i \to \infty} \sup_{z \in \mathbb{B}_z} M^p(\Phi_i(z)) = \mathcal{H}_n(\Sigma) > L_{\mathfrak{A}_0, g_0} := \sup_{z \in \partial \mathbb{B}_z} M^p(\Phi_0(z)).$$

Observe that $L(\Pi; g)$ and $L_{\mathfrak{A}_0, g_0}$ is Lipschitz in $g$ in $C^0_{\text{loc}}$ sense. Therefore, $\exists \epsilon_1 > 0$ such that for every $\|g' - g\|_{C^4} \leq \epsilon_1$, if write $g'_q := \eta g' + (1 - \eta)g$, then $L(\Pi; g'_q) > L_{\mathfrak{A}_0, g_0}$. By [Wan20] Theorem 4.3, there exists a constrained embedded minimal hypersurface $V_{g'}$ in $(\operatorname{Clos}(U), g_0)$ with $\|V_{g'}\|(M) = L(\Pi; g'_q)$. Moreover, by [Wan20] Remark 4.4, there’s a constant $N_l$ only depending on $l$ such that, we can choose $V_{g'}$ satisfying the following,

For every $p \in M$, if $\{ A_t = \mathcal{A}^{M^p}_{\rho, \eta, t}(p) \}_{t=1}^{N_l}$ be a family of disjoint annuli in $M$ centered at $p$ with $s_1 > r_1 > 4s_1 + 1$; Then $V$ is a constrained embedded stable minimal hypersurface in at least one of $\{ A_t \}$.

From this fact, we see that if $g'_q \to g$ in $C^4$, then $V_{g'}$ will subconverges in varifold sense to some constrained stationary integral varifold $V_g$ in $\operatorname{Clos}(U, g)$ with the same mass as $\Sigma$, which is stable locally in every sufficiently small annuli. Hence by [Wan20] Theorem 2.11 & 3.2], $V_g$ is constrained embedded stable minimal hypersurface; And by lemma [7.1] we see that $V_g = \{ \Sigma \}$.

Also note that since $V_{g'}$ has uniformly bounded mean curvature (bounded by the mean curvature of $\partial U_k$), $\text{sp}(V_{g'}) \to \text{sp}(V_g) = \Sigma$ in Hausdorff distant sense. Therefore, for $g'$ sufficiently close to $g$, $\text{sp}(V_{g'}) \subset U_{k+2}$. In particular, $V_{g'}$ is a locally stable minimal hypersurface in $(M, g')$ and can be arbitrarily close to $\Sigma$ by taking $\|g' - g\|_{C^4}$ small enough. This completes the proof of theorem [1.2].

**Proof of theorem [1.3]** Note that for an arbitrary $f \in C^4(M)$ such that $q(f \cdot g) := -\nu(f) \cdot n/2$ is not identically 0 on $\Sigma$, and any family of Riemannian metric $g_t = g(t + 1) + o(t)$, let $\Sigma_t$ be the locally stable minimal hypersurfaces constructed in theorem [1.2] in $(M, g_t)$ which converges to $\Sigma$ as $t \to 0$.

By theorem [1.2] each subsequence of $\{ (\Sigma_t, g_t) \}$ induces a generalized Jacobi field $u$ on $\Sigma$, satisfying $L_{\gamma} u = c \cdot q(fg)$ for some $c \geq 0$. Moreover, since each tangent cone of $\Sigma_t$ is strictly minimizing and strictly stable, by taking $\sigma \in (-\gamma/2, \inf_{p \in \text{Sing}(\Sigma)} \gamma_1^+(C_p))$ in corollary [1.13] we have $\mathcal{A}(\sigma, u) \geq \sigma$ for every $p \in \text{Sing}(\Sigma)$. Hence by lemma [3.14] corollary [3.15] and [3.17], $u \in \mathcal{B}_q(\Sigma)$. Hence by nondegeneracy of $L_{\Sigma}$ and proposition [5.5] (3), $u = L_{\Sigma}^{-1}(c \cdot q(fg))$ is unique (up to a renormalization $c \neq 0$).

Now let

$$\gamma_{\text{reg}} := \{ f \in C^4(M) : \mathcal{A}_{\mathcal{B}_q}(L_{\Sigma}^{-1}(q(fg)) \geq \gamma_1^+(C_p) \text{ for every singularity } p \}$$

By lemma [3.21] $C^4(M) \setminus \gamma_{\text{reg}}$ is meager in $C^4(M)$; By corollary [1.12] for every $f \in \gamma_{\text{reg}}$, the minimal hypersurfaces in $(M, g(t + f) + o_t(t))$ are smooth for $|t| << 1$.

We shall also point out that the strict minimizing assumption in theorem [1.2] can not be dropped.

**Proposition 8.1.** There exists a Riemannian manifold $(M, g)$ and a two-sided strictly minimal mean convex hypersurface $(\Sigma, \nu) \subset (M, g)$ with only strongly isolated singularities satisfying (1) and (3) but not (2) in theorem [1.2], together with a neighborhood $U$ of $\operatorname{Clos}(\Sigma)$ and a family of metrics $g_t \to g$ in $C^\infty$ on $M$, such that there’s no stationary integral varifold supported in $U$ under each $g_t$.

Recall that by [Sim68, Lin87, Sim94, Law91], the Simons cone $C^{S,1} \subset \mathbb{R}^8$ over $S^5 \times S^1$ is strictly stable but only minimizing in one side. Using this cone as singularity model and applying the following lemma [5.2] one can derive proposition [5.1] using the same argument as in theorem [5.1] to construct piecewise smooth mean convex hypersurfaces $\{ \Sigma_t \}_{t \in (-1,1)}$ foliating $U$ with $\Sigma_0 = \Sigma$, and
such that the mean curvature vector of $\Sigma_t$ points into the same direction for $t \neq 0$. We leave the detail of this proof to readers.

**Lemma 8.2.** Let $C \subset \mathbb{R}^{n+1}$ be a stable minimal hypercone which is minimizing only in one-side. Let $E_-$ be the component of $\mathbb{R}^{n+1} \setminus C$ in which $C$ is not minimizing, $v_C$ be the normal field pointing away from $E_-$. Then the area-minimizer $S$ among all integral currents with boundary $\partial (C \cap \mathbb{B}^{n+1})$ is unique and is a smooth minimal hypersurface supported in $\text{Clos}(E_-)$ with boundary $C \cap \partial \mathbb{B}^{n+1}$. Moreover,

1. $(r \cdot S)_{r>0}$ foliates $E_-$; \n2. $S$ is a minimal graph over $C$ near $\partial S$ of some function $h$ satisfying

$$h(r, \omega) \leq 0, \quad h(1, \omega) = 0; \quad \partial_r h(1, \omega) > 0 \quad \forall \omega \in C \cap \partial \mathbb{B}^{n+1}$$

**Proof.** First note that since $C$ is area-minimizing in the other side, by comparing mass, $\text{spt}(S) \subset \text{Clos}(E_-)$. Hence by strong maximum principle and that $C$ isn’t minimizing in $E_-$, we have $\text{spt}(S) \cap C = C \cap \partial \mathbb{B}^{n+1}$. In particular, $\text{dist}(0, \text{spt}(S)) > 0$. Also, by $[\text{HS79}]$, $S$ is smooth near boundary. We may abuse the notation and do not distinguish between $S$ and $\text{spt}(S)$.

The fact that $(r \cdot S)_{r>0}$ foliates $E_-$ follows by considering $\text{sup}\{r \in (0,1) : r \cdot S \cap S = \emptyset\}$ and using strong maximum principle $[\text{Sim87}, \text{Ilm96}]$. The uniqueness of such $S$ also follows from the foliation structure.

Consider the Jacobi field $u := v_S \cdot \nabla_{\mathbb{R}^{n+1}} |x|^2/2$, where $v_S$ is the normal field of $S$ pointing away from $C$. Then since $(r \cdot S)_{r>0}$ foliates $E_-\{0\}$, $0 \leq u \leq 1$ and is not identically 0. By strong maximum principle for $L_S$, $u > 0$ on $\text{int}(S)$, hence by lemma 2.13, $\text{Sing}(S) = \emptyset$.

Also by weak maximum principle for $\Delta_S$, $u < 1$ on $\partial S$. Hence $S$ is a minimal graph over $C$ near $\partial S$. By Hopf boundary lemma, $u > 0$ on $\partial S$ and hence (8.1) holds. \qed

**List of Open Problems.** To finish this article, we list some open problems related with singular minimal hypersurface. Many of them are well known.

- **Singularity Model.** Let $C \subset \mathbb{R}^{n+1}$ be a stable minimal hypercone. Recall that $C$ is called regular if it has smooth cross section. Let

$$\mathcal{M}_k(C) := \{ \Sigma \subset \mathbb{R}^{n+1} \text{ locally stable minimal hypersurface, with codim} \geq 7 \text{ singular set} \}
\text{and let}
\mathcal{M}_k(C) := \{ \Sigma \in \bigcup_{k \geq 1} \mathcal{M}_k(C) : \Sigma \text{ is stable in } \mathbb{R}^{n+1} \}
\mathcal{M}^a(C) := \{ \Sigma \in \mathcal{M}_k(C) : \Sigma \text{ is area-minimizing in } \mathbb{R}^{n+1} \}
\mathcal{M}^{a,\text{loc}}(C) := \{ \Sigma \in \mathcal{M}_k(C) : \Sigma \text{ is area-minimizing in } U \text{ for some open subset } U \supset \Sigma \}
$$

Here are some problems regarding these classes of minimal hypersurfaces.

(P1.1) Find finite index minimal hypersurfaces in $\mathcal{M}_k(C)$ for $k \geq 2$. It seems nontrivial even when $C = \mathbb{R}^n$. When $n = 2$, this has long been studied. See $[\text{CKML17}, \text{CM18}]$ for a great survey and recent developments: When $n \geq 3$, the only connected example the author know is the higher dimensional catenoid. $[\text{Sch83}]$ show that this is unique up to scaling in $\mathcal{M}_2(\mathbb{R}^n)$; $[\text{Fl17}]$ studied the relations between $k$ (number of ends) and the index for a locally stable minimal hypersurface in $\mathcal{M}_k(\mathbb{R}^n)$. Similar index bound in terms of numbers of ends is expected for minimal hypersurfaces in $\mathcal{M}_k(C)$.

$[\text{Maz14}, \text{ABP+05}]$ construct catenoidal example in $\mathcal{M}_2(C)$ for quadratic cones $C$. Can we compute its index? Can we find similar examples for arbitrary stable/minimizing cones? Are they unique among the connected minimal hypersurfaces in $\mathcal{M}_2(C)$ up to scaling?
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[CSZ97] shows that a connected stable minimal hypersurface in \( \mathbb{R}^{n+1} \) has only one end. In particular, this implies for regular cone \( C \), every \( \Sigma \in \mathcal{M}_{\geq 2}(C) \) is unstable. Is this true when \( C \) has more singularities?

(P1.2) Find \( \Sigma \in \mathcal{M}_1(C) \) with index \( \geq 1 \)?

When \( C = C^{5,1} \subset \mathbb{R}^8 \) be the Simons cone over \( S^5 \times S^1 \), \[Maz14\] constructed two families of \( O(6) \times O(2) \)-invariant smooth minimal hypersurfaces asymptotic to \( C \) near infinity from different side, each is generated by homothety. Recall that by \[Sim74,Law91\], \( C^{5,1} \) is minimizing only in one side, hence by theorem \[4.3\], one of the two families above is unstable. Does it have index 1?

Can we find such index \( \geq 1 \) example asymptotic to an arbitrary one-sided minimizing cone? (In particular, for the isoparametric cone in \( \mathbb{R}^{18} \) mentioned in (1.1)) or more generally, a stable cone? Is it unique (up to scaling)? Note that such higher index minimal hypersurface provides an example of a family of higher index minimal hypersurface multiplicity one converges to a strictly singular minimal hypersurface, which never happen in smooth case.

(P1.3) In theorem \[4.3\] we bound the asymptotic rate of any \( \Sigma \in \mathcal{M}_1(C) \) towards infinity provide \( C \) is regular. In general, if \( \Sigma \in \mathcal{M}_1(C) \), can we bound its asymptotic rate at infinity toward \( C \) in terms of its index? A precise and more ambitious conjecture is that when \( C \) is regular, the asymptotic rate is bounded from below by \( \gamma \) (\( C \)). This is related with the finiteness of associated Jacobi field when we allow index drop. See (P2.2) below.

(P1.4) If \( C \) is regular and strictly minimizing, with cross section \( S = C \cap S^n \), then by \[Cha97\] and \[Whi89\], there’s an \( I := ind(L_S) \) dimensional family (but probably not continuously parametrized) of minimal hypersurfaces in \( \mathcal{M}^n(C) \), where \( L_S \) be the Jacobi operator of \( S \subset S^n \). More precisely, the method in \[Cha97\] actually shows that for strictly minimizing cones, there’s an \( \mathbb{R}^I \) parametrized space of "ends", to each of which one can associate a closed subset of \( \mathbb{R}^{n+1} \), being either a minimizing hypersurface or a domain bounded by 2 disjoint minimizing hypersurfaces; Moreover, every \( \Sigma \in \mathcal{M}^n(C) \) with polynomial decaying rate near infinity towards \( C \) lies in exactly one of these closed subset. Can we show that each of these closed sets is simply one minimizing hypersurfaces? A less ambitious conjecture may be that if one of such close subset has nonempty interior, show that it can be foliated by minimizing hypersurfaces.

Examples of area-minimizing hypercone has been constructed in \[Sim68,HLJ71,FKM81,FK85\], which are all isoparametric minimal cones; By \[HS85,WS94,TZ20\], every area minimizing hypercone in the family above is strictly minimizing and strictly stable. Can we determine these family of closed set in the examples above? \[SS86,ES19,Maz14\] determine \( \mathcal{M}_1(C^{p,q}) \) for Simons’ cones \( C^{p,q} \); \[So90a,So90b,So92\] study the index for the cross section of cubic and quartic isoparametric hypercones, which is the first step toward this problem.

(P1.5) When a minimizing hypercone \( C \) has singularities other than the origin, is it expected that in general an analogue of \[HS85\] is also valid (i.e. existence and uniqueness of smooth minimizing foliation with \( C \) to be a leaf)? Perhaps one should first look at cylindrical hypercones and try to prove the uniqueness. \[Loh18\] proposed an argument towards this.

• First Order Model.

(P2.1) Does every closed locally stable minimal hypersurface \( \Sigma \) satisfies the \( L^2 \)-nonconcentration property? This should be the starting point of linear theory for Jacobi operators. \[Loh19,Loh20\] studied the Martin boundary for a \( S \)-adapted Schrödinger operator, can we do the same thing for the Jacobi operator to a stable minimal hypersurface? In particular, can we get similar asymptotic behavior of Green’s functions like in section 3.3 when \( Sing(\Sigma) \) is not necessarily isolated?

(P2.2) Consider a family of pairwise different locally stable minimal hypersurfaces \( \Sigma_j \) converges in varifold sense to a multiplicity one \( \Sigma \) in \( (M,g) \). Theorem \[4.2\] tells us that when \( ind(\Sigma_j) = \)
\( \text{ind}(\Sigma) \), there exists a non-trivial Jacobi field \( u \in C^\infty(\Sigma) \) associated to \( \{ \Sigma_j \} \). Is this true when \( \Sigma \) has higher dimensional singularities? Is this true when \( \Sigma_j \) are of higher index than \( \Sigma \)? (This may be the first step to construct infinitely many closed embedded smooth minimal hypersurface in a generic eight manifold; And this is closely related with (P1.3) above.) When \( \text{Sing}(\Sigma) \) is of higher dimension, this seems non-trivial even for edge-typed singularities.

- **Neighborhood Foliation.**

(P3.1) Let \( (\Sigma, \nu) \subset (M, g) \) be a closed two-sided locally stable minimal hypersurfaces. Can we find a (piecewise smooth) foliation of some neighborhood \( U \) of \( \text{Clos}(\Sigma) \) by hypersurfaces with mean curvature of fixed sign (may be different between different leaves)?

Theorem [2] confirms this under strong assumptions on singularities. It is expected that when some singularity of \( \Sigma \) is not minimizing in one side, then there exists a family of mean concave collar neighborhood with boundary foliates some collar neighborhood of \( \Sigma \).

In [BBN10, Son18], when \( \Sigma \) is smooth and uniquely locally minimizing in one side, a smooth mean convex foliation for some collar neighborhood of \( \Sigma \) is constructed and used to show the min-max solution in the manifold with cylindrical end must have portions inside the core manifold. Does similar smooth mean convex foliation exist when \( \text{Sing}(\Sigma) \neq \emptyset \)?

(P3.2) A natural approach to (3.1) is by mean curvature flow. Can we construct mean convex ancient flow from or immortal flow towards \( \Sigma \) when \( \text{Sing}(\Sigma) \neq \emptyset \)? Are they smooth when sufficiently close to \( \Sigma \)? See also [Bro86, Problem 5.7] and [Din20].

- **Deformations and Local Moduli.**

(P4.1) In theorem [2] we see the existence of nearby minimal hypersurfaces under perturbed metrics if (1)-(3) is satisfied by \( \Sigma \). Assumption (3) is conjectured to be dropped.

Are these nearby minimal hypersurfaces unique in a fixed metric? A local version of this problem seems also unclear, even for Simons cones: Given a regular area minimizing hypercone \( C \subset \mathbb{R}^{n+1} \), for sufficiently small \( C^4 \) perturbation \( S' \) of \( \partial B^{n+1} \cap C \), is the area-minimizing hypersurface with boundary \( S' \) in \( \mathbb{R}^{n+1} \) unique? What about strictly minimizing or strictly stable cones? What about Simons cones?

(P4.2) A less ambitious problem is, if \( \{ g_t \}_{t \in [-1,1]} \) is a smooth family of Riemannian metric sufficiently close to \( g \), can we find a continuous family of locally stable minimal hypersurface \( \{ \Sigma_t \subset (M, g_t) \}_{t \in [-1,1]} \) close to \( \Sigma \)?

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