Coherent states of su(1,1): correlations, fluctuations, and the pseudoharmonic oscillator

John Schliemann

Institute for Theoretical Physics, University of Regensburg, D-93040 Regensburg, Germany

E-mail: john.schliemann@physik.uni-regensburg.de

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Abstract

We extend recent results on expectation values of coherent oscillator states and SU(2) coherent states to the case of the discrete representations of su(1,1). Systematic semiclassical expansions of products of arbitrary operators are derived. In particular, the leading order of the energy uncertainty of an arbitrary Hamiltonian is found to be given purely in terms of the time dependence of the classical variables. The coherent states considered here include the Perelomov–Gilmore (PG) coherent states. As an important application we discuss the pseudoharmonic oscillator and compare the PG states with the states introduced by Barut and Girardello. The latter ones turn out to be closer to the classical limit as their relative energy variance decays with the inverse square root of energy, while in the former case a constant is approached.

Keywords: coherent states, correlations, energy uncertainties, pseudoharmonic oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction

Coherent states are instrumental in semiclassical descriptions of generic quantum systems and have proven to be a versatile tool in a plethora of physical problems. The most prominent types of coherent states [1–6] are the coherent states of the harmonic oscillator, already investigated by Schrödinger [7, 8], and SU(2) coherent states living in the Hilbert space of a spin of general length $S$ [9, 10].

In both cases the corresponding coherent states fulfill a list of wellknown properties which are the basis for their prominent role in semiclassics: (i) the coherent states can be
generated by a group transformation from an appropriate reference state, (ii) they are (over-) complete, and (iii) are eigenstates of simple operators generic to the system. Moreover, (iv) they saturate uncertainty relations with respect to an obvious choice of variables, and (v) they show a coherent time evolution perfectly mimicking the classical limit under appropriate Hamiltonians.

Recently the present author has argued that one can add to the above list very general results on the coherent expectation values of products of arbitrary operators [11]. In particular, the leading-order contribution to the energy uncertainty was, for an arbitrary Hamiltonian, found to be purely given by the time dependence of the classical variables, a both very intuitive and very general result. Preliminary findings in this direction appeared also earlier and were obtained using concretely specified Hamiltonians [12, 13]. One of the purposes of the present note is to generalize the results of [11] to the case of the discrete representations of su(1,1). As we shall see below, such a generalization is possible for coherent states of the Perelomov–Gilmore (PG) type [14–16], but not for Barut–Girardello (BG) coherent states of su(1,1) [17].

A very natural physical system to be studied in connection with the algebra su(1,1) is the pseudoharmonic oscillator, see e.g. [18–25]. Most recently, Zipfel and Thiemann [26] have revisited this model under the aspect of complexifier coherent states [27], a concept originally inspired by loop quantum gravity [28], see also [29–32]. As shown in [26], demanding a complexifier coherent state to have a stable time evolution (in the sense of the above property (v)) is quite restrictive, and for one-dimensional systems (described by a single pair of canonical variables) the only two realizations are the usual harmonic oscillator and the pseudoharmonic oscillator. In the latter case the corresponding coherent states are of the BG type. For a deeper analysis regarding the most general Hamiltonian generating stable time evolutions of PG coherent states, we refer to [16]. In the present note we indeed compare PG and BG coherent states for the pseudoharmonic oscillator in terms of their energy expectation values and pertaining variances. As a result, the BG coherent states turn out to be closer to the classical limit as their relative energy variance decays with the inverse square root of energy, while for PG coherent states a constant is approached.

This paper is organized as follows: in section 2 we summarize important properties of SU(1,1) and its algebra. In particular, we derive explicit matrix representations of finite transformations as applied to su(1,1) operators. Using these findings we construct in section 3 a family of coherent states which includes the PG states. The BG coherent states are also introduced here. The results about expectation values of products of arbitrary operators within su(1,1) coherent states are derived and discussed in section 4. Section 5 is devoted to the pseudoharmonic oscillator. We close with a summary and an outlook in section 6.

2. SU(1,1): general properties

The Lie algebra su(1,1) is generated by three operators $K_i^j$, $i \in \{1, 2, 3\}$, fulfilling the commutation relations

$$[K_i^j, K_l^k] = i\epsilon^{ijk}\eta_{il}K^j_l,$$

where summation over repeated indices is understood. The metric $\eta_{ij} = \eta_{ji} = \text{diag}(1, 1, -1)$ will in the following raise and lower indices, and the global sign of the totally antisymmetric tensor $\epsilon^{ijk}$ is defined by $\epsilon^{123} = +1$. In terms of the usual complex combinations $K^\pm = K^1 \pm iK^2$ the above relations can also be formulated as
\[ [K^3, K^\pm] = \pm K^\pm, \quad [K^+, K^-] = -2K^3. \]  

(2) 

All generators commute with the Casimir invariant

\[ C = -K_iK^i = \frac{1}{2}(K^+K^- + K^-K^+) + K^3 K^3. \]  

(3) 

Elements of the pseudounitary group SU(1,1) are obtained by exponentiation

\[ U(\tau, n) = e^{in.K^i}. \]  

(4) 

with a real parameter \( \tau \) and and a real unit vector \( n^i \) which can either be ‘spacelike’, \( n_in^i = +1 \), or ‘timelike’, \( n_in^i = -1 \). Evaluating the expansion

\[ e^X e^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!} [X, Y]_m \]  

(5) 

with \([X, Y]_0 = Y \) and \([X, Y]_m = [X, [X, Y]_{m-1}] \), one finds

\[ \tilde{K}^i := e^{in.K^i}e^{-in.K^i} = (M(\tau, n))^i_j K^j, \]  

(6) 

where the matrix on the r.h.s. is given for spacelike unit vectors \( n \) as

\[ (M(\tau, n))^i_j = n'^i n_j + (-n'n_j + \delta^i_j) \cosh \tau - \epsilon^i_j n_k \sinh \tau \]  

(7) 

while for timelike \( n \) we have

\[ (M(\tau, n))^i_j = - n'n_j + (n'n_j + \delta^i_j) \cos \tau - \epsilon^i_j n_k \sin \tau. \]  

(8) 

In both cases these matrices are elements of the pseudoorthogonal group O(2,1)

\[ M^i_j \eta^{jk} (M^T)^k_l = \eta^{ij} \]  

(9) 

and the inverses are obtained by inverting either the sign of \( \tau \) or \( n \)

\[ (M^{-1}(\tau, n))^i_j = (M(-\tau, n))^j_i = (M(\tau, -n))^j_i. \]  

(10) 

By construction the transformation (4) leave the commutation relation (1) invariant,

\[ [\tilde{K}^i, \tilde{K}^j] = e^{in.K^i}[K^i, K^j]e^{-in.K^j} = i\epsilon^{ijk}\eta_{kl}\tilde{K}^l = i\epsilon^{ijk}M^k_l K^l. \]  

(11) 

In what follows we will focus on unitary representations of SU(1,1), i.e. those where all generators \( K^i \) are hermitian such that the group elements (4) are unitary. Specifically we focus on the discrete series where one can concentrate here on the ascending series as the descending one can be treated in a very similar fashion [2, 17]. These representations are labeled by a real parameter \( k > 0 \), and the Hilbert space is of countably infinite dimension and spanned by the orthonormalized states \(|m \rangle \) fulfilling

\[ C|k, m\rangle = k(k - 1)|k, m\rangle, \]  

(12) 

\[ K^3 |k, m\rangle = (k + m)|k, m\rangle, \]  

(13) 

\[ K^+ |k, m\rangle = \sqrt{(m + 1)(2k + m)}|k, m + 1\rangle, \]  

(14) 

\[ K^- |k, m\rangle = \sqrt{m(2k - 1 + m)}|k, m - 1\rangle. \]  

(15) 

In particular, \( K^- |k, 0\rangle = 0 \) and all higher states \(|k, m\rangle, m > 0 \) are obtained by applying the raising operator \( K^+ \).
3. Coherent states

Starting from the lowest-weight state $|k, 0\rangle$ one constructs via the transformation (4) the family of states

$$|\tau, n\rangle = U(\tau, n)|k, 0\rangle$$

which fulfill according to equations (6) and (13)

$$s_i K^i |\tau, n\rangle = k|\tau, n\rangle$$

with the timelike unit vector

$$s_i(\tau, n) = (M(\tau, n))^3, \quad s_is^i = -1.$$  

Equation (17) strongly resembles a defining property of SU(2) coherent states [9, 11, 12] and can therefore be viewed as coherent states of (the discrete series of representations of) su(1,1).

From equations (14), (15) and (17) one easily verifies that the states (16) have the expectation values

$$\langle \tau, n|K^i|\tau, n\rangle = -ks^i$$

ensuring $\langle \tau, n|s^i K^i|\tau, n\rangle = k$, and for products of generators one finds

$$\langle \tau, n|(e_i K^i)^2|\tau, n\rangle = (\langle \tau, n|e_i K^i|\tau, n\rangle)^2$$

for some arbitrary space- or timelike unit vector $e_i$. A simple way to prove equation (20) is to observe that it is fulfilled for the lowest-weight state $|0, n\rangle = |k, 0\rangle$

$$\langle k, 0|(e_i K^i)^2|k, 0\rangle = (\langle k, 0|e_i K^i|k, 0\rangle)^2$$

and by inserting the transformation (4) and its inverse in the above l.h.s. it follows with the help of equations (11) and (19)

$$\langle \tau, n|(e_i K^i)^2|\tau, n\rangle = \langle \tau, n|(e_i M^j K^j)^2|\tau, n\rangle$$

for some arbitrary space- or timelike unit vector $e_i$. As a consequence, the variances squared of such operators read

$$\langle \tau, n|e_i (M^{-1})^i |\tau, n\rangle$$
and two mutually orthogonal spacelike unit vectors \( u, v \) being perpendicular to \( s \), \( u_s v_s = 0 \) lead to the minimal uncertainty product

\[
\Delta (u_s) \Delta (v_s) = \frac{k}{2} \left( (\tau, n) | (\tau, n) \right) ^2
\]

A particular choice for these unit vectors are

\[
u = (M (\tau, n))^1, v = (M (\tau, n))^2.\]

The \( \mathfrak{su}(1,1) \) coherent states according to Perelomov and Gilmore (PG) \([2,3,14-16]\) can now be identified within the manifold of states (16) by choosing for \( n' \) any spacelike unit vector with \( n' = 0 \). A convenient parametrization is given by \( n' = (\sin \varphi, -\cos \varphi, 0) \) leading to the transformation operators \([2-4]\)

\[
U (\tau, \phi) = e^{i (\sin \varphi K^- - \cos \varphi K^+)} = e^{-\xi K^-} e^{\eta K^+} e^{\xi K^-}
\]

with

\[
z = \tanh \frac{\tau}{2}, \quad \eta = 2 \text{ln} \cosh \frac{\tau}{2}
\]

such that the PG coherent states read

\[
|\Phi(z)\rangle = (1 - |z|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2k + m)}{m!\Gamma(2k)}} (-\bar{z})^m |k, m\rangle.
\]

These states fulfill equation (17) with

\[
\kappa = (\sinh \tau \cos \varphi, \sinh \tau \sin \varphi, \cosh \tau)
\]

\[
= \left( \frac{2 \text{Re} \bar{z}}{1 - |z|^2}, \frac{2 \text{Im} \bar{z}}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right).
\]

A different type of \( \mathfrak{su}(1,1) \) coherent states has been introduced by Barut and Girardello (BG) \([17]\). These states are defined to be eigenstates of the lowering operator

\[
|K^-|\Psi(w)\rangle = w|\Psi(w)\rangle
\]

with some complex eigenvalue \( w \). In the standard basis used so far the BG coherent states can be formulated as

\[
|\Psi(w)\rangle = N (w, k) \sum_{m=0}^{\infty} \frac{w^m}{\sqrt{m! \Gamma(2k + m)}} |k, m\rangle,
\]

where the normalization factor

\[
N (w, k) = (|w|^{-2k+1}) I_{2k-1} (2|w|)^{-1/2}
\]

can be expressed in terms of modified Bessel functions

\[
I_\nu (x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\nu + 1 + m)} \left( \frac{x}{2} \right)^{\nu + 1 + m}.
\]

These states are clearly different from the PG coherent states since an inspection of the equation

\[
t_i K'|\Psi(w)\rangle = \kappa (w) |\Psi(w)\rangle
\]
shows that the only solutions are given by $\mu_{-1,0}$ reproducing equation (30). In particular there is no solution with a real and timelike $t_i$ as demanded by equation (17). Finally, as the BG coherent states are eigenstates of $K^- = K^1 + iK^2$ it is easy to see that the minimize the uncertainty product [26]

$$\Delta_{BG}(K^1)\Delta_{BG}(K^2) = \frac{1}{2} \langle \Psi(w)|K^2|\Psi(w)\rangle.$$  

(35)

4. Correlations and fluctuations

Let us now consider two operators $A$, $B$ being functions of the generators $K_i$. Using the completeness of the basis states $|k,m\rangle$, the expectation value of the operator product $AB$ within the states $|\tau,n\rangle$ can be formulated as

$$\langle \tau, n |AB| \tau, n \rangle = \sum_{m=0}^{\infty} \langle k, 0 |U^+AU| k, m \rangle \langle k, m |U^+BU| k, 0 \rangle$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(2k)}{m!\Gamma(2k + m)} [\langle k, 0 |[iK^+, U^+AU]| k, 0 \rangle$$

$$\langle k, 0 |[iK^-, U^+BU]| k, 0 \rangle]$$

$$\langle \tau, n |[i\tilde{K}^+, A]| \tau, n \rangle$$

$$\langle \tau, n |[i\tilde{K}^-, B]| \tau, n \rangle$$

(36)

where $\tilde{K}^\pm$ are given by equation (6). The above last equation extends results of [11], obtained there for harmonic oscillator coherent states and SU(2) coherent states, to the case of the discrete representations of su(1,1). All the iterated commutators in equation (36) are of the same order in $k$ whereas the prefactor of the $m$th term carries a product $2k(2k + 1)\cdots(2k - 1 + m)$ in the denominator. Thus, equation (36) is essentially an expansion in $1/k$, and in situations where the classical limit is approached via $k \rightarrow \infty$, it is therefore a systematic semiclassical expansion of the coherent-state expectation value of a product of two arbitrary operators. These operators are so far neither required to be hermitian nor commuting, and a different ordering would exchange the operators $\tilde{K}^\pm$ in equation (36) which in general describes a complex number.

An example for such a form of the classical limit is given by the pseudoharmonic oscillator to be discussed in section 5. The zeroth order in equation (36) is obviously just the classical result. Note also that the su(1,1) generators $\tilde{K}^x$, $\tilde{K}^y$ represent the direction perpendicular to the polarization $\sigma'$ of the coherent state $|\tau, n\rangle$. Moreover, for the variance of an hermitian operator $A$ one finds

$$(\Delta A)^2 = \sum_{m=0}^{\infty} \frac{\Gamma(2k)}{m!\Gamma(2k + m)} |\langle \tau, n |[i\tilde{K}^-, A]| \tau, n \rangle|^2.$$  

(37)

The expectation values occurring in leading order can be rewritten as

$$|\langle \tau, n |[i\tilde{K}^-, A]| \tau, n \rangle|^2 = \eta_{ij} \langle \tau, n |[i\tilde{K}^i, A]| \tau, n \rangle \langle \tau, n |[i\tilde{K}^j, A]| \tau, n \rangle$$

$$= \eta_{ij} \langle \tau, n |[i\tilde{K}^i, A]| \tau, n \rangle \langle \tau, n |[i\tilde{K}^j, A]| \tau, n \rangle,$$  

(38)

where we have observed that $|\tau, n\rangle$ is an eigenstate of $\tilde{K}^k$, and that $\tilde{K}^1$ and $K^i$ are related by a pseudoorthogonal matrix preserving the metric $\eta_{ij}$. Thus, we have
and choosing \( A = \mathcal{H} \) to be the Hamiltonian of some system, we can formulate the leading-order contribution to the energy variance as

\[
(\Delta \mathcal{H})^2 = \frac{1}{2k} \eta_{ij} \langle \tau, n | [i \mathcal{K}^i, A] | \tau, n \rangle \langle \tau, n | [i \mathcal{K}^i, A] | \tau, n \rangle + O\left( \frac{1}{k^2} \right) \tag{39}
\]

where the commutators have been replaced, according to the Heisenberg equations of motion, with time derivatives \( (\hbar = 1) \). Thus, if the system is prepared at some initial time in a coherent state \( |\tau, n\rangle \) equation (19) implies

\[
(\Delta \mathcal{H})^2 = k^2 \frac{1}{2k} \eta_{ij} \langle \tau, n | \partial_t s^i | \tau, n \rangle \langle \tau, n | \partial_t s^i | \tau, n \rangle + O\left( \frac{1}{k^2} \right) \tag{40}
\]

i.e. the leading-order contribution to the energy variance is just due to the time-dependence of the (semi-)classical coherent parameters. The results (36), (37) and (39)–(41) are in full analogy to the findings of [11] for the coherent states of the harmonic oscillator and SU(2). Moreover, for a Hamiltonian being linear in the \( \text{su}(1,1) \) generators, the energy uncertainty is, according to equation (23), just given by the leading order in equation (41), without any further correction. This observation is also in full analogy with the findings of [11], and the pseudoharmonic oscillator to be discussed in section 5 provides an example for such a situation.

On the other hand, the above derivation leading to equations (39)–(41) cannot be repeated for BG coherent states because these objects fail to be generated via \( \text{SU}(1,1) \) transformations from the lowest-weight state \(|k, 0\rangle\), as seen in equation (34). Indeed, a unitary transformation \( V(w) \)

\[
|\Psi(w)\rangle = V(w)|k, 0\rangle \tag{42}
\]

is necessarily not an element of the pertaining representation of \( \text{SU}(1,1) \), i.e. \( V(w) \) is not the form (4). As a consequence, there is still an analog of equation (36)

\[
\langle \Psi(w) | AB | \Psi(w) \rangle = \sum_{m=0}^{\infty} \frac{\Gamma(2k)}{m! \Gamma(2k + m)} \left( \langle \Psi(w) | [i \tilde{K}^+, A]_m | \Psi(w) \rangle \langle \Psi(w) | [i \tilde{K}^-, B]_m | \Psi(w) \rangle \right) \tag{43}
\]

and, in turn, of equation (37) with \( \tilde{K}^i = V^i V^+ \), but a relation of the form (38) does not hold.

5. The pseudoharmonic oscillator

The pseudoharmonic oscillator

\[
H = \frac{p^2}{2\mu} + \frac{\mu \omega^2}{2} q^2 + \frac{\lambda}{q^2} \tag{44}
\]

describes a particle of mass \( \mu \) with coordinate \( q \) in a potential whose harmonic part is characterized by a frequency \( \omega \) whereas the parameter \( \lambda \) mimics an angular momentum; for the latter fact this system is also referred to as the radial oscillator [18, 26]. Due to the divergence of the potential at \( q = 0 \) (giving also rise to the term ‘singular oscillator’ [2, 19, 22]) the dynamics can be restricted to \( q \geq 0 \). Moreover, the Hamiltonian (44) describes the relative coordinate of the Calogero–Sutherland model in the sector of to just two particles [20, 21].
Using the ladder operators of the usual harmonic oscillator

\[ a = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\mu \omega}{\hbar}} q + \frac{ip}{\sqrt{i\hbar \omega}} \right), \]  

(45)

one constructs a representation of \( \text{su}(1,1) \) as \([2, 20–22, 26]\)

\[ K^- = \frac{1}{2} a a - \frac{\lambda}{2 \hbar \omega q^2}, \quad K^+ = \frac{1}{2} a^+ a^+ - \frac{\lambda}{2 \hbar \omega q^2}, \quad K^3 = \frac{H}{2 \hbar \omega} \]  

(46)

with the Casimir operator

\[ C = -\frac{3}{16} + \frac{\mu \lambda}{2 \hbar^2} = -\frac{1}{4} + \frac{\alpha^2}{16}, \]  

(47)

where \( \alpha^2 = 8 \mu \lambda / h^2 + 1 \). The eigenstates of \( H = 2 \hbar \omega K^3 \) can be worked out in real-space representation by standard methods giving (assuming \( \alpha > 0 \)) \([2, 18, 26]\)

\[ \langle q | k, m \rangle = \sqrt{\frac{2 m!}{\hbar \Gamma \left( \frac{\alpha}{2} + 1 + m \right)}} \exp \left( \frac{\mu \omega}{\hbar} q \right) L_m^{\alpha/2} \left( \frac{\mu \omega}{\hbar} q^2 \right), \]  

(48)

where

\[ L_m^{\alpha}(x) = \binom{m + \alpha}{m} F(-m, \alpha + 1, x) \]  

(49)

is a generalized Laguerre polynomial expressed here in terms of Kummer’s function \([33]\).

These states fulfill the stationary Schrödinger equation

\[ H | k, m \rangle = 2 \hbar \omega \left( m + \frac{\alpha}{4} + \frac{1}{2} \right) | k, m \rangle \]  

(50)

showing that the above \( \text{su}(1,1) \) representation carries

\[ k = \frac{\alpha}{4} + \frac{1}{2} = \frac{1}{2} + \sqrt{\frac{\mu \lambda}{2 \hbar^2} + \frac{1}{16}}, \]  

(51)

consistent with equation (47). Note that the classical limit \( \hbar \to 0 \) implies \( k \to \infty \) with

\[ \lim_{\hbar \to 0} \hbar k = \sqrt{\mu \lambda / 2}, \]  

(52)

very similar to the classical limit of \( \text{SU}(2) \) spin systems \([11, 12]\).

The classical dynamics of the variable \( q \) is clearly restricted to either the positive or the negative axis due to the diverging potential barrier at \( q = 0 \). Accordingly, the wave functions (48) yield for \( \lambda = 0 \) only the odd eigenstates of the usual harmonic oscillator which vanish at \( q = 0 \) and have energy \( \hbar \omega (2 m + 1 + 1/2) \) \([19, 26]\). The even states are contained in wave functions obtained by changing \( \alpha \mapsto -\alpha \) in equations (48)–(51) (but still assuming \( \alpha > 0 \)) leading to \([19, 23]\)

\[ \langle q | k', m \rangle = \sqrt{\frac{2 m!}{\hbar \Gamma \left( \frac{\alpha}{2} + 1 + m \right)}} \exp \left( \frac{\mu \omega}{\hbar} q \right) L_m^{\alpha/2} \left( \frac{\mu \omega}{\hbar} q^2 \right). \]  

(53)
with

$$k' = -\frac{\alpha}{4} + \frac{1}{2} = \frac{1}{2} - \sqrt{\frac{\mu\lambda}{2\hbar^2} + \frac{1}{16}}. \quad (54)$$

These wave functions diverge at $q = 0$ for $\lambda > 0$ (i.e. $\alpha > 1$) but are still normalizable if $\alpha < 2 \Leftrightarrow k > 0$. Due to the latter restriction these states do not allow for a classical limit $\hbar \to 0$. According to equations (51) and (54) the states (48) and (53) form inequivalent representations of $\text{su}(1,1)$. If not stated otherwise we will in what follows focus on the regular eigenstates (48).

On the other hand, integrating the classical energy conservation law

$$E_{cl} = \frac{\mu}{2} q_{cl}^2 + \frac{\mu\omega^2}{2} q_{cl}^2 + \frac{\lambda}{q_{cl}^2} \quad (55)$$

one finds the general classical solution [20, 26]

$$q_{cl}(t) = \sqrt{\frac{E_{cl}}{\mu\omega^2} + \eta(E_{cl}) \cos(2\omega t + \varphi)} \quad (56)$$

with

$$\eta(E_{cl}) = \sqrt{\frac{E_{cl}^2}{\mu\omega^2} - \frac{2\lambda}{\mu\omega^2}} \quad (57)$$

and $\varphi$ being determined by the initial condition. We note that the classical energy is bounded from below by its minimum $E_{cl,\text{min}} = \sqrt{2\mu\lambda\omega^2}$.

Under the quantum Hamiltonian (44) the PG coherent states constructed from the regular eigenstates (48) evolve as

$$e^{-\frac{i}{\hbar}H\tau}|\Phi(z)\rangle = e^{-i2\omega\tau}|\Phi(ze^{i2\omega\tau})\rangle = e^{-i2\omega\tau}|\Phi(z(t))\rangle \quad (58)$$

and remain therefore on the manifold of PG coherent states, i.e. they are stable in the sense of [16, 26]. This property is completely analogous to the time evolution of the coherent states of the usual harmonic oscillator and SU(2) coherent states under appropriate Hamiltonians [1–6, 11–13]. To make further contact with the classical dynamics we investigate the expectation values of the ‘transversal’ $\text{su}(1,1)$ components

$$K^1 = \frac{\mu\omega}{2\hbar} q^2 - \frac{H}{2\hbar\omega}, \quad K^2 = -\frac{1}{4\hbar}(qp + pg) \quad (59)$$

According to equations (8) and (17) the PG coherent states fulfill

$$s_i(t)K^i|\Phi(z(t))\rangle = k|\Phi(z(t))\rangle \quad (60)$$

with

$$s_i(t) = (\sinh \tau \cos(2\omega t + \varphi), \sinh \tau \sin(2\omega t + \varphi), \cosh \tau) \quad (61)$$

$$= \left(\frac{2\text{Re } z(t)}{1 - |z|^2}, \frac{2\text{Im } z(t)}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right) \quad (62)$$

such that

$$\langle \Phi(z(t))|K^i|\Phi(z(t))\rangle = \frac{-2\lambda\text{Re } z(t)}{1 - |z|^2} \quad (63).$$
\[ \langle \Phi(z(t)) | K^2 | \Phi(z(t)) \rangle = \frac{-2k \text{Im} z(t)}{1 - |z|^2}. \]  

(64)

For these time-dependent expectation values to be identical to the corresponding classical quantities we must have
\[ \langle \Phi(z(t)) | K^2 | \Phi(z(t)) \rangle = \frac{-m}{2\hbar} \dot{q}_cl(t) q_\omega(t) = \frac{m \omega}{2\hbar} \eta(E_{cl}) \sin(2\omega t + \varphi) \]  

(65)

leading to
\[ \frac{-2k z(t)}{1 - |z|^2} = \frac{m \omega}{2\hbar} \eta(E_{cl}) \text{e}^{i(2\omega t + \varphi)} \]  

(66)

and
\[ \langle \Phi(z(t)) | K^2 | \Phi(z(t)) \rangle = \frac{m \omega}{2\hbar} \eta(E_{cl}) \cos(2\omega t + \varphi) = \frac{m \omega}{2\hbar} q_\omega^2(t) - \frac{E_{cl}}{2\hbar \omega}. \]  

(67)

An analogous observation can be made for BG coherent state where the time evolution is also stable
\[ \text{e}^{-\frac{i\hbar}{2}\eta(E_{cl})} | \Psi(w(t)) \rangle = \text{e}^{-\frac{i\omega}{2} |w|^2} | \Psi(w(t)) \rangle = \text{e}^{-\frac{i\omega}{2} |w|^2} | \Psi(w(t)) \rangle, \]  

(68)

leading to
\[ \langle \Psi(w(t)) | K^1 | \Psi(w(t)) \rangle = \text{Re} w(t), \]  

(69)

\[ \langle \Psi(w(t)) | K^2 | \Psi(w(t)) \rangle = -\text{Im} w(t). \]  

(70)

Putting now
\[ w(t) = \frac{m \omega}{2\hbar} \eta(E_{cl}) \text{e}^{-\frac{i\omega}{2} |w|^2} \]  

(71)

we have as before
\[ \langle \Psi(w(t)) | K^1 | \Psi(w(t)) \rangle = \frac{m \omega}{2\hbar} q_\omega^2(t) - \frac{E_{cl}}{2\hbar \omega}, \]  

(72)

\[ \langle \Psi(w(t)) | K^2 | \Psi(w(t)) \rangle = -\frac{m}{2\hbar} \dot{q}_cl(t) q_\omega(t). \]  

(73)

Both the PG and the BG coherent states of $\text{su}(1,1)$ perfectly mimic the classical dynamics of the pseudoharmonic oscillator. Specifically the moduli of the complex parameters are to be chosen as
\[ \frac{2k |z|}{1 - |z|^2} = |w| = \frac{m \omega}{2\hbar} \eta(E_{cl}) \]  

(74)

such that
\[ \frac{-2k z}{1 - |z|^2} = \bar{w} \iff z = \frac{1}{w} (k - \sqrt{k^2 + |w|^2}). \]  

(75)

The above observations are of course in close analogy to well-known properties of the coherent states of the usual harmonic oscillator and of SU(2) coherent states [1–6, 11–13]. The relationship between PG and BG coherent state to the classical dynamics was already investigated in [20] concentrating on the time-dependence of the modulus of the coherent-state wave functions. In particular, the imaginary parts of the coherent parameters $z$ and $\omega$ corresponding to the expectation values of $K^2$ were not considered.
On the other hand, we note that a stable time evolution mimicking the classical limit is as such not a particularly distinctive property [3, 5]. As an example consider states of the form

$$|\chi(z)\rangle = M(z) \sum_{m=0}^{\infty} c_m z^m |k, m\rangle,$$

(76)

where the complex numbers $c_m$ are chosen such that the series

$$(M(z))^{-1/2} = \sum_{m=0}^{\infty} |c_m|^2 |z|^{2m}$$

(77)

has a finite radius of convergence, but are otherwise arbitrary. Such states are obviously stable under the Hamiltonian time evolution. Moreover, let us further assume the expectation value

$$\langle \chi(z) | \mathcal{K}^{-1} | \chi(z) \rangle = \frac{\tilde{z}}{|z|} f(|z|)$$

(78)

with

$$f(|z|) = \sum_{m=1}^{\infty} \tilde{c}_m c_{m-1} \sqrt{m(2k - 1 + m)} |z|^{2m-1}$$

(79)

to be also finite. Choosing then $|z|$ according to

$$f(|z|) = \frac{m\omega}{2\hbar} \eta(E_{cl})$$

(80)

leads to expectation values of $\mathcal{K}^1, \mathcal{K}^2$ which have the identical classical time evolution as in equations (65), (67) and (72), (73). However, for general coefficients $c_m$ the resulting state can certainly not be expected to have other properties desired for semiclassical approximations such as minimum uncertainty products as realized by PG and BG coherent states. Another important feature are of course the expectation values of $\mathcal{K}^3$ which we now investigate.

For the expectation value $H = 2\hbar \omega \mathcal{K}^3$ we have within a PG coherent state from equation (19)

$$\langle \Phi(z)|H|\Phi(z)\rangle = 2\hbar \omega k \cosh \tau = \sqrt{E_{cl}^2 + (2\hbar \omega k)^2} - 2\mu \lambda \omega^2$$

(81)

which approaches $E_{cl}$ in the semiclassical regime of large energies $E_{cl} \gg \hbar \omega$, $E_{cl} \gg E_{cl}^{min} = \sqrt{2\mu \lambda \omega^2}$. The energy variance squared can be calculated via equation (23) as

$$\langle \Delta_{PG}H \rangle^2 = \frac{k}{2} \sinh^2 \tau = \frac{1}{2k} (E_{cl}^2 - 2\mu \lambda \omega^2)$$

(82)

such that the relative variance approaches a constant at large energies

$$\frac{\Delta_{PG}H}{\langle \Phi(z)|H|\Phi(z)\rangle} = \frac{1}{\sqrt{2k}} + \mathcal{O} \left( \frac{\hbar \omega}{E_{cl}^{1/2}}, \frac{\sqrt{\mu \lambda \omega^2}}{E_{cl}} \right),$$

(83)

which is certainly not the expected behavior for a state incorporating the semiclassics.

For the BG coherent states we can use the modified Bessel functions (33) to obtain

$$\langle \Psi(w)|H|\Psi(w)\rangle = 2\hbar \omega \left( k + \frac{|w|J_{2k}(2|w|)}{I_{2k-1}(2|w|)} \right)$$

(84)
Note that the above expression, differently from equation (82), contains higher orders in $k_1$.

Now employing the asymptotic expansion

$$I_n(x) = e^x \sqrt{\frac{2}{\pi x}} \left( 1 + \frac{4x^2 - 1}{8x} + O\left(\frac{1}{x}\right)^2 \right)$$

one finds for $E_{cl} \gg \hbar \omega, E_{cl} \gg E_{cl}^{\text{min}}$

$$\langle \Psi(w)|H|\Psi(w)\rangle = E_{cl} + \frac{\hbar \omega}{2} + O\left(\frac{\hbar \omega}{E_{cl}}, \frac{\sqrt{\mu \lambda \omega^2}}{E_{cl}}\right)$$

and

$$\frac{\Delta_{BG}H}{\langle \Psi(w)|H|\Psi(w)\rangle} = \sqrt{\frac{\hbar \omega}{E_{cl}}} + O\left(\frac{\hbar \omega}{E_{cl}}, \frac{\sqrt{\mu \lambda \omega^2}}{E_{cl}}\right).$$

Thus, the energy expectation value (87) contains a ‘zero-point energy’ $\hbar \omega/2$ very familiar from the standard harmonic oscillator, while the relative energy variance (88) vanishes in the semiclassical regime with the inverse square root of energy. The latter property is in contrast to the behavior (83) of the PG coherent state and an expected feature in the semiclassical
limit. To illustrate the above findings we have plotted the expressions (84) and (85) in figure 1 as a function of $E_{cl}$ for different values of $\hbar$.

The above analysis focused on coherent states constructed from the regular eigenstates (48) of the pseudoharmonic oscillator. Similarly one could employ the divergent states (53) which, however, do not possess a classical limit. More interestingly, as a closer inspection easily shows, coherent states constructed from either type of eigenstates, or linear combinations of them with fixed coherent parameters, do not reproduce for $\lambda = 0$ the well-known coherent states of the usual harmonic oscillator [1–6]. The latter statement holds both for PG and BG coherent states.

6. Summary and outlook

We have extended recent results [11] on expectation values of operator products within coherent oscillator states and SU(2) coherent states to the case of the discrete representations of $su(1,1)$. The results provide a systematic expansion of correlations and fluctuations around the classical limit. In particular, the leading order of the energy uncertainty of an arbitrary Hamiltonian is found, in full analogy to [11], to be given purely in terms of the time dependence of the classical variables. The latter finding holds for a family of coherent states including the PG states, but their derivation cannot be extended states to the BG type. Our results regarding PG coherent states are based on explicit matrix representations of SU(1,1) transformations derived in section 2.

As a typical application we have discussed the pseudoharmonic oscillator and established that the time evolution of the both the PG and BG coherent states perfectly mimic, for appropriate choices of the coherent parameters, the classical dynamics. However, departures between these types of coherent states are revealed when comparing expectation values: while the energy expectation values are close to each other, the variances show a qualitative difference: for BG states the relative variance vanishes with the inverse square root of energy whereas in the PG case a constant is approached. Thus, in contrast to the PG states, the BG coherent states show a behavior perfectly expected in the semiclassical regime. Moreover, the energy expectation values of BG coherent states contain a zero-point energy strongly reminiscent of the standard harmonic oscillator.

Possible direction of further work include the extension of the results obtained in section 4 and [11] to other (compact or noncompact) groups [2, 3], and the study of generalizations of the pseudoharmonic oscillator, especially to Hamiltonians with explicit time dependence [19, 22].

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