On a new method for controlling exponential processes

O. Kounchev and H. Render

February 28, 2007

Abstract

The first-named author has been supported by the Greek-Bulgarian bilateral project B-Gr17, 2005-2008. The second-named author is partially supported by Grant BFM2003-06335-C03-03 of the D.G.I. of Spain. Both authors acknowledge support within the project “Institutes Partnership” with the Alexander von Humboldt Foundation, Bonn.

O. Kounchev: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria. kounchev@gmx.de H. Render: Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, España. render@gmx.de

1 Introduction

With the great increase in the number of businesses making a presence on the Internet and the increase in the number of cyber-customers, the chances of computer security attacks increase daily. How much should an enterprise or organization budget to defend against Internet attacks?

As it is well known the processes modelled by ordinary differential equations and the stochastic differential equations describe many important practical events, and thus provide very efficient models for processes coming from physics, finance, etc. As is well known their solutions may be approximated efficiently by means of finite linear combinations of exponential functions,
called sometimes exponential polynomials. Let us mention some recent applications to exponential smoothing used for time series models which fit best for forecasting Internet Security attacks, [16].

Going further in this direction, if one considers a process which experiences jumps or unsmoothness, then it may be approximated by means of piecewise exponential polynomials.

For the control and design of polynomials the famous Bernstein polynomials provide a very efficient way to control their behaviour. Their further development as the Bezier curves provides a very efficient method to design a prescribed form, which is an indispensable tool for Computer Design. In a similar way the theory of $B-$splines provides us with an indispensable tool for the control and design of the classical splines.

Let us add to the above that for the purposes of fast recognition, representation and compression of curves and surfaces such tools as the Bézier curves and $B-$splines curves are very important. Since many of the real processes produce observational and surveillance data which carry exponential character, it is clear that it is very important to have an efficient tools for their representation, design and control.

It is curious to mention that unlike the classical polynomial case there has not been invented up to very recently a tool similar to the Bernstein-Bézier representation which would allow us to control the behavior of the exponential polynomials. The exponential analog to the classical Bernstein polynomials has been introduced in the recent authors’ paper [2], and this analog retains all basic properties of the classical Bernstein polynomials.

The main purpose of the present paper is to contribute in this direction, by proving some important properties of the Bernstein exponential operator which has been introduced in [2]. We also fix our attention upon some special type of exponential polynomials which are particularly important for the further development of theory of representation of Multivariate data.

Let us recall at first shortly the concept of Bézier curves and its relationship to the Bernstein polynomials: Let $b_0, \ldots, b_n$ be vectors either in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $t \in \mathbb{R}$, and interpret $b_k$ as a constant curve, i.e. that $b_k^0(t) := b_k$ for $k = 0, \ldots, n$. Then define new curves $b_k^1(t)$ for $k = 0, \ldots, n - 1$ by $b_k^1(t) = (1 - t)b_k^0(t) + tb_{k+1}^0(t)$. Repeating this process one arrives at new curves

$$b_k^r(t) = (1 - t)b_k^{r-1}(t) + tb_{k+1}^{r-1}(t) \text{ for } k = 0, \ldots, n - r$$
In the last step, i.e. for \( r = n \), one finally obtains exactly one curve

\[
b_n(t) = (1 - t) b_0^{n-1}(t) + t b_1^{n-1}(t),
\]

the so-called Bézier curve. The polygon formed by \( b_0, \ldots, b_n \) is called the Bézier polygon or control polygon. The Bézier curve has the property that the curve \( b_n(t) \) is in the convex hull generated by the points \( b_0, \ldots, b_n \). Moreover the two points \( b_0 \) and \( b_n \) are fixed, i.e. \( b_0(0) = b_0 \) and \( b_n(1) = b_n \). An explicit form for the Bézier curve is

\[
b_n(t) = \sum_{k=0}^{n} b_k p_{n,k}(t)
\]

where \( p_{n,k}(t) := \binom{n}{k} t^k (1 - t)^{n-k} \) are called the Bernstein basis polynomials. In the sequel we shall focus on generalizations of Bernstein basis polynomials which have arisen recently in Computer Aided Geometric Design for modeling parametric curves. Instead of the basic polynomials \( 1, x, \ldots, x^n \) one consider different systems of basic functions \( f_0, \ldots, f_n \), e.g.

\[
1, x, \ldots, x^{n-2}, \cos x, \sin x,
\]

which are better adapted to curves in spherical coordinates, see e.g. [6], [24], [33] and [7]. In mathematical terms it will be required that the linear span of the basis functions \( f_0, \ldots, f_n \) forms a an extended Chebyshev system. Recall that a subspace \( U_n \) of \( C^n(I) \) (the space of \( n \)-times continuously differentiable complex-valued functions on a interval \( I \)) is called an extended Chebyshev system for a subset \( A \subset I \), if \( U_n \) has dimension \( n + 1 \) and each non-zero function \( f \in U_n \) vanishes at most \( n \) times on the subset \( A \) (counted with multiplicities). A system \( p_{n,k} \in U_n, k = 0, \ldots, n \), is a Bernstein-like basis for \( U_n \) relative to \( a, b \in I \), if for each \( k = 0, \ldots, n \) the function \( p_{n,k} \) has a zero of order \( k \) at \( a \), and a zero of order \( n - k \) at \( b \).

In the following we shall consider Bernstein basis polynomials for the space of exponential polynomials \( E(\lambda_0, \ldots, \lambda_n) \) (induced by a linear differential operator \( L \)) defined by

\[
E(\lambda_0, \ldots, \lambda_n) := \{ f \in C^\infty(\mathbb{R}) : L f = 0 \},
\]

where \( \lambda_0, \ldots, \lambda_n \) are complex numbers, and \( L \) is the linear differential operator with constant coefficients defined by

\[
L := L(\Lambda) := \left( \frac{d}{dx} - \lambda_0 \right) \ldots \left( \frac{d}{dx} - \lambda_n \right).
\]
Exponential polynomials are sometimes called \( L \)-polynomials, and they provide natural generalization of classical, trigonometric, and hyperbolic polynomials (see [31]), and the so-called \( \mathcal{D} \)-polynomials considered in [28]. Exponential polynomials arise naturally in the context of a class of multivariate splines, the so-called polysplines, see [17], [20]). Important in this context are exponential splines associated with linear differential operators \( L_{2s+2} \) of order \( 2s + 2 \) of the form

\[
L_{2s+2} = \left( \frac{d}{dx} - \lambda \right)^{s+1} \left( \frac{d}{dx} + \mu \right)^{s+1}
\]

which are parametrized by real numbers \( \lambda \) and \( \mu \). According to the above notation \( \Lambda_n = (\lambda_0, ..., \lambda_n) \) we set \( n = 2s + 1 \) and define the vector

\[
\Lambda_{2s+1}(\lambda, \mu) := \begin{pmatrix} \lambda, \lambda, ..., \lambda, \mu, \mu, ..., \mu \end{pmatrix}_{s+1}^{s+1}
\]

containing \( s + 1 \) times \( \lambda \) and \( s + 1 \) times \( \mu \).

Let us return to the general theory of exponential polynomials, and let us recall the general fact (cf. [27]) that for \( \Lambda_n = (\lambda_0, ..., \lambda_n) \in \mathbb{C}^{n+1} \) there exists a unique function \( \Phi_{\Lambda_n} \in E(\lambda_0, ..., \lambda_n) \) such that \( \Phi_{\Lambda_n}(0) = ... = \Phi_{\Lambda_n}^{(n-1)}(0) = 0 \) and \( \Phi_{\Lambda_n}^{(n)}(0) = 1 \). We shall call \( \Phi_{\Lambda_n} \) the fundamental function in \( E(\lambda_0, ..., \lambda_n) \). An explicit formula for \( \Phi_{\Lambda_n} \) is

\[
\Phi_{\Lambda_n}(x) := [\lambda_0, ..., \lambda_n] e^{xz} = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{xz}}{(z - \lambda_0) ... (z - \lambda_n)} dz
\]

where \([\lambda_0, ..., \lambda_n] \) denotes the divided difference, and \( \Gamma_r \) is the path in the complex plane defined by \( \Gamma_r(t) = re^{it}, t \in [0, 2\pi] \), surrounding all the scalars \( \lambda_0, ..., \lambda_n \). The fundamental function \( \Phi_{\Lambda_n} \) is an important tool in the spline theory based on exponential polynomials (see [27]) and in the wavelet analysis of exponential polynomials, see [8], [23], [17], [18], [19].

In Section 2 we provide some basic results about the fundamental function, in particular we derive the Taylor expansion of the fundamental function \( \Phi_{(\lambda_0, ..., \lambda_n)} \). Section 3 is devoted to the analysis of the fundamental function with respect to the eigenvalues (3), and in order to have a short notation we set using (4)

\[
\Phi_{2s+1}(x) := \Phi_{\Lambda_{2s+1}(\lambda, \mu)} \text{ for } \lambda = 1 \text{ and } \mu = -1.
\]
We shall give explicit formula for $\Phi_{2s+1}(x)$. So far it turns out that the following recursion formula is more important:

$$
\Phi_{2s+3}(x) = x^2 \frac{1}{4s(s+1)} \Phi_{2s-1}(x) - \frac{2s+1}{2s+2} \Phi_{2s+1}(x).
$$

We shall prove the non-trivial fact that for each fixed $x > 0$ the sequence

$$
\frac{\Phi_{2s}(x)}{\Phi_{2s-1}(x)} \rightarrow 1 \quad (5)
$$

for $s \rightarrow \infty$; here $\Phi_{2s}(x)$ is the fundamental function with respect to the vector $\Lambda_{2s}$ consisting of $s+1$ eigenvalues 1 and $s$ eigenvalues $-1$, and similarly $\Phi_{2s-1}(x)$ is the fundamental function with respect to the vector $\Lambda_{2s}$ consisting of $s$ eigenvalues 1 and $s+1$ eigenvalues $-1$.

In Section 4 we shall determine some generating functions: we prove that

$$
\sum_{s=0}^{\infty} \Phi_{2s+1}(x) \cdot y^s = \frac{1}{\sqrt{y+1}} \sinh \left( x\sqrt{y+1} \right)
$$

and that

$$
\sum_{s=0}^{\infty} \Phi_{s}(x) \cdot y^s = \frac{1+y}{\sqrt{y^2+1}} \sinh \left( x\sqrt{y^2+1} \right) + \cosh \left( x\sqrt{y^2+1} \right).
$$

In Section 5 we give a detailed introduction to the notion of a Bernstein basis $p_{n,k}(x), k = 0, ..., n$, in the setting of exponential polynomials with arbitrary (complex) eigenvalues. We shall give a new proof of the result that there exists a Bernstein basis $p_{n,k}(x), k = 0, ..., n$, in $E_{\lambda_0,\ldots,\lambda_n}$ for points $a \neq b$ if and only if $E_{\lambda_0,\ldots,\lambda_n}$ is a Chebyshev space with respect to $a, b$. In the case that $\lambda_0, \ldots, \lambda_n$ are real it is well known that $E_{\lambda_0,\ldots,\lambda_n}$ is a Chebyshev space with respect to the interval $[a, b]$ and in this case $p_{n,k}(x)$ may be chosen strictly positive for $x$ in the open interval $(a, b)$.

In Section 6 we shall derive recursion formulas for the Bernstein basis $p_{n,k}(x), k = 0, ..., n$ for the special system of eigenvalues $\Lambda_{2s+1}$ which have been used in Section 3.

A remarkable result was proved recently in [5], [26] for certain classes of extended Chebyshev systems $U_n$: Assume that the constant function 1 is in $U_n$; clearly then there exist coefficients $\alpha_k, k = 0, ..., n$, such that $1 = \sum_{k=0}^{n} \alpha_k p_{n,k}$, since $p_{n,k}, k = 0, ..., n$, is a basis. The normalization property
proved in [5] and [26] says that the coefficients $\alpha_k$ are positive. It seems that the paper [2] addresses for the first time the question whether one can construct a Bernstein-type operator based on a Bernstein basis $p_{n,k}, k = 0, ..., n$, in the context of exponential polynomials, i.e. operators of the form

$$B_n(x) := [B_n f](x) := \sum_{k=0}^{n} \alpha_k f(t_k) p_{n,k}(x)$$

(6)

where the coefficients $\alpha_0, ..., \alpha_n$ should be positive and the knots $t_0, ..., t_n$ in the interval $[a, b]$. In [2] the following basic result was proven:

**Theorem 1** Assume that $\lambda_0, ..., \lambda_n$ are real and $\lambda_0 \neq \lambda_1$. Then there exist unique points $t_0 < t_1 < ... < t_n$ in the interval $[a, b]$ and unique positive coefficients $\alpha_0, ..., \alpha_n$ such that the operator $B_n : C[a, b] \rightarrow E(\lambda_0, ..., \lambda_n)$ defined by (6) has the following reproduction property

$$[B_n (e^{\lambda_0 x})](x) = e^{\lambda_0 x} \quad \text{and} \quad [B_n (e^{\lambda_1 x})](x) = e^{\lambda_1 x}.$$  

(7)

The positivity of the coefficients $\alpha_0, ..., \alpha_n$ is related to the above-mentioned normalization property. Theorem 1 says that property (7) can be used for defining knots $t_0, ..., t_n$ and weights $\alpha_0, ..., \alpha_n$ for an operator of the form (6). In the classical polynomial case this means that the Bernstein operator $B_n$ on $[0, 1]$ has the property that $B_n(1) = 1$ and $B_n(x) = x$ for the constant function 1 and the identity function.

It follows from the above construction that the operator $B_n$ defined by (6) and satisfying (7) is a positive operator. Using a Korovkin-type theorem for extended Chebyshev systems the following sufficient criterion for the uniform convergence of $B_n f$ to $f$, for each $f \in C[a, b]$ has been given in [2]. Here we use the more precise but lengthy notation $p_{(\lambda_0, ..., \lambda_n), k}$ instead of $p_{n,k}$ for $k = 0, ..., n$.

**Theorem 2** Let $\lambda_0, \lambda_1, \lambda_2$ be pairwise distinct real numbers and let $\Lambda_n = (\lambda_0, \lambda_1, ..., \lambda_n) \in \mathbb{R}^{n+1}$ with possibly variable $\lambda_j = \lambda_j(n)$ for $j = 3, ..., n$. For each natural number $n \geq 2$, and each $k \leq n$, define the numbers $a(n, k)$ and $b(n, k)$ as follows:

$$a(n, k) := \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)},$$

(8)

$$b(n, k) := \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_1, \lambda_3, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)}.$$  

(9)
Assume that for $n \to \infty$ uniformly in $k$ holds
\[ t_k(n) - t_{k-1}(n) \to 0, \tag{10} \]
where $t_k = t_k(n)$ are the points determined by Theorem 1, and for $n \to \infty$ uniformly in $k$ holds
\[ \frac{\log b(n, k)}{t_k - t_{k+1}} \to \lambda_2 - \lambda_0. \tag{11} \]
Then the Bernstein operator $B_{\lambda_0, \ldots, \lambda_n}$ defined in Theorem 1 converges to the identity operator on $C([a, b], \mathbb{C})$ with the uniform norm.

Theorem 1 applied to the system $\Lambda_{2s+1}$ considered in Section 3 shows that there exist unique points $t_0 < t_1 < \ldots < t_n$ in the interval $[a, b]$ and unique positive coefficients $\alpha_0, \ldots, \alpha_n$ such that the operator $B_n : C[a, b] \to E_{\Lambda_{2s+1}}$ defined by (6) has the property
\[ B_{\Lambda_{2s+1}}(e^x) = e^x \text{ and } B_{\Lambda_{2s+1}}(e^{-x}) = e^{-x}. \]
The contributions in this paper may serve to investigate the question whether the Bernstein operator $B_{\Lambda_{2s+1}}$ converges to the identity operator for $s \to \infty$. Since in this example one has only two different eigenvalues Theorem 2 has to be modified. Note that for $k = n$ the coefficient $a(n, k)$ defined in (8) is equal to
\[ \lim_{x \to b} \frac{p_{(\lambda_0, \lambda_2, \ldots, \lambda_n), n}(x)}{p_{(\lambda_1, \lambda_2, \ldots, \lambda_n), n}(x)} = \frac{\Phi_{2s}(b-a)}{\Phi_{2s-1}(b-a)} \]
and in (5) we have proved that these numbers converge to 1.

2 Taylor expansion of the fundamental function

We say that the vector $\Lambda_n \in \mathbb{C}^{n+1}$ is equivalent to the vector $\Lambda'_n \in \mathbb{C}^{n+1}$ if the corresponding differential operators are equal (so the spaces of all solutions are equal). This is the same to say that each $\lambda$ occurs in $\Lambda_n$ and $\Lambda'_n$ with the same multiplicity. Since the differential operator $L$ defined in (2) does not depend on the order of differentiation, it is clear that each permutation of the vector $\Lambda_n$ is equivalent to $\Lambda_n$. Hence the space $E_{(\lambda_0, \ldots, \lambda_n)}$ does not depend on the order of the eigenvalues $\lambda_0, \ldots, \lambda_n$. 

7
We say that the space \( E(\lambda_0, \ldots, \lambda_n) \) is closed under complex conjugation, if for \( f \in E(\lambda_0, \ldots, \lambda_n) \) the complex conjugate function \( \overline{f} \) is again in \( E(\lambda_0, \ldots, \lambda_n) \). It is easy to see that for complex numbers \( \lambda_0, \ldots, \lambda_n \) the space \( E(\lambda_0, \ldots, \lambda_n) \) is closed under complex conjugation if and only if there exists a permutation \( \sigma \) of the indices \( \{0, \ldots, n\} \) such that \( \overline{\lambda_j} = \lambda_{\sigma(j)} \) for \( j = 0, \ldots, n \). In other words, \( E(\lambda_0, \ldots, \lambda_n) \) is closed under complex conjugation if and only if the vector \( \Lambda_n = (\lambda_0, \ldots, \lambda_n) \) is equivalent to the conjugate vector \( \overline{\Lambda}_n \).

In the case of pairwise different \( \lambda_j, j = 0, \ldots, n \), the space \( E(\lambda_0, \ldots, \lambda_n) \) is the linear span generated by the functions \( e^{\lambda_0 x}, e^{\lambda_1 x}, \ldots, e^{\lambda_n x} \).

In the case when some \( \lambda_j \) occurs \( m_j \) times in \( \Lambda_n = (\lambda_0, \ldots, \lambda_n) \) a basis of the space \( E(\lambda_0, \ldots, \lambda_n) \) is given by the linearly independent functions \( x^s e^{\lambda_j x} \) for \( s = 0, 1, \ldots, m_j - 1 \).

In the case that \( \lambda_0 = \ldots = \lambda_n \) the space \( E(\lambda_0, \ldots, \lambda_n) \) is just the space of all polynomials of degree \( \leq n \), and we shall refer to this as the polynomial case, and we shall denote the fundamental function by \( \Phi_{\text{pol},n}(x) \). Obviously \( \Phi_{\text{pol},n}(x) \) is of a very simple form, namely

\[
\Phi_{\text{pol},n}(x) = \frac{1}{n!} x^n,
\]

so the Taylor expansion is evident from (12). Moreover one has a very elegant recursion, namely

\[
\Phi_{\text{pol},n+1}(x) = \frac{1}{n+1} \cdot x \cdot \Phi_{\text{pol},n}(x).
\]

Generally, recursion formulas for the fundamental function \( \Phi_{\Lambda_n} \) are not known or maybe non-existing, except the case that the eigenvalues are equidistant, cf. [21]. We emphasize that the important (and easy to prove) formula

\[
\left( \frac{d}{dx} - \lambda_{n+1} \right) \Phi_{(\lambda_0, \ldots, \lambda_{n+1})}(x) = \Phi_{(\lambda_0, \ldots, \lambda_n)}(x)
\]

is not a recursion formula from which we may compute \( \Phi_{(\lambda_0, \ldots, \lambda_{n+1})}(x) \) from \( \Phi_{(\lambda_0, \ldots, \lambda_n)}(x) \).

Later we need the Taylor expansion of the fundamental function \( \Phi_{\Lambda_n} \) which is probably a folklore result and not difficult to prove; as definition of the fundamental function we take formula (4).
Proposition 3  The function $\Phi_{\Lambda_n}$ with $\Lambda_n = (\lambda_0, \ldots, \lambda_n)$ satisfies $\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(0) = 0$ for $k = 0, \ldots, n-1$, and for $k \geq n$ the formula

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(\lambda_0,\ldots,\lambda_n)(0) = \sum_{s_0 + \ldots + s_n + n = k}^{\infty} \lambda_0^{s_0} \ldots \lambda_n^{s_n}.$$ 

holds. In particular, $\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(n)}(0) = 1$ and $\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(n+1)}(0) = \lambda_0 + \ldots + \lambda_n$, and

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(n+2)}(0) = \sum_{s_0 + \ldots + s_n = 2}^{\infty} \lambda_0^{s_0} \ldots \lambda_n^{s_n}. \quad (14)$$

Proof. Write for $z \in \mathbb{C}$ with $|z| > |\lambda_j|

$$\frac{1}{z} - \lambda_j = \frac{1}{z} \cdot \frac{1}{1 - \frac{\lambda_j}{z}} = \sum_{s=0}^{\infty} \lambda_j^s \left(\frac{1}{z}\right)^{s+1}.$$ 

Thus we have

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}(x) = \sum_{s_0=0}^{\infty} \ldots \sum_{s_n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\lambda_0^{s_0} \ldots \lambda_n^{s_n}}{z^{s_0 + \ldots + s_n + n + 1}} e^{xz} \ dz.$$ 

By differentiating one obtains

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(x) = \sum_{s_0=0}^{\infty} \ldots \sum_{s_n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\lambda_0^{s_0} \ldots \lambda_n^{s_n}}{z^{s_0 + \ldots + s_n + n + 1}} z^k e^{xz} \ dz.$$ 

For $x = 0$ the integral is easy to evaluate and the result is proven. □

Let us specialize the last result:

Proposition 4  In the case of $(\lambda_0, \ldots, \lambda_n) = \Lambda_{2s+1}(-1,1)$ the following holds

$$\Phi_{2s+1}^{(2s)}(0) = 1 \text{ and } \Phi_{2s+1}^{(2s+1)}(0) = 0 \text{ and } \Phi_{2s+1}^{(2s+3)}(0) = s + 1.$$ 

Proof. The equation $\Phi_{2s+1}^{(2s)}(0) = 1$ is clear; further $\Phi_{2s+1}^{(2s+1)}(0) = \lambda_0 + \ldots + \lambda_n = 0$. Next we use formula (14): split up the integral according to the cases $s_n = 0, 1, 2$ : then

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(n+2)}(0) = \Phi_{(\lambda_0,\ldots,\lambda_{n-1})}^{(n+1)}(0) + \sum_{j=0}^{n} \lambda_j \lambda_n = \Phi_{(\lambda_0,\ldots,\lambda_{n-1})}^{(n+1)}(0) + \lambda_n \Phi_{(\lambda_0,\ldots,\lambda_n)}^{(n+1)}(0).$$ 9
Using this formula for $\Phi^{(n+1)}_{(\lambda_0,\ldots,\lambda_{n-1})}(0)$ (instead of $\Phi^{(n+2)}_{(\lambda_0,\ldots,\lambda_n)}(0)$) one obtains

$$\Phi^{(n+2)}_{(\lambda_0,\ldots,\lambda_n)}(0) = \Phi^{(n)}_{(\lambda_0,\ldots,\lambda_{n-2})}(0) + \lambda_{n-1}\Phi^{(n)}_{(\lambda_0,\ldots,\lambda_{n-1})}(0) + \lambda_n\Phi^{(n+1)}_{(\lambda_0,\ldots,\lambda_n)}(0).$$

Applied to $\Lambda_n = \Lambda_{2s+1}(-1,1)$ and $2s + 1 = n$ we see that $\Phi^{(n+1)}_{(\lambda_0,\ldots,\lambda_n)}(0) = 0$ and $\Phi^{(n)}_{(\lambda_0,\ldots,\lambda_{n-1})}(0) = \lambda_{n-1}$, so we have

$$\Phi^{(n+2)}_{(\lambda_0,\ldots,\lambda_n)}(0) = \Phi^{(n)}_{(\lambda_0,\ldots,\lambda_{n-2})}(0) + 1.$$

By iterating one has $\Phi^{(n+2)}_{(\lambda_0,\ldots,\lambda_n)}(0) = \Phi^{(n-2j)}_{(\lambda_0,\ldots,\lambda_{n-2j})}(0) + j$. Since $n = 2s + 1$ is odd we can put $j = s$ and obtain

$$\Phi_{2s+1}^{(2s+3)}(0) = s + \Phi_{(\lambda_0)}^{(1)}(0) = s + 1$$

since $\Phi_{(\lambda_0)}(x) = e^{\lambda_0x}$ and $\lambda_0 = 1$.

We finish the section by recalling two standard facts:

**Proposition 5** The function $\Phi_{\Lambda_n}$ is real-valued if $(\lambda_0,\ldots,\lambda_n)$ is equivalent to $(\lambda_0,\ldots,\lambda_n)$.

**Proposition 6** If $\lambda_0,\ldots,\lambda_n$ are real then $\Phi_{\Lambda_n}(x) > 0$ for all $x > 0$.

**Proof.** Since $\lambda_0,\ldots,\lambda_n$ are real the space $E(\lambda_0,\ldots,\lambda_n)$ is a Chebyshev space over $\mathbb{R}$, so $\Phi_{\Lambda_n}$ has at most $n$ zeros in $\mathbb{R}$. Since $\Phi_{\Lambda_n}$ has exactly $n$ zeros in 0, it has no other zeros. By the norming condition $\Phi^{(n)}_{\Lambda_n}(0) = 1$ it follows that $\Phi_{\Lambda_n}(x) > 0$ for small $x > 0$, hence $\Phi_{\Lambda_n}(x) > 0$ for all $x > 0$.

The following result is a simple consequence of the definition of $\Phi_{\Lambda_n}$

**Proposition 7** If $\Lambda_n = (\lambda_0,\ldots,\lambda_n)$ and $c + \Lambda_n := (c + \lambda_0,\ldots,c + \lambda_n)$ for some $c \in \mathbb{C}$ then

$$\Phi_{c+\Lambda_n}(x) = e^{cx}\Phi_{\Lambda_n}(x).$$

If $c\Lambda_n = (c\lambda_0,\ldots,c\lambda_n)$ for $c \neq 0$ then

$$\Phi_{c\Lambda_n}(x) = \frac{1}{e^{cn}}\Phi_{\Lambda_n}(cx).$$

10
3 The fundamental function for $\Lambda_{2s+1}(\lambda, \mu)$

Let $\Gamma_r(t) = re^{it}$ for $t \in [0, 2\pi]$ and fixed $r > 0$. Let $\lambda$ and $\mu$ be two real numbers and assume that $r > 0$ be so large that $\lambda$ and $\mu$ are contained in the open ball of radius $r$ and center 0. In the following we want to compute and analyze the fundamental function

$$\Phi_{\Lambda_s(\lambda, \mu)}(x) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{xz}}{(z - \lambda)^{s+1} (z - \mu)^{s+1}} dz.$$ 

By Proposition 7 it is sufficient to consider the case $\lambda = 1$ and $\mu = -1$, so we define

$$\Phi_{2s+1}(x) := \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{xz}}{(z - 1)^{s+1} (z + 1)^{s+1}} dz$$

This integral can be evaluated by the residue theorem, giving two summands according to the poles $-1$ and 1. By a simple substitution argument we see that

$$\Phi_{2s+1}(x) = e^x P_0^s(x) - e^{-x} P_0^s(-x).$$

Now for an integer $\alpha$ we define the polynomials

$$P_\alpha^s(x) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{e^{xz}}{z^{s+1} (z + 2)^{s+1+\alpha}} dz$$

and we see that

$$\Phi_{2s+1}(x) = e^x P_\alpha^s(x) - e^{-x} P_\alpha^s(-x).$$

Using equality $\cosh x = \frac{1}{2} (e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2} (e^x - e^{-x})$ it is easy to see that

$$\Phi_{2s+1}(x) = \cosh(x) [P_\alpha^s(x) - P_\alpha^s(-x)] + \sinh(x) [P_\alpha^s(x) + P_\alpha^s(-x)].$$
For example, it is easy to see that
\[ \Phi_1(x) = \sinh x \text{ and } \Phi_3(x) = \frac{1}{2} (x \cosh x - \sinh x). \]
A similar consideration shows that for the function \( \Phi_s(x) \) defined as
\[
\Phi_s(x) := \frac{1}{2\pi i} \int_{\Gamma_s} \frac{e^{zx}}{(z - 1)^{s+1} (z + 1)^s} \, dz
\]
we have
\[
\Phi_2s(x) = e^x \frac{1}{2\pi i} \int_{\Gamma_1} \frac{e^{zx}}{z^{s+1} (z + 2)^s} \, dz + e^{-x} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{e^{x}}{z^{s+1} (z - 2)^s} \, dz
\]
\[
= e^x P_{s-1}^0(x) + e^{-x} P_{s-1}^1(-x).
\]
Here for the case \( s = 0 \) we use the convention \( P_{-1}^0(x) := 0 \). The following is straightforward:

**Lemma 8** The functions \( P_s^0(x) \) are polynomials in the variable \( x \) of degree \( s \) given by the formula
\[
P_s^0(x) = (-1)^s \frac{1}{2^{2s+1}} \sum_{k=0}^{s} \frac{1}{k! (s-k)! (s)!} (2s-k)! (-2x)^k.
\]

**Proof.** From residue theory it is known that
\[
P_s^0(x) = \frac{1}{s!} \left[ \frac{d^s}{dz^s} \left[ e^{zx} (z + 2)^{-s-1} \right] \right]_{z=0}.
\]
The rule of Leibniz gives
\[
\frac{d^s}{dz^s} \left[ e^{zx} (z + 2)^{-s-1} \right] = \sum_{k=0}^{s} \binom{s}{k} \frac{d^k}{dz^k} (z + 2)^{-s-1} \cdot \frac{d^{s-k}}{dz^{s-k}} e^{zx}.
\]
Note that \( \frac{d^k}{dz^k} (z + 2)^{-s-1} = (-s-1) \cdots (-s-k) (z + 2)^{-s-1-k} \), and clearly
\[ \frac{d^{s-k}}{dz^{s-k}} e^{zx} = e^{zx} x^{s-k}, \] so
\[
\frac{d^s}{dz^s} \left[ e^{zx} (z + 2)^{-s-1} \right] = e^{zx} \sum_{k=0}^{s} \binom{s}{k} (-1)^k \frac{(s+k)!}{s!} x^{s-k} (z + 2)^{-s-1-k}.
\]
Reverse now the summation, and we arrive at
\[
\frac{d^s}{dz^s} \left[ e^{xz} (z + 2)^{-s-1} \right] = e^{xz} \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{(2s-k)!}{s!} x^k (z + 2)^{-s-1-(s-k)}.
\]

Take now \( z = 0 \).

The following is a short list of the first four polynomials for \( \alpha = 0 \):

\[
\begin{align*}
P_0^0 (x) &= \frac{1}{2}, \\
P_1^0 (x) &= \frac{1}{4} (x - 1), \\
P_2^0 (x) &= \frac{1}{25} (2x^2 - 6x + 6), \\
P_3^0 (x) &= \frac{1}{26} \left( -20 + 20x - 8x^2 + \frac{4}{3} x^3 \right), \\
P_4^0 (x) &= \frac{1}{29} \left( 70 - 70x + 30x^2 - \frac{20}{3} x^3 + \frac{2}{3} x^4 \right).
\end{align*}
\]

The polynomial \( P_2^0 (x) \) is strictly positive on the real line, hence the polynomials \( P_s^0 (x) \) are not orthogonal polynomials with respect to any measure on the real line. However the following is true:

**Theorem 9** The polynomials \( P_s^0 \) satisfy the following recurrence relation:
\[
4s (s + 1) P_{s+1}^0 (x) = x^2 P_{s-1}^0 (x) - 2s (2s + 1) P_s^0 (x).
\] (17)

This can be derived by a direct but somewhat tedious calculation. For the fundamental function \( \Phi_{2s+1} = e^{x} P_s^0 (x) - e^{-x} P_s^0 (-x) \) we obtain by a straightforward calculation the recurrence relation (18) below. Since we shall derive this recurrence relation from Theorem 28 in Section 5 by a different method, we omit the proof of (17),

**Corollary 10** The fundamental function \( \Phi_{2s+1} (x) \) satisfies the recursion
\[
\Phi_{2s+3} (x) = x^2 \frac{1}{4s (s + 1)} \Phi_{2s-1} (x) - \frac{2s + 1}{2s + 2} \Phi_{2s+1} (x),
\] (18)

and the following estimate holds for all \( x > 0 \):
\[
0 \leq \frac{\Phi_{2s+1} (x)}{\Phi_{2s-1} (x)} \leq \frac{x^2}{2s (2s + 1)}.
\] (19)
Proof. Since $\Phi_{2s+3} (x) > 0$ the equation (18) implies that
\[
\frac{2s+1}{2s+2} \Phi_{2s+1} (x) < x^2 \frac{1}{4s(s+1)} \Phi_{2s-1} (x)
\]
from which (19) is immediate. ■

Let us recall that after formula (5) we defined $\Phi_{2s} (x)$ as the fundamental function for the vector with $s + 1$ many $1$ and $s$ many $-1$. By $\Phi_{2s-1} (x)$ we denoted the fundamental function with respect to the vector with $s + 1$ many $-1$ and $s$ many $1$. We shall denote sometimes $\Phi_{2s} (x)$ also by $\Phi_{2s+1} (x)$ in order to facilitate some formulas and to underline the difference to $\Phi_{2s-1} (x)$. The following simple identity
\[
\frac{1}{(z-1)^{s+1} (z+1)^s} - \frac{1}{(z-1)^s (z+1)^{s+1}} = 2 \frac{1}{(z-1)^{s+1} (z+1)^{s+1}}
\]
implies the formula
\[
\Phi_{2s} (x) - \Phi_{2s-1} (x) = 2\Phi_{2s+1} (x).
\]
This formula can also be derived by summing up the following two identities in Theorem 11 which we shall derive from Theorem 27.

Theorem 11 The following two recursions hold:
\[
\Phi_{2s+1} (x) = \Phi_{2s} (x) - \frac{1}{2s} x \cdot \Phi_{2s-1} (x),
\]
\[
\Phi_{2s+1} (x) = -\Phi_{2s-1} (x) + \frac{1}{2s} x \cdot \Phi_{2s-1} (x).
\]

Proof. We derive the result from Theorem 27: We choose $k = 2s - 1$ in the relation
\[
A_{\pm} p_{2s+1,2s+1} (x) = x \cdot p_{2s-1,2s-1} - 2s \cdot p_{2s,2s} (x)
\]
or which is the same (up to notation)
\[
A_{\pm} \Phi_{2s+1} (x) = x \cdot \Phi_{2s-1} - 2s \cdot \Phi_{2s,2s} (x).
\]
By Proposition 3
\[
\Phi_{2s-1} (0) = \lambda_0 + \ldots + \lambda_{2s-1} = 0
\]
\[
\Phi_{2s+1} (0) = \lambda_0 + \ldots + \lambda_{2s,2s} = \pm 1.
\]
So $A_{\pm} = (2s + 1) \Phi_{2s-1} (0) - 2s \Phi_{2s,2s} (0) = \pm 1 (-2s)$. ■
Corollary 12 The following limit exists

$$\lim_{s \to \infty} \frac{\Phi_{2s}(x)}{\Phi_{2s-1}(x)} \to 1.$$ 

Proof. By Theorem 11 we have

$$\frac{\Phi_{2s}(x)}{\Phi_{2s-1}(x)} = \frac{\Phi_{2s+1}(x) + \frac{1}{2s} x \cdot \Phi_{2s-1}(x)}{\frac{1}{2s} x \cdot \Phi_{2s-1}(x) - \Phi_{2s+1}(x)}.$$ 

Let us define $y_s := \Phi_{2s+1}(x) / \Phi_{2s-1}(x)$, then

$$\frac{\Phi_{2s}(x)}{\Phi_{2s-1}(x)} = \frac{s \cdot y_s + \frac{1}{2} x}{\frac{2}{s} x - s \cdot y_s}. \tag{20}$$

From Theorem 10 we see that $s \cdot y_s$ converges to 0, so (20) converges to 1. \[ \blacksquare \]

We mention that one can derive also recursion formula for the derivatives, e.g. the following identity holds:

**Theorem 13** The derivatives $\frac{d}{dx} P_s^0(x)$ of the polynomials $P_s^0(x)$ can be computed by

$$\frac{d}{dx} P_s^0(x) = \frac{1}{s} x \frac{P_s^0(x)}{2} - P_s^0(x).$$

4 Generating functions

The Lagrange inversion formula, see e.g. [1], is another way to investigate the polynomials defined in (15)

$$P_n^\alpha(x) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{e^{xz}}{z^{n+1} (z+2)^{n+1+\alpha}} dz, \tag{21}$$

where $\alpha$ is a fixed integer. Since this powerful method is somewhat technical, let us recall the basic facts. In our case we put $\varphi(z) = 1/(z+2)$ and $f'(z) := e^{xz} / (z+2)^\alpha$. The fundamental idea of Lagrange inversion is based on the observation that

$$a_n := \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f'(z)}{z^{n+1} [\varphi(z)]^{n+1}} dz. \tag{22}$$

15
can be seen as the $n$-th Taylor coefficient of a holomorphic function which will be constructed from $f'$ and $\varphi$. More generally, we may assume that $\varphi$ is a holomorphic function in a neighborhood of 0 such that $\varphi(0) \neq 0$, and $f$ is holomorphic in a neighborhood of 0, and we define $a_n$ by the expression (22). Consider the function

$$y(z) := \frac{z}{\varphi(z)}$$

which is holomorphic in a neighborhood of 0 (since $\varphi(0) \neq 0$) with $y(0) = 0$. Since obviously $y(z)\varphi(z) = z$ we obtain $\varphi(z) y'(z) + y(z) \varphi'(z) = 1$, so $y'(0) \neq 0$. Hence $y$ is injective in a neighborhood; let $y^{-1}$ be the inverse map. Since $y^{-1} \circ y(z) = z$ one has $\frac{d}{dz}y^{-1}(y(z)) \cdot \frac{d}{dz}y(z) = 1$, and using this formula one arrives at

$$\left(\frac{d}{dy} (f \circ y^{-1})\right)(y) = f'(y^{-1}(y)) \cdot \left(\frac{d}{dz}y^{-1}(y)\right) = f'(z) \cdot \frac{1}{\frac{d}{dz}y(z)}.$$

for $y = y(z)$. Thus we obtain

$$\frac{f'(z) \varphi(z)^{n+1}}{z^{n+1}} = \frac{f'(z)}{y(z)^{n+1}} = \frac{\frac{d}{dy} (f \circ y^{-1})(y(z)) \cdot \frac{d}{dz}y(z)}{y(z)^{n+1}}. \quad (23)$$

Let now $\gamma(t) = re^{it}$ for $r > 0$ sufficient small, and put $z = \gamma(t)$. Note that $\Gamma(t) = y(\gamma(t))$ is a path surrounding zero. Insert $z = \gamma(t)$ in (23) and multiply it with $\gamma'(t)/2\pi i$, so we obtain

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{z^{n+1}} [\varphi(z)]^{n+1} \, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{dy} (f \circ y^{-1})(y(\gamma(t))) \cdot \frac{d}{dz}y(\gamma(t)) \gamma'(t) \, dt$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{d}{dy} (f \circ y^{-1})(y) \frac{1}{y^{n+1}} \, dy.$$

Let us define $F(y) = \frac{d}{dy} (f \circ y^{-1})(y)$, then by the Cauchy formula for the Taylor coefficients one obtains

$$a_n = \frac{1}{n!} F^{(n)}(0) = \frac{1}{n!} \frac{d^n}{dy^n} F(0).$$
Since the Taylor series of $F(y)$ is equal to $F(y)$ we obtain

$$
\sum_{n=0}^{\infty} a_n y^n = F(y) = \frac{d}{dy} \left( f \circ y^{-1} \right)(y) = f' \left( y^{-1}(y) \right) \frac{dy^{-1}}{dy}(y). \quad (24)
$$

So this means that for the computation of the unknown sum $\sum_{n=0}^{\infty} a_n y^n$ we only have to compute $y^{-1}(y)$, and the derivative $\frac{dy^{-1}}{dy}(y)$ and then we have to compute the right hand side of (24). As an application we prove:

**Theorem 14** Let $P_n^\alpha(x)$ be defined by the expression (21). Then for all $y$ with $|y| < 1$

$$
F_{P_n^\alpha}(x, y) := \sum_{n=0}^{\infty} P_n^\alpha(x) y^n = e^{-x} \frac{e^{x\sqrt{y+1}}}{2\sqrt{y+1}(\sqrt{y+1}+1)^\alpha}.
$$

**Proof.** Let $\varphi(z) = (z+2)^{-1}$ and $y(z) := z/\varphi(z)$, so $y(z) = z(z+2)$. We define the inverse function $y^{-1}$ of $y(z)$ by

$$
y^{-1}(y) := \sqrt{y+1} - 1.
$$

Then $y^{-1}(y(z)) = \sqrt{z(z+2)+1} - 1 = z$ and note that

$$
\frac{dy^{-1}}{dy}(y) = \frac{1}{2\sqrt{y+1}}.
$$

Define $f'(z) = e^{xz}/(z+2)^\alpha$. Then the Lagrange inversion formula tells us that

$$
\sum_{n=0}^{\infty} P_n^\alpha(x) y^n = f' \left( \sqrt{y+1} - 1 \right) \frac{1}{2\sqrt{y+1}(\sqrt{y+1}+1)^\alpha}
$$

which is exactly our claim. ■

**Remark 15** It is easy to derive a second order differential equation for the generating function $y \mapsto F_{P_n^\alpha}(x, y)$, and this can be used to give a proof for the recursion (17).

**Theorem 16** The following identity

$$
F_{odd}(x, y) = \sum_{s=0}^{\infty} \Phi_{2s+1}(x) \cdot y^s = \frac{1}{\sqrt{y+1}} \sinh \left( x\sqrt{y+1} \right)
$$

holds for all $y \in \mathbb{C}$. 

17
Proof. Since $\Phi_{2s+1} (x) = e^x P^0_s (x) - e^{-x} P^0_s (-x)$ we obtain
\[
F_{\text{odd}} (x, y) = e^x \sum_{s=0}^{\infty} P^0_s (x) y^s - e^{-x} \sum_{s=0}^{\infty} P^0_s (-x) y^s.
\]
Using Theorem 14 a short computation shows that this is equal to
\[
F_{\text{odd}} (x, y) = \frac{1}{\sqrt{y+1}} \frac{1}{2} \left( e^{x \sqrt{y+1}} - e^{-x \sqrt{y+1}} \right) = \frac{1}{\sqrt{y+1}} \sinh x \sqrt{y+1}.
\]
Since
\[
\sum_{n=0}^{\infty} P^0_n (x) y^n = \exp \left( x \left( \sqrt{y+1} - 1 \right) \right) \frac{1}{2 \sqrt{y+1}}
\]
we obtain
\[
F_{\text{odd}} (x, y) = \left( x \sqrt{y+1} \right) \frac{1}{2 \sqrt{y+1}} - \exp \left( -x \sqrt{y+1} \right) \frac{1}{2 \sqrt{y+1}}
\]
\[
= \frac{1}{\sqrt{y+1}} \sinh \left( x \sqrt{y+1} \right).
\]
Since $\sinh z$ contains only odd powers in the Taylor expansion it follows that $y \mapsto F_{\text{odd}} (x, y)$ is entire. ■

Let $\lambda_0, \lambda_1, \ldots$ be a sequence of real numbers. In the following we consider the generating function
\[
F_{\Lambda} (x, y) := \sum_{n=0}^{\infty} \Phi_{(\lambda_0, \ldots, \lambda_n)} (x) \cdot y^n
\]
for the case that $\lambda_{2j} = 1$ and $\lambda_{2j+1} = -1$ for all $j \in \mathbb{N}_0$. With the notation from the introduction we have
\[
\Phi_{(\lambda_0, \ldots, \lambda_{2s})} = \Phi_{2s+1} \quad \text{and} \quad \Phi_{(\lambda_0, \ldots, \lambda_{2s-1})} = \Phi_{2s}
\]
and now we consider
\[
F_{\Lambda} (x, y) = \sum_{s=0}^{\infty} \Phi_{2s+1} (x) \cdot y^{2s+1} + \sum_{s=0}^{\infty} \Phi_{2s} (x) \cdot y^{2s}.
\]
(25)
It is easy to see that $(\frac{d}{dx} + 1) \Phi_{2s+1} (x) = \Phi_{2s} (x)$ (cf. the general formula (13)), so we have
\[
\sum_{s=0}^{\infty} \Phi_{2s} (x) \cdot y^{2s} = (\frac{d}{dx} + 1) \sum_{s=0}^{\infty} \Phi_{2s+1} (x) \cdot y^{2s} = \left( \frac{d}{dx} + 1 \right) F_{\text{odd}} (x, y^2).
\]
Thus we have

$$F_{\Lambda}(x, y) = (1 + y) F_{\text{odd}}(x, y^2) + \frac{d}{dx} F_{\text{odd}}(x, y^2)$$

and the following is proved:

**Theorem 17** With the above notations, the generating function $F_{\Lambda}(x, y)$ in (25) is equal to

$$F_{\Lambda}(x, y) = \frac{1 + y}{\sqrt{y^2 + 1}} \sinh \left(x \sqrt{y^2 + 1}\right) + \cosh \left(x \sqrt{y^2 + 1}\right).$$

### 5 Construction of Bernstein bases

In this Section we return to the general theory of exponential polynomials where the eigenvalues $\lambda_0, ..., \lambda_n$ may be complex numbers. We shall characterize the spaces $E_{(\lambda_0, ..., \lambda_n)}$ of exponential polynomials which admit a Bernstein basis. Let us emphasize that the existence results already follow from those in [5], [12], [25], [26] in the more general context of Chebyshev spaces. We follow here our exposition in [2] which is based on a recursive definition of the Bernstein basis, and it seems that this approach is be different from those in the above cited literature. Further references on properties of Bernstein-like bases are [10] and [29].

Let us recall some terminology and notations: The $k$-th derivative of a function $f$ is denoted by $f^{(k)}$. A function $f \in C^n(I, \mathbb{C})$ has a zero of order $k$ or of multiplicity $k$ at a point $a \in I$ if $f(a) = ... = f^{(k-1)}(a) = 0$ and $f^{(k)}(a) \neq 0$. We shall repeatedly use the fact that

$$k! \cdot \lim_{x \to 0} \frac{f(x)}{(x-a)^k} = f^{(k)}(a).$$

for any function $f \in C^{(k)}(I)$ with $f(a) = ... = f^{(k-1)}(a) = 0$.

Let us recall the definition of a Bernstein basis:

**Definition 18** A system of functions $p_{n,k}$, $k = 0, ..., n$ in the space $E_{(\lambda_0, ..., \lambda_n)}$ is called Bernstein-like basis for $E_{(\lambda_0, ..., \lambda_n)}$ and $a \neq b$ if and only if each function $p_{n,k}$ has a zero of exact order $k$ at $a$ and a zero of exact order $n - k$ at $b$. 

19
In the following we want to characterize those spaces $E_{(\lambda_0,\ldots,\lambda_n)}$ which admit a Bernstein basis. For this, we need the following simple

**Lemma 19** Suppose that $f_0,\ldots,f_n \in E_{(\lambda_0,\ldots,\lambda_n)}$ have the property that $f_k$ has a zero of order $k$ at $a$ for $k = 0,\ldots,n$. Then $f_0,\ldots,f_n$ is a basis for any interval $[a,b]$.

**Proof.** It suffices to show that $f_0,\ldots,f_n$ are linearly independent since $E_{(\lambda_0,\ldots,\lambda_n)}$ has dimension $n+1$. Suppose that there exist complex numbers $c_0,\ldots,c_n$ such that

$$c_0 f_0(x) + \ldots + c_n f_n(x) = 0$$

for all $x \in [a,b]$. Since $f_0(a) \neq 0$ and $f_j(a) = 0$ for all $j \geq 1$ we obtain $c_0 = 0$. Next we take the derivative and obtain the equation

$$c_1 f_1'(x) + \ldots + c_n f_n'(x) = 0.$$

We insert $x = a$ and obtain that $c_1 = 0$ since $f_1'(a) \neq 0$ and $f_j'(a) = 0$ for all $j = 2,\ldots,n$. Now one proceeds inductively.

It follows from the Lemma 19 that a Bernstein basis $p_{n,k}, k = 0,\ldots,n$ is necessarily a basis for $E_{(\lambda_0,\ldots,\lambda_n)}$. Next we want to show that the basis functions are unique up to a factor (provided there exists such a basis).

**Proposition 20** Suppose that there exists a Bernstein basis $p_{n,k}, k = 0,\ldots,n$ for the space $E_{(\lambda_0,\ldots,\lambda_n)}$ and $a,b$. If $f \in E_{(\lambda_0,\ldots,\lambda_n)}$ has a zero of order at least $k_0$ in $a$ and of order at least $n-k_0$ at $b$ then there exists a complex number $c$ such that

$$f = c \cdot p_{n,k_0}.$$

**Proof.** Since $p_{n,k}, k = 0,\ldots,n$ is a basis we can find complex numbers $c_0,\ldots,c_n$ such that

$$f = c_0 p_{n,0} + \ldots + c_n p_{n,n}.$$ We know that $f(a) = \ldots = f^{(k_0-1)}(a) = 0$ since $f$ has an order of exact order $k_0$ in $a$. So we see that $0 = f(a) = c_0 p_{n,0}(a)$. Since $p_{n,0}(a) \neq 0$ we obtain $c_0 = 0$. We proceed inductively and obtain that

$$f = c_{k_0} p_{n,k_0} + \ldots + c_n p_{n,n}.$$ We now use the zeros at $b$: Inserting $x = b$ yields

$$0 = f(b) = c_n p_{n,n}(b).$$
Since \( p_{n,n}(b) \neq 0 \) (here we use the exact order at the point \( b \)) it follows that \( c_n = 0 \). Proceeding inductively one obtains \( f = c_{k_0} p_{n,k_0} \). 

The proof actually shows the following:

**Corollary 21** Suppose that there exists a Bernstein basis \( p_{n,k}, k = 0, \ldots, n \) for the space \( E(\lambda_0, \ldots, \lambda_n) \) and \( a, b \). If \( f \in E(\lambda_0, \ldots, \lambda_n) \) has more than \( n \) zeros in \( \{a, b\} \) then \( f = 0 \).

Let us recall the following definition:

**Definition 22** The space \( E(\lambda_0, \ldots, \lambda_n) \) is a Chebyshev system with respect to the set \( A \subset \mathbb{C} \) if each function \( f \in E(\lambda_0, \ldots, \lambda_n) \) that has more than \( n \) zeros in \( A \) (including multiplicities) is 0.

So we have seen that a necessary condition for the existence of a Bernstein basis on the interval \( [a, b] \) is the property that \( E(\lambda_0, \ldots, \lambda_n) \) is a Chebyshev system for the set \( \{a, b\} \). In the next Proposition we shall show that this property is actually equivalent to the existence.

**Proposition 23** The space \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system with respect to \( a, b \) if and only if there exists a (unique) Bernstein-like basis \( p_{n,k} : k = 0, \ldots, n \), for \( a, b \) satisfying the condition

\[
\lim_{x \to a, x > a} \frac{p_{n,k}(x)}{(x-a)^k} = (p_{n,k}^{(k)}(a)) = 1. \tag{27}
\]

The Bernstein-like basis functions \( p_{n,k}(x) \) are recursively defined by equations (28), (29), (30) and (31) below.

**Proof.** The necessity was already proved. Assume now that \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system for \( a, b \). It is convenient to use the following notation:

\[
q_k(x) := p_{n,n-k}(x), \tag{28}
\]

so \( q_k \) has a zero of order \( n - k \) at \( a \) and a zero of order \( k \) at \( b \). Define first

\[
q_0(x) := \Phi_{\lambda_n}(x-a), \tag{29}
\]

which clearly has a zero of order \( n \) at \( a \) and of order at least 0 at \( b \). Since \( q_0 \) can not have more than \( n \) zeros on \( a, b \) by our assumption we infer that
Define \( q_1 := q_0^{(1)} - \alpha_0q_0 \) for \( \alpha_0 = q_0^{(1)}(b)/q_0(b) \) which has a zero of order at least \( n-1 \) at \( a \) and a zero of order at least \( 1 \) at \( b \). Again our assumption implies that \( q_1 \) has a zero of order at \( n-1 \) at \( a \) and a zero of order at \( 1 \) at \( b \). For \( k \geq 2 \) we define \( q_k \) recursively by

\[
q_k := q_{k-1}^{(1)} - (\alpha_{k-1} - \alpha_{k-2}) \cdot q_{k-1} - \beta_k q_{k-2}
\]

with coefficients \( \alpha_{k-1} - \alpha_{k-2} \) and \( \beta_k \) to be determined. By construction we know already that \( q_{k-1} \) and \( q_{k-2} \) respectively have a zero of order \( k-1 \) and \( k-2 \) at \( b \), and a zero of order \( n-k+1 \) and \( n-k+2 \) at \( a \). So it is clear that \( q_k \) has a zero of order at least \( k-2 \) at \( b \), and a zero of order at least \( n-k \) at \( a \). The coefficients \( \alpha_{k-1} - \alpha_{k-2} \) and \( \beta_k \) are chosen in such a way that \( q_k \) will have a zero of order at least \( k \) in \( b \). This is achieved by defining

\[
\beta_k := \frac{q_{k-1}^{(k-1)}(b)}{q_{k-2}^{(k-2)}(b)} \quad \text{and} \quad \alpha_{k-1} := \frac{q_{k-1}^{(k)}(b)}{q_{k-1}^{(k-1)}(b)}.
\]

Again our assumption implies that \( q_k \) has a zero of order \( k \) at \( b \) and a zero of order \( n-k \) at \( a \). The condition (27) is easily checked using (26), and (30) together with induction. The uniqueness property follows from the above remarks.

In the rest of the paper we shall call the Bernstein-like basis provided by Proposition 23 the Bernstein basis with respect to \( a, b \).

Next we shall give a construction of the Bernstein basis \( p_{n,n-k}(x) \), \( k = 0, \ldots, n \), will is similar to constructions known from the theory of Chebyshev spaces.

**Theorem 24** Let \( (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1} \) and define for each \( k = 0, \ldots, n \) the \((k+1) \times (k+1)\) matrix \( A_{n,k}(x) \) by

\[
A_{n,k}(x) := \begin{pmatrix}
\Phi_{\lambda_n}(x) & \cdots & \Phi_{\lambda_n}^{(k)}(x) \\
\vdots & \ddots & \vdots \\
\Phi_{\lambda_n}^{(k)}(x) & \cdots & \Phi_{\lambda_n}^{(2k)}(x)
\end{pmatrix}.
\]

Then the matrices \( A_{n,k}(b-a) \) are invertible for \( k = 0, \ldots, n \) if and only if \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system with respect to \( \{a, b\} \).

**Proof.** Assume that the matrices \( A_{n,k}(b-a) \) are invertible for \( k = 0, \ldots, n \). It suffices to show that there exists a Bernstein basis with respect to
a, b. For a polynomial \( r_{\Lambda, k}(z) = r_k z^k + r_{k-1} z^{k-1} + \ldots + r_0 \) with coefficients \( r_0, \ldots, r_k \in \mathbb{C} \) and \( r_k = 1 \) let us define

\[
f_{n,k}(x) := \frac{1}{2\pi i} \int \frac{r_{\Lambda, k}(z) e^{(x-a)z}}{(z - \lambda_0) \ldots (z - \lambda_n)} dz.
\]  

(33)

It is easy to see that \( f_{n,k} \) is in \( E(\lambda_0, \ldots, \lambda_n) \) since

\[
f_{n,k}(x) = \sum_{j=0}^{k} r_j \Phi_{\Lambda, n}^{(j)}(x - a)
\]

From this representation it follows that \( f_{n,k} \) has of order at \( n - k \) at \( a \), and that \( f^{(n-k)}(a) = r_k \Phi_{\Lambda, n}^{(n)}(x - a) = 1 \). We want to choose the coefficients \( r_0, \ldots, r_{k-1} \) such that

\[
\left( \frac{d^l}{dx^l} f_{n,k} \right)(b) = 0 \text{ for all } l = 0, \ldots, k - 1
\]

and that \( \left( \frac{d^k}{dx^k} f_{n,k} \right)(b) \neq 0 \). Writing down these equations for \( l = 0, \ldots, k - 1 \) in matrix form shows that for \( c = b - a \)

\[
\begin{pmatrix}
\Phi_{\Lambda, n}(c) & \ldots & \Phi_{\Lambda, n}^{(k-1)}(c) \\
\vdots & \ddots & \vdots \\
\Phi_{\Lambda, n}^{(k-1)}(c) & \ldots & \Phi_{\Lambda, n}^{(2k-2)}(c)
\end{pmatrix}
\begin{pmatrix}
r_0 \\
\vdots \\
r_{k-1}
\end{pmatrix}
= - \begin{pmatrix}
\Phi_{\Lambda, n}^{(k)}(c) \\
\vdots \\
\Phi_{\Lambda, n}^{(2k-2)}(c)
\end{pmatrix}
\]

Since the matrix \( A_{n,k-1}(b - a) \) is invertible we can find clearly \( r_0, \ldots, r_{k-1} \in \mathbb{C} \) which solves the equation. Hence \( f_{n,k} \) has a zero of order at \( n - k \) at \( a \) and a zero of order at least \( k \) in \( b \). Suppose now that \( f(b) = \ldots = f_{n,k}^{(k)}(b) = 0 \). Then these equations say that

\[
\begin{pmatrix}
\Phi_{\Lambda, n}(c) & \ldots & \Phi_{\Lambda, n}^{(k)}(c) \\
\vdots & \ddots & \vdots \\
\Phi_{\Lambda, n}^{(k)}(c) & \ldots & \Phi_{\Lambda, n}^{(2k)}(c)
\end{pmatrix}
\begin{pmatrix}
r_0 \\
\vdots \\
r_k
\end{pmatrix}
= 0.
\]

Since the matrix \( A_{n,k}(b - a) \) is invertible it follows that \( r_0 = \ldots = r_k = 0 \). This is a contradiction to the choice \( r_k = 1 \).

Now assume that \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system with respect to \( \{a, b\} \). Suppose that the matrix \( A_{n,k}(b - a) \) is not invertible for some \( k \in \{0, \ldots, n\} \). Then we can find \( s = (s_0, \ldots, s_k) \neq 0 \) such that \( A_{n,k}(b - a) s = 0 \).
Define $f_{n,k}(x) = \sum_{j=0}^{k} s_j \Phi_{\Lambda_n}^{(j)}(x-a)$. Then clearly $f_{n,k}$ has a zero of order at least $n-k$ at $a$. Further the equation $A_{n,k}(b-a)s = 0$ implies that $f_{n,k}^{(j)}(b) = 0$ for $j = 0, ..., k$. So $f_{n,k}$ has a zero of order $n+1$ in the set $\{a, b\}$, and our assumption implies that $f_{n,k} = 0$. By Lemma 19 the system $\Phi_{\Lambda_n}, ..., \Phi_{\Lambda_n}^{(n)}$ is a basis of $E(\lambda_0, ..., \lambda_n)$. It follows that $s_0 = ... = s_n = 0$, a contradiction.

The proof of the last theorem actually shows:

**Theorem 25** Assume that the matrices $A_{n,k}(b-a)$ are invertible for $k = 1, ..., n+1$, and let $p_{n,n-k}, k = 0, ..., n$ be the Bernstein basis with respect to $a, b$. Then for each $k = 0, ..., n$ there exists a polynomial $r_{\Lambda_n,k}(z)$ of degree $k$ and leading coefficient 1 such that

$$p_{n,n-k}(x) = \frac{1}{2\pi i} \int \frac{r_{\Lambda_n,k}(z)e^{(x-a)z}}{(z-\lambda_0)...(z-\lambda_n)} dz$$

for all $x \in \mathbb{R}$.

For a given vector $\Lambda_n = (\lambda_0, ..., \lambda_n)$ the matrices $A_{n,k}(z), k = 0, ..., n$ are defined in (32) and we set

$$Z_{\Lambda_n} := \bigcup_{k=0}^{n} \{z \in \mathbb{C} : \det A_{n,k}(z) = 0\}.$$

Note that $Z_{\Lambda_n}$ is a discrete subset of $\mathbb{C}$ since $z \mapsto \det A_{n,k}(z)$ is obviously an entire function. It follows that for given $a \in \mathbb{R}$ the space $E(\lambda_0, ..., \lambda_n)$ is an extended Chebyshev system for $\{a, b\}$ for all $b \in \mathbb{R}$ except a countable discrete subset of $\mathbb{R}$. We emphasize that this does not imply that $E(\lambda_0, ..., \lambda_n)$ is an extended Chebyshev system for the interval $[a, b]$. In order to have nice properties of the basic function $p_{n,k}$ one need the assumption that $E(\lambda_0, ..., \lambda_n)$ is an extended Chebyshev system for the interval $[a, b]$. Indeed, the following result is well known (at least in the polynomial case):

**Theorem 26** Suppose that $E(\lambda_0, ..., \lambda_n)$ is an extended Chebyshev system for the interval $[a, b]$ with Bernstein basis functions $p_{n,k}, k = 0, ..., n$. Assume that $E(\lambda_0, ..., \lambda_n)$ is closed under complex conjugation. Then the basis functions $p_{n,k}(x)$ are strictly positive on $(a, b)$ for each $k = 0, ..., n$. For each $k = 1, ..., n-1$ there exists a unique $a < t_k < b$ such that

- $p_{n,k}$ is increasing on $[a, t_k]$
- $p_{n,k}$ is decreasing on $[t_k, b]$. 

24
So \( p_{n,k} \) has exactly one relative maximum. The function \( p_{n,n} \) is either increasing or there exists \( t_0 \in (a, b) \) such that \( p_{n,n} \) is increasing on \([a, t_0]\) and decreasing on \([t_0, b]\).

**Proof.** 1. Consider \( f_{n,k} (x) = p_{n,k} (x) - \overline{p_{n,k} (x)} \) for \( k = 0, \ldots, n \). Then \( f_{n,k} \) has a zero of order at least \( k + 1 \) in \( a \) and a zero of order at least \( n - k \) in \( b \). Moreover \( f \in E(\lambda_0, \ldots, \lambda_n) \) since \( E(\lambda_0, \ldots, \lambda_n) \) is closed under complex conjugation. As \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system for \( \{a, b\} \) we infer \( f_{n,k} = 0 \), hence \( p_{n,k} \) is real-valued.

2. Since \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system for the interval \([a, b]\) it follows that \( p_{n,k} \) has no zeros in the open interval \((a, b)\). By the norming condition \( p_{n,k}^{(k)} (a) = 1 \) it follows that \( p_{n,k} \) is positive on \((a, b)\). Since the derivative \( p_{n,k}' \) is again in \( E(\lambda_0, \ldots, \lambda_n) \) it follows that \( p_{n,k}' \) has at most \( n \) zeros in \([a, b]\).

3. We assume that \( 1 \leq k \leq n - 1 \). Then \( p_{n,k}' \) has a zero order \( k - 1 \geq 0 \) at \( a \) and a zero of order \( n - k - 1 \) at \( b \). Hence \( p_{n,k}' \) has at most \( 2 \) zeros in the open interval \((a, b)\). First assume that \( p_{n,k}' \) has two different zeros \( t_0 \) and \( t_1 \). Then they must be simple. It follows that \( p_{n,k}' (t) > 0 \) for \( t \in [a, t_0] \) (recall that \( p_{n,k} \) is positive, so it must increase at first). For \( t \in [t_0, t_1] \) we have \( p_{n,k}' (t) < 0 \) since the zeros are simple. Finally we have \( p_{n,k}' (t) > 0 \) for \( t \in [t_1, b] \). So \( p_{n,k} \) is increasing on \([t_1, b]\), which implies \( p_{n,k} (t_1) \leq p_{n,k} (b) = 0 \) (recall that \( k \leq n - 1 \)). So \( p_{n,k} \) has an additional zero in \( t_1 \) which is a contradiction. Now assume that \( p_{n,k}' \) has a double zero at \( t_0 \). Since it has no further zeros \( p_{n,k}' (t) > 0 \) for all \( t \in (a, b) \), so \( p_{n,k} \) is monotone increasing, which gives a contradiction to the fact that \( p_{n,k} (b) = 0 \).

In the next case we assume that \( p_{n,k}' \) has no zero in \((a, b)\). Then \( p_{n,k} \) is strictly increasing, so \( p_{n,k} (b) > 0 \), which gives a contradiction. So we see that \( p_{n,k}' \) has exactly one zero in \((a, b)\).

5. Since \( p_{n,n}' \) has a zero order \( n - 1 \) at \( a \), it has at most one zero in \((a, b)\).

If \( p_{n,n}' \) has a zero then \( p_{n,n} \) is increasing on \([a, t_0]\) and decreasing on \([t_0, b]\). If \( p_{n,n}' \) has no zero then \( p_{n,n} \) is increasing.

We mention that \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system over intervals \([a, b]\) whose length \( b - a \) is sufficiently small. Moreover, for real eigenvalues \( \lambda_0, \ldots, \lambda_n \) the space \( E(\lambda_0, \ldots, \lambda_n) \) is an extended Chebyshev system over any interval \([a, b]\).

Simple experiments show that the assumption of a Chebyshev system over the interval \([a, b]\) are essential. In the case of the six eigenvalues

\[ \pm 7i, \pm i \left(7 - \pi\right), \pm i. \]
we have found basis function $p_{n,k}$ for the interval $[0, 3]$ or $[0, 3.14]$ with several relative maxima. Even it may be possible that $p_{n,k}$ are non-negative (but they are always real-valued). The fundamental function $\varphi_6(x)$ has its first zero at 3.2.

6 Recurrence relations of the Bernstein basis

Let us recall that $\Lambda_{2s+1} := (s + 1 \times \lambda, s + 1 \times \mu)$ and let us introduce the following notations

$$
\Lambda_{2s+} := (s + 1 \times \lambda, s \times \mu), \\
\Lambda_{2s-} := (s \times \lambda, s + 1 \times \mu).
$$

In this section we want to give recursive formula for the Bernstein basis

$$
p_{2s+1,k} := p_{\Lambda_{2s+1},k} \text{ and } p_{2s-,k} := p_{\Lambda_{2s-},k}.
$$

**Theorem 27** For each $k = 0, ..., 2s - 1$ the following recursion formula holds

$$
A_{\pm} p_{2s+1,k+2}(x) = x \cdot p_{2s-1,k} - (k + 1) \cdot p_{2s\pm,k+1}(x)
$$

where the constant $A_{\pm}$ is given through

$$
A_{\pm} = (k + 2) p_{2s-1,k}^{(k+1)}(0) - (k + 1) p_{2s\pm,k+1}^{(k+2)}(0).
$$

**Proof.** Let us recall that $p_{\Lambda_n,k}$ is the unique element in $E_{\Lambda_n}$ which has $k$ zeros in 0 and $n - k$ zeros in 1 with the normalization

$$
k! \lim_{x \to 0} \frac{p_{\Lambda_n,k}(x)}{x^k} = p_{\Lambda_n}^{(k)}(0) = 1. \tag{34}
$$

Consider $f_{\pm}(x) = a \cdot x \cdot p_{2s-2,k} + b \cdot p_{2s\pm,k+1}(x)$ for coefficients $a$ and $b$ which we want to define later. Clearly $f_{\pm}$ is in $E_{\Lambda_{2s+1}}$. Note that $p_{2s-1,k}$ has a zero of order $2s - 1 - k$ in 1, and that $p_{2s\pm,k+1}(x)$ has a zero of order $2s - (k + 1)$ in 1. Hence $f_{\pm}$ has a zero of order at least $2s - 1 - k$ in 1 (just as $p_{2s+1,k+2}$). For similar reasons we see that $f_{\pm}$ has a zero of order at least $k + 1$ in 0. We choose now the constants $a$ and $b$ in such a way that $f_{\pm}$ has a zero of order $k + 2$ in 0 and that $f$ satisfies the normalization in (34). By the uniqueness we infer that $f_{\pm} = p_{2s+1,k+2}$. Clearly $f_{\pm}$ has a zero of order $k + 2$ in 0 if

$$
a \cdot \left( \frac{d^{k+1}}{dx^{k+1}} [x \cdot p_{2s-1,k}] \right)_{x=0} + b \cdot p_{2s\pm,k+1}^{(k+1)}(0) = 0.
$$
Recall that \( p_{2s+1,k+1}^{(k+1)} (0) = 1 \) and 
\[
p_{2s-1,k} (x) = \frac{1}{k!} x^k + \frac{p_{2s-1,k}^{(k+1)} (0)}{(k+1)!} x^{k+1} + \frac{p_{2s-1,k}^{(k+2)} (0)}{(k+2)!} x^{k+1} + \ldots
\]

It follows that 
\[
\left( \frac{d^{k+1}}{dx^{k+1}} [x \cdot p_{2s-1,k} (x)] \right)_{x=0} = 1
\]

Thus \( b = -a (k+1) \). The normalization condition gives the second equation 
\[
1 = p_{2s+1,k+2}^{(k+2)} (0) = a \cdot \left( \frac{d^{k+2}}{dx^{k+2}} [x \cdot p_{2s-1,k}] \right)_{x=0} - a \cdot (k+1) p_{2s+1,k+1}^{(k+2)} (0).
\]

Again we see that 
\[
\frac{d^{k+2}}{dx^{k+2}} [x \cdot p_{2s-1,k} (x)]_{x=0} = (k+2) p_{2s-1,k}^{(k+1)} (0).
\]

So we obtain 
\[
1 = a \left( (k+2) p_{2s-1,k}^{(k+1)} (0) - (k+1) p_{2s+1,k+1}^{(k+2)} (0) \right).
\]

The proof is complete. □

**Theorem 28** For each \( k = 0, \ldots, 2s-3 \) holds the following recursion formula 
\[
A_k \cdot p_{2s+1,k+4} = x^2 \cdot p_{2s-3,k} + (k+1) (k+2) p_{2s-1,k+2} + B_k \cdot x \cdot p_{2s-1,k+2} \quad (35)
\]

where the constants \( A_k \) and \( B_k \) are defined by 
\[
B_k = \frac{k+2}{k+3} \left( (k+1) p_{2s-1,k+2}^{(k+3)} (0) - (k+3) p_{2s-3,k}^{(k+1)} (0) \right),
\]
\[
A_k = (k+3) (k+4) p_{2s-3,k}^{(k+2)} (0) - (k+1) (k+2) p_{2s-1,k+2}^{(k+4)} (0) + (k+4) p_{2s-1,k+2}^{(k+3)} (0) \cdot B_k.
\]

**Proof.** Consider \( f (x) = a \cdot x^2 \cdot p_{2s-3,k} + b \cdot p_{2s-1,k+2} + c \cdot x \cdot p_{2s-1,k+2} \) for coefficients \( a, b \) and \( c \) which we want to define later. Clearly \( f \) is in \( E_{A_{2s+1}} \). Note that the functions \( p_{2s-3,k}, p_{2s-1,k+2} \) and \( p_{2s-1,k+2} \) have a zero of order at least \( 2s-3-k \) in 1. Hence \( f \) has a zero of order at least \( 2s-3-k \) in 1 (just...
as $p_{2s+1,k+4}$). Clearly $f$ has a zero of order at least $k + 2$ in 0. We choose now the constants $a, b$ and $c$ in such a way that $f$ has a zero of order $k + 4$ in 0 and that $f$ satisfies the normalization in (34). By the uniqueness we infer that $f = p_{2s+1,k+4}$. Clearly $f$ has a zero of order $k + 4$ in 0 if and only if

$$f^{(k+2)} (0) = f^{(k+3)} (0) = 0.$$  

Recall that $p^{(k+2)}_{2s-1,k+2} (0) = 1$. Since $x \cdot p_{2s-1,k+2}$ has a zero of order $k + 3$ in 0 the equation $f^{(k+2)} (0) = 0$ is equivalent to

$$a \cdot \left( \frac{d^{k+2}}{dx^{k+2}} [x^2 \cdot p_{2s-3,k}] \right)_{x=0} + b = 0.$$ 

Consider the Taylor expansions of $p_{2s-3,k} (x)$ and $p_{2s-1,k+2}$

$$p_{2s-3,k} (x) = \frac{1}{k!} x^k + p^{(k+1)}_{2s-3,k} (0) x^{k+1} + \frac{p^{(k+2)}_{2s-3,k} (0)}{(k+1)!} x^{k+2} + \ldots$$

$$p_{2s-1,k+2} = \frac{1}{(k+2)!} x^{k+2} + \frac{p^{(k+3)}_{2s-1,k+2} (0)}{(k+3)!} x^{k+3} + \ldots$$

It follows that

$$\left( \frac{d^{k+2}}{dx^{k+2}} [x^2 \cdot p_{2s-3,k} (x)] \right)_{x=0} = \frac{d^{k+2}}{dx^{k+2}} \left[ \frac{1}{k!} x^{k+2} \right]_{x=0} = (k + 1) (k + 2).$$

Hence $b = -a (k + 1) (k + 2)$. The equation $f^{(k+3)} (0) = 0$ implies that

$$a \cdot \left( \frac{d^{k+3}}{dx^{k+3}} [x^2 \cdot p_{2s-3,k}] \right)_{x=0} + b p^{(k+3)}_{2s-1,k+2} + c \left( \frac{d^{k+3}}{dx^{k+3}} [x \cdot p_{2s-1,k+2}] \right)_{x=0} = 0.$$ 

Clearly

$$\left( \frac{d^{k+3}}{dx^{k+3}} [x^2 \cdot p_{2s-3,k}] \right)_{x=0} = \left( \frac{d^{k+3}}{dx^{k+3}} p^{(k+1)}_{2s-3,k} (0) x^{k+3} \right)_{x=0}$$

$$= (k + 3) (k + 2) p^{(k+1)}_{2s-3,k} (0)$$

and similarly

$$\left( \frac{d^{k+3}}{dx^{k+3}} [x \cdot p_{2s-1,k+2}] \right)_{x=0} = \left( \frac{d^{k+3}}{dx^{k+3}} \frac{1}{(k+2)!} x^{k+3} \right)_{x=0} = (k + 3).$$
So with $b = -a(k + 1)(k + 2)$ we obtain the equation

\[
a(k + 3)(k + 2)p_{2s-3,k}^{(k+1)}(0) - a(k + 1)(k + 2)p_{2s-1,k+2}^{(k+3)}(0) + c(k + 3) = 0,
\]

which shows that

\[
c(k + 3) = a(k + 2) \left( (k + 1)p_{2s-1,k+2}^{(k+3)}(0) - (k + 3)p_{2s-3,k}^{(k+1)}(0) \right).
\]

so $c = B_k$. The normalization condition gives the third equation

\[
1 = p_{2s+1,k+4}^{(k+4)}(0) = a \cdot \left( \frac{d^{k+4}}{dx^{k+4}} \left[ x^2 \cdot p_{2s-3,k}(x) \right] \right)_{x=0} \\
- a(k + 1)(k + 2)p_{2s-1,k+2}^{(k+4)}(0) + c \left( \frac{d^{k+4}}{dx^{k+4}} \left[ x \cdot p_{2s-1,k+2}(x) \right] \right)_{x=0}.
\]

Again we see that

\[
\left( \frac{d^{k+4}}{dx^{k+4}} \left[ x^2 \cdot p_{2s-3,k}(x) \right] \right)_{x=0} = \left( \frac{d^{k+4}}{dx^{k+4}} \left[ x^2 \cdot p_{2s-3,k}(x) \right] \right)_{x=0} = \left( \frac{d^{k+4}}{dx^{k+4}} \left[ \frac{p_{2s-3,k}^{(k+2)}(0)}{(k + 2)!} x^{k+4} \right] \right)_{x=0} = (k + 3)(k + 4)p_{2s-3,k}^{(k+2)}(0),
\]

and

\[
\left( \frac{d^{k+4}}{dx^{k+4}} \left[ x \cdot p_{2s-1,k+2}(x) \right] \right)_{x=0} = \left( \frac{d^{k+4}}{dx^{k+4}} \left[ x \cdot p_{2s-1,k+2}(x) \right] \right)_{x=0} = \left( \frac{d^{k+4}}{dx^{k+4}} \left[ \frac{p_{2s-1,k+2}^{(k+3)}(0)}{(k + 3)!} x^{k+4} \right] \right)_{x=0} = (k + 4)p_{2s-1,k+2}^{(k+3)}(0).
\]

So we arrive at the equation

\[
a^{-1} = (k + 3)(k + 4)p_{2s-3,k}^{(k+2)}(0) - (k + 1)(k + 2)p_{2s-1,k+2}^{(k+4)}(0) + (k + 4)p_{2s-1,k+2}^{(k+3)}(0) B_k.
\]

\textbf{Corollary 29} Suppose that $\mu = -\lambda$. Then the fundamental function $\Phi_{2s+1}(x)$ satisfies the following recursion

\[
\Phi_{2s+1}(x) = \frac{1}{4s(s - 1)} x^2 \cdot \Phi_{2s-3}(x) - \frac{2s - 1}{2s} \Phi_{2s-1}(x).
\]
**Proof.** Let us take \( k = 2s - 3 \) in equation (35): then \( p_{2s+1,2s+1} = \Phi_{2s+1} \) and \( p_{2s-3,2s-3} = \Phi_{2s-3} \) and \( p_{2s-1,2s-1} = \Phi_{2s-1} \). By Proposition 3

\[
\Phi_{2s-3}^{(2s-2)}(0) = (s - 2) \lambda + (s - 2) \mu = 0.
\]

and similarly \( \Phi_{2s-1}^{(2s)} = 0 \). Hence for \( k = 2s - 3 \) the last summand in (35) is zero and the formula amounts to

\[
A_{2s-3} \cdot \Phi_{2s+1} = x^2 \cdot \Phi_{2s-3} - (2s - 2) (2s - 1) \Phi_{2s-1}.
\]

We have to compute the constant \( A_k \) for \( k = 2s - 3 \). Since \( \Phi_{2s-1}^{(2s)}(0) = 0 \) and \( \Phi_{2s-3}^{(2s-2)}(0) = 0 \) the constant \( B_k \) is zero and we have

\[
A_{2s-3} = 2s (2s + 1) \Phi_{2s-3}^{(2s-1)}(0) - (2s - 2) (2s - 1) \Phi_{2s-1}^{(2s+1)}(0).
\]

Proposition 4 shows that \( \Phi_{2s+1}^{(2s+3)}(0) = s + 1 \), so \( \Phi_{2s-3}^{(2s-1)}(0) = s - 1 \) and we see that

\[
A_{2s-3} = 2s (2s + 1) (s - 1) - (2s - 2) (2s - 1) s = 2s (s - 1) (2s + 1 - 2s + 1) = 4s (s - 1).
\]

Hence

\[
\Phi_{2s+1} = \frac{1}{4s (s - 1)} x^2 \cdot \Phi_{2s-3} - \frac{2s - 1}{2s} \Phi_{2s-1}.
\]

Let us use the notation \( q_{2s+1,l} = p_{2s+1,2s+1-l} \). Then the recurrence relation in this notation means (with \( l = 2s + 1 - (k + 4) \), so \( k = 2s - 3 - l \) that

\[
A_{2s-3-l} \cdot q_{2s+1,l} = x^2 \cdot q_{2s-3,l} - (2s - 2 - l) (2s - 1 - l) \cdot q_{2s-1,l} + B_{2s-3-l} \cdot x \cdot q_{2s-1,l},
\]

so we have a recurrence relation for fixed \( l \).

7 **The derivative of the Bernstein operator**

Let \( \mathbb{C}^{\mathbb{N}_0} \) be the set of all sequences \( y = (y_0, y_1, \ldots) = (y_k) \) with complex entries. For \( y \in \mathbb{C}^{\mathbb{N}_0} \) we shall use the notation

\[
(y)_k := y_k
\]
for describing the $k$-th component of the vector $y$. The finite difference operator $\Delta : \mathbb{C}^{N_0} \to \mathbb{C}^{N_0}$ is defined for $y = (y_0, y_1, \ldots) = (y_k) \in \mathbb{C}^{N_0}$ by

$$(\Delta y)_k := y_{k+1} - y_k.$$ 

Higher differences are defined inductively by setting $\Delta^{n+1}y = \Delta(\Delta^n y)$ where $\Delta^0$ is defined as the identity operator. The difference operator $\Delta$ is useful in the classical theory to describe the derivative of the Bernstein polynomial

$$B_n f (x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k},$$

namely

$$\frac{d}{dx} (B_n f)(x) = n \sum_{k=0}^{n-1} \Delta \left[ f \left( \frac{k}{n} \right) \right] \binom{n-1}{k} x^k (1-x)^{n-1-k}.$$ 

We want to derive an analog for the Bernstein exponential polynomial.

We recall from [2] the following numbers

$$d_{\Lambda_n}^{k, \lambda_j} := \lim_{x \to b} \frac{d}{dx} p_{\Lambda_n, k} (x),$$

which have been important for the construction of the Bernstein operators. We shall assume that $\lambda_0 = 0$ which facilitates the formulas. We define a difference operator $\Delta_{\Lambda_n, \lambda_j} : \mathbb{C}^{N_0} \to \mathbb{C}^{N_0}$ for $y = (y_k) \in \mathbb{C}^{N_0}$ by

$$(\Delta_{\Lambda_n, \lambda_j} y)_k := d_{\Lambda_n}^{k, \lambda_j} \cdot y_k - d_{\Lambda_n}^{k, \lambda_0} \cdot y_{k+1}$$

By $\Lambda_n \setminus \lambda_j$ we denote the vector where we have deleted $\lambda_j$. Now we can prove

**Theorem 30** Let $\lambda_0 = 0$ and $j \in \{0, 1, \ldots, n\}$. For $B_{(\lambda_0, \ldots, \lambda_n)} f$ the following identity holds:

$$(\frac{d}{dx} - \lambda_j) B_{(\lambda_0, \ldots, \lambda_n)} f (x) = \sum_{k=0}^{n-1} \Delta_{\Lambda_n, \lambda_j} \left[ f (t_k) \right] \cdot \alpha_k \cdot p_{\Lambda_n \setminus \lambda_j, k} (x)$$
**Proof.** From the definition of the Bernstein operator we immediately see that
\[
\left( \frac{d}{dx} - \lambda_j \right) B_{(\lambda_0, \ldots, \lambda_n)} f(x) = \sum_{k=0}^{n} \alpha_k f(t_k) \left( \frac{d}{dx} - \lambda_j \right) p_{(\lambda_0, \ldots, \lambda_n),k}.
\]

By the next Theorem below we obtain
\[
\left( \frac{d}{dx} - \lambda_j \right) B_{(\lambda_0, \ldots, \lambda_n)} f(x) = \sum_{k=1}^{n} \alpha_k f(t_k) \frac{d^{k-1}}{dx^{k-1}} p_{(\lambda_0, \ldots, \lambda_n),k} + \sum_{k=0}^{n-1} \alpha_k f(t_k) d^{k}_{\lambda, j} p_{(\lambda_0, \ldots, \lambda_n),k}.
\]

From the construction of the Bernstein operator in [2] the following formula is known:
\[
\alpha_{k+1} = -\alpha_k d^{k}_{\lambda,0}.
\]
The proof is accomplished by identity
\[
\alpha_{k+1} f(t_{k+1}) + \alpha_k f(t_k) d^{k}_{\lambda, j} = \alpha_k \left[ -d^{k-1}_{\lambda, j} f(t_{k+1}) + f(t_k) d^{k}_{\lambda, j} \right].
\]

From [2] we repeat the following result:

**Proposition 31** Define for \( k = 0, \ldots, n - 1 \)
\[
d^{k}_{(\lambda_0, \ldots, \lambda_n)} := \lim_{x \to b} \frac{d}{dx} p_{(\lambda_0, \ldots, \lambda_n),k} (x)
\] (36)

Then, for any \( k = 1, \ldots, n - 1 \),
\[
\left( \frac{d}{dx} - \lambda_n \right) p_{(\lambda_0, \ldots, \lambda_n),k} = p_{(\lambda_0, \ldots, \lambda_{n-1}),k-1} + d^{k}_{(\lambda_0, \ldots, \lambda_n)} p_{(\lambda_0, \ldots, \lambda_{n-1}),k}.
\] (37)

Furthermore, for \( k = 0 \) we have
\[
\left( \frac{d}{dx} - \lambda_n \right) p_{(\lambda_0, \ldots, \lambda_n),0} = d^{\lambda_{0,0}}_{(\lambda_0, \ldots, \lambda_n)} p_{(\lambda_0, \ldots, \lambda_{n-1}),0},
\] (38)

while for \( k = n \),
\[
\left( \frac{d}{dx} - \lambda_n \right) p_{(\lambda_0, \ldots, \lambda_n),n} = p_{(\lambda_0, \ldots, \lambda_{n-1}),n-1}.
\] (39)
Proof. We may assume that \( a = 0 \). Set
\[ f_k := \left( \frac{d}{dx} - \lambda_n \right) p(\lambda_0, \ldots, \lambda_n), k, \]
and let \( 1 \leq k \leq n-1 \). Using the fact that \( f_k \) has a zero of order \( k-1 \) at 0 and of order \( n-k-1 \) at \( b \), it is easy to see that \( f_k = c_k p(\lambda_0, \ldots, \lambda_{n-1}), k-1 + d_k p(\lambda_0, \ldots, \lambda_{n-1}) \) for some constants \( c_k \) and \( d_k \). We want to show that \( c_k = 1 \). Note that \( p(\lambda_0, \ldots, \lambda_n), k \) has a zero of order \( k \) in 0, so
\[
\lim_{x \to 0} \frac{f_k(x)}{x^{k-1}} = \lim_{x \to 0} \frac{1}{x^{k-1}} \frac{d}{dx} p(\lambda_0, \ldots, \lambda_n), k \ (x) = \frac{p^{(k)}(\lambda_0, \ldots, \lambda_n), k (0)}{(k-1)!} = \frac{1}{(k-1)!}
\]
where the second equality follows from (26) applied to \( p(\lambda_0, \ldots, \lambda_n), k \ (x) \) and (27). On the other hand, the equation \( f_k = c_k p(\lambda_0, \ldots, \lambda_{n-1}), k-1 + d_k p(\lambda_0, \ldots, \lambda_{n-1}) \) shows that
\[
\lim_{x \to 0} \frac{f_k(x)}{x^{k-1}} = c_k \lim_{x \to 0} \frac{p(\lambda_0, \ldots, \lambda_{n-1}), k-1 \ (x)}{x^{k-1}} = c_k \frac{1}{(k-1)!},
\]
using again Proposition 23, (27). Hence \( c_k = 1 \). Next we divide \( f_k \) by \((b - x)^{n-k-1}\) to obtain
\[
\lim_{x \to b} \frac{d_k p(\lambda_0, \ldots, \lambda_{n-1}), k \ (x)}{(b - x)^{n-k-1}} = \lim_{x \to b} \frac{f_k(x)}{(b - x)^{n-k-1}} = \lim_{x \to b} \frac{\partial}{\partial x} p(\lambda_0, \ldots, \lambda_n), k \ (x) - \lambda_n p(\lambda_0, \ldots, \lambda_n), k \ (x)}{(b - x)^{n-k-1}}
\]
\[
= \lim_{x \to b} \frac{\partial}{\partial x} p(\lambda_0, \ldots, \lambda_n), k \ (x)}{(b - x)^{n-k-1}}.
\]
The case \( k = 0 \) follows by noticing that \( f_0 = d_0 p(\lambda_0, \ldots, \lambda_{n-1}), 0 \), solving for \( d_0 \) and taking the limit as \( x \uparrow b \), while the case \( k = n \) is an immediate consequence of the fact that \( p(\lambda_0, \ldots, \lambda_n), n = \Phi \Lambda_n \).

8 References

References

[1] G.E. Andrews, R. Askey, R. Toy, Special functions, Cambridge University Press, Cambridge, UK, 1999.
[2] J.M. Aldaz, O. Kounchev, H. Render, *Bernstein operators for exponential polynomials*, submitted

[3] J. M. Aldaz, O. Kounchev, H. Render, *On real-analytic recurrence relations for cardinal exponential B-splines*, to appear in J. Approx. Theory.

[4] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer Verlag, New York 1995.

[5] J. M. Carnicer, E. Mainar, J.M. Peña, *Critical Length for Design Purposes and Extended Chebyshev Spaces*, Constr. Approx. 20 (2004), 55–71.

[6] P. Costantini, *Curve and surface construction using variable degree polynomial splines*, Comput. Aided Geom. Design 17 (2000), 419–446.

[7] P. Costantini, T. Lyche, C. Manni, *On a class of weak Tchebycheff systems*, Preprint.

[8] C. de Boor, R. DeVore, A. Ron, *On the construction of multivariate (pre)wavelets*, Constr. Approx. 9 (1993), 123-166.

[9] M.-M. Derrienic, *On Multivariate Approximation by Bernstein-Type Polynomials*, J. Approx. Theory 45 (1985), 155–166.

[10] R.T. Farouki, T.N.T. Goodman, *On the optimal stability of the Bernstein basis*, Math. Comp. 65 (1996), 1553–1566.

[11] A.O. Gelfond, *On the generalized polynomials of S.N. Bernstein* (in Russian), Izv. Akad. Nauk SSSR, ser., math., 14 (1950), 413–420.

[12] T.N.T. Goodman, M.L. Mazure, *Blossoming beyond Chebyshev spaces*, J. Approx. Theory 109 (2001), 48–81.

[13] D. Gonsor, M. Neamtu, *Null spaces of Differential Operators, Polar Forms and Splines*, J. Approx. Theory 86 (1996), 81–107.

[14] I.I. Hirschmann, D.V. Widder, *Generalized Bernstein polynomials*, Duke Math. 16 (1949), 433–438.

[15] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Dehli 1960.
[16] Korzyk, Alexander D., Sr, A Forecasting Model for Internet Security Attacks; Holt-Level Adjusting Exponential Smoothing.

[17] O.I. Kounchev, Multivariate Polysplines. Applications to Numerical and Wavelet Analysis, Academic Press, London–San Diego, 2001.

[18] O. Kounchev, H. Render, Wavelet Analysis of cardinal $L$-splines and Construction of multivariate Prewavelets, In: Proceedings ”Tenth International Conference on Approximation Theory”, St. Louis, Missouri, March 26-29, 2001.

[19] O. Kounchev, H. Render, The Approximation order of Polysplines, Proc. Amer. Math. Soc. 132 (2004), 455-461.

[20] O. Kounchev, H. Render, Cardinal Interpolation with polysplines on annuli, J. Approx. Theory, 137 (2005), 89–107.

[21] Y. Li, On the Recurrence Relations for B-Splines Defined by Certain $L$-splines, J. Approx. Theory 43 (1985), 359–369.

[22] G.G. Lorentz, Bernstein polynomials, Chelsea Publishing Company, New York 1986 (2nd edition).

[23] T. Lyche, L.L. Schumaker, $L$-spline wavelets, In: Wavelets: Theory, Algorithms, and Applications (Taormina, 1993), Acad. Press, San Diego, CA, 1994, pp. 197–212.

[24] E. Mainar, J.M. Peña, J. Sánchez-Reyes, Shape perserving alternatives to the rational Bézier model, Comput. Aided Geom. Design 14 (1997), 5–11.

[25] M. Mazure, Bernstein bases in Müntz spaces, Numerical Algorithms 22 (1999), 285–304.

[26] M. Mazure, Chebychev Spaces and Bernstein bases, Constr. Approx. 22 (2005), 347–363.

[27] Ch. Micchelli, Cardinal $L$-splines, In: Studies in Spline Functions and Approximation Theory, Eds. S. Karlin et al., Academic Press, NY, 1976, pp. 203-250.
[28] S. Morigi, M. Neamtu, Some results for a class of generalized polynomials, Adv. Comput. Math. 12 (2000), 133–149.

[29] J.M. Peña, On the optimal stability of bases of univariate functions, Numer. Math. 91 (2002), 305–318.

[30] L.L. Schumaker, Spline Functions: Basic Theory”, Interscience, New York, 1981.

[31] L.L. Schumaker, On hyperbolic splines, J. Approx. Theory 38 (1983), 144–166.

[32] M. Unser, T. Blu, Cardinal Exponential Splines: Part I – Theory and Filtering Algorithms, IEEE Transactions on Signal Processing, 53 (2005), 1425–1438.

[33] J. Zhang, C-curves: an extension of cubic curves, Comput. Aided Geom. Design 13 (1996), 199–217.