Mathematical foundations for field theories on Finsler spacetimes

Manuel Hohmann,1, 2 Christian Pfeifer,3 and Nicoleta Voicu4, 2

1) Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia
2) Lepage Research Institute, 17. novembra 1, 08116 Prešov, Slovakia
3) ZARM, University of Bremen, 28359 Bremen, Germany
4) Faculty of Mathematics and Computer Science, Transilvania University, Iuliu Maniu Str. 50, 500091 Brasov, Romania

(*Electronic mail: nico.voicu@unitbv.ro)
(*Electronic mail: christian.pfeifer@zarm.uni-bremen.de)
(*Electronic mail: manuel.hohmann@ut.ee)

The paper introduces a general mathematical framework for action based field theories on Finsler spacetimes. As most often fields on Finsler spacetime (e.g., the Finsler fundamental function or the resulting metric tensor) have a homogeneous dependence on the tangent directions of spacetime, we construct the appropriate configuration bundles whose sections are such homogeneous fields; on these configuration bundles, the tools of coordinate free calculus of variations can be consistently applied to obtain field equations. Moreover, we prove that general covariance of natural Finsler field Lagrangians leads to an averaged energy-momentum conservation law which, in the particular case of Lorentzian spacetimes, is equivalent to the usual, pointwise energy-momentum covariant conservation law.

Keywords: Finsler spacetime, projectivized tangent bundle, fibered manifold, Euler-Lagrange operator, energy-momentum distribution tensor
MSC2020: 83D05, 58A20, 53B40

CONTENTS

I. Introduction 2

II. Pseudo-Finsler spaces and Finsler spacetime manifolds 4
   A. The notion of Finsler spacetime 4
   B. Geometric objects on Finsler spacetimes 7
   C. Homogeneous geometric objects on $TM$ 10

III. The positively projectivized tangent bundle $PTM^+$ 13
   A. Definition and structure over general manifolds 13
      1. Definition and structure 13
      2. From $PTM^+$ to $TM$ and back 15
   B. Over Finsler spacetimes ($M, L$) 16
      1. Finsler Geometry on $PTM^+$ and volume forms 16
      2. Integration on $PTM^+$ and integration on observer space 19

IV. Fibered manifolds and fields over a Finsler spacetime 20
   A. Fibered manifolds over $PTM^+$ 20
(Pseudo-)Finsler geometry is the most general geometry admitting a parametrization-invariant arc length of curves. It generalizes Riemannian geometry by using as its fundamental, geometry-defining object, a general line element - which does not necessarily arise as the square root of any quadratic expression in the velocity components, but is just a homogeneous expression of degree one in these. Historically, already Riemann himself introduced this concept in his habilitation lecture\(^1\),\(^2\), however only Finsler investigated it more deeply\(^3\). Nowadays Finsler geometry is an established field in mathematics\(^4\),\(^5\).

In physics, pseudo-Riemannian geometry is used to describe one of the four fundamental interactions, gravity. In general relativity, gravity is encoded in the Lorentzian geometry of the four-dimensional spacetime manifold, which is determined by the matter content of spacetime via the Einstein equations\(^6\). The idea to use geometry based on non-quadratic line elements to describe physical interactions goes far back, at least to Randers\(^7\), who used, in addition to a metric, a 1-form to search for a unified geometric description of gravity and electromagnetism. Since then, numerous applications of Finsler geometry in physics emerged\(^8\),\(^9\), for example in the geometric description of fields in media\(^10\)–\(^15\), to study non-local Lorentz invariant extensions of fundamental physics\(^16\)–\(^26\), and to find extensions and modifications of general relativity for an improved description of gravity\(^27\)–\(^32\), that might explain dark matter or dark energy geometrically\(^33\)–\(^38\).

From the mathematical point of view, a major difficulty in the formulation of pseudo-Finsler geometry as generalization of pseudo-Riemannian geometry is the existence, in each tangent space, of vectors along which the geometry defining function is either non-smooth or leading to a degenerate metric tensor. One of the first attempts to construct mathematically well defined Lorentz-Finsler spacetimes goes back to Beem\(^39\). It turned out that Beem’s definition was to restrictive to cover all cases one is interested in physics and numerous extensions and refinements have been discussed\(^36\),\(^38\),\(^40\)–\(^44\).
On the basis of the improved modern Finsler spacetime definitions, it has recently been suggested that the gravitational field of kinetic gases can be described by Finsler spacetime geometry. Motivated by the coupling of the kinetic gas to Finsler spacetime geometry and by the recent example of a Finslerian vacuum action, we introduce a general framework for mathematically consistent action-based field theories over Finsler spacetimes. The main difficulty for such a construction is that, most often, fields on a Finsler spacetime (e.g., the fundamental function $L$, or the resulting metric tensor) have a non-trivial homogeneous dependence on the tangent directions of spacetime. For such homogeneous geometric objects, the naive formulation of the calculus of variations - on configuration manifolds sitting over the tangent bundle - is not well defined, since the variation itself possesses the same homogeneity as the original field; thus, imposing the variation to vanish on the boundary of the arbitrary compact integration domain would typically force the variation to identically vanish, also inside this domain. Therefore, in order to correctly apply the apparatus of the calculus of variations, one must first carefully choose the fiber bundles serving as configuration manifolds for the theory, in such a way as to naturally and consistently accommodate homogeneity.

The main goal of this article is to provide a general construction of configuration bundles, define action integrals for homogeneous fields, in the just explained sense, over Finsler spacetimes and to apply the coordinate free calculus of variations to obtain field equations for the fields in a mathematically rigorous way. Also, we prove that invariance of the corresponding field Lagrangians under lifted spacetime diffeomorphisms leads to an averaged energy-momentum conservation law, which generalizes the (pointwise) energy-momentum conservation law in general relativity.

To achieve this, we proceed as follows:

Section II reviews the notion of Finsler spacetimes and the geometry of Finsler spacetimes, on which our later construction is based. We also give a brief discussion on the different definitions, which can be found in the literature. Most importantly, we discuss the homogeneity properties of the appearing geometric objects.

Section III presents the positively projectivized tangent bundle $PTM^+$ (also called in the literature on positive definite Finsler spaces, the projective sphere bundle), which will serve as base manifold for action integrals for field theories on Finsler spacetimes. We first discuss in detail the general concept of $PTM^+$ without any geometric fields on the manifold in consideration, and then, how Finsler geometry can be understood on $PTM^+$. Albeit this is still a preparatory topic, it is treated in quite some detail, since a systematic analysis of $PTM^+$ and of the various structures it gives rise to, both on general manifolds and on pseudo-Finsler spaces, seems to be missing in the literature.

Having set the stage, we use $PTM^+$ as base manifold for general physical fields having a homogeneous dependence on the direction; these fields are modeled as sections into configuration bundles over $PTM^+$ in Section IV. We introduce the corresponding configuration bundles and fibered automorphisms thereof, that serve in deforming sections and thus give rise to variations.

Eventually, Section V combines all the previous concepts to write down the general form of well defined action integrals for homogeneous fields on Finsler spacetime. Once this is done, their field equations are then obtained by the standard techniques of calculus of variations - discussed here in a coordinate-free form.

In Section VI, we derive the response of Lagrangians to compactly supported diffeomorphisms on the spacetime manifold, which leads to the novel notion of an energy-momentum distribution tensor. It satisfies an averaged covariant conservation law and can be integrated to an energy-momentum tensor density on spacetime. Only in very special cases, in particular in the case of a Lorentzian spacetime geometry, this energy-momentum tensor density can be "un-densitized" to yield an energy-momentum tensor on the base manifold.

The necessary notions of geometric calculus of variations (jet bundles over fibered manifolds, fibered
automorphisms, the first variation formula in terms of differential forms and their Lie derivatives) are briefly presented in Appendix A.

II. PSEUDO-FINSLER SPACES AND FINSLER SPACETIME MANIFOLDS

We begin this article by presenting a precise definition of Finsler spacetimes, on which we will base the presentation and discussion of this article. We will comment on its relation to other definitions of Finsler spacetimes given in the literature, highlight the importance of the details in the definition which ensure the existence of a well defined causal structure and discuss some classes of examples. Moreover, we briefly review the geometric notions on Finsler spacetimes such as connections and curvature.

A. The notion of Finsler spacetime

Let $M$ be a connected, orientable smooth manifold and $TM$, its tangent bundle with projection $\pi_{TM}: TM \to M$. We will denote by $x^i$ the coordinates in a local chart on $M$ and by $(x^i, \dot{x}^i)$, the naturally induced local coordinates of points $(x, \dot{x}) \in TM$. Whenever there is no risk of confusion, we will omit the indices, i.e., write $(x, \dot{x})$ instead of $(x^i, \dot{x}^i)$. Commas, $\partial$, will mean partial differentiation with respect to the base coordinates $x^i$ and dots, $\cdot$, partial differentiation with respect to the fiber coordinates $\dot{x}^i$. Also, by $\mathcal{A} = TM \setminus \{0\}$, we will mean the tangent bundle of $M$ without its zero section.

A conic subbundle of $TM$ is a non-empty open submanifold $\mathcal{D} \subset TM \setminus \{0\}$, with the following properties:

- $\pi_{TM}(\mathcal{D}) = M$;
- conic property: if $(x, \dot{x}) \in \mathcal{D}$, then, for any $\lambda > 0 : (x, \lambda \dot{x}) \in \mathcal{D}$.

A pseudo-Finsler space is, a triple $(M, \mathcal{A}, L)$, where $M$ is a smooth manifold, $\mathcal{A} \subset \mathcal{O}$ is a conic subbundle and $L : \mathcal{A} \to \mathbb{R}$ is a smooth function obeying the following conditions:

1. positive 2-homogeneity: $L(x, \alpha \dot{x}) = \alpha^2 L(x, \dot{x}), \forall \alpha > 0, \forall (x, \dot{x}) \in \mathcal{A}$.
2. at any $(x, \dot{x}) \in \mathcal{A}$ and in one (and then, in any) local chart around $(x, \dot{x})$, the Hessian:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$$

is nondegenerate.

The conic subbundle $\mathcal{A}$, where $L$ is defined, smooth and with nondegenerate Hessian, is called the set of admissible vectors. In the following, we will consider as $\mathcal{A}$, the maximal set with these properties - hence, we will write simply $(M, \mathcal{A}, L)$ instead of $(M, \mathcal{A}, L)$.

Another important conic subbundle in a pseudo-Finsler space is the set of non-null admissible vectors:

$$\mathcal{A}_0 := \mathcal{A} \setminus L^{-1}\{0\}. \quad (1)$$

This is the set where we can divide by $L$ in order to adjust the homogeneity degree of geometric objects in $\dot{x}$. 
Definition 1 (Finsler spacetimes) A Finsler spacetime is a 4-dimensional, connected pseudo-Finsler space obeying the extra condition:

3. There exists a conic subbundle \( T \subset A \) with connected fibers \( T_x, x \in M \), such that, on \( T \) : \( L > 0 \), \( g \) has Lorentzian signature \((+,−,−,−)\) and \( L \) can be continuously extended as 0 to the boundary \( \partial T \).

The role of condition 3 is to ensure the existence of a proper causal structure for \((M,L)\).

In the following, though we will not specify this explicitly, we will always consider that \( L \) is continuously prolonged as 0 on \( \partial T \); in particular, \( L(0) = 0 \).

The above definition is a minor modification of the one introduced in\(^{45}\) to ensure that the cone of timelike vectors is salient at each point. It recovers, in the particular case \( A := T \), the definition of improper Finsler spacetimes in\(^{44}\). The existence and uniqueness of geodesics with given initial conditions \((x,\dot{x}) \in \mathcal{T}\), which was explicitly required in an older version of this definition in\(^{43}\), follows from the axioms 1−3 above, see\(^{44}\), and thus the definition presented here also covers the Finsler spacetimes discussed in\(^{46,47}\).

In principle it would be possible to include directions in \( \mathcal{T} \) that are not in \( A \), but just in \( A_0 \), see e.g.,\(^{49,50}\) yet, for our purposes, it will be more convenient to assume that \( \mathcal{T} \subset A \), in order to avoid unnecessary complications in variational procedures involving \( \mathcal{T} \) or subsets thereof.

Timelike vectors and the observer space.

For the application of Finsler spacetimes in physics, besides the sets of admissible (respectively, non-null admissible) directions \( A \) and \( A_0 \), the following subsets of \( TM \) play an important role:

1. The conic subbundle \( \mathcal{T} \), called the set of future pointing timelike vectors.

2. The observer space, or set of unit future-pointing timelike directions

\[ \mathcal{O} := \{(x,\dot{x}) \in \mathcal{T} \mid L(x,\dot{x}) = 1\}, \]

which satisfies the inclusion \( \mathcal{O} \subset \mathcal{T} \subset A_0 \subset A \). Moreover, due to the homogeneity of \( L \), we have at any \( x \in M \):

\[ \mathcal{T}_x = (0,\infty) \cdot \mathcal{O}_x. \]

3. The set \( \mathcal{N} := L^{-1}\{0\} \) has the meaning of set of null or lightlike vectors. By continuously extending \( L \) as zero to the boundary \( \partial \mathcal{T} \), as specified above, we always have the inclusion

\[ \partial \mathcal{T} \subset \mathcal{N}. \]

It is important to notice that the null cone \( \mathcal{N} \) might not be contained in \( A \), but just in \( \overline{A} \).

The null boundary condition (4) ensures that, for every \( x \in M \), the future timelike cone \( \mathcal{T}_x \) is actually an entire connected component of \( L^{-1}((0,\infty)) \cap T_x M \). This leads to the following consequences.

Proposition 2 At any \( x \in M \), the observer space \( \mathcal{O}_x \) is a connected component of the indicatrix \( I_x = L^{-1}(1) \cap T_x M \).

---

1 Physically, this ensures the existence of a global time orientation.
FIG. 1. Future pointing timelike vectors $\mathcal{I}$, observer space $\mathcal{O}$ and null directions $\mathcal{N}$ of a double null-cone Finsler spacetime, in the tangent space at $x \in M$.

This statement follows immediately from the connectedness and maximality of $\mathcal{I}$.

As a consequence of the maximal connectedness of $\mathcal{O}$, a result by Beem, ensures that $\mathcal{O}$ is a strictly convex hypersurface of $T_xM$ and moreover, the set

$$C_x := \mathcal{I} \cap L^{-1}([1, \infty)) = [1, \infty) \cdot \mathcal{O}_x$$

is also convex. Based on this, we can state:

**Proposition 3** In a Finsler spacetime as defined above, all future timelike cones $\mathcal{I}_x$, $x \in M$, are convex.

**Proof.** Fix $x \in M$ and consider some arbitrary $u, v \in \mathcal{I}$, $\alpha \in [0, 1]$. In order to show that $w := (1 - \alpha)u + \alpha v$ lies in $\mathcal{I}$, we rescale it by $\beta \geq \max(L(u)^{-1/2}, L(v)^{-1/2})$; this way, the endpoints $\beta u, \beta v$ lie in $C_x$ and, by the convexity of $C_x$, we find that $\beta w \in C_x \subset \mathcal{I}$. The statement then follows from the conicity of $\mathcal{I}$.

The above result generalizes a similar statement in $\cite{43}$, which was proven under the more restrictive condition that $L$ is defined and continuous on the entire $TM$. The convexity of the cones $\mathcal{I}$ ensures a well defined causal structure on a Finsler spacetime and that timelike geodesics locally extremize the length between timelike separated points.

The Finslerian pseudo-norm, which defines the canonical geometric length measure for curves on a Finsler spacetime, is defined by

$$F := \sqrt{|\mathcal{L}|},$$

which implies

$$L = \epsilon F^2, \quad \epsilon = \text{sign}(L).$$

**Examples of Finsler spacetimes**

The above definition allows for Finsler spacetimes of:

1. Lorentzian type. If $a : M \to T_x^0M$, $x \mapsto a_x = a_{ij}(x)dx^i \otimes dx^j$ is a Lorentzian metric on $M$, then, we can set $\mathcal{A} = TM$, as $L : TM \to \mathbb{R}$, $L(x, \dot{x}) = a_x(\dot{x}, \dot{x})$ is smooth on $TM$. Accordingly, $F(x, \dot{x}) = \sqrt{|a_{ij}(x)\dot{x}^i\dot{x}^j|}$. 


2. Randers type, given by $F(x, \dot{x}) = \sqrt{[a_{ij}(x)] \dot{x}^i \dot{x}^j} + b_i(x) \dot{x}^i$, where $a$ is as above and $b = b_i dx^i$ is a differential 1-form on $M$. We proved in that, if $a^{ij} b_{ij} \in (0, 1)$ then $F$ provides a Finsler spacetime structure on $M$. In the context of physics, these geometries are employed to study the motion of an electrically charged particle in an electromagnetic field, the propagation of light in static spacetimes. Recently also spinors have been constructed on Randers geometries in terms of Clifford bundles.

3. Bogoslovsky/Kropina type $F = (\sqrt{a_{ij}(x) s^i s^j})^{1/2} (b_i(x) s^i)^q$, where $q \in \mathbb{R}$; the conditions upon the 1-form $b$, such that $F$ defines a spacetime structure depend on the value of $q$; a detailed discussion is made in. In physics, approaches to quantum field theories and modifications of general relativity that are only invariant under a subgroup of the Lorentz group, based on this geometry, have been investigated under the name very special and very general relativity.

4. Polynomial $m$-th root type $F = \sqrt[2m]{G_{\alpha_1...\alpha_m}(x) \dot{x}^{\alpha_1} \ldots \dot{x}^{\alpha_m} \bar{\eta}}$, which appear in physics for example in the description of propagation in birefringent media, in the context of premetric electrodynamics and in the minimal standard model extension.

5. Anisotropic conformal transformations of pseudo-Riemannian geometry $F = e^{\alpha(x, \dot{x})} \sqrt{[a_{ij}(x)] \dot{x}^i \dot{x}^j}$ have been studied in the context of an extension of the Ehlers-Pirani-Schild axiomatic to Finsler geometry, as well as examples for Finsler spacetimes according to Beem’s definition.

6. General first order perturbations of $F = \sqrt{[a_{ij}(x)] \dot{x}^i \dot{x}^j} + \epsilon h(x, \dot{x})$ pseudo-Riemannian geometry, which are often used in the study of the physical phenomenology of Planck scale modified dispersion relations.

7. Finsler spacetime metrics $L(x, \dot{x}) = \omega(x) - \tilde{F}^2(x, \dot{x})$ obtained, by “signature-reversing” from positive definite Finsler ones $\tilde{F} : TM \to \mathbb{R}$ using a nonzero 1-form $\omega \in \Omega^1(M)$. Their (2-homogeneous) Finsler function $L$ is well defined and smooth on the entire slit tangent bundle - which is quite a rare feature among pseudo-Finsler metrics. The set $\tilde{F} = \{(x, \dot{x}) : \omega(x) \geq \tilde{F}(x, \dot{x})\}$ satisfies the requests for the set of future-pointing timelike vectors.

B. Geometric objects on Finsler spacetimes

Typical Finslerian objects in a Finsler spacetime $(M, L)$ (more generally, in a pseudo-Finsler space) are obtained similarly to the corresponding objects in positive definite Finsler spaces $(M, F)$, see, e.g., just taking care that we have to restrict them to $\mathcal{A}$ or, if necessary, to $\mathcal{A}_0$.

Apart from the Finslerian pseudo-norm $F$, the fundamental building blocks of the geometry of Finsler spacetimes are obtained from partial derivatives of the Finsler Lagrangian function $L$. Below, we briefly present the coordinate expressions of the typical Finslerian geometric objects to be used in the following.

**Hilbert form, Finsler metric tensor and Cartan tensor**

On a Finsler spacetime $(M, L)$ the Hilbert form $\omega : \mathcal{A}_0 \to T^*_1 M$, the Finslerian metric tensor $g : \mathcal{A} \to T^*_2 M$.
and the Cartan tensor $C : \mathcal{A} \to T^3_1 M$ are expressed, in every manifold induced local coordinate chart, as

$$\omega_{(x,i)} := F_i dx^i,$$

$$g_{(x,i)} := g_{ij}(x,\dot{x})dx^i \otimes dx^j,$$

$$C_{(x,i)} := C_{ijk}(x,\dot{x})dx^i \otimes dx^j \otimes dx^k,$$

$$F_i = \varepsilon g_{ij}\dot{x}^j,$$

$$g_{ij} := \frac{1}{2} L_{ij},$$

$$C_{ijk} := \frac{1}{4} L_{ijk}. \tag{9}$$

We note that the Hilbert form $\omega$ is only defined on $\mathcal{A}_0$ (as it involves derivatives of $F = \sqrt{|L|}$, which are not defined at points where $L = 0$), while the Finsler metric and the Cartan tensor are defined on $\mathcal{A}$.

A curve $c : [a,b] \to M$ is called admissible if all its tangent vectors are in $\mathcal{A}$. The arc length of a regular admissible curve $c : t \in [a,b] \mapsto c(t)$ on $M$ is calculated as

$$l(c) = \int_a^b F(c(t),\dot{c}(t)) dt, \tag{10}$$

where $\dot{c}(t) = \frac{dc}{dt}(t)$. If, moreover, $\dot{c}(t)$ is nowhere lightlike, i.e., $(c(t),\dot{c}(t)) \in \mathcal{A}_0$ for all $t$, then $l(c)$ can also be expressed in terms of the Hilbert form as:

$$l(c) = \int_a^b C^* \omega = \int_{\text{int } C} \omega, \tag{11}$$

where the symbol $C : [a,b] \to TM, t \mapsto (c(t),\dot{c}(t))$ denotes here the canonical lift of $c$ to $TM$.

**Proposition 4 (Finsler geodesics)**, see, e.g., 5: Critical points of the length functional (10) are called Finsler geodesics. In an arc length parametrization, they are determined by the Finsler geodesic equation

$$\ddot{x}(s) + 2G^i(x(s),\dot{x}(s)) = 0, \tag{12}$$

where $\dot{x}(s) = \frac{dx}{ds}(s)$; the geodesic coefficients are well defined at all points $(x,\dot{x}) \in \mathcal{A}$ and given by

$$2G^i(x,\dot{x}) = \frac{1}{2} g^{ih}(L_{h,k}\dot{x}^k - L_{hk}). \tag{13}$$

A nonlinear connection will be understood as a connection on the fibered manifold $\mathcal{A}$ in the sense of 66 (pp. 30-32), i.e., as a splitting of the tangent bundle $T\mathcal{A}$ of $\mathcal{A}$,

$$T\mathcal{A} = H\mathcal{A} \oplus V\mathcal{A}. \tag{14}$$

The vertical subbundle $V\mathcal{A} = \ker d(\pi_{TM}|_{\mathcal{A}})$ and the horizontal subbundle $H\mathcal{A}$ are vector subbundles of the tangent bundle $(T\mathcal{A}, \pi_{T\mathcal{A}}, \mathcal{A})$. The local adapted basis will be denoted by $(\delta_i, \delta^i)$, where $\delta_i := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial \dot{x}^j}$, $\dot{\delta}_i = \frac{\partial}{\partial \dot{x}^i}$ and its dual basis, by $(dx^i, \delta x^i = d\dot{x}^i + G^j_i dx^j)$. Here $G^j_i$ are general local connection coefficients.

We denote by $\delta$ and $\theta$ the horizontal and, accordingly, the vertical projector determined by the nonlinear connection; that is, for any vector $X \in T\mathcal{A}$, locally written as $X = X^i \delta_i + X^i \dot{\delta}_i$ we will have: $\delta X = X^i \delta_i$ and $\theta X = X^i \dot{\delta}_i$. 
**Cartan (canonical) nonlinear connection.** The Cartan nonlinear connection $N$ on a Finsler spacetime $(M, L)$ is defined by the local connection coefficients

$$G^i_{jk} = G^j_{ik}, \quad (14)$$

Arc-length parametrized geodesics of the Finsler spacetime $(M, L)$ are autoparallel curves of the canonical nonlinear connection.

**Nonlinear curvature tensor and Finsler Ricci scalar**

The curvature tensor of the canonical nonlinear connection on a Finsler spacetime $(M, L)$ is a tensor on $TM$, which has the coordinate expression:

$$R^i_{jk} dx^j \wedge dx^k \otimes \dot{h} = dx^j \wedge dx^k \otimes [\delta_j, \delta_k] = (\delta_k G^i_{j} - \delta_j G^i_{k}) dx^j \wedge dx^k \otimes \dot{h}. \quad (15)$$

The *Finsler-Ricci scalar* $R_0$ makes sense on $\mathcal{M}_0$ and is given by

$$R_0 = \frac{1}{L} R^i_{jk} x^i. \quad (16)$$

The way introduced it above is equal to minus the one employed in (14) (and denoted by $Ric$).

Besides the canonical nonlinear connection, it is possible to additionally define several linear connections on $\mathcal{M}$, which preserve the distributions generated by the nonlinear connection. In this article we will pick, for simplicity, one of these linear connections as a mathematical tool to ensure that all objects we are dealing with are well defined tensors. Our particular choice of the linear connection is unessential, since it is just an auxiliary tool. The whole construction is independent of the typical Finslerian linear connections that one may use.

**Chern-Rund linear connection**

The Chern-Rund linear covariant derivative on a Finsler spacetime $(M, L)$, defined on $\mathcal{M} \subset TM$, is locally given by the relations

$$D_{\delta_k} \delta_j = \Gamma^i_{jk} \dot{\delta}_i, \quad D_{\delta_k} \dot{\delta}_j = \Gamma^i_{jk} \dot{\delta}_i, \quad D_{\delta_k} \dot{\delta}_j = D_{\delta_k} \dot{\delta}_j = 0, \quad (17)$$

where $\Gamma^i_{jk} := \frac{1}{2} g^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}).$ We denote by $D$-covariant differentiation with respect to $\delta$.

The Chern-Rund linear covariant derivative allows us to introduce the *dynamical covariant derivative* in a very simple way, namely, as $\nabla : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ with $\nabla = \dot{x}^i D_i$. An important remark is that, since $\dot{x}^i \Gamma^k_{ij} = G^k_{ij}$, the dynamical covariant derivative only depends on the canonical nonlinear connection $N$, see (it can actually be introduced independently of $D$ or of any other additional structure).

The dynamical covariant derivative can be used to define a measure of the change of the Cartan tensor along horizontal curves, called the Landsberg tensor, see (4).

**Landsberg tensor**

The Landsberg tensor $P = P^i_{jk} dx^j \otimes dx^k \otimes \dot{h}$ is a tensor on $TM$, defined, in any local chart, by:

$$P^i_{jk} = g^{mi} \nabla C_{mjk} = G^i_{j} - \Gamma^i_{jk}. \quad (18)$$

Its trace is denoted by $P_i = P^i_{jj}$.

The following identities will be useful when we consider action integrals and calculus of variations on Finsler spacetimes:

$$\delta_i L = L_i = 0, \quad g_{ij} = 0, \quad \dot{x}^i_{,j} = 0, \quad (19)$$

$$\nabla L = 0, \quad \nabla g_{ij} = 0, \quad \nabla \dot{x}^i = 0, \quad (20)$$

$$P^i_{jk} \delta^k = 0, \quad P_{i} \dot{x}^i = 0.$$
They can all be proven by using the homogeneity properties of the tensors involved and the definition of the canonical nonlinear connection in terms of the Finsler Lagrangian.

(Semi)-Riemannian geometry as Finsler geometry

Choosing \( L = g_{ij}(x)x^i\dot{x}^j \), the geometry of a Finsler spacetime \((M, L)\) becomes essentially the geometry of the pseudo-Riemannian spacetime manifold \((M, g)\). In this case, \( G'_{ij} = g_{ij}(x)x^k\dot{x}^l \) and \( R'_{ijk} = r'_{ijkl}(x)\dot{x}^l \) (where we have denoted by small letters the geometric objects specific to Riemannian geometry). The relation between the Finsler-Ricci scalar \( R_0 \) and the usual Riemannian one \( r = g^{ij}r_{ij} \) is: \( g^{ij}(LR_0)_{x^j} = -2r \).

C. Homogeneous geometric objects on \( TM \)

Homogeneity is a key concept in pseudo-Finslerian geometry, as the positive homogeneity of \( L \) in \( \dot{x} \) entails the positive homogeneity of all typical Finslerian geometric objects. We will briefly present here some results on homogeneous geometric objects defined on conic subbundles \( \mathcal{Q} \subset \overset{\circ}{TM} \). The results are straightforward extensions of the results in \( \text{i} \) and \( \text{67} \), referring to objects defined on the whole slit tangent bundle.

**Definition 5 (Fiber homotheties)** By fiber homotheties on \( \overset{\circ}{TM} \), we understand the mappings \( \chi_\alpha : \overset{\circ}{TM} \to \overset{\circ}{TM} \), \( \chi_\alpha(x, \dot{x}) = (x, \alpha \dot{x}) \), where \( \alpha > 0 \).

Fiber homotheties form a 1-parameter group of diffeomorphisms of \( \overset{\circ}{TM} \), isomorphic to \( (\mathbb{R}^+, \cdot) \) and are generated by the Liouville vector field

\[
C = \dot{x}^i \partial_i.
\]

We denote the corresponding group action by \( \chi \), i.e.:

\[
\chi : \overset{\circ}{TM} \times \mathbb{R}^+ \to \overset{\circ}{TM}, \quad \chi((x, \dot{x}), \alpha) = \chi_\alpha(x, \dot{x}).
\]  

(21)

**Definition 6 (Homogeneous tensor field)** Let \( T \) be a tensor field over the conic subbundle \( \mathcal{Q} \subset \overset{\circ}{TM} \). \( T \) is called positively homogeneous of degree \( k \in \mathbb{R} \) if and only if, for all \( \alpha > 0 \), its pullback along the restriction \( \chi_\alpha : \mathcal{Q} \to \mathcal{Q} \) satisfies

\[
\chi_\alpha^* T = \alpha^k T.
\]  

(22)

**Theorem 7** A tensor field \( T \) over \( \mathcal{Q} \) is positively homogeneous of degree \( k \in \mathbb{R} \) if and only if

\[
\mathcal{L}_C T = kT.
\]  

(23)

**Proof.** See \( \text{67} \) (Lemma 4.2.9) for the proof in the special case of scalar functions and \( \text{67} \) (Lemma 4.2.14) for vector fields. In order to prove it in the general case, we momentarily reinterpret the multiplicative 1-parameter group \( \{ \chi_\alpha \} \) as the additive group \( \mathbb{R} \), by setting \( t = \log(\alpha) \in \mathbb{R} \) and \( \phi_t(x, \dot{x}) = (x, e^t \dot{x}) = \chi_\alpha(x, \dot{x}) \), for all \((x, \dot{x}) \in TM\). Assume, first, that \( T \) is \( k \)-homogeneous, which means: \( \phi_t^* T = e^{kt} T \). Then,

\[
\mathcal{L}_C T = \left. \frac{d}{dt} (\phi_t^* T) \right|_{t=0} = \left. \frac{d}{dt} (e^{kt} T) \right|_{t=0} = kT.
\]
Conversely, assume \((23)\) holds. Differentiating the identity \(\phi_t^* \phi_{t}^* T = \phi_{t}^* T\) with respect to \(\varepsilon\) at \(\varepsilon = 0\), one finds:

\[
\phi_t^* \mathcal{L}_C T = \frac{d}{dt} (\phi_t^* T), \quad \forall t.
\]

Using \((23)\), this leads to the differential equation \(\frac{d}{dt} (\phi_t^* T) = k \phi_t^* T\) in the unknown \(f(t) = \phi_t^* T\). Integrating this equation with the initial condition \(f(0) = \phi_0^* T = T\), we find \(\phi_t^* T = e^{k t} T\), which, reverting to the old notation, is precisely \(\chi_{\alpha}^* T = \alpha^k T\). \(\square\)

In particular, positive 0-homogeneity in \(\dot{x}\), i.e., invariance under the fiber rescalings \(\chi_{\alpha}\), \(\alpha > 0\), can be treated as invariance under the flow of \(\mathcal{C}\).

**Note.** In \(\chi\), \(k\)-homogeneity of vector fields is defined differently (it is, in our terms \((k-1)\)-homogeneity). We prefer, yet, this definition, which allows a unified treatment of tensor fields of any rank.

In the following, we will simply refer to positive homogeneity in \(\dot{x}\) as **homogeneity**. Some canonical examples of homogeneous structures on the tangent bundle are:

1. The Liouville vector field \(\mathcal{C}\) is homogeneous of degree 0, since \(\mathcal{L}_{\mathcal{C}} \mathcal{C} = [\mathcal{C}, \mathcal{C}] = 0\).
2. The vertical local basis vectors \(\mathcal{V}_i\) are homogeneous of degree -1, as \([\mathcal{C}, \mathcal{V}_i] = -\mathcal{V}_i\).
3. The **natural tangent structure of** \(TM\),

\[
J = dx^i \otimes \mathcal{V}_i
\]

is a globally defined, \((-1)\)-homogeneous tensor of type \((1,1)\). Homogeneity follows from:

\[
\mathcal{L}_C J = \mathcal{L}_C (dx^i) \otimes \mathcal{V}_i + dx^i \otimes [\mathcal{C}, \mathcal{V}_i] = 0 - dx^i \otimes \mathcal{V}_i = -J,
\]

where we have used 2. and \(\mathcal{L}_C (dx^i) = d\mathcal{C} dx^i + i_{\mathcal{C}} dx^i = 0\).

**Definition 8 (Homogeneous nonlinear connection)** A nonlinear connection \(T = H \mathcal{O} + V \mathcal{O}\) on the conic subbundle \(\mathcal{O} \subset TM\), is called homogeneous, if fiber homotheties preserve the horizontal subbundle, i.e., \((\chi_{\alpha})_*, X \in H \mathcal{O} \text{ for all } \alpha > 0 \text{ and all } X \in H \mathcal{O}\).

As it has been shown in Prop. 2.10.1, or in Cor. 7.5.10, that a nonlinear connection on \(TM\) is homogeneous if and only if the almost product structure \(\mathcal{Q} = \mathfrak{h} - \mathfrak{v}\) is 0-homogeneous; the result holds without modifications on \(\mathcal{O} \subset TM\).

In coordinates, homogeneity of a connection is characterized by the fact that its coefficients \(G^i_j = G^i_j(x, \dot{x})\) are 1-homogeneous functions in \(\dot{x}\). An example of a homogeneous nonlinear connection is the Cartan nonlinear connection, \((4.14)\), of a Finsler spacetime.

Almost all Finsler geometric objects discussed above are **anisotropic tensor fields**, which thus deserve a special mentioning here. These can be mapped into specific tensor fields on the tangent bundle, called **distinguished tensor fields**, or **d-tensor fields**; for the latter, homogeneity can be discussed in a natural manner.

**Definition 9**: An anisotropic tensor field on the conic subbundle \(\mathcal{O} \subset TM\) is a section of the pullback bundle \(\pi^*_TM|_\mathcal{O}(\mathcal{F}^pM)\), i.e., a smooth mapping:

\[
T : \mathcal{O} \to \mathcal{F}^pM, \quad (x, \dot{x}) \mapsto T_{(x, \dot{x})};
\]

i.e., for any \((x, \dot{x}) \in \mathcal{O}\), \(T_{(x, \dot{x})}\) is a tensor on \(M\), based at \(x = \pi^*_T M(x, \dot{x})\).
Consequently, an anisotropic tensor field will be locally expressed as: $T_{(s,x)} = T^{j_1\ldots j_q}_{i_1\ldots i_p} (x,\dot{x}) (\partial_{i_1} \otimes \ldots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q})|_{(s,x)}$.

In the presence of a nonlinear connection $N$ on $TM$ the following definition makes sense.

**Definition 10**: A d-tensor field on a conic subbundle $\mathcal{D} \subset TM$ (regarded as a manifold) is a tensor field $T \in \mathcal{F}^p_q(\mathcal{D})$, obeying the condition:

$$T (\omega_1,\ldots,\omega_p, V_1,\ldots, V_q) = T (\epsilon_1 \omega_1,\ldots,\epsilon_p \omega_p, \epsilon_{p+1} V_1,\ldots, \epsilon_{p+q} V_q),$$

for an arbitrarily fixed choice of the projectors $\epsilon_1,\ldots,\epsilon_{p+q} \in \{ h,v \}$.

For instance, if $V$ is an arbitrary vector field on $\mathcal{D}$, its horizontal and vertical components $hV$ and $vV$, taken separately, are d-tensor fields (of type $(1,0)$), as each of them acts on a single specified component $h \omega$ or $v \omega$ of a 1-form $\omega \in \Omega_1(\mathcal{D})$, whereas their sum is typically, not a d-tensor field.

With respect to the horizontal/vertical adapted local bases of $T \mathcal{D}$ and $T^* \mathcal{D}$, a d-tensor field $T$ will be expressed as a linear combination of tensor products of $\delta_i, \dot{\delta}_i, dx^i$ and $\delta x^i$, i.e., $T_{(s,x)} = T^{\delta_i \ldots \dot{\delta}_i}_{\delta x^i \ldots \delta x^j} (\delta_i \otimes \ldots \otimes \dot{\delta}_i \otimes \ldots \otimes \delta x^i \otimes \ldots \otimes \delta x^j)|_{(s,x)}$.

In the presence of a homogeneous nonlinear connection, the adapted basis elements $\delta_i$ are 0-homogeneous (and, as we have seen above, $\dot{\delta}_i$ are $(-1)$-homogeneous), hence the degree of homogeneity (if any) of a d-tensor field $T$ can be established in local coordinates, by simply evaluating the $x$-homogeneity degree of the coefficients $T^{\delta_i \ldots \dot{\delta}_i}_{\delta x^i \ldots \delta x^j}$.

**Note.** Anisotropic tensor fields can be mapped into (multiple) d-tensor fields on $\mathcal{D} \subset TM$ via horizontal or vertical lifts determined by the nonlinear connection. Yet, when doing this, one must take into account that using horizontal lifts $\partial_i \rightarrow \delta_i$, one obtains a d-tensor of different degree of homogeneity, compared to the one obtained via a vertical lift $\delta_i \rightarrow \dot{\delta}_i$, due to the $-1$-homogeneity of $\dot{\delta}_i$.

Examples of canonical homogeneous d-tensors are the Liouville vector field $C$ and the tangent structure $J$.

Further examples of homogeneous d-tensors arise once we consider a pseudo-Finsler structure on $M$. For instance, on a Finsler spacetime, all the tensor fields encountered in the previous section are homogeneous d-tensor fields on $\mathcal{A}$, of some degree $m$:

- the Finslerian metric tensor $g = g_{ij} dx^i \otimes dx^j$ ($k = 0$);
- the curvature $R = R^i_{jk} dx^j \otimes dx^k \otimes \dot{\delta}_i$ of the canonical linear connection ($k = 0$);
- the Landsberg tensor $P = P^i_{jk} dx^j \otimes dx^k \otimes \delta_i$ ($k = 0$).

Other d-tensor fields, such as the Hilbert form $\omega = F_i dx^i$, or the Reeb vector field

$$\ell = l^i \delta_i,$$

are only defined on $\mathcal{A}_0 = \mathcal{A} \setminus L^{-1}(0)$, since the functions $l^i = \dot{x}^i F^{-1}$ are only defined on $\mathcal{A}_0$. Both $\omega$ and $\ell$ are 0-homogeneous in $x$ and will play a crucial role in the following, as we will see in Section III B 1.

An important feature of both the Chern connection $D$ (and more generally, of any of the typical Finslerian connections in the literature) and of the dynamical covariant derivative $\nabla$ on $T \mathcal{A}$, is that they preserve the distributions generated by the canonical nonlinear connection $N$ and hence, they map d-tensors into d-tensors. Moreover, the degree of homogeneity of d-tensors is preserved by $D$-covariant differentiation with respect to 0-homogeneous vector fields (and is increased by 1 by dynamical covariant differentiation).
III. THE POSITIVELY PROJECTIVIZED TANGENT BUNDLE $PTM^+$

The positively projectivized tangent bundle $PTM^+$ is essential for a mathematically well defined calculus of variations on Finsler spacetimes. It also gives a nice way to understand positively homogeneous geometric objects on $TM$, such as the Finsler function, or $d$-tensors, as sections of bundles sitting over $PTM^+$, which we will discuss in detail in Section IV.

We will first introduce $PTM^+$ over general manifolds before we formulate the geometry of Finsler spacetimes on $PTM^+$. This reformulation is important to construct well defined integrals of homogeneous functions. We will show that integration on domains in $PTM^+$ is actually equivalent to integration over subsets of the observer space $O$, with the advantage that $PTM^+$ is explicitly independent of the Finsler Lagrangian $L$, whereas the observer space (and therefore, all its subsets, which one may use as integration domains) are defined in terms of $L$.

A. Definition and structure over general manifolds

We first give the definition of the positively projectivized tangent bundle, before we analyze its structure and point out how objects on $PTM^+$ are related to 0-homogeneous objects on $TM$. In the context of Finsler spacetimes, the positively projectivized tangent bundle was briefly discussed in (3). In the literature on positive definite Finsler geometry, $PTM^+$ is typically called the projective sphere bundle.

Actually, in positive definite Finsler geometry, this bundle is interchangeably used with the indicatrix bundle, as the two bundles are globally diffeomorphic. But, in Lorentzian Finsler geometry, as we will see below, this diffeomorphism does no longer exist, hence, in order to avoid any confusion, we preferred to make a clear distinction by the used terminology.

1. Definition and structure

Definition 11 (The positive, or oriented, projective tangent bundle) Let $M$ be a connected, orientable smooth manifold of dimension $n$. The positive projective tangent bundle is defined as the quotient space

$$PTM^+ := TM / \sim,$$  \hspace{1cm} (25)

where $\sim$ is the equivalence relation on $TM$ given by:

$$(x, \dot{x}) \sim (x, \dot{x}') \iff \dot{x}' = \alpha \dot{x} \text{ for some } \alpha > 0.$$  \hspace{1cm} (26)

In other words we identify the half-line $\{(x, \alpha \dot{x}) \mid \alpha > 0\}$ as a single point. We denote by

$$\pi^+: TM \rightarrow PTM^+, (x, \dot{x}) \mapsto [(x, \dot{x})]$$

the canonical projection.

The usual projectivized tangent bundle $PTM$ is obtained from $PTM^+$ by deleting the distinction between positive and negative scaling factors, in other words:

$$PTM = PTM^+ / \mathbb{Z}_2.$$

Conversely, by attaching orientations to the lines representing points of $PTM$, one gets $PTM^+$. In other words, $PTM^+$ is the canonical oriented double covering (also called orientation covering in (3), p. 394)) of
the \((2n - 1)\)-dimensional manifold \(PTM\), in particular, it is always orientable. The above discussion can be summarized as follows.

**Proposition 12** If \(M\) is a connected smooth \(n\)-dimensional manifold, then \(PTM^+\) is a smooth, orientable manifold of dimension \(2n - 1\).

The orientability of \(PTM^+\) is essential when considering integrals on \(PTM^+\).

The smooth structure of \(PTM^+\) is constructed as follows. Start with an atlas on \(TM\), induced by an atlas on \(M\) and denote by \((x^i, x^j)\) the corresponding coordinate functions; then, for each local chart domain \(U \subseteq TM\) and each \(i = 0, \ldots, n - 1\), define the open sets: 
\[
U_i = \{(x, \dot{x}) \in TU \mid \dot{x}^i > 0\}, U_f = \{(x, \dot{x}) \in U \mid \dot{x}^i < 0\}.
\]
Then, for each \(U^+ \in \{\pi^+(U_i), \pi^+(U_f)\}\) and each \([(x, \dot{x})] \in U^+\), we define the diffeomorphisms \(\phi^+: (x^i, u^0) \mapsto (x^i, u^0)\) as:
\[
(x^i, u^0) = (x^0, \ldots, x^{n-1}, \dot{x}^0, \ldots, \frac{\dot{x}^i}{\dot{x}^i}, \ldots, \frac{\dot{x}^{n-1}}{\dot{x}^i}). \tag{27}
\]
The result is a differentiable atlas \(\{(U^+, \phi^+)\}\) on \(PTM^+\).

Using these charts, a quick direct computation shows that the projection \(\pi^+: \tilde{TM} \rightarrow PTM^+\), \((x^i, \dot{x}^i) \mapsto (x^i, u^0)\) is a submersion. Since, obviously, \(\pi^+\) is surjective, it follows that \((\tilde{TM}, \pi^+, PTM^+)\) is a fibered manifold; actually, it possesses an even richer structure, as has already been pointed out in \((\ref{13})\). Let us briefly recall this result:

**Proposition 13 (The principal bundle \((\tilde{TM}, \pi^+, PTM^+, \mathbb{R}^*_+)\))** The slit tangent bundle \(\tilde{TM}\) is a principal bundle over \(PTM^+\), with fiber \((\mathbb{R}^*_+, \cdot)\).

**Proof.** Consider \(\chi\), as defined in \((\ref{21})\), as the right action of the Lie group \((\mathbb{R}^*_+, \cdot)\) on \(\tilde{TM}\). This action preserves the fibers \((\pi^+)^{-1}[(x, \dot{x})] = \{(x, \alpha \dot{x}) \mid \alpha \in \mathbb{R}^*_+\}\) of \(\pi^+\), i.e., the half-lines with direction \((x, \dot{x})\). Moreover, each of the fibers of \(\pi^+\) is obviously homeomorphic to \(\mathbb{R}^*_+\).

Actually, \(PTM^+\) is nothing but the space of orbits of the Lie group action \((\ref{21})\).

The Liouville vector field \(C\) is tangent to the fibers \((\pi^+)^{-1}[(x, \dot{x})]\) (i.e., it is \(\pi^+\)-vertical), which, taking into account that these fibers are 1-dimensional, means that \(C\) actually generates the tangent spaces to these fibers.

In its turn, \(PTM^+\) is a fibered manifold over \(M\). More precisely, we have the following result.

**Proposition 14 (Structure of the bundle \((PTM^+, \pi_M, M, S^{n-1})\))** The triple \((PTM^+, \pi_M, M)\), where \(\pi_M : PTM^+ \rightarrow M\), \([(x, \dot{x})] \mapsto x\), is a fibered manifold with fibers diffeomorphic to Euclidean spheres.

**Proof.** The projection \(\pi_M\) is obviously a surjective submersion, meaning that \((PTM^+, \pi_M, M)\) is, indeed, a fibered manifold. Its fibers \(\pi_M^{-1}(x) = \{[(x, \dot{x})] \mid x \in TM\}\) are orientation coverings of the projective tangent spaces \(PTM \simeq P\mathbb{R}^n\); but, the orientation covering of the projective space \(P\mathbb{R}^n\) is nothing but the round sphere \(S^{n-1}\).

Moreover, \(PTM^+\) is a natural bundle over \(M\), meaning that it is obtained from \(M\) via a covariant functor (see the Appendix for more details on natural bundles); the natural lift to \(PTM^+\) of a diffeomorphism
$f : M \to M$ is given by $[(x, \dot{x})] \mapsto [(f(x), df_x(\dot{x}))]$, which is well defined by virtue of the linearity of $df_x$.

On natural bundles, one can speak about general covariance of Lagrangians, which is essential in ensuring the existence of a well defined notion of energy-momentum tensor.

**Note:** As already stated above, the bundle $PTM^+$ is better known in the literature on positive definite Finsler spaces under the name of projective sphere bundle over $M$, see, e.g., though the name is very well justified by the above Proposition, we preferred to avoid this terminology, in order to avoid any confusions with the indicatrix bundle $L = 1$. This distinction is necessary since, for positive definite Finsler structures, the fibers $I_x = L^{-1}(1)$ of the indicatrix bundle are diffeomorphic to Euclidean spheres (i.e., diffeomorphic to the fibers of $PTM^+$), while in Lorentzian signature, this is no longer the case; actually, in the latter case, we have already seen that the fibers of the indicatrix bundle are generally disconnected, containing the observer spaces $O_x$ as connected components. Moreover, it is essential for our later considerations to stress that the construction of $PTM^+$ is completely independent of any pseudo-Finslerian (or pseudo-Riemannian) structure whatsoever.

2. **From $PTM^+$ to $TM$ and back**

Local computations on $PTM^+$ are much simplified if one uses local homogeneous coordinates instead of the usual local coordinates $(x^i, u^a)$ defining the manifold structure, in the same fashion as on $PTM$, see$^{55}$. For a given equivalence class $[(x, \dot{x})]$, local homogeneous coordinates$^2$ are defined as the coordinates $(\dot{x}^i, \dot{\chi}^i)$ in the corresponding chart on $TM$ of an arbitrarily chosen representative of the class $[(x, \dot{x})]$; i.e., homogeneous coordinates are only unique up to multiplication by a positive scalar of the $\dot{x}$-coordinates.

In these coordinates, local computations on $PTM^+$ will become identical to those on $TM$, just with due care that the involved expressions in $(\dot{x}^i, \dot{\chi}^i)$ - which formally correspond to geometric objects on $TM$ - should really define objects on $PTM^+$. A necessary (but not always sufficient) condition is that these formally defined geometric objects on $TM$ should be positively 0-homogeneous in $\dot{x}$, i.e., invariant under the flow of $\mathcal{C}$. Here we list the most frequently encountered examples:

- **Functions.** A function $f : TM \to \mathbb{R}, f = f(x, \dot{x})$ can be identified with a function $f^+ : PTM^+ \to \mathbb{R}$ if and only if it is positively 0-homogeneous in $\dot{x}$; more precisely, $f^+ : PTM^+ \to \mathbb{R}$ is defined by:

  $$f^+ [(x, \dot{x})] = f(x, \dot{x}),$$

  i.e., $f := f^+ \circ \pi^+$; in homogeneous coordinates, $f^+$ and $f$ have identical coordinate expressions. The function $f^+$ is differentiable at $[(x, \dot{x})]$ if and only if $f$ is differentiable at one representative $(x, \dot{x})$.

- **Vector fields.** For a vector field $X = X^i \partial_i + X^\chi \partial_\chi \in \mathcal{V}^\circ(TM)$, the projection

  $$X^+ := (\pi^+) X$$

  is a well defined vector field on $PTM^+$ if and only if $X$ is positively 0-homogeneous in $\dot{x}$, i.e., $\mathcal{L}_C X = 0$.

  This is justified as follows. Having in view that $\pi^+$ is surjective, the necessary and sufficient condition for $X^+$ to be a well defined vector field on $PTM^+$ is that the mapping $[(x, \dot{x})] \mapsto X^+_{[(x, \dot{x})]} = (\pi^+) X_{(x, \dot{x})}$ is independent on the choice of $(x, \dot{x})$ in the class $[(x, \dot{x})]$; but this means precisely 0-homogeneity of $X$.

---

$^2$ Here, the word “local” is just meant to stress that these coordinates do not make sense globally, but only over a coordinate neighborhood. (Local) homogeneous coordinates are, obviously, not local coordinates as typically defined in differential geometry, since their number is equal to $2\dim(M)$, whereas the dimension of $PTM^+$ is $2\dim(M) - 1$. 
In coordinates, this boils down to the fact that the functions $X^i$ are positively 0-homogeneous, while $\tilde{X}^i$ are 1-homogeneous in $\dot{x}$.

An interesting remark is that the correspondence $X \mapsto X^+$ is surjective, but not injective, as all vector fields of the form $X + fC$ on $TM$, where $f$ is a 0-homogeneous function, descend onto the same vector field $X^+ \in \mathcal{X}(PTM^+)$.

- **Differential forms.** For differential forms $\rho$ on $TM$, 0-homogeneity is necessary, but not sufficient in order to be identified with differential forms on $PTM^+$. The following result (derived in [43]) is just a coordinate-free restatement of a similar result on $PTM$, see [65]:

**Proposition 15** Let $\rho \in \Omega(TM)$ be defined on a conic subbundle of $TM$. Then, there exists a unique differential form $\rho^+ \in \Omega(PTM^+)$ such that $\rho = (\pi^+)^*\rho^+$ if and only if the following conditions are fulfilled:

1. $\rho$ is 0-homogeneous in $\dot{x}$, i.e.,
   $\mathcal{L}_C \rho = 0$; \hspace{1cm} (28)

2. $\rho$ is $\pi^+$-horizontal, i.e.,
   $i_C \rho = 0$. \hspace{1cm} (29)

**Remark 16** \[43\]

1. The projection $\pi^+$ is locally represented in homogeneous coordinates as the identity: $\pi^+: (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i)$. The latter relation tells us that the coordinate expressions of geometric objects on $TM$ (e.g., $f, X, \rho$) that can be identified with geometric objects $f^+, X^+, \rho^+$ etc. on $PTM^+$, will be identical to the expressions of the latter in homogeneous coordinates.

2. Exterior differentiation of forms $\rho^+ \in \Omega(PTM^+)$ can be carried out identically to exterior differentiation of the corresponding form $\rho \in \Omega(TM)$, since:

   $d \rho = d ((\pi^+)^*\rho^+) = (\pi^+)^* d \rho^+$.

   In particular, differentiation of functions on $PTM^+$ is carried out identically to the one on $TM$.

B. **Over Finsler spacetimes $(M, L)$**

After having introduced $PTM^+$ in the previous section, we now demonstrate that the geometry of a Finsler spacetime can be understood in terms of geometric objects on $PTM^+$. This eventually enables us to write down the desired action integrals for field theories in a mathematically precise way.

1. **Finsler Geometry on $PTM^+$ and volume forms**

   On a Finsler spacetime, we defined the conic subbundles $\mathcal{A}, \mathcal{A}_0, \mathcal{F}, \mathcal{N} \subset TM$, and the observer space $\mathcal{O}^*$, see Section II A. We will denote by a plus sign, e.g., $\mathcal{F}^+ = \pi^+ (\mathcal{F})$, $\mathcal{A}^+ = \pi^+ (\mathcal{A})$ etc., their images through $\pi^+: TM \rightarrow PTM^+$; also, we will always use local homogeneous coordinates on $PTM^+$. 
**Canonical nonlinear connection.**

The canonical nonlinear connection $N$ on $\mathcal{A}$, see equation (14) can be transplanted to $\mathcal{A}^+$ in a natural way, as follows. Start with an arbitrary vector $X^+ \in T\mathcal{A}^+$. As we have seen above, it always corresponds to a positively 0-homogeneous vector $X \in T\mathcal{A}$ (which is unique up to a multiple of $C$). Then, $X$ is decomposed into its $N$-horizontal and vertical components $hX = X^i \delta_i$ and $vX = \dot{X}^i \dot{\partial}_i$; both components are positively 0-homogeneous, due to the homogeneity of $N$, hence they descend back onto vectors $hX^+, vX^+$ on $T\mathcal{A}^+$. Moreover, $hX^+, vX^+$ are uniquely defined by $X^+$, as the possible multiple of $C$ appearing in the procedure will be projected back to $PTM^+$ into the zero vector. This naturally gives rise to a splitting

$$X^+ = hX^+ + vX^+,$$

i.e., to a connection $N^+ := \pi^+(\mathcal{A})$:

$$T\mathcal{A}^+ = H\mathcal{A}^+ \oplus V\mathcal{A}^+.$$  \(30\)

The vectors $hX^+ = (\pi^+)_*(hX)$ and $vX^+ = (\pi^+)_*(vX)$ are expressed in homogeneous coordinates as:

$$hX^+ = X^i \delta_i, \quad vX^+ = \dot{X}^i \dot{\partial}_i.$$  

Similarly, the Chern-Rund connection $D$ gives rise to a linear connection $D^+$ on $\mathcal{A}^+$, having the same local expression of covariant derivatives as $D$.

**Contact structure and volume form for the set of non-null admissible directions.**

In the following, we will identify a canonical volume form on the set of admissible non-null directions $\mathcal{A}_0^+ = \pi^+(\mathcal{A}_0)$. The Hilbert form $\omega = F_i dx^i$, defined on $\mathcal{A}_0$ obeys the conditions:

$$i_C\omega = 0, \quad L_C\omega = d i_C\omega + i_C d\omega = 0,$$

which means that it can be identified with a differential form $\omega^+$ on $\mathcal{A}_0^+ \subset PTM^+$, such that $(\pi^+)^*\omega^+ = \omega$; in homogeneous coordinates:

$$\omega^+ = F_i dx^i$$  \(31\)

and

$$d\omega^+ = \frac{1}{F}(\varepsilon_{ijkl} - F_i F_j) \delta \dot{x}^j \wedge dx^i.$$  \(32\)

A direct calculation, see, $^{65}$, shows that, for $\dim M = 4$,

$$\omega^+ \wedge d\omega^+ \wedge d\omega^+ \wedge d\omega^+ = 3! \frac{\det g}{L^2} i_C (d^4 x \wedge d^4 \dot{x}) = 3! \frac{\det g}{L^2} \text{Vol}_0,$$

where

$$\text{Vol}_0 = i_C (d^4 x \wedge d^4 \dot{x}),$$  \(34\)

is always nonzero. In other words, the Hilbert form $\omega^+$ defines a contact structure on $PTM^+$.

In contact geometry, the Reeb vector field $\ell^+ \in \mathcal{X}(\mathcal{A}_0^+)$ corresponding to the contact structure $\omega^+$ is uniquely defined by the conditions

$$i_{\ell^+} (\omega^+) = 1, \quad i_{\ell^+} d\omega^+ = 0.$$  \(35\)

In our case, this gives:
Proposition 17 The Reeb vector field $\ell^+$ corresponding to the contact structure $\omega^+$ on $\mathcal{M}^+_0$ is expressed in local homogeneous coordinates as:

$$\ell^+ = \ell^i \partial_i, \quad \ell^i = \frac{\dot{x}^i}{\sqrt{F}}.$$  

Proof. We have $i_\ell^+ \omega^+ = 1$ and

$$i_\ell^+ d\omega^+ = F^{-1}(\varepsilon g_{ij} - F_i F_j)\ell^i (\delta \dot{x}^i \wedge dx^i) = F^{-1}(\varepsilon g_{ij} - F_i F_j)\ell^i \delta \dot{x} = 0,$$

where for the last equality we used that $F_i \ell^i = 1$ and $\varepsilon g_{ij} \ell^i = F_i$. □

The importance of the Reeb vector field is given by the following result.

Proposition 18 Let $c : [a, b] \to M$, $s \mapsto x(s)$ be a non-lightlike admissible curve parametrized by arc length and $C : [a, b] \to \mathcal{M}^+_0$, $s \mapsto [(x(s), \dot{x}(s))]$, its canonical lift. Then, $C$ is an integral curve of $\ell^+$ if and only if $c$ is an arc-length parametrized geodesic of $(M, L)$.

Proof. In homogeneous coordinates, $C = x^i(s) \partial_i + \delta_i \dot{x}^i(s) \partial_i$, where $\delta_i \dot{x}^i(s) = \dot{x}^i(s) + 2G^i(\dot{x}^j(s), \dot{x}^j(s))$; that is, $C$ is an integral curve of $\ell^+$ is and only if:

$$\dot{x}^i(s) = l^i, \quad \delta_i \dot{x}^i(s) = 0.$$

The first condition above is trivially satisfied by any curve parametrized by arc length, since $L(x, \dot{x}(s)) = \text{sign}(L)$; taking into account the properties of the canonical nonlinear connection, the second condition is equivalent to the fact that $c$ is a arc-length parametrized geodesic of $(M, L)$, see (12). □

The contact structure $\omega^+$, now enables us to identify a canonical volume form on $\mathcal{M}^+_0$.

Definition 19 (Canonical volume form) Let $(M, L)$ be a Finsler spacetime, $\mathcal{M}^+_0 \subset \mathcal{P}T^+$, the set of its admissible, non-null directions and $\omega^+$, the Hilbert form on $\mathcal{M}^+_0$. Then

$$d\Sigma^+: = \frac{\text{sign}(|\det g|)}{3!} \omega^+ \wedge (d\omega^+)^3 = \frac{|\det g|}{L^2} \text{Vol}_0, \quad (37)$$

where $\text{Vol}_0$ is as in (34), is called the canonical volume form on $\mathcal{M}^+_0$.

Note that, on $\mathcal{M}^+_0$, $g$ is nondegenerate, so, $d\Sigma^+$ is well defined.

With respect to this volume form, the divergence of horizontal and vertical vector fields, $X = X^i \partial_i$ and $Y = Y^i \partial_i$, on $\mathcal{M}^+_0$, is [33]:

$$\text{div}(X) = (X^i |_i - P_i X^i), \quad (38)$$

$$\text{div}(Y) = (Y^i + 2C^i Y^j - \frac{4}{L} Y^j \dot{x}_i), \quad (39)$$

where $P_i$ is the trace of the Landsberg tensor (17) and $C_i$ is the trace of the Cartan tensor (9). For any $f : \mathcal{M}^+_0 \to \mathbb{R}$, the above equations imply

$$\nabla f = \text{div}(f \ell) = \text{div}(f \ell \partial_i). \quad (40)$$
2. Integration on $PTM^+$ and integration on observer space

In positive definite Finsler spaces, the unit sphere bundle $L^{-1}(1)$ is globally diffeomorphic to $PTM^+$. But, passing to Finsler spacetimes, this is no longer true; this is easy to see since the fibers $L(x) = L^{-1}(1) \cap T_xM$ are non-compact (they are, even in the simplest case of Lorentzian metrics, hyperboloids), while the fibers of $PTM^+$ are compact. Still, we will be able to establish a correspondence between the observer space $\mathcal{O}$ and the set of future pointing timelike directions $\mathcal{T}^+ := \pi^+(\mathcal{T})$. A preliminary result, proven in \cite{43}, refers to compact subsets of $\mathcal{T}$.

**Proposition 20**, see \cite{43}:

1. For any admissible compact, connected subset $D \subset L^{-1}(1)$, the projection $\pi^+: D \rightarrow \pi^+(D) \subset \pi^+(L^{-1}(1))$ is a diffeomorphism.

2. For any connected, admissible and non-null compact subset $D^+ \subset \pi^+(\mathcal{O})$ and any differential form $\rho^+$ on $PTM^+$:

\[
\int_{D^+} \rho^+ = \int_D \rho,
\]

where $\rho = (\pi^+)^* \rho^+$ is a differential form on $TM$ and $D := (\pi^+)^{-1}(D^+) \cap L^{-1}(1)$.

The above result will be mostly applied to pieces $D \subset \mathcal{O} \subset L^{-1}(1)$, where, by a piece $D \subset X$, we will understand, \cite{43}, a compact $n$-dimensional submanifold of $X$ with boundary. Yet, it can be extended to the whole observer space, as long as we integrate compactly supported differential forms.

**Proposition 21** In any Finsler spacetime:

1. The mapping $\pi^+: \mathcal{O} \rightarrow \mathcal{T}^+$ is a diffeomorphism;

2. for any compactly supported 7-form $\rho^+$ on $\mathcal{T}^+$:

\[
\int_{\mathcal{T}^+} \rho^+ = \int_{\mathcal{O}} \rho,
\]

where $\rho = (\pi^+)^* \rho^+$.

**Proof.**

1. **Injectivity**: Assume $\pi^+(x, \dot{x}) = \pi^+(x', \dot{x}')$ for some $(x, \dot{x}), (x', \dot{x}') \in \mathcal{O}$. It follows that $[(x, \dot{x})] = [(x', \dot{x}')]$, i.e., $x = x'$ and there exists an $\alpha > 0$ such that $\dot{x}' = \alpha \dot{x}$. Applying $L$ to both hand sides, we find $L(x, x') = \alpha^2 L(x, \dot{x})$; but, on $\mathcal{O}$, $L = 1$, which means that $\alpha^2 = 1$. Since $\alpha > 0$, it follows that $(x, x') = (x, \dot{x})$.

2. For any compactly supported 7-form $\rho^+$ on $\mathcal{T}^+$:

\[
\int_{\mathcal{T}^+} \rho^+ = \int_{\mathcal{O}} \rho,
\]

where $\rho = (\pi^+)^* \rho^+$.

**Proof.**

1. **Injectivity**: Assume $\pi^+(x, \dot{x}) = \pi^+(x', \dot{x}')$ for some $(x, \dot{x}), (x', \dot{x}') \in \mathcal{O}$. It follows that $[(x, \dot{x})] = [(x', \dot{x}')]$, i.e., $x = x'$ and there exists an $\alpha > 0$ such that $\dot{x}' = \alpha \dot{x}$. Applying $L$ to both hand sides, we find $L(x, x') = \alpha^2 L(x, \dot{x})$; but, on $\mathcal{O}$, $L = 1$, which means that $\alpha^2 = 1$. Since $\alpha > 0$, it follows that $(x, x') = (x, \dot{x})$.

2. **Surjectivity**: Consider an arbitrary $[(x, \dot{x})] \in \mathcal{T}^+$. It means that $(x, \dot{x}) \in \mathcal{T}$; as $\mathcal{T}$ is a conic subbundle of $TM$, the vector $(x, \alpha \dot{x})$, with $\alpha := L(x, \dot{x})^{-1/2}$, also belongs to $\mathcal{T}$. But $L(x, \alpha \dot{x}) = 1$, hence $(x, \alpha \dot{x}) \in \mathcal{O}$. Since $\pi^+(x, \dot{x}) = \pi^+(x, \alpha \dot{x})$, it follows that $[(x, \dot{x})] \in \pi^+(\mathcal{O})$.

3. **Smoothness**: of $\pi^+$ and of its inverse follow immediately, working in homogeneous coordinates, in which $\pi^+$ is represented as the identity.

2. follows immediately from $\rho = (\pi^+)^* \rho^+$ and point 1.
In particular, the above result shows that:

\[ \mathcal{O}^+ = \mathcal{F}^+. \]  

(43)

With this section we have established that integration of differential forms on the observer space of Finsler spacetimes can be understood as integration of differential forms on (subsets of) PTM\(^+\).

IV. FIBERED MANIFOLDS AND FIELDS OVER A FINSLER SPACETIME

Having understood how integrals of homogeneous functions on Finsler spacetimes can be constructed, the next step in constructing action integrals is to understand fields (and their derivatives) as sections \( \gamma \) into fibered manifolds \( Y \) over PTM\(^+\). But, with this aim, we need to understand the structure of such fibered manifolds.

For an improved readability of the article, we give a detailed summary of jet bundles over fibered manifolds and coordinate free calculus of variations in Appendix A.

A. Fibered manifolds over PTM\(^+\)

Consider a Finsler spacetime \((M, L)\) and denote by \((Y, \Pi, PTM^+)\) an arbitrary fibered manifold of dimension \(7 + m\). Then, \(Y\) will acquire a double fibered manifold structure:

\[ Y \overset{\Pi}{\longrightarrow} PTM^+ \overset{\pi_M}{\longrightarrow} M. \]  

(44)

As a consequence, \(Y\) will admit an atlas consisting of fibered charts \((V, \psi)\), \(\psi = (x^i, u^\alpha, z^\sigma), \ i = 0, ..., 3, \ \alpha = 0, ..., 2, \ \sigma = 1, ..., m\) on \(Y\), that are adapted to both fibrations, i.e., the two projections will be represented in these charts as:

\[ \Pi: (x^i, u^\alpha, z^\sigma) \mapsto (x^i, u^\alpha), \quad \pi_M: (x^i, u^\alpha) \mapsto (x^i). \]

Further, corresponding to any induced local chart \((\Pi(V), \phi)\), \(\phi = (x^i, u^\alpha)\) on PTM\(^+\), we can introduce the homogeneous coordinates \((x^i, \dot{x}^i)\), which we will sometimes denote collectively as \((x^A)\). This way, we obtain on \(V = \Pi^{-1}(U^+)\) the coordinate functions

\[ \bar{\psi} := (x^i, \dot{x}^i, y^\sigma) = (x^A, y^\sigma) \]

on \(V\), which we will call fibered homogeneous coordinates. The corresponding fiber coordinate \(y^\sigma\) is typically not unique, its relation to the original coordinates \((x^i, u^\alpha, z^\sigma)\) may depend on the choice of representative \([(x, \dot{x})] \in PTM^+\). The precise relation will be discussed in the applications.

In fibered homogeneous coordinates, local sections (physical fields) \(\gamma: W^+ \rightarrow Y, \ [(x, \dot{x})] \mapsto \gamma([(x, \dot{x})])\) (where \(W^+ \subset PTM^+\) is open) are represented as:

\[ \gamma: (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, y^\sigma(x^i, \dot{x}^i)). \]  

(45)

The set of all such sections is denoted by \(\Gamma(Y)\).
Remark 22 Generically, the physical fields we are considering are \( k \)-homogeneous with respect to \( \dot{x} \). Hence their coordinate representation in fibered homogeneous coordinates satisfies
\[
\gamma(x, \dot{x}) = (x^i, \alpha^i \dot{x}^i, y^\alpha(x^i, \dot{x}^i)) = (x^i, \alpha^i, \dot{\alpha}^i \dot{x}^i(x^i, \dot{x}^i)).
\]
Alternatively we could represent them in the original coordinates on \( Y \) as
\[
\gamma(x^i, u^a(x, \dot{x})) = (x^i, u^a, z^\alpha(x^i, u^a)),
\]
where \( z^\alpha(x^i, u^a(x, \dot{x})) = z^\alpha(x^i, \dot{x}^i) \) is a zero-homogeneous in \( \dot{x} \) when expressed in terms of \( \dot{x} \), i.e. does not depend on the representative of \( [(x, \dot{x})] \in PT^{M^+} \).

On the jet bundle \( J^rY \), fibered charts \( (V, \Psi) \) induce the fibered charts \( (V', \Psi') \), with:
\[
\Psi' = (x^i, \dot{x}^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_{i_1}...i_k}),
\]
where, for \( k = 1, ..., r \) and \( \gamma \in \Gamma(Y) \) locally represented as in (45),
\[
y^\sigma_{j_1...i_k}(x^i, \dot{x}^i) = \frac{\partial^k}{\partial x^{j_1} \partial \dot{x}^{i_k}} (y^\sigma(x^i, \dot{x}^i))
\]
are all partial \( x, \dot{x} \)-derivatives up to the total order \( k \). The canonical projections \( \Pi^s : J^sY \to J^rY, J^r(x, \dot{x}) \gamma \mapsto J^s(x, \dot{x}) \gamma \) (with \( r > s \)), are then represented as:
\[
\Pi^s : (x^i, \dot{x}^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_{i_1}...i_k}) \mapsto (x^i, \dot{x}^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_{i_1}...i_k}),
\]
accordingly,
\[
\Pi^r : J^rY \to PT^{M^+}, (x^i, \dot{x}^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_{i_1}...i_k}) \mapsto (x^i, \dot{x}^i).
\]

In the calculus of variations, we will need two classes of differential forms on \( J^rY \), namely, horizontal forms and contact forms, see Appendix A 3 for more details.

1. **\( \Pi^r \)-horizontal forms** \( \rho \in \Omega_k(J^rY) \) are defined as forms that vanish whenever contracted with a \( \Pi^r \)-vertical vector field. In the natural local basis \( \{dx^i, d\dot{x}^i, dy^\sigma, dy^\sigma_{j_1}, ..., dy^\sigma_{j_{i_1}...i_k}\} \), they are expressed as:
\[
\rho = \frac{1}{k!} \rho_{i_1...i_k} dx^{i_1} \wedge d\dot{x}^{i_2} \wedge ... \wedge d\dot{x}^{i_k},
\]
where \( \rho_{i_1...i_k} \) are smooth functions of the coordinates on \( J^rY \). In particular, Lagrangians will be characterized as \( \Pi^r \)-horizontal 7-forms \( \lambda = \Lambda \text{Vol}_0 \) on \( J^rY \), where \( \text{Vol}_0 \) is as in (34).

Similarly, \( \Pi^{ra} \)-horizontal forms, \( 0 \leq s \leq r \) are locally generated by wedge products of \( dx^i, d\dot{x}^i, dy^\sigma, dy^\sigma_{j_1}, ..., dy^\sigma_{j_{i_1}...i_k} \).

2. **Contact forms** on \( J^rY \) are, by definition, forms \( \rho \in \Omega_k(J^rY) \) that vanish along prolonged sections, i.e., \( J^r\gamma \rho = 0, \forall \gamma \in \Gamma(Y) \). For instance,
\[
\Theta^a = dy^a - y^\sigma_{j_a} dx^j - y^\sigma_{j_ab} d\dot{x}^b \quad \text{etc.}
\]
are contact forms, composing the so-called contact basis \( \{dx^i, d\dot{x}^i, \Theta^\sigma, \Theta^\sigma_{j_1}, ..., \Theta^\sigma_{j_{i_1}...i_k}, dy^\sigma_{j_1...i_k}, ..., dy^\sigma_{j_{i_1}...i_k}\} \) of \( \Omega(J^rY) \).

---

3 We note that, since we are using homogeneous coordinates over each chart domain, the number of coordinate functions \( (y^\sigma, y^\sigma_{j_1}) \) is \( 8m \), not \( 7m \) as one would expect taking into account the dimension of the fibers of \( J^rY \to Y \).
Mathematical foundations for field theories on Finsler spacetimes

An important class of contact forms are source forms (or dynamical forms), \( \rho \in \Omega_8(J^rY) \) that can be expressed, corresponding to any fibered chart, as:

\[
\rho = \rho_\sigma \theta^\sigma \wedge \text{Vol}_0
\]  

(see the Appendix for a coordinate-free definition); Euler-Lagrange forms of Lagrangians fall into this class.

Raising to \( J^{r+1}Y \), any differential form \( \rho \in \Omega^K(J^rY) \) can be uniquely decomposed as:

\[
(\Pi^{r+1})^* \rho = h\rho + p\rho,
\]

where \( h\rho \) is horizontal and \( p\rho \) is contact. The horizontal component \( h\rho \) is what survives of \( \rho \) when pulled back by prolonged sections \( J^r\gamma \) (where \( \gamma \in \Gamma(Y) \), i.e.,

\[
J^r\gamma^* \rho = J^{r+1}\gamma^*(h\rho).
\]

The mapping \( h : \Omega(J^rY) \to \Omega(J^{r+1}Y) \) is a morphism of exterior algebras, called horizontalization. On the natural basis 1-forms, it acts as:

\[
hdx^i := dx^i, \quad hdx^i = dx^i, \quad hdy^\sigma = y^\sigma_i dx^i + y^\sigma_\lambda d\dot{x}^\lambda \quad \text{etc.}
\]  

Accordingly, for any smooth function \( f \) on \( J^rY \), we obtain:

\[
hd f = d f dx^i = d_i f dx^i + \dot{d}_i f d\dot{x}^i,
\]

where \( d_i f \) and \( \dot{d}_i f \) represent total \( x^i \)- and, accordingly, total \( \dot{x}^i \)-derivatives (of order \( r+1 \)). Using (49) for \( \rho = df \), we find:

\[
\partial_i (f \circ J^r\gamma) = d_i f \circ J^{r+1}\gamma, \quad \dot{\partial}_i (f \circ J^r\gamma) = \dot{d}_i f \circ J^{r+1}\gamma.
\]

Alternatively, one may use a nonlinear connection on \( \mathcal{A}^+ \subset PTM^+ \) (e.g., the canonical one), to introduce the total adapted derivative operators

\[
\delta_i := d_i - G^j_i \dot{d}_j,
\]

which help constructing manifestly covariant expressions. More precisely, using these operators, we can write (51) as

\[
hd f = (\delta_i f)dx^i + (\dot{d}_i f)\dot{d}\dot{x}^i.
\]

If \( f \) is a coordinate invariant scalar function, then \( \delta_i f \) and \( \dot{d}_i f \) are \( d \)-tensor components.

B. Fibered automorphisms

Variations of sections and, accordingly, of actions, are given by 1-parameter groups of fibered automorphisms of \( Y \). But, in the case of Finsler spacetimes, these will also have to take into account the doubly fibered structure of the configuration bundle \( Y \). This is why we introduce:
**Definition 23 (Automorphisms of Y)** An automorphism of a fibered manifold \((Y, \Pi, PTM^+)\) is a diffeomorphism \(\Phi: Y \to Y\) such that there exists a fibered automorphism \(\phi\) of \((PTM^+, \pi_M, M)\) with \(\Pi \circ \phi = \phi \circ \Pi\).

In particular, this means that there exists a diffeomorphism \(\phi_0: M \to M\) which makes the following diagram commute:

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi} & Y \\
\Pi \downarrow & & \downarrow \Pi \\
PTM^+ & \xrightarrow{\phi} & PTM^+ \\
\pi_M \downarrow & & \downarrow \pi_M \\
M & \xrightarrow{\phi_0} & M \\
\end{array}
\]

Locally, a fibered automorphism of \(Y\) is represented as:

\[
\tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{\xi}^i = \tilde{\xi}^i(x^j, \tilde{x}^j), \quad \tilde{\sigma} = \tilde{\sigma}(x^j, \tilde{x}^j, y^\mu).
\]

An automorphism of \(Y\) is called *strict* if it covers the identity of \(PTM^+\), i.e., \(\phi = id_{PTM^+}\).

Generators of 1-parameter groups \(\{\Phi_\varepsilon\}\) of automorphisms of \(Y\) are vector fields \(\Xi \in \mathcal{X}(Y)\) that are projectable with respect to both projections \(\Pi\) and \(\pi_M\): in fibered homogeneous coordinates, this is expressed as:

\[
\Xi = \xi^i(x^j) \frac{\partial}{\partial x^i} + \tilde{\xi}^i(x^j, \tilde{x}^j) \tilde{\partial}_i + \Xi^\sigma(x^j, \tilde{x}^j, y^\mu) \frac{\partial}{\partial y^\sigma}.
\] (56)

In particular, strict automorphisms are generated by \(\Pi\)-vertical vector fields \(\Xi = \Xi^\sigma(x^j, \tilde{x}^j, y^\mu) \partial / \partial y^\sigma\).

Given such a 1-parameter group \(\{\Phi_\varepsilon\}\), any section \(\gamma \in \Gamma(Y)\) is deformed into the section

\[
\gamma_\varepsilon := \Phi_\varepsilon \circ \gamma \circ \Phi_\varepsilon^{-1}.
\]

In first approximation, if \(\gamma\) is locally represented as: \(\gamma: (x^j, \tilde{x}^j) \mapsto (x^j, \tilde{x}^j, y^\sigma(x^j, \tilde{x}^j))\), then:

\[
\gamma_\varepsilon: (x^j, \tilde{x}^j) \mapsto (x^j, \tilde{x}^j, y^\sigma(x^j, \tilde{x}^j) + \varepsilon(\Xi^\sigma \circ J^1\gamma)|_{(x^j, \tilde{x}^j)} + \Theta(\varepsilon^2)),
\]

where

\[
\Xi^\sigma := (\Xi^\sigma - \xi^i(y^\sigma \partial / \partial x^i) - \tilde{\xi}^i(y^\sigma \tilde{\partial}_i)).
\] (57)

The functions \(\Xi \circ J^1\gamma\), defined on each local chart in the domain of \(\gamma\), are commonly (though in a somewhat sloppy manner) denoted by \(\delta y^\sigma\).

The automorphisms \(\Phi_\varepsilon: Y \to Y\) are prolonged into automorphisms \(J^r\Phi_\varepsilon\) of \(J^rY\) by the rule:

\[
J^r\Phi_\varepsilon(J_{r(x,y)}\gamma) := J_{\phi(x,y)}^r\gamma_\varepsilon.
\]

The generator of the 1-parameter group \(\{J^r\Phi_\varepsilon\}\) is called the \(r\)-th prolongation of the vector field \(\Xi\) and denoted by \(J^r\Xi\) (see the Appendix for the precise coordinate formula for \(J^r\Xi\)).
In order to apply the apparatus of calculus of variations with Finslerian geometric objects (e.g., Finsler function \(L\), metric tensor \(g\), nonlinear/linear connection, homogeneous d-tensors) as dynamical variables, we will describe these geometric objects as sections of fiber bundles \((Y, \Pi, PTM^+)\). A priori, \(k\)-homogeneous Finslerian geometric objects are (locally defined) sections \(f : TM \rightarrow \tilde{Y}\) into some fiber bundle \(\tilde{Y}\) sitting on \(\tilde{T}M\) (e.g., a bundle of tensors, or a bundle of connections over \(\tilde{T}M\) etc.). The key idea allowing us to reinterpret them as sections of a bundle \(Y\) sitting over \(PTM^+\), is that \(k\)-homogeneity can be interpreted as equivariance, with respect to the action of \((\mathbb{R}_+^*, \cdot)\) on the fiber bundle \(\tilde{Y}\) and, respectively, on the principal bundle \((\tilde{T}M, \pi^+, PTM^+, \mathbb{R}_+^*)\).

The construction of the configuration bundle \((Y, \Pi, PTM^+)\) follows essentially the same line of reasoning as the one made in\(^{54}\) (Sec 5.4), in the case of principal connections and relies on factoring out the action of \(\mathbb{R}_+^*\), from both the total space and the base of the original bundle \(\tilde{Y}\).

Consider a fiber bundle \(\tilde{Y} \overset{\Pi}{\rightarrow} \tilde{T}M\), with typical fiber \(Z\). Noticing that \(\tilde{Y}\) can be identified with the fibered product \(\tilde{T}M \times_{\tilde{T}M} \tilde{Y}\) (via the isomorphism \((\Pi, \text{id}_{\tilde{Y}})\) covering the identity of \(\tilde{T}M\)), it will be convenient to abuse the notation by explicitly mentioning the base point of any element in \(\tilde{Y}\). This means that we will identify elements \(\tilde{y} \in \tilde{Y}\) as triples \((x, \hat{x}, y)\), where \((x, \hat{x}) = \tilde{\Pi}(y)\). We assume that \((\mathbb{R}_+^*, \cdot)\) acts on \(\tilde{Y}\) by fibered automorphisms:

\[
H : \mathbb{R}_+^* \times \tilde{Y} \rightarrow \tilde{Y}, \quad H(\alpha, \cdot) = H_\alpha \in \text{Aut}(\tilde{Y})
\]

(58)
as:

\[
H_\alpha(x, \hat{x}, y) = \left(x, \alpha \hat{x}, \alpha^k y\right),
\]

(59)
for some fixed \(k \in \mathbb{R}\).

In particular, the above rule means that:

- Each automorphism \(H_\alpha \in \text{Aut}(\tilde{Y})\) covers the homothety \(\chi_\alpha : \tilde{T}M \rightarrow \tilde{T}M\), defined in Definition 5.

- The action is free and proper (properness is proven by verifying that the mapping \(f : \mathbb{R}_+^* \times \tilde{Y} \rightarrow \tilde{Y} \times \tilde{Y}, (\alpha, x, \hat{x}, y) \mapsto (x, \alpha \hat{x}, \alpha^k y, x, \hat{x}, y)\) is proper; the latter holds as the projection of a compact subset of a Cartesian product onto each factor is compact).

**Note.** In the following, we do not assume a specific form of the fiber \(Z\) of \(\tilde{Y}\), we just assume that, for a given \(k\), rescaling of fiber elements \(y\) by the power \(\alpha^k\), \(\forall \alpha > 0\), makes sense; e.g., in the case of vector bundles over \(\tilde{T}M\), this makes sense for any \(k \in \mathbb{R}\), whereas for bundles whose fibers do not admit a rescaling of elements, one is forced to choose \(k = 0\). An important example of bundles \(\tilde{Y}\) are the pullback bundles \(\pi^*_{TM} \left(\mathcal{T}_n^p M\right)\), whose sections are the anisotropic tensors introduced in Definition 9.

This way (see\(^{56}\), Ch. 21), the space of orbits of the action \(H\), i.e., the set:

\[
Y = \tilde{Y}/_\sim,
\]

(60)
where the equivalence relation \( \sim \) is given by:
\[
(x,\dot{x},y) \sim (x',\dot{x}',y') \iff \exists \alpha > 0: \ (x',\dot{x}',y') = H_\alpha(x,\dot{x},y),
\]
is a smooth manifold. Moreover, \( \hat{Y} \) becomes a principal bundle over \( Y \), with fiber \( \mathbb{R}_+^* \) and projection \( \pi_Y: \hat{Y} \to Y \), \( (x,\dot{x},y) \mapsto [x,\dot{x},y] \).

**Theorem 24**

1. The manifold \( Y = \hat{Y}/\sim \) is a fiber bundle over \( PTM^+ \), with typical fiber \( Z \).

2. \( k \)-homogeneous sections \( f: \mathcal{D} \to \hat{Y} \), where \( \mathcal{D} \subset TM^+ \) is a conic subbundle, are in a one-to-one correspondence with local sections \( \gamma: \mathcal{D}^+ \to Y \), where \( \mathcal{D}^+ = \pi^+(\mathcal{D}) \subset PTM^+ \).

**Proof.**

1. First, let us define the projection:
\[
\Pi: Y \to PTM^+, \quad [(x,\dot{x},y)] \mapsto [(x,\dot{x})].
\]
This mapping is independent of the choice of representatives in the class \([(x,\dot{x},y)]\), as \( \Pi([(x,\alpha \dot{x},\alpha^k y)]) = [(x,\alpha \dot{x})] = \Pi([(x,\dot{x},y)]) \) and surjective.

A local trivialization of \( Y \) can be obtained using the principal bundle structures of both \( \hat{Y} \) over \( Y \) and of \( \hat{TM} \) over \( PTM^+ \). More precisely, start with \( \hat{V} = \hat{\Pi}^{-1}(U) \subset \hat{Y} \), where \( U \in \{U_i,U_\rho\} \subset \hat{TM} \) is a (small enough) coordinate neighborhood on which, say, \( \hat{x}^3 \) keeps a constant sign (as introduced in Section III); then, \( \hat{V} \) is diffeomorphic to \( U \times Z \).

But, on the one hand, using the principal bundle structure of \( (TM,\pi^+,PTM^+,[\mathbb{R}_+^*]) \), the coordinate neighborhood \( U \) is diffeomorphic to \( U^+ \times \mathbb{R}_+^* \), where \( U^+ = \pi^+(U) \) and, on the other hand, using the principal bundle structure of \( \hat{Y} \) over \( Y \), the coordinate neighborhood \( \hat{V} \) is in its turn, diffeomorphic to \( V \times \mathbb{R}_+^* \), where \( V := \pi_Y(\hat{Y}) \).

This way, a trivialization of \( \hat{Y} \) can be written as follows:

\[
\begin{array}{ccc}
V \times \mathbb{R}_+^* & \longrightarrow & (U^+ \times \mathbb{R}_+^*) \times Z \\
\Pi & \searrow & \downarrow \text{proj}_1 \\
U^+ \times \mathbb{R}_+^* & \searrow & \\
\end{array}
\]
It remains to show that \( V = \Pi^{-1}(U^+) \):

\( \ast \): If \([x, \dot{x}, y] \in V = \text{proj}_y(\hat{V})\), then there exists a representative \((x, \dot{x}, y) \in \hat{V} = \Pi^{-1}(U)\) of the class \([x, \dot{x}, y]\). But then, \(\Pi_0(x, \dot{x}, y) = (x, \dot{x}) \in U\), which means \([x, \dot{x}] = \Pi((x, \dot{x}, y)) \in U^+\), i.e., \([x, \dot{x}, y] \in \Pi^{-1}(U^+)\).

\( \ast \ast \): Starting with \([x, \dot{x}, y] \in \Pi^{-1}(U^+)\), we find that \([x, \dot{x}] \in U^+\); picking a representative \((x, \dot{x}) \in U\) of the class \([x, \dot{x}]\), the corresponding representative \((x, \dot{x}, y)\) of the class \([x, \dot{x}, y]\) belongs to \(\Pi^{-1}(U) = \hat{V}\). That is, the class \([x, \dot{x}, y]\) is in \(\text{proj}_y(\hat{V}) = V\).

Using the above trivialization, the corresponding fibered coordinates on \(V \subset Y\) are then obtained by discarding the \(x^3\) coordinate from the coordinates \((x^i, u^\alpha, x^3, \varepsilon^\sigma)\) on \(Y\), i.e.,

\[
\psi = (x^i, u^\alpha, \varepsilon^\sigma).
\]

2. Let \(f : \mathfrak{D} \to \hat{V}, (x, \dot{x}) \mapsto f(x, \dot{x}) \in \hat{V}_{(x, \dot{x})}\) be a \(k\)-homogeneous section, i.e., \(f(x, \alpha \dot{x}) = \alpha^k f(x, \dot{x})\), \(\forall \alpha > 0\) and define:

\[
\gamma : \mathfrak{D}^+ \to Y, \quad \gamma((x, \dot{x})) = [x, \dot{x}, f(x, \dot{x})].
\]

The mapping \(\gamma\) is independent of the choice of representatives \((x, \dot{x}) \in \hat{V}_{(x, \dot{x})}\) by virtue of the \(k\)-homogeneity of \(f\). Moreover, \((\Pi \circ \gamma)((x, \dot{x})) = [x, \dot{x}]\) for all \((x, \dot{x}) \in \mathfrak{D}\), which makes \(\gamma\) a well defined local section of \(Y\).

The correspondence \(f \mapsto \gamma\) is obviously injective. To prove surjectivity, pick an arbitrary \(\gamma \in \Gamma(Y)\) and define \(f(x, \dot{x})\), for every representative \((x, \dot{x}) \in \hat{V}_{(x, \dot{x})}\), as the third component \(y\) of the representative \((x, \dot{x}, y) \in \gamma((x, \dot{x}))\); then, \(f(x, \alpha \dot{x}) = \alpha^k y\) by the definition of equivalence classes in \(Y\), which means that \(f\) is a \(k\)-homogeneous section of \(\gamma\).

\[\blacksquare\]

**Note:** Coming back to our discussion before Remark 22. Since we fixed the group action of \(\mathbb{R}_+^*\), defined in (58), (59), on \(Y\), on each fibered chart domain \(V = \Pi^{-1}(U^+)\), we can explicitly introduce homogeneous fibered coordinates as the local coordinates

\[
(x^i, \dot{x}^i, y^\sigma) := (x^i, x^3 u^0, \dot{x}^i u^1, x^3 u^2, \dot{x}^i; (x^3)^k \varepsilon^\sigma)
\]

of an arbitrarily chosen representative of the class \([x, \dot{x}, y]\) where \(u^0 = \frac{\dot{x}^0}{x^0}\) etc. These are, obviously unique up to positive rescaling, i.e., \((x^i, \dot{x}^i, y^\sigma)\) and \((x^i, \alpha \dot{x}^i, \alpha^k y^\sigma)\) will represent the same class.

1. **Finsler functions** \(L : \mathcal{A} \to \mathbb{R}\). In this case, \(\hat{V} = T^0 M \times \mathbb{R}\) is a trivial line bundle, which means the configuration bundle \(Y = \hat{V}_{/\sim}\) is the space of orbits of the Lie group action \(H : \mathbb{R}_+^* \times \hat{V} \to \hat{V}\) given by the fibered automorphisms:

\[
H_\alpha : \hat{V} \to \hat{V}, H_\alpha(x, \dot{x}, y) = (x, \alpha \dot{x}, \alpha^2 y), \quad \forall \alpha > 0.
\]

This way, 2-homogeneous Finsler functions are identified with local sections \(\gamma\)

\[
L \mapsto \gamma \in \Gamma(Y), \quad \gamma((x, \dot{x})) = [x, \dot{x}, L(x, \dot{x})].
\]

In homogeneous fibered coordinates, the class \([x, \dot{x}, L(x, \dot{x})]\) is represented as \((x^i, \dot{x}^i, L(x, \dot{x}))\).
2. **The 1-particle distribution function of a kinetic gas**, see \(^{31,72}\), can be understood as a 0-homogeneous mapping \( \varphi : \mathcal{D} \to \mathbb{R} \), defined on some conic subbundle \( \mathcal{D} \subset \dot{T}M \). Again \( \dot{Y} = \dot{T}M \times \mathbb{R} \) and the configuration bundle is \( Y = (TM \times \mathbb{R})/\sim \), i.e. the space of orbits of the Lie group action \( H : \mathbb{R}^+_+ \times \dot{Y} \to \dot{Y} \) is given by

\[
H_{\alpha} : \dot{Y} \to \dot{Y}, H_{\alpha}(x, \dot{x}, y) = (x, \alpha \dot{x}, y), \quad \forall \alpha > 0.
\]

The corresponding section \( \gamma : \mathcal{D}^+ \to Y, \gamma[(x, \dot{x})] = [x, \dot{x}, \varphi(x, \dot{x})] \) is represented in fibered homogeneous coordinates as: \( \gamma : (x', \dot{x}') \mapsto (x', \dot{x}', \varphi(x, \dot{x})). \)

3. **0-homogeneous metric tensors** \( g : \mathcal{A} \to T^0_2(TM), g_{(x, \dot{x})} = g_{ij}(x, \dot{x})dx^i \otimes dx^j \) (which are thus treated as tensors of type (0,2) on \( TM \), operating on horizontal vector fields \( X \in \Gamma(HTM) \), see Section \( \text{II C} \)), are obtained as sections \( \gamma \) of the bundle \( Y = \dot{Y}/\sim \), where \( \dot{Y} = T^0_2(TM) \). The Lie group action \( H : \mathbb{R}^+_+ \times \dot{Y} \to \dot{Y} \) is given by

\[
H_{\alpha} : \dot{Y} \to \dot{Y}, H_{\alpha}(x, \dot{x}, y) = (x, \alpha \dot{x}, y), \quad \forall \alpha > 0.
\]

In fibered homogeneous coordinates (which are naturally induced by the coordinates \( (x') \) on \( M \)), these sections are represented as \( \gamma : (x', \dot{x}') \mapsto (x', \dot{x}', g_{ij}(x, \dot{x})). \)

Homogeneous d-tensors of any rank and any homogeneity degree can be treated similarly.

V. **FINSLER FIELD LAGRANGIANS, ACTION, EXTREMALS**

Finally, we are in the position to explicitly construct action based field theories on Finsler spacetimes. The Finsler-related geometric notions have been introduced in Section \( \text{II} \). Afterwards, we discussed the proper base manifold, \( PTM^+ \), for action integrals having homogeneous fields as dynamical variables in Section \( \text{III} \), and we demonstrated that these homogeneous fields can be understood as sections into fiber bundles over \( PTM^+ \) in Section \( \text{IV} \).

A. **Actions for fields as sections of \( PTM^+ \)**

We now display all necessary definitions needed for well defined action based field theories on Finsler spacetimes.

**Definition 25 (Fields)** A homogeneous field on a Finsler spacetime \((M, L)\) is a local section \( \gamma \) of a fibered manifold \((Y, \Pi, PTM^+)\) over the positive projective tangent bundle \( PTM^+ \).

**Definition 26 (Lagrangians)** On a configuration bundle \((Y, \Pi, PTM^+)\) over a Finsler spacetime \((M, L)\), a Finsler field Lagrangian of order \( r \) is a \( \Pi^r \)-horizontal 7-form \( \lambda^+ \in \Omega^7(J^1Y) \).

This definition is a particular instance of the general definition of Lagrangians given in Appendix A 3.

In homogeneous fibered coordinates, any Lagrangian on \( Y \) can be expressed as:

\[
\lambda^+ = \Lambda d\Sigma^+ = \mathcal{L} \text{Vol}_0.
\]
where \( \Lambda = \Lambda(x', \dot{x}', y', y''', \ldots) \) is the Lagrange function and \( d\Sigma^+ \) is an invariant volume form on an appropriately chosen open subset \( \mathcal{D}^+ \subset PTM^+ \); for instance, one can choose the canonical volume form \( (37) \) on the set \( \mathcal{D}^+ \subset PTM^+ \) of non-null admissible directions over a Finsler spacetime; in this case, we obtain the Lagrange density
\[
\mathcal{L} = \frac{\Lambda}{L^2}. \tag{62}
\]

**Note.** The pulled back form \( J'\gamma^*\lambda^+ \) (where \( \gamma \in \Gamma(Y) \)) is a differential form on \( PTM^+ \), hence, it must be invariant under positive rescaling in \( \dot{x} \). In coordinates, this becomes equivalent to the result below.

**Proposition 27** In local homogeneous coordinates corresponding to any fibered chart \( (V', \psi') \) on \( J'Y \), any Finsler field Lagrangian function \( \Lambda : V' \to \mathbb{R} \) must obey:
\[
x^i \frac{d}{dx^i} \Lambda = 0. \tag{63}
\]

**Proof.** Pick an arbitrary section of \( \Pi \), say, \( \gamma : U \to Y \), where \( U \subset PTM^+ \) is a local chart domain. The function \( \Lambda \circ J'\gamma \) is then defined on a subset of \( PTM^+ \), hence, it must be 0-homogeneous in \( \dot{x} \); that is,
\[
x^i \frac{d}{dx^i} (\Lambda \circ J'\gamma) = 0.
\]

But, from \((52)\), \( \frac{d}{dt} (\Lambda \circ J'\gamma) = (d\Lambda) \circ J'^{-1} \gamma \). Substituting into the above relation and taking into account the arbitrariness of \( \gamma \), we get the result. \( \blacksquare \)

The action attached to the Lagrangian \((61)\) and to a piece \( D^+ \subset PTM^+ \) is the function \( S_{D^+} : \Gamma(Y) \to \mathbb{R} \), given by:
\[
S_{D^+}(\gamma) = \int_{D^+} J'\gamma^*\lambda^+. \tag{64}
\]

By Proposition 20, such action integrals on timelike domains \( D^+ \) can equivalently be understood as integrals over pieces \( D \subset \mathcal{D} \), i.e. as actions formulated on the observer space. The advantage in the representation of the action as integrals on \( PTM^+ \), is that the domain of the integral does not depend on the Finsler Lagrangian.

The preparation from the previous sections, in particular, the formulation of fields as sections of a configuration bundle \((Y, \Pi, PTM^+)\), allows us now to straightforwardly apply the coordinate-free formulation of the calculus of variations for Finsler field Lagrangians.

The *variation* of the action under the flow \( \{\Phi_t\} \) of a doubly projectable vector field \( \Xi \in \mathcal{X}'(Y) \) is given by the Lie derivative, see Appendix A:
\[
\delta_\Xi S_{D^+}(\gamma) = \int_{D^+} J'\gamma^* \mathcal{L}_{J'\gamma \Xi} \lambda^+. \tag{65}
\]
A field \( \gamma \in \Gamma(Y), (x, \dot{x}) \mapsto \gamma(x, \dot{x}) \) on a Finsler spacetime is a critical section for \( S \), if for any piece \( D^+ \subset PTM^+ \) and for any \( \Pi \)-vertical vector field \( \Xi \) such that \( \text{supp}(\Xi \circ \gamma) \subset D^+ \), \( \delta_\Xi S_D(\gamma) = 0 \).

For any Lagrangian \( \lambda^+ \in \Omega_2(J'Y) \), there exists (see \((41)\), or Appendix A) a unique source form \( \delta_{\lambda^+} \in \Omega_2(J'Y) \) with \( s \leq 2r \), called the Euler-Lagrange form of \( \lambda^+ \), such that:
\[
J'\gamma^* (\mathcal{L}_{J'\gamma \Xi} \lambda^+) = J'\gamma^* \delta_{\lambda^+} - d(J'\gamma^* \mathcal{J} \Xi), \tag{65}
\]
for some \( J^\Xi \in \Omega^6(J^*Y) \). The 6-form \( J^\Xi \) (which is interpreted as a Noether current), is only unique up to a total derivative; in integral form, the above relation reads:

\[
\int_{D^+} J^\gamma (\mathcal{L}_{J^\Xi} \lambda^+) = \int_{D^+} J^\gamma i_{\lambda^+} \varepsilon(\lambda^+) - \int_{\partial D^+} J^\gamma J^\Xi.
\]

In a local contact basis, \((47)\), \( \varepsilon_{\lambda^+} \) is thus given as:

\[
\varepsilon_{\lambda^+} = \varepsilon_{\sigma} g^\sigma \wedge \text{Vol}_0.
\]

The precise meaning of the requirement that \( \varepsilon_{\lambda^+} \) is a source form is that the interior product \( i_{\lambda^+} \varepsilon_{\lambda^+} = (\tilde{\Xi}^\sigma \varepsilon_{\sigma}) \text{Vol}_0 \),

only depends on the functions \( \tilde{\Xi}^\sigma = \Xi^\sigma - \xi_i y^i_j - \xi^j y^i_j \), not on higher order components of \( \Xi \).

In order to identify the Euler-Lagrange form, \( \Pi_{\text{vertical}} \) variation vector fields \( \lambda^+ \) are sufficient. More general transformations will, yet, be used when determining energy-momentum tensors.

The field equations of \( \lambda^+ \) are then given by \( \varepsilon_{\sigma} \circ J^\gamma = 0 \).

B. Finsler gravity sourced by a kinetic gas

As an example for a field theory on Finsler spacetimes we discuss in the jet bundle language the dynamics of a Finsler spacetime sourced by a kinetic gas - a theory which is considered as an extension of general relativity \(30, 31, 43, 72\).

We first discuss the purely geometric (vacuum) field theory, where the Finsler function \( L \) itself is the dynamical field, and then add a matter Lagrangian as source of these dynamics.

1. Finsler gravity Lagrangian

We have shown above that, for theories using the 2-homogeneous Finsler function \( L : \mathcal{A} \to \mathbb{R} \) as the dynamical variable, the appropriate configuration bundle is \((60)\), with fiber \( \mathbb{R} \); we will re-denote it here as \((Y_g, \Pi_g, PT M^+)\) and the homogeneous coordinates corresponding to a fibered chart on \( Y_g \) by \((x^i, \dot{x}^i, \hat{L})\), i.e.,

comparing to the notations in Section IV, \( y^i = \hat{L} \). The hat is meant to distinguish the last coordinate function on \( Y_g \) from mappings \( L : \mathcal{A} \to \mathbb{R}, L = L(x, \dot{x}) \), i.e., from components of sections \( \gamma \) of the configuration bundle; more precisely, \( L = L \circ \gamma \).

Briefly, we have:

\[
Y_g := (TM \times \mathbb{R})/\sim, \quad \Pi_g : [(x, \dot{x}, \hat{L})] \mapsto [(x, \dot{x})].
\]

As already said above, we identify 2-homogeneous functions \( L : \mathcal{A} \to \mathbb{R} \) (where \( \mathcal{A} \subset TM \) is a conic subbundle) with sections \( \gamma \in \Gamma(Y_g) \),

\[
L \mapsto \gamma : \mathcal{A}^+ \to Y_g, \gamma[(x, \dot{x})] = [x, \dot{x}, L(x, \dot{x})].
\]

Locally, \( \gamma \) is described as: \((x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, L(x, \dot{x})) \).
On $\mathcal{Y}_g$, a Lagrangian of order $r$ is expressed in fibered homogeneous coordinates as $\lambda^+ = \Lambda d\Sigma^+$, where $d\Sigma^+$ is the canonical volume form (37) and $\Lambda = \Lambda(\tilde{x}, \tilde{x}', \tilde{L}, \tilde{L}', \ldots)$ becomes a 0-homogeneous function of $\tilde{x}$ whenever evaluated along sections $\gamma \in \Gamma(Y)$. The local contact basis of $\Omega(J'^{Y})$ is then denoted by $\{dx^i, d\tilde{x}^i, \theta, \theta', \theta_i, \ldots, \theta_i \ldots \}$, where the (unique) first order contact form is:

$$\theta = d\tilde{L} - \tilde{L}_i d\tilde{x}^i.$$  (67)

On $\mathcal{Y}_g$, it is convenient to use formal adapted derivatives:

$$\delta_i := d_i - G^j_i \tilde{d}_j,$$

where the word "formal" means that $G^j_i \in \mathcal{F}^j(\mathcal{Y}_g)$ are considered as functions on a chart of the jet bundle $J'^Y$ - constructed by the usual formula from the coordinate functions $\tilde{L}, \tilde{L}', \ldots$; i.e., only when evaluated along sections $\gamma$, they become the usual canonical nonlinear connection coefficients, defined on charts of $TM$. In particular, we get:

$$\delta_i \tilde{L} = 0.$$  (68)

Using the latter relation and (54), the contact form $\theta$ can be written in a manifestly covariant form:

$$\theta = d\tilde{L} - \tilde{L}_i \delta_i \tilde{x}^i.$$  (69)

Source forms on $J'^{Y}_g$ are locally expressed as:

$$\rho = f \theta \wedge d\Sigma^+,$$

where $f = f(\tilde{x}, \tilde{x}', \tilde{L}, \tilde{L}', \ldots)$.

On the bundle $\mathcal{Y}_g$, the following Lagrangian is a natural (generally covariant) one:

$$\lambda^+_g = R_0 d\Sigma^+,$$

where, again, $R_0$ is constructed by means of the usual formula, in terms of the the coordinate functions $\tilde{L}, \tilde{L}_j$ etc.; using (13), (14) and (15), we find that $\lambda^+_g$ contains fourth order derivatives of $L$, i.e., $\lambda^+_g \in \Omega^4(J'^{Y}_g)$. Naturality of this Lagrangian follows taking into account that, along any section, both $R_0$ (which is an invariant scalar, since it is constructed using only operations with d-tensors) and $d\Sigma^+$ are invariant under coordinate changes on $TM$ induced by arbitrary coordinate changes on $M$, see also the Appendix A 3 for a discussion of natural Lagrangians.

The Euler-Lagrange form of $\lambda^+_g$ is $E = \theta \wedge d\Sigma^+$, where:

$$E = \frac{1}{2} g^{ij} (LR_0)_{i,j} - 3R_0 - g^{ij}(P_{ij} - P_i P_j + (\nabla P_i)_{j}),$$  (70)

and thus the field equation, which determines the extremal points of the action is: $E = 0$.

2. **Kinetic gas Lagrangian**

It turned out that there exists a physical field which naturally couples to Finsler geometry and can act as source for the dynamics of a Finsler spacetime. This field is the 1-particle distribution function (1PDF) $\varphi$ of a kinetic gas, which describes the dynamics of a kinetic gas on the tangent bundle of spacetime $^{73-75}$. 
Usually, the gravitational field of a kinetic gas is described in terms of the Einstein-Vlasov equations\(^{26}\), which, however, only take the averaged kinetic energy of the particles constituting the gas into account. By coupling the 1PDF of the kinetic gas directly to the Finslerian geometry of spacetime, this averaging can be omitted; thus, the velocity distribution of the gas particles contributing to the gravitational field can be fully taken into account\(^{31,72}\).

A kinetic gas is defined as a collection of a large number of particles, whose properties are encoded into 1PDF, i.e. a function

\[
\varphi : \mathcal{E} \to \mathbb{R}, \quad \varphi = \varphi (x, \dot{x}).
\]

Its interpretation is the following. The number of particles crossing a given (6-dimensional) hypersurface \(\sigma \subset \mathcal{E}\) is

\[
N[\sigma] = \int_{\sigma} \varphi \, \text{vol},
\]

where \(\text{vol} = \frac{1}{3!} d\omega \wedge d\omega \wedge d\omega\) is the canonical invariant volume form on \(\sigma\), determined by the Lorentzian (or pseudo-Finslerian) structure on spacetime. This volume form induces a coupling between the geometry of spacetime and the 1PDF. Prolonging \(\varphi\) to \(TM\) by 0-homogeneity as discussed in Section IV C, we can equivalently regard \(\varphi\) as a function defined on \(\pi^+ (\mathcal{E}) \subset PTM^+\). The partial functions \(\varphi_x = \varphi (x, \cdot)\) are all assumed to be compactly supported (which is physically interpreted as the fact that the speeds of the particles composing the gas have an upper bound lower than the speed of light).

The Lagrangian defining the dynamics of the kinetic gas on a Finsler spacetime is, see\(^{31}\),

\[
\lambda^+_0 = \frac{m |\det(g)|}{L^2} \text{Vol}_0 = \mathcal{L}_m \text{Vol}_0
\]

where \(\mathcal{L}_m = m |\det(g)| / L^2\) depends on \(x^i, \dot{x}^i, \dot{L}_i, \ldots, \dot{L}_{ij}\) and \(m\) is the mass parameter of the gas particles, here assumed all of the same mass for simplicity. The \(\dot{L}_i, \ldots, \dot{L}_{ij}\) dependence in the Lagrange density appears due to the dependence of the volume form on the Finsler Lagrangian and on the Finsler metric tensor \(g\). Yet, for the sake of uniformness (since we will couple it to \(\lambda^+_g\), which lives on \(J^4Y_g\)), we will regard \(\mathcal{L}_m\) as a function on \(J^4Y_g\), rather than on \(J^2Y_g\).

Consider on \(J^4Y_g\) the Lagrangian

\[
\lambda^+_g = \frac{1}{2\kappa^2} \lambda^+_0 + \lambda^+_m,
\]

then, calculation of the Euler-Lagrange form by variation with respect to \(L\) leads,\(^{31,72}\), to the Finsler gravity equations sourced by a kinetic gas:

\[
\frac{1}{2} g^{ij} (LR_0)_{ij} - 3R_0 - g^{ij} (P_{ij} - P_i P_j + (\nabla P_i)\cdot j) = -\kappa^2 \varphi,
\]

where \(\kappa\) is the gravitational coupling constant.

We note that the above equation determines nonzero values of \(L\); accordingly, in the construction of the actions corresponding to \(\lambda^+_0\) and \(\lambda^+_m\), one must only consider non-lightlike domains for \(L\). This is a difference from actions of metric field theories built directly over the spacetime manifold \(M\), which do not distinguish between possible causal properties of vectors.
VI. ENERGY-MOMENTUM DISTRIBUTION TENSOR

An important concept in physics, which is derived from the action of a field theory, is the energy-momentum tensor. One way to interpret the energy-momentum tensor mathematically is that it measures "the response of the matter Lagrangian to compactly supported diffeomorphisms of spacetime". This interpretation will be preserved in Finslerian field theory. In other words, naturality (general covariance, or general invariance) of Lagrangians will still be understood as invariance under (lifted) diffeomorphisms of spacetime - though, in this case, the base of our configuration bundle is not spacetime, but its positively projectivized tangent bundle. This will require an extension of the technique presented in and will result in a "weaker" (averaged) energy-momentum conservation law.

A. Generally covariant Lagrangians

To identify the energy-momentum tensor in our construction of field theories on Finsler spacetimes, we need some preparations:

1. Lifts of diffeomorphisms \( \phi_0 \) of \( M \) into doubly fibered automorphisms of \( Y \), that cover the natural lifts \( \phi^0 \) of \( \phi_0 \) to \( PTM^+ \), see the diagram (55), since a priori diffeomorphisms of \( M \) do not act on \( Y \).

2. A splitting of the total Lagrangian \( \lambda^+\) of the theory into a background (vacuum) Lagrangian \( \lambda_g^+ \) and a matter one \( \lambda_m^+ \) and, accordingly, of the variables of the theory into background and dynamical ones. The background Lagrangian (which we denote by \( \lambda_g^+ \)) will only depend on the background variables (e.g., metric components, Finsler function etc), whereas the matter Lagrangian \( \lambda_m^+ \) will depend on all the variables, see. Roughly, denoting the background coordinates by \( y_B^\sigma \) and non-background or dynamical variables \( y_D^\sigma \), we have:

\[
\lambda^+(y_B^\sigma,...,y_B^{j,...},y_D^\sigma,...,y_D^{j,...}) = \lambda_g^+(y_B^\sigma,...,y_B^{j,...}) + \lambda_m^+(y_B^\sigma,...,y_B^{j,...},y_D^\sigma,...,y_D^{j,...}).
\]

For instance, in general relativity, one has \( y_B^\sigma = g^{ij} \), whereas \( y_D^\sigma \) can be, e.g., the electromagnetic 4-potential. The names "background" vs. "dynamical" come from the fact that the Lagrangian can be split into a part \( \lambda_g^+ \), which only contains the background variables, and a part \( \lambda_m^+ \) which contains all information about the dynamical variables and their coupling to the background variables. Hence, even if one leaves \( \lambda_g^+ \) aside and fixes a value of the background fields, one can study the dynamics of the dynamical fields coupled to a fixed background.

Then, under the assumption that the matter Lagrangian \( \lambda_m^+ \) is generally covariant (see again the end of Appendix A.3), it will be invariant under any one-parameter group of canonical lifts of diffeomorphisms of the spacetime manifold \( M \), thus giving rise to conserved Noether currents \( \mathcal{J}^\Xi \) (where \( \Xi = \mathcal{Z}^\xi_0 \), is the canonical lift to \( Y \) of a diffeomorphism generating vector field \( \xi_0 \) from \( M \)). Roughly speaking, the energy-momentum tensor will be given by the correspondence \( \xi_0 \mapsto \mathcal{J}^\Xi \).

In the case of Finsler spacetimes, the fundamental background variable is the Finsler Lagrangian \( L \) itself. Yet, the whole construction can be done in a completely similar manner, e.g., for the Finsler metric tensor components \( g_{ij} \), as background variables.

---

4 Such lifts exist, e.g. when \( Y \) is a bundle of \( k \)-homogeneous d-tensors, which is the \( \mathbb{R}^*_+ \)-orbit space of a bundle \( \hat{Y} \) of d-tensors on \( \hat{T}M \). Diffeomorphisms \( \phi_0 \) of \( M \) are naturally lifted into fibered automorphisms \( d\phi_0 \) of \( TM \) and further, by tensor lifting to \( \hat{Y} \).
Consider a fibered product

\[ Y := Y_g \times_{PTM^+} Y_m \]

over \( PTM^+ \), where \( Y_g = (TM \times \mathbb{R})_\sim \) was constructed in Section V B 1 and \( Y_m \) is both a a fiber bundle over \( PTM^+ \) and a natural fiber bundle over \( M \). In particular, \( Y \) has a double fibered manifold structure:

\[ Y \xrightarrow{\Pi} PTM^+ \xrightarrow{\pi_m} M \]

We denote the homogeneous coordinates corresponding to a doubly fibered chart on \( Y \) by \((x^i, x^\lambda, \hat{L}, y^B, \hat{y}^B)\), where \( y_B = \hat{L} \) is the coordinate on the fiber of \( Y_g \) and \( y^B_D \) are local coordinates on the fiber of \( Y_m \).

As both \( Y_m \) and \( Y_g \) are natural bundles over \( M \) it follows that any vector field \( \xi_0 \in \mathcal{X}(M) \) admits a canonical lift \( \Xi \in \mathcal{X}(Y) \).

Consider a generally covariant Lagrangian \( \lambda_m^+ \in \Omega(J^r Y) \):

\[ \lambda_m^+ = \mathcal{L}_m(x^i, x^\lambda, \hat{L}, \hat{\lambda}, \hat{\lambda}_i, \ldots, \hat{\lambda}_{i_1 \ldots i_r}, y^B, \ldots, y^B_{D_1 \ldots D_r}) Vol_0, \]

which will be interpreted as the matter Lagrangian (as already mentioned above, the total Lagrangian of the theory will be obtained as \( \lambda^+ := \lambda^+_g + \lambda^+_m \)). Since \( \lambda^+_m \) is generally covariant, for any compactly supported vector field \( \xi_0 \in \mathcal{X}(M) \), \( \lambda^+_m \) is invariant under the flow of the \( r \)-th jet prolongation of the canonical lift \( \Xi := \hat{\xi}(\xi_0) \), i.e.,

\[ \mathcal{L}_{J^r \Xi} \lambda^+_m = 0. \quad (74) \]

In the following, we will explore in detail the consequences of this invariance of \( \lambda^+_m \).

### B. The energy momentum distribution tensor and the energy momentum density

We will first give the technical precise definition of the energy-momentum distribution tensor, and demonstrate the concept on the example of the kinetic gas at the end of this subsection.

Assume \( \{ \phi_{0, \epsilon} \} \) is a 1-parameter group of compactly supported diffeomorphisms of \( M \), generated by \( \xi_0 \in \mathcal{X}(M) \), \( \xi_0 = \xi^j \partial_j \). Then:

1. Each \( \phi_{0, \epsilon} \) is first naturally lifted to \( TM \), as \( \phi_{\epsilon} := d\phi_{0, \epsilon} \). The generator of \( \{ \phi_{\epsilon} \} \) is the complete lift \( \xi \in \mathcal{X}(TM) \) of \( \xi_0 \):

\[ \xi = \xi^i \partial_i + \xi^I \hat{\partial}_I, \quad \xi^i = \xi^i, \xi^I = \xi^i. \quad (75) \]

Since the canonical lift \( \xi \) is 0-homogeneous, we can identify it with a vector field on \( PTM^+ \) (more precisely, with its pushforward by \( \pi^+ \)), see Section III A 2.

2. Further, taking into account that \( Y_g = (TM \times \mathbb{R})_\sim \) is obtained as a quotient space of the trivial bundle \( TM \times \mathbb{R} \), the canonical lift, \( \Phi_{g, \epsilon} : Y_g \rightarrow Y_g \) of \( \phi_\epsilon \) is also a trivial one i.e., it acts on the fiber variable \( \hat{L} \) as the identity:

\[ \Phi_{g, \epsilon}[(x, \hat{x}, \hat{L})] = [(\phi_\epsilon(x, \hat{x}), \hat{L})]; \]
The above mapping is well defined (i.e., independent on the choice of the representative of the class \([x, \dot{x}, \dot{L}]\)), due to the linearity of \(\Phi_k\) in \(\dot{x}\). As the lifted diffeomorphisms act trivially on \(\dot{L}\), the generator \(\dot{\xi}\) is canonically lifted into a vector field \(\Xi_g \in \mathcal{X}(Y_g)\), with vanishing \(\frac{\partial}{\partial L}\) component, i.e.:

\[
\Xi_g = \dot{\xi}^i \partial_i + \dot{\xi}^0 \partial_0 + 0 \frac{\partial}{\partial L}.
\]

3. According to our first assumption at the beginning of Section VI A, there exists a canonical lift \(\dot{\xi}\) to \(Y_m\), into some vector field \(\Xi_m\) of the form \(\Xi = \dot{\xi}^i \partial_i + \dot{\xi}^0 \partial_0 + \Xi^\sigma \frac{\partial}{\partial \sigma}\). All in all, we obtain that the canonical lift of \(\xi_0 \in \mathcal{X}(M)\) to the fibered product \(Y = Y_g \times_{PT M^+} Y_m\) is expressed in a fibered chart by adding to \(\dot{\xi}\) the contributions describing the transformation of each of the fiber variables

\[
\Xi = \dot{\xi}^i \partial_i + \dot{\xi}^0 \partial_0 + 0 \frac{\partial}{\partial L} + \Xi^\sigma \frac{\partial}{\partial \sigma},
\]

where, see\(^7\), \(\Xi^\sigma\) are functions of the coordinates \(x^i, \dot{x}^i, y^\sigma_{x^i}, \ldots, y^\sigma_{x^i \dot{x}^i}\), and of a finite number of the derivatives of \(\dot{\xi}^i\).

**First variation formula.**

Accordingly, the Euler-Lagrange form \(\mathcal{E}(\lambda^+_m)\) will be split into a \(Y_g\) and a \(Y_m\)-component as

\[
\mathcal{E}(\lambda^+_m) = \mathcal{E}_g(\lambda^+_m) + \mathcal{E}_m(\lambda^+_m),
\]

where:

\[
\mathcal{E}_g(\lambda^+_m) = \frac{\delta \mathcal{L}_m}{\delta \lambda^+_m} \theta \wedge \text{Vol}_0, \quad \mathcal{E}_m(\lambda^+_m) = \frac{\delta \mathcal{L}_m}{\delta y^\sigma_{x^i}} \theta^\sigma \wedge \text{Vol}_0,
\]

and \(\theta^\sigma = dy^\sigma_{x^i} dx^i - y^\sigma_{x^i} dx^i\). Since \(h \Omega_{\nu \Xi} \lambda^+_m = 0\) (which follows from the invariance condition (74)), this leads to:

\[
h \dot{\iota}_{\nu \Xi} \mathcal{E}_g(\lambda^+_m) + h \dot{\iota}_{\nu \Xi} \mathcal{E}_m(\lambda^+_m) - h d \mathcal{F}^Z = 0.
\]

But, on-shell for the variables \(y^\sigma_{x^i}\), i.e., along sections \(\gamma := (L, y^\sigma_{x^i})\) such that the “matter field”, i.e. the section \(\gamma_m : PT M^+ \to Y_m, (x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, y^\sigma_{x^i}(x^i, \dot{x}^i))\), is critical for \(\lambda^+_m\), the \(\mathcal{E}_m\)-term above vanishes, i.e.:

\[
h \dot{\iota}_{\nu \Xi} \mathcal{E}_g(\lambda^+_m) - h d \mathcal{F}^Z \simeq_{\gamma_m} 0,
\]

where \(\simeq_{\gamma_m}\) means equality on-shell for the matter field \(\gamma_m\).

**The energy-momentum distribution tensor.**

The surviving Euler-Lagrange component \(h \dot{\iota}_{\nu \Xi} \mathcal{E}_g(\lambda^+_m)\) in (78) can again be split into a linear expression in \(\dot{\xi}^i\) and a divergence expression; the latter will couple with \(hd \mathcal{F}^Z\) into a boundary term and will provide the building block of the energy-momentum distribution tensor \(\Theta\). More precisely,

**Lemma 28** For any natural Finsler field Lagrangian \(\lambda^+_m \in \Omega_1(J^r Y)\), there exist unique \(\mathcal{F}(M)\)-linear mappings \(\Theta : \mathcal{X}(M) \to \Omega(J^r Y), \mathcal{B} : \mathcal{X}(M) \to \Omega(J^{r+1} Y)\), with \(\Pi^r\) (respectively, \(\Pi^{r+1}\))-horizontal values (where \(s \leq 2r\)) such that, for any \(\xi_0 \in \mathcal{X}(M)\):

\[
h \dot{\iota}_{\nu \Xi} \mathcal{E}_g(\lambda^+_m) = \mathcal{B}(\xi_0) + h d \Theta(\xi_0).
\]
Further, using fibered coordinate changes as the components of a tensor on $M$ where in the second equality we used:  
\[ \dot{\mathcal{L}} = \frac{\delta \mathcal{L}}{\delta \theta} \theta \wedge \text{Vol}_0 = \frac{-1}{2} \bar{\mathfrak{T}} \dot{\mathcal{L}} \theta \wedge d\Sigma^+, \]  
where $\bar{\mathfrak{T}}$ is a $0$-homogeneous scalar which acts as source term for Finsler gravity equations (70), and the factor $\dot{\mathcal{L}}$ is introduced to ensure this degree of homogeneity (as both $\dot{\mathcal{L}} \theta$ and $d\Sigma^+$ are $0$-homogeneous.)

The precise expression of $\bar{\mathfrak{T}}$ depends on the chosen volume form. For instance, if $d\Sigma^+$ is the canonical volume form (37), then:

\[ \bar{\mathfrak{T}} = -2 \frac{L^3}{|\text{det } g|} \frac{\delta \mathcal{L}}{\delta \mathcal{L}} \text{Vol} - \Sigma. \]  
Since $\lambda^+$ is a natural Lagrangian, $\frac{\delta \mathcal{L}}{\delta \mathcal{L}}$ is a scalar density and, accordingly, $\bar{\mathfrak{T}}$ is a scalar invariant. Then, inserting into $\theta$ the lift (76) of $\xi_0$, this becomes:

\[ i_{\mathcal{L}} \theta = -2 \xi_i (\dot{\xi}^i + G^j \dot{\xi}^j) = -2 \xi_i (\dot{\xi}^i, \dot{\xi}^j + G^j \dot{\xi}^i) = -2 \xi_i \nabla \xi^i, \]

where in the second equality we used: $\dot{\xi}^i = \dot{\xi}^i, \dot{\xi}^j$. We can thus rewrite (82) as:

\[ i_{\mathcal{L}} \theta = -2 \xi_i \nabla \xi^i d\Sigma^+. \]

Taking into account that $\nabla \dot{\xi}_i = 0$ and $\dot{\mathcal{L}} = 0$, this can be uniquely split into a linear term in $\xi^i$ and the divergence of a linear term in $\xi^i$:

\[ i_{\mathcal{L}} \theta = \nabla (\bar{\mathfrak{T}} \dot{\mathcal{L}} \xi^i - \xi_i \dot{\mathcal{L}} - \nabla \xi^i d\Sigma^+. \]  
Then, using $\dot{x}^i = 0$ and $\nabla = \dot{x}^j D_j$, we can rearrange the divergence term as

\[ \nabla (\bar{\mathfrak{T}} \dot{\mathcal{L}} \xi^i) = (\bar{\mathfrak{T}} \dot{\mathcal{L}} \xi_i \dot{x}^i)_{ij}, \]

which suggests the notation:

\[ \Theta^i_j := \bar{\mathfrak{T}} \dot{\mathcal{L}} \xi_i \dot{x}^i \]  
As $\bar{\mathfrak{T}}$ is a scalar invariant, the functions $\Theta^i_j$, defined on the given fibered chart, transform under induced fibered coordinate changes as the components of a tensor on $M$ (equivalently, as d-tensor components on
Also, noticing that the last term in (83) can be written as: \( \xi^i \hat{s}_i \hat{L}^{-1} \nabla \xi = \xi^i \dot{x}_i \hat{L}^{-1} \dot{\xi}^i \), this suggests to introduce the mappings \( \Theta : \mathcal{X} (M) \to \Omega_6 (J^r Y) \), \( \mathcal{R} : \mathcal{X} (M) \to \Omega_7 (J^{r+1} Y) \) given by

\[
\Theta (\xi_0) = (\Theta^j \xi_i ) \delta^j_0 \, d \Sigma^+, \quad (86)
\]

\[
\mathcal{R} (\xi_0) = -\Theta^j \delta_0^{ij} \xi^i \, d \Sigma^+, \quad (87)
\]

(where \( \xi_0 = \xi^i \delta_i \)). These mappings are well defined, i.e., independent on the chosen coordinate charts; moreover, they have \( \Pi^r \) (respectively, \( \Pi^{r+1} \))-horizontal values, they are both linear in \( \xi \) and obey (79), which completes the proof of the existence. Uniqueness of \( \mathcal{R} \) and \( \Theta \) follows from the uniqueness of the splitting (83) and the arbitrariness of \( \xi^i \).

**Note.** The proof of the above result is based on a similar idea to the one of Lemma 2 in \(^79\). The essential difference, in the Finslerian case, is that naturality of Lagrangians is based on the group of diffeomorphisms of \( M \) (and not of \( PTM^+ \) as one would have expected following\(^79\)), i.e., naturality comes from a manifold of lower dimension than the one of the base space of our configuration manifold \( Y \). This will result, as we will see below, in a “weaker” (averaged) form of the energy-momentum balance law.

Actually, taking into account (85), in homogeneous fibered coordinates, \( \Theta \) is expressed as:

\[
\Theta = \Theta^j dx^j \otimes i_\delta d \Sigma^+ = \Xi (\dot{F}_j dx^j) \otimes i_{\delta^0_\delta} d \Sigma^+, \quad (88)
\]

where \( \dot{F} = \sqrt{|L|} \); a quick computation shows that \( \dot{L}^{-1} \dot{x}_i \dot{x}_j = \dot{F}_i \dot{F}_j \), regardless of the sign of \( \dot{L} \). Equivalently, in a coordinate-free writing:

\[
\Theta = \Xi \omega^+ \otimes i_\delta d \Sigma^+, \quad (89)
\]

where we have identified, by abuse of notation, the Reeb vector field \( \ell = \ell^0 \delta_0 \in \mathcal{X} (\omega_0^+) \) with the vector field on \( J^{r+1} Y \) obtained by replacing \( \delta_0 \) with the formal total adapted derivative \( \delta_i \), i.e., with: \( \ell^i \delta_i \in \mathcal{X} (J^{r+1} Y) \). In the same fashion, the values \( \omega_{(\xi, \delta)} \) of the mapping \( \omega^+ : \omega_0^+ \to \Omega_1 (M) \) are identified with their pullbacks to \( J^{r+1} Y \).

**Definition 29 (Energy-momentum distribution tensor)** The energy-momentum distribution tensor associated to a natural Lagrangian \( \lambda_m^+ \) on a bundle \( Y = Y_s \times_{PTM^+} Y_m \), which is natural over a Finsler spacetime \( M \), is the \( \mathcal{F} (M) \)-linear mapping \( \Theta : \mathcal{F} (M) \to \Omega_6 (J^r Y) \) defined by (89).

**Definition 30 (Energy-momentum scalar)** We call the function \( \Xi : \omega_0^+ \to \mathbb{R} \), defined by the relation (80), and explicitly given by (81), the energy-momentum scalar.

We will call the \( \mathcal{F} (M) \)-linear mapping \( \mathcal{R} : \mathcal{F} (M) \to \Omega_7 (J^{r+1} Y) \) defined by (87), the balance function, as energy-momentum conservation (or energy-momentum balance) law is naturally characterized in terms of \( \mathcal{R} \), as we will see below.

**Averaged energy-momentum conservation law.** Consider, in the following, local sections \( \gamma \in \Gamma (Y) \) such that \( \text{supp}(J^r \gamma \lambda_m^+) \subset \mathcal{F}^+ \). This way, it makes sense to integrate the form \( J^r \gamma \epsilon_x \delta^0_\gamma (\lambda_m^+) \) on the entire set \( \mathcal{F}^+ = \mathcal{O}^+_x \) of timelike directions at \( x \).

Consider a piece \( D_0 \subset M \) and denote by

\[
\mathcal{F}^+ (D_0) := \bigcup_{x \in D_0} \mathcal{F}^+_x = \bigcup_{x \in D_0} \mathcal{O}^+_x,
\]
Theorem 31

We are now able to prove the following result.

Proof.

(\ref{eq:energy-momentum}) holds, and this gives:

\[
\int_{\mathcal{I}^+(D_0)} J^\gamma \iota_{\nu} \mathcal{E}_{\nu}(\lambda_m^+) = \int_{\partial \mathcal{I}^+(D_0)} J^{i+1} \gamma^i \mathcal{B}(\xi_0) + \int_{\mathcal{I}^+(D_0)} J^\gamma \Theta(\xi_0). \tag{90}
\]

But, on-shell for \( \gamma_m \), we have, according to (\ref{eq:gamma-m}): 

\[ J^\gamma \iota_{\nu} \mathcal{E}_{\nu}(\lambda_m^+) \sim \gamma_m \]

We are now able to prove the following result.

Theorem 31

Consider a bundle \( Y_m \) over \( PTM^+ \), which is natural over \( M \), and an arbitrary section \( \gamma = (L, \gamma_m) \in \Gamma(Y \times_{PTM} Y_m) \) such that \( \text{supp}(J^\gamma \lambda_m^+) \subset \mathcal{I}^+ \), then:

1. **Averaged energy-momentum conservation law**: At any \( x \in M \) and in any corresponding fibered chart:

\[
\int_{\mathcal{I}^+(x)} (\Theta_{i\lbrack j} \circ J^{i+1} \gamma) d\Sigma^+ = 0, \tag{92}
\]

where \( d\Sigma^+ = d^4x \wedge d\Sigma^+_x \).

2. **\( \Theta(\xi_0) \) is a “corrected Noether current”, i.e.,** for any \( \xi_0 \in \mathcal{I}(M) \)

\[
\int_{\partial \mathcal{I}^+(D_0)} J^\gamma \Theta(\xi_0) = \int_{\mathcal{I}^+(D_0)} J^\gamma \mathcal{Z}, \tag{93}
\]

where \( \mathcal{Z} \) denotes the canonical lift of \( \xi_0 \) to \( Y \).

**Proof.**

1. Fix \( x_0 \in M \). Consider an arbitrary piece \( D_0 \subset M \) containing \( x_0 \) as an interior point and an arbitrary \( \xi_0 \in \mathcal{I}(M) \) with support contained in \( D_0 \).

Now, let us have a look at the boundary term in (\ref{eq:boundary-term}). Since the support of the integrand, at every \( x \in M \), is strictly contained in \( \mathcal{I}^+ \), the only possible nonzero values are obtained at points \( \{ [x, \dot{x}] \} \) with \( x \in \partial D_0 \). But, at these points, \( \xi_0 \) identically vanishes (hence also \( \mathcal{Z} = 0 \), as \( \mathcal{Z} \) is built from \( \xi \) and its derivatives), which means that this boundary term is actually zero. It follows:

\[
\int_{\mathcal{I}^+(D_0)} J^{i+1} \gamma^i \mathcal{B}(\xi_0) \sim_{\gamma_m} 0. \tag{94}
\]

In coordinates, this is:

\[
\int_{\mathcal{I}^+(D_0)} (\Theta_{i\lbrack j} \circ J^{i+1} \gamma) \xi_i^j d\Sigma^+ \sim_{\gamma_m} 0.
\]

Squeezing \( D_0 \) around \( x_0 \) such that \( D_0 \) is contained into a single chart domain, the above integral can be written as an iterated integral

\[
\int_{D_0} (\mathcal{I} \circ J^{i+1} \gamma) d\Sigma^+ = \int_{D_0} (\Theta_{i\lbrack j} \circ J^{i+1} \gamma) d^4x,
\]

which, taking into account the arbitrariness of \( \xi_i^j \), leads to the result.
2. follows then immediately from (91) and 1.

Relation (93) says that, the energy-momentum tensor \( \Theta(\xi_0) \) is, at least up to a term which does not contribute to the integral (93)), the conserved Noether current \( T^\xi \) - i.e. (see also 77), it gives the correct notions of energy and momentum of the system under discussion.

**Remark 32** Taking into account that \( O^x + x = T^x + x \), the averaged conservation law can be rewritten as:

\[
\int_{O^x + x}(\Theta^{ij}(\xi_0)|_x)_{x,\dot{x}} d\Sigma^+ = 0.
\]

(95)

It is worth noting that, due to the fact that naturality of Lagrangians comes from \( M \), which is a space of lower dimension than the one of the space \( PTM^+ \) on which the action integral is considered, in the above relation, integration over \( O^x + x \) (or, equivalently, \( T^x + x \)) cannot be removed, i.e., we can typically only establish an averaged conservation law. This is a distinctive feature of Finslerian field theory.

**Energy-momentum density on M**

The mapping \( \Theta: \mathcal{X}(M) \to \Omega(Y) \) gives rise to an energy-momentum tensor density on \( M \), by averaging over observer (or timelike) directions \( O^x + x = \pi^+\left(O^x x\right) \). Then, for any section \( \gamma \in \Gamma(Y) \) such that \( \text{supp}(J^y \gamma^+ \lambda^+_m) \subset \mathcal{F}^+ \), set

\[
\mathcal{T}^y_j(x) := \int_{O^x + x}(\Theta^{ij}(\xi_0)|_x)_x d\Sigma^+,
\]

(96)

Under the above assumption this integral is finite, so the result is well defined. Moreover, given the expression of \( d\Sigma^+ \), the functions \( \mathcal{T}^y_j(x) \) represent the components of a tensor density on \( M \).

**Example: the energy momentum distribution tensor of a kinetic gas.**

The kinetic gas example, which motivated the whole above construction, has been previously presented from a somewhat pedestrian perspective. In \( Eqs.\ (42)-(43) \), the maps \( \Theta \) and \( B \), can be read off. We briefly identify these maps here from the more abstract and mathematically precise construction we presented.

In the case of a kinetic gas discussed in Section V B 2, the kinetic gas Lagrangian (72) is given by \( \lambda^+_m = m\varphi d\Sigma^+ \), where \( \varphi \) is the 1-particle distribution function, reinterpreted as a function of \( x, \dot{x}, L \) and its derivatives,

\[
\varphi(x,\dot{x}) = f(x,\dot{x},L(x,\dot{x}),...L_{ij}(x,\dot{x}))
\]

and \( d\Sigma^+ \) is chosen as the canonical volume form (37). Varying \( \lambda^+_m \), we use (81) to obtain \( \Sigma := \frac{1}{2} m\varphi \); accordingly, the energy-momentum tensor distribution \( \Theta \) has the local components, compare to (85),

\[
\Theta^y_j = \frac{1}{2L} m\varphi \dot{x}^i \dot{x}_j.
\]

For any kinetic gas, the averaged conservation law (95) holds.

In particular, for collisionless gases, it is known that \( \varphi \) is subject to the Liouville equation \( \nabla \varphi = 0 \), equivalently:

\[
D_t \varphi = 0.
\]
Taking into account that $l_{ij} = 0$, we notice that the Liouville equation is nothing else than a pointwise covariant conservation law of $\Theta$:

$$D_\delta \Theta^j_{\ j} = 0.$$ 

**Particular case: Lorentzian spaces.**

On a Lorentzian manifold $(M, a)$, the quantities

$$T^i_{\ j} = \frac{1}{\sqrt{|\det a|}} \mathcal{T}^i_{\ j}$$

represent the components of a tensor of type (1,1) on $M$, and their Levi-Civita covariant derivatives are, just the integrals of the Chern covariant derivatives of $\Theta$:

$$T^i_{\ j\ i} = (\frac{\sqrt{|\det a|}}{\epsilon^i_{\ j}}) \int J^* \gamma(\mathcal{T}^i_{\ j\ i}(x, \dot{x})d\Sigma_x.$$  

Hence, the energy-momentum conservation law (92) reads

$$T^i_{\ j\ i} = 0.$$  

In the particular case of kinetic gases on a Lorentzian spacetime, our expression (96) of the energy-momentum density agrees to the known one, see (75).

It is important to note that, in general Finsler spacetimes, we have no metric tensor on $M$, hence (97) makes no sense. All we can get is an energy-momentum tensor density on $M$, by averaging over observer directions as in (96) and, accordingly, the conservation law (95) of the energy-momentum distribution $\Theta$.

**VII. SUMMARY AND OUTLOOK**

In this article we have proposed a general framework for action based field theories on Finsler spacetimes. The starting point of our construction is the assumption that physical fields are homogeneous sections of suitable bundles defined over (conic subbundles of) the tangent bundle of a Finsler spacetime. Using the assumption of homogeneity, we have constructed an equivalent description of fields as sections of bundles over the positive projective tangent bundle $PTM^+$ instead. This step is crucial for a well-defined application of the variational principle, as it allows for variations with compact support within $PTM^+$, which is not possible in the aforementioned approach using homogeneous sections over the tangent bundle. Within this framework, we studied the implications of general covariance, and derived the corresponding conserved energy-momentum distribution. As a particular example, we studied the kinetic gas.

Since the framework we propose is kept very general, it can be applied to a wide range of conceivable theories. The most natural class of fields to study, besides the kinetic gas, would be d-tensor fields. The latter provide a simple generalization of tensor fields on the spacetime manifold, which attain a dependence on directions in the tangent space, in addition to their dependence on spacetime. This additional dependence could be employed to model a velocity-dependent interaction between such fields with observers or particles. Such a dependence would be expected in an effective description of the quantum nature of spacetime, and leads to a modified dispersion relation for highly energetic particles, which could possibly be detected in observations. An ongoing effort is to extend our construction to a well defined notion of spinors and spinor field theories on general Finsler spacetimes.

Another potential application of our proposed framework is to address the so far unexplained observations in cosmology. The well-known standard model of cosmology, coined $\Lambda$CDM model as it models 95% of the matter content of the universe as dark energy $\Lambda$ and cold dark matter (CDM), both of which
have so far eluded direct detection, is under growing tension due to discrepancies between the measured values of the Hubble parameter in different observations. The correct interpretation of these observations depends crucially on understanding the propagation of electromagnetic radiation (and, with the advent of multi-messenger astronomy, also of gravitational waves, neutrinos and high-energetic cosmic particles). A modified propagation law, as it could arise for a field propagating on a Finsler spacetime background, could therefore provide alternative explanations that might resolve the observed tension.

ACKNOWLEDGMENTS

C.P. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project Number 420243324. M.H. was supported by the Estonian Research Council grant PRG356 “Gauge Gravity” and by the European Regional Development Fund through the Center of Excellence TK133 “The Dark Side of the Universe”. The authors would like to acknowledge networking support by the COST Action QGMM (CA18108), supported by COST (European Cooperation in Science and Technology). Also, they would like to express their thanks to the anonymous JMP referee, for his/her useful comments and questions.

This article may be downloaded for personal use only. Any other use requires prior permission of the authors and AIP Publishing. This article appeared in the Journal of Mathematical Physics and may be found at https://aip.scitation.org/doi/10.1063/5.0065944.

DATA AVAILABILITY STATEMENT

The study presented in this article is of purely theoretical and mathematical nature. All results and all sources on which these results are based are cited. Data sharing is not applicable to this article as no new data were created or analyzed in this study

Appendix A: Jet bundles and the coordinate-free calculus of variations

In this appendix we briefly present the jet bundle formalism, which allows for a coordinate-free description of calculus of variations, in terms of differential forms; for more details, we mainly refer to the monograph\textsuperscript{71}.

1. Fibered manifolds and their jet prolongation

A fibered manifold is a triple $(Y, \pi, X)$, where $X, Y$ are smooth manifolds with $\dim X = n$, $\dim Y = n + m$ and $\pi: Y \to X$ is a surjective submersion. The level sets $Y_x = \pi^{-1}(x)$ are called the fibers of $Y$.

Any fibered manifold admits an atlas consisting of fibered charts. These are local charts $(V, \psi)$, $\psi = (x^A, y^\sigma) = (x^A, y^\sigma(x^A))$ such that there exists a local chart $(U, \phi)$, $\phi = (x^A)$ on $X$, with $\pi(V) = U$, in which $\pi$ is represented as $\pi: (x^A, y^\sigma) \mapsto (x^A)$.

In particular, fiber bundles, as understood in\textsuperscript{81}, are fibered manifolds that are locally trivial, i.e., in the above, each $V$ is homeomorphic to a Cartesian product $U \times Z$, where $Z$ is a manifold, called the typical fiber.

Assume, in the following, that $(Y, \pi, X)$ is a fibered manifold. Local sections $\gamma: U \to Y$ (with $U \subset X$ open) are smooth maps such that $\pi \circ \gamma = \text{id}_X$; in a fibered chart, they are represented as:

$$\gamma: (x^A) \mapsto (x^A, y^\sigma(x^A)).$$
We denote by $\Gamma(Y)$ the set of sections of $(Y, \pi, X)$. In the following, capital Latin indices $A, B, C, \ldots$ will run from 0 to $n - 1$ and Greek indices $\sigma, \mu, \nu, \rho, \ldots$ will run from 1 to $m$.

**Physical interpretation.** In field theory, these manifolds and quantities are interpreted as follows:

- The manifold $Y$ is called the **configuration space**.
- The base manifold $X$ is typically (but not always) interpreted as spacetime; a notable exception to this rule is Finslerian field theory, where $X = PTM^+$ is the positively projectivized tangent bundle of the spacetime manifold $M$ (and the naturality of Lagrangians will be discussed with respect to $M$). In the following, we will reserve the notation $M$ for manifolds to be interpreted as spacetimes and denote by $X$ generic base manifolds.
- Sections $\gamma \in \Gamma(Y)$ are interpreted as **fields**.

The jet bundle $J^rY = \{J^r_xY \mid x \in X, \gamma \in \Gamma(Y)\}$ is naturally equipped with an atlas consisting of fibered charts $(V^r, \psi^r)$, $\psi^r = (x^A, y^\sigma, x^c_1, \ldots, y^\sigma c_1 c_2 \ldots c_r)$ on $J^rY$, induced by fibered charts $(V, \psi)$, via

$$y^\sigma c_1 \ldots c_r (J^r_xY) = \frac{\partial^r y^\sigma}{\partial x^{c_1} \ldots \partial x^{c_r}} (x^A). \tag{A1}$$

Any section of $Y$ is naturally prolonged into a section $J^r\gamma$ of $J^rY$; in a chart $(V^r, \psi^r)$:

$$J^r\gamma; (x^A) \mapsto \left( x^A, y^\sigma(x^A), \frac{\partial y^\sigma}{\partial x^1}(x^A), \ldots, \frac{\partial^r y^\sigma}{\partial x^{1} \ldots \partial x^{r}}(x^A) \right).$$

When referring to local expressions of geometric objects on $J^rY$, we always understand their expressions in fibered charts $(V^r, \psi^r)$ as above.

$J^rY$ is a fibered manifold over all lower order jet bundles $J^rY$, $0 \leq s < r$ (where $J^0Y := Y$), with canonical projections

$$\pi^s: J^rY \rightarrow J^sY, \quad (x^A, y^\sigma, y^\sigma c_1, \ldots, y^\sigma c_1 c_2 \ldots c_s) \mapsto (x^A, y^\sigma, y^\sigma c_1, \ldots, y^\sigma c_1 c_2 \ldots c_s).$$

$J^rY$ is also a fibered manifold over $X$, with projection

$$\pi^r: J^rY \rightarrow X, \quad (x^A, y^\sigma, y^\sigma c_1, \ldots, y^\sigma c_1 c_2 \ldots c_r) \mapsto (x^A).$$

2. **Horizontal and contact forms**

Let us introduce the following sets on $J^rY$:

1. $\Omega_k(J^rY)$, the set of differential $k$-forms defined over open subsets $W^r \subset J^rY$
2. $\Omega(J^rY) := \bigoplus_{k \in \mathbb{N}} \Omega_k(J^rY)$ the set of all differential forms over open subsets $W^r \subset J^rY$;
3. $\mathcal{X}(J^rY) := \Gamma(TJ^rY)$ the module of vector fields on $W^r \subset J^rY$;
4. $\mathcal{F}(J^rY)$, the set of all smooth functions $f : W \rightarrow \mathbb{R}$ defined on open subsets $W \subset J^rY$. 

A differential form $\rho \in \Omega_s(J'Y)$ is $\pi^r$-horizontal, if $\iota_\Xi \rho = 0$ whenever $\Xi \in \mathcal{X}(J'Y)$ is $\pi^s$-vertical (i.e., whenever $d\pi^r(\Xi) = 0$). In a fibered chart, any $\pi^r$-horizontal form is expressed as:

$$\rho = \frac{1}{k!} \rho_{A_1A_2...A_k} dx^{A_1} \wedge dx^{A_2} \wedge ... \wedge dx^{A_k},$$  

(A2)

where $\rho_{A_1A_2...A_k}$ are smooth functions of the coordinates $x^A, y^\sigma, y^\sigma C_1, ..., y^\sigma C_1 ... C_r$ on $J'Y$. Similarly, $\pi^r$-horizontal forms, $0 \leq s \leq r$ are locally generated by $dx^A, dy^\sigma, ..., dy^\sigma C_1 ... C_r$. A particular example of horizontal forms are Lagrangians, which we define in the next subsection.

The horizontalization operator is the unique morphism of exterior algebras $h: \Omega^r(Y) \to \Omega^{r+1}(Y)$ such that, for any $f \in \mathcal{F}(J'Y)$ and any fibered chart: $hf = f \circ \pi^{r+1}$ and

$$hd f = d_A f dx^A,$$

(A3)

where $d_A f := \partial f + \frac{\partial f}{\partial y^\sigma} \gamma^A_{A_1} + ... + \frac{\partial f}{\partial \gamma^\sigma_{C_1 ... C_r}} y^\sigma_{C_1 ... C_r}$ is the total derivative (of order $r+1$) with respect to $x^A$. On the natural basis 1-forms, it acts as:

$$h dx^A := dx^A, \quad h dy^\sigma = y^\sigma_A dx^A, ..., h dy^\sigma_{C_1 ... C_k} = y^\sigma_{C_1 ... C_k} dx^A, \quad k = 1, r.$$  

(A4)

A useful property is the following. For any $f \in \mathcal{F}(J'Y), \gamma \in \Gamma(Y)$:

$$\partial_A (f \circ \pi^r \gamma) = \pi^{r+1} \gamma^A d_A f.$$  

(A5)

A differential form $\rho \in \Omega(J'Y)$ is a contact form if $J' \gamma^\rho = 0, \forall \gamma \in \Gamma(Y)$. For instance,

$$\theta^\sigma = dy^\sigma - y^\sigma_C dx^C, \quad \theta^A_{A_1} = dy^\sigma_{A_1} - y^\sigma_{A_1} dx^A, ..., \theta^A_{A_1 ... A_{r-1}} = dy^\sigma_{A_1 ... A_{r-1}} - y^\sigma_{A_1 ... A_{r-1}} dx^A,$$  

(A6)

are contact forms on a given chart domain $V' \subset J'Y$, providing a local basis $\{dx^A, \theta^\sigma, ..., \theta^\sigma_{A_1 ... A_{r-1}}, dy^\sigma_{A_1 ... A_r}\}$ of the module $\Omega_1(J'Y)$, called the contact basis.

Raising to the next “floor” $J^{r+1}Y$, any differential form can be uniquely split as

$$(\pi^{r+1})^s \rho = h \rho + p \rho,$$

where $p \rho$ is contact. Intuitively, $h \rho$ is what will survive of $\rho$ when pulled back to $X$ by prolonged sections $J^{r+1} \gamma$, where $\gamma \in \Gamma(Y)$, while $p \rho$ becomes invisible: $J^{r+1} \gamma (p \rho) = 0$.

In particular, a $k$-form $\rho \in \Omega(J'Y)$ is 1-contact if $\iota_\Xi \rho$ is a $\pi^r$-horizontal form whenever $\Xi \in \mathcal{X}(J'Y)$ is $\pi^s$-vertical; in coordinates, 1-contact forms $\rho$ can be recognized by the fact that, in their expression in the contact basis, each term contains exactly one of the contact basis 1-forms $\theta^\sigma, ..., \theta^\sigma_{A_1 ... A_k}$ defined in (A6).

A $\pi^{r0}$-horizontal, 1-contact $(n+1)$-form $\eta \in \Omega_{n+1}' Y$ is called a source form. Locally, a source form is expressed as:

$$\eta = \eta_\sigma \theta^\sigma \wedge dx^\mu,$$

(A7)

where $\eta_\sigma = \eta_\sigma (x^A, y^\mu, ..., y^\mu_{A_1 ... A_k})$.

**Fibered morphisms.**
An automorphism of a fibered manifold \((Y, \pi, X)\) is,\(^{71}\), a diffeomorphism \(\Phi : Y \to Y\) such that exists a mapping \(\phi \in \text{Diff}(X)\) with \(\pi \circ \Phi = \phi \circ \pi\), i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi} & Y \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xrightarrow{\phi} & X
\end{array}
\]

(A8)

In this case, \(\Phi\) is said to cover \(\phi\). In coordinates, these must be of the form:

\[
\begin{align*}
\phi : (x^A) &\to (\tilde{x}^A(x^B)) \\
\Phi : (x^A, y^\sigma) &\to (\tilde{x}^A(x^B), \tilde{y}^\sigma(x^B, y^\mu)).
\end{align*}
\]

(A9) (A10)

The automorphism \(\Phi\) is called strict if \(\phi = \text{id}_X\).

Any generator \(\Xi\) of a 1-parameter group \(\{\Phi_\epsilon\}\) of automorphisms of \(Y\) is a \(\pi\)-projectable vector field, i.e, \(\pi_\ast \Xi\) is a well defined vector field on \(X\); in a fibered chart, projectable vector fields are represented as:

\[
\Xi = \tilde{\xi}^A(x^B) \partial_A + \Xi^\sigma(x^B, y^\mu) \frac{\partial}{\partial y^\sigma}.
\]

(A11)

In particular, 1-parameter groups of strict automorphisms are generated by \(\pi\)-vertical vector fields \(\Xi = \Xi^\sigma(x^B, y^\mu) \frac{\partial}{\partial y^\sigma}\).

Automorphisms \(\Phi: Y \to Y\) are prolonged into automorphisms of \(J^r Y\) as: \(J^r \Phi(J^r \gamma) := J^r_{\Phi(\gamma)}(\Phi \circ \gamma \circ \Phi^{-1})\).

The generator of the 1-parameter group \(\{J^r \Phi_\epsilon\}\), with \(\Phi_\epsilon\) as above, is called the \(r\)-th prolongation of the vector field \(\Xi\) and denoted by \(J^r \Xi\). In particular, for \(r = 1\), this is given by:

\[
J^1 \Xi = \tilde{\xi}^A \partial_A + \Xi^\sigma \partial_\sigma + \Xi^\sigma_A \frac{\partial}{\partial y_\sigma}.
\]

(A12)

3. Lagrangians and first variation formula

A Lagrangian is defined as a \(\pi'\)-horizontal form \(\lambda \in \Omega^n_{\pi'} Y\) of degree \(n = \text{dim} X\); locally,

\[
\lambda = \mathcal{L} d^n x, \quad \mathcal{L} = \mathcal{L}(x^A, y^\sigma, ..., y^\sigma_{A_1...A_n}),
\]

(A12)

where \(d^n x := dx^1 \wedge ... \wedge dx^n\).

By a piece \(D \subset X\), we understand,\(^{71}\), a compact \(n\)-dimensional submanifold with boundary of \(X\). The action attached to the Lagrangian \((A12)\) and to a piece \(D \subset X\) is the function \(S_D : \Gamma(Y) \to \mathbb{R}\), given by:

\[
S_D(\gamma) = \int_D J^r \gamma \lambda.
\]

Consider an arbitrary 1-parameter group \(\{\Phi_\epsilon\}\) of automorphisms of \(Y\), with \((\pi\)-projectable) generator \(\Xi \in \mathcal{P}(Y)\). This will induce a deformation \(\gamma \mapsto \gamma_\epsilon := \Phi_\epsilon \circ \gamma \circ \Phi_\epsilon^{-1}\) of sections \(\gamma \in \Gamma(Y)\):

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi_\epsilon} & Y \\
\downarrow{\gamma} & & \downarrow{\gamma_\epsilon} \\
X & \xrightarrow{\Phi_\epsilon^{-1}} & X
\end{array}
\]

(A13)
The variation $\delta_{\varepsilon} S_D(\gamma) := \frac{d}{d\varepsilon}|_{\varepsilon=0} S_{\Phi(D)}(\gamma_{\varepsilon})$ is then expressed as the Lie derivative:

$$\delta_{\varepsilon} S_D(\gamma) = \int D \gamma' S_{p_{\varepsilon} \lambda}. \tag{A14}$$

A section $\gamma \in \Gamma(Y)$ is a critical section for $S$, if for any compact $D \subset X$ and for any $\pi$-projectable $\Xi \in \mathcal{K}(Y)$ such that $\text{supp}(\Xi \circ \gamma) \subset D$, there holds: $\delta_{\varepsilon} S_D(\gamma) = 0$.

For any Lagrangian $\lambda \in \Omega_n(JY)$ and any $\Xi \in \mathcal{K}(Y)$, there holds the first variation formula:

$$J^* \gamma'(\Lambda_{\varepsilon} \lambda) = J^* \gamma'(\Lambda_{\varepsilon} \lambda) - J^* \gamma' d \mathcal{J}, \tag{A15}$$

where:

- $\mathcal{J} \in \Omega_{n+1}(J^* Y)$ is a source form of order $s \leq 2r$, called the Euler-Lagrange form\(^5\); locally, if $\lambda = \mathcal{L}_{d^r x}$, then:

$$\mathcal{J} = E_{\sigma} \theta^\sigma \land d^r x,$$

with:

$$E_{\sigma} = \Lambda_{\sigma} \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_{\lambda} \frac{\partial \mathcal{L}}{\partial y^\lambda} + \ldots + (-1)^r d_{A_1} \ldots d_{A_r} \frac{\partial \mathcal{L}}{\partial y^{A_1 \ldots A_r}}. \tag{A16}$$

The section $\gamma \in \Gamma(Y)$ is critical for $\lambda$ if and only if $E_{\sigma} \circ J^* \gamma = 0$.

- The $(n-1)$-form $\mathcal{J} \Xi \in \Omega_{n-1}(J^* Y)$ is called the Noether current associated with $\lambda$ and to the vector field $\Xi$. If $\Xi$ is a symmetry generator for $\lambda$, i.e., if $\Lambda_{\varepsilon} \lambda = 0$, then, Noether's first theorem states that the Noether current is conserved along critical sections:

$$J^* \gamma' d \mathcal{J} \Xi \approx 0, \tag{A17}$$

where $\approx$ denotes equality on-shell, i.e., for critical sections $\gamma$.

The Euler-Lagrange form of $\lambda$ is unique, while the Noether current $\mathcal{J} \Xi$ is only unique up to an exact form $d\rho$.

In integral form, the first variation formula reads:

$$\int_D J^* \gamma'(\Lambda_{\varepsilon} \lambda) = \int_D J^* \gamma' i_{\mathcal{J} \Xi} \mathcal{J} - \int_D J^* \gamma' \mathcal{J} \Xi. \tag{A18}$$

**Remark 33**

1. The fact that $\mathcal{J} \Xi = E_{\sigma} \theta^\sigma \land d^r x$ is a source form implies that locally, only the $\frac{\partial}{\partial y^\lambda}$-components of $J^* \Xi$ will contribute to $i_{\mathcal{J} \Xi} \mathcal{J}$ (i.e., higher order components of $J^* \Xi$ will not contribute to it):

$$i_{\mathcal{J} \Xi} \mathcal{J} = (\tilde{\Xi} \theta^\sigma E_{\sigma}) d^r x, \quad \tilde{\Xi} \Xi = \Xi - y^C \Xi C. \tag{A19}$$

The functions $(\tilde{\Xi} \sigma \circ J^* \gamma) : X \to \mathbb{R}$ are commonly denoted in the literature by $\delta y^\sigma$.

---

\(^5\) The coordinate-free definition of the Euler-Lagrange form associated to a Lagrangian $\lambda$ employs the notion of Lepage equivalent of $\lambda$, see\(^1\), p. 122 and 125.; yet, for our purposes, the precise expressions of Lepage equivalents of our Lagrangians will not be necessary.
2. In order to identify the Euler-Lagrange form, it is sufficient to use $\pi$-vertical variation vector fields $\Xi \in \mathcal{X}(Y)$. Yet, general vector fields are needed in discussing general covariance and its consequence, energy-momentum conservation.

**Natural bundles and natural Lagrangians.**

Let $\mathcal{M}$ denote the category of smooth $n$-dimensional manifolds, with smooth embeddings as morphisms and $\mathcal{F}$, the category of smooth fiber bundles, whose morphisms are smooth fibered morphisms. A natural bundle functor over $n$-manifolds is, $\mathcal{F}$, a functor $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{F}$, such that:

1. For each $M \in \text{Ob}(\mathcal{M})$, $\mathcal{F}(M)$ is a fiber bundle over $M$;
2. For each embedding $\alpha : M \rightarrow M'$ in $\text{Morf}(\mathcal{M})$, the fibered manifold morphism $\mathcal{F}(\alpha) : \mathcal{F}(M) \rightarrow \mathcal{F}(M')$ covers $\alpha_0$.

If $Y = \mathcal{F}(M)$, then any automorphism $\phi$ of $M \in \text{Ob}(\mathcal{M})$ admits a canonical (or natural) lift $\Phi = \mathcal{F}(\phi)$ to $Y$. These natural lifts encode the transformations of fields - more precisely, their local expressions are identical to transition functions on $Y$ (see, e.g., [66, 82]). For instance, if $Y$ is a bundle of tensors of over $M$, then the canonical lift $\Phi = \mathcal{F}(\phi)$ of $\phi \in \text{Diff}(M)$ is given by pullback/pushforward.

Passing to infinitesimal generators, any vector field $\xi \in \mathcal{F}(M)$ admits a canonical lift $\Xi := \mathcal{F}(\xi) \in \mathcal{X}(Y)$; in a fibered chart, the components $\xi^i$ can always be expressed in terms of the components $\xi^i$ of $\xi$ and a finite number of partial derivatives thereof [77].

For example, in the case of the bundle of tensors $Y = T^p_2 M$ of type $(p, q)$ over $M$, one obtains $\Xi = \xi^i \partial_i + \xi^{ij...lp} \partial_{j...lp}$, where:

$$\xi^{ij...lp} = \xi^{ij...lp} + \xi^{l_i...lp} \alpha_{j...lp} - \xi^{i...lp} \alpha_{j...lp}$$

A globally defined Lagrangian $\lambda \in \Omega(M)$ is called natural, or generally covariant, if it is invariant under canonical lifts of arbitrary diffeomorphisms of spacetime, i.e., $J^\iota \mathcal{F}(\phi)^\iota \lambda = \lambda$ for all $\phi \in \text{Diff}(M)$ [82]. Using the formal similarity between lifts of active diffeomorphisms $\phi \in \text{Diff}(M)$ and (manifold-induced) fibered coordinate changes on $\mathcal{F}(M)$, naturality amounts to the fact that $\lambda$ must be invariant to any such coordinate changes (defined on any manifold $\mathcal{F}(M)$, where $M \in \text{Ob}(\mathcal{M})$). In terms of infinitesimal generators, this reads:

$$\Sigma_{i} J^\iota \mathcal{F}(\xi^i) \lambda = 0,$$

(A20)

for all $\xi \in \mathcal{X}(M)$. General covariance gives rise to a notion of energy-momentum tensor, $^{77}$.

---

[1] Bernhard Riemann. Über die Hypothesen, welche der Geometrie zu Grunde liegen. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 13:133–150, 1868. URL: http://www.deutschtextarchiv.de/riemann_hypothesen_1867.
[2] Bernhard Riemann. On the hypotheses which lie at the bases of geometry. *Nature*, 8:14–17, 1873. doi:10.1038/008014a0.
[3] P. Finsler. Über Kurven und Flächen in allgemeinen Räumen. PhD thesis, Georg-August Universität zu Göttingen, 1918.
[4] David Bao, S.-S. Chern, and Z. Shen. *An Introduction to Finsler-Riemann Geometry*. Springer, New York, 2000.
[5] R. Miron and I. Bucataru. *Finsler Lagrange geometry*. Editura Academiei Romane, 2007.
[6] Albert Einstein. Die Feldgleichungen der Gravitation. *Sitzung der physikalisch-mathematischen Klasse*, pages 844–847, 1915. URL: http://echo.mpiwg-berlin.mpg.de/MPIWG:ZZB2HK6W.
[7] G. S. Asanov. *Finsler Geometry, Relativity and Gauge Theories*. D. Reidel Publishing Company, 1985.
Erasmo Caponio and Antonio Masiello. On the analyticity of static solutions of a field equation in Finsler gravity.

C. Pfeifer and M. N. R. Wohlfarth. Causal structure and electrodynamics on Finsler spacetimes.

E. Minguzzi. Light cones in Finsler spacetime.

J. K. Beem. Indefinite Finsler spaces and timelike spaces.

Erasmo Caponio and Giuseppe Stancarone. Standard static Finsler spacetimes.

Manuel Hohmann, Christian Pfeifer, and Nicoleta Voicu. Cosmological Finsler spacetimes.

J. M. Lee. An introduction to smooth manifolds.

Marco Letizia and Stefano Liberati. Deformed relativity symmetries and the local structure of spacetime.

Kropina V. K. On projective two-dimensional finsler spaces with special metric.

Manuel Hohmann, Christian Pfeifer, and Nicoleta Voicu. Finsler gravity action from variational completion.

Andrea Fuster and Cornelia Pabst. Randers-Finsler spacetime.

Andrea Fuster, Cornelia Pabst, and Christian Pfeifer. Berwald spacetimes and very special relativity.

M. Elbistan, P. M. Zhang, N. Dimakis, G. W. Gibbons, and P. A. Horvathy. Geodesic motion in Bogoslovsky-Finsler spacetimes.

Norman Gurlebeck and Christian Pfeifer. Observers' measurements in premetric electrodynamics: Time and radar length.

R. K. Tavakol. Geometry of spacetime and Finsler geometry. International Journal of Modern Physics A, 24(08n09):1678 – 1685, 2009.

Florian Girelli, Stefano Liberati, and Lorenzo Sindoni. Planck-scale modified dispersion relations and Finsler geometry. Phys. Rev. D, 75:064015, 2007.

Marco Letizia and Stefano Liberati. Deformed relativity symmetries and the local structure of spacetime. Phys. Rev. D, 95(4):044007, 2017.

S. S. Chern, W. S. Chen, and Lam K. S. Trudy seminara po vektornomu i tenzornomu analizu.

Andrew G. Cohen and Sheldon L. Glashow. Very special relativity.

Miguel Ángel Javaloyes. Anisotropic tensor calculus. International Journal of Geometric Methods in Modern Physics, 16(supp02):1941001, 2019.
71D. Krupka. *Introduction to Global Variational Geometry*. Springer, Berlin Heidelberg, 2015.
72Manuel Hohmann, Christian Pfeifer, and Nicoleta Voicu. The kinetic gas universe. *Eur. Phys. J. C.*, 80(9):809, 2020. arXiv:2005.13561, doi:10.1140/epjc/s10052-020-8391-y.
73Olivier Sarbach and Thomas Zannias. Relativistic Kinetic Theory: An Introduction. *AIP Conf. Proc.*, 1548(1):134–155, 2013. arXiv:1303.2899, doi:10.1063/1.4817035.
74J. Ehlers. *General-Relativistic Kinetic Theory Of Gases*, pages 301–388. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. doi:10.1007/978-3-642-11099-3_4.
75Olivier Sarbach and Thomas Zannias. The geometry of the tangent bundle and the relativistic kinetic theory of gases. *Class. Quant. Grav.*, 31:085013, 2014. arXiv:1309.2036, doi:10.1088/0264-9381/31/8/085013.
76Hakan Andreasson. The Einstein-Vlasov System/Kinetic Theory. *Living Rev. Rel.*, 14:4, 2011. arXiv:1106.1367, doi:10.12942/lrr-2011-4.
77Mark J. Gotay and Jerrold E. Marsden. Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula. In *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 367–392. Amer. Math. Soc., Providence, RI, 1992. doi:10.1090/conm/132/1188448.
78D. Krupka and A. Trautman. General invariance of lagrangian structures. *Bull. Acad. Polon. Sci.*, 22(2):207–211, 1974.
79Nicoleta Voicu. Energy–momentum tensors in classical field theories — a modern perspective. *International Journal of Geometric Methods in Modern Physics*, 13(08):1640001, Sep 2016. URL: http://dx.doi.org/10.1142/S0219887816400016, doi:10.1142/s0219887816400016.
80M Crampin. On the construction of riemannian metrics for berwald spaces by averaging. *Houston Jour. Math.*, 40(3):737–750, 2014.
81R. S. Palais and C. L. Terng. Natural bundles have finite orders. *Topology*, 16:271–277, 1976.
82Fatibene L. *Relativistic Theories, Gravitational Theories, and General Relativity*. 2021.