Exact solutions of a nonlinear diffusion equation on polynomial invariant subspace of maximal dimension

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Abstract

The nonlinear diffusion equation \( u_t = (u^{-4/3} u_x)_x \) is reduced by the substitution \( u = v^{-3/4} \) to an equation with quadratic nonlinearities possessing a polynomial invariant linear subspace of the maximal possible dimension equal to five. The dynamics of the solutions on this subspace is described by a fifth-order nonlinear dynamical system (V.A. Galaktionov).

We found that, on differentiation, this system reduces to a single linear equation of the second order, which is a special case of the Lamé equation, and that the general solution of this linear equation is expressed in terms of the Weierstrass \( \wp \)-function and its derivative. As a result, all exact solutions \( v(x, t) \) on a five-dimensional polynomial invariant subspace, as well as the corresponding solutions \( u(x, t) \) of the original equation, are constructed explicitly.

Using invariance condition, two families of non-invariant solutions are singled out. For one of these families, all types of solutions are considered in detail. Some of them describe peculiar blow-up regimes, while others fade out in finite time.

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1 Introduction

Nonlinear diffusion equations of the form

\[
    u_t = \nabla \cdot (D(u) \nabla u),
\]

where \( u(x, t) > 0, (x, t) \in \mathbb{R}^N \times \mathbb{R} \), are found in many applications [4]. A rich variety of theoretical results are available for such equations, in particular, their group properties,
various classes of exact solutions, etc., see, for example, [5]. Especially, equations with a power-law coefficient $D(u)$,

$$ u_t = \nabla \cdot (u^m \nabla u), \quad (1.2) $$

describing the processes of “slow” (for $m > 0$) or “fast” (for $m < 0$) diffusion, are widely used, see [12, 13] and the references therein.

The present paper is devoted to the construction of a class of exact solutions for the equation

$$ u_t = (u^{-4/3} u_x)_x, \quad (1.3) $$

which was discussed by V.A. Galaktionov in [8]. By substituting

$$ u = v^{-3/4}, \quad (1.4) $$
equation (1.3) is reduced to a quadratic form

$$ v_t = vv_2 - \frac{3}{4} v_1^2 \equiv F[v], \quad v_i = \frac{\partial^i v}{\partial x^i}. \quad (1.5) $$

It was established in [8] that the operator $F$ possesses the polynomial invariant subspace

$$ W_5 = \mathcal{L}\{1, x, \ldots, x^4\}, \quad (1.6) $$
(a linear span of functions $1, x, \ldots, x^4$), that is, $F[W_5] \subseteq W_5$, of the maximal possible dimension equal to five (recall that the dimension of a linear subspace invariant with respect to a nonlinear ordinary differential operator of the order $k$ cannot exceed $2k + 1$; see [15] and [9], Ch. 2). As a result, equation (1.5) has solutions of the form

$$ v = C_0(t) + C_1(t)x + \frac{1}{2} C_2(t)x^2 + \frac{1}{6} C_3(t)x^3 + \frac{1}{24} C_4(t)x^4 \quad (1.7) $$

with coefficients $C_i(t), i = 0, 4$, satisfying the dynamical system

$$ \begin{align*}
\dot{C}_0 &= C_0 C_2 - \frac{3}{4} C_1^2 = b_0, \\
\dot{C}_1 &= C_0 C_3 - \frac{1}{2} C_1 C_2 = b_1, \\
\dot{C}_2 &= C_0 C_4 + \frac{1}{2} (C_1 C_3 - C_2^2) = b_2, \\
\dot{C}_3 &= \frac{3}{2} C_1 C_4 - \frac{1}{2} C_2 C_3 = b_3, \\
\dot{C}_4 &= C_2 C_4 - \frac{1}{2} C_3^2 = b_4, \quad (1.8) 
\end{align*} $$

where the notation $b_0, \ldots, b_4$ is introduced for the right-hand sides of the system.

The main purpose of the paper is to find all solutions of system (1.8) and, as a consequence, all solutions of the form (1.7) of equation (1.5) and the corresponding solutions of equation (1.3).
Note that equation (1.3) possesses a five-dimensional Lie algebra of point symmetries [16], which is maximal for equations of the form

$$u_t = (f(u)u_x)_x, \quad f(u) \neq \text{const}.$$  

The symmetry algebra of equation (1.5) is also five-dimensional, and some of its solutions of the form (1.7) turn out to be invariant with respect to some one-dimensional subalgebra and can be found by means of group analysis [17, 15]. Therefore, it is of particular interest to construct solutions of the form (1.7) which are not invariant.

This paper extends the study [13] where a class of radially symmetric solutions of equation (1.2) of the form

$$u = v^{1/m}$$

with a function $v$ polynomial in powers of $r = |x|$ is described and, in particular, a number of new non-invariant solutions are constructed.

The content of the paper and the main ideas are as follows.

- In Section 2, a simple criterion of invariance (non-invariance) for solutions (1.7) is obtained. Using this criterion, all the obtained solutions are classified as invariant or non-invariant.

- To construct solutions to system (1.8), one can apply classical methods of reducing its order using the symmetry algebra or first integrals (the latter are given in Section 3.1).

However, we use a simpler method based on the following remarkable property: on differentiation, system (1.8) is reduced to a single linear second-order equation

$$\ddot{y} = 2Q(t) y,$$  

where $Q(t)$ is a solution of the equation

$$\dot{Q}^2 = 4Q^3 - \tilde{g}_3, \quad \tilde{g}_3 = \text{const.}$$  

(1.10)

All functions $C_i(t), \ i = 0, 4$, must satisfy equation (1.9), and the corresponding solution (1.5) turns out to be non-invariant only if $\tilde{g}_3 \neq 0$ (Sections 2 and 3.2).

- In the case $\tilde{g}_3 \neq 0$, the function $Q(t)$ is expressed in terms of the Weierstrass $\wp$-function, $Q(t) = \wp(t; 0, \tilde{g}_3)$ (for $Q(t) = \text{const}$, system (1.8) has only the trivial solution), and equation (1.9) is a particular case of the Lamé equation [11]. In Section 3.3 it is shown that in this case the general solution of equation (1.9) is represented in the form

$$y = \alpha \frac{1}{P(t)} + \beta \frac{\dot{P}(t)}{P(t)},$$

where $P(t) = \wp(t; 0, g_3), g_3 = -\frac{1}{27} \tilde{g}_3$, and $\alpha, \beta$ are arbitrary constants. Note that this formula is much simpler than the well-known representation of the solution of the Lamé equation in terms of the Weierstrass $\sigma$-function [11].

- It follows that the corresponding solutions of equation (1.5) (in the case $\tilde{g}_3 \neq 0$) have the form

$$v = A(x) \frac{1}{P(t)} + B(x) \frac{\dot{P}(t)}{P(t)},$$

where $P(t) = \wp(t; x, g_3), g_3 = -\frac{1}{27} \tilde{g}_3$, and $A(x), B(x)$ are arbitrary functions of $x$. Note that this formula is much simpler than the well-known representation of the solution of the Lamé equation in terms of the Weierstrass $\sigma$-function [11].
where $A(x)$ and $B(x)$ are fourth degree polynomials in $x$. Having found them, we obtain two families of non-invariant solutions of equation (1.5). The first family (up to translations and dilations) is given by the formula

$$v = \frac{1}{4} \frac{1}{P(t)} \left( S_1 + S_2 x^4 \right) - \frac{1}{2} \frac{\dot{P}(t)}{P(t)} x^2,$$

(1.11)

where $S_1$ and $S_2$ are arbitrary constants such that

$$S_1 S_2 = -g_3 \neq 0.$$

The second family contains all powers of $x$ from zero to four. It is reduced to (1.11) by some transformations (generally speaking, complex) involving the inversion transformation $x \rightarrow 1/x$, $v \rightarrow v/x^4$ (Section 3.3).

• Note that solutions (1.11) are actually solutions on the three-dimensional invariant subspace $L\{1, x^2, x^4\} \subseteq W_5$. Solutions on this subspace are described by formulas (1.7), (1.8) with $C_1 = C_3 = 0$. In this case, system (1.8) is simplified and easily integrated.

In [13], a more general case of radially symmetric solutions of equation (1.2) of the form

$$u = v^{1/m}, v = a_0(t) + a_1(t) r^2 + a_2(t) r^4;$$

where $r = |x|$, was considered for the case $m = -\frac{4}{N+2}$ and arbitrary dimension $N$. The existence of such solutions was proved in [8]. Solutions of the corresponding dynamical system for the coefficients $a_0(t), a_1(t), a_2(t)$ are expressed in [13] in quadratures.

Below, in Section 4, an “explicit” representation of solutions of this dynamical system is given, and the corresponding solutions $v(x, t)$ are reduced to a form similar to (1.11), generalizing it to the case of arbitrary $N$. Conditions for invariance and non-invariance of solutions are indicated; the case $N = 1$ corresponds to non-invariant solution (1.11).

• In Section 5, all types of solutions of the form (1.11) are studied in detail. It is shown that some of them describe peculiar blow-up regimes, while others fade out in finite time.

• Section 6 discusses another remarkable King’s example, [13], concerning equation (1.2) with $N = 1, m = -\frac{3}{2}$. The substitution $u = v^{1/m}$ reduces this equation to a quadratic form, for which solutions are constructed on a four-dimensional polynomial invariant subspace. The set of obtained solutions contains non-invariant solutions. We represent these solutions in explicit form, expressing them in terms of the Weierstrass functions $\wp$ and $\zeta$, and establish their non-invariance.

• Along with non-invariant solutions, we give the corresponding invariant solutions in all cases. They are, of course, known and are presented for completeness.
2 Invariance and non-invariance conditions for solutions (1.7)

Equation (1.5) possesses a five-dimensional algebra of point symmetries with the basis

\[ \begin{align*}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v}, & X_3 &= \frac{1}{2} x^2 \frac{\partial}{\partial x} + 2xv \frac{\partial}{\partial v}, \\
X_4 &= x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, & X_5 &= \frac{\partial}{\partial x}.
\end{align*} \] (2.1)

The general form of the operator admitted by equation (1.5) is as follows:

\[ X = \alpha_1 X_1 + \cdots + \alpha_5 X_5 = \xi(t) \frac{\partial}{\partial t} + \eta(x) \frac{\partial}{\partial x} + \sigma(x) v \frac{\partial}{\partial v}, \]

where

\[ \xi(t) = \alpha_1 + \alpha_2 t, \quad \eta(x) = \frac{1}{2} \alpha_3 x^2 + \alpha_4 x + \alpha_5, \quad \sigma(x) = 2\alpha_4 - \alpha_2 + 2\alpha_3 x, \]

\(\alpha_1, \ldots, \alpha_5\) are arbitrary constants that are not equal to zero at the same time. A solution \(v = \varphi(x, t)\) of equation (1.5) is invariant with respect to the operator \(X\) if and only if this solution satisfies the condition \(\sigma v = \xi v_t + \eta v_x\) or

\[ \sigma v = \xi \left(v v_x - \frac{3}{4} v^2 x\right) + \eta v_x. \] (2.2)

For solutions of the form (1.7), condition (2.2) (after “splitting” in \(x\)) is reduced to the system

\[ \begin{align*}
\xi b_0 &= (2\alpha_4 - \alpha_2) C_0 - \alpha_5 C_1, \\
\xi b_1 &= 2\alpha_3 C_0 + (\alpha_4 - \alpha_2) C_1 - \alpha_5 C_2, \\
\xi b_2 &= 3\alpha_3 C_1 - \alpha_2 C_2 - \alpha_5 C_3, \\
\xi b_3 &= 3\alpha_3 C_2 - (\alpha_4 + \alpha_2) C_3 - \alpha_5 C_4, \\
\xi b_4 &= 2\alpha_3 C_3 - (2\alpha_4 + \alpha_2) C_4,
\end{align*} \] (2.3)

which must be fulfilled identically in \(t\).

System (2.3) is a system of linear equations with respect to the coefficients \(\xi, \alpha_2, \alpha_3, \alpha_4, \alpha_5\), and the determinant of its matrix is equal to

\[ \Delta = \frac{1}{4} \left(32 C_0^3 C_4^3 - 96 C_0^2 C_1 C_3 C_4^2 - 96 C_0^2 C_2 C_3^2 C_4 + 144 C_0^2 C_2 C_3^2 C_4 - 36 C_0^2 C_3^4 + 216 C_0 C_1 C_2 C_3^2 - 12 C_0 C_2^2 C_3 C_4 - 240 C_0 C_1 C_2 C_3 C_4 + 72 C_0 C_1 C_2 C_3^3 + 72 C_0 C_2 C_3^4 - 24 C_0 C_2^3 C_3^2 - 81 C_4^4\right) \] (2.4)

A direct verification shows that \(\dot{\Delta}\rvert_{(1.8)} = 0\), that is, \(\Delta\) is the first integral of system (1.8), \(\Delta = const\) on every solution.
In the case \( \Delta \neq 0 \), system (2.3) has only the trivial solution \( \alpha_1 = \cdots = \alpha_5 = 0 \), therefore, solution (1.7) is not invariant. If \( \Delta = 0 \), then system (2.3) can have a non-trivial constant solution \( \alpha_1, \ldots, \alpha_5 \), and then solution (1.7) will be invariant. Below, in Section 3.4, it is shown that all solutions of the form (1.7) corresponding to this case are invariant. Thus, the following criterion for the invariance or non-invariance of solution (1.7) holds.

**Proposition.** Solution (1.7) is invariant if \( \Delta = 0 \), and non-invariant if \( \Delta \neq 0 \).

3 All solutions of system (1.8) and the corresponding solutions (1.7) of equation (1.5)

3.1 Differential consequence of system (1.8) and first integrals

Differentiating equations (1.8), we obtain

\[
\ddot{C}_i = \dot{b}_i = 2 Q C_i, \quad i = 0, 4, \tag{3.1}
\]

where

\[
2 Q = C_0 C_4 - C_1 C_3 + \frac{1}{2} C_2^2 \tag{3.2}
\]

(the coefficient “2” is introduced to simplify further calculations).

It follows that for any \( i, j = 0, 4 \) it is fulfilled

\[
\dot{C}_i C_j - C_i \dot{C}_j = 0
\]

or

\[
\frac{d}{dt}(\dot{C}_i C_j - C_i \dot{C}_j) = \frac{d}{dt}(b_i C_j - C_i b_j) = 0.
\]

Therefore, system (1.8) possesses the first integrals

\[
C_j b_i - C_i b_j, \quad i \neq j.
\]

In particular, one can take the following integrals:

\[
p_1 = C_0 b_1 - C_1 b_0 = \frac{1}{4} \left( 4C_0^2 C_3 - 6C_0 C_1 C_2 + 3C_1^3 \right),
\]

\[
p_2 = C_0 b_2 - C_2 b_0 = \frac{1}{4} \left( 4C_0^2 C_4 + 2C_0 C_1 C_3 - 6C_0 C_2^2 + 3C_2^2 C_1 \right),
\]

\[
p_3 = C_0 b_3 - C_3 b_0 = \frac{3}{4} \left( 2C_0 C_1 C_4 - 2C_0 C_2 C_3 + C_1^2 C_3 \right),
\]

\[
p_4 = C_0 b_4 - C_4 b_0 = \frac{1}{4} \left( -2C_0 C_3^2 + 3C_1^2 C_4 \right)
\]

From this system, one can express the functions \( C_1(t), \ldots, C_4(t) \) in terms of \( C_0(t) \) and arbitrary constants \( p_1, \ldots, p_4 \), then, solving the equation for \( C_0(t) \), find all solutions to system (1.8) and eventually get all solutions of the form (1.7) for equation (1.5).

However, we use a different, shorter way.
3.2 Equation for \( Q(t) \) and its solutions

Differentiating (3.2) by virtue of system (1.8), yields

\[
\dot{Q} = \frac{1}{8} \left( 12 C_0 C_2 C_4 - 9 C_2^2 C_4 - 6 C_0 C_3^2 + 6 C_1 C_2 C_3 - 2 C_2^3 \right).
\]

It is directly verified that

\[
\dot{Q}^2 = 4Q^3 - \tilde{g}_3, \quad \tilde{g}_3 = \frac{1}{16} \Delta,
\]

where \( \Delta \) is defined by (2.4).

For each solution \( Q(t) \) of equation (3.3), one can solve system (3.1) with respect to the functions \( C_i(t) \) and, as a result, find all solutions (1.7) of equation (1.5).

Thus, the following cases arise (we write formulas up to translations in \( t \)):

1) \( \Delta \neq 0 \). Then the unique a non-constant solution to equation (3.3) is expressed in terms of the Weierstrass \( \wp \)-function:

\[
Q(t) = \wp(t; 0, \tilde{g}_3).
\]

In this case, the corresponding solutions (1.7) to equation (1.5) are non-invariant (see Proposition in Section 2).

Equation (3.3) also possesses constant solution \( Q(t) = \left( \frac{1}{4} \tilde{g}_3 \right)^{1/3} \neq 0 \). It is easily seen that only the zero solution \( C_0 = \cdots = C_4 = 0 \) of system (1.8) corresponds to this case, in contradiction with assumption \( \Delta \neq 0 \) (we omit the calculations).

2) \( \Delta = 0 \). Then the only solutions of equation (3.3) are the following:

\[
Q(t) \equiv 0 \quad \text{and} \quad Q(t) = t^{-2}.
\]

In these cases, all the corresponding solutions (1.7) of equation (1.5) turn out to be invariant (Section 3.4).

3.3 Case \( \Delta \neq 0 \), \( Q(t) = \wp(t; 0, \tilde{g}_3) \) (non-invariant solutions)

In this case, each equation of system (3.1) is a linear Lamé equation of the form

\[
\ddot{y} = 2Q(t) \dot{y},
\]

where \( Q(t) = \wp(t; 0, \tilde{g}_3), \tilde{g}_3 \neq 0 \). This is a special case of the general Lamé equation, the solutions of which are known to be expressed in terms of the Weierstrass \( \sigma \)-function, see [1]. For equation (1), we obtain a simpler formula for the general solution.

Equation (3.3) will be considered on the interval \((0, T)\), where \( T \) is the period of the function \( Q(t) \). Along with the function \( Q(t) \), consider the function \( P(t) = \wp(t; 0, g_3) \) with

\[
g_3 = -\frac{1}{432} \tilde{g}_3,
\]

which is a non-constant solution to equation

\[
P^2 = 4P^3 - g_3, \quad g_3 = -\frac{1}{432} \Delta.
\]
Note that the period $\tau$ of $P(t)$ is equal to $T$ if $g_3 > 0$ and $3T$ if $g_3 < 0$.

We show that the general solution of equation (3.4) is expressed in terms of the function $P(t)$ and its derivative $\dot{P}(t)$.

**Lemma.** The following formula holds:

$$Q(t) = P(t) - g_3 P^{-2}(t), \quad t \in (0, T).$$  \hspace{1cm} (3.6)

Denoting $P(t) - g_3 P^{-2}(t) \equiv \tilde{Q}(t)$, we obtain

$$\left(\tilde{Q}'\right)^2 - 4 \left(\tilde{Q}^3 + \tilde{g}_3 \equiv \left(\dot{P}^2 - 4P^3 + g_3\right)\left(1 + 2g_3 P^{-3}\right)^2,$$

so $\tilde{Q}(t)$ is a (non-constant) solution to equation (3.4). Hence, $\tilde{Q}(t) \equiv Q(t + C)$, where $C = \text{const}$. For $t \to 0$ we have $\tilde{Q}(t) \to +\infty$ and $Q(t) \to +\infty$. Therefore, $C = 0$ and $\tilde{Q}(t) \equiv Q(t)$.

**Remark 1.** If $g_3 > 0$, then for a given function $Q(t)$ there exist a unique function $P(t)$ satisfying (3.6).

If $g_3 < 0$, then there are three such functions: along with $P(t)$ in (3.6), one can also use the functions $P(t + T)$ and $P(t + 2T)$, or, equivalently, use the same function $P(t)$, taken on intervals $(T, 2T)$ and $(2T, 3T)$ (we will apply this remark in Section 5).

**Remark 2.** From (3.6), we obtain the following identity for the $\wp$-function:

$$\wp(t; 0, g_3) - g_3 \wp^{-2}(t; 0, g_3) = \wp(t; 0, -27g_3),$$  \hspace{1cm} (3.7)

which is valid on the complex plane. Transforming the right-hand side according to the well-known formula $\wp(t; 0, \varepsilon^6 g_3) = \varepsilon^2 \wp(\varepsilon t; 0, g_3)$ for $\varepsilon = i\sqrt{3}$, we obtain the identity

$$\wp(t; 0, g_3) - g_3 \wp^{-2}(t; 0, g_3) = -3 \wp(i\sqrt{3} t; 0, g_3),$$

proved in [10] in another way.

Given the representation (3.6), it can be verified that the functions $1/P(t)$ and $\dot{P}(t)/P(t)$ form a fundamental system of solutions to equation (3.4). This implies the following result.

**Theorem.** The general solution of the Lamé equation (3.4) on the interval $(0, T)$ has the form

$$y = \alpha \frac{1}{P} + \beta \frac{\dot{P}}{P},$$

where $\alpha$ and $\beta$ are arbitrary constants.

Returning to system (3.1), we find its general solution

$$C_i = \alpha_i \frac{1}{P} + \beta_i \frac{\dot{P}}{P}, \quad i = 0, 4,$$  \hspace{1cm} (3.8)

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with arbitrary constants \(\alpha_i, \beta_i\). Substituting these expressions in (1.8) and splitting the obtained identities in powers of \(\dot{P}\) leads to a system of algebraic equations, from which it is not difficult to find the coefficients \(\alpha_i, \beta_i\).

The calculations, however, can be simplified by proceeding as described below.

The solution (1.7) of equation (1.5) corresponding to (3.8) has the form

\[
v = A(x) \frac{1}{P} + B(x) \frac{\dot{P}}{P},
\]

where \(A\) and \(B\) are functions of \(x\) (polynomials of the fourth degree, but this does not need to be assumed in advance: they will turn out to be just that).

Substitution of (3.9) into (1.5) taking into account equation (3.5) and its corollary \(\ddot{P} = 6P^2\) yields

\[
-A\dot{P} + \frac{1}{2} B \left( \dot{P}^2 + 3g_3 \right) = AA'' - \frac{3}{4} A'^2 + \\
\left( AB'' + A''B - \frac{3}{2} A'B' \right) \dot{P} + \left( BB'' - \frac{3}{4} B'^2 \right) \dot{P}^2.
\]

Equating the coefficients at the powers of \(\dot{P}\) on the left- and right-hand sides, we obtain the system

\[
BB'' - \frac{3}{4} B'^2 = \frac{1}{2} B, \\
AB'' + BA'' - \frac{3}{2} A'B' = -A, \\
AA'' - \frac{3}{4} A'^2 = \frac{3}{2} g_3 B.
\]

(3.10)

To bring solutions to the simplest possible form, we will use the following symmetries of equation (1.5).

a) The dilation

\[
t = \theta \tilde{t}, \quad v = \theta^{-1} \tilde{v} \quad (\theta > 0)
\]

reduces solution (3.9) to the form

\[
\tilde{v} = \theta^3 A(x) \frac{1}{\tilde{P}} + B(x) \frac{\dot{\tilde{P}}}{\tilde{P}},
\]

where \(\tilde{P} (\tilde{t}) = \theta^2 P(t)\) satisfies the equation

\[
(\dot{\tilde{P}})^2 = 4(\tilde{P})^3 - \tilde{g}_3, \quad \tilde{g}_3 = \theta^6 g_3.
\]

For \(g_3 \neq 0\), choosing \(\theta = |g_3|^{-1/6}\), we get \(|\tilde{g}_3| = 1\). Therefore, in what follows we can assume that \(g_3 = \pm 1\). However, for ease of use, in formulas for non-invariant solutions we leave \(g_3\) an arbitrary constant.
b) The dilation

\[ x = a \tilde{x}, \quad v = a^2 \tilde{v} \quad (a > 0) \]  

reduces solution (3.9) to the form

\[ \tilde{v} = \tilde{A}(\tilde{x}) \frac{1}{P} + \tilde{B}(\tilde{x}) \frac{\dot{P}}{P}, \]

where \( \tilde{A}(\tilde{x}) = a^{-2} A(x), \) \( \tilde{B}(\tilde{x}) = a^{-2} B(x). \) System (3.10) also admits the corresponding dilation \( x = a \tilde{x}, \) \( A = a^2 \tilde{A}, \) \( B = a^2 \tilde{B}. \)

Up to translations in \( x \) and dilations (3.12), the first equation of system (3.10) has the following solutions:

\[ B = 0, \quad B = -\frac{1}{2} x^2 \quad \text{and} \quad B = \frac{1}{8} \delta (x^2 + \delta)^2, \quad \delta = \pm 1. \]

For each of these solutions, we find the function \( A(x) \) that satisfies the two remaining equations of system (3.10) and obtain the corresponding solutions (3.9) of equation (1.5). We have three cases:

1) \( B = 0. \) It follows that \( A = 0 \) and the corresponding solution (3.9) is trivial.

2) \( B = -\frac{1}{2} x^2. \) The corresponding solutions (3.9) have the form

\[ v = \frac{1}{4} \frac{1}{P} \left( S_1 + S_2 x^4 \right) - \frac{1}{2} \frac{\dot{P}}{P} x^2, \quad (3.13) \]

where \( S_1 \) and \( S_2 \) are arbitrary constants, satisfying the condition

\[ S_1 S_2 = -g_3 \neq 0. \quad (3.14) \]

Dilation (3.12) with \( a = |S_2|^{-1/2} \) reduces (3.13), (3.14) to

\[ v = \frac{\delta}{4} \frac{1}{P} \left( x^4 - g_3 \right) - \frac{1}{2} \frac{\dot{P}}{P} x^2, \quad \delta = \pm 1. \quad (3.15) \]

3) \( B = \frac{1}{8} \delta (x^2 + \delta)^2, \) \( \delta = \pm 1. \) The corresponding solutions (3.9) have the form

\[ v = \frac{1}{8} \frac{1}{P} \left( 4S_1 \left( x^3 - \delta x \right) + S_2 \left( x^4 - 6\delta x^2 + 1 \right) \right) + \frac{\delta}{8} \frac{\dot{P}}{P} \left( x^2 + \delta \right)^2, \quad (3.16) \]

where \( S_1 \) and \( S_2 \) are arbitrary constants, satisfying the condition

\[ \delta S_1^2 + S_2^2 = -g_3 \neq 0. \quad (3.17) \]

Formulas (3.13), (3.14) and (3.16), (3.17) represent two families of non-invariant solutions of equation (1.5). The first family is related to King’s non-invariant solutions, which we discuss in the next section. A complete study of this family is given in Section 5.
Remark 3. Solutions (3.13) and (3.16) are not invariant under any subalgebras of the five-dimensional algebra of continuous symmetries (2.1) of equation (1.5). But they admit some discrete symmetries of this equation. For example, any solution of the form (3.13) is invariant under the reflection

\[ x \rightarrow -x. \] (3.18)

Solution (3.15) for \( g_3 = -1 \) also admits the transformation

\[ x \rightarrow \frac{1}{x}, \ v \rightarrow \frac{v}{x^4}, \] (3.19)

and for \( g_3 = 1 \) – the composition of (3.19) and reflection

\[ t \rightarrow -t, \ v \rightarrow -v. \] (3.20)

Similarly, any solution of the form (3.16) for \( \delta = -1 \) admits transformation (3.19), and for \( \delta = 1 \) – the composition of (3.19) and reflection (3.18).

Remark 4. Note that solutions (3.16) and (3.13) are connected by some transformations, possibly complex. Indeed, consider two cases:

a) \( \delta = -1 \). Introducing constants \( \tilde{S}_1 \) and \( \tilde{S}_2 \) such that \( S_1 = \frac{1}{2}(\tilde{S}_2 - \tilde{S}_1), \ S_2 = \frac{1}{2}(\tilde{S}_2 + \tilde{S}_1) \), we rewrite solution (3.16) in the form

\[ v = \frac{1}{16} \frac{1}{P} \left( \tilde{S}_1 (x - 1)^4 + \tilde{S}_2 (x + 1)^4 \right) - \frac{1}{8} \frac{\dot{P}}{P} (x^2 - 1)^2, \ \tilde{S}_1 \tilde{S}_2 = -g_3. \]

Applying transformations \( x \rightarrow x + 1, \ (3.19) \) and \( x \rightarrow x - 1/2, \) yields

\[ v = \frac{1}{4} \frac{1}{P} \left( \frac{1}{4} \tilde{S}_1 + 4 \tilde{S}_2 x^4 \right) - \frac{1}{2} \frac{\dot{P}}{P} x^2. \]

Substituting \( \frac{1}{4} \tilde{S}_1 \rightarrow S_1, \ 4 \tilde{S}_2 \rightarrow S_2, \) we obtain solution (3.13).

b) \( \delta = 1 \). Then \( g_3 < 0 \) (see (3.17)). The complex transformation \( x \rightarrow ix, \ v \rightarrow -v, \ S_1 \rightarrow -i S_1, \ S_2 \rightarrow -S_2 \) changes the sign of \( \delta \) in formulas (3.16), (3.17) and leads to the previous case.

3.4 Case \( \Delta = 0 \) (invariant solutions)

Consider two cases. (We write down solutions up to translations in \( x \) and \( t \), dilations (3.11), (3.12) and reflections (3.18), (3.20).)

1) \( Q(t) = 0 \). In this case, the general solution of system (3.11) is given by the formulas \( C_i = \alpha_i t + \beta_i, \ i = 0, 4 \), with arbitrary constants \( \alpha_i, \beta_i \), and the corresponding solution (1.7) to equation (1.5) has the form

\[ v = A(x) t + B(x), \]
where \( A(x) \) and \( B(x) \) are some functions (polynomials of degree four). The substitution into (1.5) and splitting in \( t \) yields a system for \( A(x) \) and \( B(x) \). Solving this system, we obtain

(a) \( v = 1, \ v = x^4 \) and (b) \( v = \frac{4}{3} (x - t), \ v = \frac{4}{3} x^3(1 - x t) \).

Note that solutions in each pair are connected by transformation (3.19).

2) \( Q(t) = t^{-2} \). In this case, \( g_3 = 0 \) and from equation (3.5), we find \( P(t) = t^{-2} \). As in Section 3.3 the corresponding solution (1.7) of equation (1.5) is given by the formula (3.9),

\[ v = A(x) t^2 - 2B(x) \frac{1}{t}, \]

with \( A(x) \) and \( B(x) \) defined by system (3.10). Non-trivial solutions in this case also have the form (3.13) and (3.16), but with \( g_3 = 0 \). Having considered them, we get the following solutions:

(a) \( v = \frac{x^2}{t} \), (b) \( v = t^2 + \frac{x^2}{t} \), \( v = t^2 x^4 + \frac{x^2}{t} \),

(c) \( v = t^2 (x + 1)^4 + \frac{1}{4t} (x^2 - 1)^2 \), (d) \( v = -\frac{\delta}{4t} (x^2 + \delta)^2, \ \delta = \pm 1 \).

Here solutions (b) are related by transformation (3.19), and solution (c) is transformed by a composition of (3.19) and translations into solution (a). Solution (d) for \( \delta = -1 \) is also transformed by a composition of (3.19) and translations into solution (a), and case \( \delta = 1 \) is related with the previous one by the complex change \( x \to ix, \ v \to -v \).

It is directly seen that all solutions given in Section 3.4 are invariant.

4 King’s solutions for equation (1.2) with \( m = -\frac{4}{N+2} \)

Following [13], we consider a radially symmetric version of equation (1.2):

\[ u_t = \frac{1}{r^{N-1}} (r^{N-1} u^m u_r)_r, \]  

(4.1)

where \( N \) is the number of spatial variables, \( r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} \). By substituting \( u = v^{1/m} \), equation (4.1) is reduced to a quadratic form

\[ v_t = vv_{rr} + \frac{1}{m} v_r^2 + \frac{N - 1}{r} vv_r. \]  

(4.2)

Consider the case

\[ m = -\frac{4}{N+2}. \]  

(4.3)

Note that in this case, equation (4.2) admits the transformation

\[ r = \frac{1}{\bar{r}}, \ \bar{v} = \frac{v}{r^4}, \]  

(4.4)
which we use below when discussing a set of invariant solutions.

In [13] (pp. 41, 42) solutions of the form

\[ v = a_0(t) + a_1(t) r^2 + a_2(t) r^4 \]  

(4.5)

with coefficients satisfying the system

\[ \dot{a}_0 = 2Na_0a_1, \quad \dot{a}_1 = (N - 2)a_1^2 + 4(N + 2)a_0a_2, \quad \dot{a}_2 = 2Na_1a_2 \]  

(4.6)

were constructed for equation (4.2), (4.3). The existence of such solutions was proved in [8], Theorem 3.3, where the invariance of a three-dimensional subspace \( \mathcal{L}\{1, r^2, r^4\} \) with respect to the operator \( F[v] = vvv + \frac{1}{m} v_r^2 + \frac{N-1}{r} vvr \) is established under condition (4.3) (see also [9], Proposition 6.24).

In [13], system (4.6) is solved in quadratures. Below an “explicit” representation of its solutions is given, and the corresponding solution (4.5) is reduced to a form similar to (3.13), generalizing it to the case of arbitrary \( N \). Also, a condition for the non-invariance of the solution is formulated and both non-invariant and invariant solutions of the form (4.5) are indicated.

4.1 All solutions of system (4.6) and corresponding solutions (4.5) of equation (4.2)

System (4.6) is easily integrated. Omitting calculations, we formulate the results.

The general solution to system (4.6) is represented as

\[ a_0 = \frac{S_1}{4N} \frac{1}{y}, \quad a_1 = -\frac{1}{2N} \frac{\dot{y}}{y}, \quad a_2 = \frac{S_2}{4N} \frac{1}{y}, \]

where \( y(t) \neq 0 \) is a solution of the equation

\[ y\ddot{y} + k(\dot{y})^2 = k\alpha, \quad k = -\frac{N + 2}{2N}, \quad \alpha = S_1S_2, \]  

(4.7)

\( S_1, S_2 \) are arbitrary constants. Multiplying (4.7) by \( 2y^{2k-1}\dot{y} \) and integrating, yields

\[ (\dot{y})^2 = \alpha + \beta y^{\frac{N+2}{N}}, \]  

(4.8)

where \( \beta \) is an arbitrary constant.

The corresponding solution (4.5) of equation (4.2) has the form

\[ v = \frac{1}{4N} \frac{1}{y} \left( S_1 + S_2r^4 \right) - \frac{1}{2N} \frac{\dot{y}}{y} r^2, \quad S_1S_2 = \alpha, \]  

(4.9)

where \( S_1, S_2, \alpha, \beta \) are arbitrary constants, \( y(t) \neq 0 \) is a solution of equation (4.7) (or equation (4.8)).

It is shown below that solution (4.9) is non-invariant iff the condition \( \alpha\beta \neq 0 \) holds. In this case, for \( N = 1 \), the solution \( y(t) \) of equation (4.8) is expressed in terms of the Weierstrass \( \wp \)-function and is defined on a bounded interval of \( t \), and solution (4.9), up to dilatation and reflection, coincides with the above non-invariant solution (3.13). If \( N \geq 2 \), then the solution \( y(t) \) of equation (4.8) is defined for any real \( t \), and for \( N = 2 \) it is expressed in elementary functions (see [13]).
4.2 Condition for non-invariance of solutions (4.9)

Setting \( a_0 = C_0, a_1 = \frac{1}{2} C_2, a_2 = \frac{1}{24} C_4, \) we represent solution (4.5) in the form similar to (1.7):

\[
v = C_0(t) + \frac{1}{2} C_2(t)r^2 + \frac{1}{24} C_4(t)r^4.
\]

(4.10)

In this case, system (4.6) takes the form

\[
\begin{align*}
\dot{C}_0 &= NC_0 C_2, \\
\dot{C}_2 &= \frac{N-2}{2} C_2^2 + \frac{N+2}{3} C_0 C_4, \\
\dot{C}_4 &= NC_2 C_4
\end{align*}
\]

(4.11)

and its general solution (see (4.9)) is written as follows:

\[
C_0 = \frac{S_1}{4N} \frac{1}{y}, \quad C_2 = -\frac{1}{N} \frac{\dot{y}}{y}, \quad C_4 = \frac{6S_2}{N} \frac{1}{y}.
\]

(4.12)

The algebra of point symmetries of equation (4.2) differs for different \( N \). For \( N \geq 3 \) this algebra is three-dimensional and has a basis

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v}, \quad X_4 = r \frac{\partial}{\partial r} + 2r \frac{\partial}{\partial v}
\]

(the numbering of the operators corresponds to (2.1)). For \( N = 2 \), the algebra is four-dimensional: its basis, in addition to operators \( X_1, X_2, X_4 \) includes operator

\[
X_0 = r \ln r \frac{\partial}{\partial r} + 2(\ln r + 1)r \frac{\partial}{\partial v}.
\]

For \( N = 1 \), the algebra is five-dimensional and has basis (2.1) (this case was considered above in Section 2).

Acting as in Section 2 from the invariance condition for solution (4.10), we obtain a system of linear homogeneous equations for the coefficients of a linear combination of the basis operators. The determinant of the matrix of this system (taking into account (4.12) and (4.8)) has the form

\[
\Delta = 2(N+2)C_0C_4 \left(C_2^2 - \frac{2}{3} C_0 C_4\right) = 3\frac{N+2}{N^4} \alpha \beta \gamma^{\frac{2}{3}-3} \text{ for } N \geq 2
\]

and

\[
\Delta = 18 C_0 C_4 \left(C_2^2 - \frac{2}{3} C_0 C_4\right)^2 = 9\alpha \beta^2 \text{ for } N = 1.
\]

In the case \( \Delta \neq 0 \), solution (4.10) is non-invariant and hence the condition \( \alpha \beta \neq 0 \) is sufficient for non-invariance of solution (4.9). Since all solutions obtained from (4.9), (4.8) for \( \alpha \beta = 0 \) are invariant (see Section (1.3)), the following statement holds.

**Proposition.** Solution (4.9) is **invariant** if \( \alpha \beta = 0 \), and **non-invariant** if \( \alpha \beta \neq 0 \).
4.3 Invariant solutions of the form \((4.9)\)

Having solved equation \((4.8)\) with \(\alpha \beta = 0\), we find the corresponding solutions \((4.9)\) (where \(S \neq 0\), \(A \neq 0\), \(S_0\) are arbitrary constants):

1) \((\alpha = \beta = 0)\)
\[
v = S, \quad v = S r^4;
\]

2) \((\alpha = 0, \beta \neq 0)\)
\[
a) \quad v = S_0 e^{4At} + A r^2, \quad v = S_0 e^{4At}r^4 + A r^2, \quad \text{if } N = 2;
\]
\[
b) \quad v = S_0 |t| \frac{4N}{2-N} r^2 \frac{r^2}{2-N} t, \quad v = S_0 |t| \frac{4N}{2-N} r^4 + \frac{1}{2-N} r^2 t, \quad \text{if } N \neq 2;
\]

3) \((\alpha \neq 0, \beta = 0)\)
\[
v = \frac{1}{4NS} (r^2 - S)^2.
\]

Note that solutions in pairs 1), 2 a) and 2 b) are connected by transformation \((4.4)\), and solution 3) is transformed by \((4.1)\) into a solution of the same type, but with the replacement \(S \rightarrow 1/S\).

It is seen that all these solutions are invariant. They are all known. In particular, solutions 2 b) correspond to the solution of the “instantaneous source” and the “dipole” solution of equation \((4.1)\), see [19], [2, 3, 11], as well as [12, 13] and the references therein.

4.4 Remark on the second derivatives of \(C_i(t)\)

By virtue of system \((4.11)\), for the second derivatives of the functions \(C_i(t)\) we obtain
\[
\ddot{C}_i = \mu \dot{C}_i + \nu C_i, \quad i = 0, 2, 4,
\]
where
\[
\mu = 2(N - 1)C_2, \quad \nu = N \left( \frac{N + 2}{3} C_0 C_4 - \frac{N - 2}{2} C_2^2 \right).
\]

Therefore, all the functions \(C_0\), \(C_2\) and \(C_4\) satisfy the same second-order equation
\[
\ddot{C} = \mu \dot{C} + \nu C.
\]

Hence, the phase trajectories of system \((4.11)\) are planar curves. (This is also seen from formulas \((4.12)\)). Note that dynamical systems with a similar property appeared in [6] when studying three-dimensional equations admitting the so-called “central quadric ansatz”.

In the particular case \(N = 1\) from \((4.14)\) we obtain \(\mu = 0\) and \(\nu = C_0 C_4 + \frac{1}{2} C_2^2\) in accordance with formulas \((3.1)\), \((3.2)\) written for \(C_1 = C_3 = 0\).

Taking into account \((4.12)\), the coefficients \(\mu\) and \(\nu\) can be represented in the form
\[
\mu = -\frac{2(N - 1)}{N} \frac{\dot{y}}{y},
\]
\[
\nu = \frac{1}{2Ny^2} \left( (N + 2)\alpha - (N - 2)\dot{y}^2 \right) = \frac{1}{N} \left( \frac{2\alpha}{y^2} - \beta \frac{N - 2}{2} y^{N-1} \right),
\]
where \(y(t)\) is a solution to equation \((4.8)\). For \(\beta \neq 0\), the functions \(1/y\) and \(\dot{y}/y\) form a fundamental system of solutions to equation \((4.15)\), \((4.16)\).
5 Investigation of non-invariant solutions (3.13)

In this section we present a detailed study of all types of non-invariant solutions described by formula (3.13).

It was noted above that (3.13) is reduced by transformation (3.12) to the form (3.15), which can be rewritten as

\[ v = \frac{\delta}{4P} \left( x^4 - 2 \delta \dot{P} x^2 - g_3 \right) = \frac{\delta}{4P} \left( x^2 - \delta \dot{P} \right)^2 - 4P^3. \]  

(5.1)

Taking into account dilation (3.11), we assume that \( g_3 = \pm 1 \).

We will be interested in the values of the arguments for which condition

\[ v(x, t) > 0 \]  

(5.2)

is satisfied and, thus, function (3.14), \( u(x, t) = (v(x, t))^{-3/4} \), is defined.

Consider two cases, \( g_3 < 0 \) and \( g_3 > 0 \), with subcases corresponding to the choice \( \delta = -1 \) or \( \delta = 1 \). In each case, for clarity, we present the graphs of both functions \( v(x, t) \) and \( u(x, t) \).

To obtain the asymptotics of solutions, we use the known expansions:

\[
P(t) = \frac{1}{t^2} + \frac{g_3}{28} t^4 + o(t^4) \text{ as } t \to 0 \quad \text{and} \quad P(t) = \frac{1}{(t - \tau)^2} + \frac{g_3}{28} (t - \tau)^4 + o((t - \tau)^4) \text{ as } t \to \tau \quad (\text{the period of } P(t)).
\]

If \( g_3 < 0 \), then \( P(t) \) has two zeros on \((0, \tau)\): \( t_{01} = \tau/3 \) and \( t_{02} = 2\tau/3 \), and

\[
P(t) = (-1)^i((-g_3)^{1/2}(t - t_{0i}) + \frac{1}{2} (-g_3)(t - t_{0i})^4 + o((t - t_{0i})^4) \text{ as } t \to t_{0i}, \quad i = 1, 2.
\]

Note that \( \tau \approx 5, 2999 \) for \( g_3 = -1 \) and \( \tau \approx 3, 0599 \) for \( g_3 = 1 \).

I. Case \( g_3 < 0 \)

In accordance with Remark 1 from Section 3.3, three intervals for \( t \) should be considered: \((0, \tau/3), (\tau/3, 2\tau/3), (2\tau/3, \tau)\). However, some of these cases can be excluded, since condition (5.2) is violated for them.

Namely: for \( \delta = -1, t \in (2\tau/3, \tau) \), the inequalities \( P(t) > 0 \) and \( \dot{P}(t) > 0 \) hold, and from (5.1) we obtain \( v = -\frac{1}{4P} \left[ (x^2 + \dot{P})^2 - 4P^3 \right] < 0 \), since

\[
(x^2 + \dot{P})^2 - 4P^3 \geq \dot{P}^2 - 4P^3 = -g_3 > 0;
\]

and for \( \delta = 1, t \in (\tau/3, 2\tau/3) \), we have \( P(t) < 0 \), and from (5.1) it follows that \( v < 0 \).

Hence, it remains to consider four possibilities: \( \delta = -1, t \in (0, \tau/3) \) or \( t \in (\tau/3, 2\tau/3) \), and \( \delta = 1, t \in (0, \tau/3) \) or \( t \in (2\tau/3, \tau) \).

I.1. \( \delta = -1, t \in (0, \tau/3) \). Then \( P(t) > 0, \dot{P}(t) < 0 \) and from (5.1) we obtain

\[ v = -\frac{1}{4P} \left( x^2 - \sigma_1 \right) \left( x^2 - \sigma_2 \right), \]
where 
\[ \sigma_1(t) = -\dot{P} - 2P^{3/2}, \quad \sigma_2 = -\dot{P} + 2P^{3/2}, \quad \sigma_1 < \sigma_2. \]

Since \( \sigma_1 \sigma_2 = \dot{P}^2 - 4P^3 = -g_3 > 0 \) and \( \sigma_1 + \sigma_2 = -2\dot{P} > 0 \), then \( \sigma_1, \sigma_2 > 0 \). Condition (5.2) is met if \( x \in (-\sigma_2^{1/2}, -\sigma_1^{1/2}) \cup (\sigma_1^{1/2}, \sigma_2^{1/2}) \).

Figure 1 shows the graphs of solutions \( v(x, t) \) and \( u(x, t) \) for values of \( t \) satisfying the condition \( 0 < t_1 < t_2 < t_3 < \tau/3 \). The dotted curves represent geometric places of local maxima of the function \( v \) and, accordingly, local minima of the function \( u \).

Since the functions \( v \) and \( u \) are even in \( x \), consider the values \( x > 0 \).

When \( t \) changes from 0 to \( \tau/3 \) the interval \( X(t) = (\sigma_1^{1/2}, \sigma_2^{1/2}) \) monotonically contracts from \((0, +\infty)\) to point 1. As \( t \to +0 \), for any fixed \( x \neq 0 \), we have \( u(x, t) \to +0 \). As \( t \to \tau/3 - 0 \), the following holds (\( t_0 = \tau/3 \)):

\[ |X(t)| = \sigma_2^{1/2} - \sigma_1^{1/2} \sim 2(t_0 - t)^{3/2} \to 0, \quad \min_{X(t)} u(x, t) \sim (t_0 - t)^{-3/2} \to +\infty, \]

Therefore, solution \( u(x, t) \) describes a kind of blow-up regime: in a finite time the interval \( X(t) \) contracts to a point, and the minimum of \( u(x, t) \) on this interval tends to infinity.

**I. 1. 2.** \( \delta = -1, \quad t \in (\tau/3, 2\tau/3) \). Then \( P(t) < 0 \) and condition (5.2) is met for all \( x \in (-\infty, +\infty) \).

Figure 2 shows the graphs of solutions \( v(x, t) \) and \( u(x, t) \) for values of \( t \) satisfying the condition \( \tau/3 < t_1 < t_2 < t_3 = \tau/2 < t_4 < t_5 < 2\tau/3 \).

At \( t = \tau/2 \), the solution profiles change qualitatively: for \( t < \tau/2 \) they have three extrema, and for \( t \geq \tau/2 \) – only one extremum. The dotted curves in Figure 2 are geometric places of local minima of the function \( v \) and local maxima of the function \( u \) at \( t < \tau/2 \).
As $t \to \tau/3 + 0$, for any fixed $x$, we have

$$u(x, t) \to \begin{cases} +\infty, & x = \pm 1, \\ +0, & x \neq \pm 1. \end{cases}$$

As $t \to 2\tau/3 - 0$, the function $u(x, t)$ uniformly tends to zero on $\mathbb{R}$.

Figure 2. Case I.1.2: $g_3 = -1$, $\delta = -1$, $t \in (\tau/3, 2\tau/3)$

I. 2. 1. $\delta = 1$, $t \in (0, \tau/3)$. Then $P(t) > 0$, $\dot{P}(t) < 0$ and from (5.1) we obtain

$$v(x, t) > \frac{1}{4P} \left[ \dot{P}^2 - 4P^3 \right] = -\frac{g_3}{4P} > 0,$$

that is, condition (5.2) is met for all $x \in (-\infty, +\infty)$.

Figure 3. Case I. 2.1: $g_3 = -1$, $\delta = 1$, $t \in (0, \tau/3)$

Figure 3 shows the graphs of solutions $v(x, t)$ and $u(x, t)$ for values of $t$ satisfying the condition $0 < t_1 < t_2 < \cdots < t_5 < \tau/3$. 
As $t \to +0$, for any fixed $x$ we have:

$$u(x, t) \to \begin{cases} 
+\infty, & x = 0, \\
0, & x \neq 0.
\end{cases}$$

As $t \to \tau/3 - 0$, the function $u(x, t)$ uniformly tends to zero on $\mathbb{R}$.

**I. 2. 2.** $\delta = 1, \ t \in (2\tau/3, \tau)$. Then $P(t) > 0$, $\dot{P}(t) > 0$ and from (5.1) we obtain

$$v = \frac{1}{4P} (x^2 - \sigma_1) (x^2 - \sigma_2),$$

where

$$\sigma_1(t) = \dot{P} - 2P^{3/2}, \quad \sigma_2(t) = \dot{P} + 2P^{3/2}$$

and $0 < \sigma_1 < \sigma_2$ (as in case I. 1. 1).

Condition (5.2) is met if $x \in (-\infty, -\sigma_1^{1/2}) \cup (-\sigma_1^{1/2}, \sigma_1^{1/2}) \cup (\sigma_1^{1/2}, +\infty)$.

Figure 4 shows the graphs of solutions $v(x, t)$ and $u(x, t)$ for values of $t$ satisfying the condition $2\tau/3 < t_1 < t_2 < \tau$. The dotted curve in the left figure is a geometric place of local minima of the function $v$.

![Graph of solutions](image)

Figure 4. Case I.2.2: $g_3 = -1, \ \delta = 1, \ t \in (2\tau/3, \tau)$

When $t$ changes from $2\tau/3$ to $\tau$, $\sigma_1(t)$ decreases monotonically from 1 to 0, and $\sigma_2(t)$ increases from 1 to $+\infty$. Thus, the interval $X(t) = (-\sigma_1^{1/2}, \sigma_1^{1/2})$ contracts from $(-1, 1)$ to point 0. As $t \to \tau - 0$, we have $\sigma_1^{1/2} \sim \frac{1}{2} (\tau - t)^{3/2} \to 0$, $\sigma_2^{1/2} \sim 2(\tau - t)^{-3/2} \to +\infty$, and

$$\min_{X(t)} u(x, t) \sim 2^{3/2}(t_0 - t)^{-3/2} \to +\infty,$$

Solution $u(x, t)$ describes a blow-up regime similar to I.1.1: in a finite time interval $X(t)$ contracts to a point, and the minimum of $u(x, t)$ on this interval tends to infinity.

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II. Case \(g_3 > 0\)

In this case, the function \(P(t)\) is defined and positive on the interval \((0, \tau)\). From (5.1) we obtain

\[
v = \frac{\delta}{4P} \left( \left( x^2 - \delta \dot{P} \right)^2 - 4P^3 \right) = \frac{\delta}{4P} (x^2 - \sigma_1) (x^2 - \sigma_2),
\]

where

\[
\sigma_1(t) = \dot{\sigma} - 2P^{3/2}, \quad \sigma_2(t) = \dot{\sigma} + 2P^{3/2},
\]

and, since \(\sigma_1 \sigma_2 = \ddot{\sigma}^2 - 4P^3 = -g_3 < 0\), we have \(\sigma_1 < 0 < \sigma_2\).

Condition (5.2) takes the form \(\delta (x^2 - \sigma_2) > 0\), that is, we need to consider two possibilities: \(\delta = -1, |x| < \sigma_1^{1/2}\) and \(\delta = 1, |x| > \sigma_1^{1/2}\).

II.1. \(\delta = -1, \quad x \in (-\sigma_1^{1/2}, \sigma_1^{1/2})\). Figure 5 shows the graphs of solutions \(v(x, t)\) and \(u(x, t)\) for values of \(t\) satisfying the condition \(0 < t_1 < t_2 < t_3 = \frac{\tau}{2} < t_4 < \tau\).

At \(t = \tau/2\), the solution profiles change qualitatively: for \(t < \tau/2\), they have three extrema, and for \(t \geq \tau/2\) – only one extremum. The dotted curves in Figure 5 are geometric places of local maxima of the function \(v\) and local minima of the function \(u\) at \(t < \tau/2\).

\[
\text{Figure 5. Case II.1: } g_3 = 1, \quad \delta = -1, \quad t \in (0, \tau)
\]

When \(t\) changes from 0 to \(\tau\), the interval \(X(t) = (-\sigma_2^{1/2}, \sigma_2^{1/2})\) monotonically contracts from \((-\infty, +\infty)\) to point 0. As \(t \to +0\), we have \(u(x, t) \to +0\) for any fixed \(x \neq 0\), and \(u(0, t) \to +\infty\). As \(t \to \tau - 0\), we obtain:

\[
|X(t)| = 2\sigma_2^{1/2} \sim (\tau - t)^{3/2} \to 0, \quad \min_{X(t)} u(x, t) \sim 2^{3/2}(\tau - t)^{-3/2} \to +\infty.
\]

Solution \(u(x, t)\) describes a blow-up regime similar to I.2.2 (and I.1.1): in a finite time interval \(X(t)\) contracts to a point, and the minimum of \(u(x, t)\) on this interval tends to infinity.
II. 2. \( \delta = 1, \ x \in (-\infty, -\sigma_2^{1/2}) \cup (\sigma_2^{1/2}, +\infty) \). Figure 6 shows the graphs of solutions \( v(x,t) \) and \( u(x,t) \) for values of \( t \) satisfying the condition \( 0 < t_1 < t_2 = \frac{\tau}{2} < t_3 < t_4 < \tau \). At \( t = \tau/2 \), the profiles of solution \( v(x,t) \) change qualitatively: for \( t \leq \tau/2 \) they have one extremum, and for \( t > \tau/2 \) – three extrema.

![Figure 6. Case II. 2: \( g_3 = 1, \ \delta = 1, \ t \in (0, \tau) \)](image)

The function \( \sigma_2(t) = \dot{P} + 2P^{3/2} \) increases monotonically on the interval \( (0, \tau) \), and \( \sigma_2(+0) = +0, \ \sigma_2(\tau - 0) = +\infty \). When \( t \) changes from \( 0 \) to \( \tau \), the set \( (-\infty, -\sigma_2^{1/2}) \cup (\sigma_2^{1/2}, +\infty) \) monotonically “contracts” from \( (-\infty, 0) \cup (0, +\infty) \) to the empty set. As \( t \rightarrow +0 \), we have \( u(x,t) \rightarrow +0 \) for any fixed \( x \neq 0 \).

Summarizing, we note that in all the cases considered in this section, solutions exist for a finite time and, at any fixed \( t \), are defined for \( x \in \tilde{X}(t) \), where \( \tilde{X}(t) \) is the union of some intervals in \( \mathbb{R} \).

In cases I.1.1 and II.1, peculiar blow-up regimes are realized: as \( t \) increases, the set \( \tilde{X}(t) \) contracts to separate points, while the minimum of the solution \( u(x,t) \) on \( \tilde{X}(t) \) tends to \( +\infty \). Solution I.2.2 can be interpreted similarly. In cases I.1.2 and I.2.1, for any admissible \( t \), the set \( \tilde{X}(t) \) coincides with \( \mathbb{R} \), and \( u(x,t) \) tends to \( +0 \) uniformly, fading out in a finite time (although equation (1.3) has no zero solution).

Despite their peculiarity, the solutions obtained may be of interest to specialists studying blow-up in nonlinear evolution equations.

6 King’s solutions to equation (1.2) with \( N = 1, \ m = -\frac{3}{2} \)

The next interesting example from [13] is related to the equation

\[
    u_t = (u^{-3/2}u_x)_x,
\]

which, by substituting \( u = v^{-2/3} \), is reduced to the quadratic form

\[
    v_t = vv_2 - \frac{2}{3} v_1^2.
\]
The operator \( F[v] = vv_2 - \frac{2}{3} v_1^2 \) possesses a four-dimensional invariant subspace \( W_4 = \mathcal{L}\{1, x, x^2, x^3\} \). As a consequence, equation (6.2) has polynomial solutions
\[
v = C_0(t) + C_1(t)x + \frac{1}{2} C_2(t)x^2 + \frac{1}{6} C_3(t)x^3
\] (6.3)
with coefficients satisfying the system
\[
\begin{align*}
\dot{C}_0 &= C_0 C_2 - \frac{2}{3} C_1^2, & \dot{C}_1 &= C_0 C_3 - \frac{1}{3} C_1 C_2, \\
\dot{C}_2 &= \frac{2}{3} C_1 C_3 - \frac{1}{3} C_2^2, & \dot{C}_3 &= 0.
\end{align*}
\] (6.4)

In [13], non-invariant solutions to equation (6.2) on \( W_4 \) are constructed in quadratures. We will obtain an explicit representation for these solutions and justify their non-invariance. Note that the “method of differentiation” applied above to (1.8) does not work in the case of system (6.4). Following [13], we will solve this system by reducing its order using symmetries.

6.1 All solutions of system (6.4) and corresponding solutions (6.3) of equation (6.2)

Equation (6.2) possesses a four-dimensional algebra of point symmetries with the basis
\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad X_4 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v}.
\] (6.5)

Algebra (6.5) induces the symmetry algebra \( \{Y_1, \ldots, Y_4\} \) of system (6.4). To reduce the order of system (6.4), one can use the operator\[
Y_2 = C_1 \frac{\partial}{\partial C_0} + C_2 \frac{\partial}{\partial C_1} + C_3 \frac{\partial}{\partial C_2},
\]
corresponding to \( X_2 \). Its invariants (independent if \( C_3 \neq 0 \)) are:
\[
I_1 = \frac{1}{9} \left( C_2^2 - 2C_1 C_3 \right), \quad I_2 = -\frac{2}{9} \left( \frac{1}{3} C_2^3 - C_1 C_2 C_3 + C_0 C_3^2 \right), \quad I_3 = C_3.
\] (6.6)

On the solutions of system (6.4), we obtain
\[
\dot{I}_1 = I_2, \quad \dot{I}_2 = 6 I_1^2,
\] (6.7)
and, therefore, \( I_1 \) satisfies the equation
\[
\dot{I}_1^2 = 4 I_1^3 - g_3, \quad g_3 = \text{const}.
\] (6.8)

The third equation of system (6.4) takes the form
\[
\dot{C}_2 = -3 I_1,
\] (6.9)
while its fourth equation yields $C_3 = \text{const}$.

Consider two cases: $C_3 = 0$ and $C_3 = \text{const} \neq 0$. (We represent solutions up to translations, dilations and reflections, see Sections 3.3 and 3.4.)

1) For $C_3 = 0$, integrating system (6.4), we find the following solutions (6.3):

$$v = 1, \ v = \frac{3}{2} (x - t), \ v = \frac{3}{2} \frac{x^2}{t} \quad \text{and} \quad v = \frac{3}{2} \frac{x^2}{t} + \delta t^3, \ \delta = \pm 1. \quad (6.10)$$

2) Let $C_3 = \text{const} \neq 0$. By using dilation (3.12) and reflection (3.18), we set $C_3 = 3$, and from (6.6) we obtain

$$C_0 = \frac{1}{18} \left( \frac{1}{3} C_3^3 - 9 I_1 C_2 - 9 I_2 \right), \quad C_1 = \frac{1}{6} (C_2^2 - 9 I_1). \quad (6.11)$$

From (6.8) and (6.7) we have three possibilities: if $g_3 = 0$, then a) $I_1 = 0, I_2 = 0$ or b) $I_1 = t^{-2}, I_2 = -2t^{-3}$; if $g_3 \neq 0$, then c) $I_1 = P(t) = \varphi(t; 0, g_3), I_2 = \dot{P}(t)$.

In cases a) and b), we find solutions:

$$v = x^3, \ v = \frac{3}{2} \frac{x^2}{t} + x^3. \quad (6.12)$$

It is directly seen that all solutions (6.10) and (6.12) are invariant.

In case c), equation (6.9) yields

$$C_2 = 3 \left( Z + S \right), \ S = \text{const},$$

were $Z(t) = \zeta(t; 0, g_3)$ is the Weierstrass $\zeta$-function, $\dot{Z}(t) = -P(t)$, and from (6.11) we obtain

$$C_1 = \frac{3}{2} \left( (Z + S)^2 - P \right), \quad C_0 = \frac{1}{2} \left( (Z + S)^3 - 3P(Z + S) - \dot{P} \right).$$

The substitution into (6.3) leads to the expression

$$v = \frac{1}{2} \left( (x + Z + S)^3 - 3P(x + Z + S) - \dot{P} \right),$$

which, by translation in $x$, is reduced to the form

$$v = \frac{1}{2} \left( (x + Z)^3 - 3P(x + Z) - \dot{P} \right). \quad (6.13)$$

This is an explicit representation of King’s non-invariant solutions of equation (6.2).

Now we will show that solutions (6.13) are not invariant.

### 6.2 Condition for non-invariance of solutions (6.3).

**Non-invariance of solutions (6.13)**

Similarly to Section 2, the invariance condition for solution (6.3) leads to a system of linear homogeneous equations for the coefficients of a linear combination of basis operators (6.5).

The determinant of the matrix of this system has the form

$$\Delta = \frac{1}{3} C_3 \left[ -9C_0^2 C_3^2 + 18C_0 C_1 C_2 C_3 - 6C_0 C_3^3 - 8C_1^3 C_3 + 3C_1^2 C_2^2 \right]. \quad (6.14)$$
It is verified that \( \Delta \) is the first integral of system (6.4).

The condition \( \Delta \neq 0 \) is sufficient for the non-invariance of solution (6.3). In the case of (6.13), we have

\[
C_0 = \frac{1}{2} (Z^3 - 3PZ - \dot{P}), \quad C_1 = \frac{3}{2} (Z^2 - P), \quad C_2 = 3Z, \quad C_3 = 3.
\]

Substitution in (6.14) yields \( \Delta = \frac{81}{4} g_3 \neq 0 \), so solutions (6.13) are non-invariant.

For invariant solutions (6.10) and (6.12), we have \( \Delta = 0 \).

### 6.3 Remark on solutions (6.13)

The explicit formula allows one to study solutions (6.13) in the similar way as it was done in Section 5 for solutions (3.13). We do not present a detailed study here, confining ourselves to a remark about the roots of the cubic polynomial on the right-hand side of (6.13).

Putting \( y = x + Z(t) \), we rewrite this cubic polynomial in the form

\[
y^3 + py + q, \quad p = -3P, \quad q = -\dot{P}.
\]

Its discriminant is

\[
D = -4p^3 - 27q^2 = 27g_3,
\]

see, for example, [14]. Consequently, the number of real roots of polynomial (6.15) is determined by the sign of the invariant \( g_3 \): for \( g_3 < 0 \) (\( D < 0 \)), the polynomial has one real root, and for \( g_3 > 0 \) (\( D > 0 \)), it has three real roots.

Solution (6.13) is defined for any \( x \in \mathbb{R} \) and \( t \in (0, \tau) \), where \( \tau \) is the period of the function \( P(t) \). It vanishes for values \( x = \sigma_i(t), \ i = 1, \ldots, k \), that correspond to real roots of polynomial (6.13) \( (k = 1 \text{ for } g_3 < 0 \text{ and } k = 3 \text{ for } g_3 > 0) \). The corresponding solution \( u(x, t) = (v(x, t))^{-2/3} \) of equation (6.1) is defined for any \( t \in (0, \tau) \) and for any \( x \in \mathbb{R} \) except for the points \( x = \sigma_i(t) \), at which it tends to infinity.

### 7 Final remarks

In this paper, all solutions to equation (1.3) that have the form (1.4), (1.7) are found; in particular, explicit formulas for non-invariant solutions are obtained. Explicit representations are also given for non-invariant solutions constructed in [13] to equations (4.1) and (6.1).

In connection with the results obtained, a number of questions arise, for example, concerning the properties of the found solutions or the properties of nonlinear systems describing dynamics of solutions on invariant subspaces. Recall, for instance, the amazing properties of system (1.8) established above: despite the rather high order of this system, all its phase trajectories are planar curves, moreover, the system itself is radically simplified on differentiation. (Compare with [7], where ODEs linearizable on differentiation were discussed.)
In conclusion, we give two more examples related to equations (1.3) and (1.5).

1) Adding a source or sink of a special type to the right-hand side of equation (1.3), yields the equation
\[ u_t = \left( u^{-4/3} u_x \right)_x + \frac{3}{4} \delta u^{-1/3}, \quad \delta = \pm 1, \quad (7.1) \]
which, by substitution (1.4), \( u = v^{-3/4} \), is reduced to the quadratic form
\[ v_t = vv_2 - \frac{3}{4} v_1^2 - \delta v^2 \equiv F[v]. \quad (7.2) \]
It was established in [8] that the operator \( F[v] \) possesses a maximal (five-dimensional) non-polynomial invariant subspace: for \( \delta = 1 \) – exponential, \( W_5^+ = \mathcal{L}\{1, e^x, e^{-x}, e^{2x}, e^{-2x}\} \), and for \( \delta = -1 \) – trigonometric, \( W_5^- = \mathcal{L}\{1, \cos x, \sin x, \cos 2x, \sin 2x\} \).

Equation (7.2) is related to equation (1.5) (written for \( t, \bar{x}, \bar{v} \)) by changes of variables:
\( \bar{x} = \pm e^x, \quad \bar{v} = e^{2x} v \) for \( \delta = 1 \), or \( \bar{x} = 2 \tan (x/2), \quad \bar{v} = \cos^{-4} (x/2) v \) for \( \delta = -1 \). Note that these transformations also connect the invariant subspaces \( W_5^+ \) and \( W_5^- \) with polynomial subspace (1.6) and, thus, solutions to equations (7.2) on these subspaces can be obtained from solutions of equation (1.5) on subspace (1.6) found in Sections 3.3 and 3.4.

2) Along with equation (1.5), \( v_t = F[v] \), one can consider the wave equation \( v_{tt} = F[v] \) with the same operator \( F \) in the right-hand side. This equation also possesses five-dimensional polynomial invariant subspace (1.6), as well as five-dimensional symmetry algebra. In this case, it is also possible to consider solutions of the form (1.7), but the order of the corresponding dynamical system will double.

Similarly, one can consider the hyperbolic analogues of equations (4.2) and (6.2).

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