GLOBAL MINIMALITY OF GENERIC MANIFOLDS AND HOLOMORPHIC EXTENDIBILITY OF CR FUNCTIONS

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Introduction.

Let $M$ be a smooth generic submanifold of $\mathbb{C}^n$. Several authors have studied the property of CR functions on $M$ to extend locally to manifolds with boundary attached to $M$ and holomorphically to generic wedges with edge $M$ (cf. [14], [67], [68]). In a recent work ([69]), Tumanov has showed that CR-extendibility of CR functions on $M$ propagates along curves that run in complex tangential directions to $M$. His main result appears as a natural generalization of results by Trépreau on propagation of singularities of CR functions ([61]). Indeed, Theorem 5.1 in [69] states that the direction of CR-extendibility moves parallelly with respect to a certain differential geometric partial connection in a quotient bundle of the normal bundle to $M$, and this variation is dual to the one introduced by Trépreau, according to Proposition 7.3 in [69].

In this paper we give a new and simplified presentation of the connection introduced in Tumanov’s work. Let $M$ be a real manifold and $N$ a submanifold of $M$, $K$ a subbundle of $TM$ with the property that $K|_N \subset TN$. Then by means of the Lie bracket, we can define a $K$-partial connection on the normal bundle of $N$ in $M$ (Proposition 1.1). In general, the parallel translation associated with that partial connection will be induced by the flow of $K$-tangent sections of $TM$ (Proposition 1.2). When $M$ is a generic submanifold of $\mathbb{C}^n$ containing a CR submanifold $S$ with the same CR dimension we recover in section 2 the $T^S$-partial connection constructed by Tumanov in [69].

Recall that the CR-orbit of a point $z \in M$ is the set of points that can be reached by piecewise smooth integral curves of complex tangent vector fields. We then say that $M$ is globally minimal at a point $z \in M$ if the CR-orbit of $z$ contains a neighborhood of $z$ in $M$. Using previous results, we show that vector space generated by the directions of CR-extendibility of CR functions on $M$ exchanges by the induced composed flow between two points in a same CR-orbit (Lemma 3.5). As an application, we prove the main result of this paper, conjectured by Trépreau in [61] : for wedge extendibility of CR functions to hold at every point in the CR-orbit of $z \in M$ it is sufficient that $M$ be globally minimal at $z$ (Theorem 3.4). Up till now we can only conjecture the converse (for a local result, see [6]).
I wish to thank J.-M. Trépreau for helpful critical and simplifying remarks.

Remark : After this paper was completed, we have received a preprint by B. Jöricke Deformation of CR-manifolds, minimal points and CR-manifolds with the microlocal analytic extension property, which contains also a proof of Theorem 3.4 and Theorem 3.6. Our proof seems quite different since we obtain these results relying on Tumanov’s propagation theorems, the generic manifold $M$ being fixed, whereas B. Jöricke works with conic perturbations of the base manifold so as to produce minimal points.

§1. Partial connections associated with a system of vector fields.

Let $M$ be a real differentiable manifold of class $C^2$ of dimension $n$ and $H \to M$ a $r$-dimensional vector bundle over $M$. Recall that a connection $\nabla$ on the bundle $H \to M$ is a bilinear mapping which assigns to each pair of a vector field $X$ with domain $U$ and a section $\eta$ of $H$ over $U$ a section $\nabla_X \eta$ of $H$ over $U$ and satisfy

$$\nabla_{\phi X} = \phi \nabla_X, \quad \nabla_X (\phi \eta) = \phi \nabla_X \eta + (X \phi) \eta, \quad \phi \in C^1(M, \mathbb{R}).$$

When the covariant derivative $\nabla_X \eta$ can only be defined for vectors $X$ that belong to a subbundle $K$ of $TM$, we call the connection $\nabla$ a $K$-partial connection (cf. [69]).

If $N$ is a submanifold of $M$, let $T_NM$ be the normal bundle of $N$ in $M$, i.e.

$$T_NM = TM|_N/TN.$$

Proposition 1.1. Let $M$ be a real manifold of class $C^2$, $N \subset M$ a submanifold of class $C^2$ too and let $K$ be a $C^1$ subbundle of $TM$ with the property that $K|_N \subset TN$. Then there exists a natural $K$-partial connection $\nabla$ on the bundle $T_NM$ which is defined as follows. If $x \in N, X \in K[x]$ and $\eta$ is a local section of $T_NM$ over a neighborhood of $x$, then take

$$\nabla_X \eta = [\tilde{X}, \tilde{Y}](x) \mod T_xN$$

where $\tilde{X}$ is a $C^1$ local section of $K$ extending $X$ and $\tilde{Y}$ is a lifting of $\eta$ in $TM$ in a neighborhood of $x$.

Proof. We first check that the definition is independent of the lifting $\tilde{Y}$. In fact, when $\tilde{Y}$ is tangent to $N$, as $\tilde{X}$ is tangent to $N$ too, the Lie bracket $[\tilde{X}, \tilde{Y}]$ remains tangent to $N$ hence is zero in the quotient bundle.

Next we have to check that the definition of $\nabla$ is independent of the chosen section $\tilde{X}$ or, to rephrase, that if $\tilde{X}(x) = 0$ then $[\tilde{X}, \tilde{Y}](x)$ belongs to $T_xN$. Since $K$ is a fiber bundle we can write

$$\tilde{X} = \sum_{j=1}^{r} f_j \tilde{X}_j, \quad f_j(0) = 0, \quad j = 1, \ldots, r$$

where $r = \text{rank } K$, $(\tilde{X}_j)_{j=1, \ldots, r}$ is a frame for $K$ near $x$ and the $f_j$ are $C^1$ real valued functions defined near $x$. Noting that

$$[f \tilde{X}, \tilde{Y}] = f[\tilde{X}, \tilde{Y}] - (\tilde{Y} f) \tilde{X} \equiv f[\tilde{X}, \tilde{Y}] \mod TN$$

the result follows and the mapping $\nabla$ is well-defined. Moreover the preceding implies that if $\phi \in C^1(M, \mathbb{R})$

$$\nabla_{\phi X} \eta \equiv \phi \nabla_X \eta.$$

Last, we check that $\nabla_X$ is a derivation. Indeed

$$\nabla_X (\phi \eta) \equiv [\tilde{X}, \phi \tilde{Y}](x) \equiv (\tilde{X}, \phi) \tilde{Y} + \phi [\tilde{X}, \tilde{Y}] \equiv (X \phi) \eta + \phi \nabla_X \eta.$$
and the proof is complete.

With the connection $\nabla$ it is associated the parallel translation of fibers of $TN$ along smooth curves on the base $N$ that run in directions tangent to $K$. Let $I \ni t$ be a subinterval of $\mathbb{R}$ and $\gamma : I \to N$ be a smooth curve with the property that $\dot{\gamma}(t) \in K[\gamma(t)]$, where $\dot{\gamma} = \frac{d}{dt}\gamma(t)$. A curve $\eta(t) \in TN[\gamma(t)]$ is a horizontal lift of $\gamma$ if $\nabla_\gamma \eta = 0$. Existence and uniqueness of horizontal lifts provide linear isomorphisms

$$\Phi_{t_0,t} : TN[\gamma(t_0)] \to TN[\gamma(t)]$$

obtained by moving elements of $K$ along horizontal lifts of $\gamma$.

Recall (cf. [58]) that the Lie bracket $[\hat{X}, \hat{Y}]$ is defined as the Lie derivative $L_{\hat{X}} \hat{Y}$ of $\hat{Y}$ with respect to $\hat{X}$

$$[\hat{X}, \hat{Y}](x) = L_{\hat{X}} \hat{Y} = \lim_{h \to 0} [\hat{Y}(x) - d\hat{X}_h(\hat{Y}(\hat{X}_h(x)))]$$

where $\hat{X}_t$ is the local flow on $M$ generated in a neighborhood of $x$ by the vector field $\hat{X}$, and $d\hat{X}_t$ denotes its differential. In the assumptions of Proposition 1.1, $\hat{X}$ is of class $C^1$ so the mapping $x \to \hat{X}_t(x)$ is of class $C^1$ and the differential is a well-defined continuous mapping. When $\hat{X}$ is $K$-tangent its flow (and more generally any piecewise smooth composition of such flows) stabilizes the tangent bundle $TN$ of the manifold $N$, hence its differential induces isomorphisms of fibers of $TN$, which we denote by $dX_t$. Assume moreover that the curve $\gamma$ is an integral curve of a $C^1$ $K$-tangent vector field $\hat{X}$, (which cannot be true for most general smooth curves $\gamma$ but is sufficient enough for the applications) : $\gamma(0) = x$ and $\gamma(t) = \hat{X}_t(x)$. Then we claim that the mapping

$$dX_t : TTN[x] \to TN[\gamma(t)]$$

provides the parallel translation $\Phi_{0,t}$. Indeed let $\eta_0 \in TTN[x]$ and take $\eta(t) = dX_t(\eta_0)$. Then by the definition of the partial connection $\nabla$ and the definition of the Lie bracket we have

$$\nabla_\gamma \eta(t) \equiv 0.$$

By uniqueness of solutions of linear differential equations of order one it must be that

$$\eta(t) = \Phi_{0,t}(\eta_0).$$

**Proposition 1.2.** Under the hypotheses of Proposition 1.1, let $\gamma(t) = X_t(x_1)$ be a smooth (piecewise smooth) integral curve of a $K$-tangent vector field $X$ (a finite number of $K$-tangent vector fields) running from $x_1 \in N$ to $x_2 \in N$. Then the parallel translation along $\gamma$ associated with the $K$-partial connection $\nabla$ is induced by the differential of the flow of $X$ (composed flow).

In order to give an expression of the covariant derivatives induced by the partial connection $\nabla$, we choose coordinates on $M$, $x = (x', x'') \in \mathbb{R}^l \times \mathbb{R}^m$ such that the base point corresponds to $x = 0$ and the submanifold $N$ is defined by the equation $x'' = 0$. Let $(x, \eta) = (x', x'', \eta', \eta'')$ be the canonical coordinates on $TM$, and $(x', \eta'') \in \mathbb{R}^l \times \mathbb{R}^m$ the associated coordinates on $TN$.

If $X = \sum_{j=1}^{l+m} a_j(x) \frac{\partial}{\partial x_j}$ is a $C^1$ section of $K$, it must be tangent to $N$, so $a_j(x', 0) = 0$, $j = l+1, \ldots, l+m$. We choose a local section $\eta$ of $TN$ over a neighborhood of 0 in $N$, in fact a section $\hat{Y}$ of $TM$ of the form

$$\hat{Y} = \sum_{j=l+1}^{l+m} \eta_j(x') \frac{\partial}{\partial x_j}.$$
Recalling Proposition 1.1 we have the following expression for the covariant derivative of \(\eta\) in the direction of \(X\)

\[
\nabla_X \eta = \sum_{j=l+1}^{l+m} \sum_{k=1}^{l} a_k(x',0) \frac{\partial \eta_k}{\partial x_j}(x') \frac{\partial}{\partial x_j} - \sum_{j,l+1}^{l+m} \eta_k(x') \frac{\partial a_j}{\partial x_k}(x',0) \frac{\partial}{\partial x_j}.
\]

Given an integral curve \(\gamma(t) = (\gamma'(t),0)\) of the field \(X\), the equations for the horizontal lifts look like

\[
X.\eta_j = \dot{\eta}_j(t) = \sum_{k=l+1}^{l+m} \frac{\partial a_k}{\partial x_k}(\gamma'(t),0)\eta_k(t) \quad j = l + 1, \ldots, l + m
\]

so the curve \((\gamma'(t), \eta''(t))\) is the integral curve of the following vector field \(\dot{X}\) on \(T_N^* M\)

\[
\dot{X}(x', \eta'') = \sum_{j=1}^{l} a_j(x',0) \frac{\partial}{\partial x_j} + \sum_{j,l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x',0)\eta_k(x') \frac{\partial}{\partial \eta_j}.
\]

Alternately, the partial connection \(\nabla\) can be defined by the family of horizontal subspaces \(H(\eta) \subset T_N T_N^* M\) generated by vectors of the form \(\dot{X}\).

Let us consider the dual connection \(\nabla^*\) to the connection \(\nabla\) on the dual bundle \(T_N^* M\). Recall that the conormal bundle of \(N\) in \(M\), \(T_N^* M\), consists of forms in \(T^* M\) that vanish on \(TN\). It has fiber over a point \(x \in N\)

\[T_N^* M[x] = \{\phi \in T_x^* M; \phi|_{TN} = 0\}.\]

The dual connection \(\nabla^*\) is defined by the following relation: if \(X\) is a \(K\)-tangent vector to \(N\) at \(x\), \(\eta\) is any section of \(T_N M\) near \(x\) and \(\phi\) is any section of \(T_N^* M\)

\[X < \phi, \eta > = < \nabla^*_X \phi, \eta > + < \phi, \nabla_X \eta > .\]

It is easily checked that such a relation defines a \(K\)-partial connection on \(T_N^* M\).

Along with the coordinates on \(T_N^* M\) we introduced before we can introduce the canonical coordinates \((x', \xi'')\) on the conormal bundle \(T_N^* M\). These are dual to the coordinates \((x', \eta'')\) for the canonical duality \(<,>\) between \(T_N M\) and \(T_N^* M\)

\[< \sum_{j=l+1}^{l+m} \xi_j dx_j , \sum_{j=l+1}^{l+m} \eta_j \frac{\partial}{\partial x_j} > = \sum_{j=l+1}^{l+m} \xi_j \eta_j .\]

Using the previous definition of the dual connection we can then compute the covariant derivative of a section \(\sum \xi_j dx_j = \phi\) of \(T_N^* M\). One easily shows

\[\nabla_X \phi = \sum_{j=l+1}^{l+m} (X \xi_j + \sum_{k=l+1}^{l+m} \xi_k \frac{\partial a_k}{\partial x_j}) dx_j.\]

Hence, under the assumption of Proposition 1.2, the parallel translation associated with the connection \(\nabla^*\) is given by means of the integral curves of the following vector field on \(T_N^* M\)

\[
\dot{X} = \sum_{j=1}^{l} a_j(x',0) \frac{\partial}{\partial x_j} - \sum_{j,l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x',0)\xi_j \frac{\partial}{\partial \xi_k}.
\]

There is another way of thinking the connection \(\nabla^*\) dual to the partial connection \(\nabla\) which has been considered by Trépreau in [61].
To a general vector field $X$ on $M$ it is associated its symbol $\sigma(X)$ which is an invariantly defined function on the cotangent bundle $T^*M$ of $M$. To a function $f$ of class $C^1$ on $T^*M$ it is associated its hamiltonian field $H_f$.

Let $X_j$, $j = 1, \ldots, r$ be a local basis of $K$-tangent sections of $TM$. Let $\Sigma_K$ be the orthogonal complement of $K$ in $T^*M$. If $X = \sum_{j=1}^r \phi_j X_j$ is a $C^1$ section of $K$ we have

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K} + \sum_{j=1}^r \sigma(X_j) H_{\phi_j}|_{\Sigma_K}.$$ 

Since $\sigma(X_j)$, $j = 1, \ldots, r$ is zero on $\Sigma_K$, we deduce that the restricted hamiltonian field

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K}$$

depends only on the value of $X$ at the base point and not on the chosen section. If $X$ is tangent to $N$, $H_{\sigma(X)}$ when restricted to $T^*_N M$ is tangent to $T^*_N M$. Hence we have constructed another vector field on $T^*_N M$ which is in fact the same as the one associated with the connection dual to the partial connection $\nabla$.

Indeed, let as before $(x', \xi')$ be the canonical coordinates on the conormal bundle $T^*_N M$. Recall that the hamiltonian field of a function $f = f(x, \xi)$ just looks like

$$H_f = \sum_{j,k=1}^{l+m} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_k} - \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_k}.$$ 

The symbol of the section

$$X = \sum_{j=1}^{l+m} a_j \frac{\partial}{\partial x_j} \quad a_j(x', 0) = 0, \quad j = l + 1, \ldots, l + m$$ 

of $K$ being $\sigma(X) = \sum a_j \xi_j$ we can compute

$$H_{\sigma(X)}|_{T^*_N M} = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \xi_j \frac{\partial}{\partial \xi_k}$$

and the last expression proves that $H_{\sigma(X)}$ is the same vector field on $T^*_N M$ as $\tilde{X}$ computed previously, so the set of restricted hamiltonian fields $H_{\sigma(X)}|_{T^*_N M}$ defines the same family of horizontal subspaces for the partial connection $\nabla^*$. 

The next section is devoted to the application of the preceding results to the geometry of CR submanifolds of $C^n$.

§2. Application to generic submanifolds of $C^n$

In this section we apply results of section 1 in the context of differential geometry in the complex euclidean space $C^n$. Afterwards we check that our definitions recover those of Trépreau [61] and Tumanov [62].

Let $TC^n$ be the real tangent bundle of $C^n$ and $J$ be the standard complex structure operator on $TC^n$. Let $T^*C^n$ be the bundle of holomorphic (C-linear) 1-forms on $C^n$. In the canonical coordinates $z = (z_1, \ldots, z_n)$ its fiber over a point $z$ consists of $(1,0)$-forms $\omega = \sum_{j=1}^n \zeta_j dz_j$, $\zeta_j \in C$, $j = 1, \ldots, n$. Then
\( T^*C^n \) is a complex manifold. It can be (and it is usually) identified with the real dual bundle of \( TC^n \) introducing the real duality defined by

\[
(\omega, X) \in T^*C^n \times TC^n \quad (\omega, X) \mapsto \text{Im} \langle \omega, X \rangle.
\]

In other words we identify real and holomorphic forms by \( \text{Im} \omega \leftrightarrow \omega \).

Now, let \( M \) be a real submanifold of \( C^n \). In this identification, the conormal bundle \( T^*_M C^n \) is a subbundle of \( T^*C^n \) and it has fiber spaces

\[
T^*_M C^n[z] = \{ \omega \in T^*C^n ; \text{Im} \omega|_{T_zM} = 0 \}.
\]

Hence the bundles \( T_M C^n = TC^n|_M/TM \) and \( T^*_M C^n \) are in duality by

\[
(\omega, X \mod TM) \mapsto \text{Im} \langle \omega, X \rangle.
\]

Assume moreover that \( M \) is generic (that is \( TM + JT M = TC^n|_M \)) and let \( \Sigma_M \) be the orthogonal complement of the complex tangent bundle \( T^cM \) in the cotangent bundle \( T^*M \). In the terminology of linear partial differential equations it is the characteristic set (and since \( T^cM \) is a fiber bundle, the characteristic manifold) of the system of CR vector fields. It is easily checked that \( \Sigma_M \) and \( T_M/T^cM \) are in duality in the same way.

Since \( M \) is CR, \( \Sigma_M \) is a fiber bundle and there is a canonical bundle epimorphism

\[
\theta : T^*_M C^n \rightarrow \Sigma_M,
\]

defined by \( \theta(\omega) = \iota_M^* \omega \) where \( \iota_M : M \rightarrow C^n \) is the natural injection. Since \( M \) is generic, \( \theta \) is an isomorphism.

On the other hand the complex structure \( J \) induces an isomorphism, still denoted by \( J \)

\[
J : TM/T^cM \rightarrow T_M C^n.
\]

**Lemma 2.1.** \( \theta \) is the transposed of \( J \), i.e.

\[
(\omega, JX) = (\theta(\omega), X)
\]

for every \( \omega, X \).

(Indeed \( < \omega, JX > = i < \omega, X > \).

From now on we let \( S \subset M \) be a CR submanifold of \( M \) with the property that \( CRdim S = CRdim M \). Equivalently it is required that \( T^cS = T^cM|_S \). By restriction analogous pairs of bundles remain isomorphic when \( TM/T^cM \) is replaced by \( T_S M \), \( \Sigma_M \) is replaced by \( T^*_S M \), \( T^*_M C^n \) is replaced by \( TC^n|_S \) / \( (TM|_S + JT S) = E \), and \( T^*_M C^n \) is replaced by \( T^*_M C^n \cap iT^*_N C^n = E^* \), but now \( T^cS \)-partial connections can be defined by means of the isomorphisms \( J \) and \( \theta \) on the two new bundles \( E \) and \( E^* \). Note that the duplication essentialy deals with complex differential geometry.

First, the results of the previous section apply with \( K = T^cM = TM \cap JT M \) and \( N = S \) and produce a \( T^cS \)-partial connection \( \nabla \) on \( T_S M \) together with the dual connection \( \nabla^* \) on \( T^*_S M \). On the other hand, the push forward by \( J \) of \( \nabla \) defines a \( T^cS \)-partial connection \( \Theta \) on \( E \); its action on a section \( \vartheta \) of \( E \) in the direction of a complex tangent vector \( X \) is simply

\[
\Theta_X \vartheta := J \nabla_X (J^{-1} \vartheta).
\]
Similarly, the pull-back of the $T^S$-partial connection $\nabla^*$ by $\theta$ defines a $T^S$-partial connection $\Theta^*$ on $E^*$, and $\Theta^*$ is the connection dual to $\Theta$ since $\theta$ is the transposed of $J$ (lemma 2.1).

Recall from section 1 that if $X$ is a section of $T^S M$ then $\hat{X} = H_{\tau(X)}|_{T^S_M}$ is tangent to $T^S M$. In [61], Trépreau showed that $E^*$ is a CR manifold, using a lemma which states that given such a vector field $\hat{X}$ tangent to $T^S M$ with horizontal part $X$ complex tangent to $M$, there exists a unique vector field $\tilde{X}$ complex tangent to $E^*$ with the same horizontal part $X$. Moreover Trépreau states that

$$\tilde{X} = d\theta(\hat{X})$$

Hence we deduce that the $T^S$-partial connection $\Theta^* = \theta^*\nabla^*$ can alternately be given, as is originally done in [61], by means of the vector fields of the form $\hat{X}$, i.e. horizontal subspaces of $\Theta^*$ are spanned by tangent vectors to integral curves of $\hat{X}$. We then have checked that the parallel translation in $E^*$ introduced by Trépreau with the assumption of Proposition 1.2 is the same as the one associated with the $T^S$-partial connection $\Theta^*$ previously defined starting, as in section 1, with the partial connection associated with the bundle of complex tangents to $M$, $K = T^S M$. Moreover, since $\hat{X}$ is complex tangent to $E^*$, we see that $T^S E^*$ is the set of horizontal subspaces for the $T^S$-partial connection $\Theta^*$. This has been noticed in [69] and will be useful in the next section when proving Theorem 3.4.

§3. Orbits and the extension of CR functions.

In this section, it is assumed that $M$ is a generic submanifold of $\mathbb{C}^n$ of smoothness class $C^2$, and we let $X$ be the set of $C^1$ sections over open subsets of $M$ of $T^S M$. If $z \in M$, the subset of $M$ consisting of points of $M$ which can be reached by piecewise $C^1$-smooth integral curves of elements of $X$, starting at $z$, is called the CR-orbit of $z$, and is denoted by $\mathcal{O} [z]$.

If $U$ is an open subset of $M$, $X|_U$ denotes the set of elements of $X$ restricted to $U$. It is well-known (cf. [59], [6], [61]) that

$$\lim_{U} \mathcal{O}(X|_U, z)$$

where $U$ runs over the open neighborhoods of $z$ in $M$ defines the germ at $z$ of the unique CR-submanifold of $M$ with the same CR dimension as $M$ of minimal dimension passing through $z$, which is called the local CR-orbit of $z$ and is denoted by $\mathcal{O}^{loc} [z]$. When considering $\mathcal{O}^{loc} [z]$ in the following we shall mean such a submanifold of a neighborhood of $z$ in $M$, i.e. an actual representative of the germ. It plays the crucial role in the study of automatic extendibility of CR functions (cf. Theorem 3.1 below).

Recall that a smooth complex-valued function on $M$ is called a CR function if it is annihilated by every antiholomorphic tangent vector field on $M$. A continuous function can be thought CR in the sense of distribution theory. We denote by $CR(M)$ the set of all continuous CR functions on $M$.

For completeness we recall definitions from [61] and [69]. We say that a manifold $M$ with boundary is attached to $M$ at $(m, u)$, $m \in M$, $u \neq 0$, $u \in T_M \mathbb{C}^n [m]$ if $bM \cap U = M \cap U$ for some neighborhood $U$ of $m$, and $u$ is represented by a vector $u_1 \in T_m M$ directed inside $M$.

Let $f$ be a CR function on $M$; we say that $f$ is CR-extendible at $(m, u)$ if it extends continuously to be CR on some $\hat{M}$ attached to $M$ at $(m, u)$. When there is a CR submanifold $S$ of $M$ through $m$ and a manifold $\hat{M}$ attached to $M$ at $(m, u)$, $u \in T_M \mathbb{C}^n [m]$, we also say that $\hat{M}$ is attached to $M$ at $(m, \eta)$, if $u$ represents $\eta \in E_m$, $\eta \neq 0$ ($E$ is the bundle defined in section 2). Similarly it makes sense to consider CR-extendibility at $(m, \eta)$, $m \in S, \eta \in E_m$. But it should be noted that given $\eta \neq 0$ in $E_m$ does not determine $\hat{M}$ unambiguously unless $S$ is complex
From now on we will require that $M$ belong to the class $C^{(k,\alpha)}$, $k \geq 2, 0 < \alpha < 1$. This regularity assumption can be justified since it behaves well when proving the strongest local results on CR-extendibility (In fact, it behaves well through the so-called Bishop equation, [68], Theorem 1.), and constructing wedges with ribs and an edge having such a regularity (cf. [4]). Moreover, we need manifolds of class at least $C^2$ in order to apply Proposition 1.1. Since it will be of use in the proof of Theorem 3.4 we recall the following theorem due to Tumanov ([68])

**Theorem 3.1.** (A. E. Tumanov) Let $M$ be a generic submanifold of $\mathbb{C}^n, n = p + q$, with dim $M = 2p + q$, $\text{CRdim } M = p$, and of smoothness class $C^{k,\alpha}$ $(k \geq 2), 0 < \alpha < 1$. For every point $z \in M$ there exist $r = r(z) = \dim \mathcal{O}_{\theta}^{loc} - 2$ CRdim $M$ manifolds with boundary $\tilde{M}_1, \ldots, \tilde{M}_r$, attached to $M$ at $z$, of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$ such that

(a) Every CR function on $M$ is CR-extendible to $\tilde{M}_1, \ldots, \tilde{M}_r$.

(b) $\sum_{j=1}^{r} T_{\gamma} \tilde{M}_j = T_{\gamma} M + JT_{\gamma} \mathcal{O}_{\theta}^{loc}$ whenever $z'$ close to $z$ in $\mathcal{O}_{\theta}^{loc}$.

Moreover the manifold germ $\mathcal{O}_{\theta}^{loc}$ is of class $C^{(k,\beta)}$ whenever $0 < \beta < \alpha$.

Note that $\mathcal{O}_{\theta}^{loc}$ is at least of class $C^2$ so it can play the role of $N$ in Propositions 1 and 2. Using the connections constructed in section 2 we can reinterpret the main result on propagation of analyticity for CR functions recently proved by Tumanov.

According to Tumanov ([69], Proposition 7.3), the connection dual to the one that is constructed during the paper has the property that its horizontal subspaces are exactly fibers of the complex tangent bundle $T^c E^*$, hence, concludes Tumanov, the induced parallel translation need be the same as the one introduced on $E^*$ by Trépreau. We have shown in section 2 that our connection $\Theta$ has as a dual connection a connection $*=\Phi$ with the same property; so $=J_* \nabla$ coincides with the connection constructed by Tumanov.

Proposition 1.2 together with Theorem 5.1 in [69] leads to

**Theorem 3.2.** Let $M \subset \mathbb{C}^n$ be a generic manifold and $S \subset M$ a CR submanifold of $M$ with the property that CRdim $S = \text{CRdim } M$. Let $\gamma$ be a piecewise smooth integral curve of $T^c M$ running from $z' \in S$ to $z'' \in S$ and let $\Phi_\gamma$ be the associated composed flow. Then for every $\epsilon > 0$, every $\eta' \in E_{z'}$ and every manifold $M'$ attached to $M$ at $(z', \eta')$, there exists another manifold $M''$ attached to $M$ at $(z'', \eta'')$, $\eta'' \in E_{z''}$ such that

(a) $|\eta'' - Jd\Phi_\gamma(z).J^{-1}\eta'| < \epsilon$

(b) if a CR function on $M$ extends to be CR on $\tilde{M}'$ it extends to be CR on $\tilde{M}''$

(c) if $M, \tilde{M}'$ belong to $C^{k,\alpha}$ $(k \geq 2), 0 < \gamma < \alpha < 1$ then there exists such a $\tilde{M}'' \in C^{(k,\gamma)}$.

Theorem 3.2 shows that the so-called propagation of analyticity for CR functions is intrinsically related to the geometry of the base manifold $M$. Moreover, it fundamentally means that the study of extendibility for CR functions is closely related to the study of sections of the complex tangent space to $M$.

Following Sussmann ([59]), we begin with some adapted terminology and recalls. Let $X \in \mathbf{X}$ be a local section of $T^c M$. The $C^1$ integral curves $t \rightarrow \gamma(t)$ of $X$ generate local diffeomorphisms of $M$ where they are defined (the so-called flow of $X$) which we will denote by $z \rightarrow X_t z$. Composites of several maps of the form $X_t$ can produce local diffeomorphisms of neighborhoods of points that are far from each other in a same CR-orbit. If $X = (X_1, \ldots, X_m)$ is an element of $\mathbf{X}^m$ such that for $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, the map $z \rightarrow X_{m,t_m} \cdots X_{1,t_1} z$ is well defined in a neighborhood of $z$, we will still denote it for convenience by $X_t$ or $\Phi$ (cf. Proposition 1.2).

Let $\Delta_\mathbf{X}$ be the distribution spanned by $\mathbf{X}$, i.e. the mapping which to $z \in M$ assigns the linear hull of vectors $X(z)$ where $X$ belongs to $\mathbf{X}$: it is just the distribution associated with the complex tangent bundle of $M$. We let $P_\mathbf{X}$ denote the smallest distribution which contains $\Delta_\mathbf{X}$ and is invariant under complex-flow.
diffeomorphisms, or for short the smallest X-invariant distribution which contains \( \Delta_X \). Precisely, \( P_X(z) \) is the linear hull of vectors of the form \( dX_t(v) \) where \( v \in \Delta_X(z') \) and \( z = X_t(z') \). A \( C^1 \) distribution \( P \) on \( M \) has the maximal integral manifold property if for every \( z \in M \) there exists a submanifold \( S \) of \( M \) such that \( z \in S \) and for every \( z' \in S \), \( T_{z'}S = P(z') \). Moreover, \( S \) is said to be a maximal integral manifold of \( P \) if \( S \) is an integral manifold of \( P \) such that every connected integral manifold of \( P \) which intersects \( S \) is an open submanifold of \( S \).

Then the results of Sussmann, which extend to the \( C^2 \) case tell us that \( \mathcal{O} \ [z] \) is a (connected) maximal integral submanifold of \( P_X \) (perhaps with a finer topology) and admits a unique differentiable structure making the injection \( i : \mathcal{O} \ [z] \to M \) an immersion of class \( C^1 \).

We now introduce the following definitions.

**Definition 3.3.** Let \( M \) be a generic submanifold of \( \mathbb{C}^n \) and \( z \in M \). \( M \) is called minimal at \( z \) if \( \mathcal{O}^{loc} \ [z] \) contains a neighborhood of \( z \) in \( M \). It is called globally minimal at \( z \) if \( \mathcal{O} \ [z] \) contains a neighborhood of \( z \).

In view of the global results of Sussmann definition 3.3 means that the generic manifold \( M \) is globally minimal at a point \( z \) if and only if there exist a finite number of points \( z'_l \), \( l = 1, ..., d \) in the CR-orbit of \( z \) and composed flow diffeomorphisms \( \Phi_l \), \( l = 1, ..., d \) of a neighborhood of \( z'_l \) in \( M \) on a neighborhood of \( z \) in \( M \) respectively such that

\[
T_zM = \sum_{l=1}^{d} d\Phi_l(z'_l) \cdot (T^{\mathbb{C}}_{z'_l}M).
\]

We are now able to prove the theorem conjectured by Trépreau in [61] which is the natural generalization of a celebrated theorem of Tumanov ([67]). Here is the substance of this paper.

**Theorem 3.4.** Let \( M \) be a generic submanifold of \( \mathbb{C}^n \) of smoothness class \( C^{(k, \alpha)} \), \( k \geq 2, 0 < \alpha < 1 \) which is globally minimal at a point \( z \in M \). Then for every \( z' \) in the CR-orbit of \( z \) there exists a wedge \( W \) of edge \( M \) at \( z' \) such that

\[(*) \ every \ CR \ function \ on \ M \ extends \ holomorphically \ into \ W.\]

**Proof.** We shall make use of the following abuse of language: we will say that a CR-function \( u \) is CR-extendible in the direction \( v \in TM/T^cM[z] \) if it is in fact CR-extendible in the direction of \( Jv \). Let us consider the set

\[ H_z = Vect \{ v \in T_zM/T^c_zM; \ u \ is \ CR-extendible \ at \ (z, v) \ \}\]

and its preimage under the natural surjection \( \pi : TM \to TM/T^cM \)

\[ \hat{H}_z = \pi^{-1}(H_z) \subset T_zM. \]

**Lemma 3.5.** Let \( X \) be a \( C^1 \) section of \( T^cM \) over a neighborhood of \( z \in M \) and let \( \Phi_t \) be the flow of \( X \) and \( \Phi = \Phi_t \) for some \( t \). Then, if \( v \in T_zM \),

\[ v \in \hat{H}_z \iff d\Phi(z).v \in \hat{H}_{\Phi(z)}. \]

**Proof.** Since the statement is a symmetric and a transitive one we can assume that \( z \) and \( z' \) are so close that \( z' := \Phi(z) \) is contained in a CR submanifold \( S \) of \( M \) with \( CRdimS = CRdimM \) which is minimal at \( z \) (for instance take for \( S \) the local CR-orbit of \( z \)) and such that \( z' \) belongs to the boundary of the manifolds whose existence comes from Theorem 3.1. Hence

\[(*) \ \hat{H}_{z'} \supset T_{z'}S.\]

9
So if \( v \) belongs to \( T_zS \) there is nothing to add. On the other hand, if \( \xi = pr_{T_zM}v \neq 0 \) we apply the propagation result Theorem 3.2 and obtain that for every \( \epsilon > 0 \) \( u \) is CR-extendible at \( (z', \xi'') \), where \( \xi'' \) is \( \epsilon \)-close in Euclidean norm to \( \xi' = d\Phi(x)\xi \); so letting \( \epsilon \) decrease to zero, since every finite-dimensional vector space is closed, we have \( \xi' \in pr_{T_zM}(H_{z'}) \). Because of (*) the indetermination on the specific representative of \( \xi' \) is removed whence

\[
d\Phi(z).v \in \hat{H}_{z'}
\]

and the lemma is proved.

END OF PROOF OF THEOREM 3.4. The global lemma 3.5 and the condition of global minimality imply immediately that

\[
\hat{H}_{z'} = T_{z'}M
\]

for every \( z' \) in the (global) CR-orbit of \( z \). The conclusion follows by the edge-of-the-wedge theorem and the proof is complete.

Theorem 4.1 admits an obvious generalization which involves the concept of \( \mathcal{W}_r \)-wedges. Recall that a \( \mathcal{W}_r \)-wedge at \( z \) with edge \( M \) is locally the general intersection of a wedge of edge \( M \) at \( z \) and a generic manifold containing \( M \) as a submanifold of codimension \( r \).

**Theorem 3.6.** Let \( M \) be a generic submanifold of \( \mathbb{C}^n \) of smoothness class \( C^{(k,\alpha)} \), \( k \geq 2, 0 < \alpha < 1 \), and let \( r = \dim O\ [z] - 2CRdim M \). Then for every \( z' \) in the CR-orbit of \( z \), every \( \gamma \) with \( 0 < \gamma < \alpha \), there exists a \( \mathcal{W}_r \)-wedge \( \mathcal{W} \) of edge \( M \) at \( z' \) and of smoothness class \( C^{k,\gamma} \) such that

\[
(\ast) \text{ every CR function on } M \text{ extends to be CR on } \mathcal{W}.
\]

Moreover, the tangent space to \( \mathcal{W} \) at \( z' \) spans \( T_{z'}M + JT_{z'}O\ [z] \).

**Proof.** The same argument runs in proving that \( \hat{H}_{z'} \) contains \( T_{z'}M + JT_{z'}O\ [z] \) and the conclusion then follows by the edge-of-the wedge theorem of Ayrapetyan ([4]), in the classes \( C^{(k,\alpha)} \), \( k \geq 2, 0 < \alpha < 1 \).
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