On SO(N) Spin Vertex Models

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ABSTRACT

We describe the Boltzmann weights of the $D_k$ algebra spin vertex models. Thus, we find the SO($N$) spin vertex models, for any $N$, completing the $B_k$ case found earlier. We further check that the real (self–dual) SO($N$) models obey quantum algebras, which are the Birman–Murakami–Wenzl (BMW) algebra for three blocks, and certain generalizations, which include the BMW algebra as a sub–algebra, for four and five blocks. In the case of five blocks, the $B_4$ model is shown to satisfy additional twenty new relations, which are given. The $D_6$ model is shown to obey two additional relations.
1. Introduction.

Solvable lattice models in two dimensions are a fruitful ground to test phase transitions, universality, integrability [1] and conformal field theory [2]. For reviews see [3, 4].

We will concentrate here on a type of solvable lattice models which are called vertex models. Well known among these are the six, eight and nineteen vertex models [3, 4]. For recent works on vertex models see, e.g., [5, 6, 7, 8, 9]. Our purpose here is to introduce vertex models based on the algebra $D_k$ and the spin representation. This completes the SO($N$) spin vertex models for all $N$, where the $B_k$ models were described before in ref. [10].

We are also interested in the algebraic structure underlying these models. We use the more general results of [11, 12], which describe the three, four and five blocks algebras (where the number of blocks is the degree of polynomial equation obeyed by the Boltzmann weights), assuming only a certain ansatz for the Baxterization, described in [13], and the Yang–Baxter equation. We describe and check numerically, the algebras of $B_4$, which is a five blocks theory, and the algebra of $D_6$, which is a four blocks theory.

The algebras include a version of the Birman–Murakami–Wenzl algebra (BMW) [14, 15], along with two new relations for four blocks and twenty new relations for the five blocks theory, which are given here for $B_4$. We check that the BMW algebra is obeyed for $D_k$, for any small even $k$, with a different skein relation.
2. $D_k$ spin vertex models.

We wish to describe a vertex model based on the algebra $D_k = SO(2k)$ and the spin representation. This solution is an element of $\text{End}(V \otimes V)$ where $V$ is the spin representation of $D_k$. We denote by $\alpha_n = \epsilon_n - \epsilon_{n+1}$, for $n = 1, 2, \ldots, k-1$ and $\alpha_k = \epsilon_{k-1} + \epsilon_k$ the simple roots of $D_k$, where $\epsilon_i$ are orthogonal unit vectors. The spin representation, denoted by $S$ has the weights $\sum_{i=1}^{k} p_i \epsilon_i/2$, where $p_i = \pm 1$ and $\prod_{i=1}^{k} p_i = 1$. The last product is $-1$ for the anti-spinor representation, denoted by $\bar{S}$. We find it useful to add $1/2$ to these weights, and to represent weights of the spinor (anti-spinor) representation by the vector $m$, where $m_i = 0$ or $1$.

To start constructing the vertex model, we need a solution which commutes with the co-product of $U_q^2(SO(2k))$. We find it convenient to first describe a solution for the larger representation $\tilde{V} = S \oplus \bar{S}$, namely the sum of the spinor and anti-spinor representations. Such a solution was described recently in [16]. It is the element $C$ of $\text{End}(\tilde{V} \otimes \tilde{V})$, given by

$$C_{m,n}^{b,c} = \sum_{j=1}^{k} \delta_{m_j,1-n_j} (-q^2)^{\{m-n\}_j} \delta_{b,\bar{m}_j} \delta_{c,\bar{n}_j},$$

where

$$\{m\}_j = \sum_{r=1}^{j} m_r,$$

and $\bar{n}_j$ is equal to $n_j$ except at the $j$th coordinate where it is $1-n_j$. Here $m, n, b, c = 0$ or $1$ are weights of the spin or anti-spin representations shifted by $1/2$. The eigenvalues of the matrix $C$ were computed in ref. [16], and are

$$\lambda_j = \pm s(k-j), \quad \text{for } j = 0, 1, \ldots, k,$$

where

$$s(x) = \frac{q^{2x} - q^{-2x}}{q^2 - q^{-2}}.$$ 

The solution $C$ has the disadvantage of mapping both the spin and anti-spin
representations. We note, however, that $C$ maps the representation $S \otimes S$ to $\tilde{S} \otimes \tilde{S}$, and vice versa. Thus, to get a solution in $\text{End}(S \otimes S)$ all we need to do is to square the matrix $C$ and to equate to zero all the $C_{m,n}^{b,c}$ for weights $m, n, b, c$ which are not in $S$. Thus, $C^2$ gives the solution we want. Of course, since $C$ commutes with the co–product, so does $C^2$.

Since the matrix $C^2$ commutes with the co–product, it has the same eigenvectors as our desired solution which obeys the Yang–Baxter equation, but not the same eigenvalues. Thus, we define the projection operators

$$
(P_{a}^{b,c})_{m,n} = \prod_{p \neq a} \left[ \frac{C^2 - \lambda_p^2 I}{\lambda_a^2 - \lambda_p^2} \right],
$$

(2.5)

where the product is in $\text{End}(S \otimes S)$ and $I$ is the identity map.

We note that for even $k$, $P^a = 0$ for $a$ which is odd, whereas for odd $k$, $P^a = 0$ for $a$ which is even. The $j$th eigenvalue corresponds to the representation $V_j = \wedge^j v$, where $v$ is the vector representation, i.e., the anti–symmetric product of $j$ vector representations [16]. The highest weight of the representation $V_j$ is $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_j$. Thus, the non–zero $P^a$ are in one to one correspondence with the representations that appear in the tensor product,

$$
S \times S = \sum_{j=0}^{k} V_j,
$$

(2.6)

as they should. Thus, the projection $P^a$ projects onto the representation $V_a$.

We wish to make the connection between the solution $C^2$ and the $D_k$ WZW conformal model. For explanation of conformal field theory see the book [2], and references therein. To do this we define,

$$
q^2 = \exp[\pi i/(r + g)],
$$

(2.7)
Here $r$ is the level of the representation and

$$g = 2k - 2,$$  \hspace{1cm} (2.8)

is the dual Coxeter number.

The dimension of the highest weight $\Lambda$ in a WZW theory is given by

$$\Delta_\Lambda = \frac{\Lambda(\Lambda + 2\rho)}{2(r + g)},$$  \hspace{1cm} (2.9)

where $\rho$ is half the sum of positive roots and $C_\Lambda = \Lambda(\Lambda + 2\rho)$ is the Casimir of the representation. The Casimir of the representation $V_j$ is given by

$$C(V_j) = C_j = j(2k - j).$$  \hspace{1cm} (2.10)

As explained in [10], the eigenvalues of the $R$ matrix are given by

$$\beta_j = p_j e^{-i\pi \Delta_j} = p_j q^{-C(V_j)},$$  \hspace{1cm} (2.11)

where $p_j = \pm 1$ is some sign which corresponds to whether the product in eq. (2.6) is symmetric or anti-symmetric. In our case, the sign is given by

$$p_j = (-1)^{(k-j)/2}.$$  \hspace{1cm} (2.12)

Thus, since we know the eigenvalues of the $R$ matrix and the projection operators from eq. (2.5), we may construct the $R$ matrix as

$$R_{m,n}^{a,b} = \sum_{j=0}^{k} \beta_j(P_j)^{a,b}_{m,n}.$$  \hspace{1cm} (2.13)

It can be verified that this $R$ matrix satisfies the Yang–Baxter equation (YBE) which for the $R$ matrix is the braiding relation,

$$\sum_{\alpha,\beta,\gamma} R_{j,k}^{\beta,\alpha} R_{i,\beta}^{l,\gamma} R_{m,\alpha}^{n} = \sum_{\alpha,\beta,\gamma} R_{i,j}^{\alpha,\beta} R_{\beta,k}^{\gamma,n} R_{\alpha,\gamma}^{l,m}.$$  \hspace{1cm} (2.14)

We checked that this $R$ matrix obeys the YBE, numerically for $k = 2, 3, 4, 5, 6$ and it holds, indeed, for various weights and for general $q$.  

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Now, we wish to define a trigonometric solution for the YBE. For this purpose, we use the same general ansatz for Baxterization as in \[10, 13\]. First, we need to decide on the order of the primary fields in eq. (2.6). The order which solves the YBE is given by

\[(h_0, h_1, \ldots, h_{k/2}) = (0, 2, 4, \ldots, k), \quad (2.15)\]

for even \(k\). For odd \(k\) the order is

\[(h_0, h_1, \ldots, h_{(k-1)/2}) = (k, k-2, k-4, \ldots, 1). \quad (2.16)\]

The parameters are given by \[10, 13\],

\[\hat{\zeta}_j = \pi(\Delta_{h_{j+1}} - \Delta_{h_j})/2, \quad (2.17)\]

for \(j = 0, 1, \ldots, m - 1\), where \(m = k/2\) for even \(k\) and \(m = (k - 1)/2\) for odd \(k\). Thus, the \(D_k\) theory is an \(m + 1\) blocks theory. We thus define the parameters as

\[\zeta_j = (C_{h_{j+1}} - C_{h_j})/2. \quad (2.18)\]

We define

\[p(x) = q^x - q^{-x}. \quad (2.19)\]

Then the trigonometric solution to the YBE assumes the form \[10, 13\],

\[R_{m,n}^{a,b}(u) = \sum_{j=0}^{m} f_j(u)(P^{h_j})_{m,n}^{a,b}, \quad (2.20)\]

where

\[f_a(u) = \left[\prod_{j=1}^{a} p(\zeta_{j-1} - u) \prod_{j=a+1}^{m} p(\zeta_{j-1} + u) \right] / \left[\prod_{j=1}^{m} p(\zeta_{j-1}) \right], \quad (2.21)\]

where \(a = 0, 1, \ldots, m\).
For example, for $k = 6$, which is a four blocks theory, the parameters are $(\zeta_0, \zeta_1, \zeta_2) = (10, 6, 2)$. The crossing parameter is $\lambda = \zeta_0$. The $D_k$ vertex models with $k$ even are real (self–dual) as $S = S^\ast$. For odd $k$ the theories are not real (not self–dual), as $S \neq S^\ast$.

We can check that the solution, eqs. (2.20, 2.21), obeys the Yang–Baxter equation, which is

$$\sum_{\alpha,\beta,\gamma} R_{j,k}^{\beta,\alpha}(u)R_{i,j}^{l,\gamma}(u+v)R_{\gamma,\alpha}^{m,n}(v) = \sum_{\alpha,\beta,\gamma} R_{i,j}^{\alpha,\beta}(v)R_{\beta,k}^{n}(u+v)R_{\alpha,\gamma}^{l,m}(u).$$

We checked this equation, numerically, for $k = 2, 3, 4, 5, 6$ and various values of $u, v$ and $q$ and various heights. It is indeed obeyed. This gives the trigonometric $D_k$ spin vertex model.

3. BMW' algebra and SO($N$) spin vertex models.

We repeat here the definition of the BMW’ algebra following [10]. We find it convenient to use an operator form for the $R$ matrix. We define the matrix, following [4],

$$X_i(u) = \sum_{m,n,a,b} R_{m,n}^{a,b}(u) I^{(1)} \otimes \ldots \otimes I^{(i-1)} e_{am}^{(i)} \otimes e_{bn}^{(i+1)} \otimes I^{(i+2)} \otimes \ldots \otimes I^{(f)},$$

where $I^{(i)}$ is the identity matrix at position $i$ and $(e_{rs})_{lm} = \delta_{rl} \delta_{sm}$. The YBE, eq. (2.22), then assumes a more compact form,

$$X_i(u)X_j(v) = X_j(v)X_i(u), \quad \text{if } |i - j| \geq 2,$$

$$X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u).$$

Let us denote the number of blocks by $n$. For the $D_k$ models, this is $n = m + 1 = k/2 + 1$ ($k$ even), or $n = m + 1 = (k + 1)/2$ (for odd $k$). In this section,
we will assume that $k$ is even, so that the theory is real (self-dual). It is assumed that the number of blocks is greater or equal to three, $n \geq 3$. The algebras of non-real theories are also interesting, but we shall not describe it here. We define the limit of the matrix $X_i(u)$ as

$$X_i = \lim_{u \to i\infty} e^{(n-1)u} X_i(u), \quad X_i^t = \lim_{u \to -i\infty} e^{-i(n-1)u} X_i(u). \quad (3.3)$$

We define the operators,

$$G_i = 2^{n-1} e^{-i(n-1)\zeta_0/2} \left[ \prod_{r=1}^{n-1} \sin(\zeta_{r-1}) \right] X_i, \quad (3.4)$$

$$G_i^{-1} = 2^{n-1} e^{i(n-1)\zeta_0/2} \left[ \prod_{r=1}^{n-1} \sin(\zeta_{r-1}) \right] X_i^t, \quad (3.5)$$

and

$$E_i = X_i(\zeta_0), \quad 1_i = X_i(0), \quad (3.6)$$

where $\zeta_i$ are the parameters defined in eq. (2.17). $G_i^{-1}$, so defined, is the inverse of $G_i$, or $G_i G_i^{-1} = 1_i$.

From the ansatz, eqs. (2.20, 2.21), and from the YBE, eq. (2.22), we can prove the following relations of the operators $G_i$, $G_i^{-1}$ and $E_i$,

$$E_i E_{i+1} E_i = b E_i, \quad E_i^2 = b E_i, \quad E_i E_j = E_j E_i \quad \text{if} \ |i-j| \geq 2, \quad (3.7)$$

$$b = \prod_{r=1}^{n-1} \frac{\sin(\zeta_0 + \zeta_{r-1})}{\sin(\zeta_{r-1})}, \quad (3.8)$$

which is the Temperley–Lieb algebra [17], and

$$G_i G_j = G_j G_i \quad \text{if} \ |i-j| \geq 2, \quad G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1}, \quad (3.9)$$
which is the braiding algebra. We can also prove the relations,

\[ G_i E_i = E_i G_i = l^{-1} E_i, \]

where

\[ l = i^{n-1} \exp \left[ i(n-1)\zeta_0/2 + i \sum_{r=0}^{n-2} \zeta_r \right]. \]

The following is the skein relation which stems from the definition of the projection operators along with the ansatz, eqs. (2.20, 2.21),

\[ G_i^{n-2} = a E_i + \sum_{r=-1}^{n-3} b_r G_i^r, \]

where \( a \) and \( b_r \) are some coefficients, which can be expressed in terms of the parameters, \( \zeta_r \). From the skein relation we prove,

\[ G_{i\pm 1} G_i E_{i\pm 1} = E_i G_{i\pm 1} G_i. \]

The above relation, eqs. (3.7–3.13), are part of the Birman–Murakami–Wenzl algebra (BMW) [14, 15]. The rest of the relations of the BMW algebra are also obeyed, except of the skein relation, eq. (3.12), which is different for more than three blocks. These are

\[ G_{i\pm 1} G_i E_{i\pm 1} = E_i E_{i\pm 1}, \quad G_{i\pm 1} E_i G_{i\pm 1} = G_i^{-1} E_{i\pm 1} G_i^{-1}, \]

\[ G_{i\pm 1} E_i E_{i\pm 1} = G_i^{-1} E_{i\pm 1}, \quad E_{i\pm 1} E_i G_{i\pm 1} = E_{i\pm 1} G_i^{-1}, \]

\[ E_i G_{i\pm 1} E_i = l E_i, \quad E_i G_{i\pm 1}^{-1} E_i = l^{-1} E_i. \]

We have verified that the full BMW’ algebra (BMW with a different skein relation) is obeyed by the \( D_k \) model, with \( k = 4 \) or \( 6 \). We did this numerically,
using various heights and general $q$. We note that the BMW' algebra is obeyed also by the $B_k$ spin vertex models [10], while substituting the relevant parameters $\zeta_r$. For more than three blocks there are additional relations, except from the skein relation. These are described in the next two sections, for $B_4$ and $D_6$.

4. $n = 5$ blocks case and $B_4$ vertex models.

It was noticed in [10] that the structure of n-CB algebra, which follows from the Baxterization of IRF models, is also applicable for vertex models. In our case, the connection is between n-CB algebra and $B_{n-1}$ models. This algebra was checked, in particular, for $B_3$ models obeying 4-CB algebra [10]. In the 5-block case, the n-CB algebra reduces to a set of 20 relations in addition to BMW' sub-algebra. We get these relations by expanding the YBE, eq. (2.22) and assuming the ansatz, eqs. (2.20, 2.21). The 5-CB relations are general for all the five blocks models obeying the ansatz for Baxterization, eq. (2.20, 2.21). We specify the algebra here only for the $B_4$ spin vertex models for calculation reasons. In this section we summarize the 5-CB relations for $B_4$ spin vertex model. We give the shorter relations explicitly, here. The complete list of 5-CB relations for $B_4$ models can be found in the attached Mathematica file.

For the general values of the parameters, the 5-CB skein relation as well as the explicit projectors have been found in [12]. The skein relation reads

$$G_i^3 = \alpha 1_i + \beta E_i + \gamma G_i + \delta G_i^{-1} + \mu G_i^2,$$

where denoting $s_k = q^{\xi_k}$ the parameters are
\[\alpha = -s_1\left(s_1^2s_2^2s_3 - s_2^2s_3^2 + s_3^2 - 1\right)\frac{s_0^3s_2s_3^3}{s_0s_1s_2s_3},\]
\[\beta = \left(s_1^2 - 1\right)\left(s_2^2 - 1\right)\left(s_0^2s_1s_2 + 1\right)\left(s_3^2 - 1\right)\left(s_0^2s_1s_2s_3 - 1\right)\frac{s_0^3s_1s_2^2}{s_0^3s_2^2 - 1s_3^2(s_0^3s_3^2 - 1)},\]
\[\gamma = \frac{s_1^2s_3^2s_2^4 + s_1^2s_2^2 - s_1^2s_3^2 + s_2^2 - s_2 + 1}{s_0^2s_2^2s_3^3},\]
\[\delta = -s_1^2\frac{s_0^4s_2^2}{s_0s_1s_2s_3}, \quad \mu = -s_2^2s_1 + s_2^2s_3s_1^2 + s_1^2 - 1 \frac{s_0s_1s_2s_3}{s_0s_1s_2s_3}.
\]

In the case of \(B_4\) models the crossing parameters are
\[\zeta_0 = 7, \quad \zeta_1 = 3, \quad \zeta_2 = -1, \quad \zeta_3 = -5\] (4.3)

Using the results of [12] with the above explicit parameters the desired 5-CB algebra relations for \(B_4\) models can be found. The \(B_4\) skein relation reads explicitly
\[G_i^3 = \left(\frac{q^{16} + q^{12} - q^{10} - q^6 + q^4 + 1}{q^{14}}\right)G_i - \frac{1}{q^{12}}G_i^{-1} + \frac{(q^2 - 1)(q^4 + 1)(q^6 - q^2 - 1)}{q^{10}}G_i^2 + \frac{(q^{12} - q^6 - q^2 + 1)}{q^{14}}1_i + \frac{(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)(q^{12} - q^6 + 1)(q^2 - 1)^2}{q^{38}}E_i.
\]

To shorten notation, we denote below by \(a_{l,m,n}\) the elements of the algebra \(A_l[i]A_m[i+1]A_n[i]\) and by \(b_{l,m,n}\) the elements of the algebra \(A_l[i+1]A_m[i]A_n[i+1]\), where \(A_l[r]\) stands for \(G_r, G_r^{-1}, E_r, G_r^2\) or \(1_r\) according to whether \(l = 1, 2, 3, 4, 5\), respectively. A few of the relations, which are sufficiently short, are listed below:

\[1) \left(\frac{q^{12} - q^{10} - q^6 + q^4 + 1}{q^{10}}\right) a_{5,2,3} + \left(\frac{q^{12} - q^{10} - q^6 + q^4 + 1}{q^{10}}\right) a_{5,3,1} - \left(\frac{q^{12} - q^{10} - q^6 + q^4 + 1}{q^{10}}\right) a_{5,1,3} - \left(\frac{q^{12} - q^{10} - q^6 + q^4 + 1}{q^{10}}\right) a_{5,3,2} - \left(\frac{q^{10} + q^6 - q^4 + q^2 - 1}{q^{20}}\right) \left(\frac{q^{12} - q^6 + 1}{q^{10}}\right) a_{5,5,3} - a_{4,3,3} - a_{5,3,4} + a_{5,4,3} + \left(\frac{q^{10} + q^6 - q^4 + q^2 - 1}{q^{20}}\right) \left(\frac{q^{12} - q^6 + 1}{q^{20}}\right) b_{5,5,3} + b_{3,3,4} = 0
\]
2) \[
\frac{(q^4 + 1)(q^{12} - q^6 + 1)}{q^{20}} a_{5,5.3} - \frac{(q^2 - 1)(q^4 + 1)(q^6 + q^4 - 1)}{q^{20}} b_{5,2.3} + \\
+ \frac{(q^2 - 1)(q^4 + 1)(q^6 + q^4 - 1)}{q^{20}} b_{5,3.2} + \frac{b_{5,4.3} - b_{5,3.4}}{q^6} - \\
- \frac{(q^4 + 1)(q^{12} - q^6 + 1)}{q^{20}} b_{5,5.3} + \frac{(q^2 - 1)(q^4 + 1)(q^6 - q^2 - 1)}{q^{16}} b_{5,3.1} - \\
- \frac{(q^2 - 1)(q^4 + 1)(q^6 - q^2 - 1)}{q^{16}} b_{5,1.3} + b_{1,4.3} - a_{3,4.1} = 0
\]

3) \[
\frac{(q^{10} + q^6 - q^4 + q^2 - 1)}{q^{10}} (q^{12} - q^6 + 1) \left( a_{5,4.3} - a_{5,3.4} + b_{5,3.4} - b_{5,4.3} \right) \left( a_{5,4.3} - a_{5,3.4} + b_{5,3.4} - b_{5,4.3} \right) - \\
- \frac{(q^{10} + q^6 - q^4 + q^2 - 1)^2 (q^{12} - q^6 + 1)^2 (a_{5,5.3} - b_{5,5.3})}{q^{30}} + \\
+ \frac{(q^{10} + q^6 - q^4 + q^2 - 1)(q^{12} - q^6 + 1)(b_{5,1.3} + b_{5,3.2} - b_{5,2.3} - b_{5,3.1})}{q^{20}} + \\
+ \frac{q^{10}(a_{4,3.4} - b_{4,3.4})}{q^{12} - q^{10} - q^6 + q^4 + 1} - a_{1,3.4} + a_{2,3.4} - b_{4,3.2} + b_{4,3.1} = 0
\]

4) \[
\frac{(q^2 - 1)(q^4 + 1)(q^{12} - q^6 + 1)(q^{18} - q^{16} + q^{10} - q^8 - 1)}{q^{20}} (b_{5,3.1} - b_{5,1.3}) + \\
+ \frac{(q^2 - 1)(q^4 + 1)(q^6 - q^2 - 1)}{q^{4}} (b_{5,2.3} - b_{5,3.2}) + b_{2,4.3} + \\
+ \frac{(q^4 + 1)(q^{12} - q^6 + 1)(q^{12} - q^{10} + q^8 - q^2 + 1)}{q^{6}} (a_{5,5.3} - b_{5,5.3}) + \\
+ (q^{12} - q^{10} - q^6 + q^4 + 1) q^{2}(b_{5,4.3} - b_{5,3.4}) - a_{3,4.2} = 0
\]

5) \[
\frac{(q^2 - 1)(q^4 + 1)(q^{12} - q^6 + 1)(q^{18} - q^{16} + q^{10} - q^8 - 1)}{q^{20}} (a_{5,1.3} - a_{5,3.1}) + \\
+ \frac{(q^4 + 1)(q^{12} - q^6 + 1)(q^{12} - q^{10} + q^8 - q^2 + 1)}{q^{6}} (a_{5,5.3} - b_{5,5.3}) + \\
+ \frac{(q^2 - 1)(q^4 + 1)(q^6 - q^2 - 1)}{q^{4}} (a_{5,3.2} - a_{5,2.3}) + \\
+ (q^{12} - q^{10} - q^6 + q^4 + 1) q^{2}(a_{5,3.4} - a_{5,4.3}) - a_{2,4.3} + b_{3,4.2} = 0
\]
We find that the whole list of 19 5-CB relations, which can be found in the attached Mathematica file, is fulfilled for the Boltzmann weights of $B_4$ models. The Boltzmann weights are stated in [10]. We checked the relations numerically for a general value of the parameter $q$ and substituting various heights.

5. 4–CB relations for $D_6$.

We wish to check the 4–CB algebra for $D_6$ which is a four blocks model. The four blocks relations were given in [11]. The parameters for $D_6$ are, eq. (2.18).

$$\zeta_0 = 10, \quad \zeta_1 = 6, \quad \zeta_2 = 2,$$

and $q$ is given by eq. (2.7).

The skein relation is [11],

$$G_i^2 = -iq^{-\frac{1}{2}}(-\zeta_0 - \zeta_1 - \zeta_2) \left(1 - q^{2\zeta_1} + q^{2\zeta_1+2\zeta_2}\right) G_i - iq^{-\frac{3}{2}}(-\zeta_0 + \zeta_1 - \zeta_2) G_i^{-1}$$

$$+ \frac{q^{-2\zeta_0 - 2\zeta_1 - 2\zeta_2} - \left(1 + q^{2\zeta_0 + 2\zeta_1 + 2\zeta_2}\right) \left(q^{2\zeta_2} - 1\right)}{\left(q^{2\zeta_0 + 2\zeta_2} - 1\right)} E_i$$

(5.2)
\[-q^{-\zeta_0 - 2\zeta_2} (1 - q^{2\zeta_2} + q^{2\zeta_1 + 2\zeta_2}) \].

The single additional relation is

\[ g(i, i + 1, i) = g(i + 1, i, i + 1), \quad (5.3) \]

where

\[ g = a_{1,2,4} + a_{1,3,1} + a_{4,2,1} - iq^{-\zeta_0/2 + \zeta_1 - \zeta_2}(a_{1,3,4} + a_{4,2,4} + a_{4,3,1}) \]

\[ -iq^{\zeta_0/2 - \zeta_1 + \zeta_2}(a_{2,3,4} + a_{4,1,4} + a_{4,3,2}) - \]

\[ i \frac{q^{\zeta_1 + \zeta_2}}{(q^{2\zeta_1} - 1)(q^{2\zeta_2} - 1)} \left(q^{\zeta_0/2}a_{1,2,1} + q^{-\zeta_0/2}a_{2,1,2}\right) + za_{4,3,4}, \quad (5.4) \]

where

\[ z = \frac{q^{-\zeta_0 - 2\zeta_1 - 2\zeta_2}(q^{2\zeta_1} - 1)(q^{2\zeta_2} - 1)}{q^{2\zeta_0 + 2\zeta_2} - 1} \times \]

\[ (2q^{2\zeta_0 + 2\zeta_2} + 2q^{2\zeta_0 + 2\zeta_1 + 2\zeta_2} + q^{4\zeta_0 + 2\zeta_1 + 4\zeta_2 + 1}). \quad (5.5) \]

We denoted by \( a_{i,j,k}(r, s, t) \) the element of the algebra \( a_i[r]a_j[s]a_k[t] \) where \( a_i[r] \)

is \( G_r, G_r^{-1}, E_r \) or \( 1_r \), if \( i = 1, 2, 3, 4 \), respectively.

Finally, we proceed to check these two relations, for the \( D_6 \) vertex model substituting the explicit Boltzmann weights, eqs. (2.20, 2.21). Indeed they hold for various values of the heights and for general value of \( q \).

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