Physics in traditions: the ritual of the owners of the Mexican sky.

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Abstract. Mexico is characterized by its immense cultural diversity, one culture of many is in the state of Veracruz where the Totonac culture lived. His developed a ritual known as “The Papantla flyers”[1], which is related to fertility of the earth, this ritual consists of a group of five men who climb to the top of a 31-meter pole, tie a rope around their waist and throw themselves headlong into the void with open arms. A child, observing this ritual asks his father, “at the top, the pole has an engine that unrolls the rope and causes men to descend?” In this work, we study the equations of motion associated to the trajectory of the men who are thrown into the void, through the design and construction of an experimental model. To compare the experimental results with the analytical solution obtained by proposing the analytical equation that describes its movement, at the same time emphasizing the use of computational tools for simulation of the model using Easy Java Simulation.

1. Introduction
The ritual of the Papantla Flyers is a prehispanic tradition related to the fertility of the Earth, it was practiced by the Totonac culture that lived in Veracruz, Mexico (300-1519). Currently it is a well-known mexican ritual. Each of the five men called “Birdman” are dressed in its traditional clothes: red breeches, white shirts, red traditional needlework apron with striking colors used in the chest and waist and a colorful penacho used in the head.
Throughout the ritual a mast of approximately 30 meters of height is used with a spinning device on its top, where four ropes are tangled. The five men climb to the top of the mast, they sit in the limits of the spinning device, then they entangle the four ropes to their waist meanwhile the fifth man plays a flute and a drum, dancing in the top of the mast. Once the dance is over, the fifth man sits and the other four men throw themselves out in the air, falling backwards to the void; the owners of the Mexican sky fall. The spinning device initiates its movement as the men fall; creating a helical trajectory with an acceleration that does not correspond to the gravitational pull of the Earth [2].
Based in this ritual, in this paper we study the movement of one of the flyers, given the fact that this ritual obeys the behavior of a conical pendulum with variable length, and it has been used for teaching intermediate mechanics and applied dynamics in universities because it offers a lot of applications of the theory, so as the simple pendulum, double pendulum and the string-mass systems. This problem is also a good example to understand a time dependent length.

Using the Lagrangian Mechanics we present a solution to the system that offers four important aspects for a university course: 1) Theory, theoretical development of the properties and understanding of the problem, 2) Experimental development, model design, data collection, data analysis and outcome interpretations, 3) Programming the solution and simulation of the equations, either in Mat-Lab, Mathematica, EJS, or any other software that allows it. 4) Cultural knowledge and investigation, a student can acquire information of the Totonaca culture and the relation between the ritual and science.

In the next sections we will discuss the mentioned aspects to solve the problem of the conical pendulum with variable length and mass, based in the ritual of the owners of the Mexican sky.

2. Theory

In order to follow the movement of one of the falling men, it is necessary to observe the diagram used in figure 1, where the inertial frame of the system is located. At the top of the mast, and using spherical coordinates, a position vector is drawn to the center of mass of the falling man, given \( \rho(t) \) as the coordinate related to the rope’s length, \( \theta(t) \) being the zenithal angle and \( \phi(t) \) as the azimuthal angle.

Figure 2. Coordinate system to follow the movement of the birdman.

Hence, the vector \( \mathbf{r}(t) \), is determined in spherical coordinates as follows:

\[
\mathbf{r}(t) = \rho(t)\sin(\theta(t))\cos(\phi(t))\mathbf{i} + \rho(t)\sin(\theta(t))\sin(\phi(t))\mathbf{j} - \rho(t)\cos(\theta(t))\mathbf{k}
\]  

(1)
If we continue to effectuate the first derivative and afterwards the inner product of \( \dot{r}(t) \) with itself, we obtain \( \dot{r}(t)^2 \) in order to describe the kinetic energy. The potential energy in this system is just the gravitational potential energy, that depends on the distance of the coordinate system and the zero potential energy reference. The Lagrangian of the system is obtained using the standard formulation \([3, 4]\), \( L = K - U \), and then applying the equation of Euler-Lagrange: \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \) the equations of motion are calculated. However, it is necessary at this point to consider two possible alternatives regarding the zenithal angle: 1) It is considered constant, 2) It is considered variable.

2.1. Constant zenithal angle.

We first consider the zenithal angle constant throughout the movement, which means that we’ll obtain two equations of motion corresponding to variables \( \rho, y, \phi \). The kinetic and potential energy are given as equations (2-3) show:

\[
K = \frac{1}{2} m \left( \dot{\rho}^2 + \dot{\phi}^2 \rho^2 \sin^2(\theta) \right) \\
U = -mg \rho \cos(\theta)
\]

And the equations of motion are given by the equations (4-5):

\[
\dot{\rho} = 2\rho \dot{\phi}^2 \sin(\theta)^2 + g \cos(\theta) \\
\ddot{\phi} = -\frac{2\dot{\rho} \dot{\phi}}{\rho}
\]

Equation (5) is calculated using the Euler-Lagrange equation for coordinate \( \phi \), but this coordinate is a ignorable coordinate, because the Lagrangian has no explicit dependency in \( \phi \), which means that some property must conserve, being this property the angular momentum \( L_z \), that is given by (6), similarly as the reference \([5]\):

\[
L_z = m \rho^2 \dot{\phi} \sin(\theta)
\]

2.2. Variable zenithal angle

Now we consider the variable zenithal angle, meaning that three equations of motion are calculated, corresponding to each of the coordinate variables. The kinetic energy is then given by (7) and the potential energy does not change:

\[
K = \frac{1}{2} m \left( \dot{\rho}^2 + \dot{\theta}^2 + \dot{\phi}^2 \rho^2 \sin^2(\theta)^2 \right) \\

\text{The equations of motion (8-10):}
\]

\[
\dot{\rho} = g \cos(\theta(t)) + \rho \dot{\theta}^2 + \sin(\theta)^2 \rho \dot{\phi}^2 \\
\ddot{\phi} = -2\cot(\theta) \dot{\theta} \dot{\phi} \frac{2\dot{\rho} \dot{\phi}}{\rho}
\]
\[
\dot{\theta} = \cos(\theta)\sin(\theta)\phi^2 - \frac{g\sin(\theta)}{\rho} - \frac{2\dot{\theta}\dot{\phi}}{\rho}
\]

(10)

The equations of motion are calculated the same way, using Euler-Lagrange equation and derivatives, where \( \phi \) is ignorable too, which means that the angular momentum of the particle is constant in time and follows equation (6).

**Variable mass**

Adding another variable to the system, considering a conical pendulum with variable length and mass, is possible, where we now propose that the mass variation as given by the next expression:

\[
m(t) = \begin{cases} 
m_c + m_g (1 - \lambda t) & \text{If } t \leq \lambda \\
m_c & \text{If } t > \lambda
\end{cases}
\]

(11)

Where \( m_c \) is the container’s mass, \( m_g \) is the variable mass (a granular mass must be considered for the experimentation), \( \lambda \) is the emptying speed of the mass exiting the container, which will depend of the geometric properties of the container. With this additional consideration and using the Euler-Lagrange equations, the equations of motion are calculated:

\[
\ddot{\rho} = g\cos(\theta(t)) + \rho\dot{\theta}^2 - \frac{\dot{m}(t)}{m(t)}\dot{\rho} + \sin(\theta)\rho^2
\]

(12)

\[
\dot{\phi} = -\frac{\dot{m}(t)}{m(t)}\phi - 2\cot(\theta)\dot{\theta}\dot{\phi} - \frac{2\dot{\rho}\dot{\phi}}{\rho}
\]

(13)

\[
\ddot{\theta} = \cos(\theta)\sin(\theta)\phi^2 - \frac{g\sin(\theta)}{\rho} - \frac{\dot{m}(t)}{m(t)}\dot{\theta} - \frac{2\dot{\theta}\dot{\phi}}{\rho}
\]

(14)

Expression (6) for the angular momentum does not change, we only need to add the time dependent mass \( m(t) \). A quick observation of equations (12-14) would imply that the term \( \dot{m}(t)/m(t) \) in the three equations is conjugated by its own respective speed, this being the reason of why we used red color. The linear and quadratic terms in the velocities in the equations are the same as in equations (8-10). Remembering the damped harmonic oscillator, the equation of motion is: \( \ddot{\theta} = -\gamma \dot{\theta} - \omega^2 \theta \) and the factor \( \gamma \) corresponds to the damping factor, then it is possible to associate expression \( \gamma(t) = \dot{m}(t)/m(t) \) as a damping factor of the conical pendulum with variable length and mass.

3. Experimental development

3.1. Experimental Model.

For the experimental model, elements in table 1 were used, we tried to create a similar model used by the Papantla flyers. All the parts used for the model are shown in figure 3, and the spinning device was 3D printed. The final model is put together with 7 pieces, shown in table 1, and following the next steps: 1) Introduction of the pulley in the mast, adjusting the pulley, and fixing the mast where the rope that supports the ball will be entangled. 2) Above the pulley two bearings are fixed, one behind the pulley and one in the top of the mast. 3) The spinning device fits to the bearings, allowing the free rotation of the base as the masses fall. 4) One end of each of the ropes is introduced to one of the four holes in the pulley. 5) Finally, once the four ropes are placed in the pulley, at the other end of the rope we place the balls.
Table 1. Necessary materials for the experimental model.

|   | Material     | Measurements ± 0.01 m |
|---|--------------|-----------------------|
| 1 | Pedestal     | Base                  | (0.20-0.50) m         |
|   |              | Mast                  | (1.80-2.10) m         |
| 2 | Device       | Rotatory Base         | (0.13-0.17) m         |
|   |              | 1 pulley              | (0.03-0.05) m         |
|   |              | 2 bearings            | (0.05-0.06) m         |
| 3 | Adjuncts     | 4 balls               | (0.05-0.06) m         |
|   |              | rope                  | (1.75-2.05) m         |

The system of four balls supported by the ropes of each side of the base was entangled in the pulley to the point where the balls can no longer come up to the top of the mast. Then, the four balls are released, causing the spinning base to rotate, and each of the balls fall, following a helical trajectory, with a variation on its rope’s length over time. For the case in which we use variable mass, it was necessary to drill a hole in the ball of approximately 0.01 m of diameter, and then filling the ball with sand.

3.2. Data

Once the model is finished, the videos were filmed, using two video cameras (Nikon D3200 con 60 cps and Sony HXR-NX30 con 60 fps) located in the front of the system and other one above the system for an aerial shot in order to follow the rotation. The videos were analyzed using the free software Tracker (Video Analysis and Modeling Tool) and then the 3 best videos were chosen to demonstrate that the behavior of the system is the same.

4. Simulation.

Given the fact that equations (4 – 5), (8-10) y (12-14) are nonlinear second order differential equations, establishing an analytical solution is a hard and long task with advanced mathematics and methods, so an alternative solution using numerical methods is proposed. The standard method that offers great outcomes for systems like the one being studied, is method Runge-Kutta 4, the proposition is that the student learns to simulate and program the numerical method code, also learning to solve differential equations.
equations. In the first case, equations (4 and 5) are coupled, in the second case there exists a coupling as well in equations (8,9,10), for the last case in equations (12, 13, 14) the same thing occurs. However, in the last case it is considered that when the container is empty, the behavior must tend to be the one with only variable length.

Then it’s possible to choose two alternatives: 1) Writing the simulation code using the Runge-Kutta method. 2) Using Easy Java Simulation (EJS), and to develop a numerical solution.

4.1. Simulation Code

To develop the simulations, we must set the initial conditions, we must know \( \rho(t=0) = \rho_0 \) y \( \dot{\rho}(t=0) = \dot{\rho}_0 \), as well as \( \varphi(t=0) = \varphi_0 \) y \( \dot{\varphi}(t=0) = \dot{\varphi}_0 \). Then we must effectuate a change of variable.

As an example, we choose the constant zenithal angle in equation (4). The change of variable for \( \rho \) is shown in equations (15 y 16):

\[
f_1(t, \rho, \dot{\rho}, \varphi, \dot{\varphi}) = \dot{\rho} \\
f_2(t, \rho, \dot{\rho}, \varphi, \dot{\varphi}) = 2\rho\dot{\varphi}\sin(\theta)^2 + g\cos(\theta)
\]

In a similar way, for variable \( \varphi \) the change of variable will be (17 y 18):

\[
g_1(t, \rho, \dot{\rho}, \varphi, \dot{\varphi}) = \varphi \\
g_2(t, \rho, \dot{\rho}, \varphi, \dot{\varphi}) = -\frac{2\rho\dot{\varphi}}{\rho}
\]

This method requires the explicit dependency of the variables, then an integration interval is set to create the cycle within the integration interval, the pace step for each cycle is calculated using \( \delta h = (b - a)/N \), where \( b \) is the final value of the interval and \( a \) is the initial value, \( N \) is a integer number, chosen by the user, \( \delta h \) is the temporal unit in the simulation, meaning how fast the time will go by, this time is not real, it’s just the necessary simulation time to execute the method. Later, the 8 functions are defined to calculate the properties of interest, in this case we want to \( \rho \) y \( \dot{\rho} \) these 8 functions are defined by equations (19) with index: \( i = 1, 2, 3,4 \) and \( j = 1 \) y \( 2 \) respectively:

\[
k_{ij} = \delta h f_j(t, \rho, \dot{\rho}, \varphi, \dot{\varphi}) \\
k_{ij} = \delta h f_j\left(t + \frac{\delta h}{2}, \rho + \frac{k_{ij-1} k_{ij+1}}{2}, \dot{\rho} + \frac{k_{ij+2}}{2}, \varphi, \dot{\varphi}\right) \\
k_{ij} = \delta h f_j\left(t + \delta h, \rho + k_{ij-1}, \dot{\rho} + k_{ij+1}, \varphi, \dot{\varphi}\right)
\]

The properties of interest are then obtain using an average value (therefore we need to set the initial values):

\[
\rho_{i+1} = \rho_i + \frac{1}{6}(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}) \\
\dot{\rho}_{i+1} = \dot{\rho}_i + \frac{1}{6}(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4})
\]

This process continues throughout the integration interval. The same process is made for each of the differential equations. The student must consider and make notes for the cases where the zenithal angle is variable and for the case where the variable mass is considered in order to solve the 6 coupled equations.
4.2. Easy Java Simulation

The other alternative is to simulate using ESJ, which is a software that offers a graphic interface, through a few Windows where you can define: parameters, variables, initial values, system evolution, fixed relations, elements, and a HtmlView of the elements in 2D and 3D associated to the model. It’s necessary to be clear of which variables, parameters, integration interval and equations of motion will be used. The proposed change of variable in equations (15-18) is used in the “evolution of the system” window. One advantage that the software offers is that it’s not necessary to program the numerical method, in this alternative the student must focus in giving the right instructions to visualize the model.

The reference [6] explains each of the windows and can be used to initiate in the learning of any physical system simulation using EJS. In the [7] reference, there exists a repertoire of many executables for many physical systems, that will help the understanding of the systems. These tools help the student to acquire new abilities in the realization of system simulations, additionally we comment that the student can find many videos of the use and understanding of EJS in the internet [8]. So, the student has all the tools and help needed to do a simulation project.

5. Results

In this section we present the outcomes for the conical pendulum with variable length. The student must develop the methods and equations to get the next outcomes.

Figure 4. Show the x evolution in the time the two cases: 1) constant mass red colour, and 2) variable mass, green colour. In this graph is observed the damped produced for the variable mass.
Figure 5. Experimental solution for the Conical Pendulum of variables Length and mass.

5.1. Numerical Results

The simulation runs for an integration interval between 0 and 0.1, and was divided in N = 1000, so the pace size is \( \delta h = 8 \times 10^{-5} \), with error \( \delta \varepsilon = 8 \times 10^{-20} \). The initial values are shown in table 2, using \( g = 9.78 \text{ m/s}^2 \), which is the value of the gravitational pull of the Earth in Mexico City. We execute 35 runs with the same initial conditions in all variables except in the zenithal angle, that was changed 2 units each run. This simulation was done for the cases when the zenithal angle is constant and variable.

Figure 7. Show the phase diagram (\( q \) vs. \( \dot{q} \)) for the three variables \( (\rho, \theta, \varphi) \) respectively, for the case of the variable zenithal angle.

Figure 6. Show the trajectory of the balloon for the two cases shown in the figure 5.

5.2. Experimental Results

The experimentation requires the elaboration of the prototype that was described earlier. The difficulty in the experimentation resides in the data collection and using the Tracker software to prepare the scene and suitably place the scale and manage the light in order to follow the ball’s motion. It’s important to be really careful because Tracker does not consider any perspective in the system, even if the videos are taken in 3D, the camera will recognize it as 2D. Another option is to analyze the movement through sensors (accelerometer, compass and gyroscope) that allow us to collect data of the positions \((x,y,z)\) of the balls. We are currently working in the development of a device that will allow us to collect these coordinates. The outcomes are shown in the figures. ()

Figure 8. Comparison between numerical results for the zenithal angle constant and the experimental results for the variables \( x \) and \( y \).
Figure 9. Show the Developing Experimental for the constant zenithal angle

Figure 10. Show the x, y and z coordinate for the Conical Pendulum of variable Length to the variable zenithal angle.

Figure 11. Show the angular moment $L_z$, is constant, when the zenithal angle is variable

Figure 12. Comparison for the z coordinate between numerical results and experimental results. The points in the graphic are the experimental
results, the different curves are the running of numerical method.

Figure 13. Show simulation with EJS, in the first graph, shown the diagram phase for r, in second graph shown the evolution of the y-coordinate, in the third graph shown the trajectory of the four Papantla Flyers.

6. Conclusions

The ritual of the Papantla Flyers has been briefly presented, with its origin in the Totonaca culture. Based on this ritual, a study is presented considering the Conical pendulum with variable length and mass, as a proposal to students who are studying intermediate mechanics; proposing symbolic and iconic solutions, in accordance to its classification and we then propose a theoretical (Lagrange and Euler-Lagrange), experimental (prototype and movement tracker) and numerical solution (Mathematica and EJS programming), in order to compare and discuss the results.

7. References

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