The effect of magnetization and electric polarization on the anomalous transport coefficients of a chiral fluid

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Abstract

The effects of finite magnetization and electric polarization on dissipative and non-dissipative (anomalous) transport coefficients of a chiral fluid are studied. First, using the second law of thermodynamics as well as Onsager’s time-reversal symmetry principle, the complete set of dissipative transport coefficients of this medium is derived. It is shown that the properties of the resulting shear and bulk viscosities are mainly affected by the anisotropy induced by external electric and magnetic fields. Then, using the fact that the anomaly induced currents do not contribute to entropy production, the corresponding algebro-differential equations to non-dissipative anomalous transport coefficients are derived in a certain derivative expansion. The solutions of these equations show that, within this approximation, anomalous transport coefficients are, in particular, given in terms of the electric susceptibility of the medium.

1. Introduction

Symmetry principles play an important role in physics. Although the laws of nature are often dictated by these principles, as it turns out, many physical phenomena are due to various mechanisms of symmetry breaking. The latter includes explicit, spontaneous and anomalous symmetry breaking mechanisms. Among them, anomalous symmetry breaking is a pure quantum mechanical phenomenon, which has no counterpart in classical (non-relativistic) physics. If a global symmetry is anomalous, it implies that the symmetry of the classical Lagrangian of the theory is not obeyed in the quantum theory. In other words, the Noether current associated with that anomalous global symmetry is no longer conserved on the quantum level. This anomaly can be determined, e.g. by putting the theory in a background electromagnetic field, which couples to this anomalous current, and probes in this way the properties of the medium.

Recently, there have been many attempts to observe the effects of quantum chiral anomalies in experiments. In particular, heavy ion collisions (HICs) are of enormous interest, mainly because of the possible creation of very strong magnetic fields at an early stage of these collisions [1, 2]. The new state of matter, the plasma of quarks and gluons, which is produced at the same stage, is believed to include asymptotically free chiral fermions. The interplay between quantum anomalies with external magnetic fields results in a variety of novel non-dissipative anomalous transport phenomena in such a system (for a recent review see [3–5] and the references therein). These phenomena can principally be observed in HIC experiments [6, 7].

Some of these anomaly induced phenomena consist of a chiral magnetic effect (CME) [1, 8], chiral separation effect (CSE) [9, 10], chiral vortical effect (CVE) [11–14], or chiral vortical separation effect (CVSE) [5]. Whereas CME refers to the generation of an electric current induced by the chirality imbalance of the medium in the presence of an external magnetic field, CSE is characterized by an axial vector current which is generated along the external magnetic field. This is quite similar to the Ohm’s law, where an electric current is generated along an external electric field. In contrast to the Ohm’s coefficient, however, the transport coefficients corresponding to CME and CSE currents are non-dissipative, and entirely dictated by chiral...
anomaly. In analogy to the external magnetic field, a global rotation of the medium, quantified by its vorticity, leads, in a medium with non-vanishing electric and axial chemical potential, to non-dissipative vector and axial vector currents, which are proportional to the vorticity of the medium. The corresponding proportionality factors are referred to as CVE and CVSE coefficients.

In [14, 15], these anomalous transport coefficients are determined within a relativistic hydrodynamical approach in the presence of an external magnetic field. Using the second law of thermodynamics and the fact that the anomaly induced currents do not contribute to entropy production [16], certain algebro-differential equations are derived, whose solutions yield the anomalous transport coefficients corresponding to the aforementioned effects. The main purpose of this paper is to extend the method introduced originally in [14, 15] to a medium with finite magnetization and electric polarization. To the best of our knowledge, these in-medium modifications of anomalous transport coefficients, including the linear response of the medium to external electromagnetic fields, are not yet studied in the literature. They seem, however, to be important not only from a theoretical point of view, but also because the experimentally relevant quark-gluon plasma turns out to have finite magnetic [17] and electric susceptibilities [18].

The present paper is organized as follows: in section 2, we introduce the ideal electromagnetohydrodynamical (EMHD) framework with finite magnetization and electric polarization. We essentially follow the method previously used in [19, 20]. However, in contrast to these works, where the effect of the external electric field is neglected, we consider the case of non-vanishing magnetic and electric fields, and derive the relevant thermodynamic relations in the presence of finite magnetization and electric polarization. Moreover, an anomalous current will be considered, which includes the anomalous transport coefficients.

In section 3.1, we first derive the dissipative transport coefficients of the anomalous EMHD by using the second law of thermodynamics and the Onsager’s time-reversal symmetry principle. In particular, we show that the dissipative part of the electric current as well as the viscous stress tensor include a large number of thermal and electric conductivities, as well as shear and bulk viscosities. Their properties are mainly affected by the anisotropies induced by external electric and magnetic fields. Our results are therefore a completion of the results presented in [19, 20], where the dissipative coefficients are derived from the electric field in the dissipative currents are absent. Moreover, our results include certain, previously discarded, dissipative coefficients which arise from the interplay between external electric and magnetic fields (see (3.17)).

In section 3.2, we use the fact that the anomaly induced currents do not contribute to entropy production, and derive certain algebro-differential equations that eventually lead to anomalous transport coefficients of this medium. We show that, within a certain second-order derivative expansion, these equations include, in particular, the electric susceptibility of the medium. We then follow the method introduced in [14], and determine the anomalous transport coefficients by solving the above mentioned equations analytically. The effect of gravitational anomaly [21, 22] is not considered in the present work, neither in the case of vanishing nor in the case of non-vanishing susceptibilities. Section 4 is devoted to a summary and a number of concluding remarks.

2. Ideal anomalous electro-magnetohydrodynamics

Electro-magnetohydrodynamics addresses all phenomena related to the interaction of electric and magnetic fields with an electrically conducting magnetized fluid. An ideal and locally equilibrated relativistic fluid is characterized by its long-wavelength degrees of freedom, the four-velocity, the temperature, and the chemical potential. The four-velocity $u^\mu$ and $\mu(x)$, respectively. The four-velocity $u^\mu = \gamma(1, v)$, with $\gamma \equiv \sqrt{1 - v^2}$, is defined by the variation of the four-coordinate $x^\mu$ with respect to the proper-time $\tau$, and satisfies $u_\mu u^\mu = 1$.

In the absence of external electromagnetic fields, the physical observables, the entropy and baryonic currents $s^{(0)}_\mu$ and $n^{(0)}_{b\mu}$, as well as the fluid energy-momentum tensor $T^{(0)}_{\mu\nu}$, are expressed in terms of $u^\mu$ as

$$s^{(0)}_\mu \equiv su^\mu, \quad n^{(0)}_{b\mu} \equiv n_b u^\mu, \quad T^{(0)}_{\mu\nu} \equiv eu^\mu u^\nu - p\Delta^{\mu\nu},$$

(2.1)

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, the projector onto the direction perpendicular to $u^\mu$. In (2.1), $e, p, s, n_b$ are the local energy density, thermodynamic pressure, entropy, and baryon number densities of the ideal fluid, respectively. In an ideal and locally equilibrated fluid with no electromagnetic fields and sources, the quantities presented in (2.1) are conserved

$$\partial_\mu s^{(0)}_\mu = 0, \quad \partial_\mu n^{(0)}_{b\mu} = 0, \quad \partial_\mu T^{(0)}_{\mu\nu} = 0.$$

(2.2)

In the present work, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

The subscript $0$ denotes the case of an ideal fluid.
In the presence of electromagnetic fields, however, the total energy-momentum tensor of the ideal fluid is to be modified as \[ T^{\mu\nu}_{\text{(0)}} = T^{\mu\nu}_{\text{F(0)}} + T^{\mu\nu}_{\text{EM}} \] (2.3)

where the fluid and electromagnetic energy-momentum tensors, \( T^{\mu\nu}_{\text{F(0)}} \) and \( T^{\mu\nu}_{\text{EM}} \), read as

\[
T^{\mu\nu}_{\text{F(0)}} = T^{\mu\nu}_{\text{F(0)}} - \frac{1}{2}(\mu^{\mu\nu}\varepsilon^{\kappa\lambda}F_{\kappa\lambda} + \mu^{\nu\lambda}F_{\nu\lambda}), \quad T^{\mu\nu}_{\text{EM}} = -\mu^{\nu\lambda}F_{\nu\lambda} + \frac{1}{4}e^{\mu\nu\rho\kappa}F_{\rho\kappa}F_{\mu\nu},
\] (2.4)

where \( T^{\mu\nu}_{\text{F(0)}} \), defined in (2.1), is the energy-momentum tensor of an ideal un-magnetized and un-electropolarized fluid.\(^4\) The antisymmetric field strength and polarization tensors, \( E^{\mu\nu} \) and \( M^{\mu\nu} \) in (2.4), are expressed in terms of electric and magnetic fields, \( E \) and \( B \), as

\[
E^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\kappa}B_{\rho\kappa}, \quad M^{\mu\nu} \equiv -\mu^{\nu\rho}E_{\rho} - \mu^{\mu\rho}B_{\rho},
\] (2.5)

with

\[
e^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\kappa}E_{\rho\kappa}, \quad b^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\kappa}B_{\rho\kappa},
\] (2.6)

as well as \( e^{\mu\nu} \equiv \frac{\mu^{\rho\kappa}}{E} \) and \( b^{\mu\nu} \equiv \frac{\mu^{\rho\kappa}}{B} \). In a frame where the fluid is moving with the velocity \( u^{\mu} \), the four-vector of electric and magnetic fields are given by \( E^{\mu} \equiv \mu^{\mu\nu}u_{\nu} \) and \( B^{\mu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}F_{\kappa\lambda}u_{\lambda} \). Here, \( \varepsilon^{\mu\nu\rho\kappa} \) is the totally antisymmetric Levi-Civita tensor. In the rest frame of the fluid, with \( u^{\mu} = (1, 0) \), we therefore have \( E^{\mu} = (0, E) \) and \( B^{\mu} = (0, B) \). Here, \( E^{\mu} \equiv \mu\nu \) and \( B^{\mu} \equiv -\varepsilon^{\mu\nu\rho\kappa}F_{\kappa\lambda}/2 \), as in non-relativistic electrodynamics. The strength of the electric and magnetic fields, \( E \) and \( B \), are given by the normalization relations \( E^{\mu}E_{\mu} = -E^2 \) and \( B^{\mu}B_{\mu} = -B^2 \), which lead immediately to \( e^{\mu\nu}e_{\mu\nu} = b^{\mu\nu}b_{\mu\nu} = -1 \).

The antisymmetric polarization tensor \( M^{\mu\nu} \) in (2.5) describes the response of the system to an applied field strength \( F^{\mu\nu} \), and leads through the relation \( F^{\mu\nu} - M^{\mu\nu} \equiv H^{\mu\nu} \) to the induced field strength tensor \( H^{\mu\nu} \), which in terms of \( e^{\mu\nu} \) and \( b^{\mu\nu} \), is given by

\[
H^{\mu\nu} \equiv D_{\nu}^{\mu\nu} - \mu^{\nu\rho}B_{\rho},
\] (2.7)

where, in analogy to \( E^{\mu} \) and \( B^{\mu} \), we define the four-vectors of induced electric and magnetic fields, \( D^{\mu} \equiv H^{\mu\nu}u_{\nu} \), as \( H^{\mu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}H_{\nu\rho\kappa} \). They are in relation to the four-vectors of electric polarization \( P^{\mu} \equiv -\mu^{\nu\rho}u_{\rho} \) and magnetization \( M^{\mu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}M_{\nu\rho\kappa} \). In the rest frame of the fluid, \( D^{\mu} \equiv (0, D) \) and \( H^{\mu} \equiv (0, H) \). Similarly, we have \( P^{\mu} \equiv (0, P) \) and \( M^{\mu} \equiv (0, M) \). In (2.5) and (2.7), \( V \equiv |V| \), with \( V \equiv \{E, B, P, M, D, H\} \) is used. This fixes the normalization relations \( V^{\mu}V_{\mu} = -V^2 \), with \( V \equiv \{E, B, P, M, D, H\} \).

The purpose of this paper is to study the effect of electric polarization \( P \) and magnetization \( M \) on dissipative and non-dissipative transport coefficients of an electromagnetized chiral fluid affected by the quantum anomaly. To this purpose, we introduce the electric and magnetic susceptibilities, \( \chi_e \) and \( \chi_m \), arising in linearized relations between \( P \) and \( E \) as well as \( M \) and \( B \), \( P = \chi_e E \) and \( M = \chi_m B \). Moreover, to bring our derivation in connection with quantum anomalies, we will assume

\[
e^\nu B^{\mu\nu} = 0. \quad (2.8)
\]

In the rest frame of the fluid, (2.8) is equivalent to parallel electric and magnetic fields, \( e \parallel b \), with \( e \equiv E/E \) and \( b \equiv B/B \) denoting the directions of external electric and magnetic fields. Apart from (2.8),

\[
e^\nu \partial_{\nu} B^{\mu\nu} = -b^{\mu\nu} \partial_{\nu} e^\nu, \quad \partial_{\nu} \chi_e = \partial_{\nu} \chi_m = 0 \text{ are assumed.}
\]

In what follows, the in-medium Maxwell equations will be used to determine the relevant thermodynamic relations for an ideal and locally equilibrated fluid. A number of useful relations will be also derived. We will keep our notations similar to what is presented in \([19]\), where, in contrast to our presentation, the effect of electric field is neglected.

Introducing, in analogy to \( n_{\nu}^{\mu}_{\text{(0)}} \) from (2.1), the four-vector of electric current \( n_{\nu}^{\mu} \equiv n_{\nu}u^{\mu} \), the inhomogeneous Maxwell equation reads

\[
\partial_{\nu} H^{\mu\nu} = n_{\nu}^{\mu},
\] (2.9)

where \( H^{\mu\nu} \) is defined in (2.7) and \( n_{\nu}^{\mu} \equiv (n_{\nu}, n_{\nu}) \). Here, \( n_{\nu} \) is the electric charge density and \( n_{\nu} \) is the corresponding electric current. As expected, (2.9) is consistent with the conservation relation of the electric current, \( \partial_{\nu} n_{\nu}^{\mu} = 0 \). Together with the conservation relations \( \partial_{\nu} s_{\nu}^{\mu} = 0 \) and \( \partial_{\nu} n_{\nu}^{\mu} = 0 \) from (2.2), an ideal fluid in the presence of electromagnetic fields is particularly described by the conservation of the full energy-momentum tensor \( T^{\mu\nu}_{\text{(0)}} \) from (2.3),

\(^4\) It is important to note that the subscript (0) on \( T^{\mu\nu}_{\text{(0)}} \) in (2.4) refers only to the zeroth-order hydrodynamical derivative expansion of the fluid part of the energy-momentum tensor, \( T^{\mu\nu}_{\text{F(0)}} = \mu^{\nu\rho}u_{\rho} - \mu^{\nu} \Delta \mu^{\nu} \) from (2.1). Later we will add higher order terms in the derivative expansion to this part of \( T^{\mu\nu}_{\text{(0)}} \) from (2.3), which will contribute to dissipation. We will then determine dissipative coefficients up to first-order derivative expansion.
\[ \partial_\mu T_{\mu\nu}^{EM} = 0. \]  \tag{2.10}

Using \( T_{\mu\nu}^{EM} \) from (2.4) and the inhomogeneous Maxwell equation (2.9), we arrive at
\[ \partial_\mu T_{\mu\nu}^{EM} = n_{\text{tot}}^\nu F_{\mu\nu}, \]  \tag{2.11}

with
\[ n_{\text{tot}}^\nu \equiv n_e^\nu + \partial_\nu M^\nu, \]  \tag{2.12}

with \( M^\mu = F_{\mu\nu} - H_{\mu\nu} \). To derive (2.11), we have used the homogeneous Maxwell equation, \( \delta^\mu_{\alpha\beta} \partial_\nu E_{\alpha\beta} = 0 \). Plugging \( F_{\mu\nu} \) from (2.5) into this equation, and using the standard relation \( \delta_{\alpha\beta}^\mu \partial_\nu \delta_{\alpha\beta} = -2(\partial_{\rho} \delta_\alpha^\mu - \delta_\alpha^\rho), \) the homogeneous Maxwell equation is equivalently given by\[ \partial_\alpha (\epsilon_{\alpha\beta\gamma\delta} E_{\beta\gamma} u_\delta) + \partial_\nu (B_\nu u_\beta - B_\beta u_\nu) = 0. \]  \tag{2.13}

We use this equation to derive two useful relations, which play important roles in the rest of this work. First, contracting (2.13) with \( u_\mu \), we obtain\[ \partial_\mu B^\mu = u \cdot DB + 2E \cdot \omega, \]  \tag{2.14}

where \( D \equiv u^\nu \partial_\nu \) and \( a \cdot b \equiv u^\nu b_\nu \). The four-vector \( \omega^\mu \), on the rhs of (2.14), is the vorticity of the fluid, defined by \[ \omega^\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu \partial_\alpha u_\beta. \]  \tag{2.15}

Following the steps described in appendix A, another useful relation
\[ \partial_\nu E^\nu = u \cdot DE - \frac{2(1 - \chi_e)}{(1 + \chi_e)} B \cdot \omega + \frac{n_e}{1 + \chi_e}, \]  \tag{2.16}

can be derived (see (A.13)–(A.16) for the proof of (A.6) and set \( \partial_\alpha \chi_e = 0 \) to arrive at (2.16)). Contracting further (2.13) with \( b^\mu \), and using (2.8) as well as \( \epsilon_{\mu\nu\alpha\beta} \partial_\beta (E_\alpha u_\beta) = 0 \), we arrive at another useful relation\[ \mathcal{D} \ln B + \theta = u^\nu b^\mu \partial_\nu b_\mu = 0, \]  \tag{2.17}

where \( \theta \equiv \partial_\mu u^\mu \). The last useful relation reads\[ \mathcal{D} \ln E + \theta = u^\nu e^\mu \partial_\nu e_\mu = 0. \]  \tag{2.18}

To derive (2.18), let us first consider the energy-momentum tensor (2.3). Plugging \( F_{\mu\nu} \) and \( M_{\mu\nu} \) from (2.5) into \( T_{\mu\nu}^{(0)} \) and \( T_{\mu\nu}^{EM} \) from (2.4), we arrive after some algebraic manipulations at\[ T_{\mu\nu}^{(0)} = \epsilon' u^\mu u^\nu - p_\rho \Xi_\rho^\nu + p_\rho \Xi_\rho^\mu - E \epsilon' e^\nu, \]
\[ T_{\mu\nu}^{EM} = \frac{1}{2} B^2 (u^\mu u^\nu - \Xi_\rho^\nu + \Xi_\rho^\mu) - \frac{1}{2} E^2 (u^\mu u^\nu - \Xi_\rho^\nu - \Xi_\rho^\mu), \]  \tag{2.19}

where, \( \epsilon' \equiv \epsilon + EP \), \( p_\rho \equiv p - BM \), \( p_\rho \equiv p, \Xi_\rho^\nu \equiv \Delta_\rho^\nu + \nu_\rho b^\nu, \Xi_\rho^\mu \equiv \Delta_\rho^\mu - \nu_\rho b^\mu, \) and \( \Xi_\rho^\mu \equiv \Delta_\rho^\mu + \nu_\rho e^\mu \). We further consider \( \partial_\nu T_{\mu\nu}^{EM} = n_{\text{tot}}^\nu F_{\mu\nu} \) from (2.11). Contracting this relation with \( u^\nu \), plugging \( T_{\mu\nu}^{EM} \) from (2.19) into the lhs of the resulting expression, and eventually using (2.8) as well as \( b^\nu \partial_\nu e_\nu = 0 \) and \( \partial_\alpha \chi_e = 0 \), leads to\[ (\partial_\mu M^\mu) E_\nu = -\chi_e E^2 (\mathcal{D} \ln E + \theta - u_\mu e^\mu \partial_\mu e_\nu), \]
and we arrive at
\[ (1 + \chi_e) (\mathcal{D} \ln E + \theta - u_\mu e^\mu \partial_\mu e_\nu) = 0. \]

The latter yields (2.18) for \( \chi_e \equiv -1. \)

To check the consistency of the thermodynamic relations including electric and magnetic fields, let us contract (2.10) with \( u^\nu \). Using (2.18) and \( \partial_\mu T_{\mu\nu}^{(0)} = -\partial_\nu T_{\nu\alpha}^{EM} \) which arises from (2.10) combined with the definition of \( T_{\mu\nu}^{(0)} \) from (2.4), it is possible to show that \( u_\mu \partial_\nu T_{\mu\nu}^{(0)} = 0 \). Plugging then \( T_{\mu\nu}^{EM} \) from (2.19) into this relation, and using \( u^\nu b_\mu = u^\nu e_\mu = 0 \) and \( u_\mu \partial_\nu u^\nu = b_\nu \partial_\nu b^\nu = e_\nu \partial_\nu e^\nu = 0 \), we arrive eventually at
\[ \mathcal{D} \epsilon + (\epsilon + p) \theta + M DB + E DP = 0. \]  \tag{2.20}

The relevant thermodynamic relation for \( \mathcal{D} \epsilon \) is then derived by using
\[ \epsilon + p = TS + \mu_b n_b + \mu_e n_e, \]  \tag{2.21}

where \( \mu_b \) and \( \mu_e \) are the chemical potentials related to the baryon and electric number densities, \( n_b \) and \( n_e \). Plugging \( \epsilon + p \) from (2.21) into (2.20), and using the conservation laws of baryon number density \( \partial_\mu n_b^\mu = 0 \), electric number density \( \partial_\mu n_e^\mu = 0 \), and the entropy density current \( \partial_\mu S^\mu = 0 \), we arrive after some algebraic manipulations at...
\[ \mathcal{D} \epsilon = T \mathcal{D} s + \mu_B \mathcal{D} n_B + \mu_e \mathcal{D} n_e - M DB - E DP, \] (2.22)

which is consistent with the standard thermodynamic relation \( d\epsilon = Tds + \mu_B dn_B + \mu_e dn_e - M dB - EdP \) [23].

Combining at this stage (2.21) with (2.22), the Gibbs–Duhem relation is given by
\[ \mathcal{D} p = s \mathcal{D} T + n_B \mathcal{D} \mu_B + n_e \mathcal{D} \mu_e + M DB + E DP. \] (2.23)

To obtain the standard Gibbs–Duhem relation in the presence of electric and magnetic fields, we define \( \tilde{\rho} \equiv p - EP \). Using this definition in (2.23), we arrive at
\[ \mathcal{D} \tilde{\rho} = s \mathcal{D} T + n_B \mathcal{D} \mu_B + n_e \mathcal{D} \mu_e + M DB - PDE. \] (2.24)

Let us reiterate that the main aim of this paper is to study the effect of quantum (axial) anomaly on the EMHD equations once electric polarization \( P \) and magnetization \( M \) of the underlying fluid are not neglected. To do this, we introduce at this stage a \( U(1) \) axial vector current,
\[ n_{\mu(0)} \equiv n_\alpha u^\mu, \] (2.25)

which satisfies the classical conservation law
\[ \partial_\mu n_{\mu(0)} = 0, \] (2.26)
in the chiral limit.\(^5\) Denoting the chemical potential associated with \( n_\alpha \) by \( \mu_\alpha \), the thermodynamic relations (2.21) and (2.22) turn out to be
\[ \epsilon + p = Ts + \mu_B n_B + \mu_e n_e + \mu_\alpha n_{\alpha}, \] (2.27)

and
\[ \mathcal{D} \epsilon = T \mathcal{D} s + \sum_{i=[B,E]} \mu_i \mathcal{D} n_i - M DB - E DP, \] (2.28)

respectively. In the next section, we will consider the axial anomaly of the axial vector current,
\[ \partial_\mu n_{\mu(0)} = -\frac{e^2}{8\pi^2} F_{\mu\nu} F_{\mu\nu} = -CE \cdot B, \] (2.29)

with \( F_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} F_{\sigma\tau} \) and \( C \equiv \frac{e^2}{4\pi} \), and study its effect on non-dissipative transport coefficients of an electromagnetized relativistic fluid.\(^6\)

### 3. Dissipative and anomalous transport coefficients of a chiral fluid

In the first part of this section, section 3.1, we will derive the complete set of dissipative transport coefficients in the presence of electric and magnetic fields. To do this, we will follow the formalism of dissipative fluid dynamics from [19]. In section 3.2, by taking into account the fact that the anomaly induced current is non-dissipative [16], we will then derive the algebro-differential equations leading to anomalous transport coefficients. Similar equations are derived originally in [14, 15], where the in-medium effects are neglected. These equations will then be solved in a medium with vanishing (section 3.2.1) and non-vanishing (section 3.2.2) electric and magnetic susceptibilities. In what follows, we will first derive the general structure of \( \partial_\mu s^{\mu} \), with \( s^{\mu} \) the current of the entropy density of a dissipative anomalous fluid.

Following the method presented in [19, 20], we start by introducing the first-order dissipative and non-dissipative corrections to the conserved quantities of the ideal fluid, \( T^{\mu\nu}_{(0)}, n_{\mu(0)}^{(0)}, n_{\sigma(0)}^{(0)}, \) and \( s_{(0)}^{(0)}, \)
\[ T^{\mu\nu} = T^{\mu\nu}_{(0)} + \tau^{\mu\nu}, \]
\[ n^{\mu} = n_{\mu(0)} + j_{\mu}, \]
\[ n^{\mu} = n_{\mu(0)} + j_{\mu}, \]
\[ s^{\mu} = su^{\mu} + j_{\mu} + D_\mu \omega^{\mu} + D_B B^{\mu} + D_E E^{\mu}. \] (3.1)

Here, the total energy-momentum tensor of the ideal fluid, \( T^{\mu\nu}_{(0)} \), is defined in (2.3) and (2.4), and \( n_B, n_e, \) and \( s \) are the baryonic and electric number densities as well as the entropy density of the ideal fluid, respectively. The coefficients \( D_\alpha, D_B, \) and \( D_E \) in \( s^{\mu} \) are associated with the anomaly. The coefficients \( D_\alpha \) and \( D_B \) are originally introduced in [14], where the entropy density current \( s^{\mu} \) is expanded only in terms of the vorticity \( \omega^\alpha \) from (2.15) and the external magnetic field \( B^\mu \). Here, in the presence of magnetic and electric fields, we have also considered the effect of the external electric field \( E^\mu \), and introduced \( D_E \) as its coefficient. Later, we will

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\(^5\) We assume that the quark matter, which is effectively described by our ideal fluid, consists of massless quarks.

\(^6\) The sign in front of \( C \) in (2.29) is fixed by assuming parallel electric and magnetic fields aligned in the third direction. This leads, according to our definitions, to \( F_{12} = -B \) and \( F_{00} \equiv E \).
later, we will see that from the considerations in [14, 15], and will later be used to determine the algebro-differential equations leading to anomalous transport coefficients.

Let us notice that $j^\mu$, $i = b, e, a$ are orthogonal to $u^\mu$, i.e. $u_\mu j^\mu = 0$ for $i = b, e, a$. Moreover, $\tau^{\mu
u}$ is a symmetric rank-two tensor, satisfying the orthogonality condition $u_\mu \tau^{\mu\nu} = 0$.

At this stage, we will use the conservation relations associated with the anomalous EMHD

$$
\partial_\mu T^{\mu\nu} = 0,
\partial_\mu n^\mu = 0,
\partial_\mu j^\mu = -CE \cdot B,
$$

(3.3)

together with the second law of thermodynamics, $T \partial_\nu s^{\mu\nu} \geq 0$, in order to arrive at an appropriate relation for $T \partial_\nu s^{\mu\nu}$. To do this, we shall first consider the conservation relation (2.11) for the electromagnetic energy-momentum tensor $T^{\mu\nu}_{EM}$ from (2.19). Replacing the ideal electric current $n^\mu$ on the rhs of (2.11) by $n_e^\mu$ from (3.1), and contracting the resulting expression with $u_\nu$, we arrive at

$$
D \ln E + \theta - u_\nu e^\nu \partial_\mu e^\mu = \frac{E \cdot j^\nu}{1 + \chi_e E^2}.
$$

(3.4)

This relation replaces (2.18) of the ideal magnetized fluid. To derive (3.4), the relations (2.17), together with

$$
u_\nu \partial_\nu T^{\mu\nu}_{EM} = E^2 (D \ln E + \theta - u_\nu e^\nu \partial_\mu e^\mu), (\partial_\nu M^{\mu\nu}) e_\nu = -\chi_e E^2 (D \ln E + \theta - u_\nu e^\nu \partial_\mu e^\mu),
$$

(3.5)

are used. Then, using the conservation of the total energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ from (3.3) with $T^{\mu\nu}$ from (3.1), and following the same steps leading from $u_\nu \partial_\nu T^{\mu\nu}_{EM} = 0$ to (2.20), we obtain

$$
D \epsilon + (\epsilon + p) \theta + MD^2 + EDP + E \cdot j^\nu + u_\nu \partial_\mu \tau^{\mu\nu} = 0.
$$

(3.6)

To arrive at (3.6), the identity (3.4) and

$$
u_\nu \partial_\nu T^{\mu\nu}_{EM} = E \cdot j^\nu + \frac{E_\nu}{1 + \chi_e},
$$

(3.7)

are used. Then, expressing $j^\mu$ as a linear combination of $j_b^\mu$, $j_e^\mu$, and $j_a^\mu$, as in [19],

$$
j^\mu = -\alpha_b j^\mu_b - \alpha_e j^\mu_e - \alpha_a j^\mu_a,
$$

(3.8)

and using (2.27) to express $(\epsilon + p)$ in (3.6) in terms of all the other thermodynamic variables, we arrive at

$$
D \epsilon - \sum_{i \in \{b, e, a\}} \mu_i D n_i - TDs + MD^2 + EDP + \sum_{i \in \{b, e, a\}} \mu_i \partial_\nu (n_i u^\nu) + T \partial_\nu (su^\nu) + E \cdot j^\nu - \tau^{\mu\nu} \partial_\mu u_\nu = 0.
$$

(3.9)

The first four terms on the lhs of (3.9) vanish by making use of (2.28) from ideal EMHD. Replacing $n_i u^\mu$ and $s u^\mu$ on the lhs of (3.9) by their definitions $n_i u^\mu = n_i^\mu - j_i^\mu$ for $i = b, e, a$ and $s u^\mu = s^\mu - j_a^\mu - D_a \omega^\mu - D_b B^\mu - D_b E^\mu$ from (3.1), and using the conservation relations for $n_i^\mu$, $i = b, e, a$ from (3.3), as well as the expansion of $j_i^\mu$ in terms of the other dissipative currents from (3.8), we arrive after some straightforward algebraic manipulations at

$$
T \partial_\nu s^{\mu\nu} = \tau^{\mu\nu} w_{\nu\nu} + (\mu_a - T \alpha_a) \partial_\mu j^\mu_a + (\mu_b - T \alpha_b) \partial_\mu j^\mu_b + (\mu_c - T \alpha_c) \partial_\mu j^\mu_c - j^\mu_a T \nabla_\mu \alpha_a - j^\mu_b T \nabla_\mu \alpha_b
$$

- $j^\mu_c (T \nabla_\mu \alpha_e + E_a) + \mu_e C E_B \cdot B + T \partial_\mu (D_a \omega^\mu + D_b B^\mu + D_b E^\mu))
$$

(3.10)

Here, $\nabla_\mu \equiv \Delta \mu \partial_\mu$ and $\omega^{\mu\nu} \equiv \frac{1}{2} (\nabla^\mu w^\nu + \nabla^\nu w^\mu)$ are introduced. To satisfy the positivity condition of $T \partial_\nu s^{\mu\nu}$, the expression on the rhs of (3.10) is to be non-negative. This leads immediately to

$$
\alpha_i = \frac{\mu_i}{T}, \quad \text{for} \quad i = a, b, e,
$$

(3.11)

7 Later, we will see that $j_a^\mu$ consists of a dissipative and a non-dissipative part [14, 15].
and the general ansatz
\[
\begin{align*}
    j^\mu_b &= -\sigma_b^{\mu\nu} T \nabla_\nu \alpha_b, \\
    j^\mu_e &= -\sigma_e^{\mu\nu} (T \nabla_\nu \alpha_e + E_\nu) + \kappa_\omega \omega^\mu + \kappa_B B^\mu + \kappa_E E^\mu, \\
    \nu^\mu &= \eta^{\mu\nu\rho\sigma} \omega_{\nu\rho\sigma},
\end{align*}
\]  

(3.12)

where \(\sigma_b^{\mu\nu}\), \(\sigma_e^{\mu\nu}\), and \(\eta^{\mu\nu\rho\sigma}\) include dissipative transport coefficients, and \(\kappa_i\) and \(\xi_i\), \(i = B, E\), \(\omega\) are non-dissipative coefficients. The latter can be expressed in terms of anomalous transport coefficients. Let us notice that the dissipative transport coefficients \(\sigma_b^{\mu\nu}\), \(\sigma_e^{\mu\nu}\), and \(\eta^{\mu\nu\rho\sigma}\) are orthogonal to \(\omega^\mu\), and are symmetric under \(\mu \leftrightarrow \nu\). In the next two sections, we will use the Onsager’s time-reversal principle to first determine \(\sigma_b^{\mu\nu}\), \(\iota = b, e\) and \(\tau^{\mu\nu}\) in terms of thermal conductivity as well as longitudinal and transverse shear and bulk viscosities. This will generalize the standard formulation of magnetohydrodynamics presented in [19] to the case of a non-vanishing electric field. We will then consider the anomalous contributions to \(j^\mu_i\) and \(j^\mu_e\) in (3.10) proportional to \(\kappa_i\) and \(\xi_i\), \(i = B, E\), \(\omega\), and by combining them [15], we will arrive at the algebrao-differential equations leading to \(\kappa_i\) and \(\xi_i\), \(i = B, E\), \(\omega\) as well as \(D_b\), \(i = B, E\), \(\omega\) in a dense and hot quark matter in the presence of constant \(E\) and \(B\) fields. This will generalize the results presented in [14, 15] to the case of a fluid with finite magnetization \(M\) and electric polarization \(P\).

3.1. Dissipative currents of an anomalous chiral fluid

According to the Onsager’s principle for transport coefficients [24], the thermal conductivity, \(\sigma_b^{\mu\nu}\), \(i = b, e\), corresponding to the diffusive fluxes of the baryonic and electric number density \(n_b\) and \(n_e\) shall satisfy
\[
\sigma_b^{\mu\nu} (E, B) = \sigma_b^{\mu\nu} (E, -B), \quad i = b, e.
\]

Moreover, \(\sigma_{b/e}^{\mu\nu}\) have to satisfy the orthogonality condition \(u_{\mu} \sigma_{b/e}^{\mu\nu} = 0\) for \(i = b, e\), and are to be symmetric under \(\mu \leftrightarrow \nu\). The relation \(\sigma_{b/e}^{\mu\nu} = 0\) from (2.8) is also to be taken into account.

To build \(\sigma_{b/e}^{\mu\nu}\), we expand it in terms of independent irreducible rank-two tensors, which are built from \(u^\mu\), \(g^{\mu\nu}\), \(b^\mu\), and \(e^\mu\) [19]. The only relevant tensors that are compatible with the above mentioned conditions are thus given by
\[
\Delta^{\mu\nu}, \ b^\mu b^\nu, \ e^\mu e^\nu.
\]

(3.14)

Other rank-two tensors like \(b^{\mu\nu}\), \(e^{\mu\nu}\), \(e^{\mu\nu} b^\nu\), \ldots are excluded because of the aforementioned conditions. Here, \(A^{\mu\nu} B^{\rho\sigma} = A^{\mu\rho} B^{\nu\sigma} - A^{\mu\sigma} B^{\nu\rho}\). Introducing at this stage three independent thermal conductivity coefficients \(\sigma_{b/e}^{(i)} \ i = 1, 2, 3\) associated with the relevant tensors from (3.14), the dissipative rank-two tensors \(\sigma_b^{\mu\nu}\) and \(\sigma_e^{\mu\nu}\) from (3.12) read
\[
\sigma_b^{\mu\nu} = \sigma_b^{(1)} \Delta^{\mu\nu} + \sigma_b^{(2)} b^\mu b^\nu + \sigma_b^{(3)} e^\mu e^\nu, \quad i = b, e.
\]

(3.15)

Then, plugging \(\sigma_b^{\mu\nu}\), \(i = b, e\) from (3.15) into (3.12), the dissipative part of the baryonic and electric currents reads
\[
\begin{align*}
    j^\mu_b &= -\sigma_b^{(1)} (T \nabla^\mu \alpha_b) - \sigma_b^{(2)} b^\mu b^\nu T \nabla_\nu \alpha_b - \sigma_b^{(3)} e^\mu e^\nu T \nabla^\mu \alpha_b, \\
    j^\mu_e &= -\sigma_e^{(1)} (T \nabla^\mu \alpha_e + E^\mu) - \sigma_e^{(2)} b^\mu b^\nu (T \nabla_\nu \alpha_e + E_\nu) - \sigma_e^{(3)} e^\mu e^\nu (T \nabla^\mu \alpha_e + E^\mu).
\end{align*}
\]

(3.16)

The coefficients \(\sigma_b^{(1)}\) and \(\sigma_b^{(2)}\), \(i = 1, 2\) are previously introduced in [19]. The remaining coefficients arise once the effects of the external electric field are not neglected.

Concerning the viscous stress tensor \(\tau^{\mu\nu\rho\sigma}\) from (3.12), we apply, as in [19], the Onsager’s principle
\[
\eta^{\mu\nu\rho\sigma} (E, B) = \eta^{\sigma\mu\nu\rho\sigma} (E, -B),
\]

on the rank-four tensor \(\eta^{\mu\nu\rho\sigma}\) appearing in (3.12). All relevant tensors, compatible with this principle and expressed in terms of \(u^\mu\), \(b^\mu\), and \(e^\mu\), as well as \(g^{\mu\nu}\), \(b^{\mu\nu}\), and \(e^{\mu\nu}\) are listed in table 1 in four different series. All these bases fulfill the orthogonality condition, and are symmetric under \(\rho \leftrightarrow \sigma\). The bases appearing in series I and II are previously introduced in [19]. The new bases appearing in series III include only \(e^\mu\) and \(e^{\mu\nu}\), and those in series IV include both electric and magnetic fields. Using these bases, and following the method presented in [20] (see appendix B for more detail), the viscous stress tensor is then given by
Table 1. Relevant combinations contributing to $\eta^{\mu\nu}$. They are all compatible with the Onsager’s principle $\eta^{\mu\nu}(E, B) = \eta^{\mu\nu}(E - B)$ and the orthonormality condition $\eta_{i} \eta^{i\nu} = 0$. Moreover, they are all symmetric under $\rho \leftrightarrow \sigma$.

| Series I | Series II |
|----------|----------|
| $\Delta^{\mu}_{\nu} \Delta^{\nu}_{\mu}$ | $\Delta^{\mu}_{\nu} b^{\nu} b^{\mu} + \Delta^{\mu}_{\nu} b^{\nu} b^{\mu}$ |
| $\Delta^{\mu}_{\nu} \Delta^{\nu}_{\mu} + \Delta^{\mu}_{\nu} \Delta^{\nu}_{\mu}$ | $\Delta^{\mu}_{\nu} b^{\nu} b^{\mu} + \Delta^{\mu}_{\nu} b^{\nu} b^{\mu} + \Delta^{\mu}_{\nu} b^{\nu} b^{\mu} + \Delta^{\mu}_{\nu} b^{\nu} b^{\mu}$ |

| Series III | Series IV |
|----------|----------|
| $\Delta^{\mu}_{\nu} \epsilon^{\nu}\epsilon^{\nu}$ + $\Delta^{\mu}_{\nu} \epsilon^{\nu}\epsilon^{\nu}$ | $\epsilon^{\nu\nu} b^{\nu} b^{\nu} + \epsilon^{\nu\nu} b^{\nu} b^{\nu}$ |
| $\epsilon^{\nu\nu} \epsilon^{\nu}\epsilon^{\nu} + \epsilon^{\nu\nu} \epsilon^{\nu}\epsilon^{\nu} + \epsilon^{\nu}\epsilon^{\nu} \epsilon^{\nu}\epsilon^{\nu}$ | $\epsilon^{\nu\nu} b^{\nu} b^{\nu} + \epsilon^{\nu\nu} b^{\nu} b^{\nu} + \epsilon^{\nu\nu} b^{\nu} b^{\nu}$ |

$$\tau^{\mu\nu} = 2\eta_{B}(w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta) + \eta_{E}^{(1)} \left( \Delta^{\mu\nu} - \frac{3}{2} \frac{\theta}{\phi_{B}} \right) + 2\eta_{E}^{(2)} \left( b_{\mu} b_{\nu} \omega_{\mu\nu} - b_{\mu} b_{\nu} \omega_{\mu\nu} \right) + 3 \zeta_{E}^{(1)} b_{\mu} b_{\nu} \varphi_{E} + 3 \zeta_{E}^{(2)} (b_{\mu} b_{\nu} \varphi_{E} + b_{\mu} b_{\nu} \varphi_{E}) + 6 \zeta_{EB}^{(2)} b_{\mu} b_{\nu} \varphi_{EB} + 3 \zeta_{EB}^{(1)} (2 \Delta^{\mu\nu} \varphi_{EB} - \theta b^{\mu} b^{\nu}).$$

(3.17)

Here, $\phi_{B/E} \equiv \Xi^{\mu\nu}_{B/E} w_{\mu\nu}, \varphi_{B} \equiv b^{\mu} b^{\nu} \omega_{\mu\nu}, \varphi_{E} \equiv \epsilon^{\nu\nu} \epsilon^{\nu\nu} \varphi_{\nu},$ and $\varphi_{EB} \equiv b^{\mu} b^{\nu} \varphi_{\mu\nu}. $ Moreover $A^{\mu\nu} = A^{\nu\mu} + A^{\nu\mu}$. As expected, $\tau^{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$ and satisfies $\eta^{\mu\nu} = 0.$ According to our descriptions in B, shear viscosities $\eta_{\nu}^{(i)}, i = 1, \ldots, 4,$ $\eta_{E}^{(i)}, i = 1, 2$ and $\eta_{EB}^{(i)}, i = 1, \ldots, 6$ correspond to traceless rank-two tensors, while the tensors proportional to bulk viscosities $\zeta_{B}^{(i)}, \zeta_{E}^{(i)}, i = E, B,$ $\zeta_{EB}^{(1), (2)}, i = 1, 2,$ and $\zeta_{EB}^{(i)}, i = 1, 2, 3$ have non-vanishing traces. Shear viscosities $\eta_{\nu}^{(i)}, i = 1, \ldots, 4$ as well as bulk viscosities $\zeta_{B}^{(1), (2)}$ and $\zeta_{E}^{(i)}$ appear originally in [24] and as well as in [19]. Here, we have completed the list of dissipative transport coefficients by considering the additional effect of an external electric field. Let us also note that all bases including the combination $\epsilon^{\nu\nu} \epsilon^{\nu\nu}$ are excluded once the condition (2.8) is taken into account.

Multiplying at this stage the expressions on the rhs of (3.16) and (3.17) with $T \nabla_{\mu} \alpha_{\nu}, (T \nabla_{\mu} \alpha_{\nu} + E_{\mu})$ and $w_{\mu\nu}$, respectively (see (3.10)), we arrive at

$$T \partial_{\mu} s_{\mu} = 2\eta_{B}(w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta) + \eta_{E}^{(1)} \left( \Delta^{\mu\nu} - \frac{3}{2} \frac{\theta}{\phi_{B}} \right) + 2\eta_{E}^{(2)} \left( b_{\mu} b_{\nu} \omega_{\mu\nu} - b_{\mu} b_{\nu} \omega_{\mu\nu} \right) + 3 \zeta_{E}^{(1)} b_{\mu} b_{\nu} \varphi_{E} + 3 \zeta_{E}^{(2)} (b_{\mu} b_{\nu} \varphi_{E} + b_{\mu} b_{\nu} \varphi_{E}) + 6 \zeta_{EB}^{(2)} b_{\mu} b_{\nu} \varphi_{EB} + 3 \zeta_{EB}^{(1)} (2 \Delta^{\mu\nu} \varphi_{EB} - \theta b^{\mu} b^{\nu}).$$

(3.18)
In order to build a non-negative expression for $T\partial_\mu s^\mu$, we have set $\eta^{(1)}_{EB} = -\eta^{(2)}_{EB}$ and chosen $\xi^{(1)}_E = \xi^{(1)}_{EB} = \xi^{(2)}_{EB} = \xi^{(2)}_E$ as well as $\zeta^{(1)}_E = \zeta^{(1)}_{EB} = \zeta^{(2)}_{EB}$. We also observe that $\eta^{(i)}_B, i = 3, 4, \eta^{(i)}_{EB}, i = 3, 4, 5$, and $\zeta^{(i)}_{EB}$ do not contribute to the entropy production.

### 3.2. Anomalous transport coefficients

Let us now consider the remaining terms

$$-j^\mu_H T\partial_\mu \alpha_a - j^\mu_B (T\partial_\mu \alpha_e - E_\mu) + \mu_a CE \cdot B + T\partial_\mu (D_\omega \omega^\mu + D_B B^\mu + D_E E^\mu),$$

appearing in $T\partial_\mu s^\mu$ from (3.10). In what follows, we will determine the anomalous coefficients $D_\mu$, $\kappa$, and $\xi_\mu$, $i = B, E, \omega$, introduced in $j_H^\mu$ and $j_B^\mu$ from (3.12).

To do this, let us insert $j_H^\mu$ and the anomalous part of $j_B^\mu$ from (3.12) into the above expression and set the resulting expression equal to zero. We arrive at

$$- T\xi_\omega \omega \partial \alpha_a - T\xi_E B \cdot \partial \alpha_a - T\xi_E E \cdot \partial \alpha_a
- T\kappa \omega \omega \partial \alpha_e - T\kappa E B \cdot \partial \alpha_e - T\kappa E E \cdot \partial \alpha_e
- \kappa_B B \cdot E - \kappa_E E \cdot \omega - \kappa_E E \cdot E + C\alpha_\mu TB \cdot E
+ T\omega \omega \partial D_\omega + T B \cdot \partial D_B + TE \cdot \partial D_E
+ TD_\omega \partial \omega^\mu + TD_B \partial B^\mu + TD_E \partial E^\mu = 0.$$  

(3.19)

At this stage, we have to insert the corresponding expressions to $\partial_\mu \omega^\mu, \partial_\mu B^\mu$, and $\partial_\mu E^\mu$ into (3.19), and after reordering the resulting expression in terms of independent bases

$$1, B^\mu, E^\mu, \omega^\mu, B \cdot \omega, E \cdot \omega, E \cdot B, E \cdot B,$$

(3.20) determine their coefficients. In appendix A, we have determined the exact values of $\partial_\mu B^\mu, \partial_\mu E^\mu, \partial_\mu \omega^\mu$ (see (A.11), (A.12) and (A.22)). Linearizing these expressions in terms of independent bases (3.20), and keeping only the terms in the second-order derivative expansion, we arrive at

$$\partial_\mu B^\mu \approx b_1 (E \cdot \omega) - b_2 B^\mu \partial_\mu p - b_3 (B \cdot E),$$

$$\partial_\mu E^\mu \approx e_1 (B \cdot \omega) - e_2 E^\mu \partial_\mu p - e_3 (E \cdot E) + e_4 n_e,$$

$$\partial_\mu \omega^\mu \approx v_1 (E \cdot \omega) + v_2 \omega^\mu \partial_\mu p,$$  

(3.21)

with

$$b_1 = 2, \quad b_2 = (\epsilon + p)^{-1}, \quad b_3 = n_e (1 + \chi_e)^{-1} b_2,$$

$$e_1 = -2 (1 - \chi_m)(1 + \chi_e)^{-1} b_2, \quad e_2 = b_2, \quad e_3 = n_e (1 + \chi_e)^{-1} b_2, \quad e_4 = (1 + \chi_e)^{-1},$$

$$v_1 = -2 n_e (1 + \chi_e)^{-1} b_2, \quad v_2 = -2 b_2.$$  

(3.22)

Here, we have used the fact that $B^\mu, E^\mu, \omega^\mu, M$, and $P$ are $O(\partial^2)$ and that $\omega_\mu, \epsilon, \omega_\nu \mu$, and $p$ are $O(1)$ (see also [14] for the same power counting). In (3.21), we have kept only terms in $O(\partial^2)$, and discarded all remaining terms. This defines our second-order derivative expansion. Let us also note that the only effect of the medium, within this second-order derivative expansion, is reflected in the appearance of non-vanishing electric and magnetic susceptibilities, $\chi_e$ and $\chi_m$ in (B.4). Thus, setting $\chi_m = \chi_m = 0$ in (3.21), the results for $\partial_\mu B^\mu$ and $\partial_\mu \omega^\mu$ presented in [14] for $\partial_\mu B^\mu$ and $\partial_\mu \omega^\mu$ are reproduced. Plugging (3.21) into (3.19), and simplifying the resulting expressions, we arrive at

$$0 = + TB^\mu [\partial_\mu B^\mu - b_2 D_B \partial_\mu p - \kappa_B \partial_\mu \alpha_e - \kappa_E \partial_\mu \alpha_a] + TE^\mu [\partial_\mu E^\mu - e_2 D_E \partial_\mu p - \kappa_E \partial_\mu \alpha_e - \kappa_E \partial_\mu \alpha_a]
+ T\omega^\mu [\partial_\mu \omega^\mu - v_2 D_\omega \partial_\mu p - \kappa_\omega \partial_\mu \alpha_e - \kappa_\omega \partial_\mu \alpha_a]
+ T (B \cdot B) e_1 D_E + T (E \cdot E) \left[ v_1 D_\omega + b_1 D_B - \frac{K_E}{T} \right] - T (E \cdot E) \left( e_1 D_E + \frac{K_E}{T} \right) + T e_4 n_e D_E.$$  

(3.23)

Using then the fact that the bases (3.20) are linear independent, the expressions in front of them can be set independently equal to zero. We arrive immediately at $D_B = \kappa_E = \xi_\omega = 0$, as expected from the symmetry. We conclude that the coefficients proportional to $E^\mu$ in $\omega^\mu$, $j_H^\mu$, and $j_B^\mu$ do not receive any contribution from anomaly. All the other anomalous transport coefficients satisfy the following algebra-differential equations:

$$\frac{\partial_\mu D_B}{(\epsilon + p)} - \frac{D_B \partial_\mu p}{(\epsilon + p)} - \kappa_B \partial_\mu \alpha_e - \xi_B \partial_\mu \alpha_a = 0, \quad \frac{\partial_\mu D_\omega}{(\epsilon + p)} - \frac{2 D_\omega \partial_\mu p}{(\epsilon + p)} - \kappa_\omega \partial_\mu \alpha_e - \xi_\omega \partial_\mu \alpha_a = 0,$$

$$\frac{n_e D_B}{(1 + \chi_e)(\epsilon + p)} + \frac{\kappa_B}{T} - C\alpha_a = 0, \quad \frac{2 n_e D_\omega}{(1 + \chi_e)(\epsilon + p)} - 2 D_B - \frac{\kappa_\omega}{T} = 0.$$  

(3.24)

Let us reiterate at this stage that to derive the above algebra-differential equations a number of constraints as $e_\mu b^{\mu \nu} = 0$ from (2.8) as well as $e_\mu \partial_\mu b^{\mu \nu} = 0, \partial_\mu \chi_e = 0$ and $\chi_e = 1$ are made. These kinds of constraints, especially those related to (2.8), are used to derive the thermodynamical equations in section 2. The latter are
then used in Appendix A to derive general expressions for $\partial_\mu B^\mu$, $\partial_\mu E^\mu$, and $\partial_\mu \omega^\mu$ in (A.11), (A.12), and (A.22), respectively. Approximating these relations in an appropriate way (see above) leads to $D_k = \xi_k = \kappa_k = 0$ as well as to (3.24), whose solutions yield anomalous transport coefficients $D_k$, $\xi_k$, $\kappa_k$ with $k = B$, $\omega$. This describes the role played by these constraints to determine these anomalous transport coefficients in this paper.

In a medium with vanishing $\chi_e$, the above equations (3.24) reduce to the equations appearing in [15],

$$\partial_\mu D_k = \frac{D_k \partial_\mu p}{(e + p)} - \kappa_B \partial_\mu \alpha_e - \xi_B \partial_\mu \alpha_a = 0, \quad \partial_\mu D_\omega - \frac{2D_\omega \partial_\mu p}{(e + p)} - \kappa_\omega \partial_\mu \alpha_e - \xi_\omega \partial_\mu \alpha_a = 0,$$

$$\frac{n_k D_k}{(e + p)} + \frac{\kappa_B}{T} \alpha_a = 0, \quad \frac{2n_k D_\omega}{(e + p)} - \frac{\kappa_\omega}{T} = 0. \quad (3.25)$$

A comparison between (3.24) and (3.25) shows that, within this second-order approximation, only the algebraic equation receives contribution from $\chi_e$, the electric susceptibility of the medium. The magnetic susceptibility, $\chi_m$, thus plays no role in modifying the anomalous transport coefficients in this approximation. In what follows, we first consider the algebro-differential equations (3.25), and solve them to determine $D_k$, $\xi_k$, and $\kappa_k$ for $k = B$, $\omega$ in terms of thermodynamical quantities $\epsilon$, $p$, $\alpha_e$, and $\alpha_a$. We then present the solutions of (3.24) in an electrically polarized hot and dense medium in the presence of an external magnetic field.

### 3.2.1. Anomalous transport coefficients in a medium with vanishing $\chi_e$ and $\chi_m$

To solve (3.25), we use the same method as in [14, 15]. Introducing $\alpha_i = \frac{\mu_i}{T}$, $i = a$, $e$ and replacing $d\mu_i = T da_i + \alpha_i dT$ in the Gibbs–Duhem equation $d\mu = sdT + \sum_{i = (e,a)} n_i d\mu_i$ arising from $\epsilon + p = Ts + \sum_{i = (e,a)} n_i \alpha_i$, we arrive at

$$d\epsilon = T ds + \sum_{i = (e,a)} \mu_i d\alpha_i,$$

we obtain

$$\partial_\mu \epsilon = \frac{\epsilon + p}{T} \partial_\epsilon T + n_e T \partial_\epsilon \alpha_e + n_a T \partial_\epsilon \alpha_a. \quad (3.26)$$

We therefore have,

$$\left. \frac{d\mu}{dT} \right|_{\alpha_e \alpha_a} = \sum_{i = (e,a)} (s + n_i \alpha_i) = \frac{\epsilon + p}{T}, \quad \left. \frac{d\mu}{d\alpha_e} \right|_{\alpha_e T} = n_e T, \quad \left. \frac{d\mu}{d\alpha_a} \right|_{\alpha_a T} = n_a T. \quad (3.27)$$

At this stage, plugging

$$\partial_\mu D_k = \left( \frac{\partial D_k}{\partial \epsilon} \right) \partial_\epsilon T + \left( \frac{\partial D_k}{\partial \alpha_e} \right) \partial_\epsilon \alpha_e + \left( \frac{\partial D_k}{\partial \alpha_a} \right) \partial_\epsilon \alpha_a, \quad (3.28)$$

for $k = B$, $\omega$, and $\partial_\mu \epsilon$ from (3.26) into the first two differential equations in (3.25), we obtain

$$\frac{\partial D_k}{\partial T} = \frac{D_k}{T} \frac{\partial D_k}{\partial \alpha_e} = \frac{n_k D_k T}{(e + p)} + \kappa_B \frac{\partial D_k}{\partial \alpha_a} = \frac{n_k D_k T}{(e + p)} + \xi_B. \quad (3.29)$$

as well as

$$\frac{\partial D_\omega}{\partial T} = \frac{D_\omega}{T} \frac{\partial D_\omega}{\partial \alpha_e} = \frac{2n_\omega D_\omega T}{(e + p)} + \kappa_\omega \frac{\partial D_\omega}{\partial \alpha_a} = \frac{2n_\omega D_\omega T}{(e + p)} + \xi_\omega. \quad (3.30)$$

Combining the last two algebraic relations (3.25) with the differential equations from (3.29) and (3.30), we obtain

$$D_k = T[\kappa_{\alpha_e \alpha_a} + \gamma_B(\alpha_e)], \quad D_\omega = T^2[\kappa_{\alpha_e \alpha_a} + 2\alpha_e \gamma_B(\alpha_e) + \gamma_\omega(\alpha_e)], \quad (3.31)$$

where $\gamma_k(\alpha_k)$, $k = B$, $\omega$ are constants of integration [15]. To arrive at (3.31), the identities (3.27) have been used. Plugging $D_k$ and $D_\omega$ from (3.31) into (3.29) and (3.30), we arrive at

$$\kappa_B = C_B \mu_a \left( 1 - \frac{n_k \mu_a}{\epsilon + p} \right) \frac{\epsilon + p}{\epsilon + p},$$

$$\xi_B = C_B \mu_a \left( 1 - \frac{n_k \mu_a}{\epsilon + p} \right) + T \left( \frac{\gamma_B}{\partial \alpha_a} - n_a T \gamma_B \right),$$

$$\kappa_\omega = 2C_\omega \mu_a \left( 1 - \frac{n_k \mu_a}{\epsilon + p} \right) + 2T^2 \left( 1 - \frac{2n_\omega \mu_a}{\epsilon + p} \right) \gamma_B \frac{n_k T \gamma_\omega}{\epsilon + p},$$

$$\xi_\omega = C_\omega \mu_a \left( 1 - \frac{2n_\omega \mu_a}{\epsilon + p} \right) + 2\mu_a T \left( \frac{\gamma_B}{\partial \alpha_a} - 2n_\omega T \gamma_B \right) + T^2 \left( \frac{\gamma_\omega}{\partial \alpha_a} - n_k T \gamma_\omega \right). \quad (3.32)$$
which are consistent with the results presented in [14], provided the integration constants \( \gamma_i, \ i = 1, 2 \) are set to be zero (see below). Here, \( C = \frac{e^2}{4\pi^2} \) is the coefficient of the axial anomaly from (2.29). Let us reiterate that the first term in \( \kappa_{B} \) is the same coefficient of the chiral magnetic effect, arising originally in [1, 8]. Moreover, \( \kappa_{c} \) is the coefficient for the chiral vortical effect [11–14]; meanwhile, \( \xi_{B} \) and \( \xi_{c} \), the coefficients of chiral vortical as well as chiral vortical separation effects, appeared first in [9, 10] as well as in [3].

Let us also note that the contributions from gravitational anomaly to anomalous transport coefficients are not considered in the present work. These kinds of corrections are computed in [21, 22] using an appropriate Kubo formalism (see also [25] for a kinetic theory approach). They are shown to appear as additional \( T^2 \)-dependent terms in \( \kappa_{B}, \xi_{B}, \kappa_{c} \) with \( k = B, \omega \). These terms can also be interpreted as contributions from the aforementioned integration constants \( \gamma_{B} \) and \( \gamma_{c} \) [26]. Their determination turns out to be strongly frame-dependent [27, 28] (see also [29] for a recent review). In what follows, we will determine \( D_{k}, \xi_{B}, \kappa_{B} \) with \( k = B, \omega \) for the case of non-vanishing \( \chi_{c} \). Requiring that these results lead to the corresponding expressions for \( \chi_{c} = 0 \), new constants of integrations will be brought in connection with \( \gamma_{B} \) and \( \gamma_{c} \). The contributions from gravitational anomaly will not be considered in this framework, as in the case of \( \chi_{c} = 0 \).

3.2.2. Anomalous transport coefficients in a medium with non-vanishing \( \chi_{c} \) and \( \chi_{m} \)

In this section, we will use the method introduced in the previous part to solve the differential equations (3.24) for a system with non-vanishing \( \chi_{c} \). To do this, we first use the Gibbs–Duhem relation (2.23), or equivalently

\[
\partial_{\mu}p = \frac{(e + p)_{\mu}}{T} \partial_{\mu}T + n_{e}T \partial_{\mu}n_{e} + n_{a}T \partial_{\mu}n_{a} + M \partial_{\mu}B + E \partial_{\mu}P,
\]

(3.33)
as well as

\[
\partial_{\mu}D_{k} = \left( \frac{\partial D_{k}}{\partial T} \right) \partial_{\mu}T + \left( \frac{\partial D_{k}}{\partial n_{e}} \right) \partial_{\mu}n_{e} + \left( \frac{\partial D_{k}}{\partial n_{a}} \right) \partial_{\mu}n_{a} + \left( \frac{\partial D_{k}}{\partial B} \right) \partial_{\mu}B + \left( \frac{\partial D_{k}}{\partial P} \right) \partial_{\mu}P,
\]

(3.34)
with \( k = B, \omega \), and then rewrite the differential equations appearing in (3.24). We arrive again at (3.29) and (3.30) as well as at \( \frac{\partial D_{B}}{\partial B} = \frac{\partial D_{B}}{\partial P} = \mathcal{O}(\partial^{p}) \) for \( k = B, \omega \). To solve these sets of equations together with the two algebraic equations in (3.24), let us start with the second equation in (3.29). Replacing \( \kappa_{B} = C\alpha_{B}T = \frac{n_{e}T}{(1 + \chi_{c})(c + p)} \) from the first algebraic equation in (3.24), and defining a new variable \( w \equiv e + p \), we obtain for constant \( \alpha_{a} \) and \( T \),

\[
\frac{\partial D_{B}}{\partial \alpha_{c}} = \frac{\chi_{c}}{1 + \chi_{c}} \frac{\partial w}{D_{B}} \frac{D_{B}}{w} = C\alpha_{B}T = 0.
\]

(3.35)
To arrive at (3.35), we have used \( \frac{\partial w}{\partial \alpha_{c}} = n_{e}T \) from (3.33). For constant electric susceptibility, \( \chi_{e} \), the solution of (3.35) reads

\[
D_{B} = w^{\frac{\chi_{c}}{1 + \chi_{c}}} \left( A + C\alpha_{B}T \int d\alpha_{c}w^{\frac{\chi_{c}}{1 + \chi_{c}}} \right).
\]

(3.36)
The integration constant \( A = A(\alpha_{a}, T, \chi_{e}) \) is fixed by requiring that \( D_{B} \) from (3.36) satisfies the first equation of (3.29). We arrive at

\[
D_{B} = w^{\frac{\chi_{c}}{1 + \chi_{c}}} \left( T^{\frac{\chi_{c}}{1 + \chi_{c}}} \gamma_{B} + C\alpha_{B}T \int d\alpha_{c}w^{\frac{\chi_{c}}{1 + \chi_{c}}} \right),
\]

(3.37)
where \( \gamma_{B}(\alpha_{a}, \chi_{c}) \) is a constant of integration. Requiring further that \( D_{B} \) from (3.37) for \( \chi_{c} = 0 \) is given by \( D_{B} \) from (3.31), we obtain \( \gamma_{B}(\alpha_{a}, \chi_{c} = 0) = \gamma_{B}(\alpha_{a}) \), with \( \gamma_{B} \) as the integration constant arising in \( D_{B} \) from (3.31). To arrive at (3.37), we have used

\[
T \frac{\partial}{\partial T} w^{\chi_{c}} = \pm \frac{n\chi_{c}}{1 + \chi_{c}} w^{\chi_{c}},
\]

(3.38)
where \( n = 1 \), which arises from \( \frac{\partial w}{\partial T} = \frac{w}{T} \) from (3.33). Plugging \( D_{B} \) from (3.37) into the first algebraic equation of (3.24), we arrive at

\[
\kappa_{B} = C\alpha_{B} \left( 1 - \frac{n_{e}T}{1 + \chi_{c}} w^{\chi_{c}} \int d\alpha_{c}w^{\chi_{c}} \right) \frac{n_{e}T^{\chi_{c}}}{(1 + \chi_{c})} w^{\chi_{c}} = \frac{n_{e}T^{\chi_{c}}}{(1 + \chi_{c})} w^{\chi_{c}},
\]

(3.39)
which reduces to \( \kappa_{B} \) from (3.32) for \( \chi_{c} = 0 \), provided \( \gamma_{B}(\alpha_{a}, \chi_{c} = 0) = \gamma_{B}(\alpha_{a}) \) from (3.31). Differentiating \( D_{B} \) from (3.37) with respect to \( \alpha_{a} \), leads, according to (3.29), to
\[ \xi_B = CT \left( 1 - \frac{1}{(1 + \chi_e)} \frac{n_B \alpha_e T}{w} \right) \int \frac{dx e w^{\frac{\gamma e}{1 + \chi_e}}}{w^{\frac{\gamma e}{1 + \chi_e}}} - \frac{Cn_B \alpha_e T^3 \chi_e}{1 + \chi_e} w^{\frac{\gamma e}{1 + \chi_e}} \int dx e w^{\frac{\gamma e}{1 + \chi_e}} + T \frac{\partial \gamma_B}{\partial \alpha_e} \frac{\gamma_B}{(1 + \chi_e) w}. \]

\[ (3.40) \]

Here \( \frac{\partial \gamma_B}{\partial \alpha_e} = n_B T \) from (3.33) is used. For \( \chi_e = 0 \), \( \xi_B \) from (3.40) leads, as expected, to \( \xi_B \) from (3.32), if \( \gamma_B|_{\chi_e=0} = \gamma_B \) and \( \frac{\partial \gamma_B}{\partial \alpha_e}|_{\chi_e=0} = \frac{\partial \gamma_B}{\partial \alpha_e} \).

At this stage, plugging \( \kappa_{\omega} = 2D_B T - \frac{2n_B T D_B_0}{w(1 + \chi_e)} \) from the second algebraic equation of (3.24) into the rhs of the second differential equation of (3.30), and using \( \frac{\partial \gamma_B}{\partial \alpha_e} = n_e T \) from (3.33), we arrive at the differential equation

\[ \frac{\partial D_\omega}{\partial \alpha_e} = \frac{2 \chi_e}{1 + \chi_e} \frac{\partial w}{\partial \alpha_e} \frac{D_\omega}{w} - 2TD_B = 0, \]

whose solution for constant \( \chi_e \) reads

\[ D_\omega = \frac{\gamma_B}{w^{\frac{\gamma_B}{1 + \chi_e}}} \left( 1 - \frac{1}{(1 + \chi_e)} \frac{n_B \alpha_e T}{w} \right) \int \frac{dx e w^{\frac{\gamma e}{1 + \chi_e}}}{w^{\frac{\gamma e}{1 + \chi_e}}} + \gamma_B. \]

\[ (3.42) \]

with \( D_B \) from (3.37). As in the previous case, the integration constant is chosen so that \( D_\omega \) from (3.42) satisfies the first differential equation in (3.29). Moreover, for \( \chi_e = 0 \), \( D_\omega \) from (3.42) is given by \( D_B \) from (3.31) provided \( \gamma_B|_{\chi_e=0} = \gamma_B. \) To determine \( \kappa_{\omega} \), we use the second algebraic equation in (3.24), and arrive at

\[ \kappa_{\omega} = 2D_B T - \frac{2n_B T D_B_0}{w(1 + \chi_e)} \]

\[ (3.43) \]

with \( D_B \) from (3.37) and \( D_\omega \) from (3.42). Setting \( \chi_e = 0 \), and using \( D_B(\chi_e = 0) \) and \( D_\omega(\chi_e = 0) \) from (3.31), we immediately arrive at \( \kappa_{\omega} \) from (3.32). Finally, differentiating \( D_\omega \) from (3.42) with respect to \( \alpha_e \) and using the last equation of (3.29), we arrive at

\[ \xi_\omega = \frac{2n_B T^2 (1 - \chi_e)}{1 + \chi_e} \int \frac{dx e D_B w^{\frac{\gamma_B}{1 + \chi_e}}}{w^{\frac{\gamma_B}{1 + \chi_e}}} - \frac{2n_B T D_B_0}{w(1 + \chi_e)} + 2TD_B w^{\frac{\gamma_B}{1 + \chi_e}} \frac{\partial \gamma_B}{\partial \alpha_e} \]

\[ + w^{\frac{\gamma_B}{1 + \chi_e}} T \frac{\partial \gamma_B}{\partial \alpha_e} \]

\[ (3.44) \]

where \( D_B, \gamma_B, \) and \( D_\omega \) are given in (3.37), (3.40), and (3.42), respectively. Assuming \( \frac{\partial \gamma_B}{\partial \alpha_e}|_{\chi_e=0} = \frac{\partial \gamma_B}{\partial \alpha_e}, \) \( \xi_\omega \) from (3.44) reduces to \( \xi_\omega \) from (3.32), as expected.9

4. Concluding remarks

The anomaly induced effects on magnetized chiral fluids have attracted much attention in recent years. They are all characterized by non-dissipative vector and axial vector currents, which are proportional to either the background magnetic field or the vorticity of the medium. The proportionality factors, whose values are dictated by axial anomaly, represent non-dissipative transport coefficients. They arise naturally within relativistic hydrodynamics, as shown originally by Son and Surowka in [14]. In this paper, we have extended the method previously used in [19, 20] to the case of a non-vanishing electric field. In addition, an anomalous current has been considered, which includes anomalous transport coefficients as in [14, 15]. This brings our derivation in connection with quantum anomalies. In this way, the work of Son and Surowka is generalized to the case of an electromagnetized chiral fluid, which linearly responds to the external electromagnetic field through finite magnetization and electric polarization. We have shown that, within certain approximation, the anomalous transport coefficients are, in particular, given in terms of the electric susceptibility of the medium. Other ingredients are the energy density and thermodynamic pressure of the medium, as well as electric and axial charge densities. They are all functions of the temperature \( T \), finite electric and axial chemical potential \( \mu_e \) and \( \mu_a \) of the fluid, as well as external electric and magnetic fields \( E \) and \( B \) acting on the

9 Let us emphasize that the integrations over \( \alpha_e \) in the analytical expressions for the anomalous coefficients \( D_B, \xi_\omega, \kappa_k \) with \( k = B, \omega \) (3.37)–(3.44) are indefinite.
fluid. As a by product, we have also determined the complete set of dissipative transport coefficients arising in the dissipative part of the electric current, as well as the viscous stress tensor. This completes the set of coefficients previously obtained in [19, 20], where the dissipative coefficients arising from the external electric field were neglected.

This work can be extended in many ways. One possibility is to assume a certain thermodynamic potential for a chiral QCD-like effective model in the presence of parallel electric and magnetic fields. Using standard thermodynamical relations, it is then possible to explicitly determine the energy density, pressure, and electric susceptibility of this model in terms of a given set of thermodynamical parameters $T$, $\mu_\pi$, $\mu_\rho$, $E$, and $B$. The non-dissipative anomalous transport coefficients, which are presented in this works by certain integration over $\alpha_\pi = \frac{\mu_\pi}{T}$, can then be determined by numerically performing these integrals for a given set of $T$, $\mu_\pi$, $\mu_\rho$, $E$, and $B$.

It would be interesting to explore, for instance, the dependence of the anomalous transport coefficients on this set of parameters, especially when the chiral model exhibits a chiral phase transition. The behavior of the anomalous transport coefficients in the vicinity of a chiral critical point might be interesting for the physics of quark matter under extreme conditions, and may have phenomenological consequences in HIC experiments. We will postpone these kinds of studies for our future works.

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Appendix A. Determination of $\partial_{\mu}B^\mu$, $\partial_{\mu}E^\mu$, and $\partial_{\mu}\omega^\mu$

A.1. Determination of $\partial_{\mu}B^\mu$ and $\partial_{\mu}E^\mu$ 

Let us consider the fluid energy-momentum tensor from (2.19). Using the definitions of $\epsilon'$, $p_{\perp}$, and $p_{||}$ it can equivalently be given as

$$ T^{\mu\nu}_{\text{F}} = \epsilon u^\mu u^\nu - p\Delta_{\mu\nu} + EP(u^\mu u^\nu - \epsilon' u^\nu) + MB\Xi_{\mu\nu}^{\text{ff}}, \quad (A.1) $$

where $\Xi_{\mu\nu}^{\text{ff}} = \Delta_{\mu\nu} + b^\nu b^\mu + b^\nu b^\mu\epsilon$. To determine $\partial_{\mu}B^\mu$ and $\partial_{\mu}E^\mu$, let us consider the combination

$$ \Delta_{\mu\nu}\partial_{\nu}T^{\mu\nu}_{\text{F}} = \Delta_{\mu\nu}\partial_{\nu}T^{\mu\nu}_{\text{F}} + \Delta_{\mu\nu}\partial_{\nu}T^{\mu\nu}_{\text{EM}} = 0. $$

Using the relations (2.11)–(2.18) as well as the properties (2.8) and $u_\nu e^\nu = u_\nu b^\nu = 0$, we arrive first at

$$ \Delta_{\mu\nu}\partial_{\nu}T^{\mu\nu}_{\text{F}} = (\epsilon' + p_{\perp}) \partial_{\nu}u_{\nu} - \nabla_{\nu}p_{\perp} - \partial_{\nu}(EP\epsilon' u_{\nu} - BMBb_{\nu}) + u_\nu (PDE - MDB) + u_\nu \theta (EP - BM), $$

and

$$ \Delta_{\mu\nu}\partial_{\nu}T^{\mu\nu}_{\text{EM}} = -n_e E_{\mu} + F_{\mu\nu} \partial_{\nu}M^{\nu\mu}, $$

where $\nabla_{\nu} = \partial_{\nu} - \partial_{\nu}u_{\nu}$, and $E_{\mu} = F_{\mu\nu} u_{\nu}$. Combining these two relations, we get

$$ (\epsilon' + p_{\perp}) \partial_{\nu}u_{\nu} - \nabla_{\nu}p_{\perp} - \partial_{\nu}(EP\epsilon' u_{\nu} - BMBb_{\nu}) + u_\nu (PDE - MDB) + u_\nu \theta (EP - BM) - E_{\mu} n_e + F_{\mu\nu} \partial_{\nu}M^{\nu\mu} = 0. \quad (A.4) $$

Multiplying at this stage (A.4) with $B^\nu$, and using the definition of $P^{\mu\nu}$ from (2.5), we obtain

$$ (\epsilon' + p_{\perp}) (B \cdot D u) - B^\nu \partial_{\nu}p_{\perp} - B^\nu \partial_{\nu}(EP\epsilon' u_{\nu} - BMBb_{\nu}) - n_e (E \cdot B) - (E \cdot B) u_\nu \partial_{\nu}M^{\nu\mu} = 0. \quad (A.5) $$

Plugging $B \cdot D u = -u_\nu \cdot DB$ from (2.14) into this relation, and using the following two relations

$$ E \cdot Du = - \frac{2(1 - \chi_m)}{1 + \chi_m} (B \cdot \omega) + \frac{n_e - \frac{E\Pi_{\mu}E_{\mu}}{1 + \chi_m}}{1 + \chi_m} - \partial_{\nu}E_{\nu}, $$

and

$$ u_\nu \partial_{\nu}M^{\nu\mu} = \frac{2(\chi_e + \chi_m)(B \cdot \omega)}{1 + \chi_e} - \frac{E\Pi_{\mu}E_{\mu} + n_e \chi_m}{1 + \chi_e}, \quad (A.7) $$

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we arrive after some straightforward computations at
\[
\partial_\mu B^\mu + \frac{\chi_e (E \cdot B)}{\epsilon' + p} \partial_\lambda E^\nu = \left( \frac{(\epsilon' + p)_{\nu}}{(\epsilon' + p)} \right) \left\{ 2(E \cdot \omega) - \frac{B^\rho}{(\epsilon' + p)} \left( \partial_\rho p_\lambda + \frac{n_e E_\rho}{(1 + \chi_e)} + B_\rho M \right) + E^\rho \partial_\mu (x_\lambda E^\nu) + \frac{2E_\rho (x_\lambda + \chi_m) (B \cdot \omega)}{(1 + \chi_e)} - \frac{E_\rho E^\nu \partial_\mu x_\lambda}{(1 + \chi_e)} \right\}.
\] (A.8)

Multiplying at this stage (A.4) with \( E^\nu \), we arrive first at
\[
(\epsilon' + p)_{\nu} (E \cdot D_{\nu}) - E^\rho \partial_\rho (E E^\nu) - B M^\nu - n_\epsilon (E \cdot E_{\nu}) - (E \cdot E) \partial_\nu M^\mu = 0.
\] (A.9)

Then, using (A.6) and (A.7), we obtain
\[
\partial_\mu E^\nu - \frac{\chi_m (E \cdot B)}{\epsilon' + p_{\nu}} \partial_\lambda B^\mu = \left( \frac{(\epsilon' + p)_{\nu}}{(\epsilon' + p)} \right) \left\{ \frac{2(1 - \chi_m)(B \cdot \omega)}{(1 + \chi_e)} - \frac{n_e E_\rho}{(1 + \chi_e)} + \frac{E^\rho \partial_\mu x_\lambda}{(1 + \chi_e)} \right\},
\] (A.10)

Combining now (A.8) with (A.10), we arrive at
\[
\partial_\mu B^\mu = \left( 1 + \frac{\chi_e x_m (E \cdot B)^2}{(\epsilon' + p)(\epsilon' + p_{\nu})} \right)^{-1} \left( \frac{(\epsilon' + p)_{\nu}}{(\epsilon' + p)} \right) \left\{ 2(E \cdot \omega) - \frac{B^\rho}{(\epsilon' + p)} \left( \partial_\rho p_\lambda + E^\nu \partial_\mu (x_\lambda E^\nu) + B_\rho M + \frac{n_e E_\rho}{(1 + \chi_e)} \right) + \frac{2E_\rho (x_\lambda + \chi_m)(B \cdot \omega)}{(1 + \chi_e)} - \frac{E_\rho E^\nu \partial_\mu x_\lambda}{(1 + \chi_e)} \right\},
\] (A.11)

and
\[
\partial_\mu E^\nu = - \left( 1 + \frac{\chi_e x_m (E \cdot B)^2}{(\epsilon' + p)(\epsilon' + p_{\nu})} \right)^{-1} \left( \frac{(\epsilon' + p)_{\nu}}{(\epsilon' + p)} \right) \left\{ \frac{2(1 - \chi_m)(B \cdot \omega)}{(1 + \chi_e)} - \frac{n_e E_\rho}{(1 + \chi_e)} + \frac{E^\rho \partial_\mu x_\lambda}{(1 + \chi_e)} \right\},
\] (A.12)

To prove (A.6), we have used the definition of \( E^\nu = F^{\mu \nu} u_\mu \) to get \( E^\nu D_{\nu} = (n^\nu F^{\nu \mu}) u_\mu \partial_\nu u_\mu \). To determine \( n^\nu F^{\nu \mu} \), we multiply \( B^\mu = \frac{1}{2} e^{\mu \nu \rho} F_{\rho \nu} u_\mu \) with \( e^{\nu \rho \mu} e^{\nu' \rho' \mu'} \), and use
\[
e^{\mu \nu \rho} e^{\mu \nu \rho} = - \{ s_{\nu' \rho} (s_{\rho' \mu} s_{\nu'' \mu'} - s_{\nu'' \rho} s_{\rho' \mu'}) - s_{\nu'' \nu'} (s_{\rho' \rho} s_{\nu'' \nu'} - s_{\nu'' \nu'} s_{\rho' \rho'}) + s_{\rho' \rho} (s_{\nu' \nu} s_{\rho'' \rho'} - s_{\nu' \nu} s_{\rho'' \rho'}) \},
\] (A.13)
to obtain \( F^{\nu \rho} u_\rho = - e^{\mu \nu \rho} B_{\mu} + F^{\nu \rho} u_\rho + F^{\nu \rho} u_\rho \). We arrive first at
\[
\epsilon \cdot Du = (n^\nu F^{\nu \mu}) u_\mu \partial_\nu u_\mu = - 2(B \cdot \omega) + F^{\nu \mu} \partial_\nu u_\mu.
\] (A.14)

Here, \( u_\mu u_\mu = 1 \), \( u_\mu \partial_\nu u_\mu = 0 \) and the definition of the vorticity \( \omega^\nu = \frac{1}{2} e^{\mu \nu \rho} u_\mu \partial_\nu u_\mu \) are used. Considering \( F^{\nu \mu} \partial_\mu u_\nu \) on the rhs of (A.14), and replacing \( F^{\nu \mu} \) with \( H^{\nu \mu} + M^{\nu \mu} \), we arrive, upon using the equation of motion (2.9), at \( \partial_\nu F^{\nu \mu} = n_\epsilon (0) + M^{\mu \nu} \). Plugging this relation into (A.14), we obtain
Using the definition of $M^{\mu\nu}$ from (2.5), and performing some straightforward computations, we obtain

$$u_\mu \partial_\mu M^{\mu\nu} = -E^\mu \partial_\mu X_\nu - \chi_\nu u \cdot D E + 2\chi_m (B \cdot \omega).$$  

(A.16)

Here, the definition of magnetic susceptibility $\chi_m = \frac{M}{B}$ is used. Combining (A.15) with (A.16), we arrive finally at (A.6) and (A.7).

### A.2. Determination of $\partial_\mu \omega^\mu$

To determine $\partial_\mu \omega^\mu$, let us multiply (A.4) with $\omega^\rho$ to arrive at first

$$(\epsilon' + p_1) (\omega \cdot Du) + \omega^\rho \partial_\rho E_\mu + n_\mu (E \cdot \omega) + \omega^\mu \partial_\mu (BM^\mu B_\rho - E P^\mu p_\rho) + \omega^\rho F_{\rho\mu} \partial_\rho M^{\mu\nu} = 0.$$  

(A.17)

To determine $\omega^\rho Du_\rho$, we use the definition of the vorticity $\omega^\mu = \frac{1}{2} \varepsilon^{\mu\rho\sigma\delta} \partial_\delta u_\rho \eta_{\rho\sigma}$ and rewrite the combination $\eta_{\rho\sigma} \equiv u_\mu \partial_\mu u_\rho$ in $\omega^\rho Du_\rho = \frac{1}{2} \varepsilon^{\mu\rho\sigma\delta} \partial_\delta u_\rho \eta_{\rho\sigma}$ in terms of the vorticity $\omega^\rho$. We obtain

$$\omega^\rho Du_\rho = -2\varepsilon^{\mu\rho\sigma\delta} \partial_\delta u_\rho \omega_\sigma + \eta_{\rho\sigma} - \eta_{\rho\sigma} - \eta_{\rho\sigma} + \eta_{\rho\sigma}.$$  

(A.18)

Plugging $\eta_{\rho\sigma}$ from (A.18) into

$$\omega^\rho Du_\rho = \frac{1}{2} \varepsilon^{\mu\rho\sigma\delta} \partial_\delta u_\rho \eta_{\rho\sigma},$$

and using symmetry arguments as well as the normalization property of $u_\mu$, we obtain

$$u \cdot D \omega = \frac{1}{2} \partial_\mu \omega^\mu.$$  

(A.19)

To determine $\omega^\rho F_{\rho\mu} \partial_\rho M^{\mu\nu}$, we use the definitions of $\omega^\rho$ and $F_{\rho\mu}$, and after a lengthy but straightforward computation, where in particular

$$B_\rho B_\mu = -B^2 \varepsilon^{\rho\sigma\mu\nu} \omega_\nu B_\mu + \frac{1}{2} \left[ -u_\mu B \cdot Du + B^\rho \partial_\rho u_\mu - B^\rho \partial_\mu u_\rho \right],$$

$$B_\rho \partial_\rho B_\mu = B^\rho \partial_\rho B_\mu - B^\rho \partial_\rho B_\mu + B^\rho B^\sigma Du_\sigma + u_\mu B^\rho \omega^\nu \partial_\nu u_\sigma - u_\mu B^\sigma Du_\sigma - (B \cdot B) Du_\rho,$$

are used, we arrive at

$$\omega^\rho F_{\rho\mu} \partial_\rho M^{\mu\nu} = -(E \cdot \omega) u_\rho \partial_\rho X_\mu + (B \cdot B) \omega^\mu \partial_\mu X_\rho - (B \cdot B) \omega^\rho \partial_\rho X_\mu - \chi_\nu E^\rho \partial_\nu u_\mu - \chi_\nu \omega^\rho \partial_\nu u_\mu$$

$$+ \chi_\nu B^\sigma \omega^\rho \partial_\nu u_\sigma + \chi_\mu B^\rho \omega^\nu \partial_\nu u_\sigma - \chi_\mu \omega^\rho \partial_\nu u_\sigma - \chi_\mu B^\sigma \omega^\nu \partial_\nu u_\sigma.$$  

(A.20)

Adding this expression to the remaining terms in (A.17), and using (2.14), (A.6), (A.7), and (A.19), we arrive finally at

$$\partial_\mu \omega^\mu = -\frac{2}{\epsilon' + p} \left\{ \omega^\rho [ \partial_\rho E_\mu + \chi_\nu E^\rho \partial_\nu E_\mu - \chi_\mu B^\rho \partial_\rho B_\mu - (B \cdot B) \partial_\rho X_\mu ] + n_\mu (E \cdot \omega)$$

$$+ \frac{\chi_\nu}{2} B^\rho \left[ \partial_\rho E^\nu - \frac{(n_\nu - E^\nu \partial_\nu X_\rho)}{1 + \chi_\nu} \right] + \chi_\mu B^\rho \partial_\rho B_\mu (B \cdot \omega)$$

$$- \frac{\chi_\mu}{2} (B^\rho E^\nu \partial_\nu u^\mu - B^\rho E^\nu \partial_\nu u^\mu) \right\}.$$  

(A.22)

Here, $\partial_\mu B^\mu$ and $\partial_\mu E^\mu$ are to be read from (A.11) and (A.12), respectively. In section 3.2, we will use a certain approximation in the second-order derivative expansion to simplify these relations, as well as $\partial_\rho \omega^\mu$ from (A.22). The resulting expressions (3.21) will then be used to determine the anomalous transport coefficients $D_k$, $\zeta_k$ as well as $\kappa_k$ with $k = B$, $\omega$, $E$.

### Appendix B. Relevant bases for the shear and bulk viscosities from (3.17)

In this section we present the bases used to build the rank-two tensors appearing in $\tau^{\mu\nu}$ from (3.17).

In order to build the coefficients of the shear viscosities $\eta_\rho$, $\eta_\rho^B$, $i = 1, 2, 3$, following traceless combinations are used [19]
Other relevant combinations with non-vanishing trace are used

\[ (a) \Delta_{\mu\nu} \Delta^{\alpha\beta} + \Delta_{\alpha\beta} \Delta^{\mu\nu} - \frac{2}{3} \Delta_{\mu\nu} \Delta_{\alpha\beta}, \]

\[ (b) \left( \Delta_{\mu\nu} - \frac{3}{2} \xi_{\mu}^{\nu} \right) \left( \Delta_{\alpha\beta} - \frac{3}{2} \xi_{\alpha}^{\beta} \right), \]

\[ (c) - \left( \xi^{\mu\nu} b^{\alpha} b^{\beta} + \xi^{\alpha\beta} b^{\mu} b^{\nu} + \xi^{\mu\alpha} b^{\nu} b^{\beta} + \xi^{\nu\beta} b^{\mu} b^{\alpha} \right), \]

\[ (d) \xi^{\mu\nu} b^{\alpha} b^{\beta} + \xi^{\alpha\beta} b^{\mu} b^{\nu} + \xi^{\mu\alpha} b^{\nu} b^{\beta} + \xi^{\nu\beta} b^{\mu} b^{\alpha}, \]

\[ (e) b^{\mu} b^{\nu} b^{\alpha} + b^{\mu} b^{\nu} b^{\beta} + b^{\mu} b^{\alpha} b^{\nu} + b^{\mu} b^{\alpha} b^{\beta}. \]  

(B.1)

In addition to these combinations, there are other relevant traceless combinations, leading to the coefficients for \( \eta_{ii}^{(1)} \), \( i = 1, 2 \)

\[ (a) \left( \Delta_{\mu\nu} - \frac{3}{2} \xi_{\mu}^{\nu} \right) \left( \Delta_{\alpha\beta} - \frac{3}{2} \xi_{\alpha}^{\beta} \right), \]

\[ (b) - \left( \xi^{\mu\nu} e^{\alpha} e^{\beta} + \xi^{\alpha\beta} e^{\mu} e^{\nu} + \xi^{\mu\alpha} e^{\nu} e^{\beta} + \xi^{\nu\beta} e^{\mu} e^{\alpha} \right), \]  

(B.2)

and to \( \eta_{EE}^{(1)} \), \( i = 1, \ldots, 6 \)

\[ (a) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (b) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (c) b^{\mu} b^{\nu} b^{\alpha}, \]

\[ (d) e^{\mu} e^{\nu} e^{\alpha}. \]  

(B.3)

Let us notice that the combinations (c) and (d) from (B.3) are only traceless when the assumption \( c_{E} \quad b^{\mu
nu} = 0 \) from (2.8) is taken into account. Otherwise, only the sum of these two combinations will be traceless.

To build the bulk viscosities \( \zeta_{ij}^{(1)} \) following combinations with non-vanishing trace are used

\[ (a) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (b) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (c) b^{\mu} b^{\nu} b^{\alpha}, \]

\[ (d) e^{\mu} e^{\nu} e^{\alpha}. \]  

(B.4)

Other relevant combinations with non-vanishing trace are

\[ (a) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (b) \xi_{\mu}^{\nu} \xi_{\alpha}^{\beta}, \]

\[ (c) e^{\mu} e^{\nu} b^{\alpha} b^{\beta}. \]  

(B.5)

These combinations are used to build the coefficients \( \zeta_{i}^{(1)} \), \( i = 1, 2 \) and \( \zeta_{ii}^{(1)} \), \( i = 1, 2, 3 \) appearing in (B.2).

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