A uniform $L^\infty$ estimate
for complex Monge-Ampère equations

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0. Introduction.

Inspired by works on the Kähler-Ricci flow [TZ], [ST1] and on regularity of complex Monge-Ampère equations [K1], the second named author made the following conjecture: Let $\pi : X \to Y$ be a holomorphic mapping between compact Kähler manifolds with $\dim X = n \geq m = \dim Y$. Let $\omega_X$ be a Kähler metric on $X$ and $\omega_Y$ be a Kähler metric on $Y$. Suppose that $F \in L^\infty(M)$ and $u_t \in PSH(\omega_t)$ (which means that the current $\omega_t + dd^c u_t$ is weakly positive) be a solution of

\begin{equation}
(\omega_t + dd^c u_t)^n = ct^{n-m}F \omega_X^n, \quad \max_X u_t = 0,
\end{equation}

where $\omega = \pi^*\omega_Y$, $t \in (0, 1)$, and $c_t$ is defined by

$$
\frac{\int_X \omega_X^n}{t^{n-m} \int_X F \omega_X^n}.
$$

Then $u_t$ are uniformly bounded. Note that $\omega$ is a weakly positive $(1, 1)$ form on $X$ and $\omega_t = \omega + t\omega_X$ are Kähler forms for any $t > 0$. The main purpose of this note is to affirm this conjecture under certain technical conditions on the singular set of $\pi$. We do not know if these conditions are satisfied for any holomorphic fibration as above. However, we will check that these conditions are indeed true for many holomorphic fibrations.

Now let us state our conditions. Assume that for an analytic sets $S \subset X$ and $A \subset Y$ the form $\omega$ is non degenerate away from $S \subset \pi^{-1}(A)$. Consider the stratification of $A$, into $A_1, A_2, ..., A_{N_1}$ so that $A_0$ is the set of regular points of $A$, $A_1$ is the regular part of $A \setminus A_0$ and so on. Analogously we stratify $S$ into $S_1, S_2, ..., S_{N_2}$. Let $g_{q,j,k}$ be a fixed local system of generators of the ideal sheaf of $\bar{A}_q$ in coordinate patch $V_j$ and let $\theta_{j}$ be smooth functions on $Y$ such that $\text{supp} \ \theta_j \subset V_j$, $0 \leq \theta_j \leq 1$ and the interiors $V_j'$ of $\{\theta_j = 1\}$ cover $Y$. In the same way we define local generators $h_{q,j,k}$ for $S_q$ in $U_j \subset X$ and smooth functions $\zeta_j$ which are equal to 1 on $U_j'$.

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It may be sufficient to assume that $F \in L^p(M)$ for some $p > 1$. 
Assume that for some $a \in (0, 1), c_0 > 0$ and any fixed $c \in (0, c_0)$ we have

\[
\pi^* \left[ \frac{1}{2} \omega_Y + i c a^2 \sum_{q,k} |g_{q,j_0,k}|^{2(a-1)} \partial g_{q,j_0,k} \wedge \bar{\partial} g_{q,j_0,k} \right]^m \\
\wedge \left[ \frac{1}{2} \omega_X + i c a^2 \sum_{q,k} |h_{q,j_1,k}|^{2(a-1)} \partial h_{q,j_1,k} \wedge \bar{\partial} h_{q,j_1,k} \right]^{n-m} \geq H(\epsilon) \omega^n_X,
\]

on the set $U'_{j_1} \cap \pi^{-1}(V'_{j_0}) \cap S^\epsilon$, where $S^\epsilon = \{ \max_{j,k} \zeta_j |h_{1,j,k}| < \epsilon \}$, with some function $H$ satisfying $\lim_{\epsilon \to 0} H(\epsilon) = \infty$.

Note that a bit more explicit inequality (on the same sets, with the same $a, c_0, H$ as in (0.2)):

\[
\pi^* \left[ i \sum_{q,k} |g_{q,j_0,k}|^{2(a-1)} \partial g_{q,j_0,k} \wedge \bar{\partial} g_{q,j_0,k} + \omega_Y \right] \wedge \omega^{m-1} \\
\wedge \left[ i \sum_{q,k} |h_{q,j_1,k}|^{2(a-1)} \partial h_{q,j_1,k} \wedge \bar{\partial} h_{q,j_1,k} + \omega_X \right] \wedge \omega^{n-m-1} \geq H(\epsilon) \omega^n_X,
\]

entails (0.2).

The following theorem confirms the above conjecture under the assumption (0.2).

**Theorem 1.** Suppose that $F \in L^\infty(M)$ and the fibration $\pi : X \mapsto Y$ satisfies (0.2), then the solutions $u_t$ of (0.1) are uniformly bounded.

For its application, we consider the expanding Kähler-Ricci flow on $X$

\[
\frac{\partial \omega(t, \cdot)}{\partial t} = -Ric(\omega(t, \cdot)) - \omega(t, \cdot), \quad \omega(0, \cdot) = \omega_0.
\]

Let $\pi : X \mapsto Y$ be a holomorphic fibration such that $c_1(X) = \pi^* \omega_Y$. It follows that smooth fibers of $\pi$ are smooth Calabi-Yau manifolds. Let $Y_0$ be the dense-open subset of $Y$ over which fibers of $\pi$ are smooth, then there is an induced holomorphic map $f$ from $Y_0$ into a moduli space of Calabi-Yau manifolds. This map assigns each $y \in Y_0$ to the Calabi-Yau manifold $\pi^{-1}(y)$. By [TZ], for any $\omega_0$, then (0.4) has a global solution $\omega(t, \cdot)$. Moreover, there is a smooth family of functions $\varphi(t, \cdot)$ satisfying

\[
\omega(t, \cdot) = e^{-t} \omega_0 + (1 - e^{-t}) \pi^* \omega_Y + i \partial \bar{\partial} \varphi(t, \cdot).
\]

It was expected (cf. [ST2]) that $\omega(t, \cdot)$ converges to a generalized Kähler-Einstein metric on $Y$ in a suitable sense as $t$ tends to $\infty$. Combining estimates in [ST1] and the above theorem, we have
Theorem 2. Suppose that the fibration $\pi : X \to Y$ has no multiple fibers and satisfies $(0.2)$, then there is a positive current $\omega_\infty$ on $Y$ with properties:

1. $\omega_\infty = \omega_Y + i\partial\bar{\partial}\varphi_\infty$ for some bounded function $\varphi_\infty$ on $Y$;
2. $\varphi_\infty$ is smooth on $Y_0$;
3. $\varphi(t, \cdot)$ converges to $\pi^*\varphi_\infty$ in $C^{1,1}$-topology on any compact subsets contained in $\pi^{-1}(Y_0)$;
4. On $Y_0$, $\omega_\infty$ satisfies the equation for generalized Kähler-Einstein metrics

$$Ric (\omega_\infty) = -\omega_\infty + f^*\omega_{WP},$$

where $\omega_{WP}$ denotes the Weil-Petersson metric on the moduli of Calabi-Yau manifolds.

This theorem can be proved by those estimates developed in [ST1] without using Theorem 1 if the base is 1-dimensional or $X$ is of dimension 2. The rest of this note is organized as follows: In Section two, we give a proof of Theorem 1. In Section three, we prove Theorem 2 using Theorem 1 and results in [SZ1] and [SZ2]. In last section, we verify the assumptions $(0.2)$ for certain holomorphic fibrations, including any fibration with 1-dimensional base and generic 3-folds with 2-dimensional base and tori as fibers.
1. Proof of Theorem 1.

Proof. In what follows $C$ denotes different positive constants independent of $t$. Note that

$$
\lim_{t \to 0} c_t = \left(\frac{n}{m}\right) \int_X \omega^m \wedge \omega^{n-m} - \int_X F \omega_X^n > 0.
$$

Lemma 1. For $(Y, \omega_Y), \theta_j, V_j'$ as above there exists $c_1 > 0$ such that for all $c \in (0, c_1), a \in (0, 1),$ and

$$
\psi_a = \sum_{q,j,k} \theta_j^2 |g_{q,j,k}|^{2a},
$$

we have

$$(1.1) \quad dd^c \psi_a \geq -\frac{1}{2} \omega_Y + ca^2 i \sum_{q,k} |g_{q,j,k}|^{2(a-1)} \partial g_{q,j,k} \wedge \bar{\partial} g_{q,j,k}$$

on the set $V_j'$.

Proof. By computation (comp. [DP, Lemma 2.1])

$$
i \partial \bar{\partial} \psi_a = 2i \sum_{q,j,k} |g_{q,j,k}|^{2a} (\theta_j \partial \bar{\theta}_j - \partial \theta_j \wedge \bar{\theta}_j)$$

$$+ i \sum_{q,j,k} |g_{q,j,k}|^{2a} (2\partial \theta_j + a \frac{\theta_j}{g_{q,j,k}} \partial g_{q,j,k}) \wedge (2\partial \bar{\theta}_j + a \frac{\theta_j}{g_{q,j,k}} \bar{\partial} g_{q,j,k}).$$

Since $\theta_j$ are smooth the first sum exceeds $-\frac{1}{2c} \omega_Y$ (in the sense of currents) for $c$ small enough. All terms in the second sum are positive. On $V_j'$ we have $\theta_j = 1$, the form $\partial \theta_j$ vanishes, and thus we obtain (1.1).

We apply Lemma 1 also on $X$ for

$$\tilde{\psi}_a = \sum_{j,k} \tilde{\zeta}_j^2 |h_{jk}|^{2a},$$

and fix $c > 0$ so that (1.1) holds for $\psi_a$ and the corresponding inequality on $X$ is true for $\tilde{\psi}_a$. Define

$$(1.2) \quad \varphi_t = c \psi_a \circ \pi + ct \tilde{\psi}_a.$$

By (1.1)

$$(1.3) \quad \varphi_t \in PSH(\frac{1}{2} \omega_t).$$

By the Calabi-Yau theorem [Y] and its non smooth version [K1] it is no loss of generality to assume that $F$ and $u_t$ are smooth (provided that a priori estimates do not depend on derivatives of $u_t$ and $F$). We treat two cases separately:
**CASE 1.** There exist $\epsilon > 0$ and $\delta > 0$ such that for any $t \in (0,1)$ and any $s \in (0,-s_t)$

$$\int_{U(t,s,2\epsilon)} \omega^n_X \geq \delta \int_{U(t,s)} \omega^n_X,$$

where $U(t,s) = \{ u_t < \varphi_t + s_t + s \}$, $s_t = \inf(u_t - \varphi_t)$, $U(t,s,\epsilon) = U(t,s) \setminus \bar{S}^\epsilon$.

We seek for a uniform bound on $s_t$. For $s \in (0,-s_t]$ define

$$\Phi_t(s) = \frac{s}{(\int_{U(t,s)} \omega^n_X)^{1/n}}.$$

It is enough to find a bound for $\Phi_t(s)$ since

$$\Phi_t(-s_t) = |s_t|/(\int_X \omega^n_X)^{1/n}.$$

First we show that

$$\text{(1.4)} \quad \limsup_{s \to 0} \Phi_t(s) \leq C.$$

On the set $X \setminus S^\epsilon$ we have

$$\text{(1.5)} \quad C(\epsilon) \omega^{n-m}_X \wedge \omega^m \geq \omega^n_X.$$

Hence if $t^{n-m} F\omega^n_X = f_t \omega_t^n$ then $f_t \leq CF$ on this set. Since $u_t$ is smooth and $\varphi_t$ is smooth on $X \setminus S^\epsilon$ we have for $\zeta \in X \setminus S^\epsilon$, where $u_t - \varphi_t$ attains its minimum,

$$D^2(u_t - \varphi_t)(\zeta) \geq 0$$

and so, having $\varphi_t$ in $PSH(\frac{1}{2} \omega_t)$,

$$C \omega^n_t(\zeta) \geq f_t(\zeta) \omega^n_t(\zeta) = (\omega_t + dd^c u_t)^n(\zeta) \geq 2^{1-n} dd^c(u_t - \varphi_t) \wedge \omega^{n-1}_t(\zeta) \geq 2^{1-n} dd^c(u_t - \varphi_t) \wedge \omega^{n-1}_t(\zeta).$$

Therefore, interpreting this inequality in local coordinates diagonalizing $\omega_t(\zeta)$ we get

$$|D^2(u_t - \varphi_t)(\zeta)| \leq C.$$

Now, in geodesic coordinates around $\zeta$, consider (for $s$ close to 0) the maximal ball $B(\zeta, r(s))$ contained in $U(t,s)$. By the estimate on $D^2(u_t - \varphi_t)(\zeta)$ and the Taylor expansion for $z \in \partial B(\zeta, r(s)) \cap \partial U(t,s)$

$$s = u_t(z) - \varphi_t(z) - s_t \leq C|z - \zeta|^2 = Cr(s)^2.$$

So

$$\int_{U(t,s)} \omega^n_X \geq \int_{B(\zeta, r(s))} \omega^n_X \geq C r(s)^{2n} \geq C s^n$$

and $\Phi_t(s) \leq C$ for $s$ close to zero.
Having (1.4) it is enough to find a uniform bound for \( \Phi_t(a_t) = \max \Phi_t \). For such \( a_t \) we have

\[
(1.6) \quad \int_{U(t,a_t)} \omega_X^n \leq 2^n \int_{U(t,a_t/2)} \omega_X^n.
\]

Using Stokes’ theorem we get

\[
\int_{U(t,a_t)} (\omega_t + \dd c_{u_t}) \wedge (\omega_t + \dd c_{\varphi_t})^{n-1}
\]

\[
\leq \int_{U(t,a_t)} (\omega_t + \dd c_{u_t})^2 \wedge (\omega_t + \dd c_{\varphi_t})^{n-2} \leq \ldots
\]

\[
\leq \int_{U(t,a_t)} (\omega_t + \dd c_{u_t})^n.
\]

Integrating by parts, applying (1.3) and the above inequality one arrives at

\[
\int_{U(t,a_t)} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega_t^{n-1}
\]

\[
= \int_{U(t,a_t)} (s_t + a_t - u_t + \varphi_t) \dd c(u_t - \varphi_t) \wedge \omega_t^{n-1}
\]

\[
\leq \int_{U(t,a_t)} (s_t + a_t - u_t + \varphi_t) (\omega_t + \dd c_{u_t}) \wedge \omega_t^{n-1}
\]

\[
\leq 2^n a_t \int_{U(t,a_t)} (\omega_t + \dd c_{u_t}) \wedge (\omega_t + \dd c_{\varphi_t})^{n-1}
\]

\[
\leq 2^n a_t \int_{U(t,a_t)} (\omega_t + \dd c_{u_t})^n
\]

\[
= 2^n a_t t^{n-m} \int_{U(t,a_t)} c_t F \omega_X^n.
\]

From (1.5) it follows that for any (1,0) form \( \gamma \)

\[
i \gamma \wedge \bar{\gamma} \wedge \omega_t^{n-1} \geq C t^{n-m} i \gamma \wedge \bar{\gamma} \wedge \omega_X^{n-1}
\]

on \( U(t,a_t, \epsilon) \). By this and the previous inequality one obtains

\[
i t^{n-m} \int_{U(t,a_t, \epsilon)} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega_X^{n-1}
\]

\[
\leq C \int_{U(t,a_t, \epsilon)} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega_t^{n-1}
\]

\[
\leq C \int_{U(t,a_t)} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega_t^{n-1} \leq C a_t t^{n-m} \int_{U(t,a_t)} \omega_X^n.
\]
Thus, by the assumption of Case 1,

\[(1.7) \int_{U(t,a_t,\epsilon)} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega_X^{n-1} \leq Ca_t \int_{U(t,a_t)} \omega_X^n \leq Ca_t \int_{U(t,a_t,\epsilon)} \omega_X^n.\]

We cover \(X \setminus S^{2\epsilon}\) by a finite number of unit cubes (in local coordinates) \(W_j, j = 1, 2, ..., N_0\) such that \(W_j \cap S^{\epsilon} = \emptyset\). For one of them, say \(W_1\), we have

\[(1.8) \int_{U(t,a_t/2,2\epsilon) \cap W_1} \omega_X^n \geq (1/N_0) \int_{U(t,a_t/2,2\epsilon)} \omega_X^n.\]

Let \((x_1, x_2, ..., x_{2n})\) denote real coordinates in \(W_1 = \{x : |x_j| < 1\}\). Set \(\Omega_1 = W_1 \cap U(t,a_t,\epsilon) = W_1 \cap U(t,a_t), \Omega_2 = W_1 \cap U(t,a_t/2,2\epsilon)\) and let \(\pi_j\) denote the projection

\(\pi_j(x) = (x_1, ..., x_{j-1}, x_j+1, ..., x_{2n}).\)

By an isoperimetric inequality from [LW] there is \(j\) (and we take \(j = 1\)) such that

\[(1.9) V_{2n-1}(\pi_1(\Omega_2)) \geq V_{2n}(\Omega_2)^{n-1/n},\]

where \(V_k\) denotes Euclidean volume in \(\mathbb{R}^k\). We shall prove our estimate in two subcases separately.

**Case 1A.** For \(G = \{y \in \pi_1(\Omega_2) : \pi_1^{-1}(y) \cap \partial \Omega_1 \neq \emptyset\}\) we have

\[V_{2n-1}(G) < \frac{1}{2} V_{2n-1}(\pi_1(\Omega_2)).\]

Observe that for \(y \in \pi_1(\Omega_2) \setminus G\) we have \(\pi_1^{-1}(y) \cap W_1 \subset \Omega_1\). Therefore, by Fubini’s theorem

\[V_{2n-1}(\pi_1(\Omega_2)) \leq 2V_{2n}(\Omega_1).\]

Applying the assumption of Case 1, (1.6), (1.8), (1.9) and the last inequality one obtains

\[V_{2n}(\Omega_1)^{n-1/n} \leq CV_{2n}(\Omega_2)^{n-1/n} \leq CV_{2n-1}(\pi_1(\Omega_2)) \leq CV_{2n}(\Omega_1).\]

So \(V_{2n}(\Omega_1) \geq C\) which, via (1.6), gives a lower bound for \(\int_{U(t,a_t/2)} \omega_X^n\). Therefore

\[\int_{U(t,a_t/2)} (-u_t) \omega_X^n \geq C(||s_t||/2 - ||\varphi_t||_{\infty}).\]

On the other hand, since \(\max u_t = 0\) there exists (by the standard Green function argument) a constant \(C_0\) such that
\[ \int_X (-u_t) \omega^n_X \leq C_0 \]

for all \( t \in (0, 1) \). Combining the last two inequalities we obtain a uniform bound for \( |s_t| \) and further \( \Phi_t(a_t) \leq C \).

**Case 1B.** Now

\[ V_{2n-1}(G) \geq \frac{1}{2} V_{2n-1}(\pi_1(\Omega_2)). \]

Let us denote \( d(y) = V_1(G(y)) \), \( G(y) = \{ x_1 \in [-1, 1] : (x_1, y) \in \Omega_1 \} \), and observe that for \( y \in G \) the set \( G(y) \times \{ y \} \) contains an open interval in \( \Omega_1 \) joining points from \( \Omega_2 \) and \( \partial \Omega_1 \). Then the integral of \( \left| \frac{\partial (u_t - \varphi_t)}{\partial x_1} \right| \) over this interval exceeds \( a_t/2 \).

We use this fact and the Schwarz inequality to justify the fourth in the following chain of inequalities. The first one follows from (1.7), the third one from Fubini’s theorem, the fifth from the Schwarz inequality, the sixth from (1.9), (1.10), the seventh from (1.6), (1.8) and the assumption of Case 1, and the last one again from the assumption of Case 1.

\[
\begin{align*}
& a_t \int_{U(t,a_t,\epsilon)} \omega^n_X \geq C \int_{U(t,a_t,\epsilon) \cap W_1} d(u_t - \varphi_t) \wedge d^c(u_t - \varphi_t) \wedge \omega^{n-1}_X \\
& \geq C \int_{\Omega_1} \left| \frac{\partial (u_t - \varphi_t)}{\partial x_1} \right|^2 dV_{2n} \\
& \geq C \int_G \int_{G(y)} \left| \frac{\partial (u_t - \varphi_t)}{\partial x_1} \right|^2 dx_1 dy \geq C a_t^2 \int_G \frac{1}{d(y)} dy \\
& \geq C a_t^2 \frac{V_{2n-1}(G)}{V_{2n}(\Omega_1)} \geq C a_t^2 \frac{V_{2n-1}(\Omega_1)}{V_{2n}(\Omega_1)} \geq C a_t^2 \left( \int_{U(t,a_t,\epsilon)} \omega^n_X \right)^{1-1/n} \\
& \geq C a_t^2 \left( \int_{U(t,a_t)} \omega^n_X \right)^{1-1/n}.
\end{align*}
\]

So

\[ \Phi_t(a_t) = \frac{a_t}{\left( \int_{U(t,a_t)} \omega^n_X \right)^{1/n}} \leq C, \]

which finishes the proof of Case 1. Note that the bound of \( \Phi_t \) does not depend on derivatives of \( u_t \) and so the additional assumption that \( u_t \) be smooth may be dropped.

**Case 2.** If the assumption of Case 1 is not satisfied then for any \( \epsilon > 0 \) and \( \delta > 0 \) there exist sequences \( t(j) \to 0 \) and \( s(j) \in (0, -s_t) \) (those sequences depend on \( \epsilon \) and \( \delta \) which is omitted in the notation) such that
\( (\omega_t + dd^c \varphi_t)^n = [\pi^*(dd^cz_a + \omega_Y) + t(dd^c \tilde{z}_a + \omega_X)]^n \)

\[ \geq t^{n-m} [\pi^*(dd^cz_a + \omega_Y)]^m \wedge (dd^c \tilde{z}_a + \omega_X)^{n-m} \]

\[ \geq t^{n-m} \pi^* \left[ \frac{1}{2} \omega_Y + ica^2 \sum_{q,k} |g_{q,j_0,k}|^{2(a-1)} \partial g_{q,j_0,k} \wedge \bar{\partial} \bar{g}_{q,j_0,k} \right]^m \]

\[ \wedge \left[ \frac{1}{2} \omega_X + ica^2 \sum_{q,k} |h_{q,j_1,k}|^{2(a-1)} \partial h_{q,j_1,k} \wedge \bar{\partial} \bar{h}_{q,j_1,k} \right]^{n-m} \]

\[ \geq Ct^{n-m} H(\epsilon) \omega_X^n. \]

Using this and the comparison principle [K2] one obtains

\[ Ct^{n-m} \int_{U(t,s)} \omega_X^n \geq t^{n-m} \int_{U(t,s)} F \omega_X^n = \int_{U(t,s)} (\omega_t + dd^c u_t)^n \]

\[ \geq \int_{U(t,s) \cap S^\epsilon} (\omega_t + dd^c \varphi_t)^n \geq Ct^{n-m} H(\epsilon) \int_{U(t,s) \cap S^\epsilon} \omega_X^n. \]

Combine it with (1.11) to arrive at

\[ \int_{U(t,s) \cap S^\epsilon} \omega_X^n \geq (1 - \delta) \int_{U(t,s)} \omega_X^n \]

\[ \geq (1 - \delta) CH(\epsilon) \int_{U(t,s) \cap S^\epsilon} \omega_X^n, \]

which contradicts the assumption \( \lim_{\epsilon \to 0} H(\epsilon) = \infty \) for small \( \epsilon \). Thus Case 2 never occurs.
2. Proof of Theorem 2.

We adopt notations in the introduction. First observe that
\[ \omega(t, \cdot) = e^{-t}\omega_0 + (1 - e^{-t})\pi^*\omega_Y + i\partial\bar{\partial}\varphi(t, \cdot). \]

Denote by \( \omega_t \) the Kähler form \( e^{-t}\omega_0 + (1 - e^{-t})\pi^*\omega_Y \). Adding appropriate constants to \( \varphi(t, \cdot) \), one can show
\[
\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-m)t}(\omega_t + i\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi, \quad \varphi(0, \cdot) = 0,
\]
where \( \Omega \) is a volume form on \( X \) determined by \( \text{Ric}(\Omega) = -\pi^*\omega_Y \). Then there is a constant \( c \) such that
\[
e^{(n-m)t}\omega_t^n = \sum_{j=1}^{m} \binom{n}{j}(1 - e^{-t})^j e^{(j-m)t}\pi^*\omega_Y^j \wedge \omega_0^{n-j} \leq c\Omega.
\]

Therefore, by the Maximum principle, we have \( \varphi \leq \max\{\log c, 0\} \). Differentiating the above equation on \( t \), we get
\[
\frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} \right) = \Delta_t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} + \text{tr}_t(e^{-t}(\pi^*\omega_Y - \omega_0)) + (n - m).
\]

Again by using the Maximum principle, one can show that \( \frac{\partial \varphi}{\partial t} \) is uniformly bounded from above.

Set \( s = e^{-t}/(1 - e^{-t}) \), we then have
\[
(s\omega_0 + \pi^*\omega_Y + i\partial\bar{\partial}\psi)^n = s^{n-m}F\omega_0^n,
\]
where \( \psi = (1 - e^{-t})^{-1}\varphi \) and
\[
F = (1 - e^{-t})^{-m}e^{\varphi + \frac{\partial \varphi}{\partial t}} \frac{\Omega}{\omega_0^n}.
\]

Clearly, \( F \) is uniformly bounded, so we can apply Theorem 1 to conclude that \( \psi \) is uniformly bounded, so does \( \varphi \).

The rest of the proof is exactly the same as that in [ST1] since all other estimates there are dimension free (also see [ST2]). This finishes the proof of Theorem 2.
3. Examples.

In this section, we verify the assumptions (0.2) for two examples. We will adopt notations in the introduction. First we assume that \( \pi : X \mapsto Y \) is a holomorphic fibration with \( \dim Y = 1 \), that is, all fibers are hypersurfaces in \( X \). In this case, any point of \( Y \) has a neighborhood which can be identified with an open subset \( U \) in the complex plane and \( \pi \) is given by a holomorphic function \( h \). By (0.3), we only need to show for some \( a \in (0, 1) \)

\[
|h - h(x)|^{2(a-1)} \partial h \wedge \bar{\partial} h \wedge \omega_X^{n-1} \geq H(\epsilon)\omega_X^n.
\]

in a small neighborhood of \( x \), which may depend on \( \epsilon \), where \( x \) is any fixed point in \( \pi^{-1}(U) \) and \( H(\epsilon) \) was given in (0.2). By a result of Łojasiewicz [Lo], there are constants \( \theta \in (0, 1/2) \) and \( \sigma_1 \) such that for any \( \xi \) with \( d(x, \xi) \leq \sigma_1 \),

(3.1) \[ |\nabla h|(\xi) \geq |h(\xi) - h(x)|^{1-\theta}. \]

Now choose \( a < \theta \), then (3.1) holds for \( h \). Therefore the assumptions in (0.2) hold for any 1-dimensional base.

Before we discuss the second example we state a simple lemma regarding to the assumptions (0.2).

**Lemma 2.** Let \( \pi : X \mapsto Y \) be the product of a holomorphic fibration \( \pi_1 : X_1 \mapsto Y_1 \) and a complex manifold \( U \), that is, \( X = X_1 \times U \), \( Y = Y_1 \times U \) and \( \pi = \pi_1 \times \text{Id}_U \). If the fibration \( \pi_1 : X_1 \mapsto Y_1 \) satisfies (0.2), then so does \( \pi : X \mapsto Y \).

It follows directly from (0.2) since \( \pi \) is non-degenerate along \( U \)-directions.

Now we consider the second example. Let \( \pi : X \mapsto Y \) be a generic fibration over a complex surface \( Y \) with elliptic curves as fibers. Let \( A \) be the set of \( y \in Y \) such that \( \pi^{-1}(y) \) is a singular elliptic curve. Since the fibration is generic, \( A \) is a divisor in \( Y \) and each singular fiber has either a node or an ordinary cusp.

**Proposition 3.** Such an elliptic fibration \( \pi : X \mapsto Y \) over a surface satisfies (0.2).

The rest of this section is devoted to the proof of this proposition.

Set \( A_0 \) to be the subset of \( A \) over which singular fibers have nodes and \( A_1 \) to be the subset of those points \( y \in A \) such that \( \pi^{-1}(y) \) has a cusp. Clearly, \( A_1 \) consists of finitely many isolated points. As usual, let \( S \) be the set of singular points in the singular fibers of \( \pi : X \mapsto Y \). Then \( S_0 = \pi^{-1}(A_0) \) and \( S_1 = \pi^{-1}(A_1) \). By Lemma 2, clearly, (0.2) holds for any point in \( S_0 \). So we only need to check (0.2) for any \( p \in S_1 \).

Fix each such a \( p \), we can find local coordinates \( x, y, z \) of \( X \) near \( p \) and local coordinates \( s, t \) of \( Y \) near \( \pi(p) \) satisfying: \( \pi(x, y, z) = (z, y^2 - x^3 + zx) = (s, t) \in Y \). Furthermore, we may assume

\[
\omega_X = i(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z})
\]
and

\[ \omega_Y = i(ds \wedge d\bar{s} + dt \wedge d\bar{t}) \].

Then we have

\[ \omega = \pi^* \omega_Y = i(dz \wedge d\bar{z} + [(z - 3x^2)dx + 2ydy] \wedge [(z - 3x^2)dx + 2ydy]) \]

and

\[ \omega^2 = -2dz \wedge d\bar{z} \wedge [(z - 3x^2)dx + 2ydy] \wedge [(z - 3x^2)dx + 2ydy]. \]

So

\[ \omega^2 \wedge \omega_X = (-2i)(|z - 3x^2|^2 + 4|y|^2)dx \wedge d\bar{x} \wedge dy \wedge d\bar{y} \wedge dz \wedge d\bar{z}. \]

The singular set is

\[ S = \{ y = 0 \} \cap \{ z = 3x^2 \}. \]

Since \( \pi(x, 0, 3x^2) = (3x^2, 2x^3) \) the image of \( S \) lies in

\[ A = \{ 4s^3 = 27t^2 \}. \]

As the generator for \( A \) close to origin we take \( g(s, t) = 4s^3 - 27t^2 \), and as generators for \( S \) we take

\[ h_1(x, y, z) = y, \quad h_2(x, y, z) = z - 3x^2. \]

Define, for positive \( a < 1/100 \)

\[
\gamma_1 = \pi^*(i|y|^{a-2}\partial g \wedge \bar{\partial} \bar{g} \wedge \omega_Y) \\
= \pi^*(36i|4s^3 - 27t^2|^{a-2}(4|s|^4 + 81|t|^2)ds \wedge d\bar{s} \wedge dt \wedge d\bar{t}).
\]

and

\[
\gamma_2 = i(|h_1|^{a-2}\partial h_1 \wedge \bar{\partial} \bar{h}_1 + |h_2|^{a-2}\partial h_2 \wedge \bar{\partial} \bar{h}_2) + \omega_X \\
\geq i|y|^{a-2}dy \wedge d\bar{y} + i|z - 3x^2|^{a-2}(dz - 6xdx) \wedge (dz - 6xdx) + idx \wedge d\bar{x}.
\]

Wedging \( \gamma_1 \) and \( \gamma_2 \) we can forget about differentials in \( \gamma_2 \) containing \( dz \) or \( d\bar{z} \). Thus we seek for the lower bound for \( \gamma_1 \wedge \gamma_3 \), where

\[ \gamma_3 = i|y|^{a-2}dy \wedge d\bar{y} + i(36|x|^2|z - 3x^2|^{a-2} + 1)dx \wedge d\bar{x}. \]

We have

\[
\gamma_1 \wedge \gamma_3 \geq \pi^*[i|4s^3 - 27t^2|^{a-2}(4|s|^4 + 81|t|^2)]dz \wedge d\bar{z} \wedge \pi^*[i\partial t \wedge \partial \bar{t}] \wedge \gamma_3 \\
= \pi^*[i|4s^3 - 27t^2|^{a-2}(4|s|^4 + 81|t|^2)]dz \wedge d\bar{z} \\
\vee i[(z - 3x^2)dx + 2ydy] \wedge [(z - 3x^2)dx + 2ydy] \wedge \gamma_3 \\
\geq \pi^*[i|4s^3 - 27t^2|^{a-2}(4|s|^4 + 81|t|^2)] \\
\times [(z - 3x^2)|y|^{a-2} + 4|y|^2(36|x|^2|z - 3x^2|^{a-2} + 1)](1/6)\omega_X^3
\]
Let us denote
\[ f_1(x, y, z) = \pi^*(|4s^3 - 27t^2|), \]
\[ f_2(x, y, z) = \pi^*(4|s|^4 + 81|t|^2), \]
and writing \( w = z - 3x^2 \):
\[ f_3(x, y, z) = |z - 3x^2|^2|y|^{a-2} + 4|y|^2(36|x|^2|z - 3x^2|^2 + 1) \]
\[ = |w|^2|y|^{a-2} + 4|y|^2(36|x|^2|w|^{a-2} + 1). \]

To check the hypothesis of the theorem we need to show that
\[ \lim_{(x, y, z) \to \text{origin}} f_1^{-2/3-2a} f_2 f_3 = \infty \]
when \((x, y, z)\) tends to the origin. Since
\[ f_1 \leq 100 \max(|s|^3, |t|^2) \]
we conclude that
\[ f_1^{-4/3-2a} f_2 \to \infty. \]

To finish the verification we need to prove that \( f_1^{-2/3+2a} f_3 \) is bounded away from zero. It is enough to check it for points satisfying \( f_3 < 1 \). So we assume that
\[ |w|^2 < |y|^{2-a} \quad \text{and so} \quad |w|^{a-2} \geq |y|^{-5/3}. \]

We have
\[ f_1 = |w|^2(4w + 9x^2) - 27y^2(4x^3 + 2wx + y^2)|. \]

Hence
\[ f_1 \leq M := 500 \max(|w|^3, |w^2x^2|, |y^2x^3|, |y|^4). \]

Accordingly consider four cases:
1) \( M = 500|w|^3 \) (then \( |w^3| \geq |y^4| \)). So
\[ 500 f_1^{-2/3+2a} f_3 \geq |w|^6 a^2 |w|^2 |y|^{a-2} \geq |w|^{6a-1} |y|^{a-2/3} \to \infty. \]

2) \( M = 500|xw|^2 \) (then \( |w| \leq |x^2| \)). So (using \( |y^2 \geq |w|^{2+2a} \) which follows from (3.2))
\[ 500 f_1^{-2/3+2a} f_3 \geq |wx|^{4a-4/3} |xy|^2 |w|^{a-2} \geq |x|^{4a-1/3} |w|^{7a-5/6} \to \infty. \]

3) \( M = 500|x^3y^2| \) (then \( |y^2| \leq |x^3| \)). So (see point 1))
\[ 500 f_1^{-2/3+2a} f_3 \geq |y|^{4a-4/3} |x|^{6a-2} |xy|^2 |w|^{a-2} \geq |y|^{4a-4/3} |y|^{2+4a} |w|^{a-2} \]
\[ \geq |y|^{-1+8a} \to \infty. \]

4) \( M = 500|y^4| \). So (see (1))
\[ 500 f_1^{-2/3+2a} f_3 \geq |y|^{8a-8/3} |y|^2 \to \infty. \]

Thus Proposition 3 is proved.
References

[DP] J.-P. Demailly, M. Paun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. Math., 159 (2004), 1247-1274.

[K1] S. Kołodziej, The complex Monge-Ampère equation, Acta Math. 180 (1998), 69-117.

[K2] S. Kołodziej, Stability of solutions to the complex Monge-Ampère on compact Kähler manifolds, Indiana U. Math. J. 52 (2003), 667-686.

[Lo] S. Lojasiewicz, Ensembles semi-analtiques, I.H.E.S. notes (1965).

[LW] L.H. Loomis, H. Whitney, An inequality related to the isoperimetric inequality, Bull. Amer. Math. Soc. 55 (1949), 961-962.

[ST1] J. Song and G. Tian, The Kähler-Ricci flow on surfaces of positive Kodaira dimension, To appear in Inventiones Math.

[ST2] J. Song and G. Tian, Generalized Kähler-Einstein metrics, in preparation.

[TZ] G. Tian and Z. Zhang, On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. B 27 (2) (2006), 179-192.

[Y] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, Comm. Pure and Appl. Math. 31 (1978), 339-411.