Localization of weakly interacting Bose gas in quasiperiodic potential

To cite this article: Sayak Ray et al 2016 New J. Phys. 18 013013

View the article online for updates and enhancements.

You may also like

- Focus on Gold Atoms in Optical Lattices
  Immanuel Bloch and Peter Zoller

- Superfluid and insulator of dirty Bose-Fermi mixture
  H Mori and S Ikeda

- Characterizing localization properties of two spinless electrons in a one-dimensional Harper model with concurrence
  Gong Long-Yan and Tong Pei-Qing
Localization of weakly interacting Bose gas in quasiperiodic potential

Sayak Ray, Mohit Pandey, Anandamohan Ghosh and Subhasis Sinha
Indian Institute of Science Education and Research-Kolkata, Mohanpur, Nadia-741246, India
E-mail: subhasis@iiserkol.ac.in

Keywords: quantum gas, superfluidity, localization

Abstract
We study the localization properties of weakly interacting Bose gas in a quasiperiodic potential. The Hamiltonian of the non-interacting system reduces to the well known ‘Aubry–André model’, which shows the localization transition at a critical strength of the potential. In the presence of repulsive interaction we observe multi-site localization and obtain a phase diagram of the dilute Bose gas by computing the superfluid fraction and the inverse participation ratio. We construct a low-dimensional classical Hamiltonian map and show that the onset of localization is manifested by the chaotic phase space dynamics. The level spacing statistics also identify the transition to localized states resembling a Poisson distribution that are ubiquitous for both non-interacting and interacting systems. We also study the quantum fluctuations within the Bogoliubov approximation and compute the quasiparticle energy spectrum. Enhanced quantum fluctuation and multi-site localization phenomenon of non-condensate density are observed above the critical coupling of the potential. We briefly discuss the effect of the trapping potential on the localization of matter wave.

1. Introduction

In recent years ‘Anderson localization’ of particles and waves has regained interest in quantum many particle physics. ‘Anderson localization’ is a remarkable quantum phenomenon for which the propagating wave becomes exponentially localized in the presence of disorder [1, 2] and is a well studied subject in the context of electronic systems in the presence of disorder. Like particles, waves can also localize in a disordered medium. Localization of ‘matter wave’ has recently been observed in experiments on ultracold Bose gases in bichromatic optical lattices [3] and in the presence of the speckle potential [4–6]. For certain parameters, bosons in a bichromatic lattice can be mapped on to the Aubry–André (AA) model [7] with ‘quasi-periodic’ potential. Localization of light in the AA model has been observed experimentally [8]. Unlike the Anderson model with random disorder, the AA model exhibits a localization transition in one-dimension at a critical strength of the potential. The localization length can be calculated theoretically by constructing a suitable classical Hamiltonian map (CHM) [9–11]. Apart from that the quasi-periodicity of the AA model gives rise to various interesting spectral properties [12]. In recent experiments [13–15] the effect of interaction on the AA model has been studied. Repulsive interaction plays an important role in the formation of correlated phases like ‘Bose glass’ phase [16–20]. The localization of spinless fermions in a quasiperiodic potential has also been studied using Luttinger liquid theory [21] and DMRG [22]. The dynamics and diffusion of interacting Bose gas in the presence of disorder have been investigated both experimentally [23–26] and theoretically [27, 28]. In recent years, ‘many body localization’ [29] and localization at finite temperature [30–32] have generated an impetus to study disordered Bose gas.

In this work we investigate localization of weakly interacting Bose gas in the AA potential. Localization of wavefunction for both the non-interacting AA model and interacting system is studied by using an alternative approach of a classical dynamical map. The interplay between interaction and localization is investigated for dilute condensate and Bogoliubov quasi-particles. The paper is organized as follows: in section 2, we discuss non-interacting Bose gas in a quasiperiodic potential. In section 2.1, we review the AA model and discuss the single particle localization properties. Localization in a non-interacting system using the dynamical map approach is presented in section 2.2. In section 3, a phase diagram of weakly interacting Bose gas within mean-field theory is calculated from physical quantities like inverse participation ratio (IPR) and superfluid fraction.
(SFF). The effect of nonlinearity due to interactions on the dynamical map is studied. We compute the Bogoliubov quasiparticle energies and amplitudes to investigate disorder induced quantum fluctuations and to study the localization of quasi-particle amplitudes and non-condensate densities. We also investigate the localization of Bogoliubov excitations from spectral statistics. The crossover to disorder induced localization is studied for Bose gas with weak attractive interaction. The effect of the trapping potential on localization and effect of disorder on the center of mass (COM) motion are presented in section 4. Finally we summarize our results in section 5.

2. Non-interacting bosons in quasiperiodic potential

In the experiment [3] localization of ultracold bosons has been studied using a bichromatic optical lattice which can be mapped on to the AA model within the tight binding approximation and for a certain parameter regime [33]. The AA model is defined by the Hamiltonian

$$H = -t \sum_{n}(\ket{n} \bra{n+1} + \ket{n+1} \bra{n}) + \lambda \sum_{n} \cos(2\pi/\beta n) \ket{n} \bra{n},$$

where \ket{n} is a Wannier state at lattice site \(n\), \(t\) is the nearest neighbor hopping strength, \(\lambda\) is the strength of the onsite potential and the period of the potential is determined by \(\beta\). In the rest of the paper, we would be working in the units in which \(t = 1\).

For \(\lambda = 2\), the Hamiltonian 1 is equivalent to the Harper model [34] describing the motion of an electron in a square lattice in the presence of a perpendicular magnetic field, where the flux \(\Phi\) through each plaquette gives rise to the known Hofstadter butterfly [35]. The Hamiltonian given in equation (1) poses very interesting properties for irrational values of \(\beta\). When \(\beta\) is chosen to be a ‘Diophantine number’, the AA model undergoes a localization transition at a critical value of the potential strength \(\lambda = 2\) [7, 36]. In contrast the localization transition is absent in the one-dimensional Anderson model with random disorder. The quasiperiodic potential generates a correlated disorder in the AA model.

2.1. Single particle localization

In order to study localization transition in the AA model, we choose \(\beta = (\sqrt{5} - 1)/2\), which is the inverse of the ‘golden mean’ and a Diophantine number. This is particularly helpful since a rational approximation of \(\beta\) can be done by the Fibonacci numbers. Although incommensurability of the potential with the underlying lattice plays a crucial role in the AA model, for numerical studies one can approximate \(\beta = F_{n-1}/F_{n}\), where \(F_{n}\) is \(n\)th Fibonacci number for a sufficiently large value \(n\). This rational approximation of \(\beta\) fixes the lattice size \(N_s = F_{n}\) to impose periodic boundary condition [37].

Duality in the AA model can be shown by introducing new basis states in momentum space. This is an interesting aspect due to which the form of the Hamiltonian in dual momentum space remains the same (as given in equation (1)) but the coupling constant changes from \(\lambda\) to \(2/\lambda\) [7]. It is important to note that the AA model becomes self-dual at a critical coupling \(\lambda = 2\), where the localization transition occurs. To obtain the eigenvalues and eigenfunctions of the Hamiltonian we expand the state vector in terms of Wannier states, \(\ket{\psi} = \sum_{n} \psi_n \ket{n}\), where \(\psi_n\) is the wavefunction at \(n\)th lattice site. The eigenvalue equation of the Hamiltonian in equation (1) is reduced to a discrete Schrödinger equation

$$-\left(\psi_{n+1} + \psi_{n-1}\right) + \lambda \cos(2\pi/\beta n) \psi_n = \epsilon \psi_n,$$

where \(\epsilon\) is the energy eigenvalue. The degree of localization of a normalized state \(\ket{\psi}\) can be quantified by the IPR \(I\),

$$I = \sum_{n} \left|\psi_n\right|^4.$$

The wavefunction of an extremely localized particle at a site \(n_0\) is given by \(\psi_n = \delta_{n,n_0}\), for which the IPR becomes unity. On the other hand, for the completely delocalized wavefunction \(\psi_n = 1/\sqrt{N_s}\), the IPR is \(1/N_s\) which vanishes in the thermodynamic limit. For the AA model all energy eigenfunctions in real space are exponentially localized and the IPR sharply increases to unity above the critical coupling \(\lambda = 2\). Due to the duality of the AA model, the localization of wavefunction in real space and in dual momentum space shows opposite behavior. In real space the wavefunctions are localized above \(\lambda = 2\), whereas localization in dual momentum space occurs for \(\lambda \leq 2\) [37].

The spectrum of the AA model also shows various interesting properties. For \(\lambda = 0\) it has the usual single band energy spectrum of a periodic lattice. Addition of the potential generates a quasiperiodic structure and destroys the band like dispersion. Successive rational approximation of \(\beta\) generates more periodicity over the underlying lattice, which in turn breaks the original energy band into many subbands and leads to opening of band gaps. The energy spectrum of the AA model is obtained by numerical diagonalization and is depicted in
figure 1(a) for increasing values of $\lambda$. The variation of energy gaps with the coupling strength can be noticed in this figure. The spectral statistics and distribution of energy gaps are analyzed in [12]. Mathematically it can be shown that the energy spectrum of the AA model forms a Cantor set [38]. Self-similarity in energy levels can be understood from the integrated level density

$$N(\epsilon) = \sum_i \theta(\epsilon - \epsilon_i),$$

where $\epsilon_i$ is the $i$th eigenvalue and $\theta(x)$ is the Heaviside step function. The integrated density of states $N(\epsilon)$ shows the ‘Devil’s staircase’ like fractal structure, which is evident from figure 1(b), where we plotted the normalized integrated density of states $N(\epsilon)$ as a function of scaled energy within the interval of zero to one. For different values of $\lambda$, the Devil’s staircase structures of $N(\epsilon)$ do not overlap, which indicates that the fractal dimension changes with the strength of the potential $\lambda$. The localization transition in AA model can also be captured from the energy level spacing distribution. In the localized regime the normalized level spacing follows a Poisson distribution [39]. To obtain the spectral distribution directly, a large set of energy levels are required. To analyze the spectral distribution avoiding such numerical difficulty, we calculate a dimensionless quantity $r_i$ which is defined as [40, 41],

$$r_i = \frac{\min(\delta_{i+1}, \delta_i)}{\max(\delta_{i+1}, \delta_i)}$$

where $\delta_i = \epsilon_{i+1} - \epsilon_i$ is the spacing between the consecutive energy levels. For a Poisson distribution of level spacing, it can be shown that $\langle r_i \rangle = 2 \ln 2 - 1 \approx 0.386$. To obtain a good statistics we consider the disorder potential $\lambda \cos(2\pi/\beta n + \phi)$ with a random phase $\phi$ ranged from 0 to $\pi$. A typical behavior of $r_i$ averaged over 100 realizations on $\phi$ chosen uniformly between 0 to $\pi$ for each value of $\lambda$ is shown in figure 1(c). We note that $\langle r_i \rangle$ rises sharply around $\lambda = 2$ and attains the value $\langle r_i \rangle \approx 0.386$, which indicates that the critical coupling strength for this localization transition is $\lambda = 2$.

Transport properties also show significant changes in the localization transition. Due to spatial localization of single particle states transport coefficients vanish in the thermodynamic limit. In electronic systems the conductivity vanishes exponentially with the system size due to ‘Anderson localization’. For a neutral superfluid the corresponding physical quantity is SFF, which is measured by generating a superflow by applying a phase twist [42]. In the presence of the phase twist the Hamiltonian in equation (1) becomes

$$H_0 = -\sum_{n=1}^{N} (e^{-i\phi} |n\rangle \langle n + 1| + \text{h.c.}) + \lambda \sum_{n=1}^{N} \cos(2\pi/\beta n) |n\rangle \langle n|,$$

where $\Phi$ is an arbitrarily small phase difference across the boundary. For a one-dimensional system with a periodic boundary condition this is similar to a quantum ring in the presence of a flux which generates supercurrent through the ring. The SFF $f_i$ is defined as [43],

$$f_i = N_i \frac{E_0(\Theta) - E_0(0)}{\Theta^2},$$

where $E_0(\Theta)$ is the ground state energy of the Hamiltonian $H_0$ with an arbitrary small value of $\Theta$, and $E_0(0)$ is the ground state energy of the original Hamiltonian given in equation (1). To obtain the ground state energy up to $\Theta^2$ order, the Hamiltonian $H_0$ is expanded as
the usual kinetic energy and $(\lambda = \ldots)$ are eigenvalues and eigenfunctions respectively of Hamiltonian given in equation (8),

$$H_{00} \approx H + \frac{\Theta}{N_s} \hat{\epsilon} - \frac{\Theta^2}{2N_s^2} \hat{T},$$

where we define a current operator $\hat{\epsilon} = -i \sum_{n} (|n\rangle \langle n + 1| - \text{h.c.})$ and the usual kinetic energy $\hat{T} = -\sum_{n} (|n\rangle \langle n + 1| + \text{h.c.})$. Using second order perturbation, the SFF in equation (7) can be written as

$$f_s = -\frac{1}{2} \left| \langle \psi_0 | \hat{T} | \psi_0 \rangle + \sum_{i=0}^{\infty} \frac{\langle \psi_0 | \hat{\epsilon} | \psi_0 \rangle}{\epsilon_i - \epsilon_0} \right|^2,$$

where $\epsilon_i$ and $|\psi_i\rangle$ are eigenvalues and eigenfunctions respectively of Hamiltonian given in equation (1) and $|\psi_0\rangle$ is the ground state wavefunction. Variation of SFF with the coupling strength of the potential is shown in figure 2. For increasing values of $\lambda$ the SFF decreases from unity and vanishes at the critical point $\lambda = 2$.

### 2.2. Hamiltonian map

Localization phenomena can also be studied by an alternative method of CHM [9, 10], which is very useful to obtain an analytical estimate of the localization length [11]. The discrete Schrödinger equation given in equation (2) can be written as the following dynamical map

$$p_{n+1} = p_n + (\lambda \cos(2\pi/b) - \epsilon - 2)x_n,$$

$$x_{n+1} = x_n + p_{n+1},$$

where the classical dynamical variables are given by, $x_n = \psi_n$ and $p_n = \dot{\psi}_n = \psi_{n-1}$. Here the wavefunction $\psi_n$ plays the role of position in CHM and the number of lattice site becomes the number of iteration or time axis of the dynamics. We can choose initial real wavefunctions $\psi_0$ and $\psi_1$ (or equivalently $x_1$ and $p_1$) and evolve the dynamical variables by the transfer matrix

$$T_n = \begin{pmatrix} (\lambda \cos(2\pi/b) - \epsilon - 1) & 1 \\ (\lambda \cos(2\pi/b) - \epsilon - 2) & 1 \end{pmatrix}.$$

By iteration of this map we obtain the asymptotic behavior of the dynamical variables for a fixed energy eigenvalue $\epsilon$. The typical solution of equation (2) will be either $\psi \sim e^{\pm ibn}$ or $\psi \sim e^{i\beta n}$. The former solution clearly indicates the delocalized regime. On the other hand the physical solution $\psi \sim e^{-\beta n}$ corresponds to the localized state resulting in the chaotic dynamics of the phase space variables governed by equations (10) and (11). To understand the localization transition in the AA model, we choose an eigenvalue close to the band center and calculate the phase space trajectories of the Hamiltonian map for an ensemble of random initial values of $x$ and $p$. The phase portraits of equations (10) and (11) are shown in figure 3 for increasing values of $\lambda$. In the delocalized regime with small disorder strength $\lambda$, the phase portrait consists of closed elliptic orbits as seen from figure 3(a). Increasing the strength of quasiperiodic potential $\lambda$ leads to diffusive behavior at the outer part of the phase space and regular portion of phase space with periodic orbits is reduced (see figures 3(b) and (c)). Finally for $\lambda$ close to the critical value, an instability sets in the CHM and phase space trajectories in figure 3(d) show chaotic behavior which indicates localization transition.

In dynamical systems, the ‘Lyapunov exponent’ (LE) is a measure to quantify chaos. Since the number of lattice sites plays the role of time in the classical map, the LE corresponds to the inverse localization length of the wavefunction. For periodic motion the LE vanishes and in the chaotic regime the non-vanishing LE gives a finite localization length $\xi$. To calculate the LE we construct the matrix $U_N = T_N T_{N-1} \ldots T_2 T_1$ by multiplying the
transfer matrices $T_n$ sequentially for $N_s$ iterations. The LE $l$ can be obtained by using the formula

$$ l = \lim_{N_s \to \infty} \frac{\log (\lambda_{\max})}{N_s}, \quad (13) $$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $U_N$ and the localization length is obtained from $\xi = 1/l$. The localization length can also be calculated by using the Thouless formula [44]. Using the duality of the AA model the analytical expression of the LE is given by $\xi = \log (\lambda/2)$ [7], which is independent of energy. Within the CHM approach we numerically compute the localization length $\xi$ from equation (13) using the QR decomposition method [45]. The numerically obtained localization length as a function of $\lambda$ for different energy eigenvalues is compared with the analytical result in figure 4.

For the localization transition in the AA model with a quasiperiodic potential, the choice of $\beta$ as the irrational number (particularly a Diophantine number) plays a crucial role [36]. To understand this mathematical condition, we analyze the phase space of CHM for successive rational approximation of $\beta$, which is shown in figure 5. Using a Fibonacci series, the rational approximation of $\beta$ can be written as $\beta_n = F_{n-1}/F_n$ and the
potential has periodicity \( F_n \) for successive integers \( n \). For a sufficiently large value of \( n \), \( \beta_n \) approaches the inverse of the ‘golden mean’ and the potential becomes quasiperiodic. In the localized regime, for \( \lambda = 2.2 \) the phase space trajectories of CHM with increasing order of rational approximation of \( \beta \) are presented in figures 5(a)–(c) for fixed initial conditions. Even in the localized regime phase space contains periodic orbits for \( 5b \). As shown in figures 5(b)–(d), the periodic orbits break as \( \beta \) approaches the inverse of the ‘golden mean’ by successive rational approximation and finally the dynamics become chaotic which is consistent with the localization phenomena. In figure 5(d) we have taken different initial conditions chosen randomly between 0 and 1 to compare with figure 3(d) where the onset of localization has been shown by the chaotic phase space dynamics.

3. Localization of weakly interacting Bose gas

Interacting bosons in the presence of the quasiperiodic potential can be described by the Bose–Hubbard model [16],

\[
H = -\sum_{\langle ij \rangle} (a_i^\dagger a_j + \text{h.c.}) + \lambda \sum_i \cos(2\pi\beta_i)\hat{n}_i + \frac{U}{2} \sum_i (\hat{n}_i - 1),
\]

where \( a_i^\dagger(a_i) \) are creation (annihilation) operators for bosons at site \( i \), \( \hat{n}_i = a_i^\dagger a_i \), and \( U \) is the strength of the onsite repulsive interaction. The above quantum many-body Hamiltonian can capture various correlated phases of strongly interacting bosons which undergo quantum phase transitions [16]. It is known that the Bose–Hubbard model in the presence of random disorder can give rise to the ‘Bose glass’ phases [16]. The momentum distribution of the glassy phase of the AA-model also shows interesting features [46]. For sufficiently weak interaction strength \( U \) and for high average density of bosons a ‘quasi-condensate’ may form in one-dimensional systems [47]. In this regime, one can replace the bosonic operators \( a_i \) by a classical field \( \psi_i \) which represents the macroscopic wavefunction of the ‘quasi-condensate’. Minimization of the classical energy corresponding to equation (14) leads to the ‘discrete nonlinear Schrödinger equation’ (DNLS)

\[
-\left( \psi_{i+1} + \psi_{i-1} \right) + \lambda \cos(2\pi\beta_i)\psi_i + U \left| \psi_i \right|^2 \psi_i = \mu \psi_i,
\]

where \( \mu \) is the chemical potential, and the normalization of the wavefunction of \( N_b \) number of bosons gives \( \sum_i \left| \psi_i \right|^2 = N_b \).

We numerically calculate the ground state wavefunction of equation (15) by the imaginary time propagation method. For weak disorder, the wavefunction is extended and numerical convergence is fast, whereas close to the localization transition the ground state wavefunction needs to be calculated carefully since many metastable states appear in this region. To obtain the degree of localization of the ground state wavefunction in the presence of repulsion we calculate the IPR in real space using equation (3). The variation of the IPR of the condensate
wavefunction with increasing strength of disorder \(\lambda\) for different values of effective interaction \(UN_b\) is shown in figure 6(a). The IPR becomes nonzero for \(\lambda > 2\) and increases with a slower rate compared to the non-interacting system, indicating spreading of the wavefunction due to the repulsive interaction. The degree of localization decreases with increasing strength of the repulsive interaction \(UN_b\). In figure 6(b) the spatial variation of the wavefunction with disorder strength \(\lambda\) is represented by a color scale plot. It is evident from this figure that single site localization is not energetically favorable and the wavefunction is localized at almost degenerate but spatially separated sites. This particular feature of the fragmented condensate has also been studied in [48]. Due to the multi-site localization the IPR is much less than unity and increases very slowly with \(\lambda\). A change in the slope of the IPR with increasing \(\lambda\) occurs when the number of localized sites decreases and we have checked that finally at a very large value of \(\lambda\) the wavefunction becomes localized at a single site.

To study the interplay between disorder and interaction in the transport properties of dilute Bose gas, we calculate the SFF \(f_s\). To generate a superflow we introduce a small amount of phase twist \((0.1 \, \pi)\) in the hopping term of the DNLS (in equation (15)) similar to equation (6) and then the SFF is computed using the formula

\[
f_s = N_c^2 \frac{E_d(\Theta) - E_d(0)}{\Theta^2},
\]

where \(E_d(\Theta)\) is the classical energy corresponding to the Hamiltonian in equation (14),

\[
E_d(\Theta) = - \sum_{(ij)} \left[ e^{i\Theta/N_c} \phi_i^* \phi_j + \text{h.c.} \right] + \lambda \sum_i \cos(2\pi\beta) \left| \phi_i \right|^2 + \frac{UN_b}{2} \sum_i \left| \phi_i \right|^4.
\]

The wavefunction \(\phi_i\) minimizes \(E_d(\Theta)\) and is normalized to unity. The SFF \(f_s\) as a function of \(\lambda\) for different values of repulsive interactions \(UN_b\) is shown in figure 7(a). The SFF of weakly interacting Bose gas obtained from the ‘density matrix renormalization group’ also shows similar behavior [49]. Due to the repulsive interaction, \(f_s\) vanishes at larger strength of quasiperiodic potential \(\lambda > 2\), however the IPR rises from zero at
This behavior is different from the noninteracting AA model where the onset of localization and vanishing of superfluidity occur at the same point $\lambda = 2$.

From the above results we calculate the phase diagram of weakly interacting Bose gas which is depicted in figure 7(b). The first region represents the delocalized phase with non-vanishing superfluidity. The onset of localization of wavefunction as indicated by the IPR is represented by the phase boundary between I and II. Due to multi-site localization of the condensate wavefunction, the IPR grows in region II, although SFF does not vanish. The boundary of localized phase III is obtained from vanishing of SFF. For dilute Bose gas the localization of wavefunction starts around $\lambda \approx 2$, whereas the mass transport vanishes at larger strength of disorder.

We also investigated the localization properties of the DNLS by the Hamiltonian map approach [10, 11]. Equation (15) can be written in the form of a nonlinear classical map

$$P_{i+1} = p_i + (\lambda \cos(2\pi/\beta) - \mu - 2)x_i + UN_b x_i^3,$$

$$x_{i+1} = x_i + p_{i+1}$$

where, $x_i = \phi_i$ and $p_i = \dot{\phi}_i - \phi_{i-1}$. The repulsive nonlinear potential $\sim -\frac{UN_b}{4}x_i^4$ results in unstable phase space trajectories. To avoid this problem we choose a relatively small region of phase space within which the potential is metastable, and study the effect of disorder in the phase space dynamics. In figure 8, for a fixed value of nonlinearity $UN_b$ we show the phase portrait with an ensemble of initial configurations for increasing values of disorder strength $\lambda$. Similar qualitative features like a non-interacting system are also seen in the phase space dynamics of the DNLS, however the classical periodic orbits are modified due to the nonlinearity. The results obtained from CHM of an interacting system also support the fact that the localization of the wavefunction starts around $\lambda \approx 2$.

### 3.1. Fluctuations within Bogoliubov approximation

So far we have studied weakly interacting bosons within the mean-field approximation using a macroscopic wavefunction for the condensate. It is also important to analyze the quantum fluctuations induced by the quasiperiodic disorder potential. Within the Bogoliubov approximation [50], the quantum field operators can be approximated by

$$a_i = e^{-i\omega t}\left[\psi_i + \sum_{\nu} (u_{\nu}^i b_{\nu} e^{-i\omega_{\nu} t} + v_{\nu}^i b_{\nu}^{\dagger} e^{i\omega_{\nu} t}) \right],$$

where $\psi_i$ is the macroscopic wavefunction of the condensate and satisfies equation (15), $u_{\nu}^i, v_{\nu}^i$ are amplitudes corresponding to the $\nu$th eigenmode with bosonic operators $b_{\nu}, b_{\nu}^{\dagger}$. Bogoliubov quasiparticle energies $\omega_{\nu}$ can be obtained from
and the normalization condition gives
\[ \sum \psi_i^2 = 1. \]

For a given interaction strength \( U \) and average density of bosons \( n_b = N_b / N \), we first numerically obtain the ground state macroscopic wavefunction \( \psi_i \), then diagonalize equations (21) and (22) to calculate Bogoliubov quasiparticle energies and amplitudes \( u_i \), \( v_i \). The Bogoliubov energy spectrum for increasing strength of disorder \( \lambda \) is depicted in figure 9(a) for a fixed value of interaction strength \( U N_b = 1 \) and \( N = 144 \). A typical variation of quasiparticle amplitudes with the increasing disorder strength are shown in figures 9(c) and (d). We also study the localization of the Bogoliubov quasiparticles from the level spacing distribution of excitation energies. As explained in section 2, we compute the variation of the average value of the ratios of level spacing \( \langle \sigma_\lambda \rangle \) (see equation (5)) with \( \lambda \), which is shown in figure 9(b) for different values of interaction strengths. For higher interaction strengths, \( \langle \sigma_\lambda \rangle \) reaches the value \( \sim 0.386 \) with a slower rate indicating the Poisson statistics of the localized excitations.

Due to the quantum fluctuations, depletion of the condensate occurs and the non-condensate density \( \rho_{nc} \) at zero temperature can be obtained from the Bogoliubov theory

\[ \rho_{nc} = \sum \psi_i^2 \]

The condensate fraction is given by \( N_c / N_b = 1 - \sum \psi_i^2 \) \( / N_b \). In a one-dimensional system the noncondensate fraction diverges as \( \log(N_c) \), which prohibits the formation of condensate in the thermodynamic limit. However, for a sufficiently weak interaction a quasi condensate can form in a quasi one-dimensional and finite system [47]. To investigate the disorder induced quantum fluctuation in a quasi-condensate in a finite lattice with \( N_c = 144 \), we calculate the condensate fraction with increasing disorder strength \( \lambda \), which is shown in figure 10(a). In the absence of disorder the quantum depletion is small in weakly interacting gas of bosons and increases with the interaction strength. With the increase in disorder strength, \( \lambda \), the condensate fraction remains close to unity in the delocalized regime and then rapidly decreases around the critical point \( \lambda \approx 2 \). This qualitative feature (as shown in figure 10(a)) indicates that enhanced quantum fluctuations near the localization transition can destroy the quasi-condensate above \( \lambda \approx 2 \) and strongly correlated phases can appear. The
variation of non-condensate density $\rho_{nc}$ with disorder strength $\lambda$ is depicted in figure 10(c) by a color scale plot. It is interesting to note that the distinct feature of multi-site localization for $2l \gg 1$ is observed even for non-condensate density. Also the IPR of normalized non-condensate density shows similar behavior as that of the condensate wavefunction and increases from $2l \approx 1$ (as seen from figure 10(b)). Localization of Bogoliubov quasiparticles and enhancement of quantum fluctuations in the presence of a quasiperiodic potential particularly for $\lambda \gg 2$ are clearly evident from this analysis.

### 3.2. Effect of disorder on Bose gas with attractive interaction

In one-dimensional Bose gas the attractive interparticle interaction ($U < 0$) leads to the formation of a bright soliton [50]. A bright soliton is a self-bound object which can move with a constant velocity without dissipation and keeping its shape unchanged. The wavefunction of a bright soliton at rest is given by, $\psi(x) = N \text{sech}(x/b)$ where $N$ is the normalization constant and the parameter $b \sim 1/|U| N_b$ determines the size of the soliton [50]. Although the attractive condensate is localized within a length $\sim b$, a crossover to disorder induced localization can be studied for a sufficiently weak interaction strength $U$ (so that $b$ is much larger than the lattice spacing). We numerically obtain the ground state wavefunction from equation (15) with $U < 0$, and calculate the IPR and the width of the ground state wavefunction from equation (16) for $\lambda < 0$, and compute the SFF of the attractive Bose gas by using equation (16), which is also shown in the same figure 11. It is interesting to note that as a result of single site localization due to attractive interaction, the onset of localization of the wavefunction (as indicated by the IPR and width) and vanishing of the transport coefficient (SFF) occur at the same value of disorder strength. In contrast for a condensate with repulsive interaction the

![Figure 10](image1.png)

**Figure 10.** Variation of (a) condensate fraction $N_c/N_b$ and (b) inverse participation ratio $I$ of non-condensate density $\rho_{nc}$ with $\lambda$ for different interaction strengths $U N_b$ as shown in the figure. (c) Spatial distribution of $\rho_{nc}$ with increasing disorder strength $\lambda$ keeping $U N_b = 1$. For all the above figures, $N_b = 144$ and $N_c = 500$.

![Figure 11](image2.png)

**Figure 11.** Width of the ground state wavefunction $\sigma$ (red line) scaled by $\sigma$ for $\lambda = 0$, SFF (blue line) and the IPR (black line) are shown as a function of $\lambda$ for $U N_b = 0.2$. 

In free space the motion of a bright soliton is dissipationless and only quantum effects can give rise to small damping [51]. Whereas in this system with a fixed phase twist the average velocity of the self bound state decreases with increasing disorder strength. To quantify the transport property we compute the SFF of the attractive Bose gas by using equation (16), which is also shown in the same figure 11.
localization of the wavefunction starts around $\lambda \approx 2$ (as indicated by the IPR), however, the SFF vanishes at larger $\lambda$ due to the multi-site localization.

We also study the degree of localization from the Hamiltonian map with attractive nonlinearity. Numerically we iterate the nonlinear classical map given in equation (19) for $U < 0$. For a fixed value of the interaction strength $UN_0$, the change in the behavior of the phase space dynamics with the increasing disorder strength $\lambda$ is shown in figure 12.

Although the qualitative features of the phase space trajectories remain same compared to that of the non-interacting system, the chaos sets in much before $\lambda = 2$ indicating the exponential localization of the wavefunction. This also supports the fact that localization for the attractive interaction occurs at a smaller disorder strength as observed from the IPR and SFF.

4. Localization in the presence of a trap

Although the localization transition in the AA model occurs in the thermodynamic limit, in a real experimental setup a weak trapping potential is always present in order to confine the ultracold atoms. The main features of the localization can also be observed in a trapped system provided the length scale of the trapping potential is larger compared to the localization length. Additionally some interesting effects due to the trap can also be seen. A harmonic trap is introduced by adding a potential $V_i = \frac{1}{2}\omega_{10}^2(i - ic)^2$ in the Hamiltonian (equation (14)), where $\omega_{10}$ is related to the trapping frequency, $i$ is the site index, and $ic$ is the center of the trap.

Although in a trap the wavefunction is always localized, in a weak trapping potential the width of the wavepacket is sharply reduced due to disorder induced localization. The density profile of the condensate $D_{\text{HH}}$ in the presence of a harmonic trap for different disorder strengths $\lambda$ is shown in figure 13(b). We calculate the IPR of the ground state wavefunction with increasing disorder strength $\lambda$ and an increase of the IPR around $\lambda \approx 2$ is observed as expected. However the IPR does not vanish in the delocalized regime $\lambda < 2$ and takes a small value due to the tapping potential. We have noticed earlier that in the absence of a trap the IPR increases very slowly after the localization transition due to the repulsive interaction and the wavefunction is localized at spatially separated sites with quasi-degenerate energies. This quasi-degeneracy of onsite energies can be lifted by introducing a harmonic trap which leads to enhancement of the degree of localization which is elucidated in figure 13(a) where a rapid increase of the IPR to unity is shown by increasing the trap frequency by a small amount.

Collective oscillation of a Bose–Einstein condensate in a trap can also show its superfluid properties. A small displacement of the condensate from the center of the trap generates a COM oscillation, which is a well studied collective mode of a Bose–Einstein condensate. Here we numerically study the motion of the COM of a trapped condensate in the presence of quasi periodic disorder. Since the superfluidity is affected by the disorder, we
calculate the coherence factor $\Psi$ of the oscillating condensate which is given by [52]

$$\Psi = \sum_i \psi_i^\ast \psi_{i+1}. \quad (24)$$

We can notice that the above expression contains information of relative phase difference between neighboring sites and $|\Psi|^2$ gives a quantitative measure of the overall phase coherence of the oscillating condensate.

In figures 14 (a) and (b), we have shown the COM motion of the trapped condensate and coherence factor of corresponding macroscopic wavefunction for increasing disorder strength $\lambda$. It is clear that the coherence of the time dependent wavefunction decreases with increasing disorder. It is also interesting to study the variation of condensate fraction with disorder. In a one-dimensional trapped condensate the divergence of the non-condensate fraction is less severe due to the finiteness of the trap. Like the homogeneous system, we expect enhancement of quantum fluctuation due to disorder, which may destroy the condensate around a critical disorder strength $\lambda \approx 2$.

**5. Conclusions**

In conclusion, we have investigated localization of both non-interacting bosons as well as a weakly interacting quasi-condensate in the presence of a quasiperiodic potential. Apart from calculating various physical properties, understanding localization transition through the approach of CHM is one of the main results of this work.

In the non-interacting AA model the localization transition can be identified by the vanishing "SFF" and the rise of the IPR at the critical strength of quasiperiodic potential $\lambda = 2$. Manifestation of localization of eigensates has also been investigated from the spectral properties of the AA model. A clear signature of Poissonian level spacing distribution for localized states above $\lambda = 2$ has become evident from the average of ratios of consecutive level spacing as a function of $\lambda$. A CHM is constructed from the Schrödinger equation. The
phase space trajectories of the corresponding classical map show periodic orbits for small disorder strength. With the increasing disorder strength, chaotic behavior is observed at the outer region of the phase space, and finally near the critical value $\lambda = 2$ all periodic orbits are destroyed due to the onset of chaos indicating the localization transition. The choice of the parameter $\beta = (\sqrt{5} - 1)/2$ (inverse of the golden mean) as an irrational Diophantine number for the localization transition has been elucidated from the CHM approach. In the localized regime with $\lambda > 2$, the successive rational approximation of $\beta$ leads to the destruction of periodic orbits and finally the phase portrait becomes chaotic in nature.

To study the interplay between interaction and localization we calculated a phase diagram of weakly interacting Bose gas in the presence of a quasiperiodic potential within the mean-field approach. For $\lambda > 2$, multi-site localization of the condensate wavefunction occurs due to the repulsive interparticle interaction, which is manifested by a much slower rate of increase of the IPR. The wavefunction is localized at many spatially separated lattice sites with quasi-degenerate energies and the number of localized sites decreases with increasing disorder strength. Unlike the non-interacting case, the vanishing of SFF and localization of wavefunction as indicated by the IPR do not take place at the same strength of disorder $\lambda$ due to multi-site localization. In the CHM, the repulsive interaction gives rise to an unstable nonlinear potential due to which the stable region of phase space decreases. In the phase portrait the stable region containing the periodic orbits decreases with increasing $\lambda$ and finally the onset of chaos occurs at $\lambda \approx 2$, which signifies the localization of the wavefunction. Both from the IPR and CHM we observe that the onset of localization of the condensate wavefunction takes place around $\lambda \approx 2$, however, the transport coefficient SFF vanishes at a larger strength of disorder. Further, the Bogoliubov quasi-particle spectrum has been calculated numerically. We observe a similar multi-site localization phenomenon for quasi-particle amplitudes and non-condensate densities above the critical strength of disorder. The localization of quasi-particles is also seen from the statistical properties of level spacing of the Bogoliubov spectrum with increasing disorder strength. For larger repulsive interactions, the average of the ratio of consecutive level spacing $r_{ij}$ increases with a slower rate and saturates at the value $\sim 0.386$, which corresponds to the Poissonian level spacing distribution of the localized states. Disorder enhances the quantum fluctuations due to which the condensate fraction of a finite system of bosons decreases rapidly around $\lambda \approx 2$. This indicates the possible formation of a glassy phase and multi-site localized insulators.

Localization of Bose gas has also been investigated in the presence of a weak attractive interaction. Although attractive Bose gas in one-dimension can form a localized soliton-like state, for a sufficiently weak attraction (for a large size of the self-bound state) the crossover to the disorder induced localization can be observed from the IPR and the width of the ground state. As expected, the localization occurs at a smaller value of disorder strength due to the attractive interaction, which is also evident from the trajectories of CHM. Since the attractive interaction favors single site localization, the localization of the ground state wavefunction and the vanishing of the transport coefficient SFF take place at same value of disorder strength. This scenario can clearly be contrasted with the multi-site localization phenomenon induced by the repulsive interaction.

Finally, we also considered the effect of a trapping potential on the localization transition. In the localized regime, the number of sites over which the wavefunction is localized reduces due to the presence of a trap which has been shown from the rapid increase in the IPR by tuning the trap frequency. The COM motion of the condensate in a harmonic trap also shows the signature of localization. The COM oscillations become incoherent with increasing disorder. To summarize, the present study provides a clear picture of localization of non-interacting and weakly interacting Bose gas in the presence of quasiperiodic disorder and reveals various interesting features, which are interesting for both academic point of view as well for future experiments.

References

[1] Anderson P W 1958 Absence of diffusion in certain random lattices Phys. Rev. 109 1492
[2] Lee P A and Ramakrishnan T V 1985 Disordered electronic systems Rev. Mod. Phys. 57 287
[3] Roati G, D’Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 Anderson localization of a non-interacting Bose–Einstein condensate Nature 453 895
[4] Billy J, Josse V, Zuo Z, Bernard A, Hambrecht B, Lugan P, Clément D, Sanchez–Palencia L, Bouyer P and Aspect A 2008 Direct observation of Anderson localization of matter waves in a controlled disorder Nature 453 891–4
[5] Jendrzejewski F, Bernard A, Muller K, Cheinet P, Josse V, Piraud M, Pezze L, Sanchez–Palencia L, Aspect A and Bouyer P 2012 Three-dimensional localization of ultracold atoms in an optical disordered potential Nature 8 398–403
[6] Semeghini G, Landini M, Castillo P, Roy S, Spagnoli G, Trenkwalder A, Fattori M, Inguscio M and Modugno G 2015 Measurement of the mobility edge for 3D Anderson localization Nature 11 554–9
[7] Aubry S and André G 1980 Analyticity breaking and Anderson localization in incommensurate lattices Ann. Israel. Phys. Soc. 3 133
[8] Lahini Y, Pugatch R, Pozzi F, Sorel M, Morandotti R, Davidson N and Silberberg Y 2009 Observation of a localization transition in quasiperiodic photonic lattices Phys. Rev. Lett. 103 013901
[9] Izrailev F M, Krokhin A A and Makarov N M 2012 Anomalous localization in low-dimensional systems with correlated disorder Phys. Rep. 512 125
[10] Izrailev F M, Kottos T and Tsironis G 1995 Hamiltonian map approach to resonant states in paired correlated binary alloys Phys. Rev. B 52 3274
[11] Dossetti-Romero V, Izrailev F M and Krokhin A A 2004 Transport properties of 1D tight-binding disordered models: the Hamiltonian map approach Physica E 25 13–22
[12] Evangelou S N and Pichard J L 2000 Critical quantum chaos and the one-dimensional Harper model Phys. Rev. Lett. 84 1643
[13] Deissler Benjamin, Zaccanti M, Roati G, D’Errico C, Fattori M, Modugno M, Modugno G and Inguscio M 2010 Delocalization of a disordered bosonic system by repulsive interactions Nat. Phys. 6 354
[14] D’Errico C, Lucioni E, Tanzi L, Gori L, Roux G, Mc Culloch I P, Giamarchi T, Inguscio M and Modugno G 2014 Observation of a disordered bosonic insulator from weak to strong interactions Phys. Rev. Lett. 113 095301
[15] Tanzi L, Lucioni E, Chaudhuri S, Gori L, Kumar A, D’Errico C, Inguscio M and Modugno G 2013 Transport of a Bose gas in 1D disordered lattices at the fluid–insulator transition Phys. Rev. Lett. 111 115301
[16] Fisher M P A, Weichman P B, Grinstein G and Fisher D S 1989 Boson localization and the superfluid–insulator transition Phys. Rev. B 40 546
[17] Lugan P, Clément D, Bouyer P, Aspect A, Lewenstein M and Sanchez–Palencia L 2007 Ultracold Bose gases in 1D disorder: from Lifshits glass to Bose–Einstein condensate Phys. Rev. Lett. 98 170403
[18] Fallani L, Lye J E, Guarrera V, Fort C and Inguscio M 2007 Ultracold atoms in a disordered crystal of light: towards a Bose glass Phys. Rev. Lett. 98 130404
[19] Pasienski M, McKay D, White M and DeMarco B 2010 A disordered insulator in an optical lattice Nat. Phys. 6 677
[20] Roux G, Barthel T, Mc Culloch I P, Kollath C, Schollwöck U and Giamarchi T 2008 Quasiperiodic Bose–Hubbard model and localization in one-dimensional cold atomic gases Phys. Rev. A 78 023628
[21] Vidal J, Mouhanna D and Giamarchi T 2001 Interacting fermions in self-similar potentials Phys. Rev. B 65 014201
[22] Tezuka M and García–García A M 2012 Testing the universality of the many-body metal-insulator transition by time evolution of a disordered one-dimensional ultracold fermionic gas Phys. Rev. A 85 031602(R)
[23] Lye J E, Fallani L, Fort C, Guarrera V, Modugno M, Wiersma D S and Inguscio M 2007 Effect of interactions on the localization of a Bose–Einstein condensate in a quasiperiodic lattice Phys. Rev. Lett. 95 060403(R)
[24] Clem ent D, Varón A F, Hughart M, Retter J A, Bouyer P, Sanchez–Palencia L, Gargard D M, Shlyapnikov G V and Aspect A 2005 Suppression of transport of an interacting elongated Bose–Einstein condensate in a random potential Phys. Rev. Lett. 95 170409
[25] Lucioni E, Deissler B, Tanzi L, Roati G, Zaccanti M, Modugno M, Larcher M, Dalfovo F, Inguscio M and Modugno G 2011 Observation of a disordered interacting system Phys. Rev. Lett. 106 230403
[26] D’Errico C, Moratti M, Lucioni E, Tanzi L, Deissler B, Inguscio M, Modugno G, Plenio M B and Caruso F 2013 Quantum diffusion with disorder, noise and interaction New J. Phys. 15 045007
[27] Larcher M, Dalfovo F and Modugno M 2009 Effects of interaction on the diffusion of atomic matter waves in one-dimensional quasiperiodic potentials Phys. Rev. A 80 053606
[28] Larcher M, Lapi yeva T Y, Bodyfelt J D, Dalfovo F, Modugno M and Flach S 2012 Subdiffusion of nonlinear waves in quasiperiodic potentials New J. Phys. 14 103036
[29] Pal A and Huse D A 2010 Many-body localization phase transition Phys. Rev. B 82 174411
[30] Oganesyan V and Huse D A 2007 Localization of interacting fermions at high temperature Phys. Rev. B 75 155111
[31] Aleiner I, Altshuler B and Shlyapnikov G 2010 A finite-temperature phase transition for disordered weakly interacting bosons in one-dimensional Nat. Phys. 6 900
[32] Michal V P, Altshuler B L and Shlyapnikov G V 2014 Delocalization of weakly interacting bosons in a 1D quasiperiodic potential Phys. Rev. Lett. 113 045304
[33] Modugno M 2009 Exponential localization in one-dimensional quasi-periodic optical lattices New J. Phys. 11 033023
[34] Harper P G 1955 Single band motion of conduction electrons in a uniform magnetic field Proc. Phys. Soc. A 68 874
[35] Hofstädter D R 1976 Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields Phys. Rev. B 14 2259
[36] Jitomirskaya S 1999 Metal-insulator transition for the almost Mathieu operator Ann. Math. 150 1159
[37] Aubach C, Wobst A, Ingold G L, Hänggi P and Varga I 2004 Phase–space visualization of a metal–insulator transition New J. Phys. 6 70
[38] Avila A and Jitomirskaya S 2009 The ten martini problem Ann. Math. 170 303
[39] Shkolowskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 Statistics of spectra of disordered systems near the metal–insulator transition Phys. Rev. B 47 11487
[40] Jot er S, Oganesyan V, Refael G and Huse D 2013 Many-body localization in a quasiperiodic system Phys. Rev. B 87 134202
[41] Papanicolaou C, Bogomolov E, Giraud O and Roux G 2013 Distribution of the ratio of consecutive level spacings in random matrix ensembles Phys. Rev. Lett. 110 084101
[42] Fisher M E, Barber M N and Jasnow D 1973 Helicity modulus, superfluidity, and scaling in isotropic systems Phys. Rev. A 8 1111
[43] Roth R and Burnett K 2003 Phase diagram of bosonic atoms in two-color superlattices Phys. Rev. A 68 023604
[44] Thouless DJ 1983 Bandwidths for a quasiperiodic tight-binding model Phys. Rev. B 28 4272
[45] Geist K, Parlitz U and Lauterborn W 1990 Comparison of different methods for computing Lyapunov exponents Prog. Theor. Phys. 83 5
[46] Deng X, Citro R, Orignac E and Minguzzi A 2008 Phase diagram and momentum distribution of an interacting Bose gas in a bichromatic lattice Phys. Rev. A 78 013625
[47] Petrò D S, Gangardt D M and Shlyapnikov G V 2004 Low-dimensional trapped gases J. Phys. IV France 116 5
[48] Lellouch S and Sanchez–Palencia L 2014 Localization transition in weakly interacting Bose superfluids in one-dimensional quasiperiodic lattices Phys. Rev. A 90 061602(R)
[49] Deng X, Citro R, Minguzzi A and Orignac E 2009 Superfluidity and Anderson localisation for a weakly interacting Bose gas in a quasiperiodic potential Eur. Phys. J. B 68 435
[50] Pitaevskii Lev P and Stringari S 2003 Bose–Einstein Condensation (Oxford: Oxford University Press)
[51] Sinha S, Cherny A Yu, Kovrizhin D and Brand J 2006 Friction and diffusion of matter–wave bright solitons Phys. Rev. Lett. 96 030406
[52] Smerzi A, Trombettoni A, Kevekás G P and Bishop A R 2002 Dynamical superfluid–insulator transition in a chain of weakly coupled Bose–Einstein condensates Phys. Rev. Lett. 89 170402