A comparative study of ideals in fuzzy orders

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Abstract

This paper presents a comparative study of three kinds of ideals in fuzzy order theory: forward Cauchy ideals (generated by forward Cauchy nets), flat ideals and irreducible ideals, including their role in connecting fuzzy order with fuzzy topology.

Keywords: Fuzzy order, Fuzzy topology, forward Cauchy ideal, Flat ideal, Irreducible ideal, Class of fuzzy sets, Scott \(\mathcal{Q}\)-topology, Scott \(\mathcal{Q}\)-cotopology

1. Introduction

The notion of ideals (i.e., directed lower sets) in a partially ordered set is a primitive one in domain theory. Domains (continuous partially ordered sets) and Scott topology are both postulated in terms of ideals and their suprema. For a partially ordered set \(P\), let \(\text{Idl}(P)\) denote the set of ideals in \(P\) with the inclusion order, and \(y : P \to \text{Idl}(P)\) be the map that assigns each \(x \in P\) to the principal ideal \(\downarrow x\). Then \(P\) is directed complete if \(y\) has a left adjoint \(\text{sup} : \text{Idl}(P) \to P\) that sends each ideal to its supremum; \(P\) is a domain if it is directed complete and the left adjoint of \(y\) has a left adjoint. A Scott open set of \(P\) is an upper set \(U\) such that for each ideal \(I\) in \(P\), if the supremum of \(I\) is in \(U\) then \(I\) intersects with \(U\).

In order to establish a theory of fuzzy domains (or, quantitative domains), the first step is to find an appropriate notion of ideals for fuzzy orders (or, \(\mathcal{Q}\)-orders, where \(\mathcal{Q}\) is the truth-value quantale). The problem seems simple, but, it turns out to be a very intricate one because of the complication of the table of truth-values — the quantale \(\mathcal{Q}\). In fact, we have many choices when postulating this notion in the fuzzy setting. This paper presents a comparative study of three kinds of them: forward Cauchy ideals, flat ideals and irreducible ideals.

Before summarizing related attempts in the literature and explaining what we will do in this paper, we recall some equivalent reformulations of ideals in a partially ordered set. Let \(P\) be a partially ordered set. A net \(\{x_i\}\) in \(P\) is eventually monotone if there is some \(i\) such that \(x_j \leq x_k\) whenever \(i \leq j \leq k\). Let \(I\) be a non-empty lower set in \(P\). The following are equivalent:

- \(I\) is an ideal, that is, for any \(x, y\) in \(I\), there is some \(z\) in \(I\) such that \(x, y \leq z\).
- There exists an eventually monotone net \(\{x_i\}\) such that \(I = \bigcup_i \bigcap_{j \geq i} \downarrow x_j\).

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• *I* is flat in the sense that for any upper sets $G, H$ of $P$, if $I$ intersects with both $G$ and $H$, then $I$ intersects with $G \cap H$.

• *I* is irreducible in the sense that for any lower sets $B, C$ of $P$, if $I \subseteq B \cup C$ then either $I \subseteq B$ or $I \subseteq C$.

The net-approach is extended to fuzzy orders in [3, 39, 40], resulting in the notions of forward Cauchy net and Yoneda completeness (a.k.a lim inf completeness). Fuzzy lower sets generated by forward Cauchy nets are called ideals in [6, 7]. They will be called forward Cauchy ideals in this paper, in order to distinguish them from flat ideals and irreducible ideals. Yoneda completeness, as a version of quantitative directed completeness, has received wide attention in the study of fuzzy orders, including generalized metric spaces as a special case, see e.g. [3, 6, 7, 10, 11, 16, 17, 22, 23, 25, 26, 34, 40].

The extension of the flat-approach to the fuzzy setting originates in the work of Vickers [37] in the case the truth-value quantale is Lawvere’s quantale $([0, \infty]^{op}, +)$ (which is isomorphic to the unit interval endowed with the product $t$-norm). This approach results in the notions of flat ideal (called flat left module in [37]) and flat completeness of fuzzy orders. It is shown in [37] that for Lawvere’s quantale, flat completeness is equivalent to Yoneda completeness.

The recent paper [47] extends the irreducible-approach to the fuzzy setting in the study of sobriety of fuzzy cotopological spaces. This approach yields the notions of irreducible ideal and irreducible completeness of fuzzy orders.

Forward Cauchy ideals, flat ideals and irreducible ideals in a fuzzy ordered set are all natural generalizations of the notion of ideals in a partially ordered set; the resulting completeness for fuzzy orders are natural extensions of directed completeness in order theory. We note in passing that, from a category theory perspective, such completeness for fuzzy orders is an example of the theory of cocompleteness in enriched category theory with respect to a class of weights [1, 18, 19].

This paper aims to present a comparative study of forward Cauchy ideals, flat ideals and irreducible ideals, hence of the resulting completeness notions. Since all of them are intended to play the role of directed lower sets in fuzzy order theory, before comparing them with each other, we propose the following criteria for a class $\Phi$ of fuzzy sets (see Definition [3, 7]) that are meant for the role of ideals in fuzzy orders:

1. If the truth-value quantale $Q$ is the two-element Boolean algebra, then for each partially ordered set $A$, $\Phi(A)$ is the set of ideals in $A$. This is to require that $\Phi$ is a generalization of the class of ideals.

2. $\Phi$ is saturated. Saturatedness of $\Phi$ guarantees that for each $Q$-ordered set $A$, $\Phi(A)$ is the free $\Phi$-continuous $Q$-ordered set generated by $A$. So, for a saturated class $\Phi$ of fuzzy sets, there exist enough $\Phi$-continuous $Q$-ordered sets.

3. $\Phi$ generates a functor from the category of $Q$-ordered sets and $\Phi$-cocontinuous maps to that of $Q$-topological spaces and/or $Q$-cotopological spaces. This functor is expected to play the role of the functor in domain theory that sends each partially ordered set to its Scott topology. As in the classical case, such functors are of fundamental importance in the theory of fuzzy domains.

Besides the interrelationship between the class $\mathcal{W}$ of forward Cauchy ideals, the class $\mathcal{F}$ of flat ideals and the class $\mathcal{I}$ of irreducible ideals, their saturatedness and their connection to fuzzy topology will also be discussed in this paper.

The contents are arranged as follows.
Section 2 recalls some basic ideas that are needed in the subsequent sections.

Section 3 concerns the relationship among forward Cauchy ideals, flat ideals and irreducible ideals. The main results are: (i) All forward Cauchy ideals are irreducible if and only if the quantale $Q$ is dually meet continuous. (ii) All forward Cauchy ideals are flat if and only if the quantale $Q$ is meet continuous. (iii) For the quantale obtained by equipping $[0,1]$ with a left continuous t-norm $\&$, irreducible ideals coincide with forward Cauchy ideals. (iv) For a prelinear quantale, every irreducible ideal is flat. (v) For a quantale that satisfies the law of double negation, flat ideals coincide with irreducible ideals.

Section 4 proves that for every quantale, both the class of flat ideals and that of irreducible ideals are saturated. As for forward Cauchy ideals, it is shown in [8] that for a completely distributive value quantale (see [7, 8] for definition), the class of forward Cauchy ideals is saturated. The conclusion is extended in [25] to the case that $Q$ is a continuous and integral quantale.

Section 5 concerns the connection between fuzzy orders and fuzzy topological spaces. For each subclass $\Phi$ of irreducible ideals, a full and faithful functor is constructed from the category of $Q$-ordered sets and $\Phi$-cocontinuous maps to that of stratified $Q$-cotopological spaces. For each subclass $\Phi$ of flat ideals, a functor is constructed from the category of $Q$-ordered sets and $\Phi$-cocontinuous maps to that of stratified $Q$-topological spaces. This shows that irreducible ideals are related to closed fuzzy sets ($Q$-cotopology), whereas flat ideals are related to open fuzzy sets ($Q$-topology). We would like to remind the reader that in general there is no natural way to switch between closed fuzzy sets and open fuzzy sets. This lack of “duality” between closed sets and open sets demands that we need different kinds of fuzzy ideals to connect fuzzy orders with fuzzy topological spaces and/or fuzzy cotopological spaces. This is the raison d’être for flat ideals and irreducible ideals.

2. Preliminaries

In this preliminary section, we recall briefly some basic ideas of complete lattices [9], quantales [33], and $Q$-orders that will be needed.

A quantale $Q$ is a monoid in the monoidal category of complete lattices and join-preserving maps [33]. Explicitly, a quantale $Q$ is a monoid $(Q, \&)$ such that $Q$ is a complete lattice and

$$p \& \bigvee_{j \in J} q_j = \bigvee_{j \in J} p \& q_j, \quad \left( \bigvee_{j \in J} q_j \right) \& p = \bigvee_{j \in J} q_j \& p,$$

for all $p \in Q$ and $\{q_j\}_{j \in J} \subseteq Q$. The unit 1 of the monoid $(Q, \&)$ is in general not the top element of $Q$. If it happens that the unit element coincides with the top element of $Q$, then we say that $Q$ is integral. If the operation $\&$ is commutative then we say $Q$ is a commutative quantale. A quantale $(Q, \&)$ is meet continuous if the underlying lattice $Q$ is meet continuous.

Standing Assumption. Throughout this paper, if not otherwise specified, all quantales are assumed to be integral and commutative.

Since the semigroup operation $\&$ distributes over arbitrary joins, it determines a binary operation $\rightarrow$ on $Q$ via the adjoint property

$$p \& q \leq r \iff q \leq p \rightarrow r.$$

The binary operation $\rightarrow$ is called the implication, or the residuation, corresponding to $\&$. 
Some basic properties of the binary operations & and \( \rightarrow \) are collected below, they can be found in many places, e.g. [2, 33].

**Proposition 2.1.** Let \( Q \) be a quantale. Then

1. \( 1 \rightarrow p = p \).
2. \( p \leq q \iff 1 = p \rightarrow q \).
3. \( p \rightarrow (q \rightarrow r) = (p \& q) \rightarrow r \).
4. \( p \& (p \rightarrow q) \leq q \).
5. \( \bigvee_{j \in J} p_j \rightarrow q = \bigwedge_{j \in J} (p_j \rightarrow q) \).
6. \( p \rightarrow (\bigwedge_{j \in J} q_j) = \bigvee_{j \in J} (p \rightarrow q_j) \).
7. \( p = \bigwedge_{q \in Q} ((p \rightarrow q) \rightarrow q) \).

We often write \( \neg p \) for \( p \rightarrow 0 \) and call it the *negation* of \( p \). Though it is true that \( p \leq \neg \neg p \) for all \( p \in Q \), the inequality \( \neg \neg p \leq p \) does not always hold. A quantale \( Q \) satisfies the *law of double negation* if

\[
(p \rightarrow 0) \rightarrow 0 = p
\]

for all \( p \in Q \).

**Proposition 2.2.** (2) Suppose that \( Q \) is a quantale that satisfies the law of double negation. Then

1. \( p \rightarrow q = \neg (p \& \neg q) = \neg q \rightarrow \neg p \).
2. \( p \& q = \neg (q \rightarrow \neg p) = \neg (p \rightarrow \neg q) \).
3. \( \neg (\bigwedge_{i \in I} p_i) = \bigvee_{i \in I} \neg p_i \).

The quantales with the unit interval \([0, 1]\) as underlying lattice are of particular interest in fuzzy set theory [12, 20]. In this case, the semigroup operation & is exactly a left continuous t-norm on \([0, 1]\) [20]. A continuous t-norm [20] on \([0, 1]\) is a left continuous t-norm & on \([0, 1]\) that is continuous with respect to the usual topology.

**Example 2.3.** (20) Some basic t-norms:

1. The t-norm min: \( a \& b = a \wedge b = \min\{a, b\} \). The corresponding implication is given by

\[
a \rightarrow b = \begin{cases} 1, & a \leq b; \\ b, & a > b. \end{cases}
\]

2. The product t-norm: \( a \& b = a \cdot b \). The corresponding implication is given by

\[
a \rightarrow b = \begin{cases} 1, & a \leq b; \\ b/a, & a > b. \end{cases}
\]

3. The Lukasiewicz t-norm: \( a \& b = \max\{a + b - 1, 0\} \). The corresponding implication is given by

\[
a \rightarrow b = \min\{1, 1 - a + b\}.
\]

In this case, \(([0, 1], \&)\) satisfies the law of double negation.
(4) The nilpotent minimum t-norm:

\[ a \& b = \begin{cases} 
0, & a + b \leq 1; \\
\min\{a, b\}, & a + b > 1.
\end{cases} \]

The corresponding implication is given by

\[ a \rightarrow b = \begin{cases} 
1, & a \leq b; \\
\max\{1 - a, b\}, & a > b.
\end{cases} \]

In this case, \([0, 1], \&\) satisfies the law of double negation.

The following theorem, known as the ordinal sum decomposition theorem, is of fundamental importance in the theory of continuous t-norms.

**Theorem 2.4.** ([5, 31]) For each continuous t-norm \& on \([0, 1]\), there is a set of disjoint open intervals \(\{(a_i, b_i)\}\) of \([0, 1]\) that satisfy the following conditions:

(i) For each \(i\), both \(a_i\) and \(b_i\) are idempotent and the restriction of \& on \([a_i, b_i]\) is either isomorphic to the Lukasiewicz t-norm or to the product t-norm;

(ii) \(x \& y = \min\{x, y\}\) if \((x, y) \notin \bigcup[a_i, b_i]^2\).

A \(\mathcal{Q}\)-order (or an order valued in the quantale \(\mathcal{Q}\)) \([39, 45]\) on a set \(A\) is a reflexive and transitive \(\mathcal{Q}\)-relation on \(A\). Explicitly, a \(\mathcal{Q}\)-order on \(A\) is a map \(R: A \times A \rightarrow Q\) such that \(R(x, x) = 1\) and \(R(y, z) \& R(x, y) \leq R(x, z)\) for any \(x, y, z \in A\). The pair \((A, R)\) is called a \(\mathcal{Q}\)-ordered set. A \(\mathcal{Q}\)-ordered set is also called a \(\mathcal{Q}\)-category in the literature, since it is precisely a category enriched over the symmetric monoidal category \(\mathcal{Q}\). As usual, we write \(A\) for the pair \((A, R)\) and \(A(x, y)\) for \(R(x, y)\) if no confusion would arise.

Two elements \(x, y\) in a \(\mathcal{Q}\)-ordered set \(A\) are isomorphic if \(A(x, y) = A(y, x) = 1\). We say that \(A\) is separated if isomorphic elements in \(A\) are equal, that is, \(A(x, y) = A(y, x) = 1\) implies that \(x = y\).

If \(R: A \times A \rightarrow Q\) is a \(\mathcal{Q}\)-order on \(A\), then \(R^{\text{op}}: A \times A \rightarrow Q\), given by \(R^{\text{op}}(x, y) = R(y, x)\), is also a \(\mathcal{Q}\)-order on \(A\) (by commutativity of \&), called the opposite of \(R\).

**Example 2.5.** This example belongs to the folklore in fuzzy order theory. For all \(p, q \in Q\), let

\[ d_L(p, q) = p \rightarrow q. \]

Then \((Q, d_L)\) is a separated \(\mathcal{Q}\)-ordered set. The opposite of \((Q, d_L)\) is given by \((Q, d_R)\), where

\[ d_R(p, q) = q \rightarrow p. \]

Both \((Q, d_L)\) and \((Q, d_R)\) play important roles in the theory of \(\mathcal{Q}\)-ordered sets.

**Example 2.6.** [2] Let \(X\) be a set. A map \(\lambda: X \rightarrow Q\) is called a fuzzy set (valued in \(Q\)) of \(X\), the value \(\lambda(x)\) is interpreted as the membership degree of \(x\). The map

\[ \text{sub}_X: Q^X \times Q^X \rightarrow Q, \]

given by

\[ \text{sub}_X(\lambda, \mu) = \bigwedge_{x \in X} \lambda(x) \rightarrow \mu(x), \]
defines a separated \( Q \)-order on \( Q^X \). Intuitively, the value \( \text{sub}_X(\lambda, \mu) \) measures the degree that \( \lambda \) is a subset of \( \mu \). Thus, \( \text{sub}_X \) is called the fuzzy inclusion order on \( Q^X \). The opposite of \( \text{sub}_X \) is called the converse fuzzy inclusion order on \( Q^X \). In particular, if \( X \) is a singleton set then the \( Q \)-ordered sets \((Q^X, \text{sub}_X)\) and \((Q^X, \text{sub}^{op}_X)\) degenerate to the \( Q \)-ordered sets \((Q, d_L)\) and \((Q, d_R)\), respectively.

A map \( f : A \rightarrow B \) between \( Q \)-ordered sets is \( Q \)-order-preserving if

\[
A(x_1, x_2) \leq B(f(x_1), f(x_2))
\]

for any \( x_1, x_2 \in A \). We write

\[
Q-\text{Ord}
\]

for the category of \( Q \)-ordered sets and \( Q \)-order preserving maps.

Let \( f : A \rightarrow B \) and \( g : B \rightarrow A \) be \( Q \)-order-preserving maps. We say \( f \) is left adjoint to \( g \) (or, \( g \) is right adjoint to \( f \)), \( f \dashv g \) in symbols, if

\[
A(x, g(y)) = B(f(x), y)
\]

for all \( x \in A \) and \( y \in B \).

Let \( A, B \) be \( Q \)-ordered sets. A \( Q \)-distributor \( \phi : A \rightarrow B \) from \( A \) to \( B \) is a map \( \phi : A \times B \rightarrow Q \) such that

\[
B(b, b') \land \phi(a, b) \land A(a', a) \leq \phi(a', b')
\]

for any \( a, a' \in A \) and \( b, b' \in B \). It is clear that the set \( Q-\text{Dist}(A, B) \) of all \( Q \)-distributors from \( A \) to \( B \) form a complete lattice under the pointwise order.

**Example 2.7.** (Fuzzy lower sets as \( Q \)-distributors) A fuzzy lower set \([24]\) of a \( Q \)-ordered set \( A \) is a map \( \phi : A \rightarrow Q \) such that

\[
\phi(y) \land A(x, y) \leq \phi(x).
\]

It is obvious that \( \phi : A \rightarrow Q \) is a fuzzy lower set if and only if \( \phi : A \rightarrow (Q, d_R) \) preserves \( Q \)-order. If we write \( * \) for the terminal object in the category \( Q-\text{Ord} \), that is, \( * \) is a \( Q \)-ordered set with only one element, then for each fuzzy lower set \( \phi : A \rightarrow Q \) of \( A \), the map

\[
\phi^\land : A \times \{*\} \rightarrow Q, \quad \phi^\land(x, *) = \phi(x)
\]

is a \( Q \)-distributor \( \phi^\land : A \rightarrow * \). This process establishes a bijection between fuzzy lowers set of \( A \) and \( Q \)-distributors from \( A \) to \( * \). In category theory, a \( Q \)-distributor of the form \( A \rightarrow * \) are called weights or presheaves of \( A \) \([19, 35]\).

Dually, a fuzzy upper set \([24]\) of \( A \) is a map \( \psi : A \rightarrow Q \) such that

\[
A(x, y) \land \psi(x) \leq \psi(y);
\]

or equivalently, \( \psi : A \rightarrow (Q, d_L) \) preserves \( Q \)-order. For each fuzzy upper set \( \psi \) of \( A \), the map

\[
\psi^\lor : \{*\} \times A \rightarrow Q, \quad \psi^\lor(*, x) = \psi(x)
\]

is a \( Q \)-distributor \( \psi : * \rightarrow A \). Such a \( Q \)-distributor is called a co-weight or a co-presheaf of \( A \) in category theory.
Lemma 2.8. Let $\phi$ be a fuzzy lower set (fuzzy upper set, resp.) of a $Q$-ordered set $A$, $p \in Q$.

1. Both $p \& \phi$ and $p \rightarrow \phi$ are fuzzy lower sets (fuzzy upper sets, resp.) of $A$.
2. $\phi \rightarrow p$ is a fuzzy upper set (fuzzy lower set, resp.) of $A$ and $\phi = \bigwedge_{q \in Q}(\phi \rightarrow q) \rightarrow q$.

Let $\mathcal{P}A$ denote the set of fuzzy lower sets of $A$ endowed with the fuzzy inclusion order. Explicitly, elements in $\mathcal{P}A$ are $Q$-order-preserving maps $A \rightarrow (Q, d_R)$, and

$$\mathcal{P}A(\phi_1, \phi_2) = \text{sub}_A(\phi_1, \phi_2) = \bigwedge_{x \in A} (\phi_1(x) \rightarrow \phi_2(x)).$$

Dually, let $\mathcal{P}^\dagger A$ denote the set of fuzzy upper sets of $A$ endowed with the converse fuzzy inclusion order. Explicitly, elements in $\mathcal{P}^\dagger A$ are $Q$-order-preserving maps $A \rightarrow (Q, d_L)$, and

$$\mathcal{P}^\dagger A(\psi_1, \psi_2) = \text{sub}_A(\psi_2, \psi_1) = \bigwedge_{x \in A} (\psi_2(x) \rightarrow \psi_1(x)).$$

It is clear that $(\mathcal{P}^\dagger A)^{\text{op}} = \mathcal{P}(A^{\text{op}})$.

For each $a \in A$, $A(\cdot, a)$ is a fuzzy lower set of $A$. Moreover,

$$\mathcal{P}A(A(\cdot, a), \phi) = \phi(a)$$

for all $a \in A$ and $\phi \in \mathcal{P}A$. This fact is a special case of the Yoneda lemma in enriched category theory. The Yoneda lemma ensures that the assignment $a \mapsto A(\cdot, a)$ defines an embedding $y : A \rightarrow \mathcal{P}A$, which is known as the Yoneda embedding.

The correspondence $A \mapsto \mathcal{P}A$ gives rise to a functor $\mathcal{P} : Q\text{-Ord} \rightarrow Q\text{-Ord}$ that sends a $Q$-order-preserving map $f : A \rightarrow B$ to $\mathcal{P} f = f^\rightarrow : \mathcal{P}A \rightarrow \mathcal{P}B$, where

$$f^\rightarrow(\phi)(y) = \bigvee_{x \in A} \lambda(x) \& B(y, f(x)).$$

Moreover, $f^\rightarrow : \mathcal{P}A \rightarrow \mathcal{P}B$ has a right adjoint given by $f^\leftarrow : \mathcal{P}B \rightarrow \mathcal{P}A$, where $f^\leftarrow(\psi) = \psi \circ f$. This means for all $\phi \in \mathcal{P}A$ and $\psi \in \mathcal{P}B$,

$$\text{sub}_B(f^\rightarrow(\phi), \psi) = \text{sub}_A(\phi, f^\leftarrow(\psi)).$$

The adjunction $f^\rightarrow \dashv f^\leftarrow$ is a special case of the enriched Kan extension in category theory.

For $Q$-distributors $\phi : A \leftrightarrow B$ and $\psi : B \leftrightarrow C$, the composite $\psi \circ \phi : A \leftrightarrow C$ is given by

$$(\psi \circ \phi)(a, c) = \bigvee_{b \in B} \psi(b, c) \& \phi(a, b).$$

It is clear that $(Q\text{-Dist}(\cdot, \cdot), \circ)$ is a quantale and is isomorphic to $Q = (Q, \&)$. So, we identify $(Q\text{-Dist}(\cdot, \cdot), \circ)$ with $Q$ in this paper.

For a fuzzy lower set $\phi : A \rightarrow Q$ and a fuzzy upper set $\psi : A \rightarrow Q$ of a $Q$-ordered set $A$, let

$$\phi \otimes \psi$$

be the composite of $Q$-distributors

$$\phi \uparrow \circ \uparrow \psi : \cdot \leftrightarrow A \leftrightarrow \cdot.$$
That means, $\phi \otimes \psi$ is an element of the quantale $Q$ given by $\phi \otimes \psi = \bigvee_{x \in A} \phi(x) \& \psi(x)$. Intuitively, the value $\phi \otimes \psi$ measures the degree that the fuzzy lower set $\phi$ intersects with the fuzzy upper set $\psi$. The correspondence

$$(\psi, \phi) \mapsto \phi \otimes \psi$$

defines a $Q$-distributor

$$\otimes : \mathcal{P}^\dagger A \to \mathcal{P}A.$$  

In particular, for each fuzzy upper set $\psi$ of $A$, the correspondence $\phi \mapsto \phi \otimes \psi$ defines a fuzzy upper set of $\mathcal{P}A$:

$$- \otimes \psi : \mathcal{P}A \to Q.$$  

The following lemma exhibits a close relationship between the $Q$-distributor $\otimes : \mathcal{P}^\dagger A \to \mathcal{P}A$ (intersection degree) and the fuzzy inclusion order (subset degree).

**Lemma 2.9.** Let $A$ be a $Q$-ordered set.

1. For each fuzzy lower set $\phi$ and each fuzzy upper set $\psi$ of $A$,

$$\phi \otimes \psi = \bigwedge_{a \in Q} (\text{sub}_A(\phi, \psi \to a) \to a).$$

In particular, if $Q$ satisfies the law of double negation, then $\phi \otimes \psi = \neg(\text{sub}_A(\phi, -\psi)).$

2. For any fuzzy lower sets $\phi_1, \phi_2$ of $A$,

$$\text{sub}_A(\phi_1, \phi_2) = \bigwedge_{a \in Q} (\phi_1 \otimes (\phi_2 \to a) \to a).$$

In particular, if $Q$ satisfies the law of double negation, then $\text{sub}_A(\phi_1, \phi_2) = \neg(\phi_1 \otimes (\neg \phi_2)).$

**Proof.** (1) By Proposition 2.1(7), it holds that

$$\phi \otimes \psi = \bigvee_{x \in A} \phi(x) \& \psi(x)$$

$$= \bigwedge_{a \in Q} \left[ \left( \bigvee_{x \in A} \phi(x) \& \psi(x) \right) \to a \right]$$

$$= \bigwedge_{a \in Q} \left[ \bigwedge_{x \in A} \left( \phi(x) \to (\psi(x) \to a) \right) \to a \right]$$

$$= \bigwedge_{a \in Q} (\text{sub}_A(\phi, \psi \to a) \to a).$$

(2) The proof is similar, so, we omit it here.\[ Q.E.D. \]

A supremum of a fuzzy lower set $\phi$ of a $Q$-ordered set $A$ is an element of $A$, say $\text{sup} \phi$, such that

$$A(\text{sup} \phi, x) = \text{sub}_A(\phi, y(x))$$

for all $x \in A$. It is clear that, up to isomorphism, every fuzzy lower set has at most one supremum. So, we’ll speak of the supremum of a fuzzy lower set. A $Q$-order-preserving map $f : A \to B$ preserves the supremum of a fuzzy lower set $\phi$ of $A$ if, whenever $\text{sup} \phi$ exists, $f(\text{sup} \phi)$ is a supremum of $f^{-1}(\phi)$. It is well-known that left adjoints preserve suprema.
Example 2.10. Let $A$ be a $Q$-ordered set. Then every fuzzy lower set of $\mathcal{P}A$ has a supremum. Actually, for each fuzzy lower set $\Lambda$ of $\mathcal{P}A$, $\sup \Lambda = \bigvee_{\phi \in \mathcal{P}A} \Lambda(\phi) \land \phi$.

Example 2.11. (Intersection degree as supremum) For each fuzzy lower set $\phi$ and each fuzzy upper set $\psi$ of a $Q$-ordered set $A$, the intersection degree of $\phi$ with $\psi$ is the supremum of $\psi^{-}(\phi)$ in $(Q, d_{L})$ (recall that $\psi : A \rightarrow (Q, d_{L})$ is a $Q$-order-preserving map), i.e., $\phi \circ \psi = \sup \psi^{-}(\phi)$. This is because for all $q \in Q$,

$$
\text{sub}_{Q}(\psi^{-}(\phi), d_{L}(-, q)) = \text{sub}_{A}(\phi, d_{L}(\psi(-), q)) \\
= \bigwedge_{x \in A} (\phi(x) \rightarrow (\psi(x) \rightarrow q)) \\
= d_{L} \left( \bigvee_{x \in A} \phi(x) \land \psi(x), q \right) \\
= d_{L}(\phi \otimes \psi, q).
$$

In particular, letting $\psi$ be the identity map on $(Q, d_{L})$ one obtains that for each fuzzy lower set $\phi$ of $(Q, d_{L})$, $\sup \phi = \bigvee_{q \in Q} q \land \phi(q)$.

Example 2.12. (Inclusion degree as supremum) For any fuzzy lower sets $\phi, \psi$ of a $Q$-ordered set $A$, the inclusion degree $\text{sub}_{A}(\phi, \psi)$ is the supremum of $\psi^{-}(\phi)$ in $(Q, d_{R})$ (recall that $\psi : A \rightarrow (Q, d_{R})$ is a $Q$-order-preserving map), i.e., $\text{sub}_{A}(\phi, \psi) = \sup \psi^{-}(\phi)$. This is because for all $q \in Q$,

$$
\text{sub}_{Q}(\psi^{-}(\phi), d_{R}(-, q)) = \text{sub}_{A}(\phi, d_{R}(\psi(-), q)) \\
= \bigwedge_{x \in A} (\phi(x) \rightarrow (q \rightarrow \psi(x))) \\
= \bigwedge_{x \in A} (q \rightarrow (\phi(x) \rightarrow \psi(x))) \\
= d_{R}(\text{sub}_{A}(\phi, \psi), q).
$$

In particular, letting $\psi$ be the identity map on $(Q, d_{R})$ one obtains that for each fuzzy lower set $\phi$ of $(Q, d_{R})$, the supremum of $\phi$ in $(Q, d_{R})$ is given by $\bigwedge_{q \in Q} (\phi(q) \rightarrow q)$.

3. Forward Cauchy ideals, flat ideals and irreducible ideals

A net $\{x_{i}\}$ in a $Q$-ordered set $A$ is forward Cauchy if

$$
\bigvee_{i} \bigwedge_{i \leq j \leq k} A(x_{j}, x_{k}) = 1.
$$

Forward Cauchy nets are clearly a $Q$-analogue of eventually monotone nets in partially ordered sets. A Yoneda limit (a.k.a. liminf) of a forward Cauchy net $\{x_{i}\}$ in $A$ is an element $a$ in $A$ such that

$$
A(a, y) = \bigvee_{i} \bigwedge_{i \leq j} A(x_{j}, y)
$$

for all $y \in A$. It is clear that Yoneda limit is a $Q$-version of least eventual upper bound. Yoneda limits of a forward Cauchy net, if exists, are unique up to isomorphism.
Lemma 3.1. If \( \{a_i\} \) is a forward Cauchy net in \((Q, d_L)\), then \( \bigvee_i \bigwedge_{j \geq i} a_j \) is a Yoneda limit of \( \{a_i\} \) and
\[
\bigvee_i \bigwedge_{j \geq i} a_j = \bigwedge_i \bigvee_{j \geq i} a_j.
\]

Proof. The first half is Proposition 2.30 in [40]. It remains to check the equality
\[
\bigvee_i \bigwedge_{j \geq i} a_j = \bigwedge_i \bigvee_{j \geq i} a_j.
\]
Since \( \bigvee_i \bigwedge_{j \geq i} a_j \) is a Yoneda limit of \( \{a_i\} \), it follows that for all \( x \in Q \),
\[
(\bigvee_i \bigwedge_{j \geq i} a_j) \to x = d_L \left( \bigvee_i \bigwedge_{j \geq i} a_j, x \right)
= \bigwedge_i d_L(x, a_j)
= \bigwedge_i (a_j \to x)
\leq \left( \bigwedge_i \bigvee_{j \geq i} a_j \right) \to x.
\]
Letting \( x = \bigvee_i \bigwedge_{j \geq i} a_j \) we obtain that
\[
\bigvee_i \bigwedge_{j \geq i} a_j \geq \bigwedge_i \bigvee_{j \geq i} a_j.
\]
The inequality ‘\( \leq \)’ is trivial, so, the equality is valid. \( \square \)

Proposition 3.2. ([40], Theorem 3.1) For each forward Cauchy net \( \{\phi_i\} \) in \( PA \), the fuzzy lower set \( \bigvee_i \bigwedge_{j \geq i} \phi_j \) is a Yoneda limit of \( \{\phi_i\} \). That is, for each fuzzy lower set \( \phi \) of \( A \),
\[
\operatorname{sub}_A \left( \bigvee_i \bigwedge_{j \geq i} \phi_j, \phi \right) = \bigwedge_i \bigvee \operatorname{sub}_A (\phi_j, \phi).
\]

The following proposition says that every Yoneda limit of forward Cauchy net \( \{x_i\} \) is a supremum of a fuzzy lower set generated by \( \{x_i\} \).

Proposition 3.3. ([8], Lemma 46) An element \( a \) in a \( Q \)-ordered set \( A \) is a Yoneda limit of a forward Cauchy net \( \{x_i\} \) if and only if \( a \) is a supremum of the fuzzy lower set \( \bigvee_i \bigwedge_{j \leq i} A(-, x_j) \) generated by \( \{x_i\} \).

A fuzzy set \( \lambda : A \to Q \) is inhabited [27] if \( \bigvee_{a \in A} \lambda(a) = 1 \). Inhabited fuzzy sets are a fuzzy version of non-empty sets.

Definition 3.4. Let \( A \) be a \( Q \)-ordered set, \( \phi : A \to Q \) a fuzzy lower set of \( A \).

(1) \( \phi \) is a forward Cauchy ideal if there exists a forward Cauchy net \( \{x_i\} \) in \( A \) such that
\[
\phi = \bigvee_i \bigwedge_{i \leq j} A(-, x_j).
\]
\( \phi \) is a flat ideal if it is inhabited and is flat in the sense that
\[
\phi \otimes (\psi_1 \land \psi_2) = \phi \otimes \psi_1 \land \phi \otimes \psi_2
\]
for all fuzzy upper sets \( \psi_1, \psi_2 \) of \( A \).

(3) \( \phi \) is an irreducible ideal if it is inhabited and is irreducible in the sense that
\[
\operatorname{sub}_A(\phi, \phi_1 \lor \phi_2) = \operatorname{sub}_A(\phi, \phi_1) \lor \operatorname{sub}_A(\phi, \phi_2)
\]
for all fuzzy lower sets \( \phi_1, \phi_2 \) of \( A \).

**Remark 3.5.** Forward Cauchy ideals, flat ideals and irreducible ideals in a \( \mathcal{Q} \)-ordered set are all natural extensions of ideals in a partially ordered set.

The study of forward Cauchy ideals dates back to Wagner \([39, 40]\). For more information on forward Cauchy ideals the reader is referred to \([6, 7, 8, 16, 17, 23, 25, 48]\), besides the works of Wagner.

The notion of flat ideals originates in the paper \([37]\) of Vickers in the case that \( \mathcal{Q} \) is Lawvere’s quantale \((\mathbb{L}[0, \infty]^\text{op}, +)\), under the name of flat left module. It is extended to the general case in \([38]\). It is shown in \([38]\) that if the quantale \( \mathcal{Q} = (Q, \& ) \) is a frame, i.e., \( \& = \land \), then a fuzzy lower set \( \phi \) of a \( \mathcal{Q} \)-ordered set \( A \) is flat if and only if for any \( x, y \in A \),
\[
\phi(x) \land \phi(y) \leq \bigvee_{z \in A} \phi(z) \land A(x, z) \land A(y, z).
\]
Hence, in the case that \( \mathcal{Q} = (Q, \& ) \) is a frame, flat ideals in a \( \mathcal{Q} \)-ordered set \( A \) coincides with ideals of \( A \) in the sense of \([25]\), Definition 5.1. In this case, based on flat ideals (under the name of fuzzy ideals), a theory of frame-valued directed complete orders and frame-valued domains have been developed in \([30, 42, 43, 44]\).

**Example 3.6.** For each \( a \) in a \( \mathcal{Q} \)-ordered set \( A \), the fuzzy lower set \( A(\neg, a) \) is a forward Cauchy ideal, a flat ideal and an irreducible ideal.

**Definition 3.7.** \((25)\) By a class of fuzzy sets we mean a functor \( \Phi : \mathcal{Q} \text{-Ord} \rightarrow \mathcal{Q} \text{-Ord} \) such that

(1) for each \( \mathcal{Q} \)-ordered set \( A \), \( \Phi(A) \) is a subset of \( \mathcal{P}A \) with the \( \mathcal{Q} \)-order inherited from \( \mathcal{P}A \);
(2) for all \( \mathcal{Q} \)-ordered set \( A \) and all \( a \in A \), \( y(a) \in \Phi(A) \);
(3) \( \Phi(f) = \mathcal{P}f = f^\rightarrow \) for every \( \mathcal{Q} \)-order preserving map \( f : A \rightarrow B \).

The second condition ensures that \( A \) can be embedded in \( \Phi(A) \) via the Yoneda embedding. We also write \( y \) for the embedding \( A \rightarrow \Phi(A) \) if no confusion will arise.

It should be noted that the notion of a class of fuzzy sets is a special case of that of a class of weights in category theory \([1, 19]\).

**Lemma 3.8.** Let \( f : A \rightarrow B \) be a \( \mathcal{Q} \)-order-preserving map. Then for each forward Cauchy ideal (flat ideal, irreducible ideal, resp.) \( \phi \) of \( A \), \( f^\rightarrow(\phi) \) is a forward Cauchy ideal (flat ideal, irreducible ideal, resp.) of \( B \).

**Proof.** We check, for example, that if \( \phi \) is irreducible then so is \( f^\rightarrow(\phi) \). For all fuzzy lower sets \( \phi_1, \phi_2 \) of \( B \), thanks to Equation \((2.1)\), we have
\[
\operatorname{sub}_B(f^\rightarrow(\phi), \phi_1 \lor \phi_2) = \operatorname{sub}_A(\phi, (\phi_1 \lor \phi_2) \circ f) \\
= \operatorname{sub}_A(\phi, \phi_1 \circ f) \lor \operatorname{sub}_A(\phi, \phi_2 \circ f) \\
= \operatorname{sub}_B(f^\rightarrow(\phi), \phi_1) \lor \operatorname{sub}_B(f^\rightarrow(\phi), \phi_2),
\]

hence \( f^\rightarrow(\phi) \) is irreducible. \( \square \)
Therefore, forward Cauchy ideals, flat ideals and irreducible ideals are all examples of class of fuzzy sets.

Let \( \Phi \) be a class of fuzzy sets. A \( \mathcal{Q} \)-ordered set \( A \) is \( \Phi \)-complete\(^1\) if each \( \phi \in \Phi(A) \) has a supremum. It is clear that \( A \) is \( \Phi \)-complete if and only if \( y : A \to \Phi(A) \) has a left adjoint. In this case, the left adjoint of \( y \) sends each \( \phi \in \Phi(A) \) to its supremum \( \sup \phi \). A \( \mathcal{Q} \)-order-preserving map \( f : A \to B \) is \( \Phi \)-cocontinuous if \( f(\sup \phi) = \sup f^{-1}(\phi) \) for all \( \phi \in \Phi(A) \) whenever \( \sup \phi \) exists.

**Definition 3.9.** A \( \mathcal{Q} \)-ordered set \( A \) is

1. **Yoneda complete** (a.k.a. liminf complete) if each forward Cauchy ideal of \( A \) has a supremum (which is equivalent to that every forward Cauchy net in \( A \) has a Yoneda limit);
2. **irreducible complete** if each irreducible ideal of \( A \) has a supremum;
3. **flat complete** if each flat ideal of \( A \) has a supremum.

Yoneda complete, irreducible complete, and flat complete are all natural extension of directed complete to the fuzzy setting.

The rest of this section concerns the relationship among forward Cauchy ideals, irreducible ideals, and flat ideals.

A complete lattice \( L \) is meet continuous [9] if for all \( a \in L \) and all directed subset \( D \) of \( L \),

\[
a \wedge \bigvee D = \bigvee_{d \in D} (a \wedge d).
\]

A complete lattice is dually meet continuous if its opposite is meet continuous. A quantale \( \mathcal{Q} = (Q, \& ) \) is (dually, resp.) meet continuous if the complete lattice \( \mathcal{Q} \) is (dually, resp.) meet continuous.

**Theorem 3.10.** For a dually meet continuous quantale \( \mathcal{Q} \), every forward Cauchy ideal in a \( \mathcal{Q} \)-ordered set is an irreducible ideal.

**Lemma 3.11.** If \( \{a_i\} \) is a forward Cauchy net in the \( \mathcal{Q} \)-ordered set \( (Q, d_R) \), then \( \bigvee_i \bigwedge_{j \geq i} a_j \) is a Yoneda limit of \( \{a_i\} \) and

\[
\bigvee_i \bigwedge_{j \geq i} a_j = \bigwedge_{i \geq j} \bigvee a_j.
\]

**Proof.** First, we show that \( \bigwedge_{i \geq j} \bigvee a_j \) is a Yoneda limit of \( \{a_i\} \). That is, for all \( x \in Q \),

\[
d_R \left( \bigwedge_{i \geq j} \bigvee a_j, x \right) = \bigvee_{i \geq j} d_R(a_j, x).
\]

On one hand, since \( \{a_i\} \) is a forward Cauchy net in \( (Q, d_R) \),

\[
\bigvee_i \bigwedge_{i \leq j \leq k} (a_k \rightarrow a_j) = \bigvee_i \bigwedge_{i \leq j \leq k} d_R(a_j, a_k) = 1,
\]

then

\[
\bigvee_{i \geq j} \bigwedge_{i \geq k} (a_l \rightarrow a_j) = 1,
\]

\(^1\) “\( \Phi \)-cocomplete” would be better from the viewpoint of category theory. However, following the tradition in domain theory, we choose “\( \Phi \)-complete”.

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12
hence
\[ \bigvee_i \bigwedge_j \left( \bigwedge_k \bigvee_l a_l \to a_j \right) = 1. \]

So,
\[
d_R \left( \bigwedge_i \bigvee_j \geq i a_j, x \right) = \left( x \to \bigvee_i \bigwedge_k a_k \right) \land \bigvee_i \bigwedge_j \left( \bigwedge_k \bigvee_l a_l \to a_j \right)
\]
\[ \leq \bigvee_i \bigwedge_j (x \to a_j) \]
\[ = \bigvee_i \bigwedge_j d_R(a_j, x). \]

On the other hand, since for each \( i \) we always have
\[ x \to \bigwedge_j \geq i a_j \geq \bigwedge_k \bigvee_l \geq k a_l \to a_j \]

it follows that
\[
d_R \left( \bigwedge_i \bigvee_j \geq i a_j, x \right) = \bigwedge_i \left( x \to \bigvee_j \geq i a_j \right) \geq \bigvee_i \bigwedge_j (x \to a_j) = \bigvee_i \bigwedge_j d_R(a_j, x). \]

Therefore, \( \bigwedge_i \bigvee_j \geq i a_j \) is a Yoneda limit of \( \{ a_i \} \).

Next, we prove the equality
\[ \bigvee_i \bigwedge_j a_j = \bigwedge_i \bigvee_j a_j. \]

Since \( \bigwedge_i \bigvee_j a_j \) is a Yoneda limit of \( \{ a_i \} \) in \((Q, d_R)\), it follows that for all \( x \in Q \),
\[
x \to \bigvee_i \bigwedge_j a_j = d_R \left( \bigwedge_i \bigvee_j a_j, x \right)
\]
\[ = \bigvee_i \bigwedge_j d_R(a_j, x) \]
\[ = \bigvee_i \bigwedge_j (x \to a_j) \]
\[ \leq x \to \bigvee_i \bigwedge_j a_j. \]

Letting \( x = 1 \), we obtain that
\[ \bigwedge_i \bigvee_j a_j \leq \bigvee_i \bigwedge_j a_j. \]

The converse inequality is trivial, so the equality is valid.

Lemma 3.1 and Lemma 3.11 imply that a forward Cauchy net in either \((Q, d_L)\) or \((Q, d_R)\) is order convergent. But, \((Q, d_L)\) and \((Q, d_R)\) may have different forward Cauchy nets. For example, the sequence \( \{ n \} \) is a forward Cauchy in \([0, \infty], d_R\) but not in \([0, \infty], d_L\).
Proof of Theorem 3.10. Let \( \{x_i\} \) be a forward Cauchy net in a \( Q \)-ordered set \( A \) and \( \varphi = \bigvee_i \bigwedge_{j \geq i} A(-, x_j) \). We show that \( \varphi \) is an irreducible ideal.

**Step 1.** \( \varphi \) is inhabited. This is easy since
\[
\bigvee_{x \in A} \varphi(x) \geq \bigvee_{i} \bigwedge_{j \geq i} A(x_i, x_k) \geq \bigvee_{i} A(x_i, x_k) = 1.
\]

**Step 2.** For each fuzzy lower set \( \phi \) of \( A \),
\[
\text{sub}_A(\varphi, \phi) = \bigwedge_i \bigvee_{j \geq i} \phi(x_j).
\]
Since \( \phi \) is a fuzzy lower set, \( \{\phi(x_j)\} \) is a forward Cauchy net in \((Q, d_R)\). Then,
\[
\text{sub}_A(\varphi, \phi) = \text{sub}_A \left( \bigvee_i \bigwedge_{j \geq i} A(-, x_j), \phi \right) = \bigwedge_i \bigvee_{j \geq i} \phi(x_j).
\]

**Step 3.** For all fuzzy lower sets \( \phi_1, \phi_2 \) of \( A \), \( \text{sub}_A(\varphi, \phi_1 \lor \phi_2) = \text{sub}_A(\varphi, \phi_1) \lor \text{sub}_A(\varphi, \phi_2) \).
Since \( Q \) is dually meet continuous, we have
\[
\text{sub}_A(\varphi, \phi_1) \lor \text{sub}_A(\varphi, \phi_2) = \bigwedge_i \bigvee_{j \geq i} \phi_1(x_j) \lor \bigwedge_i \bigvee_{j \geq i} \phi_2(x_j) = \bigwedge_i \bigvee_{j \geq i} (\phi_1(x_j) \lor \phi_2(x_j)),
\]

therefore the conclusion holds. \( \square \)

Interestingly, the dual meet continuity of \( Q \) is also necessary for Theorem 3.10.

**Proposition 3.12.** If every forward Cauchy ideal in every \( Q \)-ordered set is irreducible, then the quantale \( Q \) is dually meet continuous.

**Proof.** We show that for each \( a \in Q \) and each filtered set \( F \) in \( Q \),
\[
a \lor \bigwedge_{x \in F} x = \bigwedge_{x \in F} (a \lor x).
\]
Consider the fuzzy lower set \( \phi = \bigvee_{x \in F} d_R(-, x) \) of the \( Q \)-ordered set \((Q, d_R)\). Since
\[
\phi = \bigvee_{x \in F} \bigwedge_{y \in F, y \leq x} d_R(-, y),
\]
it follows that \( \phi \) is a forward Cauchy ideal, hence an irreducible ideal by assumption.
Since both the identity map \(\text{id}_Q\) on \(Q\) and the constant map \(a : Q \rightarrow Q\) with value \(a\) are fuzzy lower sets of \((Q,d_R)\), then

\[
a \lor \bigwedge_{x \in F} x = \text{sub}_Q(\phi,a) \lor \text{sub}_Q(\phi,\text{id}_Q) = \text{sub}_Q(\phi,a \lor \text{id}_Q) = \bigwedge_{x \in F} (a \lor x).
\]

This finishes the proof. \(\square\)

Irreducible ideals need not be forward Cauchy in general. Let \(Q = \{0,a,b,1\}\) be the Boolean algebra with four elements. Assume that \(A\) is the \(Q\)-ordered set with points \(x,y\) and

\[A(x,x) = A(y,y) = 1, \quad A(x,y) = A(y,x) = 0.\]

Then the map \(\phi\), given by \(\phi(x) = a\) and \(\phi(y) = b\), is an irreducible ideal in \(A\). But, \(\phi\) cannot be generated by any forward Cauchy net in \(A\). This example is essentially Note 3.12 in [47].

The following conclusion is important in the theory of fuzzy orders based on left continuous t-norms.

**Theorem 3.13.** If \(\&\) is a left continuous t-norm and \(Q = ([0,1],\&)\), then irreducible ideals coincide with forward Cauchy ideals.

**Proof.** By Theorem 3.10 we only need to prove that every irreducible ideal \(\phi\) of a \(Q\)-ordered set \(A\) is a forward Cauchy ideal.

Let

\[C\phi = \{(x,r) \in X \times [0,1) \mid \phi(x) > r\}.
\]

Define a relation \(\sqsubseteq\) on \(C\phi\) by

\[(x,r) \sqsubseteq (y,s) \iff A(x,y) \to r \leq s.
\]

We claim that \((C\phi,\sqsubseteq)\) is a directed set. Before proving this, we note that if \((x,r) \sqsubseteq (y,s)\) then \(r < A(x,y)\) and \(r \leq s\). That \(\sqsubseteq\) is reflexive and transitive is easy; it remains to check that it is directed. For any \((x,r),(y,s) \in C\phi\), consider the fuzzy lower sets \(\psi_1 = A(x,-) \to r\) and \(\psi_2 = A(y,-) \to s\). Since \(\phi\) is an irreducible ideal,

\[
\text{sub}_X(\phi,\psi_1 \lor \psi_2) = \text{sub}_X(\phi,A(x,-) \to r) \lor \text{sub}_X(\phi,A(y,-) \to s) = \text{sub}_X(A(x,-),\phi \to r) \lor \text{sub}_X(A(y,-),\phi \to s) = (\phi(x) \to r) \lor (\phi(y) \to s).
\]

Since \((\phi(x) \to r) \lor (\phi(y) \to s) < 1\), there exists some \(z\) such that

\[\phi(z) \to [(A(x,z) \to r) \lor (A(y,z) \to s)] < 1.
\]

Let \(t = (A(x,z) \to r) \lor (A(y,z) \to s)\), then \((z,t) \in C\phi\) and \((x,r) \sqsubseteq (z,t),(y,s) \sqsubseteq (z,t)\). Hence \((C\phi,\sqsubseteq)\) is a directed set.

From now on, we also write an element in \(C\phi\) as a pair \((x_i,r_i)\). Define a net

\[\mathbf{r} : C\phi \to A\]
by \( r(x_i, r_i) = x_i \). We prove in two steps that \( r \) is a forward Cauchy net and it generates \( \phi \), hence \( \phi \) is a forward Cauchy ideal.

**Step 1.** \( r \) is a forward Cauchy net.

Let \( t < 1 \). Since \( \phi \) is inhabited, there is some \( (x_i, r_i) \in C\phi \) such that \( t \leq r_i \). Then for all \( (x_k, r_k) \sqsupseteq (x_j, r_j) \sqsupseteq (x_i, r_i) \), we have \( A(x_j, x_k) \rightarrow r_j \leq r_k < 1 \), hence

\[
A(x_j, x_k) > r_j \geq r_i \geq t.
\]

By arbitrariness of \( t \) we obtain that \( r \) is forward Cauchy.

**Step 2.** \( \phi \) is generated by \( r \), i.e.,

\[
\phi(x) = \bigvee_{(x_i, r_i)} \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j)
\]

for all \( x \in A \).

Take \( x \in A \) and \( r < \phi(x) \). For all \( (x_j, r_j) \in C\phi \), if \( (x, r) \sqsubseteq (x_j, r_j) \), then \( A(x, x_j) > r \), hence, by arbitrariness of \( r \),

\[
\phi(x) \leq \bigvee_{r < \phi(x)} \bigwedge_{(x_j, r_j) \sqsupseteq (x, r)} A(x, x_j) \leq \bigvee_{(x_i, r_i)} \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).
\]

For the converse inequality, we show that for each \( (x_i, r_i) \in C\phi \),

\[
\phi(x) \geq \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).
\]

Let \( t \) be an arbitrary number that is strictly smaller than

\[
\bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).
\]

Since \( \& \) is left continuous and \( \phi \) is inhabited, there is some \( (x_k, r_k) \in C\phi \) such that

\[
t \leq r_k \& \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).
\]

Take some \( (x_l, r_l) \in C\phi \) such that \( (x_i, r_i), (x_k, r_k) \sqsubseteq (x_l, r_l) \). Then

\[
\phi(x) \geq \phi(x_l) \& A(x, x_l) \geq r_k \& A(x, x_l) \geq t.
\]

Therefore, by arbitrariness of \( t \),

\[
\phi(x) \geq \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).
\]

The proof is completed.

A slight improvement of the argument shows that the above theorem is valid for all linearly ordered quantales. That is, if \( Q \) is a linearly ordered quantale, then irreducible ideals coincide with forward Cauchy ideals.

As an application of the above theorem, the following corollary characterizes the irreducible ideals in the \( Q \)-ordered sets \( ([0, 1], d_L) \) and \( ([0, 1], d_R) \), where \( Q \) is the interval \([0, 1]\) coupled with a left continuous t-norm. This characterization will be useful in Section 5.
Corollary 3.14. Let & be a left continuous t-norm and \( Q = ([0, 1], \&). \)

1. A fuzzy lower set \( \phi \) of the \( Q \)-ordered set \( ([0, 1], d_L) \) is an irreducible ideal if and only if either \( \phi(x) = x \to a \) for some \( a \in [0, 1] \) or \( \phi(x) = \bigvee_{b<a}(x \to b) \) for some \( a > 0 \).

2. A fuzzy lower set \( \psi \) of the \( Q \)-ordered set \( ([0, 1], d_R) \) is an irreducible ideal if and only if either \( \psi(x) = a \to x \) for some \( a \in [0, 1] \) or \( \psi(x) = \bigvee_{b>a}(b \to x) \) for some \( a < 1 \).

Proof. (1) Sufficiency is easy since the fuzzy lower set \( \phi(x) = \bigvee_{b<a}(x \to b) \) is generated by the forward Cauchy sequence \( \{a - 1/n\} \). As for necessity, suppose that \( \phi \) is an irreducible ideal of \( ([0, 1], d_L) \). Then there is a forward Cauchy net \( \{x_i\} \) in \( ([0, 1], d_L) \) such that

\[
\phi(x) = \bigvee_{i, j \geq i}(x \to x_j).
\]

Let \( a = \bigvee_i \bigwedge_{j \geq i} x_j \). Then

\[
\phi(x) = \bigvee_{i, j \geq i}(x \to x_j) = \bigvee_i \left( x \to \bigwedge_{j \geq i} x_j \right),
\]

hence either \( \phi(x) = x \to a \) or \( \phi(x) = \bigvee_{b<a}(x \to b) \).

(2) Similar to (1).

Theorem 3.15. For a meet continuous quantale \( Q \), every forward Cauchy ideal in a \( Q \)-ordered set is flat.

Proof. We only need to show that if \( \{x_i\} \) is a forward Cauchy net in a \( Q \)-ordered set \( A \) and

\[
\varphi = \bigvee_i \bigwedge_{j \geq i} A(\neg, x_j),
\]

then \( \varphi \) is flat. We do this in two steps.

Step 1. For each fuzzy upper set \( \psi \) of \( A \),

\[
\varphi \otimes \psi = \bigvee_i \bigwedge_{j \geq i} \psi(x_j).
\]

We calculate:

\[
\varphi \otimes \psi = \bigwedge_{p \in Q} \left( \text{sub}_X(\varphi, \psi \to p) \to p \right) \quad \text{(Lemma 2.9)}
\]

\[
= \bigwedge_{p \in Q} \left( \text{sub}_X \left( \bigvee_i \bigwedge_{j \geq i} A(\neg, x_j), \psi \to p \right) \to p \right) \quad \text{(Proposition 3.2)}
\]

\[
= \bigwedge_{p \in Q} \left( \bigvee_i \bigwedge_{j \geq i} \text{sub}_A(A(\neg, x_j), \psi \to p) \to p \right) \quad \text{(Yoneda lemma)}
\]

\[
= \bigwedge_{p \in Q} \left( \bigvee_i \bigwedge_{j \geq i} (\psi(x_j) \to p) \to p \right) \quad \text{(Lemma 3.1)}
\]

\[
= \bigvee_{i, j \geq i} \psi(x_j) \quad \text{(Yoneda lemma)}
\]
Step 2. \( \varphi \) is flat. For any fuzzy upper sets \( \psi_1 \) and \( \psi_2 \),

\[
\varphi \otimes \psi_1 \land \phi \otimes \psi_2 = \bigvee_i \bigwedge_{j \geq i} \psi_1(x_j) \land \bigvee_i \bigwedge_{j \geq i} \psi_2(x_j) \quad \text{(Step 1)}
\]

\[
= \bigvee_i (\psi_1(x_j) \land \psi_2(x_j)) \quad \text{(Q is meet continuous)}.
\]

The proof is completed.

Similar to Proposition 3.12, it can be shown that the meet continuity of \( Q \) is also necessary for Theorem 3.15.

**Proposition 3.16.** If every forward Cauchy ideal in every \( Q \)-ordered set is flat, then the quantale \( Q \) is meet continuous.

**Example 3.17.** Consider the quantale \( Q = ([0, 1], \land) \) and the \( Q \)-ordered set \( ([0, 1], d_L) \). By linearity of \([0, 1]\), every fuzzy lower set \( \phi \) of \(([0, 1], d_L)\) satisfies that

\[
\phi(x) \land \phi(y) \leq \bigvee_{z \in [0,1]} \phi(z) \land d_L(x, z) \land d_L(y, z),
\]

hence every inhabited fuzzy lower set \( \phi \) of \(([0, 1], d_L)\) is a flat ideal by Remark 3.5. In particular, the map \( \phi : [0, 1] \to [0, 1] \), given by

\[
\phi(x) = \begin{cases} 
1 - x, & x \leq 1/2, \\
1/2, & x > 1/2,
\end{cases}
\]

is a flat ideal of \(([0, 1], d_L)\). But, it is not irreducible by Corollary 3.14, hence not forward Cauchy by Theorem 3.10.

In the case that \( Q \) is the interval \([0, 1]\) coupled with a continuous t-norm, we are able to present a sufficient and necessary condition for flat ideals to be forward Cauchy.

**Theorem 3.18.** Let \( Q \) be the unit interval coupled with a continuous t-norm \&. The following are equivalent:

1. \& is Archimedean, i.e., \& has no non-trivial idempotents.
2. Every flat ideal is a forward Cauchy ideal.

We prove a lemma first. A quantale \( Q = (Q, \&), \) is called divisible \([12, 13]\) if

\[
x \& (x \to y) = x \land y
\]

for all \( x, y \in Q \). It is known that the underlying lattice of a divisible quantale is a frame, hence a distributive lattice, see e.g. \([13]\). Let \( Q \) be a divisible quantale and \( b \) an idempotent element. Then for all \( x \in Q \),

\[
b \land x = b \& (b \to x) = b \& b \& (b \to x) \leq b \& x \leq b \land x,
\]

hence \( b \land x = b \& x \).
Lemma 3.19. Suppose $Q = (Q, \&)$ is a divisible quantale, $b$ is an idempotent element of $\&$. Then for each $a \in Q$, the map $\phi(x) = b \lor (x \to a)$ is a flat ideal of the $Q$-ordered set $(Q, d_L)$.

Proof. It is clear that $\phi$ is a fuzzy lower set of $(Q, d_L)$ and $\bigvee_{x \in Q} \phi(x) = 1$. It remains to check that $\phi \otimes (\psi_1 \land \psi_2) = (\phi \otimes \psi_1) \land (\phi \otimes \psi_2)$ for all upper sets $\psi_1, \psi_2$ of $(Q, d_L)$.

Because for $i = 1, 2$,

$$\phi \otimes \psi_i = \bigvee_{x \in Q} ((b \& \psi_i(x)) \lor ((x \to a) \& \psi_i(x)))$$

$$= (b \land \psi_i(1)) \lor \psi_i(a) \quad \text{ (by Theorem 3.10, a contradiction.)}$$

$$= (b \lor \psi_i(a)) \land \psi_i(1), \quad \text{ (is an irreducible ideal.)}$$

therefore

$$(\phi \otimes \psi_1) \land (\phi \otimes \psi_2) = (b \lor \psi_1(a)) \land \psi_1(1) \land (b \lor \psi_2(a)) \land \psi_2(1)$$

$$= (b \lor (\psi_1(a) \land \psi_2(a))) \land (\psi_1(1) \land \psi_2(1))$$

$$= \phi \otimes (\psi_1 \land \psi_2).$$

The proof is completed. □

Proof of Theorem 3.18. (1) ⇒ (2) This is contained in Proposition 7.9 in [29]. If $\&$ is isomorphic to the product t-norm, an equivalent version of this implication can also be found in Vickers [37].

(2) ⇒ (1) Suppose $b$ is a non-trivial idempotent element of $\&$. Take some $a \in (0, b)$. Since $[0, 1]$ together with a continuous t-norm is a divisible quantale [2, 12], it follows from Lemma 3.19 that $\phi(x) = b \lor (x \to a)$ is a flat ideal of the $Q$-ordered set $([0, 1], d_L)$. But, $\phi$ is not irreducible, because neither $\phi(x) \leq b$ for all $x$ nor $\phi(x) \leq x \to a$ for all $x$. So, $\phi$ is not forward Cauchy by Theorem 3.10, a contradiction. □

Finally, we discuss the relationship between irreducible ideals and flat ideals.

A quantale $Q$ is prelinear if $(p \to q) \lor (q \to p) = 1$ for all $p, q \in Q$. It is known that $Q$ is prelinear if and only if $(p \land q) \to r = (p \to r) \lor (q \to r)$ for all $p, q, r \in Q$, see e.g. [2].

Proposition 3.20. If $Q$ is prelinear, then every irreducible ideal is flat.

Proof. Assume that $\phi$ is an irreducible ideal. For all fuzzy upper sets $\psi_1, \psi_2$,

$$\phi \otimes (\psi_1 \land \psi_2) = \bigwedge_{p \in Q} \left(\text{sub}_A(\phi, (\psi_1 \land \psi_2) \to p) \to p\right)$$

$$= \bigwedge_{p \in Q} \left(\text{sub}_A(\phi, (\psi_1 \to p) \lor (\psi_2 \to p)) \to p\right)$$

$$= \bigwedge_{p \in Q} \left(\text{sub}_A(\phi, (\psi_1 \to p) \to p) \land (\text{sub}_A(\phi, (\psi_2 \to p) \to p)\right)$$

$$= (\phi \otimes \psi_1) \land (\phi \otimes \psi_2),$$

hence $\phi$ is flat. □

Proposition 3.21. If $Q$ satisfies the law of double negation then flat ideals coincide with irreducible ideals.
Proof. If \( Q \) satisfies the law of double negation, then, by Lemma \([2,9]\) for all fuzzy lower sets \( \phi, \varphi \) and fuzzy upper set \( \psi \),
\[
\text{sub}_A(\phi, \varphi) = \phi \otimes (\varphi \rightarrow 0) \rightarrow 0
\]
and
\[
\phi \otimes \psi = \text{sub}_A(\phi, \psi \rightarrow 0) \rightarrow 0.
\]
The conclusion follows easily from these equations. \( \square \)

**Corollary 3.22.** Let \( Q \) be the unit interval coupled with a left continuous \( t \)-norm \&. If \( Q \) satisfies the law of double negation, then for each fuzzy lower set \( \phi \) of a \( Q \)-ordered set, the following are equivalent:

1. \( \phi \) is a forward Cauchy ideal.
2. \( \phi \) is an irreducible ideal.
3. \( \phi \) is a flat ideal.

4. **Saturatedness**

Let \( \Phi \) be a class of fuzzy sets. A \( Q \)-ordered set \( A \) is \( \Phi \)-continuous if it is \( \Phi \)-complete and the left adjoint \( \text{sup} : \Phi(A) \rightarrow A \) of \( y : A \rightarrow \Phi(A) \) has a left adjoint. This kind of postulation is standard in order theory \([41]\). In the case that \( \Phi = \mathcal{P} \) (the biggest class of fuzzy sets), \( \Phi \)-continuous \( Q \)-ordered sets are the completely distributive (or, totally continuous) \( Q \)-categories in \([32, 36]\). We write \( \Phi \text{-Cont} \) for the category of \( \Phi \)-continuous \( Q \)-ordered sets and \( \Phi \)-cocontinuous maps. The category \( \Phi \text{-Cont} \) is the subject of fuzzy domain theory. So, a natural question is whether there exist such \( Q \)-ordered sets. As we will see, saturatedness of \( \Phi \) guarantees that there exist enough such things.

A class of fuzzy sets \( \Phi \) is **saturated** \([19, 25]\) if for all \( Q \)-ordered set \( A \) and \( \Lambda \in \Phi(\Phi(A)) \),
\[
\bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi \in \Phi(A).
\]

**Theorem 4.1.** Let \( \Phi \) be a saturated class of fuzzy sets.

1. For each \( Q \)-ordered set \( A \), \( \Phi(A) \) is \( \Phi \)-continuous.
2. For each \( Q \)-order-preserving map \( f : A \rightarrow B \), \( \Phi(f) \) is \( \Phi \)-cocontinuous.
3. The functor \( \Phi : Q \text{-Ord} \rightarrow \Phi \text{-Cont} \), which sends each \( Q \)-order-preserving map \( f \) to \( \Phi(f) \), is a left adjoint of the forgetful functor \( \Phi \text{-Cont} \rightarrow Q \text{-Ord} \).

**Proof.** (1) For each \( \Lambda \in \Phi(\Phi(A)) \), since \( \Phi \) is saturated, \( \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi \) belongs to \( \Phi(A) \). It is easy to verify that for all \( \psi \in \Phi(A) \),
\[
\mathcal{P}\Phi(A)(\Lambda, \Phi(A)(-, \psi)) = \Phi(A) \left( \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi, \psi \right)
\]
hence \( \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi \) is a supremum of \( \Lambda \) in \( \Phi(A) \), i.e.,
\[
\sup_{\phi \in \Phi(A)} \Lambda = \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi.
\]
This shows that $\Phi(A)$ is $\Phi$-complete.

Next, we show that $\Phi(A)$ is $\Phi$-continuous, that is, $\sup_{\Phi(A)} : \Phi(\Phi(A)) \to \Phi(A)$ has a left adjoint. To this end, write $y_A$ for the Yoneda embedding $A \to \Phi(A)$. For each $\Lambda \in \Phi(\Phi(A))$, since $\Lambda : \Phi(A) \to (Q,d_R)$ preserves $Q$-order, it follows that for all $x \in A$ and $\phi \in \Phi(A)$,

$$\phi(x) = \text{sub}_A(y_A(x),\phi) \leq \Lambda(\phi) \to \Lambda(y_A(x)),$$

hence

$$\sup_{\Phi(A)}(A)(x) = \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi(x) = \Lambda \circ y_A(x).$$

This means that $\sup_{\Phi(A)}$ is the map obtained by restricting the domain and codomain of

$$y_A^\top : \mathcal{P}(A) \to \mathcal{P}(A)$$

to $\Phi(\Phi(A))$ and $\Phi(A)$, respectively. Therefore, $\sup_{\Phi(A)}$ has a left adjoint, given by restricting the domain and codomain of $y_A^\top : \mathcal{P}(A) \to \mathcal{P}(A)$ to $\Phi(\Phi(A))$ and $\Phi(\Phi(A))$, respectively.

(2) and (3) are a special case of Theorem 4.7 in [25], which is again a special case of a general result in category theory [1, 18, 19].

The above theorem shows that if $\Phi$ is a saturated class of fuzzy sets, then for each $Q$-ordered set $A$, $\Phi(A)$ is the free $\Phi$-continuous $Q$-ordered set generated by $A$. A category-minded reader will recognize soon that a saturated class of fuzzy sets is an example of KZ-monads [21, 49].

This section concerns the saturatedness of the classes of forward Cauchy ideals, irreducible ideals, and flat ideals.

A quantale $Q = (Q, \&)$ is completely distributive (continuous, resp.) if the complete lattice $Q$ is a completely distributive lattice (a continuous lattice, resp.). So, each completely distributive quantale is a continuous quantale and each continuous quantale is a meet continuous quantale. For continuous lattices and completely distributive lattices, we refer to the monograph [9].

The following proposition was first proved in [8] when $Q$ is a completely distributive value quantale, the version presented below was proved in [24] making use of Lemma 4.3.

**Proposition 4.2.** If $Q$ is a continuous quantale, then the class $W$ of forward Cauchy ideals is saturated.

**Lemma 4.3.** ([24]) Let $Q$ be a continuous quantale and $\phi$ be an inhabited fuzzy lower set of a $Q$-ordered set $A$. The following are equivalent:

1. $\phi$ is a forward Cauchy ideal.
2. If $r \ll \phi(x)$ and $s \ll \phi(y)$, then for every $t \ll 1$, there is some $z \in A$ such that $t \ll \phi(z)$, $r \ll A(x,z)$ and $s \ll A(y,z)$.

The saturatedness of the classes $I$ and $F$ is a special case of a general result in enriched category theory, namely, Proposition 5.4 in Kelly and Schmidt [19]. However, in order to make this paper self-contained, a direct verification in this special case is included here.

**Proposition 4.4.** The class $I$ of irreducible ideals is saturated.
Proof. It suffices to show that for each \( \mathbb{Q} \)-ordered set \( A \) and each irreducible ideal \( \Lambda : \mathcal{I}A \to \mathbb{Q} \) of \( (\mathcal{I}A, \text{sub}_A) \), the map \( \sup \Lambda : A \to \mathbb{Q} \), given by

\[
\sup \Lambda(x) = \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) \& \phi(x),
\]

is an irreducible ideal of \( A \).

**Step 1.** \( \bigvee_{x \in A} \sup \Lambda(x) = 1 \). This is easy since

\[
\bigvee_{x \in A} \sup \Lambda(x) = \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) \& \phi(x) = \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) = 1.
\]

**Step 2.** For any fuzzy lower sets \( \phi_1, \phi_2 \) of \( A \),

\[
\text{sub}_A(\sup \Lambda, \phi_1 \vee \phi_2) = \text{sub}_A(\sup \Lambda, \phi_1) \vee \text{sub}_A(\sup \Lambda, \phi_2).
\]

To see this, for a fuzzy lower set \( \phi \) of \( A \), consider the fuzzy lower set of \( (\mathcal{I}A, \text{sub}_A) \):

\[
\text{sub}_A(-, \phi) : \mathcal{I}A \to \mathbb{Q}.
\]

Then

\[
\text{sub}_{\mathcal{I}A}(\Lambda, \text{sub}_A(-, \phi)) = \bigwedge_{\psi \in \mathcal{I}A} (\Lambda(\psi) \to \text{sub}_A(\psi, \phi))
\]

\[
= \bigwedge_{\psi \in \mathcal{I}A} \left( \Lambda(\psi) \to \bigwedge_{x \in A} (\psi(x) \to \phi(x)) \right)
\]

\[
= \bigwedge_{x \in A} \left( \bigvee_{\psi \in \mathcal{I}A} \Lambda(\psi) \& \psi(x) \to \phi(x) \right)
\]

\[
= \text{sub}_A(\sup \Lambda, \phi).
\]

Therefore,

\[
\text{sub}_A(\sup \Lambda, \phi_1 \vee \phi_2) = \text{sub}_{\mathcal{I}A}(\Lambda, \text{sub}_A(-, \phi_1 \vee \phi_2))
\]

\[
= \text{sub}_{\mathcal{I}A}(\Lambda, \text{sub}_A(-, \phi_1) \vee \text{sub}_A(-, \phi_2))
\]

\[
= \text{sub}_{\mathcal{I}A}(\Lambda, \text{sub}_A(-, \phi_1) \vee \text{sub}_{\mathcal{I}A}(\Lambda, \text{sub}_A(-, \phi_2))
\]

\[
= \text{sub}_A(\sup \Lambda, \phi_1) \vee \text{sub}_A(\sup \Lambda, \phi_2),
\]

where the second equality holds since each element in \( \mathcal{I}A \) is irreducible; the reason for the third equality is that \( \Lambda \) is irreducible. \( \square \)

**Proposition 4.5.** The class \( \mathcal{F} \) of flat ideals is saturated.

**Proof.** We only need to show that for each \( \mathbb{Q} \)-ordered set \( A \) and each flat ideal \( \Lambda : \mathcal{F}A \to \mathbb{Q} \) of \( (\mathcal{F}A, \text{sub}_A) \), the map \( \sup \Lambda : A \to \mathbb{Q} \), given by

\[
\sup \Lambda(x) = \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) \& \phi(x),
\]

is a flat ideal of \( A \).
Step 1. $\bigvee_{x \in A} \sup \Lambda(x) = 1$. This is easy since

$$\bigvee_{x \in A} \sup \Lambda(x) = \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) \land \phi(x) = \bigvee_{\phi \in \mathcal{F}A} \bigvee_{x \in A} \Lambda(\phi) \land \phi(x) = \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) = 1.$$  

Step 2. For any fuzzy upper sets $\psi_1, \psi_2$ of $A$,

$$\sup \Lambda \otimes (\psi_1 \land \psi_2) = (\sup \Lambda \otimes \psi_1) \land (\sup \Lambda \otimes \psi_2).$$

To see this, for each fuzzy upper set $\psi$ on $A$, consider the fuzzy upper set of $(\mathcal{F}A, \text{sub}_A)$ (see Equation (2.2)):

$$- \otimes \psi : \mathcal{F}A \rightarrow Q.$$  

Then

$$\Lambda \otimes (- \otimes \psi) = \bigvee_{\phi \in \mathcal{F}A} \left( \Lambda(\phi) \land \bigvee_{x \in A} (\phi(x) \land \psi(x)) \right) = \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{F}A} (\Lambda(\phi) \land \phi(x)) \land \psi(x) = \bigvee_{x \in A} \sup \Lambda(x) \land \psi(x) = \sup \Lambda \otimes \psi.$$  

Therefore,

$$\sup \Lambda \otimes (\psi_1 \land \psi_2) = \Lambda \otimes (- \otimes (\psi_1 \land \psi_2)) = \Lambda \otimes ((- \otimes \psi_1) \land (- \otimes \psi_2)) = (\Lambda \otimes (- \otimes \psi_1)) \land (\Lambda \otimes (- \otimes \psi_2)) = (\sup \Lambda \otimes \psi_1) \land (\sup \Lambda \otimes \psi_2).$$

The proof is completed. \hfill \square

5. Scott $Q$-topology and Scott $Q$-cotopology

The connection between partially ordered sets and topological spaces is the essence of domain theory. The fuzzy version of Alexandroff topology has been investigated in [24]. This section concerns the extension of Scott topology to the fuzzy setting.

We recall some basic definitions first.

**Definition 5.1.** A $Q$-topology on a set $X$ is a subset $\tau$ of $Q^X$ subject to the following conditions:

(O1) $p_X \in \tau$ for all $p \in Q$;

(O2) $\lambda \land \mu \in \tau$ for all $\lambda, \mu \in \tau$;

(O3) $\bigvee_{j \in J} \lambda_j \in \tau$ for each subset $\{\lambda_j\}_{j \in J}$ of $\tau$.

A $Q$-topological space in the sense of the above definition is also called a *weakly stratified* $Q$-topological space in the literature, see e.g. [14, 15]. A $Q$-topology $\tau$ is *stratified* [14, 15] if

(O4) $p \land \lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$. 

23
A \( Q \)-topology \( \tau \) is \textit{co-stratified} if
\[(O5) \quad p \rightarrow \lambda \in \tau \quad \text{for all} \quad p \in Q \quad \text{and} \quad \lambda \in \tau.\]

A \( Q \)-topology is \textit{strong} if it is both stratified and co-stratified.

It is clear that if \( Q = (Q, \&) \) is a frame, i.e., if \( \& = \wedge \), then every \( Q \)-topology is stratified.

**Definition 5.2.** A \( Q \)-cotopology on a set \( X \) is a subset \( \tau \) of \( Q^X \) subject to the following conditions:

(C1) \( p_X \in \tau \) for all \( p \in Q \);
(C2) \( \lambda \vee \mu \in \tau \) for all \( \lambda, \mu \in \tau \);
(C3) \( \bigwedge_{j \in J} \lambda_j \in \tau \) for each subset \( \{\lambda_j\}_{j \in J} \) of \( \tau \).

A \( Q \)-cotopology \( \tau \) is \textit{stratified} if

(C4) \( p \rightarrow \lambda \in \tau \) for all \( p \in Q \) and \( \lambda \in \tau \).

A \( Q \)-cotopology \( \tau \) is \textit{co-stratified} if

(C5) \( p \& \lambda \in \tau \) for all \( p \in Q \) and \( \lambda \in \tau \).

A \( Q \)-cotopology \( \tau \) is \textit{strong} if it is both stratified and co-stratified.

Let \( Q \) be a quantale that satisfies the law of double negation. If \( \tau \) is a stratified (co-stratified, resp.) \( Q \)-cotopology on a set \( X \), then

\[\neg(\tau) = \{\neg \lambda \mid \lambda \in \tau\}\]

is a stratified (co-stratified, resp.) \( Q \)-topology on \( X \), where \( \neg \lambda(x) = \neg(\lambda(x)) \) for all \( x \in X \). Conversely, if \( \tau \) is a stratified (co-stratified, resp.) \( Q \)-topology on \( X \), then

\[\neg(\tau) = \{\neg \lambda \mid \lambda \in \tau\}\]

is a stratified (co-stratified, resp.) \( Q \)-cotopology on \( X \).

In general, there does not exist a natural way to switch between closed fuzzy sets and open fuzzy sets. So, we need to consider the open-set version and the closed-set version when generalizing Scott topology to \( Q \)-ordered sets. Interestingly, flat ideals and irreducible ideals are related to the open-set and the closed-set version, respectively.

**Definition 5.3.** Let \( \Phi \) be a class of fuzzy sets. A fuzzy set \( \psi : A \to Q \) of a \( Q \)-ordered set \( A \) is \( \Phi \)-closed if it is a fuzzy upper set and

\[\text{sub}_A(\phi, \psi) \leq \psi(\sup \phi)\]

for all \( \phi \in \Phi(A) \) whenever \( \sup \phi \) exists.

**Proposition 5.4.** A fuzzy set \( \psi \) of \( A \) is \( \Phi \)-closed if and only if \( \psi : A \to (Q, d_R) \) is \( \Phi \)-cocontinuous.

**Proof.** By Example\(\text{2.12} \), \( \text{sub}_A(\phi, \psi) \) is the supremum of \( \psi^-(\phi) \), hence \( \sup \psi^-(\phi) \leq \psi(\sup \phi) \).

The converse inequality is trivial. \(\square\)

**Lemma 5.5.** Let \( \Phi \) be a class of fuzzy sets, \( A \) a \( Q \)-ordered set.

(1) Each constant fuzzy set is \( \Phi \)-closed.
If $\psi$ is a $\Phi$-closed fuzzy set of $A$, then so is $p \rightarrow \psi$ for all $p \in Q$.

(3) The meet of each set of $\Phi$-closed fuzzy sets is $\Phi$-closed.

(4) If $\Phi(A) \subseteq I(A)$, then the join of two $\Phi$-closed fuzzy sets is $\Phi$-closed.

A class of fuzzy sets $\Phi$ is said to be a subclass of irreducible (flat, resp.) ideals if $\Phi(A) \subseteq I(A)$ ($\Phi(A) \subseteq F(A)$, reso.) for every $Q$-ordered set $A$. Theorem 3.10 says that for a dually meet continuous quantale, the class $W$ of forward ideals is a subclass of irreducible ideals.

**Proposition 5.6.** Let $\Phi$ be a subclass of irreducible ideals. Then for every $Q$-ordered set $A$, the $\Phi$-closed fuzzy sets form a stratified $Q$-cotopology on $A$, called the $\Phi$-Scott $Q$-cotopology and denoted by $\Sigma^{co}_\Phi(A)$. In particular, if $Q$ is dually meet continuous, the $W$-closed fuzzy sets of each $Q$-ordered set form a stratified $Q$-cotopology.

**Remark 5.7.** Let $\{x_i\}$ be a forward Cauchy net in a $Q$-ordered set $A$ and

$$\phi = \bigvee_i \bigwedge_{j \geq i} A(-, x_j).$$

By Proposition 3.2 (or, Step 2 in the argument of Theorem 3.10),

$$\text{sub}_A(\phi, \psi) = \bigvee_i \bigwedge_{j \geq i} \psi(x_j)$$

for each fuzzy lower set $\psi$ of $A$. By Proposition 3.3 Yoneda limits of $\{x_i\}$ are exactly the suprema of the fuzzy lower set

$$\phi = \bigvee_i \bigwedge_{j \geq i} A(-, x_j).$$

So, a fuzzy lower set $\psi$ of $A$ is $W$-closed if and only if

$$\bigvee_i \bigwedge_{j \geq i} \psi(x_j) \leq \psi(x)$$

for every forward Cauchy net $\{x_i\}$ with a Yoneda limit $x$. This shows that $W$-closed fuzzy sets are exactly the Scott closed fuzzy sets in the sense of Wagner ([40], Definition 4.4).

**Example 5.8.** Let $Q = ([0, 1], \&)$ with $\&$ being a left continuous t-norm. By Corollary 3.14 a fuzzy lower set $\psi$ of $([0, 1], d_R)$ is an irreducible ideal if and only if either $\psi(x) = a \rightarrow x$ for some $a \in [0, 1]$ or $\psi(x) = \bigvee_{b>a} (b \rightarrow x)$ for some $a < 1$. Since the supremum of $\bigvee_{b>a} (b \rightarrow x)$ in $([0, 1], d_R)$ is (see Example 2.12)

$$\bigwedge_{x \in [0,1]} \left( \bigvee_{b>a} (b \rightarrow x) \right) = \bigwedge_{b>a} \bigwedge_{x \in [0,1]} ((b \rightarrow x) \rightarrow x) = a,$$

it follows that a fuzzy lower set $\phi$ of $([0, 1], d_R)$ is $I$-closed if and only if for all $a < 1$,

$$\text{sub}_{[0,1]} \left( \bigvee_{b>a} (b \rightarrow x), \phi \right) = \bigwedge_{b>a} \phi(b) \leq \phi(a).$$

Hence, $\phi$ is $I$-closed in $([0, 1], d_R)$ if and only if

(i) $\phi : ([0, 1], d_L) \rightarrow ([0, 1], d_L)$ preserves $Q$-order; and
(ii) $\phi : [0, 1] \rightarrow [0, 1]$ is right continuous.

If $\&$ is a continuous t-norm, we have more: the $\mathcal{I}$-Scott $\mathcal{Q}$-cotopology on the $\mathcal{Q}$-ordered sets $([0, 1], d_R)$ is the strong $\mathcal{Q}$-cotopology on $[0, 1]$ generated by the identity map on $[0, 1]$. To see this, we only need to show that the strong $\mathcal{Q}$-cotopology $\tau$ on $[0, 1]$ generated by the identity map consists of maps $[0, 1] \rightarrow [0, 1]$ that satisfy (i) and (ii). In the case that $\&$ is the t-norm min, or the Łukasiewicz t-norm, or the product t-norm, this conclusion is contained in [47]. Here we give a proof for the general case.

The set $K$ of maps $[0, 1] \rightarrow [0, 1]$ that satisfy (i) and (ii) is clearly a strong $\mathcal{Q}$-cotopology on $[0, 1]$ and contains the identity map, so, $\tau \subseteq K$. It remains to show that $K \subseteq \tau$. We do this in two steps.

**Step 1.** If $\phi \in K$ and $\phi \geq \text{id}$, then $\phi \in \tau$.

Since $\&$ is a continuous t-norm, by the ordinal sum decomposition theorem (see Theorem 2.4) there is a set of disjoint open intervals $\{(a_i, b_i)\}$ such that

- for each $i$, both $a_i$ and $b_i$ are idempotent and the restriction of $\&$ on $[a_i, b_i]$ is either isomorphic to the Łukasiewicz t-norm or to the product t-norm;
- $x \& y = \min\{x, y\}$ if $(x, y) \notin \bigcup_i [a_i, b_i]^2$.

For each $x \in [0, 1]$, define $g_x : [0, 1] \rightarrow [0, 1]$ by

$$g_x(y) = \begin{cases} \phi(x) \lor ((\phi(x) \rightarrow x) \rightarrow y), & (x, \phi(x)) \in (a_i, b_i)^2 \text{ for some } i \text{ and } \phi(x) \neq x, \\ \phi(x) \lor (b_i \rightarrow y), & (x, \phi(x)) \in (a_i, b_i)^2 \text{ for some } i \text{ and } \phi(x) = x, \\ \phi(x) \lor (x \rightarrow y), & (x, \phi(x)) \notin (a_i, b_i)^2 \text{ for any } i. \end{cases}$$

Each $g_x$ is clearly a member of $\tau$, so, in order to see that $\phi \in \tau$, it suffices to show that for all $y \in [0, 1]$,

$$\phi(y) = \bigwedge_{x \in [0, 1]} g_x(y).$$

Before proving this equality, we list here some facts about the maps $g_x$, the verifications are left to the reader.

(M1) $\phi(y) \leq g_x(y)$ whenever $y \leq x$.

(M2) If $(x, \phi(x)) \in (a_i, b_i)^2$ for some $i$ and $\phi(x) \neq x$, then $a_i < \phi(x) \rightarrow x < b_i$, $g_x(x) = (\phi(x) \rightarrow x) \rightarrow x = \phi(x)$, and $g_x(y) = (\phi(x) \rightarrow x) \rightarrow y \geq \phi(y)$ for all $y > x$.

(M3) If $(x, \phi(x)) \in (a_i, b_i)^2$ for some $i$ and $\phi(x) = x$, then $\phi(y) = y = g_x(y)$ whenever $x \leq y < b_i$ and $g_x(y) = 1 \geq \phi(y)$ for all $y \geq b_i$.

(M4) If $(x, \phi(x)) \notin (a_i, b_i)^2$ for any $i$, then for all $y \geq x$, $g_x(y) = \phi(x) \lor (x \rightarrow y) = 1 \geq \phi(y)$.

It follows immediately from these facts that for all $y \in [0, 1]$, $\phi(y) \leq \bigwedge_{x \in [0, 1]} g_x(y)$. For the converse inequality, we distinguish three cases.

**Case 1.** $(y, \phi(y)) \in (a_i, b_i)^2$ for some $i$ and $\phi(y) \neq y$. Then by fact (M2), $g_y(y) = \phi(y)$, hence $\phi(y) \geq \bigwedge_{x \in [0, 1]} g_x(y)$.

**Case 2.** $(y, \phi(y)) \in (a_i, b_i)^2$ for some $i$ and $\phi(y) = y$. Then by fact (M3), $g_y(y) = \phi(y)$, hence $\phi(y) \geq \bigwedge_{x \in [0, 1]} g_x(y)$.

**Case 3.** $(y, \phi(y)) \notin (a_i, b_i)^2$ for any $i$. In this case, if we can show that $g_x(y) = \phi(x)$ for all $x > y$, then we will obtain $\phi(y) = \bigwedge_{x > y} \phi(x) \geq \bigwedge_{x \in [0, 1]} g_x(y)$ by right continuity of $\phi$. The proof is divided into four subcases.

Subcase 1. $y \in (a_i, b_i)$ for some $i$ and $\phi(y) \geq b_i$. If $x \leq b_i$, then $x \rightarrow y \leq b_i$ and $\phi(x) \geq \phi(y) \geq b_i$, hence $g_x(y) = \phi(x) \lor (x \rightarrow y) = \phi(x)$. For $x > b_i$,
• if \((x, \phi(x)) \in (a_j, b_j)^2\) for some \(j\) and \(\phi(x) \neq x\), then
\[
g_x(y) = \phi(x) \vee ((\phi(x) \to x) \to y) = \phi(x) \vee y = \phi(x);
\]
• if \((x, \phi(x)) \in (a_j, b_j)^2\) for some \(j\) and \(\phi(x) = x\), then
\[
g_x(y) = \phi(x) \vee (b_j \to y) = \phi(x) \vee y = \phi(x);
\]
• if \((x, \phi(x)) \notin (a_j, b_j)^2\) for any \(j\), then
\[
g_x(y) = \phi(x) \vee (x \to y) = \phi(x) \vee y = \phi(x).
\]

Subcase 2. \(y \notin [a_i, b_i]\) for any \(i\). In this case, since \(t \to y = y\) for all \(t > y\), it follows that
\[
g_x(y) = \phi(x) \vee y = \phi(x).
\]

Subcase 3. \(y = a_i\). For \(x < b_i\),
• if \(x < \phi(x) < b_i\), then \((x, \phi(x)) \in (a_i, b_i)^2\), hence \(g_x(y) = \phi(x) \vee ((\phi(x) \to x) \to a_i) = \phi(x);
• if \(\phi(x) \geq b_i\), then \(g_x(y) = \phi(x) \vee (x \to y) = \phi(x)\) since \(x \to y = x \to a_i \leq b_i;\)
• if \(\phi(x) = x\), then \(g_x(y) = \phi(x) \vee (b_i \to y) = \phi(x) \vee (b_i \to a_i) = \phi(x).
\]

For \(x > b_i\),
• if \((x, \phi(x)) \in (a_j, b_j)^2\) for some \(j\) and \(\phi(x) \neq x\), then \(a_i < a_j\), hence
\[
g_x(y) = \phi(x) \vee ((\phi(x) \to x) \to a_i) = \phi(x) \vee a_i = \phi(x);
\]
• if \((x, \phi(x)) \in (a_j, b_j)^2\) for some \(j\) and \(\phi(x) = x\), then \(g_x(y) = \phi(x) \vee (b_j \to a_i) = \phi(x);
• if \((x, \phi(x)) \notin (a_j, b_j)^2\) for any \(j\), then \(g_x(y) = \phi(x) \vee (x \to a_i) = \phi(x).
\]

Subcase 4. \(y = b_i\). If \(a_j = b_i\) for some \(j\), then the conclusion holds by Subcase 3. Otherwise, the argument for Subcase 2 can be applied to show that \(g_x(y) = \phi(x)\).

**Step 2.** \(K \subseteq \tau\).

Let \(\phi \in K\). Since \(\phi(1) \to \phi \in K\) and \(id \leq \phi(1) \to \phi\), it follows that \(\phi(1) \to \phi \in \tau\) by Step 1. Since \(\tau\) is strong and \& is continuous, then \(\phi = \phi(1) \& (\phi(1) \to \phi) \in \tau\).

**Corollary 5.9.** Let \(Q = ([0,1], \&\) with \& being a left continuous \(t\)-norm. Then following are equivalent:

1. \& is a continuous \(t\)-norm.
2. The \(I\)-Scott \(Q\)-cotopology on each \(Q\)-ordered set is a strong \(Q\)-cotopology.

**Proof.** (1) \(\Rightarrow\) (2) We only need to check that if \(\phi\) is a \(I\)-closed fuzzy set of a \(Q\)-ordered set \(A\), then so is \(a\&\phi\) for all \(a \in [0,1]\). By Proposition 5.4, \(\phi : A \to ([0,1], d_R)\) is \(I\)-cocontinuous. By Example 5.8, \(a\&id : ([0,1], d_R) \to ([0,1], d_R)\) is \(I\)-cocontinuous. Thus, \(a\&\phi = (a\&id) \circ \phi : A \to ([0,1], d_R)\) is \(I\)-cocontinuous, hence \(I\)-closed.

(2) \(\Rightarrow\) (1) If \& is not continuous, by Proposition 1.19 in [20] there is some \(a \in [0,1]\) such that \(a\&id : [0,1] \to [0,1]\) is not right continuous, hence not a \(I\)-closed set in \(([0,1], d_R)\).

Since the identity map \(id\) is \(I\)-closed in \(([0,1], d_R)\), the \(I\)-Scott \(Q\)-cotopology on \(([0,1], d_R)\) cannot be a strong \(Q\)-cotopology. \(\Box\)
Example 5.10. Let $Q = ([0, 1], \&)$ with $\&$ being a left continuous t-norm. By Corollary 3.14, a fuzzy lower set $φ$ of $([0, 1], d_L)$ is an irreducible ideal if and only if either $φ(x) = x \to a$ for some $a \in [0, 1]$ or $φ(x) = \bigvee_{b < a} (x \to b)$ for some $a > 0$. Since the supremum of $\bigvee_{b < a} (x \to b)$ is $a$, a fuzzy lower set $ψ$ of $([0, 1], d_L)$ is $I$-closed if and only if for all $a > 0$,

$$\text{sub}_{[0,1]} \left( \bigvee_{b < a} (x \to b), ψ \right) = \bigwedge_{b < a} ψ(b) \leq ψ(a).$$

Therefore, a fuzzy set $ψ$ of $[0, 1]$ is $I$-closed in $([0, 1], d_L)$ if and only if $ψ$ is a fuzzy lower set of $([0, 1], d_L)$ and, as a map, $ψ : [0, 1] \to [0, 1]$ is left continuous.

For the class $W$ of forward Cauchy ideals, the following lemma is Proposition 4.15 in [40].

Lemma 5.11. Let $Φ$ be a class of fuzzy lower sets. For each map $f : A \to B$ between $Q$-ordered sets, the following are equivalent:

(1) $f : A \to B$ is $Φ$-cocontinuous.

(2) For each $Φ$-closed fuzzy set $φ$ of $B$, $φ \circ f$ is a $Φ$-closed fuzzy set of $A$.

Proof. (1) $\Rightarrow$ (2) This follows from Lemma 5.1 immediately, since the composite of $Φ$-cocontinuous maps is $Φ$-cocontinuous.

(2) $\Rightarrow$ (1) First, we show that $f$ preserves $Q$-order. For any $a_1, a_2 \in A$, since $ψ = B(-, f(a_2))$ is a $Φ$-closed fuzzy set of $B$, then $ψ \circ f = B(f(-), f(a_2))$ is $Φ$-closed, hence

$$A(a_1, a_2) = B(f(a_2), f(a_2)) \& A(a_1, a_2) = ψ \circ f (a_2) \& A(a_1, a_2) \leq ψ \circ f (a_1) = B(f(a_1), f(a_2)),$$

showing that $f$ preserves $Q$-order.

Second, we show that for each $φ \in Φ(A)$, if $\text{sup}_A φ$ exists, then for all $b \in B$,

$$\text{sub}_B(f^+(φ), B(−, b)) = B(f(\text{sup}_A φ), b),$$

hence $f(\text{sup}_A φ)$ is a supremum of $f^+(φ)$. On one hand, since $B(−, b)$ is $Φ$-closed fuzzy set of $B$, $B(−, b) \circ f$ is a $Φ$-closed fuzzy set of $A$, hence

$$\text{sub}_B(f^+(φ), B(−, b)) = \text{sub}_A(φ, B(−, b) \circ f) \leq B(f(\text{sup}_A φ), b).$$

On the other hand, we have by definition that

$$1 = A(\text{sup}_A φ, \text{sup}_A φ) = \text{sub}_A(φ, A(−, \text{sup}_A φ)) = \bigwedge_{a \in A} (φ(a) \to A(a, \text{sup}_A φ)),$$

then $φ(a) \leq A(a, \text{sup}_A φ)$ for all $a \in A$. Since $f$ preserves $Q$-order, it follows that

$$B(f(\text{sup}_A φ), b) \& φ(a) \leq B(f(\text{sup}_A φ), b) \& B(f(a), f(\text{sup}_A φ)) \leq B(f(a), b).$$

Therefore,

$$B(f(\text{sup}_A φ), b) \leq \bigwedge_{a \in A} (φ(a) \to B(f(a), b)) = \text{sub}_A(φ, B(−, b) \circ f) = \text{sub}_B(f^+(φ), B(−, b)).$$

This completes the proof. □
Therefore, for a subclass $\Phi$ of irreducible ideals, assigning each $\mathcal{Q}$-ordered set $A$ to the $\mathcal{Q}$-cotopological space $\Sigma^\mathcal{Q}_\Phi(A)$ defines a full and faithful functor $\Sigma^\mathcal{Q}_\Phi$ from the category of $\mathcal{Q}$-ordered sets and $\Phi$-cocontinuous maps to that of stratified $\mathcal{Q}$-cotopological spaces.

**Definition 5.12.** Given a class $\Phi$ of fuzzy sets, a fuzzy set $\psi : A \to \mathcal{Q}$ of a $\mathcal{Q}$-ordered set $A$ is $\Phi$-open if it is a fuzzy upper set and for all $\phi \in \Phi(A)$,

$$\psi(\sup \phi) \leq \phi \otimes \psi = \bigvee_{x \in A} \phi(x) \& \psi(x)$$

whenever $\sup \phi$ exists.

Since $\phi \otimes \psi$ is the supremum of $\psi^\frown(\phi)$ (see Example 2.11), it is easy to see that $\psi$ is $\Phi$-open if and only if $\psi : A \to (\mathcal{Q}, d_L)$ is $\Phi$-cocontinuous.

**Lemma 5.13.** Let $\Phi$ be a class of fuzzy sets, $A$ a $\mathcal{Q}$-ordered set.

1. Each constant fuzzy set is $\Phi$-open.
2. If $\psi$ is a $\Phi$-open fuzzy set of $A$, then so is $p \& \psi$ for all $p \in \mathcal{Q}$.
3. The join of each set of $\Phi$-open fuzzy sets is $\Phi$-open.
4. If $\Phi(A) \subseteq \mathcal{F}(A)$, then the meet of two $\Phi$-open fuzzy sets is $\Phi$-open.

**Proposition 5.14.** Let $\Phi$ be a subclass of flat ideals. Then for each $\mathcal{Q}$-ordered set $A$, the $\Phi$-open fuzzy sets of $A$ form a stratified $\mathcal{Q}$-topology on $A$, called the $\Phi$-Scott $\mathcal{Q}$-topology and denoted by $\Sigma_\Phi(A)$. In particular, if $\mathcal{Q}$ is meet continuous, the $W$-open fuzzy sets of each $\mathcal{Q}$-ordered set form a stratified $\mathcal{Q}$-cotopology.

**Proposition 5.15.** Let $\Phi$ be a subclass of flat ideals. Then for each $\Phi$-cocontinuous map $f : A \to B$ between $\mathcal{Q}$-ordered sets, $f : (A, \Sigma_\Phi(A)) \to (B, \Sigma_\Phi(B))$ is continuous.

Therefore, for a subclass $\Phi$ of flat ideals, assigning each $\mathcal{Q}$-ordered set $A$ to the $\mathcal{Q}$-cotopological space $\Sigma_\Phi(A)$ yields a functor $\Sigma_\Phi$ from the category of $\mathcal{Q}$-ordered sets and $\Phi$-cocontinuous maps to that of stratified $\mathcal{Q}$-cotopological spaces. But, we do not know whether $\Sigma_\Phi$ is a full functor (c.f. Lemma 5.11).

**Remark 5.16.** (1) Let $\{x_i\}$ be a forward Cauchy net in a $\mathcal{Q}$-ordered set $A$ and

$$\varphi = \bigvee_i \bigwedge_{j \geq i} A(-, x_j).$$

For each fuzzy upper set $\psi$ of $A$, by the argument of Theorem 3.15, we have

$$\varphi \otimes \psi = \bigvee_i \bigwedge_{j \geq i} \psi(x_j).$$

Thus, a fuzzy upper set $\psi$ of $A$ is $W$-open if and only if

$$\bigvee_{i} \bigwedge_{j \geq i} \psi(x_j) \geq \psi(x)$$

for every forward Cauchy net $\{x_i\}$ with a Yoneda limit $x$. This shows that $W$-open fuzzy sets are the Scott open fuzzy sets in the sense of Wagner ([40], Definition 4.1). In particular, if $\mathcal{Q}$
is meet continuous, $W$-open fuzzy sets (i.e., Scott open fuzzy sets in the sense of Wagner) of $A$ form a stratified $Q$-topology on $A$.

(2) Lemma 4.6 in [40] claims that if the fuzzy sets $\phi, \psi$ of a $Q$-ordered set $A$ are $W$-open, then so is the fuzzy set $\phi \& \psi : A \to Q$ given by $(\phi \& \psi)(x) = \phi(x) \& \psi(x)$. This is not true in general. Let $Q$ be the unit interval $[0, 1]$ coupled with the product t-norm $\&$. For each class of fuzzy sets $\Phi$, the identity map $id$ is clearly $\Phi$-open in the $Q$-ordered set $([0, 1], d_L)$, but, $id \& id$ is not a fuzzy upper set of $([0, 1], d_L)$.

(3) If $Q = (Q, \&)$ is a frame, i.e., $\& = \land$, then a fuzzy set $\psi$ of a $Q$-ordered set $A$ is $F$-open if and only if it is fuzzy Scott open in the sense of Yao ([43], Definition 2.10) since the fuzzy ideals in $[43]$ are exactly the flat ideals, as noted in Remark 3.5.

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