LIOUVILLE INTEGRABILITY OF A CLASS OF INTEGRABLE SPIN CALOGERO-MOSER SYSTEMS AND EXPONENTS OF SIMPLE LIE ALGEBRAS

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ABSTRACT. In previous work, we introduced a class of integrable spin Calogero-Moser systems associated with the classical dynamical r-matrices with spectral parameter, as classified by Etingof and Varchenko for simple Lie algebras. Here the main purpose is to establish the Liouville integrability of these systems by a uniform method.

1. Introduction.

Systems of spin Calogero-Moser (CM) type are Hamiltonian systems with very rich structures. After the initial example of Gibbons and Hermen [GH], a variety of such systems have appeared in the literature over the years. (See, for example, [BAB1, BAB2, FP, HH, L1, L3, LX1, LX2, MP, Pech, P, Wo, Y] and the references therein.) This is a testimonial to the relevance of such systems in various areas of mathematics and physics. In [LX1, LX2], as a by-product of an effort to understand conceptually the calculations in [BAB1, BAB2], we introduced a class of spin CM systems associated with the classical dynamical r-matrices with spectral parameter, as defined and classified by Etingof and Varchenko for complex simple Lie algebras [EV]. The classical dynamical r-matrices with spectral parameter in [EV] are solutions of the classical dynamical Yang-Baxter equation (CDYBE) with spectral parameter, which was first introduced and studied by Felder [F]. While Felder studied CDYBE in the context of conformal field theory, we showed how to make use of the solutions of this equation to construct and to study our spin systems. Indeed, in [L2], we showed how to obtain the explicit solutions of the integrable spin CM systems in [LX2] by using the factorization method developed in [L1]. That this is possible is due to some remarkable geometric structures underlying the so-called modified dynamical Yang-Baxter equation (mDYBE) [L1]. This work is a sequel.
to [LX2] and [L2]. Our main purpose here is to establish the Liouville integrability of the integrable spin CM systems in [LX2] on generic symplectic leaves.

The spin CM systems constructed in [LX1, LX2] are of three types: rational, trigonometric, and elliptic, as in the case of their spinless counterparts in [OP]. In the rational case, recall that we have a family of rational spin CM systems parametrized by subsets $\Delta' \subset \Delta$ which are closed with respect to addition and multiplication by $-1$. Here $\Delta$ is the root system associated with a complex simple Lie algebra $\mathfrak{g}$ and a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In the trigonometric case, there is also a family but now the systems are parametrized by subsets $\pi'$ of a fixed simple system $\pi \subset \Delta$. Finally we have an elliptic spin CM system for each complex simple Lie algebra. In [LX1,LX2], generalized Lax operators taking values in the dual bundles $A^\Gamma$ of the corresponding coboundary dynamical Lie algebroids $A^\ast \Gamma$ were constructed for these systems. If we let $H$ denote a connected Lie group corresponding to $\mathfrak{h}$, then recall that in each case, the phase space $P$ of the spin CM system is a Hamiltonian $H$-space (with equivariant momentum map $J$) which admits an $H$-equivariant realization in the corresponding $A^\Gamma$ and the Hamiltonian is the pullback of a natural invariant function on $A^\Gamma$ under the realization map. It is a characteristic of these systems that the pullback of natural invariant functions on $A^\Gamma$ to $P$ do not Poisson commute everywhere, but they do so on $J^{-1}(0)$ in all cases. Hence we can obtain the integrable spin systems on $J^{-1}(0)/H$ by Poisson reduction. It should be pointed out in our setup, the Lax operator $L$ is only part of the generalized Lax operator. Indeed, for the rational (resp. trigonometric) case, if $\Delta' \neq \Delta$ (resp. $< \pi' > \neq \Delta$), the equation of motion for $L$ only carries partial information about the dynamics and it is necessary to obtain the missing piece from the other parts of the generalized Lax operator [L2]. Nevertheless, as the reader will see, the Lax operator suffices when we consider Liouville integrability.

The method we use to establish the Liouville integrability of the integrable spin CM systems can be explained as follows. Let us begin with a familiar situation. In the usual classical $R$-matrix theory, it is well-known that if the Lax operator $L$ of a finite-dimensional system takes values in a loop algebra $L\mathfrak{g}$, (i.e., $L$ depends on a spectral parameter), then one can obtain integrals in involution by pulling back the ad-invariant functions on $L\mathfrak{g}$ using $L$. However, due to the finite-dimensionality of the system, we cannot expect the integrals obtained in this way to be functionally independent or nonzero even though the ring of ad-invariant functions of $L\mathfrak{g}$ has an infinite number of generators. In the case when $\mathfrak{g} \subset gl(n, \mathbb{C})$ for some $n$, say, then of course there is a standard way to construct a finite collection of Poisson commuting
integrals. Namely, one simply writes down the characteristic polynomial of $L(z)$ and in this case, the completeness of the integrals can be addressed by algebro-geometric means provided certain additional conditions are satisfied [RSTS]. (For an example where there exists a family of additional integrals besides the ones given by the characteristic polynomial of $L(z)$, see [DLT].) In our case, we can of course take concrete representations of the simple Lie algebras, however, an intrinsic way to construct the necessary integrals which works for all simple Lie algebras is clearly preferred. What we essentially do in this work is to substitute the elementary symmetric functions in the matrix case by the primitive invariant polynomials of Chevalley, which are homogeneous with degrees related to the exponents of the simple Lie algebras. (See, for example, [C, K2, V].) To obtain the quantities of interest, we simply evaluate the primitive invariants on the Lax operator and these give Poisson commuting integrals on $J^{-1}(0)$ by the general theory in [LX2, L1].

To count the number of nontrivial integrals obtained in this manner, our basic realization is that we can appeal to a theorem of Shephard and Todd [ST] relating the sum of exponents of a complex simple Lie algebra to the number of roots of $(\mathfrak{g}, \mathfrak{h})$. Of course, there remains the task of showing that the nontrivial integrals are functionally independent on an open, dense set of the phase space. As the reader will see, we can also accomplish this in a uniform way due to some common structure which exists among the three types of spin CM systems. To conclude, we remark that the method which we develop here to construct and count the number of integrals is a general method. In principle, it should work for other systems associated with simple Lie algebras and with spectral parameter dependent Lax operators. Thus what we show in this work is just an illustration of this general method. Furthermore, some of our analysis involving invariant polynomials (see Lemma 3.1 and Theorem 6.4) may also be of independent interest in Lie theory.

The paper is organized as follows. In Section 2, we present for the most part some background material for the reader’s convenience, we also take the opportunity to set up the notations. In the first subsection, we begin by summarizing some basic facts about the invariant polynomials of Chevalley and the exponents of simple Lie algebras which are of relevance here. We also recall some of the tools which we find useful in dealing with these invariant polynomials. (Further tools will be developed in subsequent sections.) In the second subsection, we recall the construction of the class of spin Calogero-Moser systems associated with the classical dynamical r-matrices with spectral parameter. We then explain how Poisson reduction gives rise to the associated integrable models. At the end of the subsection, we conclude
with our first result, namely, the connection between the dimension of the maximal dimensional phase spaces of our systems and the exponents of the complex simple Lie algebras. In Section 3, we construct the integrals for the rational case by evaluating the primitive invariants on the Lax operators and we count the number of nontrivial integrals. As it turns out, for each primitive invariant $I_k$, exactly one of the quantities which arise in the expansion of $I_k(L(z))$ in $z$ is identically zero in this case while another one is a Casimir function. In Section 4 and 5, we do the same for the trigonometric case and the elliptic case. Finally, in Section 6, we show that the integrals constructed in Sections 3-5 are functionally independent, thus proving the Liouville integrability of the systems on generic symplectic leaves.

2. Preliminaries.

In [LX2], we introduced a class of integrable spin Calogero-Moser systems associated with the classical dynamical $r$-matrices with spectral parameter, as classified by Etingof and Varchenko [EV] for complex simple Lie algebras. Our goal in this section is to establish Proposition 2.2.6 which gives the dimension of the maximal dimensional phase spaces of such systems in terms of the exponents of the complex simple Lie algebras. For the reader’s convenience, we will provide some background material, we will also take the opportunity to set up the notations. In the first subsection, we will begin by summarizing a number of basic facts about the invariant polynomials of Chevalley and the exponents of simple Lie algebras. We will also collect here some of the tools which we find useful in dealing with these polynomials. In the second subsection, we will recall the class of integrable spin Calogero-Moser systems in [LX2]. Then we will present our first result which we alluded to above, thus tying together the two subsections.

2.1 The invariant polynomials of Chevalley and the exponents.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $N$, and let $G$ be a connected Lie group with $Lie(G) = \mathfrak{g}$. We recall that the group $G$ acts on the algebra $\mathcal{P}(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$ by $g \cdot P = P^g$, $g \in G$, $P \in \mathcal{P}(\mathfrak{g})$, where

$$P^g(x) = P(Ad_g^{-1}x), \quad x \in \mathfrak{g}. \quad (2.1.1)$$

Let $I(\mathfrak{g})$ denote the ring of polynomial functions on $\mathfrak{g}$ invariant under the above action of $G$. Then the well-known theorem of Chevalley [C] asserts that $I(\mathfrak{g})$ is generated by $N$ algebraically independent homogeneous polynomials $I_1, \ldots, I_N$. 
In other words, if $C[Y_1, \ldots , Y_N]$ denotes the polynomial ring in the $N$ variables $Y_1, \ldots , Y_N$, then

$$I(g) = C[I_1, \ldots , I_N].$$

(2.1.2)

Let us denote by $\mathfrak{h}$ a fixed Cartan subalgebra of $g$, and let $W$ be the Weyl group of the pair $(g, \mathfrak{h})$ generated by reflections in the hyperplanes in $\mathfrak{h}$. Then indeed the restriction of $I(g)$ to $\mathfrak{h}$ is an algebra isomorphism of $I(g)$ onto the algebra $P(\mathfrak{h})^W$ of polynomials on $\mathfrak{h}$ which are invariant under $W$. Write

$$\text{deg } I_k = d_k, \quad k = 1, \ldots , N.$$  

(2.1.3)

We will assume that the $I_k$’s are ordered in the sense that

$$d_1 \leq d_2 \leq \ldots \leq d_N.$$  

(2.1.4)

Following Kostant [K2], we will refer to the $I_k$’s as the primitive invariants. The numbers $m_k = d_k - 1$, $k = 1, \ldots , N$, are called the exponents of $g$ and are the basic invariants of $g$ [B]. (See also [CM] and the references therein.) For the purpose in this work, we will need the following results due to Shephard and Todd [ST].

**Theorem 2.1.1** [ST]. Let $m_k$ be the exponents of $g$, $k = 1, \cdots , N$. Then

$$\sum_{k=1}^{N} m_k = \# \text{ of reflections in } W$$

$$= \frac{1}{2}(\# \text{ of roots of } (g, \mathfrak{h}))$$

(2.1.5)

$$= \frac{1}{2}(\dim g - N).$$

As the reader will see, (2.1.5) is crucial in counting the total number of integrals which we construct by evaluating the primitive invariants on the Lax operators of the integrable spin Calogero-Moser systems. We will explain this in the next subsection below.

While the problem of computing the exponents was originally motivated by the problem of computing the Betti numbers of complex simple Lie groups, it turns out that there is a different way to describe these numbers which is relevant for us. For this purpose, let us introduce some notation. First of all, we will assume from now on that $g$ is simple with Cartan subalgebra $\mathfrak{h}$ and Killing form $(\cdot , \cdot )$, and let $g = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} g_\alpha$ be the root space decomposition of $g$ with respect to $\mathfrak{h}$. We fix a simple system of roots $\pi = \{\alpha_1, \ldots , \alpha_N\}$, and denote by $\Delta^\pm$ the corresponding
positive/negative system. If \( \alpha \in \Delta^+ \), recall that we can express \( \alpha \) uniquely as a sum of simple root \( \sum_{i=1}^{N} n_i \alpha_i \), where \( n_i \) are non-negative integers. The height of \( \alpha \) is defined to be the number
\[
\text{ht}(\alpha) = \sum_{i=1}^{N} n_i.
\]

**Theorem 2.1.2 [K1].** If \( b_j \) is the number of \( \alpha \in \Delta^+ \) such that \( \text{ht}(\alpha) = j \), then

(a) \( b_j - b_{j+1} \) is the number of times \( j \) appears as an exponent of \( \mathfrak{g} \).

(b) \( N = b_1 \geq b_2 \geq \cdots \geq b_{h-1} = 1 \), where \( h \) is the Coxeter number. Moreover, the partition of \( r = |\Delta^+| \) as defined by the above sequence of numbers is conjugate to the partition \( h - 1 = m_N \geq m_{N-1} \geq \cdots \geq m_1 = 1 \).

In the rest of the subsection, we will summarize a number of basic facts from [K2] that we will use in Section 3 and more significantly in Section 6 below. To begin with, we recall that the symmetric algebra \( S = S(\mathfrak{g}^*) \) can be identified with \( P(\mathfrak{g}) \)

On the other hand, we can associate to each \( x \in \mathfrak{g} \) a differential operator \( \partial_x \) on \( \mathfrak{g} \), defined by
\[
\partial_x f(y) = \left. \frac{d}{dt} \right|_{t=0} f(y + tx), \quad f \in C^\infty(\mathfrak{g}).
\]

In this way we have a linear map \( x \mapsto \partial_x \) which can be extended to an isomorphism from the symmetric algebra \( S_* = S(\mathfrak{g}) \) to the algebra of differential operators \( \partial \) with constant coefficients on \( \mathfrak{g} \). From now onwards we will identify the two spaces and with this identification, we have a nondegenerate pairing between \( S_* \) and \( S \) given by
\[
\langle \partial, f \rangle = \partial f(0),
\]
where \( \partial \in S_* \), \( f \in S \), and \( \partial f(0) \) denotes the value of the function \( \partial f \) at 0 \( \in \mathfrak{g} \). It is obvious that both \( S_* \) and \( S \) are graded: \( S_* = \oplus_{j \geq 0} S_j \), \( S = \oplus_{j \geq 0} S^j \). If \( f \in S^m \) and \( x \in \mathfrak{g} \), it follows from the Taylor expansion that
\[
\left\langle \left( \frac{\partial_x}{m!} \right)^m, f \right\rangle = f(x).
\]

Now \( S \) is a \( G \)-module by (2.1.1). On the other hand, it is clear that the adjoint action of \( G \) on \( \mathfrak{g} \) can be naturally extended to an action of \( G \) on \( S_* \). Therefore in view of (2.1.1), we have
\[
\langle g \cdot \partial, f \rangle = \langle \partial, f \rangle,
\]
for all \( g \in G \), \( \partial \in S_* \) and \( f \in S \). By differentiation, \( S \) and \( S^* \) become \( \mathfrak{g} \)-modules and the actions of \( \mathfrak{g} \) on both spaces are by derivations. Therefore we have the “product rule” and the “power rule”:
\[
x \cdot (\partial \delta) = (x \cdot \partial) \delta + \partial(x \cdot \delta),
\]
for all \( g \in G \), \( \partial \in S_* \) and \( f \in S \). By differentiation, \( S \) and \( S^* \) become \( \mathfrak{g} \)-modules and the actions of \( \mathfrak{g} \) on both spaces are by derivations. Therefore we have the “product rule” and the “power rule”:
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for all \( g \in G \), \( \partial \in S_* \) and \( f \in S \). By differentiation, \( S \) and \( S^* \) become \( \mathfrak{g} \)-modules and the actions of \( \mathfrak{g} \) on both spaces are by derivations. Therefore we have the “product rule” and the “power rule”:
\[ x \cdot \partial^n = n \partial^{n-1}(x \cdot \partial), \quad (2.1.12) \]

for all \( x \in g, \partial, \delta \in S_* \), and \( n \in \mathbb{N} \). For \( y \in g \), we also have

\[ x \cdot \partial_y = \partial_{[x,y]}. \quad (2.1.13) \]

Since the pairing between \( S_* \) and \( S \) obeys (2.1.10), it follows that

\[ \langle x \cdot \partial, f \rangle + \langle \partial, x \cdot f \rangle = 0 \quad (2.1.14) \]

for all \( x \in g, f \in S \). In particular, this implies that

\[ \langle x \cdot \partial, f \rangle = 0, \quad \text{for all } f \in I(g) \quad (2.1.15) \]

since \( x \cdot f = 0 \) for \( f \in I(g) \).

Now let \( x_0 \) be the unique element in \( h \) such that \( \alpha_i(x_0) = 1, \quad i = 1, \cdots, N \). Then \( \alpha(x_0) = \text{ht}(\alpha) \), and \( [x_0, e_\alpha] = \text{ht}(\alpha)e_\alpha \) for all \( \alpha \in \Delta \). For each \( j \in \mathbb{Z} \), let

\[ S_*^{(j)} = \{ \partial \in S_*| x_0 \cdot \partial = j\partial \}. \quad (2.1.16) \]

Then \( S_* = \sum_{j \in \mathbb{Z}} S_*^{(j)} \), and if \( \partial \in S_*^{(j)} \), we will say \( \partial \) has weight \( j \). Clearly, we have

\[ \partial_{e_\alpha} \in S_*^{(\text{ht}(\alpha))}, \quad \partial_p \in S_*^{(0)} \quad (2.1.17) \]

for \( \alpha \in \Delta, \quad p \in h \). Also,

\[ S_*^{(j)} S_*^{(j)} \subseteq S_*^{(i+j)}. \quad (2.1.18) \]

The following consequence of (2.1.15) is very important to us in Section 6 below. If \( \partial \in S_*^{(j)} \) for \( j \neq 0 \), then \( \partial = \frac{1}{j}x_0 \cdot \partial \) and hence we have

\[ \langle \partial, f \rangle = 0 \quad \text{for all } f \in I(g). \quad (2.1.19) \]

Analogously, we let \( g^{(j)} \) be the eigenspace of \( \text{ad} x_0 \) for the eigenvalue \( j \). Then \( g = \bigoplus_{j \in \mathbb{Z}} g^{(j)} \) and

\[ [g^{(i)}, g^{(j)}] \subseteq g^{(i+j)}. \quad (2.1.20) \]

### 2.2 A class of integrable spin Calogero-Moser systems and their phase spaces.

We recall that \( g \) is a complex simple Lie algebra with Cartan subalgebra \( fh \), and \( \Delta^\pm \) are the positive/negative system relative to a fixed simple system \( \pi \) of roots. For each positive root \( \alpha \in \Delta^+ \), let \( e_\alpha \in g_\alpha \) and \( e_{-\alpha} \in g_{-\alpha} \) be root vectors which
are dual with respect to $(\cdot, \cdot)$ so that $[e_\alpha, e_{-\alpha}] = H_\alpha$, where the latter is the unique element in $\mathfrak{h}$ which corresponds to $\alpha$ under the isomorphism induced by the Killing form $(\cdot, \cdot)$. We also fix an orthonormal basis $(x_i)_{1 \leq i \leq N}$ of $\mathfrak{h}$, and write $p = \sum_i p_i x_i$, $\xi = \sum_i \xi_i x_i + \sum_{\alpha \in \Delta} \xi_\alpha e_\alpha$ for $p \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$. Lastly, we let $H$ be a connected Lie subgroup of $G$ with $\text{Lie}(H) = \mathfrak{h}$.

Let $\rho$ be a classical dynamical $r$-matrix with spectral parameter in the sense of [EV], with coupling constant equal to 1. We will fix a simply connected set $U \subset \mathfrak{h}$ on which $r(\cdot, z)$ is holomorphic. By Proposition 4.5 of [LX2], we can construct the associated $H$-equivariant classical dynamical $r$-matrix $R : U \rightarrow L(L_\mathfrak{g}, L_\mathfrak{g})$, where $L_\mathfrak{g}$ is the loop algebra of $\mathfrak{g}$ and $L(L_\mathfrak{g}, L_\mathfrak{g})$ is the space of linear maps on $L_\mathfrak{g}$. Indeed, it was established in [LX2] that $R$ is a solution of the modified dynamical Yang-Baxter equation (mDYBE). Hence we can equip $A^* \Gamma = T^* U \times L_\mathfrak{g}^* \simeq TU \times L_\mathfrak{g}$ (where $L_\mathfrak{g}^*$ is the restricted dual of $L_\mathfrak{g}$) with a Lie algebroid structure, the so-called coboundary dynamical Lie algebroid associated to $R$. Our construction of the class of spin Calogero-Moser system and its realization is based on the following result.

**Theorem 2.2.1 [LX2].** The map $\rho = (m, \tau, L) : A^* \Omega \simeq TU \times \mathfrak{g} \rightarrow TU \times L_\mathfrak{g} \simeq A^\Gamma$ given by

$$\rho(q, p, \xi) = (q, -\Pi_\mathfrak{h} \xi, p + r^\#(q)\xi) \quad (2.2.1)$$

is an $H$-equivariant Poisson map, when the domain is equipped with the Lie-Poisson structure corresponding to the trivial Lie algebroid $A\Omega \simeq TU \times \mathfrak{g}$, and the target is equipped with the Lie-Poisson structure corresponding to $A^\Gamma$. Here, $H$ acts on $TU \times \mathfrak{g}$ and $TU \times L_\mathfrak{g}$ by acting on the second factors with the adjoint action and the map $r^\#(q) : \mathfrak{g} \rightarrow L_\mathfrak{g}$ is defined by

$$((r^\#(q)\xi)(z), \eta) = (r(q, z), \eta \otimes \xi) \quad (2.2.2)$$

for $\xi, \eta \in \mathfrak{g}$.

**Remark 2.2.2.** (a) The Lie-Poisson structure on the dual of the trivial Lie algebroid $A\Omega \simeq TU \times \mathfrak{g}$ is given by $\{\phi, \psi\}_{A^\Gamma}(q, p, \xi) = (\delta_2 \phi, \delta_1 \psi) - (\delta_1 \phi, \delta_2 \psi) + (\xi, [\delta \phi, \delta \psi])$ [L1]. Thus the Poisson structure is a product structure, where $T^* U \simeq TU$ is equipped with the canonical structure, and $\mathfrak{g}^* \simeq \mathfrak{g}$ is equipped with the Lie-Poisson structure. Moreover, the $H$-action on $TU \times \mathfrak{g}$ above is a canonical action with equivariant momentum map $J : TU \times \mathfrak{g} \rightarrow \mathfrak{h}, (q, p, \xi) \mapsto -\Pi_\mathfrak{h} \xi$, where $\Pi_\mathfrak{h}$ is the projection map to $\mathfrak{h}$ relative to the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.

(b) As a special case of Proposition 3.1 in [L1], the dual bundle $A^\Gamma$ equipped with
the Lie-Poisson structure and $H$-action as defined in the above theorem is also a Hamiltonian $H$-space. The corresponding equivariant momentum map $\gamma : A\Gamma \rightarrow \mathfrak{h}$ is given by the simple formula $\gamma(q,p,X) = p$.

(c) The map $\rho = (m, \tau, L)$ in the above theorem is to be regarded as the generalized Lax operator of the corresponding spin CM system whose construction we recall below. The map $L$, on the other hand, is the Lax operator.

Let $Q$ be the quadratic function

$$Q(\xi) = \frac{1}{2} \oint_c (\xi(z), \xi(z)) \frac{dz}{2\pi i z}$$

where $c$ is a small circle around the origin. Clearly, $Q$ is an ad-invariant function on $Lg$.

**Definition 2.2.3** [LX2]. Let $r$ be a classical dynamical $r$-matrix with spectral parameter with coupling constant equal to 1. Then the Hamiltonian system on $A^*\Omega \simeq TU \times g$ (equipped with the Lie-Poisson structure as in Theorem 2.2.1) generated by the $H$-invariant Hamiltonian

$$\mathcal{H}(q,p,\xi) = (L^*Q)(q,p,\xi) = \frac{1}{2} \oint_c (\xi(z), \xi(z)) \frac{dz}{2\pi i z}$$

is called the spin Calogero-Moser system associated to $r$.

Note that the pullback of ad-invariant functions on $Lg$ by the Lax operator $L$ do not Poisson commute everywhere. In order to construct the integrable spin CM systems, we have to invoke Poisson reduction [MR, OR]. For this purpose, let $Pr_i$ be the projection map onto the $i$-th factor of $U \times \mathfrak{h} \times Lg \simeq A\Gamma, i = 1, 2, 3$, and let $\pi_0 : J^{-1}(0) \rightarrow J^{-1}(0)/H, \pi_H : \gamma^{-1}(0) \rightarrow \gamma^{-1}(0)/H$ be the canonical projections. If $f$ is an ad-invariant function on $Lg$, the unique function on $\gamma^{-1}(0)/H$ determined by $Pr_3^*f|_{\gamma^{-1}(0)}$ will be denoted by $\tilde{f}$, while the unique function on $J^{-1}(0)/H$ determined by $L^*f|_{J^{-1}(0)}$ will be denoted by $\mathcal{F}_0$. Because the map $\rho$ is an $H$-equivariant Poisson map, it induces a unique Poisson map $\hat{\rho} : J^{-1}(0)/H \rightarrow \gamma^{-1}(0)/H$ characterized by $\pi_H \circ \rho|_{J^{-1}(0)} = \hat{\rho} \circ \pi_0$. From the various definitions, we have

$$\mathcal{F}_0 = \hat{\rho}^* \tilde{f}, \quad \mathcal{F}_0 \circ \pi_0 = L^*f|_{J^{-1}(0)}.$$  

In particular, the Hamiltonian $\mathcal{H}$ of the spin CM system in (2.2.4) drops down to $\mathcal{H}_0 = \hat{\rho}^* \tilde{Q}$ on $J^{-1}(0)/H$. 
Theorem 2.2.4 [LX2, L1]. (a) The pullback of ad-invariant functions on $L_{g}$ by $L$ Poisson commute on $J^{-1}(0)$.

(b) Functions $F_{0} = \hat{\rho}^{*} \bar{\tilde{f}}$ corresponding to ad-invariant functions $f$ on $L_{g}$ Poisson commute on the reduced space $J^{-1}(0)/H$.

Remark 2.2.5. (a) The reduced spaces $J^{-1}(0)/H$ and $\gamma^{-1}(0)/H$ are Poisson varieties in the sense of [OR].

(b) The existence of the Poisson map $\hat{\rho}$ and the formulation of Theorem 2.2.4 (b) follow from general result in [L1]. In [LX2], we made an additional assumption, we also did not have $\gamma^{-1}(0)/H$ available at the time. Thus the reduction picture obtained there was not an intrinsic one.

(c) In [L1], we obtain an intrinsic expression for the Lie-Poisson structure on the dual bundle of a coboundary dynamical Lie algebroid, from which it is clear that functions which are obtained as pullback of ad-invariant functions under the map $Pr_{3}$ Poisson commute on $\gamma^{-1}(0)$. Thus in hindsight, the result in Theorem 2.2.4 (a) is just a consequence of this fact, Theorem 2.2.1, and the property that $\rho(J^{-1}(0)) \subset \gamma^{-1}(0)$.

We now restrict to a smooth component of $J^{-1}(0)/H$ and for this purpose, we consider the following open submanifold of $g$:

$$U = \{ \xi \in g \mid \xi_{-\alpha} = (\xi, e_{\alpha}) \neq 0, \quad i = 1, \ldots, N \}. \quad (2.2.6)$$

(Note the convention in (2.2.6) is opposite to that in [LX2].) Then the $H$-action above induces a Hamiltonian $H$-action on $TU \times U$ and we denote the corresponding momentum map also by $J$ so that $J^{-1}(0) = TU \times (h^{\perp} \cap U)$. Now recall from [LX2] that there exists an $H$-equivariant map $g : U \rightarrow H$. Using this map, we can identify the reduced space $J^{-1}(0)/H = TU \times (h^{\perp} \cap U/H)$ with $TU \times g_{\text{red}}$, where $g_{\text{red}} = \epsilon + \sum_{\alpha \in \Delta_{-\pi}} \mathbb{C} e_{\alpha}$, and $\epsilon = \sum_{j=1}^{N} e_{-\alpha_{j}}$. Indeed, the identification map is given by

$$(q, p, [\xi]) \mapsto (q, p, Ad_{g(\xi)^{-1}} \xi). \quad (2.2.7)$$

Thus the natural projection $\pi_{0} : J^{-1}(0) \rightarrow TU \times g_{\text{red}}$ is the map

$$(q, p, \xi) \mapsto (q, p, s = Ad_{g(\xi)^{-1}} \xi). \quad (2.2.8)$$

Consequently, by Poisson reduction [MR], the Poisson structure on $TU \times g_{\text{red}}$ is a product structure, where the second factor $g_{\text{red}}$ is equipped with the reduction (at 0) of the Lie-Poisson structure on $U$. Thus the symplectic leaves of $TU \times g_{\text{red}}$ are of the form $TU \times O_{\text{red}}$, where $O_{\text{red}} = (O \cap U \cap h^{\perp})/H$ and $O$ is an orbit in $g$. 
Proposition 2.2.6. The generic symplectic leaves in $TU \times g_{\text{red}}$ have dimension equal to $\dim g - N = 2 \sum_{k=1}^{N} m_k$.

Proof. Clearly, the generic symplectic leaves in $TU \times g_{\text{red}}$ correspond to generic orbits in $g$. So let $\mathcal{O}$ be a generic orbit in $g$ and let $\mathcal{O}_{\text{red}}$ be the corresponding reduction in $g_{\text{red}}$. It is well-known that $\dim \mathcal{O} = \dim g - N$. (See, for example, [K2].) Therefore, $\dim \mathcal{O}_{\text{red}} = \dim \mathcal{O} - 2N = \dim g - 3N$. Consequently,

$$\text{dimension of } TU \times \mathcal{O}_{\text{red}} = 2N + \dim \mathcal{O}_{\text{red}}$$

$$= \dim g - N. \quad (2.2.9)$$

To complete the proof, it remains to establish the equality $\dim g - N = 2 \sum_{k=1}^{N} m_k$. But this is just the assertion in (2.1.5). $\square$

According to the above proposition, in order to establish the Liouville integrability of the integrable models associated with our spin Calogero-Moser systems, we have to exhibit $\sum_{k=1}^{N} m_k$ nontrivial integrals in involution which are functionally independent on open dense sets of the generic symplectic leaves of $TU \times g_{\text{red}}$. But as the reader will see, each of the primitive invariants $I_k$ when evaluated on the Lax operators will give rise to $m_k = d_k - 1$ such integrals. Hence the total number of nontrivial conserved quantities with the required properties is exactly $\sum_{k=1}^{N} m_k$. This explains the importance of (2.1.5).

3. The rational spin Calogero-Moser systems.

The rational spin Calogero-Moser systems are associated with the rational dynamical $r$-matrices with spectral parameter

$$r(q, z) = \frac{\Omega}{z} + \sum_{\alpha \in \Delta'} \frac{1}{a(q)} e_\alpha \otimes e_{-\alpha}, \quad (3.1)$$

where $\Delta' \subset \Delta$ is any set of roots which is closed with respect to addition and multiplication by $-1$, and $\Omega \in (S^2 g)^{\theta}$ is the Casimir element corresponding to the Killing form $(\cdot, \cdot)$. Therefore, the Hamiltonians are given explicitly by

$$\mathcal{H}(q, p, \xi) = \frac{1}{2} \sum_{i} p_i^2 - \frac{1}{2} \sum_{\alpha \in \Delta'} \frac{\xi_\alpha \xi_{-\alpha}}{a(q)^2} \quad (3.2)$$

and the corresponding Lax operators are of the form

$$L(q, p, \xi)(z) = p + \sum_{\alpha \in \Delta'} \frac{\xi_\alpha}{a(q)} e_\alpha + \frac{\xi}{z}. \quad (3.3)$$
From the homogeneity of $I_k$ and the form of $L(q,p,\xi)(z)$ above, $I_k(L(q,p,\xi)(z))$ can be expanded as

$$I_k(L(q,p,\xi)(z)) = \sum_{j=0}^{d_k} I_{kj}(q,p,\xi) z^{-j}. \quad (3.4)$$

As the reader will see, the $I_{kj}$'s are actually identically zero on $J^{-1}(0)$. In order to demonstrate this for all cases, we need to establish the following lemma which is a refinement of (2.1.19) using just weights.

**Lemma 3.1.** Let $X \subset \Delta$. Then for all $f \in I(\mathfrak{g})$, $p \in \mathfrak{h}$,

$$\left\langle \partial_n \prod_{\alpha \in X} \partial_{e_\alpha}^m, f \right\rangle = 0 \quad (3.5)$$

unless $\sum_{\alpha \in X} m_\alpha \alpha = 0$.

**Proof.** For any $h \in \mathfrak{h}$, it follows from (2.1.15) that

$$\left\langle h \cdot \left( \partial_n \prod_{\alpha \in X} \partial_{e_\alpha}^m \right), f \right\rangle = 0. \quad (3.6)$$

Since $\mathfrak{g}$ acts on $S_*$ by derivation, we can use (2.1.11) and (2.1.12) and (2.1.13) to expand the the left hand side of (3.6). This gives

$$\left\langle h \cdot \left( \partial_n \prod_{\alpha \in X} \partial_{e_\alpha}^m \right), f \right\rangle = n \left\langle \partial_p^{n-1}(h \cdot \partial_p) \prod_{\alpha \in X} \partial_{e_\alpha}^m, f \right\rangle + \sum_{\alpha \in X} m_\alpha \left\langle \partial_p^n \prod_{\beta \neq \alpha} \partial_{e_\beta}^m \partial_{e_\alpha}^{m_\alpha-1}(h \cdot \partial_{e_\alpha}), f \right\rangle$$

$$= n \left\langle \partial_p^{n-1}\partial_{[h,p]} \prod_{\alpha \in X} \partial_{e_\alpha}^m, f \right\rangle + \sum_{\alpha \in X} m_\alpha \left\langle \partial_p^n \prod_{\beta \neq \alpha} \partial_{e_\beta}^m \partial_{e_\alpha}^{m_\alpha-1}\partial_{[h,e_\alpha]}, f \right\rangle$$

$$= \left( \sum_{\alpha \in X} m_\alpha \alpha(h) \right) \left\langle \partial_p^n \prod_{\alpha \in X} \partial_{e_\alpha}^m, f \right\rangle.$$  

Therefore if $\sum_{\alpha \in X} m_\alpha \alpha \neq 0$, we must have $\left\langle \partial_p^n \prod_{\alpha \in X} \partial_{e_\alpha}^m, f \right\rangle = 0$. \qed
Proposition 3.2. For each $1 \leq k \leq N$, $I_{k,d}(q,p,\xi) = I_k(\xi)$. Moreover, for $(q,p,\xi) \in J^{-1}(0)$, we have $I_{k1}(q,p,\xi) = 0$. Hence the number of nontrivial integrals $I_{kj}(q,p,s)$ which Poisson commute on $TU \times O_{\text{red}}$ is equal to $\sum_{k=1}^{N} m_k$, where $O_{\text{red}}$ is the reduction of a generic orbit $O$ in $\mathcal{U}$.

Proof. From the homogeneity of $I_k$ and the relation $L(q,p,\xi)(z) = L(q,p,\xi)z$, we have

$$I_k(L(q,p,\xi)(z)) = I_k(L(q,p,\xi)(\infty)) + \frac{1}{z^d_k} I_k(\xi)$$

from which it is immediate that $I_{k,d}(q,p,\xi) = I_k(\xi)$. From the same expansion above and the definition of $I_{k1}$, we also have

$$I_{k1}(q,p,\xi) = \lim_{z \to \infty} z[I_k(L(q,p,\xi)(z)) - I_k(L(q,p,\xi)(\infty))]$$

$$= \frac{d}{dt} \bigg|_{t=0} I_k(L(q,p,\xi)(\infty) + t\xi)$$

$$= (\delta I_k(L(q,p,\xi)(\infty)), \xi).$$

Let us first consider the case where $\Delta' = \Delta$. For $(q,p,\xi) \in J^{-1}(0)$, it is clear that we have

$$\xi = \left[ q, \sum_{\alpha \in \Delta} \frac{\xi_{\alpha}}{\alpha(q)} e_\alpha \right].$$

Therefore, upon substituting into the above expression for $I_{k1}(q,p,\xi)$, we find

$$I_{k1}(q,p,\xi) = (\delta I_k(L(q,p,\xi)(\infty)), [q, L(q,p,\xi)(\infty)])$$

$$= 0$$

as $I_k$ is invariant. In the other case where $\Delta' \neq \Delta$, let $\bar{\Delta} = \Delta \setminus \Delta'$ be the complement of $\Delta'$, then

$$I_{k1}(q,p,\xi) = \left( \delta I_k(L(q,p,\xi)(\infty)), \sum_{\beta \in \bar{\Delta}} \xi_{\beta} e_\beta \right)$$

(3.8)

because $\left( \delta I_k(L(q,p,\xi)(\infty)), \sum_{\beta \in \bar{\Delta}} \xi_{\beta} e_\beta \right) = 0$ by the same reasoning as in the previous case. Now it is clear that the right hand side of (3.8) is linear in $\sum_{\beta \in \bar{\Delta}} \xi_{\beta} e_\beta$. In view of this, it suffices to show that

$$(\delta I_k(L(q,p,\xi)(\infty)), e_\beta) = 0 \text{ for all } \beta \in \bar{\Delta}.$$  (3.9)
To this end, observe that
\[ (\delta I_k(x), y) = \frac{1}{(d_k - 1)!} \langle \partial_{x}^{d_k-1} \partial_y, I_k \rangle \] (3.10)
for all \( x, y \in \mathfrak{g} \). If we put \( x = L(q, p, \xi)(\infty) \) and \( y = e_\beta \) in the above expression and invoke the multinomial expansion to calculate \( \partial_{x}^{d_k-1} L(q, p, \xi)(\infty) \), the result is
\[ (\delta I_k(L(q, p, \xi)(\infty)), e_\beta) = \sum_{m+\sum_{\alpha \in \Delta'} m_\alpha = d_k-1} \frac{\prod_{\alpha \in \Delta'} (\frac{\xi_\alpha}{\alpha(q)})^{m_\alpha}}{m! \prod_{\alpha \in \Delta'} m_\alpha!} \langle \partial^m_p \prod_{\alpha \in \Delta'} \partial_{e_\alpha} \partial_{e_\beta}, I_k \rangle . \] (3.11)
But for \( \beta \in \bar{\Delta} \), we have
\[ \beta + \sum_{\alpha \in \Delta'} j_\alpha \alpha \neq 0 \] (3.12)
for any choice of \( j_\alpha \in \mathbb{N}, \alpha \in \Delta' \). Hence it follows from Lemma 3.1 that each individual term of the sum in (3.11) is equal to zero. This completes the proof that \( I_{k_1}(q, p, \xi) = 0 \) for \( (q, p, \xi) \in J^{-1}(0) \). On the other hand, it is a consequence of Proposition 6.8 in Section 6 that all the other \( I_{k_j}'s \) are not identically zero. Finally, since \( I_k(\xi) \) are Casimir functions for \( 1 \leq k \leq N \), the number of nontrivial integrals for each \( k \) is \( d_k - 1 = m_k \). □

Remark 3.3. (a) Because the height function \( ht : \Delta \rightarrow \mathbb{Z} \) is not one-to-one, for this reason, we cannot conclude from (3.12) that \( ht (\beta) + \sum_{\alpha \in \Delta'} j_\alpha \alpha (\alpha) \neq 0 \) for \( \beta \in \bar{\Delta}, j_\alpha \in \mathbb{N}, \alpha \in \Delta' \). This is why it is necessary to use Lemma 3.1.

(b) For the rational spin CM systems considered in this section, it was pointed out in [L2] that there exists a second realization in the dual bundle of a coboundary dynamical Lie algebroid. More precisely, define \( R : U \rightarrow L(\mathfrak{g}, \mathfrak{g}) \) by
\[ R(q)\xi = -\sum_{\alpha \in \Delta'} \frac{\xi_\alpha}{\alpha(q)} e_\alpha, \] (3.13)
then \( R \) is a solution of the CDYBE. Let \( A^* \Omega \simeq TU \times \mathfrak{g} \) be the coboundary dynamical Lie algebroid associated with \( R \) and let \( A\Omega \simeq TU \times \mathfrak{g} \) be the trivial Lie algebroid. Then according to [L1],
\[ \mathcal{R} : A^* \Omega \rightarrow A\Omega, (q, p, \xi) \mapsto (q, \Pi_\delta \xi, -p + R(q)\xi) \] (3.14)
is a morphism of Lie algebroids. Consequently, the dual map \( \mathcal{R}^* \) is an \( H \)-equivariant Poisson map, when the domain and target are equipped with the corresponding Lie-Poisson structures. Explicitly,
\[ \mathcal{R}^*(q, p, \xi) = (q, -\Pi_\delta \xi, p - R(q)\xi) = (q, -\Pi_\delta \xi, L(q, p, \xi)(\infty)). \] (3.15)
We would like to point out that the (spectral parameter independent) Lax operator $L^\infty(q,p,\xi) = L(q,p,\xi)(\infty)$ coming out from this picture is of no use in proving Liouville integrability. This is because the number of integrals it gives is far from sufficient. The same remark also applies to the Lax operators of the hyperbolic spin CM systems in [L1] and the Lax operators of the spin CM systems associated with the Alekseev-Meinrenken dynamical r-matrices [AM] in [FP].

4. The trigonometric spin Calogero-Moser systems.

The trigonometric spin Calogero-Moser systems are the Hamiltonian systems in Definition 2.2.3 associated to the following trigonometric dynamical r-matrices with spectral parameter:

$$
\begin{align*}
\rho(q,z) &= c(z) \sum_i x_i \otimes x_i - \sum_{\alpha \in \Delta} \phi_\alpha(q,z) e_\alpha \otimes e_{-\alpha} \\
&= p + c(z) \sum_i x_i - \sum_{\alpha \in \Delta} \phi_\alpha(q,z) \xi_\alpha e_\alpha \\
&= p + c(z) \xi + \sum_{\alpha \in \Delta} \psi_\alpha(q) \xi_\alpha e_\alpha
\end{align*}
$$

(4.1)

where

$$
c(z) = \cot z
$$

(4.2)

and

$$
\begin{cases}
-\frac{\sin(\alpha(q)+z)}{\sin \alpha(q) \sin z}, & \alpha \in \pi' > \\
-e^{i\pi z}, & \alpha \in \pi'^+ \\
-\frac{e^{-i\pi z}}{\sin z}, & \alpha \in \pi'^-
\end{cases}
$$

(4.3)

In (4.3) above, $\pi'$ is an arbitrary subset of the simple system $\pi \subset \Delta$, $\pi' >$ is the root span of $\pi'$ and $\pi'^\pm = \Delta^\pm \setminus \pi' > \pm$. Accordingly, the Lax operators are given by

$$
L(q,p,\xi)(z) = p + c(z) \sum_i x_i - \sum_{\alpha \in \Delta} \phi_\alpha(q,z) \xi_\alpha e_\alpha
$$

(4.4)

where

$$
\psi_\alpha(q) = \begin{cases}
\alpha(q), & \alpha \in \pi' > \\
-i, & \alpha \in \pi'^+ \\
i, & \alpha \in \pi'^-
\end{cases}
$$

(4.5)

Hence we have a family of dynamical systems parametrized by subsets $\pi'$ of $\pi$ with
Hamiltonians of the form:
\[
H(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{2} \sum_{\alpha \in \pi'} \left( \frac{1}{\sin^2 \alpha(q)} - \frac{1}{3} \right) \xi_\alpha \xi_{-\alpha} - \frac{5}{6} \sum_{\alpha \in \Delta \setminus \pi'} \xi_\alpha \xi_{-\alpha} - \frac{1}{3} \sum_i \xi_i^2.
\] (4.6)

Now, from the homogeneity of \(I_k\) and (4.4), we have the expansion
\[
I_k(L(q, p, \xi)(z)) = \sum_{j=0}^{d_k} I_{kj}(q, p, \xi)(c(z))^j.
\] (4.7)

**Proposition 4.1.** For each \(1 \leq k \leq N\), \(I_{k, d_k}(q, p, \xi) = I_k(\xi)\). If in addition, \((q, p, \xi) \in J^{-1}(0)\), then the following relation holds:
\[
\sum_{j \text{ odd}} I_{kj}(q, p, \xi) i^j = 0.
\] (4.8)

Therefore, the number of nontrivial integrals \(I_{kj}(q, p, s), j \neq 1\), which Poisson commute on \(TU \times O_{\text{red}}\) is equal to \(\sum_{k=1}^m m_k\), where \(O_{\text{red}}\) is the reduction of a generic orbit \(O\) in \(U\).

**Proof.** As in the proof of Proposition 3.2, it is easy to show that \(I_{k, d_k}(q, p, \xi) = I_k(\xi)\) and this is a Casimir function for each \(k\). To establish the relation (4.8) for \((q, p, \xi) \in J^{-1}(0)\), we divide into two cases. First, consider \(\pi' = \pi\). In this case, we have
\[
L(q, p, \xi)(\pm i\infty) = p + \sum_{\alpha \in \Delta} c(\alpha(q)) \xi_\alpha e_\alpha \mp i\xi.
\]
Therefore, on using the relation \((c(\alpha(q)) - i)e^{2i\alpha(q)} = c(\alpha(q)) + i\), we find that
\[
Ad_{e^{2i\gamma}} L(q, p, \xi)(i\infty) = L(q, p, \xi)(-i\infty).
\]
As a consequence, we obtain
\[
I_k(L(q, p, \xi)(i\infty)) = I_k(L(q, p, \xi)(-i\infty))
\]
from which (4.8) follows upon using (4.7). Now, consider the case \(\pi' \neq \pi\). We will establish (4.8) in this case through a limiting procedure. For this purpose, we define
\[
L_{q_0}(q, p, \xi)(z) = p + \sum_{\alpha \in \Delta} c(\alpha(q - q_0)) \xi_\alpha e_\alpha + c(z)\xi, \quad q_0 \in \mathfrak{h}.
\]
Then as above, if
\[ I_k(L_{q_0}(q, p, \xi))(z) = \sum_{j=0}^{d_k} I_{k_j}^{q_0}(q, p, \xi)(c(z))^j, \]
we have
\[ \sum_{j \text{ odd}} I_{k_j}^{q_0}(q, p, \xi) i^j = 0. \]
Now, let \( \omega_1, \ldots, \omega_N \) be the fundamental weights (with respect to \( \pi \)). We set \( q_0 = q_0(t) = -it \sum_{\alpha_j \notin \pi} H_{\omega_j} \) (cf. [EV]). By using the relation
\[ \alpha_i(H_{\omega_j}) = (\alpha_i, \omega_j) = (\alpha_i, \alpha_j) \frac{1}{2} \delta_{ij}, \]
we find that
\[ \lim_{t \to \infty} c(\alpha(q - q_0(t))) = \psi_{\alpha}(q). \]
Therefore,
\[ \lim_{t \to \infty} L_{q_0(t)}(q, p, \xi)(z) = L(q, p, \xi)(z) \]
and so \( I_{k_j}(q, p, \xi) = \lim_{t \to \infty} I_{q_0(t)}(q, p, \xi) \). Hence we obtain (4.8) upon passing to the limit as \( t \to \infty \) in the relation \( \sum_{j \text{ odd}} I_{k_j}^{q_0(t)}(q, p, \xi) i^j = 0 \). By the same reason as in Proposition 3.2, all the \( I_{k_j} \)'s are not identically zero in this case. Finally, since we can express \( I_{k_1} \) in terms of \( I_{k_3}, \cdots \) through (4.8), the count follows. \( \square \)

5. The elliptic spin Calogero-Moser systems.

Let \( \wp(z) \) be the Weierstrass \( \wp \)-function with periods \( 2\omega_1, 2\omega_2 \in \mathbb{C} \), and let \( \sigma(z), \zeta(z) \) be the related Weierstrass sigma-function and zeta-function, respectively.

The elliptic spin Calogero-Moser system is the spin Calogero-Moser system associated with the elliptic dynamical r-matrix with spectral parameter
\[ r(q, z) = \zeta(z) \sum_i x_i \otimes x_i - \sum_{\alpha \in \Delta} l(\alpha(q), z) e_\alpha \otimes e_{-\alpha} \] (5.1)
where
\[ l(w, z) = -\frac{\sigma(w + z)}{\sigma(w)\sigma(z)}. \] (5.2)
Explicitly, the Hamiltonian is given by
\[ \mathcal{H}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{2} \sum_{\alpha \in \Delta} \wp(\alpha(q)) \xi_\alpha \xi_{-\alpha} \] (5.3)
and its Lax operator is of the form
\[ L(q, p, \xi)(z) = p + \zeta(z) \sum_i \xi_i x_i - \sum_{\alpha \in \Delta} l(\alpha(q), z) \xi_\alpha e_\alpha. \] (5.4)
From now onwards, we will restrict our attention to \((q, p, \xi) \in J^{-1}(0)\).
Proposition 5.1. For each $1 \leq k \leq N$, $I_k(L(q,p,\xi)(z))$ is an elliptic function of $z$ with poles of order $d_k$ at the points of the rank 2 lattice
\[ \Lambda = 2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}. \] (5.5)
Hence $I_k(L(q,p,\xi)(z))$ can be expanded in the form
\[ I_k(L(q,p,\xi)(z)) = I_{k0}(q,p,\xi) + \sum_{j=2}^{d_k} \frac{(-1)^j}{(j-1)!} I_{kj}(q,p,\xi)\psi^{(j-2)}(z). \] (5.6)

Proof. Let $\eta_i = \zeta(\omega_i), i = 1, 2$. Then from $l(\alpha(q), z + 2\omega_i) = e^{2\eta_i} l(\alpha(q), z)$ and $e^{2\eta_i} e_\alpha = Ad_{e^{2\eta_i}} e_\alpha$, we have $L(q,p,\xi)(z + 2\omega_i) = Ad_{e^{2\eta_i}} L(q,p,\xi)(z), i = 1, 2$. Therefore, $I_k(L(q,p,\xi)(z))$ is a doubly-periodic function of $z$. As $L(q,p,\xi)(z)$ is meromorphic with simple poles at the points of the lattice $\Lambda = 2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}$, it follows from the homogeneity of $I_k$ that $I_k(L(q,p,\xi)(z))$ is an elliptic function of $z$ with poles of order $d_k$ at the points of $\Lambda$. The expansion of $I_k(L(q,p,\xi)(z))$ then follows from standard argument in the theory of elliptic functions. \hfill \Box

Proposition 5.2. For each $1 \leq k \leq N$, $I_{k,d_k}(q,p,\xi) = I_k(\xi)$. Hence the number of nontrivial integrals $I_{kj}(q,p,s)$ which Poisson commute on $TU \times \mathcal{O}_{\text{red}}$ is equal to $\sum_{k=1}^N m_k$, where $\mathcal{O}_{\text{red}}$ is the reduction of a generic orbit $\mathcal{O}$ in $\mathcal{U}$.

Proof. In a deleted neighborhood of $z = 0$, we have
\[ l(\alpha(q), z) = -\frac{1}{z} + \zeta(\alpha(q)) + \text{higher order terms} \]
from which it follows that
\[ L(q,p,\xi)(z) = p + \frac{\xi}{z} + \sum_{\alpha \in \Delta} \zeta(\alpha(q)) \xi_\alpha e_\alpha + \text{higher order terms}. \] (5.7)

Therefore, on invoking the homogeneity of $I_k$, we obtain the following expansion in a deleted neighborhood of $z = 0$:
\[ I_k(L(q,p,\xi)(z)) = \frac{1}{z^{d_k}} I_k(\xi) + O(1). \]

But on the other hand, we have
\[ \psi^{(j-2)}(z) = (-1)^j \frac{(j-1)!}{z^j} + O(1) \]
for $j = 2, \ldots, d_k$. Consequently, it follows from (5.6) that we also have
\[ I_k(L(q,p,\xi)(z)) = z^{-d_k} I_{k,d_k}(q,p,\xi) + O(1) \]
in a deleted neighborhood of $z = 0$. Comparing the two expansions of $I_k(L(q,p,\xi)(z))$, the first assertion follows. The second assertion is now obvious as none of the coefficients in the expansion (5.6) is identically zero by Proposition 6.8. \hfill \Box
6. Functional independence of the integrals and Liouville integrability.

As the reader will see, we can establish the functional independence of the integrals for all three cases in a uniform way. For \((q, p, \xi) \in J^{-1}(0)\), we begin with the observation (see (3.3), (4.4) and (5.7)) that the Lax operator can be expressed in the following form

\[
L(q, p, \xi) = p + h(z)\xi + k_0(q, \xi) + k_1(q, \xi, z)
\]  

(6.1)

in a deleted neighborhood of 0, where

\[
h(z) = \begin{cases} 
\frac{1}{z}, & \text{in the rational/elliptic case} \\
c(z), & \text{in the trigonometric case}
\end{cases}
\]

(6.2)

and

\[
k_0(q, \xi) = \begin{cases} 
\sum_{\alpha \in \Delta} \frac{\xi_{\alpha}}{a(q)} e_{\alpha}, & \text{in the rational case} \\
\sum_{\alpha \in \Delta} \psi_{\alpha}(q) \xi_{\alpha} e_{\alpha}, & \text{in the trigonometric case} \\
\sum_{\alpha \in \Delta} \zeta(a(q)) \xi_{\alpha} e_{\alpha}, & \text{in the elliptic case}
\end{cases}
\]

(6.3)

and lastly,

\[
k_1(q, \xi, z) = \begin{cases} 
0, & \text{in the rational/trigonometric case} \\
\sum_{i=1}^{\infty} k_{1i}(q, \xi) z^i, & \text{in the elliptic case}
\end{cases}
\]

(6.4)

By using (2.1.9) and the above, it follows from the multinomial expansion that

\[
I_k(L(q, p, \xi)(z)) = \sum_{a+b+j=d_k} \frac{1}{j!a!b!} \left\langle \frac{\partial^a p}{\partial \xi^j} (\partial k_0(q, \xi) + \partial k_1(q, \xi, z))^b, I_k \right\rangle h(z)^j.
\]

(6.5)

We will split the second line of the above expression into a sum of two terms

\[
I_k(L(q, p, \xi)(z)) = F_k(p, \xi, z) + R_k(q, p, \xi, z)
\]

(6.6)

where

\[
F_k(p, \xi, z) = \sum_{a+j=d_k} \frac{1}{j!a!} \left\langle \frac{\partial^a p}{\partial \xi^j}, I_k \right\rangle h(z)^j
\]

(6.7)

and

\[
R_k(q, p, \xi, z) = \sum_{a+b+j=d_k} \sum_{b \geq 1} \frac{1}{j!a!b!} \left\langle \frac{\partial^a p}{\partial \xi^j} (\partial k_0(q, \xi) + \partial k_1(q, \xi, z))^b, I_k \right\rangle h(z)^j.
\]

(6.8)
Clearly, we have
\[ F_k(p, \xi, z) = \sum_{j=0}^{d_k} F_{kj}(p, \xi) h(z)^j \]  
(6.9)

where
\[ F_{kj}(p, \xi) = \frac{1}{j!(d_k - j)!} \left\langle \partial_p^{d_k-j} \partial_\xi^j, I_k \right\rangle \]  
(6.10)
for each \( j \). Therefore these functions are the same in all three cases and the degree of \( F_{kj}(p, \xi) \) in the variable \( p \) is equal to \( d_k - j \). On the other hand,
\[ R_k(q, p, \xi, z) = \sum_{j=0}^{d_k} R_{kj}(q, p, \xi) h(z)^j + R'_k(q, p, \xi, z), \]  
(6.11)

where \( R'_k(q, p, \xi, z) \) is identically zero in the rational/trigonometric case and is given by a power series in \( z \) which vanishes at 0 in the elliptic case. From the formulas in (6.3), (6.4), it is clear that \( R_{kj} \) is given by a different formula for each of the three cases. However, these play no role in our analysis. For us, the only piece of information which is needed is the degree of \( R_{kj}(q, p, \xi) \) in the variable \( p \) and according to (6.8) and (6.4), this is at most equal to \( d_k - j - 1 \) (and hence is less than that of \( F_{kj} \)). We next turn to the definitions of the \( I_{kj} \)'s in (3.4), (4.7) and (5.6) for the three cases. By comparing these expressions with (6.6), (6.9)-(6.11), we find that
\[ I_{kj}(q, p, \xi) = F_{kj}(p, \xi) + R_{kj}(q, p, \xi), \quad j = 1, \cdots, d_k \]  
(6.12)
in all three cases. The relation also holds for \( j = 0 \) for the rational/trigonometric case but for the elliptic case, we have
\[ I_{k0}(q, p, \xi) \equiv F_{k0}(p, \xi) + R_{k0}(q, p, \xi) \]  
(6.13)
where \( \equiv \) means the two sides differ by a linear combination of \( I_{kj}(q, p, \xi) \) for \( j \geq 4 \) and even. That this is so is due to contributions from the constant terms in the Laurent series expansions of \( \varphi^{(j-2)}(z) \) on the right hand side of (5.6) for \( j \geq 4 \) and even.

**Proposition 6.1.** The functional independence of \( F_{kj}(p, \xi), \ j = 0,1, \cdots, d_k, \ k = 1, \cdots, N \) on an open dense set of \( \mathfrak{h} \times (U \cap \mathfrak{h}^\perp) \) implies the functional independence of \( I_{kj}(q, p, \xi), \ j = 0,1, \cdots, d_k, \ k = 1, \cdots, N \) on an open dense set of \( TU \times (U \cap \mathfrak{h}^\perp) \).

**Proof.** Suppose the \( I_{kj} \)'s are functionally dependent. Then there exists an analytic function \( f(u_1, \cdots, u_d) \) depending on \( d = \frac{1}{2}(\dim \mathfrak{g} + N) \) variables such that
Let $f(I_{kj}(q,p,\xi)) = 0$. Fix a point $q = q_0 \in U$, then $f(I_{kj}(q_0,p,\xi)) = 0$ is a functional dependence relation among the polynomials $I_{kj}^{q_0}(p,\xi) := I_{kj}(q_0,p,\xi)$ in $p$ and $\xi$. Since analytic dependence implies algebraic dependence for polynomials (see, for example, [W] and the references therein), we can assume that $f$ is a polynomial in the variables $u_1, \ldots, u_d$. Now the highest order term in $p$ in the expression $f(I_{kj}(q_0,p,\xi))$ is of the form $g(F_{kj}(p,\xi))$ for a summand $g$ of $f$, since for each monomial $I_{10}^{n_1} \cdots I_{Nd}^{n_d}$, the highest order term in $p$ is given by $F_{n_1}^{p_1} \cdots F_{n_d}^{p_d}$. Furthermore, since $f$ is not identically zero, neither is $g$. But $f(I_{kj}(q,p,\xi)) = 0$ implies $g(F_{kj}(p,\xi)) = 0$, hence the $F_{kj}$’s are functionally dependent.

In what follows, we will establish the functional independence of $F_{kj}(p,\xi)$, $j = 0, \hat{1}, \ldots, d_k$, $k = 1, \ldots, N$ on an open dense set of $\mathfrak{h} \times (U \cap \mathfrak{h}^\perp)$. The following is a lemma which is very useful in some of our calculations.

**Lemma 6.2.** Let $f \in I(\mathfrak{g})$, then for all $x,y,z \in \mathfrak{g}$, and all $m,n \geq 0$, we have

$$
\langle \partial_x^m \partial_{[x,y]}^n f \rangle = \frac{n}{m+1} \langle \partial_x^{m+1} \partial_{[y,z]}^{n-1} f \rangle,
$$

(6.14)

where by convention the right hand side of the formula is zero when $n = 0$.

**Proof.** By using (2.1.13), (2.1.12) back and forth and (2.1.11), we find for $n \geq 1$ that

$$
\langle \partial_x^m \partial_{[x,y]}^n f \rangle = - \langle \partial_x^m (y \cdot \partial_x) \partial_x^n f \rangle
$$

$$
= - \frac{1}{m+1} \langle (y \cdot \partial_x^{m+1}) \partial_x^n f \rangle
$$

$$
= - \frac{1}{m+1} \langle y \cdot (\partial_x^{m+1} \partial_x^n) f \rangle + \frac{1}{m+1} \langle \partial_x^{m+1} (y \cdot \partial_x^n) f \rangle
$$

$$
= \frac{n}{m+1} \langle \partial_x^{m+1} (y \cdot \partial_x) \partial_x^{n-1} f \rangle
$$

$$
= \frac{n}{m+1} \langle \partial_x^{m+1} \partial_{[y,z]}^{n-1} f \rangle
$$

where we have used (2.1.15) in addition to the “power rule” in going from the third line to the fourth line. When $n = 0$, the calculation stops in the second line for we can invoke (2.1.15) to conclude that the resulting expression is equal to zero. □

**Proposition 6.3.** For all $1 \leq k \leq N$, $(p,\xi) \in \mathfrak{h} \times (U \cap \mathfrak{h}^\perp)$,

(a) $F_{k0}(p,\xi) = I_k(p),$

(b) $F_{k1}(p,\xi) = 0,$

(c) $F_{k,d_k}(p,\xi) = I_k(\xi).$
Proof. The assertions in (a) and (c) are obvious. For (b), we use the representation in (6.10) together with the fact that $\partial_\xi$ has no weight zero part for $\xi \in h^\perp$. The assertion therefore is a consequence of (2.1.19).

In order to set up our calculation, we will arrange the variables and the functions $F_{k_j}$ in some definite order. Note that for each $1 \leq k \leq N$, the number of $F_{k_j}(p, \xi)$'s with $j \neq 1$ is equal to $d_k$. Therefore we have a partition given by the sequence

$$h = d_N \geq d_{N-1} \geq \cdots \geq d_1 = 2. \quad (6.15)$$

Since $d_k = m_k + 1$, it is easy to show from Theorem 2.1.2 that the above sequence is conjugate to the partition

$$N = b_0 = b_1 \geq b_2 \geq \cdots \geq b_{h-1} = 1. \quad (6.16)$$

The ordering of the $F_{k_j}$'s which we will use is the following:

$$F_{10}, \cdots, F_{N0}; F_{12}, \cdots, F_{N2}; F_{n-b_2+1,3}, \cdots, F_{N3}; \cdots; F_{N,d_N}. \quad (6.17)$$

Clearly, for each value of $j \geq 2$, the number of functions in each group $\{F_{k_j}\}$ is precisely $b_{j-1}$ from our discussion above. Now for each $1 \leq j \leq h-1$, let us denote the roots with height equal to $j$ by $\alpha_{j,i}, i = 1, \cdots, b_j$. We will order the variables as depicted in the following:

$$p_1, \cdots, p_N; \xi_{\alpha_1}, \cdots, \xi_{\alpha_N}; \xi_{\alpha_{2,1}}, \cdots, \xi_{\alpha_{2,b_2}}; \cdots; \xi_{\alpha_{h-1,1}}. \quad (6.18)$$

**Theorem 6.4.** The functions $F_{k_j}(p, \xi), j = 0, 1, \cdots, d_k, k = 1, \cdots, N$ are functionally independent on an open dense set of $h \times (U \cap h^\perp)$.

To prove this assertion, we will compute the coefficient of

$$dp_1 \wedge \cdots \wedge dp_N \wedge d\xi_{\alpha_1} \wedge \cdots \wedge d\xi_{\alpha_N} \wedge \cdots \wedge d\xi_{\alpha_{h-1,1}}$$

in the expression for

$$dF_{10} \wedge \cdots \wedge dF_{N0} \wedge dF_{1,2} \wedge \cdots \wedge dF_{N2} \wedge \cdots \wedge dF_{N,d_N}$$

at the points of $h \times (\epsilon + n)$, where $\epsilon$ is as in Section 2.2 and $n$ is the nilpotent sub-algebra $\sum_{\alpha \in \Delta^+} g_{\alpha}$. Note that the choice of $\epsilon + n$ follows Kostant in [K2]. Indeed, if $e_+ = \sum_{\alpha \in \pi} c_\alpha e_\alpha, c_\alpha \neq 0$ for all $\alpha \in \pi$, then Kostant showed that the $N$-dimensional plane $\mathfrak{v} = \epsilon + g^{e_+} \subset \epsilon + n$ is a global cross-section of the generic orbits in $\mathfrak{g}$ in the
sense that each such orbit intersects $\mathfrak{o}$ at precisely one point and no two distinct points in $\mathfrak{o}$ are conjugate. This is the reason why it suffices to consider $\mathfrak{h} \times (\epsilon + n)$.

**Remark 6.5.** For $\mathfrak{g} = sl(N + 1, \mathbb{C})$, the generic orbits can be characterized as those orbits through matrices whose characteristic polynomial and minimal polynomial coincide. In this case, we can take $\mathfrak{o}$ to be the set of companion matrices and the result of Kostant which we quoted above is well-known in matrix theory. (See, for example, [HJ].)

The computation which we referred to above will be achieved in a sequence of propositions. First of all, the coefficient which we want to compute is the determinant of a square (block) matrix $D$ of partial derivatives whose diagonal blocks are given by

$$D_0 = \left( \frac{\partial F_{l0}}{\partial p_i} \right)_{l,i=1}^N, \quad D_j = \left( \frac{\partial F_{N-b_j+l,j+1}}{\partial \xi_{\alpha,j,i}} \right)_{l,i=1}^{b_j}, \quad j = 1, \cdots, h-1, \quad (6.19)$$

in that order.

**Proposition 6.6.** At the points $(p, \xi) \in \mathfrak{h} \times (\epsilon + n)$,
(a) $F_{k0}$ does not depend on $\xi_\alpha$ for all $\alpha \in \Delta^+$,
(b) for $j \geq 2$, $F_{kj}$ does not depend on $\xi_\alpha$ for those $\alpha$ with $ht(\alpha) \geq j$, and it depends linearly on $\xi_\alpha$ for those $\alpha$ with $ht(\alpha) = j - 1$.
(c) $D$ is block lower-triangular, i.e.,

$$D = \begin{pmatrix}
D_0 & & & \\
D_1 & D_2 & 0 & \\
& D_3 & \cdots & \\
& & & \cdots \\
& & \star & \cdots \\
& & & \cdots \\
& & & \cdots
\end{pmatrix}, \quad (6.20)$$

and the square blocks $D_j$ defined in (6.19) depend only on $p$.

**Proof.** (a) This is just a consequence of Proposition 6.3 (a).
(b) This part follows from weight consideration. Apply (6.10) with $\xi = \epsilon + \xi^+ = \epsilon + \sum_{\alpha \in \Delta^+} \xi_\alpha e_\alpha$ and apply the binomial expansion to calculate $(\partial_\epsilon + \partial_{\xi^+})^j$, we have

$$F_{kj}(p, \xi) \equiv \langle \partial_p^{d_{k-j}} \partial_\epsilon^j, I_k \rangle + j \sum_{\alpha \in \Delta^+} \xi_\alpha \langle \partial_p^{d_{k-j}} \partial_\epsilon^j \partial_{e_\alpha}^{j-1}, I_k \rangle + O(\xi^2). \quad (6.21)$$

Here the notation $a \equiv b$ is a shorthand for $a = \lambda b$ for some $\lambda \neq 0$ and we will henceforth use this shorthand. On the other hand, the reminder term $O(\xi^2)$ involves terms which are at least quadratic in the components of $\xi^+$. From (2.1.17) and
(2.1.18), $\partial_p^{d_k-j}$ has weight 0 while $\partial^j_\alpha$ has weight $-j$. Therefore the first term in (6.21) is zero by (2.1.19). If $\text{ht}(\alpha) \geq j$, then $\partial^j_\alpha \partial_\epsilon$ has weight strictly bigger than 0 and therefore the corresponding term $\langle \partial_p^{d_k-j} \partial^j_\alpha \partial_\epsilon, I_k \rangle$ in (6.21) is zero by (2.1.19). On the other hand, if $\text{ht}(\alpha) = j - 1$, the operator $\partial^j_\alpha \partial_\epsilon$ has weight 0 and therefore the corresponding $\xi_\alpha$ appears linearly in $F_{kj}$. Finally, it is clear that the term $O(\xi^2)$ does not depend on $\xi_\alpha$ for $\alpha$ with height greater or equal to $j$. This completes the argument.

(c) This immediately follows from the assertions in (a), (b) and (6.19). \qed

We next compute the values of the determinants $|D_j|$, $j = 0, \ldots, h - 1$. For this purpose, we have to study the diagonal blocks of $D$ more closely.

**Proposition 6.7.** At the points $(p, \xi) \in \mathfrak{h} \times (\mathfrak{e} + \mathfrak{n})$, the following properties hold.

(a) For $1 \leq l, i \leq b_j$, the element $D_j(l, i)$ of $D_j$ in the $(l, i)$ position has degree $d_{N-b_j+i} - j - 1$ in $p$.

(b) The first $b_j - b_{j+1}$ rows of $D_j$ are constants. (When $b_{j+1} = b_j$, this just means that there are no constant rows.) Indeed, when $b_j - b_{j+1} > 0$, we have the formula

$$D_j(l, i) \equiv \langle \partial^j_\epsilon \partial_\epsilon^{\alpha_{j-i}}, I_{N-b_j+i} \rangle, \text{ for } 1 \leq l \leq b_j - b_{j+1}.$$  

(6.22)

**Proof.** (a) For $j = 0$, the assertion is clear because $F_{l0}$ is homogeneous of degree $d_l$ in $p$ by Proposition 6.3 (a). For $j \geq 2$, it follows from (6.10), (6.19) and (6.21) that

$$D_j(l, i) = \frac{\partial F_{N-b_j+i,j+1}}{\partial \epsilon^{\alpha_{j-j}}} = \langle \partial_p^{d_{N-b_j+i}-j-1} \partial^j_\epsilon \partial_\epsilon^{\alpha_{j-i}}, I_{N-b_j+i} \rangle.$$  

(6.23)

Hence the degree of $D_j(l, i)$ in $p$ is $d_{N-b_j+i} - j - 1$.

(b) If $b_j - b_{j+1} > 0$, we have $m_k = j$ for $N - b_j + 1 \leq k \leq N - b_{j+1}$ from Theorem 2.1.2 (a) which implies $d_{N-b_{j+1}} = j + 1$ for $l = 1, \ldots, b_j - b_{j+1}$. Thus $D_j(l, i)$ is of degree 0 in $p$ for $l = 1, \ldots, b_j - b_{j+1}$ from (6.23), i.e., they are constants. \qed

**Proposition 6.8.** Let $\Delta_j^+$ denote the set of positive roots of height $j$. Then on $\mathfrak{h}$, we have the recursion relations:

(a) $|D_1| \prod_{i=1}^N \alpha_i \equiv |D_0|$, 

(b) $|D_j| \prod_{\alpha \in \Delta_j^+} \alpha \equiv |D_{j-1}|$ for $j \geq 2$,

where the proportionality constants in (a) and (b) are independent of $p \in \mathfrak{h}$.

Therefore

$$|D_j|(p) \equiv \prod_{ht(\alpha) > j} \alpha(p), j = 0, 1, \ldots, h - 1$$  

(6.24)
with the convention that $|D_{h-1}(p)| \equiv 1$. Hence $|D_j|(p) \neq 0$ for $p \in \mathfrak{h}'$, $j = 0, 1, \cdots, h - 1$, where $\mathfrak{h}'$ is the open, dense set of regular points of $\mathfrak{h}$.

Proof. It is a classical result that $|D_0|(p) = \prod_{\alpha \in \Delta^+} \alpha(p)$ and the regular points of $\mathfrak{h}$ are precisely those points where $|D_0|(p) \neq 0$. (See [S] and [K2].) Therefore, if we can establish the recursion relations, it will follow from this result that $|D_j|(p) \neq 0$ for $p \in \mathfrak{h}'$, $j = 0, 1, \cdots, h - 1$.

(a) Let $H_i = H_{\alpha_i}, i = 1, \cdots, N$. Since the $H_i$’s form a basis of $\mathfrak{h}$, the determinant $|D_0|$ in (6.19) can be computed in this basis up to a nonzero scale. That is,

$$|D_0|(p) = \left| \left( \langle \partial^{d_i-1}_p \partial_{\alpha_i} \rangle \right)_{l,i} \right| = \left| \left( \langle \partial^{d_i-1}_p \partial_{H_i} \rangle \right)_{l,i} \right|.$$

But from (6.23), (6.14) and the relation $[e_{\alpha_i}, \epsilon] = H_i$, we have

$$\alpha_i(p)D_1(l, i)(p) \equiv \alpha_i(p)\langle \partial^{d_i-2}_p \partial_{\epsilon} \partial_{e_{\alpha_i}}, I_i \rangle = \langle \partial^{d_i-2}_p \partial_{[p,e_{\alpha_i}]}, I_i \rangle \equiv \langle \partial^{d_i-1}_p \partial_{e_{\alpha_i}}, I_i \rangle = \langle \partial^{d_i-1}_p \partial_{H_i}, I_i \rangle.$$

Hence the formula follows from the property of determinants.

(b) Consider the root vector $e_{\alpha_j}$. Clearly we have $e_{\alpha_j} \in \mathfrak{g}^{(j)}$ and $\epsilon \in \mathfrak{g}^{(-1)}$. (See the definition at the end of Section 2.1.) Therefore $[e_{\alpha_j}, \epsilon] \in \mathfrak{g}^{(j-1)}$ by (2.1.20).

Hence we can write

$$[e_{\alpha_j}, \epsilon] = \sum_{n=1}^{b_{j-1}} a_{j,n,i} e_{\alpha_{j-1},n},$$

(6.25)

where the coefficients on the right hand side are not all zero. Indeed, it follows from (4.4.3) and the proof of Proposition 19 in [K1] that $\ker(ad \epsilon) \cap \mathfrak{n} = 0$ and therefore the $b_{j-1} \times b_j$ matrix $A_j = (a_{j,n,i})_{n,i}$ is of full rank. Now, by making use of the formula for $D_j(l, i)$ in (6.23), it follows by applying (6.14) and (6.25) that

$$\alpha_{j,i}(p)D_j(l, i)(p) \equiv \alpha_{j,i}(p) \left\langle \partial^{d_N-b_j+i-j-1}_{p} \partial_{e_{\alpha_{j,i}}} \partial_{e_{\alpha_{j-1,i}}}, I_{N-b_{j+1}} \right\rangle = \left\langle \partial^{d_N-b_j+i-j}_{p} \partial_{e_{\alpha_{j,i}}}, I_{N-b_{j+1}} \right\rangle \equiv \left\langle \partial^{d_N-b_j+i-j}_{p} \partial_{e_{\alpha_{j-1,i}}}, I_{N-b_{j+1}} \right\rangle = \sum_{N=1}^{b_{j-1}} a_{j,n,i} \left\langle \partial^{d_N-b_j+i-j}_{p} \partial_{e_{\alpha_{j-1,n}}}, I_{N-b_{j+1}} \right\rangle = \sum_{n=1}^{b_{j-1}} a_{j,n,i} D_{j-1}(l + b_{j-1} - b_j, n)(p).$$

(6.26)
We now divide the proof into two cases.

Case 1. $b_{j-1} = b_j$

In this case, we have

$$D_j(p) \text{ diag} \left( \alpha_{j,1}(p), \cdots, \alpha_{j,b_j}(p) \right) \equiv D_{j-1}(p)A_j$$  \hspace{1cm} (6.27)

from (6.26) above and the matrix $A_j$ is invertible. Therefore, when we take the determinant of both sides of (6.27), we obtain the desired formula.

Case 2. $b_{j-1} > b_j$

In this case, (6.26) can be rewritten as

$$D_j(p) \text{ diag} \left( \alpha_{j,1}(p), \cdots, \alpha_{j,b_j}(p) \right) \equiv D'_{j-1}(p)A_j$$  \hspace{1cm} (6.28)

where $D'_{j-1}(p)$ is the $b_j \times b_{j-1}$ submatrix of $D_{j-1}(p)$ obtained by deleting its first $b_{j-1} - b_j$ rows. Now, recall that the first $b_{j-1} - b_j$ rows of $D_j$ are constants in this case by Proposition 6.7 (b). Consequently, for $1 \leq l \leq b_{j-1} - b_j$, it follows by using (6.22) and by reversing the steps in the kind of calculation in (6.26) that

$$\sum_{n=1}^{b_{j-1}} a_{j,n,i}D_{j-1}(l,n)(p) = \sum_{n=1}^{b_{j-1}} a_{j,n,i}(\partial^{j-1}_{\epsilon} \partial_{\alpha_{j-1,n,i}}, I_{N-b_j-1+l})$$

$$= (\partial^{j-1}_{\epsilon} \partial_{\alpha_{j-1,i}}, I_{N-b_j-1+l})$$

$$= 0$$

(6.29)

where we have used (6.14) in the $n = 0$ case and (6.25) in going from the first line to the second line. By combining (6.28) and (6.29), we conclude that

$$\begin{pmatrix} 0 \\ D_j(p) \text{ diag} \left( \alpha_{j,1}(p), \cdots, \alpha_{j,b_j}(p) \right) \end{pmatrix} = D_{j-1}(p)A_j.$$  \hspace{1cm} (6.30)

But since the $b_{j-1} \times b_j$ matrix $A_j$ has full rank, we can extend it to an invertible $b_{j-1} \times b_{j-1}$ matrix $\tilde{A}_j$ by adjoining $b_{j-1} - b_j$ column vectors from the canonical basis of $\mathbb{C}^{b_{j-1}}$ on the right hand side of $A_j$. In this way, we obtain from (6.30) that

$$\begin{pmatrix} 0 \\ D_j(p) \text{ diag} \left( \alpha_{j,1}(p), \cdots, \alpha_{j,b_j}(p) \right) \end{pmatrix} \# * = D_{j-1}(p)\tilde{A}_j.$$  \hspace{1cm} (6.31)

Therefore, on taking the determinants of both sides of (6.31), we again obtain the desired formula. \hfill \Box
This proves Theorem 6.4 as

\[
|D|(p) = \prod_{j=0}^{h-2} \left( \prod_{ht(\alpha) > j} \alpha(p) \right) = \prod_{\alpha \in \Delta^+} \alpha(p)^{ht(\alpha)} \neq 0
\] 

(6.32)

for \( p \in \mathfrak{h}' \).

As a consequence of Theorem 6.4 and Propositon 6.1, we obtain the following corollary.

**Corollary 6.9.** The Poisson commuting integrals \( I_{kj}(q,p,\xi), j = 0, \hat{1}, \cdots, d_k, k = 1, \cdots, N \) on \( TU \times (U \cap \mathfrak{h}^\perp) \) are functionally independent on an open dense set of \( TU \times (U \cap \mathfrak{h}^\perp) \).

Finally we are ready to state the main theorem of this work.

**Theorem 6.10.** The reduction of the rational, trigonometric and elliptic spin Calogero-Moser systems to \( J^{-1}(0)/H \approx TU \times \mathfrak{g}_{\text{red}} \) are Liouville integrable on the generic symplectic leaves of \( TU \times \mathfrak{g}_{\text{red}} \).

**Proof.** With the identification \( J^{-1}(0)/H \approx TU \times \mathfrak{g}_{\text{red}} \), the conserved quantities in involution are given by \( I_{kj}(q,p,s) \), where \( s \in \mathfrak{g}_{\text{red}} \). Therefore the number of non-trivial integrals required for Liouville integrability is exactly one-half the dimension of the generic symplectic leaves of \( TU \times \mathfrak{g}_{\text{red}} \) for each of the three cases. (See Proposition 3.2, 4.1 and 5.2.) Finally, the functional independence of the integrals follows from Corollary 6.9 above. \( \square \)
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