REFINED ENUMERATIONS OF ALTERNATING SIGN TRIANGLES

FLORIAN AIGNER

Abstract. This article introduces and investigates a refinement of alternating sign trapezoids by means of Catalan objects and Motzkin paths. Alternating sign trapezoids are a generalisation of alternating sign triangles that were recently introduced by Ayyer, Behrend and Fischer. We show that the number of alternating sign trapezoids associated with a Catalan object (resp. Motzkin path) is a polynomial function in the length of the shorter base of the trapezoid. We also study the rational roots of the polynomials and formulate several conjectures and derive some partial results. Further we deduce a constant term identity for the refined counting of alternating sign trapezoids.

1. Introduction

In the 1980s Mills, Robbins and Rumsey [17, 19] introduced alternating sign matrices (ASMs) and thereby initiated a new branch in combinatorics. Over time more combinatorial objects were introduced which are equinumerous to ASMs, e.g. fully packed loops (FPLs). FPLs led in a natural way to a refined enumeration by means of Catalan objects. This refinement gave rise to a variety of important results, one of them is the Razumov-Stroganov-Cantini-Sportiello Theorem [8, 18]. In [6] Ayyer, Behrend and Fischer introduced alternating sign triangles (ASTs) and showed that they are equinumerous to ASMs. While there exists an easy bijection between ASMs and FPLs, finding a bijection between ASMs and ASTs is still an open problem.

In this paper we introduce and investigate a refined enumeration of ASTs by centred Catalan sets, objects enumerated by the Catalan numbers, and by Motzkin paths due to Ayyer [5]. As we will see in Section 6 it makes sense to look at both refinements although the Motzkin path refinement is coarser since we deduce statements and make conjectures for this refinement which are not true for the centred Catalan refinement.

By looking at the structure of ASTs and their associated centred Catalan set, it is very natural to introduce AS-trapezoids. AS-trapezoids arise by generalising ASTs from a triangular to a trapezoidal shape and have by definition the following property: Taking an AST of order \( n \) and putting an \((m, 2n)\)-AS-trapezoid on top of it one obtains an AST of order \( m+n \). Further the associated centred Catalan set (resp. Motzkin path) of the AST of order

Key words and phrases. Alternating sign triangles, alternating sign trapezoids, polynomial enumeration formula, centred Catalan sets, Motzkin paths, constant term identity.

Supported by the Austrian Science Foundation FWF, START grant Y463.
\(m + n\) is the concatenation of the centred Catalan set (resp. Motzkin path) associated to the AST of order \(n\) and the \((m, 2n)\)-AS-trapezoid. In Lemma 3.2 we show that the converse is also true: every AST whose associated centred Catalan set (resp. Motzkin path) can be split into two parts allows a splitting of the AST into a smaller AST corresponding to the first centred Catalan set (resp. Motzkin path) and an AS-trapezoid corresponding to the second. Following [5] we define the weight function \(w_l(S)\) (resp. \(w_l(M)\)) as the number of \((n, l)\)-AS-trapezoids with associated centred Catalan set \(S\) of size \(n - 1\) (Motzkin path \(M\) of length \(n\) resp). Exploiting this splitting property of ASTs into an AST and AS-trapezoids leads to our first main result.

**Proposition 1.1.** Let \(S_1, S_2\) be centred Catalan sets and \(M_1, M_2\) be Motzkin paths. We have

\[
\begin{align*}
  w_l(S_1 \circ S_2) &= w_l(S_1)w_{l+2|S_1|-2}(S_2), \\
  w_l(M_1 \circ M_2) &= w_l(M_1)w_{l+2|M_1|}(M_2),
\end{align*}
\]

i.e. the weight functions are multiplicative.

In Theorem 2.8 we present an enumeration formula for the number of AS-trapezoids which was first conjectured by Behrend [7] and later independently by the author. The proof was found by Behrend and Fischer [7] using the six-vertex model and by Fischer [13] using the operator formula.

We establish a bijection between AS-trapezoids corresponding to a given centred Catalan set \(S\) to \((s, t)\)-trees, which were defined in [15], however already appeared as partial monotone triangles in [12], and are a generalisation of monotone triangles. By analysing the structure of the \((s, t)\)-trees we can deduce our second main result.

**Theorem 1.2.** Let \(S\) be a centred Catalan set. The weight \(w_l(S)\) is a polynomial in \(l\) of degree \(|\lambda(S)/\mu(S)| = \text{area}(M(S))\) with leading coefficient \(\frac{f_{\lambda(S)/\mu(S)}}{|\lambda(S)/\mu(S)|!}\).

It is very remarkable that there exists a very analogous theorem for the refined enumeration of FPLs for a specific class of noncrossing matchings, which was conjectured in [20] and proven in [3, 9]. A detailed elaboration is provided in Remark 4.12.

Using the enumeration formula for \((s, t)\)-trees from [15] we present a constant term formula for the refined enumeration of AS-trapezoids which generalises essentially a constant term formula for refined ASTs [15, Theorem 2].

**Theorem 1.3.** Let \(S = \{s_1, \ldots, s_u, 0, s_{u+1}, \ldots, s_n\}\) be a centred Catalan set with \(s_1 < \ldots < s_u \leq -1, 1 \leq s_{u+1} < \ldots < s_n\). The number \(w_l(S)\) of \((n, l)\)-AS-trapezoids with centred Catalan set \(S\) is equal to the constant term of

\[
\prod_{i=1}^{n} x_i^{-n-s_i} \prod_{i=u+1}^{n} x_i^{2-l(1+x_i)^l} \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 + x_i + x_ix_j),
\]

in \(x_1, \ldots, x_n\).
The final part of this paper describes the rational roots of the weight functions $w_l(S)$ and $w_l(M)$. It appears that both have generally a rich structure of rational roots which is not totally understood at this point. We present the results of a first analysis of their structure in form of conjectures and partial results. The goal would be to find a description of all rational roots in a combinatorial way similar as it is done in [16] for the number of FPLs associated with certain families of link patterns.

The paper is structured in the following way. In Section 2 we define centred Catalan sets, Motzkin paths, ASTs and AS-trapezoids and establish the relations between them. In Section 3 we present a relation between the inner structure of an AS-trapezoid and the centred Catalan path it corresponds to. This is needed to deduce the splitting Lemma (Lemma 3.2) of which our first main result is a direct consequence. In Section 4 we relate AS-trapezoids to $(s,t)$-trees and deduce our second main result. Section 5 contains the derivation of the constant term identity for the refined enumeration of AS-trapezoids. Finally, Section 6 presents first conjectures and partial results concerning the rational roots of $w_l(S)$ and $w_l(M)$.

An extended abstract of this paper was published in the proceedings of FPSAC 2017 [2].

2. Preliminaries

2.1. Centred Catalan sets, Dyck and Motzkin paths. We introduce new combinatorial objects, which are enumerated by the Catalan numbers, and explore their relation to Dyck paths.

Definition 2.1. A centred Catalan set $S$ of size $n$ is an $n$-subset of $\{-n + 1, -n + 2, \ldots, n - 1\}$ such that $|S \cap \{-i, -i + 1, \ldots, i\}| \geq i + 1$ for all $0 \leq i \leq n - 1$, in particular $0 \in S$.

Centred Catalan sets of size $n$ are in bijection with Dyck paths of length $2n$. For a given centred Catalan set $S$, we construct a Dyck path $D(S)$ in the following way. We read the integers $-n + 1, \ldots, n - 1, n$ in the order $0, -1, 1, -2, 2, \ldots, -n + 1, n - 1, n$ and draw a north-east step if the number is in $S$ and a south-east step otherwise, see Figure 1. In fact, there are $2^{n-1}$ different bijections of the above kind between centred Catalan sets of size $n$ and Dyck paths of length $2n$. For every $1 \leq i \leq n - 1$ we can switch the order of reading $-i, i$ in the above algorithm and obtain a new bijection.
Let $l$ be an integer and define the dilation operator $s_l : \mathbb{Z} \to \mathbb{Z}$ as

$$s_l(x) = \begin{cases} 
  x + l & x > 0, \\
  0 & x = 0, \\
  x - l & x < 0.
\end{cases}$$

By abuse of notation we write $s_l : 2\mathbb{Z} \to 2\mathbb{Z}$, $s_l(A) = \{s_l(x) | x \in A\}$. The concatenation $S_1 \circ S_2$ of two centred Catalan sets $S_1, S_2$ is defined as

$$S_1 \circ S_2 := S_1 \cup s_{|S_1|^{-1}}(S_2).$$

We call a centred Catalan set $S$ irreducible if there exist no centred Catalan sets $S_1, S_2$ of size at least 2 such that $S = S_1 \circ S_2$. It is not hard to convince oneself that every centred Catalan set can be written uniquely as a concatenation of irreducible centred Catalan sets. For two centred Catalan sets $S_1, S_2$ the Dyck path $D(S_1 \circ S_2)$ is obtained by deleting the last step of $D(S_1)$, the first step of $D(S_2)$ and concatenating both paths, i.e., it is not just adding the paths $D(S_1)$ and $D(S_2)$, see Figure 2.

A Motzkin path of length $n$ is a path on the half-plane $y \geq 0$ starting at $(0, 0)$ and ending at $(n, 0)$ with step-set $\{(1, 1), (1, 0), (1, -1)\}$. In the following we encode a Motzkin path $M$ of length $n$ by a sequence $M = (m_1, \ldots, m_n)$ with $m_i \in \{1, 0, -1\}$ for $1 \leq i \leq n$, where $m_i$ stands for the step $(1, m_i)$. The concatenation of two Motzkin paths is given by attaching the second path at the end of the first one. A Motzkin path is called irreducible if it cannot be written as a concatenation of non-empty Motzkin paths.

We define $M(D)$ as the Motzkin path obtained by “averaging” the steps in the Dyck path $D$: the $i$-th step of $M(D)$ is the average of the $(2i)$-th and $(2i + 1)$-st step of $D$. By averaging we mean that two north-east steps result in a north-east step, a north-east step and a south-east step in an east step and two south-east steps in a south-east step. This map is a surjection from Dyck paths of length $2n$ to Motzkin paths of length $n - 1$. For a centred Catalan set $S$ of size $n$ the Motzkin path $M(S) := M(D(S)) = (m_1(S), \ldots, m_{n-1}(S))$ is given by

$$m_i(S) := |\{-i, i\} \cap S| - 1.$$ 

It is easy to see that concatenating centred Catalan sets or Motzkin paths respectively and mapping Catalan sets to Motzkin paths commute, i.e., $M(S_1 \circ S_2) = M(S_1) \circ M(S_2)$. Further every irreducible component of $M(S)$ corresponds to an irreducible component of $S$ and vice-versa. For an example see Figure 2.

2.2. Alternating sign triangles. Alternating sign triangles were recently introduced by Ayyer, Behrend and Fischer [6] and are the newest member in a family of combinatorial objects with the same enumeration formula.

**Definition 2.2.** An alternating sign triangle (AST) of order $n$ is a configuration of $n$ centred rows where the $i$-th row, counted from the bottom, has $2i - 1$ elements. The entries are $-1, 0$ or $1$ such that in all rows and
columns the non-zero elements are alternating, all row-sums are 1 and in every column the first non-zero entry from top is positive.

The following is an example of an AST of order 6

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & \\
1 & -1 & 1 & \\
1 & 
\end{array}
\]

Theorem 2.3 \([6]\). The number of ASTs of order \(n\) is given by

\[
\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!},
\]

i.e. order \(n\) ASTs and \(n \times n\) ASMs are equinumerous.

We label the columns of an AST \(A\) of order \(n\) form left to right with \(-n+1, \ldots, n-1\) and the rows from bottom to top with \(1, \ldots, n\).

Proposition 2.4. Let \(A\) be an AST of order \(n\) and \(S(A)\) be the set of columns with positive column-sum. Then \(S(A)\) is a centred Catalan set of size \(n\). Conversely for all centred Catalan sets \(S\) of size \(n\) there exists an AST \(A\) of order \(n\) with \(S(A) = S\).

Proof. The set \(S(A)\) is an \(n\)-subset of \((-n+1, \ldots, n-1)\). Define \(S_i(A)\) as the set of columns \(j\) such that \(|j| < i\) and the partial column-sum of elements below the \((i+1)\)-st row is positive. We have the following relations between \(S_i(A)\) and \(S(A)\)

\[S_i(A) \subseteq S(A) \cap \{-i+1, -i+2, \ldots, i-1\}.
\]

Since the partial column-sums can only have the values 1, 0, -1 and the sum of the partial column-sums of elements below the \((i+1)\)-st row is \(i\) we obtain \(|S_i(A)| \geq i\). Hence we have

\[i \leq |S_i| \leq |S(A) \cup \{-i+1, \ldots, i-1\},
\]

which implies the first claim.

Now let \(S\) be given. By definition we can choose a sequence \((s_i)_{1 \leq i \leq n}\) such that \(S = \{s_i : 1 \leq i \leq n\}\) and \(|s_i| < i\). We construct an AST \(A\) by
Figure 3. A Dyck path and its corresponding Motzkin path.

setting the entry in column $s_i$ of the $i$-th row to 1 for all $1 \leq i \leq n$ and the other entries to 0. Then $A$ is an AST and satisfies $S(A) = S$. \hfill \Box

The following refinement of ASTs by Motzkin paths is due to Ayyer [5].

**Corollary 2.5.** The map $M(A) := M(S(A))$ is a surjection from ASTs of order $n$ to Motzkin paths of length $n - 1$.

The object of our interest is the weight function $w(S)$ (resp. $w(M)$) of a centred Catalan set $S$ (resp. Motzkin path $M$) which is defined as the number of ASTs $A$ with $S(A) = S$ (resp. $M(A) = M$).

### 2.3. Alternating sign trapezoids

Following the definition of ASTs, we introduce a generalisation of ASTs to a trapezoidal shape.

**Definition 2.6.** Let $n, l$ be positive integers. An $(n, l)$-AS-trapezoid is an array of $n$ centred rows where the $i$-th row from bottom has $l + 2i - 1$ entries, filled with $-1$, $0$ or $1$ such that all row-sums are 1, the column-sums are 0 for the central $l - 1$ columns, the non-zero entries in all rows and columns are alternating and in every row the first non-zero entry from top is positive.

*Alternating sign trapezoids* where first introduced in [2] with bases of odd length. The above definition is more generally also allowing bases of even length. The term $(n, l)$-AS-trapezoid in [2] corresponds in this article to an $(n, 2l)$-AS-trapezoid.

ASTs of order $n + 1$ and $(n, 2)$-AS-trapezoids are in bijection by deleting the bottom 1 of an AST or adding a 1 at the bottom of an $(n, 2)$-AS-trapezoid. We label the rows of an $(n, l)$-AS-trapezoid from bottom to top with $1, \ldots, n$ and the columns from left to right by $-n + 1, \ldots, l + n - 1$. Let $A$ be an $(n, l)$-AS-trapezoid. We define $S(A)$ as the set obtained by the following four steps: take the set of columns of $A$ with positive column sum; subtract 1 from the columns with a non-positive label; subtract $l - 1$ from the columns with positive labels; finally add 0 to the set. One can prove analogously to Proposition 2.4 that $S(A)$ is a centred Catalan set of size $n - 1$.

The following is an example of a $(4, 6)$-AS-trapezoid $A$ with $S(A) = \{-2, -1, 0, 1, 3\}$

```
0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
0 0 0 0 1 0 -1 0 0 0 1
1 0 0 -1 0 1 0 0 0
1 0 0 0 -1 0 1
```

labels: $-3 -2 -1 0 1 2 3 4 5 6 7 8 9$
Analogously to ASTs we define $M(A) := M(S(A))$ for an $(n, l)$-AS-trapezoid $A$. For a centred Catalan set $S$ of size $n + 1$ we define $w_1(S)$ (resp. $w_1(M)$) as the number of $(n, l)$-trapezoids $A$ with $S(A) = S$ (resp. $M(A) = M$).

**Remark 2.7.** It is easy to see that the bijection between ASTs of order $n + 1$ and $(n, 2)$-AS-trapezoids commutes with the maps $S$ and $M$ which maps ASTs or AS-trapezoids respectively to centred Catalan sets or Motzkin paths. This implies

$$w_2(S) = w(S),$$

$$w_2(M) = w(M).$$

Since our interest in ASTs in this paper is with respect to their centred Catalan set and Motzkin path refinement, we can treat ASTs without loss of generality as a special case of AS-trapezoids.

By using a computer system to calculate the number of $(n, l)$-AS-trapezoids for $1 \leq n \leq 9$ and treating $l$ as a variable it is possible to guess the formula enumerating $(n, l)$-AS-trapezoids. The formula was first conjectured in a more general way by Behrend [7] and later independently by the author. Behrend found together with Fischer [7] a proof using the six-vertex model and Fischer [13] further found a proof based on her operator formula for monotone triangles.

**Theorem 2.8 ([7, 13]).** The number of $(n, l)$-AS-trapezoids is

$$2^{|\frac{n+1}{2}| - |\frac{n+2}{2}|} \prod_{i=1}^{\frac{n+1}{2}} \frac{(i - 1)!}{(n - i)!} \prod_{i \geq 0} \left( \frac{l}{2} + 3i + 2 \right)^{\left| \frac{n - 4i - 1}{2} \right|} \left( \frac{l}{2} + 3i + 2 \right)^{\left| \frac{n - 4i - 2}{2} \right|} \prod_{i \geq 0} \left( \frac{l}{2} + 2\left| \frac{n}{2} - i + 1 \right| \right)^{\left| \frac{n - 4i}{2} \right|} \left( \frac{l}{2} + 2\left| \frac{n - 1}{2} - i + \frac{3}{2} \right| \right)^{\left| \frac{n - 4i - 3}{2} \right|}. \quad (3)$$

Remarkably the above formula appears in two further places. Firstly it is the same as the determinant

$$\det_{0 \leq i, j \leq n-1} \left( \begin{pmatrix} l + i + j \end{pmatrix} + q \delta_{i,j} \right), \quad (4)$$

for $q = 1$. This determinant appeared for $q = 1$ in [4] where it was used to count descending plane partitions with parts less than $n$ by setting $l = 2$. In [10] it was shown that for $q$ being a sixth root of unity this determinant gives the weighted enumeration of cyclically symmetric lozenge tilings of a hexagon with side lengths $n, n + l, n, n + l, n, n + l$ with a central triangular hole of side length $l$. Secondly, the above formula is the $(-q)^{-n}$-th multiple of

$$\det_{0 \leq i, j \leq n-1} \left( \begin{pmatrix} l + i + j \end{pmatrix} 1 - (-q)^{j+k-1} \right), \quad (5)$$

where $k = 3$ and $q$ is a primitive third root of unity. This determinant appeared first in [13] in the special case $k = 1$ and enumerates for $k = 1$ and $l = 0$ the number of ASMs of size $n$. The above form appears in [1] where this determinant is evaluated for $k \in \mathbb{Z}$ and for $q$ being either an indeterminate, a fourth or a sixth root of unity and used to present an alternative proof of the ASM Theorem based on the operator formula. We will study possible
interpretation of \( l \) in (5) and possible connections between (3), (4) and (5) in forthcoming work.

3. The structure of AS-trapezoids

**Lemma 3.1.** Let \( S \) be a centred Catalan set of size \( n+1 \), \( A \) an \((n,l)\)-trapezoid with \( S(A) = S \) and write \( M(S) = (m_1, \ldots, m_n) \). Then the number of 1’s in the \( i \)-th row is at most \( 1 + \sum_{j=1}^{i} m_j \). Further for \( l \geq 2 \) there exists an \((n,l)\)-AS-trapezoid such that these bounds are sharp.

**Proof.** We define an \textit{allowed position} for a 1 in the \( i \)-th row of \( A \) as a position such that the next non-zero entry below is negative or all entries below are 0 and the label of the column corresponds to an element of \( S \). Denote by \( a_i \) the number of allowed positions for a 1 in the \( i \)-th row. It follows that \( a_1 = 1 + m_1 \). Since every 1 (resp. \(-1\)) in the \( i \)-th row cancels out (resp. adds) an allowed position for a 1 in the \((i+1)\)-th row and there is one more 1 than \(-1\) in every row, the number of allowed positions in the central \( l + 2i - 1 \) columns of the \((i+1)\)-th row is \( a_i - 1 \). There are two new columns in row \( i + 1 \) of which \( m_{i+1} + 1 \) have a label corresponding to an element in \( S \). Hence we obtain \( a_{i+1} = a_i + m_{i+1} \) and therefore

\[
a_i = a_1 + \sum_{j=2}^{i} m_j = 1 + \sum_{j=1}^{i} m_i.
\]

We construct an \((n,l)\)-AS-trapezoid \( A' \) with \( S(A') = S \) and a maximal number of 1’s in a recursive manner. Place in the \( i \)-th row a 1 in all allowed positions and put a \(-1\) between the 1 entries. Since the allowed positions are either the most left or right positions of a row or above a \(-1\) from the row before, two allowed positions are by induction not direct neighbours. Therefore it is always possible to place a new row by the above algorithm. By the above formula there is only one allowed position in the top row. If there exists a column in \( A' \) with a \(-1\) as first non-zero entry form top there would be a second allowed position in the top row. An allowed position can appear in the central \( l - 1 \) columns only above a \(-1\). Hence the column-sum \( \sum_{j=1}^{l-1} a_j \) is for the central \( l - 1 \) columns zero and the resulting array is an \((n,l)\)-AS-trapezoid.

**Lemma 3.2.** Let \( S_1, S_2 \) be centred Catalan sets, \( A_1 \) an \((|S_1| - 1, l)\)-AS-trapezoid and \( A_2 \) an \((|S_2| - 1, l + 2|S_1| - 2)\)-AS-trapezoid with \( S(A_i) = S_i \) for \( i = 1, 2 \). By placing \( A_2 \) centred above \( A_1 \) we obtain an \((|S_1 \circ S_2| - 1, l)\)-AS-trapezoid with centred Catalan set \( S_1 \circ S_2 \). Further every \((|S_1 \circ S_2| - 1, l)\)-AS-trapezoid \( A \) with \( S(A) = S_1 \circ S_2 \) is of the above form.

**Proof.** It is obvious that this construction yields an \((|S_1| + |S_2| - 2, l)\)-AS-trapezoid with centred Catalan set \( S_1 \circ S_2 \).

On the other hand let \( A \) be an \((|S_1 \circ S_2| - 1, l)\)-AS-trapezoid with \( S(A) = S_1 \circ S_2 \). We split \( A \) into a bottom part \( A_1 \) consisting of the first \(|S_1| - 1 \) rows from bottom and a top part \( A_2 \) consisting of the remaining rows. By Lemma 3.1 there is only one allowed position for a 1 in the top row of \( A_1 \). If \( A_1 \) had a column whose first non-zero entry from top is negative, there would be a second allowed position for a 1 in the top row of \( A_1 \). Since the
centred \( l - 1 \) columns of \( A \) have column-sum zero and the first entry from top is positive, their partial column-sums of the bottom \(|S_1| - 1\) rows can not be positive. Hence the centred \( l - 1 \) columns of \( A_1 \) have column-sum zero which implies that \( A_1 \) is an \((|S_1| - 1, l)\)-AS-trapezoid with \( S(A_1) = S_1 \). One of the centred \( l + 2|S_1| - 3 \) columns of \( A \) has a positive column-sum if its column label corresponds to an element in \( S_1 \) which implies that \( A_1 \) has a positive column-sum in this column. Therefore the column-sums of the central \( l + 2|S_1| - 3 \) columns of \( A_2 \) are zero, which implies that \( A_2 \) is an \((|S_2| - 1, l + 2|S_1| - 2)\)-AS-trapezoid. \( \square \)

**Proof of Proposition 1.1.** The theorem is a direct consequence of the above lemma. \( \square \)

### 4. A refined enumeration of AS-trapezoids

#### 4.1. A different perspective on AS-trapezoids

The aim of this section is to prove that the weight function \( w_l(S) \) is a polynomial in \( l \). First we need the following definition which is due to Fischer [15].

**Definition 4.1.** Let \( 1 \leq u < v \leq n \), \( s = (s_1, \ldots, s_u) \) be a weakly decreasing sequence of non-negative integers, \( t = (t_v, \ldots, t_n) \) a weakly increasing sequence of non-negative integers and \( k = (k_1, \ldots, k_n) \) an increasing sequence of integers. An \((s, t)\)-tree with bottom entries \( k \) is a triangular array of integers with the following properties:

- The entries are weakly increasing in north-east and south-east direction and strictly increasing in east direction.
- For \( 1 \leq i \leq u \) the bottom \( s_i \) elements in the \( i\)-th north-east diagonal are deleted and the bottom entry of this north-east diagonal is \( k_i \). This entry does not have to be less than its right neighbour.
- For \( v \leq i \leq n \) the bottom \( t_i \) elements in the \( i\)-th south-east diagonal are deleted. The bottom entry of this south-east diagonal is \( k_i \). This entry does not have to be greater than its left neighbour.
- The entries in the bottom row are \( k_{u+1}, \ldots, k_{v-1} \).
Let $f$ be a function in $x$. We define

$$E_x(f)(x) := f(x + 1) \quad \text{shift operator},$$

$$\Delta x := E_x - \text{Id} \quad \text{forward difference},$$

$$\Delta^{-1} x := \text{Id} - E_x^{-1} \quad \text{backward difference}.$$

**Theorem 4.2** (\cite{12, Section 4]). Set

$$M_n(x_1, \ldots, x_n) := \prod_{1 \leq p < q \leq n} (\text{Id} + \Delta x_p \Delta x_q + \Delta x_q) \prod_{1 \leq p < q \leq n} \frac{x_q - x_p}{q - p}. \quad (6)$$

The number of $(s, t)$-trees with bottom entries $k = (k_1, \ldots, k_n)$ is given by

$$\left( \prod_{i=1}^u (-\Delta x_i)^{s_i} \prod_{i=v}^n \Delta x_i \right) M_n(x_1, \ldots, x_n) \bigg|_{x = k}. \quad (7)$$

Let $S = \{s_1, \ldots, s_u, 0, s_{u+1}, \ldots, s_n\}$ be a centred Catalan set of size $n + 1$, $s_1 < \ldots < s_u \leq -1$ and $1 \leq s_{u+1} < \ldots < s_n$. Let $s = (-s_1 - 1, \ldots, -s_u - 1, t) = (s_{u+2} - 1, \ldots, s_{n} - 1)$ and $k = (s_1 + 1, \ldots, s_u + 1, l + s_{u+1} - 1, \ldots, l + s_n - 1)$. The following algorithm is a bijection between $(n, t)$-AS-trapezoids $A$ with $S(A) = S$ and $(s, t)$-trees with bottom entries $k$, for an example see Figure 5. First we construct a triangular array $T_A$. We fill the $i$-th row from bottom of $T_A$ by the column labels of $A$, for which the first non-zero entry above the $i$-1-th row is positive. Thereby we write the numbers in an increasing order from left to right. The bottom row of $T_A$ is $k$. Since in every row of the trapezoid there is one more 1 than -1 the number of entries in a row of $T_A$ is one less than the number of entries in the row below. Further it is easy to see that the entries of $T_A$ are weakly increasing in north-east and south-east direction. Since the column-sum of the $(s_1 + 1)$-th column of $A$ is 1, the first $-s_i$ entries of the $i$-th north-east diagonal of $T_A$ for $1 \leq i \leq u - 1$ will be $s_i + 1$. Hence we can delete without loss of information $-s_i - 1$ of them. Analogously the first $s_i$ entries of the $i$-th south-east diagonal of $T_A$ will be $l + s_i - 1$ for $u + 2 \leq i \leq n$ and we can delete $s_i - 1$ of them. The resulting array $T_A$ is an $(s, t)$-tree with bottom entries $k$. On the other hand it is not difficult to see that every such $(s, t)$-tree with bottom entries $k$ is of the form $T_A$. This proves the following proposition.

**Proposition 4.3.** Let $S$ be an irreducible centred Catalan set of size $n + 1$ with $n \geq 2$. The map $A \mapsto T_A$ from $(n, t)$-AS-trapezoids to $(s, t)$-trees with bottom entries $k$ as described above is a bijection.
In order to compact the notation we call an \((s,t)\)-tree with bottom entries \(k\) corresponding to an \((n,l)\)-AS-trapezoids with associated centred Catalan set \(S\) an \((S,l)\)-tree.

### 4.2. Equivalence classes of \((S,l)\)-trees

In the following we introduce an equivalence relation for AS-trapezoids and translate it to an equivalence relation for \((S,l)\)-trees. The equivalence relation is more intuitively accessible in the AS-trapezoid picture, however it is more handy for \((S,l)\)-trees.

**Definition 4.4.** Let \(n,l\) be integers and \(A, B\) two \((n,l)\)-AS-trapezoids. Among the central \(l-1\) columns of \(A\) (resp. \(B\)) denote the columns which have non-zero entries from left to right by \(c_1(A), \ldots, c_f(A)\) (resp. \(c_1(B), \ldots, c_g(B)\)). We call \(A\) equivalent to \(B\) iff the following holds

- the columns with labels less than or equal to 0 are identical for \(A\) and \(B\),
- the columns with labels greater than or equal to \(l\) are identical for \(A\) and \(B\),
- \(f = g\) and \(c_i(A) = c_i(B)\) for all \(1 \leq i \leq f\).

We call the columns \(c_1(A), \ldots, c_f(A)\) the free columns of \(A\).

**Example 4.5.** Let \(A, B, C\) be the \((4,4)\)-AS-trapezoids defined as below. The columns with labels less than or equal to 0 and the columns with labels greater than or equal to \(l = 4\) are identical for \(A, B, C\). The free columns are marked in blue and green.

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 & 1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & 1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 & 1
\end{bmatrix}.
\]

The free columns of \(A\) and \(C\) are

\[
c_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \end{bmatrix},
\]

hence \(A\) and \(C\) are equivalent. The free columns of \(B\) are \(c_2, c_1\). Since they are in the opposite order than the free columns of \(A\), the AS-trapezoids \(A\) and \(B\) are not equivalent.
Let \( A, B \) be \((n, l)\)-AS-trapezoids. If \( A \) and \( B \) are equivalent then \( S(A) = S(B) \). This is true since the centred Catalan set \( S(A) \) only depends on the columns of \( A \) with labels less than or equal to 0 and columns with labels greater than or equal \( l \), which are by definition fixed by the equivalence relation. Hence the equivalence relation on \((n, l)\)-AS-trapezoids induces an equivalence relation for \((S, l)\)-trees and is given in the following way.

**Definition 4.6.** Let \( c = (c_{i,j}), c' = (c'_{i,j}) \) be two \((S, l)\)-trees. We call \( c \) and \( c' \) equivalent iff the following holds.

1. For all \( i, i', j, j' \) holds \( c_{i,j} < c_{i', j'} \) iff \( c'_{i,j} < c'_{i', j'} \).
2. For all \( i, j \) with \( c_{i,j} \leq 0, c_{i,j} \geq l \) holds \( c_{i,j} = c'_{i,j} \).

We define the number of free columns of an \((S, l)\)-tree \( c \) as \( |\{c_{i,j} : 0 < c_{i,j} < l\}| \). It follows from the definition that the number of free columns is constant on equivalence classes.

**Example 4.7.** Let \( A \) be the first \((4, 4)\)-AS-trapezoid in Example 4.5. The \((\{-2, -1, 0, 1, 2\}, 4)\)-tree \( T_A \) corresponding to \( A \) and the equivalence class \( T_A \) of \( T_A \) are

\[
T_A = \begin{array}{ccc}
2 & 3 & b \\
-1 & 2 & 5 \\
0 & 4 &
\end{array}, \quad T_A = \begin{array}{ccc}
b & a & b \\
a & 5 & 0 \\
0 & 4 &
\end{array},
\]

with \( 0 < a < b < 4 \). The number of free columns of \( T_A \) is 2.

**Lemma 4.8.** Let \( S \) be a centred Catalan set, \( l > 0 \) and \( c \) an \((S, l)\)-tree with \( f \) free columns. Then the size of the equivalence class \( \overline{c} \) of \( c \) is given by

\[
|\overline{c}| = \binom{l-1}{f}.
\]

*Proof.* Let \( x_1, \ldots, x_f \) be entries in \( c \) such that \( 0 < x_1 < \ldots < x_f < l \). Every element of the equivalence class \( \overline{c} \) is uniquely described by the sequence of these values \( (x_1, \ldots, x_f) \). Hence

\[
|\overline{c}| = \left| \{(x_1, \ldots, x_f) : 0 < x_1 < \ldots < x_f < l \} \right| = \binom{l-1}{f}. \quad \square
\]

Let \( S = \{s_1, \ldots, s_u, 0, s_u+1, \ldots, s_n\} \) be a centred Catalan set of size \( n+1 \) with \( s_1 < \ldots < s_u < 0 < s_{u+1} < \ldots < s_n \). We define two Young diagrams \( \lambda(S), \mu(S) \) via

\[
\lambda(S) = (u + 1 - s_{u+1}, \ldots, n - s_u),
\]

\[
\mu(S) = (-s_1 - u, \ldots, -s_u - 1),
\]

where \( \lambda' \) denotes the conjugate Young diagram, see Figure 6. Alternatively one can describe \( \lambda(S) \) and \( \mu(S) \) as the smallest Young diagrams such that \( \lambda(S)/\mu(S) \) is the area enclosed between the paths \( \lambda(S) = (\lambda_i)_{1 \leq i \leq n} \) and \( \mu(S) = (\mu_i)_{1 \leq i \leq n} \) respectively which are defined as:
Figure 6. The (skew-shaped) Young diagrams corresponding to $S_1 = \{-4, -2, -1, 0, 1, 3, 4\}$, $S_2 = \{-3, -1, 0, 1, 2\}$ and $S_1 \circ S_2$.

Using this description it is easy to see that $\mu(S)$ is included in $\lambda(S)$ and that

$$|\lambda(S)/\mu(S)| = \text{area}(M(S)).$$

Remark 4.9. Let $S_1, S_2$ be two centred Catalan sets. It follows from the definition of the Young diagrams $\lambda, \mu$ that the diagrams corresponding to the concatenation $S_1 \circ S_2$ are given by

$$\lambda(S_1 \circ S_2) = (\lambda(S_2) + |S_1| - 1, \lambda(S_1)),$$

$$\mu(S_1 \circ S_2) = (\mu(S_2) + |S_1| - 1, \mu(S_1)).$$

For an example see Figure 6. Hence the size of the skew-shaped Young diagram is given by

$$|\lambda(S_1 \circ S_2)/\mu(S_1 \circ S_2)| = |\lambda(S_1)/\mu(S_1)| + |\lambda(S_2)/\mu(S_2)|,$$

and the number of standard Young tableaux of skew shape $|\lambda(S_1 \circ S_2)/\mu(S_1 \circ S_2)|$ is given by

$$f^{\lambda(S_1 \circ S_2)/\mu(S_1 \circ S_2)} = f^{\lambda(S_1)/\mu(S_1)} f^{\lambda(S_2)/\mu(S_2)}.$$

Lemma 4.10. Let $S$ be an irreducible centred Catalan set of size $n$. The number of free columns of an $(S, l)$-tree is at most $|\lambda(S)/\mu(S)|$; the number of equivalence classes of $(S, l)$-trees is for $l > |\lambda(S)/\mu(S)|$ independent of $l$ and the number of equivalence classes with maximal free columns is for $l > |\lambda(S)/\mu(S)|$ equal to $f^{\lambda(S)/\mu(S)}$, the number of standard Young tableaux of skew shape $\lambda(S)/\mu(S)$.

Since the above lemma plays a fundamental role in the proof of Theorem 1.2 we want to illustrate it first by an example before proving it.
Figure 7. On the left the general form of a (4, l)-AS-trapezoid with central Catalan set \( S = \{-2, -1, 0, 1, 2\} \), where there is a box around every entry of the trapezoid which is not fixed by definition. On the right the skew-shaped Young diagram \( \lambda(S)/\mu(S) \) with \( S = \{-2, -1, 0, 1, 2\} \).

Figure 8. The transformation from the order relation between \( a, b, c \) and \( d \) to a standard Young diagram.

Let \( S = \{-2, -1, 0, 1, 2\} \) as in Example 4.7. Every \((S, l)\)-tree \( T \) has the form

\[
    T = \begin{array}{ccc}
        c \\
        a & d \\
        b & l + 1 \\
        0 & l \\
    \end{array}
\]

with \( a \leq b, c \leq d \). In the proof of Lemma 4.10 we show that the skew shaped Young tableau \( \lambda(S)/\mu(S) \) can be constructed by putting a box around the entries of \( T \) which are not fixed by definition, i.e., in our case around \( a, b, c, d \), and rotate the resulting shape by 45° clockwise, see Figure 7.

The number of free columns of \( T \) is \( \{|1, \ldots, l - 1|\} \leq 4 \). The equivalence class of \( T \) is fully determined by the order relation between the entries \( a, b, c \) and \( d \) and their exact value iff the value is in the set \( \{z \in \mathbb{Z} | z \leq 0 \text{ or } z \geq l\} \). Hence it is obvious that all possible equivalence classes appear for \( |\{1, \ldots, l - 1\}| \geq 4 \) which is equivalent to \( l > 4 \).

If \( T \) has a maximal number of free columns, the equivalence class of \( T \) is determined solely by the order relation between \( a, b, c, d \) which can be either \( a < b < c < d \) or \( a < c < b < d \). We forget about all other entries of the AS-trapezoid and replace \( a, b, c, d \) by the numbers 1, 2, 3, 4 regarding their ordering. This is possible since we have a maximal number of free columns which means that all values of \( a, b, c, d \) must be pairwise different. In the next step we rotate the array by 45° clockwise and obtain a standard Young tableaux of skew shape \( \lambda(S)/\mu(S) \), see Figure 8.
Figure 9. Schematic representation of an \((S, l)\)-tree and the skew-shaped Young diagram \(\lambda(S)/\mu(S)\) for \(S = \{-2, -1, 0, 1, 2, -4\}\).

Proof. Given an \((S, l)\)-tree \(T\), we draw around all entries which are not fixed by definition a box, an example can be seen on the left side of Figure 9. Denote by \(b_i\) the number of boxes in the \(i\)-th row from bottom of \(T\). This implies \(b_1 = 0\) and \(b_{i+1} = b_i + |\{-i, i\} \cap S| - 1\) for \(1 \leq i \leq |S| - 2\). If \(-i \in S\) then the box complex is extended in the \((i + 1)\)-th row to the left, if \(i \in S\) the box complex is extended in the \((i + 1)\)-th row to the right. Hence the outer shape is determined by the two paths \(\lambda'\) and \(\mu'\), where the \(i\)-th steps \(\lambda'_i, \mu'_i\) of \(\lambda', \mu'\) are given by

\[
\begin{array}{c|cc}
\{-i, i\} \subseteq S & \lambda'_i & \mu'_i \\
-i \in S, i \notin S & NE & NW \\
i \in S, -i \notin S & NW & NE \\
-i, i \notin S & NW & NE \\
\end{array}
\]

By rotating the box complex by 45° clockwise we obtain the skew-shaped diagram \(\lambda(S)/\mu(S)\) whose entries are the entries of \(T\) which are not fixed by definition. It follows that the maximal number of free columns is hence \(|\lambda(S)/\mu(S)|\).

The equivalence class of \(T\) is completely determined by the entries with value less than or equal to 0 or greater than or equal to \(l\) and the order between the elements with values between 1 and \(l - 1\). It is easy to see that the number of equivalence classes of \((S, l)\)-trees with \(k\) free columns is for \(l > k\) not depending in \(l\). Hence for \(l > |\lambda(S)/\mu(S)|\) the total number of equivalence classes is not depending in \(l\).

For an equivalence class with a maximal number of free columns, all entries in the boxes are different. Hence we can order them according to their values. We record the order in the skew shape Young diagram. Since the entries in an \((s, t)\)-tree are strictly increasing along rows from left to right and weakly increasing along south-west north-east diagonals from bottom to top, the filling of the skew shaped Young diagram is strictly increasing along columns and rows, i.e., it is a skew shape standard Young tableau. Hence every equivalence class with maximal free columns corresponds to a standard Young tableau of skew shape \(\lambda(S)/\mu(S)\). It is obvious that the converse is also true.

\(\square\)
With the above preparation we can prove our second main result.

Proof of Theorem 1.2. By Proposition 1.1 and Remark 4.9 it suffices to assume \( S \) to be irreducible. As a consequence of Theorem 1.2 and Proposition 4.3 the number \( w_l(S) \) of \((S,l)\)-trees is a polynomial in \( l \). Let \( d(S) \) be the degree of \( w_l(S) \). We show by induction on \( \text{area}(M(S)) \) that \( d(S) \leq \text{area}(M(S)) \). Write \( S \) in the form \( S = \{s_1, \ldots, s_u, 0, s_{u+1}, \ldots, s_n\} \) with \( s_1 < \ldots < s_u < 0 < s_{u+1} < s_n \). The degree \( d(S) \) is at most the degree of the polynomial \( M_n(x_1, \ldots, x_n) \) minus the number of \( \Sigma, \Delta \) operators appearing in (7) and therefore

\[
d(S) \leq \binom{n}{2} - \sum_{i=1}^{u} (-s_i - 1) - \sum_{i=n+1}^{u} (s_i - 1) = \binom{n}{2} - \sum_{i \in S \setminus \{0\}} (|i| - 1). \tag{9}
\]

We claim that

\[
\binom{n}{2} = \sum_{i \in S \setminus \{0\}} (|i| - 1) + \text{area}(M(S)), \tag{10}
\]

which will imply together with (9) that \( d(S) \leq \text{area}(M(S)) \). If \( \text{area}(M(S)) = 0 \), exactly one of \( i \) and \(-i\) is in \( S \) for all \( 1 \leq i \leq n \), hence

\[
\sum_{i \in S \setminus \{0\}} (|i| - 1) = \sum_{i=1}^{n} (i - 1) = \binom{n}{2}.
\]

If \( \text{area}(M(S)) > 0 \) denote by \( i_0 \) the largest integer with \( 1 \leq i_0 \leq n - 1 \) and \( \{-i_0, i_0\} \subseteq S \). Then the centred Catalan set

\[
S' := \begin{cases} (S \setminus \{i_0\}) \cup \{i_0 + 1\} & (i_0 + 1) \notin S, \\ (S \setminus \{i_0\}) \cup \{-i_0 + 1\} & -(i_0 + 1) \notin S, \end{cases}
\]

is well defined. The paths \( M(S) \) and \( M(S') \) differ only in the \( i_0 \)-th and \((i_0 + 1)\)-th step as shown in Figure 10. This implies that \( \text{area}(M(S')) = \text{area}(M(S)) - 1 \). On the other hand the sum over all \( i \in S \setminus \{0\} \) in (9) is one less than the sum over all \( i \in S' \setminus \{0\} \) which proves (10).

By Lemma 4.8 an equivalence class of \((S,l)\)-trees with \( f \) free columns contributes \( \binom{f}{l-1} \) to \( w_l(S) \). Lemma 4.10 states that there are \( f^{\lambda(S)/\mu(S)} \) equivalence classes with \( |\lambda(S)/\mu(S)| = \text{area}(M(S)) \) free columns. Hence \( w_l(S) \) has degree \( \text{area}(M(S)) \) and leading the coefficient is \( f^{\lambda(S)/\mu(S)} \). \( \square \)

Since \( w_l(M) = \sum_{S: M(S) = M} w_l(S) \) the following is a direct consequence of the above Theorem.

Figure 10. The two possibilities of changes between \( M(S) \) and \( M(S') \) at the positions \( i_0, i_0 + 1, i_0 + 2 \).
We define \( S_n \) noncrossing matching of size \( n \).

**Remark 4.12.** For a centred Catalan set of size \( w \) let \( \lambda \) be a polynomial in \( l \) of degree \( \text{area}(M) \) with leading coefficient \( \frac{1}{\text{area}(M)!} \sum_{S:M(S)=S} f^{\lambda(S)/\mu(S)} \).

It is unknown to the author if there exists a closed formula for the leading coefficient of \( w_1(M) \) than the one in the above corollary.

**Corollary 4.11.** Let \( M \) be a Motzkin path. The weight function \( w_1(M) \) is a polynomial in \( l \) of degree \( \text{area}(M) \) with leading coefficient \( \frac{1}{\text{area}(M)!} \sum_{S:M(S)=S} f^{\lambda(S)/\mu(S)} \).

Hence Theorem 1.2 implies that \( w(S(m)) \) is a polynomial in \( m \) of degree \( |\lambda(S_1)/\mu(S_1)| + |\lambda(S_2)/\mu(S_2)| \) and leading coefficient \( 2^{|\lambda(S_1)/\mu(S_1)| + |\lambda(S_2)/\mu(S_2)|} \frac{f^{\lambda(S_1)/\mu(S_1)} f^{\lambda(S_2)/\mu(S_2)}}{\lambda(S_1)/\mu(S_1)! \cdot \lambda(S_2)/\mu(S_2)!} \).

An analogous theorem [9] Theorem 1.9 exists for fully packed loops (FPLs) which are equinumerous to ASTs. Denote by \( \pi(m) := (\pi(S_1))_m, \pi(S_2) \) the noncrossing matching which consists of \( \pi(S_1) \) enclosed by \( m \) ‘nested arches’ concatenated with \( \pi(S_2) \), see Figure 11. Then this theorem states that the number of FPLs associated to \( \pi(m) \) is a polynomial in \( m \) of degree \( |\lambda(\pi(S_1))| + |\lambda(\pi(S_2))| \) and leading coefficient \( \frac{f^{\lambda(\pi(S_1))} f^{\lambda(\pi(S_2))}}{|\lambda(\pi(S_1))|! \cdot |\lambda(\pi(S_2))|!} \).

where \( \lambda(\pi) \) is a Young diagram associated to the noncrossing matching \( \pi \). This is remarkable because of three reasons. Firstly the refinements of ASTs or FPLs respectively by Catalan objects are of different nature, however we
have in both cases a polynomiality theorem. Secondly the degree and the leading coefficient are for both almost the same with the major difference that they are linked to Young diagrams for FPLs while they are linked to skew-shaped Young diagrams for ASTs. The last reason is that the two refinements seem to be in some sense ‘dual’: comparing the polynomiality results they differ by the dual notions ‘nested arches’ and ‘small arches’; further the number of FPLs linked to a noncrossing matching is minimal if it consists of nested arches and is maximal if it consists of small arches while ASTs seem to have the opposite behaviour.

5. A CONSTANT TERM EXPRESSION FOR AS-TRAPEZOIDS

Using Theorem 4.2 we are going to yield a constant term expression for \( w_l(S) \). Our proof follows the same steps as the proof of Theorem 7 in [15]. The following identities will be needed later. The first one is an easy consequence of the definitions of the operators \( E_x, \Delta x, \Delta_x \) while the second one is stated in [11, Lemma 5].

\[
\text{Id} + \Delta_y + \Delta_x \Delta_y = E_x E_y (\text{Id} - \Delta_x + \Delta_x \Delta_y) = E_y (\text{Id} + \Delta_x \Delta_y), \quad (11)
\]

\[
M_n(x_1, \ldots, x_n) = (-1)^{n-1} M_n(x_2, \ldots, x_n, x_1 - n). \quad (12)
\]

**Proof of Theorem 1.3.** We set \( s = \{-s_1 - 1, \ldots, -s_u - 1\} \), \( t = \{s_{u+1} - 1, \ldots, s_n - 1\} \), \( k = \{s_1 + 1, \ldots, s_u + 1, l + s_{u+1} - 1, \ldots, l + s_n - 1\} \). Since \((n,l)\)-alternating sign trapezoids correspond to \((s,t)\)-trees with bottom entries \(k\), Theorem 4.2 states

\[
w_l(S) = \left( \prod_{i=1}^{u} (-\Delta_{x_i})^{-s_i-1} \prod_{i=u+1}^{n} \Delta_{x_i+1}^{s_i} \right) M_n(x_1, \ldots, x_n) \bigg|_{x=k}
\]

\[
= \left( \prod_{i=1}^{u} (-\Delta_{x_i})^{-s_i-1} E_{x_i}^{s_i+1} \prod_{i=u+1}^{n} \Delta_{x_i}^{s_i-1} E_{x_i}^{l+s_i} \right) M_n(x_1, \ldots, x_n) \bigg|_{x=0}
\]

\[
= \left( \prod_{i=1}^{u} (-\Delta_{x_i})^{-s_i-1} \prod_{i=u+1}^{n} \Delta_{x_i}^{s_i-1} E_{x_i} \right) M_n(x_1, \ldots, x_n) \bigg|_{x=0}.
\]

Using (11) and (12), we can write \( M_n(x_1, \ldots, x_n) \) as

\[
M_n(x_1, \ldots, x_n) = (-1)^{u(n-1)} M_n(x_{u+1}, \ldots, x_n, x_1 - n, \ldots, x_u - n)
\]

\[
= (-1)^{u(n-1)} \prod_{i=1}^{u} E_{x_i}^{-n} \prod_{1 \leq i < j \leq u} E_{x_i} E_{x_j} (\text{Id} - \Delta_{x_i} + \Delta_{x_i} \Delta_{x_j})
\]

\[
\times \prod_{u+1 \leq i < j \leq n} (\text{Id} + \Delta_{x_j} + \Delta_{x_i} \Delta_{x_j}) \prod_{i=1}^{u} \prod_{j=u+1}^{n} E_{x_i} (\text{Id} + \Delta_{x_i} \Delta_{x_j}) (-1)^{u(n-1)}
\]

\[
\times \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i} = \text{Op} \prod_{i=1}^{u} E_{x_i}^{-1} \prod_{1 \leq i, j \leq n} \left( \frac{x_i}{j - 1} \right), \quad (13)
\]
By \[13\] we obtain for \(w_l(S)\) the following expression

\[
w_l(S) = \text{Op} \sum_{\sigma \in S_n} \sum_{i=1}^u \prod_{i=1}^u \left( \text{sgn}(\sigma) \prod_{i=1}^u \left( \frac{x_i^l}{(\sigma(i) + s_l)} \right) \prod_{i=1}^u \left( \frac{x_i + l}{(\sigma(i) - s_l)} \right) \right)_{x=0}.
\]

Since \((-\Delta_x^{(x+s)})^{-1}(-)^s = E_{(x+s)}^{-1}(x+s)\) and \(\Delta_x^{(x+l)} = E_{(x+l)}\) we can replace \(\text{Op}\) by \(\text{Op}'\), where \(\text{Op}'\) is \(\text{Op}\) with every \(-\Delta_x\) replaced by an \(E_{s_i}^{-1}\) and every \(\Delta_x\) replaced by an \(E_{s_j}\). As a consequence of this we can evaluate now \[14\] at \(x_1, \ldots, x_n = 0\) and obtain

\[
w_l(S) = \text{Op}' \left( (-1)^{u+\sum_{i=1}^u s_i} \sum_{\sigma \in S_n} \prod_{i=1}^u \left( \frac{s_i}{\sigma(i) + s_i} \right) \prod_{i=1}^n \left( \frac{l}{\sigma(i) - s_i} \right) \right).
\]

In order to evaluate this expression we use the following identity. Let \(f: \mathbb{Z}^n \rightarrow \mathbb{R}\) be a function such that the generating function \(F(x_1, \ldots, x_n) := \sum_{a \in \mathbb{Z}^n} f(a)x^a\) is a Laurent series and let \(p(x_1, \ldots, x_n)\) be a Laurent polynomial. Then

\[p(E_{a_1}, \ldots, E_{a_n})f(a) = \text{CT}_x \prod_{i=1}^n x_i^{-a_i}p(x_1^{-1}, \ldots, x_n^{-1})F(x_1, \ldots, x_n).\]

This identity can be proven easily if \(p\) is a monomial. The general statement follows immediately since \(p\) is in general a linear combination of monomials. We apply it for the function

\[f(s_1, \ldots, s_n) := (-1)^{u+\sum_{i=1}^u s_i}\]

\[
\times \sum_{\sigma \in S_n} \prod_{i=1}^u \left( \frac{s_i}{\sigma(i) + s_i} \right) \prod_{i=1}^n \left( \frac{l}{\sigma(i) - s_i} \right).
\]

By using

\[
\sum_{s \in \mathbb{Z}} (-1)^s \binom{s}{\sigma + s} x^s = (-x)^{-1}(-1 - x^{-1})^{-\sigma - 1},
\]

\[
\sum_{s \in \mathbb{Z}} \binom{l}{\sigma - s} x^s = x^{\sigma - l}(1 + x)^l.
\]
we obtain for \( F \)
\[
F(x_1, \ldots, x_{n-1}) = (-1)^u \sum_{\sigma \in \Theta_n} \text{sgn}(\sigma) \\
\times \prod_{i=1}^u (-x_i)^{-1}(1 - x_i^{-1})^{\sigma(i)-1} \prod_{i=u+1}^n x_i^{\sigma(i)-1}(1 + x_i)^i \\
= (-1)^u \det_{1 \leq i, j \leq n} \left( \begin{array}{cc} (-x_i)^{-1}(1 - x_i^{-1})^{j-1} & i \leq u \\ x_i^{-j}(1 + x_i)^i & i > u \end{array} \right) \\
= \prod_{i=1}^u x_i^{-u} \prod_{i=u+1}^n x_i^{1-i}(1 + x_i)^i \prod_{1 \leq i < j \leq n} (x_j - x_i) \prod_{i=1}^n \prod_{j=u+1}^n (1 + x_i^{-1} + x_j),
\]
where we evaluated in the last step the determinant by the Vandermonde determinant evaluation. Hence we obtain
\[
w_l(S) = \text{CT}_{x_1, \ldots, x_n} \prod_{i=1}^u x_i^{-n-s_i} \\
\prod_{i=u+1}^n x_i^{-n-l-s_i+2}(1 + x_i)^i \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 + x_i + x_ix_j). \quad \Box
\]

6. Roots of \( w_l(S) \) and \( w_l(M) \)

Studying \( w_l(S) \) and \( w_l(M) \) for small centred Catalan sets \( S \) or Motzkin paths \( M \) shows that \( w_l(S) \) and \( w_l(M) \) often have rational roots, where almost all of these are even integer roots. In Appendix A we list \( w_l(S), w_l(M) \) for all irreducible centred Catalan sets up to size 6 and irreducible Motzkin paths up to length 5. The following section contains results and conjectures, based on a first analysis of a slightly larger data set, which can be divided into three kinds:

- Conjecture 6.1 is determining which rational numbers are appearing as roots of weight functions.
- The conjectures in section 6.1 state relations between the rational roots of \( w_l(S) \) and the beginning of \( M(S) \).
- In Section 6.2 rational roots of \( w_l(M) \) are conjecturally connected to the ending of the Motzkin path \( M \).

The ultimate goal would be a description of all rational roots of \( w_l(S) \) and \( w_l(M) \) similar to [16] Conjecture 1.2

**Conjecture 6.1.**

1. Let \( M \) be an irreducible Motzkin path of length \( n \geq 8 \). The rational roots of \( w_l(M) \) lie in \( \{-1, -2, \ldots, -2n+2\} \) and for every integer in this set exists a Motzkin path \( M \) such that it is a root of \( w_l(M) \).

2. Let \( S \) be an irreducible centred Catalan set of size \( n \geq 11 \). The rational roots of \( w_l(S) \) lie in \( \{-1, \ldots, -2n + 4, \frac{n^2 - 5n + 7}{(n-3)}\} \) and for every integer in this set exists a centred Catalan set \( S \) such that it is a root of \( w_l(S) \). Further \( \frac{n^2 - 5n + 7}{(n-3)} \) is a root of \( w_l(S) \) iff \( S = \{-n+2, -1, 0, 1, \ldots, n-3\} \) or \( S = \{-n+3, \ldots, -1, 0, 1, n-2\} \).
6.1. Rational roots of $w_l(S)$.

**Conjecture 6.2.** Let $S$ be a centred Catalan set. Then

$$\{-i, \ldots, i\} \subseteq S \iff \prod_{k=0}^{\lfloor \frac{|S|}{2} \rfloor} (l + 1 + 3k)i_{i-2k}$$

divides $w_l(S)$,

where $(x)_k := x(x+1) \cdots (x+k-1)$.

The above conjecture implies for Motzkin paths that their weight function has certain integer roots depending on the number of consecutive north-east steps at the beginning of the path. We can prove the above conjecture for $i = 1, 2$, however our proof technique does not extend to $i > 2$.

**Proposition 6.3.** Let $S$ be a centred Catalan set, then the following holds:

$$\{-1, 1\} \subset S \iff (l + 1) | w_l(S),$$

$$\{-2, -1, 1, 2\} \subset S \iff (l + 1)(l + 2) | w_l(S).$$

**Proof.** By Proposition 1.1 the weight function $w_l(S)$ factorises for $S = S_1 \circ S_2$ into $w_l(S) = w_l(S_1)w_{l+2}(S_1)w_{l-2}(S_2)$. By Lemma 3.1 there exists for $l \geq 2$ always an $(|S| - 1, l)$-AS-trapezoid with associated centred Catalan set $S$, i.e., $w_l(S) \neq 0$ for $l \geq 2$. Hence it suffice to prove the Proposition for irreducible $S$.

The proof is based on the following fact. Let $p(x)$ be a polynomial in $x$ and define

$$P(x) = \begin{cases} \sum_{i=0}^{x} p(i) & x \geq 0, \\ 0 & x = -1, \\ -\sum_{i=x+1}^{-1} p(i) & x < 0. \end{cases}$$

Then $P(x)$ is again a polynomial in $x$. Let $\{-1, 1\} \subset S$. Then an $(S, l)$-tree has the form

$$\ldots a \ldots 0 \ l \ldots,$$

where $0 \leq a \leq l$. Denote by $f(l, x)$ the number of $(S, l)$-trees with the bottom row removed and with $a = x$. By Theorem 4.2, $f(l, x)$ is a polynomial in $l$ and $x$. The weight $w_l(S)$ is given by

$$w_l(S) = \sum_{x=0}^{l} f(l, x).$$

Hence for $l = -1$ equation (15) becomes

$$w_{-1}(S) = \sum_{x=0}^{-1} f(-1, x) = 0.$$
Let $\{-2, -1, 1, 2\} \subseteq S$. Then an $(S, l)$-tree has the form

\[
\begin{array}{cccc}
\vdots & a & b & \vdots \\
-1 & c & l + 1 \\
0 & l
\end{array}
\]

with $-1 \leq a \leq c \leq b \leq l + 1$, $0 \leq c \leq l$ and $a < b$. Denote by $f(l, x, y)$ the number of $(S, l)$-trees with the bottom two rows deleted and $a = x, b = y$.

The weight function is given by

\[
w_l(S) = \sum_{c=0}^{l} \left( \sum_{a=-1}^{c} \sum_{b=c+1}^{l+1} f(l, a, b) + \sum_{a=-1}^{c-1} f(l, a, c) \right)
\]

For $l = -2$ the sum $\sum_{c=0}^{-2} f(-2, a, b)$ is equal to $-\sum_{c=-1}^{-2}$ hence we obtain for the weight function

\[
w_{-2}(S) = -\left( \sum_{a=-1}^{-1} \sum_{b=0}^{a-1} f(-2, a, b) + \sum_{a=-1}^{-2} f(-2, a, -1) \right)
\]

\[= -\left( \sum_{a=-1}^{-1} 0 + 0 \right) = 0.\]

Assume $\{-2, -1, 1, 2\}$ is not a subset of $S$. We can extend our definition of an $(S, l)$-tree using the generalisation of (15). The enumeration then becomes a weighted enumeration, where the weight of an $(S, l)$-tree is $(-1)$ to the power of encounters of entries $x_{i,j} > x_{i,j+1}$. Hence the $(S, -2)$-tree have the form

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & -1 \\
0 & -2 & 0 & -2
\end{array}
\]

where the left corresponds to an $(S, -2)$-tree with $\{-2, -1, 0, 1\} \subseteq S$ and the right one to $\{-1, 0, 1, 2\} \subseteq S$ and their weight is $-1$. By deleting the bottom row and adding 1 to all entries we obtain an $(S', 0)$-tree with $S' = \{i + 1 \mid i \in S, i < -2\} \cup \{-1, 0, 1\} \cup \{i - 1, i \in S, i > 2\}$. Hence in both cases hold

\[-w_{-2}(S) = |w_0(S')|.\]

By definition of an $(S, l)$-tree it is obvious that for $l \geq 0$ there is at least one $(S, l)$-tree, hence $w_0(S') \neq 0$ which proves the claim.

6.2. Rational roots of $w_l(M)$.

**Proposition 6.4.** Let $M$ be a Motzkin path of length $n$ ending with a southeast step and denote by $M'$ the Motzkin path of length $n + 1$ obtained by putting an east step in front of the last step of $M$. The weight $w_l(M')$ is given by

\[w_l(M') = (l + 2n)w_l(M).\]
Proof. Let \( A' \) be an \( (n+1,l) \)-AS-trapezoid with \( M(A') = M' \). If \( A' \) doesn’t have a \(-1\) in the second last row from top, the last two rows have up to horizontal and vertical reflection of the inner \( l+2n-1 \) columns the form

\[
\begin{align*}
0 & \quad 1 \quad 0 \ldots 0 \quad 0 \quad 0 \ldots 0 \\
0 & \quad 0 \ldots 0 \quad 1 \quad 0 \ldots 0 \\
\end{align*}
\]

By reflecting the top two rows in such a way such that one obtains the above form and deleting the top row we obtain an \( (n,l) \)-AS-trapezoid \( A \) with \( M(A) = M \). Now assume \( A' \) has a \(-1\) entry in its second last row from top. Then the two top rows have up to horizontal reflection the form

\[
\begin{align*}
0 & \quad 0 \ldots 0 \quad 1 \quad 0 \ldots 0 \quad 0 \ldots 0 \\
1 & \quad 0 \ldots 0 \quad -1 \quad 0 \ldots 0 \quad 1 \quad 0 \ldots 0 \\
\end{align*}
\]

If we delete the top row and delete in the second row from top the left \( 1 \) and the \(-1\) we obtain again an \( (n,l) \)-AS-trapezoid \( A \) with \( M(A) = M \). On the other hand starting from an \( (n,l) \)-AS-trapezoid \( A \) we can construct 4 or \( l+2(n-2) \) AS-trapezoids \( A' \) with no \(-1\) or one \(-1\) respectively in the second row from top and \( M(A') = M' \), which proves the claim. \( \square \)

Conjecture 6.5. Let \( M = (\ldots, e_1, \ldots, e_k) \) be a Motzkin path of length \( n+k \) whose last \( k \) steps are \( e_1, \ldots, e_k \), then holds

\[
M = (\ldots, 1, 0, -1, -1) \Rightarrow (l + 2n + 5)w_l(M),
\]

\[
M = (\ldots, 1, 0, 0, -1, -1) \Rightarrow (l + 2n + 7)w_l(M),
\]

\[
M = (\ldots, 1, 1, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 7)(l + 2n + 8)w_l(M),
\]

\[
M = (\ldots, 1, 1, -1, 0, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 7)w_l(M),
\]

\[
M = (\ldots, 1, 1, -1, 0, 0, -1, -1) \Rightarrow (l + 2n + 2)w_l(M),
\]

\[
M = (\ldots, 1, 1, 0, -1, -1, -1) \Rightarrow (l + 2n + 8)w_l(M),
\]

\[
M = (\ldots, 1, 1, 0, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 8)^2w_l(M),
\]

\[
M = (\ldots, 1, 1, 0, -1, 0, -1, -1) \Rightarrow (l + 2n + 2)w_l(M),
\]

\[
M = (\ldots, 1, 1, 0, 0, -1, -1, -1) \Rightarrow (l + 2n + 2)w_l(M),
\]

\[
M = (\ldots, 1, 1, 0, -1, -1, -1) \Rightarrow (l + 2n + 2)(l + 2n + 10)w_l(M).
\]

It seems that the list of conjectures of the above kind can be continued. Experiments suggest that the steps \( e_1, \ldots, e_k \) at the end have to satisfy \( e_1 = 1 \) and \( \sum_{i=1}^{k} e_i = -1 \), i.e., the Motzkin path before these steps “ends” at height 1. Further it is interesting that all rational roots of the shortest Motzkin path with one of the above endings, except the 7, 9, 12-th ending, are explained by the above conjecture and Conjecture 6.2.

Acknowledgements

The author wants to thank Arvind Ayyer for giving him access to his conjectures on the Motzkin path refinement of ASTs which was a major influence to this work.
Appendix A. Tables for $w_I(S)$ and $w_I(M)$

The following two tables list the weight functions of all irreducible centred Catalan sets up to the reflection $S \mapsto \{-s : s \in S\}$ of size less than 6 and the weight functions of irreducible Motzkin paths up to length 5.

| $S$       | $w_I(S)$                                                     |
|-----------|--------------------------------------------------------------|
| $\{0, 1\}$ | 1                                                            |
| $\{-1, 0, 1\}$ | $(l + 1)$                                                   |
| $\{-1, 0, 1, 2\}$ | $\frac{1}{2}(l + 1)(l + 4)$                             |
| $\{-2, -1, 0, 1, 2\}$ | $\frac{1}{12}(l + 1)(l + 2)(l + 6)(l + 7)$                 |
| $\{-3, -1, 0, 1, 2\}$ | $\frac{1}{6}(l + 1)(l + 6)(2l + 7)$                        |
| $\{-1, 0, 1, 2, 3\}$ | $\frac{1}{6}(l + 1)(l + 5)(l + 6)$                        |
| $\{-2, -1, 0, 1, 2, 3\}$ | $\frac{1}{144}(l + 1)(l + 2)(l + 6)(l + 7)(l^3 + 23l^2 + 168l + 360)$ |
| $\{-2, -1, 0, 1, 2, 4\}$ | $\frac{1}{24}(l + 1)(l + 2)(l + 6)(l + 7)(l + 8)$     |
| $\{-3, -1, 0, 1, 2, 3\}$ | $\frac{1}{24}(l + 1)(l^4 + 25l^3 + 226l^2 + 864l + 1176)$ |
| $\{-1, 0, 1, 2, 3, 4\}$ | $\frac{1}{24}(l + 1)(l + 6)(l + 7)(l + 8)$                |
| $\{-2, -1, 0, 1, 3, 4\}$ | $\frac{1}{24}(l + 1)(l + 6)(3l^2 + 37l + 92)$              |
| $\{-3, -1, 0, 1, 2, 4\}$ | $\frac{1}{24}(l + 1)(l + 6)(5l^2 + 55l + 132)$            |
| $\{-4, -1, 0, 1, 2, 3\}$ | $\frac{1}{24}(l + 1)(l + 6)(l + 8)(3l + 13)$              |

| $M$       | $w_I(M)$                                                     |
|-----------|--------------------------------------------------------------|
|           | 2                                                            |
| $\uparrow$ | $(l + 1)$                                                   |
| $\uparrow$ | $(l + 1)(l + 4)$                                           |
| $\uparrow$ | $\frac{1}{12}(l + 1)(l + 2)(l + 6)(l + 7)$                 |
| $\uparrow$ | $(l + 1)(l + 4)(l + 6)$                                     |
| $\uparrow$ | $\frac{1}{12}(l + 1)(l + 2)(l + 6)(l + 7)(l^3 + 23l^2 + 168l + 360)$ |
| $\uparrow$ | $\frac{1}{12}(l + 1)(l + 2)(l + 6)(l + 7)(l + 8)$          |
| $\uparrow$ | $\frac{1}{12}(l + 1)(l^4 + 25l^3 + 226l^2 + 864l + 1176)$  |
|           | $(l + 1)(l + 4)(l + 6)(l + 8)$                             |

References

[1] F. Aigner. Evaluations of determinants related to alternating sign matrices leading to a new variation of a proof of the asm theorem. In preparation.
[2] F. Aigner. Redefined enumerations of alternating sign triangles. Sém. Lothar. Combin., 78B:Art. 60, 2017.
[3] F. Aigner. Fully packed loop configurations: polynomiality and nested arches. Elect. J. Combin., 25(1), 2018.
[4] G. E. Andrews. Plane partitions (III): The weak Macdonald conjecture. Invent. Math., 53:193–225, 1979.
[5] A. Ayyer. private communication.
[6] A. Ayyer, R. Behrend, and I. Fischer. Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order. arXiv:1611.03823, 2016.
[7] R. Behrend and I. Fischer. private communication.
[8] L. Cantini and A. Sportiello. Proof of the Razumov-Stroganov conjecture. *J. Combin. Theory, Ser. A*, 118(5):1549–1574, 2011.

[9] F. Caselli, C. Krattenthaler, B. Lass, and P. Nadeau. On the Number of Fully Packed Loop Configurations with a Fixed Associated Matching. *Elect. J. Combin.*, 11(2), 2004.

[10] M. Ciucu, T. Eisenkölbl, C. Krattenthaler, and D. Zare. Enumeration of Lozenge Tilings of Hexagons with a Central Triangular Hole. *J. Combin. Theory Ser. A*, 95(2):251–334, 2001.

[11] I. Fischer. A new proof of the refined alternating sign matrix theorem. *J. Combin. Theory Ser. A*, 114(2):253–264, 2007.

[12] I. Fischer. Refined enumerations of alternating sign matrices: monotone \((d,m)\)-trapezoids with prescribed top and bottom row. *J. Alg. Combin.*, 33:239–257, 2011.

[13] I. Fischer. Alternating sign trapezoids and a constant term approach. *arXiv: 1804.08681*, 2018.

[14] I. Fischer. Constant term formulas for refined enumerations of Gog and Magog trapezoids. *arXiv:1804.07054*, 2018.

[15] I. Fischer. Enumeration of alternating sign triangles using a constant term approach. *arXiv:1804.03630*, 2018.

[16] T. Fonseca and P. Nadeau. On some polynomials enumerating Fully Packed Loop configurations. *Adv. Appl. Math.*, 47(3):434–462, 2011.

[17] W. H. Mills, D. P. Robbins, and H. C. Rumsey Jr. Alternating Sign Matrices and Descending Plane Partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.

[18] A. V. Razumov and Y. G. Stroganov. Combinatorial nature of ground state vector of O(1) loop model. *Theor. Math. Phys.*, 138(3):333–337, 2001.

[19] D. P. Robbins and H. C. Rumsey Jr. Determinants and alternating sign matrices. *Adv. Math.*, 62(2):169–184, 1986.

[20] J. B. Zuber. On the Counting of Fully Packed Loop Configurations: Some new conjectures. *Elect. J. Combin.*, 11(1), 2004.

FLORIAN AIGNER, UNIVERSITÄT WIEN, FAKULTÄT FÜR MATHEMATIK, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA

E-mail address: florian.aigner@univie.ac.at