BINARY CURVES OF SMALL FIXED GENUS AND GONALITY WITH MANY RATIONAL POINTS

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Abstract. We determine the maximum number of rational points on a curve over $\mathbb{F}_2$ with fixed gonality and small genus. This document differs from the published edition in *Journal of Algebra* in one significant way: it contains Appendix B on quadratic forms over finite fields and their associated orthogonal groups.

1. Introduction

Let $C/\mathbb{F}_q$ be a smooth complete algebraic curve of genus $g$ over the finite field with $q$ elements. The Weil bound gives an upper limit on the number of rational points on this curve in terms of $q$ and $g$:

$$\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}.$$ 

It is natural to ask how close we can get to this bound as we vary the curve $C$. Serre, Oesterlé, Drinfeld-Vlăduţ, and others gave improvements to this bound in a variety of settings and constructed examples of curves that in some cases meet these improved bounds. For a brief survey of this work, see [24]; a more thorough treatment is given in [13]. From a computational point of view, van der Geer and van der Vlugt [22] built the first comprehensive table of maximal values for small genus and field size. Their table evolved into manypoints.org.

We restrict our attention almost entirely to the binary field $\mathbb{F}_2$ in the present work. Write $N_2(g)$ for the maximum number of rational points on a curve of genus $g$ over $\mathbb{F}_2$. The values in Table 1 come from manypoints.org.

| $g$ | $N_2(g)$ |
|-----|-----------|
| 0   | 3         |
| 1   | 5         |
| 2   | 6         |
| 3   | 7         |
| 4   | 8         |
| 5   | 9         |

Table 1. Maximum number of rational points on binary curves with fixed genus

The gonality of a curve $C$ over a field $k$ is the minimum degree of a $k$-morphism $C \to \mathbb{P}^1$. As special cases, one often refers to curves with genus $g \geq 2$ and gonality 2 as hyperelliptic, and curves with gonality 3 as trigonal. We caution the reader that some authors use the word “gonality” to mean the gonality of $\bar{C} = C \times_{\text{Spec} k} \text{Spec} k^{\text{sep}}$, and some authors use the word “trigonal” to mean “admits a morphism to $\mathbb{P}^1$ of degree 3”, without the requirement that no morphism of degree less than 3 exists. We will also say that “$C$ has a $g^1_d$” if it has a
Gal($k^{\text{sep}}/k$)-invariant divisor $D$ of degree-$d$ such that $\dim |D| \geq 1$; if true, then the gonality of $C$ is at most $d$.

Van der Geer [21] asked, 

**What is the maximum number of rational points on a curve of genus $g$ and gonality $\gamma$ defined over $\mathbb{F}_q$?**

We found no evidence in the literature that anyone has tried to answer this question, so we are embarking on a program of filtering Table 1 according to the gonality invariant. To that end, define $N_2(g, \gamma)$ to be the supremum of the number of rational points on a smooth proper connected curve over $\mathbb{F}_2$ with genus $g$ and gonality $\gamma$. We use the supremum in order to be able to make statements like $N_2(1, 3) = -\infty$; this means there is no genus-1 curve with gonality 3. With regard to non-existence results, we found some valuable insight in [5] on which gonalties can occur for a curve of small genus over finite fields with odd characteristic.

The remainder of the paper will be dedicated to explaining the entries that appear in Table 2, where we have limited our attention to curves of genus at most 5. While this is somewhat arbitrary, it reflects the limitation of our understanding of the geometry of curves of genus $g \geq 6$. But even for genera that we are explicitly considering, certain entries like $N_2(1, 3)$ do not appear: this is because of the general fact that a curve of genus $g$ over a finite field has gonality at most $g + 1$ (Proposition 2.1). We omit entries beyond this bound from Table 2. Beyond our systematic approach to genus $g \leq 5$, some additional values of $N_2(g, \gamma)$ can be obtained from a variety ad hoc methods; we give examples in Appendix A.

| $g$ | $\gamma$ | $N_2(g, \gamma)$ | Reference |
|-----|----------|------------------|-----------|
| 0   | 1        | 3                | $\mathbb{P}^1$ |
| 1   | 2        | 5                | Equation (3.1) |
| 2   | 2        | 6                | Theorem 3.1  |
| 3   | 2        | 6                | Theorem 3.1  |
|     | 3        | 7                | Theorem 4.1  |
|     | 4        | 0                | Theorem 4.2  |
| 4   | 2        | 6                | Theorem 3.1  |
| 3   | 8        | 5                | Theorem 5.2  |
| 4   | 5        | 0                | Theorem 5.3  |
| 5   | 0        | 0                | Theorem 5.4  |
| 5   | 2        | 6                | Theorem 3.1  |
| 3   | 8        | 5                | Theorem 6.2  |
| 4   | 9        | 5                | Theorem 6.7  |
| 5   | 3        | 0                | Theorem 6.8  |
| 6   |          | $-\infty$        | Theorem 6.9  |

**Table 2.** Supremum of the number of rational points on a binary curve with fixed genus and gonality.

Given a curve $C/\mathbb{F}_2$ with gonality $\gamma$, and a morphism $\pi : C \to \mathbb{P}^1$ of degree $\gamma$, we observe that every rational point of $C$ must lie over a rational point of $\mathbb{P}^1$, and that over each such point there are at most $\gamma$ rational points of $C$:

$$\#C(\mathbb{F}_2) \leq \#\mathbb{P}^1(\mathbb{F}_2) \cdot \gamma = 3\gamma.$$  \hspace{1cm} (1.1)
This “gonality-point inequality” is substantially weaker than the truth when $g$ is small; for example, $N_2(5,4) = 9$. It is also much weaker than the Weil bound. If $C$ were an example of equality in (1.1), we would have

$$3\gamma = \#C(\mathbb{F}_2) \leq 3 + 2g\sqrt{2} \quad \Rightarrow \quad g \geq \frac{3(\gamma - 1)}{2\sqrt{2}}.$$  

We conjecture that small genus is the only obstruction to achieving equality:

**Conjecture 1.1.** Fix $\gamma \geq 2$. For $g$ sufficiently large, $N_2(g, \gamma) = 3\gamma$.

We give a simple construction that proves this conjecture in the case $\gamma = 2$ (Theorem 3.1). The case $\gamma = 3$ should require a bit more ingenuity since a plane curve over $\mathbb{F}_2$ has at most 7 rational points.\footnote{Shortly before this paper was accepted for publication, we learned that Vermeulen has addressed the case $\gamma = 3$ using curves on toric surfaces [23].}

A curve $C$ over a field $k$ has gonality at most $d$ if it admits a morphism $f : C \to \mathbb{P}^1$ of degree $d$. If we write $D = f^{-1}(\infty)$, then $D$ is a Galois-invariant divisor and the complete linear system $|D|$ has degree $d$ and dimension at least 1. In this way, we see that computing the gonality of $C$ can be reduced to looking for Galois-invariant effective divisors that move in families. A canonical divisor $K$ on $C$ is one candidate, and Galois orbits of points in $C(k^{\text{sep}})$ provide others. In Section 2, we assemble a number of results (most of which are known) that will allow us to identify the gonality of curves of small genus.

We treat curves of genus at most 3 in Sections 3 and 4. The arithmetic of a non-hyperelliptic curve of genus $g \geq 4$ is intimately tied to the quadric hypersurfaces in $\mathbb{P}^{g-1}$ on which it lives. It is fruitful to understand these hypersurfaces via the classification of quadratic forms. As we could not find a single self-contained reference that gives all of the results we need in characteristic 2, we assemble the necessary facts in Appendix B. Curves of genus 4 and 5 are studied in Sections 5 and 6.

Our work on curves of genus at most 4 lives squarely in the realm of pure mathematics: we provide an upper bound for the number of rational points on a curve with specified $g$ and $\gamma$ and then exhibit a curve that achieves this bound. This also works for genus-5 curves when the gonality is at most 4. However, we were unable to maintain this pattern for $g = 5$ and $\gamma \geq 5$. Instead, we provide a certain amount of theoretical scaffolding, and then complete the edifice with an exhaustive computation in Sage [19]. A curious non-existence result comes out of these computations:

**Theorem 1.2.** There is no curve of genus 5 and gonality 6 over $\mathbb{F}_2$.

General theory of gonality on curves over finite fields shows that a curve of genus $g$ has gonality at most $g + 1$ (see §2), but there is no guarantee that any curve of gonality $g + 1$ exists. In fact, Weil’s (lower) inequality in tandem with Corollary 2.5 proves that no curve of genus 5 and gonality 6 over $\mathbb{F}_q$ exists as soon as $q > 4$. The above theorem deals with one of the cases the Weil bound misses. We note that the question of whether the above theorem holds over $\mathbb{F}_3$ appears in [5, Rem. 3.14].

We draw attention to two results in §6.2 that we expect to be of value to a wider audience. Over an algebraically closed field, it is known that a canonically embedded curve of genus 5 is the complete intersection of quadric hypersurfaces if and only if it is non-trigonal. With the
help of Galois cohomology, we extend this result to an arbitrary perfect field (Theorem 6.3). In particular, this shows that a curve of genus 5 over a perfect field is trigonal if and only if it is geometrically trigonal (Corollary 6.4). We believe this to be a new result for fields of cohomological dimension larger than 1.

While our results are focused on binary curves, many of the results in this article apply equally well to curves over more general fields. We explicitly state when a result applies more broadly. Unless otherwise specified, in this article a curve $C/k$ is a smooth proper geometrically irreducible scheme of dimension 1 over a field $k$. A divisor on $C$ will always be defined over the ground field $k$. For a divisor $D$ on $C/k$, we write $L(D)$ for the $k$-vector space of rational functions $f$ whose divisor satisfies $\text{div}(f) \geq -D$. The set of effective divisors that are linearly equivalent to $D$ is denoted $|D|$, and we recall that when it is nonempty, it admits the structure of a projective space with $\dim |D| = \dim L(D) - 1$.

2. Generalities on gonality

Throughout this section, we work over a fixed finite field $\mathbb{F}_q$.

The following results are well known, at least in some form — see the appendix of [14], for example. We collect them here for ease of use later. We begin with general bounds for the gonality in terms of the genus, thus limiting the number of entries that must appear in Table 2.

**Proposition 2.1.** Let $C/\mathbb{F}_q$ be a curve of genus $g$ and gonality $\gamma$.

1. If $g = 0$, then $\gamma = 1$.
2. If $g \in \{1, 2\}$, then $\gamma = 2$.
3. If $g \geq 2$ and $C(\mathbb{F}_q) \neq \emptyset$, then $\gamma \leq g$.
4. In general, the gonality satisfies $\gamma \leq g + 1$.

**Proof.** Every curve of genus zero over a finite field has a rational point. (Recall that the anticanonical linear system $|-K|$ has degree 2 and dimension 2, so a genus-zero curve is cut out by an irreducible quadratic form in 3 variables. Every such form has a nonzero solution over a finite field.) Linear projection through the rational point gives an isomorphism between the curve and $\mathbb{P}^1$. That is, $\gamma = 1$ when $g = 0$.

Every curve of genus 1 over a finite field also has a rational point. Indeed, the Weil bound takes the form

$$|\#C(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q},$$

and $q + 1 > 2\sqrt{q}$ for all $q > 1$. Choosing $P \in C(\mathbb{F}_q)$, Riemann-Roch implies that the linear system $|2P|$ has dimension 1. Hence, $\gamma = 2$.

For a curve of genus 2, Riemann-Roch shows that the canonical linear system $|K|$ has degree 2 and dimension 1. So $\gamma = 2$.

Suppose that $g \geq 2$, and let $P \in C(\mathbb{F}_q)$. The linear system $|K - (g - 2)P|$ has degree $g$. Since $|(g - 2)P|$ is nonempty, Riemann-Roch gives the inequality

$$\dim |K - (g - 2)P| = \dim |(g - 2)P| + 1 \geq 1.$$

Hence, $\gamma \leq g$. 

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Finally, we show that $\gamma \leq g + 1$ in general for a curve $C$ over a finite field. This statement appears as [20, Cor. 4.2.18], though the proof is incomplete.\footnote{The authors of [20] assert that every curve over a finite field $\mathbb{F}_q$ has a point defined over an extension of degree $(g + 1)$. While this is true, it does not follow from the Weil bound when $q = 2$ and $2 \leq g \leq 6$. We thank Felipé Voloch for pointing us toward Schmidt’s proof.} A result of F.K. Schmidt from 1931 [20, Cor. 3.1.12] asserts that there is a (Galois-invariant) divisor $D$ on $C$ of degree 1. The linear system $|(g + 1)D|$ has degree $g + 1$, and Riemann-Roch shows it has dimension
\[
\dim |(g + 1)D| = \dim |K - (g + 1)D| + 2 \geq 1.
\]

We now give some refinements of this result which will help us to determine the gonality of certain examples we come across.

**Proposition 2.2.** Let $C/\mathbb{F}_q$ be a curve of genus $g \geq 2$. Then $C$ admits a $g^1_g$ if and only if there is an effective divisor $D$ on $C$ of degree $g - 2$.

**Proof.** If $C$ has a $g^1_g$, then there is an effective divisor $D_0$ of degree $g$ such that $\dim |D_0| \geq 1$. By Riemann-Roch, we see that
\[
\dim |K - D_0| = \dim |D_0| - 1 \geq 0.
\]
Let $D$ be an effective divisor in the linear system $|K - D_0|$. Then $D$ has degree $g - 2$, as desired. Conversely, if $D$ is an effective divisor on $C$ of degree $g - 2$, then
\[
\dim |K - D| \geq \dim |K| - \deg(D) = 1.
\]
That is, $|K - D|$ contains a $g^1_g$. \hfill $\square$

**Corollary 2.3.** Let $C/\mathbb{F}_q$ be a non-hyperelliptic curve of genus 3. If $C(\mathbb{F}_q)$ is nonempty, then $C$ has gonality 3; otherwise, $C$ has gonality 4.

**Proof.** Proposition 2.2 implies that $C$ has a $g^1_3$ if and only if it has an effective divisor of degree 1 — i.e., a rational point. If $C$ does not have gonality 3, it must have gonality 4 by Proposition 2.1. \hfill $\square$

**Corollary 2.4.** Let $C/\mathbb{F}_q$ be a curve of genus 4 that is not hyperelliptic or trigonal. Then $C$ has gonality 4 if $C(\mathbb{F}_{q^2}) \neq \emptyset$, and otherwise it has gonality 5.

**Proof.** We claim that $C$ has a $g^1_4$ if and only if $C(\mathbb{F}_{q^2}) \neq \emptyset$. Proposition 2.2 asserts that $C$ has a $g^1_4$ if and only if it admits an effective divisor $D$ of degree 2. If $D$ is supported at a single point $P$, then $P$ is rational. Otherwise, the support of $D$ is a single Galois orbit of quadratic points. This proves the claim.

Since $C$ is not hyperelliptic or trigonal, we see that it has gonality 4 if $C(\mathbb{F}_{q^2}) \neq \emptyset$, and otherwise it has gonality 5 by Proposition 2.1. \hfill $\square$

**Corollary 2.5.** Let $C$ be a curve of genus 5 over $\mathbb{F}_q$ with gonality at least 5. Then $C$ has gonality 5 if $C(\mathbb{F}_{q^2}) \neq \emptyset$, and otherwise it has gonality 6.

**Proof.** As in the previous proof, it suffices to show that $C$ has a $g^1_5$ if and only if $C(\mathbb{F}_{q^2}) \neq \emptyset$. Proposition 2.2 asserts that $C$ has a $g^1_5$ if and only if there is an effective divisor $D$ on $C$ of degree 3. If $D$ is supported at a single point $P$, then that point must be rational. If $D$ is supported at two distinct points $P$, $Q$, then without loss we have $D = 2P + Q$, and
the Galois invariance of $D$ implies that both $P$ and $Q$ are rational. Finally, suppose that
$D = P + Q + R$ for three distinct points $P, Q, R$. If none of these points is rational, then
they constitute a Galois orbit of length 3. In all cases, we find that $C(F_q) \neq \emptyset$. □

3. Curves of genus at most 2

Proposition 2.1 allows us to address curves of small genus rather quickly. A curve $C/F_2$ of
 GENUS 0 is isomorphic to $P^1/F_2$, and hence, $#C(F_2) = 3$. That is, $N_2(0, 1) = 3$.
A curve $C/F_2$ of genus 1 has gonality 2. The Weil bound shows that $#C(F_2) \leq 2 + 1 + 2\sqrt{2} < 6$. Hence $N_2(1, 2) \leq 5$. The following example of an elliptic curve shows that this bound is
sharp:
\[ E/F_2 : y^2 + y = x^3 + x. \] (3.1)
A curve $C/F_2$ of genus 2 is hyperelliptic. The following result, applied when $g = 2$, tells us
that $N_2(g, 2) = 6$.

**Theorem 3.1.** Fix an integer $g \geq 2$, and let $\delta \in \{0, 1\}$ satisfy $\delta \equiv g \pmod{2}$. The curve
$C/F_2$ with affine plane equation
\[ y^2 + (x^{g+1} + x^g + 1)y = [x(x + 1)]^{g-\delta} \]
is hyperelliptic of genus $g$ and has 6 rational points. In particular, $N_2(g, 2) = 6$.

**Proof.** A model for $C$ near infinity is given by setting $y = z/w^{g+1}$ and $x = 1/w$:
\[ z^2 + [w^{g+1} + w + 1]z = w^{2+2\delta}(1 + w)^{g-\delta}. \]
The description of hyperelliptic curves in [12, Prop. 7.4.24] shows that $C$ is smooth of genus $g$, and one sees immediately that $C$ has 4 affine rational points and 2 rational points at infinity (with $w = 0$).
This example shows that $N_2(g, 2) \geq 6$. The opposite inequality follows from the gonality-
point inequality (1.1). □

4. Curves of genus 3

Proposition 2.1 shows that a curve $C/F_2$ of genus 3 has gonality at most 4. The case of
a hyperelliptic curve is immediately dispensed with by Theorem 3.1. Every other curve of
genre 3 is canonically embedded as a quartic curve in $P^2$; conversely, the adjunction formula
shows that any smooth quartic curve in $P^2$ has genus 3. (In general, the canonical class is
defined over the ground field, so the canonical embedding will be as well.)

**Theorem 4.1.** $N_2(3, 3) = 7$.

**Proof.** Dickson observed that the plane curve
\[ C/F_2 : x^3y + x^2y^2 + xz^3 + x^2z^2 + y^3z + yz^3 = 0 \]
is the unique nonsingular quartic (up to isomorphism) that passes through all 7 rational
points of the projective plane [7, §5]. As it is smooth, it has genus 3. Corollary 2.3 shows
that $C$ has gonality 3. Thus, $N_2(3, 3) = 7$.
For the reverse inequality, let $C$ be any curve of genus 3 and gonality 3, which we may
identify with its image in $P^2$ under a canonical embedding. The projective plane over $F_2$ has
7 rational points, so we obtain $N_2(3, 3) \leq 7$. □
Theorem 4.2. $N_2(3, 4) = 0$.

Proof. In [7, §4], Dickson observed that the plane quartic curve

$$C/F_2 : x^4 + y^4 + z^4 + x^2y^2 + x^2z^2 + y^2z^2 + x^2yz + xy^2z + xyz^2 = 0$$

has no $F_2$-rational point. One verifies easily that it is nonsingular, which means $C$ has genus 3. Corollary 2.3 shows that $C$ has gonality 4, so $N_2(3, 4) ≥ 0$. The same proposition shows that $N_2(3, 4) ≤ 0$.

5. CURVES OF GENUS 4

For a curve of genus 4, Proposition 2.1 shows that the gonality is at most 5. Our approach will serve as a paradigm for the more difficult case of genus-5 curves.

Hyperelliptic curves are handled by Theorem 3.1, where we learned that $N_2(4, 2) = 6$. The key geometric facts about non-hyperelliptic curves of genus 4 are contained in the following result. We sketch a proof as we could not easily reconstruct one from the available literature.

(The result is stated, for example, in [6, §4.4].)

Lemma 5.1. Let $C$ be a genus-4 curve over a finite field $F_q$. If $C$ is not hyperelliptic, then the canonical linear system $|K|$ embeds $C$ into $P_{F_q}^5 = \text{Proj} \ F_q[x, y, z, w]$ as the intersection of a unique quadric surface $S$ and a cubic surface. Exactly one of the following is true, up to automorphism of $\mathbb{P}^3$:

1. $C$ admits a $g_3^1$, say $|D|$, such that $D \not\sim K - D$. Then $S = \{xy + zw = 0\}$, and $S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding. The two linear systems $|D|$ and $|K - D|$ correspond to the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$, and $C$ corresponds to a smooth curve of bidegree $(3, 3)$.
2. $C$ admits a $g_3^1$, say $|D|$, such that $D \sim K - D$. Then $S = \{xy + z^2 = 0\}$, a quadric cone, and the family of lines through the singularity cuts out the unique $g_3^1$ on $C$.
3. $C$ admits no $g_3^1$. In this case, $S = \{xy + N(z, w) = 0\}$, where $N(z, w)$ is the norm form for $F_q/\mathbb{F}_q$, as in (B.1).

Proof. As $C$ is not hyperelliptic, the canonical linear system is very ample of degree 6 and dimension 3. The image of the corresponding morphism $C \hookrightarrow \mathbb{P}^3$ is the intersection of a unique quadric surface $S = \{Q = 0\}$ and a cubic surface $\{F = 0\}$ [10, IV.5.2.2]. The classification of quadratic forms in at most 4 variables (Theorem B.5) shows that, up to linear change of coordinates and rescaling, $Q$ may be taken to be among the following

- (4 variables) $xy + zw$ or $xy + N(z, w)$;
- (3 variables) $xy + z^2$;
- (2 variables) $xy$ or $N(x, y)$;
- (1 variable) $x^2$.

The forms in 1 or 2 variables are geometrically reducible; as $C$ is not contained in a hyperplane of $\mathbb{P}^3$, none of these cases can occur. The three remaining surfaces are pairwise non-isomorphic over $\mathbb{F}_q$.

Suppose that $C$ admits a $g_3^1$, say $|D|$. By Riemann-Roch, $|K - D|$ is also a $g_3^1$. As we have assumed that $C$ is not hyperelliptic, neither of these linear systems has a basepoint. We need to consider separately the cases $D \sim K - D$ and $D \not\sim K - D$. 

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Assume that \( D \) is a \( g^3_3 \) such that \( D \sim K - D \). Let \( f \in L(D) \) be a nonconstant rational function with \( \text{div}(f) \geq -D \). The divisor of poles of \( f \) is precisely \( D \), else \( C \) would be rational or hyperelliptic. Then the Riemann-Roch space \( L(2D) \cong L(K) \) has dimension 4, and it contains the linearly independent functions 1, \( f \), and \( f^2 \). Let \( h \in L(2D) \) complete this to a basis. Define a morphism by

\[
C \setminus \text{supp}(D) \rightarrow \mathbb{P}^3 = \text{Proj} \ \mathbb{F}_q[x, y, z, w] \\
P \mapsto (-1, f^2(P), f(P), h(P)).
\]

Note that the image lies on the surface \( xy + z^2 = 0 \). As \( C \) is smooth, this morphism extends over all of \( C \), and it is clear by its definition that it yields the canonical embedding of \( C \).

Now assume that \( D \) is a \( g^3_3 \) with \( D \not\sim K - D \). Consider the morphism \( C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) induced by \( |D| \) and \( |K - D| \). Applying the Segre embedding gives a composition

\[
C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3,
\]

and if we choose the coordinates correctly, the image lies on the quadric surface \( \{xy + zw = 0\} \). Intersecting a hyperplane with this surface gives a bidegree-(1, 1) divisor on \( \mathbb{P}^1 \times \mathbb{P}^1 \), and pulling it back to \( C \) gives \( D + K - D = K \). That is, this composition corresponds to the canonical linear system. In order to show that the map \( C \rightarrow \mathbb{P}^3 \) is a canonical embedding, it remains to check that it corresponds to the complete linear system \( |K| \), or equivalently, that the image of \( C \) does not lie on a hyperplane in \( \mathbb{P}^3 \). Suppose otherwise. Since a hyperplane pulls back to a (1, 1)-divisor on \( \mathbb{P}^1 \times \mathbb{P}^1 \), we conclude that the image of \( C \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is a (1, 1)-curve, say \( C_0 \). Write \( \psi : C \rightarrow C_0 \) for the induced morphism. By the adjunction formula, the curve \( C_0 \) has genus 0. Write \( \pi_1, \pi_2 \) for the morphisms \( C_0 \rightarrow \mathbb{P}^1 \) induced by the component maps on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Since \( C_0 \) is a rational curve, \( \pi_1^{*}[\infty] \sim \pi_2^{*}[\infty] \), and we find that

\[
D \sim \psi^{*}\pi_1^{*}[\infty] \sim \psi^{*}\pi_2^{*}[\infty] \sim K - D,
\]

which is a contradiction. Thus \( C \) does not lie on a hyperplane in \( \mathbb{P}^3 \).

Finally, we are left with the case where \( C \) has no \( g^3_3 \). If, under the canonical embedding, \( C \) lies on \( xy + zw = 0 \), then it would have a \( g^3_3 \) coming from either of the rulings on this surface. If instead, \( C \) lies on \( xy + z^2 = 0 \), then the lines through the singularity of this quadric cone give rise to a pencil of degree-3 divisors on \( C \), and hence a \( g^3_3 \). So \( C \) must lie on the remaining surface \( \{xy + N(z, w) = 0\} \).

We now return to the case of curves over \( \mathbb{F}_2 \). Note that the norm form for \( \mathbb{F}_4/\mathbb{F}_2 \) is given by \( N(z, w) = z^2 + zw + w^2 \). Consequently, any canonically embedded curve must lie on one of the quadrics \( \{xy + zw = 0\} \), \( \{xy + z^2 = 0\} \), or \( \{xy + z^2 + zw + w^2 = 0\} \).

**Theorem 5.2.** \( N_2(4, 3) = 8 \)

**Proof.** Consider the genus-4 curve \( C \subset \mathbb{P}^3_{\mathbb{F}_2} \) described by the following equations:

\[
xy + zw = 0 \\
x^2y^2 + y^3 + x^2z + y^2z + xz^2 + x^2w + y^2w + xw^2 = 0.
\]

One verifies that \( C \) passes through 8 of the nine points on the quadric surface \( xy + zw = 0 \). Lemma 5.1 shows that \( C \) has gonality 3, so \( N_2(4, 3) \geq 8 \). Serre has shown that \( N_2(4) \leq 8 \) [18]. In particular, \( N_2(4, 3) \leq 8 \). \( \square \)
Theorem 5.3. $N_2(4, 4) = 5$.

Proof. We begin with the genus-4 curve $C_{/F_2}$ in $\mathbb{P}^3$ given by the equations

$$
xy + z^2 + zw + w^2 = 0 \\
xy^2 + x^2z + y^2z + yz^2 + x^2w + z^2w = 0.
$$

A quick calculation shows that the cubic form vanishes at all five rational points of the quadratic form. Thus $#C(\mathbb{F}_2) = 5$. Lemma 5.1 shows that it does not admit a morphism to $\mathbb{P}^1$ of degree 3. Thus, its gonality must be at least 4. On the other hand, Proposition 2.1 shows that, since it has a rational point, its gonality is at most 4. This proves $N_2(4, 4) \geq 5$.

For the reverse inequality, observe that every non-hyperelliptic genus-4 curve can be embedded in $\mathbb{P}^3$ using the canonical linear system. By Lemma 5.1, we may assume it lies on the quadric surface $xy + z^2 + wz + w^2 = 0$ (else it would admit a degree-3 map to the projective line). This surface has only 5 rational points, so $N_2(4, 4) \leq 5$. □

Theorem 5.4. $N_2(4, 5) = 0$.

Proof. Consider the curve $C_{/F_2}$ in $\mathbb{P}^3$ cut out by the equations

$$
xy + z^2 + zw + w^2 = 0 \\
x^3 + y^3 + z^3 + yz^2w + xzw = 0.
$$

They define a curve of genus 4, and by Lemma 5.1, $C$ has gonality at least 4.

Direct search using the above equations shows that $#C(\mathbb{F}_4) = 0$. Corollary 2.4 shows that $C$ has no $g_1^1$, and hence gonality 5. That is, $N_2(4, 5) \geq 0$.

For the reverse inequality, we recall that any curve of genus 4 with a rational point has gonality at most 4. It follows that a curve of gonality 5 has no rational point, and $N_2(4, 5) \leq 0$. □

6. Curves of genus 5

According to Proposition 2.1, a curve of genus 5 can have gonality $\gamma$ satisfying $2 \leq \gamma \leq 6$. The hyperelliptic case is taken care of by Theorem 3.1: $N_2(5, 2) = 6$. A trigonal curve of genus 5 can be expressed as a singular plane quintic; we use this fact to compute $N_2(5, 3)$ in the next subsection. Non-trigonal curves of genus 5 over an algebraically closed field are well understood, and with some additional effort we are able to extend this description to arbitrary fields. (This issue is touched upon in [5, p.36], but they abdicate responsibility in the case of characteristic 2.) We give a general discussion in §6.2, and we use this theory to execute the calculation of $N_2(5, 4)$, $N_2(5, 5)$, and $N_2(5, 6)$ in §6.3 and §6.4.

6.1. Trigonal curves of genus 5. Here is a useful fact about trigonal curves; see [11, §2.2]. Note that this is incorrectly stated in Exercise IV.5.5 of [10]: the cuspidal case can actually occur.

Lemma 6.1. Every trigonal curve $C$ of genus 5 over a field $k$ is birational to a plane quintic $C' \subset \mathbb{P}^2_k$ with a unique $k$-rational singularity of multiplicity 2. After moving the singularity to $(0 : 0 : 1) \in \mathbb{P}^2(k)$, a homogeneous equation for $C'$ can be given by an irreducible polynomial

$$
f(x, y, z) = f_2(x, y)z^2 + g(x, y, z),
$$

where $f_2(x, y)$ is a homogeneous polynomial of degree 2 in $x$ and $y$.
where \( f_2 \) is a quadratic form, \( g \) is a quintic form that vanishes to order 3 at \((0 : 0 : 1)\), and one of the following holds:

- Cusp: \( f_2(x, y) = x^2 \) and \( g \) has nonzero coefficient on the \( y^3z^2 \)-term;
- Split node: \( f_2(x, y) = xy \);
- Nonsplit node: \( f_2(x, y) \) is irreducible over \( k \).

Conversely, the normalization of any quintic curve in \( \mathbb{P}^2 \) satisfying the above conditions is a trigonal curve of genus 5.

**Proof.** This is a consequence of the theorems of Riemann-Roch and Clifford. See [11, Prop. 2.2] for the case of odd characteristic; the even characteristic case is virtually identical. □

**Theorem 6.2.** \( N_2(5, 3) = 8 \).

**Proof.** Consider the singular curve \( C' \subset \mathbb{P}^2 = \text{Proj} \mathbb{F}_2[x, y, z] \) given by

\[
xyz^3 + x^3z^2 + y^3z^2 + x^4z + xy^3z + y^4z + x^4y + x^2y^3 = 0.
\]

One verifies that its only singularity is the node at \((0 : 0 : 1)\) and that it passes through all 7 rational points of the plane. After blowing up the node, we obtain a curve \( C \) of genus 5 with 8 rational points. By the converse part of Lemma 6.1, \( C \) is trigonal, so that \( N_2(5, 3) \geq 8 \).

For the reverse inequality, let \( C/\mathbb{F}_2 \) be a trigonal curve of genus 5, and let \( C' \) be a quintic in \( \mathbb{P}^2 \) that is birational to \( C \). Without loss, we may move the unique singularity of \( C' \) to \((0:0:1)\). Let \( \pi : C \to C' \) be the normalization morphism. Since \( \#\mathbb{P}^2(\mathbb{F}_2) = 7 \), it follows that the number of rational points on \( C \) is at most 6 plus the number of rational points on \( \pi^{-1}(0 : 0 : 1) \). Blowing up the singularity at the origin, we see that \( \pi^{-1}(0 : 0 : 1) \) contains two rational points in the split nodal case, no rational point in the nonsplit nodal case, and a unique rational point in the cuspidal case. In particular, \( N_2(5, 3) \leq 8 \). □

### 6.2. The canonical embedding of genus-5 curves

Throughout this section, \( k \) denotes a perfect field with algebraic closure \( \overline{k} \). Given a variety \( Y \), we write \( \overline{Y} \) for the base extension \( Y \times_{\text{Spec} k} \text{Spec} \overline{k} \). Our goal for this section is to prove the following result:

**Theorem 6.3.** Suppose \( C/k \) has genus 5 and is not hyperelliptic. Identify \( C \) with its image in \( \mathbb{P}^4 \) under a canonical embedding. The \( k \)-vector space of global sections of \( \mathcal{O}(2) \) (i.e., quadratic forms) that vanish on \( C \) is 3-dimensional. Write \( \mathcal{Q} \) for the corresponding 2-dimensional space of quadrics in \( \mathbb{P}^4 \).

- If \( C \) is trigonal, then the intersection of all quadrics in \( \mathcal{Q} \) is an irreducible ruled surface that is \( k \)-isomorphic to the blowing up of \( \mathbb{P}^2 \) at a rational point.
- If \( C \) is not trigonal, then the intersection of the quadrics in \( \mathcal{Q} \) is \( C \).

This result is well known when \( k \) is algebraically closed and can be assembled from the work of Max Noether and Enriques/Babbage/Petri in characteristic zero [1, III.3] and Saint-Donat in positive characteristic [16]. Since the space of quadrics \( \mathcal{Q} \) is defined over \( k \), we obtain the following interesting consequence:

**Corollary 6.4.** If \( C \) is a genus-5 non-hyperelliptic curve over a perfect field \( k \), then \( C \) is trigonal if and only if it is geometrically trigonal.
When $k = \mathbb{F}_q$, the corollary follows easily from the fact that a $g_3^1$ on a genus-5 curve is unique and that every Galois invariant divisor class over $\mathbb{F}_q$ admits an $\mathbb{F}_q$-rational divisor. (See, e.g., [5, Rem. 2.4] or [9, Lem. 6.5.3].) To the best of our knowledge, we are the first to extend this to arbitrary perfect fields.

We expend the remainder of this section in proving Theorem 6.3.

Let $\mathcal{I}$ be the homogeneous ideal sheaf of $C$ in $\mathbb{P}^4$, and write $\iota: C \to \mathbb{P}^4$ for the canonical closed immersion. Twist the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^4} \to \iota_* \mathcal{O}_C \to 0$$

by $\mathcal{O}(2)$ and consider the first part of the long exact sequence on sheaf cohomology:

$$0 \to H^0(\mathcal{I}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(\mathcal{O}_C(2)).$$

(6.1)

The final homomorphism is surjective after base extending to $\bar{k}$ by Max Noether’s theorem in characteristic zero or Saint-Donat’s theorem for positive characteristic [16, p.157]. Since all of these vector spaces are defined over $k$, it follows that (6.1) is already surjective before passing to $\bar{k}$.

Counting quadratic forms in 5 variables shows that

$$\dim H^0(\mathcal{O}_{\mathbb{P}^4}(2)) = \binom{2 + 4}{2} = 15,$$

while the Riemann-Roch formula implies that

$$\dim H^0(\mathcal{O}_C(2)) = \dim H^0(2K) = 2 \deg(K) + 1 - 5 = 12.$$

Thus, by (6.1), we see that $H^0(\mathcal{I}(2))$ has dimension 3. This is precisely the subspace of global sections of $\mathcal{O}(2)$ that vanish on $C$, so the first part of the theorem is proved.

Write $\mathcal{D} = \mathcal{D} \otimes_k \bar{k}$ for the space of quadrics in $\mathbb{P}^4_k$ that contain $\bar{C}$. The results of Enriques, as formulated by Saint-Donat in [16, (4.13)], say the following:

- If $\bar{C}$ is trigonal, then the intersection of the quadrics in $\mathcal{D}$ is isomorphic to the ruled surface $\mathbf{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Moreover, the linear series cut out by the ruling on $\mathbf{F}_1$ is a $g_3^1$.

- If $\bar{C}$ is not trigonal, then $\bar{C}$ is the intersection of the quadrics in $\mathcal{D}$.

To complete the proof of Theorem 6.3, we must show that this result descends to $C$.

Suppose first that $C$ is trigonal. Then $\bar{C}$ is also trigonal. Let $Y$ be the intersection of the quadrics in $\mathcal{D}$. The above result shows that $\bar{Y} \cong \mathbf{F}_1$. The ruled surface $\mathbf{F}_1$ is isomorphic to the blowing up of $\mathbb{P}^2_k$ at a rational point [10, Example V.2.11.5], and hence has no nontrivial twist by Lemma 6.5 below. That is, $Y$ is isomorphic to $\mathbf{F}_1$ over $k$, and a $g_3^1$ is given by restring the ruling on $\mathbf{F}_1$ to $C$.

Conversely, suppose that $C$ is not trigonal. Let $Y$ be the intersection of the quadrics in $\mathcal{D}$. If $\bar{C}$ is trigonal, then the argument in the preceding paragraph applies to show that $Y$ is $k$-isomorphic to $\mathbf{F}_1$. But then the ruling on $\mathbf{F}_1$ gives a $g_3^1$ on $C$, a contradiction. So $C$ is not trigonal, and we find that $\bar{C}$ is the intersection of the quadrics in $\mathcal{D}$. But the latter space has a $k$-basis, so $C$ is the intersection of the quadrics in $\mathcal{D}$.

**Lemma 6.5.** Let $k$ be a perfect field. The surface $S$ given by blowing up $\mathbb{P}^2_k$ at a rational point has no nontrivial twist.
Proof. Twists are in bijection with $H^1(k, \text{Aut}(\mathcal{S})) := H^1(\text{Gal}(\overline{k}/k), \text{Aut}(\mathcal{S}))$. Since $\mathcal{S}$ has a unique curve with self-intersection $-1$, this curve must be stabilized by every automorphism of $\mathcal{S}$. Blowing down the $(-1)$-curve yields a homomorphism $\psi : \text{Aut}(\mathcal{S}) \to \text{Aut}(\mathbb{P}^2_k; p)$, where $p$ is the rational point of $\mathbb{P}^2$ that we blew up to obtain $\mathcal{S}$. In fact, $\psi$ is an isomorphism because every automorphism of $\mathbb{P}^2_k$ that fixes $p$ maps lines through $p$ to lines through $p$.

Without loss, we may suppose that $p = (1 : 0 : 0)$. The subgroup of $\text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(\overline{k})$ that fixes $p$ is

$$G = \left\{ \left( \begin{smallmatrix} 1 & v & w \\ 0 & A \\ & & & \end{smallmatrix} \right) : v \in \overline{k} \oplus \overline{k}, A \in \text{GL}_2(\overline{k}) \right\}. $$

Evidently $G$ fits into a (split) short exact sequence

$$0 \to \overline{k}^2 \to G \to \text{GL}_2(\overline{k}) \to 0,$$

and the associated long exact sequence on Galois cohomology contains the exact sequence of pointed sets

$$H^1(k, \overline{k}^2) \to H^1(k, G) \to H^1(k, \text{GL}_2(\overline{k})). $$

The first term is trivial because $\overline{k}$ has no cohomology, and the last term is trivial by Hilbert 90. Hence, $H^1(k, G)$ is trivial, and $\mathcal{S}$ has no nontrivial twist. \hfill \square

6.3. Gonality 4. Every non-trigonal curve of genus 5 is cut out by three linearly independent quadratic forms. It turns out that one can detect the presence of a $g^1_4$ based on the equivalence class of these forms. This is well known in the algebraically closed setting [1, p.207–8]; we give a proof that is agnostic to the field.

Lemma 6.6. Suppose that $C \subset \mathbb{P}_q^4 = \text{Proj} \mathbb{F}_q[v, w, x, y, z]$ is a non-trigonal canonically embedded genus-5 curve. Then $C$ has gonality 4 if and only if it lies on a quadric hypersurface isomorphic to $vw + x^2 = 0$ or $vw + xy = 0$.

Proof. The proof is quite similar to the one used for Lemma 5.1. Suppose $C$ lies on a quadric hypersurface isomorphic to $S = V(vw + x^2)$; without loss, we may assume $C \subset S$. This hypersurface is a cone over the singular quadric cone $S_0 \subset \mathbb{P}^3 = \text{Proj} \mathbb{F}_q[v, w, x, y]$ given by the same equation. Since $S_0$ contains a 1-parameter family of lines through its singularity, we find that $S$ contains a 1-parameter family of 2-planes through the cone point $(0 : 0 : 0 : 1)$. Write $\{P_\lambda\}$ for this family of planes. Since $C$ is not trigonal, it is the intersection of three quadric surfaces; say $C = V(Q_1, Q_2, Q_3)$, where $Q_1 = vw + x^2$. It follows that

$$P_\lambda \cap C = P_\lambda \cap V(Q_1) \cap V(Q_2) \cap V(Q_3) = P_\lambda \cap V(Q_2) \cap V(Q_3),$$

and the latter intersection is visibly a zero-cycle of degree 4. That is, the family of divisors $\{P_\lambda \cap C\}$ is a $g^1_4$ on $C$.

The same argument applies if $C$ lies on the quadric hypersurface $xy + vw = 0$. Such a hypersurface is isomorphic to a cone over a quadric surface in $\mathbb{P}^3$ given by the same equation — which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ — and this surface has a 1-parameter family of lines on it.

For the converse, let $D$ be an effective divisor on $C$ such that $|D|$ contains a $g^1_4$. We must consider two cases. Suppose first that $D \sim \mathcal{K} - D$. Let $f \in L(D)$ be a nonconstant rational function. Then $1, f, f^2 \in L(2D)$. The common poles of these three functions have
different orders, so they are linearly independent. Since $2D \sim K$ and $\dim L(K) = 5$, there are rational functions $h, j$ such that $1, f, f^2, h, j$ are a basis for $L(2D)$. Define a rational map

\[
C \to \mathbb{P}^4 = \text{Proj } \mathbb{F}_q[v, w, x, y, z] \\
P \mapsto (-1, f^2(P), f(P), h(P), j(P)).
\]

The image lies on $vw + x^2 = 0$, and the fact that $C$ is smooth shows that this morphism extends over $\text{supp}(D)$. Evidently, this is a canonical embedding of $C$.

Finally, suppose that $D \not\sim K - D$. Then $|K - D|$ is also a $g^1_1$, and $\dim |D| = \dim |K - D|$ by Riemann-Roch. Let $D' \sim K - D$ be an effective divisor, and let $f, h$ be nonconstant rational functions in $L(D)$ and $L(D')$, respectively. It follows that $1, f, h, fh$ are in $L(D + D')$, and there is a function $j \in L(D + D')$ not in the span of these four. Define a rational map

\[
C \to \mathbb{P}^4 = \text{Proj } \mathbb{F}_q[v, w, x, y, z] \\
P \mapsto (f(P), h(P), -1, f(P)h(P), j(P)).
\]

The image lies on $vw + xy = 0$, and it extends to a morphism on $C$. To show that the morphism gives a canonical embedding of $C$, it remains to prove that $1, f, h, fh$ are linearly independent. The argument is identical to the one in the proof of Lemma 5.1.

Consider the morphism $\psi : C \to \mathbb{P}^1 \times \mathbb{P}^1$ induced by $|D|$ and $|D'|$; in coordinates, it is $P \mapsto (1, f(P)) \times (1, h(P))$. If $1, f, h, fh$ are linearly dependent, then the image $C_0$ of $\psi$ is a $(1, 1)$-curve on $\mathbb{P}^1 \times \mathbb{P}^1$. Such a curve has arithmetic genus 0 by the adjunction formula, and hence geometric genus 0. The component projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induce a birational morphism $C_0 \to \mathbb{P}^1$, and they give two compositions

\[
C \to C_0 \to \mathbb{P}^1,
\]

induced by $|D|$ and $|D'|$, respectively. But this implies $D \sim D' \sim K - D$, which is a contradiction. \hfill \Box

**Theorem 6.7.** $N_2(5, 4) = 9$

**Proof.** Consider the subvariety of $\mathbb{P}^4 = \text{Proj } \mathbb{F}_2[v, w, x, y, z]$ cut out by the equations

\[
vw + xy = 0 \\
vx + z(v + w + z) = 0 \\
(x + y)^2 + y(v + w) = 0.
\]

A Gröbner basis calculation (in Sage, say) shows that these equations determine a smooth curve of genus 5, and one verifies directly that it has 9 rational points. Hence, $N_2(5, 4) \geq 9$. For the upper bound, Serre showed that $N_2(5) \leq 9$ [18], which completes the proof. \hfill \Box

**6.4. Gonality at least 5.** In this section, we describe a computer calculation that allowed us to inspect every curve of genus 5 over $\mathbb{F}_2$ of gonality at least 5. This turned out to be necessary for two reasons. First, we were unable to produce an upper bound on the number of rational points on a curve with gonality 5 by purely theoretical means. Second, we have no satisfactory explanation for the non-existence of a curve of gonality 6; it is simply a phenomenon we observed in our data.

Let $C$ be a genus-5 curve of gonality at least 5, which we identify with its image under a canonical embedding in $\mathbb{P}^4 = \text{Proj } \mathbb{F}_2[v, w, x, y, z]$. Theorem 6.3 shows that $C$ is the vanishing
locus of a 3-dimensional space \( W \) of quadratic forms. If \( Q \in W \) is a nonzero form, then it must be geometrically irreducible as \( C \) does not lie on a hyperplane. The classification of quadratic forms (Theorem B.5) shows that, after an appropriate linear change of variable, any geometrically irreducible quadratic form is equivalent to one of the following:

I. \( vw + x^2 \) (singular line and 15 rational points)
II. \( vw + xy \) (isolated singularity and 19 rational points)
III. \( vw + x^2 + xy + y^2 \) (isolated singularity and 11 rational points)
IV. \( vw + xy + z^2 \) (smooth and 15 rational points)

The parenthetic statements describe the associated quadric hypersurface. We will say that a form \( Q \) has type I if it is equivalent to \( vw + x^2 \), and similarly for types II, III, and IV. Lemma 6.6 shows that every nonzero quadratic form in \( W \) is of type III or IV.

To organize our search, we now argue that, up to linear change of variable, \( W \) admits a special kind of basis \( \{Q_1, Q_2, Q_3\} \). This involves two cases, depending on whether \( W \) contains a form of type III.

**Case \( W \) contains a form of type III.** Every nonzero form in \( W \) must be of type III or IV. Set \( Q_1 = vw + x^2 + xy + y^2 \). The orthogonal group \( O(Q_1) \) acts on the set of quadratic forms of type III or IV; choose a set \( A(Q_1) \) of orbit representatives for this action. Let us discard \( Q_1 \) from the set \( A(Q_1) \), as well as any \( Q \) such that the linear span of \( Q_1 \) and \( Q \) contains a nonzero form that is not of type III or IV. Take \( B(Q_1) \) to be the set of all forms of type III or IV. We may take a basis for \( W \) of the form \( \{Q_1, Q_2, Q_3\} \) with \( Q_2 \in A(Q_1) \) and \( Q_3 \in B(Q_1) \).

**Case \( W \) contains no form of type III.** Every nonzero form in \( W \) must be of type IV. Set \( Q_1 = vw + xy + z^2 \). The orthogonal group \( O(Q_1) \) acts on the set of quadratic forms of type IV; choose a set \( A(Q_1) \) of orbit representatives for this action. Let us discard \( Q_1 \) from the set \( A(Q_1) \), as well as any \( Q \) such that the linear span of \( Q_1 \) and \( Q \) contains a nonzero form that is not of type IV. Take \( B(Q_1) \) to be the set of all forms of type IV. Evidently, we may take a basis for \( W \) of the form \( \{Q_1, Q_2, Q_3\} \) with \( Q_2 \in A(Q_1) \) and \( Q_3 \in B(Q_1) \).

From an algorithmic standpoint, it is worth noting two things:

- The type of a quadratic form can be determined by calculating the dimension of its singular locus and by counting its rational points. The former amounts to computing the rank of a \( 5 \times 5 \) matrix over \( \mathbb{F}_2 \), while the latter can be accomplished with a search over the 31 points of \( \mathbb{P}^4(\mathbb{F}_2) \). This should be done once for all quadratic forms in \( \mathbb{F}_2[v, w, x, y, z] \) and stored; there are \( 2^{15} - 1 = 32,767 \) quadratic forms.
- We need to compute the orthogonal groups \( O(vw + xy + z^2) \) and \( O(vw + x^2 + xy + y^2) \). This can be accomplished with the techniques in §B.2.

We implemented Algorithm 1 in Sage. For each of the standard forms \( Q_1 \), we computed the orthogonal group and the sets \( A(Q_1) \) and \( B(Q_1) \) and saved them to disk for later use; this required a small number of minutes of compute time. The loop over pairs \( (Q_2, Q_3) \in A(Q_1) \times B(Q_1) \) involves a number of commutative algebra computations that are handled by Singular, and constitute the bulk of the runtime of the algorithm. Table 3 summarizes the outcome of this computation.
Algorithm 1 — Compute a list of genus 5 curves over \( \mathbb{F}_2 \) containing all those of gonality at least 5, up to isomorphism

1: initialize an empty list \texttt{curves}.  
2: \textbf{for} \( Q_1 \in \{vw + x^2 + xy + y^2, vw + xy + z^2\} \) \textbf{do}  
3: compute the sets of quadratic forms \( A(Q_1) \) and \( B(Q_1) \).  
4: \textbf{for} \( (Q_2, Q_3) \in A(Q_1) \times B(Q_1) \) \textbf{do}  
5: \textbf{if} every nonzero member of the linear span of \( \{Q_1, Q_2, Q_3\} \) has type at least that of \( Q_1 \), and the variety \( V(Q_1, Q_2, Q_3) \) is irreducible and smooth of dimension 1 \textbf{then}  
6: append \( (Q_1, Q_2, Q_3) \) to \texttt{curves}.  
7: \textbf{end if}  
8: \textbf{end for}  
9: \textbf{end for}  
10: \textbf{return} \texttt{curves}

| \( Q_1 \) | \#O(\( Q_1 \)) | \#A(\( Q_1 \)) | \#B(\( Q_1 \)) | Curves | Wall Time |
|---|---|---|---|---|---|
| \( vw + x^2 + xy + y^2 \) | 1,920 | 17 | 19,096 | 30,296 | 371min |
| \( vw + xy + z^2 \) | 720 | 10 | 13,888 | 8,296 | 190min |

Table 3. Counts and timing in our search for all canonically embedded genus-5 curves over \( \mathbb{F}_2 \) of gonality at least 5, up to linear isomorphism. Every isomorphism class is represented by at least one curve that we found, though we make no claim of uniqueness of representation.

Each of the curves in the output of Algorithm 1 is presented as the intersection of three quadratic forms. This makes it possible to count their rational points by a straightforward search over points of \( \mathbb{P}^4(\mathbb{F}_2) \). The results of this procedure are presented in Table 4.

| \( Q_1 \) \textbackslash \#C(\( \mathbb{F}_2 \)) | 0 | 1 | 2 | 3 | \( \geq 4 \) |
|---|---|---|---|---|---|
| \( vw + x^2 + xy + y^2 \) | 11,864 | 13,184 | 5,248 | 0 | 0 |
| \( vw + xy + z^2 \) | 0 | 0 | 0 | 8,296 | 0 |

Table 4. Number of curves found with Algorithm 1 with a given \( Q_1 \) and number of rational points.

Theorem 6.8. \( N_2(5, 5) = 3. \)

\textit{Proof.} Consider the genus-5 curve \( C \subset \mathbb{P}^4_{\mathbb{F}_2} = \text{Proj} \ \mathbb{F}_2[v, w, x, y, z] \) described by the following equations:

\begin{align*}
vw + xy + z^2 &= 0 \\
vw + xy + z^2 &= 0 \\
x^2 + wy + xy + vz + xz &= 0.
\end{align*}

This is one of the curves discovered by Algorithm 1, so it has gonality at least 5. One verifies by a direct search that it has three rational points. In particular, it has gonality 5 by the
third part of Proposition 2.1. Hence, $N_2(5,5) \geq 3$. Looking at the data in Table 4, we see that no curve has more than 3 rational points, so $N_2(5,5) \leq 3$.

**Theorem 6.9.** Every curve of genus 5 over $\mathbb{F}_2$ has gonality less than or equal to 5. In particular, $N_2(5,6) = -\infty$.

*Proof.* Algorithm 1 yielded 11,864 pointless curves of gonality at least 5. On each such curve $C$, we located a cubic point. Corollary 2.5 shows that each of these curves has gonality 5. □

**Appendix A. Miscellaneous values of $N_2(g, \gamma)$ for $6 \leq g \leq 10$**

We determine $N_2(g, \gamma)$ in a number of additional cases by techniques that do not fit the main narrative of the above article. Table 5 summarizes our findings. Our main tool is the “gonality-point inequality” from (1.1):

$$C \text{ has gonality } \gamma \implies \#C(\mathbb{F}_2) \leq 3 \gamma.$$

| $g$ | $\gamma$ | $N_2(g, \gamma)$ |
|-----|----------|------------------|
| 6   | 2        | 6                |
|     | 3        | 9                |
|     | 4        | 10               |
|     | 5–6      | $< 10$           |
|     | 7        | $\leq 0$         |
| 7   | 2        | 6                |
|     | 3        | 9                |
|     | 4        | 10               |
|     | 5–7      | $\leq 10$        |
|     | 8        | $\leq 0$         |
| 8   | 2        | 6                |
|     | 3        | 9                |
|     | 4        | 11               |
|     | 5–9      | $\leq 11$        |
|     | 9        | $\leq 0$         |
| 9   | 2        | 6                |
|     | 3        | 9                |
|     | 4        | 12               |
|     | 5–9      | $\leq 12$        |
|     | 10       | $\leq 0$         |
| 10  | 2        | 6                |
|     | 3        | 9                |
|     | 4–10     | $\leq 13$        |
|     | 11       | $\leq 0$         |

**Table 5.** Maximum number of rational points on binary curves with fixed genus and gonality.

All of the entries in Table 5 with $\gamma = 2$ follow from Theorem 3.1. Each of the entries with $\gamma = 3$ must satisfy $N_2(g,3) \leq 9$ by the gonality-point inequality. The following examples
achieve this bound. We discovered them via a naive search for polynomials \( f \in \mathbb{F}_2[x, y] \) with \( y \)-degree 3 and small \( x \)-degree. In each case, the equation \( \{ f = 0 \} \) gives a (typically singular) affine plane model for the curve \( C \), and the \( x \)-coordinate function provides a morphism to \( \mathbb{P}^1 \) of degree 3. We used Magma [3] to compute the genus and count the rational points on the smooth model; each example has 9 rational points. Note that if \( C \) were hyperelliptic, then \( \#C(\mathbb{F}_2) \leq 6 \) by the gonality-point inequality. In all cases, we conclude that \( N_2(g, 3) = 9 \).

Example A.1 (genus 6, gonality 3, 9 rational points).
\[
C/\mathbb{F}_2 : (x^3 + x)y^3 + (x^4 + x + 1)y^2 + (x^4 + x^3 + 1)y + x^3 + x^2 = 0
\]

Example A.2 (Genus 7, gonality 3, 9 rational points).
\[
C/\mathbb{F}_2 : y^3 + (x^5 + x^2 + x + 1)y^2 + (x^7 + x^6 + x^2)y + x^7 + x^6 = 0
\]

Example A.3 (Genus 8, gonality 3, 9 rational points).
\[
C/\mathbb{F}_2 : y^3 + (x^6 + 1)y^2 + (x^7 + x^6 + x^2)y + x^7 + x^6 = 0
\]

Example A.4 (Genus 9, gonality 3, 9 rational points).
\[
C/\mathbb{F}_2 : y^3 + (x^6 + 1)y^2 + (x^7 + x^5 + x)y + x^7 + x^6 = 0
\]

Example A.5 (Genus 10, gonality 3, 9 rational points).
\[
C/\mathbb{F}_2 : y^3 + (x^7 + x^6 + x^5 + 1)y^2 + (x^7 + x^5 + x)y + x^6 + x^5 = 0
\]

The remaining known entries of Table 5 have gonality \( \gamma = 4 \). We look through the entries with small genus over \( \mathbb{F}_2 \) on manypoints.org to find additional curves that meet our needs.

Example A.6 (Genus 6, gonality 4, 10 rational points). Steve Fischer gave the following example in 2014:
\[
X/\mathbb{F}_2 : (x^3 + x^2)y^3 + (x^2 + x + 1)y + 1 = 0
\]
\[
C/\mathbb{F}_2 : z^2 + x^2 + (x^3 + x) = 0.
\]
Here \( X \) is a curve of genus 2 with 6 rational points, and \( C \) is a double cover of \( X \) with 10 rational points. Serre showed that \( N_2(6) = 10 \). Since \( C \) is a double cover of a genus-2 curve, it must have gonality at most 4. If its gonality \( \gamma \) were at most 3, then the gonality-point inequality would show \( \#C(\mathbb{F}_2) \leq 9 \). Hence its gonality is exactly 4, and we have \( N_2(6, 4) = 10 \).

Example A.7 (Genus 7, gonality 4, 10 rational points). Another example by Steve Fischer:
\[
X/\mathbb{F}_2 : (x^3 + x)y^3 + (x^3 + x^2 + x)y + 1 = 0
\]
\[
C/\mathbb{F}_2 : z^2 + z + x^3 + x = 0.
\]
Just as in the case of genus 6, we find \( N_2(7, 4) \geq 10 \). Serre showed that \( N_2(7) \leq 10 \); therefore, \( N_2(7, 4) = 10 \).

Example A.8 (Genus 8, gonality 4, 11 rational points). Isabel Pirsic found the following example in 2012:
\[
C/\mathbb{F}_2 : (x^8 + x^4 + 1)y^4 + (x^9 + x^6 + x^5 + x + 1)y^2
\]
\[
+ (x^9 + x^8 + x^6 + x^5 + x^4 + x)y + (x^8 + x^7 + x^4 + x^3) = 0
\]
The $x$-coordinate function gives a degree-4 morphism to $\mathbb{P}^1$, and $C$ cannot have gonality smaller than 4 by the gonality-point inequality. Hence, $N_2(8, 4) \geq 11$. Serre showed that $N_2(8) = 11$, so $N_2(8, 4) = 11$.

**Example A.9 (Genus 9, gonality 4, 12 rational points).** Another example by Isabel Pirsic:

$$C_{/\mathbb{F}_2} : (x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1)y^4 + (x^{14} + x^{13} + x^{11} + x^9 + x^7 + x^5 + x^3 + x + 1)y^2 + (x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x)y + x^{11} + x^9 + x^7 + x^5 = 0$$

Just as in the genus 8 case, we find that $N_2(9, 4) \geq 12$. Serre showed that $N_2(9) \leq 12$, so we find that $N_2(9, 4) = 12$.

One would like to extend the same kind of reasoning to genus 10. However, $N_2(10) = 13$, so any example that achieves this bound necessarily has gonality at least 5. Steve Fischer produced a curve with 13 rational points via iterated double covers that has a natural morphism to $\mathbb{P}^1$ of degree 6. We conclude that $N_2(10, 5) = 13$ or $N_2(10, 6) = 13$, but we are unable to determine which of these is the truth without additional work.

Finally, we address the entry in the table with $g = 6$ and $\gamma > 4$. In [15], the author proves that there are exactly two curves of genus 6 with 10 rational points, up to isomorphism. Her arguments show that one of the curves, say $C_1$, is a double cover of an elliptic curve, and hence has gonality 4; it does not seem immediately obvious how to suss out the gonality of the other. We claim that the other curve, $C_2$, has gonality 4 as well. On manypoints.org, Steve Fischer gave the following example of a genus-6 curve over $\mathbb{F}_2$ with 10 rational points:

$$X : (x^3 + x^2) y^3 + (x^2 + x + 1) y + 1 = 0$$

$$C_2 : z^2 + x^2 z + (x^3 + x^2) y = 0.$$  

Here $X$ is a genus-2 curve with 6 rational points, and $C_2$ is a double cover of it. That is, $C_2$ has gonality 4. One can verify, using Magma say, that $\#C_2(\mathbb{F}_{32}) = 20$. Since $\#C_1(\mathbb{F}_{32}) = 25$, as one can read off of the data for $a(X)$ in [15, Thm. 2.4], we see that $C_1$ and $C_2$ must be non-isomorphic. It follows that both of the isomorphism classes of curves with 10 rational points have gonality 4, and hence any curve of genus 6 with gonality 5, 6, or 7 must have fewer than 10 points.

**Appendix B. Quadratic Forms over Finite Fields**

Literature on the classification of non-degenerate quadratic forms in odd characteristic is abundant, but it is much harder to find a self-contained reference for general quadratic forms in all characteristics. Once one lets go of the idea that the associated bilinear form should retain all of the information about the quadratic form, the theory becomes quite streamlined in all characteristics. Following unpublished notes of Bill Casselman [4] — who essentially follows the treatment in [8] — we give a summary of all of the results that we need. For additional reference, see [17] or [2].
B.1. Classification of quadratic forms.

Definition B.1. Let $\mathbb{F}_q$ be a finite field and let

$$Q(x) = \sum_{1 \leq i \leq j \leq n} c_{i,j} x_i x_j$$

be a quadratic form over $\mathbb{F}_q$ in $n$ variables, where $x = (x_1, \ldots, x_n)$ and $c_{i,j} \in \mathbb{F}_q$ are not all zero. Here $n$ is the dimension of $Q$. The associated bilinear form is

$$\langle x, y \rangle = Q(x + y) - Q(x) - Q(y).$$

The radical of the bilinear form $\langle \cdot, \cdot \rangle$ is defined to be

$$\text{rad} = \{ a \in \mathbb{F}_q^n : \langle a, \mathbb{F}_q^n \rangle = 0 \}.$$ 

A quadratic form is called strictly non-degenerate if $\text{rad} = 0$, or equivalently, if $\langle \cdot, \cdot \rangle$ is a perfect pairing.

Remark B.2. If we write $e_i$ for the $i$-th standard basis vector of $\mathbb{F}_q^n$, then

$$\langle a, e_i \rangle = \frac{\partial Q}{\partial x_i}(a),$$

Consequently, the radical is precisely the set of $\mathbb{F}_q$-rational points at which all partial derivatives of $Q$ vanish.

For the remainder of this section, we fix $\alpha \in \mathbb{F}_q^2$ such that $\mathbb{F}_q^2 = \mathbb{F}_q(\alpha)$. The norm form,

$$N(x_1, x_2) := N_{\mathbb{F}_q^2/\mathbb{F}_q}(x_1 + \alpha x_2) = x_1^2 + (\alpha + \alpha^q)x_1 x_2 + \alpha^{q+1} x_2^2,$$ (B.1)

is a strictly non-degenerate quadratic form over $\mathbb{F}_q$.

We say that two quadratic forms $Q_1, Q_2$ are equivalent if they agree up to a linear change of variables on their domain.

Proposition B.3 (Strictly non-degenerate forms). Let $Q$ be a strictly non-degenerate quadratic form over $\mathbb{F}_q$ of dimension $n$.

- If $n$ is even, then $Q$ is equivalent to one of the following two forms:
  $$x_1 x_2 + x_3 x_4 + \cdots + x_{n-1} x_n$$
  $$x_1 x_2 + x_3 x_4 + \cdots + x_{n-3} x_{n-2} + a N(x_{n-1}, x_{n-2}),$$

  where $a \in \mathbb{F}_q^\times$. The first and second forms are never equivalent, and two of the latter type of form with final coefficients $a, a'$ are equivalent if and only if $a/a'$ is a square in $\mathbb{F}_q^\times$.

- If $n$ is odd, then $q$ must also be odd, and $Q$ is equivalent to
  $$x_1 x_2 + x_3 x_4 + \cdots + x_{n-2} x_{n-1} + a x_n^2$$

  for some $a \in \mathbb{F}_q^\times$. Two such forms with final coefficients $a, a'$ are equivalent if and only if $a/a'$ is a square in $\mathbb{F}_q^\times$. 

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Now we deal with the general case. Fix a quadratic form $Q$ over $\mathbb{F}_q$ of dimension $n$. Take a linear complement $U$ to $\text{rad}$ in $\mathbb{F}_q^n$, so that $\mathbb{F}_q^n = U \oplus \text{rad}$. Since every element of $\mathbb{F}_q^n$ is orthogonal to the radical for $\langle \cdot, \cdot \rangle$, this is an orthogonal direct sum, and the restriction of $Q$ to $U$ is strictly non-degenerate. Moreover, we see that if $u \in U$ and $v \in \text{rad}$, then

$$Q(u + v) = \langle u, v \rangle + Q(u) + Q(v) = Q(u) + Q(v).$$

So the quadratic form decomposes additively over this direct sum. The above proposition characterizes $Q|_U$, and the next proposition characterizes the restriction of $Q$ to the radical.

**Proposition B.4** (Totally degenerate forms). Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n$ such that $\text{rad} = \mathbb{F}_q^n$.

- If $q$ is odd, then $Q = 0$.
- If $q$ is even, then either $Q = 0$ or $Q$ is equivalent to $x_1^2$.

**Proof.** All partial derivatives of $Q$ vanish identically by our assumption on the radical. In the odd characteristic case, this shows $Q(x) = 0$. In the even characteristic case, $Q(x) = \sum c_i x_i^2$ for some $c_i \in \mathbb{F}_q$. The squaring map is bijective on finite fields of even characteristic, so for each $i$ with $c_i \neq 0$, we may replace $x_i$ with $x_i/\sqrt{c_i}$ in order to assume all of the $c_i \in \{0, 1\}$. Then

$$Q(x) = \sum_{i=1}^n c_i x_i^2 = \left( \sum_{i=1}^n c_i x_i \right)^2.$$

If all $c_i = 0$, we are finished. Otherwise, we move $c_1 x_1 + \cdots + c_n x_n$ to $x_1$ to obtain the result. \qed

Combining the previous two results gives a complete classification:

**Theorem B.5** (Classification of quadratic forms). Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n$. There is $m \leq n$ for which $Q$ is equivalent to one of

\begin{align*}
& x_1 x_2 + x_3 x_4 + \cdots + x_{m-1} x_m, \hspace{1cm} (B.2) \\
& x_1 x_2 + x_3 x_4 + \cdots + x_{m-3} x_{m-2} + a N(x_{m-1}, x_m), \hspace{1cm} (B.3) \\
& x_1 x_2 + x_3 x_4 + \cdots + x_{m-2} x_{m-1} + ax_m^2 \hspace{1cm} (B.4)
\end{align*}

where $a \in \mathbb{F}_q^\times$.

- If $Q$ is equivalent to (B.2) or (B.3), or if $q$ is odd and $Q$ is equivalent to (B.4), then $\text{rad} = \{ x_1 = x_2 = \cdots = x_m = 0 \}$.
- If $q$ is even and $Q$ is equivalent to (B.4), then $\text{rad} = \{ x_1 = x_2 = \cdots = x_{m-1} = 0 \}$.

**Proof.** The result is immediate from the two preceding propositions when $q$ is odd or when $q$ is even and $Q|_{\text{rad}} = 0$. So we are reduced to considering the case where $q$ is even, $\mathbb{F}_q^n = U \oplus \text{rad}$ with $\dim U = m - 1$, and $Q|_{\text{rad}} = x_m^2$. Note that $Q|_U$ is strictly non-degenerate. If $Q|_U$ is equivalent to $x_1 x_2 + \cdots + x_{m-2} x_{m-1}$, we are finished. Suppose instead that $Q|_U$ is equivalent to

$$x_1 x_2 + x_3 x_4 + \cdots + x_{m-4} x_{m-3} + a N(x_{m-2}, x_{m-1})$$

for some $a \in \mathbb{F}_q^\times$. Since the squaring map is surjective, we may absorb $\sqrt{a}$ into $x_{m-2}$ and $x_{m-1}$ in order to assume $a = 1$. \hfill \square
To complete the proof, it suffices to show that the form $N(x, y) + z^2$ is equivalent to $xy + z^2$. If $N(x, y) = x^2 + Axy + By^2$, then the linear change of variables

$$x \mapsto x + y, \quad y \mapsto \frac{1}{A} y, \quad z \mapsto x + \frac{\sqrt{B}}{A} y + z$$

does the trick. Note that $A \neq 0$ and $\sqrt{B} \in \mathbb{F}_q$ since $q$ is even. \qed

B.2. Orthogonal groups. Now we look at the group of linear transformations that preserves a given quadratic form.

**Definition B.6.** Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n$. The orthogonal group of $Q$ is defined to be

$$O(Q) = \{ g \in \text{GL}_n(\mathbb{F}_q) : Q(g(x)) = Q(x) \}.$$  

An immediate consequence of the definition is that the associated bilinear form is also preserved by elements of the orthogonal group:

$$\langle g(x), g(y) \rangle = \langle x, y \rangle \quad \text{for all } g \in O(q).$$

Consequently, the orthogonal group preserves the radical of $\langle \cdot, \cdot \rangle$.

**Theorem B.7** (Witt Extension Theorem). Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n$ with radical $\text{rad}$, and let $U_1, U_2 \subset \mathbb{F}_q^n$ be subspaces such that $U_1 \cap \text{rad} = U_2 \cap \text{rad} = 0$. Then any isometry $U_1 \to U_2$ may be extended to an element of the orthogonal group of $Q$.

We will only need to use this result on 1-dimensional subspaces, or equivalently, on points of projective space. We define two special linear subvarieties of interest. Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n + 1$. Denote the set of $\mathbb{F}_q$-rational points on the quadric hypersurface $V(Q) \subset \mathbb{P}^n$ by

$$\mathcal{Z} = \{ a \in \mathbb{P}^n(\mathbb{F}_q) : Q(a) = 0 \}.$$

The projectivization of the radical will be denoted

$$\mathcal{R} = \mathbb{P}(\text{rad})(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q).$$

The linear subvariety $\mathbb{P}(\text{rad})$ will be referred to as the **radical locus**. Note that the orthogonal group acts on both $\mathcal{Z}$ and $\mathcal{R}$.

Looking at 1-dimensional subspaces of $\mathbb{F}_q^{n+1}$, the Witt Extension Theorem implies that if $a, b \in \mathcal{Z} \setminus \mathcal{R}$, then there exists $g \in O(Q)$ such that $g(a) = g(b)$. We would like to augment this result to include a simultaneous transitivity statement on $\mathcal{R}$. There is one small wrinkle that must be addressed before stating the result.

As the orthogonal group acts on $\mathcal{Z}$ and $\mathcal{R}$, it also acts on their intersection

$$\mathcal{S} = \mathcal{Z} \cap \mathcal{R} \subset \mathbb{P}^n(\mathbb{F}_q).$$

This is the set of $\mathbb{F}_q$-rational points at which $Q$ and all of its partial derivatives vanish; that is, $\mathcal{S}$ is the set of $\mathbb{F}_q$-rational singular points of the hypersurface $V(Q)$. Our transitivity result is the best possible result on points that takes into account these subspaces:
Theorem B.8 (Transitivity). Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n + 1$, and let $\mathcal{Q}, \mathcal{R}, \mathcal{I}$ be the $\mathbb{F}_q$-rational points of $V(Q)$, of the radical locus, and of the singular locus of $V(Q)$, respectively. Write $\mathcal{V}$ for the product of all of the nonempty sets among $(\mathcal{Q} \setminus \mathcal{S}), (\mathcal{R} \setminus \mathcal{S}), \mathcal{S}$. The orthogonal group $O(Q)$ acts transitively on $\mathcal{V}$.

Example B.9. Suppose that $Q$ is equivalent to (B.4) for some $m \leq n + 1$. If $q$ is odd and $m = n + 1$, then $\mathcal{R} = \mathcal{I} = \emptyset$ and $\mathcal{V} = \mathcal{Q}$. At the other extreme, if $q$ is even and $1 < m < n + 1$, then $\emptyset \subset \mathcal{I} \subset \mathcal{R}, \mathcal{Q} \neq \mathcal{S}$, and $\mathcal{V} = (\mathcal{Q} \setminus \mathcal{S}) \times (\mathcal{R} \setminus \mathcal{S}) \times \mathcal{S}$.

Recall that $Q$ is additive when restricted to $\text{rad}$, since $Q(a + b) = \langle a, b \rangle + Q(a) + Q(b) = Q(a) + Q(b)$.

The subset of $\text{rad}$ on which $Q$ vanishes is therefore an $\mathbb{F}_q$-subspace, which we will call $\mathcal{S}$. It follows that $\mathcal{S} = P(\mathcal{S})(\mathbb{F}_q).$ Before proving the theorem, we require two lemmas: the first describes how the singular locus sits inside the radical locus, and the second gives a block decomposition of $O(Q)$ along $\mathcal{S}$ and its orthogonal complement.

Lemma B.10 (Singular Locus). Let $\mathcal{S} \subset \text{rad}$ be the subspace on which $Q$ vanishes. Then $\text{codim}(\mathcal{S}, \text{rad}) \leq 1$.

Moreover, $\text{codim}(\mathcal{S}, \text{rad}) = 1$ precisely when $q$ is even and $Q$ is equivalent to

$$x_1 x_2 + \cdots + x_{m-2} x_{m-1} + x_m^2$$

for some $m \leq n + 1$.

Proof. Decompose $\mathbb{F}_q^{n+1} = U \oplus \text{rad}$, and let $m = \dim U$. Suppose first that $q$ is odd, or that $q$ is even and $m$ is even. Then the Classification of Quadratic Forms shows that $Q$ is equivalent to one of

$$x_1 x_2 + \cdots + x_{m-1} x_m,$$

$$x_1 x_2 + \cdots + x_{m-3} x_{m-2} + a N(x_{m-1}, x_m),$$

$$x_1 x_2 + \cdots + x_{m-2} x_{m-1} + a x_m^2,$$

for some $a \in \mathbb{F}_q^\times$. The latter case can only occur when $q$ is odd. In these coordinates, we have

$$U = \{x_{m+1} = \cdots = x_{n+1} = 0\}$$

$$\text{rad} = \{x_1 = \cdots = x_m = 0\}$$

Evidently, $\mathcal{S} = \text{rad}$.

By the Classification of Quadratic Forms, the only remaining case is when $q$ is even, $m$ is odd, and $Q$ is equivalent to

$$x_1 x_2 + \cdots + x_{m-2} x_{m-1} + x_m^2.$$ (Since squaring is onto in characteristic 2, we can absorb the coefficient $a$ in (B.4).) In these coordinates, we have

$$U = \{x_m = \cdots = x_{n+1} = 0\}$$

$$\text{rad} = \{x_1 = \cdots = x_{m-1} = 0\}$$
Here, we find $S = \{x_1 = \cdots = x_m = 0\}$, so that $S$ has codimension 1 inside $\text{rad}$.

**Lemma B.11** (Block Decomposition of Isometries). Let $Q$ be a quadratic form over $\mathbb{F}_q$ of dimension $n+1$. Let $S$ be the maximal subspace of $\text{rad}$ on which $Q$ vanishes. If we decompose $\mathbb{F}_q^{n+1}$ as $U \oplus S$, then an element $g \in \text{GL}_{n+1}(\mathbb{F}_q)$ lies in the orthogonal group of $Q$ if and only if it admits a decomposition as

$$g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where $A \in O(Q|_U)$, $B : U \to S$ is an arbitrary linear map, and $C \in \text{GL}(S)$.

**Proof.** One implication is just a computation: if $g$ has the desired form, $u \in U$ and $s \in S$, then

$$Q(g(u + s)) = Q(A(u) + B(u) + C(s)) = Q(A(u)),$$

since $Q$ is additive across $U \oplus S$ and kills $S$.

Turning to the other direction, we take $g \in O(Q)$ and write it in block form

$$g = \begin{pmatrix} A & D \\ B & C \end{pmatrix},$$

where $A : U \to U$, $D : S \to U$, $B : U \to S$, and $C : S \to S$ are all linear. We must show that $A \in O(Q|_U)$, that $D = 0$, and that $C$ is invertible.

Using the fact that $Q$ is additive across $U \oplus S$ again, for any $u \in U$ and $s \in S$ we have

$$Q(u) = Q(u + s) = Q(g(u + s)) = Q(A(u) + D(s) + B(u) + C(s)) = Q(A(u) + D(s)) = \langle A(u), D(s) \rangle + Q(A(u)) + Q(D(s)).$$

Setting $s = 0$ shows that $A \in O(Q|_U)$. In particular, we obtain the equation

$$\langle A(u), D(s) \rangle = -Q(D(s)) \quad (u \in U, s \in S). \quad (B.5)$$

For the sake of a contradiction, suppose that $D \neq 0$ and fix $s_0 \in S$ such that $D(s_0) \neq 0$. Setting $u = 0$ in (B.5) shows that $Q(D(s_0)) = 0$. Since $A$ is invertible, it is onto, and we see from (B.5) that $\langle u, D(s_0) \rangle = 0$. Consequently, $D(s_0) \in U \cap \text{rad}$.

If $\text{rad} = S$, then $D(s_0) \in U \cap S = 0$, a contradiction. Thus, we must have $\text{rad} \neq S$. The Singular Locus lemma shows that $q$ is even, and that in an appropriate choice of coordinates we have

$$Q|_U = x_1x_2 + \cdots + x_{m-2}x_{m-1} + x_m^2.$$ 

As $D(s_0) \in U \cap \text{rad}$, we find that $D(s_0) = ae_m$, where $a \in \mathbb{F}_q^\times$ and $e_m$ is the $m$-th standard basis vector. But now $Q(D(s)) = a^2 \neq 0$, another contradiction. We must concede that our initial assertion, namely $D \neq 0$, was false.

Finally, we see that $C$ must be invertible, for otherwise it would have some nonzero element $s$ in its kernel, and then the equation $g(s) = C(s) = 0$ would contradict the invertibility of $g$. \qed
Proof of the Transitivity Theorem. Decompose $\mathbb{F}_q^{n+1} = U \oplus S$, where $S$ is the subspace of $\text{rad}$ on which $Q$ vanishes. Let us, if necessary, replace the standard basis with one that has its first $m$ vectors in $U$ and its last $n+1-m$ vectors in $S$. By the definition of $S$, we know $Q|_S = 0$, so $Q$ depends only on $x_1, \ldots, x_m$.

In the remainder of the proof, if $a \in \mathbb{F}_q^n(\mathbb{F}_q)$, we will write $\tilde{a}$ for a fixed choice of lift to $\mathbb{F}_q^{n+1}$. If $\tilde{a}$ is a lift, so is $t \cdot \tilde{a}$ for any $t \in \mathbb{F}_q^\times$.

Consider first the case where $S = \text{rad}$. Then $\mathcal{R} = \mathcal{S}$. Choose $u_1, u_2 \in \mathcal{D} \setminus \mathcal{R}$ and $v_1, v_2 \in \mathcal{R}$. We wish to produce an isometry $g \in O(Q)$ such that $g(u_1) = u_2$ and $g(v_1) = v_2$. To that end, we may write $\tilde{a}_i = \tilde{a}_i + \tilde{b}_i$ with $\tilde{a}_i \in U$ and $\tilde{b}_i \in S$. Since $u_i \not\in \mathcal{R}$, we have $\tilde{a}_i \not= 0$. Since $S = \text{rad}$, we know that $Q|_U$ is strictly non-degenerate. By the Witt Extension Theorem applied to $Q|_U$, there is $A \in O(Q|_U)$ such that $A(\tilde{a}_1) = \alpha \tilde{a}_2$ for some $\alpha \in \mathbb{F}_q^\times$. Choose any $C \in \text{GL}(S)$ such that $C(\tilde{v}_1) = \tilde{v}_2$. Finally, since $\tilde{a}_1 \not= 0$, we can choose a linear map $B$ such that $B(\tilde{a}_1) = \alpha \tilde{b}_2 - C(\tilde{b}_1)$. Define $g$ to be a block matrix with entries $A, 0, B, C$ as in the lemma. Then we have

$$g(\tilde{a}_1) = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \tilde{b}_1 \end{pmatrix} = A(\tilde{a}_1) + B(\tilde{a}_1) + C(\tilde{b}_1) = \alpha \tilde{a}_2 + \alpha \tilde{b}_2 = \alpha \tilde{a}_2,$$

and similarly, $g(\tilde{v}_1) = \tilde{v}_2$. This completes the proof of the theorem when $S = \text{rad}$.

In the remaining case, we have $q$ even and $S \subset \text{rad}$ has codimension 1. In particular, after an appropriate choice of coordinates, our quadratic form is

$$Q = x_1 x_2 + \cdots + x_{m-2} x_{m-1} + x_m^2,$$

where $m \leq n+1$. In these coordinates, we have

$$U = \{x_{m+1} = \cdots = x_{n+1} = 0\},$$

$$S = \{x_1 = \cdots = x_m = 0\},$$

$$\text{rad} = \{x_1 = \cdots = x_{m-1} = 0\}.$$

Choose $u_1, u_2 \in \mathcal{D} \setminus \mathcal{R}$, $s_1, s_2 \in \mathcal{S}$, and $v_1, v_2 \in \mathcal{R} \setminus \mathcal{S}$. Choose lifts of all of these. Write $\tilde{u}_i = \tilde{a}_i + \tilde{b}_i$ with $\tilde{a}_i \in U$ and $\tilde{b}_i \in S$. Since $u_i \not\in \mathcal{R}$, it follows that $\tilde{a}_i \not\in \text{rad}$. By Witt’s Extension Theorem, there is $A \in O(Q|_U)$ such that $A(\tilde{a}_1) = \alpha \tilde{a}_2$ for some $\alpha \in \mathbb{F}_q^\times$. Choose any $C \in \text{GL}(S)$ such that $C(\tilde{s}_1) = \tilde{s}_2$.

Now let’s look at $\tilde{v}_1$ and $\tilde{v}_2$. As these lie in $\text{rad}$, but not in $S$, it follows that the $m$-th entry of each of them is nonzero. For convenience, we may rescale their lifts so that they both have $m$-th entry equal to 1. Now $\tilde{v}_1 = e_m + w_1$ and $\tilde{v}_2 = e_m + w_2$, where $e_m$ is the $m$-th standard basis vector for $\mathbb{F}_q^{n+1}$ and $w_1, w_2 \in S$. (We do not decorate the $w_i$ with tildes because one or both of them could be identically zero, and hence may not be lifts of elements of $\mathcal{S}$.)

Now observe that $U \cap \text{rad}$ is the radical of $Q|_U$. In particular, $A$ preserves this subspace, so we must have $A(e_m) = \beta e_m$ for some $\beta \in \mathbb{F}_q^\times$. From the previous paragraph, we saw that $\tilde{a}_1 \not\in \text{rad}$, and hence $\tilde{a}_1$ and $e_m$ are linearly independent. Thus, we are able to choose a linear map $B : U \to S$ such that

$$B(\tilde{a}_1) = \alpha \tilde{b}_2 - C(\tilde{b}_1)$$

$$B(e_m) = \beta w_2 - C(w_1).$$

If we define $g$ to be a block matrix with entries $A, 0, B, C$ as in the lemma, then it follows as before that $g(\tilde{u}_1) = \alpha \tilde{u}_2$, $g(\tilde{v}_1) = \beta \tilde{v}_2$, and $g(\tilde{s}_1) = \tilde{s}_2$, which completes the proof. $\square$
We close with a discussion of how to compute the orthogonal group \( O \) of a quadratic form \( Q \) of dimension \( n+1 \). The first sensible thing to do is to loop over the \((n+1) \times (n+1)\) matrices \( g \) with coefficients in \( \mathbb{F}_q \) and test whether \( g \) is invertible and whether \( Q(g(x)) = Q(x) \). Since there are \( q^{(n+1)^2} \) such matrices, this is only practical for small \( n \) and small \( q \).

**Example B.12.** Consider the quadratic form over \( \mathbb{F}_2 \) of dimension 5 given by

\[
Q(v, w, x, y, z) = vw + x^2.
\]

There are \( 2^{25} \approx 10^{7.5} \) matrices to search through. Running Sage on a 2.6 GHz Intel Core i5, the naive search took approximately 54 minutes to compute \( O(Q) \).

Typically, a more efficient method for computing \( O(Q) \) can be given by using our Transitivity Theorem.

**Step 1.** Compute \( \mathcal{D} \). To accomplish this, we loop over the the \( \frac{q^{n+1}-1}{q-1} \) points \( x \in \mathbb{P}^n(\mathbb{F}_q) \) and keep \( x \) if \( Q(x) = 0 \).

**Step 2.** Compute \( \mathcal{R} \) and \( \mathcal{S} \). The set \( \mathcal{R} \) can be computed by linear algebra: it is the vanishing locus of the \( n+1 \) partial derivatives of \( Q \). Then we set \( \mathcal{S} = \mathcal{D} \cap \mathcal{R} \).

**Step 3.** Compute \( O(Q) \). Let \( \mathcal{Y} \) be as in the Transitivity Theorem; let \( i \in \{1, 2, 3\} \) be the number of sets in the product defining \( \mathcal{Y} \). Fix \( p_0 \in \mathcal{Y} \). For \( g \in \text{GL}_{n+1}(\mathbb{F}_q) \), write \( g.p_0 \) for the image of \( p_0 \) in \( (\mathbb{P}^n)^i \). For any \( p \in \mathcal{Y} \), the equation \( g.p_0 = p \) provides \( i(n+1) \) linear equations in \( (n+1)^2 + i \) unknowns: \( (n+1)^2 \) from the entries of \( g \) and \( i \) coming from the scaling factors inherent in working with \( i \) points of \( \mathbb{P}^n \). Linear algebra gives a space of solutions of dimension \( (n+1)^2 - in \). Looping over \( (n+1) \times (n+1) \) matrices \( g \) that satisfy these conditions and recording those that are invertible and satisfy \( Q(g(x)) = Q(x) \), we obtain \( O(Q) \).

The computations in Steps 1 and 2 are negligible compared to the search in Step 3. The search space in this latter step has size

\[
\# \mathcal{Y} \cdot q^{(n+1)^2-in},
\]

which — depending on \( Q \) — may or may not compare favorably to the naive bound given by \( \# \text{GL}_{n+1}(\mathbb{F}_q) \). By computing \( \mathcal{Y} \), we can determine in advance whether this is a better strategy before doing the search.

**Example B.13.** Returning to Example B.12, we find that \( \#(\mathcal{D} \setminus \mathcal{S}) = 12 \), \( \#(\mathcal{R} \setminus \mathcal{S}) = 4 \), and \( \# \mathcal{S} = 3 \). Thus, \( \# \mathcal{Y} = 144 \), and the search space involved in Step 3 of the above algorithm is \( 144 \cdot 2^{25-34} = 1179648 \approx 10^{6.07} \). This search space is \( 10^{7.5-6.07} \approx 28.4 \) times smaller than the naive search search. Running Sage on the same computer as in Example B.12, this improved algorithm took around 4 minutes to compute \( O(Q) \).

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