AN EQUIDISTRIBUTION THEOREM FOR BIRATIONAL MAPS OF $\mathbb{P}^k$

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ABSTRACT. We prove an equidistribution theorem of positive closed currents for a certain class of birational maps $f_+ : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \geq 2$ satisfying $\bigcup_{n \geq 0} f_+^n(I^+) \cap \bigcup_{n \geq 0} f_+^n(I^-) = \emptyset$, where $f_-$ is the inverse of $f_+$ and $I^\pm$ are the sets of indeterminacy for $f_\pm$, respectively.

1. INTRODUCTION

Let $\omega$ be a Fubini-Study form chosen so that $\int_{\mathbb{P}^k} \omega^k = 1$. For an integer $1 \leq p \leq k$, $\mathcal{C}_p$ denotes the space of positive closed $(p, p)$-currents of unit mass on $\mathbb{P}^k$ where the mass is defined by $\|S\| := \langle S, \omega^{k-p} \rangle$ for positive closed $(p, p)$ current $S$ on $\mathbb{P}^k$. For an open subset $W \subseteq \mathbb{P}^k$, $\mathcal{C}_p(W)$ is the set of currents $S \in \mathcal{C}_p$ with supp$S \subseteq W$.

In [2], the following equidistribution theorem for regular polynomial automorphisms of $\mathbb{C}^k$ was proved.

**Theorem 1.1** (Theorem 1.3 in [2], See also [1], [8]). Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a regular polynomial automorphism of $\mathbb{C}^k$ of degree $d \geq 2$ and $s > 0$ an integer such that $\dim I^+ = k - s - 1$ and $\dim I^- = s - 1$ where $I^\pm$ are the sets of indeterminacy of $f$, $f^{-1}$, respectively. Then, for an integer $0 < p \leq s$, for $S \in \mathcal{C}_p$, whose super-potential $\mathcal{U}_S$ of mean 0 is continuous near $I^-$, $d^{-pn}(f^n)^* S$ converges to the Green $(p, p)$-current $T_p^+$ for $f$ in the sense of currents where $T^+_p = \lim_{n \to \infty} d^{-pn}(f^n)^* \omega^p$.

(For the notion of the local continuity and Hölder continuity of super-potentials, see Section 3) The motivation of this note is to further study Theorem 1.1 in the case of certain birational maps of $\mathbb{P}^k$.

Let $f_+ : \mathbb{P}^k \to \mathbb{P}^k$ be a birational map of algebraic degree $d \geq 2$ and $f_-$ its inverse. Let $\delta$ denote the algebraic degree of $f_-$. Let $I^\pm$ denote the indeterminacy sets of $f_\pm$ and $I^\pm_\infty := \bigcup_{n \geq 0} f_\pm^n(I^\pm)$, respectively. Let $s > 0$ be an integer such that $\dim I^+ = k - s - 1$ and $\dim I^- = s - 1$. The main theorem of this note is as follows:

**Theorem 1.2.** Let $f_+ : \mathbb{P}^k \to \mathbb{P}^k$ be a birational map of algebraic degree $d \geq 2$ such that $I^-_\infty \cap I^+_\infty = \emptyset$ and that there exists an open subset $V \subset \mathbb{P}^k$ such that $V \cap I^+_\infty = \emptyset$ and $I^-_\infty \subset f_+(V) \subseteq V$. Then, for an integer $0 < p \leq s$, for every $S \in \mathcal{C}_p$, whose super-potential is Hölder continuous in $V$, then we have $d^{-pn}(f^n_+)^* S$ converges to $T^+_p$ in the sense of currents where $T^+_p = \lim_{n \to \infty} d^{-pn}(f^n_+)^* \omega^p$.

If we assume $I^-_\infty$ is attracting for $f_+$, we obtain

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Theorem 1.3. Assume the hypotheses in Theorem 1.2 and further that $I^-_\infty$ is attracting for $f_+$. Then, for an integer $0 < p \leq s$, for a generic analytic subset $H$ of pure dimension $k - p$ which means $H \cap I^-_\infty = \emptyset$, we have

$$d^{-pn}(f^+_n)^*[H] \to cT^p_+$$

in the sense of currents where $c$ is the degree of $H$.

Here, a compact subset $A$ of $\mathbb{P}^k$ is called an attracting set if it has an open neighborhood $U$, called a trapping neighborhood, such that $f_+(U) \subseteq U$ and $A = \bigcup_{n \geq 0} f_+^n(U)$ where $f_+^n := f_+ \circ \cdots \circ f_+$, $n$-times.

Among numerous studies on the birational maps on $\mathbb{P}^k$, listing some works related to equidistribution of inverse images of positive closed currents, in \cite{4}, Diller proved that in $\mathbb{P}^2$, the equidistribution is true for $S \in \mathcal{C}_1$ with $\text{supp} S \cap I^-_\infty = \emptyset$. In \cite{6}, Dinh-Sibony defined notions of regular birational maps and $PC_p(V)$-currents for an open subset $V$ of $\mathbb{P}^k$, which is equivalent to its super-potential $\mathcal{K}$ being continuous in $\mathbb{P}^k \setminus V$ and proved that if the initial current $S \in \mathcal{C}_p$ satisfies a regularity condition in terms of $PC_p(V)$, then the equidistribution is true in a certain open subset of $\mathbb{P}^k$ where $V$ is an open neighborhood of $I^+_\infty$. In \cite{10}, De Thélin-Vigny prove that for $f_+$ with $I^-_\infty \cap I^+_\infty = \emptyset$, outside a super-polar set of $\mathcal{C}_s$, equidistribution in Theorem 1.2 holds. They gave a sufficient condition in terms of super-potentials for the equidistribution. The condition in \cite{10} is not stated in terms of the set $I^-_\infty$. In this note we focus on a sufficient condition in terms of the set $I^-_\infty$ of critical values of $f_+$ as in \cite{4} and equidistribution on the whole $\mathbb{P}^k$.

The condition $I^+_\infty \cap I^-_\infty = \emptyset$ was introduced in \cite{4} and \cite{10}. From the dynamical viewpoint, that is, considering iteration of $f_\pm$, it seems reasonable to regard $I^-_\infty$ for birational maps as a generalization of $I^-$ in Theorem 1.1 rather than $I^-$ alone. Along the same lines, the regularity condition in Theorem 1.1 may be translated into $I^-_\infty \cap I^-_\infty = \emptyset$ for birational maps. If we compare the class of birational maps in this note, in \cite{10} and in \cite{6}, ours contains the case of \cite{6} and ours belongs to the case of \cite{10}.

For the proof, we basically follow and refine the proof of Theorem 1.1 and 1.4 in \cite{2}. The main difficulty is to get uniform estimates of $\int_W S \wedge U_{W^c}(R)$ with respect to $n \in \mathbb{N}$ in a neighborhood $W$ of $I^-_\infty$ for smooth $R \in \mathcal{C}_{k-p+1}$ where $U_{W^c}$ denotes the Green quasi-potential of a given current and $\Lambda$ is a constant multiple of the operator $(f_+)^*$. For this, we use the idea in the proof of Theorem 1.1 in \cite{2}, which essentially means that a Green quasi-potential can be approximated from below by a negative closed current with a small error. (See the proof of Proposition 2.3.6 in \cite{8}.) Also, there are subtle differences between Theorem 1.2 and the case of regular polynomial automorphisms of $\mathbb{C}^k$. Firstly, we do not know whether for every $U \in \mathbb{P}^k \setminus I^+_\infty$, a super-potential of $T^p_+$ is continuous in $U$. This is needed to bound the dynamical super-potentials from above. As a replacement for this, we will use a convergence appearing in a proof of Theorem 3.2.4 in \cite{10}. Next, in general, $I^-$ may not be invariant under $f_+$ and $I^-_\infty$ may not be an attracting set. For the former part, we construct another invariant current for $f_+$ which is denoted by $R_\infty$ in Proposition 5.9 and for the latter part, the assumption of the existence of the neighborhood $V$ such that $V \cap I^-_\infty = \emptyset$ and $I^-_\infty \subset f_+(V) \Subset V$ resolves the difficulty.
In this note, the \( C^\alpha \)-norm \( \| \cdot \|_{C^\alpha} \), the uniform norm \( \| \cdot \| \), the uniform norm \( \| \cdot \|_U \) on a set \( U \) are computed in terms of the sum of the coefficients of a given form with respect to a fixed finite atlas.

2. Currents

In this note, we assume some familiarity of the reader to pluripotential theory and currents. For details, consult \([3]\) and \([11]\) for instance. In this section, we introduce some notions and notations that we will use in this note.

Let \( 1 \leq q \leq k \) and \( W \) an open subset of \( \mathbb{P}^k \). The following spaces and norms are useful in the study of currents. For instance, see \([9]\), \([5]\), \([2]\). Let \( D_q \) be the real vector space spanned by \( \mathcal{C}_q \) and \( D_q^0(W) \) the subspace of \( D_q \) of currents \( R \) which are cohomologous to 0 and satisfy \( \text{supp}R \subseteq W \). We define \( \| R \|_* := \inf \{ \| R+ \| : R = R_+ - R_- \text{ positive and closed} \} \) on \( D_q^0(W) \). Let \( \tilde{D}_q^0(W) := \{ R \in D_q^0(W) : \| R \|_* \leq 1 \} \).

The topology on \( \tilde{D}_q^0(W) \) is the subspace topology of the space of currents in \( W \). Note that the space \( \tilde{D}_q^0(W) \) is compact. The norm \( \| \cdot \|_* \) bounds the mass norm. So, \( \tilde{D}_q^0(W) \) is metrizable. More precisely, if \( \gamma > 0 \) is a constant, we define for \( R \in \tilde{D}_q^0(W) \)

\[
\| R \|_{-\gamma} := \sup \{ \| \langle R, \phi \rangle \| : \phi \text{ is a test form of bi-degree } (k-q, k-q) \text{ with } \| \phi \|_{C^{\gamma}} \leq 1 \}.
\]

In a similar fashion, we have

**Definition 2.1** (See \([7]\)). Let \( \phi : \mathbb{P}^k \to \mathbb{P}^k \) be an \( L^1 \)-function. We say that \( \phi \) is a DSH function if outside a pluripolar set, \( \phi \) can be written as a difference of two quasi-plurisubharmonic functions. Two DSH functions are identified if they are equal to each other outside a pluripolar set.

If \( \phi \) is a DSH function on \( \mathbb{P}^k \), we define the DSH-norm of \( \phi \) by

\[
\| \phi \|_{\text{DSH}} := \| \phi \|_{L^1} + \| d\bar{d} \phi \|_{*}.
\]

On \( \mathbb{P}^k \), we have a good smooth approximation of positive closed currents. The following is from \([8]\). We will simply call it the standard regularization or the \( \theta \)-regularization of a current. Since \( \text{Aut}(\mathbb{P}^k) \cong \text{PGL}(k+1, \mathbb{C}) \), we choose and fix a holomorphic chart such that \( |y| < 2 \) and \( y = 0 \) at id \( \in \text{Aut}(\mathbb{P}^k) \). We denote by \( \tau_y \) the automorphism corresponding to \( y \). We choose a norm \( |y| \) of \( y \) so that it is invariant under the involution \( \tau \to \tau^{-1} \). Fix a smooth probability measure \( \rho \) with compact support in \( \{ y : |y| < 1 \} \) such that \( \rho \) is radial and decreasing as \( |y| \) increases. Then, the involution \( \tau \to \tau^{-1} \) preserves \( \rho \). Let \( h_\theta(y) := \theta y \) denote the multiplication by \( \theta \in \mathbb{C} \) and for \( |\theta| \leq 1 \) define \( \rho_\theta := (h_\theta)_* \rho \). Then, \( \rho_\theta \) becomes the Dirac mass at \( \text{id} \in \text{Aut}(\mathbb{P}^k) \). We define for \( R \in \mathcal{C}_q \),

\[
R_\theta := \int_{\text{Aut}(\mathbb{P}^k)} (\tau_y)_* R_{\rho_\theta}(y) = \int_{\text{Aut}(\mathbb{P}^k)} (\tau_{\rho_\theta})_* R_{\rho_\theta}(y) = \int_{\text{Aut}(\mathbb{P}^k)} (\tau_{\rho_\theta})^* R_{\rho_\theta}(y).
\]

Note that \( R_\theta \in \mathcal{C}_q \).

**Proposition 2.2** (Proposition 2.1.6 in \([8]\)). If \( \theta \neq 0 \), then \( R_\theta \in \mathcal{C}_q \) is a smooth form which depends continuously on \( R \). Moreover, for every \( \alpha \geq 0 \) there is a constant \( c_\alpha \) independent of \( R \) such that

\[
\| R_\theta \|_{C^\alpha} \leq c_\alpha \| R \| |\theta|^{-2k^2 - 4k - \alpha}.
\]
3. Super-potentials

For the details of super-potentials on \( \mathbb{P}^k \), we refer the reader to [8]. For the reader’s convenience, we summarize some definitions and properties of super-potentials on \( \mathbb{P}^k \).

**Definition 3.1.** Let \( 0 < q \leq k \) be an integer. For smooth \( S \in \mathcal{C}_q \) the super-potential \( \mathcal{U}_S \) of \( S \) of mean 0 is a function defined on \( \mathcal{C}_{k-q+1} \) by

\[
\mathcal{U}_S(R) = \langle U_S, R \rangle
\]

where \( R \in \mathcal{C}_{k-q+1} \) and \( U_S \) is a quasi-potential of \( S \) of mean 0, which is a \((q-1, q-1)\)-current such that \( S - \omega^k = dd^c U_S \) and \( \langle U_S, \omega^{k-q+1} \rangle = 0 \).

For a general current \( S \in \mathcal{C}_q \),

\[
\mathcal{U}_S(R) = \lim_{\theta \to 0} \mathcal{U}_S \mathcal{U}_\theta(R)
\]

where \( \mathcal{U}_\theta \) is the standard regularization of \( S \) as in Section 2 and \( \mathcal{U}_S \) is its super-potential of \( S_\theta \) of mean 0.

Among various quasi-potentials, there is a good one for a computational purpose. It is called the Green quasi-potential and given by an integral formula.

**Proposition 3.2** (Proposition 2.3.2 in [8]). Let \( \Delta \) be the diagonal submanifold of \( \mathbb{P}^k \times \mathbb{P}^k \) and \( \Omega \) a closed real smooth \((k, k)-\)form cohomologous to \([\Delta]\). Then, there is a negative \((k-1, k-1)\)-form \( K \) on \( \mathbb{P}^k \times \mathbb{P}^k \) smooth outside \( \Delta \) such that \( dd^c K = [\Delta] - \Omega \) which satisfies the following inequality near \( \Delta \):

\[
\|K(\cdot)\|_\infty \lesssim -\text{dist}(\cdot, \Delta)^{2-2k} \log \text{dist}(\cdot, \Delta) \quad \text{and} \quad \|\nabla K(\cdot)\|_\infty \lesssim \text{dist}(\cdot, \Delta)^{1-2k}.
\]

Moreover, there is a negative dsh function \( \eta \) and a positive closed \((k-1, k-1)\)-form \( \Theta \) smooth outside \( \Delta \) such that \( K \geq \eta \Theta \), \( \|\Theta(\cdot)\|_\infty \lesssim \text{dist}(\cdot, \Delta)^{2-2k} \) and \( \eta - \log \text{dist}(\cdot, \Delta) \) is bounded near \( \Delta \).

Here, the inequalities are up to a constant multiple independent of the point in \( \mathbb{P}^k \times \mathbb{P}^k \setminus \Delta \). The norm \( \|\nabla K\|_\infty \) is the sum \( \sum_j |\nabla K_j| \), where the \( K_j \)’s are the coefficients of \( K \) for a fixed atlas of \( \mathbb{P}^k \times \mathbb{P}^k \).

We consider a fixed kernel \( K \) throughout the rest of the note. The Green quasi-potential \( U_S \) of \( S \) is defined by

\[
U_S(z) := \int_{z \neq \zeta} K(z, \zeta) \wedge S(\zeta).
\]

Using the notion of super-potentials, we can define the operator \( f_+^* \) on \( \mathcal{C}_q \) where \( f_+ \) is a birational map in Theorem 1.2.

**Definition 3.3** (Definition 5.1.4 in [8]). We say that \( S \in \mathcal{C}_q \) is \( f_+^* \)-admissible if there is a current \( R_0 \in \mathcal{C}_{k-q+1} \) which is smooth on a neighborhood of \( I^+ \), such that the super-potential of \( S \) are finite at \( \Lambda_{k-q+1}(R_0) \).

**Proposition 3.4** (Proposition 5.1.8 in [8]). Let \( S \) be an \( f_+^* \)-admissible current in \( \mathcal{C}_p \). Let \( \mathcal{U}_S \) and \( \mathcal{U}_{L(\omega^p)} \) be super-potentials of \( S \) and \( L_p(\omega^p) \). Then, we have

\[
\lambda_p(f_+)^{-1}\lambda_{p-1}(f_+)^{-1} \mathcal{U}_S \circ \Lambda_{k-p+1} + \mathcal{U}_{L_p(\omega^p)}
\]

is equal to a super-potential of \( L_p(S) \) for \( R \in \mathcal{C}_{k-p+1} \), smooth in a neighborhood of \( I^+ \).
Lemma 3.5 (Lemma 3.1.5 in [10]). Let $S \in \mathcal{C}_q$ for $0 < q \leq k$. Let $n > 0$ be such that $S$ is $(f^n_q)^*$-admissible then for all $j$ with $0 \leq j \leq n - 1$, $L^j(S)$ is well defined, $f^*_q$-admissible and $L^{j+1}(S) = L_{j+1}(S)$. In particular, $L^n(S) = L_n(S)$.

4. Locally regularity of super-potentials

In [2], the notions of locally bounded/continuous superpotentials were given as below. Similarly, we define local Hölder continuity. The notion of the Hölder continuity of superpotentials was given in [5]. We will write $\mathcal{U}_S$ for the super-potential of a current $S \in \mathcal{C}_q$ of mean 0.

Definition 4.1. Let $1 \leq q \leq k$. Let $S \in \mathcal{C}_q$ and $W$ an open subset of $\mathbb{P}^k$. The super-potential $\mathcal{U}_S$ of $S$ of mean $m$ is said to be bounded in $W$ if there exists a constant $C_S > 0$ such that for any smooth current $R \in \tilde{D}^{0}_{k-q+1}(W)$, we have

$$|\mathcal{U}_S(R)| \leq C_S.$$  

The super-potential $\mathcal{U}_S$ of $S$ of mean $m$ is said to be continuous in $W$ if $\mathcal{U}_S$ continuously extends to $\tilde{D}^{0}_{k-q+1}(W)$ with respect to the subspace topology of the space of currents in $W$.

The super-potential $\mathcal{U}_S$ of $S$ of mean $m$ is said to be Hölder continuous in $W$ if $\mathcal{U}_S$ continuously extends to $\tilde{D}^{0}_{k-q+1}(W)$ and Hölder continuous with respect to one of the norms $\| \cdot \|_{-\gamma}$ on $\tilde{D}^{0}_{k-q+1}(W)$.

Note that when $W = \mathbb{P}^k$, our notion coincides with the definition in [5]. Since $\tilde{D}^{0}_{k-q+1}(W)$ is compact, $\mathcal{U}_S$ is continuous in an open subset $W \subset \mathbb{P}^k$, then it is bounded in an open subset $W \subset \mathbb{P}^k$.

Remark 4.2. By interpolation theory, for any $\gamma \geq \gamma' > 0$, there is a constant $c > 0$ such that $\| \cdot \|_{-\gamma} \leq \| \cdot \|_{-\gamma'} \leq c(\| \cdot \|_{-\gamma})^{\gamma'/\gamma}$. So, if $\mathcal{U}_S$ is Hölder continuous for one $\| \cdot \|_{-\gamma}$, then it is Hölder continuous for all $\| \cdot \|_{-\gamma}$.

Remark 4.3. For $S \in \mathcal{C}_q$, its super-potential $\mathcal{U}_S$ is continuous in an open subset $W \subset \mathbb{P}^k$ if and only if $S$ is $PC_q(\mathbb{P}^k \setminus W)$. The equivalence can be observed via the Green quasi-potential kernel in Proposition 3.2.

Proposition 4.4. If a super-potential $\mathcal{U}_S$ of $S \in \mathcal{C}_q$ is bounded in an open subset $W \subset \mathbb{P}^k$ and if $R \in \mathcal{C}_{k-q+1}$ is smooth outside a compact subset $K \subset W$, then $\mathcal{U}_S(R)$ is finite.

Proof. Let $\chi : \mathbb{P}^k \rightarrow [0,1]$ be a smooth cut-off function such that $\text{supp}\, \chi \subset W$ and $\chi \equiv 1$ on $K$. Then, $dd^c((1-\chi)U_R)$ is a smooth $(k-q+1, k-q+1)$-current and $dd^c(\chi U_R)$ is a current in $D^{0}_{k-q+1}(W)$. Hence,

$$\mathcal{U}_S(R) = \mathcal{U}_S(dd^c((1-\chi)U_R)) + \mathcal{U}_S(dd^c(\chi U_R)) + \mathcal{U}_S(\omega^{k-p+1})$$

and so, it is finite as desired. □

5. Birational Maps

In this section, we summarize well-known properties of birational maps on $\mathbb{P}^k$. For details, see [10] for instance.

Let $f_+ : \mathbb{P}^k \rightarrow \mathbb{P}^k$ be a birational map of algebraic degree $d \geq 2$ and $f_-$ its inverse. Let $d$ denote the algebraic degree of $f_-$. Let $I^\pm$ denote the indeterminacy sets of $f_\pm$ and
\[ I^\pm_\infty := \bigcup_{n \geq 0} f^n_\pm(I^\pm), \] respectively. Let \( s > 0 \) be an integer such that \( \dim I^+ = k - s - 1 \) and \( \dim I^- = s - 1 \). Let \( C^\pm \) be the critical sets for \( f^\pm \):

\[
C^+ := f^+_1(I^-) \quad \text{and} \quad C^- := f^-_1(I^+).
\]

Then, we have \( I^+ \subset C^+ \), \( I^- \subset C^- \) and \( f_+ : \mathbb{P}^k \setminus C^+ \to \mathbb{P}^k \setminus C^- \) is a biholomorphism (p. 42 in [10]). We also have

\[
f \implies f^- \circ f_+ = \text{id} \quad \text{on} \quad \mathbb{P}^k \setminus C^+; \quad f_+ \circ f^- = \text{id} \quad \text{on} \quad \mathbb{P}^k \setminus C^-.
\]

Note that the operators \( L_q := (\lambda_q(f^+_\pm))^{-1} f_+^* \) and \( \Lambda_q := (\lambda_{k-q}(f^-_\pm))^{-1} f_+^* \).

For \( 0 \leq q \leq k \) and \( n > 0 \), we define \( \lambda_q(f^+_{\xi}) \) by

\[
\lambda_q(f^+_{\xi}) := ||(f^+_{\xi})^*(\omega^q)|| = ||(f^+_{\xi})*_n(\omega^{k-q})||.
\]

**Proposition 5.1** (Proposition 3.1.2 in [10]). We have \( \lambda_q(f_+) = d^q \) for \( q \leq s \) and \( \lambda_q(f_-) = \delta^{k-q} \) for \( q \geq s \). In particular, \( d^s = \delta^{k-s} \).

**Proposition 5.2** (Corollary 3.1.4 in [10]). We have \( (f^+_{\xi})^n = (f^+_{\xi})* \) for smooth currents in \( \mathcal{C}_q \) and \( \lambda_q(f^+_{\xi}) = (\lambda_q(f_+))^n \) for all \( 0 \leq q \leq k \).

We define two operators acting on \( \mathcal{C}_q \):

\[
L_q := (\lambda_q(f_+))^{-1} f_+^* \quad \text{and} \quad \Lambda_q := (\lambda_{k-q}(f_+))^{-1} f_+^*.
\]

Note that the operators \( L_q \) and \( \Lambda_q \) are well-defined for currents in \( \mathcal{C}_q \) which are smooth near \( I^- \) and \( I^+ \), respectively.

**Proposition 5.3.** Let \( 0 < q \leq k \). Let \( R \) be a smooth current of bidegree \( (q, q) \). Then,

\[
(f_+)^* R = (f_-)^* R
\]

as a current on \( \mathbb{P}^k \) and \( \text{supp}((f_+)^* R) = (f_-)^* R \subseteq (f_+)_{\text{supp}} \setminus C^- = f_-(\text{supp} R \setminus C^-) \).

**Proof.** Notice that \( (f_+)^* R \) and \( (f_-)^* R \) are both forms with \( L^1 \)-coefficients. So, they do not charge any algebraic sets of dimension \( \leq k - 1 \). Let \( \varphi \) be a smooth test form of bidegree \( (k - q, k - q) \). Then, we have

\[
\langle (f_+)^* R, \varphi \rangle = \langle R, (f_-)^* \varphi \rangle = \langle R, (f_+)^* \varphi \rangle_{\mathbb{P}^k \setminus C^+}.
\]

Since \( f_+ : \mathbb{P}^k \setminus C^- \to \mathbb{P}^k \setminus C^+ \) is a biholomorphism, the change of coordinates by \( f_- \) implies

\[
\langle R, (f_+)^* \varphi \rangle_{\mathbb{P}^k \setminus C^-} = \langle (f_-)^* R, (f_-)^* (f_+)^* \varphi \rangle_{\mathbb{P}^k \setminus C^-} = \langle (f_-)^* R, \varphi \rangle_{\mathbb{P}^k \setminus C^-} = \langle (f_-)^* R, \varphi \rangle.
\]

The second last inequality is from \( f_+ \circ f_- : \mathbb{P}^k \setminus C^- \to \mathbb{P}^k \setminus C^- \) being identity on \( \mathbb{P}^k \setminus C^- \).

The support property is from direct computations together with the fact that the currents \( (f_+)^* R \) and \( (f_-)^* R \) have \( L^1 \)-coefficients.

**Corollary 5.4.** For \( R \in \mathcal{C}_{k-q+1} \) smooth outside \( I^- \), \( \Lambda^n_{k-q+1}(R) \) is smooth outside \( I^- \).

Together with Lemma 3.5 we obtain the following proposition:

**Proposition 5.5.** If \( S \in \mathcal{C}_q \) admits a super-potential bounded in a neighborhood of \( I^- \), then for every \( m, n \geq 0 \), \( L^n_m(S) \) is well-defined and \( L^n_m(S) = L^n_m(L^n_q(S)) \).
Proof. Let \( R \in \mathcal{C}_{k-q+1} \) be a smooth current. Then, by Corollary \([5.4]\) and Lemma \([3.5]\), \( \Lambda_{k-q+1}^n(R) \) is well defined and smooth outside \( I_{\infty}^- \). So, by Proposition \([4.4]\), the super-potential of \( S \) is finite at \( \Lambda_{k-q+1}^n(R) \) for every \( n \). Definition \([3.3]\) and Lemma \([3.5]\) finish the proof.

Now, we consider the Green current of order \( q \) associated to \( f_+ \). We further assume the existence of an open subset \( V \subset \mathbb{P}^k \) such that \( V \cap I_{\infty}^+ = \emptyset \) and \( I_{\infty}^- \subset f_+(V) \subset V \) as in Theorem \([1.2]\). Then, there is a strictly positive distance between \( I_{\infty}^+ \) and \( I_{\infty}^- \). Then, this satisfies Hypothesis 3.1.6 in \([10]\). As a result of it, we have the existence of the Green current of order \( q \) as below:

**Theorem 5.6** (Theorem 3.2.2 in \([10]\)). Let \( 0 < q \leq s \). The sequence \( (L_q^m(\omega^q)) \) converges in the Hartog's sense to the Green current \( T_q^m \) of order \( q \) of \( f \). Further, \( \mathcal{M}_{T_q^m}(I^-) \geq -\infty \).

The following proposition can be obtained via a slight modification of Lemma 5.4.2 and Lemma 5.4.3 in \([8]\).

**Proposition 5.7.** Suppose that there is an open subset \( V \subset \mathbb{P}^k \) such that \( V \cap I_{\infty}^+ = \emptyset \) and \( I_{\infty}^- \subset f_+(V) \subset V \) as in Theorem \([1.2]\). Let \( 0 < q \leq s \). Then, the Green current \( T_q^m \) of order \( q \) is Hölder continuous on \( \mathcal{C}_{k-q+1}(V) \).

Here, the Hölder continuity is with respect to the same \( \| \cdot \|_{-\gamma} \)-norm (\( \gamma > 0 \)) as in Definition 4.1, but the difference is that we are taking a different set \( \mathcal{C}_{k-q+1}(V) \) of currents other than \( \tilde{D}_{k-q+1}(V) \).

In the rest of this section, we construct a \((k-p+1,k-p+1)\)-current \( R^p \) such that \( \Lambda_{k-p+1}(R^p) = R^p \) and \( \text{supp} R^p \subset T_{\infty}^- \) where \( \Lambda_{k-s+1} = d^{-(p-1)}(f_+)^{s} \), for \( 0 < p \leq s \).

We will first construct such a current \( R^p \) of bidegree \((k-s+1,k-s+1)\) and then consider the case of bidegree \((k-p+1,k-p+1)\). The current \([I^-]\) denotes the current of integration on the regular part of \( I^- \). It is not difficult to prove the following proposition:

**Proposition 5.8.** Suppose that \( I_{\infty}^- \cap I_{\infty}^+ = \emptyset \). For all \( i = 0,1,2,\ldots \), the currents \( \Lambda_{k-s+1}^i([I^-]) \) are well-defined positive closed currents of bidegree \((k-s+1,k-s+1)\) and they have the same mass as \([I^-]\) does. Also, we have \( \text{supp} \Lambda_{k-s+1}^i([I^-]) \subset f_{+}([I^-]) \).

Consider the following sequence of currents:

\[
R_n := (n+1)^{-1} \sum_{i=0}^{n} \Lambda_{k-s+1}^i([I^-])
\]

Then, the sequence \( \{R_n\} \) has bounded mass. So, there exists a convergent subsequence \( \{R_{n_j}\} \) in the sense of currents. Let \( R_{\infty}^s \) denote one of its limit currents.

**Proposition 5.9.**

\[
\Lambda_{k-s+1}(R_{\infty}^s) = R_{\infty}^s, \quad \text{supp} R_{\infty}^s \subset T_{\infty}^-. 
\]

**Proof.** The first part is clear from \( \text{supp} R_{n_j} \subset I_{\infty}^- \) for all \( j \in \mathbb{N} \). For a convergent subsequence \( \{R_{n_j}\} \), we have

\[
\Lambda_{k-s+1}(R_{n_j}) \rightarrow R_{n_j} = (n_j+1)^{-1}(\Lambda_{k-s+1}^{n_j+1}([I^-]) - [I^-])
\]

as \( j \rightarrow \infty \). Since \( \Lambda_{k-s+1} \) is continuous for currents in \( \mathcal{C}_{k-s+1} \) smooth near \( I^+ \) and the mass of \( \Lambda_{k-s+1}^{n_j} \) is bounded in \( j \in \mathbb{N} \), we see that \( \Lambda_{k-s+1}(R_{\infty}^s) = R_{\infty}^s \) by letting \( j \rightarrow \infty \). \( \square \)
For $0 < p \leq s$, from Proposition 5.7, a super-potential $\mathcal{U}_{T_{+}^{s-p}}$ of $T_{+}^{s-p}$ is Hölder continuous in $V$. Since $\operatorname{supp} R_{\infty}^{p} \subset I_{\infty}^{+} \subset V$, the current $R_{\infty}^{p} := (T_{+}^{s-p}) \wedge R_{\infty}^{s}$ is well-defined.

**Proposition 5.10.** Let $0 < p \leq s$. Assume the existence of the neighborhood $V$ of $I_{\infty}^{+}$ in Theorem 1.2. Then, we have

$$\Lambda_{k-p+1}(R_{\infty}^{p}) = R_{\infty}^{p}.$$  

**Proof.** By use of the standard regularization and the Hartog’s convergence, we may assume that $T_{+}^{s-p}$ is smooth. Note that, $R_{\infty}^{p}$ has support in $V$. Then, we have

$$\Lambda_{k-p+1}(R_{\infty}^{p}) = d^{-(p-1)}(f_{+})_{\ast}((T_{+}^{s-p}) \wedge R_{\infty}^{s}) = d^{-(p-1)}(f_{+})_{\ast}(d^{-(s-p)}(f_{+})^{s}(T_{+}^{s-p}) \wedge R_{\infty}^{s})$$

$$= d^{-(s-1)}T_{+}^{s-p} \wedge (f_{+})_{\ast}R_{\infty}^{s} = T_{+}^{s-p} \wedge R_{\infty}^{s} = R_{\infty}^{p}.$$  

\[\square\]

We also obtain the following corollary from Proposition 5.7.

**Corollary 5.11.** Let $0 < p \leq s$. Assume the existence of the neighborhood $V$ of $I_{\infty}^{+}$ in Theorem 1.2 then the value $\mathcal{U}_{T_{+}^{p}}(R_{\infty}^{p})$ is finite.

The argument in p.53 of [10] works for any $0 < p \leq s$. So, we obtain

**Proposition 5.12.** The sequence

$$d^{-n}\mathcal{U}_{T_{+}^{p}} \circ \Lambda_{k-p+1}^{n}$$

goes to 0 on smooth forms in $\mathcal{C}_{k-p+1}.$

### 6. Proof of Theorem 1.2 and Theorem 1.3

For the rest of the note, the current $S \in \mathcal{C}_{p}$ denotes the current in Theorem 1.2. For the proof of Theorem 1.2, we set up environment as in [8] and [2]. For simplicity, we will write $L$ and $\Lambda$ for $L_{p}$ and $\Lambda_{k-p+1}$, respectively. Also, $S_{n} := L^{n}(S) = d^{-p_{n}}(f_{+})^{n}(S)$ and $T_{n} := T_{n}^{p}$.

**Definition 6.1.** For $S \in \mathcal{C}_{p}$, we define the dynamical super-potential $\mathcal{V}_{S}$ by

$$\mathcal{V}_{S} := \mathcal{U}_{S} - \mathcal{U}_{T_{n}} - c_{S}, \quad \text{where } c_{S} := \mathcal{U}_{S}(R_{\infty}) - \mathcal{U}_{T_{n}}(R_{\infty})$$

and the dynamical Green quasi-potential of $S$ by

$$V_{S} := U_{S} - U_{T_{n}} - (m_{S} - m_{T_{n}} + c_{S})\omega^{p-1}$$

where $U_{S}, U_{T_{n}}$ are the Green quasi-potentials of $S, T_{n}$ and $m_{S}, m_{T_{n}}$ are their mean, respectively.

The lemma below can be proved in the same way as in Lemma 5.5.5 in [8].

**Lemma 6.2** (See Lemma 5.5.5 in [8]).

1. $\mathcal{V}_{S}(R_{\infty}^{p}) = 0$,
2. $\mathcal{V}_{S}(R) = \langle V_{S}, R \rangle$ for smooth $R \in \mathcal{C}_{k-p+1}$ and
3. $\mathcal{V}_{L(S)} = d^{-1}\mathcal{V}_{S} \circ \Lambda$ for currents in $\mathcal{C}_{k-p+1}$ smooth near $I_{\infty}^{+}$.
Different from the case of Theorem\[1.1\] since we cannot say that a super-potential $\mathcal{U}_T$ of $T$ is continuous in an open subset $W \in \mathbb{P}^k \setminus I_\infty$, we cannot say that $\mathcal{U}_S - \mathcal{V}_S$ is bounded from above on $\mathcal{C}_k-p+1(W)$ by a constant independent of $S$ in general. However, we have Proposition \[5.12\] as an alternative for this.

We further introduce some more notations. Let $R \in \mathcal{C}_k-p+1$ be a smooth current. We choose and fix a constant $\lambda$ such that $1 < \lambda < d$ throughout the proof. Let $\eta_n := \min\{\eta, -\lambda^n\} + \lambda^n$ where $\eta$ is a DSH function in Proposition \[3.2\]. Then, the DSH-norm of $\eta_n$ is bounded in dependent of $n$ and $K \geq \eta \Theta \geq \eta_n \Theta - \lambda^n \Theta$ where $\eta$ and $\Theta$ are a function and a current in Proposition \[3.2\].

For a positive or negative current $S'$ and each $n \in \mathbb{N}$, define
\[
U''_{n, S'} := \int_{\xi \neq z} \lambda^n \Theta(\zeta, z) \wedge S'(\zeta), \quad U''_{n, S} := \int_{\xi \neq z} \eta_n(\zeta, z) \Theta(\zeta, z) \wedge S'(\zeta)
\]
and
\[
U_n(R) := \int_{\xi \neq z} \lambda^n \Theta(\zeta, z) \wedge \Lambda^n(R)(\zeta), \quad U''_n(R) := \int_{\xi \neq z} \eta_n(\zeta, z) \Theta(\zeta, z) \wedge \Lambda^n(R)(\zeta).
\]
Note that if $S'$ is closed, then $U''_{n, S'}$ is closed and its mass is $c_m \lambda^n \|S'\|$ for a constant $c_m > 0$ which is independent of $n$ and $S'$.

For the proof of Theorem \[1.2\] we need the following estimate. Observe that Lemma 2.3.9 in \[8\] does not need closedness of the current. Its proof consists of disintegration and singularity estimate. So, we have

**Lemma 6.3** (Lemma 2.3.9 in \[8\]). Let $S'$ be a positive current of bidegree $(p, p)$ with bounded mass. Then, we have
\[
\left| \int U''_{n, S'} \wedge \omega^{k-p+1} \right| \leq e^{-\lambda^n} \|S'\|.
\]
The inequality is up to a constant multiple independent of $n$ and $S'$.

Now, we start to prove Theorem \[1.2\]. It is a direct consequence of the following proposition.

**Proposition 6.4.** Assume the hypotheses in Theorem \[1.2\]. Let $R \in \mathcal{C}_k-p+1$ be a smooth current. Then, we have
\[
\mathcal{V}_{S_n}(R) = d^{-n} \mathcal{V}_S(\Lambda^n(R)) \to 0
\]
in the sense of currents.

We begin with an estimate near $I_\infty$ in Lemma \[6.10\].

**Lemma 6.5.** There exist open subsets $W_3 \subset W_2 \subset W_1 \subset W_0 \subset V$ such that $f_+(W_i) \subset W_i$.

**Proof.** Note that $f_+$ is holomorphic outside $I^+$. Since $V \cap I^+ = \emptyset$, $f_+(\overline{V})$ is compact in $V$. So, simply take $f_+(\overline{V}) \subset W_3 \subset W_2 \subset W_1 \subset W_0 \subset V$. $\square$

Let $\chi : \mathbb{P}^k \to [0, 1]$ be a cut-off function such that $\chi \equiv 1$ on $W_4$ and $\text{supp} \chi \subset W_0$. Let $M > 1$ be a constant such that $\|Df_-\|_{\mathbb{P}^k \setminus W_3} < M$. Here, $\| \cdot \|_{\mathbb{P}^k \setminus W_3}$ denotes the uniform norm of coefficients on $\mathbb{P}^k \setminus W_3$ with respect to a fixed finite atlas of $\mathbb{P}^k$. 
Lemma 6.6. Let $R \in \mathcal{C}_{k-p+1}$ be smooth outside $I^-_\infty$. Then, there exists a constant $c > 0$ independent of $R$ and $n$ such that
\[
\|dd^c(\chi U_{\Lambda^n(R)})\|_* \leq cM^{3kn}\|R\|_{\mathbb{P}^k\setminus W_3}
\]
where $U_{(\cdot)}$ denotes the Green quasi-potential of a given current.

Proof. We can write
\[
dd^c(\chi U_{\Lambda^n(R)}) = dd^c\chi \wedge U_{\Lambda^n(R)} + d\chi \wedge dd^c U_{\Lambda^n(R)} + dd^c\chi + \chi (\Lambda^n(R) - \omega^{k-p+1}).
\]
The last term is bounded below by $\omega^{k-p+1}$. Since the first three terms are all smooth since $\Lambda^n(R)$ is smooth outside $I^-_\infty$. They are all bounded by the $C_1$-norm of $U_{\Lambda^n(R)}$ on the support of $d\chi$ or $dd^c\chi$ which is compact outside $W_3$. Hence, by Proposition 5.3
\[
\|\Lambda^n(R)\|_{\mathbb{P}^k\setminus W_3} \leq c_1 M^{3kn}\|R\|_{\mathbb{P}^k\setminus W_3}
\]
for some $c_1 > 0$. Then, due to Lemma 2.3.5 in [8], we get the desired estimate.

Since the above estimate only depends on the uniform norm of $\Lambda^n(R)$ on the support of $d\chi$ and $dd^c\chi$, we see that there exists a constant $\delta_0 > 0$, which only depend on the distance between $W_3$ and $\mathbb{P}^k \setminus W_1$, such that the same estimate as in Lemma 6.6 holds for all $\delta > 0$ with $\delta < \delta_0$ and for all $n \in \mathbb{N}$:
\[
\|dd^c(\chi U_{(\Lambda^n(R))\delta})\|_* \leq cM^{3kn}\|R\|_{\mathbb{P}^k\setminus W_3}.
\]
So, we have $c^{-1}M^{-3kn}\|R\|_{\mathbb{P}^k\setminus W_3}^{-1} dd^c(\chi U_{(\Lambda^n(R))\delta}) \in \tilde{D}_{k-p+1}(W_1)$ for $\delta > 0$ with $|\delta| < \delta_0$.

Lemma 6.7. For $0 < \delta < \delta_0$, we have
\[
\|dd^c(\chi U_{\Lambda^n(R)}) - dd^c(\chi U_{(\Lambda^n(R))\delta})\|_{-2} \lesssim \delta.
\]
The inequality is up to a constant multiple independent of $n$ and $\delta$.

Proof. Let $\varphi$ be a smooth test form with $\|\varphi\|_{C^2} \leq 1$. We have
\[
\langle dd^c(\chi U_{\Lambda^n(R)}) - dd^c(\chi U_{(\Lambda^n(R))\delta}), \varphi \rangle = \langle U_{\Lambda^n(R)} - U_{(\Lambda^n(R))\delta}, \chi dd^c \varphi \rangle
\]
\[
= \langle \Lambda^n(R) - (\Lambda^n(R))\delta, U_{\chi dd^c \varphi} \rangle \leq \langle \Lambda^n(R), U_{\chi dd^c \varphi} \rangle - \langle U_{\chi dd^c \varphi} \rangle
\]
Hence, by Lemma 2.3.5 in [8], we have
\[
\|\langle dd^c(\chi U_{\Lambda^n(R)}) - dd^c(\chi U_{(\Lambda^n(R))\delta}), \varphi \rangle\| \lesssim \|U_{\chi dd^c \varphi}\|_{C^1} \delta \lesssim \|\chi dd^c \varphi\|_{C^1} \delta \lesssim \delta.
\]

Lemma 6.8. Let $S_\theta$ be a standard regularization of $S$ for sufficiently small $0 < |\theta| \ll 1$. For $0 < \delta < \delta_0$, we have
\[
\left| \int U_{S_{\theta \delta}} \wedge dd^c(\chi U_{\Lambda^n(R))\delta}) \right| \lesssim \delta^{-2k^2-4k-2} e^{-\Lambda^n} + \chi^n.
\]
Here, the inequality is independent of $\theta$, $\delta$ and $n$.

Indeed, the $\theta$ is chosen so that $\text{supp}(dd^c \chi \wedge U'_{\Lambda^n(R))\delta})_{\theta} \subseteq V$. This condition is completely determined by the function $\chi$. 
Proof. We have
\[
\int U_{S_\theta} \wedge dd^c(\chi U_{(\Lambda^n(R))_S}) = \int S_\theta \wedge \chi U_{(\Lambda^n(R))_S} - \int \chi \omega^p \wedge U_{(\Lambda^n(R))_S}
\]
\[
\geq \int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} + \int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} - \int \chi \omega^p \wedge U_{(\Lambda^n(R))_S}
\]
We estimate the first integral \( \int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} \). Note that \( U''_{n,(\Lambda^n(R))_S} \) is closed and its mass is a constant multiple of \( \lambda^n \).
\[
\int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} = \int U_{S_\theta} \wedge dd^c \chi \wedge U''_{n,(\Lambda^n(R))_S} + \int \omega^s \wedge \chi U''_{n,(\Lambda^n(R))_S} \gtrsim -\lambda^n.
\]
Since \( \text{supp} \, dd^c \chi \wedge U''_{n,(\Lambda^n(R))_S} \subset W_0 \) and \( \| dd^c \chi \wedge U''_{n,(\Lambda^n(R))_S} \|_s \lesssim \lambda^n \), the first integral is estimated from the boundedness of \( \mathcal{W}_{S_\theta} \) in \( W_0 \subset V \). The second integral is bounded by the mass of \( U''_{n,(\Lambda^n(R))_S} \). So, we get the last inequality.

We estimate the second integral \( \int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} \). From the negativity of \( \eta_n \) and the positivity of \( \Theta \), we have
\[
\int S_\theta \wedge \chi U''_{n,(\Lambda^n(R))_S} = \int \chi(z) S_\theta(z) \wedge \eta_n(z, \zeta) \wedge \Theta(z, \zeta) \wedge (\Lambda^n(R))_S(\zeta)
\]
\[
\geq \delta^{-2k^2-4k-2} \int_{\mathbb{P}^k \times \mathbb{P}^k} \chi(z) S_\theta(z) \wedge \eta_n(z, \zeta) \wedge \Theta(z, \zeta) \wedge \omega^{k-s+1}(\zeta)
\]
\[
= \delta^{-2k^2-4k-2} \int U''_{n,S_\theta} \wedge \omega^{k-p+1} \gtrsim -\delta^{-2k^2-4k-2} e^{-\lambda^n}.
\]
The last inequality is from Lemma 6.3.

From the hypothesis on \( S \in \mathcal{C}_p \) in Theorem 1.2, let \( \alpha > 0 \) and \( C_\alpha > 0 \) be two constants such that for all \( \theta \in \mathbb{C} \) with sufficiently small \( |\theta| \) as in Lemma 6.8
\[
|\mathcal{W}_{S_\theta}(R) - \mathcal{W}_{S_\theta}(R')| \leq C_\alpha (\| R - R' \|_{-2})^\alpha \text{ for } R, R' \in \mathcal{D}_0^{k-p+1}(W_0).
\]

Lemma 6.9. Let \( R \in \mathcal{C}_{k-p+1} \) be a current smooth outside \( I^\infty \). Let \( S_\theta \) be a standard regularization of \( S \) for sufficiently small \( |\theta| \) as in Lemma 6.8. We have
\[
\int_{W_1} S_\theta \wedge U_{\Lambda^n(R)} \gtrsim -\lambda^n
\]
for all sufficiently large \( n \). The inequality is independent of \( \theta \) and \( n \).

Proof. From the negativity of the Green quasi-potential, we have
\[
\int_{W_1} S_\theta \wedge U_{\Lambda^n(R)} \geq \int \chi S_\theta \wedge U_{\Lambda^n(R)}
\]
\[
= \int U_{S_\theta} \wedge dd^c(\chi U_{\Lambda^n(R)}) + \int \chi \omega^p \wedge U_{\Lambda^n(R)}
\]
Let \( \delta > 0 \) be a small constant to be determined later. Then, the last quantity can be written as
\[
\int U_{S_\theta} \wedge dd^c(\chi U_{(\Lambda^n(R))_S}) + \int U_{S_\theta} \wedge (dd^c(\chi U_{\Lambda^n(R)}) - dd^c(\chi U_{(\Lambda^n(R))_S})) + \int \chi \omega^p \wedge U_{\Lambda^n(R)}
\]
From the Hölder continuity of $\mathcal{U}_{S_0}$ in $W_0$ with Lemma 6.6 and Lemma 6.7, the second integral can be estimated as follows:

\[
\int U_{S_0} \wedge (dd^c(\chi U_{\Lambda^n(R)}) - dd^c(\chi U_{\Lambda^n(\theta)})) \leq C_n c M^{3kn} \| R \|_{\mathbb{P}^k W_3} \frac{\delta}{c M^{3kn} \| R \|_{\mathbb{P}^k W_3}} \alpha.
\]

Since the mass of quasi-potential is uniformly bounded, the third integral is uniformly bounded. From Lemma 6.8, the first integral can be approximated by

\[
\int U_{S_0} \wedge dd^c(\chi U_{\Lambda^n(\theta)})) \geq -\delta^{-2k^2-4k-2} e^{-\lambda^2} - \lambda^2.
\]

Altogether, if we choose $\delta = 1/(2M^{3k})^{n/\alpha}$, we have

\[
\int_{W_1} S_0 \wedge U_{\Lambda^n(\theta)} \geq -\lambda^2
\]

for all sufficiently large $n$. \qed

**Lemma 6.10.** Let $R \in \mathcal{G}_{k-p+1}$ be a current smooth outside $I_-$. Let $S_0$ be a standard regularization of $S$ for sufficiently small $|\theta|$ as in Lemma 6.8. We have

\[
\int_{W_2} U_{S_0} \wedge \Lambda^n(R) \geq -\lambda^2
\]

for all sufficiently large $n$. The inequality is independent of $\theta$ and $n$.

**Proof.**

\[
\int_{W_2} U_{S_0} \wedge \Lambda^n(R) = \int_{z \in W_2} \int_{\mathbb{P}^k \setminus \{z\}} S_0(\zeta) \wedge K(z, \zeta) \wedge \Lambda^n(R)(z)
\]

\[
= \int_{z \in W_2} \int_{\mathbb{P}^k \setminus \{z\}} S_0(\zeta) \wedge K(z, \zeta) \wedge \Lambda^n(R)(z)
\]

\[
+ \int_{z \in W_2} \int_{\mathbb{P}^k \setminus \{z\}} S_0(\zeta) \wedge K(z, \zeta) \wedge \Lambda^n(R)(z)
\]

From the estimate of $K$ in Proposition 3.2, the second integral is bounded by a constant independent of $\theta$ and $n$. From the negativity of the Green quasi-potential, the first integral is bounded by

\[
\int_{W_1} S_0 \wedge U_{\Lambda^n(R)}.
\]

Hence, by Lemma 6.9, we get the estimate. \qed

**Proof of Proposition 6.4.** Let $R \in \mathcal{G}_{k-p+1}$ be a smooth current. By Lemma 6.2, we can write

\[
\mathcal{U}_{S_n}(R) = d^{-n} \mathcal{U}_S(\Lambda^n(R)).
\]

By the definition, we have $\mathcal{U}_S(\Lambda^n(R)) = \mathcal{U}_S(\Lambda^n(R)) - \mathcal{U}_S(\Lambda^n(R)) - c_S \|\Lambda^n(R)\|$. Since the super-potentials on $\mathbb{P}^k$ are upper semicontinuous on $\mathcal{G}_{k-s+1}$ which is compact, $\mathcal{U}_S(\Lambda^n(R))$ is bounded from above. So, Proposition 5.12 implies that $\limsup_{n \to 0} d^{-n} \mathcal{U}_S(\Lambda^n(R)) \leq 0$. So, we only consider the estimate of $d^{-n} \mathcal{U}_S(\Lambda^n(R))$ from below.

We consider $\mathcal{U}_S(\Lambda^n(R))$. From the definition of the super-potential, we have

\[
\mathcal{U}_S(\Lambda^n(R)) = \lim_{\theta \to 0} \mathcal{U}_{S_0}(\Lambda^n(R)).
\]
Hence, we estimate \( \mathcal{U}_{S_\theta}(\Lambda^n(R)) \) for \( \theta \in \mathbb{C} \) with sufficiently small \( |\theta| \).

In the rest of the proof, the inequalities \( \lesssim, \gtrsim \) are up to a constant multiple independent of \( \theta \) and \( n \).

Let \( \varepsilon_n > 0 \) be a sufficiently small positive number to be determined later and \( U_{\cdot} \) denotes the Green quasi-potential of a given current with respect to a fixed Green quasi-potential kernel in Proposition 3.2. Since \( S_\theta \) is smooth and has the same mean and mass as \( S \) does, we can write

\[
\mathcal{U}_{S_\theta}(\Lambda^n(R)) = \int U_{S_\theta} \wedge \Lambda^n(R) - m_S \|\Lambda^n(R)\|
\]

From the negativity of the Green quasi-potential, we have

\[
\mathcal{U}_{S_\theta}(\Lambda^n(R)) \gtrsim \int U_{S_\theta} \wedge (\Lambda^n(R) - (\Lambda^n(R))_{\varepsilon_n}) + \int U_{S_\theta} \wedge (\Lambda^n(R))_{\varepsilon_n} - m_S \|\Lambda^n(R)\|
\]

for all sufficiently large \( n \).

We estimate the first integral. From Lemma 6.10, we have

\[
\int_{W_2} U_{S_\theta} \wedge \Lambda^n(R) \gtrsim -\lambda^n
\]

for the second integral, we will use the fact that the mass of the Green quasi-potential is uniformly bounded. Proposition 5.3 implies that \( \Lambda^n(R) \) is smooth in \( \mathbb{P}^k \setminus W_3 \), and from \( \|Df_{-}\|_{\mathbb{P}^k \setminus W_3} < M \) and \( f_+(W_3) \subset W_3 \), we have \( \|\Lambda^n(R)\|_{\mathbb{P}^k \setminus W_3} \lesssim M^{3kn} \). So, we have

\[
\|\Lambda^n(R) - (\Lambda^n(R))_{\varepsilon_n}\|_{\mathbb{P}^k \setminus W_3} \lesssim M^{-3kn}\varepsilon_n
\]

and since the mass of \( U_{S_\theta} \) is uniformly bounded, the second integral is

\[
\int_{\mathbb{P}^k \setminus W_2} U_{S_\theta} \wedge (\Lambda^n(R) - (\Lambda^n(R))_{\varepsilon_n}) \gtrsim -M^{-3kn}\varepsilon_n
\]

For the last integral, thanks to Lemma 3.2.10 in [8] and Proposition 2.1.6 in [8], we have

\[
\int U_{S_\theta} \wedge (\Lambda^n(R))_{\varepsilon_n} \gtrsim \log \varepsilon_n.
\]

If we choose \( \varepsilon_n := \min\{1/2, M^3/2\}^{kn} \), then we have \( \mathcal{U}_S(\Lambda^n(R)) \gtrsim -\lambda^n \) for all sufficiently large \( n \). Since \( 1 < \lambda < d \), we have \( \liminf_{n \rightarrow 0} d^{-n} \mathcal{U}_S(\Lambda^n(R)) \geq 0 \). Together with Proposition 5.12 again, we get \( \liminf_{n \rightarrow 0} d^{-n} \mathcal{U}_S(\Lambda^n(R)) \geq 0 \)

**Proof of Theorem 1.2** Let \( \varphi \) be a smooth test form of bidegree \((k - p, k - p)\). Since \( \varphi \) is smooth, there exists \( m_\varphi > 0 \) such that \( m_\varphi \omega^{k-s+1} + dd^c \varphi \geq 0 \). Then, we have

\[
\langle S_n - T^p, \varphi \rangle = \langle dd^c V S_n, \varphi \rangle = \langle V S_n, dd^c \varphi \rangle
\]

\[
= \langle V S_n, m_\varphi \omega^{k-p+1} + dd^c \varphi \rangle - \langle V S_n, m_\varphi \omega^{k-p+1} \rangle
\]

If we apply Proposition 6.4 to both terms, we see the desired convergence.
Proof of Theorem 1.3. Let $0 < p \leq s$. Let $H$ be an analytic subset of pure dimension $k - p$ and suppose that $H \cap I_{\infty}^c = \emptyset$. Let $c_H$ denote the degree of $H$. Let $W$ be a subset of the trapping neighborhood $U$ of $I_{\infty}$ such that $H \cap W = \emptyset$. Then, since $U$ is a trapping neighborhood, there exists an $N$ such that $I_{\infty}^c \subseteq f_N^+(W) \subseteq W$. Indeed, there exists $N$ such that $I_{\infty}^c \subseteq f_N^+(U) \subseteq W$. Note that $I_{\infty}^c$, $I_{\infty}$, and $T^p$ remain the same if we replace $f$ by $f^N$.

In the proof of Proposition 6.4, by applying the same lemmas and propositions to $(f_+)^N$, $\lambda^N$ and $\Lambda^j(R)$ in place of $f_+$, $\lambda$ and $R$, we obtain $\mathcal{W}_{c_H}^{-1}(H)((\Lambda^N)^n(\Lambda^j(R))) \geq -(\Lambda^N)^n$ for each $j = 0, \ldots, N - 1$. By considering a subsequence, Proposition 5.12 implies that $d^{-Nn}\mathcal{W}_{T^n}((\Lambda^N)^n(\Lambda^j(R))) \to 0$ as $n \to \infty$ for each $j = 0, \ldots, N - 1$. Applying the same argument in Proposition 6.4, we see that for a given smooth $R \in \mathcal{G}_{k-p+1}$, $\mathcal{Y}_{L, N^k(c_H^{-1}[H])}(\Lambda^j(R)) \to 0$ as $n \to \infty$ holds for each $j = 0, \ldots, N - 1$, which means $d^{-p(Nn+j)}((f_+^{Nn+j})^*(c_H^{-1}[H]))$ converges to $T^p$ for each $j = 0, 1, \ldots, N - 1$. So, the only limit points of $d^{-m}(f_+^m)^*(c_H^{-1}[H])$ is $T^p$, which completes the proof.

Remark 6.11. In our method, one obstacle against measuring the speed of convergence is lack of the speed of convergence in Proposition 5.12. In the case of [6], adapting their notation, the convergence is exponentially fast on $U^+$. In this case, simply Hölder continuity of the super-potential of $T^p$ replaces Proposition 5.12.

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