Small flow-time representation of fermion bilinear operators

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Fermion bilinear operators of mass dimension 3, such as the axial-vector and vector currents, the pseudo-scalar and scalar densities, whose normalizations are fixed by Ward–Takahashi (WT) relations, are related to small flow-time behavior of composite operators of fermion fields evolved by Lüscher’s flow equation. The representations can be useful in lattice numerical simulations, as recently demonstrated by the WHOT QCD collaboration for the chiral condensation of the $N_f = 2 + 1$ quantum chromodynamics (QCD) at finite temperature.

Subject Index \quad B01, B31, B32, B38
1. Introduction

It is a remarkable fact [1, 2] that, under the conventional parameter renormalization in gauge theory, a simple one-parameter evolution of bare gauge and bare fermion fields by flow equations in Refs. [2–4] renders composite operators of these fields renormalized. See also Ref. [5] for an earlier study. Because of this renormalization property, the flow can be employed in lattice gauge theory for instance to set a mass scale, to define a non-perturbative running coupling, and to define the topological charge. Refs. [6, 7] are reviews and Ref. [8] provides a pedestrian proof of the above renormalization property and an extensive list of recent related works.

Although any composite operator of fields evolved by the flow (which we term “flowed fields” in what follows) is always a renormalized one, how to relate it to desired renormalized quantities in the original gauge theory is a different issue. One possible rather versatile method to do this is the small flow-time expansion [1]. The flow is parametrized by the flow-time \( t > 0 \) and the small flow-time expansion expresses for \( t \sim 0 \) a composite operator of flowed fields by an asymptotic series of local operators. Since the flow-time \( t \) possesses the mass dimension \( -2 \), the expansion is a series of local operators of increasing mass dimensions.

On the basis of this small flow-time expansion, representations of renormalized Noether currents, such as the energy–momentum tensor [3, 10] and the flavor non-singlet axial vector current [11], in terms of flowed fields have been constructed. Since composite operators of flowed fields are renormalized ones and independent of the regularization, those representations are thought to be useful in lattice gauge theory. The validity of the representation of the energy–momentum tensor has been numerically examined in Refs. [12–17] with promising results.

In this paper, as further extension of the above idea, we construct a small flow-time representation of various fermion bilinear operators of mass dimension 3. These include the axial-vector and vector currents, the pseudo-scalar and scalar densities, both flavor non-singlet and flavor singlet ones. Our basic principle is that the normalizations of the operators are fixed by Ward–Takahashi (WT) relations associated with the relevant symmetry. Although it is not a priori clear whether those small flow-time representations are practically useful, a preliminary numerical study of the chiral condensation in the \( N_f = 2 + 1 \) QCD at finite temperature [17] on the basis of the small flow-time representation suggests that our representations are practically useful; this observation encouraged us to publish the present work.

\(^1\) Up to multiplicative renormalization factors determined by the number of fermion fields contained in a composite operator. The use of “ringed fermion variables” in Eqs. (2.11) and (2.12) avoids this renormalization.
In this paper, we consider the vector-like gauge theory with $N_f$ flavor fermions. The classical action is given by

$$S = \frac{1}{4g_0^2} \int d^Dx \, F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \int d^Dx \, \bar{\psi}(x)(\not{D} + M_0)\psi(x),$$

(1.3)

where $g_0$ and $M_0$ are the bare gauge coupling and the bare mass matrix, respectively; the flavor index is almost always suppressed and we assume that the mass matrix $M_0$ is flavor-diagonal. The field strength is defined by

$$F_{\mu\nu}^a(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + [A^a_\mu(x), A^a_\nu(x)],$$

(1.4)

from the gauge potential $A^a_\mu(x) = A_\mu^a(x) T^a$ and $F_{\mu\nu}^a(x) = F_{\mu\nu}^a T^a$. The covariant derivative on the fermion is given by

$$D_\mu = \partial_\mu + A^a_\mu.$$  

(1.5)

Our flow equations are identical to those of Refs. [2–4]. The flow of the gauge field along the flow-time $t$ is defined by

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x), \quad B_\mu(t = 0, x) = A_\mu(x),$$

(1.6)

where

$$D_\mu = \partial_\mu + [B_\mu,] \quad \text{and} \quad G_{\mu\nu}(t,x) \equiv \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)],$$

(1.7)

and the flow for the fermion fields is defined by

$$\partial_t \chi(t,x) = D^2 \chi(t,x), \quad \chi(t = 0, x) = \psi(x),$$

(1.8)

$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x) \tilde{D}^2, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x),$$

(1.9)

where the covariant derivatives on the fermion fields are defined by

$$D_\mu = \partial_\mu + B_\mu, \quad \tilde{D}_\mu \equiv \partial_\mu - B_\mu.$$  

(1.10)

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Here is our notation: Without noting otherwise, repeated indices are understood to be summed over. The generators $T^a$ of the gauge group $G$ are anti-hermitian and the structure constants are defined by $[T^a, T^b] = f^{abc} T^c$. Quadratic Casimirs are defined by $f^{acd} f^{bde} = C_2(G) \delta^{ab}$ and, for a gauge representation $R$, $\text{tr}_R(T^a T^b) = -T(R) \delta^{ab}$ and $T^a T^a = -C_2(R) 1$. We also denote $\text{tr}_R(1) = \dim(R)$. For the fundamental $N$ representation of $SU(N)$ for which $\dim(N) = N$, our choice is

$$C_2(SU(N)) = N, \quad T(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}.$$  

(1.1)

Our $\gamma$ matrices are all Hermitian and for the trace over the spinor index we set $\text{tr}(1) = 4$ for any spacetime dimension $D$. The chiral matrix is defined by $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ for any $D$ and thus

$$\text{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = \begin{cases} 4 \epsilon_{\mu\nu\rho\sigma}, & \mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise,} \end{cases}$$

(1.2)

where the totally anti-symmetric tensor is normalized as $\epsilon_{0123} = 1$.  

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2. Small flow-time representation of fermion bilinear operators

2.1. One-loop coefficients in the small flow-time expansion

Because of the gauge and flavor symmetries, the small flow-time expansion \[1\] of a dimension 3 fermion bilinear operator assumes the form

\[
\bar{\chi}(t, x)\mathcal{M}\chi(t, x) \xrightarrow{t \to 0} \zeta_0(t)\mathbb{1} + \zeta_1(t)\bar{\psi}(x)\mathcal{M}\psi(x) + O(t),
\]

where \(\mathcal{M}\) is a certain constant matrix in the spinor and flavor spaces and \(\mathbb{1}\) denotes the identity operator. The coefficients in Eq. (2.1) are expanded in the loop expansion as

\[
\zeta_0(t) = \sum_{\ell=1}^{\infty} \zeta_0^{(\ell)}(t), \quad \zeta_1(t) = 1 + \sum_{\ell=1}^{\infty} \zeta_1^{(\ell)}(t).
\]

The computation of the one-loop coefficients \(\zeta_0^{(1)}\) and \(\zeta_1^{(1)}\) is straightforward; see Refs. [10, 11, 18] for actual one-loop calculations. Assuming the dimensional regularization with \(D = 4 - 2\epsilon\), we find

\[
\zeta_0^{(1)}(t) = -\frac{1}{(4\pi)^2} \dim(R) \text{tr} \left( \mathcal{M} \mathcal{M} \left\{ \frac{1}{2t} + M^2 \left[ \gamma + \ln(2M^2t) \right] + O(t) \right\} \right),
\]

where \(M = M_0[1 + O(g^2)]\) is the renormalized mass matrix, \(\gamma\) denotes the Euler constant, and

\[
\zeta_1^{(1)}(t) = \frac{g^2}{(4\pi)^2} C_2(R) \begin{cases} 
(\text{-6}) & \left[ \frac{1}{\epsilon} + \ln(8\pi\mu^2t) \right] - 2 & \text{when } M \propto 1 \\
(\text{-3}) & \left[ \frac{1}{\epsilon} + \ln(8\pi\mu^2t) \right] + \frac{1}{2} & \text{when } M \propto \gamma_\mu \\
(\text{-2}) & \left[ \frac{1}{\epsilon} + \ln(8\pi\mu^2t) \right] & \text{when } M \propto \gamma_\mu\gamma_\nu \\
(\text{-3}) & \left[ \frac{1}{\epsilon} + \ln(8\pi\mu^2t) \right] - \frac{7}{2} & \text{when } M \propto \gamma_\mu\gamma_5 \\
(\text{-6}) & \left[ \frac{1}{\epsilon} + \ln(8\pi\mu^2t) \right] - 10 & \text{when } M \propto \gamma_5 \end{cases} + O(t),
\]

where we have used the gauge coupling renormalization \(g_0^2 = \mu^{2\epsilon}g^2[1 + O(g^2)]\). This is obtained by evaluating the flow Feynman diagrams depicted in Fig. 1.

**Fig. 1** The flow Feynman diagrams which contribute to Eq. (2.4). The blob denotes the composite operator in the left-hand side of Eq. (2.1). The convention here is identical to that in Refs. [10, 11]: The single straight line and the doubled straight line denote the fermion propagator and the fermion heat kernel, respectively; the wavy line denotes the gauge propagator. The filled circle denotes the vertex in the original gauge theory, while the open circle denotes the vertex originating from non-linear terms in the fermion flow equations.
2.2. Flavor non-singlet axial-vector current and the pseudo-scalar density

Our first example is the flavor non-singlet axial-vector current \( j_{5\mu}^A(x) \) and the flavor non-singlet pseudo-scalar density. These have been already considered in Ref. [11] but we recapitulate the basic reasoning and the results here for completeness and for later use.

The correctly normalized flavor non-singlet axial-vector current \( j_{5\mu}^A(x) \) and the flavor non-singlet renormalized pseudo-scalar density are characterized by the following chiral WT relation (the partially conserved axial-vector current (PCAC) relation):

\[
\left\langle \left[ \partial_\mu j_{5\mu}^A(x) - \{ \bar{\psi} \gamma_5 \{ t^A, M \} \psi \}_R(x) \right] \prod_i \psi(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu_k}(w_k) \right\rangle
\]

\[
= - \sum l \delta(x - y_l) \left\langle \prod_i \psi(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu_k}(w_k) \right\rangle_{\psi(y_i) \rightarrow \gamma_5 t^A \psi(y_i)}
- \sum l \delta(x - z_l) \left\langle \prod_i \psi(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu_k}(w_k) \right\rangle_{\psi(z_j) \rightarrow \gamma_5 \gamma_5 t^A },
\]

where \( t^A \) denotes the (anti-hermitian) generator of the flavor group. An analysis of the one-loop diagrams in Fig. 2 shows that, under the dimensional regularization, such operators are given by (see §13.5 of Ref. [19])

\[
j_{5\mu}^A(x) = \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(R)(-4) + O(g^4) \right] \bar{\psi}(x) \gamma_5 t^A \psi(x),
\]

\[
\left\{ \bar{\psi} \gamma_5 \{ t^A, M \} \psi \}_R(x) = \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(R)(-8) + O(g^4) \right] \bar{\psi}(x) \gamma_5 \{ t^A, M_0 \} \psi(x).
\]

We thus express bare operators on the right-hand sides of these expressions by composite operators of flowed fermion fields. This can be carried out by inverting the small flow-time expansion (2.1) with Eq. (2.2) for \( t \) small. Using Eq. (2.4), we have

\[
j_{5\mu}^A(x) = \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) 3 \left[ \frac{1}{\epsilon} + \ln(8\pi \mu^2 t) - \frac{1}{6} \right] + O(g^4) \right\} \bar{\chi}(t, x) \gamma_5 t^A \chi(t, x) + O(t),
\]

\[
\left\{ \bar{\psi} \gamma_5 \{ t^A, M \} \psi \}_R(x) = \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) 6 \left[ \frac{1}{\epsilon} + \ln(8\pi \mu^2 t) + \frac{1}{3} \right] + O(g^4) \right\} \bar{\chi}(t, x) \gamma_5 \{ t^A, M_0 \} \chi(t, x) + O(t).
\]

Fig. 2  The one-loop diagrams which contribute to the chiral WT identity with two external fermion lines. The blob stands for a composite operator.

\footnote{In writing this expression, we assume that the gauge fixing is made.}
We further rewrite the resulting expressions in terms of the renormalized mass in the minimal subtraction (MS) scheme,

\[ M_0 = \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(R)(-3) \frac{1}{\epsilon} + O(g^4) \right] M, \] (2.10)

and the “ringed variables” \[10\]

\[ \hat{\chi}_r(t, x) = \left[ \frac{-2 \text{dim}(R)}{(4\pi)^2 t^2 \left\langle \bar{\nabla} \chi_r(t, x) \right\rangle} \right] \chi_r(t, x), \] (2.11)

\[ \ddot{\chi}_r(t, x) = \left[ \frac{-2 \text{dim}(R)}{(4\pi)^2 t^2 \left\langle \bar{\nabla} \chi_r(t, x) \right\rangle} \right] \ddot{\chi}_r(t, x), \] (2.12)

where \( \bar{D}_\mu = D_\mu - \hat{D}_\mu \) and it is understood that the sum over the flavor index \( r \) is not taken, by using \[10\]

\[ \frac{-2 \text{dim}(R)}{(4\pi)^2 t^2 \left\langle \bar{\nabla} \chi_r(t, x) \right\rangle} = \frac{1}{(8\pi t)^2} \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) \left[ 3 \frac{1}{\epsilon} + 3 \ln(8\pi \mu^2 t) - \ln(432) \right] + O(t) + O(g^4) \right\}. \] (2.13)

In this way, we have

\[ j_{5\mu}^A(x) = \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] + O(g^4) \right\} \ddot{\chi}(t, x) \gamma_5 \gamma_5 t^A \ddot{\chi}(t, x) + O(t), \] (2.14)

\[ \{ \bar{\psi} \gamma_5 \{ t^A, M \} \psi \} R(x) = \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) \left[ 3 \ln(8\pi \mu^2 t) + 2 + \ln(432) \right] + O(g^4) \right\} \ddot{\chi}(t, x) \gamma_5 \{ t^A, M \} \ddot{\chi}(t, x) + O(t). \] (2.15)

The ringed variables, \( \ddot{\chi}(t, x) \) and \( \dddot{\chi}(t, x) \), are free from the wave function renormalization of flowed fermion fields \[2\] and the above expressions \[2.11\] and \[2.15\] are manifestly finite as they should be.

Finally, a renormalization group argument \[10, 11\] says that when renormalized parameters in these expressions are replaced by running parameters, \( g \to \bar{g}(\mu) \) and \( M \to \bar{M}(\mu) \), then the expressions are independent of the renormalization scale \( \mu \). By setting \( \mu = 1/\sqrt{8t} \), perturbation theory is justified in the limit \( t \to 0 \) owing to the asymptotic freedom. Thus, we have

\[ j_{5\mu}^A(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] \right\} \ddot{\chi}(t, x) \gamma_5 \gamma_5 t^A \ddot{\chi}(t, x), \] (2.16)

\[ \{ \bar{\psi} \gamma_5 \{ t^A, \bar{M} \} \psi \} R(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ 3 \ln \pi + 2 + \ln(432) \right] \right\} \ddot{\chi}(t, x) \gamma_5 \{ t^A, \bar{M}(1/\sqrt{8t}) \} \ddot{\chi}(t, x). \] (2.17)

These are expressions obtained in Ref. \[11\]. If one adopts the modified MS (\( \overline{\text{MS}} \)) scheme instead of the MS scheme, the factor \( \ln \pi \) in the last expression is replaced by \( \gamma - 2 \ln 2 \).
2.3. Flavor non-singlet vector current

Our next example is the flavor non-singlet vector current. The flavor non-singlet vector current $j^A_\mu(x)$ is characterized by the WT relation,

\[
\left\langle \partial_\mu j^A_\mu(x) \prod_i \bar{\psi}(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu k}(w_k) \right\rangle
\]

\[
= - \sum_l \delta(x - y_l) \left\langle \prod_i \bar{\psi}(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu k}(w_k) \right\rangle \bigg|_{\psi(y_l) \to t^A \bar{\psi}(y_l)}
\]

\[
- \sum_l \delta(x - z_l) \left\langle \prod_i \bar{\psi}(y_i) \prod_j \bar{\psi}(z_j) \prod_k A^{a_k}_{\mu k}(w_k) \right\rangle \bigg|_{\bar{\psi}(z_l) \to -\bar{\psi}(z_l) t^A}.
\] (2.18)

The dimensional regularization preserves the corresponding $SU(N_f)_V$ symmetry and thus the correctly normalized current is simply given by

\[
j^A_\mu(x) = \bar{\psi}(x) \gamma_\mu t^A \psi(x),
\] (2.19)

assuming the dimensional regularization.

Thus from Eqs. (2.1), (2.2), (2.4), (2.11), (2.12), and (2.13), in conjunction of the renormalization group argument, we have

\[
j^A_\mu(x) = \lim_{t \to 0} \left\{ 1 + \tilde{g}(1/\sqrt{8})^2 C_2(R) \left[ \frac{1}{2} - \ln(432) \right] \right\} \tilde{\chi}(t, x) \gamma_\mu t^A \chi(t, x).
\] (2.20)

Now, there is another interesting method to obtain the expression (2.20). We may require that the flavor non-singlet axial-vector current $j^A_5(x)$ and the flavor non-singlet vector current $j^A_\mu(x)$ form a current algebra. We can use this requirement as the definition of the vector current. That is, we would require that the vector current of the form

\[
\{ \bar{\psi} \gamma_\mu [t^A, t^B] \psi \}_R(x)
\] (2.21)

is given by the infinitesimal chiral transformation

\[
\psi(x) \to (1 + \alpha \gamma_5 t^B) \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)(1 + \alpha \gamma_5 t^B),
\] (2.22)

of the axial-vector current $j^A_{5\mu}(x)$ (2.16).

The chiral transformation (2.22), through the initial conditions of the flow equations (1.8) and (1.9), induces the chiral transformation on the flowed fermions fields,

\[
\chi(t, x) \to (1 + \alpha \gamma_5 t^B) \chi(t, x), \quad \bar{\chi}(t, x) \to \bar{\chi}(t, x)(1 + \alpha \gamma_5 t^B).
\] (2.23)

We now claim that the composite operator of the flowed fields on the right-hand side of Eq. (2.16) transforms in a naive way under the chiral transformation (2.23). That is, we claim that no nontrivial renormalization is required under the transformation. That a composite operator of the flowed fermion fields, such as $\bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x)$, transforms in a naive way under the flavor non-singlet chiral transformation is explained in detail in Sect. 4.1 and Sect. 4.2 of Ref. [2]; see Eqs. (4.14) and (4.15) there. One possible explanation is that the meaning of a composite operator such as $\bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x)$ is independent of the regularization. Then one may adopt regularization (such as the overlap fermion) that manifestly
preserves the flavor non-singlet chiral symmetry. Then the transformation law of such a composite operator can be derived in a naive way.

Applying the chiral transformation (2.22) to both sides of Eq. (2.16) thus would yield
\[
\{ \overline{\psi} \gamma_{\mu} [t^A, t^B] \psi \}_R (x) = \lim_{t \to 0} \left\{ 1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] \right\} \tilde{\chi}(t, x) \gamma_{\mu} [t^A, t^B] \tilde{\chi}(t, x). \tag{2.24}
\]
By setting \([t^A, t^B] \to t^A\), this coincides with Eq. (2.20) which was obtained by a direct small flow-time expansion. This coincidence supports the above argument on the basis of the naive transformation law of composite operators of flowed fields.

### 2.4. Flavor singlet vector current

For the flavor singlet vector current (i.e., the fermion number current)
\[
j_{\mu}(x) = \{ \overline{\psi} \gamma_{\mu} \psi \}_R (x), \tag{2.25}
\]
the argument in the second half of the preceding subsection does not apply, because this current is not related to the flavor non-singlet axial-vector current (which fulfills the WT relation without anomaly) by the chiral transformation. Still, since the expression \(j_{\mu}(x) = \overline{\psi}(x) \gamma_{\mu} \psi(x)\) with the dimensional regularization provides the correctly normalized current, we can directly use the small flow-time expansion. Again from Eqs. (2.1), (2.2), (2.4), etc., we have
\[
j_{\mu}(x) = \lim_{t \to 0} \left\{ 1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] \right\} \tilde{\chi}(t, x) \gamma_{\mu} \tilde{\chi}(t, x). \tag{2.26}
\]

### 2.5. Scalar density

As for the vector current (2.24), we may obtain the small flow-time representation of the scalar density of the form
\[
\{ \overline{\psi} \{ \{ t^A, M \}, t^B \} \psi \}_R (x) \tag{2.27}
\]
by requiring it to be obtained by the chiral transformation (2.22) of the correctly normalized pseudo-scalar density (2.17). Apart from the chiral limit \(M \to 0\), however, there exists a subtlety in the small flow-time limit\(^4\) and our argument should be somewhat modified as follows.

We assume that the composite operator of the flowed fermion fields in Eq. (2.15) transforms in a naive way under the chiral transformation. This assumption is the same as a previous subsection and, in terms of the integrated form of the chiral WT relation (2.5), this is expressed as
\[
\int_{\partial R} d^{D-1}y_{\mu} \left\langle j_{\mu}^{\{ t^A, M \}} (y) \tilde{\chi}(t, x) \gamma_5 \{ t^A, M \} \tilde{\chi}(t, x) \cdots \right\rangle
\]
\[
- \int_{R} d^Dy \left\langle \{ \overline{\psi} \gamma_5 \{ t^B, M \} \psi \}_R (y) \tilde{\chi}(t, x) \gamma_5 \{ t^A, M \} \tilde{\chi}(t, x) \cdots \right\rangle
\]
\[
= - \left\langle \tilde{\chi}(t, x) \{ \{ t^A, M \}, t^B \} \tilde{\chi}(t, x) \cdots \right\rangle, \tag{2.28}
\]
\(^4\)We would like to thank Tetsuya Onogi and Hidenori Fukaya for discussion regarding this subtlety.
where the bounded integration region $\mathcal{R}$ contains the point $x$. $\partial \mathcal{R}$ is the boundary of the region $\mathcal{R}$ and $dy^\mu$ is the area element on $\partial \mathcal{R}$. The abbreviated terms ($\cdots$) stand for other operators contained in the correlation function. The small flow-time expansion of the operator on the right-hand side yields, from Eqs. (2.1), (2.2), and (2.3),

\[
\tilde{\chi}(t, x)\{\{t^A, M\}, t^B\} \tilde{\chi}(t, x) \\
\quad \tilde{\sim} 0 \left[-\frac{1}{(4\pi)^2} 4 \text{dim}(R) \frac{1}{2t} \text{tr}(\{\{t^A, M\}, t^B\} M) + O(t^0) + O(g^2)\right] 1 \\
\quad + [1 + O(g^2)] \bar{\psi}(x)\{\{t^A, M\}, t^B\} \psi(x) + O(t). \quad (2.29)
\]

The first term on the right-hand side which is proportional to the identity operator $1$ behaves as $\sim 1/t$ for $t \to 0$. In Eq. (2.28), although the small flow-time expansion of the pseudo-scalar density $\tilde{\chi}(t, x)\gamma_5\{t^A, M\} \chi(t, x)$ on the left-hand side starts from an $O(t^0)$ term as Eq. (2.29) shows, the equal point ($y = x$) product in the second line makes the right-hand side more singular as $t \to 0$.

Thus we cannot simply take the $t \to 0$ limit in Eq. (2.29) because of the presence of the identity operator. To avoid this, we define the renormalized scalar density by the small flow-time limit of the chiral transformation of the pseudo-scalar density (2.15) after the vacuum expectation value subtracted. That is, we set

\[
\{\bar{\psi}\{\{t^A, M\}, t^B\}\psi\}_R(x) \\
\quad \equiv \lim_{t \to 0} \left\{1 + \frac{g(1/\sqrt{8t})}{(4\pi)^2} C_2(R) [3 \ln \pi + 2 + \ln(432)]\right\} \\
\quad \times \left[\tilde{\chi}(t, x)\{\{t^A, \bar{M}(1/\sqrt{8t})\}, t^B\} \tilde{\chi}(t, x) - \langle \tilde{\chi}(t, x)\{\{t^A, \bar{M}(1/\sqrt{8t})\}, t^B\} \tilde{\chi}(t, x) \rangle\right]. \quad (2.30)
\]

One might think that this subtraction, which makes the chiral condensation in the vacuum, $\langle \{\bar{\psi}\{t^A, M\}, t^B\}\psi\rangle_R(x)$, always vanishing is rather ad hoc. We should note, however, that the flavor singlet part of the scalar density possesses the quantum number identical to the identity operator and its $c$-number component (i.e., the mixing with the identity operator) is ambiguous without supplementing a certain prescription (except for the chiral limit $M \to 0$).

It appears that the definition (2.30) is the most natural one for massive fermions from this perspective. For the chiral limit $M \to 0$, for which the mixing with the identity operator in Eq. (2.29) vanishes for dimensional reason, we may adopt another definition of the scalar density (setting $M = mT$ with a constant matrix $T$),

\[
\lim_{m \to 0} \left\{\bar{\psi}\{\{t^A, T\}, t^B\}\psi\right\}_R(x) \\
\quad \equiv \lim_{t \to 0} \lim_{m \to 0} \left\{1 + \frac{g(1/\sqrt{8t})}{(4\pi)^2} C_2(R) [3 \ln \pi + 2 + \ln(432)]\right\} \\
\quad \times \left[\tilde{\chi}(t, x)\{\{t^A, \bar{M}(1/\sqrt{8t})\}, t^B\} \tilde{\chi}(t, x)\right], \quad (2.31)
\]

which might be employed to compute the chiral condensation in the vacuum.
For the $N_f = 2 + 1$ quantum chromodynamics (QCD), setting

$$M = \begin{pmatrix} m_{ud} & 0 & 0 \\ 0 & m_{ud} & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad t^A = t^B = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(2.32)

Eq. (2.30) gives (since $C_2(R) = 4/3$ for the fundamental representation of $G = SU(3)$)

$$\{ \bar{\psi}_u \psi_u \}_R(x) + \{ \bar{\psi}_d \psi_d \}_R(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} \frac{4}{3} [3 \ln \pi + 2 + \ln(432)] \right\} \frac{\bar{m}_{ud} (1/\sqrt{8t})}{m_{ud}} \times \left[ \tilde{\chi}_u (t, x) \chi_u (t, x) + \tilde{\chi}_d (t, x) \chi_d (t, x) - \text{VEV} \right].$$

(2.33)

Similarly, by setting

$$t^A = t^B = \frac{i}{2} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

(2.34)

and using Eq. (2.33), we have

$$\{ \bar{\psi}_s \psi_s \}_R(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} \frac{4}{3} [3 \ln \pi + 2 + \ln(432)] \right\} \frac{\bar{m}_{s} (1/\sqrt{8t})}{m_{s}} \times \left[ \tilde{\chi}_s (t, x) \chi_s (t, x) - \text{VEV} \right].$$

(2.35)

The representations (2.33) and (2.35) provide a possible method to compute the chiral condensation at finite temperature, as demonstrated recently by the WHOT QCD collaboration for the $N_f = 2 + 1$ QCD [17].

3. Flavor singlet axial-vector current and the topological charge density

The flavor singlet axial-vector current $j_{5\mu R}(x)$ requires a special treatment because its renormalization is ambiguous due to the axial (or chiral) anomaly. The chiral WT relation associated with the flavor-singlet chiral symmetry is

$$\left\langle \left[ \partial_\mu j_{5\mu R}(x) - 2 \left\{ \bar{\psi} \gamma_5 M \psi \right\}_R(x) + 4N_f T(R) q_R(x) \right] \prod_i \bar{\psi}(y_i) \prod_j \psi(z_j) \prod_k A_{\mu_k}^a(w_k) \right\rangle$$

$$= - \sum_l \delta(x - y_l) \left\langle \prod_i \bar{\psi}(y_i) \prod_j \psi(z_j) \prod_k A_{\mu_k}^a(w_k) \right|_{\psi(y_i) \to \gamma_5 \psi(y_i)}$$

$$- \sum_l \delta(x - z_l) \left\langle \prod_i \bar{\psi}(y_i) \prod_j \psi(z_j) \prod_k A_{\mu_k}^a(w_k) \right|_{\bar{\psi}(z_l) \to \bar{\psi}(z_l) \gamma_5},$$

(3.1)

where $q_R(x)$ stands for the topological charge density induced by the axial anomaly.
An analysis of one-loop diagrams, Figs. 2–4, by employing the dimensional regularization shows that the following combinations

\[ j_{5\mu R}(x) = \left[1 + \frac{g^2}{(4\pi)^2} C_2(R)(-4) + O(g^4)\right] \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x), \] (3.2)

\[ \{\bar{\psi} \gamma_5 M \psi\}_R(x) = \left[1 + \frac{g^2}{(4\pi)^2} C_2(R)(-8) + O(g^4)\right] \bar{\psi}(x) \gamma_5 M_0 \psi(x), \] (3.3)

\[ q_R(x) = \mu^{D-4} \left[1 + O(g^4)\right] \frac{1}{64\pi^2} \epsilon_{\mu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) + \left[\frac{g^4}{(4\pi)^4} C_2(R) \left(-3\frac{1}{\epsilon} + \text{finite}\right) + O(g^6)\right] \partial_\mu [\bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x)], \] (3.4)

are finite and fulfill the WT relation (3.1). For Eq. (3.1) to be meaningful for any \( D \), we have required that \( q_R \) in Eq. (3.4) is of mass dimension \( D \) and supplemented the factor \( \mu^{D-4} \) by using the renormalization scale. Equation (3.4) shows that the flavor-singlet axial vector current requires infinite renormalization from the two-loop order through the axial anomaly \([20]\), a property quite different from the flavor non-singlet axial vector current.

![Fig. 3](image1.png)

**Fig. 3** One loop diagrams which contribute to the renormalization of the topological charge density which is denoted by the blob.

![Fig. 4](image2.png)

**Fig. 4** One loop diagram which gives the operator mixing between the topological charge density and the axial vector current.

As far as the finiteness and the WT identity \([3.1]\) are concerned, however, renormalized operators are to a large extent arbitrary. That is, the redefinition of renormalized operators,

\[ j_{5\mu R}(x) \rightarrow [1 - 4N_f T(R)z] j_{5\mu R}(x), \quad q_R(x) \rightarrow q_R(x) + z \partial_\mu j_{5\mu R}(x), \] (3.5)

where \( z \) is a finite constant of the order \( O(g^2) \), does not affect the finiteness and the WT relation (3.1). Thus, in what follows, we will consider only the combinations which are
independent of this ambiguity. From Eqs. (3.2)–(3.4), they are given by
\[
\partial_\mu j_{5\mu R}(x) + 4N_fT(R)q_R(x)
= \left[ 1 + \frac{g^2}{(4\pi)^2}C_2(R)(-4) + \frac{g^4}{(4\pi)^4}C_2(R)N_fT(R)\left( -12\frac{1}{\epsilon} + \text{finite} \right) + O(g^6) \right]
\times \partial_\mu [\bar{\psi}(x)\gamma_\mu \gamma_5\psi(x)]
+ N_fT(R)\mu^{D-4} \left[ 1 + O(g^4) \right] \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{a}_{\mu\nu}(x) F^{a}_{\rho\sigma}(x),
\]
\[
\{\bar{\psi}\gamma_5 M\psi\}_R(x) = \left[ 1 + \frac{g^2}{(4\pi)^2}C_2(R)(-8) + O(g^4) \right] \bar{\psi}(x)\gamma_5 M_0 \psi(x).
\] (3.6)

We now express bare operators on the right-hand sides of these expressions by the small flow-time expansion of composite operators of flowed fields.

Let us start with the topological charge density \( \epsilon_{\mu\nu\rho\sigma} F^{a}_{\mu\nu}(x) F^{a}_{\rho\sigma}(x) \). The one-loop computation of this expansion in the pure Yang–Mills theory is given in Appendix B of Ref. 18. By comparing the right-hand side of Eq. (3.8) and Eq. (3.4), we find that this is a renormalized finite quantity. This is consistent with the fact that the composite operator of the flowed gauge field on the left-hand side must be a renormalized quantity. Substituting this into Eq. (3.6), we have
\[
\partial_\mu j_{5\mu R}(x) + 4N_fT(R)q_R(x)
= \left[ 1 + \frac{g^2}{(4\pi)^2}C_2(R)(-4) + O(g^4) \right] \partial_\mu [\bar{\psi}(x)\gamma_\mu \gamma_5\psi(x)]
+ N_fT(R)\mu^{D-4} \left[ 1 + O(g^4) \right] \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} G^{a}_{\mu\nu}(t,x) G^{a}_{\rho\sigma}(t,x) + O(t).
\] (3.9)

For the axial-vector current \( \bar{\psi}(x)\gamma_\mu \gamma_5\psi(x) \) on the right-hand side, the computation is identical to the one for the flavor non-singlet axial vector current in Eq. (2.14) except for the absence of the group generator \( t^A \). Thus,
\[
\partial_\mu j_{5\mu R}(x) + 4N_fT(R)q_R(x)
= \left\{ 1 + \frac{g^2}{(4\pi)^2}C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] + O(g^4) \right\} \partial_\mu [\bar{\chi}(t,x)\gamma_\mu \gamma_5\chi(t,x)]
+ N_fT(R) \left[ 1 + O(g^4) \right] \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} G^{a}_{\mu\nu}(t,x) G^{a}_{\rho\sigma}(t,x) + O(t),
\] (3.10)

where we have set \( D \rightarrow 4 \).
The computation for the pseudo-scalar density on the right-hand side of Eq. (3.7) is again identical to the one for Eq. (2.15) except for the absence of the group generator $t^A$. We have

$$\left\{ \bar{\psi} \gamma_5 M \psi \right\}_R(x) = \left\{ 1 + \frac{g^2}{(4\pi)^2} C_2(R) \left[ 3 \ln(8\pi\mu^2 t) + 2 + \ln(432) \right] + O(g^4) \right\} \tilde{\chi}(t, x) \gamma_5 M \tilde{\chi}(t, x) + O(t).$$  

(3.11)

Finally, repeating the renormalization group argument as in previous sections, we obtain the desired expressions,

$$\partial_\mu j_{5\mu}(x) + 4N_f T(R) q_R(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ -\frac{1}{2} + \ln(432) \right] \right\} \partial_\mu [\tilde{\chi}(t, x) \gamma_5 \gamma_{5\mu}(x)] + N_f T(R) \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x),$$  

(3.12)

$$\left\{ \bar{\psi} \gamma_5 M \psi \right\}_R(x) = \lim_{t \to 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[ 3 \ln \pi + 2 + \ln(432) \right] \right\} \tilde{\chi}(t, x) \gamma_5 M(1/\sqrt{8t}) \tilde{\chi}(t, x).$$  

(3.13)

Let us discuss implication of the representation (3.12). The topological charge $Q$, whose normalization is consistent with the chiral WT relation (3.1), is defined by

$$Q \equiv \int d^4x \left[ q_R(x) + \frac{1}{4N_f T(R)} \partial_\mu j_{5\mu}(x) \right] = \int d^4x q_R(x).$$  

(3.14)

According to our representation (3.12), this can be expressed as

$$Q = \lim_{t \to 0} \int d^4x \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x).$$  

(3.15)

On the other hand, it follows from the flow equation (1.6) that

$$\partial_t \left[ \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \right] = \partial_\mu W_\mu(t, x), \quad W_\mu(t, x) \equiv 4\epsilon_{\mu\nu\rho\sigma} D_\lambda G_{\nu\rho}^a(t, x) G_{\lambda\sigma}^a(t, x).$$  

(3.16)

Since $W_\mu(t, x)$ is a gauge-invariant current, we see that the spacetime integral in Eq. (3.15) is independent of the flow-time $t$ and, for any $t > 0$\footnote{Note that in this paper we are assuming that the cutoff is sent to infinity after the renormalization. With a finite cutoff such as a finite lattice spacing, the story is different and for instance the topological charge $Q$ can depend on the flow-time $t$.}

$$Q = \int d^4x \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x).$$  

(3.17)

This is precisely the definition of the topological charge advocated in Ref. [4] and this definition has been extensively employed recently to compute the topological susceptibility in quenched and full QCD. Our representation (3.12) confirms that the topological charge (3.17) is in fact normalized in a consistent manner with the chiral WT relation (3.11); this statement can also be found in Ref. [22].
4. Conclusion

In this paper, we constructed small flow-time representations of various fermion bilinear operators of mass dimension 3, such as the axial-vector and vector currents, the pseudo-scalar and scalar densities, flavor non-singlet and flavor singlet ones. We believe that these representations, not only provide universal expressions being independent of regularization, also practically are useful in actual lattice Monte Carlo simulations. We treated various operators case by case, but we found a very simple pattern in the representations: For vector and axial-vector currents, we replace \( \psi(x) \rightarrow \tilde{\chi}(t,x) \), \( \bar{\psi}(x) \rightarrow \bar{\tilde{\chi}}(t,x) \), and multiply it by the factor

\[
1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R) \left[ -\frac{1}{2} + \ln(432) \right],
\]

and consider the \( t \rightarrow 0 \) limit. For the scalar and pseudo-scalar densities, after \( \psi(x) \rightarrow \tilde{\chi}(t,x) \), \( \bar{\psi}(x) \rightarrow \bar{\tilde{\chi}}(t,x) \), we multiply it by the factor

\[
1 + \frac{g(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R) \left[ 3\ln \pi + 2 + \ln(432) \right],
\]

and replace the mass matrix \( M \) by the running one \( \tilde{M}(1/\sqrt{8t}) \), and consider the \( t \rightarrow 0 \) limit; for the flavor singlet scalar density, before the \( t \rightarrow 0 \) limit, we subtract the vacuum expectation value or take the limit \( M \rightarrow 0 \). Although there should exist an underlying mechanism for the above simple “universal” rule, at this moment we do not have a convincing understanding. Presumably, the WT relations in terms of the flowed fermion fields provide the clue.

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