On the controlled eigenvalue problem for stochastically perturbed multi-channel systems

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Abstract In this paper, we consider the problem of minimizing the asymptotic exit rate of diffusion processes from an open connected bounded set pertaining to a multi-channel system with small random perturbations. Specifically, we establish a connection between: (i) the existence of an invariant set for the unperturbed multi-channel system with respect to certain class of state-feedback controllers; and (ii) the asymptotic behavior of the principal eigenvalue and the solution of the Hamilton-Jacobi-Bellman (HJB) equations corresponding to a family of singularly perturbed elliptic operators. Finally, we provide a sufficient condition for the existence of a Pareto equilibrium (i.e., a set of optimal exit rates with respect to each of input channels) for the HJB equations – where the latter correspond to a family of nonlinear controlled eigenvalue problems.

Keywords Diffusion process · HJB equations · multi-channel system · principal eigenvalue · optimal exit time · small random perturbations

1 Introduction

Consider the following continuous-time multi-channel system

\[ \dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t), \quad x(0) = x_0, \]  

(1)

where \( x(t) \in X \subseteq \mathbb{R}^d \) is the state of the system, \( u_i(t) \in U_i \subseteq \mathbb{R}^{r_i} \) is the control input to the \( i \)-th-channel, and \( A \in \mathbb{R}^{d \times d} \) and \( B_i \in \mathbb{R}^{d \times r_i} \), for \( i = 1, 2, \ldots, N \), are constant matrices.

1 This work is, in some sense, a continuation of our previous paper [2].
Let $D \subset \mathcal{X}$ be an open connected bounded set with smooth boundary. For the multi-channel system in (1), we consider the following class of state-feedback controllers
\[
K \subseteq \left\{ (K_1, K_2, \ldots, K_N) \in \prod_{i=1}^{N} \mathbb{R}^r_i \times d \mid \Lambda(D) \neq \emptyset \right\}, \tag{2}
\]
where $\Lambda(D) \subseteq D \cup \partial D$ is the maximal invariant set (under the action of $T^t \triangleq \exp\{ (A + \sum_{i=1}^{N} B_i K_i) t \}, \ t \geq 0$) such that
\[
T^t \Omega = \Omega \subset \Lambda(D), \quad \forall t \geq 0, \tag{3}
\]
for any set $\Omega$.

In what follows, we provide a connection between the existence of an invariant set for the system $T^t$ in $D \cup \partial D$ and the asymptotic behavior of the principal eigenvalue for singularly perturbed elliptic operator which is associated with the following stochastic differential equation (SDE)
\[
dX^\epsilon(t) = \left( A + \sum_{i=1}^{N} B_i K_i \right) X^\epsilon(t) dt + \sqrt{\epsilon} \sigma(X^\epsilon(t)) dW(t), \quad X^\epsilon(0) = x_0, \tag{4}
\]
where
- $X^\epsilon(\cdot)$ is an $\mathbb{R}^d$-valued diffusion process, $\epsilon$ is a small parameter lying in an interval $(0, \epsilon^*)$ (which represents the level of random perturbation in the system), and $(K_1, K_2, \ldots, K_N)$ is an $N$-tuple of state-feedbacks from the class $K$,
- $\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is Lipschitz continuous with the least eigenvalue of $\sigma(\cdot)\sigma^T(\cdot)$ uniformly bounded away from zero, i.e.,
  \[
  \sigma(x)\sigma^T(x) \geq \kappa I_{d \times d}, \quad \forall x \in \mathbb{R}^d,
  \]
  for some $\kappa > 0$, and
- $W(\cdot)$ (with $W(0) = 0$) is an $m$-dimensional standard Wiener process.

For any fixed $\epsilon \in (0, \epsilon^*)$, let $\tau^\epsilon$ be the first exit time of the diffusion process $X^\epsilon(t)$ from the set $D$, i.e.,
\[
\tau^\epsilon = \inf \left\{ t > 0 \mid X^\epsilon(t) \notin D \right\}. \tag{5}
\]
Further, let $P^\epsilon_{x_0}\{A\}$ and $E^\epsilon_{x_0}\{\xi\}$, as usual, denote the probability of an even $A$ and the expectation of a random variable $\xi$, respectively, for the diffusion process $X^\epsilon(t)$ starting from $x_0 \in D$.

Here, it is worth mentioning that, in system reliability analysis and other studies, one often requires to confine a controlled diffusion process $X^\epsilon(t)$ to a given open connected bounded set $D$ as long as possible. A standard formulation for such a problem is to maximize the mean exit time $E^\epsilon_{x_0}\{\tau^\epsilon\}$ from the set $D$. We also observe that a more suitable objective would be to minimize the asymptotic rate with which the diffusion process $X^\epsilon(t)$ exits from the set $D$. Further, this suggests minimizing the principal eigenvalue $\lambda_{\epsilon,0}$
\[
\lambda_{\epsilon,0} = -\limsup_{t \to \infty} \frac{1}{t} \log P^\epsilon_{x_0}\{\tau^\epsilon > t\}. \tag{6}
\]
with respect to certain class of admissible controls.

The following lemma provides a condition under which the maximal invariant set $\Lambda(D)$ is nonempty (e.g., see [2]).
Lemma 1

If for some \( x_0 \in D \),
\[
- \limsup_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log p^{x_0}_\epsilon \{ \tau^\epsilon > t \} < \infty,
\]
(7)
then the maximal invariant set \( \Lambda(D) \) is nonempty.

(a) If for some \( x_0 \in D \),
\[
\limsup_{\epsilon \to 0} \log e^{x_0}_\epsilon \{ \tau^\epsilon \} = \infty,
\]
(8)
then the maximal invariant set \( \Lambda(D) \) is nonempty.

Proof Suppose that the maximum closed invariant set \( \Lambda(D) \) is empty (i.e., the invariant set for \( T^t \) in \( D \cup \partial D \), with respect to \( K \), is empty). Then, there exists an open bounded domain \( \tilde{D} \supset D \cup \partial D \) such that the corresponding set \( \Lambda(\tilde{D}) \) is also empty.

Note that it is easy to check that if \( D_2 \subset D_1 \), then \( \Lambda(D_2) \subset \Lambda(D_1) \). Take the following sequence \( \{ D_m \} \) of open domains such that
\[
D_1 \supset D_2 \supset D_3 \supset \cdots \quad \text{and} \quad \bigcap_{m \geq 1} D_m = D \cup \partial D.
\]
(9)
If \( \Lambda(D_m) \neq \emptyset \) for all \( m \geq 1 \), then
\[
\Lambda = \bigcap_{m \geq 1} \Lambda(D_m).
\]
(10)
Moreover, since \( \Lambda(D_m) \) is closed, we have
\[
\Lambda(D_1) \supset \Lambda(D_2) \supset \Lambda(D_3) \supset \cdots.
\]
(11)
Note that \( \Lambda \) is an invariant closed set with respect to the unperturbed multi-channel dynamical system and \( \Lambda \supset D \cup \partial D \). Thus, \( \emptyset \neq \Lambda \subset \Lambda(D) \). This contradicts our earlier assumption.

Then, for some \( m_0 \geq 1 \), we have
\[
\Lambda(D_{m_0}) = \emptyset.
\]
(12)
Let \( \tilde{D} = D_{m_0} \), for any \( \epsilon \in (0, \epsilon^*) \) and \( x_0 \in \tilde{D} \cup \partial \tilde{D} \), introduce the following
\[
\tau^\epsilon_{\tilde{D}} = \inf \{ t > 0 \mid X^\epsilon(t) \notin \tilde{D} \}.
\]
(13)
Then, we can show that \( \tau^\epsilon_{\tilde{D}} < \infty \) for any \( x_0 \in \tilde{D} \).

Notice that, if \( \tau^\epsilon_{\tilde{D}} = \infty \), then \( \tilde{D} \cup \partial \tilde{D} \supset \exp \left\{ (A + \sum_{i=1}^{N} B_i K_i) t \right\} x_0 \) for all \( t \geq 0 \). Then, we have the following
\[
\left\{ \exp \left\{ (A + \sum_{i=1}^{N} B_i K_i) t \right\} x_0 \mid t \geq 0 \right\} \subset \Lambda(\tilde{D}) = \emptyset,
\]
(14)
which show that \( \tau^\epsilon_{\tilde{D}} \) is finite.

Note that, from upper-semicontinuity of \( \tau^\epsilon_{\tilde{D}} \), we have
\[
T = \sup_{x_0 \in D \cup \partial D} \tau^\epsilon_{\tilde{D}} < \infty.
\]
(15)
Moreover, for any $\delta > 0$, let
\[
\lim_{\varepsilon \to 0} \sup_{x_0 \in D} E^{x_0}_\varepsilon \left\{ \text{dist} \left( X^\varepsilon(t), \exp \left\{ \left( A + \sum_{i=1}^N B_i K_i \right) t \right\} x_0 \right) > \delta \right\} = 0, \quad t \geq 0.
\] (16)
From Equations (13)–(16), we have
\[
P^{x_0}_\varepsilon \{ \tau^\varepsilon > T \} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\] (17)
Then, using the Markov property, we have
\[
P^{x_0}_\varepsilon \{ \tau^\varepsilon > \ell T \} = E^{x_0}_\varepsilon \chi_{\tau^\varepsilon > T} E^{x_0}_\varepsilon \chi_{\tau^\varepsilon > T} \cdots E^{x_0}_\varepsilon \chi_{\tau^\varepsilon > T},
\] (18)
where $\chi_A$ is the indicator for the event $A$.
Since $P^{x_0}_\varepsilon \{ \tau^\varepsilon > t \}$ decreases in $t$, then we have
\[
\lim_{t \to \infty} \frac{1}{t} \log P^{x_0}_\varepsilon \{ \tau^\varepsilon > t \} \leq \frac{1}{T} \log P^{x_0}_\varepsilon \{ \tau^\varepsilon > T \}.
\] (19)
Taking into account Equation (17), then, for any $x_0 \in D$, we have the following
\[
- \lim_{t \to \infty} \frac{1}{t} \log P^{x_0}_\varepsilon \{ \tau^\varepsilon > t \} \to \infty \quad \text{as} \quad \varepsilon \to 0.
\] (20)
Hence, our assumption that $A(D) = \emptyset$ is inconsistent.
To proof the part (ii), notice that \[E^{x_0}_\varepsilon \{ \tau^\varepsilon \} \leq T \sum_{\ell=1}^\infty P^{x_0}_\varepsilon \{ \tau^\varepsilon > (\ell - 1)T \}.
\] (21)
Assumption $A(D) = \emptyset$ gives, in view of Equations (17), (18) and (21) for sufficiently small $\varepsilon > 0$ and for any $x_0 \in D$, that
\[
E^{x_0}_\varepsilon \{ \tau^\varepsilon \} \leq T \sum_{\ell=1}^\infty P^{x_0}_\varepsilon \{ \tau^\varepsilon > (m - 1)T \},
\] (22)
which contradicts with Equation (7). This completes the proof of Lemma [1].
Next, we introduce a definition for minimal action state-feedbacks based on the notion of action functional $I_T(\varphi)$ over a set $\Psi$ consisting of all absolutely continuous functions $\varphi \in C_T([0, T], \mathbb{R}^d)$, with $\varphi(0) = x_0$, a compact set in $D$ (e.g., see [3] Chapter 14) or [4] for additional discussions).

**Definition 1** The $N$-tuple $(K^*_1, K^*_2, \ldots, K^*_N) \in \mathcal{K}$ is said to be minimal action state-feedbacks if
\[
\mathcal{K} \ni (K^*_1, K^*_2, \ldots, K^*_N) \in \arg \min I(x_0, \partial D),
\] (23)
where the action functional $I(x_0, \partial D)$ is given by
\[
I(x_0, \partial D) = \lim_{T \to \infty} \inf_{\varphi \in \Psi} \frac{1}{T} \left\{ I_{0:T}(\varphi(t)) \big| \varphi(t) \in D \cup \partial D, \ t \in [0, T] \right\},
\] (24)
with \((K_1, K_2, \ldots, K_N) \in K\) and

\[
I_T(\varphi(t)) = \frac{1}{2} \int_0^T \left[ \frac{d\varphi(t)}{dt} - (A + \sum_{i=1}^N B_i K_i) \varphi(t) \right]^T \left( \sigma(\varphi(t)) \sigma^T(\varphi(t)) \right)^{-1} \times \left[ \frac{d\varphi(t)}{dt} - (A + \sum_{i=1}^N B_i K_i) \varphi(t) \right] dt. \tag{25}
\]

**Remark 1** Notice that the principal eigenvalue \(\lambda_{e,0}\), which is associated with the singularly perturbed elliptic operator (e.g., see [9], [4] or [5]; see also [3])

\[
- L_\epsilon(\cdot)(x) = \nabla \cdot (A + \sum_{i=1}^N B_i K_i^*) x + \frac{\epsilon}{2} \tr \left\{ \sigma(x) \sigma^T(x) \nabla^2 (\cdot) \right\}, \tag{26}
\]

is given by

\[
\lambda_{e,0}^* = - \limsup_{t \to \infty} \frac{1}{t} \log P_{x_0}^{\tau_e} \{ \tau_e > t \}. \tag{27}
\]

Moreover, it is the unique solution to the following eigenvalue problem

\[
L_\epsilon \psi^*(x) + \lambda_{e,0}^* \psi^*(x) = 0, \quad \forall x \in D \\
\psi^*(x) = 0, \quad \forall x \in \partial D \tag{28}
\]

where \(\psi^* \in W^{2,p}_{\text{loc}}(D) \cap C(D \cup \partial D)\), for \(p > 2\), with \(\psi^* > 0\) on \(D\).

In the following section, we present our main results – where we establish a connection between the asymptotic exit rate with which the controlled diffusion process (with respect to each of the input channels) exits from the set \(D\) and the asymptotic behavior of principal eigenvalues for a family of singularly perturbed elliptic operators with zero boundary condition on \(\partial D\). Later, such a formulation allows us to provide a sufficient condition for the existence of a Pareto equilibrium (i.e., a set of optimal exit rates with respect to each of the input channels) for the HJB equations – where the latter correspond to a family of nonlinear controlled eigenvalue problems (e.g., see [3] or [6, Chapter 8] for additional discussions on eigenvalue problems).

## 2 Main Results

In this section, we consider a family of SDEs (with respect to each of input channels)

\[
dX^{\epsilon,i}(t) = \left( A + \sum_{j \neq i} B_j K_j \right) X^{\epsilon,i}(t) dt + B_j u_j(t) + \sqrt{\epsilon} \sigma(X^{\epsilon,i}(t)) dW(t), \quad X^{\epsilon,i}(0) = x_0, \quad i = 1, 2, \ldots, N, \tag{29}
\]

where \(u_i(\cdot)\) is a \(U_i\)-valued measurable control process to the \(i\)th-channel (i.e., an admissible control from the set \(U_i \subset \mathbb{R}^{r_i}\)) such that for all \(t > s\), \(W(t) - W(s)\) is independent of \(u_i(\nu)\) for \(\nu > s\).

Next, let

\[
\lambda_\epsilon = (\lambda_{\epsilon,1}, \lambda_{\epsilon,2}, \ldots, \lambda_{\epsilon,N}), \tag{30}
\]
with
\[ \lambda_{e,i} = - \limsup_{t \to \infty} \frac{1}{t} \log P_{e,i}^{x_{0},u_{i}} \{ \tau_{i}^{e} > t \}, \quad i = 1, 2, \ldots, N, \]
where the probability \( P_{e,i}^{x_{0},u_{i}} \{ \cdot \} \) is conditioned on the initial point \( x_{0} \in D \) as well as on the admissible control \( u_{i} \in \mathcal{U}_{i} \). Moreover, \( \tau_{i}^{e} \) is the first exit time for the diffusion process \( X_{e,i}(t) \) from the set \( D \), i.e., \( \tau_{i}^{e} = \inf \{ t > 0 \mid X_{e,i}(t) \notin D \} \) for \( i = 1, 2, \ldots, N \).

Further, let us introduce the following set
\[ \Sigma^{e} = \left\{ \lambda_{e} \in \mathbb{R}_{+}^{N} \bigg| (K_{1}, K_{2}, \ldots, K_{N}) \in \mathcal{K} \right\}. \tag{31} \]

**Remark 2** Notice that the set \( \Sigma^{e} \) (with respect to the class of state-feedbacks \( \mathcal{K} \)) is a closed subset of \( \mathbb{R}_{+}^{N} \).

Define the partial ordering \( \prec \) on \( \Sigma^{e} \) by
\[ (\lambda_{e,1}, \lambda_{e,2}, \ldots, \lambda_{e,N}) \prec (\lambda_{e}', 1, 2, \ldots, \lambda_{e,N}), \tag{32} \]
if \( \lambda_{e,i} \leq \lambda_{e,i}' \) for all \( i = 1, 2, \ldots, N \), with strict inequality for at least one \( i \). Further, we say that \( \lambda_{e}^{*} \in \Sigma^{e} \) is a Pareto equilibrium (with respect to the class of state-feedbacks \( \mathcal{K} \)) if there is no \( \lambda_{e} \in \Sigma^{e} \) for which \( \lambda_{e} \prec \lambda_{e}^{*} \).

Then, we have the following proposition that provides a sufficient condition for the existence of a Pareto equilibrium \( \lambda_{e}^{*} \in \Sigma^{e} \).

**Proposition 1** Suppose that the statement in Definition \( \mathbf{1} \) holds true for some minimal state-feedbacks \( (K_{1}^{*}, K_{2}^{*}, \ldots, K_{N}^{*}) \in \mathcal{K} \). Then, there exists a Pareto equilibrium \( \lambda_{e}^{*} \in \Sigma^{e} \) such that
\[ (\lambda_{e,1}^{*}, \lambda_{e,2}^{*}, \ldots, \lambda_{e,N}^{*}) \prec (\lambda_{e,1}, \lambda_{e,2}, \ldots, \lambda_{e,N}) \quad \text{on} \quad \Sigma^{e}, \tag{33} \]
where the principal eigenvalues \( \lambda_{e,1}^{*} \) are the unique solutions for the HJB equations corresponding to the following family of nonlinear controlled eigenvalue problems
\[ \max_{u_{i} \in \mathcal{U}_{i}} \left\{ \mathcal{L}_{e,i} \psi_{u_{i}}(x, u_{i}) + \lambda_{e,i}^{*} \psi_{u_{i}}(x) = 0 \right\}, \quad \forall x \in D \] \[ \psi_{u_{i}}(x) = 0, \quad \forall x \in \partial D \tag{34} \]
with \( \psi_{u_{i}} \in W^{2,p}(D) \cap C(D \cup \partial D) \), for \( p > 2 \), with \( \psi_{u_{i}}^{*} > 0 \) on \( D \), and
\[ -\mathcal{L}_{e,i}(\cdot) = \left\{ \nabla (\cdot), (A + \sum_{j \neq i} B_{j}K_{j}^{*} )x + B_{i}u_{i} \right\} + \frac{\epsilon}{2} \text{tr} \left\{ \sigma(x)\sigma^{T}(x)\sigma^{2}(\cdot) \right\}, \tag{35} \]
for \( i = 1, 2, \ldots, N \).

**Proof** Suppose there exists an \( N \)-tuple of state-feedbacks \( (K_{1}^{*}, K_{2}^{*}, \ldots, K_{N}^{*}) \in \mathcal{K} \) that satisfies the statement in Definition \( \mathbf{1} \). Then, our first claim for \( \psi_{u_{i}}^{*} \in W^{2,p}_{loc}(D) \cap C(D \cup \partial D) \), with \( p > 2 \), follows from Equation (34) (cf. \( \mathbf{8} \), Theorems 1.1, 1.2 and 1.4). That is, if \( u_{i}^{*} \) for \( i \in \{1, 2, \ldots, N\} \), is a measurable selector of \( \arg \max \left\{ \mathcal{L}_{e,i} \psi_{u_{i}}(x) \right\} \), with \( x \in D \).

Then, by the uniqueness claim for the eigenvalue problem (cf. Equation (28)), we have
\[ \lambda_{e,i}^{*} = - \limsup_{t \to \infty} \frac{1}{t} \log P_{e,i}^{x_{0},u_{i}^{*}} \{ \tau_{i}^{e} > t \}, \tag{36} \]
where the probability $E_{\epsilon,i}^{\hat{\psi}^*_u \lambda^*_\epsilon} \{ \cdot \}$ is conditioned with respect to $x_0$, $u^*_i$ as well as $(K^*_1, K^*_2, \ldots, K^*_N)$. Moreover, for any other admissible controls $v_i \in \mathcal{U}_i$, we have

$$L_{\epsilon,i} \psi^*_u (x, v_i) + \lambda^*_\epsilon \psi^*_u (x) \leq 0, \quad \forall t \geq 0. \tag{37}$$

Let $Q \subset \mathbb{R}^d$ be a smooth bounded open domain containing $D \cup \partial D$. Let $\hat{\psi}$ and $\hat{\lambda}_{\epsilon,i}$ be the principal eigenfunction-eigenvalue pairs for the eigenvalue problem of $L_{\epsilon,i}$ on $\partial Q$. Further, let

$$\hat{\tau}_i^\epsilon = \inf \left\{ t > 0 \mid X^{\epsilon,i}(t) \notin Q \right\}. \tag{38}$$

Then, under $v_i$, for $i \in \{1, 2, \ldots, N\}$ and for some $\hat{x} \in D$, we have

$$\hat{\psi}(\hat{x}) \geq L_{\epsilon,i}^{\hat{x}} \psi^*_u \left( \exp \left( \hat{\lambda}_{\epsilon,i} t \right) \psi \left( x(t) \right) 1 \{ \hat{\tau}_i^\epsilon > t \} \right),$$

$$\geq \inf_{x \in D} \left| \psi \left( x(t) \right) \right| \exp \left( \hat{\lambda}_{\epsilon,i} t \right) P_{\epsilon,i}^{x,u_i} \{ \hat{\tau}_i^\epsilon > t \}, \tag{39}$$

Leading to

$$\hat{\lambda}_{\epsilon,i} \geq - \limsup_{t \to \infty} \frac{1}{t} \log P_{\epsilon,i}^{x,u_i} \{ \hat{\tau}_i^\epsilon > t \}. \tag{40}$$

Letting $Q$ shrink to $D$ and using Proposition 4.10 of [8], then we have $\hat{\lambda}_{\epsilon,i} \to \lambda^*_{\epsilon,i}$. Thus, we have

$$\lambda^*_{\epsilon,i} \leq - \limsup_{t \to \infty} \frac{1}{t} \log P_{\epsilon,i}^{x,u_i} \{ \hat{\tau}_i^\epsilon > t \}. \tag{41}$$

Combining with Equation (39), this establishes the optimality of $u^*_i$ and the fact that $\lambda^*_{\epsilon,i}$ is the optimal exit rate.

Conversely, let $v^*_i$ be an admissible optimal control, then we have

$$L_{\epsilon,i} \psi^*_v \left( x, v^*_i \right) + \lambda^*_\epsilon \psi^*_v \left( x \right) = 0 \tag{42}$$

and

$$L_{\epsilon,i} \psi^*_u \left( x, v^*_i \right) + \lambda^*_\epsilon \psi^*_u \left( x \right) \leq 0, \quad \forall t \geq 0, \tag{43}$$

with $\hat{\lambda}_{\epsilon,i} = \lambda^*_{\epsilon,i}$, for $i = 1, 2, \ldots, N$.

Further, notice that $\psi^*_u$ is a scalar multiple of $\psi^*_v$ and, at some $\hat{x} \in D$ (cf. [8, Theorem 1.4(a)]). Then, we see that $v^*_i$ is also a maximizing measurable selector in Equation (34).

On the other hand, for a fixed small $\epsilon$, define the following continuous functional (i.e., a utility function over a closed set $\Sigma^* \subset \mathbb{R}^N$ that can be convexified)

$$\Sigma^* \ni \lambda \mapsto U^\epsilon(\lambda) \triangleq \langle \omega, \lambda \rangle \in \mathbb{R}, \tag{44}$$

where $\omega_i > 0$ for $i = 1, 2, \ldots, N$. 


Note that the utility function $U^\epsilon$ satisfies the property $U^\epsilon(\lambda^1) < U^\epsilon(\lambda^2)$, whenever $\lambda^1 \preceq \lambda^2$ on $\Sigma^\epsilon$ with respect to the class of state-feedbacks $\mathcal{K}$. Then, from the Arrow-Barankin-Blackwell theorem (see [1]), one can see that the set in

$$\left\{ \lambda \in \Sigma^\epsilon \mid \exists \omega_i > 0, \quad i = 1, 2, \ldots, N, \quad \min_{\lambda \in \Sigma^\epsilon} \langle \omega, \lambda \rangle = \langle \omega, \lambda^*_\epsilon \rangle \right\},$$

(45)

is dense in the set of all Pareto equilibria. This further implies that, for any choice of $\omega_i > 0, \ i = 1, 2, \ldots, N$, the minimizer $\langle \omega, \lambda^*_\epsilon \rangle = \sum_{i=1}^{N} \omega_i \lambda^*_{\epsilon,i}$ over $\Sigma^\epsilon$ satisfies the Pareto equilibrium condition with respect to some minimal state-feedbacks $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$. This completes the proof of Proposition 1.

We conclude this section with the following corollary that gives a lower-bound for the action functional $I^\epsilon(x_0, \partial D)$ in (24) (with respect to some minimal state-feedbacks $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$) (cf. [5, Corollary 11.2, pp. 377]).

**Corollary 1** Suppose that the statement in Proposition 1 holds. Then, the minimal action functional $I^\epsilon(x_0, \partial D)$ in (24), with respect to the $N$-tuple of state-feedbacks $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}$, satisfies

$$I^\epsilon(x_0, \partial D) \geq \max_{i \in \{1, 2, \ldots, N\}} I_i(x_0, \partial D),$$

(46)

where

$$I_i(x_0, \partial D) = - \lim_{\epsilon \to 0} \epsilon \log \lambda^*_{\epsilon,i}.$$  

(47)

**Remark 3** Note that, for each $i \in \{1, 2, \ldots, N\}$, we have (cf. Equations (6) and (30))

$$\lambda^*_{\epsilon,0} - \lambda^*_{\epsilon,i} \geq 0, \quad \forall \epsilon \in (0, \epsilon^*), \quad (K_1, K_2, \ldots, K_N) \in \mathcal{K},$$

which suggests that the above corollary is useful for selecting the most appropriate minimal action state-feedbacks that confine the diffusion process $X^\epsilon(t)$ to the prescribed set $D$ for a longer duration.

\[ ^2 \text{Notice that} \]

$$I^\epsilon(x_0, \partial D) = \limsup_{T \to \infty} \inf_{\varphi(t) \in \Psi} \frac{1}{T} \left\{ I^*_T(\varphi(t)) \mid \varphi(t) \in D \cup \partial D, \ t \in [0, T] \right\},$$

with

$$I^*_T(\varphi(t)) = \frac{1}{2} \int_0^T \left[ \frac{d\varphi(t)}{dt} - \left( A + \sum_{i=1}^{N} B_i K_i^* \right) \varphi(t) \right]^{T} \left( \sigma(\varphi(t)) \sigma^T(\varphi(t)) \right)^{-1} \left[ \frac{d\varphi(t)}{dt} - \left( A + \sum_{i=1}^{N} B_i K_i^* \right) \varphi(t) \right] dt.$$
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