SCALING OF CONFORMAL BLOCKS AND GENERALIZED THETA FUNCTIONS OVER $\overline{M}_{g,n}$

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Abstract. There is an identification, over smooth curves, of conformal blocks and generalized theta functions. We show that for conformal blocks in type A associated to projective varieties of minimal degree, if this interpretation extends to stable curves, identities between first Chern classes of vector bundles of conformal blocks will be satisfied. Examples show the extension can fail on $\overline{M}_2$, and hold on $\overline{M}_{0,n}$.

1. Introduction

Conformal blocks are vector spaces associated to stable curves together with certain Lie theoretic data. These vector spaces fit together to give vector bundles on the moduli stacks $\overline{M}_{g,n}$ parameterizing stable $n$-pointed curves of genus $g$. Given such a nonzero vector bundle $V$ on $\overline{M}_{g,n}$, and any point $x \in \overline{M}_{g,n}$ representing a smooth $n$-pointed curve of genus $g$, there is a canonical isomorphism $V^i_c \cong H^0(X_x, L_x)$, where $X_x$ is a moduli space determined by the Lie data, and $L_x$ is a canonical ample line bundle on it. These isomorphisms are compatible with multiplication operations on global sections and the algebra structure on conformal blocks. The global sections $H^0(X_x, L_x)$ are called generalized theta functions. It is natural to wonder whether such canonical isomorphisms exist for all points of $\overline{M}_{g,n}$, and in this work, we consider the following:

Question 1. Given a vector bundle of conformal blocks $V$ on $\overline{M}_{g,n}$, is there an extension of the identification of the conformal block $V|_x$ with generalized theta functions, for boundary points $x \in \overline{M}_{g,n} \setminus M_{g,n}$?

We study this problem in settings where we would know something about extensions if they exist, working with bundles $V[m]$, for $m \in \mathbb{N}$, obtained from $V$ under an operation called stretching (Def 2.8). Given $x \in \overline{M}_{g,n}$, information about the pair $(X_x, L_x)$ is determined by the ranks of $V[m]$, including its so-called $\Delta$-invariant (Def 2.5). Fujita has shown that if $(X_x, L_x)$ has $\Delta$-invariant zero, then $L_x$ is very ample, embedding $X_x$ as a projective variety of minimal degree. In other words, one can realize $X_x$ as a rational normal curve, a projective space, a quadric hypersurface, a rational normal scroll, a Veronese surface, or a polarized variety obtained from coning over one of the other types.

If for some $x \in \overline{M}_{g,n}$, one has $\overline{V}|_x^i \cong H^0(X_x, L_x)$, for a pair $(X_x, L_x)$ of $\Delta$-invariant zero, and if the answer to Question 1 is yes for $V$, then $c_1(V[m]) = \sum_{i=1}^D A_i(m)c_1(V[i])$, where $D = \text{vol}(X_x)$, and the $A_i(m)$ are polynomials in $m$ of degrees $\leq d = \text{rk}(V) - 1$ (Cor 3.6). We give explicit formulas for the $A_i(m)$. In Thm 6.1, conditions are given, which if satisfied by a particular bundle, identify it as having multiples whose first Chern classes are governed by the scaling identity of Cor 3.6.

For projective spaces (as moduli) we show more. If $V$ has projective space rank scaling: $\text{rk}(V[m]) = \binom{m+d}{d}$, then $\binom{d+d}{d+1} c_1(V) = c_1(V[m]) + D_m$, where $D_m$ is an effective divisor supported on the boundary, and the answer to Question 1 is yes for $V$ if and only if $D_m = 0$. In Example 3.8 (also see Example 5.8) we exhibit a bundle on $\overline{M}_2$ where $D_m \neq 0$, showing the answer to Question 1 is no in this case. In Thm 5.9, bundles for which $D_m = 0$ are identified, and an infinite family on $\overline{M}_{0,n}$ is given in Example 5.14.

Incidentally, if the rank of $V$ is one, then by a quantum generalization of a conjecture of Fulton (see [BGM14b] and the references therein), the rank of $V[m]$ is one as well; projective rank scaling can be viewed as a further generalization (Remark 5.3).

1The problem is stated more concisely, after sufficient notation is given, as Question 2.10 in Section 2.3.
In the cases we have determined that the answer to Question 1 is yes, we know for \( x \in \overline{M}_{g,n} \setminus M_{g,n} \) one will have that \((X_x, L_x) \equiv (P^d, O(1))\). However, we would like to know more about these polarized varieties \((X_x, L_x)\), and we ask the following question.

**Question 2.** Given a vector bundle of conformal blocks \( V \) on \( \overline{M}_{g,n} \), for which there an extension of the identification of the conformal block \( V_{|x} \) with generalized theta functions at boundary points \( x \in \overline{M}_{g,n} \setminus M_{g,n} \), can one describe the pair \((X_x, L_x)\)? For example, does \( X_x \) have a modular interpretation?

We have not come across a bundle on \( \overline{M}_{0,n} \), satisfying \( \Delta \)-invariant zero rank scaling (Def 3.5), but for which the divisor identities do not hold, and we wonder if one exists. We also raise the following question, based on our examples and results:

**Question 3.** Given a vector bundle of conformal blocks \( V \) of type A on \( \overline{M}_{g,n} \), is there an extension of the identification of the conformal block \( V_{|x} \) with generalized theta functions, for points \( x \in \overline{M}_{g,n} \setminus \Delta_{irr} \)? These are points \( x \) corresponding to stable curves in the locus of rational tails (one component of the dual graph has genus \( g \)), or curves in the interior of the component of the boundary whose generic point has a non-separating node.

**Remark 1.1.** We note that very few moduli spaces of parabolic bundles have been explicitly identified in the literature. Using techniques of Schubert calculus, we have been able to extend the list of examples.

Before giving the definitions and notations we use, we briefly attempt to put Question 1, and our approach to it, into context.

### 1.1. History of Question 1

The Verlinde formula gives a closed expression for the dimension of spaces of generalized theta functions. Although there are independent proofs in special cases, see [Ber93, Tha94, Zag95], the theory of conformal blocks plays an essential role in the derivation of the general formula. We recommend the survey article of [Sor96] for a good account.

While admittedly leaving out important contributions, one can summarize the two crucial aspects of the proof:

1. There is a canonical identification between conformal blocks and generalized theta functions over smooth curves [Fal94, BL94, KNR94, DS95, Pau96, LS97].
2. The factorization theorem of Tsuchiya-Ueno-Yamada allows for a decomposition of conformal blocks at stable pointed curves \( x \in \overline{M}_{g,n} \setminus \Delta_{irr} \).

Several interesting questions arising from the above picture have been investigated. One theme, originating in [NR93, DW93, Ram96], is to try to remove the reference to conformal blocks. The idea is that if the space of generalized theta functions could be factorized geometrically, then one would obtain a “finite dimensional” proof of the Verlinde formula. The goal is then to find pairs \((X_x, L_x)\) for points \( x \in \overline{M}_{g,n} \setminus \Delta_{irr} \) such that global sections have suitable factorization properties. This motif has been continued in the work of Kausz and Sun [Sun00, Sun03, Kau05]. To the best of our knowledge, these factorizations of generalized theta functions (which in a suitable sense are for GL(r) rather than SL(r)) have not been related to conformal blocks. We also note the related papers [Fal96, Tel98].

Our outlook is to suppose that there is an extension of the identification of conformal blocks and generalized thetas at points on the boundary, and to ask what consequences may follow. In type A, there are usually no descent problems, and so it seems reasonable to ask for an interpretation over moduli spaces rather than stacks. Projective varieties of minimal degree have the advantage that they will degenerate to other projective varieties of minimal degree. The unique resolutions associated to their ideal sheaves is our main tool. We also find, in special cases, geometric interpretations over the boundary by studying factorization properties of conformal blocks, and their algebras [TUY89, Man09].
Remark 1.2. It may be feasible to use our methods for bundles in other types. Modification may have to be made when one has to work with powers (expected to be 2 or less for classical groups) of $L_i$ to get past issues of descent.

2. Notation and basic definitions

2.1. Conformal blocks. For a positive integer $\ell$ (called the level), we let $P_\ell(\mathfrak{sl}_{r+1})$ denote the set of dominant integral weights $\lambda$ with $(\lambda, \theta) \leq \ell$. Here $\theta$ is the highest root, and $(\ ,\ )$ is the Killing form, normalized so that $(\theta, \theta) = 2$. To a triple $(\mathfrak{sl}_{r+1}, \tilde{\lambda}, \ell)$, such that $\tilde{\lambda} \in P_\ell(\mathfrak{sl}_{r+1})^n$, there corresponds a vector bundle $V$ of conformal blocks on the stack $\overline{M}_{g,n}$ [TUY89,Fak12]. Throughout the paper, as our notation is fixed, we usually just refer to such bundles as $V$.

Remark 2.1. In the notation for the vector bundle, there isn’t any indication for the genus of the curve, which should be understood from the context.

We often use standard intersection theoretic computations on $\overline{M}_{g,n}$ and the factorization formulas of [TUY89]. We recommend the Bourbaki article of Sorger [Sor96] for some of the background on conformal blocks. In many of the examples and computations done here, we find the ranks of conformal blocks bundles using the following cohomological form of Witten’s Dictionary, which expresses these ranks as the intersection numbers of particular classes (depending on the bundle) in the small quantum cohomology ring of certain Grassmannian varieties.

Theorem 2.2. Let $V = V(\mathfrak{sl}_{r+1}, \tilde{\lambda}, \ell)$ be a vector bundle on $\overline{M}_{g,n}$ such that $\sum_{i=1}^n |\lambda_i| = (r+1)(\ell + s)$ for some integer $s$.

1. If $s > 0$, then let $\lambda = \ell \omega_1$. The rank of $V$ is the coefficient of $q^s \sigma_{\ell \omega_{r+1}}$ in the quantum product
   \[ \sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_{\lambda}^s \in \text{QH}^*(\text{Gr}(r+1, r + 1 + \ell)). \]

2. If $s \leq 0$, then the rank of $V$ is the multiplicity of the class of a point $\sigma_{k \omega_{r+1}}$ in the product
   \[ \sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \cdots \cdot \sigma_{\lambda_n} \in \text{H}^*(\text{Gr}(r+1, r + 1 + k)), \]
   where $k = \ell + s$.

The relation to quantum cohomology follows from [Wit95] and the twisting procedure of [Bel08], see Eq (3.10) from [Bel08]. Examples of such rank computations were done using Witten’s Dictionary in [BGM14], [BGM14b], and [Kaz14].

Remark 2.3. In this work, we also use the operation of plussing, described in Def 2.4, which can often turn a quantum computation of ranks into a classical one. We note that while plussing preserves the rank of a bundle, it does not preserve other invariants such as the first Chern class.

Definition 2.4. Let $\sigma_j : \omega_i \mapsto \omega_{i+j \pmod{r+1}}$ be a cyclic permutation of the affine Dynkin diagram of $\mathfrak{sl}_{r+1}$. Then $\text{rk } V(\mathfrak{sl}_{r+1}, \{\lambda_1, \ldots, \lambda_n\}, \ell) = \text{rk } V(\mathfrak{sl}_{r+1}, \{\sigma_j \lambda_1, \ldots, \sigma_j \lambda_n\}, \ell)$ if $j_1 + \cdots + j_n$ is divisible by $r+1$ [FS99]. We refer to this operation as “plussing”.

Note that one can use the operation of plussing to take a bundle of rank $R$ on $\overline{M}_{g,n}$ to define a bundle of rank $R$ on $\overline{M}_{g,n+k}$ as follows. To do this, for $i \in \{1, \ldots, k\}$, put $\lambda_{n+i} = \ell \omega_{j_i}$ for any integer $0 \leq j_i \leq r+1$ such that $\sum_{i=1}^k j_i$ is divisible by $r+1$. 
2.2. The \(\Delta\)-invariant and projective varieties of minimal degree.

**Definition 2.5.** Let \(X\) be an irreducible projective variety, and \(L\) an ample line bundle on \(X\). The \(\Delta\)-invariant of the polarized variety \((X, L)\) is given by the formula
\[
\Delta(X, L) = \dim(X) + L^\dim(X) - h^0(X, L).
\]
We refer to \(\Delta(X, L)\) as the \(\Delta\)-**genus** or as the \(\Delta\)-**invariant** of the pair \((X, L)\).

We note that by [Fuj90, Chapter 1, (4.2), (4.12)], and [BS95, Theorem 3.1.1], \(\Delta(X, L) \geq 0\), and if \(\Delta(X, L) = 0\), then \(H^0(X, L)\) generates the algebra \(\oplus_{m \geq 0} H^0(X, L^m)\) and hence \(L\) is very ample, giving rise to an embedding into a projective space
\[
X \hookrightarrow \text{Proj}(\mathcal{O}_X) = \mathbb{P}^N, \quad \mathcal{O}_X = \oplus_{m \geq 0} \text{Sym}^m(H^0(X, L)),
\]
so that its image is a (nondegenerate) variety of degree equal to \(L^\dim(X)\). We will write \(D = L^\dim(X)\) (the definition of the self intersection \(L^\dim(X)\) can be found in [Fuj90, 3.11], for example). The quantity \(L^\dim(X)\) is both called deg\(L\) and the volume of the polarized variety \((X, L)\).

For example, if \(X\) has dimension one and \(\Delta(X, L) = 0\) then \((X, L) = (\mathbb{P}^1, \mathcal{O}(D))\) for some \(D > 0\). Indeed, \(\Delta(\mathbb{P}^1, \mathcal{O}(D)) = 0\). Conversely, \(0 = \Delta(X, L) = 1 + \deg(L) - h^0(L)\). Riemann-Roch gives that \(\chi(X, L) = 1 - \text{deg}(X) + \deg(L)\). Hence \(h^1(X, L) = -\text{deg}(X) = 0\), where \(\text{deg}(X)\) is the arithmetic genus of \(X\).

**Definition 2.6.** A nondegenerate variety \(X \subset \mathbb{P}^N\) always has \(\text{deg}(X) \geq 1 + \text{codim}(X)\), and is said to be of **minimal degree** if \(\text{deg}(X) = 1 + \text{codim}(X)\).

Therefore, polarized varieties \((X, L)\) with \(\Delta(X, L) = 0\) correspond to projective varieties of minimal degree. A description of these varieties has been given by many authors including the classification of minimal surfaces given by Del Pezzo in 1886 [DP86] (also [Nag60]), and the higher dimensional cases by Bertini in 1907 [Ber07], and subsequent treatments by Harris, Xambó, Griffiths and Harris, Fujita, and Beltrametti and Sommese [Har81, Xam81, GH78, Fuj90, BS95]. We recommend [EH87] for a good historical account.

**Proposition 2.7.** [Fuj90, Chapter 1, (5.10), (5.15)], [BS95, Proposition 3.1.2] We suppose that \(\Delta(X, L) = 0\) and \(d = \dim X \geq 2\):

1. \((X, L) \cong (\mathbb{P}^d, \mathcal{O}_\mathbb{P}(1))\) if \(L^d = 1\);
2. \((X, L) \cong (\mathbb{P}^d, \mathcal{O}_\mathbb{P}(1))\), where \(Q\) is a not necessarily smooth quadric in \(\mathbb{P}^{d+1}\), if \(L^d = 2\);
3. \((X, L)\) is a \(\mathbb{P}^{d-1}\) bundle over \(\mathbb{P}^1\), \(X \cong \mathbb{P}(E)\), for a vector bundle \(E\) on \(\mathbb{P}^1\) which is a direct sum of line bundles of positive degrees;
4. \((X, L) \cong (\mathbb{P}^2, \mathcal{O}_\mathbb{P}(2))\); or
5. \((X, L)\) is a generalized cone over a smooth subvariety \(V \subset X\) with \(\Delta(V, L_V) = 0\), where \(L_V\) denotes the restriction of \(L\) to \(V\).

For (1) in Proposition 2.7, we note the related paper [Gor68].

2.3. \(V[m]\) and the algebra of conformal blocks.

**Definition 2.8.** For \(\lambda_i = \sum_{j=1}^r c_i \omega_j \in P(\mathfrak{sl}_{r+1})\), and \(m \in \mathbb{N}\), set \(m \lambda_i = \sum_{j=1}^r (mc_i) \omega_j \in P_{m \ell}(\mathfrak{sl}_{r+1}).\) Given \(V = \mathbb{V}(\mathfrak{sl}_{r+1}, \ell, \ell)\), set
\[
\mathbb{V}[m] = \mathbb{V}(\mathfrak{sl}_{r+1}, m \lambda, m \ell),
\]
where \(m \lambda = (m \lambda_1, \ldots, m \lambda_n) \in P_{m \ell}(\mathfrak{sl}_{r+1})^n\). We often refer to the new bundles \(\mathbb{V}[m]\) as **multiples** of \(\mathbb{V}\), and we say they are obtained by **stretching** the Lie data used to form \(\mathbb{V}\).

Using the \(\mathbb{V}[m]\), one can form a flat sheaf of algebras (see [Fal94, p. 368], [Man09]):
\[
(2.1) \quad \mathcal{A} = \oplus_{m \geq 0} \mathcal{A}_m = \oplus_{m \geq 0} \mathbb{V}[m]^*,
\]
over \( \overline{M}_{g,n} \). At any interior point \( x \in M_{g,n} \), one has a natural identification of graded algebras

\[
\mathcal{A}_x = \oplus_{m \in \mathbb{Z}_{\geq 0}} (\mathcal{A}_m)_x = \oplus_{m \geq 0} (V[m])_x = \oplus_{m \geq 0} H^0(X, L^\otimes m).
\]

Here, for \( x = (C, p_1, \ldots, p_n) \in M_{g,n} \), one has that \( X_x = X_x(s_{l+r+1}, \tilde{\lambda}, t) \) is a (GIT) moduli space of parabolic bundles of rank \( r + 1 \) with trivial determinant on the curve \( C \) with parabolic structures at \( p_1, \ldots, p_n \), and \( L_x = L_x(s_{l+r+1}, \tilde{\lambda}, t) \) an ample line bundle.

**Remark 2.9.** We write \( V_x^\ast \cong H^0(X, L_x) \) to indicate that \( X_x = X_x(s_{l+r+1}, \tilde{\lambda}, t) \) and \( L_x = L_x(s_{l+r+1}, \tilde{\lambda}, t) \) for \( x \in M_{g,n} \).

We ask if a similar description for the algebra of conformal blocks holds at points \( x \in \overline{M}_{g,n} \setminus M_{g,n} \):

**Question 2.10.** Given a vector bundle of conformal blocks \( \mathcal{V} \) on \( \overline{M}_{g,n} \) in type A, and \( x \in \overline{M}_{g,n} \setminus M_{g,n} \) is there a polarized scheme \((X_x, L_x)\) such that (2.2) holds as graded algebras?

Note that by Remark 2.12, such \( X_x \) (if they exist) are necessarily reduced and irreducible. Moreover, in this question we do not require that \((X_x, L_x)\) fit together into a flat family. Question 2.10 is a point-wise version of the question of existence of a family \( \pi : \mathcal{X} \to \overline{M}_{g,n} \) with relatively ample bundles \( L \) such that \( \pi_* L^\otimes m = V[m] \) consistent with multiplication operations. One could then hope to (recursively) control the Chern classes of \( V[m] \) by applications of the Grothendieck-Riemann-Roch formula. In the situation when \( \Delta(X_x, L_x) = 0 \) for \( x \in M_{g,n} \), the point-wise question is implied by the family question by cohomology vanishing ([Fuj90, Chapter 1, (5.1)] and Remark 3.1).

**Definition 2.11.** Given a vector bundle of conformal blocks \( \mathcal{V} \) on \( \overline{M}_{g,n} \), if the answer to Question 2.10 is yes for \( \mathcal{V} \), then we will say that \( \mathcal{V} \) has geometric interpretations at boundary points \( x \in \overline{M}_{g,n} \setminus M_{g,n} \).

**Remark 2.12.** We note that \( \mathcal{A}_x \) for \( x \in \overline{M}_{g,n} \) is an integral domain, since it is a subalgebra (formed by suitable Lie-algebra invariants) of the algebra of sections of a line bundle on the ind-integral affine Grassmannian times an n-fold product of complete flag varieties (see e.g., [LS97, Section 10]). Therefore, \( X_x \) are necessarily irreducible varieties when they exist.

The algebras \( \mathcal{A}_x \) are finitely generated for \( x \in M_{g,n} \), since the coordinate ring of a polarized variety is finitely generated. We do not know if \( \mathcal{A}_x \) are finitely generated for \( x \in \overline{M}_{g,n} \setminus M_{g,n} \). If \( \mathcal{A}_x \) are finitely generated for \( x \in \overline{M}_{g,n} \setminus M_{g,n} \) then, \( X_x \) (if they exist) will coincide with \( \text{Proj}(\mathcal{A}_x) \). However \( \text{Proj}(\mathcal{A}_x) \) need not carry an ample line bundle \( L_x \) whose section ring equals \( \mathcal{A}_x \).

3. **Divisor class formulas in case extensions exist**

### 3.1. Resolutions of ideals of projective varieties of minimal degree.

Let \( I_\bullet \subset O_P \) be the ideal associated to a projective variety \( X_\bullet \subset \text{Proj}(B_\bullet) = \mathbb{P}^N \), where \( B_\bullet = \oplus_{m \geq 0} B_m \), and \( B_m = \text{Sym}^m(A_1) \). The Hilbert Syzygy Theorem says that the graded ideal \( I_\bullet \) has a finite graded free resolution

\[
0 \to \mathcal{F}_k \to \cdots \to \mathcal{F}_{k-1} \to \cdots \to \mathcal{F}_0 \to I_\bullet \to 0,
\]

where \( k \leq N + 1 \).

If the pair \((X, L)\) is a polarized variety having \( \Delta \)-invariant zero, then \( L \) is very ample on \( X \), then from \( L \) one obtains an embedding \( X \hookrightarrow \text{Proj}(B_\bullet) \), as a variety of minimal degree. In particular, the ideal \( I_X \) of \( X \) is the kernel of the surjective map

\[
\phi : B_\bullet = \oplus_{m \in \mathbb{Z}_{\geq 0}} \text{Sym}^m(H^0(X, L)) \to A_\bullet = \oplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, L^\otimes m),
\]

and \( I_X \) is known to have a minimal free resolution of the form

\[
0 \to W_D \otimes O(-D) \to \cdots \to W_3 \otimes O(-3) \to W_2 \otimes O(-2) \to I_X \to 0
\]

where the \( W_i \) for \( i \in \{1, \ldots, D\} \), are vector spaces of dimension \( (i - 1)(i^2)/2 \), and \( D = \deg(L) \) [Nag07, EG84].
Remark 3.1. Let \( F_k \) be the image of the map \( W_{k+1} \otimes O(-k-1) \to W_k \otimes O(-k) \) in (3.2). Then the exactness of (3.2) implies that \( H^i(P^N, F_k(m)) = 0 \) for all \( m \geq 0 \) and \( i > 0 \), and for \( i = 0 \) and \( m < k + 1 \). This is proved by induction using the exact sequences

\[
0 \to F_k \to W_k \otimes O(-k) \to F_{k-1} \to 0.
\]

Note that this also implies that \( H^i(P^N, I_X(m)) = H^i(X, L^{\otimes m}) = 0 \) for all \( i \geq 1 \), and \( m \geq 0 \) and \( H^0(X, L^{\otimes m}) = H^0(P^N, O(m))/H^0(P^N, I_X(m)) \).

Minimal free resolutions in the context of graded rings are known to be unique up to unique isomorphisms. The same technique shows that the complex (3.2) is unique up to a unique isomorphism. Twisting (3.2) by \( O(k) \) with \( 2 \leq k \leq D \), and then global sections gives an exact sequence (see Remark 3.1)

\[
0 \to W_k \to W_{k-1} \otimes \text{Sym}^1(B_1) \to \cdots \to W_2 \otimes \text{Sym}^{k-2}(B_1) \to H^0(P^N, I(k)) \to 0.
\]

Therefore, \( W_k \) can be reconstructed uniquely, as a kernel, by induction.

It is easy to see that we can form the above process and produce spaces \( W_i \) even if we do not have a resolution of \( I_X \) of the form (3.2). The end product is a complex which may not be exact. The following lemma is now immediate: (For a graded abelian group \( M \), \( M^{(-s)} \) is a graded abelian group with \( M^{(-s)}_m = M_{m-s} \)).

Lemma 3.2. Given a pair \((X,L)\), having \( \Delta \)-invariant zero, with notation as above, there are canonical vector spaces \( W_i \), for \( 2 \leq i \leq D \), and a long exact sequence of the form:

\[
(3.4) 0 \to W_D \otimes B_m(-D) \xrightarrow{\phi_1} \cdots \to W_3 \otimes B_m(-3) \xrightarrow{\phi_1} W_2 \otimes B_m(-2) \xrightarrow{\phi_0} I_m \to 0,
\]

giving a graded resolution of the graded ideal \( I_m = \oplus_{m \geq 0} I_m \) with \( I_m = H^0(P^N, I_X(m)) \). In particular

\[
(3.5) 0 \to W_D \otimes B_{m-D}(-D) \xrightarrow{\phi_1} \cdots \to W_3 \otimes B_{m-3}(-3) \xrightarrow{\phi_1} W_2 \otimes B_{m-2}(-2) \xrightarrow{\phi_0} I_m \to 0
\]

is a canonical resolution of the degree \( m \) part \( I_m \).

Returning to the setting of a conformal blocks bundle \( V \) on \( \overline{M}_{g,n} \), suppose we have a map \( \pi : X \to \overline{M}_{g,n} \) and a relatively ample line bundle \( L \) on \( X \), so that \( \pi_* L^{\otimes m} = V[m]^* \) (consistent with stretching), and \( \Delta(X, L) = 0 \). Then the canonicality of the spaces \( W_i \) in Lemma 3.2 then can be expected to lead to a resolution of the form (3.7) below of \( V[m]^* \) in terms of \( V \) and a finite number of unknown bundles \( W_k \). This gives recursions for the first Chern classes.

But we choose to work with a hypothesis of point-wise existence of pairs \((X,x)\), without assuming that they glue together into a flat family. This hypothesis is weaker than the families assumption above when the \( \Delta \)-invariant is zero (see Section 2.3).

3.2. Corollary 3.6. Let \( \mathcal{A}_m = \oplus_{m \geq 0} \mathcal{B}_m \) be a sheaf of algebras on a Deligne-Mumford stack \( \mathcal{N} \), with \( \mathcal{B}_m \) vector bundles, and \( \mathcal{O}_0 = O_N \). Let \( \mathcal{B}_m = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{B}_m \), where \( \mathcal{B}_m = \text{Sym}^m \mathcal{A}_1 \). Consider \( \phi_m : \mathcal{B}_m = \text{Sym}^m_{\mathcal{O}_N} (\mathcal{A}_1) \to \mathcal{A}_m \) and \( I_m = \ker(\phi_m) \). Clearly \( I_m \) is a graded \( \mathcal{B}_m \) ideal.

Proposition 3.3. With the notation from above, there is a complex of sheaves of graded \( \mathcal{B}_m \)-modules \( F_i \)

\[
(3.6) \cdots \to \mathcal{F}_k \to \cdots \to \mathcal{F}_{k-1} \to \cdots \to \mathcal{F}_0 \to I_m
\]

For a given \( V \), we let \( \mathcal{A} = \oplus_{m \geq 0} \mathcal{B}_m \), where \( \mathcal{B}_m = V[m]^* \). Suppose that geometric interpretations for \( V \) exist (Def 2.11), so that for \( x \in \mathcal{N} = \overline{M}_{g,n} \) we have \( \mathcal{A}_x \cong \oplus_{m \geq 0} H^0(X, L^{\otimes m}) \), and suppose further that \((X_x, L_x)\) has \( \Delta \)-invariant zero for some (and hence every) \( x \in \overline{M}_{g,n} \). Then (3.6) takes the form of

\[
(3.7) 0 \to W_D \otimes \text{Sym}^{m-D}_{\mathcal{O}_N} (\mathcal{A}_1) \to \cdots \to W_2 \otimes \text{Sym}^{m-2}_{\mathcal{O}_N} (\mathcal{A}_1) \to \text{Sym}^m_{\mathcal{O}_N} (\mathcal{A}_1) \to V[m]^* \to 0,
\]

with fibers equal to the resolution of the \( \ker(\phi_m) \) given in Lemma 3.2. Furthermore, (3.7) is exact and \( W_i \) are locally free of finite rank.
Proof. Let \( I^0 = I \). We note that \( I^0 \) is a graded \( B \)-module. Because \( I^0 \) is an ideal, there is a morphism \( I^0 \otimes \mathcal{O}_{M_{g,n}} \rightarrow I \), and we set \( I^1_m = \text{ker}(a_{m}^0) \). Now we consider the morphism \( I^1_m \otimes \mathcal{O}_{M_{g,n}} \rightarrow I^2_m \), and we set \( I^2_m = \text{ker}(a_{m}^1) \). At the \( i \)-th step we consider the morphism \( I^i_m \otimes \mathcal{O}_{M_{g,n}} \rightarrow I^{i+1}_m \), and set \( I^{i+1}_m = \text{ker}(a_{m}^{i+1}) \).

The terms in (3.1) are defined as follows. The sheaf \( \mathcal{F}_0 = \oplus_m I^0_m \otimes \mathcal{O}_{M_{g,n}} \) with \( \mathcal{F}_i = \oplus_m I^i_m \otimes \mathcal{O}_{M_{g,n}} \) for \( i \geq 0 \). It is easy to see how to define the maps in (3.1) by continuing this process. The general process is the following: if \( f : M \rightarrow N \) is a map of graded modules for a graded ring \( R \), then for any positive integer \( \psi \), \( \text{ker}(f) \otimes R(-s) \rightarrow M \rightarrow N \) is a complex (possibly non-exact).

Working on the (reduced) scheme \( \{ x \} \), we form a complex similar to (3.1). Let \( \mathcal{A}_m = (\mathcal{A}_m) \otimes k(x) \).

\[
\cdots \rightarrow \mathcal{F}'_{k} \rightarrow \cdots \rightarrow \mathcal{F}'_{k-1} \rightarrow \cdots \rightarrow \mathcal{F}'_{0} \rightarrow \mathcal{F}'_{1}. 
\]

Here \( \mathcal{F}'_i \) is ker\( (\phi^0_m) \) where \( \phi^0_m : B' = \text{Sym}^0_{\mathcal{O}_M} (\mathcal{A}'_m) \rightarrow \mathcal{A}'_m \). From our assumption on \( \Delta \)-invariants (and Section 3.1) it follows that (3.8) is exact, and that the maps \( \phi^0_m \) are surjective. Note that for \( k \geq 2 \), \( \mathcal{F}_k \) coincides with \( \mathcal{W}_k \) in Lemma 3.2 (as a suitable kernel from (3.3)). Now \( \mathcal{I}_m \) is a kernel of a surjective map of vector bundles since \( \phi^0_m \) are surjective, and the map \( \mathcal{I}_m \otimes k(x) \rightarrow \mathcal{I}_m \) is an isomorphism.

For the inductive step, fix a point \( x \), and suppose \( \psi : \mathcal{G}_m \rightarrow \mathcal{H}_m \) is morphism of graded \( B \)-modules. Let \( \mathcal{F}_m \) be the kernel of \( \psi \) and \( \mathcal{F}'_m \) the kernel of \( \psi \otimes k(x) \) and \( s \in \mathbb{Z} \). Assume

- \( \mathcal{G}_m \) and \( \mathcal{H}_m \) are vector bundles for all \( m \);
- images of \( \psi_m \) are locally free subbundles of \( \mathcal{H}_m \) in the localization \( \text{Spec}(O_x) \); and
- the three term sequence on fibers is exact: \( \mathcal{F}'_s \otimes \mathcal{B}'_*(-s) \rightarrow \mathcal{G}_m \otimes k(x) \rightarrow \mathcal{H}_m \otimes k(x) \).

Then \( \mathcal{F}_m \otimes \mathcal{B}_m[\cdot - s] \rightarrow \mathcal{G}_m \rightarrow \mathcal{H}_m \) is exact over \( \text{Spec}(O_x) \), and hence \( \mathcal{F}_m \) is the image of \( \mathcal{F}_s \otimes \mathcal{B}_m \) in \( \mathcal{G}_m \), and is a subbundle of the latter. The next step in the induction is with \( \psi \) the map \( \mathcal{F}_s \otimes \mathcal{B}_m \rightarrow \mathcal{G}_m \).

We work over \( \text{Spec}(O_x) \). Note that \( 0 \rightarrow \mathcal{F}_m \rightarrow \mathcal{G}_m \rightarrow \text{im}(\psi)_m \rightarrow 0 \) is an exact sequence, and therefore, \( \mathcal{F}_m \) are vector bundles. Now tensoring it by \( k(x) \) we obtain another exact sequence. Note that \( \text{im}(\psi)_m \otimes k(x) \rightarrow \mathcal{H}_m \otimes k(x) \) is injective since \( \text{im}(\psi)_m \rightarrow \mathcal{H}_m \) is a subbundle. This tells us that \( \mathcal{F}_m \otimes k(x) \rightarrow \mathcal{F}'_m \) is an isomorphism. Finally \( \mathcal{F}_s \otimes \mathcal{B}_m \rightarrow \mathcal{F}_m \) is a surjection of vector bundles because it is so after tensoring with \( k(x) \) by our third hypothesis. \( \square \)

We set \( \text{Sym}^i \mathcal{V} = 0 \) for \( i < 0 \), and \( \text{Sym}^0 \mathcal{V} = \mathbb{C} \).

**Remark 3.4.** If \( \mathcal{V} \) is a vector bundle of rank \( R \) then \( \text{rk}(\text{Sym}^m \mathcal{V}) = \binom{m+R-1}{R-1} \), and \( c_1(\text{Sym}^m \mathcal{V}) = \binom{m+R-1}{R}c_1(\mathcal{V}) \).

**Definition 3.5.** We say that \( \mathcal{V} \) has \( \Delta \)-invariant zero rank scaling if \( \Delta(X_x, L_x) = 0 \) for \( x \in \mathcal{M}_{g,n} \) such that \( \mathcal{V}^0_x \cong H^0(X_x, L_x) \). The function \( f(m) = \text{rk}(\mathcal{V}^m) \) determines \( \Delta(X_x, L_x) \): \( f(1) = h^0(X_x, L_x) \), the dimension of \( X_x \) is the degree of \( f(m) \) and the degree \( \text{dim}_{X_x} L_x \) is \( \text{dim} X_x \) times the leading coefficient of \( f(m) \). We will refer to \( L_x \) as the volume of the block \( \mathcal{V} \).

Note that the higher cohomology \( H^i(X_x, L_x^m) \) vanishes for \( i > 0, m \geq 0 \) by [Tel00, Theorem 9.6], so the Hilbert polynomial is determined by the first few values of \( \text{rk}(\mathcal{V}^m) \). The dimension of \( X_x \) can be bounded above, in genus \( 0 \) it is no more than the dimension of the corresponding flag variety. There is also an explicit formula for \( L_x^d \) due to Witten [Wit91] (also see Sections 3, 4 in [BBV15] for examples).

Using Proposition 3.3, and Remark 3.4, we obtain
Corollary 3.6. Suppose that \( V \) has \( \Delta \)-invariant zero rank scaling, and assume that geometric interpretations exist for \( V \) at all points (see Definition 2.11). Then,

\[
(3.9) \quad c_1(V[m]) = \left( \frac{m + R - 1}{R} \right) c_1(V) + \sum_{i=2}^{D} (-1)^{i-1} \left( \frac{D}{i} \left( \frac{m - i + R - 1}{R} \right) \right) c_1(W_i).
\]

Remark 3.7. For the validity of (3.9), we only need geometric interpretations on the generic points of all boundary divisors, which would imply that (3.9) holds in \( \text{Pic}(\overline{M}_{g,n} - Z) \) with \( Z \) of codimension 2. Since Picard groups are unaffected by removal of codimension 2 substacks, (3.9) would then hold on \( \overline{M}_{g,n} \).

One can determine the \( c_1(W_i) \) in terms of \( c_1(V), \ldots, c_1(V[m]) \) by setting \( m = 2, \ldots, D \) in (3.9). When one substitutes \( m = i \) with \( 2 \leq i \leq D \), then (3.9) is of the form \( (-1)^i c_1(W_i) \) plus a linear expression in \( c_1(W_2), \ldots, c_1(W_{i-1}) \), and \( c_1(V) \). These identities, which follow if geometric interpretations for \( V \) exist at boundary points, can be used to show that geometric interpretations do not extend across the boundary (see Examples 3.8 and 5.8). While positive examples are inconclusive with regard to geometric interpretations, they do give information about the location of multiples in the nef cone (see Examples 3.9 and 10.1). Formulas are given throughout the first half of the paper (see Corollaries 7.3 (quadric hypersurfaces), 8.3 (the Veronese surface), 9.1 (rational normal scrolls), and 10 (rational normal curves)). In the cases of projective space and quadric rank scaling we give direct proofs of such identities (see Lemma 5.7 and Corollary 7.3).

3.3. Examples. In Example 3.8 (also see Example 5.8) we give a bundle that satisfies projective space rank scaling but for which the scaling identity of Theorem 5.9 does not hold. In Example 3.9 we give the formulas a bundle \( V \) must satisfy if it has rational normal curve scaling in degree 2, and if geometric interpretations exist.

Example 3.8. Consider \( V = V(sl_2, 1) \) on \( \overline{M}_2 \). By [NR69], for \( x \in M_2 \) one has \( V_x^\ast \cong H^0(X_x, L_x) \), where \( (X_x, L_x) \cong (\mathbb{P}^3, O(1)) \). By Corollary 3.6 (also see Lemma 5.7), if \( V \) has geometric interpretation at boundary points (see Definition 2.11), then

\[
(3.10) \quad c_1(V[m]) = \left( \frac{m + 3}{4} \right) c_1(V) = \frac{(m + 3)(m + 2)(m + 1)m}{24} \cdot c_1(V)
\]

which we can show fails by intersecting with F-curves.

There are two types of F-curves on \( \overline{M}_2 \). The first is the image of a clutching map from \( \overline{M}_{0,4} \) for which points are identified in pairs. The second is the image of a map from \( \overline{M}_{1,1} \) given by attaching a point \( (E, p) \in M_{1,1} \), gluing the curves at the marked points. We will see a contradiction when we intersect with either type of F-curve.

Intersecting both sides of Equation 3.10 with the F-curve defined by \( \overline{M}_{0,4} \). By the formulas of Fakhruddin [Fak12], we can see that if \( D_m = 0 \), then the intersection of Equation 3.10 with a pigtail type F-curve would imply the following identity:

\[
(3.11) \quad \sum_{0 \leq \lambda, \mu \leq m} \deg V(sl_2, \{ \lambda, \lambda, \mu, \mu \}, m) = \frac{(m + 3)(m + 2)(m + 1)m}{24}.
\]
Here we note \( \lambda^* = \lambda \) for \( \mathfrak{sl}_2 \). But Equation 3.11 doesn’t hold, for example, for \( m = 2 \), the right hand side equals 5, whereas there are three non-zero terms on the left hand side:

\[
\deg \mathcal{V}(\mathfrak{sl}_2, \{w_1, aw_1, 2aw_1, 2w_1\}, 2) = \deg \mathcal{V}(\mathfrak{sl}_2, \{2aw_1, 2aw_1, aw_1, w_1\}, 2) = 1, \quad \deg \mathcal{V}(\mathfrak{sl}_2, \{2aw_1, 2aw_1, 2aw_1, 2w_1\}, 2) = 2,
\]

which add up to 4. We note that using Witten’s Dictionary, one can verify that

\[
\sum_{0 \leq \lambda, \mu \leq m} \text{rk} \mathcal{V}(\mathfrak{sl}_2, \{\lambda, \lambda, \mu, \mu\}, m) = \frac{(m+3)(m+2)(m+1)}{6}.
\]

Intersecting both sides of Equation 3.10 with the F-curve defined by \( \overline{M}_{1,1} \). If Equation 3.10 held, then by pulling back along the map from \( \overline{M}_{1,1} \) onto the second type of F-curve, one would have that

\[
\sum_{\mu=0}^{m} r_{\mu}^{(m)} c_{\mu}^{(m)} = \frac{(m+3)(m+2)(m+1)m}{24} \cdot (r_{1}^{(1)} c_{1}^{(1)} + r_{0}^{(0)} c_{0}^{(0)}) = -\frac{(m+3)(m+2)(m+1)m}{24} \cdot 2 \cdot \frac{1}{6},
\]

where \( r_{\mu}^{(m)} \) and \( c_{\mu}^{(m)} \) are ranks and first Chern classes of the bundle \( \mathcal{V}(\mathfrak{sl}_2, \mu, m) \) on \( \overline{M}_{1,1} \), respectively. We know that \( r_{\mu}^{(m)} = 0 \) if \( \mu \) is odd and equals \( m+1-\mu \) if \( \mu \) is even. The degree \( c_{\mu}^{(m)} = c_{\mu}^{(m)} \) is given by [Fak12, Corollary 6.2]: if \( \mu \) is even, \( c_{\mu}^{(m)} = -\left(\frac{m^2-3m+2m^2+2m}{24}\right) \). Consider \( m = 2 \), then

\[
\sum_{\mu=0}^{m} r_{\mu}^{(m)} c_{\mu}^{(m)} = r_{0}^{(2)} c_{0}^{(2)} + r_{2}^{(2)} c_{2}^{(2)} = -\frac{3}{2} - \frac{2}{24} = -\frac{19}{12}.
\]

The right hand side of (3.13) is \( -\frac{19}{6} \). We again see that (3.10) fails.

As we explain later in Example 4.4, \( \mathcal{V} \) has a socle which fails to have projective space rank scaling, a property on our list of conditions which will guarantee, for a particular bundle, that divisor class identities hold, and in the case that \( \mathcal{V} \) has projective space rank scaling, the answer to Question 1 is yes.

**Example 3.9.** If \( \mathcal{V} \) satisfies \((\mathcal{P}^1, \mathcal{O}(2))\) scaling, and if geometric interpretations exist for \( \mathcal{V} \) at boundary points, then by Corollary 3.6

\[
\text{c}^{(1)}(\mathcal{V}[m]) = \left(\begin{pmatrix} m+2 \\ 3 \end{pmatrix} - 4 \begin{pmatrix} m \\ 2 \end{pmatrix} - \begin{pmatrix} m \\ 3 \end{pmatrix}\right) \text{c}^{(1)}(\mathcal{V}) + \begin{pmatrix} m \\ 2 \end{pmatrix} \text{c}^{(1)}(\mathcal{V}[2]).
\]

For example on \( \overline{M}_{1,1} \), let \( \mathcal{V}_k^2 = \mathcal{V}(\mathfrak{sl}_2, 2k\omega_1, 2(k+1)) \). By [Fak12, p. 27], if \( i \) is even, then the rank of \( \mathcal{V}(\mathfrak{sl}_2, i\omega_1, \ell) \), is \( \ell + 1 - i \). So for \( \mathcal{V}[m]_k^2 = \mathcal{V}(\mathfrak{sl}_2, m(2k)\omega_1, m(2(k+1))) \), one has \( \text{rk}(\mathcal{V}[m]_k^2) = 2m + 1 \), and one has that \( \mathcal{V}_k^2 \) satisfies \((\mathcal{P}^1, \mathcal{O}(2))\) scaling. By [Fak12, Corollary 6.2],

\[
\deg(\mathcal{V}[m]_k^2) = -\frac{m(2m+1)(k+1)}{12},
\]

and using this one can easily verify Equation (3.14) (which also follows Proposition 7.5).

**Remark 3.10.** By [KP95, Theorem E], given \( \mathcal{V} \) on \( \overline{M}_{g,n} \), with \( n = 0 \), the corresponding polarized varieties \((\mathcal{X}_x, \mathcal{L}_x)\) cannot be of Delta invariant zero, for genus \( g \geq 3 \), \( x \in \mathcal{M}_g \).

### 4. Free restriction data and quasi rank one restriction behavior

**Definition 4.1.** Given a bundle \( \mathcal{V} = \mathcal{V}(\mathfrak{sl}_{r+1}, \lambda, \ell) \) on \( \overline{M}_{g,n} \), we say that the **restriction data for \( \mathcal{V} \) is free**, if given any boundary point \( x \in \Delta_{g,1} = \Delta_{g-1,1} \), or \( x \in \Delta_{irr} \), and \( \alpha_1, \ldots, \alpha_P \) the \( P \) weights of restriction data for \( \mathcal{V}(\mathfrak{sl}_{r+1}, \lambda, \ell) \) at \( x \), then

\[
\sum_{i=1}^{P} a_i \alpha_i = 0 \implies \sum_{i=1}^{P} a_i \neq 0.
\]
Definition 4.2. Suppose that $V = V(sl_{r+1}, \lambda, \ell)$ on $\overline{M}_{g,n}$ is a bundle of rank $R$ with $\Delta$-invariant zero rank scaling. We say that $V$ satisfies quasi rank one factorization if given any boundary point $x \in \Delta_{g,l} = \Delta_{g,\bar{g},f'}$, (or $x \in \Delta_{\text{irr}}$), and $\alpha_1, \ldots, \alpha_P$ the $P$ weights of restriction data for $V(sl_{r+1}, \lambda, \ell)$ at $x$, then at most one of the $P$ restrictions
\[ \text{rk } V(sl_{r+1}, \lambda(f^c) \cup \{ \alpha_i^* \}, \ell), \text{ if } x \in \Delta_{g,l} = \Delta_{g,\bar{g},f'} \]
or
\[ \text{rk } V(sl_{r+1}, \lambda \cup \{ \alpha_i, \alpha_i^* \}, \ell), \text{ if } x \in \Delta_{\text{irr}} \]
can be greater than one. A restriction of rank greater than one is called the socle.

Definition 4.3. Suppose that $V = V(sl_{r+1}, \lambda, \ell)$ on $\overline{M}_{g,n}$ is a bundle of rank $R$ which satisfies $\Delta$-invariant zero rank scaling. We say that each socle satisfies $\Delta$-invariant zero rank scaling with the same volume as that of $V$ (see Definition 3.5), if
\begin{enumerate}
  \item For any (generic) boundary point $x \in \Delta_{g,l} = \Delta_{g,\bar{g},f'}$, with socle given by restriction data $\alpha_1$, the polarized variety $X_x(sl_{r+1}, \lambda(f^c) \cup \{ \alpha_i \}, \ell) \times X_x(sl_{r+1}, \lambda(f^c) \cup \{ \alpha_i^* \}, \ell)$ with the corresponding product of ample line bundles $L_x(sl_{r+1}, \lambda(f) \cup \{ \alpha_i \}, \ell) \boxtimes L_x(sl_{r+1}, \lambda(f^c) \cup \{ \alpha_i^* \}, \ell)$ has $\Delta$-invariant zero with the same volume as that of $V$.
  \item For any (generic) $x \in \Delta_{\text{irr}}$, with socle given by restriction data $\alpha_1$, the polarized variety $(X_x(sl_{r+1}, \lambda \cup \{ \alpha_1, \alpha_1^* \}, \ell), L_x(sl_{r+1}, \lambda \cup \{ \alpha_1, \alpha_1^* \}, \ell))$ has $\Delta$-invariant zero with the same volume as that of $V$.
\end{enumerate}

Example 4.4. We saw in Example 3.8, that for $V = V(sl_2,1)$ on $\overline{M}_2$, the divisor scaling identity fails, and so the answer to Question 1 is no for this bundle. To see that $V$ has a socle which fails to have projective space rank scaling, we find the restriction data a generic point $x$ in $\Delta_{1,\bar{1}}$, which can be represented by nodal curve where each arm has genus one. There is one piece of valid restriction data at $x$, given by $\alpha = 0$, and when restricted to each arm, the bundle $V(sl_2,[0],1)$ has rank 2 [Fak12]. The polarized variety associated to the socle is $(\mathbb{P}^1 \times \mathbb{P}^1, O(1) \boxtimes O(1)) = (Q, O(1))$ where $Q$ is a quadric in $\mathbb{P}^3$. This polarized variety is of $\Delta$-invariant zero, and degree 2, and not 1 as required for an application of Theorem 5.9.

5. Projective spaces

Definition 5.1. There is a natural morphism coming from the algebra structure on the sheaf of conformal blocks $T_m : \mathcal{M}[m] \to \text{Sym}^m(\mathcal{V})$, after dualizing.

Lemma 5.2. Set $d = \text{rk } V - 1$, and suppose that $\mathcal{V}_x^\ell \equiv H^0(X_x, L_x)$ for $x \in M_{g,n}$. The following are equivalent:
\begin{enumerate}
  \item $\text{rk } (\mathcal{V}[m]) = (m+d) \binom{m+d}{d}$ for all $m$.
  \item $(X_x, L_x) = (\mathbb{P}^d, O(1))$ for all $x \in M_{g,n}$.
  \item The maps $T_m$ are isomorphisms over $M_{g,n}$.
\end{enumerate}

Remark 5.3. If $d = 0$, the condition Lemma 5.2 (a) is automatically true; it is a consequence of a quantum generalization of Fulton’s conjecture in representation theory (see [BGM14b] and the references therein). Statement (c) in this case was proved in [BGM14b, Corollary 2.2].

Proof. It is easy to see that (a) implies (b) by Proposition 2.7,(1). For (c), note that on fibers over $x \in M_{g,n}$, $T_m$, from Definition 5.1, is dual to the maps
\[ \phi_m : \text{Sym}^m(H^0(X_x, L_x)) \to H^0(X_x, L_x^\otimes m) \]
which are easily verified to be isomorphisms under the assumption (b). So (b) implies (c). It is easy to see that (c) implies (a) by counting dimensions. \qed

Definition 5.4. $V$ has projective space rank scaling if the equivalent conditions of Lemma 5.2 hold.
By Proposition 2.7, \( \mathcal{V} \) has projective space rank scaling if and only if it has \( \Delta \)-invariant zero rank scaling with degree (i.e., volume) 1.

**Proposition 5.5.** If \( \mathcal{V} \) has projective space rank scaling, then for \( m \geq 1 \)

\[
(5.1) \quad \binom{m + d}{d + 1} c_1(\mathcal{V}) = c_1(\mathcal{V}[m]) + D_m,
\]

where \( D_m \) is an effective Cartier divisor supported on \( \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \). Note that \( D_1 = 0 \).

**Proof.** It follows from Lemma 5.2 that if a bundle \( \mathcal{V} \) has projective space rank scaling, the maps \( T_m \) are isomorphisms on \( \mathcal{M}_{g,n} \). Taking determinants, we find a map \( \det \mathcal{V}[m] \to \det \text{Sym}^m(\mathcal{V}) \). We obtain a global section of \( (\det \mathcal{V}[m])^{-1} \otimes \det \text{Sym}^m(\mathcal{V}) \), and can write \( (\det \mathcal{V}[m])^{-1} \otimes \det \text{Sym}^m(\mathcal{V}) = \mathcal{O}(D_m) \) for an effective Cartier divisor \( D_m \) supported on \( \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \). This gives the assertion. \( \square \)

**Definition 5.6.** We will call the divisor \( D_m \) in Equation 5.1 the **projective space stretching anomaly**.

We make the following basic observation:

**Lemma 5.7.** Suppose \( \mathcal{V} \) has projective space rank scaling. The following are equivalent.

(a) For each \( x \in \overline{\mathcal{M}}_{g,n} \), there exists a pair \( (\mathcal{X}_x, \mathcal{L}_x) \) of a projective scheme and an ample line bundle such that \( \mathcal{A}_x \cong \bigoplus_{m \geq 0} H^0(\mathcal{X}_x, \mathcal{L}_x^\otimes m) \).

(b) \( D_m = 0 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \) for all \( m \geq 2 \).

(c) The maps \( T_m \) are isomorphisms, so that \( \mathcal{V}[m] = \text{Sym}^m(\mathcal{V}) \) for all \( m \).

**Proof.** If (a) holds then as in Lemma 5.2, \( (\mathcal{X}_x, \mathcal{L}_x) = (\mathbb{P}^d, \mathcal{O}(1)) \) for all \( x \in \overline{\mathcal{M}}_{g,n} \) and \( T_m \) is dual to the maps \( \text{Sym}^m(H^0(\mathcal{X}_x, \mathcal{L}_x)) \to H^0(\mathcal{X}_x, \mathcal{L}_x^\otimes m) \), which are isomorphisms since \( (\mathcal{X}_x, \mathcal{L}_x) = (\mathbb{P}^d, \mathcal{O}(1)) \). Therefore \( T_m \) is an isomorphism over \( \overline{\mathcal{M}}_{g,n} \) and \( D_m = 0 \). Therefore (b) and (c) hold. If (b) holds then since \( D_m \) is an effective Cartier divisor, we conclude that \( \mathcal{O}(D_m) = \mathcal{O} \) and hence \( T_m \) is an isomorphism. If (c) holds we can take \( (\mathcal{X}_x, \mathcal{L}_x) = (\mathbb{P}(\mathcal{V}_x), \mathcal{O}(1)) \), and therefore (a) holds. \( \square \)

**Example 5.8.** In Example 3.8 we can compute the projective stretching anomalies \( D_m \) as follows. Write \( D_m = \alpha(m)\Delta_0 + \beta(m)\Delta_{1,0} \). By [AC98], the intersection numbers of \( \Delta_0 \) and \( \Delta_{1,0} \) with the pig-tail F-curve, are \(-2\) and \(1\) respectively; and with the other F-curve (the \( \overline{\mathcal{M}}_{1,1} \) type) are \(1\) and \(-1/12\) respectively. We can compute the intersection of \( D_m \) with these curves as in Example 3.8. Note that in [Fak12, Corollary 6.2], the degree of the Hodge bundle is \( 1/12 \).

In Example 3.8, we get the equations \(-2\alpha(2) + \beta(2) = 5 - 4 = 1 \) and \( \alpha(2) = 1/12 \beta(2) = -10/5 + 19/12 = -1/12 \). Therefore \( \alpha(2) = 0 \) and \( \beta(2) = 1 \).

Using [Fak12, Corollary 6.2] and B. Alexeev’s formula [Swi09, Lemma 3.3], we obtain \( \alpha(m) = 0 \), and that \( \beta(m) \) is equal to \( \frac{s(s+1)(2s^2+2s-1)}{6} \) if \( m = 2s \) is even, and \( \frac{s(s+1)^2(s+2)}{3} \) if \( m = 2s + 1 \) is odd. Therefore geometric interpretations extend across \( \Delta_0 \) (the corresponding factorization is not quasi-rank one (Definition 4.2)).

**Theorem 5.9.** Suppose \( \mathcal{V} \) is a bundle on \( \overline{\mathcal{M}}_{g,n} \) such that

(1) \( \mathcal{V} \) has projective space rank scaling (Def 5.4);
(2) the restriction data for \( \mathcal{V} \) is free (Def 4.1);
(3) \( \mathcal{V} \) satisfies quasi rank one factorization (Def 4.2);
(4) and such that each socle satisfies projective space rank scaling.

Then \( \mathcal{V} \) satisfies Chern class scaling:

\[
(5.2) \quad c_1(\mathcal{V}[m]) = \binom{d + m}{d + 1} c_1(\mathcal{V}).
\]
Proof. The proof proceeds by showing that the natural maps $T_m : V[m] \to \text{Sym}^m(V)$ are isomorphisms for all $m \geq 0$ on fibers over all points $x \in \overline{\mathcal{M}_{g,n}}$. First, by Lemma 5.2, it suffices to show that $T_m$ is an isomorphism over marked curves which have exactly one node. Second, since the two sides of $T_m$ have the same ranks, we only need argue the map is an injection. We make the argument for points in $\Delta_{g,1} = \Delta_{g-\frac{1}{2},2}$, as the argument for points in $\Delta_0$ is analogous. For simplicity we write $\Delta_f$ for $\Delta_{g,1}$.

If $x \in \Delta_f$ and the curve corresponding to $x$ is $C_1 \cup C_2$ with “normalization” $\tilde{C} = C_1 \cup C_2$ with two extra marked points $a$ and $b$ we have a factorization

$$V[m]|_x = \oplus_{\mu \in P(m)} \tilde{V}(sl_{r+1}, \{m \lambda, \mu, \mu^*\}, m\ell)|_y$$

where $y$ is the pointed curve $\tilde{C}$. Here the $\tilde{V}$ are conformal blocks for $\tilde{C}$, these break up into tensor products of blocks for $C_1$ and $C_2$. There are natural maps

$$(5.3) \quad \tilde{V}^\ast(sl_{r+1}, \{m \lambda, \mu, \mu^*\}, m\ell) \otimes \tilde{V}^\ast(sl_{r+1}, \{m' \lambda, \nu, \nu^*\}, m'\ell) \to \tilde{V}^\ast(sl_{r+1}, \{(m + m') \lambda, (\mu + \nu), (\mu + \nu)^*\}, (m + m')\ell),$$

inducing an algebra structure on $\oplus V[m]|_x$ which we denote by $\bar{A}_x$. Manon showed that the algebra $\bar{A}_x$ is a degeneration of $A_x$ [Man09, Prop 3.3]. Therefore, it suffices to show that the map $\text{Sym}^m(V|_x) \to V[m]|_x$ is an isomorphism in the algebra $\bar{A}_x$.

Assuming that the socle exists for our stratum (otherwise take $B = C$ below), and is $\tilde{V}(sl_{r+1}, \{\lambda, \alpha_1, \alpha_1^*\}, \ell)$ of rank $R_1$, we let $\alpha_2, \ldots, \alpha_p$ be the other terms in the factorization. Pick non-zero generators $y_2, \ldots, y_p$ of the spaces $\tilde{V}^\ast(sl_{r+1}, \{\lambda, \alpha_i, \alpha_i^*\}, \ell), i > 2$. Let $S$ be the conformal blocks algebra of the socle

$$S = \oplus_{m \geq 0}(\tilde{V}^\ast(sl_{r+1}, \{m \lambda, m \alpha_1, m \alpha_1^*\}, m\ell)|_y).$$

$S$ is a graded ring, the total section ring of a polarized pair $(M, L)$ ($\tilde{C}$ is smooth). Since the socle satisfies projective space rank scaling, we find that $(M, L) = (\mathbb{P}^{R_1-1}, O(1))$, and hence $S$ is a polynomial algebra in $R_1$ variables.

We may form a new graded ring $C = S[Y_2, \ldots, Y_p]$ with graded pieces

$$C_m = \oplus_{m_1 + \ldots + m_p = m} S_{m_1} Y_{m_2}^2 \ldots Y_{m_p}^p.$$ 

Therefore $C$ is a polynomial algebra and the maps $\text{Sym}^m(C_1) \to C_m$, are isomorphisms. Now note that the natural map of algebras $C \to \bar{A}_x$ is an isomorphism. To see this, write $\bar{A}_x = \oplus \bar{A}_m$. The natural algebra map $C \to \bar{A}_x$ sends:

$$S_{m_1} Y_{m_2}^2 \ldots Y_{m_p}^p \to (\tilde{V}^\ast(sl_{r+1}, \{m \lambda, \mu, \mu^*\}, m\ell))|_y,$$

with $\mu = m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_p \alpha_p, m = \sum m_i$. It follows that different tuples $(m_1, \ldots, m_p)$ map to different direct summands in $\bar{A}_m$. Now the rank of $\bar{A}_m$ is the same as that of $C_m$ since both satisfy projective space rank scaling. It therefore suffices to note that the map $S_{m_1} Y_{m_2}^2 \ldots Y_{m_p}^p$ to $(\tilde{V}^\ast(sl_{r+1}, \{m \lambda, \mu, \mu^*\}, m\ell))|_y$ is injective (by [BGM14b, Proposition 2.1]). Note that the surjection part of [BGM14b, Proposition 2.1] is valid for any genus $g$, because we may replace the use of invariants by the integral section rings of line bundles over ind-integral affine Grassmannians.

For future use, we record the following observation (here $x$ is a general point of a boundary divisor, $\mathcal{A}_x$ the algebra of Conformal blocks, and $\mathcal{A}_x$ its degeneration via the factorization formula as above).

Lemma 5.10. Suppose $\mathcal{A}_x$ is the algebra of sections of a polarized variety of $\Delta$-invariant zero. Then $\mathcal{A}_x$ is also the algebra of sections of a (possibly different) polarized variety of $\Delta$-invariant zero.

Proof. We form a family of algebras (by [Man09]) $C_x$ over $\mathbb{A}^1$ such that for $t \neq 0 \in \mathbb{A}^1$, the algebra $(C_x)_t$ is isomorphic to $\mathcal{A}_x$, and for $t = 0$ is isomorphic to $\mathcal{A}_x$. Note that the vector bundles $C_m$ are constant here, only the algebra operation varies with $t$. 

Since $\widetilde{A}_x$ is generated in degree 1, by Nakayama’s lemma and the fact that the algebras are constant for $t \neq 0$, we see that $C_*$ is generated in degree 1. We can therefore form $P = \text{Proj}(C_*)$ over $\mathbb{A}^1$. Since $C_*$ is a flat sheaf of algebras over $\mathbb{A}^1$, we get a flat family $\pi : P \to \mathbb{A}^1$ with an relatively ample line bundle $O(1)$, and maps $C_m \to \pi_*O(m)$.

The higher cohomology $H^i(\pi^{-1}(t), O(m) |_{\pi^{-1}(t)}), i > 0, m \geq 0$ vanishes for $t = 0$ because of our assumption on $\Delta$ invariants (and Remark 3.1) the polarized variety carrying the algebra $\widetilde{A}_x$ is necessarily the fiber of $\pi$ over $t = 0$, and hence for all $t$ by semi-continuity. The map $C_m \to \pi_*O(m)$ is an isomorphism on fibers at $t = 0$, and hence in a neighborhood of $t = 0$, and hence over $\mathbb{A}^1$. Therefore $\widetilde{A}_x$ which is integral, is the algebra of sections of the polarized variety $(P, O(m)|_P)$ for $t \neq 0$. The $\Delta$-invariant is constant in this family, and the desired statement follows.

**Remark 5.11.** If $V$ is a bundle of rank $R$ that satisfies projective space rank scaling, so that $\text{rk} V[m] = (\frac{m+R-1}{R-1})$, then one can use the formulas of Fakhruddin [Fak12], to reduce the proof of Theorem 5.9 to the statement for $n = 4$ by showing that divisors on both sides of the purported identity intersect all $F$-curves in the same degree. The proof breaks into cases depending on the rank of the bundle when restricted to the particular $F$-curve. Namely, (1) the restriction of $V$ to the spine = 1 but there can be a unique leg whose restriction has rank > 1; and (2) the restriction of $V$ to the legs is one but the spine rank can be > 1. The only part of the proof that requires an inductive hypothesis, is in Case 2.

**Examples.** Below we give families of examples on $\overline{M}_{1,n}$ and on $\overline{M}_{0,n}$ of bundles $V$ for which the answer to Question 1 is yes.

**Example 5.12.** On $\overline{M}_{1,1}$ we consider $V(sl_2, \lambda = 0, \ell = 1)$. As we will show, this is an example where both rank and degree scaling hold, and so for this bundle there is a geometric interpretation of the conformal block $V(sl_2, 0, 1)_{|_x}$ (see Definition 2.11), for all points $x \in \overline{M}_{1,1} \setminus M_{1,1}$. By factorization, $\text{rk} V[m] = |P_m| = m + 1$. Therefore we have projective space moduli with $d = 1$ Geometric interpretations for $V$ exist at all boundary points provided:

$$\text{deg}(V[m]) = \left(\frac{m+1}{2}\right) \text{deg} V = \frac{m(m+1)}{2} \cdot \text{deg}(V).$$

By Corollary 6.2 in [Fak12] (with $\lambda = 0$), $\text{deg}(V[m]) = \frac{m(m+1)}{12}$. Therefore $\text{deg}(V) = -\frac{1}{6}$ and (5.4) holds.

**Example 5.13.** Let $n > 1$. We define a vector bundle on $\overline{M}_{1,n}$ as follows. If $n$ is odd, let $\lambda = (2k \omega_1, ((2k + 1) \omega_1)^{n-1})$, and if $n$ is even, let $\lambda = (\omega_1, ((2k + 1) \omega_1)^{n-1})$. Define $V = V(sl_2, \lambda, 2k + 1)$. By Remark 2.3, $\text{rk} V[m] = \text{rk} V(sl_2, 2km \omega_1, m(2k + 1)) = m + 1$, so $V$ satisfies projective space rank scaling. The restriction data for $V$ at a point $x$ in the boundary $\Delta_{irr}$ is given by $\mu_1 = \mu_1^* = k \omega_1$ and $\mu_2 = \mu_2^* = (k + 1) \omega_1$, so it is free, and we have $\text{rk} V(sl_2, \mu, \mu^*, \lambda, 2k + 1) = 1$ for $i = 1, 2$. When restricting to a point $x \in \Delta_{1,\ell} = \Delta_{0,\ell}$, one also gets free restriction data (which differs depending on the parity of $\ell$), and all restrictions have rank one.

Therefore, by Theorem 5.9, a bundle $V$ satisfies projective space Chern class scaling, i.e. $c_1(mV) = \frac{(m+1)^2}{2}c_1(V)$, and there is a geometric interpretation of the conformal block $V(sl_2, \lambda, 2k + 1)_{|_x}$ (see Definition 2.11), for all points $x \in \overline{M}_{1,n} \setminus M_{1,n}$.

**Example 5.14.** On $\overline{M}_{0,4}$ let $V = V(sl_{r+1}, (\omega_i + \omega_{r+1-i}), 2)$. We will check below, using Littlewood-Richardson, that $\text{rk} V[m] = \frac{(d+m)}{m}$, where $d = 2i \leq r + 1 - 2i$ (so $i \leq \frac{d}{2}$), so these vector bundles satisfy projective space rank scaling. These bundles are $S_4$-invariant and so there is just one boundary restriction, up to symmetry. In particular, the restriction data at the boundary point is free, given by the $d + 1$ weights $\alpha_j = \omega_j + \omega_{r+1-j}$, where $1 \leq j \leq d$, and $\alpha_{d+1} = 0$. Moreover, this bundle has quasi rank one factorization, since one can check that $\text{rk} V(sl_{r+1}, (\omega_i + \omega_{r+1-i}, \omega_i + \omega_{r+1-i}, \alpha_j, 2) = \text{rk} V(sl_{r+1}, (\omega_i + \omega_{r+1-i}, \omega_i + \omega_{r+1-i}, \alpha_j, 2) = 1$, for all $j \in \{1, \ldots, d + 1\}$. 


To check the ranks of $W[m]$, we will first apply the plussing operation (see Remark 2.3), which preserves the ranks but makes these bundles into ones whose ranks can be computed using Schubert calculus. We obtain the bundles $W[m] = V(sl_{r+1}, \beta, 2m)$, where $\beta = \{(\omega_{2}), (\omega_{r+1+i})\}$. We note that $\sum_{i=1}^{4} |\mu_{i}| = 2m(r + 1)$, and so by Witten’s Dictionary, the $rk W[m]$ is the coefficient of a point in the intersection $\sigma_{m\omega_{2}}^{2} \cdot \sigma_{m\omega_{r+1}}^{2}$ in $H^{*}(Gr(r + 1, C^{r+1+2m})).$

This is the same as the coefficient of $\sigma_{(m\omega_{r+1})}^{2}$ in the product $\sigma_{m\omega_{2}}^{2} \cdot \sigma_{m\omega_{r+1}}^{2}$ in $H^{*}(Gr(r + 1, C^{r+1+2m})).$ Namely, by Littlewood-Richardson ([BCFF99] and [Ful00, page 63]), letting $\lambda = m\omega_{r+1+i}$ and $\mu = m\omega_{2i}$, $\sigma_{\lambda} \cdot \sigma_{\mu} = \sum_{\nu} N_{\sigma_{\lambda}, \sigma_{\mu}}^{\nu} \sigma_{\nu}$, where $N_{\sigma_{\lambda}, \sigma_{\mu}}^{\nu}$ is the number of reverse lattice ordered words (RLOWs), with content $\mu$ in the skew shape $\nu/\lambda$. Recall the definition of an RLOW of type $N^{w}_{\lambda, \mu}$, with content $\mu$ in the skew shape $\nu/\lambda$. A tableau of content $\mu = (\mu^{1},\ldots,\mu^{r+1})$ is a numbering of the boxes of $\nu/\lambda$ with $\mu^{1}$ 1’s, $\mu^{2}$ 2’s, $\ldots$, $\mu^{r+1}$ $r + 1$’s, which are weakly increasing across rows and strictly increasing down columns. The word of a tableau is a list of its entries, read from left to right in rows, from bottom to top. A word is reverse lattice if, from any point in it to the end, there are at least as many 1’s as there are 2’s, at least as many 2’s as 3’s, and so forth.

We note that the total shape we are filling will be a union of two Young diagrams which are adjacent to the right and below the $\lambda$ which resides in the upper left corner of a box of size $2m$ (across) and $r + 1$ (down). Suppose $A$ is the diagram to the right of $\lambda$. The last entry in the top row of $A$ will have to be filled with a 1 since $\mu$ has the same number of 1’s, 2’s, $\ldots$, $2i$’s. This will force us to fill the entire first row with 1’s since the labeling must be weakly increasing across rows. Moreover, since we must fill the boxes so that the numbers are strictly increasing downwards, we must fill the far right entry of the second row with a 2 (it can’t be larger than 2 or we will ruin the RLOW). This will now force the entire second row to be filled with 2’s. Proceeding in this way, we argue that row $j$ will be filled entirely by $j$’s ending only at row 2$r$. We see that $B$, the diagram below $\lambda$ is therefore determined by $A$ (it is $(A')^{T}$). Hence to determine the rank, we just count the number of such fillings of $A$. This is equal to the number of tableaux that one can fit in an $m \times 2i$ grid. This number is known to be $\binom{m+2i}{m}$.

Finally, when we intersect each of the $\binom{m+2i}{m}$ terms by our last class, $\sigma_{m\omega_{2}}^{2}$ then, by a similar argument, we see that the coefficient in each summand of the class of $\sigma_{(m\omega_{r+1})}^{2}$ is one. This gives the rank of $W[m]$ which is equal to the rank of $V[m]$ is $\binom{m+2i}{m}$.

We have checked that we may apply Theorem 5.9 to conclude that first Chern class scaling identities hold for this bundle $V$, and using Fakhruddin’s formula [Fak12], one can see (by a straightforward, but somewhat involved combinatorial argument), that $deg(V) = \frac{d(d+1)}{2} = (d+1)^{2}$. So $deg V[m] = \frac{(m+1)(d+1)}{2}$.

**Example 5.15.** Below we consider the family bundles $V$ on $\overline{M}_{0,n}$, for $n \geq 5$, that give $(\mathbb{P}^{d}, O(1))$, for every $d$. As before, each of these bundles satisfies the conditions of Theorem 5.9, so $D_{m} = 0$ and the geometric interpretation of conformal blocks extends across the boundary for each of these.

- For $n = 5$, we take $V(sl_{r+1}, ((\omega_{i} + \omega_{r+1+i})^{2}, \omega_{i-1} + \omega_{r+i}, 2\omega_{1}), 2)$;
- For $n \geq 6$, we let $\lambda_{j} = (\omega_{i} + \omega_{r+1+i})$ where $i \leq \left\lfloor \frac{r+1}{2} \right\rfloor$ for $j = 1, \ldots, 4$. Let $\lambda_{1} = 2\omega_{1}$ for $j = 5, \ldots, n - 1$, $s \equiv (r - n + 6) \mod (r + 1)$, and $\lambda_{n} = 2\omega_{s}$. We can define $V = V(sl_{r+1}, [\bigwedge], 2)$.

Since the bundles are obtained by plussing (see Remark 2.3), we conclude from Example 5.14 that in both cases, $rk V = \binom{d+n}{m}$, where $d = 2i \leq r + 1 - 2i$.

To check that one can apply Theorem 5.9 to conclude that the projective space scaling identities hold for divisor classes, one must compute the restriction data for the $V$ on $\Delta_{r}$ for all $I$. Treatment of $n = 5$ and $n \geq 6$ differ slightly. We outline the approach for $n \geq 6$, which breaks into the following cases:

1. $\{1,2,3,4\} \subset I$. In this case the restriction data is given by $\mu = 2\omega_{k}$, where $k \equiv (r + 1 - \frac{1}{2} \sum_{j \in I} |\lambda_{j}|) \mod (r + 1)$. There is one piece of restriction data, so it is free and just as before we have $rk V(sl_{r+1}, [\bigwedge(I)], \mu, 2) = d + 1$, and $rk V(sl_{r+1}, [\bigwedge(I)^{c}], \mu^{*}, 2) = 1$. An argument analogous
to that given in Example 5.14 shows \( \text{rk } V(\mathfrak{sl}_{r+1}, [m \vec{\lambda}(I), m \mu], 2m) = (\frac{d+m}{m}) \), so that the socle satisfies projective space rank scaling.

(2) \( a \in I, b, c, d \notin I \), where \( \{a, b, c, d\} = \{1, 2, 3, 4\} \). In this case the restriction data is given by \( \mu = a \omega_{1+s} + a \omega_{r+1-j+s} \), where the indices are modulo \( r+1 \), and \( s \) is such that \( \frac{1}{2} \sum_{i \in I, j \neq 1} |\lambda_i| \equiv (r+1-s) \pmod (r+1) \).

Clearly, the restriction data is free, moreover, we have \( \text{rk } V(\mathfrak{sl}_{r+1}, [m \vec{\lambda}(I), m \mu], 2m) = 1 \), and by reasoning as above in Example 5.14, one can conclude \( \text{rk } V(\mathfrak{sl}_{r+1}, [m \vec{\lambda}(I), m \mu], 2m) = (\frac{d+m}{m}) \).

(3) \( \{a, b\} \in I, c, d \notin I \), where \( \{a, b, c, d\} = \{1, 2, 3, 4\} \). In this case the restriction data is given by \( \mu_j = a_j + s + a_{r+1-j+s} \), where indices are taken modulo \( r+1 \), \( 0 \leq j \leq d \), and \( s \) is such that \( \frac{1}{2} \sum_{i \in I, j \neq 1} |\lambda_i| \equiv (r+1-s) \pmod (r+1) \). Restriction data is free, and for all \( j \in \{1, \ldots, d+1\} \) we have \( \text{rk } V(\mathfrak{sl}_{r+1}, \vec{\lambda}(I), \alpha_j], 2) = \text{rk } V(\mathfrak{sl}_{r+1}, \vec{\lambda}(I^C), \alpha_j^C), 2) = 1 \).

6. Theorem 6.1

In the next result we give general conditions which if satisfied by a bundle \( V \) which has \( \Delta \)-invariant 0 rank scaling, guarantee that multiples \( V[m] \) will satisfy \( \Delta \)-invariant 0 first Chern class scaling identities (3.9) given in Corollary 3.6.

**Theorem 6.1.** Given a conformal blocks bundle \( V \) on \( \overline{M}_{g,n} \) such that

1. \( V \) has \( \Delta \)-invariant 0 rank scaling (Def 3.5);
2. the restriction data for \( V \) is free (Def 4.1);
3. \( V \) satisfies quasi rank one factorization (Def 4.2), and
4. each socle satisfies \( \Delta \)-invariant 0 rank scaling with the same degree as \( V \) (Def 4.3).

Then the corresponding first Chern class scaling identities (3.9) hold for \( V \) on \( \overline{M}_{g,n} \).

**Proof.** To prove Theorem 6.1, we generalize the proof of Theorem 5.9. Use notation from the proof of Theorem 5.9. By Lemma 5.10, we need to show that \( \mathcal{A}_x \) is the algebra of sections of a polarized variety of \( \Delta \)-invariant 0 when \( x \) is the generic point of a boundary divisor. If \( S \) is the conformal blocks algebra of the socle we will need to verify that the algebra \( C \) has the same graded ranks as \( \mathcal{A}_x \). (The map \( C \rightarrow \mathcal{A}_x \) is injective for the same reason as in the proof of Theorem 5.9.)

The graded algebra \( C \) is again the algebra of sections of a polarized variety of \( \Delta \)-invariant zero (by coning). We claim that the triple of rank, degree and the dimension of this polarized variety is the same as that for \( \mathcal{A}_x \). If the triple is \( (R_1, D, d_1) \) for the socle, then the triple for the cone is \( (R_1 + p-1, D, d_1 + p-1) \). Now \( R_1 + p-1 \) equals the rank of \( (\mathcal{A}_x)_1 \), and the desired equality holds (using the \( \Delta \)-invariant zero condition). \( \square \)

**Example 6.2.** For \( n \geq 5 \), on \( \overline{M}_{0,n} \), let \( V = V(\mathfrak{sl}_4, [a_1+3a_2+a_3, 3a_1+a_2+a_3, 2a_1+2a_2+2a_3, (7a_1)^{n-4}, 7a_3], 7) \), where \( s \equiv (-n) \pmod 4 \). One computes ranks of \( V[m] \) by noticing that \( V \) is obtained by applying the plussing operation to \( V = V(\mathfrak{sl}_4, [a_1+3a_2+a_3, 3a_1+a_2+a_3, 2a_1+2a_2+2a_3, 7a_1^{n-4}, 7a_3], 7) \) on \( \overline{M}_{0,3} \) which satisfies \( \text{rk}(V[m]_3) = 2 \cdot (\frac{m+2}{3}) + (\frac{m+2}{2}) \). The ranks of the first few multiples \( V[m] \) can be computed using [Swi10]. However, to verify that it has quadric hypersurface rank scaling for \( d = 3 \), and that all the conditions of Claim 6.1 were met, we computed ranks using Witten’s Dictionary together with the plussing operation and Littlewood-Richardson counting arguments like that done in Example 5.14. So by Lemma 7.1, one has that \( V^* = H^d(X, \mathcal{L}) \), where \( \mathcal{L} \) embeds \( X \) as a quadric hypersurface in \( \mathbb{P}^4 \). Given any \( x \in \delta_1 \) for any \( I \) there will always be a unique socle with quadric rank scaling. For example, it will have restriction weight \( \mu = 0 \) at \( x \in \delta_{123} \), and the rank of \( V(\mathfrak{sl}_4, [\lambda_1, \lambda_2, \lambda_3, 0], 7) \) has quadric hypersurface scaling (for \( d = 3 \)). Therefore \( V \) satisfies the hypothesis of Theorem 6.1, so multiples \( c_1(V[m]) \) are governed by first Chern class scaling identities given in Corollary 3.6.
7. Quadric hypersurfaces

Recall that if \((X_{\lambda}, \mathcal{L}_{\lambda})\) is a polarized variety, such that the degree \(\mathcal{L}^{d\dim(X_{\lambda})} = 2\) and \(\Delta(X_{\lambda}, \mathcal{L}_{\lambda}) = 0\), then by [BS95, Prop 3.1.2], \((X_{\lambda}, \mathcal{L}_{\lambda}) = (Q, \mathcal{O}_Q(1))\) where \(Q\) is a quadric hypersurface in projective space. In this section we consider rank scaling properties and divisor class identities governing such bundles.

**Lemma 7.1.** Set \(d = \text{rk} \mathcal{V} - 2\), and suppose that \(\mathcal{V}_{|_x}^1 \equiv H^0(X_{\lambda}, \mathcal{L}_{\lambda})\) for \(x \in M_{g,n}\). The following are equivalent:

(a) \(\text{rk}(\mathcal{V}[m]) = 2^{(m+d-1)} + (m+d-1)\).

(b) \((X_{\lambda}, \mathcal{L}_{\lambda}) = (Q, \mathcal{O}_Q(1))\), where \(Q\) is a quadric hypersurface in \(\mathbb{P}^{d+1}\).

**Proof.** Assume (a). It follows that the dimension of \(X_{\lambda}\) is \(d\) and the degree \(\mathcal{L}_{\lambda}^d = 2\). Therefore \(\Delta(X_{\lambda}, \mathcal{L}_{\lambda}) = d + 2 - (d + 2) = 0\), and (b) follows from [BS95, Prop 3.1.2]. The implication \((b) \implies (a)\) is proved using the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^{d+1}}(-2) \xrightarrow{\mathcal{Q}} \mathcal{O}_{\mathbb{P}^{d+1}} \to \mathcal{O}_Q \to 0
\]

\(\square\)

**Definition 7.2.** We say that a bundle \(\mathcal{V}\) has **quadric rank scaling** if the equivalent conditions of Lemma 7.1 hold.

By Proposition 2.7, \(\mathcal{V}\) has quadric rank scaling if and only if it has \(\Delta\)-invariant zero rank scaling with degree (i.e., volume) 2.

**Corollary 7.3.** If \(\mathcal{V}\) has **quadric rank scaling** and if a geometric interpretation exists for \(\mathcal{V}\) on the boundary of \(\overline{M}_{g,n}\), then

\[
c_1(\mathcal{V}[m]) = \left(\begin{pmatrix} m + 2d + 1 \\ d + 2 \end{pmatrix} - \begin{pmatrix} m - 2 + d + 1 \\ d + 2 \end{pmatrix} - (d + 3) \begin{pmatrix} m - 2 + d + 1 \\ d + 1 \end{pmatrix}\right)c_1(\mathcal{V}) + \begin{pmatrix} m - 2 + d + 1 \\ d + 1 \end{pmatrix}c_1(\mathcal{V}[2]).
\]

**Proof.** This follows from Corollary 3.6. We indicate an alternate direct proof here. Define \(I_m\) to the kernel of \(\mu_m : \text{Sym}^m \mathcal{V}^* \to \mathcal{V}[m]^*\). There are maps \(\nu_m : I_2 \otimes \text{Sym}^{m-2} \mathcal{V}^* \to I_m\). Note that \(\mu_m\) and \(\nu_m\) are defined on \(\overline{M}_{g,n}\). In general \(I_m\) may not be vector bundles.

If geometric interpretations exist, \(\nu_m\) are isomorphisms for all \(m > 2\), and \(\mu_m\) are surjections for all \(m\) by working over fibers, and using the exact sequence (7.1) (tensoed with \(\mathcal{O}_{\mathbb{P}^{d+1}}(m)\)). Furthermore \(I_2\) would be a line bundle on \(\overline{M}_{g,n}\). Therefore we obtain formulas for \(m > 2\)

\[
c_1(I_2) = c_1(\text{Sym}^2 \mathcal{V}^*) - c_1(\mathcal{V}[2]^*)
\]

and

\[
c_1(I_2) \text{ rank}(\text{Sym}^{m-2} \mathcal{V}^*) + c_1(\text{Sym}^{m-2} \mathcal{V}^*) = c_1(I_m) = c_1(\text{Sym}^m(\mathcal{V}^*)) - c_1(\mathcal{V}[m]^*)
\]

Putting all these together gives us the desired formula for \(c_1\mathcal{V}[m]\). \(\square\)

The hypotheses of Theorem 6.1 are not the only ones under which \(\widetilde{\mathcal{A}}_x\) is the algebra of sections of a polarized variety of \(\Delta\)-invariant 0.

**Definition 7.4.** Given a bundle \(\mathcal{V} = \mathcal{V}(s_{i+1}, \lambda, f)\) on \(\overline{M}_{g,n}\) and \(x \in \Delta_{s_1,j} = \Delta_{s_1,j'}\) or \(x \in \Delta_{\text{irr}}\), we say that the restriction data \(\alpha_1, \ldots, \alpha_p\) for \(\mathcal{V}\) at \(x\) has a single relation in degree 2 if

- There is a degree 2 relation of the form \(\alpha_{i_1} + \alpha_{i_2} = \alpha_{i_3} + \alpha_{i_4}\) with \(i_3 \neq i_4\) (but \(i_1\) could equal \(i_2\)), \([i_1, i_2] \cap [i_3, i_4] = \emptyset\).
- Any relation \(\sum_{i=1}^p a_i \alpha_i = 0\), with \(\sum a_i = 0\) and \(a_i \in \mathbb{Z}\) is an integer multiple of \(\alpha_{i_1} + \alpha_{i_2} - \alpha_{i_3} - \alpha_{i_4} = 0\).
Proposition 7.5. Let $V$ be a bundle on $\overline{M}_{g,n}$ such that $V$ has quadric rank scaling (Definition 7.2), and quasi rank one factorization (Definition 4.2). We do not impose the condition of freeness, but allow for the following dichotomy: For each boundary point $x$, a generic point of a boundary divisor, assume that either of the following conditions holds

1. The socle satisfies quadric rank scaling, and the restriction data is free, or
2. The socle satisfies projective space rank scaling, and the restriction data for $V$ at $x$ has a single relation in degree 2.

Then the quadric first Chern class identity of Proposition 7.3 holds for $V$ on $\overline{M}_{g,n}$.

Proof. To verify this claim, we show that under the second condition $\tilde{A}$ is the algebra of sections of a polarized variety of $\Delta$-invariant zero. Proceeding as in Theorem 5.9, we find a graded morphism $q : C \to \tilde{A}$ with $C$ a polynomial algebra. The argument is divided into two steps: Let $I$ be the kernel.

1. $I_2$ is not zero and generated by a single element $Q$.
2. The map $q : C/\langle Q \rangle \to \tilde{A}$ is injective, and hence surjective (counting ranks).

Since we have a relation in degree two, we see that the multiplication map $\text{Sym}^2(\tilde{A})_1 \to (\tilde{A})_2$ takes two graded pieces injectively into the same graded part of $(\tilde{A})_2$. This multiplication map should have a kernel since $(\tilde{A})_2$ has rank one less than that of $\text{Sym}^2(\tilde{A})_1$, and since one of the graded pieces has rank one, is one dimensional.

Using the relation $Q$, we can write $C/\langle Q \rangle$ as a direct sum: of lifts to $C$: Suppose for simplicity that the relation is $a_1 + a_2 = a_3 + a_4$ (with the socle $a_1$), and write $C' = \oplus B_{m_1} Y_{2}^{m_2} \cdots Y_{p}^{m_p}$ where we restrict to weights where either $m_3 = 0$ or $m_4 = 0$. Now $C'$ maps isomorphically (as vector spaces) to $C/\langle Q \rangle$ since we can trade $Y_3 Y_4$ for elements in $B_1 Y_2$ in $C/\langle Q \rangle$ (which shows surjectivity of $C' \to C/\langle Q \rangle$, and we compare graded ranks).

Each summand of $C'$ maps injectively to $\tilde{A}$ by [BGM14b, Proposition 2.1], and different summands of $C'$ map to different weight summands of $\tilde{A}$. Therefore, the kernel of $C' \to \tilde{A}$ is zero, and we are done. $\square$

Examples. In Example 6.2 we gave an example of a bundle $V$ with Quadric hypersurface scaling (of dimension 3 in $\mathbb{P}^4$), where the hypothesis of Theorem 6.1 held, and so first Chern classes multiples $V[m]$ are governed by the equations given in Corollary 3.6. We give a few more examples below.

Quadric hypersurfaces of dimension 2 in $\mathbb{P}^3$. Conformal blocks bundles with quadric scaling for $d = 2$ have rank sequence $(m + 1)^2$. If an extension of the geometric interpretation to the boundary holds, then for all $m \geq 3$, there will be a polynomial identity that holds between $c_1(V[m]), c_1(V)$, and $c_1(V[2])$. Here are a few examples: $c_1(V[3]) = -6c_1(V) + 4c_1(V[2]); c_1(V[4]) = -20c_1(V) + 10c_1(V[2]); c_1(V[5]) = -45c_1(V) + 20c_1(V[2]);$ and $c_1(V[6]) = -84c_1(V) + 35c_1(V[2]).$

Example 7.6. First, let $n = 5$. Let $V[m] = V(s_{l_2}, (m\omega_1)^2, (2m\omega_1)^3, 4m)$, we will show that $\text{rk } V[m] = (m+1)^2$, i.e. satisfies the formula $H^0(X, L_{x_0}^{km}) = 2^m d + 1(2m+1)^2$ for $d = 2$. We will use Witten's Dictionary to compute $\text{rk } V[m]$. Since $2 \cdot m + 3 \cdot 2m + 8m = 2 \cdot 2m$, $\text{rk } V$ is equal to the coefficient of $d_{2m\omega_1+2m\omega_2}$ is the classical product $s_{m\omega_1} \cdot s_{m\omega_1} \in H^* \text{Gr}(2, 4m + 2)$. Using Pieri we obtain that $s_{m\omega_1} \cdot s_{m\omega_1} = \sum_{i=0}^{2m} \sigma_i(2m_2+2m-1)$, and $\sigma_i(2m_2+2m-1) = \sigma_i(2m_2+2m-1 + 1) + \sigma_i(2m_2+2m-1 + 2)$ + other terms. Thus $\text{rk } V[m] = \sum_{i=0}^{2m} (\min[i, 2m - i] + 1) = (m+1)^2$.

Let $n > 5$ be an odd integer. Let $V[m] = V(s_{l_2}, (m\omega_1)^2, (2m\omega_1)^3, (4m\omega_1)^{n-5}, 4m)$. Define $j_k = 0$ for $k = 1, \ldots, 5$, and $j_k = 1$ for $k = 6, \ldots, n$, then $\sum_{k=1}^{n} j_k = n - 5$, which is even. Using the fact from above, we
obtain that
\[
\text{rk } V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3, (4m\omega_1)^{n-5}], 4m)
\]
\[
= \text{rk } V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3, (\alpha_1(4m\omega_1))^{n-5}], 4m)
\]
\[
= \text{rk } V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3, (\omega_0)^{n-5}], 4m) = \text{rk } V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3], 4m),
\]
which is \((m+2)^2\). So \(\text{rk } V[m] = 2 \left( \frac{m+2}{d} \right) + \left( \frac{m+2}{d-1} \right)\) for \(d = 2\).

We next show this bundle satisfies the hypothesis of Proposition 7.5. For this bundle, for points \(x \in \Delta_{12}\) the restriction data is free, and the bundle has quasi rank one factorization: with restriction data \(\mu = 0\), the ranks are one, and for \(\mu = 2\omega_1,\)
\[
\text{rk}(V(sl_2, [3\omega_1, 3\omega_1, \mu], 4)) = 1, \quad \text{and } \text{rk}(V(sl_2, [2\omega_1, 2\omega_1, 2\omega_1, \mu], 3)) = 3.
\]

For boundary points \(x \in \Delta_{i,j}\) for \(i \in [1,2]\), and \(j \in [3,4,5]\), the restriction data is free, and the bundle has one rank one factorization (the restriction data is \(\mu_1 = \omega_1\), and \(\mu_2 = 2\omega_1\), and \(\text{rk}(V(sl_2, [3\omega_1, 2\omega_1, \mu_1], 3)) = 1\), while \(\text{rk}(V(sl_2, [3\omega_1, 2\omega_1, 2\omega_1, \mu_2], 3)) = 2\), for both \(i \in [1,2]\). Moreover the degree two bundle has projective space rank scaling. For points \(x \in \Delta_{i,j}\), for \(ij \in [345]\), the restriction data is not free (it is \(\mu_1 = 0, \mu_2 = 2\omega_1, \mu_3 = 4\omega_1\), and there is a unique relation in degree two given by \(2\mu_2 = \mu_1 + \mu_3\). The rank two component has projective space rank scaling).

Let \(n > 5\) be an even integer. Let \(V[m] = V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3, (4m\omega_1)^{n-5}], 4m)\). Define \(j_k = 0\) for \(k = 1, \ldots, 4\), and \(j_k = 1\) for \(k = 5, \ldots, n\), then \(\sum_{k=1}^{n} j_k = n - 4\), which is even. As in the even case above, we obtain that the rank of \(V[m]\) is equal to \(\text{rk } V(sl_2, [(m\omega_1)^2, (2m\omega_1)^3], 4m)\) which is \((m+1)^2\). Thus \(\text{rk } V[m] = 2 \left( \frac{m+2}{d} \right) + \left( \frac{m+2}{d-1} \right)\) for \(d = 2\). The bundle satisfies the hypothesis of Proposition 7.5.

**Quadric hypersurfaces of dimension 3 in \(\mathbb{P}^4\)**. If there is an extension of the geometric interpretation to the boundary then
\[
(7.3) \quad c_1(V[m]) = \left( \frac{m+4}{5} \right) - \left( \frac{m+2}{5} \right) - 6 \left( \frac{m+2}{4} \right) c_1(V[2]).
\]
So \(c_1(V[3]) = (-10) c_1(V) + 5 c_1(V[2]);\) \(c_1(V[4]) = (-40) c_1(V) + 15 c_1(V[2]);\) and \(c_1(V[5]) = (-105) c_1(V) + 35 c_1(V[2]).\)

**Example 7.7.** \(V = V(sl_3, [(2\omega_1 + \omega_2)^2, 2\omega_1, \omega_1 + 2\omega_2], 4)\) is a critical level bundle on \(\overline{M}_{0,4}\) which satisfies quadric rank scaling for dimension 3. At a point \(x \in \Delta_{12}\) one obtains restriction data \(\alpha_1 = \omega_1, \alpha_2 = 2\omega_2, \alpha_3 = \omega_1 + 3\omega_2,\) and \(\alpha_4 = 2\omega_1 + \omega_2,\) satisfying the relations \(\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4\). This bundle has quasi rank one factorization, the socle has rank 2, and has projective space rank scaling. One can check the bundle satisfies the hypothesis of Proposition 7.5, and hence first Chern classes of \(V[m]\) are governed by the identities in Equation 7.3.

**Quadric hypersurfaces of dimension 4 in \(\mathbb{P}^5\)**. If a geometric interpretation exists on the boundary of \(\overline{M}_{g,n}\) for a bundle \(V\) that gives a quadric hypersurfaces of dimension 4 in \(\mathbb{P}^5\), then
\[
(7.4) \quad c_1(V[m]) = \left( \frac{m+5}{6} \right) - \left( \frac{m+3}{6} \right) - 7 \left( \frac{m+3}{5} \right) c_1(V[2]).
\]
So \(c_1(V[3]) = (-15) c_1(V) + 6 c_1(V[2]);\) \(c_1(V[4]) = (-70) c_1(V) + 21 c_1(V[2]);\) and \(c_1(V[5]) = (-210) c_1(V) + 56 c_1(V[2]).\)

**Example 7.8.** \(V(sl_3, [(3\omega_1 + \omega_2)^2, 2\omega_1 + \omega_2, \omega_1 + 3\omega_2], 5)\) satisfies quadric rank scaling for dimension 4. We next show that this bundle satisfies the conditions of Proposition 7.5. Because of symmetry, there are only two boundary restrictions. Let \(\lambda_1 = \lambda_2 = 3\omega_1 + \omega_2,\) \(\lambda_3 = 2\omega_1 + \omega_2,\) and \(\lambda_4 = \omega_1 + 3\omega_2.\) If one takes a point \(x \in \Delta_{12}\), then there are 4 pieces of restriction data: \(\alpha_1 = 2\omega_1, \alpha_2 = 3\omega_1 + \omega_2, \alpha_3 = \omega_1 + 2\omega_2,\) and \(\alpha_4 = 2\omega_1 + 3\omega_2.\) One can check that \(\text{rk } V(sl_3, [\lambda_1, \lambda_2, \alpha_1], 5) = \text{rk } V(sl_3, [\lambda_3, \lambda_4, \alpha_1], 5) = 1,\)

for $i \in \{1, 4\}$: \(\text{rk} V(sl_3, \{\lambda_1, \lambda_2, \alpha_2\}, 5) = 2\), \(\text{rk} V(sl_3, \{\lambda_3, \lambda_4, \alpha_3\}, 5) = 1\), \(\text{rk} V(sl_3, \{\lambda_1, \lambda_2, \alpha_3\}, 5) = 1\), and \(\text{rk} V(sl_3, \{\lambda_3, \lambda_4, \alpha_4\}, 5) = 2\). One has a single relation satisfied by the restriction data in degree 2 \(\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3\); and one can check that the rank 2 components satisfy projective space rank scaling. A similar check holds for a point \(x \in \Lambda_{4,4}\), where there are 4 pieces of restriction data: \(\alpha_1 = 3\omega_1, \alpha_2 = \omega_1 + \omega_2, \alpha_3 = 2\omega_1 + 2\omega_2, \) and \(\alpha_4 = 3\omega_2\).

8. Veronese surfaces

**Lemma 8.1.** Suppose that \(V\mid_x \cong H^0(X, L_x)\) for \(x \in \overline{M}_{g,n}\). The following are equivalent:

(a) \(\text{rk}(V[m]) = (2m + 2) = (m + 1)(2m + 1)\);

(b) \((X_x, L_x) = (P^2, O(2));\)

**Proof.** It is easy to see that (b) implies (a). Assume (a). It follows that \(X_x\) is two dimensional, the degree \(L_x = 4\), and \(H^0(X, L_x) = 6\), so \(\Delta(X_x, L_x) = 0\). Now (b) follows from Proposition 2.7. \(\square\)

**Definition 8.2.** \(V\) has Veronese surface rank scaling if the conditions of Lemma 8.1 hold.

The following is a special case of Corollary 3.6:

**Corollary 8.3.** If \(V\) has Veronese surface rank scaling and if a geometric interpretation exists for \(V\) on the boundary of \(\overline{M}_{g,n}\), then for all \(m \geq 1\),

\[
c_1(V[m]) = A_1(m)c_1(V) + A_2(m)c_1(V[2]) + A_3(m)c_1(V[3]) + A_4(m)c_1(V[4]),
\]

where

1. \(A_1(m) = -7(\frac{m+3}{5}) + 20(\frac{m+2}{5}) - 23(\frac{m+1}{5}) - 6(\frac{m+3}{6}) + 8(\frac{m+2}{6}) - 3(\frac{m+1}{6}) + (\frac{m+5}{6});\)

2. \(A_2(m) = (\frac{m+3}{5}) - 6(\frac{m+2}{5}) + 15(\frac{m+1}{5});\)

3. \(A_3(m) = (\frac{m+2}{5}) - 6(\frac{m+1}{5});\)

4. \(A_4(m) = (\frac{m+1}{5}).\)

**Examples.** In Examples 8.4 and 8.5 bundles are given for which the first few predicted identities of Corollary 8.3 hold, but for which the hypothesis of Claim 6.1 are not satisfied. It is therefore likely that if the time is taken, a more general result can be found.

**Example 8.4.** On \(\overline{M}_{0,4}\) we consider the bundle \(V = V\mid_{sl_3, \{4\omega_1 + 3\omega_2, 4\omega_1 + 4\omega_2, 2\omega_1 + 4\omega_2, 4\omega_1\}}\), for which one can compute, using [Swi10], that \(\deg(V) = 15\); \(\deg(V[2]) = 75\); \(\deg(V[3]) = 210\); \(\deg(V[4]) = 450\); \(\deg(V[5]) = 825\); \(\deg(V[6]) = 1365\); and \(\deg(V[7]) = 2100\), and \(V\) satisfies the first three predicted relations of Corollary 8.3. We note that this bundle does not satisfy the hypothesis of Theorem 6.1. There are three necessary boundary restrictions, at the points \(\delta_{12}, \delta_{13}, \text{and} \delta_{14}\). There are exactly six pieces of restriction data at \(\delta_{12}, \delta_{13}, \text{and} \delta_{14}\), and so ranks of restricted bundles are one. But at \(\delta_{13}\), there are four: \(\mu \in \{4\omega_1, 5\omega_2, 3\omega_1 + 2\omega_2, 2\omega_1 + 4\omega_2\}\), and by [Swi10], \(\text{rk}(V\mid_{sl_3, \{\lambda_1, \lambda_3, \mu\}}) = 2\), for \(\mu \in \{3\omega_1 + 2\omega_2, 2\omega_1 + 4\omega_2\}\).

**Example 8.5.** On \(\overline{M}_{0,5}\) we consider the bundle \(V = V\mid_{sl_2, \{\omega_1, 3\omega_1, 4\omega_1, 5\omega_1, 6\omega_1\}}\), which in the non-adjacent basis consisting of \(\delta_{13}, \delta_{14}, \delta_{24}, \delta_{25}, \text{and} \delta_{35}\), one can use [Swi10] to check that the first five multiples have the following expressions:

\[
c_1(V) = \delta_{14} + 2\delta_{24} + \delta_{25} + 3\delta_{35}; \ c_1(V[2]) = 4\delta_{14} + 11\delta_{24} + 4\delta_{25} + 15\delta_{35};
\]

\[
c_1(V[3]) = 10\delta_{14} + 32\delta_{24} + 10\delta_{25} + 42\delta_{35}; \ c_1(V[4]) = 20\delta_{14} + 70\delta_{24} + 20\delta_{25} + 90\delta_{35};
\]

\[
c_1(V[5]) = 35\delta_{14} + 130\delta_{24} + 35\delta_{25} + 165\delta_{35}. \ A \text{ simple calculation shows that} \ c_1(V[5]) = -15c_1(V) + 20c_1(V[2]) - 15c_1(V[3]) + 6c_1(V[4]), \text{as is predicted.} \]

**Example 8.6.** At a point on the boundary \(\delta_{123}\), one finds three pieces of restriction data given by \(\mu \in \{2\omega_1, 4\omega_1, 6\omega_1\}\), and by [Swi10], \(\text{rk}(V\mid_{sl_2, \{\lambda_1, \lambda_2, \lambda_3, \mu\}}) = 2\) for those weights.
9. Rational normal scrolls

Suppose \( a_1, \ldots, a_d \) are strictly positive integers, and let \( E = \oplus_{i=1}^d O(a_i) \), a vector bundle on \( \mathbb{P}^1 \). Let \( X = S(a_1, \ldots, a_d) = \mathbb{P}(E) \), a projective bundle over \( \mathbb{P}^1 \). Let \( L = O(1) \) be the natural ample line bundle on \( X \). It is known that \( \Delta(X, L) = 0 \). Let \( D = \sum a_i \) and \( N = D + d - 1 \). Set \( V = H^0(X, L)^* \), it is easy to see that \( \dim V = N + 1 \), and clearly \( X \hookrightarrow \mathbb{P}^N = \mathbb{P}(V) \). The varieties \( S(a_1, \ldots, a_d) \) are called rational normal scrolls (See Section A2H in [Eis05]).

When some \( a_i \) are zero and the rest positive, \( O(1) \) is base point free but not ample on \( \mathbb{P}(E) \). The image in \( \mathbb{P}(H^0(\mathbb{P}(E), O(1))) \) is again denoted by \( S(a_1, \ldots, a_d) \). It is known that \( \Delta(S(a_1, \ldots, a_d), O(1)) = 0 \) in this case.

The vector bundles \( \oplus_{i=1}^d O(a_i) \) with fixed \( d \) and \( \sum a_i \) lie in the same component of the moduli stack of bundles on \( \mathbb{P}^1 \). Therefore, the polarized varieties \( (S(a_1, \ldots, a_d), O(1)) \) are deformation equivalent for fixed \( d \) and \( \sum a_i \), and such varieties will have the same rank sequence.

A special case \( S(1, 2) \). It is easy to check that for \( m \geq 0 \),

\[
h^0(S(a, b), O(m)) = (m+1)(1 + \frac{m(a+b)}{2}).
\]

Suppose \( \mathcal{V} \) satisfies \( (S(1, 2), O(1)) \) scaling: \( \text{rk} \mathcal{V}[m] = (m+1)(1 + \frac{3m}{2}) \) for all positive integers \( m \). Assume geometric interpretations exist.

**Proposition 9.1.** For all \( m \geq 1 \), we have \( c_1(\mathcal{V}[m]) = A_1(m)c_1(\mathcal{V}) + A_2(m)c_1(\mathcal{V}[2]) + A_3(m)c_1(\mathcal{V}[3]) \), where

1. \( A_1(m) = \binom{m+4}{5} - 6\binom{m+2}{4} + 12\binom{m+1}{4} - 3\binom{m+2}{5} + 2\binom{m+1}{5} \);
2. \( A_2(m) = \binom{m+2}{4} - 5\binom{m+1}{4} \);
3. \( A_3(m) = \binom{m+3}{5} \).

For example, putting \( m = 4 \), we get \( c_1(\mathcal{V}[4]) = 10c_1(\mathcal{V}) - 10c_1(\mathcal{V}[2]) + 5c_1(\mathcal{V}[3]) \).

**Example 9.2.** Let \( \mathcal{V} = \mathcal{V}(s_{L_2}, (2\omega_1)^4, 4\omega_1), 5 \), then using Witten’s Dictionary and Quantum Pieri, one can check that \( \text{rk} \mathcal{V}[m] = \frac{m+1}{2}(3m+2) \). In particular, \( \mathcal{V} \) gives rise to the surface \( S(1, 2) \) or its degeneration \( S(3, 0) \). One can use [Swi10] to compute coefficients of the first Chern classes of multiples \( \mathcal{V}[m] \) in the nonadjacent bases \( \{\delta_{13}, \delta_{14}, \delta_{24}, \delta_{25}, \delta_{35} \} \): \( c_1(\mathcal{V}) = 2(\delta_{14} + \delta_{25} + \delta_{35}), c_1(\mathcal{V}[2]) = 9(\delta_{14} + \delta_{25} + \delta_{35}), c_1(\mathcal{V}[3]) = 24(\delta_{14} + \delta_{25} + \delta_{35}) \), and \( c_1(\mathcal{V}[4]) = 50(\delta_{14} + \delta_{25} + \delta_{35}) \). The formula from Section 9, it was calculated that if the extension of the interpretation of conformal blocks and theta functions holds on stable curves, then \( c_1(\mathcal{V}[4]) = 10c_1(\mathcal{V}) - 10c_1(\mathcal{V}[2]) + 5c_1(\mathcal{V}[3]) \), for example. Since \( 20 - 90 + 5 \cdot 24 = 50 \), the predicted divisor identity given in Section 9, holds for \( c_1(\mathcal{V}[4]) \).

10. Rational normal curves

While rational normal curves are often left off of lists classifying projective varieties of minimal degree, we include a short section here with the rank scaling identities and a few examples to show what the divisor identities will look like in case we have geometric interpretations of conformal blocks at boundary points (see Definition 2.11). We also note Example 3.9 for Veronese curves of degree 2, done previously.

One says that \( \mathcal{V} \) satisfies \( (\mathbb{P}^1, O(d)) \) scaling if \( \text{rk} \mathcal{V}[m] = dm + 1 \) for all positive integers \( m \). In this case, at smooth points \( x \in M_{g,n} \), one has \( \mathcal{V}^*_x \cong H^0(X_x, L^*) \), where \( (X_x, L_x) \cong (\mathbb{P}^1, O(d)) \), and \( L_x \) embeds \( X_x \) as a rational normal curve in \( \mathbb{P}^d \) in its Veronese embedding.

For such \( \mathcal{V} \), if geometric interpretations exist at boundary points, then Corollary 3.6 gives identities which govern first Chern classes of \( \mathcal{V}[m] \). We consider a few examples, starting with the twisted cubic.
Twisted cubics. Suppose $\mathcal{V}$ satisfies $(\mathbb{P}^1, \mathcal{O}(3))$ scaling, and if geometric interpretations exist at boundary points, then from Corollary 3.6, we get that for all $m \geq 1$,

\begin{equation}
(10.1) \quad c_1(\mathcal{V}[m]) = A_1(m)c_1(\mathcal{V}) + A_2(m)c_1(\mathcal{V}[2]) + A_3(m)c_1(\mathcal{V}[3]),
\end{equation}

where $A_1(m) = (m^4 + 3m^3 - 3m^2 + 8m + 2)/4$, $A_2(m) = (m^4 + 3m^3 - 4m)/3$, and $A_3(m) = m/3$. For example, for $m = 4$, the expression reads $c_1(\mathcal{V}[4]) = 4c_1(\mathcal{V}) - 6c_1(\mathcal{V}[2]) + 4c_1(\mathcal{V}[3])$, and for $m = 5$ one has $c_1(\mathcal{V}[5]) = c_1(\mathcal{V}) - 20c_1(\mathcal{V}[2]) + 10c_1(\mathcal{V}[3])$.

Example 10.1. On $\overline{M}_{0,4}$ consider $\mathcal{V} = \mathcal{V}(\mathfrak{sl}_2, \{6\omega_1, 5\omega_1, 5\omega_1, 6\omega_1\}, 9)$, which one can check using Pieri satisfies $\text{rk}\mathcal{V}[m] = 3m + 1$, and using [Swi10] has degree sequence starting 8, 28, 60, 104, 160, 228, 308. There are two necessary boundary restrictions, at $\delta_{12} = \delta_{13}$ and $\delta_{14}$. At the former there are four pieces of restriction data $\{\omega_1, 3\omega_1, 5\omega_1, 7\omega_1\}$, and all restricted bundles have rank one. At the latter there are four pieces of restriction data $\{0, 2\omega_1, 4\omega_1, 6\omega_1\}$, and all restricted bundles have rank one. In particular, the bundle satisfies quasi rank one factorization, but there are relations between the restriction data in degree two. The first four relations predicted by Corollary 3.6, as given in Equation 10.1 hold for this bundle. For example: $\deg(\mathcal{V}[4]) = 4\deg(\mathcal{V}) - 6\deg(\mathcal{V}[2]) + 4\deg(\mathcal{V}[3]) = 104$, and $\deg(\mathcal{V}[5]) = 15\deg(\mathcal{V}) - 20\deg(\mathcal{V}[2]) + 10\deg(\mathcal{V}[3]) = 160$.

Example 10.2. On $\overline{M}_{1,1}$, let $\mathcal{V}_k^d = \mathcal{V}(\mathfrak{sl}_2, 2\omega_1, d + 2k)$. By [Fak12, p. 27], if $i$ is even, the rank of $\mathcal{V}(\mathfrak{sl}_2, io_1, \ell)$, is $\ell + 1 - i$, and one has that $\text{rk}(\mathcal{V}_k^d[m]) = dm + 1$. While the ranks of multiples do not depend on $k$, we from [Fak12, Corollary 6.2] that degrees do:

\[ \deg(\mathcal{V}_k^d) = -\frac{m(dm + 1)(k + d)}{12}. \]

One checks that this bundle satisfies the identity for $d = 3$ in Equation 10.1 that one gets for $m = 4$: $4\deg(\mathcal{V}_k^d[4]) = 4\deg(\mathcal{V}_k^d) - 6\deg(\mathcal{V}_k^d[2]) + 4\deg(\mathcal{V}_k^d[3])$. There is only one necessary boundary restriction, at the point in the image of the map from $\overline{M}_{0,3}$ given by attaching the first two markings together. As all $\mathfrak{sl}_2$ bundles restricted to $\overline{M}_{0,3}$ will have rank zero or one, one has that $\mathcal{V}_k^d$ has quasi rank one factorization.

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