Higher Order Topological Insulators in Amorphous Solids

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(Dated: February 4, 2019)

We identify the possibility of realizing higher order topological (HOT) phases in noncrystalline or amorphous materials. Specifically, starting from two and three dimensional crystalline HOT insulators, accommodating topological corner states, we gradually enhance structural randomness in the system. Within a parameter regime, as long as amorphousness is confined by outer crystalline boundary, the system continues to host corner states, realizing an amorphous HOT insulator. However, as structural disorder percolates to the edges, corner states start to dissolve into amorphous bulk, and ultimately the system becomes a trivial insulator (devoid of corner modes), when amorphousness plagues the entire system. These outcomes are further substantiated from the scaling of the quadrupolar (octupolar) moment in two (three) dimensions with the scrambling radius. Therefore, HOT phases can be realized in amorphous solids, when wrapped by a thin crystalline layer.

Introduction: The bulk-boundary correspondence is the hallmark signature of any topological phase of matter, and typically a $d$-dimensional topological system supports gapless boundary modes of codimension ($d_c$) one [1–3]. For example, two-dimensional quantum Hall and spin Hall insulators support one dimensional edge modes, whereas a three-dimensional topological insulator accommodates massless Dirac fermions on two-dimensional surfaces [4–7]. A similar bulk-boundary correspondence is also operative for the gapless systems, such as three-dimensional Dirac and Weyl semimetals [8]. Recently a higher order generalization of topological systems has been proposed [9], which can be realized, for example, in Bi [10], phononic [11] and photonic [12, 13] systems, and electrical circuits [14]. Namely, an $n$th order topological phase supports boundary states of codimension $d_c=n$, with corner and hinge modes standing as its paradigmatic representatives [15–36]. In this language the traditional topological phases are first order in nature. While the boundary modes in conventional topological materials are guaranteed by fundamental symmetries (such as time-reversal, charge-conjugation) or no symmetry at all, the existence of corner or hinge modes crucially relies on the crystalline, such as the discrete four-fold rotational ($C_4$), symmetry of the system.

Therefore, in the context of higher order topological (HOT) phases the following set of questions arises quite naturally. (a) Can HOT phases be realized in a non-crystalline environment, such as in amorphous solids?, (b) If so, how robust are the corresponding boundary (hinge, corner) states? We here provide an affirmative answer to the first question by introduce the concept of amorphous higher order topological insulators (HOTIs) in two and three dimensions and systematically analyze the stability of the corner states in such a medium.

Our main results can be summarized as follows. Starting from a two-dimensional crystalline HOTI, supporting four corner states (with $d_c=2$), we systematically introduce structural disorder around its center, see Fig. 1. When (a) amorphousness is confined within the interior of the system and (b) the scale of structural disorder is smaller than the band gap, the corner states remain sharp and we realize a second order amorphous HOTI, see Fig. 2. In this regime the quadrupolar moment ($Q_{xy}$), characterizing a two-dimensional HOTI (amorphous or crystalline), remains pinned at its quantized value of 0.5 (within numerical accuracy), see Fig. 3. However, the corner modes start to melt in the amorphous bulk when structural disorder destroys the crystalline symmetry of the boundaries, see Fig. 2 (fourth column) and $Q_{xy}$ starts to deviate from 0.5. Ultimately, the system becomes a trivial insulator when structural disor-
FIG. 2: Energy spectra (top row) and the local density of states (LDoS) for zero energy modes (bottom row) with increasing scrambling radius $R_s$ (from left to right) in a system with linear dimension $L = 20$ (see Fig. 1). We set $R = 4$, $M = 0$, $r_a = 1$, $t_1 = t_2 = g = 1$. In the top row $n$ is the index for energy eigenvalues $E_n$. The first column is devoted to a crystalline HOTI, supporting four in-gap corner modes at zero energy, well separated from the bulk states. With increasing structural disorder or $R_s$ the bulk gap shrinks, but the system continues to support ‘sharp’ corner states, as long as $R_s \lesssim L/2$. The system then represents an amorphous HOTI. By contrast, when $R_s > L/2$, the bulk gap closes and the corner states melt into the glassy bulk, see the fourth column when the entire system is scrambled. The system then represents a trivial insulator. The scaling of corresponding quadrupolar momentum $(Q_{xy})$ with $R_s$ is shown in Fig. 3.

der plagues the entire system. Nonetheless, a significant portion of the global phase diagram of a two-dimensional amorphous insulator is occupied by HOTI, see Fig. 4. We arrive at (qualitatively) similar conclusions for a three-dimensional amorphous HOTI, see Fig. 5. To summarize, it is conceivable to realize HOTIs in amorphous solids, in contrast to first order phases where boundary modes can be supporting four in-gap corner modes at zero energy, well separated from the bulk states. With increasing structural disorder or $R_s$ the bulk gap shrinks, but the system continues to support ‘sharp’ corner states, as long as $R_s \lesssim L/2$. The system then represents an amorphous HOTI. By contrast, when $R_s > L/2$, the bulk gap closes and the corner states melt into the glassy bulk, see the fourth column when the entire system is scrambled. The system then represents a trivial insulator. The scaling of corresponding quadrupolar momentum $(Q_{xy})$ with $R_s$ is shown in Fig. 3.

Model (2D): The Hamiltonian operator, describing a two-dimensional crystalline HOTI is

$$\hat{h} = t_1 [\sigma_3 \tau_1 S_1 + \sigma_0 \tau_2 S_2] + [M + t_2 (2 - C_1 - C_2)] \sigma_0 \tau_3 + g (C_1 - C_2) \sigma_1 \tau_1,$$

where $S_j \equiv \sin(k_j a)$, $C_j \equiv \cos(k_j a)$, $k_1$ and $k_2$ are two components of spatial momenta. We set the lattice spacing $a = 1$. Two sets of Pauli matrices $\{\tau_\mu\}$ and $\{\sigma_\mu\}$ with $\mu = 0, 1, 2, 3$ respectively operate on the orbital and spin degrees of freedom. In real space, the above Hamiltonian operator can be equivalently characterized by a hopping matrix $T_{j\alpha,k\beta}(|r|, \phi)$, acting between two sites $j$ and $k$ with spin/orbit indices $\alpha$ and $\beta$, respectively. Here $|r|$ is the distance between two sites and the $\phi$ is the polar angle. For $|r| = 0$, the hopping matrix element is described by the onsite quantity $T(0, 0) = \sigma_0 \tau_3 (M + 2t_2)$, while for $|r| > 0$ it reads $T(|r|, \phi) = \frac{t(|r|)}{|r|} \times f(\phi)$, where

$$f(\phi) = -it_i [\sigma_3 \tau_1 C_\phi + \sigma_0 \tau_2 S_\phi] - t_2 \sigma_\alpha \tau_3 + ge^{i2\phi} \sigma_1 \tau_1,$$

with $t(|r|) = \Theta(R - |r|) \exp (-|r|/r_o)$, $C_\phi = \cos(\phi)$ and $S_\phi = \sin(\phi)$. The hopping elements are bounded by a hard cut-off $R$ on the hopping radius, accompanied by an exponential decay (controlled by $r_o$), since in an amorphous solid (the system of ultimate interest) sites are randomly distributed, see Fig. 1.

For $g = 0$, the system describes a quantum spin Hall insulator, supporting two counter-propagating one dimensional edge modes for two opposite spin projections. The term proportional to $g$ also known as the Wilson mass, changes sign under the $C_4$ rotation and anticommutes with rest of the Hamiltonian operator. Hence, for $|g| \neq 0$, the boundary of the system is effectively described by one-dimensional gapless Dirac fermions in the presence of a domain wall mass. A generalized Jackiw-Rebbi index theorem assures the presence of four corner states (with $d_c = 2$), where the Wilson mass changes its sign. We then realize a second order topological insulator. Note that the Wilson mass breaks both time-reversal ($T$) and $C_4$ symmetries, but remains invariant under a composite symmetry operation $TC_4$, protecting the corner modes. Next we investigate the stability of such corner states in an amorphous system, lacking the $C_4$ symmetry.

Amorphous system: The structural disorder or amorphousness in a two-dimensional system of linear dimension $L$ is introduced by replacing the sites of a regular square lattice by random lattice points within a region of radius $R_s$ around the center of the system, see Fig. 1. For $R_s < L/2$, structural disorder remains confined within the interior of the system and it first reaches the boundary through the center of four edges when $R_s = L/2$,
leaving the local crystalline environment around the four corners (connected via $C_4$ rotations) unaffected. Finally, the entire system becomes amorphous (while retaining the overall square shape) when $R_s \geq L/\sqrt{2}$. We numerically diagonalize $T(|r|, \phi)$ for various choices of the scrambling radius $R_s$ with open boundary in both directions and search for the localized corner states. The results are displayed in Fig. 2.

In a crystalline system (when $R_s = 0$), the energy spectra display a sharp gap with precisely four states at half-filling or zero energy. They correspond to four corner states and the system represents a crystalline HOTI. With increasing $R_s$, the bulk gap starts to shrink, but remains finite as long as $R_s \leq L/2$, and four corner states remain sharp. The system then stands as an example of amorphous HOTI. Upon further increasing the scrambling radius the corner modes start to dissolve, and ultimately disappear when $R_s \geq L$, indicating onset of a trivial insulator. Most importantly, the corner modes start to disappear as soon as the edges loose $C_4$ symmetry (i.e., when $R_s \geq L/2$), although four corners still remain buried in local crystalline environment. Therefore, amorphous HOTI is a stable phase of matter as long as the boundary of the system possesses discrete rotational symmetry, despite the bulk being plagued by structural disorder. This is the central result of our work. We further anchor this outcome from the scaling of the quadrupolar moment ($Q_{xy}$) with the scrambling radius $R_s$.

**Quadrupolar moment:** A two-dimensional HOTI is identified from a quantized quadrupolar moment $Q_{xy} = 1/2$ (modulo 1) [9, 45, 46], which we compute in an amorphous insulator as follows. We consider an operator of the form $U = \exp[iO]$, where

$$O = \frac{2\pi}{L^2} \sum_{i\alpha} f(x_{i\alpha}, y_{i\alpha}) n_{i\alpha}, \quad (3)$$

and $i(\alpha)$ is the site (orbital/spin) index. Now from the set of $N_o$ occupied states, schematically represented as

$$|j\rangle = \sum_{i\alpha} \psi_{i\alpha} |i\alpha\rangle, \quad (4)$$

where $N$ is the total number of states and $j = 1, \cdots, N_o$, we construct a $N \times N_o$ dimensional matrix $U$ by arranging $N_o$ eigenvectors columnwise. Subsequently, we introduce another matrix operator $W$, where

$$W_{i\alpha,j} = \exp \left[\frac{i}{2\pi} \frac{L^2}{f(x_{i\alpha}, y_{i\alpha})} \right] U_{i\alpha,j}. \quad (5)$$

The quantity of our interest $n$ (modulo 1) is given by

$$n = -\frac{i}{2\pi} \text{Tr} \left[\ln (U^\dagger W)\right]. \quad (6)$$

To capture the topological content one needs to subtract its contribution (modulo 1) in the atomic limit, given by

$$n_{al} = n_f \sum_{i\alpha} \frac{1}{L^2} f(x_{i\alpha}, y_{i\alpha}), \quad (7)$$

where $n_f = 1/2$ is the filling in the system. To compute the quadrupolar moment, defined as $Q_{xy} = n - n_{al}$ (modulo 1), we take $f(x_{i\alpha}, y_{i\alpha}) = x_{i\alpha} y_{i\alpha}$. We numerically compute $Q_{xy}$ with both periodic and open boundaries, which, however, do not affect the results (qualitatively).
As shown in Fig. 3, $Q_{xy} = 1/2$ in a two-dimensional crystalline HOTI ($R_s = 0$). In amorphous systems, $Q_{xy}$ remains pinned at 0.5 (within numerical accuracy) as long as $R_s \leq L/2$, confirming the realization of an amorphous HOTI for sufficiently small (in comparison to the linear dimensionality of the system) scrambling radius. By contrast, when $R_s \gtrsim L/2$, $Q_{xy}$ starts to deviate from its quantized value of 0.5, indicating a gradual disappearance of an amorphous HOTI. For $R_s \gg L/2$, the quadrupolar moment fluctuates wildly and the system becomes a trivial insulator. These findings are in qualitative agreement with the explicit computation of the corner states in a system with open boundaries, see Fig. 2. Therefore, $Q_{xy}$ continues to be an indicator for HOTI even in amorphous solids, which we further exploit to construct the global phase diagram associated with the model, introduced in Eq. (1).

Global phase diagram: Now we proceed to unearth the global phase diagram of an amorphous insulator in the $(M, g)$ plane [see Eq. (1)] for a fixed scrambling radius $R_s = 6$, which we show in Fig. 4 by computing $Q_{xy}$ in a system with linear dimension $L = 20$. For any non-trivial value of $g$, there are only two possible phases: (a) an amorphous HOTI and (b) a trivial insulator. Respectively deep inside these two phases $Q_{xy} = 0.5$ and 0.

For $|M| \gg 0$, the system is always a trivial insulator, irrespective of the strength of $g$. By contrast, around $M = 0$ system describes an amorphous HOTI (a trivial insulator) for small (large) value $|g|$. This outcome is in stark contrast to that in a crystalline environment, where the system remains in the HOTI phase for arbitrary value of $g$ (around $M = 0$), which solely arises from inevitable presence of structural disorder in any amorphous system that gets amplified with increasing $|g|$. As $|g|$ gets stronger the scale of structural disorder eventually becomes comparable to the band gap, and corner modes diffuse into the bulk states, indicating onset of a trivial insulator. Consequently, the parameter region over which an amorphous HOTI can be realized shrinks with increasing value of $|g|$. Also note that as one approaches a trivial insulator (with $Q_{xy} = 0$) from deep interior of an amorphous HOTI (with $Q_{xy} = 0.5$), the quadrupolar moment $Q_{xy}$ vanishes smoothly. Such an (almost) continuous variation of $Q_{xy}$ in amorphous HOTI is starkly distinct from the jump of the Bott index by an integer amount in an amorphous Chern insulator across a topological phase transition [37, 38]. These features of the global phase diagram of an amorphous insulator are qualitatively independent of the choices of (a) the extent of the hopping, characterized by $R$ and $r_0$, and (b) the scrambling radius ($R_s$). However, with increasing $R_s$ the available space for the amorphous HOTI in the $(M, g)$ plane decreases for fixed $L$.

Three dimensions: Finally, we introduce a three-dimensional amorphous HOTI, supporting eight corner modes ($d_c = 3$). Once again we start from a crystalline HOTI, for which the Hamiltonian operator reads [9]

$$\hat{h}_{3D} = \sum_{j=1}^{3} [M + t_1 S_j] \Gamma_j + t_2 \sum_{j=1}^{3} C_j \Gamma_{3+j},$$

where $\Gamma_j$s are mutually anticommuting eight dimensional Hermitian matrices. We implement the above Hamiltonian operator in real space following a similar approach, introduced earlier for two dimensional systems, and set $M = 0$ and $t_1 = t_2 = 1$ for numerical analyses.

Our conclusions regarding the amorphous HOTI in three dimensions are qualitatively similar to the ones reported for its two-dimensional counterpart. Namely, an amorphous HOTI, accommodating eight corner states, is realized when the structural disorder is confined within the interior of the system and boundary respects the cubic symmetry, see Fig. 5(b). However, the corner states start to melt in the amorphous bulk when the edges become amorphous as well, see Fig. 5(c). Finally, the system becomes a trivial insulator when $R_s \sim L$.

These outcomes are also substantiated from the scaling of the octupolar moment ($Q_{xyz}$) with the scrambling radius $R_s$, see Fig. 5(d) [47]. In a crystalline HOTI $Q_{xyz} = 1/2$ [45, 46]. With increasing $R_s$, $Q_{xyz}$ remains pinned at 0.5 (within numerical accuracy) as long as $R_s \leq L/2$, supporting the notion of a three-dimensional amorphous HOTI. However, for $R_s \gtrsim L/2$, the octupolar moment deviates from its quantized value, and the
system gradually approaches a trivial insulating phase.

Discussion & Outlook: To summarize, we here establish that HOT phases can be realized even when the crystalline symmetry is absent in the bulk of the system. In particular, we demonstrate that two- and three-dimensional HOTIs, supporting topological corner states, are stable in amorphous solids as long as they are coated by, for example, lithographically grown crystalline material, such that boundary of the system is devoid of any structural disorder. Our results should also be applicable (at least qualitatively) for HOTIs, supporting hinge modes, as well as HOT semimetals and superconductors. We hope that the present work will motivate further works toward systematic exploration of amorphous HOT phases both theoretically and experimentally.

Acknowledgements: AA acknowledges many enlightening discussions and related collaborations with Vijay B. Shenoy and Subhro Bhattacharjee. AA thanks Gil Young Cho for discussions. AA acknowledges MPG for funding.

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