Overpartitions with bounded part differences

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Abstract. We generalize recent results of Breuer and Kronholm, and Chern on partitions and overpartitions with bounded differences between largest and smallest parts. We prove the generalization analytically and combinatorially.

Keywords. Partition, overpartition, bounded difference between largest and smallest parts, combinatorial proof.

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1. Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum is $n$. For example, there are three partitions of 3: 3, 2 + 1, and 1 + 1 + 1. Recently, Andrews et al. [2] obtained the generating function for partitions with the difference between largest and smallest parts being a given positive integer $t$. Motivated by the work of Andrews et al., Breuer and Kronholm [3] studied partitions in which the difference between largest and smallest parts is at most $t$, and they showed that the generating function for such partitions is

$$
\sum_{n \geq 1} p_t(n)q^n = \frac{1}{1 - q^t} \left( \frac{1}{(q^t)_t} - 1 \right),
$$

where $p_t(n)$ counts the number of partitions of $n$ with part differences at most $t$.

Here and in the sequel, we use the standard $q$-series notation

$$(a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

The proof of Breuer and Kronholm has geometric flavor and their main tool used in the proof is polyhedral cones. Subsequently, Chapman [4] also provided a simpler proof, which involves $q$-series manipulations.

An overpartition of $n$ is a partition of $n$ where the first occurrence of each distinct part may be overlined. For example, there are eight overpartitions of 3: 3, $\overline{3}$, 2 + 1, $\overline{2} + 1$, $\overline{2} + 1$, $\overline{1} + 1 + 1$, and $\overline{1} + 1 + 1$. Recently, motivated by the works of Andrews et al., Breuer and Kronholm, and Chapman, the first author [5] considered an overpartition analogue with bounded differences between largest and smallest parts.

To obtain a generating function analogous to (1.1), apart from requiring the difference between largest and smallest parts being at most a given positive integer

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let \( t \), he added the following restriction: if the difference between largest and smallest parts is exactly \( t \), then no largest parts can be overlined. Let \( g_t(n) \) count the number of such overpartitions of \( n \), then its generating function is

\[
\sum_{n \geq 1} g_t(n) q^n = \frac{1}{1 - q^t} \left( \frac{(-q)_t}{(q)_t} - 1 \right).
\]  

(1.2)

The first proof in his paper uses heavy \( q \)-series manipulation, which originates from [2]. His second proof, which consists of many combinatorial ingredients such as over \( q \)-binomial coefficient introduced by Dousse and Kim [6], however, still needs some nontrivial computation, and hence is also not completely combinatorial.

The main purpose of this paper is to provide a completely combinatorial and transparent proof of (1.2). More precisely, we prove the following refined result.

**Theorem 1.1.** For a positive integer \( t \), let \( g_t(m,n) \) count the number of overpartitions of \( n \) in which there are exactly \( m \) overlined parts, the difference between largest and smallest parts is at most \( t \), and if the difference between largest and smallest parts is exactly \( t \), then no largest parts are overlined. Then

\[
\sum_{n \geq 1} \sum_{m \geq 0} g_t(m,n) z^m q^n = \frac{1}{1 - q^t} \left( \frac{(-zq)_t}{(q)_t} - 1 \right).
\]  

(1.3)

We remark that (1.1) and (1.2) follow immediately by taking \( z \to 0 \) and \( z \to 1 \) respectively.

**1.1. Notation and terminologies.** Throughout this paper, \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{> 0} \) denote the set of nonnegative integers and positive integers, respectively. Given a partition or an overpartition \( \lambda \) of \( n \), let \( \ell(\lambda) \) be the number of parts of \( \lambda \) and \( |\lambda| = n \) be the sum of the parts of \( \lambda \). When \( \lambda \) is an overpartition, we use \( o(\lambda) \) to count the number of overlined parts in \( \lambda \). We write parts in weakly decreasing order.

For a positive integer \( t \), we denote by \( \mathcal{P}_t \) the set of (nonempty) overpartitions with parts less than or equal to \( t \) and no parts equal to \( t \) overlined, and by \( \mathcal{G}_t \) the set of (nonempty) overpartitions with differences between largest and smallest parts at most \( t \) and no largest parts overlined when the difference between largest and smallest parts is exactly \( t \). Also, \( \mathcal{B}_t \) denotes the set of bipartitions where the first subpartition, which can be an empty partition, consists of only parts equal to \( t \), all not overlined, and the second subpartition is a nonempty overpartition with the largest part at most \( t \).

The rest of this paper is organized as follows. In Section 2.1, we first construct a weight preserving map \( \phi \) from \( \mathcal{G}_t \) to \( \mathcal{P}_t \). In Section 2.2, we then construct another weight preserving map \( \psi \) from \( \mathcal{P}_t \) to \( \mathcal{B}_t \). Finally, by combining these two maps, we will deduce that \( \mathcal{G}_t \) and \( \mathcal{B}_t \) have the same generating functions:

\[
\sum_{\pi \in \mathcal{G}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\beta \in \mathcal{B}_t} z^{o(\beta)} q^{|eta|},
\]

which is indeed equivalent to Theorem 1.1. In Section 3, a \( q \)-series proof of Theorem 1.1 will be given.

**2. A combinatorial approach**

**2.1. Partition Sets \( \mathcal{G}_t \) and \( \mathcal{P}_t \).** For an overpartition \( \pi = (\pi_1, \pi_2, \ldots, \pi_\ell) \) in \( \mathcal{G}_t \), let \( s(\pi) = |\pi_\ell|/t \), where \( |a| \) denotes the largest integer not exceeding \( a \), and let
Let \( k(\pi) \) be a positive integer \( k \) such that \( \pi_k \geq (s(\pi) + 1)t \) and \( \pi_{k+1} < (s(\pi) + 1)t \). If there is no such \( k \), then we let \( k(\pi) = 0 \).

We now define a map \( \phi : \mathcal{G}_t \rightarrow \mathcal{F}_t \) as follows. For an overpartition \( \pi \in \mathcal{G}_t \), let \( \ell(\pi) = \ell, s(\pi) = s \) and \( k(\pi) = k \). Then
\[
\phi: (\pi_1, \pi_2, \ldots, \pi_t) \mapsto (t, t, t, \ldots, t, \pi_{k+1} - st, \ldots, \pi_{\ell} - st, \pi_1 - (s + 1)t, \ldots, \pi_k - (s + 1)t),
\]
where all the parts equal to \( t \) are not overlined, and if \( \pi_i \) is overlined, then \( \pi_i - st \) (or \( \pi_i - (s + 1)t \) depending on the value of \( i \)) is overlined. Here we note that there may be parts equal to 0 in \( \phi(\pi) \). If there are any parts equal to 0, then we delete them so that \( \phi(\pi) \) has positive parts only.

**Theorem 2.1.** \( \phi \) is a weight preserving map from \( \mathcal{G}_t \) to \( \mathcal{F}_t \).

**Proof.** Since \( \pi_1 - \pi_{\ell} \leq t \), \( s = \lfloor \pi_{\ell}/t \rfloor \), and \( \pi_k \geq (s + 1)t > \pi_{k+1} \), we have
\[
\frac{t}{\pi_{k+1} - st} \geq \cdots \geq \frac{\pi_k - (s + 1)t}{t} \geq \frac{\pi_1 - (s + 1)t}{t}.
\]
Thus the parts of \( \phi(\pi) \) are less than or equal to \( t \), and if there are overlined parts, they are less than \( t \).

We now show that no more than one part of the same size is overlined. Since \( \pi \) is an overpartition, at most one part of the same size is overlined in \( \pi \). Hence, of \( \pi_1 - st, \ldots, \pi_k - st \), if there are overlined parts, then they must be of different sizes. For the same reason, of \( \pi_{k+1} - (s + 1)t, \ldots, \pi_{\ell} - (s + 1)t \), overlined parts must be of different sizes. Thus, if \( \pi_{\ell} - st > \pi_1 - (s + 1)t \), then it is clear that all the overlined parts of \( \phi(\pi) \) have different sizes.

Let us suppose that \( \pi_{\ell} - st = \pi_1 - (s + 1)t \). Then, we have \( \pi_1 - \pi_{\ell} = t \). By the definition of \( \mathcal{G}_t \), we know that all the parts equal to \( \pi_1 \) are not overlined. Thus, there is at most one overlined part in \( \phi(\pi) \) that is equal to \( \pi_{\ell} - st = \pi_1 - (s + 1)t \).

Therefore, \( \phi(\pi) \in \mathcal{F}_t \).

We also note that the map \( \phi \) preserves the weight of \( \pi \), that is, \( |\phi(\pi)| = |\pi| \). \( \square \)

As we see in the following example, the map \( \phi \) is not a bijection.

**Example 2.1.** Let \( t = 3, \pi_1 = (7, 3) \) and \( \pi_2 = (3, 4, 3) \). Then
\[
s(\pi_1) = 1, \quad k(\pi_1) = 1, \quad \phi(\pi_1) = (3, 3, 3, 1, 1), \quad |\phi(\pi_1)| = |\pi_1| = 11;
\]
\[
s(\pi_2) = 1, \quad k(\pi_2) = 0, \quad \phi(\pi_2) = (3, 3, 3, 1, 1), \quad |\phi(\pi_2)| = |\pi_2| = 11.
\]

However, \( \phi \) is a surjection since \( \mathcal{F}_t \) is a subset of \( \mathcal{G}_t \) and \( \phi(\pi) = \pi \) for any \( \pi \in \mathcal{F}_t \). So, we will count how many pre-images each \( \mu \in \mathcal{F}_t \) has under \( \phi \).

Let \( \pi \in \mathcal{G}_t \). We describe how to recover \( \pi \) from \( \phi(\pi) \). First, note that it is clear from the definition of \( s(\pi) \) and \( k(\pi) \) that \( \pi_i - (s(\pi) + 1)t \) and \( \pi_j - s(\pi)t \) are the remainders of \( \pi_i \) and \( \pi_j \) when divided by \( t \) for \( 1 \leq i \leq k(\pi) \) and \( j > k(\pi) \). If the remainders are equal to 0, then they are deleted in \( \phi(\pi) \). Thus if we know the number of such deleted remainders, we can determine \( \ell(\pi) \). Also, one of the deleted remainder may have been overlined.

We then need to find \( s(\pi) \) and \( k(\pi) \), where \( s(\pi) \) is the quotient of the smallest part of \( \pi \) when divided by \( t \) and \( k(\pi) \) counts the number of parts whose quotients are equal to \( s(\pi) + 1 \). Therefore, once we have \( \ell(\pi), k(\pi), \) and \( s(\pi) \) along with the information on existence of an overlined deleted remainder, it is clear that we
can recover \( \pi \). Thus possible choices for \( \ell(\pi), k(\pi), \) and \( s(\pi) \) with having a deleted remainder overlined or not will determine the number of pre-images under \( \phi \).

In the following lemma, we will see the range for \( \ell(\pi) \). For any \( \mu \in \overline{\mathcal{P}}_t \), we use \( m(\mu) = m_4(\mu) \) to count the number of parts of \( \mu \) that equal \( t \).

**Lemma 2.2.** Let \( \pi \) be a nonempty overpartition in \( \overline{\mathcal{G}}_t \) and \( \mu = \phi(\pi) \) in \( \overline{\mathcal{P}}_t \). Then we have

(i) \( \ell(\pi) \leq \ell(\mu) \);

(ii) \( \ell(\pi) \geq \ell(\mu) - m(\mu) + \delta_{\ell(\mu),m(\mu)} \), where \( \delta_{\ell(\mu),m(\mu)} \) is the Kronecker delta.

**Proof.** Let \( \ell(\pi) = \ell, s(\pi) = s, k(\pi) = k \). We first note that, if \( s \geq 1 \), then

\[
\ell(\mu) \geq s(\ell - k) + (s + 1)k = s\ell + k \geq \ell.
\]

If \( s = 0 \), since \( \pi\ell > 0 \), all of \( \pi_{k+1} - st, \ldots, \pi\ell - st \) are nonzero in \( \mu \). Hence

\[
\ell(\mu) \geq s(\ell - k) + (s + 1)k + (\ell - k) = \ell.
\]

This completes the proof of (i).

Next, we prove (ii). If all of the parts of \( \mu \) are \( t \), i.e., \( \ell(\mu) = m(\mu) \), then

\[
\ell(\mu) - m(\mu) + \delta_{\ell(\mu),m(\mu)} = 1 \leq \ell,
\]

where the last inequality follows from the fact that \( \pi \) is nonempty.

We now suppose that \( \mu \) has a part not equal to \( t \), i.e., \( \ell(\mu) - m(\mu) \geq 1 \). From the definition of \( \phi \), we know that the parts of \( \mu \) not equal to \( t \) are the positive remainders of the parts of \( \pi \), so at most \( \ell \) parts of \( \mu \) are not equal to \( t \). Hence

\[
\ell(\mu) - m(\mu) + \delta_{\ell(\mu),m(\mu)} = \ell(\mu) - m(\mu) \leq \ell.
\]

This completes the proof of (ii). \( \square \)

It follows from Lemma 2.2 that

\[
\delta_{\ell(\mu),m(\mu)} \leq \ell(\pi) - (\ell(\mu) - m(\mu)) \leq m(\mu), \tag{2.1}
\]

where \( \ell(\pi) - (\ell(\mu) - m(\mu)) \) is the number of multiples of \( t \) in \( \pi \).

**Lemma 2.3.** Let \( n \) be a fixed positive integer, and \( n' \) a fixed nonnegative integer. Then the following system of equations

\[
\begin{cases}
  x + y &= n, \\
  s \ x + (s + 1)y &= n'
\end{cases} \tag{2.2}
\]

has exactly one simultaneous solution \( (x, y, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \).

**Proof.** We readily see that \( y = n' - s \cdot n \). Also, since \( x > 0 \) and \( y \geq 0 \), it follows from the first equation that \( 0 \leq y < n \). Hence

\[
\frac{n'}{n} - 1 < s \leq \frac{n'}{n},
\]

from which it follows that \( s = \lfloor n'/n \rfloor \). Therefore, there is only one solution \( (x, y, s) \). \( \square \)

We are now ready to determine how many pre-images an overpartition in \( \overline{\mathcal{P}}_t \) has.
Theorem 2.4. Let $\lambda$ be a nonempty overpartition $\in \overline{\mathcal{P}}_t$.

(i) If $\lambda$ has parts equal to $t$ only, then there are exactly $2m(\lambda)$ pre-images in $\overline{\mathcal{G}}_t$ under $\phi$. Moreover, of those pre-images, exactly $m(\lambda)$ pre-images have no overlined parts, and the other $m(\lambda)$ pre-images have one of the smallest parts overlined.

(ii) If $\lambda$ has parts not equal to $t$, then there are exactly $2m(\lambda) + 1$ pre-images in $\overline{\mathcal{G}}_t$ under $\phi$. Moreover, of those pre-images, exactly $m(\lambda) + 1$ pre-images have the same number of overlined parts as $\lambda$ and the other $m(\lambda)$ pre-images have one more overlined part than $\lambda$ does.

Proof. Let $\pi$ be a pre-image of $\lambda$. By Lemma 2.2, we know that

$$\ell(\lambda) - m(\lambda) + \delta_{\ell(\lambda), m(\lambda)} \leq \ell(\pi) \leq \ell(\lambda).$$

Hence, for any integer $\ell$ in this range, we want to know how many $\pi \in \overline{\mathcal{G}}_t$ with $\ell(\pi) = \ell$ can be pre-images of $\lambda$.

In order for $\pi$ to be a pre-image of $\lambda$ with $\ell(\pi) = \ell$, $s(\pi)$ and $k(\pi)$ must satisfy

$$s(\pi)(\ell - k(\pi)) + (s(\pi) + 1)k(\pi) = m(\lambda).$$

By the definition of $k(\pi)$, it should be less than $\ell(\pi)$, i.e., $\ell - k(\pi) > 0$. Thus, (2.4) is equivalent to that $(\ell - k(\pi), k(\pi), s(\pi))$ is a solution to (2.2) with $n = \ell$ and $n' = m(\lambda)$, which is unique.

(i) Suppose that $\lambda$ has parts equal to $t$ only, i.e., $\ell(\lambda) = m(\lambda)$. By (2.3), there are $m(\lambda)$ choices for $\ell$. For a fixed $\ell$, $k(\pi)$ and $s(\pi)$ are uniquely determined as seen above. With these $(\ell, k(\pi), s(\pi))$, we can construct $\pi$, in which parts are multiples of $t$ differing by at most $t$ and there are no overlined parts.

For each $\pi$, by having one of the smallest parts overlined, we obtain a different pre-image. Therefore, the total number of pre-images must be equal to $2m(\lambda)$ as claimed. Also, $m(\lambda)$ pre-images have no overlined parts and the other $m(\lambda)$ pre-images have one overlined smallest part.

(ii) Suppose that $\lambda$ has parts not equal to $t$, i.e., $\ell(\lambda) > m(\lambda)$. By (2.3), there are $(m(\lambda) + 1)$ choices for $\ell$. For a fixed $\ell$, $k(\pi)$ and $s(\pi)$ are uniquely determined. With these $(\ell, k(\pi), s(\pi))$, we can construct $\pi$, in which no multiples of $t$ are overlined.

Note that if $\ell(\pi) > \ell(\lambda) - m(\lambda)$, then $\pi$ must have a multiple of $t$ as a part. For such $\pi$, by having one of the smallest multiples of $t$ overlined, we obtain a different pre-image.

Therefore, the total number of pre-images must be equal to $(2m(\lambda) + 1)$ as claimed. Also, $(m(\lambda) + 1)$ pre-images have the same number of overlined parts as $\lambda$ and the other $m(\lambda)$ pre-images have one more overlined part than $\lambda$ does. \qed

Theorem 2.4 yields

$$\sum_{\pi \in \mathcal{D}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\lambda \in \overline{\mathcal{P}}_t} \left( (1 - \delta_{\ell(\lambda), m(\lambda)}) + (1 + z)m(\lambda) \right) z^{o(\lambda)} q^{\ell(\lambda)}. \quad (2.5)$$

In the following example, we present how to find all the pre-images $\pi$ of $\lambda$.

Example 2.2. Let $t = 3$.

(i) Let $\lambda = (3, 3, 3)$. Since $\ell(\lambda) = m(\lambda) = 3$, by Lemma 2.2

$$1 \leq \ell(\pi) \leq 3.$$

By solving (2.4), we have $(\ell(\pi), k(\pi), s(\pi)) = (1, 0, 3), (2, 1, 1), (3, 0, 1)$, which yield

(9), (9),
(6, 3), (6, 3),
(3, 3, 3), (3, 3, 3), respectively. There are 2m(λ) pre-images.

(ii) Let λ = (3, 3, 3, 1, 1). Since ℓ(λ) = 5 and m(λ) = 3, by Lemma 2.2

2 ≤ ℓ(π) ≤ 5.

By solving (2.4), we have (ℓ(π), k(π), s(π)) = (2, 1, 1), (3, 0, 1), (4, 3, 0), (5, 3, 0), which yield

(7, 4), (4, 4, 3), (4, 4, 3), (4, 3, 3, 1), (4, 3, 3, 1),
(3, 3, 3, 1, 1), (3, 3, 3, 1, 1), respectively. Thus, there are 2m(λ) + 1 pre-images.

2.2. Partition Sets \( \overline{P}_t \) and \( \overline{B}_t \). Let us recall the definition of \( \overline{B}_t \), from which it is clear that

\[
\sum_{\beta \in \overline{B}_t} z^{o(\beta)} q^{j(\beta)} = \left(1 + q^t + q^{2t} + \cdots\right) \left(\frac{(-zq)_t}{(q)_t} - 1\right)
\]

\[
= \frac{1}{1 - q^t} \left(\frac{(-zq)_t}{(q)_t} - 1\right), \tag{2.6}
\]

where \( o(\beta) \) denotes the number of overlined parts in \( \beta \), which is indeed the number of overlined parts in the second subpartition of \( \beta \).

We now construct a map \( \psi : \overline{B}_t \rightarrow \overline{P}_t \) as follows:

1. First collect all parts equal to \( t \) in both subpartitions and replace an overlined \( t \) by a non-overlined \( t \);
2. and then append the remaining parts in the second subpartition to the parts collected in (1).

For example, \([3, (3, 3, 1, 1)]\) and \([3, (3, 3, 1, 1)]\) are both mapped to \((3, 3, 3, 1, 1)\) under \( \psi \).

Let \( \lambda \in \overline{P}_t \). Suppose that \( \ell(\lambda) = m(\lambda) \), i.e., \( \lambda \) has parts equal to \( t \) only. Then, its pre-image \( \beta \) must be a bipartition of this form

\[
[(t_1, \ldots, t), (t_1, \ldots, t)]
\]

for some \( x > 0 \) with at most one of \( t \)'s in the second subpartition overlined. Thus there are \( 2m(\lambda) \) pre-images of \( \lambda \) in \( \overline{B}_t \) under \( \psi \). Of those pre-images, \( m(\lambda) \) pre-images have the same number of overlined parts as \( \lambda \), and the other \( m(\lambda) \) pre-images have one more overlined part than \( \lambda \).

Suppose that \( \ell(\lambda) > m(\lambda) \), i.e., \( \lambda \) has a part not equal to \( t \). Then, its pre-image \( \pi \) must be a bipartition of this form

\[
[(t_1, \ldots, t), (t_1, \ldots, t, \lambda_{m(\lambda)+1}, \ldots)]
\]

for some \( x \geq 0 \) with at most one of \( t \)'s in the second subpartition overlined. Thus there are \( 2m(\lambda) + 1 \) pre-images of \( \lambda \) in \( \overline{B}_t \) under \( \psi \). Of those pre-images, \( (m(\lambda) + 1) \)
pre-images have the same number of overlined parts as $\lambda$, and the other $m(\lambda)$
pre-images have one more overlined part than $\lambda$.

Therefore, it follows from the map $\psi$ that
\[
\sum_{\lambda \in \mathcal{P}_t} ((1 - \delta_{t(\lambda),m(\lambda)}) + (1 + z)m(\lambda)) z^{o(\lambda)} q^{|\lambda|} = \sum_{\beta \in \mathcal{B}_t} z^{o(\beta)} q^{|\beta|}. \tag{2.7}
\]

By (2.5), (2.6), and (2.7),
\[
\sum_{n \geq 1} \sum_{m \geq 0} g_t(m, n) z^m q^n = \sum_{n \in \mathcal{V}_t} z^{o(\pi)} q^n = \sum_{\beta \in \mathcal{B}_t} z^{o(\beta)} q^{\beta} = \frac{1}{1 - q} \left( \frac{(-zq)_t}{(q)_t} - 1 \right),
\]
which completes the proof of Theorem 1.1.

3. Final remarks

We remark that, by slightly modifying the first proof of [5, Theorem 2.1], we can
also prove Theorem 1.1 analytically.

Let
\[
\phi_s(a_0, a_1, a_2, \ldots; q, z) := \sum_{n \geq 0} \frac{(a_0; q)_n(a_1; q)_n \cdots (a_s; q)_n}{(q; q)_n(b_0; q)_n \cdots (b_s; q)_n} \left( (-1)^n q^{n(3)} \right)^{s-r} z^n.
\]

Then we will need the following identities later.

Lemma 3.1 (First $q$-Chu–Vandermonde Sum [1, Eq. (17.6.2)]). We have
\[
2\phi_1 \left( \frac{a, q^{-n}}{c}; q, cq^n/a \right) = \left( \frac{e/a; q}_n \right) \left( \frac{c; q}_n \right), \tag{3.1}
\]

Lemma 3.2 (1, Eq. (17.9.6)). We have
\[
3\phi_2 \left( \frac{a, b, c}{d, e}; q, de/(abc) \right) = \left( \frac{e/a; q}_\infty \right) \left( \frac{d/(bc); q}_\infty \right) \left( \frac{a, d/b, d/c}{e; q}_\infty \right) \left( \frac{d, de/(bc); q}_\infty \right) 3\phi_2 \left( \frac{a, d/b, d/c}{e; q}_\infty \right). \tag{3.2}
\]

First, note that the generating function for partitions in $\mathcal{U}_t$ with smallest part
equal to $r$ is
\[
\frac{(1 + z)q^r 1 + zq^{r+1}}{1 - q} \cdots \frac{1 + zq^{r+t-1}}{1 - q^{r+t-1}} \frac{1}{1 - q^{r+t}},
\]
in which the coefficient of $z^m q^n$ counts the number of such overpartitions of $n$ with
exactly $m$ overlined parts. Hence
\[
\sum_{n \geq 1} \sum_{m \geq 0} g_t(m, n) z^m q^n = \sum_{r \geq 1} \frac{(1 + z)q^r 1 + zq^{r+1}}{1 - q^r} \cdots \frac{1 + zq^{r+t-1}}{1 - q^{r+t-1}} \frac{1}{1 - q^{r+t}}
= (1 + z) \sum_{r \geq 1} \frac{(q)_r(1-q^{r+t})}{(q)_r(-q)(q)_r} q^r
= (1 + z)q \sum_{r \geq 0} \frac{(q)_r(-zq)r_t}{(q)_r(-q)(q)_r} q^r
= \frac{(1 + z)q(-zq)_t}{(1 + z)q(q)_{t+1}} \sum_{r \geq 0} \frac{(q)_r(-zq)(q)_{r+t+1}}{(q)_r(q^{t+2})(q)_r} q^r
= \frac{(1 + z)q(-zq)_t}{(1 + z)q(q)_{t+1}} 3\phi_2 \left( \frac{q, q, -zq^{t+1}}{-zq^2, q^{t+2}; q, q} \right)
(by Eq. (3.2)) $= \frac{(1 + z)q(-q)_t}{(1 + z)(q)_t} \frac{(q^{t+1})_\infty (q^2)_\infty}{(q^t+2)_\infty (q)_\infty} 3 \phi_2 \left( \frac{q, -zq, q^{1-t}}{-zq^2, q^2 : q, q^{t+1}} \right)$

$= \frac{(1 + z)q(-q)_t q^{t+1}}{(1 - q)(1 + zq)(q)_t} \sum_{r \geq 0} \frac{(-zq)_r(q^{1-t})_r}{(-zq^2)_r (q^2)_r} q^{r(t+1)}$

$= -\frac{(-zq)_t}{(1 - q^t)(q)_t} \sum_{r \geq 0} \frac{(-z)_r(q^{-t})_r}{(-zq^t)_r (q)_r} q^{r(t+1)(t+1)}$

(by Eq. (3.1)) $= -\left\{ -\frac{(-zq)_t}{(1 - q^t)(q)_t} \left( -\frac{q^{-t}}{-zq} : q, q^{t+1} \right) - 1 \right\}$

$= \frac{1}{1 - q^t} \left( -\frac{(-zq)_t}{(q)_t} - 1 \right)$.

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