THE BROWNIAN CONGA LINE

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Abstract

We introduce a new model called the Brownian Conga Line. It is a random curve evolving in time, generated when a particle performing a two dimensional Gaussian random walk leads a long chain of particles connected to each other by cohesive forces. We approximate the discrete Conga line in some sense by a smooth random curve and subsequently study the properties of this smooth curve.

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1 Introduction

The Conga Line is a Cuban carnival march that has become popular in many cultures over time. It consists of a queue of people, each one holding onto the person in front of him. The person at the front of the line can move as he will, and the person holding onto him from behind follows him. The third person in the queue follows the second, and so on. Often people keep on joining the line over time by attaching themselves to the last person in the line. As the Conga Line grows in time, it displays interesting motion patterns where the randomness in the motion of the first person propagates down the line, diminishing in influence as it moves further down. In this article, we devise a mathematical formulation of the Conga Line and study its properties.

The formulation is as follows.

Let $Z_k, k \geq 1$, be i.i.d standard 2-dimensional normal random variables. Fix some $\alpha \in (0, 1)$. Let $X_1(0) = 0$ and $X_1(n) = \sum_{i=1}^{n} Z_i$ for $n \geq 1$. This denotes the leading particle, or the tip of the Conga line.

Now, we define processes $X_k$ inductively as follows. Suppose that $\{X_k(n), n \geq 0\}$ have already been defined for $1 \leq k \leq j$. Then we let $X_{j+1}(0) = 0$ and

$$X_{j+1}(n) = (1 - \alpha)X_{j+1}(n-1) + \alpha X_j(n-1) \quad (1.1)$$

for $n \geq 1$. Here, the process $X_k$ denotes the motion over time of the particle at distance $k$ from the leading particle. The relation (1.1) describes the manner in which a particle $X_{j+1}$ follows the preceding particle $X_j$. It is easy to check from (1.1) that $X_j(n) = 0$ for all $j > n$. These represent the particles at rest at the origin at time $n$. Note that the $j$-th particle $X_j$ joins the Conga Line at time $j$. See Figure 1 for the construction of the Conga line for $n = 1, 2, 3, 4$.

The Conga line at time $n$ is defined as the collection of random variables $\{X_k(n), k \leq n\}$.

One can also think of this model as a discrete version of a long string or molecule whose tip is moving randomly under the effect of an erratic force and the rest of it performs a constrained motion governed by the tip together with the cohesive forces. Burdzy and Pal [8] performed some simulations (see Figure 2) which led them to make the following observations:

1. For a fixed large $n$, the locations of the particles $\{X_k(n), k \geq 1\}$ sufficiently away from the tip look like a ‘smooth’ curve, and the smoothness increases as we move away from the tip.
2. For $k$ significantly larger than 1, there is very little variability in the location of the particles over short periods of time.
3. The small loops in the curve tend to die out over time. Just before death, they look ‘elongated’ and their death site forms a cusp.
4. The particles near the origin seem to freeze showing very little movement over time.

All the above observations need precise mathematical formulations. Once the rigorous foundations are established, we can ask the correct questions and try to answer them. This, broadly, is the goal of the article.

We give a brief outline of the content of each section.
In Section 2, we try to make mathematical sense of the statement ‘the process looks like a smooth curve’. This is the toughest challenge as the Conga line, unlike most known stochastic processes which can be approximated by continuous models, does not seem to have an interesting scaling limit. This is because if we look at the Conga Line for any fixed $n$, the distance between the particles decays exponentially as we move away from the tip, i.e., increase $k$. But it is precisely why this is a novel model, which exhibits particles moving in different scales ‘in the same picture’. The particles near the tip are wider spaced and their paths mostly resemble a Gaussian random walk, but those for large $k$ are more closely packed and the Conga Line looks very smooth in this region (see Figure 2). To circumvent this problem, we describe a coupling between our discrete process $\{X_k(n), k \leq n\}$ and a smooth random process $\{u(x, t) : (x, t) \in \mathbb{R}^2\}$ such that, when observed sufficiently away from the tip, more precisely for $k \geq n^\epsilon$ for any fixed $\epsilon > 0$ and large $n$, the points $X_{k+1}(n)$ are uniformly close to the points $u(k, n)$. Thus, $u$ serves as a smooth approximation to the discrete process $X$ in a suitable sense. The $x$ variable of $u$ represents distance from the tip and the $t$ variable represents time. Future references to the Conga line refer to this smooth version $u$. We close the section by presenting another smooth process $\overline{u}$ that also serves as an approximation in the same sense, and is more intuitive when considering the motion of individual particles, i.e. trajectories of the form $\{X_k(n) : n \geq k\}$ for fixed $k$. It is also used in Section 6 to study the phenomenon of freezing of the Conga line near the origin.

In Section 3, we study the properties of the continuous, one dimensional Conga line $u$. First, we investigate the phenomenon of the particles at different distances from the tip moving in ‘different scales’ suggested by their different order of variances. The particles near the tip wiggle wildly indicated by their variance being $O(t)$, while those far away from the tip show very little movement, indicated by exponentially decaying variances. Furthermore, there exists a cutoff near $x = \alpha t$, where the variance shows a sharp transition from ‘very large to very small’. We identify this and study the fine changes in variance around this point.
Next, using the scaling properties of Brownian motion, we show that for fixed $t$, the Conga line can be scaled so that the space variable $x$ runs in $[0, 1]$. We call this scaled version $u_t$ and study its analytical properties. Upper bounds on the growth rate of the derivatives show that $u_t$ is real analytic. We also make a detailed study of the covariance structure of the derivatives. This turns out to be a major tool in studying the subsequent properties like critical points, length, loops, etc.

With the basic framework of the Conga line established, we set out to investigate its finer properties. We investigate the distribution of critical points of $u_t$, i.e., points at which the derivative vanishes. The number of critical points in an interval serves as a measure of how wiggly the Conga line looks on that interval. The critical points are distributed as a point process on the real line and we show using an expectation meta-theorem for smooth Gaussian fields (see [1] p. 263) that its first intensity at $x$ (for a large time $t$) is approximately of the form $\sqrt{tx}^{-1/2}$. This shows that, though the typical number of these points in a given interval is $O(\sqrt{t})$ for large $t$, the proportion of critical points around $x$ decreases as $x^{-1/2}$ as we go farther away from the tip. We also show subsequently using second moment estimates that the critical points are reasonably well-spaced and they do not tend to crowd around any point. Furthermore, we show that the first intensity is a good estimate of the point process itself as for a given interval $I$ sufficiently away from the ends $x = 0$ and $x = 1$, the ratio $N_t(I)/E[N_t(I)]$ goes to one in probability as $t$ grows large.

In Section 4 we study properties of the scaled two dimensional Conga line, like length and number of loops. We also investigate a strange phenomenon. Although the mechanism of subsequent particles following the preceding ones and 'cutting corners' results in progressively smoothing out the roughness of the Gaussian random walk of the tip, we see that as $t$ increases, the scaled Conga line looks more and more like Brownian motion in that the sup-norm distance between them on $[0, 1]$ is roughly of order $t^{-1/4}$ (with a log correction term). This can be explained by the fact that the noticeable smoothing of the paths of the unscaled Conga line takes place in a window of width $\sqrt{t}$ around each point, which translates to a window of width $t^{-1/2}$ as we scale space and time by $t$. Thus in the scaled version, the smoothing window becomes smaller with time, resulting in this phenomenon. Thus, the scaled Conga line $u_t$ for large $t$ serves as a smooth approximation to Brownian motion which smooths out microscopic irregularities but retains its macroscopic characteristics.

In Section 5 we study the evolution of loops in the sequence of paths that the particles at successively larger distances from the tip trace out. We study this evolution under a metric similar to the Skorohod metric. It turns out that with probability one, every singularity, i.e a point where the speed of the curve becomes zero, in a particle path is a cusp singularity (looks like the graph of $y = x^{2/3}$ in a local co-ordinate frame). Furthermore, there is a bijection between dying loops and cusp singularities in the sense that small loops die (i.e. the end points of the loop merge and loop shrinks to a point) creating cusp singularities, and conversely, if such a singularity appears in the path of some particle, we can find a loop in the path of the immediately preceding particles, and it dies creating the singularity.

Finally, in Section 6, we investigate the phenomenon of freezing near the origin. We work with the smooth approximation $\overline{u}$, and show that for an appropriate choice of a sequence $x_t$ of distances from the tip such that the particles at these distances remain sufficiently close to the origin, $\overline{u}(x_t, t)$ converges almost surely and in $L^2$, and find the limiting function.

**Notation:** Before we proceed, we clarify the notation that we will be using here:

(i) If $g$ is a random function and $V$ is a random variable with distribution function $F$ and independent of $g$, then

$$\mathbb{E}_V g(V) = \int g(v) dF(v)$$
denotes the expectation with respect to $V$ for a fixed realisation of $g$.

(ii) $\Phi$ denotes the normal distribution function and $\Phi = 1 - \Phi$.

(iii) For any function $f$ of several variables, $\partial_x^k f$ denotes the partial derivative of $f$ with respect to the variable $x$ taken $k$ times.

(iv) For functions $f, g : [0, \infty) \to \mathbb{R}^+$, $f(t) \sim g(t)$ means that there $f$ and $g$ have the same growth rate in $t$, i.e., there exists a constant $C$ such that

$$\frac{f(t)}{g(t)} \vee \frac{g(t)}{f(t)} \leq C$$

for all sufficiently large $t$.

(v) For a family of real-valued functions $\{f_t: t \in (0, \infty)\}$ defined on a compact set $I \subseteq \mathbb{R}^k$ and a function $a : (0, \infty) \to [0, \infty)$, we say

$$f_t = O_\infty (a(t)) \text{ on } I$$

if

$$\sup_{t \in (0, \infty)} \frac{\sup_{x \in I} |f_t(x)|}{a(t)} \leq C$$

for some constant $C < \infty$. Sometimes, (by abuse of notation) we will write

$$f_t(x) = O_\infty (a(t)) \text{ for } x \in I$$

to denote the same.

2 The Discrete Conga Line

We set out by finding a neater expression for $X_k(n)$ in terms of $X_1(n)$.

Let $T_1, T_2, \ldots$ be i.i.d Geom($\alpha$) and let

$$\Theta_j = \sum_{i=1}^{j} T_i.$$

Then $\Theta_j \sim NB(j, 1 - \alpha)$, where $NB(a, b)$ represents the Negative Binomial distribution with parameters $a$ and $b$. It is easy to see from the recursion relation (1.1) that one can write

$$X_k(n) = \mathbb{E}_{T_1} X_{k-1}(n - 1 - T_1).$$

By induction, we get

$$X_k(n) = \mathbb{E}_{S} X_1(n - k + 1 - \Theta_{k-1}) = \sum_{m=0}^{n-k+1} \binom{m + k - 2}{m} (1 - \alpha)^m \alpha^{k-1} X_{1}(n - k + 1 - m).$$

(2.1)
2.1 Approximation by a smooth process

Here we show that for any fixed $\epsilon > 0$, the discrete one dimensional Conga line can be approximated uniformly in $k$, for $n^\epsilon \leq k \leq n$, for large $n$, by a smooth process that arises as a smoothing kernel acting on Brownian motion.

Let $B_l \sim \text{Bin}(l, \alpha)$. From (2.1) and the fact that
\[ P(\Theta_{k-1} \leq l - k + 1) = P(B_l \geq k - 1), \]
we get
\[
X_k(n) = \mathbb{E}X_1(n - k + 1 - \Theta_{k-1}) = \sum_{j=0}^{n-k+1} P(\Theta_{k-1} = j) \sum_{l=0}^{n-k+1-j} Z_l
\]
\[
= \sum_{l=0}^{n-k+1} Z_l \sum_{j=0}^{n-k+1-l} P(\Theta_{k-1} = j) = \sum_{l=k-1}^{n} P(\Theta_{k-1} \leq l - k + 1)Z_{n-l}
\]
\[
= \sum_{l=k-1}^{n} P(B_l \geq k - 1)Z_{n-l}
\]

The next step is the key to the approximation. We obtain a coupling between a Brownian motion and our process $X$. Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Brownian motion $W$. Then
\[
X_k(n) = \sum_{l=k-1}^{n} P(B_l \geq k - 1)(W(n-l) - W(n-1))
\]
gives the desired coupling on this space. Note that we can write
\[
X_{k+1}(n) = \int_0^n g(k, z) dW^z
\]
where
\[ g(k, z) = P(B_{[z]} \geq k), \]
$[\cdot]$ being the greatest integer function, and $W^z_t = W(t) - W(t - z)$, $0 \leq z \leq t$, is the time reversed Brownian motion from time $t$.

Let $\sigma = \sqrt{\alpha(1 - \alpha)}$. Consider the “space-time” process
\[
u(x, t) = \int_0^t \Phi \left( \frac{x - \alpha z}{\sigma \sqrt{z}} \right) dW^z_t
\]
\[
= \int_0^t W(t - z)(\sqrt{2\pi})^{-1} \left( \frac{x + \alpha z}{2\sigma z^{3/2}} \right) \exp \left( -\frac{(x - \alpha z)^2}{2\sigma^2 z} \right) dz
\]
(We obtain the second expression from the first by an application of the Stochastic Fubini Theorem, see [7]).

We prove in what follows that for large $n$, and for $n^\epsilon \leq k \leq n$ for any fixed $\epsilon > 0$, the points $X_{k+1}(n)$ are
“uniformly close” to the points \( u(k, n) \) (\( u \) evaluated at integer points) for the given range of \( k \). Our strategy is to first consider a discretized version of the process \( u(x, t) \), given by

\[
\hat{u}(k, n) = \sum_{l=0}^{n} \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) (W(n - l) - W(n - l - 1))
\]

In Lemma 1, we give a bound on the \( L^2 \) distance between \( X_{k+1}(n) \) and \( \hat{u}(k, n) \) for large \( n \) when \( n^\epsilon \leq k \leq n \). In Lemma 2, a similar bound is achieved for the \( L^2 \) distance between \( \hat{u}(k, n) \) and \( u(k, n) \). In Theorem 1, we prove using a Borel Cantelli argument that for large \( n \) the two processes \( X \) and \( u \) (evaluated at integer points) come uniformly close on \( n^\epsilon \leq k \leq n \).

In the following, \( C_1, C_2, \ldots \) represent absolute constants, \( C_{\epsilon}, C'_{\epsilon} \) denote constants that depend only on \( \epsilon \), \( C_p \) denotes a constant depending only on \( p \) and \( D_{\epsilon,p}, D'_{\epsilon,p} \) denote constants depending upon both \( \epsilon \) and \( p \).

**Lemma 1** Fix \( \epsilon > 0 \). For \( n^\epsilon \leq k \leq n \),

\[
\sum_{l=0}^{n} \left[ P(B_l \geq k) - \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) \right]^2 \leq C_{\epsilon} \frac{\sqrt{\log k}}{\sqrt{k}}
\]

where \( \sigma = \sqrt{\alpha(1 - \alpha)} \). Consequently,

\[
\mathbb{E}(X_{k+1}(n) - \hat{u}(k, n))^2 \leq C_{\epsilon} \frac{\sqrt{\log k}}{\sqrt{k}}
\]

uniformly on \( n^\epsilon \leq k \leq n \).

**Proof:** Choose \( C > 0 \) such that \( \epsilon(C - 1) \geq 2 \) and \( C \geq \frac{3}{2} \). Take \( L_k = \lfloor \alpha^{-1} \sqrt{Ck \log k} \rfloor \), where \( \lfloor \cdot \rfloor \) represents the greatest integer function. Then, we can write

\[
\sum_{l=0}^{n} \left[ P(B_l \geq k) - \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) \right]^2 \leq \sum_{l=0}^{\lfloor k/\alpha \rfloor - L_k} \left[ P(B_l \geq k) - \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) \right]^2
\]

\[
+ \sum_{l=\lfloor k/\alpha \rfloor - L_k}^{\lfloor k/\alpha \rfloor + L_k} \left[ P(B_l \geq k) - \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) \right]^2
\]

\[
+ \sum_{l=\lfloor k/\alpha \rfloor + L_k}^{n} \left[ P(B_l \geq k) - \Phi \left( \frac{k - \alpha l}{\sigma \sqrt{l}} \right) \right]^2
\]

\[
= S_1^{(k)} + S_2^{(k)} + S_3^{(k)}.
\]

Here, \( S_1^{(k)} \) and \( S_3^{(k)} \) correspond to the *tails* of the distribution functions, and we shall show that they are negligible compared to \( S_2^{(k)} \). To this end, note that

\[
S_1^{(k)} \leq 2k\alpha^{-1} \left( P^2(B_{\lfloor k/\alpha \rfloor - L_k} \geq k) + \Phi^2 \left( \frac{\alpha L_k}{\sigma \sqrt{\frac{k}{\alpha}}} \right) \right).
\]
Now, by Bernstein’s Inequality,

\[
P(B_{\frac{k}{\alpha}} - L_k \geq k) \leq \exp \left( -\frac{\alpha^2 L_k^2}{2} \cdot \frac{1}{\alpha\left(\frac{k}{\alpha} - L_k + \alpha L_k/3\right)} \right) \leq \exp \left( -\frac{\alpha^2 L_k^2}{2k} \right).
\]

We also have

\[
\Phi \left( \frac{\alpha L_k}{\sigma \sqrt{\frac{L_k}{\alpha}}} \right) \leq \frac{\sigma \sqrt{\frac{L_k}{\alpha}}}{\alpha^{3/2} L_k} \exp \left( -\frac{\alpha^3 L_k^2}{2\sigma^2 k} \right).
\]

Therefore, for large \( k \),

\[
S_1^{(k)} \leq 4k \exp \left( -\frac{\alpha^2 L_k^2}{k} \right) \leq \frac{C_1}{k^{C-1}} \leq \frac{C_1}{\sqrt{k}}.
\]

Similarly, for \( S_3^{(k)} \), we get

\[
S_3^{(k)} \leq 2(n - [k\alpha^{-1}]) \left( P^2(B_{\frac{k}{\alpha}} + L_k < k) + \Phi^2 \left( -\frac{\alpha L_k}{\sigma \sqrt{2k}} \right) \right) \leq 4(n - [k\alpha^{-1}]) \exp \left( -\frac{\alpha^2 L_k^2}{2k} \right) \leq \frac{C_3(n - [k\alpha^{-1}])k^{C/2}}{\sqrt{kn^{C-1}/2}} \leq \frac{C_3}{\sqrt{k}}.
\]

Now, for \( S_2^{(k)} \), we use the Berry Esseen Theorem (see [3]).

\[
S_2^{(k)} \leq \sum_{l=[\frac{k}{\alpha}] - L_k}^{[\frac{k}{\alpha}] + L_k} \frac{1}{l} \leq C_4 \sqrt{\frac{C \log k}{\sqrt{k}}}.
\]

The above yield the lemma.

**Lemma 2** \( \mathbb{E}((\hat{u}(k, n) - u(k, n))^2) \leq C_4 \sqrt{\frac{\log k}{\sqrt{k}}} \) uniformly on \( n^\epsilon \leq k \leq n \).

**Proof:** Write \( \hat{u}(k, n) = \int_0^n \hat{f}(k, z) dW_z^n \) and \( u(k, n) = \int_0^n f(k, z) dW_z^n \) where

\[
\hat{f}(k, z) = \sum_{j=0}^n \Phi \left( \frac{k - \alpha j}{\sigma \sqrt{j}} \right) \mathbb{I}(j \leq z < j + 1) \quad \text{and} \quad f(k, z) = \Phi \left( \frac{k - \alpha z}{\sigma \sqrt{z}} \right).
\]

Then, we can decompose \( \mathbb{E}((\hat{u}(k, n) - u(k, n))^2) \) as in the proof of Lemma[1] as follows:

\[
\mathbb{E}((\hat{u}(k, n) - u(k, n))^2) = \int_0^n \left[ \sum_{j=0}^n \left( \Phi \left( \frac{k - \alpha z}{\sigma \sqrt{z}} \right) - \Phi \left( \frac{k - \alpha j}{\sigma \sqrt{j}} \right) \right) \mathbb{I}(j \leq z < j + 1) \right]^2 \, dz
\]
≤ \int_{0}^{\left[\frac{k}{\alpha}\right]-L_{k}} \left[ \sum_{j=0}^{n} \left( \Phi \left( \frac{k-\alpha z}{\sigma \sqrt{z}} \right) - \Phi \left( \frac{k-\alpha j}{\sigma \sqrt{j}} \right) \right) I(j \leq z < j + 1) \right]^{2} dz \\
+ \int_{\left[\frac{k}{\alpha}\right]-L_{k}}^{\left[\frac{k}{\alpha}\right]+L_{k}} \left[ \sum_{j=0}^{n} \left( \Phi \left( \frac{k-\alpha z}{\sigma \sqrt{z}} \right) - \Phi \left( \frac{k-\alpha j}{\sigma \sqrt{j}} \right) \right) I(j \leq z < j + 1) \right]^{2} dz \\
+ \int_{\left[\frac{k}{\alpha}\right]+L_{k}}^{n} \left[ \sum_{j=0}^{n} \left( \Phi \left( \frac{k-\alpha z}{\sigma \sqrt{z}} \right) - \Phi \left( \frac{k-\alpha j}{\sigma \sqrt{j}} \right) \right) I(j \leq z < j + 1) \right]^{2} dz \\
= I_{1}^{(k)} + I_{2}^{(k)} + I_{3}^{(k)}.

Now,
\frac{\partial f}{\partial z}(k, z) = (\sqrt{2\pi})^{-1} \left( \frac{k + \alpha z}{2\sigma z^{3/2}} \right) \exp \left( - \frac{(k - \alpha z)^2}{2\sigma^2 z} \right). \quad (2.3)

We can follow the same argument as in the proof of Lemma 1 and verify that \( I_{1}^{(k)} \) and \( I_{3}^{(k)} \) are bounded above by \( C_{6}(\sqrt{k})^{-1} \). To handle the second term, note that by (2.3), we have

\[ |\frac{\partial f}{\partial z}(k, z)| \leq C_{6}(\sqrt{k})^{-1}. \]
on \left[\frac{k}{\alpha}\right] - L_{k} \leq z \leq \left[\frac{k}{\alpha}\right] + L_{k}. \]So,

\[ I_{2}^{(k)} \leq \int_{\left[\frac{k}{\alpha}\right]-L_{k}}^{\left[\frac{k}{\alpha}\right]+L_{k}} \left( C_{6}(\sqrt{k})^{-1} \right)^2 dz = 2C_{6} \frac{\sqrt{\log k}}{\sqrt{k}}. \]

This proves the lemma. \( \square \)

So, by the preceding lemmas, we have proved that

\[ \mathbb{E}(X_{k+1}(n) - u(k, n))^2 \leq C_{\epsilon}' \frac{\sqrt{\log k}}{\sqrt{k}} \]

uniformly on \( n^{\epsilon} \leq k \leq n \).

Now, \( X_{k+1}(n) - u(k, n) = \int_{0}^{n} (g(k, z) - f(k, z)) dW_{z}^{n} \). Now, by the fact that this is a centred Gaussian random variable,

\[ \mathbb{E}|X_{k+1}(n) - u(k, n)|^{2p} \leq C_{p} \left[ \int_{0}^{n} (g(k, z) - f(k, z))^{2} dz \right]^{p} \leq C_{p} C_{\epsilon}^{p} \left( \frac{\log k}{k^{p/2}} \right). \quad (2.4) \]

We use this to obtain the following theorem.

**Theorem 1**  For any \( \delta > 0, \epsilon > 0 \) and \( \eta > 0 \), define the following events:

\[ A_{k,n} = \left\{ |X_{k+1}(n) - u(k, n)| > \delta k^{-\frac{1}{4} - \eta} \right\} \]
and

\[ B_n^\varepsilon = \bigcup_{k=n}^{n} A_{k,n} \]

Then \( P(\limsup_n B_n) = 0 \).

**Proof:** By Chebyshev type Inequality (for \( 2p \)-th moment) and (2.4), we get for any \( p \geq 1 \),

\[
P(A_{k,n}) \leq \frac{C_p C_{\rho p}^p}{\delta^{2p}} \left( k^{p/2-2\eta p} \left( \frac{\log k}{k} \right)^{p/2} \right) \leq D c_p k^{-(2\eta p-1)}.
\]

Hence,

\[
P(B_n^\varepsilon) \leq D c_p \sum_{k=n}^{n} k^{-(2\eta p-1)} \leq \frac{D c_p}{2\eta p - 2} n^{-(2\eta p-2)}.
\]

Now, choose \( p \) large enough such that \( P(B_n^\varepsilon) \leq C n^{-2} \) for some constant \( C \). The result now follows by the Borel Cantelli lemma.

**Note:** The above theorem suggests that the distance between the points \( X_k(n) \) and \( u(k,n) \) decreases as we increase \( k \). This is exactly what is suggested by Figure 2.

### 2.2 A related process

Here we give another smooth approximation \( \overline{u} \) to \( X \) given by

\[
\overline{u}(x, t) = E_Z W \left( t - \frac{x}{\alpha} - Z_{\rho^2 x} \right),
\]

that proves to be more useful and intuitive while investigating the paths of individual particles and study the phenomenon of freezing near the origin. Note that \( \overline{u} \) has the following properties:

- Increasing \( x \) (i.e. going away from the tip) results in the paths \( \overline{u}(x, \cdot) \) being progressively smoother, indicated by the increasing variance of the \( Z \) variable.
- For fixed \( x \), varying \( t \) represents the path of the particle at distance \( x \) from the tip. As successive particles ‘cut corners’, the paths become smoother.

**Theorem 2** The result of Theorem 7 holds with \( u \) replaced by \( \overline{u} \).

**Proof:** Let \( \rho = \sigma/\alpha^{3/2} \). Consider another continuous process \( u^* \) given by

\[
u^*(x, t) = \int_0^t \int_{\sqrt{2\pi x \rho}} \Phi \left( \frac{x - w}{\rho \sqrt{x}} \right) dW_s^t.
\]

Rewrite \( u^* \) as follows:

\[
u^*(x, t) = \int_0^t \int_{-\infty}^{\max\{w,0\}} \Phi \left( \frac{x - w}{\rho \sqrt{x}} \right) dW_s^t
\]

Theorem 2 The result of Theorem 7 holds with \( u \) replaced by \( \overline{u} \).
\[
\begin{align*}
\int_0^t \frac{1}{\sqrt{2\pi x_0}} \phi \left( \frac{x_0 - w}{\rho \sqrt{\xi}} \right) W(t - w) dw + W(t) \Phi \left( \frac{\sqrt{\xi}}{\alpha \rho} \right) \\
\int_{-\infty}^t \frac{1}{\sqrt{2\pi x_0}} \phi \left( \frac{x_0 - w}{\rho \sqrt{\xi}} \right) W(t - w) dw \\
- \int_{-\infty}^0 \frac{1}{\sqrt{2\pi x_0}} \phi \left( \frac{x_0 - w}{\rho \sqrt{\xi}} \right) W(t - w) dw + W(t) \Phi \left( \frac{\sqrt{\xi}}{\alpha \rho} \right)
\end{align*}
\]

\( = \Phi(x, t) + e_1(x, t) + e_2(x, t). \)

Clearly,
\[
\mathbb{E}(e_2^2(k, n)) \leq \frac{\alpha^2 \rho^2 n}{k} \exp \left\{ - \frac{k}{\alpha^2 \rho^2} \right\}.
\]

So, on \( n^\varepsilon \leq k \leq n \),
\[
\mathbb{E}(e_2^2(k, n)) \leq \alpha^2 \rho^2 n \exp \left\{ - \frac{n^\varepsilon}{\alpha^2 \rho^2} \right\}.
\]

Furthermore, it is easy to verify that
\[
\mathbb{E}(e_1^2(x, t)) = t \Phi^2 \left( \frac{\sqrt{\xi}}{\alpha \rho} \right) + \int_0^\infty \Phi^2 \left( \frac{s + \sqrt{\xi}}{\rho \sqrt{\xi}} \right) ds.
\]

So, on \( n^\varepsilon \leq k \leq n \),
\[
\begin{align*}
\mathbb{E}(e_1^2(k, n)) & \leq n \Phi^2 \left( \frac{n^{t/2}}{\rho \alpha} \right) + \int_0^\infty \Phi^2 \left( \frac{s + n^{t/2}}{\rho \sqrt{n}} \right) ds \\
& \leq \frac{\alpha^2 \rho^2}{n^\varepsilon} \exp \left\{ - \frac{n^\varepsilon}{\alpha^2 \rho^2} \right\} + \rho \sqrt{n} \int_{n^{t/2}}^\infty \frac{1}{y^2} e^{-y^2} dy \\
& \leq \frac{\alpha^2 \rho^2}{n^\varepsilon} \exp \left\{ - \frac{n^\varepsilon}{\alpha^2 \rho^2} \right\} + \frac{\rho \sqrt{n}}{2 \left( \frac{n^{t/2}}{\rho \alpha} \right)^3} \exp \left\{ - \frac{n^\varepsilon}{\alpha^2 \rho^2} \right\}.
\end{align*}
\]

Now, by calculations similar to those in the proof of Lemma 1,
\[
\mathbb{E}(u(k, n) - u^*(k, n))^2 = \int_0^n \left[ \Phi \left( \frac{k - \alpha s}{\sigma \sqrt{s}} \right) - \frac{\Phi \left( k - \alpha s \right)}{\alpha \rho \sqrt{k}} \right]^2 ds \\
\leq C \frac{(\log k)^{\frac{3}{2}}}{\sqrt{k}}.
\]

Now, the same proof as Theorem 1 yields the result. \( \square \)

3 \hspace{1cm} THE CONTINUOUS CONGA LINE

Here we investigate properties of the continuous one dimensional Conga line \( u \) obtained in the previous section as an approximation to the discrete Conga line \( X \).
3.1 Particles moving in different scales

It is not hard to observe by estimating

\[ \text{Var} \left( u(x, t) \right) = \int_0^t \Phi^2 \left( \frac{x - \alpha y}{\sigma \sqrt{y}} \right) dy \]

that particles at distances \( ct \) from the leading particle have variance \( O(t) \) if \( c < \alpha \) and \( o(1) \) (in fact, the variance decays exponentially with \( t \)) if \( c > \alpha \). Also in a window of width \( c \sqrt{t} \) about \( \alpha t \), the variance is \( O(\sqrt{t}) \). In particular, this indicates that there is a small window somewhere between \( 0 \) and \( \alpha t \) where the variance changes from being 'very small to very large', i.e., there is a cut-off below which the variance goes to zero with \( t \), and above which the variance grows to infinity.

**Theorem 3**

(i) For \( \lambda > 0 \) and \( 1/2 \leq \beta < 1 \), \( \text{Var}(u(\alpha t - \lambda t^\beta, t)) \sim t^\beta \).

(ii) \( \text{Var}(u(\alpha t + \sigma \sqrt{t \log t}, t)) \sim \frac{t^{1/2-\lambda}}{(\log t)^{3/2}} \).

**Proof:** (i) Take any \( c > \lambda/\alpha \). Then, decomposing the variance,

\[ \text{Var}(u(\alpha t - \lambda t^\beta, t)) = \int_0^t \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) dy = \int_0^{t-ct^\beta} \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) dy + \int_{t-ct^\beta}^t \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) dy. \]

The first integral satisfies

\[ \int_0^{t-ct^\beta} \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) dy \leq C t^{1/2} \int_0^{t-ct^\beta} \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) \left( \frac{\alpha t - \lambda t^\beta + \alpha y}{2\sigma y \sqrt{y}} \right) dy \]

\[ \leq C t^{1/2} \int_{\alpha c - \lambda t^\beta}^\infty \Phi^2(z) dz \]

\[ \leq C t^{1/2} \exp \left( -\frac{(\alpha c - \lambda)^2 t^{2\beta - 1}}{\sigma^2} \right), \]

where the second step above follows from a change of variables.

It is easy to check that

\[ \int_{t-ct^\beta}^t \Phi^2 \left( \frac{\alpha t - \lambda t^\beta - \alpha y}{\sigma \sqrt{y}} \right) dy \sim t^\beta \]

proving part (i).

(ii) We decompose the variance as

\[ \text{Var}(u(\alpha t + \sigma \sqrt{t \log t}, t)) = \int_0^{t/2} \Phi^2 \left( \frac{\alpha t + \sigma \sqrt{\lambda t \log t} - \alpha y}{\sigma \sqrt{y}} \right) dy + \int_{t/2}^t \Phi^2 \left( \frac{\alpha t + \sigma \sqrt{\lambda t \log t} - \alpha y}{\sigma \sqrt{y}} \right) dy. \]
The first integral decays like $e^{-Ct}$ for some constant $C$. For the second integral, we make a change of variables similar to (i) and standard estimates for the normal c.d.f. to get

$$
\int_{t/2}^{t} \Phi^2 \left( \frac{\alpha t + \sigma \sqrt{\lambda \log t - \lambda y}}{\sigma \sqrt{y}} \right) dy \sim t^{1/2} \int_{\sqrt{\lambda \log t}}^{\infty} \Phi^2(z) dz \sim t^{1/2} \int_{\sqrt{\lambda \log t}}^{\infty} z^{-2} \exp(-z^2) dz
$$

$$
\sim \frac{t^{1/2}}{(\log t)^{3/2}} \int_{\sqrt{\lambda \log t}}^{\infty} 2z \exp(-z^2) dz \sim \frac{t^{1/2-\lambda}}{(\log t)^{3/2}},
$$

proving (ii).

Part (ii) of the above theorem has the following interesting consequence, demonstrating a cut-off phenomenon for the variance of $u(x, t)$ in the vicinity of $x = \alpha t + \sigma \sqrt{t \log t}$.

**Corollary 3.1** As $t \to \infty$

(i) $\text{Var}(u(\alpha t + \sigma \sqrt{\lambda \log t}, t)) \to 0$ if $\lambda \geq 1/2$.

(ii) $\text{Var}(u(\alpha t + \sigma \sqrt{\lambda \log t}, t)) \to \infty$ if $\lambda < 1/2$.

(iii) For $0 < \delta < \infty$, $\text{Var}(u(\alpha t + \sigma \sqrt{t(1/2 \log t - (3/2 \log \log t - \log \delta), t))) \sim \delta$.

So, the variance exhibits a sharp jump around $\alpha t + \sigma \sqrt{t(1/2 \log t - (3/2 \log \log t))}$.

The proof follows easily from part (ii) of Theorem 3.

### 3.2 Analyticity of the scaled Conga Line

For a fixed time $t$, the Conga line satisfies the following equality in distribution:

$$
\left\{ \int_{0}^{t} \Phi \left( \frac{tx - \alpha s}{\sigma \sqrt{s}} \right) dW_{s} : 0 \leq x \leq 1 \right\} \overset{d}{=} \sqrt{t} \int_{0}^{1} \Phi \left( \frac{x - \alpha s}{\sigma \sqrt{s}} \right) dW_{s} : 0 \leq x \leq 1 \right\}.
$$

(3.1)

where $\sigma_t = \frac{\sigma}{\sqrt{t}}$. This can be checked from the fact that both are Gaussian processes with covariance function $K$ where

$$
K(x_1, x_2) = \int_{0}^{t} \Phi \left( \frac{tx_1 - \alpha s}{\sigma \sqrt{s}} \right) \Phi \left( \frac{tx_2 - \alpha s}{\sigma \sqrt{s}} \right) ds = t \int_{0}^{1} \Phi \left( \frac{x_1 - \alpha s}{\sigma t \sqrt{s}} \right) \Phi \left( \frac{x_2 - \alpha s}{\sigma t \sqrt{s}} \right) ds.
$$

So, to study the Conga line for fixed $t$, we study the scaled process

$$
\left\{ u_t(x) = \int_{0}^{1} \Phi \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) dW_{s} : 0 \leq x \leq 1 \right\}.
$$

Now, we take a look at the derivatives of this process. It is easy to check that we can differentiate under the integral. Thus,

$$
\partial_x u_t(x) = - \int_{0}^{1} \frac{1}{\sigma t \sqrt{s}} \phi \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) dW_{s}.
$$
\[
\partial^2_{xx} u_t(x) = - \int_0^1 \frac{1}{\sigma_t^2} \phi^\prime \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) dW_s^1 \\
= \int_0^1 \frac{1}{\sigma_t^2} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) dW_s^1.
\]

In general, the \((n + 1)\)th derivative takes the following form:

\[
\partial^{n+1}_{xx} u_t(x) = \int_0^1 (\sigma_t \sqrt{s})^{-(n+1)} (-1)^{n+1} \He_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) dW_s^1,
\]

where \(\He_n\) is the \(n\)-th Hermite polynomial (probabilist version) given by

\[
\He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.
\]

In the following lemma, we give an upper bound on the growth rate of the derivatives. Using this, we will prove that, for fixed \(t\), \(u_t\) is real analytic on the interval \((0, 1)\), and the radius of convergence around \(x_0\) is comparable to \(|x_0|\). This is natural as the Conga line gets smoother as we move away from the tip. We start off with the following lemma.

**Lemma 3** For \(0 < \epsilon < \frac{x}{\alpha}\),

\[
|\partial^{n+1}_{xx} u_t(x)| \leq (2\pi)^{1/4} ||W|| \left\{ \left( \frac{\sqrt{2}}{x - \alpha \epsilon} \right)^{n+1} + \frac{1}{x} \left( \frac{\sqrt{2}}{x - \alpha \epsilon} \right)^n \right\} (n + 1)!
\]

\[
+ ||W|| \left\{ \left( \frac{1}{\sigma_t \sqrt{\epsilon}} \right)^{n+1} + \frac{n + 1}{x} \left( \frac{1}{\sigma_t \sqrt{\epsilon}} \right)^n \right\} \sqrt{(n + 1)!},
\]

where \(||W|| = \sup_{0 \leq s \leq 1} |W_s|\).

**Proof:** In the proof, we consider \(C\) as a generic positive constant whose value might change in between steps.

Let

\[
K^n_t(x, s) = (\sigma_t \sqrt{s})^{-(n+1)} (-1)^{n+1} \He_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right)
\]

Then

\[
\partial^{n+1}_{xx} u_t(x) = \int_0^1 (\sigma_t \sqrt{s})^{-(n+1)} (-1)^{n+1} \He_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) dW_s^1
\]

\[
= \int_0^1 W(1 - s) \partial_s K^n_t(x, s) ds.
\]

So, \(|\partial^{n+1}_{xx} u_t(x)| \leq ||W|| \int_0^1 |\partial_s K^n_t(x, s)| ds\). So, we have to estimate the integral \(\int_0^1 |\partial_s K^n_t(x, s)| ds\).

Now,

\[
\partial_s K^n_t(x, s) = (-1)^n \frac{n + 1}{2\sigma_t^{n+1}s^{n+3/2}} \He_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right)
\]
\[ +(-1)^{n+1} \frac{x + \alpha s}{2\sigma t s^{3/2}} (\sigma_t \sqrt{s})^{-(n+1)} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right). \]

So,

\[
\int_0^1 |\partial_s K_t^n(x, s)| \, ds \leq \frac{n + 1}{x} \int_0^1 (\sigma_t \sqrt{s})^{-n} \frac{x + \alpha s}{2\sigma t s^{3/2}} \left| \He_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \right| \, ds + \int_0^1 (\sigma_t \sqrt{s})^{-(n+1)} \frac{x + \alpha s}{2\sigma t s^{3/2}} \left| \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \right| \, ds.
\]

From the above, it is clear that estimating the second integral suffices.

\[
\int_0^1 (\sigma_t \sqrt{s})^{-(n+1)} \frac{x + \alpha s}{2\sigma t s^{3/2}} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \, ds \leq I_t^x + J_t^x,
\]

where

\[
I_t^x = \int_0^\epsilon (\sigma_t \sqrt{s})^{-(n+1)} \frac{x + \alpha s}{2\sigma t s^{3/2}} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \, ds
\]

\[
= \int_0^\epsilon (x - \alpha s)^{-(n+1)} \frac{x + \alpha s}{2\sigma t s^{3/2}} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \, ds
\]

\[
\leq (x - \alpha \epsilon)^{-(n+1)} \int_0^\epsilon \frac{x + \alpha s}{2\sigma t s^{3/2}} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \, ds
\]

\[
= (x - \alpha \epsilon)^{-(n+1)} \int_0^\epsilon s^{n+1} \He_{n+1}(s) |\phi(s)| \, ds
\]

\[
\leq (x - \alpha \epsilon)^{-(n+1)} \left( \int_0^\epsilon \frac{s^{2n+2} \phi(s)}{2\sigma t s^{3/2}} \, ds \right)^{1/2} \left( \int_0^\epsilon \He_{n+1}^2(s) \phi(s) \, ds \right)^{1/2}
\]

\[
\leq (x - \alpha \epsilon)^{-(n+1)} (2\pi)^{1/4} \sqrt{(n+1)!} \left( \int_0^\epsilon s^{2n+2} \phi(s) \, ds \right)^{1/2}
\]

\[
\leq (2\pi)^{1/4} \left( \frac{\sqrt{2}}{x - \alpha \epsilon} \right)^{n+1} (n+1)!
\]

Here we use the facts that

\[
\int_{-\infty}^\infty \He_n^2(s) \phi(s) \, ds = n!
\]

and

\[
\int_0^\infty s^{2n+2} \phi(s) \, ds = \frac{(2n + 2)!}{2^{n+1}(n+1)!} < 2^{n+1}(n+1)!
\]

Similarly,

\[
J_t^x = \int_\epsilon^1 (\sigma_t \sqrt{s})^{-(n+1)} \frac{x + \alpha s}{2\sigma t s^{3/2}} \He_{n+1} \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \, ds
\]

\[
\leq (\sigma_t \sqrt{\epsilon})^{-(n+1)} \int_{-\infty}^\infty |\He_{n+1}(s)| \phi(s) \, ds
\]

\[
\leq (\sigma_t \sqrt{\epsilon})^{-(n+1)} \left( \int_{-\infty}^\infty \He_{n+1}^2(s) \phi(s) \, ds \right)^{1/2} \left( \int_{-\infty}^\infty \phi(s) \, ds \right)^{1/2}
\]
\[
\Delta_n^t(x_0, \delta) = \left( \sup_{x \in (x_0 - \delta, x_0 + \delta)} |\partial_x u_t(x)| \right) \frac{\delta^{n+1}}{(n+1)!}
\]

The n-th order Taylor polynomial based at \(x_0\) is given by

\[
T_n^t(x) = \sum_{i=0}^{n} \frac{\partial^i u_t(x_0)}{i!} (x - x_0)^i.
\]

By Taylor’s Inequality,

\[
\sup_{x \in (x_0 - \delta, x_0 + \delta)} |u_t(x) - T_n^t(x)| \leq \Delta_n^t(x_0, \delta).
\]

From the above lemma, we know that, for \(\epsilon < \frac{x_0}{\alpha}\),

\[
\Delta_n^t(x_0, \delta) \leq (2\pi)^{1/4} \|W\| \left\{ \left( \frac{\sqrt{2\delta}}{x_0 - \delta - \alpha \epsilon} \right)^{n+1} + \frac{\delta}{x_0 - \delta} \left( \frac{\sqrt{2\delta}}{x_0 - \delta - \alpha \epsilon} \right)^n \right\} + \|W\| \left\{ \left( \frac{\delta}{\sigma \sqrt{\epsilon}} \right)^{n+1} + \frac{(n+1)\delta}{x_0 - \delta} \left( \frac{1}{\sqrt{\epsilon}} \right)^n \right\} \frac{1}{\sqrt{(n+1)!}}.
\]

The above error will go to zero only when \(\frac{\sqrt{2\delta}}{x_0 - \delta - \alpha \epsilon} < 1\), i.e., \(\delta < \frac{x_0 - \alpha \epsilon}{1 + \sqrt{2}}\). We can make \(\epsilon\) arbitrarily small to get the following:

**Corollary 3.2** The scaled Conga line \(u_t\) is real analytic on \((0, 1)\). The power series expansion of \(u_t\) around \(x_0 \in (0, 1)\) converges in \(\left( \frac{\sqrt{2x_0}}{1 + \sqrt{2}}, \min\{\sqrt{2x_0}, 1\} \right)\).

We are going to use this property of the Conga line multiple times in this article.

### 3.3 Covariance structure of the derivatives

In the following sections, we will analyse the finer properties of the Conga line like distribution of critical points, length and shape and number of loops. For all of these, fine estimates on the covariance structure of the derivatives are of utmost importance. This section is devoted to finding these estimates.

The next lemma is about the covariance between the first derivatives at two points. In what follows, we write \(L_t^x(M) = \alpha^{-1} \sqrt{-M \frac{x}{t} \log \frac{x}{t}}\).

**Lemma 4** For \(\delta \leq x, y \leq \alpha\), \(\text{Cov}(u'_t(x), u'_t(y)) \geq 0\) and satisfies

\[
\text{Cov}(u'_t(x), u'_t(y)) = \exp \left\{ -\frac{2\alpha t}{\sigma^2} \left( \sqrt{\frac{x^2 + y^2}{2} - \frac{x + y}{2}} \right) \right\} \left( \frac{\sqrt{t}}{2\sqrt{\pi \alpha \sigma} \left( \frac{x^2 + y^2}{2} \right)^{1/4}} \right) \left( 1 + O^{\infty} \left( \frac{\log t}{t} \right) \right).
\]
Consequently, the correlation function \( \rho_t(x, y) = \text{Corr}(u'_t(x), u'_t(y)) \) is always non-negative and has the following decay rate

\[
\rho_t(x, y) \geq \exp \left\{ -C_1 t (x - y)^2 \right\} \left( \frac{(xy)^{1/4}}{(x^2+y^2)^{1/4}} \right) \left( 1 + O^\infty \left( \sqrt{\frac{\log t}{t}} \right) \right),
\]

\[
\rho_t(x, y) \leq \exp \left\{ -C_2 t (x - y)^2 \right\} \left( \frac{(xy)^{1/4}}{(x^2+y^2)^{1/4}} \right) \left( 1 + O^\infty \left( \sqrt{\frac{\log t}{t}} \right) \right),
\]

where constants \( C_1, C_2 \) depend only on \( \delta \) and \( \alpha \).

**Proof:** To prove this lemma, note that, by completing squares in the exponent, we get

\[
\text{Cov}(u'_t(x), u'_t(y)) = \int_0^1 \frac{1}{\sigma_t^2 s} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) ds
\]

\[
= \exp \left\{ -2\alpha \left( \sqrt{\frac{x^2+y^2}{2} - \frac{x+y}{2}} \right) \right\} \int_0^1 \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{\sqrt{x^2+y^2} - \alpha s}{\sigma_t \sqrt{s}} \right) ds.
\]

Now we want to estimate the integral \( \int_0^1 \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds \) where \( \delta \leq x \leq \alpha \). By choosing \( M \) large enough, we can ensure that

\[
\int_0^1 \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds = \int_{\frac{\delta}{\alpha} - L_t^\delta(M)}^{\frac{\delta}{\alpha} + L_t^\delta(M)} \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds + O^\infty \left( \frac{1}{t} \right).
\]

Notice that

\[
\int_{\frac{\delta}{\alpha} - L_t^\delta(M)}^{\frac{\delta}{\alpha} + L_t^\delta(M)} \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds = \frac{1}{2\pi} \int_{\frac{\delta}{\alpha} - L_t^\delta(M)}^{\frac{\delta}{\alpha} + L_t^\delta(M)} x + \alpha s \frac{2\sqrt{s}}{2\sigma_t s^{3/2} \sigma_t (x + \alpha s)} \exp \left\{ - \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right)^2 \right\} ds
\]

\[
= \frac{1}{2\pi \sigma_t \sqrt{\alpha x}} \int_{\frac{\delta}{\alpha} - L_t^\delta(M)}^{\frac{\delta}{\alpha} + L_t^\delta(M)} \frac{x + \alpha s}{2\sigma_t s^{3/2}} \exp \left\{ - \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right)^2 \right\} ds
\]

\[
+ O^\infty (\sqrt{\log t})
\]

\[
= \frac{1}{2\pi \sigma_t \sqrt{\alpha x}} \int_{-\infty}^\infty e^{-s^2} ds + O^\infty (\sqrt{\log t})
\]

\[
= \frac{\sqrt{\pi}}{2\sigma \sqrt{\alpha x}} + O^\infty (\sqrt{\log t}).
\]

Substituting \( \sqrt{\frac{x^2+y^2}{2}} \) in place of \( x \) proves the lemma. \( \square \)

**Lemma 5** For \( \delta \leq x \leq \alpha \),

\[
\text{Var}(u''_t(x)) = \frac{\sqrt{\alpha} t^{3/2}}{4\sqrt{\pi} \sigma^3 x^{3/2}} \left( 1 + O^\infty \left( \sqrt{\frac{\log t}{t}} \right) \right).
\]
Proof: Follows along the same lines as the proof of Lemma 4.

\textbf{Lemma 6} For \( \delta \leq x \leq \alpha \),

\[
\text{Cov}(u_t'(x), u_t''(x)) = O^\infty(\sqrt{t \log t}).
\]

\textbf{Proof:}

\[
\text{Cov}(u_t'(x), u_t''(x)) = \int_0^1 \frac{1}{(\sigma t \sqrt{s})^3} \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right)^2 \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) ds
\]

\[
= \frac{1}{\pi} \int_{\frac{x}{\alpha} - L_t^X(M)}^{\frac{x}{\alpha} + L_t^X(M)} \frac{x + \alpha s}{2\sigma t s^{3/2} \sigma_t^2(x + \alpha s)} \exp \left\{ - \frac{(x - \alpha s)^2}{\sigma t \sqrt{s}} \right\} ds
\]

\[
= \frac{1}{\pi} \int_{\frac{x}{\alpha} - L_t^X(M)}^{\frac{x}{\alpha} + L_t^X(M)} \frac{x + \alpha s}{2\sigma t s^{3/2} \sigma_t^2 x} (1 + f_t(x, s))
\]

\[
\times \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) \exp \left\{ - \frac{(x - \alpha s)^2}{\sigma t \sqrt{s}} \right\} ds
\]

\[
+ O^\infty(t^{-1}),
\]

where \( f_t(x, s) = \frac{2x}{x + \alpha s} - 1. \)

Using the fact that \( \int_{-\infty}^{\infty} s \exp(-s^2)ds = 0 \), we get

\[
\frac{1}{\pi} \int_{\frac{x}{\alpha} - L_t^X(M)}^{\frac{x}{\alpha} + L_t^X(M)} \frac{x + \alpha s}{2\sigma t s^{3/2} \sigma_t^2 x} \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) \exp \left\{ - \frac{(x - \alpha s)^2}{\sigma t \sqrt{s}} \right\} ds = O^\infty(t^{-1}) (3.4)
\]

choosing sufficiently large \( M. \)

Also note that

\[
f_t(x, s) = O^\infty \left( \sqrt{\frac{\log t}{t}} \right)
\]

for \( \delta \leq x \leq \alpha, \frac{x}{\alpha} - L_t^X(M) \leq s \leq \frac{x}{\alpha} + L_t^X(M), \) which yields

\[
\frac{1}{\pi} \int_{\frac{x}{\alpha} - L_t^X(M)}^{\frac{x}{\alpha} + L_t^X(M)} \frac{x + \alpha s}{2\sigma t s^{3/2} \sigma_t^2 x} f_t(x, s) \left( \frac{x - \alpha s}{\sigma t \sqrt{s}} \right) \exp \left\{ - \frac{(x - \alpha s)^2}{\sigma t \sqrt{s}} \right\} ds = O^\infty(\sqrt{t \log t}). (3.5)
\]

(3.4) and (3.5) prove the lemma.

\textbf{Corollary 3.3}

\[
\text{Corr}(u_t'(x), u_t''(x)) = O^\infty \left( \sqrt{\frac{\log t}{t}} \right).
\]
Proof: This follows from Lemmas 4, 5 and 6. □

Let $\Sigma_t(x, y)$ be the covariance matrix of $(u'_t(x), u'_t(y))$. We need the following technical lemma to estimate the determinant of the matrix. It turns out to be crucial in certain second moment computations in Subsection 3.4.

Lemma 7 There exist constants $C^*, C_1, C_2$ such that, for $\delta \leq x, y \leq \alpha$ with $|x - y| \leq \frac{C^*}{\sqrt{t}}$,

$$C_1 t^2(y - x)^2 \leq \det \Sigma_t(x, y) \leq C_2 t^2(y - x)^2$$

Proof: We fix $x \in [\delta, \alpha]$ and consider the function $\Psi_{t,x}(y) = \det \Sigma_t(x, y)$. Consider the function $g_t(y) = \text{Var } u'_t(y) = \int_0^1 \frac{1}{\sigma_t^2 s} \phi^2 \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) ds$. Let $H_n$ denote the $n$-th Hermite polynomial (physicist version) given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = 2^n/2 He_n(\sqrt{2}x).$$

Then we can write the $n$-th derivative of $g_t$ as

$$g_t^{(n)}(x) = (-1)^n \int_0^1 \frac{1}{\sigma_t^2 s} \frac{1}{\sigma_t \sqrt{s}} H_n \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \phi^2 \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds.$$

Using the fact that $\int_{-\infty}^{\infty} H_n(s) \exp\{-s^2\} ds = 0$ and the same technique as the proof of Lemma 6, one can show that, for $n \geq 1$,$$

\begin{equation}

g_t^{(n)}(\cdot) = O^\infty(t^{n/2} \sqrt{\log t})
\end{equation}

$$

Let $\eta = \sqrt{\frac{x^2 + y^2}{2}}$.

Consider the functions

$$E_{t,x}(y) = g_t(x)g_t(y) - g_t^2(\eta)$$

and

$$F_{t,x}(y) = \left[ 1 - \exp \left\{ - \frac{4\alpha t}{\sigma^2} \left( \sqrt{\frac{x^2 + y^2}{2}} - \frac{x + y}{2} \right) \right\} \right] g_t^2(\eta).$$

Then, writing down $\text{Cov}(u'_t(x), u'_t(y))$ as in the proof of Lemma 4 we get

$$\Psi_{t,x}(y) = E_{t,x}(y) + F_{t,x}(y).$$

(3.7)

It is easy to check that

$$E_{t,x}(x) = E'_{t,x}(x) = 0.$$ 

(3.8)

The double derivative of $E_{t,x}$ takes the form

$$E''_{t,x}(y) = g_t(x)g''_t(y) - 2(g'_t(\eta))^2(\partial_y \eta)^2 - 2(g_t(\eta))(g''_t(\eta))(\partial_y \eta)^2 - 2(g_t(\eta))(g'_t(\eta))(\partial_{y}^2 \eta).$$

Using (3.6) we deduce

$$E''_{t,x}(\cdot) = O^\infty(t^{3/2} \sqrt{\log t}),$$

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which, along with (3.8) yields

$$|E_{t,x}(y)| \leq Ct^{3/2} \sqrt{\log t} (y - x)^2$$

(3.9)

for some constant $C < \infty$.

Now, to estimate $F_{t,x}$, note that in the region $\delta \leq x, y \leq \alpha$,

$$\exp \left\{ -\frac{\alpha t (y - x)^2}{2\delta \sigma^2} \right\} \leq \exp \left\{ -\frac{4\alpha t}{\sigma^2} \left( \sqrt{\frac{x^2 + y^2}{2}} - \frac{x + y}{2} \right) \right\} \leq \exp \left\{ -\frac{t(y - x)^2}{2\sigma^2} \right\}. \quad (3.10)$$

Using (3.10) along with the fact that $e^{-C}x \leq 1 - e^{-x} \leq x$ on $0 \leq x \leq C$, and Lemma 4, we get

$$C^* t^2 (y - x)^2 \leq F_{t,x}(y) \leq Ct^2 (y - x)^2,$$

(3.11)

where $C, C^*$ are positive, finite constants.

(3.9) and (3.11) together prove the lemma. □

3.4 Analyzing the distribution of critical points

Let $N_t(I)$ denote the number of critical points of $u_t$ in an interval $I \in [\delta, \alpha]$. Then $N_t$ defines a simple point process on $[\delta, \alpha]$. Our first goal is to find out the first intensity of this process. For this, we use the Expectation meta-theorem for smooth Gaussian fields (see [1] p. 263), which implies the following:

$$\mathbb{E} (N_t(I)) = \int_I \mathbb{E} \left( |u''_t(y)| \left| u'_t(y) = 0 \right. \right) p^y_t(0) dy,$$

(3.12)

where $p^y_t$ is the density of $u'_t(y)$. Before we go further, we remark that the meta-theorem from [1] mentioned above is a very general theorem which holds in a much wider set-up under a set of assumptions. In our case, it is easy to check that all the assumptions hold. Now, we utilize (3.12) and the developments in subsection 3.3 to derive a nice expression for the first intensity.

Lemma 8 The first intensity $\rho_t$ for $N_t$ satisfies

$$\rho_t(x) = \frac{\sqrt{\alpha t}}{\pi \sigma \sqrt{2x}} \left( 1 + O^\infty \left( \sqrt{\frac{\log t}{t}} \right) \right)$$

(3.13)

for $\delta \leq x \leq \alpha$.

Proof: By standard formulae for normal conditional densities and the lemmas proved in Subsection 3.3, we manipulate (3.12) as follows:

$$\mathbb{E} (N_t(I)) = \sqrt{\frac{2}{\pi}} \int_I \left[ \frac{\text{Var}(u'_t(y)) \cdot \text{Var}(u''_t(y)) - \text{Cov}^2(u'_t(y), u''_t(y))}{\text{Var}(u'_t(y))} \right]^{\frac{1}{2}} \frac{1}{\sqrt{2\pi \text{Var}(u'_t(y))}} dy$$

$$= \sqrt{\frac{2}{\pi}} \int_I \left( 1 + O^\infty \left( \frac{\log t}{t} \right) \right) \frac{\text{Var}(u'_t(y))}{\text{Var}(u'_t(y))} \frac{1}{\sqrt{2\pi \text{Var}(u'_t(y))}} dy$$

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\[
\int_I \left(1 + O^\infty \left(\sqrt{\frac{\log t}{t}}\right)\right) \left[\frac{\alpha t}{2\sigma^2 y}\right]^{1/2} dy = \int_I \left(1 + O^\infty \left(\sqrt{\frac{\log t}{t}}\right)\right) \sqrt{\frac{\alpha t}{\sqrt{2\pi}\sqrt{\gamma}}} dy.
\]

Here, we use Corollary 3.3 for to get the second equality and the estimates proved in Lemma 4 and Lemma 5 to get the third equality. This proves the lemma.

Thus \(\hat{\rho}_t(x) = \frac{\sqrt{\alpha t}}{\pi \sigma \sqrt{2x}}\) gives us the approximate first intensity for \(N_t\). From this, we see that the expected number of critical points in a small interval \([x, x + h]\) is approximately \(\frac{\sqrt{\alpha th}}{\pi \sigma \sqrt{2x}}\).

Now that we know the first intensity reasonably accurately, we can ask finer questions about the distribution of critical points, such as

(i) What can we say about the spacings of the critical points? Are there points in \([\delta, \alpha]\) around which there is a large concentration of critical points, or are they more or less well-spaced?

(ii) Given an interval \(I \in [\delta, \alpha]\), how good is \(\mathbb{E} N_t(I)\) as an estimate of \(N_t(I)\)?

The next lemma answers (i) by estimating the second intensity of \(N_t\). First we present a formula for the second intensity of \(N_t\) taken from [1].

\[
\mathbb{E} \left(N_t(I)^2 - N_t(I)\right) = \int_{I \times I} \mathbb{E} \left(\left|u''_t(y)\right| \left|u''_t(z)\right| \left|u'_t(y) = 0, u'_t(z) = 0\right| \right) p_{t}^{y,z}(0) dy dz, \quad (3.14)
\]

where \(p_{t}^{y,z}\) is the joint density of \((u'_t(y), u'_t(z))\).

In the following, \(C^*\) represents a positive constant.

**Lemma 9** For \(t > \frac{4(1 + \sqrt{2})^2}{2\delta^2}\), and \(\delta \leq y, z \leq \alpha\) with \(|y - z| \leq \frac{C^*}{\sqrt{t}}\),

\[
\mathbb{E} \left(\left|u''_t(y)\right| \left|u''_t(z)\right| \left|u'_t(y) = 0, u'_t(z) = 0\right| \right) \leq C(y - z)^{4\delta/2}
\]

for some constant \(C > 0\).

**Proof:** The hypothesis of the lemma tells us that \(y\) and \(z\) lie in the region of analyticity of each other, i.e. we can write

\[
u'_t(y) = \sum_{n=0}^{\infty} u^{(n+1)}_t(z) \frac{(y - z)^n}{n!},
\]

and the same holds with \(y\) and \(z\) interchanged. If we know that \(u'_t(y) = 0\) and \(u'_t(z) = 0\), the above equation becomes

\[
0 = \sum_{n=1}^{\infty} u^{(n+1)}_t(z) \frac{(y - z)^n}{n!}.
\]
By Stirling’s Formula,

\[ \Gamma(n) = \sqrt{2\pi} \left( \frac{n}{e} \right)^n (1 + O(n^{-1})) , \]

and the same holds with \( y \) and \( z \) interchanged. Thus, the conditional expectation in (3.14) becomes

\[
\mathbb{E} \left[ -\sum_{n=1}^{\infty} u_t^{(n+2)}(z) \frac{(y-z)^n}{(n+1)!} \right] - \sum_{n=1}^{\infty} u_t^{(n-2)}(y) \frac{(y-z)^n}{(n+1)!} \mid u'_t(y) = 0, u'_t(z) = 0 \]  

\[
= (y-z)^2 \mathbb{E} \left[ \left| -\sum_{n=0}^{\infty} u_t^{(n+3)}(z) \frac{(y-z)^n}{(n+2)!} \right| - \sum_{n=0}^{\infty} u_t^{(n+3)}(y) \frac{(y-z)^n}{(n+2)!} \mid u'_t(y) = 0, u'_t(z) = 0 \right].
\]

Now, by the Cauchy-Schwarz inequality and the fact that the conditional variance is bounded above by the total variance, we have

\[
\mathbb{E} \left( |u_t^{(m)}(y)||u_t^{(n)}(z)| \mid u'_t(y) = 0, u'_t(z) = 0 \right) \leq \sqrt{\mathbb{E} \left( u_t^{(m)}(y)^2 \right)} \sqrt{\mathbb{E} \left( u_t^{(n)}(z)^2 \right)} \mathbb{E} \left( u'_t(y) = 0, u'_t(z) = 0 \right)
\]

\[
= \sqrt{\mathbb{E} \left( u_t^{(m)}(y)^2 \right)} \mathbb{E} \left( u_t^{(n)}(z)^2 \right). \]

We know that

\[
u_t^{(m+3)}(y) = \int_0^1 \left( \sigma_t \sqrt{s} \right)^{-(m+3)} (-1)^{m+3} \text{He}_{m+2} \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) dW_s^1.
\]

So, using the same techniques as in the proof of Lemma 3 for \( 0 < \epsilon < \frac{\delta}{2\alpha} \), we estimate the variance as

\[
\mathbb{E} \left( u_t^{(m+3)}(y)^2 \right) = \int_0^1 \left( \sigma_t \sqrt{s} \right)^{-2(m+3)} \text{He}_{m+2}^2 \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) \phi^2 \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) ds
\]

\[
\leq \frac{C_1 t}{\sigma_t^2 (y - \alpha s)^{2m+4}} \int_0^\infty s^{2m+3} \text{He}_{m+2}^2 (s) \phi^2(s) ds + \frac{C_2}{(\sigma_t \sqrt{\epsilon})^{2m+3}} \int_0^\infty \text{He}_{m+2}^2 (s) \phi^2(s) ds.
\]

To estimate the first integral, we note that the function

\[ g_m(s) = s^{2m+3} \exp\{ -s^2/2 \} \]

is maximised at \( s = \sqrt{2m + 3} \). So, \( g_m(s) \leq (2m + 3)^{(2m+3)/2} \exp\{ -(2m + 3)/2 \} \).

By Stirling’s Formula,

\[ \Gamma(n) = \sqrt{\frac{2\pi}{n}} \left( \frac{n}{e} \right)^n (1 + O(n^{-1})) , \]

where \( \Gamma(\cdot) \) is the Gamma function. Using this, we get \( g_m(s) \leq C \sqrt{m 2^m (m + 1)!} \). So,

\[
\int_0^\infty s^{2m+3} \text{He}_{m+2}^2 (s) \phi^2(s) ds \leq C \sqrt{m 2^m (m + 1)!} \int_0^\infty \text{He}_{m+2}^2 (s) \phi(s) ds \leq C 2^m \{ (m + 2)! \}^2.
\]
The second integral is easier to estimate. Finally, we get
\[
E \left( u^{(m+3)}_t(y) \right)^2 \leq \frac{C_1 t}{\sigma^2(y - \alpha \epsilon)^2} 2^{m+4} (m + 2)!^2 + \frac{C_2}{(\sigma_t \sqrt{\epsilon})^{2m+5}} \{ (m + 2)! \}.
\]
Therefore,
\[
\sqrt{E \left( u^{(m+3)}_t(y) \right)^2} |y - z|^m \leq C \left[ \left( \frac{\sqrt{2} |y - z|}{y - \alpha \epsilon} \right)^{2m} \left( \frac{1}{y - \alpha \epsilon} \right)^4 + \frac{1}{(m + 2)!} \left( \frac{\sqrt{t} |y - z|}{\sigma \sqrt{\epsilon}} \right)^{2m} \left( \frac{\sqrt{t}}{\sigma \sqrt{\epsilon}} \right)^{5/2} \right]^{1/2} \leq C t^{5/4} a_m(t, y, z),
\]
where, by the assumptions of the lemma, \( \sum_{m=0}^{\infty} a_m(t, y, z) \leq C \), where \( C \) is a constant that does not depend on \( t, y, z \). Thus, we have
\[
E \left( |u''_t(y)| |u''_t(z)| |u'_t(y) = 0, u'_t(z) = 0 \right) \leq (y - z)^2 \sum_{m,n=0}^{\infty} \sqrt{E \left( u^{(m+3)}_t(y) \right)^2} (m + 2)! |y - z|^m \sqrt{E \left( u^{(n+3)}_t(y) \right)^2} (n + 2)! |y - z|^n \leq C(y - z)^2 t^{5/2} \left( \sum_{m=0}^{\infty} a_m(t, y, z) \right)^2,
\]
which proves the lemma. \( \square \)

We know that \( p_t^{y,z}(0) = \frac{1}{2\pi \sqrt{\det \Sigma_t(y, z)}} \). Using Lemmas [7] and [9] we get

**Lemma 10** For \( t > \frac{4(1 + \sqrt{2})^2}{2\delta^2} \), and \( h \leq C^* t^{-1/2} \),
\[
E(N_t^2([x, x + h]) - N_t([x, x + h])) \leq C(\delta) h^3 t^{3/2}.
\]
In particular, we get
\[
E(N_t([x, x + h])\mathbb{I}(N_t([x, x + h]) \geq 2)) \leq C(\delta) h^3 t^{3/2}.
\]

Using this lemma, we can deduce that if we divide \([\delta, \alpha] \) into subintervals of sufficiently small length, the number of critical points in any of these should not exceed one. The following corollary makes this precise.
Corollary 3.4 Let \( \{a_t\} \) be any sequence such that \( a_t = o(t^{-1/4}) \). Divide the interval \( [\delta, \alpha] \) into subintervals \( I_1, \ldots, I_{[\sqrt{t}/a_t]+1} \) of length at most \( \frac{a_t}{\sqrt{t}} \). Then

\[
P \left( \max_{1 \leq j \leq [\sqrt{t}/a_t]+1} N_t(I_j) \geq 2 \right) \leq C(\delta)a_t^2t^{1/2} \to 0
\]
as \( t \to \infty \).

This follows easily from Lemma 10 using the union bound.

Now, we answer (ii).

Note that for a Poisson point process, the first intensity determines the whole process. The Conga line lacks the Markov property. We can think of it as a process that ‘gains smoothness at the cost of Markov property’. But Lemma 4 tells us that there is exponential decorrelation, i.e. pieces of the Conga line that are reasonably far apart are almost independent. Thus, we expect that the first intensity of \( N_t \) should give us a lot of information about the process \( N_t \) itself. We conclude this section on critical points by giving basis to this intuition by showing the following:

Lemma 11 Let \( I \subseteq [\delta, \alpha] \) be an interval. Then

\[
\frac{N_t(I)}{\mathbb{E}N_t(I)} \xrightarrow{P} 1
\]
as \( t \to \infty \).

Proof: It suffices to prove the result for \( I = [\delta, \alpha] \).

Consider a collection of intervals

\[
\mathcal{C} = \{I_j : 1 \leq j \leq C[\sqrt{t}/r]\}
\]
where each interval is of length \( \frac{1}{\sqrt{t}} \) in \( [\delta, \alpha] \), and \( d(I_j, I_k) \geq \frac{r}{\sqrt{t}} \) for a sufficiently large \( r \) (which can be a function of \( t \)), whose optimal choice will be made later, and \( d(A, B) \) represents the usual distance between sets \( A \) and \( B \). Using the long range independence of the Conga line (see Lemma 4), we will prove that \( \text{Var} \left( N_t \left( \bigcup_{j=1}^{C[\sqrt{t}/r]} I_j \right) \right) \) is very small compared to \( \mathbb{E} \left( N_t \left( \bigcup_{j=1}^{C[\sqrt{t}/r]} I_j \right) \right)^2 \). The proof is completed by covering \( [\delta, \alpha] \) with \( [r] \) translates \( C_1, \ldots, C_{[r]} \) of such collections and an application of Chebychev Inequality.

Note that all the constants used in this proof depend on \( \delta \).

We begin by computing \( \mathbb{E} (N_t(I_1)N_t(I_2)) \) using an analogue of the Expectation meta-theorem (which can also be derived from the second intensity formula 3.14) as follows:

\[
\mathbb{E} (N_t(I_1)N_t(I_2)) = \int_{I_1 \times I_2} \mathbb{E} \left( |u''_t(y)u''_t(z)| \mid u'_t(y) = 0, u'_t(z) = 0 \right) \frac{1}{2\pi \sqrt{\det \Sigma_t(y, z)}} dydz \tag{3.15}
\]
where \( \Sigma_t(y, z) \) is the covariance matrix for \( (u'_t(y), u'_t(z)) \). We know that if \( (u''_t(y), u''_t(z), u'_t(y), u'_t(z)) \sim N(0, \Sigma), \)
then
\[(u''_i(y), u''_i(z) \mid u'_i(y) = 0, u'_i(z) = 0) \sim N(0, \Sigma^*),\]
where \(\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\) and \(\Sigma^* = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\). Now

\[
\text{Cov} \left( |u''_i(y)|, |u''_i(z)| \mid u'_i(y) = 0, u'_i(z) = 0 \right) = \frac{2}{\pi} \sqrt{\sigma_{11}^* \sigma_{22}^*} \left( \rho_{12}^* \arcsin \rho_{12}^* + \sqrt{1 - \rho_{12}^2} \right)
\leq \frac{2}{\pi} \sqrt{\sigma_{11}^* \sigma_{22}^*} \left( \rho_{12}^* \arcsin \rho_{12}^* \right)
\leq \sigma_{12}^* = \sigma_{12} - \left( \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \right)_{12}.
\]

Take \((y, z) \in I_j \times I_k\), where \(I_j, I_k \in \mathcal{C}\) with \(j \neq k\).

The proof of Lemma 3 shows that for \(\eta = \sqrt{\frac{y^2 + z^2}{2}}\),

\[
\text{Cov} \left( u'_i(y), u'_i(z) \right) \leq \exp \{-Ct(y - z)^2\} \text{Var}(u'_i(\eta))
\]
and

\[
\text{Var}(u'_i(\eta)) \leq C_1 \sqrt{t}.
\]

So, as \(|y - z| \geq \frac{r}{\sqrt{t}}\),

\[
\text{Cov} \left( u'_i(y), u'_i(z) \right) \leq \exp \{-Cr^2\} \text{Var}(u'_i(\eta))
\leq C_1 \sqrt{t} \exp \{-C_2r^2\}.
\]

Calculations similar to those in the proof of Lemma 3 show

\[
\text{Cov} \left( u'_i(y), u'_i(z) \right) = -\int_0^1 \frac{1}{(\sigma_t \sqrt{s})^3} \left( \frac{z - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right) \phi \left( \frac{z - \alpha s}{\sigma_t \sqrt{s}} \right) ds
\]

\[
= -\exp \left\{ -2\alpha \frac{\rho_{12}}{\sigma_t^2} \left( \sqrt{\frac{y^2 + z^2}{2}} - \frac{y + z}{2} \right) \right\}
\times \int_0^1 \frac{1}{(\sigma_t \sqrt{s})^3} \left( \frac{z - \alpha s}{\sigma_t \sqrt{s}} \right)^2 \phi \left( \frac{\sqrt{\frac{y^2 + z^2}{2} - \alpha s}}{\sigma_t \sqrt{s}} \right) ds.
\]

and thus

\[
|\text{Cov} \left( u'_i(y), u'_i(z) \right) | \leq C_1 \exp \{-C_2r^2\} \left( \int_0^1 |\eta - \eta| \frac{1}{(\sigma_t \sqrt{s})^3} \phi^2 \left( \frac{\eta - \alpha s}{\sigma_t \sqrt{s}} \right) ds \right.
\]

\[
+ \left. \int_0^1 \frac{|\eta - \alpha s|}{(\sigma_t \sqrt{s})^4} \phi^2 \left( \frac{\eta - \alpha s}{\sigma_t \sqrt{s}} \right) ds \right)
\leq C_1 \exp \{-C_2r^2\} t^{3/2}.
\]

Also, from Lemma 6

\[
|\text{Cov} \left( u'_i(y), u'_i(y) \right) | \leq C \sqrt{t \log t}.
\]
Similar calculations also show

\[
\left| \text{Cov} \left( u''(y), u''(z) \right) \right| \leq C_1 \exp \left\{ -C_2 r^2 \right\} \int_0^1 \frac{1}{(\sigma_t \sqrt{s})^4} \left| \frac{y - \alpha s}{\sigma_t \sqrt{s}} \right| \left| \frac{z - \alpha s}{\sigma_t \sqrt{s}} \right| \phi^2 \left( \frac{\sqrt{y^2 + z^2} - \alpha s}{\sigma_t \sqrt{s}} \right) ds
\]

writing \( \frac{y - \alpha s}{\sigma_t \sqrt{s}} = \frac{y - \eta}{\sigma_t \sqrt{s}} + \frac{\eta - \alpha s}{\sigma_t \sqrt{s}} \) and similarly for \( \frac{z - \alpha s}{\sigma_t \sqrt{s}} \). Furthermore, we see that

\[
\det \Sigma_{22} = \det \Sigma_t(y, z) = \text{Var}(u'_t(y)) \text{Var}(u'_t(z)) - \text{Cov}^2 \left( u'_t(y), u'_t(z) \right)
\]

\[
\geq \text{Var}(u'_t(y)) \text{Var}(u'_t(z)) - \exp \left\{ -C_2 r^2 \right\} \text{Var}^2 \left( u'_t(\eta) \right)
\]

\[
\geq \frac{1}{2} \text{Var}(u'_t(y)) \text{Var}(u'_t(z))
\]

for sufficiently large \( r \). Using equations (3.17), . . . . (3.21) to estimate the right side of equation (3.16), we see that there is a \( K > 0 \) for which

\[
\text{Cov} \left( \left| u''(y) \right|, \left| u''(z) \right| \left| u'_t(y) = 0, u'_t(z) = 0 \right) \leq C_1 \exp \left\{ -C_2 r^2 \right\} t^K.
\]

Plugging this into the expression (3.15), we get

\[
\mathbb{E} \left( N_t(I_1)N_t(I_2) \right) \leq C \int_{I_1 \times I_2} \frac{C_1 \exp \left\{ -C_2 r^2 \right\} t^K + \frac{2}{\pi} \sqrt{\text{Var}(u''(y)) \text{Var}(u''(z))}}{2 \pi \sqrt{\text{Var}(u'_t(y)) \text{Var}(u'_t(z))}} dydz.
\]

We know from Lemma 5 that

\[
\text{Var}(u''(y)) \leq Ct^{3/2}.
\]

Thus

\[
\frac{2}{\pi} \sqrt{\text{Var}(u''(y)) \text{Var}(u''(z))} = O(t^{3/2}).
\]

If we choose \( r = \sqrt{M \log t} \) for a large enough \( M \), then

\[
C_1 \exp \left\{ -C_2 r^2 \right\} t^K < \frac{2}{\pi} \sqrt{\text{Var}(u''(y)) \text{Var}(u''(z))}.
\]

Consequently, from (3.22),

\[
\mathbb{E} \left( N_t(I_1)N_t(I_2) \right) \leq C \int_{I_1 \times I_2} \frac{\sqrt{\text{Var}(u''(y)) \text{Var}(u''(z))}}{\pi^2 \sqrt{\text{Var}(u'_t(y)) \text{Var}(u'_t(z))}} dydz
\]

\[
= C \left( \int_{I_1} \frac{\sqrt{\text{Var}(u''(y))}}{\pi \sqrt{\text{Var}(u'_t(y))}} dy \right) \left( \int_{I_2} \frac{\sqrt{\text{Var}(u''(z))}}{\pi \sqrt{\text{Var}(u'_t(z))}} dz \right)
\]

\[
= \left( 1 + O \left( \sqrt{\frac{\log t}{t}} \right) \right) \mathbb{E} \left( N_t(I_1) \right) \mathbb{E} \left( N_t(I_2) \right),
\]

where the last step above follows from (3.12) using Corollary 3.3 (see the proof of Lemma 8).

Thus,

\[
\text{Cov} \left( N_t(I_1), N_t(I_2) \right) = O \left( \sqrt{\frac{\log t}{t}} \right)
\]

(3.23)
for this choice of \( r \).

Now, we have all we need to compute the variance of \( N_t \left( \bigcup_{j=1}^{C[\sqrt{t}/r]} I_j \right) \).

\[
\text{Var} \, N_t \left( \bigcup_{j=1}^{C[\sqrt{t}/r]} I_j \right) = C[\sqrt{t}/r] \sum_{j=1}^{C[\sqrt{t}/r]} \text{Var} \, N_t(I_j) + 2 \sum_{i<j} \text{Cov} \, (N_t(I_i), N_t(I_j)) \\
\leq C_1 \sqrt{\frac{t}{r}} + C_2 (\sqrt{t \log t}) r^{-2},
\]

where we used Lemma 10 crucially in putting the constant bound on \( \text{Var} \, N_t(I_j) \).

With our choice of \( r = \sqrt{M \log t} \), the above becomes

\[
\text{Var} \, N_t \left( \bigcup_{j=1}^{C[\sqrt{t}/r]} I_j \right) \leq C \sqrt{\frac{t}{\log t}}. \tag{3.24}
\]

Finally, we have, for small \( \epsilon > 0 \),

\[
P \left( \frac{|N_t([\delta, \alpha])|}{\mathbb{E}N_t([\delta, \alpha])} - 1 \geq \epsilon \right) = P \left( |N_t([\delta, \alpha]) - \mathbb{E}N_t([\delta, \alpha])| \geq \epsilon \mathbb{E}N_t([\delta, \alpha]) \right) \\
\leq \sum_{l=1}^{\sqrt{M \log t}} P \left( \left| N_t \left( \bigcup_{j \in C_l} I_j \right) - \mathbb{E}N_t \left( \bigcup_{j \in C_l} I_j \right) \right| \geq \epsilon \mathbb{E}N_t \left( \bigcup_{j \in C_l} I_j \right) \right) \\
\leq \sum_{l=1}^{\sqrt{M \log t}} \text{Var} \, N_t \left( \bigcup_{j \in C_l} I_j \right) \\
\leq C \frac{\sqrt{\log t}}{\epsilon^2} \frac{\sqrt{\frac{t}{\log t}}}{\sqrt{\frac{t}{\log t}}} = C \frac{\log t}{\epsilon^2 \sqrt{t}},
\]

which goes to zero as \( t \to \infty \). \( \square \)

4 \hspace{1em} The two dimensional Conga line

Here we study properties of the two dimensional continuous Conga line which is the approximation of our original discrete model in the plane.

4.1 Analyzing length

The length of the Conga line in the interval \([\delta, \alpha]\) is given by \( l_t = \int_{\delta}^{\alpha} |u'(x)| \, dx \). In this section, we give estimates for the expected length and its concentration about the mean.

Lemma 12 \( \mathbb{E}(l_t) \sim t^{1/4} \).
Proof: From Lemma 4, we see that
\[
\mathbb{E}(l_t) = \int_\delta^\alpha \mathbb{E}|u'_t(x)|dx = \int_\delta^\alpha \left( \frac{t^{1/4}}{(\pi^3\alpha\sigma^2)^{1/4}x^{1/4}} \right) dx + O\left( \frac{\sqrt{\log t}}{t^{1/4}} \right)
\]
\[
= \int_\delta^{4t^{1/4}/3} \left( \frac{\alpha^{3/4}}{(\pi^3\alpha\sigma^2)^{1/4}} \right) dx + O\left( \frac{\sqrt{\log t}}{t^{1/4}} \right).
\]

But this gives us only a rough estimate of the behaviour of length for large \( t \). To get a better idea of how the length behaves for large time \( t \), we need higher moments and, if possible, some form of concentration about the mean. The next lemma gives us an estimate of the variance of \( l_t \).

Lemma 13

\[ \text{Var}(l_t) = O(1). \]

Proof: From Lemma 4 we know that, for \( \delta \leq x, y \leq \alpha \),
\[
0 \leq \text{Cov}(u'_t(x), u'_t(y)) \leq C_1 \exp \left\{ -C_2 t(x - y)^2 \right\} \left( \frac{\sqrt{7}}{x^2 + y^2} \right)^{1/4} \left( 1 + O \left( \frac{\sqrt{\log t}}{t} \right) \right).
\]

Also, note that
\[
\text{Cov}(|u'_t(x)|, |u'_t(y)|) = \frac{2}{\pi} \sqrt{\text{Var}(u'_t(x))} \sqrt{\text{Var}(u'_t(y))} \left[ \rho_t(x, y) \arcsin(\rho_t(x, y)) + \sqrt{1 - \rho_t^2(x, y)} - 1 \right]
\]
\[
\leq \text{Cov}(u'_t(x), u'_t(y)).
\]

Thus,
\[
\text{Var}(l_t) = \int_\delta^\alpha \int_\delta^\alpha \text{Cov}(|u'_t(x)|, |u'_t(y)|)dx dy
\]
\[
\leq \int_\delta^{4t^{1/4}/3} \int_\delta^{4t^{1/4}/3} C \exp \left\{ -C_2 t(x - y)^2 \right\} \left( \frac{\sqrt{7}}{x^2 + y^2} \right)^{1/4} dx dy
\]
\[
= O(1).
\]

Thus, although the expected length grows like \( t^{1/4} \), the variance is bounded. This already tells us that the actual length cannot deviate much from the expected length.

In what we do next, we get Gaussian concentration of length about the mean in a window of scale \( O(\sqrt{\log t}) \).

We know that for most useful concentration results, we need some ‘independence’ in our model. Our strategy here is to construct a new process \( \hat{u}_t \) which is very ‘close’ to the original process \( u_t \) and is nicer to
analyze as $\hat{u}_t(x)$ and $\hat{u}_t(x')$ are independent whenever $x$ and $x'$ are sufficiently far apart. As this yields a useful tool which is going to be used in later sections, we give a detailed construction.

**Construction of $\hat{u}_t$**

By Lemma 14 we see that the correlation between points $x$ and $y$ in the Conga line with $|x-y| = \frac{\lambda}{\sqrt{t}}$ decays like $e^{-C\lambda^2}$ as $\lambda$ increases. We make use of this fact.

Divide the interval $[\delta - \frac{\sqrt{M \log t}}{t}, \alpha + \frac{\sqrt{M \log t}}{t}]$ into subintervals

$$I_k = [y_k, y_{k+1}], \quad -1 \leq k \leq \left(\frac{\sqrt{t}}{\sqrt{M \log t}}\right) + 1$$

of length at most $\frac{\sqrt{M \log t}}{\sqrt{t}}$. Define the process $\hat{u}_t$ as

$$\hat{u}_t(x) = \int_{y_k/\alpha}^{y_{k+1}/\alpha} x + \frac{\alpha s}{2\sigma t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \left[ W(1 - s) - W(1 - y_k/\alpha) \right] ds$$  \hspace{1cm} (4.1)

if $x \in [y_{k+1}, y_{k+2}]$.

In the following, all constants $C, C_1, C_2, \ldots$ depend only on $\delta$ and $\alpha$.

**Lemma 14** $\hat{u}_t$ satisfies the following properties:

(i) $\hat{u}_t$ is smooth everywhere except possibly at the points $y_k$.

(ii) $\{\hat{u}_t(x) : x \in I_k\}$ is independent of $\{\hat{u}_t(x) : x \in I_{k+3}\}$ for all $k$.

(iii) For $x \in I_{k+1}$,

$$|\hat{u}_t(x) - u_t(x) + W \left( 1 - \frac{y_k}{\alpha} \right) | \leq Ct^{-M/2} ||W||$$

and

$$|u'_t(x) - \hat{u}'_t(x)| \leq \frac{C}{\sqrt{M}} \frac{2^{M+\frac{1}{2}}}{t} ||W||.$$

**Proof:** Properties (i) and (ii) follow from the definition of $\hat{u}_t$.

To prove property (iii) notice that, for $x \in I_{k+1}$,

$$u_t(x) - \hat{u}_t(x) = \int_{[y_k/\alpha, y_{k+3}/\alpha] \cap [0,1]} x + \frac{\alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) W(1 - s) ds$$

$$+ \left( x + \frac{\alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \right) W(1 - y_k/\alpha) ds.$$

From (4.5) and a similar equation for the derivatives of $\hat{u}_t$, we obtain

$$u'_t(x) - \hat{u}'_t(x) = \int_{[y_k/\alpha, y_{k+3}/\alpha] \cap [0,1]} \partial_s K^\alpha_t(x, s) [W(1 - s) - W(1 - y_k/\alpha)] ds.$$
\[-W(1 - y_k/\alpha) \int_1^\infty \partial_s K^0_t(x, s) ds,\]

where \(K^0_t\) is defined as in (3.2). If \(x \in I_{k+1}\), then \(x/\alpha - L^x_t(M) \geq y_k/\alpha\) and \(x/\alpha + L^x_t(M) \leq y_k + 3/\alpha\). So, the above differences yield part (iii). \(\square\)

Consequently, if \(\hat{l}_t\) is the length of the curve \(\hat{u}_t\) restricted to \([\delta, \alpha]\), then

\[
P(|\hat{l}_t - \hat{l}_t| \geq r) \leq C_1 \exp \left\{ -C_2 r^2 \frac{t}{M^2} \right\} \tag{4.2}
\]

and

\[
E(\hat{l}_t) = \frac{4t^{1/4}}{3(\pi^3\alpha^2)} \left( \alpha^{3/4} - \delta^{3/4} \right) + O \left( \frac{\sqrt{\log t}}{t^{1/4}} \right) + O \left( \frac{1}{t^{1/4}} \right). \tag{4.3}
\]

So, to find the concentration of the length around the mean at time \(t\), we look at the length of the curve \(\hat{u}_t\).

Let \(W_k\) be the Brownian motion defined on \(I = [0, 3\alpha\sqrt{M\log t}/t]\) by

\[W_k(s) = W \left( 1 - \frac{y_k}{\alpha} - s \right) - W \left( 1 - \frac{y_k}{\alpha} \right).\]

For each \(k\), the Brownian motions \(W_k\) and \(W_{k+3}\) so defined are clearly independent.

As length is an additive functional, we can find the length on subintervals \(I_k\) and add them together. Heuristically, we can see that this gives us concentration as the length of the curve on every third interval is independent of each other, and as these are summed up, the errors get averaged out.

Now, we give the rigorous arguments. In the following, we fix the probability space \((\Omega, \mathcal{B}(\Omega), \mathcal{P})\), where

\[\Omega = C \left[ 0, \frac{3}{\alpha} \sqrt{\frac{M \log t}{t}} \right] \]

denote the set of continuous complex valued functions on \(I\) equipped with the sup-norm metric \(d\), and \(\mathcal{P}\) is the Wiener measure.

We need some concepts from Concentration of Measure Theory. See [5] for an excellent survey of techniques in this area. We give a very brief outline of the concepts we need.

**Transportation Cost Inequalities and Concentration:** Let \((\chi, d)\) be a complete separable metric space equipped with the Borel sigma algebra \(\mathcal{B}(\chi)\). Consider the \(p\)-th Wasserstein distance between two probability measures \(P\) and \(Q\) on this space, defined as

\[W_p(P, Q) = \inf_{\pi} [\mathbb{E}d(X, X')^p]^{1/p},\]

where the infimum is over all couplings \(\pi\) of a pair of random elements \((X, X')\) with the marginal of \(X\) being \(P\) and that of \(X'\) being \(Q\).

Now, fix a probability measure \(P\). Suppose there is a constant \(C > 0\) such that for all probability measures \(Q \ll P\), we have

\[W_p(P, Q) \leq \sqrt{2CH(Q \mid P)},\]

where \(H\) refers to the relative entropy \(H(Q \mid P) = \mathbb{E}^Q \log(dQ/dP)\). Then we say that \(P\) satisfies the \(L^p\) Transportation Cost Inequality. In short, we write \(P \in T_p(C)\).

Now, we present one of the key results which connects Transportation Cost Inequalities and Concentration of Measures.
Lemma 15 Suppose $P$ is a probability measure on $(\chi, \mathcal{B}(\chi))$. Suppose further that each $P$ is in $\mathcal{T}_1(C)$. Then, for any 1-Lipschitz map $F : \chi \to \mathbb{R}$ and any $r > 0$,
\[ P \left( |F - \int FdP| > r \right) \leq \exp \left\{ -\frac{r^2}{2C} \right\}. \]

It is easy to see that $\mathcal{T}_2(C)$ implies $\mathcal{T}_1(C)$. But the main advantage in dealing with $\mathcal{T}_2(C)$ comes from its tensorization property described in the following lemma.

Lemma 16 Suppose $P_i, i = 1, 2, \ldots n$ are probability measures on $(\chi, \mathcal{B}(\chi))$. Suppose further that each $P_i$ is in $\mathcal{T}_2(C)$. On $\chi^n$, define the distance between $x^n = (x_1, x_2, \ldots)$ and $y^n = (y_1, y_2, \ldots)$ by
\[ d^n(x^n, y^n) = \sqrt{\sum_{i=1}^{n} d^2(x_i, y_i)}. \]

Then $\bigotimes_{i=1}^{n} P_i \in \mathcal{T}_2(C)$ on $(\chi^n, d^n)$.

The following lemma, which follows from the developments in [6], is of key importance to us.

Lemma 17 The Wiener measure on $C[0, T]$ satisfies the transportation inequality $\mathcal{T}_2(T)$ with respect to the sup-norm metric.

These tools are all we need to establish a concentration result for $l_t$.

Let us define the function $T^k_t : \Omega \to \mathbb{R}$ as follows:
\[ T^k_t(f) = \int_{y_{k+1}}^{y_{k+2}} \int_{y_k}^{y_{k+3} - y_k} \partial_s K^0_t(x, s + \frac{y_k}{\alpha}) f(s) ds \, dx. \]

Notice that $\tilde{t} = \sum_{k=0}^{\sqrt{M \log t}} T^k_t(W_k)$. Suppose we prove that $T^k_t$ is Lipschitz with respect to $d$ with Lipschitz constant $C_t$. Then, with $N = \left[ \frac{\alpha - \delta}{3} \sqrt{\frac{t}{M \log t}} \right]$, the functions $\{T^{(i)}_t : \Omega^N \to \mathbb{R} : i = 0, 1, 2\}$ defined by
\[ T^{(i)}_t(f) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} T^{3k+i}_{t}(f_{k+1}) \]

where $f = (f_1, \ldots, f_N)$, is also Lipschitz with respect to $d^N$ with the same constant $C_t$.

Lemma 18 For each $k$, $T^k_t$ is Lipschitz on $(\Omega, d)$ with Lipschitz constant $C \sqrt{M \log t}$ where $C$ is a constant depending only on $\delta$ and $\alpha$.

Proof: For $f_1, f_2$ in $\Omega$,
\[ |T^k_t(f_1) - T^k_t(f_2)| \leq \int_{y_{k+1}}^{y_{k+2}} \int_{y_k}^{y_{k+3} - y_k} \left| \partial_s K^0_t \left(x, s + \frac{y_k}{\alpha}\right) \right| |f_1(s) - f_2(s)| ds dx. \]
\[
\leq \|f_1 - f_2\| \int_{y_{k+1}}^{y_{k+2}} \int_0^{y_{k+3} - y_k} \left| \partial_s K_t^0 \left( x, s + \frac{y_k}{\alpha} \right) \right| ds dx.
\]

By the estimates obtained in the proof of Lemma 3,

\[
\int_{y_{k+1}}^{y_{k+2}} \int_0^{y_{k+3} - y_k} \left| \partial_s K_t^0 \left( x, s + \frac{y_k}{\alpha} \right) \right| ds dx \leq C \sqrt{t} \int_{y_{k+1}}^{y_{k+2}} \frac{y_k}{\alpha} \log t.
\]

This proves the lemma.

Now, for any \( f \in \Omega^N \), define the following functions in \( \Omega^N \):

- \( f^{(1)} = (f_1, f_4, \ldots, f_{3N-2}) \)
- \( f^{(2)} = (f_2, f_5, \ldots, f_{3N-1}) \)
- \( f^{(3)} = (f_3, f_6, \ldots, f_{3N}) \)

Notice that

\[
\hat{l}_t = \sqrt{N} \left( T_t^{(1)}(\tilde{W}^{(1)}) + T_t^{(2)}(\tilde{W}^{(2)}) + T_t^{(3)}(\tilde{W}^{(3)}) \right)
\]

where \( \tilde{W} = (W_0, W_1, \ldots, W_{3N-1}) \in \Omega^{3N} \). Using this fact and Lemmas 16 and 15, we get for any \( r > 0 \),

\[
P(|\hat{l}_t - El_t| \geq r \sqrt{M \log t}) \leq C_1 \exp \left\{ -C_2 r^2 \right\}
\]

This, along with (4.2) and (4.3) gives us our main conclusion:

**Theorem 4** We have

\[
P(|l_t - El_t| \geq r \sqrt{M \log t}) \leq C_1 \exp \left\{ -C_2 r^2 \right\},
\]

where \( C_1, C_2 \) are constants depending only on \( \delta \) and \( \alpha \).

### 4.2 How close is the scaled Conga Line to Brownian motion?

Though the unscaled Conga line seen far away from the tip ‘smoothes out’ Brownian motion more and more with increasing \( t \), we see that in the simulations of the scaled Conga line, making \( t \) larger actually makes the curve rougher and resemble Brownian motion more and more. Closer analysis reveals that this in fact results from the scaling. Again, before we supply the rigorous arguments, we give a heuristic reasoning.

Looking at equation (3.1), we see that although the scaling takes the Brownian motion \( W_t \) on \([0, t]\) to a Brownian motion \( W_1 \) on \([0, 1]\), the width of the window on which the smoothing takes place in the unscaled Conga line, which is comparable to \( \sqrt{t} \), is taken to \( O(t^{-1/2}) \) in the scaled version, which shrinks with time \( t \).

In the following, we consider the sequence of random curves \( u_t(\cdot) \) indexed by \( t \), and \( L_t^x = \alpha^{-1} \sqrt{-M \sigma_t^2 x \log \sigma_t^2 x} \).

**Theorem 5** There exists a deterministic constant \( C_1 \) such that, almost surely, there is \( T = T(\omega) > 0 \) for which

\[
\left| u_t(x) - W \left( 1 - \frac{x}{\alpha} \right) \right| \leq C_1 \sqrt{-\left( \frac{x}{t} \right)^{1/2} \log \left( \frac{x}{t} \right)}
\]

(4.4)

for all \( x \in (0, \alpha] \) satisfying \( x > \alpha L_t^x \) for all \( t \geq T \). In particular, for any fixed \( \beta < 1 \), the above holds almost surely for \( x \in [t^{-\beta}, \alpha] \) for all \( t \geq T \).
Thus, the scaled Conga line is close to Brownian motion for large $t$ although the unscaled one is not, as can be seen from the right side of equation (4.4). This subsection is devoted to proving the above theorem.

For any continuous function $f : [0, 1] \to \mathbb{C}$, define

$$P_tf(x) = \int_0^1 \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) f(1 - s) ds.$$ 

Note that the Conga line is given by $u_t(x) = P_t W(x)$. $P_tf$ can be thought of as a smoothing kernel acting on the function $x \mapsto f(1 - x/\alpha)$. The following lemma shows that if $f$ is Lipschitz, then for large $t$, $P_tf(x)$ is close to $f(1 - x/\alpha)$.

**Lemma 19** If $f$ is Lipschitz with constant $C$, then for large enough $t$ and for $x \in (0, \alpha]$ satisfying $x > \alpha L_t^x$, 

$$|P_tf(x) - f \left( 1 - \frac{x}{\alpha} \right)| \leq C \sigma_t \sqrt{x}. \tag{4.5}$$

Note that

$$|P_tf(x) - f \left( 1 - \frac{x}{\alpha} \right)| \leq C \alpha^{-1} \int_0^1 \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) |x - \alpha s| ds$$

$$= I_t^x + J_t^x + S_t^x$$

where

$$I_t^x = C \alpha^{-1} \int_0^{(x/\alpha)-L_t^x} \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) |x - \alpha s| ds \leq C \alpha^{-1} \int_0^{(x/\alpha)-L_t^x} \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) ds$$

$$\leq C \alpha^{-1} \Phi \left( \sqrt{-M \log \sigma_t^2 x} \right) \leq C \alpha^{-1} (\sigma_t \sqrt{x})^M$$

and

$$J_t^x = C \alpha^{-1} \int_{(x/\alpha)-L_t^x}^{\min((x/\alpha)+L_t^x, 1)} \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) |x - \alpha s| ds$$

$$\leq C \alpha^{-1} \sigma_t \sqrt{\frac{2x}{\alpha}} \int_{(x/\alpha)-L_t^x}^{\min((x/\alpha)+L_t^x, 1)} \frac{x + \alpha s}{2\sigma_t s^{3/2}} \phi \left( \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right) \left| \frac{x - \alpha s}{\sigma_t \sqrt{s}} \right| ds$$

$$\leq C \alpha^{-1} \sigma_t \sqrt{\frac{2x}{\alpha}} \int_{-\infty}^{\infty} |s| \phi(s) ds. \tag*{\square}$$

Similarly as $I_t^x$, $S_t^x$ is small compared to $J_t^x$.

Now, Brownian motion is not Lipschitz, but it can be uniformly approximated on $[0, 1]$ by piecewise linear random functions whose Lipschitz constants can be controlled using Levy’s Construction of Brownian motion which we now briefly describe following [2]. Define the $n$-th level dyadic partition $D_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\}$ and let $D = \bigcup_{n=0}^{\infty} D_n$. Let $\{ Z_n : n \in \mathbb{N} \}$ be i.i.d standard normal random variables. Define the random piecewise linear functions $F_n$ as follows:

$$F_n(x) = \begin{cases} 
2^{-\frac{n+1}{2}} Z_x & x \in D_n \setminus D_{n-1} \\
0 & x = 0 \\
linear & \text{in between}
\end{cases}$$
With this, Levy’s construction says that a Brownian motion \( W \) can be constructed via

\[
W(x) = \sum_{n=0}^{\infty} F_n(x).
\]

for \( x \in [0, 1] \).

Let \( W_N(x) = \sum_{n=0}^{N} F_n(x) \). This function serves as the piecewise linear (hence Lipschitz) approximation to \( W \). From Lemma 19, for any \( N \),

\[
\left| P_t W(x) - W \left( 1 - \frac{x}{\alpha} \right) \right| \leq C \sum_{n=0}^{N} \sigma \sqrt{x} \frac{\|F_n\|_{\infty}}{t} + 2 \sum_{n=N}^{\infty} ||F_n||_{\infty}.
\]

Fix \( c > \sqrt{2 \log 2} \). Let \( N^* = \inf \{ n : |Z_d| \leq c \sqrt{n} \ \forall \ d \in \mathcal{D} \} \).

\[
\sum_{n=0}^{\infty} P \left( \text{there exists } d \in \mathcal{D}_n \text{ with } |Z_d| \leq c \sqrt{n} \right) \leq \sum_{n=0}^{\infty} (2^n + 1) \exp \left( \frac{-c^2 n}{2} \right) < \infty.
\]

So, by Borel-Cantelli Lemma, \( P(N^* < \infty) = 1 \).

Now, for \( n > N^* \), \( ||F_n||_{\infty} \leq c \sqrt{n} 2^{-n/2} \) and \( ||F_n'||_{\infty} \leq \frac{2||F_n||_{\infty}^2}{2-n} \leq 2c \sqrt{n} 2^{n/2} \). So, for \( l > N^* \), we get

\[
\left| P_t W(x) - W \left( 1 - \frac{x}{\alpha} \right) \right| \leq C \sum_{n=0}^{N^*} \sigma \sqrt{x} \frac{\|F_n||_{\infty}}{t} + 2 \sum_{n=N^*}^{l} \sigma \sqrt{x} c \sqrt{n} 2^{n/2} + 2 \sum_{n=l}^{\infty} c \sqrt{n} 2^{-n/2}.
\]

(4.6)

Now, take \( t \) large enough that, for every \( x \in (0, \alpha] \), the first term is less than \( \sqrt{-\left( \frac{x}{t} \right)^{1/2} \log \left( \frac{x}{t} \right) } \) and \( \sqrt{\frac{x}{t}} \in (2^{-l}, 2^{-l+1}) \) for some \( l > N^* \). Plugging this \( l \) into equation (4.6) and using the fact that the last sum above is dominated by its leading term, we get the result of the theorem.

### 4.3 Analyzing number of loops

A loop \( L \) in a continuous curve \( f : \mathbb{R} \to \mathbb{C} \) is defined as a restriction of the form \( f|_{[a,b]} \) where \( f(a) = f(b) \) and \( f \) is injective on \([a, b] \). Note that \( L \) divides the plane into a bounded component and an unbounded component. Define the size of the loop

\[
s(L) = \max \{ R > 0 : \exists x \in \text{the bounded component } B \text{ of } L \text{ such that } B(x, R) \subseteq B \}.
\]

It can be shown (the quick way is to look at the expectation meta-theorem from [1]) that if \( f \) is a continuously differentiable Gaussian process, then with probability one, it has no singularities (points where the first derivative of both \( \text{Re } f \) and \( \text{Im } f \) vanish). Using this fact, it is easy to see that if \( I \) is a compact interval on which \( f \) is not injective, then \( f|_I \) has at least one loop \( L \) of positive size.
As the number of loops is bounded above by the number of critical points of \( \text{Re } f \) (equivalently \( \text{Im } f \)), we see that by Lemma 11 for a large fixed \( t \), the number of loops in the Conga line is bounded above by \( C \sqrt{t} \) with very high probability. This section is dedicated to achieving a lower bound. The simulation (Figure 2) shows a number of loops, most of them being small. In the following, we obtain a lower bound for the number of small loops, which differs from the upper bound by a logarithmic factor. For this, our main ingredient is the Support Theorem for Brownian Motion which says the following:

**Theorem 6** If \( f : [0, 1] \to \mathbb{C} \) is continuous and \( W \) is a complex Brownian motion on \([0, 1]\), then for any \( \epsilon > 0 \),

\[
P(\|W - f\| < \epsilon) > 0,
\]

where \( \|g\| = \sup_{x \in [0,1]} |g(x)| \).

We will not prove the theorem, but this can be proved either by approximating \( f \) by piecewise linear functions and using Levy’s construction of Brownian motion, or by an application of the Girsanov Theorem (see [7]).

We also need to exploit the exponentially decaying correlation between \( u_t(x) \) and \( u_t(x') \) as \( |x - x'| \) increases (see Lemma 4) by bringing into play the approximation of \( u_t \) by the process \( \hat{u}_t \) introduced in Subsection 4.1.

Now, we state the main theorem of this section.

**Theorem 7** Choose \( R > 6C_1 \), where \( C_1 \) is the constant in Theorem 5. Let \( N^l_t \) be the number of loops of size less than or equal to \( 2R \left( \frac{\log t}{t} \right)^{1/4} \) in the (scaled) Conga line \( u_t \) in \([\delta, \alpha]\) at time \( t \). Then there exist constants \( C \) and \( C' \) such that

\[
P \left( C \sqrt{\frac{t}{\log t}} \leq N^l_t \leq C' \sqrt{t} \right) \to 1
\]
as \( t \to \infty \).

**Proof:** The upper bound follows from Lemma 11.

Proving the lower bound is more involved.

Our strategy is to choose a function \( f \) which has a loop and run the Brownian motion \( W \) in a narrow tube around \( f \), which, by Theorem 6, we can do with positive probability. Now by Theorem 5 we know that for large \( t \), \( u_t \) is ‘close’ to the Brownian motion \( W \) with very high probability, and thus the curve \( u_t \) is forced to run in a narrow sausage around \( f \) thereby inducing a loop.

Such a function is \( f(x) = C((4x - 2)^3 - (4x - 2), 1 - (4x - 2)^2) \) for \( x \in [0,1] \), where \( C \) is a suitably chosen constant to make the size of the loop in \( f \) to be \( R \). Let us denote the continuous functions restricted to the \( \epsilon \)-sausage around \( f \) \( |[a,b]| \) as

\[
S(f; \epsilon, [a,b]) = \{g \in C[a,b] : \|f - g\| < \epsilon\}.
\]
Fix $\alpha'$ such that $\delta < \alpha' < \alpha$. For $x \in [\delta, \alpha']$, define

$$f_x^{(t)}(s) = \left( \frac{M \log t}{t} \right)^{1/4} f \left( \frac{1}{\alpha} \sqrt{\frac{t}{M \log t}} (s - x) \right), \quad x \leq s \leq x + \alpha \sqrt{\frac{M \log t}{t}}.$$ 

Then for any continuously differentiable complex-valued Gaussian process $g$, defined on a subset of $[0, 1]$ containing $[x, x + \alpha \sqrt{M \log t}]$, and any complex number $c, g \in S \left( c + f_x^{(t)} ; \frac{R}{2} \left( \frac{M \log t}{t} \right)^{1/4}, [x, x + \alpha \sqrt{M \log t}] \right)$ implies that $g$ has a self-intersection on $[x, x + \alpha \sqrt{M \log t}]$ and thus, due to absence of singularities with probability one, $g$ has a loop of positive size on this interval.

We break up the proof into parts:

(i) In Lemma 20 we prove that the probability of $u_t \left[ x, x + \alpha \sqrt{M \log t} \right]$ having a loop of size comparable to $\left( \frac{\log t}{t} \right)^{1/4}$ is bounded below uniformly for all $x \in [\delta, \alpha']$ by a fixed positive constant $p$ independent of $x$ and $t$.

(ii) We use part (iii) of Lemma 14 and Lemma 20 to deduce that the probability of $\hat{u}_t$ having a loop of size comparable to $\left( \frac{\log t}{t} \right)^{1/4}$ on each interval $I_{k+1}$ is bounded below by $p/2$.

(iii) We use the independence of $\hat{u}_k | I_k$ and $\hat{u}_k | I_{k+1}$ for every $k$ to deduce in Lemma 21 that the total number of such loops in $\hat{u}_t$ is bounded below by $\frac{p}{8} \left( \alpha' - \delta \right) \sqrt{\frac{M \log t}{t}}$ with very high probability.

(iv) We finally use part (iii) of Lemma 14 again to translate the result of Lemma 21 to the original process $u_t$ in Lemma 22.

**Lemma 20** There is a constant $p > 0$ independent of $x$ and $t$ such that

$$P \left( u_t \left[ x, x + \alpha \sqrt{M \log t} \right] \right) \leq S \left( W \left( 1 - \frac{x}{\alpha} \right) + f_x^{(t)} ; \frac{R}{2} \left( \frac{M \log t}{t} \right)^{1/4}, [x, x + \alpha \sqrt{M \log t}] \right) \geq p > 0$$

for all $x \in [\delta, \alpha']$, for all sufficiently large $t$.

**Proof:** Choose and fix any $x \in [\delta, \alpha']$. By Theorem 5 and by the translation and scaling invariance of Brownian motion, we get for $R > 6C_1$ (here $C_1$ is the constant in Theorem 5) and large $t$,

$$P \left( u_t \left[ x, x + \alpha \sqrt{M \log t} \right] \right) \leq S \left( W \left( 1 - \frac{x}{\alpha} \right) + f_x^{(t)} ; \frac{R}{2} \left( \frac{M \log t}{t} \right)^{1/4}, [x, x + \alpha \sqrt{M \log t}] \right)$$

$$\geq P \left( \sup_{s \in [x, x + \alpha \sqrt{M \log t}]} \left| u_t(s) - W \left( 1 - \frac{s}{\alpha} \right) \right| \right) \leq C_1 \left( \frac{M \log t}{t} \right)^{1/4}$$

and

$$\sup_{s \in [x, x + \alpha \sqrt{M \log t}]} \left| W \left( 1 - \frac{s}{\alpha} \right) - W \left( 1 - \frac{x}{\alpha} \right) - f_x^{(t)}(s) \right| \leq \frac{R}{3} \left( \frac{M \log t}{t} \right)^{1/4}$$

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\[
\begin{align*}
&\geq P \left( \sup_{s \in [x, x + \alpha \sqrt{\frac{M \log t}{t}}]} \left| W \left( \frac{s - x}{\alpha} \right) - f_x(t)(s) \right| \leq \frac{R}{3} \left( \frac{M \log t}{t} \right)^{1/4} \right) \\
&\quad - P \left( \sup_{s \in [x, x + \alpha \sqrt{\frac{M \log t}{t}}]} \left| u_t(s) - W \left( 1 - \frac{s}{\alpha} \right) \right| > C_1 \left( \frac{M \log t}{t} \right)^{1/4} \right) \\
&= P \left( \sup_{s \in \left[0, \alpha \sqrt{\frac{M \log t}{t}}\right]} \left| W_0(t)(s) - f_0(t)(s) \right| \leq \frac{R}{3} \left( \frac{M \log t}{t} \right)^{1/4} \right) \\
&\quad - P \left( \sup_{s \in [x, x + \alpha \sqrt{\frac{M \log t}{t}}]} \left| u_t(s) - W \left( 1 - \frac{s}{\alpha} \right) \right| > C_1 \left( \frac{M \log t}{t} \right)^{1/4} \right) \\
&\geq P \left( \sup_{s \in [0, 1]} \left| W(s) - f(s) \right| \leq \frac{R}{3} \right) \\
&\quad - P \left( \sup_{s \in [\delta, \alpha]} \left| u_t(s) - W \left( 1 - \frac{s}{\alpha} \right) \right| > C_1 \left( \frac{M \log t}{t} \right)^{1/4} \right) \\
&\geq p > 0.
\end{align*}
\]

Here we used Theorem 5 and Theorem 6 for the last step. By virtue of the second last step above, we can choose \( p \) independent of \( x \) and \( t \), and the above lower bound works uniformly for all \( x \in [\delta, \alpha'] \). \( \square \)

Recall that by part (iii) of Lemma 14, we know that for \( x \in I_{k+1} \),

\[
\left| \hat{u}_t(x) - u_t(x) + W \left( 1 - \frac{y_k}{\alpha} \right) \right| \leq C(\delta) t^{-M/2} ||W||.
\]

Define the event

\[
A_k = \left\{ \hat{u}_t \mid_{I_{k+1}} \in S \left( W(1 - y_{k+1}/\alpha) - W(1 - y_k/\alpha) + f_{y_{k+1}}^{(t)}; \frac{R}{2} \left( \frac{M \log t}{t} \right)^{1/4}, I_{k+1} \right) \right\}.
\]

If \( A_k \) holds, then \( \hat{u}_t \) has a loop in \( I_{k+1} \). Write

\[
S_t = \sum_{k=1}^{(\alpha' - \delta) \sqrt{\frac{t}{M \log t}}} I_{A_k}.
\]

Then the following holds.

**Lemma 21**

\[
P \left( S_t < \frac{p}{4} \left( (\alpha' - \delta) \sqrt{\frac{t}{M \log t}} \right) \right) \leq 3 \exp \left\{ -\frac{p^2}{8} \left( (\alpha' - \delta) \sqrt{\frac{t}{M \log t}} \right) \right\}.
\]
Proof: By Lemma 20, it is easy to see that
\[
P(A_k) \geq P \left\{ u_t |_{I_{k+1}} \in S \left( W(1-y_{k+1}/\alpha) + f(t_{y_{k+1}}), \left( \frac{R}{2} - \epsilon \right) \left( \frac{M \log t}{t} \right)^{1/4}, I_{k+1} \right) \right\}
\]
and \( ||W|| \leq \frac{\epsilon t^{M/2}}{C(\delta)} \left( \frac{M \log t}{t} \right)^{1/4} \),

\[
\geq \frac{p}{2} > 0
\]
for large enough \( t \) and small enough \( \epsilon \). Thus we see that \( ES_t \geq \frac{p}{2} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \).

Now, as \( \hat{u}_t \) is independent on every third interval, so \( A_k \) is independent of \( A_{k+3} \) for every \( k \). The result now follows by Bernstein’s Inequality.

The above implies that with very high probability \( S_t \geq \frac{p}{4} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \).

Define the event
\[ B_k = \{ u_t \text{ has a loop on } I_{k+1} \} \]
and the corresponding sum
\[ \hat{N}_t^l = \sum_{k=1}^{\lfloor (\alpha' - \delta) \sqrt{t/M \log t} \rfloor} \mathbb{I}_{B_k}. \]

Our final lemma is the following.

Lemma 22
\[
P \left( N_t^l \geq \frac{p}{4} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \right) \to 1 \quad \text{as } t \to \infty.
\]

Proof: Note that \( N_t^l \geq \hat{N}_t^l. \)

By part (iii) of Lemma 14 we note that for small enough \( \epsilon > 0 \), the events \( A_k \) and \( \left\{ ||W|| \leq \frac{\epsilon t^{M/2}}{C(\delta)} \left( \frac{M \log t}{t} \right)^{1/4} \right\} \) imply that \( B_k \) holds. We see that, for large \( t \),
\[
P \left( \hat{N}_t^l \geq \frac{p}{4} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \right) \geq P \left( S_t \geq \frac{p}{4} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \text{ and } ||W|| \leq \frac{\epsilon t^{M/2}}{C(\delta)} \left( \frac{M \log t}{t} \right)^{1/4} \right)
\]
\[
\geq 1 - P \left( S_t < \frac{p}{4} \left[ (\alpha' - \delta) \sqrt{t/M \log t} \right] \right) - P \left( \frac{||W||}{\frac{\epsilon t^{M/2}}{C(\delta)} \left( \frac{M \log t}{t} \right)^{1/4}} \right),
\]
which goes to one as \( t \to \infty \) by Lemma 21.

The proof of the lower bound in Theorem 7 follows from the above lemmas.
5 LOOPS AND SINGULARITIES IN PARTICLE PATHS

We start off this section by describing an interesting phenomenon that one notices in simulations of the paths of the individual particles in the discrete Conga line. The leading particle \((k = 1)\) performs an erratic Gaussian random walk. But as \(k\) increases, the successive particles are seen to cut corners in the paths of the preceding particles making them smoother. This can be heuristically explained by the fact that a particle following another one in front directs itself along the shortest path between itself and the preceding particle (see equation (1.1)), and hence cuts corners. This phenomenon is captured by the process \(\overline{u}\) described in Subsection 2.2. So, we use the approximation of the discrete Conga line \(X\) by the smooth process \(u\) in this section.

Recall that a singularity of a curve \(\gamma : \mathbb{R} \to \mathbb{C}\) is a point \(t_0\) at which its speed vanishes, i.e. \(|\gamma'(t_0)| = 0\). A singularity \(t_0\) of an analytic curve \(\gamma\) is called a cusp singularity if there exists a translation and rotation of co-ordinates taking \(\gamma(t_0)\) to the origin, under which, \(\gamma\) has the representation \(\gamma^* = (\gamma_1^*, \gamma_2^*)\) with the following power series expansions:

\[
\gamma_1^*(t) = \sum_{i=2}^{\infty} a_i t^i
\]
\[
\gamma_2^*(t) = \sum_{i=3}^{\infty} b_i t^i
\]

with \(a_2 \neq 0, b_3 \neq 0\), for \(t\) in a neighborhood \([t_0 - \delta, t_0 + \delta]\) around \(t_0\) for some \(\delta > 0\). Intuitively, this means that the graph of \(\gamma\) locally around \(t_0\) looks like \(y = \frac{x^2}{3}\) under a rigid motion of co-ordinates taking \(\gamma(t_0)\) to the origin.

Making a change of variables \(p = t - \frac{x}{\alpha}\) and \(\tau = \rho^2 x\), we can rewrite \(\overline{u}\) as

\[
f(p, \tau) = \bar{u}(x, t) = E_Z W(p - Z_\tau)
\]

We restrict our attention to \(p > 0, \tau > 0\). Fixing \(\tau\) and varying \(p\) in the above expression for \(f\) yields the path of the particle at distance \(x = \tau/\rho^2\) from the tip. Note that this is precisely the solution to the heat equation with the initial function being the Brownian motion \(W\), the space variable represented by \(p\) and time by \(\tau\).

Another interesting observation, which was described briefly in the Introduction, is the evolution of loops as we look at the paths of successive particles. If a particle in the (two dimensional) Conga line goes through a loop, the particle following it, which cuts corners and tries to “catch it”, will go through a smaller loop. This is suggested by the simulations, where small loops are seen to die, and just before death, they look somewhat ‘elongated’, and the death site looks like a cusp singularity. Other loops are seen to break after some time, that is, their end points come apart. Figure 2, though not depicting the particle paths, gives an idea of the loops in various stages of evolution. In this section, we investigate evolving loops in the paths of successive particles, especially the relationship between dying loops and formation of singularities.

Before we can start off, we give some definitions that will be useful in describing the evolution of loops.

We define a metric space \((\mathcal{M}, d)\), with a metric similar to the Skorohod metric on RCLL paths, on which we want to study loop evolutions:
Lemma 23
With probability one, any singularity looks is a cusp singularity. The proof of this statement is based on the fact that for any singularity in the Conga line at some time \( \tau_0 \), there exists a loop evolution \( L : [0, T) \to \mathcal{M} \) of a loop starting from \( L = L_0 \).

Proof:
To show this, define a compact set

\[
\mathcal{M} = \{ f : [a, b] \to \mathbb{C}; -\infty < a < b < \infty \}
\]

If \( f : [a_1, b_1] \to \mathbb{C}, g : [a_2, b_2] \to \mathbb{C} \in \mathcal{M} \), define

\[
d(f, g) = \inf \{ \| \lambda_1 - \lambda_2 \| + \| f \circ \lambda_1 - g \circ \lambda_2 \| \mid \lambda_i : [0, 1] \to [a_i, b_i] \text{ is a homeomorphism} \}
\]

where \( \| \cdot \| \) denotes the sup-norm metric. It can be easily checked that \((\mathcal{M}, d)\) is a metric space.

Define the evolution of a loop \( L \) as a continuous function \( L : [0, T) \to \mathcal{M} \) such that \( L_0 = L \) and \( L_t \) is a loop for every \( 0 \leq t < T \). If \( f : \mathbb{R} \to \mathbb{C} \) is a space-time process, and \( L_t = f(\cdot, t) \mid_{[a, b]} \) is a loop evolution, we say that \( L \) is a loop evolution of \( f \) (starting from \( L \)). Say that a loop \( L \) of \( f \) vanishes after time \( T \) if \( T \) is the maximal time such that there exists a loop evolution \( L : [0, T) \to \mathcal{M} \) of \( f \) starting from \( L \).

**Note:** Although \( f \) has no singularities for a fixed time \( \tau \), it can be easily verified by an application of the expectation meta-theorem of \([1]\) that the expected number of singularities of \( f(\cdot, \tau) \) for \( (p, \tau) \) lying in a compact set \( K = [a, b] \times [c, d] \) is positive, and thus singularities do occur with positive probability if we allow both space and time to vary.

It is easy to see that if a loop dies at a site \((p_0, \tau_0)\), then \( p_0 \) is a singularity for the curve \( f(\cdot, \tau_0) \). We prove in Lemma 23 that with probability one, any singularity looks is a cusp singularity. In Lemma 23, we prove that for any (cusp) singularity in the Conga line at some time \( \tau_0 \), there exists a loop in some small time interval \((\tau_0 - \delta, \tau_0)\).

**Lemma 23** With probability one, any singularity \( p_0 \) of the (analytic) curve \( f(\cdot, \tau_0) \) is a cusp singularity.

**Proof:** Write \( f = (f^1, f^2) \). It suffices to prove the lemma for \((p_0, \tau_0)\) lying in a rectangle \( K = [a, b] \times [c, d] \). Our first step is showing the following:

\[
P(\exists (p_0, \tau_0) \in K \text{ such that } \partial_p f(p_0, \tau_0) = 0 \text{ and the vectors } (\partial_p^2 f^1(p_0, \tau_0), \partial_p^3 f^1(p_0, \tau_0)) \text{ and } (\partial_p^2 f^2(p_0, \tau_0), \partial_p^3 f^2(p_0, \tau_0)) \text{ are linearly dependent}) = 0 \tag{5.1}
\]

To show this, define \( A_n \) to be the event which holds when all the following are satisfied:

(i) There exists \((p_0, \tau_0) \in K\) for which \( \partial_p f(p_0, \tau_0) = 0 \) and

\[
(\partial_p^2 f^2(p_0, \tau_0), \partial_p^3 f^2(p_0, \tau_0)) = \lambda (\partial_p^2 f^1(p_0, \tau_0), \partial_p^3 f^1(p_0, \tau_0))
\]

for some \( \lambda \in [-n, n] \).

(ii) The Lipschitz constants of the functions \( \{ \partial_p^i f^j(p, \tau) : (p, \tau) \in K; i = 1, 2, 3; j = 1, 2 \} \) are less than or equal to \( n \).

We will show that \( P(A_n) = 0 \) which will yield \( 5.1 \).

Partition the rectangle into a grid of sub-rectangles of side length \( \leq \epsilon \), where \( \epsilon \) is small. Call the set of grid points \( \hat{K} \).

Now, suppose \( A_n \) holds. Let \((p_0, \tau_0)\) lie in a sub-rectangle \( R \) and let \((p_i, \tau_j) \in \hat{K} \) be a grid point adjacent to \( R \). Note that as the Lipschitz constants of the above functions and \( \lambda \) are bounded by \( n \), the following event holds:

\[
A_{nij}^j = \{ |\partial_p f(p_i, \tau_j)| \leq \sqrt{2} n \epsilon \text{ and }\]

\[
| (\partial_p^2 f^2(p_i, \tau_j), \partial_p^3 f^2(p_i, \tau_j)) - \lambda (\partial_p^2 f^1(p_i, \tau_j), \partial_p^3 f^1(p_i, \tau_j)) | \leq 4n^2 \epsilon
\]

for some \( \lambda \in [-n, n] \).

Thus we have

\[
A_n \subseteq \bigcup_{i,j} A_{nij}^{ij}.
\]

We show that there is a constant \( C \) depending on \( n \) such that \( P \left( A_{nij}^{ij} \right) \leq C \epsilon \).

To save us notation, call

\[
X = (X_1, X_2, X_3) = (\partial_p f^1(p_i, \tau_j), \partial_p^2 f^1(p_i, \tau_j), \partial_p^3 f^1(p_i, \tau_j))
\]

and similarly \( Y \) for \( f^2 \). \( X \) and \( Y \) are independent and each follows a centred trivariate normal distribution. Let us call the density function of \( X \) \( p_{ij} \) and the distribution of \( Y \) as \( Q_{ij} \). Then, as \( X \) and \( Y \) have uniformly bounded densities,

\[
P \left( A_{nij}^{ij} \right) \leq \int_{x \in [-\sqrt{2}n \epsilon, \sqrt{2}n \epsilon] \times \mathbb{R}^2} p_{ij}(x) \times Q_{ij} \left( |Y_1| \leq \sqrt{2}n \epsilon; (Y_2, Y_3) \in \text{the } 4n^2 \epsilon \text{ neighbourhood of the linear span of } (x_2, x_3) \right) \, dx
\]

\[
\leq \int_{x \in [-\sqrt{2}n \epsilon, \sqrt{2}n \epsilon] \times \mathbb{R}^2} p_{ij}(x)C_{ij}^\epsilon \epsilon^2 \, dx \leq C_{ij} \epsilon^3,
\]

where \( C_{ij}^\epsilon, C_{ij} \) depend on \( n \). Note that the determinants of the covariance matrices of \( X \) and \( Y \) are continuous and do not vanish at any point on the compact set \( K \). Thus we can bound \( C_{ij} \) by \( C \) (which depends on \( n \)) uniformly over \( i, j, \epsilon \). Using these facts, we get

\[
P(A_n) \leq \sum_{i,j} P \left( A_{nij}^{ij} \right) \leq C \epsilon.
\]

As \( \epsilon \) is arbitrary, we get \( P(A_n) = 0 \).

Now if \( p_0 \) is a singularity occurring at time \( \tau_0 \), i.e. \( (p_0, \tau_0) \in K \) for which \( \partial_p f(p_0, \tau_0) = 0 \), we can apply a rigid motion of co-ordinates such that \( f(p_0, \tau_0) \) is the new origin and the rotation angle \( \theta \) is chosen to satisfy the equation

\[
A_{\theta} \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_2 & a_3 \\ 0 & b_3 \end{bmatrix}
\]

where \( A_{\theta} \) is the rotation matrix corresponding to \( \theta \). By (5.1), we see that \( a_2 \) and \( b_3 \) are non-zero. Then, the result follows by taking this new co-ordinate frame. \( \square \)

**Lemma 24** If \((p_0, \tau_0)\) is a (cusp) singularity, then \( \exists \delta > 0 \) such that \( f(\cdot, \tau) \) has a loop on some interval containing \( p_0 \) for all \( \tau \in [\tau_0 - \delta, \tau_0] \).

**Proof:** First we show that \( f \) is jointly analytic in \((p, \tau)\). For this, note that

\[
\partial_p^n f(p, \tau) = \int_0^\infty (-1)^n \tau^{-(n+1)/2} \text{He}_n \left( \frac{y - p}{\sqrt{\tau}} \right) \phi \left( \frac{y - p}{\sqrt{\tau}} \right) W(y) \, dy.
\]
So, by using the fact that \( \lim_{y \to \infty} \frac{W(y)}{y} = 0 \) almost surely, we get that with probability one,

\[
|\partial^p_n f(p, \tau)| \leq \frac{C}{\tau^{n/2}} \sqrt{n!(p^2 + \tau)}
\]

for some random constant \( C \). From this, we know that \( f(., \tau_0) \) has an analytic representation in the space variable \( p \) as

\[
f(p, \tau_0) = \sum_{i=0}^{\infty} a_i(p_0, \tau_0)(p - p_0)^i.
\]

Now, we prove joint analyticity. Note that if we can prove

\[
\sum_{i=0}^{\infty} \mathbb{E}_Z |a_i(p_0, \tau_0)||p - p_0| + |Z_\delta|^i < \infty
\]

for \( p \in \mathbb{R} \) and \( 0 < \delta < \epsilon \) for some \( \epsilon > 0 \), then we can write

\[
f(p, \tau) = \sum_{i=0}^{\infty} \mathbb{E}_Z a_i(p_0, \tau_0) ((p - p_0) - Z_{\tau - \tau_0})^i
\]

for \( p \in \mathbb{R} \) and \( \tau \in [\tau_0, \tau_0 + \epsilon) \) and it follows that \( f \) is jointly analytic on \( \mathbb{R} \times [\tau_0, \tau_0 + \epsilon) \). Joint analyticity on \( \mathbb{R} \times \mathbb{R}^+ \) is immediate as a result.

From the bound (5.2) on \( \partial^p_n f(p, \tau) \), we see that (5.4) holds when

\[
\sum_{i=0}^{\infty} \frac{C}{\tau_0^{i/2}} \sqrt{i!}(p_0^2 + \tau_0)2^i(|p - p_0|^i + 2^{i/2}\sqrt{i!}\delta^{i/2}) < \infty
\]

which is satisfied for \( \delta < \frac{\tau_0}{8} \) and all \( p \in \mathbb{R} \).

Now, if \( f \) has a cusp singularity at \( (p_0, \tau_0) \), then by the rigid motion of co-ordinates used in Lemma 23 and joint analyticity around \( (p_0, \tau_0) \), we can write \( f = (f_1, f_2) \) in the new co-ordinate frame, where

\[
\begin{align*}
f^1(p, \tau_0 - s) &= a_2(p^2 - s) + E^1(p, s) \\
f^2(p, \tau_0 - s) &= b_3(p^3 - 3ps) + E^2(p, s)
\end{align*}
\]

for small enough \( s \). It can be checked that

\[
g_s(p) = (a_2(p^2 - s), b_3(p^3 - 3ps))
\]

has a loop in \([ -\sqrt{3s}, \sqrt{3s}] \). So, if we choose and fix \( M > \sqrt{3} \), then we see that there is \( \delta > 0 \) such that for all \( s < \delta \), and \( p \in [ -M\sqrt{s}, M\sqrt{s}] \),

\[
|E^1(p, s)| \leq Cs^{3/2}
\]

and

\[
|E^2(p, s)| \leq Cs^2
\]

for some random constant \( C \). This, along with the fact that \( g_s \) has a loop on \([ -M\sqrt{s}, M\sqrt{s}] \) forces \( f \) to have a loop on this interval. \( \square \)
From Lemma 23 and Lemma 24, it is clear that there is a bijection between dying loops and singularities in the particle path evolution.

**Shape of a dying loop**

To understand the limiting shape of the dying loop, we rescale the space co-ordinate $p$ as $\sqrt{s}P$, and rescale $f^1$ by $s$, $f^2$ by $s^{3/2}$. It can be easily checked that as $s \to 0$, this new scaled function

$$\hat{f}_s(P) = \left( s^{-1}f^1(\sqrt{s}P), s^{-3/2}f^2(\sqrt{s}P) \right)$$

converges uniformly on $[-M, M]$ to the function

$$\hat{g}(P) = (a_2(P^2 - 1), b_3(P^3 - 3P))$$

which contains a loop $\hat{g}|_{[-\sqrt{3}, \sqrt{3}]}$. It is also not too hard to check that any loop in $\hat{f}_s$ on $[-M, M]$ converges to the loop of $\hat{g}$ in the topology on the space $(\mathcal{M}, d)$ described above.

This gives us the limiting shape of a dying loop. The difference in the scaling exponent explains why the loops look elongated before death.

6 **Freezing in the tail**

This section addresses Observation 4 of Burdzy and Pal [8]. The tail of the Conga line refers to the particles at distance $x > \alpha t$ from the leading particle.

So far, we have studied the behaviour for $x \leq \alpha t$. This part is more dynamic in time and the particles in this region have variances going to infinity with time, indicating appreciable motion. Also, a particle at any fixed distance from the tip eventually steps into this regime.

The tail on the other hand seems to freeze in time and the angle at which the Conga line comes out of the origin shows very little change with time after a while (see Figure 2). Further, the very small variance of any particle in the tail region for large $t$ indicates that we indeed need to rescale the tail to study its properties. In other words, the tail behaves in a very different manner compared to particles near the tip.

To study the phenomenon of ‘Freezing in the tail’, we use the continuous version $\overline{\pi}$ described in Subsection 2.2. For any fixed $\eta > 0$, we choose a sequence of distances from the tip

$$x_t = \alpha(1 + \eta)t$$

and study

$$\overline{\pi}_t(\eta) = \overline{\pi}(x_t, t).$$

The distances $x_t$ from the tip increasing with time $t$ ensures that these particles remain in the tail region for all times. We rescale $\overline{\pi}_t$ as

$$v_t(\eta) = \sqrt{2\pi\rho^2\alpha(1 + \eta)t} \exp \left\{ \frac{\eta^2t}{2\rho^2\alpha(1 + \eta)t} \right\} \overline{\pi}_t(\eta).$$

Also define

$$v(\eta) = \int_0^\infty \exp \left\{ -\frac{8\eta}{\rho^2\alpha(1 + \eta)} \right\} W(s)ds,$$

where $W$ is the driving Brownian motion in expression 2.5.
Theorem 8 For any fixed $\eta > 0$,

$$v_t(\eta) \rightarrow v(\eta)$$

almost surely and in $L^2$ as $t \rightarrow \infty$.

Proof: In the following, $C_1, C_2, \ldots$ represent finite, positive constants.

It follows from (2.5) that

$$v_t(\eta) = \int_0^{(1-\eta)t} W(s) \exp \left\{ -\frac{s^2}{2\rho^2 t\alpha(1+\eta)} - \frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} ds.$$  

Almost sure convergence follows from the fact that

$$\int_0^\infty |W(s)| \exp \left\{ -\frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} ds < \infty$$

and the Dominated Convergence Theorem.

To prove $L^2$ convergence, note that

$$v(\eta) - v_t(\eta) = \int_0^{(1-\eta)t} g_t(s) W(s) ds + \int_{(1-\eta)t}^\infty \exp \left\{ -\frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} W(s) ds,$$  

where

$$g_t(s) = \exp \left\{ -\frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} \left( 1 - \exp \left\{ -\frac{s^2}{2\rho^2 t\alpha(1+\eta)} \right\} \right).$$

It is easy to see that

$$g_t(s) \leq \frac{C_1}{\rho^2 \alpha(1+\eta)t} s^2 \exp \left\{ -\frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} \mathbb{I}(s \leq \sqrt{t}) + C_2 \exp \left\{ -\frac{s\eta}{\rho^2 \alpha(1+\eta)} \right\} \mathbb{I}(s > \sqrt{t}).$$  

Now, notice that the first term in (6.3) can be written as

$$E_1(t) = \int_0^{(1-\eta)t} \left( \int_a^{(1-\eta)t} g_t(s) ds \right) dW(a).$$

From (6.4), we get

$$\int_a^{(1-\eta)t} g_t(s) ds \leq \frac{C_3}{t} \mathbb{I}(a < \sqrt{t}) + C_4 \exp \left\{ -\eta(\sqrt{t} \vee a) \right\}.$$  

Thus

$$\mathbb{E}(E_1(t))^2 = \int_0^{(1-\eta)t} \left( \int_a^{(1-\eta)t} g_t(s) ds \right)^2 da \leq \frac{C_5}{t^{3/2}}.$$  

Similarly, for the second term in (6.3), say $E_2(t)$, we get

$$\mathbb{E}(E_2(t))^2 \leq C_6 t \exp \left\{ -\frac{2\eta(1-\eta)t}{\rho^2 \alpha(1+\eta)} \right\}.$$  


$L^2$ convergence follows from (6.5) and (6.6). □

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7 Simulations

Figure 2: 10000 steps (red) and 10000-100 steps (blue) of the discrete two dimensional Conga line. (courtesy: Krzysztof Burdzy)

Figure 3: 2000 steps of the discrete one dimensional Conga line with $\alpha = 0.5$. (courtesy: Shirshendu Ganguly)
Figure 4: Near the tip of the Conga line. Showing the first twenty particles. (courtesy: Shirshendu Ganguly)

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