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To cite this version:
Laurent Bakri. Quantitative uniqueness for Schrödinger operator. Indiana University Mathematics Journal, 2012, 61 (4), pp.1565-1580. hal-01981175

HAL Id: hal-01981175
https://hal.science/hal-01981175
Submitted on 14 Jan 2019
Quantitative uniqueness for Schrödinger operator

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Abstract
We give an upper bound on the vanishing order of solutions to Schrödinger equation on a compact smooth manifold. Our method is based on Carleman type inequalities, and gives a generalisation to a result of H. Donnelly and C. Fefferman [DF88] on eigenfunctions. It also sharpens previous results of I. Kukavica [Kuk98].

1 Introduction

Let \((M,g)\) be a smooth, compact and connected, \(n\)-dimensional Riemannian manifold. It is well known that, if \(u\) is a non trivial solution of second order linear elliptic equation on \(M\), then all zeros of \(u\) are of finite order ([Aro57, HS89]). The aim of this paper is to obtain quantitative estimate on the vanishing order of (non trivial) solutions to

\[ \Delta u = Wu, \]  

when \(W \in C^1(M)\). In the particular case of eigenfunctions of the Laplacian \((i.e. W = \lambda \text{ is a constant})\), it has been shown by H. Donnelly and C. Fefferman [DF88] that the vanishing order is bounded by \(C\sqrt{\lambda}\). In [Kuk98], I. Kukavica established some quantitative results for solution to (1.1). When \(W\) is a bounded function he obtained that the vanishing order of solutions to (1.1) is everywhere less than \(C(1 + \sqrt{\|W\|_\infty} + (\text{osc}(W))^2)\), where \(\text{osc}(W) = \sup W - \inf W\) and \(C\) a constant depending only on \(M\). If \(W\) is \(C^1\) he got the upper bound \(C(1 + \|W\|_{C^1})\), with \(\|W\|_{C^1} = \|W\|_\infty + \|\nabla W\|_\infty\). Our main result is the following

**Theorem 1.1.** The vanishing order of solutions to (1.1) is everywhere less than

\[ C_1 \sqrt{\|W\|_{C^1}} + C_2, \]

where \(C_1\) and \(C_2\) are positive constants depending only on \(M\).
More precisely theorem 1.1 is a direct consequence of the following doubling inequality on solutions (theorem 3.2):

\[ \|u\|_{L^2(B_r(x_0))} \leq e^{C_1\sqrt{\|W\|_{C^1} + C_2\|u\|_{L^2(B_r(x_0))}}}. \]  

(1.2)

The exponent 1/2 on \(\|W\|_{C^1}\) in this result is sharp and agrees with the result of H. Donnelly and C. Fefferman [DF88] when \(W\) is constant. Indeed, consider the homogeneous polynomials \(f_k(x_1, x_2, \cdots, x_{n+1}) = \Re(x_1 + ix_2)^k\) defined in \(\mathbb{R}^{n+1}\). Set \(Y_k\) the restriction of \(f_k\) to \(S^n\). \(Y_k\) is a sequence of spherical harmonics and \(-\Delta_{S^n}Y_k = k(k + n - 1)Y_k = \lambda_k Y_k\). The vanishing order at the north pole \(N = (0, \cdots, 0, 1)\) of \(Y_k\) is \(k \geq C\sqrt{\lambda_k}\).

Let us now discuss briefly the methods usually used to deal with quantitative uniqueness for linear partial differential equations. There are two principal methods: the first one is based on Carleman-type estimates [Aro57, DF88, DF90, Hör07, JK85, JL99] and the second one relies on the frequency function of solutions [Don92, GL86, Kuk98, Lin91]. The goal of both methods is to control the local behaviour of solutions. In the original works of Donnelly and Fefferman [DF88], the authors wrote down a Carleman estimate on the operator \(\Delta + \lambda\). Later, several authors (F.-H. Lin [Lin91], D. Jerison & G. Lebeau [JL99], I. Kukavica [Kuk95], ...) obtained some generalizations and simplifications in the proof. In particular, if \(u\) is an eigenfunction of the Laplace operator on \(M\), with eigenvalue \(\lambda\), then the function \(\tilde{u}\) defined on \(M \times [-T, T]\) by

\[ \tilde{u}(x, t) = \cosh(\sqrt{\lambda} t)u(x) \]

satisfies \((\Delta + \frac{\partial^2}{\partial t^2})\tilde{u} = 0\). The problem is then simplified since one has only to deal with the 0 eigenvalue of the operator \(\Delta + \frac{\partial^2}{\partial t^2}\) on \(M \times [-T, T]\). For example, in [JL99], D. Jerison and G. Lebeau established a Carleman estimate on \(\Delta + \frac{\partial^2}{\partial t^2}\). However, it was pointed out by I. Kukavica [Kuk98] that, the method of [JL99] doesn’t seem to extend easily when studying the more general equation (1.1). Despite this, the point of our paper is that one can successfully establish a Carleman estimate directly on the operator \(\Delta + W\). Furthermore, when \(W\) is \(C^1\), it leads to a better upper bound on the vanishing order of solutions to (1.1), in terms of its dependence on \(\|W\|_{C^1}\).

The paper is organised as follows. In section 2 we establish Carleman estimate for the operator \(\Delta + W\). Our method will differ slightly from [DF88], we will discuss it briefly in the remarks following theorem 2.1. In section 3 we deduce, in a standard manner, three balls theorem for solutions to (1.1), then using compactness we derive doubling inequality which gives immediately theorem 1.1. In a forthcoming paper we study the vanishing order of solutions when \(W\) is only a bounded function.
Acknowledgement. I would like to thank the referee for helpful remarks. I would also like to thank my PhD. advisor, R. Regbaoui, for his help and support.

2 Carleman estimates

Fix $x_0$ in $M$, and let $r = r(x) = d(x, x_0)$ the Riemannian distance from $x_0$. We denote by $B_r(x_0)$ the geodesic ball centered at $x_0$ of radius $r$. We will denote by $\| \cdot \|$ the $L^2$ norm. Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau \phi}$, where the function $\phi$ satisfies some convexity properties. Let us now define the weight function we will use.

For a fixed number $\epsilon$ such that $0 < \epsilon < 1$ and $T_0 < 0$, we define the function $f$ on $(-\infty, T_0]$ by $f(t) = t - e^{\epsilon t}$. One can check easily that, for $|T_0|$ great enough, the function $f$ verifies the following properties:

$$1 - \epsilon e^{\epsilon T_0} \leq f'(t) \leq 1 \quad \forall t \in [-\infty, T_0],$$

$$\lim_{t \to -\infty} -e^{-t} f''(t) = +\infty.$$ (2.1)

Finally we define $\phi(x) = -f(\ln r(x))$. Now we can state the main result of this section:

**Theorem 2.1.** There exist positive constants $R_0, C_0, C_1, C_2$, which depend only on $M$ and $\epsilon$, such that, for any $W \in C^1(M), x_0 \in M, u \in C_0^\infty(B_{R_0}(x_0) \setminus \{x_0\})$ and $\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2$, one has

$$C \left\| r^2 e^{\tau \phi} (\Delta u + W u) \right\| \geq \tau^\frac{3}{2} \left\| r^\frac{3}{2} e^{\tau \phi} u \right\| + \tau^\frac{1}{2} \left\| r^{1+\frac{3}{2}} e^{\tau \phi} \nabla u \right\|.$$ (2.2)

Moreover, if $\text{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\}$,

then

$$C \left\| r^2 e^{\tau \phi} (\Delta u + W u) \right\| \geq \tau^\frac{3}{2} \left\| r^\frac{3}{2} e^{\tau \phi} u \right\| + \tau \delta \left\| r^{-1} e^{\tau \phi} u \right\| + \tau^\frac{1}{2} \left\| r^{1+\frac{3}{2}} e^{\tau \phi} \nabla u \right\|.$$ (2.3)

**Remark 2.2.** This inequality can be seen as a generalization of previous Carleman type estimates in the case that $W$ is a constant (see [DF88]). Indeed when $W = \lambda$ one has $\sqrt{\|W\|_{C^1}} = \sqrt{\lambda}$. The point is that since $W$ is $C^1$ we will be allowed to integrate by parts, but then we have to take care of the derivatives of $W$.

**Remark 2.3.** In [DF88] the authors used a local conformal change, due to Aronszajn [Aro57], $\tilde{g}_{ij} = e^{-2\nu r^2} g_{ij}$ to obtain the positiveness of a certain tensor (see [DF88] lemma 2.3 p.167 and the computation of $J_4$ p.168). Instead of doing this, we introduce a small function $(f'')$ to handle some terms involving the derivatives of the metric, see by example the computation of $J_2$ below (2.13).
Remark 2.4. The constants $C, C_1, C_2$ go to infinity as $\varepsilon$ goes to zero. Therefore the case $\varepsilon = 0$ is not included in our theorem.

Remark 2.5. In the inequalities (2.2) and (2.3) the gradient terms are not necessary to the purpose of this paper. We choose to include them for a more general statement.

Proof. Hereafter $C, C_1, C_2$ and $c$ denote positive constants depending only upon $M$, though their values may change from one line to another. Without loss of generality, we may suppose that all functions are real. We now introduce the polar geodesic coordinates $(r, \theta)$ near $x_0$. Using Einstein notation, the Laplace operator takes the form:

$$r^2 \Delta u = r^2 \partial_r^2 u + r^2 \left( \partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r u + \frac{1}{\sqrt{\gamma}} \partial_t(\sqrt{\gamma} \gamma^{ij} \partial_j u),$$

where $\partial_i = \frac{\partial}{\partial \theta^i}$ and for each fixed $r$, $\gamma_{ij}(r, \theta)$ is a metric on $\mathbb{S}^{n-1}$ and $\gamma = \det(\gamma_{ij})$.

Since $(M, g)$ is smooth, we have for $r$ small enough:

$$\partial_r(\gamma^{ij}) \leq C(\gamma^{ij}) \quad \text{(in the sense of tensors)};$$
$$|\partial_r(\gamma)| \leq C;$$
$$C^{-1} \leq \gamma \leq C. \quad (2.4)$$

Set $r = e^t$, we have $\frac{\partial}{\partial r} = e^{-t} \frac{\partial}{\partial t}$. Then we will consider that the function $u$ has support in $]-\infty, T_0[\times \mathbb{S}^{n-1}$, where $|T_0|$ will be chosen large enough. In this new variables, we can write:

$$e^{2t} \Delta u = \partial_t^2 u + (n-2+\partial_t \ln \sqrt{\gamma}) \partial_t u + \frac{1}{\sqrt{\gamma}} \partial_t(\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

The conditions (2.4) become

$$\partial_t(\gamma^{ij}) \leq C e^t(\gamma^{ij}) \quad \text{(in the sense of tensors)};$$
$$|\partial_t(\gamma)| \leq C e^t;$$
$$C^{-1} \leq \gamma \leq C. \quad (2.5)$$

Now we introduce the conjugate operator :

$$L_T(u) = e^{2t} e^{\tau \phi} \Delta (e^{-\tau \phi} u) + e^{2t} W u$$
$$= \partial_t^2 u + (2 \tau f' + n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u$$
$$+ \left( \tau^2 f'' + \tau f' + (n-2) \tau f' + \tau \partial_t \ln \sqrt{\gamma} f' \right) u$$
$$+ \Delta_{\phi} u + e^{2t} W u, \quad (2.6)$$
\[ \Delta \theta u = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) . \]

It will be useful for us to introduce the following \( L^2 \) norm on \([-\infty, T_0] \times S^{n-1}:

\[ \| V \|_f^2 = \int_{-\infty, T_0} V^2 \sqrt{\gamma} f r^{-3} \, dt \, d\theta , \]

where \( d\theta \) is the usual measure on \( S^{n-1} \). The corresponding inner product is denoted by \( \langle \cdot, \cdot \rangle_f \), i.e.

\[ \langle u, v \rangle_f = \int uv \sqrt{\gamma} f r^{-3} \, dt \, d\theta . \]

We will estimate from below \( \| L_\tau u \|_f^2 \) by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of \( \tau \) and exponential decay when \( t \) goes to \(-\infty\). First by triangular inequality one has

\[ \| L_\tau (u) \|_f^2 \geq I - II , \quad \text{(2.7)} \]

with

\[ I = \left\| \partial_t^2 u + 2f' \partial_t u + \tau f^2 u + e^{2t} W u + \Delta \theta u \right\|_f^2 , \]

\[ II = \left\| (n-2)f' u + (n-2)\tau f u + \tau \partial_t \ln \sqrt{\gamma} f' u \right\|_f^2 + \left\| (n-2)\partial_t u + \partial_t \ln \sqrt{\gamma} \partial_t u \right\|_f^2 . \quad \text{(2.8)} \]

We will be able to absorb \( II \) later. Then we compute \( I^2 \):

\[ I^2 = I_1 + I_2 + I_3 , \]

with

\[ I_1 = \left\| \partial_t^2 u + (\tau f^2 u + e^{2t} W u + \Delta \theta u \right\|_f^2 , \]

\[ I_2 = \| 2\tau f' \partial_t u \|^2_f , \]

\[ I_3 = 2 \left\langle 2\tau f' \partial_t u, \partial_t^2 u + \tau f^2 u + e^{2t} W u + \Delta \theta u \right\rangle_f . \quad \text{(2.9)} \]

In order to compute \( I_3 \) we write it in a convenient way:

\[ I_3 = J_1 + J_2 + J_3 , \quad \text{(2.10)} \]

where the integrals \( J_i \) are defined by:

\[ J_1 = 2\tau \int f' \partial_t (|\partial_t u|^2) f r^{-3} \sqrt{\gamma} dtd\theta \]

\[ J_2 = 4\tau \int f' \partial_t u \partial_t (\sqrt{\gamma} \gamma^{ij} \partial_j u) f r^{-3} dtd\theta \]

\[ J_3 = \int (2\tau^3 f' + 2\tau f' e^{2t} W + 2u \partial_t u f r^{-3}) \sqrt{\gamma} dtd\theta . \quad \text{(2.11)} \]
Now we will use integration by parts to estimate each term of (2.11). Note that $f$ is radial and that $2\partial_t u \partial_t^2 u = \partial_t (|\partial_t u|^2)$. We find that:

$$J_1 = \int (4\tau f'') |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta$$
$$- \int 2\tau f' \partial_t \ln \sqrt{\gamma} |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta.$$ 

The conditions (2.5) imply that $|\partial_t \ln \sqrt{\gamma}| \leq C e^\tau$. Then properties (2.1) on $f$ gives, for large $|T_0|$ that $|\partial_t \ln \sqrt{\gamma}|$ is small compared to $|f''|$. Then one has

$$J_1 \geq -c \tau \int |f''| \cdot |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.12)$$

In order to estimate $J_2$ we first integrate by parts with respect to $\partial_i$:

$$J_2 = -2 \int 2\tau f' \partial_t \partial_i u \gamma^{ij} \partial_j u f^{-3} \sqrt{\gamma} dt d\theta.$$

Then we integrate by parts with respect to $\partial_t$. We get:

$$J_2 = -4\tau \int f'' \gamma^{ij} \partial_i u \partial_j u f^{-3} \sqrt{\gamma} dt d\theta$$
$$+ \int 2\tau f' \partial_t \ln \sqrt{\gamma} \gamma^{ij} \partial_i u \partial_j u f^{-3} \sqrt{\gamma} dt d\theta$$
$$+ \int 2\tau f' \partial_t (\gamma^{ij}) \partial_i u \partial_j u f^{-3} \sqrt{\gamma} dt d\theta.$$

We denote $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$. Now using that $-f''$ is non-negative and $\tau$ is large, the conditions (2.1) and (2.5) gives for $|T_0|$ large enough:

$$J_2 \geq 3\tau \int |f''| \cdot |D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.13)$$

Similarly computation of $J_3$ gives:

$$J_3 = -2 \int \tau^3 \partial_t \ln (\sqrt{\gamma}) u^2 \sqrt{\gamma} dt d\theta$$
$$- (4f' - 4f'' + 2\tau \partial_t \ln \sqrt{\gamma}) \tau e^{2t} W u^2 f^{-3} \sqrt{\gamma} dt d\theta$$
$$- 2\tau f' e^{2t} \partial_t W |u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.14)$$

Now we assume that

$$\tau \geq C_1 \sqrt{||W||c_1} + C_2. \quad (2.15)$$

From (2.1) and (2.5) one can see that if $C_1$, $C_2$ and $|T_0|$ are large enough, then

$$J_3 \geq -c \tau^3 \int e^t |u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.16)$$

Thus far, using (2.12), (2.13) and (2.16), we have:

$$I_3 \geq 3\tau \int |f''| |D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta - c \tau^3 \int e^t |u|^2 f^{-3} \sqrt{\gamma} dt d\theta$$
$$- c \tau \int |f''| |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.17)$$
Now we consider $I_1$:

$$I_1 = \left\| \partial_t^2 u + \left( \tau^2 f'^2 + e^{2t}W \right) u + \Delta_\theta u \right\|^2_f.$$

Let $\rho > 0$ a small number to be chosen later. Since $|f''| \leq 1$ and $\tau \geq 1$, we have:

$$I_1 \geq \frac{\rho}{\tau} I'_1,$$  \hspace{1cm} (2.18)

where $I'_1$ is defined by:

$$I'_1 = \left\| \sqrt{|f''|} \left( \partial_t^2 u + \left( \tau^2 f'^2 + e^{2t}W \right) u + \Delta_\theta u \right) \right\|^2_f$$  \hspace{1cm} (2.19)

and one has

$$I'_1 = K_1 + K_2 + K_3,$$  \hspace{1cm} (2.20)

with

$$K_1 = \left\| \sqrt{|f''|} \left( \partial_t^2 u + \Delta_\theta u \right) \right\|^2_f,$$

$$K_2 = \left\| \sqrt{|f''|} \left( \tau^2 f'^2 + e^{2t}W \right) u \right\|^2_f,$$

$$K_3 = 2 \left\langle \left( \partial_t^2 u + \Delta_\theta u \right) |f''|, \left( \tau^2 f'^2 + e^{2t}W \right) u \right\rangle_f.$$  \hspace{1cm} (2.21)

Integrating by parts gives:

$$K_3 = 2 \int f'' \left( \tau^2 f'^2 + e^{2t}W \right) |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$$

$$+ 2 \int \partial_t \left[ f'' \left( \tau^2 f'^2 + e^{2t}W \right) \right] \partial_t uu \sqrt{\gamma} f'^{-3} dt d\theta$$

$$- 6 \int f'' \left( \tau^2 f'^2 + e^{2t}W \right) \partial_t uu \sqrt{\gamma} f'^{-3} dt d\theta$$

$$+ 2 \int f'' \left( \tau^2 f'^2 + e^{2t}W \right) \partial_t \ln \sqrt{\gamma} \partial_t uu f'^{-3} \sqrt{\gamma} dt d\theta$$

$$+ 2 \int f'' \left( \tau^2 f'^2 + e^{2t}W \right) |\partial_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$$

$$+ 2 \int f'' e^{2t} \partial_t W \cdot \gamma^{ij} \partial_j uu f'^{-3} \sqrt{\gamma} dt d\theta.$$  \hspace{1cm} (2.22)

The condition $\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2$ implies,

$$\int |f''| e^{2t} |\partial_t W \gamma^{ij} \partial_j uu| \sqrt{\gamma} dt d\theta \leq c \tau^2 \int |f''| |\partial_\theta u|^2 + |u|^2 |f'^{-3} \sqrt{\gamma} dt d\theta.$$

Now since $2\partial_t uu \leq u^2 + |\partial_t u|^2$, we can use conditions (2.1) and (2.5) to get

$$K_3 \geq -c \tau^2 \int |f''| \left( |\partial_\theta u|^2 + |D_\theta u|^2 + |u|^2 \right) f'^{-3} \sqrt{\gamma} dt d\theta$$  \hspace{1cm} (2.23)

We also have
Thus we obtain:

\[ K_2 \geq c\tau^4 \int |f''||u|^2 f^{-3} \sqrt{\gamma} dt d\theta \]  \hspace{1cm} (2.24)

and since \( K_1 \geq 0 \),

\[ I_1 \geq -\rho c\tau \int |f''| (|\partial_t u|^2 + |D_\theta u|^2) f^{-3} \sqrt{\gamma} dt d\theta + C\tau^3 \rho \int |f''||u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \]  \hspace{1cm} (2.25)

Then using (2.17) and (2.25)

\[ I^2 \geq 4\tau^2 ||f'\partial_t u||_f^2 + 3\tau \int |f''||D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta + C\tau^3 \rho \int |f''||u|^2 f^{-3} \sqrt{\gamma} dt d\theta - \rho c\tau \int |f''| (|\partial_t u|^2 + |D_\theta u|^2) f^{-3} \sqrt{\gamma} dt d\theta. \]  \hspace{1cm} (2.26)

\[ c \tau \int |f''||\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta \]

Now one needs to check that every non-positive term in the right hand side of (2.26) can be absorbed in the first three terms.

First fix \( \rho \) small enough such that

\[ \rho c\tau \int |f''| \cdot |D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta \leq 2\tau \int |f''|^2 \cdot |D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta \]

where \( c \) is the constant appearing in (2.26). The other terms in the last integral of (2.26) can then be absorbed by comparing powers of \( \tau \) (for \( C_2 \) large enough). Finally since conditions (2.1) imply that \( \epsilon^t \) is small compared to \( |f''| \), we can absorb \(-c\tau^3 \epsilon^t |u|^2 \) in \( C\tau^3 \rho |f''||u|^2 \).

Thus we obtain:

\[ I^2 \geq C\tau^3 \int |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''||D_\theta u|^2 f^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''||u|^2 f^{-3} \sqrt{\gamma} dt d\theta \]  \hspace{1cm} (2.27)

As before, we can check that \( I \) can be absorbed in \( I \) for \( |T_0| \) and \( \tau \) large enough. Then we obtain

\[ ||L_\tau u||_f^2 \geq C\tau^3 ||\sqrt{|f'|}|u||_f^2 + C\tau^2 ||\partial_t u||_f^2 + C\tau ||\sqrt{|f''|}D_\theta u||_f^2. \]  \hspace{1cm} (2.28)

Note that, since \( \tau \) is large and \( \sqrt{|f'|} \leq 1 \), one has

\[ ||L_\tau u||_f^2 \geq C\tau^3 ||\sqrt{|f'|}|u||_f^2 + c\tau \sqrt{|f''|} ||\partial_t u||_f^2 + C\tau ||\sqrt{|f''|}D_\theta u||_f^2, \]  \hspace{1cm} (2.29)

and the constant \( c \) can be choosen arbitrary smaller than \( C \). If we set \( v = e^{-\tau \phi} u \), then we have

\[ ||e^{2t \tau \phi} (\Delta v + Wv)||_f^2 \geq C\tau^3 ||\sqrt{|f''|} e^{\tau \phi} v||_f^2 + C\tau^3 ||\sqrt{|f''|} f' e^{\tau \phi} v||_f^2 \]

\[ + \frac{\tau}{2} ||\sqrt{|f''|} e^{\tau \phi} \partial_t u||_f^2 + C\tau ||\sqrt{|f''|} e^{\tau \phi} D_\theta u||_f^2. \]

Finally since \( f' \) is close to 1 one can absorb the negative term to obtain
\[
\|e^{2t e^{\tau \phi}} (\Delta v + Wv)\|_f^2 \geq C \tau^3 \|\sqrt{\gamma} e^{\tau \phi} v\|_f^2 + C \tau \|\sqrt{\gamma} e^{\tau \phi} \partial_t v\|_f^2. \tag{2.30}
\]

It remains to get back to the usual \(L^2\) norm. First note that since \(f'\) is close to 1 (2.1), we can get the same estimate without the term \((f')^{-3}\) in the integrals. Recall that in polar coordinates \((r, \theta)\) the volume element is \(r^{n-1} \sqrt{\gamma} dr d\theta\), we can deduce from (2.27) by substitution that:

\[
\|r^2 e^{\tau \phi} (\Delta v + Wv) r^{-\frac{n}{2}} \|_f^2 \geq C \tau^3 \|r^\frac{1}{2} e^{\tau \phi} v r^{-\frac{n}{2}} \|_f^2 + C \tau \|r^{\frac{n}{2}} e^{\tau \phi} \nabla v r^{-\frac{n}{2}} \|_f^2. \tag{2.31}
\]

Finally one can get rid of the term \(r^{-\frac{n}{2}}\) by replacing \(\tau\) with \(\tau + \frac{n}{2}\). Indeed from \(e^{\tau \phi} r^{-\frac{n}{2}} = e^{(\tau + \frac{n}{2}) \phi} r^n\) one can check easily that, for \(r\) small enough

\[
\frac{1}{2} e^{(\tau + \frac{n}{2}) \phi} \leq e^{\tau \phi} r^{-\frac{n}{2}} \leq e^{(\tau + \frac{n}{2}) \phi}.
\]

This achieves the proof of the first part of theorem 2.1.

Now suppose that \(\text{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\}\) and define \(T_1 = \ln \delta\).

Cauchy-Schwarz inequality apply to

\[
\int \partial_t (u^2) e^{-t} \sqrt{\gamma} dt d\theta = 2 \int u \partial_t u e^{-t} \sqrt{\gamma} dt d\theta
\]
gives

\[
\int \partial_t (u^2) e^{-t} \sqrt{\gamma} dt d\theta \leq 2 \left( \int (\partial_t u)^2 e^{-t} \sqrt{\gamma} dt d\theta \right)^{\frac{1}{2}} \left( \int u^2 e^{-t} \sqrt{\gamma} dt d\theta \right)^{\frac{1}{2}}. \tag{2.32}
\]

On the other hand, integrating by parts gives

\[
\int \partial_t (u^2) e^{-t} \sqrt{\gamma} dt d\theta = \int u^2 e^{-t} \sqrt{\gamma} dt d\theta - \int u^2 e^{-t} \partial_t (\ln(\sqrt{\gamma})) \sqrt{\gamma} dt d\theta. \tag{2.33}
\]

Now since \(|\partial_t \ln \sqrt{\gamma}| \leq C e^t\) for \(|T_0|\) large enough we can deduce:

\[
\int \partial_t (u^2) e^{-t} \sqrt{\gamma} dt d\theta \geq c \int u^2 e^{-t} \sqrt{\gamma} dt d\theta. \tag{2.34}
\]

Combining (2.32) and (2.34) gives

\[
c^2 \int u^2 e^{-t} \sqrt{\gamma} dt d\theta \leq 4 \int (\partial_t u)^2 e^{-t} \sqrt{\gamma} dt d\theta \leq 4e^{-T_1} \int (\partial_t u)^2 \sqrt{\gamma} dt d\theta.
\]
Finally, dropping all terms except $\tau^2 \int |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta$ in (2.27) gives:

$$C' I^2 \geq \tau^2 \delta^2 \|e^{-t} u\|^2_f.$$

Inequality (2.27) can then be replaced by:

$$I^2 \geq C \tau^2 \int |\partial_t u|^2 f^{-3} \sqrt{\gamma} dt d\theta + C \tau^3 \int |f''| \cdot |u|^2 f^{-3} \sqrt{\gamma} dt d\theta + C \tau^2 \delta^2 \int |u|^2 f^{-3} \sqrt{\gamma} dt d\theta. \quad (2.35)$$

The rest of the proof follows in a similar way than the first part.

3 Doubling inequality

In this section we prove a doubling inequality for solutions of (1.1). First we deduce from Carleman estimate a three balls theorem for solutions. The standard way to do so is to apply such estimate, to $\psi u$ where $\psi$ is an appropriate cut off function, $u$ a solution, and then make a appropriate choice of the parameter $\tau$ (see [JL99]). We give a proof, following the method of Donnelly and Fefferman [DF88], adapted to our particular choice of weight functions in the Carleman estimate.

**Proposition 3.1** (Three balls inequality). There exist positive constants $R_1$, $C_1$, $C_2$ and $0 < \alpha < 1$ which depend only on $M$ such that, if $u$ is a solution to (1.1) with $W$ of class $C^1$, then for any $R < R_1$, and any $x_0 \in M$, one has

$$\|u\|_{B_R(x_0)} \leq e^{C_1 \sqrt{\|W\|_{C^1} + C_2 \|u\|^\alpha_{B_{2R}(x_0)}}} \|u\|_{B_{2R}(x_0)}^{1-\alpha}. \quad (3.1)$$

**Proof.** Let $x_0$ a point in $M$. Let $u$ be a solution to (1.1) and $R$ such that $0 < R < \frac{R_0}{2}$ with $R_0$ as in theorem 2.1. Recall that $r(x)$ is the riemannian distance between $x$ and $x_0$ and $B_r$ the geodesic ball centered at $x_0$ of radius $r$. If $v$ is a function defined in a neighborhood of $x_0$, we denote by $\|v\|_R$ the $L^2$ norm of $v$ on $B_R$ and by $\|v\|_{R_1, R_2}$ the $L^2$ norm of $v$ on the set $A_{R_1, R_2} := \{ x \in M; R_1 \leq r(x) \leq R_2 \}$. Let $\psi \in C_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a function with the following properties:

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or $r(x) > \frac{5R}{3}$,
- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla \psi(x)| \leq \frac{C}{R}$,
- $|\nabla^2 \psi(x)| \leq \frac{C}{R^2}$.
First since the function $\psi u$ is supported in the annulus $A_{\frac{R}{3}, \frac{5R}{3}}$, we can apply estimate (2.3) of theorem 2.1. In particular we have:

$$ C \left\| r^2 e^{\tau \phi} (\Delta \psi u + 2 \nabla u \cdot \nabla \psi) \right\| \geq \tau \left\| e^{\tau \phi} \psi u \right\|. \quad (3.2) $$

Assume that $\tau \geq 1$, and use properties of $\phi$ to get:

$$ \left\| e^{\tau \phi} u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \leq C \left( \left\| e^{\tau \phi} u \right\|_{\frac{R}{2}, R} + \left\| e^{\tau \phi} u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \right) + C \left( R \left\| e^{\tau \phi} \nabla u \right\|_{\frac{R}{2}, R} + R \left\| e^{\tau \phi} \nabla u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \right). \quad (3.3) $$

Recall that $\phi(x) = -\ln r(x) + r(x)$. In particular $\phi$ is radial and decreasing (for small $r$). Then one has,

$$ \left\| e^{\tau \phi} u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \leq C \left( e^{\tau \phi \left( \frac{3R}{2} \right)} \left\| u \right\|_{\frac{R}{2}, R} + e^{\tau \phi \left( \frac{5R}{2} \right)} \left\| u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \right) + C \left( R e^{\tau \phi \left( \frac{3R}{2} \right)} \left\| \nabla u \right\|_{\frac{R}{2}, R} + R e^{\tau \phi \left( \frac{5R}{2} \right)} \left\| \nabla u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \right). $$

Now we recall the following elliptic estimates : since $u$ satisfies (1.1) then it is not hard to see that :

$$ \left\| \nabla u \right\|_{(1-a)R} \leq C \left( \frac{1}{(1-a)R} + \left\| W \right\|_{\infty}^{1/2} \right) \left\| u \right\|_{B_R}, \quad 0 < a < 1. \quad (3.4) $$

Moreover since $A_{R_1, R_2} \subset B_{R_2}$, using formula (3.4) and properties of $\phi$ gives

$$ e^{\tau \phi \left( \frac{3R}{2} \right)} \left\| \nabla u \right\|_{\frac{3R}{2}, \frac{5R}{2}} \leq C \left( \frac{1}{R} + \left\| W \right\|_{\infty}^{1/2} \right) e^{\tau \phi \left( \frac{3R}{2} \right)} \left\| u \right\|_{2R}. $$

Using $(3.3)$ one has :

$$ \left\| u \right\|_{\frac{R}{2}, R} \leq C (\left\| W \right\|_{\infty}^{1/2} + 1) \left( e^{\tau \phi \left( \frac{R}{2} \right) - \phi(R)} \left\| u \right\|_{\frac{R}{2}} + e^{\tau \phi \left( \frac{3R}{2} \right) - \phi(R)} \left\| u \right\|_{2R} \right). $$

Let $A_R = \phi \left( \frac{R}{2} \right) - \phi(R)$ and $B_R = - \left( \phi \left( \frac{3R}{2} \right) - \phi(R) \right)$. From the properties of $\phi$, we have $0 < A^{-1} \leq A_R \leq A$ and $0 < B \leq B_R \leq B^{-1}$ where $A$ and $B$ don’t depend on $R$. We may assume that $C(\left\| W \right\|_{\infty}^{1/2} + 1) \geq 2$. Then we can add $\left\| u \right\|_{\frac{R}{2}}$ to each member and bound it in the right hand side by $C(\left\| W \right\|_{\infty}^{1/2} + 1)e^{\tau A} \left\| u \right\|_{\frac{R}{2}}$. We get :

$$ \left\| u \right\|_{R} \leq C (\left\| W \right\|_{\infty}^{1/2} + 1) \left( e^{\tau A} \left\| u \right\|_{\frac{R}{2}} + e^{-\tau B} \left\| u \right\|_{2R} \right). \quad (3.5) $$

Now we want to find $\tau$ such that

$$ C (\left\| W \right\|_{\infty}^{1/2} + 1)e^{-\tau B} \left\| u \right\|_{2R} \leq \frac{1}{2} \left\| u \right\|_{R} $$

which is true for $\tau \geq -\frac{1}{B} \ln \left( \frac{1}{2C(\left\| W \right\|_{\infty}^{1/2} + 1) \left\| W \right\|_{2R}} \right)$. Since $\tau$ must also satisfy

$$ \tau \geq C_1 \sqrt{\left\| W \right\|_{C^1} + C_2}, $$

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we choose
\[
\tau = -\frac{1}{B} \ln \left( \frac{1}{2C(\|W\|_{C^1}^{1/2} + 1)} \left\| u \right\|_{2R} \right) + C_1 \sqrt{\left\| W \right\|_{c^1}^1} + C_2. \tag{3.6}
\]
Since \(\|W\|_\infty \leq \|W\|_{C^1}\) we can deduce from (3.5) that:
\[
\left\| u \right\|_{B^{\frac{B+A}{A+B}}R} \leq e^{C_1 \sqrt{\left\| W \right\|_{C^1}^1} + C_2} \left\| u \right\|_{2R}^{\frac{A}{A+B}} \left\| u \right\|_{B^R}^B.
\tag{3.7}
\]
Finally define \(\alpha = \frac{A}{A+B}\) and taking exponent \(\frac{B}{A+B}\) of (3.7):
\[
\left\| u \right\|_{B^\alpha R} \leq e^{C_1 \sqrt{\left\| W \right\|_{C^1}^1} + C_2} \left\| u \right\|_{B^R}^{\frac{\alpha}{2}} \left\| u \right\|_{2R}^{\frac{1-\alpha}{2}}.
\]

From now on we assume that \(M\) is compact. Thus we can derive from three balls theorem above uniform doubling estimate on solutions.

**Theorem 3.2** (doubling estimate). There exist two positive constants \(C_1\) and \(C_2\), depending only on \(M\) such that: if \(u\) is a solution to (1.1) on \(M\) with \(W\) of class \(C^1\) then for any \(x_0 \in M\) and any \(r > 0\), one has
\[
\left\| u \right\|_{B^R(x_0)} \leq e^{C_1 \sqrt{\left\| W \right\|_{C^1}^1} + C_2} \left\| u \right\|_{B^R(x_0)}.
\tag{3.8}
\]

**Remark 3.3.** Using standard elliptic theory to bound the \(L^\infty\) norm of \(|u|\) by a multiple of its \(L^2\) norm (see [GT01] theorem 8.17, or [CFG86]) and rescaling arguments gives for \(\delta > 0\):
\[
\left\| u \right\|_{L^\infty(B_\delta(x_0))} \leq (C_1 \|W\|_{C^1} + C_2) \delta^{-n/2} \left\| u \right\|_{L^2(B_2\delta(x_0))}.
\]
Then one can see that the doubling estimate is still true with the \(L^\infty\) norm:
\[
\left\| u \right\|_{L^\infty(B_{2\delta}(x_0))} \leq e^{C_1 \sqrt{\left\| W \right\|_{C^1}^1} + C_2} \left\| u \right\|_{L^\infty(B_\delta(x_0))}.
\tag{3.9}
\]

**Remark 3.4.** We recall also that, it is necessary to assume that \(M\) is compact to obtain an uniform upper bound on the vanishing order, and therefore a doubling estimate, on solutions. Indeed, consider the harmonic function, \(f_k = \text{Re}(x_1 + ix_2)^k\) defined in \(\mathbb{R}^2\), so \(f_k\) satisfies (1.1) with \(W = 0\). The functions \(f_k\) can vanish at arbitrary high order at 0.

To prove the theorem 3.2 we need to use the standard overlapping chains of balls argument ([DF88, JL99, Kuk98]) to show:

**Proposition 3.5.** For any \(R > 0\) there exists \(C_R > 0\) such that for any \(x_0 \in M\), any \(W \in C^1(M)\) and any solutions \(u\) to (1.1):
\[
\left\| u \right\|_{B^R(x_0)} \geq e^{-C_R(1 + \sqrt{\left\| W \right\|_{C^1}^1})} \left\| u \right\|_{L^2(M)}.
\]

Proof. We may assume without loss of generality that $R < R_0$, with $R_0$ as in the three balls inequality (proposition 3.1). Up to multiplication by a constant, we can assume that $\|u\|_{L^2(M)} = 1$. We denote by $\bar{x}$ a point in $M$ such that $\|u\|_{B_R(\bar{x})} = \sup_{x \in M} \|u\|_{B_R(x)}$. This implies that one has $\|u\|_{B_R(\bar{x})} \geq D_R$, where $D_R$ depend only on $M$ and $R$. One has from proposition (3.1) at an arbitrary point $x$ of $M$:

$$\|u\|_{B_{R/2}(x)} \geq e^{-c(1+\sqrt{\|W\|_{c_1}})}\|u\|_{B_R(x)}^{1/2}. \quad (3.10)$$

Let $\gamma$ be a geodesic curve between $x_0$ and $\bar{x}$ and define $x_1, \ldots, x_m = \bar{x}$ such that $x_i \in \gamma$ and $B_{R/2}(x_{i+1}) \subset B_R(x_i)$, for any $i$ from 0 to $m - 1$. The number $m$ depends only on diam($M$) and $R$. Then the properties of $(x_i)_{1 \leq i \leq m}$ and inequality (3.10) give for all $i$, $1 \leq i \leq m$:

$$\|u\|_{B_{R/2}(x_i)} \geq e^{-c(1+\sqrt{\|W\|_{c_1}})}\|u\|_{B_{R/2}(x_{i+1})}^{1/2}. \quad (3.11)$$

The result follows by iteration and the fact that $\|u\|_{B_R(\bar{x})} \geq D_R$. \hfill \Box

**Corollary 3.6.** For all $R > 0$, there exists a positive constant $C_R$ depending only on $M$ and $R$ such that at any point $x_0$ in $M$ one has

$$\|u\|_{L^2(M)} \geq e^{-C_R(1+\sqrt{\|W\|_{c_1}})}\|u\|_{L^2(M)}.$$  

Proof. Recall that $\|u\|_{L^2(M)} = \|u\|_{L^2(A_{R/2})}$ with $A_{R/2} := \{x \in M; R \leq d(x, x_0) \leq 2R\}$. Let $R < R_0$ where $R_0$ is from proposition 3.3, note that $R_0 \leq \text{diam}(M)$. Since $M$ is geodesically complete, there exists a point $x_1$ in $A_{R/2}$ such that $B_{R/2}(x_1) \subset A_{R/2}$. From proposition 3.5 one has $\|u\|_{B_{R/2}(x_1)} \geq e^{-C_R(1+\sqrt{\|W\|_{c_1}})}\|u\|_{L^2(M)}$ which gives the result. \hfill \Box

**Proof of theorem 3.2.** We proceed as in the proof of three balls inequality (proposition 3.3) except for the fact that now we want the first ball to become arbitrary small in front of the others. Let $R = \frac{R_0}{4}$ with $R_0$ as in the three balls inequality, let $\delta$ such that $0 < 3\delta < \frac{R}{8}$, and define a smooth function $\psi$, with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or if $r(x) > R$,
- $\psi(x) = 1$ if $r(x) \in [\frac{5\delta}{4}, \frac{R}{2}]$,
- $|\nabla \psi(x)| \leq \frac{C}{\delta}$ and $|\nabla^2 \psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$,
- $|\nabla \psi(x)| \leq C$ and $|\nabla^2 \psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$. 

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Thus, the theorem is proved for all $r$

$$\|r^{\frac{2}{3}} e^{\tau \theta} \psi u\| + \tau \delta \|r^{-1} e^{\tau \theta} \psi u\| \leq C \left( \|r^{2} e^{\tau \theta} \nabla u \cdot \nabla \psi\| + \|r^{2} e^{\tau \theta} \Delta \psi u\| \right).$$

Using properties of $\psi$, one has

$$\|r^{\frac{2}{3}} e^{\tau \theta} u\| \frac{g}{R}, \frac{D}{\tau} + \|e^{\tau \theta} u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} \leq C \left( \delta \|e^{\tau \theta} \nabla u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} + \|e^{\tau \theta} \nabla u\| \frac{g}{R}, \frac{D}{\tau} \right) + C \left( \|e^{\tau \theta} u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} + \|e^{\tau \theta} u\| \frac{g}{R}, \frac{D}{\tau} \right).$$

Using (3.4) and properties of $\phi$, we get

$$e^{\tau \phi(\frac{R}{2})} \|u\| \frac{R}{\pi, \frac{D}{\tau}} + e^{\tau \phi(3\delta)} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} \\leq C \left(1 + \|W\|_{1/2} \right) \left( e^{\tau \phi(4)} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} + e^{\tau \phi(\frac{R}{2})} \|u\| \frac{g}{R}, \frac{D}{\tau} \right).$$

Then adding $e^{\tau \phi(3\delta)} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau}$ to each side leads to

$$e^{\tau \phi(\frac{R}{2})} \|u\| \frac{R}{\pi, \frac{D}{\tau}} + e^{\tau \phi(3\delta)} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} \\leq C \left(1 + \|W\|_{1/2} \right) \left( e^{\tau \phi(4)} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} + e^{\tau \phi(\frac{R}{2})} \|u\| \frac{g}{R}, \frac{D}{\tau} \right).$$

Now we want to choose $\tau$ such that

$$C \left(1 + \|W\|_{1/2} \right) e^{\tau \phi(\frac{R}{2})} \|u\| \frac{R}{\pi, \frac{D}{\tau}} \leq \frac{1}{2} e^{\tau \phi(\frac{R}{2})} \|u\| \frac{R}{\pi, \frac{D}{\tau}}.$$ 

For the same reasons than before we choose

$$\tau = \frac{1}{\phi(\frac{R}{2}) - \phi(\frac{R}{2})} \ln \left( \frac{1}{2C \left(1 + \|W\|_{1/2} \right) \|u\| \frac{R}{\pi, \frac{D}{\tau}}} \right) + C \left(1 + \sqrt{\|W\|_{c_1}} \right).$$

Define $D_{R} = (\phi(\frac{R}{2}) - \phi(\frac{R}{2}))^{-1}$; like before one has $0 < A \leq -D_{R} \leq A^{-1}$.

Dropping the first term in the left hand side, one has

$$\|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau} \leq e^{C \left(1 + \sqrt{\|W\|_{c_1}} \right)} \left( \frac{\|u\| \frac{R}{\pi, \frac{D}{\tau}}}{\|u\| \frac{g}{R}, \frac{D}{\tau}} \right)^{-A} \|u\| \frac{\delta \tau}{4}, \frac{3\delta}{\tau}.$$ 

Finally from corollary 3.6, define $r = \frac{\delta \tau}{2}$ to have:

$$\|u\| \frac{\delta \tau}{2r} \leq e^{C \left(1 + \sqrt{\|W\|_{c_1}} \right)} \|u\| \frac{\delta \tau}{2r}.$$ 

Thus, the theorem is proved for all $r \leq \frac{R_{0}}{10}$. Using proposition 3.5 we have for $r \geq \frac{R_{0}}{10}$:

$$\|u\|_{B_{s_{0}}(r)} \geq \|u\|_{B_{s_{0}}(\frac{R_{0}}{10})} \geq e^{-C_{0} \left(1 + \sqrt{\|W\|_{c_1}} \right)} \|u\|_{L^{2}_{0}(M)} \geq e^{-C_{1} \left(1 + \sqrt{\|W\|_{c_1}} \right)} \|u\|_{B_{s_{0}}(2r)}. \square$$
As stated before, the upper bound on vanishing order of solutions (theorem 1.1) is a direct consequence of theorem 3.2 for non trivial solutions to (1.1).

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