ON THE RESTRICTION OF THE MODULI PART TO A
REDUCED DIVISOR

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Abstract. Let \( f : (X, \Delta) \to Y \) be a fibration such that \( K_X + \Delta \) is torsion along the fibres of \( f \). Assume that \( Y \) has dimension 2, or that \( Y \) has dimension 3 and the fibres have dimension at most 3. Then the restriction of the moduli part to its augmented base locus is semiample.

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1. Introduction

In this paper we study fibrations \( f : (X, \Delta) \to Y \) such that \( K_X + \Delta \) is the pullback of a \( \mathbb{Q} \)-Cartier divisor \( D \) on \( Y \). Those arise naturally, as the abundance conjecture predicts that every log canonical pair is birational to either a Mori fibre space or a pair \((X, \Delta)\) with \( K_X + \Delta \) semiample. The
induced fibration \( f : X \to Y \) is such that \( K_X + \Delta \sim_f D \) for an ample \( \mathbb{Q} \)-Cartier divisor \( D \) on \( Y \). The canonical bundle formula is a way of writing \( D \) as the sum of three divisors: the canonical divisor of \( Y \), a divisor \( B_Y \) called discriminant defined in terms of the singularities of the fibration, and a nef (on a birational model of \( Y \)) divisor \( M_Y \) called moduli part or moduli divisor, describing the variation in moduli of the fibres. For example, by [Amb05, Theorem 3.3, 3.5] if the moduli part is numerically zero and \( (X, \Delta) \) is klt, then the fibration is essentially a product.

The theory of the canonical bundle formula has its roots in the work by Kodaira and Ueno on elliptic surfaces. It has been developed and generalised in [Kaw81, Amb04, Amb05, FM00, Kol07a].

The idea of considering divisors of the form \( K_Y + B + M \) where \( K_Y \) is the canonical divisor, \( (Y, B) \) satisfies certain regularity conditions and \( M \) is nef on a higher model of \( Y \) is central in the works by Birkar–Zhang, Birkar who consider generalised polarised pairs instead of pairs.

The most important conjecture on the canonical bundle formula has been formulated in [PS09, Conjecture 7.13]:

**B-Semiampleness Conjecture.** Let \( (X, \Delta) \) be a pair and let \( f : (X, \Delta) \to Y \) be an lc-trivial fibration to an \( n \)-dimensional variety \( Y \), where the divisor \( \Delta \) is effective over the generic point of \( Y \). If \( Y \) is an Ambro model of \( f \), then the moduli divisor \( M_Y \) is semiample.

Several special cases of the conjecture are proved, mainly when the dimension of the fibre is at most two by the classical work of Kodaira and by [PS09, Fuj03, Fil18] and if the moduli part is numerically zero by [Amb05, Flo14]. For the klt case, if the moduli part is torsion, then by [Amb05, Theorem 3.3] the variation of \( f \) is zero.

In this paper we consider a connected divisor \( T = \bigcup T \) and assume the B-Semiampleness Conjecture in lower dimension. In [FL19] we proved that the divisor \( M_Y|T \) is semiample for every \( T \). In this work we study the gluing of the global sections of \( mM_Y|T \) to obtain global sections of \( mM_Y|T \).

The main result of this paper is the following:

**Theorem A.** Let \( (X, \Delta) \) be a pair and let \( f : (X, \Delta) \to Y \) be a klt-trivial fibration to a surface \( Y \), where the divisor \( \Delta \) is effective over the generic point of \( Y \). Assume that \( Y \) is an Ambro model for \( f \) and that \( M_Y \) is big.

Then there is a birational base change \( Y' \to Y \) such that the restriction of \( M_Y \) to the augmented base locus is torsion.

The semiampleness of the moduli part turns out to be deeply related to the variation of the fibres of \( f \). The variation, introduced by Viehweg [Vie83] is roughly speaking the dimension of the moduli space of fibres of \( f \) in the sense of birational geometry (see Definition 2.23 for a precise definition). The Kodaira dimension of the moduli part is at most the variation of \( f \), and conjecturally they coincide. On the other hand, for a fibration of maximal
variation there should be only a finite number of fibres birational to a given one:

**Conjecture 1.1.** Let $X$ be a $\mathbb{Q}$-factorial variety. Let $f: (X, \Delta) \to Y$ be a klt-trivial fibration of maximal variation. Then there is an open set $U \subseteq Y$ such that for every $y \in U$ the set

$$\{ z \in U \mid (f^{-1}y, \Delta^h|_{f^{-1}y}) \text{ crepant birational to } (f^{-1}z, \Delta^h|_{f^{-1}z}) \}$$

is finite, where $\Delta^h$ denotes the horizontal part of $\Delta$.

Conjecture 1.1 is true for fibrations of relative dimension at most 2. Using this fact we are able to prove

**Theorem B.** Let $(X, \Delta)$ be a pair and let $f: (X, \Delta) \to Y$ be a klt-trivial fibration to a variety $Y$ of dimension 3 and $\dim X \leq \dim Y + 3$, where the divisor $\Delta$ is effective over the generic point of $Y$. Assume that $Y$ is an Ambro model for $f$ and that $M_Y$ is big.

Then there is a birational base change $Y' \to Y$ such that the restriction of $M_{Y'}$ to the augmented base locus is semiample.

For the proof of Theorem A and Theorem B, we embrace the approach developed in [Kol13] and successfully applied in [HX13] to the study of the semiampleness of the log canonical divisor of a slc pair (roughly speaking a simple normal crossings divisor in a smooth variety).

By [FL19] we are in the following setting: we have a line bundle $L$ on a reduced, non irreducible variety $T$ which is semiample on every irreducible component of $T$. We want to prove that $L$ is semiample on $T$. The approach consists in translating the semiampleness of a line bundle into the finiteness of a certain equivalence relation. For the sake of simplicity, assume that $T = T_1 \cup T_2$. Let $\phi_i: T_i \to V_i$ be the fibration induced by $L$ for $i = 1, 2$. We say that $x_1 \in V_1$ is equivalent to $x_2 \in V_2$ if $\phi_1^{-1}(x_1) \cap \phi_2^{-1}(x_2) \neq \emptyset$ and we take the closure of this equivalence relation. This is the natural relation to consider. Indeed, if $L|_{T_1 \cup T_2}$ is semiample and $\phi: T_1 \cup T_2 \to V$ is the induced fibration, then $\phi_1^{-1}(x_1)$ and $\phi_2^{-1}(x_2)$ are sent to the same point by $\phi$.

By considering the union of the fibres of $\phi_1$ and $\phi_2$ which intersect, we construct subsets of $T_1 \cup T_2$ called pseudofibres.

The reason why we cannot fully apply Kollár’s gluing theory is that many of the required regularity hypotheses are not satisfied in our setup.

We now describe the structure of the paper as well as the techniques used in every section. Section 2 contains some preliminary results as well as some refinements of results on the canonical bundle formula. Section 3 is a semiampleness criterion for a line bundle on a simple normal crossings surface. In section 4 we recall the basic notions on equivalence relations and prove some technical lemmas necessary for the study of the equivalence relation $R_L$, which is done in section 5. In section 6 we gather some results from [Sta83] and we apply them in section 7 where we develop a criterion for the triviality a line bundle on a simple normal crossing variety. Section
uses techniques from the minimal model program and is a study of the
restriction of the moduli part to higher codimensional log canonical centres.

In section 9 we prove that, assuming the B-Semiampleness Conjecture
in dimension $n - 1$ and Conjecture 1.1 in dimension $d - 1$, the equivalence
relation is finite for $L = \mathcal{O}(mM_Y)$ for $Y$ of dimension $n$ and $X$ of dimension
$d + n$. In section 10 we prove that the restriction of $L$ to a simple normal
crossings pseudofibre is torsion.

The last section contains the proofs of Theorems A and B.

2. Preliminary results

We work over the complex numbers. For the notions on the minimal
model program and singularities of pairs we refer to [KM92]. We will use
without defining them the notions of log canonical, klt and dlt singularities,
as well as of centre of a log canonical singularity. We refer to [Kol97] and
[KM92] for a presentation of these concepts and to [FL19, Definition 2.5] for
a summary of all the required notions in our setup.

We recall that a pair $(X, \Delta)$ is the data of a normal projective variety $X$
and a $\mathbb{Q}$-Weil divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. In this paper we
do not require $\Delta$ to be an effective divisor.

We say that a closed subvariety $S$ of $X$ is a minimal log canonical centre of
$(X, \Delta)$ over $\mathbb{Z}$ if $S$ is a minimal log canonical centre of $(X, \Delta)$ (with respect
to inclusion) which dominates $Z$.

2.1. Semistable morphisms. In this paragraph we recall the definition of
semistable morphisms and the statement of the semistable reduction theo-
rem, proved in [ALT18], which will be crucial in the proof of our main
results, Theorem 9.2 and 10.1. We refer to [Ogu18, section II.1.1] for the
definition of log scheme and morphism of log schemes, and to [Stack, Defi-
nition 15.52.1, Proposition 15.52.3] for the definition of quasi excellent rings
and the first properties.

Definition 2.1 (4.2.1 [ALT18]). A morphism of log schemes $f: X \to B$, $f^\#: f^{-1}O_Y \to O_X$ is semistable if the following conditions hold:

1. $X$ and $B$ are regular and the log structures are given by normal
crossings divisors $Z \subseteq X$ and $W \subseteq B$.
2. Étale-locally at any $x \in X$ with $b = f(x)$ there exist regular param-
eters $t_1, \ldots, t_n, t'_1, \ldots, t'_{n'} \in \mathcal{O}_{X,x}$ and $\pi_1, \ldots, \pi_l, \pi'_1, \ldots, \pi'_{l'} \in \mathcal{O}_{B,b}$
such that $Z = V(t_1 \cdots t_n)$ at $x$, $W = V(\pi_1 \cdots \pi_l)$ at $b$,
$f^\#(\pi_j) = t_{n_{j+1}} \cdots t_{n_{l+1}}$ for $0 = n_1 < n_2 < \cdots < n_{l+1} \leq n$.
3. $f$ is log smooth.

In characteristic zero, the third condition can be replaced by the condition
that $f^\#(\pi'_j) = t'_j$ for $1 \leq j \leq l'$.

The following semistable reduction theorem is proved in [ALT18] and uses
a finer toroidalization proved in [ATW20].
**Theorem 2.2** (Theorem 4.7 [ALT18]). Assume that \(X \to B\) is a dominant morphism of finite type between quasi excellent integral schemes of characteristic zero and \(Z \subseteq X\) is a closed subset. Then there exists a stack-theoretic modification \(b: B' \to B\), a projective modification \(a: X' \to (X \times_B B')^{\text{proj}}\), and divisors \(W' \subseteq B'\), \(Z' \subseteq X'\) such that:

1. \(a^{-1}Z \cup f^{-1}W' \subseteq Z'\) and the morphism \(f': (X', Z') \to (B', W')\) is semistable. In particular, \(X', B'\) are regular and \(Z', W'\) are snc.
2. If a regular open \(B_0 \subseteq B\) is such that \(X_0 = X \times_B B_0 \to B_0\) is smooth and \(Z_0 = Z \times_B B_0 \to B_0\) is a relative divisor over \(B_0\) with normal crossings (in other words, \(f: (X_0, Z_0) \to (B_0, W_0)\) is semistable), then \(a\) and \(b\) are isomorphisms over \(X_0\) and \(B_0\), respectively.

**Remark 2.3.** Let \(f: (X, Z) \to (B, W)\) be a semistable map, and let \(S \subseteq X\) be a stratum of \(Z\). Let \(C = f(S)\). It follows from the definition that \(f|_S: S \to C\) is semistable. Moreover, if \(f|_S: S \xrightarrow{h} C' \xrightarrow{\tau} C\) is the Stein factorisation, then \(h\) is semistable.

### 2.2. Groups of crepant birational automorphisms

In this paragraph we state two results on the group of crepant birational selfmaps of a pair. The first one is the finiteness of pluricanonical representations [Con13, Theorem 4.5] and the second one is a generalisation to pairs of the finiteness of the group of selfmaps of a manifold of general type.

**Definition 2.4.** Let \(f_1: (X_1, \Delta_1) \to Y\) and \(f_2: (X_2, \Delta_2) \to Y\) be two fibrations of pairs to the same base \(Y\). A birational map \(\theta: X_1 \dashrightarrow X_2\) is **crepant birational over** \(Y\) if \(a(E, X_1, \Delta_1) = a(E, X_2, \Delta_2)\) for every geometric valuation \(E\) over \(X_1\) and \(X_2\) and we have the commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\theta} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y & & Y
\end{array}
\]

The map \(\theta\) is **crepant birational** if \(Y\) is a point.

The set of all crepant birational maps of a pair \((X, \Delta)\) to itself is a group, denoted by \(\text{Bir}^c(X, \Delta)\). For a positive integer \(m\) such that \(m(K_X + \Delta)\) is Cartier, every \(\sigma \in \text{Bir}^c(X, \Delta)\) defines an automorphism of \(H^0(X, m(K_X + \Delta))\), and hence the **pluricanonical representation**

\[
\rho_m: \text{Bir}^c(X, \Delta) \to \text{GL} \left( H^0(X, m(K_X + \Delta)) \right).
\]

**Remark 2.5.** If the condition \(p^*(K_{X_1} + \Delta_1) = q^*(K_{X_2} + \Delta_2)\) is true for one resolution of the indeterminacy, then it is true for every resolution of indeterminacy. Indeed, let \((p', q'): W' \to X_1 \times X_2\) be another resolution of the indeterminacy. Let \((\nu, \mu): \hat{W} \to W \times W'\) be a dominating birational model. Then \(\nu^*p^*(K_{X_1} + \Delta_1) = \nu^*q^*(K_{X_2} + \Delta_2)\). By commutativity, \(\nu^*p^*(K_{X_1} + \Delta_1) = \mu^*p^*(K_{X_1} + \Delta_1)\) and \(\nu^*q^*(K_{X_1} + \Delta_1) = \mu^*q^*(K_{X_1} + \Delta_1)\). We conclude by pushing forward with \(\nu\).
Theorem 2.6. Let \((X, \Delta)\) be a klt pair such that \(K_X + \Delta \sim_\mathbb{Q} 0\). Then for every \(m\), the image of the pluricanonical representation \(\rho_m\) is finite. In particular, there is a positive integer \(\ell\) such that the image of \(\rho_\ell\) is trivial.

Proof. The first statement is [Gon13, Theorem 4.5], and then the second statement is straightforward. \(\square\)

2.3. Canonical bundle formula. In this subsection we define lc-trivial fibration and recall several fundamental results. We refer the reader to [FL20] for a survey of the general results on the canonical bundle formula.

Definition 2.7. Let \((X, \Delta)\) be a pair and let \(\pi: X' \to X\) be a log resolution of the pair. A morphism \(f: (X, \Delta) \to Y\) to a normal projective variety \(Y\) is a klt-trivial, respectively lc-trivial, fibration if \(f\) is a surjective morphism with connected fibres, \((X, \Delta)\) has klt, respectively log canonical, singularities over the generic point of \(Y\), there exists a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(D\) on \(Y\) such that \(K_X + \Delta \sim_\mathbb{Q} f^*D\), and if \(f' = f \circ \pi\), then

\[ \text{rk} f'_* \mathcal{O}_X([K_{X'} - \pi^*(K_X + \Delta)]) = 1, \]

respectively

\[ \text{rk} f'_* \mathcal{O}_X([K_{X'} - \pi^*(K_X + \Delta) + \sum_{a(E, X, \Delta) = -1} E]) = 1. \]

Remark 2.8. This last condition in the previous definition is verified, for instance, if \(\Delta\) is effective on the generic fibre, which is mostly the case in this paper.

Definition 2.9. Let \(f: (X, \Delta) \to Y\) be an lc-trivial fibration, and let \(P \subseteq Y\) be a prime divisor with the generic point \(\eta_P\). The log canonical threshold of \(f^*P\) with respect to \((X, \Delta)\) is

\[ \gamma_P = \sup \{ t \in \mathbb{R} \mid (X, \Delta + tf^*P) \text{ is log canonical over } \eta_P \}. \]

The discriminant of \(f\) is

\[ (1) \quad B_f = \sum_P (1 - \gamma_P)P. \]

This is a Weil \(\mathbb{Q}\)-divisor on \(Y\), and it is effective if \(\Delta\) is effective. Fix \(\varphi \in \mathbb{C}(X)\) and the smallest positive integer \(r\) such that \(K_X + \Delta + \frac{1}{r} \text{div} \varphi = f^*D\). Then there exists a unique Weil \(\mathbb{Q}\)-divisor \(M_f\), the moduli part of \(f\), such that

\[ (2) \quad K_X + \Delta + \frac{1}{r} \text{div} \varphi = f^*(K_Y + B_f + M_f). \]

The formula (2) is the canonical bundle formula associated to \(f\).

Remark 2.10. As in [FL19], we adopt here the notation \(B_f, M_f\) for the discriminant and moduli part of \(f\) instead of the usual one \(B_Y, M_Y\). We will occasionally write \(B_Y, M_Y\) when the fibration is clear from the context.
Remark 2.11. If $f_1: (X_1, \Delta_1) \to Y$ and $f_2: (X_2, \Delta_2) \to Y$ are two lc-trivial fibrations over the same base which are crepant birational over $Y$, then $f_1$ and $f_2$ have the same discriminant and moduli part.

The canonical bundle formula satisfies several desirable properties. The first is the base change property, [Amb04, Theorem 0.2] and [Kaw98, Theorem 2].

Theorem 2.12. Let $f: (X, \Delta) \to Y$ be a klt-trivial fibration. Then there exists a proper birational morphism $Y' \to Y$ such that for every proper birational morphism $\pi: Y'' \to Y'$ we have:

(i) $K_{Y'} + B_{Y'}$ is a $\Q$-Cartier divisor and $K_{Y''} + B_{Y''} = \pi^*(K_{Y'} + B_{Y'})$,

(ii) $M_{Y'}$ is a nef $\Q$-Cartier divisor and $M_{Y''} = \pi^*M_{Y'}$.

In the context of the previous theorem, we say that $M_Y$ descends to $Y'$, and we call $Y'$ an Ambro model for $f$. One of the reasons why base change property is important is the following inversion of adjunction [Amb04, Theorem 3.1].

Moreover, by [Kol07a, Proposition 8.4.9, Definition 8.3.6, Theorem 8.5.1] if $f: (X, \Delta) \to Y$ is an lc-trivial fibration such that the non-smooth locus $\Sigma$ of the fibration is a simple normal crossings divisor and $f^{-1}\Sigma + \Delta$ is simple normal crossings, then $Y$ is an Ambro model.

Remark 2.13. Theorem 2.12 implies in particular that the moduli part is always pseudoeffective, even when it is not nef, as it is the push-forward of a nef divisor by a birational model.

We prove now that if the moduli part descends on $Y$, then it descends on $Y'$ with $Y' \to Y$ generically finite.

Lemma 2.14. Let $f: (X, \Delta) \to Y$ be an lc-trivial fibration and $\tau: Y' \to Y$ be a generically finite map. If $Y$ is an Ambro model, then $Y'$ is an Ambro model for the fibration obtained by base change.

Proof. By taking the Stein factorisation, it is enough to consider $\tau$ finite. Let $\tilde{\nu}: \tilde{Y} \to Y'$ be a birational map. By [PL19, Lemma 2.4] there is a diagram

\[
\begin{array}{ccc}
Y' & \leftarrow & \tilde{Y} \\
\downarrow \tau & & \downarrow \tilde{\nu}' \downarrow \pi' \\
Y & \leftarrow & W'
\end{array}
\]

such that $\nu$ and $\nu'$ are birational and $(\nu')^{-1}$ is an isomorphism along the generic point of every $\tilde{\nu}$-exceptional divisor.

Let $(p, q): \tilde{W} \to W' \times \tilde{Y}$ be a resolution of the indeterminacies. Then we have

$M_{\tilde{W}} = p^*\pi'^*\nu^*M_Y = q^*\tilde{\nu}'^*\tau^*M_Y = q^*\tilde{\nu}^*M_{Y'}$,

which implies $M_{\tilde{Y}} = q_*M_{\tilde{W}} = \tilde{\nu}^*M_{Y'}$. 

\[\Box\]
Theorem 2.15. Let \( f : (X, \Delta) \to Y \) be an lc-trivial fibration, and assume that \( Y \) is an Ambro model for \( f \). Then \((Y, B_Y)\) has klt, respectively log canonical, singularities in a neighbourhood of a point \( y \in Y \) if and only if \((X, \Delta)\) has klt, respectively log canonical, singularities in a neighbourhood of \( f^{-1}(y) \).

The following is [Amb05, Theorem 3.3]. It will be a key result in the proof of both Theorem 9.2 and 10.1.

Theorem 2.16. Let \( f : (X, \Delta) \to Y \) be a klt-trivial fibration between normal projective varieties such that \( \Delta \) is effective over the generic point of \( Y \). Then there exists a diagram

\[
\begin{array}{ccc}
(X, \Delta) & \xymatrix{\ar[r] & (X^+, \Delta^+) \\
Y & \tilde{Y} & Y^+ \\
\ar@{_{(}->}[u] & \ar@{_{(}->}[u]_{\vartheta} & \ar@{_{(}->}[u]^\chi
}
\end{array}
\]

such that:

1. \( f^+ : (X^+, \Delta^+) \to Y^+ \) is a klt-trivial fibration,
2. \( \vartheta \) is generically finite and surjective, and \( \chi \) is surjective,
3. there exists a non-empty open set \( U \subseteq \tilde{Y} \) and an isomorphism
   \[
   (X, \Delta) \times_Y U \cong (X^+, \Delta^+) \times_{Y^+} U
   \]
4. the moduli part \( M_{f^+} \) is big and, after possibly a birational base change, we have \( \vartheta^* M_f = \chi^* M_{f^+} \).

The following remark will be useful at the end of this section.

Lemma 2.17. Notation as in Theorem 2.16. Assume that \( Y \) is an Ambro model and \( M_f \) is semiample, let \( \phi : Y \to V \) be the fibration induced by \( M_f \). Then there is \( \vartheta \) such that \( \text{Exc}(\vartheta) \) is vertical with respect to \( \phi \circ \vartheta \). Moreover there is a generically finite map \( \lambda : Y^+ \to V \).

Proof. Since \( M_f \) is semiample, \( M_{f^+} \) is semiample as well. Let \( \phi^+ : Y^+ \to V^+ \) be the fibration defined by \( M_{f^+} \). We notice that as \( M_{f^+} \) is big, the fibration \( \phi^+ \) is birational. Since \( \phi^+ \circ \chi \) is a fibration, there is a finite map \( V^+ \to V \). We set \( \lambda : Y^+ \to V \) the induced generically finite map.

By the proof of [Amb05, Theorem 2.2], we have \( \vartheta = \varepsilon \circ \sigma \circ p \) where

- \( p : Y' \to Y \) is birational, such that \( Y' \) is smooth and the period map extends to a fibration \( q : Y' \to Y_0 \) and \( p \) can be taken as a composition of blow-ups along smooth centres;
- \( \sigma : Y'' \to Y' \) is finite and such that, if \( \sigma_0 \circ \alpha \) is the Stein factorisation of \( q \circ \sigma \) then \( \alpha : Y'' \to Y^+ \) admits a section;
• $\varepsilon$ is a desingularisation of $Y''$ and $\chi = \alpha \circ \varepsilon$.

We prove first that $p \operatorname{Exc}(p)$ is $\phi$-vertical. Indeed, let $C \subseteq \operatorname{Exc}(p)$ be a curve not contracted by $p$ but contracted by $q$. Let $\tilde{C} \subseteq Y''$ be such that $\sigma(\tilde{C}) = C$. Since $q \circ \sigma = \sigma_0 \circ \alpha$, the image $\alpha(\tilde{C})$ is a point. Then $\sigma \circ p \circ \phi(\tilde{C}) = \lambda \circ \alpha(\tilde{C})$ is a point. Therefore $C$ is contracted by $\phi$. This implies that the indeterminacy locus of $q \circ p^{-1}$ is $\phi$-vertical. Therefore the indeterminacy locus of $q \circ p^{-1}$ is $\phi$-vertical and we can find $p, q$ such that the exceptional locus of $p, q$ is $\phi \circ p$-vertical.

The morphism $\sigma$ is generically étale, therefore the singularities of $Y''$ are vertical with respect to $\alpha$. Therefore we can chose $\varepsilon$ which is an isomorphism over the generic point of $V$. $\square$

The following is [Amb05, Proposition 4.4], and it allows to extend the isomorphism from Theorem 2.16(iii) to a suitable bigger open subset.

**Proposition 2.18.** Let $f : (X, \Delta) \to Y$ be a klt-trivial fibration of normal projective varieties such that there exists an isomorphism 
\[
\Phi : (X, \Delta) \times_Y U \to (F, \Delta_F) \times U
\]
over a non-empty open subset $U \subseteq Y$. Then $\Phi$ extends to an isomorphism over 
\[
Y^0 = Y \setminus (\operatorname{Supp} B_1 \cup \operatorname{Sing}(Y) \cup f(\operatorname{Supp} \Delta_v^{<0})),
\]
where $\Delta_v^{<0}$ consists of the vertical components of $\Delta$ with negative coefficients in $\Delta$.

The following two lemmas were written in collaboration with V. Lazić.

**Lemma 2.19.** Let $S, T, \tilde{T}$ be quasi-projective varieties, assume that $T$ is smooth. Let $h : S \to T$ be a projective fibration and let $\vartheta : \tilde{T} \to T$ be a finite map. Let 
\[
\begin{array}{ccc}
S & \xleftarrow{\tau} & F \times \tilde{T} \\
\downarrow{h} & & \downarrow{\tilde{h}} \\
T & \xrightarrow{\vartheta} & \tilde{T}
\end{array}
\]
be a base change where $\tilde{h}$ is the second projection. Let $G$ be a reduced fibre of $h$. Let $y \in \tilde{T}$ be such that $\tau(y) = x$. Then $\tau : F \times \{y\} \to G$ is an isomorphism.

**Proof.** After cutting the base with $\dim T - 1$ hyperplane sections through $x$, we can assume that $\dim T = 1$.

The morphisms $\vartheta$ and $\tau$ have the same degree, set $d = \deg \tau = \deg \vartheta$. Let $x \in \tilde{T}$ be such that $G = h^*x$. Write $\tau^*G = \sum a_i F \times \{y_i\}$ and $\vartheta^*x = \sum e_i y_i$. Thus $\tau^*h^*x = \tau^*G = \sum a_i F \times \{y_i\} = \tilde{h}^* \vartheta^*x = \tilde{h}^* \sum e_i y_i = \sum e_i F \times \{y_i\}$. It follows, perhaps after renumbering the $y_i$, that $a_i = e_i$ for all $i$. Moreover, $d = \sum a_i \deg(F \times \{p_i\} \to G) = \sum e_i \deg(F \times \{p_i\} \to G) \geq \sum e_i = d$. Thus $\deg(F \times \{p_i\} \to G) = 1$ for all $i$. $\square$
Lemma 2.20. Let \( g: (Z, \Delta_Z) \to T \) be a klt-trivial fibration, where \( \Delta_Z \geq 0 \) and the discriminant \( B_g \) is a reduced divisor. Assume we have a base change diagram

\[
\begin{array}{c}
(Z, \Delta_Z) \\
g \downarrow \\
T
\end{array}
\quad \begin{array}{c}
\cong \\
g \downarrow \\
\cong \\
\alpha \downarrow
\end{array}
\quad \begin{array}{c}
(\overline{Z}, \Delta_{\overline{T}}) \\
\overline{g} \downarrow \\
\overline{T}
\end{array}
\]

where \( \alpha \) and \( \overline{\alpha} \) are finite morphisms and \( \overline{g} \) is weakly semistable in codimension 1. Let \( B_{\overline{g}} \) be the discriminant of \( \overline{g} \) and assume that \( \Delta_{Z,v} = (g^*B_g)_{\text{red}} \). Then there exists an open subset \( U \subseteq \overline{T} \) with complement of codimension at least 2 in \( \overline{T} \) such that:

(i) \( g^* \Delta_{Z,v} = (\overline{\alpha}^* \Delta_{Z,h} + (\overline{\alpha}^* \Delta_{Z,v})_{\text{red}} - (\overline{\alpha}^* R'_{\overline{T}}))_{\overline{g}^{-1}(U)} \), where \( R'_{\overline{T}} \geq 0 \) and \( R'_{\overline{T}} \) and \( \overline{\alpha}^* B_{\overline{T}} \) have no common components.

(ii) \( \Delta_{\overline{Z}} = \overline{\alpha}^* \Delta_Z - R_Z = (\overline{\alpha}^* \Delta_{Z,h} + (\overline{\alpha}^* \Delta_{Z,v})_{\text{red}} - R'_Z) \).

In particular, \( \Delta_{\overline{Z}} \geq 0 \) and if \( T \) is a curve, then \( \Delta_{\overline{Z}} \geq 0 \).

Proof. Step 1. Note that \( B_g \geq 0 \) since \( \Delta_Z \geq 0 \). Let \( R_T \subseteq T \) and \( R_Z \subseteq Z \) be the ramification divisors of the finite maps \( \alpha \) and \( \overline{\alpha} \), respectively. We have

\[ \text{Supp } R_Z \subseteq \overline{g}^{-1}(\text{Supp } R_T) \]

since the base change by an étale map is étale. We can write

\[ R_T = R'_T + \alpha^* B_T - (\alpha^* B_T)_{\text{red}}, \]

where \( R'_T \geq 0 \), and \( R'_T \) and \( \alpha^* B_T \) have no common components. By [Amb04, Lemma 5.1] we have

\[ K_{\overline{T}} + B_{\overline{g}} = \alpha^*(K_T + B_g) \quad \text{and} \quad M_{\overline{g}} = \alpha^* M_g, \]

where \( M_{\overline{g}} \) is the moduli part of \( \overline{g} \). Then [11] gives

\[ B_{\overline{g}} = \alpha^* B_g - R_T = (\alpha^* B_g)_{\text{red}} - R'_T. \]

Similarly, we can write

\[ R_Z = R'_{Z} + \overline{\alpha}^* \Delta_{Z,v} - (\overline{\alpha}^* \Delta_{Z,v})_{\text{red}}, \]

where \( R'_Z \geq 0 \), and \( R'_Z \) and \( \overline{\alpha}^* \Delta_{Z,v} \) have no common components. Then [11] implies

\[ \Delta_{\overline{Z}} = \overline{\alpha}^* \Delta_Z - R_Z = \overline{\alpha}^* \Delta_{Z,h} + \overline{\alpha}^* \Delta_{Z,v} - R_Z = \alpha^* \Delta_{Z,h} + (\overline{\alpha}^* \Delta_{Z,v})_{\text{red}} - R'_Z. \]

We claim that for a prime divisor \( P \subseteq \alpha(\text{Supp } R'_T) \),

\[ g^* P \text{ is reduced over the generic point of } P. \]

Indeed, otherwise we would have \( P \subseteq \text{Supp } B_g \) by the definition of the discriminant. However, this would contradict the fact that \( R'_T \) and \( \alpha^* B_g \) have no common components.
Let $U \subseteq T$ be a big open subset with the following property: $\overline{g}$ is weakly semistable over $U$, and if a prime divisor $D \subseteq \overline{g}^{-1}(\text{Supp } R_T')$ is $\overline{g}$-exceptional, then $\overline{g}(D) \cap U = \emptyset$. We show in Steps 2 and 3 that $U$ is satisfies (i) and (ii).

**Step 2.** To show (i), by (6) it is enough to prove
\begin{equation}
\overline{g}'((\alpha^* B_T)_{\text{red}})|_{\overline{g}^{-1}(U)} = ((\alpha^* \Delta_{Z,v})_{\text{red}})|_{\overline{g}^{-1}(U)}.
\end{equation}
For (10), we have
\begin{equation}
((\alpha^* \Delta_{Z,v})_{\text{red}}) = (\alpha^* g^* B_g)_{\text{red}} = (\overline{g}' \alpha^* g^*)_{\text{red}},
\end{equation}
where the first equality follows by pulling back the relation $\Delta_{Z,v} = (g^* B_g)_{\text{red}}$ by $\alpha$ and taking the reduced part, and the second equality by the base change diagram. Since $\overline{g}'((\alpha^* B_g)_{\text{red}})|_{\overline{g}^{-1}(U)}$ is reduced, we have
\begin{equation}
(\overline{g}' \alpha^* B_g)_{\text{red}}|_{\overline{g}^{-1}(U)} = \overline{g}'((\alpha^* B_g)_{\text{red}})|_{\overline{g}^{-1}(U)},
\end{equation}
which proves (i).

**Step 3.** Finally, we show (ii). By (8), it suffices to show
\begin{equation}
R'_Z|_{\overline{g}^{-1}(U)} = \overline{g}' R'_T|_{\overline{g}^{-1}(U)}.
\end{equation}
By (3), (4), (7) and (10) we have
\begin{equation}
\text{(Supp } R'_Z)|_{\overline{g}^{-1}(U)} \cup (\text{Supp } \overline{\alpha^* \Delta_{Z,v}})|_{\overline{g}^{-1}(U)}
\end{equation}
\begin{equation}
= (\text{Supp } R_Z)|_{\overline{g}^{-1}(U)} \subseteq \overline{g}^{-1}(\text{Supp } R_T|U)
\end{equation}
\begin{equation}
\subseteq \overline{g}^{-1}(\text{Supp } R'_T|U) \cup (\text{Supp } \overline{g}' \alpha^* B_T)|_{\overline{g}^{-1}(U)}
\end{equation}
\begin{equation}
= \overline{g}^{-1}(\text{Supp } R'_T|U) \cup (\text{Supp } \overline{\alpha^* \Delta_{Z,v}})|_{\overline{g}^{-1}(U)}.
\end{equation}
Since $R'_Z$ and $\overline{\alpha^* \Delta_{Z,v}}$ have no common components, this implies
\begin{equation}
(\text{Supp } R'_Z)|_{\overline{g}^{-1}(U)} \subseteq \overline{g}^{-1}(\text{Supp } R'_T|U).
\end{equation}
Therefore, for (11) it is enough to show – by the definition of $U$ – that for each prime divisor $D \subseteq \overline{g}^{-1}(\text{Supp } R'_T)$ such that $\overline{g}(D)$ is a divisor in $T$ we have
\begin{equation}
\text{mult}_D R'_Z = \text{mult}_D \overline{g}' R'_T.
\end{equation}
Fix such a prime divisor $D$. Denote $Q := \overline{g}(D)$ and $P := \alpha(Q)$, and let $e_Q = \text{mult}_Q \alpha^* P$. Then
\begin{equation}
\text{mult}_D \overline{g}' R'_T = \text{mult}_D \overline{g}' (\alpha^* P - (\alpha^* P)_{\text{red}})
\end{equation}
\begin{equation}
= (e_Q - 1) \text{mult}_D \overline{g}' Q = e_Q - 1,
\end{equation}
where the last equality follows since $\overline{g}' Q$ is reduced over the generic point of $Q$ by the assumption on weak semistability. Furthermore, by the commutativity of the base change diagram, we also have
\begin{equation}
\text{mult}_D \overline{\alpha^* g^*} P = \text{mult}_D \overline{g} \alpha^* P = e_Q \text{mult}_D \overline{g} Q = e_Q.
\end{equation}
Since $g^*P$ is reduced over the generic point of $P$ by (13), this shows that the ramification index of $\ov{\alpha}$ along $D$ is $e_Q$, which together with (13) gives
\[
\text{mult}_D R_Z = \text{mult}_D \ov{\alpha} R'_T.
\]
To finish the proof of (12) and of (ii), by (7) we only need to show that $D \not\subseteq \text{Supp} \ov{\alpha}^{*\Delta_Z,e}$. Assume otherwise: then $Q \subseteq \text{Supp} \alpha^{*B_T}$ by (10), hence $Q$ would not be a component of $R'_T$ by the construction of $R'_T$ in Step 1, a contradiction.

**Proposition 2.21.** Let $f : (X, \Delta) \to Y$ be a klt-trivial fibration of normal projective varieties with $X$ $\mathbb{Q}$-factorial. Assume $\Delta$ effective over the generic point of $Y$ and $\Delta - f^*B_f \geq 0$. Assume that $Y$ is an Ambro model and $M_f$ is semiample, let $\phi : Y \to V$ be the fibration induced by $M_f$. Let $Y_r$ be the set of points $x \in Y$ such that $f^{-1}x$ is reduced. Then there are a non empty open set $V_0 \subseteq V$, an open subset $V_0 \subseteq Y$ with complement of codimension at least 2 and a set $I(Y) \supseteq \phi^{-1}V_0 \cap Y \cap Y_r$ with the following property: for every $x_1, x_2 \in I(Y)$ such that $\phi(x_1) = \phi(x_2)$, if $(F_i, \Delta_i)$ is the fibre over $x_i$ with $\Delta_i = \Delta^h|_{F_i}$, then $(F_1, \Delta_1) \cong (F_2, \Delta_2)$.

**Proof.** We apply Theorem 2.16 and find $\vartheta$ and $\chi$ and a diagram such that $\vartheta^*M_f = \chi^*M_f$. In particular both $\vartheta^*M_f$ and $M_f$ are semiample. After passing to the Stein factorisation we can assume that $\chi$ has connected fibres. Let $\bar{X}$ be the main component of the normalisation of $X \times_Y \bar{Y}$ with the natural morphism $\tau : \bar{X} \to X$ and let $\bar{\Delta}$ be defined by $K_{\bar{X}} + \bar{\Delta} = \tau^*(K_X + \Delta)$.

By Theorem 2.16 there is an open set $\bar{U} \subseteq \bar{Y}$ and an isomorphism
\[
(\bar{X}, \bar{\Delta}) \times_{\bar{Y}} \bar{U} \cong (X^+, \Delta^+) \times_{Y^+} \bar{U}.
\]
By Proposition 2.18 the isomorphism extends to
\[
(\bar{X}, \bar{\Delta} - f^*B_f) \times_{\bar{Y}} \bar{V}_0 \to (X^+, \Delta^+) \times_{Y^+} \bar{V}_0
\]
with $\bar{V}_0 = \bar{Y} \setminus f(\text{Supp}(\bar{\Delta}_v - f^*B_f)^{<0})$.

There is a diagram
\[
\begin{array}{ccc}
Y & \overset{\vartheta}{\to} & \bar{Y} \\
\phi \downarrow & & \downarrow \bar{\phi} \\
V & \underset{\sigma}{\leftarrow} & \bar{V}
\end{array}
\]
where $\sigma \circ \bar{\phi}$ is the Stein factorisation of $\phi \circ \theta$.

By Lemma 2.17 there is a generically finite map $\lambda : Y^+ \to \bar{V}$, and it is birational because $\bar{\phi}$ is a fibration.

After passing to an open set $U^+$ of $Y^+$ we can assume that $\lambda$ is an isomorphism and let $V_0 = \lambda(U^+)$.

By Lemma 2.17 and Lemma 2.20 after possibly shrinking $\bar{V}_0$ further, we can assume that the complement of $f(\text{Supp}(\bar{\Delta}_v - f^*B_f)^{<0}) \cap \bar{\phi}^{-1}\bar{V}_0$ has codimension at least 2 in $\bar{\phi}^{-1}\bar{V}_0$. 

\[
\mult_D R_Z = \mult_D \ov{\alpha} R'_T.
\]
Then, for \( \tilde{x}_1, \tilde{x}_2 \in \tilde{\phi}^{-1}\tilde{V}_0 \) in the same fibre of \( \tilde{\phi} \), the two corresponding fibres are isomorphic, together with the boundaries.

Let \( x_1, x_2 \in Y_r \cap \vartheta^{-1}\tilde{\phi}^{-1}\tilde{V}_0 = Y_r \cap \varphi^{-1}\sigma^{-1}\tilde{V}_0 \). If \( \varphi(x_1) = \varphi(x_2) \), then there are \( \tilde{x}_1, \tilde{x}_2 \in \varphi^{-1}\tilde{V}_0 \) such that \( \theta(\tilde{x}_1) = x_1 \) and \( \tilde{\phi}(\tilde{x}_1) = \tilde{\phi}(\tilde{x}_2) \). By Lemma 2.19 the restriction of \( \tau \) to \( \tilde{f}^{-1}(\tilde{x}_i) \) is an isomorphism, concluding the proof. \( \square \)

We also need the following [Amb05, Theorem 3.5]; see also [Flo14, Theorem 1.2] for a sharper version.

**Theorem 2.22.** Let \( f : (X, \Delta) \to Y \) be a klt-trivial fibration, and assume that the moduli part \( M_Y \) descends to \( Y \). If \( M_Y \equiv 0 \), then \( M_Y \sim_{\mathbb{Q}} 0 \).

### 2.4. Variation of a klt-trivial fibration

In this section we give the definition and some properties of the variation of a fibration. For the original definition with \( \Delta = 0 \) and some further discussion of the properties see [Vie83, Kol87, Fuj03].

**Definition 2.23.** Let \( (X, \Delta) \) be a pair and let \( f : (X, \Delta) \to Y \) be a fibration. We define the variation of \( f \), denoted by \( \text{Var}(f) \) as

\[
\min \left\{ \dim Y^+ \mid \begin{array}{l}
\exists \vartheta : \tilde{Y} \to Y \text{ generically finite} \\
\exists \chi : \tilde{Y} \to Y^+, f^+ : (X^+, \Delta^+) \to Y^+ \text{ fibrations} \\
such that the fibration induced by } f, \vartheta \text{ by fibre} \\
\text{ product and the fibration induced by } f, \chi \text{ by fibre} \\
\text{ product are birational over } \tilde{Y}.
\end{array} \right\}
\]

The following is a generalisation of [Fuj03, Theorem 3.8] to the case \( \Delta \neq 0 \), the proof is essentially the same.

**Proposition 2.24.** Let \( f : (X, \Delta) \to Y \) be an lc-trivial fibration, assume \( Y \) is an Ambro model. Then \( \kappa(M_f) \leq \text{Var}(f) \).

**Proof.** Thet \( \vartheta, \chi \) be such that \( \text{Var}(f) = \text{transdeg}_k(Y^+) \). Then, after perhaps passing to higher models of \( \tilde{Y} \) and \( Y^+ \), we have \( \theta^*M_f = \chi^*M_{f^+} \). Therefore \( \kappa(M_f) = \kappa(M_{f^+}) \leq \dim Y^+ = \text{Var}(f) \).

**Proposition 2.25.** Let \( (X, \Delta) \) be a pair with \( X \) \( \mathbb{Q} \)-factorial and \( \Delta \geq 0 \) and with coefficients in \( \mathbb{Q} \). Let \( f : (X, \Delta) \to Y \) be a fibration such that \( \text{Var}(f) = \dim Y \). Then there is a countable union \( E \) of closed subsets of \( Y \) and an open set \( U \subseteq Y \) such that for every \( y \in Y \setminus E \) the set

\[
\{ z \in U \mid (f^{-1}y, \Delta^h_{|f^{-1}y}) \cong (f^{-1}z, \Delta^h_{|f^{-1}z}) \}
\]

is a finite set.

**Proof.** Set \( F = f^{-1}y \). We fix a polarisation \( A = p_F^*A_F + p_X^*A_X \) on \( F \times X \), where \( p_F \) and \( p_X \) are the two projections. Then there is a quasi projective scheme \( \text{Emb}(F, X) \subseteq \text{Hilb}(F \times X) \) representing the functor \( \text{Hilb}_p(F \times X) \)
where $P$ is the Hilbert polynomial of the graph of $F \to f^{-1}y \subseteq X$. There is also a universal family $u: \text{Univ}(F, X) \to \text{Emb}(F, X)$ and a diagram

$$
\begin{array}{ccc}
\text{Univ}(F, X) & \xrightarrow{u} & F \times X \times \text{Emb}(F, X) \\
& \downarrow & \downarrow f \\
\text{Emb}(F, X) & \rightarrow & Y.
\end{array}
$$

After perhaps replacing the polarisation $A$ with $A + p_X^* f^* A_Y$ for a sufficiently ample divisor $A_Y$ on $Y$, all the fibres of $u$ are contracted by $f \circ p_X$. By the rigidity lemma there is $\phi: \text{Emb}(F, X) \to Y$. Its image is $\{ z \in Y | f^{-1}z \cong F \}$.

Let $k \in \mathbb{N}$ be such that $k\Delta^h$ is a Cartier divisor. Set $D = k\Delta^h|_F$. Then there is a locally closed subscheme $\text{Emb}((F, D), (X, D))$ of $\text{Emb}(F, X)$ representing the functor $\mathcal{E}mb((F, D), (X, D))$ with

$$
\mathcal{E}mb((F, D), (X, D))(Z) = \left\{ \begin{array}{l}
\text{$Z$-morphisms } \varphi: F \to X, \psi: D \to D \\
\text{flat over $Z$} \\
\text{such that } \varphi \circ i = j \circ \psi, \varphi \text{ embedding}
\end{array} \right\}
$$

together with a universal family

$$
u: \text{Univ}((F, D), (X, D)) \to \text{Emb}((F, D), (X, D)).
$$

Therefore

$$
\{ z \in U | (f^{-1}y, \Delta^h|_{f^{-1}y}) \cong (f^{-1}z, \Delta^h|_{f^{-1}z}) \} = \phi(\text{Emb}((F, D), (X, D)))
$$
is the image of an algebraic set. By [Kol87, Theorem 2.6] for $y$ in the complement of a countable union of closed sets in $Y$ the left hand side is at most countable. Therefore it is a finite set. \hfill \Box

3. Semiample line bundles on simple normal crossings surfaces

In this section we establish a criterion of semiampleness of certain line bundles on simple normal crossings surfaces. For later use and for this section we introduce different notions of simple normal crossings varieties

Let $Z$ be a variety with irreducible components $\{ Z_i : i \in I \}$. Assume that $\dim Z_i = k$ for every $i \in I$. We say that $Z$ is a simple normal crossing variety [Kol14, Definition 6] if the $Z_i$ are smooth and every point $p \in Z$ has an open (Euclidean) neighborhood $p \in U_p \subseteq Z$ and an embedding $U_p \to \mathbb{C}^{k+1}$ such that the image of $U_p$ is an open subset of the union of coordinate hyperplanes $(z_1 \cdot \ldots \cdot z_n = 0)$ with $n \leq k + 1$. A stratum of $Z$ is any irreducible component of an intersection $\cap_{i \in J} Z_i$ for some $J \subseteq I$.

Assume now $Z = \cup_k Z^{(k)}$ where $Z^{(k)}$ is the union of irreducible components of dimension $k$. We say that $Z$ is a simple normal crossing variety if $Z^{(k)}$ is simple normal crossings in the above sense for every $k$ and for every stratum $Z$ of $Z^{k-1} = \cup_{j<k} Z^{(j)}$ we have that $Z^{(k)}$ does not contain $Z$ and every point $p \in Z^{(k)} \cap Z$ has an open (Euclidean) neighborhood
$p \in U_p \subseteq \mathbb{Z}^{(k)} \cup \mathbb{Z}$ and an embedding $U_p \rightarrow \mathbb{C}^{k+1}$ such that the image of $U_p$ is an open subset of the union of coordinate hyperplanes $(z_1 \ldots z_n = 0)$ and the image of $\mathbb{Z}$ is an open subset of $(z_{n+1} = \ldots = z_m = 0)$ with $n < m \leq k+1$ [Kol07b, Definition 3.24].

Finally, a curve is said to be seminormal if every point has a neighbourhood if and only if it is analytically isomorphic to the union of the $n$ coordinate axes in $\mathbb{A}^n$ [Kol13, Example 10.12].

Remark 3.1. If $\mathbb{Z} \subseteq \mathbb{Y}$ has pure codimension 1 and is a simple normal crossings variety, then it is a simple normal crossings divisor.

If $\mathbb{Z}$ is a connected simple normal crossings variety of pure dimension 1, then it is a semistable curve.

A divisor on a simple normal crossings variety can be recovered from its restrictions to its irreducible components plus a gluing condition on the intersections:

Let $\mathbb{Z}$ be a simple normal crossings variety of pure dimension $k$. A divisor on $\mathbb{Z}$ is the data of a divisor $D_Z$ on every irreducible component $Z$ of $\mathbb{Z}$ with the property that, if $Z_1$ and $Z_2$ are two irreducible components of $\mathbb{Z}$, then $D_{Z_1}|_{Z_1 \cap Z_2} = D_{Z_2}|_{Z_1 \cap Z_2}$.

From now on, we assume that $\mathbb{S}$ is a simple normal crossings variety of pure dimension 2. We refer to $\mathbb{S}$ as a simple normal crossings surface.

Lemma 3.2. Let $\mathbb{S}$ be a connected simple normal crossings surface. Assume that there is an integral curve $Q$ and a surjective morphism with connected fibres $\varphi: \mathbb{S} \rightarrow Q$, that for every $S \subseteq \mathbb{S}$ irreducible component $\varphi(S)$ is an irreducible curve. For an irreducible component $S$ of $\mathbb{S}$, we denote by $\varphi|_{S}: S \rightarrow C(S) \rightarrow Q$ the Stein factorisation. Let $D$ be an effective divisor on $\mathbb{S}$ such that $\varphi(\text{Supp } D) \subseteq Q^\text{smooth}$. Then there is a positive integer $m$ such that $mD$ is the pullback of a Cartier divisor in $Q$ if and only if for every irreducible component $S$ of $\mathbb{S}$ there is a positive integer $d$ such that the restriction of $dD$ to $S$ is the pullback of a divisor in $C(S)$.

Proof. If there is a positive integer $m$ such that $mD$ is the pullback of a divisor in $Q$, then the statement on the restrictions of $D$ to the irreducible components of $\mathbb{S}$ is obvious.

We assume now that for every irreducible component $S$ of $\mathbb{S}$ there is a positive integer $d$ such that the restriction of $dD$ to $S$ is the pullback of a divisor in $C$.

By hypothesis, there are $p_1, \ldots, p_k$ in the smooth locus of $Q$ such that the support of $D$ is contained in $\varphi^{-1}\{p_1, \ldots, p_k\}$. We prove the statement by induction on $k$. If $k = 0$, there is nothing to prove. Assume now that the statement holds for $k - 1$. Let $Q_1$ be the irreducible component of $Q$ such that $p_1 \in Q_1$. Let $S$ be an irreducible component of $\mathbb{S}$ such that $\varphi(S) = Q_1$. We set $D|_S = \sum_{\ell} \sum_j a_{\ell,j} F_{\ell,j}$ where for every $\ell$ the union $\bigcup_j F_{\ell,j}$ is a connected component of $\text{Supp } D|_S$. Without loss of generality, we can
assume that $\cup_j F_{1,j}$ is contained in $\varphi^{-1}p_1$. Let $\alpha$ be such that $\varphi^*(\alpha p_1)|_S = \sum_j a_{1,j}F_{1,j} + \sum_h \sum_j b_{h,j}F_{h,j}$. We want to prove that $\text{Supp} D - \varphi^*(\alpha p_1) \subseteq \varphi^{-1}\{p_2, \ldots, p_k\}$.

Assume that this is not the case, that is, assume that there is $S'$ and an irreducible component $F$ of $\varphi^{-1}p_1 \cap S'$ such that $\text{coeff}_F(D - \varphi^*(\alpha p_1))$ is not zero. The fibre $\varphi^{-1}p_1$ is connected, thus there are $S = S_0, S_1, \ldots, S_N = S'$ and for every $i$ a subvariety $\cup_j F^i_j$ of $S_i$ and a point $q_i \in S_i$ with the following properties:

- $\cup_j F^i_j$ is the support of a fibre of $f_{S_i}$;
- $\cup_j F^0_j = \cup_j F_{1,j}$;
- $F \subseteq \cup_j F^N_j$;
- $q_i \in (\cup_j F^i_j) \cap (\cup_j F^{i+1}_j)$.

We have that for every $j$

$$\text{coeff}_{F^0_j}(D - \varphi^*(\alpha p_1)) = 0 \quad \text{and} \quad \text{coeff}_{F^N_j}(D - \varphi^*(\alpha p_1)) \neq 0.$$  

Then there is $i$ such that for every $j$

$$\text{coeff}_{F^i_j}(D - \varphi^*(\alpha p_1)) = 0 \quad \text{and} \quad \text{coeff}_{F^{i+1}_j}(D - \varphi^*(\alpha p_1)) \neq 0.$$  

This is a contradiction as $D - \varphi^*(\alpha p_1)$ is a divisor on $S$ and $D - \varphi^*(\alpha p_1)|_{S_i}$ but $D - \varphi^*(\alpha p_1)|_{S_{i+1}}$ do not coincide on the intersection $S_i \cap S_{i+1}$. \hfill \Box

**Lemma 3.3.** Let $S$ be a connected simple normal crossings surface. Assume that there is a seminormal curve $Q$ and a surjective morphism with connected fibres $\varphi: S \rightarrow Q$ and that for every $S \subseteq S$ irreducible component $\varphi(S)$ is an irreducible curve. Let $L$ be a line bundle on $S$ such that for every fibre $F$ of $\varphi$ the restriction $L_{|F}^{(p)}$ has a nowhere vanishing section. Then there is a positive integer $m$ and a line bundle $M$ on $Q$ such that $L^{\otimes m} \sim \varphi^*M$.

**Proof.** Let $Q = \cup_{i} Q_i^\nu$ be the decomposition of $Q$ into irreducible components. Let $S_i$ be the union of the irreducible components $S$ of $S$ such that $\varphi(S) = Q_i^\nu$ and let $\varphi: S_i \overset{f_i}{\rightarrow} Q_i^\nu \overset{\nu_i}{\rightarrow} Q_i$ be the Stein factorisation. The morphism $\nu_i$ is birational and finite. We prove that $Q_i$ is normal. Indeed for every irreducible component $S_{i,j}$ of $S_i$ the restriction of $f_i$ to $S_{i,j}$ factors through the normalisation $Q_i^\nu$ of $Q_i$ and there is $f_{i,j}: S_{i,j} \rightarrow Q_i^\nu$. As $S_i$ has simple normal crossings, the restriction of $f_i$ to $S_{i,h} \cap S_{i,k}$ factors through $Q_i^\nu$. Thus, if and if $x \in S_{i,h} \cap S_{i,k}$, we have $f_{i,h}(x) = f_{i,k}(x)$ and there is a morphism $f_i': S_i \rightarrow Q_i^\nu$. By the uniqueness of the Stein factorisation $Q_i = Q_i^\nu$.

Since it is a curve, it is a smooth projective curve and the morphism $f_i$ is flat.

The sheaf $f_{is}(L^\nu|_{S_i})$ has generically rank 1 on $Q_i$ and, by semicontinuity, all its stalks are non zero. Let $A_i$ be an ample line bundle on $Q_i$. After possibly replacing $A_i$ with a multiple, we can assume that $f_{is}(L^\nu) \otimes A_i$ is globally generated and has therefore a non-zero global section. Moreover,
by the projection formula, as $A_i$ is locally free, we have

$$H^0(S_i, \mathcal{L}^\vee \otimes f_i^*A_i) = H^0(Q_i, f_i*(\mathcal{L}^\vee \otimes f_i^*A_i)) = H^0(Q_i, f_i*(\mathcal{L}^\vee \otimes A_i)).$$

Then there is a non-zero global section $s \in H^0(S_i, \mathcal{L}^\vee \otimes f_i^*A_i)$ inducing an isomorphism of line bundles $\mathcal{L}^\vee \otimes f_i^*A_i \sim \mathcal{O}(D_i)$ with $D_i$ an effective Cartier divisor on $S_i$. On the general fibre the morphism $\mathcal{L} \to f_i^*A_i$ is an isomorphism, therefore $D_i$ is supported on fibres of $f_i$.

For an irreducible component $S$ of $S_1$, we denote by $f_i|_S : S \xrightarrow{f_i} C \to Q_i$ the Stein factorisation. By Zariski’s lemma [BPVdV84, Lemma 8.2], for every irreducible component $S$ of $S_1$, the restriction $D|_S$ is proportional to fibres of $f_S$. By Lemma 3.2, the divisor $D_i$ is proportional to fibres of $f_i$ and $\mathcal{O}(D_i) \sim_{Q_i} f_i^*\delta_i$ with $\delta_i \geq 0$. After tensoring $A_i$ with a higher multiple, we can assume that $\text{Supp} \delta_i \subseteq \nu_i^{-1}Q_{\text{reg}}$.

For $p \notin Q_{\text{reg}}$ let $F_p = \varphi^{-1}p$. We notice that $F_p$ is a semistable curve. As $Q$ is seminormal, there are $A_i$ and $\delta$ on $Q$ such that $A|_{Q_i} = A_i$ and $\delta|_{Q_i} = \delta_i$ for every $i$ such that $f_i^*A_i|_{F_p} = \mathcal{L}|_{F_p} = f_i^*\delta|_{F_p}$ for every $p \notin Q_{\text{reg}}$.

It follows that $\mathcal{L} \sim \varphi^*A(-\delta)$. \hfill $\square$

**Remark 3.4.** To prove that $Q_i$ is normal we could also have argued in the following way. Let $\sqcup V_{i,j}$ be the normalisation of $S_i$ and let for every $i,j$ be $f_i : S_{i,j} \to V_{i,j}$ the Stein factorisation of $\varphi_{S_{i,j}}$. Then $Q_i$ is the quotient of $\sqcup V_{i,j}$ by the relation, for $x \in V_{ih}$ and $y \in V_{ik}$, $x \sim y$ if and only if $\sigma_{ih}(x) = \sigma_{ik}(y)$. This equivalence relation is finite, equidimensional and $\sqcup V_{i,j}$ is normal. By [Kol13, Proposition 9.14] the curve $Q_i$ is normal as well.

**Theorem 3.5.** Let $S$ be a connected simple normal crossings surface. Assume that there is an integral seminormal curve $Q$ and a surjective morphism with connected fibres $\varphi : S \to Q$. Let $\mathcal{L}$ be a line bundle on $S$ such that for every $S \subseteq S$ the restriction $\mathcal{L}|_S$ is semialpamle and the Stein factorisation of $\varphi|_S$ is the morphism induced by $\mathcal{L}|_S$. Assume that for every fibre $F$ of $\varphi$ the restricted line bundle $\mathcal{L}|_{(F)_{\text{red}}}$ has a non-zero section. Then $\mathcal{L}$ is semialpamle.

**Proof.** Let $x \in S$ be a point. We want to prove that there is a global section of $\mathcal{L}$ non zero along $x$.

We write $S = S_0 \cup S_1$ where

$$S_0 = \{ S | \mathcal{L}|_S \text{ has Kodaira dimension } 0 \}$$

$$S_1 = \{ S | \mathcal{L}|_S \text{ has Kodaira dimension } 1 \}$$

Let $\varphi : S_1 \xrightarrow{L} Q' \xrightarrow{\nu} Q_i$ be the Stein factorisation. The morphism $\nu$ is birational and finite. By Lemma 3.3 there is a positive integer $m$ and a line bundle $\mathcal{M}$ on $Q'$ such that $\mathcal{L} \otimes^m \sim f^*\mathcal{M}$. The line bundle $\mathcal{M}$ is ample on $Q'$. After maybe taking a multiple of $m$, there is a global section $s$ of $\mathcal{M}$ which is non zero on every irreducible component of $Q'$, such that if $\nu(p_1) = \nu(p_2)$ then $s(p_1) = s(p_2)$ and such that $s(\varphi(x)) \neq 0$. 


Set $\varphi(S_0) = \{q_1, \ldots, q_k\}$ and $F_i = \varphi^{-1}q_i$ taken with the reduced structure. Thus for every $i$ we chose a global section $s_i$ of $L|_{F_i}$ agreeing with $\varphi^*s$ on $F_i \cap S_1$. Thus the data $s_1, \ldots, s_k, f^*s$ define a global section of $L$ which does not vanish on $x$.

\[\square\]

4. Profinite equivalence relations

Let $X$ be a scheme. A relation on $X$ is the data of a scheme $S$ and an embedding $\sigma: S \to X \times X$ [Kol13, Definition 9.1]. It is finite if the projections $\sigma_i: S \to X$ are finite for $i = 1, 2$. A set theoretic equivalence relation, or equivalence relation for short, is a relation such that $\sigma$ is geometrically injective, $S$ contains the diagonal (reflexive), is invariant by the involution of $X \times X$ exchanging the two factors (symmetric) and is transitive, that is, if we consider

$$S \times_X S \longrightarrow S \quad \begin{array}{c}\sigma_2 \\
\sigma_1 \end{array} \quad \begin{array}{c}
S \longrightarrow \\
X \end{array}$$

then there is a natural morphism $\sigma: S \times_X S \to X \times X \times X$ and $(\pi_1, \pi_2) \circ \sigma (\text{red}(S \times_X S)) \to X \times X$ factors through $S$ [Kol13, Definition 9.2].

**Remark 4.1.** If $\mathcal{R}$ is a finite equivalence relation on an algebraic variety (not necessarily irreducible) and $Z \subset X$ a subvariety then $SZ = \{z \in X | \text{there is } z' \in Z \text{ with } (z, z') \in \mathcal{R}\}$ is a finite union of subvarieties of $X$. Indeed, we have $\mathcal{R}Z = \sigma_2\sigma_1^{-1}Z$.

**Definition 4.2.** Let $\mathcal{R}$ be an equivalence relation on $X$. A subset $Z \subset X$ is invariant by $\mathcal{R}$ if one of the following equivalent condition is verified:

- for every $x \in X$, if there is $z \in Z$ which is equivalent to $x$, then $x \in Z$;
- for every $x \in X$, if there is $z \in Z$ such that $(x, z) \in \mathcal{R}$ then $x \in Z$;
- $\sigma_2\sigma_1^{-1}Z \subseteq Z$;
- $\sigma_1\sigma_2^{-1}Z \subseteq Z$.

**Construction 4.3.** (Equivalence closure) The equivalence closure $\langle S \rangle$ of a relation $S$ is the smallest equivalence relation containing it. We refer to [Kol13, 9.3] for the complete construction, which consists in making $S$ reflexive, symmetric and transitive. We recall just that if $S_1, S_2 \subseteq S$ are irreducible components, then in order to make $S$ transitive, we “add” to $S$ the variety $S_3 = (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S_1 \times_X S_2)$, where $\pi_i$ are the natural
projections

\[ S_1 \times_X S_2 \]

An equivalence relation is called \textit{profinite} if it is the closure of a finite relation.

**Notation 4.4.** We denote by \( S_K \) (resp. \( S_{\leq K} \)) the union of all the irreducible components of \( S \) of dimension \( K \) (resp. \( \leq K \)).

**Lemma 4.5.** Let \( X \) be a normal variety of maximal dimension \( D \). Notation as in Construction 4.3. Then

1. \( \dim S_3 \leq \min \{ \dim S_1, \dim S_2 \} \);
2. if \( S_1, S_2 \subseteq S_D \) and \( \sigma_2(S_1) = \sigma_1(S_2) \), then every component of \( S_3 \) has dimension \( \dim S_1 = \dim S_2 \);
3. \( \sigma_1(S_3) \subseteq \sigma_1(S_1), \sigma_2(S_3) \subseteq \sigma_2(S_2) \).

**Proof.** Both \( \pi_1 \) and \( \pi_2 \) are finite morphisms as they are the base change of \( \sigma_1 \) and \( \sigma_2 \) respectively, which are finite. Thus

\[ \dim S_1 \times_X S_2 \leq \min \{ \dim S_1, \dim S_2 \}. \]

If \( \sigma_2(S_1) = \sigma_1(S_2) \), then \( \dim S_1 = \dim S_2 \) because \( \sigma_i \) is finite for \( i = 1, 2 \).

Assume that \( S_1, S_2 \) have dimension \( D \). Then their image \( X_1 \) in \( X \) is an irreducible component, and therefore normal. By Chevalley’s criterion \( \text{[Gro67, 14.4.4]} \) and \( \text{[Kol13, Definition 1.44]} \) the morphism \( \sigma_2: S_1 \rightarrow X_1 \) is universally open. Then \( \sigma_2': S_1 \times_{X_1} S_2 \rightarrow S_1 \) is open and finite.

Let \( S_1 \times_{X_1} S_2 = W_1 \cup \ldots \cup W_\ell \) be the decomposition into irreducible components. Then \( U_1 = W_1 \cap (W_2 \cup \ldots \cup W_\ell) \) is open in \( S_1 \times_{X_1} S_2 \) and its image \( \sigma_2'(U_1) \) is open in \( S_2 \). Thus \( \dim W_1 = \dim U_1 = \dim \sigma_2'(U_1) = \dim S_2 \).

Since \( \pi_i \) and \( \sigma_i \) are finite for \( i = 1, 2 \), \( (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2) \) is finite as well, proving (1) and (2).

As for (3), we have \( \sigma_i(S_3) = \sigma_i((\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)S_1 \times_X S_2) = (\sigma_i \circ \pi_i)(S_1 \times_X S_2) \subseteq \sigma_i(S_i) \).

\[ \square \]

**Remark 4.6.** We denote by \( \Delta_X \) the diagonal of \( X \times X \) By Lemma 4.5 if \( D = \dim (S \setminus \Delta_X) \), then \( \langle S_D \rangle_D = \langle S \rangle_D \).

The following lemma is a slight generalisation of \( \text{[ByB04, 2.7]} \). It is a consequence of Lemma 4.5(2) for which we followed closely the proof of \( \text{[Kol13, Lemma 9.14]} \).
Lemma 4.7. Let $X$ be a normal variety of dimension $D$. Let $S$ be a finite relation on $X$. Then $\langle S_D \rangle = \langle S \rangle_D$. In particular $\langle S \rangle_D$ is an equivalence relation.

Proof. By Remark 4.3 it is enough to prove that $\langle S_D \rangle_D = \langle S \rangle_D$. One inclusion is obvious. For the other one, if $S_1, S_2$ are two components of $S_D$, since $X$ is normal and $D = \dim X$, then either $\sigma_1(S_1) = \sigma_2(S_2)$ or $\sigma_1(S_1) \cap \sigma_2(S_2) = \emptyset$. Therefore by Lemma 4.5(2), every irreducible component of $S_1 \times X S_2$ and of its projection in $X \times X$ has dimension $D$. 

\[
\square
\]

Lemma 4.8. Let $X$ be a normal variety of dimension $D$. Let $S$ be a finite reflexive and symmetric relation on $X$ and $R$ the equivalence closure of $S$. Assume that $R_D$ is finite. Then $X_1 = R_D(\sigma_1(S_{\leq D-1}) \cup \sigma_2(S_{\leq D-1}))$ is $R$-invariant.

Proof. The set $X_1$ is $R_D$-invariant. It is enough to prove that $\sigma_j(R_{\leq D-1}) \subseteq X_1$ for $j = 1, 2$. Set $S^i = (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)S^{i-1} \times_X S^{i-1}$. By [Kol13, 9.3], the equivalence closure of $S$ is $R = \cup S^i$. We will prove by induction on $i$ that $\sigma_j((S^i)_{\leq D-1}) \subseteq X_1$ for $j = 1, 2$.

We have

\[
S^i = (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1} \times_X S^{i-1})
\]

\[
= (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1} \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_D \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_{\leq D-1})
\]

By Lemma 4.5(2) we have $(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_D) \subseteq R_D$ and by Lemma 4.5(1)

\[
(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1} \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_D \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_{\leq D-1}) \subseteq R_{\leq D-1}.
\]

Therefore

\[
S^i_{\leq D-1} = (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1} \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_D \cup S^{i-1}_{\leq D-1} \times_X S^{i-1}_{\leq D-1})
\]

By Lemma 4.5(3) we have $\sigma_2(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1}) \subseteq \sigma_2 S^i_{\leq D-1}$ and by induction $\sigma_2 S^i_{\leq D-1} \subseteq X_1$, proving

\[
(14) \quad \sigma_2(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1}) \subseteq X_1.
\]

As for $\sigma_1(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1})$, we have

\[
\sigma_1(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1}) \subseteq \sigma_1(\sigma_2|_{R_D})^{-1} \sigma_1 S^{i-1}_{\leq D-1}
\]

\[
\subseteq \sigma_1(\sigma_2|_{R_D})^{-1} X_1 \subseteq X_1
\]

where the first inclusion is because

\[
(\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2)(S^{i-1}_D \times_X S^{i-1}_{\leq D-1}) = \{(x, y) \mid \exists z \in X, (x, z) \in S^{i-1}_D, (z, y) \in S^{i-1}_{\leq D-1}\}
\]

\[
\subseteq \{(x, y) \mid \exists z \in \sigma_1(S^{i-1}_{\leq D-1}), (x, z) \in R_D\}
\]
and the image via $\sigma_1$ of the last set coincides with $\sigma_1 (\sigma_2 |_{R_D})^{-1} \sigma_1 S_{\leq D-1}^{i-1}$. The second inclusion follows by induction and the third because $X_1$ is $R_D$-invariant.

A very similar proof implies that $\sigma_j (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2) S_{\leq D-1}^{i-1} \times X S_{\leq D-1}^{i-1} \subseteq X_1$ for $j = 1, 2$.

Again by Lemma 4.8, we have $\sigma_j (\sigma_1 \circ \pi_1, \sigma_2 \circ \pi_2) (S_{\leq D-1}^{i-1} \times X S_{\leq D-1}^{i-1}) \subseteq \sigma_2 S_{\leq D-1}^{i-1}$ for $j = 1, 2$ and $\sigma_2 S_{\leq D-1}^{i-1} \subseteq X_1$ by induction. □

**Definition 4.9.** Let $S \to X \times X$ be a finite relation and $g: \tilde{X} \to X$ a finite morphism. The pullback of $S$ by $g$ is $g^* S = S \times_{X \times X} \tilde{X} \times \tilde{X}$.

**Lemma 4.10.** Let $S$ be a finite relation on a variety $X$ and let $D = \dim S \setminus \Delta_X$. Let $g: \tilde{X} \to X$ be a finite surjective morphism. If $\langle S_D \rangle_D$ is infinite, then $\langle g^* S_D \rangle_D$ is infinite.

**Proof.** For every $D$-dimensional component $S$ of $\langle S_D \rangle_D$ the pull back $\tilde{S}$ in $\tilde{X} \times \tilde{X}$ has dimension $D$. □

**Definition 4.11.** A profinite equivalence relation $R$ on an equidimensional variety $X$ is equidimensional if every irreducible component of $R$ projects onto a connected component of $X$.

The definition coincides with what is called wide in [ByB04, Definition 2.1].

**Proposition 4.12.** Let $S$ be a finite relation on a normal variety $X$, let $R$ be the equivalence closure of $S$. If $R$ is not finite then there are

1. a subrelation $R' \subseteq R$
2. $Z_1, \ldots, Z_k$ subvarieties of $X$

such that $\bigcup Z_i$ is $R'$-invariant, $R'|_{\bigcup Z_i}$ is an infinite equidimensional relation and the set of infinite equivalence classes is dense in $\bigcup Z_i$.

**Proof.** We prove the statement by induction on $D = \dim X$. If $R_D$ is not finite, we let $Z_i$ be the irreducible components of $X$ of dimension $D$ which are dominated by infinitely many components of $R_D$ and we set $R' = R_D$. We assume now that $R_D$ is finite. We set $X_1 = R_D (\sigma_1 (S_{\leq D-1}) \cup \sigma_2 (S_{\leq D-1}))$. By Lemma 4.8, the subvariety $X_1$ is $R$-invariant.

By Lemma 4.10, the pullback of the restriction of $R$ to $X_1$ via the normalisation of $X_1$ is not finite. We conclude by induction as the dimension of the normalisation of $X_1$ is at most $D - 1$. □

### 5. Gluing bases of fibrations

Throughout this section, $\mathcal{L}$ will be a line bundle with the property that $\mathcal{L}|_T$ is semiample for every irreducible component $T \subseteq T$. For every $T$ we denote by $\phi_T: T \to V$ the fibration induced by a multiple of $\mathcal{L}$. 
Definition 5.1. The equivalence relation $R_L$ on the set $\bigcup_{T \in \mathcal{T}} V$ is the closure of the relation

$$x_1 \sim x_2 \iff \exists T, T' \subseteq \mathcal{T}, \exists y \in T \cap T' \quad \phi_T(y) = x_1, \phi_{T'}(y) = x_2.$$ 

Remark 5.2. Assume that $\mathcal{T}$ is a simple normal crossing divisor. Let $\nu: \sqcup T \to \mathcal{T}$ be the normalisation. Let $\Xi^n$ be the normalisation of the non-normal locus of $\mathcal{T}$. Then there is an involution $(\zeta_1, \zeta_2): \Xi^n \to \sqcup T \times \sqcup T$. Let $\nu: \Xi^n \to \sqcup W$ be the fibration induced by $L|_{\Xi}$. The morphism $(\zeta_1, \zeta_2)$ induces a morphism $(\xi_1, \xi_2): \sqcup W \to \sqcup V$. Then the equivalence relation $(\xi_1, \xi_2): \sqcup W \to \sqcup V$ coincides with $R_L$.

Notation 5.3. Let $\nu: \sqcup T \to T$ be the normalisation. For a subset $S \subseteq \sqcup V$ we will denote by $\phi^{-1}[S]$ the set $\nu(\sqcup \phi^{-1}T(S \cap V))$.

Remark 5.4. If the line bundle $L$ restricted to $T$ is base point free, then the relation $\sim$ is finite and $L$ induces a morphism $\phi: T \to (\bigcup_{T \in \mathcal{T}} V)/R_L$.

Definition 5.5. Let $\mathcal{T} \subseteq Y$ be a divisor and let $L$ be a line bundle such that $L|_{T}$ is base point free for every $T \subseteq \mathcal{T}$ for every irreducible component. Let $\phi_T: T \to V$ be the morphism induced by $L$. For an equivalence class $[x]$ of $R_L$ we set the pseudo-fibre as

$$\mathcal{T}[x] = \sqcup_{x' \in [x]} \phi^{-1}_T(x') = \phi^{-1}[x].$$

Remark 5.6. The relation $R_L$ is finite if and only if $\mathcal{T}[x]$ is an algebraic variety for every $[x]$. Indeed $R_L$ is finite if and only if $[x]$ is a finite set for every $x$.

Proposition 5.7. Let $Y$ be a normal variety and let $\mathcal{T} \subseteq Y$ be a divisor. Let $L$ be a line bundle on $Y$ which is semiample on the irreducible components of $\mathcal{T}$. Let $\tau: \overline{Y} \to Y$ be a finite map and $\overline{\mathcal{T}} = \tau^{-1}\mathcal{T}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\bigcup_{T \in \mathcal{T}} \mathcal{T} & \xrightarrow{\tau} & \bigcup_{T \in \mathcal{T}} T \\
(\phi_T) \downarrow & & \downarrow (\phi_T) \\
\bigcup_{V} & \xrightarrow{\sigma} & \bigcup V
\end{array}
$$

with $\sigma$ a finite map. Moreover for every $x \in \sqcup V$ we have $\sigma^{-1}[x] = \sqcup_{\sigma(x) = x} [x]$. In particular $R_{\tau^*L} = \sigma^*R_L$.

Proof. Assume that $\tau(T) = T$. There is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\tau} & T \\
\phi_T \downarrow & & \downarrow \phi_T \\
\overline{V} & \xrightarrow{\sigma_V} & \overline{V}
\end{array}
$$

where $\sigma_V \circ \phi_T = \phi_T \circ \tau$. And the $\sigma_V$ define a finite map $\sigma: \sqcup \overline{V} \to \sqcup \overline{V}$. 
Let $\bar{x}, \bar{x}' \in \bigcup V$ such that $\bar{x} \sim \bar{x}'$. Then there is $\bar{y} \in T \cap \bar{T}$ such that $\phi_T (\bar{y}) = \bar{x}$ and $\phi_{\bar{T}} (\bar{y}) = \bar{x}'$. By the commutativity of the diagram $\phi_T (\tau \bar{y}) = \sigma(\bar{x})$ and $\phi_{\bar{T}} (\tau \bar{y}) = \sigma(\bar{x}')$. Therefore $\sigma[\bar{x}] \subseteq [\sigma(\bar{x})]$.

On the other hand let $\bar{x} \in \sigma^{-1}[x]$. We want to prove that $\bar{x}$ is equivalent to a point in $\sigma^{-1}x$. The point $\sigma \bar{x}$ is equivalent to $x$. Therefore there are $\sigma \bar{x} \sim x_1 \sim \ldots \sim x_k = x$. We prove our statement by induction on $k$. If $k = 1$, the statement is obvious. We assume from now on that $k > 1$. Then $\sigma \bar{x} \sim x_1$ if and only if $\phi_T^{-1}(\sigma \bar{x}) \cap T^{-1}(x_1) \neq \emptyset$. Therefore $\phi_T^{-1}(\bar{x}) \cap \tau^{-1} \phi_{\bar{T}}^{-1}(x_1) \neq \emptyset$.

Let $\bar{y}_1 \in \phi_T^{-1}(\bar{x}) \cap \tau^{-1} \phi_{\bar{T}}^{-1}(x_1)$ and $\bar{x}_1 = \phi_T \bar{y}_1$. Then $\sigma(\bar{x}_1) = x_1$ and we can conclude by the inductive hypothesis.

\[ \square \]

**Proposition 5.8.** Let $Y$ be a normal variety and let $T \subseteq Y$ be a divisor. Let $\mathcal{L}$ be a line bundle on $Y$ which is semiample on the irreducible components of $T$. Let $\varepsilon : \bar{Y} \to Y$ be a birational map which is an isomorphism on the irreducible components of $T$ and $T'$. Let $\bar{T}$ be the strict transform of $T$. Then there is a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{T \subseteq T} \bar{T} & \xrightarrow{\varepsilon} & \bigcup_{T \subseteq T} T \\
\downarrow (\phi_T) & & \downarrow (\phi_T) \\
\bigcup V & \xrightarrow{\sigma} & \bigcup V.
\end{array}
\]

with $\phi_T = \phi_T \circ \varepsilon$. Then $R_{\varepsilon^* \mathcal{L}} = R_{\mathcal{L}}$.

**Proof.** It is enough to prove that the equivalence classes coincide. Let $x_1, x_2 \in \bigcup V$ be such that there is $\bar{y} \in \bar{T}_1 \cap \bar{T}_2$ with $\phi_{\bar{T}_i} (\bar{y}) = x_i$. The divisor $\bar{T}_i$ is the strict transform of $T_i \subseteq Y$. Then $y = \varepsilon(\bar{y})$ is such that $\phi_{T_i} (y) = x_i$. This proves that $R_{\varepsilon^* \mathcal{L}} \subseteq R_{\mathcal{L}}$.

Let $x_1, x_2 \in \bigcup V$ be such that there is $y \in T_1 \cap T_2$ with $\phi_{T_i} (y) = x_i$. Let $\bar{T}_i$ be the strict transform of $T_i \subseteq Y$. As $\varepsilon$ is an isomorphism on the generic point of $T_1 \cap T_2$, the intersection $T_1 \cap T_2 \cap \varepsilon^{-1} y$ is non empty. If $\bar{y}$ is in the intersection, then $\phi_{\bar{T}_i} (\bar{y}) = \phi_{T_i}(\varepsilon y) = x_i$. This proves that $R_{\varepsilon^* \mathcal{L}} \supseteq R_{\mathcal{L}}$, concluding the proof. \[ \square \]

**Corollary 5.9.** Let $Y$ be a normal variety and let $T \subseteq Y$ be a divisor. Let $\mathcal{L}$ be a line bundle on $Y$ which is semiample on the irreducible components of $T$. Let $\theta : \bar{Y} \to Y$ be a generically finite map such that $\theta \mathrm{Exc}(\theta)$ does not contain the generic points of $T \cap T'$ for every $T, T'$ irreducible component of $T$. Let $\bar{T}$ be the strict transform of $T$.

Then there is a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{T \subseteq T} \bar{T} & \xrightarrow{\theta} & \bigcup_{T \subseteq T} T \\
\downarrow (\phi_T) & & \downarrow (\phi_T) \\
\bigcup V & \xrightarrow{\sigma} & \bigcup V.
\end{array}
\]
with $\sigma$ a finite map such that $\mathcal{R}_{\theta^*\mathcal{L}} = \sigma^*\mathcal{R}_{\mathcal{L}}$.

Proof. Let $\theta = \varepsilon \circ \tau$ be the Stein factorisation. Let $\tilde{T} = \tau^{-1}T$. By Proposition 5.7 there is a diagram

\[
\begin{array}{ccc}
\bigcup \tilde{T} & \xrightarrow{\tau} & \bigcup T \\
(\phi_T) & & (\phi_T) \\
\bigcup \tilde{V} & \xrightarrow{\sigma} & \bigcup V.
\end{array}
\]

with $\sigma$ a finite map such that $\mathcal{R}_{\tau^*\mathcal{L}} = \sigma^*\mathcal{R}_{\mathcal{L}}$. By Proposition 5.8 there is a diagram

\[
\begin{array}{ccc}
\bigcup T & \xrightarrow{\varepsilon} & \bigcup \tilde{T} \\
(\phi_T) & & (\phi_T) \\
\bigcup \tilde{V} & \xrightarrow{\sigma} & \bigcup V.
\end{array}
\]

\[\square\]

Lemma 5.10. Let $T \subseteq Y$ be a reduced and connected divisor and let $\mathcal{L}$ be a line bundle such that $\mathcal{L}|_T$ is base point free for every $T \subseteq \mathcal{T}$ for every irreducible component. Let $\phi_T: T \to V$ be the morphism induced by $\mathcal{L}$. Then $\mathcal{T}_{[x]}$ is connected.

Proof. Let $y_1, y_2 \in \mathcal{T}_{[x]}$. Then there are $x_2, \ldots, x_r$ such that $\phi_1(y_1) = x_1 \sim x_2 \sim \cdots x_{r+1} = \phi_2(y_2)$. Let $V_i$ be such that $x_i \in V_i$ and $T_i$ with $\phi_i: T_i \to V_i$. Then there are $y_{i,i+1} \in T_{i,i+1}$ such that $\phi_i(y_{i,i+1}) = x_i$ and $\phi_{i+1}(y_{i,i+1}) = x_{i+1}$. Thus

\[y_1, y_2 \in \bigcup_{i=1}^{r+1} \phi_i^{-1}(x_i) \subseteq \mathcal{T}_{[x]}\]

and $\bigcup_{i=1}^{r+1} \phi_i^{-1}(x_i)$ is connected as for every $i$ there is $y_{i,i+1} \in \phi_i^{-1}(x_i) \cap \phi_{i+1}^{-1}(x_{i+1})$.

\[\square\]

6. Graph theory

We recall here a few basic notions of graph theory. We follow the presentation of [Stan3].

A graph $\Gamma$ consists of two sets $E$ and $V$ (edges and vertices), and two functions $E \to E$, $e \mapsto \bar{e}$ and $E \to V$, $e \mapsto i(e)$: for each $e \in E$, there is an element $\bar{e} \in E$, and an element $i(e) \in V$. The function $\bar{e}$ is such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$. The vertex $i(e)$ is called the initial vertex of $e$, the vertex $t(e) = i(\bar{e})$ is called the terminal vertex of $e$.

We call a graph finite if both $V$ and $E$ are finite sets.
A map of graphs \( f : \Gamma_1 \rightarrow \Gamma_2 \) consists of a pair of functions, edges to edges, vertices to vertices, preserving the structure. A map of graphs is surjective if it is surjective on vertices and on edges.

We recall that pull-backs exist in the category of graphs: given \( f_1 : \Gamma_1 \rightarrow \Delta \) and \( f_2 : \Gamma_2 \rightarrow \Delta \) two maps of graphs, there is a graph \( \Gamma_1 \times_\Delta \Gamma_2 \) together with surjective maps \( g_1 : \Gamma_1 \times_\Delta \Gamma_2 \rightarrow \Gamma_1 \) and \( f_2 \circ g_1 = f_2 \circ g_2 \).

A path in a graph \( \Gamma \) is an \( n \)-tuple of edges \( (e_1, \ldots, e_n) \) in \( E^n \) such that \( t(e_i) = i(e_{i+1}) \). The vertices \( i(e_1) \) and \( t(e_n) \) are the initial vertex and terminal vertex of the path.

A circuit is a path whose initial and terminal vertex coincide. Equivalently, we define \( C_n \) the standard circuit of length \( n \) as the regular polygon with \( n \) edges and a circuit in \( \Gamma \) is a map of graphs \( C_n \rightarrow \Gamma \). A circuit is proper if the map \( C_n \rightarrow \Gamma \) is injective on the vertices. The standard arc of length \( n \) \( A_n \) can be described as the interval \( [0, n] \) subdivided at the integral points. The vertices are \( V = \{0, \ldots, n\} \), the edges are the oriented segments \( [i, i+1] \) and \( [i+1, i] \) between \( i \) and \( i+1 \). The involution \( \bar{\cdot} \) exchanges \( [i, i+1] \) and \( [i+1, i] \).

The homotopy equivalence on paths is the relation generated by
\[
(e_1, \ldots, e_n) \sim (e_1, \ldots, e_i, e, e_{i+1}, \ldots, e_n)
\]
and the set of paths starting and ending at a same vertex \( v \) modulo homotopy is denoted by \( \pi_1(\Gamma, v) \) and called the fundamental group of \( \Gamma \). It has a natural group structure with respect to the concatenation of paths.

A path is reduced if it contains no sub-paths of the form \( e \bar{e} \) and one can prove that every path is homotopic to a reduced one.

Let \( v \) be a vertex of the graph \( \Gamma \). The star of \( v \) in \( \Gamma \) is the set
\[
St(v, \Gamma) = \{ e \in E \mid i(e) = v \}.
\]

A map of graphs \( f : \Gamma_1 \rightarrow \Gamma_2 \) is a covering if for each vertex \( v \) of \( \Gamma_1 \) the natural function
\[
f_v : St(v, \Gamma_1) \rightarrow St(f(v), \Gamma_2)
\]
is bijective. By [Sta83, 4.1(d)] if \( f : \Gamma_1 \rightarrow \Gamma_2 \) is a covering, then \( f : \pi_1(\Gamma_1, v) \rightarrow \pi_1(\Gamma_2, f(v)) \) is an injective homomorphism and if the graphs are finite then \( f \pi_1(\Gamma_1, v) \subseteq \pi_1(\Gamma_2, f(v)) \) has finite index equal to the cardinality of \( f^{-1}f(v) \).

This last remark combined with [Sta83 3.3] and [Sta83 4.4], gives the following proposition

**Proposition 6.1.** If \( f : \Gamma_1 \rightarrow \Gamma_2 \) is a surjective maps of finite graphs, then \( f \pi_1(\Gamma_1, v) \subseteq \pi_1(\Gamma_2, f(v)) \) has finite index \( i \leq |f^{-1}f(v)| \).

We conclude this section with an easy but useful lemma.

**Lemma 6.2.** Let \( \Gamma \) be a finite graph. Then there is a standard circuit \( C_N \) and a surjective morphism \( C_N \rightarrow \Gamma \).

**Proof.** We construct recursively a morphism \( f : A_N \rightarrow \Gamma \). We notice that if \( f \) is surjective on the edges then it is surjective on the vertices and that it is
Definition 7.1. Let $E = \{f[i, i+1]|e\}$ or such that $\bar{e} = f[i, i+1]$. We set $\pi: E \to \tilde{E}$ the quotient by the action of $\mathbb{Z}/2\mathbb{Z}$ sending $e$ to $\bar{e}$. Let $e \in E$. We set $f[0, 1] = e$. Assume we have $f: A_k \to \Gamma$. If $\hat{E} \setminus \pi(\{f[i, i+1]|e\})$ is not empty, then we pick $e \in \hat{E} \setminus \{f[i, i+1], \hat{f}[i, i+1]|e\}$, we pick a path $(e_1, \ldots, e_n)$ from $t(f[k-1, k])$ to $i(e)$ and we set
\[
\begin{align*}
\{ f[k + i - 1, k + i] &= e_i \text{ for } i \leq n \\
\lfloor f[k + n + 1, k + n + 2] &= e.
\end{align*}
\]
If $\hat{E} \setminus \pi(\{f[i, i+1]|e\})$ is empty, then we pick a path $(e_1, \ldots, e_n)$ from $t(f[k-1, k])$ to $i(f[0, 1])$ and we set $f[k+i-1, k+i] = e_i$ for $i \leq n$.

\[\square\]

7. Trivial line bundles on simple normal crossings varieties

In this section we discuss a triviality condition for line bundles on reducible varieties and develop the tools for the proof of Theorem 10.1. We are mostly concerned with the case of simple normal crossings varieties in the sense of Definition 3. Lemmas 7.5, 7.8 and 7.10 can be seen as a refinement of [BLR90] Example 9.2.8.

Definition 7.1. Let $\mathcal{Z} = \cup \mathcal{Z}$ be a reducible variety. We define the incidence graph $\Gamma^i(\mathcal{Z})$ of $\mathcal{Z}$ by $\Gamma^i = \{\mathcal{Z}| \mathcal{Z} \text{ irreducible component of } \mathcal{Z}\}$ with an edge between $\mathcal{Z}$ and $\mathcal{Z}'$ for every connected component of $\mathcal{Z} \cap \mathcal{Z}'$.

Notation 7.2. A circuit $C$ in $\Gamma^i(\mathcal{Z})$ will be denoted by

$$(\{Z_1, \ldots, Z_k\}, Z_{i_1}, Z_{i_2} \ldots Z_{i_{k+1}})$$

or $$(\{Z_i\}, Z_{i_{i+1}})$$

for short where the $Z_i$ are irreducible components of $\mathcal{Z}$ and for every the variety $Z_{i_{i+1}}$ is a connected component of $Z_i \cap Z_{i+1}$, and $Z_{k,1}$ is a connected component of $Z_1 \cap Z_k$.

We will refer to $\cup Z_i$ as the support of the circuit $C$.

Remark 7.3. If $\mathcal{Z}$ is a divisor with simple normal crossing support, then $\Gamma^i(\mathcal{Z})$ coincides with the 1-skeleton of the dual complex of $\mathcal{Z}$ (see [dFKX17], Section 2).

Throughout this subsection $Y$ will be a normal connected variety and $\mathcal{Z} \subseteq Y$ a reducible reduced and connected subvariety of $Y$. We will consider $L$ a line bundle on $Y$ such that $L|Z \sim O_Z$ for every irreducible component $Z$ of $\mathcal{Z}$.

Definition 7.4. Let $C = (\{Z_i\}, Z_{i_{i+1}})$ be a circuit in $\Gamma^i(\mathcal{Z})$. A section of the restriction of $L$ to $C$ (or of $L|C$) is the data of $s_i \in H^0(Z_i, L)$ such that $s_i|Z_{i_{i+1}} = s_{i+1}|Z_{i_{i+1}}$.

Lemma 7.5. Let $\mathcal{Z}$ be a connected reduced simple normal crossings variety of pure dimension $k$. Let $L$ be a line bundle on $\mathcal{Z}$ such that $L|Z \sim O_Z$ for every irreducible component $Z$ of $\mathcal{Z}$. Then $L|Z \sim O_Z$ if and only if for every
Proof. If $\mathcal{L}$ is trivial, then it has a nowhere vanishing global section $s \in H^0(Z, \mathcal{L})$. Then for any circuit $\mathcal{C} = (\{Z_i\}, Z_{1,2}, \ldots, Z_{k,1})$ in $\Gamma^i(Z)$ it is enough to set $s_i = s|_{Z_i}$.

Conversely, let $\mathcal{Z} = \bigcup Z_i$ be the decomposition of $Z$ into its irreducible components. By possibly relabeling, we may assume that for any $i > 1$ the subvariety $Z_i$ meets $Z_{i-1} := \bigcup_{j \leq i} Z_j$. Let $Z_i = Z_{i-1} \cup Z_i$. For each $i \neq j$ with $Z_i \cap Z_j \neq \emptyset$, we fix $p_{i,j} \in Z_i \cap Z_j$. Fix $s_1 \in H^0(Z_1, \mathcal{L}) \setminus \{0\}$. We construct inductively a nowhere-vanishing section $\sigma_i \in H^0(Z_i, \mathcal{L})$ such that $\sigma_i|_{Z_1} = s_1$.

For $i > 1$ we assume there is a section $\sigma_{i-1} \in H^0(Z_{i-1}, \mathcal{L})$. Choose the largest $r < i$ such that $Z_i \cap Z_r \neq \emptyset$, and let $s_i \in H^0(Z_i, \mathcal{L}) \setminus \{0\}$ be the unique section such that
\[
s_i(p_{i,r}) = \sigma_{i-1}|_{Z_r}(p_{i,r}).
\]

If $Z_j \cap Z_r = \emptyset$ for all $j < i$ with $j \neq r$ and $Z_i \cap Z_r$ is connected, then (15) defines a nowhere-vanishing section $\sigma_i \in H^0(Z_i, \mathcal{L})$.

Otherwise, there exists $Z_s$ with $s < i$ and a point $p_{i,s} \in Z_i \cap Z_s$. Then there exists a circuit $\mathcal{C} = (Z_{i_1}, \ldots, Z_{i_k}, Z_{1,2}, \ldots, Z_{k,1})$ such that $Z_{i_1} = Z_s$, $Z_{k-1} = Z_r$, $Z_{k,s} = Z_i$, $p_{i,r} \in Z_{k-1,k}$ and $p_{i,s} \in Z_{k,1}$. By assumption there exists a non-trivial global section of $\mathcal{L}|_C$, which is the data of $\theta_i \in H^0(Z_{i,j}, \mathcal{L})$ for $j = 1, \ldots, k$. By rescaling, we may assume that $\theta_1 = \sigma_{i-1}|_{Z_{i_1}}$. Then, by the construction above, for every $1 \leq j \leq k-1$ we have $\theta_j = \sigma_{i-1}|_{Z_j}$ and $\theta_k = s_i$, and in particular
\[
s_i(p_{i,s}) = \sigma_{i-1}|_{Z_s}(p_{i,s}).
\]

Since this holds for any choice of $p_{i,s} \in Z_i \cap Z_s$, (15) and (16) define a nowhere-vanishing section $\sigma_i \in H^0(Z_i, \mathcal{L})$.

\[\square\]

Definition 7.6. Let $\mathcal{Z}$ be a connected reduced simple normal crossings variety of pure dimension $k$. Let $\mathcal{L}$ be a line bundle on $\mathcal{Z}$ such that $\mathcal{L}|_Z \sim O_Z$ for every irreducible component $Z$ of $\mathcal{Z}$. Let $\mathcal{C} = (\{Z_1, \ldots, Z_k\}, Z_{i,i+1})$ be a circuit in $\Gamma^i(\mathcal{Z})$. We chose $s_1 \in H^0(Z_1, \mathcal{L}) \setminus \{0\}$ and for every $i > 1$ we set $s_i \in H^0(Z_i, \mathcal{L}) \setminus \{0\}$ as the unique section such that
\[
s_i|_{Z_{i-1,i}} = s_{i-1}|_{Z_{i-1,i}}.
\]

We define then
\[
\Phi_{\mathcal{L},\mathcal{C}}: H^0(Z_1, \mathcal{L}) \to H^0(Z_1, \mathcal{L})
\]
\[
s \mapsto s \cdot s_{i+1}/s_1,
\]

Remark 7.7. The map $\Phi_{\mathcal{L},\mathcal{C}}$ is the identity if and only if the restriction of $\mathcal{L}$ to $\mathcal{C}$ admits a nowhere vanishing global section.

It is easy to see that this does not depend on the choice of $s_1$. Moreover, if $\mathcal{C}, \mathcal{C}'$ are circuits based in $Z_1$ and they are homotopically equivalent, then $\Phi_{\mathcal{L},\mathcal{C}} = \Phi_{\mathcal{L},\mathcal{C}'}$. If $C_1, C_2$ are circuits based in $Z_1$ and $\mathcal{C} = C_1 \ast C_2$ is their
concatenation, then $\Phi_{L,C} = \Phi_{L,C_2} \circ \Phi_{L,C_1}$. All these remarks prove the following lemma.

Lemma 7.8. Let $Z$ be a connected reduced simple normal crossings variety of pure dimension $k$, let $Z_1 \subseteq Z$ be an irreducible component. Let $L$ be a line bundle on $Z$ such that $L|_Z \sim O_Z$ for every irreducible component $Z$ of $Z$. There is a group homomorphism

$$
\Phi_L: \pi_1(\Gamma^i(Z), Z_1) \to GL(H^0(Z_1, L)) \cong \mathbb{C}^*
$$

which is trivial if and only if $L \sim O_Z$.

Remark 7.9. In the context of the previous definition, for all $m$ we have

$$
\Phi_{L \otimes m, C} = \Phi_{L, C} \circ \cdots \circ \Phi_{L, C},
$$

Lemma 7.10. Let $Z$ be a connected reduced simple normal crossings variety of pure dimension $k$, let $Z_1 \subseteq Z$ be an irreducible component. Let $L$ be a line bundle on $Z$ such that $L|_Z \sim O_Z$ for every irreducible component $Z$ of $Z$. Then $L$ is a torsion line bundle if and only if the image of $\Phi_L$ is a finite subgroup of $\mathbb{C}^*$.

Proof. If $L$ is torsion, then there is a positive integer $m$ such that $L^m \sim O_Z$. Therefore for every circuit $C$ based in $Z_1$ the map $\Phi_{L \otimes m, C}$ is the identity. The conclusion follows from Remark 7.9.

Conversely, let $m$ be a positive integer such that the image of $\Phi_L$ is contained in the $m$-th roots of $1$. Then for every circuit $C$ in $\Gamma^i(Z)$ based in $Z_1$ the map $\Phi_{L \otimes m, C}$ is the identity. By Remark 7.9, this map is $\Phi_{L \otimes m, C}$, and then the restriction of $L^{\otimes m}$ to every circuit $C$ in $\Gamma^i(Z)$ based in $Z_1$ admits a global section. The statement follows from Lemma 7.5 and from the fact that every circuit is homotopic to a circuit based in $Z_1$.

\[\square\]

7.1. Trivial line bundles on divisors.

Definition 7.11. Let $\tau: Y \to Y$ be a finite map of normal projective varieties and let $Z$ be a connected subvariety of $Y$. Let $\overline{Z}$ be the preimage of $Z$ under $\tau$. Set $\overline{Z} = \tau^{-1}Z$. We define a graph $\Gamma^i(\overline{Z}, \tau) \subseteq \Gamma^i(\overline{Z})$ having as vertices the vertices of $\Gamma^i(\overline{Z})$ and having an edge between $\overline{Z}$ and $\overline{Z}'$ if and only if there is an edge between $\overline{Z}$ and $\overline{Z}'$ in $\Gamma^i(\overline{Z})$ and $\tau(\overline{Z}) \neq \tau(\overline{Z}')$.

Construction 7.12. Let $\tau: Y \to Y$ be a finite map of normal projective varieties and let $Z$ be a connected subvariety of $Y$. Let $\overline{Z}$ be the preimage of $Z$ under $\tau$. Then there is a natural map of graphs

$$
\tau: \Gamma^i(\overline{Z}, \tau) \to \Gamma^i(\overline{Z})
$$

defined on vertices by $\tau(v_{\overline{Z}}) = v_{\tau(\overline{Z})}$. To an edge $e$ of $\Gamma^i(\overline{Z})$ corresponding to a connected component $Z_0$ of $\overline{Z} \cap \overline{Z}'$ the map $\tau$ associates the unique connected component of $\tau\overline{Z} \cap \tau\overline{Z}'$ containing $\tau Z_0$. 
Lemma 7.13. Let $\tau : \overline{Y} \to Y$ be a finite map of normal projective varieties and let $Z$ be a connected subvariety of $Y$. Let $\overline{Z}$ be the preimage of $Z$ under $\tau$. The map in Construction 4.12 is surjective and for every $v$ vertex of $\Gamma^i(Z)$ we have $|\tau^{-1}v| \leq \deg \tau$.

Proof. The map is clearly surjective on vertices. Let $Z_0$ be a connected component of $Z \cap Z'$. Let $\overline{Z}$ be an irreducible component of $\overline{Z}$ such that $\tau \overline{Z} = Z$. The set $\tau^{-1}Z_0$ is not empty and it is contained in $\overline{Z} \cap \tau^{-1}Z'$. Then there is an irreducible component $\overline{Z}'$ of $\tau^{-1}Z'$ meeting $\overline{Z}$. Let $\overline{Z}_0$ be a connected component of $\overline{Z} \cap \overline{Z}'$. Then $\tau$ sends the edge corresponding to $\overline{Z}_0$ to the edge corresponding to $Z_0$. \)

Combining Lemma 7.13 and Proposition 6.1 we get

Corollary 7.14. Let $\tau : \overline{Y} \to Y$ be a finite map of normal projective varieties and let $Z \subseteq Y$ be a simple normal crossings divisor. Let $\overline{Z}$ be the preimage of $Z$ under $\tau$ and fix an irreducible component $\overline{Z}_1$ of $\overline{Z}$. Then $\tau \pi_1(\Gamma^i(\overline{Z}, \tau), \overline{Z}_1)$ has finite index $k$ in $\pi_1(\Gamma^i(Z), \tau \overline{Z}_1)$. Moreover $k \leq \deg \tau$.

Lemma 7.15. Let $\tau : \overline{Y} \to Y$ be a finite map of degree $d$ of normal projective varieties and let $Z \subseteq Y$ be a simple normal crossings divisor. Let $\overline{Z}$ be the preimage of $Z$ under $\tau$. Let $\mathcal{L}$ be a line bundle on $Y$ such that $\mathcal{L}|_Z \sim \mathcal{O}_Z$ for every component $Z$ of $Z$. If $\mathcal{L}|_Z \sim \mathcal{O}_Z$ then $\tau^*\mathcal{L}|_{\overline{Z}} \sim \mathcal{O}_{\overline{Z}}$. If $\tau^*\mathcal{L}|_{\overline{Z}} \sim \mathcal{O}_{\overline{Z}}$, then $\mathcal{L}|_{\overline{Z}} \sim \mathcal{O}_{\overline{Z}}$.

Proof. If $\mathcal{L}|_Z \sim \mathcal{O}_Z$, then the pullback of the nowhere vanishing global section of $\mathcal{L}|_Z$ by $\tau$ gives a nowhere vanishing global section of $\tau^*\mathcal{L}|_{\overline{Z}}$, settling the first part of the statement.

Conversely, we assume that $\tau^*\mathcal{L}|_{\overline{Z}} \sim \mathcal{O}_{\overline{Z}}$. Fix an irreducible component $\overline{Z}_1$ of $\overline{Z}$ and set $Z_1 = \tau \overline{Z}_1$. We want to prove that for every circuit $\mathcal{C}$ in $\Gamma^i(Z)$ based on $Z_1$, the morphism $\Phi_{\mathcal{L}, \mathcal{C}}^k$ is the identity. As $\pi_1(\Gamma^i(Z), Z_1)$ is finitely generated, the result will follow from Lemma 7.10.

By Corollary 7.11, the group $\tau \pi_1(\Gamma^i(\overline{Z}, \tau), \overline{Z}_1)$ is a subgroup of $\pi_1(\Gamma^i(Z), Z_1)$ of index $k \leq d$. Therefore there exist a circuit $\mathcal{C} = ([Z_{i}], [Z_{i,i+1}])$ in $\Gamma^i(\overline{Z})$ such that, if we denote by $\mathcal{C}^k$ the concatenation of $\mathcal{C}$ with itself $k$ times, the circuits $\mathcal{C}^k$ and $\tau \mathcal{C}$ are homotopically equivalent. By Remark 7.9 it is enough to prove that $\Phi_{\mathcal{L}, \tau \mathcal{C}} = \Phi_{\mathcal{L}, \mathcal{C}}^k$ is the identity. We notice that if $Z_i = \tau(\overline{Z}_i)$ for $i = 1, 2$ and $Z_{1,2} = \tau(\overline{Z}_{1,2})$, we have a commutative diagram

\[
\begin{array}{ccc}
H^0(Z_1, \mathcal{L}) & \xrightarrow{i} & H^0(Z_1, \tau^*\mathcal{L}) \\
\downarrow & & \downarrow \\
H^0(Z_1, \tau^*\mathcal{L}) & \xrightarrow{i} & H^0(Z_1, \tau^*\mathcal{L})
\end{array}
\]

\[
\begin{array}{ccc}
H^0(Z_1, \mathcal{L}) & \xrightarrow{i} & H^0(Z_1, \tau^*\mathcal{L}) \\
\downarrow & & \downarrow \\
H^0(Z_1, \tau^*\mathcal{L}) & \xrightarrow{i} & H^0(Z_1, \tau^*\mathcal{L})
\end{array}
\]

where the horizontal arrows are the restriction isomorphisms and the vertical arrows are isomorphisms induced by the pullback by $\tau$. 

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Les $s$ be a global section for $\tau^*L|_{\Xi}$. Then, with the identifications of the previous diagram, the linear map $\Phi^{k}_{L,\tau_{\text{C}}}$ is the multiplication by $s/s = 1$, therefore it is the identity.

Then $\Phi^{k}_{L,\tau_{\text{C}}}$ is the identity, and so is $\Phi^{d}_{L,\tau_{\text{C}}}$, proving the statement. \(\square\)

### 7.2. Trivial line bundles on semistable curves.

In this subsection we present an analog of Construction 7.12 and Lemma 7.13 for curves.

**Definition 7.16.** Let $\tau: \bar{Y} \to Y$ be a generically finite map of normal projective varieties and let $Z$ be a connected curve in $Y$. Let $\bar{Y}$ be the curve in $Z$ such that $\tau Z = Z$. We define a graph $\Gamma^i(Z, \tau) \subseteq \Gamma^i(\bar{Z})$ having as vertices the vertices of $\Gamma^i(Z)$ and having an edge between $\bar{Z}$ and $\bar{Z}'$ if and only if there is an edge between $Z$ and $Z'$ in $\Gamma^i(Z)$ and either

- $\tau Z$ and $\tau Z'$ are curves in $Y$, or
- $\tau Z$ is a curve in $Y$, and $\tau Z' = p \in \tau Z$, or
- $\tau Z = \tau Z' = p \in Y$.

**Construction 7.17.** Let $\tau: \bar{Y} \to Y$ be a generically finite map of normal projective varieties and let $Z$ be a a simple normal crossings curve in $Y$. Let $\bar{Z}$ be a simple normal crossings curve in $Z$ such that $\tau Z = Z$. Let $\bar{Z}_1 \subseteq \bar{Z}$ be such that $Z_1 = \tau Z_1$ is a curve in $Y$.

Then there a homomorphism of groups

$$
\tau: \{\text{circuits in } \Gamma^i(\bar{Z}, \tau) \text{ based at } \bar{Z}_1\} \to \{\text{circuits in } \Gamma^i(Z) \text{ based at } Z_1\}
$$

defined in the following way. Let $C = (\{Z_1, \ldots, Z_k\}, Z_{i,i+1})$ be a circuit in $\Gamma^i(\bar{Z}, \tau)$. If $\tau(Z_{i})$ is a curve for every $i$, we set $\tau C = ((\tau Z_1, \ldots, \tau Z_k), \tau Z_{i,i+1})$. Otherwise let $i_j$ and $h_j$ be such that $h_j > 0$ and $i + h_j + 1 = i + 1$, and

- \(\text{for every } s = 1, \ldots, h_j \text{ we have } \tau(Z_{i,j+1}) = \tau(Z_{i,j+s}) \text{ is a point in } Y\),
- \(\tau(Z_{i,j}) \text{ and } \tau(Z_{i,j+h_j+1}) \text{ are curves in } Y\).

If $\tau(Z_{i,j}) = \tau(Z_{i,j+h_j+1})$ we set

$$
(\bar{Z}_{i,j}, \bar{Z}_{i,j+1}, \ldots, \bar{Z}_{i,j+h_j+1}) \mapsto (\tau(Z_{i,j})).
$$

If $\tau(Z_{i,j}) \neq \tau(Z_{i,j+h_j+1})$ we set

$$
(\bar{Z}_{i,j}, \bar{Z}_{i,j+1}, \ldots, \bar{Z}_{i,j+h_j+1}) \mapsto (\tau(Z_{i,j}), \tau(Z_{i,j+h_j+1}))
$$

with the edge $\tau(Z_{i,j})$ between $\tau(Z_{i,j})$ and $\tau(Z_{i,j+h_j+1})$.

**Lemma 7.18.** Notation as in Construction 7.17, The map of Construction 7.17 respects the homotopy of loops and defines thus a homomorphism of groups $\tau: \pi_1(\Gamma^i(\bar{Z}, \tau), \bar{Z}_1) \to \pi_1(\Gamma^i(Z), Z_1)$.

**Proof.** It is enough to prove that the two circuits

$$
\bar{C} = (\{\bar{Z}_1, \ldots, \bar{Z}_k\}, \bar{Z}_{i,i+1}) \quad \text{and}
$$

$$
\bar{C}' = (\{\bar{Z}_1, \ldots, \bar{Z}_j, \bar{Z}_j, \ldots, \bar{Z}_k\}, \bar{Z}_{1,2}, \ldots, \bar{Z}_{j-1,j}, \bar{Z}_j, \bar{Z}_{j,j+1}, \ldots, \bar{Z}_{k,1})
$$

satisfy the conditions for being homomorphisms. This is clear.
have homotopically equivalent images. If $\tau Z_j$ and $\tau Z$ are curves, then it
is clear. If $\tau Z$ is a point, then the path $(Z_j, Z, Z_j)$ has the same image as
$(Z_j)$. If $\tau Z$ is a curve and $\tau Z_j$ a point, then let $h < j < k$ be such that $\tau Z_h$
and $\tau Z_k$ are curves and $\tau Z_i$ is a point for every $h < i < k$.
If $\tau Z_h = \tau Z_k$, then
$$(Z_h, \ldots, Z_k) \mapsto (\tau Z_h).$$
If moreover $\tau Z_h \neq \tau Z$, then
$$(Z_h, \ldots, Z_j, Z, Z_j, \ldots, Z_k) \mapsto (\tau Z_h).$$
If $\tau Z_h \neq \tau Z_k$, then
$$(Z_h, \ldots, Z_j, Z, Z_j, \ldots, Z_k) \mapsto (\tau Z_h, \tau Z_k).$$
In both cases we get homothopically equivalent circuits.

If $\tau Z_h \neq \tau Z_k$, then
$$(Z_h, \ldots, Z_k) \mapsto (\tau Z_h, \tau Z_k).$$
Then either $\tau Z_h = \tau Z$ or $\tau Z_k = \tau Z$, and in both cases
$$(Z_h, \ldots, Z_j, Z, Z_j, \ldots, Z_k) \mapsto (\tau Z_h, \tau Z_k).$$

\[\Box\]

7.3. Trivial line bundles and pullbacks. We prove in this subsection that, if we have a generically finite morphism between two immersed simple normal crossings varieties, then a line bundle is trivial on the first variety if and only if its pullback is trivial on the second.

Lemma 7.19. Let $Z$ be a connected simple normal crossings variety of
dimension at least 1 and let $L$ be a line bundle on $Z$ which is trivial for
every irreducible component of $Z$. Then there is a simple normal crossings
curve $K \subseteq Z$ such that the restriction of $L$ to $K$ is trivial if and only if $L$ is
trivial.

Proof. If $L$ is trivial, then for every curve $K \subseteq Z$, the restriction of $L$ to $K$ is trivial.

For the other implication, we proceed by induction on
$$\dim Z = \max\{\dim Z \mid Z \text{ irreducible component of } Z\}.$$ If $\dim Z = 1$, then we set $K = Z$. We assume now the existence of such a
curve for connected simple normal crossings varieties of dimension $k - 1$. Let $Z$
be a connected simple normal crossings variety of dimension $k$. Let $Z^{(k)}$
be the union of all the irreducible component of dimension $k$ and let $Z_{k-1}$ be the
union of all the irreducible component of dimension at most $k - 1$. Let $A$
be a section of a very ample divisor on $Z^{(k)}$ such that $H^1(Z^{(k)}, O(-A)) = \{0\}$
and $H^1(Z^{(k)}, L(-A)) = \{0\}$. In particular, for every connected component $Z$
of $Z^{(k)}$ the intersection $A \cap Z$ is connected. Assume moreover that $A \subseteq Z^{(k)} \cap Z^{(1)}$ and that $A \cup Z_{k-1}$ is a simple normal crossings variety.
We set $W = A \cup Z_{k-1}$. For every irreducible component $Z$ of $Z_{k-1}$ such that $Z \cap Z^{(k)} \neq \emptyset$, we have $Z \cap A \neq \emptyset$. Indeed, if $\dim Z \cap Z^{(k)} \geq 1$, it is true because $A$ is ample. If $\dim Z \cap Z^{(k)} = 0$, then it is true by construction of $A$.

Then $W$ is a connected simple normal crossings variety of dimension $k-1$. In order to conclude, it is enough to prove that if the restriction of $L$ to $W$ is trivial, then $L$ is trivial.

If the restriction of $L$ to $W$ is trivial, then there is a section $s \in H^0(W, L)\setminus \{0\}$. As $H^1(Z^{(k)}, L(-A)) = \{0\}$, there is a section $s^k \in H^0(Z^{(k)}, L)$ such that $s^k|_A = s|_A$.

We want to show that $s^k$ and $s|_{Z_{k-1}}$ glue to a section of $Z$. This happens if and only if $(s^k, s|_{Z_{k-1}})$ is in the kernel of

$$\alpha: H^0(Z^{(k)}, L) \oplus H^0(Z_{k-1}, L) \to H^0(Z^{(k)} \cap Z_{k-1}, L) \quad (s_1, s_2) \mapsto s_1 - s_2.$$ 

We have a commutative diagram

$$\begin{array}{ccc}
H^0(Z^{(k)}, L) \oplus H^0(Z_{k-1}, L) & \xrightarrow{\alpha} & H^0(Z^{(k)} \cap Z_{k-1}, L) \\
i & \downarrow & \downarrow
\end{array}$$

$$\begin{array}{ccc}
H^0(A, L) \oplus H^0(Z_{k-1}, L) & \xrightarrow{\beta} & H^0(A \cap Z_{k-1}, L)
\end{array}$$

Since $(s|_A, s|_{Z_{k-1}})$ is in the kernel of $\beta$, it follows that $(s^k, s|_{Z_{k-1}})$ is in the kernel of $\alpha$.

$$\square$$

Lemma 7.20. Let $\varepsilon: \bar{Y} \to Y$ be a generically finite map of normal projective varieties and let $Z$ be a connected simple normal crossings subvariety of $Y$. Assume that the preimage $\bar{Z}$ of $Z$ under $\varepsilon$ is a simple normal crossings variety. Let $L$ be a line bundle on $Y$ such that $L|_Z \sim O_Z$ for every irreducible component $Z$ of $Z$. Then $L|_Z$ is torsion if and only if $\varepsilon^*L|_{\bar{Z}}$ is torsion.

Proof. If $L|_Z \sim O_Z$, then $\varepsilon^*L|_{\bar{Z}} \sim O_{\bar{Z}}$. By Lemma 7.19 there is a semistable curve $K \subseteq Z$ such that the restriction of $L$ to $K$ is trivial if and only if $L$ is trivial.

Claim 7.21. There is a semistable curve $\bar{K} \subseteq \bar{Y}$ such that $\tau\bar{K} = K$ and the image of the homomorphism $\tau: \pi_1(\Gamma^i(\bar{K}, \tau), \bar{K}_1) \to \pi_1(\Gamma^i(K), K_1)$ has finite index in $\pi_1(\Gamma^i(K), K_1)$.

Assuming the claim, we conclude the proof.

Fix an irreducible component $\bar{K}_1$ of $\bar{K}$ such that $K_1 = \tau\bar{K}_1$ is a curve. We want to prove that there is a positive integer $h$ such that for every circuit $\mathcal{C}$ in $\Gamma^i(K)$ based in $K_1$, morphism $\Phi^h_{\mathcal{C}}$ is the identity. As $\pi_1(\Gamma^i(K), K_1)$ is finitely generated, the result will follow from Lemma 7.19.
By Claim 7.21 the group $\tau (\pi_1(\Gamma^i(K, \tau), \bar{K}_i))$ is a subgroup of $\pi_1(\Gamma^i(K), K_1)$ of finite index $k$. Therefore, there exists a circuit $\mathcal{C}$ in $\Gamma^i(K)$ such that, if we denote by $\mathcal{C}^k$ the concatenation of $\mathcal{C}$ with itself $k$ times, the circuits $\mathcal{C}^k$ and $\tau \mathcal{C}$ are homotopically equivalent. By Remark 7.4 it is enough to prove that $\Phi_{\mathcal{C}, \tau \mathcal{C}} = \Phi_{\mathcal{C}^k, \mathcal{C}^k}$ is the identity. Let $K_i, K_{i+1}$ be curves in $\tau \mathcal{C}$ with the edge $p_i$ between them. Let $K_i, \bar{K}_{i,j}, \bar{K}_{i,j+1}$ be curves in $\mathcal{C}$ with $K_h = \tau K_h$ for $h = i, i + 1$ and $\tau \bar{K}_{i,j} = p_i$ for $j = 1, \ldots, \ell_i$, where for every $j = 0, \ldots, \ell_i$ we denote by $\bar{p}_{i,j}$ the edge between $\bar{K}_{i,j}$ and $\bar{K}_{i,j+1}$, with $\bar{K}_{i,0} = K_i$ and $\bar{K}_{i,\ell_i+1} = \bar{K}_{i+1}$. We have commutative diagrams

$$
\begin{array}{ccc}
H^0(K_i, \mathcal{L}) & \longrightarrow & H^0(p_i, \mathcal{L}) \\
\downarrow & & \downarrow \\
H^0(\bar{K}_i, \tau^* \mathcal{L}) & \longrightarrow & H^0(\bar{p}_{i,0}, \tau^* \mathcal{L}) \longleftarrow H^0((\cup_j \bar{K}_{i,j}), \tau^* \mathcal{L})
\end{array}
$$

and

$$
\begin{array}{ccc}
H^0(p_i, \mathcal{L}) & \Longleftarrow & H^0(K_{i+1}, \mathcal{L}) \\
\downarrow & & \downarrow \\
H^0(\cup_j \bar{K}_{i,j}, \tau^* \mathcal{L}) & \longrightarrow & H^0(\bar{p}_{i,\ell_i}, \tau^* \mathcal{L}) \longrightarrow H^0(\bar{K}_{i+1}, \tau^* \mathcal{L})
\end{array}
$$

where the horizontal arrows are the restriction isomorphisms and the vertical arrows are isomorphisms induced by the pullback by $\tau$.

Let $s$ be a global section for $\tau^* \mathcal{L}|_{\mathcal{Z}}$. Then, with the identifications of the previous diagram, the linear map $\Phi_{\mathcal{C}, \tau \mathcal{C}}$ is the multiplication by $s/s = 1$, therefore it is the identity.

Then $\Phi_{\mathcal{C}, \tau \mathcal{C}}$ is the identity, proving the statement.

We are left with the proof of Claim 7.21

Let

$$
\begin{align*}
\mathcal{K}_f &= \{K \subseteq K \text{ irreducible component} \mid K \not\subseteq \tau \text{Exc}(\tau)\} \\
\mathcal{K}_e &= \{K \subseteq K \text{ irreducible component} \mid K \subseteq \tau \text{Exc}(\tau)\}.
\end{align*}
$$

For $K \in \mathcal{K}_f$, let $\bar{K}$ be its strict transform.

Let $K \in \mathcal{K}_e$. For every irreducible component $Z \subseteq \tau^{-1}K$ surjecting onto $K$, let $H_i$ be hyperplane sections such that $K_Z = Z \cap H_i$ is a reduced curve.

We can moreover find the $H_i$ such that if $\cup Z_i$ is connected, then $\bar{K} = \cup Z K_Z$ is connected.

If $p \in \mathcal{K}^{\text{sing}} \cap \tau \text{Exc}(\tau)$, and $\{p\} = K_1 \cap K_2$, for every irreducible component $Z \subseteq \tau^{-1}p$ let $H_i$ be hyperplane sections such that $K_Z = Z \cap H_i$ is a reduced curve and has the following property: if $\bar{K}_1, \bar{K}_2$ are such that $\tau \bar{K}_i = K_i$, then $\bar{K}_i \cap \tau^{-1}p \subseteq K_p$. We can moreover find the $H_i$ such that if $\cup Z_i$ is connected, then the union $K_p = \cup Z K_Z$ is connected.
Finally, we set
\[ \overline{\mathcal{K}} = \cup \{ \overline{K} \mid K \in \mathcal{K}_f \cup \mathcal{K}_e \} \cup \{ K_p \mid p \in \mathcal{K}^{sing} \cap \tau E\text{xc}(\tau) \}. \]

By the generality of \( \mathcal{K} \), we can assume that \( \overline{\mathcal{K}} \) is a simple normal crossings curve.

We want to prove now that \( \tau: \pi_1(\Gamma^i(\overline{\mathcal{K}}, \tau), \overline{K}_1) \to \pi_1(\Gamma^i(\mathcal{K}), K_1) \) has finite index in \( \pi_1(\Gamma^i(\mathcal{K}), K_1) \).

Let \( \mathcal{C} = (K_1, \ldots, K_\ell, p_i) \) be a circuit in \( \Gamma^i(\mathcal{K}) \). Let \( \overline{K}_1 \subseteq \overline{\mathcal{K}} \) be such that \( \tau \overline{K}_1 = K_1 \). Let \( \mathcal{N} \) be the number of curves in \( \mathcal{K} \) surjecting onto \( K_1 \). We construct a circuit \( \overline{\mathcal{C}} \) in \( \Gamma^i(\overline{\mathcal{K}}, \tau) \) such that \( \tau \overline{\mathcal{C}} = m \mathcal{C} \) in the group of circuits based on \( K_1 \) with \( m \) dividing \( \mathcal{N} \).

We assume now that we have \( \overline{K}_i \) for \( i = 1, \ldots, r+q_\ell \), and \( \overline{K}_{i,j} \) for \( i = 1, \ldots, r-1+q_\ell \) and \( j = 1, \ldots, \ell_i \) and edges \( \overline{q}_{i,j} \in \overline{K}_{i,j} \cap K_{i,j} \) for \( j = 0, \ldots, \ell_i \) such that \( \tau \overline{K}_i = K_i \), where \( i \) is the remainder of the euclidean division of \( i \) by \( \ell \), and \( \tau \overline{K}_{i,j} = p_i \).

If \( q_r \not\in \tau E\text{xc}(\tau) \), then we let \( \overline{K}_{r+1+q_\ell} \) be a curve such that
\[ \tau \overline{K}_{r+1+q_\ell} = K_{r+1} \text{ and } \tau^{-1} K_{r,r+1} \cap \overline{K}_{r+1+q_\ell} \cap \overline{K}_{r+q_\ell} \neq \emptyset. \]

We set \( \overline{q}_{i+q_\ell,0} \) as a point in \( \tau^{-1} K_{r,r+1} \cap \overline{K}_{r+1+q_\ell} \cap \overline{K}_{r+q_\ell} \).

If \( q_r \in \tau E\text{xc}(\tau) \), then let \( \overline{K}_{r+1} \) such that \( \tau \overline{K}_{r+1} = K_{r+1} \) and \( \overline{K}_{r+1} \) meets a connected component of \( \tau^{-1} q_r \) meeting \( \overline{K}_{r+q_\ell} \).

Let \( \overline{K}_{r+q_\ell, j} \) be such that
\[ \cdot \overline{K}_{r+q_\ell, j} \cap \overline{K}_{r+q_\ell, j+1} \neq \emptyset, \]
\[ \cdot \overline{K}_{r+q_\ell} \cap \overline{K}_{r+q_\ell, 1} \neq \emptyset, \] and
\[ \cdot \overline{K}_{r+q_\ell, \ell+q_\ell} \cap \overline{K}_{r+1} \neq \emptyset. \]

We set \( \overline{p}_{r+q_\ell, 0} \in \overline{K}_{r+q_\ell} \cap \overline{K}_{r+q_\ell, 1}, \overline{p}_{r+q_\ell, j} \in \overline{K}_{r+q_\ell, j} \cap \overline{K}_{r+q_\ell, j+1} \) and \( \overline{p}_{r+q_\ell, \ell} \in \overline{K}_{r+q_\ell, \ell+q_\ell} \cap \overline{K}_{r+1} \). Finally, we set \( \overline{K}_{r+1+q_\ell} = \overline{K}_{r+1} \).

Then there are \( q_1 < q_2 \) with \( q_2 - q_1 \leq \mathcal{N} \) such that \( \overline{K}_{1+q_1 \ell} = \overline{K}_{1+q_2 \ell} \). Then we set \( \gamma = (K_1, \ldots, K_{q_1 \ell}) \) and \( \overline{C} = \gamma \ast (K_{q_1 \ell}, \ldots, K_{q_1 \ell}) \ast \gamma^{-1} \). We have \( \tau \overline{C} = (q_2 - q_1) \mathcal{C} = \mathcal{C} \ast \ldots \ast \mathcal{C} \).

\[ \Box \]

8. Restriction of the moduli part to log canonical centres

The goal of this section is to describe the restriction of the moduli part to a log canonical centre of \( (Y, \Sigma_f) \). Part of the results can be seen as a higher codimensional version of [FL19, Proposition 4.2]. We refer to [Hu20] for similar results.

**Definition 8.1.** (Definition 3.12 [FL19]) Let \( f: (X, \Delta) \to Y \) be an lc-trivial (respectively klt-trivial) fibration. Then \( f \) is acceptable if there exists another lc-trivial (respectively klt-trivial) fibration \( \tilde{f}: (\overline{X}, \overline{\Delta}) \to Y \) such that \( \overline{\Delta} \) is effective on the generic fibre of \( \tilde{f} \), and a birational morphism \( \mu: X \to \overline{X} \) such that \( f = \tilde{f} \circ \mu \) and \( K_X + \Delta \sim_Q \mu^*(K_{\overline{X}} + \overline{\Delta}) \). Note that
then the horizontal part of $\Delta^<0$ with respect to $f$ is $\mu$-exceptional. Note also that any birational base change of an acceptable lc-trivial (respectively klt-trivial) fibration is again an acceptable lc-trivial (respectively klt-trivial) fibration.

\[
(X, \Delta) \xrightarrow{\mu} (\overline{X}, \overline{\Delta}) \xrightarrow{\bar{f}} Y
\]

**Definition 8.2.** (Definition 3.10 [FL19]) Let $f: (X, \Delta) \to Y$ be an lc-trivial fibration, where $(X, \Delta)$ is log smooth and $Y$ is smooth. Fix a prime divisor $T$ on $Y$. An $(f, T)$-bad divisor is any reduced divisor $\Sigma_{f, T}$ on $Y$ which contains:

(a) the locus of critical values of $f$,
(b) the closed set $f(\text{Supp } \Delta_v) \subseteq Y$, and
(c) the set $\text{Supp } B_f \cup T$.

The next result is a corollary of [FL19, Proposition 4.2].

**Proposition 8.3.** Let $f: (X, \Delta) \to Y$ be an acceptable klt-trivial fibration, where $(X, \Delta)$ is a log smooth log canonical pair and $Y$ is a smooth Ambro model for $f$. Assume that there exists an $(f, 0)$-bad divisor $\Sigma_f \subseteq Y$ which has simple normal crossings, and such that the divisor $\Delta + f^* \Sigma_f$ has simple normal crossings support. Denote $\Delta_X = \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma_\Gamma f^* \Gamma$, where $\gamma_\Gamma$ are the generic log canonical thresholds with respect to $f$ as in Definition 2.9.

Let $Z = T_1 \cap \ldots \cap T_k$ be a log canonical centre of $(Y, \Sigma_f)$. Denote $\Xi_Z := (\Sigma_f - \sum T_i)|_Z$.

Let $S$ be a minimal log canonical centre of $(X, \Delta_X)$ over $Z$, which exists by [FL19, Lemma 4.1]. Let $f|_S: S \xrightarrow{h} Z' \xrightarrow{\tau} Z$ be the Stein factorisation, and let $R$ denote the ramification divisor of $\tau$ on $Z'$. Then:

(i) if $K_S + \Delta_S = (K_X + \Delta_X)|_S$, then $h: (S, \Delta_S) \to Z'$ is a klt-trivial fibration with $B_h \geq 0$,
(ii) $\tau^*(M_f|_Z) \sim_{Q} M_h + R' + E$, where $M_f$ is chosen so that $Z \not\subseteq M_f$ and $R' = \sum_{\Gamma \subseteq \Xi_Z} (\text{mult}_\Gamma R) \cdot \Gamma$ and $E = \sum_{\Gamma \subseteq \Xi_Z} (\text{mult}_\Gamma B_h) \cdot \Gamma$.

**Proof.** The proof follows the same line as [FL19, Proposition 4.2]. In particular, Steps 1-5 are the same: we find a birational map $\rho: (X, \Delta_X) \to (W, \Delta_W)$ over $Y$ such that, if $\psi: (W, \Delta_W) \to Y$ is the induced lc-trivial fibration.
fibration, then $(\phi^*\Sigma_f)_{\text{red}} \leq \Delta_{W,v}$. After replacing $T$ with $Z$ in Step 5, the fibration $h: (S, \Delta_S) \rightarrow Z'$ is klt-trivial.

**Step 6.** Let $T_1, \ldots, T_h$ be components of $\Sigma_f$ such that $Z = T_1 \cap \ldots \cap T_h$. By equation [FL19 (13)] every component $D_i$ of $\psi^*T_i$ which dominates $T_i$ has coefficient 1 in $\Delta_W$. Denote $\Delta_{D_i} := (\Delta_W - D_i)|_{D_i}$, so that the Stein factorisation of $\psi|_{D_i}: (D_i, \Delta_{D_i}) \rightarrow T_i$ gives an lc-trivial fibration. Let $\Xi_{T_i} = (\Sigma_f - T_i)|_{T_i}$ and let $P$ be a component of $(\psi|_{D_i})^*\Xi_{T_i}$. Since $(\psi|_{D_i})^*\Xi_{T_i} = (\psi^*\Sigma_f - \psi^*T_i)|_{D_i}$, and each component of $\psi^*\Sigma_f$ is a component of $\Delta_W$ by [FL19 (12) and (13)], this implies that $P$ is a component of $(\Delta_W^1 - D_i)|_{D_i} = \Delta_{D_i}^1$. In other words,

$$((\psi|_{D_i})^*\Xi_{T_i})_{\text{red}} \leq \Delta_{D_i}^1.$$

Assume that for $i > 1$ there are components $D_1, \ldots, D_i$ such that $\phi(D_j) = T_j$ and $((\psi|_{D_1 \cap \ldots \cap D_i})^*\Xi_{T_1 \cap \ldots \cap T_i})_{\text{red}} \leq \Delta_{D_1 \cap \ldots \cap D_i}^1$, where $\Xi_{T_1 \cap \ldots \cap T_i} = (\Sigma_f - T_1 - \ldots - T_i)|_{T_1 \cap \ldots \cap T_i}$ and $\Delta_{D_1 \cap \ldots \cap D_i} := (\Delta_W^1 - D_1 - \ldots - D_i)|_{D_1 \cap \ldots \cap D_i}$.

There is a component $D_{i+1}$ of $\psi^*T_{i+1}$ which has coefficient 1 in $\Delta_W$. Denote $\Delta_{D_1 \cap \ldots \cap D_{i+1}} = (\Delta_W - D_1 - \ldots - D_{i+1})|_{D_1 \cap \ldots \cap D_{i+1}}$, so that the Stein factorisation of $\psi|_{D_1 \cap \ldots \cap D_{i+1}}: (D_1 \cap \ldots \cap D_{i+1}, \Delta_{D_1 \cap \ldots \cap D_{i+1}}) \rightarrow T_1 \cap \ldots \cap T_{i+1}$ gives an lc-trivial fibration. Let $\Xi_{T_1 \cap \ldots \cap T_{i+1}} = (\Sigma_f - T_1 - \ldots - T_{i+1})|_{T_1 \cap \ldots \cap T_{i+1}}$ and let $P$ be a component of $(\psi|_{D_1 \cap \ldots \cap D_{i+1}})^*\Xi_{T_1 \cap \ldots \cap T_{i+1}}$. As before,

$$((\psi|_{D_1 \cap \ldots \cap D_{i+1}})^*\Xi_{T_1 \cap \ldots \cap T_{i+1}})_{\text{red}} \leq \Delta_{D_1 \cap \ldots \cap D_{i+1}}^1.$$

We proved by induction that there are $D_1, \ldots, D_h$ such that

$$((\psi|_{D_1 \cap \ldots \cap D_h})^*\Xi_Z)_{\text{red}} \leq \Delta_{D_1 \cap \ldots \cap D_h}^1.$$

Now, by [FL19 Proposition 2.6] there are components $D_1, \ldots, D_h$ of $\Delta_W$ and $S_1, \ldots, S_k$ of $\Delta_D^1$, where $D = D_1 \cap \ldots \cap D_h$, such that $S_W$ is a component of $S_1 \cap \ldots \cap S_k$, and note that the $S_i$ dominate $Z$. This and (17) imply

$$((\psi|_{D})^*\Xi_Z)_{\text{red}} \leq \Delta_D^1 - S_1 - \ldots - S_k,$$

hence

$$((\psi|_{S_W})^*\Xi_Z)_{\text{red}} \leq (\Delta_D^1 - S_1 - \ldots - S_k)|_{S_W} \leq \Delta_{S_W}^1.$$

Thus, for every prime divisor $P \subseteq \text{Supp} \tau^*\Xi_Z$, the generic log-canonical threshold $\gamma_P$ of $(S_W, \Delta_{S_W})$ with respect to $h^*_WP$ is zero. If we define

$$E := \sum_{\tau^*\Xi_Z \cap \Gamma} (\text{mult}_\Gamma B_{hW}) \cdot \Gamma = \sum_{\tau^*\Xi_Z \cap \Gamma} (\text{mult}_\Gamma B_h) \cdot \Gamma,$$

where the second equality follows from [FL19 (17)], then

$$B_{hW} = (\tau^*\Xi_Z)_{\text{red}} + E.$$

Finally, Steps 7 is the same after replacing $T$ with $Z$. \qed
Proposition 8.4. Let $f: (X, \Delta) \to Y$ be an acceptable klt-trivial fibration. Assume that $Y$ is an Ambro model for $f$ and that there exists a simple normal crossings divisor $R$ on $Y$ such that the support of the divisor $\Delta + f^{-1}\Sigma_f$ has simple normal crossings. Assume that $f$ is semistable. Set

$$\Delta_X = \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma_\Gamma f^* \Gamma$$

where $\gamma_\Gamma$ are the generic log canonical thresholds with respect to the klt-fibration $f$ as in Definition 2.9. Then there exists a birational map $\rho: X \rightarrow W$ and a fibration $\psi: W \to Y$ such that:

(a) the pair $(W, \Delta_W)$ is $\mathbb{Q}$-factorial dlt, where $\Delta_W := \rho_* \Delta_X$, and $\Delta_W \geq 0$;
(b) $\psi: (W, \Delta_W) \to Y$ is a klt-trivial fibration;
(c) $\rho: (X, \Delta_X) \to (W, \Delta_W)$ is crepant birational;
(d) the discriminant of $\psi$ is $\Sigma_f$ and the moduli part is $M_f$;
(e) $\Delta_{W,v} = \psi^* \Sigma_f$.

Let $Z$ be a log canonical centre of $(Y, \Sigma_f)$ and let $S$ be a minimal log canonical centre of $(W, \Delta_W)$ over $Z$. Let $\psi|_S: S \xrightarrow{h} Z' \xrightarrow{\tau} Z$ be the Stein factorisation.

(i) If $K_S + \Delta_S = (K_W + \Delta_W)|_S$, then $h: (S, \Delta_S) \to Z'$ is a klt-trivial fibration.
(ii) Assume that $\tau^* M_f|_Z = M_h$. Then $\Delta_{S,v} = h^* B_h$ and $B_h = (\tau^* \Sigma Z)|_{\text{red.}}$.
(iii) Let $Z$ be a component of $\Sigma_f$ such that $M_f|_Z \equiv 0$. Then $\tau^* M_f|_Z = M_h \sim_\mathbb{Q} 0$ and $\Delta_{S,v} = h^* B_h$.
(iv) If either $\tau^* M_f|_Z = M_h$ or $M_f|_Z \equiv 0$, then $h$ has reduced fibres over an open set meeting all the irreducible components of $B_h$.

Proof. Step 1. The existence of $\rho$ satisfying (a), (b), (c), (d) follows from Steps 2 and 3 of the proof of [FL19, Proposition 4.2]. We have then

(20) $$K_W + \Delta_W \sim_\mathbb{Q} \psi^*(K_Y + \Sigma_f + M_f).$$

The divisor $\Delta_{W,v}$ is reduced, and by [FL19, Proposition 4.2, (13)] $\Delta_{W,v} = (\psi^* \Sigma_f)|_{\text{red.}}$. As for (e) every component $D$ of $\Delta_{W,v}$ is a log canonical centre of $(W, \Delta_W)$. By [FL19, Lemma 2.8] there is a centre $D_X$ of $(X, \Delta_X)$ such that $\rho$ induces a birational map $\rho|_{D_X}: D_X \dashrightarrow D$. Therefore

$$\Delta_{W,v} = \sum_{i=1}^n D_i = \sum_{i=1}^n \rho_* D_i, \Delta_X = \rho_* f^* \Sigma_f$$

the last equality following from the semistability of $f$. Let $(p, q): Z \to X \times W$ be a resolution of the indeterminacy of $\rho$. Then $\rho_* f^* \Sigma_f = q_* p^* f^* \Sigma_f = q_* q^* \psi^* \Sigma_f = \psi^* \Sigma_f$ proving (e).

Step 2. The proof of (i) follows the same lines as [FL19, Proposition 4.2], which has slightly different hypotheses. We recall it here for completeness.
By restricting the equation (20) to $S$ we obtain
\begin{equation}
K_{S} + \Delta_{S} \sim_{\mathbb{Q}} (\psi|_{S})^{\ast}(K_{Z} + \Xi_{Z} + M_{f}|_{Z}),
\end{equation}
where $\Xi_{T} = (\Sigma_{f} - T)|_{T}$. Thus $h$ is an lc-trivial fibration, and moreover, it is a klt-trivial fibration. Indeed, if there existed a log canonical centre $\Theta$ of $(S, \Delta_{S})$ which dominated $T'$, then $\Theta$ would be a log canonical centre of $(W, \Delta_{W})$ by [Fuj07, Proposition 3.9.2], which contradicts the minimality of $S$. This proves (1).

Step 3. In order to show (ii) and (iii), denote by $M_{h}$ and $B_{h}$ the moduli part and the discriminant of $h$. From (21) we have
\begin{equation}
\tau^{\ast}(K_{Z} + \Xi_{Z} + M_{f}|_{Z}) = K_{Z'} + B_{h} + M_{h}.
\end{equation}
By [FL19] Lemma 2.8, there is a centre $S_{X}$ of $(X, \Delta_{X})$ such that $\rho$ induces a birational map $\rho: S_{X} \dashrightarrow S$. Moreover, if we define $\Delta_{S_{X}}$ by $K_{S_{X}} + \Delta_{S_{X}} = (K_{X} + \Delta_{X})|_{S_{X}}$, by (e) the restriction $\rho: (S_{X}, \Delta_{X}) \dashrightarrow (S, \Delta_{S})$ is crepant birational.

If $f|_{S_{X}} = \tau_{X} \circ h_{X}$ is the Stein factorisation, then we claim that $\tau_{X} = \tau$. Indeed, let $(p, q): W \to S_{X} \times S$ be the resolution of indeterminacies of the birational map $\rho|_{S_{X}}: S_{X} \dashrightarrow S$. Both $p$ and $q$ have connected fibres by Zariski’s main theorem, since $S_{X}$ and $S$ are normal. Then every curve contracted by $p$ is contracted by $h \circ q$, and thus $f|_{S_{X}}$ factors through $T'$ by the Rigidity lemma [Deb01, Lemma 1.15]. This proves the claim.

By [19] there exists an effective divisor $E$ such that
\begin{equation*}
B_{h} = (\tau^{\ast}\Xi_{Z})_{\text{red}} + E.
\end{equation*}
Write the Hurwitz formula for $\tau$ as $K_{Z'} = \tau^{\ast}K_{Z} + R$. Then
\begin{equation}
\tau^{\ast}(K_{Z} + \Xi_{Z}) = K_{Z'} + B_{h} - E + R + \tau^{\ast}\Xi_{Z} - (\tau^{\ast}\Xi_{Z})_{\text{red}}.
\end{equation}
We notice moreover that
\begin{equation*}
\tau^{\ast}\Xi_{Z} - (\tau^{\ast}\Xi_{Z})_{\text{red}} \leq R.
\end{equation*}

Step 4. We assume that $\tau^{\ast}M_{f}|_{Z} = M_{h}$ and we prove that $\Delta_{S_{v}} = (h^{\ast}B_{h})_{\text{red}}$. Then (21) becomes $\tau^{\ast}(K_{Z} + \Xi_{Z}) = K_{Z'} + B_{h}$. Equation (23) implies that $-E - R + \tau^{\ast}\Xi_{Z} - (\tau^{\ast}\Xi_{Z})_{\text{red}} = 0$. In particular $E = 0$ and
\begin{equation}
B_{h} = (\tau^{\ast}\Xi_{Z})_{\text{red}}.
\end{equation}
Therefore, by (e) by the fact that $S$ is a minimal log canonical centre of $(W, \Delta_{W})$ over $T$ and by (24) we have
\begin{equation}
\Delta_{S_{v}} = (h^{\ast}B_{h})_{\text{red}}.
\end{equation}

Step 5. We assume that $M_{f}|_{Z} \equiv 0$ and we prove that $\Delta_{S_{v}} = (h^{\ast}B_{h})_{\text{red}}$ and $\tau^{\ast}M_{f}|_{Z} = M_{h} \sim_{\mathbb{Q}} 0$.

Equations (22) and (24) imply that $\tau^{\ast}(M_{f}|_{Z}) \geq M_{Z'}$. Since $M_{f}|_{Z} \equiv 0$ and $M_{h}$ is pseudoeffective by Theorem 2.12 and Remark 2.13 we get $\tau^{\ast}(M_{f}|_{Z}) = M_{Z'}$. In particular, $M_{h} \equiv 0$, hence $M_{h} \sim_{\mathbb{Q}} 0$ by Theorem 2.12.
Moreover, \( \tau^*(K_Z + \Xi_Z) = K_Z + B_h \) and \( E = 0 \), proving that \( B_h = (\tau^*\Xi_Z)_{\text{red}} \). Therefore, by [e] by the fact that \( S \) is a minimal log canonical centre of \((W, \Delta_W)\) over \( T \) and by [24], we have

\[
\Delta_{S,v} = (h^*B_h)_{\text{red}}.
\]

**Step 6.** Assuming that \( \Delta_{S,v} = (h^*B_h)_{\text{red}} \), we prove that \( \Delta_{S,v} = h^*B_h \). By Remark 2.3 the fibration \( h \) has reduced fibres.

To prove [ii] we reason as in [e]. Let \( D \) be an irreducible component of \( \Delta_{S,v} \). Then \( D \) is a log canonical centre of \((S, \Delta_S)\) and therefore of \((W, \Delta_W)\). By [PL19, Lemma 2.8] there is a log canonical centre \( D_X \) of \((X, \Delta_X)\) such that \( \rho \) induces a birational map \( D_X \to D \).

Then

\[
\Delta_{S,v} = \sum_{i=1}^{n} D_i = \sum_{i=1}^{n} (\rho|_{S_X})_*D_{i,X} = (\rho|_{S_X})_* (\Delta_{S_X,v})_{1} = (\rho|_{S_X})_*h_X^*B_h = h^*B_h.
\]

**Step 7.** Finally, [iv] follows directly from Step 6, as \( h^*B_h \) is a reduced divisor.

\[\square\]

9. **Finiteness of the equivalence relation for the moduli part**

This section is devoted to the proof of the finiteness of the equivalence relation induced by \( \mathcal{O}_Y(mM_f) \) on a connected divisor \( T \).

**Assumption 9.1.** We consider the following set of assumptions on a triple \((f: (X, \Delta) \to Y, T, \Sigma_f)\) or \((f, T, \Sigma_f)\) for short.

1. \( f: (X, \Delta) \to Y \) is an acceptable klt-trivial fibration;
2. \( \Sigma_f \) is a simple normal crossings divisor and is an \((f, T)\)-bad for every \( T \subseteq T \);
3. for every \( T \subseteq T \) the restriction \( \mathcal{O}_T(mM_f) \) is semiample and we denote by \( \phi_T \) the induced fibration;
4. \( f \) is semistable.

In particular by [Kol07a, Proposition 8.4.9, Definition 8.3.6, Theorem 8.5.1] the base \( Y \) is an Ambro model and \( T \) is simple normal crossing.

**Theorem 9.2.** Let \((f: (X, \Delta) \to Y, T, \Sigma_f)\) be a triple satisfying Assumption 9.1. Let \( m \) be a positive integer such that \( mM_f \) is a Cartier divisor and let \( \mathcal{L} = \mathcal{O}(mM_f) \). Assume Conjecture 1.1. Then the equivalence relation \( R_\mathcal{L} \) is finite.

The following lemma is a higher-codimensional version of [FL19, Proposition 4.4] (see also [Hu20]).

**Lemma 9.3.** Let \((f, T, \Sigma_f)\) be a triple satisfying Assumption 9.1(1,2,3). Let \( \mathcal{P}, \overline{\mathcal{P}} \) be two sets of log-canonical centres of \( \Sigma_f \) such that
(i) if $P, Q \in \mathcal{P}$ (resp. $\overline{P}, \overline{Q} \in \overline{\mathcal{P}}$) then $P \subseteq Q$ implies $P = Q$ (resp. $\overline{P} \subseteq \overline{Q}$ implies $\overline{P} = \overline{Q}$).

(ii) for every $\overline{P} \in \overline{\mathcal{P}}$ there is $P \in \mathcal{P}$ such that $\overline{P} \subseteq P$.

(iii) whenever $\overline{P} \subseteq P \subseteq T$ we have $\phi_T(\overline{P}) = \phi_T(P)$.

Let $\mathcal{P} \to \text{Nklt}(X, \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma \tau f^* \Gamma)$ be a function such that $P \mapsto S_X(P)$ and $S_X(P)$ is minimal over $P$. For every pair $(P, \overline{P})$ such that $\overline{P} \subseteq P$ let $R_X(P, \overline{P})$ be a log-canonical centre of $(X, \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma \tau f^* \Gamma)$ minimal over $\overline{P}$ and such that $R_X(P, \overline{P}) \subseteq S_X(P)$. Then there is a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\eta} & X \\
\downarrow f' & & \downarrow f \\
Y_0 & \xrightarrow{\varepsilon} & Y
\end{array}
\]

where $\varepsilon$ is a birational morphism with the following properties. For every $P \in \mathcal{P}$ (resp. $\overline{P} \in \overline{\mathcal{P}}$) let $P_0$ (resp. $\overline{P}_0$) be the strict transform of $P$ and $S_0$ the strict transform of $S_X(P)$ (resp. $R_X(P, \overline{P})$). Let $f_0|_{S_0}: S_0 \xrightarrow{h} P_0$ (resp. $f_0|_{R_0}: R_0 \xrightarrow{g} \overline{P}_0$) be the Stein factorisation. Then the following hold:

1. $\varepsilon$ is an isomorphism at the generic point of every subvariety $P \in \mathcal{P}$.
2. $\varepsilon$ is an isomorphism at the generic point of $T \cap T'$ for every $T, T' \subseteq \mathcal{T}$.
3. $\eta$ is a desingularisation of the fibre product which is an isomorphism over $Y' \setminus \text{Exc}(\varepsilon)$.
4. for every $P \in \mathcal{P}$ we have $M_h = \tau^* M_f|_{P_0}$ and $P_0'$ is an Ambro model.
5. for every $\overline{P} \in \overline{\mathcal{P}}$ we have $M_g = \sigma^* M_f|_{\overline{P}_0}$ and $\overline{P}_0'$ is an Ambro model.
6. $\varepsilon^{-1} \Sigma_f$ has simple normal crossings.

\textbf{Proof.} We say that $P \in \mathcal{P}$ satisfies $(\star)$ if, denoting by $f|_{S_X}: S_X \xrightarrow{h_X} P' \xrightarrow{\tau_X} P$ the Stein factorisation, we have $M_h \subseteq \tau^* M_f|_P$ and $P'$ is an Ambro model. We prove by induction on the cardinality of

$$\mathcal{P}' = \{P \in \mathcal{P} | P \text{ does not satisfy } (\star)\}$$

that there is $\varepsilon$ satisfying (1-4) and (6). If the cardinality of $\mathcal{P}'$ is zero, there is nothing to prove.

Otherwise, we pick $P \in \mathcal{P}$. By [Fuj07, Proposition 3.9.2] there are $D_1, \ldots, D_{\ell} \subseteq \text{Supp} \Delta^{=1}$ such that $S_X = D_1 \cap \ldots \cap D_{\ell}$. We set $\Delta_{S_X} = (\Delta_{X_2} - \sum D_i)|_{S_2}$.

Let $f|_{S_X}: S_X \xrightarrow{h_X} P' \xrightarrow{\tau_X} P$ be the Stein factorisation. By Proposition 3.3 the morphism $h_X$ is a klt-trivial fibration and there is an effective divisor $E$ such that $M_h = \tau^* M_f|_P - E$. Let $P \subseteq T$, let $C$ be a general curve in $P$.
contained in a fibre of $\phi_T$ and let $\tilde{C}$ be a curve in $P'$ such that $\tau_X(\tilde{C}) = C$. Then

$$0 \leq M_{h_X} \cdot \tilde{C} = M_f \cdot C - E \cdot \tilde{C} \leq 0.$$ 

Therefore $E$ is a vertical divisor with respect to $\phi_T \circ \tau_X$. We call $E$ the union of the components of $\tau_X(\text{Supp } E)$ which are not of components of $T \cap T'$ for some $T, T' \subseteq T$.

We let $\varepsilon: Y_0 \to Y$ be the composition of the blow up $\mu: Y_1 \to Y$ of $E$ with a log resolution of $(Y_0, \mu^{-1}\Sigma_f)$ centered in the singular locus. Let $X_0$ be a normalisation of the main component of the base change followed by a desingularisation centered in the singular locus, with the natural map $f_0: X_0 \to Y_0$. Since $\tau_X(\text{Supp } E)$ is vertical with respect to $\phi_T$, the divisor $E$ satisfies the same property. Therefore, if $P \subseteq P$, the morphism $\varepsilon$ is an isomorphism on the generic point of $P$ as this subvariety is such that $\phi_T(P) = \phi_T(P)$. If $P \not\subseteq P$ or $Q \in P$ and $Q \neq P$, the morphism $\varepsilon$ is obviously an isomorphism on the generic point of $P$ or $Q$. Moreover, it is an isomorphism at the generic point of the intersections $T \cap T'$.

Following the proof of [FL19, Proposition 4.4], replacing $T$ with $P$ and [FL19 Proposition 4.2(ii)] with Proposition [S3,3(ii)], we have that, if $S_0$ is the strict transform of $S_X$ in $X_0$, $P_0$ is the strict transform of $P$ in $Y_0$ and $S_0 \to P_0' \to P_0$ is the Stein factorisation, then $\tau_0^*M_{f_0} = M_{h_0}$ and $P_0'$ is an Ambro model.

Let $Q \in P$ satisfying property (\star). There is a diagram

$$
\begin{array}{ccc}
S(Q)_0 & \xrightarrow{\eta} & S_X(Q) \\
\downarrow h_0 & & \downarrow h_X \\
Q'_0 & \xrightarrow{\zeta} & Q' \\
\downarrow \tau_0 & & \downarrow \tau_X \\
Q_0 & \xrightarrow{\varepsilon} & Q
\end{array}
$$

By applying $\zeta^*$ to $\tau^*M_f|Q = M_{h_X}$ we get

$$\tau_0^*M_{f_0}|Q_0 = \tau_0^*\varepsilon^*M_f|Q = \zeta^*\tau_X^*M_f|Q = \zeta^*M_h = M_{h_0}.$$ 

Since $Q'$ is an Ambro model and $\zeta$ is birational, $Q'_0$ is one too.

Let $P_0$ be the set of strict transforms of elements of $P$. Then the cardinality of the set $\{P \in P_0| P$ does not satisfy (\star)\} is at most $|P'| - 1$ and we conclude by induction.

As for (5), the proof is completely analogous. □

Proof of Theorem 9.2. Assume that $R_\mathcal{L}$ is not a finite equivalence relation. By Proposition 4.112 there is $Z \subseteq \sqcup V$ and a subrelation $R' \subseteq R_\mathcal{L}$ such that $Z$ is $R'$-invariant $R'|_Z$ is equidimensional and the set of infinite equivalence classes is dense in $Z$.
Let $P \subseteq \phi^{-1}Z$ be an irreducible component surjecting onto an irreducible component of $Z$. Then $\mathcal{L}|_P$ is not big. Indeed, if it were big, then $\phi|_P$ would be a birational morphism and generically on $\phi(P)$ the induced equivalence relation would be the gluing $\sqcup T \to \mathcal{T}$, thus finite.

**Step 1.** We can assume that every irreducible component of $\phi^{-1}Z$ is a log canonical centre of $(Y, \Sigma_f)$.

Indeed, let $\phi^{-1}Z = W_1 \cup \ldots \cup W_k$ be the decomposition into irreducible components. We can assume that there is $h$ such that $W_i$ is a centre of $(Y, \Sigma_f)$ for $i > h$. Let $\delta: Y_1 \to Y$ be such that $\delta^{-1}(W_1 \cup \ldots \cup W_h \cup \Sigma_f)$ has simple normal crossings. The morphism $\delta$ is an isomorphism over the generic point of $T$ and $T \cap T'$ for every $T, T' \subseteq \mathcal{T}$. Let $\eta': X' \to X$ be the natural morphism followed by a desingularisation of the main component of $X \times_Y Y_1$ and set $K_{X'} + \Delta' = \eta''(K_X + \Delta)$. Let $\eta_1: X_1 \to X'$ be a log resolution of $(X', \Delta')$. We can assume that the birational morphism $X_1 \to X$ is an isomorphism on $Y \setminus \delta \text{Exc}(\delta)$. Let $f_1: X_1 \to Y_1$ be the natural morphism and we define $\Delta$ by $\Delta = \Delta_1 = \eta_1^*(K_{X_1} + \Delta_1)$. Indeed, let $\Delta' = a^*(K_{X_1} + \Delta)$ and $\eta = \varepsilon_X \circ a$. Thus $(\tilde{X}, \tilde{\Delta})$ is log smooth, $\tilde{f}: (\tilde{X}, \tilde{\Delta}) \to \tilde{Y}$ is acceptable and $\tilde{f}^{-1}\Sigma_f$ has simple normal crossings. Thus $\Sigma_f$ has simple normal crossings.

We apply Theorem 8.5.1 to $X_1, Y_1$, with $Z = \text{Supp} \Delta_{X_1} \cup f_1^{-1}\delta^{-1}\Sigma_f$. We get $a, b: (\tilde{X}, \tilde{Y}) \to (X_1, Y_1)$ étale outside $\text{Exc}(\delta)$. Let $\Sigma_{\tilde{f}} = b^{-1}\delta^{-1}\Sigma_f$. Then $\tilde{f}^{-1}\Sigma_{\tilde{f}} \cup a^{-1}\text{Supp} \Delta_1$ has simple normal crossings support. Define $\tilde{\Delta}$ by $K_{\tilde{X}} + \tilde{\Delta} = a^*(K_{X_1} + \Delta_1)$ and $\eta = \varepsilon_X \circ a$. Thus $(\tilde{X}, \tilde{\Delta})$ is log smooth, $\tilde{f}: (\tilde{X}, \tilde{\Delta}) \to \tilde{Y}$ is acceptable and $\tilde{f}^{-1}\Sigma_{\tilde{f}}$ has simple normal crossings. Thus $\Sigma_f$ has simple normal crossings.

We let $\tilde{T}$ be the strict transform of $\mathcal{T}$. By [Kol07a, Proposition 8.4.9, Definition 8.3.6, Theorem 8.5.1] the variety $\tilde{Y}$ is an Ambro model. Then $(\tilde{f}: (\tilde{X}, \tilde{\Delta}) \to \tilde{Y}, \tilde{T}, \Sigma_{\tilde{f}})$ satisfies Assumption 9.1. We set $\theta = b \circ \delta$. Then $\theta$ is a generically finite morphism satisfying the hypothesis of Corollary 9.8. If $\sigma$ is as in Corollary 9.9 then $\sigma^*\mathcal{R}' \subseteq \mathcal{R}_\theta \mathcal{L}$ and $\sigma^{-1}Z$ is $\sigma^*\mathcal{R}'$-invariant. We have $\tilde{\phi}^{-1}\sigma^{-1}Z = \theta^{-1}\phi^{-1}Z = b^{-1}\delta^{-1}\phi^{-1}Z$.

By our construction $\delta^{-1}\phi^{-1}Z$ is a union of log canonical centres of the log smooth pair $(Y_1, \delta^{-1}\Sigma_f)$. Since $\Sigma_{\tilde{f}} = b^{-1}\delta^{-1}\Sigma_f$ has simple normal crossings, the set $b^{-1}\delta^{-1}\phi^{-1}Z$ is a union of log canonical centres of $(\tilde{Y}, \Sigma_{\tilde{f}})$.

**Step 2.** Let $P, Q$ be irreducible components of $\phi^{-1}Z$ such that either there exists $T$ with $P, Q \subseteq T$ and $\phi_T(P) = \phi_T(Q)$ or $P \subseteq T, Q \subseteq T'$ and $\phi_T(P) = \phi_T(Q \cap T)$. Let $H_\alpha$ be ample divisors such that $\Sigma_f + \sum H_\alpha$ has simple normal crossings and the restriction of $\phi_T$ to $P \cap Q \cap \cap H_\alpha$ is generically finite and surjective. We set $\mathcal{T}' = P \cap Q \cap \cap H_\alpha$. By replacing $\Delta$ with $\Delta + \sum \alpha f^*H_\alpha$ and $\Sigma_f$ with $\Sigma_f + \sum \alpha H_\alpha$ we can assume that $\mathcal{T}'$ is a log canonical centre of $(Y, \Sigma_f)$.

We set

$$
P = \{ P \subseteq \phi^{-1}Z \text{ irreducible component} \}$$

$$
\mathcal{P} = \{ \mathcal{P} \subseteq \phi^{-1}Z \text{ log canonical centre of } (Y, \Sigma_f) \text{ such that } \phi_{\mathcal{P}} \text{ is finite} \}. $$
ON THE MODULI PART

Let \( P \to Nklt(X, \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma_{\Gamma} f^* \Gamma) \) be a function such that \( P \to S_X(P) \) and \( S_X(P) \) is minimal over \( P \). For every pair \( (P, \overline{P}) \) such that \( \overline{P} \subseteq P \) let \( R_X(P, \overline{P}) \) be a log-canonical centre of \( (X, \Delta + \sum_{\Gamma \subseteq \Sigma_f} \gamma_{\Gamma} f^* \Gamma) \) minimal over \( \overline{P} \) and such that \( R_X(P, \overline{P}) \subseteq S_X(P) \).

Then \( P \) and \( \overline{P} \) satisfy the hypotheses of Lemma [4.3] and there is a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\theta} & X \\
\downarrow f_0 & & \downarrow f \\
Y_0 & \xrightarrow{\epsilon} & Y \\
\end{array}
\]

with \( \epsilon \) birational and such that the exceptional locus does not contain any of the \( P \in P, \overline{P} \in \overline{P} \) or \( T \cap T' \) and for every \( P \in P \) we have \( M_h = \tau^* M_{f_0} |_{H} \); for every \( \overline{P} \in \overline{P} \) we have \( M_g = \sigma^* M_{f_0} |_{\overline{P}} \) (notation as in Lemma [4.3]). We define \( \Delta_0 \) by \( K_{X_0} + \Delta_0 = \eta^*(K_X + \Delta) \).

We apply Theorem 2.2 to \( X_0, Y_0, \) with \( Z = \text{Supp} \Delta_0 \cup f_0^{-1} \varepsilon^{-1} \Sigma_f \). We get \( a, b : (\tilde{X}, \tilde{Y}) \to (X_0, Y_0) \) étale outside \( \text{Exc}(\varepsilon) \). Let \( \Sigma_f = b^{-1} \varepsilon^{-1} \Sigma_f \). Then \( \tilde{f}^{-1} \Sigma_f \cup a^{-1} \text{Supp} \Delta_0 \) has simple normal crossings support. Define \( \tilde{\Delta} \) by \( K_{\tilde{X}} + \tilde{\Delta} = a^*(K_{X_0} + \Delta_0) \). Thus \( (\tilde{X}, \tilde{\Delta}) \) is log smooth, \( \tilde{f} : (\tilde{X}, \tilde{\Delta}) \to \tilde{Y} \) is acceptable and \( \tilde{f}^{-1} \Sigma_f \) has simple normal crossings. This implies that \( \Sigma_f \) has simple normal crossings. By [Kol07a] Proposition 8.4.9, Definition 8.3.6, Theorem 8.5.1 \( \) the variety \( \tilde{Y} \) is an Ambro model.

We let \( \overline{T} \) be the strict transform of \( T \). Then \( (\tilde{f} : (\tilde{X}, \tilde{\Delta}) \to \tilde{Y}, \tilde{T}, \Sigma_f) \) satisfies Assumption [9.4]. We set \( \theta = b \circ \varepsilon \). Then \( \theta \) is a generically finite morphism satisfying the hypothesis of Corollary 5.9. If \( \sigma \) is as in Corollary 5.9 then \( \sigma^* R' \subseteq R_{\theta^* \xi} \) and \( \sigma^{-1} Z \) is \( \sigma^* R' \)-invariant. We have \( \phi^{-1} \sigma^{-1} Z = \theta^{-1} \phi^{-1} Z = b^{-1} \delta^{-1} \phi^{-1} Z \).

As \( \varepsilon \) is an isomorphism on the general point of every component of \( \phi^{-1} Z \), the preimage \( \varepsilon^{-1} \phi^{-1} Z \) is a union of log canonical centres of \( (Y_0, \varepsilon^{-1} \Sigma_f) \). Moreover \( \varepsilon^{-1} \Sigma_f \) has simple normal crossings by Lemma [9.3]. Since \( \Sigma_f = b^{-1} \varepsilon^{-1} \Sigma_f \) has simple normal crossings, the set \( b^{-1} \delta^{-1} \phi^{-1} Z \) is a union of log canonical centres of \( (\tilde{Y}, \Sigma_f) \).

We prove now that for every \( P \in P \), if \( P_1 \) is the strict transform of \( P \) in \( \tilde{Y} \) and \( S_1 \) is the strict transform of \( S_X(P) \) in \( X_1 \) and \( \tilde{f}^1 |_{S_1} : S_1 \xrightarrow{h_1} P_1 \rightarrow X_1 \) is the Stein factorisation, then \( M_{h_1} = \tau^* M_{f_1} |_{P_1} \) and \( P_1 \) is an Ambro model.

(The same proof will imply that for every \( \overline{P} \in \overline{P} \) if \( \overline{P}_1 \) is the strict transform in \( \tilde{Y} \) and \( R_1 \) is the strict transform of \( R_X(P, \overline{P}) \) in \( X_1 \) and \( f_0 |_{R_0} : R_0 \xrightarrow{g} \overline{P}_0 \xrightarrow{\sigma} \overline{P}_0 \) is the Stein factorisation, then \( M_g = \sigma^* M_{f_0} |_{\overline{P}_0} \).)
We have a diagram

\[
\begin{array}{ccc}
S_1 & \xrightarrow{\alpha} & S_0 \\
\downarrow h_1 & & \downarrow h_0 \\
P'_1 & \quad & P'_0 \\
\downarrow \tau_1 & & \downarrow \tau_0 \\
P_1 & \xleftarrow{\varepsilon} & P_0
\end{array}
\]

Every curve contracted by \( h_1 \) is contracted by \( h_0 \circ \alpha \). Therefore by the Rigidity lemma there is a generically finite morphism \( P'_1 \to P'_0 \). By Lemma 2.14 \( P'_1 \) is an Ambro model. Then

\[
\tau_1^* M_f |_{P_1} = \tau_1^* b^* M_{f_0} |_{P_0} = \nu^* \sigma^* \tau_0^* M_{f_0} |_{P_0} = \nu^* \sigma^* M_{h_0} = M_{h_1}.
\]

By replacing \((X, \Delta)\) with \((\bar{X}, \bar{\Delta} + \sum_{i \in \Sigma} \gamma_i \bar{f}_* \Gamma), \Sigma_f \) with \( \Sigma_{\bar{f}} \) we can make the following

**Assumption 9.4.**  
1. Every irreducible component of \( \phi^{-1} Z \) is a log canonical centre of \((Y, \Sigma_f)\).
2. for every \( P \in \mathcal{P} \) we have \( M_{h_X} = \tau_X^* (M_f |_P) \),
3. for every \( \bar{P} \in \overbar{\mathcal{P}} \) we have \( M_{g_X} = \sigma_X^* (M_f |_{\overbar{\mathcal{P}}}) \)

**Step 3.** We run now an MMP with scaling as in [FL19 Proposition 4.2]. By Proposition 8.4 there is \( \rho: (X, \Delta_X) \to (W, \Delta_W) \) such that \( \psi^* \Sigma_f = \Delta_{W,v} \).

By [FL19 Lemma 2.8] for every \( P, \overbar{P} \) there are log canonical centres \( S \) and \( R \) of \((W, \Delta_W)\) with birational morphisms induced by \( \rho \)

\[
\rho|_{S_X(P)}: S_X(P) \to S \quad \rho|_{R_X(P, \overbar{P})}: R_X(P, \overbar{P}) \to R.
\]

Let \( P \) be a component of \( \phi^{-1} Z, \overbar{P} \subseteq P \subseteq T \) as above and let \( S \) be the strict transform of \( S_X(P) \), \( R \) of \( R_X(P, \overbar{P}) \) and \( \Delta_S, \Delta_R \) defined by adjunction.

Let \( \phi|_S: S \xrightarrow{h} P' \xrightarrow{\tau} P \) and \( \phi|_R: R \xrightarrow{g} \overbar{P}' \xrightarrow{\sigma} \overbar{P} \) be the Stein factorisations. Then \( M_h = \tau^* (M_f |_P) = \tau^*_X (M_{g_X} |_{\overbar{P}}) \) and \( M_g = \sigma^* (M_f |_{\overbar{P}}) = \sigma^*_X (M_{g_X} |_{\overbar{P}}) \).

By Proposition 8.4(ii) we have \( \Delta_S - h^* B_h \geq 0 \).

Then we can apply Proposition 2.21 and there are non empty sets \( Z_0, P'_0, P'_r \), where \( P'_r \) be the set of points \( x \) such that \( h^{-1} x \) is reduced, \( Z_0 \) and \( P'_0 \) are open, the complement of \( P'_0 \) in \( P' \) has codimension at least 2 and \( I(P') \supseteq P'_0 \cap T \) with the following property: for every \( x_1, x_2 \in I(P') \) such that \( \phi_T(x_1) = \phi_T(x_2) \), if \( (F_i, \Delta_i) \) is the fibre over \( x_1 \) with \( \Delta_i = \Delta^h |_{F_i} \), then \( (F_1, \Delta_1) \equiv (F_2, \Delta_2) \).

We claim that \( \tau^{-1} \overbar{P} \) meets the set \( I(P') \) and that \( R \) is a connected component of \( h^{-1} \tau^{-1} \overbar{P} \). We prove the claim in Step 4. Assuming the claim, we finish the proof.
We denote by $\Lambda_{\mathcal{P}} \subseteq \mathcal{P}$ the locus where $g: R \to \mathcal{P}$ has non-reduced fibres. Every fibre over $\mathcal{P} \setminus \Lambda_{\mathcal{P}}$ is isomorphic, with the boundary, to a fibre over $P \setminus \mathcal{P}$.

Let $\mathcal{P} \subseteq P_1 \cap P_2$, let $R_i$ be the strict transform of $R_X(P_i, \mathcal{P})$. By [Kol13, 4.45(1) and 4.45.8] there is a crepant birational map $(R_1, \Delta_{R_1}) \to (R_2, \Delta_{R_2})$ over $\mathcal{P}$. Let $g_i: (R_i, \Delta_{R_i}) \to \mathcal{P}$ for $i = 1, 2$ be the induced klt-trivial fibrations. For $x \in \mathcal{P}$ general the fibre of $g_1$ over $x$ is crepant birational to the fibre of $g_2$ over $x$.

Consider the set

$$\Lambda = \bigcup_{\mathcal{P} \subseteq P \subseteq T} \mathcal{R}' \phi_T(\Lambda_{\mathcal{P}}).$$

The set $\Lambda$ is a countable union of proper closed subsets of $\mathcal{Z}$. Since it is closed under $\mathcal{R}'$, the infinite equivalence classes $[x]$ of $\mathcal{R}'$ such that $[x] \subseteq \mathcal{Z} \setminus \Lambda$ form a dense subset of $\mathcal{Z} \setminus \Lambda$.

Fix $\mathcal{P}$ and let $R$ be a minimal log canonical centre of $(W, \Delta_W)$ over $\mathcal{P}$, with $g: (R, \Delta_R) \to \mathcal{P}'$ the klt-trivial fibration. By the discussion above, if $[x] \subseteq \mathcal{Z} \setminus \Lambda$ then for every $x_1, x_2 \in \phi^{-1}[x] \cap \mathcal{P}$ the fibres over $x_1$ and $x_2$ are crepant birational to each other, with their boundaries.

Since the classes $[x] \subseteq \mathcal{Z} \setminus \Lambda$ form a dense subset of $\mathcal{Z} \setminus \Lambda$, the union of the intersections $\phi^{-1}[x] \cap \mathcal{P}$ is a dense subset of $\mathcal{P}$. By construction, if $[x] \subseteq \mathcal{Z}$ then $\phi^{-1}[x] \cap \mathcal{P}$ is an infinite set.

On the other hand, we have by construction $M_g = \sigma^* M_f|_{\mathcal{P}} = \sigma^* \phi_T^* A$ where $A$ is an ample divisor on $V$. As $\phi_T|_{\mathcal{P}}$ is generically finite, $M_g$ is big.

By Proposition 2.24 the variation of $g$ is maximal.

If $\dim R - \dim \mathcal{P} = \dim W - \dim Y$, then the crepant birational fibres are in fact isomorphic and by Proposition 2.23 there is a finite number of fibres isomorphic to a fixed general one.

If $\dim R - \dim \mathcal{P} = \dim W - \dim Y$, then by Conjecture 1.1 there is a finite number of fibres crepant birational to a fixed general one.

**Step 4.** We prove that $\tau^{-1}\mathcal{P}$ meets the set $I(P')$ and that $R$ is a connected component of $h^{-1}\tau^{-1}\mathcal{P}$. Let $P = T_1 \cap \ldots \cap T_k$ with $T_i \subseteq \Sigma_f$ and $\Xi_P = (\Sigma_f - T_1 - \ldots - T_k)|_P$.

First, we prove the following statement:

**Claim 9.5.** Let $Q$ be a component of $\Xi_P$ such that $\phi_T(Q) = \phi_T(P)$. Then every irreducible component of $\tau^{-1}Q$ meets $P_0' \cap \tau^{-1}\phi^{-1}Z_0 \cap P'$ and every connected component of $h^{-1}\tau^{-1}Q$ is irreducible and a minimal log canonical centre over $Q$.

Since $M_h = \tau^* M_\psi|_P$, by Proposition 8.4(ii) we have $B_h = (\tau^* \Xi_P)|_{\text{red}}$. Thus $\tau^{-1}Q \subseteq \text{Supp} B_h$. Let $Q' \subseteq \tau^{-1}Q$. Since the complement of $P_0'$ in $P'$ has codimension 2, $Q'$ meets $P_0'$. Since $\phi_T(Q) = \phi_T(P)$, $Q'$ meets $\tau^{-1}\phi^{-1}Z_0$.

Finally, every irreducible component of $h^{-1}Q'$ is a log canonical centre of $(W, \Delta_W)$, therefore $\rho$ is an isomorphism at its generic point and the
restriction of $\psi$ to it has generically reduced fibre. We proved that $Q'$ meets $I(P')$.

Let $K$ be a connected component of $h^{-1}Q'$. Then the general fibre of $\psi|_{K}$ is isomorphic to a fibre of $h$. Thus $K$ is irreducible and $(K, \Delta_K)$ is generically klt over $Q$, ending the proof of Claim 9.5.

We prove now the statement on $\overline{P}$ by induction on the codimension of $\overline{P}$ in $P$. If the codimension is 1, it follows from Claim 9.5. If the codimension is at least 2, there is a component $Q$ of $\Xi_P$ such that $\overline{P} \subseteq Q$. By 9.3, every connected component $K$ of $h^{-1}\tau^{-1}Q$ is irreducible and a minimal log canonical centre of $(W, \Delta_W)$ over $Q$. Let $\psi|_K : S \xrightarrow{\ell} Q' \xrightarrow{\vartheta} Q$ be the Stein factorisation, let $\vartheta : Q' \to P'$ be the induced finite map. By Proposition 10.2, $\Delta_K - \ell^*B_\vartheta \geq 0$. Then we can apply Proposition 2.21 and there is a set $I(Q')$. We notice that $I(Q') = \theta^{-1}I(P')$. By the inductive hypothesis $\theta^{-1}\tau^{-1}\overline{P}$ meets $I(Q')$. Thus $\tau^{-1}\overline{P}$ meets $I(P')$.

$\square$

10. Triviality of the moduli part on pseudo-fibres

This section is entirely devoted to the proof of our second main technical result: if the moduli part is numerically zero along a simple normal crossings reducible connected variety, then it is torsion along it.

**Theorem 10.1.** Let $f : (X, \Delta) \to Y$ be an acceptable klt-trivial fibration, where $(X, \Delta)$ is a log smooth log canonical pair and $Y$ is a smooth Ambro model for $f$. Let $T$ be a connected divisor such that there is a simple normal crossings $(f, T)$-bad divisor $\Sigma_f$ and such that the restriction of $M_f$ to $T$ is semistable for every $T \subseteq T$. Let $m$ be a positive integer such that $mM_T$ is a Cartier divisor.

Set $L = O_Y(mM_f)$. Assume that $R_L$ is a finite equivalence relation. Then for a general equivalence class $[x]$ of $R_L$ the restriction of $L$ to $T_{[x]}$ is torsion.

**Proof.** Step 1. Since $R_L$ is a finite equivalence relation, the set $T_{[x]}$ is a finite union of irreducible subvarieties of $Y$.

As $[x]$ is general, the subvariety $T_{[x]}$ has simple normal crossings in the sense of $\mathbb{3}$.

Let $\varepsilon : Y' \to Y$ be a birational morphism such that $\varepsilon^{-1}T_{[x]}$ is divisorial and $\varepsilon^*\Sigma_f$ has simple normal crossings support. By Lemma $[\tilde{\mathcal{F}}, 20]$ the restriction $L|_{T_{[x]}}$ is torsion if and only if $\varepsilon^*L|_{\varepsilon^{-1}T_{[x]}}$ is torsion. Let $X'$ be a normalisation of the main component of $X \times_Y Y'$ with $\varepsilon_X : X' \to X$ and $f' : X' \to Y'$ the induced morphisms. By Ambro04, Proposition 5.5] we have $M_{f'} = \varepsilon^*M_f$. Then for every $T \subseteq \varepsilon^{-1}T_{[x]}$ we have $M_{f'}|_T \equiv 0$.

Let $(a, b) : (\tilde{X}, \tilde{Y}) \to (X', Y')$ be a semistable reduction such that $b^{-1}\varepsilon^{-1}\Sigma_f$ and $a^{-1}\varepsilon^{-1}(\Delta + f^*\Sigma_f)$ have simple normal crossings supports. By Lemmas $[\tilde{\mathcal{F}}, 13]$ and $[\tilde{\mathcal{F}}, 20]$ the pullback $\varepsilon^*L|_{\varepsilon^{-1}T_{[x]}}$ is torsion if and only if $b^*\varepsilon^*L|_{b^{-1}\varepsilon^{-1}T_{[x]}}$
is. After replacing $X, Y, f$ with $\tilde{X}, \tilde{Y}, \tilde{f}$ we can assume that $f$ is semistable. After replacing $\mathcal{T}$ with $b^{-1} \varepsilon^{-1} t_{[\varepsilon]}$ we have to prove that the restriction of $M_f$ to the divisor $\mathcal{T}$ is torsion. By Proposition 8.4 (iii) for every irreducible component $T \subseteq \mathcal{T}$ we have $L|_T \sim_{Q, 0} 0$. After replacing $m$ by a multiple, we can assume that for every irreducible component $T \subseteq \mathcal{T}$ we have $L|_T \sim 0$.

**Step 2.** We fix a circuit $C = \{(T_1, \ldots, T_k), (T_{i, j+1})\}$ in $\Gamma'(T)$. By Lemma 7.5 it is enough to prove that $\Phi|_{L, C}$ has finite order. We set

$$\Delta_X = \Delta + \sum_{I \subseteq \Sigma_f} \sigma f^* \Gamma$$

and run an MMP as in Proposition 8.4. We get a crepant birational map $\rho: (X, \Delta_X) \rightarrow (W, \Delta_W)$ over $Y$ and a klt-trivial fibration $\psi_0: (W, \Delta_W) \rightarrow Y$. For every $i$ let $S_i$ be a log canonical centre of $(W, \Delta_W)$ minimal over $T_i$. We let $S_i^0$ and $S_i^1$ be log canonical centres of $(W, \Delta_W)$ minimal over $T_{i, i+1}$ and with $S_i^1 \subseteq S_{i, i+1}$. Let $\Delta_{S_i^1}$ be the boundary defined by $(K_W + \Delta_W)|_{S_i^1} = K_{S_i^1} + \Delta_{S_i^1}$. The varieties sit in the following diagram

$$\begin{array}{c}
S_i \leftarrow S_i^0 & \rightarrow & S_i^1 \rightarrow S_{i, i+1} \\
T_i \leftarrow T_{i, i+1} & \rightarrow & T_{i+1}
\end{array}$$

The fibration $\psi_0: (W, \Delta_W) \rightarrow Y$ is a crepant, dlt, log structure in the sense of [Kol13] Section 4.4. By [Kol13] 4.45(1) and 4.45.8 there is a crepant birational map $\lambda_0: (S_i^0, \Delta_{S_i^0}) \rightarrow (S_i^1, \Delta_{S_i^1})$.

By [FL19] Lemma 2.8 there are centres $S_{X,i}^\ell$ of $(X, \Delta_X)$ such that the restriction of $\rho$ induces a birational map $\rho: S_{X,i}^\ell \rightarrow S_i^\ell$. We let $\psi_{S_i^\ell}: S_i^\ell \rightarrow Q_i$ and $f|_{S_{X,i}^\ell}: S_{X,i}^\ell \rightarrow Q_i$ be the Stein factorisation. Let $V \subseteq T_{i, i+1}$ be a non-empty open set such that over $\sigma_i^{-1} V$ the map $\rho|_{S_{X,i}^\ell}$ is defined at every generic point of every fibre over $q \in V$ and does not extract any component of the fibres of $f|_{S_{X,i}^\ell}$ for $\ell = 0, 1$. In particular,

the fibres of $g_i$ over points of $\sigma_i^{-1} V$ are reduced because they are

(27) push forward of fibres of $g_{X,i}$, and those are reduced by

Remark 2.3. Set $K_i = T_{i, i+1} \setminus V$.

**Step 3.** Let $\Delta_{S_i}$ be defined by $(K_W + \Delta_W)|_{S_i} = K_{S_i} + \Delta_{S_i}$. Let $\psi|_{S_i}: S_i \rightarrow T_i$ be the Stein factorisation. By [FL19] Proposition 4.2, $h_i$ is a klt-trivial fibration. By Proposition 8.4 (iii) we have $\tau^*(M_f)|_{T} \sim_{Q, 0} M_h$. By Proposition 8.4 (iii) we have $\Delta_{S_i} - h_i^* B_{h_i} \geq 0$. Moreover $h_i$ has reduced fibres over the generic points of every component of $B_{h_i}$ by Proposition 8.4 (iv).
By Theorem 2.16 there is a diagram

\[
 \begin{array}{ccc}
 S_i & \xrightarrow{h_i} & F_i \\
 \downarrow & & \downarrow \\
 T'_i & \xrightarrow{\vartheta_i} & \tilde{T}_i \\
 & & \xrightarrow{\rho_i} \{x_i\}
\end{array}
\]

where \( \vartheta_i \) is a finite map. Let \( \tilde{S}_i \) be the normalisation of the main component of \( S_i \times_{T'_i} \tilde{T}_i \) with the natural map \( \tilde{h}_i: \tilde{S}_i \to \tilde{T}_i \). By Theorem 2.16 there is a birational map \( \eta: (\tilde{S}_i, \Delta_{\tilde{S}_i}) \to (F_i, \Delta_i) \times \tilde{T}_i \). After possibly composing \( \vartheta_i \) with a finite map (or by the proof of Theorem 2.16 [Amb05, Theorem 3.3]), we can assume that \( \tilde{h}_i \) is weakly semistable in codimension 1. By Lemma 2.20 we have \( (\Delta_{\tilde{S}_i} - \tilde{h}_i^*B_{\tilde{h}_i})|_{\tilde{T}_i^{-1}U} \geq 0 \) with \( U \) an open set of \( \tilde{T}_i \) meeting \( \vartheta_i^{-1}\tau_i^{-1}T_{i,i+1} \) and \( \vartheta_i^{-1}\tau_i^{-1}T_{i-1,i} \) non trivially. We set \( J_i = \tilde{T}_i \setminus U \) and \( J'_i = J_i \cap \tilde{T}_i^{sing} \). By Proposition 2.18 the birational map \( \eta \) can be extended to an isomorphism \( \eta: (\tilde{S}_i, \Delta_{\tilde{S}_i} - \tilde{h}_i^*B_{\tilde{h}_i}) \to (F_i, \Delta_i) \times \tilde{T}_i \) over \( \tilde{T}_i \setminus J_i \). It follows that

\[
\mathcal{O}(mM_{\tilde{h}_i}|_{\tilde{T}_i \setminus J_i}) \sim \tilde{h}_i^*\mathcal{O}(\pi^*_i(m(K_{F_i} + \Delta_{F_i})))|_{\tilde{T}_i \setminus J_i}
\]

where \( \pi_i: F_i \times \tilde{T}_i \to F_i \) is the first projection.

We fix \( q_i \in T_{i,i+1} \) with \( q_i \notin K_i \), \( \vartheta_i^{-1}\tau_i^{-1}q_i \notin J_i \), \( \vartheta_i^{-1}\tau_i^{-1}q_i \notin J_{i+1} \). We also let \( p_i^0 \in \tau_i^{-1}(q_i) \) be a point such that \( p_i^0 \notin \vartheta_i(J_i) \) and \( p_i^1 \in \tau_i^{-1}(q_{i+1}) \) be a point such that \( p_i^1 \notin \vartheta_{i+1}(J_{i+1}) \).

By (27), by our choice of \( p_i^\ell \) the fibre \( G_i^\ell \) of \( h_{i+\ell} \) over \( p_i^\ell \) is reduced. By Lemma 2.19 we have \( (G_{i}^{\ell}, (\Delta_{\tilde{S}_i} - \tilde{h}_i^*B_{\tilde{h}_i})|_{G_{i}^{\ell}}) \cong (F_{i+\ell}, \Delta_{i+\ell}) \). Thus we have a canonical isomorphism

\[
\mathcal{O}_{T_{i}}(mM_f)_{q_i} \cong H^0(F_i, m(K_{F_{i}} + \Delta_{F_{i}})).
\]

Step 4. By our choice of \( p_i^\ell \) the crepant birational map \( \lambda_i: (S_i^0, \Delta_{S_i^0}) \to (S_i^1, \Delta_{S_i^1}) \) restricts to a crepant birational map \( \lambda_i: (G_i^0, \Delta_{G_i^0}) \to (G_i^1, \Delta_{G_i^1}) \).

The map \( \lambda_i \) composed with the isomorphisms with \( F_i \) and \( F_{i+1} \) gives a crepant birational map \( \chi_{i,i+1}: (F_i, \Delta_i) \to (F_{i+1}, \Delta_{i+1}) \) such that there is a diagram

\[
\begin{array}{ccc}
\mathcal{O}_{W_{q_i}^+}(m(K_W + \Delta_W)) & \xrightarrow{R_{i+1}} & \mathcal{O}_{W_{q_i}^-}(m(K_W + \Delta_W)) \\
\downarrow & & \downarrow \\
\mathcal{O}_{F_{i+1}}(m(K_{F_{i+1}} + \Delta_{F_{i+1}})) & \cong & \mathcal{O}_{F_i}(m(K_{F_{i}} + \Delta_{F_{i}}))
\end{array}
\]

where \( W_{q_i} \) is the fibre of \( \psi \) over \( q_i \) and \( R_i \) and \( R_{i+1} \) are the Poincaré residue maps existing by [Kol13 4.45(4)] restricted to \( W_{q_i} \).

Then

\[
\Phi_{\mathcal{E},c} = \chi^*_{1,2} \circ \cdots \circ \chi^*_{k,1}.
\]
Thus $\Phi_{L,C}$ is in the image of the crepant birational representation

$$\text{Bir}^n(F_1, \Delta_{F_1}) \to \text{GL}(H^0(F_1, m(K_{F_1} + \Delta_{F_1})))$$

which is finite by Theorem 2.6.

11. Proof of the main results

We are now ready to prove our main results.

Proof of Theorem A. Let $T$ be a connected component of $B_+(M_Y)$. For every component $T \subseteq T$, the restriction $M_Y|_T$ is a torsion divisor. Therefore $\phi_T$ contracts $T$ to a point $p_T$. If $L = O(mM_Y)|_T$, then the relation $R_L$ is finite because it is a subset of $\square\{p_T\} \times \square\{p_T\}$. By Theorem 10.1 the line bundle $L$ is torsion. □

Proof of Theorem B. Set $L = O(mM_Y)|_T$.

Conjecture 1.1 is true for fibrations of relative dimension at most 2. Indeed, let $(F_1, \Delta_1), (F_2, \Delta_2)$ be crepant birational fibres. If $\dim F_1 = 1$, then $(F_1, \Delta_1)$ and $(F_2, \Delta_2)$ are isomorphic and the Conjecture follows from Proposition 2.25. If $\dim F_1 = 2$, then let $(p_1, p_2): G \to F_1 \times F_2$ be a resolution of the indeterminacy such that $K_G = p_1^*(K_{F_1} + \Delta_i) + \sum a_j E_j$, where the $a_j$ do not depend on $i$ by the definition of crepant birational map. Set $\Delta_G = \sum_{a_j < 0} -a_j E_j$. Thus $B_-(K_G + \Delta_G) = \cup_{a_j > 0} E_j$ and $(F_1, \Delta_1)$ and $(F_2, \Delta_2)$ are minimal models of $(G, \Delta_G)$. Thus they are connected by flops. As $\dim F_1 = 2$, they are isomorphic. The Conjecture then follows from Proposition 2.25.

We can assume that the augmented base locus is a simple normal crossings divisor $T$.

By [FL19 Corollary D] for every irreducible component $T$, the restriction $L|_T$ is semiample. We denote by $\phi_T: T \to V$ the induced fibration. By Theorem 10.2 $R_L$ is a finite equivalence relation. The relation is therefore stratifiable by [Kol13 Remark 9.20]. We notice that, as $\dim Y = 3$, the normal variety $\bigsqcup V$ is such that $\dim V \in \{0, 1\}$. The strata of the stratification have dimension 0 or 1. Therefore the stratification satisfies the regularity hypotheses (HN) and (HSN) [Kol13 Definition 9.8]. By [Kol13 Theorem 9.21] the quotient $\pi: \bigsqcup V \to Q$ for $R_L$ exists and is reduced because $\pi$ is surjective, separated by [Kol12 Definition 47, Corollary 48]. Moreover $Q$ is seminormal and there is a fibration $\phi: T \to Q$ whose fibres are the pseudofibres.

If $L|_T$ is torsion for every $T$, then $\dim V = 0$ for every $V$ and $Q$ is a point, hence projective. Then the claim follows from Theorem 10.1. □

Otherwise, for every component $Q_0$ of $Q$ of dimension 1, there is $V_0 \subseteq \bigsqcup V$ together with a finite surjective morphism $\pi: V_0 \to Q_0$. By [Har77 Proposition II.6.8] $Q$ is complete, by [Har77 Proposition II.6.7] $Q$ is projective.

Let $\epsilon: Y' \to Y$ be a birational morphism such that $\epsilon \text{Exc}(\epsilon) \subseteq B_+(M_Y)$ and every (set-theoretic) fibre of the restriction of $\phi \circ \epsilon$ to $\epsilon^{-1} B_+(M_Y)$ is simple normal crossing in the sense of Section 6.
We have $\varepsilon^{-1}B_+(M_Y) = B_+(\varepsilon^*M_Y)$. The latter is the augmented base locus of the moduli part $M_Y$ of the base changed fibration, because $Y$ is an Ambro model.

We replace thus $Y$ with $Y'$ and $\phi$ with $\phi \circ \varepsilon$.

By Theorem 10.1 for every fibre $F$ of $\phi$, the restriction of $L$ to the reduced part of $F$ is torsion.

After replacing $L$ with $L \otimes m$ for $m$ divisible enough, we can assume that for every fibre $F$ of $\phi$, the restriction of $L$ to the reduced part of $F$ is trivial.

By Theorem 3.5 the line bundle $L$ is semiample.

□

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