NEW TOPICS IN ERGODIC THEORY

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Abstract. The entangled ergodic theorem concerns the study of the convergence in the strong, or merely weak operator topology, of the multiple Cesaro mean

\[
\frac{1}{N^{k-1}} \sum_{n_1, \ldots, n_k = 0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} ,
\]

where \( U \) is a unitary operator acting on the Hilbert space \( \mathcal{H} \), \( \alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\} \) is a partition of the set made of \( m \) elements in \( k \) parts, and finally \( A_1, \ldots, A_{2k-1} \) are bounded operators acting on \( \mathcal{H} \).

While reviewing recent results about the entangled ergodic theorem, we provide some natural applications to dynamical systems based on compact operators.

Namely, let \( (\mathfrak{A}, \alpha) \) be a \( C^* \)–dynamical system, where \( \mathfrak{A} = \mathcal{K}(\mathcal{H}) \), and \( \alpha = \text{Ad}_U \) is an automorphism implemented by the unitary \( U \).

We show that

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n = E ,
\]

pointwise in the weak topology of \( \mathcal{K}(\mathcal{H}) \). Here, \( E \) is a conditional expectation projecting onto the \( C^* \)–subalgebra

\[
\left( \bigoplus_{z \in \sigma_{pp}(U)} E_z \mathcal{B}(\mathcal{H}) E_z \right) \cap \mathcal{K}(\mathcal{H}) .
\]

If in addition \( U \) is weakly mixing with \( \Omega \in \mathcal{H} \) the unique up to a phase, invariant vector under \( U \) and \( \omega = \langle \cdot, \Omega, \Omega \rangle \), we have the following recurrence result. If \( A \in \mathcal{K}(\mathcal{H}) \) fulfils \( \omega(A) > 0 \), and \( 0 < m_1 < m_2 < \cdots < m_l \) are natural numbers kept fixed, then there exists an \( N_0 \) such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} \omega(A_0^{n_{m_1}}(A) A_1^{n_{m_2}}(A) \cdots A_{m_l}^{n_{m_l}}(A)) > 0
\]

for each \( N > N_0 \).

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1. INTRODUCTION

Recently, it was shown that some ergodic properties of classical dynamical systems fail to be true by passing to noncommutative setting. It is then of interest to understand among the various ergodic properties, which ones survive by passing from the classical to the quantum case. We mention the pivotal paper [7], where such an investigation is carried out for some basic recurrence, as well as multiple mixing properties.

Notice that it is in general unclear what should be the right quantum counterpart of a classical ergodic property. As an example, we mention the property of the convergence to the equilibrium (i.e. ergodicity for an invariant state $\omega$)

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega \left( B^* \alpha^n (A) B \right) = \omega (B^* B) \omega (A)$$

suggested by the quantum physics, and the standard definition of ergodicity

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega (A \alpha^n (B)) = \omega (A) \omega (B).$$

We refer the reader to [4], Proposition 1.1 for further details.

A notion which is meaningful in quantum setting is that of entangled ergodic theorem, introduced in [1] in connection with the central limit theorem for suitable sequences of elements of the group $C^*$–algebra of the free group $\mathbb{F}_\infty$ on infinitely many generators.

The entangled ergodic theorem can be clearly formulated in the following way. Let $U$ be a unitary operator acting on the Hilbert space $\mathcal{H}$, and for $m \geq k$, $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ a partition of the set $\{1, \ldots, m\}$ in $k$ parts. The entangled ergodic theorem concerns the convergence in the strong, or merely weak operator topology, of the multiple Cesaro mean

$$\left(1.1\right) \frac{1}{N^k} \sum_{n_1, \ldots, n_k = 0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)},$$

$A_1, \ldots, A_{m-1}$ being bounded operators acting on $\mathcal{H}$.

Notice that expressions like (1.1) naturally appear also in [7] relatively to the study of the behaviour of the multiple correlations. Just by considering the simplest case of the partition of the empty set, the limit of the Cesaro mean in (1.1) reduces itself to the well–known mean
ergodic theorem due to John von Neumann (cf. [8])

\[ s-lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n = E_1, \]

\( E_1 \) being the selfadjoint projection onto the eigenspace of the invariant vectors for \( U \).

Some applications of the entangled ergodic theorem are discussed below. Apart from the other potential applications to the study of the ergodic properties of quantum dynamical systems, the entangled ergodic theorem is a fascinating self-contained mathematical problem. It is certainly true if the spectrum \( \sigma(U) \) of \( U \) is finite. Some very special cases for which it holds true are listed in [6].

The first part of the present paper, based on [2, 3], is devoted to review the known results on the entangled ergodic theorem.

We start by considering the sufficiently general situation when the operators \( A_1, \ldots, A_{m-1} \) in (1.1) are compact (cf. [2]).

Then we pass to the case when the unitary \( U \) is almost periodic (i.e. \( \mathcal{H} \) is generated by the eigenvectors of \( U \)), and \( \alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\} \) a pair–partition, without any condition on the operators \( A_1, \ldots, A_{2k-1} \) (cf. [3]).

Another interesting situation arises from the generalization to the noncommutative setting, of the ergodic theorem of H. Furstenberg relative to diagonal measure (cf. [3, 5]). By using such a result, we can treat the following situation. Let \( \mathcal{M} \) be a von Neumann algebra equipped with the adjoint action of an ergodic unitary \( U \), and a standard vector \( \Omega \) which is invariant under \( U \). Let \( \mathcal{M}' \) be the commutant von Neumann algebra of \( \mathcal{M} \). In this situation, the Cesaro mean

\[ \frac{1}{N} \sum_{n=0}^{N-1} U^n AU^n \]

converges in the strong operator topology for each \( A \in \mathcal{M} \cup \mathcal{M}' \) (cf. [3]). Notice that (1.3) is the particular case of (1.1) relative to the trivial pair–partition of two elements.

The second part of the present paper concerns the application of the entangled ergodic theorem, as well as some lines of its proof, to the investigation of ergodic properties of \( C^* \)-dynamical systems based on compact operators. More precisely, let \( (\mathfrak{A}, \alpha) \) be a \( C^* \)-dynamical system, where \( \mathfrak{A} = \mathcal{K}(\mathcal{H}) \), and \( \alpha = \text{Ad}_U \) is an automorphism implemented
by the unitary $U$. We show that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n = E,$$

pointwise in the weak topology of $\mathcal{K}(\mathcal{H})$. Here, $E$ is a conditional expectation projecting onto the $C^*$–subalgebra

$$\left( \bigoplus_{z \in \sigma_{pp}(U)} E_z \mathcal{B}(\mathcal{H}) E_z \right) \cap \mathcal{K}(\mathcal{H}).$$

If in addition $U$ is weakly mixing with $\Omega \in \mathcal{H}$ the unique up to a phase, invariant vector under $U$, we can consider the weakly mixing $C^*$–dynamical system $(\mathfrak{A}, \alpha, \omega)$ where $\omega = \langle \Omega, \Omega \rangle$. We prove the following recurrence result. If $A \in \mathcal{K}(\mathcal{H})$ satisfies $\omega(A) > 0$, and $0 < m_1 < m_2 < \cdots < m_l$ are natural numbers kept fixed, then there exists an $N_0$ such that

$$\frac{1}{N} \sum_{n=0}^{N-1} \omega(A \alpha^{nm_1}(A) \alpha^{nm_2}(A) \cdots \alpha^{nm_l}(A)) > 0$$

for each $N > N_0$.

We end the present section with some notations and definitions useful in the sequel.

The convergence in the weak, respectively strong operator topology (see e.g. [10, 11]) of a net $\{A_{\alpha}\}_{\alpha \in J} \subset \mathcal{B}(\mathcal{H})$ is denoted respectively as

$$\text{w-lim} \ A_{\alpha} = A, \quad \text{s-lim} \ A_{\alpha} = A.$$

Let $U$ be a unitary operator acting on $\mathcal{H}$. Consider the resolution of the identity $\{E(B) : B \text{ Borel subset of } \mathbb{T}\}$ of $U$ (cf. [12], Section VII.7). Denote with an abuse of notation, $E_z := E(\{z\})$. Namely, $E_z$ is nothing but the selfadjoint projection on the eigenspace corresponding to the eigenvalue $z$ in the unit circle $\mathbb{T}$. Denote $\sigma_{pp}(U) := \{z \in \mathbb{T} : z \text{ is an eigenvalue of } U\}$ (cf. [8]).

The unitary $U$ is said to be ergodic if the fixed–point subspace $E_1 \mathcal{H}$ is one dimensional. By the mean ergodic theorem (1.2), it is equivalent to the existence of a unit vector $\xi_0 \in \mathcal{H}$ such that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n \xi = \langle \xi, \xi_0 \rangle \xi_0,$$

or equivalently,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle U^n \xi, \eta \rangle = \langle \xi, \xi_0 \rangle \langle \xi_0, \eta \rangle.$$
The unitary $U$ is said to be weakly mixing if there exists a unit vector $\xi_0 \in \mathcal{H}$ such that
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \langle U^n \xi, \eta \rangle - \langle \xi, \langle \xi_0, \eta \rangle \rangle \right| = 0.
\]

A unitary $U$ is weakly mixing if and only if $\sigma_{pp}(U) = \{1\}$ and $E_1 = \langle \cdot, \xi_0 \rangle \xi_0$, see e.g. [7], Proposition 5.4.

The unitary $U$ is said to be almost periodic if $\mathcal{H} = \mathcal{H}_{ap}$, $\mathcal{H}_{ap}$ being the closed subspace consisting of the vectors having relatively norm-compact orbit under $U$. It is seen in [7] that $U$ is almost periodic if and only if $\mathcal{H}$ is generated by the eigenvectors of $U$.

For a (discrete) $C^*$-dynamical system we mean a pair $(\mathfrak{A}, \alpha)$ consisting of a $C^*$-algebra $\mathfrak{A}$, and an automorphism $\alpha$ of $\mathfrak{A}$. If in addition, a state $\omega \in S(\mathfrak{A})$ invariant under the action of $\alpha$ is kept fixed, we consider also $C^*$-dynamical systems consisting of a triplet $(\mathfrak{A}, \alpha, \omega)$.

A $C^*$-dynamical system $(\mathfrak{A}, \alpha, \omega)$ is said to be ergodic if for each $A, B \in \mathfrak{A}$,
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(A \alpha^n(B)) = \omega(A) \omega(B).
\]

It is said to be weakly mixing if
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \omega(A \alpha^n(B)) - \omega(A) \omega(B) \right| = 0
\]
for each $A, B \in \mathfrak{A}$.

Let $(\mathcal{H}, \pi, U, \Omega)$ be the GNS covariant representation (cf. [11], Section I.9) canonically associated to the dynamical system under consideration. Then $(\mathfrak{A}, \alpha, \omega)$ is ergodic (respectively weakly mixing) if and only if $U$ is ergodic (respectively weakly mixing), see e.g. [7].

2. THE ENTANGLED ERGODIC THEOREM

The present section, based on [2, 3], is devoted to review the known results on the entangled ergodic theorem.

2.1. case of compact operators.

We start with the entangled ergodic theorem for general partitions of any finite set $\{1, \ldots, m\}$, and for compact operators $\{A_1, \ldots, A_{m-1}\}$.
Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator, and for $m \geq k$, $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ a partition of the set $\{1, \ldots, m\}$ in $k$ parts.\(^1\)

**Theorem 2.1.** (cf. [2], Theorem 2.6)

For $m \geq k$, let $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ be a partition of the set $\{1, \ldots, m\}$. If $\{A_1, \ldots, A_{m-1}\} \subset \mathcal{K}(\mathcal{H})$, then the ergodic average

$$\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)}$$

converges in the weak operator topology to some bounded operator $S_{\alpha; A_1, \ldots, A_{m-1}} \in \mathcal{B}(\mathcal{H})$.

**Proof.** Define

$$\Gamma_N := \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)}.$$

Notice that

$$\|\Gamma_N\| \leq \prod_{j=1}^{m-1} \|A_j\|.$$

By Theorem II.1.3 of [11], it is then enough to show that the $\langle \Gamma_N x, y \rangle$ converges for each fixed $x, y \in \mathcal{H}$. On the other hand, we can approximate the $A_j$ by finite rank operators. Namely, put $K := \max_{1 \leq j \leq m-1} \|A_j\|$. Choose finite rank operators $A_j^\varepsilon$, such that $\|A_j^\varepsilon\| \leq K$ and $\|A_j - A_j^\varepsilon\| < \frac{\varepsilon}{4(m-1)K^{m-2}\|x\|\|y\|}, \; j = 1, \ldots, m-1$. We have with obvious notations

$$|\langle \Gamma_N x, y \rangle - \langle \Gamma_M x, y \rangle| \leq \|\Gamma_N - \Gamma_M\| + \|\Gamma_M - \Gamma_M^\varepsilon\|$$

$$+ |\langle \Gamma_M^\varepsilon x, y \rangle - \langle \Gamma_M^\varepsilon x, y \rangle| \leq \frac{\varepsilon}{2} + \|\Gamma_N x, y \rangle - \langle \Gamma_M^\varepsilon x, y \rangle|.$$

Thus, it is enough to show that

$$\left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)} x, y \right\rangle$$

\(^1\)A partition $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ of the set made of $m$ elements in $k$ parts is nothing but a surjective map, the parts of $\{1, \ldots, m\}$ being the preimages $\{\alpha^{-1}(\{j\})\}_{j=1}^k$. 
ergodic theory

converges for every \(x, y \in \mathcal{H}\), whenever the \(A_j\) are rank one operators.

By using the explicit computations in [6], we obtain in this situation,

\[
\left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)} x, y \right\rangle
\]

\[
= \prod_{j=1}^k \frac{1}{N} \sum_{n_j=0}^{N-1} \prod_{\{p : \alpha(p) = j\}} \langle U^{n_j} x_{p,j}, y_{p,j} \rangle
\]

\[
= \prod_{j=1}^k \int \cdots \int \mathbb{T}^{(\alpha^{-1}(j))} \left( \frac{1}{N} \sum_{n_j=0}^{N-1} \left( \prod_{\{p : \alpha(p) = j\}} z_p \right)_{n_j} \right) \prod_{\{p : \alpha(p) = j\}} \langle E(dz_p) x_{p,j}, y_{p,j} \rangle
\]

\[
\rightarrow_{\mathcal{N}} \prod_{j=1}^k \int \cdots \int \mathbb{T}^{(\alpha^{-1}(j))} \chi_\{1\} \left( \prod_{\{p : \alpha(p) = j\}} z_p \right) \prod_{\{p : \alpha(p) = j\}} \langle E(dz_p) x_{p,j}, y_{p,j} \rangle
\]

where we have used the Lebesgue dominated convergence theorem.

Here, the \(x_{p,j}, y_{p,j}\) are vectors uniquely determined by the rank one operators \(A_1, \ldots, A_{m-1}\) and vectors \(x, y\), and finally \(\chi_\Delta\) denotes the indicator of the set \(\Delta\).

\[\square\]

It was shown in [2] that if \(\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}\) is a pair–partition, we can explicitly write the formula for \(S_\alpha; A_1, \ldots, A_{2k-1} \in \mathcal{B}(\mathcal{H})\).

Namely, define

\[\sigma^\alpha_{pp}(U) := \{z \in \sigma_{pp}(U) : zw = 1 \text{ for some } w \in \sigma_{pp}(U)\}\]

Then we have

\[
w-\lim_N \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)} \right\}
\]

\[
= \sum_{z_1, \ldots, z_k \in \sigma^\alpha_{pp}(U)} E_{z_{\alpha(1)}} A_1 E_{z_{\alpha(2)}} \cdots E_{z_{\alpha(m-1)}} A_{m-1} E_{z_{\alpha(m)}} ,
\]

(2.1)

Here, the pairs \(z^\#_{\alpha(i)}\) are alternatively \(z_j\) and \(\bar{z}_j\) whenever \(\alpha(i) = j\), \(E_z\) is the selfadjoint projection on the eigenspace corresponding to the eigenvalue \(z \in \sigma_{pp}(U)\),\(^2\) and finally the sum in the r.h.s. is understood as the limit in the weak operator topology of the net obtained by considering the finite truncations of the r.h.s. of (2.1) (cf. [2], Proposition

\[\text{2If for example, } \alpha \text{ is the pair–partition } \{1, 2, 1, 2\} \text{ of four elements, }
\]

\[S_\alpha; A,B,C = \sum_{z,w \in \sigma^\alpha_{pp}(U)} E_z A E_w B E_z C E_w .\]
2.3). Notice that (2.1) cannot be extended to the whole $\mathcal{B}(\mathcal{H})$, see the example in pag. 8 of [7].

2.2. almost periodic case.

Another case for which the entangled ergodic theorem can be proved is the almost periodic case, that is when the Hilbert space is generated by the eigenvectors of the unitary $U$. In this situation, we have no conditions on the bounded operators appearing in (1.1).

**Theorem 2.2.** (cf. [3], Theorem 2.6)

Suppose that the dynamics induced by the unitary $U$ on $\mathcal{H}$ is almost periodic. Then for each $A_1, \ldots, A_{2k-1} \in \mathcal{B}(\mathcal{H})$,

$$
\lim_{N \to \infty} \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(2k-1)}} A_{2k-1} U^{n_{\alpha(2k)}} \right\} = S_{\alpha;A_1,\ldots,A_{2k-1}}.
$$

**Proof.** To simplify, we treat the case of the partition $\{1, 2, 1, 3, 2, 3\}$, the general case follows the same lines of this case. Fix $\varepsilon > 0$, and suppose that $A, B, C, D, F \in \mathcal{B}(\mathcal{H})$ have norm one. Let $I_\varepsilon$ be such that

$$
\left\| x - \sum_{\sigma \in I_\varepsilon} E_\sigma x \right\| < \varepsilon.
$$

For each $\sigma \in I_\varepsilon$, let $I_\varepsilon(\sigma)$ be such that

$$
\left\| FE_\sigma x - \sum_{\tau \in I_\varepsilon(\sigma)} E_\tau FE_\sigma x \right\| < \frac{\varepsilon}{|I_\varepsilon|}.
$$

Choose $N_\varepsilon$ such that

$$
\left\| \left( \frac{1}{N} \sum_{n=0}^{N-1} (\sigma U)^n - E_\sigma \right) DE_\sigma FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|},
$$

whenever $N > N_\varepsilon$ and $\sigma \in I_\varepsilon, \tau \in I_\varepsilon(\sigma)$. For each $\sigma \in I_\varepsilon, \tau \in I_\varepsilon(\sigma)$, let $I_\varepsilon(\sigma, \tau)$ be such that

$$
\left\| CE_\sigma DE_\tau FE_\sigma x - \sum_{\rho \in I_\varepsilon(\sigma, \tau)} E_\rho CE_\sigma DE_\tau FE_\sigma x \right\| < \frac{\varepsilon}{\sum_{\sigma \in I_\varepsilon} |I_\varepsilon(\sigma)|}.
Then

\[
\left\| \frac{1}{N^3} \sum_{k,m,n=0}^{N-1} U^k A U^m B U^n C U^m D U^n F U^n x - S_{\alpha;A,B,C,D,F} x \right\| \\
\leq 5\varepsilon + \sum_{\sigma \in \mathcal{I}} \sum_{\tau \in \mathcal{I}(\sigma)} \sum_{\rho \in \mathcal{I}(\sigma,\tau)} \left\| \left( \frac{1}{N} \sum_{k=0}^{N-1} (\rho U)^k \right) A \left( \frac{1}{N} \sum_{m=0}^{N-1} (\tau U)^m \right) \right. \\
\times B E_{\rho} C E_{\sigma} D E_{\tau} F E_{\sigma} x - E_{\rho} A E_{\sigma} B E_{\rho} C E_{\sigma} D E_{\tau} F E_{\sigma} x \right\|.
\]

Taking the limsup on both sides, we obtain the assertion by the mean ergodic theorem (1.2). \( \square \)

2.3. diagonal measures.

We treat the natural generalization to the quantum case of the celebrated result due to H. Furstenberg relative to the diagonal measures (cf. [5], Section 4.4).

We start with a \( C^* \)-dynamical system \( (\mathfrak{A}, \alpha, \omega) \), together with its GNS covariant representation \( (\mathcal{H}, \pi, U, \Omega) \). Denote \( M := \pi(\mathfrak{A})'' \), the von Neumann algebra acting on \( \mathcal{H} \) generated by the representation \( \pi \). The commutant von Neumann algebra is denoted as \( M' \). Suppose further that the support \( s(\omega) \) in \( \mathfrak{A}^{**} \) is central. The last property simply means that \( \Omega \) is separating for \( \pi(\mathfrak{A})'' \), see e.g. [10], Section 10.17.

Let \( \mathfrak{M} := M \otimes_{\text{max}} M' \) be the completion of the algebraic tensor product \( \mathfrak{M} := M \otimes M' \) w.r.t. the maximal \( C^* \)-norm (cf. [11], Section IV.4). It is easily seen that on \( \mathfrak{M} \) the following two states are automatically well-defined. The first one is the canonical product state

\[
\varphi(A \otimes B) := \langle A \Omega, \Omega \rangle \langle B \Omega, \Omega \rangle , \quad A \in M , \; B \in M'.
\]

The second one is uniquely defined by

\[
\psi(A \otimes B) := \langle AB \Omega, \Omega \rangle , \quad A \in M , \; B \in M'.
\]

The state \( \psi \) can be considered the (quantum analogue of the) “diagonal measure” of the “measure” \( \varphi \).

On \( \mathfrak{M} \) is also uniquely defined the automorphism

\[
\gamma := \text{Ad}_U \otimes \text{Ad}_{U^2} ,
\]

see [11], Proposition IV.4.7. Of course, \( (\mathfrak{M}, \gamma, \varphi) \) is a \( C^* \)-dynamical system whose GNS covariant representation is precisely \( (\mathcal{H} \otimes \mathcal{H}, \text{id} \otimes \text{id}, U \otimes U^2, \Omega \otimes \Omega) \). Denote \( E_1 \) the selfadjoint projection onto the invariant vectors under \( U \otimes U^2 \). Notice that the \( * \)-subalgebra \( \mathfrak{M} \) is globally stable under the action of \( \gamma \).
In addition, again by Proposition IV.4.7 of [11],

$$\sigma(A \otimes B) := AB, \quad A \in M, \ B \in M'.$$

uniquely defines a representation of $\mathfrak{M}$ on $\mathcal{H}$ such that $(\mathcal{H}, \sigma, \Omega)$ is precisely the GNS representation of the state $\psi$.

Let $A \in M, \ B \in M'$. Then by the mean ergodic theorem (1.2),

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \psi(\gamma^n(A \otimes B)) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle AU^n B \Omega, \Omega \rangle \equiv \langle A \Omega, \Omega \rangle \langle B \Omega, \Omega \rangle \equiv \varphi(A \otimes B).$$

According with Definition 4.1 of [3] (see also [5], Definition 4.4), this means that the state $\psi \in S(\mathfrak{M})$ is generic for $(\mathfrak{M}, \gamma, \varphi)$ w.r.t. $\mathfrak{N}$. In addition, define

$$\Sigma := \{(z, w) \in \sigma_{pp}(U) \times \sigma_{pp}(U) : zw^2 = 1\}.$$ 

Then by Lemma 4.18 of [5],

$$E_1 = \bigoplus_{s \in \Sigma} E^U_s \otimes E^U_w,$$

$E^U_s$ being the selfadjoint projection onto the eigenspace of $U$ corresponding to the eigenvalue $z$. As $U$ is supposed to be ergodic, by Proposition 2.2 of [3], $E^U_s \mathcal{H}$ is one dimensional, and $E^U_s \mathcal{H}$ and $E^U_w \mathcal{H}$ are generated by $V_z \omega, W_w \omega$, where $V_z$ and $W_w$ are unitaries of $M_z := \{A \in M : \ UAU^{-1} = zA\}$, $(M')_w := \{B \in M' : UB^{-1} = wB\}$ respectively. Thus, $E^U_s \mathcal{H} \otimes E^U_w \mathcal{H}$ is one dimensional, and it is generated by $V_z \omega \otimes W_w \omega$. This means that $\mathfrak{N} \Omega \cap E_1 \mathcal{H} \otimes \mathcal{H}$ is dense in $E_1 \mathcal{H} \otimes \mathcal{H}$. Then the map

$$\sum_j A_j \Omega \otimes B_j \Omega \in \mathfrak{N} \Omega \cap E_1 \mathcal{H} \otimes \mathcal{H} \mapsto \sum_j A_j B_j \Omega \in \mathcal{H}$$

uniquely defines a partial isometry $V : \mathcal{H} \otimes \mathcal{H} \mapsto \mathcal{H}$ such that $V^* V = E_1 \mathcal{H} \otimes \mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, such an isometry has the form

$$V(\xi \otimes \eta) = \sum_{\{z, w \in \sigma(U) : zw^2 = 1\}} \langle \xi, V_z \omega \rangle \langle \eta, W_w \omega \rangle V_z W_w \omega, \tag{2.2}$$

where $V_z \omega, V_z$ unitary of $M_z$ (equivalently $W_z \omega, W_z$ unitary of $(M')_z$) generates the one dimensional subspace $E^U_s \mathcal{H}$ for $z \in \sigma_{pp}(U)$.
Theorem 2.3. (cf. [3], Theorem 5.2)

Let \((\mathfrak{A}, \alpha, \omega)\) be an ergodic C*-dynamical system such that its support \(c(\omega)\) in \(\mathfrak{A}^{**}\) is central. Then for each \(A \in M \cup M',\)

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n = V (A \Omega \otimes \cdot),
\]

where \(V\) is the isometry given in (2.2).

Proof. As \(\psi\) is generic for \((\mathfrak{M}, \gamma, \phi)\) w.r.t. \(M \otimes M'\) and the last *-algebra is left stable by \(\text{Ad}_U \otimes \text{Ad}_{U^2}\), we can apply Theorem 4.5 of [3] (see also [5], Theorem 4.14 for the Abelian case) obtaining for \(X \in M, Y \in M'\)

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n X U^n Y = V (X \Omega \otimes Y \Omega).
\]

If \(A \in M\) the proof follows as \(\Omega\) is cyclic for \(M'\). By exchanging the role between \(M\) and \(M'\), we obtain the result whenever \(A \in M'\)

If \((\mathfrak{A}, \alpha, \omega)\) is weakly mixing and \(0 < m_1 < m_2\) natural numbers, we prove following the same lines of Theorem 2.3, but in a different way from [7], that

\[
(2.3) \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^{nm_1} A U^{nm_2} = \langle A \Omega, \Omega \rangle \langle \cdot, \Omega \rangle \Omega
\]

for each \(A \in M \cup M'.\)

3. Applications

We start with the following recurrence result which is a direct consequence of Theorem 1.3 of [7]. By (2.3), we then have an alternative proof of it.

Proposition 3.1. Let \((\mathfrak{A}, \alpha, \omega)\) be a weakly mixing C*-dynamical system such that its support \(c(\omega)\) in \(\mathfrak{A}^{**}\) is central, and \(0 < m_1 < m_2\) natural numbers. Consider \(A \in \mathfrak{A}\) such that \(\omega(A) > 0\)

Then there exists an \(N_0\) such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} \omega(A \alpha^{nm_1}(A) \alpha^{nm_2}(A)) > 0
\]

for each \(N > N_0\).
Proof. We have by (2.3),
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(A^{n_1}(A)A^{n_2}(A)) \\
\equiv \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle AU^{nm_1}A^{n(m_2-m_1)}A\Omega, \Omega \rangle \\
= \omega(A)^3 > 0.
\]
\[
\Box
\]
Now we pass to some interesting applications concerning compact operators.

Proposition 3.2. Let $U$ be a weakly mixing unitary acting on the Hilbert space $\mathcal{H}$, $A_1, \ldots, A_{k-1} \in \mathcal{K}(\mathcal{H})$, and finally $m_1, \ldots, m_k$ fixed nonnull natural numbers. Then
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^{nm_1}A_1U^{nm_2}A_2 \cdots U^{nm_{k-1}}A_{k-1}U^{nm_k} = E_1A_1E_1A_2 \cdots E_1A_{k-1}E_1.
\]
(3.1)

Proof. As $U$ is weakly mixing, $\sigma_{pp}(U) = \{1\}$ and $E_1 = \langle \cdot, \Omega \rangle \Omega$ for a unique up to a phase unit vector. In addition, we can approximate the $A_j$ by finite rank operators as explained in Theorem 2.1. We now decompose $U$ as
\[
U = \langle \cdot, \Omega \rangle \Omega + E_1^+U,
\]
(3.2)
where $E_1^+$ is the selfadjoint projection onto the closed subspace on which $U$ has purely continuous spectrum. By inserting (3.2) in (3.1), we obtain an addendum containing in all place the piece $\langle \cdot, \Omega \rangle \Omega$, the last coinciding with $E_1A_1E_1A_2 \cdots E_1A_{k-1}E_1$. As we reduced the matter to the case when the $A_j$ are rank one operators, the remaining addenda contain a multiplicative factor of the form
\[
G_N := \iint \cdots \int_{T_j} \left( \frac{1}{N} \sum_{n=0}^{N-1} (z_1^{m_1} \cdots z_j^{m_j})^n \right) d\mu_1(z_1) \cdots d\mu_j(z_j).
\]
(3.3)

In (3.3) $1 \leq j \leq k$ is fixed and depends on the addendum under consideration, and
\[
d\mu_l(z_l) := \langle E(dz_l)x_l, y_l \rangle, \quad 1 \leq l \leq j
\]
are bounded signed Borel measure without atoms. As
\[ \frac{1}{N} \sum_{n=0}^{N-1} (z_1^{m_1} \cdots z_j^{m_j})^n \longrightarrow \chi_{\{1\}}(z_1^{m_1} \cdots z_j^{m_j}) \]
pointwise, by taking the limit in (3.3), we obtain by Lebesgue dominated convergence theorem and Fubini theorem,
\[ \lim_{N \to +\infty} G_N = \int \cdots \int_{\mathbb{T}^j} f(z_1, \ldots, z_{j-1}) \, d\mu_1(z_1) \cdots d\mu_{j-1}(z_{j-1}) \]
where
\[ f(z_1, \ldots, z_{j-1}) := \mu_j(\{ z_j : z_1^{m_1} \cdots z_j^{m_j} = 1 \}) . \]
The proof follows as, for fixed \( z_1, \ldots, z_{j-1} \in \mathbb{T} \),
\[ \# \{ z_j : z_1^{m_1} \cdots z_j^{m_j} = 1 \} = m_j . \]
\[ \square \]
Notice that, if \( \sigma_{pp}(U) = \emptyset \), then
\[ \text{w-lim}_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^{nm_1} A_1 U^{nm_2} A_2 \cdots U^{nm_{k-1}} A_{k-1} U^{nm_k} = 0 . \]
The proof of the next lemma is the same as Lemma 2.2 of [2].

**Lemma 3.3.** The net \( \{ \sum_{z \in F} E_z A E_z \mid F \text{ finite subset of } \sigma_{pp}(U) \} \) converges in the strong operator topology.

We symbolically write for such a limit
\[ \text{s-lim}_{F \uparrow \sigma_{pp}(U)} \sum_{z \in F} E_z A E_z =: \sum_{z \in \sigma_{pp}(U)} E_z A E_z . \]

**Proposition 3.4.** Let \( U \) be a unitary acting on the Hilbert space \( \mathcal{H} \), and \( A \in \mathcal{K}(\mathcal{H}) \). Then
\[ \text{w-lim}_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^{-n} = \sum_{z \in \sigma_{pp}(U)} E_z A E_z . \]

**Proof.** By approximating \( A \) with a finite rank operator \( A_\varepsilon \), we have
\[ \left| \left\langle \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^{-n} x, y \right\rangle - \left\langle \sum_{z \in \sigma_{pp}(U)} E_z A E_z x, y \right\rangle \right| \]
\[ \leq \varepsilon + \left| \left\langle \frac{1}{N} \sum_{n=0}^{N-1} U^n A_\varepsilon U^{-n} x, y \right\rangle - \left\langle \sum_{z \in \sigma_{pp}(U)} E_z A_\varepsilon E_z x, y \right\rangle \right| . \]
So, it is enough to check (3.4) for rank one operators $A = \langle \cdot , \xi \rangle \eta$.

In this situation, we have

$$
\left \langle \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^{-n} x , y \right \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \langle U^n \eta , y \rangle \langle U^{-n} x , \xi \rangle
$$

$$
= \int_T \left \langle \frac{1}{N} \sum_{n=0}^{N-1} (z \bar{w})^n \right \rangle \langle E(dz) \eta , y \rangle \langle E(dw) x , \xi \rangle
$$

$$
\to \left \langle \sum_{z \in \sigma_{pp}(U)} E_z A E_z x , y \right \rangle
$$

as

$$
\frac{1}{N} \sum_{n=0}^{N-1} (z \bar{w})^n \to \chi_1(z \bar{w})
$$

pointwise. See Proposition 2.4 of [2] for further details. □

4. $C^*$–DYNAMICAL SYSTEMS BASED ON COMPACT OPERATORS

The present section is devoted to the study of some interesting ergodic properties of $C^*$–dynamical systems based on compact operators.

Following the same lines of the previous results, we pass to the study of the convergence of Cesaro mean of automorphisms $\alpha$ of the $C^*$–algebra $K(H)$ consisting of all the compact operators acting on $H$. Consider the double transpose $\alpha^{**} \in \text{Aut}(\mathcal{B}(H))$. As such an automorphism $\alpha^{**}$ is inner (cf. [10], Corollary 8.11), there exists a unitary $U$ acting on $H$ such that $\alpha = \text{Ad}_U$. Namely, each automorphism of $K(H)$ is implementable on $H$.

Lemma 3.3 allows us to define $E : \mathcal{B}(H) \mapsto \mathcal{B}(H)$ as

$$
E(A) := \sum_{z \in \sigma_{pp}(U)} E_z A E_z
$$

The properties of $E$ are collected in the following

**Proposition 4.1.** The map $E$ is a conditional expectation projecting onto the $C^*$–subalgebra $\bigoplus_{z \in \sigma_{pp}(U)} E_z \mathcal{B}(H) E_z$.

**Proof.** Following the same line of Lemma 2.1 of [2], we see that $\|E\| = 1$. In addition

$$
E(E(A)) = \sum_{z,w \in \sigma_{pp}(U)} E_w E_z A E_z E_w = \sum_{z \in \sigma_{pp}(U)} E_z A E_z \equiv E(A)
$$
Namely, \( \mathcal{E} \) is a norm one projection onto the the \( C^* \)-subalgebra \( \bigoplus_{z \in \sigma_{pp}(U)} E_z \mathcal{B}(\mathcal{H}) E_z \), hence a conditional expectation, see [9], Theorem 9.1.

Notice that the identity \( \mathcal{E}(I) \) of the range of \( \mathcal{E} \) is precisely

\[ E_{pp} := \sum_{z \in \sigma_{pp}(U)} E_z, \]

the selfadjoint projection onto the closed subspace of \( \mathcal{H} \) generated by the eigenvectors of \( U \).

Now we specialize the matter to the case when \( A \) is a compact operator.

**Lemma 4.2.** If \( A \in \mathcal{K}(\mathcal{H}) \) then \( \mathcal{E}(A) \in \mathcal{K}(\mathcal{H}) \).

**Proof.** We have by Schwarz, Holder and Bessel inequalities,

\[
|\langle \mathcal{E}(A - B)x, y \rangle| \leq \sum_{z \in \sigma_{pp}(U)} |\langle (A - B)E_z x, E_z y \rangle| \\
\leq \|A - B\| \sum_{z \in \sigma_{pp}(U)} \|E_z x\| \|E_z y\| \\
\leq \|A - B\| \left( \sum_{z \in \sigma_{pp}(U)} \|E_z x\|^2 \right)^{1/2} \left( \sum_{z \in \sigma_{pp}(U)} \|E_z y\|^2 \right)^{1/2} \\
\leq \|A - B\| \|x\| \|y\|.
\]

Thus, we can approximate \( A \) by a finite rank operator. In addition, for a rank one operator \( A = \langle \cdot, y \rangle x \), we have by polarization,

\[ A = \frac{1}{4} \sum_{\{z \in \mathbb{T} : z^4 = 1\}} z \langle \cdot, x + zy \rangle (x + zy). \]

Namely, we can reduce the matter to the case when \( A \) is the rank one positive operator \( \langle \cdot, x \rangle x \). We now compute

\[ \langle \mathcal{E}(A)x, x \rangle = \sum_{z \in \sigma_{pp}(U)} \|E_z x\|^2. \]

As the last sum is convergent, there exists an at most countable set \( z_1, z_2, \ldots \subset \sigma_{pp}(U) \) depending on \( A \), such that \( E_z x = 0 \) if \( z \neq z_j, j = 1, 2, \ldots \). In addition, \( \lim \|E_z x\| = 0 \). Put \( \lambda_j := \|E_{z_j} x\|^2 \) and \( y_j := \frac{E_{z_j} x}{\|E_{z_j} x\|}, j = 1, 2, \ldots \). We have

\[
\mathcal{E}(A) = \sum_j \langle \cdot, E_{z_j} x \rangle E_{z_j} x = \sum_j \lambda_j \langle \cdot, y_j \rangle y_j.
\]
It readily seen that \( \sum_{j=1}^{N} \lambda_j \langle \cdot, y_j \rangle y_j \) converges in norm, that is \( \mathcal{E}(A) \) is a compact operator. \( \square \)

**Proposition 4.3.** The restriction \( E := \mathcal{E} \mid \mathcal{K}(\mathcal{H}) \) of the map in (4.1) gives rise to a conditional expectation projecting onto the \( C^* \)-subalgebra

\[
\left( \bigoplus_{z \in \sigma_{pp}(U)} E_z \mathcal{B}(\mathcal{H}) E_z \right) \cap \mathcal{K}(\mathcal{H}) .
\]

*Proof.* Lemma 4.2 tells us that \( E \) maps the compact operators into the compact ones. Moreover, \( (\mathcal{E} \mid \mathcal{K}(\mathcal{H}))^{**} = \mathcal{E} \) and the proof follows. \( \square \)

**Theorem 4.4.** Let \( \alpha \) be an automorphism of \( \mathcal{K}(\mathcal{H}) \), with \( U \) the unitary acting on \( \mathcal{H} \) implementing \( \alpha \). Then

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n = E,
\]

pointwise in the weak topology of \( \mathcal{K}(\mathcal{H}) \), \( E \) being the conditional expectation given in Proposition 4.3.

*Proof.* By taking into account (3.4),

\[
(4.2) \quad \text{w-} \lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) = E(A)
\]

whenever \( A \) is compact. Let now \( T \) be a trace class operator and \( T_\varepsilon \) be a finite rank operator such that \( \text{Tr}(|T - T_\varepsilon|) \leq \varepsilon \), \( \text{Tr} \) being the unique normal faithful semifinite trace on \( \mathcal{B}(\mathcal{H}) \). Then

\[
\text{Tr} \left( T \left( \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) - E(A) \right) \right) \leq \text{Tr} \left( (T - T_\varepsilon) \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \right) \]

\[
+ |\text{Tr}((T - T_\varepsilon)E(A))| + \text{Tr} \left( T_\varepsilon \left( \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) - E(A) \right) \right) \]

\[
\leq 2\varepsilon \|A\| + \text{Tr} \left( T_\varepsilon \left( \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) - E(A) \right) \right) .
\]

Thus, we reduce the matter when \( T \) is finite rank. The proof now follows by (4.2). \( \square \)

Notice that if \( \sigma_{pp}(U) = \emptyset \),

\[
\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n \rightarrow 0,
\]
and if $\sigma_{pp}(U) = \{1\}$ with $\Omega$ the unique up to a phase invariant vector for $U$, that is in the case of weakly mixing $C^*$-dynamical systems based on compact operators,

$$\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n \longrightarrow \omega(\cdot)E_1,$$

$\omega$ being the vector state $\langle \cdot, \Omega, \Omega \rangle$.

We end the present section with a recurrence result which is an immediate corollary of Proposition 3.2.

Let $(\mathfrak{A}, \alpha, \omega)$ be a weakly mixing $C^*$-dynamical system, where $\mathfrak{A} = K(H)$, $\alpha = \text{Ad}_U$, and $\omega = \langle \cdot, \Omega, \Omega \rangle$, with $\Omega$ invariant under the action of the unitary operator $U$.

**Proposition 4.5.** Under the above conditions, if $\omega(A) > 0$, and $0 < m_1 < m_2 < \cdots < m_l$ are natural numbers kept fixed, then there exists an $N_0$ such that

$$\frac{1}{N} \sum_{n=0}^{N-1} \omega(A\alpha^{nm_1}(A)\alpha^{nm_2}(A)\cdots\alpha^{nm_l}(A)) > 0$$

for each $N > N_0$.

**Proof.** By Proposition 3.2, we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega(A\alpha^{nm_1}(A)\alpha^{nm_2}(A)\cdots\alpha^{nm_l}(A))$$

$$= \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle A(U^{nm_1}A^{m_2-m_1})\cdots(AU^{n(m_l-m_1-1)}A\Omega, \Omega)$$

$$= \langle AE_1AE_1\cdots AE_1A\Omega, \Omega \rangle \equiv \omega(A)^{l+1} > 0.$$

\[ \square \]

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