HYPERSPACES WITH A COUNTABLE CHARACTER OF CLOSED SUBSETS

CHUAN LIU AND FUCAI LIN

Abstract. For a regular space $X$, the hyperspace $(CL(X), \tau_F)$ (resp., $(CL(X), \tau_V)$) is the space of all nonempty closed subsets of $X$ with the Fell topology (resp., Vietoris topology). In this paper, we give the characterization of the space $X$ such that the hyperspace $(CL(X), \tau_F)$ (resp., $(CL(X), \tau_V)$) with a countable character of closed subsets. We mainly prove that $(CL(X), \tau_F)$ has a countable character on each closed subset if and only if $X$ is compact metrizable, and $(CL(X), \tau_F)$ has a countable character on each compact subset if and only if $X$ is locally compact and separable metrizable. Moreover, we prove that $(K(X), \tau_V)$ have the compact-$G_\delta$ property if and only if $X$ have the compact-$G_\delta$ property and every compact subset of $X$ is metrizable.

1. Introduction

In this paper, the base space $X$ is always supposed to be $T_1$ and regular. Let $\mathbb{N}$ and $\omega$ denote the sets of all positive integers and all non-negative integers, respectively. Given a topological space $X$, the following collections of subsets of $X$ are the hyperspaces that we will consider.

$$F_n(X) = \{A \subset X : |A| \leq n, A \neq \emptyset\},$$

$$F(X) = \bigcup \{F_n(X) : n \in \mathbb{N}\},$$

$$K(X) = \{K \subset X : K \text{ is a non-empty, compact subset in } X\} \text{ and}$$

$$CL(X) = \{H \subset X : H \text{ is non-empty, closed in } X\}.$$

For any open subsets $U_1, \ldots, U_k$ of space $X$, let

$$\langle U_1, \ldots, U_k \rangle = \{H \in CL(X) : H \subset \bigcup_{i=1}^{k} U_i \text{ and } H \cap U_j \neq \emptyset, 1 \leq j \leq k\}.$$  

We endow $CL(X)$ with Vietoris topology defined as the topology generated by

$$\{\langle U_1, \ldots, U_k \rangle : U_1, \ldots, U_k \text{ are open subsets of } X, k \in \mathbb{N}\}.$$  

For any open subset $U$ of $X$, let

$$U^- = \{H \in CL(X) : H \cap U \neq \emptyset\}$$

and

$$U^+ = \{H \in CL(X) : H \subset U\}.$$  

We endow $CL(X)$ with Fell topology defined as the topology generated by the following two families as a subbase:

$$\{U^- : U \text{ is any non-empty open in } X\}$$

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\begin{align*}
\{(K^c)^+ : K \text{ is a compact subset of } X, K \neq X\}.
\end{align*}

We denote the hyperspace with the Vietoris topology and the Fell topology by $\langle CL(X), \tau_V \rangle$ and $\langle CL(X), \tau_F \rangle$ respectively. For $T_2$ space, it is well known that $F_1(X)$ is homeomorphic to $X$ in hyperspaces with the Vietoris topology or the Fell topology, so we consider all hyperspaces have a closed copy of $X$.

Let $X$ be a topological space and $A \subseteq X(A \subset \langle CL(X), \tau_V \rangle)$ be a subset of $X(\langle CL(X), \tau_V \rangle)$. Then the closure of $A(A)$ in $X(\langle CL(X), \tau_V \rangle)$ is denoted by $\bar{A}(\bar{CL}(A))$, and $NI(X)$ and $I(X)$ are the sets of all non-isolated points and isolated points of $X$ respectively. For undefined notations and terminologies, the reader may refer to [4, 5] and [13].

It is well known that the topics of the hyperspace has been the focus of much research, see [2, 6, 13, 17]. There are many results on the hyperspace $CL(X)$ of a topological space $X$ equipped with various topologies. In this paper, we endow $CL(X)$ with the Vietoris topology $\tau_V$ and the Fell topology $\tau_F$ respectively. In 1997, Holá and Levi in [8, Corollary 1.8] gave a characterization of those spaces $X$ such that $(CL(X), \tau_V)$ is first countable; in 2003, Holá, Pelant and Zsilinszky in [7, Theorem 3.1] proved that $(CL(X), \tau_V)$ is developable iff $(CL(X), \tau_V)$ is Moore iff $(CL(X), \tau_V)$ is metrizable iff $(CL(X), \tau_V)$ has a $\sigma$-discrete network iff $X$ is compact and metrizable. Recently, F. Lin, R. Shen and C. Liu in [12] considered the following two problems, and gave some partial answers to Problems 1.1 and 1.2 respectively.

**Problem 1.1.** [12] Problem 1.1] Let $C$ be a proper subclass of the class of first-countable spaces, and let $P$ be a topological property. If $(CL(X), \tau_V) \in C$, does $X$ have the property $P$?

**Problem 1.2.** [12] Problem 1.2] Let $C$ be a class of generalized metrizable spaces. If $(CL(X), \tau_V) \in C$, is $X$ compact and metrizable?

Recall that a space $X$ is a $D_1$-space [1] if every closed subset of $X$ has a countable local base and $X$ is a $D_0$-space [18] if every compact subset of $X$ has a countable local base. Clearly, each $D_1$-space is a $D_0$-space, each $D_0$-space is first-countable, and each Moore space or space with point-countable base or $\gamma$-space is a $D_0$-space [3]. In [3], M. Dai and C. Liu discussed a characterization, some covering properties and the metrization of $D_1$-spaces. In [12], F. Lin, R. Shen and C. Liu proved that under (MA+$\neg$CH), $(CL(X), \tau_V)$ is a $\gamma$-space if and only if $(CL(X), \tau_V)$ is a $D_0$-space if and only if $X$ is a separable metrizable space and $NI(X)$ of $X$ is compact. We listed some properties of $D_1$-spaces in [3, 12] as follows.

**Fact 1:** If $X$ is a $D_1$-space, then $NI(X)$ is countably compact, see [3];

**Fact 2:** If $X$ is a $D_1$-space, then $X$ is metrizable if and only if $NI(X)$ is metrizable, see [3].

**Fact 3:** If $(CL(X), \tau_V)$ is a $D_1$-space, then $X$ is compact metrizable, see [12].

It was proved that if $X$ is a Moore space or space with a point-countable base, so is $(K(X), \tau_V)$, see [15, 16] and [10, Theorem 3.11] respectively.

Our paper is organized as follows. In Section 2, we mainly discuss the $D_1$-property of the hyperspaces. We first prove that, for a space $X$, the hyperspace $(CL(X), \tau_F)$ is a $D_1$-space if and only if $X$ is compact metrizable. Then we prove that, for a space $X$, $(K(X), \tau_V)$ is a $D_1$-space if and only if $(F_2(X), \tau_V)$ is a $D_1$-space if and only if $X$ is

\footnote{A space $(X, \tau)$ is a $\gamma$-space there exists a function $g : \omega \times X \to \tau$ such that (i) $\{g(n, x) : n \in \omega\}$ is a base at $x$; (ii) for each $n \in \omega$ and $x \in X$, there exists $m \in \omega$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$.}
discrete or compact metrizable. In Section 3, we mainly discuss the $D_0$-property of the hyperspaces. We prove that, for a space $X$, $(CL(X), \tau_F)$ is a $D_0$-space if and only if $(F_2(X), \tau_F)$ is a $D_0$-space if and only if $X$ is locally compact and separable metrizable. We also prove that, for a space $X$, $(K(X), \tau_V)$ is a $D_0$-space if and only if $(F_2(X), \tau_V)$ is a $D_0$-space if and only if $X$ is a $D_0$-space and each compact subset of $X$ is metrizable. Finally, we discuss the $G_\delta$-property of hyperspaces and prove that $(K(X), \tau_V)$ have the compact-$G_\delta$ property if and only if $X$ have the compact-$G_\delta$ property and every compact subset of $X$ is metrizable.

2. The $D_1$-property of the hyperspaces

In this section, we mainly study the $D_1$-property of the hyperspaces $(CL(X), \tau_F)$ and $(K(X), \tau_F)$ of those spaces $X$ respectively. First, we give a characterization of a space $X$ such that $(CL(X), \tau_F)$ is a $D_1$-space. In order to show that, we need some lemmas.

Lemma 2.1. [13] Lemma 2.3.1 Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_m$ be subsets of a space $X$. Then, in Vietoris topology $(CL(X), \tau_V)$, we have $\langle U_1, \ldots, U_n \rangle \subset \langle V_1, \ldots, V_m \rangle$ if and only if $\bigcup_{j=1}^m U_j \subset \bigcup_{j=1}^m V_j$ and for every $V_i$ there exists a $U_k$ such that $U_k \subset V_i$.

Lemma 2.2. [13] Lemma 2.3.2 Let $U_1, \ldots, U_n$ be subsets of a space $X$. Then $Cl(\langle U_1, \ldots, U_n \rangle) = \langle \overline{U}_1, \ldots, \overline{U}_n \rangle$, where $Cl(\langle U_1, \ldots, U_n \rangle)$ denotes the closure of the set $\langle U_1, \ldots, U_n \rangle$ in $(K(X), \tau_V)$.

We will need the following lemma, the proof of which we include for the sake of the completeness.

Lemma 2.3. Let $K$ be a compact subset of a topological space $X$ and $\{U_i : i \leq k\}$ be an open cover of $K$. Then, for each $i \leq k$, there exists a compact subset $K_i$ of $X$ such that $K_i \subset U_i$, $K = \bigcup_{i \leq k} K_i$ and $K_i \neq \emptyset$ whenever $U_i \cap K \neq \emptyset$.

Proof. For each $x \in K$, pick an open subset $V_x$ such that $x \in V_x \subset \overline{V}_x \subset U_i$ for some $i \leq k$. Since $K$ is compact and $\{V_x : x \in K\}$ is an open cover of $K$, there is a finite subcover $\{V_{x_j} : j \leq m\}$ of $K$ such that for each $i \leq k$ we have $V_{x_j} \subset U_i$ for some $j \leq m$, where $m \in \mathbb{N}$. For each $i \leq k$, let

$$K_i = K \cap \bigcup \{\overline{V}_{x_j} : \overline{V}_{x_j} \subset U_i\};$$

then $K_i$ is compact and $K_i \subset U_i$. Clearly, we have $K = \bigcup\{K_i : i \leq k\}$. Moreover, it is obvious that $K_i \neq \emptyset$ whenever $U_i \cap K \neq \emptyset$. \hfill \Box

Lemma 2.4. Let $D$ be an infinite countable discrete space. Then $(F_2(D), \tau_F)$ is not a $D_1$-space.

Proof. Enumerate $D$ as $\{d_i : i \in \mathbb{N}\}$. In order to prove this result, we first give the following two Claims.

Claim 1: Each $\{d_i : i \in \mathbb{N}\}$ is a non-isolated point in $(F_2(D), \tau_F)$.

Suppose not, then there exists $i \in \mathbb{N}$ such that $\{d_i\}$ is open in $(F_2(D), \tau_F)$, hence we can pick finitely many open subsets $\{U_i : i \leq n\}$ and a compact $K$ of $D$ such that $\bigcap\{U_i^- : i \leq n\} \cap (K^c)^+ \subset \{d_i\}$. Clearly, $d_i \notin K$ and $K \cup \{d_i\} \neq D$. Pick any $d \in D \setminus (K \cup \{d_i\})$. It easily see that $\{d, d_i\} \in \bigcap\{U_i^- : i \leq n\} \cap (K^c)^+$, thus $\{d, d_i\} \subset \{d_i\}$, that is, $\{d, d_i\} = \{d_i\}$, this is a contradiction. Hence each $\{d_i\}$ is a non-isolated point in $(F_2(D), \tau_F)$.

Claim 2: $\{\{d_i\} : i \in \mathbb{N}\}$ has no cluster point in $(F_2(D), \tau_F)$. 


Indeed, suppose that \( A \) is a cluster point of \( \{ \{ d_i \} : i \in \mathbb{N} \} \). Pick any \( x \in A \); then \( \{ x \}^- \cap \mathcal{F}_2(D) \) is a neighborhood of \( A \) in \((\mathcal{F}_2(D), \tau_F)\), but \( \{ x \}^- \) meets at most one \( \{ d_i \} \) in \( \mathcal{F}_2(D) \). Thus \( A \) is not a cluster point of \( \{ \{ d_i \} : i \in \mathbb{N} \} \), which is a contradiction.

Now we assume that \((\mathcal{F}_2(D), \tau_F)\) is a \( D_1 \)-space. By Fact 1, \( NI(\mathcal{F}_2(D), \tau_F) \) is countably compact; however, by Claims 1 and 2, this is impossible since \( \{ \{ d_i \} : i \in \mathbb{N} \} \) has no cluster point in \((\mathcal{F}_2(D), \tau_F)\).

\[ \square \]

**Lemma 2.5.** If \((\mathcal{F}_2(X), \tau_F)\) or \((\mathcal{F}_2(X), \tau_V)\) is perfect\(^2\), then \( X \) has a \( G_\delta \)-diagonal\(^3\).

**Proof.** We only prove the case for \((\mathcal{F}_2(X), \tau_F)\) since the case of \((\mathcal{F}_2(X), \tau_V)\) can be shown by a similar way. Clearly, it suffices to prove that \( \Delta = \{ (x, y) : x \in X \} \) is the intersection of countably many open subsets of \( X^2 \). Let \( f : X^2 \to (\mathcal{F}_2(X), \tau_F) \) defined by \( f((a, b)) = \{ a, b \} \); then \( f \) is continuous. Indeed, for an open subset \( U \) and compact subset \( K \) of \( X \), we have

\[
f^{-1}(U^- \cap \mathcal{F}_2(X)) = (U \times X) \cup (X \times U)
\]

and

\[
f^{-1}((K^c)^+ \cap \mathcal{F}_2(X)) = (X \setminus K) \times (X \setminus K),
\]

which are open in \( X^2 \). Hence \( f \) is continuous. Since \((\mathcal{F}_2(X), \tau_F)\) is perfect and \( \mathcal{F}_1(X) = \{ \{ x \} : x \in X \} \) is a closed subset of \((\mathcal{F}_2(X), \tau_F)\), it follows that \( \mathcal{F}_1(X) = \bigcap_{n \in \mathbb{N}} \mathcal{G}_n \), where \( \mathcal{G}_n \) is open in \((\mathcal{F}_2(X), \tau_F)\) for each \( n \in \mathbb{N} \). Clearly, \( \Delta = f^{-1}(\mathcal{F}_1(X)) = \bigcap_{n \in \mathbb{N}} f^{-1}(\mathcal{G}_n) \).

Hence \( X \) has a \( G_\delta \)-diagonal. \( \square \)

Now we can prove one of main theorems in this section.

**Theorem 2.6.** The following statements are equivalent for a space \( X \).

1. \((CL(X), \tau_F)\) is a \( D_1 \)-space;
2. \((K(X), \tau_F)\) is a \( D_1 \)-space;
3. \((\mathcal{F}_n(X), \tau_F)\) is a \( D_1 \)-space for some \( n \geq 2 \);
4. \( X \) is compact metrizable.

**Proof.** Note that a \( D_1 \)-space is a property closed under taking closed subspaces, (1) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (3) are trivial. Note that if \( X \) is compact, then \((K(X), \tau_F) = (CL(X), \tau_F) \cong (CL(X), \tau_V) = (K(X), \tau_V)\), the implications (4) \( \Rightarrow \) (1) and (4) \( \Rightarrow \) (2) follow from \([13]\). We only need to prove (3) \( \Rightarrow \) (4).

It suffices to consider the case when \( n = 2 \). Assume that \((\mathcal{F}_2(X), \tau_F)\) is a \( D_1 \)-space. Then \( X \) is a \( D_1 \)-space. By Fact 1 and Lemma 2.5, \( NI(X) \) is a countably compact subspace with a \( G_\delta \)-diagonal. Then \( NI(X) \) is metrizable \([5\text{ Theorem 2.14}]\), hence it is compact. Now we prove that \( X \) is compact. Indeed, let \( \mathcal{U} \) be any open cover of \( X \); then we can find a finite subfamily \( \mathcal{U}' \subset \mathcal{U} \) such that \( NI(X) \subset \bigcup \mathcal{U}' \). Let \( D = X \setminus \bigcup \mathcal{U}' \); then \( D \subset I(X) \). We claim that \( |D| < \omega \). Otherwise, without loss of generality, \( D \) is an infinite countable closed discrete subset of \( X \), then it follows from Lemma 2.4 that \((\mathcal{F}_2(D), \tau_F)\) is not a \( D_1 \)-space, this is a contradiction. Therefore, \( X \) is covered by finitely many elements of \( \mathcal{U} \), thus \( X \) is compact.

By Lemma 2.5, \( X \) has a \( G_\delta \)-diagonal, then \( X \) is metrizable by \([5\text{ Theorem 2.13}]\). \( \square \)

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\(^2\)A space \( X \) is called perfect if every closed subset of \( X \) is a \( G_\delta \)-set.

\(^3\)A space \( X \) is said to have a \( G_\delta \)-diagonal if, there is a sequence \( \{ \mathcal{U}_n \} \) of open covers of \( X \), such that, for each \( x \in X \), \( \{ x \} = \bigcap_{n \in \mathbb{N}} \text{st}(x, \mathcal{U}_n) \).
Remark 2.7. However, if $X$ is a discrete space, then it is easily verified that $(\mathcal{K}(X), \tau_V)$ is a discrete space, hence $(\mathcal{F}_n(X), \tau_V)$ is discrete for any $n \in \mathbb{N}$, thus all are $D_1$-spaces. Therefore, Theorem 2.6 does not hold for the case of the hyperspace with Vietoris topology. However, we have the following theorem.

In [12], we have proved that for a space $X$ if $(CL(X), \tau_V)$ is a $D_1$-space then $X$ is compact and metrizable. Therefore, it is natural to characterize $X$ such that the subspace $(\mathcal{K}(X), \tau_V)$ of $(CL(X), \tau_V)$ is a $D_1$-space, see the following theorem.

Theorem 2.8. The following statements are equivalent for a space $X$.

1. $(\mathcal{K}(X), \tau_V)$ is a $D_1$-space;
2. $(\mathcal{F}_n(X), \tau_V)$ is a $D_1$-space for some $n \geq 2$;
3. $X$ is discrete or compact metrizable.

Proof. The implication of $(1) \Rightarrow (2)$ is trivial since a closed subspace of a $D_1$-space is a $D_1$-space. It suffices to prove that $(3) \Rightarrow (1)$ and $(2) \Rightarrow (3)$.

$(3) \Rightarrow (1)$. If $X$ is discrete, then it is easy to see that $(\mathcal{K}(X), \tau_V)$ is discrete; if $X$ is compact metrizable, then $(\mathcal{K}(X), \tau_V)$ is compact metrizable by [13]. Hence $(\mathcal{K}(X), \tau_V)$ is a $D_1$-space.

$(2) \Rightarrow (3)$. We only consider the case for $n = 2$. Assume $(\mathcal{F}_2(X), \tau_V)$ is a $D_1$-space. Then $X$ is a $D_1$-space. By fact 1, $NI(X)$ is countably compact, then it follows from Lemma 2.5 that $X$ is metrizable. Suppose $X$ is neither discrete nor compact, there exist a closed infinite countable discrete subset $\{x_i : i \in \mathbb{N}\} \subset I(X)$ and a non-trivial sequence $\{x_n \in \mathbb{N}\}$ of $X$ converging to $x \in NI(X)$. Without loss of generality, we may assume that $\{d_i : i \in \mathbb{N}\} = \emptyset$, $d_n \neq d_m$ and $x_n \neq x_m$ for any $n \neq m$. Clearly, for each $i \in \mathbb{N}$, the set $\{d_i, x \in NI(\mathcal{F}_2(X), \tau_V)\}$ since $\{d_i, x_n \} \rightarrow \{d_i, x\}$ in $(\mathcal{F}_2(X), \tau_V)$ as $n \rightarrow \infty$. In order to obtain a contradiction, we prove the following Claim 3. Indeed, since $(\mathcal{F}_2(X), \tau_V)$ is a $D_1$-space, it follows that $NI(\mathcal{F}_2(X), \tau_V)$ is countably compact; however, the set $\{\{d_i, x \} : i \in \mathbb{N}\}$ is discrete in $NI(\mathcal{F}_2(X), \tau_V)$.

Claim 3 The set $\{\{d_i, x \} : i \in \mathbb{N}\}$ is discrete in $NI(\mathcal{F}_2(X), \tau_V)$.

Take any $K \in NI(\mathcal{F}_2(X), \tau_V)$. If $|K| = 2$, we write $K = \{a_1, a_2\}$, and let $V_j$ be a neighborhood of $a_j$ for each $j \leq 2$ such that $|\langle V_1 \cup V_2 \rangle \cap \{d_i : i \in \mathbb{N}\}| \leq 1$ and $V_1 \cap V_2 = \emptyset$ (this is possible since $K \in NI(\mathcal{F}_2(X), \tau_V)$ which implies that at least one of the points $a_1, a_2$ is not a non-isolated point in $X$). Then it is easily verified that $|\langle V_1, V_2 \rangle \cap \{d_i, x : i \in \mathbb{N}\}| \leq 1$. If $|K| = 1$, then let $K = \{a\}$ and let $V$ be an open neighborhood of $a$ in $X$ with $|V \cap \{d_i : i \in \mathbb{N}\}| \leq 1$. Then $\langle V \rangle$ is a neighborhood of $K$ and $|\langle V \rangle \cap \{d_i, x : i \in \mathbb{N}\}| \leq 1$. Therefore, $\{\{d_i, x \} : i \in \mathbb{N}\}$ is discrete in $NI(\mathcal{F}_2(X), \tau_V)$. \qed

However, for the cases of $(\mathcal{F}(X), \tau_V)$ and $(\mathcal{F}(X), \tau_F)$, the situations are different, see the following two theorems.

Theorem 2.9. Let $X$ be a space. Then $(\mathcal{F}(X), \tau_V)$ is a $D_1$-space if and only if $X$ is discrete.

Proof. If $X$ is discrete, it is easy to see that $(\mathcal{F}(X), \tau_V)$ is discrete, hence it is a $D_1$-space. If $(\mathcal{F}(X), \tau_V)$ is a $D_1$-space, then $(\mathcal{F}_2(X), \tau_V)$ is a $D_1$-space, by Theorem 2.8 $X$ is discrete or compact metrizable. If $X$ is compact metrizable and non-discrete, then it contains a convergent sequence $S = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ with $x_n \rightarrow x$ as $n \rightarrow \infty$, where $x$ is a non-isolated point in $X$. Let $\mathcal{A} = \{\{x\} \cup \{x_i : i \leq n\} : n \in \mathbb{N}\}$. Then it is obvious that $\mathcal{A} \subset NI(\mathcal{F}(S))$. We claim that $\mathcal{A}$ is discrete in $\mathcal{F}(S)$. 

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Suppose not, \( A \) has a cluster point \( B \) in \( F(S) \). Let \( B = \{y_1,...,y_k\} \). If \( x \notin B \), then \( \{\{y_1\},...,\{y_k\}\} \) is an open neighborhood of \( B \), \( \{\{y_1\},...,\{y_k\}\} \cap A = \emptyset \), this is a contradiction. If \( x \in B \), say \( y_1 = x \), pick an open neighborhood \( V = \{x\} \cup \{x_n: n > k + 1\} \) of \( y_1 \) in \( S \). Clearly, \( |S \setminus V| > k + 1 \). Then \( \{V,\{y_2\},...,\{y_k\}\} \) is an open neighborhood of \( B \). Since \( \langle V,\{y_2\},...,\{y_k\}\rangle \cap \{\{x\} \cup \{x_i: i \leq n\} : n > k + 1\} = \emptyset \).

Indeed, if \( \{x,x_1,...,x_m\} \in \{V,\{y_2\},...,\{y_k\}\} \cap \{\{x\} \cup \{x_i: i \leq n\} : n > k + 1\} \) for some \( m > k + 1 \), \( \{x,x_1,...,x_m\} \subset V \cap \{y_2,...,y_k\} \), it implies that \( \{x_1,...,x_{k+1}\} \subset \{y_2,...,y_k\} \), which is impossible. It follows that \( \langle V,\{y_2\},...,\{y_k\}\rangle \cap A = \emptyset \), which is a contradiction.

Therefore, \( A \) is discrete in \( F(S) \). From fact 1, \( F(S) \) is not a \( D_1 \)-space, which implies that \( (F(X),\tau_V) \) is not a \( D_1 \)-space, this is a contradiction. Hence \( X \) is not compact metrizable. Thus \( X \) is discrete. \( \square \)

**Theorem 2.10.** Let \( X \) be a space. Then \( (F(X),\tau_F) \) is a \( D_1 \)-space if and only if \( X \) is finite.

**Proof.** If \( X \) is finite, then it is easy to see that \( (F(X),\tau_F) \) is a \( D_1 \)-space.

If \( (F(X),\tau_F) \) is a \( D_1 \)-space, then \( (F_2(X),\tau_F) \) is a \( D_1 \)-space, hence \( X \) is compact metrizable by Theorem 2.8. From [2] Exercise 5.1, problem 3, it follows that

\[
(CL(X),\tau_V) = (CL(X),\tau_F),
\]

then \( (F(X),\tau_V) = (F(X),\tau_F) \) is a \( D_1 \)-space, which shows that \( X \) is discrete by Theorem 2.9 hence \( X \) is finite since a discrete compact space is finite. \( \square \)

As some applications of above results, we have the following remark.

**Remark 2.11.** Let \( X \) be an arbitrary infinite compact metrizable space. Then \( (CL(X),\tau_V) \) is a compact metrizable space, thus \( (CL(X),\tau_V) \) is a \( D_1 \)-space and \( (F(X),\tau_V) \) is metrizable; however, \( (F(X),\tau_V) \) is not a \( D_1 \)-space by Theorem 2.4. Therefore, there exists a closed subset \( F \) of \( (F(X),\tau_V) \) such that \( F \) has no countable character at \( (F(X),\tau_V) \). Moreover, \( (K(X),\tau_V) \) is a \( D_1 \)-space; however, \( (F(X),\tau_F) \) is not a \( D_1 \)-space.

Finally we consider the space \( X \) such that the subspace \( (K(X) \setminus F(X),\tau_V) \) is a \( D_1 \)-space. First, we need two lemmas.

A subset \( P \) of \( X \) is called a *sequential neighborhood* of \( x \in X \), if each sequence converging to \( x \) is eventually\(^4\) in \( P \). A subset \( U \) of \( X \) is called *sequentially open* if \( U \) is a sequential neighborhood of each of its points. A subset \( F \) of \( X \) is called *sequentially closed* if \( X \setminus F \) is sequentially open. The space \( X \) is called a *sequential space* if each sequentially open subset of \( X \) is open.

**Lemma 2.12.** Let \( X \) be a sequential space. Then \( (F(X),\tau_V) \) is open in \( (K(X),\tau_V) \) if and only if \( X \) is discrete.

**Proof.** It suffices to prove the necessity. Assume that \( (F(X),\tau_V) \) is open in \( (K(X),\tau_V) \), and assume that \( X \) is not discrete. Then there exists a non-trivial sequence \( \{x_n\} \) converging to \( x \) as \( n \to \infty \). Since \( \{x\} \in F(X) \), there exists an open neighborhood \( U \) of \( x \) such that \( (U) \subset F(X) \). However, since \( x \in U \) and \( \{x_n\} \) converging to \( x \) as \( n \to \infty \), we can find \( m \in \mathbb{N} \) such that \( \{x\} \cup \{x_n: n \geq m\} \subset U \), then \( \{x\} \cup \{x_n: n \geq m\} \in (U) \subset F(X) \), which is a contradiction since \( \{x\} \cup \{x_n: n \geq m\} \notin F(X) \).

**Lemma 2.13.** Let \( X \) be a space. Then any point of \( (K(X) \setminus F(X),\tau_V) \) is not isolated.

\(^4\)A sequence \( \{x\} \cup \{x_i: i \in \mathbb{N}\} \) with \( x_i \to x \) is called *eventually* in some subset \( P \) if there exists \( k \in \mathbb{N} \) such that \( \{x\} \cup \{x_i: i \geq k\} \subset P \).
Proof. Take any $K \in \mathcal{K}(X) \setminus \mathcal{F}(X)$. Then $K$ is an infinite compact subset of $X$. Suppose that $K$ is an isolated point in $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{V})$. Then there exists a basic neighborhood $\langle U_1, \ldots, U_k \rangle$ of $K$ in $(\mathcal{K}(X), \tau_\mathcal{V})$ such that $\langle U_1, \ldots, U_k \rangle \cap (\mathcal{K}(X) \setminus \mathcal{F}(X)) = \{K\}$, where the family $\{U_i : i \leq k\}$ is a disjoint collection of open subsets of $X$. Clearly, $K = \bigcup_{i=1}^{k} U_i$; otherwise, we can pick any point $a \in \bigcup_{i=1}^{k} U_i \setminus K$, then $K \cup \{a\} \in \langle U_1, \ldots, U_k \rangle \cap (\mathcal{K}(X) \setminus \mathcal{F}(X))$, which is a contradiction. Since $K$ is infinite, there exists $j \leq k$ such that $|K \cap U_j| \geq \omega$. Since $X$ is Hausdorff, there exist a point $b \in K \cap U_j$ and an open neighborhood $V \subset U_j$ of $b$ such that $|(K \setminus V) \cap U_j| \geq \omega$, hence $K \setminus V \in (\mathcal{K}(X) \setminus \mathcal{F}(X))$, which is a contradiction. Therefore, $K$ is an isolated point in $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{V})$.

Since Fell topology is coarser than the Vietoris topology, it follows from Lemmas 2.12 and 2.13 that we have the following two corollaries.

Corollary 2.14. Let $X$ be a sequential space. Then $(\mathcal{F}(X), \tau_\mathcal{F})$ is open in $(\mathcal{K}(X), \tau_\mathcal{V})$ if and only if $X$ is discrete.

Corollary 2.15. Let $X$ be a space. Then any point of $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{F})$ is not isolated.

Now we can prove the following theorem.

Theorem 2.16. Let $X$ be a sequential space with a $G^*_\delta$-diagonal\footnote{A space $X$ is said to have a $G^*_\delta$-diagonal if, there is a sequence $\{\mathcal{U}_n\}$ of open covers of $X$, such that, for each $x \in X$, $\{x\} = \bigcap_{n \in \mathbb{N}} \text{st}(x, \mathcal{U}_n)$}. Then $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{V})$ is a $D_1$-space if and only if $X$ is discrete.

Proof. It suffices to prove the necessity. Suppose that $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{V})$ is a $D_1$-space, and suppose that $X$ is non-discrete. Then $\mathcal{K}(X) \setminus \mathcal{F}(X)$ is countably compact by Fact 1 and Lemma 2.13. Since $X$ has a $G^*_\delta$-diagonal, it follows that $(\mathcal{K}(X), \tau_\mathcal{V})$ has a $G^*_\delta$-diagonal by \cite{claim} Theorem 1, thus $\mathcal{K}(X) \setminus \mathcal{F}(X)$ has a $G^*_\delta$-diagonal too. Therefore, $\mathcal{K}(X) \setminus \mathcal{F}(X)$ is compact by \cite{claim} Theorem 2.14, then $\mathcal{K}(X) \setminus \mathcal{F}(X)$ is closed in $(\mathcal{K}(X), \tau_\mathcal{V})$, which shows that $\mathcal{F}(X)$ is open in $(\mathcal{K}(X), \tau_\mathcal{V})$. However, by Lemma 2.12, $\mathcal{F}(X)$ is not open in $(\mathcal{K}(X), \tau_\mathcal{V})$, which is a contradiction. Therefore, $X$ is discrete.

However, it is still unknown for us if Theorem 2.16 holds for the case of the Fell topology. Therefore, we have the following two questions.

Question 2.17. Let $X$ be a sequential space with a $G^*_\delta$-diagonal. If $(\mathcal{K}(X) \setminus \mathcal{F}(X), \tau_\mathcal{F})$ is a $D_1$-space, is $X$ discrete?

Question 2.18. Let $X$ be a space. Give a characterization of $X$ such that $(\mathcal{K}(X), \tau_\mathcal{F})$ or $(\mathcal{CL}(X), \tau_\mathcal{F})$ has a $G^*_\delta$-diagonal.

The following theorem gives a characterization of a space $X$ such that $(\mathcal{CL}_0(X), \tau_\mathcal{F})$ is a $D_1$-space, where $\mathcal{CL}_0(X) = \mathcal{CL}(X) \cup \{\emptyset\}$.

Theorem 2.19. The following statements are equivalent for space $X$.

1. $(\mathcal{CL}_0(X), \tau_\mathcal{F})$ is a $D_1$-space;
2. $(\mathcal{F}_n(X) \cup \{\emptyset\}, \tau_\mathcal{F})$ is a $D_1$-space for some $n \geq 2$;
3. $X$ is locally compact, separable and metrizable.

Proof. By \cite{claim} Theorem 5.1.5, we have (3) $\Rightarrow$ (1). The implication of (1) $\Rightarrow$ (2) is obvious. It suffices to prove that (2) $\Rightarrow$ (3). Assume $(\mathcal{F}_n(X) \cup \{\emptyset\}, \tau_\mathcal{F})$ is a $D_1$-space for some $n \geq 2$, then $(\mathcal{F}_2(X) \cup \{\emptyset\}, \tau_\mathcal{F})$ is a $D_0$-space, hence $(\mathcal{F}_2(X), \tau_\mathcal{F})$ is a $D_0$-space, by Theorem 3.1, $X$ is locally compact, separable and metrizable. \hfill $\square$
3. The $D_0$-property of hyperspaces

In this section, we mainly discuss the $D_0$-property of the hyperspaces. We first give a characterization of a space $X$ such that $(\text{CL}(X), \tau_F)$ is a $D_0$-space.

**Theorem 3.1.** The following statements are equivalent for space $X$.

1. $(\text{CL}(X), \tau_F)$ is metrizable;
2. $(\text{CL}(X), \tau_F)$ is a $\gamma$-space;
3. $(\text{CL}(X), \tau_F)$ is a $D_0$-space;
4. $(K(X), \tau_F)$ is a $D_0$-space;
5. $(\mathcal{F}(X), \tau_F)$ is a $D_0$-space;
6. $(\mathcal{F}_2(X), \tau_F)$ is a $D_0$-space;
7. $X$ is locally compact, separable and metrizable.

**Proof.** The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial. (7) $\Leftrightarrow$ (1) by [2, Theorem 5.1.5]. We only prove (6) $\Rightarrow$ (7).

Assume that $(\mathcal{F}_2(X), \tau_F)$ is a $D_0$-space, then $X$ is a $D_0$-space. Fix any $x \in X$; then let \( \{V_n \cap (K_{n,x})^+ \cap \mathcal{F}_2(X) : n \in \mathbb{N}\} \) be a countable local base at \( \{x\} \) in $(\mathcal{F}_2(X), \tau_F)$, where each $V_n$, $K_{n,x}$ is open, compact in $X$ respectively. For any compact $K \subset \mathcal{K}(X \setminus \{x\})$, $(K^c)^+$ is an open neighborhood of \( \{x\} \), there exists $n \in \mathbb{N}$ such that $V_n^- \cap (K_{n,x}^c)^+ \cap \mathcal{F}_2(X) \subset (K^c)^+ \cap \mathcal{F}_2(X)$, then $K \subset K_{n,x}$. Therefore, \( \{K_{n,x} : n \in \mathbb{N}\} \) is cofinal in $\mathcal{K}(X \setminus \{x\})$.

**Claim 4** $X$ is locally compact.

Fix $x \in X$, pick any $y \neq x$ and let \( \{U_n : n \in \mathbb{N}\} \) be a decreasingly local base at $x$ in $X$ with $y \notin U_n$ for each $n \in \mathbb{N}$. We claim that there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $U_k \subset K_{n,y}$; otherwise, for any $n \in \mathbb{N}$, pick $x_n \in U_n \setminus K_{n,y}$; then $x_n \rightarrow x$ as $n \rightarrow \infty$ in $X$ and $S = \{x\} \cup \{x_n : n \in \mathbb{N}\} \subset \mathcal{K}(X \setminus \{y\})$, then $S \subset K_{m,y}$ for some $m \in \mathbb{N}$ since \( \{K_{m,y} : n \in \mathbb{N}\} \) is cofinal in $\mathcal{K}(X \setminus \{y\})$. This is a contradiction. Hence $X$ is locally compact.

Fix any compact subset $K$ of $X$. Since $(\mathcal{F}_2(K), \tau_F)$ is a compact $D_0$-space, then $(\mathcal{F}_2(K), \tau_F)$ is perfect. By Lemma [2,25 and [5, Theorem 2.13], $K$ is metrizable. Hence $X$ locally metrizable by Claim 4. By the above proof, $X$ is also a $\sigma$-compact space, thus it is separable and locally compact metrizable.\]

**Remark 3.2.** By Theorem 3.1, $(\text{CL}(\mathbb{R}), \tau_F)$ is a $D_0$-space (indeed, it is metrizable, thus perfect normal), where $\mathbb{R}$ is the real number with the usual topology; however, $(\text{CL}(\mathbb{R}), \tau_F)$ is not a $D_1$-space by Theorem [2.6].

A space $(X, \tau)$ is a $\gamma$-space if there exists a function $g : \omega \times X \rightarrow \tau$ such that (i) \( \{g(n, x) : n \in \omega\} \) is a base at $x$; (ii) for each $n \in \omega$ and $x \in X$, there exists $m \in \omega$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$. By [5, Theorem 10.6(iii)], each $\gamma$-space is a $D_0$-space.

In [12], F. Lin, R. Shen and C. Liu proved that under (MA+¬CH), $(\text{CL}(X), \tau_\gamma)$ is a $\gamma$-space if and only if $(\text{CL}(X), \tau_\gamma)$ is a $D_0$-space if and only if $X$ is a separable metrizable space and $NI(X)$ is compact. Therefore, we have the following question.

**Question 3.3.** If $(\text{CL}(X), \tau_\gamma)$ is a $D_0$-space, is $(\text{CL}(X), \tau_\gamma)$ a $\gamma$-space?

For any non-separable and metrizable space $X$, it follows from [12, Theorem 5.17] and the following Theorem 3.5 that $(\text{CL}(X), \tau_\gamma)$ is not a $\gamma$-space and $(\mathcal{K}(X), \tau_\gamma)$ is a $D_0$-space.
Lemma 3.4. [14] The following are equivalent for a space $X$.

1. $X$ is an open compact-covering image of a metric space;
2. Each compact subset of $X$ is metrizable and has a countable neighborhood base in $X$;
3. Every compact subset of $X$ has a countable outer base in $X$.

Theorem 3.5. The following statements are equivalent for a space $X$.

1. $(K(X), \tau_V)$ is an open compact-covering image of a metric space.
2. $(K(X), \tau_V)$ is a $D_0$-space;
3. $(F(X), \tau_V)$ is a $D_0$-space;
4. $(F_2(X), \tau_V)$ is a $D_0$-space;
5. $X$ is a $D_0$-space and every compact subset of $X$ is metrizable.

Proof. By Lemma 3.4, we have (1) $\Rightarrow$ (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. We only need to prove (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (5). Assume that $(F_2(X), \tau_V)$ is a $D_0$-space, then $X$ is a $D_0$-space. For any compact subset $H$ of $X$, $(F_2(H), \tau_V)$ is a compact, $D_0$-subspace of $(F_2(X), \tau_V)$, then it is perfect. By Lemma 2.5, $H$ has a $G_\delta$-diagonal, then $H$ is metrizable by Theorem 2.14.

(5) $\Rightarrow$ (2). Fix any compact subset $K$ of $(K(X), \tau_V)$. We prove that $K$ has a countable base in $(K(X), \tau_V)$.

Indeed, let $K = \bigcup K_i$. Then $K$ is compact in $X$ by [13, Theorem 2.5.2], hence $K$ has a countable base and is metrizable. By Lemma 3.4, $K$ has a countable outer base $B$. Let $\Gamma = \{(B^i) : B^i \in B^{\leq \omega}\}$, $\Delta = \{\bigcup C : C \in \Gamma^{\leq \omega}\}$; then $|\Delta| \leq \omega$. Now it suffices to prove that $\Delta$ is a countable open base in $(K(X), \tau_V)$.

Take any open neighborhood $\hat{U}$ of $K$ in $(K(X), \tau_V)$; then there exist finitely many basic open neighborhoods $\{\hat{U}_i : i \leq l\}$ such that $K \subset \bigcup\{\hat{U}_i : i \leq l\} \subset \hat{U}$, where we may assume that $\{\hat{U}_i : i \leq l\}$ is a minimal cover of $K$. Let $\hat{U} = \bigcup i \leq l K_i$ and $K_i \subset \hat{U}_i$ for each $i \leq l$. We will prove that each $K_i$ is contained in an element of $\Delta$ itself contained in $\hat{U}$, which shows that $K$ is contained in an element of $\Delta$ that is contained in $\hat{U}$. Therefore, without loss of generality, we may assume $\hat{U} = \langle U_1, \ldots, U_k \rangle$, where each $U_i (i \leq k)$ is an open subset in $X$.

Since $(K(X), \tau_V)$ is regular, there exist finitely many open subsets $\{\hat{V}(i) : i \leq n\}$ of $(K(X), \tau_V)$ such that

$K \subset \bigcup_{i \leq n} \hat{V}(i) \subset \bigcup_{i \leq n} \text{Cl}(\hat{V}(i)) \subset \hat{U},$

where $\hat{V}(i) = (V_i(i), \ldots, V_{r_i}(i))$, each $r_i \in \mathbb{N}$ and each $V_j(i)$ is open in $X$.

Fix any $i \leq n$. Then $\text{Cl}(\hat{V}(i)) \subset \hat{U}$. For each $t \leq k$, it follows from Lemmas 2.1 and 2.2 that $U_t \supset \hat{V}(i)(t)$ for some $i(t) \leq r_i$. Let $A = \{i(t) : t \leq k\}$ and $B = \{1, \ldots, r_i\} \setminus A$. Put

$K(i) = \bigcup\{K : K \in K \cap \text{Cl}(\hat{V}(i))\}.$
Then $K(i)$ is compact in $X$ by [13, Theorem 2.5.2].

For any $K' \in \{K : K \in \mathbf{K} \cap \text{Cl}(\hat{V}(i))\}$, we have $K' \cap \overline{V_j(i)} \neq \emptyset$ for $j \leq r_i$ and $K' \subset \bigcup \{\overline{V_j(i)} : j \leq r_i\}$. It is obvious that $K(i) \subset \bigcup \{\overline{V_j(i)} : j \leq r_i\}$. Note that $K' \subset K(i)$, then $K(i) \cap \overline{V_j(i)} \neq \emptyset$ for each $j$. Then, by Lemma 2.2 we have

$$K(i) \in \text{Cl}(\hat{V}(i) = \overline{(V_1(i),...,V_{r_i}(i))}).$$

For each $j \in A$,

$$K(i) \cap \overline{V_j(i)} \subset \bigcap \{U_t : t \leq k, i(t) = j\} = W_j.$$

For $x \in K(i) \cap \overline{V_j(i)}$, pick $B_x \in \mathcal{B}$ such that $x \in B_x \subset W_j$. Then there is a finite subfamily $\mathcal{B}(j)$ of $\{B_x : x \in K(i) \cap \overline{V_j(i)}\}$ such that

$$K(i) \cap \overline{V_j(i)} \subset \bigcup \mathcal{B}(j) \subset W_j.$$

For any $j \in B$, choose a finite subfamily $\mathcal{B}(j) \subset \mathcal{B}$ such that

$$K(i) \cap \overline{V_j(i)} \subset \bigcup \mathcal{B}(j) \subset \bigcup U_t.$$

Now put

$$\mathcal{R}(i) = \left\{ \bigcup \mathcal{B}'(j) : \emptyset \neq \mathcal{B}'(j) \subset \mathcal{B}(j), j \leq r_i \right\}.$$ 

Clearly, $|\mathcal{R}(i)| < \omega$. Then

$$\mathbf{K} \cap \text{Cl}(\hat{V}(i)) \subset \bigcup \{\langle \mathcal{W} \rangle : \mathcal{W} \in \mathcal{R}(i)\} \subset \hat{U},$$

where $\langle \mathcal{W} \rangle = \langle B_1,...,B_m \rangle$ for $\mathcal{W} = \langle B_1,...,B_m \rangle$. Indeed, for any $K \in \mathbf{K} \cap \text{Cl}(\hat{V}(i))$, then $K \subset K(i)$. For each $j \leq r_i$, pick $\mathcal{C}(j) \subset \mathcal{B}(j)$ such that $K \cap \overline{V_j(i)} \cap C \neq \emptyset$ for each $C \in \mathcal{C}(j)$ and $K \cap \overline{V_j(i)} \subset \bigcup \mathcal{C}(j)$. Enumerate $\bigcup_{j \leq r_i} \mathcal{C}(j)$ as $\{C_1,...,C_p\}$; then

$$\mathcal{W}_i = \{C_1,...,C_p\} \in \mathcal{R}(i).$$

Then $K \cap C_l \neq \emptyset$ for each $l \leq p$, $K \subset \bigcup_{i \leq p} C_i$. Therefore, $K \in \langle \mathcal{W}_i \rangle \subset \hat{U}$.

Hence

$$\mathbf{K} = \bigcup_{i \leq n} (\mathbf{K} \cap \text{Cl}(\hat{V}(i)) \subset \bigcup \{\langle \mathcal{W}_i \rangle : \mathcal{W}_i \in \mathcal{R}(i), i \leq n\} \subset \hat{U}.$$ 

Note that $\mathcal{R}(i) \subset \mathcal{B}$ for $i \leq n$ and $|\Delta| \leq \omega$. Hence $\mathbf{K}$ has a countable base in $(\mathcal{K}(X), \tau_\nu)$.

(2) $\Rightarrow$ (1). By Lemma 3.4 we only prove that each compact subset $\mathcal{H}$ of $(\mathcal{K}(X), \tau_\nu)$ is metrizable. Let $\mathcal{H} = \bigcup \mathbf{H}$. Then $\mathcal{H}$ is compact in $X$ by [13, Theorem 2.5.2], hence it is metrizable since (2) implies (5). Therefore, $(\mathcal{K}(H), \tau_\nu)$ is compact and metrizable, thus $\mathcal{H} \subset (\mathcal{K}(H), \tau_\nu)$ is metrizable.

Since $\gamma$-space $X$ is a $D_0$-space, and every compact subset of $X$ is metrizable, $(\mathcal{K}(X), \tau_\nu)$ is a $D_0$-space. But, the following question still open.

**Question 3.6.** If $X$ is a $\gamma$-space, is $(\mathcal{K}(X), \tau_\nu)$ a $\gamma$-space?

4. The $G_\delta$-property of hyperspaces

Finally we consider the characterization of hyperspaces which have countable pseudocharacter. We say that a space $X$ has a compact-$G_\delta$-property if every compact subset of $X$ is a $G_\delta$-set of $X$.

**Theorem 4.1.** Let $X$ be a space. Then $(\mathcal{K}(X), \tau_\nu)$ has a compact-$G_\delta$-property if and only if $X$ has a compact-$G_\delta$ property and every compact subset of $X$ is metrizable.
Definition 4.6. The pseudocharacter of a point $x$ in a space $X$ is defined as the smallest cardinal number of the form $|U|$, where $U$ is a family of open subsets of $X$ such that $\bigcap U = \{x\}$; this cardinal number is denoted by $\psi(x, X)$. The pseudocharacter of a space $X$ is defined as the supremum of all numbers $\psi(x, X)$ for $x \in X$; this cardinal number is denoted by $\psi(X)$. If $\psi(X) = \omega$, we say that $X$ has a countable pseudocharacter.

Proof. Necessity. Assume that $(K(X), \tau_V)$ has the compact-$G_\delta$-property. Obviously, $X$ has the compact-$G_\delta$-property. Fix any compact subset $K$ of $X$. Then the hyperspace $K(K)$ is compact by [13]. Since $K(K)$ is regular and has compact-$G_\delta$ property, it follows that $K(K)$ is a $D_0$-space, hence $D_1$-space (note that every closed subset of $K(K)$ is compact since $K(K)$ is compact). By Theorem 2.6 $K$ is compact metrizable.

Sufficiency. Let $K$ be a compact subset of $K(X)$, and let $H = \bigcup K$. Then $H$ is compact in $X$. By the assumption, $H$ is metrizable, hence $K(H)$ is compact metrizable. Let $B = \{(U_1(i), ..., U_{k_i}(i)) : i \in \mathbb{N}\}$ be a countable base of $K(H)$, where each $k_i \in \mathbb{N}$, each $U_j(i)$ ($i \in \mathbb{N}$ and $j \leq k_i$) is open in $H$. For any $i \in \mathbb{N}$ and $j \leq k_i$, let $V_j(i)$ be an open subset of $X$ such that $U_j(i) = V_j(i) \cap H$. Let $\{W_n : n \in \mathbb{N}\}$ be open subsets of $X$ with $H = \bigcap W_n$. For each $n \in \mathbb{N}$, put

\[ P_n = \{(V_1(i) \cap W_n, ..., V_{k_i}(i) \cap W_n) : (V_1(i), ..., V_{k_i}(i)) \cap K \neq \emptyset, i, n \in \mathbb{N}\}; \]

we say that $(V_1(i) \cap W_n, ..., V_{k_i}(i) \cap W_n)$ is associated with $(U_1(i), ..., U_{k_i}(i))$ for each $i \in \mathbb{N}$. Put

\[ Q_n = \{\bigcup P' : K \subset \bigcup P', P' \in P_n^<\omega\} \]

for each $n \in \mathbb{N}$; then let $Q = \bigcup_{n \in \mathbb{N}} Q_n$. Clearly, $|Q| \leq \omega$. We claim that $K = \bigcap Q$.

Indeed, it is trivial to verify that $K \subset \bigcap Q$. Let $K$ be an any compact subset of $X$ with $K \notin K$. We prove that there is $\bigcup P' \in Q$ such that $K \notin \bigcup P'$.

Case 1: $K \setminus H \neq \emptyset$.

Then there exists $j \in \mathbb{N}$ such that $K \setminus W_j \neq \emptyset$, hence we can pick a $\bigcup P' \in Q_j$, then $K \notin \bigcup P'$.

Case 2: $K \subset H$.

Since $B$ is a countable base of $K(H)$ and $K \notin K$, it is easily checked that there is a finite family $B' \subset B$ such that $K \notin \bigcup B'$, $K \subset \bigcup B'$ and each element of $B'$ meets $K$. Fix any $n \in \mathbb{N}$, and let $P'$ be the set of elements of $P_n$ that are associated with the elements of $B'$. Then it is easily verified that $K \notin \bigcup P'$.

□

Remark 4.2. From Theorems 3.3 and 4.1 we conclude that there exists a space $X$ such that $(K(X), \tau_V)$ has the compact-$G_\delta$-property and is not a $D_0$-space, such as, the Butterfly space [11] Example 1.8.3]. Moreover, the following questions are still unknown for us.

Question 4.3. If $X$ has the compact-$G_\delta$ property and every compact subset of $X$ is metrizable, does $(CL(X), \tau_V)$ have a compact-$G_\delta$-property?

Question 4.4. If $X$ has the compact-$G_\delta$ property and every compact subset of $X$ is metrizable, does $(K(X), \tau_F)$ or $(CL(X), \tau_F)$ have a compact-$G_\delta$-property?

Question 4.5. Characterise spaces $X$ such that $(K(X), \tau_F)$, $(CL(X), \tau_F)$, $(CL(X), \tau_V)$ and $(K(X), \tau_V)$ are perfect, respectively?

In order to give a characterization of $(CL(X), \tau_F)$ with the countable pseudocharacter, we need to prove an equality relating the pseudocharacter of hyperspace $\psi(CL(X), \tau_F)$ to other cardinal functions. First, we recall some concepts.
Definition 4.7. Let $X$ be a space. A collection of nonempty open sets $U$ of $X$ is called a $\pi$-base if for every nonempty open set $O$, there exists an $U \in U$ such that $U \subset O$. The $\pi$-weight of $X$ is defined as the infimum of all the cardinal numbers of the $\pi$-bases of $X$; this cardinal number is denoted by $\pi w(X)$. For a space $X$, define

$$\pi w_K(X) = \sup\{\pi w(A) : A \text{ is an arbitrary non-empty closed subset of } X\}.$$ 

Definition 4.8. Let $X$ be a space and $U$ be open in $X$. Denote

$$k_U = \inf\{|K| : K \text{ is a family of compact subsets of } X \text{ such that } U = \bigcup K\}$$

and

$$k_o = \sup\{k_U : U \text{ is open in } X\}.$$ 

Theorem 4.9. For any space $X$, we have $\psi(CL(X), \tau_F) = \pi w_K(X) \cdot k_o(X)$.

Proof. Suppose $\psi(CL(X), \tau_F) = \kappa$. We first prove that $\pi w(A) \leq \kappa$ for any $A \in CL(X)$, which implies that $\pi w_K(X) \leq \kappa$. Fix any $A \in CL(X)$. Then

$$A = \bigcap\{W_{\alpha_1} \cap \ldots \cap W_{\alpha_k} \cap (K_\alpha^+) : \alpha < \kappa\},$$

where each $W_{\alpha_j}$ is open in $X$, $K_\alpha$ is compact in $X$ and $k_\alpha \in \mathbb{N}$. Let $V_{\alpha_j} = A \cap W_{\alpha_j}$. Put

$$V = \{V_{\alpha_j} : j \leq k_\alpha, \alpha < \kappa\}.$$

We claim that $V$ is a $\pi$-base for $A$. Indeed, let $U'$ be an any non-empty open subset of $A$; then it can take an open subset $U$ in $X$ such that $U' = U \cap A$. Let $H = A \setminus U$. Without loss of generality, we may assume $H \neq \emptyset$, then $H \in CL(X)$. Hence there exists an $\alpha < \kappa$ such that $H \notin W_{\alpha_1} \cap \ldots \cap W_{\alpha_k} \cap (K_\alpha^+)$. Since $H \subset A \subset X \setminus K_\alpha$, it follows that $H \in (K_\alpha^+)$, which implies that $H \cap W_{\alpha_j} = \emptyset$ for some $j \in \mathbb{N}$. It follows that $V_{\alpha_j} \subset U'$. Hence $\pi w(A) \leq \kappa$. Therefore, $\pi w_K(X) \leq \kappa$.

Now we prove that $k_o(X) \leq \kappa$. Let $U$ be an any open subset of $X$ such that $B = X \setminus U \in CL(X)$. From our assumption, it follows that

$$B = \bigcap\{W_{\alpha_1} \cap \ldots \cap W_{\alpha_k} \cap (K_\alpha^+) : \alpha < \kappa\},$$

where each $W_{\alpha_j}$ is open in $X$, $K_\alpha$ is compact in $X$ and $k_\alpha \in \mathbb{N}$. We claim that $U = \bigcup_{\alpha < \kappa} K_\alpha$. Indeed, for any $x \in U$, if $x \notin \bigcup_{\alpha < \kappa} K_\alpha$, then $\{x\} \cup B \in (K_\alpha)^c$ for each $\alpha < \kappa$, hence it follows that

$$\{x\} \cup B \in \bigcap\{W_{\alpha_1} \cap \ldots \cap W_{\alpha_k} \cap (K_\alpha^+) : \alpha < \kappa\},$$

which is a contradiction. Therefore, $k_o(X) \leq \kappa$.

Conversely, suppose $\pi w_K(X) \leq \kappa$ and $k_o(X) \leq \kappa$. Fix any $A \in CL(X)$. Let $V' = \{V_\alpha : \alpha < \kappa\}$ be a $\pi$-base for $A$. For each $\alpha < \kappa$, let $V_\alpha$ be an open set of $X$ with $V_\alpha \cap A = V'_\alpha$. Put $V = \{V_\alpha : \alpha < \kappa\}$, and let

$$\Delta(V) = \{W : W \subset V, |W| < \omega\}.$$

From our assumption, let $X \setminus A = \bigcup_{\beta < \kappa} K_\beta$, where each $K_\beta$ is compact. We claim that

$$\{A\} = \bigcap\{V_{1}^- \cap \ldots \cap V_{k}^- \cap (K_\beta^+) : \{V_1, \ldots, V_k\} \in \Delta(V), \beta < \kappa\}.$$

Indeed, it is obvious that $A \in V_{1}^- \cap \ldots \cap V_{k}^- \cap (K_\beta^+)$ for each $\{V_1, \ldots, V_k\} \in \Delta(V)$ and $\beta < \kappa$. Take any

$$B \in \bigcap\{V_{1}^- \cap \ldots \cap V_{k}^- \cap (K_\beta^+) : \{V_1, \ldots, V_k\} \in \Delta(V), \beta < \kappa\}.$$
We prove that

Theorem 4.10. If $\text{CL}(X)$ is Hausdorff and has countable pseudocharacter, then $\text{CL}(X), \tau_F$ is first-countable.

Proof. By Theorem 4.9, every closed subset of $X$ is separable and $X$ is hemicompact. We prove that $X$ is first-countable. Since $\text{CL}(X), \tau_F$ is Hausdorff, it follows from [2] Proposition 5.1.2 that $X$ is locally compact. Then $X$ is first-countable since a Hausdorff, locally compact space with a countable pseudocharacter is first-countable. By [2] Corollary 7(1), $(\text{CL}(X), \tau_F)$ is first-countable.

Corollary 4.11. If $\text{CL}(X), \tau_F$ is Hausdorff and has compact-$G_\delta$ property, then $\text{CL}(X), \tau_F$ is a $D_0$-space.

Proof. Since $(\text{CL}(X), \tau_F)$ is Hausdorff. Then $X$ is locally compact [2] Proposition 5.1.2]. By Theorem 4.9, every closed subset of $X$ is separable and $X$ is hemicompact, hence $X$ is paracompact. It is easy to see that each compact subset is metrizable by Lemma 2.5 then $X$ is metrizable. Therefore, $(\text{CL}(X), \tau_F)$ is a $D_0$-space by Theorem 3.1.

Remark 4.12. There exists a hyperspace $(\text{CL}(X), \tau_F)$ of a space $X$ such that $(\text{CL}(X), \tau_F)$ has a countable pseudocharacter, but $(\text{CL}(X), \tau_F)$ does not have compact-$G_\delta$-property. Indeed, let $X$ be the Alexandroff double-arrow space, see [11] Example 1.8.9. It is well known that $X$ is Hausdorff, first-countable and compact. Then it follows from [9] Example 8 that $(\text{CL}(X), \tau_F)$ is first-countable, hence it has a countable pseudocharacter; however, from Theorem 4.1, it follows that $(\text{CL}(X), \tau_F)$ does not have compact-$G_\delta$-property because $X$ is compact and non-metrizable.

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(Chuan Liu) Department of Mathematics, Ohio University Zanesville Campus, Zanesville, OH 43701, USA
Email address: liuc1@ohio.edu

(Fucai Lin) 1. School of mathematics and statistics, Minnan Normal University, Zhangzhou 363000, P. R. China; 2. Fujian Key Laboratory of Granular Computing and Application, Minnan Normal University, Zhangzhou 363000, P. R. China
Email address: linfucai@mnnu.edu.cn