Harmonic measure of the outer boundary of colander sets

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Abstract

We present two companion results: Phragmén-Lindelöf type tight bounds on the minimal possible growth of subharmonic functions with recurrent zero set, and tight bounds on the maximal possible decay of the harmonic measure of the outer boundary of colander sets.

1 Introduction

Given a non-decreasing positive concave function $R(r) = o(r), \ r \to \infty$, and a non-increasing positive function $\varepsilon(r)$ we say that a closed set $E \subset \mathbb{R}^d$ is $(\varepsilon, R)$-recurrent if for every $x \in \mathbb{R}^d$:

$$C_d(B(x, R(|x|)) \cap E) > R(|x|) \varepsilon(|x|),$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$, and $C_d(A)$ denotes the capacity of the set $A$ (the Newtonian capacity for $d \geq 3$ and the logarithmic capacity for $d = 2$). For the precise definition of capacity that we use in this paper, we refer the reader to chapter 5 in [9]. We compare the relative capacity of the set $E$ inside the ball $B(x, R(|x|))$ with the capacity of a ball of radius $\varepsilon(|x|)$, which is its radius. Condition (1) can be rewritten as

$$\frac{C_d(B(x, R(|x|)) \cap E)}{C_d(B(x, R(|x|)))} > C_d(B(x, \varepsilon(|x|))).$$
While the function $R$ determines the distribution of our set in space, the function $\varepsilon$ determines the relative size of it.

We say a set $\Omega$ is $(\varepsilon, R)$-colander if there exists $\rho > 0$ and an $(\varepsilon, R)$-recurrent set $E$ so that $\Omega = B_\rho \setminus E$, where $B_\rho := \{|x| < \rho\}$.

In this paper we are considering two related objects in potential theory: the decay of the harmonic measure of the outer boundary of colander sets, and Phragmén-Lindelöf type theorems for subharmonic functions with a recurrent zero set.

The question of the optimal growth of subharmonic functions with $(\varepsilon, R)$-recurrent zero set, originated in a joint work with Buhovsky, Logunov and Sodin from 2017 (see [4]). The harmonic measure counterpart is a natural sequel.

We give accurate asymptotic estimates for the harmonic measure $\omega(0, \partial B_\rho; \Omega)$ for colander sets $\Omega$, showing that this harmonic measure decays like

$$\exp\left(-c \int_1^\rho \frac{1}{R(t) \sqrt{-k_d(\varepsilon(t))}} dt\right),$$

for $k_d(t) := \begin{cases} \log(t), & d = 2 \\ \frac{-1}{t^{d-2}}, & d \geq 3 \end{cases}$, while also giving precise bounds on the growth of subharmonic functions, whose zero set is recurrent showing that

$$\log M_u(r) := \log\left(\max_{|x| \leq r} u(x)\right) \sim \int_1^\rho \frac{1}{R(t) \sqrt{-k_d(\varepsilon(t))}} dt,$$

where the notation $A \sim B$ indicates that

$$A \lesssim B \text{ and } B \lesssim A,$$

and $A \lesssim B$ if there exists a constant so that $A \leq C \cdot B$.

Other than being an interesting object on their own, harmonic measures arise in the context of Brownian motion, and they are useful tools in estimating the optimal growth of subharmonic
functions that are bounded on sets with analytic boundary. For those reasons and more, harmonic measures have been the center of interest for many people. The matter of how small must a well distributed set be in order to be ignored has been investigated in many different contexts. One aspect is whether these sets are avoidable or not, i.e is the harmonic measure supported on this set, and assigns zero to the boundary of the disk. The question of when a set is avoidable has been investigated by many mathematicians: Akeroyd [2], Carrol and Ortega-Cerdà [5], O’Donovan [13], Gardiner and Ghergu [7], Pres [16], Hansen and Netuka [8] and more... An overturn of this question would be to ask when are such sets so small that the remaining harmonic measure, the one restricted to the boundary of the disk, is comparable with Lebesgue’s measure. Questions such as these have been answered by Volberg [19], Essén [6] and by Aikawa and Lundh [1]. Other aspects are the Hausdorff dimension of the support of harmonic measures, and their density. These aspects have been investigate throughout by Øksendal [14], Bourgain [3], Jones and Wolf [11], Jones and Makarov [10], Ortega-Cerdà and Seip [15], and more... Though the density of harmonic measures seems to be closely related to the question of bounds on the harmonic measure of the outer boundary of colander sets, while density theorems deal with smaller and smaller scales, our result seem to consider larger and larger scales.

1.1 Results

Before stating the results, we remind the reader the definition of the capacity kernel used in [9]

\[ k_d(t) := \begin{cases} 
\log(t) & , d = 2 \\
\frac{1}{t^{d-2}} & , d \geq 3 
\end{cases} \]
For every $n$ we denote by $\log_{[n]}(t)$ the $n$th iterated logarithm of $t$, i.e

$$
\log_{[n]}(t) := \begin{cases} 
  t, & n = 0 \\
  \log \left( \log_{[n-1]}(t) \right), & n \geq 1
\end{cases}
$$

We say a function $f$ is a gauge function if $\sup_{t \in \mathbb{R}} \frac{f(t)}{t} < \infty$, and

$$
f(t) = A \prod_{n=0}^{\infty} \log_{[n]}^\alpha(t),
$$

where $A > 0$, $\alpha_0 \in [0, 1]$, $\alpha_n \in \mathbb{R}$, $n \geq 1$, and $\# \{ n, \alpha_n \neq 0 \} < \infty$.

We are now ready to present our results: let

$$
\varphi(t) = \varphi_{\epsilon,R}(t) := \frac{1}{R(t) \sqrt{-k_d(\epsilon(t))}}.
$$

**Theorem 1.1**

(A) If $\limsup_{t \to \infty} \frac{1}{t \varphi_{\epsilon,R}(t)} < 1$ then every non-constant subharmonic function $u$ in $\mathbb{R}^d$ whose zero set is $(\epsilon, R)$-recurrent satisfies

$$
\liminf_{\rho \to \infty} \frac{\log M_u(\rho)}{\int_1^\rho \varphi_{\epsilon,R}(t) dt} > 0.
$$

(B) If $\frac{1}{\varphi_{\epsilon,R}(t)}$ is a gauge function, then there exists a non-constant subharmonic function, $u$ in $\mathbb{R}^d$ whose zero set is $(\epsilon, R)$-recurrent, while

$$
\limsup_{\rho \to \infty} \frac{\log (M_u(\rho))}{\int_1^\rho \varphi_{\epsilon,R}(t) dt} < \infty.
$$

As mentioned earlier, there are two objects at play here: subharmonic functions, and harmonic measures. The relationship between the two is more conceptional than formal, though there is also some formal connection as we will soon see.

The following theorem describes tight bounds for the decay of the harmonic measure $\omega(0, \partial B_\rho; \Omega)$ for $(\epsilon, R)$-colander sets $\Omega$. 

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Theorem 1.2  (A) If \( \limsup_{t \to \infty} \frac{1}{t} \varphi_{\varepsilon,R}(t) < 1 \), then there exist constants \( C, c > 0 \) so that for every \((\varepsilon, R)\)-recurrent set \( E \), for every \( \rho > 0 \) the \((\varepsilon, R)\)-colander set, \( \Omega := B_\rho \setminus E \) satisfies

\[
\omega(0, \partial B_\rho; \Omega) \leq C \exp \left( -c \int_1^\rho \varphi_{\varepsilon,R}(t) dt \right).
\]

(B) If \( \frac{1}{\varphi_{\varepsilon,R}(t)} \) is a gauge function, then there exist constants \( c, C > 0 \), and an \((\varepsilon, R)\)-recurrent set \( E \), so that for every \( \rho > 0 \) the \((\varepsilon, R)\)-colander set, \( \Omega := B_\rho \setminus E \) satisfies

\[
\omega(0, \partial B_\rho; \Omega) \geq C \exp \left( -c \int_1^\rho \varphi_{\varepsilon,R}(t) dt \right).
\]

Remark 1.3 Wiener’s criterion for thin sets states that for \( \gamma \in (1, \infty) \) and

\[
E_n := E \cap \{ \gamma^{n-1} \leq |z| < \gamma^n \},
\]

the set \( E \) is thin at infinity if and only if the following series converges

\[
\sum_{n=1}^{\infty} k_d(\gamma^n) - k_d(-C_d(E_n)) = \begin{cases} 
\log(\gamma) \sum_{n=1}^{\infty} \frac{n}{\log(\gamma) / \log(c_d(k_n))}, & d = 2 \\
\sum_{n=1}^{\infty} \gamma^{n(d-2)}C_d(E_n)^{d-2}, & d \geq 3 
\end{cases}
\]

Heuristically, if Theorem 1.2 (A) holds in general, without the additional condition that \( \limsup_{t \to \infty} \frac{1}{t \varphi(t)} < 1 \), then the set \( E \) is thin at infinity if for any \( \varepsilon > 0 \)

\[
\varphi(t) < \begin{cases} 
\frac{1}{t \log^{d+1}(t)} & , d = 2 \\
\frac{1}{t^{d-2+\varepsilon}} & , d \geq 3 
\end{cases}
\]

This means there is still a gap between the case where our set is thin and the restriction posed in Theorem 1.2 (A), and moreover this gap becomes bigger the higher our dimension.

1.2 An overview of the paper: Methods and Tools

Theorems 1.1 and 1.2, though conceptually equivalent, do not formally imply one another. Nevertheless, there is no need to prove both theorems:
Lemma 1.4  
(i) If Theorem 1.2 (A) holds then Theorem 1.1 (A) holds.

(ii) If Theorem 1.1 (B) holds then Theorem 1.2 (B) holds.

Proof. To prove (i), let $u$ be a non-constant subharmonic function whose zero set, $Z_u := \{u \leq 0\}$, is $(\varepsilon, R)$-recurrent. Since $u$ is not constant, we may assume that $u(0) \geq 1$, for otherwise we will use a translation of $u$. The set $B_\rho \setminus Z_u$ is an $(\varepsilon, R)$-colander set, and by Theorem 1.2 (A), there exist $0 < c < C < \infty$ satisfying

$$\omega(0, \partial B_\rho; B_\rho \setminus Z_u) \leq C \exp \left( -c \int_1^\rho \varphi(t) dt \right), \quad \forall \rho > 0.$$  

By the definition of harmonic measure, for every $\rho > 0$

$$1 \leq u(0) \leq \int_{\partial(B_\rho \setminus Z_u)} u(x) d\omega(0, x; B_\rho \setminus Z_u) \leq M_u(\rho) \cdot \omega(0, \partial B_\rho; B_\rho \setminus Z_u) \leq CM_u(\rho) \cdot \exp \left( -c \int_1^\rho \varphi(t) dt \right)$$

$$\Rightarrow \frac{1}{C} \exp \left( c \int_1^\rho \varphi(t) dt \right) \leq M_u(\rho).$$

We conclude that

$$\liminf_{\rho \to \infty} \frac{\log M_u(\rho)}{\int_1^\rho \varphi(t) dt} > 0.$$  

To prove (ii), let $u$ be the non-constant subharmonic function constructed in Theorem 1.1 (B), whose zero set, $Z_u = \{u \leq 0\}$, is $(\varepsilon, R)$-recurrent and for some constants $c, C > 0$

$$M_u(\rho) \leq C \cdot \exp \left( c \int_1^\rho \varphi(t) dt \right).$$

For every $\rho > 0$ define the set $\Omega := B_\rho \setminus Z_u$. Since $Z_u$ is $(\varepsilon, R)$-recurrent, the set $\Omega$ is $(\varepsilon, R)$-colander. As before we assume that $u(0) \geq 1$, and using the same inequality:

$$1 \leq u(0) \leq \int_{\partial \Omega} u(x) d\omega(0, x; \Omega) \leq M_u(\rho) \cdot \omega(0, \partial B_\rho; \Omega)$$

$$\leq C \exp \left( c \int_1^\rho \varphi(t) dt \right) \cdot \omega(0, \partial B_\rho; \Omega)$$

$$\Rightarrow \omega(0, \partial B_\rho; \Omega) \geq \frac{1}{C} \exp \left( -c \int_1^\rho \varphi(t) dt \right),$$
concluding the proof.

We will in fact prove Theorem 1.2 (A), and Theorem 1.1 (B). As a corollary to the lemma above, we obtain the proofs for both Theorem 1.1 and Theorem 1.2.

1.2.1 The sketch of proof of Theorem 1.2 (A)

In Section 3 we describe an example of an \((\varepsilon, R)\)-recurrent set in dimension \(d = 2\) with optimal bounds on the harmonic measure of the outer boundary of the colander sets generated by it. This example, though particular, is what paved the path for the general solution.

The first key element of the proof is observing that to understand the asymptotic decay of the harmonic measure it is enough to understand its decay on a sequence of annuli. The strong connection between harmonic measures and Brownian motion gives analysts intuition when it comes to estimating harmonic measures. In probability, the simplest case to deal with is when you have independence between events, allowing you to tightly bound the probability of each event separately to obtain a tight bound on the intersection. Would it not be wonderful if we had independence between different layers\(^1\) of our set?

The strong Markov property of Brownian motion tells us that even when we do not have independence there is some kind of relatively weak connection between layers. This was definitely known and used by experts in the field. We formally describe this ”folklore” statement in the following Proposition:

**Proposition 1.5** For every closed set \(E\) and for every sequence of domains \(0 \in D_1 \Subset D_2 \Subset \cdots\),

\[
\exp \left( -c \sum_{k=1}^{n} \sup_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right) \leq \omega(0, \partial D_n; D_n \setminus E) \leq \exp \left( -\sum_{k=1}^{n} \inf_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right),
\]

\(^1\)Here and elsewhere in the paper we use *layer* to refer to the intersection between the set \(E\) and some annuli.
provided that:

\[
\sup_k \sup_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \leq 1 - \frac{1}{c}.
\]

For the reader’s convenience, the proof of this Proposition can be found in Section 2.

To conclude the proof, it is therefore left choose the sets \( D_k \) and bound \( \omega(x, E; D_{k+1} \setminus E) \) tightly from above and bellow on \( \partial D_k \).

The second key component in the proof is that the number of layers, \( E \cap (D_k \setminus D_{k-1}) \), one needs to consider in order to obtain a meaningful bound, depends on the function \( \varepsilon \). Though the number of layers is not a lot (proportionally to the total number of layers), it is not enough to take just one layer. This implies we do have some dependence between layers, which is quite surprising.

To conclude the proof, we bound the harmonic measure of the required layers, by comparing the harmonic measure with a special subharmonic function bounding it from bellow. As the capacity kernel is different for \( d = 2 \), the proof diverges here and we use slightly different methods for the case \( d = 2 \) and for higher dimensions. The high dimensional case, is simpler and some ideas used there are inspired by the work of Tom Carroll and Joaquim Ortega-Cerdà in [5].

The proof of the example can be found in Section 3, and the proofs of the general cases can be found in Section 4.

1.2.2 The sketch of proof of Theorem 1.1 (B)

We conclude the paper with a construction of a subharmonic function whose zero set is \((\varepsilon, R)\)-recurrent, proving Theorem 1.1 (B) in Section 5.

The idea of the construction is based on the fact that the function \( z \mapsto e^{C|z|} \) is subharmonic and its Laplacian is very big for \( C \gg 1 \), making the measure \( \Delta u \) much larger than the original function.

We can use this extra growth to compensate for adding large portion of space where \( u \leq 0 \).

Now it is only a matter of finding the appropriate candidate to replace the constant \( C \), and use
some glueing techniques of Poisson integrals to create these large areas where \( u \leq 0 \).

One of the nice things about this method is though the constants depend on the dimension, it is
possible to write one proof for all dimensions \( d \).

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2 Layers and Independence- the proof of Proposition 1.5

In order to prove Proposition 1.5, we will need the following Observation, which is a direct corollary
of the strong Markov property of Brownian motion:

**Observation 2.1** Let \( D \) be a domain, \( F \subset D \) be some set, and let \( D' \subset D \) be a subdomain. Define \( F' := D' \cap F \). Then for every \( x \in D' \):

\[
\inf_{y \in \partial D'} \omega(y, \partial D; D \setminus F) \leq \frac{\omega(x, \partial D; D \setminus F)}{\omega(x, \partial D'; D' \setminus F')} \leq \sup_{y \in \partial D'} \omega(y, \partial D; D \setminus F).
\]

**Proof.** Since for every \( x \) fixed, harmonic measure is a probability measure, the left hand side of
the inequality holds if and only if

\[
1 - \sup_{y \in \partial D'} \omega(y, F; D \setminus F) = \inf_{y \in \partial D'} 1 - \omega(y, F; D \setminus F) \leq \frac{1 - \omega(x, F; D \setminus F)}{\omega(x, \partial D'; D' \setminus F')}
\]

\[
\iff 1 - \omega(x, F'; D' \setminus F') - \omega(x, \partial D'; D' \setminus F') \cdot \sup_{y \in \partial D'} \omega(y, F; D \setminus F) \leq 1 - \omega(x, F; D \setminus F)
\]

\[
\iff \omega(x, F; D \setminus F) - \omega(x, F'; D' \setminus F') \leq \omega(x, \partial D'; D' \setminus F') \cdot \sup_{y \in \partial D'} \omega(y, F; D \setminus F).
\]
Both \( \omega (\cdot, F; D \setminus F) \) and \( \omega (\cdot, F'; D' \setminus F') \) are harmonic functions on \( D' \setminus F' \), and so

\[
\omega (x, F; D \setminus F) - \omega (x, F'; D' \setminus F') = \int_{\partial (D' \setminus F')} \omega (y, F; D \setminus F) - \omega (y, F'; D' \setminus F') \, d\omega (x, y; D' \setminus F') \leq \sup_{y \in \partial D'} \omega (y, F; D \setminus F) \cdot \omega (x, \partial D'; D' \setminus F'),
\]

where (*) holds since on \( F' \) both functions are equal to 1, while on \( \partial D' \), \( \omega (\cdot, F'; D' \setminus F') = 0 \) implying that

\[
\omega (y, F; D \setminus F) - \omega (y, F'; D' \setminus F') = \omega (y, F; D \setminus F).
\]

This concludes the proof of the left hand side inequality.

To prove the right hand side inequality, we use again the fact that harmonic measures are also harmonic functions, and so

\[
\omega (x, \partial D; D \setminus F) = \int_{\partial (D' \setminus F')} \omega (y, \partial D; D \setminus F) \, d\omega (x, y; D' \setminus F') = \int_{\partial D'} \omega (y, \partial D; D \setminus F) \, d\omega (x, y; D' \setminus F') \leq \omega (x, \partial D'; F' \setminus F') \cdot \sup_{y \in \partial D'} \omega (y, \partial D; D \setminus F),
\]

concluding the proof.

\[\square\]

**Proof of Proposition 1.5**

*Proof.* We will use the observation recursively beginning with the sets:

\[
F = E, \quad D = D_n, \quad D' = D_{n-1}.
\]

As these sets satisfy the assumptions of Observation 2.1,

\[
\inf_{x \in \partial D_{n-1}} \omega (x, \partial D_n; D_n \setminus E) \leq \frac{\omega (0, \partial D_n; D_n \setminus E)}{\omega (0, \partial D_{n-1}; D_{n-1} \setminus E)} \leq \sup_{x \in \partial D_{n-1}} \omega (x, \partial D_n; D_n \setminus E).
\]
We continue to apply Observation 2.1 over and over with the sets \( D = D_{n-k} \), \( D' = D_{n-k-1} \), \( E = E \cap D_{n-k} \). Doing so recursively, we obtain the lower bound:

\[
\omega(0, \partial D_n; D_n \setminus E) \geq \prod_{k=1}^{n-1} \left( 1 - \sup_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right)
\]

\[
= \exp \left( \sum_{k=1}^{n} \log \left( 1 - \sup_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right) \right)
\]

\[
\geq \exp \left( -c \sum_{k=1}^{n} \sup_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right),
\]

since for \( x \in [0, 1 - \frac{1}{e}] \) we know that \( \log(1 - x) \geq -x \cdot c \).

A similar computation shows that:

\[
\omega(0, \partial D_n; D_n \setminus E) \leq \cdots \leq \exp \left( -\sum_{k=1}^{n} \inf_{x \in \partial D_k} \omega(x, E; D_{k+1} \setminus E) \right).
\]

\[\square\]

### 3 Example for \( d = 2 \) with tight bounds

We will start by presenting an example in two dimensions. We will later see that though this example describes a very particular case, it sheds light on the main ideas used in the proof of the general case both in dimension \( d = 2 \) and higher.

In this example, we construct an \((\varepsilon, R)\)-recurrent set \( E \) and tightly bound the harmonic measure of the outer boundary of the set \( B_{\rho_n} \setminus E \) for some sequence \( \{\rho_n\} \). In fact, the set \( E \) satisfies that

\[
\forall z \in \mathbb{C}, \ m(\overline{B(z, R(|z|)) \setminus E}) > \pi (|z|) \cdot R(|z|)^2,
\]

where \( m \) denotes Lebesgue’s measure on \( \mathbb{C} \), which is stronger than being \((\varepsilon, R)\)-recurrent.
3.1 Notation, the set $E$, and preliminaries

Given a concave differentiable monotone increasing function $R : [1, \infty) \to [1, \infty)$ so that

$$R(t) = o(t) \quad \text{and} \quad R'(0) = \sup_{x \in \mathbb{R}_+} R'(x) < 1,$$

define the sequence

$$\rho_0 := 0, \quad \rho_{n+1} = \rho_n + R(\rho_n).$$

Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$, and for every $k \in \mathbb{N}$ define the sets

$$A_0 := \frac{R(0)}{2} \mathbb{D},$$

$$A_k := \left\{ z, \ |z| \in \left[ \frac{1}{2} (\rho_{k-1} + \rho_k), \frac{1}{2} (\rho_k + \rho_{k+1}) \right] \right\}, \quad k \geq 1,$$

$$\Theta_k(j) := \left\{ z \in \mathbb{C}, \ \text{Arg}(z) \in \left[ \frac{2\pi j}{m_k} - \frac{\pi}{m_k}, \frac{2\pi j}{m_k} + \frac{\pi}{m_k} \right) \right\}, \quad j \in \{0, 1, \ldots, m_k - 1\}$$

for

$$m_k := \left\lceil \frac{\rho_k}{R(\rho_k)} \right\rceil.$$  

For every $k$ we let

$$\Lambda_k := \left\{ \rho_k e_k(j), \ e_k(j) = \exp \left( \frac{2\pi i j}{m_k} \right), \ j \in \{0, \ldots, m_k - 1\} \right\}.$$

Define the disks $D_n = \frac{1}{2} (\rho_{n+1} + \rho_n) \mathbb{D}$ and the set

$$E := \bigcup_{k=0}^{\infty} E_k, \quad \text{where} \quad E_k := \Lambda_k + \varepsilon_k R_k \mathbb{D},$$

for $\varepsilon := \varepsilon(\rho_k), R_k := R(\rho_k).$
What can be said about how much the sequence \( \{R_n\} = \{R(\rho_n)\} \) oscillates? The following claim, describing the relative oscillations in the sequence \( \{R_n\} \), will be used in obtaining lower and upper bounds for the harmonic measure of the example as well as in the proof of Theorem 1.2(A):

**Claim 3.1** If

\[
\frac{1}{\rho_n \varphi(\rho_n)} = e(\rho_n) < 1
\]

then

\[
\frac{R_n}{R_{n-\sqrt{-k_d(\varepsilon_n)R}}(1-\varepsilon)} \leq \frac{1}{1 - e(\rho_n)}.
\]

**Proof.** Since \( R \) is a concave function, its derivative is monotone decreasing and so

\[
\frac{R'(t)}{R(t)} = \frac{R'(t)}{R(t) - R(0) + R(0)} = \frac{R'(t)}{t \cdot R'(\xi_t) + R(0)} \leq \frac{R'(t)}{t \cdot R'(t) + R(0)} \leq \frac{1}{t}.
\]

\((\dagger)\)

\( R \) is a monotone increasing function. We deduce that for every \( t \geq 1 \) and \( \varepsilon \in (0, 1) \):

\[
1 \leq \frac{R(t)}{R(t (1-\varepsilon))} = 1 + \frac{R(t) - R(t (1-\varepsilon))}{R(t (1-\varepsilon))} = 1 + \frac{R'(\xi_t) \cdot \varepsilon \cdot t}{R(t (1-\varepsilon))} \leq 1 + \varepsilon \cdot t \cdot \frac{R'(t (1-\varepsilon))}{R(t (1-\varepsilon))} \leq 1 + \frac{\varepsilon}{1-\varepsilon} = \frac{1}{1-\varepsilon}.
\]

\((\dagger\dagger)\)
By the way $\varphi$ was defined,

$$1 - e(\rho_n) = 1 - \frac{1}{\rho_n \varphi(\rho_n)} = 1 - \frac{\sqrt{-k_d(\varepsilon_n) R_n}}{\rho_n} \leq \frac{\rho_n - \sqrt{-k_d(\varepsilon_n)}}{\rho_n} \leq 1.$$ 

Combining this with estimate (††), we obtain that

$$\frac{R_n}{R_{n^2}} \frac{R(\rho_n)}{R(\rho_n - \sqrt{-k_d(\varepsilon_n) R_n})} \leq \frac{R(\rho_n)}{R(\rho_n (1 - e(\rho_n)))} \leq \frac{1}{1 - e(\rho_n)}. \quad \square$$

**Remark 3.2** In fact, we only use the condition $\limsup_{t \to \infty} \frac{1}{t \varphi(t)} < 1$ to get an upper bound on the quotient $\frac{R_n}{R_{n^2}}$. If this quotient is known to be bounded (for example if $R$ is constant), we do not need this condition at all.

To conclude this subsection we would like to relate the estimates done in Proposition 1.5 with the sequence $\{\rho_n\}$:

**Corollary 3.3** Let $E \subset \mathbb{R}^d$ and let $D_1 \subset D_2 \subset \cdots$ be a sequence of sets so that

$$\partial D_k \subset \{ x \in \mathbb{R}^d, \alpha \rho_k \leq |x| \leq \beta \rho_k \}$$

for some uniform constants $0 < \alpha < \beta < \infty$. Let $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous monotone non-increasing functions satisfying that for every $k$ and every $x \in \partial D_k$:

$$g_1(\rho_k) \leq \omega(x, E; D_{k+1} \setminus E) \leq g_2(\rho_k) \leq 1 - \frac{1}{c},$$

for some uniform constant $c > 0$. Then there exists constants $0 < A, B$ so that for every $n$:

$$\exp \left(-A \int_1^{\rho_n} \frac{g_2(t)}{R(t)} dt\right) \leq \omega(0, \partial D_n; D_n \setminus E) \leq \exp \left(-B \int_1^{\rho_n} \frac{g_1(t)}{R(t)} dt\right).$$

**Proof.** Using Proposition 1.5,

$$\exp \left(-c \sum_{k=1}^{n} \sup_{\xi \in \partial D_k} \omega(\xi, E; D_{k+1} \setminus E)\right) \leq \omega(0, \partial D_n; D_n \setminus E) \leq \exp \left(-\sum_{k=1}^{n} \inf_{\xi \in \partial D_k} \omega(\xi, E; D_{k+1} \setminus E)\right).$$
Combined with the assumptions of the lemma,
\[ \exp \left( -c \sum_{k=1}^{n} g_2(\rho_k) \right) \leq \omega (0, \partial D_n, D_n \setminus E) \leq \exp \left( -\sum_{k=1}^{n} g_1(\rho_k) \right). \]

To conclude the proof it is therefore enough to bound the sums
\[ \sum_{k=1}^{n} g_j(\rho_k) \]
from above for \( j = 2 \) and from below for \( j = 1 \).

To do so we would like to use integration, for which will need an estimate on \( \rho^{-1} \). Even though we do not have a useful formula for \( \rho_n \), we do have a formula for the asymptotic behavior of \( \rho^{-1} \):

Define the function
\[ \Phi : \mathbb{R}_+ \to \mathbb{R}_+, \Phi(x) := \int_{0}^{x} \frac{1}{R(t)} dt. \]

For every \( n \in \mathbb{N} \):
\[ \Phi(\rho_n) = \int_{0}^{\rho_n} \frac{1}{R(t)} dt = \sum_{k=1}^{n} \int_{\rho_{k-1}}^{\rho_k} \frac{1}{R(t)} dt. \]

Since \( R \) is monotone increasing and concave, for every \( k \)
\[ 1 \geq \frac{R_{k-1}}{R_k} = 1 - \frac{R_k - R_{k-1}}{R_k} = 1 - \frac{\sup_{x \in \mathbb{R}_+} R'(x)R_k}{R_k} \]
\[ \geq 1 - \frac{\sup_{x \in \mathbb{R}_+} R'(x)}{R_k} = 1 - \frac{R'(0)}{R_k} = c_R > 0, \]

and therefore
\[ c_R \leq \frac{R_{k-1}}{R_k} = \frac{\rho_k - \rho_{k-1}}{R_k} \leq \int_{\rho_{k-1}}^{\rho_k} \frac{1}{R(t)} dt \leq \frac{\rho_k - \rho_{k-1}}{R_{k-1}} = 1 \]
implying that
\[ c_R \cdot n \leq \Phi(\rho_n) \leq n \Rightarrow \Phi^{-1}(c_R \cdot n) \leq \rho_n \leq \Phi^{-1}(n), \]
since \( \Phi \) is monotone increasing. The function \( g_2 \) is monotone non-increasing therefore
\[ \sum_{j=1}^{n} g_2(\rho_k) \leq \sum_{j=1}^{n} g_2(\Phi^{-1}(c_R \cdot j)) \leq \sum_{j=1}^{n} \int_{j-1}^{j} g_2(\Phi^{-1}(c_R \cdot t)) dt = \int_{0}^{n} g_2(\Phi^{-1}(c_R \cdot t)) dt. \]
We use the following change of variables:
\[ \tau = \Phi^{-1}(c_R \cdot t) \Rightarrow t = \frac{1}{c_R} \Phi(\tau) \quad \text{and} \quad \frac{d}{dt} = \frac{1}{c_R} \Phi'(\tau) d\tau = \frac{1}{c_R \cdot R(\tau)} d\tau. \]

Then
\[ \sum_{j=1}^{n} g_2(\rho_j) \leq \cdots \leq \int_{0}^{n} g_2(\Phi^{-1}(c_R \cdot t))dt \overset{c.o.v}{=} \int_{\Phi^{-1}(0)}^{\Phi^{-1}(c_Rn)} \frac{g_2(\tau)}{c_R R(\tau)} d\tau \leq \frac{1}{c_R} \int_{\rho_0}^{\rho_n} g_2(\tau) d\tau. \]

For a lower bound, we use a similar change of variables:
\[ \sum_{j=1}^{n} g_1(\rho_j) \geq \sum_{j=1}^{n} g_1(\Phi^{-1}(j)) \geq \sum_{j=1}^{n} \int_{j}^{j+1} g_1(\Phi^{-1}(t)) dt \geq \int_{1}^{n} g_1(\Phi^{-1}(t)) dt \]
\[ \overset{c.o.v}{=} \int_{\Phi^{-1}(1)}^{\Phi^{-1}(n)} \frac{g_1(\tau)}{R(\tau)} d\tau \geq \int_{\rho_{[\frac{1}{2} \cdot 1]}}^{\rho_n} \frac{g_1(\tau)}{R(\tau)} d\tau \]

concluding our proof.

\[ \Box \]

### 3.2 Bounding each layer

In this subsection we shall present estimates for the harmonic measure of \( E \) within the set \( D_n := \frac{1}{2} (\rho_{n+1} + \rho_n) \mathbb{D} \). Instead of rescaling everything all the time, We will be working with the set \( D_n \setminus E \), but one should think of this as working on \( \mathbb{D} \setminus \frac{2}{\rho_{n+1} + \rho_n} E \) so that the harmonic measure is well defined.

For any \( c \in (0, 1) \) we let \( A_c := \{ 1 - c < |z| < 1 \} \). We abuse the notation of \( A_c \) and use it also to denote \( D_n \setminus (1 - c)D_n \).

**Lemma 3.4** Under the assumption that
\[ \frac{m_0 R_{n-m_0}}{\rho_{n-m_0}} = o(1), \]
for every \( z \in \partial D_{n-1} \),
\[ \frac{m_0}{m_0^2 + \log \left( \frac{1}{\varepsilon_n} \right)} \lesssim \omega(z, E \cap A_c; D_n \setminus E) \lesssim \frac{m_0}{m_0 + \log \left( \frac{1}{\varepsilon_{n-m_0}} \right)}, \]
for \( c = \frac{\rho_{n-m_0}}{2(\rho_n + \rho_{n-1})} \).
Proof. Define the function
\[ u(z) := \sum_{j=1}^{m_0} u_j(z), \]
where
\[ u_j(z) := \log \left| 1 - \left( \frac{z}{\rho_{n-j}} \right)^{m_{n-j}} \right|. \]

For every \( 1 \leq j \leq m_0 \) the function \( u_j \) is harmonic in \( D_n \setminus E \), and therefore \( u \) is harmonic in \( D_n \setminus E \) as a sum of such functions. We will use the function \( u \) to tightly bound the harmonic measure.

The first step is to find a tight estimate for \( u \) on \( \partial (D_n \setminus E) \):

**Bounding \( u \) on \( \partial D_n \):** For every \( j \)
\[ \log \left( \left( \frac{1}{2} \left( \rho_{n+1} + \rho_n \right) \right)^{m_{n-j}} - 1 \right) \leq u_j(z) \leq \log \left( \left( \frac{1}{2} \left( \rho_{n+1} + \rho_n \right) \right)^{m_{n-j}} + 1 \right). \]

Following Claim 3.1,
\[ \frac{m_0 R_{n-m_0}}{\rho_{n-m_0}} = o(1) \Rightarrow \frac{R_n}{R_{n-m_0}} \leq 1 + o(1), \]
and therefore \( R_k \geq R_n(1 - o(1)) \) and \( R_n \leq R_k(1 + o(1)) \) whenever \( n - m_0 \leq k \leq n \). Since for every \( x \geq 0, \log(1 + x) \leq x \) and by the way the sequence \( \{m_k\} \) was defined:
\begin{align*}
\left( \frac{1}{2} \left( \rho_{n+1} + \rho_n \right) \right)^{m_{n-j}} &= \left( 1 + \frac{\rho_n - \rho_{n-j} + \frac{1}{2} R_n}{\rho_{n-j}} \right)^{m_{n-j}} \\
&\leq \left( 1 + \frac{\left( j + \frac{1}{2} \right) R_n}{\rho_{n-j}} \right)^{m_{n-j}} \\
&\leq \exp \left( m_{n-j} \log \left( 1 + \frac{\left( j + \frac{1}{2} \right) R_{n-j} (1 + o(1))}{\rho_{n-j}} \right) \right) \leq \exp \left( m_{n-j} \cdot \frac{\left( j + \frac{1}{2} \right) R_{n-j} (1 + o(1))}{\rho_{n-j}} \right) \\
&\leq \exp \left( \frac{\rho_{n-j}}{R_{n-j}} + 1 \right) \cdot \frac{R_{n-j}}{\rho_{n-j}} \left( j + \frac{1}{2} \right) (1 + o(1)) = \exp \left( \left( j + \frac{1}{2} \right) (1 + o(1)) \right). \end{align*}
For a lower bound we use the fact that if $0 < x$ then $\log(1 + x) \geq x - \frac{x^2}{2} = x \left(1 - \frac{x}{2}\right)$ to conclude that
\[
\left(\frac{1}{2} \left(\rho_{n+1} + \rho_n\right) / \rho_{n-j}\right)^{m_{n-j}} \geq \left(1 + \left(\frac{j + \frac{1}{2}}{\rho_{n-j}}\right) R_{n-j}\right)^{m_{n-j}} = \exp\left(m_{n-j} \log \left(1 + \left(\frac{j + \frac{1}{2}}{\rho_{n-j}}\right) R_{n-j}\right)\right)
\geq \exp\left(m_{n-j} \cdot \left(\frac{j + \frac{1}{2}}{\rho_{n-j}}\right) R_{n-j} \left(1 - \left(\frac{j + \frac{1}{2}}{2\rho_{n-j}}\right) R_{n-j}\right)\right)
\geq \exp\left(\left(j + \frac{1}{2}\right) \left(1 - o(1)\right)\right),
\]

since $\frac{(j + \frac{1}{2}) R_{n-j}}{2\rho_{n-j}} \leq \frac{m_0 R_{n-m_0}}{\rho_{n-m_0}} = o(1)$. Over all, we conclude that
\[
\exp\left(\left(j + \frac{1}{2}\right) \left(1 - o(1)\right)\right) \leq \left(\frac{1}{2} \left(\rho_{n+1} + \rho_n\right) / \rho_{n-j}\right)^{m_{n-j}} \leq \exp\left(\left(j + \frac{1}{2}\right) \left(1 + o(1)\right)\right),
\]
and therefore
\[
u_j(z) \leq \log \left(\left(\frac{1}{2} \left(\rho_{n+1} + \rho_n\right) / \rho_{n-j}\right)^{m_{n-j}} + 1\right) \leq j + \frac{1}{2} + o(1)
\]
\[
u_j(z) \geq \log \left(\left(\frac{1}{2} \left(\rho_{n+1} + \rho_n\right) / \rho_{n-j}\right)^{m_{n-j}} - 1\right) \geq j + \frac{1}{2} - o(1).
\]

Then, for every $\xi \in \partial D_n$ we have that
\[
\left|u(\xi) - \frac{m_0 (m_0 + 2)}{2}\right| \leq \sum_{j=1}^{m_0} \left|u_j(\xi) - \left(j + \frac{1}{2}\right)\right| = \sum_{j=1}^{m_0} o(1) = o(m_0).
\]

**Bounding $u$ on $\partial E \cap A_\cdot$:** For every $j$, if $z \in E_{n-m}$ for $m \neq j$, then a similar computation to the one done for $\partial D_n$ gives us that
\[
m - j - o(1) \leq u_j(z) \leq m - j + o(1) \quad \text{,} \quad m > j
\]
\[
u_j(z) = \Theta(e^{-(j-m)}) \quad \text{,} \quad m < j.
\]
Let $z \in E_{n-j}$, and assume without loss of generality that $z = \rho_{n-j} + \varepsilon_{n-j} R_{n-j} e^{it}$ for some $t \in [0, 2\pi)$.

Then

$$
\left( \frac{\rho_{n-j} + \varepsilon_{n-j} R_{n-j} e^{it}}{\rho_{n-j}} \right)^{m_{n-j}} \leq \left( 1 + \frac{\varepsilon_{n-j} R_{n-j}}{\rho_{n-j}} \right)^{m_{n-j}} \leq 1 + 2 m_{n-j} \cdot \frac{\varepsilon_{n-j} R_{n-j}}{\rho_{n-j}} \leq 1 + 2 \varepsilon_{n-j},
$$

$$
\left( \frac{\rho_{n-j} + \varepsilon_{n-j} R_{n-j} e^{it}}{\rho_{n-j}} \right)^{m_{n-j}} \geq \left( 1 - \frac{\varepsilon_{n-j} R_{n-j}}{\rho_{n-j}} \right)^{m_{n-j}} \geq 1 - 2 m_{n-j} \cdot \frac{\varepsilon_{n-j} R_{n-j}}{\rho_{n-j}} \geq 1 - 2 \varepsilon_{n-j},
$$

since for every $x \in \left(0, \frac{1}{m}\right)$

$$
1 - 2mx \leq (1 - x)^m \leq (1 + x)^m \leq 1 + 2mx.
$$

We conclude that for every $z \in E_{n-j}$

$$
\log(\varepsilon_n) - \log(3) \leq \log(\varepsilon_{n-j}) - \log(3) \leq u_j(z) \leq \log(\varepsilon_{n-j}) + \log(3) \leq \log(\varepsilon_{n-m_0}) + \log(3).
$$

Overall, on the set $E \cap A_c = E \cap D_n \setminus (1 - c)D_n$ we get that

$$
\log(\varepsilon_n) - \log(3) \leq u(z) \leq \log(\varepsilon_{n-m_0}) + \frac{m_0(m_0 - 1)}{2} + o(m_0).
$$

We denote by $e_n$ the maximum over the two errors we get from bounding $u$ on $\partial D_n$ and on $E \cap A_c$.

We know that $e_n = o(m_0)$.

**The functions:** Define the functions:

$$
u_+(z) := \frac{m_0(m_0+2)}{2} + e_n - u(z) = \frac{m_0(m_0+2)}{2} + e_n + \frac{1}{\varepsilon_{n-m_0}} - \frac{5}{2} m_0 + 2e_n + \log \left( \frac{1}{\varepsilon_{n-m_0}} \right)
$$

$$
u_-(z) := \frac{m_0(m_0+2)}{2} - e_n - u(z) = \frac{m_0(m_0+2)}{2} - e_n + \log \left( \frac{1}{\varepsilon_n} \right).
$$

Then following the maximum principle

$$
u_-(z) \leq \omega(z, E \cap A_c; D_n \setminus E) \leq u_+(z).
$$
To conclude the proof we will bound the function $u$ on the set $\partial D_{n-1}$: a similar computation to the one done to estimate $u$ on $\partial D_n$ gives-

$$u(z) = \sum_{j=0}^{m_0} u_j(z) \leq \sum_{j=1}^{m_0} j - \frac{1}{2} + o(m_0) = \frac{1}{2}m_0^2 + o(m_0)$$

$$u(z) = \sum_{j=0}^{m_0} u_j(z) \geq \sum_{j=1}^{m_0} j - \frac{1}{2} - o(m_0) = \frac{1}{2}m_0^2 - o(m_0)$$

implying that

$$\omega(z, E \cap A_c; D_n \setminus E) \leq \frac{m_0(m_0+2)}{2} + \epsilon_n + o(m_0) - \frac{1}{2}m_0^2 \leq \frac{m_0 \left(1 + 2 \frac{\epsilon_n}{m_0}\right)}{5m_0 + 2\epsilon_n + \log \left(\frac{1}{\epsilon_n - m_0}\right)}$$

and

$$\omega(z, E \cap A_c; D_n \setminus E) \geq \frac{m_0(m_0+2)}{2} - \epsilon_n - o(m_0) - \frac{1}{2}m_0^2 \geq \frac{m_0 \left(1 - 2 \frac{\epsilon_n}{m_0}\right)}{m_0(m_0+2) - \epsilon_n + \log \left(\frac{1}{\epsilon_n}\right)}$$

concluding our proof since

$$\frac{\epsilon_n}{m_0} = o(1) < \frac{1}{2},$$

for $n$ large enough.

\[\square\]

Remark 3.5 If we choose $m_0$ to be a constant, it means we are only considering a finite number of layers. The terms on both the left hand side inequality and the right hand side inequality in Lemma 3.4, are equal $\frac{1}{\log(\frac{1}{\epsilon})}$ up to multiplication by two different constants. In this case the harmonic measure is roughly the sum of measures.

On the other side of the spectrum, if we choose $m_0$ to be very large, then the right hand side of the inequality tends to 1 while the left hand one tends to 0, which means the inequality is not very useful, but still true.

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3.3 A tight bound for the harmonic measure

**Proposition 3.6** There exist $c, C > 0$ constants so that for every $n$:

$$\exp\left(-C \int_{\rho_n}^\rho \varphi(t) dt\right) \leq \omega(0, \partial D_n, D_n \setminus E) \leq \exp\left(-c \int_{\rho_n}^\rho \varphi(t) dt\right),$$

provided that

$$\limsup_{t \to \infty} \frac{1}{t \varphi(t)} = 0.$$

**Proof.** We note that while Lemma 3.4 can be used to obtain an upper bound, it is not entirely clear how to obtain a lower bound as $A_c \not\subseteq D_n$. Nevertheless, the following observation shows that as long as $c$ is chosen correctly, considering $E \cap A_c$ is indeed enough:

**Observation 3.7** Let $E \subset B := \{x \in \mathbb{R}^d, |x| < 1\} \subset \mathbb{R}^d$ be any set. Then for every $c \in (0, 1)$, and every $|\xi| = 1 - \delta$, $\delta \ll c$,

$$0 \leq \omega(\xi, E; B \setminus E) - \omega(\xi, E \cap A_c; B \setminus (E \cap A_c)) \lesssim \frac{\delta}{c},$$

where the asymptotic constants depend on the dimension alone.

Let us first conclude the proof of Proposition 3.6 assuming the observation above holds.

For $m_0 = \sqrt{\log \left(\frac{1}{\varepsilon_n}\right)}$, if $\delta = 1 - \frac{1}{2} \frac{(\rho_{n+1} - \rho_{n-1})}{(\rho_n + \rho_{n+1})}$ and $c = \frac{\rho_n - \rho_{n-m_0}}{\frac{1}{2} (\rho_n + \rho_{n+1})}$, then for every $n$ large enough,

$$\frac{\delta}{c} = \frac{1}{2} \frac{(\rho_{n+1} - \rho_{n-1})}{\rho_n - \rho_{n-m_0}} \leq \frac{R_n}{m_0 R_{n-m_0}} = \frac{R_n}{m_0 R_n (1 - o(1))} \leq \frac{2}{m_0} = \frac{2}{\sqrt{\log \left(\frac{1}{\varepsilon_n}\right)}}.$$

Now, for every $n$ large enough,

$$\frac{m_0 R_{n-m_0}}{\rho_n - \rho_{n-m_0}} = o(1) < 1 \Rightarrow m_0 \leq \frac{\rho_n - \rho_{n-m_0}}{R_n - m_0} \leq \frac{(n - m_0) R_{n-m_0}}{R_n} = n - m_0 \Rightarrow m_0 \leq \frac{n}{2}.$$

Following the observation above and Lemma 3.4, for every $z \in \partial D_{n-1}$,

$$\omega(z, E; D_n \setminus E) \lesssim \omega(\xi, E \cap A_c; B \setminus (E \cap A_c)) + \frac{\delta}{c} \lesssim \frac{1}{\sqrt{\log \left(\frac{1}{\varepsilon_n-m_0}\right)}} \leq \frac{1}{\sqrt{\log \left(\frac{1}{\varepsilon_n}\right)}},$$

$$\omega(z, E; D_n \setminus E) \lesssim \omega(\xi, E \cap A_c; B \setminus (E \cap A_c)) + \frac{\delta}{c} \lesssim \frac{1}{\sqrt{\log \left(\frac{1}{\varepsilon_n-m_0}\right)}} \leq \frac{1}{\sqrt{\log \left(\frac{1}{\varepsilon_n}\right)}},$$

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while
\[
\omega(z, E; D_n \setminus E) \geq \omega(\xi, E \cap A_c; B \setminus (E \cap A_c)) \gtrsim \frac{1}{\sqrt{\log \left( \frac{1}{\varepsilon_n} \right)}}.
\]

To conclude the proof, we apply Corollary 3.3 to the sequence of sets \( \{D_n\} \) and the functions
\[
g_1(t) = \frac{C_1}{\sqrt{\log \left( \frac{1}{\varepsilon(t)} \right)}}, \quad g_2(t) = \frac{C_2}{\sqrt{\log \left( \frac{1}{\varepsilon(t)^2} \right)}},
\]
which are continuous and monotone non-increasing (as \( \varepsilon \) is monotone non-increasing).

\( \square \)

**Proof of Observation 3.7:**

**Proof.** Following the maximum principle,
\[
\omega(\xi, E \cap A_c; B \setminus (E \cap A_c)) \leq \omega(\xi, E; B \setminus E) = \omega(\xi, E \cap (1-c)B; B \setminus E) + \omega(\xi, E \cap A_c; B \setminus E)
\]
\[
\leq \omega(\xi, E \cap (1-c)B; B \setminus E) + \omega(\xi, E \cap A_c; B \setminus (E \cap A_c)).
\]

To conclude the proof, it is therefore enough to bound \( \omega(\xi, E \cap (1-c)B; B \setminus E) \) from above:
\[
\omega(\xi, E \cap (1-c)B; B \setminus E) \overset{(i)}{\leq} \omega(\xi, E \cap (1-c)B; B \setminus (E \cap (1-c)B))
\]
\[
= 1 - \omega(\xi, \partial B; B \setminus (E \cap (1-c)B)) \overset{(i)}{\leq} 1 - \omega(\xi, \partial B; A_c)
\]
\[
= \omega(\xi, \partial (1-c)B; A_c) \overset{(ii)}{\sim} \frac{\delta}{c},
\]
where \( (i) \) is by inclusion and subordination principle of harmonic measures, and \( (ii) \) is a known estimate about the harmonic measure of an annulus in any dimension.

\( \square \)

**4 A lower bound- the general case**

In this section we will prove Theorem 1.2(A): we show that for every two functions \( R \) and \( \varepsilon \) there exist \( c > 0 \) so that every \( (\varepsilon, R) \)-recurrent set \( E \), every \( (\varepsilon, R) \)-colander set, \( \Omega = B_\rho \setminus E \), satisfies:
\[
\omega(0, \partial B_\rho; \Omega) \leq \exp \left( -c \int_1^\rho \varphi(t) dt \right).
\]
We continue to use the notation from the previous section, Section 3. We will first observe that it is enough to prove \((\star)\) for the sequence \(\{\rho_n\}\): for every \(\rho > \rho_1\) there exists \(n\) so that \(\rho_n \leq \rho < \rho_{n+1}\).

Using the maximum principle:

\[
\omega \left(0, \partial B_\rho; B_\rho \setminus E \right) \leq \omega \left(0, \partial B_{\rho_n}; B_{\rho_n} \setminus E \right) \leq \exp \left( -c \int_1^{\rho_n} \varphi(t) dt \right) \leq \exp \left( -c' \int_1^{\rho} \varphi(t) dt \right),
\]

as \(\varphi\) is monotone non-increasing.

We start with a discussion of the idea of the proof, which is a generalization of what we saw in Section 3.

### 4.1 The skeleton of the proof

We begin our proof using Corollary 3.3 with the sequence of sets \(D_n = B(0, \varrho_n)\) for some sequence \(\{\varrho_n\}\), that will be a bit different from the one used in the example in the case \(d = 2\). Following the corollary, it is therefor enough to show that for every \((\varepsilon, R)\)-recurrent set \(E\) there exists a constant \(c\) so that

\[
\inf_{\xi \in \partial D_{n-1}} \omega (\xi, D_n \setminus E) \geq \frac{c}{\sqrt{-k_d(\varepsilon_n)}}.
\]

To prove the latter, we partition \(\mathbb{R}^d\) and look at the capacity of \(E \cap P\) for every set \(P\) in the partition \(\mathcal{P}\). We will then choose a sub-partition of the partition \(\mathcal{P}\), denoted \(\hat{\mathcal{P}}\), of sets contained in the annulus \(D_n \setminus D_{n-m_0}\) for some \(m_0 \ll n\) to be chosen properly. These 'fragments' of \(E\), denoted \(E_P\), will be used to define a subharmonic function that will bound the harmonic measure from below:

\[
\inf_{\xi \in \partial D_{n-1}} \omega (\xi, D_n \setminus E) \geq \inf_{\xi \in \partial D_{n-1}} \omega \left( \xi; \bigcup_{P \in \mathcal{P}} E_P; D_n \setminus \bigcup_{P \in \mathcal{P}} E_P \right) \geq \frac{1}{\sqrt{-k_d(\varepsilon_n)}}.
\]

Let \(\mu_P\) denote the equilibrium measure of the set \(E_P\), and let \(u_P\) be some variation on the potential, \(-p_{\mu_P}\). What kind of variation will depend on the dimension.
Let \( u = \sum_{P \in \tilde{\mathcal{P}}} u_P \). We will then bound \( u \) from above

1. On \( \partial D_n \) by \( |B_n| := \Theta(m_0^2) \).

2. On \( \bigcup_{P \in \tilde{\mathcal{P}}} E_P \) by \( B_E := c_d \left( \frac{1}{-k_d(\varepsilon_n)} + m_0^2 \right) \).

Define the function

\[
v(x) := \frac{u(x) - B_n}{B_E - B_n}.
\]

Then \( v \) is harmonic on \( D_n \setminus E \) as a sum of harmonic functions, and subharmonic on \( D_n \), while it satisfies that

\[
v(x) \leq \omega \left( x, \bigcup_{P \in \tilde{\mathcal{P}}} E_P; D_n \setminus \bigcup_{P \in \tilde{\mathcal{P}}} E_P \right)
\]

on \( \partial \left( D_n \setminus \bigcup_{P \in \tilde{\mathcal{P}}} E_P \right) \) and by the maximum principle on \( D_n \setminus \bigcup_{P \in \tilde{\mathcal{P}}} E_P \). To conclude the proof we only need to bound \( u - B_n \) from below on \( \partial D_{n-1} \) by \( \Theta(m_0) \).

We will then obtain that

\[
\inf_{\xi \in \partial D_{n-1}} \omega \left( \xi, \bigcup_{P \in \tilde{\mathcal{P}}} E_P; D_n \setminus \bigcup_{P \in \tilde{\mathcal{P}}} E_P \right) \geq \frac{m_0}{m_0^2 - k_d(\varepsilon_n)}.
\]

Lastly, we choose \( m_0 = \sqrt{-k_d(\varepsilon_n)} \), and use Corollary 3.3 with the functions \( g_1(0) = 0, \ g_2(t) := \frac{1}{\sqrt{-k_d(\varepsilon(t))}} \) to conclude the proof of the lower bound.

Although the result is the same for any dimension, the proof of the case \( d = 2 \) differs from the general case. This is because the kernel is different and so the variation that we choose on the potentials to define the functions \( u_P \) will be different as well as the partition \( \mathcal{P} \) and the sub-partition \( \tilde{\mathcal{P}} \).
4.2 The case $d = 2$

4.2.1 The Setup

To obtain a general lower bound, we will describe a setup which is a little different than the one described in 3.1, for the Example.

For every $k \in \mathbb{N}$ define the sets

\[ A_0 := 4R(0)\mathbb{D}, \]
\[ A_k := \{ z, |z| \in [4(\rho_{k-1} + \rho_k), 4(\rho_k + \rho_{k+1})], k \geq 1, \]
\[ \Theta_j(k) := \{ z \in \mathbb{C}, \text{Arg}(z) \in \left[ \frac{2\pi j}{2^\ell_k} - \frac{\pi}{2^\ell_k}, \frac{2\pi j}{2^\ell_k} + \frac{\pi}{2^\ell_k} \right], j \in \{0, 1, \ldots, 2^\ell_k - 1\} \]

for $2^\ell_k$ satisfying that

\[ 2^\ell_k \leq \frac{\rho_k}{R_k} < 2^\ell_{k+1}. \]

For every $k$ we let

\[ \Lambda_k := \left\{ 8\rho_k e_k(j), e_k(j) = \exp \left( \frac{2\pi i j}{2^\ell_k} \right), j \in \{0, \ldots, 2^\ell_k - 1\} \right\}. \]

For brevity we will denote by $\varepsilon_\lambda = \varepsilon(|\lambda|)$, $R_\lambda = R(|\lambda|)$, and abuse, the notation $R_k$ to denote $R(\rho_k)$, and $\varepsilon_k$ to denote $\varepsilon(\rho_k)$. Note that for every $\lambda \neq \mu \in \Lambda := \bigcup_{k=n-m_0}^{n} \Lambda_k$ we have

\[ |\lambda - \mu| \geq 8 \max \{R_\lambda, R_\mu\}. \]

Let

\[ \tilde{E} := \bigcup_{\lambda \in \Lambda} E_\lambda, \text{ for } E_\lambda := E \cap B(\lambda, R_\lambda). \]

Following Condition (1), for every $\lambda$ we know that

\[ C_2(E_\lambda) \geq \varepsilon_\lambda \cdot R_\lambda. \]
We define the sets \( D_n := \varrho_n \mathbb{D} \) for \( \varrho_n := 4 (\rho_{n+1} + \rho_n) \) to be used when applying Corollary 3.3.

For every \( \lambda \in \Lambda \) let \( \mu_\lambda \) denote the equilibrium measure for the set \( E_\lambda \). Let \( \limsup_{t \to \infty} \frac{1}{t \varphi(t)} < A < 1 \).

Following Claim 3.1, if \( n \) is large enough, then
\[
\frac{R_n}{R_n - \sqrt{k \xi_n}} \leq \frac{1}{1 - \frac{1}{\rho_n \varphi(n)}} \leq \frac{1}{1 - A} := C.
\]

Define the function
\[
u(z) = \sum_{k=n-m_0}^{n-C} \sum_{\lambda \in \Lambda_k} (-p_{\mu_\lambda}(z) + \log(|\lambda|)) = \sum_{k=n-m_0}^{n-C} u_k(z),
\]
where \( p_{\mu} \) denotes the logarithmic potential of the measure \( \mu \).

### 4.2.2 The Origami Lemma

We will start with an auxiliary lemma describing the cancelations we get for the logarithmic kernel.

**Lemma 4.1** Fix \( R \) and let
\[
\Lambda = \left\{ R \cdot \exp\left(\frac{2\pi i k}{2^\ell}\right), 1 \leq k \leq 2^\ell \right\}.
\]

for some \( \ell \in \mathbb{N} \). Then for every \( z = |z| e^{i \frac{\pi t}{2^\ell}} \)
\[
\sum_{\lambda \in \Lambda} \log(|z - \lambda|) = \frac{1}{2} \log \left( \left( |z|^{2^\ell} - R^{2^\ell} \right)^2 + 4 |z|^{2^\ell} \cdot R^{2^\ell} \sin^2(\pi t) \right).
\]

**Proof.** We first note that without loss of generality \( |\text{Arg}(z)| \leq \frac{\pi}{2^\ell} \). For otherwise, \( \text{Arg}(z) = \frac{2\pi k}{2^\ell} + \varphi \)
for some \( |\varphi| \leq \frac{\pi}{2^\ell} \) and \( 1 \leq k \leq 2^\ell \). By the symmetry of the set \( \Lambda \),
\[
\sum_{\lambda \in \Lambda} \log(|z - \lambda|) = \sum_{\lambda \in \Lambda} \log(|z e^{i \varphi} - \lambda|). 
\]

Next, if \( z = r e^{i \varphi} \) and \( \lambda = R \cdot \exp\left(\frac{2\pi i k}{2^\ell}\right) \) then
\[
|z - \lambda|^2 = (r - R)^2 + 4r \cdot R \sin^2\left(\frac{\varphi - \frac{2\pi k}{2^\ell}}{2}\right).
\]
If \( \text{Arg}(z) = \varphi = \frac{2\pi t}{2^\ell} \) for \(|t| \leq \frac{1}{2} \), then we have

\[
\sum_{\lambda \in \Lambda} \log(|z - \lambda|) = \sum_{k=1}^{2^\ell} \log \left( a + b \cdot \sin^2 \left( \frac{\pi (t - k)}{2^\ell} \right) \right) := f(t),
\]

for \( a = (r - R)^2 \), \( b = 4r \cdot R \).

Since the sum is finite we can differentiate every summand separately:

\[
\frac{\partial}{\partial t} f(t) = \sum_{k=1}^{2^\ell} \frac{\partial}{\partial t} \left( \log \left( a + b \cdot \sin^2 \left( \frac{\pi (k - t)}{2^\ell} \right) \right) \right) = -\frac{\pi b}{2^\ell} \sum_{k=1}^{2^\ell} \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(k-t)}{2^\ell} \right)}.
\]

We will use the following properties of sine for \( k \geq 0 \):

\[
\sin \left( \frac{\pi (2^\ell - 1 + k - t)}{2^{\ell-1}} \right) = \sin \left( \pi + \frac{\pi (k - t)}{2^{\ell-1}} \right) = -\sin \left( \frac{\pi (k - t)}{2^{\ell-1}} \right)
\]

\[
\sin \left( \frac{\pi (2^\ell - 1 + k - t)}{2^\ell} \right) = \sin \left( \frac{\pi}{2} + \frac{\pi (k - t)}{2^\ell} \right) = \cos \left( \frac{\pi (k - t)}{2^\ell} \right).
\]

By using these identities and induction,

\[
\sum_{k=1}^{2^\ell} \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(k-t)}{2^\ell} \right)} = \sum_{k=1}^{2^\ell-1} \left( \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(k-t)}{2^\ell} \right)} + \frac{\sin \left( \frac{\pi(2^\ell-1+k-t)}{2^{\ell-1}} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(2^\ell-1+k-t)}{2^\ell} \right)} \right)
\]

\[
= \sum_{k=1}^{2^\ell-1} \left( \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(k-t)}{2^\ell} \right)} - \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a + b \cdot \cos^2 \left( \frac{\pi(k-t)}{2^\ell} \right)} \right)
\]

\[
= \frac{b}{2} \sum_{k=1}^{2^\ell-1} \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-2}} \right)}{a^2 + a \cdot b + \frac{b^2}{4} \sin^2 \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}.
\]

We can apply the same argument over and over with \( a_{m+1} = a_m (a_m + b_m) \), \( b_{m+1} = \frac{b_m^2}{4} \) obtaining

\[
\sum_{k=1}^{2^\ell} \frac{\sin \left( \frac{\pi(k-t)}{2^{\ell-1}} \right)}{a_0 + b_0 \cdot \sin^2 \left( \frac{\pi(k-t)}{2^\ell} \right)} = \ldots = \frac{1}{2^\ell} \cdot \left( \prod_{m=0}^{\ell-1} b_m \right) \cdot \frac{\sin(-2\pi t)}{a_\ell + b_\ell \sin^2(\pi t)}.
\]

Integrating both sides of the equality

\[
f(t) = \int_0^t f'(s) ds = \int_0^t -\pi b \frac{2^\ell}{2^\ell} \sum_{k=1}^{2^\ell} \frac{\sin \left( \frac{\pi(k-s)}{2^\ell} \right)}{a + b \cdot \sin^2 \left( \frac{\pi(k-s)}{2^\ell} \right)} ds
\]

\[
= -\pi b \frac{2^\ell}{2^\ell} \int_0^t \prod_{m=0}^{\ell-1} b_m \cdot \frac{\sin(-2\pi s)}{a_\ell + b_\ell \sin^2(\pi s)} ds = \frac{b}{b_\ell} \cdot \frac{1}{4^\ell} \left( \prod_{m=0}^{\ell-1} b_m \right) \cdot \log (a_\ell + b_\ell \sin^2(\pi t)) + \text{Const}.
\]
To conclude the proof note that by induction

\[ b_ℓ = 4 r^{2ℓ} R^{2ℓ}, \quad a_ℓ = \left( r^{2ℓ} - R^{2ℓ} \right)^2, \]

and since \( f(0) = 2^ℓ \log(R) \Rightarrow Const = 0 \) and

\[ \sum_{λ ∈ Λ} \log (|z - λ|) = \frac{1}{2} \log \left( \left( r^{2ℓ} - R^{2ℓ} \right)^2 + 4r^{2ℓ} \cdot R^{2ℓ} \sin^2(πt) \right). \]

\[ \square \]

4.2.3 An error is an error no matter how small:

We know almost nothing about the probability measures \( \{μ_λ\}_{λ ∈ Λ} \). The Origami Lemma, Lemma 4.1, only applies to sums over \( \log (|z - λ|) \) for \( λ ∈ Λ_k \), while here we have sums over integrals of \( \log (|z - w|) \). What is the error when taking \( λ \) instead of any \( w ∈ E_λ \)? For \( z \) so that \( |z - w| > 2R_λ \):

\[ \left| \log \left( \frac{|λ|}{|z - w|} \right) - \log \left( \frac{|λ|}{|z - λ|} \right) \right| = \left| \log \frac{|z - λ|}{|z - w|} \right| \leq \log \left( 1 + \frac{2R_λ}{|z - λ|} \right) \leq \frac{2R_λ}{|z - λ|}. \]

Even thought the error seems small, it is not as we are summing over order of \( \frac{2R_λ}{|z - λ|} \) elements each step and we have \( m_0 \) steps. We obtain a total error of order \( m_0 \log(n) \), which is a huge error. Our salvation comes from the fact that the distortion is the same distortion whether you take \( u(z) \) or \( u(w) \). We will therefore bound the ”relative” distortion in \( u \), in a sense, with respect to two points at the same time. Formally, we will bound the distortion of the function \( u(z) - u(w) \):

**Proposition 4.2** Let

\[ ψ(z) := \sum_{λ ∈ Λ, \ z \notin B(λ, R_λ)} \log \left( \frac{|λ|}{|z - λ|} \right), \]

and define the function

\[ \tilde{u}(z) = u(z) - ψ(z) + [p_{μ_λ}(z) - \log (|λ|)] 1_{B(λ, R_λ)}(z). \]
For every $|z - w| \in [4R_n, 10R_n \cdot m_0]$ satisfying that

$$z, w \in \overline{D_n \setminus \left( D_{n - m_0} \cup \bigcup_{\lambda \in \Lambda} (B(\lambda, 4R_\lambda) \setminus B(\lambda, R_\lambda)) \right)},$$

we have:

$$|\tilde{u}(z) - \tilde{u}(w)| \lesssim \frac{|z - w|}{R_n} \left( \frac{|z - w|}{R_n} + \log(m_0) \right).$$

Figure 2: The gray area is the set $\overline{D_n \setminus \left( D_{n - m_0} \cup \bigcup_{\lambda \in \Lambda} (B(\lambda, 4R_\lambda) \setminus B(\lambda, R_\lambda)) \right)}$. Note that about each point there is a small gray disk surrounded by a white annulus.

Proof. For every $\lambda$ for which $z, w \not\in B(\lambda, R_\lambda)$ $\Rightarrow$ $z, w \not\in B(\lambda, 4R_\lambda)$. If $\xi \in B(\lambda, R_\lambda)$ then by the triangle inequality $|z - \xi|, |w - \xi| \geq 2R_\lambda$ and so the ”joint distortion” in every component is bounded by

$$\left| \log \left( \frac{|\lambda|}{|z - \xi|} \right) - \log \left( \frac{|\lambda|}{|z - \lambda|} \right) - \log \left( \frac{|\lambda|}{|w - \xi|} \right) - \log \left( \frac{|\lambda|}{|w - \lambda|} \right) \right| \leq \frac{2R_\lambda |z - w|}{|z - \lambda| |w - \lambda|}.$$

Define the sets-

$$\Lambda_1 := \{ \lambda \in \Lambda, |z - \lambda| \leq 2|z - w| \text{ and } |z - \lambda|, |w - \lambda| > R_\lambda \}, \quad \Lambda_2 := \{ \lambda \in \Lambda \setminus \Lambda_1, |w - \lambda| > R_\lambda \}.$$
For every $\lambda \in \Lambda_1$ either $|z - \lambda| \geq \frac{|z - w|}{2}$ or $|w - \lambda| \geq \frac{|z - w|}{2}$, for otherwise we get a contradiction to the triangle inequality. On the other hand, by definition of the set $\Lambda_1$ we know that $w, z \notin \bigcup_{\lambda \in \Lambda_1} B(\lambda, R_\lambda)$. Then

$$\frac{2R_\lambda |z - w|}{|z - \lambda| |w - \lambda|} \leq \frac{4R_\lambda |z - w|}{4R_\lambda |z - w|} = 1.$$ 

As $\#\Lambda_1 \sim \frac{|z-w|^2}{R_{n-m_0}}$ we conclude that

$$\sum_{\lambda \in \Lambda_1} \frac{2R_\lambda |z - w|}{|z - \lambda| |w - \lambda|} \lesssim \#\Lambda_1 \lesssim \frac{|z - w|^2}{R_{n-m_0}}.$$ 

For every $\lambda \in \Lambda_2$ we have that $|z - \lambda| \geq 2 |w - z|$. Using triangle inequality,

$$\frac{|z - \lambda|}{|w - \lambda|} \leq \frac{|z - \lambda|}{|w - \lambda| - |w - z|} = 1 + \frac{|z - w|}{|z - \lambda| - |w - z|} \leq 1 + \frac{|z - w|}{|z - w|} = 2.$$ 

Then

$$\sum_{\lambda \in \Lambda_2} \frac{1}{|z - \lambda| |w - \lambda|} = \sum_{\lambda \in \Lambda_2} \frac{1}{|z - \lambda|^2} \cdot \frac{|z - \lambda|}{|w - \lambda|} \leq 2 \sum_{\lambda \in \Lambda_2} \frac{1}{|z - \lambda|^2}.$$ 

To conclude the proof it is therefore enough to bound

$$\sum_{k = n-m_0}^{n-C} \sum_{\lambda \in \Lambda_k \atop |z - \lambda| > 2 |z - w|} \frac{1}{|z - \lambda|^2}.$$ 

By monotonicity of the function $t \mapsto \frac{1}{t^2}$ we see that-

$$\sum_{k = n-m_0}^{n-C} \sum_{\lambda \in \Lambda_k \atop |z - \lambda| > 2 |z - w|} \frac{1}{|z - \lambda|^2} \sim \frac{1}{R_{n-m_0}} \sum_{k = n-m_0}^{n-C} \int_{\max\{|z-w|, \text{dist}(z, \Lambda_k)\}}^{\rho_k} \frac{1}{t^2} dt \leq \frac{1}{R_{n-m_0}} \sum_{k = n-m_0}^{n-C} \frac{1}{\max\{|z-w|, \text{dist}(z, \Lambda_k)\}} \leq \frac{\# \{1 \leq k \leq m_0, \text{dist}(z, \Lambda_k) < |z - w| \}}{R_{n-m_0} |z - w|} + \frac{1}{R_{n-m_0}^2} \log(m_0) \lesssim \frac{1}{R_n^2} \log(m_0).$$

Combining everything together we conclude that

$$|\tilde{u}(z) - \tilde{u}(w)| \leq \cdots \lesssim \frac{|z - w|^2}{R_n^2} + \frac{R_n |z - w|}{R_n^2} \log(m_0) \leq \frac{|z - w|}{R_n} \left( \frac{|z - w|}{R_n} + \log(m_0) \right).$$
4.2.4 Bounding \((B_E - B_n)\)

The first step will be to find an upper bound for \(\psi\) on \(\partial D_n\) and on \(E\).

We begin by showing that for every \(z \in \partial D_n\) we have

\[
\psi(z) \sim -m_0^2.
\]

Using the Origami Lemma, Lemma 4.1, we see that for every \(k\) and \(z \in \partial D_n\)

\[
\log \left( \frac{\theta_n}{8\rho_k} \right) - 1 \leq \sum_{\lambda \in \Lambda_k} \log \left( \frac{|z - \lambda|}{|\lambda|} \right) \leq \log \left( \frac{\theta_n}{8\rho_k} \right) + 1.
\]

Then the above holds if we will show that

\[
\exp \left( \frac{1}{4} \left( n - k + \frac{1}{2} \right) \right) \leq \frac{\theta_n}{8\rho_k} \leq \exp \left( C \left( n - k + \frac{1}{2} \right) \right),
\]

for \(C = \frac{1}{1 - A}\) defined earlier.

To see an upper bound note that since for \(x > 0\), \(\log(1 + x) \leq x\):

\[
\left( \frac{\theta_n}{8\rho_k} \right)^{2^{k}} = \left( \frac{4(\rho_{n+1} + \rho_n)}{8\rho_k} \right)^{2^{k}} = \left( \frac{\rho_{n+1} + \rho_n}{2\rho_k} \right)^{2^{k}} = \left( 1 + \frac{\rho_{n+1} + \rho_n - 2\rho_k}{2\rho_k} \right)^{2^{k}}
\]

\[
= \left( 1 + \frac{\rho_n - \rho_k + \frac{1}{2}R_n}{\rho_k} \right)^{2^{k}} = \exp \left( 2^{k} \log \left( 1 + \frac{\rho_n - \rho_k + \frac{1}{2}R_n}{\rho_k} \right) \right)
\]

\[
\leq \exp \left( 2^{k} \cdot \frac{\rho_n - \rho_k + \frac{1}{2}R_n}{\rho_k} \right) \leq \exp \left( \frac{(n - k + \frac{1}{2}) R_n}{R_k} \right) \sim \exp \left( C \left( n - k + \frac{1}{2} \right) \right).
\]

While since \(\log(1 + x) > \frac{x}{2}\) for \(x \in (0, 1)\)

\[
\left( \frac{\theta_n}{8\rho_k} \right)^{2^{k}} = \cdots = \exp \left( 2^{k} \log \left( 1 + \frac{\rho_n - \rho_k + \frac{1}{2}R_n}{\rho_k} \right) \right)
\]

\[
\geq \exp \left( 2^{k-1} \cdot \frac{\rho_n - \rho_k + \frac{1}{2}R_n}{\rho_k} \right) \geq \exp \left( \frac{(n - k + \frac{1}{2}) R_k}{4R_k} \right) = \exp \left( \frac{1}{4} \left( n - k + \frac{1}{2} \right) \right).
\]

Overall we conclude that on \(\partial D_n\),

\[
\psi(z) \sim - \sum_{k=n-m_0}^{n-C} (n - k) = - \sum_{j=C}^{m_0} j \sim -m_0^2.
\]
A similar computation shows that for every $z \in E_\lambda$, $\lambda \in \Lambda_{k'}$ we have:

$$1 \lesssim \sum_{k \neq k'} \left( \sum_{\lambda \in \Lambda_k} \log \left( \frac{|z - \lambda|}{\rho_k} \right) \right) \lesssim m_0^2.$$  

It is left to bound $\sum_{\lambda \in \Lambda_k} \log \left( \frac{|z - \lambda|}{\rho_k} \right)$ on $E_{\lambda_0}$ for $\lambda_0 \in \Lambda_k$. Following rotation invariance of the set $\Lambda_k$, we assume without loss of generality that $\lambda_0 = \rho_k$. Following Frostman’s theorem

$$-p_{\mu_{\lambda_0}}(w) = \log \left( \frac{1}{R_k \varepsilon_k} \right), \text{ for } w \in E_{\lambda_0}.$$  

To bound the rest of the sum, we cannot use the Origami Lemma, since we only have a partial sum here. We will therefore look at two cases:

**Case 1:** If $z = \lambda_0$, then it is well known that

$$\sum_{\lambda \in \Lambda_k, \lambda \neq \lambda_0} \log \left( \frac{|\lambda|}{|\lambda_0 - \lambda|} \right) = - \sum_{\lambda \in \Lambda_k, \lambda \neq \lambda_0} \log \left( 1 - \frac{|\lambda|}{|\lambda_0|} \right) = \log \left( \prod_{j=1}^{2\ell_k-1} \left| 1 - e^{2\pi i j/2\ell_k} \right| \right)$$

$$= - \log (2^{\ell_k}) \leq - \log \left( \frac{\rho_k}{R_k} \right) + \log(2).$$

**Case 2:** If $z = \lambda_0 + R_k \cdot w$ for $w \in \mathbb{D} \setminus \{0\}$, then by adding and removing the first component in the sum, we may use the Origami lemma:

$$\sum_{\lambda \in \Lambda_k, \lambda \neq \lambda_0} \log \left( \frac{|\lambda|}{|z - \lambda|} \right) = \sum_{\lambda \in \Lambda_k} \log \left( \frac{|\lambda|}{|z - \lambda|} \right) - \log \left( \frac{|\lambda|}{|w| R_k} \right)$$

$$= 2^{\ell_k} \log(\rho_k) - \sum_{\lambda \in \Lambda_k} \log |z - \lambda| - \log \left( \frac{\rho_k}{R_k} \right) + \log |w|$$

$$\leq - \log \left| 1 - \left( 1 - |w| \cdot \frac{R_k}{\rho_k} \right)^{2^{\ell_k}} \right| - \log \left( \frac{\rho_k}{R_k} \right) + \log |w|$$

$$\leq - \log \left| 1 - e^{-|w|} \right| + \log(2) - \log \left( \frac{\rho_k}{R_k} \right) + \log |w| \leq - \log \left( \frac{\rho_k}{R_k} \right) + \log(3),$$

since for $t \in (0, 1)$ we have

$$1 + t \leq e^t \leq 1 + t + t^2.$$
Overall, there exists a constant $c > 0$ so that for every $z \in E_{\lambda_0}$,

$$
\log(\rho_k) - p_{\mu_{\lambda_0}}(z) + \sum_{\lambda \in \Lambda_k, \lambda \neq \lambda_0} \log \left( \frac{|\lambda|}{|z - \lambda|} \right) \leq \log \left( \frac{\rho_k}{R_k \varepsilon_k} \right) - \log \left( \frac{\rho_k}{R_k} \right) + c
$$

$$
= \log \left( \frac{1}{\varepsilon_k} \right) + c \leq \log \left( \frac{1}{\varepsilon_n} \right) + c.
$$

Note that by the way $\psi$ and $\tilde{u}$ were defined we know that

$$
u(z) - \tilde{u}(z) = \psi(z) + [\log (|\lambda|) - p_{\mu_{\lambda}}(z)] 1_{B(\lambda, R_{\lambda})(z)}.
$$

Combining these together we conclude that if $B_E = u(w_0) = u(|w_0|e^{it})$ and $w_0 \in B(\lambda_0, R_{\lambda_0})$ then

$$
B_E - B_n = u(|w_0|e^{it}) - B_n = u(|w_0|e^{it}) - u(\varrho_n e^{it}) + u(\varrho_n e^{it}) - B_n \leq u(|w_0|e^{it}) - u(\varrho_n e^{it})
$$

$$
= u(|w_0|e^{it}) - \tilde{u}(|w_0|e^{it}) + \tilde{u}(|w_0|e^{it}) - \tilde{u}(\varrho_n e^{it}) + \tilde{u}(\varrho_n e^{it}) - u(\varrho_n e^{it})
$$

$$
= \psi(|w_0|e^{it}) + \log (|\lambda_0|) - p_{\mu_{\lambda_0}}(|w_0|e^{it}) + \tilde{u}(|w_0|e^{it}) - \tilde{u}(\varrho_n e^{it}) - \psi(\varrho_n e^{it})
$$

$$
\lesssim \rho_0^2 + \log \left( \frac{1}{\varepsilon_n} \right) + \tilde{u}(|w_0|e^{it}) - \tilde{u}(\varrho_n e^{it}) \lesssim \rho_0^2 + \log \left( \frac{1}{\varepsilon_n} \right),
$$

since by Proposition 4.2,

$$
\tilde{u}(|w_0|e^{it}) - \tilde{u}(\varrho_n e^{it}) \lesssim \frac{\rho_n - |w_0|}{R_n} \left( \frac{\rho_n - |w_0|}{R_n} + \log(m_0) \right) \leq 2m_0^2.
$$

4.2.5 The function

Define the function

$$
v(z) := \frac{u(z) - B_n}{B_E - B_n} \geq \frac{u(z) - \max_{w \in \partial D_n} u(w)}{c \left( \log \left( \frac{1}{\varepsilon_n} \right) + m_0^2 \right)}.
$$

$v$ is harmonic on $D_n \setminus \tilde{E}$ and, according to the estimates done above, as long as $c$ is chosen to be large enough, $v \leq 0$ on $\partial D_n$ while $v \leq 1$ on $\tilde{E}$. By the maximum principle

$$
v(z) \leq \omega \left( z, \tilde{E}, D_n \setminus \tilde{E} \right).
$$
To conclude the proof, let us bound \( v \) from below on \( \partial D_{n-1} \): informally, \( \partial D_{n-1} \) is closer to the set \( \tilde{E} \) and so the origami lemma gives us the intuition that for every layer \( \Lambda_k \) the sum of logarithmic differences is bounded from below by some constant.

Formally, fix \( k \), and for every \( 0 \leq j \leq 2^\ell_k - 1 \) let \( \lambda_j := 8 \rho_k e_k(j) \). Assume that \( z = \varrho_n, \ w = \varrho_{n-1} e_k(j_0) \) for some \( j_0 \in \{0, \ldots, 2^\ell_k - 1\} \). Then by pairing \( \lambda_j \) with \( \lambda_{j+j_0} \) we obtain that

\[
\sum_{j=0}^{2^\ell_k-1} \log \left( \frac{|z - \lambda_j - R_k|}{|w - \lambda_j e_k(j_0)| + R_k} \right) = \sum_{j=0}^{2^\ell_k-1} \log \left( \frac{|\varrho_n - \lambda_j|}{|\varrho_{n-1} - \lambda_j| + R_k} \right)
\]

\[
\geq \frac{1}{2} \sum_{j=0}^{2^\ell_k-1} \log \left( 1 + \frac{|\varrho_n - \lambda_j|^2 - |\varrho_{n-1} - \lambda_j|^2 - 2R_k(|\varrho_n - \lambda_j| + |\varrho_{n-1} - \lambda_j|)}{|\varrho_{n-1} - \lambda_j|^2 + 2R_k|\varrho_{n-1} - \lambda_j| + R_k^2} \right)
\]

\[
\geq \frac{1}{2} \sum_{j=0}^{2^\ell_k-1} \log \left( 1 + \frac{2R_k|\varrho_n + \varrho_{n-1} - 2|\lambda_j|)}{4|\varrho_{n-1} - \lambda_j|^2} \right) \geq \frac{1}{2} \sum_{j=0}^{2^\ell_k-1} \log \left( 1 + \frac{R_k^2(n-k)}{|\varrho_{n-1} - \lambda_j|^2} \right).
\]

On the other hand, using Claim 3.1, since \( k \leq n - C \)

\[
\frac{R_k}{|\varrho_{n-1} - \lambda_j|} \leq \frac{R_k}{8(\rho_n - \rho_k) - 4R_n} \leq \frac{1}{8(n-k) - 4\frac{R_n}{R_k}} \leq \frac{1}{8(n-k) - 4C} \leq \frac{1}{4(n-k)},
\]

and therefore we know that

\[
\left( \frac{R_k}{|\varrho_{n-1} - \lambda_j|} \right) \left( n-k \right) \leq \frac{16(n-k)^2}{16} \Rightarrow \log \left( 1 + \frac{R_k^2(n-k)}{|\varrho_{n-1} - \lambda_j|^2} \right) \geq \frac{R_k^2(n-k)}{2|\varrho_{n-1} - \lambda_j|^2}.
\]

Combining (†) with the estimate above, we obtain that

\[
\frac{1}{2} \sum_{j=0}^{2^\ell_k-1} \log \left( 1 + \frac{R_k^2(n-k)}{|\varrho_{n-1} - \lambda_j|^2} \right) \geq \frac{R_k^2(n-k)}{4} \sum_{j=0}^{2^\ell_k-1} \frac{1}{|\varrho_{n-1} - \lambda_j|^2} \geq R_k(n-k) \int_{\text{dist}(\varrho_{n-1}, \Lambda_k)}^\rho \frac{1}{t^2} dt = R_k(n-k) \left( \frac{1}{\text{dist}(\varrho_{n-1}, \Lambda_k)} - \frac{1}{\rho_k} \right) \gtrsim 1,
\]

by monotonicity of the function \( t \mapsto \frac{1}{t^2} \).

Since for all \( n - m_0 \leq k \leq n - C \)

\[
\{ e_{n-m_0}(j), j \in \{0, 1, \ldots, 2^\ell_{n-m_0} - 1\} \} \subset \{ e_k(j), j \in \{0, 1, \ldots, 2^\ell_k - 1\} \},
\]

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we may apply the argument above for any $w = \varrho_{n-1} e_{n-m_0}(j_0)$ and $z = \varrho_n e_{n-m_0}(j_1)$. We conclude that

$$u(w) - u(z) \gtrsim m_0.$$ 

What about different $z', w'$? For every $w' \in \Theta_{n-m_0}(j_0) \cap \partial D_{n-1}$

$$|w - w'|^2 = 4\varrho_{n-1}^2 \sin^2 \left( \frac{\text{Arg}(w) - \text{Arg}(w')}{2} \right) = 4\varrho_{n-1}^2 \sin^2 \left( \frac{2\pi}{2^{\ell_{n-m_0}+1}} \right) \sim \left( \frac{\varrho_{n-1}}{2^{\ell_{n-m_0}}} \right)^2 \tag{5}$$

Claim 3.1 $R_n^2 \left( \frac{8\varrho_n - 4R_{n-1}}{\rho_{n-m_0}} \right) \sim \Theta(R_n^2)$.

A similar estimate holds for every $z' \in \Theta_{n-m_0}(j_0) \cap \partial D_n$. We note that for any $z, z' \in \partial D_n$:

$$|\psi(z) - \psi(z')| \leq \sum_{k=n-m_0}^{n-C} \left[ \log \left( \left( \frac{\varrho_n}{8\varrho_k} \right)^{2^{\ell_k}} + 1 \right) - \log \left( \left( \frac{\varrho_n}{8\varrho_k} \right)^{2^{\ell_k}} - 1 \right) \right]$$

$$= \sum_{k=n-m_0}^{n-C} \log \left( 1 + \frac{2}{\left( \frac{\varrho_n}{8\varrho_k} \right)^{2^{\ell_k}} - 1} \right) \leq \sum_{k=n-m_0}^{n-C} \exp \left( -c \left( n - k + \frac{1}{2} \right) \right) = \text{Const}$$

$$\Rightarrow |u(z) - \tilde{u}(z) - (u(z') - \tilde{u}(z'))| = |\psi(z) - \psi(z')| = \text{Const}.$$ 

An equivalent computation shows the same holds for any $w, w' \in \partial D_{n-1}$. We conclude that if $w' \in \Theta_{n-m_0}(j_0) \cap \partial D_{n-1}$ and $w = \varrho_{n-1} e_{n-m_0}(j_0)$, then using Proposition 4.2,

$$|u(w) - u(w')| \leq |\tilde{u}(w) - \tilde{u}(w')| + |(u(w) - \tilde{u}(w)) - (u(w') - \tilde{u}(w'))|$$

$$= |\tilde{u}(w) - \tilde{u}(w')| + |\psi(w) - \psi(w')|$$

$$\lesssim \frac{|w - w'|}{R_n} \left( \frac{|w - w'|}{R_n} + \log(m_0) \right) + \text{Const} = o(\log(m_0)).$$

The same computation shows that if $z' \in \Theta_{n-m_0}(j_1) \cap \partial D_n$ and $z = \varrho_n e_{n-m_0}(j_1)$, then

$$|u(z) - u(z')| = o(\log(m_0)).$$

We conclude that based on the lower bound for $u(w) - u(z)$ proven for $z = \varrho_n e_{n-m_0}(j_1)$ and $w = \varrho_{n-1} e_{n-m_0}(j_0)$,

$$u(w') - u(z') = u(w') - u(w) + u(w) - u(z) + u(z) - u(z') \geq u(w) - u(z) - o(m_0) \gtrsim m_0.$$
Overall, for every $w \in \partial D_{n-1}$ and $z \in \partial D_n$ we know that $u(w) - u(z) \gtrsim m_0$ and so

$$\inf_{\xi \in \partial D_{n-1}} \omega(\xi, \tilde{E} ; \mathbb{D} \setminus \tilde{E}) \geq \inf_{\xi \in \partial D_{n-1}} v(\xi) \gtrsim \frac{m_0}{\log\left(\frac{1}{\varepsilon n}\right) + m_0^2}.$$ 

Let $m_0 := \sqrt{\log\left(\frac{1}{\varepsilon n}\right)} = o\left(\log\left(\frac{1}{\varepsilon n}\right)\right)$, then using the estimate above

$$\inf_{\xi \in \partial D_{n-1}} \omega(\xi, E ; \mathbb{D} \setminus E) \geq \inf_{\xi \in \partial D_{n-1}} \omega(\xi, \tilde{E} ; \mathbb{D} \setminus \tilde{E}) \gtrsim \frac{1}{\sqrt{\log\left(\frac{1}{\varepsilon n}\right)}}.$$ 

We conclude the proof by using Corollary 3.3 with the functions $g_1(t) = 0$, $g_2(t) = \frac{1}{\sqrt{\log\left(\frac{1}{\varepsilon n}\right)}}$.

### 4.3 The case $d \geq 3$

#### 4.3.1 The Setup

The proof in $\mathbb{R}^d$ is simpler because the kernel is simpler.

As in the example in Section 3, we let $D_n := \{|x| < \varrho_n\}$ where $\varrho_n := \frac{1}{2} (\rho_{n+1} + \rho_n)$. We will bound the harmonic measure from bellow at the point $\varrho_{n-1}$. Because the condition on the set $E$ is invariant with respect to rotations, this simplifies the proof on one hand, while one can apply the same argument to every element in $\partial D_{n-1}$ by rotating the set and using the fact that harmonic measure is rotation invariant.

Observation 3.7 shows that if your relative distance from the outer boundary is $c$, then for every $\delta \ll c$

$$0 \leq \omega(x, E, B \setminus E) - \omega(x, E \cap B(x, \delta), B \setminus (E \cap B(x, \delta))) \lesssim \frac{\delta}{c}.$$ 

Then to bound the harmonic measure at the point $\varrho_{n-1}$ we will only be interested in a relatively small neighborhood of it, and the function we construct will be custom-made for this point. Never the less, the construction is the same for any point and the constants are uniform, so the same proof can be applied to any part of $\partial D_{n-1}$ giving us a uniform lower bound.
We note that we are only interested in the asymptotic growth up to multiplication by constants. Moreover, as noted above, the asymptotic decay of the harmonic measure is effected only by a relatively small neighborhood of the point $\varrho_{n-1}$ where the functions $R$ and $\varepsilon$ do not change much, under the assumption that $\frac{1}{t \varphi(t)} < 1$, following Claim 3.1. This implies that we loose only a multiplication by a constant when working with $R_n = R(\rho_n)$ and $\varepsilon_n = \varepsilon(\rho_n)$ instead of $R(t)$, $\varepsilon(t)$.

To describe the function, used to bound the harmonic measure, we start by covering $\mathbb{R}^d$ by half-open half-closed cubes of edge-length $R_n$ enumerated by the set of multi-indexes $I := \{(i_1, i_2, \ldots, i_d), \ i_j \in \mathbb{Z}\}$.

That is, for $I = (i_1, \cdots, i_d)$ we let $\lambda_I$ denote the center of the cube $C_I$:

$$C_I = \prod_{j=1}^{d} [i_j \cdot R_n, (i_j + 1) R_n) = \lambda_I + [0, R_n)^d,$$

and define $E_I = E \cap B(\lambda_I, R(|\lambda_I|))$. Define the sets

$$U_{m_0} := B(0, \varrho_{n-1}) \cap B(\varrho_n, m_0 \cdot R_n), \quad \tilde{E} := \bigcup_{I, C_I \subset U_{m_0}} E_I,$$

for some $m_0 \ll n$ that will be chosen later (see Figure 3 below). For every $I \subset U_{m_0}$

$$\text{Cap}(E_I) \geq \varepsilon(|\lambda_I|) \cdot R(|\lambda_I|) \geq \varepsilon_n R_{n-m_0} \gtrsim \varepsilon_n R_n$$

according to the definition of $(\varepsilon, R)$-recurrent set combined with Claim 3.1. Define the function

$$u(x) = R_n^{d-2} \sum_{I \text{ s.t. } C_I \subset U_{m_0}} - p_{\mu_I}(x),$$

where $p_{\mu_I}$ is the potential of the equilibrium measure of the set $E_I$ with kernel $k_d$.

4.3.2 The Error

Like in the two dimensional case, the only thing known about the equilibrium measures, $\mu_I$, is the fact that these are probability measures supported on $E_I$. For that reason, we would like to approximate the integrals, appearing in the definition of $u$, by integrals with respect to Lebesgue’s
Figure 3: The set $U_{m_0}$ is the checkered area in the picture.

measure.

Let $x$ be so that $|x - \lambda_I| \geq 2d\sqrt{d}R_n$. Then for every $y \in C_I$

\[
\frac{1}{|x - y|^{d-2}} \leq \frac{1}{|x - \lambda_I|^{d-2}} \left( \frac{|x - \lambda_I|}{|x - \lambda_I| - |\lambda_I - y|} \right)^{d-2} \leq \frac{1}{|x - \lambda_I|^{d-2}} \left( 1 + \frac{2\sqrt{d}R_n}{|x - \lambda_I|} \right)^{d-2} \leq \frac{1}{|x - \lambda_I|^{d-2}} \left( 1 + \frac{4\sqrt{d}R_n}{|x - \lambda_I|} \right),
\]

(3) since for every $x < \frac{1}{d}$ we have

\[
1 - 2dx \leq (1 - x)^d \leq (1 + x)^d \leq 1 + 2dx.
\]

We deduce that since $\mu_I$ are probability measures, for every $x$ for which $\text{dist}(x, C_I) \geq 2d\sqrt{d}R_n$:

\[
-p_{\mu_I}(x) = \int_{E_I} \frac{1}{(|x - y|)^{d-2}} d\mu_I(y) \leq \left( 1 + \frac{4\sqrt{d}R_n}{|x - \lambda_I|} \right) \frac{1}{|x - \lambda_I|^{d-2}}
\]

\[
\leq \left( 1 + \frac{4\sqrt{d}R_n}{|x - \lambda_I|} \right)^2 \frac{1}{m_d(C_I)} \int_{C_I} \frac{1}{|x - y|^{d-2}} dm_d(y) \leq \frac{1}{m_d(C_I)} \int_{C_I} \frac{1}{|x - y|^{d-2}} dm_d(y).
\]

As a corollary, we can bound the function $u$ by

\[
u(x) = R_n^{d-2} \sum_{I : C_I \subset U_{m_0}} -p_{\mu_I}(x) \leq \frac{1}{R_n^2} \int_{\{I : C_I \subset U_{m_0} \}} \frac{1}{|x - y|^{d-2}} dm_d(y) + A_d(x),
\]

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where

\[ A_d(x) = R_n^{d-2} \sum_{I \text{ s.t. } C_I \subset U_{m_0}} - p_{\mu_I}(x). \]

### 4.3.3 Bounding \((B_E - B_n)\)

Recall that for \( \tilde{E} = \bigcup_{C_I \subset U_{m_0}} E \cap C_I \) we defined

\[ B_E := \max_{x \in \tilde{E}} u(x), \quad B_n := \max_{x \in \partial D_n} u(x). \]

To find suitable bounds for \( u \), we will use an idea similar to the one used by Carroll and Ortega-Cerdà in [5]. In this paper they approximate a similar function by summing approximations of it on annuli where the center of the annuli is the point where you are estimating the function, instead of the origin. To estimate the integral component of \( u \), we will use (without proof) the following observation:

**Observation 4.3** For every

\[ x \in U_{m_0}^+ := \{ x, \text{dist}(x, U_{m_0}) \leq R_n \} \]

for every \( j \):

\[ m_d(A_x(R_n(j - 1), R_n \cdot j)) \sim R_n^d \cdot j^{d-1}, \]

where

\[ A_x(a, b) := (B(x, b) \setminus B(x, a)) \cap B(0, \varrho_n), \]

and the constants are uniform, and depend only on the dimension \( d \).
Bounding $B_n$ from above: We note that $\text{dist}(U_{m_0}, \partial D_n) \geq R_n$ and therefore for every $x \in \partial D_n$:

$$A_d(x) \leq R_n^{d-2} \# \left\{ I \text{ s.t } C_I \subset U_{m_0} \right\} \cdot \max_{y \in U_{m_0}} \frac{1}{|x - y|^{d-2}} \leq C.$$

Integrating over $U_{m_0}$ instead of over just $C_I \subset U_{m_0}$ we bound $u$ from above by

$$u(x) \leq 1 + \frac{1}{R_n^2} \int_{U_{m_0}\backslash B(x, 2d\sqrt{d}R_n)} \frac{1}{|x - y|^{d-2}} dm_d(y) \leq 1 + \sum_{j=1}^{m_0} \frac{m_d(U_{m_0} \cap A_{\phi_n}(jR_n, (j+1)R_n))}{j^{d-2}R_n^d} \leq 1 + \sum_{j=1}^{m_0} \frac{m_d(U_{m_0} \cap A_{\phi_n}(jR_n, (j+1)R_n))}{j^{d-2}R_n^d} \leq \left(1 + \frac{1}{j}\right)^{d-2} (j+1) \lesssim j.$$ 

Because of the way the set $U_{m_0}$ was defined, it is clear that the integral is maximized for $x = \varphi_n$.

Let us bound each term separately by using Observation 4.3-

$$\frac{m_d(U_{m_0} \cap A_{\phi_n}(jR_n, (j+1)R_n))}{j^{d-2}R_n^d} \lesssim \frac{(j+1)^{d-1}R_n^d}{j^{d-2}R_n^d} \leq \left(1 + \frac{1}{j}\right)^{d-2} (j+1) \lesssim j.$$

Over all we conclude that

$$u(x) \lesssim 1 + \sum_{j=1}^{m_0} \frac{m_d(U_{m_0} \cap A_{\phi_n}(jR_n, (j+1)R_n))}{j^{d-2}R_n^d} \lesssim \sum_{j=1}^{m_0} j \sim m_0^2.$$

Bounding $B_E$ from above: In this case $A_d(x)$ can not bounded by a constant, but by using Frostman’s theorem

$$A_d(x) \lesssim \# \left\{ I \text{ s.t } C_I \subset U_{m_0} \right\} \cdot \max_{y \in \varepsilon_n} \frac{1}{|x - y|^{d-2}} \lesssim \frac{1}{\varepsilon_n^{d-2}}.$$
To bound the rest of the sum, we use the same method used to bound $u$ on $\partial D_n$. We conclude that

$$u(x) \leq C \left( \frac{1}{\varepsilon_n^{d-2}} + m_0^2 \right).$$

### 4.3.4 The function

Define the function

$$v(x) := u(x) - B_n = \frac{u(x) - \max_{y \in \partial D_n} u(y)}{C(B_E - B_n)} \geq \frac{C}{\varepsilon_n^{d-2}} + C'm_0^2.$$

The function $v$ is harmonic on $D_n \setminus \bar{E}$, subharmonic on $D_n$, and by the maximum principle for every $x \in D_n \setminus \bar{E}$

$$v(x) \leq \omega(x, E; D_n \setminus \bar{E}) \leq \omega(x, E; D_n \setminus E).$$

To conclude the proof let us bound $v$ from below on $\varrho_n - 1$. We note that since $\varrho_n, \varrho_n - 1 \in \mathbb{R}_+$, for every $y \in \mathbb{R}^d$:

$$|\varrho_n - y| - |\varrho_n - 1 - y| = \frac{|\varrho_n - y|^2 - |\varrho_n - 1 - y|^2}{|\varrho_n - y| + |\varrho_n - 1 - y|} = \frac{\varrho_n^2 - 2y_1\varrho_n - \varrho_n^2 + 2y_1\varrho_n - 1}{|\varrho_n - y| + |\varrho_n - 1 - y|} \geq \frac{R_n (\varrho_n - 1 - y_1)}{|\varrho_n - y| + |\varrho_n - 1 - y|}.

By the way $U_{m_0}$ was defined, the latter is always positive, and if in addition we require that

$$\varrho_{n-1} - y_1 \geq \frac{1}{100} |\varrho_{n-1} - y|,$$  \hspace{1cm} (4)

then

$$|\varrho_n - y| \geq |\varrho_n - 1 - y| + \frac{R_n (\varrho_n - 1 - y_1)}{|\varrho_n - y| + |\varrho_n - 1 - y|} \geq |\varrho_{n-1} - y| + \frac{R_n (\varrho_n - 1 - y_1)}{2 |\varrho_{n-1} - y| + R_n}

\geq |\varrho_{n-1} - y| + \frac{R_n (\varrho_{n-1} - y_1)}{3 |\varrho_{n-1} - y|} \geq |\varrho_{n-1} - y| + \frac{R_n}{300}.

This implies that for every measure $\mu$ in our sum, if

$$spt(\mu) \subset \{ y, |\varrho_{n-1} - y| \leq 100 (\varrho_{n-1} - y_1) \},$$

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then
\[-p_\mu(\varrho_{n-1}) - (-p_\mu(\varrho_n)) = \int \frac{1}{|\varrho_{n-1} - y|^{d-2}} d\mu(y) - \int \frac{1}{|\varrho_n - y|^{d-2}} d\mu(y)\]
\[= \int \frac{1}{|\varrho_n - y|^{d-2}} \left( \left( \frac{|\varrho_n - y|}{|\varrho_{n-1} - y|} \right)^{d-2} - 1 \right) d\mu(y)\]
\[\geq \int \frac{1}{|\varrho_n - y|^{d-2}} \left( 1 + \frac{R_n}{300 |\varrho_{n-1} - y|} \right)^{d-2} - 1 \right) d\mu(y)\]
\[\geq \frac{R_n(d-2)}{300} \int \frac{1}{|\varrho_n - y|^{d-1}} d\mu(y).\]

To estimate the latter we will use the same technique we used to bound $B_n$: Let
\[G := \left\{ I \subset U_{m_0}, \forall y \in I, y_1 \leq \varrho_{n-1} - \frac{m_0 \cdot R_n}{100} \right\}.\]

Every $y \in I \in G$ satisfies (4), and therefore
\[u(\varrho_{n-1}) - u(\varrho_n) = R_n^{d-2} \left( \sum_{I \in G} - p_\mu(\varrho_{n-1}) - (-p_\mu(\varrho_n)) + \sum_{I \notin G} - p_\mu(\varrho_{n-1}) - (-p_\mu(\varrho_n)) \right)\]
\[\geq R_n^{d-2} \sum_{I \in G} - p_\mu(\varrho_{n-1}) - (-p_\mu(\varrho_n)) \geq R_n^{d-1} \sum_{I \in G} \int I \frac{1}{|\varrho_n - y|^{d-1}} d\mu(y)\]
\[\geq \frac{1}{R_n} \sum_{I \in G} \int I \frac{1}{|\varrho_n - y|^{d-1}} |\varrho_n - \varrho_{n-1}|^{d-1} d\mu(y),\]

using a similar estimate to the one in (3). To conclude the proof, we will use Observation 4.3, while noting that for every $j \geq \frac{m_0}{2}$ at least $\frac{1}{10}$ of the cubes $I$ in the intersection between the annuli and the disk satisfy that $y_1 \leq \varrho_{n-1} - \frac{m_0 \cdot R_n}{100}$ for all $y \in I$. We conclude that
\[u(\varrho_{n-1}) - u(\varrho_n) \geq \cdots \geq \frac{1}{R_n} \sum_{j=\frac{m_0}{2}}^{m_0 - 1} \frac{R_n^d}{R_n^{d-1} \cdot j^{d-1}} \sim m_0.\]

Let $m_0 := \frac{1}{\varepsilon_n^{d-1}}$, then using the estimate above applied to every point on $\partial D_{n-1}$:
\[\omega(\varrho_{n-1}, E; D_n \setminus E) \geq v(\varrho_{n-1}) \geq \varepsilon_n^{\frac{d}{2} - 1} \Rightarrow \inf_{x \in \partial D_{n-1}} \omega(x, E; D_n \setminus E) \geq \varepsilon_n^{\frac{d}{2} - 1}.\]

We conclude the proof by using Corollary 3.3 with the functions $g_1(t) = 0$ and $g_2(t) = \varepsilon_n^{\frac{d}{2} - 1}(t)$, which are continuous and monotone non-increasing.
5 A subharmonic function- the proof of Theorem 1.1(B)

In this section we will construct a subharmonic function, for which the set \( \{ u \leq 0 \} \) is \((\varepsilon, R)\)-recurrent and

\[
\log M_u(\rho) := \log \left( \max_{|x|=\rho} u(x) \right) \lesssim \int_1^{\rho} \varphi(t) dt,
\]

thus proving Theorem 1.1(B). Before presenting our construction, we will show that all gauge functions have bounded derivatives:

**Claim 5.1** Let \( f \) be a gauge function. Then \( \frac{df}{dt} < \infty \).

**Proof.** Let \( f(t) = A \prod_{n=0}^{\infty} \log^{\alpha_n}[n](t) \), and assume without loss of generality that \( A = 1 \). Inductively using the chain rule, one can see that for every \( n \in \mathbb{N} \)

\[
\frac{d}{dt} \left( \log_{[n]}(t) \right) = \frac{1}{\log_{[n-1]}(t)} \cdot \frac{d}{dt} \left( \log_{[n-1]}(t) \right) = \cdots = \prod_{k=0}^{n-1} \frac{1}{\log_{[k]}(t)}.
\]

Let \( N \) be the maximal index so that \( \alpha_n \neq 0 \), and denote by

\[
\alpha^* := \max_{0 \leq n \leq N} |\alpha_n|.
\]

Then

\[
\frac{d}{dt} f = \frac{\alpha_0}{t^{1-\alpha_0}} \prod_{n=1}^{N} \log_{[n]}^{\alpha_n}(t) + \frac{\alpha_0}{t^{1-\alpha_0}} \sum_{n=1}^{N} \alpha_n \log_{[n]}^{\alpha_n-1}(t) \cdot \frac{d}{dt} \left( \log_{[n]}(t) \right) \prod_{\ell \geq 1, \ell \neq n} \log_{[\ell]}^{\alpha_{\ell}}(t).
\]

If \( \alpha_0 < 1 \), then the above is bounded by

\[
\frac{d}{dt} f \leq \cdots \leq \frac{1}{t^{1-\alpha_0}} \cdot (N + 1)\alpha^* \log^{N \cdot \alpha^*}(t),
\]

which is bounded.

If \( \alpha_0 = 1 \) then either \( f \) is linear, in which case its derivative is constant, or there exists \( m_0 \geq 1 \) a
minimal index so that $\alpha_{m_0} \neq 0$. As $f$ is a gauge function, $f(t) \leq t \Rightarrow \alpha_{m_0} < 0$, which implies

$$\frac{d}{dt} f = \cdots \leq \prod_{n=m_0+1}^{N} \frac{\log^\alpha_n(t)}{\log_{[n]}(t)} + \frac{1}{\log_{[m_0]}(t)} \sum_{n=m_0+1}^{N} \alpha_n \log_{[n]}^{\alpha_n-1}(t) \cdot \prod_{k=1}^{n-1} \frac{1}{\log_{[k]}(t)} \prod_{\ell \geq m_0+1} \log_{[\ell]}^\alpha(t),$$

which is bounded as well. \hfill \square

**Remark 5.2** In fact, we will only use the fact that $\frac{d}{dt} \left( \frac{1}{\varphi(t)} \right)$ is bounded in our proof.

### 5.1 The Construction

For every $k$ let $\Lambda_k$ be a collection of points on $2\rho_k S^{d-1}$, the sphere of radius $\rho_k$, satisfying

1. For every $\mu \neq \lambda \in \Lambda_k$, $B(\lambda, R(|\lambda|) + 1) \cap B(\mu, R(|\mu|) + 1) = \emptyset$.

2. $2\rho_k S^{d-1} \subset \bigcup_{\lambda \in \Lambda_k} B(\lambda, 4R_{\lambda})$.

We allow freedom to choose this collection, and it is clear that such a collection exists. Let $\Lambda := \bigcup_{k=1}^{\infty} \Lambda_k$ for $\Lambda_k$ defined above. Define the function:

$$v(x) = \exp \left( C \int_{1}^{\frac{1}{\varphi(t)}} \varphi(t) dt \right),$$

where $C > 0$ will be a large constant which depends on the functions $R(\cdot)$, $\varepsilon(\cdot)$ and the dimension.

Note that this function is radial, and

$$\Delta v(x) = \frac{\partial^2 v}{\partial r^2}(x) + \frac{d-1}{|x|} \cdot \frac{\partial v}{\partial r}(x) \geq C v(x) \varphi^2(|x|) \left( C - \frac{d}{dt} \left( \frac{1}{\varphi} \right) (|x|) \right).$$

Since $\frac{1}{\varphi}$ is a gauge function, $\frac{d}{dt} \left( \frac{1}{\varphi} \right)$ is bounded by Claim 5.1, and for large enough $C$ the function $v$ is subharmonic.

For every $\lambda \in \Lambda$ we let $\varepsilon_\lambda := \varepsilon(|\lambda|)$, $R_\lambda := R(|\lambda|)$, and $v_\lambda(\xi) = v(\lambda + R_\lambda \cdot \xi)$, and for every
\( x \in B(\lambda, R_\lambda) \) we associate \( \xi(x) \in B(0, 1) \) so that \( x = \lambda + R_\lambda \cdot \xi(x) \). Define the function

\[
\begin{cases}
  P_B v_\lambda(\xi(x)) + A_\lambda \cdot \tilde{k}_d \left( \frac{|x-\lambda|}{R_\lambda} \right), & \lambda \in \Lambda \text{ and } x \in B(\lambda, R_\lambda) \\
v(x), & \text{otherwise}
\end{cases}
\]

where \( B = B(0, 1) = \{ |x| < 1 \} \) and

\[
\tilde{k}_d := \begin{cases}
  k_d(x), & d = 2 \\
  1 + k_d(x), & d \geq 3
\end{cases}
\]

This function should satisfy two conditions:

1. For every \( \lambda \), for every \( x \in B(\lambda, R_\lambda \cdot \varepsilon_\lambda) \), \( u(x) \leq 0 \).

2. \( u \) is subharmonic.

**The first condition:** \( u|_{B(\lambda, \varepsilon_\lambda, R_\lambda)} \leq 0 \).

To address the first condition, let us use the maximum principle:

\[
\max_{|x-\lambda| = \rho} P_B v_\lambda(\xi(x)) + A_\lambda \tilde{k}_d \left( \frac{|x-\lambda|}{R_\lambda} \right) \leq \max_{|\xi| = 1} v_\lambda(\xi) + A_\lambda \tilde{k}_d \left( \frac{\rho}{R_\lambda} \right) = v(|\lambda| + R_\lambda) + A_\lambda \tilde{k}_d \left( \frac{\rho}{R_\lambda} \right) \leq 0
\]

\[
\iff \frac{\rho}{R_\lambda} \leq \left( \frac{\tilde{k}_d}{R_\lambda} \right)^{-1} \left( -v\left( |\lambda| + R_\lambda \right) A_\lambda \right).
\]

**The second condition:** \( u \) is subharmonic.

To address the subharmonicity condition, we note that by the way we defined the function \( u \), it is subharmonic on \( \mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} \partial B(\lambda, R_\lambda) \).

To show it is subharmonic in \( \mathbb{R}^d \) we will use a glueing argument, followed by Poisson-Jenssen’s formula:
Claim 5.3 Let $\Omega \subseteq \mathbb{R}^d$ be a domain, and let $\Omega_1 \subset \subset \Omega$ be a domain, so that $\partial \Omega_1$ is an orientable, smooth, $d-1$ manifold. Every function $u$ which is continuous on $\Omega$ and subharmonic on $\Omega_1 \cup \Omega \setminus \overline{\Omega_1}$, is subharmonic on $\Omega$ if on $\partial \Omega_1$ it satisfies

$$\frac{\partial u}{\partial n_1} \leq \frac{\partial u}{\partial n_2},$$

where $n_1$ is the outer normal to $\Omega_1$ along $\partial \Omega_1$ and $n_2$ is the outer normal to $\Omega \setminus \Omega_1$ along $\partial \Omega_1$.

We will use this claim with $\Omega := B \left( \lambda, R + \frac{1}{2} \right)$, and $\Omega_1 := B \left( \lambda, R \right)$. In this case, $\frac{\partial u}{\partial n_2}$ is known. To calculate $\frac{\partial u}{\partial n_1}$ we will use Poisson-Jenssen’s formula. The problem is then reduced to showing that for every $\xi \in S^{d-1}$:

$$\frac{\partial u}{\partial n_1} (\lambda + R\xi) = \lim_{r \to 1^-} \frac{v(\lambda + R\xi) - P_B v_\lambda(r\xi)}{1 - r} + A_\lambda \max \{1, d - 2\}$$

$$= \frac{\partial v}{\partial n} (\lambda + R\xi) + \lim_{r \to 1^-} \frac{G_B(r\xi)}{1 - r} + A_\lambda \max \{1, d - 2\}$$

$$= \frac{\partial v}{\partial n} (\lambda + R\xi) - \frac{\partial G_B}{\partial r} (\xi) + A_\lambda \max \{1, d - 2\} \leq \frac{\partial u}{\partial n_2} (\lambda + R\xi) = \frac{\partial v}{\partial n} (\lambda + R\xi),$$

where if $g_B$ denote Green’s function for the unit ball, $B = B(0, 1)$, then

$$G_B(\xi) := c_d \int_B g_B(\xi, y) \Delta v_\lambda(y) dm_d(y), m_d$$ is Lebegue’s measure in $\mathbb{R}^d$.

We conclude that in order to show subharmonicity, it is enough to restrict $A_\lambda$ so that

$$A_\lambda \max \{1, d - 2\} \leq \frac{\partial G_B}{\partial r} (\xi), \forall \xi \in S^{d-1}.$$

Let us begin by finding a lower bound for $\frac{\partial G_B}{\partial r} (\xi)$ to get an upper bound on $A_\lambda$. Because the collection $\left\{ \frac{\partial g_B}{\partial n} (\xi, \cdot), \xi \in S^{d-1} \right\}$ is uniformly integrable we may switch between the integral and
\[
\frac{\partial G_B}{\partial r}(\xi) = \frac{\partial}{\partial r} \left( c_d \int_B g_B(\xi, y) \Delta v_\lambda(y) dm_d(y) \right) = c_d \int_B \frac{\partial g_B(\cdot, y)}{\partial r}(\xi) \Delta v_\lambda(y) dm_d(y) \\
\geq c_d \int_B \frac{\partial g_B(\cdot, y)}{\partial r}(\xi) C v(\lambda + R_\lambda y) \varphi^2(|\lambda + R_\lambda y|) \left( C - \frac{d}{dt} \left( \frac{1}{\varphi} \right) (|\lambda + R_\lambda y|) \right) dm_d(y) \\
\geq \frac{C^2}{2} v(|\lambda| - R_\lambda) R_\lambda^2 \varphi^2(|\lambda| + R_\lambda) c_d \int_B \frac{\partial g_B(\cdot, y)}{\partial r}(\xi) dm_d(y) \\
\geq C^2 \cdot \frac{v(|\lambda| + R_\lambda)}{-k_d(\varepsilon_\lambda)} \cdot \exp \left( \frac{-\Theta(1)C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right).
\]

In order to get a maximal radius \( \rho_\lambda \) where \( u|_{B(\lambda, \rho_\lambda)} \leq 0 \), we need to choose \( A_\lambda \) to be as large as possible, that is

\[
A_\lambda \max\{d - 2, 1\} = A_1 \cdot \frac{C^2 v(|\lambda| + R_\lambda)}{-k_d(\varepsilon_\lambda)} \cdot \exp \left( \frac{-A_2 C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right),
\]

for appropriate constants \( A_1, A_2 \).

We then get that the maximal radius where \( u|_{B(\lambda, \rho_\lambda)} \leq 0 \) is

\[
\frac{\rho_\lambda}{R_\lambda} = \left( k_d \right)^{-1} \left( -\frac{v(|\lambda| + R_\lambda)}{A_\lambda} \right) \\
= \left( k_d \right)^{-1} \left( \exp \left( \frac{A_2 C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right) \frac{\max\{1, d - 2\} k_d(\varepsilon_\lambda)}{A_1 C^2} \right) \\
\geq \left( k_d \right)^{-1} \left( k_d(\varepsilon_\lambda) \left( \frac{\max\{1, d - 2\} \exp \left( \frac{A_2 C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right)}{A_1 C^2} - \frac{1}{k_d(\varepsilon_\lambda)} \right) \right).
\]

Now, remember that \( k_d, k_d^{-1} \) are monotone increasing functions, and \( k_d(t) < 0 \) for \( t < 1 \). Since without loss of generality \( \varepsilon(t) < 1 \) for all \( t \in (0, \infty) \), then \( k_d(\varepsilon_\lambda) < 0 \) and so

\[
\frac{\rho_\lambda}{R_\lambda} \geq \cdots \geq \left( k_d \right)^{-1} \left( k_d(\varepsilon_\lambda) \left( \frac{\max\{1, d - 2\} \exp \left( \frac{A_2 C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right)}{A_1 C^2} - \frac{1}{k_d(\varepsilon_\lambda)} \right) \right) \\
\geq \left( k_d \right)^{-1} \left( k_d(\varepsilon_\lambda) \cdot \frac{\max\{1, d - 2\} \exp \left( \frac{A_2 C}{\sqrt{-k_d(\varepsilon_\lambda)}} \right)}{A_1 C^2} \right) \geq \left( k_d \right)^{-1} (k_d(\varepsilon_\lambda)) = \varepsilon_\lambda.
\]

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as long as
\[
\frac{\max \{1, d-2\} \exp \left( \frac{A_2 C}{\sqrt{-k_d(\epsilon \lambda)}} \right)}{A_1 C^2} \leq 1,
\]
which holds for every $|\lambda|$ large enough, where how large $|\lambda|$ should be depends on $C$, and the constants $A_1$, $A_2$.

Let $\rho_*$ be so that for all $|\lambda| \geq \rho_*$ the above holds, and so that $v|_{\rho_* \cdot S^{d-1}}$ is radial (formally, $\min_k |\rho_* - \rho_k| - R_k > 0$). To take care of the set where $|\lambda| < \rho_*$, that is when $|\lambda|$ is too small, define
\[
v_1(x) := \begin{cases} 
\tilde{k}_d \left( \frac{|x|}{\rho_*} \right), & |x| \leq \rho_* \\
C_1 \cdot v(x) - C_2, & \text{otherwise}
\end{cases}
\]
where $C_1$ is chosen so that
\[
\frac{\partial v_1}{\partial r} \bigg|_{\{|x| = \rho_*\}} = C_1 \cdot \frac{\partial v}{\partial r} \bigg|_{\{|x| = \rho_*\}} \geq \max\{d-2, 1\} \frac{\partial}{\partial r} \left( k_d \left( \frac{|x|}{\rho_*} \right) \right) \bigg|_{\{|x| = \rho_0\}},
\]
while $C_2$ is chosen so that $v_1|_{\{|x| = \rho_*\}} = 0$, which is possible since on $\rho_* \cdot S^{d-1}$ the function $v$ is radial.

By using Claim 5.3 we conclude that $v_1$ is subharmonic, while for every $\lambda \in \Lambda$ the function $v_1$ satisfies that $v_1|_{B(\lambda, \epsilon \lambda \cdot R \lambda)} \leq 0$ as needed.

**Remark 5.4** Note that the set $Z_u := \{u \leq 0\}$ satisfies that if $m_d$ denotes the $d$-dimensional Lebesgue’s measure on $\mathbb{R}^d$ then
\[
\forall x \in \mathbb{R}^d, \frac{m_d(B(x, R(|x|)) \cap E)}{m_d(B(x, R(|x|)))} > m_d(B(x, \epsilon(|x|))).
\]

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