A new expression for the Moore–Penrose inverse of a class of matrices

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Abstract

An expression for the Moore–Penrose inverse of a matrix of the form $M = XNY$, where $X$ and $Y$ are nonsingular, has been recently established by Castro-González et al. [1, Theorem 2.2]. The expression plays an essential role in developing explicit expressions for the Moore–Penrose inverse of a two-by-two block matrix. In this paper, we present a new expression for the Moore–Penrose inverse of this class of matrices, which improves the result in [1].

Keywords: Moore–Penrose inverse; Matrix product; Orthogonal projector

1. Introduction

We first introduce some notations and concepts which are frequently used in the subsequent content. Let $\mathbb{N}^+$ and $\mathbb{C}$ denote the set of all positive integers and the field of complex numbers, respectively. Let $\mathbb{C}^{m \times n}$ be the set of all $m \times n$ complex matrices. The identity matrix of order $n$ is denoted by $I_n$ or $I$ when its size is clear in the context. For a matrix $A \in \mathbb{C}^{m \times n}$, $A^*$ denotes the conjugate transpose of $A$. We denote by $R(A)$ and $N(A)$ the range and null space of $A$, respectively, namely, $R(A) := \{ y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n \}$ and $N(A) := \{ x \in \mathbb{C}^n : Ax = 0 \}$. The Moore–Penrose inverse of $A$ is denoted by $A^\dagger$, which is defined as the unique matrix $Z \in \mathbb{C}^{n \times m}$ satisfying the following equations:

(a) $AZA = A$,  \hspace{1cm} (b) $ZA Z = Z$,  \hspace{1cm} (c) $(AZ)^* = AZ$,  \hspace{1cm} (d) $(ZA)^* = Z A$.

The symbols $E_A := I - AA^\dagger$ and $F_A := I - A^\dagger A$ stand for the orthogonal projectors onto $N(A^*)$ and $N(A)$, respectively. A matrix $Z \in \mathbb{C}^{n \times m}$ is called an inner inverse of $A$ if it satisfies the equality (a).

For a matrix $M \in \mathbb{C}^{m \times n}$ which can be decomposed as $M = XNY$, where $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{n \times n}$ are nonsingular, the equality $M^\dagger = Y^{-1}N^\dagger X^{-1}$ may fail. Several conditions validating $M^\dagger = Y^{-1}N^\dagger X^{-1}$ are presented in [2]. Recently, Castro-González et al. [1] obtained an explicit expression for $M^\dagger$, provided that $XE_N = E_N$ and $F_N Y = F_N$. More concretely, it is proved by Castro-González et al. [1, Theorem 2.2] that

$$M^\dagger = (I + L_0^*)(I + L_0 L_0^*)^{-1}Y^{-1}N^\dagger X^{-1}(I + R_0^*R_0)^{-1}(I + R_0^*), \tag{1.1}$$

where $R_0 := E_N(I - X^{-1})$ and $L_0 := (I - Y^{-1})F_N$. The expression (1.1) is a crucial result in [1], which can be exploited to establish explicit expressions for the Moore–Penrose inverse of a two-by-two block matrix.
Assume that the singular value decomposition (SVD) of \( N \in \mathbb{C}^{m \times n} \) is
\[
N = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*,
\]
where \( \Sigma \in \mathbb{C}^{r \times r} \) is a diagonal matrix with positive diagonal entries, \( r \) is the rank of \( N \),
and both \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) are unitary. Let \( X \in \mathbb{C}^{m \times m} \) and \( Y \in \mathbb{C}^{n \times n} \). We now give two assumptions \( A_1 \) and \( A_2 \) as follows:

\[
A_1 : X = U \begin{pmatrix} X_1 & 0 \\ X_2 & X_4 \end{pmatrix} U^*,
\]
where \( X_1 \in \mathbb{C}^{r \times r} \), \( X_2 \in \mathbb{C}^{(m-r) \times r} \), and \( X_4 \in \mathbb{C}^{(m-r) \times (m-r)} \);

\[
A_2 : Y = V \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_4 \end{pmatrix} V^*,
\]
where \( Y_1 \in \mathbb{C}^{r \times r} \), \( Y_3 \in \mathbb{C}^{r \times (n-r)} \), and \( Y_4 \in \mathbb{C}^{(n-r) \times (n-r)} \).

In this paper, we further investigate explicit expressions for the Moore–Penrose inverse of this class of matrices. A new expression under weakened conditions for \( M^\dagger \) is derived, which has enhanced the expression (1.1). More specifically, if the assumptions \( A_1 \) and \( A_2 \) are satisfied, then we have

\[
M^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger(I + R^*R)^{-1}(I + R^*),
\]
(2.1)

where \( R := XE_NX^{-1}(E_N - I) \) and \( L := (F_N - I)Y^{-1}F_NY \).

The rest of this paper is organized as follows. In Section 2, we first introduce a useful lemma which gives an explicit expression for the Moore–Penrose inverse of a two-by-two block matrix, and then give some specific conditions to validate \( A_1 \) and \( A_2 \). In Section 3, we present a new and improved expression (i.e., (1.2)) for \( M^\dagger \) based on the assumptions \( A_1 \) and \( A_2 \).

2. Preliminaries

In this section, we first introduce a useful lemma, which provides an explicit expression for the Moore–Penrose inverse of a two-by-two block matrix; see [3]. It is worth mentioning that some improved results of this lemma can be found in [1].

**Lemma 2.1.** Let \( M \) be a two-by-two block matrix as the form \( M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \). Assume that \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \), \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \), and \( D - BA^\dagger C = 0 \). Then \( M^\dagger \) can be given by

\[
M^\dagger = \begin{pmatrix} I \\ (A^\dagger C)^* \end{pmatrix} \Psi A^\dagger \Phi \begin{pmatrix} I & (BA^\dagger)^* \end{pmatrix},
\]

where \( \Phi = (I + (BA^\dagger)^*BA^\dagger)^{-1} \) and \( \Psi = (I + A^\dagger C(A^\dagger C)^*)^{-1} \).

Next, we give several specific conditions to guarantee the assumptions \( A_1 \) and \( A_2 \).
Lemma 2.2. Let \( N \in \mathbb{C}^{m \times n} \) have the singular value decomposition \( N = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* \), where \( \Sigma \in \mathbb{C}^{r \times r} \) is a diagonal matrix with positive diagonal entries, \( r \) is the rank of \( N \), and both \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) are unitary. Let \( X \in \mathbb{C}^{m \times m} \) be an arbitrary matrix. Suppose that one of the following conditions holds:

\( C_1 \): \( NN^*X \) is normal;
\( C_2 \): For any \( 0 \neq c_1 \in \mathbb{C} \), there exists \( k_1 \in \mathbb{N}^+ \) such that \( (NN^*X)^{k_1} = c_1 NN^* \);
\( C_3 \): For any \( 0 \neq c_2 \in \mathbb{C} \) and \( \ell \in \mathbb{N}^+ \), there exists \( k_2 \in \mathbb{N}^+ \) such that \( (NN^*X)^{k_2} = c_2 (NN^*)^\ell \);
\( C_4 \): \( XE_N \) is normal;
\( C_5 \): For any \( 0 \neq c_3 \in \mathbb{C} \) and \( X \in \mathbb{C}^{(m-r) \times (m-r)} \), there exists \( k_3 \in \mathbb{N}^+ \) such that \( (XE_N)^{k_3} = c_3 E_N \);
\( C_6 \): \( NN^*XE_N = 0 \);
\( C_7 \): There exists \( k_4 \in \mathbb{N}^+ \) such that \( (NN^*)^{k_4}XE_N = 0 \).

Then \( X \) must be of the form
\[
X = U \begin{pmatrix} X_1 & 0 \\ X_2 & X_4 \end{pmatrix} U^*,
\]
where \( X_1 \in \mathbb{C}^{r \times r}, X_2 \in \mathbb{C}^{(m-r) \times r} \), and \( X_4 \in \mathbb{C}^{(m-r) \times (m-r)} \).

**Proof.** Based on the SVD of \( N \), the expressions of \( N^* \) and \( E_N \) can be given by
\[
N^* = V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* \quad \text{and} \quad E_N = U \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} U^*.
\]

Partition \( U^*XU \) as \( U^*XU = \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix} \), where \( X_1 \in \mathbb{C}^{r \times r}, X_2 \in \mathbb{C}^{(m-r) \times r}, X_3 \in \mathbb{C}^{r \times (m-r)} \), and \( X_4 \in \mathbb{C}^{(m-r) \times (m-r)} \). Then \( X = U \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix} U^* \).

(i) The condition \( C_1 \) states that
\[
NN^*X = U \begin{pmatrix} \Sigma^2 X_1 & \Sigma^2 X_3 \\ 0 & 0 \end{pmatrix} U^*
\]
is normal, which yields that \( \Sigma^2 X_1 \) is normal and \( \Sigma^2 X_3 = 0 \). It follows from the non-singularity of \( \Sigma \) that \( X_3 = 0 \).

(ii) We have known that \( NN^*X = U \begin{pmatrix} \Sigma^2 X_1 & \Sigma^2 X_3 \\ 0 & 0 \end{pmatrix} U^* \). Then, for any \( k_1 \in \mathbb{N}^+ \), we have
\[
(NN^*X)^{k_1} = U \begin{pmatrix} (\Sigma^2 X_1)^{k_1} & (\Sigma^2 X_3)^{k_1} \Sigma^2 X_3 \\ 0 & 0 \end{pmatrix} U^*.
\]
In addition, it is easy to see that

\[ c_1 NN^\dagger = U \begin{pmatrix} c_1 I & 0 \\ 0 & 0 \end{pmatrix} U^*. \]

Hence, \( C_2 \) implies that \((\Sigma^2 X_1)^{k_1} = c_1 I \) and \((\Sigma^2 X_1)^{k_1-1} \Sigma^2 X_3 = 0 \). Due to the facts that \( c_1 \neq 0 \) and \( \Sigma \) is nonsingular, it follows that \( X_1 \) is nonsingular and \( X_3 = 0 \).

(iii) Direct calculation yields

\[(NN^* X)^{k_2} = U \begin{pmatrix} \Sigma^2 X_1^{k_2} & (\Sigma^2 X_1)^{k_2-1} \Sigma^2 X_3 \\ 0 & 0 \end{pmatrix} U^*,\]

\[c_2 (NN^*)^\ell = U \begin{pmatrix} c_2 \Sigma^{2\ell} & 0 \\ 0 & 0 \end{pmatrix} U^* .\]

Because \( c_2 \neq 0 \) and \( \Sigma \) is nonsingular, we deduce from \( C_3 \) that \( \Sigma^2 X_1 \) is nonsingular and \((\Sigma^2 X_1)^{k_2-1} \Sigma^2 X_3 = 0 \). Hence, \( X_3 = 0 \).

(iv) Straightforward calculation shows

\[ XE_N = U \begin{pmatrix} 0 & X_3 \\ 0 & X_4 \end{pmatrix} U^*. \]

If \( XE_N \) is normal, then we get that \( X_4 \) is normal and \( X_3 = 0 \).

(v) Direct computation yields

\[(XE_N)^{k_3} = U \begin{pmatrix} 0 & X_3 X_4^{k_3-1} \\ 0 & X_4 \end{pmatrix} U^*.\]

It follows from \( C_5 \) that \( X_4^{k_3} = c_3 I \) and \( X_3 X_4^{k_3-1} = 0 \). By \( c_3 \neq 0 \), we derive that \( X_4 \) is nonsingular. Hence, we obtain from \( X_3 X_4^{k_3-1} = 0 \) that \( X_3 = 0 \).

(vi) It is easy to compute that

\[ NN^\dagger XE_N = U \begin{pmatrix} 0 & X_3 \\ 0 & 0 \end{pmatrix} U^*. \]

Therefore, \( NN^\dagger XE_N = 0 \) if and only if \( X_3 = 0 \).

(vii) Direct calculation yields

\[(NN^*)^{k_4} XE_N = U \begin{pmatrix} 0 & \Sigma^{2k_4} X_3 \\ 0 & 0 \end{pmatrix} U^*.\]

Due to the fact that \( \Sigma \) is nonsingular, it follows that \((NN^*)^{k_4} XE_N = 0 \) is equivalent to \( X_3 = 0 \). Consequently, if one of the conditions \( C_1 - C_7 \) holds, then \( X \) must be of the from

\[ X = U \begin{pmatrix} X_1 & 0 \\ X_2 & X_4 \end{pmatrix} U^*. \]
which completes the proof.

Analogously, we can prove the following lemma. Its detailed proof is omitted due to limited space.

**Lemma 2.3.** Let $Y \in \mathbb{C}^{n \times n}$ and let $N \in \mathbb{C}^{m \times n}$ be the same as in Lemma 2.2. Assume that one of the following conditions holds:

- $C'_1$: $YN^*N$ is normal;
- $C'_2$: For any $0 \neq c'_1 \in \mathbb{C}$, there exists $k'_1 \in \mathbb{N}^+$ such that $(YN^*N)^{k'_1} = c'_1 N^* N$;
- $C'_3$: For any $0 \neq c'_2 \in \mathbb{C}$ and $\ell' \in \mathbb{N}^+$, there exists $k'_2 \in \mathbb{N}^+$ such that $(YN^*N)^{k'_2} = c'_2 (N^*N)^{\ell'}$;
- $C'_4$: $FN$ is normal;
- $C'_5$: For any $0 \neq c'_3 \in \mathbb{C}$, there exists $k'_3 \in \mathbb{N}^+$ such that $(FNY)^{k'_3} = c'_3 FN$;
- $C'_6$: $FNYN^*N = 0$;
- $C'_7$: There exists $k'_4 \in \mathbb{N}^+$ such that $FNY(N^*N)^{k'_4} = 0$.

Then $Y$ must be of the form

$$Y = V \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_4 \end{pmatrix} V^*,$$

where $Y_1 \in \mathbb{C}^{r \times r}$, $Y_3 \in \mathbb{C}^{r \times (n-r)}$, and $Y_4 \in \mathbb{C}^{(n-r) \times (n-r)}$.

**Remark 2.4.** Notice that Lemma 2.2 (resp., Lemma 2.3) does not need the non-singularity of $X$ (resp., $Y$). In addition, the reader can give other conditions to ensure that $A_1$ and $A_2$ hold.

### 3. Main results

In order to prove our main result, we first consider explicit expressions for $(XN)^\dagger$ and $(NY)^\dagger$. The following theorem provides two applicable formulas for $M_1^\dagger$ and $M_2^\dagger$, where $M_1 = XN$ and $M_2 = NY$.

**Theorem 3.1.** Let $N \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{m \times m}$, $Y \in \mathbb{C}^{n \times n}$, $M_1 = XN$, and $M_2 = NY$. Suppose that $X$ and $Y$ are nonsingular.

1. If the assumption $A_1$ holds, then
   $$M_1^\dagger = N^\dagger X^{-1} NN^\dagger (I + R^*R)^{-1}(I + R^*),$$
   where $R = XE_NX^{-1}(E_N - I)$.

2. If the assumption $A_2$ holds, then
   $$M_2^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger NY^{-1}N^\dagger,$$
   where $L = (F_N - I)Y^{-1}F_N Y$.  


Proof. (1) The assumption $A_1$ reads $X = U \begin{pmatrix} X_1 & 0 \\ X_2 & X_4 \end{pmatrix} U^*$, where $X_1 \in \mathbb{C}^{r \times r}$ and $r$ is the rank of $N$. It follows from the non-singularity of $X$ that both $X_1 \in \mathbb{C}^{r \times r}$ and $X_4 \in \mathbb{C}^{(m-r) \times (m-r)}$ are nonsingular. We define $R := XE_NX^{-1}(E_N - I)$. By simple computation, we can get

$$R = U \begin{pmatrix} 0 & 0 \\ X_2X_1^{-1} & 0 \end{pmatrix} U^* = U \begin{pmatrix} 0 & 0 \\ G & 0 \end{pmatrix} U^*,$$

where $G := X_2X_1^{-1}$. Because $U$ and $V$ are unitary matrices and

$$M_1 = XN = U \begin{pmatrix} X_1\Sigma & 0 \\ X_2 \Sigma & 0 \end{pmatrix} V^*,$$

we obtain

$$M_1^\dagger = V \begin{pmatrix} X_1\Sigma & 0 \\ X_2 \Sigma & 0 \end{pmatrix}^\dagger U^*.$$

Note that $X_1\Sigma$ is nonsingular. Using Lemma 2.1, we obtain

$$\begin{pmatrix} X_1\Sigma & 0 \\ X_2 \Sigma & 0 \end{pmatrix}^\dagger = \begin{pmatrix} I \\ 0 \end{pmatrix} \Sigma^{-1}X_1^{-1}(I + G^*G)^{-1} \begin{pmatrix} I & G^* \end{pmatrix}.$$

Hence,

$$M_1^\dagger = V \begin{pmatrix} I \\ 0 \end{pmatrix} \Sigma^{-1}X_1^{-1}(I + G^*G)^{-1} \begin{pmatrix} I & G^* \end{pmatrix} U^*.$$

Straightforward computation yields

$$N^\dagger X^{-1} = V \begin{pmatrix} I \\ 0 \end{pmatrix} \Sigma^{-1}X_1^{-1} \begin{pmatrix} I & 0 \end{pmatrix} U^*,$$

$$NN^\dagger(I + R^*R)^{-1}(I + R^*) = U \begin{pmatrix} I \\ 0 \end{pmatrix} (I + G^*G)^{-1} \begin{pmatrix} I & G^* \end{pmatrix} U^*.$$

It can be easily seen that $M_1^\dagger = N^\dagger X^{-1}NN^\dagger(I + R^*R)^{-1}(I + R^*)$ holds.

(2) Applying the formula (3.1) to the matrix $Y^*N^*$, we obtain

$$(M_2^\dagger)^* = (N^*)^\dagger(Y^*)^{-1}N^*(N^\dagger)^\dagger(I + \hat{R}^*\hat{R})^{-1}(I + \hat{R}^*),$$

where

$$\hat{R} = Y^*E_N^*(Y^*)^{-1}(E_N^* - I) = Y^*(F_N^*)^{(Y^*)^*(F_N^* - I)}.$$

We define $L := (F_N^* - I)Y^{-1}F_NY$. Then,

$$(M_2^\dagger)^* = (M_2^\dagger)^\dagger = (N^\dagger)^*(Y^{-1})^*N^*(N^\dagger)^*(I + LL^*)^{-1}(I + L).$$

Therefore, we drive that $M_2^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger NY^{-1}N^\dagger$. \qed
Using Theorem 3.1, we can easily obtain the following expressions for the orthogonal projectors onto $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2^*)$.

**Corollary 3.2.** Under the same conditions as in Theorem 3.1.

1. If the assumption $A_1$ is valid, then
   \[ M_1M_1^\dagger = (I + R)NN^\dagger(I + R^*R)^{-1}(I + R^*). \]  
   (3.2)

2. If the assumption $A_2$ is valid, then
   \[ M_2^\dagger M_2 = (I + L^*)(I + LL^*)^{-1}N^\dagger N(I + L). \]  
   (3.3)

**Proof.** According to the equality (3.1), it follows that
\[ M_1M_1^\dagger = XNN^\dagger X^{-1}NN^\dagger(I + R^*R)^{-1}(I + R^*). \]  
(3.4)

Notice that
\[ (I + R)NN^\dagger = \left(I - NN^\dagger + XNN^\dagger X^{-1}NN^\dagger\right)NN^\dagger = XNN^\dagger X^{-1}NN^\dagger. \]  
(3.5)

Inserting (3.5) into (3.4) gives $M_1M_1^\dagger = (I + R)NN^\dagger(I + R^*R)^{-1}(I + R^*)$. Similarly, we can prove the equality (3.3). \qed

Based on the expressions (3.2) and (3.3) for orthogonal projectors $M_1M_1^\dagger$ and $M_2^\dagger M_2$, we can establish the following main result.

**Theorem 3.3.** Let $N \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{m \times m}$, $Y \in \mathbb{C}^{n \times n}$, and $M = XNY$. Assume that $X$ and $Y$ are nonsingular. If the assumptions $A_1$ and $A_2$ are satisfied, then
\[ M^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger(I + R^*R)^{-1}(I + R^*), \]
where $R = XENX^{-1}(EN - I)$ and $L = (FN - I)Y^{-1}FY$.  

**Proof.** Note that $Y^{-1}N^\dagger X^{-1}$ is an inner inverse of $M$. Then we have
\[ M^\dagger = M^\dagger M(Y^{-1}N^\dagger X^{-1})MM^\dagger. \]

Let $M_1 = XN$ and $M_2 = NY$. We claim that $MM^\dagger = M_1M_1^\dagger$ and $M^\dagger M = M_2^\dagger M_2$. In fact, it is clear that $MM^\dagger$ is the orthogonal projector onto $\mathcal{R}(M)$. Because $Y$ is nonsingular and $M = M_1Y$, it follows that $\mathcal{R}(M) = \mathcal{R}(M_1)$. Hence, $MM^\dagger$ is also an orthogonal projector onto $\mathcal{R}(M_1)$. Using the uniqueness of orthogonal projectors, we get that $MM^\dagger = M_1M_1^\dagger$. Similarly, we can verify that $M^\dagger M = M_2^\dagger M_2$. Therefore, we have
\[ M^\dagger = M_2^\dagger M_2(Y^{-1}N^\dagger X^{-1})M_1M_1^\dagger. \]
Under the assumptions of this theorem, by Corollary 3.2, we have

\[ M^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger N(I + L)Y^{-1}N^\dagger X^{-1}(I + R)NN^\dagger(I + R^*R)^{-1}(I + R^*). \]

Using \( R = XE_NX^{-1}(E_N - I) \) and \( L = (F_N - I)Y^{-1}F_NY \), we obtain

\[
(I + L)Y^{-1}N^\dagger X^{-1}(I + R) = Y^{-1}N^\dagger X^{-1} + Y^{-1}N^\dagger X^{-1}R + LY^{-1}N^\dagger X^{-1} + LY^{-1}N^\dagger X^{-1}R
= Y^{-1}N^\dagger X^{-1},
\]

where we have applied the facts that \( N^\dagger E_N = 0 \) and \( F_NN^\dagger = 0 \). Consequently, we infer that

\[ M^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger(I + R^*R)^{-1}(I + R^*). \]

This completes the proof. \( \square \)

**Corollary 3.4.** Under the same conditions as in Theorem 3.3. If both \( XE_N \) and \( F_NY \) are Hermitian, then

\[ M^\dagger = (I + L^*)(I + LL^*)^{-1}Y^{-1}N^\dagger X^{-1}(I + R^*R)^{-1}(I + R^*). \]  \hfill (3.6)

**Proof.** Because \( XE_N \) and \( F_NY \) are Hermitian, by Lemmas 2.2 and 2.3, the assumptions \( A_1 \) and \( A_2 \) are clearly satisfied. An application of Theorem 3.3 gives

\[ M^\dagger = (I + L^*)(I + LL^*)^{-1}N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger(I + R^*R)^{-1}(I + R^*). \]  \hfill (3.7)

Due to both \( XE_N \) and \( F_NY \) are Hermitian, it follows that \( XE_N = E_NX^* \) and \( F_NY = Y^*F_N \). Then, \( E_N(X^*)^{-1} = X^{-1}E_N \) and \( (Y^*)^{-1}F_N = F_NY^{-1} \). Notice that

\[
N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger = Y^{-1}N^\dagger X^{-1} - F_NY^{-1}N^\dagger X^{-1} - Y^{-1}N^\dagger X^{-1}E_N + F_NY^{-1}N^\dagger X^{-1}E_N.
\]

Using \( (Y^*)^{-1}F_N = F_NY^{-1} \) and \( F_NN^\dagger = 0 \), we can derive that \( F_NY^{-1}N^\dagger X^{-1} = 0 \). By \( E_N(X^*)^{-1} = X^{-1}E_N \) and \( N^\dagger E_N = 0 \), we have \( Y^{-1}N^\dagger X^{-1}E_N = 0 \). Consequently,

\[ N^\dagger N(Y^{-1}N^\dagger X^{-1})NN^\dagger = Y^{-1}N^\dagger X^{-1}. \]  \hfill (3.8)

By substituting (3.8) into (3.7), we obtain the formula (3.6). \( \square \)

**Remark 3.5.** If \( XE_N = E_N \) and \( F_NY = F_N \), the conditions in Corollary 3.4 are obviously satisfied because \( E_N \) and \( F_N \) are orthogonal projectors. In this case,

\[
R = XE_NX^{-1}(E_N - I) = E_N(X^{-1}E_N - X^{-1}) = E_N(E_N - X^{-1}) = E_N(I - X^{-1}) = R_0,
L = (F_N - I)Y^{-1}F_NY = (F_NY^{-1} - Y^{-1})F_N = (F_N - Y^{-1})F_N = (I - Y^{-1})F_N = L_0,
\]

where \( R_0 \) and \( L_0 \) are defined as in expression (1.1). Therefore, Corollary 3.4 has extended the expression (1.1).
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