Feasibility of random basis function approximators for modeling and control

Ivan Yu. Tyukin and Danil V. Prokhorov

Abstract—We discuss the role of random basis function approximators in modeling and control. We analyze the published work on random basis function approximators and demonstrate that their favorable error rate of convergence $O(1/n)$ is guaranteed only with very substantial computational resources. We also discuss implications of our analysis for applications of neural networks in modeling and control.

I. INTRODUCTION

Efficient modeling and control of complex systems in the presence of uncertainties is important for modern engineering. This is especially true in the domain of intelligent systems that are designed to operate in uncertain environments. Uncertainties in such systems are usually quantitative and their homogenous structure due to efficiency of approximators in modeling and control, crucially simplifying analysis of properties of the system.

In the domain of modeling and control of intelligent systems the multilayer perceptrons (MLP) and radial basis functions (RBF) networks are popular function approximators [3]. The MLP uses a basis in the form of the sigmoids with global support. For one-hidden layer MLP, its output is determined by

$$f_n(x) = \sum_{i=1}^{n} c_i \frac{1}{1 + e^{(w_i^T x + b_i)}}$$  (1)

Typically, both nonlinear ($w_i$ and $b_i$) and linear ($c_i$) parameters, or weights, are subject to training on data specific to the problem at hand (full network training).

The RBF networks use a basis in the form of the Gaussians with local (but not compact) support:

$$f_n(x) = \sum_{i=1}^{n} c_i e^{-\|w_i^T x + b_i\|^2}$$  (2)

Though all parameters may be trained in principle, typically only linear weights $c_i$ of the RBF network are trained. The locations and the widths of the Gaussians are usually set on a uniform or nonuniform grid covering the operating domain of the system.

The popularity of approximators (1), (2) is not only due to their approximation capabilities (see e.g., [2], [4], [16]) and their homogenous structure but also due to efficiency of approximators (1), (2) in high dimensions. In particular, if all parameters $w_i$, $c_i$, $b_i$ are allowed to vary, the rate of convergence of the approximation error of a target function $f \in C^{[0,1]^d}$ as a function of $n$ (the number of elements in the network) is shown to be independent of the input dimension $d$ [14], [1]. Furthermore, the achievable rate of convergence of the $L_2$-norm of $f(x) - f_n(x)$ is shown to be of order $O(1/n)$.

Despite these advantageous features of approximators (1), (2), viz. favorable independence of the convergence rates on the input dimension of the function to be approximated, the issue is how to achieve the convergence rate of order $O(1/n)$ in practice. Even though [14], [1] offer a constructive procedure for optimal selection of basis functions, each step of these procedures involves a nonlinear optimization routine searching for the best possible values of $w_i$, $b_i$ (see Section II for details). It is also shown in [1] that if only linear parameters of (1), (2) are adjusted the approximation error cannot be made smaller than $1/n^{2/d}$ uniformly for functions satisfying the same smoothness constraints.

The necessity to adjust nonlinear parameters of (1), (2), restricts practical application of these models to those problems in which such optimization is feasible. Adaptive control with nonlinearly parameterized models remains a challenging issue; see e.g., [17], [19], [18], [15].

Though published quite a while ago, the paper by Igelhik and Pao [5] (see also comments [6]) has recently received numerous citations in a variety of intelligent control publications; see, e.g., [8]–[13]. The paper advocates the use of random basis in the MLP (1) and RBF (2) networks. That is, the nonlinear parameters $w_i$ and $b_i$ are to be set randomly at the initialization, rather than through training. The only trainable parameters are those which enter the network equation linearly ($c_i$).

The paper [5] provides mathematical justification to the use of linear-in-parameters function approximators for modeling and control, crucially simplifying analysis of properties of the closed-loop control system featuring such approximators. While analysis simplification is attractive, it entails a number of issues which are important to consider whenever

Ivan Tyukin is with Department of Mathematics, University of Leicester, UK, Department of Automation and Control Processes, St. Petersburg State University of Electrical Engineering, Russia, Perceptual Dynamics Lab, RIKEN BSI, Japan i.tyukin@leicester.ac.uk

Danil Prokhorov is with Toyota Research Institute NA, Ann Arbor, MI 48105, USA dvprokhorov@gmail.com
planning to apply such random basis function approximators in practice. We show that the rate of convergence of order $O(1/n)$ for such approximators is achievable only for large $n$, and it is probabilistic in nature. The latter feature may require introduction of a supervisory mechanism in the control system to re-initialize the network if the required accuracy is not met.

The paper is organized as follows. In Section II we analyze the reasoning in [5] and compare these results with [14], [1]. We show that, although these results may seem inconsistent (i.e., the lower approximation bound $1/n^{2/d}$ derived in [1] for any linear-in-parameter approximator vs. the rate of convergence of order $O(1/n)$ and independent of $d$ in [5]), they are derived for different asymptotics (for every $n$ in [14], [1] vs. for large $n$ in [5]) and use different convergence criteria (deterministic in [14], [1] vs. statistical in [5]). Implications of our analysis are illustrated in Section III with a simple example, followed by a discussion in Section IV. Section V concludes the paper.

II. FUNCTION APPROXIMATION CONCEPTS

In this section we review and compare two results for function approximation with neural networks. The first result is the so-called greedy approximation upon which the famous Barron’s construction is based [1]. In this framework a function is approximated by a sequence of linear combinations of basis functions. Each basis function is to satisfy certain optimality condition, and as a result the overall rate of convergence is optimized as well.

The second result is the random function approximation also known as the Random Vector Functional-Link (RVFL) network [5] in which the basis functions are randomly chosen, and only their linear parameters are optimized. Both results enjoy the convergence rates that do not depend on the input dimension $d$ of the target functions. However, there are differences important for practical use of these results.

First, as we show below the number of practically required approximation elements (network size) that guarantees given approximation quality differ substantially. Second, the quality criteria are also different: in the framework of greedy approximation this is merely the $L_2$-norm which is a deterministic functional, whereas in the RVFL framework the criterion is statistical.

A. Approximation problem

Consider the following class of problems. Let $f : [0, 1]^d \subset \mathbb{R}^d \to \mathbb{R}$ be a continuous function, and

$$\|f\|^2 = \langle f, f \rangle = \int_{[0,1]^d} f(x)f(x)dx,$$

be the $L_2$-norm of $f$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\|g(\cdot)\| \leq M, \ M \in \mathbb{R}_{>0},$$

and that

$$f \in \text{convex hull} \ \{g(w^T x + b)\}, \ w \in \mathbb{R}^d, \ b \in \mathbb{R},$$

In other words there is a sequence of $w_i, b_i$, and $c_i$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i g(w_i^T x + b_i), \ \sum_{i=1}^{\infty} c_i = 1$$

Let

$$f_n(x) = \sum_{i=1}^{n} c_i g(w_i^T x + b_i)$$

be a superposition of functions $g(w_i^T x + b_i)$. The question is how many elements do we need to pick in (3) to assure that the approximation error does not exceed certain specified value?

B. Greedy approximation and Jones Lemma

In order to answer the question above one needs first to determine the error of approximation. It is natural for functions from $L_2$ to define the approximation error as follows:

$$e_n = \|f_n - f\|$$

The classical Jones iteration [14] (refined later by Barron [1]) provides us with the following estimate of achievable convergence rate:

$$e_n^2 \leq \frac{M'^2 e_0^2}{n^2 + M''^2}, \ M' \in \mathbb{R}_{>0},$$

$$M' > \sup_{g} \|g\| \|f\|.$$  

The rate of convergence depends on $d$ only through the $L_2$-norms of $f_0$, $g$, and $f$. The iteration itself is deterministic and can be described as follows:

$$f_{n+1} = (1 - \alpha_n)f_n + \alpha_n g_n$$

$$\alpha_n = \frac{e_n^2}{M''^2 + e_n^2}, \ M'' > M'$$

where $g_n$ is chosen such that the following condition holds

$$\langle f_n - f, g_n - f \rangle < \frac{(M''^2 - (M')^2)e_n^2}{2(M'')^2},$$

This choice is always possible (see [14] for details).

According to (5) the rate of convergence of such approximators is estimated as

$$\|e_n\|^2 = O(1/n).$$

This convergence estimate is guaranteed because it is the upper bound for the approximation error at the $n$th step of iteration (6).

C. Approximation with randomly chosen basis functions

We now turn our attention to the result in [5]. In this approximator the original function $f(\cdot)$ is assumed to have the following integral representation

$$f(x) = \lim_{\alpha \to \infty} \lim_{\Omega \to \infty} \int_{\mathbb{R}^d} F_{\alpha,\Omega}(\omega)g(\alpha w^T x + b)d\omega,$$

We keep the original notation of [5] which uses both $\omega$ and $w$ for the sake of consistency.
where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a non-trivial function from \( L_2 \):

\[
0 < \int_{\mathbb{R}} g^2(s) ds < \infty,
\]

where \( \omega = (y, w, u) \in \mathbb{R}^d \times \mathbb{R}^d \times [-\Omega, \Omega], \Omega \in \mathbb{R}_{>0} \), \( W^d = [-2d\Omega; 2d\Omega] \times I^d \times V^d \), \( V^d = [0; \Omega] \times [-\Omega; \Omega]^{d-1}, \) \( b = -\langle \alpha u^T y + u \rangle \) and

\[
F_{\alpha,\Omega}(\omega) \sim \frac{\alpha \prod_{i=1}^{d} w_i}{\Omega^{2d-1}} f(y).
\]

See [5], [6] for more detailed description. Function \( g(\cdot) \) induces a parameterized basis. Indeed if we were to take integral (8) in quadratures for sufficiently large values of \( \alpha \) and \( \Omega \), we would then express \( f(x) \) by the following sums of parameterized \( g(\alpha u^T x + b) \) [5]:

\[
f_n(x) \approx \sum c_i g(\alpha w_i^T x + b_i), \quad b_i = -\langle u_i^T y_i + u_i \rangle \tag{9}
\]

The summation in (9) is taken over points \( \omega_i \) in \( W^d \), and \( c_i \) are weighting coefficients. Variables \( \alpha \) in (8) and \( \alpha_n \) in (6) play different roles in each approximations schemes. In (6) the value of \( \alpha_n \) is set to ensure that the approximation error is decreasing with every iteration, and in (8) it stands for a scaling factor of random sampling.

The main idea of [5] is to approximate integral representation (8) of \( f(x) \) using the Monte-Carlo integration method as

\[
\int_{W^d} f(x) dx \sim \frac{4\Omega^d}{n} \lim_{\alpha \rightarrow \infty} \lim_{\Omega \rightarrow \infty} \sum_{k=1}^{n} F_{\alpha,\Omega}(\omega_k) g(\alpha u_k^T x + b_k)
\]

\[
= \frac{4}{n} \lim_{\alpha \rightarrow \infty} \lim_{\Omega \rightarrow \infty} \sum_{k=1}^{n} c_k g(\alpha \omega_k^T x + b_k) \tag{10}
\]

\[
= f_{n,\alpha,\Omega}(x),
\]

where the coefficients \( c_k(\alpha, \omega_k) \) are defined as

\[
c_k(\alpha, \omega_k) = \frac{\alpha \prod_{i=1}^{d} w_i}{\Omega^{2d-1}} f(y_k) \tag{11}
\]

and \( \omega_k = (y_k, w_k, u_k) \) are randomly sampled in \( W^d \) (domain of parameters, i.e., weights and biases of the network).

When the number of samples, \( n \), i.e., the network size, is large, then the expectation \( E_\omega(n, x) \)

\[
E_\omega(n, x) = f(x) - \frac{4}{n} \sum_{k=1}^{n} c_k g(\alpha \omega_k^T x + b_k)
\]

converges to zero for large \( n \) (Theorem 1 in [5]):

\[
\lim_{n \rightarrow \infty} E_\omega(n, x) = 0.
\]

The advantage of the Monte-Carlo integration, and hence the approximation techniques that are based upon this method, is its order of convergence for large \( n \). It is known that if \( W^d \) is bounded (i.e., its volume is bounded) then the variance of the estimate (10) is bounded pointwise from above:

\[
\text{Var}_\omega(n, x) = \lim_{n \rightarrow \infty} \left| W^d \right| \frac{\sigma_f^2(x)}{n} \tag{12}
\]

where

\[
\sigma_f^2(x) = \int_{W^d} (c_k(\alpha, \omega) g(\alpha u_k^T x + b_k) - f(x))^2 d\omega.
\]

In this sense the order of Monte-Carlo approximation for large number of processing elements of the approximator (network nodes) \( n \) may be made similar to that of the greedy approximation.

**D. Comparison**

There are, however, important points that make this method different from the greedy approximation:

- Approximation “error” (12) is statistical, whereas the approximation error (11) is deterministic. This means that \( f_{n,\omega,\Omega}(x) \) is not at all guaranteed to be close to \( f(x) \) for every randomly chosen set of length \( n \). We can, however, conclude that for sufficiently small \( \gamma = \sigma_f^2(x)/([N \varepsilon]^2) \) the probability that \( f_{n,\omega,\Omega}(x) \) is close to \( f(x) \) approaches 1 (from the Chebyshev inequality):

\[
P \left( \left| \frac{1}{n} \sum_{k=1}^{n} c_k g(\alpha \omega_k^T x + b_k) - f(x) \right| < \varepsilon \right) \geq 1 - \gamma
\]

- For the Monte-Carlo based scheme (10)–(12) to converge, one needs to ensure that \( W^d \) is bounded. This, however, conflicts with the requirement that \( \Omega \rightarrow \infty \) (8). Hence the class of functions to which the scheme applies is restricted. In order to mitigate this restriction, it is proposed to consider functions \( g(\cdot) \) with compact support, and for this class of functions dimension-independent (statistical) rate of convergence (12) is guaranteed.

- The relatively fast rate of convergence (12) is guaranteed only for large \( n \).

These points are summarized in Table 1.

**III. Example**

In order to illustrate the main difference between greedy and RVFL approximators, we consider the following example in which a simple function is approximated by both methods, greedy approximation (5–6) and approximation based on...
the Monte-Carlo integration \([8]–[12]\). Let \(f(x)\) be defined as follows:
\[
f(x) = 0.2e^{-10(x-4)^2} + 0.5e^{-(80x-40)^2} + 0.3e^{-(8x-20)^2}
\]
The function \(f(x)\) is shown in Fig. 1 (top panel). Clearly, \(f(\cdot)\) belongs to the convex hull of \(G\), and hence to its closure.

First, we implemented greedy approximation \([3]–[6]\) in which we searched for \(g_n\) in the following set of functions
\[
G = \{e^{-(w^T x + b)^2}\},
\]
where \(w \in [0, 200], b \in [-100, 0]\). The procedure for constructing \(f_n\) was as follows. Assuming \(f_0(x) = 0\), \(e_0 = -f\) we started with searching for \(w_1, b_1\) such that
\[
\begin{align*}
(0 - f(x), g(w_1 x + b) - f(x)) &= \\
- (f(x), g(w_1 x + b) + \|f(x)\|^2) < \varepsilon.
\end{align*}
\]
where \(\varepsilon\) was set to be small (\(\varepsilon = 10^{-6}\) in our case). When searching for a solution of \([13]\) (which exists because the function \(f\) is in the convex hull of \(G\) \([14]\)), we did not utilize any specific optimization routine. We sampled the space of parameters \(w_i, b_i\) randomly and picked the first values of \(w_i, b_i\) which satisfy \([13]\). Integral \([13]\) was evaluated in quadratures over a uniform grid of 1000 points in \([0, 1]\).

The values of \(\alpha_1\) and the function \(f_1\) were chosen in accordance with \([6]\) with \(M'' = 2, M' = 1.5\) (these values are chosen to assure \(M'' > M' > \sup_g \|g\| + \|f\|\)). The iteration was repeated, resulting in the following sequence of functions
\[
f_n(x) = \sum_{i=1}^{n} c_i g(w_i^T x + b_i),
\]
\[
c_i = \alpha_i(1 - \alpha_{i+1})(1 - \alpha_{i+2}) \cdots (1 - \alpha_n)
\]
Evolution of the normalized approximation error
\[
\bar{e}_n = \frac{e_n^2}{\|f\|^2} = \frac{\|f_n - f\|^2}{\|f\|^2},
\]
for 100 trials is shown in Fig. 1 (middle panel). Each trial consisted of 100 iterations \([3]–[6]\), thus leading to the networks of 100 elements at the 100th step. We observe that the values of \(\bar{e}_n\) monotonically decrease as \(O(1/n)\), with the behavior of this approximation procedure consistent across trials.

Second, we implemented an approximator based on the Monte-Carlo integration. At the \(n\)th step of the approximation procedure we pick randomly an element from \(G\), where \(w \in [0, 200], b \in [-200, 200]\) (uniform distribution). After an element is selected, we add it to the current pool of basis functions
\[
P_{n-1} = \{g(w_1^T x + b_1), \ldots, g(w_{n-1} x + b_{n-1})\}.
\]
Then the weights \(c_i\) in the superposition
\[
f_n = \sum_{i=1}^{n} c_i g(w_i^T x + b_i)
\]
are optimized so that \(\|f_n - f\| \to \min\). Evolution of the normalized approximation error \(\bar{e}_n\) \([14]\) over 100 trials is shown in Fig. 1 (bottom panel). As can be observed from the figure, even though the values of \(\bar{e}_n\) form a monotonically decreasing sequence, they are far from \(1/n\), at least for \(1 \leq n \leq 100\). Behavior across trials is not consistent, at least for the networks smaller than 100 elements, as indicated by a significant spread among the curves.

Fig. 1. Practical speed of convergence of function approximators that use greedy algorithm (middle panel) and Monte-Carlo based random choice of basis functions (bottom panel). The target function is shown on the top panel.

Overall comparison of these two methods is provided in Fig. 2 in which the errors \(\bar{e}_n\) are presented in the form of a box plot. Black solid curves depict the median of
the error as a function of the number of elements, \( n \), in the network; blue boxes contain 50% of the data points in all trials; “whiskers” delimit the areas containing 75% of data, and red crosses show the remaining part of the data. As we can see from these plots, random basis function approximators, such as the RVFL networks, mostly do not match performance of greedy approximators for networks of reasonable size. Perhaps, employing integration methods with variance minimization could improve the performance. This, however, would amount to using prior knowledge about the target function \( f \), making it difficult to apply the RVFL networks to problems in which the function \( f \) is uncertain.

Now we demonstrate performance of an MLP trained to approximate this target function. The NN is trained by a gradient based method described in [20]. At first, the full network training is carried out for several network sizes \( n = 20, 40, 60, 80 \), and 100 and input samples randomly drawn from \( x \in [0, 1] \). The values of \( \bar{e}_n \) are \( 1.5 \cdot 10^{-4} \) for all the network sizes (as confirmed in many training trials repeated to assess sensitivity to weight initialization). This suggests that training and performance of much smaller networks should be examined. The networks with \( n = 2, 4, 6, 8, 10 \) are trained, resulting in \( \bar{e}_n = 0.5749, 0.1416, 0.0193, 0.0011, 0.0004 \), respectively, averaged over 100 trials per the network size. Next, we train only the linear weights \((c_i)\) of the MLP, fixing the nonlinear weights \( w_i \) and \( b_i \) to random values. The results for \( \bar{e}_n \) averaged over 100 trials are shown in Fig. 2 bottom panel (black curve). Remarkably, the results of random basis network with \( n = 100 \) are worse than those of the MLP with \( n \geq 4 \) and full network training. These results indicate that both the greedy and the Monte-Carlo approximation results shown in Fig. 2 are quite conservative. Furthermore, the best of those two, i.e., the greedy approximation’s, can be dramatically improved by a practical gradient based training.

IV. DISCUSSION

We just analyzed theoretically and illustrated on a simple example what may happen if the basis for function approximation is chosen at random. We wish to discuss recent result presented in [8] regarding the use of the random basis function approximators. We choose this work because it is representative of a recent trend in neural network control literature exemplified by [9]–[13]. In this trend, the purpose of one or several neural networks implementing random basis is to account for (and ideally - cancel asymptotically) an unknown bounded modeling nonlinearity.

While ensuring that the tracking errors are bounded asymptotically, the main theorem in [8] and its proof do not imply performance improvement. Instead the proof attempts to relate design parameters \( \gamma_i \) with magnitudes of disturbances and weights of neural networks. Though the disturbance magnitude may indeed be known a priori, one cannot assume sufficiently small bounds on the values of weights because the weights may need to be large in order to compensate for residual modeling errors from randomly assigned basis functions. Furthermore, the larger the weights or the farther the system of basis functions from an orthogonal one (ill-conditioning), the more time is needed for an adaptive system to converge into the desired domain; see, e.g., [7]. In fact, in the example of Section III we observed values of the hidden layer weights as large as

![Fig. 2. Box plots of convergence rates for function approximators that use greedy algorithm (top panel) and Monte-Carlo random choice of basis functions (middle and bottom panels). The middle panel corresponds to the case in which the basis functions leading to ill-conditioning were discarded. The bottom panel shows performance of the MLP trained by the method in [20] which is effective at counteracting ill-conditioning while adjusting the linear weights only. The red curve shows the upper bound for \( \bar{e}_n \) calculated in accordance with (5). We duplicated the average performance of the greedy algorithm (grey solid curve) in the middle and bottom panels for convenience of comparison.](image-url)
200. However, large bounds on the weights force the control system designer to decrease design parameters $\gamma$, which, in turn, results in an increase of the region of uniform ultimate boundedness (UUB) (determined by equations (A.5-A.9) in [8]). Ironically, the region of UUB in [8] may not shrink to zero even in the ideal case of zero disturbances. The UUB depending on such uncontrollable quantities as weights makes it impossible to provide practically valuable guarantees of the closed-loop system performance.

V. Conclusion

In this work we demonstrate that, despite increasing popularity of random basis function networks in control literature, especially in the domain of intelligent/adaptive control, one needs to pay special attention to practical aspects that may affect performance of these systems in applications.

First, as we analyzed in Section II and showed in our example, although the rate of convergence of the random basis function approximator is qualitatively similar to that of the greedy approximator, the rate of the random basis function approximator is achievable only when the number of elements in the network is sufficiently large. Second, approximators which are motivated by the Monte-Carlo integration method offer only statistical measure of approximation quality. In other words, small approximation errors are guaranteed here in probability. This means that, for practical adaptive control in which the RVFL networks are to model or compensate system uncertainties, employment of a re-initialization with a supervisory mechanism monitoring quality of the RVFL network is necessary. Unlike network training methods that adjust both linear and nonlinear weights of the network, such mechanism may have to be made robust against numerical problems (ill-conditioning) which often occurs in the Monte-Carlo method.

Our conclusion about the random basis function approximators is also consistent with the following intuition. If the approximating elements (network nodes) are chosen at random and not subsequently trained, they are usually not placed in accordance with the density of the input data. Though computationally easier than for nonlinear parameters, training of linear parameters becomes ineffective at reducing errors “inherited” from the nonlinear part of the approximator. Thus, in order to improve effectiveness of the random basis function approximators one could combine unsupervised placement of network nodes according to the input data density with subsequent supervised or reinforcement learning values of the linear parameters of the approximator. However, such a combination of methods is not-trivial because in adaptive control and modeling one often has to be able to allocate approximation resources adaptively – and the full network training seems to be the natural way to handle such adaptation.

ACKNOWLEDGMENT

The authors are grateful to Prof. A.N. Gorban for useful comments and numerous technical discussions during preparation of this work. The first author’s research was supported by a Royal Society International Joint Project grant, and partially supported by RFBR grant 8-08-00103-a.

REFERENCES

[1] A. R. Barron. Universal approximation bounds for superposition of a sigmoidal function. IEEE Trans. on Information Theory, 39(3):930–945, 1993.
[2] G. Cybenko. Approximation by superpositions of a sigmoidal function. Math. of Control, Signals and Systems, 2:303–314, 1989.
[3] S. Haykin. Neural Networks: A Comprehensive Foundation. Prentice Hall, 1999.
[4] K. Hornik, K. Stinchcombe, and H. White. Universal approximation of an unknown function and its derivatives using multilayer neural networks. Neural Networks, (3):551–560, 1990.
[5] B. Igelnik and Y.-H. Pao. Stochastic choice of basis functions in adaptive function approximation and the functional-link net. IEEE Trans. Neural Networks, 6(6):1320–1329, 1995.
[6] J.-Y. Lin and W. S. 'Chow. Comments on “Stochastic choice of basis functions in adaptive function approximation and the functional-link net”’. IEEE Trans. Neural Networks, 8(2):452–454, 1997.
[7] M. French. Cs. Szepesvari and E. Rogers, Uncertainty, Performance, and Model Dependency in Approximate Adaptive Nonlinear Control. IEEE Trans. Automatic Contr., 45(2):353–358, 2000.
[8] P. He and S. Jagannathan. Reinforcement learning based output feedback control of nonlinear systems with input constraints. IEEE Trans. Systems, Man and Cybernetics, 35(1):150–154, 2005.
[9] O. Kuljaca and S. Tesnjak and V. Koroman. Performance Analysis of Adaptive Neural Network Frequency Controller for Thermal Power Systems. Proceedings of the 9th WSEAS International Conference on Automatic Control, Modeling & Simulation, Istanbul, Turkey, May 27–29, 2007, pp. 127–132.
[10] F. Lewis and S. Ge. Neural Networks in Feedback Control Systems. In M. Kutz (Ed.) Mechanical Engineers’ Handbook, Volume 2, Third Edition, Wiley, 2006.
[11] J. Sarangapani. Neural Network Control of Nonlinear Discrete-Time Systems. Control Engineering Series, CRC Press, 2006.
[12] F. Lewis and J. Campos and R. Selmic Neuro-fuzzy control of industrial systems with actuator nonlinearities. SIAM, 2002.
[13] W. Liu and J. Sarangapani and G. Venayagamoorthy and L. Liu and D. Wunsch and M. Crow and D. Cartes. Decentralized Neural Network-based Excitation Control of Large-scale Power Systems. International Journal of Control, Automation, and Systems, vol. 5, no. 5., pp. 526–538, October 2007.
[14] L. K. Jones. A simple lemma on greedy approximation in Hilbert space and convergence rates for projection pursuit regression and neural network training. The Annals of Statistics, 20(1):608–613, 1992.
[15] Ai-Poh Loh, A.M. Annaswamy, and F.P. Skantze. Adaptation in the presence of general nonlinear parameterization: An error model approach. IEEE Trans. on Automatic Control, 44(9):1634–1652, 1999.
[16] Park and I. W. Sandberg. Approximation and radial basis function networks. Neural Computation, 5(2):305–316, 1993.
[17] D.V. Prokhorov, V.A. Terekhov, and I.Yu. Tyukin. On the applicability conditions for the algorithms of adaptive control in nonconvex problems. Automation and Remote Control, 63(2):262–279, 2002.
[18] I.Yu. Tyukin, D.V. Prokhorov, and C. van Leeuwen. Adaptation and parameter estimation in systems with unstable target dynamics and nonlinear parametrization. IEEE Trans. on Automatic Control, 52(9):1543–1559, 2007.
[19] L. A. Feldkamp, D. V. Prokhorov, C. F. Eagen, and F. Yuan. Enhanced multi-stream Kalman filter training for recurrent networks. In J. Suykens and J. Vandewalle (eds), Nonlinear Modeling: Advanced Black-Box Techniques, pp. 29–53, Kluwer Academic Publishers, 1998.