On geometry of some pseudo-semisimple groups

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Abstract

Based on the work of Canrad-Gabber-Prasad, the paper deals with the geometry of particular pseudo-semisimple groups, namely those which can be written as quotient of Weil restriction of semisimple groups. We establish that these groups are retract rational when they are split, and give results on their Picard groups.

Keywords: Extension groups, Grothendieck cohomologies, Imperfect fields, Linear algebraic groups, Picard groups, Pseudo-reductive groups, Pseudo-semisimple groups, Retract rationality, Weil restriction.

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1. Introduction

**Context.** Pseudo-reductive groups generalize reductive groups. A smooth connected linear algebraic group $G$ over a field $k$ is *pseudo-reductive* if $G$ does not contain any non trivial subgroup defined over $k$ which is smooth, connected, normal and unipotent; in other words, the $k$-unipotent radical of $G$ is the trivial subgroup $1 \subseteq G$. For a smooth connected linear algebraic group, reductivity implies pseudo-reductivity. The converse holds over perfect base fields but is false when the base field is imperfect. Pseudo-reductive groups were first studied by Tits (years 1991-92 and 1992-93 of [Tit13]) and then Conrad, Gabber and Prasad pursued the work which led to the wonderful books [CP15] and [CP16]. For a great introduction to pseudo-reductive groups following [CGP15], see the Bourbaki talk by Remy.

In this paper we will be interested in pseudo-semisimple groups (that is a perfect pseudo-reductive group). Two geometric aspects of pseudo-semisimple groups are studied.

First of all, we study retract rationality of basic examples of pseudo-semisimple groups. It happens that pseudo-reductive groups are not necessarily unirational over a separable closed fields and may have infinite Picard groups. But pseudo-semisimple groups are unirational and their Picard groups are finite and additive as those of reductive groups ([Ros20, Prop. 3.3]). Retract rationality was introduced for fields by Saltman in [Sal84] and by considering function fields of varieties one defines retract rationality for varieties. For a variety, retract rationality is weaker than rationality and than stable rationality, but it is stronger than unirationality. In practice, retract rationality is closer to rationality than to unirationality and it is often possible to state results for retract rational varieties primarily stated for rational varieties. However, retract rationality can be checked via a practical criterion and which makes it easier to show. To study rationality questions it is thus interesting to begin by studying retract rationality. This is what we do in Section 2. This section is a first step in the study of retract rationality of pseudo-semisimple groups.

In a second part, we study group of extensions by the multiplicative group for pseudo-semisimple group (or their Picard groups). Using [CP16, §5], things work the same way as for semisimple groups via their universal cover. Along this part, we shall need to determine the character groups of Weil restrictions of groups of multiplicative type, so we propose a partial generalization of [Oes84, Th. II.2.4] which can be found in the appendix.

We know that the results in this paper are not very deep. But we’d like to propose a first step in the study of geometrical properties of pseudo-semisimple groups, such as rationality problem, and we hope to pursue this paper in the future.
and bring these questions to the linear algebraic group community.

Content. In Section 2 we review the basics of retract rationality following [Mer17] and study pseudo-semisimple groups of a specific form to show they are retract rational. More precisely,

**Proposition 1.1** (Prop. 2.10). Let $k'/k$ be a finite, purely inseparable field extension. Let $G'$ be a (connected) split simply connected semisimple $k'$-group; let $\mu'$ be a central $k'$-subgroup of $G'$. Then $R_{k'/k}(G')/R_{k'/k}(\mu)$ is retract rational over $k$.

In particular, if $G'$ is not assumed split, then $R_{k'/k}(G')/R_{k'/k}(\mu)$ is retract rational over a separable closure of $k$.

In Section 3, we are interested in the groups of extension by the multiplicative group $\mathbb{G}_m$ for pseudo-semisimple groups. For this purpose we exploit [CP16, §5] where Conrad and Prasad study central extensions of perfect groups. From loc. cit. we derive Proposition 3.4 which leads us to two tangible applications:

**Proposition 1.2** (Prop. 3.6). Let $k'/k$ be a finite, purely inseparable field extension. Write $p$ for the characteristic exponent of $k$ and define $h$ to be the least non negative integer such that $(k')^{p^h} \subseteq k$. Let $G'$ be a semisimple, simply connected $k'$-group and let $\mu$ be a central subgroup of $G'$. Then, writing $G := R_{k'/k}(G')/R_{k'/k}(\mu)$, one has an isomorphism

\[ \text{Ext}^1(G, \mathbb{G}_m) \simeq p^h \cdot \text{Hom}_{k'-\text{groups}}(\mu, \mathbb{G}_m). \]

and:

**Proposition 1.3** (Prop. 3.9). Let $G$ be a standard pseudo-semisimple $k$-group which is pseudo-split. For any separable extension $K/k$ (finite or not), the scalar extension homomorphism $\text{Ext}^1(G, \mathbb{G}_m) \to \text{Ext}^1(G_{K}, \mathbb{G}_m)$ is an isomorphism.

In Appendix A we determine the character group of Weil restriction $R_{k'/k}(M_{k'})$ where $k'/k$ is finite and purely inseparable, and $M/k$ is a group of multiplicative and finite type. This is Proposition A.2. The latter is used to complete calculations for Propositions 3.6 and 3.9 from Section 3.

Conventions, notation and vocabulary. The letter $k$ denote the ground field over which (almost) every algebro-geometric object of the paper is defined.

Let $K$ be a field. By *algebraic $K$-group* we mean a group scheme of finite type over $k$ and by *linear algebraic $k$-group* an affine group scheme of finite type over $k$. The $K$-character group of an algebraic $K$-group $G$ is denoted by $\hat{G}(K)$. A *$K$-variety* is a separated and geometrically integral $K$-scheme of finite type over $K$.
We recall some basic definitions from [CGP15] to be used below. A pseudo-reductive \( k \)-group is a smooth connected linear algebraic \( k \)-group which has a trivial unipotent \( k \)-radical. A pseudo-semisimple group is a perfect pseudo-reductive group. A smooth linear algebraic \( k \)-group is said pseudo-split if it contains a maximal torus which is split.

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2. Retract rationality of some pseudo-semisimple groups

In this section we review some basics on retract rationality (following [Mer17]). Then we prove that this property is satisfied by a peculiar family of pseudo-semisimple groups.

The base field is \( k \).

2.1. Retract rational varieties

Definition 2.1. Let \( l/k \) be a field extension. A \( k \)-variety \( X \) is said \( l \)-retract rational or retract rational over \( l \) if there exist

- a non empty open \( U \) of an affine space \( \mathbb{A}^n_l \),
- a non empty open \( V \) of \( X_l \) and,
- two morphisms \( f : U \to V \), \( r : V \to U \) defined over \( l \) such that \( f \circ r = \text{id}_U \).

For an arbitrary local ring \( A \), let’s denote by \( \bar{A} \) the residual field of \( A \).

Proposition 2.2 ([Mer17, Prop. 3.1]). Let \( X \) be a \( k \)-variety. The following assertions are equivalent:

1. \( X \) is retract rational over \( k \);

2. For every local \( k \)-algebra \( A \), there exists a non empty open \( V \subseteq X \) such that \( V(A) \to V(\bar{A}) \) is onto;
3. For every local $k$-algebra with $\bar{A}$ infinite, there exists a non empty open $V \subseteq X$ such that $V(A) \to V(\bar{A})$ is onto;

4. For every local $k$-algebra $A$ with $\bar{A} \simeq k(X)$ as $k$-algebras, there exists a non empty open $V \subseteq X$ such that $V(A) \to V(\bar{A})$ is onto.

Actually, from the proof of [Mer17, Prop. 3.1] or the proof of [CTS07, Prop. 1.2], one can extract a more specific criterion :

**Lemma 2.3.** Let $X$ be a $k$-variety. Let $A = k[X_1, \ldots, X_m]_p$ be a polynomial $k$-algebra localized in a prime ideal $p$ such that $\bar{A} \simeq k(X)$. Then $X$ is retract rational over $k$ if, and only if, there exists a non empty open scheme $V \subseteq X$ such that $V(A) \to V(\bar{A})$ is onto. \(\square\)

Here are the links between the different rationality properties known for varieties.

**Proposition 2.4 ([Mer17, Prop. 3.4]).** Let $X$ be a $k$-variety. Consider the properties :

(i) $X$ is $k$-rational,

(ii) $X$ is stably rational over $k$,

(iii) $X$ is $k$-retract rational,

(iv) $X$ is $k$-unirational.

Then the following implications hold : $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

An instance for which rationality can be replace by retract rationality is the following proposition. This is a straightforward generalization of [San81, Lem. 6.6].

**Proposition 2.5.** Let $X$ and $Y$ be two smooth $k$-varieties. Assume that $Y$ is retract rational over a separable closure of $k$ and that $Y(k) \neq \emptyset$. Then the projections $X \times_k Y \to X, Y$ induce a group isomorphism

$$\Pic(X) \oplus \Pic(Y) \overset{\sim}{\longrightarrow} \Pic(X \times_k Y).$$

Proof. \(\square\)
The case of algebraic groups. For an algebraic group, being retract rational can be check for the whole space and not just an open. This is a convenient fact for proofs.

**Lemma 2.6.** Let $G$ be a smooth and connected algebraic $k$-group and let $A$ be a local $k$-algebra. Assume that the set of $k$-points $G(k)$ is dense in $G$ (this is the case when $G$ is $k$-unirational and $k$ is infinite). If there exists a dense open subscheme $V$ of $G$ such that $V(A) \to V(\bar{A})$ is onto, then $G(A) \to G(\bar{A})$ is also onto.

**Proof.** This is just a homogeneity argument. First of all, $\bigcup_{g \in G(k)} (gV(A)) \to \bigcup_{g \in G(k)} (gV(\bar{A}))$ is onto. But since $G(k)$ is dense in $G$, the union of all of the $gV$ for $g$ describing $G(k)$ equals $G$. Thus, $G(\bar{A}) = \bigcup_{g \in G(k)} (gV(\bar{A}))$ because $\bar{A}$ is a field, so $\bigcup_{g \in G(k)} (gV(A)) \to G(\bar{A})$ in onto, where $\bigcup_{g \in G(k)} (gV(A)) \subseteq G(A)$ (the equality does not necessarily hold).

**Consequence 2.7.** If $k$ is infinite and $G$ is $k$-retract rational, then for every local $k$-algebra $A$, the map $G(A) \to G(\bar{A})$ is onto.

### 2.2. Examples of retract rational pseudo-semisimple groups

#### 2.2.1 Non retract rational pseudo-reductive groups

Pseudo-reductive groups are not always retract rational. Here is an example from [CGP15].

Assume $k$ is imperfect of characteristic $p > 0$. Let $q = p^r$ be a power of $p$ such that $q > 2$ and let $t \in k \setminus k^p$. As explained in [CGP15, Ex. 11.3.1],

- the smooth and connected $k$-subgroup $U = \{(x, y) \in \mathbb{G}_a \times_k \mathbb{G}_a \mid y^q = x - tx^p\}$ de $\mathbb{G}_a \times_k \mathbb{G}_a$ is not unirational over $k$;

- any $k$-group $C$ in an exact sequence

$$1 \to \mathbb{G}_m \to C \to U \to 1$$

is commutative and pseudo-reductive unless $C$ is $\mathbb{G}_m \times_k U$;

- there exist $k$-groups $C$ as above which are not isomorphic to $\mathbb{G}_m \times_k U$.

Thus there are commutative pseudo-reductive $k$-groups $C$ which have a quotient isomorphic to $U$. These groups are not $k$-unirational, since $U$ is not; they are neither unirational over a separable closure $k_s$ of $k$.

However, all smooth and connected perfect linear algebraic $k$-groups are $k$-unirational ([CGP15, Prop. A.2.11]). This applies in particular to pseudo-semisimple $k$-groups.
2.2.2 Some retract rational pseudo-semisimple groups

For Proposition 2.10, we will need the following two lemmas. By \textit{fppf} we mean the "finitely presented and faithfully flat" Grothendieck topology on a scheme. Flat cohomology groups will be denoted by $H^d_{\text{fppf}}(\cdot,\cdot)$.

**Lemma 2.8** ([DG11, Prop. 8.2]). Let $S \to S'$ be a scheme morphism. Then for every fppf group sheaf $G'$ on $S'$, the map $H^1_{\text{fppf}}(S,R_{S'/S}(G')) \to H^1_{\text{fppf}}(S',G')$ is one-to-one, and its image is the set formed by the classes of $G'$-torsors that become trivial over a fppf cover $R \times_S S'$ of $S'$ induced by a fppf cover $R$ of $S$.

**Lemma 2.9.** Let $l/k$ be a finite field extension and let $A$ be a local $k$-algebra. Then $A \otimes_k l$ is a semi-local ring. In particular its Picard group is trivial.

\begin{proof}
Actually the lemma is more generally true for $A$ semi-local and for all finite $A$-algebra. So let $B$ be a finite $A$-algebra. Replacing $A$ by its quotient with respect to the kernel of $A \to B$, we may assume that $A$ is a subring of $B$. Thus $A \to B$ is an injective and integral ring extension, so the Going-up Theorem implies that every maximal ideal of $B$ lies over a (unique) maximal ideal of $A$. Furthermore, since $A \hookrightarrow B$ is a finite ring extension, there are only finitely many maximal ideals of $B$ lying over a given maximal ideal of $A$. Then fact that $A$ has only finitely many prime ideals implies that it is also the case for $B$.

Regarding the Picard group, we just need to recall that projective modules of finite type and of constant rank over any semi-local ring are free by [Bou06, II.§5.3. Prop. 5].
\end{proof}

Here is the main result of this section.

**Proposition 2.10.** Let $k'/k$ be a finite, purely inseparable field extension. Let $G'$ be a (connected) split simply connected semisimple $k'$-group; let $\mu'$ be a central $k'$-subgroup of $G'$. Then $R_{k'/k}(G')/R_{k'/k}(\mu)$ is retract rational over $k$.

In particular, if $G'$ is not assumed split, then $R_{k'/k}(G')/R_{k'/k}(\mu)$ is retract rational over a separable closure of $k$.

\begin{proof}
Write $G$ for the algebraic $k$-group $R_{k'/k}(G')/R_{k'/k}(\mu)$ and fix a local $k$-algebra $A$. First we notice that $k$ may be assumed to be infinite because a finite field is perfect, so in the case $k$ is finite, then $k' = k$ and the group $R_{k'/k}(G')/R_{k'/k}(\mu)$ is a split semisimple group, hence a rational variety (the assumption "$k$ infinite" is useful to invoke Consequence 2.7 at some points of the proof). We want to show that $G(A) \to G(\bar{A})$ is onto. By Proposition 2.2 this will imply that $G$ is retract rational over $k$.

Consider the exact sequence of fppf sheaves over $A$ and $\bar{A}$

$$1 \to R_{k'/k}(\mu) \to R_{k'/k}(G') \to G \to 1.$$
Taking $fppf$ cohomology yields the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
R_{k'/k}(\mu)(A) & \longrightarrow & R_{k'/k}(G')(A) & \longrightarrow & G(A) & \longrightarrow & H^1_{fl}(A, R_{k'/k}(\mu)) & \longrightarrow & H^1_{fl}(A, R_{k'/k}(G')) \\
\phi_1 & & \phi_2 & & \phi_3 & & \phi_4 & & \phi_5 \\
R_{k'/k}(\mu)(\bar{A}) & \longrightarrow & R_{k'/k}(G')(\bar{A}) & \longrightarrow & G(\bar{A}) & \longrightarrow & H^1_{fl}(\bar{A}, R_{k'/k}(\mu)) & \longrightarrow & H^1_{fl}(\bar{A}, R_{k'/k}(G'))
\end{array}
\]

The surjective part of the Five Lemma would imply that $G(\bar{A}) \to G(\bar{A})$ is onto. In order to apply the Five Lemma we want to show that $\phi_2$ and $\phi_4$ are onto and that $\phi_5$ is one-to-one.

Actually, we don’t need to show that $\phi_5$ is one-to-one. Indeed, $G'$ is split, so there is a split torus $T' \subseteq G'$ such that $\mu$ sits in $T'$. Then, $H^1_{fl}(A, R_{k'/k}(\mu)) \to H^1_{fl}(A, R_{k'/k}(G'))$ factorises through $H^1_{fl}(A, R_{k'/k}(T'))$. But the latter set can be realized as a subset of $H^1_{fl}(A', T')$ by Lemma 2.8 and $H^1_{fl}(A', T')$ is a singleton since $T'/k'$ is a split torus and $A'$ is a semilocal ring by Lemma 2.9. Thus the arrow $H^1_{fl}(A, R_{k'/k}(\mu)) \to H^1_{fl}(A, R_{k'/k}(G'))$ is the zero map. Note that the same result also holds for $\bar{A}$ instead of $A$. We are then reduced to apply the Five Lemma to the commutative diagram with exact rows:

\[
\begin{array}{cccccc}
R_{k'/k}(\mu)(A) & \longrightarrow & R_{k'/k}(G')(A) & \longrightarrow & G(A) & \longrightarrow & H^1_{fl}(A, R_{k'/k}(\mu)) & \longrightarrow & 1 \\
\phi_1 & & \phi_2 & & \phi_3 & & \phi_4 & & \phi_5 \\
R_{k'/k}(\mu)(\bar{A}) & \longrightarrow & R_{k'/k}(G')(\bar{A}) & \longrightarrow & G(\bar{A}) & \longrightarrow & H^1_{fl}(\bar{A}, R_{k'/k}(\mu)) & \longrightarrow & 1
\end{array}
\]

◊ First of all, $\phi_5$ is one-to-one.

◊ Secondly, since $G'$ is a split reductive group, it is rational over $k'$. So its Weil restriction through $k'/k$ is rational over $k$ and is in particular retract rational over $k$. By Consequence 2.7 $\phi_2$ must be onto.

◊ Let’s show that $\phi_4$ is onto.

The $k'$-group $\mu$ is diagonalizable and finite. So it is a product of groups of roots of unity $\mu_{l,k'}$ for $l \in \mathbb{N}$. For factors $\mu_{l,k'}$ with $l$ prime to $p$, the $k'$-group $\mu_{l,k'}$ is smooth, hence by [CGP15, Cor. A.5.4(3)] the following sequence of algebraic $k$-groups is exact:

\[
1 \to R_{k'/k}(\mu_{l,k'}) \to R_{k'/k}(\mathbb{G}_m) \to R_{k'/k}(\mathbb{G}_m) \to 1.
\]

The associated long exact sequence for $fppf$ cohomology over a local $k$-algebra $B$ leads to

\[
H^1_{fl}(B, R_{k'/k}(\mu_{l,k'})) = R_{k'/k}(\mathbb{G}_m)(B)/(R_{k'/k}(\mathbb{G}_m)(B)).
\]
Since $R_{k'/k}(\mathbb{G}_m)$ is $k$-retract rational, the map $R_{k'/k}(\mathbb{G}_m)(A) \to R_{k'/k}(\mathbb{G}_m)(\bar{A})$ is onto. Thus

$$H^1_{fl}(A, R_{k'/k}(\mu_{l,k'})) \to H^1_{fl}(\bar{A}, R_{k'/k}(\mu_{l,k'}))$$

is onto.

For factors $\mu_{q,k'}$ with $q = p^r$, the following sequence of algebraic $k$-groups is exact:

$$(1) \quad 1 \to R_{k'/k}(\mu_{q,k'}) \to R_{k'/k}(\mathbb{G}_m) \to R_{k(k^{p^r})/k}(\mathbb{G}_m) \to 1,$$

where the arrow $R_{k'/k}(\mu_{q,k'}) \to R_{k'/k}(\mathbb{G}_m)$ is inclusion and the arrow $R_{k(k^{p^r})/k}(\mathbb{G}_m)$ is elevating to the $p^r$-th power. Considering the sequence (1) for $B = A$ and $\bar{A}$ yields a commutative diagram with exact rows

$$(\bar{A} \otimes_k k^{(p^r)})^* \quad H^1_{fl}(\bar{A}, R_{k'/k}(\mu)) \quad H^1_{fl}(A, R_{k'/k}(\mathbb{G}_m)) \quad \phi_4 \quad H^1_{fl}(\bar{A}, R_{k'/k}(\mathbb{G}_m))$$

Groups on the right column are zero by Lemmas 2.8 and 2.9. Moreover, the left vertical arrow is surjective because $R_{k(k^{p^r})/k}(\mathbb{G}_m)$ is $k$-retract rational. Thus we find that $H^1_{fl}(A, R_{k'/k}(\mu_{q,k'})) \to H^1_{fl}(\bar{A}, R_{k'/k}(\mu_{q,k'}))$ is onto, that is $\phi_4$ is onto. □

**Example 2.11.** Let $k'/k$ be a finite and purely inseparable field extension.

- Let $l$ be a positive integer, coprime to $p$. The algebraic $k$-group

$$R_{k'/k}(\text{SL}_l,k')/R_{k'/k}(\mu_{l,k'})$$

is isomorphic to $R_{k'/k}(\text{PGL}_l,k')$ ([CGP15, Cor. A.5.4(3)]), which is actually rational over $k$.

- Let $q \neq 1$ be a power of $p$. Proposition 2.10 says that $R_{k'/k}(\text{SL}_q,k')/R_{k'/k}(\mu_{q,k'})$ is retract rational over $k$.

### 3. Extension groups of pseudo-semisimple groups

We determine the group of extensions by $\mathbb{G}_m$ for some pseudo-semisimple groups. Thanks to the great result [CP16, Th. 5.1.3] we can proceed as in the case of semisimple groups via the notion of universal covers.
3.1. Universal cover of perfect groups

In [CP16], Conrad and Prasad establish a result on tame central extensions of smooth connected linear algebraic groups which are perfect. It generalizes the notion of universal cover for semisimple groups. Let’s precise that an affine group scheme $\mu$ over $k$ is tame if $\mu$ has no unipotent subgroup except for $1$ ([CP16, Def. 5.1.1]).

Here we sum up [CP16, Th. 5.1.3].

**Definition 3.1** ([CP16, Def. 51.1]). A tame extension of a perfect smooth and connected linear algebraic $k$-group $G$ is a central extension of affine $k$-group schemes

$$1 \to \mu \to H \to G \to 1$$

for a perfect smooth $H$ and where $\mu$ is tame.

Recall the following construction: given a smooth and connected linear algebraic group $H$ over a field $K$, if the radical $\bar{R}$ of $H_{\bar{K}}$ is defined over $K$, i.e. comes from a group $R/K$, then the semisimple quotient of $H$ is simply $H^{ss} := H/R$ — in that case $H/R$ is a semisimple $K$-group.

**Theorem 3.2** ([CP16, Th. 5.1.3]). Let $G$ be a perfect smooth and connected linear algebraic $k$-group $G$. Take $K/k$ to be the definition field of the unipotent radical $R_u(G_{\bar{k}})$ of $G_{\bar{k}}$. Then there exists a tame extension $E_0$ (unique up to extension isomorphism)

$$(E_0) \quad 1 \to \mu \to \tilde{G} \to G \to 1,$$

which has the following property: for any other tame extension $E$

$$(E) \quad 1 \to \nu \to H \to G \to 1,$$

the semisimple group $(\tilde{G}_K)^{ss}$ is a covering of $(H_K)^{ss}$ and the set of (isomorphism classes of) morphisms $E_0 \to E$ is in a one-to-one correspondance with the set of homomorphisms $(\tilde{G}_K)^{ss} \to (H_K)^{ss}$.

The extension $E_0$ is characterized by the fact that $\tilde{G}_K^{ss}$ is the universal covering of the semisimple group $G_K^{ss}$.

The extension $E_0$ (or just the group $\tilde{G}$) is called the universal tame extension/covering of $G$.

**Remark 3.3.** Let $k'/k$ be a finite, purely inseparable field extension. Let $G'$ be a semisimple, simply connected $k'$-group and let $\mu$ be a central subgroup of $G'$. According to [CGP15, Prop. 1.3.4], the $k$-group $G := R_{k'/k}(G')/R_{k'/k}(\mu')$ is pseudo-semisimple. In particular it is perfect. In this case the universal tame extension of $G$ is

$$(E_0) \quad 1 \to R_{k'/k}(\mu') \to R_{k'/k}(G') \to G \to 1.$$
3.2. Extension groups of perfect groups

Let $G$ be a perfect smooth and connected linear algebraic $k$-group. Theorem 3.2 yields a tame extension

$$(E_0)\ 1 \to \mu \to \tilde{G} \to G \to 1.$$

As what is done for semisimple groups, one defines a homomorphism $\Theta : \hat{\mu}(k) \to \text{Ext}^1(G, \mathbb{G}_m)$ as follows: for any group homomorphism $\chi : \mu \to \mathbb{G}_m$, consider the push forward of $E_0$ with respect to $\chi$, that is the extension

$$(E_\chi)\ 1 \to \mathbb{G}_m \to H \to G \to 1$$

where $H$ is the quotient of $\tilde{G} \times_k \mathbb{G}_m$ by $Z_\chi := \{(x, \chi(x)^{-1}) | x \in \mu\}$. There is a commutative diagram

(2)\[\begin{array}{c}
1 \longrightarrow \mu \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \\
\downarrow \chi \quad \downarrow \quad \downarrow \\
1 \longrightarrow \mathbb{G}_m \longrightarrow H \longrightarrow G \longrightarrow 1
\end{array}\]

The map $\Theta$ is then defined to be the map that sends $\chi \in \hat{\mu}(k)$ to the isomorphism class $E_\chi$ of $E_\chi$. The map $\Theta$ is easily seen to be a group homomorphism.

**Proposition 3.4.** Let $G$ be a perfect smooth and connected linear algebraic $k$-group. Denote the kernel of its universal tame extension by $\mu$. Then the homomorphism $\Theta : \hat{\mu}(k) \to \text{Ext}^1(G, \mathbb{G}_m)$ is an isomorphism.

**Proof.** If $\chi \in \hat{\mu}(k)$ is such that $E_\chi$ is isomorphic to the trivial extension of $G$ by $\mathbb{G}_m$, then $E_\chi$ admits a retraction $r : H \to \mathbb{G}_m$ ($H$ defined as above). Write $f : \tilde{G} \to H$ for the homomorphism obtained by pushing forward $E_0$, and $i$ for $\mu \to \tilde{G}$. Since Diagram (2) is commutative, the equality $\chi = j \circ f \circ i$ holds. But $j \circ f$ is a character of $\tilde{G}$ which is a perfect group, so $j \circ f = 1$ whence $\chi = 1$.

We now show $\Theta$ is surjective. Let

$$(E) :\ 1 \to \mathbb{G}_m \to H \to G \to 1$$

be an extension. Considering the descending chain of derived subgroups of $H$, we find a smooth perfect group $H_0$ of $H$. Since $G$ is perfect, the restriction $H_0 \to G$ of $H \to G$ is also surjective, whence a tame extension

$$(E') :\ 1 \to \nu \to H_0 \to G \to 1$$
obtained by restricting $E$ where $\nu$ is a subgroup of $G_m$. Theorem 3.2 says that there is a homomorphism $E_0 \to E'$ and composing it with the embedding $E' \to E$ yields a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & G_m & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]

One easily checks that the bottom row $E$ is obtained by pushing forward the top row $E_0$ with respect to the homomorphism $\chi : \mu \to \nu \hookrightarrow G_m$. This means that $E \simeq E_\chi$, whence $\Theta$ surjective.

**Remark 3.5.** For $G$ as in Proposition 3.4, the canonical inclusion $\text{Ext}^1(G, G_m) \hookrightarrow \text{Pic}(G)$ is an isomorphism, so $\text{Pic}(G) \simeq \hat{\mu}(k)$. This follows either by Proposition 2.5 and Proposition 2.10, or by [Ros20, Prop. 3.2].

**3.3. Application to some pseudo-semisimple groups**

Pseudo-semisimple groups defined as quotients $R_{k'/k}(G')/R_{k'/k}(\mu)$ (where $k'/k$ is a finite, purely inseparable extension, $G'/k'$ is semisimple and $\mu \subset G'$ is central) form a wide family of non reductive groups. We have seen that the kernels of their universal tame extension is of the form $R_{k'/k}(\mu)$ for a group $\mu$ which is finite and of multiplicative type. But thanks to Theorem A.2, the character groups of such $R_{k'/k}(\mu)$ is known and we get the following proposition.

**Proposition 3.6.** Let $k'/k$ be a finite, purely inseparable field extension. Write $p$ for the characteristic exponent of $k$ and define $h$ to be the least non negative integer such that $(k')^{p^h} \subseteq k$. Let $G'$ be a semisimple, simply connected $k'$-group and let $\mu$ be a central subgroup of $G'$. Then, writing $G := R_{k'/k}(G')/R_{k'/k}(\mu)$, one has an isomorphism

\[
\text{Ext}^1(G, G_m) \simeq p^h \hat{\mu}(k').
\]

**Example 3.7.** Let $m$ be a non negative integer and let $k'/k$ be a finite, purely inseparable extension. Recall the notation $p$ and $h$ from Proposition 3.6. One has a group isomorphism:

\[
\text{Ext}^1(R_{k'/k}(\text{SL}_{p^m,k'}))/R_{k'/k}(\mu_{p^m,k'}), G_m) \simeq p^h \mathbb{Z}/p^m \mathbb{Z},
\]

where $\mu_{p^m,k'}$ is the group of $p^m$-th roots of unity inside the special linear group $\text{SL}_{p^m,k'}/k'$. Notice that when $h \geq m$ then the extension group is trivial.

**Remark 3.8.** Groups of the form $R_{k'/k}(G')/R_{k'/k}(\mu)$ as above are not all pseudo-semisimple groups. However, every standard absolutely pseudo-simple group has this form for $G'$ absolutely simple ([CGP15, Rem. 5.3.6]).
3.4. Rigidity for the Picard groups of pseudo-semisimple groups

For reductive groups, it is straightforward (thanks to [CT08, Prop. 3.3]) to show that their Picard groups are invariant under a purely inseparable field extension, and the same property holds for any field extension when groups are moreover split. Here is an analogue of the latter statement for standard pseudo-semisimple groups.

**Proposition 3.9.** Let $G$ be a standard pseudo-semisimple $k$-group which is pseudo-split. For any separable extension $K/k$ (finite or not), the scalar extension homomorphism $\text{Ext}^1(G, \mathbb{G}_m) \to \text{Ext}^1(G_K, \mathbb{G}_m)$ is an isomorphism.

**Remark 3.10.** Since pseudo-semisimple groups are perfect, they are unirational ([CGP15, Prop. A.2.11]). Thus, by [Ros20, Prop. 3.1], their groups of extensions by $\mathbb{G}_m$ are equal to their Picard groups.

First of all, let’s state the following lemma which is a gathering of many results from [CGP15]. The content of the lemma is actually used to prove [CGP15, Prop. 5.3.1], but we explain it here with some differences.

**Lemma 3.11.** Let $G$ be as in Proposition 3.9 above. Then there exist

- an integer $r$,
- finite and purely inseparable field extensions $k'_1, \cdots, k'_r$, and
- split absolutely simple, simply connected groups $G'_1/k'_1, \cdots, G'_r/k'_r$,

such that $G$ is isomorphic to a quotient of

$$R_{k'_1/k}(G'_1) \times_k \cdots \times_k R_{k'_r/k}(G'_r)$$

by a central subgroup.

**Proof.** For a subgroup of $G$, consider the property of being smooth and connected perfect normal subgroups which are $\neq 1$. By [CGP15, Prop. 3.1.8], there are only finitely many subgroups of $G$ which are minimal with the former property. Let’s call them $N_1, \cdots, N_r$. By ibid., the scheme morphism

$$\phi : N_1 \times_k \cdots \times_k N_r \to G$$

given by product in $G$ is a surjective group homomorphism such that its kernel $Z$ is a central subgroup.

The discussion following [CGP15, Prop. 3.1.8], together with [CGP15, Lem. 3.1.4]), tells us that the $N_i$’s are absolutely pseudo-simple groups. Moreover, for
any split torus $T$ of $G$, the intersection $N_i \cap T$ is a maximal torus for $N_i$ ($i = 1, \ldots, r$) according to [CGP15, Cor. A.2.7], so the $N_i$’s are also pseudo-split. At last, applying [CGP15, Prop. 5.2.6], we know that the $N_i$’s are standard pseudo-reductive since $G$ is.

Altogether, we know that $G$ is a central quotient of $N := N_1 \times_k \cdots \times_k N_r$ where the $N_i$’s are standard absolutely pseudo-simple groups and are pseudo-split.

\begin{itemize}
\item Let’s fix $i \in \{1, \ldots, r\}$. According to [CGP15, Rem. 5.3.6] and the discussion/construction before the remark, we know there exist a finite and purely inseparable field extension $k'_i/k$, an absolutely simple, simply connected $k'_i$-group $G'_i$ such that $N_i$ can be inserted into the projection homomorphism $R_{k'_i/k}(G'_i) \rightarrow R_{k'_i/k}(G'_i)/R_{k'_i/k}(\mu'_i)$ where $\mu'_i$ denotes the center of $G'_i$. That is, we have two homomorphisms
\[
R_{k'_i/k}(G'_i) \xrightarrow{\pi_i} N_i
\]
and $N_i \xrightarrow{\xi_i} R_{k'_i/k}(G'_i)/R_{k'_i/k}(\mu'_i)$
whose composition is the canonical projection. Moreover, still according to \textit{loc. cit.}, $\pi_i$ and $\xi_i$ are surjective, with central kernel and $G'_i/\mu'_i$ is the adjoint quotient of $(N_i)_{k'_i}$.

Then $G'_i$ is split as a reductive group. Indeed, $N_i$ contains a split maximal torus, so the same holds for the quotient $G'_i/\mu'_i$ of $(N_i)_{k'_i}$. Thus, since $G'_i \rightarrow G'_i/\mu'_i$ is a central isogeny, $G'_i$ has also a split maximal torus.

To sum up : for all $i = 1, \ldots, r$, the group $N_i$ is a quotient of $R_{k'_i/k}(G'_i)$ modulo a central subgroup $Z_i$, where $k'_i/k$ is a finite and purely inseparable field extension and $G'_i$ is a split, absolutely simple and simply connected $k'_i$-group.

\begin{itemize}
\item Write $H$ for the algebraic $k$-group
\[
R_{k'_i/k}(G'_i) \times_k \cdots \times_k R_{k'_r/k}(G'_r),
\]
and consider the homomorphism $j : H \rightarrow G$ obtained by composing the homomorphism $\phi$ (from the beginning of the proof) with the projection $p : H \rightarrow N_1 \times_k \cdots \times_k N_r$. To prove the lemma, it remains to show that the kernel of $j$ is a central subgroup. By [CGP15, Prop. 2.2.12], the center $Z_N$ of $N$ is the image of $Z_H$ through $p$, the center of $H$. But $Z_H$ is the poduct of the many $R_{k'_i/k}(\mu'_i)$’s. Thus, since $p$ has a central kernel, one has $p^{-1}(Z_N) = Z_H$, so the kernel of $j$, which is equal to $p^{-1}(Z)$ for $Z \subseteq Z_N$, is a central subgroup of $H$. 
\end{itemize}

\textit{Proof of Proposition 3.9.} With the notation from Lemma 3.11 above, $G$ is isomorphic to $\tilde{G}/Z$ where $\tilde{G} = R_{k'_i/k}(G'_i) \times_k \cdots \times_k R_{k'_r/k}(G'_r)$ and $Z$ is a central subgroup of $\tilde{G}$. The group $Z$ is a subgroup of $R_{k'_i/k}(\mu'_i) \times_k \cdots \times_k R_{k'_r/k}(\mu'_r)$, where $\mu'_i$ is the
center of $G'_i$. The $\mu'_i/k'_i$ are finite and diagonalizable (because the $G'_i$ are split), so $Z$ is a tame $k$-group. Thus

$$1 \to Z \to \tilde{G} \to G \to 1$$

is the universal tame extension of $G$ and

$$1 \to Z_K \to \tilde{G}_K \to G_K \to 1$$

is that of $G_K$ ($K/k$ is the separable extension given in the statement of the proposition). Let’s call $\phi$ the homomorphism $\widehat{Z}(k) \to \text{Ext}^1(G, \mathbb{G}_m)$ from Proposition 3.4 for $G$, and $\phi_K$ the similar homomorphism for $G_K$. There is a commutative diagram

$$
\begin{array}{ccc}
\widehat{Z}(k) & \phi \to & \text{Ext}^1(G, \mathbb{G}_m) \\
\downarrow & & \downarrow \\
\widehat{Z}(K) & \phi_K \to & \text{Ext}^1(G_K, \mathbb{G}_m)
\end{array}
$$

where the vertical arrows are obtained by scalar extension and the horizontal ones are isomorphisms by Proposition 3.4. So it remains to show that $\widehat{Z}(k) \to \widehat{Z}(K)$ is an isomorphism.

Since the extensions $k'_i/k$ are finite and purely inseparable, the finite diagonalizable groups $\mu'_i$ are defined over $k$: $\mu'_i = (\mu_i)_{k'_i}$ for a finite diagonalizable $k$-group $\mu_i$. Write

$$\mu := \mu_1 \times_k \cdots \times_k \mu_r$$

and

$$\mu^\dagger := R_{k'_1/k}(\mu'_1) \times_k \cdots \times_k R_{k'_r/k}(\mu'_r).$$

The quotient $\mu^\dagger/\mu$ is unipotent (see Consequence A.5), so $Z/(Z \cap \mu)$ is also unipotent. Thus, for all field extension $L/k$, the restriction homomorphism $\widehat{Z}(L) \to (\widehat{Z} \cap \mu)(L)$ is injective (see again A.5 and its proof). But $Z \cap \mu$ is diagonalizable, so the scalar extension homomorphism $(\widehat{Z} \cap \mu)(k) \to (\widehat{Z} \cap \mu)(K)$ is an isomorphism. One gets a commutative diagram

$$
\begin{array}{ccc}
\widehat{Z}(k) & \to & (\widehat{Z} \cap \mu)(k) \\
\downarrow & & \downarrow \cong \\
\widehat{Z}(K) & \to & (\widehat{Z} \cap \mu)(K)
\end{array}
$$

and this shows that $\widehat{Z}(k) \to \widehat{Z}(K)$ is an isomorphism.

In conclusion, the scalar extension homomorphism $\text{Ext}^1(G, \mathbb{G}_m) \to \text{Ext}^1(G_K, \mathbb{G}_m)$ is indeed an isomorphism.
A. Character groups of Weil restrictions

We describe here the character groups of groups $R_{k'/k}(M_{k'})$ in terms of those of $M$, for any $k$-group of finite type and of multiplicative type, and any finite, purely inseparable field extension $k'/k$.

Comment. There is a similar result in [Oes84], namely Théorème II.2.4. Oesterlé states and shows the link between the character group of $G'$ over $K'$ and the one of $R_{K'/K}(G')$ over $K$ where $K'/K$ is a finite extension of global fields and $G'$ is a smooth and connected linear algebraic $K'$-group. But Oesterlé only mentions the case of a general purely inseparable extension $K'/K$, he is actually only interested in global field extensions and in this case the high of a purely inseparable extension $K'/K$ is exactly the degree $[K':K]$, that is to say that $h = [K':K]$ is the least integer such that $(K')^p^h \subseteq K$ where $p$ is the characteristic of $K$.

First recall that if $G$ is a linear algebraic $k$-group and $K/k$ is a field extension, then $G$ can be seen as a subgroup of $R_{K/k}(G_K)$ in a canonical way: considering the functors of points, the closed embedding $G \hookrightarrow R_{K/k}(G_K)$ is the functor morphism given by the maps $G(A) \rightarrow G(K \otimes_k A)$ induced by $A \rightarrow K \otimes_k A, a \mapsto 1 \otimes a$ for all $k$-algebra $A$.

Notation A.1. For any abstract group $G$ and non negative integer $m$, we denote the image of $G \rightarrow G, g \mapsto m \cdot g$ by $mG$.

Proposition A.2. Let $k'/k$ be a finite, purely inseparable field extension. Call $p$ the characteristic exponent of $k$ and $h$ the least non negative integer such that $(k')^{p^h} \subseteq k$. Let $M/k$ be a group of multiplicative type and also of finite type; write $M_0$ for $R_{K/k}(M_{k'})$ ($M$ is a subgroup of $M_0$). Then the restriction homomorphism $\hat{M}_0(k) \rightarrow \hat{M}(k)$ is injective and identifies $\hat{M}_0(k)$ with $p^h\hat{M}(k) \subseteq \hat{M}(k)$.

Remark A.3. Suppose $M$ is split. Then $M$ is isomorphic to a product of finitely many $\mathbb{G}_m$ and $\mu_n$ ($\mu_n$ is the group of $p$-th roots of unity). Take $m$ to be the character which is the map $x \mapsto x^m$ on each factor $\mathbb{G}_m$ and $\mu_n$.

If $M$ is a subgroup of $\mathbb{G}_m$, then $m$ coincides with the usual notation for the character $x \mapsto x^m$.

We have then two subgroups with the same notation, because $m\hat{M}(k)$ may mean "the subgroup generated by $m$ as above" and also "the image of the iteration map $\hat{M}(k) \rightarrow \hat{M}(k)$". But they are easily seen to be the same subgroup.

The proof for A.2 is split in different cases :

1. $\hat{M}_0(k) \rightarrow \hat{M}(k)$ is injective for arbitrary $M$;
2. $M = \mathbb{G}_m$;
3. $M$ is a group of roots of unity $\mu_n$ for any $n \in \mathbb{N} \setminus \{0\}$;

4. $M$ is split (i.e. diagonalizable);

5. $M$ is arbitrary.

Proof of Proposition A.2 - The beginning

Fix $k'/k$ and $M$ as in the proposition’s statement for all the proof and recall the notation $h$ and $M_0$.

- **Injection between the character groups.** The following lemma can be found in [Oes84] as an application of [Oes84, Lem. Chap. 6, §5.1] or [Oes84, Cor. A.3.5], but we are in an easier situation.

**Lemma A.4.** Let $T$ be a $k$-torus. Then the subtorus $T$ of $R_{k'/k}(T_{k'})$ is the maximal torus and the quotient $R_{k'/k}(T_{k'})/T$ is unipotent.

**Proof.** We may assume that $k$ equals its separable closure $k_s$. In that case, the quotient $R_{k'/k}(T_{k'})/T$ is $R_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m$ and one has

$$R_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m(k) = R_{k'/k}(\mathbb{G}_m)(k)/\mathbb{G}_m(k) = ((k')^x)^s/(k^x)^s.$$

Hence all $k$-points of $R_{k'/k}(T_{k'})/T$ are killed to the $p^h$-power. This implies that $R_{k'/k}(T_{k'})/T$ is unipotent and thus $T$ is the maximal torus. \qed

Lemma A.4 implies:

**Consequence A.5.** The quotient group $M_0/M$ is unipotent and the restriction homomorphism $\widehat{M}_0(k) \to \widehat{M}(k)$ is injective.

**Proof.** We may assume that $k = k_s$. Thus $M$ is split, in other words $M$ is a product of finitely many copies of $\mathbb{G}_m$ and of finitely many groups of roots of unity $\mu_n$. Also $M_0$ is a product of the same number of copies of $R_{k'/k}(\mathbb{G}_m)$ and of groups $R_{k'/k}(\mu_n)$. According to Oesterlé’s Lemma just above, the quotient group $R_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m$ is unipotent and so is $R_{k'/k}(\mu_n)/\mu_n$ (the latter is indeed a subgroup of the first group). This shows that $M_0/M$ is unipotent.

We no longer assume $k = k_s$ and show that $\widehat{M}_0(k) \to \widehat{M}(k)$ is injective. Let $\chi \in \widehat{M}_0(k)$ becoming trivial when restricted to $M \hookrightarrow M_0$. Then $\chi$ may be quotiented to yield a homomorphism $\tilde{\chi} : M_0/M \to \mathbb{G}_m$. Since $M_0/M$ is unipotent, [DG70, Exp. XVII, Prop. 2.4] implies that $\tilde{\chi}$ is trivial. Thus $\chi$ is trivial, whence the result. \qed
• **The case** \( M = \mathbb{G}_m \). Here we treat the case when the algebraic \( k \)-group \( M \) of finite type and of multiplicative type, is the multiplicative group \( \mathbb{G}_m \).

There is an identification \( \mathbb{Z} \xrightarrow{\sim} \mathbb{G}_m(k), r \mapsto (x \mapsto x^r) \). We want to show that the image of the one-to-one homomorphism \( R : R_{k'/k}(\mathbb{G}_m,k')(k) \hookrightarrow \mathbb{G}_m(k) \) is exactly the subgroup \( p^h\mathbb{Z} \).

First of all, \( p^h \) is in the image of \( R \) : Indeed, for every \( k \)-algebra \( A \), the map

\[
\begin{cases}
(A \otimes_k k')^\times & \rightarrow A^\times \\
\sum_i x_i \otimes \lambda_i & \mapsto \sum_i x_i^{p^h} \lambda_i^{p^h}
\end{cases}
\]

is well-defined and is a group homomorphism. All these homomorphisms together form an algebraic group homomorphism \( R_{k'/k}(\mathbb{G}_m) \rightarrow \mathbb{G}_m \) whose restriction to \( \mathbb{G}_m \) is \( x \mapsto x^{p^h} \).

Secondly, \( R_{k'/k}(\mathbb{G}_m,k')(k) \) is a subgroup of \( \mathbb{Z} \) and has a generator \( \rho \). The integer \( r = R(\rho) \) is non zero and we may assume that it is a positive

To show that \( r = p^h \), write \( r = p^s s \) for some positive integer \( s \) prime to \( p \). Since \( p^h \in r\mathbb{Z} = \text{Im}(R) \), \( s \) must be \( 1 \) and \( \alpha \leq h \). Thus, both homomorphisms \( (\cdot)^{p^h}, \rho^{p^h-\alpha} : R_{k'/k}(\mathbb{G}_m) \rightarrow \mathbb{G}_m \) become equal when restricted to \( \mathbb{G}_m \subset R_{k'/k}(\mathbb{G}_m) \), so they are equal. Looking at \( k \)-points we find

\[
\forall x \in (k')^\times, x^{p^h} = \rho(x)^{p^h-\alpha}.
\]

But the Frobenius homomorphism is injective, so

\[
\forall x \in (k')^\times, x^{p^\alpha} = \rho(x).
\]

This shows that \( (k')^\times)^{p^\alpha} = \rho(k')^\times \subseteq k^\times \), whence \( \alpha = h \).

In conclusion, Proposition A.2 is verified for the multiplicative group.

• **The case** \( M = \mu_n \). We treat the case when \( M = \mu_n \), the algebraic \( k \)-group of \( n \)-th roots of unity for some positive integer \( n \). By the Chinese Remainder Theorem, \( \mu_n \) is a product of \( \mu_{n_0} \) and \( \mu_{n_1} \) where \( n_0 \) is prime to \( p \) and \( n_1 \) is a non negative integer. We have thus to treat two cases : the case \( n \) is prime to \( p \) and \( n \) is a \( p \)-power. Assume for the remaining that we are in one of these situations.

Similarly to the previous point, there is an identification \( \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \hat{\mu}_n(k), r \mapsto (x \mapsto x^r) \). We want to show that the image of the one-to-one homomorphism \( R : R_{k'/k}(\mu_n)(k) \hookrightarrow \hat{\mu}_n(k) \) is exactly the subgroup \( p^h\mathbb{Z}/n\mathbb{Z} \).

We define a character of \( R_{k'/k}(\mu_n) \) whose image in \( R \) is \( p^h \) as follows : for all \( k \)-algebra \( A \), the map

\[
\begin{cases}
\{ a \in (A \otimes_k k')^\times \mid a^n = 1 \} & \rightarrow \{ a \in A^\times \mid a^n = 1 \} \\
\sum_i x_i \otimes \lambda_i & \mapsto \sum_i x_i^{p^h} \lambda_i^{p^h}
\end{cases}
\]
is well defined and all these maps together form a group homomorphism $\chi_0 : R_{k'/k}(\mu_n) \to \mu_n$ such that $R(\chi_0) = p^h \in \mathbb{Z}/n\mathbb{Z}$.

Let $\phi$ be a character of $R_{k'/k}(\mu_n)$. Write $r$ for the representative of $R(\phi) \in \mathbb{Z}/n\mathbb{Z}$ such that $0 \leq r \leq n - 1$ and let’s show that $r$ is a multiple of $p^h$ in $\mathbb{Z}/n\mathbb{Z}$. The key argument to establish Proposition A.2 for $\mu_n$ is:

**Lemma A.6.** The character $\phi$ can be extended to the whole group $R_{k'/k}(\mathbb{G}_{m,k'})$.

*Proof.\diamond$ In case $n$ is coprime to $p$, then $\mu_n$ is smooth and according to [CGP15, Cor. A.5.4(3)] the sequence of algebraic groups

$$1 \to R_{k'/k}(\mu_n) \to R_{k'/k}(\mathbb{G}_{m,k'}) \to R_{k'/k}(\mathbb{G}_{m,k'}) \to 1 \tag{3}$$

is exact. Pushing (3) with respect to $\phi$ yields a commutative diagram with exact rows

$$1 \to R_{k'/k}(\mu_{n,k'}) \to R_{k'/k}(\mathbb{G}_{m,k'}) \to R_{k'/k}(\mathbb{G}_{m,k'}) \to 1.$$

But $R_{k'/k}(\mathbb{G}_{m,k'})$ has trivial Picard group (it is an open subscheme of the affine space $A^{\mathbb{G}_{m,k'}}_k$), so the second row of (4) is isomorphic to the trivial extension of $R_{k'/k}(\mathbb{G}_{m,k'})$ by $\mathbb{G}_m$. Thus there exists a retraction $H \to \mathbb{G}_m$ of $\mathbb{G}_m \to H$ and its composite with $f$ yields a scheme morphism $\tilde{\phi} : R_{k'/k}(\mathbb{G}_{m,k'}) \to \mathbb{G}_m$ extending $\phi$. By Rosenlicht’s Lemma ([San81, Lem. 6.5]), it comes that $\tilde{\phi}$ is a group homomorphism.

$\diamond$ In case $n = p^m$, we consider the exact sequence of algebraic groups

$$1 \to R_{k'/k}(\mu_{n,k'}) \to R_{k'/k}(\mathbb{G}_{m,k'}) \to R_{k'[k^{p^m}]/k}(\mathbb{G}_{m,k[k^{p^m}]}) \to 1 \tag{5}$$

and we argue as in the first case. $\square$

We obtain this way a group homomorphism $\tilde{\rho}$ from $R_{k'/k}(\mathbb{G}_{m,k'})$ into $\mathbb{G}_m$ that extends $\rho$. On one hand, $\tilde{\rho}$ corresponds to an integer $s$ equal to $r$ modulo $n$ via $R_{k'/k}(\mathbb{G}_{m,k'})(k) \to \mathbb{G}_m(k) \cong \mathbb{Z}$. On the other hand, as it has been seen for the case $M = \mathbb{G}_m$, the integer $p^h$ must divide $s$. Modulo $n$ we get what was wanted and finish the proof of Proposition A.2 for groups of roots of unity.

- **The case $M$ is split.** If $M$ is a split $k$-group of multiplicative and finite type, then $M$ is a product of finitely many $\mathbb{G}_m$ and $\mu_n$ (group of roots of unity). Proposition A.2 has been showed for $\mathbb{G}_m$ and for all $\mu_n$, whence the result for the split $M$. 

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• **The case \(M\) is arbitrary.** Every \(k\)-group \(M\) of multiplicative and finite type is split over \(k_s\). Thus, according to what has been done above, the image of

\[
\hat{M}_0(k_s) \rightarrow \hat{M}(k_s)
\]

is \(p^h\hat{M}(k_s)\). Restricting (6) to elements that are invariant under the absolute Galois group of \(k\), we find that the image of \(\hat{M}_0(k) \rightarrow \hat{M}(k)\) is \(\left(\hat{M}(k_s)\right) \cap \left(p^h\hat{M}(k_s)\right)\), that is \(p^h\hat{M}(k)\).

**Proof of Proposition A.2 - The end**

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