Response of Composite Laminate Beam Using Higher Order Beam Theory

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Abstract. The present work proposes the composite laminate beam behaviour using higher order beam theory (HBT). The higher order displacement field with twelve degrees of freedom and nine node isoparametric element is used for the finite element formulation. The finite element model for linear static analysis is developed using MATLAB code. The numerical results are obtained for the deflection of beam for arbitrary boundary conditions, stacking angle, aspect ratio and different loading conditions. The results are validated using the literature and it shows good agreement.

1. Introduction

Now a days requirement is a material which have high strength to weight and stiffness to weight ratios for boosting the air, space and naval industries and this need is fulfilled by the composite materials. Composites Laminate seems to be the best materials for different applications in engineering. Their strain energy reserves are insufficient as well as no yield limit, but different local failure modes can store a large amount of energy. Most of the structures in these industries can be modelled as a beam. Generally composite laminate beams were analysed using classical or first order beam theories [1]. In Classical beam theory (CBT) the effect of transverse shear deformation was neglected so that deformation is underestimated. Classical beam theory has been modified called as first-order shear deformation beam theory (FBT). Since FBT considers top and bottom surfaces are stress free one needs to add shear correction factor to avoid discrepancies in solution [2]. As the influence of shear correction factor in CBT and FBT was noticeable, it cannot be neglected. That is why shear deformation theory of higher order for composites laminates has been developed to avoid use of shear correction factor but the higher order terms were considered in displacement field model. Different approaches can be used to develop beam theories of higher order. In-plane displacement with respect to thickness and nonlinear distribution were considered by for developing higher order beam theory [3]. They used up to cubic power of \( z \) for in plane displacement. The higher order theory for plates was developed which considers parabolic distribution of shear strains in thickness direction [4-6]. To evaluate the behaviour of symmetric and unsymmetric composite laminates the efficient higher order model for displacement field was developed [7]. From the literature reviewed it can be concluded that the higher order beam theory gives exact results as compared to CBT and FBT for laminated composites. Most of the study till now is done for laminated composite plates.
The present work focuses on the HBT for laminated composite beam. The displacement field up to higher order cubic power of $z$ is considered.

2. General Formulation

The displacement field can be expanded to cubic power of thickness coordinate as follows

$$
\begin{align*}
    u(x, y, z) &= u_0(x, y) + z \theta_x(x, y) + z^2 f_x(x, y) + z^3 \xi_x(x, y) \\
    v(x, y, z) &= v_0(x, y) + z \theta_y(x, y) + z^2 f_y(x, y) + z^3 \xi_y(x, y) \\
    w(x, y, z) &= w_0(x, y) + z \theta_z(x, y) + z^2 f_z(x, y) + z^3 \xi_z(x, y)
\end{align*}
$$

where along the $(x, y, z)$ coordinates, $(u, v, w)$ are the displacements, the parameter $u_0, v_0, w_0, \theta_x, \theta_y, \theta_z$ are the mid-plane displacements and rotations, $\xi_x, \xi_y, \xi_z$ are the corresponding higher order terms in the Taylor's series expansion.

The corresponding vectors for strain are

$$
\begin{align*}
    \epsilon_x &= \frac{\partial u}{\partial x}, \\
    \epsilon_y &= \frac{\partial v}{\partial y}, \\
    \epsilon_z &= \frac{\partial w}{\partial z} \\
    \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \\
    \gamma_{xz} &= \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}, \\
    \gamma_{yz} &= \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}
\end{align*}
$$

The linear vectors for strain are

$$
\{\epsilon\} = \{\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^T
$$

Associated strains for assumed displacement field are

$$
\begin{align*}
    \epsilon_x &= \xi_{x0} + z k_x + z^2 \xi_{x0}^2 + z^3 k_x^2 \\
    \epsilon_y &= \xi_{y0} + z k_y + z^2 \xi_{y0}^2 + z^3 k_y^2 \\
    \epsilon_z &= \xi_{z0} + z k_z + z^2 \xi_{z0}^2 + z^3 k_z^2 \\
    \gamma_{xy} &= \xi_{xy0} + z k_{xy} + z^2 \xi_{xy0}^2 + z^3 k_{xy}^2 \\
    \gamma_{xz} &= \xi_{xz0} + z k_{xz} + z^2 \xi_{xz0}^2 + z^3 k_{xz}^2 \\
    \gamma_{yz} &= \xi_{yz0} + z k_{yz} + z^2 \xi_{yz0}^2 + z^3 k_{yz}^2
\end{align*}
$$

where,

$$
\begin{align*}
    \xi_{x0} &= \frac{\partial u_0}{\partial x}, \\
    \xi_{y0} &= \frac{\partial v_0}{\partial y}, \\
    \xi_{z0} &= \frac{\partial w_0}{\partial z}, \\
    k_x &= \frac{\partial \theta_x}{\partial x}, \\
    k_y &= \frac{\partial \theta_y}{\partial y}, \\
    k_z &= \frac{\partial \theta_z}{\partial z}, \\
    k_{xy} &= 2 \phi_x, \\
    k_{xz} &= 3 \xi_x, \\
    k_{yz} &= \frac{\partial \phi_y}{\partial z}
\end{align*}
$$


\[ k_{xy}^* = \frac{\partial \xi_x}{\partial y} + \frac{\partial \xi_y}{\partial x}, \quad \xi_{y0} = \theta_y + \frac{\partial w_0}{\partial y}, \quad k_{yc} = 2\phi_y + \frac{\partial \phi_y}{\partial y} \]

\[ \xi_{yc0} = 3\xi_y + \frac{\partial \phi_y}{\partial y}, \quad k_{zc} = 2\phi_z + \frac{\partial \phi_z}{\partial x} \]

Vectors of strain corresponding to middle plane are

\[ \{\varepsilon\} = \left\{ \xi_{x0}, \xi_x, \xi_{y0}, \xi_y, \xi_{z0}, \xi_z, \xi_{xy0}, \xi_{xy}, \xi_{yz0}, \xi_{yz}, \xi_{xz0}, \xi_{xz} \right\} \tag{6} \]

Vectors of displacement in middle plane are

\[ \{\Delta\} = \left\{ u_0, v_0, w_0, \theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z, \psi_x, \psi_y, \psi_z \right\} \tag{7} \]

The relation of stress to strain for a lamina can be written as

\[ \sigma_x = \begin{bmatrix} \sigma_1 & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{12} & \sigma_2 & Q_{23} & Q_{24} & 0 & 0 \\ Q_{13} & Q_{23} & \sigma_3 & Q_{34} & 0 & 0 \\ Q_{14} & Q_{24} & Q_{34} & \sigma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55} & \sigma_{56} \\ 0 & 0 & 0 & 0 & \sigma_{56} & \sigma_{66} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{8} \]

Here \( \sigma_i, \sigma_j, \sigma_k, \tau_{xy}, \tau_{yz}, \tau_{xz} \) and \( \xi_i, \xi_j, \xi_k, \gamma_{xy}, \gamma_{yz}, \gamma_{xz} \) are stresses and strains with respect to \((x, y, z)\). Where \( Q_{ij} \) is elastic material constants and \( \theta_k \) is the angle of fibre orientation, of lamina as described in Reddy’s book [9].

3. Formulation for Finite Element

The matrix of stiffness for element \( k^{(e)} \) can be represented as

\[ k^{(e)} = \int_{A^{(e)}} \{ B \}^T [D] \{ B \} dA \tag{9} \]

The vector for displacement \( \{\Delta\} \) can be given as

\[ \{\Delta\} = \sum_{i=1}^{MM} N_i \{\Delta_i\} \tag{10} \]

where, \( MM \) = total number of nodes per element, \( N_i \) are the shape functions for \( i^{th} \) node.

The displacement and strain are related as

\[ [B] = [L] N_i \tag{11} \]

where \([L]\) is operator for differentiation.

\([D]\) is a matrix for property of material and calculated as
\[ D = \sum_{k=1}^{NL} \int_{z_{k-1}}^{z_k} [T] \{Q\} \{T\} dz \]  

(12)

\[
[D] = \begin{bmatrix}
A_{ij} & B_{ij} & D_{ij} & BE_{ij} & 0 & 0 & 0 & 0 \\
B_{ij} & D_{ij} & E_{ij} & BF_{ij} & 0 & 0 & 0 & 0 \\
D_{ij} & E_{ij} & F_{ij} & BG_{ij} & 0 & 0 & 0 & 0 \\
AE_{ij} & AF_{ij} & AG_{ij} & H_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & AA_{ij} & BB_{ij} & AA_{ij} & AA_{ij} \\
0 & 0 & 0 & 0 & BB_{ij} & DD_{ij} & EE_{ij} & FF_{ij} \\
0 & 0 & 0 & 0 & DD_{ij} & EE_{ij} & FF_{ij} & GG_{ij} \\
0 & 0 & 0 & 0 & EE_{ij} & FF_{ij} & GG_{ij} & HH_{ij}
\end{bmatrix}
\]  

(13)

with,

\[
\begin{bmatrix}
A_{ij} & B_{ij} & D_{ij} & E_{ij} & F_{ij} & G_{ij} & H_{ij}
\end{bmatrix} = \int_{-h/2}^{h/2} Q_{ij} \left(1, z, z^2, z^3, z^4, z^5, z^6\right) dz \quad i, j = 1, 2, 3, 4
\]  

(14)

\[
\begin{bmatrix}
AA_{ij} & BB_{ij} & DD_{ij} & EE_{ij} & FF_{ij} & GG_{ij} & HH_{ij}
\end{bmatrix} = \int_{-h/2}^{h/2} Q_{ij} \left(1, z, z^2, z^3, z^4, z^5, z^6\right) dz \quad i, j = 5, 6
\]  

(15)

The stiffness matrix for element in natural coordinate system with respect to \((\xi, \eta)\) can be represented as

\[
[k^{(e)}] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] d\xi d\eta
\]  

(17)

Since Jacobean \([J]\) is

\[
[J] = \begin{bmatrix}
\frac{dx}{d\xi} & \frac{dy}{d\xi} \\
\frac{dx}{d\eta} & \frac{dy}{d\eta}
\end{bmatrix}
\]  

(18)
4. Solution Technique

The potential energy approach is used here which gives

\[ P.E. = \sum_{e=1}^{ME} U^{(e)} = \sum_{e=1}^{ME} U_1^{(e)} + \sum_{e=1}^{ME} U_2^{(e)} \]  \hspace{1cm} (19)

The strain energy with respect to \( U_1^{(e)} \) is given by

\[ U_1^{(e)} = \frac{1}{2} \int_A \left[ \{ \sigma \}^T \{ D \} \{ \sigma \} \right] dA \]  \hspace{1cm} (20)

The strain in the middle plane with respect to displacements expressed as

\[ \{ \sigma \} = [L] \{ \Delta \} \]  \hspace{1cm} (21)

Equation can be expressed as

\[ U_1^{(e)} = \frac{1}{2} \int_A \left[ \{ \Delta \}^T [L]^T \{ D \} [L] \{ \Delta \} \right] dA \]  \hspace{1cm} (22)

\[ U_1^{(e)} = \frac{1}{2} \int_A \left( \sum_{i=1}^{NN} \{ \Delta_i \}^T [L]^T N_i \right) \{ D \} \left( \sum_{i=1}^{NN} [L] \{ \Delta_i \} N_i \right) dA \]  \hspace{1cm} (23)

\[ U_1^{(e)} = \frac{1}{2} \int_A \left( \{ q \}^T [B]^T \{ D \} [B] \{ q \} \right) dA \]  \hspace{1cm} (24)

\[ U_1^{(e)} = \frac{1}{2} \int_A \{ \Delta \}^T \left[ \overline{F} \right] dA \]  \hspace{1cm} (25)

Where \( \left\{ \overline{F} \right\} \) vector for load corresponding to every DOF.

For current transverse loading case \( \left. \left\{ \overline{F} \right\} \right. \) written as

\[ \left. \left\{ \overline{F} \right\} \right. = \{ 0 \ 0 \ F_z \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \}^T \] \hspace{1cm} (26)

Substituting for \( \{ \Delta \} \)

\[ U_2^{(e)} = \int_A \left( \sum_{i=1}^{NN} N_i \{ \Delta_i \}^T \right) \{ \overline{F}_i \}^{(e)} dA \] \hspace{1cm} (27)

\[ = \{ q \}^{(e)T} \{ F_i \}^{(e)} \]

\[ \left\{ F_i \right\}^{(e)} = \int_A \left\{ N_i \right\}^{(e)T} \left\{ \overline{F} \right\}^{(e)} dA \] \hspace{1cm} (28)

The present problem can be solved using equation of equilibrium for composite laminate governed by

\[ \left[ K_f \right] \left\{ q \right\}_f = \left\{ F_i \right\} \] \hspace{1cm} (29)

where \( \left[ K_f \right] = \sum_{e=1}^{NE} [k_{ij}^{(e)}] \) called matrix for global stiffness, \( \left\{ F_i \right\} = \sum_{e=1}^{NE} \left\{ F_i \right\}^{(e)} \) is a global force vector, \( \left\{ q \right\}_f = \sum_{e=1}^{NE} \left\{ q \right\}^{(e)} \) is a vector for deflection global.
Eq. (29) is then determined for composite laminate beam under various conditions of transverse loading to get displacements.

5. Numerical Result Tables

Table 1 to 5 shows comparison of deflection of laminated composite beam obtained by using different boundary conditions, loading conditions, lamination schemes and aspect ratios with the results available in various literature. The results show good agreement with the results of literature.

Table 1. Non dimensionalised deflection of middle span (\(\bar{w}\)) for lamination scheme \([0/90/0]\) beam for multiple boundary conditions under action of load distributed uniformly

\[
\left( \frac{E_1}{E_2} = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25 \right) \left( \bar{w} = \frac{(100E_2h^3)}{qL^4}w_{max} \right)
\]

| \(a/h\) | H-H | C-C | C-H |
|-------|-----|-----|-----|
| 5     | 2.398 | 1.538 | 1.946 |
|       | 2.412 | 1.537 | 1.952 |
|       | 2.432 | 1.69  | 2.0714 |
| 10    | 1.09  | 0.532 | 0.738 |
|       | 1.096 | 0.532 | 0.740 |
|       | 1.107 | 0.555 | 0.767 |
| 50    | 0.661 | 0.147 | 0.279 |
|       | 0.665 | 0.147 | 0.280 |
|       | 0.666 | 0.1476 | 0.290 |

Table 2. Non-dimensionalised deflections for clamped end-free end composite laminate beam with point load \((P)\) at the free end

\[
\left( \frac{E_1}{E_2} = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25 \right) \left( \bar{w} = \frac{(100E_2h^3)}{qL^4}w_{max} \right)
\]

| Laminate \(a/h\) | \(10\)  | \(20\)  | \(100\)  |
|-------------------|--------|--------|--------|
| \((90/0)_{s}\)    | Present | 24.02  | 19.4537 | 18.214 |
|                   | Reddy[11] | 21.58  | 19.01  | 18.18  |
| \((45/-45)_{s}\)  | Present | 403    | 169.76 | 143.44 |
|                   | Reddy[11] | 87.86  | 86.28  | 85.86  |
Table 3. Non-dimensionalised deflection of middle span ($\bar{w}$) for lamination scheme [0/90] beam for multiple boundary conditions with load distributed uniformly

| $a/h$ | Murthy[8] | Present | Murthy[8] | Present | Murthy[8] | Present |
|-------|-----------|----------|-----------|----------|-----------|----------|
| 5     | 4.750     | 3.4151   | 3.668     | 2.1016   | 3.336     | 1.6513   |
| 10    | 1.924     | 2.2394   | 1.007     | 1.0634   | 0.681     | 0.6801   |
| 50    | 2.855     | 2.875    | 1.736     | 1.5911   | 1.343     | 1.1728   |

Table 4. Non-dimensionalised deflections for clamped-clamped laminated composite beam under action of uniform distributed load

$$\left(\frac{E_1}{E_2} = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, v_{12} = 0.25\right) \left(\bar{w} = \frac{(100E_2h^3}{qL^4})w_{\text{max}}\right)$$

| Laminate | UDL | Point Load |
|----------|-----|------------|
| $a/h$    | 10  | 20 | 100 | 10 | 20 | 100 |
| 0        | Present | 0.4136 | 0.1986 | 0.1277 | 1.228 | 0.445 | 0.255 |
|          | Reddy[4] | 0.425 | 0.200 | 0.128 | 0.85 | 0.4 | 0.256 |
|          | Ferreira[10] | 0.425 | 0.200 | 0.128 | - | - | - |
| 90       | Present | 3.8644 | 3.3077 | 3.1039 | 8.39 | 6.694 | 6.208 |
|          | Reddy[4] | 3.875 | 3.312 | 3.132 | 7.75 | 6.625 | 6.265 |
|          | Ferreira[10] | 3.875 | 3.312 | 3.132 | - | - | - |
| (90/0),  | Present | 0.6553 | 0.2756 | 0.1472 | 1.908 | 0.62 | 0.295 |
|          | Reddy[4] | 0.570 | 0.249 | 0.146 | 1.141 | 0.498 | 0.292 |
|          | Ferreira[10] | - | - | - | - | - | - |
| (45/-45),| Present | 4.512 | 1.083 | 0.995 | - | - | - |
|          | Reddy[4] | 2.217 | 1.895 | 1.793 | - | - | - |
|          | Ferreira[10] | 2.177 | 1.856 | 1.759 | - | - | - |

Figure 1 shows deformation of laminated composite beam for different boundary conditions and aspect ratio obtained by using present approach. Figure 2 shows deformation of laminated composite beam for [0/θ/0] lamination scheme and boundary conditions. The results show that the under clamped-free boundary conditions laminated composite beam shows more deformation for all lamination angles.
Table 5. Non-dimensionalised deflections for Hinged-Hinged laminated composite beam under action of uniform distributed load and point load

| Laminate | UDL       |   | Point Load       |   |   |
|----------|-----------|---|------------------|---|---|
|          | $a/h$     |   |                  |   |   |
| 0        | Present   | 0.9249 | 0.7000 | 0.6278 | 1.97 | 1.19 | 1.005 |
|          | Reddy[4]  | 0.9250 | 0.7000 | 0.6280 | 1.6  | 1.15 | 1.001 |
|          | Ferreira[10] | 0.925 | 0.700 | 0.628 | -    | -    | -    |
| 90       | Present   | 16.375 | 15.810 | 15.6047 | 27.15 | 25.44 | 24.96 |
|          | Reddy[4]  | 16.375 | 15.813 | 15.633 | 26.5 | 25.37 | 25.01 |
|          | Ferreira[10] | 16.373 | 15.814 | 15.624 | -    | -    | -    |
| (90,0)$_s$ | Present   | 1.2628 | 0.8487 | 0.7149 | 2.7  | 1.479 | 1.147 |
|          | Reddy[4]  | 1.137  | 0.816  | 0.7130 | 1.991 | 1.348 | 1.143 |
|          | Ferreira[10] | -     | -    | -    | -    | -    | -    |
| (45/-45)$_s$ | Present   | 6.507  | 6.1274 | 5.979 | -    | -    | -    |
|          | Reddy[4]  | 9.371  | 9.049  | 8.947 | -    | -    | -    |
|          | Ferreira[10] | 9.175 | 8.857 | 8.708 | -    | -    | -    |

Figure 1. Graph of Deformation for different aspect ratio($a/h$).

Figure 2. Graph of Deformation for various Lamination Scheme.

6. Summary and Conclusion

- The present study concludes that the higher order shear deformation beam theory using twelve degrees of freedom and nine node isoparametric element shows accurate results as compared to classical theory and first order shear deformation theory.
- The deformation for angle ply laminates shows slight discrepancies in the results.
• Lamination scheme with angle $\theta = 0$ shows less deformation as compared to other lamination scheme.
• Symmetric laminates show good agreement with results.
• For angle ply lamination scheme certain multiplying factor needs to be introduced to match the results with accurate solution available in literatures.
• The deformation obtained for uniform distributed loading condition is less as compared to concentrated point load condition.
• As aspect ratio ($a/h$) increases the deformation shows decrease in value of results.

References
[1] Maiti DK and Sinha PK 1994, Bending and free vibration analysis of shear deformable laminated composite beams by finite element method, Compos. Struct., 29, pp 421-31.
[2] Shao D, Hu S and Wang Q 2017, Free vibration of refined higher-order shear deformation composite laminated beams with general boundary conditions, Compos. Part B, 108, pp 75-90.
[3] Lo KH and Christensen RM 1977, A high-order theory of plate deformation, A J. of Appl. Mech., 44, pp 669-76.
[4] Reddy JN (1984), A simple higher-order theory for laminated composite plates, A J. of Appl. Mech.s, 51, pp 745-52.
[5] Gadade AM, Lal A and Singh BN 2016, Finite element implementation of Puck’s failure criterion for failure analysis of laminated plate subjected to biaxial loadings, Aerospace Sci. and Tech., 55, pp 227-41.
[6] Gadade AM, Lal A and Singh BN 2014, Progressive failure analysis of laminated composite plate by using higher order shear deformation theory, Appl. Mech. and Mat., 52, pp 1151-54
[7] Manjunatha BS and Kant T 1993, Different numerical techniques for the estimation of multiaxial stresses in symmetric/unsymmetric composite and sandwich beams with refined theories, J. of Reinf. Plastics and Compos., 12, pp 1-37.
[8] Murthy MVVS, Mahapatra DR, Badrinarayana K and Gopalkrishnan S 2005, A refined higher order finite element for asymmetric composite beams, Compos. Struct., 67, pp 27-35.
[9] Khdeir AA and Reddy JN 1997, An exact solution for the bending of thin and thick cross-ply laminated beams, Compos. Struct., 37, pp 195-203.
[10] Ferreira AJM, Roque CMC and Martins PALS 2004, Radial basis functions and higher-order shear deformation theories in the analysis of laminated composite beams and plates, Compos. Struct., 66, pp 287-93.
[11] Reddy JN 2004, Mechanics of laminated composite plates and shell: theory and analysis, 2nd edition, CRC Press LLC, USA, p 194