Area spectrum of near-extremal SdS black holes via the new interpretation of quasinormal modes

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Motivated by the recent work about a new physical interpretation of quasinormal modes by Maggiore, we investigate the quantization of near-extremal Schwarzschild-de Sitter black holes in the four dimensional spacetime. Following Kunstatter’s method, we derive the area and entropy spectrum of near-extremal Schwarzschild-de Sitter black holes which differs from Setare’s result. Furthermore, we find that the derived a universal area spectrum is $2\pi n$ which is equally spaced.

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I. INTRODUCTION

The quantization of the black hole horizon area is a fascinating subject. Since an equally spaced entropy spectrum was firstly predicted by Bekenstein in 1974, there have been many attempts to derive the entropy spectrum directly from the dynamical modes of the classical theory. However, there has been little known about the direct physical connection between the classical dynamical quantities that give rise to Bekenstein-Hawking entropy and the corresponding microscopic degrees of the quantum black hole. An important step in this direction was made by Hod by a semiclassical consideration of the macroscopic oscillation modes of black holes. In particular, he assumed an equally discrete area spectrum and used the existence of a unique quasinormal mode frequency in the large damping limit.
to uniquely fix the spacing. Dreyer \cite{9} demonstrated that Hod’s result could be recovered in loop quantum gravity if the relevant group is taken to be $SO(3)$ rather than $SU(2)$. These developments have spurred lots of subsequent activity \cite{10,26}.

Recently, a new physical interpretation for the quasinormal modes of black holes was presented by Maggiore \cite{27}. According to Maggiore’s proposal, in order to overcome or at least alleviate some difficulties raised by the Hod’s proposal \cite{8} in the interpretation of quasinormal frequencies, the Black hole perturbations are modeled in terms of a collection of damped harmonic degrees of freedom. In addition, he indicated that the real frequencies of the equivalent damped harmonic oscillators were $(\omega_R^2 + \omega_I^2)^{1/2}$, rather than simply $\omega_R$. Motivated by Maggiore’s work, Vagenas \cite{28} utilized the new proposal to the interesting case of Kerr black hole and proposed a new interpretation as a result of Maggiore’s idea, for the frequency that appears in the adiabatic invariant of a black hole. In a more recent paper \cite{29}, a universal form for the Kerr and Schwarzschild quantum area spectra was established by Medved by presenting a simple but vital modification to a recent treatment on the Kerr (or rotating black hole) spectrum. Although the above considerations are still somewhat speculative, they certainly propose a reasonable physical interpretation on the spectrum of the black hole quasinormal modes. It is, therefore, of interest to study the area spectrum of other black holes, in particular, near-extremal black holes for which the quasinormal frequencies are quite different from that of non-extremal black holes.

In this article our aim is to investigate the area and entropy spectrum of near-extremal Schwarzschild-de Sitter black holes in four dimensional spacetime by adopting Maggiore’s proposal. According to Maggiore’s work, the frequency of the harmonic oscillator is $\omega_0 = (\omega_R^2 + \omega_I^2)^{1/2}$. Although the author of Ref. \cite{22} has discussed the area and entropy spectrum, the most interesting case is that of highly excited quasinormal modes, whose frequency is $\omega_0 = |\omega_I|$ rather than simply $\omega_0 = \omega_R$. Since, for $\omega_I \gg \omega_R$, this observation will change the physical understanding of the black hole spectrum and examine the various results of the literature. In the next section, we will try to derive the area and entropy spectrum by extending Kunstatter’s method.
II. QUASINORMAL MODES OF NEAR EXTREMAL SDS BLACK HOLES

The near-extremal Schwarzschild-de Sitter (SdS) spacetime in four dimensions is a non-trivial case with a non-asymptotically flat spacetime. General SdS spacetimes have a metric of the form

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_2^2, \]

with

\[ f(r) = 1 - \frac{2M}{r} - \frac{r^2}{L_{ds}^2}, \]

where \( M \) denotes the black hole mass and \( L_{ds}^2 \) is the de Sitter curvature radius, which related to the cosmological constant \( \Lambda \) by \( L_{ds}^2 = 3/\Lambda \). The spacetime possesses two horizons: the usual black hole horizon locates at \( r = r_+ \) and the cosmological horizon locates at \( r = r_c \), where \( r_+ < r_c \). We assume that the three roots of the equation \( f(r) = 0 \) are \( r_+ \), \( r_c \), and \( r_0 \) respectively. In terms of these roots, \( f(r) \) can be rewritten as

\[ f(r) = \frac{1}{L_{ds}^2 r}(r - r_+)(r_c - r)(r - r_0), \]

with \( r_0 = -(r_+ + r_c) \). In addition, \( M \) and \( L_{ds}^2 \) as functions of these roots can be expressed as

\[ L_{ds}^2 = r_+^2 + r_+ r_c + r_c^2, \]
\[ 2ML_{ds}^2 = r_+ r_c (r_+ + r_c). \]

Defined by the relation \( \kappa_+ \equiv \frac{1}{2}(df/dr)|_{r=r_+} \), the surface gravity \( \kappa_+ \) can be written as

\[ \kappa = \frac{(r_c - r_+)(r_+ - r_0)}{2L_{ds}^2 r_+}. \]

Let us now specialize to a non-trivial case with a non-asymptotically flat spacetime called the near-extremal SdS black hole. As for this case, the cosmological horizon \( r_c \) is very close to the black hole horizon \( r_+ \). Hence, one can make the following approximations:

\[ r_0 \sim -2r_+, \quad L_{ds}^2 \sim 3r_+^2, \quad \kappa_+ \sim \frac{r_c - r_+}{2r_+}. \]

Cardoso and Lemos studied firstly the analytical quasinormal mode spectrum for the near-extremal SdS black hole \[16], and they concluded that the asymptotic quasinormal
frequencies of near-extremal SdS black hole are given by the simple expression

$$\omega = \kappa_+ \left[ \sqrt{\frac{\nu_0}{\kappa_+^2}} - 1/4 - i(n + 1/2) \right], \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (7)

where

$$\nu_0 = \kappa_+^2 l(l + 1),$$ \hspace{1cm} (8)

for scalar and electromagnetic perturbations, and

$$\nu_0 = \kappa_+^2 (l + 2)(l - 1),$$ \hspace{1cm} (9)

for gravitational perturbations, $l$ is the angular quantum number. Very recently, by computing the Lyapunov exponent which is the inverse of instability timescale associated with the geodesic motion, Cardoso et al presented that quasinormal modes of black holes are determined by the parameters of the circular null geodesics in the eikonal limit ($l \gg 1$) \[30\]. And then, they found a simple analytical quasinormal modes for the near-extremal SdS black holes in $d = 4$:

$$\omega_{BQNM} = \kappa_+ [l - i(n + 1/2)].$$ \hspace{1cm} (10)

Though the form of Eq. (10) is similar with Eq. (7), there exists an important difference, namely, Eq. (10) is valid for the low-lying modes $n \ll l$ with $l \gg 1$ while Eq. (7) is valid in the limit $n \to \infty$. Here we are interested in highly excited black holes, hence the area spectrum of near-extremal SdS black holes should be discussed in the next section with the aid of Eq. (7) found in \[16\].

III. AREA SPECTRUM OF NEAR EXTREMAL SDS BLACK HOLES

We now utilize a new physical interpretation of quasinormal modes proposed by Maggiore \[27\] to the near-extremal SdS black hole and attempt to derive the area spectrum by following Kunstatter’s method.

Given a system with energy $E$, one can show the first law of black hole thermodynamics

$$dM = \frac{1}{4} T_H dA,$$ \hspace{1cm} (11)
here we adopt the mass definition $E = M$ according to the Ref. [31]. Employing Kunstatter’s method [11], the adiabatically invariant integral can be written as

$$I = \int \frac{dE}{\omega(E)}. \quad (12)$$

At this point we should clarify what the frequency $\omega$ in the denominator should be. As for a harmonic oscillator, the frequency is the vibrational frequency that corresponds to the system’s energy $E$ under a slow variation of a parameter related to the energy, hence a small variation energy $dE$ can be created and the quantity $E/\omega$ is an adiabatic invariant. According to Maggiore’s proposal, one has to model the Black hole perturbations in terms of a collection of damped harmonic degrees of freedom if one wants to avoid the difficulties raised by Hod’s conjecture in the interpretation of quasinormal frequencies [27]. In addition, Maggiore indicates that the real frequencies of the equivalent damped harmonic oscillators should be written as

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2}, \quad (13)$$

where the frequency of the harmonic oscillator becomes $\omega_0 = \omega_R$ for the case of long-lived quasinormal modes $\omega_I \to 0$ and becomes $\omega_0 = |\omega_I|$ for the most interesting case of highly excited quasinormal modes $\omega_I \gg \omega_R$. We are interested in highly excited black holes, i.e. $n$ is large, hence the proper frequency should be $\omega_0 = |\omega_I|$. In this framework, we consider the transition $n \to n - 1$ for a near-extremal SdS black hole. Thus according to Eq. (12) and (13), the absorbed energy is

$$\delta M = h[(\omega_0)_n - (\omega_0)_{n-1}]$$
$$= h[(\omega_I)_{n-1} - (\omega_I)_n]$$
$$= h\kappa_+, \quad (14)$$

i.e. the transition frequency is

$$\omega = [(\omega_I)_{n-1} - (\omega_I)_n] = \kappa_+. \quad (15)$$

Therefore, the adiabatic invariant (12) can be rewritten as

$$I = \int \frac{dE}{\omega} = \int \frac{dM}{\omega} = \frac{M}{\kappa_+} + c, \quad (16)$$
where \( c \) is a constant. Bohr-Sommerfeld quantization then implies that the mass spectrum is equally spaced, namely

\[
M = n\hbar \kappa_+.
\] (17)

We set \( r_c - r_+ = \Delta r \). By using Eqs. (4) and (14) we get

\[
\delta M = \frac{3r_+ \Delta r \delta r_+}{2L_{ds}^2} = \hbar \kappa_+,
\] (18)

On the other hand, the black hole horizon area is given by

\[
A_+ = 4\pi r_+^2,
\] (19)

Implementing the variation of the black hole horizon and using Eq. (18), we have

\[
\delta A_+ = 8\pi r_+ \delta r_+ = 8\pi \frac{2L_{ds}^2 \hbar \kappa_+}{\Delta r}.
\] (20)

Substituting Eq. (6) into Eq. (20), one can obtain

\[
\delta A_+ = 8\pi \hbar.
\] (21)

Hence, the area spectrum of near-extremal SdS black hole has the following form:

\[
A_n = 8\pi n \hbar.
\] (22)

According to the definition of the Benkenstein-Hawking entropy, we have

\[
S = \frac{A_n}{4\hbar} = 2\pi n.
\] (23)

Thus, in the highly damping limit, we have derived the area and entropy spectrum of the near-extremal SdS black hole. In particular, we find that the spectra is equally spaced. In addition, for the eikonal limit \( l \gg 1 \), large-\( l \) modes will dominate the black hole’s response to perturbations. Using Eq. (10) and the new proposal, we have \( \omega_0 = \omega_R \) and the transition frequency \( \omega = (\omega_R)_l - (\omega_R)_{l-1} = \kappa_+ \) for \( l \gg n \). In this framework, we could also obtain the same area spectrum.

### IV. CONCLUSION

In this paper, we have succeeded in utilizing a new physical interpretation of quasinormal modes which was proposed by Maggiore [27] to the near-extremal Schwarzschild de Sitter
black hole. According to Maggiore’s proposal, we consider $\omega = |\omega_I|_n - |\omega_I|_{n-1}$ to be the frequency of the adiabatic invariant of a black hole that appears in the context of Kunstatter’s method, since we are working with the highly damped quasinormal modes. In this framework, we have obtained the area and entropy spectrum of event horizon which is different from Setare’s result [22]. In addition, it is noteworthy that the derived area spectrum of near-extremal SdS black holes in the four dimensional spacetime is equally spaced which coincides with Bohr-Sommerfeld quantization condition. On the other hand, the derived quantum area is $\Delta A = 8\pi \hbar$. As one can see in Ref. [27, 28], $\Delta A = 8\pi \hbar$ for Schwarzschild and Kerr black holes. Therefore, consistent with claim of the paper [32], the quantum area $\Delta A$ may be universal for all black holes. Furthermore, it would be of interest to generalize our work to other black holes. Our result will also be supported by observable effects in the future.

Acknowledgments

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