Informational properties of the family of cubic rank transmuted distributions

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Abstract

Recently, cubic rank transmuted (CRT) distribution was introduced and studied by Granzotto et al. (2017). In this work, we consider CRT distribution and establish some of its information theoretic properties. First, divergence measures between CRT distribution and each of its components have been studied. In this regard, we consider Kullback-Leibler, harmonic mean and chi-square divergence measures. Further, we derive Shannon entropy, CRT Shannon entropy, Gini information, CRT Gini information and Fisher information matrix for CRT distribution.

Keywords: CRT Shannon entropy; CRT Gini information; Kullback-Leibler divergence; Harmonic mean divergence; Jeffreys’ divergence.

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1 Introduction

In recent statistical literature, the class of transmuted distributions has received a lot of attention from various researchers. Shaw and Buckley (2009) introduced a new family of distributions, dubbed as transmuted family of distributions. These authors used quadratic rank transmutation map \( k(u) = u + \lambda u(1 - u), \ u \in [0, 1], \ |\lambda| \leq 1 \) to obtain a transmuted family of distributions from a given baseline cumulative distribution function (CDF). Let \( F \) be the CDF of a baseline distribution. Then, a transmuted distribution’s CDF is given by (see Shaw and Buckley (2009))

\[
F_T(x) = (1 + \lambda)F(x) - \lambda F^2(x), \quad |\lambda| \leq 1, \quad x \in \mathcal{R},
\]

(1.1)

where \( \mathcal{R} \) denotes set of real numbers. From (1.1), it is clearly observed that the CDF of a transmuted family of distributions is a linear mixture of \( F \) and \( F^2 \), where \( F^2 \) is the CDF

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of maximum of a random sample of sample size two drawn from a population with CDF $F$. Recently, Granzotto et al. (2017) introduced another flexible family of distributions, known as family of CRT distributions. Similar to the quadratic rank transmutation map, Granzotto et al. (2017) have employed cubic ranking transmutation map to develop a new class of distributions, which is presented in the following definition.

**Definition 1.1.** Let $F$ and $f$ be the CDF and probability density function (PDF) of a baseline distribution. An absolutely continuous random variable $X$ is said to have a CRT distribution if its CDF is given by

$$F_{CT}(x) = \lambda_1 F(x) + (\lambda_2 - \lambda_1)F^2(x) + (1 - \lambda_2)F^3(x), \quad x \in \mathbb{R},$$

(1.2)

where $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$. The PDF of CRT distribution is

$$f_{CT}(x) = f(x) \left( \lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x) \right),$$

(1.3)

which can be easily obtained after differentiating (1.2) with respect to $x$.

The model presented in Definition 1.1 is more flexible than the model proposed by Shaw and Buckley (2009). Next, we briefly discuss stochastic representation of (1.2). For details, the readers are referred to Granzotto et al. (2017). Let $X_1$, $X_2$, and $X_3$ be independent and identically distributed (i.i.d) random variables with common CDF and PDF $F$ and $f$, respectively. The PDFs and CDFs of the ordered random variables $X_{1:3} = \min\{X_1, X_2, X_3\}$, $X_{2:3} = \max\{X_1, X_2, X_3\}$, and $X_{3:3} = \max\{X_1, X_2, X_3\}$, where $X_{1:3} \leq X_{2:3} \leq X_{3:3}$ are respectively given by (see Arnold et al. (2008))

$$\begin{cases} f_{\min}(x) = f(x)(3 - 6F(x) + 3F^2(x)), \\ f_{\max}(x) = 3f(x)F^2(x), \\ f_{\max}(x) = 3f(x)F^2(x), \\ f_{\max}(x) = 3f(x)F^2(x), \\ f_{\max}(x) = 3f(x)F^2(x), \end{cases}$$

(1.4)

and

$$\begin{cases} F_{\min}(x) = 1 - (1 - F(x))^3, \\ F_{\max}(x) = F(x) + F^2(x) + F^3(x). \end{cases}$$

(1.5)

Now, the CDF given by (1.2) can be expressed in terms of $F$, $F_{2:3}$, and $F_{\max}$ as

$$F_{CT}(x) = \lambda_1 F(x) + \frac{1}{3}(\lambda_2 - \lambda_1)F_{2:3}(x) + \frac{1}{3}(3 - \lambda_2 - 2\lambda_1)F_{\max}(x),$$

(1.6)

where $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$. Granzotto et al. (2017) further considered two special cases based on well known Weibull and log-logistic distributions, and proposed estimates of the unknown parameters. Besides this, many other authors have studied CRT distribution based on several particular baseline distributions. For example, Ansari and Eledum (2018) have considered CRT Pareto distribution, and examined its statistical properties. Celik (2018) has developed CRT Fréchet distribution, CRT Gumbel distribution, and CRT
Gompertz distribution, and examined their hazard rate properties. Recently, Chhetri et al. (2022) developed CRT Lindley distribution, and studied its various mathematical properties.

Modern information theory has a long history. It has started to develop since the middle of the twentieth century, particularly after the path-breaking work by Shannon (1948). Since then, it evolved into a vigorous branch of mathematics fostering the development of other scientific fields, such as statistics, biology, behavioral science, neuroscience, and statistical mechanics. After that in literature we find several formula proposed by other researchers to quantify the amount of uncertainty in a probability distribution, among which the most popular one is Rényi entropy introduced by Rényi (1961). Gini mean difference, also known as Gini information was first introduced by Gini (1912) as a measure of variability. Fisher information was proposed by Fisher (1929), provides a way of quantifying the amount of information that observations of a random sample carry about an unknown parameter or vector of parameters in a statistical model. Further, it is an important issue in many applications to find an appropriate measure of distance (or difference or discrimination) between two probability models. For this purpose, a number of divergence measures have been proposed by Kullback-Leibler divergence (see Kullback and Leibler (1951)), harmonic mean divergence (see Dacunha-Castelle et al. (2006)) and chi-square divergence (Nielsen and Nock (2013)).

Due to the importance in modelling various data sources, Kharazmi and Balakrishnan (2021) studied transmuted distribution from information theory view point. These authors have established various informational properties of transmuted distribution. In particular, they proposed extensions of Shannon entropy and Gini’s information measures. In this paper, we develop various informational properties of a more general CRT distribution. We study Kullback-Leibler, harmonic mean and chi-square divergences between a general flexible transmuted model and its components. In addition to these, we also study Shannon entropy of the CRT density function. We propose CRT Shannon entropy. Gini information for CRT distribution and CRT Gini information are also explored.

The rest of the article is arranged as follows. In Section 2, we derive Kullback-Leibler divergence between CRT distribution and its components. Some properties are discussed. We further study harmonic mean divergence and chi-square divergence. In Section 3, we study Shannon entropy of CRT distribution. A new CRT Shannon entropy, which is an extension of Shannon entropy has been proposed. We have studied Gini’s information for CRT distribution and CRT Gini’s information. Fisher information matrix for CRT distribution has been presented. In Section 4, some concluding remarks are added. Finally, some expressions are presented in Appendix.

2 Divergence measures

In this section, we present three divergence measures between the CRT density function \(f_{XCT}\) and its components \(f, f_{23},\) and \(f_{max}\). Mainly, we consider Kullback-Leibler, harmonic
mean and chi-square divergence measures. First, the results for Kullback-Leibler divergence measure are discussed in the following subsection. Let \( U_1, U_2, \) and \( U_3 \) be independent uniform random variables in the interval \((0, 1)\). Then, \( U_{\min} \leq U_{2:3} \leq U_{\max} \) be the order statistics of \( \{U_1, U_2, U_3\} \). Further, let the baseline distribution in \((1.2)\) be uniform distribution in \((0, 1)\). Then, the PDFs of CRT uniform distribution, \( U_{\min} \), and \( U_{2:3} \) are respectively given by

\[
\begin{align*}
f_{U_{\text{CT}}}(u) &= \lambda_1 + 2(\lambda_2 - \lambda_1)u + (1 - \lambda_2)u^2, \quad u \in (0, 1), \\
f_{U_{\min}}(u) &= (3 - 6u + 3u^2), \quad u \in (0, 1), \\
f_{U_{2:3}}(u) &= 6(u - u^2), \quad u \in (0, 1).
\end{align*}
\]

2.1 Kullback-Leibler divergence

We prove the following result, which shows that Kullback-Leibler divergence measures between CRT distribution and its components are equal to that between CRT uniform distribution and its components. Let \( f_1 \) and \( f_2 \) be the PDFs of two absolutely continuous random variables, say \( X_1 \) and \( X_2 \), respectively. Then, the Kullback-Leibler divergence between \( f_1 \) and \( f_2 \) is given by (see Kullback and Leibler [1951])

\[
KL(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x) \log \left( \frac{f_2(x)}{f_2(x)} \right) dx,
\]

where ‘log’ stands for natural logarithm with base \( e \). Note that the Kullback-Leibler divergence is not symmetric, that is, in general, \( KL(f_1, f_2) \neq KL(f_2, f_1) \). Further, it can be established that Kullback-Leibler divergence takes nonnegative values. Indeed, it may be infinity.

**Theorem 2.1.** Suppose \( X \) is a random variable with PDF \( f \). Further, let the CRT random variable \( X_{\text{CT}} \) have PDF \( f_{X_{\text{CT}}} \). Then,

(a) \( KL(f, f_{X_{\text{CT}}}) = KL(f_U, f_{U_{\text{CT}}}) \),

(b) \( KL(f_{X_{\text{CT}}}, f) = KL(f_{U_{\text{CT}}}, f_U) \),

where uniform and CRT uniform random variables, denoted by \( U \) and \( U_{\text{CT}} \) in the interval \((0, 1)\) have PDFs \( f_U \) and \( f_{U_{\text{CT}}} \), respectively.

**Proof.** (a) The Kullback-Leibler divergence between \( f \) and \( f_{X_{\text{CT}}} \) is given by

\[
KL(f, f_{X_{\text{CT}}}) = \int_{-\infty}^{\infty} f(x) \log \left\{ \frac{f(x)}{f_{X_{\text{CT}}}} \right\} dx
\]

\[
= \int_{-\infty}^{\infty} f(x) \log \left\{ \frac{f(x)}{f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x))} \right\} dx
\]

\[
= - \int_0^1 \log \left\{ (\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2) \right\} du \quad (\text{using } u = F(x))
\]

\[
= KL(f_U, f_{U_{\text{CT}}}).
\]

(b) The second part can be proved similarly to the first part, and thus it is omitted. \( \square \)
A random variable $X$ is said to have beta distribution if its PDF is of the following form

$$g(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathbb{B}(a,b)}, \quad 0 < x < 1, \ a, b > 0,$$  

(2.5)

where $\mathbb{B}(a,b)$ denotes complete beta function. Henceforth, we denote $X \sim \text{Beta}(a,b)$, if its PDF is given by (2.5).

**Theorem 2.2.** Suppose $X_1$, $X_2$, and $X_3$ are i.i.d as $X$ with PDF $f$. Further, let $f_{2:3}$ be the PDF of $X_{2:3}$. Then,

(a) $KL(f_{2:3}, f_{X_{CT}}) = KL(f_{U_{2:3}}, f_{U_{CT}})$,

(b) $KL(f_{2:3}, f_{\max}) = KL(f_{U_{2:3}}, f_{W})$,

where $f_{W}$ is the PDF of $W \sim \text{Beta}(3,1)$, $U_{CT}$ is defined in Theorem 2.1 and $f_{U_{2:3}}$ is the PDF of $U_{2:3}$.

**Proof.** (a) The Kullback-Leibler divergence between $f_{2:3}$ and $f_{X_{CT}}$ is given by

$$KL(f_{2:3}, f_{X_{CT}}) = \int_{-\infty}^{\infty} f_{2:3}(x) \log \left( \frac{f_{2:3}(x)}{f_{X_{CT}}(x)} \right) dx$$

$$= \int_{-\infty}^{\infty} f_{2:3}(x) \log \left( \frac{6f(x)(F(x) - F^2(x))}{f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x))} \right) dx$$

$$= \int_{0}^{1} 6(u - u^2) \log \left( \frac{6(u - u^2)}{\lambda_1 + 2(\lambda_2 - \lambda_1)u + (1 - \lambda_2)u^2} \right) du \quad \text{(using } u = F(x))$$

$$= KL(f_{U_{2:3}}, f_{U_{CT}}).$$

(b) The Kullback-Leibler divergence between $f_{2:3}$ and $f_{\max}$ is

$$KL(f_{2:3}, f_{\max}) = \int_{-\infty}^{\infty} f_{2:3}(x) \log \left( \frac{f_{2:3}(x)}{f_{\max}(x)} \right) dx$$

$$= \int_{-\infty}^{\infty} 6f(x)(F(x) - F^2(x)) \log \left( \frac{6f(x)^2(F(x) - F)}{3f(x)F^2(x)} \right) dx$$

$$= \int_{0}^{1} 6(u - u^2) \log \left( \frac{6(u - u^2)}{3u^2} \right) du \quad \text{(using } u = F(x))$$

$$= KL(f_{U_{2:3}}, f_{W}).$$

Thus, the proof of the theorem is completed. \qed

**Theorem 2.3.** Suppose $X_1$, $X_2$, and $X_3$ are i.i.d as $X$ with PDF $f$. Then,

(a) $KL(f_{X_{CT}}, f_{\max}) = KL(f_{U_{CT}}, f_{W})$,

(b) $KL(f_{\max}, f_{X_{CT}}) = KL(f_{W}, f_{U_{CT}}),$
(c) $KL(f_{max}, f) = KL(f_W, f_U)$,

where $f_{UCT}$ and $f_W$ are mentioned in Theorem 2.1 and Theorem 2.2, respectively.

Proof. (a) We have

\[
KL(f_{XCT}, f_{max}) = \int_{-\infty}^{\infty} f_{XCT}(x) \log \left\{ \frac{f_{XCT}(x)}{f_{max}(x)} \right\} dx
\]

\[
= \int_{-\infty}^{\infty} f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x)) \log \left\{ \frac{f_{XCT}(x)}{3f(x)F^2(x)} \right\} dx
\]

\[
= \int_{0}^{1} (\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2) \log \left\{ \frac{(\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2)}{3u^2} \right\} du (\text{using } u = F(x))
\]

\[
= KL(f_{UCT}, f_W).
\]

The proofs for (b) and (c) are similar, and thus these are omitted. \qed

The closed form expressions for Kullback-Leibler divergence measures between various combinations of $U$, $U_{CT}$, $U_{2:3}$ and $W$ can be obtained (see Appendix A.1), from which it is clear that the Kullback-Leibler divergence between the CRT density function $f_{XCT}$ and the density functions of each $f$, $f_{2:3}$, $f_{max}$, and $f_{min}$ are free from the underlying distribution $F$.

The Jeffreys’ divergence between two density functions $f_1$ and $f_2$ is defined as (see Jeffreys (1946))

\[
J_d(f_1, f_2) = \int_{-\infty}^{\infty} \{f_1(x) - f_2(x)\} \log \left\{ \frac{f_1(x)}{f_2(x)} \right\} dx = KL(f_1, f_2) + KL(f_2, f_1). \tag{2.6}
\]

We recall that the Kullback-Leibler divergence measure is not symmetric, but the Jeffreys’ divergence is symmetric, that is,

\[
J_d(f_1, f_2) = J_d(f_2, f_1).
\]

From the following remark, it is revealed that the Jeffreys’ divergences between $f_{XCT}$, $f_{max}$, and $f_{XCT}$, $f$ are equal to that between $f_{UCT}$, $f_W$, and $f_{UCT}$, $f_U$, respectively. This statement can be proved using the results presented in Theorem 2.1 and Theorem 2.3.

Remark 2.1. We have

\[
J_d(f_{XCT}, f_{max}) = J_d(f_{UCT}, f_W) = KL(f_W, f_{UCT}) + KL(f_{UCT}, f_W) \text{ and }
\]

\[
J_d(f_{XCT}, f) = J_d(f_{UCT}, f_U) = KL(f_U, f_{UCT}) + KL(f_{UCT}, f_U).
\]

Note that like Kullback-Leibler divergence, here, the Jeffreys’ divergence is also free from the underlying distribution $F$. 

6
Now, we obtain Kullback-Leibler divergence between a mixture distribution and CRT distribution in the following example.

**Example 2.1.** Suppose the PDF of a general mixture distribution is \( f_{\text{mix}}(x) = vf(x) + 3(1-v)f(x)F^2(x), \ v \in [0, 1]. \) The Kullback-Leibler divergence between \( f_{\text{mix}} \) and \( f_{X_{\text{CT}}} \), and \( f_{X_{\text{CT}}} \) and \( f_{\text{mix}} \) are respectively obtained as

\[
\begin{align*}
(a) \ KL(f_{\text{mix}}, f_{X_{\text{CT}}}) &= \log \left\{ \frac{3 - 2v}{3 - \lambda_1 - \lambda_2} \right\} + \frac{(1-v)}{9(1 - \lambda_2)^2} (4\lambda_1^2 + 13\lambda_2^2 - 5\lambda_1\lambda_2 - 3\lambda_1 \\
&\quad - 15\lambda_2 + 6) + \frac{4v^2}{3\sqrt{3(1-v)}} \arctan^{-1} \left( \frac{3(1-v)}{v} \right) - \frac{2}{3}(1-2v) - \xi(p) + \xi(q), \\
(b) \ KL(f_{X_{\text{CT}}}, f_{\text{mix}}) &= \log \left\{ \frac{3 - \lambda_1 - \lambda_2}{3 - 2v} \right\} - \frac{v}{3(1-v)} \left( 2(1 - \lambda_2) - (\lambda_2 - \lambda_1) \log \left\{ \frac{v}{3 - 2v} \right\} \right) \\
&\quad - 2 \left( \lambda_1 - \frac{v(1 - \lambda_2)}{3(1-v)} \right) \sqrt{\frac{v}{3(1-v)}} \arctan^{-1} \left( \frac{3(1-v)}{v} \right) \\
&\quad + \frac{2}{9(1 - \lambda_2)} \left( \lambda_1^2 + 13\lambda_2^2 - 2\lambda_1\lambda_2 - 18\lambda_2 + 6 \right) - \frac{1}{3}(4 + 5\lambda_1 - 9\lambda_2) \\
&\quad - \phi(p) + \phi(q),
\end{align*}
\]

where \( \xi(x) = \frac{1}{2r} \left[ 2v(\lambda_2 - \lambda_1)x + 6v(1 - \lambda_2)x^2 + 2(\lambda_2 - \lambda_1)(1-v)x^3 + 6(1-v)(1 - \lambda_2)x^4 \right] \log \left\{ \frac{x}{x-1} \right\}, \)

\( r = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 - 3\lambda_1}, \) and the expression of \( \phi(x) \) is provided in Appendix A.1.

Using the values of \( KL(f_{\text{mix}}, f_{X_{\text{CT}}}) \) and \( KL(f_{X_{\text{CT}}}, f_{\text{mix}}) \), the Jeffreys’ divergence between a mixture distribution and CRT distribution can be easily computed.

### 2.2 Harmonic mean divergence

In this subsection, we obtain harmonic mean divergence for \( f_{X_{\text{CT}}} \) with the components of CRT distribution, say \( f, f_{\text{max}}, \) and \( f_{2:3}. \) The harmonic mean divergence between \( f_1 \) and \( f_2 \) is defined as (see Dacunha-Castelle et al. (2006))

\[
H_M(f_1, f_2) = \int_{-\infty}^{\infty} \frac{(f_1(x) - f_2(x))^2}{f_1(x) + f_2(x)} \, dx.
\]

(2.7)

From (2.7), it is clear that the harmonic mean divergence is symmetric in nature, that is, \( H_M(f_1, f_2) = H_M(f_2, f_1). \)

**Theorem 2.4.** Suppose \( f_{X_{\text{CT}}} \) is the PDF of a cubic rank transmuted distribution, and its components’ PDFs are \( f, f_{\text{max}}, \) and \( f_{2:3}. \) Then,

\[
\begin{align*}
(a) \ H_M(f_{X_{\text{CT}}}, f) &= H_M(f_{U_{\text{CT}}}, f_U), \\
(b) \ H_M(f_{X_{\text{CT}}}, f_{\text{max}}) &= H_M(f_{U_{\text{CT}}}, f_W),
\end{align*}
\]
(c) $H_M(f_{X_{CT}}, f_{2:3}) = H_M(f_{U_{CT}}, f_{U_{2:3}})$.

Proof. We present the proof of Part (a). Other parts can be proved similarly, and thus they are not presented here. We have

$$H_M(f_{X_{CT}}, f) = \int_{-\infty}^{\infty} \frac{(f_{X_{CT}}(x) - f(x))^2}{f_{X_{CT}}(x) + f(x)} dx$$

$$= \int_{-\infty}^{\infty} \frac{(f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x)) - f(x))^2}{f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x)) + f(x)} dx$$

$$= \int_{0}^{1} \frac{((\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2) - 1)^2}{(\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2 + 1)} du \text{ (using } u = F(x))$$

$$= H_M(f_{U_{CT}}, f_U).$$

Next, we derive the closed form expressions of the harmonic mean divergence measures, say $H_M(f_{U_{CT}}, f_U)$, $H_M(f_{U_{CT}}, f_W)$, and $H_M(f_{U_{CT}}, f_{U_{2:3}})$. The detailed derivations are omitted for brevity.

$$H_M(f_{U_{CT}}, f_U) = -2 + \frac{4}{r_1} \tanh^{-1}\left(\frac{r_1}{1 - \lambda_2}\right)$$

$$H_M(f_{U_{CT}}, f_W) = -2 - 18\xi(q_1, q_2) - \frac{3}{r_2} \left(6(q_2^3 - q_1^3) + 3(q_2^2 - q_1^2) + (q_2 - q_1)\right),$$

$$H_M(f_{U_{CT}}, f_{U_{2:3}}) = -\frac{\lambda_1}{s} \tanh^{-1}(\lambda_1) - \mu(a) + \mu(b) - P_1(b^3 - a^3) - P_2(b^2 - a^2) - P_3(b - a),$$

where the unknown constants are provided in Appendix A.2.

Remark 2.2. From Theorem 2.4 and the expressions of $H_M(f_{U_{CT}}, f_U)$, $H_M(f_{U_{CT}}, f_W)$, and $H_M(f_{U_{CT}}, f_{U_{2:3}})$, we notice that the harmonic mean divergence measures are free from the underlying distribution $F$.

2.3 Chi-square divergence

In this subsection, we derive some results related to chi-square divergence (denoted by $\chi^2_{CT}$) for the CRT distribution. The chi-square divergence between $f_1$ and $f_2$ is given by

$$\chi^2(f_1, f_2) = \int_{-\infty}^{\infty} \frac{(f_1(x) - f_2(x))^2}{f_2(x)} dx. \quad (2.8)$$

For details about chi-square divergence, we refer to Nielsen and Nock (2013).

Theorem 2.5. Suppose $X_1$, $X_2$ and $X_3$ are i.i.d as $X$ with PDF $f$. Then, the chi-square divergence between $f_{X_{CT}}$ with $f$ and $f_{\max}$ are obtained as

(a) $\chi^2_{CT}(f_{X_{CT}}, f) = \chi^2_{CT}(f_{U_{CT}}, f_U)$,
(b) \( \chi_{CT}^2(f, f_{XCT}) = \chi_{CT}^2(f_U, f_{UCT}) \),
\( \psi_{CT}(f, f_{XCT}) = \psi_{CT}(f_U, f_{UCT}) \).

**Proof.** (a) We have
\[
\chi_{CT}^2(f_{XCT}, f) = \int_{-\infty}^{\infty} \frac{(f_{XCT}(x) - f(x))^2}{f(x)} dx
\]
\[
= \int_{-\infty}^{\infty} \frac{(f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1 - \lambda_2)F^2(x) - f(x))^2}{f(x)} dx
\]
\[
= \int_{0}^{1} ((\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2) - 1)^2 du \quad \text{(using } u = F(x))
\]
\[
= \chi_{CT}^2(f_{UCT}, f_U).
\]

Other parts can be proved similarly. \( \square \)

The closed form expressions for \( \chi_{CT}^2(f_{UCT}, f_U), \chi_{CT}^2(f_U, f_{UCT}), \) and \( \chi_{CT}^2(f_W, f_{UCT}) \) can be obtained as

(a) \( \chi_{CT}^2(f_{UCT}, f_U) = \frac{1}{15}(12 - 15\lambda_1 - 9\lambda_2 + 5\lambda_1\lambda_2 + 5\lambda_1^2 + 2\lambda_2^2) \),

(b) \( \chi_{CT}^2(f_U, f_{UCT}) = -1 + \frac{1}{p} \tan^{-1}(\lambda) \),

(c) \( \chi_{CT}^2(f_W, f_{UCT}) = -1 + \frac{1}{3(1 - \lambda_2)^3}(4\lambda_1^2 + 10\lambda_2^2 - 8\lambda_1\lambda_2 - 9\lambda_2 + 3) - 9\sigma(p, q) \),

where \( \lambda = \frac{r}{\lambda_2} \) and \( \sigma(p, q) = \frac{1}{\lambda_2} (p^4 \log \left\{ \frac{p}{p-1} \right\} - q^4 \log \left\{ \frac{q}{q-1} \right\}) \).

**Remark 2.3.** From Theorem 2.2 and the expressions of \( \chi_{CT}^2(f_{UCT}, f_U), \chi_{CT}^2(f_U, f_{UCT}), \) and \( \chi_{CT}^2(f_W, f_{UCT}) \), we notice that the chi-square divergence measures are free of the underlying distribution function \( F \).

The Symmetric \( \chi^2 \)-divergence between two density functions \( f_1 \) and \( f_2 \) is defined as (see Dragomir et al. (2000))
\[
\psi(f_1, f_2) = \int_{-\infty}^{\infty} \frac{(f_1(x) - f_2(x))^2(f_1(x) + f_2(x))}{f_1(x)f_2(x)} dx.
\] (2.9)

Note that the chi-square divergence is non-symmetric, but the symmetric chi-square divergence is symmetric, that is, in general \( \psi(f_1, f_2) = \psi(f_2, f_1) \). In the following remark, we state that the symmetric chi-square divergence between \( f_{XCT} \) and \( f \) is equal to that between \( f_{UCT} \) and \( f_U \).

**Remark 2.4.** We have
\[
\psi(f_{XCT}, f) = \chi^2(f_{UCT}, f_U) + \chi^2(f_U, f_{UCT}) = \psi(f_{UCT}, f_U).
\]

It is clear that the symmetric chi-square divergence between \( f_{XCT} \) and \( f \) is free from the underlying distribution \( F \).
3 Information measures

This section addresses various information measures of the CRT random variable $X_{CT}$. First, we present Shannon entropy of $X_{CT}$. The Shannon entropy of a random variable $X$ with PDF $f$ is defined as

$$H(f) = - \int_{-\infty}^{\infty} f(x) \log f(x) dx. \quad (3.1)$$

3.1 Shannon entropy

It is well-known that the Shannon entropy plays an important role in many real life situations to measure the uncertainty contained in a probability distribution (see Shannon (1948)). Here, we discuss Shannon entropy of CRT random variable. The weighted Shannon entropy of $X$ is defined as

$$H^\psi(f) = - \int_{-\infty}^{\infty} \psi(x) f(x) \log f(x) dx, \quad (3.2)$$

where $\psi$ is a nonnegative real-valued measurable function. When $\psi(x) = x$, (3.2) reduces to a length-biased information measure (see Di Crescenzo and Longobardi (2007)).

Theorem 3.1. Let $X_1, X_2,$ and $X_3$ be i.i.d random variables with a common CDF $F$ and PDF $f$. Then, the Shannon entropy of a CRT random variable $X_{CT}$ is given by

$$H(f_{X_{CT}}) = \lambda_1 H(f) + (1 - \lambda_2) H(f_{\max}) + 2(\lambda_2 - \lambda_1) H^F(f) - (1 - \lambda_2) H(f_W) + H(f_{U_{CT}}),$$

where $U_{CT}$ is the CRT uniform random variable, $W \sim Beta(3,1)$, and $H^F(X)$ means weighted Shannon entropy defined by (3.2) with weight function $F$.

Proof. The Shannon entropy of the CRT random variable $X_{CT}$ is

$$H(f_{X_{CT}}) = - \int_{-\infty}^{\infty} f_{X_{CT}}(x) \log f_{X_{CT}}(x) dx = I_1 + I_2, \quad (3.3)$$

where

$$I_1 = - \int_{-\infty}^{\infty} (f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1) F(x) + 3(1 - \lambda_2) F^2(x))) \log f(x) dx,$$

$$I_2 = - \int_{-\infty}^{\infty} (f(x)(\lambda_1 + 2(\lambda_2 - \lambda_1) F(x) + 3(1 - \lambda_2) F^2(x))) \log (\lambda_1 + 2(\lambda_2 - \lambda_1) F(x)) + 3(1 - \lambda_2) F^2(x) dx.$$
Using transformation $u = F(x)$ in (3.3) and after some calculations, we obtain

$$H(f_{\text{CRT}}) = \lambda_1 H(f) + (1 - \lambda_2)H(f_{\text{max}}) + (1 - \lambda_2) \int_0^1 3u^2 \log(3u^2)du$$

$$- \int_0^1 (\lambda_1 + 2(\lambda_2 - \lambda_1)u + 3(1 - \lambda_2)u^2) \log(\lambda_1 + 2(\lambda_2 - \lambda_1)u)$$

$$+ 3(1 - \lambda_2)u^2)du - 2(\lambda_2 - \lambda_1) \int_{-\infty}^\infty F(x)f(x) \log f(x)dx$$

$$= \lambda_1 H(f) + (1 - \lambda_2)H(X_{\text{max}}) - (1 - \lambda_2)H(f_W) + H(f_{\text{CRT}}) + 2(\lambda_2 - \lambda_1)H^F(f).$$

Hence the result follows.

**Remark 3.1.** Define $\theta(\lambda_1, \lambda_2) = H(f_{\text{CRT}}) - (1 - \lambda_2)H(f_W)$. Then, after some calculations, we obtain the closed form expression for $\theta(\lambda_1, \lambda_2)$ as

$$\theta(\lambda_1, \lambda_2) = -\left[\log(3 - \lambda_1 - \lambda_2) - \phi(p) + \phi(q) - \frac{2}{3}(3 + \lambda_1 - 4\lambda_2)ight.$$  

$$+ \frac{2}{9(1 - \lambda_2)}(\lambda_1^2 + 13\lambda_2^2 - 18\lambda_1\lambda_2 + 6)] - (1 - \lambda_2)\left(\frac{2}{3} - \log 3\right),$$

where

$$\phi(x) = \frac{1}{2r} \left[2(\lambda_2 - \lambda_1)^2x^2 + 10(\lambda_2 - \lambda_1)(1 - \lambda_2)x^3 + 12(1 - \lambda_2)^2x^4\right] \log \frac{x}{x - 1},$$

$$r = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 - 3\lambda_1},$$

$$p = \frac{\lambda_1 - \lambda_2 + r}{3(1 - \lambda_2)},$$

$$q = \frac{\lambda_1 - \lambda_2 - r}{3(1 - \lambda_2)}.$$

Because of the complex form of $\theta(\lambda_1, \lambda_2)$, it is difficult to analyze from theoretical point of view. To get an idea related to it’s codomain, we plot the function with respect to $\lambda_1$ when $\lambda_2$ is fixed, and with respect to $\lambda_2$ when $\lambda_1$ is fixed (see Figure 1). From the figure, it is clearly seen that $\theta(\lambda_1, \lambda_2)$ takes both positive as well as negative values.

**Example 3.1.** Let $X_1$, $X_2$, and $X_3$ be i.i.d as Pareto distribution with PDF $f(x) = 1 - x^{-\alpha}$, $x \geq 1$, $\alpha > 0$. Then, the Shannon entropy for CRT random variable $X_{\text{CRT}}$ is

$$H(f_{\text{CRT}}) = \lambda_1 \left(\frac{\alpha + 1}{\alpha^2} - \log \alpha\right) + (1 - \lambda_2) \left(\frac{1}{6} - \frac{7}{6\alpha} - \log(3\alpha)\right)$$

$$+ (\lambda_2 - \lambda_1) \left(\frac{1}{2\alpha}(\alpha + 1)(3 - 2\alpha) - \log \alpha\right) + \theta(\lambda_1, \lambda_2),$$

where $\theta(\lambda_1, \lambda_2)$ is given in Remark 3.1.
Example 3.2. Let $X_1$, $X_2$ and $X_3$ be i.i.d as exponential distribution with PDF $f(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$. Then, the Shannon entropy of $X_{CT}$ is

$$H(f_{X_{CT}}) = \lambda_1 (1 - \log \beta) + (1 - \lambda_2) \left( \log(3\beta) - \frac{5}{2} \right) - (\lambda_2 - \lambda_1) \log \beta + \theta(\lambda_1, \lambda_2),$$

where $\theta(\lambda_1, \lambda_2)$ is given in Remark 3.1.

3.2 Cubic rank transmuted Shannon entropy

In this subsection, we introduce an extension of the Shannon entropy of a CRT distribution as well as Jensen-Shannon divergence, and we call it as the CRT Shannon entropy.

Definition 3.1. Suppose $X$ is a continuous random variable with PDF $f$ and $X_{CT}$ is the corresponding CRT random variable with PDF $f_{X_{CT}}$. Then, the CRT Shannon (CTS) entropy between $f_{X_{CT}}$ in (1.6) and its components $f$, $f_{2:3}$, and $f_{\text{max}}$ is given by

$$CTS(f, f_{2:3}, f_{\text{max}}; \lambda_1, \lambda_2) = H(f_{X_{CT}}) - \lambda_1 H(f) - C_1 H(f_{2:3}) - C_2 H(f_{\text{max}}),$$

where $C_1 = \frac{1}{3}(\lambda_2 - \lambda_1)$, $C_2 = \frac{1}{3}(3 - \lambda_2 - 2\lambda_1)$, $\lambda_1 \in [0, 1]$, and $\lambda_2 \in [-1, 1)$.

Now, we will derive an important theorem of CRT Shannon entropy. The following theorem shows that $CTS(f, f_{2:3}, f_{\text{max}}; \lambda_1, \lambda_2)$ defined in Definition 3.1 can be expressed in terms of the Kullback-Leibler divergence measures between $f_{\text{max}}$ and $f$, $f_{2:3}$ and $f_{\text{max}}$, and $f_{X_{CT}}$ and $f$.

Theorem 3.2. The CRT Shannon entropy is expressed as

$$CTS(f, f_{2:3}, f_{\text{max}}; \lambda_1, \lambda_2) = (1 - \lambda_2) KL(f_{\text{max}}, f) + \frac{1}{3}(\lambda_2 - \lambda_1) KL(f_{2:3}, f_{\text{max}}) - KL(f_{X_{CT}}, f) - (\log 3 - 1)(\lambda_1 - \lambda_2).$$

(3.5)
Proof. Substituting $H(f_{X_C})$ from Theorem 3.1 in (3.4), we obtain

$$CTS(f, f_{2:3}, f_{max}; \lambda_1, \lambda_2) = \frac{2}{3}(\lambda_1 - \lambda_2)H(f_{max}) - \frac{1}{3}(\lambda_2 - \lambda_1)H(f_{2:3}) + H(f_{U_C})$$

$$- (1 - \lambda_2)H(f_W) + 2(\lambda_2 - \lambda_1)H^F(f)$$

$$= \frac{1}{3}(\lambda_1 - \lambda_2)(2H(f_{max}) + H(f_{2:3}) - 6H^F(f)) + H(f_{U_C})$$

$$- (1 - \lambda_2)H(f_W)$$

$$= \frac{1}{3}(\lambda_1 - \lambda_2)(KL(f_{2:3}, f_{max}) + 3(\log 3 - 1)) + H(f_{U_C})$$

$$- (1 - \lambda_2)H(f_W).$$

(3.6)

Now, utilizing the following relations (see Theorem 2.1)

$$KL(f_{X_C}, f) = KL(f_{U_C}, f_U) = -H(f_{U_C})$$

and

$$KL(f_{max}, f) = KL(f_W, f_U) = -H(f_W)$$

in (3.6), the desired result follows. Hence, the theorem is proved.

Further, bounds of CRT Shannon entropy can be found easily in terms of the Kullback-Leibler divergence measures between various components, which are given by

$$CTS(f, f_{2:3}, f_{max}; \lambda_1, \lambda_2) \begin{cases} \geq (1 - \lambda_2)KL(f_{max}, f) + \frac{1}{3}(\lambda_2 - \lambda_1)KL(f_{2:3}, f_{max}) \\ -KL(f_{X_C}, f), \text{ if } \lambda_2 \geq \lambda_1 \\ \leq (1 - \lambda_2)KL(f_{max}, f) + \frac{1}{3}(\lambda_2 - \lambda_1)KL(f_{2:3}, f_{max}) \\ -KL(f_{X_C}, f), \text{ if } \lambda_2 < \lambda_1. \end{cases}$$

(3.7)

Remark 3.2. It is also clear that the CRT Shannon entropy is free from the underlying distribution $F$.

Example 3.3. Let $X_1$, $X_2$ and $X_3$ be i.i.d random variables having a common PDF $f$. Then, the CRT Shannon entropy is expressed as

$$CTS(f, f_{2:3}, f_{max}; (\lambda_1, \lambda_2)) = (1 - \lambda_2)KL(f_W, f_U) + \frac{1}{3}(\lambda_2 - \lambda_1)KL(f_{U_{2:3}}, f_W)$$

$$-KL(f_{U_C}, f_U) - (\log 3 - 1)(\lambda_1 - \lambda_2).$$

Remark 3.3. Note that the Jensen-Shannon (JS) entropy between for $f$, $f_{2:3}$, and $f_{max}$ is a special case of CRT Shannon entropy when $\lambda_2 \geq \lambda_1$ and $\lambda_1, \lambda_2 \in [0, 1)$. Denote $\delta_1 = \lambda_1$, $\delta_2 = \frac{1}{3}(\lambda_2 - \lambda_1)$, and $\delta_3 = \frac{1}{3}(3 - \lambda_2 - 2\lambda_1)$. Then, clearly

$$CTS(f, f_{2:3}, f_{max}; \lambda_1, \lambda_2) = JS(f, f_{2:3}, f_{max}; \delta),$$

where

$$JS(f, f_{2:3}, f_{max}; \delta) = H(\delta_1 f + \delta_2 f_{2:3} + \delta_3 f_{max}) - \{\delta_1 H(f) + \delta_2 H(f_{2:3}) + \delta_3 H(f_{max})\}.$$
3.3 Gini information

In this subsection, we investigate Gini’s mean difference (GMD) for CRT distribution. The Gini’s mean difference is also known as Gini information. For a random variable with CDF \( F \) and survival function \( \bar{F} \), the Gini information is given by

\[
GMD(F) = 2 \int_{-\infty}^{\infty} F(x) \bar{F}(x) dx. \tag{3.8}
\]

The following result shows that the Gini information of CRT distribution can be expressed in terms of the Gini information of \( F_{\max} \) and \( F \).

**Theorem 3.3.** Let \( F_{X_{CT}} \) be the CRT distribution function. The CDFs of its components with PDFs \( f \) and \( f_{\max} \) are respectively denoted by \( F \) and \( F_{\max} \). Then,

\[
GMD(F_{X_{CT}}) = (1 - \lambda_2)^2 GMD(F_{\max}) + (\lambda_1^2 + \lambda_1 - \lambda_2) GMD(F) + 2 \int_0^1 \left( \frac{Au + Bu^3 + Cu^4 - Du^5}{f(F^{-1}(u))} \right) du,
\]

where \( A = (\lambda_2^2 - \lambda_1^2) \), \( B = \lambda_2(1 - \lambda_2) - 2\lambda_1(\lambda_2 - \lambda_1) \), \( C = (1 - \lambda_2)(3 - 3\lambda_2 - 2\lambda_1) - (\lambda_2 - \lambda_1)^2 \), and \( D = (1 - \lambda_2)(3 - 2\lambda_1 - \lambda_2) \).

**Proof.** We have

\[
\frac{1}{2} GMD(F_{X_{CT}}) = \int_{-\infty}^{\infty} F_{X_{CT}}(x) \bar{F}_{X_{CT}}(x) dx
\]

\[
= (1 - \lambda_2)^2 \int_{-\infty}^{\infty} F^3(x) \bar{F}^3(x) dx + (\lambda_1^2 + \lambda_1 - \lambda_2) \int_{-\infty}^{\infty} F(x) \bar{F}(x) dx
\]

\[
+ \int_{-\infty}^{\infty} (AF(x) + BF^3(x) + CF^4(x) - DF^5(x)) dx
\]

\[
= \frac{1}{2} (1 - \lambda_2)^2 GMD(F_{\max}) + \frac{1}{2} (\lambda_1^2 + \lambda_1 - \lambda_2) GMD(F)
\]

\[
+ \int_0^1 \left( \frac{Au + Bu^3 + Cu^4 - Du^5}{f(F^{-1}(u))} \right) du. \tag{3.9}
\]

Thus, the proof is completed after multiplying 2 both sides of (3.9).

Now, we present lower and upper bounds of the Gini information of CRT distribution as follows:

\[
GMD(F_{X_{CT}}) \begin{cases}
\geq (1 - \lambda_2)^2 GMD(F_{\max}) + (\lambda_1^2 + \lambda_1 - \lambda_2) GMD(F), & \text{if } I^* \geq 0 \\
\leq (1 - \lambda_2)^2 GMD(F_{\max}) + (\lambda_1^2 + \lambda_1 - \lambda_2) GMD(F), & \text{if } I^* \leq 0,
\end{cases} \tag{3.10}
\]

where \( I^* = \int_0^1 (Au + Bu^3 + Cu^4 - Du^5)/(f(F^{-1}(u))) du \) and \( A, B, C, D \) are given in Theorem 3.3. Using Theorem 3.3 in the next example, we compute Gini information of the CRT distribution with baseline as power distribution.
Example 3.4. Suppose $X$ is a random variable following power distribution with CDF $F(x) = (\frac{x}{l})^\kappa$, for $x \in (0, l)$ and $\kappa > 0$. Then, the Gini information of CRT distribution is obtained as

$$GMD(F_{XCT}) = 2l(1 - \lambda_2)^2\left(\frac{1}{3\kappa + 1} - \frac{3}{4\kappa + 1} + \frac{3}{5\kappa + 1} - \frac{1}{6\kappa + 1}\right) + (\lambda_1^2 + \lambda_1 - \lambda_2)\left(\frac{2l\kappa}{(\kappa + 1)(2\kappa + 1)}\right) + 2l\left(\frac{A}{\kappa + 1} + \frac{B}{3\kappa + 1} + \frac{C}{4\kappa + 1} - \frac{D}{5\kappa + 1}\right).$$

For $l = 2$ and $\kappa = 2$, the term $I^*(\lambda_1, \lambda_2) = \frac{A}{\kappa + 1} + \frac{B}{3\kappa + 1} + \frac{C}{4\kappa + 1} - \frac{D}{5\kappa + 1}$ has been plotted in Figure 2 for some regions of $\lambda_1$ and $\lambda_2$ to show that $I^*$ assumes both positive and negative values. From Figure 2, we notice that

$$I^*(\lambda_1, \lambda_2) \begin{cases} 
\geq 0, & \text{if } \lambda_1 \in [0, 0.15] \text{ and } \lambda_2 \in [-0.3, 1], \\
\leq 0, & \text{if } \lambda_1 \in [0.15, 1] \text{ and } \lambda_2 \in [-1, -0.3]. 
\end{cases} \quad (3.11)$$

3.4 Cubic rank transmuted Gini information

In this subsection, we introduce an extension of the Gini’s mean difference as well as Jensen-Gini divergence, and call it as CRT Gini’s mean divergence.

Definition 3.2. Let $X$ and $X_{CT}$ be continuous random variable and CRT random variable with respective CDFs $F$ and $F_{XCT}$. Then, the CRT Gini’s mean divergence (CTG) between $F_{XCT}$ in (1.5) and its components $F$, $F_{1:3}$ and $F_{max}$ is defined as

$$CTG(F, F_{1:3}, F_{max}; \lambda_1, \lambda_2) = GMD(F_{XCT}) - \lambda_1GMD(F) - C_1GMD(F_{1:3}) - C_2GMD(F_{max}),$$

where $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$, and $C_1$ and $C_2$ are defined in Definition 3.1.

Now, we derive an important theorem for CRT Gini’s mean divergence. The theorem shows that $CTG(F, F_{1:3}, F_{max}; \lambda_1, \lambda_2)$ defined in Definition 3.1 can be expressed in terms of the energy distances between $F$ and $F_{1:3}$, $F$ and $F_{max}$, and $F_{max}$ and $F_{1:3}.$

Definition 3.3. Suppose that $X$ and $Y$ are two continuous random variables with CDFs $F_1$ and $F_2$, respectively. Then, the energy distance between $F_1$ and $F_2$ is

$$CD(F_1, F_2) = \int_{-\infty}^{\infty} (F_1(x) - F_2(x))^2dx.$$ 

We note that $CD(F_1, F_2)$ is also known as Cramer’s distance, which was introduced by Cramér (1928).
For Upper bound of GMD

For Lower bound of GMD

Figure 2: Plots for $I^*(\lambda_1, \lambda_2)$ as in Example 3.4.
Theorem 3.4. The CRT Gini’s mean difference can be represented through energy distances as

\[
\text{CTG}(F, F_{2:3}, F_{\text{max}}; \lambda_1, \lambda_2) = \frac{1}{2} \lambda_1 C_1 CD(F, F_{2:3}) + \frac{1}{2} \lambda_1 C_2 CD(F, F_{\text{max}}) \\
+ \int_{0}^{1} \left( \frac{R_1 u^2 + R_2 u^3 + R_3 u^4 + R_4 u^5 + R_5 u^6}{f(F^{-1}(u))} \right) du \\
+ \frac{1}{2} C_1 C_2 CD(F_{\text{max}}, F_{2:3}),
\]

where \(C_1\) and \(C_2\) are defined in Definition 3.1. \(R_1 = \frac{3}{2} \lambda_1 (1 - \lambda_1)\), \(R_2 = 3 \lambda_1 (\lambda_1 - \lambda_2)\), \(R_3 = \frac{3}{2} (4 - 5 \lambda_1 - 5 \lambda_2 + 4 \lambda_1 \lambda_2 - \lambda_1^2 + 3 \lambda_2^2)\), \(R_4 = -(2 - 3 \lambda_1 - \lambda_2 + \lambda_1 \lambda_2 + \lambda_2^2)\), and \(R_5 = -\frac{3}{2} (2 \lambda_1 - 3 \lambda_2 + \lambda_2^2)\).

Proof. From Definition 3.2 we have

\[
\text{CTG}(F, F_{2:3}, F_{\text{max}}; \lambda_1, \lambda_2) = (1 - \lambda_2)^2 GMD(F_{\text{max}}) + (\lambda_1^2 + \lambda_1 - \lambda_2) GMD((F)) \\
+ 2 \int_{-\infty}^{\infty} (AF(x) + BF^3(x) + CF^4(x) - DF^5(x)) dx \\
- \lambda_1 GMD(F) - C_1 GMD(F_{2:3}) - C_2 GMD(F_{\text{max}}) \\
= 2 \lambda_1 (1 - \lambda_1) \int_{-\infty}^{\infty} F^2(x) dx + 4 \lambda_1 (\lambda_1 - \lambda_2) \int_{-\infty}^{\infty} F^3(x) dx \\
+ (6 - 10 \lambda_1 - 6 \lambda_2 + 8 \lambda_1 \lambda_2 + 4 \lambda_2^2 - 2 \lambda_1^2) \int_{-\infty}^{\infty} F^4(x) dx \\
- (6 - 12 \lambda_1 + 4 \lambda_1 \lambda_2 + 2 \lambda_2^2) \int_{-\infty}^{\infty} F^5(x) dx \\
+ (6 \lambda_2 - 4 \lambda_1 - 2 \lambda_2^2) \int_{-\infty}^{\infty} F^6(x) dx \tag{3.12}
\]

Further,

\[
\frac{1}{2} \lambda_1 C_1 CD(F, F_{2:3}) + \frac{1}{2} \lambda_1 C_2 CD(F, F_{\text{max}}) + \frac{1}{2} C_1 C_2 CD(F_{\text{max}}, F_{2:3}) = \frac{1}{2} (\lambda_1 - \lambda_2^2) \int_{-\infty}^{\infty} F^2(x) dx \\
- (\lambda_1 \lambda_2 - \lambda_1^2) \int_{-\infty}^{\infty} F^3(x) dx - \frac{1}{2} (\lambda_2^2 + \lambda_2^2 - 4 \lambda_1 \lambda_2 + 5 \lambda_1 - 3 \lambda_2) \int_{-\infty}^{\infty} F^4(x) dx \\
- (3 \lambda_2 - 3 \lambda_1 + \lambda_1 \lambda_1 - \lambda_2^2) \int_{-\infty}^{\infty} F^5(x) dx - \frac{1}{2} (2 \lambda_1 - 3 \lambda_2 + \lambda_2^2) \int_{-\infty}^{\infty} F^6(x) dx. \tag{3.13}
\]

Combining (3.12) and (3.13), the desired result follows. \(\square\)
3.5 Fisher information matrix

In this subsection, we present Fisher information matrix for the CRT distribution, which is derived as

\[ I(\Lambda) = \begin{bmatrix} E(\tau \rho_1) & E(\tau \rho_2) \\ E(\tau \rho_2) & E(\tau \rho_3) \end{bmatrix}, \]

where \( \tau = \frac{f(x)}{f_{XCT}(x)} \), \( \rho_1 = (1 - 2F(x))^2 \), \( \rho_2 = (2F(x) - 7F^2(x) + 6F^3(x)) \), \( \rho_3 = (2F(x) - 3F^2(x^2))^2 \), and \( \Lambda = (\lambda_1, \lambda_2) \).

Note that it is difficult to obtain closed form expressions of the elements of Fisher information matrix \( I(\Lambda) \) when baseline distribution is a general CDF. Due to this reason, in the following, we consider an example with uniform baseline distribution in the interval (0, 1) and discuss the Fisher information matrix.

**Example 3.5.** Let the baseline distribution for the CRT distribution be uniform distribution in (0, 1). Then, the Fisher information matrix of CRT uniform distribution is obtained as

\[ I(\Lambda) = \begin{bmatrix} E(\tau \rho_1) & E(\tau \rho_2) \\ E(\tau \rho_2) & E(\tau \rho_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2 \tanh^{-1}(\lambda) + \omega_1} + \frac{4}{3(1 - \lambda_2)} & -\omega_2 - \frac{4(1 - \lambda_1)}{3(1 - \lambda_2)^2} \\ -\omega_2 - \frac{4(1 - \lambda_1)}{3(1 - \lambda_2)^2} & \frac{1}{3(1 - \lambda_2)^2} (4 \lambda_1^2 - 8 \lambda_1 + 3 \lambda_2 + 1) - \omega_3 \end{bmatrix}, \]

where \( r, p, q \) are given in Remark 3.1, \( \lambda = r/\lambda_2 \), and

\[ \omega_1 = \frac{2}{r} \left( (p - p^2) \log \frac{p}{p-1} - (q - q^2) \log \frac{q}{q-1} \right), \]

\[ \omega_2 = \frac{1}{2r} \left( (2p - 7p^2 + 6p^3) \log \frac{p}{p-1} - (2q - 7q^2 + 6q^3) \log \frac{q}{q-1} \right), \]

\[ \omega_3 = \frac{1}{2r} \left( (4p^2 - 12p^3 + 9p^4) \log \frac{p}{p-1} - (4q^2 - 12q^3 + 9q^4) \log \frac{q}{q-1} \right). \]

4 Concluding remarks

In this paper, the informational properties of a flexible CRT distribution have been studied. We have computed divergence between CRT distribution and densities of each of its components using Kullback-Leibler, harmonic mean, and chi-square divergence measures. Some properties have been discussed. Further, we have derived some informational properties of Shannon entropy and Gini’s mean difference. The CRT Shannon entropy and CRT Gini’s information have been introduced. Finally, the Fisher information matrix of the CRT distribution is obtained.
Note that the CDF in (1.2) can also be written based on $F$, $F_{2:3}$ and $F_{\text{min}}$ as

$$F_{CT}(x) = (3 - \lambda_1 - \lambda_2)F(x) + \frac{1}{3}(\lambda_1 + 2\lambda_2 - 3)F_{2:3}(x) + \frac{1}{3}(2\lambda_1 + \lambda_2 - 3)F_{\text{min}}(x),$$

and the corresponding PDF is

$$f_{CT}(x) = (3 - \lambda_1 - \lambda_2)f(x) + \frac{1}{3}(\lambda_1 + 2\lambda_2 - 3)f_{2:3}(x) + \frac{1}{3}(2\lambda_1 + \lambda_2 - 3)f_{\text{min}}(x),$$

where $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$. Results similar to this paper can be obtained when we use the CRT distribution with CDF (4.1).

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Appendix A.1

\[
KL(f_U, f_{UCT}) = 2 - \varphi(p) + \varphi(q) - \log(3 - \lambda_1 - \lambda_2),
\]

\[
KL(f_{UCT}, f_U) = \log(3 - \lambda_1 - \lambda_2) - \phi(p) + \phi(q) - \frac{2}{3}(3 + \lambda_1 - 4\lambda_2)
+ \frac{2}{9(1 - \lambda_2)}(\lambda_2^2 + 13\lambda_2^2 - 2\lambda_1\lambda_2 - 18\lambda_2 + 6),
\]

\[
KL(f_{U_2:3}, f_{UCT}) = \log \left(\frac{6}{3 - \lambda_1 - \lambda_2}\right) - \frac{5}{3} - \eta(p) + \eta(q)
- \frac{1}{9(1 - \lambda_2)^2}(55\lambda_2^2 - 8\lambda_1^2 - 44\lambda_1\lambda_2 - 78\lambda_2 + 60\lambda_1 + 15),
\]

\[
KL(f_{UCT}, f_W) = -\phi(p) + \phi(q) - \log \left(\frac{3}{3 - \lambda_1 - \lambda_2}\right) + \frac{1}{3}(\lambda_1 + 9\lambda_2 - 4)
+ \frac{2}{9(1 - \lambda_2)}(\lambda_2^2 + 13\lambda_2^2 - 2\lambda_1\lambda_2 - 18\lambda_2 + 6),
\]

\[
KL(f_W, f_{UCT}) = \gamma(q) - \gamma(p) + \log \left(\frac{3}{3 - \lambda_1 - \lambda_2}\right) + \frac{(\lambda_2 - \lambda_1)}{9(1 - \lambda_2)^2}(3 - 7\lambda_2 + 4\lambda_1)
+ \frac{2}{9(1 - \lambda_2)^2}(4\lambda_1^2 + 10\lambda_2^2 - 8\lambda_1\lambda_2 - 9\lambda_2 + 3) - \frac{2}{3},
\]

\[
KL(f_{U_2:3}, f_W) = \log 2,
\]

where,

\[
\varphi(x) = \frac{1}{r}[(\lambda_2 - \lambda_1)x + 3(1 - \lambda_2)x^2] \log \left(\frac{x}{x-1}\right),
\]

\[
\phi(x) = \frac{1}{2r}[2(\lambda_2 - \lambda_1)^2x^2 + 10(\lambda_2 - \lambda_1)(1 - \lambda_2)x^3 + 12(1 - \lambda_2)^2x^4] \log \left(\frac{x}{x-1}\right),
\]

\[
\eta(x) = \frac{1}{2r}[6(\lambda_2 - \lambda_1)x^2 - (18 + 4\lambda_1 - 22\lambda_2)x^3 + 12(1 - \lambda_2)x^4] \log \left(\frac{x}{x-1}\right),
\]

20
\[ \gamma(x) = \frac{1}{2s}[2(\lambda_2 - \lambda_1)x^3 + 6(1 - \lambda_2)x^4] \log \left( \frac{x}{x-1} \right). \]

**Appendix A.2**

\[
\begin{align*}
    r_1 &= \sqrt{(\lambda_1 - \lambda_2)^2 - 3(1 + \lambda_1)(1 - \lambda_2)}, \\
    r_2 &= \sqrt{(\lambda_1 - \lambda_2)^2 - 3\lambda_1(2 - \lambda_2)}, \\
    s &= \sqrt{(\lambda_2 - \lambda_1 + 3)^2 + 3\lambda_1(1 + \lambda_2)}, \\
    q_1 &= \frac{\lambda_1 - \lambda_2 + r_2}{3(2 - \lambda_2)}, \quad q_2 = \frac{\lambda_1 - \lambda_2 - r_2}{3(2 - \lambda_2)}, \\
    a &= \frac{\lambda_2 - \lambda_1 + 3 + s}{3(1 + \lambda_2)}, \quad b = \frac{\lambda_2 - \lambda_1 + 3 - s}{3(1 + \lambda_2)}, \\
    A^* &= 2(\lambda_2 - \lambda_1 - 3), \quad B^* = 3(3 - \lambda_2), \quad \lambda_h = \frac{s}{\lambda_2 + 3}, \\
    P_1 &= \frac{(B^*)^2}{2s}, \quad P_2 = \frac{1}{4s}(4A^*B^* + (B^*)^2), \\
    P_3 &= \frac{1}{6s}\left(3((A^*)^2 + 2B^*\lambda_1 + 3A^*B^*) + (B^*)^2\right), \\
    \mu(x) &= \frac{1}{2s}\left(2A^*\lambda_1 x + ((A^*)^2 + 2B^*\lambda_1)x^2 + 2A^*B^*x^3 + (B^*)^2x^4\right) \log \left( \frac{x}{x-1} \right), \\
    \xi(q_1, q_2) &= \frac{1}{2r_2}\left(q_1^4 \log \left( \frac{q_1}{q_1 - 1} \right) - q_2^4 \log \left( \frac{q_2}{q_2 - 1} \right) \right).
\end{align*}
\]