Algebraic vs. topological vector bundles on spheres

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Abstract
We prove that all rank 2 topological complex vector bundles on smooth affine quadrics of dimension 11 over the complex numbers are algebraizable.

1 Introduction

If $X$ is a smooth complex algebraic variety, write $V_n(X)$ for the set of isomorphism classes of rank $n$ algebraic vector bundles on $X$, and $V_n^{top}(X)$ for the set of isomorphism classes of rank $n$ complex topological vector bundles on $X^{an} := X(C)$ (viewed as a complex manifold). There is a function “forget the algebraic structure”
\[ \mathfrak{R}_{n,X} : V_n(X) \rightarrow V_n^{top}(X). \]
A complex topological vector bundle of rank $n$ on $X^{an}$ is called algebraizable if it lies in the image of $\mathfrak{R}_{n,X}$.

Let $Q_{2n-1}$ be the smooth quadric hypersurface in $\mathbb{A}^{2n}$ cut out by the equation $\sum_{i=1}^{n} x_i y_i = 1$. The underlying complex manifold $Q_{2n-1}(C)$ is homotopy equivalent to the sphere $S^{2n-1}$. The goal of this note is to establish the following result.

Theorem 1. The map $\mathfrak{R}_{2,Q_{11}}$ is surjective, i.e., every rank 2 topological vector bundle on $Q_{11}$ admits an algebraic structure.

Remark 2. With some additional analysis, it is possible to prove that $\mathfrak{R}_{r,Q_{2n-1}}$ is actually surjective for arbitrary $r$ and $n \leq 6$, but the case $r = 2$ and $n = 6$ is most interesting for reasons we now explain.

Remark 3. For a general smooth complex affine $X$ the map $\mathfrak{R}_{r,X}$ need not be either injective or surjective: injectivity can fail for $r = 1$ on smooth complex affine curves, and surjectivity can fail for $r = 1$ and smooth complex affine surfaces.

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Remark 4. It is an open problem to determine whether, when \( X = \mathbb{P}^n, n \geq 4 \), all rank 2 vector bundles are algebraizable (see, e.g., [OSS11, Chapter 1 §6.5]). There is a smooth surjective morphism \( Q_{2n-1} \to \mathbb{P}^{n-1} \). While we do not know, e.g., whether all rank 2 vector bundles on \( \mathbb{P}^5 \) are algebraizable, Theorem 1 implies that any rank 2 topological complex vector bundle on \( \mathbb{P}^5 \) becomes algebraic after pullback to \( Q_{11} \). The problem of whether the original bundle on \( \mathbb{P}^5 \) is algebraizable can then be viewed as a question in descent theory.

The set \( \mathcal{V}^{	ext{top}}(Q_{2n-1}(\mathbb{C})) \) is, by means of the homotopy equivalence \( Q_{2n-1}(\mathbb{C}) \cong S^{2n-1} \), in bijection with the set of free homotopy classes of maps \([S^{2n-1}, BU(r)]\). Because \( BU(r) \) is simply connected, the canonical map from pointed to free homotopy classes of maps is a bijection, i.e., \( \pi_{2n-1}(BU(r)) \to [S^{2n-1}, BU(r)] \) is a bijection. On the other hand, for \( n \geq 3 \), the map \( \pi_{2n-1}(BSU(r)) \to \pi_{2n-1}(BU(r)) \) is an isomorphism. In the special case where \( r = 2 \) we know that \( \pi_{2n-1}(BSU(2)) \cong \pi_{2n}(SU(2)) \cong \pi_{2n}(S^3) \).

Fix a perfect field \( k \) and write \( \mathcal{H}(k) \) for the Morel-Voevodsky \( \mathbb{A}^1 \)-homotopy category of \( k \)-schemes. F. Morel gave an algebraic analog of Steenrod’s celebrated homotopy classification of vector bundles: there is an \( \mathbb{A}^1 \)-homotopy classification of algebraic vector bundles on smooth affine schemes; see [Mor12, Theorem 8.1] for a precise statement. As explained in the introduction to [AF12a], by a procedure analogous to that described in the previous paragraph, there is a canonical bijection

\[
\mathcal{V}^O_r(Q_{2n-1}) \cong \pi_{n-1,n+1}^A(SL_2),
\]

where \( \mathcal{V}^O_r(X) \) is the set of isomorphism classes of oriented vector bundles on \( X \), i.e., vector bundles with a chosen trivialization of the determinant.

The comparison between the results of the previous two paragraphs is facilitated by “complex realization,” which provides a homomorphism

\[
\pi_{n-1,n+1}^A(SL_2)(\mathbb{C}) \longrightarrow \pi_{2n}(S^3).
\]

To establish the result above, it suffices to prove the displayed homomorphism is surjective and the above algebraizability question boils down to a question regarding \( \mathbb{A}^1 \)-homotopy sheaves of \( SL_2 \). Theorem 1 is then a consequence of the following result.

**Theorem 5** (See Theorem 4.2.1). The homomorphism

\[
\pi_{4,6}^A(SL_2)(\mathbb{C}) \longrightarrow \pi_{10}(S^3) \cong \mathbb{Z}/15.
\]

is surjective.

That \( \pi_{10}(S^3) \cong \mathbb{Z}/15 \) is classical (see, e.g., [Tod62]), so it suffices simply to produce a lift of a generator. The group \( \pi_{10}(S^3) \) is especially interesting because it is the first place where 5-torsion appears in homotopy groups of \( S^3 \). To prove the main result, we introduce a spectral sequence whose \( E_1 \)-page involves \( \mathbb{A}^1 \)-homotopy sheaves of punctured affine spaces and that converges to \( \mathbb{A}^1 \)-homotopy sheaves of symplectic group; this is achieved in Subsection 2.2. In Section 3, we establish some results about the Hopf map \( \nu \). In particular, we show that it is stably essential over \( \text{Spec} \mathbb{Z} \); we use this to deduce stable non-triviality of an auxiliary class \( \delta \). In Section 4, we analyze the “symplectic spectral sequence” constructed above to produce the lift of a generator of \( \pi_{10}(S^3) \).

Finally, Section 5 begins to study the problem of constructing explicitly the rank 2 algebraic vector
bundle whose existence is guaranteed by Theorem 1. We show that any rank 2 algebraic vector bundle whose associated classifying map corresponds to an element of $\pi_{2,0}^A(SL_2)(\mathbb{C})$ not lying in the kernel of the map in Theorem 5, remains non-trivial after adding a trivial bundle of rank $\leq 3$, but becomes trivial after adding a trivial summand of rank 4.

Notation/Preliminaries

Throughout we fix a base-field $k$. This note uses much notation from [AF12a, AF12b] and [AF12c], and our conventions and notation will follow those papers. In particular, if we write $Sm_k$ for the category of schemes that are separated, finite type and smooth over $\text{Spec} \, k$, $H^\bullet(k)$ for the Morel-Voevodsky pointed $A^1$-homotopy category, and if $(\mathcal{X}, x)$ is a pointed simplicial Nisnevich sheaf on $Sm_k$, we define $\pi_{i,j}^A(\mathcal{X}, x)$ as the Nisnevich sheaf on $Sm_k$ associated with the presheaf

$$U \mapsto \text{Hom}_{\mathcal{L}_n(k)}(S^j_{\text{HS}} \wedge G^{ij}_{m-n} \wedge U_+, (\mathcal{X}, x)).$$

We write $K_i^Q$ for the Nisnevich sheafification of the Quillen K-theory presheaves $U \mapsto K_i(U)$, and $GW^j_i$ for the Nisnevich sheafification of the Grothendieck-Witt groups $GW^j_i(U, \mathcal{O}_U)$. For a discussion of complex realization, we refer the reader to [MV99, §3.3.2].

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2 Some spectral sequences

The goal of this section is to construct some spectral sequences whose $E_1$-pages are homotopy sheaves of punctured affine spaces and that converge to algebraic $K$-theory sheaves or Grothendieck-Witt sheaves. These spectral sequences are the algebro-geometric cousins of the “orthogonal spectral sequence” studied by Mahowald (see the discussion just subsequent to [Rav86, Diagram 1.5.14]).

2.1 The linear spectral sequence

For any integer $n \geq 1$ there are $A^1$-fiber sequences of the form

$$GL_{n-1} \rightarrow GL_n \rightarrow \mathbb{A}^n \setminus 0.$$ 

For any integer $j \geq 0$, these $A^1$-fiber sequences induce long exact sequences in $A^1$-homotopy sheaves of the form

$$\pi_{i,j}^A(GL_{n-1}) \rightarrow \pi_{i,j}^A(GL_n) \rightarrow \pi_{i,j}^A(\mathbb{A}^n \setminus 0) \rightarrow \pi_{i-1,j}^A(GL_{n-1}) \rightarrow \cdots.$$
Putting these sequences together yields an exact couple of the form

\[
\bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(GL_n) \to \bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(GL_n) \to \bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(A^n_2 \setminus 0),
\]

and the structure of the associated spectral sequence is summarized in the next result.

**Proposition 2.1.1.** For any integer \( j \geq 0 \), there is a spectral sequence with

\[
E^1_{p,q}(j) := \pi_{p+q,j}^{A_1}(A_p \setminus 0)
\]

and converging to \( \pi_{p+q,j}^{A_1}(GL) = K^Q_{p+q-j+1} \).

**Proof.** The only thing that remains to be checked is the convergence statement, which follows from [Wei94, Theorem 5.9.7]. \( \square \)

### 2.2 The symplectic spectral sequence

For any integer \( n \geq 1 \), there are \( A^1 \)-fiber sequences of the form

\[
Sp_{2n-2} \to Sp_{2n} \to A^{2n}_2 \setminus 0.
\]

For any integer \( j \geq 0 \), these \( A^1 \)-fiber sequences induce long exact sequences in \( A^1 \)-homotopy sheaves of the form

\[
\pi_{i,j}^{A_1}(Sp_{2n-2}) \to \pi_{i,j}^{A_1}(Sp_{2n}) \to \pi_{i,j}^{A_1}(A^{2n}_2 \setminus 0) \to \pi_{i-1,j}^{A_1}(Sp_{2n-2}) \to \cdots.
\]

Putting these sequences together, we get an exact couple

\[
\bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(Sp_{2n}) \to \bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(Sp_{2n}) \to \bigoplus_{n \geq 1, i \geq 0} \pi_{i,j}^{A_1}(A^{2n}_2 \setminus 0),
\]

Regarding the associated spectral sequence, we have the following result, whose proof is identical to that of Proposition 2.1.1.

**Proposition 2.2.1.** For any integer \( j \geq 0 \), there is a spectral sequence with

\[
E^1_{p,q}(j) := \pi_{p+q,j}^{A_1}(A^{2p}_2 \setminus 0)
\]

and converging to \( \pi_{p+q,j}^{A_1}(Sp) = GW^{2-j}_{p+q-j+1} \).

The next result is an immediate consequence of [Mor12, Corollary 6.39].

**Lemma 2.2.2.** For any \( j \geq 0 \), \( E^{p,q}(j)_{1} = 0 \) for i) \( p < 0 \), and ii) \( q \leq p - 2 \).

The above vanishing statement, together with the identification \( \pi_1^{A_1}(A^2_2 \setminus 0) \cong K^{MW}_2 \) immediately implies the following result.

**Corollary 2.2.3.** There are low-dimensional isomorphisms \( GW_1^2 \cong 0 \) and \( GW_2^2 \cong K^{MW}_2 \).
2.3 The anti-symmetric spectral sequence

Similarly, for any integer \( n \geq 1 \), there are \( \mathbb{A}^1 \)-fiber sequences of the form

\[
X_n \longrightarrow X_{n+1} \longrightarrow \mathbb{A}^{2n+1} \setminus 0,
\]

where \( X_n = GL_{2n}/Sp_{2n} \). As above, these \( \mathbb{A}^1 \)-fiber sequences yield, for any integer \( j \geq 0 \), long exact sequences in \( \mathbb{A}^1 \)-homotopy sheaves of the form

\[
\pi_{i,j}^{A^1}(X_n) \longrightarrow \pi_{i,j}^{A^1}(X_{n+1}) \longrightarrow \pi_{i,j}^{A^1}(\mathbb{A}^{2n+1} \setminus 0) \longrightarrow \pi_{i-1,j}(X_n) \longrightarrow \cdots.
\]

As above, one can put these sequences together to obtain an exact couple, and regarding the associated spectral sequence, one has the following result; again, the proof is identical to Proposition 2.1.1.

**Proposition 2.3.1.** For any integer \( j \geq 0 \), there is a spectral sequence with

\[
E^1_{p,q}(j) := \pi_{p+q,j}^{A^1}(\mathbb{A}^{2p-1} \setminus 0)
\]

and converging to \( \pi_{p+q,j}^{A^1}(GL/Sp) = GW_{p+q-j+1}^{3} \).

**Lemma 2.3.2.** For any \( j \geq 0 \), \( E^1_{p,q}(j) = 0 \) for i) \( q \leq p - 3 \), ii) \( p < 0 \), and iii) \( p = 1 \) and \( q > 0 \).

The above vanishing statement, together with \( \pi_{0}^{A^1}(G_m) \cong G_m \), and Morel’s computation \( \pi_{2}^{A^1}(\mathbb{A}^3 \setminus 0) \) immediately implies the following result.

**Corollary 2.3.3.** There are low-dimensional isomorphisms \( GW_{1}^{3} \cong G_m \), \( GW_{2}^{3} \cong 0 \) and \( GW_{3}^{3} \cong K_{3}^{MW} \).

3 The Hopf map \( \nu \)

The goal of this section is to establish that the class \( \delta \) described in [AF12c, §5] is stably essential; we review the construction of \( \delta \) in Subsection 3.5. We also observed in [AF12c, Corollary 5.3.1] that, under complex realization, \( \delta \) is mapped to a suspension of the classical topological Hopf map \( \nu_{top} : S^7 \rightarrow S^4 \) and \( \nu_{top} \) is known to be stably essential. Thus, over any field that admits a complex embedding, one knows that \( \delta \) is stably essential. Nevertheless, it is possible to give a purely algebraic proof of this fact and we do this here. We begin by giving an explicit morphism of quadrics representing \( \nu \) and then we proceed by mimicking the classical topological argument studying Steenrod operations on the cone of \( \nu \).

3.1 The \( \mathbb{A}^1 \)-homotopy type of \( Q_4 \)

Let \( Q_4 \) be the quadric defined by \( x_1y_1 + x_2y_2 = z(z + 1) \). The \( \mathbb{A}^1 \)-homotopy type of \( Q_4 \) was effectively described in [AD08]; we review the argument here. Consider the closed subscheme \( E_2 \) of \( Q_4 \) defined by \( x_1 = 0, x_2 = 0, z = -1 \). Observe that \( E_2 \) is isomorphic to \( \mathbb{A}^2 \). Let \( Y_4 \subset Q_4 \) be the (open) complement of \( E_2 \). In [AD08], it is observed that there is a \( G_a \)-torsor \( \mathbb{A}^5 \rightarrow Y_4 \), and
Therefore $Y_4$ is $\mathbb{A}^1$-contractible (in fact, over $\text{Spec} \mathbb{Z}$). The homotopy purity theorem [MV99, §3.2, Theorem 2.23] yields a cofiber sequence of the form

$$Y_4 \longrightarrow Q_4 \longrightarrow Th(\nu_{E_2/Q_4}) \longrightarrow \Sigma^1 Y_4 \longrightarrow \cdots.$$ 

The normal bundle to $E_2 \subset Q_4$ is trivial, and picking a trivialization of $\nu_{E_2/Q_4}$ yields an $\mathbb{A}^1$-weak equivalence $Th(\nu_{E_2/Q_4}) \cong \mathbb{P}^{1\wedge 2} \wedge E_2$. Since $E_2$ is itself $\mathbb{A}^1$-contractible we see that $E_2 \cong S_k^0$, and therefore $Th(\nu_{E_2/Q_4}) \cong \mathbb{P}^{1\wedge 2}$. Thus, there is an induced map $Q_4 \to \mathbb{P}^{1\wedge 2}$. Since $Y_4$ is $\mathbb{A}^1$-contractible, the next result follows from the fact that pushouts of $\mathbb{A}^1$-weak equivalences along cofibrations are again $\mathbb{A}^1$-weak equivalence, i.e., properness of the $\mathbb{A}^1$-local model structure [MV99, §2 Theorem 3.2].

**Proposition 3.1.1** (Asok, Doran). For any field $k$, the map $Q_4 \to \mathbb{P}^{1\wedge 2}$ is an $\mathbb{A}^1$-weak equivalence over $\text{Spec}(\mathbb{Z})$.

**Remark 3.1.2.** More generally, let $Q_{2n}$ be the smooth affine quadric defined by the equation

$$\sum_i x_i x_{n+i} = x_{2n+1}(1 + x_{2n+1}).$$

It is straightforward to check that $Q_2 \cong SL_2/G_n$ and is therefore $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$. Let $E_n \subset Q_{2n}$ be the closed subscheme defined by $x_1 = \cdots = x_n = 0, z = -1$, and let $Y_{2n}$ be its open complement. The same argument as above gives a map $Q_{2n} \to \mathbb{P}^{1\wedge n}$. If one knew that $Y_{2n}$ was $\mathbb{A}^1$-contractible, then it would follow that $Q_{2n} \to \mathbb{P}^{1\wedge n}$ is an $\mathbb{A}^1$-weak equivalence. It is known that $Y_{2n}$ cannot be the base space of a unipotent group torsor, so the techniques of [AD08] cannot be applied.

### 3.2 A geometric Hopf map and fibration

Given a pair of $2 \times 2$-matrices $A$ and $B$ consider the equation $\det A - \det B = 1$; the result is a quadric $Q'_7 \subset \mathbb{A}^8$. Let $\mu : M_2 \times M_2 \to M_2$ be multiplication of $2 \times 2$-matrices. Define a function

$$h_\mu(A, B) := (\mu(A, B), \det B).$$

Observe that if $\det A - \det B = 1$, then since $\det(AB) = \det(A) \det(B)$, it follows that $\det(AB) = \det(B)(1 + \det B)$, i.e., $h_\mu$ restricts to a morphism $Q'_7 \longrightarrow Q_4$. If $Q_7$ is the quadric from the introduction, then there is an isomorphism $Q_7 \cong Q'_7$ obtained by changing signs.

**Definition 3.2.1.** The Hopf map $\nu : Q_7 \to Q_4$ is the map $h_\mu$ precomposed with the isomorphism $Q_7 \cong Q'_7$ described above.

Define an action of $SL_2$ on pairs $(A, B)$ by means of the formula

$$C \cdot (A, B) = (AC, C^{-1}B).$$

If $C \in SL_2$, then $\det(AC) - \det(C^{-1}B) = \det(A) - \det(B)$, so this action preserves $Q'_7$. Moreover, $\mu(AC, C^{-1}B) = \mu(A, B)$ and $\det(C^{-1}B) = \det B$. Therefore, $h_\mu(C \cdot (A, B)) = h_\mu(A, B)$. In fact, this action makes $Q'_7$ into an $SL_2$-torsor over $Q_4$ (see the proof of [AD08, Corollary 3.1] for details). Because $SL_2$-torsors give rise to $\mathbb{A}^1$-fiber sequences, we deduce the following result from the identification $SL_2 \simeq Q_3$.
Proposition 3.2.2. There is an $A^1$-fibration sequence of the form
\[ Q_3 \to Q'_7 \to Q_4. \]

Remark 3.2.3. There is a homotopically simpler but less geometric description of the Hopf map. Indeed, the multiplication map $SL_2 \times SL_2 \to SL_2$ yields a morphism $\Sigma^1_4 SL_2 \wedge SL_2 \to \Sigma^1_4 SL_2$, which provides another candidate for $\nu$ (see [Mor12, p. 189] for further discussion of this map). The morphism we called $\nu$ above and this Hopf map should agree (perhaps up to a sign).

3.3 The cone of $\nu$

If $\eta : A^2 \setminus 0 \to \mathbb{P}^1$ is the Hopf map given by the usual projection morphism, it is a classical fact that the cone of $\eta$, computed in $\mathcal{H}_*(k)$ is isomorphic to $\mathbb{P}^2$. To see this, one takes $\mathbb{P}^2$ and considers the standard open cover by two open sets isomorphic to $A^2$ and $\mathbb{P}^2 \setminus 0$. The inclusion of the intersection gives a map $A^2 \setminus 0 \to \mathbb{P}^2 \setminus 0$ that under the $A^1$-weak equivalence $\mathbb{P}^2 \setminus 0 \to \mathbb{P}^1$ coincides with the Hopf map. Since $A^2$ is contractible, the Mayer-Vietoris square gives the required computation of the cone. The benefit of this computation is that the cohomology of $\mathbb{P}^2$ is well understood.

We now provide an analogous computation for $\nu$. To this end, consider the spaces $HP^n$ defined by Panin and Walter in [PW10, §1]. The space $HP^n$ is a smooth affine scheme of dimension $4n$ that behaves in a fashion very similar to the quaternionic projective spaces one considers in topology.

Remark 3.3.1. One can check that $HP^1$ coincides with the quadric $Q_4$ we considered. The varieties $HP^n$ can all be constructed as quotients of the split smooth affine quadric $Q_{4n+3}$ by a free action of $SL_2$, generalizing the construction of $Q_4$ as a quotient of $Q_7$ by a free action of $SL_2$. In fact, the varieties $HP^n$ can all be seen to be smooth over $\text{Spec } \mathbb{Z}$.

Roughly speaking, $HP^n$ admits a “cell decomposition” with cells of dimension $4i$. More precisely, there exist smooth locally closed (in general, quasi-affine) subschemes $Z_{2i}$ in $HP^n$ of codimension $2i$, such i) $Z_{2n} = A^{2n}$, ii) each $Z_{2i}$ is an $A^1$-contractible variety realized as the quotient $A^{4n-2i+1}$ by a free action of $G_{2i}$, and iii) the closure $\overline{Z}_{2i}$ is a vector bundle of rank $2i$ over $HP^{n-i}$ [PW10, Theorem 1.1]. Given this notation, the next result describes the cone of $\nu$.

Proposition 3.3.2. The cone of $\nu$ in $\mathcal{H}_*(k)$ is $HP^2$.

Proof. We know that $HP^2$ has a cell-decomposition with cells $Z_0$, $Z_2$ and $Z_4$, where $Z_{2i}$ has codimension $2i$, and the closure of $Z_2$ is a rank 2 vector bundle over $Q_4 = HP^1$ [PW10, Theorem 3.2]. Since $Z_0$ is $A^1$-contractible, with complement $\overline{Z}_2$, the Thom isomorphism theorem, combined with the cofiber sequence attached to the inclusion $Z_0 \hookrightarrow HP^2$ yields an $A^1$-weak equivalence

\[ HP^2 \cong Th(N_{Z_2/HP^2}). \]

Now, by definition the Thom space of $N_{Z_2/HP^2}$ is the quotient $N_{Z_2/HP^2} / N_0^0 / Z_2/HP^2$, where $N_0^0$ denotes the complement of the zero section. We now describe these spaces more explicitly.

The total space $N_{Z_2/HP^2}$ is a rank 2 vector bundle over $\overline{Z}_2$, which is itself a rank 2 vector bundle over $HP^1$. Therefore, the composite map yields an $A^1$-weak equivalence

\[ N_{Z_2/HP^2} \to HP^1. \]
On the other hand, the space $N_{Z_2/\text{HP}^2}^2$ admits the following description. The space $\overline{Z_2}$ is affine and $\mathbb{A}^1$-weakly equivalent to $Q_4$. If $\nu$ is the Hopf map, then $\nu$ is an $SL_2$-torsor over $Q_4$, and we can form the associated $\mathbb{A}^2 \setminus 0$-bundle $Z := Q_7 \times^{SL_2} \mathbb{A}^2 \setminus 0 \to Q_4$. The map

$$Q_7 \cong Q_7 \times^{SL_2} SL_2 \to Q_7 \times^{SL_2} \mathbb{A}^2 \setminus 0$$

is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^1$ and is therefore an $\mathbb{A}^1$-weak equivalence. Therefore, the induced map $Z \to Q_4$ coincides with $\nu$ under the $\mathbb{A}^1$-weak equivalence $Q_7 \simeq Q_7 \times^{SL_2} \mathbb{A}^2 \setminus 0 = Z$. One can check that $N_{Z_2/\text{HP}^2}^2$ is precisely the pullback of $Z$ to $\overline{Z_2}$ along the vector bundle $\overline{Z_2} \to Q_4$.

Combining these two facts, we see that, up to $\mathbb{A}^1$-weak equivalence, the inclusion of the complement of the zero section of the normal bundle to $\overline{Z_2}$ into the total space is $\nu$.

\[\square\]

### 3.4 Stable non-triviality of $\nu$

Recall that, in topology, given a map $g : S^{4n-1} \to S^{2n}$, one can form the CW complex $C(g) = D^{4n} \cup_f S^{2n}$, which has two cells of dimension $4n$ and $2n$. If $g$ is homotopically trivial, this complex is simply $S^{4n} \vee S^{2n}$, and this completely determines (say) the cohomology of $C(g)$ (even, say, as modules over the Steenrod algebra). One way to detect that $C(g)$ is non-trivial is to study its cohomology ring or Steenrod operations.

In algebraic geometry, one may replace $C(g)$ by the cone of the map $g$ and perform all the same arguments. Suppose given an element of $f \in \mathbb{A}^{2n} \setminus 0, \mathbb{P}^{1 \wedge n} \mathbb{A}^1$. We can form the cone $C(f)$ in $\mathcal{H}_k$. If $f$ is $\mathbb{A}^1$-homotopically constant, then $C(f) \cong \mathbb{P}^{1 \wedge n} \vee \Sigma_1^n(A^{2n} \setminus 0)$. In particular, we have

$$\tilde{H}^{*,*}(\mathbb{P}^{1 \wedge n} \vee \Sigma_1^n(A^{2n} \setminus 0), \mathbb{Z}/2) \cong \tilde{H}^{*,*}(\mathbb{P}^{1 \wedge n}, \mathbb{Z}/2) \oplus \tilde{H}^{*,*}(\Sigma_1^n(A^{2n} \setminus 0), \mathbb{Z}/2),$$

where we write $\tilde{H}^{*,*}(-, \mathbb{Z}/2)$ for reduced motivic cohomology with $\mathbb{Z}/2$-coefficients. If $A^{*,*}$ is the (mod 2) motivic Steenrod algebra studied in [Voe03, §11], then $\tilde{H}^{*,*}(-, \mathbb{Z}/2)$ is a module over $A^{*,*}$, and the above direct sum decomposition is a decomposition as modules over $A^{*,*}$.

Note that $\tilde{H}^{*,*}(\Sigma_1^n(A^{2n} \setminus 0), \mathbb{Z}/2) \cong \tilde{H}^{*,*}(\text{Spec } k)[\xi]/\xi^2$, where $\xi$ is a class of bidegree $(4n, 2n)$, and that $\tilde{H}^{*,*}(\mathbb{P}^{1 \wedge n}, \mathbb{Z}/2) \cong \tilde{H}^{*,*}(\text{Spec } k)[\tau]/\tau^2$, where $\tau$ is a class of bidegree $(2n, n)$. For this reason, we will limit our attention to the subring $\tilde{H}^{2*,*}(-, \mathbb{Z}/2)$, which we view as a $\mathbb{Z}/2$-vector space. Now, the algebra $A^{*,*}$ does not preserve $\tilde{H}^{2*,*}(-, \mathbb{Z}/2)$. However, if we write $A^{*,*}/\beta$ for the quotient of $A^{*,*}$ by the 2-sided ideal generated by the Bockstein, then $A^{*,*}/\beta$ actually does preserve $\tilde{H}^{2*,*}$ (see [Bro03, §11] for a discussion of this fact, in our context it follows from [Voe03, Theorem 10.2] upon observing that $Sq^{2i+1} = \beta Sq^{2i}$). The action of $A^{*,*}/\beta$ on $\tilde{H}^{2*,*}(-, \mathbb{Z}/2)$ is $\mathbb{Z}/2$-linear by construction.

If $f \in [\mathbb{A}^{2n} \setminus 0, \mathbb{P}^{1 \wedge n}]_{\mathbb{A}^1}$ is $\mathbb{A}^1$-homotopically constant, then the $A^{*,*}/\beta$-module structure on $\tilde{H}^{2*,*}(\Sigma_1^n(A^{2n} \setminus 0), \mathbb{Z}/2)$ is trivial, since every Steenrod operation acts trivially on $\xi$. Similarly, the $A^{*,*}/\beta$-module structure on $\tilde{H}^{*,*}(\mathbb{P}^{1 \wedge n}, \mathbb{Z}/2)$ is trivial. Thus, to prove non-triviality of $f$, it suffices to prove that the action of $A^{*,*}/\beta$ on $C(f)$ is non-trivial. Since the operations we consider are all stable with respect to both simplicial and $G_m$-suspension, it follows that if the $A^{*,*}/\beta$-module structure on $C(f)$ is non-trivial, then $f$ remains non-trivial after both simplicial and $G_m$-suspension, so is non-trivial in the stable $\mathbb{A}^1$-homotopy category of $\mathbb{P}^1$-spectra (see [Mor04, §5] for details regarding the latter category).
Remark 3.4.1. One would like to just describe the $A^{*,*}$-module structure on the motivic cohomology of $H^{*,*}(\text{Spec } k)[\xi]/\xi^2$ directly, but there are some technical difficulties preventing an easy statement. The main problem is that if $X$ is a scheme, the action of $A^{*,*}$ on $\tilde{H}^{*,*}(X, \mathbb{Z}/2)$ is not $H^{*,*}(\text{Spec } k, \mathbb{Z}/2)$-linear (see [Voe03, p. 41]). This is the reason we consider $A^{*,*}/\beta$.

Remark 3.4.2. If $f \in \pi_{2n-1,2n}(\mathbb{P}^1^{\wedge n})$ is as above, and we look at $H^{2*,*}(C(f), \mathbb{Z})$ instead, then we see that
\[ H^{2*,*}(C(f), \mathbb{Z}) \cong \mathbb{Z}[\xi, \tau]/(\xi^2, \xi \tau, \tau^2 - h_f \xi) \]
with $h_f \in \mathbb{Z}$ for dimensional reasons. One can check that the function
\[ H : \pi_{2n-1,2n}^{A^1}(\mathbb{P}^1^{\wedge n}) \rightarrow \mathbb{Z} \]
given by the assignment $f \mapsto h_f$ is actually a group homomorphism, just as in topology and defines a motivic analog of the classical Hopf invariant [Whi50]. Since this invariant depends only on the ring structure of the motivic cohomology of $C(f)$, it is an unstable invariant.

Loosely following the notation of Morel [Mor04, §5], we write
\[ \pi^{A^1}_{i,j}(S_k^0) := \colim_n \pi^{A^1}_{i+n,j+n}(\mathbb{P}^1^{\wedge n}); \]
in words, this sheaf is the bidegree $(i,j)$-stable $A^1$-homotopy sheaf of the motivic sphere spectrum. Iterated $\mathbb{P}^1$-suspension of $\nu$ gives rise to an element of $\pi^{A^1}_{1,2}(S_k^0)(k)$.

Theorem 3.4.3. The element $\nu \in \pi^{A^1}_{1,2}(S_k^0)(k)$ is non-trivial.

Proof. Since $\nu$ is defined over $\text{Spec } \mathbb{Z}$, it suffices by a base-change argument to show that it is non-trivial over the prime field. Since the prime field is perfect, we can use motivic cohomology to detect non-triviality. We saw that $C(\nu) = \text{HP}^2$ in Proposition 3.3.2. By [PW10, Theorem 8.1], we know that $H^{*,*}(C(\nu), \mathbb{Z}/2) \cong \mathbb{Z}/2[\zeta]/\zeta^3$, where $\zeta$, a class of bidegree $(4,2)$, is the first Pontryagin class of a canonical symplectic line bundle over $\text{HP}^2$. In particular, $Sq^4(\zeta) = \zeta^2$ by [Voe03, Lemma 9.8], so $\tilde{H}^{*,*}(\text{HP}^2, \mathbb{Z}/2)$ has a non-trivial $A^{*,*}/\beta$-module structure, and the required stable non-triviality of $\nu$ follows. \(\square\)

3.5 Stable non-triviality of $\delta$

Recall that in [AF12c, §5.2] we defined a map $\delta$ as the composite
\[ \Sigma^3_{\mathcal{G}_{m}} G^{\wedge 5} \rightarrow \Omega^1_{\Sigma^3_{\mathcal{G}_{m}}} G^{\wedge 5} \rightarrow A^3 \setminus 0 \]
where the second map is the connecting homomorphism in the fiber sequence
\[ SL_4/Sp_4 \rightarrow SL_6/Sp_6 \rightarrow A^5 \setminus 0 \]
that we discussed in Subsection 2.3. For any field $k$ having characteristic unequal to 2, by [AF12c, Proposition 5.2.1], $\delta$ provides a generator of $\pi_{3,5}^{A^1}(A^3 \setminus 0)(k)$ as a $GW(k)$-module. Note that, by construction, $\Sigma_{\mathcal{G}_{m}} \nu$ is an element of $\pi_{3,5}^{A^1}(A^3 \setminus 0)$.

Corollary 3.5.1. If $k$ is a field having characteristic unequal to 2, the element $\delta$ is $\mathbb{P}^1$-stably non-trivial.

Proof. We know that $\Sigma_{\mathcal{G}_{m}} \nu$ is a multiple of $\delta$. The former is stably non-trivial by Theorem 3.4.3 and it follows that the latter is also non-trivial. \(\square\)
4 Some results on odd primary torsion in $\mathbb{A}^1$-homotopy groups

Serre showed [Ser53] that, if $p$ is a prime, then the first $p$-torsion in the higher homotopy groups of $S^3$ appears in $\pi_{2p}(S^3)$. The classical proof of this result relies on an analysis of the Serre spectral sequence for the fibration $BS^1 \to S^3(3) \to S^3$ and Serre class theory. The goal of this section is to begin to lift this computation to unstable $\mathbb{A}^1$-homotopy theory, in which neither of the two tools just mentioned are available.

4.1 Odd primary torsion and the topological symplectic spectral sequence

If $Sp(2n)$ is the compact real form of the symplectic group, there are topological fiber sequences of the form

$$Sp(2n - 2) \to Sp(2n) \to S^{4n-1}.$$ 

Putting the long exact sequences in homotopy groups associated with these fibrations together yields an exact couple and an associated spectral sequence with $E^1_{p,q} = \pi_{p+q}(S^{4p-1})$ that converges to the homotopy groups of $Sp(\infty)$. We will refer to this spectral sequence as the topological symplectic spectral and the differentials appearing in this spectral sequence will bear a superscript “$\text{top}$” to distinguish them from those appearing in the spectral sequence constructed in Subsection 2.2. The homotopy groups of $Sp(\infty)$ are known by Bott periodicity and we now use this to interpret the $p$-torsion in $\pi_{2p}(S^3)$ in terms of differentials in this spectral sequence.

Proposition 4.1.1. Suppose $p$ is an odd prime. The generator of the $p$-torsion of $\pi_{2p}(S^3)$ is the image of an element of $\pi_{2p+1}(S^{2p+1})$ under the differential $d^{\text{top}}_{(p-1)/2}$ in the topological symplectic spectral sequence.

Proof. First, note that if $F \to E \to B$ is a Serre fibration, then there is a corresponding long exact sequence in homotopy groups mod $n$ for any integer $n$. In particular, the exact couple associated with the long exact sequence in homotopy of the fiber sequences $Sp(2n - 2) \to Sp(2n) \to S^{4n-1}$ yields a $\mod n$ topological symplectic spectral sequence [Nei80]. Now, by Serre we know that $p$-torsion appears in $\pi_{n+k}(S^n)$ appears only if $k \geq 2p - 3$. On the other hand, by Bott periodicity [Bot59, Corollary to Theorem II], the $p$-completion of $\pi_i(Sp(\infty))$ is non-trivial only if $i$ is congruent to 3 mod 4. In particular, $\pi_{2p}(Sp(\infty), \mathbb{Z}/p)$ is always trivial since $2p$ is even. Combining these vanishing statements, It follows that the only non-trivial differential in the topological symplectic spectral sequence that lands on $\pi_{2p}(S^3)$ is the one indicated in the theorem statement, and this differential is necessarily an isomorphism since $\pi_{2p}(Sp(\infty), \mathbb{Z}/p)$ is trivial.

Lemma 4.1.2. The map

$$\ker(\pi_{11}(S^{11}) \to \pi_{10}(S^7)) \xrightarrow{d^{\text{top}}_2} \pi_{10}(S^3)/\text{im}(\pi_{11}(S^7))$$

is surjective. Moreover, $\pi_{10}(S^3)$ is generated by $d^{\text{top}}_2(24\iota)$, where $\iota$ is a generator of $\pi_{11}(S^{11})$.

Proof. By [Tod62] one knows that $\pi_{10}(S^3) \cong \mathbb{Z}/15$ and $\pi_{11}(S^7) = 0$. Therefore, it suffices to show that $d^{\text{top}}_2$ is surjective after reduction modulo 3 and reduction modulo 5. The statement after reduction modulo 5 follows immediately from Proposition 4.1.1. After reduction modulo 3, observe
that \( d_1 : \pi_{10}(S^7, \mathbb{Z}/3) \to \pi_9(S^3, \mathbb{Z}/3) \) is a map \( \mathbb{Z}/3 \to \mathbb{Z}/3 \). Serre showed that the generator of \( \pi_9(S^3) \) is precisely the composite of a 2-fold suspension of \( \nu \), generating \( \pi_9(S^6) \) and \( \nu : S^6 \to S^3 \), which comes from the connecting map \( S^7 \to BSp(2) \). Since the \( d_1 \) differential at this stage is precisely induced by composition with the connecting map, it follows that \( d_1 \) is an isomorphism after reduction modulo 3. It follows that the induced map \( \pi_{11}(S^{11}) \to \pi_{10}(S^3) \) is surjective after reduction modulo 3 as well.

Remark 4.1.3. The group \( \pi_{14}(S^3) \cong \mathbb{Z}/84 \times \mathbb{Z}/2 \times 2 \) is not cyclic, and therefore the differential \( d_3^{\text{top}} \) cannot be surjective integrally.

### 4.2 Lifting some odd primary torsion from topology to \( \mathbb{A}^1 \)-homotopy

Complex realization gives a morphism from the exact couple giving rise to the symplectic spectral sequence we considered in Subsection 2.2 to the exact couple that gives rise to the topological symplectic spectral sequence described in Subsection 4.1. As a consequence, there is an induced morphism of spectral sequences. By the discussion of the introduction, the next result implies Theorem 1 from the introduction.

**Theorem 4.2.1.** If \( p = 3, 5 \), then the homomorphism \( \pi_{p-1,p+1}(\mathbb{A}^2 \setminus 0)(\mathbb{C}) \to \pi_{2p}(S^3) \) is surjective.

**Proof.** For \( p = 3 \), this is worked out in [AF12b, §7], so we treat the case \( p = 5 \). We study the map from the \( E_2 \)-page of the symplectic spectral sequence to the \( E_2 \)-page of the topological symplectic spectral sequence whose existence is guaranteed by the discussion just prior to the theorem statement. In particular, there is a commutative diagram of the form

\[
\begin{align*}
\ker(\pi_{5,6}^A(\mathbb{A}^6 \setminus 0)(\mathbb{C})) & \to \pi_{4,6}(\mathbb{A}^4 \setminus 0)(\mathbb{C}) \\
\downarrow & \\
\ker(\pi_{11}(S^{11}) \to \pi_{10}(S^7)) & \to d^{\text{top}}_2 \pi_{10}(S^3)/\im(\pi_{5,6}^A(\mathbb{A}^4 \setminus 0)(\mathbb{C}))
\end{align*}
\]

We now analyze this diagram.

Observe that \( d^{\text{top}}_2 \) is surjective (with a precise generator identified) by Lemma 4.1.2. It follows from [Mor12, Corollary 6.39] that \( \pi_{n,n+1}^A(\mathbb{A}^{n+1} \setminus 0)(\mathbb{C}) \cong K_0^{MW}(\mathbb{C}) \cong \mathbb{Z} \) and a generator of the group \( \pi_{n,n+1}^A(\mathbb{A}^{n+1} \setminus 0)(\mathbb{C}) \) is sent to a generator of \( \pi_{2n+1}(S^{2n+1}) \) under complex realization, so the left vertical map is an injection. We will show that it is split injective.

There is also a commutative diagram of the form

\[
\begin{align*}
\pi_{4,5}^A(\mathbb{A}^5 \setminus 0)(\mathbb{C}) & \to \pi_{3,5}^A(\mathbb{A}^3 \setminus 0)(\mathbb{C}) \\
\downarrow & \\
\pi_{p,p+1}^A(\mathbb{A}^{p+1} \setminus 0)(\mathbb{C}) & \to \pi_{p-1,p+1}(\mathbb{A}^{p-1} \setminus 0)(\mathbb{C})
\end{align*}
\]

where the vertical arrows are given by the \( \mathbb{P}^1 \)-suspension maps. The left hand map is an isomorphism because the identification \( \pi_{n,n+1}^A(\mathbb{A}^{n+1} \setminus 0)(\mathbb{C}) \cong K_0^{MW}(\mathbb{C}) \cong \mathbb{Z} \) of [Mor12, Corollary
 Proposition 5.2.1] (see Subsection 3.5 for the construction of \( \delta \)). Moreover, by analyzing the proof of [AF12c, Proposition 5.1], one observes that \( \delta \) is the image of a generator \( \nu \) of \( \pi_{5,7}(\mathbb{A}^3 \setminus 0)(\mathbb{C}) \) as a \( GW(\mathbb{C}) = \mathbb{Z} \)-module under the top horizontal map; in other words, \( \nu \) is sent to a generator of \( \mathbb{Z}/24 \). Since \( \delta \) is stably non-trivial, it follows that \( \Sigma_p \delta \) is non-zero in \( \pi_{p-1,p+1}(\mathbb{A}^{p-1} \setminus 0)(\mathbb{C}) \) under the right vertical map and commutativity shows that \( \Sigma_p \delta \) is the image of \( \Sigma_p \delta \).

We also observed in [AF12c, Corollary 5.3.1] that \( \delta \) is sent to the suspension of \( \nu^{top} \) under complex realization, and by compatibility of complex realization and suspension, it follows that complex realization sends the \( \mathbb{P}^1 \)-suspension of \( \delta \) to a threefold suspension of \( \nu^{top} \). This provides the splitting mentioned in the previous paragraph. Combining these two observations, we obtain the splitting mentioned two paragraphs above.

Since the generator \( 24\Sigma P \alpha \) of \( 24\mathbb{Z} \subset \pi_{p-1,p+1}(\mathbb{A}^{p-1} \setminus 0)(\mathbb{C}) \) is sent to a generator of \( 24\mathbb{Z} \subset \pi_{2p+1}(S^{2p+1}) \) under complex realization, and the latter is sent by \( d_{top} \) in the topological symplectic spectral sequence to a generator of \( \pi_{10}(S^3) \), it follows that \( d_{top}(24\Sigma P \alpha) \) lifts this generator in \( \pi_{p-1,p+1}(\mathbb{A}^2 \setminus 0) \).

\[ \square \]

### 4.3 Complements

It is possible to establish a result like Theorem 4.2.1 using the anti-symmetric spectral sequence. Since the proof is essentially identical to proof of Theorem 4.2.1 it will only be sketched.

**Theorem 4.3.1.** The homomorphism \( \pi_{5,7}(\mathbb{A}^3 \setminus 0) \rightarrow \pi_{12}(S^5) \cong \mathbb{Z}/30 \) is surjective.

**Proof.** If \( \Gamma_n := U(2n)/Sp(2n) \), then there are fiber sequences of the form \( \Gamma_n \rightarrow \Gamma_{n+1} \rightarrow S^{2n+1} \). The long exact sequences in homotopy fit together to yield an exact couple that is the topological analog of the anti-symmetric spectral sequence considered in Subsection 2.3. This spectral sequence converges to the homotopy groups of \( U/Sp \), which are known by Bott periodicity. Observe that \( \pi_{2i}(U/Sp) \cong \pi_{2i-2}(Sp) \) and so vanishes by explicit computation. Again, complex realization yields a morphism from the anti-symmetric spectral sequence to its topological counterpart.

Consider then the commutative diagram:

\[
\begin{array}{ccc}
\ker(\pi_{5,7}(\mathbb{A}^3 \setminus 0)(\mathbb{C})) & \rightarrow & \pi_{5,7}(\mathbb{A}^3 \setminus 0)(\mathbb{C})) \\
\downarrow & & \downarrow \\
\ker(\pi_{13}(S^{13}) \rightarrow \pi_{12}(S^9)) & \rightarrow & \pi_{12}(S^9)/\im(\pi_{13}(S^9)).
\end{array}
\]

Again, \( \pi_{13}(S^9) = 0 \) and one observes that \( d_{top} \) is surjective in a fashion identical to Lemma 4.1.2. The remainder of the analysis is analogous to the end of the proof of Theorem 4.2.1. \[ \square \]

### 5 Building explicit representatives

Given the existence of at least 15 non-trivial rank 2 algebraic vector bundles on \( Q_{11} \) it would be interesting to construct explicit representatives of these bundles. It follows from the results of
[AF12a], which we review below, that all such bundles are stably trivial. Corollary 5.2.2 demonstrates that non-trivial rank 2 algebraic vector bundles on $Q_{11}$ whose associated topological bundles are non-trivial, remain algebraically non-trivial after forming the direct sum with trivial bundles of rank $\leq 3$.

### 5.1 Stable triviality results

The inclusion of $M \in SL_n(R)$ in $SL_{n+1}(R)$ as block diagonal matrices of the form $\text{diag}(M,1)$ gives a morphism of spaces $BSL_n \to BSL_{n+1}$. If $X$ is a smooth affine scheme, then the induced map

$$[X, BSL_n]_{k^1} \to [X, BSL_{n+1}]_{k^1}$$

corresponds to the operation of adding a trivial rank 1 bundle. By means of the identifications mentioned in the introduction, when $X = Q_{2i-1}$, the above function corresponds to a morphism

$$\Phi_{1,n} : \pi_{i-1,n}^1(BSL_n) \to \pi_{i-1,n}^1(BSL_{n+1}).$$

Write $\Phi_{m,n}$ for the composite morphism $\Phi_{1,n+m-1} \circ \cdots \circ \Phi_{1,n+1} \circ \Phi_{1,n}$.

To answer the question of whether a bundle on $Q_{11}$ becomes trivial after successively adding trivial bundles of rank 1 amounts to studying whether a class $\xi \in \pi_{5,6}^1(BSL_2)(\mathbb{C})$ is sent to 0 under $\Phi_{m,2}$. The next result shows that $\Phi_{m,2}$ is the zero map for $m \geq 4$.

**Lemma 5.1.1** ([AF12a, Corollary 4.7]). If $n \geq 1$, and $m \geq n$, $\mathcal{V}_m(Q_{2n-1}) = \ast$.

### 5.2 Adding trivial summands of small rank

The topological analog of the sequence of homomorphisms considered in the previous section can be analyzed using classical results. Precisely, we have the following result.

**Proposition 5.2.1.** The homomorphisms $\pi_{11}(BSU(2)) \to \pi_{11}(BSU(m))$ are injective for $3 \leq m \leq 5$.

**Proof.** One knows that $\pi_{11}(BSU(3)) \cong \mathbb{Z}/30$ [MT64], $\pi_{11}(BSU(4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/5!$ [Ker60], and $\pi_{11}(BSU(5)) \cong \mathbb{Z}/5!$ [Bot58, Theorem 5]. For injectivity when $m = 3$, consider the portion of the long exact sequence in homotopy groups associated with $S^5 \to BSU(2) \to BSU(3)$

$$\pi_{11}(S^5) \to \pi_{11}(BSU(2)) \to \pi_{11}(BSU(3)).$$

We know that $\pi_{11}(S^5) \cong \mathbb{Z}/2$, so the left hand map is zero, and map in the middle must be injective.

Next, consider the long exact sequence in homotopy groups associated with $S^7 \to BSU(3) \to BSU(4)$. In that case, we have

$$\pi_{11}(S^7) \to \pi_{11}(BSU(3)) \to \pi_{11}(BSU(4)) \to \pi_{10}(S^7).$$

In this case, $\pi_{11}(S^7) = 0$, so the composite map $\pi_{11}(BSU(2)) \to \pi_{11}(BSU(4))$ is the composite of a pair of injective maps and injectivity for $m = 4$ is settled.

For the case $m = 5$, first observe that since $\pi_{11}(BSU(2)) \cong \mathbb{Z}/15$ its image in $\pi_{11}(BSU(4))$ is necessarily contained in the summand isomorphic to $\mathbb{Z}/5!$. It follows from the computations of Ker-vaire that the map $\pi_{11}(BSU(4)) \to \pi_{11}(BSU(5))$ is the projection onto the summand isomorphic to $\mathbb{Z}/5!$ and the result follows. □
The next result is a straightforward consequence of Lemma 5.1.1, Proposition 5.2.1, Theorem 4.2.1 and the functoriality of complex realization.

**Corollary 5.2.2.** If $\xi$ an element of $\pi_{5,6}^{A_1}(BSL_2)(\mathbb{C})$ that does not lie in the kernel of the (surjective) complex realization map $\pi_{5,6}^{A_1}(BSL_2)(\mathbb{C}) \to \pi_{11}(BSU(2))$, then the image of $\xi$ under the homomorphism

$$\pi_{5,6}^{A_1}(BSL_2)(\mathbb{C}) \longrightarrow \pi_{5,6}^{A_1}(BSL_m)(\mathbb{C})$$

is non-trivial for $3 \leq m \leq 5$.

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