H^p estimation for the Cauchy problem for nonlinear elliptic equation

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ABSTRACT
In this paper, we investigate the Cauchy problem for a ND nonlinear elliptic equation in a bounded domain. As we know, the problem is severely ill-posed. We apply the Fourier truncation method to regularize the problem.

Key words: nonlinear elliptic equation, ill-posed problem, regularization, truncation method

INTRODUCTION
In this paper, we consider the Cauchy problem for a nonlinear elliptic equation in a bounded domain. The problem has the form

\[\begin{align*}
\Delta u &= F(x', x_N, u(x', x_N)), & (x', x_N) &\in \Omega \times (0, T), \\
u(x', x_N) &= 0, & (x', x_N) &\in \partial \Omega \times (0, T), \\
u(x', T) &= \varphi(x'), & x' &\in \Omega \\
u_{x_N}(x', T) &= 0, & x' &\in \Omega
\end{align*}\]

Where \(T\) is a positive constant, \(\Omega = (0, \pi)^{N-1}\), \(N\) is a natural number and \(N \geq 2\), the function \(\varphi \in L^2(\Omega)\) is known and \(F\) is called the source function. It is well-known the above problems is severely ill-posed in the sense of Hadamard. In fact, for a given final data, we are not sure that a solution of the problem exists. In the case a solution exists, it may not depend continuously on the final data. The problem has many various applications, for example in electrocardiography [7], astrophysics [6] and plasma physics [15, 16].

In the past, there have been many studies on the Cauchy problem for linear homogeneous elliptic equations, [1, 5, 9, 10, 12]. However, the literature on the nonlinear elliptic equation is quite scarce. We mention here a nonlinear elliptic problem of [13] with globally Lipschitz source terms, where authors approximated the problem by a truncation method. Using the method in [13,14], we study the Cauchy problem for nonlinear elliptic in multidimensional domain.

The paper is organized as follows. In Section 2, we present the solution of equation (1). In Section 3, we present the main results on regularization theory for local Lipschitz source function. We finish the paper with a remark.

SOLUTION OF THE PROBLEM
Assume that problem (1) has a unique solution \(u(x', x_N)\). By using the method of separation of variables, we can show that solution of the problem has the form
Indeed, let \( u(x', x_N) = \sum_{n_1=1}^{\infty} \ldots \sum_{n_{N-1}=1}^{\infty} u_{n_1 \ldots n_{N-1}} (x_N) \phi_{n_1 \ldots n_{N-1}} (x') \) be the Fourier series in \( L^2(\Omega) \) with orthonormal basis \( \phi_{n_1 \ldots n_{N-1}} (x') = \frac{1}{\sqrt{\pi}} \sin(n_1 x_1) \sin(n_2 x_2) \ldots \sin(n_{N-1} x_{N-1}) \). From (1), we can obtain the following ordinary differential equation

\[
\begin{align*}
\frac{d^2}{dx_N^2} u_{n_1 \ldots n_{N-1}} (x_N) &= -n_1^2 + n_2^2 + \ldots + n_{N-1}^2 \ u_{n_1 \ldots n_{N-1}} (x_N) = F_{n_1 \ldots n_{N-1}} (u) \quad \text{for} \quad x_N \in [0, T],
\end{align*}
\]

where \( F_{n_1 \ldots n_{N-1}} (u) = \int \mathcal{F}(x', x_N, u(x', x_N) \phi_{n_1 \ldots n_{N-1}} (x') dx' \), \( \varphi_{n_1 \ldots n_{N-1}} = \int \varphi(x') \phi_{n_1 \ldots n_{N-1}} (x') dx' \) and \( u_{n_1 \ldots n_{N-1}} = \int u(x', x_N) \phi_{n_1 \ldots n_{N-1}} (x') dx' \).

The equation (3) is ordinary differential equations. It is easy to see that its solution is given by

\[
\begin{align*}
\frac{d^2}{dx_N^2} u_{n_1 \ldots n_{N-1}} (x_N) &= -n_1^2 + n_2^2 + \ldots + n_{N-1}^2 \ u_{n_1 \ldots n_{N-1}} (x_N) = F_{n_1 \ldots n_{N-1}} (u) \quad \text{for} \quad x_N \in [0, T],
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Let the function \( F: \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) such that: for each \( M > 0 \) and for any \( u, v \) satisfying \(|u|, |v| \leq M \), there holds
\[
|F(x', x_N, u) - F(x', x_N, v)| \leq K_p(M)|u - v|,
\]
where \((x', x_N) \in \Omega \times [0, T] \) and
\[
K_p(M) := \sup \left\{ \frac{|F(x', x_N, u) - F(x', x_N, v)|}{|u - v|}; \quad |u|, |v| \leq M, \quad u \neq v, (x', x_N) \in \Omega \times [0, T] \right\} < +\infty.
\]
We note that \( K_p(M) \) is increasing and \( \lim_{M \rightarrow +\infty} K_p(M) = +\infty \). For all \( M > 0 \), we approximate \( F \) by \( F_M \) defined by
\[
F_M(x', x_N, u(x', x_N)) = \begin{cases} 
F(x', x_N, M), & u(x', x_N) > M, \\
F(x', x_N, u(x', x_N)), & -M \leq u(x', x_N) \leq M, \\
F(x', x_N, -M), & u(x', x_N) < -M.
\end{cases}
\]
For each \( \varepsilon > 0 \), we consider a parameter \( M_{\varepsilon} \rightarrow +\infty \) as \( \varepsilon \rightarrow 0 \). We shall use the following well-posed problem
\[
\begin{aligned}
\Delta v &= F_M, & x', x_N, v, x', x_N, \\
v(x', x_N) &= 0, & x', x_N \in \Omega \times 0, T, \\
v(x', T) &= P_C w = x', & v(x', T) = 0, \quad x' \in \Omega,
\end{aligned}
\]
where
\[
P_C w = \sum_{n_1, n_2, \ldots, n_K \geq 1} \left< w, \phi_{n_1, n_2, \ldots, n_K} \right> \phi_{n_1, n_2, \ldots, n_K} \quad \text{for all } w \in L^2(\Omega).
\]
We show that the solution \( u_{\varepsilon, \psi} \) of problem (6) satisfies the following integral equation
\[
u_{\varepsilon, \psi}(x', x_N) = \sum_{n_1, n_2, \ldots, n_K \geq 1} \left[ \cosh((T - x_N) \sqrt{n_1^2 + n_2^2 + \cdots + n_K^2}) \psi_{n_1, n_2, \ldots, n_K} + \right.
\]
\[
+ \int_{x_N}^{T} \frac{\sinh((T - x_N) \sqrt{n_1^2 + n_2^2 + \cdots + n_K^2})}{\sqrt{n_1^2 + n_2^2 + \cdots + n_K^2}} \left. F_M(n_1, n_2, \ldots, n_K, (\psi'))dT \right] \phi_{n_1, n_2, \ldots, n_K}(x'),
\]
Lemma 1. For \( u_1(x', x_N), u_2(x', x_N) \), we have
\[
\left| F_M(x', x_N, u_1(x', x_N)) - F_M(x', x_N, u_2(x', x_N)) \right| \leq K_p(M) \left| u_1(x', x_N) - u_2(x', x_N) \right|.
\]
Proof. If \( u_1(x', x_N) < -M \) and \( u_2(x', x_N) < -M \) then
\[
\left| F_M(x', x_N, u_1(x', x_N)) - F_M(x', x_N, u_2(x', x_N)) \right| = 0.
\]
If \( u_1(x', x_N) < -M \leq u_2(x', x_N) \leq M \) then
\[
\left| F_M(x', x_N, u_1(x', x_N)) - F_M(x', x_N, u_2(x', x_N)) \right| = \left| F_M(x', x_N, u_1(x', x_N)) - F_M(x', x_N, -M) \right| \leq K_p(M) \left| u_2(x', x_N) - u_1(x', x_N) \right|.
\]
If \( u_1(x', x_N) < -M < u_2(x', x_N) \) then

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\[
\left| F_M'(x', x_N, u(x', x_N)) - F_M'(x', x_N, u(x', x_N)) \right| = \left| F_M'(x', x_N, M) - F_M'(x', x_N, -M) \right| \\
\leq K_p(M) \left| u(x', x_N) - u(x', x_N) \right|.
\]

If \( -M \leq u(x', x_N), u(x', x_N) \leq M \) then
\[
\left| F_M'(x', x_N, u(x', x_N)) - F_M'(x', x_N, u(x', x_N)) \right| = \left| F(x', x_N, u(x', x_N)) - F(x', x_N, u(x', x_N)) \right| \\
\leq K_p(M) \left| u(x', x_N) - u(x', x_N) \right|.
\]

This completes the proof.

**Lemma 2.** Let \( u \) be the exact solution to problem (1). Then we have the following estimate
\[
\left\| u(x', x_N) - P_{\epsilon^2} u(x_N) \right\|_{L^2([0, T]; L^2([0, T])}) \leq 2 \exp(2(T - x_N)C) \left\| \phi^2 - \phi \right\|_{L^2([0, T]; L^2([0, T]))} \\
+ 2K_p^2(M) \int_{x_N}^{T} \exp(2(T - x_N)C) \left\| u_{\epsilon^2}^2(\tau) - u(\tau) \right\|_{L^2([0, T]; L^2([0, T]))} d\tau.
\]

Proof. From the definition of \( u_{\epsilon^2} \) and \( u \), we have
\[
\left\| u_{\epsilon^2}^2(x_N) - P_{\epsilon^2} u(x_N) \right\|_{L^2([0, T]; L^2([0, T])}) \leq 2 \sum_{n_1, n_2, \ldots, n_{N - 1}\geq 1} \left( \frac{\sinh((T - x_N)\sqrt{n_1^2 + n_2^2 + \ldots + n_{N - 1}^2})}{\sqrt{n_1^2 + n_2^2 + \ldots + n_{N - 1}^2}} \right)
\]
\[
\leq 2 \exp(2(T - x_N)C) \left\| \phi^2 - \phi \right\|_{L^2([0, T]; L^2([0, T]))} + 2(T - x_N) \int_{x_N}^{T} \exp(2(T - x_N)C) \left\| F_M' (\tau, u_{\epsilon^2}^2(\tau)) - F(\tau, u(\tau)) \right\|_{L^2([0, T]; L^2([0, T]))} d\tau.
\]

Since \( \lim_{\epsilon \to 0} M_\epsilon = +\infty \), for a sufficiently small \( \epsilon > 0 \), there exists \( M_\epsilon \) such that \( M_\epsilon \geq \|u\|_{L^2([0, T]; L^2([0, T]))} \).

For \( M_\epsilon \), we have \( F_M'(x', x_N, u(x', x_N)) = F(x', x_N, u(x', x_N)) \). Using the Lipschitz property of \( F_M' \) as in Lemma 1, we get
\[
\left\| F_M' (\tau, u_{\epsilon^2}^2(\tau)) - F(\tau, u(\tau)) \right\|_{L^2([0, T]; L^2([0, T]))} \leq K_p^2(M) \left\| u_{\epsilon^2}^2(\tau) - u(\tau) \right\|_{L^2([0, T]; L^2([0, T]))}.
\]

Combining (8) and (9), we complete the proof of Lemma 2. □

**Theorem 1.** Let \( \epsilon > 0 \) and let \( F \) be the function defined in (5). Then the problem (6) has a unique solution \( u_{\epsilon^2} \in C([0, T]; L^2([0, T])) \).

Proof. We prove the equation (7) has a unique solution \( u_{\epsilon^2} \in C([0, T]; L^2([0, T])) \). Put
\[
\Phi(u_{\epsilon^2}) = \psi(x', x_N) + G(x', x)
\]
where
\[
\psi(x', x_N) = \sum_{n_1, n_2, \ldots, n_{N - 1}\geq 1} \sinh((T - x_N)\sqrt{n_1^2 + n_2^2 + \ldots + n_{N - 1}^2}) \phi_{n_1, n_2, \ldots, n_{N - 1}} (x')
\]
\[
G(x', x) = \sum_{n_1, n_2, \ldots, n_{N - 1}\geq 1} \cosh((T - x_N)\sqrt{n_1^2 + n_2^2 + \ldots + n_{N - 1}^2}) \phi_{n_1, n_2, \ldots, n_{N - 1}} (x')
\]

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and
\[ G(x', x_N) = \sum_{n, n_1, \ldots, n_{k+1} \geq 0} \left( \int_{x_N} \frac{\sinh((\tau - x_N)\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2})}{\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2}} F_{M_{n, n_1, \ldots, n_{k+1}}} (u_{\tau, v_{x'}}(\tau)) d\tau \right) \delta_{n, n_1, \ldots, n_{k+1}} (x') \]

We claim that
\[ \left\| F^p(v_{x'}, w_{x'}) (x_N) - F^p(w_{x'}, v_{x'}) (x_N) \right\|_{C^0} \leq \sqrt{K^2(M) T \exp(2T C)} \frac{p!}{p} \left\| v_{x'} - w_{x'} \right\| \tag{10} \]

for \( p \geq 1 \), where \( \| \| \) is the sup norm in \( C([0, T]; L^2(\Omega)) \). We shall prove the above inequality by induction.

For \( p = 1 \), using the inequality
\[ \int_{x_N} \frac{\sinh((\tau - x_N)\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2})}{\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2}} d\tau \leq \exp(2\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2} T) T \]

and using Lemma 1, we have
\[ \left\| F(v_{x'}, v_{x'}) (x_N) - F(w_{x'}, v_{x'}) (x_N) \right\|_{C^0} = \sum_{n, n_1, \ldots, n_{k+1} \geq 0} \left( \int_{x_N} \frac{\sinh((\tau - x_N)\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2})}{\sqrt{n_1^2 + n_2^2 + \cdots + n_{k+1}^2}} \left( F_{M_{n, n_1, \ldots, n_{k+1}}} (v_{x'}(\tau)) - F_{M_{n, n_1, \ldots, n_{k+1}}} (w_{x'}(\tau)) \right) d\tau \right)^2 \]

\[ \leq \exp(2T C) \int_{x_N} \sum_{n=1}^{k+1} \sum_{n_1=1}^{n} \sum_{n_{k+1}=1}^{n} \left| F_{M_{n, n_1, \ldots, n_{k+1}}} (v_{x'}(\tau)) - F_{M_{n, n_1, \ldots, n_{k+1}}} (w_{x'}(\tau)) \right|^2 d\tau \]

\[ \leq \exp(2T C) \int_{x_N} F_{M_{n, n_1, \ldots, n_{k+1}}} (\tau, v_{x'}(\tau)) - F_{M_{n, n_1, \ldots, n_{k+1}}} (\tau, w_{x'}(\tau)) \right\|_{C^0} \leq K^2(M) \exp(2T C) T \left\| v_{x'} - w_{x'} \right\|. \]

Thus (10) holds for \( p = 1 \). Suppose that (10) holds for \( p = k \). We prove that (10) holds for \( p = k + 1 \).

We have
\[ \left\| F^{k+1}(v_{x'}, w_{x'}) (x_N) - F^{k+1}(w_{x'}, v_{x'}) (x_N) \right\|_{C^0} \leq \exp(2T C) \int_{x_N} F_{M_{n, n_1, \ldots, n_{k+1}}} (\tau, \Phi^k(v_{x'}(\tau))) - F_{M_{n, n_1, \ldots, n_{k+1}}} (\tau, \Phi^k(w_{x'}(\tau))) \right\|_{C^0} \leq K^2(M) \exp(2T C) T \left\| v_{x'} - w_{x'} \right\|. \]

Therefore, we get

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\begin{align*}
\|\Phi^p(v_{e_{\omega^p}})(x_N) - \Phi^p(w_{e_{\omega^p}})(x_N)\|_{L^\infty} & \leq \sqrt{\frac{K_p^2 (M_p) T \exp(2TC_p)}{p!}} \|v_{e_{\omega^p}} - w_{e_{\omega^p}}\|, \quad (11)
\end{align*}
for all $v_{e_{\omega^p}}, w_{e_{\omega^p}} \in C([0,T];L^2(\Omega))$.
Let us consider $\Phi : C([0,T];L^2(\Omega)) \rightarrow C([0,T];L^2(\Omega))$. It is easy to see that
\begin{align*}
\lim_{p \to +\infty} \sqrt{\frac{K_p^2 (M_p) T \exp(2TC_p)}{p!}} = 0.
\end{align*}
As a consequence, there exists a positive integer number $p_0$ such that $\Phi^{p_0}$ is a contraction. It follows that
the equation $\Phi^{p_0}(u) = u$ has a unique solution $u_{e_{\omega^p}} \in C([0,T];L^2(\Omega))$. We claim that $\Phi(u_{e_{\omega^p}}) = u_{e_{\omega^p}}$.
In fact, one has $\Phi(\Phi^{p_0}(u_{e_{\omega^p}})) = \Phi(u_{e_{\omega^p}})$. By the uniqueness of the fixed point of $\Phi^{p_0}$, one has
$\Phi(u_{e_{\omega^p}}) = u_{e_{\omega^p}}$, i.e., the equation $\Phi(u) = u$ has a unique solution $u_{e_{\omega^p}} \in C([0,T];L^2(\Omega))$.

To show error estimates between the exact solution and the regularized solution, we need the exact solution belonging
 to the Gevrey space.

Definition 1. (Gevrey-type space). (see [2, 3]) The Gevrey class of functions of order $s > 0$ and index $\sigma > 0$ is
denoted by $G^{s,\sigma}_2$ and is defined as
\begin{align*}
G^{s,\sigma}_2 = \{ f \in L^2(\Omega) : \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \ldots \sum_{n_{N-1}=1}^{n_{N-2}} (n_1^n + n_2^n + \ldots + n_{N-1}^n)^{s/2} \exp\left(2\sigma \sqrt{n_1^n + n_2^n + \ldots + n_{N-1}^n}\right) | \left\{ f, \phi_{n_1^{n_1} \ldots n_{N-1}^{n_{N-1}}} \right\} |^2 < \infty \}.
\end{align*}
It is a Hilbert space with the following norm
\begin{align*}
\|f\|_{G^{s,\sigma}_2} = \left( \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \ldots \sum_{n_{N-1}=1}^{n_{N-2}} (n_1^n + n_2^n + \ldots + n_{N-1}^n)^{s/2} \exp\left(2\sigma \sqrt{n_1^n + n_2^n + \ldots + n_{N-1}^n}\right) | \left\{ f, \phi_{n_1^{n_1} \ldots n_{N-1}^{n_{N-1}}} \right\} |^2 \right)^{1/2}.
\end{align*}
For a Hilbert space $H$, we denote $L^\infty(0,T;H) = \{ f : [0,T] \rightarrow H \mid \text{ess sup}_{0 \leq t \leq T} |f(t)|_H < \infty \}$ and
\begin{align*}
\|f\|_{L^\infty(0,T;H)} = \text{ess sup}_{0 \leq t \leq T} |f(t)|_H.
\end{align*}
We consider some assumptions on the exact solution as the following:
\begin{align*}
\text{ess sup}_{0 \leq x_y \leq T} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \ldots \sum_{n_{N-1}=1}^{n_{N-2}} (n_1^n + n_2^n + \ldots + n_{N-1}^n)^{s/2} \exp\left(2\sigma \sqrt{n_1^n + n_2^n + \ldots + n_{N-1}^n}\right) | \left\{ f, \phi_{n_1^{n_1} \ldots n_{N-1}^{n_{N-1}}} \right\} |^2 \leq I_1, \quad (12)
\end{align*}
\begin{align*}
\text{ess sup}_{0 \leq x_y \leq T} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \ldots \sum_{n_{N-1}=1}^{n_{N-2}} \exp(2(x_y + \alpha)\sqrt{n_1^n + n_2^n + \ldots + n_{N-1}^n}) u_{\phi_{n_1^{n_1} \ldots n_{N-1}^{n_{N-1}}}^2}(x_N) \leq I_2, \quad (13)
\end{align*}
for all $x_y \in [0,T)$, where $\alpha, \beta, I_1, I_2$ are positive constants.
Lemma 3. For any $w \in G^{s,\sigma}_2$, we have the following inequality
\begin{align*}
\left\| w - P_{\omega^p} w \right\|_{L^\infty(\Omega)} \leq C^{1/2} e^{-\sigma C^{1/2}} \left\| w \right\|_{G^{s,\sigma}_2}.
\end{align*}
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Proof. For \( w \in C^k_e \), we get
\[
\left\| w - P_{e} \phi \right\| \leq \sum_{n_1, n_2, \ldots, n_k \geq 1} \left\| w, \phi_{n_1, n_2, \ldots, n_k} \right\| e^{-2\pi n_1^2 + n_2^2 + \cdots + n_k^2},
\]
where \( C \) depends on the specific problem. This completes the proof.

The following theorem provides some error estimates in the \( L^2 \) norm when the exact solution belongs to the Gevrey space.

**Theorem 2.** Assume that the problem (1) has a unique solution \( u \) which satisfies (12). If \( C_s \) and \( M \) are chosen such that
\[
\lim_{n \to 0} e^{-TC_s} = 0 \quad \text{and} \quad \lim_{n \to 0} \exp(2K^2(M)T^2)e^{-TC_s} = 0,
\]
then we have
\[
\left\| u(x_N) - u(x) \right\|_{L^2} \leq 2C_s e^{-2\pi C_s} \exp(2K^2(M)T^2)e^{-\pi C_s}.
\]

**Proof.** Since \( w \in C^k_e \), then using Lemma 3, we get
\[
\left\| w - P_{e} \phi \right\| = \sum_{n_1, n_2, \ldots, n_k \geq 1} \left\| w, \phi_{n_1, n_2, \ldots, n_k} \right\| e^{-2\pi n_1^2 + n_2^2 + \cdots + n_k^2},
\]
which leads to the desired result
\[
\left\| u(x_N) - u(x) \right\|_{L^2} \leq 2C_s e^{-2\pi C_s} \exp(2K^2(M)T^2)e^{-\pi C_s}.
\]

Multiplying (15) by \( e^{2\pi C_s} \) and applying Gronwall’s inequality, we get
\[
e^{2\pi C_s} \left\| u(x_N) - u(x) \right\|_{L^2} \leq \left( 2C_s \sup_{0 \leq x \leq T} \left\| u(x_N) \right\| + 4e^{2\pi C_s} \exp(2K^2(M)T^2) e^{-\pi C_s} \right),
\]
which leads to the desired result
\[
\left\| u(x_N) - u(x) \right\|_{L^2} \leq \sqrt{2C_s T^2 + 4e^{2\pi C_s} e^{-\pi C_s} 2K^2(M)T^2 e^{-\pi C_s}}.
\]

This completes the proof.
The next theorem provides an error estimate in the Hilbert scales \( \{ H^p(\Omega) \}_{p \in \mathbb{R}} \) which is equipped with a norm defined by
\[
\left\| f \right\|_{H^p(\Omega)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \ldots \sum_{n_p=1}^{N_p} \left( n_1^2 + n_2^2 + \ldots + n_p^2 \right)^p \left\| f, \phi_{n_1,n_2,\ldots,n_p} \right\|^p .
\]

Theorem 3. Assume that the problem (1) has a unique \( u \) which satisfies (13). Let us choose \( C_\varepsilon \) and \( M_\varepsilon \) such that
\[
\lim_{\varepsilon \to 0} C_\varepsilon e^{r_\varepsilon C} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} e^{K_\varepsilon(M_\varepsilon)^2} e^{-\varepsilon C} = \lim_{\varepsilon \to 0} e^{K_\varepsilon(M_\varepsilon)^2} e^{r_\varepsilon C} = 0 ,
\]
then we have
\[
\left\| u_{\varepsilon} - P_{\varepsilon} u(x) \right\|_{H^p(\Omega)} \leq \left( \sqrt{2} + 1 \right) e^{K_\varepsilon(M_\varepsilon)^2} e^{-\varepsilon C} I_3 + 2 e^{K_\varepsilon(M_\varepsilon)^2} e^{r_\varepsilon C} C_\varepsilon e^{-\varepsilon C} , \quad x_N \in [0,T] .
\]

Proof. First, we have
\[
\left\| u_{\varepsilon} - P_{\varepsilon} u(x) \right\|_{H^p(\Omega)} = \sum_{n_1,n_2,\ldots,n_p = 1}^{N_1,N_2,\ldots,N_p} n_1^2 + n_2^2 + \ldots + n_p^2 \left\| u_{\varepsilon}(x',x_N) - u(x',x_N), \phi_{n_1,n_2,\ldots,n_p} (x') \right\|^p \leq C^p_\varepsilon \left\| u_{\varepsilon}(x) - u(x) \right\|_{H^p(\Omega)} .
\]

It follows from theorem 2 that
\[
\left\| u_{\varepsilon} - P_{\varepsilon} u(x) \right\|_{H^p(\Omega)} \leq \exp(2K_\varepsilon(M_\varepsilon)^2) C^p_\varepsilon e^{-\varepsilon C} 2 e^{-2\varepsilon C} \sup_{0 \leq x_\varepsilon \leq T} \left\| u(x) \right\|_{L^\infty} + 4 e^{2\varepsilon C} , \quad (16)
\]

On the other hand, we consider the function
\[
G(\xi) = \xi^2 e^{-\xi C} , \quad D>0 .
\]
Since \( G'(\xi) = \xi e^{-\xi C} (p - D \xi) \), it follows that \( G \) is decreasing when \( D \xi \geq p \) . Thus if \( \xi \leq \frac{\varepsilon(T-\alpha)}{2D} \), i.e., \( 2(x_N + \alpha)C_\varepsilon \geq p \), then for \( n_1^2 + n_2^2 + \ldots + n_p^2 \geq C_\varepsilon \), we get
\[
n_1^2 + n_2^2 + \ldots + n_p^2 \exp -2(x_N + \alpha)\sqrt{n_1^2 + n_2^2 + \ldots + n_p^2} \leq C_\varepsilon e^{-2(x_N + \alpha)C_\varepsilon} ,
\]
and
\[
\left\| u(x_N) - P_{\varepsilon} u(x_N) \right\|_{H^p(\Omega)} = \sum_{n_1,n_2,\ldots,n_p = 1}^{N_1,N_2,\ldots,N_p} n_1^2 + n_2^2 + \ldots + n_p^2 \left\| u(x',x_N), \phi_{n_1,n_2,\ldots,n_p} (x') \right\|^p \leq C^p_\varepsilon \exp -2(x_N + \alpha)C_\varepsilon \sum_{n_1,n_2,\ldots,n_p = 1}^{N_1,N_2,\ldots,N_p} \exp 2(x_N + \alpha)\sqrt{n_1^2 + n_2^2 + \ldots + n_p^2} \left\| u(x',x_N), \phi_{n_1,n_2,\ldots,n_p} (x') \right\|^p \leq C^p_\varepsilon e^{-2(x_N + \alpha)C_\varepsilon} \sup_{0 \leq x_\varepsilon \leq T} \left\| u(x_N) \right\|_{L^\infty} .
\]

Therefore
\[
\left\| u(x_N) - P_{\varepsilon} u(x_N) \right\|_{H^p(\Omega)} \leq C^p_\varepsilon e^{-2(x_N + \alpha)C_\varepsilon} \sup_{0 \leq x_\varepsilon \leq T} \left\| u(x_N) \right\|_{L^\infty} . \quad (17)
\]
Combining (16) and (17), we get

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The inequality $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$ leads to
$$\|u_{\varepsilon, y} (x_N) - u(x_N)\|_H^2(0) \leq \exp \{ 2K^2_y (M) T^2 \} \sup_{0 \leq x_N \leq \gamma} \|u(x_N)\|_{H^2(0)} + 4e^{2\gamma^2 T C} \sup_{0 \leq x_N \leq \gamma} \|u(x_N)\|_{H^2(0)} e^{-\gamma C} C^2 e^{-\gamma C}$.

Remark 1. In theorem 2, let us choose $C_\epsilon = \frac{\gamma}{T} \ln \left( \frac{1}{\epsilon} \right)$, for $\gamma \in (0,1) \quad$ and $\quad M_\epsilon$ such that
$$K_\epsilon M_\epsilon = \frac{1}{\sqrt{2T}} \sqrt{\beta - r \ln \left( \frac{\gamma}{T} \ln \frac{1}{\epsilon} \right)},$$
for $r \in [0, \beta]$. It is easy to check that $\lim_{\gamma \to 0} \exp (2K^2_y (M) T^2) C_\epsilon^{-\beta} = \lim_{\gamma \to 0} \exp (2K^2_y (M) T^2) e^{-\gamma C} = 0$.

Then (14) becomes
$$\|u_{\varepsilon, y} (x_N) - u(x_N)\|_H^2(0) \leq 2T^2 + 4e^{2\gamma^2 T C} \left( \frac{\gamma}{T} \ln \left( \frac{1}{\epsilon} \right) \right)^2 \left( \frac{\gamma}{T} \ln \left( \frac{1}{\epsilon} \right) \right) e^{-\gamma C} \frac{\gamma}{T} \ln \left( \frac{1}{\epsilon} \right).$$

CONCLUSION

In this paper, we investigate the Cauchy problem for a ND nonlinear elliptic equation in a bounded domain. We apply the Fourier truncation method for regularizing the problem. Error estimates between the regularized solution and exact solution are established in $H^p$ space under some priori assumptions on the exact solution. In future, we will consider the Cauchy problem for a coupled system for nonlinear elliptic equations in three dimensions.

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Đánh giá $H^p$ cho bài toán Cauchy cho phương trình elliptic phi tuyến

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TÓM TÁT

Trong bài báo này, chúng tôi nghiên cứu bài toán Cauchy cho phương trình elliptic phi tuyến trên miền bị chặn trong không gian nhiều chiều. Như đã biết, bài toán này là không chỉnh. Chúng tôi sử dụng phương pháp chặt chẽ Fourier để chỉnh hóa nghiệm của bài toán. Đánh giá sai số giữa nghiệm chỉnh hóa và nghiệm chính xác được thiết lập trong không gian $H^p$ với các giả thiết cho trước về tính trơn của nghiệm chính xác.
Từ khóa: phương trình elliptic phi tuyến, bài toán không chỉnh, chỉnh hóa, phương pháp chặt cụt

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