Diluted one-dimensional spin glasses with power law decaying interactions

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We introduce a diluted version of the one dimensional spin-glass model with interactions decaying in probability as an inverse power of the distance. In this model varying the power corresponds to change the dimension in short-range models. The spin-glass phase is studied in and out of the range of validity of the mean-field approximation in order to discriminate between different theories. Since each variable interacts only with a finite number of others the cost for simulating the model is drastically reduced with respect to the fully connected version and larger sizes can be studied. We find both static and dynamic evidence in favor of the so-called replica symmetry breaking theory.

Mean-field spin glass models are known to have a rather complex low-temperature phase [1], which has not been clearly observed so far in numerical simulations of finite-dimensional models with short range interactions. Theories alternative to the mean-field one have been proposed [2], but short-range systems with quenched disorder are very tough to study analytically [3]. Numerical simulations have been, thus, extensively employed, developing more and more refined algorithms over the years, though with no conclusive indication on the nature of the spin-glass phase in finite dimension.

Long-range models are such that their lower critical dimension is lower than the one of the corresponding short range model. In particular, one can have a phase transition even in one dimensional systems, provided the range of interaction is large enough. One dimensional spin glass models with power-law decaying interactions actually allow to explore both long- and short-range regimes by changing the power [4,5,6,7]. These models would be perfect candidates for comparing the spin glass phase in and out of the range of validity of the mean field approximation. Unfortunately, since each variable interacts with all the others, numerical simulations are very computer demanding and it is hard to get a clear numerical evidence supporting a specific spin glass theory [6,7].

We, therefore, introduce a diluted version of the model, where the mean coordination number is fixed (see also Ref. [8]). In diluting, the run time grows as the size $N$ of the system, rather than proportionally to $N^2$. This is a fundamental issue because finite volume effects are strong in these models: previous studies were restricted to $N \leq 512$, while we can now thermalize systems up to $N = 16384$, thus keeping these effects under control.

We are interested in analyzing the differences among the predictions on the spin glass phase of the droplet theory [2], the trivial-non-trivial (TNT) scenario [3] and the replica symmetry breaking (RSB) theory [1]. Studying the thermodynamics, we focus on site and link overlaps, providing strong evidence that both fluctuate in the infinite volume limit. From the dynamic behavior we learn that the four-point correlation function goes to zero at large distances when extrapolated at infinite times. In this framework we are able to identify a characteristic length-scale $\ell(T; t)$ for such a decay.

The model investigated is a one dimensional chain of $N = L$ Ising spins ($\sigma_i = \pm 1$) whose Hamiltonian reads

\[ \mathcal{H} = -\sum_{i<j} J_{ij} \sigma_i \sigma_j . \] (1)

The quenched random couplings $J_{ij}$ are independent and identically distributed random variables taking a non zero value with a probability decaying with the distance between spins $\sigma_i$ and $\sigma_j$, $r_{ij} = |i-j| \mod (L/2)$, as

\[ P[J_{ij} \neq 0] \propto r_{ij}^{-\rho} . \] (2)

Non-zero couplings take value $\pm 1$ with equal probability. We choose an average coordination number $z = 6$ and periodic boundary conditions.

The universality class depends on the value of the exponent $\rho$, and it turns out to be equal to the one of the fully connected version of the model, where bonds are Gaussian distributed with zero mean and a variance depending on the distance as $\overline{J_{ij}^2} \propto \overline{r_{ij}^{-\rho}}$ [4,5,6,7]. The underline denotes the average over quenched disorder.

As $\rho$ varies this model is known to display different statistical mechanics behaviors. For the diluted case they are reported in Tab. 1 in the limit $\rho \rightarrow 0$ the model is a spin-glass on a Bethe lattice $[10,11]$, at variance with the fully connected version where this limit is ill-defined for

| $\rho$ | Behavior |
|-------|----------|
| $\rho < 1$ | Bethe lattice like |
| $1 < \rho \leq 4/3$ | 2nd order transition, mean-field (MF) |
| $4/3 < \rho < 2$ | 2nd order transition, infrared divergence (IRD) |
| $\rho = 2$ | Kosterlitz-Thouless or $T = 0$ phase transition |
| $\rho > 2$ | no phase transition |

TABLE I: From infinite range to short range behavior of the spin-glass model defined in Eqs. (1,2).
any \( \rho < 1 \). If the decay is gentle enough (\( \rho \leq 4/3 \)), the mean-field (MF) approximation is exact. As it becomes steeper (\( \rho > 4/3 \)), the MF approximation breaks down because of infrared divergences (IRD). For \( \rho = 2 \) one does not expect a finite temperature phase transition, though power law correlations might still be there \([23]\). This special case deserves further investigation.

The \( \rho = 4/3 \) case corresponds to the upper critical dimension of short-range spin-glasses in absence of an external magnetic field (\( D = 6 \)), whereas \( \rho = 2 \) plays the role of the lower critical dimension. An approximate relationship between \( \rho \) and the dimension \( D \) of short-range models can be identified as follows. In long-range models, the free theory in the replica space reads \([6]\)

\[
\mathcal{H} = \frac{N}{4} \int \frac{dk}{2\pi} (k^{\rho-1} + m_0^2) \sum_{a \neq b} |Q_{ab}(k)|^2,
\]

(3)

where \( a \) and \( b \) are replica indices and \( \tilde{Q}_{ab}(k) \) is the Fourier transform of the distance-dependent overlap matrix element \( Q_{ab}(r_{ij}) \). Comparing the critical scaling \((m_0 \propto |T - T_c| = 0)\) of Eq.\((3)\) with that of the free theory for short-range spin glass models in \( D \) dimensions \((\mathcal{H} \sim \int dk k^{2-D} T^2 Q^2)\) the following equation turns out to hold close to the upper critical dimension

\[
\rho = 1 + 2/D.
\]

(4)

We study the equilibrium properties of the diluted long-range model both in \( (\rho = 5/4) \) and out \( (\rho = 3/2, 5/3) \) of the MF regime for sizes up to \( L = 16384 \) \( 2^{14} \). We simulate two replicas \( a_i^{(1,2)} \) using the parallel tempering algorithm \([12]\) with 20 temperatures and we measure site and link overlaps, respectively defined as

\[
q = \frac{1}{N} \sum_{i=1}^{N} a_i^{(1)} a_i^{(2)}, \quad q_l = \frac{1}{zN} \sum_{i,j} J_{0i}^{(1)} a_i^{(1)} a_j^{(2)} a_j^{(1)} a_i^{(2)}.
\]

(5)

as well as the correlation length \([13]\):

\[
\xi_L = \frac{1}{2 \sin(\pi/L)} \left[ \frac{\chi_{\text{sg}}}{\chi(2\pi/L)} - 1 \right]^{\frac{1}{\nu-1}},
\]

(6)

where \( \chi_{\text{sg}} = L^{d_0} \) is the spin glass susceptibility \((\langle \cdots \rangle \text{ denotes the thermal average and } \langle \cdot \cdots \rangle \text{ denotes the average over the disorder)} \) and \( \chi(k) \) is the Fourier transform of the four-point correlation function \((\chi(0) = \chi_{\text{sg}})\). Averages over the disorder are taken on \( O(3 \times 10^5) \) samples in the smallest lattices and over \( O(2 \times 10^4) \) samples in the larger ones. In order to compute critical temperatures, critical exponents and the Finite Size Scaling (FSS) corrections we have used the quotient method \([14]\). We have computed the exponent \( \nu \) using the scaling of the temperature derivative of \( \xi_L/L \) and \( \eta \) from the scaling of \( \chi_{\text{sg}} \).

As a typical case, we show in Fig.\(1\) the temperature and size dependence of \( \chi_{\text{sg}} \) and \( \xi_L \). In the quotient method the estimates of the critical exponent still depend on the lattice size: the extrapolation to infinite volume provides both their asymptotic values and the \( \omega \) exponent of the leading FSS correction, \( O(L^{-\omega}) \).

The results are summarized in Table I. The \( \eta \) exponent coincides with the theoretical prediction \( \eta = 3 - \rho \) \((\eta \text{ is not renormalized in the IRD regime} \ [4, 6])\). Due to strong finite size effects this check failed in previous works \([7]\). The \( \nu \) exponent is consistent with the theoretical prediction, \( \nu = 1/(\rho - 1) \), in the MF case. In the IRD regime, thermodynamic fluctuations dominate and a renormalization is necessary: at present only one loop calculations are available \([4, 6]\), but the estimate of \( \nu \) is too rough to compare with numerical data.

In the spin glass phase \((T < T_c)\), site and link overlap distributions, \( P(q) \) and \( P_l(q) \), can be used as hallmarks to discriminate among different theories for finite dimensional spin glasses. Indeed, three cases are contemplated in the literature.

1. Droplet theory: one state; both distributions delta-shaped.
2. TNT scenario: many states \((q \text{ fluctuates}), \text{ but droplet-like excitations} \((q_l \text{ fluctuations vanish for large sizes}); \text{ non-trivial} \( P(q) \text{ and trivial} \( P_l(q) \).
3. RSB theory: many states with space-filling excitations; both distributions broad.

Distributions \( P(q) \) and \( P_l(q) \) for \( T \approx 0.4 T_c \) are plotted in Figs.\(2\) and \(3\) in a model case where MF is exact \((\rho = 5/4)\) and in an IRD case \((\rho = 3/2)\), respectively. In both cases, we see two peaks in the \( P_l(q) \) for large sizes (insets). Such a result would have been impossible to
The simpler relation between the two overlaps is equivalent measures of the distance among states \([15]\).

\[
A \to \infty \quad \rho = \frac{3}{2}, \quad L = 2^{12} \text{ and } T = 0.7. \quad \text{Inset: } A \text{ and } B \text{ vs. } L^{-0.72} \text{ obtained by measuring the Kullback-Leibler divergence between the two distributions for } L = 2^\alpha, \quad \kappa = 6, 8, 10, 12.
\]

(KL) divergence \([18]\) between the distribution of \(q_l\) and that of \(q_{\text{aux}}\). In Fig. 4 for \(L = 2^{12}\), we compare the optimal distributions, which should coincide if Eq. (7) held. Eq. (7) provides a strong evidence for a non-trivial link overlap distribution as long as the \(B\) parameter converges to a non zero value for large size, as one can verify in the inset of Fig. 3 where we show such an extrapolation for \(A\) and \(B\) plotted vs. an inverse power of \(L\) (and the optimal KL divergence go to zero, as expected).

As a complementary approach we look at the off-equilibrium four-point correlation function

\[
C_q(x,t) = \frac{1}{L} \sum_{i=1}^{L} \langle \sigma_{i}^{(1)}(t)\sigma_{i+x}^{(2)}(t)\sigma_{i+x}^{(1)}(t)\sigma_{i}^{(2)}(t) \rangle. \quad (8)
\]

For very large distances the fastest decay expected goes like \(x^{-\rho}\), because of long-range interactions. For intermediate distances, up to an effective crossover length \(\ell(t)\), we observe a slower decay \(x^{-\alpha}\), with \(0 < \alpha < \rho\), which is incompatible with the onset of a plateau at \(q_{\text{EA}}^{0}\) in the large times limit. This suggests to use the function

\[
A x^{-\alpha}\left[1 + \left(\frac{x}{\tau}\right)^{\delta(\rho-\alpha)}\right]^{-1/\delta}. \quad (9)
\]

to interpolate \(C_q(x,t)\) data at a fixed time \(t\). The fits are very good and their quality can be appreciated in Fig. 4 for an IRD \((\rho = 3/2)\) system of size \(L = 2^{12}\). The crossover length \(\ell\) plays a role similar to the correlation (or coherence) length in short range spin glasses \([13, 20, 21]\). We allow the fitting parameters \(A, \alpha\) and \(\delta\) to depend on time. Nonetheless we observe (see inset of Fig. 4) that they become stationary for large times: this is a strong evidence that Eq. (9) is a significant and robust behavior.

The growth of \(\ell(t)\) with time at different temperatures below \(T_c\) is plotted in Fig. 6. The length \(\ell(t)\) reaches very large values (> \(10^3\)) with respect to pre-

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**FIG. 2:** \(P(q)\) and \(P(q_l)\) at \(\rho = 5/4\) (MF). \(T = 0.9 \simeq 0.4T_c\), with \(L = 2^\alpha\) and \(\kappa = 7, \ldots, 13\). Inset: \(P(q)\) for \(L = 2^{13}\).

**FIG. 3:** \(P(q)\) and \(P(q_l)\) at \(\rho = 3/2\) (IRD). \(T = 0.7 \simeq 0.4T_c\), with \(L = 2^\alpha\) and \(\kappa = 7, \ldots, 13\). Inset: \(P(q)\) for \(L = 2^{12}\).

**FIG. 4:** Distributions of \(q_l\) (line) and \(q_{\text{aux}}\) (empty squares) for \(\rho = 3/2, \quad L = 2^{12} \text{ and } T = 0.7\). Inset: \(A\) and \(B\) vs. \(L^{-0.72}\) obtained by measuring the Kullback-Leibler divergence between the two distributions for \(L = 2^\alpha, \quad \kappa = 6, 8, 10, 12\).
and
FIG. 6: Crossover length \( \ell(x) \) for \( \rho = 3/2 \), \( L = 2^{17} \) at \( T = 0.7 \sim 0.4T_c \) and different times \( t = 2^8, 2^{13}, 2^{18} \). The curves are fits to Eq. (10). Inset: data collapse by plotting \( C(x; t) \) vs. \( x/\ell(t) \), for all times between \( 2^8 \) and \( 2^{18} \).

Previously studied spin glass models [19, 20, 21]. In this region \( \ell(t) \) is very well fitted by the phenomenological law \( a(T) \exp(b(T) \sqrt{T \log t}) \), with \( a \) and \( b \) not very dependent on the temperature; this seems reasonable since in activated processes the typical scaling variable is \( T \log(t) \).

We also tried to fit the previous \( \ell(t) \) data with a generalized droplet scaling [21] \( \tau(t) = A(T) e^{x c} \exp(\tau(T)e^{\psi}) \), where the power-law factor dominates near the transition \( (\lim_{T \to T_c} \tau(T) = 0) \) and the exponential term governs the low temperature regime. The critical exponents \( x_c \) and \( \psi \) are predicted not to depend on \( T \). The data shown in Fig. 6 are not compatible with this scaling law for any temperature-dependent \( A(T) \) and \( \Upsilon(T) \).

In conclusion, we have introduced a model which is easy to simulate and allow to probe the spin glass phase beyond mean-field. In this regime, from the analysis of thermodynamics, we observe that both the site and the link overlap parameter fluctuate for large sizes. In the large times limit the out-of-equilibrium four-point function \( C_q(x, t) \) tends to a well defined function that displays a power-law decay to zero and is incompatible with the onset of a plateau at any large \( x \). These observations are consistent with the clustering properties of the RSB theory. The bond diluteness of the model under investigation strongly reduces simulation times and allows to thermalize systems of sizes large enough to clearly discern the double peak structure of \( P(q) \). Both droplet and TNT proposal appear, in conclusion, not consistent with a FSS analysis over large sizes and with the behavior of the four-point correlation function and the related coherence length. This model is well suited to study the spin glass transition in a magnetic field and work is in progress in this direction.

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