Hawking radiation and the Stefan-Boltzmann law: The effective radius of the black-hole quantum atmosphere

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
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It has recently been suggested [S. B. Giddings, Phys. Lett. B 754, 39 (2016)] that the Hawking black-hole radiation spectrum originates from an effective quantum “atmosphere” which extends well outside the black-hole horizon. In particular, comparing the Hawking radiation power of a (3 + 1)-dimensional Schwarzschild black hole of horizon radius \( r_H \) with the familiar Stefan-Boltzmann radiation power of a (3 + 1)-dimensional flat space perfect blackbody emitter, Giddings concluded that the source of the Hawking semi-classical black-hole radiation is a quantum region outside the Schwarzschild black-hole horizon whose effective radius \( r_A \) is characterized by the relation \( \Delta r \equiv r_A - r_H \sim r_H \). It is of considerable physical interest to test the general validity of Giddings’s intriguing conclusion. To this end, we study the Hawking radiation of \((D+1)\)-dimensional Schwarzschild black holes. We find that the dimensionless radii \( r_A/r_H \) which characterize the black-hole quantum atmospheres, as determined from the Hawking black-hole radiation power and the \((D+1)\)-dimensional Stefan-Boltzmann radiation law, are a decreasing function of the number \( D+1 \) of spacetime dimensions. In particular, it is shown that radiating \((D+1)\)-dimensional Schwarzschild black holes are characterized by the relation \( (r_A/r_H)/r_H \ll 1 \) in the large \( D \gg 1 \) regime. Our results therefore suggest that, at least in some physical cases, the Hawking emission spectrum originates from quantum excitations very near the black-hole horizon.

I. INTRODUCTION

The Hawking evaporation process of black holes seems, at first glance, to be characterized by a non-unitary evolution of quantum fields in curved spacetimes [1]. In particular, according to Hawking’s original analysis, matter fields in a pure quantum state may collapse to form a black hole which eventually evaporates into a mixed thermal state [1]. Since a unitary temporal evolution of quantum states is one of the cornerstones of quantum mechanics, it is widely believed that the semi-classical Hawking radiation spectra of evaporating black holes should be modified in order to restore quantum unitarity [2, 3].

What is the characteristic lengthscale associated with these yet unknown quantum modifications? It is commonly believed that the semi-classical Hawking radiation spectra of evaporating black holes originate from quantum excitations in the near-horizon \( \Delta r = r - r_H \ll r_H \) region [1, 2]. It is therefore widely expected [3] that the required quantum modifications of the semi-classical Hawking radiation spectra would also be characterized by this relatively short lengthscale \( \Delta r \ll r_H \).

However, in a very interesting work, Giddings [2] has recently suggested that the radiation spectrum of an evaporating black hole originates from an effective quantum “atmosphere” which extends well outside the black-hole horizon. In particular, by comparing the numerically computed Havking radiation power \( P_{BH} \) of an evaporating (3 + 1)-dimensional Schwarzschild black hole of horizon radius \( r_H \) with the familiar Stefan-Boltzmann radiation power \( P_{BB} = \sigma A T^4 \) of a (3 + 1)-dimensional flat space perfect blackbody emitter of radius \( r_A \), Giddings concluded that the source of the Hawking radiation is a quantum region (the effective black-hole atmosphere) located outside the black-hole horizon and whose effective radius \( r_A \) is characterized by the relation

\[
\Delta r \equiv r_A - r_H \sim r_H .
\]

As emphasized in [2], the relation (1), which characterizes the (3+1)-dimensional Schwarzschild black hole, is consistent with the existence of an effective emitting atmosphere which extends well outside the black-hole horizon.

It is of physical interest to test the general validity of Giddings’s intriguing conclusion (1). In particular, one naturally wonders whether the relation (1), which characterizes the effective quantum atmosphere of the (3 + 1)-dimensional Schwarzschild black hole, is a generic feature of all evaporating black holes?

In order to address this important question, in this paper we shall study the Hawking radiation powers of \((D+1)\)-dimensional Schwarzschild black holes. In particular, following [2] we shall define the effective radii \( r_A(D) \) of the black-hole quantum atmospheres by equating the Hawking radiation powers of the \((D+1)\)-dimensional black holes with the corresponding Stefan-Boltzmann radiation powers of flat space perfect blackbody emitters. Below we shall explicitly show that the dimensionless radii \( r_A/r_H \), which characterize the effective black-hole quantum atmospheres, are a decreasing function of the number \( D+1 \) of spacetime dimensions. In particular, our results (to be presented below)
suggest that radiating \((D+1)\)-dimensional Schwarzschild black holes are characterized by the relation \((r_A - r_H)/r_H \ll 1\) [see Eq. (17) below] in the large \(D \gg 1\) regime.

II. THE HAWKING RADIATION SPECTRA OF \((D+1)\)-DIMENSIONAL SCHWARZSCHILD BLACK HOLES

We study the Hawking emission of massless scalar fields from \((D+1)\)-dimensional Schwarzschild black holes. The semi-classical Hawking radiation power for one bosonic degree of freedom is given by the integral relation \[\text{Eq. (17)}\]

\[
P_{\text{BH}} = \frac{\hbar}{2^{D-1} \pi^{D/2} (D/2)!} \sum_j \int_0^{\infty} \frac{\Gamma(\omega)}{e^{\hbar \omega/T_{\text{BH}}} - 1} d\omega,
\]

where \(j\) denotes the angular harmonic indices of the emitted field modes, and

\[
T_{\text{BH}} = \left(\frac{D-2}{4\pi} \hbar^2 r_H^D\right)^{1/(D-1)}
\]

is the semi-classical Bekenstein-Hawking temperature of the black hole. Here \(r_H\) is the horizon radius of the black hole \([10, 11]\). The dimensionless coefficients \(\Gamma = \Gamma(\omega; j, D)\), which are known as the greybody factors \([8]\) of the composed black-hole-field system, quantify the interaction of the emitted fields with the curved black-hole spacetime.

III. THE EFFECTIVE RADIUS OF THE BLACK-HOLE QUANTUM ATMOSPHERE

As pointed out by Giddings \([2]\), one may define the effective radius of the black-hole quantum atmosphere by equating the Hawking radiation power \(P_{\text{BH}}\) of the emitting black hole with the corresponding radiation power \(P_{\text{BB}}\) of a flat space perfect blackbody emitter. The scalar radiation power of a spherically-symmetric blackbody (BB) of temperature \(T\) and radius \(R\) in \(D+1\) spacetime dimensions is given by the generalized Stefan-Boltzmann radiation law \([12]\)

\[
P_{\text{BB}} = \sigma A_{D-1}(R) T^{D+1},
\]

where

\[
\sigma = \frac{D \Gamma(D/2) \zeta(D+1)}{2 \pi^{D/2} \hbar^D}
\]

is the generalized [(\(D+1\))-dimensional] Stefan-Boltzmann constant and

\[
A_{D-1}(R) = \frac{2\pi^{D/2}}{\Gamma(D/2)} R^{D-1}
\]

is the surface area of the \((D+1)\)-dimensional emitting body.

Following \([2]\), we shall define the effective radius \(r_A\) of the black-hole quantum atmosphere from the relation \([13, 14]\)

\[
P_{\text{BH}}(r_H, T_{\text{BH}}) = P_{\text{BB}}(r_A, T_{\text{BH}}).
\]

Taking cognizance of Eqs. \((3), (4), (5), (6),\) and \((7)\), one finds

\[
r_A = \left[\frac{\pi}{D \zeta(D+1)} \left(\frac{4\pi}{D-2}\right)^{D+1} \tilde{P}_{\text{BH}}\right]^{1/4} \times r_H
\]

for the effective radiating radius of the \((D+1)\)-dimensional Schwarzschild black hole, where

\[
\tilde{P}_{\text{BH}} \equiv P_{\text{BH}} \times \frac{r_H^2}{\hbar}
\]

is the scaled Hawking radiation power of the black hole.

Our main goal is to determine the functional dependence \(r_A = r_A(D)\) of the effective radius \([8]\) of the black-hole quantum atmosphere on the spacetime dimension \(D+1\) of the radiating black hole.
IV. THE RADIUS OF THE BLACK-HOLE QUANTUM ATMOSPHERE: NUMERICAL AND ANALYTICAL RESULTS

In the present section we shall study the functional dependence $\bar{r}_A = \bar{r}_A(D)$ of the dimensionless ratio

$$\bar{r}_A \equiv \frac{r_A - r_H}{r_H}$$

which characterizes the effective quantum atmospheres of the radiating $(D + 1)$-dimensional black holes.

A. The $(3 + 1)$-dimensional case

The Hawking radiation power of scalar quanta from a $(3 + 1)$-dimensional Schwarzschild black hole is given by [15]

$$P_{BH}(D = 3) = 2.976 \times 10^{-4} \frac{\hbar}{r_H^2}.$$  (11)

Substituting (11) into (8), one finds

$$r_A = 2.679 \times r_H$$  (12)

for the effective radius of the black-hole quantum atmosphere. This relation yields

$$\bar{r}_A = 1.679$$  (13)

for the dimensionless radius (10) which characterizes the effective atmosphere of the $(3+1)$-dimensional Schwarzschild black hole.

B. $(D + 1)$-dimensional black holes: Intermediate $D$-values

In the previous subsection we have used numerical data to demonstrate the fact that the dimensionless radii $\bar{r}_A(D)$, which characterize the effective quantum atmospheres of the radiating $(D + 1)$-dimensional Schwarzschild black holes, are a decreasing function of the number $D$ of spatial dimensions.

We shall now show that the dimensionless radii $\bar{r}_A(D)$ [see Eqs. (8) and (10)], which characterize the effective quantum atmospheres of the radiating $(D + 1)$-dimensional Schwarzschild black holes, are a decreasing function of the number $D + 1$ of spacetime dimensions.

The Hawking radiation powers of scalar quanta from $(D + 1)$-dimensional Schwarzschild black holes were computed numerically in [8, 16]. In Table I we present, for intermediate values of the number $D + 1$ of spacetime dimensions, the dimensionless radii $\bar{r}_A(D)$ [see Eqs. (8) and (10)] which characterize the effective quantum atmospheres of the $(D + 1)$-dimensional radiating black holes. The results presented in Table I reveal that the dimensionless radii $\bar{r}_A(D)$ of the black-hole quantum atmospheres are a decreasing function of the number $D + 1$ of spacetime dimensions.

| $(r_A - r_H)/r_H$ | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-------------------|----|----|----|----|----|----|----|
| 0.982             | 0.972 | 0.959 | 0.952 | 0.349 | 0.391 | 0.355 |

TABLE I: The dimensionless radii $\bar{r}_A(D) \equiv (r_A - r_H)/r_H$ which characterize the effective quantum atmospheres of the $(D + 1)$-dimensional radiating black holes. Here $r_H$ is the radius of the black-hole horizon and $r_A$ is the effective radius of the black-hole quantum atmosphere as defined from (8). One finds that the dimensionless radii $\bar{r}_A(D)$ of the black-hole quantum atmospheres are a decreasing function of the number $D + 1$ of spacetime dimensions.

C. $(D + 1)$-dimensional black holes: The large-$D$ regime

In the previous subsection we have used numerical data to demonstrate the fact that the dimensionless radii $\bar{r}_A(D)$, which characterize the quantum atmospheres of the radiating $(D + 1)$-dimensional Schwarzschild black holes, are a
The Hawking radiation spectrum of \((D+1)\)-dimensional Schwarzschild black holes is characterized by the frequency distribution \(\omega^D/(e^{\hbar\omega/T_{\text{BH}}}-1)\) [see Eq. (1)]. One finds that the frequency dependent function has a peak at

\[
\frac{\hbar\omega_{\text{peak}}}{T_{\text{BH}}} = D + W(-De^{-D}),
\]

where \(W(x)\) is the Lambert function and \(T_{\text{BH}}\) is the black-hole temperature as given by (3). Substituting (3) into (14), one finds the strong inequality (17)

\[
\frac{\lambda_{\text{peak}}}{r_{\text{H}}} = \frac{8\pi^2}{D^2}[1 + O(D^{-1})] \ll 1
\]

which characterizes the Hawking radiation spectra of the higher-dimensional Schwarzschild black holes in the large \(D \geq 1\) regime. The asymptotic large-D relation (15) reflects the fact that the characteristic wavelengths emitted by the higher-dimensional Schwarzschild black holes in the large \(D \geq 1\) regime are very short on the lengthscale \(r_{\text{H}}\) set by the horizon radii of the radiating black holes.

As shown in (17), the strong inequality (15), which characterizes the radiating \((D+1)\)-dimensional Schwarzschild black holes in the large \(D \geq 1\) regime, implies that the corresponding Hawking emission spectra of these higher-dimensional black holes are described extremely well by the geometric-optics (short wavelengths) approximation. In particular, in this large-D regime (18), the effective radiating radius \(r_{A}\) of the black hole is determined by the high-energy (short wavelengths) absorptive radius of the hole [8, 16, 17], which is given by the \((D+1)\)-dimensional geometric-optics relation [8, 16, 17]

\[
\frac{r_{A}}{r_{\text{H}}} = \left(\frac{D}{2}\right)^{\frac{D}{D-2}} \sqrt{\frac{D}{D-2}}.
\]

From Eq. (16) one finds

\[
\frac{r_{A}}{r_{\text{H}}} \to 1 + O\left(\frac{\ln D}{D}\right) \quad \text{for} \quad D \gg 1
\]

in the large-D regime. The relation (17) reveals the fact that, in the large \(D \gg 1\) regime, the effective radius \(r_{A}\) of the black-hole quantum atmosphere approaches the horizon radius \(r_{\text{H}}\) of the higher-dimensional black hole.

V. SUMMARY AND DISCUSSION

In a very interesting paper, Giddings [2] has recently suggested that the Hawking radiation spectrum which characterizes an evaporating semi-classical black hole originates from an effective quantum “atmosphere” which extends well outside the black-hole horizon. In particular, Giddings has provided evidence that, for a \((3+1)\)-dimensional Schwarzschild black hole, the source of the Hawking radiation is a quantum region outside the black-hole horizon whose effective radius \(r_{A}\) is characterized by the relation \(\Delta r \equiv r_{A} - r_{\text{H}} \sim r_{\text{H}}\) [see Eq. (1)]. It is certainly of physical interest to test the general validity of Giddings’s intriguing conclusion (1). In particular, one naturally wonders whether the relation \(r_{A} - r_{\text{H}} \sim r_{\text{H}}\) suggested by Giddings for the effective quantum atmosphere of the \((3+1)\)-dimensional Schwarzschild black hole is a generic characteristic of all radiating black holes?

In order to address this important question, we have studied in this paper the Hawking radiation spectra of \((D+1)\)-dimensional Schwarzschild black holes. Interestingly, it was found that the dimensionless radii \(r_{A}/r_{\text{H}}\), which characterize the effective black-hole quantum atmospheres, are a decreasing function of the number \(D+1\) of spacetime dimensions. In particular, we have shown that the effective quantum atmospheres of \((D+1)\)-dimensional Schwarzschild black holes are characterized by the relation [see Eqs. (10) and (17)]

\[
\tilde{r}_{A}(D) \to 0 \quad \text{for} \quad D \gg 1
\]

in the large-D regime.

It is interesting to note that the large-D eikonal (geometric-optics) relation (16) provides a remarkably accurate description of the effective black-hole quantum atmospheres for all \(D\)-values. In Table II we present the dimensionless ratio \(r_{A}\text{numerical}/r_{A}\text{analytical}\) between the numerically computed radii \(r_{A}\text{numerical}(D)\) of the black-hole quantum atmospheres [see Eq. (8)] and the analytically predicted radii \(r_{A}\text{analytical}(D)\) of the large-D eikonal relation (16). One finds a
TABLE II: The dimensionless ratio $r_A^{\text{numerical}} / r_A^{\text{analytical}}$ between the numerically computed radii $r_A^{\text{numerical}}(D)$ of the black-hole quantum atmospheres [see Eq. (8)] and the analytically predicted radii $r_A^{\text{analytical}}(D)$ of the large-D eikonal (geometric-optics) relation (16). Remarkably, one finds a good agreement between the accurate (numerically computed) radii (8) of the black-hole quantum atmospheres and the analytically predicted radii of the eikonal relation (16) for all values of the number $D + 1$ of spacetime dimensions.

remarkably good agreement between the accurate (numerically computed) radii (8) and the analytically predicted radii of the eikonal relation (16) for all values of the number $D + 1$ of spacetime dimensions.

The results presented in this paper reveal that, at least in some physical cases, the effective radii of the black-hole quantum atmospheres are characterized by the relation $\bar{r}_A \ll 1$ [see Eq. (18)]. This fact suggests that in these cases, the Hawking radiation originates from quantum excitations very near the black-hole horizon.

| $D + 1$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------|---|---|---|---|---|---|----|----|
| $r_A^{\text{numerical}} / r_A^{\text{analytical}}$ | 1.031 | 0.991 | 0.986 | 0.986 | 0.988 | 0.990 | 0.990 | 0.992 |
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