Semi Bornological Groups

Anwar N. Imran\textsuperscript{1} and Sh. K. Said Husain\textsuperscript{2}

\textsuperscript{1}Deparment of Mathematics, Faculty of Sciences, University Diyala, Iraq
\textsuperscript{2}Department of Mathematics, Faculty of Science and Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, Malaysia

anwarnooraldeen@yahoo.com

Abstract. The motivation for this work is to find the sufficient conditions to bornologize any group and solve the problem of boundedness for any group, by constructing new structure semi bornological groups. In particular, we show that every left (right) translations in semi bornological groups are bornological isomorphisms and therefore the semi bornological groups structures are homogeneous, and this property from the semi bornological group is not shared with bornological semigroups. Further, the boundedness of a semi bornological group can be deduced from its boundedness at the identity. So, the problem of boundedness for any group can be solved.

1. Introduction

A bornological structure is very useful to solve the problem of boundedness of set and function, that means to determine the limits of the area. The formal study of fundamental construction of bornological groups (BG) was begun in 2012 by D. P. Pombo [7], the idea of the bornological group came from starting to solve the equation of boundedness of the group instead of a set of elements. (BG) is set with two structure group and bornology such that the product map and inverse map are bounded. Note that every group can be turned into a bornological group by providing it with the discrete bornology. However, the problem of existence of non-discrete bornologies on infinite groups which would make them into bornological groups. This problem was solved by [4] in 2017, by introducing new structure bornological semigroups (BSG). (BSG) can be easily done by taken semigroup instead of the group and require that just the product map is bounded, the condition of a bornological semigroup is weaker than the condition of a bornological group, so the problem of boundedness for this kind of groups, which cannot bornologize because the inverse map is not bounded was solved. In this paper, we deal with the problem of boundedness for kind of groups, which cannot bornologize because the product map is not bounded. So, we construct semi bornological group which it is consist of group and bornology with the condition that left (right) translation are bounded and the motivation to introduce this kind of structure is that left (right) translation to be a bornological isomorphism. So, our new structure is homogenous. Additionally, one of the main features of homogenous is that they behave in the same way at any point. It follows that if we know how the bornology of a semi bornological group behaves at the identity, we know this bornology where ever, these observations suggest a certain approach to bornologize any group. Furthermore, that left (right) translation is bornological isomorphism just in semi bornological group. So, this structure is totally different from bornological semigroup, for more detail in bornological group we refer to ([1], [2], [5],[6]).
2. Semi Bornological Groups

The problem is to understand what the sufficient condition should be imposed on bornology to bornologize any group and solve the problem of boundedness for any group. In the present section we addressed the above questions. However, we need to construct new structure semi bornological group:

**Definition 2.1.** A triple \((G, *, \beta)\) is called semi bornological group (SBG) if:

(i) \((G, *)\) is a group and \((G, \beta)\) is a bornological set;

(ii) The group operation \(* : G \times G \to G\) that maps \((g_1, g_2)\) to \(g_1 * g_2\), is bounded in each variable separately.

That means, the map \(* : G \times G \to G\) is bounded in the variable \(g_1\) when the map \(r_g : G \to G\) defined by

\[ r_g : G \to G \]

\[ g \to g_1 * g \]

is bounded for all \(g\) in \(G\). Similarly, \(*\) is bounded in \(g_2\) when the map

\[ l_g : G \to G \]

\[ g_2 \to g * g_2 \]

is bounded for all \(g\) in \(G\).

**Proposition 2.1.** If \((G, \beta)\) is a semi bornological group. Then, every left translation \(l_a\) and right translation \(r_a\) in a semi bornological group is a bornological isomorphism from \((G, \beta)\) to \((G, \beta)\) for all \(a \in G\).

**Proof.**

As we know, every left translation \(l_a\) is a bijection and every element \(a \in G\) determines a bounded translation \(l_a\) of a bornological group \(G\) into itself there is the left translation map

\[ l_a : G \to G \]

\[ l_a(h) = ah. \]

Clearly \(l_a h = l_a \circ l_h\) and \(l_a^{-1} = (l_a)^{-1}\). Since

\[ (l_a \circ l_a^{-1})(x) = l_a(l_a^{-1}(x)) = l_{aa^{-1}}(x) = le(x) = x = id_G, \]

in which it is the identity mapping. It follows that the inverse of \(l_a\) is also bounded, that is, \(l_a\) is a bornological isomorphism of a semi bornological group \(G\) onto itself. A similar argument applies in the case of right translation of a bornological group \(G\).

**Definition 2.2.** A bornological set \(X\) is said to be a bornological homogeneous if for each \(x, y \in X\), there exists a bornological isomorphism \(f\) of the bornological set \(X\) onto itself such that \(f(x) = y\).
An immediate and useful result we get from proposition 2.1, is as follow, where we obtain immediately, that semi bornological group structure is homogeneous space.

**Theorem 2.1.** Every semi bornological group is a bornological homogeneous.

**Proof.** Take any elements \( g_1 \) and \( g_2 \) in \( G \), and put \( g = g_2g_1^{-1} \). Then

\[
l_g(g_1) = gg_1 = g_2g_1^{-1}g_1 = g_2.
\]

Since, by proposition 2.1, every left translation in semi bornological group is a bornological isomorphism, thus \( l_g \) is a bornological isomorphism which implies that the semi bornological group \( G \) is homogenous.

Given a group \( G \), it follows from theorem 2.1 that to make the group \( G \) into semi bornological group, we can only use homogenous bornology.

On the contrary, a bornological semigroup, even if it has identity, need not be homogenous, then the example below shows that Theorem 2.2 cannot be extended to bornological semigroup.

**Example 2.1.** Take the bounded unit interval \( L = [0, 1] \), and put \( ab = \max\{a, b\} \) for all \( a, b \in L \).

Clearly, \( L \) with the canonical bornology and this product operation, is a bornological semigroup (with 0 in the role of identity). However, \( L \) is not a homogeneous space since no isomorphism of \( L \) takes, 0 to 1/2.

**Remark 2.1.** If \((G, \beta)\) is a semi bornological group, then \( B \in \beta \) if and only if \( B^{-1} \in \beta \).

As we mentioned earlier a semi bornological group is a set with two structures group and bornology such that the group operation \( \ast : G \ast G \rightarrow G \) that maps \((g_1, g_2)\) to \( g_1 \ast g_2 \), is bounded in each variable separately.

The next definition gives an equivalent concept to semi bornological group in terms of bounded set.

**Definition 2.3.** A group \((G, \ast)\) with the bornology \( \beta \) is said to be a semi bornological group if and only if for each \( g_1, g_2 \in G \) and each bounded sets \( B_1, B_2 \) containing \( g_1, g_2 \) respectively, there is a bounded set \( B \) contains \( g_1 \ast g_2^{-1} \) in \( G \) such that \( B_1 \ast B_2^{-1} \subset B \).

We describe here some simple facts on families of bounded sets in semi bornological groups.

**Proposition 2.2.** If \( G \) is a semi bornological group, then for any bounded subset \( L \) of \( G \) and any subset \( H \) of \( G \), the sets \( LH \) and \( HL \) are bounded.

**Proof.** Every left translation of semi bornological group \( G \) into itself is a bornological isomorphism, by proposition 2.1. Since \( HL = \bigcup_{h \in H} l_h(L) \), the conclusion follows.

A similar argument applies in the case of every right translation of semi bornological group. So, if \( G \) is a semi bornological group then, for any bounded subset \( L \) of \( G \) and any subset \( H \) of \( G \), the set \( LH \) is bounded.

We obtain immediately the following corollary.

**Corollary 2.1.** Suppose that a subgroup \( H \) of a semi bornological group \( G \) contained in a bounded subset of \( G \). Then \( H \) is bounded in \( G \).
Proof. Let $B$ be a bounded subset of $G$ with $H \nabla B$. A semi bornological group consists of a group $G$ and a bornology $\beta$ on $G$ since $\beta$ is stable under hereditary, i.e. if $B$ is a bounded subset of bornological group $G$ and $H \nabla B$, then $H \nabla G$. Therefore, the set $H \nabla B$ is bounded in bornological group $G$.

**Theorem 2.2.** Let $f : G \rightarrow H$ be a homomorphism, where $(G, \ast, \beta_G)$ and $(H, \ast, \beta_H)$ are semi bornological groups. If $f$ is a bounded map at the identity element $e_G$ of $G$, so $f$ is a bounded map on $G$.

**Proof.** Let $g \in H$ be an arbitrary element, and let $B$ be a bounded set containing $g$ in $G$. Since the left translations in $G$ are bounded maps, there is a bounded set $B_1$ containing identity element $e_G$ of $G$ such that

$$l_g(B_1) = g * B_1 \nabla B.$$  

Since $f$ is a bounded map at $e_G$ of $G$, it follows the existence of a bounded set $B_2 \nabla H$ containing $e_H$ such that $B_2 \nabla f(B_1)$. But

$$l_g1 = f(g) : G \rightarrow G$$

is a bounded mapping, so that the set $f(g) * B_2$ is a bounded set containing $g_1 = f(g)$, for which we have

$$f(g) * B_2 \nabla f(g) * f(B_1) \nabla f(g * B_1) \nabla f(B).$$

Hence $f$ is a bounded map at $g$ of $G$, and since $g$ is an arbitrary element in $G$, $f$ is bounded map on $G$. 

Next we will discuss embedding of a group.

**Definition 2.4.** Given that for all $\alpha \in I$ we have $G_\alpha = G$, we denote the product

$$\prod_{\alpha \in I} G_\alpha = G \times G \times G \ldots \times G$$

as $G^I$. We can view $G$ as the set of all maps from $I$ to $G$. Consider the element $g \in G$ as $g = (g_{\lambda_1}, g_{\lambda_2}, g_{\lambda_3}, \ldots, g_{\lambda_i})$ we can associate this element with the function $f_g : I \rightarrow G^I$ that maps $\lambda_1$ to $g_{\lambda_i}$ for all $\alpha \in I$.

**Proposition 2.3.** Let $G$ be a semi bornological group. There exists a bijection from $G$ onto a subset of $G^I$. This map is embedding of $G$.

**Proof.** Define map from $G$ to $G^G$ by $\psi_r : g \mapsto r_g$ and $\psi_l : g \mapsto l_g$. In this proof we will use $\psi_r$ but the left translation also work. In the beginning, we have to show that $\psi_r$ is injective, so assume that $\psi_r = \psi_l$ for some $g_1 / g_2 = G$. It follows that $r_{g_1} = r_{g_2}$, so $gg_1 = gg_2$ for all $g \in G$, multiplying by $g^{-1}$ on the left gives us $g_1 = g_2$ as required. Then $\psi_r : G \rightarrow \psi_r(G)$ is a bijection. We call $\psi_r$ and $\psi_l$ the right and left canonical embedding of $G$ into $G^G$.

We say that $G$ is a quasi-bornological group if it is a semi bornological group and the inverse map is bounded.

3. Conclusion

We have constructed new structure semi bornological group to restrict the condition of boundedness for any group and give theoretical solution for many existing problem to boundedness any groups. Furthermore, investigated their properties, and established their differences from bornological semigroups. For this purpose we employed left (right) translation to be bounded instead of product map to be bounded in the case of bornological semigroups.
4. Acknowledgement
The authors are very grateful to Prof. Dr. Isamiddin S. Rakhimov as Fellow Researcher from INSPEM, Universiti Putra Malaysia, person who provided help during our research and preparation of the manuscript. This research was funded by Grant 01-01-18-2032FR Kementerian Pendidikan Malaysia.

References
[1] Bambozzi, F 2013 On a Generalization of Affinoid Varieties Ph.D Thesis (Universit degli Studi di Padova, Dipartimento di Matematica)
[2] Bambozzi, F 2015 Closed graph theorems for bornological spaces arXiv preprint arXiv:1508.01563.
[3] Hogbe-Nlend, H 1977 Bornologies and Functional Analysis Notas de Mathematica 62 North - Holland
[4] Imran, Anwar. N., Rakhimov, I. S. and Said Husain, Sh. K. 2017 Journal of Mathematical Sciences 102 (8) pp 1647-1661
[5] Imran, Anwar. N., Rakhimov, I. S. 2017 Applied Mathematics and Information Sciences 11 (4) pp 1235-1240
[6] Pombo Jr. and D. P. 2012 Indian J. Math 54 pp 225-258