The Asymptotic Normalized Linear Complexity of Multisequences

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Abstract

We show that the asymptotic linear complexity of a multisequence \( a \in (\mathbb{F}_q^M)^\infty \) that is \( I := \liminf_{n \to \infty} L_{a(n)} \) and \( S := \limsup_{n \to \infty} L_{a(n)} \) satisfies the inequalities

\[
\frac{M}{M + 1} \leq S \leq 1 \quad \text{and} \quad M(1 - S) \leq I \leq 1 - \frac{S}{M},
\]

if all \( M \) sequences have nonzero discrepancy infinitely often, and all pairs \((I, S)\) satisfying these conditions are met by \( 2^{\aleph_0} \) multisequences \( a \). This answers an Open Problem by Dai, Imamura, and Yang.

Keywords: Linear complexity, multisequence, Battery Discharge Model, isometry.
1 Introduction

Given $M$ formal power series

$$G_m = \sum_{t=1}^{\infty} a_{m,t} x^{-t} \in \mathbb{F}_q[[x^{-1}]], 1 \leq m \leq M$$

with $a = (a_{m,t}) \in (\mathbb{F}_q^M)^\infty$, the linear complexity $L_a(n)$ is defined as the smallest degree $\deg(v)$ of a denominator polynomial $v \in \mathbb{F}_q[x]$, which approximates all $G_m$’s up to $x^{-n}$:

$$\exists u_1, \ldots, u_M \in \mathbb{F}_q[x]: G_m = \frac{u_m(x)}{v(x)} + o(x^{-n}).$$

Typically $L_a(n) \approx n \cdot \frac{M}{M+1}$, and we define the linear complexity deviation

$$d := d_a(n) := L_a(n) - \left\lceil \frac{M}{M+1} \cdot n \right\rceil.$$

In Section 2, we recall Dai and Feng’s multi–Strict Continued Fraction Algorithm (mSCFA) and our Battery–Discharge–Model (BDM) [9][2], which keeps track of the linear complexity deviation of all multisequences in $(\mathbb{F}_q^M)^\infty$ simultaneously.

The normalized linear complexity is defined as $\overline{L}_a(n) = L_a(n)/n$ with $0 \leq \overline{L}_a(n) \leq 1$, typically $\overline{L}_a(n) \approx M/(M+1)$, similarly the normalized deviation $\overline{d}_a(n) = d_a(n)/n$ is typically $\overline{d} \approx 0$, and in Section 3, we show bounds for the possible values for $I := \liminf \frac{L_a(n)}{n}$ and $S := \limsup \frac{L_a(n)}{n}$.

In Section 4 we give an algorithm to construct an $M$–multisequence (over any finite field) with any allowed parameters $I, S$.

The final Section 5 considers the cardinality, Hausdorff dimension, and measure of the set of multisequences matching a given pair $(I, S)$. Niederreiter and Wang [6, 7, 10] recently have shown that with measure one we have $I = S = M/(M + 1)$. We shall see however that all the other points $(I, S)$ matching the conditions are also met by $2^{30} = |(\mathbb{F}_q^M)^\infty|$ sequences $a$, leading to a set of positive Hausdorff dimension at least for $S < 1$.

This answers an Open Problem posed by Dai, Imamura, and Yang [5], and extends the work in [4] for $M = 1$ to arbitrary parallelism $M$. 
2 Diophantine Approximation of Multisequences

We start with the multi–Strict Continued Fraction Algorithm (mSCFA) by Dai and Feng [3]. The mSCFA calculates a best simultaneous approximation to a set of $M$ formal power series $G_m = \sum_{t=1}^{\infty} a_{m,t} x^{-t} \in \mathbb{F}_q[[x^{-1}]]$, $1 \leq m \leq M$. It computes a sequence $(u_{m,n}(x)/v_{m,n}(x))$ of approximations in $\mathbb{F}_q(x)$ in the order $(m, n) = (M, 0), (1, 1), (2, 1), \ldots, (M, 1), (1, 2), (2, 2), \ldots$ with

$$G_k = \sum_{t \in \mathbb{N}} a_{k,t} \cdot x^{-t} = \frac{u_{k,m,n}(x)}{v_{k,m,n}(x)} + o(x^{-n}), \forall 1 \leq m, k \leq M, \forall n \in \mathbb{N}_0.$$ 

We will denote the degree of $v_{m,n}(x)$ by $\deg(m, n) \in \mathbb{N}_0$ instead of $d$ as in [3] (we will use $d$ differently). Then the multisequence $a = (a_{m,n}) \in (\mathbb{F}_q^M)^{\infty}$ has the linear complexity profile $(\deg(M, n))_{n \in \mathbb{N}_0} = (L_a(n))_{n \in \mathbb{N}_0}$.

The mSCFA also uses $M$ auxiliary degrees $w_1, \ldots, w_M \in \mathbb{N}_0$. The update of these values depends on a so–called “discrepancy” $\delta(m, n) \in \mathbb{F}_q$. $\delta(m, n)$ is zero if the current approximation predicts correctly the value $a_{m,n}$, and $\delta(m, n)$ is nonzero otherwise. Furthermore, the polynomials $u_m(x)$ and $v(x)$ are updated, crucial for the mSCFA, but of no importance for our concern.

Algorithm 1. mSCFA [3]

```
deg := 0; w_m := 0, 1 \leq m \leq M
FOR n := 1, 2, \ldots
  FOR m := 1, \ldots, M
    compute $\delta(m, n)$ //discrepancy
      IF $\delta(m, n) = 0$: \{ // do nothing, [3] Thm. 2, Case 2a
        \}
      ELSE $\delta(m, n) \neq 0$ AND $n - \deg -w_m \leq 0$: \{ // [3] Thm. 2, Case 2c
        \}
      ELSE $\delta(m, n) \neq 0$ AND $n - \deg -w_m > 0$: // [3] Thm. 2, Case 2b
        deg_copy := deg
        deg := n - w_m
        w_m := n - deg_copy
      ENDIF
  ENDFOR
ENDFOR
```

The linear complexity grows like $\deg(M, n) \approx \left\lceil n \cdot \frac{M}{M+1} \right\rceil$ (exactly, if always $\delta(m, n) \neq 0$), and the $w_m \approx \left\lfloor \frac{n}{M+1} \right\rfloor$. We therefore extract the deviation
from this average behaviour as
\[ d := \deg - \left\lceil n \cdot \frac{M}{M + 1} \right\rceil, \tag{1} \]
the linear complexity deviation or degree deviation, which we call the “drain” value, and
\[ b_m := \left\lfloor n \cdot \frac{1}{M + 1} \right\rfloor - w_m, \quad 1 \leq m \leq M, \tag{2} \]
the deviation of the auxiliary degrees, which we call the “battery charges”.

We establish the behaviour of \( d \) and \( b_m \) in two steps. First we treat the change of \( d, b_m \) when increasing \( n \) to \( n + 1 \) (keeping \( \deg, w_m \) fixed for the moment):
\[
\deg - \left\lceil (n+1) \cdot \frac{M}{M + 1} \right\rceil = \begin{cases} 
\deg - \left\lceil n \cdot \frac{M}{M + 1} \right\rceil - 1, & n \not\equiv M \mod (M + 1), \\
\deg - \left\lceil n \cdot \frac{M}{M + 1} \right\rceil, & n \equiv M \mod (M + 1), 
\end{cases} \tag{3}
\]
and
\[
\left\lfloor (n+1) \cdot \frac{1}{M + 1} \right\rfloor - w_m = \begin{cases} 
\left\lfloor n \cdot \frac{1}{M + 1} \right\rfloor - w_m, & n \not\equiv M \mod (M + 1), \\
\left\lfloor n \cdot \frac{1}{M + 1} \right\rfloor - w_m + 1, & n \equiv M \mod (M + 1). 
\end{cases} \tag{4}
\]

Hence, by (3) we have to decrease \( d \) in all steps, except when \( n \equiv M \to n \equiv 0 \mod (M + 1) \), and only here we increase all \( M \) battery values \( b_m \), by (4).

With \( d(M,0) = b_m(M,0) := 0, \forall m \), initially, we obtain the invariant
\[ d(M,n) + \left( \sum_{m=1}^{M} b_m(M,n) \right) + n \mod (M + 1) = 0. \tag{5} \]

Now, for \( n \) fixed, the \( M \) steps of the inner loop of the mSCFA change \( w_m \) and \( \deg \) only in the case of \( \delta(m,n) \not= 0 \) and \( n - \deg - w_m > 0 \) that is
\[ n - \deg - w_m > 0 \iff n - \left( d + \left\lceil n \cdot \frac{M}{M + 1} \right\rceil \right) - \left( \left\lfloor n \cdot \frac{1}{M + 1} \right\rfloor - b_m \right) > 0 \]
\[ \iff b_m > d. \] In this case \( \delta \not= 0 \) and \( b_m > d \), the new values are (see mSCFA)
\[ \deg^+ = n - w_m \quad \text{and} \quad w_m^+ = n - \deg \tag{6} \]
and thus in terms of the BDM variables:
\[ d^+ \overset{(1;6)}{=} (n - w_m) - \left\lfloor \frac{n \cdot M}{M + 1} \right\rfloor \overset{(2)}{=} \left\lfloor \frac{n}{M + 1} \right\rfloor + b_m - \left\lfloor \frac{n}{M + 1} \right\rfloor = b_m \]

and
\[ b_m^+ \overset{(2;6)}{=} \left\lfloor \frac{n}{M + 1} \right\rfloor - n + \deg \overset{(1)}{=} - \left\lfloor \frac{n \cdot M}{M + 1} \right\rfloor + \left( d + \left\lfloor \frac{n \cdot M}{M + 1} \right\rfloor \right) = d, \]

an interchange of the values \( d \) and \( b_m \). We say in this case that “battery \( b_m \) discharges the excess charge into the drain”. A discharge does not affect the invariant (5), which is thus valid for every timestep \((m, n)\).

In the limit, as \( n \to \infty \), we want to obtain \( d \) as a probability distribution over all multisequences in \((\mathbb{F}_q^M)^\infty\). Since we do not actually compute with a given multisequence \( a \in (\mathbb{F}_q^M)^\infty \), we have to model the distinction between \( \delta = 0 \) and \( \delta \neq 0 \) probabilistically.

**Proposition 2.** In any given position \((m, n), 1 \leq m \leq M, n \in \mathbb{N}\) of the formal power series, exactly one choice for the next symbol \( a_{m,n} \) will yield a discrepancy \( \delta(m, n) = 0 \), all other \( q - 1 \) symbols from \( \mathbb{F}_q \) result in some \( \delta(m, n) \neq 0 \).

**Proof.** The current approximation \( u_{m,n}(x)/v_{m,n}(x) \) determines exactly one approximating coefficient sequence for the \( m \)-th formal power series \( G_m \). The (only) corresponding symbol leads to \( \delta = 0 \), all other symbols lead to \( \delta \neq 0 \). \( \square \)

In fact, for every position \((m, n)\), each discrepancy value \( \delta(m, n) \in \mathbb{F}_q \) occurs exactly once for some \( a_{m,n} \in \mathbb{F}_q \), in other words (see [1, 8] for \( M = 1 \)):

**Fact** The mSCFA induces an isometry on \((\mathbb{F}_q^M)^\infty\).

Hence, we can model \( \delta = 0 \) as occurring with probability \( 1/q \), and \( \delta \neq 0 \) as having probability \( (q - 1)/q \). In terms of \( d, b_m \), we have the following equivalent probabilistic formulation of the mSCFA:
Algorithm 3. Battery-Discharge-Model (probabilistic mSCFA)

\[ d := 0; b_m := 0, 1 \leq m \leq M \]

FOR \( n := 1, 2, \ldots \)
  IF \( n \equiv M \mod M + 1 \):
    \[ b_m := b_m + 1, 1 \leq m \leq M \]
  ELSE
    \[ d := d - 1 \]
  ENDIF

FOR \( m := 1, \ldots, M \)
  IF \( b_m > d \):
    WITH prob. \((q - 1)/q\):
      \[ \text{swap}(b_m, d) \] // Discharge of battery \( b_m \)
    WITH prob. \(1/q\):
      \{ \} // Do nothing, since \( \delta = 0 \)
  ELSE
    \{ \} // Do nothing, since \( b_m \leq d \)
  ENDIF
ENDFOR
ENDFOR

3 Normalized Linear Complexity I: Bounds for \( \lim \inf \) and \( \lim \sup \)

We need the following facts about the mSCFA and BDM:
1. \( 0 \leq L_a(n)/n \leq 1 \).
2. The invariant (5).
3. Proposition 2 that is, after each prefix we can enforce both \( \delta(m, n) = 0 \) and \( \delta(m, n) \neq 0 \) by choosing an appropriate \( a_{m,n} \), for any finite field \( \mathbb{F}_q \).

Definition. Asymptotic Normalized Bounds

We denote the asymptotic lower bound for the normalized linear complexity by

\[
I := \liminf_{n \to \infty} \frac{L_a(n)}{n} = \liminf_{n \to \infty} \frac{\deg(M, n)}{n}
\]
and for the normalized drain or linear complexity deviation by

$$\tilde{I} := \liminf_{n \to \infty} \frac{d(M, n)}{n} = I - \frac{M}{M + 1},$$

similarly the asymptotic upper bounds are

$$S := \limsup_{n \to \infty} \frac{L_a(n)}{n} \quad \text{and} \quad \tilde{S} := \limsup_{n \to \infty} \frac{d(M, n)}{n} = S - \frac{M}{M + 1}.$$ 

**Definition.** *Active Series*

We call a formal power series $G_m$ active, if $\delta(m, n) \neq 0$ infinitely often and denote the number of active series by $K$ ($0 \leq K \leq M$).

**Proposition 4.** $K$ is the number of $F_q(x)$–independent irrational series that is

$$K = \dim_{F_q(x)} < 1, G_1, \ldots, G_M > -1.$$ 

**Proof.** If the discrepancy sequence of a series is ultimately zero, this series will be either rational or dependent (as $F_q(x)$–linear combination) on the active series. Thus $K$ is the number of $F_q(x)$–independent irrational series, where including 1 as generating element of the vector space, and decrementing the dimension removes any effect of ultimately periodic (rational) series.

Since nonactive series do not change the linear complexity profile, we shall in fact assume for the purpose of deriving bounds that all $M$ series are active. After proving a technical lemma, we will obtain bounds for $I$, $S$, $\tilde{I}$, and $\tilde{S}$, which will turn out to be tight in the next section.

**Lemma 5.** (i) If $G_m$ is active, and if there is an $n_0$ with $\tilde{I} \leq d(m, n)/n \leq \tilde{S}$ for all $n \geq n_0$, then there is also an $n_1$ with $\tilde{I} \leq b_m(m, n)/n \leq \tilde{S}$ for all $n \geq n_1$.

(ii) Asymptotically, the normalized drain and batteries sum up to zero, $\lim_{n \to \infty} d(m, n)/n + \sum_{k=1}^M b_k(m, n)/n = 0, \forall 1 \leq m \leq M$.

**Proof.** (i) Let $n_1$ be the first time after $n_0$ where $b_m$ discharges (since $G_m$ is active, such an $n_1$ exists). Then, we have $\tilde{I} \leq d(m, n_1)/n_1 \leq \tilde{S}$ by assumption, and also $\tilde{I} \leq d^+(m, n_1)/n_1 = b_m(m, n_1)/n_1 \leq \tilde{S}$ after the discharge. The same holds for every $n^* > n_1$ where $b_m$ discharges and as $G_m$ is active, infinitely many such $n^*$ exist. Also, between $n_1$ and $n^*$, $b_m/n$ has to stay between $\tilde{I}$ and $\tilde{S}$ since otherwise it would make $d/n$ leave this interval at discharge. Hence, not only $d/n$, but all $b_m/n$ for active batteries $b_m$ are eventually bounded by $\tilde{I}$ and $\tilde{S}$.

(ii) Since $(n \mod (M + 1))/n \to 0$, this follows from the invariant (5).

Theorem 6. Let \( a \in (\mathbb{F}_q^M)^\infty \) with \( \delta(m,n) \neq 0 \) infinitely often for all \( 1 \leq m \leq M \) (all series active). Then \( I, S, \bar{I}, \bar{S} \) satisfy conditions

\[
\frac{M}{M + 1} \leq S \leq 1, \quad 0 \leq \bar{S} \leq \frac{1}{M + 1},
\]

and

\[
M(1 - S) \leq I \leq 1 - \frac{S}{M}, \quad -M \cdot \bar{S} \leq \bar{I} \leq -\frac{\bar{S}}{M}.
\]

(7)

Proof. We show the four inequalities in turn:

a) \( S \leq 1 \) or \( \bar{S} \leq 1/(M + 1) \):

Since \( L_a(n) \leq n \), the normalized linear complexity stays below or at 1, and the normalized drain below or at \( \bar{S} \leq 1 - \frac{M}{M + 1} = \frac{1}{M + 1} \).

b) \( M/(M + 1) \leq S \) or \( 0 \leq \bar{S} \):

The maximum of the \( b_m \) and \( d \) is larger than or equal to the average over all \( b_m \) and \( d \), which is zero. From time to time, \( d \) assumes this maximum after discharging the currently largest \( b_m \) (all \( G_m \) are active). Hence \( \bar{S} \geq 0 \) and \( S \geq \frac{M}{M + 1} \).

c) \( M(1 - S) \leq I \) or \( -M \cdot \bar{S} \leq \bar{I} \):

For all \( \varepsilon > 0 \) and \( n \geq n_1 \) for some \( n_1 \), \( b_m/n \leq \bar{S} + \varepsilon \). So \( \sum b_m/n \leq M \cdot (\bar{S} + \varepsilon) \), and with \( d/n + \sum b_m/n \to 0 \) (Lemma 7 (ii)), we have \( d/n \geq -M \cdot (\bar{S} + \varepsilon) \). With \( n \to \infty, \varepsilon \to 0 \), therefore \( \bar{I} \geq -M \cdot \bar{S} \). Now, \( I = \bar{I} + \frac{M}{M + 1} \geq -M \cdot \left( S - \frac{M}{M + 1} \right) + \frac{M}{M + 1} = (M + 1) \frac{M}{M + 1} - M \cdot S = M(1 - S) \).

d) \( I \leq 1 - S/M \) or \( \bar{I} \leq -\bar{S}/M \):

Asymptotically, the drain and all (active) batteries stay above \( I \) by Lemma 5(i). The normalized values thus satisfy

\[
\forall \varepsilon_1 > 0, \exists n_1, \forall n > n_1, \forall m, \forall k: \frac{d(m,n)}{n} \geq I - \varepsilon_1, \frac{b_k(m,n)}{n} \geq \bar{I} - \varepsilon_1.
\]

Also, there are infinitely many timesteps where the normalized drain value \( d/n \) is arbitrarily near \( \bar{S} \) after a discharge. Some battery, \( b_{m^*} \) say, is involved in infinitely many of these discharges and hence itself was near \( \bar{S} \) before those discharges:

\[
\forall \varepsilon_2 > 0, \exists m^*, \forall n, \exists n_1 > n: \frac{b_{m^*}(m^*,n_1)}{n_1} > \bar{S} - \varepsilon_2.
\]
Therefore, at the infinitely many timesteps \((m^*, n_1)\), we have with Lemma 5(ii):

\[
0 \leftarrow \frac{b_{m^*}(m^*, n_1)}{n_1} + \frac{d(m^*, n_1)}{n_1} + \sum_{k=1 \atop k \neq m^*}^{M} \frac{b_k(m^*, n_1)}{n_1} \geq (\bar{S} - \varepsilon_2) + (1 + (M-1))(\bar{I} - \varepsilon_1).
\]

Letting \(n \to \infty\) and \(\varepsilon_1, \varepsilon_2 \to 0\) gives \(0 \geq \bar{S} + M \cdot \bar{I} \iff \bar{I} \leq -\frac{S}{M}\), and thus

\[
I = \bar{I} + \frac{M}{M+1} \leq -\frac{\bar{S}}{M} + \frac{M}{M+1} = -\frac{S - \frac{M}{M+1}}{M} + \frac{M}{M+1} = 1 - \frac{S}{M}.
\]

Now, again incorporating the possibility of inactive sequences, we may state as a corollary:

**Theorem 7.** For any multisequence \(a \in (\mathbb{F}_q^M)^\infty\), the bounds \(I, S, \bar{I}, \bar{S}\) satisfy

\[
\frac{K}{K+1} \leq S \leq 1, \quad 0 \leq \bar{S} \leq \frac{1}{K+1}
\]

and

\[
K(1 - S) \leq I \leq 1 - \frac{S}{K}, \quad -K \cdot \bar{S} \leq \bar{I} \leq -\frac{\bar{S}}{K}
\]

for some \(1 \leq K \leq M\),

or \(a\) is ultimately periodic, hence \(K = 0, I = S = 0, \) and \(\bar{I} = \bar{S} = -\frac{M}{M+1}\).

**Proof.** If all series have ultimately periodic coefficient sequences, \(L_a(n) = O(1)\) and thus \(L_a(n)/n \to 0\). Otherwise, apply Theorem 6 with \(M := K\), since the \(M-K\) inactive series asymptotically do not affect \(L_a, \deg, \) or \(d\). \(\square\)

We visualize all allowed pairs \((I, S)\) in Figure 1.

The allowed parameters lie on the point \((0, 0)\) for \(K = 0\), on the line \(I + S = 1, I \leq S\) for \(K = 1\), and on overlapping triangles with endpoints \((0, 1), (\frac{K}{K+1}, \frac{K}{K+1})\) and \((\frac{K-1}{K}, 1)\) for \(2 \leq K \in \mathbb{N} (K = 0, \ldots, 5\) shown). The allowed area thus is not convex, not even connected. The points on the diagonal \(I = S\) are just the values \((\frac{K}{K+1}, \frac{K}{K+1}), K \in \mathbb{N}_0\), for convergent normalized complexities, and almost all multisequences (in the sense of Haar measure) can be found here \([6, 7, 10]\).

For \(M\) sequences in parallel, all cases \(0 \leq K \leq M\) are allowed (see (8)).
4 Normalized Linear Complexity II: Existence of Multisequences
Meeting any Allowed \( \lim \inf \) and \( \lim \sup \)

We next show that all pairs \((I, S)\) satisfying the conditions (7), resp. (8) actually occur for some multisequence \( a \in (F_q^M)^\infty \), for any finite field \( F_q \).

We construct a discrepancy sequence \( \delta(m, n) \) which leads to the specified behaviour of the normalized linear complexity. From the sequence \( \delta(m, n) \) one
can then obtain the actual coefficient sequence \((a_{m,n})\) applying the mSCFA. We first assume \(K = M\) that is all sequences are active.

Since only the asymptotic behaviour is of importance, small effects from the integrality of all numbers can be ignored, and we assume from now on \(d, b_m \in \mathbb{R}\). Also, we shall use \(b_m(t), d(t)\) to mean \(b_m(M, t), d(M, t)\), since the precise internal timestep does not matter any longer. The trajectories of the values for \(d(t)\) and \(b_m(t)\) shall follow a hexagon or butterfly pattern (see Figure 2).

![Diagram](https://via.placeholder.com/150)

**Fig. 2**

Drain \(d\): boldface,
Battery \(b_1\): solid,
Batteries \(b_2 \ldots b_M\): dashed (or “buried” in the \(d\) trajectory),
Asymptotics \(\tilde{S} \cdot t, \tilde{I} \cdot t\): dotted

The example shown uses the following values (for \(A\) see (9)):

\[
M = K = 3, t_0 = 96, S = 0.85, \tilde{S} = 0.1, I = 0.6, \tilde{I} = -0.15, A = 0.025
\]

| \(t\) | \(t_0 = 96\) | \(t_1 = 132\) | \(t_x = 160\) | \(t_2 = 172\) | \(t_* = 256\) |
|---|---|---|---|---|---|
| \(d\) | 0 | -3 | -24 \(\rightarrow\) 16 | 7 | 0 |
| \(b_1\) | 0 | 9 | 16 \(\rightarrow\) -24 | -15 | 0 |
| \(b_2, b_3\) | 0 | -3 | 4 | 7 | 0 |

**Description of the Construction:**

We will stack an infinite sequence of these hexagonal patterns one after the other, where each hexagon \(H\) starts at time \(t_0^{(H)}\) and finishes at \(t_*^{(H)}\).

We consider 5 moments and 4 time intervals:
At \(t_0\), all batteries and the drain are at zero (this is always possible for
\(t_0 := M\), with all discrepancies nonzero up to this point).

\((t_0, t_1)\): \(b_1\) grows \((\delta = 0)\), while \(b_2 = \cdots = b_M = d\) by discharging \((\delta \neq 0)\).

At \(t_1\), batteries \(b_2, \ldots, b_M\) stop to discharge.

\((t_1, t_x)\): All batteries grow \((\delta \neq 0)\).

At \(t_x\), battery \(b_1\) has reached the value \(\tilde{S} \cdot t_x\), while \(d\) is at value \(\tilde{I} \cdot t_x\). Now \(b_1\) discharges, and thus \(d\) becomes \(\tilde{I} \cdot t_x\). It is at these points \(t_x\), where \(d\) assumes both limiting values and thus assures the asymptotic behaviour.

\((t_x, t_2)\): All batteries are less than \(d\) and thus inhibited to discharge, irrespective of \(\delta\).

At \(t_2\), \(d = b_2 = \cdots = b_M\).

\((t_2, t_*)\): All batteries except \(b_1\) have to discharge, \(\delta \neq 0\), to ensure \(b_2 = \cdots = b_M = d\).

At \(t_*\), again all batteries and the drain are at zero.

How are the different timesteps related:

\(t_x\): Since battery \(b_1\) grows (all \(\delta(1, n) = 0\)) with slope \(\frac{1}{M+1}\) (by (4)) until touching the asymptotical line \(\tilde{S} \cdot t\) in \(t = t_x\), we have

\[
(t_x - t_0) \cdot \frac{1}{M + 1} = \tilde{S} \cdot t_x \iff t_x = \frac{t_0}{1 - \tilde{S}(M + 1)}.
\]

\(A\): We require \(d(t_x) = \tilde{I} \cdot t_x\) and \(b_1(t_x) = \tilde{S} \cdot t_x\). Assuming \(b_2 = \cdots = b_M\), we then have \(\tilde{I} + \tilde{S} + (M - 1) \cdot b_2/t_x = 0\) from (5), and thus

\[
A \cdot t_x := b_m(t_x) = \frac{-\tilde{I} - \tilde{S}}{M - 1} \cdot t_x, \quad 2 \leq m \leq M.
\] (9)

\(t_1\): We reach the point \((t_x, A \cdot t_x)\) from \((t_0, 0)\) following batteries \(b_2 \ldots b_m\):

\[
A \cdot t_x = \frac{-1}{M(M + 1)}(t_1 - t_0) + \frac{1}{M+1}(t_x - t_1)
\]

\[
\iff t_x \cdot AM(M + 1) = -t_1 + t_x(1 - \tilde{S}(M + 1)) + Mt_x - Mt_1
\]

\[
\iff t_1 \cdot (M + 1) = t_x(M + 1)(1 - \tilde{S} - AM)
\]

\[
\iff t_1 = t_x \left(1 - \tilde{S} - AM\right) = t_x \left(1 + \tilde{I} - A\right).
\]

\(t_2\): Between \(t_2\) and \(t_x\), the initial difference \((\tilde{S} - a)t_x\) between \(b_1\) and \(b_2\) is overcome by \(b_1\) with slope \(-\frac{M}{M+1}\) and \(b_2\) with slope \(\frac{1}{M+1}\), thus

\[
(\tilde{S} - A)t_x = (t_2 - t_x) \left(\frac{M}{M + 1} + \frac{1}{M + 1}\right) \iff t_2 = t_x(1 + \tilde{S} - A).
\]
The final time $t_*$ follows from

$$(t_* - t_0) \frac{1}{M + 1} = (\tilde{S} - \tilde{I})t_x \iff t_* = t_0 + (M + 1)\frac{(\tilde{S} - \tilde{I})t_0}{1 - \tilde{S}(M + 1)}$$

by following the trajectory of $b_1$ with (always) slope $1/(M + 1)$ by (3).

The quotient $t_*/t_0$ is (excluding the case $S = 1$, see Theorem 9 below)

$$\frac{t_*}{t_0} = \frac{1 - \tilde{I}(M + 1)}{1 - \tilde{S}(M + 1)} = \frac{1 - I}{1 - S},$$

and we obtain a geometric progression

$$t_*^{(H-1)} = t_0^{(H)} = c_0 \cdot \left(\frac{1 - I}{1 - S}\right)^H$$

when stacking hexagon $H$ directly after hexagon $H - 1$, $H \in \mathbb{N}_0$, starting in $c_0$.

Case $K < M$: Let now $0 \leq K \leq M$. We construct a discrepancy sequence $\delta(m, n), n \in \mathbb{N}, 1 \leq m \leq K$, as before, which can be mapped via the mSCFA to $K$ formal power series $G_1, \ldots, G_K$ matching the bounds $I$ and $S$. The $M - K$ other formal power series are set to $G_m = 0, K + 1 \leq m \leq M$, not affecting the behaviour of $L_a$ or $d$. 

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Algorithm 8. hexagon
INPUT \( \tilde{I}, \tilde{S} \in \mathbb{R}, M \in \mathbb{N} \)
IF \( M = 1 \) THEN \( A := \tilde{I} \) ELSE \( A := (-\tilde{S} - \tilde{I})/(M - 1) \)
\( t_1 := M + 1 \)
\( t_x := M + 1 \)
\( t := 0 \)
FOREVER
\( (t < t_1) \)
\( t++ \)
Do Not Discharge \( b_1 \): \( \delta(1, t) := 0 \)
Discharge \( b_2 \ldots b_M \): \( \delta(m, t) := 1, \forall 2 \leq m \leq M \)
END
\( (t < t_x) \)
\( t++ \)
Do Not Discharge: \( \delta(m, t) := 0, \forall 1 \leq m \leq M \)
END
\( (\exists b_m \neq 0) \) // \( t_x \ldots t_2 \ldots t_1 \)
\( t++ \)
Discharge All: \( \delta(m, t) := 1, \forall 1 \leq m \leq M \)
END
//Optionally: Discharge All for \( M + 1 \) additional timesteps
//to obtain different multisequences for the same \((I, S)\)
\( t_0 := t \)
\( t_x := t_0/(1 - \tilde{S}(M + 1)) \) // \( \tilde{S} \neq 1/(M + 1), 0 \)
\( t_1 := t_x(1 + \tilde{I} - A) \)
END

Theorem 9. Algorithm hexagon produces the discrepancy sequence of a multisequence \( a \in (\mathbb{F}_q^M)^\infty \) with \( \lim \inf L_a(n)/n = \tilde{I} \) and \( \lim \sup L_a(n)/n = \tilde{S} \), provided that \( I, S \) satisfy (7).

Proof. We already have shown by construction that the discrepancy sequence produced by hexagon corresponds to a multisequence \( a \in (\mathbb{F}_q^M)^\infty \) with asymptotic normalized linear complexities \( I \) and \( S \), provided \( t \to \infty \).

It remains to be verified that the algorithm indeed proceeds with \( t \to \infty \). This is not the case, only if \( \tilde{S} = 0 \), hence \( t_x = t_0 \), or for \( \tilde{S} = 1/(M + 1) \), leading to \( t_x = \infty \). In these cases, hexagon has to be adapted as follows: Instead of \( \tilde{S} \), use \( \tilde{S} = 1/t_0 \) or \( \tilde{S} = 1/(M + 1) - 1/t_0 \), respectively, and otherwise follow the same algorithm. Since \( \tilde{S} \to \tilde{S} \), we obtain the same asymptotics. \( \square \)
5 Cardinalities, Hausdorff Dimensions, Measures

Let $\mathcal{A}(I, S) \subset (\mathbb{F}_q^M)^\infty$ be the set of multisequences $a$ with asymptotic behaviour $I = \liminf_{n \to \infty} L_a(n)/n$ and $S = \limsup_{n \to \infty} L_a(n)/n$.

Cardinality: For every admissible pair $(I, S)$, $|\mathcal{A}(I, S)| = 2^{\aleph_0} = |(\mathbb{F}_q^M)^\infty|$.

Between every $t_*$ and the next $t_0$, we may choose to include $M + 1$ steps with $\delta(m, n) \neq 0$ (outcommented lines in Algorithm 8), leaving us again in $b_m = d = 0, \forall m$. Following immediately with the next hexagon would imply $\delta(1, n) = 0$ at $b_1 < 0$, leading to different multisequences.

Measure: Niederreiter and Wang \[6, 7, 10\] recently have shown for all $M \in \mathbb{N}$ that

$$\mu(\mathcal{A}(I, S)) = \begin{cases} 1, & I = S = M/(M + 1), \tilde{I} = \tilde{S} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hausdorff dimension: We map $\mathcal{A}(I, S)$ to the real unit interval $[0, 1]$ by $\iota: (\mathbb{F}_q^M)^\infty \ni a \mapsto \sum_{t=1}^\infty \sum_{m=1}^M a_{t,m} \cdot q^{-(M(t-1)+m)} \in [0, 1] \subset \mathbb{R}$, where we identify the set $\mathbb{F}_q$ with $\{0, 1, \ldots, q-1\} \subset \mathbb{Z}$ by some fixed bijection, and denote its Hausdorff dimension by $D_H(\mathcal{A}(I, S)) := D_H(\iota(\mathcal{A}(I, S)))$.

**Theorem 10. (Hausdorff Dimension)**

Given a multiness $M$ and a pair $(I, S)$ of asymptotic limits, let $K'$ be the largest $K \leq M$, such that $(I, S)$ lies within the $K'$-th triangle $((0, 1), (\frac{K-1}{K}, 1), (\frac{K}{K+1}, \frac{K}{K+1}))$ (or on the point $(0, 0)$, or on the segment $(0, 1), (\frac{1}{2}, \frac{1}{2})$ for $K' = 0, 1$, resp.).

If no such $K'$ exists, $(I, S)$ is not admissible for that $M$ and $\mathcal{A}(I, S)$ is empty. Otherwise the Hausdorff dimension of $\mathcal{A}(I, S)$ within $(\mathbb{F}_q^M)^\infty$ is bounded by

$$\frac{K'}{M} \cdot \frac{1 - S}{(M+1)(1-I)^2} \leq \mathcal{A}(I, S) \leq \frac{K'}{M}.$$

In particular, for $S < 1$ the Hausdorff dimension is positive.

**Proof.** There may be at most $K'$ active sequences, since this is the largest value permitted for $(I, S)$. We shall initially assume $M = K'$ and later generalize to $M \geq K'$.

We define a subset of $\mathcal{A}(I, S)$ with discrepancy sequences that alternate between hexagons according to Algorithm 8, $H_n, n \in \mathbb{N}$ and “fill”, $F_n, n \in \mathbb{N}$, where the sequence may behave arbitrarily while staying within the $(I, S)$ interval.

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Assume that we have at least \((q^M)^{t_{N-1} \cdot \left(1 - \frac{1}{N}\right)}\) sequence prefixes up to \(t_{N-1}\) (always possible for \(N = 1\) at \(t_0 = 0\) with the single (empty) sequence \(\varepsilon\)). We now want to append a hexagon. Since at the end of the fill phase, \(d\) and the \(b_m\) may be anywhere within \((I, S)\), we first discharge until \(d = b_m = 0\). This takes at most a time from \(t_x := t_{N-1}\) to the corresponding \(t_s\). Thereafter, we are ready to add another full hexagon which ensures the limiting behaviour. With \(t_x = t_0 \cdot 1/(1-\tilde{S}(M+1)) = t_0/((M+1)(1-S))\) and \(t_s = t_0 \cdot (1-I)/(1-S)\), we obtain \(t_s/t_x = (M + 1) \cdot (1 - I)\) for the “half” hexagon and a total time of

\[
t_{N-1} \cdot (M + 1) \cdot (1 - I) \cdot \frac{1 - I}{1 - S} = t_{N-1} \cdot (M + 1) \cdot \frac{(1 - I)^2}{1 - S}
\]

to reach the end of the full hexagon. During the hexagon phase, we allow only a single extension (putting \(\delta = 1\), whenever \(\delta \neq 0\) is required) and thus produce a single well-defined discrepancy sequence. We then still have

\[
(q^M)^{t_{N-1} \cdot \left(1 - \frac{1}{N}\right)} \cdot 1^{t_{N-1} \cdot \left((M+1) \frac{(1-I)^2}{1-S} - 1\right)} = \left[(q^M)^{t_{N-1} \cdot (M+1) \frac{(1-I)^2}{1-S}} \right]^{(1-1/N)(1-S)/(M+1)(1-I)^2}
\]

prefixes of length \(t_{N-1} \cdot (M + 1) \frac{(1-I)^2}{1-S}\) in \(A(I, S)\), which leads to a Hausdorff dimension at least \(\frac{1 - S}{(M + 1)(1-I)^2}\).

By [6, 10], almost all sequences in \((\mathbb{F}_q^{K'})^\infty\) lead to \(I = S = \frac{M}{M+1}\) or \(\bar{I} = \bar{S} = 0\) and thus can be used to fill between hexagons without leaving the bounds \(I\) and \(S\). Hence it is possible to reach some \(t_N\) at the end of fill \(F_N\) with at least \((q^M)^{t_N \cdot \left(1 - \frac{1}{M+1}\right)}\) different prefixes. The Hausdorff dimension of \(A(I, S)\) thus is lowerbounded by the number of prefixes at the end of the hexagons, with \(n \to \infty\) thus

\[
D_H \geq \frac{1 - S}{(M + 1)(1-I)^2}.
\]

Finally, with \(M > K'\), only \(K'\) sequences may be active, the other \(M - K'\) depending \(\mathbb{F}_q(x)\)–linearly on them. Letting the first \(K'\) sequences fix \(I\) and \(S\), gives as before \(\frac{1-\bar{S}}{(M+1)(1-I)^2} \leq D_H \leq 1\) in \((\mathbb{F}_q^{K'})^\infty\) and thus \(\frac{K'}{M} \cdot \frac{1-\bar{S}}{(M+1)(1-I)^2} \leq D_H \leq \frac{K'}{M}\) in \((\mathbb{F}_q^M)^\infty\). The remaining sequences are \(\mathbb{F}_q(x)\)–dependent, hence increase the number of feasible sequences only by a factor of \((M/K') \cdot |\mathbb{F}_q(x)|^{(M-K')K'} = \aleph_0\), too few to change \(D_H\).

\[\square\]
Conclusion

We have determined all possible values for the asymptotic behaviour of the normalized linear complexity of multisequences. We have also given an algorithm to actually produce a sequence of any multiness $M$ with prescribed infimum $I$ and supremum $S$ of its normalized linear complexity. This gives a positive answer to the question posed by Dai, Imamura and Yang, whether the well–known equality $\lim \inf L_a(n)/n + \lim \sup L_a(n)/n = 1$ in the case of one sequence has a generalization.

We finished with the cardinality, Hausdorff dimension, and measure of the set $A(I, S)$ of sequences attaining the prescribed bounds, obtaining that all sets $A(I, S)$ have $2^{\aleph_0}$ elements, and, at least for $S \neq 1$, positive Hausdorff dimension.

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