On the Impulsive Formation Control of Spacecraft Under Path Constraints

AMIR SHAKOURI
Sharif University of Technology, Tehran, Iran

This paper deals with the impulsive formation control of spacecraft in the presence of constraints on the position vector and time. Determining a set of path constraints can increase the safety and reliability in an impulsive relative motion of spacecraft. Specially, the feasibility problem of the position norm constraints is considered in this paper. Under assumptions, it is proved that if a position vector be reachable, then the reach time and the corresponding time of impulses are unique. The trajectory boundedness of the spacecraft between adjacent impulses are analyzed using the Gerschgorin and the Rayleigh–Ritz theorems as well as a finite form of the Jensen’s inequality. Some boundaries are introduced regarding the Jordan–Brouwer separation theorem which are useful in checking the satisfaction of a constraint. Two numerical examples (approximate circular formation keeping and collision-free maneuver) are solved in order to show the applications and visualize the results.

I. INTRODUCTION

The relative spacecraft dynamics has considerably drawn the attentions due to its applications both in the formation flying and the rendezvous missions. In the simplest form, two spacecrafts are considered in which a chaser spacecraft (CS) is the actuated system and a target spacecraft (TS) is located at the origin. The spacecrafts are assumed to be point masses that are governed by a central gravitational force. In this context, the CS follows a path relative to the TS which is constrained dynamically and/or geometrically. Some constraints are essential to make the mission possible, while some others can be considered in order to raise the safety and reliability of the mission.

The relative motion of spacecraft can be developed as a linear time-invariant state-space model as initially proposed in [1] that is called the Clohessy–Wiltshire (CW) system. The CW model is straightforward to be implemented and optimized in the unconstrained cases [2], [3], while the simplification assumptions are not much away from reality. However, many different models are proposed for the relative motion of spacecraft in the presence of perturbations and in the vicinity of circular or elliptical orbits [4], [5].

Relative control of spacecraft is widely discussed in the literature and many different schemes are proposed. Optimal impulsive approaches based on the primer vector solutions are investigated in [6]–[8]. Gao et al. discussed the robust $H_\infty$ control of relative motion [9], while solutions to the matrix inequalities are proposed by Tian and Jia [10]. Mesbahi and Hadaegh studied the formation flying control via graphs, matrix inequalities, and switching [11]. Constraints can be applied on the spacecraft state and/or actuator in different forms. Path constrained problems are discussed by Taur et al. [12] with impulsive time-fixed actuation. Soileau and Stern developed some necessary and sufficient conditions [13]. A method for constrained trajectory generation for micro-satellite formations is investigated by Milan et al. [14] and a finite thrust solution for state constraints is discussed by Beard and Hadaegh [15]. A model predictive control for handling the constraints is proposed by Weiss et al. [16] and Chen et al. analyzed the non-holonomic constraints [17]. For the spacecraft rendezvous with actuator saturation, a gain scheduled control is developed by Zhou et al. [18]. The global stabilization of CW system by saturated linear feedback is discussed in [19]. A covariance-based rendezvous design method is developed in [20] by Shakouri et al. at which several constraints on the control effort, maximum impulse value, flight time, and safe zones are taken into account.

The spacecraft actuation can be modeled by impulses in which the burning duration is negligible with respect to the time interval between adjacent burning instances. These low duration accelerations can be modeled by an impulse as a momentary change in velocity vector [6], [7], [12], [21]. The use of hybrid impulsive-continuous actuation is studied by Sobiesiak and Damaren [22]. In [23] a multi-objective optimization is implemented by Luo et al. in order
to achieve safe collision-free trajectories with admissible control efforts.

In this paper, the impulsive behavior of the CS under equality and/or inequality path constraints are investigated. The impulse times and positions are assumed as the decision variables instead of impulse values. This point of view has advantages in considering the path constraints and disadvantages in the ignorance of the optimal solution. Under certain assumptions, it is shown that if a position vector is reachable, then the time of the next impulse and the corresponding reach time are unique (Theorem 1). Furthermore, using the Gerschgorin circle theorem and some results from the Rayleigh–Ritz theorem beside several spectral facts, an upper norm bound for the CS’s trajectory subject to two-impulse maneuver is found (Theorem 2) which constructs the primary contribution of this paper. Afterwards, using the Jensen’s inequality, some bounding cones are introduced (Theorem 3). The Jordan–Brouwer separation theorem is used to show that how an area can be unreachable for the CS. Two numerical examples are provided used to show the applications of the results. First, an approximate circular formation keeping (CFK) problem is considered in which the CS finds those reachable areas as its impulse positions such that the keep-out and the approach circles constraints are not violated. It is shown how the CS can stay in an approximately circular trajectory just by two impulses. The second example is a collision-free maneuver (CFM) where it is shown that how several impulse positions can be selected such that the CS do not collide with a keep-out circle with any times of impulse. This point of view can result in CFMs which are insensitive to the impulse times.

The rest of this paper is organized as follows. First, the preliminaries regarding the notations, basic formulations, assumptions, some spectral facts, and the essential definitions are presented. Then, in Section III, the main theoretical results of the paper are derived. In Section IV, two numerical examples are provided in order to visualize the theoretical results. Next, some discussions are presented about the consequences and the applications. Finally, Section V is dedicated to the concluding remarks.

II. PRELIMINARIES

A. Notation

This section briefly introduces the notations that are used throughout this paper.

Let $\mathbb{M}^{m,n}$ denote the space of $m \times n$ real (or complex) matrices and $\mathbb{M}^{n}$ its square analog. In addition, let $\mathbb{S}^{n}$ denote the space of $n$-dimensional real symmetric matrices, and $\mathbb{R}^{n}$ denotes the space of $n$-dimensional real vectors. The $(i,j)$th entry of a matrix $M \in \mathbb{M}^{m,n}$ is referred to by $M_{i,j}$ and the $i$th entry of vector $r \in \mathbb{R}^{n}$ is referred to by $r_{i}$. Upper and lower case letters are used to denote matrices and vectors, respectively. Greek letters are used to denote the scalars. For matrix $M$, we note by $M^{T}$ its transpose, by $M^{-1}$ its inverse (if exists), by null($M$) its nullity, and by rank($M$) its rank.

The symbol $\| \cdot \|$ denotes a norm and specially $\| \cdot \|_{p}$ denotes the $p$-norm of a vector. The boundary of a set $S \subset \mathbb{R}^{n}$ is denoted by $\partial S$. We use the element-wise inequality $x \geq 0$ to show that $x_{i} \geq 0$, $i = 1, \ldots, n$, and $x \geq y$ is equivalent to $x - y \geq 0$. The symbol $\mathbf{1}$ is used to denote a vector with all elements equal to 1.

Let $t_{1}, t_{2} \in [0, \infty)$ such that $t_{1} < t_{2}$, then the following notation is used to define the time sets:

$$\mathcal{T}_{t_{1}} = \{ \tau \in \mathbb{R} | 0 < \tau < t_{1} \}.$$ So, $\mathcal{T}_{\infty} = \{ \tau \in \mathbb{R} | 0 < \tau < \infty \}$ and $\mathcal{T}_{t_{2}} - \mathcal{T}_{t_{1}} = \{ \tau \in \mathbb{R} | t_{1} < \tau < t_{2} \}$. Subscript $i \in \mathbb{N}$ is used when referring to the time steps, and subscript $l \in \mathbb{N}$ is used to distinct the constraints.

B. System Model and Essentials

In spacecraft relative motion, the CS is the actuated system and the TS defines the final state that needs to be reached. Let us introduce several assumptions that are made in the rest of this paper.

ASSUMPTION 1 Let $a_{TS} \in \mathbb{R}$ denote the semimajor axis of the TS’s orbit and $r \in \mathbb{R}^{3}$ denote the relative position of the CS with respect to the TS in an arbitrary TS-centered coordinate system. The following assumptions are made.

1) The two-body gravitational force is governing and no perturbations exist.

2) The TS is in a circular orbit.

3) $\|r\|/a_{TS} \ll 1$.

Let $r_{i} \in \mathbb{R}^{3}$ and $v_{i} \in \mathbb{R}^{3}$ denote the position and velocity vectors at step $i \in \mathbb{N}$, respectively. Suppose $r_{i}$ and $v_{i}$ are relative position and velocity of the CS defined in the RSW coordinate system of the TS (The RSW coordinate systems is defined such that its $x$-axis is in the direction of the position vector of the associated spacecraft, the $z$-axis toward the orbital angular momentum vector, and the $y$-axis completes the right-handed coordinate system). Let $t_{i} \in \mathcal{T}_{\infty}$ denote the time $t_{i+1} = t_{i} + \Delta t$, and $F_{r}(\cdot) : \mathcal{T}_{\infty} \mapsto \mathbb{M}^{3}$ (similarly for $F_{vr}$, $F_{vv}$, and $F_{rv}$). Considering Assumption 1 holds, the solution of the CW equations are as follows:

$$r_{i+1} = F_{rr}(\Delta t_{i+1},i)r_{i} + F_{rv}(\Delta t_{i+1},i)v_{i}$$

$$v_{i+1} = F_{rv}(\Delta t_{i+1},i)r_{i} + F_{vv}(\Delta t_{i+1},i)v_{i}$$

in which

$$F_{rr} = \begin{bmatrix} 4 - 3 \cos \kappa_{i} & 0 & 0 \\ 6(\sin \kappa_{i} - \kappa_{i}) & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{rv} = \frac{1}{\kappa} \begin{bmatrix} \sin \kappa_{i} & 2(1 - \cos \kappa_{i}) & 0 \\ -2(1 - \cos \kappa_{i}) & 4 \sin \kappa_{i} - 3 \kappa_{i} & 0 \\ 0 & 0 & \sin \kappa_{i} \end{bmatrix}$$

$$F_{vr} = \frac{d(F_{rr})}{d(\Delta t_{i+1},i)} = \kappa \begin{bmatrix} 3 \sin \kappa_{i} & 0 & 0 \\ 6(\cos \kappa_{i} - 1) & 0 & 0 \\ 0 & 0 & -\sin \kappa_{i} \end{bmatrix}$$
\[ F_{\nu\nu} = \frac{d(F_{\nu\nu})}{d(\Delta t_{i+1,i})} = \begin{bmatrix} \cos \kappa_i & 2 \sin \kappa_i & 0 \\ -2 \sin \kappa_i & 4 \cos \kappa_i & -3 \cos \kappa_i \\ 0 & 0 & \cos \kappa_i \end{bmatrix} \]

where \( \kappa_i = \kappa \Delta t_{i+1,i}, \kappa = \sqrt{\mu/\alpha^3_{TS}} \) is the mean motion of the TS, and \( \mu \) stands for the central body (earth) gravitational parameter. We shall use \( F_{\nu\nu}/r_i \) instead of \( F_{i/r_i}(\cdot) \) for simplicity. Here, another assumption is made to avoid singularities in the matrices defined above.

**Assumption 2** The time interval between any two subsequent impulses should be less than \( \pi/\kappa \), i.e., \( \forall i \in \{1, \ldots, n-1\}, \Delta t_{i+1,i} \in T_{\pi/\kappa}. \)

It is worth mentioning that for a relative motion of spacecraft with a total flight time of \( \Delta t_{\text{total}} \in [0, \infty) \), if the number of impulses satisfy \( n \geq \min \{ \lceil \Delta t_{\text{total}}/(\pi/\kappa) \rceil + 1, 2 \} \), then Assumption 2 can be satisfied.

Let us introduce the position and velocity vectors \( r_i^n \) and \( v_i^n \) which define the position and velocity vectors before applying an impulse vector \( \Delta v_i \in \mathbb{R}^3 \) at \( r \)th step. After applying the impulse vector, the position and velocity vectors become \( r_i^{n+1} \equiv r_i \) and \( v_i^{n+1} = v_i^n + \Delta v_i \). So, the position and velocity after the \( i+1 \)th impulse are found from (1)

\[
\begin{align*}
    r_{i+1} &= F_{rr} r_i + F_{rv} v_i^n + F_{rv}(v_i^n + \Delta v_i) \\
    v_{i+1} &= F_{rv} r_i + F_{vv} v_i^n + F_{rv}(v_i^n + \Delta v_i).
\end{align*}
\]

Therefore, form (2) knowing \( r_i, r_{i+1} \) and \( v_i, v_{i+1} \) the impulse vector \( \Delta v_i \) is

\[
\Delta v_i = F_{rv}^{-1}(r_{i+1} - F_{rr} r_i) - v_i.
\]

**Remark 1** For an \( n \)-impulse relative motion between known \( r_i, r_n, v_i^n \), and \( v_n^n \), which is usually the case, the decision variables can be chosen to be one of the following sets:

1) \( \{ r_i \in \mathbb{R}^3 \mid i = 2, \ldots, n-1 \} \cup \{ \Delta t_{i+1,i} \in T_{\pi/\kappa} \mid i = 1, \ldots, n-1 \} \).
2) \( \{ \Delta v_i \in \mathbb{R}^3 \mid i = 1, \ldots, n-2 \} \cup \{ \Delta t_{i+1,i} \in T_{\pi/\kappa} \mid i = 1, \ldots, n-1 \} \).
3) \( \{ r_i \in \mathbb{R}^3 \mid i = 2, \ldots, n-1 \} \cup \{ \| \Delta v_i \|_2 \in \mathbb{R}^3 \mid i = 1, \ldots, n-1 \} \).

Each of the above sets has \( 2n-3 \) members. So, for an \( n \)-impulse trajectory, a number of \( 2n-3 \) decision variables (\( 2n \) vectors and \( n-1 \) scalars) are needed to determine the whole trajectory.

Consider the following matrices:

\[
F_2(t, r) = F_{rv}(t) F_{rv}^{-1}(r) \quad (6)
\]

\[
F_1(t, r) = F_{rr}(t) - F_2(t, r) F_{rv}(r) \quad (7)
\]

in which \( t \in T_{\pi/\kappa} \) and \( \tau \in T_{\tau} \). Suppose \( \tau \) to be fixed, so we use \( F_1(t, \tau) \equiv F_1(t) \) and \( F_2(t, \tau) \equiv F_2(t) \) for simplicity. From (1) to (4) it can be shown that how the CS’s trajectory behaves in the time domain subjected to fixed initial \( (r_1) \) and final positions \( (r_{i+1}) \) as well as the total flight time \( (\Delta t_{i+1,i}) \). Using the forms defined in (6) and (7) we get

\[
r(t, \Delta t_{i+1,i}) = F_1(t, \Delta t_{i+1,i}) r_i + F_2(t, \Delta t_{i+1,i}) r_{i+1} \quad (8)
\]

in which we simply use \( r \equiv r(t, \Delta t_{i+1,i}) \) for fixed \( \Delta t_{i+1,i} \).

### C. Spectral Analysis

The spectral properties of some matrices are needed to be analyzed to be used further in obtaining the results. Let \( \lambda_{11}(t), \lambda_{22}(t) \in \mathbb{R}^3 \) denote the eigenvalues of \( F_1(t) F_1(t) \) and \( F_2(t) F_2(t) \), respectively, that are sorted in vectors such that \( \lambda_{ii}(j)(t) \leq \lambda_{ii}(j+1)(t) \) for \( i, j \in \{1, 2\} \). Now, consider the following \( 6 \times 6 \) symmetric block form matrix:

\[
\hat{F}(t) = \begin{bmatrix} F_1(t) F_1(t) & F_1(t) F_2(t) \\ F_2(t) F_1(t) & F_2(t) F_2(t) \end{bmatrix} \quad (9)
\]

Denote the eigenvalues of \( \hat{F}(t) \) by \( \hat{\lambda}_{11}(t) \leq \hat{\lambda}_{22}(t) \leq \cdots \leq \hat{\lambda}_6(t) \). The following facts are analytically or numerically evaluated:

**FACT 1** Under Assumptions 1 and 2, the following statements hold for matrices \( F_1 \) and \( F_2 \):

1) \( \text{Rank}[F_1(t) F_1(t)] = 3, \text{rank}[F_2(t) F_2(t)] = 3 \), and both have three real nonzero eigenvalues.

2) At any \( t \in T_{\tau} \), the following property holds for the eigenvalues of \( F_1 F_1 \) and \( F_2 F_2 \):

\[
\lambda_{11}(t) - \lambda_{22}(\tau - t) = 0 \quad (10)
\]

3) For \( i \in \{1, 2\} \), at any \( t \in T_{\tau} \), \( \lambda_{ii}(3) \leq 1 \) for \( \tau < (\pi/\kappa)/2, \lambda_{ii}(3) = 1 \) for \( \tau = (\pi/\kappa)/2 \), and \( \lambda_{ii}(3) > 1 \) for \( \pi/\kappa < \tau < \pi/\kappa \).

**FACT 2** Under Assumptions 1 and 2 the following statements hold for matrix \( \hat{F} \):

1) \( \text{Rank}[\hat{F}(t)] = 3, \text{null}[\hat{F}(t)] = 3, \) and it has three eigenvalues of zero, i.e., \( \hat{\lambda}_{11}(t) = \hat{\lambda}_{22}(t) = \hat{\lambda}_{33}(t) = 0 \).

2) At any \( t \in T_{\tau} \), the entries of \( \hat{F}(t) \) have the following property for \( i = 1, 2, 3, 6 \):

\[
\sum_{j=1}^{6} \| \hat{F}_{(i,j)}(t)(\tau - t) \|_1 = 0 \quad (11)
\]

3) The eigenvalues of \( \hat{F}(t) \) are constant over time or they have a single extremum at \( t = \tau/2 \).

4) At any \( t \in T_{\tau} \), \( \hat{\lambda}_{66}(t) < 1 \) for \( 0 < \tau < (\pi/\kappa)/2, \hat{\lambda}_{66}(t) = 1 \) for \( \tau = (\pi/\kappa)/2 \), and \( \hat{\lambda}_{66}(t) > 1 \) for \( \pi/\kappa < \tau < \pi/\kappa \).

### D. Definitions

**NOTATION 1** Let \( \Gamma_{ij}^l(t) \) denote a trajectory such that \( n \) impulses are used starting from index \( i \) and ending in \( j \) such that the first and the last impulses are applied at \( i \) and \( j \), respectively. For example, \( \Gamma_{11}^3(2) \) denotes a two-impulse trajectory that starts from \( r_1, v_1 \) at \( t_1 \), and ends in \( r_3 \) at \( t_3 \).

It should be noted that the symbol \( \Gamma_{ij}^l(t) \) does not give any knowledge about the decision variables and the initial/final states of the trajectory. So, \( \Gamma_{ij}^l(t) \) alone cannot
define a relative spacecraft motion trajectory even for two-impulse missions.

**Definition 1** Let \( r \in \mathbb{R}^3 \) denote the position vector at \( t \in T_\infty \). Consider the sets \( \tilde{T}_l \subset T_\infty \) and \( \tilde{R}_l \subset \mathbb{R}^3 \) at \( l \in \{1, 2, \ldots, m\} \) for \( m \in \mathbb{N} \). Path constraints (PCs) are those constraints that can be stated as follows:

\[
\forall t \in \tilde{T}_l : r \in \tilde{R}_l, \quad (12)
\]

In this paper, we are dealing with a special kind of PCs. Suppose \( \tilde{r}_l \in \mathbb{R}^3, \rho'_l, \rho''_l \in \mathbb{R}^3, \) and \( \tilde{r}_l \in T_\infty \) are predefined parameters at \( l = 1, \ldots, m \). The general PCs defined in (12) can be reduced to an inequality form such that

\[
\tilde{T}_l = \tilde{T}_l, \quad \tilde{R}_l = \{ r \in \mathbb{R}^3 | \rho'_l \leq \| r - \tilde{r}_l \|_2 \leq \rho''_l \}. \quad (13)
\]

Each PC of the form (13) restricts the CS’s path between two spheres at a time interval. The form of inequality PCs introduced in (13) can be used for trajectories that avoid collisions independent of the transfer time. The following feasibility problem, states the main subject of this paper.

**Problem 1** Find the decision variables (discussed in Remark 1) such that it satisfies a PC of the form (13).

**Remark 2** In (13), assuming \( \rho''_l \to \infty \), the PC defines a forbidden region at \( t \in \tilde{T}_l \). This region is bounded by \( \| r - \tilde{r}_l \|_2 = \rho'_l \). This kind of PCs can be used to define collision-free trajectories that are robust with respect to actuator fault and failure.

The inequality form of (13) can turn to equality, if \( \tilde{T}_l = \{ \tilde{r}_l \} \) and \( \rho'_l = \rho''_l = 0 \). So, an equality PC can be stated in the following form:

\[
\tilde{T}_l = N, \quad \tilde{R}_l = \{ \tilde{r}_l \}. \quad (14)
\]

**Remark 3** A spacecraft trajectory with known initial and final positions \( r_1, r_n \) which is usually the case, essentially has two PCs of the equality form; the first is \( \| r - \tilde{r}_l \|_2 = 0 \) at \( t = 0 \) and the second is \( \| r - \tilde{r}_l \|_2 = 0 \) at \( t = t_n \).

A two-point constrained single-impulse reachability (or simply "reachability") can be defined in the context of this paper which is an impulsive reachability that is constrained in order to achieve an initial and a final position.

**Definition 2** Let \( t_j - t_i \in T_{\pi/k} \) and \( \tau \in [t_i, t_j] \). A position vector \( r = r(t) \) is called reachable in \( \Gamma' \) at \( t \in [t_i, t_j] \) if there exists \( \bar{v}_i \in \mathbb{R}^3 \) such that \( r = r(t_i) \in [t_i, t_j] \). If there exists \( \bar{v}_i \in \mathbb{R}^3 \) such that \( r = r(t_i) \) subject to \( r = r_i \) at \( t = t_i \) and \( r = r_j \) at \( t = t_j \). A position vector that is not reachable is called unreachable.

**E. Time Uniqueness**

In this section, it is proved that if a point in the space under Assumptions 1 and 2 is reachable, so the corresponding total time of flight and the current time are unique. Consider the following results:

**Lemma 1** Let \( \mathcal{T}, \mathcal{T}' \subset T_{\pi/k} \) and \( \mathcal{R}, \mathcal{R}' \subset \mathbb{R}^3 \), then functions \( v_i() : \mathcal{T} \mapsto \mathcal{R}, \quad r_i() : \mathcal{T}' \times \mathcal{R} \mapsto \mathcal{R}_t, \) and \( r_i() : \mathcal{T} \times \mathcal{T}' \mapsto \mathcal{R}_t \) are injective such that

\[
v_i(t) = F^i \cdot v_i(t) - v_i(t) \quad (15)
\]

\[
r_i(t_1, t_2) = F_i(t_1, t_2) \quad (16)
\]

\[
r_i(t_1, t_2) = F_i(t_1, t_2) - F_i(t_2, t_1) \quad (17)
\]

**Proof** First it should be noted that \( F_i(t_1, t_2) \) and \( F_i(t_2, t_1) \) are injective maps from \( \mathcal{T} \) to some subsets of \( \mathbb{R}^3 \). Function \( v_i(t) \) generates an impulse vector to transfer the CS from \( r_i \) and \( v_i \) at \( t_1 \) to \( r_j \) and \( v_j \) at \( t_2 \). From the physics of the problem, obviously, each impulse (\( t' \in T_{\infty} \)) has a unique time to transfer and (15) is injective. To show the uniqueness of (16) consider \( r_i(t_1, t_2) = r_i(t_2, t_1) \) that leads to \( F_i(t_1, t_2) = F_i(t_2, t_1) \). Given that \( \forall t' \in T_{\infty}, \) rank\( [F_i(t')] \) = 3, according to the rank-nullity theorem null\( [F_i(t')] = 3 \). So, the only solution to \( F_i(t_1, t_2) = F_i(t_2, t_1) \) is \( t_1 = t_2 \).

**Lemma 2** Let \( i, j, n \in \mathcal{R}, \mathcal{R}_i, \mathcal{R}_j \subset \mathbb{R}^3 \), and \( t_i, t_j, t \in T_{\infty} \) where \( t_i < t < t_j \) and \( \Delta t_{i,j} = t_j - t_i \), such that

\[
\mathcal{R}_{i,j}(t) = \{ r \in \mathbb{R}^3 | \| r - \tilde{r}_i \|_2 = 0 \} \quad (18)
\]

then \( r_j \) is reachable in \( \Gamma_i(t) \) at \( t_i, t \) if and only if \( r \in \mathcal{R}_{i,j}(t) \).

**Proof** It can be directly concluded from (8) that if \( r \in \mathcal{R}_{i,j}(t) \) then \( \exists \Delta t_{i,j} \in T_{\pi/k} \) such that the CS reaches \( r \). For the other direction, we know that if \( r \notin \mathcal{R}_{i,j}(t) \) then the CS cannot reach \( r \) at \( t, \forall \Delta t_{i,j} \in T_{\pi/k} \).

**Remark 4** In Lemma 2, the elements of set \( \mathcal{R}_{i,j}(t) \) are the outputs of a nonlinear mapping from \( \tilde{T}_l \) to a higher dimension in \( \mathbb{R}^3 \). In fact, the locus of the elements of \( \mathcal{R}_{i,j}(t) \) is a three-dimensional (3-D) curve that starts from \( r \to r_j \) at \( \Delta t_{i,j} \) and ends in some infinite \( r \) at \( \Delta t_{i,j} \). From (14) if \( \Delta t_{i,j} \neq \Delta t_{i,j} \), then \( \exists \Delta t_{i,j} \neq \Delta t_{i,j} \) that no equality exists and if \( \Delta t_{i,j} = \Delta t_{i,j} \), then \( r \in \mathcal{R}_{i,j} \) which demonstrates a contradiction.

**Proposition 1** Let \( t_1, t_2 \in T_{\pi/k} \) then \( \mathcal{R}_{1,2}(t_1) \cap \mathcal{R}_{1,2}(t_2) \neq \emptyset \) if and only if \( t_1 = t_2 \).

**Proof** The necessity is obvious. For sufficiency, a proof by contradiction is used. Suppose \( t_1 \neq t_2 \) and \( \mathcal{R}_{1,2}(t_1) \cap \mathcal{R}_{1,2}(t_2) \neq \emptyset \), then there exists a position \( r \) and two time intervals \( \Delta t_{i,j} \) and \( \Delta t_{i,j} \) such that \( r(t_1, t_2) = r(t_2, t_1) \). Thus, from Lemma 1 and (14) if \( \Delta t_{i,j} = \Delta t_{i,j} \), then no equality exists and if \( \Delta t_{i,j} 
eq \Delta t_{i,j} \), then \( r \in \mathcal{R}_{i,j} \) which demonstrates a contradiction.

**Theorem 1** Suppose Assumptions 1 and 2 hold. If the position vector \( r \) is reachable in \( \Gamma_{i,1} \) at \( t \in [t_i, t_{i+1}] \), then the corresponding \( t \) and \( \Delta t_{i,j} \) are unique.

**Proof** From the result of Lemma 2, the reachability of \( r \) in \( \Gamma_{i,1} \) at \( t \in [t_i, t_{i+1}] \) is guaranteed if and only if there exist an \( t \) and \( \Delta t_{i,j} \) such that \( r = r_j(\Delta t_{i,j}, t) \). From
Lemma 1, \( r_i(\cdot) \) is injective. Therefore, having the output, the corresponding inputs are unique.

Theorem 1 individually concludes that a two-impulse rendezvous maneuver which is restricted to reach a point in the space (except of initial and final locations), has a unique solution. This can be used for cases that we need to hit a target between initial and final locations. Furthermore, Theorem 1 can be used for an initial relative orbit determination using three vectors (similar to the Gibbs method in two-body problem) which can be a subject for the future researches.

Regarding Theorem 1, the reader can refer to the work done by Wen et al. in [24] at which the reachability problem of impulsive maneuvers for nonlinear unperturbed problems is investigated through analytic geometry.

III. MAIN RESULTS

A. Trajectory Boundedness

In this section an upper bound on the CS’s trajectory subjected to fixed initial and final positions are presented. It is shown that for a two-impulse transfer between \( r_i \) at \( t_i \) and \( r_{i+1} \) at \( t_{i+1} \) the trajectory has upper bounds on the norm of the position vector, depending on the initial (\( r_i \)) and final positions (\( r_{i+1} \)) as well as the total flight time \((\Delta t_{i+1,i})\). The results of this section can be directly used for the design of the decision variables in a constrained formation of spacecraft. First we need the following two lemmas.

**LEMMA 3 (GERSHGORIN)** Let \( M \in \mathbb{M}^n \) with associated eigenvalues of \( \mu_i, i = 1, \ldots, n \), and let
\[
\rho_i(M) = \sum_{j=1, j \neq i}^{n} \| M_{i,j} \|_1, \quad 1 \leq i \leq n
\]
denote the deleted absolute row sums of \( M \). Then all the eigenvalues of \( M \) are located in the union of \( n \) circles (i.e., \( \forall i \in 1, \ldots, n : \mu_i \in \mathcal{G}(M) \))
\[
\mathcal{G}(M) = \bigcup_{i=1}^{n} \{ z \in \mathbb{R} : \| z - M_{i,i} \|_1 \leq \rho_i(M) \}.
\]

**PROOF** A proof can be found in [25], pp. 344.

**LEMMA 4 (RAYLEIGH–RITZ)** Let \( M \in \mathbb{M}^n \) be Hermitian and \( x \in \mathbb{R}^n \). Let \( \mu_{\text{max}} \) and \( \mu_{\text{min}} \) denote the maximum and the minimum eigenvalue of \( M \), respectively. Then
\[
\mu_{\text{min}}\|x\|_2^2 \leq x^TMx \leq \mu_{\text{max}}\|x\|_2^2.
\]

**PROOF** A proof can be found in [25], pp. 176.

**THEOREM 2** Suppose Assumptions 1 and 2 hold. The two-impulse trajectory of the CS subject to \( r_i \) at \( t_i \) and \( r_{i+1} \) at \( t_{i+1} \) is bounded by a sphere centered at the origin with a radius of \( \delta_{i+1,i} \), i.e., \( \forall t \in \mathcal{T}_{i+1} - \mathcal{T}_i : \|r\|_2 \leq \delta_{i+1,i} \), such that
\[
\delta_{i+1,i} = \sigma(\Delta t_{i+1,i})\sqrt{\|r_i\|_2^2 + \|r_{i+1}\|_2^2} \quad \text{ (22)}
\]
\[
\sigma(t) = \begin{cases} 1 & t \in (0, 0.5\pi/\kappa] \\ 0.5\sqrt{2}\sec(0.5\kappa t) & t \in [0.5\pi/\kappa, \pi/\kappa). \end{cases} \quad \text{ (23)}
\]

**PROOF** The two-impulse trajectory subjected to fixed initial \( (r_i) \) and final \( (r_{i+1}) \) positions with a fixed time of flight \((\Delta t_{i+1,i})\), has a position vector \( r \) at \( t \) that is introduced previously in (10). From (10), the squared 2-norm of \( r \) can be written as
\[
\|r\|_2^2 = r^TR = \begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}^T\tilde{P}\begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix} \quad \text{ (24)}
\]

From Lemma 4 the above identity has the following upper bound:
\[
\lambda_{\text{max}}(t)\|\begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}\|_2^2 = \lambda_{\text{max}}(t)(\|r_i\|_2^2 + \|r_{i+1}\|_2^2) \quad \text{ (25)}
\]

in which \( \lambda_{\text{max}}(t) \) is the maximum eigenvalue of \( \tilde{P} \) at \( t \in \mathcal{T}_{i+1,i} \). Let us determine an upper bound for \( \lambda_{\text{max}}(t) \) in order to make the formulas independent from time. Suppose
\[
\sigma^2 = \sup_{t \in \mathcal{T}_{i+1,i}} \lambda_{\text{max}}(t) \quad \text{ (26)}
\]

So, using (24) to (26):
\[
\|r\|_2^2 \leq \sigma^2(\|r_i\|_2^2 + \|r_{i+1}\|_2^2) = \delta_{i+1,i}^2 \quad \text{ (27)}
\]

From Facts 1 and 2 it can be concluded that \( \sigma^2 = 1 \) at \( \Delta t_{i+1,i} \leq (\pi/\kappa)/2 \), since at the initial and final positions all of the positive eigenvalues reach unity which is the solution of (26). At \((\pi/\kappa)/2 < \Delta t_{i+1,i} < \pi/\kappa, \sigma^2 > 1 \) and is equal to its extremum value at \( t = \Delta t_{i+1,i}/2 \). Lemma 3 provides an upper bound on the eigenvalues at \( t = \Delta t_{i+1,i}/2 \) in which from Fact 2, the upper boundary of Gerschgorin circle has also an extremum at \( t = \Delta t_{i+1,i}/2 \) and is equal to the following equation:
\[
\max_{k=1,\ldots,6} \{ \tilde{P}(k,k)(\Delta t_{i+1,i}/2) + \rho_k(\tilde{P}(\Delta t_{i+1,i}/2)) \}
\]
\[
= \frac{1}{2}\sec^2\left(\frac{\kappa}{2}\Delta t_{i+1,i}\right) \quad \text{ (28)}
\]

Thus, at \((\pi/\kappa)/2 < \Delta t_{i+1,i} < \pi/\kappa, \sigma(\Delta t_{i+1,i}) = 0.5\sqrt{2}\sec(0.5\kappa \Delta t_{i+1,i}) \), and the theorem can be proved considering this result by taking the square root of (27).

Equation (23) of Theorem 2 checks that even if the true anomaly of the TS has been changed less than 90° or more. Note that the TS can be just a reference for the coordination of the CS and can be assumed a virtual point of reference. Thus, without loss of generality, one can assume \( \|r_{i+1}\| = 0 \) which concludes that for a spacecraft rendezvous under Assumption 1, if the decision variable \( \Delta t_{i+1,i} \) be considered less than \( 0.5\pi/\kappa \), then the spacecraft will not increase its distance from the destination. If Assumption 1 approximately holds for real case applications, this kind of maneuver can assure that the approximation error would not grow.
Both upper and lower bounds for $s^Tr$ can restrict the CS’s trajectory to lie inside or outside a cone with the apex on the origin. First, consider a special finite form of the Jensen’s inequality [26] in which the proof is omitted.

**Lemma 5 (Jensen)** Let $s, r \in \mathbb{R}^n$, and $s \succeq 0$. Consider a real function $f : \mathbb{R} \rightarrow \mathbb{R}$. If $f$ is convex [27] then

$$f \left( \frac{s^Tr}{1^Ts} \right) \leq \sum_{i} s_{(i)} f(r_{(i)}) \frac{1^Ts}{1^Ts}$$  \hspace{1cm} (30)

and if $f$ is concave ($-f$ is convex) then the above inequality holds with a change in its direction.

From the Jensen’s inequality in Lemma 5, upper and/or lower bounds can be found for $s^Tr$ by choosing any convex (or concave) function $f$. Suppose $f$ be a convex nondecreasing invertible function (for example $f(x) = x^t, y \geq 1$). Then inequality (30) simplifies to

$$s^Tr \leq (1^Ts)^{-1} \left[ \sum_{i} s_{(i)} f(r_{(i)}) \right].$$  \hspace{1cm} (31)

Inequality (31) defines an upper bound for $s^Tr$, if some priory knowledge about the $r$ exists. For example, suppose it is known that the elements of $r$ are bounded above, i.e., $r \leq r^+$. So, from the nondecreasing property of $f$ we have $f(r_{(i)}) \leq f(r^+_{(i)})$. Thus,

$$s^Tr \leq (1^Ts)^{-1} \left[ \sum_{i} s_{(i)} f(r^+_{(i)}) \right].$$  \hspace{1cm} (32)

For concave nondecreasing invertible functions (for example $f(x) = \log_{\gamma}(x), y \geq 1$) a similar result is obtainable in which the lower bound of $r$ should be used, as $r \geq r^-$. If only the direction of $s = \|s\|_2 e_s$ ($\|e_s\|_2 = 1$) is important, without loss of generality, $s$ can be defined as $s = (\sum_{i} e_i) e_s$, so we can eliminate the terms of $1^Ts$ in the previous inequalities. Therefore, (31) can be reduced to

$$s^Tr \leq (1^Ts)^{-1} \left[ \sum_{i} s_{(i)} f(r_{(i)}) \right].$$  \hspace{1cm} (33)

Considering $f(x) = \|x\|$ as a convex function, Lemma 5 leads to the following theorem:

**Theorem 3** Let $s \in \mathbb{R}^3$ be a unit vector such that $e_s \succeq 0$. Assume that in the time interval of $T$, it is known that the 2-norm of the CS’s trajectory is bounded below such that $\rho^- = \inf_{r \in T} (\|r\|_2)$. Moreover, assume that the maximum distance of an element of $r$ from the origin is bounded above by $\rho^+ = \sup_{r \in T} (\|r\|_1)$. Then, $\theta = \angle(s, r)$ is restricted by the following inequality:

$$\|\cos \theta\|_1 \leq \min \left\{ 1, \frac{e^T r^+}{\rho^-} \right\}.$$  \hspace{1cm} (34)

**Proof** From the result of Lemma 5, considering $f$ to be any norm function $f(x) = \|x\|$, since $x$ is a scalar variable, the function equivalently reduces to $f(x) = \|x\|_1$. Thus, we have

$$\left\| \frac{e^T r}{1^Te_s} \right\|_1 \leq \frac{\sum_{i} e_{(i)} \|r_{(i)}\|_1}{1^T e_s}.$$  \hspace{1cm} (35)
such that 

\[
\|e_i^T r\|_1 \leq \sum e_{i(i)}\|r_{0(i)}\|. 
\]

The left-hand side of (36) is equal to \(\|e_i^T r\|_1 = \|r\|_2 \cos \theta\|_1\). Therefore, taking \(\|r\|_2\) into the denominator of the right-hand side of (36) and considering the upper bound of the right-hand side by replacing the supremum of nominator and the infimum of denominator over time, inequality (34) is proved. \(\blacksquare\)

Conic bounds introduced in Theorem 3 particularly may be applicable in spacecraft formation flying as restricts the CS in a special TS’s field of view which can be considered as a beneficial property for missions equipped by vision-based relative navigation sensors.

**Remark 6** Theorem 3 defines a restricting cone which is also a function of \(e_i\). In order to obtain the smallest set for \(\theta\), the vector \(e_i\) in the inequality (34) should be selected in order to minimize the term \(\sum e_{i(i)}\|\rho_{0(i)}^+\|_1\). Let \(i^*\) be the non-trivial solution of \(\min_{\rho_{0(i)}^+}(\|\rho_{0(i)}^+\|_1) = \min_{\|\rho_{0(i)}^+\|_1}(\|\rho_{0(i)}^+\|_1)\). Then, the vector \(e_i\) that is constructed as \(e_{i(i^*)} = 1\) and \(e_{i(\neq i^*)} = 0\) yields the smallest set for \(\theta\) (and consequently for \(r\)).

### B. Impulse Design Under PC

In this section, it is shown how the impulse positions and times can be determined in order to satisfy the PCs subject to the CW system. Some boundaries are introduced regarding the Jordan–Brouwer separation theorem, in which the satisfaction of the constraints is guaranteed by considering those boundary sets.

The two-impulse trajectory of the CS subject to the initial and final positions is determined by two parameters \(t_1\) and \(\Delta t_{i+1,i}\), that is formulated in (8). The set \(R_{i,i+1}(t)\) contains all position vectors which can be reached at \(t = T_{\Delta t_{i+1}}\) by any \(\Delta t_{i+1,i} \in T_{\pi/k}\). Now, consider the following set which is used further:

\[
Q_{i,i+1} = \bigcup_{t \in T_{\pi/k}} R_{i,i+1}(t). 
\]

Set \(Q_{i,i+1}\) is a 3-D surface in which subject to \(r = r_i\) at \(t_i\) and \(r = r_{i+1}\) at \(t_{i+1}\) as the trivial constraints (using two impulses), \(r \in Q_{i,i+1}\) at \(\forall t \in \Delta t_{i+1,i}\). Roughly speaking, the CS’s trajectory entirely lies in the set \(Q_{i,i+1}\).

**Lemma 6 (Jordan–Brouwer)** Let \(\partial S\) be a connected surface that is closed as a subset of \(\mathbb{R}^3\). Then \(\mathbb{R}^3 - \partial S\) has exactly two connected components (\(S\) and its complement \(\mathbb{R}^3 \setminus S\)) whose common boundary is \(\partial S\).

**Proof** See proof in [28, Th. 4.16]. \(\blacksquare\)

A result of Lemma 6 is that every continuous path connecting a point in \(S\) to a point in \(\mathbb{R}^3 \setminus S\) intersects somewhere with \(\partial S\). Thus, the following proposition can be directly concluded.

**Proposition 2** Let \(t \in T_{\pi/k}\) and \(r_i \in \mathbb{R}^3\) be reachable in \(\Gamma_i^T(2)\) at \(t = T_{\pi/k}\). Let \(S \subset \mathbb{R}^3\) be a set of position vectors with \(\partial S \subset \mathbb{R}^3\) as a boundary such that \(\partial S\) is a connected closed surface and \(r_i \notin \partial S\). If the boundary of \(S\) is unreachable in \(\Gamma_i^T(2)\) at any \(t \in T_{\pi/k}\), then any member of \(S\) is unreachable in \(\Gamma_i^T(2)\) at any \(t \in T_{\pi/k}\), i.e.,

\[
\partial S \cap Q_{i,i+1} = \emptyset \Rightarrow S \cap Q_{i,i+1} = \emptyset. 
\]

**Proof** The set \(Q_{i,i+1}\) is a continuous surface in which the assumption \(r_i \notin S\) states that the surface has at least one point in \(\mathbb{R}^3 \setminus S\). So, according to Lemma 6, if there exists a \(t \in T_{\pi/k}\) in which \(\partial S \cap R_{i,i}(t) \neq \emptyset\) then the curve intersects by \(\partial S\) somewhere. Therefore, its contrapositive is equivalently true. \(\blacksquare\)

The above-mentioned results can be subjected to the inequality PCs of type (13) to conclude the following corollary.

**Corollary 1** Suppose Assumptions 1 and 2 hold. Assume it is known that a position vector \(r_s \in \{r \in \mathbb{R}^3\} = \|r - \hat{r}_i\|_2 \leq \rho_i^f\) is reachable in \(\Gamma_i^T(2)\) at any \(\forall t \in T_{\pi/k}\). An inequality PC of type (13) is satisfied in \(\Gamma_i^T(2)\) if any \(r \in \mathbb{R}^3\) is unreachable in \(\Gamma_i^T(2)\) at \(\forall t \in T_{\pi/k}\).

**Proof** It can be simply proved by substituting \(S = \{r \in \mathbb{R}^3\} = \|r - \hat{r}_i\|_2 \leq \rho_i^f\) and consequently \(\partial S = \{r \in \mathbb{R}^3||r - \hat{r}_i||_2 = \rho_i^f\}\) in Proposition 2. \(\blacksquare\)

**Remark 7** A direct result from the above corollary can be presented for special kinds of inequality PCs. Suppose Assumptions 1 and 2 hold. Assume it is known that a position vector \(r_s \in \{r \in \mathbb{R}^3\} = \|r - \hat{r}_i\|_2 \leq \rho_i^f\) is reachable in \(\Gamma_i^T(2)\) at \(\forall t \in T_{\pi/k}\). Consider an inequality PC of type (13) such that \(\rho_i^f \rightarrow \infty\) and \(t = \pi/k\). Then, the PC is satisfied in \(\Gamma_i^T(2)\) if \(\forall t \in \mathbb{R}^3\) is unreachable in \(\Gamma_i^T(2)\) at \(\forall t \in T_{\pi/k}\).

**Remark 8** As an example, the vector \(r_i\) that is used in Corollary 1 can be selected to be the position at \(t_1 = 0\), \(r_s = r_i\), if \(r_1\) is located inside \(\hat{R}_i\). This condition checks that if the CS is initially located inside or outside the set \(\hat{R}_i\).

Another result can be obtained from Lemma 6 which helps to find those impulse times (associated with fixed impulse positions) that the corresponding trajectory satisfies the constraints.

**Proposition 3** Suppose Assumptions 1 and 2 hold. Assume it is known that a point \(r_s \in \mathbb{R}^3\) is reachable in \(\Gamma_i^T(2)\) at \(t \in [t_1, t_{i+1}]\) corresponding to \(\Delta t_{i+1,i} = \Delta t_s\). Consider two time intervals of \(\Delta t_a\) and \(\Delta t_b\) such that \(\Delta t_a < \Delta t_b\) and any point in \(S\) is unreachable in \(\Gamma_i^T(2)\) at \(t \in [t_1, t_{i+1}]\) corresponding to both \(\Delta t_{i+1,i} = \Delta t_a\) and \(\Delta t_{i+1,i} = \Delta t_b\). Then, the following statements hold.

1) If \(\Delta t_a \in [\Delta t_a, \Delta t_b]\) then any \(r \in \mathbb{R}^3\) is unreachable in \(\Gamma_i^T(2)\) at any \(t \in [t_1, t_{i+1}]\) corresponding to \(\Delta t_{i+1,i} \in [0, \Delta t_a] \cup [\Delta t_b, \pi/k]\).
2) If $\Delta t_s \in [0, \Delta t_a] \cup [\Delta t_b, \pi/k]$ then any $r \in S$ is unreachable in $\Gamma_{2+1}^1(2)$ at any $t \in [t_i, t_{i+1}]$ corresponding to $\Delta t_{i+1} \in [\Delta t_a, \Delta t_b]$. 

**Proof** Denote the set $R_{i,i+1}(t)$ corresponding to $\Delta t_{i+1,i} = \Delta t_{a/b}$ by $R_{i,a/b}(t)$. From Theorem 1 (the time uniqueness property), we know that $R_a(t)$ and $R_b(t)$ do not intersect, unless at $\Delta t_{i+1,i} = 0$. The union of $R_a(t)$ and $R_b(t)$ can construct a boundary for a set of position vectors in $Q_{i,i+1}$ which we refer to it by $\partial \tilde{R}_{a/b}(t) \subset Q_{i,i+1}$ (and its interior by $\tilde{R}_{a/b} \subset Q_{i,i+1}$). According to Proposition 2, since $R_a(t)$ and $R_b(t)$ do not intersect by $S$, the set $\tilde{R}_{a/b}$ have no intersections by $S$, i.e., $\tilde{R}_{a/b} \cap S = \emptyset$, if $r \notin \tilde{R}_{a/b}$ (and $S \subset \tilde{R}_{a/b}$ if $r \in \tilde{R}_{a/b}$). Roughly speaking $\tilde{R}_{a/b}$ is the set of those trajectories that $\Delta t_{i+1,i} \in [\Delta t_a, \Delta t_b]$. Item (ii) can be proved similarly by considering the fact that if $\Delta t_s \leq \Delta t_a$ or $\Delta t_b \leq \Delta t_s$, then $R_s(t) \cap \tilde{R}_{a/b} = \emptyset$. 

It is worth mentioning that an $n$-impulse mission can be divided into a number of $n-1$ two-impulse missions and consequently any result about the two-impulse trajectories can be used for $n$-impulse cases just by reusing the result for $n-1$ times.

**IV. NUMERICAL EXAMPLES**

**A. Approximate CFK**

In this section the impulsive approximate CFK problem is analyzed by the use of numerical analysis. The CFK problem seeks for control solutions in order to keep the CS on a circular path around the TS. The term “approximate” is used to show that the CS is not going to lie on an exact circle (since is impossible with finite number of impulses), but should lie in a ring that is restricted by two circles; the approach circle ($\|r\| = \rho'$), and the keep-out circle ($\|r\| = \rho''$).

A finite number of polar grids are used for numerical computations (see Fig. 3). The time is also approximated by discrete time instances. The numerical analysis decreases the accuracy of results based on how many nodes are used. However, increasing the number of nodes confronts the problem with the curse of dimensionality.

A 2-D problem in $xy$ plane is used as an example. The time step in the simulation is 10 s and the TS is located at a circular orbit with an altitude of 400 km. The radius of the approach and the keep-out circles are $\rho' = 0.9$ km and $\rho'' = 1.1$ km, respectively. The position vectors for the impulses are considered to be $r_i = [\cos \beta_i \ \sin \beta_i \ 0]^T$, $i = 2, 3, \ldots, n-1$. It is assumed that $\beta_i \in \mathbb{N}$, $\beta_i = 360^\circ$. Fig. 4 shows those values of $\beta_2$ and $t_2$ in which the CS’s trajectory in $\Gamma_2^1(2)$ satisfies the constrained problem is highlighted in gray. The area between dashed lines contains those positions that are unreachable in $\Gamma_2^1(2)$ at any $t \in T_{t_i}$ corresponding to any $t_2 \in T_{\pi/k}$ subject to the problem constraints. Therefore, the positions that are located between the dashed lines cannot be reached by two impulses and may become reachable by three impulses or more. Fig. 5 shows four typical trajectories corresponding to the points that are specified in Fig. 4.

Fig. 6 shows those values of $\beta_2$ that are reachable from $i = 1$ such that the $\beta_2 = \beta_0 = 0$ is reachable from $i = 2$ subject to the problem constraints. The highlighted areas of Fig. 6 can be divided into two categories. Consider the first category to be the union of two separated areas with $\Delta t_{3,2} \leq 740$ s at the left side (up and down) of the figure, and the second category to be the area with $1570 \leq \Delta t_{3,2} \leq 1860$ s at the middle of the figure. The
first category contains those trajectories that satisfy the constraints of the problem while the CS do not visit all values of the polar angles $0^\circ \leq \beta \leq 360^\circ$ with respect to the TS. The second category includes those trajectories that the CS visits all values of $\beta$ and satisfy the constraints of the problem.

Two unreachable areas are distinguished in Fig. 6; the unreachable two- and three-impulse area, and the unreachable two-impulse area. The former includes the same unreachable points that are defined in Fig. 6, and the latter contains the new results. The two- and three-impulse unreachable points are those values of $\beta_2$ that cannot be reached from $i = 1$. The two-impulse unreachable area are those values of $\beta_2$ that $\beta_3 = \beta_4 = 0$ becomes unreachable from $i = 2$.

Fig. 7 shows three typical trajectories corresponding to the points that are specified in Fig. 6.

B. Collision-Free Maneuver

In this section the impulsive CFM is analyzed numerically. The CFM is achieved by implementing those impulse positions at $i$ and $i+1$ such that a sphere area (problem constraint) should not be violated for every value of $\Delta t_{i+1, i}$. In this example, fixed impulse position is found such that the CS accomplish its mission while the constraints be satisfied independent from the transfer times. From Propositions 2 and 3, we know that if $r_i$ and $r_{i+1}$ be considered such that the CS’S trajectory corresponding to $\Delta t_{i+1, i} = 0$ and $\Delta t_{i+1, i} = \pi/\kappa$ do not collide with the the constraint sphere, then the CFM is solved as well.

The trajectory of the CS corresponding to $\Delta t_{i+1, i} = 0$ is a straight line from $r_i$ to $r_{i+1}$. The trajectory of the CS assuming $\Delta t_{i+1, i} = \pi/\kappa$ becomes singular, therefore, we approximate its locus by $\Delta t_{i+1, i} = \pi/\kappa - \epsilon$, such that $\epsilon > 0$ is a small value to be determined.

As an example, consider a 2-D problem in $xy$ plane. Suppose the CS is located initially at $r_1 = [1 \ 0 \ 0]^T$ km. The problem asks to find those values of $r_i$, $i \geq 2$, in which a CFM can be accomplished such that the CS observes all the polar angles with respect to the TS and a keep-out circle with a radius of $\rho' = 0.5$ km be satisfied.

Assuming $\epsilon = 1$ s, Fig. 8 shows two different choices of $r_2 = [-0.1 \ 0 \ 0]^T$ km and $r_2 = [0 \ 1 \ 0]^T$ km beside the reachable area in $\Gamma_i^2(2)$ starting from $r_1$ and ending in $r_2$. In Fig. 8, it is shown that $r_2 = [-0.1 \ 0 \ 0]^T$ km leads to a CFM since both trajectories corresponding to $\Delta t_{i+1, i} = 0$ and $\Delta t_{i+1, i} = \pi/\kappa - \epsilon$ do not collide with the keep-out circle. Instead, selecting the second impulse position to be $r_2 = [0 \ 1 \ 0]^T$ km the CFM cannot be constructed.

Fig. 9 shows four impulse positions that can be considered as a solution to our CFM problem. Taking these four positions, for any time intervals $\Delta t_{i+1, i} \in T_{\pi/\kappa}, i = 1, 2, 3,$
TABLE I
Upper Norm Bounds $\delta_{i+1,1}$ (Obtained From Theorem 2), for the Two-Impulse Numerical Examples of Section IV

| No. | $\beta_i$ (deg.) | $\beta_{i+1}$ (deg.) | $\Delta t_{i+1,1}$ (s) | $\delta_{i+1,1}$ (km) |
|-----|------------------|----------------------|------------------------|------------------------|
| 1   | 5                | 20                   | 200                    | $\sqrt{2}$             |
| 2   | 20               | 340                  | 200                    | $\sqrt{2}$             |
| 3   | 20               | 600                  | 1000                   | $\sqrt{2}$             |
| 4   | 20               | 200                  | 200                    | $\sqrt{2}$             |
| 5   | 0                | 20                   | 200                    | $\sqrt{2}$             |
| 6   | 20               | 200                  | 200                    | $\sqrt{2}$             |
| 7   | 20               | 200                  | 1700                   | $\approx 1.19\sqrt{2}$ |
| 8   | 20               | 180                  | 1700                   | $\approx 1.19\sqrt{2}$ |
| 9   | 0                | 0                    | 1700                   | $\approx 1.19\sqrt{2}$ |
| 10  | 0                | 270                  | 1700                   | $\approx 1.19\sqrt{2}$ |
| 11  | 0                | 270                  | 1700                   | $\approx 1.19\sqrt{2}$ |
| 12  | 0                | 1428                 | 1700                   | $\approx 1.19\sqrt{2}$ |
| 13  | 0                | 1428                 | 1700                   | $\approx 1.19\sqrt{2}$ |
| 14  | 0                | 1428                 | 1700                   | $\approx 1.19\sqrt{2}$ |

Note: $r_i = [\cos \beta_i \sin \beta_i 0]^T$

TABLE II
Conic Bounds, $\| \cos \theta \|_1 \leq c_\theta$ (Obtained From Theorem 3), for the Two-Impulse Numerical Examples of Section IV

| No. | $c_\theta$ (deg.) | $\rho^+$ (km) | $\rho^-$ (km) | $c_\theta$ (km) |
|-----|-------------------|---------------|---------------|-----------------|
| 1   | 0                 | 0.9           | 1.0           | 5/9             |
| 2   | 0                 | 0.9           | 1.0           | 5/9             |
| 3   | 0                 | 0.9           | 1.0           | 5/9             |
| 4   | 0                 | 0.9           | 1.0           | 5/9             |
| 5   | 0                 | 0.9           | 1.0           | 5/9             |
| 6   | 0                 | 0.9           | 1.0           | 5/9             |
| 7   | 0                 | 0.9           | 1.0           | 5/9             |
| 8   | 0                 | 0.9           | 1.0           | 5/9             |
| 9   | 0                 | 0.9           | 1.0           | 5/9             |
| 10  | 0                 | 0.9           | 1.0           | 5/9             |
| 11  | 0                 | 0.9           | 1.0           | 5/9             |
| 12  | 0                 | 0.9           | 1.0           | 5/9             |
| 13  | 0                 | 0.9           | 1.0           | 5/9             |
| 14  | 0                 | 0.9           | 1.0           | 5/9             |

Note: $\rho^+$ and $\rho^-$ are the estimated values of $\rho^+$ and $\rho^-$. The keep-out circle is not violated and a periodic motion around the TS can be accomplished as well.

C. Discussions

In this section, some relations between Theorems 2 and 3 with the numerical examples of Section IV are discussed in more detail.

Theorem 2 introduces an upper norm bound for CS’s trajectory. This upper norm bound is tabulated for every trajectory example of Section IV in Table I. In Figs. 5, 7–9, the impulse positions are located on a unit circle, therefore if $\Delta t_{i+1,1} < (\pi/\kappa)/2 \approx 1428$ s then the upper bound (according to Theorem 2) is simply $\sqrt{2}$ (such as numbers 1–8 and 11–14 in Table I). If $\Delta t_{i+1,1} > (\pi/\kappa)/2 \approx 1428$ s then the upper bound exceeds $\sqrt{2}$ and is computed by (22) and (23) (such as numbers 9 and 10 in Table I).

Theorem 3 introduces conic bounds on the CS’s trajectory. Using the optimal index value $i^*$ (which is discussed in Remark 6), the conic bounds for every trajectory example of Section IV is presented in Table II. Each number corresponds to a two-impulse trajectory which is previously defined in Table I. For numbers 3, 4, and 9–14, the upper bound of $\| \cos \theta \|_1$ (i.e., $c_\theta$) are equal to 1 that are obtained using (34). Therefore, the conic bound of Theorem 3 is useless for these scenarios. For numbers 1, 2, and 5–8, the upper bound has a value less than unity which restricts the CS’s trajectory to lie outside a double cone with an obtuse aperture.

In Fig. 10, a 3-D example is used for the numerical evaluation of the trajectory upper-bound found in Theorem 2. The TS is located at the origin with an altitude of 400 km above earth. The initial location of the CS is formulated as $r_1 = \| r_1 \|_2 [\cos(\phi) \cos(\psi) \cos(\phi) \sin(\psi) \sin(\phi)]^T$ in which the simulations are done for $\phi \in [-\pi, \pi]$, $\psi = [0, 2\pi]$, and $t = [0, t_2]$, and the maximum reached distance defined as $\max_\phi,\psi, t (\| r_1 \|_2)$ is evaluated for each amount of $\| r_1 \|_2$ from 0.1 to 5. This method is accomplished separately for $t_2 = 0.5\pi/\kappa$ and $t_2 = 0.75\pi/\kappa$.

V. CONCLUSION

The relative spacecraft motion has been analyzed under PC using the CW equations. Initially, the time uniqueness of the spacecraft’s trajectory is analyzed under assumptions and the main result is proved in a theorem. The spectral analysis of the CW equations demonstrates some facts which are used to determine upper norm bounds for the spacecraft position between adjacent impulses. Moreover, a finite form of the Jensen’s inequality is implemented to develop a conic bound for the spacecraft path which needs additional priory estimations about the position time-history. Furthermore, it is shown that the unreachability of a set of continuous position vectors can be proven by considering the unreachability of some boundary positions. Finally, two numerical examples in the $xy$ plane are presented. The first example is an approximate CFK which seeks those impulse positions and times such that the chaser spacecraft’s trajectory lies in a ring. The second example is a CFM in which the impulse positions are found such that the chaser spacecraft does not violate a keep-out circle with any choice of impulse times, i.e., attaining a set of trajectories that are robust in terms of impulse times.

ACKNOWLEDGMENT

The author would like to thank Dr. Nima Assadian for his helpful advice.
REFERENCES

[1] W. H. Clohessy and R. S. Wiltshire
Terminal guidance system for satellite rendezvous
"J. Aerosp. Sci.," vol. 27, no. 9, pp. 653–658, 1960, doi: 10.2514/8.8704.

[2] T. E. Carter
Optimal impulsive space trajectories based on linear equations
"J. Optim. Theory Appl.," vol. 70, no. 2, pp. 277–297, 1991, doi: 10.1007/BF00940627.

[3] T. E. Carter and J. Briet
Linearized impulsive rendezvous problem
"J. Optim. Theory Appl.," vol. 86, no. 3, pp. 553–584, 1995, doi: 10.1007/BF02191259.

[4] J. Sullivan, S. Grimberg, and S. D’Amico
Comprehensive survey and assessment of spacecraft relative motion dynamics models
"J. Guid., Control, Dyn.," vol. 40, no. 8, pp. 1837–1859, 2017, doi: 10.2514/1.G002309.

[5] H. Cho, S. Y. Park, H. E. Park, and K. H. Choi
Analytic solution to optimal reconfigurations of satellite formation flying in circular orbit under $J_2$ perturbation
"IEEE Trans. Aerosp. Electron. Syst.," vol. 48, no. 3, pp. 2180–2197, Jul. 2012, doi: 10.1109/TAES.2012.6237587.

[6] J. E. Prussing and J. Chiu
Optimal multiple-impulse time-fixed rendezvous between circular orbits
"J. Guid., Control, Dyn.," vol. 9, no. 1, pp. 17–22, 1986, doi: 10.2514/3.20060.

[7] L. Riggio and S. D’Amico
Optimal impulsive closed-form control for spacecraft formation flying and rendezvous
In "Proc. Amer. Control Conf.," 2016, pp. 5854–5861.

[8] R. Serra, D. Arzelier, and A. Rondepierre
Analytical solutions for impulsive elliptic out-of-plane rendezvous problem via primer vector theory
"IEEE Trans. Control Syst. Technol.," vol. 26, no. 1, pp. 207–221, Jan. 2018, doi: 10.1109/TCST.2017.2656022.

[9] H. Gao, X. Yang, and P. Shi
Multi-objective robust $H_{\infty}$ control of spacecraft rendezvous
"IEEE Trans. Control Syst. Technol.," vol. 17, no. 4, pp. 794–802, Jul. 2009, doi: 10.1109/TCST.2008.2012166.

[10] X. Tian and Y. Jia
Analytical solutions to the matrix inequalities in the robust control scheme based on implicit Lyapunov function for spacecraft rendezvous on elliptical orbit
"IET Control Theory Appl.," vol. 11, no. 12, pp. 1983–1991, 2017, doi: 10.1049/iet-cta.2017.0176.

[11] M. Mesbahi and F. Hadaegh
Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching
"J. Guid., Control, Dyn.," vol. 24, no. 2, pp. 369–377, 2001, doi: 10.2514/2.4721.

[12] D. Taur, V. Coverstone-Carroll, and J. E. Prussing
Optimal impulsive time-fixed orbital rendezvous and interception with path constraints
"J. Guid., Control, Dyn.," vol. 18, no. 1, pp. 54–60, 1995, doi: 10.2514/3.56656.

[13] K. M. Soileau and S. A. Stern
Path-constrained rendezvous: Necessary and sufficient conditions
"J. Spacecraft Rockets," vol. 23, no. 5, pp. 492–498, 1986, doi: 10.2514/3.25835.

[14] M. Milan, N. Petit, and R. Murray
Constrained trajectory generation for micro-satellite formation flying
In "Proc. AIAA Guid., Navigation, Control Conf. Exhibit," 2001, pp. 328–333.

[15] R. W. Beard and F. Y. Hadaegh
Finite thrust control for satellite formation flying with state constraints
In "Proc. Amer. Control Conf.," 1999, pp. 4383–4387.

[16] A. Weiss, M. Baldwin, R. S. Erwin, and I. Kolmanovsky
Model predictive control for spacecraft rendezvous and docking: Strategies for handling constraints and case studies
"IEEE Trans. Control Syst. Technol.," vol. 23, no. 4, pp. 1638–1647, Jul. 2015, doi: 10.1109/TCST.2014.2379639.

[17] J. Chen, D. Sun, J. Yang, and H. Chen
Leader-follower formation control of multiple non-holonomic mobile robots incorporating a receding-horizon scheme
"Int. J. Robot. Res.," vol. 29, no. 6, pp. 727–747, 2010, doi: 10.1177/0278364909104290.

[18] B. Zhou, Q. Wang, Z. Lin, and G. Duan
Gain scheduled control of linear systems subject to actuator saturation with application to spacecraft rendezvous
"IEEE Trans. Control Syst. Technol.," vol. 22, no. 5, pp. 2031–2038, Sep. 2014, doi: 10.1109/TCST.2013.2296044.

[19] B. Zhou and J. Lam
Global stabilization of linearized spacecraft rendezvous system by saturated linear feedback
"IEEE Trans. Control Syst. Technol.," vol. 25, no. 6, pp. 2185–2193, Nov. 2017, doi: 10.1109/TCST.2016.2632529.

[20] A. Shakouri, M. Kiani, and S. H. Pourtakdoust
Covariance-based multiple-impulse rendezvous design
"IEEE Trans. Aerosp. Electron. Syst.," 2018, doi: 10.1109/TAES.2018.2882939.

[21] M. Brentari, S. Urbina, D. Arzelier, C. Louembet, and L. Zaccarian
A hybrid control framework for impulsive control of satellite rendezvous
"IEEE Trans. Control Syst. Technol.," pp. 1–15, 2018, doi: 10.1109/TCST.2018.2812197.

[22] L. A. Sobiesiak and C. J. Damaren
Lorentz-augmented spacecraft formation reconfiguration
"IEEE Trans. Control Syst. Technol.," vol. 24, no. 2, pp. 514–524, Mar. 2016, doi: 10.1109/TCST.2015.2461593.

[23] Y. Luo, G. Tang, and Y. Lei
Optimal multi-objective linearized impulsive rendezvous
"J. Guid., Control, Dyn.," vol. 30, no. 2, pp. 383–389, 2007, doi: 10.2514/1.21433.

[24] C. Wen, Y. Zhao, P. Shi, and Z. Hao
Orbital accessibility problem for spacecraft with a single impulse
"J. Guid., Control, Dyn.," vol. 37, no. 4, pp. 1260–1271, 2014, doi: 10.2514/1.62629.

[25] R. A. Horn and C. R. Johnson
"Matrix Analysis," 1st ed. New York, NY, USA: Cambridge Univ. Press, 1985.

[26] G. H. Hardy, J. E. Littlewood, and G. Polya
"Inequalities." New York, NY, USA: Cambridge Univ. Press, 1952.

[27] S. Boyd and L. Vandenberghe
"Convex Optimization." Cambridge, U.K.: Cambridge Univ. Press, 2004.

[28] S. Monteil and A. Ros
"Curves and Surfaces," 2nd ed. Providence, RI, USA: Amer. Math. Soc., 2005.

Author’s photograph and biography not available at the time of publication.