GENERALIZED EQUIVARIANT COHOMOLOGIES OF SINGULAR TORIC VARIETIES

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Abstract. A complex $n$-dimensional toric variety is equipped with a natural action of compact torus $T^n$. In this paper, we introduce the notion of an almost simple polytope and compute the $T^n$-equivariant K-theory, cobordism theory and cohomology theory of toric varieties associated to almost simple polytopes with rational coefficients. Moreover, we introduce the concept of divisive toric variety generalizing the similar concept for weighted projective spaces.

1. Introduction

A toric variety of complex dimension $n$ is a complex algebraic variety with an action of algebraic torus $(\mathbb{C}^\ast)^n$ having an open dense orbit, and it has been one of the main attraction in algebraic and symplectic geometry from 1970s. One of the reasons is their rich interaction with different fields of mathematics, such as representation theory and combinatorics. For instance, one may get a lattice polytope $P$ from a projective toric variety $X$ by the convexity theorem \cite{Ati82, GS82} and vice versa by the Delzant’s construction \cite{Del88, Gu94}. Moreover, the lattice points in $P$ give a weight decomposition of $H^0(X, \mathcal{L})$ as a torus representation, where $\mathcal{L}$ is a very ample line bundle over $X$.

Another interplay of toric varieties has been revealed from the theory of Newton–Okounkov bodies \cite{KK12, LM09}. The results of \cite{And13, HK15} show that a projective algebraic variety $X$ admits a flat degeneration to the toric variety $X_0$ over a corresponding Newton–Okounkov body of $X$, where $X_0$ is singular in general.

From the topological point of view, it is natural to ask how to compute invariants of a toric variety in terms of the associated combinatorics. There are vast literatures which discuss this question for several invariants. For example, we refer to \cite{Dan78, DJ91, Jur85, Mor93, BB00, BR98} for non-equivariant cohomology theories, and \cite{Bag07, VV03, HHRW16} for equivariant cohomology theories. However, most of the computations are focused on smooth toric varieties.

Turning the gear to singular toric varieties, the computation of invariants becomes much complicated. Indeed, their ordinary cohomology may not vanish in odd degrees in general, which does not simplify some spectral sequences such as Leray–Serre or Atiyah–Hirzebruch spectral sequences. More essential reasons can

2010 Mathematics Subject Classification. Primary 14F43, 55N91; Secondary 19L47, 57R85, 57R91.

Key words and phrases. generalized equivariant cohomology, cohomology, K-theory, complex cobordism, toric variety.
be found from the absence of nice enough cell structures on singular toric varieties. However, recent works of [HHH05], [HHRW16], [BFR09] and [BNSS] make a breakthrough in the computation of several cohomology theories.

The authors of [HHH05] considered \( G \)-invariant stratified spaces satisfying certain conditions, and developed the generalized GKM theory which generalizes many known results, see [HHH05] Remark 3.4. It provides a combinatorial way to describe complex oriented torus equivariant cohomologies, say \( E^*_T \) [May96, Chapter XIII]. Then, the results of [HHRW16] provide nice descriptions of \( E^*_T \) for divisive weighted projective spaces in terms of piecewise algebras. Though a divisive weighted projective space is a singular toric variety, it is equipped with a nice torus invariant stratification obtained by the structure of iterated Thom spaces, which allows to apply the theory of [HHH05]. For general singular toric varieties, it is hard to find such a nice stratification.

The main purpose of this paper is to overcome such a difficulty for certain class of singular toric varieties with the aid of orbifold vector bundles. Then we give a description of complex oriented generalized torus equivariant cohomologies, but sacrificing integral coefficients, namely, we consider \( E^*_T(\cdot) \otimes \mathbb{Z} \mathbb{Q} \) in general. Here, \( T \) stands for the compact torus acting on a toric variety \( X \) and \( E^*_T \) can be Borel equivariant cohomology \( H^*_T \), complex equivariant \( K \)-ring \( K^*_T \) following [Seg68], and equivariant complex cobordism ring \( MU^*_T \) following [LD70].

This paper is organized as follows. In Section 2 we employ orbifold \( G \)-bundles to extend the \( G \)-invariant stratification of [HHH05] to an orbifold stratification. Using the orbifold version of Thom classes and Euler classes, we verify how the generalized GKM theory of [HHH05] can be extended to orbifold stratifications.

Section 3 is devoted to give a combinatorial characterization of toric varieties for which our main results hold. Such a class of toric varieties may have arbitrary toric singularities beyond orbifold singularities. Here, we bring the idea of retraction sequence [BSS17] of a convex polytope, which is a sequence of fixed points of \( X \) having orbifold singularities in certain invariant subspaces of \( X \). Then, the application of generalized GKM theory of [HHH05] naturally follows from the discussion of Section 2.

In Section 4 we summarize the concept of piecewise algebras associated to a fan, which is studied in [HHRW16] Section 4. Then, we establish Theorem 4.3 which describes \( E^*_T(X) \otimes \mathbb{Q} \) where \( X \) is a singular toric variety discussed in Section 3.

Finally, generalizing the idea of divisive weighted projective spaces, we introduce the notion of a divisive toric variety in Section 5 to compute generalized equivariant cohomologies with integer coefficients. Such a variety satisfies all hypothesis which are necessary to apply the main theorem of [HHRW16]. The conclusion is stated in Proposition 5.3.

**Acknowledgements.** The first author would like to thank the international relation office of IIT-Madras. The second author has been supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B0704840). He also has been supported by the POSCO Science Fellowship of POSCO TJ Park Foundation. The authors would like to express sincere gratitude to Tony Bahri for his encouragement and helpful suggestions.
2. Generalized Equivariant cohomologies over \( \mathbb{Q} \)

The goal of this section is to extend the notion of generalized equivariant cohomologies to orbifold stratifications. Let \( G \) be a topological group, and \( X \) a Hausdorff \( G \)-space which may have a \( G \)-invariant stratification. Let \( E^*_G \) be a \( G \)-equivariant cohomology theory as defined in [May96] Chapter XIII associated to a ring \( G \)-spectrum \( E \) which leads \( E^*_G(X) \) to a \( \mathbb{Z} \)-graded commutative ring. Throughout this paper, the generalized equivariant cohomology theory is considered over \( \mathbb{Q} \), i.e., \( E^*_G(X) \otimes \mathbb{Q} \), unless it is specified. For simplicity, we use \( E^*_G(X) \) in place of \( E^*_G(X) \otimes \mathbb{Q} \).

Let \( \xi : V \to B \) be an \( E \)-orientable \( G \)-vector bundle (see [May96] p. 177). Let \( K \) be a finite group acting linearly on \( V \), which commutes with \( G \)-action on \( V \), preserves the fiber and \( E \)-orientation of \( \xi \). Then, one may consider the induced fiber bundle \( \xi^K : V/K \to B \), which we call a simple orbifold \( G \)-bundle. The associated disc bundle \( D(V) \to B \) and the sphere bundle \( S(V) \to B \) are invariant under \( K \)-action, as it acts linearly. Hence, one can define a \( q \)-disc bundle \( D(V/K) = D(V)/K \to B \) and \( q \)-sphere bundle \( S(V/K) = S(V)/K \to B \) in the usual manner, which yield the Thom space \( Th(V/K) := D((V/K)/S(V/K)) \) of \( \xi^K \) and the map

\[
\tilde{\xi}^K : Th(V/K) \to B.
\]

Similar to the usual \( G \)-vector bundle, one can define \( E \)-orientations or Thom classes of a simple orbifold \( G \)-bundle. Briefly, an element \( u \in E^*_G(Th(V/K)) \) is called an \( E \)-orientation or Thom class of a simple orbifold \( G \)-bundle \( \xi^K : V/K \to B \) if for each closed subgroup \( H \) and \( x \in B^H \), the restriction of \( u \) to \( (V/K)_{|G \cdot x} = (V_{|G \cdot x})/K \) is a generator of

\[
E^*_G(Th(V/K)_{|G \cdot x}) \cong E^*_H(D(V_{|x})/K, S(V_{|x})/K).
\]

Note that the Thom class \( u \) is natural under pullbacks. The restriction of the Thom class \( u \) to the base \( F \) via the zero section is called \( G \)-equivariant Euler class \( e_G(\xi^K) := s^*(u) \in E^*_G(B) \) for the zero section \( s : B \to V/K \).

Notice that the collapsing map \( X \to \{ pt \} \) gives a morphism \( E^*_G(pt) \to E^*_G(X) \), which yields an \( E^*_G(pt) \)-algebra structure on \( E^*_G(X) \).

**Proposition 2.1 (Thom isomorphism).** Let \( E^*_G \) is one of \( H^*_G \), \( K^*_G \), and \( MU^*_G \) and \( V \) be a finite dimensional complex \( G \)- and \( K \)-representations such that these two representations commute. Then \( E^*_G(Th(V/K)) \) is rank one free module over \( E^*_G(pt) \) and it vanishes in odd degree.

**Proof.** Since \( K \) acts on \( V \) linearly and preserving the orientation, this follows from the equivariant Thom isomorphism [May96] Theorem 9.2, Chapter XVI. \( \square \)

We note that the non-equivariant version of the above proposition for singular cohomology has been studied in [PS10] Section 5.

Now we consider the following \( G \)-invariant stratification

\[
(2.1) \quad X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k \cdots
\]

of a \( G \)-space \( X \) such that \( X = \bigcup_{i \geq 1} X_i \) and each successive quotient \( X_j/X_{j-1} \) is homeomorphic to the Thom space \( Th(V_j/K_j) \) of a simple orbifold \( G \)-bundles \( \xi^{K_j} : V_j/K_j \to B_j \). Therefore \( X \) can be built from \( X_1 \) inductively by attaching \( q \)-disc bundles \( D(V_j/K_j) \) to \( X_{j-1} \) via some \( G \)-equivariant map

\[
\phi_j : S(V_j/K_j) \to X_{j-1},
\]
which gives us the following cofibration

\[ X_{j-1} \to X_j \to Th(V_j/K_j). \]

Now, one gets the following proposition about generalized equivariant cohomologies with rational coefficients by the induction on the stratification and by a similar arguments of the proof of \[\text{HHH05, Theorem 2.3}\].

**Proposition 2.2.** \[\text{HHH05, Theorem 2.3}\] Let \(X\) be a \(G\)-space with a stratification as in (2.1) and \(E_G^*\) a generalized cohomology theory. If each equivariant Euler class \(e_G(\xi^{K_j}) \in E_G^*(B_j)\) of the associated simple orbifold \(G\)-bundle is not a zero divisor, then the inclusion \(\iota: \bigsqcup B_j \hookrightarrow X\) induces an injection

\[ \iota^*: E_G^*(X) \to \prod_j E_G^*(B_j). \]

An approach to compute the image of \(\iota^*\) in (2.2) has been discussed in \[\text{HHH05, Section 3}\]. We briefly outline that in the rest of this section according to our setting. Let \(X\) be a \(G\)-space with the \(G\)-stratification as in (2.1) which satisfies the following assumptions.

(A1) The simple orbifold bundles \(\xi^{K_i}: V_j/K_j \to B_j\) are \(E\)-orientable and has a decomposition

\[ (\xi^{K_i}: V_j/K_j \to B_j) \cong \bigoplus_{s<j} (\xi^{K_{js}}: V_{js}/K_{js} \to B_j) \]

into \(E\)-orientable simple orbifold bundles \(\xi^{K_{js}}\), possibly \(V_{js}\) can be trivial.

(A2) The restriction of the attaching map \(\phi_j: S(V_j/K_j) \to X_{j-1}\) to \(S(V_{js}/K_{js})\) satisfies

\[ \phi_j|_{S(V_{js})} = f_{js} \circ \xi^{K_{js}} \]

for some \(G\)-equivariant map \(f_{js}: B_j \to B_s\), identifying \(B_s\)'s with their images in \(X_{j-1}\) for \(s < j\).

(A3) The equivariant Euler classes \(e_G(\xi^{K_{js}})\) are not zero divisors and pairwise relatively prime in \(E_G^*(B_s)\).

We remark that the \(G\)-invariant stratifications with trivial \(K_j\)'s are studied in \[\text{HHH05}\]. Nevertheless, under the above assumptions on a \(G\)-space \(X\) with the property as in (2.1), one may obtain the following proposition after slight modification in the proof of \[\text{HHH05, Theorem 3.1}\] with rational coefficients.

**Proposition 2.3.** Let \(X\) be a \(G\)-space with the \(G\)-stratification as in (2.1) and satisfying assumptions (A1) to (A3). Then the image of \(\iota^*: E_G^*(X) \to \prod_j E_G^*(B_j)\) is

\[ \Gamma_X = \left\{ (x_j) \in \prod_j E_G^*(F_j) \mid e_G(\xi^{K_{js}}) \text{ divides } x_j - f_{js}^*(x_j) \text{ for all } s < j \right\}. \]

3. Toric varieties over almost simple polytopes

In this section, we give a combinatorial characterization of toric varieties which is essential for the main results of this paper. Let \(\Sigma\) be a full dimensional rational polytopal fan in \(\mathbb{R}^n\) and \(P\) the lattice polytope whose normal fan is \(\Sigma\). The corresponding toric variety \(X_\Sigma\) is equipped with an action of compact torus \(T^n \subset (\mathbb{C}^*)^n\). Here, we identify \(\mathbb{R}^n\) with the Lie algebra of \(T^n\).
Following the result of [Jur81] (we also refer to [CLS11, Theorem 12.2.5]), there is a $T^n$-equivariant homeomorphism

$$f: X_\Sigma \cong (T^n \times P)/\sim,$$

where $(t, p) \sim (s, q)$ whenever $p = q$ and $t^{-1}s$ is an element in the subtorus $T_{F(p)} \subseteq T^n$ whose Lie algebra is generated by the outward normal vectors of the codimension-1 faces of $P$ which contain $p$ if $p$ is not in the interior of $P$. When $p$ is in the interior of $P$, we consider $T_{F(p)}$ to be trivial.

Here, we notice that $T^n$-action on $(T^n \times P)/\sim$ is induced from the multiplication on the first factor of $T^n \times P$ and the corresponding orbit map,

$$(3.1) \eta: (T^n \times P)/\sim \rightarrow P,$$

is given by $[t, p]_\sim \mapsto p$, where $[t, p]_\sim$ denotes the equivalence class of $(t, p)$. Therefore, the topology of a toric variety is completely determined by the combinatorics of the orbit space $P$ and its geometric data, namely, outward normal vectors of codimension 1 faces of $P$.

We discuss the combinatorics of $P$ in Subsection 3.1 and study some topological information of $X_\Sigma$ obtained from the geometry of $P$ in Subsection 3.2.

### 3.1. Retraction sequence of a convex polytope.

The goal of this subsection is to introduce a combinatorial characterization of certain convex polytopes. Let $\ell$ denote the number of vertices of a convex polytope $P$. We consider a finite sequence

$$(3.2) P = P_\ell \supset P_{\ell-1} \supset \cdots$$

of subspaces of $P$ defined inductively as follows: We set the initial term $P_\ell = P$. Given $P_i$, we choose a vertex $v_i$ of $P_i$ having $v_i$ has a neighborhood $U_{v_i}$ in $P_i$ such that $U_{v_i}$ is homeomorphic to $\mathbb{R}^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i \geq 0, \ i = 1, \ldots, k\}$ as manifolds with corners for some $1 \leq k \leq \dim P_i$. A vertex $v_i$ with this property is called a free vertex. Then, we construct $P_{i-1}$ by deleting all faces containing $v_i$ from $P_i$. We call a sequence $P$ of subspaces of $P$ constructed as above a retraction sequence of $P$, if the sequence ends up with a vertex of $P$.

**Example 3.1.** Let $P$ be a 3-dimensional polytope given by the following system of inequalities;

$$0 \leq x \leq 1 \leq y \leq z$$

It is a 3-dimensional example of Gelfand–Zetlin polytopes which play an important role in many researches, such as [And13, CKO, Kir10]. Figure 1 describes a retraction sequence of $P$.

**Example 3.2.** A cone $C(P)$ on a simple polytope $P$ has a retraction sequence. Indeed, it is shown in [BSS17, Proposition 2.3] that every simple polytope has at least one retraction sequence. Let $P_\ell \supset \cdots \supset P_1$ be a retraction sequence of $P$. Then, $C(P_\ell) \supset \cdots \supset C(P_1) \supset \ast$ is a retraction sequence for $C(P)$, where $\ast$ is the apex of $C(P)$. Moreover, if $P$ is a lattice polytope and the apex $\ast$ is chosen to
Figure 1. A retraction sequence of 3-dimensional Gelfand–Zetlin polytope.

be a lattice point, then $C(P)$ is lattice polytope which is not simple unless $P$ is a simplex.

Example 3.3. Some retraction sequences for 3-dimensional Bruhat interval polytopes (see [TW15]) are described in [LM, Figures 25, 27].

Now, we introduce the following definition, which we shall mainly consider throughout this paper.

Definition 3.4. A convex polytope $P$ is called almost simple if it admits at least one retraction sequence.

Notice that polytopes in Examples 3.1, 3.2 and 3.3 are almost simple, but not simple polytopes. We also note that not every convex polytope has a retraction sequence, for instance the convex hull of $\{\pm (1,0,0), \pm (0,1,0), \pm (0,0,1)\}$. It is a 3-dimensional convex polytope with 6 vertices, and each of vertices does not have any neighborhood homeomorphic to $\mathbb{R}^3$.

3.2. Torus equivariant stratifications. From now on, we consider a toric variety $X$ whose orbit space, via the orbit map $\eta: X \rightarrow P$ defined in (3.1), is an almost simple polytope $P$.

Proposition 3.5. A retraction sequence $P_\ell \supset P_{\ell-1} \supset \cdots \supset P_1$ yields a $T^n$-equivariant stratification of $X$

(3.3)

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\ell = X.$$ 

Furthermore, the quotient $X_j/X_{j-1}$ is homeomorphic to the Thom space $Th(\xi^{K_j})$ of the simple orbifold $T^n$-bundle

$$\xi^{K_j}: \mathbb{C}^{k_j}/K_j \to \eta^{-1}(v_j),$$

for some $k_j \in \mathbb{N}$ and finite abelian group $K_j$, where $v_j \in V(P)$ denotes the free vertex of $P_j$ to define $P_{j-1}$, for $j = \ell, \ldots, 2$.

Proof. Given a retraction sequence $P_\ell \supset \cdots \supset P_j \supset P_{j-1} \supset \cdots \supset P_1$, we define $X_j := \eta^{-1}(P_j)$ for $j = 1, \ldots, \ell$. Then, $X_{j-1} \subset X_j$ as $P_j \supset P_{j-1}$. Since $\eta$ is the orbit map with respect to $T^n$-action, the stratification (3.3) is $T^n$-equivariant.

To prove the second assertion, we consider the unique maximal face $E_j$ of $P_j$ which contains $v_j$, and denote by $U_j \subset E_j$ the union of all relative interiors of faces in $E_j$ containing $v_j$. For instance, the colored faces in Figure 1 are $U_j$ for $j = 1, \ldots, 7$. Then, one can see from the property of a free vertex that $U_j$ is
homeomorphic to $\mathbb{R}^k_{\geq}$ as manifolds with corners, where $k_j := \dim U_j = \dim E_j$. Also, note that

$$X_j - X_{j-1} = \eta^{-1}(U_j).$$

Let $\mathbb{R}^k_{E_j}$ be the $k_j$-dimensional subset of $\mathbb{R}^n$ generated by the normal vectors of facets whose intersection is $E_j$. Since $e_j$ is a free vertex of $E_j$, there are $k_j$-many facets, say $F_1, \ldots, F_{k_j}$, such that $v_j = \bigcap_{i=1}^{k_j} (E_j \cap F_i)$. Consider the projection

$$(3.4) \quad \mathbb{Z}^n \to \mathbb{Z}^n/((\mathbb{R}^k \cap \mathbb{Z}^n) \cong \mathbb{Z}^{k_j}$$

and $\mu_1, \ldots, \mu_{k_j}$, the images of primitive outward normal vectors of $F_1, \ldots, F_{k_j}$, respectively.

Now, the result of [BNSS] Proposition 4.4 shows

$$(3.5) \quad \eta^{-1}(U_j) \cong D^{2k_j}/K_j,$$

where

$$(3.6) \quad K_j = \ker(\exp[\mu_1 | \cdots | \mu_{k_j}] : T^{k_j} \to T^{k_j}).$$

Here, one can regard the space $(3.5)$ as the q-disc bundle of a simple orbifold $T^{k_j}$-bundle

$$(3.7) \quad \xi^{K_j} : \mathbb{C}^{k_j}/K_j \to \eta^{-1}(v_j),$$

where the standard $T^{k_j}$-action on $\mathbb{C}^{k_j}$ induces the action on $\mathbb{C}^{k_j}/K_j$, and $T^{k_j}$ acts on the fixed point $\pi^{-1}(v_j)$ trivially. Note that $\mathbb{Z}^n$ acts on this bundle via the projection of $\mathbb{T}^n \to T^{k_j}$ determined by $(3.4)$. Hence we have the following $\mathbb{T}^n$-equivariant homotopy equivalences

$$X_j/X_{j-1} \simeq \eta^{-1}(E_j)/\pi^{-1}(E_{j-1} \cap P_j) \simeq Th(\xi^{G_j}).$$

$$\square$$

**Proposition 3.6.** The $\mathbb{T}^n$-equivariant stratification as in $(3.3)$ satisfies assumptions (A1), (A2) and (A3) in Section 2.

**Proof.** Recall that $\xi_j$ in $(3.3)$ is the fixed point and the bundle $\xi^{K_j} : V_j/K_j \to B_j$ in (A1) is the bundle in $(3.7)$ for $j = 1, \ldots, \ell$.

Observe that the total space in $(3.7)$ is a quotient of a $\mathbb{T}^n$-representation on $\mathbb{C}^{k_j}$ where $K_j$ is a subgroup of $\mathbb{T}^n$. Therefore $\mathbb{C}^{k_j}$ can be decomposed into 1-dimensional representations as

$$\mathbb{C}^{k_j} \cong \mathbb{C}(\alpha_1) \oplus \cdots \oplus \mathbb{C}(\alpha_{k_j})$$

for some characters $\alpha_i : \mathbb{T}^n \to S^1$. Since each $\mathbb{C}(\alpha_i)$ is invariant under $K_j$, we have

$$(\xi^{K_j} : \mathbb{C}^{k_j}/K_j \to B_j) \cong \bigoplus_{s=1}^{k_j} (\xi^{H_j} : \mathbb{C}(\alpha_s)/H_{js} \to B_j)$$

for some subgroups $H_1, \ldots, H_{k_j}$ of $K_j$. This proves assumption A(1).

The quotient of 1-dimensional representation $\mathbb{C}(\alpha_i)$ by $\mathbb{T}^n$-action is identical to $\mathbb{R}_{\geq 0}$ which corresponds to an edge $e_i$ of $U_j$, since $\eta^{-1}(U_j) \to \eta^{-1}(v_j)$ is the associated q-disc bundle of $\xi^{K_j}$. So the bundle $\mathbb{C}(\alpha_i)/H_i \to B_j$ is isomorphic to the bundle $\eta^{-1}(e_i) \to \eta^{-1}(v_j)$. Note that one can write the attaching map $\phi_j$ explicitly following the proof of [BSS17] Theorem 4.1. Therefore, $\phi_j|_{\mathbb{C}(\alpha_i)}$ is a vertex which is another vertex of $e_i$ in $E_j$. A pictorial explanation is given.
in Figure 2. Considering $f_{js}: B_j \to B_s$ as a map between two fixed points, we conclude assumption $A(2)$.

Assumption $A(3)$ follows from [HHH05, Lemma 5.2], as the vectors $\mu_1, \ldots, \mu_k$ defined by (3.4) are linearly independent. \hfill \Box

The following is an application of Proposition 2.3 to the category of toric varieties.

**Proposition 3.7.** Let $X$ be a toric variety over an almost simple polytope and $|X^I| = \ell$. Generalized $T^n$-equivariant cohomologies $E^*_T(X)$ for $E^*_T = K^*_T$, $MU^*_T$, $H^*_T$ can be given by

$$E^*_T(X) = \left\{ (x_j) \in \prod_j E^*_T(pt) \mid e_T(\xi^{f^*_j}) \text{ divides } x_j - f^*_j(x_j) \text{ for all } s < j \right\}.$$  

We note that $H^*_T(pt)$ and $K^*_T(pt)$ are isomorphic to the ring of polynomials and the ring of Laurent polynomials with $n$-variables, respectively. For $MU^*_T(pt)$, though its structure is unknown, it is referred as the ring of $T^n$-cobordism forms in [HHRW16].

### 4. Piecewise algebras and Applications

We begin this section with summarizing the concept of some piecewise algebras associated to a fan, studied in [HHRW16, Section 4]. The authors apply those algebras to weighted projective spaces to get a description of generalized equivariant cohomologies with integral coefficients. Here, we generalize their several results to a wider class of singular toric varieties discussed in Section 3.

Recall that if $\sigma$ is a cone in a fan $\Sigma$, then all of the faces of $\sigma$ belongs to $\Sigma$. This leads us to form a small category $\text{cat}(\Sigma)$ whose objects are elements of $\Sigma$ and morphisms are face inclusions. The zero cone $\{0\}$ is the initial object of this category.

Let $\Sigma$ be an $n$-dimensional rational fan in $\mathbb{R}^n$, namely, one-dimensional cones are generated by rational vectors in $\mathbb{R}^n$. Here we may identify $\mathbb{R}^n$ with the Lie algebra of $T^n$. Given a $k(\leq n)$-dimensional cone $\sigma \in \Sigma$, we consider a subtorus $T_\sigma$ generated by primitive vectors spanning 1-dimensional cones in $\sigma$. For the category $T^n$-$\text{top}$ of $T^n$-spaces, we define a diagram

$$\mathcal{V}: \text{cat}(\Sigma) \to T^n$-$\text{top}$

by $\mathcal{V}(\sigma) := T^n/T_\sigma$ and $\mathcal{V}(\sigma \subseteq \tau) = (T^n/T_\tau \to T^n/T_\sigma)$, where the projection $T^n/T_\tau \to T^n/T_\sigma$ is induced from the natural inclusion $T_\sigma \subseteq T_\tau$. Then, the toric variety $X_{\Sigma}$ associated to $\Sigma$ is homotopy equivalent to the homotopy colimit $\text{hocolim}\mathcal{V}$.
Next, regarding \( E_{T^n} \) as a functor from \( T^n \text{-top} \) to the category \( \text{gcalg}_E \) of graded commutative \( E_{T^n} \)-algebras, we consider the following composition

\[
\mathcal{EV} : \text{cat}(\Sigma) \xrightarrow{\mathcal{V}} T^n \text{-top} \xrightarrow{E_{T^n}} \text{gcalg}_E,
\]

which leads us to the following definition.

**Definition 4.1.** [HHRW16 Definition 4.6] Let \( \Sigma \) be a rational fan in \( \Sigma \), we call \( \lim \mathcal{EV} \) the piecewise algebra over \( E_{T^n} \). Furthermore, as the theory \( E_{T^n} \) is chosen to be \( H_{T^n}^*, K_{T^n}^* \) and \( MU_{T^n}^* \), we denote the associated piecewise algebra by

- \( PP[\Sigma] \), the algebra of piecewise polynomials,
- \( PL[\Sigma] \), the algebra of piecewise Laurent polynomials, and
- \( PC[\Sigma] \), the algebra of piecewise \( T^n \)-cobordism forms, respectively.

We note that the object \( \mathcal{EV}(\sigma) \) can be calculated explicitly as follows. The natural action of \( T^n \) on \( V(\sigma) = T^n / T_\sigma \) yields a \( T^n \)-representation \( \eta_\sigma \) on which \( T_\sigma \) acts trivially. Since \( T^n \) is abelian, \( \eta_\sigma \) is decomposed into 1-dimensional representations, say \( \eta_\sigma \cong \bigoplus_{i=1}^{n-k} \eta_{\sigma(i)} \). We denote by \( S^1_{\eta_\sigma(i)} \) the corresponding circle for each \( i = 1, \ldots, n-k \). The inclusion of \( S^1_{\eta_\sigma(i)} \) into the unit disc \( D_{\eta_\sigma(i)} \) gives an equivariant cofiber sequence

\[
(4.2) \quad \quad S^1_{\eta_\sigma(i)} \to D_{\eta_\sigma(i)} \to D_{\eta_\sigma(i)}/S^1_{\eta_\sigma(i)}.
\]

Regarding each term of (4.2) as an \( S^1 \)-bundle, disc bundle over a point and the associated Thom space, respectively, we may consider the equivariant Euler class \( e_{T^n}(\eta_\sigma(i)) \) in \( E_{T^n}^* \) for each \( i = 1, \ldots, n-k \).

**Proposition 4.2.** [HHRW16 Section 4, (4.11)]

\[
\mathcal{EV}(\sigma) \cong E_{T^n}^*(T^n/T_\sigma) \cong E_{T^n}^*(pt)/(e_{T^n}(\eta_\sigma(1)), \ldots, e_{T^n}(\eta_\sigma(n-k))).
\]

The proof of the following theorem is almost same as the proof of [HHRW16 Theorem 5.5] with very few modifications in notation. To be more precise, one needs to replace the equivariant Euler classes [HHRW16 (3.8)] of weighted projective space by the Euler classes for simple orbifold bundles defined in Section 2.

**Theorem 4.3.** Let \( X_P \) be a toric variety over an almost simple polytope \( P \) and \( \Sigma_P \) the normal fan of \( P \). Then,

1. \( H_{T^n}^*(X_P) \) is isomorphic to \( PP[\Sigma_P] \) as an \( H_{T^n}^*(pt) \)-algebra;
2. \( K_{T^n}^*(X_P) \) is isomorphic to \( PL[\Sigma_P] \) as a \( K_{T^n}^*(pt) \)-algebra;
3. \( MU_{T^n}^*(X_P) \) is isomorphic to \( PC[\Sigma_P] \) as an \( MU_{T^n}^*(pt) \)-algebra.

5. **Divisive toric varieties**

In this section, we introduce the notion of divisive toric varieties. This concept is motivated by weighted projective spaces [BFR09] and defined on the category of toric varieties. Then we follow the same arguments as we discussed in Section 4. Here all cohomology theories can be considered with integral coefficients.

**Definition 5.1.** Let \( X \) be a toric variety satisfying the hypothesis of Proposition 4.5. If the finite groups \( K_j \)'s in (3.7) are trivial, then we call \( X \) a divisive toric variety.
Example 5.2. Recall the 3-dimensional Gelfand–Zetlin polytope $P$ described in Example 3.1. The outward normal vectors of facets intersecting $v_7$ in $P_3$ of Figure 1 are $(-1, 0, 0)$, $(0, 1, 1)$ and $(0, 1, 0)$, which form an integral basis of $\mathbb{Z}^3$. Hence, the finite group $K_7$ defined in (3.6) is trivial. To compute $K_6$, we consider the facet given by $\{x = 1\}$ whose primitive outward normal vector is $(1, 0, 0)$. In this case, the map (3.4) yields the projection $\pi^3 : T_2$ onto the last two coordinates. Hence, $K_6 = \ker(\rho : T^2 \to T^2)$, where $\rho(t_1, t_2) = (t_1 t_2, t_3)$, which is trivial. We refer to [BSS17, Proposition 4.3] for the general statement about this computation. Finally, one can conclude by similar computations for the other vertices $v_5, \ldots, v_2$ that the associated toric variety $X_P$ is divisive.

Note that [HHRW16] defines the piecewise algebras of a fan with integral coefficients which is summarized in Section 4 with rational coefficients. We denote the algebra of piecewise polynomials, the algebra of piecewise Laurent polynomials and the algebra of piecewise cobordism forms with integral coefficients by $PP_{\mathbb{Z}}[\Sigma]$, $PL_{\mathbb{Z}}[\Sigma]$ and $PC_{\mathbb{Z}}[\Sigma]$ respectively. If a toric variety is divisive, then it is equipped with a $T^n$-equivariant stratification in the sense of [HHH05, Section 2]. So one can apply their results to divisive toric varieties, which yields the following proposition with integral coefficients.

Proposition 5.3. Let $X$ be a divisive toric variety over an almost simple polytope $Q$ and $\Sigma_Q$ the normal fan of $Q$. Then,

1. $H^*_T(X_Q; \mathbb{Z})$ is isomorphic to $PP_{\mathbb{Z}}[\Sigma_Q]$ as an $H^*_T(\mathbb{pt}; \mathbb{Z})$-algebra;
2. $K^*_T(X_Q; \mathbb{Z})$ is isomorphic to $PL_{\mathbb{Z}}[\Sigma_Q]$ as a $K^*_T(\mathbb{pt}; \mathbb{Z})$-algebra;
3. $K^*_T(X_Q; \mathbb{Z})$ is isomorphic to $PC_{\mathbb{Z}}[\Sigma_Q]$ as an $MU^*_T(\mathbb{pt}; \mathbb{Z})$-algebra.

REFERENCES

[And13] Dave Anderson. Okounkov bodies and toric degenerations. Math. Ann., 356(3):1183–1202, 2013.
[Ati82] Michael. F. Atiyah. Convexity and commuting Hamiltonians. Bull. London Math. Soc., 14(1):1–15, 1982.
[Bag07] Silvano Baggio. Equivariant K-theory of smooth toric varieties. Tohoku Math. J. (2), 59(2):203–231, 2007.
[BB00] Anthony Bahri and Martin Bendersky. The KO-theory of toric manifolds. Trans. Amer. Math. Soc., 352(3):1191–1202, 2000.
[BFR09] Anthony Bahri, Matthias Franz, and Nigel Ray. The equivariant cohomology ring of weighted projective space. Math. Proc. Cambridge Philos. Soc., 146(2):395–405, 2009.
[BNS88] Anthony Bahri, Dietrich Notbohm, Soumen Sarkar, and Jongbaek Song. On integral cohomology of certain orbifolds. [arXiv:1711.07718]
[BR98] Victor. M. Buchstaber and N. Ray. Toric manifolds and complex cobordisms. Uspekhi Mat. Nauk, 53(2(320)):139–140, 1998.
[BSS17] Anthony Bahri, Soumen Sarkar, and Jongbaek Song. On the integral cohomology ring of toric orbifolds and singular toric varieties. Algebr. Geom. Topol., 17(6):3779–3810, 2017.
[CKO] Yunhyung Cho, Yoosik Kim, and Yong-Geun Oh. Lagrangian fibers of Gelfand-Cetlin systems. [arXiv:1704.02713]
[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.
[Dan78] Vladimir. I. Danilov. The geometry of toric varieties. Uspekhi Mat. Nauk, 33(2(200)):85–134, 247, 1978.
[Del88] Thomas Delaun. Hamiltoniens périodiques et images convexes de l’application moment. Bull. Soc. Math. France, 116(3):315–339, 1988.
GENERALIZED EQUIVARIANT COHOMOLOGIES OF SINGULAR TORIC VARIETIES

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62(2):417–451, 1991.

[Fra10] Matthias Franz. Describing toric varieties and their equivariant cohomology. *Colloq. Math.*, 121(1):1–16, 2010.

[GS82] Victor Guillemin and Shlomo Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[Gui94] Victor Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.

[HHH05] Megumi Harada, André Henriques, and Tara S. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197(1):198–221, 2005.

[HHRW16] Megumi Harada, Tara S. Holm, Nigel Ray, and Gareth Williams. The equivariant K-theory and cobordism rings of divisive weighted projective spaces. *Tohoku Math. J.*, (2), 68(4):487–513, 2016.

[HK15] Megumi Harada and Kiumars Kaveh. Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.*, 202(3):927–985, 2015.

[Jur81] Jerzy Jurkiewicz. On the complex projective torus embedding, the associated variety with corners and Morse functions. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 29(1-2):21–27, 1981.

[Jur85] Jerzy Jurkiewicz. Torus embeddings, polyhedra, J*-actions and homology. *Dissertationes Math. (Rozprawy Mat.*), 236:64, 1985.

[Kir10] Valentina Kiritchenko. Gelfand-Zetlin polytopes and flag varieties. *Int. Math. Res. Not. IMRN*, (13):2512–2531, 2010.

[KK12] Kiumars Kaveh and Askold Khovanskii. Algebraic equations and convex bodies. In *Perspectives in analysis, geometry, and topology*, volume 296 of *Progr. Math.*, pages 263–282. Birkhäuser/Springer, New York, 2012.

[LM] Eunjeong Lee and Mikiya Masuda. Generic torus orbit closures in Schubert varieties. arXiv:1807.02907.

[LM09] Robert Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.

[May96] J. Peter May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[Mor93] Robert Morelli. The K-theory of a toric variety. *Adv. Math.*, 100(2):154–182, 1993.

[PS10] Mainak Poddar and Soumen Sarkar. On quasitoric orbifolds. *Osaka J. Math.*, 47(4):1055–1076, 2010.

[Seg68] Graeme Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[tD70] Tammo tom Dieck. Bordism of G-manifolds and integrality theorems. *Topology*, 9:345–358, 1970.

[TW15] Emmanuel Tsukerman and Lauren Williams. Bruhat interval polytopes. *Adv. Math.*, 285:766–810, 2015.

[VV03] Gabriele Vezzosi and Angelo Vistoli. Higher algebraic K-theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003.

[WZZ99] Volkmar Welker, Günter M. Ziegler, and Rade T. Zivaljević. Homotopy colimits—comparison lemmas for combinatorial applications. *J. Reine Angew. Math.*, 509:117–149, 1999.

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