Symmetry Reductions and Exact Solutions of Shallow Water Wave Equations

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Abstract.

In this paper we study symmetry reductions and exact solutions of the shallow water wave (SWW) equation

\[ u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \]

(1)

where \( \alpha \) and \( \beta \) are arbitrary, nonzero, constants, which is derivable using the so-called Boussinesq approximation. Two special cases of this equation, or the equivalent nonlocal equation obtained by setting \( u_x = U \), have been discussed in the literature. The case \( \alpha = 2\beta \) was discussed by Ablowitz, Kaup, Newell and Segur [Stud. Appl. Math., 53 (1974) 249], who showed that this case was solvable by inverse scattering through a second order linear problem. This case and the case \( \alpha = \beta \) were studied by Hirota and Satsuma [J. Phys. Soc. Japan, 40 (1976) 611] using Hirota’s bi-linear technique. Further the case \( \alpha = \beta \) is solvable by inverse scattering through a third order linear problem.

In this paper a catalogue of symmetry reductions is obtained using the classical Lie method and the nonclassical method due to Bluman and Cole [J. Math. Mech., 18 (1969) 1025]. The classical Lie method yields symmetry reductions of (1) expressible in terms of the first, third and fifth Painlevé transcendents and Weierstrass elliptic functions. The nonclassical method yields a plethora of exact solutions of (1) with \( \alpha = \beta \) which possess a rich variety of qualitative behaviours. These solutions all like a two-soliton solution for \( t < 0 \) but differ radically for \( t > 0 \) and may be viewed as a nonlinear superposition of two solitons, one travelling to the left with arbitrary speed and the other to the right with equal and opposite speed. These families of solutions have important implications with regard to the numerical analysis of SWW and suggests that solving (1) numerically could pose some fundamental difficulties. In particular, one would not be able to distinguish the solutions in an initial value problem since an exponentially small change in the initial conditions can result in completely different qualitative behaviours.

We compare the two-soliton solutions obtained using the nonclassical method to those obtained using the singular manifold method and Hirota’s bi-linear method.

Further, we show that there is an analogous nonlinear superposition of solutions for two 2 + 1-dimensional generalisations of the SWW equation (1) with \( \alpha = \beta \). This yields solutions expressible as the sum of two solutions of the Korteweg-de Vries equation.
1 Introduction.

In this paper we discuss symmetry reductions and exact solutions for the shallow water wave (SWW) equation
\[ \Delta \equiv u_{xxx} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \] (1.1)
where \( \alpha \) and \( \beta \) are arbitrary, nonzero, constants, which can be derived from the classical shallow water theory in the Boussinesq approximation (cf., [26]). This equation may be written in the nonlocal form (set \( u_x = U \))
\[ U_{xxx} + \alpha UU_t - \beta U_x \partial^{-1}_x U_t - U_t - U_x = 0, \] (1.2)
where \( (\partial^{-1}_x f)(x) = \int^\infty_x f(y) \, dy \); in fact this is the form of the physical model derived in [26].

Two special cases of (1.2) have attracted some attention in the literature, namely \( \alpha = 2 \beta \) and \( \alpha = \beta \). In their seminal paper on soliton theory, Ablowitz, Kaup, Newell and Segur [2] show that
\[ U_{xxx} + 2\betaUU_t - \beta U_x \partial^{-1}_x U_t - U_t - U_x = 0, \] (1.3)
which is (1.2) with \( \alpha = 2\beta \) is solvable by inverse scattering. Further, Ablowitz et al [2] remark that (1.3) reduces in the long wave, small amplitude limit to the celebrated Korteweg-de Vries (KdV) equation
\[ u_t + uu_{xxx} + 6uu_x = 0, \] (1.4)
which was the first equation to be solved by inverse scattering [27], and they also comment that (1.3) also has the desirable properties of the regularized long wave (RLW) equation [8,47]
\[ u_{xxx} + uu_x - u_t - u_x = 0, \] (1.5)
sometimes known as the Benjamin-Bona-Mahoney equation, in that it responds feebly to short waves. Additionally, we note that (1.5) and (1.3) have the same linear dispersion relation \( \omega(k) = -k/(1 + k^2) \) for the complex exponential \( u(x,t) \sim \exp\{ikx + \omega(k)t\} \}. However, in contrast to (1.3), the RLW equation (1.5) is thought not to be solvable by inverse scattering (cf., [42]).

Hirota and Satsuma [34] studied both (1.3) and
\[ U_{xxx} + \beta UU_t - \beta U_x \partial^{-1}_x U_t - U_t - U_x = 0, \] (1.6)
i.e. (1.2) with \( \alpha = 2\beta \), using Hirota’s bi-linear method [32] and obtained \( N \)-soliton solutions for both equations (see also [43]).

In the sequel, we shall refer to (1.1) with \( \alpha = 2\beta \)
\[ u_{xxx} + 2\beta u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \] (1.7)
as the SWW-AKNS equation and (1.1) with \( \alpha = \beta \)
\[ u_{xxx} + \beta u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \] (1.8)
as the SWW-HS equation.

The scattering problem for the SWW-AKNS equation (1.7) is the second order problem [2]
\[ \psi_{xx} + \frac{1}{2} \beta u_x \psi = \lambda \psi, \] (1.9)
with associated time-dependence
\[ (4\lambda - 1)\psi_t = (1 - \beta u_t)\psi_x + \frac{1}{2} \beta u_{xt} \psi, \] (1.10)
where $\lambda$ is the constant eigenvalue, and $\psi_{xxx} = \psi_{uxx}$ if and only if $u$ satisfies (1.7) We note that (1.9) is the time-independent Schrödinger equation which is also the scattering problem for the KdV equation (1.4) [27]. In contrast, the scattering problem for the SWW-HS equation (1.8) is the third order problem [23]

$$\psi_{xxx} + \left(\frac{1}{2}\beta u_x - 1\right) \psi_x = \lambda \psi,$$

with associated time-dependence

$$3\lambda \psi_t = (1 - \beta u_t)\psi_{xxx} + \beta u_{xt}\psi_x.$$  

We remark that (1.11) is similar to the scattering problem

$$\psi_{xxx} + \frac{1}{4}(1 + 6u)\psi_x + \frac{3}{4} \left[u_x - i\sqrt{3} \partial_x^{-1}(v_t)\right] \psi = \lambda \psi$$

which is the scattering problem for the Boussinesq equation

$$u_{xxxx} + 3(u^2)_{xx} + u_{xx} = u_{tt},$$

and which has been comprehensively studied by Deift, Tomei and Trubowitz [24].

The SWW equation (1.1) was also discussed by Hietarinta [31] who shows that it can be expressed in Hirota’s bi-linear form [32] if and only if either (i), $\alpha = \beta$, when it reduces to (1.8), or (ii), $\alpha = 2\beta$, when it reduces to (1.7). Further, in [19] it is shown that the SWW equation (1.1) satisfies the necessary conditions of the Painlevé tests due to Ablowitz, Ramani and Segur [3,4] and Weiss, Tabor and Carnevale [55] to be completely integrable if and only if either $\alpha = \beta$ or $\alpha = 2\beta$. These results strongly suggest that the SWW equation (1.1) is completely integrable if and only if it has one of the two special forms (1.7) or (1.8), which are both known to be solvable by inverse scattering.

The classical method for finding symmetry reductions of partial differential equations (PDEs) is the Lie group method of infinitesimal transformations (cf., [10,44]). Though this method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually, and so symbolic manipulation programs have been developed, for example in MACSYMA, MAPLE, MATHEMATICA, MUMATH and REDUCE, to facilitate the calculations. An excellent survey of the different packages presently available and a discussion of their strengths and applications is given by Hereman [30].

In recent years the nonclassical method due Bluman and Cole [9] (in the sequel referred to as the nonclassical method) and the direct method of Clarkson and Kruskal [17] have been used to generate many new symmetry reductions and exact solutions for several physically significant PDEs that are not obtainable using the classical Lie method, which represents important progress (cf., [16,29] and references therein).

Symmetry groups and associated reductions and exact solutions have several different important applications in the context of differential equations (see, for example, [10,16,44] for further details and references):

- **Derive new solutions from old solutions.** Applying the symmetry group of a differential equation to a known solution yields a family of new solutions. Quite often interesting solutions can be obtained from trivial ones.
- **Integration of ODEs.** Symmetry groups of ODEs can be used to reduce the order of the equation, for example to reduce a second order equation to first order.
- **Reductions of PDEs.** Symmetry groups of PDEs are used to reduce the total number of dependent and independent variables, for example from a PDE with two independent and one dependent variables to an ODE.
- **Linearisation of PDEs.** Symmetries groups can be used to discover whether or not a PDE can be linearised and to construct an explicit linearisation when one exists.
• **Classification of equations.** Symmetry groups can be used to classify differential equations into equivalence classes and to choose simple representatives of such classes.

• **Asymptotics of solutions of PDEs.** It is known that as solutions of PDEs asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be effectively used to study properties such as asymptotics and “blow-up”.

• **Numerical methods and testing computer coding.** Symmetry groups and exact solutions of physically relevant PDEs are used in the design, testing and evaluation of numerical algorithms; these solutions provide an important practical check on the accuracy and reliability of such integrators.

• **Conservation Laws.** The application of symmetries to conservation laws dates back to the work of Noether who proved the remarkable result that for systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property.

• **Further Applications.** There are several other important applications of symmetry groups including bifurcation theory, control theory, special function theory, boundary value problems and free boundary problems.

The method used to find solutions of the determining equations for the infinitesimals in both the classical and nonclassical case is that of Differential Gröbner Bases (DGBs), defined to be a basis $\mathcal{B}$ of the differential ideal generated by the system such that every member of the ideal pseudo-reduces to zero with respect to $\mathcal{B}$. This method provides a systematic framework for finding integrability and compatibility conditions of an overdetermined system of PDEs. It avoids the problems of infinite loops in reduction processes, and yields, as far as is currently possible, a “triangulation” of the system from which the solution set can be derived more easily (cf., [18,41,48,49]). In a sense, a DGB provides the maximum amount of information possible using elementary differential and algebraic processes in a finite time.

The triangulations of the systems of determining equations for infinitesimals arising in the classical and nonclassical methods in this article were all performed using the MAPLE package diffgrob2 [40]. This package was written specifically to handle fully nonlinear equations of polynomial type; packages such as those in [50,51,53] have been developed for the study of linear equations. All calculations are strictly “polynomial”, that is, there is no division. Implemented there are the Kolchin-Ritt algorithm, the differential analogue of Buchberger’s algorithm [14] using pseudo-reduction instead of reduction, and extra algorithms needed to calculate a DGB (as far as possible using the current theory), for those cases where the Kolchin-Ritt algorithm is not sufficient [41]. Designed to be used interactively as well as algorithmically, the package has proved useful for solving some fully nonlinear systems. As yet, however, algorithmic methods for finding the most efficient orderings, the best method of choosing the sequence of pairs to be cross-differentiated, for deciding when to integrate and read off coefficients of independent functions in one of the variables, for finding the best change of coordinates, and so on, are still the subject of much investigation.

In §2 we find the classical Lie group of symmetries and associated reductions of (1.1), which are expressible in terms of the first, third and fifth Painlevé transcendents and Weierstrass elliptic functions. Then in §3 we discuss the nonclassical symmetries and reductions of (1.1). In particular, the nonclassical symmetry reductions obtained for (1.8) generate a wide variety of interesting exact analytical solutions of the equations which we plot using MAPLE. In §4 we compare the two-soliton solutions generated in §3 with those obtained using the singular manifold method and Hirota’s bi-linear method. In §5 we show that there are analogous symmetry reductions and exact solutions also occurs for two $2 + 1$-dimensional generalisations of the SWW equation (1) with $\alpha = \beta$. We discuss our results in §6.
2 Classical Symmetry Reductions of the SWW Equation.

To apply the classical method to the SWW equation (1.1) we consider the one-parameter Lie group of infinitesimal transformations in \((x, t, u)\) given by

\[
\begin{align*}
\tilde{x} &= x + \varepsilon \xi(x, t, u) + O(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon \tau(x, t, u) + O(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon \phi(x, t, u) + O(\varepsilon^2),
\end{align*}
\]

(2.1a) (2.1b) (2.1c)

where \(\varepsilon\) is the group parameter. Then one requires that this transformation leaves invariant the set

\[
\mathcal{S}_{\Delta} \equiv \{u(x, t) : \Delta = 0\},
\]

(2.2)

of solutions of (1.1). This yields an overdetermined, linear system of equations for the infinitesimals \(\xi(x, t, u), \tau(x, t, u)\) and \(\phi(x, t, u)\). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

\[
v = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u.
\]

(2.3)

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equations

\[
\begin{align*}
\frac{dx}{\xi(x, t, u)} &= \frac{dt}{\tau(x, t, u)} = \frac{du}{\phi(x, t, u)},
\end{align*}
\]

(2.4)

which is equivalent to solving the invariant surface condition

\[
\psi \equiv \xi(x, t, u) u_x + \tau(x, t, u) u_t - \phi(x, t, u) = 0.
\]

(2.5)

The set \(\mathcal{S}_{\Delta}\) is invariant under the transformation (2.1) provided that

\[
\text{pr}^{(4)} v(\Delta) \bigg|_{\Delta=0} = 0,
\]

(2.6)

where \(\text{pr}^{(4)} v\) is the fourth prolongation of the vector field (2.3), which is given explicitly in terms of \(\xi, \tau\) and \(\phi\) (cf., [44]). This yields a system of fourteen determining equations, as calculated using the MACSYMA package symmgrp.max [15]. A triangulation or standard form [18,48] of these determining equations is the following system of eight equations,

\[
\begin{align*}
\xi_u &= 0, & \xi_t &= 0, & \xi_{xx} &= 0, & \tau_u &= 0, & \tau_x &= 0, \\
\alpha \phi_x - 2 \xi_x &= 0, & \beta \phi_t - \tau_t - \phi_x &= 0, & \phi_u + \xi_x &= 0,
\end{align*}
\]

from which we easily obtain the following infinitesimals,

\[
\begin{align*}
\xi &= \kappa_1 x + \kappa_2, & \tau &= g(t), & \phi &= -\kappa_1 \left(u - \frac{2x}{\alpha} - \frac{t}{\beta}\right) + \frac{g(t)}{\beta} + \kappa_3,
\end{align*}
\]

(2.7)

where \(g(t)\) is an arbitrary differentiable function and \(\kappa_1, \kappa_2\) and \(\kappa_3\) are arbitrary constants. The associated Lie algebra is spanned by the vector fields

\[
\begin{align*}
v_1 &= x \partial_x - \left(u - \frac{2x}{\alpha} - \frac{t}{\beta}\right) \partial_u, & v_2 &= \partial_x, & v_3 &= \partial_u, & v_4(g) &= g(t) \left(\partial_t + \beta^{-1} \partial_u\right).
\end{align*}
\]

Solving (2.4) with \(\xi, \tau\) and \(\phi\) given by (2.7), or equivalently solving (2.5), we obtain the following two canonical symmetry reductions.
Case 2.1 $\kappa_1 \neq 0$. In this case we set $g(t) = f(t)/\dot{f}(t)$, $\kappa_1 = 1$ and $\kappa_2 = \kappa_3 = 0$ (with $\dot{f} \equiv df/dt$). Hence we obtain the symmetry reduction

$$u(x,t) = f(t)w(z) + \frac{x}{\alpha} + \frac{t}{\beta}, \quad z = xf(t),$$

(2.8)

where $w(z)$ satisfies

$$zw''' + 4w'' + (\alpha + \beta)zw'' + \beta w'' + 2\alpha (w')^2 = 0,$$

(2.9)

where $' = d/dz$. It is straightforward to show using the algorithm of Ablowitz et al [4] that this equation is of Painlevé-type, i.e., its solutions have no movable singularities other than poles, only if either (i), $\alpha = \beta$ or (ii), $\alpha = 2\beta$. These two special cases (2.9) are solvable in terms of solutions of the third Painlevé equation [36]

$$\frac{d^2y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + ay^3 + \frac{by^2 + c}{x} + \frac{d}{y},$$

(2.10)

and the fifth Painlevé equation,

$$\frac{d^2y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} \left( \frac{y-1}{x} \right)^2 \left( \frac{ay + b}{y} \right) + \frac{cy}{x} + \frac{dy(y+1)}{y-1},$$

(2.11)

with $a, b, c$ and $d$ constants (see [19] for details). Hence the Painlevé Conjecture [3,4] predicts that a necessary condition for (1.1) to be completely integrable is that $(\alpha - \beta)(\alpha - 2\beta) = 0$, i.e., only if (1.1) has one of the two special forms (1.8) or (1.7).

Case 2.2 $\kappa_1 = 0$. In this case we set $g(t) = 1/\dot{f}(t)$, $\kappa_2 = 1$ and $\kappa_3 = -1/\beta$ and obtain the symmetry reduction

$$u(x,t) = w(z) + t/\beta, \quad z = x - f(t),$$

(2.12)

where $W(z) = w'(z)$ satisfies

$$(W')^2 + \frac{1}{3}(\alpha + \beta)W^3 = AW + B,$$

(2.13)

with $A$ and $B$ arbitrary constants. This equation is equivalent to the Weierstrass elliptic function equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

(2.14)

where $g_2$ and $g_3$ are arbitrary constants [56].

We remark that the vector field $\mathbf{v}_4(g)$ shows that (1.1) is invariant under the following variable coefficient transformation

$$\tilde{x} = x, \quad \tilde{t} = g(t), \quad \tilde{u} = u + [g(t) - t]/\beta,$$

(2.15)

i.e., if $u(x,t)$ is a solution of (1.1), then so is $\tilde{u}(\tilde{x},\tilde{t})$. Further, the associated Lie algebra $\mathfrak{g} = \{\mathbf{v}_4(g)\}$, with $g(t) \in C^\infty(\mathbb{R})$, is a Virasoro algebra since

$$[\mathbf{v}_4(g_1), \mathbf{v}_4(g_2)] = \mathbf{v}_4 \left( g_1 \dot{g}_2 - g_2 \dot{g}_1 \right),$$

where $[A,B] = AB - BA$ is the standard commutator. Virasoro algebras frequently arise in the study of symmetry reductions for $2+1$-dimensional equations such as the Kadomtsev-Petviashvili and Davey-Stewartson equations (see, for example, [57] and the references therein), but not $1+1$-dimensional equations such as (1.1).
It might be thought that the presence of the arbitrary function \( g(t) \) in the infinitesimals associated with the SWW equation (1.1) is an artefact of the fact that we are considering (1.1) rather than (1.2), which is the physical equation. In view of the nonlocal term, we write the nonlocal equation (1.2) as the system

\[
U_{xxt} + \alpha UU_t + \beta VU_x - U_t - U_x = 0, \quad V_x = U_t, \tag{2.16}
\]

where \( \alpha \) and \( \beta \) are arbitrary, nonzero constants. To apply the classical method to (2.16), we consider the one-parameter Lie group of infinitesimal transformations in \((x, y, U, V)\) given by

\[
\tilde{x} = x + \varepsilon \xi_1(x, y, U, V) + O(\varepsilon^2), \\
\tilde{y} = y + \varepsilon \xi_2(x, y, U, V) + O(\varepsilon^2), \\
\tilde{U} = U + \varepsilon \phi_1(x, y, U, V) + O(\varepsilon^2), \\
\tilde{V} = V + \varepsilon \phi_2(x, y, U, V) + O(\varepsilon^2),
\]

where \( \varepsilon \) is the group parameter. Solving the resulting determining equations yields the infinitesimals

\[
\xi = \kappa_1 x + \kappa_2, \quad \tau = g(t), \quad \phi_1 = -2\kappa_1(U - 1/\alpha), \quad \phi_2 = -\left(\kappa_1 + \frac{dg}{dt}\right)(V - 1/\beta),
\]

where \( \kappa_1 \) and \( \kappa_2 \) are arbitrary constants and \( g(t) \) is an arbitrary differentiable function and the associated vector fields are

\[
w_1 = x \partial_x - 2(U - 1/\alpha) \partial_U + (v - 1/\beta) \partial_V, \quad w_2 = \partial_x, \quad w_3(g) = g(t) \partial_t + \frac{dg}{dt}(V - 1/\beta) \partial_V.
\]

This means that the system (2.16) is invariant under the transformation

\[
\tilde{x} = x, \quad \tilde{t} = g(t), \quad \tilde{U} = U, \quad \tilde{V} = \frac{\beta V + \dot{g}(t) - t}{\beta \dot{g}(t)}.
\]

Hence if \( \{U(x, t), V(x, t)\} \) are solutions of (2.16), then so are \( \{\tilde{U}(\tilde{x}, \tilde{t}), \tilde{V}(\tilde{x}, \tilde{t})\} \).

### 3 Nonclassical Symmetry Reductions of the SWW Equation.

There have been several generalisations of the classical Lie group method for symmetry reductions. Bluman and Cole [9], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions. This method involves considerably more algebra and associated calculations than the classical Lie method. In fact, it has been suggested that for some PDEs, the calculation of these nonclassical reductions might be too difficult to do explicitly [45], especially if attempted manually since the associated determining equations are now an overdetermined, nonlinear system. For some equations such as the KdV equation (1.4), the nonclassical method does not yield any additional symmetry reductions to those obtained using the classical Lie method, while there are PDEs which do possess symmetry reductions not obtainable using the classical Lie group method. It should be emphasised that the associated vector fields arising from the nonclassical method do not form a vector space, still less a Lie algebra, since the invariant surface condition (2.5) depends upon the particular reduction. Subsequently, these methods were further generalised by Olver and Rosenau [45,46] to include “weak symmetries” and, even more generally, “side conditions” or “differential constraints”, and they concluded that “the unifying theme behind finding special solutions of PDEs is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of PDEs”. 
In the nonclassical method one requires only the subset of $\mathcal{S}_\Delta$ given by
\[
\mathcal{S}_{\Delta,\psi} = \{ u(x,t) : \Delta(u) = 0, \psi(u) = 0 \},
\] (3.1)
where $\mathcal{S}_\Delta$ is as defined in (2.2) and $\psi = 0$ is the invariant surface condition (2.5), is invariant under the transformation (2.1). The usual method of applying the nonclassical method (e.g., as described in [39]), to the SWW equation (1.1) involves applying the prolongation $\text{pr}^{(4)} v$ to the system of equations given by (1.1) and the invariant surface condition (2.5) and requiring that the resulting expressions vanish for $u \in \mathcal{S}_{\Delta,\psi}$, i.e.,
\[
\text{pr}^{(4)} v(\Delta) \big|_{\Delta=0,\psi=0} = 0, \quad \text{pr}^{(1)} v(\psi) \big|_{\Delta=0,\psi=0} = 0.
\] (3.2)
It is easily shown that
\[
\text{pr}^{(1)} v(\psi) = -(\xi u u_x + \tau u_t - \phi u) \psi,
\]
which vanishes identically when $\psi = 0$ without imposing any conditions upon $\xi$, $\tau$ and $\phi$. However as shown in [20], this procedure for applying the nonclassical method can create difficulties, particularly when implemented in symbolic manipulation programs. These difficulties often arise for equations such as (1.1) which require the use of differential consequences of the invariant surface condition (2.5). In [20] we proposed an algorithm for calculating the determining equations associated with the nonclassical method which avoids many of the difficulties commonly encountered; we use that algorithm here.

In the canonical case when $\tau \not\equiv 0$ we set $\tau = 1$. (We shall omit the special case $\tau \equiv 0$; in that case one obtains a single condition for $\phi$ with 424 summands, and even considering the subcase $\phi_u = 0$ leads to an equation more complex than the one we are studying.) Eliminating $u_t$, $u_{xt}$ and $u_{xxx}$, in (1.1) using the invariant surface condition (2.5) yields
\[
\tilde{\Delta} \equiv \phi_u u_{xxx} + 3\phi uu_x u_{xx} + 3\phi u_x u_{xx} + \phi u_{uu} u_x^2 + 3\phi u_{uu} u_x^2 + 3\phi u_{xxx} u_x + \phi_{xxx} \\
- \xi_{xxx} u_x - 3\xi_{xx} u_{xx} - 3\xi_x u_{xxx} - \xi_{uuu} u_x^4 - 3\xi_{xuu} u_x^3 - 6\xi_{uu} u_x^2 u_{xx} \\
- 3\xi_{xuu} u_x^2 - 9\xi_x u_x u_{xx} - \xi_{x} (4u_x u_{xxx} + 3u_x^2) - \xi u_{xxx} \\
+ (\alpha u_x - 1) [\phi_x + \phi_u u_x - \xi_x u_x - \xi_x u_x^2 - \xi u_{xx}] + u_{xx} [\beta (\phi - \xi u_x) - 1] = 0,
\] (3.3)
with $t$ a parameter, which involves the infinitesimals $\xi$ and $\phi$ that are to be determined. Now we apply the classical Lie algorithm to this equation using the fourth prolongation $\text{pr}^{(4)} v$ and eliminate $u_{xxx}$ using (3.3). This yields the following overdetermined, nonlinear system of equations for $\xi$ and $\phi$ (contrast the classical case discussed in §2 above where they are linear).
\[
\xi_u = 0,
\] (3.4i)
\[
\phi_{uu} = 0,
\] (3.4ii)
\[
(\alpha + \beta)(\phi_u + \xi_x) = 0,
\] (3.4iii)
\[
\xi \phi_{tu} + 3\xi^2 \xi_{xx} + 3\xi \xi_x - 3\xi \xi_{xt} - 3\xi^2 \phi_{xx} - \xi \phi_u = 0,
\] (3.4iv)
\[
\alpha \xi \phi_{tu} - \alpha \xi_t \phi_u + \alpha \xi \phi_{tu} - 2\beta \xi \phi_{xx} - \xi \phi_{uu} + \beta \xi \phi_{xx} - \alpha \xi \xi_x^2 + \alpha \xi \xi_x - \alpha \xi \xi_xt = 0,
\] (3.4v)
\[
3\xi \phi_x \phi_{xx} + \beta \xi \phi_{xx} - \xi \phi_{xx} - 3\xi \phi_x \phi_x - \phi_{xx} + \alpha \xi \phi_x^2 + 3\xi \phi_{xxx} \phi_x - \xi \phi_{xxx} \phi_x \\
- 2\xi \phi_x \phi_x + \xi \phi_x \phi_{xxx} - \xi \phi_{xxx} - \xi \phi_{xxx} \phi_x = 0,
\] (3.4vi)
\[
3\xi \phi_u \phi_{xx} - 6\xi \phi_x \phi_{xx} + 3\xi \phi_{xx} - \alpha \xi \phi_x - 5\xi \phi_x^2 \phi_x + \beta \phi \phi_{xxx} - 3\xi \phi_x \phi_x + 3 \phi_{xxx} + \beta \phi \phi_t \\
+ \beta \xi \xi \phi - \phi_x - \xi \phi_{xxx} + 6\xi \phi_{xx} \phi_x + 3 \xi \phi_x + 2 \xi^2 \phi_x - \xi \phi_x - 3\xi \xi_{xt} + \xi_{xt} = 0,
\] (3.4vii)
\[
9\xi \xi_{xxx} \phi_x + \beta \xi \phi_x + \alpha \xi \phi_x + 3 \xi \phi_x \phi_x - 2 \alpha \phi \phi_x - 6 \xi \phi_{xx} - \alpha \phi \phi_{xx} \phi_x - \xi \phi_{xxx} \\
+ 2 \xi \phi_x + \xi \phi_{xxx} \phi_x - 3 \xi \phi_x \phi_{xxx} - 3 \xi^2 \phi_{xx} - 2 \beta \phi \phi_{xxx} - 3 \phi_{xxx} + 2 \xi \phi_x - \xi \phi_x - 2 \xi^2 \phi_x \\
- \xi_t \phi_u + \xi \phi_{tu} + \xi \phi_x - \xi \phi_{xx} + \beta \xi \phi_x + \phi_x \phi_{xxx} - \alpha \phi \phi_x \\
- \xi_t \xi_x - \xi \xi_x + \xi \phi_{xx} = 0,
\] (3.4viii)
These equations were calculated using the MACSYMA package symmgrp.max [15]. We then used the method of DGBs as outlined in [18,20] to solve this system.

**Case 3.1** \( \alpha + \beta \neq 0 \). In this case it is straightforward to obtain from (3.4i–v), the condition

\[
\xi_{xx} \xi^2 (\alpha + \beta)(3\beta - 2\alpha) = 0.
\]

The case \( 3\beta - 2\alpha = 0 \) leads to no solutions different from those obtained using the classical method.

**Subcase 3.1.1** \( \xi_x \neq 0 \). This is the generic case which has the solution

\[
\xi = (\kappa_1 x + \kappa_2) f(t), \quad \phi = -\kappa_1 f(t) \left( u - \frac{2x}{\alpha} + \frac{\kappa_2 - t}{\beta} \right) + \frac{1}{\beta},
\]

where \( f(t) \) is an arbitrary differentiable function and \( \kappa_1 \neq 0 \) and \( \kappa_2 \) are arbitrary constants. These are equivalent to the infinitesimals (2.7) obtained using the classical method.

**Subcase 3.1.2** \( \xi_x = 0 \). In this case it is easy to obtain the condition

\[
\phi_{xx} \xi^3 (\beta - \alpha)(\alpha + \beta) = 0.
\]

There are two subcases to consider.

(i) \( \alpha \neq \beta, \xi_x = 0 \). In this case the solution is

\[
\xi = f(t), \quad \phi = \kappa_3 f(t) + 1/\beta,
\]

where \( f(t) \) is an arbitrary differentiable function and \( \kappa_3 \) is an arbitrary constant, which is equivalent to the infinitesimals (2.7) obtained using the classical method in the case when \( \kappa_1 = 0 \).

(ii) \( \alpha = \beta, \xi_x = 0 \). In this case, we obtain the following DGB for \( \xi, \phi \)

\[
\begin{align*}
\xi_u &= 0, \quad \xi_x = 0, \quad \phi_u = 0, \\
\xi \phi_{xxx} - (\xi + 1) \phi_{xx} + \beta \phi_{xx} + \beta \phi_x^2 &= 0, \\
\beta \xi^2 \phi_x - \beta \xi \phi_t + \beta \xi_t \phi - \xi_t &= 0.
\end{align*}
\]

Solving these yields

\[
\xi = \frac{df}{dt}, \quad \phi = 2V(\zeta) \frac{df}{dt} + \frac{1}{\beta}, \quad (3.5)
\]

where \( \zeta = x + f(t) \), \( f(t) \) is an arbitrary differentiable function and \( V(\zeta) \) satisfies

\[
V_{\zeta \zeta} + \beta V^2 - V = \kappa_4 \zeta + \kappa_5, \quad (3.6)
\]

with \( \kappa_4 \) and \( \kappa_5 \) arbitrary constants. If \( \kappa_4 \neq 0 \), then this equation is equivalent to the first Painlevé equation [36]

\[
\frac{d^2 y}{dx^2} = 6y^2 + x, \quad (3.7)
\]

otherwise it is equivalent to the Weierstrass elliptic function equation (2.14).

Thus solving the characteristic equations (2.4) yields the nonclassical reduction

\[
u(x,t) = v(\zeta) + w(z) + t/\beta, \quad \zeta = x + f(t), \quad z = x - f(t), \quad (3.8)
\]

where \( f(t) \) is an arbitrary function and \( V(\zeta) = v_\zeta \) and \( W(z) = w_z \) satisfy

\[
V_{\zeta \zeta} + \beta V^2 - V = -\lambda \zeta + \mu_1, \quad (3.9a)
\]
and

\[ W_{zz} + \beta W^2 - W = \lambda z + \mu_2, \tag{3.9b} \]

respectively, where \( \mu_1 \) and \( \mu_2 \) are arbitrary constants and \( \lambda \) is (effectively) a “separation” constant. If \( \lambda \neq 0 \) then these equations are equivalent to the first Painlevé equation (3.7), whilst if \( \lambda = 0 \) then they are equivalent to the Weierstrass elliptic function equation (2.14).

In particular, if \( \lambda = 0 \) (we set \( \beta = 1 \) without loss of generality) then equations (3.9) possess the special solutions

\[ V(\zeta) = \{6\kappa_1^2 \sech^2(\kappa_1 \zeta) + \frac{1}{2} - 2\kappa_1^2\}, \quad W(z) = \{6\kappa_2^2 \sech^2(\kappa_2 z) + \frac{1}{2} - 2\kappa_2^2\}, \]

where \( \kappa_1 = \frac{1}{2}(1 + 4\mu_1)^{1/4} \) and \( \kappa_2 = \frac{1}{2}(1 + 4\mu_2)^{1/4} \). Hence we obtain the exact solution of (1.8) given by

\[ u(x,t) = \{6\kappa_1 \tanh \{\kappa_1 \left[x + f(t)\right]\} + 6\kappa_2 \tanh \{\kappa_2 \left[x - f(t)\right]\} + x(1 - 2\kappa_1^2 - 2\kappa_2^2) + 2f(t)(\kappa_2^2 - \kappa_1^2) + t\}, \tag{3.10} \]

where \( f(t) \) is an arbitrary differentiable function.

If \( \mu_1 = \mu_2 = 0 \) then \( \kappa_1 = \kappa_2 = \frac{1}{2} \) (with \( \beta = 1 \)) and (3.10) simplifies to

\[ u(x,t) = \{3 \tanh \left\{ \frac{1}{2} \left[x + f(t)\right]\} + 3 \tanh \left\{ \frac{1}{2} \left[x - f(t)\right]\} + t\}. \tag{3.11} \]

In Figure 1 we plot \( u_x \) with \( u \) given by (3.11) for various choices of the arbitrary function \( f(t) \). This is one of the simplest, nontrivial family of solutions of (1.1) with \( \alpha = \beta = 1 \), using this reduction. In Figure 1, \( f(t) \) is chosen so that \( f(t) \sim t + t_0 \), as \( t \to -\infty \), where \( t_0 \) is a constant. Consequently all the solutions plotted in Figure 1 have a similar asymptotic behaviour as \( t \to -\infty \). However the asymptotic behaviours as \( t \to \infty \) are radically different.

In the special case when \( f(t) = ct \), then choosing \( \kappa_1 = \frac{1}{2}(1 + 1/c)^{1/2} \) and \( \kappa_2 = \frac{1}{2}(1 - 1/c)^{1/2} \) in (3.10) yields the two-soliton solution for (1.8) given by

\[ u(x,t) = \frac{3}{\sqrt{c}} \left\{ \sqrt{c + 1} \tanh \left[ \frac{1}{2} \sqrt{1 + 1/c} \left(x + ct\right)\right] + \sqrt{c - 1} \tanh \left[ \frac{1}{2} \sqrt{1 - 1/c} \left(x - ct\right)\right]\right\}. \tag{3.12} \]

This solution is of special interest since such two-soliton solutions are normally associated with so-called Lie-Bäcklund transformations (cf., [7]), whereas (3.12) has arisen from a Lie point symmetry, albeit nonclassical. Plots of (3.12) and its \( x \)-derivative are given in Figure 2(a) for \( c = 3 \). These should be compared with the plots in Figure 2(b) of the \( x \)-derivative of solution (4.6) below for the SWW-IHS equation with \( \kappa_1 = 2, \kappa_2 = 1.7 \) and \( \kappa_1 = \frac{3}{2}, \kappa_2 = \frac{3}{2} \) (so that \( A_{12} = 1 \) in both cases).

We stress that the “decoupling” of the nonclassical reduction (3.8) into a function of \( \zeta = x + f(t) \) and a function of \( z = x - f(t) \) occurs for the SWW equation (1.1) only in this special case when \( \alpha = \beta \).

Case 3.2 \( \alpha + \beta = 0 \). This case also leads to no solutions different from those obtained using the classical method in §2.

To conclude this section we briefly consider the equations

\begin{align*}
&u_{xxxx} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{3.13} \\
&u_{xxtt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{3.14}
\end{align*}

where \( \alpha \) and \( \beta \) are arbitrary, nonzero, constants, which are variants of the SWW equation (1.1). Both these equations are thought to be nonintegrable, i.e. not solvable by inverse scattering, for all values of \( \alpha \) and \( \beta \) since it is straightforward to show that neither satisfies the Painlevé PDE test due to Weiss et al [55].
Applying the classical Lie method to equations (3.13) and (3.14) yields the infinitesimals

\[
\begin{align*}
    \xi & = \kappa_1 x + \kappa_2, \\
    \tau & = (3\kappa_1 + \kappa_3)t + \kappa_4, \\
    \phi & = (\kappa_1 + \kappa_3)u - \frac{\kappa_3 x}{\alpha} + \frac{2\kappa_1 t}{\beta} + \kappa_5,
\end{align*}
\]

respectively, where \(\kappa_1, \kappa_2, \ldots, \kappa_5\) are arbitrary constants. In contrast to the SWW equation (1.1), these symmetry groups are finite dimensional.

If \(\alpha = \beta\), then equations (3.13) and (3.14) both possess nonclassical reductions of the form

\[
u(x, t) = v(\zeta) + w(z), \quad \zeta = x + ct, \quad z = x - ct
\]

where for (3.13), \(V(\zeta) = v_\zeta\) and \(W(z) = w_z\) satisfy

\[
\begin{align*}
    V_{\zeta\zeta} - (1 + c)V + \beta c V^2 & = -\lambda \zeta + \lambda_1, \tag{3.15i} \\
    W_{zz} + (c - 1)W - \beta c W^2 & = \lambda z + \lambda_2, \tag{3.15ii}
\end{align*}
\]

and for (3.14), \(V(\zeta) = v_\zeta\) and \(W(z) = w_z\) satisfy

\[
\begin{align*}
    c^2 V_{\zeta\zeta} - (1 + c)V + \beta c V^2 & = -\lambda \zeta + \lambda_1, \tag{3.16i} \\
    c^2 W_{zz} + (c - 1)W - \beta c W^2 & = \lambda z + \lambda_2, \tag{3.16ii}
\end{align*}
\]

with \(\lambda, \lambda_1\) and \(\lambda_2\) arbitrary constants. If \(\lambda \neq 0\), then equations (3.15,3.16) are solvable in terms of the first Painlevé equation (3.7), whilst if \(\lambda = 0\) then they are equivalent to the Weierstrass elliptic function equation (2.14).

Setting \(\lambda = \lambda_1 = \lambda_2 = 0\), and solving equations (3.15) and (3.16) yields the exact solutions of (3.13) and (3.14) with \(\alpha = \beta\) given by

\[
\begin{align*}
    u(x, t) & = \frac{3}{\beta c} \left\{ \sqrt{1 + c} \tanh \left[ \frac{1}{2} \sqrt{1 + c} (x + ct) \right] - \sqrt{1 - c} \tanh \left[ \frac{1}{2} \sqrt{1 - c} (x - ct) \right] \right\}, \tag{3.17} \\
    u(x, t) & = \frac{3}{\beta} \left\{ \sqrt{1 + c} \tanh \left[ \frac{1}{2c} (x + ct) \right] - \sqrt{1 - c} \tanh \left[ \frac{\sqrt{1 - c}}{2c} (x - ct) \right] \right\}, \tag{3.18}
\end{align*}
\]

respectively, which are analogues of (3.12). Plots of the solutions (3.17) and (3.18) and their derivatives with respect to \(x\) are given in Figures 3 and 4, respectively. The solutions (3.17) and (3.18) are plotted in Figures 3(i) and 4(i) and look like the elastic interaction of a “kink” and an “anti-kink” solution. The \(x\)-derivatives of (3.17) and (3.18) are plotted in Figures 3(ii) and 4(ii) and look like the elastic interaction of two “soliton” solutions. These plots are very similar to the plots of (3.12) and its \(x\)-derivative given in Figure 2, however whereas (1.8) is completely integrable, both (3.13) and (3.14) are seemingly non-integrable. These solutions (3.17) and (3.18) are of particular interest since such solutions are normally associated with integrable equations, whereas they arise here for nonintegrable equations. Furthermore, like (3.12), these “two-soliton” solutions have also arisen from nonclassical reductions. As mentioned above, such solutions are normally associated with Lie-Bäcklund transformations (cf., [7]).
4 Soliton Solutions of the SWW-HS and SWW-AKNS Equations.

4.1 The SWW-HS Equation. Exact solutions of the SWW-HS equation (1.8) can be obtained using the so-called singularity manifold method which uses truncated Painlevé expansions [54,55]. If we seek a solution of (1.8) in the form

\[ u(x,t) = \frac{6}{\beta} \frac{\phi(x,t)}{\phi(x,t)} \tag{4.1} \]

and then equate coefficients of powers of \( \phi \) to zero, we find that \( \phi(x,t) \) satisfies the overdetermined system

\[
\begin{align*}
\phi_{xxxx} - \phi_{xx} &= 0, \\
\phi_t \phi_{xx} - 3\phi_{xt}\phi_{xx} + 3\phi_x \phi_{xxt} - \phi_x (\phi_x + \phi_t) &= 0. 
\end{align*} \tag{4.2a, 4.2b}
\]

(A DGB analysis of this system leads to some very complex expressions. Although it does yield some ODEs in \( x \) for \( \phi \) in the various subcases they appear difficult to solve.) Now suppose we seek a solution of these equations in the form

\[ \phi(x,t) = \alpha_1 \exp \{\kappa_1 x + \mu_1 t\} + \alpha_2 \exp \{\kappa_2 x + \mu_2 t\} + \alpha_0, \tag{4.3} \]

where \( \alpha_0, \alpha_1, \alpha_2, \kappa_1, \kappa_2, \mu_1 \) and \( \mu_2 \) are constants such that \( \alpha_0 \alpha_1 \alpha_2 \neq 0 \). It is straightforward to show that equations (4.2) have a solution of the form (4.3) provided that \( \mu_1 = \kappa_1/(\kappa_1^2 - 1) \), \( \mu_2 = \kappa_2/(\kappa_2^2 - 1) \) and \( \kappa_1 \) and \( \kappa_2 \) satisfy the constraint

\[ \kappa_1^2 - \kappa_1 \kappa_2 + \kappa_2^2 = 3. \tag{4.4} \]

Thus we obtain the following exact solution of the SWW-HS equation (1.8) given by

\[
\begin{align*}
u(x,t) &= \frac{6}{\beta} \frac{\alpha_1 \kappa_1 \exp \{\kappa_1 x + \kappa_1 t/(\kappa_1^2 - 1)\} + \alpha_2 \kappa_2 \exp \{\kappa_2 x + \kappa_2 t/(\kappa_2^2 - 1)\}}{\alpha_0 + \alpha_1 \exp \{\kappa_1 x + \kappa_1 t/(\kappa_1^2 - 1)\} + \alpha_2 \exp \{\kappa_2 x + \kappa_2 t/(\kappa_2^2 - 1)\}}, 
\end{align*} \tag{4.5}
\]

provided \( \kappa_1 \) and \( \kappa_2 \) satisfy (4.4). It should be noted that (4.5) and (3.12) are fundamentally different solutions of the SWW-HS equation (1.8) as we shall now demonstrate.

The general two-soliton solution of (1.8) is given by

\[ \phi(x,t) = 1 + \alpha_1 \exp (\eta_1) + \alpha_2 \exp (\eta_2) + A_{12} \exp (\eta_1 + \eta_2) \tag{4.6a} \]

where

\[
\begin{align*}
\eta_1 &= \kappa_1 x + \kappa_1 t/(\kappa_1^2 - 1), \quad \eta_2 = \kappa_2 x + \kappa_2 t/(\kappa_2^2 - 1), \\
A_{12} &= \alpha_1 \alpha_2 \frac{(\kappa_1 - \kappa_2)^2 (\kappa_1^2 - \kappa_1 \kappa_2 + \kappa_2^2 - 3)}{(\kappa_1 + \kappa_2)^2 (\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2 - 3)},
\end{align*} \tag{4.6b, 4.6c}
\]

with \( \alpha_1, \alpha_2, \kappa_1 \) and \( \kappa_2 \) arbitrary constants [34] (see also [43]). A straightforward method of obtaining this solution is to substitute (4.1) into SWW-HS equation (1.8) and then integrate once. This yields the bi-linear equation

\[ \phi \phi_{xxxx} - \phi_t \phi_{xx} + 3(\phi_x \phi_{xt} - \phi_x \phi_{xxt}) - \phi (\phi_{xx} + \phi_{xt}) + \phi_x (\phi_x + \phi_t) = 0, \tag{4.7} \]

where we have set the function of integration to zero. Now, by seeking a solution of this equation, in the form

\[ \phi(x,t) = 1 + \alpha_1 \exp (\eta_1) + \alpha_2 \exp (\eta_2) + A_{12} \exp (\eta_1 + \eta_2), \]
with \( \eta_1 = \kappa_1 x + \mu_1 t \) and \( \eta_2 = \kappa_2 x + \mu_2 t \), it is easy to show that necessarily \( \mu_1 = \kappa_1 / (\kappa_1^2 - 1) \), \( \mu_2 = \kappa_2 / (\kappa_2^2 - 1) \) and \( A_{12} \) is given by (4.6c); this technique may be viewed as a variant of Hirota's bi-linear method [32]. Four plots of the \( x \)-derivative of (4.6), i.e. solutions of (1.6), are given in Figure 5 for (i), \( \kappa_1 = 2, \kappa_2 = 1.7 \), (ii), \( \kappa_1 = 3/4, \kappa_2 = 2/3 \), (iii), \( \kappa_1 = 1.1, \kappa_2 = (11 + 3\sqrt{33})/20 \) (so that \( A_{12} = 0 \)), and (iv), \( \kappa_1 = 0.8, \kappa_2 = \sqrt{2 + 3\sqrt{7}} /2 \) (so that \( A_{12} = 0 \)). Plots 5(i) and (ii) illustrate “standard” two-soliton interaction whilst plots 5(iii) and (iv) are in the special case when \( A_{12} = 0 \). Plots of the \( x \)-derivative of (4.6) for Figure 2(b) of the x-derivative of solution (4.6) below for the SWW-HS equation, i.e. solutions of (1.6), for \( \kappa_1 = 2, \kappa_2 = 1.7 \) and \( \kappa_1 = 3/4, \kappa_2 = 2/3 \) (so that \( A_{12} = 1 \) in both cases).

The solution (4.5) is the special case of (4.6) with \( A_{12} = 0 \); Hirota and Ito [33] refer to this as being the “resonant state” where either a single soliton splits into two solitons, see Figure 5(iii), or two solitons fuse together after colliding with each other, see Figure 5(iv). On the other hand, (3.12) is the special case of (4.6) with \( A_{12} = 1 \) where two solitons pass through each other with no phase shift as a consequence of the interaction. Thus whereas both solutions are asymptotically equivalent as \( t \to -\infty \), they are qualitatively very different as \( t \to \infty \). This shows that the nonclassical method and the singular manifold method do not, in general, yield the same solution set.

We remark that the SWW-HS equation (1.8) also possesses the solution

\[
u(x,t) = \frac{6}{\beta} \frac{\kappa_1 [\alpha_1 \exp(\eta_1) - \alpha_2 \exp(-\eta_1)] - \kappa_2 B_{12} \sin(\eta_2 + \delta_0)}{\alpha_1 \exp(\eta_1) + \alpha_2 \exp(-\eta_1) + B_{12} \cos(\eta_2 + \delta_0)},
\]

where

\[
\eta_1 = \kappa_1 x + \frac{(\kappa_1^2 + \kappa_2^2 - 1)t}{(\kappa_1^2 - 2\kappa_1 + \kappa_2^2 + 1)(\kappa_1^2 + 2\kappa_1 + \kappa_2^2 + 1)};
\]

\[
\eta_2 = \kappa_2 x - \frac{(\kappa_1^2 + \kappa_2^2 + 1)t}{(\kappa_1^2 - 2\kappa_1 + \kappa_2^2 + 1)(\kappa_1^2 + 2\kappa_1 + \kappa_2^2 + 1)};
\]

\[
B_{12} = 2(\alpha_1 \alpha_2)^{1/2} \kappa_1 \kappa_2 \left(\frac{3\kappa_1^2 - \kappa_2^2 - 3}{3\kappa_2^2 - \kappa_1^2 + 3}\right)^{1/2},
\]

with \( \alpha_1, \alpha_2, \kappa_1, \kappa_2 \) and \( \delta_0 \) arbitrary constants. Leble and Ustinov [38] show that solutions of the form (4.8) exist for several of PDEs that are solvable by inverse scattering through third order linear problems. Two plots of the \( x \)-derivative of (4.8), i.e. solutions of (1.6), are given in Figure 6 for (i), \( \kappa_1 = 1, \kappa_2 = 1.17 \) and (ii), \( \kappa_1 = 1, \kappa_2 = 1.24 \).

### 4.2 The SWW-AKNS Equation.

The general two-soliton solution of the SWW-AKNS equation (1.7) is given by

\[
u(x,t) = \frac{6}{\beta} \frac{\kappa_1 \alpha_1 \exp(\eta_1) + \kappa_2 \alpha_2 \exp(\eta_2) + (\kappa_1 + \kappa_2) A_{12} \exp(\eta_1 + \eta_2)}{1 + \alpha_1 \exp(\eta_1) + \alpha_2 \exp(\eta_2) + A_{12} \exp(\eta_1 + \eta_2)},
\]

where

\[
\eta_1 = \kappa_1 x + \kappa_1 t/(\kappa_1^2 - 1), \quad \eta_2 = \kappa_2 x + \kappa_2 t/(\kappa_2^2 - 1), \quad A_{12} = \alpha_1 \alpha_2 \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2.
\]

with \( \alpha_1, \alpha_2, \kappa_1 \) and \( \kappa_2 \) arbitrary constants [34] (see also [43]). Two plots of the \( x \)-derivative of (4.9), i.e. solutions of (1.3), are given in Figure 7 for (i), \( \kappa_1 = 2, \kappa_2 = 1.7 \) and (ii), \( \kappa_1 = \frac{4}{5}, \kappa_2 = \frac{4}{5} \).

To apply the singularity manifold method to the SWW-AKNS equation (1.7), we seek a solution in the form

\[
u(x,t) = \frac{4}{\beta} \frac{\phi_x(x,t)}{\phi(x,t)}.
\]
Equating coefficients of powers of \( \phi \) to zero, yields the overdetermined system
\[
\begin{align*}
\phi_{xxxxx} - \phi_{xxx} - \phi_{xx} &= 0, \\
\phi_t \phi_{xxx} + 4 \phi_x \phi_{xxxt} - 2 \phi_{xx} \phi_{xxt} - 3 \phi_x \phi_{xx} - \phi_t \phi_{xx} - 2 \phi_x \phi_{xt} &= 0, \\
2 \phi_t \phi_x \phi_{xx} - 2 \phi_{xx} \phi_x \phi_{xt} &= 0.
\end{align*}
\]
(4.11a) (4.11b) (4.11c)

We note that this is a system of three equations in contrast to that for the SWW-HS equation (1.8) where \( \phi(x,t) \) satisfies a system of two equations (4.2). It is easily shown that there exist no solutions of (4.11) in the form (4.3) if \( \alpha \neq 0 \). Additionally substituting (4.10) into (1.7) yields a tri-linear equation for \( \phi(x,t) \), whereas the analogous equation for (1.8), i.e. (4.7), is bi-linear.

5 Nonlinear Superposition for 2+1-dimensional Equations.

In this section we show that nonlinear superposition of solutions can also occur for two 2+1-dimensional equations of the SWW-HS equation (1.8). Two 2+1-dimensional generalisations of the SWW equation (1.1) are the following equations
\[
\begin{align*}
u_{yy} + \alpha u_x u_y + \beta u_y u_{xx} - u_{xxxy} &= 0, \\
u_{xt} + \alpha u_x u_y + \beta u_y u_{xx} - u_{xxxy} &= 0.
\end{align*}
\]
(5.1) (5.2)

We remark that these two equations differ only in the first term, which belongs to the linear part, yet their symmetries differ remarkably. Further both (5.1) and (5.2) reduce to the KdV equation (1.4) if \( y = x \).

Boiti, Leon, Manna and Pempinelli [13] developed an inverse scattering scheme to solve the Cauchy problem for (5.1) with \( \alpha = \beta \), i.e.
\[
u_{yt} + u_{xxx} + \beta u_{x} u_{yy} + \beta u_{x} u_{yy} = 0,
\]
(5.3)

which is a 2+1-dimensional generalisation of the SWW-HS equation (1.8), for initial data decaying sufficiently rapidly at infinity. This inverse scattering scheme is formulated as a nonlocal Riemann-Hilbert problem and involves a so-called “weak-Lax pair”. We note that both (5.3) and SWW-HS (1.8) are reductions of the 3+1-dimensional equation
\[
u_{yt} + u_{xxx} + \beta u_{x} u_{yy} + \beta u_{x} u_{yy} - u_{xx} = 0,
\]
(5.4)

which was introduced by Jimbo and Miwa [37] as the second equation in the so-called Kadomtsev-Petviashvili hierarchy, though (5.4) is not completely integrable in the usual sense (cf., [25]). Bogoyavlenskii [11,12] discusses the inverse scattering method of solution for (5.2) with \( \alpha = 2 \beta \), i.e.,
\[
u_{xt} + u_{xxx} + \beta u_{x} u_{yy} + 2 \beta u_{x} u_{xy} = 0,
\]
(5.5)

which is a 2+1-dimensional generalisation of the SWW-AKNS equation (1.7); it should be noted that the 2+1-dimensional generalisation of (1.7) discussed by Gilson et al [28] is different to (5.2).

It is routine to show that (5.1) satisfies the necessary conditions of the Painlevé PDE test due to Weiss et al [55] to be completely integrable if and only if \( \alpha = \beta \), i.e. when it reduces to (5.3), and (5.2) satisfies these necessary conditions if and only if \( \alpha = 2 \beta \), i.e. when it reduces to (5.5). This strongly suggests that (5.1) and (5.2) are solvable by inverse scattering only in these two special cases, both of which are known to be completely integrable ([13] and [11,12], respectively).

5.1 Classical Symmetries. To apply the classical method to (5.1), we consider the one-parameter Lie group of infinitesimal transformations in \( (x,y,t,u) \) given by
\[
\begin{align*}
\tilde{x} &= x + \varepsilon \xi_1(x,y,t,u) + O(\varepsilon^2), \\
\tilde{y} &= y + \varepsilon \xi_2(x,y,t,u) + O(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon \xi_3(x,y,t,u) + O(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon \phi(x,y,t,u) + O(\varepsilon^2),
\end{align*}
\]
where \( \varepsilon \) is the group parameter. Solving the resulting determining equations yields the infinitesimals
\[
\xi_1 = \kappa_1 x + f_1(t), \quad \xi_2 = g(y), \quad \xi_3 = 3\kappa_1 t + \kappa_2, \quad \phi = -\kappa_1 u + \frac{xf_1}{\alpha} + f_2(t),
\]
if \( \alpha \neq \beta \) and
\[
\xi_1 = x \frac{df_1}{dt} + f_2(t), \quad \xi_2 = g(y), \quad \xi_3 = 3f_1(t), \quad \phi = -u \frac{df_1}{dt} + \frac{x^2}{2\alpha} \frac{d^2 f_1}{dt^2} + \frac{xf_2}{\alpha} + f_3(t),
\]
if \( \alpha = \beta \), where \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are arbitrary constants and \( f_1(t), f_2(t), f_3(t) \) and \( g(y) \) are arbitrary differentiable functions \([21]\) (classical symmetries in the case \( \alpha = \beta \) are also discussed in \([52]\)).

Applying the classical Lie method to (5.2) yields the infinitesimals
\[
\xi_1 = \kappa_1 x + f_1(t), \quad \xi_2 = \kappa_2 t + \kappa_3 y + \kappa_4, \quad \xi_3 = (2\kappa_1 + \kappa_3) t + \kappa_5, \quad \phi = -\kappa_1 u + \frac{y df_1}{\beta} + \frac{\kappa_2}{\alpha} + f_2(t),
\]
if \( \alpha \neq 2\beta \) and
\[
\xi_1 = (\kappa_1 t + \kappa_2) x + f_1(t), \quad \xi_2 = (2\kappa_1 y + \kappa_3) t + \kappa_4 y + \kappa_5, \quad \xi_3 = 2\kappa_1 t^2 + (2\kappa_2 + \kappa_4) t + \kappa_6, \quad \phi = -(\kappa_1 t + \kappa_2) u + \frac{y}{\beta} \left( \kappa_1 x + \frac{df_1}{dt} \right) + \frac{\kappa_3 x}{2\beta} + f_2(t),
\]
if \( \alpha = 2\beta \), where \( \kappa_1, \kappa_2, \ldots, \kappa_6 \) are arbitrary constants and \( f_1(t) \) and \( f_2(t) \) are arbitrary differentiable functions \([21]\).

### 5.2 Nonclassical Reductions

Applying the nonclassical method to (5.1) yields an additional reduction in the case when \( \alpha = \beta \), when it reduces to (5.3). This nonclassical reduction is given by
\[
u(x, y, t) = v(\eta, \tau) + w(\zeta, \tau) - \frac{f(t)g^2(y)}{2\beta} \frac{df}{dt}, \quad \eta = x + f(t)g(y), \quad \zeta = x - f(t)g(y), \quad \tau = t, \tag{5.6}
\]
where \( V(\eta, \tau) = v_\eta \) and \( W(\zeta, \tau) = w_\zeta \) satisfy the variable-coefficient KdV equations
\[
V_{\eta\eta} - 2\beta VV_\eta - V_\tau - F(\tau)V + \frac{\eta}{2\beta} \left\{ \frac{dF}{d\tau} + 2F^2(\tau) \right\} + Q(\tau) = 0, \tag{5.7}
\]
\[
W_{\zeta\zeta} - 2\beta WW_\zeta - W_\tau - F(\tau)W + \frac{\zeta}{2\beta} \left\{ \frac{dF}{d\tau} + 2F^2(\tau) \right\} + Q(\tau) = 0, \tag{5.8}
\]
with \( F(\tau) = \dot{f}(t)/f(t) \) and \( Q(\tau) \) an arbitrary function. It is easily shown that (5.7) and (5.8) satisfy the necessary conditions of the Painlevé test due to Weiss et al \([55]\) to be completely integrable and both equations can be transformed into the usual KdV equation (1.4). In particular, if in (5.7) and (5.8) we set \( f(t) = \kappa \), a constant, then we obtain solutions to (5.1) with \( \alpha = \beta \) in terms of sums of solutions of the KdV equation (1.4), with arguments \( x \pm g(y) \), where \( g(y) \) is an arbitrary function; this is perhaps the simplest family of solutions to (5.3) found using this nonclassical reduction.

Applying the nonclassical method to (5.2) yields an additional reduction in the case when \( \alpha = \beta \), given by
\[
u(x, y, t) = v(\eta, \tau) + w(\zeta, \tau), \quad \eta = x + \kappa y, \quad \zeta = x - \kappa y, \quad \tau = t, \tag{5.9}
\]
where \( V(\eta, \tau) = v_\eta \) and \( W(\zeta, \tau) = w_\zeta \) satisfy
\[
\kappa V_{\eta\eta} - \beta \kappa VV_\eta - V_\tau + Q(\tau) = 0, \tag{5.10}
\]
\[
\beta \kappa W_{\zeta\zeta} - \beta \kappa WW_\zeta - W_\tau + Q(\tau) = 0, \tag{5.11}
\]
with \( Q(\tau) \) an arbitrary function.
6 Discussion.

In this paper we have discussed symmetry reductions and exact solutions for the shallow water wave equation (1.1). In particular, for the special case of (1.1) given by the SWW-HS equation (1.8), using the nonclassical symmetry reduction method originally proposed by Bluman and Cole [9], we obtained a family of solutions (3.10) which have a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function \( f(t) \). One can choose \( f_1(t) \) and \( f_2(t) \) such that \( |f_1(t) - f_2(t)| \) is exponentially small as \( t \to -\infty \), yet \( f_1(t) \) and \( f_2(t) \) are quite different as \( t \to -\infty \), so that as \( t \to -\infty \) the two solutions are essentially the same, yet as \( t \to \infty \) they are radically different. In Figure 1 we show that by a judicious choice of \( f(t) \) we can exhibit a plethora of different solutions. We believe that these results suggest that solving the SWW-HS equation (1.8) numerically for initial conditions such as those in the solutions plotted in Figure 1 could pose some fundamental difficulties. An exponentially small change in the initial data yields a fundamentally different solution as \( t \to \infty \). How can any numerical scheme in current use cope with such behaviour?

The solution (3.10) appears to be a nonlinear superposition of solutions suggesting that the SWW-HS equation (1.8) may be linearisable through a transformation to a linear PDE, analogous to the linearisation of Burgers’ equation

\[
\frac{\partial u}{\partial t} = u_{xx} + 2uu_x,
\]

which is mapped to the linear heat equation through the Cole-Hopf transformation [22,35]. If so then the solution (3.10) could be viewed as an artefact of the fact that the SWW-HS equation (1.8) is linearisable. However the SWW-HS equation (1.8) can be expressed as the compatibility condition of the third order spectral problem (1.11,1.12). Further the associated scattering problem (1.11) is very similar to that for the Boussinesq equation which has been thoroughly studied by Deift et al [24]. This strongly suggests the SWW-HS equation (1.8) is solvable by inverse scattering. The spatial part of the inverse scattering formalism (1.11) only defines \( u \) up to an arbitrary additive function of \( t \); this arbitrary function may be incorporated into \( u \) using the variable-coefficient transformation (2.15). Indeed the initial value problem for the SWW-HS equation (1.8) is not well-posed without the imposition of an additional constraint since if \( u(x,t) \) is a solution of (1.8) satisfying the initial condition \( u(x,0) = \phi(x) \), then so is

\[
\tilde{u}(\tilde{x},\tilde{t}) = u(x,t) + [g(t) - t]/\beta, \quad \tilde{x} = x, \quad \tilde{t} = g(t),
\]

where \( g(t) \) is any differentiable function such that \( g(0) = 0 \). It appears likely that the inverse scattering formalism for the SWW-HS equation (1.8) will require that \( u(x,t) \) satisfies a constraint of the form

\[
\int_{-\infty}^{\infty} |u(x,t)| \, dx < \infty,
\]

for all \( t \). It is well-known that such constraints are required in the inverse scattering formalism for 2 + 1-dimensional equations such as the Kadomtsev-Petviashvili equation (cf., [6], see also [1] and the references therein).

Since the shallow water wave equation (1.1) is invariant under the variable-coefficient transformation (2.15) for all nonzero \( \alpha \) and \( \beta \), one can take any solution of (1.1) and using (2.15) generate some interesting solutions. For example, one can apply the transformation (2.15) to the two-soliton solution (4.9) and thus generate some exotic solutions for the SWW-AKNS equation (1.7) analogous to the solution (3.10) of the SWW-HS equation (1.8). Consequently the above remarks about difficulties in solving (1.8) numerically apply to (1.1) in general. Additionally, the inverse scattering formalism for the SWW-AKNS equation (1.8) will probably require that \( u(x,t) \) satisfies a constraint such as (6.3).

Recently Ablowitz, Schober and Herbst [5] have shown that the focusing nonlinear Schrödinger equation

\[
\frac{\partial u}{\partial t} + u_{xx} + |u|^2u = 0,
\]
exhibits numerical chaos created by small errors on the order of roundoff. The results of Ablowitz et al [5] together with those given in this paper suggest that numerical analysts need to take care to ensure the accuracy of their programs. Numerical predictions of chaos may not always be what they seem!

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Figure captions

Figure 1. The solution (3.11) where
(i), \( f(t) = \frac{1}{2} t \),
(ii), \( f(t) = \frac{1}{2} t - \exp \left( \frac{1}{5} t - 1 \right) \),
(iii), \( f(t) = \frac{1}{2} t + [1 - \tanh t] \sin t \),
(iv), \( \frac{1}{2} t - \exp \left( -t^2/100 \right) \sin t \),
(v), \( f(t) = \frac{1}{2} t + 6\pi^{-1} \tan^{-1} t \),
(vi), \( f(t) = \frac{1}{2} \sqrt{t^2 + 20} \),
(vii), \( f(t) = f(t) = \frac{1}{4} t(1 - \tanh t) + \frac{1}{2}[2\sin(\frac{2}{3} t) + 3](1 + \tanh t) \),
(viii), \( f(t) = \frac{1}{4} t(1 - \tanh t) \).

Figure 2. (a) The solution (3.12) of the SWW-HS equation (1.8) and its \( x \)-derivative for \( c = 3 \).

Figure 2. (b) The \( x \)-derivative of solution (4.6) of the SWW-HS equation (1.8) for \( \kappa_1 = 2, \kappa_2 = 1.7 \)
and \( \kappa_1 = \frac{3}{4}, \kappa_2 = \frac{2}{3} \).

Figure 3. The solution (3.17) of equations (3.13) and its \( x \)-derivative for \( c = \frac{1}{4} \).

Figure 4. The solution (3.18) of equations (3.14) and its \( x \)-derivative for \( c = \frac{1}{4} \).

Figure 5. The solution (4.6) of the SWW-HS equation (1.8).
(i), \( \kappa_1 = 2, \kappa_2 = 1.7 \)
(ii), \( \kappa_1 = \frac{3}{4}, \kappa_2 = \frac{2}{3} \),
(iii), \( \kappa_1 = 1, \kappa_2 = (11 + 3\sqrt{93})/20 (A_{12} = 0) \),
(iv), \( \kappa_1 = 0.8, \kappa_2 = \frac{1}{5}(2 + 3\sqrt{7}) (A_{12} = 0) \).

Figure 6. The solution (4.9) of the SWW-HS equation (1.8).

Figure 7. Two soliton solutions (4.10) of the SWW-AKNS equation (1.7).