Skyrmions in integral and fractional quantum Hall systems

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Numerical results are presented for the spin excitations of a two-dimensional electron gas confined to a quantum well of width \( w \). Spin waves and charged skyrmion excitations are studied for filling factors \( \nu = 1, 3 \), and \( \frac{1}{3} \). Phase diagrams for the occurrence of skyrmions of different size as a function of \( w \) and the Zeeman energy are calculated. For \( \nu = 3 \), skyrmions occur only if \( w \) is larger than about twice the magnetic length. A general necessary condition on the interaction pseudopotential for the occurrence of stable skyrmion states is proposed.

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Introduction. In a numerical study of reversed spin excitations of the spin polarized fractional quantum Hall (FQH) state near filling factor \( \nu = \frac{1}{3} \). Rezayi \([2]\) found that for charged excitations the lowest lying band of states had total angular momentum \( L \) equal to the total spin \( S \) when the Zeeman energy \( E_Z \) was taken to be zero. In this case the minimum energy occurred at \( L = 0 \) and corresponded to a “spin texture” containing \( K = \frac{1}{2} N \) spin flips for an \( N \) electron system. Sondhi et al. \([3]\) and others \([4]\) investigated the \( \nu = 1 \) state and found that for \( E_Z \) less than a critical value \( \tilde{E}_Z \), the lowest charged excitation was a skyrmion \([5]\) containing a number \( K \) of reversed spins that increased as \( E_Z \) decreased from \( \tilde{E}_Z \) to zero. Skyrmions have been observed both in magnetization and transport studies for the \( \nu = 1 \) state \([6]\) and, when \( E_Z \) was sufficiently decreased by application of hydrostatic pressure, for the \( \nu = \frac{1}{3} \) state \([5]\). The form of the Coulomb pseudopotential in higher Landau levels (LL’s) suggested that skyrmions would not occur at \( \nu = 3, 5, \ldots \) \([8]\). However, when allowance was made for the softening of the pseudopotential associated with finite well width \( w \), skyrmions at \( \nu = 3 \) were predicted \([9]\) and observed \([10]\) in sufficiently wide quantum wells.

In this paper we demonstrate the similarities between electron and composite Fermion (CF) spin excitations in the integral and fractional quantum Hall systems. We present phase diagrams (in the \( w - E_Z \) plane) for the number of spin flips \( K \) in the lowest energy charged excitation of the \( \nu = 1, 3 \), and \( \frac{1}{3} \) fillings. In addition, we propose necessary conditions on the pseudopotentials (applicable both to integral and fractional filling) for low energy skyrmions at \( E_Z = 0 \).

Model. We perform numerical calculations for a system of \( N \) electrons confined to a spherical surface \([12]\) of radius \( R \). The radial magnetic field is produced by a Dirac monopole at the center, whose strength \( 2Q \) is given in units of the quantum of flux, \( \phi_0 = hc/e \), so that \( 4\pi R^2 B = 2Q\phi_0 \). The single particle states \( |Q, l, m\rangle \), called monopole harmonics \([3,11]\), are eigenfunctions of the orbital angular momentum \( l \) and its \( z \)-component \( m \). They form degenerate LL’s labeled by \( n = l - Q \). The cyclotron energy \( \hbar \omega_c \propto B \) is assumed to be much larger than the Coulomb energy scale \( E_C = e^2/|C| \propto \sqrt{B} \) (where \( \lambda^2 = \hbar c/eB \) is the square of the magnetic length). However, the ratio \( \eta = E_Z/E_C \) is taken as an arbitrary parameter. Finite well width enters the problem only through modifying the quasi-2D interaction by replacing \( e^2/r \) (where \( r \) is the in-plane separation) by \( \nu \langle(r) \rangle = e^2 \int dz dz' \xi^2(z)\xi^2(z')/\sqrt{r^2 + (z - z')^2} \). Here \( \xi(z) \) is the envelope function for the lowest subband of the quantum well. This change modifies the Coulomb pseudopotential \([14]\) \( V^{(n)}(\mathcal{R}) \), defined as the interaction energy of a pair of electrons in the \( n \)th LL as a function of their relative angular momentum \( \mathcal{R} \). There are four conserved quantum numbers: \( L, \) the total orbital angular momentum, \( S \), the total spin, and their projections \( L_z \) and \( S_z \). The eigenvalues depend only on \( L \) and \( S \), and they are therefore \( (2L + 1)(2S + 1) \) fold degenerate.

Integral filling. In Fig. 1(a) and (b) we present the low energy spectra of the \( \nu = 1 \) and \( \nu = 1^- \) (a single hole in \( \nu = 1 \)) states, respectively. In this and all other spectra, only the lowest state at each \( L \) and \( S \) is shown,
monotonically with the skyrmion band denoted by \( E \). band of states with 0 closed (e.g., by applying hydrostatic pressure), the \( \nu \) one reversed spin electron in a Fig. 1(b) can be viewed as containing either one hole or the \( \nu \) lowest LL, the \( \nu \) lowest energy excitation for a given value of either \( E \) of spin flips away from the fully polarized ground state. The (near) linearity of \( E(K) \) for this band of states (denoted by \( W_K \)) suggests that it consists of \( K \) SW’s, each with \( L = 1 \), which are (nearly) noninteracting. As shown with the dot-dash lines connecting different states of the same number \( K \) of \( L = 1 \) SW’s, only the \( L = K \) state (in which the SW’s have parallel angular momenta) is noninteracting, and all others (at \( L < K \)) are repulsive.

We have compared the linear \( W_K \) energy bands calculated for different electron numbers \( N \leq 14 \), and found that they all have the same slope \( u \approx 1.15 \varepsilon^2/\lambda \) when plotted as a function of the “relative” spin polarization \( \zeta = K/N \). The fact that \( E - E_0 = u\zeta \) for the \( W_K \) band for every value of \( N \) has two noteworthy consequences in the \( N \to \infty \) limit: (i) For any value of \( E_Z \neq 0 \), the interaction energy of each \( W_K \) state, \( E - E_0 \propto K/N \), is negligibly compared to its total Zeeman energy, \( K E_Z \). (ii) The gap for spin excitations at \( \nu = 1 \) equals \( E_Z \); if this gap can be closed (e.g., by applying hydrostatic pressure), the \( \nu = 1 \) ferromagnet becomes gapless and the density of states for the \( W_K \) excitations becomes continuous.

Because of the exact particle–hole symmetry in the lowest LL, the \( \nu = 1 \) state whose spectrum appears in Fig. 1(b) can be viewed as containing either one hole or one reversed spin electron in a \( \nu = 1 \) ground state. The band of states with \( 0 \leq L \leq 5 \) and \( S = L \) (dotted line) is the skyrmion band denoted by \( S_K \). Its energy increases monotonically with \( S \) and \( L \). For \( 6 \leq L \leq 12 \) the single SW band (dashed line) and band of \( K \) SW’s each with \( L = 1 \) (solid line) resemble similar bands in Fig. 1(a), except that their angular momenta are added to that of the hole which has \( l_h = l = 6 \).

Fig. 1 completely ignores the Zeeman energy. The total Zeeman energy measured from the fully polarized state is proportional to \( K \). The total energy of the skyrmion band is \( E(K) = E_S(K) + K E_Z \) and the lowest \( S_K \) state occurs when \( E(K) \) has its minimum. If we roughly approximate the skyrmion energy in a finite system by \( E_S(K) \approx E_S(\frac{1}{2}N) + \beta S^2 \), where \( \beta \geq 0 \) is a constant, this minimum occurs at \( K = \frac{1}{2}(N - E_Z/\beta) \) spin flips. This vanishes when \( E_Z = \beta N \), defining the critical value, \( E_Z \), and it reaches its maximum value \( (K = \frac{1}{2}N \) or complete depolarization) when \( E_Z = 0 \). At such \( E_Z \) that the ground state at \( \nu = 1^\pm \) is a finite size skyrmion, its gap for spin excitations is much smaller than (and largely independent of) \( E_Z \). This is in contrast to the exact \( \nu = 1 \) filling and allows spin coupling of the electron system to the magnetic ions, nuclei, or charged excitons.

In Fig. 2 we show the numerical results analogous to those in Fig. 1 but for the \( n = 1 \) LL. Features clearly apparent in the lowest LL are now absent. For example, the \( W_K \) band in Fig. 2(a) departs noticeably from linearity, and it does not generally lie below the single SW band (dashed line). More striking is the fact that the \( S_K \) band of Fig. 2(b) goes above the single hole state at \( L = 6 \), in contrast to the behavior in Fig. 2(b). Therefore skyrmions are not the lowest energy charged excitations in excited LL’s even when \( E_Z = 0 \). This effect was first predicted by Jain and Wu [8].

The only difference between the filling factors \( \nu = 3 \), 5, . . . and \( \nu = 1 \) is that the monopole harmonics \( |Q, l = Q + n, m \rangle \) correspond to the excited LL instead of the lowest. Matrix elements of the Coulomb interaction \( e^2/r \) between these higher monopole harmonics give a different pseudopotential \( V^{(n)}(R) \) from that for \( n = 0 \). Though one might expect skyrmions to be the lowest energy charged excitations in this case, the change in the pseudopotential from \( V^{(0)}(R) \) to \( V^{(n)}(R) \) with \( n \geq 1 \) causes the charged spin flip excitations to have higher energy than the single hole or reversed spin electron.

**Fractional filling.** Since the CF picture [1] describes the FQH effect in terms of integral filling of effective CF levels, it is interesting to ask [12] if spin excitations analogous to the SW’s and skyrmions occur at Laughlin fractional fillings \( \nu = (2p + 1)^{-1} \) (where \( p = 1, 2, \ldots \)). In Fig. 3 we display numerical results for \( \nu \approx \frac{1}{3} \). The values of \( N \) and \( 2l \) in frames (b), (a), and (c) correspond to a Laughlin \( \nu \approx \frac{1}{3} \) condensed state, Laughlin quasihole (QH), and Laughlin quasielectron (QE) or reversed spin quasielectron (QE_R), respectively. For each of these cases the lowest CF LL has a degeneracy of seven. Clearly the single SW dispersion (dashed line) and the linear \( W_K \) band (solid line) both appear in Fig. 3(b).
The $S_K$ bands beginning at $L = 0$ lie below the single QH state (a) and below the single QER state (c). The solid and dashed lines at $3 \leq L \leq 6$ in Fig. 3(a) and (c) are completely analogous to those in Fig. 1(b), and correspond to the single SW band and the $W_K$ band, except that their angular momenta are added to $l_{QH} = 3$ or $l_{QER} = 3$. What is clearly different from the $\nu = 1$ case is the smaller energy scale, and a noticeable difference between the $\nu = \frac{1}{1} - 1$ (QH) and $\nu = \frac{1}{2}$ (QER) spectra. Since the QH–QH and QER–QER interactions are known to be different [17], this lack of QH–QER symmetry is not unexpected. It implies a lack of symmetry between the CF skyrmion (QER + K SW) and CF antiskyrmion (QH + K SW) states in contrast to the skyrmion–antiskyrmion symmetry of $\nu = 1$. Because the CF skyrmion energy scale is so much smaller than $E_C$ at $\nu = 1$, the critical $E_Z$, at which skyrmions are stable is correspondingly smaller [7].

Effect of finite well width. As suggested by Cooper [9] and confirmed experimentally by Song et al. [10], skyrmions become the lowest energy charged excitations in higher LL’s if the quantum well is sufficiently wide. The finite well width $w$ can be accounted for by using effective potential $V_e(r)$ and selecting a subband wave function $\xi(z)$ appropriate to the depth and width of the quantum well. In Fig. 4(a) we show the $e-e$ and $e-h$ pseudopotentials for the lowest (ac) and excited (bd) LL as a function of a parameter $d$ which is proportional to $w$. In the calculation we have used a potential $V_d(r) = e^2/\sqrt{r^2 + d^2}$ as done earlier by He et al. [11]. Comparing the resulting pseudopotentials with those obtained using an envelope function $\xi_0(z) \propto \cos(\pi z/w)$ appropriate to the lowest subband of an infinitely deep quantum well gives $w \approx 5d$ ($w$ is slightly larger than the actual width of a finite depth well). It is clear that finite width must have the largest effect on those pseudopotential coefficients corresponding to the smallest average $e-e$ or $e-h$ separation. As a result, increasing $w$ causes suppression of the maxima of $V_{ee}(R)$ and of the minima of $V_{eh}(k)$ characteristic of the excited LL’s. This makes the $n = 1$ pseudopotentials (and, in consequence, the many-body spectra of Fig. 2) more similar to those of the lowest LL.

In Fig. 5(a) we show the same energy spectra as given in Fig. 2 but for the Coulomb pseudopotential in the $n = 1$ LL replaced by one appropriate for $w = 3\lambda$. The $W_K$ band in Fig. 2(a) is now much closer to linear with $K$, and the $S_K$ band in Fig. 2(b) now has $E < E_0$. We have done similar calculations for the $n = 2$ LL with similar results. These results show that skyrmions are the lowest excitations in higher LL’s if $w$ is sufficiently large.

In Fig. 6 we sketch the phase diagrams (in the $w-E_Z$ plane) for charged excitations at the integral filling of the lowest and excited LL’s ($\nu = 1$ and 3), and at the fractional filling $\nu = \frac{1}{2}$. For $n = 0$ ($\nu = 1$ or $\frac{1}{2}$), skyrmions are the lowest charged excitations below a critical value of $E_Z$ which is relatively insensitive to the width $w$. As $E_Z$ is decreased, larger skyrmions (with increasing $K$) become the lowest energy states. For $n = 1$, no skyrmions occur unless $w \leq 2\lambda$ (we have checked that this value remains correct for small skyrmions in the $N \to \infty$ limit).

Pseudopotentials and skyrmion stability. Only the leading pseudopotential coefficients $V(0)$, $V(1)$, $V(2)$, corresponding to small average in-plane $e-e$ separations, are strongly influenced by finite $w$. In fact, the change
in the energy $E_S(K) - E_0$ of the skyrmion band from positive to negative occurred in the $n = 1$ LL only when the pseudopotential coefficient $V(2)$ was quite strongly affected by the increase in $w$. For this reason, we investigate $E_S(K)$ for a simple model pseudopotential with the following properties: (i) $V(0)$ is sufficiently large to cause decoupling of the many body states that avoid all $\mathcal{R} = 0$ pairs (i.e., the skyrmion states) from all other states that contain some $\mathcal{R} = 0$ pairs; (ii) behavior of $V(\mathcal{R})$ between $\mathcal{R} = 1$ and 3 can be varied similarly to how $V^{(1)}$ varies with increasing $w$. We choose the simplest possible model pseudopotential with these properties by defining $U_x(\mathcal{R})$ as follows: $U_x(0) = \infty$, $U_x(1) = 1$, $U_x(2) = x$, and $U_x(\mathcal{R}) = 0$ for $\mathcal{R} > 2$. This choice of $U_x$ guarantees that skyrmions are its only finite energy eigenstates, and their energy depends on one free parameter $x$. A simple relation between the fractional grandparentage coefficients $G_K(\mathcal{R})$ at $\mathcal{R} = 0, 1, 2$ yields $E_S(K) - E_0 = G_K(2)(x - \alpha)$, where $G_K(2)$ is a positive constant and $\alpha^{-1} = 2 - (N - 1)^{-1}$. Since for every value of $K$, $E_S(K) - E_0$ changes sign at $x = \alpha$, and $\alpha = \frac{1}{2}$ in large systems, we conclude that skyrmions are the lowest charged excitations when $U_x(2)$ drops below half of $U_x(1)$, and $U_x$ becomes superlinear between $\mathcal{R} = 1$ and 3. Therefore we suggest that for both integral and fractional quantum Hall states the stability of skyrmion states requires an effective pseudopotential for which (i) $V(0)$ is large enough to cause Laughlin correlations, and (ii) $V(2)$ is less than or equal to half of $V(1) + V(3)$.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_diagram.pdf}
\caption{Phase diagrams for the occurrence of skyrmions with different $K$ as a function of well width $w$ and Zeeman energy $E_Z$, calculated at $\nu = 1^{-}$ (a), $3^{-}$ (b), $\frac{1}{2}^{-}$ (c), and $\frac{1}{2}^{+}$ (d). Numbers in top-left corners give the upper bounds of the vertical axes (the lower bound are zero in all frames).}
\end{figure}