A SPECIAL CASE OF THE TWO-DIMENSIONAL JACOBIAN
CONJECTURE

VERED MOSKOWICZ

Abstract. Let $f : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ be a $\mathbb{C}$-algebra endomorphism having an
invertible Jacobian.

We show that for such $f$, if, in addition, the group of invertible elements
of $\mathbb{C}[f(x), f(y), x][1/v] \subset \mathbb{C}(x,y)$ is contained in $\mathbb{C}(f(x), f(y)) - 0$, then $f$ is
an automorphism. Here $v \in \mathbb{C}(f(x), f(y)) - 0$ is such that $y = u/v$, with
$u \in \mathbb{C}[f(x), f(y), x] - 0$.

Keller’s theorem (in dimension two) follows immediately, since Keller’s con-
dition $\mathbb{C}(f(x), f(y)) = \mathbb{C}(x,y)$ implies that the group of invertible elements of
$\mathbb{C}[f(x), f(y), x][1/v]$ is contained in $\mathbb{C}(x,y) - 0 = \mathbb{C}(f(x), f(y)) - 0$.

1 Introduction

Throughout this note, $f : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ is a $\mathbb{C}$-algebra endomorphism that
satisfies $\text{Jac}(p, q) \in \mathbb{C}^*$, where $p := f(x)$ and $q := f(y)$.

Formanek’s field of fractions theorem [6, Theorem 2] in dimension two says that
$\mathbb{C}(p, q, x) = \mathbb{C}(x,y)$. From this it is not difficult to obtain that $y = u/v$, for some
$u \in \mathbb{C}[p, q, x] - 0$ and $v \in \mathbb{C}[p, q] - 0$.

We show in Theorem 3.1 that for such $f$, if, in addition, the group of invertible
elements of $\mathbb{C}[p, q, x][1/v]$ is contained in $\mathbb{C}(p, q) - 0$, then $f$ is an automorphism.

Our proof of Theorem 3.1 is almost identical to the proof of Formanek’s auto-
morphism theorem [5, Theorem 1]; we did only some slight changes in his proof,
and also used Formanek’s field of fractions theorem and Wang’s intersec-
tion theorem [15, Theorem 41 (i)].

Keller’s theorem in dimension two follows immediately from our theorem: As-
sume that $\mathbb{C}(p, q) = \mathbb{C}(x,y)$. Then our condition of Theorem 3.1 is satisfied, be-
cause the group of invertible elements of $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x,y)$ is contained in
$\mathbb{C}(x,y) - 0 = \mathbb{C}(p, q) - 0$.

2 Preliminaries

Our Theorem 3.1 deals with the two-dimensional case only. However, the re-

sults we rely on are valid in any dimension $n$, so we add the following nota-
tion: $F : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is a $\mathbb{C}$-algebra endomorphism that satisfies
$\text{Jac}(F_1, \ldots, F_n) \in \mathbb{C}^*$, where $F_1 := F(x_1), \ldots, F_n := F(x_n)$. When $n = 2$ we will
keep the above notation, namely, $x_1 = x, x_2 = y, F_1 = p, F_2 = q$.

Theorem 2.1 (Formanek’s automorphism theorem). Suppose that there is a poly-
nomial $W$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that $\mathbb{C}[F_1, \ldots, F_n, W] = \mathbb{C}[x_1, \ldots, x_n]$. Then
$\mathbb{C}[F_1, \ldots, F_n] = \mathbb{C}[x_1, \ldots, x_n]$, namely, $F$ is an automorphism.

Proof. See [5, Theorem 1] and [4, page 13, Exercise 9].

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• If there exists \( w \in \mathbb{C}[x, y] \) such that \( \mathbb{C}[p, q, w] = \mathbb{C}[x, y] \), then \( \mathbb{C}[p, q] = \mathbb{C}[x, y] \), namely, \( f \) is an automorphism.

**Theorem 2.2** (Formanek’s field of fractions theorem).
\[
\mathbb{C}(F_1, \ldots, F_n, x_1, \ldots, x_{n-1}) = \mathbb{C}(x_1, \ldots, x_n).
\]

*Proof.* See [6, Theorem 2].

• \( \mathbb{C}(p, q, x) = \mathbb{C}(x, y) \) and \( \mathbb{C}(p, q, y) = \mathbb{C}(x, y) \).

Formanek remarks that when \( n = 2 \), \( \mathbb{C}(p, q, w) = \mathbb{C}(x, y) \), where \( w \) is the image of \( x \) under any automorphism of \( \mathbb{C}[x, y] \); see [6, page 370, just before Theorem 6].

The two-dimensional case was already proved by Moh [12, page 151] and by Hamann [7, Lemma 2.1, Proposition 2.1(2)]. Moh and Hamann assumed that \( p \) is monic in \( y \), but this is really not a restriction.

It is easy to see that:

**Corollary 2.3.** There exist \( u \in \mathbb{C}[p, q, x] - 0 \) and \( v \in \mathbb{C}[p, q] - 0 \) such that \( y = u/v \).

*Proof.* \( y \in \mathbb{C}(x, y) = \mathbb{C}(p, q, x) = \mathbb{C}(p, q)(x) \). Since \( x \) is algebraic over \( \mathbb{C}(p, q) \), we have \( \mathbb{C}(p, q)(x) = \mathbb{C}(p, q)[x] \) (see [14, Remark 4.7]). Hence, \( y \in \mathbb{C}(p, q)[x] \). Therefore, there exist \( a_i, b_i \in \mathbb{C}[p, q] \) \((b_i \neq 0)\) such that \( y = \sum (a_i/b_i) x^i \). Then if we denote \( B = \prod b_i \) and \( B_i \) the product of the \( b_j \)'s except \( b_i \), we get \( y = (1/B) \sum B_i a_i x^i \). Just take \( v := B \) and \( u := \sum B_i a_i x^i \).

**Theorem 2.4** (Wang’s intersection theorem). \( \mathbb{C}(F_1, \ldots, F_n) \cap \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[F_1, \ldots, F_n] \).

*Proof.* See [15, Theorem 41 (i)] and [4, Corollary 1.1.34 (ii)].

Wang’s intersection theorem has a more general version due to Bass [2, Remark after Corollary 1.3, page 74], [4, Proposition D.1.7]; we will not need the more general version here.

• \( \mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q] \).

The following is immediate:

**Corollary 2.5.** \( \mathbb{C}(p, q) \cap R = \mathbb{C}[p, q] \), for any \( \mathbb{C}[p, q] \subseteq R \subseteq \mathbb{C}[x, y] \). In particular, \( \mathbb{C}(p, q) \cap \mathbb{C}[p, q, x] = \mathbb{C}[p, q] \).

*Proof.* \( \mathbb{C}(p, q) \cap R \subseteq \mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q] \). The other inclusion, \( \mathbb{C}(p, q) \cap R \supseteq \mathbb{C}[p, q] \), is trivial.

**Theorem 2.6** (Keller’s theorem). If \( \mathbb{C}(F_1, \ldots, F_n) = \mathbb{C}(x_1, \ldots, x_n) \), then \( F \) is an automorphism.

\( F \) as in Keller’s theorem is called birational (\( F \) has an inverse formed of rational functions).

*Proof.* See [9], [4, Corollary 1.1.35] and [1, Theorem 2.1].

• If \( \mathbb{C}(p, q) = \mathbb{C}(x, y) \), then \( f \) is an automorphism.

**Remark 2.7.** Notice that the above results are dealing with \( k[x_1, \ldots, x_n] \), where \( k \) is:

• \( \mathbb{C} \): Formanek’s field of fractions theorem.
• a field of characteristic zero: Formanek’s automorphism theorem.
• any field: Keller’s theorem.
• a UFD: Wang’s intersection theorem.
We have not checked if Formanek’s field of fractions theorem is valid over a more general field than $\mathbb{C}$; if, for example, it is valid over any algebraic closed field of characteristic zero, then our Theorem 3.1 is valid over any algebraic closed field of characteristic zero, not just over $\mathbb{C}$.

Anyway, working over $\mathbb{C}$ is good enough in view of [4, Lemma 1.1.14].

### 3 A new proof of Keller’s theorem in dimension two

Our proof of Theorem 3.1 relies heavily on the proof of Formanek’s automorphism theorem; we did only some slight changes in his proof, changes that seem quite natural in view of Corollary 2.3.

Although we do not know if $\mathbb{C}[p, q, x] = \mathbb{C}[x, y]$ (if so, then $f$ is an automorphism by Formanek’s automorphism theorem), we do know that $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$ (by Formanek’s field of fractions theorem), so by Corollary 2.3, $y = u/v$ for some $u \in \mathbb{C}[p, q, x] - 0$ and $v \in \mathbb{C}[p, q] - 0$. Therefore, it seems natural to consider $\beta : \mathbb{C}[U_1, U_2, U_3][1/V] \to \mathbb{C}[p, q, x][1/v]$, where $V = v(U_1, U_2)$.

This $\beta$ has $x$ and $y$ in its image, so most of Formanek’s proof can be adjusted here, except that the group of invertible elements of $\mathbb{C}[p, q, x][1/v]$ is not as easily described as the group of invertible elements of $\mathbb{C}[x, y]$, which is obviously $\mathbb{C}^*$. Only after adding a condition on the group of invertible elements of $\mathbb{C}[p, q, x][1/v]$, we are able to show that $f$ is an automorphism.

Now we are ready to bring our theorem; we recommend the reader to first read the proof of Formanek’s automorphism theorem, and then read our proof, with $p, q, x$ in our proof instead of $F_1, F_2, F_3$ in his proof.

**Theorem 3.1** (Main Theorem). If the group of invertible elements of $\mathbb{C}[p, q, x][1/v]$ is contained in $\mathbb{C}(p, q) - 0$, then $f$ is an automorphism.

**Proof.** By Corollary 2.3, there exist $u \in \mathbb{C}[p, q, x] - 0$ and $v \in \mathbb{C}[p, q] - 0$ such that $y = u/v$.

Let $U_1, U_2, U_3$ be independent variables over $\mathbb{C}$. Define $\alpha : \mathbb{C}[U_1, U_2, U_3] \to \mathbb{C}[p, q, x]$ by $\alpha(U_1) := p$, $\alpha(U_2) := q$, $\alpha(U_3) := x$. Clearly, $\alpha$ is surjective.

Claim: The kernel of $\alpha$ is a principal prime ideal of $\mathbb{C}[U_1, U_2, U_3]$.

Proof of claim: $\mathbb{C}(U_1, U_2, U_3)$ has transcendence degree 3 over $\mathbb{C}$, and $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$ has transcendence degree 2 over $\mathbb{C}$. From [11, Theorem 5.6], $\mathbb{C}(U_1, U_2, U_3)$ is of Krull dimension 3 and $\mathbb{C}[p, q, x]$ is of Krull dimension 2. Hence, the kernel of $\alpha$ is of height 1, and in a Noetherian UFD a height one prime ideal is principal, see [3, Theorem 15.9].

Denote by $H$ a generator of the kernel of $\alpha$: $H = H_1U_3^r + \ldots + H_1U_3 + H_0$, where $H_j \in \mathbb{C}[U_1, U_2]$ and $r \geq 1$. $H$ is a product of the minimal polynomial for $x$ over $\mathbb{C}(p, q)$ by some element $H_r$ of $\mathbb{C}[U_1, U_2]$ which clears the denominators of the minimal polynomial for $x$ over $\mathbb{C}(p, q)$. Notice that $r = 0$ is impossible, since then $H = H_0(U_1, U_2)$:

- If $H_0(U_1, U_2) \equiv 0$, then $H(U_1, U_2, U_3) \equiv 0$, so the kernel of $\alpha$ is zero, but then we have $\mathbb{C}[U_1, U_2, U_3] \cong \mathbb{C}[p, q, x]$, which is impossible from considerations of Krull dimensions.
- If $H_0(U_1, U_2) \not\equiv 0$, then $0 = \alpha(H) = \alpha(H_0(U_1, U_2)) = H_0(p, q)$ is a non-trivial algebraic dependence of $p$ and $q$ over $\mathbb{C}$. But $p$ and $q$ are algebraically independent over $\mathbb{C}$, because $\text{Jac}(p, q) \not\equiv 0$; see [10, pages 19-20] or [14, Proposition 6A.4].

Since we do not know if $y$ is in the image of $\alpha$, we define the following (surjective)

$\beta : \mathbb{C}[U_1, U_2, U_3][1/V] \to \mathbb{C}[p, q, x][1/v]$ by $\beta(U_1) := p$, $\beta(U_2) := q$, $\beta(U_3) := x,$
\(\beta(1/V) := 1/(\beta(V))\), where \(V := v(U_1, U_2)\), namely, in \(v \in C[p, q, x]/0\) replace \(p\) by \(U_1\) and \(q\) by \(U_2\) and get \(V\). It is clear that \(\beta(V) = v\), so \(\beta(1/V) = 1/v\).

Notice that \(V \in C[U_1, U_2]\); the fact that the \(U_3\)-degree of \(V\) is zero will be crucial in what follows.

Now, \(y\) is in the image of \(\beta\); indeed, let \(U := u(U_1, U_2, U_3)\), namely, in \(u \in C[p, q, x]/0\) replace \(p\) by \(U_1\), \(q\) by \(U_2\) and \(x\) by \(U_3\), and get \(U\). Then clearly \(\beta(U/V) = u/v = y\).

Take: \(T_1 := U_3\) and \(T_2 := U/V\). Then, \(\beta(T_1) = \beta(U_3) = x\), and \(\beta(T_2) = \beta(U/V) = u/v = y\).

Each of the following three elements lie in the kernel of \(\beta\): \(U_1 - p(T_1, T_2), U_2 - q(T_1, T_2)\) and \(U_3 - x(T_1, T_2) = U_3 - T_1 = 0\). Indeed, \(\beta(U_1 - p(T_1, T_2)) = \beta(U_1) - \beta(p(T_1, T_2)) = p - p = 0\) and \(\beta(U_2 - q(T_1, T_2)) = \beta(U_2) - \beta(q(T_1, T_2)) = q - q = 0\).

Claim: The kernel of \(\beta\) is a principal prime ideal of \(C[U_1, U_2, U_3]/[1, V]\), generated by exactly the same \(H \in C[U_1, U_2, U_3]\) that generates the kernel of \(\alpha\).

Proof of claim: Assume that \(R/V^2\) is in the kernel of \(\beta\), where \(R \in C[U_1, U_2, U_3]\). We have \(0 = \beta(R/V^2) = \beta(R)/\beta(V^2) = \beta(R)/v^2\), hence \(0 = \beta(R)\). Since \(\beta\) restricted to \(C[U_1, U_2, U_3]/\alpha\), we get that \(R\) belongs to the kernel of \(\alpha\), hence \(R = \bar{R}H\), for some \(\bar{R} \in C[U_1, U_2, U_3]\). So, \(R/V^2 = \bar{R}H/V^2 = (\bar{R}/V^2)H\), as claimed.

Therefore, there exist \(R_1, R_2 \in C[U_1, U_2, U_3]\) (\(R_1 = 0\)) and \(n, m \geq 0\) such that \(U_1 - p(T_1, T_2) = (R_1/V^n)H\) and \(U_2 - q(T_1, T_2) = (R_2/V^m)H\). So, \(U_1 = p(T_1, T_2) + (R_1/V^n)H\) and \(U_2 = q(T_1, T_2) + (R_2/V^m)H\) (and \(U_3 = T_1\)).

Differentiating these three equations with respect to \(U_1, U_2, U_3\) and using the Chain Rule, we get similar matrices to those in Formanek’s proof; the difference is that instead of \(R_1, R_2, R_3\) of Formanek’s proof, we have here \(R_1/V^n, R_2/V^m, 0\).

Applying \(\beta\) gives a matrix equation over \(C[p, q, x]/[1, v]\), similar to the matrix equation (2) of Formanek’s proof.

Cramer’s Rule shows that \(\beta(\partial H/\partial U_3) = \lambda/d\), where \(\lambda = \text{Jac}(p, q) \in C^*\) and \(d \in C[p, q, x]/[1, v] - 0\) is the determinant of the matrix on the left.

\(d\) belongs to the group of invertible elements of \(C[p, q, x]/[1, v]\), hence, by our assumption, \(d\) belongs to \(C(p, q) - 0\).

On the one hand, \(d \in C[p, q, x]/[1, v] - 0\), hence \(d = \tilde{d}/v^l\) for some \(\tilde{d} \in C[p, q, x]/[1, v]\) and \(l \geq 0\). On the other hand, \(d \in C(p, q) - 0\), hence \(d = a/b\) for some \(a, b \in C[p, q] - 0\). Combining the two we get, \(\tilde{d}/v^l = a/b\), so \(\tilde{C}[p, q, x]/[1, v] - 0 \ni \tilde{d} = v^l(a/b) \in C(p, q) - 0\).

From Corollary 2.5 we get that \(\tilde{d} \in C(p, q) - 0\).

(Remark: Actually, one can use Wang’s intersection theorem directly, without Corollary 2.5, and still get \(\tilde{d} \in C[p, q] - 0\), as long as one observes that \(C[p, q, x]/[1, v] = C[x, y]/[1, v]\). Indeed, \(d \in C[p, q, x]/[1, v] = C[x, y]/[1, v]\), hence \(d = \tilde{d}/v^l\) for some \(\tilde{d} \in C[x, y]/[1, v]\) and \(l \geq 0\), etc.).

So \(d = \tilde{d}/v^l\), with \(\tilde{d} \in C[p, q] - 0\). Let \(D = d(U_1, U_2) = \tilde{d}(U_1, U_2)/v^l(U_1, U_2) = \tilde{d}(U_1, U_2)/V^l\). Clearly, \(\beta(D) = d\).

For convenience, multiply the above equation \(\beta(\partial H/\partial U_3) = \lambda/d\) by \(d\) and get \(d\beta(\partial H/\partial U_3) = \lambda\). Then \(\beta(D) \beta(\partial H/\partial U_3) = \lambda\), so \(\beta(D\partial H/\partial U_3) = \beta(\lambda)\). Therefore, \(D\partial H/\partial U_3 - \lambda\) is in the kernel of \(\beta\).

We have seen that the kernel of \(\beta\) is a principal ideal of \(C[U_1, U_2, U_3]/[1, V]\), generated by \(H \in C[U_1, U_2, U_3]\), hence there exist \(S \in C[U_1, U_2, U_3]\) and \(t \geq 0\) such that \(D\partial H/\partial U_3 - \lambda = (S/V^t)H\). Replace \(D\) by \(\tilde{d}(U_1, U_2)/V^t\) and get, \((\tilde{d}(U_1, U_2)/V^t)\partial H/\partial U_3 - \lambda = (SH)/V^t\). Multiply both sides by \(V^{t+1}\) and get, \(V^{t+1}\tilde{d}(U_1, U_2)\partial H/\partial U_3 - \lambda V^{t+1} = V^t(SH)\).

Now, as promised above, we use the fact that the \(U_3\)-degree of the right side is at least \(r (=\text{ that of } H\), which is exactly \(r\), plus that of \(S\), which is \(\geq 0\)), while the \(U_3\)-degree of the left side is exactly \(r - 1\) (= that of \(\partial H/\partial U_3\)).
It follows that $S = 0$ and $r - 1 = 0$, so $r = 1$ and $H = H_1(U_1, U_2)U_3 + H_0(U_1, U_2)$.

Apply $\beta$ and get $0 = H_1(p, q)x + H_0(p, q)$, so $x = -H_0(p, q)/H_1(p, q) \in \mathbb{C}(p, q)$.

By Wang’s intersection theorem, $x \in \mathbb{C}[p, q]$. Then obviously, $\mathbb{C}[p, q][y] = \mathbb{C}[x, y]$.

Finally, Formanek’s automorphism theorem implies that $\mathbb{C}[p, q] = \mathbb{C}[x, y]$, namely $f$ is an automorphism.

All the arguments and known results we use do not depend on Keller’s theorem, hence we have a new proof of Keller’s theorem in dimension two:

**Theorem 3.2** (Keller’s theorem). Let $f : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ be a $\mathbb{C}$-algebra endomorphism that satisfies $\text{Jac}(p, q) \in \mathbb{C}^*$. If $\mathbb{C}(p, q) = \mathbb{C}(x, y)$, then $f$ is an automorphism.

**Proof.** The group of invertible elements of $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x, y)$ is contained in $\mathbb{C}(x, y) - 0 = \mathbb{C}(p, q) - 0$. Now apply Theorem 3.1.

Notice that the converse of Theorem 3.1 is trivially true: If $f$ is an automorphism, then $\mathbb{C}[p, q] = \mathbb{C}[x, y]$, so $\mathbb{C}(p, q) = \mathbb{C}(x, y)$, hence the group of invertible elements of $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x, y)$ is contained in $\mathbb{C}(x, y) - 0 = \mathbb{C}(p, q) - 0$.

Another argument: If $f$ is an automorphism, then we can take $u = y$ and $v = 1$. Then $\mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y]$, and its group of invertible elements is $\mathbb{C}^*$, which is contained in $\mathbb{C}(p, q) - 0$.

Therefore, the condition in Keller’s theorem is equivalent to our condition, not just implies our condition:

**Proposition 3.3.** TFAE:

1. $f$ is an automorphism, i.e. $\mathbb{C}[p, q] = \mathbb{C}[x, y]$.
2. $f$ is birational, i.e. $\mathbb{C}(p, q) = \mathbb{C}(x, y)$.
3. The group of invertible elements of $\mathbb{C}[p, q, x][1/v]$ is contained in $\mathbb{C}(p, q) - 0$.

We do not know how to show directly that (iii) implies (ii).

## 4 Further discussion

We wish to bring some related ideas.

**First idea:** We have already mentioned in the Preliminaries that Formanek remarks that $\mathbb{C}(p, q, w) = \mathbb{C}(x, y)$, where $w$ is the image of $x$ under any automorphism of $\mathbb{C}[x, y]$. Therefore, we can obtain similar theorems to Theorem 3.1 with $x$ replaced by any image of $x$ under an automorphism of $\mathbb{C}[x, y]$.

More elaborately, take any automorphism $g : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ and denote $g_1 := g(x)$ and $g_2 := g(y)$.

We have $\mathbb{C}(p, q, g_1) = \mathbb{C}(x, y) = \mathbb{C}(g_1, g_2)$; the first equality follows from Formanek’s remark, while the second equality trivially follows from $\mathbb{C}[x, y] = \mathbb{C}[g_1, g_2]$. Then, $g_2 \in \mathbb{C}(p, q)(g_1) = \mathbb{C}(p, q)[g_1]$, because $g_1$ is algebraic over $\mathbb{C}(p, q)$. It is easy to obtain $g_2 = u_p/v_q$, where $u_p \in \mathbb{C}[p, q, g_1] - 0$ and $v_q \in \mathbb{C}[p, q] - 0$.

**Theorem 4.1.** If the group of invertible elements of $\mathbb{C}[p, q, g_1][1/v_q]$ is contained in $\mathbb{C}(p, q) - 0$, then $f$ is an automorphism.

**Proof.** In the proof of Theorem 3.1 replace $x$ and $y$ by $g_1$ and $g_2$, do the appropriate adjustments, and get a proof for the new theorem. Notice that now, instead of considering $p$ and $q$ as functions of $x$ and $y$, one has to consider $p$ and $q$ as functions of $g_1$ and $g_2$.

**Second idea:** For $v$ as in Corollary 2.3 write $v = v_1 \cdots v_m$, where $v_1, \ldots, v_m \in \mathbb{C}[p, q]$ are irreducible elements of $\mathbb{C}[p, q]$. There are two options, either one (or
more) of the $v_j$’s becomes reducible in $\mathbb{C}[x, y]$ or all the $v_j$’s remain irreducible in $\mathbb{C}[x, y]$.

If one (or more) of the $v_j$’s becomes reducible in $\mathbb{C}[x, y]$, then it is possible to show that our condition of Theorem 3.1 is not satisfied, and hence $f$ is not an automorphism: Assume that $v_1 = v_1 \cdots v_l$, where $w_1, \ldots, w_l \in \mathbb{C}[x, y]$ are irreducible in $\mathbb{C}[x, y]$, $l > 1$. It is not difficult to see (use Wang’s intersection theorem) that at least two factors are in $\mathbb{C}[x, y] - \mathbb{C}[p, q]$, w.l.o.g. $w_1$ and $w_2$. We claim that $w_1$ is invertible in $\mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$. Indeed, $1 = v/v = v_1 \cdots v_m/v = w_1 w_2 \cdots w_l v_1 \cdots v_m/v = w_1 (w_2 \cdots w_l v_1 \cdots v_m/v)$.

Clearly, $w_1 \notin \mathbb{C}(p, q)$, because otherwise, $w_1 \in \mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q]$, but $w_1 \in \mathbb{C}[x, y] - \mathbb{C}[p, q]$.

Actually, if one (or more) of the $v_j$’s becomes reducible in $\mathbb{C}[x, y]$, then it is immediate that $f$ is not an automorphism, since an automorphism satisfies $\mathbb{C}[p, q] = \mathbb{C}[x, y]$, so trivially every irreducible element of $\mathbb{C}[p, q]$ is an irreducible element of $\mathbb{C}[x, y]$.

Next, if all the $v_j$’s remain irreducible in $\mathbb{C}[x, y]$, then our condition of Theorem 3.1 is satisfied:

**Theorem 4.2** (A special case of the main theorem). If $v_1, \ldots, v_m$ remain irreducible in $\mathbb{C}[x, y]$, then $f$ is an automorphism.

Of course, since $\mathbb{C}[p, q] (\mathbb{C}[x, y])$ is a UFD, every irreducible element of $\mathbb{C}[p, q]$ ($\mathbb{C}[x, y]$) is prime.

**Proof.** By assumption, $v_1, \ldots, v_m \in \mathbb{C}[p, q]$ are irreducible elements of $\mathbb{C}[x, y]$, hence, $v_1, \ldots, v_m$ are prime elements of $\mathbb{C}[x, y]$.

Claim: The condition of Theorem 3.1 is satisfied.

Proof of claim: Let $a \in \mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$ be an invertible element, so there exists $b \in \mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$ such that $ab = 1$. We can write $a = r/v^k$ and $b = s/v^l$, for some $r, s \in \mathbb{C}[x, y] - 0$ and $k, l \geq 0$. Then $ab = 1$ becomes $rs = v^{k+l} = (v_1 \cdots v_m)^{k+l}$. Since $v_1, \ldots, v_m$ are prime elements of $\mathbb{C}[x, y]$, we obtain that $r = v_1^{a_1} \cdots v_m^{a_m}$ and $s = v_1^{b_1} \cdots v_m^{b_m}$, where $a_j + b_j = k + l$, $1 \leq j \leq m$. Therefore, $r, s \in \mathbb{C}[p, q] - 0$, so $a = r/v^k \in \mathbb{C}(p, q) - 0$, and we are done. \hfill \Box

Notice that in Theorem 4.2 we demand that each of the irreducible factors $v_1, \ldots, v_m \in \mathbb{C}[p, q]$ of $v$ remain irreducible in $\mathbb{C}[x, y]$, but we do not demand that other irreducible elements of $\mathbb{C}[p, q]$ remain irreducible in $\mathbb{C}[x, y]$.

If one demands that every irreducible element of $\mathbb{C}[p, q]$ remains irreducible in $\mathbb{C}[x, y]$, then, without relying on Theorem 3.1, one can get that $f$ is an automorphism, thanks to the result [8, Lemma 3.2] of Jedrzejewicz and Zieliński.

Their result says the following: Let $A$ be a UFD. Let $R$ be a subring of $A$ such that $R^* = A^*$. The following conditions are equivalent:

(i) Every irreducible element of $R$ remains irreducible in $A$.

(ii) $R$ is factorially closed in $A$.

(Recall that a sub-ring $R$ of a ring $A$ is called factorially closed in $A$ if whenever $a_1, a_2 \in A$ satisfy $a_1 a_2 \in R - 0$, then $a_1, a_2 \in R$.)

In [8, Lemma 3.2] take $A = \mathbb{C}[x, y], R = \mathbb{C}[p, q]$; since we now assume that every irreducible element of $\mathbb{C}[p, q]$ remains irreducible in $\mathbb{C}[x, y]$, we obtain that $\mathbb{C}[p, q]$ is factorially closed in $\mathbb{C}[x, y]$, and we are done by the following easy lemma:

**Lemma 4.3.** If $\mathbb{C}[p, q]$ is factorially closed in $\mathbb{C}[x, y]$, then $f$ is an automorphism.

**Proof.** Let $H$ be as in the proof of Theorem 3.1, and denote $h_j := H_j(p, q), 0 \leq j \leq r$. Obviously, $h_0 \neq 0$ by the minimality of $r$. 

We have \( x(h_r x^{r-1} + h_{r-1} x^{r-2} + \ldots + h_1) = -h_0 + \mathbb{C}[p, q] = 0 \). By assumption \( \mathbb{C}[p, q] \) is factorially closed in \( \mathbb{C}[x, y] \), hence \( x \in \mathbb{C}[p, q] \) (and \( h_r x^{r-1} + h_{r-1} x^{r-2} + \ldots + h_1 \in \mathbb{C}[p, q] \)).

Then \( \mathbb{C}[p, q, y] = \mathbb{C}[x, y] \), and \( f \) is an automorphism by Formanek’s automorphism theorem.

Notice that in the proof of Lemma 4.3, \( h_r x^{r-1} + h_{r-1} x^{r-2} + \ldots + h_1 \in \mathbb{C}[p, q] \) also yields that \( f \) is an automorphism, because by the minimality of \( r \), we must have \( r = 1 \), so \( h_1 x + h_0 = 0 \). Then \( x = -h_0/h_1 \in \mathbb{C}(p, q) \), and by Wang’s intersection theorem, \( x \in \mathbb{C}[p, q] \), etc.

**Third idea:** Notations as in the second idea, another special case is when all the \( v_j \)’s are primes in \( \mathbb{C}[p, q, x] \); this special case is dealt with in [13]: It is shown in [13, Theorem 2.2] that if all the \( v_j \)’s are primes in \( \mathbb{C}[p, q, x] \), then \( \mathbb{C}[p, q, x] \) is a UFD, and it is shown in [13, Theorem 2.1] that if \( \mathbb{C}[p, q, x] \) is a UFD, then \( f \) is an automorphism.

It is not yet clear to us what happens in the more general case when all the \( v_j \)’s are irreducibles in \( \mathbb{C}[p, q, x] \). It may happen that some (or all) of the \( v_j \)’s are not primes in \( \mathbb{C}[p, q, x] \), since we just know that \( \mathbb{C}[p, q, x] \) is an integral domain (if we knew it is a UFD, then \( f \) is an automorphism by [13, Theorem 2.1]).

**Fourth idea:**

We do not know if a similar result to Theorem 3.1 holds in higher dimensions. Even if the answer is positive, the proof should be somewhat different from the proof of the two-dimensional case. For example, already in the three-dimensional case some problems may arise when trying to generalize the proof of the two-dimensional case:

Let \( f : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z] \) be a \( \mathbb{C} \)-algebra endomorphism having an invertible Jacobian. Denote \( p := f(x), q := f(y), r := f(z) \).

It is not difficult to generalize Corollary 2.3:

**Corollary 4.4.** There exist \( u \in \mathbb{C}[p, q, r, x, y] = 0 \) and \( v \in \mathbb{C}[p, q, r] = 0 \) such that \( z = u/v \).

**Proof.** By Formanek’s field of fractions theorem, \( \mathbb{C}(p, q, r, x, y) = \mathbb{C}(x, y, z) \).

Since \( x \) and \( y \) are algebraic over \( \mathbb{C}(p, q, r) \), a generalization of [14, Remark 4.7] implies that \( \mathbb{C}(p, q, r)[x, y] = \mathbb{C}(p, q, r)/(x, y) \). Then, \( \mathbb{C}(p, q, r)[x, y] = \mathbb{C}(x, y, z) \). From this it is not difficult to obtain that \( z = u/v \), where \( u \in \mathbb{C}[p, q, r, x, y] = 0 \) and \( v \in \mathbb{C}[p, q, r] = 0 \). □

Define: \( \alpha : \mathbb{C}[U_1, U_2, U_3, U_4, U_5] \to \mathbb{C}[p, q, r, x, y] \) by \( \alpha(U_1) := p, \alpha(U_2) := q, \alpha(U_3) := r, \alpha(U_4) := x, \alpha(U_5) := y \). Clearly, \( \alpha \) is surjective.

We can define \( \beta : \mathbb{C}[U_1, U_2, U_3, U_4, U_5][1/V] \to \mathbb{C}[p, q, r, x, y][1/v] \). It is clear that \( z \in \mathbb{C}[p, q, r, x, y][1/v] \).

The kernel of \( \alpha \) is a height two prime ideal; indeed, \( \mathbb{C}[U_1, U_2, U_3, U_4, U_5] \) is of Krull dimension 5 and \( \mathbb{C}[p, q, r, x, y] \) is of Krull dimension 3, hence the kernel of \( \alpha \) is of height two.

From Krull’s principal ideal theorem [3, Theorem 8.42], the kernel of \( \alpha \) is generated by at least two elements.

Assume for the moment that the kernel of \( \alpha \) is generated by exactly two elements, hence the kernel of \( \beta \) is generated by the same two elements. The matrix equation (2) in Formanek’s proof will involve the product of a \( 5 \times 5 \) matrix with a \( 5 \times 5 \) matrix, but Cramer’s Rule seems not to help here. We do not know yet if it is possible to overcome this problem.
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Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.
E-mail address: vered.moskowicz@gmail.com