Cartesian closed coreflective subcategories of topological spaces determined by monotone convergence classes

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Abstract We introduce the notion of an operation $\mathcal{P}$ and a $\mathcal{P}$-determined space. It is shown that a category is a coreflective full subcategory of Top if and only if it is equal to $\text{Top}_\mathcal{P}$ for some idempotent and consistent operation $\mathcal{P}$, where $\text{Top}_\mathcal{P}$ is the category of all $\mathcal{P}$-determined spaces. As concrete examples of $\mathcal{P}$-determined spaces, several classes of topological spaces determined by monotone convergence classes are investigated in detail. By the tool of $\mathcal{C}$-generated spaces, it is shown uniformly that categories of these examples of $\mathcal{P}$-determined spaces are all cartesian closed. Moreover, the exponential objects and categorical products of some categories in domain theory are shown to be closely related to those of $\text{DTop}$, the category of directed spaces.

Keywords $\mathcal{P}$-determined space · coreflective subcategory · cartesian closed category · monotone determined space · $\mathcal{C}$-generated space

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1 Introduction

Monotone determined spaces were introduced by Erné in [3]. An interesting result for monotone determined spaces is that the lattice of all open sets is algebraic if and only if it is hyperalgebraic [3, 16]. By means of setting a final topology on a $T_0$ space, Yu and Kou in [17] gave a more natural definition of a monotone determined space and called it a directed space. Moreover, they showed that $\text{DTop}$, the category consisting of directed spaces together with continuous maps is cartesian closed, which includes all posets endowed with the Scott topology as a full subcategory. Later, the category $\text{Top}_\text{md}$ consisting of monotone determined spaces was also shown to be cartesian closed [12] in an essentially similar way as in [17].

Noticing that the topology of a monotone determined space is determined by only a part of the monotone nets and limits in the space, we investigate the spaces determined by a class of pairs of convergent nets and limits. We call $\mathcal{P}$ an operation if it determines a class $\mathcal{P}_X$ of pairs of convergent nets and limits for every topological space $X$. A new finer topology on $X$, denoted by $\mathcal{P}(X)$, can be defined through $\mathcal{P}_X$. We call a topological space $X$ a $\mathcal{P}$-determined space if $\mathcal{P}(X)$ is equal to the topology of $X$. We show that for any idempotent and consistent operation $\mathcal{P}$, $\text{Top}_\mathcal{P}$ is a coreflective subcategory of Top and vice versa, where Top is the category of all topological spaces and $\text{Top}_\mathcal{P}$ is the category of all $\mathcal{P}$-determined spaces.

$\mathcal{C}$-generated space is a powerful tool to construct cartesian closed categories [4]. It is closely related to coreflective subcategories of Top [4,8]. Kelley spaces and sequential spaces are the well known examples. Sequential spaces can also be viewed as the topological spaces determined by convergent sequences. Monotone determined spaces are topological spaces determined by convergent monotone nets. There is a close relation between monotone determined spaces and $\mathcal{C}$-generated spaces. We find out that a series of $\mathcal{P}$-determined spaces, including monotone determined spaces, are some kind of $\mathcal{C}$-generated spaces on the other hand. Using the tool of $\mathcal{C}$-generated spaces, we can show uniformly

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that these \( \mathcal{P} \)-determined spaces form cartesian closed subcategories of \( \text{Top} \) and give the descriptions for their categorical product and exponential objects.

Finally, we investigate the relationships between these \( \mathcal{P} \)-determined spaces and some classical structures in domain theory, such as depots endowed with the Scott topology, the weak Scott topology and the Alexandroff topology. In particular, these categorical products and exponential objects are shown to be deeply related to those in the category \( \text{DTop} \) of directed spaces.

## 2 Preliminaries

We assume some basic knowledge of domain theory, topology and category theory, as in, e.g., [1,2,6,7].

For any pre-ordered set \((\mathcal{P}, \leq)\) and \(\mathcal{A} \subseteq \mathcal{P}\), we set \(\mathcal{A}^\uparrow = \{ x \in \mathcal{P} : \forall a \in \mathcal{A}, a \leq x \}\), \(\mathcal{A}^\downarrow = \{ x \in \mathcal{P} : \forall a \in \mathcal{A}, x \leq a \}\). A nonempty subset \(\mathcal{D}\) of \(\mathcal{P}\) is directed iff every nonempty finite subset of \(\mathcal{D}\) has an upper bound in \(\mathcal{D}\). If \(\mathcal{P}\) is directed, then we say that \(\mathcal{P}\) is a directed set. For any poset \(\mathcal{P}\), we use \(\sigma(\mathcal{P})\), \(v(\mathcal{P})\) and \(A(\mathcal{P})\) to denote the Scott topology, the upper topology, and the Alexandroff topology on \(\mathcal{P}\) respectively.

Given any topological space \(X\), its topology is denoted by \(\mathcal{O}(X)\). For any net \(\xi = \{ x_j \}_{j \in J}\) where \(J\) is a directed set, we write it \(\{ x_j \}_{j \in J}\) for short. Given any \(x \in X\), \(\{ x_j \}_{j \in J}\) is called converging to \(x\), denoted by \(\xi \rightarrow x\) or \(\{ x_j \}_{j \in J} \rightarrow x\), if \(\{ x_j \}_{j \in J}\) is eventually in every open neighborhood of \(x\). Give any directed set \(\mathcal{D}\), it can be viewed as a monotone net \(\{ d \}_{d \in \mathcal{D}}\). The specialization order on \(X\) is defined by \(x \preceq y\) if \(x \in \overline{\{ y \}}\). Usually, the specialization order is a pre-order. When \(\mathcal{X}_0\) is directed, its specialization order is a partial order. For any poset \((\mathcal{P}, \leq)\) and a topology \(\tau\), the specialization order \(\preceq\) of the topology \(\tau\) agrees with the order \(\leq\) iff \(\tau\) is coarser than the Alexandroff topology and finer than the upper topology. We do not distinguish \(\preceq\) and \(\subseteq\) if they coincide. Given any two \(\mathcal{X}_0\) topological spaces \(X\) and \(Y\), we denote \(X^Y\) to be the poset of all continuous maps from \(X\) to \(Y\) endowed with the pointwise order. Denote \([X \rightarrow Y]_p\) to be the topological space as follows: the underlying set is same to \(Y^X\); the topology is the pointwise convergence topology. Then the specialization order of \([X \rightarrow Y]_p\) is equal to the pointwise order.

A category \(\mathcal{C}\) is called cartesian closed if it has finite categorical products and exponential objects for all objects. We use \(X \otimes Y\) and \([X \rightarrow Y]\) to denote the categorical product and exponential object of \(X\) and \(Y\) respectively. Let \(\mathcal{C}\) be a collection of topological spaces, referred to as generating spaces. A map \(k : C \rightarrow X\) is called a probe over \(X\), of \(\mathcal{C}\)-generated topology (also called the final topology) on \(X\) [4]. \(X\) equipped with its \(\mathcal{C}\)-generated topology is denoted by \(\mathcal{C}X\). A topological space \(X\) is called a \(\mathcal{C}\)-generated space if \(X = \mathcal{C}X\). All \(\mathcal{C}\)-generated spaces together with continuous maps as morphisms form a category, denoted by \(\text{Top}_c\).

We denote \(\text{Top}\) the category of all topological spaces. A topological space \(X\) is core compact if \(\mathcal{O}(X)\) is a continuous lattice, which is equivalent to that \(X\) is exponentiable in the category \(\text{Top}\). A collection \(\mathcal{C}\) of topological spaces is called a productive class if all topological spaces in \(\mathcal{C}\) are core compact and every topological product \(C_1 \times C_2\), where \(C_1, C_2 \in \mathcal{C}\), is \(\mathcal{C}\)-generated.

**Theorem 1.** [4] If \(\mathcal{C}\) is a productive class of topological spaces and contains at least one non-empty space \(C_0\), then \(\text{Top}_c\) is cartesian closed.

A topological space \(X\) is said to be a monotone determined space if for any subset \(\mathcal{U} \subseteq X\), \(U\) is open iff every monotone net \(\{ x_i \}\) (satisfying \(i \leq j \Rightarrow x_i \leq x_j\)) that converges to a point in \(U\) is eventually in \(U\) [3,12]. In \(\mathcal{T}_0\) topological spaces, the notion of monotone determined spaces is equivalent to directed spaces [17]. Let \(X\) be a \(\mathcal{T}_0\) topological space. Set \(\mathcal{D}_X = \{ (D, x) : D \subseteq X, \mathcal{D} \text{ is directed, } D \rightarrow x \in X \}\). A subset \(\mathcal{U} \subseteq X\) is called directed-open if for all \((D, x) \in \mathcal{D}_X, x \in \mathcal{U}\) implies \(D \cap \mathcal{U} \neq \emptyset\). All directed-open subsets form a topology on \(X\), called the directed topology, denoted by \(d(X)\) or \(d(\mathcal{O}(X))\). We denote \((X, d(X))\) briefly by \(\mathcal{D}X\).

A topological space \(X\) is said to be a directed space if it is \(\mathcal{T}_0\) and every directed-open set is open, i.e., \(d(X) = \mathcal{O}(X)\). For a directed space \(X\), we also denote \(d(X)\) to be the topology of \(X\). All directed spaces together with continuous maps form a category, denoted by \(\mathcal{D}\text{Top}\).

The following are some basic properties of directed spaces.

**Proposition 2.** [17] Let \(X\) be a \(\mathcal{T}_0\) topological space.

1. For any \(U \subseteq d(X)\), \(U = \uparrow U\).
2. \(X\) equipped with \(d(X)\) is a \(\mathcal{T}_0\) topological space such that \(\sqsubseteq_d = \sqsubseteq\), where \(\sqsubseteq_d\) is the specialization order relative to \(d(X)\).
3. For a directed subset \(D\) of \(X\), \(D \rightarrow x\) iff \(D \rightarrow_d x\) for all \(x \in X\), where \(D \rightarrow_d x\) means that \(D\) converges to \(x\) with respect to the topology \(d(X)\).
4. \(d(\mathcal{D}X) = d(X)\), i.e., \(\mathcal{D}X\) is a directed space.
Theorem 3. [17] The category $\mathbf{DTop}$ of all directed spaces is cartesian closed. Let $X, Y$ be in $\mathbf{DTop}$ and $X \times Y$ be the topological product of $X$ and $Y$.

1. The categorical product $X \otimes_d Y = \mathcal{D}(X \times Y)$;
2. The exponential object $[X \to Y]_d = \mathcal{D}([X \to Y])$.

Proposition 4. [17] Let $X, Y, Z$ be directed spaces.

1. A function $f : X \otimes_d Y \to Z$ is continuous if it is separately continuous, i.e., continuous with respect to each variable.
2. In $X \otimes_d Y$, a directed subset $D$ converges to $(x, y)$ iff $\pi_1(D) \to x$ in $X$ and $\pi_2(D) \to y$ in $Y$, where $\pi_1, \pi_2$ are the projections from $X \otimes_d Y$ to $X, Y$ respectively.

Directed spaces contain many interesting spaces in domain theory such as posets with the Scott topology, posets with the Alexandroff topology and c-spaces, and they were shown to be appropriate topological extensions of dcpos (see [12,15,17]).

3 $\mathcal{P}$-determined spaces

It is sometimes convenient to define a topology through convergent nets. Some topological properties, like compactness, can be characterized by the convergent nets. We first illustrate some basic relationships between convergent nets and topologies. Then, we define a special kind of topological spaces, called $\mathcal{P}$-determined spaces, which are motivated by the Scott topology on dcpos and directed spaces. Monotone determined spaces are concrete examples of $\mathcal{P}$-determined spaces. We will show that any kind of $\mathcal{P}$-determined spaces form a coreflective subcategories of $\mathbf{Top}$. Conversely, every coreflective subcategory of $\mathbf{Top}$ can be viewed as a category of some kind of $\mathcal{P}$-determined spaces.

We say that $\mathcal{E}$ is a convergence class on $X$, if $\mathcal{E} \subseteq \{(\xi, x) : \xi$ is a net in $X, x \in X\}$ for a set $X$. A convergence class $\mathcal{E}$ on $X$ naturally induce a topological space, denoted by $(X, \mathcal{E}(X))$ or $\mathcal{E}X$ for short; as follows: $U$ is open in $\mathcal{E}X$ iff for any $(\xi, x) \in \mathcal{E}$, $x \in U$ implies $\xi$ is eventually in $U$. We say that the topology $\mathcal{E}(X)$ and the topological space $\mathcal{E}X$ are determined by $\mathcal{E}$. On the other hand, given any topological space $(X, \tau)$, we define $\mathcal{C}(X, \tau)$ (or $\mathcal{C}_X$) to be $\{(\xi, x) : \xi \to x \in X\}$. Obviously, for any topological space $(X, \mathcal{E}(X))$, we have $\mathcal{E} \subseteq \mathcal{C}(X, \mathcal{E}(X))$. We call a convergence class $\mathcal{E}$ topological if $\mathcal{E} = \mathcal{C}(X, \mathcal{E}(X))$.

There is a one-to-one correspondence between topological convergence classes and topologies on a set $X$ [11]. When $\mathcal{E}$ is a topological convergence class, for any subset $F$ of $X$, its closure $\overline{F} = \{x \in X : (\xi, x) \in \mathcal{E}, \xi \in F\}$. This equality does not always hold for general convergence classes $\mathcal{E} \subseteq \{(\xi, x) : \xi$ is a net in $X, x \in X\}$. However, for some special $\mathcal{E}$, we can gain its closure by transfinite induction.

Definition 5. Let $P$ be a set. For any convergence class $\mathcal{E} \subseteq \{(\xi, x) : \xi$ is a net in $P, x \in P\}$, any subset $F \subseteq P$, and any ordinal $\alpha \in \text{ORD}$, we define $F^\alpha$ and $F^*$ as follows:

\[
F^0 = F
F^\alpha = \{x \in P : \exists \xi \subseteq (\cup_{\beta < \alpha} F^\beta), (\xi, x) \in \mathcal{E}\}
F^* = \cup_{\alpha \in \text{ORD}} F^\alpha
\]

Proposition 6. Let $(P, \leq)$ be a poset (resp., pre-ordered set) and $\mathcal{E}$ be any convergence class such that $\{(y, x) : x, y \in P, x \leq y\} \subseteq \mathcal{E} \subseteq \{(D, x) : D$ is a directed subset of $P, x \in D^h\}$.

1. For any $F \subseteq P$, we have $\overline{F} = F^*$ in the topological space $(P, \mathcal{E}(P))$.
2. The specialization order $\subseteq$ of $(P, \mathcal{E}(P))$ is equal to $\leq$.

Proof. (1) It is easy to see that if $\alpha < \beta$, then $F^\alpha \subseteq F^\beta$, and if $F^\alpha = F^{\alpha+1}$, then $F^* = F^\alpha$. For any subset $F$, there exists some $\alpha$ such that $F^\alpha = F^{\alpha+1} = F^*$ since the cardinal of $F^*$ is less than or equal to that of $P$. We need only to show that $P \setminus F^*$ is open. For any $(D, x) \in \overline{F}$ such that $x \in P \setminus F^*$, if $D \not\subseteq F^*$, then $x \in F^*$, a contradiction. If $D \not\subseteq F^*$, then $D \cap (P \setminus F^*) \neq \emptyset$. By $\{(y, x) : x, y \in P, x \leq y\} \subseteq \mathcal{E}$, we know that $P \setminus F^*$ is an upper subset of $P$. Therefore, $P \setminus F^*$ is eventually in $P \setminus F^*$, i.e., $P \setminus F^*$ is open.

(2) Given any $y \in P$, assume that $(D, x) \in \mathcal{E}$ and $x \in X \setminus y$. Then $D \cap (X \setminus y) \neq \emptyset$. Otherwise, $x \in D^h \subseteq \setminus y$, a contradiction. Therefore, $P \setminus y$ is open in $(P, \mathcal{E}(P))$ and then $x \subseteq y$ implies $x \leq y$. If $x \leq y$, for any open subset $U$ such that $x \in U$, we have $y \in U$ by $\{(y, x) : x, y \in P, x \leq y\} \subseteq \mathcal{E}$. Thus, $x \leq y$ implies $x \subseteq y$. \qed

The conclusion of Proposition 6 does not always hold for general $\mathcal{E}$. For example, let $N^+\tau$ be the poset of natural numbers adding a top element $\top, \mathcal{E} = \{(N, \tau)\}$ and $A = \{2n+1 : n \in N\}$. Then $A^* = A$ and it is not closed in $(N^+, \mathcal{E}(N^+))$.

For a topological space $X$ determined by a convergence class $\mathcal{E}$, the continuous maps from $X$ can be characterized by the preservation for $\mathcal{E}$. We have the following statement.
Lemma 7. Let $Y$ be any topological space, $X$ a set and $\mathcal{E}, \mathcal{F} \subseteq \{(\xi, x) : \xi$ is a net in $X, x \in X\}$.

1. If $\mathcal{E} \subseteq \mathcal{F}$, then $(X, \mathcal{F}(X))$ is coarser than $(X, \mathcal{E}(X))$.

2. A map $f$ from $(X, \mathcal{E}(X))$ into $Y$ is continuous iff for any $(\xi, x) \in \mathcal{E}$, $f(\xi) \rightarrow f(x)$ in $Y$ holds, i.e., $f$ preserves $\xi \rightarrow x$ in $\mathcal{E}X$ for each $(\xi, x) \in \mathcal{E}$.

Proof. (1) Let $U$ be an open subset in $(X, \mathcal{F}(X))$, then for any $(\xi, x) \in \mathcal{F}$ and $x \in U$, $\xi$ is eventually in $U$. If $\mathcal{E} \subseteq \mathcal{F}$, we have that for any $(\xi, x) \in \mathcal{E}$ and $x \in U$, $\xi$ is eventually in $U$, i.e. $U$ is open in $(X, \mathcal{E}(X))$.

(2) The necessity is obvious. For sufficiency, letting $U$ be an open subset of $Y$, we need only to show that $f^{-1}(U)$ is open in $(X, \mathcal{E}(X))$. For any $(\xi, x) \in \mathcal{E}$ and $x \in f^{-1}(U)$, we have that $\xi$ is eventually in $f^{-1}(U)$. Otherwise, $f(\xi)$ is not eventually in $U$, a contradiction. □

Definition 8. We call $\mathcal{P}$ an operation if for any topological space $X$, it determines a convergence class $\mathcal{P}_X$ such that $\mathcal{P}_X \subseteq \mathcal{C}_X$. If $\mathcal{P}_X \subseteq \mathcal{P}_X(\mathcal{P}_X)$ holds for every topological space $X$, then we say that $\mathcal{P}$ is idempotent. For any idempotent $\mathcal{P}$, we define $\mathcal{P}X = (X, \mathcal{P}_X(X))$. $X$ is called a $\mathcal{P}$-determined space iff $X = \mathcal{P}X$.

Lemma 9. Let $X$ be a topological space.

1. For any idempotent operation $\mathcal{P}$, $\mathcal{P}X$ is a $\mathcal{P}$-determined space.

2. For any two idempotent operation $\mathcal{P}, \mathcal{Q}$, if $\mathcal{P}Y \subseteq \mathcal{Q}Y$ holds for every topological space $Y$ (called that $\mathcal{Q}$ is larger than $\mathcal{P}$), then every $\mathcal{P}$-determined space is a $\mathcal{Q}$-determined space.

Proof. (1) For any topological space $X$, since $\mathcal{P}_X \subseteq \mathcal{P}_X$ we have that $\mathcal{P} \mathcal{P}X$ is coarser than $\mathcal{P}X$ by Lemma 7. It is obvious that $\mathcal{P}X$ is coarser than $\mathcal{P} \mathcal{P}X$, so $\mathcal{P}X$ is a $\mathcal{P}$-determined space.

(2) Assuming that $Y$ is a $\mathcal{P}$-determined space, then $\mathcal{P}Y = Y$ and then it is coarser than $\mathcal{Q}Y$. By the assumption that $\mathcal{P}Y \subseteq \mathcal{Q}Y$ and Lemma 7, we have that $\mathcal{Q}Y$ is coarser than $\mathcal{P}Y = Y$. Thus, $Y$ is also a $\mathcal{Q}$-determined space. □

Definition 10. Let $X, Y$ be two topological spaces and $\mathcal{P}$ be an idempotent operation. A map $f : X \rightarrow Y$ is said to be $\mathcal{P}$-continuous if $f : \mathcal{P}X \rightarrow \mathcal{P}Y$ is continuous.

Definition 11. An operation $\mathcal{P}$ is said to be consistent if for any two topological spaces $X, Y$ and any continuous map $f : X \rightarrow Y$, $(\xi, x) \in \mathcal{P}_X$ implies $(f(\xi), f(x)) \in \mathcal{P}_Y$.

Lemma 12. Let $\mathcal{P}$ be an idempotent and consistent operation. Given any topological space $Y$ and any $\mathcal{P}$-determined space $X$, a map $f : X \rightarrow Y$ is continuous iff $f : X \rightarrow \mathcal{P}Y$ is $\mathcal{P}$-continuous.

Proof. Suppose that $f : X \rightarrow Y$ is continuous. Since $\mathcal{P}$ is consistent, $(\xi, x) \in \mathcal{P}_X$ implies $(f(\xi), f(x)) \in \mathcal{P}_Y$. By lemma 7, $f : \mathcal{P}X \rightarrow \mathcal{P}Y$ is continuous, that is, $f : X \rightarrow \mathcal{P}Y$ is continuous. Since $\mathcal{P} \mathcal{P}Y = \mathcal{P}Y$, we have $f : X \rightarrow \mathcal{P}Y$ is $\mathcal{P}$-continuous. For converse, suppose that $f : X \rightarrow \mathcal{P}Y$ is $\mathcal{P}$-continuous, i.e., $f : \mathcal{P}X \rightarrow \mathcal{P} \mathcal{P}Y$ is continuous. Since $\mathcal{P}X = X$ and $\mathcal{P} \mathcal{P}Y = \mathcal{P}Y$ by Lemma 9, we get that $f : X \rightarrow \mathcal{P}Y$ is continuous. By that $Y$ is coarser than $\mathcal{P}Y$, we know that $f : X \rightarrow Y$ is continuous. □

Corollary 13. Let $\mathcal{P}$ be an idempotent and consistent operation and $X, Y$ be $\mathcal{P}$-determined spaces. A map $f : X \rightarrow Y$ is continuous iff it is $\mathcal{P}$-continuous.

For any idempotent and consistent operation $\mathcal{P}$, we denote $\mathbf{Top}_\mathcal{P}$ the category of all $\mathcal{P}$-determined spaces with $\mathcal{P}$-continuous maps as morphisms. Since a map between $\mathcal{P}$-determined spaces is continuous iff it is $\mathcal{P}$-continuous, $\mathbf{Top}_\mathcal{P}$ is a full subcategory of $\mathbf{Top}$.

We now define $\overline{\mathcal{P}}$ as follows:

$$\forall X \in \text{Ob}(\mathbf{Top}), \overline{\mathcal{P}}(X) = \mathcal{P}X,$$

$$\forall f \in \text{Mor}(\mathbf{Top}), \overline{\mathcal{P}}(f)(x) = f(x).$$

Because $\mathcal{P}X$ is a $\mathcal{P}$-determined space for any topological space $X$, and the continuity is equivalent to $\mathcal{P}$-continuity for maps between $\mathcal{P}$-determined spaces, we know that $\overline{\mathcal{P}}$ is a functor from $\mathbf{Top}$ to $\mathbf{Top}_\mathcal{P}$. Moreover, for every idempotent and consistent $\mathcal{P}$, we show that $\mathbf{Top}_\mathcal{P}$ is a coreflective subcategory of $\mathbf{Top}$ and $\overline{\mathcal{P}}$ is the coreflector.

Definition 14. [2] Let $A$ be a subcategory of $\mathbf{B}$ and let $B$ be a $\mathbf{B}$-object.

1. An $A$-coreflection for $B$ is a $\mathbf{B}$-morphism $A \xrightarrow{\delta} B$ from an $A$-object $A$ to $B$ with the following universal property: for any $\mathbf{B}$-morphism $A' \xrightarrow{\delta'} B$ from some $A$-object $A'$ to $B$ there exists a unique $A$-morphism $f : A' \rightarrow A$ such that $\delta = \delta' \circ f$. Such $A$ is called a $A$-coreflection for $B$.

2. $A$ is called a coreflective subcategory of $\mathbf{B}$ provided that each $\mathbf{B}$-object has an $A$-coreflection.

Theorem 15. Let $\mathcal{P}$ be an idempotent and consistent operation. Then $\mathbf{Top}_\mathcal{P}$ is a coreflective subcategory of $\mathbf{Top}$. 

Proof. Given any $Y$ in Top, the identity map $id : PY \to Y$ is continuous. For any $X$ in Top$_P$ and continuous map $f : X \to Y$, $f : X \to PY$ is continuous by Lemma 12 and Corollary 13. Therefore, PY is the Top$_P$-coreflection for Y.

Example 16. These are some examples of $P$-determined spaces for idempotent and consistent $P$.

Then we can know directly that they are coreflective subcategories of Top.

1. Discrete spaces are 1-determined, where $1_X = \{(x), x \in X\}$ for any space $X$.
2. Alexandroff spaces are $S$-determined, where $S_X = \{(y), x \in X, x \subseteq y\}$.
3. Sequential spaces are $N''$-determined, where $N''_X = \{(M, x), x \in X, M$ is a sequence in $X, M \to x\}$.

Conversely, Given any coreflective subcategory $C$ of Top, there is an idempotent and consistent operation $P$ such that $C = Top_P$.

Proposition 17. [2] If $A$ is a coreflective subcategory of $B$ and for each $B$-object $B$, $A_B \subseteq B$ is an $A$-coreflection, then there exists a unique functor $C : B \Rightarrow A$ (called a coreflector for $A$) such that the following conditions are satisfied:

1. $C(B) = A_B$ for each $B$-object $B$;
2. for each $B$-morphism $f : B \to B'$ the diagram below commutes.

\[
\begin{array}{ccc}
C(B) & \xrightarrow{c_B} & B \\
C(f) \downarrow & & \downarrow f \\
C(B') & \xleftarrow{c_{B'}} & B'
\end{array}
\]

Theorem 18. [8] A subcategory $C$ of Top is coreflective in Top iff for each space $(X, \tau)$ in Top, there exists a topology $\eta_X$ on $X$ with the following properties:

1. $\eta_X$ is finer than $\tau$;
2. $(X, \eta_X)$ is in $C$;
3. $\eta_X$ is the coarsest topology on $X$ satisfying (1) and (2);
4. for each space $(Y, \phi)$ and each continuous map $f : (X, \tau) \to (Y, \phi)$, the same set map $f : (X, \eta_X) \to (Y, \eta_Y)$ is continuous.

From Proposition 17 and Theorem 18, we know that $C(X)$, the $C$-coreflection for $(X, \tau)$ in Top, is $(X, \eta_X)$ up to isomorphism. Then we have the following statement.

Proposition 19. Let $C$ be a coreflective subcategory of Top. Taking $P$ an operation such that $P_X = C_{C(X)}$, then $P$ is idempotent and consistent. Moreover, for any topological space $X$, $C(X) = P_X$, i.e., $P$ is the coreflector for $C$.

Proof. By Theorem 18, given any two topological spaces $X_1, X_2$, if $X_1$ is coarser than $X_2$, then $P_{X_2} = C_{C(X_2)} \subseteq C_{C(X_1)} = P_{X_1}$. Since $P_X = C_{C(X)}$, we have $PX = PX X = C_{C(X)} X = C(X)$. Obviously, for any space $X$ in $C$, $C(X) = X$ and then $PX = C(X) = X$. Thus, $P_X = P_{P_X}$, that is, $P$ is idempotent. Given any continuous map $f : X \to Y$ in Top, by Theorem 18, $f : C(X) \to C(Y)$ is continuous. Then $(\xi, x) \in C_{C(X)}$, implies $(f(\xi), f(x)) \in C_{C(Y)}$, i.e., $P$ is consistent.

By Proposition 15 and Proposition 19, we know that there is a correspondence between idempotent and consistent operations and coreflective subcategories. We define an equivalence relation on idempotent and consistent operations that two operation $P, Q$ are equivalent iff $QX = PX$ for every topological space $X$. The equivalent class of $P$ is denoted by $[P]$. It is easily seen that there is a largest one among all operations in $[P]$, which determines a convergence class $C_{P_X}$ for each topological space $X$, where the largest means that $C_{P_X}$ contains the most pairs compared to other operations in $[P]$. Then there is a one-to-one correspondence between all equivalent classes of idempotent and consistent operations and coreflective subcategories.

It is easy to check that for any two idempotent and consistent operations $P, Q$, if $Q$ is larger than $P$, i.e., $P_X \subseteq Q_X$ for every topological space $X$, then Top$_P$ is a coreflective subcategory of Top$_Q$. The following are some basic properties of coreflective subcategories of Top.

Theorem 20. [8] The following statements hold:

1. A subcategory of Top is coreflective in Top iff it is invariant under the formation of disjoint topological unions and topological quotient spaces in Top.
2. Every coreflective subcategory of Top is cocomplete and is a cocomplete subcategory of Top.
3. Every coreflective subcategory $C$ of Top is complete and the limit in $C$ is the $C$-coreflection for the limit in Top.
Given a dcpo $L$, a subset $U$ is Scott open iff it is an upper subset and for any directed subset $D \subseteq L$, $\sup D \in U$ implies $D \cap U \neq \emptyset$. A map is said to be Scott continuous if it preserves all suprema of directed subsets. From the viewpoint of convergence, we can define the Scott topology to be the topology determined by convergence class $\{\{D, x\} : D \subseteq L, \ D \text{ is directed}, \ x \leq \sup D\}$. A deep relation between domain theory and topology is that for a map $f : L \to M$ between two dcpos $L, M$, $f$ is Scott continuous equivalent to that $f : \Sigma L \to \Sigma M$ is continuous, which is very natural by Lemma 7. The notion of $P$-determined spaces is a generalization for monotone determined spaces. We say a convergence class is monotone if all nets in the pairs in it are monotone. Some simple examples of $P$-determined spaces are presented in Example 16. We introduce some other concrete examples, which are all determined by monotone convergence classes and are the main research objects in this paper.

Given any pre-ordered set $P$, we denote a pre-chain to be a directed subset of $P$ such that any two elements are comparable. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called monotone if $i \leq j$ implies $x_i \leq x_j$.

**Definition 21.** We define six operations $D, D', I, I', N, N'$ as follows. Let $X$ be any topological space.

1. $D_X = \{(D, x) : D \text{ is directed in } X, \ D \to x\}$
2. $D'_X = \{(D, x) : D \text{ is directed in } X, \ x \in \downarrow D \text{ or } D \to x, \ D \leq x\}$
3. $I_X = \{(I, x) : I \text{ is a pre-chain in } X, \ I \to x\}$
4. $I'_X = \{(I, x) : I \text{ is a pre-chain in } X, \ x \in \downarrow I \text{ or } I \to x, \ I \leq x\}$
5. $N_X = \{(M, x) : M \text{ is a monotone sequence in } X, \ M \to x\}$
6. $N'_X = \{(M, x) : M \text{ is a monotone sequence in } X, \ x \in \downarrow M \text{ or } M \to x, \ M \leq x\}$

We use operations $P_i (1 \leq i \leq 6)$ to denote these operations for convenience. The following result can be checked easily, so we omit the proof.

**Theorem 22.** The six operations above are all idempotent and consistent. Hence, $\mathbf{Top}P_1$ is a coreflective subcategory of $\mathbf{Top}$ for $1 \leq i \leq 6$.

By definition, $D$-determined spaces are just the monotone determined spaces. Spaces determined by operations $P_i (2 \leq i \leq 6)$ are all included in $D$-determined spaces. They all form coreflective subcategories of $\mathbf{Top}$ and then are cocomplete and complete. We denote $\omega D_X = \{(D, x) \in D_X : D \text{ is countable}\}$, then it is easy to see that given any topological space $X$, $\omega D(X) = NX$. Further properties of these spaces will be investigated by using the tool of $C$-generated spaces in the next section.

**Remark 23.** Obviously, if we restrict all topological spaces in $T_0$ spaces, the above statements still hold. When $X$ is $T_0$, $P_X$ is also $T_0$ for any operation $P$. Directed spaces are just all the $T_0$ $D$-determined spaces. Denote $\mathbf{Top}_0$ the category of all $T_0$ topological spaces. We also have the conclusion that $D\mathbf{Top}$ is a coreflective subcategory of $\mathbf{Top}_0$.

**Lemma 24.** Let $X$ be a $T_0$ topological space and $D$ be a directed subset of $X$.

1. If $D \subseteq \downarrow x$ and $D \to x$ for $x \in P$, then $x = \sup D$.
2. If $D \to x$ and $\sup D$ exists, then $x \leq \sup D$.

**Proof.** (1) Suppose that $D \subseteq \downarrow x$ and $D \to x$ for $x \in X$. Assume that there exists $y \in X$ such that $D \subseteq \downarrow y$ and $x \not\leq y$. Then $x \in P \downarrow y \in O(X)$. Thus $D \cap (X \downarrow y) \neq \emptyset$, a contradiction. Hence, $x = \sup D$.

(2) Since $D \to x$, given any open subset $U$ such that $x \in U$, there exists some $d \in D$ such that $d \in U$ and then $\sup D \subseteq U$. Therefore, $x \leq \sup D$.

By Lemma 24, we know that when $X$ is $T_0$, $D_X' = \{(D, x) : D \text{ is directed in } X, \ x \in \downarrow D \text{ or } D \to x, \ \sup D = x\}$. It is easy to check that $(L, \sigma(L))$, the dcpo $L$ endowed with the Scott topology, is $D'$-determined.

4 Monotone determined spaces as $C$-generated spaces

In this section, we will establish the relationships between these $P$-determined spaces defined in the previous section and $C$-generated spaces. By the tool of $C$-generated spaces, we can uniformly show that they are all cartesian closed.

Some well known spaces such as Kelly spaces, sequential spaces are some kind of $C$-generated spaces. Given any class $C$ of topological spaces, $\mathbf{Top}_C$ is a coreflective subcategory of $\mathbf{Top}$ [4]. From Example 16, we notice that some $C$-generated spaces are closely related to $P$-determined spaces: (1) discrete spaces are also one point space generated; (2) Alexandroff spaces are $S$-generated, where $S$ is the Sierpinski space; (3) sequential spaces are sequentially generated [4].

Similarly, we define some class of generating spaces and show that spaces generated by them coincide with these spaces determined by $P_i (1 \leq i \leq 6)$ respectively.
Definition 25. Let $\mathcal{D}$ denote the class of all directed posets, $\mathcal{J}$ denote the class of all chains and $\mathcal{N}$ denote the poset of natural numbers. Given any poset $P$, let $\infty$ be an element that is not in $P$. We define two topologies on $P \cup \{\infty\}$ as follows: $\beta_P$ is the topology by taking all $\uparrow x$ and $\uparrow x \cup \{\infty\}$ for $x \in P$ as a subbase; $\gamma_P$ is the topology by taking all $\uparrow x \cup \{\infty\}$ for $x \in P$ as a subbase. Particularly, we define six classes of generating spaces as follows:

1. $\mathcal{D} = \{D \cup \{\infty\}, \beta_D : D \in \mathcal{D}\}$;
2. $\mathcal{D}' = \{D \cup \{\infty\}, \gamma_D : D \in \mathcal{D}\}$;
3. $\mathcal{I} = \{(I \cup \{\infty\}, \beta) : I \in \mathcal{J}\}$;
4. $\mathcal{I}' = \{(I \cup \{\infty\}, \gamma_I) : I \in \mathcal{J}\}$;
5. $\mathcal{N} = \{(N \cup \{\infty\}, \beta_N)\}$;
6. $\mathcal{N}' = \{(N \cup \{\infty\}, \gamma_N)\}$.

Lemma 26. Let $X, Y$ be topological spaces. A map $f : X \to Y$ is monotone iff $f$ preserves $\{y \to x \mid x \leq y \in X \}$ for any $x \leq y$ in $X$ iff $f$ preserves preserves $D \to x$ for any directed subset $D$ of $X$ such that $x \in \downarrow D$.

Proof. For any topological space $X$ and $x, y \in X$, $x \leq y$ iff $\{y \to x\} \to x$. Assuming that $f$ is monotone, for any directed subset $D$ and $x \in D$, if $x \leq y \in \downarrow D$, then $f(D)$ is directed in $Y$ and $f(x) \preceq f(y)$. It follows that $f(D) \to f(x)$.

If $f$ preserves $D \to x$ in $X$ for any directed subset $D$ of $X$ and $x \in \downarrow D$, then $f$ preserves $\{y \to x \mid x \leq y \in X \}$ for any $x \leq y$. Let $x \leq y$ in $X$, then $\{y \to x\} \to x$ and $\{f(y) \to f(x)\} \to f(x)$. Thus, $f(x) \preceq f(y)$.

For convenience, we denote $\mathcal{C}_i$ ($1 \leq i \leq 6$) the six classes in Definition 25. Then we have the following statement.

Proposition 27. Given any topological space $X$, we have $\mathcal{P}_iX = \mathcal{C}_iX$ for $1 \leq i \leq 6$.

Proof. (1) For any $(D \cup \{\infty\), \beta_D) \in \mathcal{D}$, it is easy to check that $D \to \infty$ and $\{d \to e \mid e \leq d \in D \to e$ for any $e \leq d \in D$. Let $E$ be the convergence class consisting of these pairs of convergent nets and limits, then it determines a topology $\mathcal{E}(D \cup \{\infty\}$ and we have $\mathcal{E}(D \cup \{\infty\} = \beta_D$. Given any open set $U \in \mathcal{E}(D \cup \{\infty\}$, and $d \in U$, since $\{e \to d \mid d \leq e \in \mathcal{E}$, we know $\uparrow d \subseteq U$.

By Definition 25, we have $\mathcal{C}_iX = \mathcal{P}_iX$ for $1 \leq i \leq 6$.

Remark 28. Denote $\text{Seq}$ the category of sequential spaces and $\text{TS}$ the category of transfinite sequential spaces [9]. Let $\mathcal{D}'' = \{(D \cup \{\infty\), \delta_D) : D \in \mathcal{D}\}$, $\mathcal{D}_X'' = \mathcal{C}_X$ for any topological space $X$;

$\mathcal{I}'' = \{(I \cup \{\infty\), \delta_I) : I \in \mathcal{J}\}$, $\mathcal{I}_X'' = \{(\{x_i\}_{i \in I}, x) : x \in X, \{x_i\} \to x \}$;

$\mathcal{N}'' = \{(N \cup \{\infty\), \delta_N)\}$, where for any directed poset $D, \delta_D$ is the topology by taking all $\uparrow x \cup \{\infty\}$ and $\{x \} \to D$ as a subbase.

Similarly, we have the conclusion that $\text{Top} = \text{Top}_{\mathcal{D}''}, \text{Top}_D = \text{Top}_{\mathcal{D}''}, \text{Seq} = \text{Top}_{\mathcal{I}''}, \text{Top}_{\mathcal{N}''} = \text{Top}_{\mathcal{N}''}$.

Corollary 29. Let $X$ be a topological space.

1. $X$ is $\mathcal{D}$-generated if and only if $X$ is monotonically determined.
2. $X$ is $\mathcal{D}$-generated and $T_0$ if and only if $X$ is a directed space.

We now show that all these classes $\mathcal{C}_i$ ($1 \leq i \leq 6$) are productive.
Proof. We firstly claim that all these topological spaces are locally compact and hence core compact. Given any \((D \cup \{\infty\}, \beta_D)\) in \(D\), since \(D\) is a directed set, it is easy to check that all \(\mathcal{V}\) and \(\mathcal{V} \cup \{\infty\}\) for \(x \in D\) form a base of \(\beta_D\). Then for any open subset \(U \in \beta_D\) and \(x \in U\); if \(x \in D\), then \(\mathcal{V}\) is a compact neighbourhood of \(x\) contained in \(U\); if \(x = \infty\), there must exist an element \(y \in U\) with \(y \neq x\). Then \(\mathcal{V} \cup \{\infty\}\) is a compact neighbourhood of \(x\) contained in \(U\). Therefore, \((D \cup \{\infty\}, \beta_D)\) is locally compact. The proof for other cases is similar.

Then, we need only to prove that in each class \(C_i\), every topological product of any two generating spaces is \(C_i\)-generated.

1. Let \(X = (D_1 \cup \{\infty\}, \beta_{D_1}) \times (D_2 \cup \{\infty\}, \beta_{D_2})\). By Proposition 27, to show \(X = DX\), we only need to check that \(X \in DX\). Obviously, \((a, b) \rightarrow (c, d)\) for every \((c, d) \in X\), \((d, b) \in D_2 \rightarrow (\infty, b)\) for every \(b \in D_2 \cup \{\infty\}\), and \((a, d) \in D_2 \rightarrow (a, \infty)\) for every \(a \in D_1 \cup \{\infty\}\). Conversely, we claim that these pairs of convergent nets and limits determines \(X\). It suffices to show that the topology determined by these bases, denoted by \(E(X)\), is coarser than \(O(X)\). Given any \(U \in E(X)\), it is an upper set since \((a, b) \rightarrow (c, d)\) for every \((c, d) \subseteq (a, b)\) in \(X\). Let \((a, b) \in U\). There are four cases:

   - Case 1. \(a \in D_1\), \(b \in D_2\): \((a, b) \rightarrow (c, d)\) for every \((c, d) \subseteq (a, b)\) in \(X\).
   - Case 2. \(a \in D_1\), \(b = \infty\): \((a, d) \rightarrow (a, \infty)\) for every \(d \in D_2\)
   - Case 3. \(a = \infty\), \(b \in D_2\): similarly to case 2, there exists some \(c_0\) such that \((\{c_0 \cup \{\infty\}\} \times b) \subseteq U\).
   - Case 4. \(a = \infty\), \(b = \infty\): \((\{\infty, \infty\} d) \rightarrow (\infty, \infty)\) for every \(d \in D_2\).

Thus, \(E(X) \subseteq O(X)\). Since these pairs of convergent nets and limits of \(E\) are contained in \(D_X\), we have \(O(X) \subseteq D(X) \subseteq O(X) \subseteq O(X), i.e., X = DX\).

2. Let \(X = (D_1 \cup \{\infty\}, \beta_{D_1}) \times (D_2 \cup \{\infty\}, \gamma_{D_2})\). Similarly, \(X\) is the topological space determined by the following pairs of nets and limits: \((a, b) \rightarrow (c, d)\) for every \((c, d) \subseteq (a, b)\) in \(X\), \((d, b) \in D_2 \rightarrow (\infty, b)\) for every \(b \in D_2 \cup \{\infty\}\), and \((a, d) \in D_2 \rightarrow (a, \infty)\) for every \(a \in D_1 \cup \{\infty\}\). Thus, \(E(X) \subseteq O(X)\), so \(X = D'X\).

The proofs for \(C_i\) (3 \(\leq i \leq 6\)) are similar.

Then we can get the following statement by combining Theorem 1 and Proposition 30.

Theorem 31. Each category \(Top_{C_i}\) is cartesian closed for \(1 \leq i \leq 6\).

Lemma 32. \([4]\) A space is \(C_i\)-generated iff it is a colimit in \(Top\) of generating spaces iff it is a quotient of disjoint sums of generating spaces.

The inclusion relation of the categories of topological spaces is as follows. We use \(A \subseteq B\) to denote that \(A\) is a full subcategory of \(B\) and \(A \subseteq B\) and \(A \neq B\) to denote \(A \subseteq B\) and \(A \neq B\).

Proposition 33. \((1)\) \(Top_{B} \subseteq Top_{P} \subseteq Top_{D}\), \(Top_{P'} \subseteq Top_{P} \subseteq Top'_{D}\).

\((2)\) \(Top_{G} \subseteq Top_{P} \subseteq Top \subseteq \mathcal{S}\), \(Top_{P} \subseteq \mathcal{T}\), \(Top_{P} \subseteq \mathcal{S}\).

Proof. \((1)\) By Lemma 32, \(N \subseteq \mathcal{I} \subseteq \mathcal{D}\) implies \(Top_{N} \subseteq Top_{P} \subseteq Top_{D}\). We give an example to show the strictness. Let \(D = P_{fin}(R)\), the set of all finite subsets of real numbers \(R\). Considering \(\mathcal{D}(U \cup \{\infty\}, \beta_D)\), we claim that \(\{\infty\}\) is an open subset of it. Given any chain \(J\) such that \(J \subseteq D\), since any element \(x \in J\) is a finite set, it is a subset of a finite set, \(J\) is countable. Thus, there exists some \(x \in D\) such that \(x \notin J\). It follows that \(\bigcap_{x \notin J}(x \cup \{\infty\}) = \emptyset\) and \(J \neq \infty\). Therefore, \(\{\infty\}\) is open in \(\mathcal{D}(U \cup \{\infty\}, \beta_D)\). Since \(\{\infty\}\) is not open in \(\mathcal{D}(U \cup \{\infty\}, \beta_D)\), then \(\mathcal{D}(U \cup \{\infty\}, \beta_D)\) is not in \(Top_{P}\) but in \(Top_{P}\). The proof for \(Top_{P} \subseteq Top_{P'}\) is similar, by replacing \(D = N_1\), where \(N_1\) is the least countable ordinal.

Similarly, \(N' \subseteq \mathcal{I}' \subseteq \mathcal{D}'\) implies \(Top_{P} \subseteq Top_{P'} \subseteq Top_{D}\). \((P_{fin}(R) \cup \{\infty\}, \gamma_{P_{fin}(R)})\) is an example for \(Top_{P} \neq Top_{P'}\) and \((\mathcal{N}, \gamma_{\mathcal{N}})\) is an example for \(Top_{P} \neq Top_{P}\).

\((2)\) \(Top_{G} \subseteq Top_{D} \subseteq \mathcal{T}\) (resp., \(Top_{P} \subseteq \mathcal{T}\), \(Top_{P} \subseteq \mathcal{S}\)) can be gained by combining Lemma 7, Proposition 27, Remark 28 and the fact that for any topological space \(X\), \(\mathcal{D}_X \subseteq \mathcal{N}_X \subseteq \mathcal{B}_X\) (resp., \(\mathcal{D}_X \subseteq \mathcal{I}_X \subseteq \mathcal{I}_X\), \(\mathcal{N}_X \subseteq \mathcal{N}_X \subseteq \mathcal{N}_X\)).

Let \(X = (\mathcal{N} \cup \{\infty\}, \beta_{\mathcal{N}})\). Then \(\{\infty\}\) is open in \(\mathcal{D}\), but not open in \(X\). Then \(Top_{G} \subseteq \mathcal{T}\).

It is also an example for \(Top_{P} \subseteq \mathcal{T}\), and \(Top_{P} \subseteq \mathcal{T}\). Similarly, let \(Y = (\mathcal{N} \cup \{\infty\}, \delta_{\mathcal{N}})\). It is also an example for \(Top_{P} \subseteq \mathcal{T}\), and \(Top_{P} \subseteq \mathcal{T}\).
Example 34. Let \( R \) be the set of all real numbers, \( P_{fin}(R) \) be the set of all finite subsets of \( R \) and \( \mathcal{M} \) be the set of all the maximal chain of \( P_{fin}(R) \). For each \( I \in \mathcal{M} \), we define an element \( \alpha_I \) that is different from \( R \) and all elements in \( P_{fin}(R) \). Let \( S = \{ R \} \cup P_{fin}(R) \cup \{ \alpha_I \}_{I \in \mathcal{M}} \). We define the order \( \leq \) on \( S \) as follows:

1. For \( x, y \in P_{fin}(R) \cup \{ R \} \): the order is the inclusion order of sets, i.e., \( x \leq y \iff x \subseteq y \);
2. For \( x \in P_{fin}(R) \) and \( y = \alpha_I: x \leq y \iff \exists z \in I, x \subseteq z \);
3. For \( x, y \in \{ R \} \cup \{ \alpha_I \}_{I \in \mathcal{M}} \): \( x \leq y \iff x = y \).

It is easy to check that \( (S, \leq) \) is a well defined poset. We show that \( P_{fin}(R) \) is a chain closed subset, but not a Scott closed subset. By definition, for any infinite chain in \( P_{fin}(R) \), like \( \{ 1 \}, \{ 1, 2 \}, \{ 1, 2, 3 \}, \ldots \), there is an upper bound \( \alpha_I \) for some \( I \). So \( R \) and any \( \alpha_I \) are both not a supremum of the chain. Hence, \( P_{fin}(R) \) is chain closed. For any \( \alpha_I \), since \( \alpha_I \) is countable, there exists an element \( y \in P_{fin}(R) \) such that \( y \not\leq \alpha_I \), i.e., \( \alpha_I \) is not an upper bound of \( P_{fin}(R) \). Hence, \( R \) is the supremum of \( P_{fin}(R) \) and then \( P_{fin}(R) \) is not Scott closed. In fact, every \( P_{fin}(R) \cup A \), where \( A \) is a subset of \( \{ \alpha_I \}_{I \in \mathcal{M}} \), is chain closed but not Scott closed.

As mentioned above, we know that for any dcpo \( L \), \( \mathbb{D}(L, v(L)) = \mathbb{D}(L, v(L)) = (L, \sigma(L)) \). It is also easy to check that \( \Gamma(L, v(L)) = \mathbb{D}(L, v(L)) \) and \( \mathbb{D}(L, v(L)) = \mathbb{D}'(L, v(L)) \). However, letting \( X \) be a topology space whose underlying set is a complete lattice relative to its specialization order: (1) \( \mathbb{D}X \) may be different from \( \mathbb{D}X \); (2) \( \Gamma X \) may be different from \( \Gamma X \); (3) \( \mathbb{D}'X \) may be different from \( \mathbb{D}'X \). The space \( (P_{fin}(R) \cup \{ \infty \}, \gamma_{P_{fin}(R)}) \) in the proof for Proposition 33 is the example for (1) and (2). For (3), we give an example as follows.

Example 35. Let \( E = N \cup \{ \infty, a, \bot \} \), where \( N \) is the set of natural numbers and \( \infty, a, \bot \notin N \). The topology \( O(E) \) is by taking \( E, \emptyset, \{ \infty \} \), and all \( \uparrow n \), \( \uparrow n \cup \{ \infty \} \), \( \uparrow n \cup \{ \infty, a \} \) for \( n \in N \) as open subsets, where \( \uparrow n = \{ x \in N : n \leq x \} \). It is easy to check that \( O(E) \) is indeed a topology. We have \( \mathbb{D}E = E \), but \( \{ \infty, a \} \) is open in \( \mathbb{D}E \).

5 Cartesian closed properties in \( T_0 \) spaces

In domain theory, \( T_0 \) topological spaces are the main research objects since the specialization order of a \( T_0 \) topological space is a partial order. However, these spaces generated by classes \( C_i \) (\( 1 \leq i \leq 6 \)) may not be \( T_0 \) topological spaces. When these spaces are restricted in \( T_0 \) topological spaces, they also form cartesian closed categories. There are three approaches to show it:

1. show that Theorem 1 still holds by replacing topological spaces to be \( T_0 \) topological spaces and \( \text{Top}_C \) to be the category of all \( T_0 \) \( C \)-generated topological spaces. In fact, this is trivial;
2. construct the categorical products and exponential objects within the similar process of the proof for the cartesian closed property of \( \text{DTop} \) \([12, 17]\);
3. show that any cartesian closed full subcategory of \( \text{Top} \) is also cartesian closed when restricted in \( \text{Top}_0 \) by a simple proof.

Here, we take approach 3 to get a uniform proof for the cartesian closed properties of these categories \( \text{Top}_P \), restricted in \( T_0 \) spaces. Approach 3 is the most general since it applies to all cartesian closed full subcategories of \( \text{Top} \). The following lemma is obvious.

Lemma 36. Let \( C \) be a cartesian closed full subcategory of \( \text{Top} \). \( \text{Top}_0 \cap C \) is cartesian closed if for any \( T_0 \) space \( X, Y \in C \), \( X \otimes Y \) and \( [X \to Y] \) in \( C \) are both \( T_0 \) spaces.

Theorem 37. \([7]\) Let \( C \) be any full subcategory of \( \text{Top} \) with finite products, and assume that \( 1 = \{ \ast \} \) is an object of \( C \). Let \( X, Y \) be two objects of \( C \) that have an exponential object \([X \to Y] \) in \( C \).

Then there is a unique homeomorphism \( \theta : [X \to Y] \to Y^X \), for some unique topology on \( Y^X \), such that \( cv(h, x) = \theta(h)(x) \) for all \( h \in [X \to Y], x \in X \).

Moreover, \( A_X(f)(z) \) is the image by \( \theta^{-1} \) of \( f(z, \_ ) \) for all \( f : Z \times X \to Y, z \in Z \).

Proposition 38. Let \( C \) be a cartesian closed full subcategory of \( \text{Top} \). If \( X, Y \) are \( T_0 \) spaces in \( C \), then \( X \otimes Y \) and \( [X \to Y] \) are both \( T_0 \).

Proof. \( X \otimes Y \): Let \( Z = \{ \ast \} \). In Diagram (1), for any element \( x \in X \) and \( y \in Y \), we define \( f_x \) and \( f_y \) to be the constant map from \( X \) to \( Y \) with the image \( x, y \) respectively. Then, by the existence and uniqueness of \( (f_x, f_y) \), we know that the underlying set of \( X \otimes Y \) is isomorphic to \( X \times Y \) and \( \pi_1, \pi_2 \) must be projections. By the continuity of \( \pi_1, \pi_2 \), we know that the topology of \( X \otimes Y \) is finer than
The categorical products

Letting \( T \)

Corollary 40.

\( \otimes \)

\( X \)

finer than \([X,Y]_0\)

Diagram (1)

\[ X \rightarrow Y : \] We claim that for any \( X \otimes Y \), if \( \{x_i\} \rightarrow x \) in \( X \), then \( \forall y \in Y, \{ (x_i, y) \} \rightarrow (x, y) \) in \( X \otimes Y \). For any \( y \in Y \), let \( Z = X \) in Diagram (1), \( f_1 \) be the identity map \( id \) and \( f_2 = f_y \). By the continuity of \( \langle id, f_y \rangle \), if \( \{x_i\} \rightarrow x \), then \( \{ \langle id, f_y \rangle(x_i) \} \rightarrow \langle id, f_y \rangle(x) \), i.e., \( \{ (x_i, y) \} \rightarrow (x, y) \).

If \( \{ f_i \} \rightarrow f \) in \( [X \rightarrow Y] \), then \( \forall x \in X, \{ (f_i, x) \} \rightarrow (f, x) \) in \( [X \rightarrow Y] \otimes X \). By the continuity of \( ev \), we get \( \{ f_i(x) \} \rightarrow f(x) \) in \( Y \). Thus, the topology on the exponential object \([X \rightarrow Y]_0\) is finer than the pointwise convergence topology and then it is \( T_0 \).

Combining Lemma 36 and Proposition 38, we have the following statement.

Theorem 39. If \( C \) is a cartesian closed full subcategory of \( Top \), then \( Top_0 \cap C \) is also a cartesian closed category.

Corollary 40. All categories \( Top_p \) (\( 1 \leq i \leq 6 \)) are cartesian closed when restricted to \( T_0 \) spaces.

For \( T_1 \) and \( T_2 \) separation properties, there is a similar conclusion: The category consisting of all \( T_1 \) or \( T_2 \) spaces in \( C \), where \( C \) is a cartesian closed full subcategory of \( Top \), is cartesian closed. Letting \( C = Top_4 \), when restricted in \( T_1 \) spaces, it is in fact just \( Disc \), the category of discrete topological spaces [17]. Next, we give a uniform description for categorical products and exponential objects of these categories \( Top_p \).

Theorem 41. For any idempotent and consistent operation \( P \), we have the following statements.

1. The categorical products \( X \otimes Y \) in \( Top_p \), if exists, must be \( P(X \times Y) \). \( X \otimes Y \) exists if \( \xi, (x, y) \in P_{X \times Y} \Leftrightarrow (\pi_1, x) \in P_X, (\pi_2, y) \in P_Y \).
2. If \( Top_p \) is closed under finite product and satisfies that for any objects \( X, Y, Z \) of \( Top_p \) and any map \( f : X \otimes Y \rightarrow Z \), \( f \) is continuous if and only if it is separately continuous, then the exponential object \([X \rightarrow Y]_p \) in \( Top_p \) must be \( P[X \rightarrow Y]_p \) if it exists.

Proof. (1) Suppose that \( X \otimes Y \) exists. In diagram (1), let \( Z = P(X \times Y) \), \( f_1, f_2 \) be projections \( \pi_1 \) and \( \pi_2 \), respectively. Then by that \( \langle \pi_1, \pi_2 \rangle = id : P(X \times Y) \rightarrow X \otimes Y \) is continuous, we have that \( X \otimes Y \) is coarser than \( P(X \times Y) \), \( X \otimes Y \) is finer than \( X \otimes Y \). When \( \langle \pi_1, \pi_2 \rangle = id : P(X \times Y) \rightarrow X \otimes Y \) is continuous, we have that \( X \otimes Y \) is finer than \( X \otimes Y \). Thus, \( X \otimes Y = P(X \times Y) \). Now, supposing that \( \xi, (x, y) \in P_{X \times Y} \Leftrightarrow (\pi_1, x) \in P_X, (\pi_2, y) \in P_Y \), we show \( P(X \times Y) \) is the categorical product of \( X \) and \( Y \) in \( Top_p \). Given any \( Z \) and \( f_1, f_2 \) in diagram (1), we need only to show that \( \langle f_1, f_2 \rangle \) is continuous. Since \( Z \) is a \( P \)-determined space, we need only to show that for any \( \xi, z \in P_Z, \langle (f_1, f_2) \xi, (f_1, f_2) z \rangle \in P_{X \times Y} \). This is obvious by the continuity of \( f_1, f_2 \) and the supposing condition.

\[ X \rightarrow Y \]

Diagram (2)

(2) In diagram (2), let \( Z = P[X \rightarrow Y]_p \) and \( f \) be the evaluation map \( ev \). It is easy to see that \( f \) is separately continuous. Hence, \( f \) is continuous. By the continuity of \( A_X(f) \times id_X \), we have that \( X \rightarrow Y \) is coarser than \( P[X \rightarrow Y]_p \). From the proof for Proposition 38, we see that \( X \rightarrow Y \) is finer than \( X \rightarrow Y \), and then it is finer than \( P[X \rightarrow Y]_p \). Hence, \( X \rightarrow Y = P[X \rightarrow Y]_p \).}

Corollary 42. For each category \( Top_p \) (\( 1 \leq i \leq 6 \)), the exponential object \([X \rightarrow Y] \) is \( P_i([X \rightarrow Y]_p) \), and the categorical product \( X \otimes Y \) is \( P_i(X \times Y) \) up to isomorphism.

Directed spaces are similar to dcpo in many aspects. It is well known that for any cartesian closed full subcategory \( C \) of \( Dcpo \), the categorical product and exponential objects in \( C \) agree with those in \( Dcpo \) [10]. Naturally there arises a question: for a cartesian closed full subcategory \( C \) of \( DTop \), do the categorical products and exponential objects in \( C \) coincide with those in \( DTop \)? For products, the answer is positive. However, the exponential objects may differ.
Proof. Suppose that \( C \) is a cartesian closed full subcategory of \( \text{Top}_0 \) (resp. \( \text{DTop} \)), then the finite products in \( C \) coincide with those in \( \text{Top}_0 \) (resp. \( \text{DTop} \)).

**Definition 47.** Let \( A \) and \( B \) be two topologies on \( X \times Y \). The categorical product \( A \times B \) is defined as follows:

\[
\langle a, b \rangle = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle = \langle a_1, b_2 \rangle = \langle a_2, b_1 \rangle
\]

Example 44. Let \( \text{Poset}_A \) be the category of posets endowed with the Alexandroff topology. Then it is cartesian closed and the exponential object of \( X, Y \), denoted by \( [X \rightarrow Y]_A \), is as follows: the underlying set is \( \{ f : f \) is a monotone map from \( X \) to \( Y \} \), endowed with the pointwise order; the topology is the Alexandroff topology. But \( [X \rightarrow Y]_A \) is not always equal to the exponential object \( [X \rightarrow Y]_d \) in \( \text{DTop} \). Let \( N \) be the poset of natural numbers. For any \( n \in N \), we define \( f_n(x) = n, \forall x \in N \). Then for any \( x \in N \), \( \{ f_n(x) \}_{n \in N} \rightarrow \text{id}(x) \). Therefore \( \{ f_n \}_{n \in N} \rightarrow \text{id} \) in \( [N \rightarrow N]_A \). But for any \( n, d \), \( n \neq d \), and then \( \{ f_n \}_{n \in N} \neq \text{id} \) in \( [N \rightarrow N]_d \). Hence \( [N \rightarrow N]_A \) is different from \( [N \rightarrow N]_d \).

Core compactness is closely related to exponential objects in topological spaces. By the tool of \( C \)-generated space, we give a characterization of core compactness of a directed space.

**Theorem 45.** [4] Let \( C \) be a productive class. If \( X \) and \( Y \) are \( C \)-generated spaces with \( Y \) core compact, then the topological product \( X \times Y \) is \( C \)-generated, i.e., \( X \times Y = C(X \times Y) \).

**Corollary 46.** Let \( X, Y \) be two directed spaces and \( Y \) be core compact. Then \( X \times Y = X \otimes_d Y \).

6 Order compatible topologies in directed spaces

In this section, we discuss the conditions for convergence classes to determine compatible directed topologies and investigate the relationships between directed spaces and some classical structures in domain theory such as posets endowed with the Scott topology, the weak Scott topology and the Alexandroff topology.

**Definition 47.** [6] Let \( (P, \leq) \) be a poset. A topology \( \tau \) on \( P \) is called order compatible if the specialization order relative to \( \tau \) agrees with \( \leq \), the original order on \( P \). Two \( T_0 \) topologies \( \tau_1, \tau_2 \) on a set \( X \) are called order compatible if they have the same the specialization order.

**Lemma 48.** Let \( P \) be an idempotent and consistent operation. Then \( P \) is monotone, i.e., \( \tau_1 \subseteq \tau_2 \) implies that \( P(X, \tau_1) \) is coarser than \( P(X, \tau_2) \) for any two topologies \( \tau_1, \tau_2 \) on a set \( X \).

Proof. Let \( P \) be an idempotent and consistent operation and \( \tau_1, \tau_2 \) be two topologies on \( X \) with \( \tau_1 \subseteq \tau_2 \). Since the identity map \( \text{id} : (X, \tau_2) \rightarrow (X, \tau_1) \) is continuous, then \( P(X, \tau_2) \subseteq P(X, \tau_1) \) by that \( P \) is consistent. Then \( P(X, \tau_1) \) and \( P(X, \tau_2) \) are \( P \)-determined spaces and \( P(X, \tau_1) \) is coarser than \( P(X, \tau_2) \) by Lemma 7.
The relationship between the exponential object \([\mathcal{D}(P,\tau)]\) of any poset \(P\), \(x \in \mathcal{D}^D\). \((P,\mathcal{A}(P))\) is determined by \(\mathcal{A} = \{(D,x) : x \in \downarrow D, D \subseteq P, D \text{ is directed}\}\). They are separately the coarsest and finest order compatible directed spaces with \(P\). Any convergence class \(U\) with \(A \subseteq U \subseteq V\) determines an order compatible directed space with \(P\). Conversely, for any order compatible directed space \((P,\tau)\) with \(P\), we have \(A \subseteq \mathcal{D}(P,\tau) \subseteq V\).

**Proof.** Given any order compatible directed topology \(\tau\) on \(P\), \((v(\mathcal{P})) \subseteq \tau \subseteq A(P)\). Then By Lemma 48, \(d(\mathcal{P}(P)) \subseteq d(\tau) = \tau \subseteq A(P) = d(A(P))\). It is easy to check that \(\mathcal{D}(P,v(\mathcal{P})) = V\) and \(\mathcal{D}(P,A(P)) = A\). Thus, \(d(\mathcal{P}(P))\) is determined by \(V\), \(A(P)\) is determined by \(A\) and they are separately the coarsest and finest order compatible directed topology on \(P\). Let \(U\) be a convergence class such that \(A \subseteq U \subseteq V\). We have \(U \subseteq \mathcal{D}(UP)\) and then \(D(UP) = UP\). Therefore, \(U\) determines an order compatible directed space with \(P\). The converse is trivial.

\(d(\mathcal{P}(P))\) is called the weak Scott topology \(\sigma_2(P)\) by Erné in [3]. Similarly, we have the conclusion for general idempotent and consistent operations. The proof is similar to that for Proposition 49, so we omit it.

**Proposition 50.** Let \(P\) be a poset. For any idempotent and consistent operation \(Q\), the coarsest and finest order compatible \(Q\)-determined spaces with \(P\) are \((P,\mathcal{Q}(P,v(\mathcal{P})))\) and \((P,\mathcal{Q}(P,A(P)))\) respectively.

**Corollary 51.** For any poset \(P\), \((P,\sigma(P))\) is determined by \(\Sigma = \{(D,y) : D \text{ is directed in } P, x \in \downarrow D \text{ or } x = \sup \mathcal{D}(P)_1\}\). \((P,\sigma(P))\) and \((P,\mathcal{A}(P))\) are the coarsest and finest order compatible directed spaces with \(P\) in \(\text{Top}_{\mathcal{D}}\) respectively.

We call a \(T_0\) \(\mathcal{D}\)-determined space a strictly directed space. Denote \(\text{SDDTop}\) the category of all strictly directed spaces. We mentioned three kind of topologies above, the Scott topology, the Alexandroff topology and the weak Scott topology. When do these topologies agree? It is obvious that for two directed spaces \(X_1 = (X, \mathcal{O}_1(X))\) and \(X_2 = (X, \mathcal{O}_2(X))\), \(\mathcal{O}_1(X) = \mathcal{O}_2(X)\) if \(\mathcal{D}_1 = \mathcal{D}_2\) if for any directed subset \(D, \mathcal{cl}_{X_1}(D) = \mathcal{cl}_{X_2}(D)\). Next, we have the following statement.

**Proposition 52.** Given any poset \(P\), \(\sigma(P) = A(P)\) iff \(\sup D \in D\) for all directed subsets \(D\) of \(P\) with \(\sup D\) existing. \(\sigma(P) = \sigma_2(P)\) iff for every directed subset \(D\), \(\mathcal{cl}_D(D) = D^\delta\).

**Proof.** Assume that \(\sigma(P) = A(P)\) and \(D\) is a directed subset of \(P\) such that \(\sup D = c \notin D\). Then \(\uparrow c\) is open in \(A(P)\) but not open in \(\sigma(P)\), a contradiction. Conversely, suppose that for all directed subsets \(D\) of \(P\) with \(\sup D\) existing, \(\sup D \in D\). Given any element \(d \in P\), \(\uparrow d\) is open in \(\sigma(P)\) and \(\mathcal{cl}_D(D) = D^\delta\). Next, we need only to show that \(\mathcal{cl}_{\sigma_2}(D) = D^\delta\) for any directed subset \(D\). By definition, letting \(x \in D^\delta\), if \(D \not\rightarrow x\), then there exists a finite subset \(F\) of \(P\), \(D \subseteq \downarrow F, x \in P \setminus F\). Since \(F\) is finite and \(D\) is directed, we have \(3a \in F, D \subseteq \downarrow a\). Since \(x \notin \downarrow a\), it contradicts to \(x \in D^\delta\). Thus, \(D^\delta \subseteq \mathcal{cl}_{\sigma_2}(D)\). For converse, assume that \(D \rightarrow x\). Given any element \(y\) such that \(D \subseteq \downarrow y\), \(\downarrow y\) is closed, and then \(x \in \downarrow y\), i.e., \(x \leq y\). Therefore, \(D^\delta = \mathcal{cl}_{\sigma_2}(D)\).

Given any dcpo \(L\), we have \(\mathcal{S}(L) = \sigma_2(L)\), since \(\mathcal{cl}_D(D) = \downarrow \sup D = D^\delta\) for any directed subset \(D\). We denote \(\text{Poset}_{\sigma_2}\) the category of posets endowed with the weak Scott topology, together with continuous maps as morphisms. Is \(\text{Poset}_{\sigma_2}\) cartesian closed? Given \(X, Y \in \text{Ob}(\text{Poset}_{\sigma_2})\), what is the relationship between the exponential object \([X \rightarrow Y]_{\sigma_2}\) in \(\text{Poset}_{\sigma_2}\) and the one \([X \rightarrow Y]_d\) in \(\text{DTop}\)? Before answering these questions, we investigate some basic properties of \(\text{Topo}\).

**Lemma 53.** For any cartesian closed full subcategory \(\mathcal{C}\) of \(\text{Topo}\), the underlying sets of \(X \otimes Y\) and \([X \rightarrow Y]\) are equal to \(X \times Y\) and \(Y^X\) respectively. Moreover, the specialization order of \(X \otimes Y\) agrees with the one of the topological product, and the specialization order of \([X \rightarrow Y]\) agrees with the pointwise order.

**Proof.** By Theorem 37 and the proof of Proposition 38, the underlying sets of \(X \otimes Y\) and \([X \rightarrow Y]\) are \(X \times Y\) and \(Y^X\) respectively. Since \(\otimes\) is finer than the topological product \(X \times Y\), \((x_1, y_1) \subseteq (x_2, y_2)\) in \(X \otimes Y\) implies \((x_1, y_1) \leq (x_2, y_2)\). For the converse, assume that \(z_1 < z_2\) in \(Z\) and \((x_1, y_1) \leq (x_2, y_2)\) in \(X \times Y\). Define \(f_1, f_2\) as follows.

\[
\begin{align*}
  f_1(z) &= \begin{cases} x_1, & z \leq z_1 \\ x_2, & \text{otherwise} \end{cases} \\
  f_2(z) &= \begin{cases} y_1, & z \leq z_1 \\ y_2, & \text{otherwise} \end{cases}
\end{align*}
\]

Then \((f_1, f_2)\) maps \(z_1\) onto \((x_1, y_1)\) and \(z_2\) onto \((x_2, y_2)\). By the continuity of this map, \((x_1, y_1) \subseteq (x_2, y_2)\) in \(X \otimes Y\). If such \(Z\) does not exist, then all objects in \(\mathcal{C}\) are \(T_1\), it is trivial.

For \([X \rightarrow Y]\), assume that \(f_1 \leq f_2\) in \([X \rightarrow Y]_d\) and \(z_1 < z_2\) in \(Z\). Define

\[
f(z, x) = \begin{cases} f_1(x), & z \leq z_1 \\ f_2(x), & \text{otherwise} \end{cases}
\]
Given any open subset $U$ of $Y$, $f^{-1}(U) = \{(z, x) : z \in \downarrow z_1, f_1(x) \in U\} \cup \{(z, x) : z \in Z \uparrow z_2, f_2(x) \in U\}$ and it is open in $Z \otimes X$. Then $f : Z \otimes X \to Y$ is continuous. By the continuity of $\lambda X, f_1 = \lambda X(z_1) \subseteq \lambda X(z_2) = f_2$ in $[X \to Y]$. Conversely, by the fact that $[X \to Y]$ is finer than $[X \to Y]_p$, we have that $f_1 \subseteq f_2$ implies $f_1 \leq f_2$. □

From Proposition 43, we know that for any cartesian closed full subcategory $C$ of $D\text{Top}$, the categorical products in $C$ agree with those in $D\text{Top}$. By Example 44, the exponential objects in $C$ and $D\text{Top}$ may differ. However, for any cartesian closed full subcategory of $\text{Poset}_{\tau_2}$, we show that the exponential objects coincide with those in $D\text{Top}$ as well. Moreover, we give a necessary and sufficient condition for which a full subcategory of $\text{Poset}_{\tau_2}$ is cartesian closed.

**Theorem 54.** Let $C$ be a full subcategory of $\text{Poset}_{\tau_2}$. Then the following statements are equivalent.

1. $C$ is cartesian closed.
2. The categorical products and exponential objects in $C$ exist and coincide with those in $D\text{Top}$.
3. Let $X, Y$ be any two objects of $C$.
   (a) The upper bound of any directed subset $D$ of $X$ exists.  
   (b) For any directed subset $\{f_i\}$ of $Y^X$, $g \in \{f_i\}^\delta$ implies $\forall x \in X$, $\{f_i(x)\} \to g(x)$ in $Y$.
   (c) $(X \times Y, \sigma_1(X \times Y))$ and $(Y^X, \sigma_2(Y^X))$, where $Y^X$ endowed with the pointwise order, are objects of $C$.

**Proof.** Supposing that $X, Y$ are objects in $C$, by Lemma 53, we have: if the cartesian product $X \otimes Y$ in $C$ exists, then $X \otimes Y = (X \times Y, \sigma_2(X \times Y))$ up to homeomorphism; if the exponential object $[X \to Y]$ exists, then $[X \to Y] = (Y^X, \sigma_2(Y^X))$ up to homeomorphism. And by Proposition 43, $X \otimes Y = X \otimes_Y Y$.

$(1) \Rightarrow (2)$: Suppose that $C$ is a cartesian closed full subcategory of $\text{Poset}_{\tau_2}$. Then the exponential object $[X \to Y]$ in $C$ must be $(Y^X, \sigma_2(Y^X))$ up to homeomorphism. We need only to show that $(Y^X, \sigma_2(Y^X))$ is in $[X \to Y]_d$. By the continuity of the evaluation map $ev$, we know $d([X \to Y]_p) \subseteq \sigma_2(Y^X)$. Since $\sigma_2(Y^X)$ is the coarsest directed topology on $Y^X$, then $\sigma_2(Y^X) \subseteq d([X \to Y]_p)$. Thus $(Y^X, \sigma_2(Y^X)) = [X \to Y]_d$.

$(2) \Rightarrow (3)$: Let $C$ be a cartesian closed full subcategory of $\text{Poset}_{\tau_2}$. (c) is obvious.

(a) Suppose that there exists a $X$ in $C$ and a directed subset $D$ of $X$ such that the upper bound of $D$ does not exist. We compare $\sigma_2(X \times X)$ and $D(X \times X)$. Let $a \in X$, then $X \neq \downarrow a$, considering $A = \{(x, a) : x \in D\}$. In $X \times X$, an element $(x, y)$ is an upper bound of a subset $A$ iff $x$ is an upper bound of $\pi_1 D$ and $y$ is an upper bound of $\pi_2 D$. Since the upper bound of $D$ does not exist, then $D^\delta = X$ in $X$ and $A^\delta = X \times X$ in $X \times X$. Thus, for any element $z \notin A$, $A \neq (z, z)$ in $\sigma_2(X \times X)$ but $A \neq (z, z)$ in $D(X \times X)$ by definition, i.e., $\sigma_2(X \times X) \neq D(X \times X)$, a contradiction to (2).

(b) Since $d([X \to Y]_p) = \sigma_2(Y^X)$ and they are directed topologies, they must contain the same pairs of convergent directed subsets and limits. Let $\{f_i\}$ be a directed subset of $Y^X$ and $g \in Y^X$. By definition, $\{f_i\} \to g$ in $d([X \to Y]_p)$ iff $\forall x \in X, f_i(x) \to g(x)$ in $X$; $\{f_i\} \to g$ in $\sigma_2(Y^X)$ iff $g \in \{f_i\}^\delta$. Thus, $g \in \{f_i\}^\delta$ if $\forall x \in X, f_i(x) \to g(x)$ in $Y$.

$(3) \Rightarrow (1)$: We need only to show that $\sigma_2(X \times Y) = d(X \otimes Y)$ and $\sigma_2(Y^X) = d([X \to Y]_p)$ for any two objects $X, Y$ of $C$. Given any directed subset $\{(x_i, y_i)\}$ in $X \times Y$, since the upper bound of every directed subset exists in $X$ and $Y$ respectively, we have $(x, y) \in \{x_i, y_i\}^\delta$ if $x \in \{x_i\}^\delta$ and $y \in \{y_i\}^\delta$. Thus, $\{(x, y_i)\} \to (x, y)$ in $\sigma_2(X \times Y)$ iff $\{(x_i, y)\} \to (x, y)$ in $d(X \otimes Y)$, that is, $\sigma_2(X \times Y) = d(X \otimes Y)$. Naturally, $\sigma_2(Y^X) \subseteq d([X \to Y]_p)$. To show $d([X \to Y]_p) = \sigma_2(Y^X)$, we need only to show that $d([X \to Y]_p) \subseteq \sigma_2(Y^X)$. It is equivalent to that for any directed subset $\{f_i\}$ and any $g \in \{f_i\}^\delta$, $\{f_i(x)\} \to g(x), \forall x \in X$. □

**Corollary 55.** $\text{Poset}_{\tau_2}$ is not cartesian closed.

**Proof.** Let $N$ be the poset of natural numbers. Then the upper bound of $N$ is empty. By Theorem 54 (2), $\text{Poset}_{\tau_2}$ is not cartesian closed. □

$\text{Poset}_A$ is cartesian closed. By Proposition 43, we know that the categorical product of $A, B$ in $\text{Poset}_A$ is same as the one in $D\text{Top}$. Moreover, the categorical product is equal to the topological product by Corollary 46, since any object in $\text{Poset}_A$ is core compact. However, the exponential objects may differ by Example 44. Generally, the topology of $[A \to B]_A$ is finer than the topology of $[A \to B]_d$.

Denote $\text{Poset}_r$ the category of posets endowed with the Scott topology, together with continuous maps as morphisms. It is equivalent to $\text{Poset}$, the category of all posets with Scott continuous maps as morphisms. $\text{Poset}$ is not Cartesian Closed [5]. For convenience, given any poset $P$, we also view it as $\Sigma P$, the objects in $D\text{Top}$. We denote $[P \to Q]_A$ the exponential object, if exists, for $\Sigma P, \Sigma Q$.
in $\mathbf{Poset}_\sigma$ and $[P \to Q]_d$ the exponential object for $\Sigma P, \Sigma Q$ in $\mathbf{DTop}$. A subcategory $\mathbf{C}$ of $\mathbf{Poset}$ is cartesian closed iff it is closed under products and function spaces, and the all existing directed sups in the function spaces are pointwise [18]. We have the following statement.

**Proposition 56.** Let $P, Q$ be two posets. The following two conditions are equivalent.

1. $d([P \to Q]_d) \subseteq \sigma(Q^P)$
2. All existing directed sups in $[P \to Q]_\sigma$ are pointwise.

Moreover, if $Q$ is a dcpo, then $d([P \to Q]_d) = \sigma(Q^P)$, that is, $[P \to Q]_d = [P \to Q]_\sigma$.

**Proof.** Recall that the carrier sets of $[P \to Q]_d$ and $[P \to Q]_\sigma$ are equal to the set of all Scott continuous functions from $P$ into $Q$, and the specialization order on $[P \to Q]_d$ is the pointwise order. So, regarded as posets, $[P \to Q]_d$ and $[P \to Q]_\sigma$ coincide.

(1) $\Rightarrow$ (2): Suppose that $S \subseteq Q^P$ is directed and $\sqcup S$ exists. Set $f = \sqcup S$. Then $S \to f$ relative to the Scott topology $\sigma(Q^P)$. Since $d([P \to Q]_d) \subseteq \sigma(Q^P)$, we have $S \to f$ relative to the directed topology $d([P \to Q]_d)$. Hence, for any $x \in P$, we have $\{s(x) : s \in S\} \to f(x)$ by Theorem 3. Obviously, $f(x)$ is an upper bound of $\{s(x) : s \in S\}$ for $f = \sqcup S$. Thus $f(x) = \sqcup_{s \in S} s(x)$ by Lemma 24. It follows that $\sqcup_{s \in S} s(x)$ exists and $f(x) = \sqcup_{s \in S} s(x)$ for any $x \in P$.

(2) $\Rightarrow$ (1): Suppose that $U \in d([P \to Q]_d)$ and $S \subseteq Q^P$ is directed with a existing sup $\sqcup S$. Since $\sqcup S$ is pointwise, $\sqcup_{s \in S} s(x)$ exists in $Q$ and $(\sqcup S)(x) = \sqcup_{s \in S} s(x)$ for any $x \in P$. Thus, for any $x \in P$, $(s(x))_{s \in S} \to (\sqcup S)(x)$ relative to the Scott topology $\sigma(Q)$. It follows that $S$ converges to $\sqcup S$ in $d([P \to Q]_d)$ by Theorem 3. Since $U$ is directed-open, $\sqcup S \subseteq U$ imples $U \cap S \neq \emptyset$, i.e., $U$ is Scott open. Hence, $d([P \to Q]_d) \subseteq \sigma(Q^P)$.

Suppose that $Q$ is a dcpo. Then $Q^P$ is a dcpo and all directed sups are pointwise. Hence, $d([P \to Q]_d) \subseteq \sigma(Q^P)$ by the above proof and $[P \to Q]_\sigma = (Q^P, \sigma(Q^P))$ exists. Note that since $Q^P$ is a dcpo, then $\sigma(Q^P)$ is the coarsest order compatible directed topology on $Q^P$. Therefore, $[P \to Q]_d = [P \to Q]_\sigma$ holds.

From Proposition 56, we can see that for $\mathbf{Dcpo}_\sigma$, the categorical exponential objects agree with those in $\mathbf{DTop}$. Let $\mathcal{F}$ be the functor from $\mathbf{Dcpo}$ to $\mathbf{DTop}$ such that $\mathcal{F}(P) = \Sigma P, \mathcal{F}(f)(x) = f(x)$ for $P \in \text{Ob}(\mathbf{Dcpo})$ and $f \in \text{Mor}(\mathbf{Dcpo})$. Since given any cartesian closed full subcategory $\mathbf{C}$ of $\mathbf{Dcpo}$, the categorical products and exponential objects of $\mathbf{C}$ coincide with those in $\mathbf{Dcpo}$ [1, 10], we have the following statement.

**Proposition 57.** The embedding functor $\mathcal{F} : \mathbf{Dcpo} \rightarrow \mathbf{DTop}$ preserves finite products and exponential objects. Moreover, for any cartesian closed full subcategory $\mathbf{C}$ of $\mathbf{Dcpo}$, $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{DTop}$ preserves finite products and exponential objects.

Generally, given a cartesian closed full subcategory $\mathbf{C}$ of $\mathbf{Poset}_\sigma$, the exponential object in $\mathbf{C}$ is finer than the one in $\mathbf{DTop}$. We have the following question and conjecture.

**Problem 58.** Let $\mathbf{C}$ be a cartesian closed full subcategory of $\mathbf{Poset}_\sigma$. Given any two objects $X, Y$ of $\mathbf{C}$, must the exponential object $[X \to Y]$ in $\mathbf{C}$ be equal to the one $[X \to Y]_d$ in $\mathbf{DTop}$?

**Conjecture 59.** $\mathbf{DTop}$ is the largest cartesian closed full subcategory of $\mathbf{Top}_0$ satisfying the following two conditions:

1. $\mathbf{Dcpo}_\sigma$ is a full subcategory of $\mathbf{C}$.
2. The categorical products and exponential objects in $\mathbf{Dcpo}_\sigma$ agree with those in $\mathbf{C}$.

We present the inclusion relation of these categories discussed in this paper.

7 topological convergence classes in directed spaces

For a dcpo, given any net $(x_i)_{i \in I}$, we say that $y$ is a eventual lower bound of $(x_i)_{i \in I}$ if there exists $k \in I$ such that $\forall i \geq k, y \leq x_i$. The convergence classes $\mathcal{S}$ is defined by $((x_i)_{i \in I}, x) \in \mathcal{S}$ iff there exists a directed subset $D$ of eventual lower bound of $(x_i)_{i \in I}$ such that $\sup D = x$. The following result is well known.

**Theorem 60.** [6] The convergence class $\mathcal{S}$ on a dcpo $L$ is topological iff $L$ is a domain.
It can be extended to general topological spaces very smoothly. A topological space $X$ is called a continuous space if for any $x \in X$, there exists a directed subset $D$ such that $D \to x$, sup $D = x$ and $\forall d \in D. d \ll_n x$, where $d \ll_n x$ means that for any net $(x_i)_{i \in I} \to x$, there exists a $i_0 \ I$ such that $d \leq x_{i_0}$. The notion of a continuous space is equivalent to the notion of a c-space [14].

**Definition 61.** Let $X$ be a $T_0$ topological space. Denote $EL((x_i)_{i \in I})$ the eventual lower bound of $(x_i)_{i \in I}$.

We define a convergence class $S_X$ on $X$ as follows: $(x_i)_{i \in I}, x \in S_X$ iff there exists a directed subset $D$ of eventual lower bound of $(x_i)_{i \in I}$ such that $D \to x$ with respect to $O(X)$.

$S_X$ is called topological iff $S_X$ contains all pairs of convergent nets and limits in $S(X)$, where $S(X)$ is the topology on $X$ determined by $S_X$, i.e., $U \in S(X)$ iff for all $(x_i)_{i \in I} \in S_X, x \in U$, we have $(x_i)_{i \in I}$ is eventually in $U$.

**Lemma 62.** Let $X$ be a $T_0$ space. Then $U \in S(X)$ iff $U \in D(X)$.

**Proof.** Suppose that $U \in S(X)$ and $D$ be a directed subset of $X$ with $D \to x \in U$ relative to $O(X)$. Then $(D, x) \in S$. Thus, $D$ is eventually in $U$, $D \cap U \neq \emptyset$. $U$ is in $D(X)$ (or simply: $D_X \subseteq S_X \Rightarrow S(X) \subseteq D(X)$).

Suppose that $U \in D(X)$ and $(x_i)_{i \in I} \in S_X$ with $x \in U$. We need only to show that $(x_i)_{i \in I}$ is eventually in $U$. By definition of $S_X$, there exists a directed subset $D \subseteq EL((x_i)_{i \in I})$ such that $D \to x$ relative to $O(X)$. Then $D \to x$ relative to $D(X)$ and there exists a $d \in D$ such that $d \in U$. Therefore, $(x_i)_{i \in I}$ is eventually in $U$.

**Proposition 63.** Let $L$ be a continuous space. Then $(x_i)_{i \in I, x} \in S$ iff $(x_i)_{i \in I} \to x$ with respect to $O(X)$.

**Proof.** Since a continuous space is a directed space, if $(x_i)_{i \in I, x} \in S$, then $(x_i)_{i \in I} \to x$ with respect to $D(X) = O(X)$. Conversely, suppose that $(x_i)_{i \in I} \to x$ relative to $O(X)$. For each $y \in \downarrow x$, we have $\downarrow y$ is an open set containing $x$. Thus, $(x_i)_{i \in I}$ is eventually in $\downarrow y$ and $y \in EL((x_i)_{i \in I})$. Since $\downarrow x$ is directed and converges to $x$, we have $(x_i)_{i \in I, x} \in S_X$.

**Lemma 64.** Let $X$ be a directed space. If convergence class $S_X$ is topological, then $X$ is a continuous space.

**Proof.** By Lemma 62, $S(X) = D(X)$. If $S_X$ is topological, then $(x_i)_{i \in I, x} \in S$ iff $(x_i)_{i \in I} \to x$ with respect to $D(X) = O(X)$. Let $x \in X$, define

$$I = \{(U, n, a) \in N(X) \times N \times L : a \in U\},$$

where $N(x)$ consists of all open sets containing $x$. Define an order on $I$ to be the lexicographic order on the first two coordinates, that is, $(U, m, a) < (V, n, b)$ iff $V$ is a proper subset of $U$ or $U = V$ and $m < n$. For each $i = (U, n, a)$, let $x_i = a$. Then $(x_i)_{i \in I}$ converges to $x$ relative to $O(X)$. Thus, $(x_i)_{i \in I, x} \in S$. It follows that there exists a directed set $D \subseteq EL((x_i)_{i \in I})$ such that $D \to x$ relative to $O(X)$. Given any $d \in D$, there exists a $k = (U, m, a) \in I$ such that $(V, n, b) \geq k$ implies $d \leq b$. Since $\forall b \in U, (U, m + 1, b) \geq (U, m, a)$, we have $d \leq U$. Thus, $x \in (\downarrow d)$. Then $D$ is a directed subset with $\forall d \in D, d \ll_n x$ and $D \to x$. Thus, $X$ is a continuous space.

Combining Proposition 63 and Lemma 64, we have the following statement.

**Theorem 65.** Let $X$ be a directed space. The convergence class $S_X$ is topological iff $X$ is a continuous space.

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