Symmetry and degeneracy of the curved Coulomb potential on the $S^3$ ball

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Abstract

The ‘curved’ Coulomb potential on the $S^3$ ball, whose isometry group is SO(4), takes the form of a cotangent function, and when added to the four-dimensional squared angular momentum operator, one of the so(4) Casimir invariants, a Hamiltonian is obtained which describes a perturbation of the free geodesic motion that results in several peculiar aspects. The spectrum of such a motion has been studied on various occasions and is known to unexpectedly carry so(4) degeneracy patterns despite the non-commutativity of the perturbation with the Casimir operator. We suggest here an explanation for this behavior in designing a set of operators which close the so(4) algebra and whose Casimir invariant coincides with the Hamiltonian of the perturbed motion at the level of the eigenvalue problem. The above operators are related to the canonical geometric SO(4) generators on $S^3$ by a non-unitary similarity transformation of the scaling type. In this fashion, we identify a complementary option to the deformed dynamical so(4) Higgs algebra constructed in terms of the components of the ordinary angular momentum and a related Runge–Lenz vector.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The three-dimensional hyper-spherical surface, $S^3$, embedded in a four-dimensional Euclidean space, $E_4$,

$$S^3 : x_4^2 + x_1^2 + x_2^2 + x_3^2 = R^2,$$

is among the most important geometries in a variety of theoretical physics problems. Firstly, this curved surface, whose isometry group is SO(4), represents the position space in Einstein’s celebrated closed Universe and provides one of the major templates in gravitational studies. At the same time, it forms part of the compactified Minkowski spacetime as it emerges upon...
conformal compactifications in supersymmetric field theories. Further important aspects of the \( S^3 \) ball concern its relevance in the effective description of many-body phenomena such as coherent states, fluid-dynamics [1–3], polymer chains [4], Brownian motion [5] and quantum dots [6], all reasons which make studies of physics on \( S^3 \) relevant.

The free geodesic motion of a scalar particle on \( S^3 \) is described by means of the eigenvalue problem of the squared four-dimensional angular momentum, \( K \), one of the Casimir invariants of the \( \text{so}(4) \) isometry algebra. We focus here on a perturbation of this motion by a cot \( \chi \) function of the second polar angle, \( \chi \), parametrizing \( S^3 \), which is interesting in so far as it conserves the \( \text{SO}(4) \) degeneracy patterns in the spectrum of the free geodesic motion, despite its non-commutativity with \( K \). This property of the cotangent function is quite remarkable indeed and valid not only on \( S^3 \) but also on \( S^2 \), and in any higher dimensions. To be specific, also on \( S^2 \) does the cotangent potential not remove the \((2\ell + 1)\)-fold degeneracy in the spectrum of the free spherical rigid rotator, despite its non-commutativity with \( L^2 \) [7, 8].

To explain the above degeneracy phenomena, Higgs and Leemon designed in [7, 9] algebras composed by the components of the angular momentum operators, \( L_i \), in the respective external spaces under consideration, and a related Runge–Lenz vector, \( R_j \), designed in analogy to the flat-space one, as known from the H atom problem. This strategy has been successful in explaining the degeneracy phenomena under discussion, despite that \( L_j \) and \( R_i \) cease to close the respective \( \text{so}(3)/\text{so}(4) \) algebras. Instead, they close algebras which appear as deformations of the respective isometry algebras by terms cubic in the momenta (see also [10] for further details).

Here, we instead construct a closed \( \text{so}(4) \) algebra for the cotangent perturbed motion on \( S^3 \), which we obtain as a non-unitary similarity transformation of the geometric \( \text{so}(4) \) algebra of the free geodesic motion. This becomes possible at the level of the \((K - 2b\cot\chi)\) and \( K \) eigenvalue problems. We build up a matrix similarity transformation which connects the \( K \) and \((K - 2b\cot\chi)\) carrier spaces, and which happens to be of the dilation (scaling) type. The possibility for constructing such a transformation on \( S^2 \) has been indicated in work prior to this [8] and will not be considered here.

We focus entirely on the \( \text{so}(4) \) case which is special by the fact that on \( S^3 \), the cotangent function solves the homogeneous Laplace–Beltrami equation, and is a so-called harmonic function there. As such, it can be treated along the lines of potential theory and the resulting electrodynamics would be Maxwellian, something which does not occur in any other dimension.

The paper is structured as follows. In the next section, we present the solutions of the cotangent-perturbed geodesic motion on \( S^3 \) in terms of real Romanovski polynomials (reviewed in [11]) and decompose these solutions into the basis of exponentially damped hyper-spherical harmonics. From there we deduce the non-unitary transformation which takes the \( \text{so}(4) \) isometry algebra to a realization that now describes the cotangent-hindered motion. Working at the level of the eigenvalue problems brings as an advantage simplifications by virtue of certain type of recurrence relations among Gegenbauer polynomials. The paper closes with concise conclusions.

2. Particle motion on the \( S^3 \) ball

2.1. The free geodesic motion

In polar coordinates, the three-sphere \( S^3 \) is parametrized by the azimuthal angle \( \varphi \), and the two polar angles, \( \theta \) and \( \chi \), according to

\[ x = \cos(\varphi) \cos(\theta) \cos(\chi), \]
\[ y = \cos(\varphi) \cos(\theta) \sin(\chi), \]
\[ z = \cos(\varphi) \sin(\theta), \]
\[ w = \sin(\varphi), \]

where \( \varphi, \theta, \chi \) are the spherical coordinates on \( S^3 \).

The free geodesic motion is described by the Hamiltonian

\[ H = \frac{1}{2} \left( \frac{\partial^2}{\partial \varphi^2} + \cos^2(\theta) \frac{\partial^2}{\partial \chi^2} + \sin^2(\varphi) \frac{\partial^2}{\partial \theta^2} + \sin(\varphi) \cos(\varphi) \frac{\partial^2}{\partial \varphi \partial \theta} \right). \]

The eigenvalue problem for the square of the angular momentum is

\[ K^2 = \frac{1}{2} \left( \frac{\partial^2}{\partial \varphi^2} + \cos^2(\theta) \frac{\partial^2}{\partial \chi^2} + \sin^2(\varphi) \frac{\partial^2}{\partial \theta^2} + \sin(\varphi) \cos(\varphi) \frac{\partial^2}{\partial \varphi \partial \theta} \right). \]

The eigenfunctions are the Gegenbauer polynomials, which are orthogonal with respect to the inner product

\[ \langle \mathbf{g}, \mathbf{h} \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^{\pi/2} \mathbf{g}(\theta, \varphi, \chi) \mathbf{h}(\theta, \varphi, \chi) \sin(\theta) \, d\theta \, d\varphi \, d\chi. \]

The eigenvalues are

\[ K^2 = \ell(\ell + 1), \]

where \( \ell \) is a non-negative integer. The eigenfunctions are

\[ C_\ell^m(\cos(\theta)), \]

where \( C_\ell^m(\cos(\theta)) \) are the Gegenbauer polynomials.

The free geodesic motion is then given by

\[ H = \frac{1}{2} K^2, \]

and the energy levels are

\[ E = \ell(\ell + 1). \]

The free geodesic motion on \( S^3 \) is a one-dimensional problem, and the solutions can be expressed in terms of the Gegenbauer polynomials.

The cotangent function can be introduced into the Hamiltonian as a perturbation

\[ H = \frac{1}{2} K^2 + b \cot(\chi), \]

where \( b \) is a constant. The eigenvalue problem for this perturbed Hamiltonian is

\[ (K^2 - 2b\cot(\chi)) \psi_n = E_n \psi_n, \]

where \( E_n \) are the energy levels and \( \psi_n \) are the eigenfunctions.

The eigenfunctions are given by

\[ \psi_n = C_\ell^m(\cos(\theta)) \cos^n(\chi), \]

where \( n \) is a non-negative integer.

The eigenvalues are

\[ E_n = \ell(\ell + 1) + nb, \]

and the eigenfunctions are

\[ \psi_n = C_\ell^m(\cos(\theta)) \cos^n(\chi). \]
where \( R \) will be treated as a constant and will be set equal to 1 for simplicity. The Casimir operator, \( K \), of the so(4) algebra is associated with the squared four-dimensional angular momentum, and is given by \[ K = L^2 + N^2, \]

where the components \( L_i \) and \( N_j \) of the respective angular momentum, \( L \), and the Euclidean boost operator, \( N \), have the property to close the so(4) algebra. The quantum-mechanical \( K \) eigenvalue problem is well known and given by

\[
\frac{\hbar^2}{2M} K Y_{K\ell m}(\chi, \theta, \varphi) = \frac{\hbar^2}{2M} [(K + 1)^2 - 1] Y_{K\ell m}(\chi, \theta, \varphi),
\]

where the constant \( K \) determines the value of the four-dimensional angular momentum in the \((K + 1)^2\)-dimensional representation space of the hyper-spherical harmonics, \( Y_{K\ell m}(\chi, \theta, \varphi) \), \( G_{K-\ell}^{\ell+1}(\cos \chi) \) denote the Gegenbauer polynomials and \( Y^{\ell m}(\theta, \varphi) \) are the standard three-dimensional spherical harmonics. We have chosen to work in dimensionless units, setting \( \hbar = 1 \) and \( 2M = 1 \), with \( M \) standing for the mass of the particle under consideration. It is obvious that the spectrum of the free geodesic motion on \( S^3 \) in (4) is characterized by a \((K + 1)^2\)-fold degeneracy of the states, as it should be given the fact that SO(4) is the isometry group of the three-ball. The immediate conspicuous analogy that comes to one’s mind is that modulo the level spacings, the degeneracy patterns of the free geodesic motion on \( S^3 \) are the same as those appearing in the inverse distance potential problem which shapes the H atom spectrum. Yet, the reasons for the two phenomena are indeed to some extent different. The common denominator of both degeneracy patterns is an underlying so(4) symmetry algebra of the associated Hamiltonians. The difference lies in the qualitatively distinct realizations of this very algebra in the respective cases. While the so(4) algebra of the free geodesic motion on \( S^3 \) is purely geometric in the sense that its elements, \( L_i \) and \( N_j \), in (3) exclusively depend on the position on the curved surface under consideration, the so(4) algebra in the inverse distance potential problem is dynamical as its elements need to be defined over the full phase space. Indeed, the so(4) algebra of the Coulomb potential contains besides the components \( L_1 \), \( L_2 \) and \( L_3 \), of ordinary angular momentum, the three components of the Runge–Lenz vector, \( R_1 \), \( R_2 \) and \( R_3 \) [12]. Below, this concept will acquire special importance in the investigation of the perturbed problem.

Combining equations (3) and (4), the eigenvalue problem of the squared four-dimensional angular momentum operator becomes

\[
\left[ -\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} + \frac{L^2}{\sin^2 \chi} \right] Y_{K\ell m}(\chi, \theta, \varphi) = K(K + 2) Y_{K\ell m}(\chi, \theta, \varphi),
\]

with the hyper-spherical harmonics, \( Y_{K\ell m}(\chi, \theta, \varphi) \), being defined in (4) above. In what follows, we introduce the short-hand

\[
S_K^{\ell}(\chi) = \sin^\ell \chi G_{K-\ell}^{\ell+1}(\cos \chi),
\]

in terms of which the hyper-spherical harmonics equivalently rewrite to

\[
Y_{K\ell m}(\chi, \theta, \varphi) = S_K^{\ell}(\chi) Y^{\ell m}(\theta, \varphi), \quad K \in [0, \infty), \quad \ell \in [0, K), \quad m \in [-\ell, \ell].
\]
1. Figure 1. The dependence of the energies, $\epsilon\!,\!_K$, in (11), of the levels within the ‘curved’ Coulomb potential on the strength $b$ of the cot $\chi$ perturbation. The figure shows that the cotangent interaction mainly affects the gap between the ground state and its first excitations by increasing it. For moderate values of the potential strength around $b \sim 1$, the higher lying states practically remain unaltered by the perturbation due to the rapid flattening of the exponential factor in (12) with the increase of $K$. The exponential damping factor furthermore reduces the de-excitation probabilities to the ground state, the most affected being the first excitation.

2.2. The ‘curved’ Coulomb potential problem on $S^3$

For the sake of self-sufficiency of the presentation and fixing notations, we briefly review the exact solutions of the ‘curved’ Coulomb (-like) potential on $S^3$, given by

$$ (K - 2b \cot \chi) \Psi_{K\tilde{m}\tilde{\ell}}(\chi, \theta, \varphi) = \epsilon \Psi_{K\tilde{m}\tilde{\ell}}(\chi, \theta, \varphi), \quad (8) $$

before going to the heart of our study in the next subsection, namely, to their expansions in the basis of the hyper-spherical harmonics. In (8), $\epsilon$ is the energy, $E$, here in dimensionless units, $\epsilon = 2ME/\hbar^2$ and $b$ defines the (equally dimensionless) strength of the cotangent potential. We furthermore admitted in (8) for the possibility that $\tilde{m}$ and $\tilde{\ell}$ may not necessarily be the same as $m$ and $\ell$ in (4) and (5).

Upon changing variable in (8) to

$$ \Psi_{K\tilde{m}\tilde{\ell}}(\chi, \theta, \varphi) = \frac{U^\tilde{\ell}_K(\chi)}{\sin \chi} Y^\tilde{m}_\ell(\theta, \varphi), \quad (9) $$

equation (8) transforms into the well-known one-dimensional Schrödinger equation with the so-called trigonometric Rosen–Morse potential [13], $V_{\text{RM}}(\chi)$, in reality introduced by Schrödinger [14, 15] as a ‘curved’ Coulomb-, better, Coulomb-like potential [16],

$$ -\frac{d^2 U^\tilde{\ell}_K(\chi)}{d\chi^2} + V_{\text{RM}}(\chi)U^\tilde{\ell}_K(\chi) - (\epsilon + 1)U^\tilde{\ell}_K(\chi) = 0, $$

$$ V_{\text{RM}}(\chi) = -2b \cot \chi + \frac{\ell(\ell + 1)}{\sin^2 \chi}, \quad (10) $$

depending on the choice for the $b$ value. The latter equation is exactly solvable because it can be reduced to the hyper-geometric differential equation [13], and the spectrum is such that the cotangent perturbation does not remove the $(K + 1)^2$-fold degeneracy of the free geodesic motion, as visible from the expression for the energy,

$$ \epsilon_K + 1 = (K + 1)^2 - \frac{\hbar^2}{(K + 1)^3}. \quad (11) $$

The dependence of the excitation energies in equation (11) on the strength $b$ of the cotangent perturbation is displayed in figure 1.
Among others, the solutions in (9) have been independently reproduced also in [17] in terms of non-classical Romanovski polynomials, $R_{n}^{\alpha,\beta}(\cot \chi)$, as

$$
\Psi_{K\tilde{m}}(\chi, \theta, \varphi) = e^{-\frac{\alpha \chi}{2}} \sin^{K} \chi R_{K-\ell}^{\alpha,\beta}(\cot \chi) Y_{\ell}^{\tilde{m}}(\theta, \varphi),
$$

$$
\beta_{K} = -(n + \tilde{m}) = -K, \quad \alpha_{K} = \frac{2b}{\ell + 1}.
$$

(12)

The Romanovski polynomials (reviewed in [11]) satisfy the following differential hyper-geometric equation:

$$
(1 + x^{2}) \frac{d^{2}R_{n}^{\alpha,\beta}}{dx^{2}} + 2 \left( \frac{\alpha}{2} + \beta x \right) \frac{dR_{n}^{\alpha,\beta}}{dx} - n(2\beta + n - 1)R_{n}^{\alpha,\beta} = 0.
$$

(13)

They are obtained from the following weight function:

$$
\omega_{\alpha,\beta}(x) = (1 + x^{2})^{-\beta} \exp \left( -\frac{\alpha}{\ell + 1} x \right),
$$

(14)

and the Rodrigues formula,

$$
R_{n}^{\alpha,\beta}(x) = \frac{1}{\omega_{\alpha,\beta}(x)} \frac{d^{n}}{dx^{n}} \left[ (1 + x^{2})^{\beta} \omega_{\alpha,\beta}(x) \right].
$$

(15)

Upon introducing the short-hand

$$
\psi_{K\tilde{m}}^{\ell}(\chi) = \sin^{K} \chi R_{K-\ell}^{\alpha,\beta}(\cot \chi),
$$

(16)

one arrives at the final form of the solutions to equation (8),

$$
\Psi_{K\tilde{m}}(\chi, \theta, \varphi) = e^{-\frac{\alpha \chi}{2}} \psi_{K}^{\ell}(\chi) Y_{\ell}^{\tilde{m}}(\theta, \varphi).
$$

(17)

2.3. The perturbed motion in terms of damped hyper-spherical harmonics

The goal of this section is to find finite decompositions of the exact solutions of the ‘curved’ Coulomb(-like) potential on $S^{3}$ in the basis of the canonical hyper-spherical harmonics describing the free geodesic motion. We begin with seeking to relate the $\chi$ dependent parts, $\psi_{K}^{\ell}(\chi)$ and $S_{K}^{\ell}(\chi)$, of the wavefunctions of the respective perturbed and free motions in (16) and (6), as

$$
\psi_{K}^{\ell}(\chi) = \sum_{\ell=\tilde{m}}^{K} C_{\ell} S_{\ell}^{\ell}(\chi) = \sum_{\ell=\tilde{m}}^{K} C_{\ell} \sin^{\ell} \chi G_{K-\ell}^{\ell+1}(\cos \chi).
$$

(18)

The latter equation in fact represents a new ansatz for constructing $\psi_{K}^{\ell}(\chi)$ in (16), and it is indeed possible to write down a series of conditions which fix the constants $C_{\ell}$. However, one can equally well take advantage of already knowing $\psi_{K}^{\ell}(\chi)$, and encounter the expansion coefficients in (18) using the orthogonality properties of the hyper-spherical harmonics. Both ways are eligible. We here opt for the second one, and encounter the expansions given in table 1. Substituting them into equation (17), and making use of the identity

$$
S_{K}^{\ell}(\chi) \equiv \frac{e^{-\text{imp}}}{P_{\ell}^{\ell}(\cos \theta)} Y_{K\ell\ell}(\chi, \theta, \varphi)
$$

(19)

allows us to cast the decompositions in the following matrix form:

$$
\begin{pmatrix}
\Psi_{100}(\chi, \theta, \varphi) \\
\Psi_{111}(\chi, \theta, \varphi) \\
\Psi_{200}(\chi, \theta, \varphi) \\
\Psi_{211}(\chi, \theta, \varphi) \\
\Psi_{222}(\chi, \theta, \varphi)
\end{pmatrix} = e^{-\frac{\pi \varphi}{2}} A_{1}(\theta, \varphi) \begin{pmatrix}
Y_{100}(\chi, \theta, \varphi) \\
Y_{111}(\chi, \theta, \varphi) \\
Y_{200}(\chi, \theta, \varphi) \\
Y_{211}(\chi, \theta, \varphi) \\
Y_{222}(\chi, \theta, \varphi)
\end{pmatrix},
$$

(20)

$$
\begin{pmatrix}
\Psi_{100}(\chi, \theta, \varphi) \\
\Psi_{111}(\chi, \theta, \varphi) \\
\Psi_{200}(\chi, \theta, \varphi) \\
\Psi_{211}(\chi, \theta, \varphi) \\
\Psi_{222}(\chi, \theta, \varphi)
\end{pmatrix} = e^{-\frac{\pi \varphi}{2}} A_{2}(\theta, \varphi) \begin{pmatrix}
Y_{100}(\chi, \theta, \varphi) \\
Y_{111}(\chi, \theta, \varphi) \\
Y_{200}(\chi, \theta, \varphi) \\
Y_{211}(\chi, \theta, \varphi) \\
Y_{222}(\chi, \theta, \varphi)
\end{pmatrix},
$$

(21)
The explicit expressions for the $A_3(\theta, \varphi)$ matrices for the lowest $K$ values are given by

$$\begin{align*}
A_1(\theta, \varphi) &= \begin{pmatrix}
1 & \frac{b e^{-i\alpha}}{P_1} \\
0 & 0
\end{pmatrix}, \\
A_2(\theta, \varphi) &= \begin{pmatrix}
1 & \frac{b e^{-i\alpha}}{P_1} \\
0 & 0
\end{pmatrix}, \\
A_3(\theta, \varphi) &= \begin{pmatrix}
1 & \frac{b e^{-i\alpha}}{P_1} \\
0 & 0
\end{pmatrix},
\end{align*}$$

where we have dropped the $\cos \theta$ argument of the associated Legendre functions, $P_l^m(\cos \theta)$, for the sake of simplifying the notations. In noting that $P_l^m(\cos \theta) \sim \sin^l \theta$, we observe that the matrices connecting the representation functions of the perturbed and free motions on $S^3$ are invertible in the entire open interval, $\theta \in (0, \pi)$, being singular only at the poles. The invertible matrices $A_K(\theta, \varphi)$ at the poles can be obtained by replacing $P_l^m(\cos \theta)$ in equations (24)–(26), through either $P^0_l(\cos \theta)$ ("North" pole) or $P^0_l(\cos \pi)$ ("South" pole), because the special values of the Legendre polynomials are finite at the ends of the interval under consideration. On $S^3$, the related matrices depend on the azimuthal angle $\varphi$ alone and are completely free from singularities.
2.4. Scaling similarity transformation of the geometric so(4) algebra on $S^3$ to the algebra of the perturbed motion

Substituting (18) into (8) and dragging the exponential factor from the very right to the very left results in

$$e^{-\frac{\alpha_2}{2}\chi} \left( -\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} + \ell (\ell + 1) - \frac{\alpha_2^2}{4} + \alpha K D_\ell \right) \sum_{\ell=0}^{K} C_\ell S^K_\ell (\chi)$$

$$= \left( -\frac{\alpha_2^2}{4} + K (K + 2) \right) \sum_{\ell=0}^{K} C_\ell e^{-\frac{\alpha_2}{2}\chi} S^K_\ell (\chi). \quad (27)$$

Here, we introduced the differential operator $D_\ell$ as

$$D_\ell = \left( \frac{\partial}{\partial \chi} - K \cot \chi \right), \quad (28)$$

and used $\alpha K = 2b/(K+1)$. For $\ell$ taking the maximal value, $\ell = K$, the operator $D_\ell$ coincides with the raising operator of the so(4) algebra and nullifies $S^K_K (\chi) \sim \sin^K \chi$. In effect, one encounters the identity

$$[\mathcal{K} - 2b \cot \chi] \Psi_{KK\tilde{m}} (\chi, \theta, \phi) = \left[ e^{-\frac{\alpha_2}{2}\chi} \left( \mathcal{K} - \frac{\alpha_2^2}{4} \right) e^{\frac{\alpha_2}{2}\chi} \right] \Psi_{KK\tilde{m}} (\chi, \theta, \phi)$$

$$= \left( K (K + 2) - \frac{\alpha_2^2}{4} \right) \Psi_{KK\tilde{m}} (\chi, \theta, \phi). \quad (29)$$

The latter expression shows that up to a non-unitary scaling transformation, and a shift by a constant, the eigenvalue problem of the cotangent perturbed geodesic motion on $S^3$ for $|KK\tilde{m}\rangle$ has results equivalent to the eigenvalue problem of the free geodesic motion. This in fact is valid for any set of the quantum numbers, $|K\tilde{m}\rangle$, by virtue of the following recurrence relations among $S^K_\ell (\chi)$ functions, which translate into recurrence relations among Gegenbauer polynomials:

$$D_1 S^0_1 (\chi) = 2 \csc^2 \chi S^1_1 (\chi), \quad D_3 S^3_1 (\chi) = 6 \csc^2 \chi S^1_3 (\chi),$$

$$D_2 S^2_2 (\chi) = 2 \csc^2 \chi S^1_2 (\chi), \quad D_3 S^3_1 (\chi) = \frac{20}{3} \csc^2 \chi S^3_3 (\chi),$$

$$D_2 S^1_2 (\chi) = 4 \csc^2 \chi S^2_2 (\chi), \quad D_K S^K_\ell (\chi) = 0, \quad \forall K, \text{ etc}, \quad S^K_\ell (\chi) = \sin^K \chi G^{\ell+1}_{K-\ell} (\cos \chi). \quad (30)$$

In order to illustrate the role of the recurrence relations, we present here the simple example of $\Psi_{100} (\chi, \theta, \phi)$. In this case, and according to table 1, equation (18) reduces to

$$\psi^0_1 (\chi) = S^0_1 (\chi) + b S^1_1 (\chi), \quad S^0_1 (\chi) = -2 \sin \chi \cot \chi, \quad S^1_1 (\chi) = \sin \chi. \quad (31)$$

Upon substitution of (31) into (27), it is straightforward to show that by virtue of the first relation in (30), the $\alpha_1 D_1 S^0_1 (\chi)$ term produces the exact centrifugal term of the second component, $b S^1_1 (\chi)$, so that the net action of $[\mathcal{K} - 2b \cot \chi]$ on $e^{-\frac{\alpha_2}{2}\chi} \psi^0_1 (\chi)$ becomes

$$[\mathcal{K} - 2b \cot \chi] e^{-\frac{\alpha_2}{2}\chi} (S^0_1 (\chi) + b S^1_1 (\chi))$$

$$= \left[ e^{-\frac{\alpha_2}{2}\chi} \left( \mathcal{K} - \frac{\alpha_2^2}{4} \right) e^{\frac{\alpha_2}{2}\chi} \right] [e^{-\frac{\alpha_2}{2}\chi} (S^0_1 (\chi) + b S^1_1 (\chi))]$$

$$= \left( 3 - \frac{\alpha_2^2}{4} \right) e^{-\frac{\alpha_2}{2}\chi} \psi^0_1 (\chi). \quad (32)$$
In figures 2 and 3, the modification of the shapes of hyper-spherical and damped hyper-spherical harmonics, corresponding to the case in equation (31), is shown for illustrative purposes.

Therefore, at the general level of the total wavefunctions, one finds

\[
[K - 2b \cot \chi] \Psi_{100}(\chi, \theta, \phi) = \left[ e^{-\frac{\alpha \chi}{2}} \left( K - \frac{\alpha^2}{4} \right) e^{\frac{\alpha \chi}{4}} \right] \Psi_{100}(\chi, \theta, \phi) = \left( K(K+2) - \frac{\alpha^2}{4} \right) \Psi_{100}(\chi, \theta, \phi). \tag{33}
\]

In fact, the recurrence relations in (30) guarantee the validity of

\[
\left( \frac{\ell(\ell+1)}{\sin^2 \chi} + \alpha K K \right) \sum_{\ell=0}^{\ell=K} C_\ell S_{\ell K}^{\ell} = \sum_{\ell=0}^{\ell=K} \ell(\ell+1) \frac{C_\ell S_{\ell K}^{\ell}}{\sin^2 \chi}, \tag{34}
\]

which, upon substitution into (27), allows us to generalize the transformation in (29) and (33) to any \( \Psi_{K\ell m}(\chi, \theta, \phi) \). In consequence, the similarity transformation between the \((K - 2b \cot \chi)\)
and the $\mathcal{K}$ eigenvalue problems can be cast in the following matrix form:

$$
(K - 2b \cot \chi) X_K(\chi, \theta, \phi) = \left[ e^{\frac{\alpha \ell}{2}} A_K(\theta, \phi) \left( K - \frac{\sigma_0^2}{4} \right) e^{\frac{\alpha \ell}{2}} A_K^{-1}(\theta, \phi) \right] X_K(\chi, \theta, \phi)
$$

$$
X_K(\chi, \theta, \phi) = \begin{pmatrix}
\Psi_{K00}(\chi, \theta, \phi) \\
\Psi_{K1m}(\chi, \theta, \phi) \\
\vdots \\
\Psi_{KK\tilde{m}}(\chi, \theta, \phi)
\end{pmatrix}.
$$

The latter equation shows that the eigenvalue problem of the cotangent perturbed motion on $S^3$ can be represented as the eigenvalue problem of a similarity-transformed Casimir operator, $K$, of the ordinary $\text{so}(4)$ isometry algebra, shifted by a constant (within the representation space of interest). The transformation is non-unitary and of the dilation type. As long as the similarity transformation of the Casimir invariant of the geometric algebra can be viewed as the result of subjecting the $\text{so}(4)$ elements $L_i$ and $N_j$ in (7) to the same transformation, we have designed a representation of the geometric $\text{so}(4)$ algebra for a ‘curved’ Coulomb potential on $S^3$.

3. Conclusions

The interest of the present study is that modulo a shift by a constant, a non-unitary scaling similarity transformation converts the Casimir invariant of the $\text{so}(4)$ isometry algebra of the three-ball $S^3$ into the Coulomb potential problem there. The transformation, presented in (35), was concluded from decomposing the eigenfunctions of the ‘curved’ Coulomb potential problem in the bases of exponentially scaled (damped) hyper-spherical harmonics in (20)–(23). It was furthermore shown to emerge by virtue of specific recurrence relations (30) among Gegenbauer polynomials.

The particle motion on $S^3$ considered here had the peculiarity that the perturbation left the degeneracy patterns characterizing the free motion intact, though the isometry group symmetry has been broken. The breaking of the initial SO(4) rotational invariance by the ‘curved’ Coulomb potential occurred at the level of the representation functions, thus avoiding the more severe breakdown at the level of the algebra through deformations [7, 9, 10, 18]. This subtle type of symmetry breaking is visualized in figure 4 by the deformation of the metric of the $S^3$ sphere through the damping exponential factor under consideration. It might be interesting to note that such a metric deformation also emerges in effect of a conformal symmetry breaking by the dilaton mass [19]. The method proposed can easily be extended toward hyperbolic spaces through complexification of the polar angle $\chi$ in (2) as $\chi \rightarrow i \chi$. 

Figure 4. Deformation of the spherical metric, $|Y_{000}(\chi, 0, \phi)|$ (left) versus the exponentially scaled one, $|\tilde{Y}_{000}(\chi, 0, \phi)|$ (right) for $b = 1$. 
Simultaneously changing $b \rightarrow -ib$ in (10) takes one to the Coulomb problem on a hyperboloid. Quantum problems on hyperbolic spaces, pioneered by Dirac [20] through his consideration of the electron wave equation in the De-Sitter universe, also acquire importance within the context of pure relativistic descriptions of bound systems, as within the context of gravitational studies.

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