The No-ghost Theorem for AdS$_3$ and the Stringy Exclusion Principle

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Abstract

A complete proof of the No-ghost Theorem for bosonic and fermionic string theories on AdS$_3$, or the group manifold of $SU(1,1)$, is given. It is then shown that the restriction on the spin (in terms of the level) that is necessary to obtain a ghost-free spectrum corresponds to the stringy exclusion principle of Maldacena and Strominger.

1 Introduction

It has been conjectured recently that there exists a duality between supergravity (and string theory) on $(D+1)$-dimensional anti-de Sitter space, and a conformal field theory that lives on the $D$-dimensional boundary of anti-de Sitter space [1]. This proposal has been further elaborated in [2, 3], where a relation between the correlation functions of the two theories has been proposed, and many aspects of it have been analysed (see [4] and references therein).

In this paper we shall consider a specific example of the above class of proposals, in which type IIB string theory on AdS$_3 \times S^3 \times M^4$ (where $M^4$ is either K3 or $T^4$) is conjectured to be dual to a two-dimensional conformal field theory whose target manifold is a symmetric product of a number of copies of $M^4$. This example is of special interest as both partners of the dual pair are fairly well-understood theories, and
and it should therefore be possible to subject the proposal to non-trivial tests. On the string theory side, for instance, we have $\text{AdS}_3 \cong \text{SL}(2, \mathbb{R}) \cong \text{SU}(1,1)$ and $S^3 \cong \text{SU}(2)$, and since these are both group manifolds we should be able to determine the spectrum of string states exactly. The conjectured duality together with the S-duality of type IIB relates this string theory to a superconformal field theory with target space $\text{Sym}_Q M^4$, where $Q$ appears as the level of each WZW model, and $Q$ is presumed to be large. It was pointed out in [5] that in the dual conformal field theory, there are only finitely many chiral primary states, and as a consequence this must be somehow reflected in the corresponding string theory. This has lead to the proposal that there is a “stringy exclusion principle” which removes certain states from the string spectrum. It is the purpose of this paper to shed some light on this proposal. In particular, we shall explain below how this restriction has a natural interpretation in terms of the no-ghost-theorem for a string theory on $\text{SU}(1,1)$.

The question of consistency of string theories on $\text{SU}(1,1)$ has a long history [6]-[15]. There are effectively two different approaches which are in a sense orthogonal to each other. In the first approach, advocated some time ago by Hwang and collaborators [9]-[13] following the earlier work of [7, 8], the Fock space of states that is analysed is the space on which all generators of the Kac-Moody algebra of $\text{su}(1,1)$ are well-defined, whereas this is not true in the second approach advocated by Bars [14] and more recently Satoh [15] in which free-field-like Fock spaces are introduced. These spaces are therefore at best different dense subspaces of the space of string states, and since very little is known about possible completions, it is not clear how these approaches are related, if at all.

In this paper we shall follow the first approach, in which it is necessary to restrict the set of $\text{SU}(1,1)$ representations to those whose spin (in the case of the discrete series) is essentially bounded by the level, as first proposed in [7, 8]. This restriction guarantees that certain negative norm physical states are removed from the spectrum, and it is this condition that we show to correspond to the stringy exclusion principle. Unfortunately, the various arguments for the positivity of the physical states under this restriction that have been given in the literature are not quite satisfactory: for example, the “proof” in [9, 10] is clearly incomplete, the restriction in [10] is too strong, and there is a gap in the proof in [12]. We shall therefore give a complete description of the proof. In the bosonic case, our argument follows closely the approach of [12] (which in turn follows the old argument of Goddard & Thorn [16]) together with the result of [17]. We then give what we believe to be the first correct statement and proof of the corresponding result for the fermionic case.

It should be stressed that these arguments only guarantee that the string theory is free of ghosts at the free level. To get a consistent (ghost free) interacting theory, it would be necessary to show that crossing symmetric amplitudes can be defined whose fusion rules close among the ghost-free representations. This is a rather difficult problem as the fusion rules of the $\text{SU}(1,1)$ WZW model are not well understood. On the other hand, one may regard the fact that this theory with the appropriate truncation appears as the dual pair of a very well understood conformal field theory as evidence that it is
indeed consistent.

The paper is organised as follows. In section 2, we describe our conventions and give the proof of the no-ghost-theorem in the bosonic case. Section 3 is devoted to a similar analysis of the fermionic case. In section 4, we explain in detail that the bound that arises in the no-ghost-theorem corresponds to the stringy exclusion principle. Section 5 contains our conclusions and open problems, and in the appendix we give explicit examples of physical states for the fermionic theory which demonstrate that the bound on the spin is necessary to ensure positivity.

2 The bosonic theory

2.1 SU(1, 1) WZW models and Strings

We should first emphasize the intrinsic interest of string theory on $SU(1, 1) \cong SL(2, \mathbb{R})$ (quite independent of the spectacular recent developments already mentioned). The standard procedure for deciding whether a given string background is consistent is to check for quantum conformal invariance of the world-sheet sigma-model, as given by the vanishing of appropriate $\beta$-functions. It is not hard to see that these conditions are insensitive to some vital properties, however: by these criteria a flat ‘spacetime’ with 13 timelike and 13 spacelike directions would be a perfectly consistent background for the bosonic string with $c = 26$, and yet there will clearly be physical states of negative norm in such a theory. If there is a single time-direction, the no-ghost theorem [18, 16] for the bosonic string in flat Minkowski spacetime, Mink$_d$ ensures that there are no negative-norm states for $d \leq 26$, and this can immediately be extended to backgrounds of the type Mink$_d \times \mathcal{M}$ with $2 \leq d \leq 26$ provided $\mathcal{M}$ corresponds to a unitary CFT of appropriate central charge. But if we are considering a background whose geometry involves a time-like direction in an essential way, then unitarity and the absence of ghosts is something which must be scrutinized very carefully.

To examine such issues it is natural to turn to the simplest string models which one can hope to solve exactly, namely those for which the backgrounds are group manifolds [19]. If we require only a single time-like direction then we are led to the non-compact group $SU(1, 1) \cong SL(2, \mathbb{R})$, or its covering space, as a laboratory for testing these basic ideas about string theory [6]. In this section we shall consider string theory on $SU(1, 1) \times \mathcal{M}$ where $\mathcal{M}$ is some unspecified target space corresponding to a unitary conformal field theory. We now proceed to define the string theory and its physical states in terms of an $SU(1, 1)$ WZW model at level $k$.

The Kac-Moody algebra corresponding to $su(1, 1)$ is defined by

$$[J^a_m, J^b_n] = if^{abc}c_{m+n} + km\eta^{ab}\delta_{m,-n},$$

where $\eta^{ab} = \text{diag}(+1, +1, -1)$, and

$$f^{abc} \equiv f^{abc}d\eta^{dc} = \varepsilon^{abc}.$$
We can then define $J_n^\pm = J_n^1 \pm iJ_n^2$, and in terms of these modes the commutation relations are
\[
[J_m^+, J_n^-] = -2J_{m+n}^3 + 2km\delta_{m,-n}
\]
\[
[J_m^3, J_n^\pm] = \pm J_{m+n}^\pm
\]
\[
[J_m^3, J_n^3] = -km\delta_{m,-n}.
\]
(2.2)
The adjoint operator of $J_m^a$ is $J_{-m}^a$, and thus
\[
(J_m^\pm)^* = J_{-m}^\mp \quad (J_m^3)^* = J_{-m}^3.
\]
(2.3)
The Sugawara expression for the Virasoro algebra is
\[
L_n = \frac{1}{2(k - 1)} \sum_l : \left[ \frac{1}{2} \left( J_{n+l}^+ J_l^- + J_{n+l}^- J_l^+ \right) - J_{n+l}^3 J_l^3 \right] :,
\]
which satisfies the Virasoro algebra
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m,-n}
\]
with
\[
c = \frac{3k}{k - 1}.
\]
(2.5)
Furthermore we have
\[
[L_n, J_m^\pm] = -mJ_{n+m}^\pm \quad [L_n, J_m^3] = -mJ_{n+m}^3.
\]
(2.7)
In the following we shall always consider the case $k > 1$, as then $c > 0$ and the group manifold has one time-like and two space-like directions.

The Kac-Moody algebra contains the subalgebra of zero modes $J_0^a$ for which we introduce the quadratic Casimir as
\[
Q = \frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3.
\]
(2.8)
Representations of the $su(1,1)$ zero mode algebra are characterised by the value of $Q$ and $J_0^3$ on a cyclic state $|j, m\rangle$,
\[
Q|j, m\rangle = -j(j + 1)|j, m\rangle \quad J_0^3|j, m\rangle = m|j, m\rangle.
\]
(2.9)
In the following we shall mainly be concerned with the unitary representations $D_j^+$ of the $su(1,1)$ algebra for which a cyclic state can be chosen to be of the form $|j, j\rangle$, where $j \in \{-1/2, -1, -3/2, \ldots\}$, and $J_0^+|j, j\rangle = 0$. There exists also another discrete series ($D_j^-$) whose cyclic state is of the form $|j, -j\rangle$ with $j \in \{-1/2, -1, -3/2, \ldots\}$, and $J_0^-|j, -j\rangle = 0$. In addition, there exist the continuous (unitary) series for which the states $|j, m\rangle$ have $j = -1/2 + ik$, and $m \in \mathbb{Z}$ ($C_0^j$) or $m \in \mathbb{Z} + 1/2$ ($C_j^{1/2}$); and also there exists the exceptional representations with $-1/2 \leq j < 0$ and $m \in \mathbb{Z}$. Finally, we should not forget the trivial representation consisting of the single state with $j = m = 0$. 

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The only unitary representations of the $SU(1,1)$ group are those we have listed above. If we consider the universal covering group of $SU(1,1)$ however, then there are more general representations of the type $D^\pm_j$ in which $j$ and $m$ need not be half-integral (although the allowed values of $m$ within any irreducible representation always differ by integers) and similarly there are additional continuous representations where $m$ need not be half-integral. The group manifold of $SU(1,1)$ is topologically $\mathbb{R}^2 \times S^1$ (with the compact direction being timelike) and this is responsible for the quantisation of $m$ in units of half integers. By contrast, the simply-connected covering group is topologically $\mathbb{R}^3$. As will become apparent in section 4, the conjectured duality mentioned in the introduction would seem to involve a string theory defined on $SU(1,1)$ itself, rather than on its covering space, and so this is the case on which we shall concentrate. This seems to be similar to what was found in the case of AdS$_5$ in [20]. Our proof of the no-ghost theorem given below applies equally well to either $SU(1,1)$ or its covering space, however.

For a bosonic string on $SU(1,1) \times \mathcal{M}$ the world-sheet conformal field theory has a chiral algebra generated by two commuting subalgebras: one is the Kac-Moody algebra corresponding to $su(1,1)$, and the other subalgebra corresponds to a unitary conformal field theory. The Virasoro generators of the whole theory are then of the form $L_n = L_{su(1,1)}^n + L_0^n$, where $L_{su(1,1)}^n$ and $L_0^n$ commute. We shall consider the case where the total conformal charge

$$c = c_{su(1,1)} + c_0 = 26,$$

which is necessary for the BRST operator $Q$ to satisfy $Q^2 = 0$. Let us denote by $\mathcal{H}$ the Fock space that is generated from the ground states (that form a representation of the zero modes of the whole theory) by the action of the negative modes. The conformal weight of a ground state is $h_{su(1,1)} + h_0$, where $h_{su(1,1)}$ is the conformal weight of the $su(1,1)$ ground state representation, and by the assumption on the unitarity of the commuting subtheory, $h_0 \geq 0$. The physical states in the Fock space are defined to be those that satisfy the Virasoro primary condition

$$L_n \psi = 0 \quad n > 0,$$

and the mass shell condition

$$L_0 \psi = \psi.$$

Suppose then that the Casimir operator of the $su(1,1)$ ground state representation takes the value $-j(j + 1)$. If $\psi$ is a descendant at grade$^1 N$, the second condition becomes

$$\frac{-j(j + 1)}{2(k - 1)} + N \leq 1.$$

It follows immediately that for the continuous unitary representations of $su(1,1)$, this condition can only be satisfied for $N = 0$, as $-j(j + 1) = 1/4 + \kappa^2 > 0$; in this case

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$^1$We shall use the terminology grade for what is usually called level in string theory, and reserve the term level for the central term of the affine algebra.
all states satisfy (2.10), and the norms are by construction unitary. We can therefore concentrate on the discrete unitary representations.

2.2 The no-ghost theorem in the discrete case

Let us now consider the case where the ground states transform according to the discrete representation $D_j^-$ of $su(1,1)$. (The case where the representation is $D_j^+$ can be treated similarly.) It is easy to find states in the Verma module constructed from these ground state representations which are Virasoro primary and satisfy the mass-shell condition and yet which have negative norms for certain values of $j$ and $k$ \[6, 7, 8\]. It would seem, therefore, as though the no-ghost theorem fails in this case. Following the proposal of \[6, 7, 8\], however, we shall show that if we impose the additional restriction

$$0 < -j < k$$ \hspace{1cm} (2.13)

then all physical states in $\mathcal{H}$ indeed have positive norm. Notice that this restriction together with the mass-shell condition (2.11) implies severe restrictions on the allowed grades for physical states. In particular since $j + 1 > 1 - k$ and $j < 0$ we find that

$$N \leq 1 + \frac{j(j + 1)}{2(k - 1)} < 1 - \frac{j}{2} < 1 + \frac{k}{2}.$$ \hspace{1cm} (2.14)

The last bound implies that for fixed level $k$, the physical states only arise at a finite number of grades. (However, there are infinitely many physical states at every allowed grade, since the unitary representations of $SU(1,1)$ are infinite dimensional.)

Let us now turn to the proof of the no-ghost theorem, following the general strategy of Hwang \[12\] and \[16\]. We denote by $\mathcal{F}$ the subspace of $\mathcal{H}$ that is spanned by states $\psi \in \mathcal{H}$ for which

$$J_3^n \psi = 0 \quad L_n \psi = 0 \quad \text{for } n > 0.$$ \hspace{1cm} (2.15)

We also denote by $\mathcal{H}^{(N)}$ the subspace of $\mathcal{H}$ that consists of states whose grade is less or equal to $N$. In a first step we want to prove the following Lemma

**Lemma.** If $c = 26$ and $-k < j < 0$, the states of the form

$$|\{\lambda, \mu\}, f\rangle := L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} (J_3^1)^{\mu_1} \cdots (J_3^m)^{\mu_m} |f\rangle,$$ \hspace{1cm} (2.16)

where $f \in \mathcal{F}$ with $L_0 |f\rangle = h_f |f\rangle$ is at grade $L$ and $\sum_r r \lambda_r + \sum_s s \mu_s + L \leq N$, form a basis for $\mathcal{H}^{(N)}$.

**Proof.** The proof proceeds in two steps. First, we prove that the states of the form (2.16) are linearly independent. Let us define the Virasoro algebra corresponding to the $U(1)$ theory generated by $J^3$ as

$$L_n^3 = -\frac{1}{2k} \sum_{m} : J_n^3 J_{n-m}^3 :,$$ \hspace{1cm} (2.17)

\[2\]We shall always ignore the exceptional representations, since they do not occur in the Peter-Weyl decomposition of the $L^2$ space, and therefore should not contribute in string theory.
whose corresponding central charge is \( c^3 = 1 \). We can then define
\[
L_c^n = L_n - L_3^n,
\]
and by construction the \( L_c^n \) commute with \( J_3^m \), and therefore with \( L_3^m \), and define a Virasoro algebra with \( c = 25 \). Using (2.18), we can then rewrite the states of the form
\[
|f⟩ \quad \text{(2.16)}
\]
in terms of states where \( L_r \) is replaced by \( L_c^r \). It is clear that this defines an isomorphism of vector spaces, and it is therefore sufficient to prove that these modified states are linearly independent. Since \( L_c^n \) and \( J_3^m \) commute, the corresponding Kac-determinant is then a product of the Kac-determinant corresponding to the \( U(1) \) theory (which is always non-degenerate), and the Kac-determinant of a Virasoro highest weight representation with \( c = 25 \) and highest weight
\[
h^c = h_f + \frac{m^2}{2k},
\]
where \( h_f \) and \( m \) are the \( L_0 \)-eigenvalue and the \( J_3^0 \) eigenvalue of the state \(|f⟩\). If \( f \) is at grade \( M \), then
\[
h^c = -\frac{j(j+1)}{2(k-1)} + M + \frac{m^2}{2k} + h^0
\]
and since \( j < 0 \), \( j + k > 0 \) and \( j - m + M \geq 0 \), and \( h^0 \geq 0 \) it follows that \( h^c > 0 \). Since the only degenerate representations of the Virasoro algebra at \( c = 25 \) arise for \( h \leq 0 \) (see e.g. [21]) it follows that the Kac-determinant is non-degenerate, and the states of the form (2.16) are indeed linearly independent.

The final step, completing the proof, is to establish by induction on \( N \) (as in [16]) that these states form a basis of \( \mathcal{H}^{(N)} \) for all \( N \geq 0 \). The induction start \( N = 0 \) is trivial. Suppose then that we have proven the statement for \( N - 1 \), and let us consider the states at grade \( N \). Let us denote by \( \mathcal{G}^{(N)} \) the subspace of \( \mathcal{H}^{(N)} \) that is generated by the states of the form (2.16) with \( L < N \). We have shown above that \( \mathcal{G}^{(N)} \) does not contain any null states, and this implies that \( \mathcal{H}^{(N)} \) is the direct sum of \( \mathcal{G}^{(N)} \) and its orthogonal complement (in \( \mathcal{H}^{(N)} \)). By the induction hypothesis it follows that every state in the orthogonal complement of \( \mathcal{G}^{(N)} \) is annihilated by \( L_n \) and \( J_3^m \) (with \( n \geq 0 \), and therefore that the orthogonal complement consists of states in \( \mathcal{F} \). This completes the proof of the Lemma.

Let us call a state *spurious* if it is a linear combination of states of the form (2.16) for which \( \lambda \neq 0 \). Any given physical state \( ψ \) can then be written as a spurious state \( ψ_s \) plus a linear combination of states of the form (2.16) with \( \lambda = 0 \), i.e.
\[
ψ = ψ_s + χ.
\]
For \( c = 26 \), following the argument of Goddard and Thorn [16], \( L_1ψ_s \) and \( \tilde{L}_2ψ_s = (L_2 + 3/2L_1^2)ψ_s \) are again spurious states, and it follows that \( χ \) must also be a physical
state, i.e. that $L_n \chi = 0$ for $n > 0$. The next Lemma fills the gap in the argument given previously in [12].

**Lemma.** Let $0 > j > -k$. If $\chi$ is a physical state of the form (2.16) with $\lambda = 0$, then $\chi \in \mathcal{F}$.

**Proof.** For fixed $|f\rangle \in \mathcal{F}$, let us denote by $\mathcal{H}_f$ the Fock space that is generated by the action of $J_3$ from $|f\rangle$, and by $\mathcal{H}_f^{\text{vir}}$ the Fock space that is generated by the action of $L_3$ from $|f\rangle$. Since $L_3$ can be expressed as a bilinear in terms of $J_3$ (2.17), it is clear that $\mathcal{H}_f^{\text{vir}}$ is a subspace of $\mathcal{H}_f$. On the other hand $\mathcal{H}_f^{\text{vir}}$ is a Virasoro Verma module for $c = 1$ whose ground state has conformal weight $-m^2/2k$ (where $m$ is the $J_3^0$ eigenvalue of $|f\rangle$), and it follows from the Kac-determinant formula that $\mathcal{H}_f^{\text{vir}}$ does not contain any null states unless $m = 0$ [21]. Provided that $m \neq 0$, it is then easy to see that $\mathcal{H}_f^{\text{vir}}$ and $\mathcal{H}_f$ contain the same number of states at each grade, and this then implies that $\mathcal{H}_f^{\text{vir}} = \mathcal{H}_f$. Since $\mathcal{H}_f^{\text{vir}}$ does not contain any null states (with respect to the Virasoro algebra) it then follows that $\mathcal{H}_f$ does not contain any Virasoro primary states other than $|f\rangle$ itself. It therefore only remains to show that all physical states have $m \neq 0$.

The physical states at fixed grade $N$ form a representation under the zero mode $su(1,1)$ algebra since $J_0^+ \psi$ and $J_3^0 \psi$ are physical states provided that $\psi$ is. If the ground states form a representation $D_{j_{-}}$ of the $su(1,1)$ zero mode algebra, then the possible representations at grade $N$ are of the type $D_{j_{-}N}$ with $J = j + N, \ldots, j - N$, and therefore $m \leq j + N$ for all physical states at grade $N$. To prove the lemma it therefore suffices to show that the mass shell condition (2.12) together with $j + k > 0$ and $j < 0$ implies that $j + N < 0$.

Let us consider more closely those grades which are allowed by the mass-shell condition (2.11) and the spin-level restriction (2.13). If $0 > j > -1$ then the mass-shell condition alone implies that $N = 0$ is the only possibility. For $-1 \geq j \geq -2$ we claim that $N < 2$. To see this note that $N \geq 2$ implies $k \geq 2$ because of (2.14). But then $j(j+1)/2(k-1) < 1$ and so $N \geq 2$ is still forbidden by (2.14). We have therefore shown that $j + N < 0$, as required, if $0 > j \geq -2$. But also if $j \leq -2$ (which allows $N \geq 2$) then we find that $j + N < 0$ directly from (2.14). This completes the proof.

One may think that the Lemma should also hold under weaker assumptions, but it is maybe worth mentioning that if there was no restriction on $j$ and if $J_3$ was spacelike, the corresponding statement would not hold: indeed there exists a state $[(J_{-3}^3)^2 - m J_{-2}^3]|j, m\rangle$ with $m = \sqrt{-k/2}$ which is annihilated by all Virasoro positive modes, but which is not annihilated by $J_2^3$.

**Theorem:** For $c = 26$ and $0 < -j < k$, every physical state $\psi$ differs by a spurious physical state from a state in $\mathcal{F}$. Consequently, the norm of every physical state is non-negative.

**Proof.** This follows directly from the previous two lemmas and the fact that $\mathcal{F}$ is a subspace of the coset space corresponding to $su(1,1)/u(1)$ which has been shown to be unitary for $0 > j > -k$ by Dixon et.al. [17].
We should mention that the above argument can also be used to give a proof of the no-ghost theorem in the flat case. In this case, the coset module is positive definite (without any restrictions on the momenta) and only the calculations that demonstrate that $h^c$ is positive and that the conformal weight of the ground state of $H_f$ is negative need to be modified. This can easily be done (see also [12]).

Finally, since the norms of states based on $D_j^-$ are continuous functions of $j$ and $k$, the arguments above actually show that the representations with $0 > j \geq -k$ do not contain any negative norm physical states. Furthermore, there are certainly physical states with negative norm whenever $j < -k$ (see e.g. [6, 7, 8]), and so our result cannot be improved.

3 The supersymmetric theory

A fermionic string theory on $SU(1,1)$ is defined by a supersymmetric WZW model on this group manifold. The supersymmetric Kac-Moody algebra corresponding to $su(1,1)$ is generated by $J^a_m$ and $\psi^a_r$, where $a = \pm, 3$, $n \in \mathbb{Z}$, and $r$ is a half-integer in the NS sector (which we shall consider in the following). The (anti-)commutation relations are

$$
[J^a_m, J^b_n] = i f^{abc} J^c_{m+n} + km \eta^{ab} \delta_{m,-n}
$$

$$
[J^a_m, \psi^b_r] = i f^{abc} \psi^c_{m+r}
$$

$$
\{\psi^a_r, \psi^b_s\} = k \eta^{ab} \delta_{r,-s},
$$

where $f^{abc}$ and $\eta^{ab}$ are the same structure constants and metric, respectively as before for the bosonic case. We shall use the metric (and its inverse) to raise (and lower) indices.

The universal algebra that is generated from $J^a$ and $\psi^a$ is isomorphic to the direct (commuting) sum of a bosonic Kac-Moody algebra and three free fermions. Indeed, if we define

$$
\tilde{J}^a_m = J^a_m + \frac{i}{2k} f^{abc} \sum_r \psi^b_{m-r} \psi^c_r,
$$

then

$$
[\tilde{J}^a_m, \tilde{J}^b_n] = i f^{abc} \tilde{J}^c_{m+n} + \tilde{k} m \eta^{ab} \delta_{m,-n}
$$

$$
[\tilde{J}^a_m, \psi^b_r] = 0,
$$

where $\tilde{k} = k+1$. We can thus introduce a Virasoro algebra by the Sugawara construction,

$$
L_m = \frac{1}{2(k-1)} \eta_{ab} \sum_t \tilde{J}^a_{m-t} \tilde{J}^b_t + \frac{1}{2k} \eta_{ab} \sum_r \psi^a_{m-r} \psi^b_r,
$$

3 The same argument cannot be applied to the limit $j = 0$ however. At this value there are new physical states at grade $N = 1$ with $h^0 = 0$ (which are excluded by the mass-shell condition for $j < 0$), and the theorem does indeed fail: the norms of the two physical states $J^+_0|0\rangle$ and $J^-_1|0\rangle$ have opposite sign.
which satisfies
\[
[L_m, J^a_n] = -n J^a_{m-n}, \\
[L_m, \psi^a_r] = -\left(\frac{m}{2} + r\right) \psi^a_{m+r}
\] (3.5)

and the Virasoro algebra (2.5) with central charge
\[
c = \frac{3k}{k-1} + 3 \frac{2}{2k} = \frac{3k+2}{2k}.
\] (3.6)

The theory has actually a super Virasoro symmetry, where the additional generator is defined by
\[
G_r = \frac{1}{k} \eta_{ab} \sum_s \tilde{J}_r^a \psi^b_s - \frac{i}{6k2f_{abc}} \sum_{s,t} \psi^a_{r-s-t} \psi^b_s \psi^c_t,
\] (3.7)

and satisfies
\[
G_r, J^a_n \] = -n \psi^a_{r+n}, \\
\{G_r, \psi^a_s\} = J^a_{r+s}.
\] (3.8)

The supersymmetric central charge is usually defined by
\[
\hat{c} = \frac{2}{3} c = \frac{3k+2}{k}.
\]

The modes $L_n$ and $G_r$ satisfy the $N = 1$ supersymmetric Virasoro algebra
\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m,-n}, \\
[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r,-s}.
\] (3.9)

As before we want to consider the case of a theory whose chiral algebra is generated by two commuting subalgebras, where one subalgebra is the above supersymmetric Kac-Moody algebra and the other defines a (supersymmetric) unitary conformal field theory. The Virasoro generators of the whole theory are then of the form $L_n = L^{su(1,1)}_n + L^0_n$, where $L^{su(1,1)}_n$ and $L^0_n$ commute, and the total central charge is
\[
c = c^{su(1,1)} + c^0 = 15.
\] (3.10)

The physical states are those states that satisfy
\[
L_n \phi = G_r \phi = 0 \quad \text{for} \quad n, r > 0,
\] (3.11)

together with the mass-shell condition
\[
L_0 \phi = \frac{1}{2} \psi.
\] (3.12)
If the ground states transform in a representation of $su(1,1)$ whose Casimir takes the value $-j(j+1)$ then the mass-shell condition implies (as $\hbar^0 \geq 0$)

$$-rac{j(j+1)}{2k} + N \leq \frac{1}{2},$$  \hfill (3.13)

It is then again clear that for the continuous representations (where $-j(j+1) = \frac{1}{4} + \kappa^2$) only the ground states can satisfy the mass shell condition, and the corresponding states have positive norm. The only interesting cases are therefore the discrete representations $D_j^\pm$. In the following we shall analyse in detail the case of $D_j^-$; the situation for $D_j^+$ is completely analogous.

In this section we want to show that the physical states in the Fock space whose ground states transform in the $D_j^-$ representation of $su(1,1)$ have positive norm provided that

$$0 > j > -k - 1.$$  \hfill (3.14)

The argument will be very similar to the argument in the bosonic case. Let us denote by $\mathcal{F}$ the subspace of the Fock space $\mathcal{H}$ that consist of states $\phi \in \mathcal{H}$ for which $J_3^\ell n \phi = 0$ for $n > 0$ $\psi_3^\ell r \phi = 0$ for $r > 0$, (3.15)

and denote by $\mathcal{H}^{(N)}$ the subspace of the Fock space that consists of states whose grade is less or equal to $N$. In a first step we prove the

**Lemma.** If $c = 15$ and $0 > j > -k - 1$, then the states of the form

$$\{|\{\varepsilon, \lambda, \delta, \mu\}, f\} := G_{-\ell - \frac{1}{2}}^{\ell_1} \cdots G_{-a+\frac{1}{2}}^{\ell_a} L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} (\psi_{-\ell - \frac{1}{2}}^{\ell_1})^{\delta_1} \cdots (\psi_{-a+\frac{1}{2}}^{\ell_a})^{\delta_a} (J_{-1}^{3})^{\mu_1} \cdots (J_{-m}^{3})^{\mu_m} |f\},$$  \hfill (3.16)

where $f \in \mathcal{F}$ is at grade $L$, $\varepsilon_b, \delta_b \in \{0,1\}$, and $\sum_b \varepsilon_b (b-1/2) + \sum_c \delta_c (c-1/2) + \sum_r r \lambda_r + \sum_s s \mu_s + L \leq N$, form a basis for $\mathcal{H}^{(N)}$.

**Proof.** Let us define

$$L_3^c = -\frac{1}{2k} \sum_m : J_{n-m}^3 J_m^3 : ,$$  \hfill (3.17)

and

$$G_r^3 = -\frac{1}{k} \sum_s J_{r-s}^3 \psi_s^3 ,$$  \hfill (3.18)

which satisfy the $N = 1$ supersymmetric algebra (3.9) with $c = 3/2 (\hat{c} = 1)$, and (3.3) and (3.8), respectively, for $a = 3$. We can then define

$$L_n^c = L_n - L_3^c \quad G_r^c = G_r - G_r^3 ,$$  \hfill (3.19)

and, by construction, $L_n^c$ and $G_r^c$ commute (or anticommute) with $J_3^c$ and $\psi_r^3$, and therefore with $L_3^c$ and $G_3^3$. This implies that $L_n^c$ and $G_r^c$ define a $N = 1$ supersymmetric algebra (3.3) with $c = 27/2 (\hat{c} = 9)$.

\footnote{This is slightly stronger than the statement in \cite{ref}}
Using (3.17) and (3.18), we can rewrite the states in (3.16) in terms of states where $L_n$ is replaced by $L^c_n$ and $G_r$ by $G^c_r$, and it is clear that this transformation defines an isomorphism of vector spaces. In a first step we want to prove that the states of the form (3.16) are linearly independent, and to this end it is sufficient to do this for the modified states. As $L^c$ and $G^c$ commute (or anticommute) with $J^3$ and $\psi^3$, the Kac-determinant is then a product of the Kac-determinant corresponding to the supersymmetric $U(1)$ theory (which is always non-degenerate), and the Kac-determinant of a supersymmetric Virasoro highest weight representation with $\hat{c} = 9$ and highest weight

\[ h^c = h_f + \frac{m^2}{2k}, \]  

where $h_f$ and $m$ are the $L_0$-eigenvalue and $J^3_0$-eigenvalue of the corresponding ground state $|f\rangle$. If $f$ is at grade $M$, then

\[ h^c = \frac{-j^2 - j + m^2 + 2Mk}{2k} + h^0, \]  

where $h^0 \geq 0$ is the eigenvalue of $|f\rangle$ with respect to $L^0_0$. It is known that the degenerate representations at $\hat{c} = 9$ only arise for $h \leq 0$ \cite{21}, and it therefore remains to show that the first term is always positive.

For $M = 0$, $m \leq j$, and (3.21) is clearly positive, and for $M = 1/2$, $m \leq j + 1$, and the numerator of the first term in (3.21) is bounded by $j + 1 + k > 0$. For $M \geq 1$, we observe that the possible values of $m$ are bounded by $m \leq j + M + 1/2$, and it is therefore useful to consider the two cases (I) $j + M + 1/2 < 0$, and (II) $j + M + 1/2 \geq 0$ separately. In case (II), (3.21) is minimal for $m = 0$, and we can rewrite the numerator of the first term on the right-hand side as

\[ -j^2 - j + 2Mk = -j(j + k + 1) + k(2M + j). \]  

The first term is strictly positive for $0 > j > -k - 1$, and the second term is non-negative (as for $M \geq 1$, $2M \geq M + 1/2$).

In case (I), (3.21) is minimal for $m = j + M + 1/2$, and then the numerator of the first term on the right-hand side simplifies to

\[ -j^2 - j + (M + j + 1/2)^2 + 2Mk = 2M(j + k + 1) + (M - 1/2)^2. \]  

This is also strictly positive, and we have thus shown that the states of the form (3.16) are linearly independent.

We can then follow the same argument as in the Lemma of the previous section to show that the states of the from (3.16) span the whole Fock space. This completes the proof of the Lemma.

Let us call a state spurious if it is a linear combination of states of the from (3.16) for which $\lambda \neq 0$ or $\varepsilon \neq 0$. Because of the Lemma, every physical state $\phi$ can be written
as a spurious state $\phi_s$ plus a linear combination of states of the form (3.16) with $\lambda = 0$ and $\varepsilon = 0$, i.e.

$$\phi = \phi_s + \chi. \quad (3.24)$$

For $c = 15$, following the argument of Goddard and Thorn [16], $\phi_s$ and $\chi$ are separately physical states, and $\phi_s$ is therefore null. Next we want to prove the

**Lemma.** Let $0 > j > -k - 1$. If $\chi$ is a physical state of the form (3.16) with $\lambda = 0$ and $\varepsilon = 0$, then $\chi \in F$.

**Proof.** For fixed $|f\rangle$, let us denote by $H_f$ the Fock space that is generated by the action of $J^3$ and $\psi^3$ from $|f\rangle$, and by $H^\text{svir}_f$ the Fock space that is generated by the action of $L^3$ and $G^3$ from $|f\rangle$. Because of (3.17) and (3.18), it is clear that $H^\text{svir}_f$ is a subspace of $H_f$. On the other hand $H^\text{svir}_f$ is the Verma module for the $N = 1$ superconformal algebra with $c = 3/2$ whose ground state has conformal weight $-m^2/2k$ where $m$ is the $J^3_0$ eigenvalue of $|f\rangle$. It then follows from the Kac-determinant formula that $H^\text{svir}_f$ does not contain any null states unless $m = 0$ [21]. Provided that $m \neq 0$, it is easy to see that $H^\text{svir}_f$ and $H_f$ contain the same number of states at each grade, and this implies that $H_f = H^\text{svir}_f$. Since $H^\text{svir}_f$ does not contain any null states (with respect to the superconformal algebra), it then follows that $H_f$ does not contain any physical states other than possibly $|f\rangle$ itself. It therefore remains to check whether there are physical states with $m = 0$, and if so whether they are in $F$.

The physical states at fixed grade $N$ form a representation under the zero mode $su(1,1)$ algebra since $J^+_0\phi$ and $J^3_0\phi$ are physical states if $\phi$ is. If the ground states form a representation $D^-_J$ of the $su(1,1)$ zero mode algebra, then the possible representations at grade $N$ are of the type $D^-_{J+}$, where $J$ is at most $j+N+1/2$. Because of the restriction on $j$, the mass-shell condition implies

$$N \leq \frac{1}{2} + \frac{j(j+1)}{2k} < \frac{1}{2} - \frac{j}{2} < \frac{1}{2} + \frac{k+1}{2}. \quad (3.25)$$

For $0 > j > -1$, the first inequality implies $N = 0$, and then $m \leq j < 0$. For $j = -1$, $N = 0$ and $N = 1/2$ are allowed; all of the corresponding physical states satisfy $m < 0$, except the state $|A.1\rangle$ in appendix A for which $m = 0$ (and $J = 0$, $h^0 = 0$). This state is however clearly in $F$. For $-1 > j > -2$ it follows that $N \leq 1$, since if $N \geq 3/2$, then $k > 1$ by the last bound in (3.25). Hence $j(j+1)/2k < 1$, but this contradicts the first inequality in (3.25). Thus $m \leq j+1 < 0$.

Finally, for $j \leq -2$, then (3.25) implies directly that $N+j+1/2 < 1+j/2 \leq 0$, and thus again $m \leq N+j+1/2 < 0$. This proves the claim.

We are now ready to prove

**Theorem.** For $c = 15$ and $0 > j > -k - 1$, every physical state $\phi$ differs by a spurious physical state from a state in $F$. Consequently, the norm of every physical state is non-negative.
Proof. This follows directly from the previous two lemmas, and the fact that the coset $su(1,1)/u(1)$ is unitary if $0 > j > -\tilde{k}$, as can be established by a slight modification of the argument in [17]. (The Kac-determinant of the full Fock space is the product of the expression [17] (4.8) with $k$ replaced by $\tilde{k}$ and the fermionic contributions. Apart from the fermionic part (which is manifestly positive), the Kac-determinant of the coset model is then given by [17] (4.9), where all $k$’s are replaced by $\tilde{k}$’s, except for the $k$ in the factor $k^{-r_2(N)}$. This determinant is then positive for $0 > j > -\tilde{k}$ by the same arguments as in [17].)

Again, it is easy to see how to generalise the above argument to other backgrounds (including the flat case). It is also clear by continuity, as in the bosonic case, that the representations with $0 > j \geq -k - 1$ do not contain any physical states of negative norm. Furthermore, there always exist states of negative norm if this condition is violated; we give examples in appendix A. The analysis for the representations $D_j^+$ is similar, and we find that all physical states have positive norm provided that $0 > j \geq -k - 1$.

4 The relation to the conformal field theory bound

We now return to the conjectured relation between type IIB string theory on $AdS_3 \times S^3 \times M^4$ (where $M^4$ is either K3 or $T^4$) and two-dimensional conformal field theory whose target is a symmetric product of a number of copies of $M^4$ [1]. This relation can be understood by considering the string theory in the background of $Q_1$ D-strings and $Q_5$ D5-branes. The theory on the world-volume of the D-strings is a conformally-invariant sigma-model that has, in a certain limit, target space $Sym_Q M^4$, where $Q = Q_1 Q_5$ for $M^4 = T^4$ and $Q = Q_1 Q_5 + 1$ for $M^4 = K3$ [22].

By S-duality, the background of the D1-D5 system is related to a conventional IIB string theory on $SU(1,1) \times SU(2) \times M^4$, where the level of the two WZW models is the same so that the total central charge of the six-dimensional part of the theory is indeed

$$c = c_{su(2)}(k) + c_{su(1,1)}(k) = \frac{3}{2} \left( \frac{3k - 2}{k} + \frac{3k + 2}{k} \right) = 9. \quad (4.26)$$

According to [3, 23, 24], the level of the $SU(1,1)$ and the $SU(2)$ WZW model is $Q_5$, and one may therefore think that $k = Q_5/2$ (taking into account that $k$ is half-integral, and $Q_5$ integral). However, this assignment is somewhat delicate, as the $Q_1$ D-strings are mapped to $Q_1$ fundamental strings in the dual theory, and this interpretation is therefore only simple for $Q_1 = 1$, in which case $k = Q/2$. Nevertheless, for more general $Q_1 > 1$ one should anticipate that the bound on the allowed values of the $U(1)$ charge will be $Q_1$ times what it is for $Q_1 = 1$, and this means that the effective level is again $k = Q/2$. This is the assumption we shall make in extending our analysis from $Q_1 = 1$ to $Q_1 > 1$.

The superconformal field theory on $Sym_Q M^4$ has a $(4,4)$ superconformal algebra with $c = 6Q$ [25]. The level of the $su(2)$ subalgebra is then $\ell = Q/2$ (in our conventions where the level is half-integral) [26], and the possible values of the $U(1)$ charge of primary
su(2) highest weight fields are therefore \( m = 0, 1/2, \ldots, \ell \). The primary fields that are chiral with respect to a \( N = 2 \) subalgebra (and that correspond to the BPS states of the dual string theory) satisfy in addition \( h = m \). There are therefore only finitely many (namely \( 2\ell + 1 = Q + 1 \)) different chiral primary fields, and this must thus be reflected in the dual string theory; this is the content of the “stringy exclusion principle” of Maldacena and Strominger [5].

In terms of the string theory on \( SU(1,1) \times SU(2) \times M^4 \), the different values of \( h(= m) \) are to be identified with the different values of \(-j\), the eigenvalue of \( J_0^3 \) of a \( su(1,1) \) highest weight in the \( D_j^+ \) representation [5]. The above bound then transforms into the condition that \( 0 \geq j \geq -\ell \). As is explained in [3] (see [28]), a stability analysis on AdS_3 suggests that \( j \leq -1/2 \). The above bound (together with the stability bound) therefore gives \( Q/2 \geq -j \geq 1/2 \). For the case of K3, \( Q/2 = k + 1/2 \), and we therefore obtain precisely the range of allowed representations \( k + 1 > -j > 0 \) which we have shown to be ghost-free. In the case of \( T^4 \), however, we obtain \( k > -j > 0 \), which is a more restrictive condition, corresponding to a proper subset of the ghost-free representations. A priori we have no grounds for expecting the two restrictions to coincide except in the limit of large \( Q \). It is encouraging that this is indeed what occurs, and particularly interesting that the bounds coincide exactly for the case of K3.

5 Conclusions

In this paper we have analysed the no-ghost theorem for string theory on \( SU(1,1) \). We have filled the gap in the proof of [12] in the bosonic case, and extended the argument to the fermionic case. We have also shown that the restriction on the spin (in terms of the level) that is necessary to obtain a ghost-free spectrum corresponds to the stringy exclusion principle of Maldacena and Strominger [5]. Among other things, we regard this as evidence that the \( SU(1,1) \) model with the restriction on the set of allowed representations defines a consistent string theory.

There are many interesting questions which need to be addressed. In order to get a consistent string theory the amplitudes must be crossing symmetric, and it is not clear whether this can be achieved with the restricted set of representations. This is a rather difficult problem as the fusion rules of the \( SU(1,1) \) WZW model are not well understood (see however recent progress on an understanding of the fusion rules of the \( SU(2) \) WZW model at fractional level which is technically similar [29]). Furthermore, in order to get a modular invariant theory, additional representations (that correspond to winding states along the compact direction in \( SU(1,1) \)) presumably have to be considered [11, 13], for which the \( L_0 \) spectrum is not bounded from below. Finally, the set of ghost-free representations contains a continuum, the so-called continuous representations of the global \( SU(1,1) \), and thus problems similar to those faced in Liouville theory [30] arise.

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5 Here we have taken into account that in the usual conventions, the \( U(1) \) generator of the \( N = 2 \) subalgebra is twice the \( T^3 \) generator of the \( su(2) \) algebra of the \( N = 4 \) algebra [26].

6 The only other ghost-free representation occurs for \(-j = k + 1\) but differs from the others in that it contains null vectors; it presumably does not occur in a modular-invariant partition function [11].
Nevertheless, it is quite suggestive that the representations that are allowed by the no-ghost-theorem are those representations whose Verma module does not contain any null-vectors \[13\] and this may ultimately be sufficient to prove that the restricted representations define a consistent interacting theory. One may also hope that the structure of the dual superconformal field theory could shed light on some of these questions.

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### Appendix

#### A Some illustrative examples for the supersymmetric case

Let us determine the norms of the physical states at the various grades. At every grade, we shall look for physical highest weight states that generate the representation \(D^\pm_j\).

**Grade 1/2**

\(\mathbf{J} = j + 1\): There is one physical state

\[
P_{1/2} := \psi^+_{1/2}|j, j\rangle \quad \|P_{1/2}\|^2 = 2k.
\] (A.1)

This state has positive norm.

\(\mathbf{J} = j\): A physical state only exists for \(j = -1\), in which case it is given as

\[
P_{1/2} := \left(\psi^3_{-1/2} + \frac{1}{2}\psi^+_{-1/2}J_0^-\right)|-1, -1\rangle.
\] (A.2)

The norm of this state is 0.

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\[7\]This is the case for the continuous representations, and for the discrete representations \(D^-_j\) if we impose the strict inequality \(0 > j > -k\) in the bosonic case (and \(0 > j > -k - 1\) in the fermionic case) and similarly for \(D^+_j\); strictly speaking the no-ghost-theorem allows also \(j \geq -k\) and \(j \geq -k - 1\), respectively.
J = j − 1: There is one physical state of the form
\[ P_{\frac{j}{2},-1} := \left( \psi_{-1/2}^− - \frac{1}{j} \psi_{-1/2}^3 J_0^- + \frac{1}{2 j (2 j - 1)} \psi_{-1/2}^+ J_0^− J_0^− \right) |j, j⟩, \]  
and its norm square is
\[ ||P_{\frac{j}{2},-1}||^2 = 2 k \left( \frac{2 j + 1}{2 j - 1} \right). \]  
This is positive as the mass-shell condition implies j ≤ −1.

Grade 1

J = j + 1: There is one physical state of the form
\[ P_{1,1} := \left( J_{-1}^+ + \frac{1}{j + 1} \psi_{-1/2}^+ \psi_{-1/2}^3 \right) |j, j⟩, \]  
whose norm square is
\[ ||P_{1,1}||^2 = 2 \frac{(k + j + 1)(j(j + 1) - k)}{(j + 1)^2}. \]  
The second bracket in the numerator is non-negative because of the mass-shell condition (3.12) at grade N = 1, and the expression is therefore non-negative if and only if j ≥ −k − 1 holds.

J = j: There is one physical state of the form
\[ P_{1,0} := \left( \psi_{-1/2}^+ \psi_{-1/2}^− - \frac{1}{j} \psi_{-1/2}^+ \psi_{-1/2}^3 J_0^− \right) |j, j⟩, \]  
and its norm square is
\[ ||P_{1,0}||^2 = 4 k^2 \frac{j + 1}{j}. \]  
This is positive as the mass-shell condition implies j < −1.

J = j − 1: There is one physical state whose norm square is
\[ ||P_{1,-1}||^2 = -2 \frac{(2 j + 1)(k - j)(j - j(j + 1))}{j^2 (2 j - 1)}. \]  
Because of the mass shell condition (3.12) with N = 1, the last bracket in the numerator is non-positive and j ≤ −3/2. Thus the norm is non-negative.
\textbf{Grade 3/2}

\(J = j + 2\): There is one physical state of the form

\[P_{\frac{3}{2},2} = \psi^+_{-1/2} J^+_{-1} |j,j\rangle, \quad (A.10)\]

whose norm square is

\[||P_{\frac{3}{2},2}||^2 = 4k(j + k + 1). \quad (A.11)\]

This is non-negative if \(j \geq -k - 1\).

\(J = j + 1\): There is one physical state whose norm square is

\[||P_{\frac{3}{2},1}||^2 = -2jk(j + 2)(2k - j(j + 1)). \quad (A.12)\]

This is non-negative since \(j \leq -2\) (for \(j = -3/2\) only \(N = 1\) is possible), and \(2k - j(j + 1) \leq 0\) because of (3.12) with \(N = 3/2\).

\(J = j\): There is a two-dimensional space of physical states. The determinant of the \(2 \times 2\) inner product matrix is

\[
\text{Det} = -64k^2 j (2j + 3)(2j - 1)(j + 1)(j + k + 1) \times \left(2k - j(j + 1)\right)(k - j) \frac{k(3k + 2) + j(j + 2)(j + 1)(j - 1)}{(3k + j(j + 1))^2},
\]

which is manifestly positive. As the two eigenvalues are positive for large \(j\) and \(k\), the only negative norm states can occur if the determinant vanishes, which can happen for \(k = j(j + 1)/2\) and \(k = -1 - j\). In the former case, the trace of the inner product matrix is then

\[
\text{Trace}(k = j(j + 1)/2) = \frac{2}{25}(2j + 3)(j + 2)(j + 1)(j - 1)(2j - 1)j(7j(j + 1) - 4), \quad (A.13)
\]

which is non-negative for \(j \leq -2\), and in the second case the trace is

\[
\text{Trace}(k = -j - 1) = 4(16j(j + 1) + 5)\frac{j(2j - 1)(j + 2)(j + 1)(j^2 + 1)}{(j - 3)^2}, \quad (A.14)
\]

which is also non-negative for \(j \leq -2\). This demonstrates that there are no negative norm states in this case.

\textbf{References}

[1] J. Maldacena, \textit{The large N limit of superconformal field theories and supergravity}, \url{hep-th/9711200}.

[2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, \textit{Gauge theory correlators from non-critical string theory}, \url{hep-th/9802109}.
[3] E. Witten, *Anti-de Sitter space and holography*, hep-th/9802150.

[4] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, hep-th/9803131.

[5] J. Maldacena, A. Strominger, AdS$_3$ black holes and a stringy exclusion principle, hep-th/9804083.

[6] J. Balog, L. O’Raifeartaigh, P. Forgács, A. Wipf, *Consistency of string propagation on curved spacetimes. An SU(1, 1) based counterexample*, Nucl. Phys. B 325, 225 (1989).

[7] P.M.S. Petropoulos, *Comments on SU(1, 1) string theory*, Phys. Lett. B 236, 151 (1990).

[8] N. Mohammedi, *On the unitarity of string propagation on SU(1, 1)*, Int. Journ. Mod. Phys. A5, 3201 (1990).

[9] S. Hwang, *No ghost theorem for SU(1, 1) string theories*, Nucl. Phys. B 354, 100 (1991).

[10] M. Henningson, S. Hwang, *The unitarity of su(1, 1) fermionic strings*, Phys. Lett. B 258, 341 (1991).

[11] M. Henningson, S. Hwang, P. Roberts *Modular invariance of su(1, 1) strings*, Phys. Lett. B 267, 350 (1991).

[12] S. Hwang, *Cosets as gauge slices in SU(1, 1) strings*, Phys. Lett. B 276, 451 (1992); hep-th/9110039.

[13] S. Hwang, P. Roberts, *Interaction and modular invariance of strings on curved manifolds*, hep-th/9211073.

[14] I. Bars, *Ghost-Free Spectrum of a Quantum String in SL(2,R) Curved Spacetime*, Phys. Rev. D 53, 3308 (1996); hep-th/9503205. I. Bars, *Solution of the SL(2, R) string in curved space-time*, hep-th/9511187.

[15] Y. Satoh, *Ghost-free and modular invariant spectra of a string in SL(2, R) and three dimensional black hole geometry*, Nucl. Phys. B 513, 213 (1998); hep-th/9705208.

[16] P. Goddard, C.B. Thorn, *Compatibility of the dual pomeron with unitarity and the absence of ghosts in the dual resonance model*, Phys. Lett. B 40, 235 (1972).

[17] L.J. Dixon, M.E. Peskin, J. Lykken, *N = 2 superconformal symmetry and SO(2, 1) current algebra*, Nucl. Phys. B 325, 329 (1989).

[18] R.C. Brower, *Spectrum-generating algebra and no-ghost theorem for the dual model*, Phys. Rev. D 6, 1655 (1972).
[19] D. Gepner, E. Witten, *String theory on group manifolds*, Nucl. Phys. B 278, 493 (1986).

[20] G.T. Horowitz, H. Ooguri, *Spectrum of large $N$ gauge theory from supergravity*, hep-th/9802110.

[21] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer (1997).

[22] M. Bershadsky, V. Sadov, C. Vafa, *D-Branes and Topological Field Theories*, Nucl. Phys. B 463, 420 (1996); hep-th/9511222.

[23] H. Ooguri, C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, Nucl. Phys. B 463, 55 (1996); hep-th/9511164.

[24] M. Cvetič, A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, Phys. Lett. B 366, 95 (1996); hep-th/9510097;
A. Tseytlin, *Extreme dyonic black holes in string theory*, Mod. Phys. Lett. A 11, 689 (1996), hep-th/9601177.

[25] A. Strominger, C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. B 379, 99 (1996); hep-th/9601029.

[26] A. Schwimmer, N. Seiberg, *Comments on the $N = 2, 3, 4$ superconformal algebras in two dimensions*, Phys. Lett. B 184, 191 (1987).

[27] W. Lerche, C. Vafa, N.P. Warner, *Chiral rings in $N = 2$ superconformal algebras*, Nucl. Phys. B 324, 427 (1989).

[28] P. Breitenlohner, D.Z. Freedman, *Positive energy in anti-de-Sitter backgrounds and gauged extended supergravity*, Phys. Lett. B 115, 197 (1982); *Stability in gauged extended supergravity*, Ann. Phys. 144, 249 (1982).

[29] M.R. Gaberdiel, M.A. Walton, *in preparation*.

[30] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. B 477, 577 (1996); hep-th/9506136.