Formation of electrostatic solitary and periodic waves in dusty plasmas in the light of Voyager 1 and 2 spacecraft and Freja satellite observations

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Abstract
Motivated by the observations of Voyager 1 and 2 spacecraft and Freja satellite observations in Saturn’s magnetosphere, the formation of dust-acoustic (DA) localized and periodic waves in a complex plasma having superthermal electrons and ions are reported. In this regard, a modified Kadomtsev–Petviashvili (mKP) equation is derived by employing the weak turbulence theory for studying the characteristics of the nonlinear dust-acoustic waves (DAWs) in the model under consideration. The localized and periodic wave solutions to the mKP equation are derived using ansatz method in terms of Jacobi elliptic functions (JEFs). It is reported that the phase velocity of the DAWs in the Saturn’s magnetosphere is lower for kappa distributed ions and electrons by comparison with regions of space plasmas where the electrons and ions follow the Maxwellian distribution. The conditions for the existence of both localized and periodic waves are also presented. Estimates are also given of the spatial scales over which the dust-acoustic solitary/periodic structures form in Saturn’s magnetosphere.

Keywords
Dusty plasmas, dust-acoustic solitary and periodic structures, Jacobi elliptic function expansion method, modified Kadomtsev–Petviashvili equation

Introduction
Dusty/complex plasmas have received a lot of attention on account of their omnipresence in the universe over the past few decades. It has been revealed that such types of plasmas are abundant in different regions of space, in astrophysical environments, for example, as interstellar molecular clouds, solar nebulae, the Earth’s ionosphere, planetary rings, and comets, etc.1–5 and also in laboratory (e.g., fusion devices, plasma devices, solar cells, semiconductor chips etc.).

A dusty plasma consists of normal electrons and ions with the addition of some large size positive or negative dust particles.5 According to several experimental and theoretical studies5–10 the interaction of charged dust with electromagnetic and gravitational forces plays a momentous role in generating new low-frequency waves like dust ion-acoustic waves (DIAWs), dust lattice waves (DLWs), DAWs, DA cyclotron waves (DACWs), and dust drift mode etc.

Rao et al.11 studied theoretically linear/nonlinear modes of the DAWs and Barkan et al.12 provided experimental verification of DAWs. These waves are caused by the dynamic dust mass while electrons and ions provide the restoring
force. Since then, many studies related to the complex plasmas have been carried out to understand the basic characteristics of these localized structures in laboratory and space plasmas. During the last few years, DA solitary waves (DASWs) in complex plasmas have been examined by many researchers.

To investigate the formation of nonlinear structures in a dusty plasma, massless charged species are generally presumed to have a Maxwellian distribution. However, satellite missions have indicated that there are many regions in space plasmas where charged species deviate from Maxwellian behavior. These superthermal (kappa) species have been found in the solar wind and planetary magnetospheres, interstellar medium, auroral zone plasma, and also in the terrestrial magnetosheath. Vasyliunas used the superthermal velocity distribution for the first time as an empirical formula to fit data from the spacecraft OGO 1 and OGO 3 in the terrestrial magnetosphere. Since then, the non-Maxwellian kappa distribution follows the power law has been used to fit data from spacecraft in Saturn, solar wind, Earth’s magnetospheric plasma and Jupiter. For larger spectral index values (i.e., $\kappa \to \infty$), kappa distribution can be recovered the thermal (Maxwellian) distribution. The data obtained by Voyager 1 and 2 spacecraft from Saturn’s magnetosphere revealed that ions follow power law at high energies. Krimigis et al. used kappa distributions to fit data observations for ions in the Saturn magnetosphere with spectral index values ranging from 6 to 8. In addition, the Cassini team collected data from spacecraft orbiting Saturn and covering distances ranging from $5.4 - 18 R_s$, where $R_s$ is the radius of Saturn ($R_s \approx 60268 km$). The observed data are well fitted by superthermal electrons in Saturn’s magnetosphere.

In the laboratory and space plasmas, one-dimensional (1D) non-planar geometry is unsuitable because of the observed geometry aberrations in the wave. Most importantly, the waves detected in space and in the laboratory are not confined to one-dimension (1D). A completely 1D-model, according to Franz et al. cannot explain observable patterns inside the auroral zone, particularly at higher polar altitudes. The transverse perturbations are supported by higher dimensions. Introducing the transverse perturbation results in the generation of an anisotropy in the system, which has an impact on wave propagation. Kadomtsev and Petviashvili proposed a universal two-dimensional wave equation known as Kadomtsev Petviashvili (KP) equation after their names to investigate the stability of solitary waves under transverse perturbations. The nonlinearity arising in KP equation is in quadratic form. The reductive perturbation technique (RPT) is always used to derive small amplitude wave equations like Korteweg-de Vries (KdV), KP and modified mKP, etc. However, when quadratic nonlinearity disappears, the amplitude of a solitary wave becomes infinite rendering the equations governed by quadratic nonlinearity invalid. Then, at the critical condition, we proceed to the next order of nonlinearity and derive the mKP equation.

In dusty plasmas, two-dimensional nonlinear equations have been investigated in a variety of physical situations of interest. Gill et al. studied the KP equation in a dusty plasma and obtained compressive (positive) and rarefactive (negative) solitons. Pakzad derived KP equation in a complex plasma having two-types of ions with different temperatures and found that at the critical condition, the nonlinearity coefficient vanishes and, therefore, no soliton solutions of the KP equation can be obtained. Duan analyzed the KP equation in a warm complex plasma and found that the system supported only the formation of negative solitons. Lin and Duan studied KP, mKP, and coupled KP (CKP) equations and reported that smaller dust grains cover more distances than larger dust grains owing to their higher speed. Dorranian and Sabetkar derived both KP and mKP equations for nonlinear DASWs in a complex plasma having two different types of nonthermal ions.

In this work, we shall investigate the behavior of DA nonlinear structures including DASWs and DA periodic waves (DAPWs) in a complex plasma having inertialess superthermal species (electrons and ions) by deriving the mKP equation which involves cubic nonlinearity. The critical condition under which mKP equation is valid will also be determined. In the limit $\kappa \to \infty$, the kappa distribution will be shown to reduce to the Maxwellian/thermal case. The following is an overview of how the manuscript is organized: In The physical model and derivation of the evolution equation, for the dusty plasma model, we present a basic set of fluid equations. Also, in The physical model and derivation of the evolution equation, the mKP equation is derived using the RPT. In Solution of the MKP equation, the ansatz method is employed to find the periodic and localized (solitary) wave solutions in the form of Jacobi elliptic functions (JEFs). Results and discussion deals with the numerical study of non-Maxwellian dusty plasma using the data from Saturn’s magnetosphere. In Conclusion, main findings of the work are succinctly written.

The physical model and derivation of the evolution equation

Here, we proceed to investigate the propagation of the DAWs in the $x - y$ plane in a homogeneous, unmagnetized complex plasma composed of inertial cold dust particles and superthermal massless species (ions and electrons). For this plasma model, the equilibrium condition is written in the form: $n_i = Z_d n_{i0} + n_{e0}$, where $n_{i0}$ represents the equilibrium number density of the plasma species “$s$” ($s = d, e$, and $i$ for the dust grains, electrons, ions, respectively), and $Z_d$ denotes the number
of charges. For describing and understanding the dynamics of the DAWs, the normalized set of fluid equations of the present model are introduced as[21]

\[
\frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} + \frac{\partial (n_d v_d)}{\partial y} = 0,
\]

(1)

\[
\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} + v_d \frac{\partial u_d}{\partial y} = \frac{\partial \Phi}{\partial x},
\]

(2)

\[
\frac{\partial v_d}{\partial t} + u_d \frac{\partial v_d}{\partial x} + v_d \frac{\partial v_d}{\partial y} = \frac{\partial \Phi}{\partial y},
\]

(3)

and

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = n_d + \Gamma n_e - \Gamma_1 n_i,
\]

(4)

where \( \Gamma = P/(1 - P) \), \( \Gamma_1 = 1/(1 - P) \) and \( P = n_{e\phi}/n_{\phi} \) denotes the electron concentration. The normalizations for above dusty plasma model are as follows: All number densities are normalized by the respective equilibrium values such as \( n_{d0} \rightarrow n_d \), \( n_{e0}/n_{\phi} \rightarrow n_e \) and \( n_{i0}/n_{\phi} \rightarrow n_i \), the dust velocities are scaled by the DA speed \( (u_{d0}v_{d0})/C_{sd} \rightarrow (u_d,v_d) \), the electrostatic wave potential \( \Phi \) is scaled by \( e\Phi/T_i \rightarrow \Phi \), the space and time coordinates are scaled by \( \lambda_{Dd} \) and \( \omega_{pd} \), respectively. The dust acoustic speed \( C_{sd} = (Z_dT_i/m_d)^{1/2} \), Debye length \( \lambda_{Dd} = (T_i/4\pi Z_d n_{d0} e^2)^{1/2} \), and dust plasma frequency \( \omega_{pd}^{-1} = (m_d/4\pi Z_d^2 n_{d0} e^2)^{1/2} \).

The ions and electrons total normalized number densities according to the kappa distribution read[41]

\[
n_i = \left(1 + \frac{\Phi}{\kappa_i}\right)^{-b_i} = 1 - A_i \Phi + B_i \Phi^2 - C_i \Phi^3,
\]

(5)

\[
n_e = \left(1 - \frac{\Phi}{\kappa_e}\right)^{-b_e} = 1 + \delta A_e \Phi + \delta^2 B_e \Phi^2 + \delta^3 C_e \Phi^3,
\]

(6)

with \( A_{i,e} = b_{i,e} a_{i,e} \), \( b_{i,e} = b_{i,e} (b_{i,e} + 1)/(2a_{i,e}^2) \), and \( C_{i,e} = b_{i,e} (b_{i,e} + 1)/(6a_{i,e}^3) \), where \( \kappa_i \) and \( \kappa_e \) represent the spectral indices of the ions and electrons, respectively, \( \delta = T_i/T_i \) indicates the temperature ratio, \( a_{i,e} = (\kappa_{i,e} - 3/2) \), \( b_{i,e} = (\kappa_{i,e} - 1/2) \), and for \( (\kappa_i,\kappa_e) \rightarrow \infty \), the superthermal distribution reduces to the well-known thermal one which leads to \( (A_{i,e},B_{i,e},C_{i,e}) = (1,1,1/6) \).

The RPT is applied for deriving the mKP equation. Based on this technique, the space and time coordinates are stretched as[21,39]

\[
\zeta = \epsilon(x - \lambda t), \eta = \epsilon^2 y \text{ and } \tau = \epsilon^3 t,
\]

(7)

where \( \lambda \) is the normalized phase velocity of the DAWs and \( \epsilon \) is the real and smallness parameter \( (0 < \epsilon < 1) \) that defines the weakness of dispersion. The perturbed quantities \( (n_d,u_d,v_d,\Phi) \) are expanded as[21]

\[
n_d = 1 + \epsilon^2 n_{d1} + \epsilon^4 n_{d2} + \epsilon^6 n_{d3} + \cdots,
\]

\[
u_d = \epsilon u_{d1} + \epsilon^3 u_{d2} + \epsilon^5 u_{d3} + \cdots,
\]

\[
\Phi = \epsilon \Phi^{(1)} + \epsilon^3 \Phi^{(2)} + \epsilon^5 \Phi^{(3)} + \cdots.
\]

(8)

The value of epsilon \( \epsilon \) can be taken to be larger than one but then we enter the regime of strong turbulent theory[42–44] which is beyond the scope of this work. Substituting equations (5)–(8) in the normalized set of fluid equations, that is, Equations (1)–(4), the lowest order-terms of \( \epsilon \) give us

\[
u_{d1} = \lambda n_{d1} = -\frac{\Phi^{(1)}}{\lambda},
\]

\[
n_{d1} = -(\Gamma_0 A_i + \Gamma_1 A_e) \Phi^{(1)}.
\]

(9)
Solving system (9), the value of $\lambda$ is obtained as

$$\lambda = \sqrt{\frac{1}{(\Gamma \delta A_i + \Gamma_1 A_e)}}$$ (10)

The expressions obtained by Xue and Lin and Duan for the Maxwellian and non-Maxwellian limits can be recovered from expression of the phase velocity of the DAW given in equation (10).

Going to the next-higher orders of $\epsilon$ and collecting them from equations (1)–(4), we obtain

$$-\lambda \frac{\partial n_{d2}}{\partial \xi} + \frac{\partial u_{d2}}{\partial \xi} = -\frac{\partial n_{d1} u_{d1}}{\partial \xi},$$

$$-\lambda \frac{\partial u_{d2}}{\partial \xi} - \frac{\partial \Phi^{(2)}}{\partial \xi} = -u_{d1} \frac{\partial u_{d1}}{\partial \xi},$$

$$n_{d2} + \Gamma_2 \Phi^{(2)} = -\Gamma_3 (\Phi^{(1)})^2,$$ (11)

where $\Gamma_2 = (\Gamma \delta A_i + \Gamma_1 A_e)$ and $\Gamma_3 = (\Gamma \delta^2 B_i - \Gamma_1 B_e)$. After solving system (11) simultaneously, we have

$$\lambda^4 = \frac{-3}{2\Gamma_3}.$$ (12)

In multicomponent plasmas, this is the critical condition at which the quadratic nonlinearity ceases to exist. Thus, in order to solve the system, we should go to the next-higher order of nonlinearity. Consequently, the next-orders of $\epsilon$ give us

$$-\lambda \frac{\partial n_{d3}}{\partial \xi} + \frac{\partial u_{d3}}{\partial \xi} = -\frac{\partial n_{d1} u_{d1}}{\partial \xi} - \frac{\partial n_{d2} u_{d2}}{\partial \xi} - \frac{\partial u_{d1} u_{d1}}{\partial \xi},$$

$$-\lambda \frac{\partial u_{d3}}{\partial \xi} - \frac{\partial \Phi^{(3)}}{\partial \xi} = -u_{d1} \frac{\partial u_{d1}}{\partial \xi},$$

$$n_{d3} + \Gamma_3 \Phi^{(3)} = \frac{\partial \Phi^{(1)}}{\partial \xi^2} - 2\Gamma_2 \Phi^{(1)} \Phi^{(2)} - (C_i + \delta^3 \Gamma_1 C_e) (\Phi^{(1)})^3.$$ (13)

Solving system (13), we finally obtain the required mKP equation

$$\frac{\partial}{\partial \xi} \left[ \frac{\partial \Phi^{(1)}}{\partial \tau} + D \left( \Phi^{(1)} \right)^2 \frac{\partial \Phi^{(1)}}{\partial \xi} + E \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} \right] + F \frac{\partial^2 \Phi^{(1)}}{\partial \eta^2} = 0,$$ (14)

where $D$, $E$, and $F$ are, respectively, the coefficients of the nonlinear, dispersive, and diffraction terms

$$D = \frac{3}{2 \lambda^2} \left[ (1 - \lambda^4 \Gamma_3) - \lambda^6 (\Gamma \delta^3 C_i + \Gamma_1 C_e) \right],$$

$$E = \frac{\lambda^3}{2} \& F = \frac{\lambda}{2}.$$ (15)

Solution of the mKP equation

Several techniques have been suggested to find some exact solutions to different types of differential and integral equations such as homogeneous balance method, the tangent hyperbolic method, sine-cosine method, the trial function method, fractal solitary theory and the nonlinear transformation method and many other effective analytical and numerical techniques. All these methods are employed to calculate the solitary/shock solutions. The advantage of using the ansatz method or Jacobi elliptic function expansion method (JEFM) is that one can also obtain periodic solutions of the nonlinear equation besides the solitary and shock wave solutions. Recently, Tian et al. have carried out research to
understand the basic characteristics of nonlinear localized structures in laboratory and space plasmas using the fractal space which holds a lot of potential for further investigations.\textsuperscript{68,69}

**Analytic solutions to the evolution equation using JFEM**

The solution of equation (14) is obtained by going into a co-moving frame $\chi = k(l\xi + w\eta - ut)$ such that

$$\Phi^{(1)}(x,t) = \Phi^{(1)}(\chi),$$

where $k$ is the wave number, $u$ is the velocity of nonlinear wave, while $l$ and $w$ are the direction cosines, respectively. Using equation (16) in equation (14), the following ODE is obtained

$$kl \frac{\partial}{\partial \chi} \left[ -kU \frac{\partial \Phi^{(1)}}{\partial \chi} + D(kl\Phi^{(1)}) \frac{\partial^3 \Phi^{(1)}}{\partial \chi^3} + E(kl) \frac{\partial^3 \Phi^{(1)}}{\partial \chi^3} \right] + F(kw)^2 \frac{d^2 \Phi^{(1)}}{d\chi^2} = 0.$$  

(17)

Further simplification of equation (17) yields the following expression

$$(Fw^2 - IU)\Phi^{(1)} + \frac{DI^2}{3} [\Phi^{(1)}]^3 + Ek^2l^2 \frac{d^2 \Phi^{(1)}}{d\chi^2} = 0.$$  

(18)

To find a solution to equation (18), we suppose the solution in the following ansatz form\textsuperscript{70}

$$\Phi^{(1)}(\chi) = \sum_{j=0}^{n} a_j cn[j\chi].$$  

(19)

The value of $j$ can be obtained with the help of harmonic balance method (the balance between the higher-order dispersion and nonlinearity of Eq. (18), that is, $3n = n + 2 \Rightarrow n = 1$) which yields (see appendix for details)

$$\Phi^{(1)} = N cn[\chi],$$  

(20)

where $N \equiv a_1$.

Inserting solution (20) into equation (18), we finally get

$$N = -\sqrt{\frac{6(U - Fw^2)m^2}{DL^2(2m^2 - 1)}},$$  

(21)

and

$$k = \sqrt{\frac{(U - Fw^2)}{EL^2(2m^2 - 1)}}.$$  

(23)

where the modulus $m$ has the range $0 < m < 1$ for the periodic waves (cnoidal waves) while for localized waves (solitary waves), $m \rightarrow 1$. Note that for $0 < m \leq 0.7$, we can get periodic structures when $U < \frac{Fw^2}{l}$, however, for $0.7 < m < 0.99$, we can get periodic structures for $U > \frac{Fw^2}{l}$. However, for $m \rightarrow 1$, the periodic solution (20) reduces to the solitary wave solution

$$\Phi^{(1)} = N \sec h[j\chi],$$  

(22)

with

$$N = -\sqrt{\frac{6 \left( U - \frac{Fw^2}{l} \right)}{DL}},$$  

(23)

and

$$k = \sqrt{\frac{(U - \frac{Fw^2}{l})}{EL^2}}.$$  

(23)
For the amplitude $N$, it is seen that the nonlinear structures exist only when $U > \frac{E_{w}^{2}}{D}$ and $D > 0$. Unlike the case of periodic structures, here, no solitary structures can be obtained for the condition $U < F_{W}^{2}/l$. It is shown from equation $(18)$ (more specifically the part $U - F_{W}^{2}/l$) that the choice of the velocity of the nonlinear structure $U$ gets modified as the ratio $F_{W}^{2}/l$ changes which in turn has a direct bearing on the existence regimes of the solitary structures as $U$ must always be greater than $F_{W}^{2}/l$ to yield real value.

**Results and discussion**

Here, we proceed to study the linear dispersion characteristics of the DAWs and the formation of solitary and periodic wave solutions to the mKP equation (i.e., equation $(14)$). It is pertinent to mention here that we have written the model equations in the light of recent satellite observations and studies which show that the spectral observations of electrons and ions can be explained in a satisfactory manner by assuming them to be kappa-distributed. Krimigis et al.\(^34\) assumed ions to be kappa-distributed and showed that they fitted the ion spectral observations (taken from Voyager 1 and 2 spacecraft) in the magnetosphere of Saturn extremely well. It was shown that the kappa values for ions are in the range of $6–8$ in Saturn’s magnetosphere. The measurements of Saturn’s magnetosphere by the Cassini–Huygen’s mission team\(^35\) reveal that the electrons distribution can be adequately fitted by considering them to be kappa distributed. It is clear that the kappa fits are observed over large swaths of Saturn’s magnetosphere. It is noted that the kappa values vary with increasing $R$ with reference to the $R_{S}$, where $R_{S}$ is the Saturn’s radius and are found to lie in the range 2–9. Furthermore, for the numerical investigation of mKP equation, we used the data from the Saturn’s magnetosphere which gives us the following values: $n_{i0} \sim 0.1–1 \text{ cm}^{-3}$ and $n_{e0} \sim (1 – 4) \times 10^{3} \text{ cm}^{-3}$ and of magnetic field is $B_{0} = 0.04 G$. Note that we considered the DAWs which primarily travel along the magnetic field lines and, therefore, ignore the effect of Lorentz force here. The value of ion to electron temperature ratio $\delta$ has been taken in the range $0.1–0.01$ with $Z_{d} = 100.\(^{17,41,71,72\}$ In a nutshell, the numerical investigation has been done in strict accordance with the satellite observations.

Figure 1 depicts the variation of the phase velocity $\lambda$ of the DAWs against the electron concentration $P$ and the different cases of the electron and ion distributions and the temperature ratio $\delta$. It is shown from Figure 1a that an increase in $P$ mitigates the phase velocity in the entire range of $P$. This happens because an increase in the electron concentration leads to a decrease of the dust concentration, thereby causing the observed decrement in the phase velocity of the DAWs. Moreover, it is found that the phase velocity of the DAWs becomes maximum for the Maxwelian electrons and ions, whereas it is least for kappa-distributed electrons and ions. It means that the phase velocity of the DAWs in Saturn’s magnetosphere is lower in regions of space plasmas where the ions and electrons are Maxwelian distributed. Also, it is found that the phase velocity of the DAWs for non-Maxwellian electrons and Maxwelian ions is greater than the converse combination. Figure 1b demonstrates the variation of phase velocity of the DAWs against the electron concentration $P$ and the temperature ratio $\delta$. One can see that the increase of $\delta$ leads to a decrease in the phase velocity.

In Results and discussion, we have shown that the mKP equation admits two-types of nonlinear structures, namely, solitary and periodic waves, for different conditions of the modulus of elliptic functions $m$. Here, we investigate the effects of plasma parameters on the propagation characteristics of solitary and periodic structures. Figure 2 shows the effect of spectral indices $(\kappa_{i},\kappa_{e})$ on the profile of the DASWs. It is clear from Figure 2a that the increase in the superthermal ions $\kappa_{i}$ leads to the increase in the amplitude and width of the DASWs. Also, the increase in the electron spectral index $\kappa_{e}$ enhances the amplitude and width of the DASWs as shown in Figure 2b. It is observed that the effect of the electron spectral index $\kappa_{e}$ be more significant by comparison with the ion spectral index $\kappa_{i}$. It means that superthermal electrons energize the DASWs

![Figure 1](image_url)

**Figure 1.** The variation of the phase velocity of the DAWs against electron concentration $P$ and (a) the spectral indices $(\kappa_{i},\kappa_{e})$ for $\delta = 0.1$ and (b) the temperature ratio $\delta$ for $(\kappa_{i},\kappa_{e}) = (6, 5)$. 

![Figure 2](image_url)
more than superthermal ions. In Figure 3, we make a detailed comparison of the effects of different electron and ion distributions on the structure of DASWs. It is observed that the amplitude of the DASWs becomes maximum when both ions and electrons are considered to follow Maxwellian distribution whereas becomes minimum when both electrons and ions follow the kappa distribution function. Note that the DASWs form over a much longer spatial scale when both charged species follow Maxwellian distribution function by comparison with the case when the ions and electrons are assumed to follow kappa distribution. Figure 4 manifests the behavior of DASWs against the dust concentration $\mu_d = 1 - P$. It is seen that the enhancement of the dust concentration $\mu_d$ leads to the expansion of the width of the solitary structure while the amplitude experience a slight change. This is due to the fact that we are studying the DAWs, which is driven by dust mass, and, therefore, increasing the dust concentration brings about the observed increase. Figure 5 depicts the behavior of the

![Figure 3](image3.png)

**Figure 3.** A comparison between the electron and ion different distributions and their impact on the structure of DASWs for $(\mu_d, \delta) = (0.12, 0.1)$. 

![Figure 4](image4.png)

**Figure 4.** The profile of the DASWs is plotted against the dust number density for $(\tau, \kappa_i, \kappa_e) = (0.01, 6, 3)$. 

![Figure 2](image2.png)

**Figure 2.** The effect of (a) electron spectral index $\kappa_e$ and (b) ion spectral index $\kappa_i$ on DA solitary structures for $\delta = 0.1, \tau = 0.01, \mu_d = 0.12$ and $U = 0.1$. 
DASWs against the temperature ratio $\delta$. We note that, for fixed values of other plasma parameters, increasing the temperature ratio $\delta$ mitigates the amplitude and width of the DASWs. This happens due to a complex interplay of nonlinearity and dispersive coefficients. The objective of plotting Figure 6 is to show that the factor $Fw^2/l$ changes with the change in the plasma parameters of the system under consideration. Note that the increase of the value of the factor $Fw^2/l$ implies that the solitary structures would form for higher values of the nonlinear velocity $U$ whereas the decrease of the factor means that the solitary structure would form for a lower value of $U$. In short, the changing values of the factor $Fw^2/l$ modify the existence regimes of the DASWs. Figure 7a shows the DAPWs for the case when $0.7 < m < 0.99$ and $U > Fw^2/l$.

**Figure 5.** The profile of DASWs is plotted against the temperature ratio $\delta$ for $(\kappa_i, \kappa_e, \mu_d) = (6, 3, 0.12)$.

**Figure 6.** Exploring how $Fw^2/l$ changes with the change in the plasma parameters of the system under consideration.

**Figure 7.** (a) A comparison between the electron and ion different distributions and their impact on the DAPWs for specific range of $m$ and $U$ with $\mu_d = 0.12$ and $\delta = 0.1$. (b) The effect of the $m$ and $U$ ranges on the DA periodic structures of dust in comparison to 7a.
whereas Figure 7b illustrates the propagation of DAPWs when \( 0 < m \leq 0.7 \) and \( U < F_w^2/l \). It is found that the DAPWs formed for the case 2 (i.e., \( 0 < m \leq 0.7 \) and \( U < F_w^2/l \)) have a longer spatial scale by comparison with the case when \( 0.7 < m < 0.99 \) and \( U > F_w^2/l \). Observe that in Figure 7a, we used the same plasma parameters as used in Figure 3 except the difference in the value of the modulus \( m \). It is found that the amplitude of the periodic structures shows the same trend as observed in Figure 3. Finally, Figure 8 presents the comparison of the spatial scale over which the dust acoustic rarefactive solitary structures form in the Saturn’s magnetosphere for both mKP and KP equations. It is observed that the solitary structures form over a longer spatial scale (\( \sim 77\text{km} \)) for mKP equation as compared to the ones obtained for the KP equation, which are found to form over a spatial scale of \( \sim 46\text{km} \).

**Conclusion**

In this paper, the dust-acoustic periodic (cnoidal) and localized (solitary) waves in an unmagnetized complex plasma having cold inertial dust grains and inertialess kappa-distributed ions and electrons in the light of Voyager 1 and 2 spacecraft and Freja satellite observations of Saturn’s magnetosphere have been investigated. In the small amplitude limit, the mKP equation, which represents the two-dimensional propagation of nonlinear structure has been derived. The corresponding periodic and localized wave solutions of the mKP equation have been obtained via the Jacobi elliptic function expansion technique. In the linear regime, the phase velocity has been found to be maximum for the Maxwellian case and minimum for kappa-distributed ions and electrons. It has also been observed that an increase in the temperature ratio \( \delta \), mitigates the phase velocity of the dust acoustic waves (DAWs). Moreover, it has been noted that the increasing \( \delta \) (i.e., the ion to electron temperature ratio) leads to the mitigation of the phase velocity of the DAWs. Using the plasma parameters of Saturn’s magnetosphere, the nonlinear periodic and localized (solitary) structures for the DAWs have been found to form on a longer spatial scale for Maxwellian ions and electrons as opposed to their kappa distributed counterparts. Furthermore, it has been noticed that both amplitude and width of the DASWs increase with the enhancement of the dust concentration. It has been observed that increasing the temperature ratio leads to the reduction of the amplitude of the DASWs. It has been seen that the velocity of nonlinear structures gets modified both in the case of the solitary and periodic structures. The existence conditions of localized and periodic structures have also been discussed at length. Importantly, it has been found that the mKP (driven by cubic nonlinearity) solitary structures form over a longer spatial scale (\( \sim 77\text{km} \)) as compared to their KP (driven by quadratic nonlinearity) counterparts, which form over a spatial scale of \( \sim 46\text{km} \).

Future work: The unmodulated multi-soliton solutions to one-dimensional modified KdV (mKdV) equation is considered a hot topic which can be studied in future. Also, the nonplanar rogue wave approximate solutions to the nonplanar nonlinear Schrödinger equation are very important research topics that occupy the minds of many researchers, trying to deeply understand the mechanisms of propagation and generation of these waves in different physical systems.

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References
1. Horanyi M and Mendis DA. The dynamics of charged dust in the tail of comet Giacobini-Zinner. J Geophys Res 1986; 91: 355.
2. Carlile RN, Gaha S, O’Hanlon JF, et al. Electrostatic trapping of contamination particles in a process plasma environment. Appl Phys Lett 1991; 59: 1167.
3. Verheest F and Burton W.B.. Waves in Dusty Plasma. Dordrecht: Kluwer Academic, 2000.
4. Bingham R, De Angelis U, Tsytovich VN, et al. Electromagnetic wave scattering in dusty plasmas. Phys Fluids B 1991; 3: 811–817.
5. Shukla PK and Mamun AA. Introduction to Dusty Plasma Physics. Bristol, UK: Institute of Physics, 2002.
6. Shukla PK. Nonlinear waves and structures in dusty plasmas. Phys Plasmas 2003; 10: 1619.
7. Mamun AA and Shukla PK. Electrostatic solitary and shock structures in dusty plasmas. Phys Scripta 2002; 98: 107.
8. El-Tantawy SA, Alharbey RA, and Salas Alvaro H. Novel approximate analytical and numerical cylindrical rogue wave and breathers solutions: An application to electronegative plasma. Chaos, Solitons & Fractals 2022; 155: 111776.
9. El-Tantawy SA, Salas Alvaro H, Abu Hammad M, et al. Impact of dust kinematic viscosity on the breathers and rogue waves in a complex plasma having kappa distributed particles. Waves in Random and Complex Media 2021; 61: 1708–1728.
10. Wedad A, El-Tantawy SA, and Salas Alvaro H. On the rogue wave solution in the framework of a Korteweg–de Vries equation. Results Phys 2021; 30: 104847.
11. Rao NN, Shukla PK, and Yu MY. Dust-acoustic waves in dusty plasmas. Planet Space Sci 1990; 38: 543–546.
12. Barkan A, Merlino RL, and D’Angelo N. Laboratory observation of the dust-acoustic wave mode. Phys Plasmas 1995; 2: 3563.
13. Shukla PK, Mendis DA, and Chow VW. The Physics of Dusty Plasmas. Singapore: World Scientific, 1996.
14. Shukla PK. A survey of dusty plasma physics. Phys Plasmas 2001; 8: 1791.
15. Duan WS. The Kadomtsev–Petviashvili (KP) equation of dust acoustic waves for hot dust plasmas. Chaos, Solitons & Fractals 2002; 14: 503.
16. Gill TS, Saini NS, and Kaur H. The Kadomstev–Petviashvili equation in dusty plasma with variable dust charge and two temperature ions. Chaos, Solitons & Fractals 2006; 28: 1106.
17. Masood W, Rizvi H, Hasnain H, et al. Rotation induced nonlinear dispersive dust drift waves can be the progenitors of spokes. Phys Plasmas 2012; 19: 032112.
18. Masood W and Ahmad A. Coupled dispersive drift acoustic modes in inhomogeneous dusty plasmas with different nonthermal distributions for electrons and ions. Astrophys Space Sci 2012; 340: 357–372.
19. Masood W, Rizvi H, Hasnain H, et al. Dust drift shock waves with non-Maxwellian ion population in nonuniform collisional dusty plasmas in planetary environments. Astrophys SpaceSci 2013; 345: 49–55.
20. Saha A. and Chatterjee P. Bifurcations of dust acoustic solitary waves and periodic waves in an unmagnetized plasma with nonextensive ions. Astrophys Space Sci 2014; 351: 533–537.
21. Shohaib M, Masood W, Jahangir R, et al. On the dynamics of nonlinear propagation and interaction of the modified KP solitons in multicomponent complex plasmas. J Ocean Eng Sci 2021, DOI: 10.1016/j.joes.2021.10.005.
22. Liu Z and Du J. Dust acoustic instability driven by drifing ions and electrons in the dust plasma with Lorentzian kappa distribution. Phys Plasmas 2009; 16: 123707.
23. Feldman WC, Asbridge JR, Bame SJ, et al. Solar wind electrons. J Geophys Res 1975; 80: 4181.
24. Lazar M, Schlickeiser R, Poedts S, et al. Counterstreaming magnetized plasmas with kappa distributions–I. Parallel wave propagation. Mon Not R Astron Soc 2008; 390: 168.
25. Leubner MP. Fundamental issues on kappa-distributions in space plasmas and interplanetary proton distributions. *Phys Plasmas* 2004; 11: 1308.

26. Mendis DA and Rosenberg M. Cosmic dusty plasma. *Ann Rev Astron Astrophys* 1994; 32: 419.

27. Masood W, Schwartz SJ, Maksimovic M, et al. Electron velocity distribution and lion roars in the magnetosheath. *Ann Geophys* 2006; 24: 1725.

28. Masood W and Schwartz SJ. Observations of the development of electron temperature anisotropies in Earth’s magnetosheath. *J Geophys Res* 2008; 113: A01216.

29. Qureshi MNS, Nasir W, Masood W, et al. Terrestrial lion roars and non-Maxwellian distribution. *Geophys Res Space Phys* 2014; 119: 10.

30. Qureshi MNS, Nasir W, Bruno R, et al. Whistler instability based on observed flat-top two-component electron distributions in the Earth’s magnetosphere. *Mon Not R Astron Soc* 2019; 488: 954.

31. Vasyliunas VMA. A survey of low-energy electrons in the evening sector of the magnetosphere with OGO 1 and OGO 3. *J Geophys Res* 1968; 73: 2839.

32. Armstrong TP, Paonessa MT, Bell EV, et al. Voyager observations of Saturnian ion and electron phase space densities. *J Geophys Res* 1983; 88: 8893.

33. Leubner MP. On Jupiter’s whistler emission. *J Geophys Res* 1982; 87: 6335.

34. Krimigis SM, Carberry JF, Keath EP, et al. General characteristics of hot plasma and energetic particles in the Saturnian magnetosphere: Results from the Voyager spacecraft. *J Geophys Res* 1983; 88: 8871.

35. Schippers P, Blanc M, André N, et al. Multi-instrument analysis of electron populations in Saturn’s magnetosphere. *J Geophys Res* 2008; 113: A07208.

36. Franz JR, Kintner PM, and Pickett JS. POLAR observations of coherent electric field structures. *Geophys Res Lett* 1998; 25: 2041.

37. Kadomtsev BB and Petviashvili VI. On the stability of solitary waves in weakly dispersing media. *Sov Phys Dokl* 1970; 15: 539–541.

38. Pakzad HR. Soliton energy of the Kadomtsev–Petviashvili equation in warm dusty plasma with variable dust charge, two-temperature ions, and nonthermal electrons. *Astrophys Space Sci* 2010; 326: 69–75.

39. Mai-mai L and Wen-shan D. The Kadomtsev–Petviashvili (KP), MKP, and coupled KP equations for two-ion-temperature dusty plasmas. *Chaos, Solitons & Fractals* 2005; 23: 929–937.

40. Dorranian D and Sabetkar A. Dust acoustic solitary waves in a dusty plasma with two kinds of nonthermal ions at different temperatures. *Phys Plasmas* 2012; 19: 013702.

41. Baluku TK and Hellberg MA. Dust acoustic solitons in plasmas with kappa-distributed electrons and/or ions. *Phys Plasmas* 2008; 15: 123705.

42. Anjum N, He JH, Ain QT, et al. Li-He’s modified homotopy perturbation method for doubly-clamped electrically actuated microbeams-based microelectromechanical system. *Facta Univ Ser Mech Eng* 2021; 19: 601–612.

43. He JH and El-Dib YO. The enhanced homotopy perturbation method for axial vibration of strings. *Facta Univ Ser Mech Eng* 2021; 19: 735–750.

44. He JH, El-Dib YO, and Mady AA. Homotopy Perturbation Method for the Fractal Toda Oscillator. *Fractal Fract* 2021; 5: 93.

45. Ju-Kui X. A spherical KP equation for dust acoustic waves. *Phys Lett A* 2003; 314: 479.

46. Lin MM and Duan WS. Dust acoustic solitary waves in a dusty plasma with nonthermal ions. *Chaos, Solitons & Fractals* 2007; 33: 1189.

47. Wang ML. Solitary wave solutions for variant Boussinesq equations. *Phys Lett A* 1995; 199: 169.

48. Wang ML, Zhou YB, and Li ZB. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Phys Lett A* 1996; 216: 67.

49. Yang L, Zhu Z, and Wang Y. Exact solutions of nonlinear equations. *Phys Lett A* 1999; 260: 55.

50. Yang L, Liu J, and Yang K. Exact solutions of nonlinear PDE, nonlinear transformations and reduction of nonlinear PDE to a quadrature. *Phys Lett A* 2001; 278: 267.

51. Parkes EJ and Duffy BR. Travelling solitary wave solutions to a compound KdV-Burgers equation. *Phys Lett A* 1997; 229: 217.

52. Fan E. Extended tanh-function method and its applications to nonlinear equations. *Phys Lett A* 2000; 277: 212.

53. Yan C. A simple transformation for nonlinear waves. *Phys Lett A* 1996; 224: 77–84.

54. Hirota R. Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices. *J Math Phys* 1973; 14: 810.

55. Kudryashov NA. Exact solutions of the generalized Kuramoto-Sivashinsky equation. *Phys Lett A* 1990; 147: 287.

56. Wang KL. Exact solitary wave solution for fractal shallow water wave model by He’s variational method. *Mod Phys Lett B* 2022, DOI: 10.1142/S0217984921506028.
The solution of equation (14) is obtained by going into a co-moving frame

\[ \Phi^{(1)}(x,t) = \Phi^{(1)}(\chi), \]

where \( k \) is the very, \( u \) is the velocity of nonlinear wave, while \( l \) and \( w \) are the direction cosines, respectively. Using equation (16), we can convert the nonlinear partial differential equations (PDEs), that is, equation (14) into an ordinary differential equations (ODEs) and upon integration obtain the following expression

\[ kl \frac{d}{d\chi} \left[ -kU \frac{\partial \Phi^{(1)}}{\partial \chi} + D\left( kU \Phi^{(1)} \right) \frac{\partial^2 \Phi^{(1)}}{\partial \chi^2} + E(kU)^2 \frac{\partial^3 \Phi^{(1)}}{\partial \chi^3} \right] + F(kw)^2 \frac{d^2 \Phi^{(1)}}{d\chi^2} = 0. \]

Further simplification of equation (25) yields the following expression

\[ (Fw^2 - kU) \Phi^{(1)} + \frac{D^2}{3} [\Phi^{(1)}]^3 + Ek^2 l^2 \frac{d^2 \Phi^{(1)}}{d\chi^2} = 0. \]

To find a solution to equation (26), the following ansatz is introduced

\[ \Phi^{(1)}(\chi) = \sum_{j=0}^{n} a_j c n^j |\chi|. \]

The value of \( j \) is obtained with the help of harmonic balance method (the balance between the higher-order nonlinearity and dispersion of Eq. (18), that is, \( 3n = n + 2 \Rightarrow n = 1 \)) which yields

\[ \Phi^{(1)} = a_0 + a_1 c |\chi|. \]
By inserting solution (28) into equation (18), we obtain
\[(Fw^2 - IU)(a_0 + a_1cn[\chi]) + \frac{Dl^2}{3}(a_0 + a_1cn[\chi])^3 + Ek^2l^4(a_0 + a_1cn[\chi])\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = 0,\]
\[\frac{d(a_0 + a_1cn[\chi])}{dx} = -a_1sn[\chi]dn[\chi],\]  \hspace{1cm} (30)
\[\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = -a_1cn[\chi]dn^2[\chi] - a_1sn[\chi]d^2n[\chi] + a_1cn[\chi]dn[\chi],\] \hspace{1cm} (31)
\[\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = -a_1cn[\chi](1 - m^2sn^2[\chi]) - a_1sn[\chi](-m^2sn[\chi]cn[\chi]),\]
\[\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = -a_1cn[\chi](1 - m^2(1 - cn^2[\chi])) + a_1m^2(1 - cn^2[\chi])cn[\chi],\]
\[\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = a_1(m^2 - 1)cn[\chi] - a_1m^2cn^3[\chi] + a_1m^2cn^3[\chi] - a_1m^2cn^3[\chi],\]
\[\frac{d^2(a_0 + a_1cn[\chi])}{dx^2} = a_1(2m^2 - 1)cn[\chi] - 2a_1m^2cn^3[\chi].\]

Comparing the coefficient of \(cn[\chi] \cdot cn^0\)
\[(Fw^2 - IU)a_0 + \frac{Dl^2a_0^2}{3} = 0,\] \hspace{1cm} (33)
\(cn^1\)
\[(Fw^2 - IU)a_1 + \frac{2Dl^2a_0a_1}{3} + Ek^2l^4a_1(2m^2 - 1) = 0,\] \hspace{1cm} (34)
\(cn^2\)
\[\frac{2Dl^2a_0a_1^2}{3} = 0 \Rightarrow a_0 = 0,\] \hspace{1cm} (35)
\(cn^3\)
\[\frac{Dl^2a_0^2a_1^3}{3} - 2Ek^2l^4a_1m^2 = 0.\] \hspace{1cm} (36)

From equation (36)
\[a_1 = \sqrt{\frac{6Ek^2l^4m^2}{Dl^2}},\] \hspace{1cm} (37)

From equation (34)
\[k = \sqrt{\frac{(IU - Fw^2)}{El^4(2m^2 - 1)}},\] \hspace{1cm} (38)

substituting (38) in (37)
\[a_1 = \sqrt{\frac{6(IU - Fw^2)m^2}{Dl^2(2m^2 - 1)}},\] \hspace{1cm} (39)
using (39), (38) and (34) in (28)

\[
\Phi^{(1)} = \sqrt{\frac{6(U - Fw^2)m^2}{Dl^2(2m^2 - 1)}} \text{cn} \left[ \sqrt{\frac{(U - Fw^2)}{E\nu^4(2m^2 - 1)}} (l\zeta + w\eta - U\tau) \right].
\]

(40)

For \( m \to 1 \), the cnoidal wave solution given in equation (40) reduces to the following soliton solution

\[
\Phi^{(1)} = \sqrt{\frac{6(U - Fw^2)}{Dl^2}} \text{sech} \left[ \sqrt{\frac{(U - Fw^2)}{E\nu^4}} (l\zeta + w\eta - U\tau) \right].
\]

(41)

Solution (41) represent the soliton solution to mKP equation. We can also write it in compact form as follows

\[
\Phi^{(1)} = N \text{ cn}[\chi],
\]

(42)

with

\[
N = -\sqrt{\frac{6(U - Fw^2)}{Dl}}.
\]