Transport of Solar Energetic Particles along Stochastic Parker Spirals

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Abstract

It was recently shown that, owing to the turbulent nature of the solar wind, the interplanetary magnetic field lines can be well described by stochastic Parker spirals. These are realizations of Brownian diffusion on a sphere of increasing radius, superimposed on the angular drift due to the solar rotation. In this work, we present a model for the transport of solar energetic particles along stochastic Parker spirals in the inner heliosphere. The transport model is governed by a set of four stochastic differential equations for the heliographic position \((r, \alpha = \cos \theta, \phi)\) of the guiding centers and the cosine of the pitch angle between the velocity vector and the Parker field. The model accounts for the role played by the combination of pitch angle scattering and magnetic focusing in the interplanetary medium. The effects of the dynamical evolution of the turbulence are included in the model by taking the field line angular diffusivity to be a function of the radial distance from the Sun. The heliolongitudinal distribution of particles propagating along stochastic Parker spirals is given by the wrapped Gaussian distribution. This angular distribution can also well be represented by the von Mises distribution that interpolates between the Gaussian distribution at small angular spread and the uniform distribution at large distances from the acceleration region of energetic particles in the aftermath of a solar eruption.

Unified Astronomy Thesaurus concepts: Interplanetary turbulence (830); Interplanetary magnetic fields (824)

1. Introduction

Understanding the transport of high-energy charged particles in turbulent astrophysical plasmas has been the subject of a long endeavor, starting with the work of Fermi (1949). The elucidation of the physical mechanisms responsible for solar energetic particle (SEP) events (Reames 1999; Cliver 2000; Klein & Dalla 2017), more typically observed in the near-Earth environment, is among the common objectives of the two recently launched Parker Solar Probe and Solar Orbiter missions (McComas et al. 2019; Rodriguez-Pacheco et al. 2020). Charged particles are guided by magnetic fields and, thus, the transport of SEPs depends on the structure of the interplanetary magnetic field. In the original work of Parker (1958), the field is assumed to be carried radially outward with the bulk solar wind that emanates from the rotating Sun, yielding the basic spiral structure of the interplanetary magnetic field. Jokipii & Parker (1969) argued that, however, owing to the turbulent nature of the solar wind plasma, the interplanetary magnetic field acquires a stochastic component. They combined the concept of field line stochasticity and Leighton’s (1964) theory of diffusive flux transfer at the Sun to provide a sound explanation for the angular spread of “solar cosmic rays” (Meyer et al. 1956) inferred from measurements by the Pioneer missions (Fan et al. 1968). Due to the stochastic component of the field, the location of spacecraft magnetic footpoints at the Sun is expected to develop an angular distribution, reflecting the probabilistic nature of the magnetic connection between the observer position and the region where the energetic particles are released in the solar wind. After the observational study of Fan et al. (1968), the angular dispersion of SEPs has been the subject of many investigations based either on single-spacecraft or multispacecraft measurements in longitude (Van Hollebeke et al. 1975; Cane et al. 1986; Shea & Smart 1990; Reames 1999; Lario et al. 2006; Reames et al. 2013; Mewaldt et al. 2013; Wiedenbeck et al. 2013; Dressing et al. 2014; Dröge et al. 2014; Richardson et al. 2014; Cohen et al. 2017; Hu et al. 2017, 2018) and latitude (Zhang et al. 2001; Dalla et al. 2003; Zhang et al. 2003). Cohen et al. (2017) conducted a statistical survey of the longitudinal spread of energetic ion species, showing that the latter only weakly depends on the charge to mass ratio. This suggests that the angular dispersion of SEPs is governed by the same turbulent processes that are at the origin of the angular dispersion of the guiding magnetic field lines in the solar wind. The Parker Solar Probe and the Solar Orbiter spacecraft, in combination with the fleet of spacecraft orbiting the Sun, now provide an unprecedented source of stereoscopic observations in the inner heliosphere, with their common vantage points that will soon be approaching the acceleration region of energetic particles (and of the solar wind) as close as ~10 solar radii. Bian & Li (2021) have recently argued that the turbulent interplanetary magnetic field lines can well be represented by stochastic Parker spirals. They are realizations of a spherical random walk superimposed on the standard deterministic angular drift due to solar rotation. This stochastic process is aimed at capturing the physical essence of the cartoon drawn by Jokipii & Parker (1969) representing the meandering of the magnetic field lines in the heliosphere. Combined, all these considerations motivate us to extend the work of Jokipii & Parker (1969) by developing a model for the transport of SEPs along stochastic Parker spirals. In Section 2, we expose the general features of the drift-kinetic transport model that forms the basis of our subsequent study. In Section 3, we analyze the idealized situation where the energetic particles emitted at the Sun propagate along a Parker spiral connecting the source to the observer location in the solar wind. The scatter-free and strongly scattered regimes of particle transport are discussed in more detail. In Section 4 we include the effect of turbulent magnetic fluctuations in the transport model, concentrating on the case where these...
fluctuations are driven by the turbulent motions of the magnetic footpoints on the solar wind source surface. In Section 5 we develop a model for the propagation of SEPs along stochastic Parker spirals in the solar wind. The forms of the resulting heliolongitudinal distributions of energetic particles, at a given distance from the Sun, are discussed in Section 6. A comprehensive summary of the main results and a conclusion are given in Section 7.

2. The Transport Model

Analysis of the transport of energetic particles typically is carried out within the framework of drift-kinetic theory (Skilling 1971, 1975; Isenberg 1997; Schlickeiser 2002; Webb et al. 2009; Smolyakov & Garbet 2010; Zank 2014; Achterberg & Norman 2018). A relatively complete description of the three-dimensional evolution of the gyrotropic distribution function of particles can be obtained from the transport equation

$$\frac{\partial f}{\partial t} + \nabla \left( \frac{d r}{d t} f \right) + \frac{\partial}{\partial \mu} \left( \frac{d \mu}{d t} f \right) = \frac{\partial}{\partial \mu} \left( D_{\mu} \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial r_{j}} \left( D_{j} \frac{\partial f}{\partial r_{j}} \right).$$

(1)

The phase-space distribution $f(r, \mu, t)$ depends on the guiding-center position $r$ and on $\mu$, the cosine of the pitch angle between the particle velocity and the guiding magnetic field $\vec{B}$. In Equation (1) we ignore the effects of adiabatic cooling, which is of the order of the ratio of the solar wind speed to particle speed, i.e., $V_{sw}/v$. Consequently, the particle speed is treated as a constant parameter in this model. Particle speed only affects the arrival time of a particle at a particular location and does not affect the angular spreading of particles in a stochastic Parker field. Because we focus on the angular spreading of energetic particles, we therefore do not consider the adiabatic cooling effect in this work. Note that including the adiabatic cooling can be important to understand (late) time intensities of protons and ions in, e.g., large SEP events. This effect is particularly noticeable if the source spectrum is very steep. This is because a proton observed at 1 au with energy of, say 100 MeV may correspond to a proton with an energy of 110 MeV near the Sun (or at the shock front), and a very steep source spectrum will result in a significant delay in the late time-intensity profiles (e.g., Zhang et al. 2009). The inclusion of adiabatic cooling in the focused transport equation can be found in Ruffolo (1995; le Roux et al. 2005; Zhang et al. 2009; Le Roux & Webb 2009, 2012). Despite this simplification, Equation (1) still accounts for a number of important transport effects.

The second term on the left side of Equation (1) describes the change in the guiding-center position $r$ as a result of streaming parallel to the guiding magnetic field and of guiding-center drifts,

$$\frac{d r}{d t} = v_{||} B + v_{d},$$

(2)

where $v_{||} = \mu v$ is the component of the velocity parallel to the guiding magnetic field, $\vec{B} = \vec{B}/|\vec{B}|$ is the unit vector along the guide field and $v_{d}$ is the guiding-center drift velocity. In this work, we consider the situation where the main source of particle drift is the presence of turbulent magnetic fluctuations in the solar wind. For this reason, we decompose the magnetic field into a mean component $\vec{B}$ and transverse fluctuations $\delta B_{\perp}$, according to

$$\vec{B} = \vec{B} + \delta \vec{B}.$$  

(3)

The expression for the turbulent magnetic drift velocity $v_{d}$ resulting from the fluctuations $\delta B_{\perp}$ can be obtained from the equations describing the magnetic field lines. Magnetic field lines are the instantaneous family of solutions of the three ordinary differential equations (see, e.g., Longcope 2005),

$$\frac{d r}{d s} = \vec{B} + \delta \vec{B}.$$  

(4)

where $s$ is the field-aligned coordinate. Inserting the relation expressing the streaming of the guiding center along the guide field, which is

$$\frac{d s}{d t} = \mu v,$$

(5)

into the magnetic-field line Equation (4), yields the guiding-center equation of motion (2) with the drift velocity in the form given by

$$v_{d} = \mu v \left( \frac{\delta \vec{B}}{\vec{B}} \right).$$

(6)

The third term on the left side of Equation (1) accounts for the secular change in pitch angle, which is imposed by the conservation of the first adiabatic invariant,

$$\frac{p_{\perp}^{2}}{2 m |B|} = m (1 - \mu^{2}) v^{2} = cst,$$

(7)

where $p_{\perp}$ is the perpendicular momentum and $m$ is the mass of the particle. Conservation of the magnetic moment $p_{\perp}/2m\vec{B}$ implies that the amount of magnetic flux $\vec{B} \times \pi r_{p}^{2}$ enclosed by a gyro-orbit is conserved, with $r_{p} = c p_{\perp}/q|\vec{B}|$ being the particle gyroradius. Conservation of the magnetic moment can either result in magnetic mirroring or in magnetic focusing depending on whether the particles move in the direction of increasing or decreasing magnetic field strength $\vec{B}$. Taking the time derivative of Equation (7) in the frame comoving with the particle, yields

$$\frac{d \mu}{d t} = - \frac{v}{2} (1 - \mu^{2}) \frac{\partial \ln |\vec{B}|}{\partial s}.$$  

(8)

This shows that the mirroring/focusing effect is determined by the local rate of exponential growth/decrease of the magnetic field strength per unit of the field-aligned distance $s$, or equivalently by the so-called mirroring/focusing length $L_{B}$, whose inverse is essentially the divergence of the unit vector along the guide field

$$\frac{1}{L_{B}} = - \frac{\partial \ln |\vec{B}|}{\partial s} = \nabla \cdot \vec{B}.$$  

(9)

The first term on the right side of Equation (1) is stochastic and describes random changes in the pitch angle of the particles with respect to the guiding magnetic field. The pitch angle
diffusion coefficient $D_{\mu\nu}$ is related to the scattering frequency $\nu$ (e.g., Kulsrud & Pearce 1969) by

$$D_{\mu\nu} = \frac{1}{2} \left( 1 - \mu^2 \right) \nu. \quad (10)$$

Finally, the second term on the right side of the transport Equation (1) describes the spatial diffusion of the guiding centers. In general, this process is characterized by a diffusion tensor $D_{ij}$ (Zhang et al. 2009; Effenberger et al. 2012; Dröge et al. 2014; Strauss & Fichtner 2015; Strauss et al. 2017), which derives from a positive semidefinite matrix $B_{ij}$ (Gardiner 1986; Pei et al. 2010), via the relation

$$D_{ij} = \frac{1}{2} B_{im} B_{jm}. \quad (11)$$

There can be various sources of spatial diffusion in the interplanetary plasma. In the following, we shall consider in more detail the process of spatial transport of particles that is due to field line diffusion (Jokipii 1966; Jokipii & Parker 1969; Rechester & Rosenbluth 1978; Mattheus et al. 1995; Bian et al. 2011; Laitinen & Dalla 2019) (see also the recent review by Shalchi 2020). In the situation where this diffusion is isotropic and transverse to the guide field, the diffusion tensor can be written as

$$D_{ij} = D_{\perp} (\delta_{ij} - B_i B_j), \quad (12)$$

where $D_{\perp}$ is a scalar coefficient characterizing the diffusion across the mean field.

Recasting the pitch angle diffusion operator (the first term on the right-hand side of Equation (1)) as

$$\frac{\partial}{\partial \mu} \left( D_{\mu\nu} \frac{\partial f}{\partial \mu} \right) = \frac{\partial}{\partial \mu} \left( (D_{\mu\nu} f) - \frac{\partial D_{\mu\nu} f}{\partial \mu} \right), \quad (13)$$

and the spatial diffusion operator (the last term in Equation (1)) as

$$\frac{\partial}{\partial r_i} \left( D_{ij} \frac{\partial f}{\partial r_j} \right) = \frac{\partial}{\partial r_i} \left( \frac{\partial D_{ij} f}{\partial r_j} - \frac{\partial D_{ij} f}{\partial r_j} \right) = \frac{\partial}{\partial r_i} \left( \frac{1}{2} \frac{\partial B_{im} B_{jm} f}{\partial r_j} \right) \quad (14)$$

the kinetic Equation (1) is effectively cast into the Fokker-Planck form:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r_i} \left( \mu v_i \delta_{i} + n_i + \frac{\partial D_{ii} f}{\partial r_i} \right) + \frac{\partial}{\partial \mu} \left( \frac{\partial (D_{\mu\nu} f)}{\partial \mu} \right) f$$

$$+ \frac{\partial}{\partial \mu} \left[ \frac{(1 - \mu^2 \nu)}{2L_B(r, \cos \chi)} + \frac{\partial D_{\mu\nu} f}{\partial \mu} \right] f$$

$$= \frac{\partial^2 (D_{\mu\nu} f)}{\partial \mu^2} + \frac{\partial^2}{\partial r_i \partial r_j} \left( \frac{1}{2} B_{im} B_{jm} f \right). \quad (15)$$

Equivalently then, the transport of particles is governed by a set of four stochastic differential equations for the guiding-center coordinates and the pitch angle coordinate as

$$\frac{dr_i}{dt} = \mu v_i + n_i + \frac{\partial D_{ii} f}{\partial r_i} + B_{ij} \zeta_j(t), \quad (16)$$

$$\frac{d\mu}{dt} = \frac{1 - \mu^2}{2L_B} \nu + \frac{\partial D_{\mu\nu} f}{\partial \mu} + \sqrt{2D_{\mu\nu} \zeta_\mu(t)}, \quad (17)$$

where the functions $\zeta$ are independent Gaussian white noises with the following properties,

$$\langle \zeta(t) \rangle = 0, \quad \langle \zeta(t) \zeta(t') \rangle = \delta(t - t'). \quad (18)$$

In the absence of both drifts and diffusion, $n_i = 0$ and $D_{ij} = 0$, the spatial transport of guiding centers effectively becomes one-dimensional along the guiding magnetic field lines that connect to the observer. In this case, a convenient coordinate system is the field-aligned coordinate. The one-dimensional distribution $f(s, \mu, t)$ of particles then obeys the focused transport equation (Roelof 1969; Earl 1976; Kunstmann 1979) (see also the recent review by van den Berg et al. 2020) in the form given by

$$\frac{df}{dt} + \nu \frac{df}{ds} + \frac{(1 - \mu^2)}{2L_B} \frac{df}{d\mu} = \mu \frac{\partial}{\partial \mu} \left( D_{\mu\nu} \frac{df}{d\mu} \right). \quad (19)$$

The focused transport Equation (19) is equivalent to the set of only two differential equations, comprising Equation (5) for the field-aligned coordinate of the particles and the stochastic differential Equation (17) for their pitch angle coordinate. The description of the field-aligned transport of scatter-free particles, when $D_{\mu\nu} = 0$ in Equation (19), can be reduced to the single ordinary differential Equation (5) in the form given by

$$\frac{ds}{dt} = \pm \sqrt{ \frac{1 - \mu^2}{B(s) B(s_0)} (1 - \mu^2)} \nu, \quad (20)$$

where the expression for $\mu(s, \mu_0)$ (under the square root bracket in Equation (20)) is obtained from Equation (7) and is a consequence of the conservation of the magnetic moment. The positive (negative) sign in Equation (20) corresponds to cases where particles are injected with $\mu = +\mu_0$ ($\mu = -\mu_0$) at $s = s_0$. For instance, taking the field strength $B(s)$ to be a monotonically decreasing function of the field-aligned coordinate $s$, corresponding to $L_\mu(s) > 0$, the group of particles injected with $\mu = +\mu_0$ will experience magnetic focusing and their velocity vector will become aligned with $\mu = 1$ along the guide field. On the contrary, those particles injected in the opposite direction, with $\mu = -\mu_0$, will first experience an increase in the field strength and thus will be mirrored at the turning point $s$ defined as $\mu(s_\mu, \mu_0) = 0$, i.e., where $ds/dt = 0$. The quantity inside the square root bracket in Equation (20) is always positive and goes to zero at the mirroring points where the parallel velocity changes sign. In the absence of scattering, the special group of particles injected with $\mu = \pm 1$, having a vanishing gyroradius $(r_g = 0)$, neither experience the effect of magnetic focusing nor mirroring, and thus, they propagate at a constant speed $v$ in both directions along the guide field.

A small amount of pitch angle scattering may however change the above picture, even for a beam-like distribution of energetic particles (Bian & Emslie 2019, 2020; Li & Lee 2019), and more importantly, when $\nu T \gg 1$, where $T$ is the dynamical propagation timescale toward the observer.

Indeed, in the opposite case of the strongly scattered regime of transport, the pitch angle distribution of the particle has
reached a quasi-steady state consisting of the dynamical equipartition between secular and random changes in pitch angle. In this limit (Litvinenko 2012), $d\mu/dt = 0$ in Equation (17), yields
\[ \mu = \frac{(1 - \mu^2)}{2\nu L_B} v + \frac{(1 - \mu^2)}{\nu} \zeta(t). \] (21)

The field-aligned transport of strongly scattered particles is thus governed by the single stochastic differential equation
\[ \frac{d\zeta}{dt} = \frac{(1 - \mu^2)}{2\nu L_B} v + \frac{(1 - \mu^2)}{\nu} \zeta(t), \] (22)
representing a random walk superimposed on a drift of the guiding centers along the guide field. Expressions for the field-aligned drift velocity $V_\parallel$ (cm s$^{-1}$) and the field-aligned diffusion coefficient $D_\parallel$ (cm$^2$ s$^{-1}$) are obtained by averaging over pitch angle as
\[ V_\parallel = \frac{d\langle \Delta s \rangle}{dt} = \frac{(1 - \mu^2)}{2\nu L_B} v, \]
\[ D_\parallel = \frac{1}{2} \frac{d\langle \Delta s^2 \rangle}{dt} = \frac{(1 - \mu^2)}{2\nu} v^2, \] (23)
and thus the parallel drift and the parallel diffusion coefficients are related by
\[ V_\parallel = \frac{D_\parallel}{L_B}. \] (24)

The sign of the parallel drift velocity $V_\parallel$ is identical to that of $L_B$ and $V_\parallel \to 0$ when $L_B \to \infty$, corresponding to a uniform magnetic field. In terms of the parallel mean free path $\lambda_\parallel$, we have
\[ D_\parallel = \frac{1}{3} \lambda_\parallel v, \quad V_\parallel = \frac{1}{3} \left( \frac{\lambda_\parallel}{L_B} \right) v. \] (25)

We notice that the drift-diffusion equation describing the transport of strongly scattered particles can equivalently be obtained by first decomposing the distribution function $f(s, \mu, t)$ entering the kinetic Equation (19) into the complete basis of Legendre polynomials $P_n(\mu)$ as
\[ f(s, \mu, t) = \sum_{n=0}^{\infty} f_n(s, t) P_n(\mu), \] (26)
and then making the so-called $P_1$ approximation. This approximation consists of truncating the sum at the order $n = 1$, as $f(s, \mu, t) = f_0(s, t) + \mu f_1(s, t)$ and in taking $\partial f_1/\partial t = 0$. The spatial flux of particles is then related to the particle density via Fick’s law, resulting in the spatial diffusion of the particles along the guide field. As discussed in Bian et al. (2017), various forms of nonlocal transport effects, involving convolution products that relate the flux of particles to the density gradient, can arise from more refined approximation schemes adapted to the situation where the mean free path is large enough$^1$ that the standard diffusion approximation breaks down. In this case, the transport of the guiding centers is more accurately described by continuous-time random walks (Metzler & Klafter 2000) along the guiding magnetic field lines.

In the next section, we shall focus on the idealized situation when the transport of particles occurs along the nominal Parker spirals and only consider in more detail the two opposite limits of scatter-free and strongly scattered regimes of transport.

3. Transport along the Nominal Parker Spirals

In the original work of Parker (1958), the solar wind is accelerated at the Sun and emanates from a spherical source surface with a uniform radial velocity. The angular separation between any given two nearby fluid elements accelerated at the source surface is therefore a constant. Taking the magnetic field to be frozen in the wind expanding from the rotating Sun yields the basic spiral geometry of the interplanetary magnetic field lines (see, e.g., the recent review by Lhotka & Narita 2019). In this situation, the interplanetary magnetic field is given, on average, by the Parker (1958) field that, in the heliographic coordinate system, can be represented as
\[ \mathbf{B} = \mathbf{B}_r (u_r - \tan \chi u_\theta), \quad \mathbf{B}_r = B_0 \frac{r^2}{r^2}, \] (27)
with
\[ \tan \chi = \frac{\Omega r \sin \theta}{V_{sw}}. \] (28)

Here, $V_{sw}$ is the solar wind speed, $\Omega$ is the equatorial rotation rate of the Sun, and $\chi$ is the angle between the Parker field and the radial direction, i.e.,
\[ \cos \chi = \frac{\mathbf{B}_r}{\mathbf{B}} = \frac{1}{\sqrt{1 + \left( \frac{\Omega r \sin \theta}{V_{sw}} \right)^2}}. \] (29)

Therefore, the unit vector $\mathbf{B}$ along the mean field and the magnetic focusing/mirroring length $L_B$ in the heliosphere can be both expressed in terms of the angle $\chi$ as
\[ \mathbf{B} = \cos \chi u_r - \sin \chi u_\theta, \]
\[ L_B(r, \cos \chi) = \frac{r}{\cos \chi (1 + \cos^2 \chi)}. \] (30)

The small spiral angle approximation, which is tantamount to taking the unit vector $\mathbf{B}$ and the focusing length $L_B$ as
\[ \mathbf{B} = u_r - \chi u_\theta, \quad L_B = \frac{r}{2}, \] (31)
is a relatively accurate representation of the Parker (1958) field in the ecliptic plane $\theta = \pi/2$, within a distance of 1 AU (at the poles, the field is purely radial and hence, $\chi = 0$).

In the heliographic coordinate system, the interplanetary magnetic field lines are the instantaneous family of solutions of the set of two first-order differential equations
\[ r \frac{d\theta}{dr} = \frac{\mathbf{B}_\theta}{\mathbf{B}_r} = 0, \quad r \sin \theta \frac{d\phi}{dr} = \frac{\mathbf{B}_\phi}{\mathbf{B}_r} = -\tan \chi. \] (32)

They can be integrated to provide the following two angular constraints resulting from the working hypothesis that particles propagate around specific curves given here by the Parker

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$^1$ A well-known example is given by the telegrapher equation (Joseph & Preziosi 1989).
The initial conditions are here chosen such that
\[ \theta = \theta_0, \quad \phi = -\frac{\Omega}{V_{sw}}(r - r_0) + \phi_0. \]  
(33)

Inserting the field-aligned streaming relation (5) for the particles into the relation that stands for the definition of the spiral angle,
\[ \frac{dr}{ds} = \cos \chi, \]  
(34)

the heliospheric transport of the guiding centers propagating along Parker spirals is thus described by the equations
\[ \frac{dr}{dt} = \mu v \cos \chi, \quad \theta(t) = \theta_0, \]  
\[ \phi(t) = -\frac{\Omega}{V_{sw}}(r(t) - r_0) + \phi_0, \]  
(35)

In the scatter-free regime of transport, the problem of obtaining the particle trajectories along Parker spirals is reduced to integrating the single differential equation
\[ \frac{dr}{dt} = \pm \cos \chi \sqrt{1 - \frac{r_0^2}{r^2} \left(1 - \mu_0^2\right)} v, \]  
(36)

where the positive (negative) sign in Equation (36) corresponds to the antisunward (sunward) direction of the initial propagation. In the small spiral angle approximation, corresponding to \( \cos \chi \sim 1 \), the radial component (36) of the guiding-center equations of motion is readily integrated to give
\[ r(t) = \sqrt{r_0^2 \left(1 - \mu_0^2\right) + \left(\mu_0 r_0 + vt\right)^2}, \]  
(37)

and thus
\[ \mu(t) = \sqrt{1 - \frac{r_0^2}{r(t)^2} \left(1 - \mu_0^2\right)} \frac{r_0}{\mu_0 r_0 + vt}. \]  
(38)

The initial conditions are here chosen such that \( \mu(r=r_0) = +\mu_0 \) corresponds to the antisunward direction of propagation for the particles escaping the acceleration region. Overall, this shows that in the absence of scattering, energetic particles injected outward from the Sun have their velocity vector becoming increasingly aligned with the Parker field, and hence, \( \mu(r) \to 1 \) when \( r/r_0 \gg 1 \), despite the weakening of the Parker field focusing strength with radial distance. At large time \( t \gg r_0/v \), the particles all become focused with \( \mu = 1 \) along the guiding magnetic field and only then is their transport ballistic with \( r=vt \). We notice that taking \( \mu_0 = 0 \) in Equation (37) results in a linear relation between the travel distance and the arrival time at the observer position, which forms the basis of the standard velocity dispersion analysis of the onset time of SEP events (Lin 1974; Lin et al. 1981; Krucker & Lin 2000; Tylka et al. 2003; Sáiz et al. 2005; Kahler & Ragot 2006; Reames 2009; Wang et al. 2016). The group of particles injected in the opposite hemisphere, in the sunward direction, is first mirrored at the turning points located at
\[ r_s = r_0 \sqrt{1 - \mu_0^2}, \]  
(39)

and afterward, they propagate antisunward according to
\[ r(t) = \sqrt{r_s^2 + (vt)^2}, \quad \mu(t) = \sqrt{1 - \frac{r_s^2}{r(t)^2 + (vt)^2}}, \]  
(40)

and eventually become focused along the field.

In the opposite limit of the strongly scattered regime of transport, the radial motion is governed by the drift-diffusion equation
\[ \frac{dr}{dt} = V_r + \sqrt{2D_r \zeta(t)}, \]  
(41)

where the radial drift velocity \( V_r \) and the radial diffusion coefficient \( D_r \) are given by
\[ V_r = \frac{\langle 1 - \mu^2 \rangle v^2 \cos \chi}{2\nu L_B(r, \cos \chi)}, \]  
\[ D_r = \frac{\langle 1 - \mu^2 \rangle v^2 \cos^2 \chi}{2\nu}. \]  
(42)

These two Fokker–Planck coefficients are sufficient to describe the transport of strongly scattered particles along the Parker spirals. For simple geometrical reasons, they differ from their parallel counterparts, \( V_{r\parallel} \) and \( D_{r\parallel} \) in Equation (23), by a factor of \( \cos \chi \) and \( \cos^2 \chi \), respectively. The sign of the magnetic mirroring/focusing length \( L_B \) associated with the Parker field yields an antisunward drift velocity \( V_r \). In the small-\( \chi \) approximation, the drift-diffusion Equation (41) takes the form
\[ \frac{dr}{dt} = \frac{2}{3} \left( \frac{\lambda_r}{r} \right) v + \sqrt{\frac{2}{3} \frac{\lambda_r}{r} v \zeta(t)}, \]  
(43)

where \( \lambda_r \) is the radial mean free path.

The structure of the interplanetary magnetic field can however substantially deviate from the nominal Parker field as a consequence of turbulence in the solar wind and at its source surface, the effects of which on the transport of SEPs we shall now consider.

### 4. Effect on Transport of Turbulent Magnetic Field Fluctuations

The form of the guiding-center drift velocities \( v_d \) resulting from the presence of turbulent magnetic fluctuations \( \delta B_{\perp} \) in the solar wind is obtained by combining Equations (6), (27), and (29) to yield
\[ v_d = \mu v \cos \chi \left( \frac{r^2}{r_0^2} \right) \left( \frac{\delta B_{\perp}}{B_0} \right). \]  
(44)

The transverse field fluctuations \( \delta B_{\perp} \) are here assumed to solely derive from fluctuations \( \delta A_{\parallel} \) in the radial component of the magnetic vector potential as
\[ \delta B_{\perp} = \nabla_\perp \times (\delta A_{\parallel} u_\parallel), \]  
(45)

where \( \nabla_\perp \) denotes the restriction to the sphere of radius \( r \) of the \( \nabla \) operator. Hence, the fluctuating components of the
interplanetary magnetic field\textsuperscript{2} are written as
\[
\delta B_\theta = \frac{1}{r \sin \theta} \frac{\partial \delta A_\theta}{\partial \phi}, \quad \delta B_\phi = - \frac{1}{r} \frac{\partial \delta A_\phi}{\partial \theta}. \quad (46)
\]

Denoting by \( q \) the cosine of the heliographic colatitude \( \theta \), i.e., \( \alpha = \cos \theta \), the turbulent magnetic field lines are thus the instantaneous family of solutions of the two first-order differential equations:
\[
d\alpha \over dr = - \frac{1}{B_0^2 \rho_0^2} \frac{\partial \delta A_\alpha}{\partial \phi}, \quad d\phi \over dr = - \frac{\Omega}{V_w} + \frac{1}{B_0^2 \rho_0^2} \frac{\partial \delta A_\phi}{\partial \alpha}. \quad (47)
\]

We notice that they take the form of Hamilton equations with
\[
H_m(\alpha, \phi, r) = - \frac{\Omega}{V_w} \alpha + \frac{\delta A_\alpha}{B_0 \rho_0^2}, \quad (48)
\]

where the two canonically conjugate variables are the cosine of the heliographic latitude \( \alpha = \cos \theta \) and the longitude \( \phi \). It is understood that time only plays the role of a parameter in the field line Hamiltonian (48). Parameterizing the perturbed magnetic field lines by the field-aligned coordinate \( s \) along the Parker field, we have
\[
dr \over ds = \cos \chi, \quad d\alpha \over ds = - \cos \chi \frac{\partial \delta A_\alpha}{\partial \phi}, \quad d\phi \over ds = - \cos \chi \frac{\Omega}{V_w} + \cos \chi \frac{\partial \delta A_\phi}{\partial \alpha}. \quad (49)
\]

Inserting the field-aligned streaming relation (5) for the particles into the magnetic-field line Equation (49), the spatial transport of particles in the turbulent field becomes governed by a set of three differential equations:
\[
\frac{dr}{dt} = \mu \nu \cos \chi, \quad \frac{d\alpha}{dt} = - \mu \nu \cos \chi \frac{\partial \delta A_\alpha}{\partial \phi}, \quad \frac{d\phi}{dt} = - \mu \cos \chi \left( \frac{V}{V_w} \right) \Omega + \frac{\mu \nu \cos \chi}{B_0 \rho_0^2} \frac{\partial \delta A_\phi}{\partial \alpha}. \quad (50)
\]

This transport model reduces to the one studied in Section 3 when \( \delta A = 0 \), corresponding to vanishing magnetic fluctuations. In the heliographic coordinate system, a natural representation of the turbulent magnetic field can be obtained by decomposing its fluctuating component into a complete set of spherical harmonics, according to
\[
\delta A_n = \sum_{m=1}^{N} \sum_{m=-n}^{n} \delta A_n^m(r, t) Y_n^m(\theta, \phi)
= \sum_{n=1}^{N} \sum_{m=-n}^{n} (-1)^m \left[ \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right]^{1/2} \times \delta A_n^m(r, t) e^{im\phi} P_n^m(\alpha), \quad (51)
\]

where \( Y_n^m(\theta, \phi) \) are the spherical harmonics and \( P_n^m(\alpha) \) are the associated Legendre polynomials. There are various sources of magnetic fluctuations in the solar wind, due to turbulence, eruptions, electromagnetic instabilities, and also including those produced by the energetic particles themselves. We now focus on the particular case where the braiding of the interplanetary magnetic field is produced by the turbulent motions of the magnetic footpoints on the solar wind source surface. The source surface is the inner boundary of the solar wind magnetic field. In the solar corona, the magnetic field lines can be regarded as an admixture of closed and open field lines all originating from the solar photosphere. Potential field source surface models (Alscher & Newkirk 1969; Schatten et al. 1969; Schrijver & De Rosa 2003; Gombosi et al. 2018) are commonly used to describe the coronal expansion of the magnetic field from the photosphere to the source surface. Such a coronal expansion of the field is not the subject of the present work. Magnetic footpoints refer here to those at the solar wind source surface, which for the purpose of modeling their turbulent motions is identified with the photosphere. Moreover, the footpoint motions are here fully specified by the source surface stream function \( \Phi \) (cm\textsuperscript{2}s\textsuperscript{-1}), according to
\[
\frac{d\theta}{dt} = \frac{1}{r_0^2 \sin \theta} \frac{\partial \Phi}{\partial \phi}, \quad \frac{d\phi}{dt} = \Omega - \frac{1}{r_0^2 \sin \theta} \frac{\partial \Phi}{\partial \theta}. \quad (52)
\]

The typical structure of the toroidal component of the photospheric convection spectrum is the network of vortices depicted in Figure 2 of Rincon et al. (2017). There are three important parameters that can be extracted from the vortical component of photospheric turbulence. These parameters are the typical amplitude of the velocity fluctuations \( V_{rms} = \Phi_{rms}/r_0 \text{cm} \text{s}^{-1} \), the correlation length \( \lambda_c = r_0 \Omega_c / \text{cm} \), and the correlation time \( \tau_c(s) \) of the turbulence, where \( \Phi_{rms} = \sqrt{\left( \Phi^2 \right)} \) is the rms amplitude of the fluctuating stream function \( \Phi \) and \( \Omega_c \) is the angular size of the turbulent eddies on the source surface. These three independent parameters \( \Phi_{rms}, \Omega_c, \) and \( \tau_c \) can all be derived from a more fundamental quantity, which is the covariance function \( C_\Phi(\Theta, \tau) \) of a source surface eddy. The covariance function is sufficient to characterize the vortical component of the surface flows provided the stream function \( \Phi \) does not significantly deviate from a stationary isotropic Gaussian random field on the sphere. The turbulent deviations of the footpoint motions around the solid body rotation are uniquely controlled by the dimensionless Kubo number
\[
K = \frac{\tau_c V_{rms}}{\lambda_c} = \frac{\tau_c}{\tau_{cross}}, \quad (53)
\]

which represents the typical lifetime \( \tau_c \) of a source surface eddy in units of the crossover time \( \tau_{cross} = V_{rms}/\lambda_c \) (or turnover time) of a footpoint trapped in such an eddy. Due to the very high electrical conductivity of the solar wind plasma, the magnetic field can be considered to be frozen in the solar wind. Thus, the solar wind velocity provides a map between the footpoint stream function and the radial component of the vector potential in the solar wind. Assuming the solar wind velocity to be purely radial and uniform yields the form of the electromagnetic fluctuations in the Giacalone & Jokipii (2004) model (see also Giacalone 1999, 2001) of

\textsuperscript{2} The role of the fluctuating radial component of the field will be considered separately in a forthcoming paper.
boundary-driven solar wind turbulence as
\[
\delta A_r(r, \theta, \phi, t) = -\frac{B_0}{V_{sw}} \Phi(\theta, \phi, t),
\]
\[
r' = t - \frac{r - r_0}{V_{sw}}.
\] (54)

The magnetic fluctuations \(\delta B \) and the drift velocity \(v_d\) of the SEPs are thus both related in this case to the fluctuating velocity \(v_{fp}\) of the magnetic footpoints on the source surface by
\[
\frac{\delta B}{B_0} = -\left(\frac{v_{fp}}{V_{sw}}\right)\left(\frac{r_0}{r}\right)v_d.
\]
\[
v_d = -\mu \cos \chi \left(\frac{v}{V_{sw}}\right)\left(\frac{r}{r_0}\right)v_{fp}.
\] (55)

The drift velocity \(v_d\) of the particles increases linearly with radial distance from the Sun as a result of the boundary-driven magnetic fluctuation being a decreasing function of \(r\), as \(r^{-1}\). The variance of the magnetic fluctuations is related to \(V_{rms}\) on the source surface by
\[
\langle \delta B^2 \rangle = B_0^2 \left(\frac{V_{rms}}{V_{sw}}\right)^2 \left(\frac{r_0}{r}\right)^2,
\] (56)
decreasing as \(r^{-2}\) as originally proposed by Jokipii & Kota (1989). The quasi-static boundary-driven models of Giacalone (1999, 2001) and Giacalone & Jokipii (2004) have been used by several other authors to study how the low-frequency component of the solar wind magnetic field fluctuations impacts the interplanetary transport of SEP’s (Pei et al. 2006; Kelly et al. 2012; Moradi & Li 2019). We have here derived a drift-kinetic version of the model that naturally includes the effects of magnetic focusing and pitch angle scattering on the transport of energetic particles. The drift-kinetic approach adopted here provides a useful alternative to the full orbit description of the transport of SEPs in the fluctuating magnetic fields produced by turbulence on the source surface. Spherical harmonic decomposition is commonly used to analyze full-disk Doppler velocity images of the photosphere (see, e.g., Hathaway et al. 2015). Writing the source surface stream function as
\[
\Phi = \sum_{n=1}^{N} \sum_{m=-n}^{n} t_n^m(t) Y_n^m(\theta, \phi),
\] (57)

it follows that the fluctuations \(\delta A_r\) of the magnetic vector potential in the solar wind, driven by source surface turbulence, can be expressed as
\[
\delta A_r(r, \alpha, \phi, t) = -\frac{B_0}{V_{sw}} \sum_{n=1}^{N} \sum_{m=-n}^{n} \left[ \frac{(2n + 1)(n - m)!}{4\pi(n + m)!} \right]^{1/2}
\times t_n^m \left( t - \frac{r - r_0}{V_{sw}} \right) e^{i m \alpha} p_n^m(\alpha).
\] (58)

The set of Equations (50) with the magnetic potential \(\delta A_r\) given by Equation (54) or by Equation (58) describe the spatial transport of energetic particles in the fluctuating magnetic fields produced by the turbulent motion of the magnetic footpoints on the solar wind source surface. Thanks to HMI high-resolution data, the wavenumber \((m, n)\) dependence of the toroidal coefficients \(t_n^m(\text{cm}^2 \text{s}^{-1})\) entering the particle transport model can be very accurately constrained from observations of the vortical component of the photospheric convection spectrum.

In the scatter-free regime of transport and in the small spiral angle approximation, the angular dispersion in both heliographic latitude and longitude of the guiding centers propagating radially outward is then described by Hamilton equations
\[
\frac{d\alpha}{dt} = -\frac{\partial H}{\partial \phi}, \quad \frac{d\phi}{dt} = \frac{\partial H}{\partial \alpha},
\] (59)

with the Hamiltonian \(H\) (s⁻¹),
\[
H(\alpha, \phi, t) = \mu(t) v \left( \frac{\Omega}{V_{sw}} + \frac{\delta A_r(r = r(t), \alpha, \phi, t)}{B_0 r_0^2} \right).
\] (60)

where \(r(t)\) and \(\mu(t)\) are given by Equations (37)–(38), or by Equation (40) for the group of scatter-free particles that initially were injected in the sunward direction. When the fluctuations are produced by footpoint motions on the source surface, the Hamiltonian is
\[
H(\alpha, \phi, t) = \mu(t) v \left( \frac{\Omega}{V_{sw}} + \frac{1}{t_0^m} \sum_{m=1}^{N} \sum_{n=-n}^{n} \left[ \frac{(2n + 1)(n - m)!}{4\pi(n + m)!} \right]^{1/2}
\times t_n^m \left( t - \frac{r(t) - r_0}{V_{sw}} \right) e^{i m \alpha} p_n^m(\alpha) \right],
\] (61)

involving the toroidal coefficients that can be extracted from the observed photospheric convection spectrum. We shall now focus on the situation where the presence of turbulent magnetic fluctuations in the solar wind results in the angular diffusion of the magnetic field lines in the heliosphere.

5. Transport along Stochastic Parker Spirals

In the diffusion approximation, the angular dispersion of magnetic field lines in the solar wind can be described by the following set of stochastic differential equations for the two heliographic angles as a function of radial distance,
\[
\frac{d\theta}{dr} = \frac{D_{ma}}{V_{sw}} \tan \theta + \sqrt{2D_{ma}} \zeta_{\theta}(r),
\]
\[
\frac{d\phi}{dr} = -\frac{\Omega}{V_{sw}} + \frac{1}{\sin \theta} \sqrt{2D_{ma}} \zeta_{\phi}(r),
\] (62)

where \(D_{ma}\) (cm⁻¹) is the angular diffusivity of the magnetic field lines. The magnetic field lines are here realizations of a random walk on a sphere of increasing radius. The field line angular diffusivity \(D_{ma}\) is related to the field line diffusivity \(D_m\) (cm) by
\[
D_{ma} = r^2 D_m,
\] (63)

and it generally varies with \(r\) except in the case where \(D_m\) decreases as \(r^{-2}\). This spherical random walk, superimposed on the secular drift due to the solar rotation, is aimed at capturing the essence of the cartoon originally drawn by Jokipii & Parker (1969) in their Figure 7, representing the meandering of the magnetic field lines in the heliosphere in the form of stochastic
Parker spirals. By parameterizing the field line random walk by the field-aligned distance \(s\) along the nominal Parker field, the stochastic Parker spirals can be equivalently described by

\[
\frac{dr}{ds} = \cos \psi, \quad \frac{d\theta}{ds} = \frac{D_{ma}^\prime}{\tan \theta} + \sqrt{2D_{ma}^\prime} \zeta(s),
\]

\[
\frac{d\phi}{ds} = -\cos \chi \left( \frac{\Omega}{V_{sw}} \right) + \frac{1}{\sin \theta} \sqrt{2D_{ma}^\prime} \zeta_s(s),
\]

where the coefficient \(D_{ma}^\prime\) (cm\(^{-1}\)) is the angular diffusivity of the field lines per unit of field aligned, which is related to \(D_{ma}\) by

\[
D_{ma}^\prime = \cos \chi D_{ma}.
\]

Therefore, the transport of solar energetic particles propagating along stochastic Parker spirals is governed by the set of stochastic differential equations

\[
\frac{dr}{dt} = \mu v \cos \psi, \quad \frac{d\theta}{dt} = \frac{D_a}{\tan \theta} + \sqrt{2D_a} \zeta(t),
\]

\[
\frac{d\phi}{dt} = -\mu \cos \chi \left( \frac{v}{V_{sw}} \right) \Omega + \frac{1}{\sin \theta} \sqrt{2D_a} \zeta_s(t),
\]

where the coefficient \(D_a\) (s\(^{-1}\)) represents the angular diffusivity of the guiding centers, which is related to the field line angular diffusivity \(D_{ma}\) (cm\(^{-1}\)) by

\[
D_a = |\mu| v \cos \chi D_{ma}.
\]

In the small spiral angle approximation, the angular diffusion of the guiding center propagating scatter-free along the stochastic Parker spirals, under the sole influence of magnetic focusing, is thus governed by

\[
\frac{d\theta}{dt} = \frac{D_a(t)}{\tan \theta} + \sqrt{2D_a(t)} \zeta(t),
\]

\[
\frac{d\phi}{dt} = -\mu(t) \left( \frac{v}{V_{sw}} \right) \Omega + \frac{1}{\sin \theta} \sqrt{2D_a(t)} \zeta_s(t),
\]

with the time-dependent angular diffusion coefficient

\[
D_a(t) = |\mu(t)| v D_{ma},
\]

where \(\mu(t)\) is given by Equation (38). In the work by Jokipii & Parker (1969), the diffusion of the magnetic field lines in the heliosphere is produced by the diffusion of the magnetic footpoints on the spherical source surface, described by the Leighton (1964) equations:

\[
\frac{d\theta}{dt} = \frac{\kappa_a}{\tan \theta} + \sqrt{2\kappa_a} \zeta(t),
\]

\[
\frac{d\phi}{dt} = \Omega + \frac{1}{\sin \theta} \sqrt{2\kappa_a} \zeta_s(t),
\]

where \(\kappa_a\) (s\(^{-1}\)) is the angular diffusivity of the footpoints. We notice that the footpoint diffusivity \(\kappa = \kappa_a \kappa_0^2\) (cm\(^2\) s\(^{-1}\)), entering the Leighton (1964) equations (Equation (70)) has the same physical dimension as the (turbulent) magnetic diffusivity that is commonly used in mean-field electrodynamics models of flux transfer at the solar surface; see, e.g., the reviews by Sheeley (2005) and Jiang et al. (2014). The thermal magnetic diffusivity, estimated by Kubat & Karlicky (1986) to be \(\sim 0.07\) km\(^2\) s\(^{-1}\) at the photosphere, can therefore be regarded as the minimum value that the footpoint diffusivity \(\kappa\) can take, in absence of turbulence, in the Leighton (1964) Equations (70). The calculations of Giacalone & Jokipii (2004) yield values of the footpoint diffusivity of \(\sim 1500-2000\) km\(^2\) s\(^{-1}\), much larger than the thermal diffusivity and very close to those used by Leighton (1964) to describe turbulent flux transfer on the photosphere. We further notice that some authors (Blackman & Field 2003; Brandenburg et al. 2004) advocate the use of a telegrapher equation to describe the early non-Fickian stage of turbulent dispersion processes, such as those of magnetic footpoints on the solar wind source surface. The solution of the telegrapher equation on the sphere is discussed by Broadbridge et al. (2019). A variation of the Leighton (1964) equation, accounting for the observations of subdiffusion of magnetic bright points (Cadavid et al. 1999; Abramenko et al. 2011), is discussed by Stanislavsky & Weron (2007).

The stochastic Parker spirals Equations (62) can be obtained from the Leighton (1964) Equations (70) by taking the solar wind velocity to be radial and uniform, \(dr/dt = V_{sw}\). This shows that the angular diffusivity \(D_{ma}\) of the magnetic field lines driven by the diffusion of their footpoints is related to the angular diffusivity \(\kappa_a\) of the magnetic footpoints by Jokipii & Parker (1969)

\[
D_{ma} = \frac{\kappa_a}{V_{sw}},
\]

which is independent of \(r\). Therefore, the angular diffusivity \(D_a\) of the particles propagating along the stochastic Parker spirals is itself related to the angular diffusivity \(\kappa_a\) of the footpoints by

\[
D_a = |\mu| v \frac{\kappa_a}{V_{sw}} \kappa_a.
\]

The angular diffusivity of the magnetic footpoints on the solar wind source surface is uniquely controlled by the dimensionless Kubo number (111), hence the scaling relation

\[
\kappa_a \sim \frac{\lambda^2}{\tau_0^2} K^\beta.
\]

In the quasilinear regime of footpoint transport, corresponding to \(K \ll 1\), the scaling exponent \(\beta\) in Equation (73) is equal to 2. When \(K\) tends to infinity, corresponding to quasi-stationary convective cells, footpoints do not spread a distance larger than \(\lambda\), and thus \(\kappa_a\) tends to zero. It means here that a quasi-stationary collection of vortical source surface convective cells, with \(\tau_c \gg \tau_{cross}\), is a poor mixer of the magnetic footpoints. Thus, turbulent trapping of the magnetic footpoints can be interpreted as being the fingerprint of the integrability of the Hamiltonian system (52) describing turbulent transport of the footpoints in the limit when \(K \to \infty\). Ideas borrowed from percolation theory (Gruzinov et al. 1990; Isichenko 1992; Milovanov 2009) or the decorrelation trajectory method (Vlad et al. 1998) applied to the problem of chaotic mixing by two-dimensional incompressible flows at large Kubo numbers \(K \gg 1\) yield a value slightly smaller than unity, e.g., \(\alpha \sim 0.7\). The kinetic energy spectrum of surface convective turbulence at the photosphere can be broadly decomposed into two energetically dominant spectral features associated with
supergranules at large angular scales and granules at smaller scales. The Kubo number associated with both of these components of source surface turbulence is of the order of unity (Schrijver & Martin 1990; Berger et al. 1998; Hagenaar et al. 1999; Rincon & Rieutord 2018), and hence, the angular diffusivity of the particles scales in this case as

\[ D_\theta \sim \cos \chi \left( \frac{v}{V_{Sw}} \right) \left( \frac{\lambda^2}{r_0^2} \right)^{r^{-1}}. \]  

(74)

6. Forms of the Longitudinal Distributions

In the ecliptic plane, where \( \theta = \pi/2 \), the angular diffusion of magnetic field lines is described by the Langevin equation,

\[ \frac{d\phi}{dr} = -\frac{\Omega}{V_{Sw}} + \sqrt{2D_{ma}(r)\zeta(r)}, \]  

(75)

describing a random walk on a circle of increasing radius superimposed on the deterministic angular drift due to the solar rotation. In general, the field line angular diffusivity \( D_{ma} \) is a quantity depending on the radial distance \( r \) from the Sun because of the radial evolution of the turbulence in the solar wind. The angular field line density \( f_m(\phi, r) \), representing the number of field lines per unit of longitudinal angle at a given radial distance \( r \) from the Sun, thus satisfies the drift-diffusion equation,

\[ \frac{\partial f_m}{\partial r} - \frac{\Omega}{V_{Sw}} \frac{\partial f_m}{\partial \phi} = \frac{\partial}{\partial \phi} D_{ma}(r) \frac{\partial f_m}{\partial \phi}. \]

(76)

The Green’s function solution of Equation (76) is the wrapped Gaussian distribution on a circle of radius \( r \), given by

\[ f_m(\phi, r) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=-\infty}^{+\infty} \exp \left[ -\frac{(\phi - m + 2\pi n)^2}{2\sigma^2} \right], \]  

(77)

where the mean \( m \) and the variance \( \sigma^2 \) of each Gaussian distribution appearing in the sum (77) are

\[ m = -\frac{\Omega}{V_{Sw}} \Delta r + \phi_0, \quad \sigma^2 = 2 \int_{r_0}^r D_{ma}(r')dr'. \]

(78)

Note that the Green’s function solutions (77) and (78) are for \( D_{ma}(r) \), which has an arbitrary dependence on \( r \), therefore accounting for the evolution of the turbulence with radial distance from the Sun. In the boundary-driven model of Jokipii & Parker (1969), the field line angular diffusivity is a constant independent of \( r \). In this case, the variance of the distribution grows linearly with the distance \( \Delta r = r - r_0 \) as

\[ \sigma^2 = 2D_{ma}\Delta r. \]

(79)

The radial dependence of the diffusion coefficient does not affect the basic shape of the angular distribution (77); it only changes its characteristic scale. Making use of the Poisson summation formula, the distribution (77) can also be expressed as

\[ f_m(\phi, r) = \frac{1}{2\pi} \Theta_3 \left( \frac{\phi - m}{2}, e^{-\frac{\pi i}{2}} \right), \]  

(80)

where the Jacobi theta function \( \Theta_3(\zeta, q) \) is defined as

\[ \Theta_3(\zeta, q) = \sum_{n=-\infty}^{+\infty} q^n e^{in\zeta}. \]

(81)

The longitudinal dispersion can alternatively be characterized by the cosine of the longitude \( \phi \) that satisfies the relation

\[ \langle \cos \phi \rangle = e^{-\frac{\pi^2}{4}}. \]

(82)

For a small longitudinal dispersion, i.e., when \( \sigma^2 \ll 1 \), the angular distribution can well be approximated by the \( n = 0 \) term in the sum (77), which is the Gaussian distribution

\[ f_m(\phi, r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\phi - m)^2}{2\sigma^2}}, \]

(83)

and therefore, at small angles, the mean square angular dispersion satisfies the relation

\[ \langle \phi^2 \rangle = \sigma^2. \]

(84)

For a large longitudinal dispersion, i.e., when \( \sigma^2 \gg 1 \), the angular distribution approaches the uniform distribution

\[ f_m(\phi, r) = \frac{1}{2\pi}, \]

(85)

which is the \( n = 0 \) term in the sum (80), and thus

\[ \langle \cos \phi \rangle = 0. \]

(86)

It may be convenient to represent the Green’s function solution of Equation (76) by the von Mises distribution

\[ f_m(\phi, r) = \exp[\kappa \cos(\phi - m)] \frac{1}{2\pi I_0(\kappa)}, \]

(87)

where \( I_0(\kappa) \) is the modified Bessel function and where the scale parameter \( \kappa \) is related to \( \sigma^2 \) via

\[ \frac{1}{\kappa} = \sigma^2. \]

(88)

A comparison of the circular Gaussian and the von Mises distributions is provided in Figure 1. The radial evolution of the von Mises distribution is plotted in Figure 2. The von Mises distribution has the useful property of being periodic and of reducing to the uniform distribution when \( \kappa \to 0 \). The von Mises distribution (87) belongs to a wide class of probability distribution functions that smoothly interpolate to the Gaussian distribution when the shape parameter \( \kappa \to \infty \). An example is given by the kappa distribution that provides a good fit to the velocity space distribution of accelerated electrons in solar flares (Bian et al. 2014). The energetic particles are here assumed to propagate along stochastic Parker spirals and thus their angular distribution \( f(\phi, r) \), at a given distance \( r \) from the Sun, is expected to coincide with the angular distribution \( f_m(\phi, r) \) of the field lines themselves. This is consistent with the observations of Cohen et al. (2017) showing that the angular spreads of energetic particles, at 1 au, are independent of the charge to mass ratio. In the small spiral angle approximation, the mean and the variance of the angular distribution of a focused distribution (\( \mu = 1 \)) of particles both evolve linearly
with time, according to

\[
m = - \left( \frac{v}{V_{sw}} \right) \Omega t + \phi_0, \quad \sigma^2 = 2nD_{ma}t, \tag{89}
\]

provided the field line angular diffusivity is independent of \( r \).

We note that the von Mises distribution was adopted by Leske et al. (2017) in approximating the angular distribution of ions in the 2012 July 23 event. The superposition of a Gaussian and a uniform distribution was used by Borovsky (2010) to fit long-term ACE measurements of the directional distribution (with respect to the nominal Parker field) of the solar wind magnetic field at 1 au.

### 7. Summary and Conclusions

The physical mechanisms responsible for SEP events involve particle acceleration followed by transport toward the observer. There is a large body of evidence that the acceleration of SEPs, either by second-order Fermi acceleration (Parker & Tidman 1958) of electrons (Miller et al. 1997; Bian & Browning 2008; Bian et al. 2012; Petrov et al. 2012; Bian & Kontar 2013) or diffusive shock acceleration (Malkov & Drury 2001) of protons and heavy ions (Lee & Ryan 1986; Zank et al. 2000; Lee et al. 2003, 2005; Lee et al. 2012; Zank et al. 2015), occurs at coronal or CME-driven shocks. The transport of SEPs depends on the structure of the interplanetary magnetic field that can be affected by various turbulent processes occurring in the solar wind and at its source surface. The transport to the observer of a gyrotropic distribution \( f(r, \mu, t) \) of energetic particles accelerated at the Sun can be well described by the drift-kinetic equation

\[
\frac{\partial f}{\partial t} + \nabla \cdot \left[ \left( \mu v \mathbf{B} + \mathbf{v} \right) f \right] = \frac{\partial}{\partial \mu} \left[ \frac{1}{L_B(r, \cos \chi)} \frac{\partial f}{\partial \mu} \right] = \frac{\partial}{\partial \mu} \left( D_{ma} \frac{\partial f}{\partial \mu} \right), \tag{90}
\]

including the effect on particle transport of the presence of turbulent magnetic fluctuations \( \delta B_\perp \) in the solar wind, via the turbulent drift term in the form

\[
v_d = \mu v \cos \chi \left( \frac{r^2}{r_0^2} \right) \left( \frac{\delta B_\perp}{B_0} \right), \tag{91}
\]

in addition to the secular effect of magnetic focusing due to the spatial variation of the Parker field \( \mathbf{B} \) and the random process of pitch angle scattering. The unit vector along the mean field \( \mathbf{b} \) and the heliospheric focusing length \( L_B \) can both be expressed in terms of the Parker spiral angle \( \chi \) as

\[
\mathbf{b} = \cos \chi \mathbf{u}_r - \sin \chi \mathbf{u}_\phi,
\]

\[
L_B(r, \cos \chi) = \frac{r}{\cos \chi \left( 1 + \cos^2 \chi \right)}, \quad \cos \chi = \sqrt{\frac{1}{1 + \left( \frac{r \sin \theta}{V_{sw}} \right)^2}}, \tag{92}
\]

and in the small spiral angle approximation, they reduce to \( \mathbf{b} = \mathbf{u}_r - \chi \mathbf{u}_\phi \) and \( L_B(r) = r/2 \). Consider a purely azimuthal magnetic fluctuations \( \delta B_\perp \), which is given by the radial component of the fluctuating magnetic vector potential through

\[
\delta B_\perp = \nabla_\perp \times (\delta A_\perp \mathbf{u}_r), \tag{93}
\]
then the kinetic Equation (90) can equivalently be formulated in terms of a set of four differential equations for the heliographic position \( r, \alpha = \cos \theta, \phi \) of the guiding center and for the pitch angle cosine \( \mu \) between their velocity vector and the Parker field, as functions of time,

\[
\begin{align*}
\frac{dr}{dt} &= \mu \nu \cos \chi, \\
\frac{d\alpha}{dt} &= -\frac{\mu \nu \cos \chi}{B_0 r_0^2} \frac{\partial A_r}{\partial \phi}, \\
\frac{d\phi}{dt} &= -\mu \cos \chi \left( \frac{\nu}{V_{\text{sw}}} \right) \Omega + \frac{\mu \nu \cos \chi}{B_0 r_0^2} \frac{\partial A_r}{\partial \alpha}, \quad \text{(94)} \\
\frac{d\mu}{dt} &= \frac{1 - \mu^2}{2L_B(r, \cos \chi)} + \frac{\partial D_{\mu \mu}}{\partial \mu} + \sqrt{2D_{\mu \mu}} \zeta_{\mu}(t). \quad \text{(95)}
\end{align*}
\]

The vector potential \( \delta A_r \) in Equations (94)–(95) is generally a function of both \( r \) and time \( t \). Therefore, the effects on SEP propagation due to the radial evolution of the turbulent magnetic fluctuations are implicitly included in this drift-kinetic transport model. In particular, setting the pitch angle scattering coefficient \( D_{\mu \mu} = 0 \) in Equation (95) results in Equations (94)–(95), providing a description of the scatter-free propagation of SEPs in the radially evolving magnetic turbulence in the solar wind. Moreover, in the limit of vanishing field fluctuations corresponding to \( \delta A_r = 0 \), the model given by Equations (94)–(95) describes the transport of scatter-free particles along nominal Parker spirals. In the small spiral angle approximation, the transport Equations (94)–(95), with \( \delta A_r = 0 \), can be readily integrated to give

\[
r(t) = \sqrt{r_0^2 \left( 1 - \mu_0^2 \right) + (\mu_0 r_0 + \nu t)^2},
\]

\[
\theta(t) = \theta_0, \quad \phi(t) = -\frac{\Omega}{V_{\text{sw}}} (r(t) - r_0) + \phi_0 \quad \text{(96)}
\]

and

\[
\mu(t) = \sqrt{1 - \frac{r_0^2 \left( 1 - \mu_0^2 \right)}{r_0^2 \left( 1 - \mu_0^2 \right) + (\mu_0 r_0 + \nu t)^2}}, \quad \text{(97)}
\]

describing the motion of particles injected and propagating radially upward along a Parker spiral under the sole influence of the magnetic focusing effect. The group of particles injected in the sunward direction will first be mirrored at the turning points and then propagate antisunward according to

\[
r(t) = \sqrt{r_s^2 + (\nu t)^2}, \quad \mu(t) = \sqrt{1 - \frac{r_s^2}{r_s^2 + (\nu t)^2}}, \quad \text{(98)}
\]

where \( r_s \) is the radial position of the points turning the initial sunward direction of particle transport into the antisunward direction,

\[
r_s = r_0 \sqrt{1 - \mu_0^2}, \quad \text{(99)}
\]
in the turbulent magnetic drift velocity \( v_d \).

Taking, however, the guiding-center drift velocity \( v_d \) in the kinetic Equation (90) to be related to the velocity \( v_{fp} \) of the...
magnetic footpoints on the source surface,
\[ v_\perp = -\mu \cos \chi \left( \frac{v}{V_{sw}} \right) \frac{r}{r_0} v_{fp}, \]  
(100)

provides a drift-kinetic description, including the effect of pitch angle scattering, of the spatial transport of SEPs in the fluctuating magnetic fields produced by the turbulent motion of the footpoints on the source surface. This drift-kinetic transport model is equivalent to the set of Equations (94)–(95) with the magnetic potential in the form given by
\[ \delta A_r(n, \alpha, \phi, t) = -\frac{B_0}{V_{sw}} \sum_{n=1}^{N} \sum_{m=-n}^{n} \left[ \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right]^{1/2} \times \left( 1 - \frac{r - r_0}{V_{sw}} \right) e^{i m \phi} P_n^m(\alpha), \]  
(101)

where the wavenumber \((m, n)\) dependence of the toroidal coefficients \(c_n^m\) entering the model can be accurately constrained from observations of the vortical component of the photospheric convection spectrum. This constitutes a drift-kinetic description of the transport of SEPs in the Giacalone & Jokipii (2004) model (see also Giacalone 1999, 2001) of boundary-driven solar wind turbulence.

In the case where the presence of solar wind magnetic field fluctuations yields a spatial diffusion of the particles, the formal substitution
\[ v_\perp f \rightarrow -D \nabla_\perp \overline{f} \]  
(102)
in Equation (94) results in the coarse-grained distribution of the particles \(\overline{f}(r, \mu)\) obeying the kinetic equation (omitting the overbar on \(f\) that denotes the ensemble average)
\[ \frac{\partial \overline{f}}{\partial t} + \nabla_\perp (\mu \nabla_\perp \overline{f}) = \frac{\partial}{\partial \mu} \left( 1 - \frac{\mu^2}{L^2} \right) \nabla_\perp f, \]  
(103)
where \(D \text{ (cm}^2 \text{s}^{-1}\)) is the spatial diffusivity of the particles and where the operator \(\nabla_\perp\) is the restriction to a sphere of radius \(r\) of the \(\nabla\) operator. The spatial diffusivity \(D\) of energetic particles emitted at the Sun is related to their angular diffusivity \(D_a \text{ (s}^{-1}\)) in the heliosphere by
\[ D = r^2 D_a. \]  
(104)

The kinetic Equation (103) is equivalent to the following set of stochastic differential equations
\[ \frac{dv}{dt} = \mu \cos \psi, \quad \frac{d\theta}{dt} = \frac{D_a}{\tan \theta} + \sqrt{2D_a} \zeta_\theta(t), \]  
\[ \frac{d\phi}{dt} = -\mu \cos \chi \left( \frac{v}{V_{sw}} \right) \Omega + \frac{1}{\sin \theta} \sqrt{2D_a} \zeta_\phi(t), \]  
(105)
coupled to Equation (95), which accounts for the effect of magnetic focusing and pitch angle scattering. Equation (105) describes the spatial transport of particles along stochastic Parker spirals. Stochastic Parker spirals (Bian & Li 2021) are realizations of Brownian diffusion (with longitudinal drift) on a sphere of increasing radius, and thus they obey
\[ \frac{d\theta}{dr} = \frac{D_{ma}}{\tan \theta} + \sqrt{2D_{ma}} \zeta_\theta(r), \]  
\[ \frac{d\phi}{dr} = -\frac{\Omega}{V_{sw}} + \frac{1}{\sin \theta} \sqrt{2D_{ma}} \zeta_\phi(r), \]  
(106)
where \(D_{ma} \text{ (cm}^{-1}\)) is the angular diffusivity of the magnetic field lines in the heliosphere. The effect of the radial evolution of the turbulence in the solar wind can be taken into account through a specific, model-related dependence on \(r\) of the field line angular diffusivity. The angular diffusivity \(D_a\) of the particles propagating along stochastic Parker spirals is related to the angular diffusivity \(D_{ma}\) of the magnetic field lines by
\[ D_a = |\mu| v \cos \chi D_{ma}. \]  
(107)
In the work by Jokipii & Parker (1969), the braiding of the interplanetary magnetic field is produced by the Brownian diffusion of the magnetic footpoints on the solar wind source surface, described by the Leighton (1964) equations
\[ \frac{d\theta}{dt} = \kappa_\theta \tan \theta + \sqrt{2\kappa_\theta} \zeta_\theta(t), \]  
\[ \frac{d\phi}{dt} = \Omega + \frac{1}{\sin \theta} \sqrt{2\kappa_\theta} \zeta_\phi(t), \]  
(108)
where \(\kappa_\theta \text{ (s}^{-1}\)) is the angular diffusivity of the footpoints. The braided magnetic field lines are obtained in this case by radially unfolding at the solar wind speed \(V_{sw}\) the Brownian paths followed by the footpoints on the source surface, and thus
\[ D_{ma} = \frac{\kappa_\theta}{V_{sw}}. \]  
(109)
which is independent of \(r\). As a consequence, the angular diffusivity \(D_a\) of SEPs propagating along stochastic Parker spirals that are produced by source surface turbulence is related to the angular diffusivity \(\kappa_a\) of the magnetic footpoints on the source surface by
\[ D_a = |\mu| v \cos \chi \frac{v}{V_{sw}} \kappa_a. \]  
(110)
The angular diffusion of the magnetic footpoints on the solar wind source surface is uniquely controlled by the dimensionless Kubo number
\[ K = \frac{\tau_c V_{rms}}{\lambda_c}, \]  
(111)
that represents the typical lifetime \(\tau_c\) of an eddy in units of the turnover time \(\tau_{cross} = V_{rms}/\lambda_c\) of footpoints advected on the source surface by such an eddy. The Kubo number associated with supergranular- and granular-scale convection is of the order unity, \(K \sim 1\), and thus the angular diffusivity of the magnetic field lines scales, in this case, as
\[ D_{ma} \sim \left( \frac{V_{rms}}{V_{sw}} \right) \left( \frac{\lambda_c}{r_0} \right)^2. \]  
(112)
Assuming that the accelerated particles propagate along stochastic Parker spirals, their heliolongitudinal distribution at a radial distance \(r\) from the Sun is given by the circular
Gaussian distribution

\[ f(\Delta \phi, r) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(\Delta \phi + 2\pi n)^2}{2\sigma^2(r)} \right]. \]  
\[ \Delta \phi \text{ is the deviation from the angle of best longitudinal connection between the spacecraft and its magnetic footpoint, as determined by the nominal Parker spiral. The variance } \sigma^2(r) \text{ of the distribution, at a distance } r \text{ from the Sun, is related to the field line angular diffusivity } D_{\text{nl}}(r) \text{ via} \]

\[ \sigma^2(r) = 2 \int_{r_0}^{r} D_{\text{nl}}(r')dr'. \]

The mean cosine of \( \Delta \phi \) is related to the variance \( \sigma^2 \) of the distribution via

\[ \langle \cos \Delta \phi \rangle = e^{-\frac{\sigma^2}{2}}. \]

The circular Gaussian distribution can alternatively be expressed in terms of the Jacobi theta function as

\[ f(\Delta \phi, r) = \frac{1}{2\pi} \Theta_3 \left( \frac{\Delta \phi}{2}, e^{-\frac{\sigma^2}{4}} \right). \]

For a small dispersion, the angular distribution can be well approximated by the \( n = 0 \) term in the sum (113), which is the Gaussian distribution and therefore the mean square angular deviation is given by \( \langle \Delta \phi^2 \rangle = \sigma^2 \). In the opposite situation when the longitudinal dispersion is large, the angular distribution approaches a uniform distribution and thus \( \langle \cos \Delta \phi \rangle = 0 \). In the boundary-driven model of Jokipii & Parker (1969), the angular diffusivity is a constant independent of \( r \) and therefore the variance grows linearly with the radial distance as

\[ \sigma^2 = 2 \frac{\kappa}{V_{sw}} \Delta r. \]

The circular Gaussian can be conveniently represented by the von Mises distribution

\[ f(\Delta \phi, r) = \frac{\exp(\kappa \cos \Delta \phi)}{2\pi I_0(\kappa)}, \]

where the scale parameter \( \kappa \) is related to the variance of the circular Gaussian distribution via \( \kappa^{-1} = \sigma^2 \). Notice that the effect of the radial evolution of the turbulence, via the radial dependence of the diffusion coefficient on \( r \), has no effect on the shape of these angular distributions—it only affects their scale. In directional statistics, the von Mises distribution is constructed in the same spirit as the kappa distribution. It is a distribution that interpolates between the Gaussian distribution at a small angular spread, when \( \kappa \to \infty \) and the uniform distribution when \( \kappa \to 0 \), farther away from the source of accelerated particles during solar flares.

This summarizes our generalizations of the work by Jokipii & Parker (1969), taking into account the combined role of heliospheric magnetic focusing and pitch angle scattering on the transport of energetic particles along stochastic Parker spirals in the solar wind. Combined, these models open additional windows for improving our understanding of the transport of SEPs in the interplanetary medium.
