VARIATIONAL CONSTRAINED MECHANICS ON LIE AFFGEBROIDS

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ABSTRACT. In this paper we discuss variational constrained mechanics (vako-
nomic mechanics) on Lie affgebroids. We obtain the dynamical equations and
the aff-Poisson bracket associated with a vakonomic system on a Lie affgebroid
A. We devote special attention to the particular case when the nonholonomic
constraints are given by an affine subbundle of A and we discuss the varia-
tional character of the theory. Finally, we apply the results obtained to several
examples.

1. Introduction

Lie algebroids have deserved a lot of interest in recent years (from a theoretical
and applied point of view). In the context of Mechanics, an ambitious program
was proposed by Weinstein [34] in order to develop geometric formulations of the
dynamical behavior of Lagrangian and Hamiltonian systems on Lie algebroids. In
the last years, this program has been actively developed by many authors. In fact, a
Klein’s formalism for unconstrained Lagrangian systems on Lie algebroids has been
discussed and a symplectic formulation of Hamiltonian mechanics on these objects
has been developed (see [20, 23, 24, 30]). The main notion is that of prolongation
of a Lie algebroid over a map introduced by Higgins and Mackenzie [14]. An
alternative approach, using the linear Poisson structure on the dual bundle of a Lie
algebroid, was discussed in [12].

An interesting kind of mechanical systems are those subject to external linear
constraints. For these systems, one may derive the dynamical equations using the
Lagrange-D’Alembert principle (nonholonomic mechanics) or using a constrained
variational principle (vakonomic mechanics). The resultant equations are, in gene-
ral, different. Constrained Lagrangian systems (variational or not) have application
in many different areas: engineering, optimal control theory, mathematical econom-
ics, sub-Riemannian geometry, motion of microorganisms, etc. For a geometrical
treatment of standard mechanical systems subject to external linear constraints we
remit to the monographs [2, 5] and references therein.

More recently, several authors discuss the more general class of nonholonomic La-
grangian (Hamiltonian) systems subject to linear constraints on Lie algebroids (see
[7, 8, 27, 28]. In the same Lie algebroid setting, other authors [16] consider vari-
tional constrained mechanical systems. In another direction, a unified approach of
nonholonomic and vakonomic mechanics, using general algebroids instead of just
Lie algebroids, was developed in [9].

As a consequence of all these investigations, one deduces that there are several
reasons for discussing unconstrained (constrained) Mechanics on Lie algebroids:
i) The inclusive nature of the Lie algebroid framework. In fact, under the same umbrella, one can consider standard unconstrained (constrained) mechanical systems, (nonholonomic and vakonomic) Lagrangian systems on Lie algebras, unconstrained (constrained) systems evolving on semidirect products or (nonholonomic and vakonomic) Lagrangian systems with symmetries.

ii) The reduction of a (nonholonomic or vakonomic) mechanical system on a Lie algebroid is a (nonholonomic or vakonomic) mechanical system on a Lie algebroid. However, the reduction of an standard unconstrained (constrained) system on the tangent (cotangent) bundle of the configuration manifold is not, in general, an standard unconstrained (constrained) system.

iii) The theory of Lie algebroids gives a natural interpretation of the use of quasi-coordinates (velocities) in Mechanics (particularly, in nonholonomic and vakonomic mechanics).

On the other hand, in [10, 26] an affine version of the notion of a Lie algebroid structure was introduced. The resultant geometric object is called a Lie affgebroid structure. A Lie affgebroid structure on an affine bundle $\mathcal{A}$ is equivalent to a Lie algebroid structure on the bidual bundle to $\mathcal{A}$ such that the section of the affine dual to $\mathcal{A}$ induced by the constant map on $\mathcal{A}$ equal to 1 is a 1-cocycle.

Lie affgebroid structures may be used to develop a time-dependent version of unconstrained Lagrangian and Hamiltonian equations on Lie algebroids (see [11, 17, 23, 26, 31]). In addition, in [15] the authors present a geometric description of Lagrangian and Hamiltonian systems on Lie affgebroids subject to affine nonholonomic constraints. If we apply this general theory to the particular case when the Lie affgebroid is the 1-jet bundle of local sections of a fibration $\tau : Q \rightarrow \mathbb{R}$ then one recovers some results obtained in [4, 18, 19] for standard time-dependent nonholonomic Lagrangian systems subject to affine constraints. The same reasons for discussing unconstrained (constrained) mechanics on Lie algebroids are also valid for discussing unconstrained (constrained) mechanics on Lie affgebroids.

On the other hand, in [33] the authors discuss standard time-dependent vakonomic dynamics and its relation with presymplectic geometry. More recently, in [1] a geometric approach to time-dependent optimal control problems is proposed. This formulation is based on the Skinner and Rusk formalism for Lagrangian and Hamiltonian systems. Some applications are also presented. The aim of this paper is to extend these formulations to the Lie affgebroid setting or, in other words, to discuss vakonomic mechanics on Lie affgebroids.

The paper is organized as follows. In Section 2, we recall some well-known facts about the geometry of Lie affgebroids and about the unconstrained Hamiltonian formalism on Lie affgebroids. In Section 3, we obtain the vakonomic equations and the vakonomic bracket for a constrained mechanical system on a Lie affgebroid $\mathcal{A}$. We devote special attention to the particular case when the constraints are given by an affine subbundle of $\mathcal{A}$. We also discuss, in this section, the variational character of the theory. In section 4, we apply the results obtained in the paper to several examples. In fact, we develop a Skinner-Rusk formalism on Lie affgebroids. We also consider vakonomic Mechanics on a Lie affgebroid $\mathcal{A}$, for the particular case when $\mathcal{A}$ is the 1-jet bundle of local sections of a fibration over $\mathbb{R}$. As a consequence, we recover some previous results in the literature. We also discuss optimal control systems as vakonomic systems on Lie affgebroids. The paper ends with our conclusions and a description of future research directions.
2. Hamiltonian Formalism on Lie Affgebroids

2.1. Lie affgebroids. Let $\tau_A : A \to Q$ be an affine bundle with associated vector bundle $\tau_V : V \to Q$. Denote by $\tau_A^+ : A^+ = \text{Aff}(A, \mathbb{R}) \to Q$ the dual bundle whose fibre over $x \in Q$ consists of affine functions on the fibre $A_x$. Note that this bundle has a distinguished section $1_A \in \Gamma(\tau_A^+)$ corresponding to the constant function 1 on $A$. We also consider the bidual bundle $\tau_{\bar{A}} : \bar{A} \to Q$ whose fibre at $x \in Q$ is the vector space $(A_x^+)^*$. Then, $A$ may be identified with an affine subbundle of $\bar{A}$ via the inclusion $i_A : A \to \bar{A}$ given by $i_A(a)(\varphi) = \varphi(a)$, which is an injective affine map whose associated linear map is denoted by $i_V : V \to \bar{A}$. Thus, $V$ may be identified with a vector subbundle of $\bar{A}$.

A Lie affgebroid structure on $A$ consists of a Lie algebra structure $[,]_A$ on the space $\Gamma(\tau_V)$ of the sections of $\tau_V : V \to Q$, a $\mathbb{R}$-linear action $D : \Gamma(\tau_A) \times \Gamma(\tau_V) \to \Gamma(\tau_V)$ of the sections of $A$ on $\Gamma(\tau_V)$ and an affine map $\rho_A : A \to TQ$, the anchor map, satisfying the following conditions:

- $D_X[\bar{Y},\bar{Z}]_A = [D_X\bar{Y},\bar{Z}]_A + [\bar{Y},D_X\bar{Z}]_A$,
- $D_{X+\bar{Z}}\bar{Y} = D_X\bar{Z} + [\bar{Y},\bar{Z}]_A$,
- $D_X(f\bar{Y}) = fD_X\bar{Y} + \rho_A(X)(f)\bar{Y}$,

for $X \in \Gamma(\tau_A)$, $\bar{Y}, \bar{Z} \in \Gamma(\tau_V)$ and $f \in C^\infty(Q)$ (see [10, 26]).

If $(\{[,\cdot]_A,D,\rho_A\}$ is a Lie affgebroid structure on an affine bundle $A$ then $(V,\{[,\cdot]_A,D,\rho_A\}$ is a Lie algebroid, where $\rho_V : V \to TQ$ is the vector bundle map associated with the affine morphism $\rho_A : A \to TQ$ (for the definition and properties of Lie algebroids we remit to [22]).

A Lie affgebroid structure on an affine bundle $\tau_A : A \to Q$ induces a Lie algebroid structure $(\{[,\cdot]_{\bar{A}},\rho_{\bar{A}}\}$ on the bidual bundle $\bar{A}$ such that $1_A \in \Gamma(\tau_{\bar{A}})$ is a 1-cocycle in the corresponding Lie algebroid cohomology, that is, $d^A1_A = 0$. Here, $d^A$ is the differential of the Lie algebroid $(\bar{A},\{[,\cdot]_{\bar{A}},\rho_{\bar{A}}\}$.

Conversely, let $(U,\{[,\cdot]_U,\rho_U\}$ be a Lie algebroid over $Q$ and $\phi : U \to \mathbb{R}$ be a 1-cocycle of $(U,\{[,\cdot]_U,\rho_U\}$ such that $\phi|_U \neq 0$, for all $x \in Q$. Then, $\bar{A} = \phi^{-1}\{1\}$ is an affine bundle over $Q$ which admits a Lie affgebroid structure in such a way that $(U,\{[,\cdot]_U,\rho_U\}$ may be identified with the bidual Lie algebroid $(\bar{A},\{[,\cdot]_{\bar{A}},\rho_{\bar{A}}\}$ to $A$ and, under this identification, the 1-cocycle $1_A : \bar{A} \to \mathbb{R}$ is just $\phi$. The affine bundle $\tau_A : A \to Q$ is modelled on the vector bundle $\tau_V : V \to Q = \phi^{-1}\{0\} \to Q$.

Let $\tau_A : A \to Q$ be a Lie affgebroid modelled on the Lie algebroid $\tau_V : V \to Q$. Suppose that $(x^i)$ are local coordinates on an open subset $U$ of $Q$ and that $\{e_0,e_\alpha\}$ is a local basis of $\Gamma(\tau_{\bar{A}})$ in $U$ which is adapted to the 1-cocycle $1_A$, i.e., such that $1_A(e_0) = 1$ and $1_A(e_\alpha) = 0$, for all $\alpha$. Note that if $\{e^0,e^\alpha\}$ is the dual basis of $\{e_0,e_\alpha\}$ then $e^0 = 1_A$. Moreover, since $1_A$ is a 1-cocycle, we have that

$$
\begin{align*}
[e_0,e_\alpha]_{\bar{A}} &= C^\gamma_{0\alpha} e_\gamma, &\quad [e_\alpha,e_\beta]_{\bar{A}} &= C^\gamma_{\alpha\beta} e_\gamma, &\quad \rho_{\bar{A}}(e_0) &= \rho^i_0 \frac{\partial}{\partial x^i}, &\quad \rho_{\bar{A}}(e_\alpha) &= \rho^i_\alpha \frac{\partial}{\partial x^i}.
\end{align*}
$$

Denote by $(x^i, y^0, y^\alpha)$ the corresponding local coordinates on $\bar{A}$. Then, the local equation defining the affine subbundle $A$ (respectively, the vector subbundle $V$) of $\bar{A}$ is $y^0 = 1$ (respectively, $y^0 = 0$). Thus, $(x^i, y^\alpha)$ may be considered as local coordinates on $A$ and $V$.

The standard example of a Lie affgebroid is the 1-jet bundle $\tau_{1,0} : J^1 \tau \to Q$ of local sections of a fibration $\tau : Q \to \mathbb{R}$. It is well known that $\tau_{1,0}$ is an affine bundle modelled on the vector bundle $\pi = (\pi_Q)|_{V\tau} : V\tau \to Q$, where $V\tau$ is the vertical bundle of $\tau$. Moreover, if $t$ is the usual coordinate on $\mathbb{R}$ and $\eta$ is the closed 1-form
on \(Q\) given by \(\eta = \tau^*(dt)\) then we have the identification \(J^1\tau \cong \{ v \in TQ \mid \eta(v) = 1 \}\) (see, for instance, [32]). Note that \(V\tau = \{ v \in TQ \mid \eta(v) = 0 \}\). Thus, the dual bundle \(J^1\tau\) to \(\tau_{1,0} : J^1\tau \to Q\) may be identified with the tangent bundle \(TQ\) to \(Q\) and, under this identification, the Lie algebroid structure on \(\pi_Q : TQ \to Q\) is the standard Lie algebroid structure and the 1-cocycle \(1_{\mu,\tau}\) on \(\pi_Q : TQ \to Q\) is just \(\eta\).

2.2. The Hamiltonian formalism. Suppose that \((\tau_{\mathcal{A}} : \mathcal{A} \to Q, \tau_V : V \to Q)\) is a Lie algebroid. Then, we consider the prolongation \(\mathcal{T}^A\mathcal{V}^*\) of the dual Lie algebroid \((\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\rho_{\mathcal{A}})\) over the fibre \(\tau_V : V^* \to Q\) and denote by \((\llbracket ,\llbracket ,\llbracket ,\rho_{\mathcal{A}})\) the Lie algebroid structure on \(\mathcal{T}^A\mathcal{V}^*\) (for the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a fibration, we remit to [14, 20]).

Let \(\mu : \mathcal{A}^+ \to V^*\) be the canonical projection given by \(\mu(\varphi) = \varphi^i\), for \(\varphi \in \mathcal{A}^+_\mathcal{A}\), with \(x \in Q\), where \(\varphi^i \in V^*_\mathcal{A}\) is the linear map associated with the affine map \(\varphi\) and \(h : V^* \to \mathcal{A}^+\) be a Hamiltonian section \(\eta\), that is, \(\mu \circ h = Id\).

Now, we consider the prolongation \(\mathcal{T}^A\mathcal{A}^+\) of the Lie algebroid \(\mathcal{A}\) over \(\tau_{\mathcal{A}^+} : \mathcal{A}^+ \to Q\) with vector bundle projection \(\tau_{\mathcal{A}^+} : \mathcal{T}^A\mathcal{A}^+ \to \mathcal{A}^+\). Then, we may introduce the map \(\mathcal{T}h : \mathcal{T}^A\mathcal{V}^* \to \mathcal{T}^A\mathcal{A}^+\) defined by \(\mathcal{T}h(\mathcal{a}, X_\alpha) = (\mathcal{a}, (\tau_{\mathcal{A}} h)(X_\alpha))\), for \((\mathcal{a}, X_\alpha) \in \mathcal{T}^A\mathcal{V}^*\), with \(\alpha \in V^*\). It is easy to prove that the pair \((\mathcal{T}h, h)\) is a Lie algebroid morphism between the Lie algebroids \(\tau_{\mathcal{V}^*} : \mathcal{T}^A\mathcal{V}^* \to V^*\) and \(\tau_{\mathcal{A}^+} : \mathcal{T}^A\mathcal{A}^+ \to \mathcal{A}^+\). We denote by \(\lambda_h\) and \(\Omega_h\) the sections of the vector bundles \((\mathcal{T}^A\mathcal{V}^*)^* \to V^*\) and \(\Lambda^2((\mathcal{T}^A\mathcal{V}^*)^*) \to V^*\) defined by

\[
\lambda_h = (\mathcal{T}h, h)^*(\lambda_{\mathcal{A}}), \quad \Omega_h = (\mathcal{T}h, h)^*(\Omega_{\mathcal{A}}),
\]

\((\mathcal{T}^A\mathcal{A}^+)^*\) being the Liouville section and the canonical symplectic section, respectively, associated with the Lie algebroid \(\mathcal{A}\) (see [20]). Note that \(\Omega_h = -d^{\mathcal{T}^A\mathcal{V}^*} \lambda_h\).

On the other hand, let \(\eta : \mathcal{T}^A\mathcal{V}^* \to \mathbb{R}\) be the section of \((\mathcal{T}^A\mathcal{V}^*)^* \to V^*\) given by

\[
\eta(\mathcal{a}, X_\nu) = 1_{\mathcal{A}}(\mathcal{a}), \quad \text{for } (\mathcal{a}, X_\nu) \in \mathcal{T}^A_\mathcal{V}^*, \text{ with } \nu \in V^*.
\]

We remark that if \(pr_1 : \mathcal{T}^A\mathcal{V}^* \to \mathcal{A}\) is the canonical projection on the first factor then \((pr_1, \tau_V)\) is a morphism between the Lie algebroids \(\tau_{\mathcal{V}^*} : \mathcal{T}^A\mathcal{V}^* \to V^*\) and \(\tau_{\mathcal{A}} : \mathcal{A} \to Q\) and \((pr_1, \tau_V)^*(1_{\mathcal{A}}) = \eta\). Thus, since \(1_{\mathcal{A}}\) is a 1-cocycle of \(\tau_{\mathcal{A}} : \mathcal{A} \to Q\), we deduce that \(\eta\) is a 1-cocycle of the Lie algebroid \(\tau_{\mathcal{V}^*} : \mathcal{T}^A\mathcal{V}^* \to V^*\).

Let \((x^i)\) be local coordinates on an open subset \(U\) of \(Q\) and \(\{ e_0, e_\alpha \}\) be a local basis of \(\Gamma(\tau_{\mathcal{V}^*})\) on \(U\) adapted to \(1_{\mathcal{A}}\). Denote by \((x^i, y^0, y^\alpha)\) the induced local coordinates on \(\mathcal{A}\) and by \((x^i, y_0, y_\alpha)\) the dual coordinates on \(\mathcal{A}^+\). Then, \((x^i, y_\alpha)\) are local coordinates on \(V^*\) and \(\{ y_0, y_\alpha, \mathcal{U}_\alpha \}\) is a local basis of \(\Gamma(\tau_{\mathcal{V}^*})\), where

\[
y_0(\psi) = \left( e_0(x), \rho_\alpha e^i \frac{\partial}{\partial x^i} \right), \quad y_\alpha(\psi) = \left( e_\alpha(x), \rho_\alpha e^i \frac{\partial}{\partial x^i} \right), \quad \mathcal{U}_\alpha(\psi) = \left( 0, \frac{\partial}{\partial y_\alpha} \right),
\]

for \(\psi \in V^*\). Suppose that \(h(x^i, y_\alpha) = (x^i, -H(x^i, y_\beta), y_\alpha)\) and that \(\{ y^0, y^\alpha, \mathcal{U}_\alpha \}\) is the dual basis of \(\{ y_0, y_\alpha, \mathcal{U}_\alpha \}\). Then \(\eta = y^0\) and, from [21], and the definition of the map \(\mathcal{T}h\), it follows that

\[
\Omega_h = y^0 \wedge \mathcal{U}_\alpha + \frac{1}{2} C^\gamma_{\alpha \beta} y_\gamma y^\alpha \wedge y^\beta + \left( \rho_\alpha \frac{\partial H}{\partial x^i} - C^\gamma_{\alpha \beta} y_\gamma \right) y^0 \wedge y^0 + \frac{\partial H}{\partial y_\alpha} \mathcal{U}_\alpha \wedge y^0.
\]

Thus, it is easy to prove that the pair \((\Omega_h, \eta)\) is a cosymplectic structure on the Lie algebroid \(\tau_{\mathcal{V}^*} : \mathcal{T}^A\mathcal{V}^* \to V^*\) (this means that \(d^{\mathcal{T}^A\mathcal{V}^*} \Omega_h = 0\), \(d^{\mathcal{T}^A\mathcal{V}^*} \eta = 0\) and
(η ∧ Ωₜ ∧ … ∧ Ωₙ)(ψ) ≠ 0, ∀ψ ∈ V*, where n is the rank of A). If Rₜ ∈ Γ(τₜ₄) is the Reeb section of (Ωₜ, η) (that is, iₚΩₜ = 0 and iₚη = 1), then its integral curves (i.e., the integral curves of ρₜ₄(Rₜ)) are just the solutions of the Hamilton equations for h,

\[
\frac{dx^i}{dt} = \rho^i_0 + \rho^i_α \frac{\partial H}{\partial x^α} \quad \frac{dy_α}{dt} = -\rho^i_α \frac{\partial H}{\partial y_α} + y_γ \left( C^γ_αβ \frac{\partial H}{\partial y_β} \right),
\]

for i ∈ {1, …, m} and α ∈ {1, …, n}.

Next, we will present an alternative approach in order to obtain the Hamilton equations. For this purpose, we will use the notion of an aff-Poisson structure on an AV-bundle which was introduced in [10] (see also [11]).

Let τₜ : Z → Q be an affine bundle of rank 1 modelled on the trivial vector bundle τₚ × ℝ : Q × ℝ → Q, that is, τₜ : Z → Q is an AV-bundle in the terminology of [11]. Then, we have an action of ℝ on the fibres of Z. This action induces a vector field Xₜ on Z which is vertical with respect to the projection τₜ : Z → Q.

On the other hand, there exists a one-to-one correspondence between the space of sections of τₜ : Z → Q, Γ(τₜ), and the set \{Fₜ ∈ Cₜ(Z) | Xₜ(Fₜ) = −1\}. In fact, if h ∈ Γ(τₜ) and (x, s) are local fibred coordinates on Z such that Xₜ = \frac{∂}{∂s} and h is locally defined by h(x) = (x, −H(x)), then the function Fₜ on Z is locally given by Fₜ(x, s) = −H(x) − s, (for more details, see [11]).

Now, an aff-Poisson structure on the AV-bundle τₜ : Z → Q is a bi-affine map, \{·, ·\} : Γ(τₜ) × Γ(τₜ) → Cₜ(Q), which satisfies the following properties:

i) Skew-symmetric: \{h₁, h₂\} = −\{h₂, h₁\}.

ii) Jacobi identity: \{h₁, \{h₂, h₃\}\}_V + \{h₂, \{h₃, h₁\}\}_V + \{h₃, \{h₁, h₂\}\}_V = 0, where \{·, ·\}_V is the affine-linear part of the bi-affine bracket.

iii) If h ∈ Γ(τₜ), then the map \{h, ·\} : Γ(τₜ) → Cₜ(Q) defined by \{h, ·\}(h') = \{h, h'\}, for h' ∈ Γ(τₜ), is an affine derivation.

Condition iii) implies that, for each h ∈ Γ(τₜ) the linear part \{h, ·\}_V : Cₜ(Q) → Cₜ(Q) of the affine map \{h, ·\} : Γ(τₜ) → Cₜ(Q) defines a vector field on Q, which is called the Hamiltonian vector field of h (see [11]).

In [11], the authors proved that there is a one-to-one correspondence between aff-Poisson brackets \{·, ·\} on τₜ : Z → Q and Poisson brackets \{·, ·\}_procs on Z which are Xₜ-invariant, i.e., which are associated with Poisson 2-vectors II on Z such that \mathcal{L}_{Xₜ} II = 0. This correspondence is determined by

\{h₁, h₂\} ∗ τₜ = \{Fₜ₁, Fₜ₂\}_procs, for h₁, h₂ ∈ Γ(τₜ).

Using this correspondence one may prove the following result.

**Theorem 2.1.** [17] Let τₐ : A → Q be a Lie affgebroid modelled on the vector bundle τₐ : V → Q. Denote by τₐ₊ : A₊ → Q (resp., τ₊ : V₊ → Q) the dual vector bundle to A (resp., to V) and by µ : A₊ → V₊ the canonical projection. Then:

i) µ : A₊ → V₊ is an AV-bundle which admits an aff-Poisson structure.

ii) If h : V₊ → A₊ is a Hamiltonian section then the Hamiltonian vector field of h with respect to the aff-Poisson structure is a vector field on V₊ whose integral curves are just the solutions of the Hamilton equations for h.

3. **Vakonomic mechanics on Lie affgebroids**

3.1. **Vakonomic equations and vakonomic bracket.** Let τₐ : A → Q be a Lie affgebroid of rank n over a manifold Q of dimension m. We consider an embedded
submanifold $M \subseteq A$, called the constraint submanifold, of dimension $n + m - \tilde{m}$ such that $\tau_M = \tau_A|_M : M \to Q$ is a surjective submersion.

Now, suppose that $e$ is a point of $M$, with $\tau_M(e) = x$, that $(x^i)$ are local coordinates on an open subset $U$, $x \in U$, and that $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau_{\hat{A}})$ on $U$ adapted to the 1-cocycle $1_A$. Denote by $(x^i, y^\alpha)$ (respectively, $(x^i, y^\alpha)$) the corresponding local coordinates for $A$ (respectively, $A$) on the open subset $\tau_{\hat{A}}^{-1}(U)$ (respectively, $\tau_{\hat{A}}^{-1}(U)$). Assume that

$$M \cap \tau_{\hat{A}}^{-1}(U) \equiv \{(x^i, y^\alpha) \in \tau_{\hat{A}}^{-1}(U) \mid \Phi^A(x^i, y^\alpha) = 0, \ A = 1, \ldots, \tilde{m}\}.$$  

The rank of the $(\tilde{m} \times (n + m))$-matrix $\left(\frac{\partial \Phi^A}{\partial x^i}, \frac{\partial \Phi^A}{\partial y^\alpha}\right)$ is maximum, that is, $\tilde{m}$. Then, using that $\tau_M : M \to Q$ is a submersion, we can suppose that the $(\tilde{m} \times \tilde{m})$-matrix

$$\left(\frac{\partial \Phi^A}{\partial y^\beta}\right)_{e=1, \ldots, \tilde{m}; \ B=1, \ldots, \tilde{m}}$$  

is regular. Then, we will use the following notation $(y^\alpha) = (y^\alpha, y^\alpha)$, for $1 \leq \alpha \leq n$, $1 \leq A \leq \tilde{m}$ and $\tilde{m} + 1 \leq a \leq n$.

Now, using the implicit function theorem, we obtain that the exist an open subset $\tilde{V}$ of $\tau_{\hat{A}}^{-1}(U)$, an open subset $W \subseteq \mathbb{R}^{m+n-\tilde{m}}$ and smooth real functions $\Psi^A : W \to \mathbb{R}$, $A = 1, \ldots, \tilde{m}$, such that

$$M \cap \tilde{V} \equiv \{(x^i, y^\alpha) \in \tilde{V} \mid y^A = \Psi^A(x^i, y^\alpha), \ A = 1, \ldots, \tilde{m}\}.$$  

Consequently, $(x^i, y^\alpha)$ are local coordinates on $M$.

Next, consider the Whitney sum of $A$ and $A_\alpha$ and the canonical projections $\text{pr}_1 : A_+ \oplus Q.A \to A_+^+$ and $\text{pr}_2 : A_+^+ \oplus Q.A \to A$. Let $W_0$ be the submanifold of $A_+^+ \oplus Q.A$ given by $W_0 = \text{pr}_2^{-1}(M) = A_+^+ \oplus Q.M$ and the restrictions $\pi_1 = \text{pr}_1|_{W_0}$ and $\pi_2 = \text{pr}_2|_{W_0}$. Also denote by $\nu : W_0 \to Q$ the canonical projection.

Now, we take the prolongation $\tau_{\hat{A}}^+ : \tilde{T}_{\hat{A}}A_+^+ \to A_+^+$ (respectively, $\tau_{\hat{A}}^\nu : \tilde{T}_{\hat{A}}W_0 \to W_0$) of the Lie algebroid $\hat{A}$ over $\tau_{\hat{A}}^+ : A_+^+ \to Q$ (respectively, $\nu : W_0 \to Q$). Moreover, we can prolong $\pi_1 : W_0 \to A_+^+$ to a morphism of Lie algebroids $\tilde{T}_{\pi_1} : \tilde{T}_{\hat{A}}W_0 \to \tilde{T}_{\hat{A}}A_+^+$ defined by $\tilde{T}_{\pi_1} = (Id, T_{\pi_1})$.

If $(x^i, y_0, y_\alpha)$ are the local coordinates on $A_+^+$ induced by the dual basis $\{e_0, e_\alpha\}$ of the local basis $\{e_0, e_\alpha\}$ of $\Gamma(\tau_{\hat{A}})$, then $(x^i, y_0, y_\alpha, y^\alpha)$ are local coordinates for $W_0$ and we can consider the local basis $\{y_0, y_\alpha, \mathcal{P}_0, \mathcal{P}_\alpha, \mathcal{V}_\alpha\}$ of $\Gamma(\tau_{\hat{A}}^\nu)$ defined by

$$y_0(\varphi, a) = \left(e_0(x), \rho^i_0 \frac{\partial}{\partial x^i}, \rho^\alpha_0 \frac{\partial}{\partial y^\alpha}, 0\right), \quad y_\alpha(\varphi, a) = \left(e_\alpha(x), \rho^i_\alpha \frac{\partial}{\partial x^i}, 0\right),$$  

$$\mathcal{P}_0(\varphi, a) = \left(0, \frac{\partial}{\partial y_0}, 0\right), \quad \mathcal{P}_\alpha(\varphi, a) = \left(0, \frac{\partial}{\partial y_\alpha}, 0\right), \quad \mathcal{V}_\alpha(\varphi, a) = \left(0, 0, \frac{\partial}{\partial y^\alpha}\right),$$  

for $(\varphi, a) \in W_0$ and $\nu(\varphi, a) = x$, where $\rho^i_0$ and $\rho^i_\alpha$ are the components of the anchor map $\rho^i_{\hat{A}}$ with respect to the local basis $\{e_0, e_\alpha\}$.

Now, one may consider on the Lie algebroid $\tau_{\hat{A}}^\nu : \tilde{T}_{\hat{A}}W_0 \to W_0$ the presymplectic 2-section $\Omega = (\tilde{T}_{\pi_1}, \pi_1)^* \Omega_{\hat{A}}$, where $\Omega_{\hat{A}}$ is the canonical symplectic section on $\tilde{T}_{\hat{A}}A_+^+$. The local expression of $\Omega$ is

$$\Omega = y^0 \wedge \mathcal{P}_0 + y^\alpha \wedge \mathcal{P}_\alpha + C^\gamma_{0\alpha} y^\gamma \wedge y^\alpha + \frac{1}{2} C^\gamma_{\alpha\beta} y^\gamma \wedge y^\alpha \wedge y^\beta, \quad (3.1)$$  

$\{y^0, y^\alpha, \mathcal{P}_0, \mathcal{P}_\alpha, \mathcal{V}_\alpha\}$ being the dual basis of the local basis $\{y_0, y_\alpha, \mathcal{P}_0, \mathcal{P}_\alpha, \mathcal{V}_\alpha\}$.  

On the other hand, if $pr_1 : \mathcal{T}^\lambda W_0 \to \tilde{A}$ is the canonical projection on the first factor, then we can introduce the section $\eta \in \Gamma({(\mathcal{T}^\lambda W_0)}^*)$ defined by $\eta = (pr_1, \nu)^*1_A$.

Since $1_A$ is a 1-cocycle of $\tilde{A} \to Q$, we deduce that $\eta$ is a 1-cocycle of $\mathcal{T}^\lambda W_0 \to W_0$.

Moreover, it is easy to prove that

$$\eta = \eta^0. \quad (3.2)$$

Now, let $L : A \to \mathbb{R}$ be a Lagrangian function on $A$ and denote by $\tilde{L}$ the restriction of $L$ to the constraint submanifold $M$.

The Pontryagin Hamiltonian $H_{W_0}$ is the real function in $W_0 = A^+ \oplus Q M$ given by

$$H_{W_0}(\varphi, a) = \varphi(a) - \tilde{L}(a),$$

or, in local coordinates,

$$H_{W_0}(x^i, y_0, y^a, y^\sigma) = y_0 + y_a y^a + y_A \Psi^A(x^i, y^\sigma) - \tilde{L}(x^i, y^\sigma). \quad (3.3)$$

Thus, one can consider the presymplectic 2-section $\Omega_{W_0}$ on $\mathcal{T}^\lambda W_0$ defined by

$$\Omega_{W_0} = \Omega + d\mathcal{T}^\lambda W_0 H_{W_0} \wedge \eta.$$

In local coordinates, using (3.1), (3.2) and (3.3), we deduce that

$$\Omega_{W_0} = \eta^a \wedge p_a + \left( \frac{\partial \Psi^A}{\partial x^i} - \frac{\partial \tilde{L}}{\partial x^i} \right) \rho^i_a + C_{\alpha_\sigma}^\gamma y_\gamma \left[ \eta^a \wedge (\eta^0 + y^a p_a) \wedge \eta^0 ight] 
+ \Psi^A p_a \wedge \eta^0 + \frac{1}{2} C_{\alpha_\beta}^\gamma y_\gamma \wedge \eta^\beta + \left( y_a + y_A \frac{\partial \Psi^A}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} \right) \eta^a \wedge \eta^0. \quad (3.4)$$

**Definition 3.1.** The vakonomic problem $(L, M)$ on the Lie affgebroid $A$ consists of finding the solutions for the equations

$$i_X \Omega_{W_0} = 0 \quad and \quad i_X \eta = 1, \quad with \quad X \in \Gamma({(\mathcal{T}^\lambda A)}^*). \quad (3.5)$$

First, we will obtain the local expression of the vakonomic problem. In general, a section $X$ satisfying the equations (3.5) cannot be found in all points of $W_0$. Thus, we consider the points where (3.5) have sense. We define

$$W_1 = \{ w \in W_0 | \exists Z \in \mathcal{T}^\lambda w W_0 : i_Z \Omega_{W_0}(w) = 0 \ and \ i_Z \eta(w) = 1 \}.$$

In local coordinates, we deduce that $W_1$ is characterized by the equations

$$\varphi_a = y_a + y_A \frac{\partial \Psi^A}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} = 0, \quad \hat{m} + 1 \leq a \leq n.$$

Moreover, a direct computation, using (3.2) and (3.4), proves that the local expression of any section $X$ satisfying the equations (3.5) is of the form

$$X(\Gamma_0, \Gamma^a) = y_0 + \Psi^A y_A + y^a y_a + \Gamma_0 \eta^0 + \left[ \rho^i_a \left( \frac{\partial \tilde{L}}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i} \right) 
- y_\gamma (C_{\alpha_0}^\gamma + \Psi^A C_{\alpha A}^\gamma + y^a C_{\alpha a}^\gamma) \right] \eta^a + \Gamma^a v_a,$$

with $\Gamma_0$ and $\Gamma^a$ arbitrary functions. Consequently, the vakonomic equations are

$$\begin{cases}
\dot{x}^i = \rho^i_0 + \Psi^A \rho^i_A + y^a \rho^i_a, \\
\dot{y}_A = \left( \frac{\partial \tilde{L}}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i} \right) \rho^i_A - y_\gamma (C_{\alpha 0}^\gamma + \Psi^B C_{\alpha A}^\gamma + y^a C_{\alpha a}^\gamma), \\
\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a} \right) = \left( \frac{\partial \tilde{L}}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i} \right) \rho^i_A - y_\gamma (C_{\alpha 0}^\gamma + \Psi^B C_{\alpha a}^\gamma + y^b C_{\alpha ab}^\gamma),
\end{cases} \quad (3.6)$$

for all $1 \leq i \leq m, 1 \leq A \leq \hat{m}$ and $\hat{m} + 1 \leq a \leq n$. 

Remark 3.2. The motion equations for the vakonomic mechanics may be also expressed as follows

\[
\begin{align*}
\dot{x}^i &= \rho_0^i + y^a \rho_a^i, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial x^i} &= -\lambda_A \left[ \frac{d}{dt} \left( \frac{\partial \phi^A}{\partial y^a} \right) - \rho_a^i \frac{\partial \phi^A}{\partial x^i} \right] - \lambda_A \frac{\partial \phi^A}{\partial y^a} \\
\phi^A &= 0,
\end{align*}
\]

where \(\phi^A = y^A - \Psi^A\) and \(\lambda_A = y_A - \frac{\partial L}{\partial y^A}\). Note that in contrast to equations (3.6), equations (3.7) are expressed in terms of the global Lagrangian \(L: \mathcal{A} \to \mathbb{R}\). Thus, the equations (3.6) stress how the information given by the Lagrangian \(L\) outside \(\mathcal{M}\) is irrelevant to obtain the vakonomic equations. This is in contrast with what happens in nonholonomic mechanics (see [15]). \(\diamondsuit\)

Then, we know that there exist sections \(X\) of \(\mathcal{T}^A W_0 | W_1\) satisfying (3.5). However, \(X\) doesn’t belong, in general, to \(\mathcal{T}^A W_1 \subseteq \bar{\mathcal{A}} \times TW_1\). In fact, one may prove that the restriction to \(W_1 \) of \(X_{(y_0, T^\alpha)}\) is a section of \(\mathcal{T}^A W_1 \to W_1\) if and only if

\[
[\mathcal{T}^A W_0 (\phi_b)] (V_a) = (d^\mathcal{T}^A W_0 (\phi_b)) ([\mathcal{T}^A V_a - X_{(y_0, T^\alpha)}]) | W_1, \quad \forall b.
\]

Then we have a system of \((n - \tilde{m})\) equations with \((n - \tilde{m})\) unknowns (the functions \(\mathbf{T}^a\)). Thus, if we denote by \(\mathcal{R}_{ab}\) and \(\mu_b\) the functions

\[
\mathcal{R}_{ab} = ([d^\mathcal{T}^A W_0 (\phi_b)] (V_a)] | W_1 = \left( \frac{\partial^2 \tilde{L}}{\partial y^a \partial y^b} - y_A \frac{\partial^2 \Psi^A}{\partial y^a \partial y^b} \right) | W_1,
\]

\[
\mu_b = ([d^\mathcal{T}^A W_0 (\phi_b)] (\mathcal{T}^A V_a - X_{(y_0, T^\alpha)})) | W_1,
\]

it is clear that the above system has a solution \(\mathbf{T}^a\) if the matrices \(\mathcal{R} = (\mathcal{R}_{ab})\) and \(\mathcal{R}^b = (\mathcal{R}_{ab}; \mu_a)\) have the same rank. Note that even if the above system has a unique solution (i.e., if the matrix \(\mathcal{R} = (\mathcal{R}_{ab})\) is regular), the solution \(\left( X_{(y_0, T^\alpha)} \right) | W_1\) is not, in general, unique (since the function \((\mathbf{T}^a)| W_1\) is still arbitrary).

To solve the above problem, we consider a suitable submanifold \(W'_1\) of \(W_1\) whose intrinsic definition is

\[
W'_1 = \{ w \in W_1 \mid H_{W_1}(w) = 0 \},
\]

where \(H_{W_1}: W_1 \to \mathbb{R}\) is the restriction to \(W_1\) of the Pontryagin Hamiltonian \(H_{W_0}\). In local coordinates, the submanifold \(W'_1\) is given by the equation

\[
y_0 + y_A \Psi^A(x^i, y^b) + y_a y^a - \tilde{L}(x^i, y^b) = 0.
\]

Let \(\Omega_{W'_1}\) (respectively, \(\eta_{W'_1}\)) be the restriction of \(\Omega_{W_0}\) (respectively, \(\eta\)) to \(\mathcal{T}^A W'_1\). Note that the restriction \(\nu'_1 : W'_1 \to Q\) of \(\nu : W_0 \to Q\) to \(W'_1\) is a fibration and, therefore, we can consider the prolongation \(\mathcal{T}^A W'_1\) of the Lie algebroid \(\mathcal{A}\) over \(\nu'_1\). Moreover, we have the following result.

Proposition 3.3. \((\Omega_{W'_1}, \eta_{W'_1})\) is a cosymplectic structure on \(\mathcal{T}^A W'_1\) if and only if for any system of coordinates \((x^i, y_0, y_a, y^a)\) on \(W_0\) we have that

\[
\det(\mathcal{R}_{ab}) = \det \left( \frac{\partial^2 \tilde{L}}{\partial y^a \partial y^b} - y_A \frac{\partial^2 \Psi^A}{\partial y^a \partial y^b} \right) \neq 0, \quad \text{for all point in } W'_1.
\]

Proof. It is clear that \(d^\mathcal{T}^A W'_1 \Omega_{W'_1} = 0\) and \(d^\mathcal{T}^A W'_1 \eta_{W'_1} = 0\).
Now, suppose that the matrix \((R_{ab})\) is regular. Since the rank of the Lie algebroid \(\mathcal{T}\alpha W'_1 \to W'_1\) is \((2n + 1)\), we have to prove that
\[
\ker \Omega_{W'_1}(w'_1) \cap \ker \eta_{W'_1}(w'_1) = \{0\}, \quad \forall w'_1 \in W'_1.
\]
Now, let \(Z \in \ker \Omega_{W'_1}(w'_1) \cap \ker \eta_{W'_1}(w'_1)\). From (3.3), it follows that
\[
(i_Z \Omega_{W_0}(w'_1))(\mathcal{P}^0(w'_1)) = (i_Z \Omega_{W_0}(w'_1))(\mathcal{V}_a(w'_1)) = 0, \quad \forall a.
\]
On the other hand,
\[
(d^{\alpha}W_0H_{W_0})(w'_1)(\mathcal{P}^0(w'_1)) = 1, \quad (d^{\alpha}W_0\varphi_b)(w'_1)(\mathcal{V}_a(w'_1)) = R_{ab}(w'_1), \quad \forall b.
\]
Thus, \(\mathcal{P}^0(w'_1) \not\in \mathcal{T}\alpha W'_1\), \(\mathcal{V}_a(w'_1) \not\in \mathcal{T}\alpha W'_1\) and \(Z \in \ker \Omega_{W_0}(w'_1) \cap \ker \eta(w'_1)\). This implies that \(Z = \lambda_0\mathcal{P}^0(w'_1) + \lambda^a\mathcal{V}_a(w'_1)\). Therefore, since \(Z \in \mathcal{T}\alpha W'_1\), we have that
\[
0 = \lambda_0(d^{\alpha}W_0\varphi_b)(w'_1)(\mathcal{P}^0(w'_1)) + \lambda^a(d^{\alpha}W_0\varphi_b)(w'_1)(\mathcal{V}_a(w'_1)) = \lambda^aR_{ab}(w'_1),
\]
for all \(b\), and, consequently, \(\lambda^a = 0\), for all \(a\). Thus, \(Z = \lambda_0\mathcal{P}^0(w'_1)\) and
\[
0 = \lambda_0(d^{\alpha}W_0H_{W_0})(w'_1)(\mathcal{P}^0(w'_1)) = \lambda_0,
\]
that is, \(Z = 0\).

The converse is proved in a similar way. \(\square\)

**Remark 3.4.** We remark that the condition \(\det (R_{ab}) \neq 0\) implies that the matrix
\[
\left(\frac{\partial \varphi_a}{\partial y^b}\right)_{a,b=m+1,...,n}
\]
is regular. Thus, using the implicit theorem function, we deduce that there exist open subsets \(\tilde{W}_0 \subseteq W_0\), \(\tilde{W} \subseteq \mathbb{R}^{m+n}\) and smooth real functions \(\mu^a : \tilde{W} \to \mathbb{R}, a = m + 1, \ldots, n\), such that \(W_1 \cap \tilde{W}_0\) is locally defined by the equations
\[
y^a = \mu^a(x^i, y_0), \quad a = m + 1, \ldots, n.
\]
Therefore, we may consider \((x^i, y_0, y_a)\) as local coordinates on \(W_1\) and, consequently, from (3.8), we obtain that \((x^i, y_a)\) are local coordinates on \(W'_1\). Thus, a local basis of sections of \(\mathcal{T}\alpha W'_1 \to W'_1\) is given by \(\{y_{0V}, y_{aV'}, \mathcal{P}^0\}_{V'}\), where
\[
y_{0V} = \left(y_0 + \rho^0_i \left(\frac{\partial L}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i}\right)\mathcal{P}^0\right)_{W'_1},
\]
\[
y_{aV'} = \left(y_a + \rho^i_a \frac{\partial \Psi^A}{\partial x^i}\right)_{W'_1},
\]
\[
\mathcal{P}^A_{V'} = \left(\mathcal{P}^A - \Psi^A \mathcal{P}^0 + \frac{\partial \mu^a}{\partial y_A} \mathcal{V}_a\right)_{W'_1}.
\]

Proceeding as in the proof of Proposition [3.3], we deduce the following result.

**Theorem 3.5.** If \((\Omega_{W'_1}, \eta_{W'_1})\) is a cosymplectic structure on the Lie algebroid \(\tau_{\alpha}^V : \mathcal{T}\alpha W'_1 \to W'_1\) then there exists a unique section \(\zeta_1 \in \Gamma(\tau_{\alpha}^V)\) solution of the vakonomic problem \((L, \mathcal{M})\). In fact, \(\zeta_1\) is the Reeb section of \((\Omega_{W'_1}, \eta_{W'_1})\), that is, \(\zeta_1\) is characterized by the conditions \(i_{\zeta_1} \Omega_{W'_1} = 0\) and \(i_{\zeta_1} \eta_{W'_1} = 1\).

The above results suggest us to introduce the following definition.

**Definition 3.6.** The vakonomic system \((L, \mathcal{M})\) on the Lie affgebroid \(\tau_{\alpha} : A \to Q\) is said to be regular if the pair \((\bar{\Omega}_{W'_1}, \bar{\eta}_{W'_1})\) is a cosymplectic structure on the Lie algebroid \(\tau_{\alpha}^V : \mathcal{T}\alpha W'_1 \to W'_1\).
In what follows, we will suppose that \((L, \mathcal{M})\) is a regular vakonomic system on the Lie affgebroid \(\mathcal{A}\). Then, from Theorem 3.5, we have that the vakonomic problem has a unique solution which is the Reeb section \(\zeta_1\) of the symplectic structure \((\Omega_{W_1'}, \eta_{W_1'})\).

First, we will give the local expression of the solution section \(\zeta_1\). Suppose that \((x^i, y_a)\) are local coordinates on \(W_1'\) as in Remark 3.4 and that \(\{y_0, y_{a1'}, \mathcal{P}_{a1'}\}\) is the corresponding local basis of \(\Gamma(\tau_{\mathcal{A}}^{1})\). Then, if \(\{y_0, y_{a1'}, \mathcal{P}_{a1'}\}\) is the dual basis of \(\{y_0, y_{a1'}, \mathcal{P}_{a1'}\}\), we have that (see (3.3))

\[
\Omega_{W_1'} = y_0 \land \mathcal{P}_{a1'} + \frac{1}{2} C_{\alpha \beta \gamma}^\gamma y_\alpha \land y_\beta + \Psi^A \mathcal{P}_{A1'} \land y_0.
\]

Thus, we obtain that

\[
\zeta_1(x^i, y_\alpha) = y_0 + \mu^a(x^j, y_\beta) y_{a1'} + \Psi^A(x^j, \mu^a(x^j, y_\beta)) y_{A1'}
\]

\[
- \left[ y_\gamma \left( C_{\alpha 0}^\gamma + \Psi^A(x^j, \mu^a(x^j, y_\beta)) C_{\alpha A}^\gamma + \mu^a(x^j, y_\beta) C_{\alpha a}^\gamma \right) \right] + \rho_a^i \left( y_A \frac{\partial \Psi^A}{\partial x^i} \right) = \mathcal{P}_{01'},
\]

(3.9)

Now, we will introduce an aff-Poisson bracket on the AV-bundle determined by the constraint submanifolds \(W_1\) and \(W_1'\). For this propose, we define the application \(\mu_1 : \mathcal{W}_1 \to \mathcal{W}_1'\) given by

\[
\mu_1(\varphi, a) = (\varphi - H_{W_1}(\varphi, a) 1_A(x), a),
\]

for \((\varphi, a) \in W_1 \subseteq W_0 = \mathcal{A}^+ \oplus Q \mathcal{M}\), with \(\nu_1(\varphi, a) = x \in Q\).

If \((x^i, y_0, y_a)\) (respectively, \((x^i, y_a)\)) are local coordinates on \(W_1\) (respectively, \(W_1'\)) as in Remark 3.4, we deduce that the local expression of \(\mu_1 : \mathcal{W}_1 \to \mathcal{W}_1'\) is

\[
\mu_1(x^i, y_0, y_a) = (x^i, y_a).
\]

Moreover, we have the following result.

**Theorem 3.7.** If \((L, \mathcal{M})\) is a regular vakonomic system on the Lie affgebroid \(\tau_\mathcal{A} : \mathcal{A} \to Q\), then \(\mu_1 : \mathcal{W}_1 \to \mathcal{W}_1'\) is an AV-bundle which admits an aff-Poisson structure.

**Proof.** It is easy to prove that \(\mu_1 : \mathcal{W}_1 \to \mathcal{W}_1'\) is an AV-bundle (see Section 2.2). In fact, if \(w = (\varphi, a) \in (W_1)_x\), with \(x \in Q\), and \(t \in \mathbb{R}\) then

\[
w + t = (\varphi + t 1_A(x), a).
\]

To define an aff-Poisson bracket on \(\mu_1\) we will introduce a Poisson bracket on \(W_1\) which is invariant with respect to \(X_{W_1}\). Here, \(X_{W_1}\) is the infinitesimal generator of the principle action of \(\mathbb{R}\) on \(W_1\).

Consider the prolongation \(\mathcal{T}(\pi_1)_{|W_1} : \mathcal{T}_\mathcal{A} W_1 \to \mathcal{T}_\mathcal{A} \mathcal{A}^+\) of the restriction \((\pi_1)_{|W_1} : W_1 \to \mathcal{A}^+\) to \(W_1\) of the application \(\pi_1 = p_{|W_0} : W_0 \to \mathcal{A}^+\). It is clear that \((\mathcal{T}(\pi_1)_{|W_1}, (\pi_1)_{|W_1})\) is a Lie algebroid morphism and, therefore, we can introduce the 2-section \(\Omega_{W_1} \in \Gamma(\wedge^2(\tau_{\mathcal{A}}^\circ)^*\mathcal{A}_\mathcal{A})\) defined by

\[
\Omega_{W_1} = (\mathcal{T}(\pi_1)_{|W_1}, (\pi_1)_{|W_1})^* \Omega_{\mathcal{A}},
\]

\(\Omega_{\mathcal{A}}\) being the canonical symplectic 2-section on \(\mathcal{T}_\mathcal{A} \mathcal{A}^+\). Obviously \(d^\mathcal{T}_\mathcal{A} W_1 \Omega_{W_1} = 0\).
If \((x^i, y_0, y_\alpha)\) are local coordinates on \(W_1\) as in Remark 3.4, we can consider the local basis of sections \(\{y_{01}, y_{a1}, \mathcal{P}_0^a, \mathcal{P}_1^a\}\) of \(\tilde{T}^A W_1 \rightarrow W_1\) given by

\[
\begin{align*}
y_{01} &= \left(y_0 + \rho_0 \frac{\partial \mu^a}{\partial x^i} y_a\right) |_{W_1}, \\
y_{a1} &= \left(y_a + \rho_a \frac{\partial \mu^a}{\partial x^i} y_a\right) |_{W_1}, \\
\mathcal{P}_0^a &= (\mathcal{P}_0) |_{W_1}, \\
\mathcal{P}_1^a &= (\mathcal{P}_1) |_{W_1}.
\end{align*}
\]

If \(\{y_{01}^0, y_{a1}^0, \mathcal{P}_0^1, \mathcal{P}_1^1\}\) is the dual basis of \(\{y_{01}, y_{a1}, \mathcal{P}_0^a, \mathcal{P}_1^a\}\), we obtain that

\[
\Omega_{W_1} = y_{01}^0 \wedge \mathcal{P}_{01} + y_{a1}^0 \wedge \mathcal{P}_{a1} + C_0^\gamma y_1^\gamma y_{01}^\alpha + \frac{1}{2} C_0^\gamma y_1^\gamma y_1^\alpha y_1^\beta.
\]

Moreover, the Poisson 2-vector \(\Pi_{W_1}\) determined by the bracket \(\{\cdot, \cdot\}_{W_1}\) is invariant with respect to \(X_{W_1}\). In fact, we have that \(X_{W_1} = \frac{\partial}{\partial y_0}\) and

\[
\Pi_{W_1} = \rho_0 \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_0} + \rho_a \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_\alpha} - C_0^\gamma y_1^\gamma \frac{\partial}{\partial y_\alpha} + \frac{1}{2} C_1^\gamma y_1^\gamma \frac{\partial}{\partial y_\beta}.
\]

Thus, we conclude that \(\mu_1 : W_1 \rightarrow W_1'\) admits an aff-Poisson structure which we denote by \(\{\cdot, \cdot\}_{vak} : \Gamma(\mu_1) \times \Gamma(\mu_1) \rightarrow C^\infty(W_1')\). This structure is characterized by the following condition

\[
\{h'_1, h''_1\}_{vak} \circ \mu_1 = \{F_{h'_1}, F_{h''_1}\}_{W_1}, \text{ for } h'_1, h''_1 \in \Gamma(\mu_1),
\]

\(F_{h'_1}, F_{h''_1}\) being the real functions on \(W_1\) associated with the sections \(h'_1, h''_1\) (as we know, \(X_{W_1}(F_{h'_1}) = X_{W_1}(F_{h''_1}) = -1\)).

\[
\square
\]

The aff-Poisson bracket on the AV-bundle \(\mu_1 : W_1 \rightarrow W_1'\),

\[
\{\cdot, \cdot\}_{vak} : \Gamma(\mu_1) \times \Gamma(\mu_1) \rightarrow C^\infty(W_1'),
\]

is called the vakonomic bracket associated with the regular system \((L, M)\).

On the other hand, note that the restriction \(H_{W_0}\) to \(W_1\) of the Pontryagin Hamiltonian \(H_{W_0}\) verifies that \(X_{W_1}(-H_{W_1}) = -1\). Therefore, there exists \(h_1 \in \Gamma(\mu_1)\) such that \(F_{h_1} = -H_{W_1}\). In fact, \(h_1\) is the inclusion of \(W_1'\) into \(W_1\). Moreover, we have the following result.

**Theorem 3.8.** If \(F_1 \in C^\infty(W_1')\) then the temporal evolution of \(F_1, \dot{F}_1\), is given by

\[
\dot{F}_1 = \{h_1, F_1\}_{vak}^{al},
\]

\(\{\cdot, \cdot\}_{vak}^{al}\) being the affine-linear part of the bi-affine bracket \(\{\cdot, \cdot\}_{vak}\). In other words, the Hamiltonian vector field associated with \(h_1\) with respect to the vakonomic bracket coincides with \(\rho_1^{\nu_1}(\zeta_1)\).
Then, from (3.9), (3.10), (3.11) and Remark 3.4, we deduce that this vector field is the vakonomic bracket is given by
\[ \{h_1, \cdot \}_{vak}(\varphi) \circ \mu_1 = \{F_{h_1}, \varphi \circ \mu_1 \}_{W_1}, \quad \text{for } \varphi \in C^\infty(W'_1). \] (3.11)

Proof. We know that the Hamiltonian vector field \( \{h_1, \cdot \}_{vak} \) of \( h_1 \) with respect to the vakonomic bracket is given by
\[ \{h_1, \cdot \}_{vak}(\varphi) \circ \mu_1 = \{F_{h_1}, \varphi \circ \mu_1 \}_{W_1}, \quad \text{for } \varphi \in C^\infty(W'_1). \]

Then, from (3.9), (3.10), (3.11) and Remark 3.4, we deduce that this vector field is just \( \rho_A^{\nu_1}(\zeta_1) \) (see (3.9)).

Next, let \( h'_1, h''_1 \) be two sections of \( \mu_1 : W_1 \to W'_1 \) and suppose that
\[ h'_1(x^i, y_\alpha) = (x^i, -H'_1(x^i, y_\beta, y_\alpha)) \quad \text{and} \quad h''_1(x^i, y_\alpha) = (x^i, -H''_1(x^i, y_\beta, y_\alpha)). \]

Then, using (3.10), we have that
\[ \{h'_1, h''_1\}_{vak} = \rho^i_\mu \left( \frac{\partial(H'_1 - H''_1)}{\partial x^i} \right) + \rho^i_\mp \left( \frac{\partial H'_1}{\partial y_\alpha} \frac{\partial H''_1}{\partial y_\beta} - \frac{\partial H'_1}{\partial y_\beta} \frac{\partial H''_1}{\partial y_\alpha} \right). \] (3.12)

Since \( \mathcal{A} \) is a Lie affgebroid, it follows that \( \mathcal{A}^+ \) is the total space of an AV-bundle over \( V^* \) with projection \( \mu : \mathcal{A}^+ \to V^* \) and, moreover, the linear Poisson structure \( \Pi_\mathcal{A} \) on \( \mathcal{A}^+ \) (induced by the Lie algebroid structure of \( \mathcal{A} \)) defines an aff-Poisson bracket \( \{\cdot, \cdot\} : \Gamma(\mu) \times \Gamma(\mu) \to C^\infty(V^*) \) on the AV-bundle \( \mu : \mathcal{A}^+ \to V^* \) (see Theorem 2.1).

On the other hand, we may consider the applications \((\pi_1)|_{W_1} : W_1 \to \mathcal{A}^+ \) and \( \mu \circ (\pi_1)|_{W'_1} : W'_1 \to V^* \) and it is clear that \( \mu \circ (\pi_1)|_{W_1} = \mu \circ (\pi_1)|_{W'_1} \circ \mu_1 \).

In fact, using (3.10) and Remark 3.4, we can prove the following result.

**Corollary 3.9.** If \((L, \mathcal{M}) \) is a regular vakonomic system on \( \mathcal{A} \), then the pair \(( (\pi_1)|_{W_1}, \mu \circ (\pi_1)|_{W'_1} \) \) is a local aff-Poisson isomorphism of AV-bundles, that is:

i) \( \mu \circ (\pi_1)|_{W'_1} : W'_1 \to V^* \) is a local diffeomorphism;
ii) The restriction of \((\pi_1)|_{W_1} \) to each fibre of \( \mu_1 : W_1 \to W'_1 \) is an affine isomorphism over the corresponding fibre of \( \mu : \mathcal{A}^+ \to V^* \);
iii) If \( h'_1, h''_1 \in \Gamma(\mu_1) \) and \( h', h'' \in \Gamma(\mu) \) satisfy that
\[ (\pi_1)|_{W_1} \circ h'_1 = h' \circ \mu \circ (\pi_1)|_{W'_1}, \quad (\pi_1)|_{W_1} \circ h''_1 = h'' \circ \mu \circ (\pi_1)|_{W'_1}, \]
then \( \{h'_1, h''_1\}_{vak} = \{h', h''\} \circ \mu \circ (\pi_1)|_{W'_1} \).

**Remark 3.10.** If \( \mu \circ (\pi_1)|_{W'_1} : W'_1 \to V^* \) is a global diffeomorphism then we can define the section \( h \in \Gamma(\mu) \) given by
\[ h = (\pi_1)|_{W_1} \circ h'_1 \circ (\mu \circ (\pi_1)|_{W'_1})^{-1}, \]
and it is clear that \( (\pi_1)|_{W_1} \circ h_1 = h \circ \mu \circ (\pi_1)|_{W'_1} \). This implies that the Hamiltonian vector fields of \( h_1 \) and \( h \) are \(( (\pi_1)|_{W_1}, \mu \circ (\pi_1)|_{W'_1} \)-related. Therefore, if \( \gamma'_1 : I \to W'_1 \) is a solution of the vakonomic equations for the system \((L, \mathcal{M}) \), then \( \mu \circ (\pi_1)|_{W'_1} \circ \gamma'_1 : I \to V^* \) is a solution of the Hamilton equations for \( h \). Conversely, if \( \gamma : I \to V^* \) is a solution of the Hamilton equations for \( h \) then \( (\mu \circ (\pi_1)|_{W'_1})^{-1} \circ \gamma : I \to W'_1 \) is a solution of the vakonomic equations for the system \((L, \mathcal{M}) \).

**Remark 3.11.** If \((L, \mathcal{M}) \) is a vakonomic system on a Lie affgebroid \( \mathcal{A} \) which is not regular then one may apply a constraint algorithm in order to obtain solutions of the vakonomic equations in a suitable Lie subalgebroid of \( \mathcal{T}^A W_1 \to W_1 \).
3.2. Mechanical systems subject to affine constraints on Lie affgebroids.\

The Lagrangian function $L : \mathcal{A} \rightarrow \mathbb{R}$ of a mechanical system on a Lie affgebroid $\tau_\mathcal{A} : \mathcal{A} \rightarrow Q$ is of the form

$$L(a) = \frac{1}{2} \bar{\mathcal{G}}_{\tau_\mathcal{A}(a)}(i_\mathcal{A}(a), i_\mathcal{A}(a)) - \mathcal{V}(\tau_\mathcal{A}(a)), \text{ for all } a \in \mathcal{A},$$

where $\mathcal{G}$ is a bundle metric on $\bar{\mathcal{A}}$ and $\mathcal{V}$ a function on $Q$. We also denote by $\mathcal{G}$ the bundle metric induced on $\mathcal{A}^+$ and we suppose that the 1-cocycle $1_\mathcal{A}$ has constant norm equal to 1. Moreover, using the metric, we can identify $\mathcal{A}$ and $V$. In fact, we have the affine bundle morphism $\mathcal{I} : \mathcal{A} \rightarrow V$ given by

$$\mathcal{I}(a) = i_\mathcal{A}(a) - 1_\mathcal{A}^2(\tau_\mathcal{A}(a)), \text{ for all } a \in \mathcal{A},$$

where $1_\mathcal{A}^0 \in \Gamma(\tau_\mathcal{B})$ is defined by $\varphi(1_\mathcal{A}^0(x)) = \mathcal{G}_x(1_\mathcal{A}(x), \varphi)$, for all $\varphi \in \mathcal{A}^+$. Then, if $\mathcal{G}$ is the restriction to $V$ of $\mathcal{G}$, the Lagrangian $L$ may be written as follows

$$L(a) = \frac{1}{2} \bar{\mathcal{G}}_{\tau_\mathcal{A}(a)}(\mathcal{I}(a), \mathcal{I}(a)) - \bar{\mathcal{V}}(\tau_\mathcal{A}(a)), \text{ for all } a \in \mathcal{A},$$

where $\bar{\mathcal{V}}(x) = \mathcal{V}(x) - \frac{1}{2}$, for all $x \in Q$.

Now, let $(x^i)$ be local coordinates on an open subset $U$ of $Q$. Then, since the section $1_\mathcal{A}$ has constant norm equal to 1, we can consider an orthonormal basis of sections of the vector bundle $\tau_{\mathcal{A}}^{-1}(U) \rightarrow U$ of the form $\{e^0 = 1_\mathcal{A}, e^a\}$. Thus, its dual basis $\{\alpha_0, \alpha_a\}$ is an orthonormal local basis of sections of $\bar{\mathcal{A}}$. Moreover, if $(x^i, y^0, y^a)$ are the corresponding local coordinates on $\bar{\mathcal{A}}$, the local expression of the Lagrangian function is

$$L(x^i, y^a) = \frac{1}{2}(y^a)^2 - \bar{\mathcal{V}}(x^i)$$

and the Euler-Lagrange equations (that is, the vakonomic equations for the system $(L, \mathcal{A})$) reduce to

$$\frac{dx^i}{dt} = \rho_0^i + \rho_a^i y^a, \quad \frac{dy^a}{dt} = -\rho_a^i \frac{\partial \bar{\mathcal{V}}}{\partial x^i} - (C_\gamma^a_{\alpha 0} + C_\gamma^a_{\alpha 0} y^b) y^\gamma.$$

Next, suppose that the constraint submanifold $M$ of the vakonomic system is an affine subbundle $\mathcal{B}$ of $\mathcal{A}$, that is, we have an affine bundle $\mathcal{B}$ over $Q$ with associated vector bundle $\tau_{\mathcal{B}} : \mathcal{U}_B \rightarrow Q$ and the corresponding inclusions $i_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{A}$ and $i_{\mathcal{B}_\gamma} : \mathcal{U}_B \rightarrow V$. Furthermore, assume that $1_\mathcal{A}^0 \in \Gamma(\tau_\mathcal{B})$. Then, we can choose an special coordinate system adapted to the structure of the problem as follows. In fact, we consider local coordinates $(x^i)$ on an open subset $U$ of $Q$ and an orthonormal local basis of $\Gamma(\tau_\mathcal{V})$, $\{e_\alpha, e_a\}$, adapted to the decomposition $V = U_{\mathcal{B}}^{1,3} \oplus U_{\mathcal{B}}$, $U_{\mathcal{B}}^{1,3}$ being the orthogonal subbundle to $U_{\mathcal{B}}$ with respect to the bundle metric $\mathcal{G}$. Thus, we deduce that $\{1_\mathcal{A}^0 = e_0, e_A, e_a\}$ is an orthonormal local basis of $\Gamma(\tau_\mathcal{A})$ adapted to the affine subbundle $\mathcal{B}$. Denote by $(x^i, y^0, y^A, y^a)$ the corresponding local coordinates on $\bar{\mathcal{A}}$ and by $(x^i, y_0, y_A, y_a)$ the dual local coordinates on $\mathcal{A}^+$. Note that the equations which define $\mathcal{B}$ as an affine subbundle of $\mathcal{A}$ are $y^A = 0$. Therefore, the vakonomic system $(L, \mathcal{B})$ is regular and the local expression of the vakonomic equations is

$$\begin{cases}
\frac{dx^i}{dt} = \rho_0^i + \rho_a^i y^a, \\
\frac{dy_\alpha}{dt} = -\rho_a^i \frac{\partial \bar{\mathcal{V}}}{\partial x^i} - (C_\gamma^a_{\alpha 0} + C_\gamma^a_{\alpha 0} y_\alpha) y^\gamma, \\
y^0 = 1, \quad y^A = 0, \quad y^a = y_a, \quad y_0 = \frac{1}{2}(y_a)^2 + \bar{\mathcal{V}}(x^i).
\end{cases}$$
3.3. The variational point of view. Let \( \tau_A : A \to Q \) be a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to Q \) and \( L : A \to \mathbb{R} \) be a Lagrangian function on \( A \). Next, we will show how to obtain the Euler-Lagrange equations on the Lie affgebroid \( A \) from a variational point of view.

We define the set of \( A \)-paths as follows

\[
Adm([t_0, t_1], A) = \{ a : [t_0, t_1] \to A \mid \rho_A \circ a = \frac{d}{dt} (\tau_A \circ a) \},
\]

that is, as the set of admissible curves in \( A \). Then, for two fixed points \( x, y \in Q \), denote by \( Adm([t_0, t_1], A)^y_x \) the set of \( A \)-paths with fixed base endpoints equal to \( x \) and \( y \).

Now, if \( i_V : V \to \tilde{A} \) is the canonical inclusion, we consider as infinitesimal variations the complete lifts of sections of \( \tau_V : V \to Q \) which vanish at the points \( x \) and \( y \), that is,

\[
\{ (i_V \circ \bar{X})^i_x \mid \bar{X} \in \Gamma(\tau_V), \bar{X}(x) = 0 \text{ and } \bar{X}(y) = 0 \}.
\]

Note that if \( \{ e_\alpha, \rho_\alpha \} \) is a local basis of \( \Gamma(\tau_{\tilde{A}}) \) and \( \bar{X} \in \Gamma(\tau_V) \) is locally given by \( \bar{X} = \bar{X}^\alpha e_\alpha \), then \( (i_V \circ \bar{X})^i_x \) is the vector field on \( A \) given by

\[
(i_V \circ \bar{X})^i_x = \bar{X}^\alpha \frac{\partial}{\partial x^i} + \bar{X}_\gamma \frac{\partial}{\partial y^\gamma},
\]

where \( \bar{X}^\alpha = \bar{X}^\alpha \rho_\alpha \), \( \bar{X}_\gamma = \frac{\partial \bar{X}^\alpha}{\partial y^\gamma} (\rho_\gamma + y^\beta \rho_\beta) - \bar{X}^\gamma (C^\gamma_{\alpha\beta} + y^\delta C^\delta_{\gamma\beta}) \), for all \( i \) and \( \alpha \).

On the other hand, we introduce the action functional \( \delta S : Adm([t_0, t_1], A) \to \mathbb{R} \) defined by

\[
\delta S(a) = \int_{t_0}^{t_1} L(a(t)) dt.
\]

With this definition it is not difficult to prove that the critical points of \( \delta S \) on \( Adm([t_0, t_1], A)^y_x \) are the curves \( a \in Adm([t_0, t_1], A)^y_x \) which satisfy the Euler-Lagrange equations (that is, the vakonomic equations for the system \( (L, M) \), with \( M = A \)).

Now, let \( (L, M) \) be a vakonomic system on the Lie affgebroid \( \tau_A : A \to Q \). Denote by \( Adm([t_0, t_1], M)^y_x \) the set of \( A \)-paths on \( M \) with fixed base endpoints equal to \( x \) and \( y \), respectively, that is,

\[
Adm([t_0, t_1], M)^y_x = \{ a \in Adm([t_0, t_1], A)^y_x \mid a(t) \in M, \forall t \in [t_0, t_1] \}.
\]

In this case, we are going to consider infinitesimal variations (that is, complete lifts \( (i_V \circ \bar{X})^i_x \), with \( \bar{X} \in \Gamma(\tau_V) \)) tangent to the constraint submanifold \( M \) and we assume that there exist enough infinitesimal variations of this kind (that is, we are studying the so-called normal solutions of the vakonomic problem). If \( M \) is locally given by the equations \( y^A - \Psi^A(x^i, y^a) = 0 \), for \( A = 1, \ldots, \tilde{m} \), we deduce that the infinitesimal variations must satisfy

\[
(i_V \circ \bar{X})^i_x (y^A - \Psi^A(x^i, y^a)) = 0, \quad \bar{X}(x) = 0, \quad \bar{X}(y) = 0.
\]

Note that if \( a \in Adm([t_0, t_1], M)^y_x \) then

\[
(i_V \circ \bar{X})^i_x (y^A - \Psi^A(x^i, y^a)) \circ a = 0
\]

if and only if

\[
\frac{d\bar{X}^A}{dt} = \rho_i \bar{X}^\alpha \frac{\partial \Psi^A}{\partial x^i} + \frac{d\bar{X}^a}{dt} \frac{\partial \Psi^A}{\partial y^a} - (C^a_{\gamma\beta} + C^a_{\gamma\beta} y^\beta) \bar{X}^\gamma \frac{\partial \Psi^A}{\partial y^a} 
\]

\[
+(C^A_{\gamma\beta} + C^A_{\gamma\beta} y^\beta) \bar{X}^\gamma. \tag{3.13}
\]
Thus, if we consider our infinitesimal variations then
\[
\frac{d}{ds}\big|_{s=0} \int_{t_0}^{t_1} L(a_s(t)) dt = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x^i} \dot{X}^i + \frac{\partial L}{\partial y^A} \frac{\partial \Psi^A}{\partial x^i} X^i + \frac{\partial L}{\partial y^A} \frac{\partial \Psi^A}{\partial y^a} X^a \right) dt = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x^i} \rho^i_A + \frac{\partial L}{\partial y^A} \rho^A_a \right) dt
\]

Now, let \( y_A \) be the solution of the differential equations
\[
\dot{y}_A = \left( \frac{\partial L}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i} \right) \rho^i_A - y_\gamma (C_{A0} + \Psi^B C_{AB} + y^a C_{Aa}),
\]
where
\[
y_a = \frac{\partial L}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a}.
\]

Then, from (3.13), we have that
\[
\frac{d}{dt} (y_A X^A) = y_A \dot{X}^A + y_A \ddot{X}^A = \frac{\partial L}{\partial x^i} \rho^i_A X^A + y_A \rho^i_A \frac{\partial X^A}{\partial t} + \frac{\partial L}{\partial y^A} \frac{\partial \Psi^A}{\partial y^a} + y_A \frac{d}{dt} \left( \frac{\partial \Psi^A}{\partial y^a} \right)
\]

Using this equality, we deduce that
\[
\frac{d}{ds}\big|_{s=0} \int_{t_0}^{t_1} L(a_s(t)) dt = \int_{t_0}^{t_1} \left( \frac{d}{dt} (y_A X^A) - y_A \frac{\partial \Psi^A}{\partial x^i} \rho^i_A X^A + \frac{\partial \Psi^A}{\partial y^a} \rho^A_a \right) dt
\]

Finally, using (3.14) and the fact that
\[
\dot{X}^a = \frac{d X^a}{dt} = \left( C_{A0}^a + \Psi^A C_{Aa} + y^b C_{Ab} \right) \dot{X}^\gamma,
\]
we obtain that
\[
\frac{d}{ds}\big|_{s=0} \int_{t_0}^{t_1} L(a_s(t)) dt = \int_{t_0}^{t_1} \left( \left( \frac{\partial L}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i} \right) \rho^i_A - \frac{\partial L}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a} \right) dt
\]

Since the variations \( \dot{X}^a \) are free, we conclude that the equations are
\[
\begin{align*}
\dot{x}^i &= \rho^i_0 + \Psi^A \rho^i_A + y^a \rho^i_a, \\
\dot{y}_A &= \left( \frac{\partial L}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i} \right) \rho^i_A - y_\gamma (C_{A0} + \Psi^B C_{AB} + y^a C_{Aa}), \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a} \right) &= \left( \frac{\partial L}{\partial x^i} - y_A \frac{\partial \Psi^A}{\partial x^i} \right) \rho^i_A - y_\gamma (C_{A0} + \Psi^B C_{AB} + y^a C_{Aa})
\end{align*}
\]

for all \( 1 \leq i \leq m, \, 1 \leq A \leq \bar{m} \) and \( \bar{m} + 1 \leq a \leq n \), with \( y_a = \frac{\partial L}{\partial y^a} - y_A \frac{\partial \Psi^A}{\partial y^a} \), that is, the vakonomic equations for the system \((L, \mathcal{M})\) on \( \tau_A : A \to Q \) (see (3.16)).
4. Examples

4.1. Skinner-Rusk formalism on Lie affgebroids. Consider on a Lie affgebroid \( \tau_A : \mathcal{A} \to Q \) a vakonomic system \((L, \mathcal{M})\) with \( \mathcal{M} = \mathcal{A} \), that is, a free system. In this case, \( W_0 = A^+ \oplus Q A \) and the Pontryagin Hamiltonian \( H_{W_0} : A^+ \oplus Q A \to \mathbb{R} \) is defined by \( H_{W_0}(\varphi, a) = \varphi(a) - L(a) \). Moreover, the precosymplectic structure \((\Omega_{W_0}, \eta)\) on \( \mathfrak{T} \mathfrak{A} W_0 \) is given by

\[
\Omega_{W_0} = (\mathfrak{T} \mathfrak{pr}_1, \mathfrak{pr}_1)^* \Omega_{\mathfrak{A}} + \eta^\mathfrak{A} W_0 \wedge \eta \quad \text{and} \quad \eta = (\mathfrak{p}_1, \nu)^* 1_A,
\]

where \( \mathfrak{pr}_1 : A^+ \oplus Q A \to A^+ \) is the canonical projection on the first factor, \( \mathfrak{T} \mathfrak{pr}_1 : \mathfrak{T} \mathfrak{A}(A^+ \oplus Q A) \to \mathfrak{T} \mathfrak{A} A^+ \) is its prolongation and \( \mathfrak{p}_1 : \mathfrak{T} \mathfrak{A}(A^+ \oplus Q A) \to \mathfrak{A} \) is the restriction of the projection \( \mathfrak{A} \times T(A^+ \oplus Q A) \to \mathfrak{A} \) (on the first factor) to the prolongation \( \mathfrak{T} \mathfrak{A}(A^+ \oplus Q A) \). In local coordinates, we have that

\[
H_{W_0}(x^i, y_0, y_\alpha, y^\alpha) = y_0 + y_\alpha y^\alpha - L(x^i, y^\alpha),
\]

\[
\Omega_{W_0} = y^\alpha \wedge \mathfrak{p}_\alpha + \left( C_{\alpha \beta}^\gamma y_\gamma - \rho_\alpha^i \frac{\partial L}{\partial x^i} \right) y^\alpha \wedge y^\beta + \frac{1}{2} C_{\alpha \beta \gamma} y_\gamma y^\alpha \wedge y^\beta
\]

and

\[
\eta = y^0.
\]

Then, the submanifold \( W_1' \subseteq A^+ \oplus Q A \) is locally characterized by

\[
y_0 = L(x^i, y_\alpha) - y_\alpha y^\alpha, \quad y_\alpha - \frac{\partial L}{\partial y^\alpha} = 0
\]

and the vakonomic equations reduce to

\[
\begin{cases}
\dot{x}^i = \rho_0^i + y^\alpha \rho_\alpha^i, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) = \rho_\alpha^i \frac{\partial L}{\partial x^i} - (C_{\alpha \beta}^\gamma y_\gamma + C_{\alpha \beta \gamma} y^\alpha) \frac{\partial L}{\partial y^\beta}.
\end{cases}
\]

Thus, if \( (\mathfrak{pr}_2)|_{W_1} : W_1 \to \mathcal{A} \) is the restriction to \( W_1 \) of the canonical projection on the second factor and \( \gamma_1 : I \to W_1 \) is a solution of the vakonomic equations, then \( (\mathfrak{pr}_2)|_{W_1} \circ \gamma_1 \) is a solution of the Euler-Lagrange equations for \( L \).

Note that in the standard case, that is, if \( \mathcal{A} = J^1 \tau \), this procedure is the Skinner-Rusk formulation for time-dependent mechanics (see \cite{[1, 6]}).

4.2. The 1-jet bundle of local sections of a fibration. Let \( \tau : Q \to \mathbb{R} \) be a fibration and \( \tau_{1,0} : J^1 \tau \to Q \) be the associated Lie affgebroid modelled on the vector bundle \( \pi = (\pi_Q)|_{\mathcal{V}_\tau} : \mathcal{V}_\tau \to Q \) (see Section 2.1). If \( (t, q^i) \) are local fibred coordinates on \( Q \) then \( \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial q^i} \right\} \) is a local basis of sections of \( \pi_Q : TQ \to Q \). Denote by \( (t, q^i, \dot{t}, \dot{q}^i) \) the corresponding local coordinates on \( TQ \). Then, the (local) structure functions of \( TQ \) with respect to this local trivialization are given by

\[
C_{ij}^k = 0 \quad \text{and} \quad \rho_j^i = \delta_{ij}, \quad \text{for } i, j, k \in \{0, 1, \ldots, n\}.
\]

Moreover, \( (t, q^i, \dot{q}^i) \) are the corresponding local coordinates on \( J^1 \tau \).

Now, let \( \mathcal{M} \subseteq J^1 \tau \) be a constraint submanifold such that \( \tau_{1,0}|_{\mathcal{M}} : \mathcal{M} \to Q \) is a surjective submersion and \( L : J^1 \tau \to \mathbb{R} \) be a Lagrangian function. Suppose that the constraint submanifold \( \mathcal{M} \) is locally defined by the equations \( \dot{q}^A = \Psi^A(t, q^i, \dot{q}^i) \), where we use the following notation \( (t, q^i, \dot{q}^i, \dot{q}^a) = (t, q^i, \dot{q}^A, \dot{q}^a) \).
Thus, if we apply the results of the Section 3.3 to this particular case, we recover some results obtained in [1]. In particular, using (3.6) and (4.1), it follows that the vakonomic equations reduce to

\[
\begin{align*}
\dot{p}_A &= \frac{\partial \tilde{L}}{\partial q^A} - p_B \frac{\partial \Psi^B}{\partial q^A}, \\
\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial q^A} - p_A \frac{\partial \Psi^A}{\partial q^A} \right) &= \frac{\partial \tilde{L}}{\partial q^A} - p_A \frac{\partial \Psi^A}{\partial q^A}, \\
\dot{q}^A &= \Psi^A(t, q^i, \dot{q}^i),
\end{align*}
\]

where \((t, q^i, p, \dot{p}_i)\) are the local coordinates on \(T^*Q\) induced by the local basis \(\{dt, dq^i\}\). If these equations are written using the Lagrange multipliers (see (3.7)) then they coincide with the equations obtained in [33].

On the other hand, if \((L, M)\) is a regular vakonomic system on \(\tau_{1,0} : J^1\pi \to Q\), then the AV-bundle \(\mu_1 : W_1 \to W_1^{'1}\) is locally defined by \(\mu_1(t, q^i, p, \dot{p}_i) = (t, q^i, p_i)\). Thus, if \(h_1', h_1'' : W_1' \to W_1^{''1}\) are two sections of \(\mu_1 : W_1 \to W_1^{'1}\),

\[
h_1'(t, q^i, p_i) = (t, q^i, -H_1'(t, q^i, p_j), p_i) \quad \text{and} \quad h_1''(t, q^i, p_i) = (t, q^i, -H_1''(t, q^i, p_j), p_i),
\]

then, from (3.12), we deduce that the vakonomic bracket \(\{\cdot, \cdot\}_vak : \Gamma(\mu_1) \times \Gamma(\mu_1) \to C^\infty(W_1^{'1})\) is locally given by

\[
\{h_1', h_1''\}_vak = \frac{\partial(H_1' - H_1'')}{\partial t} + \frac{\partial H_1'}{\partial q^i} \frac{\partial H_1''}{\partial p_i} - \frac{\partial H_1'}{\partial p_i} \frac{\partial H_1''}{\partial q^i}.
\]

### 4.3. Optimal control systems as vakonomic systems on Lie affgebroids

Let \(\tau_A : A \to Q\) be a Lie affgebroid and \(C\) a fibred manifold over the state manifold \(\pi : C \to Q\). We also consider a section \(\sigma : C \to A\) along \(\pi\) and an index function \(l : C \to \mathbb{R}\). The triple \((l, \pi, \sigma)\) is an optimal control system on the Lie affgebroid \(A\).

One important case happens when the section \(\sigma : C \to A\) along \(\pi\) is an embedding. In such a case, we have that the image \(M = \sigma(C)\) is a submanifold of \(A\). Moreover, since \(\sigma : C \to M\) is a diffeomorphism, we can define a Lagrangian function \(L : M \to \mathbb{R}\) by \(L = l \circ \sigma^{-1}\). Therefore, it is equivalent to analyze the optimal control problem defined by \((l, \pi, \sigma)\) (applying the Pontryagin maximum principle) that to study the vakonomic problem on the Lie affgebroid \(\tau_A : A \to Q\) defined by \((L, M)\).

In the general case (when \(\sigma : C \to A\) is not, in general, an embedding), we consider the subset \(\mathcal{J}^A C\) of the product manifold \(A \times TC\) defined by

\[
\mathcal{J}^A C = \{(a, v) \in A \times TC \mid \rho_A(a) = (T\pi)(v)\}.
\]

Next, we will show that \(\mathcal{J}^A C\) admits a Lie affgebroid structure. Let \(\tau_{A}^\pi : \mathcal{T}^A C \to C\) be the prolongation of the biadjoint Lie algebroid \(\mathcal{T}^A\) of \(A\) over the fibration \(\pi : C \to Q\).

On the other hand, let \(\phi : \mathcal{T}^A C \to \mathbb{R}\) be the section of \((\tau_{A}^\pi)^* : (\mathcal{T}^A C)^* \to C\) defined by

\[
\phi(\tilde{a}, X_p) = 1_A(\tilde{a}), \quad \text{for} \quad (\tilde{a}, X_p) \in \mathcal{T}^A_p C, \quad \text{with} \quad p \in C.
\]

Note that if \(\text{pr}_1 : \mathcal{T}^A C \to \mathcal{T}\) is the canonical projection on the first factor then \((\text{pr}_1, \pi)\) is a morphism between the Lie algebroids \(\tau_{A}^\pi : \mathcal{T}^A C \to C\) and \(\tau_{\mathcal{T}^A} : \mathcal{T} \to Q\) and, moreover, we have that \((\text{pr}_1, \pi)^* (1_A) = \phi\). Since \(1_A\) is a 1-cocycle of \(\tau_{\mathcal{T}^A} : \mathcal{T} \to Q\), we deduce that \(\phi\) is a 1-cocycle of the Lie algebroid \(\tau_{A}^\pi : \mathcal{T}^A C \to C\) and, using the fact that \((1_A)|_{\mathcal{T}^A_x} \neq 0\), for all \(x \in Q\), we have that \(\phi|_{\mathcal{T}^A_p C} \neq 0\), for all \(p \in C\).
In addition, it follows that
\[ \phi^{-1}(1) = \{ (\tilde{a}, X_p) \in T^A_q C \mid 1_A(\tilde{a}) = 1 \} = \mathcal{A}^A C. \]

On the other hand, let \( \tau^V_C : T^V C \to C \) be the prolongation of the Lie algebroid \( (V, [\cdot, \cdot], \rho^C) \) over the fibration \( \pi : C \to Q \). Then, it is easy to prove that \( \phi^{-1}(1) = T^V C \). Therefore, we conclude that \( \mathcal{A}^A C \) is an affine bundle over \( C \) with projection \( \tau^A_C : \mathcal{A}^A C \to C \) defined by \( \tau^A_C(a, v) = \pi_C(v) \), where \( \pi_C : TC \to C \) is the canonical projection. Moreover, the affine bundle \( \tau^A_C : \mathcal{A}^A C \to C \) admits a Lie affgebroid structure such that its bidual Lie algebroid is just \( (T^A C, [\cdot, \cdot], \tau^\pi_A, \rho^\pi_A) \) and it is modelled on the Lie algebroid \( \tau^V_C : T^V C \to C \) (see Section 2.1).

Thus, we can consider the constraint submanifold \( M \) of the Lie affgebroid \( \mathcal{A}^A C \) defined by
\[ M = \bigcup_{p \in C} \{ (a, X_p) \in \mathcal{A}^A p \mid \sigma(p) = a \} \]
and the Lagrangian function \( L : \mathcal{A}^A C \to \mathbb{R} \) given by \( L = l \circ \tau^A_C \). Then, \( (L, M) \) is the vakonomic system associated with the optimal control system.

If \( \mathcal{A} = J^1 \tau \) is the 1-jet bundle of local sections of a fibration \( \tau : Q \to \mathbb{R} \), it is easy to prove that the prolongation of \( J^1 \tau \cong TQ \) over \( \pi : C \to Q \) is just \( TC \).

Thus, \( \mathcal{A}^A C = \{ X \in TC \mid dt(X) = 1 \} \cong J^1(\tau \circ \pi) \), \( t \) being the usual coordinate on \( \mathbb{R} \). Under these identifications, the constraint submanifold is
\[ M = \{ X \in TC \mid T\pi(X) = \sigma(\pi_C(X)) \}. \]
Therefore, we recover the construction developed in \([1]\).

**Example 4.1.** We consider the following mechanical problem (see [3] [4] [15] [21] [29]).

A (homogeneous) sphere of radius \( r > 0 \), mass \( m \) and inertia \( mk^2 \) about any axis rolls without sliding on a horizontal table which rotates with time-dependent angular velocity about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere. The configuration space of the sphere is \( Q = \mathbb{R}^3 \times SO(3) \) and the Lagrangian of the system corresponds with the kinetic energy
\[ K(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}(m\dot{x}^2 + mg^2 + mk^2(\omega_x^2 + \omega_y^2 + \omega_z^2)), \]
where \( (\omega_x, \omega_y, \omega_z) \) are the components of the angular velocity of the sphere.

Since the ball is rolling without sliding on a rotating table then the system is subjected to the affine constraints:
\[ \dot{x} - r\omega_y = -\Omega(t)y, \quad \dot{y} + r\omega_x = \Omega(t)x, \]
where \( \Omega(t) \) is the angular velocity of the table. Moreover, it is clear that \( Q = \mathbb{R}^3 \times SO(3) \) is the total space of a trivial principal \( SO(3) \)-bundle over \( \mathbb{R}^3 \) and the bundle projection \( \pi : Q \to \mathbb{R}^3 \) is just the canonical projection on the first factor.

Therefore, we may consider the corresponding Atiyah Lie algebroid \( TQ/SO(3) \) over \( \mathbb{R}^3 \) (see [20] [22]).

Since the Atiyah Lie algebroid \( TQ/SO(3) \) is isomorphic to the product manifold \( T\mathbb{R}^3 \times so(3) \cong T\mathbb{R}^3 \times \mathbb{R}^3 \), then a section of \( TQ/SO(3) \cong T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) is a pair \( (X, u) \), where \( X \) is a vector field on \( \mathbb{R}^3 \) and \( u : \mathbb{R}^3 \to \mathbb{R}^3 \) is a smooth map. Therefore, a global basis of sections of \( T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) is
\[ \{ e_0 = (\frac{\partial}{\partial t}, 0), e_1 = (\frac{\partial}{\partial x}, 0), e_2 = (\frac{\partial}{\partial y}, 0), e_3 = (0, u_1), e_4 = (0, u_2), e_5 = (0, u_3) \}, \]
where \( u_1, u_2, u_3 : \mathbb{R}^3 \to \mathbb{R}^3 \) are the constant maps \( u_1(t, x, y) = (1, 0, 0) \), \( u_2(t, x, y) = (0, 1, 0) \) and \( u_3(t, x, y) = (0, 0, 1) \).
The anchor map $\rho_{T\mathbb{Q}/SO(3)} : T\mathbb{R}^3 \times \mathbb{R}^3 \to T\mathbb{R}^3$ is the projection over the first factor and if $[\cdot, \cdot]_{T\mathbb{Q}/SO(3)}$ is the Lie bracket on the space $\Gamma(T\mathbb{Q}/SO(3))$ then the only non-zero fundamental Lie brackets are

$$[e_3, e_4]_{T\mathbb{Q}/SO(3)} = e_5, \quad [e_4, e_5]_{T\mathbb{Q}/SO(3)} = e_3, \quad [e_5, e_3]_{T\mathbb{Q}/SO(3)} = e_4.$$  

Moreover, $\phi : T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by $\phi(t, x; \dot{t}, \dot{x}, \omega, \omega_y, \omega_z) = \dot{t}$ is a 1-cocycle in the corresponding Lie algebroid cohomology and, then, it induces a Lie affgebroid structure over $A = \phi^{-1}(1) \cong \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$. In addition, the affine bundle $\tau_A : \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ is modelled on the vector bundle $\tau_V : V = \phi^{-1}(0) \cong \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ and its bidual Lie algebroid $\tilde{A}$ is just the Atiyah Lie algebroid $T\mathbb{R}^3 \times \mathbb{R}^3$. Note that the Lie affgebroid structure on $A = \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$ is a special type of Lie affgebroid structure called Atiyah Lie affgebroid structure (see Section 9.3.1 in [17] for a general construction). Thus, $(t, x; \dot{t}, \dot{x}, \dot{y}, \omega, \omega_y, \omega_z)$ may be considered as local coordinates on $A$ and $V$.

It is clear that the Lagrangian function and the nonholonomic constraints are defined on the Atiyah Lie affgebroid $A \equiv \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$ (since the system is $SO(3)$-invariant). In fact, we have a nonholonomic system on the Atiyah Lie affgebroid $A \equiv \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$ (see [15] for more details).

After some computations the equations of motion for this nonholonomic system may be written as follows

$$\dot{x} - r\omega_y = -\Omega(t)y, \quad \dot{y} + r\omega_x = \Omega(t)x, \quad \omega_z = c, \quad (4.2)$$

where $c$ is a constant, together with

$$\dot{x} + \frac{k^2}{k^2 + r^2}(\Omega'(t)y + \Omega(t)\dot{y}) = 0, \quad \dot{y} - \frac{k^2}{k^2 + r^2}(\Omega'(t)x + \Omega(t)\dot{x}) = 0.$$  

Now, we pass to an optimization problem. Assume full control over the motion of the center of the sphere and consider the cost function

$$L(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}((\dot{x})^2 + (\dot{y})^2)$$

and the following optimal control problem: Given two points $q_0, q_1 \in Q$, find an optimal control curve $(t, x(t), y(t))$ on the reduced space that steer the system from $q_0$ and $q_1$ and minimizes $\int_0^1 \frac{1}{2}((\dot{x})^2 + (\dot{y})^2) \, dt$ subject to the constraints defined by equations (4.2).

Note that this problem is equivalent to the optimal control problem defined by the section $\sigma : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$ along $\mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\sigma(t, x, y; u^1, u^2) = (t, x, y; u^1, u^2, \frac{1}{r}(-u^2 + \Omega(t)x), \frac{1}{r}(u^1 + \Omega(t)y), c)$$

and the index function $l(t, x, y; u^1, u^2) = \frac{1}{2}((u^1)^2 + (u^2)^2)$. Since $\sigma$ is obviously an embedding, we deduce the equivalence between both problems.

A necessary condition for optimality of the problem is given by the corresponding vakonomic equations. In this case, we will denote by

$$y^1 = \dot{x}, \quad y^2 = \dot{y}, \quad y^3 = \omega_x, \quad y^4 = \omega_y, \quad y^5 = \omega_z.$$  

Therefore, the vakonomic problem is given by the Lagrangian

$$L(t, x, y; y^1, y^2, y^3, y^4, y^5) = \frac{1}{2}((y^1)^2 + (y^2)^2)$$
and the submanifold $M$ defined by the constraints

$$
y^3 = \Psi^3(t, x, y, y^1, y^2) = \frac{1}{r}(-y^2 + \Omega(t)x),$$
$$y^4 = \Psi^4(t, x, y, y^1, y^2) = \frac{1}{r}(y^1 + \Omega(t)y),$$
$$y^5 = \Psi^5(t, x, y, y^1, y^2) = c.
$$

After some computations, we obtain that the vakonomic equations are

$$
\begin{aligned}
\dot{y}_3 &= -\frac{1}{r}(y^1 + \Omega(t)y)y_5 + cy_4, \\
\dot{y}_4 &= -\frac{1}{r}(y^2 - \Omega(t)x)y_5 - cy_3, \\
\dot{y}_5 &= \frac{1}{r}(y^1 + \Omega(t)y)y_3 - \frac{1}{r}(-y^2 + \Omega(t)x)y_4, \\
\frac{d}{dt}(ry^1 - y_4) &= -\Omega(t)y_3, \\
\frac{d}{dt}(ry^2 + y_3) &= -\Omega(t)y_4, \\
y_1 = \dot{x}, & \quad y_2 = \dot{y}.
\end{aligned}
$$

Moreover, it is easy to prove that the vakonomic system is regular. Therefore, there exists a unique solution of the vakonomic equations on the submanifold $W^*_1$ which is determined by the following conditions

$$
y_1 = \frac{\partial L}{\partial y^3} - y_A\frac{\partial \Psi^4}{\partial y^1} = y^1 - \frac{1}{r}y_4, $$
$$y_2 = \frac{\partial L}{\partial y^2} - y_A\frac{\partial \Psi^4}{\partial y^2} = y^2 + \frac{1}{r}y_3, $$
$$y_0 = L - y_A\Psi^4 - y_a y^a = -\frac{1}{2}(y_1 + \frac{1}{r}y_4)^2 - \frac{1}{2}(y_2 - \frac{1}{r}y_3)^2 - \frac{\Omega(t)}{r}(xy_3 + yy_4) - cy_5.
$$

Thus, it follows that $(t, x, y; y_0, y_1, y_2, y_3, y_4, y_5)$ (respectively, $(t, x, y; y_0, y_1, y_2, y_3, y_4, y_5)$) are local coordinates on $W^*_1$ (respectively, on $W^*_1$). Then, the local expression of the Hamiltonian $H_{W^*_1}$ is

$$
H_{W^*_1}(t, x, y; y_0, y_1, y_2, y_3, y_4, y_5) = -y_0 - \frac{1}{2}(y_1 + \frac{1}{r}y_4)^2 - \frac{1}{2}(y_2 - \frac{1}{r}y_3)^2 - \frac{\Omega(t)}{r}(xy_3 + yy_4) - cy_5.
$$

and, in terms of the affine-linear part $\{\cdot, \cdot\}_{vak}$ of the vakonomic bracket $\{\cdot, \cdot\}_{vak}$ associated with the regular system $(L, M)$, the vakonomic equations are

$$
\begin{aligned}
\dot{y}_1 &= \{h_1, y_1\}_{vak} = -\frac{\Omega(t)}{r}y_3, & \dot{y}_2 &= \{h_1, y_2\}_{vak} = -\frac{\Omega(t)}{r}y_4, \\
\dot{y}_3 &= \{h_1, y_3\}_{vak} = -\frac{1}{r}\left(y_1 + \frac{1}{r}y_4 + \Omega(t)y\right)y_5 + cy_4, \\
\dot{y}_4 &= \{h_1, y_4\}_{vak} = \frac{1}{r}\left(-y_2 + \frac{1}{r}y_3 + \Omega(t)x\right)y_5 - cy_3, \\
\dot{y}_5 &= \{h_1, y_5\}_{vak} = \frac{1}{r}\left(y_1 + \frac{1}{r}y_4 + \Omega(t)y\right)y_3 + \left(y_2 - \frac{1}{r}y_3 - \Omega(t)x\right)y_4, \\
\dot{x} &= \{h_1, x\}_{vak} = y_1 + \frac{1}{r}y_4, & \dot{y} &= \{h_1, y\}_{vak} = y_2 - \frac{1}{r}y_3.
\end{aligned}
$$

$\triangle$
4.3.1. Optimal control of affine control systems. Let $\tau_A : A \rightarrow Q$ be a Lie affgebroid. Suppose that the constraint submanifold $M$ of the vakonomic system is an affine subbundle $B$ of $A$, that is, we have an affine bundle $B$ over $Q$ with associated vector bundle $\tau_B : U_B \rightarrow Q$ and the corresponding inclusions $i_{U_B} : B \rightarrow A$ and $i_{U_B} : U_B \rightarrow V$. Choose now a coordinate system adapted to this affine subbundle $B$. That is, take local coordinates $(x^i)$ on an open subset $U$ of $Q$ and an local basis of $\Gamma(\tau_V)$, $\{e_A, e_a\}$, adapted to the decomposition $V = U_B \oplus U_{B_x}$, where $U_B$ is an arbitrary complementary subspace. Thus, $\{e_0, e_A, e_a\}$ is an local basis of $\Gamma(\tau_A)$ adapted to the affine subbundle $B$, where $1_A(e_0) = 1$. Denote by $(x^i, y^0, y^A, y^a)$ the corresponding local coordinates on $A$ and by $(x^i, y_0, y_A, y_a)$ the dual local coordinates on $A^*$. Note that the equations which define $B$ as an affine subbundle of $A$ are $y^A = 0$.

The affine control problem given by the drift section $e_0$ and the input sections $e_a$ is defined by the following equation on $Q$, $\dot{x}^i = \rho^i_0 + y^a \rho^i_a$, where the coordinates $y^a$ are playing the role of the set of admissible controls.

Now, consider a function $\tilde{L} : B \rightarrow \mathbb{R}$ as a performance index. The equations of motion of the optimal control problem defined by $(\tilde{L}, B)$ are precisely the vakonomic equations. In the selected coordinate system are:

$$
\begin{aligned}
\dot{x}^i &= \rho^i_0 + y^a \rho^i_a, \\
\dot{y}^a &= \rho^a_\gamma \frac{\partial \tilde{L}}{\partial x^\gamma} - y^\gamma (C^\gamma_{a0} + y^a C^\gamma_{aa}), \\
\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial y^a} \right) &= \rho^a_\gamma \frac{\partial \tilde{L}}{\partial x^\gamma} - y^\gamma (C^\gamma_{a0} + y^b C^\gamma_{ab}),
\end{aligned}
$$

for all $1 \leq i \leq m$, $1 \leq \gamma \leq n$, $1 \leq A \leq \tilde{m}$ and $\tilde{m} + 1 \leq a \leq n$, with $y_a = \frac{\partial \tilde{L}}{\partial y^a}$.

Example 4.2. Consider a particle of unit mass in a planar inverse-square law gravitational field which has thrusters in the “x, y” directions (see [13]). Then, the equations of motion are:

$$
\begin{aligned}
\dot{q}_1 &= v_1, & \dot{q}_2 &= v_2, & \dot{v}_1 &= -q_1 (q_1^2 + q_2^2)^{-3/2} + u_1, & \dot{v}_2 &= -q_2 (q_1^2 + q_2^2)^{-3/2} + u_2
\end{aligned}
$$

defined on $M = (\mathbb{R}^2 - \{(0, 0)\}) \times \mathbb{R}^2$. The objective will be to drive the particle to a given circular orbit with minimum energy. Therefore, let us take $L = \frac{1}{2}(u_1^2 + u_2^2)$.

Now, choose a global basis of sections of $T(\mathbb{R} \times M) \rightarrow \mathbb{R} \times M$ adapted to this affine control system:

$$
\begin{aligned}
e_0 &= \frac{\partial}{\partial x^0} + v_1 \frac{\partial}{\partial q_1} + v_2 \frac{\partial}{\partial q_2} - q_2 (q_1^2 + q_2^2)^{-3/2} \frac{\partial}{\partial v_1} - q_1 (q_1^2 + q_2^2)^{-3/2} \frac{\partial}{\partial v_2}, \\
e_1 &= \frac{\partial}{\partial q_1}, & e_2 &= \frac{\partial}{\partial q_2}, & e_3 &= \frac{\partial}{\partial v_1}, & e_4 &= \frac{\partial}{\partial v_2}
\end{aligned}
$$

where $\{e_0; e_3, e_4\}$ defines a affine subbundle of $\mathbb{R} \times TM \rightarrow \mathbb{R} \times M$ determining the initial affine control system.

Applying the result developed in Subsection 4.3.1 and denoting by $y_3 = u_1$ and $y_4 = u_2$ then the equations of motion are now:

$$
\begin{aligned}
\dot{q}_1 &= v_1, & \dot{q}_2 &= v_2, & \dot{v}_1 &= -q_1 (q_1^2 + q_2^2)^{-3/2} + u_1, & \dot{v}_2 &= -q_2 (q_1^2 + q_2^2)^{-3/2} + u_2, & \\
\dot{y}_1 &= -\left( u_1 (2q_1^2 - q_2^2) + 3u_2 q_1 q_2 \right) \left( q_1^2 + q_2^2 \right)^{-5/2}, & \dot{y}_2 &= -\left( 3u_3 q_1 q_2 + u_4 (2q_2^2 - q_1^2) \right) \left( q_1^2 + q_2^2 \right)^{-5/2}, & \\
u_1 &= -y_1, & \dot{u}_2 &= -y_2.
\end{aligned}
$$
5. Conclusions and future work

Variational constrained Mechanics is discussed in the Lie affgebroid setting. We obtain the vakonomic equations and the vakonomic bracket associated with a constrained mechanical system on a Lie affgebroid. The variational character of the theory is analyzed. Vakonomic systems subjected to affine constraints are of special interest. Other examples are also discussed.

In this paper we only consider normal solutions of the vakonomic problems. It would be interesting to extend the results of the paper for abnormal solutions.

Acknowledgments

This work has been partially supported by MEC (Spain) Grants MTM 2006-03322, MTM 2007-62478, project “Ingenio Mathematica” (i-MATH) No. CSD 2006-00032 (Consolider-Ingenio 2010) and S-0505/ESP/0158 of the CAM.

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