BPS States and Minimal Surfaces

A.Mikhailov

Department of Physics, Princeton University
Princeton, NJ 08544, USA

E-mail: andrei@puhep1.princeton.edu

Abstract

It was observed recently, that the low energy effective action of the four-dimensional supersymmetric theories may be obtained as a certain limit of M theory. From this point of view, the BPS states correspond to the minimal area membranes ending on the M theory fivebrane. We prove that for the configuration, corresponding to the $SU(2)$ Super Yang-Mills theory, the BPS spectrum is correctly reproduced, and develop techniques for analyzing the BPS spectrum in more general cases. We show that the type of the supermultiplet is related to the topology of the membrane: disks correspond to hypermultiplets, and cylinders to vector multiplets. We explain the relation between minimal surfaces and geodesic lines, which shows that our description of BPS states is closely related to one arising in Type II string compactification on Calabi-Yau threefolds.

1On leave from the Institute of Theoretical and Experimental Physics, 117259, Bol. Cheremushkinskaya, 25, Moscow, Russia.
1 Introduction.

The exact solution of $N = 2$ supersymmetric gauge theories given by N. Seiberg and E. Witten in [1] played an important role in the modern development of Quantum Field Theory and String Theory. It turned out, that the low energy effective action and some massive states may be described in beautiful geometric language. The moduli space of vacua corresponds to the moduli space of certain 2-dimensional algebraic curves, and various physical quantities are given by the very natural functions on it.

Recently it turned out that this geometric language has a natural interpretation in terms of string theory. In particular, it was shown [2], that the low energy effective action may be obtained as a special limit of M-theory.

M-theory is the quantum theory whose low dimensional limit is 11D supergravity [3, 4]. Its solitons are 5-branes and 2-branes.

The configuration considered in [2] contains a single M-theory 5-brane, with the worldvolume $\mathbb{R}^4 \times \Sigma$, embedded into the 11-dimensional space-time $\mathbb{R}^{10} \times S^1$. Here $\mathbb{R}^4$ is parametrized by $x^0, x^1, x^2$ and $x^3$, and $\Sigma$ is the surface in $\mathbb{R}^3 \times S^1$. $\mathbb{R}^3$ has the coordinates $x^4, x^5$ and $x^6$, and $S^1$ is parametrized by $x^{10}$. The surface $\Sigma$ was given by the equation

$$F(x^6 + ix^{10}, x^4 + ix^5) = 0$$

and it may be identified with the Seiberg-Witten curve of [1]. The fact that $F$ is holomorphic means that the configuration preserves $1/4$ of the 11-dimensional supersymmetries [7, 8]. This gives $N = 2$ supersymmetry. Notice that for the flat 5-brane we would have $N = 4$ supersymmetry in $D = 4$. From the point of view of string theory, the curved 5-brane corresponds to a configuration of 5-branes and 4-branes, which has half of the supersymmetry of the single 5-brane or 4-brane. The asymptotical behaviour of $\Sigma$ fixes the function $F$ up to several parameters, which are identified with the moduli of $N = 2$ vacua. The metric on the moduli space is given by the kinetic energy of the motion of the 5-brane, when we change these parameters. The BPS states correspond to minimal area membranes ending on these 5-branes.

The masses of the BPS states in 4D $N = 2$ theories are given by the formula [1]:

$$m_{n_e, n_m} = |n_e a + n_m a_D|$$  \hspace{1cm} (2)

where $n_e$ and $n_m$ are the electric and magnetic charges of the BPS state, and $a(u), a_D(u)$ are the functions of the order parameters, given by the integral over the basic cycles in $H_1(\Sigma, \mathbb{Z})$ of the meromorphic 1-form. The set of allowed values of $n_e$ and $n_m$ carries important information about the dynamics. Much work has been done in determining which values are allowed [3, 4].

At some points $u$, $a/a_D$ becomes a rational number, and for $n_e/n_m = -a_D/a$, formula (2) implies that the mass becomes zero. If the corresponding BPS states exist, this means that at this point there is a singularity in the low energy description.
of the theory. But in fact not all the values of $n_e$ and $n_m$ are allowed. In particular, it turns out that massless states exist only when the corresponding homology cycles in $\Sigma$ actually contract to zero. This may be considered as one of the indications that the curve $\Sigma$ itself, not just $a$, $a_D$, has some physical meaning.

In this paper, we will study the BPS spectrum from the viewpoint of M-theory. We will show that the correct BPS spectrum is reproduced for $N = 2$ SYM with the gauge group $SU(2)$, and develop techniques for the study of BPS spectra in more general cases.

When this paper was in preparation, the related papers [8] and [9] appeared.

2 Membranes of minimal area.

The projection of the 5-brane world-sheet to $S^1 \times \mathbb{R}^3$ is the surface $\Sigma$, complex in the complex structure $(x^6 + ix^{10}, x^4 + ix^5)$. This surface is non-compact (goes through infinity). For the closed curve $\Gamma \subset \Sigma$, the winding number $W[\Gamma] = \frac{1}{2\pi} \int_{\Gamma} dx^{10}$ depends only on the homology class $[\Gamma] \in H^1_{\text{comp}}(\Sigma, \mathbb{Z})$. Here $H^1_{\text{comp}}(\Sigma)$ means $H^1(\Sigma)$ with the points at infinity omitted. If $W[\Gamma] = 0$, we can consider the membrane $M$ in $S^1 \times \mathbb{R}^3$ whose boundary is $\Gamma$. Let $S[\Gamma]$ be the minimal area of membranes with the boundary $\Gamma$. Suppose that for any homology class $\gamma \in H^1_{\text{comp}}(\Sigma, \mathbb{Z})$, the minimum $\min_{\Gamma \in \gamma} S[\Gamma]$ is realized on some curve $\Gamma_0(\gamma)$. The curve $\Gamma_0(\gamma)$ may have several connected components, then the corresponding membrane can also have several connected components probably with different topologies.

The area of the membrane is given by the integral

$$S[M] = \int_M dS = \int_M \sqrt{\sigma_{10,4}^2 + \sigma_{10,5}^2 + \sigma_{10,6}^2 + \sigma_{4,5}^2 + \sigma_{4,6}^2 + \sigma_{5,6}^2}$$

(3)

where $\sigma_{i,j} = dx^i \wedge dx^j(\sigma)$ are the coordinates of the surface bivector. Since this bivector is decomposable (it is a wedge product of two vectors, tangent to the surface), it satisfies the bilinear identity

$$\sigma_{4,6} \sigma_{10,5} - \sigma_{10,6} \sigma_{4,5} + \sigma_{10,4} \sigma_{6,5} = 0$$

(4)

we may rewrite $dS^2$ as $(s = x^6 + ix^{10}, v = x^4 + ix^5)$:

$$dS^2 = |ds \wedge dv|^2 + (\sigma_{10,6} + \sigma_{5,4})^2$$

(5)

This means that

$$S[\Gamma] \geq \left| \int_M ds \wedge dv \right| = \left| \int_{\Gamma} vd\sigma \right|$$

(6)

The BPS states correspond to the membranes for which this inequality is saturated.

This means that two conditions are satisfied:

1) $\sigma_{10,6} + \sigma_{5,4} = 0$;
2) $\text{Arg}(ds \wedge dv) = \text{Arctan} \frac{\sigma_{10,4} + \sigma_{6,5}}{\sigma_{6,4} + \sigma_{5,10}} = \text{const}$

(7)
This is equivalent to the statement that the membrane is complex in the complex structure

$$(x^4 - ix^{10}, \ x^6 + ix^5)$$

where

$$(x^4 + ix^5) = e^{i\phi}(x^4 + ix^5)$$

and $\phi$ is a fixed angle.

Let us denote $\tilde{v} = \tilde{x}^4 + i\tilde{x}^5$. It is useful to introduce the complex coordinates $(x, y)$, $x + y^\ast = s$, $x^\ast - y = \tilde{v}$. The 5-brane is holomorphic in $(s, v)$, the 2-brane is holomorphic in $(x, y)$.

For the 5-brane $ds \wedge d\tilde{v} = 0$, and $\frac{4}{i}dx \wedge dy = d\sigma$ is the surface area.

For the 2-brane $\frac{1}{i}ds \wedge d\tilde{v} = d\sigma$ is the surface area, and $dx \wedge dy = 0$.

Thus the question of classification of BPS states is reduced to the classification of the curves in $C$ which are the lines of intersection with the surface holomorphic in the other complex structure. Two surfaces, holomorphic in two complex structures, generally intersect at points. But there are exceptional cases when they may intersect at curves. (In the latter case one can show that they intersect at a right angle.)

Given any curve $\Gamma \in C$, we can in principle (by analytical continuation) continue it to the complex surface $M$. The surface $M$ will then be divided by $\Gamma$ into two parts, and we must require that one of those parts is compact (does not go to infinity).

3 The simplest examples of membranes.

The SW curve for $SU(2)$ is given by the equation

$$\cosh(s) - v^2 + u = 0$$

Consider first the weak coupling region, that is, sufficiently large $|u|$. Topologically, the curve $\Sigma$ is the torus with two points omitted. Its first homology group is generated by three cycles, having winding numbers 1, 1 and 0. They may be represented by the equations $(x^5 = 0, x^4 \in [\sqrt{u-1}, \sqrt{u+1}])$, $(x^5 = 0, x^4 \in [-\sqrt{u+1}, -\sqrt{u-1}])$ and $(x^5 = 0, x^{10} = \pi)$ respectively. For real $u > 1$, the projection of the curve onto the plane $(x^4, x^5, x^6)$ looks like
where the values of $x^{10}$ are shown. Two line intervals $X$ and $Y$ correspond to
the points where the projection is not one to one ($x^6 = 0$, so $x^{10}$ goes with $-x^{10}$.)
Consider the simplest examples of membranes:

A) Monopole. $\gamma = (1, 0)$, $m = |a_D|$.
The membrane has the topology of a disk. The equation of the membrane is:

$$x^5 + ix^{10} = i\pi;$$
$$|x^4| \leq \sqrt{u - \cosh x^6}$$  \hspace{1cm} (11)

B) Vector Boson. $\gamma = (0, 2)$, $m = |2a|$.
This membrane has the topology of a cylinder, its projection on the $(4, 5, 6)$ plane
is just an interval. The equation of the membrane is:

$$x^6 + ix^5 = 0;$$
$$|x^4| \leq \sqrt{u + \cos x^{10}}$$  \hspace{1cm} (12)

The strong coupling regime corresponds to small $|u|$. For real $u$ this means $u < 1$. In this case, the projection of the curve to $\mathbb{R}^3$ is two surfaces, touching at
two intervals. These two intervals are

$$x^5 = 0, \ x^4 = \pm \sqrt{u + \cos x^{10}}, \ x^{10} \in [-\pi + \alpha, \pi - \alpha],$$

and

$$x^4 = 0, \ x^5 = \pm \sqrt{-u - \cos x^{10}}, \ x^{10} \in [\pi - \alpha, \pi + \alpha]$$
where $\alpha = \arccos u$

The compact homology has dimension three and is generated by three cycles:

(Shown is the view along the $x^6$-axis. We have two planes glued along the cross.)

Cycles A and B have winding number zero, and cycle C has winding number 1.

To identify electric and magnetic cycles, we should follow the deformation of these cycles at $u > 1$ into the region $u < 1$. In the process of deformation, it is necessary to avoid the point $u = 1$, since at that point the magnetic cycle gets contracted, and we wouldn’t be able to distinguish between the cycle corresponding to the dyon and the cycle corresponding to the vector boson. Thus, we have to turn on the imaginary part of $u$. Since it is not very easy to draw the pictures at complex $u$, we will consider the periods of $vds$ instead. At real $u > 1$,

$$
a = i\sqrt{2} \int_0^2 \sqrt{u + \cos x^{10}} dx^{10} = i \left( 4 - \frac{u-1}{2} \log \frac{u-1}{2} + \ldots \right)
$$

$$a_D = \sqrt{2} \int_{\cosh x^6 < u} \sqrt{u - \cosh x^6} dx^6 = 2\pi \left( \frac{u-1}{2} + \ldots \right) \quad (13)$$

There is the sign ambiguity in the magnetic quantum number corresponding to the states at $u < 1$, related to the possibility to continue $a$ and $a_D$ from $u > 1$ by going above or below the point $u = 1$ [13]. Let us assume that we pass this point in the upper halfplane. At $u < 1$, the period of the differential over the A cycle is imaginary, and the period over the B cycle is real. Comparing this to (13), we see that A corresponds to the dyon $(n_e, n_m) = (2, -1)$ and B corresponds to the monopole $(n_e, n_m) = (0, 1)$. When $u \to 1$, $\alpha \to 0$, the horizontal interval shrinks, and the monopole becomes massless.

In the case of real $u$, $|u| < 1$, we can write explicit expressions for these two disks:

A.) Dyon. $x^6 + ix^5 = 0$, $|x^4| \leq \sqrt{u + \cos x^{10}}$.

B.) Monopole. $x^6 - ix^4 = 0$, $|x^5| \leq \sqrt{-u - \cos x^{10}}$.

The vector boson would correspond to C–[-C], where –C means C transformed by $x^{4,5} \to -x^{4,5}$. The corresponding membrane would have two boundaries, the topology of a cylinder. But it turns out, that such a holomorphic membrane does not exist. This follows from the arguments of the next section.
Notice that $A + B = C - [-C]$ and intuitively it is probably clear from the picture that the minimum is realized on the union of two disks with boundaries $A$ and $B$, not on the cylinder.

4 Complex $u$.

In this section we consider membranes for complex values of the order parameter $u$. Suppose that we have constructed the holomorphic membrane, corresponding to the given homology cycle in $\Sigma_u$ for the given value of $u$. We will consider what happens to this holomorphic membrane, when we start changing $u$. We will show that when $a(u)/a_D(u)$ is not real, the membrane changes smoothly. On the other hand, when it is real, that is, when $u$ crosses the curve of marginal stability, there is the possibility that the membrane decays into two membranes.

This result will enable us to construct the membranes for general $u$ starting from real $u$, and to construct membranes corresponding to dyons applying the monodromy transformations to the membrane corresponding to the monopole.

We will also show, that the membranes with the topology of a disk have moduli space $\mathbb{R}^3$ (spatial translations), and the moduli space of the cylinder is $\mathbb{R}^3 \times I$ (spatial translations plus one extra modulus, with the topology of an interval).

The membrane with the topology of a disk can be represented by the holomorphic map

$$M : D \to \mathbb{C}^2$$

from the disk $|z| < 1$ to $\mathbb{C}^2$, such that the image of the boundary of the disk, $z = e^{i\phi}$, lies on the surface of the 5-brane. When we vary $u$, the surface of the 5-brane varies, and the variation may be thought of as the section of the normal bundle of $\Sigma$:

$$\xi \in H^0(\Sigma, \mathbb{C}^2/T\Sigma)$$

Consider the restriction of $\xi$ to the boundary of the membrane. We may think of this restriction as the $\mathbb{C}^2$-valued function on the circle, modulo those functions whose values are tangent to the 5-brane. Let us prove that we can always find a representative which can be holomorphically continued inside the disk, to $|z| < 1$. This representative gives us the deformation of the surface of the membrane.

Let $(x, y)$ be the coordinates in $\mathbb{C}^2$ in which the membrane is holomorphic, $x = (s - \bar{v}^*)/2$, $y = (s^* + \bar{v})/2$. Let us denote $X_1 = (\dot{x}, \dot{y})$ the vector tangent to the boundary of the membrane, $M \frac{\partial}{\partial \phi}$. At any $\phi$, this vector is tangent to the 5-brane, since $\partial_{\phi} M \subset \Sigma$. Another vector tangent to the 5-brane is $I(\dot{x}, \dot{y})$, where $I$ is the operator of multiplication by $i$ in the complex structure $(s, v)$. Explicitly $X_2 = (i\dot{y}^*, -i\dot{x}^*)$.

The deformation vector $\xi$, restricted to the boundary,

$$\xi(\phi) = (\xi^x(\phi), \xi^y(\phi))$$
can be decomposed into the two parts one of which can be holomorphically continued inside the disk and the other outside:

\[ \xi(\varphi) = \xi_{\geq 0}(\varphi) + \xi_{< 0}(\varphi) \]  

(16)

Here \( \xi_{\geq 0} \) and \( \xi_{< 0} \) may be defined in terms of Fourier series. Given

\[ \xi(\varphi) = \sum_{n=-\infty}^{\infty} \xi_n e^{in\varphi} \]

we define

\[ \xi_{\geq 0}(\varphi) = \sum_{n=0}^{\infty} \xi_n e^{in\varphi} \]

and

\[ \xi_{< 0}(\varphi) = \sum_{n=-\infty}^{-1} \xi_n e^{in\varphi} \]

We want to show that there exist real functions \( a(\varphi), b(\varphi) \) such that

\[ \xi_{< 0} = [aX_1 + bX_2]_{< 0} \]  

(17)

Then the vector-function \( \tilde{\xi}(\varphi) = \xi - aX_1 - bX_2 \) can be analytically continued inside the disk, and the map

\[ M_\epsilon(z) = M(z) + \tilde{\xi}(z) \]  

(18)

will give the deformed membrane.

Let us first find \( b \). Given two equations,

\[ \begin{align*} 
[a \dot{x} + ib \dot{y}]_{< 0} &= \xi_x \leq 0 \\
[a \dot{y} - ib \dot{x}]_{< 0} &= \xi_y \leq 0 
\end{align*} \]  

(19)

we have

\[ \left( \dot{y} [a \dot{x} + ib \dot{y}^*]_{< 0} - \dot{x} [a \dot{y} - ib \dot{x}^*]_{< 0} \right)_{\leq 0} = (\dot{y} \xi_x \leq 0 - \dot{x} \xi_y \leq 0) \leq 0 \]  

(20)

Notice that \( \dot{x}_{\leq 0} = 0 \). Thus, for any function \( \psi(\varphi) \), we get \[ \psi_{< 0} \leq 0 = [\psi \dot{x}]_{\leq 0} \]. Taking this into account, we get:

\[ i[b(|\dot{x}|^2 + |\dot{y}|^2)]_{\leq 0} = [\dot{y} \xi_x \leq 0 - \dot{x} \xi_y \leq 0]_{\leq 0} \]  

(21)

Is this equation solvable for \( b \)? The obvious solution:

\[ ib = \frac{[\dot{y} \xi_x \leq 0 - \dot{x} \xi_y \leq 0]_{\leq 0} - [h.c.]_{> 0}}{|\dot{x}|^2 + |\dot{y}|^2} \]  

(22)
is valid modulo two potential problems, which we will discuss later. But if it works, then \( a(\varphi) \) can be found from either one of the two equations \( \diamond \) \( \Box \). (Actually, \( a_{-1} \), \( a_0 \) and \( a_1 = a_{-1}^* \) are not determined from that equation — this corresponds to the \( SL(2, \mathbb{R}) \) symmetry of the disk.)

The first problem is that the zero mode of the LHS is purely imaginary:

\[
\text{Re} \left( i[b(|\dot{x}|^2 + |\dot{y}|^2)]_0 \right) = 0
\]

So we have to prove that the zero mode of the RHS is also imaginary:

\[
[\dot{y}\xi_{<0}^x - \dot{x}\xi_{<0}^y] = \int_{\partial M} \xi^x dy - \xi^y dx \in i\mathbb{R}
\] (23)

At least locally, we may introduce in \( \mathbb{C}^2 \) the coordinate system \((s,u)\). The surfaces \( u = \text{const} \) are the curves \( \Sigma_u \), and there is a function \( \tilde{v}(s,u) \) such that the integral of \( \tilde{v}ds \) over the certain cycle in \( \Sigma_u \) is purely imaginary (that is how we defined \( \tilde{v} \) in the first section). Now

\[
\int_{\partial M} \xi^x dy - \xi^y dx = \frac{1}{4} \int_{\partial M} ((\delta s - \delta \tilde{v}^*)d(s^* + \tilde{v}) - (\delta s^* + \delta \tilde{v})d(s - \tilde{v}^*)) =
\]

\[
\int_S ds \wedge ds^* + d\tilde{v} \wedge d\tilde{v}^* - 2\text{Re}(d\tilde{v} \wedge ds)
\] (24)

where \( S \) is the strip between the membrane boundaries in \( \Sigma_u \) and \( \Sigma_{u+\delta u} \). It is clear that the first two terms are imaginary. The third one by the Stokes theorem is the real part of the difference between the integrals of \( \tilde{v}ds \) over the cycles in \( \Sigma_u \) and \( \Sigma_{u+\delta u} \), and it is zero as follows from the previous paragraph.

The second problem: it may happen that \( |\dot{x}|^2 + |\dot{y}|^2 = 0 \) at some point \( \varphi_0 \) on the boundary. This means that the boundary of the membrane develops a cusp at this point: the direction of the velocity vector on the left of \( \varphi_0 \) does not coincide with the direction of the velocity vector on the right. But this cannot happen with the boundary of a holomorphic minimal surface, unless the boundary touches itself.

Indeed, suppose that the boundary does not touch itself, and yet develops a cusp. Without loss of generality, we may consider the case when the cusp is at \( z = -1 \). Then, near the cusp, the equation of the curve may be approximated as \( \zeta = z + 1 \):

\[
\begin{align*}
x &= a\zeta^\alpha(1 + o(1)) \\
y &= b\zeta^\beta(1 + o(1))
\end{align*}
\] (25)

with \( a, b, \alpha > 1, \beta > 1 \) — some parameters. The vectors, tangent to \( \Sigma \) near the cusp of the membrane, are:

\[
(\dot{x}, \dot{y}) = i(a\alpha\zeta^{\alpha-1}, b\beta\zeta^{\beta-1}) = (A\zeta^{\alpha-1}, B\zeta^{\beta-1})
\]

\[
(i\dot{y}^*, -i\dot{x}^*) = (iB^*(\zeta^*)^{\beta-1}, -iA^*(\zeta^*)^{\alpha-1})
\] (26)

\(^2\text{Written in terms of the Fourier coefficients, this is the triangular system of equations, and always has a solution.}\)
The surface element of $\Sigma$ (normalized to unit surface) has the $xy^*$ coordinates

$$
\sigma^{xy^*} = \frac{i A^2 \zeta^{2\alpha - 2} - (B^*)^2 (\zeta^*)^{2\beta - 2}}{|A|^2 |\zeta|^{2\alpha - 2} + |B|^2 |\zeta|^{2\beta - 2}}
$$

(27)

It follows from this expression, that $\sigma^{xy^*}$ is irregular at $\zeta = 0$, that is the curve $\Sigma$ necessarily has singularity at the point where the boundary of the membrane develops cusp. For regular $\Sigma$, the cusps are impossible, except for the case when the boundary of the membrane touches itself: in this case, we cannot approximate the cusp by the simple equation (25).

In the case that the boundary touches itself, the membrane is the union of two membranes. Their boundaries represent different homology classes of $\Sigma$, and yet they are holomorphic in the same complex structure. This means that the phases of the two periods of $v ds$ coincide, that is we are on the curve of marginal stability.

The conclusion is that to the deformation of the curve $\Sigma$ corresponds the deformation of the membrane, except for possible decays when we intersect the curve of marginal stability.

This consideration do not apply to the membrane corresponding to the gauge boson, since it has the topology of a cylinder. For the cylinder, we will use somewhat different approach. Introduce the coordinate $z = \tau + i\varphi$, so that the boundaries are at $\tau = -t$ and $\tau = t$. The deformation of the boundary of the cylinder may be represented as:

$$
\xi^x = \alpha \frac{\partial y}{w} + i\beta \partial x
\xi^y = -\alpha \frac{\partial x}{w} + i\beta \partial y
$$

(28)

where we denoted

$$
w = |\dot{x}|^2 + |\dot{y}|^2 = |\partial x|^2 + |\partial y|^2
$$

(29)

and $\alpha = \alpha(\varphi)$ and $\beta = \beta(\varphi)$ are some functions. The representation (28) corresponds to decomposition of $\xi$ in the basis $X_1, X_2, iX_1, iX_2$, where $X_1 = (x, y)$ and $X^2 = IX^1$. For real $\alpha$ and $\beta$, $\xi$ is parallel to $\Sigma$ (notice that on the boundary $\dot{x} = i\partial x$). Thus, the element of the normal bundle $N\Sigma$, corresponding to $\xi$, depends only on $\text{Im}\alpha$ and $\text{Im}\beta$. Given the imaginary part of the function $\alpha$ on the boundary, we can determine the holomorphic continuation inside the cylinder unambiguously.

The only requirement is that

$$
\int d\varphi \text{ Im } \alpha(t + i\varphi) = \int d\varphi \text{ Im } \alpha(-t + i\varphi)
$$

(30)

and this follows from (28) and (24). Let us also continue the function $\beta$ inside the cylinder, not necessarily as a holomorphic function (the ambiguities in the continuation of $\beta$ correspond to the possible reparametrizations of the cylinder). Now, the formulae (28) determine some vector $\xi$ on the cylinder, and the deformed surface

$$
x'(z, \bar{z}) = x(z) + \xi^x(z, \bar{z})
y'(z, \bar{z}) = y(z) + \xi^y(z, \bar{z})
$$

(31)
turns out to be holomorphic in the complex structure \((x, y)\), although \(x'\) and \(y'\) are not holomorphic functions of \(z\). Indeed, the necessary condition for the surface \((x(z, \bar{z}), y(z, \bar{z}))\) to be holomorphic is

\[
\{x, y\} = \partial x \bar{\partial} y - \bar{\partial} x \partial y = 0
\]  
(32)

and one can see that this condition is satisfied for \((x + \xi x, y + \xi y)\) with \(\xi x\) and \(\xi y\) as in (28). It is possible to introduce a new variable \(z\) so that \(x'\) and \(y'\) are holomorphic functions of \(z\). We should not have expected that the deformed membrane is given in parametric form by \(x(z), y(z)\) with the same \(z\) as the initial membrane. Indeed, this would mean that the deformed membrane is isomorphic to the initial as complex manifold. But it is generally not true, since the cylinder has a modulus, the length of the cylinder.

Notice that the cylinder has nontrivial moduli space. Indeed, we may take \(\beta = 0\), and \(\alpha\) a real constant. This corresponds to sliding the boundary of the cylinder along \(\Sigma\), with the normal velocity at point \(\varphi\) equal

\[
\frac{\alpha}{\sqrt{|\dot{x}|^2 + |\dot{y}|^2}}
\]

where \(\dot{x}\) and \(\dot{y}\) denote the derivative of \(x\) and \(y\) with respect to \(\varphi\). In string theory language, this zero mode corresponds to the motion of the string connecting two fourbranes, in the direction \(x^6\). This motion stops when the boundary of the cylinder touches itself:

At this point, the length of the cylinder goes to zero — the cylinder degenerates into the disk. When we map a very short cylinder into the four-dimensional space, so that the image is of finite size, the derivatives \(\dot{x}\) and \(\dot{y}\) are typically very large, except for the small region near the point where the boundary touches itself, and expressions (28) become ill defined. Thus, the moduli space of the cylinder has the topology of an interval.

We will generalize the construction of the bosonic zero modes to the surfaces of more complicated topology in Section 7.

Now we can apply the considerations of [13] to find the BPS spectrum. First let us prove that the membrane corresponding to the dyon exists \((u > 1)\). Notice that
the curve $\Sigma_{-u}$ may be obtained from the curve $\Sigma_u$ by the change of variables:

$$
\begin{align*}
    y^6 &= x^6, \quad y^{10} = x^{10} + \pi, \\
    y^4 &= -x^5, \quad y^5 = x^4
\end{align*}
$$

(33)

- this is the manifestation of the $\mathbb{Z}_2$ symmetry of the moduli space of vacua.

Now the simplest membrane, with the topology of a disk, whose boundary is on $\Sigma_{-u}$,

$$
\begin{align*}
    y^5 + iy^{10} &= i\pi; \\
    y^4 &\leq \sqrt{u - \cosh y^6}
\end{align*}
$$

(34)

goes to the membrane corresponding to the dyon when we smoothly change $\Sigma_{-u}$ to $\Sigma_u$:

The boundary of the disk corresponding to the dyon has the following shape:

The membranes corresponding to the states $(n_e, n_m) = (2m, 1)$ can be constructed from monopole and dyon by moving $u$ around the circle of the large radius. Membranes with magnetic charge greater than one do not exist, because if they existed, then we would be able to move $u$ to certain point on the curve of marginal stability in such a way, that the corresponding state becomes massless — see [13] for the details. This would mean that the membrane of zero area has boundary representing the nonzero homology class in $\Sigma$.

For the strong coupling regime, we have explicitly constructed the membranes corresponding to the monopole and the dyon, and the considerations from [13] show that no other states exist. The proof goes as follows. First, it is possible to prove that the existence of $(n_e, n_m)$ implies the existence of $(n_e + 2n_m, -n_e - n_m)$. (Given $(n_e, n_m)$, we know the state with the same mass exists at $-u$, because of the $\mathbb{Z}_2$ symmetry [33], and deforming it from $-u$ to $u$ we get $(n_e + 2n_m, -n_e - n_m)$.) But
one of the states \((n_e, n_m)\) or \((n_e + 2n_m, -n_e - n_m)\) becomes massless at some point on the curve of marginal stability. So, the only possibilities are \((0, \pm 1)\) and \(\pm(2, -1)\) – the monopole and the dyon.

We have not proven that the membranes with more complicated topology do not exist. If they exist, then their zero modes may be described by the general construction, which we will discuss in Section 7.

5 Matter in mixed representation.

Besides the pure SYM, the theories with matter hypermultiplets were described in [3] in the M-theory language. The configuration of branes corresponding to the theories with hypermultiplets in fundamental representation was constructed. Also, in the case when the gauge group is \(SU(n)^k\), the hypermultiplets in the mixed representation (adjoint when \(k = 1\)) were described.

In the case of mixed representation, the string theory fivebrane is represented as a “spike” on the curve \(\Sigma\), of the form

\[
v = \frac{m}{s - s_0} + \ldots
\]

where dots represent terms regular at \(s = s_0\). The constant \(m\) is the mass of the matter hypermultiplet. The membrane corresponding to this hypermultiplet has the topology of a disk, the boundary of the disk being the small circle around \(s = s_0\). To prove the existence of such a membrane, we use the deformation argument from the previous section.

Let us deform the neighborhood of \(s = s_0\) to the neighborhood of the point \(s = s_0\) on the curve given by \(v = \frac{m}{s - s_0}\) (without any regular terms). We have to require that \(m\) is sufficiently small, so that the membrane is located in the region around \(s = s_0\), which can be deformed to the corresponding region of the curve \(v = \frac{m}{s - s_0}\). Then the membrane is given by the equation

\[
v - (s^* - s_0^*) = 0
\]

\[
|s - s_0|^2 \leq m
\]

Since the membrane has the topology of a disk, it corresponds to the hypermultiplet. The masses of these states are just \(|m|\).

6 Matter in fundamental representation.

To describe \(N = 2\) supersymmetric QCD with \(N_f\) flavors, it is necessary to introduce \(N_f\) 6-branes. The 6-brane in M-theory is described by replacing \(R^3 \times S^1\) with the Taub-NUT space. It may be obtained as the Hyper-Kahler reduction of the flat quaternionic space \(H \times H\) [21]. Introduce the quaternionic coordinates \((q, w),\)
This space is endowed with the flat quaternion-valued symplectic form:

\[-\frac{1}{2} dq \wedge d\bar{q} - \frac{1}{2} dw \wedge d\bar{w} = i\omega_I + j\omega_J + k\omega_K \quad (37)\]

The moment map for the \( \mathbb{R} \) action

\[
q \rightarrow q e^{it} = e^{it}y + e^{-it}zj, \\
w \rightarrow w + t = (u + t) + vj
\]

(38)

takes values in imaginary quaternions:

\[
\mu = \frac{1}{2} qi\bar{q} - \frac{1}{2}(w - \bar{w}) = i\mu_R + \mu_C j
\]

(39)

where

\[
\mu_R = \frac{1}{2}(|y|^2 - |z|^2 - 2Imu) \\
\mu_C = izy + v
\]

(40)

The Taub-NUT space is obtained by first putting \( \mu_R = \mu_C = 0 \) and then dividing by the action of \( \mathbb{R} \). The space \( \mathbf{H} \times \mathbf{H} \) has three natural complex structures, which become complex structures on the reduced space, as explained in Appendix A. But to describe the brane configuration, it is more convenient to fix \( \mu_C \) and then divide by the action of \( G_C = \mathbb{C}^2 \). \( \mu_C = e \) gives \( zy = e - iv \), \( \mathbb{C} \) acts as \( y \rightarrow e^{it}y, \ z \rightarrow e^{-it}z, \ u \rightarrow u + t \). To complete the reduction, we introduce the \( G_C \)-invariant functions \( Z = ze^{iu}, \ Y = ye^{-iu} \) on the constraint manifold, and get the \( \text{dim}_C = 2 \) manifold

\[\{(Z,Y,v)|ZY = e + iv\} \quad (41)\]

In this formalism, one of the three complex structures of Taub-NUT is more manifest then the other two. The surface \( \Sigma \), representing the 5-brane, is holomorphic in this complex structure.

BPS states are described as the membranes of minimal area. The relation between the area of the surface in the Kahler manifold and the integral of the holomorphic 2-form is a particular case of the Wirtinger inequality [19, 20]. It says, that the volume form on the submanifold of the Kahler manifold is greater or equal to the restriction of the appropriate power of the Kahler form. The equality is when the submanifold is complex. Let us consider the special case of the four-dimensional hyper-Kahlerian manifold \( X \). It has three complex structures, \( I, J \) and \( K \). For each complex structure, consider the corresponding Kahler form:

\[\omega_I(a, \eta) = (a, Ia\eta) \quad (42)\]

where \( (, ) \) is the (real) scalar product. Operators \( I, J \) and \( K \) generate the algebra \( su(2) \). The space of bivectors \( \Lambda^2TX \) decomposes into the direct sum of one vector and three scalar representations of this \( su(2) \). Let us denote the corresponding
projectors $P_0$ and $P_1$. For the decomposable bivector of the form $\xi \wedge \eta$, we have an analogue of the bilinear identity (4):

$$|P_0(\xi \wedge \eta)| = |P_1(\xi \wedge \eta)|$$  \hspace{1cm} (43)

where $|b|$ means the area of the bivector $b$. Indeed, this is the unique $su(2)$-invariant condition which becomes (4) for the flat space. Thus, for the area of the surface element we have:

$$|\xi \wedge \eta|^2 = 2|P_1(\xi \wedge \eta)|^2 = -\frac{1}{4} \left( \xi \wedge \eta, \sum_{a=1}^{3} I_a^2 (\xi \wedge \eta) \right) =$$

$$= \frac{3}{2} |\xi \wedge \eta|^2 - \frac{1}{2} \left( \xi \wedge \eta, \sum_{a=1}^{3} I_a \eta \wedge I_a \xi \right) =$$

$$= \frac{3}{2} |\xi \wedge \eta|^2 + \frac{1}{2} (\xi, I_a \eta)(\eta, I_a \xi)$$  \hspace{1cm} (44)

or

$$|\xi \wedge \eta|^2 = \sum_{a=1}^{3} [\omega_a(\xi, \eta)]^2$$  \hspace{1cm} (45)

where $\omega_a$ is the Kahler form corresponding to the complex structure $I_a$, and we have used the fact that the value of the Casimir operator $\sum_a I_a^2$ in the vector representation is $-8$.

Consider the membrane, holomorphic in the complex structure $Je^{-i\phi}$, where $\phi$ is some constant. The restriction on this membrane of the form

$$\omega_\phi = i\omega_I + \text{Im} \left( e^{-i\phi} \omega_C \right)$$  \hspace{1cm} (46)

is zero. Indeed, for such a membrane we may choose the vectors $\xi$ and $Je^{-i\phi} \xi$ as the basis of the tangent space, and

$$\omega_\phi(\xi, Je^{-i\phi} \xi) = i\omega_\phi(\xi, \xi) = 0$$  \hspace{1cm} (47)

It follows from (15) and (47), that the area of the holomorphic membrane is equal to

$$\int_M \text{Re} \left( e^{-i\phi} \omega_C \right) = \left| \int_{\partial M} d^{-1} \omega_C \right|$$  \hspace{1cm} (48)

Thus, to find the masses of the BPS states, we have to know an explicit expression for $\omega_C$. We will obtain it as the Hamiltonian reduction of the flat $\omega_C$ to the manifold $\mu_R = \mu_C = 0$. On the flat space,

$$\omega_C = dy \wedge dz + du \wedge dv$$  \hspace{1cm} (49)

Since on the constraint manifold $\mu_C = e$, $dv = -i(ydz + zd\bar{y})$, we get

$$\omega_C|_{\mu_C = e} = \left( du + i \frac{dy}{y} \right) \wedge dv$$  \hspace{1cm} (50)
and we kill the $G_C$-symmetry by introducing $Y = e^{-iu}y$ to get

$$\omega_C = i\frac{dY}{Y} \wedge dv$$  \hspace{1cm} (51)

The integral of this 2-form over the surface of the 2-brane is reduced to the integral of $vdY/Y$ over the boundary. The new feature in the case of Taub-NUT compared to $\mathbb{R}^3 \times S^1$ is the existence of the membranes whose boundary corresponds to odd electric charge. In the case when the membrane goes through the singularity ($v = e$), the boundary includes the small circle around $v = e$, because at this point $Y = 0$, and the differential has a singularity with the residue

$$\text{res}_{Y=0} vdY/Y = e$$  \hspace{1cm} (52)

Thus, the mass of the BPS state is given by

$$m = |iSe + na + nmD|$$  \hspace{1cm} (53)

where $S$ is the “winding number”, which for the state corresponding to the cycle $\Gamma$ is equal to $\int_{\Gamma} dY/Y$.

7 Deformations of Fivebranes and Membranes.

Here we will follow the method developed in [12] to study the deformations of fivebrane and membranes. Consider the complex submanifold $M \subset X$ of the Hyper-Kahler manifold $X$. We can describe the deformation of this manifold by the vector field

$$\xi \in \Gamma(\mathcal{N}M)$$  \hspace{1cm} (54)

where $\mathcal{N}M$ is the normal bundle of the manifold $M$, whose fiber at point $m \in M$ is $\mathcal{N}_mM = TM_t \mathcal{X} / i_\ast TM_t$, $\Gamma$ denotes the sections of the bundle over $M$. The vectors corresponding to complex deformations (that is, the deformed surface remains complex) correspond to the holomorphic sections:

$$\xi \in H^0(M, \mathcal{N}M)$$  \hspace{1cm} (55)

To study these holomorphic sections, we will follow the method of [12]. Consider the map:

$$p : \mathcal{N}M \to TM$$  \hspace{1cm} (56)

given by the contraction of $\xi$ with the holomorphic symplectic form, and then restricting to $TM$:

$$\xi \mapsto \iota_\xi \omega_C|_TM$$  \hspace{1cm} (57)

Here $\omega_C$ is the holomorphic symplectic form, corresponding to the complex structure of $M$. In this formula we use any representative of $\xi$ in $TX$. All these representatives differ by the vectors, tangent to $M$, and $\omega_C$ is zero on $M$. Thus, the right
hand side of (57) does not depend on the choice of representative, and the map is correctly defined. This is an isomorphism, since $M$ is a Lagrangian manifold (that is, a maximal manifold, on which $\omega_C$ is zero).

Thus, the deformations of complex submanifolds correspond to the holomorphic forms on them. There is a natural metric on the space of deformations:

$$||\xi||^2 \overset{\text{def}}{=} \int_M d^2\sigma ||\xi_\perp||^2$$  \hspace{1cm} (58)

where $\xi_\perp$ is defined as the representative of $\xi$, orthogonal to $M$, and $d^2\sigma$ is the surface element on $M$. This metric can be rewritten as:

$$||\xi||^2 = \frac{1}{2} \int_M p(\xi) \wedge * p(\xi)$$  \hspace{1cm} (59)

Indeed,

$$||\xi_\perp(m)||^2 = \frac{1}{2} ||(J + iK)\xi_\perp(m)||^2 = \frac{1}{2} ||\iota_{\xi_\perp} \omega_C||^2$$  \hspace{1cm} (60)

For any $\nu \in T^*_m X$, $||\nu||^2 = ||\nu|_{T_m M}||^2 + ||\nu|_{(T_m M)^\perp}||^2$. But the restriction of $\iota_{\xi_\perp} \omega_C$ on $T^\perp M$ is zero, since $T^\perp M$ is Lagrangian plane in $TX$. Thus, $\frac{1}{2}||\iota_{\xi_\perp} \omega_C||^2$ is equal to $\frac{1}{2}||\iota_{\xi_\perp} \omega_C|_{T M}||^2 = \frac{1}{2}||p(\xi)||^2$.

As an application of this formalism, consider first the possible deformations of the curve $\Sigma$, representing the fivebrane. The deformations, which have finite norm (58), correspond via the map (57) to the square-integrable meromorphic forms, which is the same as holomorphic forms (decreasing at infinity of $\Sigma$), or just square integrable harmonic forms. There are $2g$ of them, where $g$ is the genus of the (compactified) curve. The integral (59) may be rewritten using Stokes theorem:

$$||\xi||^2 = \sum_{ij} \Omega^{ij} \int_{c^i} p(\xi) \int_{c^j} \overline{p(\xi)}$$  \hspace{1cm} (61)

where $\Omega^{ij}$ is the intersection pairing in $H_1(\Sigma, \mathbb{Z})$. Since $\iota_{\xi_\perp} \omega_C$ is holomorphic, it represents the section of the cotangent bundle to the Jac($\Sigma$), and the integral can be written as

$$\int_{\text{Jac}\Sigma} \iota_{\xi_\perp} \omega_C \wedge \overline{\iota_{\xi_\perp} \omega_C} \wedge t^{r-1}$$  \hspace{1cm} (62)

where $t$ is the polarization on Jac$\Sigma$. The massless vector fields in the four-dimensional theory are also described in terms of harmonic forms on $\Sigma$ \[2\]. The metric (62) on the moduli space of five-brane agrees with the coefficients of the kinetic terms of these vector fields, as it should be in supersymmetric theory.

---

3As we explained in Section 3, the curve $\Sigma$ is topologically the genus $g$ surface $\bar{\Sigma}$ with some $n$ points omitted. The group $H^{\text{comp}}_1(\Sigma) = H_1(\bar{\Sigma}) \oplus \mathbb{R}^{n-1}$ has thus dimension $2g + n - 1$. But the cohomology class, whose pairing with the cycles from $H_1(\Sigma)$ is zero, does not have the square-integrable harmonic representative. Indeed, there are only $2g$ harmonic differentials on $\Sigma$, and they correspond to $2g$ cycles in $H_1(\Sigma)$. As was discussed in \[2\], the harmonic differential corresponding to the cycle around the point at infinity necessarily has a pole at that point.
The relation between the holomorphic symplectic form \( \omega = ds \wedge dv \) and the Witten-Seiberg differential \( \lambda \) is the following. The meromorphic differential \( \lambda \) is defined as

\[
d\lambda = \omega \tag{63}
\]

We may take \( \lambda = vds \). This differential is meromorphic, but its derivative with respect to the moduli is the holomorphic differential on the curve, up to maybe the derivative of the meromorphic function:

\[
\nu_{\text{hol.}} = i_\xi \omega = i_\xi d\lambda = -d_i \xi \lambda + L_\xi \lambda \tag{64}
\]

Here \( L_\xi \lambda \) may be thought of as the derivative with respect to the moduli.

Consider the deformations of the membranes. Again, the holomorphic deformation vectors \( \xi \) are related to holomorphic 1-forms \( p(\xi) \) via \( \omega_C \). (If the membrane is holomorphic in the complex structure \( J \), we should take \( \omega_C = \omega_K - i \omega_I \).) Since the membrane has the boundary, which is supposed to lie on the surface of the 5-brane, the appropriate boundary conditions should be imposed on \( \xi \). These boundary conditions correspond to the following boundary conditions on \( p(\xi) \). Consider the harmonic 1-form

\[
h(\xi) = p(\xi) + \overline{p(\xi)} \tag{65}
\]

The restriction of \( h(\xi) \) on the boundary of the membrane should be zero. Indeed, introduce the vector \( \eta \) directed along the intersection line of the membrane and the fivebrane. The tangent planes to the 5-brane and membrane will be \( (\eta, I\eta) \) and \( (\eta, J\eta) \). Notice that \( \iota_\xi \omega_K = \frac{1}{2} h(\xi) \). Thus, \( \xi^a_+ = \frac{1}{2} K^a_i g^{bc} h_c \) is parallel to \( K J \eta = I \eta \), that is, belongs to the tangent space to \( \Sigma \).

The number of the harmonic forms with these boundary conditions is equal to the first Betti number of the surface. For the genus \( g \) surface with \( n \) holes there are \( 2g + n - 1 \) of them. For the cylinder, there is one harmonic differential, \( h = d\tau \), where \( (\phi, \tau) \) are the coordinates on the cylinder. This harmonic differential corresponds to the zero mode, described in Section 4.

8 Membrane worldsheet theory.

In this section we will explain, from the point of view of the membrane worldsheet theory, why the type of the supermultiplet depends on the topology of the membrane. The fields of the membrane worldsheet theory \([13, 17]\) are 11 bosons \( X^\mu \) and their superpartners, the components of the \( SO(1, 10) \) Majorana fermions \( \Theta \). Let us introduce the light-cone coordinates

\[
X^\pm = \frac{X^0 \pm X^3}{\sqrt{2}} \tag{66}
\]

and choose the light-cone gauge

\[
X^+(\zeta) = X^+(0) + \tau, \quad \gamma_+ \Theta = 0 \tag{67}
\]
We have chosen one of the coordinates parallel to the 5-brane worldvolume, $x^3$, as the longitudinal coordinate.

The reality condition for fermions is

$$\Theta^* = C \Theta$$

where $C$ is the $SO(9)$ charge conjugation matrix, characterized by the property

$$C^\dagger \Gamma^A C = (\Gamma^A)^T$$

Let us choose the following representation for $SO(9)$ gamma-matrices:

$$\Gamma^{i+6} = \sigma^i \otimes \gamma^5 \otimes \rho^3, \hspace{1em} i = 1, 2, 3;$$
$$\Gamma^a = 1 \otimes \gamma^a \otimes \rho^3, \hspace{1em} a = 4, 5, 6, 10;$$
$$\Gamma^1 = 1 \otimes 1 \otimes \rho^1;$$
$$\Gamma^2 = 1 \otimes 1 \otimes \rho^2$$

where $\sigma^i$ and $\rho^i$ are Pauli matrices, and $\gamma^a$ are Euclidean gamma-matrices corresponding to $x^4, x^5, x^6, x^{10}$.

We will use the following charge conjugation matrix:

$$C = \sigma^2 \otimes C \otimes \rho^3$$

where $C$ is the four-dimensional Euclidean charge conjugation, $C^\dagger \gamma^a C = - (\gamma^a)^T$.

For the membrane with the boundary on the fivebrane the following boundary conditions should be imposed on fermions:

$$\frac{1}{2} \left( 1 + \prod_{j \in \mathcal{N}} \Gamma^j \right) S = 0$$

where $\mathcal{N}$ is the set of indices $j$ for which the boundary conditions on $X^j$ are of Neumann type. The origin of these boundary conditions is explained in [18]. In our situation, the 5-brane is not flat, thus the boundary conditions for fermions are not constant. The directions parallel to the 5-brane world-volume, are spatial directions $x^1$ and $x^2$, the vector parallel to the boundary of the membrane, and the rotation of this vector by the operator $I$ (the complex structure of the 5-brane). Thus, we have the following boundary conditions:

$$\frac{1}{2} \left( 1 + \Gamma^1 \Gamma^2 \Gamma^\parallel \right) S = 0$$

where

$$\Gamma^\parallel = 1 \otimes \gamma^\parallel \otimes 1$$
$$\gamma^\parallel = \gamma^{\parallel \nu} \gamma^{1 \nu}$$
$$\gamma^{\parallel \nu} = \frac{1}{\sqrt{|x|^2 + |y|^2}} (\dot{x} \gamma^x + \dot{y} \gamma^y + c.c.)$$
$$\gamma^{1 \nu} = \frac{i}{\sqrt{|x|^2 + |y|^2}} (\dot{y}^* \gamma^x - \dot{x}^* \gamma^y - c.c.)$$
The group $SO(4)$, acting on the tangent space to the four-manifold, can be decomposed as the product of two $SU(2)$ groups:

$$SO(4) = (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$$  \hfill (75)

where $SU(2)_L$ and $SU(2)_R$ act on the components of positive and negative chirality. For the Hyper-Kahlerian manifold, the holonomy group is $SU(2)_R$, and the group $SU(2)_L$ is generated by $I$, $J$ and $K$ — the three complex structures. We will need the following identity:

$$\gamma_a \gamma_b = \frac{1}{2} \left( g_{ab} + \sum_n \omega_n^a \hat{I}_n \right)$$  \hfill (76)

(Here $\hat{I}_1$, $\hat{I}_2$ and $\hat{I}_3$ are the gamma-matrix realizations of the complex structures, and $\omega_1$, $\omega_2$, $\omega_3$ are the corresponding Kahler forms.) Indeed, for any two vectors $\xi$, $\eta$, we have:

$$\hat{\xi} \hat{\eta} = \frac{1}{2} \left( g(\xi, \eta) 1 + [\hat{\xi}, \hat{\eta}] \right)$$  \hfill (77)

and the chiral part of the commutator can be represented as:

$$[\hat{\xi}, \hat{\eta}]_L = -\frac{1}{2} \sum_n \text{tr}_L \left( [\hat{\xi}, \hat{\eta}] I_n \right) I_n = \\
= \frac{1}{2} \sum_n \text{tr}_L \left( \hat{\xi} I_n \hat{\eta} \right) I_n = \sum_n \omega_n(\xi, \eta) I_n$$  \hfill (78)

This identity implies that the $SU(2)_L$ component of $\gamma^\parallel$ is $\gamma^\parallel_L = \hat{I}$ (the gamma-matrix realization of the 5-brane complex structure). Notice that the $SU(2)_R$ component of $\gamma^\parallel$ is non-constant (depends on the point on the boundary). Thus, the boundary conditions for chiral fermions are:

$$\Gamma^1 \Gamma^2 S = \hat{I} S$$  \hfill (79)

Fermionic zero modes are the solutions of the equations of motion for fermions, which do not depend on the worldsheet time coordinate. They satisfy the following equation \cite{13,14}:

$$dX^a \Gamma_a \wedge \nabla S = 0$$  \hfill (80)

where $\nabla$ is the pullback of the spin connection on the membrane worldsurface. Since the holonomy group is $SU(2)_R$, the positive chirality fermions are in trivial bundle. Thus, the constant spinors of positive chirality are solutions of \cite{80}. Let us show, that those constant spinors of positive chirality, which satisfy the boundary condition \cite{79}, correspond to spatial spinors.

Let us choose the tetrad $(e_x, e_y, \bar{e}_x, \bar{e}_y)$ in such a way, that $SU(2)_L$ acts on the coordinates of a vector as:

$$I \xi^x = i \xi^x, \quad I \xi^y = i \xi^y$$  

$$J \xi^x = i \xi^y, \quad J \xi^y = -i \xi^x$$  \hfill (81)
One can check that the only metric with the property $g(\xi, \eta) = g(I\xi, I\eta) = g(J\xi, J\eta)$ is

$$||\xi||^2 = G(||\xi^x||^2 + ||\xi^y||^2)$$

which means that the tetrad is orthogonal.

The groups $SU(2)_L$ and $SU(2)_R$ act on spinors as follows:

$$
\begin{align*}
E & : \gamma_y \gamma_x & & \gamma_x \gamma_y \\
H & : [\gamma_x, \gamma_x] + [\gamma_y, \gamma_y] & & [\gamma_x, \gamma_x] - [\gamma_y, \gamma_y] \\
F & : \gamma_x \gamma_y & & \gamma_y \gamma_x
\end{align*}
$$

Spinors of positive chirality have the following form$^4$

$$S^{(0)} = \gamma_x \gamma_y \phi^{(0)}_+ + \gamma_x \gamma_y \phi^{(0)}_-$$

Substituting these solutions in the boundary conditions (79) we get:

$$\phi^{(0)}_- = -\rho^3 \gamma_x \gamma_y \phi^{(0)}_+$$

Let us represent $\gamma_x$ and $\gamma_y$ in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ as follows:

$$\gamma_x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes 1, \quad \gamma_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \gamma_x = \gamma_x^T, \quad \gamma_y = \gamma_y^T$$

In this representation, the zero mode $S^{(0)}$ has the following form:

$$S^{(0)} = \begin{bmatrix} a \\ 0 \\ 0 \\ -\rho^3 a \end{bmatrix}$$

where $a$ and $b$ are arbitrary constant spinors of $SO(3)_{7,8,9} \times SO(2)_{1,2}$.

Let us impose the reality conditions (68). For the representation (86), $C = i\tau^2 \otimes \tau^1$, thus the charge conjugation matrix (71) is:

$$C = \sigma^2 \otimes (i\tau^2 \otimes \tau^1) \otimes \rho^1$$

and (88) gives the $SO(3)_{7,8,9}$ - invariant conditions

$$a^* = i(\sigma^2 \otimes \rho^2)a$$

which enable us to express the negative $SO(2)_{7,8}$-helicity component of $a$ through the positive helicity component:

$$a = \begin{bmatrix} a^+ \\ \rho^2 a^*_+ \end{bmatrix}$$

$^4$To satisfy equation (80), we could take arbitrary holomorphic $\phi^{(0)}_+(z)$ and antiholomorphic $\phi^{(0)}_-(\bar{z})$, but only the constant $\phi_{\pm}$ can satisfy the boundary conditions.
This means, that the zero modes are parametrized by the spinors \( a^\uparrow \) of the spatial \( SO(3) \) group (which is broken to \( SO(2) \) by the gauge choice).

It turns out, that for the membranes with the topology of the disk these constant \( S \) are the only solutions of the equations \((80)\), while for the surfaces with more complicated topology, non-constant solutions with negative chirality are possible.

These non-constant solutions are related to the harmonic 1-differentials on the membrane worldsurface, satisfying the boundary conditions described in the end of Section 7. For any such differential \( h \), consider the vector field

\[
\xi^i = g^{ij} h_j
\]

where \( g_{ij} \) is an induced metric on the membrane worldsurface. Consider the operator

\[
B_h = (i_s \xi)^a \Gamma_a
\]

acting on spinors. Here \( i_s \xi \) is the vector \( \xi \) on the membrane surface, considered as the vector in the ambient space. This operator has the following properties: it maps solutions of \((80)\) with positive chirality to solutions with negative chirality, and also preserves boundary conditions and the reality condition. Indeed, the vector \( i_s \xi \) is orthogonal to the tangent space of the 5-brane, thus \( B \) respects boundary conditions. The reality condition for \( B_h S \) follows from \((69)\). Let us check that \( B_h S \) is the solution to \((80)\):

\[
d X^a \Gamma_a \wedge \nabla \left( (i_s \xi)^b \Gamma_b S \right) = - \nabla \left( d X^a \Gamma_a (i_s \xi)^b \Gamma_b S \right) =
\]

\[
= - \frac{1}{2} \nabla \left( d X^a g_{ab} (i_s \xi)^b S + \sum_n d X^a \omega^n_{ab} (i_s \xi)^b \hat{I}_n S \right)
\]

(Here \( \hat{I}_1, \hat{I}_2, \hat{I}_3 \) are the gamma-matrix realizations of the complex structures, and \( \omega^1, \omega^2, \omega^3 \) are the corresponding Kahler forms.) The last row in \((93)\) is

\[
- \frac{1}{2} \nabla \left( (h + *h \hat{J}) S \right) = 0
\]

— this is zero, since both \( h \) and \( *h \) are closed forms, and \( S \) is covariantly constant.

In the light-cone gauge, the supersymmetry on the membrane worldsheet is generated by \( \alpha^- \) and \( \beta^- \) transformations \([13\, 17]\):

\[
\delta X^I = 2i\alpha^I \Gamma^I S + 2i\epsilon^{ab} \partial_a X^J \alpha^J \Gamma^I \, f \, d\tau \partial_b S
\]

\[
\delta S = - \frac{1}{2\sqrt{2} \lambda^2} \left( \dot{X}^I \Gamma_I - \frac{1}{2} \epsilon^{ab} \partial_a X^I \partial_b X^J \Gamma_{IJ} \right) \alpha + 2i\epsilon^{ab} \partial_a S \alpha^J \Gamma^I \, f \, d\tau \partial_b S + \beta
\]

Here \( \alpha \) and \( \beta \) are \( SO(9) \) Majorana spinors, they have together 32 real components. Consider the membrane, holomorphic in the appropriate complex structure. It is natural to consider separately \( \alpha \) and \( \beta \) of positive and negative chirality with respect to the group \( SO(4) \) rotating \( x^4, x^5, x^6, x^{10} \). On the non-flat Hyper-Kahlerian manifold, only \( \alpha \) and \( \beta \) of positive chirality generate rigid supersymmetry transformations. Even if the space is flat, those \( \alpha \) and \( \beta \) which have negative \( SO(4) \) chirality
give $\delta S$ which does not satisfy the boundary conditions. Of those $\alpha$ and $\beta$ which have positive $SO(4)$ chirality, half give $\delta S$ satisfying the boundary conditions. Thus, we get 8 of 32 supercharges preserved by the 5-brane. The BPS state, corresponding to the membrane, should be annihilated by 4 of these 8 supercharges.

Notice that the terms with $\partial S$ in (95) are zero for $\alpha$ with positive $SO(4)$ chirality.

Consider the variation $\delta S$. It is zero if $\beta = -\frac{1}{2\sqrt{2}P^+} \epsilon^{ab} \partial_a X^I \partial_b X^J \gamma_{IJ} \alpha$

For the holomorphic membrane,

$$\frac{1}{P^+} \partial_a X' \partial_b X^J \Gamma_{IJ} = \frac{1}{P^0} J + \text{terms from } SU(2)_R$$

where $J$ is the gamma-matrix realization of the complex structure of the membrane, and the terms from $SU(2)_R$ are non-constant.

Then, the transformation $\delta_\alpha S$ with $\alpha$ of positive chirality can be compensated by constant $\beta$ given by (96).

Consider now the variation $\delta X^a$. For the (constant) fermionic modes $S^{(0)}$ of positive chirality, the first formula in (97) gives $\delta X^a$ corresponding to the spatial translations (in the light cone gauge, there are two transverse directions $x^1$ and $x^2$). Zero modes $B_b S^{(0)}$ of negative chirality give bosonic zero modes, described in the previous section, and reparametrizations of the membrane worldsheet:

$$\delta X^a = \alpha^\dagger \Gamma^a (i_s \xi)^b \Gamma_b S^{(0)} =$$

$$= \frac{1}{2} (i_s \xi)^a (\alpha^\dagger S^{(0)}) + \sum_n (I_n i_s \xi)^a (\alpha^\dagger \hat{I}_n S^{(0)})$$

(98)

The terms with $(I_i \xi)^a$ and $(i_s \xi)^a$ are zero. Indeed, $S^{(0)}$ are can be written as $S^{(0)} = \hat{J} \alpha'$, where $\hat{I} \alpha' = -\Gamma_1 \Gamma_2 \alpha'$. Thus,

$$\alpha^\dagger \hat{I} S^{(0)} = \alpha^\dagger \hat{I} \hat{J} \alpha' = -\alpha^\dagger \Gamma_1 \Gamma_2 \hat{J} \alpha' =$$

$$= \alpha^\dagger \hat{J} \hat{I} \alpha' = \alpha^\dagger \hat{J} \Gamma_1 \Gamma_2 \alpha' = 0$$

(99)

and for the same reason $\alpha^\dagger S^{(0)} = 0$. Thus, the only remaining transformations are

$$\delta X^a = (K i_s \xi)^a$$

(100)

and

$$\delta X^a = (J i_s \xi)^a$$

(101)

The first one is of the type described in section 4 – the boundary of the membrane slides along the 5-brane. The second is an infinitesimal reparametrization of the membrane worldsheet.

For the disk, quantization of fermionic zero modes gives four bosonic states

$$|0 >, \ a_1 a_2 |0 >, \ |\tilde{0} >, \ a_1 a_2 |\tilde{0} >$$

(102)
and four fermionic states

\begin{equation}
 a_1 |0>, a_2 |0>, a_1 |\bar{0}>, a_2 |\bar{0}>
\end{equation}

where $|0>$ and $|\bar{0}>$ correspond to the disks of the opposite orientation (monopole and antimonopole). This is the hypermultiplet. Notice, that this is indeed a BPS state. We can make $\delta S = 0$ by choosing $\beta$ as in (96). Then, the first equation in (95) gives us $\delta X$ corresponding to spatial translations, and we require the wave function to be constant over $x^1, x^2$ (notice that we are working in the centre of mass frame).

For the cylinder, we have also non-constant modes of negative chirality. We will call them $b$-modes. The supersymmetry transformations of these $b$-modes give the bosonic zero mode, corresponding to the moduli space of cylinders. The BPS states correspond to the vacua of supersymmetric quantum mechanics on this moduli space [10]. There are four supercharges

\begin{equation}
 b_1 \frac{\partial}{\partial m}, b_2 \frac{\partial}{\partial m}, b_1^* \frac{\partial}{\partial m}, b_2^* \frac{\partial}{\partial m}
\end{equation}

which should annihilate the wave function. We have denoted the collective coordinate $m$. The Hamiltonian is $\frac{\partial^2}{\partial m^2}$. Since the moduli space is the interval $[m_l, m_r]$, some boundary conditions should be imposed (otherwise the Hamiltonian is not self-conjugate). As we explain in Appendix B, for the wave function

\begin{equation}
 \psi(m) = \psi_0(m) + b_1 \psi_1(m) + b_2 \psi_2(m) + b_1 b_2 \psi_{12}(m)
\end{equation}

the only possible supersymmetric boundary conditions are either

\begin{equation}
 \partial_m \psi_0 = \partial_m \psi_{12} = 0
\end{equation}

\begin{equation}
 \psi_1 = \psi_2 = 0
\end{equation}

or

\begin{equation}
 \partial_m \psi_1 = \partial_m \psi_2 = 0
\end{equation}

\begin{equation}
 \psi_0 = \psi_{12} = 0
\end{equation}

— these equalities should be satisfied at the ends of the interval, $m = m_l$ and $m = m_r$.

Since the vacuum wave function should be constant, the boundary conditions of the first type leave us with the states

\begin{equation}
 \psi_0 + b_1 b_2 \psi_{12}
\end{equation}

which have spin zero, and the second type gives

\begin{equation}
 b_1 \psi_1 + b_2 \psi_2
\end{equation}

which have spin 1/2. We should also act on these states by the $a$-modes. This gives us the hypermultiplet for the boundary conditions of the type (106), and the vector multiplet for the boundary conditions of the type (107).
Understanding which type of boundary conditions should be imposed probably requires considering the theory at short distances. There are at least two cases when it may be important to know what happens at short distances. The first case is when the modulus of $\Sigma$ intersects the curve of marginal stability. The boundary of the cylinder touches itself and the cylinder decays into two disks. These disks do not have $b$-modes, only the constant $a$-modes. We cannot explain how $b$-mode for the cylinder transforms into the $a$-mode for the disk in the low energy theory. It follows from expression (91), (92), that when the length of the cylinder goes to zero, the $b$-modes become localized in the very small region near the points where the boundary touches itself. Indeed, when the length of the cylinder is very small, $dx$ and $dy$ should be very large everywhere except for that region. Thus, the complete understanding of what happens to the fermionic zero modes when the cylinder decays requires considering the theory at short distances.

The other case is when we consider the boundary conditions for the wave function of the cylinder at the boundary of the moduli space. As we have discussed in Section 4, the boundary of the cylinder moduli space corresponds to the cylinder degenerating into the disk. At this point, the boundary touches itself and the same problem with the $b$-modes appear. It should be true that the boundary conditions of the type (107) appear.

Notice that the $SO(3)_{7,8,9}$ group, realized by $a_1a_2 + b_1b_2$, $a_1^*a_2^* + b_1^*b_2^*$ and $[a_1, a_1^*] + [a_2, a_2^*] + [b_1, b_1^*] + [b_2, b_2^*]$, acts on the states in four-dimensional theory as $SU(2)_R$ symmetry.

9 Membranes of minimal area and geodesics.

In this section we will show that the boundaries of the minimal surfaces are not necessarily geodesics. We will explain how our discussion of BPS states is related to the one in [11]. We will first consider two examples.

The simplest example arises in the theories with matter in mixed representation. Let us consider the “spike”:

$$v = m/s$$

In the case of the pure spike (without regular terms) the boundary of the membrane is geodesic. Indeed, it is described by the equation $|s|^2 = m$, and the meromorphic differential

$$\lambda = vds = mds/s$$

has constant phase (purely imaginary) on any vector tangent to the boundary. This means, that it is geodesic in the metric $|\lambda|^2$.

But, if we add regular terms, then the boundary is generally not geodesic any more. For example, consider

$$v = m/s + \epsilon s$$

$^{5}$Explicit expressions for the $b$-modes contain terms like $\partial_x / |dx|^2 + |dy|^2$, $\partial_y / |dx|^2 + |dy|^2$. — cf. [28].
where \( \epsilon \) is very small. Then, the equation for the membrane is

\[
v - s^* = \epsilon \frac{v^* + s}{2} + o(\epsilon) \tag{112}
\]

and the boundary is

\[
|s|^2 = m + o(\epsilon) \tag{113}
\]

Now

\[
v ds = mi (1 + \epsilon e^{2i\varphi} + o(\epsilon)) d\varphi \tag{114}
\]

so the phase is changing, and the boundary is not geodesic.

As the second example we take the pure \( SU(2) \). Consider a very light monopole, that is \( u \) close to 1. Put \( x^6 + ix^{10} = i\pi + s \). The monopole is located in the region of small \( s \) and \( v \), where the curve may be represented as

\[
-\frac{s^2}{2} - \epsilon s^4 - v^2 + (\rho + i\sigma) + o(\epsilon) = 0 \tag{115}
\]

(after appropriate rescaling of \( s \) and \( v \) by the same factor. Here \( \rho + i\sigma \) is related to \( u - 1 \) by the appropriate rescaling.)

We consider the equation for the membrane to the first order in \( \epsilon \). Describe the membrane by the ansatz:

\[
s^* - v = (-i + a)(s + v^*) + ib(s + v^*)^3 \tag{116}
\]

where \( a \) and \( b \) are real constants of the order \( \epsilon \). We may rewrite it as

\[
\begin{align*}
s &= [(i + a) + (3|v|^2 - |s|^2)b] s^* \\
v &= [(i - a) + (3|s|^2 - |v|^2)b] v^*
\end{align*} \tag{117}
\]

To find the boundary, we have to use these equations together with (115). After substituting (117) in (115) we get

\[
\frac{1}{2}|s|^2 + |v|^2 = \sigma \tag{118}
\]

for the imaginary part and

\[
\epsilon |s|^4 + \rho - \frac{1}{2} \left[ a + (3|v|^2 - |s|^2)b \right] |s|^2 - \left[ -a + (3|s|^2 - |v|^2)b \right] |v|^2 = 0 \tag{119}
\]

for the real part of (115). The condition that the membrane intersects the curve by the line means that (119) follows from (118). This requirement gives

\[
\begin{align*}
a &= -\frac{11}{2}b\sigma \\
b &= -\frac{1}{3}\epsilon \\
\rho &= \frac{\sigma}{2}b^2
\end{align*} \tag{120}
\]
Actually we do not need the values of these parameters in addressing the question of the phase of $v_{ds}$. Using the equations (118) and (117), we can write an equation for the boundary of the membrane, parametrized by the angle $\varphi$:

$$
\begin{align*}
 v &= \sqrt{\sigma} \cos \varphi e^{i\pi \varphi} \left( 1 + \frac{i\varphi}{2} - (3|s|^2 - |v|^2) \frac{i\varphi}{2} + o(\epsilon) \right) \\
 s &= \sqrt{2\sigma} \sin \varphi e^{i\pi \varphi} \left( 1 - \frac{i\varphi}{2} - (3|v|^2 - |s|^2) \frac{i\varphi}{2} + o(\epsilon) \right)
\end{align*}
$$

And for the differential we have:

$$
\begin{align*}
 v \frac{ds}{d\varphi} &= \sqrt{2\sigma^2} i \left[ \cos^2 \varphi \left( 1 - 2(|v|^2 + |s|^2) \frac{i\varphi}{2} \right) - \\
 &\quad - \cos \varphi \sin \varphi \frac{d}{d\varphi} (3|v|^2 - |s|^2) \frac{i\varphi}{2} \right] = \\
 &= \sqrt{2\sigma^2} i \cos^2 \varphi \left[ 1 - ib\sigma^2(1 - 4\sin^2 \varphi) \right]
\end{align*}
$$

The phase of this expression is not constant.

The boundaries of the membranes are more complicated than geodesics. They depend essentially on the embedding of the curve into $\mathbb{R}^3 \times S^1$. Consider two embeddings:

$$
cosh s + u - v^2 = 0 \quad (123)
$$

and

$$
cosh s + u - \alpha^2 v^2 = 0 \quad (124)
$$

where $\alpha$ is constant. These two curves are isomorphic as complex curves, the isomorphism given by $v \rightarrow \alpha v$. And the geodesics of the metrics $|v_{ds}|^2$ are preserved by this isomorphism. But this is not true for the boundaries of the minimal surfaces: their dependence on $\alpha$ is quite complicated. Indeed, the shape of the minimal surface depends on the metric in the ambient space. The minimal surface for the curve (124) would be related to the minimal surface for (123), if the metric in the space $\mathbb{R}^3 \times S^1$, where (124) is embedded, were not $|ds|^2 + |dv|^2$, but $|ds|^2 + |\alpha dv|^2$. Notice however the simple relation between the masses of the membranes with boundaries on (124) and the masses of the membranes ending on (123):

$$
m_\alpha = \frac{1}{|\alpha|} m_1 \quad (125)
$$

Let us prove that in the limit $\alpha \rightarrow \infty$ the boundaries of the membranes become geodesics in the metric $|v_{ds}|^2$. This means, that the value of the differential $v_{ds}$ on the vector, tangent to the boundary of the membrane, has constant phase:

$$
\text{Arg} \left( v \frac{ds}{dt} \right) = \text{const} \quad (126)
$$

We assume that $\alpha$ is real. It is useful to introduce the coordinate $V = \alpha v$. The equation for $\Sigma$ is:

$$
cosh s + P(V) = 0 \quad (127)
$$
Since the membrane is holomorphic in coordinates \((x, y)\), the restriction of the form
\[
4dx \wedge dy = ds \wedge ds^* + d\tilde{v} \wedge d\tilde{v}^* + ds \wedge d\tilde{v} + ds^* \wedge d\tilde{v}^* =
\]
\[
ds \wedge ds^* + \frac{1}{\alpha}d\bar{V} \wedge d\bar{V}^* + \frac{1}{\alpha} \left( ds \wedge d\bar{V} + ds^* \wedge d\bar{V}^* \right)
\]
(128)
on the surface of the membrane is zero. In the imaginary part of this expression, we may neglect the term \(\frac{1}{\alpha}d\bar{V} \wedge d\bar{V}^*\), small compared to \(ds \wedge ds^*\). This gives the condition:
\[
ds \wedge ds^*|_{\Lambda_{\gamma TM}} = 0
\]
(129)
meaning that the projection of the membrane on the surface \(s\) has real dimension one. In other words, the membrane lies in three-dimensional surface \(C \times C_v\), where \(C\) is some curve of real dimension one in the \(s\)-plane, and \(C_v\) is the \(v\)-plane.

The real part of (128) is:
\[
\frac{1}{\alpha} \text{Re}(ds \wedge d\bar{V}) = 0
\]
(130)
This implies, that the section of the membrane by the plane \(s = s_0\) is a line interval. If the tangent vector to \(C\) at point \(s_0\) is \(\dot{s}\), then this line interval is given by the equation:
\[
v = v_0 + i\kappa \frac{\dot{s}}{\dot{s}}
\]
(131)
where \(\kappa\) is a (real) parameter along the line. This gives the following description of the membrane boundary. Consider two solutions \(v_1(s)\) and \(v_2(s)\) of
\[
cosh s + P(v) = 0
\]
(132)
The tangent vector to the boundary of the membrane is
\[
\dot{s} = \frac{e^{i\phi}}{v_1(s) - v_2(s)}, \quad \phi = \text{const}
\]
(133)
In particular, for the \(SU(2)\) case, \(v_2 = -v_1 = -v(s)\), and we get
\[
v(s)\dot{s} = e^{i\phi}
\]
(134)
which means, that the boundary is a geodesic line.

10 Appendix A. Complex structures of the Taub-NUT space.

One of the complex structures is just
\[
I. \begin{bmatrix} \xi_Y \\ \xi_Z \\ \xi_v \end{bmatrix} = \begin{bmatrix} i\xi_Y \\ i\xi_Z \\ i\xi_v \end{bmatrix}
\]
(135)
Let us calculate the other complex structure, \( J \). On the tangent, space to \( \mathbf{H} \times \mathbf{H} \), the other complex structure acts as the left multiplication on \( j: (\delta q, \delta s) \to (j\delta q, j\delta s) \), or:

\[
J. \begin{bmatrix} \xi_y \\ \xi_z \\ \xi_u \\ \xi_v \end{bmatrix} = \begin{bmatrix} -\xi_y^* \\ \xi_z^* \\ -\xi_u^* \\ \xi_v^* \end{bmatrix}
\]  \hspace{1cm} (136)

Given the equivalence class \((Y, Z)\) of the points on the manifold \( \mu_C = 0 \) modulo \( G_C \), we can always find the corresponding point on \( \mu_C = \mu_R = 0 \) (modulo \( G \)). Let it be

\[
Z = \exp(iu)z; \ Y = \exp(-iu)y
\]  \hspace{1cm} (137)

Then, to find \( u_{im} = \text{Im} \) we may use the equation

\[
\mu_R = \frac{1}{2} \left( \exp(-2u_{im})|Y|^2 - \exp(2u_{im})|Z|^2 - 2u_{im} \right) = 0
\]  \hspace{1cm} (138)

- this is easy to see that this equation always has a solution, and the solution is unique.

So, we have the vector \((\xi_y, \xi_z)\), tangent to \( \mu_C = 0 \) at the point \((Y, Z)\). We construct the vector \((\xi_y, \xi_z) = (e^{iu}\xi_Y, e^{-iu}\xi_Z)\), at the point \((y, z)\) in \( \mu_R = \mu_C = 0 \), tangent to \( \mu_C = 0 \). We now add to \((\xi_y, \xi_z)\) the vector of the form \([ity, -itz, t, 0]\) so that acting by \( J \) on the resulting vector we get again a vector tangent to \( \mu_C = 0 \):

\[
t^* = -i\frac{ix^*_u - z\xi^*_z + y\xi^*_y}{1 + |z|^2 + |y|^2}
\]  \hspace{1cm} (139)

Now, acting by \( J \) and going to coordinates \( Z, Y, \) we get:

\[
\begin{bmatrix} \xi_Y \\ \xi_Z \end{bmatrix} \to i \begin{bmatrix} -\frac{2 + \rho}{1 + \rho} Y^* Z^* - \frac{2 + |y|^2 + \rho}{1 + \rho} YY^* \\ \frac{2 + |z|^2 + \rho}{1 + \rho} ZZ^* + \frac{2 + \rho}{1 + \rho} ZY^* \end{bmatrix} \begin{bmatrix} \xi_Y^* \\ \xi_Z^* \end{bmatrix}
\]  \hspace{1cm} (140)

where \( \rho = |y|^2 + |z|^2 \). Notice, that in this expression \(|y|^2\) and \(|z|^2\) are related to \(|Y|^2\) and \(|Z|^2\) by multiplication on \( \exp(2u_{im}) \) and \( \exp(-2u_{im}) \) correspondingly, where \( u_{im} \) may be found from the transcendental equation \((138)\).

Now \( K \) can be found from \( K = JI \). When \( e \to \infty \) and \( v \) remains finite, both \( Z \) and \( Y \) are large. We keep \( u_{im} \) finite, so \(|z|^2\) and \(|y|^2\) are also large. We find

\[
J. \begin{bmatrix} \xi_Y \\ \xi_Z \end{bmatrix} = \begin{bmatrix} -Y(Z^*\xi_Y^* + Y^*\xi_Z^*) \\ Z(Y^*\xi_Y^* + Y^*\xi_Z^*) \end{bmatrix}
\]  \hspace{1cm} (141)

Using \( \xi_v = i(Z\xi_Y + Y\xi_Z) \), and going to \( s = \log Y \), we get

\[
J.\xi_s = i\xi_v^* \hspace{1cm} J.\xi_v = -i\xi_s^*
\]  \hspace{1cm} (142)

which is the same as the \( J \) complex structure in the flat space.
11 Appendix B. Supersymmetric quantum mechanics on the interval.

Consider the wave function of the form

$$\psi = \psi_0 + b_1 \psi_1 + b_2 \psi_2 + b_1 b_2 \psi_{12}$$  \hspace{1cm} (143)

For the Hamiltonian to be self-conjugate, we have to require for the boundary values:

$$\left[ \begin{array}{c} \partial_m \psi \\ \psi \end{array} \right] \in L$$  \hspace{1cm} (144)

where $L$ is the $\mathbb{C}$-linear subspace in $\mathbb{C}^8$, such that for any two vectors $\left[ \begin{array}{c} p_1 \\ q_1 \end{array} \right]$ and $\left[ \begin{array}{c} p_2 \\ q_2 \end{array} \right]$ in $L$:

$$(p_1, q_2) - (q_1, p_2) = 0$$  \hspace{1cm} (145)

These boundary conditions should be supersymmetric. That is, the space of functions such that the boundary values $\left[ \begin{array}{c} \partial_m \psi \\ \psi \end{array} \right] \in L$ should be preserved by

$$b_1 \partial_m, \ b_2 \partial_m, \ b_1^* \partial_m, \ b_2^* \partial_m$$  \hspace{1cm} (146)

Our moduli space is an interval $m \in [m_l, m_r]$. Consider the functions, whose Fourier coefficients in the series

$$\psi(m) = \sum_n \left[ a_n \cos \left( \frac{2\pi n}{m_l - m_r} m \right) + b_n \sin \left( \frac{2\pi n}{m_l - m_r} m \right) \right]$$  \hspace{1cm} (147)

decrease sufficiently rapidly with $n$. Then,

$$\frac{\partial^2}{\partial m^2} \psi(m) = -\sum_n \left( \frac{2\pi n}{m_l - m_r} \right)^2 \left[ a_n \cos \left( \frac{2\pi n}{m_l - m_r} m \right) + b_n \sin \left( \frac{2\pi n}{m_l - m_r} m \right) \right]$$  \hspace{1cm} (148)

Let us describe all the possible subspaces in the space of boundary values, which are preserved by (146). We have to require that $L$ is preserved by the operators

$$\left[ \begin{array}{cc} 0 & \left( \frac{2\pi n}{m_l - m_r} \right)^2 b_i \\ b_i & 0 \end{array} \right], \quad \left[ \begin{array}{cc} 0 & \left( \frac{2\pi n}{m_l - m_r} \right)^2 b_i^* \\ b_i^* & 0 \end{array} \right]$$  \hspace{1cm} (149)

where $i = 1, 2$ and $n$ an arbitrary integer. This means, that if

$$\left[ \begin{array}{c} p \\ q \end{array} \right] \in L$$  \hspace{1cm} (150)
then \( L \) contains also the vectors of the form

\[
\begin{bmatrix}
\sum_i (\alpha_i b_i + \beta_i b_i^*) & 0 \\
0 & \sum_i (\alpha_i b_i + \beta_i b_i^*)
\end{bmatrix}
\]

(151)

for arbitrary \((\alpha_i, \beta_i)\). A straightforward computation shows that if \( v = v_0 + b_1 v_1 + b_2 v_2 + b_1 b_2 v_{12} \neq 0 \), then

\[ \dim_{\mathbb{C}} (Cb_1 + Cb_2 + Cb_1^* + Cb_2^*) v \geq 2 \]  

(152)

with equality if and only if \( v \) is an eigenvector of \((-1)^F\), that is, \( v \) is either of the form \( b_1 v_1 + b_2 v_2 \), or of the form \( v_0 + b_1 b_2 v_{12} \). Taking into account that the dimension should be \( \dim_{\mathbb{C}} L = 4 \), we get the only two possibilities:

1) \( L \) consists of the vectors of the form

\[
\begin{bmatrix}
p_0 + b_1 b_2 p_{12} \\
b_1 q_1 + b_2 q_2
\end{bmatrix}
\]

(153)

2) \( L \) consists of the vectors of the form

\[
\begin{bmatrix}
b_1 p_1 + b_2 p_2 \\
q_0 + b_1 b_2 q_{12}
\end{bmatrix}
\]

(154)

In both cases, \( L \) satisfies the property \( \text{(145)} \).

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