Approximate analytical solutions of the Klein–Gordon equation for the Hulthén potential with the position-dependent mass

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Abstract
The Klein–Gordon equation is solved approximately for the Hulthén potential for any angular momentum quantum number ℓ with the position-dependent mass. Solutions are obtained by reducing the Klein–Gordon equation into a Schrödinger-like differential equation using an appropriate coordinate transformation. The Nikiforov–Uvarov method is used in the calculations to get energy eigenvalues and the wavefunctions. It is found that the results in the case of constant mass are in good agreement with the ones obtained in the literature.

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1. Introduction

Exact or approximate solutions of relativistic/non-relativistic wave equations have received great attention. So far, solutions have been, in general, obtained for the case of constant mass or, at most, time-dependent mass [1, 2]. Effective mass solutions have received much attention recently. A quite general hermitian effective Hamiltonian is used to describe non-relativistic systems; such a description is applied to study semiconductor nanostructures [3]. Another interesting problem is the correct form of the kinetic energy operator for such a Hamiltonian, since the momentum and the mass operators no longer commute with each other in the case of position-dependent mass, which is related to the problem of ordering ambiguity [4]. There are some important problems related to the ordering ambiguity concept, such as the dependence of nuclear forces on the relative velocity of the two nucleons [5, 6], and the impurities of crystals [7]. In addition, many authors have studied to propose some effective Hamiltonians for the non-relativistic case taking into account the dependence of the mass on position [8].

There have been many efforts to solve the Schrödinger equation for the case of position-dependent mass using different methods or schemes for different potentials, such as exponential-type potentials [4], Natanzon potentials using a group-theoretical method [9], solutions in the case of mappings of the Morse+oscillator+Coulomb potential [10], hyperbolic-type potentials [11] and Morse and Coulomb potentials with the position-dependent mass [12, 13], \( PT \)-symmetric anharmonic oscillators [14], the Morse-like potential in the scheme of supersymmetric quantum mechanics [15], Kratzer and Scarf II potentials [16] and deformed Rosen–Morse and Scarf potentials [17]. Many authors have also solved the Klein–Gordon and Dirac equations by taking suitable mass distributions in the one- and/or three-dimensional cases for different potentials, such as Coulomb potential [18], Lorentz scalar interactions [19], hyperbolic-type potentials [20], Morse potential [21] and Pöschl–Teller potential [22].

In the present paper, we intend to solve the Klein–Gordon equation within the framework of an approximation to the centrifugal potential term. We study the effect of the mass varying with position on the energy spectra, and the eigenfunctions of the vector and scalar Hulthén potentials [23]. The potential is widely used in nuclear and particle physics, atomic physics and condensed matter and...
For this task, we use a general parametric form of the Nikiforov–Uvarov (NU) method, which is based on transforming a second-order differential equation into a hypergeometric-type equation [27].

This work is organized as follows. In section 2, we briefly give the parametric generalization of the NU method. In section 3, we give the energy eigenvalue equation and corresponding eigenfunctions for the vector and scalar Hulthén potentials for any \( \ell \) value in the position-dependent mass background. We also obtain the results for the case of the constant mass, and we summarize our conclusions in section 4.

**2. The NU method**

The Schrödinger equation can be transformed into a second-order differential equation with the following form:

\[
\sigma(s) \frac{d^2\Psi(s)}{ds^2} + \sigma(s) \tau(s) \frac{d\Psi(s)}{ds} + \sigma(s) \Psi(s) = 0,
\]

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at most, second degree, and \( \tau(s) \) is a first degree polynomial. In order to find a particular solution, we take the following form:

\[
\Psi(s) = \psi(s) \phi(s).
\]

We get from equation (1)

\[
\sigma(s) \frac{d^2\phi(s)}{ds^2} + \tau(s) \frac{d\phi(s)}{ds} + \lambda \phi(s) = 0,
\]

where \( \phi(s) \) can be written in terms of the Rodriguez formula

\[
\phi_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[ \sigma^n(s) \rho(s) \right],
\]

and the weight function \( \rho(s) \) satisfies

\[
\frac{d \rho(s)}{ds} + \sigma(s) \frac{d \rho(s)}{ds} = \tau(s).
\]

The other factor of the solution is defined as

\[
\frac{1}{\psi(s)} \frac{d}{ds} \left[ \frac{\psi(s)}{\sigma(s)} \right] = \pi(s).
\]

In the method, the polynomial \( \pi(s) \) and the parameter \( k \) are defined as [27]

\[
\pi(s) = \frac{1}{2} \left[ \sigma' s - \tau' \right] \left[ \left[ \sigma' s - \tau' \right]^2 - \tilde{\sigma}(s) + k \sigma(s) \right]^{1/2}
\]

and

\[
\lambda = k + \pi'(s),
\]

where \( \lambda \) is a constant and given in equation (3). Since the square root in the polynomial \( \pi(s) \) in equation (7) must be a square, this defines the constant \( k \). Substituting \( k \) into equation (7), we define

\[
\tau(s) = \tilde{\tau}(s) + 2\pi(s).
\]

Since \( \rho(s) > 0 \) and \( \sigma(s) > 0 \), the derivative of \( \tau(s) \) should be negative [27], which leads to the choice of the solution. If \( \lambda \) in equation (8) is

\[
\lambda = \lambda_n = -n\tau' - \frac{n(n-1)\sigma'}{2}, \quad n = 0, 1, 2, \ldots,
\]

then the hypergeometric-type equation has a particular solution with degree \( n \).

In order to explain the general parametric form of the NU method, let us take the general form of a Schrödinger-like equation including any potential

\[
[s(1 - \alpha s^2)]^2 \frac{d^2\Psi(s)}{ds^2} + [s(1 - \alpha s^2)(\alpha_1 - \alpha_2 s)]^2 \frac{d\Psi(s)}{ds} + [\xi_1 s^2 + \xi_2 s - \xi_3] \Psi(s) = 0.
\]

When equation (11) is compared with equation (1), we get

\[
\tilde{\tau}(s) = \alpha_1 - \alpha_2 s; \quad \sigma(s) = s(1 - \alpha s^2); \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3.
\]

Substituting these into equation (7), we get

\[
\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{\alpha_6 - k \alpha_7 s^2 + (\alpha_7 + k) s + \alpha_8},
\]

where the parameters in the above equation are as follows:

\[
\begin{align*}
\alpha_4 &= \frac{1}{2} (1 - \alpha_1), \quad \alpha_5 = \frac{1}{2} (\alpha_2 - 2\alpha_3), \\
\alpha_6 &= \alpha_7^2 + \xi_1, \quad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \\
\alpha_8 &= \alpha_4^2 + \xi_3.
\end{align*}
\]

In the NU method, the function under square root must be the square of a polynomial, so

\[
k_{1,2} = -(\alpha_1 + 2\alpha_3 s) \pm 2\sqrt{\alpha_6 \alpha_8},
\]

where

\[
\alpha_0 = \alpha_3 \alpha_7 + \alpha_5^2 \alpha_8 + \alpha_6.
\]

The function \( \pi(s) \) becomes

\[
\pi(s) = \alpha_4 + \alpha_5 s - \left[(\sqrt{\alpha_0} + \alpha_3 \sqrt{\alpha_8}) s - \sqrt{\alpha_8} \right]
\]

for the \( k \) value \( k = -(\alpha_1 + 2\alpha_3 s) - 2\sqrt{\alpha_6 \alpha_8} \), where we have to say that different \( k \) values lead to different \( \pi(s) \) values. We also have from equation (9)

\[
\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_3) s - 2 \left[ \sqrt{\alpha_0} + \alpha_3 \sqrt{\alpha_8} s - \sqrt{\alpha_8} \right].
\]

Thus, we impose the following for satisfying the condition that the derivative of \( \tau(s) \) must be negative:

\[
\tau'(s) = -(\alpha_2 - 2\alpha_3) - 2 \sqrt{\alpha_0} \alpha_3 \sqrt{\alpha_8} < 0,
\]

From equations (8), (9), (18) and (19) and by equating equation (8) to the condition that \( \lambda \) should satisfy given by equation (10), we obtain

\[
\alpha_3 n - (2n + 1) \alpha_3 + (2n + 1)(\sqrt{\alpha_0} + \alpha_3 \sqrt{\alpha_8}) + n(n - 1) \alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 + 2\sqrt{\alpha_6 \alpha_8} = 0.
\]
which is the energy eigenvalue equation of a given potential.

Now, let us look at the eigenfunctions of the problem with any potential. We obtain the second part of the solution from equation (4)

\[ \phi_n(s) = P^{αβ}_{ns}(s) \{ \alpha(s) \} - α_{ns}^{-1}(1 - 2α_{ns} s), \]

by using the explicit form of the weight function obtained from equation (5)

\[ ρ(s) = s^{α - 1}(1 - α_{ns} s)^{-α_{ns}^{-1}}, \]

where

\[ α_{10} = α_4 + 2α_4 + 2√α_4; \]
\[ α_{11} = α_5 - 2α_5 + 2(α_α + √α_5), \]

and \( P^{αβ}_{ns}(1 - 2α_{ns}) \) are Jacobi polynomials. From equation (6), one gets

\[ ψ(s) = s^{α - 1}(1 - α_{ns} s)^{-α_{ns}^{-1}}, \]

then the general solution \( Ψ(s) = ψ(s) φ(s) \) becomes

\[ ψ(s) = s^{α/2}(1 - α_{ns} s)^{-α_{ns}^{-1}} × P^{αβ}_{ns}(s) \{ α(s) \} - α_{ns}^{-1}(1 - 2α_{ns} s), \]

where

\[ α_{12} = α_4 + √α_5; \]
\[ α_{13} = α_5 - (√α_4 + √α_5). \]

### 3. Bound-state solutions

The Klein–Gordon equation for a particle with mass \( m \) with vector potential \( V_n(r) \) and scalar potential \( V_s(r) \) is (\( h = c = 1 \))

\[ -\nabla^2 - [E^2 - m^2(r)] + 2[m(r)V_s(r) + EV_n(r)] + [V_s^2 - V_s(r)] \Psi(r, θ, φ) = 0. \]

Using \( Ψ(r, θ, φ) = r^{-1} φ(r) Y_{mθ}(θ, φ) \), we have the radial part of the equation

\[ \frac{d^2 φ(r)}{dr^2} + \left[ \frac{E^2 - m^2}{r^2} - 2m(r)V_s(r) + EV_n(r) \right] - \frac{ξ(ξ + 1)}{r^2} \phi(r) = 0, \]

where \( Y_{mθ}(θ, φ) \) is spherical harmonics and \( ξ \) is the angular momentum quantum number.

In order to solve equation (28), we prefer to use the following mass function:

\[ m(r) = m_0 + \frac{m_1 e^{-r/r_0}}{1 - e^{-r/r_0}}, \]

where \( m_0 \) and \( m_1 \) are two arbitrary, positive constants. We have to use an approximation, given by \( 1/r^2 ≃ e^{r/r_0}/(e^{r/r_0} - 1)^2 r_0^2 \), to the centrifugal term, since the radial equation has no analytical solutions for \( ξ ≠ 0 \) [28, 29]. By taking the scalar and vector potentials as the Hulthén potential

\[ V_s(r) = -\frac{S_0}{e^{r/r_0} - 1}; \quad V_n(r) = -\frac{V_0}{e^{r/r_0} - 1}, \]

and using equation (31), we get

\[ \frac{d^2 φ(r)}{dr^2} + \left[ \frac{E^2 - m_0^2}{e^{r/r_0} - 1} + \frac{2m_1 S_0 - m_1 + V_0}{(e^{r/r_0} - 1)^2} - \frac{ξ(ξ + 1)e^{r/r_0}}{r_0^2(e^{r/r_0} - 1)^2} \right] φ(r) = 0. \]

By using a new variable \( e^{-r/r_0} = s \), equation (31) becomes

\[ \frac{d^2 φ(s)}{ds^2} + \frac{1 - s}{s(1 - s)} \frac{dφ(s)}{ds} + \left[ \frac{r_0^2(E^2 - m_0^2)}{s^2} + \frac{2m_0 S_0 - m_1 + V_0}{s(1 - s)} - \frac{ξ(ξ + 1)}{s^2} \right] φ(s) = 0. \]

By using the new parameters

\[ \alpha(m_1) = η(m_1)r_0, \]
\[ η^2(m_1) = (m_1 - m_0)^2 - E^2, \]
\[ β_1^2(m_1) = r_0^2[2E^2 - 2S_0(m_1 - m_0)], \]
\[ β_2^2(m_1) = r_0^2[2E^2 - 2m_0(m_1 - S_0)], \]
\[ ν^2 = -α^2 + α^2(m_1) + β_1^2(m_1) - β_2^2(m_1) + ν^2, \]

where \( ν(m_1)(m_1 → 0) = ν, \quad η(m_1)(m_1 → 0) = η \) and \( α = νr_0 \), and comparing equation (34) with equation (11), we get the following parameter set given in section 2:

\[ \alpha_1 = 1, \quad ξ_1 = α_2(m_1) + β_2^2(m_1) + ν^2(m_1), \]
\[ α_2 = 1, \quad ξ_2 = 2α_2 + β_2^2(m_1) - ξ(ξ + 1), \]
\[ α_3 = 1, \quad ξ_3 = α_2, \]
\[ α_4 = 0, \quad ξ_4 = -1, \]
\[ α_5 = ξ_1 + 1, \quad ξ_5 = -ξ_2, \]
\[ α_6 = ξ_3, \quad α_7 = ξ_1 - ξ_2 + ξ_3 + \frac{1}{2}, \]
\[ α_8 = ξ_1 + ξ_2, \quad α_9 = ξ_1 - ξ_2 - ξ_3 + ξ_4, \]
\[ α_{10} = 1 + 2√ξ_5, \quad α_{11} = 2 + 2\left(\sqrt{ξ_1 - ξ_2} + ξ_3 + \frac{1}{2} + ξ_4\right), \]
\[ α_{12} = √ξ_5, \quad α_{13} = -\frac{1}{2} - \left(\sqrt{ξ_1 - ξ_2} + ξ_3 + \frac{1}{2} + ξ_4\right), \]

where \( ν^2 = r_0^2(S_0^2 - V_0^2) \) and \( η^2 = m_0^2 - E^2 \) in the above equations.

We can easily get the energy eigenvalue equation of the Hulthén potential by using equation (20)

\[ α = \frac{β_2^2(m_1) - ξ(ξ + 1) - m_0^2 - (2n - 1)δ'}{2(n + δ')}, \]

where \( δ' = \frac{1}{2} + \frac{1}{2}\sqrt{(ξ+1)^2 + 4ν^2(m_1)} \). In tables 1 and 2, we list some energy eigenvalues for the case of constant mass and spatially dependent mass, respectively. To compare our results, we have used the values of the parameters given in \[30\]. The variable \( E_a \) denotes the energy eigenvalue of the particle and \( E_p \) denotes the energy eigenvalue of the antiparticle in tables 1 and 2.

According to the result obtained in equation (35), we can easily give the eigenvalue equation in the case of constant
Table 1. The energy eigenvalues of the vector and scalar Hulthén potentials for \( m_0 = 1 \) and \( m_1 = 0 \).

| \( n \) | \( \ell \) | \( E_n^a \) | \( E_n^b \) | \( E_m^a \) | \( E_m^b \) | \( E_m^c \) |
|---|---|---|---|---|---|---|
| \( V_0 = S_0 = 1 \) | 1 | 0 | -0.6000000 | 1.0000000 | -0.6000000 | 1.0000000 |
| 1 | - | - | - | - | - | - |
| \( V_0 = S_0 = 2 \) | 1 | 0 | -0.7071068 | 0.7071068 | -0.7071068 | 0.7071068 |
| 1 | - | - | - | - | - | - |
| \( V_0 = S_0 = 3 \) | 1 | 0 | -0.7637079 | 0.3021695 | -0.7637080 | 0.3021690 |
| 1 | - | - | - | - | - | - |
| \( V_0 = S_0 = 6 \) | 1 | 0 | -0.8449409 | -0.3550510 | -0.8449409 | -0.3550510 |
| 1 | - | - | - | - | - | - |

Our results.

Results obtained in [30].

Results obtained in [31, 32].

By following the same procedure, the normalization constant \( A_n' \) in the eigenfunctions for the case of constant mass is obtained as \( A_n' = A_n(\beta \rightarrow 1 + 2\delta'') \) in equation (43).
4. Conclusion

We have approximately solved the Klein–Gordon equation for the Hulthén potential for any angular momentum quantum number in the position-dependent mass background. We have found the eigenvalue equation and corresponding wavefunctions in terms of Jacobi polynomials by using the NU method within the framework of an approximation to the centrifugal potential term. We have also obtained the energy eigenvalue equation and corresponding eigenfunctions for the case of constant mass. The results for the case of constant mass are the same as those obtained in [28].

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