NEW FINITE VOLUME METHOD FOR ROTATING CHANNEL FLOWS INVOLVING BOUNDARY LAYERS

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ABSTRACT. We investigate in this article the boundary layers appearing for a fluid under moderate rotation when the viscosity is small. The fluid is modeled by rotating type Stokes equations known also as the Barotropric mode equations in the primitive equations theory. First we derive the correctors that describe the sharp variations at large Reynolds number (i.e., small viscosity). Second, thanks to a new finite volume method (NFVM) we give numerical solutions of the rotating Stokes system at small viscosity. The NFVM can be applied to a large class of singular perturbation problems.

1. INTRODUCTION

We are interested in this article in the study of boundary layers of a time dependant rotating fluid when the viscosity is small and the boundary is characteristic; this occurs for example when the boundary is solid and at rest. The boundary conditions are considered of homogeneous Dirichlet type. More precisely, we consider the flow in 3D verifying the following system:

$$\begin{align*}
\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + \omega \times u^\varepsilon + \nabla p^\varepsilon &= f, \quad \text{in } \Omega_\infty \times (0, T), \\
\text{div } u^\varepsilon &= 0, \quad \text{in } \Omega_\infty \times (0, T), \\
u^\varepsilon &= 0, \quad \text{on } \partial \Omega_\infty \\
u^\varepsilon|_{t=0} &= u_0;
\end{align*}$$

(1.1)

see [3] and [8] for more details about the theory of rotating fluids. Here \( \omega = \alpha e_3 \) where \( e_3 \) is the unit vector in the canonical basis of \( \mathbb{R}^3 \), \( \Omega_\infty = \mathbb{R}^2 \times (0, h) \) is the relevant domain, \( \Gamma = \partial \Omega_\infty = \mathbb{R}^2 \times \{0, h\} \) its boundary. The functions \( u_0 \) and \( f \) are given and supposed to be as regular as necessary. Without loss of generality, the constant \( h \) will be taken from now equal to 1.

The solution \((u^\varepsilon, p^\varepsilon)\) of the system (1.1) is such that \( u^\varepsilon(t, x, y, z) = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \in \mathbb{R}^3 \) and \( p^\varepsilon \in \mathbb{R} \), the coefficient \( \varepsilon \) is a positive constant representing the inverse of the Reynolds number. Throughout this paper the coefficient \( \varepsilon > 0 \) is intended to be small \( \varepsilon \ll 1 \). Because of the periodicity conditions (1.1) we will consider a portion of the channel \( \Omega_\infty \) that we denote by \( \Omega = (0, 2\pi) \times (0, 2\pi) \times (0, 1) \) and its boundary \( \Omega = (0, 2\pi) \times (0, 2\pi) \times \{0, 1\} \) on which all our calculations will be done. The formal limit solution \( u^0 \) of the system (1.1) is simply obtained...
by setting \( \varepsilon = 0 \) in (1.1). Hence, we have:

\[
\begin{aligned}
&\frac{\partial u^0}{\partial t} + \omega \times u^0 + \nabla p^0 = f, \quad \text{in } \Omega \times (0, T), \\
&\text{div } u^0 = 0, \quad \text{in } \Omega \times (0, T), \\
&u^0_3 = 0, \quad \text{on } \partial \Omega, \\
&u^0 \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions}, \\
&u^0_{|t=0} = u_0.
\end{aligned}
\]

(1.2)

The absence in the limit system of the Laplacian term \( -\varepsilon \Delta u^\varepsilon \) which is a regularizing term, generates a loss of regularity of the limit solution \( u^0 \). Thus some discrepancies between the viscous and inviscid solutions appear near the boundary of the domain, that is here \( z = 0, 1 \). This thin region is called boundary layer and the convergence of \( u^\varepsilon \) to \( u^0 \) is not expected there especial when we look for the convergence in \( H^s(\Omega) \) for \( s > 1 \) and \( X = \). Hence, we introduce a correcting term called corrector for which the equation must be of course simpler than the one in the original problem namely (1.1); see [4], [10], [11], [12] and [13] for more details on this notion.

2. THE CORRECTOR EQUATIONS

To study the asymptotic behavior of \( u^\varepsilon \), when \( \varepsilon \to 0 \), we propose the following asymptotic expansion of \( u^\varepsilon \):

\[ u^\varepsilon \simeq u^0 + \varphi^\varepsilon, \]

where \( \varphi^\varepsilon \) is the corrector function that will be introduced to correct the difference \( u^\varepsilon - u^0 \) at \( z = 0, 1 \). The equations verified by \( \varphi^\varepsilon \) are as follows:

\[
\begin{aligned}
&\frac{\partial \varphi^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \varphi^\varepsilon}{\partial z^2} + \omega \times \varphi^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\
&\text{div } \varphi^\varepsilon = 0, \quad \text{in } \Omega \times (0, T), \\
&\varphi^\varepsilon_{|z=0,1} = -u^0_{|z=0,1}, \\
&\varphi^\varepsilon_{|t=0} = 0.
\end{aligned}
\]

(2.1)

We now introduce an approximate function \( \bar{\varphi}^\varepsilon \) of \( \varphi^\varepsilon \) defined as the sum of \( \bar{\varphi}^{0,\varepsilon} \) and \( \bar{\varphi}^{1,\varepsilon} \) the correctors that we propose to solve the boundary layers at the boundaries \( z = 0 \) and \( z = 1 \), respectively:

\[
\bar{\varphi}^\varepsilon(t, x, y, z) = \varphi^{0,\varepsilon}(t, x, y, \frac{z}{\sqrt{\varepsilon}}) + \varphi^{1,\varepsilon}(t, x, y, \frac{1-z}{\sqrt{\varepsilon}}),
\]
where \( \bar{z} = \frac{z}{\sqrt{\varepsilon}} \) and \( \bar{z} = \frac{1 - \bar{z}}{\sqrt{\varepsilon}} \). Taking into consideration the linearity of the equations (2.1)_{1,2} and the boundary conditions (2.1)_{3}, the system verified by \( \varphi^{0,\varepsilon} \) is given by:

\[
\begin{align*}
\frac{\partial \varphi^{0,\varepsilon}}{\partial t} - \frac{\partial^2 \varphi^{0,\varepsilon}}{\partial z^2} + \omega \times \varphi^{0,\varepsilon} &= 0, \quad \text{in } \tilde{\Omega} \times (0, T), \\
\varphi^{0,\varepsilon}(\bar{z} = 0) &= -u^0(\bar{z} = 0), \\
\varphi^{0,\varepsilon} &\to 0 \quad \text{as } \bar{z} \to \infty, \\
\varphi^{0,\varepsilon}_{|t=0} &= 0.
\end{align*}
\]

(2.2)

Similarly \( \varphi^{1,\varepsilon} \) satisfies the following system:

\[
\begin{align*}
\frac{\partial \varphi^{1,\varepsilon}}{\partial t} - \frac{\partial^2 \varphi^{1,\varepsilon}}{\partial z^2} + \omega \times \varphi^{1,\varepsilon} &= 0, \quad \text{in } \tilde{\Omega} \times (0, T), \\
\varphi^{1,\varepsilon}(\bar{z} = 0) &= -u^0(\bar{z} = 0), \\
\varphi^{1,\varepsilon} &\to 0 \quad \text{as } \bar{z} \to \infty, \\
\varphi^{1,\varepsilon}_{|t=0} &= 0.
\end{align*}
\]

(2.3)

Here we denoted by \( \tilde{\Omega} \) the stretched domain, i.e., \( \tilde{\Omega} = (0, 2\pi) \times (0, 2\pi) \times (0, +\infty) \) of the system (2.2). In the following we will derive the expressions of the solutions of the systems (2.2) and (2.3). For that purpose we need the following proposition.

**Proposition 2.1.** Let \( u = u(t, x, y, z) \) be the solution of the following problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} + \omega \times u &= 0, \quad \text{in } \tilde{\Omega} \times (0, T), \\
u &= g, \quad \text{at } z = 0, \\
u &\to 0, \quad \text{as } z \to +\infty, \\
u &= 0, \quad \text{at } t = 0.
\end{align*}
\]

(2.4)

where \( g = (g_1, g_2, 0) \) is a continuous function in \( \tilde{\Omega} \times (0, T) \) and \( \omega = \alpha e_3 \). Then, the explicit expression of \( u \) is given by:

\[
u(t, x, y, z) = -\int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(g - i(e_3 \times g))(\tau, x, y, 0)e^{i\alpha(\tau - t)} + (g + i(e_3 \times g))(\tau, x, y, 0)e^{i\alpha(\tau - t)}]d\tau,
\]

where \( i \) is the complex number s.t. \( i^2 = -1 \), and \( K \) is the fundamental solution of the heat equation:

\[
K(t, z) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}.
\]

**Proof.** Let \( u = (u_1, u_2, u_3) \) the solution of (2.4). We have \( g_3 = 0 \), hence \( u = (u_1, u_2, 0) \) i.e. \( u_3 = 0 \). Taking the cross product of (2.4) with \( e_3 \), we find:

\[
\partial_t (e_3 \times u) - \partial_z^2 (e_3 \times u) - \alpha u = 0.
\]

We then set \( C^\pm = u \mp i(e_3 \times u) \), we obtain:

\[
\partial_t C^\pm - \partial_z^2 C^\pm \pm i\alpha C^\pm = 0.
\]
Denoting by $H^\pm = C^\pm e^{\pm i\alpha t}$, one arrives to the following system:

$$
\begin{cases}
\frac{\partial H^\pm}{\partial t} - \frac{\partial^2 H^\pm}{\partial z^2} = 0, \text{ in } \bar{\Omega} \times (0, T), \\
H^\pm(z = 0) = (g(z = 0) \mp i(e_3 \times g(z = 0)))e^{\pm i\alpha t}, \\
H^\pm \to 0, \text{ as } z \to +\infty, \\
H^\pm|_{t=0} = 0.
\end{cases}
$$

(2.5)

Hence $H^\pm$ satisfies a heat equation with non-homogeneous boundary conditions, then it has the following expression ([2]):

$$
H^\pm = -2 \int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(g \mp i(e_3 \times g)))(\tau, x, y, 0)]e^{\pm i\alpha \tau}d\tau.
$$

Then, we infer that:

$$
C^\pm = -2 \int_0^t \frac{\partial K}{\partial z}(t - \tau, z)[(g \mp i(e_3 \times g)(\tau, x, y, 0)]e^{\pm i\alpha(\tau - t)}d\tau.
$$

Coming back to $u$ we have:

$$
u = \frac{1}{2}(C^+ + C^-),$$

hence we deduce the explicit expression of the solution of (2.4):

$$
u = -\int_0^t \frac{\partial K}{\partial z}(t - \tau, z) \times \{(g - i(e_3 \times g)(\tau, x, y, 0))e^{i\alpha(\tau - t)} + (g + i(e_3 \times g)(\tau, x, y, 0))e^{i\alpha(t - \tau)}\}d\tau.
$$

Now, according to Proposition 2.1, the solution of (2.2) $\varphi^{0,\varepsilon}_j = (\varphi^{0,\varepsilon}_1, \varphi^{0,\varepsilon}_2, \varphi^{0,\varepsilon}_3)$ has the following expression:

$$
\begin{align*}
\varphi^{0,\varepsilon}_j &= -\int_0^t \frac{1}{\sqrt{4\pi(t - \tau)}} \frac{z}{2\sqrt{\varepsilon(t - \tau)}} e^{-\frac{z^2}{4\varepsilon(t - \tau)}} \times \{2u^0_j(\tau, x, y, 0)\cos(\alpha(\tau - t)) \\
& \quad + 2(e_3 \times u^0)(\tau, x, y, 0)\sin(\alpha(\tau - t))\}d\tau, \quad j = 1, 2,
\end{align*}
$$

for the two tangential components of $\varphi^{0,\varepsilon}$, and the normal component of $\varphi^{0,\varepsilon}$ is simply deduced using the incompressibility condition:

$$
\varphi^{0,\varepsilon}_3 = \left\{-2\int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t - \tau)}} e^{-\frac{z^2}{4\varepsilon(t - \tau)}} \times \{-2\partial_\tau u^0_3(\tau, x, y, 0)\cos(\alpha(\tau - t)) \\
- 2(\partial_x u^0_2 - \partial_y u^0_1)(\tau, x, y, 0)\sin(\alpha(\tau - t))\}d\tau \\
+ \int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t - \tau)}} e^{-\frac{z^2}{4\varepsilon(t - \tau)}} \times \{-2\partial_\tau u^0_3(\tau, x, y, 0)\cos(\alpha(\tau - t)) \\
- 2(\partial_x u^0_2 - \partial_y u^0_1)(\tau, x, y, 0)\sin(\alpha(\tau - t))\}d\tau.
\right.
$$

(2.6)
Then we write the system satisfied by $\varphi^{0, \varepsilon}$ which reads as follows:

$$
\left\{ \begin{array}{l}
\frac{\partial \varphi^{0, \varepsilon}}{\partial t} - \varepsilon \frac{\partial^2 \varphi^{0, \varepsilon}}{\partial z^2} + \omega \times \varphi^{0, \varepsilon} = (0, 0, \frac{\partial \varphi^{0, \varepsilon}}{\partial z_3} - \varepsilon \frac{\partial^2 \varphi^{0, \varepsilon}}{\partial z^2}), \text{ in } \widetilde{\Omega} \times (0, T), \\
\text{div } \varphi^{0, \varepsilon} = 0, \text{ in } \widetilde{\Omega} \times (0, T), \\
\varphi^{0, \varepsilon}(z = 0) = (-u^0_1(z = 0), -u^0_2(z = 0), \varphi^{0, \varepsilon}_3(z = 0)), \\
\varphi^{0, \varepsilon}(z = 1) = (\varphi^{0, \varepsilon}_1(z = 1), \varphi^{0, \varepsilon}_2(z = 1), 0), \\
\varphi^{0, \varepsilon}(t = 0) = 0.
\end{array} \right.
$$

(2.7)

Now, we need to estimate the right-hand side (denoted hereafter RHS) of (2.7)$_1$. First we set the change of variables $s = \frac{1}{\sqrt{1-t}}$, and we infer that:

$$
\varphi^{0, \varepsilon}_3 = - \int_{\frac{1}{\sqrt{1-t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{-\frac{x^2}{4s^2}} \times \{ -2\partial_z u^0_3(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \\
+ 2(\partial_x u^0_2 - \partial_y u^0_1)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \} ds \\
+ \int_{\frac{1}{\sqrt{1-t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{-\frac{x^2}{4s^2}} \times \{ -2\partial_z u^0_3(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \\
+ 2(\partial_x u^0_2 - \partial_y u^0_1)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \} ds.
$$

(2.8)

By differentiating (2.8) with respect to the time variable $t$, we obtain:

$$
\frac{\partial \varphi^{0, \varepsilon}_3}{\partial t} = - \frac{\sqrt{\varepsilon}}{\sqrt{\pi} t^2} e^{-\frac{x^2}{4t^2}} \times \{ -\partial_x u^0_3(0, x, y, 0) \cos(\alpha t) \\
+ (\partial_x u^0_2 - \partial_y u^0_1)(0, x, y, 0) \sin(\alpha t) \} \\
- \int_{\frac{1}{\sqrt{1-t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{-\frac{x^2}{4s^2}} \times \{ -2\partial_t^2 u^0_3(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \\
+ 2(\partial_{tx} u^0_2 - \partial_{ty} u^0_1)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) ds \} \\
- \frac{\sqrt{\varepsilon}}{\sqrt{\pi} t^2} e^{-\frac{x^2}{4t^2}} \times \{ -\partial_x u^0_3(0, x, y, 0) \cos(\alpha t) \\
+ (\partial_x u^0_2 - \partial_y u^0_1)(0, x, y, 0) \sin(\alpha t) \} - \\
- \int_{\frac{1}{\sqrt{1-t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{-\frac{x^2}{4s^2}} \times \{ -2\partial_t^2 u^0_3(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \\
+ 2(\partial_{tx} u^0_2 - \partial_{ty} u^0_1)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) ds \}.
$$

(2.9)
We denote by \( I_1 + \cdots + I_4 \) the sum of the terms appearing in the RHS of (2.9). First, we estimate the \( L^2 \)-norm of \( I_1 \), we get:

\[
\|I_1\|_{L^2(\Omega)}^2 \leq k \varepsilon \int_0^1 e^{\frac{z^2}{4\varepsilon t}} dz \\
\leq k \varepsilon \int_0^1 e^{\frac{z^2}{cz}} dz, \quad c > 0 \\
\leq k \varepsilon^{3/2}.
\]

Second, we estimate the \( L^2 \)-norm of \( I_2 \), we obtain:

\[
\|I_2\|_{L^2(\Omega)}^2 \leq k \int_0^1 \left( \int_0^\infty \frac{\varepsilon^{1/2}}{\sqrt{4\pi s^2}} e^{-\frac{s^2 z^2}{2\varepsilon^2}} ds \right)^2 dz \\
\leq (\text{using Cauchy Schwartz inequality}) \\
\leq k \varepsilon \int_0^1 \int_0^\infty \frac{1}{s^2} \int_0^\infty \frac{1}{s^2} e^{-\frac{s^2 z^2}{2\varepsilon^2}} ds dz \\
\leq k \varepsilon \int_0^1 \int_0^\infty \frac{1}{s^2} \int_0^1 e^{-\frac{c\sqrt{s}z}{\varepsilon}} ds dz \quad c > 0, \\
\leq k \varepsilon^{3/2}.
\]

Finally, combining (2.10), (2.11) and the fact that \( I_3 \) and \( I_4 \) are e.s.t. (where e.s.t. stands for quantities which are exponentially small terms in all \( H^m((0, T) \times \Omega), \ m \geq 0 \)), we conclude that:

\[
\left\| \frac{\partial \bar{\psi}_{3}^{0,\varepsilon}}{\partial t} \right\|_{L^2(\Omega)} \leq k \varepsilon^{3/4}.
\]

We will estimate in the following the \( z \)- derivative of \( \bar{\psi}_{3}^{0,\varepsilon} \) appearing second term in the RHS of (2.7). Hence by differentiating \( \bar{\psi}_{3}^{0,\varepsilon} \) with respect to the normal variable \( z \), we obtain:

\[
\frac{\partial^2 \bar{\psi}_{3}^{0,\varepsilon}}{\partial z^2} = -\int_0^t \sqrt{\frac{\varepsilon}{\pi(t-\tau)}} \int_0^{z^2} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{\tau^2}{4(t-\tau)}} \times \{2 \partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau - t)) + \\
+ 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau - t))\} d\tau - \\
-\int_0^t \frac{z^2 \varepsilon^{1/2}}{\sqrt{4\pi(t-\tau)}} 4(t-\tau) e^{\frac{z^2}{4\varepsilon(t-\tau)}} \times \{2 \partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau - t)) + \\
+ 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau - t))\} d\tau.
\]
We denote by $J_1 + J_2$ the sum of the terms in the RHS of (2.13). Multiplying $J_1$ by $z$ and setting the change of variables $s = \frac{z}{\sqrt{2\varepsilon(t - \tau)}}$, we find:

\[
zJ_1 = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varepsilon e^{-s^2} \left\{ -2\partial_z u_0^0(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0) \cos \left( \frac{\alpha z^2}{2\varepsilon s^2} \right) \\
+ 2(\partial_x u_2^0 - \partial_y u_1^0)(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0) \sin \left( \frac{\alpha z^2}{2\varepsilon s^2} \right) \right\} ds.
\]

Then, we have

\[
|zJ_1| \leq k\varepsilon \int_0^\infty e^{-s^2} ds
\leq k\varepsilon \int_0^\infty e^{-cs} ds, \quad c > 0
\leq k\varepsilon e^{-\frac{cs^2}{2}} ds,
\]

and we get the $L^2$- norm of the term $zJ_1$:

\[
\|zJ_1\|_{L^2(\Omega)}^2 \leq k\varepsilon^2 \int_0^1 e^{-\frac{cs^2}{2}} ds
\leq k\varepsilon^{5/2}.
\]

Hence, we obtain (2.14)

\[
\|zJ_1\|_{L^2(\Omega)} \leq k\varepsilon^{5/4}.
\]

Identically we multiply $J_2$ by $z$ and apply the same change of variables $s = \frac{z}{\sqrt{2\varepsilon(t - \tau)}}$, we get:

\[
zJ_2 = \frac{1}{\sqrt{\pi}} \int_0^\infty 2\varepsilon s^2 e^{-s^2} \times \left\{ -\partial_z u_0^0(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0) \cos \left( \frac{\alpha z^2}{2\varepsilon s^2} \right) \\
+ (\partial_x u_2^0 - \partial_y u_1^0)(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0) \sin \left( \frac{\alpha z^2}{2\varepsilon s^2} \right) \right\} ds.
\]

Then, we have

\[
|zJ_2| \leq \int_0^\infty k\varepsilon s^2 e^{-s^2} ds
\leq k\varepsilon (\frac{z}{2\sqrt{\varepsilon t}} e^{-\frac{s^2}{2\varepsilon t}} + \int_0^s e^{-s^2/2} ds)
\leq k\sqrt{\varepsilon} e^{-\frac{cs^2}{2\varepsilon t}} + 2\varepsilon e^{-\frac{cs^2}{2\varepsilon t}},
\]

and we obtain the $L^2$- norm of $zJ_2$,

\[
\|zJ_2\|_{L^2(\Omega)}^2 \leq k \int_0^1 \varepsilon z^2 e^{-\frac{cs^2}{2\varepsilon t}} dz + k\varepsilon^2 \int_0^1 e^{-\frac{cs^2}{2\varepsilon t}} dz
\leq k\varepsilon \int_0^1 z^2 e^{-\frac{cs^2}{2\varepsilon t}} dz + k\varepsilon^2 \int_0^1 e^{-\frac{cs^2}{2\varepsilon t}} dz
\leq k\varepsilon^{5/2}.
\]
Hence, we infer that
\begin{equation}
\|z J_2\|_{L^2(\Omega)} \leq k \varepsilon^{5/4}.
\end{equation}
Finally, combining (2.14) and (2.15), we deduce the following estimate:
\begin{equation}
\|\varepsilon \frac{\partial^2 \overline{\varphi}_{3, \varepsilon}}{\partial \varepsilon^2}\|_{L^2(\Omega)} \leq k \varepsilon^{5/4}.
\end{equation}

3. CONVERGENCE RESULT

In this section we prove the main theoretical result of this article.

**Theorem 3.1.** The solution $u^\varepsilon$ of (1.1), with $u_0$ and $f$ supposed to be sufficiently smooth, satisfies the following estimates:
\begin{equation}
\|u^\varepsilon - u^0 - \overline{\varphi}_{0, \varepsilon} - \varepsilon^{1/2} \varphi^1, \varepsilon\|_{L^\infty(0,T; L^2(\Omega))} \leq k \varepsilon^{3/4},
\end{equation}
\begin{equation}
\|u^\varepsilon - u^0 - \overline{\varphi}_{0, \varepsilon} - \varepsilon^{1/2} \varphi^1, \varepsilon\|_{L^\infty(0,T; H^1(\Omega))} \leq k \varepsilon^{1/4},
\end{equation}
where $k$ is a positive constant depending on the data but not $\varepsilon$, and $u^0$ and $\varphi^\varepsilon$ are defined respectively by (1.2) and (2.1). Here we denoted by $L^2(\Omega) = (L^2(\Omega))^3$ and $H^1(\Omega) = (H^1(\Omega))^3$.

**Proof.** First we observe that the corrector $\varphi^\varepsilon$ does not satisfy the desired boundary conditions as given by (2.1), this is due to the choice of a corrector in a simpler form. To overcome this difficulty we introduce additional (small) correctors $\tilde{\varphi}^\varepsilon$ and $\tilde{\varphi}^\varepsilon$ as follows:
\begin{equation}
\begin{cases}
-\varepsilon \Delta \tilde{\varphi}^\varepsilon + \nabla \Pi^\varepsilon = 0, & \text{in } \Omega \times (0,T), \\
\text{div} \tilde{\varphi}^\varepsilon = 0, \\
\tilde{\varphi}^\varepsilon|_{z=0} = (0,0,-\overline{\varphi}_{3, \varepsilon}|_{z=0}), \\
\tilde{\varphi}^\varepsilon|_{z=1} = (\overline{\varphi}_{1, \varepsilon}|_{z=1}, \overline{\varphi}_{2, \varepsilon}|_{z=1}, 0),
\end{cases}
\end{equation}
and
\begin{equation}
\begin{cases}
-\varepsilon \Delta \tilde{\varphi}^\varepsilon + \nabla Q^\varepsilon = 0, & \text{in } \Omega \times (0,T), \\
\text{div} \tilde{\varphi}^\varepsilon = 0, \\
\tilde{\varphi}^\varepsilon|_{z=1} = (0,0,-\overline{\varphi}_{3, \varepsilon}|_{z=1}), \\
\tilde{\varphi}^\varepsilon|_{z=0} = (\overline{\varphi}_{1, \varepsilon}|_{z=0}, \overline{\varphi}_{2, \varepsilon}|_{z=0}, 0).
\end{cases}
\end{equation}
To estimate the $L^2$- norm of the additional correctors, we set $\tilde{\varphi}^\varepsilon = \sqrt{\varepsilon} \tilde{\varphi}^\varepsilon$, $\Pi^\varepsilon = \varepsilon^{3/2} \Pi^\varepsilon$, hence $\tilde{\varphi}^\varepsilon$ satisfies the following system:
\begin{equation}
\begin{cases}
-\Delta \tilde{\varphi}^\varepsilon + \nabla \Pi^\varepsilon = 0, & \text{in } \Omega \times (0,T), \\
\text{div} \tilde{\varphi}^\varepsilon = 0, \\
\tilde{\varphi}^\varepsilon|_{z=0} = (0,0,-\frac{\overline{\varphi}_{3, \varepsilon}}{\sqrt{\varepsilon}}|_{z=0}), \\
\tilde{\varphi}^\varepsilon|_{z=1} = (\frac{\overline{\varphi}_{1, \varepsilon}}{\sqrt{\varepsilon}}|_{z=1}, \frac{\overline{\varphi}_{2, \varepsilon}}{\sqrt{\varepsilon}}|_{z=1}, 0).
\end{cases}
\end{equation}
Then we deduce from the direct estimates of the Stokes problem (see [1]) that:
\begin{align*}
\|\tilde{\varphi}^\varepsilon\|_{L^2(\Omega)} & \leq k \|\frac{\overline{\varphi}_{3, \varepsilon}}{\sqrt{\varepsilon}}|_{z=0}\|_{H^{-1/2}(\Gamma)} + k \|\frac{\overline{\varphi}_{1, \varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\|_{H^{-1/2}(\Gamma)} + k \|\frac{\overline{\varphi}_{2, \varepsilon}}{\sqrt{\varepsilon}}|_{z=1}\|_{H^{-1/2}(\Gamma)} \\
& \leq k \|\frac{\overline{\varphi}_{3, \varepsilon}}{\sqrt{\varepsilon}}\|_{L^2(\Omega)} + e.s.t.
\end{align*}
Now we will estimate the $L^2$- norm of $\frac{\varphi^{0,\varepsilon}}{\sqrt{\varepsilon}}$, hence we have:

\[ \left| \frac{\varphi^{0,\varepsilon}}{\sqrt{\varepsilon}} \right|^2 \leq k \left( \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{4\sqrt{\pi s^2}} e^{\frac{-s^2}{4\varepsilon}} ds \right)^2 \]

\[ \leq \left( \text{Using the Cauchy-Schwartz inequality} \right) \]

\[ \leq k \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} ds \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} e^{\frac{-s^2}{4\varepsilon}} ds \]

\[ \leq k \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} e^{\frac{-s^2}{4\varepsilon}} ds. \]

Therefore, we have

\[ \| \frac{\varphi^{0,\varepsilon}}{\sqrt{\varepsilon}} \|^2_{L^2(\Omega)} \leq k \int_0^{1} \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} e^{\frac{-s^2}{4\varepsilon}} ds dz \]

\[ \leq k \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} \int_0^{1} e^{\frac{-s^2}{4\varepsilon}} dz ds \]

\[ \leq k \int_{1/\sqrt{\varepsilon}}^{\infty} \frac{1}{s^2} \int_0^{1} e^{cz} dz ds, \quad c > 0 \]

\[ \leq k \sqrt{\varepsilon}, \]

Hence, we infer that

\[ \| \tilde{\theta} \|_{L^2(\Omega)} \leq k \varepsilon^{1/4}. \]

Finally, we get

(3.6) \[ \| \bar{\theta} \|_{L^2(\Omega)} \leq k \varepsilon^{3/4}. \]

In the following we will estimate the $L^2(\Omega)$ norm of the gradient of $\tilde{\theta}$, hence we find:

\[ \| \nabla \tilde{\theta} \|_{L^2(\Omega)} \leq k \| \frac{\varphi^{0,\varepsilon}}{\sqrt{\varepsilon}} \|_{z=0} \| H^{1/2}(\Gamma) \| + k \| \frac{\varphi^{1,\varepsilon}}{\sqrt{\varepsilon}} \|_{z=1} \| H^{1/2}(\Gamma) \|
\]

\[ \leq k \| \frac{\varphi^{0,\varepsilon}}{\sqrt{\varepsilon}} \|_{H^1(\Omega)} + e.s.t \]

\[ \leq k \varepsilon^{-1/4}. \]

Thus we deduce that:

(3.7) \[ \| \nabla \tilde{\theta} \|_{L^2(\Omega)} \leq k \varepsilon^{-1/4}. \]

We notice that the estimate (3.6) also holds for the time derivative of $\tilde{\theta}$, i.e.,

(3.8) \[ \| \frac{\partial \tilde{\theta}}{\partial t} \|_{L^2(\Omega)} \leq k \varepsilon^{3/4}. \]

We now define $w^\varepsilon = u^\varepsilon - u^0 - \varphi^{0,\varepsilon} - \varphi^{1,\varepsilon} - \theta - \tilde{\theta}$, and according to (1.1), (1.2), (2.7), (3.3) and (3.4), $w^\varepsilon$ verifies:
\begin{equation}
\begin{aligned}
\left\{ \begin{aligned}
\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + \omega \times w^\varepsilon + \nabla (p^\varepsilon - p^0 - \Pi^\varepsilon - Q^\varepsilon) &= \varepsilon \frac{\partial^2 \varphi^{0,\varepsilon}}{\partial x^2} + \varepsilon \frac{\partial^2 \varphi^{1,\varepsilon}}{\partial x^2} \\
+ \varepsilon \frac{\partial^2 \varphi^{0,\varepsilon}}{\partial y^2} + \varepsilon \frac{\partial^2 \varphi^{1,\varepsilon}}{\partial y^2} + \varepsilon \Delta u^0 - \omega \times \vec{\theta} - \omega \times \vec{\theta}^\varepsilon - \frac{\partial \vec{\theta}}{\partial t} - \frac{\partial \vec{\theta}^\varepsilon}{\partial t} \\
+ (0, 0, \frac{\partial \varphi^{3,\varepsilon}}{\partial t}) + (0, 0, \frac{\partial \varphi^{3,\varepsilon}}{\partial t}) + (0, 0, \varepsilon \frac{\partial \varphi^{3,\varepsilon}}{\partial z^2}) + (0, 0, \varepsilon \frac{\partial \varphi^{3,\varepsilon}}{\partial z^2}), & \text{in } \Omega_\infty \times (0, T),
\end{aligned}
\right.
\end{aligned}
\end{equation}

\( \text{div } w^\varepsilon = 0, \text{ in } \Omega_\infty \times (0, T) \),

\( w^\varepsilon = 0, \text{ at } z = 0, 1 \),

\( w^\varepsilon \) is 2\( \pi \)-periodic in the \( x \) and \( y \) directions,

\( w^\varepsilon|_{t=0} = 0. \)

We multiply (3.9), by \( w^\varepsilon \), integrate over \( \Omega \), and apply the Cauchy-Shwarz inequality, we obtain:

\[
\frac{1}{2} \frac{d}{dt} \| w^\varepsilon \|^2 + \varepsilon \| \nabla w^\varepsilon \|^2 \leq \varepsilon \| \frac{\partial^2 \varphi^{0,\varepsilon}}{\partial x^2} \| \| w^\varepsilon \| + \varepsilon \| \frac{\partial^2 \varphi^{1,\varepsilon}}{\partial x^2} \| \| w^\varepsilon \| + \varepsilon \| \frac{\partial^2 \varphi^{0,\varepsilon}}{\partial y^2} \| \| w^\varepsilon \| + \varepsilon \| \frac{\partial^2 \varphi^{1,\varepsilon}}{\partial y^2} \| \| w^\varepsilon \| + \\
+ \varepsilon \| \Delta u^0 \| \| w^\varepsilon \| + \| \vec{\theta} \| \| w^\varepsilon \| + \| \vec{\theta}^\varepsilon \| \| w^\varepsilon \| + \\
+ \| \frac{\partial \vec{\theta}}{\partial t} \| \| w^\varepsilon \| + \| \frac{\partial \vec{\theta}^\varepsilon}{\partial t} \| \| w^\varepsilon \| + \| \varepsilon \frac{\partial \varphi^{3,\varepsilon}}{\partial z^2} \| \| \nabla w^\varepsilon \| + \\
+ \| \frac{\partial \varphi^{3,\varepsilon}}{\partial t} \| \| w^\varepsilon \| + \| \frac{\partial \varphi^{3,\varepsilon}}{\partial t} \| \| w^\varepsilon \| + \| \frac{\partial \varphi^{3,\varepsilon}}{\partial t} \| \| w^\varepsilon \|.
\]

Hence according to (2.12), (2.16), (3.6) and (3.8), we have:

\[
\frac{1}{2} \frac{d}{dt} \| w^\varepsilon \|^2 + \varepsilon \| \nabla w^\varepsilon \|^2 \leq \frac{1}{2} \| w^\varepsilon \|^2 + k_1 \varepsilon^3/2 + k_2 \varepsilon^{3/2} \varepsilon^{1/2} \| \nabla w^\varepsilon \| + k_3 \varepsilon^{3/4} \varepsilon^{1/2} \| \nabla w^\varepsilon \| \\
\leq \frac{1}{2} \| w^\varepsilon \|^2 + k_1 \varepsilon^3/2 + \frac{\varepsilon}{2} \| \nabla w^\varepsilon \|^2.
\]

In conclusion, we have

\[
\frac{d}{dt} \| w^\varepsilon \|^2 + \varepsilon \| \nabla w^\varepsilon \|^2 \leq \| w^\varepsilon \|^2 + k_1 \varepsilon^3/2.
\]

Using the Gronwall inequality, we obtain

\[
\| w^\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \leq k_1 \varepsilon^{3/4} \text{ and } \| \nabla w^\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \leq k_1^{1/4}.
\]

Hence, according to (3.6), (3.7), and the triangular inequality, we deduce (3.1) and (3.2). This concludes the proof of Theorem 3.1.
4. A COLLOCATED FINITE VOLUME SCHEME WITH A SPLITTING METHOD FOR THE TIME DISCRETIZATION

We follow here the notations of [7] that we recall in this section for the reader convenience. In the following, we uniformly discretize the domain $\Omega$ by using cube finite volumes of dimensions $\Delta x \Delta y \Delta z$:

$$K_{i,j,k} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \times [z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}],$$

where:

$$x_{i+\frac{1}{2}} = i \Delta x, \quad y_{j+\frac{1}{2}} = j \Delta y, \quad z_{k+\frac{1}{2}} = k \Delta z,$$

$$\forall i = 0, \ldots, M, \forall j = 0, \ldots, N, \forall k = 0, \ldots, L.$$

The edges of the control volumes are defined by:

$$\Gamma_{i+1/2,j,k} = \{(x, y, z); x = x_{i+1/2}, y \in [y_{j-1/2}, y_{j+1/2}], z \in [z_{k-1/2}, z_{k+1/2}]\},$$

$$\Gamma_{i,j+1/2,k} = \{(x, y, z); x \in [x_{i-1/2}, x_{i+1/2}], y = y_{j+1/2}, z \in [z_{k-1/2}, z_{k+1/2}]\},$$

$$\Gamma_{i,j,k+1/2} = \{(x, y, z); x \in [x_{i-1/2}, x_{i+1/2}], y \in [y_{j-1/2}, y_{j+1/2}], z = z_{k+1/2}\},$$

$$\forall i = 0, \ldots, M, \forall j = 0, \ldots, N, \forall k = 0, \ldots, L.$$

The velocity and the pressure are approximated in the center of the cells as follows:

$$u_{i,j,k}(t) \approx \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u(x, y, z, t) dx dy dz,$$

$$p_{i,j,k}(t) \approx \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} p(x, y, z, t) dx dy dz.$$

We also define the velocity fluxes:

$$F_{ui+\frac{1}{2},j,k} \approx \frac{1}{\Delta y \Delta z} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u(x_{i+\frac{1}{2}}, y, z, t) dy dz,$$

$$F_{vi,j+\frac{1}{2},k} \approx \frac{1}{\Delta x \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} v(x, y_{j+\frac{1}{2}}, z, t) dx dz,$$

$$F_{wi,j,k+\frac{1}{2}} \approx \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} w(x, y, z_{k+\frac{1}{2}}, t) dx dy.$$

4.1. Time discretization. We start by choosing a time discretization for (1.1):

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \varepsilon \Delta u^{n+1} + 2\omega \times u^n - \omega \times u^{n-1} + 2\nabla p^n - \nabla p^{n-1} = f^{n+1}.$$

Thanks to (4.1) we are able to compute the new velocity $u^{n+1}$.

Hence, to obtain the pressure, we take the divergence of (1.1) and use the incompressibility condition (1.1) we find:

$$\Delta p = \text{div}(f + \varepsilon \Delta u + \omega \times u).$$

Thus we discretize (4.2) as follows:

$$\Delta p^{n+1} = \text{div}(f^{n+1} + \varepsilon \Delta u^{n+1} - 2\omega \times u^n - \omega \times u^{n-1}).$$
By replacing $\Delta$ by $-\nabla \times \nabla \times$ (see[7] and [9]), we rewrite (4.3) as below:

$$\Delta p^{n+1} = \text{div}(f^{n+1} - \varepsilon \nabla \times \nabla \times u^{n+1} - 2\omega \times u^n - \omega \times u^{n-1}).$$

Now, by using the relation $\Delta u = \nabla \text{div} u^{n+1} - \nabla \times \nabla \times u^{n+1}$, then (4.1) becomes:

$$f^{n+1} - \varepsilon \nabla \times \nabla \times u^{n+1} - 2\omega \times u^n - \omega \times u^{n-1} = \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \varepsilon \nabla \text{div} u^{n+1} + 2\nabla p^n - \nabla p^{n-1}.$$

Hence, we deduce from (4.4) that

$$\Delta p^{n+1} = \text{div}(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \varepsilon \nabla \text{div} u^{n+1} + 2\nabla p^n - \nabla p^{n-1}).$$

Thus, we obtain

$$\Delta(p^{n+1} - 2p^n + p^{n-1} + \varepsilon \text{div} u^{n+1}) = \text{div}(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t}).$$

Then we compute the pressure from

$$\begin{cases}
\Delta \psi^{n+1} = \text{div}(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t}), \\
\frac{\partial \psi^{n+1}}{\partial n} = 0,
\end{cases}$$

and

$$p^{n+1} = \psi^{n+1} + 2p^n - p^{n-1} - \varepsilon \text{div} u^{n+1}.$$

Concerning the boundary conditions, we have the periodicity in the $x$ and $y$ directions and the Dirichlet boundary conditions in the $z$ direction for $u^{n+1}$:

$$u^{n+1}_{i,j,1} = u^{n+1}_{N,i,j}, \quad u^{n+1}_{N+1,i,j,k} = u^{n+1}_{1,i,j,k},$$
$$u^{n+1}_{i,0,k} = u^{n+1}_{i,N,k}, \quad u^{n+1}_{i,N+1,k} = u^{n+1}_{i,1,k},$$
$$\frac{u^{n+1}_{i,j,N+1} + u^{n+1}_{i,j,N}}{2} = 0, \quad \frac{u^{n+1}_{i,j,0} + u^{n+1}_{i,j,1}}{2} = 0.$$

The Neumann boundary conditions are imposed for $\psi^{n+1}$ in the $z$ direction and the periodicity in $x$ and $y$ directions. Thus, we have

$$\psi^{n+1}_{0,j,k} = \psi^{n+1}_{N,j,k}, \quad \psi^{n+1}_{N+1,j,k} = \psi^{n+1}_{1,j,k},$$
$$\psi^{n+1}_{i,0,k} = \psi^{n+1}_{i,N,k}, \quad \psi^{n+1}_{i,N+1,k} = \psi^{n+1}_{i,1,k},$$
$$\psi^{n+1}_{i,j,N+1} = \psi^{n+1}_{i,j,N}, \quad \psi^{n+1}_{i,j,0} = \psi^{n+1}_{i,j,1}.$$

The periodicity in $x$ and $y$ for the pressure yields:

$$p_{0,j,k} = p_{M,j,k}, \quad p_{M+1,j,k}, p_{1,j,k},$$
$$p_{i,0,k} = p_{i,M,k}, \quad p_{i,M+1,k} = p_{i,1,k},$$
and for the terms $p_{i,j,0}$ and $p_{i,j,L+1}$ we use the second order compact scheme to compute them:

$$p_{i,j,0} = \frac{5}{2}p_{i,j,1} - 2p_{i,j,2} + \frac{1}{2}p_{i,j,3}, \quad p_{i,j,L+1} = \frac{5}{2}p_{i,j,L} - 2p_{i,j,L-1} + \frac{1}{2}p_{i,j,L-2}.$$
4.2. Finite volume discretization. To compute the velocity $u^{n+1}$, we discretize (4.1) and we obtain:

\[
\begin{align*}
\Delta x \Delta y \Delta z & \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \varepsilon \left[ \Delta x \Delta y \frac{u_{i,j,k+1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k-1}^{n+1}}{\Delta z} \right] \\
+ \Delta y \Delta z & \frac{u_{i+1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i-1,j,k}^{n+1}}{\Delta x} + \Delta x \Delta z \frac{u_{i,j,k+1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k-1}^{n+1}}{\Delta y} \\
+ \frac{\Delta y \Delta z}{2} & \left( p_{i+1,j,k}^n - p_{i-1,j,k}^n \right) + \frac{\Delta x \Delta z}{2} \left( p_{i,j+1,k}^n - p_{i,j-1,k}^n \right) + \frac{\Delta x \Delta y}{2} \left( p_{i,j,k+1}^n - p_{i-1,j,k}^n \right) \\
+ \Delta x \Delta y \Delta z & \left( \omega \times (2u_{i,j,k}^n - u_{i,j,k}^{n-1}) \right) = \Delta x \Delta y \Delta z f_{i,j,k}^{n+1}.
\end{align*}
\]

To compute the pressure we first compute $p^{n+1}$:

\[
\begin{align*}
\Delta x \Delta y & \frac{\psi_{i,j,k+1}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i,j,k-1}^{n+1}}{\Delta z} + \Delta y \Delta z \frac{\psi_{i+1,j,k}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i-1,j,k}^{n+1}}{\Delta x} \\
+ \Delta x \Delta z & \frac{\psi_{i,j+1,k}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i,j,k-1}^{n+1}}{\Delta y} = \frac{1}{2\Delta t} \left( \Delta y \Delta z \left( (3F_{ui+1/2jk}^{n+1} - 4F_{ui+1/2jk}^{n} + F_{ui+1/2jk}^{n-1}) \right) \\
- (3F_{ui-j-1/2jk}^{n+1} - 4F_{ui-j-1/2jk}^{n} + F_{ui-j-1/2jk}^{n-1}) \right) \\
- (3F_{uj+1/2jk}^{n+1} - 4F_{uj+1/2jk}^{n} + F_{uj+1/2jk}^{n-1}) \right) \Delta x \Delta y \left( (3F_{uj+j+1/2jk}^{n+1} - 4F_{uj+j+1/2jk}^{n} + F_{uj+j+1/2jk}^{n-1}) \right) \\
- (3F_{ujj+1/2jk}^{n+1} - 4F_{ujj+1/2jk}^{n} + F_{ujj+1/2jk}^{n-1}) \right) \Delta x \Delta y \left( (3F_{ujj-j+1/2jk}^{n+1} - 4F_{ujj-j+1/2jk}^{n} + F_{ujj-j+1/2jk}^{n-1}) \right).
\end{align*}
\]

Then, we easily obtain the pressure:

\[
p_{i,j,k}^{n+1} = p_{i,j,k}^{n} + 2p_{i,j,k}^{n} - p_{i,j,k}^{n-1} - \varepsilon \left\{ \Delta x \Delta y \Delta z \left[ 3F_{ui+1/2jk}^{n+1} - 4F_{ui+1/2jk}^{n} + F_{ui+1/2jk}^{n-1} \right] \\
+ \Delta x \Delta z \left[ 3F_{uj+j+1/2jk}^{n+1} - 4F_{uj+j+1/2jk}^{n} + F_{uj+j+1/2jk}^{n-1} \right] \right\} \\
+ \Delta x \Delta y \left[ 3F_{ujj-j+1/2jk}^{n+1} - 4F_{ujj-j+1/2jk}^{n} + F_{ujj-j+1/2jk}^{n-1} \right].
\]

4.3. Computation of the fluxes. We recall here that the simplest method to compute the fluxes (linear interpolation) does not work when the viscosity $\varepsilon$ is small. Hence the authors in [7] considered a modified interpolation method for the fluxes in two dimensional case. Now, since we aim here to study the boundary layers at small viscosity, we need, on the one hand, to adapt the discretization in [7] to the 3D dimensional case and, on the other hand, to introduce the correctors in the finite volume discretization basis that is the NFVM. Thus we first start by introducing the 3D fluxes inherited from [7]:

\[
F_{ij+1/2,k}^{n+1} = \frac{u_{i+1,j,k+1}^{n+1} + u_{i,j,k+1}^{n+1}}{2} + \frac{\Delta y \Delta z}{4\Delta t} (p_{i,j,k+1}^{n+1} - 2p_{i+1,j,k}^{n+1} + p_{i,j,k}^{n+1}) \\
- \frac{\Delta y \Delta z}{4\Delta t} (p_{i,j,k+1}^{n+1} - 2p_{i,j,k}^{n+1} + p_{i-1,j,k}^{n+1}),
\]
In this section we introduce a new finite volume schemes, that is we approximate the solution of (1.1) by:

\[
\mathbf{u}_h = \sum_{i,j=1}^{M,N} r_{i,j,0} \phi_{i,j,0} \chi_{i,j,0} + \sum_{i,j=1}^{M,N} r_{i,j,L+1} \phi_{i,j,L+1} \chi_{i,j,L+1} + \sum_{i,j=1}^{M,N} u_{i,j,k} \chi_{i,j,k},
\]

where:

\[
h = \Delta z,
\]

\[
r_{i,j,0} = \frac{u_{i,j,0} + u_{i,j,1}}{2},
\]

\[
r_{i,j,L+1} = \frac{u_{i,j,L+1} + u_{i,j,L}}{2},
\]

\[
\chi_{i,j,0} = \chi((x_{i-\frac{1}{2},j+,\frac{1}{2}} x_{i+,\frac{1}{2},j+\frac{1}{2}}) \times(0,h)),
\]

\[
\chi_{i,j,L+1} = \chi((x_{i-\frac{1}{2},j+,\frac{1}{2}} x_{i+,\frac{1}{2},j+\frac{1}{2}}) \times(L-1)h,Lh)),
\]

\[
\chi_{i,j,k} = \chi((x_{i-\frac{1}{2},j+,\frac{1}{2}} x_{i+,\frac{1}{2},j+\frac{1}{2}}) \times(z_{h-\frac{1}{2},z_{i+\frac{1}{2}}}),
\]

and

\[
\frac{\phi_{i}}{\phi_i} = - 2 \frac{1}{\sqrt{4\pi (t-\tau)}} \frac{z e^{-\frac{z^2}{2(t-\tau)}}}{2 \sqrt{\varepsilon(t-\tau)} e^{-\frac{(z^2-\varepsilon^2)}{4(t-\tau)}}}
\]

\[
\times \{2\tau \cos(\alpha(\tau-t)) - 2\tau \sin(\alpha(\tau-t))\} d\tau, \quad \forall \ i = 1, 2,
\]

\[
\frac{\phi_{i}}{\phi_i} = - 2 \frac{1}{\sqrt{4\pi (t-\tau)}} \frac{z e^{-\frac{z^2}{2(t-\tau)}}}{2 \sqrt{\varepsilon(t-\tau)} e^{-\frac{(z^2-\varepsilon^2)}{4(t-\tau)}}}
\]

\[
\times \{2\tau \cos(\alpha(\tau-t)) - 2\tau \sin(\alpha(\tau-t))\} d\tau, \quad \forall \ i = 1, 2.
\]

Multiplying (1.1) by \(\chi_{i,j,k}\), integrating over \(\Omega\), and replacing \(\mathbf{u}^\varepsilon\) by \(\mathbf{u}_h\) we find that the equations are the same as the classical finite volume scheme (4.9). Moreover the correctors verify (2.2). Hence they do not contribute to these equations. For the numerical simulations
we do not use the modified boundary layer $\tilde{\varphi}^{0,\varepsilon}$ and $\tilde{\varphi}^{1,\varepsilon}$ directly. Instead we consider another approximate form which reads as follows:

$$\tilde{\varphi}^{0,\varepsilon}(t, z) = (-\exp(-\frac{z^2}{4\varepsilon t}), -\exp(-\frac{z^2}{4\varepsilon t}), 0).$$

Indeed, the approximation $\tilde{\varphi}^{0,\varepsilon}$ is much easier to be implemented numerically in coding than the theoretical corrector $\varphi^{0,\varepsilon}$ obtained in section 2.
Due to the nodes $r_{i,j,0}$ and $r_{i,j,L+1}$, the linear system associated with this scheme is not closed. However, by adding the correctors, we ensuring the closure of the linear system corresponding to the NFVM considered. Hence, We multiply (4.1) by the corrector $\bar{\varphi}^{0,e}$ and integrate over $K_{i,j,1}$, we find:

\[
(5.1) \int_{K_{i,j,1}} \frac{3u_{n+1} - 4u_n + u_{n-1}}{2\Delta t} \bar{\varphi}^{0,e} - \varepsilon \int_{K_{i,j,1}} \Delta u_{n+1} \bar{\varphi}^{0,e} + \int_{K_{i,j,1}} \omega \times (2u_n - u_{n-1}) \bar{\varphi}^{0,e} + 2 \int_{K_{i,j,1}} \nabla p \bar{\varphi}^{0,e} - \int_{K_{i,j,1}} \nabla p_{n-1} \bar{\varphi}^{0,e} = \int_{K_{i,j,1}} f_{n+1} \bar{\varphi}^{0,e}.
\]

In the following we calculate each term of (5.1), for the first term in the LHS (left-hand side) of (5.1) we find:

\[
\int_{K_{i,j,1}} \frac{3u_{n+1} - 4u_n + u_{n-1}}{2\Delta t} \bar{\varphi}^{0,e} dx dy dz = \int_{K_{i,j,1}} \frac{3u_{i,j,1}^{n+1} - 4u_{i,j,1}^n + u_{i,j,1}^{n-1}}{2\Delta t} \bar{\varphi}^{0,e} dx dy dz.
\]

For the second term in the LHS of (5.1), we obtain:

\[
(5.2) \int_{K_{i,j,1}} \Delta u_{n+1} \bar{\varphi}^{0,e} dx dy dz = -\int_{K_{i,j,1}} \nabla u_{n+1} \nabla \bar{\varphi}^{0,e} dx dy dz + \int_{\partial K_{i,j,1}} \bar{\varphi}^{0,e} \frac{\partial u_{n+1}}{\partial n} d\Gamma,
\]

\[
= -\int_{K_{i,j,1}} \frac{\partial u_{n+1}}{\partial z} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz + \int_{\partial K_{i,j,1}} \frac{\bar{\varphi}^{0,e}}{\partial n} \frac{\partial u_{n+1}}{\partial n} d\Gamma.
\]

Now, we calculate the first term in the RHS of (5.2) we find:

\[
\int_{K_{i,j,1}} \nabla u_{n+1} \nabla \bar{\varphi}^{0,e} dx dy dz = \int_{K_{i,j,1}} \frac{\partial u_{n+1}}{\partial z} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz
\]

\[
= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{0}^{h/2} \frac{\partial u_{n+1}}{\partial z} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz
\]

\[
+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{h/2}^{h} \frac{\partial u_{n+1}}{\partial z} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz
\]

\[
= \frac{u_{i,j,1}^{n+1} - r_{i,j,1}^{n+1}}{h/2} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{0}^{h/2} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz
\]

\[
+ \frac{u_{i,j,2}^{n+1} - u_{i,j,1}^{n+1}}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{h/2}^{h} \frac{\partial \bar{\varphi}^{0,e}}{\partial z} dx dy dz
\]

\[
= \frac{2}{h} (u_{i,j,1}^{n+1} - r_{i,j,0}^{n+1}) \Delta x \Delta y (\bar{\varphi}^{0,e}(h/2) - \bar{\varphi}^{0,e}(0))
\]

\[
+ \frac{u_{i,j,2}^{n+1} - u_{i,j,1}^{n+1}}{h} \Delta x \Delta y (\bar{\varphi}^{0,e}(h) - \bar{\varphi}^{0,e}(h/2)).
\]
For the second term in the RHS of (5.2) we obtain:

\[
\int_{\partial K_{ij}} \tilde{\varphi}^{0,\varepsilon}_{1} \frac{\partial u^{n+1}}{\partial n} d\Gamma = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z=0}^{\tilde{\varphi}^{0,\varepsilon}_{1}} \left( -\frac{\partial u}{\partial z} \right) d\Gamma \\
+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z=h}^{\tilde{\varphi}^{0,\varepsilon}_{1}} \left( \frac{\partial u}{\partial z} \right) d\Gamma \\
+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{0}^{h} \int_{y=y_{j-1/2}}^{y_{j+1/2}} \tilde{\varphi}^{0,\varepsilon}_{1} \left( -\frac{\partial u}{\partial y} \right) d\Gamma \\
+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{0}^{h} \int_{x=x_{i-1/2}}^{x_{i+1/2}} \tilde{\varphi}^{0,\varepsilon}_{1} \left( -\frac{\partial u}{\partial x} \right) d\Gamma \\
+ \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{0}^{h} \int_{x=x_{i-1/2}}^{x_{i+1/2}} \tilde{\varphi}^{0,\varepsilon}_{1} \left( \frac{\partial u}{\partial x} \right) d\Gamma.
\]

Now, the third term in the LHS of (5.1), can be rewritten as bellow:

\[
\int_{K_{ij}} \omega \times (2u^{n} - u^{n-1})\tilde{\varphi}^{0,\varepsilon}_{1} dxdydz = \omega \times (2u_{i,j,1}^{n} - u_{i,j,1}^{n-1}) \Delta x \Delta y \int_{0}^{h} \tilde{\varphi}^{0,\varepsilon}_{1} dxdydz.
\]

We calculate the first component of the fourth term in the LHS of (5.1) and, we find:

\[
\int_{K_{ij}} \partial_{x} p^{n+1} \tilde{\varphi}^{0,\varepsilon}_{1} dxdydz = \frac{p_{i+1,j,1}^{n+1} - p_{i-1,j,1}^{n+1}}{2\Delta x} \Delta x \Delta y \int_{0}^{h} \tilde{\varphi}^{0,\varepsilon}_{1} dz.
\]

For the second component of the fourth term in the LHS of (5.1) we have:

\[
\int_{K_{ij}} \partial_{y} p^{n+1} \tilde{\varphi}^{0,\varepsilon}_{2} dxdydz = \frac{p_{i,j+1,1}^{n+1} - p_{i,j-1,1}^{n+1}}{2\Delta y} \Delta x \Delta y \int_{0}^{h} \tilde{\varphi}^{0,\varepsilon}_{2} dz.
\]

Concerning the first term on the RHS of (5.1), we have:

\[
\int_{K_{ij}} f^{n+1} \tilde{\varphi}^{0,\varepsilon}_{1} dxdydz = \Delta x \Delta y f_{i,j,1}^{n+1} \int_{0}^{h} \tilde{\varphi}^{0,\varepsilon}_{1} dz.
\]
Hence, we infer that
\[
\frac{1}{2\Delta t}(3u_{i,j,1}^{n+1} - 4u_{i,j,1}^{n} + u_{i,j,1}^{n-1}) \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz - \varepsilon \left[ \frac{1}{h} (-3\tilde{\phi}^{0,\varepsilon} (h/2) u_{i,j,1}^{n+1} + 
\right.
\]
\[
+ 2r_{i,j,0}^{n+1} \tilde{\phi}^{0,\varepsilon} (h/2) + u_{i,j,2}^{n+1} \tilde{\phi}^{0,\varepsilon} (h/2)) - \left( \frac{1}{(\Delta x)^2} (u_{i-1,j,1}^{n+1} - 2u_{i,j,1}^{n+1} + u_{i+1,j,1}^{n+1}) + 
\right.
\]
\[
+ \frac{1}{(\Delta y)^2} (u_{i,1,j-1,1}^{n+1} - 2u_{i,j,1}^{n+1} + u_{i,j+1,1}^{n+1})) \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz \right] + \omega \times (2u_{i,j,1}^n - u_{i,j,1}^{n-1}) \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz +
\]
\[
\left( \begin{array}{cc}
\frac{p_{i+1,j,1}^n - p_{i-1,j,1}^n}{2\Delta x} \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz \\
\frac{p_{i,j+1,1}^n - p_{i,j-1,1}^n}{2\Delta y} \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz
\end{array} \right)
\right) -
\left( \begin{array}{cc}
\frac{p_{i+1,j,1}^{n-1} - p_{i-1,j,1}^{n-1}}{2\Delta x} \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz \\
\frac{p_{i,j+1,1}^{n-1} - p_{i,j-1,1}^{n-1}}{2\Delta y} \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz
\end{array} \right)
\]
\[
= f_{i,j,1}^n \int_0^h \tilde{\phi}^{0,\varepsilon} \, dz.
\]

6. Numerical results

In this section we will compute the error approximation using the classical finite volume method and the new finite volume method, so the pressure and the source term are chosen such that:

\[
p(x, y, z, t) = \cos(2\pi x) \cos(2\pi y) \cos(\pi z) t,
\]
\[
u\varepsilon(x, y, z, t) = t \sin(2\pi y)(1 - e^{-z\sqrt{\varepsilon}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right))(1 - e^{-\frac{1-z}{\sqrt{\varepsilon}}} \cos\left(\frac{1-z}{\sqrt{\varepsilon}}\right)),
\]
\[
v\varepsilon(x, y, z, t) = t \sin(2\pi x)(1 - e^{-z\sqrt{\varepsilon}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right))(1 - e^{-\frac{1-z}{\sqrt{\varepsilon}}} \cos\left(\frac{1-z}{\sqrt{\varepsilon}}\right)),
\]
and
\[
w\varepsilon(x, y, z, t) = 0.
\]
Note that the test solution given above satisfies the equations (1.1)1,2 including the boundary and initial conditions (1.1)3,5 with \( u_0 = 0 \).
Thus, the function source is chosen using this test solution.
| N=M=L t | ε     | CFVM     | NFVM     |
|---------|-------|----------|----------|
| 10 1   | 10^{-2} | 0.03206  | 0.12836  |
| 20 1   | 10^{-2} | 0.00634  | 0.03893  |
| 30 1   | 10^{-2} | 0.00269  | 0.02553  |
| 10 1   | 10^{-3} | 0.092294 | 0.22647  |
| 20 1   | 10^{-3} | 0.033726 | 0.15753  |
| 30 1   | 10^{-3} | 0.01331  | 0.08020  |
| 10 1   | 10^{-4} | 1.61660e+03 | 0.04487 |
| 20 1   | 10^{-4} | 0.08741  | 0.010303 |
| 30 1   | 10^{-4} | 0.11722  | 0.00460  |
| 10 1   | 10^{-5} | 1.10612e+10 | 0.044901 |
| 20 1   | 10^{-5} | 4.42881e+06 | 0.01032 |
| 30 1   | 10^{-5} | 1.12960e+03 | 0.00442 |
| 10 1   | 10^{-6} | 5.26218e+62 | 0.04490 |
| 20 1   | 10^{-6} | 1.16428e+29 | 0.01032 |
| 30 1   | 10^{-6} | 6.72495e+17 | 0.00443 |
| 10 1   | 10^{-7} | 5.26218e+62 | 0.04490 |
| 20 1   | 10^{-7} | 1.16428e+29 | 0.01032 |
| 30 1   | 10^{-7} | 6.72495e+17 | 0.00443 |

**Figure 1.** The $L^2$ norm of the velocity error with classical finite volume (CFVM) and new finite volume method (NFVM) for different values of $\varepsilon$ at $t = 1$.

| N=M=L t | ε     | CFVM     | NFVM     |
|---------|-------|----------|----------|
| 10 1   | 10^{-2} | 0.02493  | 0.03178  |
| 20 1   | 10^{-2} | 0.00511  | 0.00920  |
| 30 1   | 10^{-2} | 0.00224  | 0.00533  |
| 10 1   | 10^{-3} | 0.02684  | 0.02771  |
| 20 1   | 10^{-3} | 0.00553  | 0.00907  |
| 30 1   | 10^{-3} | 0.002381 | 0.00590  |
| 10 1   | 10^{-4} | 1.48996e+02 | 0.02602 |
| 20 1   | 10^{-4} | 0.00774  | 0.00539  |
| 30 1   | 10^{-4} | 0.00655  | 0.00238  |
| 10 1   | 10^{-5} | 1.01953e+16 | 0.026016 |
| 20 1   | 10^{-5} | 2.83861e+05 | 0.00539 |
| 30 1   | 10^{-5} | 58.98117 | 0.00238  |
| 10 1   | 10^{-6} | 4.85027e+61 | 0.02601 |
| 20 1   | 10^{-6} | 7.46273e+27 | 0.005394 |
| 30 1   | 10^{-6} | 3.51186e+16 | 0.00238 |

**Figure 2.** The $L^2$ norm of the pressure error with classical finite volume (CFVM) and new finite volume method (NFVM) for different values of $\varepsilon$ at $t = 1$. 
7. Conclusion and fracture works

In this paper we have compared two different finite volume methods (CFVM) and (NFVM) when the viscosity is considered small and more precisely in the range $10^{-3} - 10^{-7}$. We derived an approximate solution of the time-dependent rotating fluid in 3D channel using the splitting methods for the time discretization and colocated space discretization. One of the novelties of this article is that we propose a new numerical approach to treat the pressure and the divergence free condition introducing correctors to solve the boundary layers. We also showed that the (NFVM) is more performing than the (CFVM) when the viscosity is small. We showed that our NFVM still perform for very large Reynolds number. To the best of our knowledge, this is the first work which gives a new finite volume scheme taking into account boundary layer variations for the linearized Navier-Stokes equations. Note that the method developed here may apply to many other problems and domains. This will be the subject of subsequent work.

8. Appendix.

In this paragraph, we give a sketch of the proof of the existence and regularity of the solution of the limit problem \((1.2)\). For complete study of the existence of system \((1.2)\) we refer the reader to , see also and . We first want to apply the Hille-Phillips-Yosida Theorem to prove the existence and uniqueness of the solution of \((1.2)\). Thus we start by introducing the adequate function spaces:

$$ H = \{ v \in (L^2(\Omega))^3; \text{div} v = 0, v_3(z = 0) = v_3(z = h) = 0, \text{and } v \text{ is } 2\pi \text{ periodic in the } x \text{ and } y \text{ directions} \}. $$

$$ D(A) = \{ v \in H; \exists p \in D'(\Omega), \text{ such } \omega \times v + \nabla p \in H \}, $$

with the norm:

$$ (8.1) \quad \| v \|_{D(A)} = (\| v \|_H^2 + \| \omega \times v + \nabla p \|_H^2)^{1/2}. $$

Then for \( v \in D(A) \) we set \( Av = \omega \times v + \nabla p \), thus we define an unbounded linear operator \( A \) which maps \( D(A) \subset H \) onto \( H \).

**Theorem 8.1** (Hille-Yosida Theorem). Let \( H \) be a Hilbert space and let \( B : D(B) \rightarrow H \) a linear unbounded operator, with domain \( D(B) \subset H \) such that \( D(B) \) is dense in \( H \) and \((-B)\) is m-dissipative. Then \((-B)\) is the infinitesimal generator of a contraction semigroup \( \{ S(t) \}_{t>0} \) in \( H \), and the solution of the following system:

$$ \begin{align*}
\frac{dv}{dt} + Bv &= f, \\
v|_{t=0} &= v_0,
\end{align*} 
$$

satisfies the following properties:

(\(H_0\)) If \( v_0 \) and \( f \in L^1(0, T; H) \), then \( v \in C([0, T]; H), \forall T > 0 \).

(\(H_1\)) If \( v_0 \in D(B) \) and \( f' \in L^1(0, T; H) \) then \( v \in C^1([0, T]; H) \cap C^0([0, T]; D(B)) \) and \( \frac{dv}{dt} \in L^\infty([0, T]; H) \), \( \forall T > 0 \).

**Remark 1.** A linear operator is dissipative in \( H \) if and only if: \( \forall u \in D(A), \forall \lambda > 0, \| u - \lambda Au \| \geq \| u \|. \)
Remark 2. A linear operator $A$ is m-dissipative if: $A$ is dissipative and $\forall f \in X, \forall \lambda > 0, \exists u \in D(A), u - \lambda Au = f$.

Proof. Now we want to show that the operator $(-A)$ is m-dissipative, hence we will prove that the following system:

(8.3) \[
\begin{align*}
\lambda \omega \times u + \lambda \nabla p + u &= f, \\
\text{div } u &= 0, \\
u_3 &= 0, \text{ en } z = 0, 1,
\end{align*}
\]

has a unique solution in $D(A)$ for all $f \in H$ and $\forall \lambda > 0$, and the solution satisfies the estimate:

(8.4) \[\|u\|_H \leq \|f\|_H, \forall f \in H.\]

We multiply (8.3) by $v \in H$, integrate over $\Omega$ and we find:

\[\lambda \int_\Omega (\omega \times u) \cdot v d\Omega + \lambda \int_\Omega \nabla p \cdot v d\Omega + \int_\Omega u \cdot v d\Omega = \int_\Omega f v d\Omega.\]

We have:

\[\int_\Omega \nabla p \cdot v = - \int_\Omega p \text{div } v + \int_{\partial \Omega} p v nd(\Gamma) = 0.\]

We set

\[a(u, v) = \lambda \int_\Omega (\omega \times u) \cdot v + \int_\Omega u \cdot v,\]

and

\[F(v) = \int_\Omega f v.\]

Here $a(\cdot, \cdot)$ is a continuous and coercive bilinear form in $H \times H$. In fact we have:

\[|a(u, v)| \leq \lambda |u|_H |v|_H + |u|_H |v|_H,\]

\[\leq k(\lambda) |u|_H |v|_H,\]

and

\[|a(u, u)| = |u|^2_H.\]

Also $F(v)$ is a continuous linear form:

\[\int_\Omega f v \leq \|f\| \|v\|.\]

Hence according to the Lax-Milligram theorem, there exists a unique $u \in H$ such that:

\[\lambda Au + u = f,\]

that is,

\[\lambda \omega \times u + \lambda \nabla p + u = f.\]

Multiplying the above equation by $u$ and integrating over $\Omega$, we find:

\[\lambda \int_\Omega (\omega \times u) u + \lambda \int_\Omega \nabla p u + \int_\Omega uu = \int_\Omega f u,\]

then the solution $u$ satisfies the estimate:

\[\|u\|_H \leq \|f\|_H.\]
Also we have:

$$\|u\|_{D(A)} = \left(\|u\|^2_H + \|\omega \times u + \nabla p\|^2_H\right)^{1/2},$$

$$\leq \|u\|_H + \|\omega \times u + \nabla p\|_H,$$

$$\leq k(\lambda)\|f\|_H.$$ 

Hence (-A) is m-dissipative operator, Moreover we have $u_0 \in H$, then according to the Hille-Yosida theorem the system (1.2) has a unique solution $u \in C([0, \infty[, H)$. Furthermore, we have:

$$\|\nabla p\|_{H^{-1}} \leq \frac{1}{\lambda} \|f\|_{H^{-1}} + \frac{1}{\lambda} \|u\|_{H^{-1}} + \|\omega \times u\|_{H^{-1}}$$

$$\leq k(\lambda)\|f\|_H.$$ 

Then, we obtain:

$$\|p\|_{L^2(\Omega)} \leq k(\lambda)\|f\|_H.$$ 

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REFERENCES

[1] C. Amrouche, Š. Nečasová, Y. Raudin, Very weak, generalized and strong solutions to the Stokes system in the half-space, J. Differential Equations 244 (2008), no. 4, 887–915.

[2] J. R. Cannon, The one-dimensional heat equation. Encyclopedia of Mathematics and its Applications, 23. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984.

[3] J. Y. Chemin, B. Desjardins, I. Gallagher, Mathematical Geophysics, An introduction to rotating fluids and the Navier-Stokes equations. Claredon press. Oxford (2006).

[4] R. F. Dressler and K. O. Friedrichs, A boundary-layer theory for elastic plates. Comm. Pure Appl. Math. 14 1961, 1-33.

[5] S. Faure, Stability of a colocated finite volume scheme for the Navier-Stokes equations. Numer. Methods Partial Differential Equations 21 (2005), no. 2, 242 – 271.

[6] S. Faure, A finite volume scheme for the nonlinear heat equation. Numer. Funct. Anal. Optim. 25 (2004), no. 1-2, 27 – 56.

[7] S. Faure, J. Laminie and R. Temam, Colocated finite volume schemes for fluid flows. Commun. Comput. Phys. 4 (2008), no. 1, 1 – 25.

[8] H. P. Greenspan, The Theory of Rotating Fluids, Cambridge Univ. Press, Cambridge (1968).

[9] J. L. Guermond and J. Shen, A new class of truly consistent splitting schemes for incompressible flows. J. Comput. Phys. 192 (2003), no. 1, 262 – 276.

[10] M. Hamouda and R. Temam, Boundary layers for the Navier-Stokes equations. The case of a characteristic boundary. Georgian Math. J. 15 (2008), no. 3, 517 – 530.

[11] M. Hamouda and R. Temam, Some singular perturbation problems related to the Navier-Stokes equations, In Advances in deterministic and stochastic analysis, pages 197-227. World Sci. Publ., Hackensack, NJ, 2007.

[12] J. L. Lions, Perturbations singulières dans les problèmes aux limites et en contrôle optimal. Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York.

[13] J. L. Lions, Selected work, Vol 1. (French) EDP Sciences, Les Ulis;Société de Mathématiques Appliquées et Industrielles, Paris, 2003.

[14] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data. Comm. Pure Appl. Math. 53 (2000), no. 4, 432 – 483.
[15] N. Masmoudi, *The Euler limit of the Navier-Stokes equations, and rotating fluids with boundary*. Arch. Rational Mech. Anal. 142 (1998), no. 4, 375 – 394.

[16] R. Temam, *Navier Stokes Equations Theory And Numerical Analysis*. North-Holland publishing company Amsterdam. New York. Oxford (1977).

[17] R. Temam and X. Wang, *Asymptotic analysis of the linearized Navier-Stokes equations in a channel*. Differential Integral Equations 8 (1995), no. 7, 1591 – 1618.

[18] R. Temam and X. Wang, *Asymptotic analysis of Oseen type equations in a channel at small viscosity*. Indiana Univ. Math. J. 45 (1996), no. 3, 863 – 916.

[19] C. Jung and R. Temam, *Finite volume approximation of one-dimensional stiff convection-diffusion equations*. J. Sci. Comput. 41 (2009), no. 3, 384 – 410.

[20] C. Jung and R. Temam, *Finite volume approximation of two-dimensional stiff problems*. Int. J. Numer. Anal. Model. 7 (2010), no. 3, 462 – 476.

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