Torsionless T-selfdual Affine NA Toda Models $^1$

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ABSTRACT

A general construction of affine Non Abelian Toda models in terms of gauged two loop WZNW model is discussed. In particular we find the Lie algebraic condition defining a subclass of $T$-selfdual torsionless NA Toda models and their zero curvature representation.

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1 Introduction

Two dimensional integrable models represent an important laboratory for testing new ideas and developing new methods for constructing exact solutions as well as for the nonperturbative quantization of 4-D non-abelian gauge theories, gravity and string theory. Among the numerous technics for constructing 2-D integrable models and their solutions [1], [2], the two loop $\hat{G}$-WZNW and gauged $\hat{G}/\hat{H}$-WZNW models [3] have the advantage of providing a simple and universal method for derivation of the zero curvature representation as well as a consistent path integral formalism for their description. The power of such method was demonstrated in constructing (multi) soliton solution of the abelian affine Toda models [3] and certain nonsingular nonabelian (NA) affine Toda models [4].

The present paper is devoted to the systematic construction of the simplest class of singular torsionless affine NA Toda models characterized by the fact that the space of physical fields $g_0^0$ lies in the coset $G_0/G^0_0 = SL(2) \otimes U(1)^{rank G-1}$. Our main result is that such models exists only for the following three affine Kac-Moody algebras, $B_n^{(1)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ under certain specific restrictions on the choice of the subgroup $G^0_0$ (i.e. equivalently the choice of gradation $Q$ and constant grade $\pm 1$ elements $\epsilon_{\pm}$). It turns out that these models are T-selfdual (i.e. the axial and the vector gauging of the $U(1)$ factor in the coset $SL(2) \otimes U(1)^{rank G-1}$ leads to identical actions). They appear to be natural generalization of the Lund-Regge (“complex Sine-Gordon”) model [5] and exactly reproduce the family of models proposed by Fateev [6]. Our construction provide a simple proof of their classical integrability.

It is important to mention that relaxing the structure of the coset $G_0/G^0_0 = SL(2) \otimes U(1)^{rank G-1}$, i.e. gauging specific combinations of the Cartan subalgebra of $G^0_0$ we find two new families of integrable models, axionic (for axial gauging) and torsionless (for vector gauging) for all affine (twisted and untwisted) Kac-Moody algebras which are T-dual (but not self dual) [7].

An important motivation for the construction of the above singular NA Toda models is the fact that their soliton solutions (for imaginary coupling) carries both electric and magnetic (topological) charges and have properties quite similar to the 4-D dyons of the Yang-Mills-Higgs model [7].

The generic NA Toda models are classified according to a $G_0 \subset G$ embedding induced by the grading operator $Q$ decomposing an finite or infinite Lie algebra $G = \oplus_i G_i$ where $[Q, G_i] = iG_i$ and $[G_i, G_j] \subset G_{i+j}$. A group element $g$ can then be written in terms of the Gauss decomposition as

$$g = NBM$$

(1.1)

where $N = \exp G_\prec \in H_-$, $B = \exp G^0_0$ and $M = \exp G_\succ \in H_+$. The physical fields $B$ lie in the zero grade subgroup $G_0$ and the models we seek correspond to the coset $H_+ \backslash G/H_-$.

For consistency with the hamiltonian reduction formalism, the phase space of the $G$-invariant WZNW model is reduced by specifying the constant generators $\epsilon_{\pm}$ of grade $\pm 1$. In order to derive an action for $B \in G_0$, invariant under

$$g \rightarrow g' = \alpha_- g \alpha_+,$$

(1.2)

where $\alpha_{\pm}(z, \bar{z}) \in H_\pm$. we have to introduce a set of auxiliary gauge fields $A \in G_\prec$ and
\[ \tilde{A} \in G > \text{ transforming as} \]
\[ A \longrightarrow A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1}, \quad \tilde{A} \longrightarrow \tilde{A}' = \alpha_+^{-1} \tilde{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+. \quad (1.3) \]

The result is
\[ S_{G/H}(g, A, \tilde{A}) = S_{WZNW}(g) - \frac{k}{2\pi} \int dz^2 \text{Tr} \left( A(\bar{\partial} g g^{-1} - \epsilon_+) + \tilde{A}(g^{-1} \partial g - \epsilon_-) + A g \tilde{A} g^{-1} \right). \]

Since the action \( S_{G/H} \) is \( H \)-invariant, we may choose \( \alpha_- = N_{-1} \) and \( \alpha_+ = M_{+1}^{-1} \). From the orthogonality of the graded subspaces, i.e. \( Tr G_{i,j} = 0, i + j \neq 0 \), we find
\[ S_{G/H}(g, A, \tilde{A}) = S_{G/H}(B, A', \tilde{A}') \]
\[ = S_{WZNW}(B) - \frac{k}{2\pi} \int dz^2 \text{Tr} [A' \epsilon_+ + \tilde{A}' \epsilon_- + A' B \tilde{A} B^{-1}], \quad (1.4) \]

where
\[ S_{WZNW} = -\frac{k}{4\pi} \int d^2 z \text{Tr}(g^{-1} \partial g g^{-1} \partial g) - \frac{k}{2\pi} \int_D \epsilon_{ijk} \text{Tr}(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g), \quad (1.5) \]

and the topological term denotes a surface integral over a ball \( D \) identified as space-time.

Action (1.4) describe the non singular Toda models among which we find the Conformal and the Affine abelian Toda models where \( Q = \sum_{i=1}^r 2 \lambda_i H_{\alpha_i}, \quad \epsilon_+ = \sum_{i=1}^r c_{\pm i} E_{\pm \alpha_i} \) and \( Q = \hbar d + \sum_{i \neq 0} 2 \lambda_i H_{\alpha_i}, \quad \epsilon_+ = \sum_{i=1}^r c_{\pm i} E_{\pm \alpha_i}^{(0)} + E_{\pm \psi}^{(\pm 1)} \) respectively, where \( \psi \) denote the highest root, \( \lambda_i \), the fundamental weights, \( h \) the coxeter number of \( G \) and \( H_i \) are the Cartan subalgebra generators in the Cartan Weyl basis, \( Tr(H_i H_j) = \delta_{ij} \).

More interesting cases arise in connection with non abelian embeddings \( G_0 \subset G \). In particular, if we suppress one of the fundamental weights from \( Q \), the zero grade subspace acquires a nonabelian structure \( sl(2) \otimes u(1)^{rank G - 1} \). Let us consider for instance \( Q = \hbar' d + \sum d_{\neq a} 2 \lambda_i H_{\alpha_i} \) where \( \hbar' = 0 \) or \( \hbar' \neq 0 \) corresponding to the Conformal and Affine nonabelian (NA) Toda respectively. The absence of \( \alpha_0 \) in \( Q \) prevents the contribution of the simple root step operator \( E_{\alpha_0}^{(0)} \) in constructing \( \epsilon_+ \). It in fact, allows for reducing the phase space even further. This fact can be understood by enforcing the nonlocal constraint \( J_{Y,H} = \tilde{J}_{Y,H} = 0 \) where \( Y \) is such that \( [Y \cdot H, \epsilon_+] = 0 \) and \( J = g^{-1} \partial g \) and \( \tilde{J} = -\partial g g^{-1} \). Those generators of \( G_0 \) commuting with \( \epsilon_\pm \) define a subalgebra \( G_0^0 \subset G_0 \). Such subsidiary constraint is incorporated into the action by requiring symmetry under \( G_0^0 \)
\[ g \longrightarrow g' = \alpha_0 g \alpha_0^* \quad (1.6) \]

where we shall consider \( \alpha_0' = \alpha_0(z, \bar{z}) \in G_0^0 \), i.e., axial symmetry (the vector gauging is obtained by choosing \( \alpha_0' = \alpha_0^{-1}(z, \bar{z}) \in G_0^0 \)). Auxiliary gauge fields \( A_0 = a_0 Y \cdot H \) and \( \tilde{A}_0 = \bar{a}_0 \cdot H \in G_0^0 \) are introduced by splitting \( A = A_0 + \tilde{A}_0 \) and \( \tilde{A} = \tilde{A}_0 + \tilde{A}_0 \) transforming as
\[ A \longrightarrow A' = \alpha_0 A \alpha_0^{-1} + \alpha_0 \partial \alpha_0^{-1}, \quad \tilde{A} \longrightarrow \tilde{A}' = \alpha_0^{-1} \tilde{A} \alpha_0 + \bar{\partial} \alpha_0^{-1} \alpha_0, \]
\[ A_0 \rightarrow A'_0 = A_0 + \alpha_0^{-1} \partial \alpha_0, \quad \bar{A}_0 \rightarrow \bar{A}'_0 = \bar{A}_0 + \bar{\partial} \alpha_0 \alpha_0^{-1}. \]

The invariant action, under transformations generated by \( H(-,0) \) (in left) and \( H(+,0) \) (in right) is given by

\[
S_{G/H}(g, A, \bar{A}) = S_{WZNW}(g) - \frac{k}{2\pi} \int d^2z Tr \left( A(\bar{\partial}g g^{-1} - \epsilon_+) \right) \\
+ A(g^{-1} \partial g - \epsilon_-) + A g A g^{-1} + A_0 \bar{A}_0. \tag{1.7}
\]

Notice that the physical fields \( g_0^f \) lie in the coset \( \mathcal{G}_0/\mathcal{G}_0^0 = sl(2) \otimes u(1)^{rank \mathcal{G} - 1}/u(1) \) of dimension \( rank \mathcal{G} + 1 \) and are classified according to the gradation \( Q \). It therefore follows that \( S_{G/H}(g, A, \bar{A}) = S_{G/H}(g_0^f, A', \bar{A}') \).

In \cite{8} a detailed study of the gauged WZNW construction for finite dimensional Lie algebras leading to Conformal NA Toda models was presented. The study of its symmetries was given in refs. \cite{9} and in \cite{10}. Here we generalize the construction of ref. \cite{8} to infinite dimensional Lie algebras leading to NA Affine Toda models characterized by the broken conformal symmetry and by the presence of solitons.

Consider the Kac-Moody algebra \( \hat{\mathcal{G}} \)

\[
[T^a_m, T^b_n] = f^{abc} T^c_{m+n} + \hat{c} m \delta_{m+n} \delta^{ab}
\]

\[
[\hat{d}, T^a_n] = n T^a_n; \quad [\hat{c}, T^a_n] = [\hat{c}, \hat{d}] = 0 \tag{1.8}
\]

The NA Toda models we shall be constructing are associated to gradations of the type \( Q_a(h') = h'd + \sum_{i \neq a} \frac{2\lambda_i H}{\alpha_i^+} \), where \( h' \) is chosen such that the zero grade subalgebra \( \mathcal{G}_0 \) defined by \( Q_a(h') \), coincide with the zero grade subalgebra \( \mathcal{G}_0 \) defined by \( Q_a(h' = 0) \) (apart from two commuting generators \( \hat{c} \) and \( \hat{d} \)). Since they commute with \( \mathcal{G}_0 \), the kinetic part decouples such that the conformal and the affine singular NA-Toda models differ only by the potential term. Due to the specific graded structure of the algebra, the following trace properties hold

\[
\text{Tr} A \partial g_0^f(g_0^f)^{-1} = \text{Tr} A_0 \bar{\partial} g_0^f(g_0^f)^{-1}, \quad \text{Tr} \bar{A}(g_0^f)^{-1}\partial g_0^f = \text{Tr} \bar{A}_0(g_0^f)^{-1}\partial g_0^f \]

\[
\text{Tr} A g_0^f A'(g_0^f)^{-1} = \text{Tr} A_0 g_0^f A(g_0^f)^{-1} + \text{Tr} A_- g_0^f \bar{A}_+(g_0^f)^{-1}. \]

Henceforth the action decomposes into three parts, i. e.,

\[
S_{G/H} = S_{WZNW}(g_0^f) + F_0 + F_\pm, \tag{1.9}
\]

where

\[
F_0 = -\frac{k}{2\pi} \int d^2z Tr \left( A_0 \bar{\partial}g_0^f(g_0^f)^{-1} + \bar{A}_0(g_0^f)^{-1}\partial g_0^f + A_0 g_0^f \bar{A}_0(g_0^f)^{-1} + A_0 \bar{A}_0 \right), \tag{1.10}
\]

\[
F_\pm = -\frac{k}{2\pi} \int d^2z Tr \left( A_- \epsilon_+ + \bar{A}_+ \epsilon_- + A_- g_0^f \bar{A}_+(g_0^f)^{-1} \right),
\]

\footnote{We are considering \( \mathcal{G}' \) to be semisimple and therefore \( a \) to be one of the end points of the Dynkin diagram of \( \mathcal{G} \), otherwise the model decomposes in two abelian Toda models coupled by \( \psi \) and \( \chi \).}
and the functional integral now factorizes into

$$Z = \int DA_0 D\tilde{A}_0 \exp(-F_0) \int DA_- D\tilde{A}_+ \exp(-F_\pm). \quad (1.11)$$

The integration over the auxiliary gauge fields $A$ and $\tilde{A}$ require explicit parametrization of $B$.

$$B = \exp(\chi E_{-a_a}) \exp(RY^j h_j + \Phi(H) + \nu \dot{c} + \eta \dot{d}) \exp(\bar{\psi} E_{a_a}) \quad (1.12)$$

where $\Phi(H) = \sum_{i=1}^{r-1} \varphi_i X^j_i / h_i + \sum_{j=1}^2 Y^j X^j_i = 0$ and $h_i = \frac{2a_i h}{\alpha^2}, i = 1, \ldots, r-1$. After gauging away the nonlocal field $R$, the factor group element becomes

$$g_0^f = \exp(\chi E_{-a_a}) \exp(\Phi(H) + \nu \dot{c} + \eta \dot{d}) \exp(\bar{\psi} E_{a_a}) \quad (1.13)$$

where $\chi = \tilde{\chi} e^{\frac{1}{2}Y^{\cdot a_a}}, \psi = \tilde{\psi} e^{\frac{1}{2}Y^{\cdot a_a}}$. We therefore get for the zero grade component

$$F_0 = -\frac{k}{2\pi} \int \left( a_0 \bar{a}_0 2Y^2 \Delta + 2 \left( \frac{\alpha_a \cdot Y}{\alpha^2} \right) (a_0 \psi \bar{\partial} \chi + a_0 \chi \bar{\partial} \psi) e^{\Phi(a_a)} \right) d^2x \quad (1.14)$$

where $\Delta = 1 + \frac{(Y^{\cdot a_a})^2}{2Y^2} \psi \chi e^{\Phi(a_a)}$ and

$$F_\pm = -\frac{k}{2\pi} \int \left( Tr(A_- - g_0^f \epsilon_-(g_0^f)^{-1}) g_0^f (\tilde{A}_+ - (g_0^f)^{-1} \epsilon_+ g_0^f) (g_0^f)^{-1} \right) d^2x \quad (1.15)$$

The effective action is obtained by integrating over the auxiliary fields $A_0, \tilde{A}_0, A_-$ and $\tilde{A}_+$,

$$Z_0 = \int DA_0 D\tilde{A}_0 \exp(F_0) \sim e^{-S_0} \quad (1.16)$$

where $S_0 = -\frac{k}{\pi} \int \frac{\sqrt{\psi \bar{\psi} \chi \chi}}{2Y^2} e^{2\Phi(a_a)}.$ Also,

$$Z_\pm = \int DA_- D\tilde{A}_+ \exp(F_\pm) \sim e^{-\frac{k}{2\pi} \int Tr \epsilon_-(g_0^f)^{-1}}$$

The total action (1.17) is therefore given as

$$S = -\frac{k}{4\pi} \int \left( Tr(\partial \Phi(H) \bar{\partial} \Phi(H)) + \frac{2 \partial \bar{\psi} \partial \chi}{\Delta} e^{\Phi(a_a)} + \partial \eta \bar{\partial} \nu + \partial \nu \bar{\partial} \eta - 2 Tr(\epsilon_+ g_0^f \epsilon_-(g_0^f)^{-1}) \right) \quad (1.18)$$

Note that the second term in (1.18) contains both symmetric and antisymmetric parts:

$$\frac{e^{\Phi(a_a)}}{\Delta} \partial \bar{\psi} \partial \chi = \frac{e^{\Phi(a_a)}}{\Delta} (g_{\mu \nu} \partial_\mu \psi \partial_\nu \chi + \epsilon_{\mu \nu} \partial_\mu \bar{\psi} \partial_\nu \chi), \quad (1.19)$$

where $g_{\mu \nu}$ is the 2-D metric of signature $g_{\mu \nu} = diag(1, -1), \partial = \partial_0 + \partial_1, \bar{\partial} = \partial_0 - \partial_1$. For $n = 1$ ($\mathcal{G} \equiv A_1$, $\Phi(\alpha_1)$ is zero) the antisymmetric term is a total derivative:

$$\epsilon_{\mu \nu} \partial_\mu \psi \partial_\nu \chi = \frac{1}{1 + \psi \chi} \frac{1}{2} \epsilon_{\mu \nu} \partial_\mu \left( \ln \{1 + \psi \chi \} \partial_\nu \ln \chi \psi \right), \quad (1.20)$$
where the "affine potential" \( J \) (subsidiary nonlocal constraint and it can be neglected. This \( A_1 \)-NA-Toda model (in the conformal case), is known to describe the 2-D black hole solution for (2-D) string theory \[1\]. The \( G_n \)-NA conformal Toda model can be used in the description of specific (n+1)-dimensional black string theories \[12\], with n-1-flat and 2-non flat directions \((g^{\mu\nu}G_{ab}(X)\partial_\mu X^a \partial_\nu X^b, X^a = (\psi, \chi, \varphi_i))\), containing axions \((\epsilon_{\mu\nu}B_{ab}(X)\partial_\mu X^a \partial_\nu X^b)\) and tachyons \((\exp \{-k_{ij}\varphi_j\})\), as well.

It is clear that the presence of the \( \epsilon^{(\alpha_a)} \) in \((1.18)\) is responsible for the antisymmetric tensor generating the axionic terms. On the other hand, notice that \( \Phi(\alpha_a) \) depend upon the subsidiary nonlocal constraint \( J_{Y;H} = J_{Y;H} = 0 \) and hence upon the choice of the vector \( Y \). It is defined to be ortononal to all roots contained in \( \epsilon_\perp \).

In ref. \[3\], the most general constant grade one element \( \epsilon_+ \) was analysed and the precise condition for the absence of axions, i.e. \( \Phi(\alpha_a) = 0 \), was established determining a subclass of torsionless conformal NA singular Toda models (no torsion theorem). For finite dimensional Lie algebras, it was shown in ref. \[8\] that the absence of axions can only occur for \( G_n = B_n \), \( a = n \) and and \( \epsilon_\perp = \sum_{i=1}^{n-2} c_i E_{\pm \alpha_i} + d_\pm E_{\pm (\alpha_n + \alpha_{n-1})} \). In such case, \( G_0^0 \) is generated by \( Y \cdot H = (2\alpha_n - \frac{2\alpha_{n-1}}{\alpha_{n-1}}) \cdot H \) and \( \Phi(H) = \sum_{i=1}^{n-2} \varphi_i h_i + \varphi_0 (\alpha_{n-1} + \alpha_n) \cdot H \). Due to the root structure of \( B_n \), we verify that \( \Phi(\alpha_a) = \alpha_a \cdot (\alpha_{n-1} + \alpha_n) \varphi_0 = 0 \)

In extending the no torsion theorem to infinite affine Lie algebras, \( h' \) is chosen by defining the gradation \( Q_0(h') \) such that preserves \( G_0 \). We consider \( \epsilon_+ = \epsilon_+ + E_\psi^{(1)} \) where \( \psi \) is the highest root of \( G \). Since conformal and the affine models differ only by the potential term, the solution for the no torsion condition is also satisfied for infinite dimensional algebras, whose Dynkin diagram possess a \( B_n \)-"tail like". An obvious solution is the untwisted \( B_n^{(1)} \) model. Two other solutions were found within the twisted affine Kac-Moody algebras \( A_{2n}^{(2)} \) and \( D_{n+1}^{(2)} \) as we shall describe in detail.

2 The \( B_n^{(1)} \) Torsionless NA Toda model

Let \( Q = 2(n - 1)d + \sum_{i=1}^{n-2} \frac{2\alpha_i H}{\alpha_i} \) decomposing \( B_n^{(1)} \) into graded subspaces. In particular \( G_0 = SL(2) \otimes U(1)^{n-1} \otimes U(1), e \otimes U(1), \hat{d} \) generated by \( \{ E^{(0)}_{\pm \alpha_i}, h_1, \ldots, h_n, \hat{c}, \hat{d} \} \). Following the no torsion theorem of ref. \[6\], we have to choose \( \epsilon_\perp = \sum_{i=1}^{n-2} c_i E_{\pm \alpha_i} + c_{(n-1)} E_{\pm (\alpha_n + \alpha_{n-1})} + c_0 E_{\pm (1)} \), where \( \psi = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) \) is the highest root of \( B_n \) and \( G_0^0 \) is generated by \( Y \cdot H = (2\alpha_n - 2\frac{\alpha_{n-1}}{\alpha_{n-1}}) \cdot H \) such that \( [Y \cdot H, \epsilon_\perp] = 0 \). The coset \( G_0/G_0^0 \) is then parametrized according to \((1.13)\) with \( \Phi(H) = \sum_{i=1}^{n-1} \mathcal{H}_i \varphi_i + \eta \hat{h} + \nu \hat{c} \) where \( \mathcal{H}_i = (\alpha_n + \cdots + \alpha_i) \cdot H \) so that \( \text{Tr}(\mathcal{H}_i \mathcal{H}_j) = \delta_{ij}, i, j = 1, \ldots, n - 1 \) and the total effective action becomes

\[
S = -\frac{k}{4\pi} \int d^2x \left( \frac{1}{4} \sum_{i=1}^{n-1} g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i + g^{\mu\nu} \frac{\partial_\mu \psi \partial_\nu \chi}{1 + \psi \chi} + \frac{1}{2} g^{\mu\nu} \partial_\mu \nu \partial_\nu \eta - 2V \right) \tag{2.21}
\]

where the "affine potential" \((n > 2)\) is

\[
V = \sum_{i=1}^{n-2} |c_i|^2 \psi_{i-1} \varphi_{i+1} + |c_{n-1}|^2 2(1 + 2\psi \chi) e^{-\varphi_{n-1}} + |c_n|^2 e^{\varphi_1 + \varphi_2 - \eta} \tag{2.22}
\]
The action (2.21) is invariant under conformal transformation
\[ z \rightarrow f(z), \quad \bar{z} \rightarrow g(\bar{z}), \quad \psi \rightarrow \psi, \quad \chi \rightarrow \chi, \]
\[ \varphi_s \rightarrow \varphi_s + s \ln f'g'; \quad s = 1, 2, \ldots, n - 1; \quad \eta \rightarrow \eta + 2(n - 1) \ln f'g' \quad (2.23) \]

We should point out that the \( \eta \) field plays a crucial role in establishing the conformal invariance of the theory. Integrable deformation of such class of theories can than, be sistematically obtained by setting \( \eta = 0 \).

For the case \( n = 2 \) we choose, \( \hat{\epsilon}_\pm = E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}, \Phi(\alpha_{n-1}) = \varphi \), i.e. \( \hat{G} = \tilde{SO}(5) \), is also special in the sense that its complexified theory, i.e.
\[ \psi \rightarrow i\psi; \quad \chi \rightarrow i\chi^*; \quad \varphi \rightarrow i\varphi \]
leads to the real action
\[ S = -\frac{k}{4\pi} \int d^2x \left( \frac{g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi^*}{(1 - \psi \psi^*)} + \frac{1}{4}g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 8(1 - 2\psi \psi^*) \cos \varphi \right) \quad (2.24) \]

3 The twisted NA Toda Models

The twisted affine Kac-Moody algebras are constructed from a finite dimensional algebra possessing a nontrivial symmetry of their Dynkin diagrams (folding). Such symmetry can be extended to the algebra by an outer automorphism \( \sigma \) \[13\], as
\[ \sigma(E_\alpha) = \eta_\alpha E_{\sigma(\alpha)} \quad (3.25) \]
where \( \eta_\alpha = \pm 1 \). For the simple roots, \( \eta_{\alpha_i} = 1 \). The signs can be consistently assign to all generators since nonsimple roots can be written as sum of two roots other roots.

The no torsion theorem require a \( B_n \)-“tail like” structure which is fulfilled only by the \( A_{2n}^{(2)} \) and \( D_{n+1}^{(2)} \) (see appendix N of ref. \[13\]). In both cases the automorphism is of order 2 (i.e. \( \sigma^2 = 1 \)).

Let us denote by \( \alpha \) the roots of the untwisted algebra \( \mathcal{G} \). For the \( A_{2n}^{(2)} \) case, the automorphism is defined by
\[ \sigma(\alpha_1) = \alpha_{2n}, \quad \sigma(\alpha_2) = \alpha_{2n-1} \cdots, \sigma(\alpha_{n-1}) = \alpha_n \quad (3.26) \]
whilst for the \( D_{n+1}^{(2)} \), the automorphism acts only in the “fish tail” of the Dynkin diagram of \( D_{n+1} \), i.e.
\[ \sigma(E_{\alpha_1}) = E_{\alpha_1}, \quad \cdots, \sigma(E_{\alpha_{n-1}}) = E_{\alpha_{n-1}}, \quad \sigma(E_{\alpha_{n}}) = E_{\alpha_{n+1}} \quad (3.27) \]

The automorphism \( \sigma \) decomposes the algebra \( \mathcal{G} = \mathcal{G}_{\text{even}} \cup \mathcal{G}_{\text{odd}} \). The twisted affine algebra is constructed from \( \mathcal{G} \) assigning an affine index \( m \in Z \) to the generators in \( \mathcal{G}_{\text{even}} \) while \( m \in Z + \frac{1}{2} \) to those in \( \mathcal{G}_{\text{odd}} \) (see appendix N of \[13\]).

The simple root step operators for \( A_{2n}^{(2)} \) are
\[ E_{\beta_1} = E_{\alpha_1}^{(0)} + E_{\alpha_{2n-1}}^{(0)}, \quad i = 1, \cdots, n \quad E_{\beta_0} = E_{-\alpha_1-\cdots-\alpha_{2n}}^{(4)} \quad (3.28) \]
corresponding to the simple and highest roots

\[ \beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i+1}) \quad i = 1, \ldots, n, \quad \psi = \alpha_1 + \cdots + \alpha_{2n} = 2(\beta_1 + \cdots \beta_n) \]  

(3.29)

respectively.

For \( D_n^{(2)} \), simple root step operators are

\[ E_{\beta_i} = E_{\alpha_i}^{(0)}, \quad i = 1, \ldots, n - 1, \quad E_{\beta_n} = E_{\alpha_n}^{(0)} + E_{\alpha_{n+1}}^{(0)} \]

\[ E_{\beta_0} = E_{-\alpha_1-\cdots-\alpha_{n-1}-\alpha_{n+1}}^{(\pm 1)} - E_{-\alpha_1-\cdots-\alpha_{n-1}-\alpha_n}^{(\pm 1)} \]  

(3.30)

corresponding to the simple and highest roots

\[ \beta_i = \alpha_i \quad i = 1, \ldots, n - 1, \quad \beta_n = \frac{1}{2}(\alpha_n + \alpha_{n+1}), \]

\[ \psi = \alpha_1 + \cdots + \alpha_{n-1} + \frac{1}{2}(\alpha_n + \alpha_{n+1}) = \beta_1 + \cdots \beta_n \]  

(3.31)

where have denoted by \( \beta \) the roots of the twisted (folded) algebra.

The torsionless affine NA Toda models are defined by

\[ Q = 2(2n - 1)\hat{d} + \sum_{i \neq n, n+1} \frac{2\lambda_i \cdot H}{\alpha_i^2} \]  

(3.32)

and

\[ Q = (2n - 2)\hat{d} + \sum_{i=1}^{n} \frac{2\lambda_i \cdot H}{\alpha_i^2} \]  

(3.33)

for \( A_{2n}^{(2)} \) and \( D_{n+1}^{(2)} \) respectively, where \( \lambda_i \) are the fundamental weights of the untwisted algebra \( G \), i.e. \( \frac{2\lambda_i \cdot \alpha_j}{\alpha_j} = \delta_{ij} \).

Both models are specified by the constant grade \( \pm 1 \) operators \( \epsilon_\pm \)

\[ \epsilon_\pm = \sum_{i=1}^{n-2} c_{\pm i} E_{\pm \beta_i} + c_{\pm(n-1)} E_{\pm(\beta_{n-1} + \beta_n)} + c_{\pm n} E_{\pm \beta_0} \]  

(3.34)

where \( \beta_i \) are the simple roots of the twisted affine algebra specified in (3.29) and in (3.31).

According to the grading generators (3.32) and (3.33), the zero grade subalgebra is in both cases \( G_0 = SL(2) \otimes U(1)^{n-1} \otimes U(1)_{\hat{c}} \otimes U(1)_{\hat{d}} \) generated by \( \{ E_{\pm \beta_n}^{(0)}, h_1, \cdots, h_n, \hat{c}, \hat{d} \} \).

Hence the zero grade subgroup is parametrized as in (1.12) where we have taken \( \eta = 0 \), responsible for breaking the conformal invariance. The factor group is given in (1.13), where \( G_0^0 \) is generated by \( Y \cdot H = (2\mu_n \beta_n - \frac{\nu_n}{\mu_n}) \cdot H \) and \( \mu_i \) are the fundamental weights of the twisted algebra i.e. \( \frac{2\mu_i \beta_i}{\alpha_i} = \delta_{ij} \) In order to decouple the \( \phi_i, \quad i = 1, \cdots, n - 1 \) we chose an orthonormal basis for the Cartan subalgebra, i.e. \( \Phi(H) = \mathcal{H}_i \phi_i + \eta \hat{h} + \nu \hat{c} \) where

\[ \mathcal{H}_i = (\alpha_i + \cdots + \alpha_{2n-i+1}) \cdot H, \quad Y \cdot H = \mathcal{H}_n, \quad Tr(\mathcal{H}_i \mathcal{H}_j) = 2\delta_{ij}, \quad i, j = 1, \cdots n \]  

(3.35)
and
\[ H_i = (\alpha_{n-i+1} + \cdots + \alpha_{n+1}) \cdot H, \quad Y \cdot H = \mathcal{H}_n, \quad Tr(\mathcal{H}_i \mathcal{H}_j) = \delta_{ij}, \quad i, j = 1, \cdots n \] (3.36)
for \( A_{2n}^{(2)} \) and \( D_{n+1}^{(2)} \) respectively.

The Lagrangean density is obtained from (1.18) leading to
\[
\mathcal{L}_{A_{2n}^{(2)}} = \frac{\partial \chi \bar{\partial} \psi}{1 + \frac{1}{2} \psi \chi} + \frac{1}{2} \sum_{i=1}^{n-1} \partial \varphi_i \bar{\partial} \varphi_i - V_{A_{2n}^{(2)}}
\] (3.37)
and
\[
\mathcal{L}_{D_{n+1}^{(2)}} = 2 \frac{\partial \chi \bar{\partial} \psi}{1 + \psi \chi} + \frac{1}{2} \sum_{i=1}^{n-1} \partial \varphi_i \bar{\partial} \varphi_i - V_{D_{n+1}^{(2)}}
\] (3.38)
where
\[
V_{A_{2n}^{(2)}} = \sum_{i=1}^{n-2} |c_i|^2 e^{-\varphi_i + \varphi_{i+1}} + \frac{1}{2} |c_n|^2 e^{2\varphi_1} + |c_{n-1}|^2 e^{-\varphi_{n-1}} (1 + \psi \chi)
\] (3.39)
and
\[
V_{D_{n+1}^{(2)}} = \sum_{i=1}^{n-2} |c_i|^2 e^{-\varphi_i + \varphi_{i+1}} + \frac{1}{2} |c_n|^2 e^{\varphi_1} + |c_{n-1}|^2 e^{-\varphi_{n-1}} (1 + 2\psi \chi)
\] (3.40)

The models described by (2.21), (3.37) and (3.38) coincide with those proposed by Fateev in [6].

4 Zero Curvature

The equations of motion for the NA Toda models are known to be of the form [14]
\[
\bar{\partial}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0, \quad \partial(\bar{\partial}BB^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0
\] (4.41)
The subsidiary constraint \( J_{Y,H} = Tr(B^{-1}\partial BY \cdot H) = \bar{J}_{Y,H} = Tr(\bar{\partial}BB^{-1}Y \cdot H) = 0 \) can be consistently imposed since \([Y \cdot H, \epsilon_+] = 0\) as can be obtained from (1.41) by taking the trace with \( Y \cdot H \). Solving those equations for the nonlocal field \( R \) yields,
\[
\partial R = \left( \frac{Y \cdot \alpha_n}{Y^2} \right) \frac{\psi \partial \chi}{\Delta} e^{\varphi(\alpha_n)}, \quad \bar{\partial} R = \left( \frac{Y \cdot \alpha_n}{Y^2} \right) \frac{\chi \partial \psi}{\Delta} e^{\varphi(\alpha_n)}
\] (4.42)
The equations of motion for the fields \( \psi, \chi \) and \( \varphi_i, i = 1, \cdots, n-1 \) obtained from (1.41) imposing the constraints (1.42) coincide precisely with the Euler-Lagrange equations derived from (3.37) and (3.38). Alternatively, (4.41) admits a zero curvature representation \( \partial \bar{A} - \bar{\partial} A + [A, A] = 0 \) where
\[
A = \epsilon_- + B^{-1}\partial B, \quad \bar{A} = -B^{-1}\epsilon_+ B
\] (4.43)
Whenever the constraints (1.42) are incorporated into \( A \) and \( \bar{A} \) in (1.43), equations (4.41) yields the zero curvature representation of the NA singular Toda models. Such argument is valid for all NA Toda models, in particular for the torsionless class of models discussed in the previous two sections.
Using the explicit parametrization of $B$ given in (3.34), (3.29) and (3.31) together with (4.42) where $Y$ is given in (3.33) and (3.36), we find, in a systematic manner, the following form for $A$ and $\tilde{A}$

\[
A_{n+2} = \sum_{i=1}^{n-2} c_i (E^{(0)}_{\alpha} + E^{(0)}_{-\alpha_{n+1}+\alpha_n+2}) + c_{n-1} (E^{(0)}_{\alpha-n-\alpha_n} + E^{(0)}_{-\alpha_{n+1}+\alpha_n})
+ c_n E^{(\frac{n}{2})}_{\alpha_1+\cdots+\alpha_2} + \partial \psi e^{-\frac{1}{2}R} (E^{(0)}_{\alpha} + E^{(0)}_{\alpha+1}) + \sum_{i=1}^{n-1} \partial \phi_i \mathcal{H}_i
+ \frac{\partial \chi}{\Delta} e^{\frac{1}{2}R} (E^{(0)}_{-\alpha} + E^{(-\alpha+1)})
\]

and

\[
\tilde{A}_{n+2} = \sum_{i=1}^{n-2} c_i e^{-\phi_i+\phi_{i+1}} E^{(0)}_{\alpha_i} + c_{n-1} e^{-\phi_n-1} (E^{(0)}_{\alpha_n} + E^{(0)}_{\alpha_{n+1}})
+ c_n e^{-\frac{1}{2}R} (E^{(0)}_{\alpha_{n+1}+\alpha_n} - E^{(0)}_{\alpha_{n+2}+\alpha_n+1}) + \partial \psi e^{-\frac{1}{2}R} (E^{(0)}_{\alpha} + E^{(0)}_{\alpha+1})
+ \sum_{i=1}^{n-1} \partial \phi_i \mathcal{H}_i + \frac{\partial \chi}{\Delta} e^{\frac{1}{2}R} (E^{(0)}_{-\alpha} + E^{(-\alpha+1)})
\]
\[-A^{(1)}_{B_n} = \sum_{i=1}^{n-2} c_i e^{-\varphi_i + \varphi_{i+1}} E^{(0)}_{\alpha_i} + c_n e^\varphi_1 + \varphi_2 E^{(1)}_{-(\alpha_1 - 2(\alpha_2 + \ldots + \alpha_n))} + 2 \chi e^{\varphi_{n-1} + \frac{1}{2} R} E^{(0)}_{\alpha_{n-1}} + c_{n-1} (1 + 2 \psi \chi) e^{\varphi_{n-1}} E^{(0)}_{\alpha_{n-1} + \alpha_n} - 2 c_n e^{-\varphi_{n-1}} - \frac{1}{4} R \psi (1 + \psi \chi) E^{(0)}_{\alpha_{n-1} + 2 \alpha_n} (4.49)\]

The zero curvature representation of such subclass of torsionless NA Toda models shows that they are in fact classically integrable field theories. The construction of the previous sections provides a systematic affine Lie algebraic structure underlying those models.

5 Conclusions

We have constructed a class of affine NA Toda models from the gauged two-loop WZNW models in which left and right symmetries are incorporated by a suitable choice of grading operator $Q$. Such framework is specified by grade $\pm 1$ constant generators $\epsilon_{\pm}$ and the pair $(Q, \epsilon_{\pm})$ determines the model in terms of a zero grade subgroup $G_0$. We have shown that for non-abelian $G_0$, it is possible to reduce even further the phase space by constraining to zero the currents commuting with $\epsilon_{\pm}$, $(J \in G_0^0)$ to the fields lying in the coset $G_0/G_0^0$ only. Moreover, we have found a Lie algebraic condition which defines a class of $T$-selfdual torsionless models, for the case $G_0^0 = U(1)$. The action for those models were systematically constructed and shown to coincide with the models proposed by Fateev [6], describing the strong coupling limit of specific 2-d models representing sine-Gordon interacting with Toda-like models. Their weak coupling limit appears to be the Thirring model coupled to certain affine Toda theories [8].

Following the same line of arguments of the previous sections, one can construct more general models, say, $G_0/G_0^0 = \frac{SL(2) \otimes U(1)^{n-1}}{U(1)^n}$, $G_0/G_0^0 = \frac{SL(2) \otimes SL(2) \otimes U(1)^{n-2}}{U(1)^n}$, $G_0/G_0^0 = \frac{SL(3) \otimes U(1)^{n-2}}{U(1)^n}$, etc. Those models represent more general NA affine Toda models obtained by considering specific gradations $Q_{a,b,\ldots} = h_{a,b,\ldots} + \sum_{i \neq a,b,\ldots} \frac{2\lambda_i H}{\alpha_i^2}$. However the important problem of the classification of all integrable models obtained as gauged two loop $G$-WZNW models remains open.

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