On the Ergodic Control of Ensembles

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Abstract

Across smart-grid and smart-city applications, there are problems where an ensemble of agents is to be controlled such that both the aggregate behaviour and individual-level perception of the system’s performance are acceptable. In many applications, traditional PI control is used to regulate aggregate ensemble performance. Our principal contribution in this note is to demonstrate that PI control may not be always suitable for this purpose, and in some situations may lead to a loss of ergodicity for closed-loop systems. Building on this observation, a theoretical framework is proposed to both analyse and design control systems for the regulation of large scale ensembles of agents with a probabilistic intent. Examples are given to illustrate our results.

1 Introductory remarks

At a very high level, smart-city related research concerns designing systems that endeavour to make the best use of limited resources across a number of domains (energy, transport, water etc). While classical control has much to offer in such application areas, there are aspects and peculiarities of many of these applications that require practitioners in Control Theory to explore new types of theoretical and practical challenges. Roughly speaking, classical control is typically concerned with regulating a single system, such that the system behaviour achieves a desired behaviour in an optimal way. In contrast, in many smart city applications, we are interested in allocating a resource among a population of agents. These might be humans or algorithmic processes that bid for access to a resource in some probabilistic manner (for example, access to part of a road network). In such applications, both the experience of the individual, and their aggregate effect are important. Further, classical control is concerned with the
control of systems, whose structure does not vary in time. On the other hand, in
smart-city applications, we typically wish to control and influence the behaviour
of large-scale populations, where the number of agents varies over time and is
not known with certainty. Additionally, there are limits to the observability of
such systems and data sets are often obtained in a closed-loop fashion; that is,
operator’s decisions are often reflected in the data sets. Finally, a fundamental
difference between classical control and smart-grid and smart-city control
is the need to study the effect of control signals on the individual agent and
its long term access to a constrained resource. Among all of these fundamental
differences, it is this last issue that is perhaps most alien to the classical
control theorist, and yet the issue that is perhaps the most pressing in real-life
applications, since the need for predictability, at the level of individual agents,
underpins an operator’s ability to write contracts.

In this paper, our starting point is the observation that many problems that are
considered in smart-grids and smart-cities can be cast in a framework, where a
large number of agents, such as people, cars, or machines, often with unknown
objectives, compete for a limited resource. The challenge of allocating this
resource in a manner that is not wasteful, which gives an optimal return on the
use of the resource for society, and which, in addition, gives a guaranteed level
of service to each of the agents competing for that resource, gives rise to a whole
host of problems, which in principle are best addressed in a control-theoretic
manner.

From the perspective of a control engineer, this statement can be decomposed
into three objectives; two of which are familiar in control, and the other one
constitutes a relatively new consideration. Our first objective is to fully utilise
the resource, which is a regulation problem. Second, we would then like to make
optimal use of the resource. While both of these objectives are concerned with
the aggregate behaviour of an agent population, they make no attempt to control
the manner in which the agents orchestrate their behaviour to achieve this
aggregate effect. Our third objective thus focuses on the effects of the control
on the microscopic properties of the agent population. Ultimately, this third
objective can be phrased in terms of properties of the stochastic process cap-
turing the share of the resource that is allocated to an individual agent. For
example, we may wish that each agent, on average, receives a fair share of the
resource over time, or, at a much more fundamental level, we wish the average
allocation of the resource to each agent over time to be a stable quantity that is
entirely predictable and which does not depend on initial conditions, and which
is not sensitive to noise entering the system. From the point of view of the
design of the feedback system, these latter concerns are related to the existence
of the unique invariant measure that governs the distribution of the resource
amongst the agents in the long run. Thus, the design of feedback systems for
deployment in multi-agent applications must consider not only the traditional
notions of regulation and optimisation, but also the guarantees concerning the
existence of this unique invariant measure. As we shall see, this is not a triv-
ial task and many familiar control strategies, in very simple situations, do not
necessarily give rise to feedback systems which possess all three of these features.

The main purpose of the present paper is to introduce a problem class which is of great interest in the particular application field. For this class, we study the ergodicity properties associated to output regulation problems. We show that the classical output regulation approach of using PI control with an integrator fails to provide ergodic properties for these systems, even when agent behaviour is benign, and despite the fact that regulation is achieved. For stable controllers, we then show that ergodicity of the system holds. This result is the extended to a variety of conditions for nonlinear systems.

The paper is organized as follows. In Section 2, we present a model, which captures the main features of the problems of interest. We then formulate the necessary concepts from the theory of Markov chains we will use, and recall the concept of coupling of invariant measures in order to state a necessary condition for ergodicity. In Section 4, we present a negative result, which shows that ergodicity may fail whenever a standard PI controller is used in the loop. In particular, the amount of a resource that can be used by a particular agent depends on the initial state of the controller. Finally, a positive result is obtained for stable linear controllers and extensions to nonlinear systems are discussed.

Comment: A preliminary version of this paper has appeared in [1]. The present paper extends beyond this preliminary paper in several ways. In particular, full proofs are given in this manuscript, and more extensive simulations are presented. Furthermore, positive results are developed for both linear and nonlinear systems.

2 Preliminaries

We now develop the general setting of this paper. The objective here is to set out our modelling framework, and to present basic results that can be useful in studying the properties of control strategies for ensembles.

2.1 Models

We consider the problem of repeatedly distributing a limited resource among multiple agents, based on some information concerning the resource, which are provided by a central authority. Throughout, we consider several constraints. First, the central authority does not observe the individual consumption of individual agents, but rather the total utilisation of the resource, or a filtered version thereof. Based on the filtered measurements of the utilisation of the resource, the central authority provides information to the agents, sets the price of utilising the resource, or similar. Second, the agents respond to information broadcast by the central authority, but have only limited communication capability, otherwise. Specifically, we assume no inter-agent communication. Third,
the agents have their own, private objectives. That is, although they receive information from the central authority, they need not pick an action the authority would deem most appropriate. As we shall see, it will be convenient to encode the selfish response of an agent to the information in a probabilistic manner. Finally, the agents may be limited to a choice from a finite set of possible requests for the resource, in our model. In an extreme case, the agents only have the possibility to turn their utilisation on or off, i.e. \( x_i \in \{0,1\} \). In a more general setting, a subpopulation might be able to choose their consumption from a continuous interval or via some local control.

With these constraints in mind, we are interested in the closed loop depicted in Figure 1 comprising a controller, a number \( N \in \mathbb{N} \) of agents, and a filter, in discrete time. A controller \( C \), which represents the central authority, produces a signal \( \pi(k) \) at time \( k \). In response, the agents, modelled by systems \( S_1, S_2, \ldots, S_N \), amend their use of the resource. We model the use \( x_i(k) \) of agent \( i \) at time \( k \) as a random variable, where the randomness can be a result of the inherent randomness in the reaction of user \( i \) to the control signal \( \pi(k) \), or the response to a control signal that is intentionally randomized [2] [3] [4]. The aggregate resource utilisation \( y(k) := \sum_{i=1}^{N} x_i(k) \) at time \( k \) is then also a random variable. The controller may not have access to either \( x_i(k) \) or \( y(k) \), but only to the error signal \( e(k) \), which is the difference of \( \hat{y}(k) \), the output of a filter \( F \), and \( r \), the desired value of \( y(k) \). Further, we assume that the controller has its private state \( x_c(k) \in \mathbb{R}^{n_c} \). The controller aims to regulate the system by providing a signal \( \pi(k) \in \Pi \subseteq \mathbb{R}^{n_s} \) at time \( k \). In the simpler static case, the signal \( \pi(k) \) is hence a function of an error signal \( e(k) \) and the controller state \( x_c(k) \), whose range is \( \Pi \).

The non-deterministic agent-specific response to the feedback signal \( \pi(k) \in \Pi \) can hence be modelled by agent-specific and signal-specific probability distributions over certain agent-specific set of actions \( \mathcal{A}_i = \{a_1, \ldots, a_L\} \subset \mathbb{R}^{n_i} \), where \( \mathbb{R}^{n_i} \) can be seen as the space of agent’s \( i \) private state. Assume that the set of possible resource demands of agent \( i \) is \( \mathcal{R}_i \), where in the case that \( \mathcal{R}_i \) is finite we denote

\[
\mathcal{R}_i := \{r_{i,1}, r_{i,2}, \ldots, r_{i,m_i}\}. \tag{1}
\]

and assume there are \( W_i \in \mathbb{N} \) state transition maps \( w_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}, j = 1, \ldots, W_i \) and \( H_i \in \mathbb{N} \) output maps \( h_{i\ell} : \mathbb{R}^{n_i} \to \mathcal{R}_i, \ell = 1, \ldots, H_i \). The evolution of the states and the corresponding demands then satisfy:

\[
x_i(k+1) \in \{w_{ij}(x_i(k)) \mid j = 1, \ldots, W_i\} \tag{2}
\]

\[
y_i(k) \in \{h_{i\ell}(x_i(k)) \mid \ell = 1, \ldots, H_i\} \tag{3}
\]

where the choice of agent \( i \)’s response at time \( k \) is governed by a probability functions \( p_{ij} : \Pi \to [0,1] \), \( j = 1, \ldots, W_i \), respectively \( p'_{i\ell} : \Pi \to [0,1] \), \( \ell = 1, \ldots, H_i \). Specifically, we have for all \( k \in \mathbb{N} \) and all \( \pi(k) \in \Pi \) that

\[
P(x_i(k+1) = w_{ij}(x_i(k))) = p_{ij}(\pi(k)), \tag{4a}
\]

\[
P(y_i(k) = h_{i\ell}(x_i(k))) = p'_{i\ell}(\pi(k)). \tag{4b}
\]
As \(p_{ij}, p'_{i\ell}\) are probabilities we have the additional constraint that for all \(\pi \in \Pi\) and all \(i\)

\[
\sum_{j=1}^{W_i} p_{ij}(\pi) = \sum_{\ell=1}^{H_i} p'_{i\ell}(\pi) = 1. \tag{4c}
\]

In particular, the probabilistic laws (4) suggest that conditioned on \(\pi(k)\), the random variables \(\{x_i(k+1)|i = 1, \ldots, n\}\), \(\{y_i(k)|i = 1, \ldots, n\}\) are stochastically independent. As we shall see, this general framework allows for surprisingly sharp results.

Our specific aim is to distribute the resource such that we achieve the following goals almost surely, i.e. with probability 1:

1. **Feasibility**: for all \(k \in \mathbb{N}\)

   \[
   \sum_{i=1}^{N} x_i(k) = y(k) \leq r. \tag{5}
   \]

   More generally, the resource could be time-varying; for the purposes of this paper it will assumed to be a constant quantity.

2. **Predictability**: for each agent \(i\) there exists a constant \(\tau_i\) such that

   \[
   \lim_{k \to \infty} \frac{1}{k+1} \sum_{j=0}^{k} x_i(j) = \tau_i, \tag{6}
   \]

   where this latter limit is independent of initial conditions.

Further optional requirements may include: **fairness**, which could be formulated by saying that all the \(\tau_i\) coincide, and **optimality** so that the vector \(\bar{\tau} = [\tau_1 \ldots \tau_N]\) is a local optimum of an underlying optimization problem. In addition, it is also of interest to achieve the goals after a transient phase, i.e. for all \(k \geq K\), where \(K\) is a constant.

While all of these goals are important from a practical perspective, the principal property of interest in this paper is the goal of **predictability**, since this latter issue defines the ability of service providers to write contracts. To consider this more formally, we consider an augmented state space \(\mathcal{X} \subset \mathbb{R}^d\), which captures the state of the controller, the filter, and the agents. Let us consider the space \(M(\mathcal{X})\) of probability measures over \(\mathcal{X}\). The behaviour of the overall system in response to the signal \(\pi(k)\) can be modelled as \(P : \mathcal{X} \times \Pi \to M(\mathcal{X})\). In order to reason about the evolution of the state, we introduce the notion of a path-space. This space, \(\mathcal{X}^\infty\), of one-sided infinite sequences is associated with the space \(M(\mathcal{X}^\infty)\) of probability measures on path-space \(\mathcal{X}^\infty\). Notice that for a particular combination of a filter \(\mathcal{F}\), controller \(\mathcal{C}\), and population \(P\), the feedback
loop can be modelled by an operator $P_k : M(X) \to M(X)$ and the associated dynamical system $(P_k)_{k \in \mathbb{N}}$. Our paper, at some level, asks what properties of $C$ and $\mathcal{F}$ make $(P_k)_{k \in \mathbb{N}}$ predictable and what properties of $C$ render it impossible for it to be predictable. Predictability is related to asymptotic convergence in probability in terms of $M(X^\infty)$ and will be conveniently characterised in the next section in terms of the existence of a unique invariant measure, in the language of Markov chains.

2.2 Markov Chains and Iterated Function Systems

The set-up described in the prequel resembles closely that of an iterated function system [5, 6, 7]. Iterated function systems are a class of stochastic dynamic systems, for which strong stability and convergence results exist. We now introduce some notation and mention some of the most important results.

To begin, let $\Sigma$ be a closed subset of $\mathbb{R}^n$ with the usual Borel $\sigma$-algebra $\mathcal{B}(\Sigma)$. We call the elements of $\mathcal{B}$ events. A Markov chain on $\Sigma$ is a sequence of (\Sigma-valued) random vectors $\{X(k)\}_{k \in \mathbb{N}}$ with the Markov property, that is the probability of an event conditioned on past events is given by conditioning on the previous event, i.e., we always have

$$\mathbb{P}(X(k+1) \in G \mid X(j) = x_j, j = 0, 1, \ldots, k) = \mathbb{P}(X(k+1) \in G \mid X(k) = x_k),$$

where $G$ is an event and $k \in \mathbb{N}$. We assume the Markov chain is time-homogeneous and the transition operator $P$ of the Markov chain is defined

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{feedback_model}
\caption{Feedback model.}
\end{figure}
by
\[ P(x, G) := \mathbb{P}(X(k + 1) \in G \mid X(k) = x). \]
If \( X_0 \) is distributed according to an initial distribution \( \lambda \), we denote by \( \mathbb{P}_\lambda \) the probability measure induced on the space of sequences with values in \( \Sigma \). Conditioned on an initial distribution \( \lambda \), the random variable \( X(k) \) is distributed according to the measure \( \lambda_k \) which is given by
\[ \lambda_{k+1}(G) := \lambda_k P(G) := \int_{\Sigma} P(x, G) \lambda_k(\mathrm{d}x), \quad (7) \]
for \( G \in \mathcal{B} \). A measure \( \mu \) on \( \Sigma \) is called invariant with respect to the Markov process \( \{X(k)\} \) if it is a fixed point for the iteration described by (7), i.e., if \( \mu P = \mu \). The existence of attractive invariant measures is intricately linked to ergodic properties of the system.

### 2.3 Invariant Measures and Ergodicity

A particular class of Markov chains that are of particular interest are the aforementioned iterated function systems (IFSs). In an iterated function system, we are given a set of maps \( \{f_j : \Sigma \to \Sigma \mid j \in \mathcal{J}\} \), where \( \mathcal{J} \) is an index set. Associated to these maps, there are probability functions \( p_j : \Sigma \to [0,1] \) such that
\[ X(k + 1) = f_j(X(k)), \]
with probability \( p_j(X(k)) \).

It is, of course, required that \( \sum_{j \in \mathcal{J}} p_j(x) = 1 \) for all \( x \in \Sigma \). IFSs have been studied to a limited extent in the control community [8, 9], in fractal image compression [10], and in networking [11].

Sufficient conditions for the existence of a unique attractive invariant measure can be given in terms of “average contractivity”. This key notion can be traced back to [3, 6, 7].

**Theorem 1** (Barnsley et al. [6]). Let \( \Sigma \subset \mathbb{R}^n \) be closed. Consider an IFS with a finite index set \( \mathcal{J} \), and Lipschitz maps \( f_j : \Sigma \to \Sigma, j \in \mathcal{J} \). Assume that the probability functions \( p_j \) are Lipschitz continuous and bounded below by \( \eta > 0 \). If there exists a \( \delta > 0 \) such that for all \( x, y \in \Sigma, x \neq y \)
\[ \sum_{j \in \mathcal{J}} p_j(x) \log \left( \frac{\|f_j(x) - f_j(y)\|}{\|x - y\|} \right) < -\delta < 0, \]
then there exists an attractive (and hence unique) invariant probability measure \( \mu \) for the IFS.

We can combine Theorem 1 with a theorem by Elton [5], to obtain that for all (deterministic) initial conditions \( x \in \Sigma \) and continuous \( g : \Sigma \to \mathbb{R} \), the limit
\[ \lim_{k \to \infty} \frac{1}{k+1} \sum_{\nu=0}^{k} g(X(\nu)) = \mathbb{E}_\mu(g) \quad (8) \]
exists almost surely ($\mathbb{P}_{x_0}$) and is independent of $x_0 \in \Sigma$. The limit is given by the expectation with respect to the invariant measure $\mu$. For more general theorems, the reader is referred to [5, 6, 7, 12, 13, 14, 15, 16, 17] and especially two recent surveys [18, 19]. Note that, albeit more general, those conditions are still only sufficient for asymptotic convergence.

**Remark 1.** From the point of view of applications in smart cities, the existence of such a limit (8) is a minimum requirement. We want to avoid situations, where the average allocation of resources to agents depends on their initial conditions, on possible initial conditions of controllers and filters, etc. In addition, it is desirable to shape the expected value so that an overall optimum is obtained. In fact, if Theorem 1 and [5] are applicable, then predictability (cf. Section 2.1 above) holds with probability one and the limit is independent of initial conditions. It is then trivial to ensure that the relevant constants $r_i$ are all positive by making sure that there is a positive probability for positive consumption of each agent. For the question of feasibility (cf. Section 2.1 above) the shaping of the expectation is essential.

There is a vast literature on invariant measures and ergodic properties of stochastic systems. In place of definition of ergodicity, we summarise some of the main results following [20].

**Proposition 1.** The following are equivalent:

- **E1** a probability measure $\pi$ on $\Sigma$ is ergodic.
- **E2** every $\pi$-invariant set is of $\pi$-measure 0 or 1, i.e., every measurable set $A \subset \Sigma^\infty$, which is invariant under $\theta$, satisfies $\mathbb{P}_\pi(A) \in \{0, 1\}$, where a measurable set $\tilde{A} \subset \Sigma$ is $\pi$-invariant if $\mathbb{P}(x, \tilde{A}) = 1$ for $\pi$-almost every $x \in \tilde{A}$.
- **E3** $\pi$ cannot be decomposed as $\pi = t\pi_1 + (1 - t)\pi_2$ with $t \in (0, 1)$ for two invariant measures $\pi_1, \pi_2$.

Additionally, the following implies that $\pi$ is ergodic:

- **F1** Markov process with transition operator $T$ has a unique invariant measure.

Additionally, if $\pi$ is ergodic, the following holds:

- **C1** $\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} g(x(k)) = E_\pi(g)$ almost surely, for $\pi$-almost all initial conditions.

- **C2** any other distinct ergodic invariant measure is singular to $\pi$.

*Proof.* E1 is equivalent to E2 by Corollary 5.11 of [20]. E1 is equivalent to E3 by Theorem 5.7 of [20]. F1 implies E1 by Corollary 5.12 of [20]. E1 implies C1 by Corollary 5.3 of [20]. E1 implies C2 by Theorem 5.7 of [20].
We note that it is possible that multiple ergodic invariant measures exist for a given system. This case is however not of interest for us. We call a system ergodic if it has a unique ergodic invariant measure, such that (8) holds for all initial conditions.

With this background, our general problem considered in this paper is modelled as a Markov chain on a state space representing all the system components. We thus let $X_S = \{(x_i)\}$ be the set of vectors representing the possible values for the agents. The spaces $X_F, X_C$ contain the possible internal states for filter and central controller. Our system thus evolves on the state space $\Sigma := X_S \times X_C \times X_F$.

2.4 Ergodic Invariant Measures and Coupling

We shall be interested in determining control strategies that destroy ergodicity. For this investigation, a discussion of coupling arguments is useful. Specifically, as we shall see coupling arguments provide criteria for the non-existence of a unique invariant measure$^1$.

To formalise this specific discussion, let us denote the space of trajectories of a $\Sigma$-valued Markov chain $\{X(k)\}_{k \in \mathbb{N}}$, i.e., the space of all sequences $(x(0), x(1), x(2), \ldots)$ with $x(k) \in \Sigma, k \in \mathbb{N}$, by $\Sigma^\infty$ (the “path space”). Recall, for example, that $P_\lambda \in M(\Sigma^\infty)$ is the probability measure induced on the path space by the initial distribution $\lambda$ of $X(0)$.

A coupling of two measures $P_{\mu_1}, P_{\mu_2} \in M(\Sigma^\infty)$ is a measure on $\Sigma^\infty \times \Sigma^\infty$ whose marginals coincide with $P_{\mu_1}, P_{\mu_2}$. To be precise, consider $\Gamma \in M(\Sigma^\infty \times \Sigma^\infty)$, i.e., a measure over the product of the two path spaces. Clearly, $\Gamma$ can be projected to the space of measures over one or the other path space $\Sigma^\infty$; we denote the projectors $\Pi^{(1)} \Gamma$ and $\Pi^{(2)} \Gamma$. The set $C(P_{\mu_1}, P_{\mu_2})$ of couplings of $P_{\mu_1}, P_{\mu_2} \in M(\Sigma^\infty)$ is then defined by

$$\{ \Gamma \in M(\Sigma^\infty \times \Sigma^\infty) : \Pi^{(1)} \Gamma = P_{\mu_1}, \Pi^{(2)} \Gamma = P_{\mu_2} \}.$$ 

We say that a coupling $\Gamma$ is an asymptotic coupling if $\Gamma$ has full measure on the pairs of convergent sequences. To make this precise consider the following set denoted $D$:

$$\{ (x_1, x_2) \in \Sigma^\infty \times \Sigma^\infty : \lim_{k \to \infty} \| x_1(k) - x_2(k) \| = 0 \}$$

$\Gamma$ is an asymptotic coupling if $\Gamma(D) = 1$. The following statement is a specialization of [23, Theorem 1.1] to our situation:

$^1$ Coupling arguments have been used since the theorem of Harris [21, 22], and are hence sometimes known as Harris-type theorems. Generally, they link the existence of a coupling with the forgetfulness of initial conditions.
Theorem 2 (Hairer et al. [23]). Let $P$ be a Markov operator admitting two ergodic invariant measures $\mu_1$ and $\mu_2$. The following are equivalent:

(i) $\mu_1 = \mu_2$.

(ii) There exists an asymptotic coupling of $P_{\mu_1}$ and $P_{\mu_2}$.

Consequently, if no asymptotic coupling of $P_{\mu_1}$ and $P_{\mu_2}$ exists, then $\mu_1$ and $\mu_2$ are distinct.

3 Descriptive Examples

To facilitate ease of exposition, we now present some examples of well known systems that can be modelled in our framework.

A. Air-quality management

Atmospheric pollutant levels have risen to a critical level in several cities around the world and pose a threat to human health and well-being. Local governments and environmental agencies have introduced restrictions to internal combustion engine (ICE) vehicles and have also promoted incentives to electric and hybrid vehicles aiming at regulating particulate matter levels in urban areas [2].

New vehicle types are emerging that offer new actuation possibilities that can be used to manage aggregate emissions. For example, using plug-in hybrid vehicles, switching between their operating modes can be orchestrated in a network of vehicles to reduce pollution levels to a desired value.

To see how this can be modelled using our framework, let us consider, for instance, a fleet of hybrid vehicles, where $x_i(k)$ determines whether or not vehicle $i$ switched into electric mode at time $k$ or not. Although measuring the pollution associated to each agent’s actions $x_i(k)$ is difficult, measurements $\hat{y}(k)$ of the aggregate level of pollution in a city are widely available from sensors. Then, based on this measurement, a central agency broadcasts a price signal $\pi(k)$, adjusted via a PI control, for example, based on the difference between a target level of pollution, and a measurement, which is then used by the agents in order to probabilistically decide switching to electric mode or not.

This problem is discussed in detail in [2], in which various control strategies are considered, one of which is based on a classic PID controller. In addition, it is shown how this simple strategy can be used to realise many policy objectives.

B. Interruptible Loads and Smart Grids

Balancing power demand and supply is challenging and expensive, especially at peak times. Traditionally, demand was considered a constant, while the supply
would be varied to match, with the most flexible generators providing the so-called ancillary services to the transmission system. Increasingly, demand-side flexibility is being leveraged as well. As of 2015, which is the most recent year for which data [24] are available, there were more than 9 million customers enrolled in a so-called demand-response management (DRM) programme in the United States. DRM can both provide ancillary services to the transmission system and address reliability needs of the distribution system [25]. For instance, it may be possible not to operate as many diesel generators, if some loads can be interrupted or reduced briefly.

Using our framework, one could consider managing loads [26], such as pool pumps or fans [27, 28] of heating, ventilation, and air conditioning (HVAC) systems, with the aim of regulating the active power to a reference value $r$. For each load $i$, $x_i(k)$ is the amount of active power supplied at time $k$. Although continuous changes to the demand are possible, in theory, many schemes consider a binary notion [26] of interruptibility $x_i(k) \in \{0, P_i\}$ for a fixed real-valued $P_i$, in practice. The aggregate consumption $y(k)$ of the ILs together with the (non-linear) losses can be observed as the total active power demand $\hat{y}$. This is compared with reference $r$ to obtain the error signal $e(k)$. In this context, many researchers [29, 27, 28] propose to use the classic PID controller.

In modelling the response of a load $i$ to signal $\pi(k)$ at time $k$, one could consider interruptible loads, where for some threshold $\pi'_i$, whenever $\pi(k) > \pi'_i$, load $x_i(k) = 0$. In the so-called incentive-based DRM mechanisms, the signal $\pi(k)$ could be seen as a price to pay. Then, the higher the $\pi(k)$ the higher the probability of $x_i(k) = 0$, but this relationship may be non-linear.

4 Controls with poles on the unit circle

To illustrate the importance of the discussion on ergodicity we now present our first main result. In many applications, controllers with integral action, such as the Proportional-Integral (PI) controller, are widely adopted [30, 31]. A simple PI control can be implemented as:

$$\pi(k) = \pi(k - 1) + \kappa [e(k) - \alpha e(k - 1)],$$

which means its transfer function from $e$ to $\pi$ is given by

$$C(z) := \frac{\hat{\pi}(z)}{\hat{e}(z)} = \kappa \frac{1 - \alpha z^{-1}}{1 - z^{-1}}.$$  

Since this transfer function is not asymptotically stable, any associated realisation matrix will not be Schur. Note that this is the case for any controller with any sort of integral action, i.e., pole at $z = 1$. 

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Theorem 3. Consider $N$ agents with states $x_i, i = 1, \ldots, N$. Assume that there is an upper bound $L$ on the different values the agents can attain, i.e., for each $i$ we have $x_i \in A_i = \{a_1, \ldots, a_{L_i}\} \subset \mathbb{R}$ for a given set $A_i$ and $1 \leq L_i \leq L$.

Consider the feedback system in Figure 1, where $F : y \mapsto \hat{y}$ is a finite-memory moving-average (FIR) filter. Assume the controller $C_L$ is a linear marginally stable single-input single-output (SISO) system with a pole $s_1 = e^{aq\pi}$ on the unit circle where $q$ is a rational number. In addition, let the probability functions $p_{ij} : \mathbb{R} \to [0, 1]$ be continuous for all $i = 1, \ldots, N, j = 1, \ldots, M_i$, i.e., if $\pi(k)$ is the output of $C_L$ at time $k$, then $P(x_i(k+1) = a_j) = p_{ij}(\pi(k))$. Then the following holds.

(i) The set $O_F$ of possible output values of the filter $F$ is finite.

(ii) If the real additive group $E$ generated by $\{r - \hat{y} \mid \hat{y} \in O_F\}$ is discrete, then the closed-loop system cannot be ergodic.

Remark 2. One implication of the theorem it that it is perfectly possible for the closed loop both to perform its regulation function well and to destroy the ergodic properties of the closed loop at the same time. Both the classical performance of the closed loop in terms of regulation and the ergodic behaviour need to be studied.

Remark 3. As we have stressed in Remark 1, the existence of a unique invariant measure is only a baseline ergodic property. Under mild but technical conditions, one could prove moment bounds [16] and a geometric rate of convergence [15]. Such results concerning the “shape” of the unique invariant measure and its changes, where it exists, would complement the present theorem.

Proof. (i) By assumption, the states of the agents $x \in \mathbb{R}^N$ can only attain finitely many values. Consequently, the set of possible values of $y$ is finite, and thus also the set of possible outputs of the filter is finite, as it is just the moving average over a history of finite length.

(ii) We denote by $E$ the additive subgroup of $\mathbb{R}$ generated by the filter outputs. By (i), the set of possible inputs to the linear part of the controller is finite at any time $k \in \mathbb{N}$. Let $(A, B, C)$ be a minimal realization of the linear controller with $A \in \mathbb{R}^{n_c \times n_c}, B, C^T \in \mathbb{R}^{n_c}$. Without any loss of generality, assume that

$$A = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$  

Here $Q$ is equal to 1, $-1$ or a $2 \times 2$ orthogonal matrix with the eigenvalues $s_1$ and $s_2$. The matrix $R$ is marginally Schur stable. We will concentrate on the first (or first two) component of the state of the controller, which we denote by $x^{(1)}$. Given an initial value $x_0^{(1)}$, these states are given by

$$x^{(1)}(k) = Q^k x_0^{(1)} + \sum_{\nu=0}^{k-1} Q^{k-\nu-1} B_1 e(\nu),$$  

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where the sequence $e(0), e(1), \ldots$ represents the output of the filter. For some power $K \geq 1$ we have by assumption that $Q^K = I_2$. We may thus rearrange the sums and just consider finitely many powers of $Q$. This induces a further summation over a subsequence of \{e(\nu)\}, which by construction lies in $E$. Thus $x^{(1)}(k)$ is an element of the set $Z(x_0)$ given by

$$\left\{ Q^k x^{(1)}(0) + \sum_{\nu=0}^{K-1} Q^\nu B_1 e_\nu \mid k = 0, \ldots, K-1, e_\nu \in E \right\}.$$ 

By assumption, this set is discrete in $\mathbb{R}$ or $\mathbb{R}^2$, as the case may be. The state space of the controller may thus be partitioned into the uncountably many equivalence classes under the equivalence relation on $\mathbb{R}^{n_c}$ given by $x \sim y$, if $y^{(1)} \in Z(x)$. These are invariant under the evolution of the Markov chain. Ergodic invariant measures that are concentrated on different equivalence classes cannot couple asymptotically, as the respective trajectories remain a positive distance apart. By Theorem 2, the Markov chain cannot be ergodic.

While the conditions of the previous result appear fairly abstract, we like to point out that they apply in many practical settings. In an implementation using standard digital computers all constants appearing in the system description are rational numbers. It is therefore of interest to observe that in this case the above theorem applies, as we note in the next result.

**Corollary 1.** In the situation of Theorem 3, assume that $A_i \subset Q$ for all $i = 1, \ldots, N$. Assume furthermore that $r \in Q$ and that the coefficients of the FIR filter $F$ are rational. Then the group $E$ is discrete. If the linear controller satisfies the assumptions of Theorem 3, the closed-loop system cannot be ergodic.

**Proof.** As $Q$ is a field, it is easy to see that the set $O_F$ is contained in $Q$. Indeed, the possible outputs are obtained by manipulation of rational numbers using linear maps with rational coefficients. It follows that the finite set of generators of the additive group $E$ is rational. It follows that $E$ is discrete and the final claim follows from Theorem 3(ii).

We note that in real implementations it may happen that the common numerator of the elements of $E$ is so small that it is below machine precision which may lead to effects not predicted by Corollary 1. But we do not pursue this question here.

Before proceeding we now give a simple example to illustrate that the previous result is not just of academic interest, but rather is also of some practical importance.

**Example:** Let us illustrate the undesirable behaviour that may arise whenever a PI controller is being used in the closed-loop system. In this example, we point out that the integral action may be heavily dependent on the controller’s initial state. In the following we choose all data so that Corollary 1 is applicable. That is, the sets $A_i$, $r$ and the coefficients of $F$ are chosen to be rational, and the
probability functions $p_{ij}$ are continuous.

Consider the feedback system depicted in Figure 1 with $N = 10$ agents, whose states $x_i$ are in the set $\{0, 1\}$; as before, if $x_i = 1$, we say that agent $i$ has taken the resource or is active.

Our main goal is to regulate the number of active agents around the reference value $r = 5$. We assume that five agents, namely $x_1$ to $x_5$, have the following probabilities of being active ($i = 1, \ldots, 5$)

\[ p_{i1}(x_i(k+1) = 1) = 0.02 + \frac{0.95}{1 + \exp(-100(\pi(k) - 5))} \]

whereas the remaining agents’ probability of consuming the resource is given by (for $i = 6, \ldots, 10$)

\[ p_{i1}(x_i(k+1) = 1) = 0.98 - \frac{0.95}{1 + \exp(-100(\pi(k) - 1))} \]

As all agents have two options we always have $p_{i0} = 1 - p_{i1}$. Note that their behaviour is, thus, complementary. Indeed, if the control signal $\pi(k) \gg 5$, then the first five agents are more likely to be active. On the other hand, if $\pi(k) \ll 1$, then remaining ones are more likely to take the resource. In this design problem, we implement two types of linear controllers $C_L$: a PI controller and its lag approximant. The PI controller implements \(9\) with $\kappa = 0.1$ and $\alpha = -4$. This controller is approximated by a lag controller with $\kappa = 0.1$, $\alpha = -4.01$ and $\beta = 0.99$. The filter $F$ is the moving average (FIR) filter defined by

\[ \hat{y}(k) = \frac{y(k) + y(k-1)}{2} \]  

(11)

Our first observation from one simulation is that the filter output, $\hat{y}$, assumes, indeed, a finite set of rational values, as shown in Figure 2. Hence, the conditions of Corollary 1 are met by both the controller and the filter in the PI case. As the closed loop system is not ergodic for this case, it is possible that undesirable characteristics may be observed during simulations; such behaviour should not be observed in the lag controller.

Figure 3 points out that the PI controller regulates the average number of active agents $\bar{y}$, whereas the lag controller presents a steady-state error (as expected). However, Figure 4 shows different average trajectories of one of the five first agents, say $\bar{x}_1$, for different initial conditions of the controller $C_L$, namely $x_c(0) = 50$ and $x_c(0) = -50$. As the figure points out, this agent’s behaviour is completely dependent on the initial value of $x_c$, when $C_L$ is the PI controller. It is important to note, however, that this undesirable behaviour vanishes, on the long run, when a lag controller is used; that is, the system becomes ergodic and, hence, predictable. We further analyse this unexpected dependency in Figure 5 which points out the influence of the initial PI controller state on the average
Figure 2: Filter output for a single simulation. Note that \( \hat{y}(k) \) assumes a finite set of rational values, verifying the conditions of Corollary 1.

state of one of the first agents, say again \( \bar{x}_1 \), on the long run. Figure 2 illustrates the dynamic response of the broadcast signal \( \pi \) for both initial conditions and both controllers; both cases converge to the same value for the lag structure and this is not observed when PI is used.

Finally, to conclude this section, we note that coupling fails in our PI example due to a lack of contractivity. Fortunately, for linear systems, the notion of contractivity needed for ergodicity is relatively easy to enforce, and we shall now provide conditions which guarantee a stable behavior, for particular combinations of agent dynamics, filter, and controller. Specifically, in the linear setting, the controller dynamics may be:

\[
C : \begin{cases} 
    x_c(k+1) & = A_c x_c(k) + B_c e(k), \\
    \pi(k) & = C_c x_c(k) + D_c e(k),
\end{cases}
\]

(12)

where \( x_c \in \mathbb{R}^{n_c} \) is the internal state of the controller in dimension \( n_c \). One could adopt a linear model for the \( n_f \)-dimensional filter \( F \), based on the classic IIR/FIR structures. Remembering that \( y \) is the sum of each agents’ output...
Figure 3: Average number of active systems. Regulation is observed for the PI and for the lag controllers, within a given precision.
Figure 4: Average trajectory of the first agent for both controllers and for both initial controller states. Predictability is lost for the PI controller but is verified for the lag controller.
Figure 5: Average value for $x_1(1000)$ for different initial conditions of both controllers.
Figure 6: Average value of the broadcast signal $\pi(k)$ for both controllers and for both initial controller states.
Then, for every stable linear controller $C$, there are scalars $\delta$, $\delta$ such that the feedback system converges in distribution to a unique invariant measure. To this end, consider the augmented state $\xi$ given in (12) and (13). Assume that each agent $i$ is a Schur matrix and $\delta$, $\delta$ are chosen, at each time step, from the sets $\{b_{ij}\} \subset \mathbb{R}^{n}$ and $\{d_{ij}\} = \mathbb{R}$ according to Dini-continuous probabilities $p_{ij}(\pi)$, resp. $p_{ij}(\pi)$ that verify (14). Assume also that $\delta, \delta' > 0$ such that $p_{ij}(\pi) \geq \delta > 0, p_{ij}(\pi) \geq \delta' > 0$ for all $(i, j)$. Then, for every stable linear controller $C$ and every stable linear filter $F$, the feedback loop converges in distribution to a unique invariant measure.

**Theorem 4.** Consider the feedback system depicted in Figure [1] with $C$ and $F$ given in (12) and (13). Assume that each agent $i \in \{1, \cdots , N\}$ has state $x_i$ with dynamics governed by the affine stochastic difference equations given in (15), where $A_i$, $B_i$ and $d_i$ are random variables that assume values in $\mathbb{R}^{n_i}$ and $\mathbb{R}$, respectively, with $\mathbb{P}(B_i = b_{ij}) = p_{ij}(\pi)$ and $\mathbb{P}(d_i = d_{ij}) = p_{ij}(\pi)$. Note that this is a generalisation of the situation when agents switch between two states - \{on, off\}. The following result gives conditions for ergodicity.

**Proof.** Following [6], the proof is centred at the construction of an iterated function system (IFS) with place-(state-)dependent probabilities that describes the feedback system. To this end, consider the augmented state $\xi := [x', y, \hat{y}, \hat{x}'f, \hat{y}, \hat{e}, \hat{x}'c, \pi]' \in \mathbb{R}^d$, with $d = (\sum_{i=1}^N n_i) + M + n_f + n_c + n_x + 3$, whose dynamic behaviour is described by the difference equation

$$\xi(k + 1) = w_\ell(x) := A\xi(k) + b_\ell,$$

where $A = \begin{bmatrix} A_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
where \( \mathbf{1} \) is the vector of ones, \( \hat{A} := \text{diag}(A_i) \), \( \hat{C} := \text{diag}(c'_i) \) and \( b_\ell \) is built from all the combinations of the vectors \( b_{ij} \), the scalars \( \hat{d}_{ij} \) and other signals. To apply Corollary 2.3 from [6], two observations must be made. First, note that each map \( w_\ell \) is chosen with probability \( p_\ell(\xi) \geq \prod_{i=1}^{N} \delta_i > 0 \) and, thus, they are bounded away from zero. Second, since \( \sigma(A) = \sigma(\hat{A}) \cup \sigma(L) \cup \sigma(A_f) \cup \sigma(A_c) \cup \{0\} \) and, by hypothesis, \( A_i, A_f \) and \( A_c \) are Schur matrices, then for any induced matrix norm there exists \( m \in \mathbb{N} \) sufficiently large such that \( \|A^m\| < 1 \). Hence, due to these two properties, the result then follows from [6]. The proof is complete.

Comment: Dini’s condition on the probabilities may, obviously, be replaced by simpler, more conservative assumptions, such as Lipschitz or Hölder conditions [6]. Also, as we have seen, the requirement \( p_{ij}(\pi) \geq \delta_i > 0 \) in the theorem statement is not an artefact of our analysis, as \( p_{ij}(\pi) = 0 \) may lead to a non-ergodic behaviour.

5 Switched linear and non-linear systems

Considering that Theorem 1 does not require linearity, it is clear that one can extend its use to non-linear and switched control systems, under suitable assumptions. Here we give some examples.

5.1 Switched controllers

In the switched control case, the designer focuses on determining the controller matrices \( (A_{ci}, B_{ci}, C_{ci}, D_{ci}), i \in \mathbb{K} := \{1, \cdots, N_s\} \), aiming to provide a robust and stable closed-loop system with respect to any possible switching signal. For this particular case, we can extend the result of Theorem 4, under some assumptions. To this end, we first prove the following lemma, which is based on the classic stability result presented in [33].

Lemma 1. Consider the switched linear system

\[
\Sigma: \quad x(k+1) = A_{\sigma(k)}x(k),
\]

where \( x \in \mathbb{R}^n \) is its state and \( \sigma : \mathbb{N} \rightarrow \mathbb{K} \) is the switching sequence. If there exist positive definite symmetric matrices \( P_1, \cdots, P_{N_s} \in \mathbb{S}_+^n \) satisfying the following linear matrix inequalities

\[
A_i'P_jA_i - P_j < 0,
\]

for all \( (i, j) \in \mathbb{K}^2 \), then \( \Sigma \) is exponentially stable and there exists \( m \in \mathbb{N} \) sufficiently large such that

\[
\|A_{i_m} \cdots A_{i_1}\| < 1
\]

holds for any sequence of indices \( i_1, \cdots, i_m \in \mathbb{K} \).
Proof. First, note that, if there exist matrices \( P_1, \ldots, P_N \), satisfying (20), then there exists a sufficiently small scalar \( \epsilon \in (0, 1) \) such that
\[
A_i' P_j A_i < (1 - \epsilon)^2 P_i
\] (22)
hold for all \((i, j) \in \mathbb{K}^2\). Now, to ease notation, define the quadratic function
\[
v(k) = x(k)' P_{\sigma(k)} x(k).
\] (23)
It is clear [34] that there exist constants \( \alpha \) and \( \beta \) such that
\[
\alpha \|x\|^2 \leq x' P_i x \leq \beta \|x\|^2
\] (24)
hold for any \( x \in \mathbb{R}^n \) and for any \( i \in \mathbb{K} \); here, \( \| \cdot \| \) can be any vector norm. Now, let us first prove that \( \Sigma \) is exponentially stable. To this end, first note that (22) implies that \( v \) satisfies
\[
v(k + 1) \leq (1 - \epsilon)^2 v(k)
\] (25)
for all \( k \in \mathbb{N} \) and any state trajectory of \( \Sigma \). Therefore, an inductive argument applied to (25) yields
\[
v(k) \leq (1 - \epsilon)^{2k} v(0),
\] (26)
for any \( k \in \mathbb{N} \) and any given initial condition \( x(0) \) in \( \mathbb{R}^n \). Using the bounds (24), it follows that
\[
\|x(k)\| \leq c(1 - \epsilon)^k \|x(0)\|
\] (27)
holds for all \( k \in \mathbb{N} \), where \( c := \sqrt{\beta/\alpha} \). Exponential stability holds – see [35]. Finally, let us move our attention to (21). Noting that
\[
\|A_{i_k} \cdots A_{i_1}\| = \max_{w \neq 0} \frac{\|A_{i_k} \cdots A_{i_1} w\|}{\|w\|},
\] (28)
one can take \( w \) as any initial condition \( x(0) \) to \( \Sigma \) and, since (27) holds for any switching sequence \( \sigma \) and any \( x(0) \), it follows that
\[
\|A_{i_k} \cdots A_{i_1}\| = \max_{x(0) \neq 0} \frac{\|x(k)\|}{\|x(0)\|} \leq c(1 - \epsilon)^k
\] (29)
holds for all \( k \in \mathbb{N} \). Since there exists a sufficiently large \( m \in \mathbb{N} \) such that \((1 - \epsilon)^m < c^{-1}\), (21) holds, completing, thus, the proof.

The reader must now remember that the main argument adopted in the proof of Theorem 4 focuses on the contractivity of a sufficiently large power of the augmented matrix \( A \) (17), extended to the case of switching systems. The result of Lemma 1 points out that, provided some matrix inequality conditions hold, there always exists a sufficiently large number of jumps that ensures the contractivity of any possible switching chain. From these observations – and from the results in [36], it remains clear that the results of Theorem 4 can be trivially extended to consider switched controllers provided: (i) at any mode, any switching function (state-dependent or not) must have a nonzero probability of reaching every possible mode in the system; and (ii) based on Lemma 1, the controller matrices \( A_{c_i}, i \in \mathbb{K} \), must satisfy the inequalities in Lemma 1.
5.2 Non-linear controllers

We now give an example of results that can be deduced for non-linear systems. A particular case of the general setup described in Figure 1 is given by systems of the following form:

\[
\begin{align*}
\{ x_i(k+1) & \in \{ w_i(x_i(k)) \mid j = 1, \ldots, W_i \} \\
y_i(k) & \in \{ h_i(x_i(k)) \mid j = 1, \ldots, H_i \}, \end{align*}
\]

\[y(k) = \sum_{i=1}^{N} y_i(k),\]

\[F : \left\{ \begin{array}{ll}
x_f(k+1) = w_f(x_f(k), y(k)) \\
\hat{y}(k) = h_f(x_f(k), y(k)),
\end{array} \right. \]

\[C : \left\{ \begin{array}{ll}
x_c(k+1) = w_c(x_c(k), \hat{y}(k), r) \\
\pi(k) = h_c(x_c(k), \hat{y}(k), r),
\end{array} \right. \]

In addition, we have Dini continuous probability functions $p_{ij}$, $p'_{il}$ so that the probabilistic laws (4) are satisfied.

If we denote by $X_i, i = 1, \ldots, N$, $X_C$ and $X_F$ the state spaces of the agents, the controller and the filter, then the system evolves on the overall state space $X := \prod_{i=1}^{N} X_i \times X_C \times X_F$ according to the dynamics

\[x(k+1) := \left(\begin{array}{c}
(x_i)_{i=1}^{N} \\
x_f \\
x_c
\end{array}\right) (k+1) \in \{ F_m(x(k)) \mid m \in \mathcal{M} \}. \quad (34)\]

where each of the maps $F_m$ is of the form

\[F_m(x(k)) = \left(\begin{array}{c}
(w_{ij}(x_i(k)))_{i=1}^{N} \\
w_f(x_f(k), \sum_{i=1}^{N} h_i(x_i(k))) \\
w_c(x_c(k), h_f(x_f(k)))
\end{array}\right). \quad (35)\]

The maps $F_m$ are indexed by indices $m$ lying in the set

\[\prod_{i=1}^{N} \{ (i, 1), \ldots, (i, W_i) \} \times \prod_{i=1}^{N} \{ (i, 1), \ldots, (i, H_i) \}. \quad (36)\]

By the independence assumption on the choice of the transition maps and output maps for the agents, for each multi-index $m = ((1, j_1), \ldots, (N, j_N), (1, l_1), \ldots, (N, l_N))$ in this set, the probability of choosing the corresponding map $F_m$ is given by

\[\mathbb{P}(x(k+1) = F_m(x(k))) = \left(\prod_{i=1}^{N} p_{ij_i}(\pi(k))\right) \left(\prod_{i=1}^{N} p'_{il_i}(\pi(k))\right) := q_m(\pi(k)). \quad (37)\]
Theorem 5. Consider the feedback system depicted in Figure 1. Assume that each agent $i \in \{1, \ldots, N\}$ has a state governed by the non-linear stochastic difference equations

$$x_i(k + 1) = w_{ij}(x_i(k))$$  \hspace{1cm} (38)

$$y_i(k) = h_{ij}(x_i(k)),$$  \hspace{1cm} (39)

where $w_{ij}$ and $h_{ij}$ are globally Lipschitz-continuous functions with Lipschitz constant $l_{ij}$, resp. $l'_{ij}$. Assume we have Dini continuous probability functions $p_{ij}, p'_{il}$ so that the probabilistic laws are satisfied. Assume furthermore that there are scalars $\delta, \delta' > 0$ such that $p_{ij}(\pi) \geq \delta > 0$, $p'_{ij}(\pi) \geq \delta' > 0$ for all $(i, j)$. Further, assume that one of the following holds:

(a) “contractivity”: for all $1 \leq i \leq N, 1 \leq j \leq J$, $l_{ij} < 1$ and $l'_{ij} < 1$.

(b) “average contractivity”: for all $1 \leq i \leq N$, $\sum_{j=1}^{J} p_{ij}(x_i) l_{ij} < 1$; for all $1 \leq i \leq N$, $\sum_{j=1}^{J} p'_{ij}(x_i) l'_{ij} < 1$.

(c) “marginal contractivity”: for all $1 \leq i \leq N, 1 \leq j \leq J$, $l_{ij} \leq 1$, with probability 1, there exist $i, j$, such that $l_{ij} < 1$. Notice $p_{ij}(x_i) \geq \delta_i > 0$ by definition. Likewise for $l'_{ij}$.

Then, for every stable linear controller $C$ and every stable linear filter $F$, the feedback loop has a unique attractive invariant measure. In particular, the system is ergodic.

Proof. This follows from Theorem 2.1 and Corollary 2.2 of [6]. There, the average contraction property is required. This is indeed the case here, in a straightforward consequence of the assumptions on the Lipschitz constants and the internal asymptotic stability of controller and filter, similarly to the proof of Theorem 4.\qed

Remark 4. Notice that Lipschitz continuity can be rephrased in many ways. For instance, there is the QUAD condition, the sector condition, or, when restricted to convex functions $w, h$, the bounded subgradient condition. The sector condition suggests that there exist constants $K$ and $\kappa$ such that the vector-valued functions $w(x) := [w_i(x)]$ and $h(x) := [h_i(x)]$ satisfy $w(x)^T [w(x) - K x] \leq 0$ and $h(x)^T [h(x) - \kappa x] \leq 0$. The bounded subgradient refers to instance-specific constants $L, L'$, such that for all norms $| \cdot |$, for all $z, z'$ in the subdifferentials of $w, h$, respectively, at all points in the domains of the respective functions, we have that $|z| \leq L, |z'| \leq L'$, where $w(x) := [w_i(x)]$ and $h(x) := [h_i(x)]$ are vector-valued functions, which are this at time only assumed both to be convex, subdifferentiable with a non-empty subdifferential throughout their domains, but not necessarily differentiable. The equivalence follows from basic convex analysis, e.g., as a corollary of Lemma 2.6 in [37].

Next, consider the case, when the agents’ actions are limited to a finite set. In this case the Lipschitz conditions in Theorem 5 cannot be satisfied except in
trivial cases. In this case we may use results in [38, 39] to obtain ergodicity results.

The general setup of the following result is that for each agent \( i \) the set \( \mathcal{A}_i \) is finite. Then, \( X_S := \prod_{i=1}^{N} \mathcal{A}_i \) is finite and we consider the directed graph \( G = (X_S, E) \), where there is an arc between vertices representing \((x_i) \in X_S\) and \((y_i) \in X_S\), if there is a choice of maps \( w_{ij} \) in (30) such that \( (w_{ij}(x_i)) = (y_i) \).

**Theorem 6.** Consider the feedback system depicted in Figure 1. Assume that \( \mathcal{A}_i \) is finite for each \( i \). Assume that each agent \( i \in \{1, \cdots, N\} \) has a state governed by the non-linear stochastic difference equations (38). Assume we have Dini continuous probability functions \( p_{ij}, p'_{il} \) so that the probabilistic laws (4) are satisfied. Assume furthermore that there are scalars \( \delta, \delta' > 0 \) such that \( p_{ij}(\pi) \geq \delta > 0, p'_{ij}(\pi) \geq \delta' > 0 \) for all \((i, j)\). Then, for every stable linear controller \( C \) and every stable linear filter \( F \) the following holds:

If the graph \( G = (X_S, E) \) is strongly connected, then there exists an invariant measure for the feedback loop. If in addition, the adjacency matrix of the graph is primitive, then the invariant measure is attractive and the system is ergodic.

**Proof.** This is a consequence of [39] and the observation that the necessary contraction properties follow from the internal asymptotic stability of controller and filter.

We note that a simple condition for the primitivity of the graph \( G = (X_S, E) \) is that for each agent the graph describing the possible transitions is primitive.

### 6 Conclusions and Further Work

Within feedback systems, a challenging area of study concerns the control of ensembles of agents. Practically important examples of such systems arise in Smart Cities. Typically, such problems deviate from classical control problems in two main ways. First, even though ensembles are typically too large to allow for a microscopic approach, they are not sufficiently large to allow for a meaningful fluid (mean-field) approximation. Second, the regulation problem concerns not only the ensemble, but also the individual agents; a certain quality of service should be provided to each agent. We have formulated this problem as an iterated function system with the objective of designing an ergodic control, and demonstrated that controls with poles on the unit circle (e.g., PI) may destroy ergodicity even for benign ensembles.

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