Limit theorems for radial random walks on $p \times q$-matrices as $p$ tends to infinity

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Abstract

The radial probability measures on $\mathbb{R}^p$ are in a one-to-one correspondence with probability measures on $[0, \infty)$ by taking images of measures w.r.t. the Euclidean norm mapping. For fixed $\nu \in M^1([0, \infty])$ and each dimension $p$, we consider i.i.d. $\mathbb{R}^p$-valued random variables $X_1^p, X_2^p, \ldots$ with radial laws corresponding to $\nu$ as above. We derive weak and strong laws of large numbers as well as a large deviation principle for the Euclidean length processes $S_k^p := \|X_1^p + \ldots + X_k^p\|$ as $k, p \to \infty$ in suitable ways. In fact, we derive these results in a higher rank setting, where $\mathbb{R}^p$ is replaced by the space of $p \times q$ matrices and $[0, \infty)$ by the cone $\Pi_q$ of positive semidefinite matrices. Proofs are based on the fact that the $(S_k^p)_{k \geq 0}$ form Markov chains on the cone whose transition probabilities are given in terms Bessel functions $J_\mu$ of matrix argument with an index $\mu$ depending on $p$. The limit theorems follow from new asymptotic results for the $J_\mu$ as $\mu \to \infty$. Similar results are also proven for certain Dunkl-type Bessel functions.

KEYWORDS: Bessel functions of matrix argument, matrix cones, Bessel functions associated with root systems, asymptotics, radial random walks, laws of large numbers, large deviations.

1 Introduction

This paper has its origin in the following problem: Let $\nu \in M^1([0, \infty])$ be a probability measure. For each dimension $p \in \mathbb{N}$ consider the time-homogeneous random walk $(S_k^p)_{k \geq 0}$ on $\mathbb{R}^p$ which starts at time $k = 0$ at $0 \in \mathbb{R}^p$ and makes a random jump at each time step with uniformly distributed direction and a size with distribution $\nu$ where the sizes and directions are independent of each other and of the earlier ones. As the distributions of the $S_k^p$ are radial, we study $\|S_k^p\|$ with the usual Euclidean norm $\|\cdot\|$. Now let $(p_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ be a sequence of dimensions with $k \to \infty$. Our aim is to find limit theorems for the $[0, \infty]$-valued random variables $\|S_k^p\|$ as $k \to \infty$ for suitable sequences $(p_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ of dimensions and suitable
measures $\nu$. This is an interesting question even for point measures $\nu = \delta_r$ for $r > 0$. We only have to exclude the trivial case $\nu = \delta_0$, which we shall do from now on.

To get a first feeling for possible results, assume that $\nu$ has second moment $\sigma^2(\nu) := \int_0^\infty r^2 \, dv(r) \in [0, \infty[.$ For each dimension $p$ there is a unique radial measure $\nu_{\rho} \in M^1(\mathbb{R}^p)$ with $\nu$ as its radial part, i.e., for the norm mapping $\varphi_{\rho} : \mathbb{R}^p \to [0, \infty[, \ x \mapsto \|x\|$, we have $\varphi_{\rho}(\nu_{\rho}) = \nu$. We then may realize the random walk $(S^p_k)_{k \geq 0}$ as $S^p_k = \sum_{l=0}^k X^p_l$ for i.i.d. $\mathbb{R}^2$-valued random variables $X^p_l$ with laws $\nu_{\rho}$ which admit second moments. The classical CLT on $\mathbb{R}^p$ and the well-known relation between the standard normal distribution on $\mathbb{R}^p$ and the $\chi^2$-distribution $\chi^2_p$ with $p$ degrees of freedom imply after some short computation that for fixed $p$ and $k \to \infty$, the variables $\frac{p}{\nu(\sigma^2(\nu))} \|S^p_k\|^2$ tend in distribution to $\chi^2_p$. Moreover, as $Z_p/p$ tends to 1 in probability for $\chi^2_p$-distributed random variables $Z_p$, we obtain that $\|S^p_k\|/\sqrt{k}$ tends to $\sqrt{\sigma^2(\nu)}$ in probability if we first take $k \to \infty$ and then $p \to \infty$. We already observed in \cite{V1} that this result remains correct for other combinations of $k, p \to \infty$:

\textbf{1.1 Theorem.} Assume that $\nu \in M^1([0, \infty[)$ has the second moment $\sigma^2(\nu) \in [0, \infty[.$ Then for each sequence $(p_k)_{k \in \mathbb{N}} \subset N$ of dimensions with $\lim_{k \to \infty} p_k = \infty$,

$$\|S^p_k\|/\sqrt{k} \to \sqrt{\sigma^2(\nu)} \quad \text{in probability.}$$

One purpose of this paper is to prove an associated strong law. For simplicity we will assume that $\nu$ has a compact support.

\textbf{1.2 Theorem.} Assume that $\nu \in M^1([0, \infty[)$ has compact support. Let $(p_k)_{k \in \mathbb{N}} \subset N$ and $(n_k)_{k \in \mathbb{N}} \subset N$ sequences of dimensions and time steps with the following properties:

1. $\lim_{k \to \infty} p_k/k^a = \infty$ for all $a \in \mathbb{N}$;
2. $\lim_{k \to \infty} p_k/(n_k^2(\ln k)^2) = \infty$;
3. $\lim_{k \to \infty} n_k/(\ln k)^2 = \infty$.

Then $\|S^p_{n_k}\|/\sqrt{n_k} \to \sqrt{\sigma^2(\nu)}$ almost surely.

For the case $n_k = k$, only condition (1) on the dimensions remains, i.e., the dimensions have to grow faster than any polynomial. Unfortunately, we are not able to get rid of this strong growth condition. We shall discuss the conditions also in Section 4 below. Besides these laws of large numbers we shall also derive a large deviation principle for $S^p_k$ in Section 5 under the condition that $p_k$ grows faster than exponentially.

Theorems \textbf{1.1} and \textbf{1.2} and, in part, also this large deviation principle will appear as special cases of extensions of these results in two directions.

The first extension concerns a higher rank setting. We consider the following geometric situation: For fixed dimensions $p, q \in \mathbb{N}$ let $M_{p,q} = M_{p,q}(F)$ denote the space of $p \times q$-matrices over one of the division algebras $F = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$ with real dimension $d = 1, 2,$ or $4$ respectively. This is a Euclidean vector space of (real) dimension $dpq$ with scalar product $\langle x, y \rangle = \mathfrak{N} tr(x^* y)$ where $x^* := \overline{x}$, $\mathfrak{N}t := \frac{1}{2}(t + \overline{t})$ is the real part of $t \in F$, and $tr$ is the trace in $M_q := M_{q,q}$. A measure on $M_{p,q}$ is called radial if it is invariant under the action of the unitary group $U_p = U_p(F)$ by left multiplication, $U_p \times M_{p,q} \to M_{p,q}$, $(u, x) \mapsto ux$. This action is orthogonal w.r.t. the scalar product above, and, by uniqueness of the polar
decomposition, two matrices $x, y \in M_{p,q}$ belong to the same $U_p$-orbit if and only if $x^*x = y^*y$. Thus the space $M_{p,q}^{U_p}$ of $U_p$-orbits in $M_{p,q}$ is naturally parameterized by the cone $\Pi_q = \Pi_q(F)$ of positive semidefinite $q \times q$-matrices over $F$. We identify $M_{p,q}^{U_p}$ with $\Pi_q$ via $U_qx \simeq (x^*x)^{1/2}$, i.e., the canonical projection $M_{p,q} \to M_{p,q}^{U_p}$ will be realized as the mapping

$$\varphi_p : M_{p,q} \to \Pi_q, \quad x \mapsto (x^*x)^{1/2}.$$ 

The square root is used here in order to ensure for $q = 1$ and $F = \mathbb{R}$ that $\Pi_1 = [0, \infty[$ and $\varphi_p(x) = \|x\|$, i.e. the setting above appears. By taking images of measures, the mapping $\varphi_p$ induces a Banach space isomorphism between the space $M_{p,q}^{U_q}(M_{p,q})$ of all bounded radial Borel measures on $M_{p,q}$ and the space $M_b(\Pi_q)$ of bounded Borel measures on the cone $\Pi_q$. In particular, for each probability measure $\nu \in M^1(\Pi_q)$ there is a unique radial probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$. We shall say that $\nu \in M^1(\Pi_q)$ admits a second moment if $\int_{\Pi_q} \|s\|^2 \, d\nu(s) < \infty$ where again, $\|s\| = (tr s^2)^{1/2}$ is the Hilbert-Schmidt norm. In this case, the second moment of $\nu$ is defined as the matrix-valued integral

$$\sigma^2(\nu) := \int_{\Pi_q} s^2 \, d\nu(s) \in \Pi_q.$$ 

With these notions, we shall derive the following generalizations of Theorems [1.1] and [1.2]:

**1.3 Theorem.** Let $\nu \in M^1(\Pi_q)$ be a probability measure with finite second moment $\sigma^2(\nu) \in \Pi_q$. For each dimension $p \in N$ consider the unique $U_p$-invariant probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$. Furthermore, let $(X_i^p)_i \in N$ be a sequence of i.i.d. $M_{p,q}$-valued random variables with law $\nu_p$. Then for each sequence $(p_k)_{k \in N} \subset N$ of dimensions with $\lim_{k \to \infty} p_k = \infty$,

$$\frac{1}{\sqrt{k}} \varphi_{p_k} \left( \sum_{i=1}^k X_i^{p_k} \right) \to \sqrt{\sigma^2(\nu)} \in \Pi_q \quad \text{in probability.}$$

**1.4 Theorem.** Let $\nu \in M^1(\Pi_q)$ be a probability measure with compact support. For each dimension $p \in N$ consider the unique $U_p$-invariant probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$. Furthermore, let $(X_i^p)_i \in N$ be a sequence of i.i.d. $M_{p,q}$-valued random variables with law $\nu_p$. Let $(p_k)_{k \in N} \subset N$ and $(n_k)_{k \in N} \subset N$ sequences of dimensions and time steps with the following properties:

1. $\lim_{k \to \infty} p_k / k^a = \infty$ for all $a \in N$;
2. $\lim_{k \to \infty} p_k / n_k^2 (\ln k)^2 = \infty$;
3. $\lim_{k \to \infty} n_k / (\ln k)^2 = \infty$.

Then $\varphi_{p_k} \left( \sum_{i=1}^{n_k} X_i^{p_k} \right) / \sqrt{n_k}$ tends to $\sqrt{\sigma^2(\nu)} \in \Pi_q$ almost surely.

We next turn to a further generalization of these theorems. Consider again the Banach space isomorphism between $M_{b,\Pi_q}^{U_q}(M_{p,q})$ and $M_b(\Pi_q)$. The usual group convolution on $M_{p,q}$ induces a Banach-$*$-algebra-structure on $M_b(\Pi_q)$ such that this isomorphism becomes a probability-preserving Banach-$*$-algebra isomorphism. The space $\Pi_q$ together with this new convolution becomes a commutative orbit hypergroup; see [2] and [3] for a general background and [3] for our specific example. It follows from Eq. (3.5) and Corollary 3.2 of
that in case $p \geq 2q$, the convolution product of two point measures on $\Pi_q$ induced from $M_{p,q}$ is given by

$$\langle \delta_r \ast \mu \delta_s \rangle(f) := \frac{1}{r\mu} \int_{D_q} f\left(\sqrt{r^2 + s^2 + svr + rv^*s}\right) \Delta(I - vv^*)^{\mu - \rho} \, dv$$  \hspace{1cm} (1.1)

with $\mu := pd/2$, $\rho := d(q - \frac{1}{2}) + 1$,

$$D_q := \{v \in M_q : v^*v < I\}$$

(where $v^*v < I$ means that $I - v^*v$ is strictly positive definite), and with the normalization constant

$$\kappa_\mu := \int_{D_q} \Delta(I - v^*v)^{\mu - \rho} \, dv.$$  \hspace{1cm} (1.2)

The convolution of arbitrary measures is just given by bilinear, weakly continuous extension.

It was observed in [R3] that Eq. (1.1) defines a commutative hypergroup actually for all indices $\mu \in \mathbb{R}$ with $\mu > \rho - 1$. In all cases, $0 \in \Pi_q$ is the identity of the hypergroup and the involution is given by the identity mapping. These hypergroup structures are closely related with a product formula for Bessel functions of index $\mu$ on the matrix cone $\Pi_q$ and are therefore called Bessel hypergroups on $\Pi_q$. Indeed, the hypergroup characters are given in terms of matrix Bessel functions $J_\mu$. We refer to the monograph [FK] for Bessel functions on cones, and to [R3] for the particular details. For general indices $\mu$, the Bessel hypergroups on $\Pi_q$ do not have a nice geometric (orbit) interpretation as in the cases $\mu = pd/2$ with integral $p$, but nevertheless the notion of random walks on these hypergroups is meaningful in the general cases just as well.

1.5 Definition. Fix $\mu > \rho - 1$ and a probability measure $\nu \in M^1(\Pi_q)$. A Bessel random walk $(S_\mu^n)_{n \geq 0}$ on $\Pi_q$ of index $\mu$ and with law $\nu$ is a time-homogeneous Markov chain on $\Pi_q$ with $S_\mu^0 = 0$ and transition probability

$$P(S_{n+1}^\mu \in A | S_n^\mu = x) = \langle \delta_x \ast \mu \nu \rangle(A)$$

for $x \in \Pi_q$ and Borel sets $A \subset \Pi_q$.

This notion is quite common on hypergroups (see [BH]) and was in particular used in [V3] for Bessel hypergroups on matrix cones and already in [K] for the one-dimensional case $q = 1$ and $F = \mathbb{R}$. The notion has its origin in the following well-known fact for the orbit cases $\mu = pd/2$ with $p \in \mathbb{N}$: If we fix a radial measure $\nu_p \in M^1(M_{p,q})$ and consider a sequence of i.i.d. $M_{p,q}$-valued random variables $(X_i^p)_{i \in \mathbb{N}}$ with law $\nu_p$, then $\left(\varphi_p\left(\sum_{i=1}^k X_i^p\right)\right)_{k \geq 0}$ is a random walk on $\Pi_q$ of index $\mu$ with law $\varphi_p(\nu_p)$. Having this in mind, we can state generalizations of Theorems 1.3 and 1.4 for such random walks on $\Pi_q$ for indices $\mu \to \infty$ and time steps $k \to \infty$. This will be done in Section 4 where we state and prove our results in this generality. The preceding limit results will then appear just as special cases.

The proofs of the limit results in Section 4 are roughly as follows: As the characters of the Bessel hypergroups on $\Pi_q$ can be expressed in terms of Bessel functions $J_\mu$, the multi-dimensional Hankel transform on $\Pi_q$ is just the hypergroup Fourier transform, and we can easily write down these transforms of the distributions of the $S_n^\mu$. On the other hand, we shall derive several uniform limit results for $J_\mu(\mu x)$ as $\mu \to \infty$. These results imply that the Hankel transforms tend to Laplace transforms of these distributions, which leads to the
stated limit theorems. We point out that the direct proofs of Theorems 1.3 and 1.4 are precisely the same as in the slightly more general setting adopted in our paper.

The organization of this paper is as follows: In Section 2 we recapitulate some known results about Bessel functions and Bessel convolutions on matrix cones from [FK] [FT] [H], and [R3]. The central part of the paper is Section 3, where we present several uniform asymptotic results for $J_\mu(\mu x)$ as $\mu \to \infty$. Except for partial results proven by one of the authors already in [V1] for $q = 1$, these results seem to be new even in the one-variable case $q = 1$. This is surprising as in the classical monograph [W] a complete chapter is devoted to $J_\mu(\mu x)$ with $\mu \to \infty$. In Section 4, the asymptotic results from Section 3 are transferred to certain classes of Dunkl-type Bessel functions associated with the root system $B_q$. Finally, the results of Section 3 are used as a basis for the proofs of the laws of large numbers in Section 5 and the large deviation principle in Section 6.

## 2 Bessel functions and Bessel hypergroups on matrix cones

In this section we collect some known facts about Bessel functions on matrix cones and the associated Bessel hypergroups. The material is mainly taken from [FK] and [R3]. We also refer to the fundamental work [H] of Herz, to [Di] and to [FT].

### 2.1 Bessel functions associated with matrix cones

Let $F$ be one of the real division algebras $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2$ or $4$ respectively. Denote the usual conjugation in $F$ by $t \mapsto \overline{t}$, the real part of $t \in F$ by $R_t = \frac{1}{2}(t + \overline{t})$, and by $|t| = (\overline{t} t)^{1/2}$ its norm.

For $p, q \in \mathbb{N}$ we denote by $M_{p,q} := M_{p,q}(F)$ the vector space of all $p \times q$-matrices over $F$ and put $M_q := M_q(F) := M_{q,q}(F)$ for abbreviation. Let further

$$H_q = H_q(F) = \{x \in M_q(F) : x = x^*\}$$

the space of Hermitian $q \times q$-matrices over $F$. All these spaces are real Euclidean vector spaces with scalar product $(x,y) := R \text{tr}(x^* y)$ and the associated norm $\|x\| = (x, x)^{1/2}$. Here $x^* := \overline{x}$ and $\text{tr}$ denotes the trace. The dimension of $H_q$ is given by $\dim_R H_q := q + \frac{d}{2} q(q - 1)$.

Let further

$$\Pi_q := \{x^2 : x \in H_q\} = \{x^* x : x \in H_q\}$$

be the set of all positive semidefinite matrices in $H_q$, and $\Omega_q$ its topological interior which consists of all strictly positive definite matrices. $\Omega_q$ is a symmetric cone, i.e. an open convex cone which is self-dual and whose linear automorphism group acts transitively; see [FK] for details.

To define the Bessel functions associated with the symmetric cone $\Omega_q$ we first introduce their basic building blocks, the so-called spherical polynomials. These are just the polynomial spherical functions of $\Omega_q$ considered as a Riemannian symmetric space. They are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q) \in \mathbb{N}_0^q$ (we write $\lambda \geq 0$ for short) and are given by

$$\Phi_\lambda(x) = \int_{U_q} \Delta_\lambda(uxu^*) du, \quad x \in H_q$$

where $du$ is the normalized Haar measure of $U_q$ and $\Delta_\lambda$ is the power function

$$\Delta_\lambda := \Delta_1(x)^{\lambda_1} \Delta_2(x)^{\lambda_2} \ldots \Delta_q(x)^{\lambda_q} \quad (x \in H_q).$$


The \( \Delta_i(x) \) are the principal minors of the determinant \( \Delta(x) \), see [FK] for details. There is a renormalization \( Z_\lambda = c_\lambda \Phi_\lambda \) with constants \( c_\lambda > 0 \) depending on the underlying cone such that

\[
(\text{tr } x)^k = \sum_{|\lambda|=k} Z_\lambda(x) \quad \text{for } k \geq 0; \tag{2.1}
\]

see Section XI.5. of [FK] where these \( Z_\lambda \) are called zonal polynomials. By construction, the \( Z_\lambda \) are invariant under conjugation by \( U_q \) and thus depend only on the eigenvalues of their argument. More precisely, for \( x \in H_q \) with eigenvalues \( \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q \), one has

\[
Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = \frac{2}{d} \tag{2.2}
\]

where the \( C_\lambda^\alpha \) are the Jack polynomials of index \( \alpha \) in a suitable normalization (c.f. [FK], [Ka], [R3]). The Jack polynomials \( C_\lambda^\alpha \) are homogeneous of degree \(|\lambda|\) and symmetric in their arguments.

The matrix Bessel functions associated with the cone \( \Omega_q \) are defined as \( _0F_1 \)-hypergeometric series in terms of the \( Z_\lambda \), namely

\[
J_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x), \tag{2.3}
\]

where for \( \lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{N}_0^q \), the generalized Pochhammer symbol \((\mu)_\lambda\) is given by

\[
(\mu)_\lambda = (\mu)^{2/d}_{\lambda} \quad \text{where} \quad (\mu)_\lambda^\alpha := \prod_{j=1}^q (\mu - \frac{1}{\alpha}(j-1))_{\lambda_j} \quad (\alpha > 0),
\]

and \( \mu \in \mathbb{C} \) is an index satisfying \((\mu)_\lambda^\alpha \neq 0\) for all \( \lambda \geq 0 \). If \( q = 1 \), then \( \Pi_q = \mathbb{R}_+ \) and the Bessel function \( J_\mu \) is independent of \( d \) with

\[
J_\mu\left(\frac{x^2}{4}\right) = j_{\mu-1}(x)
\]

where \( j_\mu(z) = _0F_1(\kappa+1; -z^2/4) \) is the usual modified Bessel function in one variable.

### 2.2 Bessel hypergroups on matrix cones

Hypergroups are convolution structures which generalize locally compact groups insofar as the convolution product of two point measures is in general not a point measure again, but just a probability measure on the underlying space. More precisely, a hypergroup \((X, \ast)\) is a locally compact Hausdorff space \( X \) together with a convolution \( \ast \) on \( M_b(X) \) (the regular bounded Borel measures on \( X \)), such that \((M_b(X), \ast)\) becomes a Banach algebra, where \( \ast \) is weakly continuous, probability preserving and preserves compact supports of measures. Moreover, one requires an identity \( e \in X \) with \( \delta_e \ast \delta_x = \delta_x \ast \delta_e = \delta_x \) for \( x \in X \), as well as a continuous involution \( x \mapsto \bar{x} \) on \( X \) such that for all \( x, y \in X \), \( e \in \text{supp}(\delta_x \ast \delta_y) \) is equivalent to \( x = \bar{y} \), and \( \delta_y \ast \delta_y = (\delta_y \ast \delta_x)^{-1} \). Here for \( \mu \in M_b(X) \), the measure \( \mu^- \) is given by \( \mu^-(A) = \mu(A^-) \) for Borel sets \( A \subset X \). A hypergroup \((X, \ast)\) is called commutative if and only if so is the convolution \( \ast \). Thus for a commutative hypergroup \((X, \ast)\), the measure space \( M_b(X) \) becomes a commutative Banach-\( \ast \)-algebra with identity \( \delta_e \). Notice that due to its
weak continuity, the convolution of measures on a hypergroup is uniquely determined by the convolution product of point measures.

On a commutative hypergroup \((X, \ast)\) there exists a (up to a multiplicative factor) unique Haar measure \(\omega\), i.e. \(\omega\) is a positive Radon measure on \(X\) satisfying

\[
\int_X \delta_x \ast \delta_y(f)d\omega(y) = \int_X f(y)d\omega(y) \quad \text{for all } x \in X, f \in C_c(X).
\]

The decisive object for harmonic analysis on a commutative hypergroup is its dual space, which is defined by

\[
\hat{X} := \{\varphi \in C_b(X) : \varphi \neq 0, \varphi(\overline{x}) = \overline{\varphi(x)}, \delta_x \ast \delta_y(\varphi) = \varphi(x)\varphi(y) \text{ for all } x, y \in X\}.
\]

The elements of \(\hat{X}\) are also called characters. As in the case of LCA groups, the dual of a commutative hypergroup is a locally compact Hausdorff space with the topology of locally uniform convergence and can be identified with the symmetric spectrum of the convolution algebra \(L^1(X, \omega)\).

The following theorem contains some of the main results of [R3].

2.1 Theorem. Let \(\mu \in \mathbb{R}\) with \(\mu > \rho - 1\). Then

(a) The assignment

\[
(\delta_r \ast \mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_{\mu}} f(\sqrt{r^2 + s^2 + sv r + rv^* s}) \Delta(I - vv^*)^{\mu - \rho} dv, \quad f \in C(\Pi_q)
\]

with \(\kappa_\mu\) as in (1.2), defines a commutative hypergroup structure on \(\Pi_q\) with neutral element 0 \(\in \Pi_q\) and the identity mapping as involution. The support of \(\delta_r \ast \mu \delta_s\) satisfies

\[
\text{supp}(\delta_r \ast \mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \leq \|r\| + \|s\|\}.
\]

(b) A Haar measure of the hypergroup \(\Pi_{q, \mu} := (\Pi_q, \ast_{\mu})\) is given by

\[
\omega_{\mu}(f) = \frac{\pi^{q\mu}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r}) \Delta(r)^\gamma dr
\]

with \(\gamma = \mu - \frac{d}{2}(q - 1) - 1\).

(c) The dual space of \(\Pi_{q, \mu}\) is given by

\[
\widehat{\Pi_{q, \mu}} = \{\varphi_s : s \in \Pi_q\}
\]

with

\[
\varphi_s(r) := \mathcal{J}_\mu(\frac{1}{4}r^2 s^2 r) = \varphi_r(s).
\]

The hypergroup \(\Pi_{q, \mu}\) is self-dual via the homeomorphism \(s \mapsto \varphi_s\). Under this identification of \(\widehat{\Pi_{q, \mu}}\) with \(\Pi_{q, \mu}\), the Plancherel measure on \(\Pi_{q, \mu}\) is \((2\pi)^{-2\mu q}\omega_{\mu}\).
3 Estimates for Bessel functions of large indices

We first recapitulate the following well known one-dimensional inequalities for the exponential function (see, for instance, Sections 3.6.2 and 3.6.3 of [Mi]):

\[(1 - z/r)^r \leq e^{-z} \text{ for } r > 0, z \in \mathbb{R}, \quad (3.1)\]
\[0 \leq e^{-z} - (1 - z/r)^r \leq r^2 e^{-z}/r \text{ for } r, z \in \mathbb{R}, r \geq 1, |z| \leq r; \quad (3.2)\]
\[(1 + z/r)^r \leq e^z \leq (1 + z/r)^{r+z/2} \text{ for } r > 0. \quad (3.3)\]

These results have the following matrix-valued extension:

3.1 Lemma. (1) For all \( \mu > 1 \) and \( v \in \sqrt{\mu} \cdot D_q \subset M_q \),

\[0 \leq e^{-\langle v, v \rangle} - \Delta(I - \frac{1}{\mu} v^*v)^\mu \leq \frac{1}{\mu} tr\left((v^*v)^2\right) \cdot e^{-\langle v, v \rangle}. \]

(2) For all \( \mu > 0 \) and \( v \in M_q \),

\[\Delta(I + \frac{1}{\mu} v^*v)^\mu \leq e^{\langle v, v \rangle} \leq \Delta(I + \frac{1}{\mu} v^*v)^\mu + \frac{m}{2}\]

where \( m \geq 0 \) is the maximal eigenvalue of \( v^*v \).

Proof. (1) The positive semidefinite matrix \( v^*v \) may be written as

\[v^*v = u \cdot \text{diag}(a_1, a_2, \ldots, a_q) \cdot u^*\]

with some \( u \in U_q \) and the eigenvalues \( a_1, \ldots, a_q \in [0, m] \) of \( v^*v \). Then

\[\Delta(I - \frac{1}{\mu} v^*v)^\mu = \prod_{k=1}^{q} (1 - a_k/\mu)^\mu\]

and \( \langle v, v \rangle = tr(v^*v) = a_1 + \ldots + a_q \). Using (3.2) and a telescope sum argument, we obtain

\[0 \leq e^{-\langle v, v \rangle} - \Delta(I - \frac{1}{\mu} v^*v)^\mu\]
\[= \sum_{l=1}^{q} \left[ \prod_{k=1}^{l-1} (1 - a_k/\mu)^\mu \cdot (e^{-a_l} - (1 - a_l/\mu)^\mu) \cdot \prod_{k=l+1}^{q} e^{-a_k} \right] \]
\[\leq \frac{1}{\mu} \sum_{l=1}^{q} \left[ \prod_{k=1}^{q} e^{-a_k} a_l^2 \right] = \frac{1}{\mu} tr\left((v^*v)^2\right) \cdot e^{-\langle v, v \rangle}\]

as claimed.

(2) is proven in the same way by use of (3.3). \( \square \)

Our first estimate for the Bessel functions \( J_\mu \) as \( \mu \to \infty \) will be based on the following integral representation of \( J_\mu \) for \( \mu > \rho - 1 \) (see Eq. (3.12) of [R3]):

\[J_\mu(x^*x) = \frac{1}{\kappa_\mu} \int_{D_q} e^{-2i\langle v, x \rangle} \Delta(I - v^*v)^{\mu-\rho} dv. \quad (3.4)\]
3.2 Proposition. There exists a constant $C = C(q,d) > 0$ such that for $\mu > \rho - 1$ and all $x \in M_q$, 
\[ |J_\mu(\mu x^* x) - e^{-(x,x)}| \leq C/\mu \] and 
\[ |\mu^{dq^2/2} \kappa_\mu - \pi^{dq^2/2}| \leq C/\mu. \]

Proof. In a first step we obtain for $x \in M_q$,
\[ D := \int_{\sqrt{\mu} D_q} e^{-i(v,x)} \Delta(I - \frac{1}{\mu} v^* v)^{\mu-\rho} \, dv - \int_{M_q} e^{-i(v,x)} e^{-(v,v)} \, dv \]
\[ \leq \int_{M_q \setminus (\sqrt{\mu} D_q)} e^{-(v,v)} \, dv + \int_{\sqrt{\mu} D_q} \left( e^{-(v,v)(1-\rho/\mu)} - \Delta(I - \frac{1 - \rho/\mu}{\mu - \rho} v^* v)^{\mu-\rho} \right) dv \]
\[ + \int_{\sqrt{\mu} D_q} e^{-(v,v)} \left( e^{(v,v)(\rho/\mu - 1)} \right) dv. \]

By Lemma 3.1(1) for $\mu - \rho$ instead of $\mu$ and with the elementary estimate 
\[ e^z - 1 \leq (e^\rho - 1)z \quad \text{for} \quad z \in [0,\rho] \]

we further obtain that for $\mu \geq 2\rho$,
\[ D \leq \frac{C_1}{\mu} + \frac{1}{\mu - \rho} \int_{\sqrt{\mu} D_q} e^{-(v,v)/2} tr((v^* v)^2) \, dv + \frac{(e^\rho - 1)\rho}{\mu} \int_{\sqrt{\mu} D_q} e^{-(v,v)} \langle v, v \rangle \, dv \]
\[ \leq \frac{C_2}{\mu} \quad (3.5) \]

with suitable constants $C_1, C_2 > 0$. We next observe that $M_q \simeq \mathbb{R}^{dq^2}$ implies
\[ \int_{M_q} e^{-i(v,x)} e^{-(v,v)} \, dv = \pi^{dq^2/2} \cdot e^{-(x,x)/4}. \quad (3.6) \]

Moreover, replacing $x$ by $(\sqrt{\mu}/2) \cdot x$ and $v$ by $(1/\sqrt{\mu}) \cdot v$ in integral representation (3.4), we obtain
\[ J_\mu(\frac{\mu}{4} x^* x) = \frac{1}{\mu^{dq^2/2} \kappa_\mu} \int_{\sqrt{\mu} D_q} e^{-i(v,x)} \Delta(I - \frac{1}{\mu} v^* v)^{\mu-\rho} \, dv. \quad (3.7) \]

We now conclude from (3.5), (3.6) and (3.7) that for $\mu \geq 2\rho$ and $x \in M_q$,
\[ |\mu^{dq^2/2} \kappa_\mu \cdot J_\mu(\frac{\mu}{4} x^* x) - \pi^{dq^2/2} \cdot e^{-(x,x)/4}| \leq C_2/\mu. \]

For $x = 0$ we in particular observe that
\[ |\mu^{dq^2/2} \kappa_\mu - \pi^{dq^2/2}| \leq C_2/\mu. \quad (3.8) \]

As $|J_\mu(\frac{\mu}{4} x^* x)| \leq 1$ by (3.4), it follows for $\mu \geq 2\rho$ and $x \in M_q$ that
\[ |J_\mu(\frac{\mu}{4} x^* x) - e^{-(x,x)/4}| \]
\[ \leq |J_\mu(\frac{\mu}{4} x^* x)| \cdot \left| 1 - \frac{\mu^{dq^2/2} \cdot \kappa_\mu}{\pi^{dq^2/2}} \right| + \frac{1}{\pi^{dq^2/2}} |\mu^{dq^2/2} \cdot \kappa_\mu \cdot J_\mu(\frac{\mu}{4} x^* x) - \pi^{dq^2/2} e^{-(x,x)/4}| \]
\[ \leq C_3/\mu \]

with some constant $C_3 > 0$. Together with (3.8), this implies the statements of the proposition in case $\mu \geq 2\rho$. Within the range $\rho - 1 < \mu \leq 2\rho$, the proposition is immediate in view of the estimate $|J_\mu(\mu x^* x)| \leq 1$ for all $x \in M_q$. \qed
In the following, we shall derive a variant of Proposition 3.2 which is based on the power series (2.3) and provides a good estimate for small arguments. We start with some basic inequalities for the zonal polynomials $Z^\lambda$:

3.3 Lemma. For all partitions $\lambda \geq 0$ and and $y \in \Pi_q$,

$$|Z^\lambda(-y)| \leq Z^\lambda(y).$$

Proof. We use the relation between the $Z^\lambda$ and the Jack polynomials $C^\alpha_{\lambda}$ in Section 2 and the well-known fact that the $C^\alpha_{\lambda}$ are nonnegative linear combinations of monomials, see [KS]. This yields for the eigenvalues $\xi = (\xi_1, \ldots, \xi_q)$ of $-y$ and $|\xi| := (|\xi_1|, \ldots, |\xi_q|)$ of $y$ that

$$|Z^\lambda(-y)| = |C^\alpha_{\lambda}(\xi)| \leq C^\alpha_{\lambda}(|\xi|) = Z^\lambda(y).$$

3.4 Lemma. For all partitions $\lambda \geq 0$, $\mu > \rho - 1$ and $\mu_{\lambda} = (\mu)^{2/d}^\lambda$, 

$$\left|1 - \frac{\mu^{|\lambda|}}{(\mu)_{\lambda}}\right| \leq dq \cdot 2^{dq(q-1)/2} \cdot \frac{|\lambda|^2}{\mu}. \quad (3.9)$$

Proof. Consider $(\mu)_{\lambda} = \prod_{j=1}^q (\mu - \frac{d}{2}(j-1))_{\lambda_j}$. In this product, each factor can be estimated below by $\mu - \frac{d}{2}(q-1)$. Moreover, precisely

$$(0 + 1 + \ldots + (q-1)) \left[\frac{d}{2}\right] = \frac{(q-1)q}{2} \cdot \left[\frac{d}{2}\right] =: r$$

of these factors are smaller than $\mu$. As $\mu > \rho - 1 = d(q - 1/2)$, this implies

$$\begin{align*}
(\mu)_{\lambda} &\geq (\mu - \frac{d}{2}(q-1))^r \cdot \mu^{|\lambda|-r} \\
&\geq (\mu/2)^r \cdot \mu^{|\lambda|-r} \geq 2^{-dq(q-1)/2} \cdot \mu^{|\lambda|},
\end{align*}$$

and thus

$$\mu^{|\lambda|}/(\mu)_{\lambda} \leq 2^{dq(q-1)/2}. \quad (3.9)$$

We now prove by induction on the length $k := |\lambda|$ that for $\mu > \rho - 1 = d(q - 1/2)$,

$$\left|1 - \frac{\mu^{|\lambda|}}{(\mu)_{\lambda}}\right| \leq \frac{dq}{2(\mu - d(q - 1/2))} \cdot 2^{dq(q-1)/2}|\lambda|^2 \quad (3.10)$$

which immediately implies the lemma. In fact, for $k = 0, 1$, the left hand side of (3.10) is equal to zero, while the right-hand side is nonnegative.

For the induction step, consider a partition $\lambda$ of length $k \geq 2$. Then there is a partition $\lambda_i$ with $|\lambda_i| = k - 1$ for which there exists precisely one $j = 1, \ldots, q$ with $\lambda_j = \lambda_i + 1$ while all the other components are equal. Hence, if we assume the inequality to hold for $\lambda_i$ and use
we may write this expansion as
\[
1 - \frac{\mu^k}{(\mu)_\lambda} = 1 - \frac{\mu^{k-1}}{(\mu)_\lambda} + \frac{\mu^{k-1}}{(\mu)_\lambda} - \frac{\mu^k}{(\mu)_\lambda}
\]
\[
\leq \frac{dq}{\mu - d(q - 1)/2} \cdot 2^{dq(q-1)/2-1} \cdot (k-1)^2 + \frac{\mu^{k-1}}{(\mu)_\lambda} \cdot \left| 1 - \frac{\mu}{(\mu - d(j-1)/2 + \lambda_j - 1)} \right|
\]
\[
\leq \frac{dq}{\mu - d(q - 1)/2} \cdot 2^{dq(q-1)/2-1} \cdot (k-1)^2 + 2^{dq(q-1)/2} \cdot \left| 1 - \frac{\mu}{(\mu - d(j-1)/2 + \lambda_j - 1)} \right|
\]
\[
\leq \frac{2^{dq(q-1)/2-1}}{\mu - d(q - 1)/2} \cdot (dq(k-1)^2 + dq + 2k - 2)
\]
\[
\leq \frac{2^{dq(q-1)/2-1}}{\mu - d(q - 1)/2} \cdot dqk^2
\]
for \(k \geq 2\). This completes the proof. \(\blacksquare\)

3.5 Proposition. There exists a constant \(C = C(q, d) > 0\) such that for \(\mu > 2\rho\) and \(y \in \Pi_q\),
\[
|J_\mu(\mu y) - e^{-tr y}| \leq C \frac{(tr y)^2}{\mu}.
\]

Proof. Using the power series (2.3) as well as (2.1) in terms of the homogeneous polynomials \(Z_\lambda\), we obtain
\[
J_\mu(\mu y) - e^{-tr y} = \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \left( \frac{\mu^{|\lambda|}}{(\mu)_\lambda} - 1 \right) \cdot Z_\lambda(-y).
\]
As
\[
(\mu)_{(1,0,\ldots,0)} = \mu, \quad (\mu)_{(2,0,\ldots,0)} = \mu(\mu + 1), \quad (\mu)_{(1,1,0,\ldots,0)} = \mu(\mu - d/2),
\]
we may write this expansion as
\[
J_\mu(\mu y) - e^{-tr y} = R_2 + R_3
\]
with
\[
R_2 = \frac{1}{2} \left( \left( \frac{\mu^2}{\mu(\mu + 1)} - 1 \right) Z_{(2,0,\ldots,0)}(-y) + \left( \frac{\mu^2}{\mu(\mu - d/2)} - 1 \right) Z_{(1,1,0,\ldots,0)}(-y) \right)
\]
and
\[
R_3 = \sum_{k \geq 3} \frac{1}{k!} \sum_{|\lambda| = k} \left( \frac{\mu^k}{(\mu)_\lambda} - 1 \right) \cdot Z_\lambda(-y).
\]
Recall from Lemma 3.3 that \(|Z_\lambda(-y)| \leq Z_\lambda(y)\) and \(Z_\lambda(y) \geq 0\). Hence Eq. (2.1) implies for \(|\lambda| = 2\) that \(|Z_\lambda(-y)| \leq (tr y)^2\). Therefore, \(|R_2| \leq M_1 \frac{(tr y)^2}{\mu}\) with a suitable constant \(M_1 > 0\). Moreover, Lemmata 3.3 and 3.4 imply that
\[
|R_3| \leq \sum_{k \geq 3} \frac{1}{k!} \sum_{|\lambda| = k} M_2 \frac{k^2}{\mu} Z_\lambda(y) = \frac{M_2}{\mu} \sum_{k \geq 3} \frac{k^2}{k!} (tr y)^k
\]
\[
\leq \frac{2M_2}{\mu} (tr y)^2 \sum_{k \geq 1} \frac{1}{k!} (tr y)^k \leq \frac{2M_2}{\mu} (tr y)^2 e^{tr y} \quad (3.11)
\]
with a constant $M_2 > 0$. In summary we have

$$|J_\mu(\mu y) - e^{-\mu y}| \leq M_3 \cdot \frac{(\mu y)^2}{\mu} \cdot (1 + e^{\mu y}).$$

Together with the estimate of Proposition 3.2 for large $y$, this yields the stated result.

Summarizing Propositions 3.2 and 3.5, we obtain:

3.6 Theorem. There exists a constant $C = C(q,d) > 0$ such that for $\mu > 2\rho$ and $y \in \Pi_q$,

$$|J_\mu(\mu y) - e^{-\mu y}| \leq \frac{C}{\mu} \cdot \min(1, (\mu y)^2).$$

We next turn to an estimate for $J_\mu(-\mu y)$ with $y \in \Pi_q$. In order to simplify formulas, we replace the factor $\mu$ in the argument by $\mu - \rho$.

3.7 Proposition. There exists a constant $C = C(q,d) > 0$ such that for $\mu > 2\rho$ and all $x \in M_q$,

$$e^{\langle x,x \rangle} \left( 1 - \frac{C}{\mu} \|x\|^4 - H(x, \sqrt{\mu - \rho}) \right) \leq J_\mu(-(\mu - \rho)x^*)x \leq e^{\langle x,x \rangle} (1 + C/\mu)$$

where

$$H(x, r) := \int_{M_q \setminus rD_q} e^{-\|v-x\|^2} dv \quad \text{for} \quad r > 0.$$

Proof. We first conclude (by analytic continuation) from integral representation (3.4) that

$$J_\mu(-(\mu - \rho)x^*)x = \frac{1}{(\mu - \rho)^{d^2/2}K_\mu} \int_{\sqrt{\mu - \rho}D_q} e^{2\langle v,x \rangle} \Delta(I - \frac{1}{\mu - \rho}v^*v)^{\mu - \rho} dv. \quad (3.12)$$

Moreover, Proposition 3.2 implies that

$$\left| \frac{1}{(\mu - \rho)^{d^2/2}K_\mu} - \pi^{-d^2/2} \right| = O(1/\mu). \quad (3.13)$$

We next estimate the integral in (3.12). For this we use Lemma 3.1(1) and observe that

$$\int_{\sqrt{\mu - \rho}D_q} e^{2\langle v,x \rangle} \Delta(I - \frac{1}{\mu - \rho}v^*v)^{\mu - \rho} dv \leq \int_{M_q} e^{2\langle v,x \rangle} e^{-\langle v,v \rangle} dv \leq \pi^{d^2/2}e^{\langle x,x \rangle}.$$

Together with Eq. (3.12) and (3.13) this yields

$$J_\mu(-(\mu - \rho)x^*)x \leq (\pi^{d^2/2} + O(1/\mu)) \cdot \pi^{d^2/2}e^{\langle x,x \rangle}.$$

This proves the upper estimate as claimed.

For the lower estimate, we use Lemma 3.1(1) again. We obtain

$$\int_{\sqrt{\mu - \rho}D_q} e^{2\langle v,x \rangle} \Delta(I - \frac{1}{\mu - \rho}v^*v)^{\mu - \rho} dv \geq \int_{\sqrt{\mu - \rho}D_q} e^{2\langle v,x \rangle - \langle v,v \rangle} \cdot (1 - \frac{1}{\mu - \rho}tr((v^*v)^2)) dv$$

$$= \int_{M_q} e^{2\langle v,x \rangle - \langle v,v \rangle} dv - I_1 - \frac{1}{\mu - \rho}I_2$$

$$= \pi^{d^2/2}e^{\langle x,x \rangle} - I_1 - \frac{1}{\mu - \rho}I_2.$$
with
\[ I_1 := \int_{M_q \setminus \sqrt{\mu - \rho} D_q} e^{2\langle v, x \rangle - \langle v, v \rangle} \, dv \]
and
\[ I_2 := \int_{\sqrt{\mu - \rho} D_q} e^{2\langle v, x \rangle - \langle v, v \rangle} \cdot tr((v^* v)^2) \, dv. \]
We have
\[ I_1 = e^{\langle x, x \rangle} \cdot \int_{M_q \setminus \sqrt{\mu - \rho} D_q} e^{-\|v-x\|^2} \, dv = e^{\langle x, x \rangle} \cdot H(x, \sqrt{\mu - \rho}) \]
and
\[ I_2 = e^{\langle x, x \rangle} \cdot \int_{\sqrt{\mu - \rho} D_q} e^{-\|v-x\|^2} \cdot tr((v^* v)^2) \, dv = e^{\langle x, x \rangle} \cdot O(\|x\|^4), \]
which finally leads to the lower estimate.

\[ \square \]

4 Estimates for Dunkl-type Bessel functions associated with root systems of type B

There is a close connection between Bessel convolutions on the cone \( \Pi_q \) and the theory of Dunkl operators associated with the root system \( B_q \) which is explained in [R3]. In this short section, we shall recall this connection and use it to obtain asymptotic relations between certain classes of Dunkl-type Bessel functions, which can be expressed as generalized hypergeometric functions in terms of Jack polynomials. This section is independent of the remaining parts of this paper and may be skipped by readers interested in the probabilistic results only. Also, we shall not go into details of Dunkl theory, but refer the reader to [DX], [R1] and [R2]. For multivariable hypergeometric functions, see e.g. [GR] and [Ka].

In the following, we always assume that \( q \geq 2 \). For a reduced root system \( R \subset \mathbb{R}^q \) and a multiplicity function \( k : R \to \mathbb{C} \) (i.e. \( k \) is invariant under the action of the corresponding reflection group), we denote by \( J_k = J_k^R \) the Dunkl-type Bessel function associated with \( R \) and \( k \). It is obtained from the Dunkl kernel by symmetrization with respect to the underlying reflection group. Dunkl-type Bessel functions generalize the spherical functions of Euclidean type symmetric spaces, which occur for crystallographic root systems and specific discrete values of \( k \). For the root system \( A_{q-1} = \{ \pm(e_i - e_j) : i < j \} \subset \mathbb{R}^q \), the multiplicity \( k \) is a single complex parameter and if \( k > 0 \), then the associated Dunkl-type Bessel function can be expressed as a generalized \( 0F_0 \)-hypergeometric function,
\[
J_k^A(\xi, \eta) = 0F_0^\alpha(\xi, \eta) := \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \frac{C_\alpha^\lambda(\xi) C_\alpha^\lambda(\eta)}{C_\alpha^\lambda(1)} \quad \text{with} \quad 1 = (1, \ldots, 1), \alpha = 1/k,
\]
due to relations (3.22) and (3.37) of [BF]. For the root system \( B_q = \{ \pm e_i, \pm e_i \pm e_j : i < j \} \), the multiplicity is of the form \( k = (k_1, k_2) \) where \( k_1 \) and \( k_2 \) are the values on the roots \( \pm e_i \) and \( \pm e_i \pm e_j \) respectively. The associated Dunkl-type Bessel function is given by
\[
J_k^B(\xi, \eta) = 0F_1^\alpha(\mu; \xi^2/2, \eta^2/2) \quad \text{with} \quad \alpha = 1/k_2, \mu = k_1 + (q-1)k_2 + 1/2
\]
where \( \xi^2 = (\xi_1^2, \ldots, \xi_q^2) \) and
\[
0F_1^\alpha(\mu; \xi, \eta) := \sum_{\lambda \geq 0} \frac{1}{(\mu)^{\lambda+1} |\lambda|!} \frac{C_\alpha^\lambda(\xi) C_\alpha^\lambda(\eta)}{C_\alpha^\lambda(1)}.
\]
Recall now that the characters of the hypergroup $\Pi_{q,\mu}$ on the matrix cone $\Pi_q$ are given by $\varphi_x(r) = J_\mu(\sqrt{r}su^2s)$. The conjugation action $x \mapsto uxx^{-1}$ of the unitary group $U_q = U_q(F)$ on $\Pi_q$ induces a new commutative hypergroup structure on the set of possible eigenvalues of matrices from $\Pi_q$ ordered by size, i.e. the $B_q$-Weyl chamber

$$\Xi_q = \{ \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0 \}.$$ 

This hypergroup on $\Xi_q$ depends on $d$ and $\mu$. Its characters are given by the functions

$$\psi_\eta(\xi) = \int_{U_q} J_\mu(\frac{1}{4}|\eta u\xi^2u^{-1}\eta|^{1/2}) du = J^B_{k(\mu,d)}(\xi, i\eta), \quad \eta \in \Xi_q$$

where $k(\mu,d) = (\mu - (d(q - 1) + 1)/2, d/2)$ (and elements from $\Xi_q$ are identified with diagonal matrices in the natural way). For details, see Section 4 of [R3]. The estimates for the matrix Bessel functions $J_\mu$ according to Theorem 3.6 imply the following estimate for the Dunkl-type Bessel function $J^B_{k(\mu,d)}$ as $\mu \to \infty$.

**4.1 Corollary.** There exists a constant $C = C(q,d) > 0$ such that for $\mu > 2\rho$ and $\xi, \eta \in \Xi_q$,

$$\left| J^B_{k(\mu,d)}(2\sqrt{\mu} \xi, i\eta) - J^A_{d/2}(\xi^2, \eta^2) \right| \leq \frac{C}{\mu} \cdot \min(1, (|\xi|^2|\eta^2|)^2)$$

where $|\xi| = \left( \sum_{i=1}^q \xi_i^2 \right)^{1/2}$ denotes the standard Euclidean norm in $\mathbb{R}^q$.

**Proof.** For $k = k(\mu,d)$ we obtain by Eq. (4.1) and Theorem 3.6 the estimate

$$\left| J^B_{k(\mu,d)}(2\sqrt{\mu} \xi, i\eta) - \int_{U_q} e^{-\text{tr}(\eta u\xi^2u^{-1}\eta)} du \right| \leq \frac{C}{\mu} \cdot \min(1, S(\xi, \eta))$$

where

$$S(\xi, \eta) = \int_{U_q} |\text{tr}(\eta u\xi^2u^{-1}\eta)|^2 du = \int_{U_q} |(\eta^2, u\xi^2u^{-1})|^2 du \leq |\xi|^2|\eta^2|^2$$

by the Cauchy-Schwarz inequality. The spherical polynomials $Z_\lambda$ satisfy the product formula

$$\frac{Z_\lambda(r)Z_\lambda(s)}{Z_\lambda(I)} = \int_{U_q} Z_\lambda(\sqrt{r}su^{-1}u^{-1}\sqrt{r}) du \quad \text{for } r, s \in \Pi_q,$$

see Prop. 5.5. of [GR]. Thus by Eq. (2.1) and (2.2) we further obtain, with $\alpha = 2/d$,

$$\int_{U_q} e^{-\text{tr}(\eta u\xi^2u^{-1}\eta)} du = \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \int_{U_q} Z_\lambda(-\eta u\xi^2u^{-1}\eta) du$$

$$= \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \frac{Z_\lambda(-\xi^2)Z_\lambda(\eta^2)}{Z_\lambda(I)} = \pi F^\alpha_0(-\xi^2, \eta^2),$$

which implies the assertion. \qed

**4.2 Remarks.**

1. It is conjectured that the statement of this corollary remains valid for arbitrary $d \in \mathbb{R}$ with $d > 0$.

2. The integral on the left side of formula (4.2) is of Harish-Chandra type. If $F = \mathbb{C}$, then by Theorem II. 5.35 of [He] it can be written as an alternating sum

$$\int_{U_q} e^{-\text{tr}(\eta u\xi^2u^{-1}\eta)} du = \prod_{i=1}^{q-1} \frac{j!}{\pi(\xi^2)\pi(\eta^2)} \sum_{w \in S_q} \text{sgn}(w) e^{-(\xi^2, w\eta^2)}$$

where $(\ldots)$ denotes the usual Euclidean scalar product in $\mathbb{R}^q$ and $\pi(\xi) = \prod_{i<j}(\xi_i - \xi_j)$ is the fundamental alternating polynomial.


5 Laws of large numbers

Let $\nu \in M^1(\Pi_q)$ be a probability measure and $\mu > \rho - 1 = d(q - 1/2)$ a fixed index. We say that a time-homogeneous Markov chain $(S^\mu_k)_{k \geq 0}$ on $\Pi_q$ is a Bessel-type random walk on $\Pi_q$ of index $\mu$ with law $\nu$ if $S^\mu_0 = 0$ and if its transition probability is given by

$$P(S^\mu_{k+1} \in A | S^\mu_k = x) = (\delta_x * \nu)(A)$$

for all $k \in \mathbb{N}_0$, $x \in \Pi_q$ and Borel sets $A \subseteq \Pi_q$. It is easily checked by induction on $k$ that the distribution of $S^\mu_k$ is just the $k$-fold convolution power $\nu^{(k,\mu)} = \nu * \mu * \nu * \ldots * \mu * \nu$ of $\nu$ with respect to the Bessel convolution of index $\mu$. As announced in the introduction, we are interested in limit theorems for the random variables $S^\mu_k$ as $k, \mu \to \infty$. Our first result in this direction is the following weak law of large numbers:

5.1 Theorem. Let $\nu \in M^1(\Pi_q)$ be a probability measure with finite second moment

$$\sigma^2(\nu) := \int_{\Pi_q} s^2 \, d\nu(s) \in \Pi_q,$$

and let $(\mu_k)_{k \in \mathbb{N}} \subseteq [\rho - 1, \infty[$ be an arbitrary sequence of indices with $\lim_{k \to \infty} \mu_k = \infty$. Let $S^\mu_k$ be the $k$-th member of the Bessel-type random walk of index $\mu_k$ with law $\nu$. Then

$$\frac{1}{\sqrt{k}} S^\mu_k \to \sqrt{\sigma^2(\nu)}$$

in probability as $k \to \infty$.

This first main result has the following consequence which was stated as Theorem 1.3 in the introduction:

5.2 Corollary. Let $\nu \in M^1(\Pi_q)$ be a probability measure with finite second moment $\sigma^2(\nu) \in \Pi_q$. For each dimension $p \in \mathbb{N}$ consider the unique $U_p$-invariant probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$ where $\varphi_p : M_{p,q} \to \Pi_q$, $x \mapsto (x^* x)^{1/2}$, is the canonical projection. Let further $(X^p_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ be a sequence of i.i.d. $M_{p,q}$-valued random variables with law $\nu_p$. Then for each sequence $(p_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ of dimensions with $\lim_{k \to \infty} p_k = \infty$, the $\Pi_q$-valued random variables

$$\frac{1}{\sqrt{k}} \varphi_{p_k} \left( \sum_{l=1}^k X^p_l \right)$$

tend in probability to the constant $\sqrt{\sigma^2(\nu)}$.

Proof. This is clear from Theorem 5.1 because $(\varphi_p(\sum_{l=1}^k X^p_l))_{k \geq 0}$ is a Bessel-type random walk on $\Pi_q$ with index $\mu = pd/2$. 

The proof of Theorem 5.1 relies on estimates for matrix Bessel functions from the preceding section and on standard properties of the Laplace transform on matrix cones. These properties are likely to be known but we include them for the reader’s convenience.

Recall that the Laplace transform $L\nu \in C_b(\Pi_q)$ of a measure $\nu \in M^1(\Pi_q)$ is defined by

$$L\nu(x) = \int_{\Pi_q} e^{-\langle x, y \rangle} \, d\nu(y), \quad x \in \Pi_q.$$ 

The Laplace transform on the cone $\Pi_q$ satisfies the following Levy-type continuity theorem.
5.3 Proposition. For probability measures $\nu$, $(\nu_k)_{k \geq 1} \in M^1(\Pi_q)$ the following statements are equivalent:

1. $\nu_k \to \nu$ weakly.
2. $L\nu_k(x) \to L\nu(x)$ for all $x \in \Pi_q$.
3. $L\nu_k(x) \to L\nu(x)$ for all $x \in \Omega_q$.

Proof. (1) $\implies$ (2) $\implies$ (3) is obvious. For (3) $\implies$ (1) observe that for $x \in \Omega_q$, the exponential function $e_x(y) := e^{-(x,y)}$ is contained in $C_0(\Pi_q)$, i.e., it vanishes at infinity. Moreover, the linear span of $\{e_x, x \in \Omega_q\}$ is a $\|\cdot\|_\infty$-dense subspace of $C_0(\Pi_q)$ by the Stone-Weierstrass theorem. It follows from (3) and a $3\varepsilon$-argument that $\int f \, d\nu_k \to \int f \, d\nu$ for all $f \in C_0(\Pi_q)$ which implies (1). 

The following result can be readily derived from the dominated convergence theorem as in the classical setting:

5.4 Lemma. Let $\nu \in M^1(\Pi_q)$ be a probability measure which admits $r$-th moments for $r \in \mathbb{N}$, i.e., $\int_{\Pi_q} \|y\|^r \, d\nu(r) < \infty$. Then $L\nu$ is $r$-times continuously differentiable on $\Pi_q$.

Using the Taylor formula at $0 \in \Pi_q$, we in particular obtain:

5.5 Corollary. Let $\nu \in M^1(\Pi_q)$ with finite second moment $\sigma^2(\nu) \in \Pi_q$. Then

\[
\int_{\Pi_q} e^{-(s x, s x)} \, d\nu(s) = 1 - tr(x \sigma^2(x)) + o(\|x\|^2) \quad \text{as } x \to 0 \text{ in } \Pi_q. \tag{5.1}
\]

Moreover, if $\nu \in M^1(\Pi_q)$ admits fourth moments, then even $O(\|x\|^4)$ is true instead of $o(\|x\|^2)$ in relation (5.1).

The following result is a variant of the preceding corollary:

5.6 Lemma. Let $\nu \in M^1(\Pi_q)$ with finite second moment $\sigma^2(\nu) \in \Pi_q$, and let $(\mu_k)_{k \geq 1} \subset [0, \infty[$ be as in Theorem 5.7. Then for each $x \in \Pi_q$,

\[
\int_{\Pi_q} J_{\mu_k} \left( \frac{k}{k x s^2} \right) \, d\nu(s) = 1 - \frac{1}{k} tr(x \sigma^2(\nu)x) + o(\|x\|^2/k) \quad \text{as } k \to \infty.
\]

Moreover, if $\nu \in M^1(\Pi_q)$ admits fourth moments, then the error term $o(1/k)$ can be replaced by

\[
O(\|x\|^4/k^2 + \|x\|^2/(k \mu_k)).
\]

Proof. We first conclude from Theorem 5.6 that for $y \in \Pi_q$,

\[
|J_{\mu_k}(\mu_ky) - e^{-tr y}| \leq \frac{c}{\mu_k} tr y.
\]

Therefore

\[
\int_{\Pi_q} \left| J_{\mu_k} \left( \frac{k}{k x s^2} \right) - e^{-\frac{1}{k} tr(xs^2x)} \right| \, d\nu(s) \leq \frac{c}{k \mu_k} \int_{\Pi_q} tr(x s^2x) \, d\nu(s)
\]

\[
\leq \frac{c\|x\|^2}{k \mu_k} \int_{\Pi_q} \|s\|^2 \, d\nu(s)
\]

\[
\leq \frac{c\|x\|^2}{k \mu_k}.
\]
with suitable constants \(c, \tilde{c} > 0\). On the other hand, we conclude from Corollary 5.5 that

\[
\int_{\Pi_q} e^{-\frac{1}{k} \text{tr}(xs^2x)} d\nu(s) = 1 - \frac{1}{k} \text{tr}(x\sigma^2(\nu)x) + o\left(\frac{\|x\|^2}{k}\right)
\]

which yields the first claim. The second statement follows readily from the second statement in Corollary 5.5.

**Proof of Theorem 5.1.** Let \(\nu^{(k,\mu_k)}\) be the \(k\)-fold Bessel convolution power of \(\nu\) with index \(\mu_k\). Then \(\nu^{(k,\mu_k)}\) is the distribution of the random variable \(S_{\nu}^{\mu_k}\). Being hypergroup characters, the matrix Bessel functions \(s \mapsto J_{\mu_k}(\frac{\mu_k}{k}xs^2x)\) are multiplicative w.r.t. the Bessel convolution of index \(\mu\). Together with the preceding lemma this implies that

\[
\lim_{k \to \infty} \int_{\Pi_q} J_{\mu_k}(\frac{\mu_k}{k}xs^2x) d\nu^{(k,\mu_k)}(s) = \lim_{k \to \infty} \left( \int_{\Pi_q} J_{\mu_k}(\frac{\mu_k}{k}xs^2x) d\nu(s) \right)^k = \lim_{k \to \infty} \left( 1 - \frac{1}{k} \text{tr}(x\sigma^2(\nu)x) + o(1/k) \right)^k = e^{-\text{tr}(x\sigma^2(\nu)x)} = A(x).
\] \hspace{1cm} (5.2)

We thus conclude from Proposition 3.2 that

\[
\lim_{k \to \infty} \int_{\Pi_q} e^{-\frac{1}{k} \text{tr}(xs^2x)} d\nu^{(k,\mu_k)}(s) = \lim_{k \to \infty} \int_{\Pi_q} J_{\mu_k}(\frac{\mu_k}{k}xs^2x) d\nu^{(k,\mu_k)}(s) = A(x)
\]

for \(x \in \Pi_q\). From this we conclude (after a quadratic transformation of the argument) that the Laplace transforms of the distributions of \( (S_{\nu}^{\mu_k})^2/k \) tend to the Laplace transform of the point measure \(\delta_{\sigma^2(\nu)}\) on \(\Pi_q\) as \(k \to \infty\). The theorem now follows from Proposition 5.3.

**5.7 Theorem.** Let \(\nu \in M^1(\Pi_q)\) be a probability measure with compact support. Let \((\mu_k)_{k \in \mathbb{N}} \subset ]\rho - 1, \infty[\) be an arbitrary sequence of indices and \((n_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) a sequence of time steps with the following properties:

1. \(\lim_{k \to \infty} \mu_k/k^a = \infty\) for all \(a \in \mathbb{N}\);
2. \(\lim_{k \to \infty} \mu_k/(n_k^2(\ln k)^2) = \infty\);
3. \(\lim_{k \to \infty} n_k/(\ln k)^2 = \infty\).

Let \(S_{n_k}^{\mu_k}\) be the \(n_k\)-th member of the Bessel-type random walk of index \(\mu_k\) with law \(\nu\). Then,

\[
\frac{1}{\sqrt{n_k}} S_{n_k}^{\mu_k} \to \sqrt{\sigma^2(\nu)}
\]

for \(k \to \infty\) almost surely.

As for the WLLN in the beginning of this section, this theorem immediately implies Theorems 1.2 and 1.4.

Recall that the dimension of \(H_q\) as a real vector space is given by \(n = q + \frac{q}{2}q(q - 1)\). The proof of Theorem 5.7 relies on the following elementary observation:
5.8 Lemma. There exist matrices \( b_1, \ldots, b_n \in \Pi_q \) such that for all \( a \in \Pi_q \) and sequences \( (a_k)_{k \in \mathbb{N}} \subset \Pi_q \) we have \( a_k \rightarrow a \) if and only if \( \langle b_j, a_k \rangle \rightarrow \langle b_j, a \rangle \) for all \( j = 1, \ldots, n \).

**Proof.** If \( b_1, \ldots, b_n \) is any \( R \)-basis of the vector space \( H_q \) of Hermitian matrices with dimension \( n = q + q(q-1)d/2 \), then obviously \( a_k \rightarrow a \) if and only if \( \langle b_j, a_k \rangle \rightarrow \langle b_j, a \rangle \) for all \( j = 1, \ldots, n \).

On the other hand, we can find a basis consisting of elements from \( \Pi_q \). For instance, we may take the \( q \) diagonal matrices of the form \( \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \) together with the matrices of the form

\[
I + \frac{1}{2}(e_{i,j} + l^*e_{j,i}) \in \Pi_q
\]

for \( 1 \leq i < j \leq q \) and the \( l \in F \) with \( |l| = 1 \) forming an \( R \)-basis of \( F \) where the \( e_{i,j} \) are the elementary matrices with 1 in the \((i,j)\)-coordinate and 0 otherwise. Notice that these matrices are positive definite by the Gershgorin criterion.

**Proof of Theorem 5.7.** Let \( \mu_k \) and \( n_k \) be given as in the theorem. By Lemma 5.8, it suffices to prove that for each \( c \in \Pi_q \),

\[
\frac{1}{n_k}(c^2, (S_{n_k}^\mu)^2) \rightarrow \langle c^2, \sigma^2(\nu) \rangle \quad \text{almost surely.}
\]

For this we shall prove for each \( \varepsilon > 0 \) that

\[
P\left( \frac{1}{n_k}(c^2, (S_{n_k}^\mu)^2) \geq \langle c^2, \sigma^2(\nu) \rangle + \varepsilon \right) = O(1/k^2)
\]

and

\[
P\left( \frac{1}{n_k}(c^2, (S_{n_k}^\mu)^2) \leq \langle c^2, \sigma^2(\nu) \rangle - \varepsilon \right) = O(1/k^2).
\]

Relation (5.4) then follows immediately from the Borel-Cantelli lemma.

We first turn to the proof of relation (5.6). Here we proceed as in the beginning of the proof of Lemma 5.6 and conclude from Theorem 3.6 that

\[
E\left(e^{-\frac{2\ln k}{c_nk} \cdot \text{tr}(c^2(S_{n_k}^\mu)^2)}\right) = \int_{\Pi_q} e^{-\frac{2\ln k}{c_nk} \cdot \text{tr}(c^2c)} \, d\nu_{(n_k, \mu_k)}(s)
\]

\[
= \int_{\Pi_q} J_{\mu_k} \left( \frac{2\mu_k \ln k}{\varepsilon \cdot n_k} \cdot cs^2c \right) \, d\nu_{(n_k, \mu_k)}(s) + O\left( \frac{1}{\mu_k} \right).
\]

Moreover, using the stronger statement of Lemma 5.6, Eq. (3.1), and the assumptions (1) and (3) of the theorem, we obtain

\[
\left( \int_{\Pi_q} J_{\mu_k} \left( \frac{2\mu_k \ln k}{\varepsilon \cdot n_k} \cdot cs^2c \right) \, d\nu(s) \right)^{n_k}
\]

\[
= \left( \frac{2\ln k}{\varepsilon \cdot n_k} \cdot \text{tr}(c^2(\nu)c) + O\left( \frac{(\ln k)^2}{n_k^2} + \frac{\ln k}{n_k \mu_k} \right) \right)^{n_k}
\]

\[
\leq e^{-\frac{2\ln k}{c_nk} \cdot \text{tr}(c^2(\nu)c) \cdot O(1)}
\]
The Markov inequality and estimates (5.7), (5.8) now lead to

\[
P \left( \frac{1}{n_k} \langle c^2, (S_{nk}^\mu)^2 \rangle \leq \langle c^2, \sigma^2(\nu) \rangle - \varepsilon \right)
\]

\[
= P \left( e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(S_{nk}^\mu)^2)} \geq e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(\nu)c) - \varepsilon} \right)
\]

\[
\leq \frac{1}{e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(S_{nk}^\mu)^2)} - e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(\nu)c) - \varepsilon}} \cdot E \left( e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(S_{nk}^\mu)^2)} \right)
\]

\[
\leq e^{\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(\nu)c)} e^{-2\ln k \left( e^{-\frac{2\ln k}{\varepsilon \mu_k} \text{tr}(c^2(\nu)c)} \cdot O(1) + O\left( \frac{1}{\mu_k} \right) \right)}
\]

\[
\leq O \left( \frac{1}{k^2} \right) + O \left( \frac{k^a}{\mu_k} \right)
\]

with a suitable constant \( a = a(\varepsilon, c) > 0 \). Condition (1) of the theorem now completes the proof of (5.6).

We now turn to the proof of relation (5.5). Assume that \( \text{supp} \nu \subset \{ x \in \Pi_q : \|x\|_2 \leq M \} \) holds for a suitable constant \( M > 0 \). Then by the support properties of the Bessel convolution on \( \Pi_q \), we have for all \( k \in \mathbb{N} \)

\[
\text{supp} \nu^{(n_k, \mu_k)} \subset \{ x \in \Pi_q : \|x\|_2 \leq n_k M \}.
\]

(5.9)

We now consider the function \( H \) and the constant \( C > 0 \) of Proposition 3.7. We conclude from Eq. (5.9) and condition (2) of the theorem that for all sequences \( s_k \in \text{supp} \nu^{(n_k, \mu_k)} \),

\[
\frac{(\ln k)^2 \|s_k\|_2^4}{\mu_k n_k^2} \rightarrow 0 \quad \text{and} \quad \frac{1}{\sqrt{\mu_k - \rho}} \cdot \sqrt{\frac{\ln k}{n_k}} \cdot s_k \rightarrow 0.
\]

Thus by the definition of \( H \) we have for each \( c \in \Pi_q \)

\[
H \left( \sqrt{\frac{2\ln k}{\varepsilon n_k}} \cdot cs_k, \sqrt{\mu_k - \rho} \right) \rightarrow 0
\]

and

\[
R_k(s) := \left( 1 - \frac{4C}{\varepsilon^2} \cdot \frac{(\ln k)^2 \|s\|_2^4}{\mu_k n_k^2} - H \left( \sqrt{\frac{2\ln k}{\varepsilon n_k}} \cdot cs_k, \sqrt{\mu_k - \rho} \right) \right)^{-1}
\]

remains bounded as \( k \rightarrow \infty \) and \( s \in \text{supp} \nu^{(n_k, \mu_k)} \). This fact together with the estimates of
Proposition 3.7 and conditions (2) and (3) of the theorem imply that
\[
E\left(e^{\frac{2\ln k}{n_k} \text{tr}(c(S_{n_k}^{uk})^2c)}\right) = \int_{\Pi_q} e^{\frac{2\ln k}{n_k} \text{tr}(cs^2c)} \nu^{(nk,\mu_k)}(s) \, ds
\leq \int_{\Pi_q} J_{\mu_k} \left(-\left(\mu_k - \rho\right)\frac{2 \ln k}{\varepsilon \cdot n_k} \cdot cs^2c\right) \nu^{(nk,\mu_k)}(s) \cdot O(1)
= \left(\int_{\Pi_q} J_{\mu_k} \left((\mu_k - \rho)\frac{2 \ln k}{\varepsilon \cdot n_k} \cdot cs^2c\right) \nu(s)\right)^{nk} \cdot O(1)
\leq \left(\int_{\Pi_q} e^{\frac{2\ln k}{n_k} \text{tr}(cs^2c)} \nu(s)\right)^{nk} \cdot (1 + C/\mu_k)^nk \cdot O(1)
\leq \left(\int_{\Pi_q} \left(1 + \frac{2 \ln k}{\varepsilon \cdot n_k} \cdot \text{tr}(cs^2c) + O((\ln k/n_k)^2)\right) \nu(s)\right)^{nk} e^{Cnk/\mu_k} \cdot O(1)
= \left(1 + \frac{2 \ln k}{\varepsilon \cdot n_k} \cdot \text{tr}(c\sigma^2(\nu)c) + O((\ln k/n_k)^2)\right)^{nk} \cdot O(1)
\leq e^{\frac{2\ln k}{n_k} \text{tr}(c\sigma^2(\nu)c)} \cdot O(1). \tag{5.10}
\]

Employing again the Markov inequality we thus obtain
\[
P\left(\frac{1}{n_k} \langle c^2, (S_{n_k}^{uk})^2 \rangle \geq \langle c^2, \sigma^2(\nu) \rangle + \varepsilon\right)
= P\left(e^{\frac{2\ln k}{n_k} \text{tr}(c(S_{n_k}^{uk})^2c)} \geq e^{\frac{2\ln k}{n_k} \text{tr}(c\sigma^2(\nu)c) + \varepsilon}\right)
\leq \frac{1}{e^{\frac{2\ln k}{n_k} \text{tr}(c\sigma^2(\nu)c) + \varepsilon}} \cdot E\left(e^{\frac{2\ln k}{n_k} \text{tr}(c(S_{n_k}^{uk})^2c)}\right)
\leq e^{-2\ln k} \cdot O(1) = O(1/k^2)
\]
as claimed. This proves Eq. (5.5) and completes the proof of the theorem.

\[\square\]

**5.9 Remarks.**

1. Let us briefly comment on the conditions of Theorem 5.7. The most interesting case appears for $n_k = k$, where only the growth condition (1) on the indices $\mu_k$ and the compact support condition for $\nu$ remain. Condition (1) is the essential condition in the end of the proof of Eq. (5.6), and we see no possibility to weaken this one. On the other hand, the compact support of $\nu$ has been used mainly in order to derive estimate (5.10) in a smooth way. We expect that here somewhat more involved estimations (for example, by using Hölders inequality in between) might also lead to the under weaker conditions on the support of $\nu$. It is however clear that any proof along our approach will need that square-exponential moments of $\nu$ exist, i.e. $\int_{\Pi_q} e^{\text{tr}(cs^2c)} \nu(s) < \infty$ for all $c \in \Pi_q$.

2. We expect that there exist also central limit theorems associated with the laws of large numbers above. In particular, the convergence of $\chi^2$-distributions to normal distributions for $q = 1$ and convergence of Wishart distributions to multidimensional normal distributions for $q \geq 2$ suggest that in a CLT normal distributions appear as limits after taking squares after suitable renormalizations.

3. Let us briefly return to the case $q = 1$ discussed in Theorems 1.1 and 1.2. In this context one might ask for limit theorems for series of random walks on series of two-point homogeneous spaces where the number of steps and the dimensions of these spaces tend to infinity. For spheres and projective spaces over $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, central limit
theorems were given in [V2] and references cited therein. It should also be interesting to study the non-compact cases, i.e. random walks on hyperbolic spaces.

(4) As explained in Section 4, there is a close connection between Bessel convolutions on the matrix cones $\Pi_q$ and the theory of Dunkl operators on a $B_q$-Weyl chamber in $\mathbb{R}^3$ for certain indices. It is clear that we may project Theorems 5.1 and 5.7 to these particular cases. We do not state this result separately. Under the hypothesis that Dunkl operators are related to commutative hypergroups on Weyl chambers for all root systems and all positive multiplicities (see [R2]), it will become an interesting question in Dunkl theory whether there exist laws of large numbers for random walks on Weyl chambers similar to Theorems 5.1 and 5.7 when the multiplicities of Dunkl theory tend to infinity.

6 A large deviation principle

In this section we derive a large deviation principle (LDP) for $q = 1$ and $\mathbb{F} = \mathbb{R}$ which fits to the laws of large numbers given in Theorems 1.1 and 1.2. Before going into details we explain the restriction $q = 1$. Our proof of a LDP will be based on the limits

$$E\left(e^{c(S_n^q)^2}\right) \quad \text{for} \quad k \to \infty \quad \text{and all} \quad c \in H_q$$

(6.1)

(in the notion of the preceding section) together with a standard result from LDP theory (see e.g. Theorem II.6.1 of Ellis [E]) which states that suitable convergence of Laplace transforms implies a LDP. Unfortunately we can prove this convergence only for matrices of the form $\pm e \in H_q$ with $c \in \Pi_q$, as our convergence proofs depend on estimates for the Bessel functions $J_\mu$ which were derived in Section 3 from the integral representation (5.4) which is not available for arbitrary matrices $c \in H_q$ for $q \geq 2$. We therefore restrict our attention to $q = 1$ and consider the Bessel-type random walks $(S_k^q)_{k \geq 0}$ on $[0, \infty] = \Pi_1$ of indices $\mu$ with fixed law $\nu \in M^1([0, \infty])$.

6.1 Proposition. Let $\nu \in M^1(\Pi_q)$ be a probability measure with compact support. Let $(\mu_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of indices and $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ a sequence of time steps with $n_k \to \infty$ and $\lim_{k \to \infty} e^{\alpha n_k}/\mu_k = 0$ for all $\alpha > 0$. Then, $c_k(t) := \frac{1}{n_k} \ln E(e^{t(S_k^q)^2})$ converges for $t \in \mathbb{R}$ and $k \to \infty$ to

$$c(t) := \ln\left(\int_0^\infty e^{ts^2} \, d\nu(s)\right).$$

6.2 Proof. We proceed as the proof of Theorem 5.7. The case $t = 0$ is trivial. Now let $t > 0$ and put $h(t) := \int_0^\infty e^{ts^2} \, d\nu(s)$. Using Theorem 3.6 twice, we obtain that

$$c_k(-t) = \frac{1}{n_k} \ln\left(\int_0^\infty e^{-ts^2} \, d\nu(n_k, \mu_k)(s)\right)$$

$$= \frac{1}{n_k} \ln\left(\int_0^\infty J_{\mu_k} (\mu_k t s^2) \, d\nu(n_k, \mu_k)(s) + O(1/\mu_k)\right)$$

$$= \frac{1}{n_k} \ln\left(\left(\int_0^\infty J_{\mu_k} (\mu_k t s^2) \, d\nu(s)\right)^{n_k} + O(1/\mu_k)\right)$$

$$= \frac{1}{n_k} \ln\left((h(-t) + O(1/\mu_k))^{n_k} + O(1/\mu_k)\right)$$

$$= \ln(h(-t) + O(1/\mu_k)) + \frac{1}{n_k} \ln\left(1 + \frac{1}{(h(-t) + O(1/\mu_k))^{n_k} \mu_k}\right) \to c(-t) \quad (6.2)$$
by the convergence conditions of the theorem. Furthermore, we obtain from the estimations in Proposition 6.7 and with the notions there that

$$
\int_0^\infty e^{ts^2} d\nu^{(n_k, \mu_k)}(s) \geq (1 + O(1/\mu_k))^{-1} \cdot \int_0^\infty J_{\mu_k}(- (\mu_k - 3/2)ts^2) d\nu^{(n_k, \mu_k)}(s)
$$

$$
= (1 + O(1/\mu_k))^{-1} \cdot \left( \int_0^\infty J_{\mu_k}(- (\mu_k - 3/2)ts^2) d\nu(s) \right)^{n_k}
$$

$$
\geq \frac{1}{1 + O(1/\mu_k)} \left( \int_0^\infty e^{ts^2}\left[ 1 - Cs^t\mu_k^2/(\mu_k - H(s\sqrt{t}, \sqrt{\mu_k - 3/2})) \right] d\nu(s) \right)^{n_k}.
$$

As [...] → 1 uniformly on the compact set suppν, it follows readily that lim inf c_k(t) ≥ c(t).

Finally, Proposition 3.7, supp ν^{(n_k, \mu_k)} ∈ [0, M\mu_k] for a suitable M > 0, and the convergence condition of the theorem imply

$$
\int_0^\infty e^{ts^2} d\nu^{(n_k, \mu_k)}(s) \leq \int_0^\infty \frac{J_{\mu_k}(- (\mu_k - 3/2)ts^2)}{1 - Cs^t\mu_k^2/(\mu_k - H(s\sqrt{t}, \sqrt{\mu_k - 3/2}))} d\nu^{(n_k, \mu_k)}(s)
$$

$$
\leq (1 + o(1/\mu_k)) \int_0^\infty J_{\mu_k}(- (\mu_k - 3/2)ts^2) d\nu^{(n_k, \mu_k)}(s)
$$

$$
= (1 + o(1/\mu_k)) \left( \int_0^\infty J_{\mu_k}(- (\mu_k - 3/2)ts^2) d\nu \right)^{n_k}
$$

$$
\leq (1 + o(1/\mu_k))(1 + O(1/\mu_k))^{n_k} h(t)^{n_k}
$$

and thus lim sup c_k(t) ≤ c(t). In summary, c_k(t) → c(t) for t > 0 which completes the proof.

Notice that the very strong convergence condition in the proposition was needed in the end of (6.2) only where h(−t) < 1 may become arbitrarily small. This is caused by the fact that the difference estimation in Theorem 3.6 does not fit well to the "multiplicative" structure of LDPs. For all other estimates in the proof above much weaker polynomial convergence conditions are sufficient.

We here notice that the free energy function c of Proposition 6.1 is precisely the same as for the classical LDP of Cramer for sums of i.i.d. random variables on [0, ∞] with common law ν (see e.g. Ch. II.4 of [E]). Moreover, Proposition 6.1 together with Theorem II.6.1 of Ellis [E] immediately imply that in the setting of Proposition 6.4 the distributions of the random variables (S_{nk})^2 have the large deviation property with scaling parameters n_k and the rate function

$$
I(s) := \sup_{t \in \mathbb{R}} (st - c(t)) \quad (s \in \mathbb{R})
$$

in the sense of Definition II.3.1 of [E]. We skip the details here.

**6.2 Remark.** If the conditions of Proposition 6.1 are satisfied, we obtain that the free energy function c is differentiable on R with c'(0) = σ^2(ν) > 0. Theorems II.6.3 and II.6.4 of [E] now imply that (after taking square roots) S_{\mu_k}^2/\sqrt{n_k} converges to √σ^2(ν) almost surely. Notice that this strong law of large numbers (SLLN) holds under conditions which are slightly different from those in Theorem 5.7 for q = 1. This SLLN can be also derived directly for arbitrary q ≥ 1 similar to the proof of Theorem 5.7. As the conditions concerning the parameters μ_k are extremely strong here, we omit details.
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