A New Family of Singular Integral Operators Whose $L^2$-Boundedness Implies Rectifiability

Petr Chunaev

Received: 14 April 2016 / Published online: 18 February 2017
© Mathematica Josephina, Inc. 2017

Abstract Let $E \subset \mathbb{C}$ be a Borel set such that $0 < \mathcal{H}^1(E) < \infty$. David and Léger proved that the Cauchy kernel $1/z$ (and even its coordinate parts $\text{Re} z/|z|^2$ and $\text{Im} z/|z|^2$, $z \in \mathbb{C}\setminus\{0\}$) has the following property: the $L^2(\mathcal{H}^1[E])$-boundedness of the corresponding singular integral operator implies that $E$ is rectifiable. Recently Chousionis, Mateu, Prat and Tolsa extended this result to any kernel of the form $(\text{Re} z)^{2n-1}/|z|^{2n}$, $n \in \mathbb{N}$. In this paper, we prove that the above-mentioned property holds for operators associated with the much wider class of the kernels $(\text{Re} z)^{2N-1}/|z|^{2N} + t \cdot (\text{Re} z)^{2n-1}/|z|^{2n}$, where $n$ and $N$ are positive integer numbers such that $N \geq n$, and $t \in \mathbb{R}\setminus(t_1, t_2)$ with $t_1, t_2$ depending only on $n$ and $N$.

Keywords Singular integrals · Rectifiability · Calderón–Zygmund kernels

Mathematics Subject Classification 42B20 (primary) · 28A75 (secondary)

1 Introduction

Let $E \subset \mathbb{C}$ be a Borel set and $B(z, r)$ an open disc with centre $z \in \mathbb{C}$ and radius $r > 0$. By $\mathcal{H}^1(E)$ we denote the one-dimensional Hausdorff measure of $E$. A set $E$ is called rectifiable if it is contained, up to an $\mathcal{H}^1$-negligible set, in a countable union of Lipschitz graphs. A set $E$ with $\mathcal{H}^1(E) < \infty$ is called purely unrectifiable if it intersects any Lipschitz graph in a set of $\mathcal{H}^1$-measure zero.
A singular integral operator $T_K$ associated with a kernel $K$ is formally defined as

$$T_K f(z) := \int_E f(\zeta) K(z - \zeta) \, d\mathcal{H}^1(\zeta),$$

(1.1)

where $K : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is a standard kernel (see its definition, for instance, in [4]) and $f$ is some reasonable function, say, $f \in L^1(\mathcal{H}^1 \lfloor E)$. The integral in (1.1) might not converge absolutely and therefore one usually considers $T_{K, \varepsilon}$, a truncated version of $T_K$, which is defined by the above-mentioned integral but over the set $E \setminus B(z, \varepsilon)$ for some $\varepsilon > 0$. The operator $T_K$ is said to be bounded on $L^2(\mathcal{H}^1 \lfloor E)$ if the operators $T_{K, \varepsilon}$ are bounded on $L^2(\mathcal{H}^1 \lfloor E)$ uniformly on $\varepsilon$. We also recall that the principal value (p.v.) of the operator $T_K$ is said to exist $\mathcal{H}^1$-a.e. on $E$ if $\lim_{\varepsilon \to 0^+} T_{K, \varepsilon} f(z)$ exists and is finite for almost every $z \in E$ and $f$ from a reasonable functional space.

The connection between the $L^2$-boundedness, existence of p.v. of $T_K$ (defined for more general measures than $\mathcal{H}^1$) on a set $E$ and the geometric properties of this set, e.g. rectifiability, is an object of intensive investigations. They were initiated by Calderón [1], who proved that the Cauchy transform, i.e. $T_K$ with $K(z) = 1/z$, is $L^2$-bounded on Lipschitz graphs with small slope. Later on, Coifman, McIntosh and Meyer [6] removed the small Lipschitz constant assumption. In [7], David fully characterized rectifiable curves $\Gamma$, for which the Cauchy transform is bounded on $L^2(\mathcal{H}^1 \lfloor \Gamma)$: they have to satisfy the linear growth condition

$$\mathcal{H}^1(\Gamma \cap B(z, r)) \leq Cr, \quad r > 0, \quad z \in \mathbb{C}.$$

These results led to further development of tools for understanding the above-mentioned connection. For more information about this topic, see the corresponding parts of [9,13–15,21].

A second wind in the area happened after the discovery of the so-called curvature method, which became very influential in the study of the Cauchy transform and analytic capacity [8,16,17,20]. We now describe the core of the method. Given pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, their Menger curvature is

$$c(z_1, z_2, z_3) = \frac{1}{R(z_1, z_2, z_3)},$$

where $R(z_1, z_2, z_3)$ is the radius of the circle passing through $z_1, z_2$ and $z_3$ (with $R(z_1, z_2, z_3) = \infty$ and $c(z_1, z_2, z_3) = 0$ if the points are collinear). It is easily seen that the curvature can be calculated in geometrical terms in different ways, e.g.

$$c(z_1, z_2, z_3) = \frac{4S(z_1, z_2, z_3)}{|z_1 - z_2||z_1 - z_3||z_2 - z_3|} = \frac{2\sin \hat{z_1z_2z_3}}{|z_1 - z_3|},$$

(1.2)

where $S(z_1, z_2, z_3)$ stands for the area of the triangle $(z_1, z_2, z_3)$ and $\hat{z_1z_2z_3}$ is the angle of this triangle opposite to the side $z_1z_3$. 
The relationship between the curvature and the Cauchy kernel originates from the following identity due to Melnikov [17]:

\[ c(z_1, z_2, z_3)^2 = \sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})(z_{s_3} - z_{s_1})}, \quad (1.3) \]

where \( S_3 \) is the group of permutations of three elements. It is very important that the quantity in the right-hand side turns out to be a non-negative real number. Let us also define the so-called curvature of a Borel measure \( \mu \):

\[ c^2(\mu) = \int \int \int c(z_1, z_2, z_3)^2 \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3), \quad (1.4) \]

introduced by Melnikov, too. One can consider \( c^2_\varepsilon(\mu) \), a truncated version of \( c^2(\mu) \), which is the above-mentioned triple integral over the set

\[ \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_k - z_j| \geq \varepsilon > 0, \quad 1 \leq k, j \leq 3, \quad j \neq k \}. \]

Clearly, \( c^2_\varepsilon(\mu) \) is always non-negative due to (1.3).

If \( \mu \) is a finite Borel measure with linear growth, i.e. \( \mu(B(z, r)) \leq Cr \) for all \( z \in \text{spt} \mu \), then the relation between the curvature and the \( L^2(\mu) \)-norm of the Cauchy transform (of \( f(\zeta) \equiv 1 \)) is specified by the following Melnikov–Verdera identity [18]:

\[ \int \left| \int_{\mathbb{C} \setminus B(z, \varepsilon)} \frac{d\mu(\xi)}{\xi - z} \right|^2 d\mu(z) = \frac{1}{6} c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \quad |O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C}). \quad (1.5) \]

Relying on the curvature method described above, Mattila, Melnikov and Verdera [16] proved that if \( E \subset \mathbb{C} \) is an Ahlfors–David regular set, i.e. \( 0 < \mathcal{H}^1(E) < \infty \) and there exists a constant \( C > 0 \) such that

\[ C^{-1}r \leq \mathcal{H}^1(E \cap B(z, r)) \leq Cr, \quad z \in E, \quad 0 < r < \mathcal{H}^1(E), \]

then the Cauchy transform is \( L^2(\mathcal{H}^1[E]) \)-bounded if and only if \( E \) is contained in an Ahlfors–David regular curve.

Later on, essentially using that the curvature is non-negative, David and Léger made the following deep contribution (see a brief exposition of the proof in Section 5).

**Theorem A** ([12]) **Given a Borel set \( E \subset \mathbb{C} \) such that \( 0 < \mathcal{H}^1(E) < \infty \), if \( c^2(\mathcal{H}^1[E]) < \infty \), then \( E \) is rectifiable. Moreover, if the Cauchy transform is \( L^2(\mathcal{H}^1[E]) \)-bounded, then \( E \) is rectifiable.**

This theorem is stated in [12] for \( \mathbb{R}^d \) but we formulate it only for \( \mathbb{R}^2 = \mathbb{C} \). Note also that its second statement is a direct corollary of the first one and (1.5).
Until recently, very few things were known in this direction beyond the Cauchy kernel. For instance, the same result is true for its coordinate parts [2]:

$$\text{Re } z/|z|^2 \text{ or } \text{Im } z/|z|^2, \quad z \in \mathbb{C}\{0\}. \quad (1.6)$$

Indeed, consider the following permutations:

$$p_K(z_1, z_2, z_3) := K(z_1 - z_2)K(z_1 - z_3) + K(z_2 - z_1)K(z_2 - z_3) + K(z_3 - z_1)K(z_3 - z_2), \quad (1.7)$$

where $K$ is some standard real kernel. Then it is not difficult to show that if $K$ is one of the kernels (1.6), then

$$p_K(z_1, z_2, z_3) = \frac{1}{4} c(z_1, z_2, z_3)^2. \quad (1.8)$$

This fact was a motivation point of the recent paper [2] by Chousionis, Mateu, Prat and Tolsa. The David–Léger result is extended there to the kernels

$$\kappa_n(z) := \frac{(\text{Re } z)^{2n-1}}{|z|^{2n}}, \quad n \in \mathbb{N}. \quad (1.9)$$

Namely, it is shown in [2] that for any given triple $(z_1, z_2, z_3) \in \mathbb{C}^3$,

$$p_{\kappa_n}(z_1, z_2, z_3) \geq 0, \quad (1.10)$$

and $p_{\kappa_n}(z_1, z_2, z_3) = 0$ if and only if the points $z_1, z_2, z_3$ are collinear. Moreover, it is proved that the permutations $p_{\kappa_n}(z_1, z_2, z_3)$ behave similarly to $c^2(z_1, z_2, z_3)$ for triangles with comparable sides, whose one side makes a big angle with the vertical line. This fact enables the authors of [2] to adapt the method from [12] to the kernels $\kappa_n$. This adaptation however requires them to make several essential modifications in crucial points, where the curvature must be exchanged for the permutations $p_{\kappa_n}$, and provide new arguments whenever the scheme of Léger does not work (see also Sect. 5 for more details).

To state the corresponding result we need the following generalization of (1.4):

$$p_K(\mu) = \iint\int p_K(z_1, z_2, z_3) \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3). \quad (1.11)$$

**Theorem B** ([2]) Let $n \geq 1$. Given a Borel set $E \subset \mathbb{C}$ such that $0 < \mathcal{H}^1(E) < \infty$, if $p_{\kappa_n}(\mathcal{H}^1|E) < \infty$, then $E$ is rectifiable. Moreover, if the operator $T_{\kappa_n}$ is $L^2(\mathcal{H}^1|E)$-bounded, then $E$ is rectifiable.

Obviously, for $n = 1$ one gets the statement for the real part of the Cauchy kernel. However, for $n \geq 2$ this is the first example of singular integral operators with the above-mentioned property, which are not directly related to the Cauchy transform.
In this paper, we consider a linear combination of the kernels (1.9), namely, the parametric kernels

\[ K_t(z) := \kappa_N(z) + t \cdot \kappa_n(z), \quad N > n, \quad n, N \in \mathbb{N}^+, \quad t \in \mathbb{R}, \quad (1.12) \]

where the parameters \( n, N \) and \( t \) are fixed. We find values of \( t \), depending on \( n \) and \( N \), such that a result, analogous to the David–Léger theorem, is valid for singular integral operators associated with the kernels \( K_t \). For this purpose we first study the sign of the permutations (1.7) for the kernels (1.12) and then, for the case when these permutations are non-negative, adapt the scheme from [2] to prove the result of David–Léger type. The next section contains the corresponding statements.

2 Main Results

First of all let us mention that the case \( t = 0 \) in the theorems below agrees with the inequality (1.10) and Theorem B, proved in [2]. We now indicate the values of \( t \) such that the permutations \( p_{K_t}(z_1, z_2, z_3) \) are non-negative for all triples \((z_1, z_2, z_3)\).

**Theorem 1** Let \( K_t \) be a kernel of the form (1.12) with \( t = 0 \) or

\[ t \in \mathbb{R} \setminus \left( -\frac{1}{2} \left( 3 + \frac{9 - 4N}{n} \right); \frac{2 - \frac{N}{n}}{2} \right), \quad n < N \leq 2n, \quad (2.1) \]

\[ t \in \mathbb{R} \setminus \left( -\frac{1}{2} \left( 3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4\frac{N}{n}} \right); \rho_{n,N} \right), \quad N \geq 2n, \quad (2.2) \]

where \( \rho_{n,N} := \left( \frac{N}{n} - 2 \right) \sqrt{N - 2n} \). In particular, \( t \in \mathbb{R} \setminus (-2; 0) \) for \( N = 2n \). Then

\[ p_{K_t}(z_1, z_2, z_3) \geq 0 \quad \text{for all} \quad (z_1, z_2, z_3) \in \mathbb{C}^3. \]

Furthermore, the range of the parameter \( t \) in the case \( N = 2n \) is sharp.

**Remark 1** The conditions (2.1) and (2.2), guaranteeing that \( p_{K_t}(z_1, z_2, z_3) \geq 0 \), cannot be weakened much in the following sense. As we will show in Sect. 4, there are triples \((z_1, z_2, z_3)\) such that \( p_{K_t}(z_1, z_2, z_3) \) can change sign if

\[ t \in \left( -\frac{N}{n}; 0 \right) \quad \text{for all} \quad n \text{ and } N, \]

\[ t \in \left( 0; \frac{2 - \frac{N}{n}}{e^{3/2}} \right) \quad \text{for} \quad N \gg n \quad \text{(i.e. } N \text{ is large enough with respect to } n). \quad (2.3) \]

Surprisingly, in this context the case \( t = 0 \) is an isolated point with non-negative permutations. Thus the curvature method, requiring the permutations to be non-negative, cannot be applied directly for \( t \) indicated in (2.3).

From the aforesaid it follows that (2.1) and (2.2) are indeed sharp for \( N = 2n \).

Figure 1 illustrates Theorem 1 and this remark for \( n = 3 \) and different \( N > 3 \) (consider the horizontal line passing through a fixed positive integer \( N \) in order to determine the corresponding \( t \)). The green area represents the values of \( t \), indicated in...
(2.1) and (2.2), i.e. those where $p_{K_t}$ are guaranteed to be non-negative (the boundaries are included). In the blue area (the boundaries are not included), obtained by computer-based exhaustive search, the permutations can change sign. Note that the part of the blue area for $t < 0$ is exactly the former in (2.3). Moreover, the part for $t > 0$ quite agrees with the latter in (2.3). The white area is not covered by our results and, generally speaking, we can say nothing about the sign of $p_{K_t}$ therein. However, computer experiments suggest that the permutations $p_{K_t}$ are non-negative everywhere except the blue area and thus (2.3) seems to give likely boundaries for $t$, whose corresponding permutations can change sign.

Relying deeply on Theorem 1, we will prove the following David–Léger type result.

**Theorem 2** Let $K_t$ be a kernel of the form (1.12) with $t$, mentioned in Theorem 1. Given a Borel set $E \subset \mathbb{C}$ such that $0 < \mathcal{H}^1(E) < \infty$, if $p_{K_t}(\mathcal{H}^1|E) < \infty$, then $E$ is rectifiable. Moreover, if the operator $T_{K_t}$ is $L^2(\mathcal{H}^1|E)$-bounded, then $E$ is rectifiable.
Remark 2 It is known that for $t = -1$ (see the red line in Fig. 1), which belongs to the area, where the permutations can change sign, the statement of Theorem 2 is not valid anymore, i.e. $L^2$-boundedness does not imply rectifiability. Indeed, it is a corollary of the following result due to Huovinen (note that all kernels $K_t$ of the form (1.12) with $t = -1$ belong to Huovinen’s class $\mathcal{H}$).

**Theorem C** ([11]) Let $K$ belong to the class $\mathcal{H}$ of odd kernels satisfying

$$|K(x - y) - K(x - z)| \leq \frac{C}{|x - y||x - z|}, \quad |K(z)| \leq \frac{C}{|z|}, \quad x, y, z \in \mathbb{C},$$

$K(r) = 0, \quad K(z) = -K(-\overline{z}), \quad r \in \mathbb{R}, \quad C = \text{const.}$

Then there exists a purely unrectifiable set $E$ with $\mathcal{H}^1(E) > 0$, such that the operator $T_K$, associated with the kernel $K$, is bounded on $L^2(\mathcal{H}^1|E)$ and, moreover, p.v. $T_K$ exists $\mathcal{H}^1$-a.e. and is finite.

For a particular family of the kernels (1.12) with $(n, N) = (1, 2)$, i.e.

$$k_t(z) := \frac{(\text{Re} z)^3}{|z|^4} + t \cdot \frac{\text{Re} z}{|z|^2},$$

even more is known. Namely, for $t = -3/4$, whose corresponding $p_{K_t}$ can change sign, there also exists a purely unrectifiable set $E$ such that $T_{K_t}$ is $L^2(\mathcal{H}^1|E)$-bounded. One can get this from the following result of Jaye and Nazarov for the kernel $k(z) := \overline{z}/z^2$, noting that $\text{Re} k(z) = 4k_t(z)$ for $t = -3/4$. We formulate it in a slightly different form than in [10].

**Theorem D** ([10]) There exists a purely unrectifiable set $E$ with $\mathcal{H}^1(E) > 0$, such that the operator $T_k$, associated with the kernel $k$, is bounded on $L^2(\mathcal{H}^1|E)$ and, moreover, p.v. $T_k$ fails to exist $\mathcal{H}^1$-a.e.

Figure 2 illustrates known results for the kernels $k_t$. By Theorems 1 and 2, if $t \in \mathbb{R} \setminus (-2; 0)$, then the permutations $p_{K_t}$ are non-negative and the $L^2(\mathcal{H}^1|E)$-boundedness of $T_{K_t}$ implies the rectifiability of $E$ (see the intervals coloured green). By the arguments from Remarks 1 and 2, the permutations $p_{K_t}$ for $t \in (-2; 0)$ change sign (the interval coloured blue) and there are two values of $t$ (the red points) such that the operator $T_{K_t}$ is $L^2(\mathcal{H}^1|E)$-bounded but $E$ is not rectifiable.

### 3 Auxiliary Results

This section is devoted to several auxiliary lemmas, which will be used to prove Theorems 1 and 2 in Sect. 5.
If a kernel $K$ is real and odd, then one can show that the permutations (1.7) are invariant under translations. This can be done, e.g. by the substitutions $u = z_1 - z_2$ and $v = z_1 - z_3$. Consequently, one point can be always fixed and it is enough to consider only permutations of the form

$$p_K(0, u, v) = K(u)K(v) + K(u)K(u - v) + K(v)K(v - u),$$

where $u, v \in \mathbb{C}\setminus\{0\}$ are distinct points. The kernels (1.9) and (1.12) that we study are real and odd and hence we can use (3.1) instead of (1.7). Furthermore, the case of collinear points $u$ and $v$ is trivial as then $p_K(0, u, v) \equiv 0$ and thus we can skip it.

We will use the following lemma many times below. Note that it can be easily generalized for any other couple of kernels instead of $\kappa_n$ and $\kappa_N$.

**Lemma 1** Given $K_t$ of the form (1.12),

$$p_{K_t}(0, u, v) = p_{K_t}(0, u, v) + \varphi_{n,N}(0, u, v)t + p_{K_t}(0, u, v)t^2,$$

where

$$\varphi_{n,N}(0, u, v) = \kappa_N(u)(\kappa_n(v) + \kappa_n(u - v)) + \kappa_N(v)(\kappa_n(u) + \kappa_n(v - u)) + \kappa_N(u - v)(\kappa_n(u) - \kappa_n(v)).$$

**Proof** We substitute (1.12) into (3.1) and get

$$p_{K_t}(0, u, v) = \kappa_N(u)\kappa_N(v) + (\kappa_N(u)\kappa_n(v)
+ \kappa_n(u)\kappa_N(v))t + \kappa_n(u)\kappa_n(v)t^2
+ \kappa_N(u)\kappa_N(u - v) + (\kappa_N(u)\kappa_n(u - v)
+ \kappa_n(u)\kappa_N(u - v))t + \kappa_n(u)\kappa_n(u - v)t^2
+ \kappa_N(v)\kappa_N(v - u) + (\kappa_N(v)\kappa_n(v - u)
+ \kappa_n(v)\kappa_N(v - u))t + \kappa_n(v)\kappa_n(v - u)t^2.$$ 

To finish the proof it is enough to group the terms and take into account (3.1). \qed

It is important that the leading coefficient of the quadratic polynomial (3.2) (with respect to $t$) is always non-negative by the property (1.10).

From now on, in order to simplify formulas we skip $(0, u, v)$ in permutations and other expressions if there is no confusion. For example, we write $p_K$ instead of $p_K(0, u, v)$. In addition, we use the following notations:

$$\lambda_1 := \frac{\text{Re }u}{|u|}, \quad \lambda_2 := \frac{\text{Re }v}{|v|}, \quad \lambda_3 := \frac{\text{Re }u - v}{|u - v|}, \quad \Lambda := \lambda_1\lambda_2\lambda_3,$$

where the denominators do not vanish as the points $u$ and $v$ are assumed to be distinct and non-collinear. Note that in these terms,

$$p_{\kappa_n} = \frac{(\lambda_1\lambda_2)^{2n-1}}{|u||v|} + \frac{(\lambda_1\lambda_3)^{2n-1}}{|u||u - v|} - \frac{(\lambda_2\lambda_3)^{2n-1}}{|v||v - u|},$$

Springer
and

\[
\varphi_{n,N} = \frac{\lambda_1^{2N-1}}{|u|} \left( \frac{\lambda_2^{2n-1}}{|v|} + \frac{\lambda_3^{2n-1}}{|u - v|} \right) \\
+ \frac{\lambda_2^{2N-1}}{|v|} \left( \frac{\lambda_1^{2n-1}}{|u|} - \frac{\lambda_3^{2n-1}}{|v - u|} \right) \\
+ \frac{\lambda_3^{2N-1}}{|u - v|} \left( \frac{\lambda_1^{2n-1}}{|u|} - \frac{\lambda_2^{2n-1}}{|v|} \right).
\] (3.6)

What is more, another representation of \(\varphi_{n,N}\) is valid.

**Lemma 2** In terms of (3.4) it holds that

\[
\varphi_{n,N} = \tau_1 p_{\kappa_n} - \tau_2,
\] (3.7)

where

\[
\tau_1 := \lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)}, \quad 0 \leq \tau_1 \leq 3,
\] (3.8)

and

\[
\tau_2 := \lambda^{2(N-n)} \left( \frac{\lambda_1 \lambda_2}{|u||v|} \right)^{2n-1} \left( \frac{\lambda_1 \lambda_3}{|u||u - v|} \right)^{2n-1} \\
- \frac{(\lambda_2 \lambda_3)}{|v||v - u|} \left( \frac{\lambda_2}{|u||u - v|} \right)^{2n-1} \left( \frac{\lambda_3}{|v|} \right)^{2n-1} \left( \frac{\lambda_1}{|u|} \right)^{2n-1} (\lambda_2 \lambda_3).
\] (3.9)

In particular, \(\tau_2 \equiv 0\) if \(N = 2n\).

**Proof** Direct multiplication of \(\tau_1\) by \(p_{\kappa_n}\) gives

\[
\left( \lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)} \right) \\
\times \left( \frac{\lambda_1 \lambda_2}{|u||v|} \right)^{2n-1} \left( \frac{\lambda_1 \lambda_3}{|u||u - v|} \right)^{2n-1} \left( \frac{\lambda_2 \lambda_3}{|v||v - u|} \right)^{2n-1} \\
= \left( \frac{\lambda_3^{2(N-n)}}{|u||v|} \right)^{2n-1} \left( \frac{\lambda_2^{2(N-n)}}{|u||u - v|} \right)^{2n-1} \\
- \left( \frac{\lambda_1^{2(N-n)}}{|v||v - u|} \right)^{2n-1} \\
+ \left( \frac{\lambda_2^{2n-1}}{|u|} \right)^{2n-1} \left( \frac{\lambda_3^{2n-1}}{|v|} \right)^{2n-1} \left( \frac{\lambda_1^{2n-1}}{|u - v|} \right)^{2n-1}.
\]

 Springer
which is exactly \( \tau_2 + \varphi_{n,N} \) by (3.6) and (3.9). \( \square \)

**Lemma 3** Given \( \kappa_n \) and \( \kappa_N \) of the form (1.9),

\[
\frac{N}{n} \cdot \Lambda^{2(N-n)} \cdot p_{\kappa_n} \leq p_{\kappa_N}, \quad 1 \leq n \leq N.
\]

(3.10)

Note that this inequality for \( n = 1 \) was obtained in [2, Proof of Lemma 2.3]. We will use the following lemma from there in order to prove the general form.

**Lemma 4** (Proof of Proposition 2.1 in [2]) One has the representation

\[
p_{\kappa_m} = \sum_{k=1}^{m} \binom{m}{k} \Lambda^{2(m-k)} h_k(u, v),
\]

where \( h_k(u, v) \geq 0 \) and are defined as follows:

\[
h_k(u, v) = (|u||v||u - v|)^{-2k} \left( (\text{Re} u \text{ Re} v)^{2k-1} (\text{Im} (u - v))^{2k} + (\text{Re} u \text{ Re} (u - v))^{2k-1} (\text{Im} v)^{2k} + (\text{Re} v \text{ Re} (v - u))^{2k-1} (\text{Im} u)^{2k} \right).
\]

**Proof** Within the settings of Lemma 4,

\[
\frac{\Lambda^{2(N-n)} p_{\kappa_n}}{p_{\kappa_N}} = \sum_{k=1}^{n} \binom{n}{k} \binom{N}{k}^{-1} \frac{H_k(u, v)}{\sum_{k=1}^{N} H_k(u, v)},
\]

where \( H_k(u, v) := \binom{N}{k} \Lambda^{2(N-k)} h_k(u, v) \geq 0 \). Furthermore,

\[
\binom{n}{k} \binom{N}{k}^{-1} = \frac{n! (N-k)!}{(n-k)! N!} = \frac{(n-k+1) \cdots n}{(N-k+1) \cdots N} \leq \frac{n}{N}, \quad 1 \leq k \leq n,
\]

and finally

\[
\frac{\Lambda^{2(N-n)} p_{\kappa_n}}{p_{\kappa_N}} \leq \frac{n}{N} \cdot \frac{\sum_{k=1}^{n} H_k(u, v)}{\sum_{k=1}^{N} H_k(u, v)} \leq \frac{n}{N}, \quad n \leq N,
\]

which is the desired result. \( \square \)
Lemmas 1, 2 and 3 enable us to obtain lower pointwise estimates for the permutations $p_{K_t}$ via the permutations $p_{\kappa_n}$ for some $t$. To do so, we will use (3.7) and (3.10) to estimate the coefficients of the quadratic polynomial (3.2). Let us start with the case $n < N \leq 2n$.

**Lemma 5** Given $K_t$ of the form (1.12) with $n < N \leq 2n$, if

$$t \in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( 3 + \sqrt{9 - 4\frac{N}{n}} \right); 2 - \frac{N}{n} \right],$$

then $p_{K_t} \geq C(t) \cdot p_{\kappa_n}$ with some $C(t) > 0$.

**Proof** To get the required estimate, we first look at the expression for $\tau_2$ in (3.9) for our case. Since $n < N \leq 2n$, from (3.5) and (3.9) we immediately get

$$\tau_2 = \Lambda^{2(N-n)} \cdot p_{\kappa_{2n-N}}, \quad 0 \leq 2n - N \leq n - 1,$$

with $\tau_2 \equiv 0$ if $N = 2n$. Consequently, by (3.2) and (3.7),

$$p_{K_t} = p_{\kappa_N} + \left( \tau_1 p_{\kappa_n} - \Lambda^{2(N-n)} p_{\kappa_{2n-N}} \right) t + p_{\kappa_n} t^2. \quad (3.11)$$

Now we show that the right-hand side of (3.11) for $t$ mentioned in the lemma is bounded from below by $p_{\kappa_n}$, multiplied by a positive constant, depending only on $t$.

Applying the inequality (3.10) to $p_{\kappa_N}$ and $p_{\kappa_{2n-N}}$ in (3.11) for $t \geq 0$ gives

$$p_{K_t} \geq \left( \frac{N}{n} \Lambda^{2(N-n)} + (\tau_1 - 2 + \frac{N}{n}) t + t^2 \right) \cdot p_{\kappa_n} = f(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n}, \quad (3.12)$$

where $\xi_j := \lambda_j^{2(N-n)} \in [0, 1]$, $j = 1, 2, 3$, and

$$f(\xi_1, \xi_2, \xi_3) := \frac{N}{n} \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3 - 2 + \frac{N}{n}) t + t^2. \quad (3.13)$$

Analysis of $\partial f / \partial \xi_j$ shows that $f$ is non-decreasing for $t \geq 0$ with respect to each $\xi_j \in [0, 1]$. Consequently,

$$f(\xi_1, \xi_2, \xi_3) \geq f(0, 0, 0) = t \left( t - 2 + \frac{N}{n} \right),$$

which is strictly positive for $t > 2 - \frac{N}{n} \geq 0$.

For $t \leq 0$ we apply (3.10) to $p_{\kappa_N}$ and use that $p_{\kappa_{2n-N}} \geq 0$ (see (1.10)). This yields

$$p_{K_t} \geq \left( \frac{N}{n} \Lambda^{2(N-n)} + \tau_1 t + t^2 \right) \cdot p_{\kappa_n} = F(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n}, \quad (3.14)$$

where the function

$$F(\xi_1, \xi_2, \xi_3) := \frac{N}{n} \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3) t + t^2 \quad (3.15)$$
is non-increasing for \( t \leq -\frac{N}{n} \) with respect to each \( \xi_j \in [0, 1] \). Consequently,

\[
F(\xi_1, \xi_2, \xi_3) \geq F(1, 1, 1) = \frac{N}{n} + 3t + t^2,
\]

where the latter expression is positive for \( t < -\frac{1}{2} \left( 3 + \sqrt{9 - 4\frac{N}{n}} \right) \leq -\frac{N}{n} \). \( \square \)

Now let \( N > 2n \). Note that the following lemma coincides with the previous one if we put \( N = 2n \).

**Lemma 6** Given \( K_t \) of the form (1.12) with \( N > 2n \), if

\[
t \in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( 3 + \rho_n,N + \sqrt{(3 + \rho_n,N)^2 - 4\frac{N}{n}} \right) ; \rho_n,N \right],
\]

\[
\rho_n,N = \left( \frac{N}{n} - 2 \right) \sqrt{N - 2n},
\]

then \( p_{K_t} \geq C(t) \cdot p_{\kappa_n} \) with some \( C(t) > 0 \).

**Proof** We will again estimate the coefficients of the polynomial (3.2) in terms of \( p_{\kappa_n} \). At first, we will estimate \( \phi_{\kappa_n,N} \). By (3.7), this will only need to estimate \( |\tau_2| \).

As we have already mentioned before Lemma 1, the permutations \( p_{K_t} \) and \( p_{\kappa_n} \) are invariant under translations. Therefore we can assume without loss of generality that all triangles \((0, u, v)\) that we consider belong to the half plane \( \text{Re} z \geq 0 \). This will be necessary in the further analysis of angles of these triangles.

From now on, we use the following notation additionally to (3.4):

\[
\sin \alpha_j := \lambda_j, \quad \lambda_j \in [-1; 1], \quad j = 1, 2, 3.
\]

We also suppose that \( \lambda_j^2 \) are pairwise distinct. One can get the other case by passage to a limit below. For the geometrical interpretation of \( \alpha_j \) see Figs. 3 and 4.

Now we aim to represent \( \tau_2 \) from (3.9) in terms of the curvature written in the form (1.2). For this purpose we will segregate the area squared \( S(0, u, v)^2 \) in the numerator and \( |u|^2|v|^2|u - v|^2 \) in the denominator of \( \tau_2 \). First, from (3.9), taking into account (3.4), we obtain

\[
\tau_2 = \frac{\Lambda^{2n-1}}{|u||v||u - v|} \frac{(\lambda_1 \lambda_2) 2(2n-N)-1|u - v| + (\lambda_1 \lambda_3) 2(2n-N)-1|v| - (\lambda_2 \lambda_3) 2(2n-N)-1|u|}{\Lambda^{2(2n-N)-1}}
\]

\[
= \frac{\Lambda^{2n-1}}{|u||v||u - v|} \left( \lambda_3^{2(N-2n)} \text{Re} (u - v) + \lambda_2^{2(N-2n)} \text{Re} v - \lambda_1^{2(N-2n)} \text{Re} u \right)
\]

\[
= \frac{\Lambda^{2n-1}}{|u||v||u - v|} \left( \text{Re} u \left( \lambda_3^{2(N-2n)} - \lambda_1^{2(N-2n)} \right) - \text{Re} v \left( \lambda_3^{2(N-2n)} - \lambda_2^{2(N-2n)} \right) \right)
\]

\[
= \frac{\Lambda^{2n-1}}{|u||v||u - v|} \left( \lambda_1 |u| \left( \lambda_3^2 - \lambda_1^2 \right) A_1(u, v) - \lambda_2 |v| \left( \lambda_3^2 - \lambda_2^2 \right) A_2(u, v) \right),
\]
where

\[
A_1(u, v) := \frac{\lambda_3^{2(N-2n)} - \lambda_1^{2(N-2n)}}{\lambda_3^{2} - \lambda_1^{2}} \quad \text{and} \\
A_2(u, v) := \frac{\lambda_3^{2(N-2n)} - \lambda_2^{2(N-2n)}}{\lambda_3^{2} - \lambda_2^{2}}.
\]  
(3.17)

Finally, we can rewrite \( \tau_2 \) as

\[
\tau_2 = \frac{\Lambda^{2n-1}}{|u|^2|v|^2|u-v|^2} \cdot A(u, v),
\]  
(3.18)

where \( A(u, v) := |u||v||u-v| \left( \lambda_1 (\lambda_3^{2} - \lambda_1^{2}) |u|A_1(u, v) - \lambda_2 (\lambda_3^{2} - \lambda_2^{2}) |v|A_2(u, v) \right) \).

By (3.16) and the formulas for the sum of sines and the sine of a double angle,

\[
\lambda_3^{2} - \lambda_1^{2} = (\sin \alpha_3 + \sin \alpha_1) (\sin \alpha_3 - \sin \alpha_1) = 2 \sin \frac{\alpha_3 + \alpha_1}{2} \cos \frac{\alpha_3 - \alpha_1}{2} \cdot 2 \sin \frac{\alpha_3 - \alpha_1}{2} \cos \frac{\alpha_3 + \alpha_1}{2} = \sin (\alpha_3 + \alpha_1) \sin (\alpha_3 - \alpha_1).
\]

Analogously, \( \lambda_3^{2} - \lambda_2^{2} = \sin (\alpha_3 + \alpha_2) \sin (\alpha_3 - \alpha_2) \). Thus

\[
A(u, v) = |u|^2|v||u-v| \sin (\alpha_3 + \alpha_1) \sin (\alpha_3 - \alpha_1) \lambda_1 A_1(u, v) \]
\[
-|u|^2|v|^2|u-v| \sin (\alpha_3 + \alpha_2) \sin (\alpha_3 - \alpha_2) \lambda_2 A_2(u, v). 
\]  
(3.19)

Now let us see how one can calculate the angles \( \angle(u, 0, v), \angle(0, u, v) \) and \( \angle(0, 0, u) \) of the triangle \((0, u, v)\), using the angles \( \alpha_j, j = 1, 2, 3 \). Recall that the triangles \((0, u, v)\) belong to the half plane \( \text{Re} \ z \geq 0 \). Thus only two cases are possible:

1. The vertexes \( u \) and \( v \) both lie in the same (first or forth) quarter of the plane.
2. The vertexes \( u \) and \( v \) lie in different quarters of the plane.

One can check that four options are realizable in the case 1 (see the examples in Fig. 3; several other situations are possible but they produce the same cases):

1a. \( \angle(u, 0, v) = \alpha_1 - \alpha_2, \angle(0, u, v) = -(\alpha_1 - \alpha_3), \angle(0, v, u) = \pi + (\alpha_2 - \alpha_3); \)
1b. \( \angle(u, 0, v) = -(\alpha_1 - \alpha_2), \angle(0, u, v) = \alpha_1 - \alpha_3, \angle(0, v, u) = \pi - (\alpha_2 - \alpha_3); \)
1c. \( \angle(u, 0, v) = \alpha_1 - \alpha_2, \angle(0, u, v) = \pi - (\alpha_1 + \alpha_3), \angle(0, v, u) = -(\alpha_2 + \alpha_3); \)
1d. \( \angle(u, 0, v) = -(\alpha_1 - \alpha_2), \angle(0, u, v) = \pi + (\alpha_1 + \alpha_3), \angle(0, v, u) = -(\alpha_2 + \alpha_3). \)

In the case 2 (see Fig. 4) one always has

\[
\angle(u, 0, v) = \pi - (\alpha_1 + \alpha_2), \\
\angle(0, u, v) = \alpha_1 - \alpha_3, \\
\angle(0, v, u) = \alpha_2 + \alpha_3.
\]
Fig. 3  Triangles in the case 1 (the proof of Lemma 6)

Fig. 4  Triangles in the case 2 (the proof of Lemma 6)
Consequently, taking into account the formulas

\[ S(0, u, v) = \frac{1}{2} |u| |v| \sin \angle(u, 0, v) \]
\[ = \frac{1}{2} |u| |u - v| \sin \angle(0, u, v) \]
\[ = \frac{1}{2} |v| |u - v| \sin \angle(0, v, u), \]

we conclude from (3.19) that,

- in the cases 1a and 1b:

\[
A(u, v) = |u| |v| \sin(\pm(\alpha_2 - \alpha_1)) |u| |u - v| \sin(\pm(\alpha_1 - \alpha_3)) \\
\times \frac{\sin(\alpha_3 + \alpha_1)}{\sin(\alpha_1 - \alpha_2)} \lambda_1 A_1(u, v) \\
- |u| |v| \sin(\pm(\alpha_2 - \alpha_1)) |v| |u - v| \sin(\pi \pm (\alpha_3 - \alpha_2)) \\
\times \frac{\sin(\alpha_3 - \alpha_2)}{\sin(\alpha_1 - \alpha_2)} \lambda_2 A_2(u, v) \\
= 4 S(0, u, v)^2 \frac{\sin(\alpha_3 + \alpha_1) \lambda_1 A_1(u, v) - \sin(\alpha_3 + \alpha_2) \lambda_2 A_2(u, v)}{\sin(\alpha_1 - \alpha_2)};
\]

- in the cases 1c and 1d:

\[
A(u, v) = |u| |v| \sin(\pm(\alpha_1 - \alpha_2)) |u| |u - v| \sin(\pi \mp (\alpha_1 + \alpha_3)) \\
\times \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\alpha_1 - \alpha_2)} \lambda_1 A_1(u, v) \\
- |u| |v| \sin(\pm(\alpha_1 - \alpha_2)) |v| |u - v| \sin(\pm(\alpha_2 + \alpha_3)) \\
\times \frac{\sin(\alpha_3 - \alpha_2)}{\sin(\alpha_1 - \alpha_2)} \lambda_2 A_2(u, v) \\
= 4 S(0, u, v)^2 \frac{\sin(\alpha_3 - \alpha_1) \lambda_1 A_1(u, v) - \sin(\alpha_3 - \alpha_2) \lambda_2 A_2(u, v)}{\sin(\alpha_1 - \alpha_2)};
\]

- in the case 2:

\[
A(u, v) = -|u| |v| \sin(\alpha_1 + \alpha_2) |u| |u - v| \sin(\alpha_1 - \alpha_3) \\
\times \frac{\sin(\alpha_3 + \alpha_1)}{\sin(\alpha_1 + \alpha_2)} \lambda_1 A_1(u, v) \\
- |u| |v| \sin(\alpha_1 + \alpha_2) |v| |u - v| \sin(\alpha_2 + \alpha_3) \\
\times \frac{\sin(\alpha_3 - \alpha_2)}{\sin(\alpha_1 + \alpha_2)} \lambda_2 A_2(u, v) \\
= -4 S(0, u, v)^2 \frac{\sin(\alpha_3 + \alpha_1) \lambda_1 A_1(u, v) + \sin(\alpha_3 - \alpha_2) \lambda_2 A_2(u, v)}{\sin(\alpha_1 + \alpha_2)}.
\]
Note that the substitutions $\alpha_1 \mapsto -\alpha_1, \alpha_2 \mapsto -\alpha_2 (\lambda_1 \mapsto -\lambda_1, \lambda_2 \mapsto -\lambda_2)$ in the expression for $A(u, v)$ for the cases 1a and 1b give $A(u, v)$ in the cases 1c and 1d. Moreover, the substitution $\alpha_2 \mapsto -\alpha_2 (\lambda_2 \mapsto -\lambda_2)$ in $A(u, v)$ for the cases 1a and 1b gives $-A(u, v)$ in the case 2. In what follows, this allows us to consider only one expression for $A(u, v)$, say, the one corresponding to 1a and 1b, instead of the three. This reduction will not affect the final result. By this reason, let

$$A(u, v) = 4S(0, u, v)^2 \cdot \frac{V(u, v)}{\sin(\alpha_1 - \alpha_2)},$$

where

$$V(u, v) := \sin(\alpha_3 + \alpha_1)\lambda_1 A_1(u, v) - \sin(\alpha_3 + \alpha_2)\lambda_2 A_2(u, v). \quad (3.20)$$

From this and (3.18) by the formula (1.2), connecting the curvature $c(0, u, v)$ and the area $S(0, u, v)$, we get

$$\tau_2 = \frac{4S(0, u, v)^2}{|u|^2|v|^2|u - v|^2} \cdot \Lambda^{2n-1} \cdot \frac{V(u, v)}{\sin(\alpha_1 - \alpha_2)} = \frac{1}{4} c(0, u, v)^2 \cdot \Lambda^{2n-1} \cdot \frac{V(u, v)}{\sin(\alpha_1 - \alpha_2)}.$$

Note that $\frac{1}{4} c(0, u, v)^2 = p_{k_1}(0, u, v)$ by (1.8). Consequently, the inequality (3.10) and the fact that $|\Lambda| \leq 1$ yield

$$|\tau_2| = n \Lambda^{2(n-1)} p_{k_2} \cdot \frac{|\Lambda|}{n} \cdot \frac{|V(u, v)|}{|\sin(\alpha_1 - \alpha_2)|} \leq p_{k_2} \cdot \frac{|V(u, v)|}{n} \cdot \frac{1}{|\sin(\alpha_1 - \alpha_2)|}. \quad (3.21)$$

Now we want to show that $|V(u, v)| \leq \text{const} \cdot |\sin(\alpha_1 - \alpha_2)|$. If we rewrite $A_1(u, v)$ and $A_2(u, v)$, defined in (3.17), using the formula

$$\sum_{v=0}^{m-1} a^{m-1-v} b^v, \quad m \in \mathbb{N}^+,$$

for $m := N - 2n \geq 1$, then (3.20) takes the form

$$V(u, v) = \sum_{v=0}^{m-1} \lambda_3^{2(m-1-v)} \left( \sin(\alpha_3 + \alpha_1) \cdot \lambda_1^{2v+1} - \sin(\alpha_3 + \alpha_2) \cdot \lambda_2^{2v+1} \right).$$

Now we substitute $\lambda_j = \sin \alpha_j, j = 1, 2, 3$, by (3.16) and apply the well-known formula

$$(\sin \theta)^{2v+1} = \frac{1}{2^{2v}} \sum_{k=0}^{v} (-1)^{v-k} \binom{2v+1}{k} \sin(2v + 1 - 2k)\theta.$$
This leads to the following representation:

\[ V(u, v) = \sum_{v=0}^{m-1} \lambda_3^{2(m-1-v)} \frac{1}{2^v} \sum_{k=0}^{v} (-1)^{v-k} \binom{2v+1}{k} B_{v,k}(\alpha_1, \alpha_2, \alpha_3), \quad (3.22) \]

where

\[ B_{v,k}(\alpha_1, \alpha_2, \alpha_3) := \sin(\alpha_3 + \alpha_1) \sin(2v + 1 - 2k)\alpha_1 - \sin(\alpha_3 + \alpha_2) \sin(2v + 1 - 2k)\alpha_2. \]

By the formulas for the product of sines and the difference of cosines we obtain

\[
B_{v,k}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2} \left( \cos(\alpha_3 - 2(v - k)\alpha_1) - \cos(\alpha_3 - 2(v - k)\alpha_2) + \cos(\alpha_3 + 2(v - k + 1)\alpha_2) \right) \\
= \sin(\alpha_3 - (v - k)(\alpha_1 + \alpha_2)) \sin((v - k)(\alpha_1 - \alpha_2)) + \sin(\alpha_3 + (v - k + 1)(\alpha_1 + \alpha_2)) \sin((v - k + 1)(\alpha_1 - \alpha_2)).
\]

Since \( |\sin rx| \leq r |\sin x|, \) \( r \geq 0, \) it follows that

\[ |B_{v,k}(\alpha_1, \alpha_2, \alpha_3)| \leq (2v - 2k + 1)|\sin(\alpha_1 - \alpha_2)|. \]

This and the obvious estimate of \(|V(u, v)|\) from (3.22) yield

\[
\frac{|V(u, v)|}{|\sin(\alpha_2 - \alpha_1)|} \leq \sigma(m) := \sum_{v=0}^{m-1} \frac{1}{2^v} \sum_{k=0}^{v} \binom{2v+1}{k} (2v - 2k + 1).
\]

One can check by successive use of the formulas (4.2.1.6), (4.2.2.13) and (4.2.3.19) from [19, Sect. 4.2] that

\[ \sigma(m) = \frac{4m^2 - 1}{3 \cdot 4^m - 1} \binom{2m - 2}{m - 1}. \]

Moreover, it can be easily proved by induction that

\[ \sigma(m) \leq m^{3/2}, \quad m \in \mathbb{N}. \]

Since \( m = N - 2n \geq 1, \) (3.21) yields

\[
|\tau_2| \leq \rho_n, N \cdot p_{kn}, \quad \rho_n, N = \frac{(N - 2n)^{3/2}}{n} = \left( \frac{N}{n} - 2 \right) \sqrt{N - 2n}, \quad N > 2n. \quad (3.23)
\]
Now we come back to the representation (3.2) from Lemma 1 and estimation of its terms. By (3.7), (3.10) and (3.23), we deduce for $t \geq 0$ that

$$p_{K_t} \geq \left( \frac{N}{n}\Lambda^{2(N-n)} + (\tau_1 - \rho_{n,N}) t + t^2 \right) \cdot p_{\kappa_n} = g(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n},$$  

(3.24)

where $\xi_j = \lambda_j^{2(N-n)} \in [0, 1]$ as in the proof of the previous lemma, and

$$g(\xi_1, \xi_2, \xi_3) := \frac{N}{n} \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3 - \rho_{n,N}) t + t^2.$$  

(3.25)

The function $g$ is non-decreasing for $t \geq 0$ with respect to each $\xi_j \in [0, 1]$, hence for $t > \rho_{n,N} > 0$ we obtain the inequality

$$g(\xi_1, \xi_2, \xi_3) \geq g(0, 0, 0) = t(t - \rho_{n,N}) > 0.$$  

For $t \leq 0$ we have

$$p_{K_t} \geq \left( \frac{N}{n}\Lambda^{2(N-n)} + (\tau_1 + \rho_{n,N}) t + t^2 \right) \cdot p_{\kappa_n} = G(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n},$$  

(3.26)

where the function

$$G(\xi_1, \xi_2, \xi_3) := \frac{N}{n} \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3 + \rho_{n,N}) t + t^2$$  

(3.27)

is non-increasing for $t \leq -\frac{N}{n}$ with respect to each $\xi_j \in [0, 1]$ and therefore

$$G(\xi_1, \xi_2, \xi_3) \geq G(1, 1, 1) = \frac{N}{n} + (3 + \rho_{n,N}) t + t^2.$$  

The roots of the latter quadratic polynomial are

$$-\frac{1}{2} \left( 3 + \rho_{n,N} \pm \sqrt{(3 + \rho_{n,N})^2 - 4\frac{N}{n}} \right),$$

so it has only positive values if $t < -\frac{1}{2} \left( 3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4\frac{N}{n}} \right) \leq -\frac{N}{n}$.  

Note that Lemmas 5 and 6 give Theorem 1 by continuity. For the proof of Theorem 2, additionally to Theorem 1, we will also need lower estimates of $p_{K_t}$ for $t$, which are the end points of the intervals excluded in (2.1) and (2.2) from the real line. And to obtain these estimates, we first introduce additional notation.

Given two distinct points $z, w \in \mathbb{C}$, we denote by $L_{z,w}$ the line passing through $z$ and $w$. Given three pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, we denote by $\angle(z_1, z_2, z_3)$ the smallest angle formed by the lines $L_{z_1,z_2}$ and $L_{z_2,z_3}$. This angle belongs to $[0; \pi/2]$. If $L$ and $L'$ are lines, then $\angle(L, L')$ is the smallest angle between them. This angle belongs to $[0; \pi/2]$, too. Also, $\theta_V(L) := \angle(L, V)$ and $\theta_H(L) := \angle(L, H)$, where $V$
and $H$ are the vertical and horizontal lines, correspondingly. Furthermore, for a fixed constant $\tau \geq 1$, we set

$$\mathcal{O}_\tau = \left\{(z_1, z_2, z_3) : \frac{|z_i - z_j|}{|z_i - z_k|} \leq \tau \text{ for pairwise distinct } i, j, k \in \{1, 2, 3\} \right\}, \quad (3.28)$$

so that all the triangles with vertexes $z_1, z_2$ and $z_3$ in $\mathcal{O}_\tau$ have comparable sides.

Given $\alpha_0 \in (0, \pi/2)$ and $(z_1, z_2, z_3)$, in what follows we will use the conditions

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \geq \alpha_0 \quad (3.29)$$

and

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \geq \alpha_0. \quad (3.30)$$

Note that (3.29) and (3.30) can be correspondingly replaced by the conditions

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \leq \frac{3}{2} \pi - \alpha_0$$

and

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leq \frac{3}{2} \pi - \alpha_0.$$

To obtain the desired result, we first prove several geometrical lemmas.

**Lemma 7** Fix $\alpha_0 \in (0, \pi/2)$. Given $(0, u, v) \in \mathcal{O}_\tau$, if the condition (3.29) is satisfied, then

$$\tau_1(0, u, v) = \lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)} \geq C_1(\alpha_0) > 0.$$

**Proof** Clearly,

$$\lambda_1^2 = \sin^2 \theta_V(L_{0,u}), \quad \lambda_2^2 = \sin^2 \theta_V(L_{0,v}), \quad \lambda_3^2 = \sin^2 \theta_V(L_{u,v}).$$

Moreover, from (3.29) it follows that at least one of the angles $\theta_V(L_{0,u}), \theta_V(L_{0,v}), \theta_V(L_{u,v})$ is not less than $\alpha_0/3$. Thus $\tau_1 \geq (\sin \frac{\alpha_0}{3})^{2(N-n)}$. $\Box$

**Lemma 8** Fix $\alpha_0 \in (0, \pi/2)$. Given $(0, u, v) \in \mathcal{O}_\tau$, if the condition (3.30) is satisfied, then

$$\Upsilon(0, u, v) := 2 + (\lambda_1 \lambda_2 \lambda_3)^2(N-n)$$

$$- \left(\lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)}\right) \geq C_2(\alpha_0, \tau) > 0.$$
Proof First we note that
\[
\Upsilon(0, u, v) \geq 2 + \lambda_1^2 \lambda_2^2 \lambda_3^2 - \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right)
\]
as the function \(2 + \xi_1 \xi_2 \xi_3 - (\xi_1 + \xi_2 + \xi_3)\) is non-increasing with respect to each \(\xi_j \in [0, 1], j = 1, 2, 3,\) and \(\lambda_j^2 \geq \lambda_j^2 (N - n)\) for \(N > n \geq 1.\)

In addition, we have
\[
\lambda_1^2 = 1 - \sin^2 \theta_H(L_0, u),
\]
\[
\lambda_2^2 = 1 - \sin^2 \theta_H(L_0, v),
\]
\[
\lambda_3^2 = 1 - \sin^2 \theta_H(L_u, v),
\]
and hence
\[
\Upsilon(0, u, v) \geq \sin^2 \theta_H(L_0, u) \sin^2 \theta_H(L_0, v) + \sin^2 \theta_H(L_0, v) \cos^2 \theta_H(L_0, u) \sin^2 \theta_H(L_u, v)
\]
\[
+ \sin^2 \theta_H(L_0, v) \sin^2 \theta_H(L_0, u) + \sin^2 \theta_H(L_0, u) \sin^2 \theta_H(L_u, v)
\]
\[
\geq \frac{2}{3} \left( \sin^2 \theta_H(L_0, v) \sin^2 \theta_H(L_0, u) + \sin^2 \theta_H(L_0, u) \sin^2 \theta_H(L_u, v) \right).
\]

Consider a triangle \((0, u, v) \in \mathcal{O}_\tau\) such that (3.30) is satisfied. Fix some \(\varepsilon \in (0; \alpha_0/3).\) Two cases are possible:

(1) amongst \(\theta_H(L_0, u), \theta_H(L_0, v), \theta_H(L_u, v),\) there exists a pair of angles, each being greater than \(\varepsilon\) and then it is easily seen that \(\Upsilon(0, u, v) \geq \frac{2}{3} \sin^4 \varepsilon;\)

(2) amongst those, there exists no pair of angles, each being greater than \(\varepsilon.\)

Let us consider the second case in detail (see Fig. 5). It is clear that at least two angles amongst \(\theta_H(L_0, u), \theta_H(L_0, v), \theta_H(L_u, v)\) are less than \(\varepsilon\) then. In other words, two sides of the triangle cut the horizontal line at angles less than \(\varepsilon.\) We call these sides \(A\) and \(B.\)

Furthermore, let the angle \(\gamma\) between \(A\) and \(B\) be acute; then obviously it is smaller than \(2\varepsilon.\) Then the acute angle between the third side \(C\) and the horizontal line is

![Fig. 5 Triangles in the case (2) (the proof of Lemma 8)](image-url)
greater than $\alpha_0 - 2\varepsilon$ and the acute angle between $A$ and $C$ is greater than $\alpha_0 - 3\varepsilon$. Consequently, the obtuse angle between $A$ and $C$ is smaller than $\pi - (\alpha_0 - 3\varepsilon)$. Thus we have for the angle $\beta$ of the triangle:

$$\alpha_0 - 3\varepsilon < \beta < \pi - (\alpha_0 - 3\varepsilon).$$

Therefore by the law of sines, the inequalities $(2/\pi) x \leq \sin x \leq x$ for $x \in [0, \pi/2]$, and (3.28), we get

$$\frac{1}{\tau^2} \leq \frac{\text{length}(C)}{\text{length}(B)} = \frac{\sin \gamma}{\sin \beta} < \frac{\sin 2\varepsilon}{\sin(\alpha_0 - 3\varepsilon)} \leq \frac{\pi\varepsilon}{\alpha_0 - 3\varepsilon}$$

$$\Rightarrow \varepsilon > \varepsilon_0(\alpha_0, \tau) := \frac{\alpha_0}{3 + \pi\tau^2}.$$

Now let the angle $\gamma$ between $A$ and $B$ be not acute (it is greater than $\pi - 2\varepsilon$). Then for one of acute angles of the triangle, say $\beta$, we have

$$\beta < \varepsilon - (\alpha_0 - 2\varepsilon) = 3\varepsilon - \alpha_0 < 0, \quad \varepsilon \in (0; \alpha_0/3),$$

which is impossible.

It follows from the aforesaid that there is a contradiction for $\varepsilon = \varepsilon_0(\alpha_0, \tau)$ in the second case and thus $\Upsilon(0, u, v) \geq \frac{2}{\tau} \sin^4 \varepsilon_0(\alpha_0, \tau)$. $\square$

We will also need the following result.

**Lemma 9** (Lemma 2.3 in [2]) Fix $\alpha_0 \in (0, \pi/2)$. Given $\kappa_n$ of the form (1.9) and $(z_1, z_2, z_3) \in O_{\tau}$, if the condition (3.29) is satisfied, then

$$p_{\kappa_n}(z_1, z_2, z_3) \geq C_3(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2$$

for some $C_3(\alpha_0, \tau) > 0$.

Now we are able to obtain necessary lower pointwise estimates for $p_K$, if $t$ are the end points of the intervals excluded in (2.1) and (2.2) from the real line. Recall that $\rho_{n,N} = (\frac{N}{n} - 2)\sqrt{N - 2n}$.

**Lemma 10** Fix $\alpha_0 \in (0, \pi/2)$. Given $K_t$ of the form (1.12) and $(z_1, z_2, z_3) \in O_{\tau}$.

(i) if the condition (3.29) is satisfied and $t = 2 - \frac{N}{n}$ for $n < N \leq 2n$ or $t = \rho_{n,N}$ for $N \geq 2n$, or

(ii) if both the conditions (3.29) and (3.30) are satisfied and $t = -\frac{1}{2} \left(3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - \frac{4N}{n}}\right)$ for $n < N \leq 2n$ or $t = -\frac{1}{2} \left(3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - \frac{4N}{n}}\right)$ for $N \geq 2n$, then

$$p_{K_t}(z_1, z_2, z_3) \geq C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2$$

for some $C(\alpha_0, \tau) > 0$. 

 Springer
Proof It is enough to prove it for triples \((0, u, v)\). What is more, the statement for \(t = 0\) in \((i)\), i.e. when \(N = 2n\), is just Lemma 9 and therefore we may exclude it. We also recall the notation \(\xi_j := \lambda_j^{2(N-n)} \in [0, 1], j = 1, 2, 3\).

Now let \(t = 2 - \frac{N}{n}\) and \(n < N < 2n\). Then for the function given in (3.13) we have

\[
f(\xi_1, \xi_2, \xi_3) = \frac{N}{n} \xi_1 \xi_2 \xi_3 + (2 - \frac{N}{n})(\xi_1 + \xi_2 + \xi_3) \geq (2 - \frac{N}{n}) \tau_1,
\]

where \(\tau_1\) is as in (3.8). If \(t = \rho_{n,N}\) and \(N > 2n\), then from (3.25) it follows that

\[
g(\xi_1, \xi_2, \xi_3) = \frac{N}{n} \xi_1 \xi_2 \xi_3 + \rho_{n,N}(\xi_1 + \xi_2 + \xi_3) \geq \frac{N}{n} \tau_1.
\]

Thus from the inequalities (3.12) for \(n < N < 2n\) and (3.24) for \(N > 2n\), both being valid for \(t \geq 0\), and Lemmas 7 and 9 (with the assumption (3.29)) we get

\[
p_{K_t}(0, u, v) \geq \frac{N}{n} \tau_1(0, u, v) \cdot p_\kappa_n(0, u, v) \geq \frac{N}{n} C_1(\alpha_0) C_3(\alpha_0, \tau) \cdot c(0, u, v)^2,
\]

which is the required result in the case \((i)\).

Now consider \((ii)\). Set \(t = -\frac{1}{2} \left(3 + \sqrt{9 - 4 \frac{N}{n}}\right)\) and \(n < N \leq 2n\). Then, by (3.15),

\[
F(\xi_1, \xi_2, \xi_3) = \frac{N}{n}(\xi_1 \xi_2 \xi_3 - 1) + \frac{1}{2}(3 - (\xi_1 + \xi_2 + \xi_3)) \left(3 + \sqrt{9 - 4 \frac{N}{n}}\right).
\]

Since \(-\frac{1}{2} \left(3 + \sqrt{9 - 4 \frac{N}{n}}\right) \leq -\frac{N}{n}\),

\[
F(\xi_1, \xi_2, \xi_3) \geq \frac{N}{n}(2 + \xi_1 \xi_2 \xi_3 - (\xi_1 + \xi_2 + \xi_3)) \geq \frac{N}{n} \Upsilon.
\]

The function (3.27) for

\[
t = t_0 := -\frac{1}{2} \left(3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4 \frac{N}{n}}\right)
\]

and \(N \geq 2n\) after some simplifications takes the form

\[
G(\xi_1, \xi_2, \xi_3) = \frac{N}{n}(\xi_1 \xi_2 \xi_3 - 1) + (\xi_1 + \xi_2 + \xi_3 - 3)t_0,
\]

and hence \(G(\xi_1, \xi_2, \xi_3) \geq \frac{N}{n} \Upsilon\) since \(t_0 \leq -\frac{N}{n}\) for \(N \geq 2n\).

Thus for the last two values of \(t\), by the inequalities (3.14) for \(n < N \leq 2n\) and (3.26) for \(N \geq 2n\), both being valid for \(t \leq -\frac{N}{n}\), and Lemma 9 (with the assumption (3.29)), we get

\[
p_{K_t}(0, u, v) \geq \frac{N}{n} \Upsilon(0, u, v) \cdot p_\kappa_n(0, u, v) \geq \frac{N}{n} \Upsilon(0, u, v) C_3(\alpha_0, \tau) \cdot c(0, u, v)^2.
\]
If (3.30) is also satisfied, then by Lemma 8 we obtain the desired inequality
\[ p_{K_t}(0, u, v) \geq \frac{N}{n} C_2(\alpha_0, \tau) C_3(\alpha_0, \tau) \cdot c(0, u, v)^2, \]
and we are done. \( \square \)

4 Examples

In this section we present triples \((0, u, v)\) such that the permutations \(p_{K_t}(0, u, v)\) change sign for \(t\) mentioned in (2.3), namely, \(t \in (-N/n; 0)\) for all \(n\) and \(N\) and \(t \in \left(0; \frac{2\epsilon_0 N}{n}\right)\) for \(N \gg n\). We use the notations of Lemma 1 below. Note that by this lemma, \(p_{K_t}\) can be calculated via \(p_{\kappa_m}\) for \(m, n\) and \(N\), and \(\varphi_{n,N}\) is calculated by (3.3).

We first show that \(p_{K_t}(0, u, v)\) is positive for any \(t\) if \(u = a + i\) and \(v = a - i\), where \(a \in \mathbb{R} \setminus \{0\}\) is suitably chosen. By (3.1) and taking into account that \(K_t(u) = K_t(v)\) and \(K_t(u - v) \equiv 0\),

\[ p_{K_t} = \left( \frac{a^{2N-1}}{(1 + a^2)^N} + t \cdot \frac{a^{2n-1}}{(1 + a^2)^n} \right)^2 = \frac{a^{4n-2}}{(1 + a^2)^2n} \left( \frac{a^{2(N-n)}}{(1 + a^2)^{N-n}} + t \right)^2, \]

which is positive for any real \(t\) if \(a\) is chosen so that the expression in the latter brackets does not vanish.

Now the aim is to show that for any fixed \(t\) from (2.3) there exist triples \((0, u, v)\) such that \(p_{K_t}(0, u, v)\) is negative. To do so, we find families of \((0, u, v)\) such that the quadratic polynomial

\[ \frac{p_{K_t}}{p_{\kappa_n}} = \frac{p_{\kappa_{N-n}}}{p_{\kappa_n}} + \frac{T_{n,N}}{p_{\kappa_n}} t + t^2, \quad p_{\kappa_n} > 0, \]

(with respect to \(t\)) has two different roots \(t_1\) and \(t_2\), depending on \(u\) and \(v\), and thus \(p_{K_t}\) (of the form (3.2)) is negative for \(t \in (t_1; t_2)\). In addition, we prove that the union of the intervals \((t_1; t_2)\) when \((0, u, v)\) runs the whole above-mentioned family is either the interval \((-N/n; 0)\) or \(\left(0; \frac{2\epsilon_0 N}{n}\right)\), indicated in (2.3).

Let us consider the case \(t < 0\).

Example 1 Set \(u = -a + i\), \(v = a + i\), where \(a \in \mathbb{R} \setminus \{0\}\). Then

\[ p_{\kappa_m} = -\frac{a^{2(2m-1)}}{(a^2 + 1)^{2m}} + \frac{a^{2m-2}}{2(a^2 + 1)^m} + \frac{a^{2m-2}}{2(a^2 + 1)^m} = \frac{a^{2m-2}((a^2 + 1)^m - a^2)}{(a^2 + 1)^{2m}}. \]
where \( m \) equals \( n \) or \( N \), and

\[
\varphi_{n,N} = - \frac{a^{2N-1}}{(a^2 + 1)^N} \left( \frac{a^{2n-1}}{(a^2 + 1)^n} - \frac{1}{2a} \right)
+ \frac{a^{2N-1}}{(a^2 + 1)^N} \left( - \frac{a^{2n-1}}{(a^2 + 1)^n} + \frac{1}{2a} \right)
- \frac{1}{2a} \left( \frac{a^{2n-1}}{(a^2 + 1)^n} - \frac{a^{2n-1}}{(a^2 + 1)^n} \right)
= a^{2n-2} \left( (a^2 + 1)^N + (a^2 + 1)^n a^{2(N-n)} - 2a^{2N} \right)
\]

From this by (3.2) we deduce that

\[
\frac{P_{K_t}}{P_{K_n}} = \frac{d_1(a)d_2(a)}{d_3(a)^2} + \frac{d_1(a) + d_2(a)}{d_3(a)} t + t^2, \quad a \neq 0,
\]

where

\[
d_1(a) := (a^2 + 1)^n \left( 1 - \frac{a^{2N}}{(a^2 + 1)^N} \right),
\]

\[
d_2(a) := (a^2 + 1)^n \left( \frac{a^{2(N-n)}}{(a^2 + 1)^{N-n}} - \frac{a^{2N}}{(a^2 + 1)^N} \right)
\]

and \( d_3(a) := (a^2 + 1)^n - a^{2n} \). The polynomial (4.1) has two different negative roots \( t_1(a) = -d_1(a)/d_3(a) \) and \( t_2(a) = -d_2(a)/d_3(a) \), where \( d_1(a) > d_2(a) > 0 \) and \( d_3(a) > 0 \). It is easy to check that the roots \( t_1(a) \) and \( t_2(a) \) run the intervals \((-N/n; -1)\) and \((-1; 0)\), correspondingly, when \( a \) runs \((0; \infty)\). Furthermore, we see by continuity that

\[
\bigcup_{a \in (0; \infty)} (t_1(a); t_2(a)) = (-N/n; 0).
\]

This means that \( P_{K_t} \) is negative for any \( t \) in \((-N/n; 0)\) for a suitably chosen.

As we have already mentioned above, this example shows that (2.2) is sharp for \( N = 2n \) in the sense of Remark 1. In addition, the left-hand side of (2.2) is also sharp for \( N = 2n + 1 \).

Now we give the example for \( t > 0 \).

**Example 2** Let \( n \) be fixed. Consider the triples \((0, u, v)\) such that

\[
u = -r(1 + \delta_N(q)i),
\]

\[
v = r \left( 1 - \frac{1}{r} + \delta_N(q)i \right),
\]

\[
\delta_N(q) := \sqrt{\frac{\ln q}{N - n}},
\]

\[
(4.2)
\]
where \( r > 0 \) and \( q \geq e \). We can also calculate \( p_{\kappa_m} \) and \( \varphi_{n,N} \) for these \((0,u,v)\) using (1.9), (3.1) and (3.3). However, the expression of \( p_{K_{1}}/p_{\kappa_n} \) obtained is too big and therefore we do not place it here. Instead, we give the following identity (the permutations are calculated for \((0,u,v)\) as in (4.2)):

\[
P(t) := \lim_{r \to \infty} \frac{p_{K_{1}}}{p_{\kappa_n}} = c_{N}(q) + b_{N}(q)t + t^{2},
\]

where

\[
c_{N}(q) = \frac{N}{n(1 + \delta_{N}(q))^{2(N-n)}} \cdot \frac{(2N - 1)\delta_{N}^{2}(q) + 1}{(2n - 1)\delta_{N}^{2}(q) + 1},
\]

and

\[
b_{N}(q) = -\frac{(2(N - n)^{2} + N - 4nN + n)\delta_{N}^{2}(q) - (n + N)}{n(1 + \delta_{N}(q))^{N-n}(2n - 1)\delta_{N}^{2}(q) + 1}.
\]

Note that

\[
c_{N}(q) \sim \frac{(2 \ln q + 1)N}{q^{2}n}, \quad b_{N}(q) \sim -\frac{(2 \ln q - 1)N}{qn}, \quad N \to \infty.
\]

The quadratic polynomial \( P \) (with respect to \( t \)) has two different positive roots \( t_{1}(N,q) \) and \( t_{2}(N,q) \) if \( N \) is large enough (as the discriminant is positive). Additionally, one can check that

\[
t_{1}(N,q) \sim \tilde{t}_{1}(q) := \frac{2 \ln q + 1}{q(2 \ln q - 1)},
\]

\[
t_{2}(N,q) \sim \tilde{t}_{2}(N,q) := \frac{(2 \ln q - 1)N}{qn}, \quad N \to \infty.
\]

Taking into account the properties

\[
\tilde{t}_{1}(q) \to 0 \text{ as } q \to \infty \quad \text{and} \quad \max_{q \in [e; \infty)} \frac{2 \ln q - 1}{q} = \frac{2}{e^{3/2}},
\]

we deduce by continuity that

\[
\bigcup_{q \in [e; \infty)} (\tilde{t}_{1}(q); \tilde{t}_{2}(q)) = \left(0; \frac{2}{e^{3/2}} \frac{N}{n}\right).
\]

Thus, \( p_{K_{1}}(0,u,v) \) with \( u \) and \( v \) as in (4.2) are negative for any \( t \) in \( \left(0; \frac{2}{e^{3/2}} \frac{N}{n}\right) \), if \( N \) (with respect to \( n \)) and \( r \) are large enough and \( q \) is suitably chosen.
5 Proof of Theorems 1 and 2

Recall that Lemmas 5 and 6 state that if \( K_t \) is of the form (1.12) and \( t \in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( 3 + \sqrt{9 - 4 \frac{N}{n}} \right); 2 - \frac{N}{n} \right], \quad n < N \leq 2n, \)

\( t \in \mathbb{R} \setminus \left[ -\frac{1}{2} \left( 3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4 \frac{N}{n}} \right); \rho_{n,N} \right], \quad N \geq 2n, \)

where \( \rho_{n,N} = (\frac{N}{n} - 2) \sqrt{N - 2n} \), then

\[ p_{K_t} \geq C(t) \cdot p_{\kappa_n}, \quad C(t) > 0. \]

These lemmas immediately give Theorem 1 by continuity if we take into account the fact that \( p_{\kappa_n}(z_1, z_2, z_3) \geq 0 \) for all \((z_1, z_2, z_3) \in \mathbb{C}^3 \) (see (1.2) and (1.10)).

What is said from now on is related to Theorem 2.

First of all we note that the latter statement of Theorem 2, i.e. the one asserting that if the operator \( T_{K_t} \) is \( L^2(\mathcal{H}^1|E) \)-bounded, then \( E \) is rectifiable, is a corollary of the fact that the \( L^2(\mathcal{H}^1|E) \)-boundedness of \( T_{K_t} \) implies that \( p_{K_t}(\mathcal{H}^1|E) < \infty \). This can be proved by generalization of the Melnikov–Verdera identity (1.5) for the kernels \( K_t \) and permutations \( p_{K_t} \); see Lemma 3.3 in [2] and also Lemma 2.1 in [16].

Now we come to the proof of the former statement in Theorem 2.

The proof for \( t \), mentioned in Lemmas 5 and 6 (see also the beginning of the current section), is direct via Theorem B, which states that if \( p_{\kappa_n}(\mathcal{H}^1|E) < \infty \), then \( E \) is rectifiable. Indeed, if \( p_{K_t}(\mathcal{H}^1|E) < \infty \) for such \( t \), then \( p_{\kappa_n}(\mathcal{H}^1|E) < \infty \) by the inequality \( p_{K_t} \geq C(t) \cdot p_{\kappa_n}, C(t) > 0 \), and thus the set \( E \) is rectifiable.

What is left is to prove the former statement in Theorem 2 for

\[ t = -\frac{1}{2} \left( 3 + \sqrt{9 - 4 \frac{N}{n}} \right), \quad t = 2 - \frac{N}{n}, \quad n < N \leq 2n, \]

\[ t = -\frac{1}{2} \left( 3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4 \frac{N}{n}} \right), \quad t = \rho_{n,N}, \quad N \geq 2n. \]

It requires some additional work and therefore for the reader’s convenience we first make several observations, which could help to clarify the forthcoming proof.

We start with a very brief exposition of the proof of Theorem A given in [12] (note that one can find a modified version of the proof from [12] in [21, Chap. 7] and follow it instead). Recall that Theorem A states that, for a Borel set \( E \subset \mathbb{C} \) such that \( 0 < \mathcal{H}^1(E) < \infty \), if \( c^2(\mathcal{H}^1|E) < \infty \), then \( E \) is rectifiable. We emphasize again that it is essential in the proof that the curvature is non-negative.

The first step is to show that there exists a compact subset \( F \) of the given set \( E \) such that, among other things, \( c^2(\mathcal{H}^1|F) \) is well-controlled and can be made very small (this is done in [12] by a quite standard uniformization procedure). Then the second and most important step follows—to prove that if \( \mu \) is a positive Radon measure on
The problem is to choose an adequate coordinate system of $\mathbb{C}$ and construct a Lipschitz function $A$ whose graph will be the one needed. For this purpose, the author of [12] first defines some functions used to measure how well the spt $\mu$ is approximated by straight lines at a given location and a given scale. It is shown that these functions are related to the $c^2(\mu)$ in the case when the measure $\mu$ does not degenerate too much. These preliminary results are then used to construct the function $A$ by stopping time arguments, which demand fine adjustments to many parameters and thresholds. Starting with choosing a point $x_0 \in \text{spt } \mu$ and fixing an approximating line $D_0$ (which will be the domain of the function $A$) such that the mean distance from spt $\mu$ to the line $D_0$ is suitably small, the author of [12] comes to cutting spt $\mu$ in four disjoint pieces $Z$, $F_1$, $F_2$ and $F_3$ such that

$$spt \mu = Z \cup F_1 \cup F_2 \cup F_3.$$  

It is shown that $Z$ is very nice for constructing the graph but the three others admit “bad events”. Then the goal is to prove that these bad pieces carry only a small part of the measure $\mu$, namely, $\mu(F_j) \leq 10^{-6} \mu(\mathbb{C})$ for each $j$ and thus $\mu(Z) \geq \frac{99}{100} \mu(\mathbb{C})$.

This allows to construct the required Lipschitz function $A : D_0 \rightarrow D_0^\perp$ such that the set $Z$ is contained in the graph of $A$.

Coming back to the initial settings, if $\mu = \mathcal{H}^1|F$, where $F$ is the above-mentioned subset of $E$, then there exists a Lipschitz graph $\Gamma$ such that $\mathcal{H}^1(\Gamma \cap F) \geq C \cdot \mathcal{H}^1(F)$. This fact is used in the last step of the proof from [12], which is as follows. Since $\mathcal{H}^1(E) < \infty$ by the assumptions, the set $E$ can be decomposed into a rectifiable and purely unrectifiable part, i.e. $E = E_{\text{rect}} + E_{\text{unrect}}$. Suppose that

$$\mathcal{H}^1(E_{\text{unrect}}) > 0. \quad (5.1)$$

Then there exists a compact set $F \subset E_{\text{unrect}}$ and Lipschitz graph $\Gamma$ such that $\mathcal{H}^1(\Gamma \cap F) \geq C \cdot \mathcal{H}^1(F)$ that contradicts the fact that $F$ is purely unrectifiable.

Let us now say a few words about the proof of Theorem B given in [2]. Recall that this theorem is an analogue of Theorem A, where the kernel $1/|z|$ and curvature squared $c^2(\mathcal{H}^1|E)$ are replaced by the kernels $\kappa_n(z) = (\text{Re } z)^{2n-1}/|z|^{2n}$, $n \in \mathbb{N}$, and corresponding permutations $\rho_n$ (see (1.9)—(1.11)). We will use the definitions given near the formula (3.28) and in the discussion of Theorem A above. First we mention that it is proved in [2] that the permutations $\rho_n(z_1, z_2, z_3)$ behave similarly to $c^2(z_1, z_2, z_3)$ for all triangles with comparable sides, whose one side makes a big angle with the vertical line. More precisely (see Lemma 9 above), it is shown there that for a fixed $\alpha_0 \in (0, \pi/2)$ and given $(z_1, z_2, z_3) \in \mathcal{O}_r$, if the condition (3.29), i.e.

$$\theta_V(L_{z_1, z_2}) + \theta_V(L_{z_2, z_3}) + \theta_V(L_{z_1, z_3}) \geq \alpha_0,$$

or

$$\theta_H(L_{z_1, z_2}) + \theta_H(L_{z_2, z_3}) + \theta_H(L_{z_1, z_3}) \leq \frac{3}{2} \pi - \alpha_0,$$
is satisfied, then

\[ p_{\kappa_n}(z_1, z_2, z_3) \geq C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2, \quad C(\alpha_0, \tau) > 0. \]

This enables the authors of [2] to use the above-described scheme from [12] in order to construct the required Lipschitz graph \( \Gamma \) in the case when the first approximating line \( D_0 \) for \( \spt \mu \) is far from the vertical line. Note that exchanging the curvature for the permutations \( p_{\kappa_n} \) still requires new arguments in several key points of the proof. Otherwise, when \( D_0 \) is close to the vertical line and the scheme from [12] does not work (as \( \mu(F_3) \) may be too big), they tune thresholds and apply some coverings so that they can use the result for \( D_0 \), being far from the vertical line, to construct countably many Lipschitz graphs, which give \( \Gamma \) after appropriate joining.

We are now at the position to finish the proof of our Theorem 2. This will be an adaptation of the arguments from [2].

On the one hand, by the clause (i) of Lemma 10, for a fixed \( \alpha_0 \in (0, \pi/2) \) and given \( (z_1, z_2, z_3) \in \mathcal{O}_\tau \), if the condition (3.29), i.e. the same as in the result for \( t = 0 \) from [2] mentioned above, is satisfied and \( t = 2 - \frac{N}{n} \) for \( n < N \leq 2n \) or \( t = \rho_{n,N} \) for \( N \geq 2n \), then we also have

\[ p_{K_t}(z_1, z_2, z_3) \geq C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2, \quad C(\alpha_0, \tau) > 0. \] (5.2)

It means that we can undeviatingly follow the scheme from [2] (exchanging \( p_{\kappa_n} \) for \( p_{K_t} \)) in order to get our result for \( t = 2 - \frac{N}{n} \), \( n < N \leq 2n \), and \( t = \rho_{n,N} \), \( N \geq 2n \).

On the other hand, by the clause (ii) of Lemma 10, the inequality (5.2) is true for

\[ t = -\frac{1}{2} \left( 3 + \sqrt{9 - 4\frac{N}{n}} \right), \quad n < N \leq 2n, \]
\[ t = -\frac{1}{2} \left( 3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4\frac{N}{n}} \right), \quad N \geq 2n, \] (5.3)

only if both the conditions (3.29) and (3.30) are satisfied, i.e.

\[ \alpha_0 \leq \theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leq \frac{3}{2} \pi - \alpha_0, \]

or

\[ \alpha_0 \leq \theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \leq \frac{3}{2} \pi - \alpha_0, \]

and thus the triangles \((z_1, z_2, z_3)\) are far from both the vertical and horizontal line.

Consequently, the scheme from [2] cannot be applied directly for \( t \) from (5.3). However, as we will see, it works after a few modifications (besides the exchange of \( p_{\kappa_n} \) for \( p_{K_t} \)) connected basically with adapting geometrical arguments to both the conditions (3.29) and (3.30). Since the cases where we are close to either the vertical or horizontal line are well-separated and similar geometrically, the arguments for the
first approximating line $D_0$, being close (far) to (from) the vertical line, can be easily transferred into the ones for $D_0$, being close (far) to (from) the horizontal line.

We now reproduce the main steps of the proof, stemming from [2], with necessary changes when our permutations and the conditions (3.29) and (3.30) are involved.

Below we consider only $t$ from (5.3). The following two propositions will then imply Theorem 2 by the same contradiction arguments as in the proof from [12] (see the arguments around (5.1) above). Note that one has to take $\mu = 40\mathcal{H}^1|F$ in Proposition 2, where the set $F$ is from Proposition 1 (it may be suitably rescaled if necessary).

**Proposition 1** (An analogue of Lemma 3.4 in [2] and Proposition 1.1 in [12]) Let $E \subset \mathbb{C}$ be a Borel set with $0 < \mathcal{H}^1(E) < \infty$ and $p_{K_t}(\mathcal{H}^1|E) < \infty$. Then for all $\eta > 0$ there exists a set $F \subset E$ such that

- $F$ is compact,
- $p_{K_t}(\mathcal{H}^1|F) \leq \eta \text{ diam } F$,
- $\mathcal{H}^1(F) > \frac{1}{40} \text{ diam } F$,
- for all $z \in F$, for all $r > 0$, $\mathcal{H}^1(F \cap B(z,r)) \leq 3r$.

**Proposition 2** For any constant $C_0 \geq 10$, there exists a number $\eta > 0$ such that if $\mu$ is any positive Radon measure on $\mathbb{C}$ satisfying

- $\mu(B(0,1)) \geq 1$, $\mu(\mathbb{C}\setminus B(0,2)) = 0$,
- for any ball $B$, $\mu(B) \leq C_0 \text{ diam } B$,
- $p_{K_t}(\mu) \leq \eta$,

then there exists a Lipschitz graph $\Gamma$ such that $\mu(\Gamma) \geq 10^{-5} \mu(\mathbb{C})$.

The rest of the paper is devoted to the proof of Proposition 2, which is an analogue of Proposition 3.1 from [2]. First, we give several definitions (we confine ourselves to those which will be needed below; see [2,12] for further ones). Let $\mu$ be a positive Radon measure on $\mathbb{C}$ and

$$p_{K_t,\tau}(\mu) = \int \int \int_{\mathcal{O}_\tau} p_{K_t}(z_1,z_2,z_3)d\mu(z_1)d\mu(z_2)d\mu(z_3),$$

see (1.11) and (3.28). For a ball $B = B(x, r)$ set

$$\delta_{\mu}(x,r) = \frac{\mu(B(x,r))}{r}.$$

We will use a small density threshold $\delta > 0$ for this quantity.

Given a fixed constant $k > 1$, for any ball $B = B(x, r) \subset \mathbb{C}$ and $D$ a line in $\mathbb{C}$, set

$$\beta_{1,\mu}^D(x,r) = \frac{1}{r} \int_{B(x,kr)} \frac{\text{dist}(y,D)}{r} d\mu(y),$$

$$\beta_{2,\mu}^D(x,r) = \left(\frac{1}{r} \int_{B(x,kr)} \left(\frac{\text{dist}(y,D)}{r}\right)^2 d\mu(y)\right)^{1/2}.$$
Geometrical notation connected with lines and angles is given near the formula (3.28) at the end of Sect. 3.

**Lemma 11** Let \( \mu \) be a measure with linear growth (with a constant \( C_0 \)), and \( B(x, r) \subset \mathbb{C} \) a ball with \( \delta_\mu(x, r) \geq \delta \). Suppose that \( \tau \) is big enough, then for any \( \varepsilon > 0 \), there exists some \( \delta_1 = \delta_1(\delta, \varepsilon) > 0 \) such that

\[
\frac{p_{K_1, \tau}(\mu | k B)}{\mu(B)} \leq \delta_1 \iff \inf_D \beta^D_{2, \mu}(x, r) \leq \varepsilon.
\]

**Proof** The proof is the same as for Lemma 4.4 in [2]. We just have to use our Lemma 10 instead of Lemma 2.3 there for the case when both the conditions (3.30) and (3.29) are satisfied, and say that in the case

\[
\theta_H(L_{z_1, z_2}) + \theta_H(L_{z_1, w}) + \theta_H(L_{z_2, w}) \leq \alpha_0
\]

we obtain the same estimate for \( \text{dist}(w, L_{z_1, z_2}) \) as in the case

\[
\theta_V(L_{z_1, z_2}) + \theta_V(L_{z_1, w}) + \theta_V(L_{z_2, w}) \leq \alpha_0.
\]

By Lemma 11, chosen a point \( x_0 \in \text{spt} \mu \), there exists an approximating line \( D_0 \) such that \( p_{D_0}(x_0, 1) \leq \varepsilon \). The next step is to construct a first Lipschitz graph in the case when \( D_0 \) is far from both the horizontal and vertical lines.

To do so, one first has to introduce a family of stopping time regions and obtain the partition \( \text{spt} \mu = Z \cup F_1 \cup F_2 \cup F_3 \) (see the exposition of the proof from [12] above). As this entirely repeats the corresponding part of [2, Sect. 5] (cf. [12, Sect. 3.1]), we omit it. We just have to mention that the thresholds \( \theta_0 \) and \( \alpha \), arising there, have to be adapted to that \( D_0 \) is far from both the horizontal and vertical lines. Namely, \( \theta_0 \) is now a threshold for both \( \theta_V(D_0) \) and \( \theta_H(D_0) \). It means that one has to distinguish not only the cases \( \theta_V(D_0) \geq \theta_0 \) and \( \theta_V(D_0) < \theta_0 \) but also \( \theta_H(D_0) \geq \theta_0 \) and \( \theta_H(D_0) < \theta_0 \). Moreover, \( \alpha \) is tuned as follows: if \( \theta_V(D_0) \) or \( \theta_H(D_0) \) are greater than \( \theta_0 \), then \( \alpha \leq \theta_0 / 10 \); if \( \theta_V(D_0) \) or \( \theta_H(D_0) \) are not greater than \( \theta_0 \), then \( \alpha = 10\theta_0 \).

Furthermore, see [2, 12] for the way how one can define the Lipschitz function \( A \) on the line \( D_0 \), using \( Z, F_1, F_2, F_3 \), and appropriate thresholds.

Now we come to the main step of the proof of Proposition 2. The following lemma is an analogue of Lemma 6.1 from [2].

**Lemma 12** Under the assumptions of Proposition 2, if furthermore

\[
\theta_0 < \theta_V(D_0) < \frac{\pi}{2} - \theta_0,
\]

then there exists a Lipschitz graph \( \Gamma \) such that \( \mu(\Gamma) \geq \frac{99}{100} \mu(\mathbb{C}) \).

For the proof one uses the above-mentioned function \( A \) to obtain the graph \( \Gamma, Z \subset \Gamma \), and show that

\[
\mu(F_1) + \mu(F_2) + \mu(F_3) \leq \frac{1}{100} \mu(\mathbb{C}).
\]
Indeed, the following lemmas are valid (recall that $\mu(\mathbb{C}) \geq 1$ by the assumptions).

**Lemma 13** *Under the assumptions of Proposition 2,*

$$\mu(F_1) \leq 10^{-6}.$$  

**Proof** This is an analogue of [2, Proposition 6.3], whose proof includes consideration of the two cases: (1) $\theta_V(D_0) > \theta_0$ and (2) $\theta_V(D_0) \leq \theta_0$ (see the proof of [2, Lemma 6.4]).

Under our settings, we have to consider three cases. Namely, the case (1) has to be exchanged for $\theta_0 < \theta_V(D_0) < \frac{\pi}{2} - \theta_0$, although the proof remains the same. The case (2) splits up into the following two: $\theta_V(D_0) \leq \theta_0$ and $\theta_V(D_0) \geq \frac{\pi}{2} - \theta_0$ (i.e. $\theta_H(D_0) \leq \theta_0$). Arguments in the latter case are the same as in the former one.  

**Lemma 14** *(An analogue of Proposition 6.2 in [2])* *Under the assumptions of Proposition 2,*

$$\mu(F_2) \leq 10^{-6}.$$  

**Lemma 15** *Under the assumptions of Lemma 12,*

$$\mu(F_3) \leq 10^{-6}.$$  

**Proof** The proof stems from the one of [2, Proposition 6.5], but with exchange of $\theta_V(D_0) > \theta_0$ for $\theta_0 < \theta_V(D_0) < \frac{\pi}{2} - \theta_0$ as in Lemma 12.  

Thus Proposition 2 is proved under the assumptions of Lemma 12. What is left is to consider the other case. The following statement satisfies the question.

**Lemma 16** *Under the assumptions of Proposition 2, if furthermore*

$$\theta_V(D_0) \leq \theta_0 \quad \text{or} \quad \theta_V(D_0) \geq \frac{\pi}{2} - \theta_0 \quad \text{(i.e. } \theta_H(D_0) \leq \theta_0),$$

*then there exists a Lipschitz graph $\Gamma$ such that $\mu(\Gamma) \geq 10^{-5} \mu(\mathbb{C})$.*

**Proof** To prove this, we repeat arguments from the proof of [2, Lemma 7.1], given for $\theta_V(D_0) \leq \theta_0$, for the case $\theta_H(D_0) \leq \theta_0$.  

6 Concluding Remarks

In this section we generalize Theorem 2 to higher dimensions. Let us introduce necessary notation first. For $d \in \mathbb{N}^+$ and $E \subset \mathbb{R}^d$ with finite length we consider the singular integral operator $T_{K_t} = (T_{K_t}^j)_{j=1}^d$ such that formally

$$T_{K_t}^j f(x) := \int_{E} f(y) K_t^j(x - y) dH^1(y),$$

$$K_t^j(x) := \kappa_{N}^j(x) + t \cdot \kappa_{n}^j(x),$$
where \( \kappa_n^j(x) := x_j^{2n-1}/|x|^{2n} \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\} \). As before, we suppose that \( N > n \), where \( n, N \in \mathbb{N}^+ \), and \( t \in \mathbb{R} \). We also need the permutations

\[
p_{K_t}(x, y, z) := \sum_{j=1}^d p_{K_t}^j(x, y, z) \quad \text{for distinct points } x, y, z \in \mathbb{R}^d,
\]

where \( p_{K_t}^j(x, y, z) \) are the same as in (1.7) with \( K_t^j \) instead of \( K \). We also define the permutation of measure \( p_{K_t}(\mu) \) analogously to (1.11).

**Theorem 3** Let \( t \) be as mentioned in Theorem 1. Given a Borel set \( E \subset \mathbb{R}^d \) such that \( 0 < \mathcal{H}^1(E) < \infty \), if \( p_{K_t}(\mathcal{H}^1|E) < \infty \), then \( E \) is rectifiable. Moreover, if the operator \( T_{K_t} \) is \( L^2(\mathcal{H}^1|E) \)-bounded, then \( E \) is rectifiable.

This result for \( t = 0 \) was recently proved in [3, Theorems 1.2(1) and 6.2]. To prove Theorem 3 for all required \( t \) we only need to use our Lemmas 5, 6 and 10 in order to show that for all \( x, y, z \in \mathbb{R}^d \) such that \( (x, y, z) \in \mathcal{O}_\tau \) and the assumptions of Lemma 10 are satisfied,

\[
p_{K_t}(x, y, z) \geq C(t, \alpha_0, \tau) p_{K_0}(x, y, z), \quad C(t, \alpha_0, \tau) > 0. \tag{6.1}
\]

See the definitions of \( \alpha_0, \tau \) and \( \mathcal{O}_\tau \) before Lemma 7. Then by [3, Proposition 3.3], adapted to the conditions (3.29) and (3.30), and the arguments similar to those in [3, Sect. 6] and our Sect. 5 we immediately get the result. Note that [3, Proposition 3.3] slightly simplifies the approach from [2] (and improves Lemma 9) in the case \( t = 0 \) as the parameter \( \tau \) is not needed anymore. In our case this parameter is still necessary because of the inequality (6.1).

To finish, it is also worth mentioning here that under Ahlfors–David regularity assumption one can expect that for \( t \) as in Theorem 1 the \( L^2 \)-boundedness of the operator associated with \( K_t \) implies uniform rectifiability (the same for the \( \mathbb{R}^d \) case). This is indeed true. This result, among others with Ahlfors–David regularity condition, has appeared in [5].

**Acknowledgements** I would like to express my sincere gratitude to Joan Mateu and Xavier Tolsa for suggesting the problem and for many stimulating conversations. I am also grateful to the Referee for his/her valuable recommendations. The research was supported by the ERC Grant 320501 of the European Research Council (FP7/2007-2013).

**References**

1. Calderón, A.P.: Cauchy integrals on Lipschitz curves and related operators. Proc. Nat. Acad. Sci. USA 74, 1324–1327 (1977)
2. Chousionis, V., Mateu, J., Prat, L., Tolsa, X.: Calderón–Zygmund kernels and rectifiability in the plane. Adv. Math. 231(1), 535–568 (2012)
3. Chousionis, V., Prat, L.: Some Calderón-Zygmund kernels and their relation to rectifiability and Wolff capacities. Math. Z. 231(1–2), 435–460 (2016)
4. Christ, M.: Lectures on Singular Integral Operators. Regional Conference Series in Mathematics, vol. 77. American Mathematical Society, Providence (1990)
5. Chunaev, P., Mateu, J., Tolsa, X.: Singular integrals unsuitable for the curvature method whose $L^2$-boundedness still implies rectifiability, arXiv:1607.07663 (2016)
6. Coifman, R., McIntosh, A., Meyer, Y.: L’intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes. Ann. Math. (2) 116(2), 361–387 (1982)
7. David, G.: Opérateurs intégraux singuliers sur certaines courbes du plan complexe. Ann. Sci. École Norm. Sup. (4) 17(1), 157–189 (1984)
8. David, G.: Unrectifiable 1-sets have vanishing analytic capacity. Rev. Mat. Iberoam. 14(2), 369–479 (1998)
9. David, G., Semmes, S.: Analysis of and on Uniformly Rectifiable Sets. Surveys and Monographs, vol. 38. American Mathematical Society, Providence (1993)
10. Jaye, B., Nazarov, F.: Three revolutions in the kernel are worse than one, arXiv:1307.3678 (2013)
11. Huovinen, P.: A nicely behaved singular integral on a purely unrectifiable set. Proc. Am. Math. Soc. 129(11), 3345–3351 (2001)
12. Léger, J.C.: Menger curvature and rectifiability. Ann. Math. 149, 831–869 (1999)
13. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Cambridge University Press, Cambridge (1995)
14. Mattila, P.: Singular integrals, analytic capacity and rectifiability, In: Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. 3, Special Issue, 797–812 (1997)
15. Mattila, P.: Singular integrals and rectifiability, In: Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), vol. Extra. Publicacions Matemàtiques, pp. 199–208 (2002)
16. Mattila, P., Melnikov, M., Verdera, J.: The Cauchy integral, analytic capacity, and uniform rectifiability. Ann. Math. (2) 144(1), 127–136 (1996)
17. Melnikov, M.: Analytic capacity: a discrete approach and the curvature of measure, Mat. Sb. 186(6), 57–76 (1995) (in Russian); Translation in Sb. Math. 186(6), 827–846 (1995)
18. Melnikov, M., Verdera, J.: A geometric proof of the $L^2$ boundedness of the Cauchy integral on Lipschitz graphs. Int. Math. Res. Not. 7, 325–331 (1995)
19. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I.: Integrals and Series. Vol. 1. Elementary functions, vol. 1. Gordon and Breach Science Publishers, New York (1986)
20. Tolsa, X.: Painlevé’s problem and the semiadditivity of analytic capacity. Acta Math. 190(1), 105–149 (2003)
21. Tolsa, X.: Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory. Progress in Mathematics, vol. 307. Birkhäuser/Springer, Cham (2014)