One-Shot Device Testing Data Analysis under Logistic-Exponential Lifetimes with an Application to Murine model with Melioidosis Data

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ARTICLE HISTORY
Compiled November 2, 2022

ABSTRACT
In the literature, the reliability analysis of one-shot devices is found under accelerated life testing in presence of various stress factors. The application of one-shot devices can be extended to the bio-medical field, where often we evidence the emergence of certain diseases under different stress factors due to environmental condition, life style aspects, presence of co morbidity etc. In this work, one-shot device data analysis is performed in application to Murine model for Melioidosis data. Two parameter logistic exponential distribution is assumed as lifetime distribution. Weighted minimum density power divergence estimators (WMDPDEs) for robust parameter estimation are obtained along with the conventional maximum likelihood estimators (MLEs). Asymptotic behaviour of the WMDPDEs and testing of hypothesis based on it are also studied. The performances of estimators are evaluated through extensive simulation experiments. Later those developments are applied to the Murine model for Melioidosis Data. Citing the importance of knowing exactly when to inspect the one-shot devices put to test, a search for optimum inspection times is performed. This optimization is designed to minimize a defined cost function which strikes a trade-off between precision of the estimation and experimental cost. The search is performed through population based heuristic optimization method Genetic Algorithm.

KEYWORDS
Density Power Divergence Estimator, Genetic Algorithm, Kullback-Leibler divergence, Logistic-Exponential distribution, One-Shot Devices.

AMS CLASSIFICATION
62F10, 62F12, 62NO2.

1. Introduction

Applications of one-shot device prevails in wide spectrum of life. One-shot devices stay in torpid state until activated and they are immediately destroyed after the operation. Amidst COVID-19 pandemic, extensive use of PPE kits, RT-PCR and Rapid Antigen testing kits etc., is evidenced, which are only of one time use. Apart from these, the sugar level testing strips, pregnancy testing kit, fire extinguishers, fuses, fuel injectors,
missiles, nuclear weapons, space probes, all of those come into the category of one-shot devices. The reliability analysis of such devices becomes quite challenging as defectiveness of the devices can be discovered only after being tested and the exact failure times of such devices cannot be recorded accurately in most instances. Hence it can only be observed whether the failure of device occurs before or after the inspection time which yields dichotomous data. With highly reliable devices, typically accelerated life tests (ALTs) are executed to conduct the reliability testing of these devices under limited time and experimental budget. With high levels of stress factors under ALT, reliability is estimated and this is extrapolated to real life operating situations. The application of one-shot devices can be extended in bio-medical field where often emergence of diseases like cancer, pulmonary infection is exhibited under different stress factors like environmental stress, life style stress, presence of co-morbidity etc.

In this work, reliability analysis of 10 groups of mice is performed which happened to be inflicted with Melioidosis, also called Whitemore’s disease, based on the study of the Murine model by West et al. (2012). They conducted an experiment to observe the Murine pulmonary infection and inflammation induced by inhalation of Burkholderia (B.) pseudomallei. B. pseudomallei is the Gram-negative soil saprophyte bacterium responsible for tropical disease Melioidosis which is potentially fatal for both the humans and animals. Melioidosis disease (White 2003) is endemic in north-east Thailand and northern Australia but is imported to the other parts of world as well through human or animal transportation. The objective of their experiment is the development of Murine model of pneumonic Melioidosis by inhalation of aerosolized B. pseudomallei which is to be useful for the study of bacterial and host factors causative to the Melioidosis. They performed a series of experiments where they aerosolized various dilutions of B. pseudomallei to the two different strain of mice namely C57BL/6 and BALB/c mice. They started with the deposition dose 56 CFU/lung and increased it up to deposition dose 1637 CFU/lung. They monitored Murine survival and reported the mortality of mice at the inspection times observed in days. The data is described in Table (13) for the convenience of the readers. In this work, under the stress factor of strain variety and different doses of B. pseudomallei, the survival analysis of the mice is studied where death due to Melioidosis disease indicates failure.

By reviewing the literature for one-shot devices it is noticed that Exponential, Gamma, Weibull distributions are commonly used as lifetime distributions. For exponentially distributed lifetime of one-shot devices, Balakrishnan et al. (2012) developed EM algorithm for point estimation under ALT and they also compared it with the Bayesian approach using normal prior developed by Fan et al. (2009). Balakrishnan et al. (2016) studied the reliability of one-shot devices in presence of competing risk factors. Balakrishnan and Ling (2014) applied gamma distributions as lifetime of one-shot devices and provided inference study. Balakrishnan and Castilla (2022) studied the reliability of one-shot devices under log-normal distributions and provided EM algorithm for estimation purposes under constant stress accelerated life tests.

Usually the lifetime distributions having bathtub curved hazard rates are considered appropriate for reliability analysis, survival analysis and other related fields (Wang, Wu, Tripathi & Lodhi 2022). But the lifetime distributions with different shapes of hazard rates are always preferred (Reed 2011). In present work, two parameter Logistic-Exponential (LE) distribution is chosen for depicting the lifetime in murine model because of its high flexibility regarding shape of hazard function. Figure (1) depicts the various hazard rate shapes of LE distribution for different values of shape and scale parameters. It can be observed that it exhibits five different hazard rate shapes which are constant, increasing, decreasing, bath-tub and upside-down-bath-tub (Ali,
As a result, LE distribution can be used as an alternative to some well-known two-parameter models like gamma, log-normal, Weibull, exponentiated exponential, inverse Gaussian, Birnbaum–Saunders distributions (Balakrishnan & Kundu, 2009) and in terms of model fitting it might work well than the stated distributions in some situations. In present work, to incorporate the stress factors, the shape and scale parameters of assumed two parameter Logistic-Exponential distribution are linked with the stress factors through a log-linear function.

For estimation of the lifetime distribution, though maximum likelihood estimation (MLE) is very popular and common method because of its well-known properties such as asymptotic efficiency, consistency, sufficiency, invariance, it may not posses the property of robustness. To overcome this drawback, weighted minimum density power divergence (WMDPD) estimator is used for estimating the lifetime distribution as an alternative. Tuning parameter is the important feature of density power divergence (DPD) measure whose optimization is studied by Ghosh and Basu (2015), and Basak et al. (2021). In the literature, it is seen that Balakrishnan et al. (2019), observed the robustness of WMDPD estimators for the analysis of one-shot device testing data under ALT based on gamma, exponential and Weibull lifetime distributions. In this work the asymptotic distribution of the WMDPD estimator is derived. Inspired by the idea of Basu et al. (2013), testing of hypothesis based on WDPD is also developed. The WMDPDEs and test based on WDPD measure satisfy robustness property without compromising the efficiency.

Another key contribution of this work is in the design aspect of the life testing experiment which consists of finding out the optimal inspection times. Optimal inspection
times are the set of inspection times which will optimize certain cost function which may be based on maximizing the precision of the estimation or minimizing the experimental cost or both. In the literature, it is found that Wu et al. (2020) utilized D-optimality criterion with cost constraint to determine the optimal inspection times. Ling and Hu (2020) minimized the asymptotic variance of MLE under normal operating conditions for Weibull distributions with respect to sample allocations and inspection times for determining the optimal designs of simple ALT for one-shot devices. Balakrishnan and Castilla (2022) studied optimal design under constant stress accelerated life tests for one-shot devices with budget constraints.

In this work, a cost function is proposed which is based on minimizing the determinants of asymptotic covariance matrix of the WMDPD estimator and the number of expected failures. Population-based heuristic optimization method Genetic Algorithm (GA) in continuous design space is used in finding optimal inspection times. Immense popularity of GA lies on the merits of its easy understanding of the concept, ability to converge global or near global optima fairly well, to avoid the trap of local optima, to work without computation of complex mathematical derivatives, to deal with various constraints, to use probabilistic transition rules, to show efficiency even when many parameters are involved.

The rest of the work proceeds as follows: Section (2) comprises of model description and likelihood function computation. Density power divergence (DPD) measure with the weighted estimating equations, its asymptotic properties and testing of hypothesis along with the power function are discussed in Section (3). Section (4) is comprised with extensive simulation study to assess the performances of the derived estimators. In section (5), the cost function is defined and search for the optimum inspection times applying GA is provided. In section (6), outcomes of previous sections are applied to the murine model with Melioidosis data. The study is concluded with a summary discussion in section (7) along with some future insights.

2. Model Description

In the context of murine model, the mice are set down to $I$ observation groups where the number of mice in $i$th group is $k_i$ which are subject to $J$ types of stress factors, quantified by $x_{ij}; j = 0, 1, 2, \ldots, J$, where $x_{i0} = 1$, and inspected at times $\tau_i$ for $i = 1, \ldots, I$. In the $i$th observation group, the number of mice deaths until inspection time $\tau_i$, say, $n_i$, are recorded. Therefore, the observed data can be represented in tabular form as given in Table (1).

| Groups | Inspection Times | Mice/Devices | Deaths/ Failures | Covariates |
|--------|------------------|--------------|-----------------|------------|
|        | $\tau_1$         | $k_1$        | $n_1$           | $x_{11}$, ..., $x_{1J}$ |
|        | $\tau_2$         | $k_2$        | $n_2$           | $x_{21}$, ..., $x_{2J}$ |
|        | \vdots           | \vdots       | \vdots          | \vdots     |
|        | $\tau_I$         | $k_I$        | $n_I$           | $x_{I1}$, ..., $x_{IJ}$ |

In this work, lifetime $T$ of a mouse is assumed to follow two parameter Logistic-Exponential distribution with shape parameter $\alpha$ and scale parameter $\lambda$. The cumu-
Relative distribution function and probability density function of $T$ are given as follows:

$$
F(t) = 1 - (1 + (e^{\lambda t} - 1)^{\alpha})^{-1}; \ t > 0, \ \alpha, \lambda > 0,
$$

$$
f(t) = \frac{\alpha \lambda e^{\lambda t} (e^{\lambda t} - 1)^{\alpha - 1}}{(1 + (e^{\lambda t} - 1)^{\alpha})^2}; \ t > 0, \ \alpha, \lambda > 0.
$$

Within each observation group, shape and scale parameters are assumed to have log-linear link with the stress factors as follows:

$$
\alpha_i = \exp \left\{ \sum_{j=0}^{J} a_j x_{ij} \right\}
$$

$$
\lambda_i = \exp \left\{ \sum_{j=0}^{J} b_j x_{ij} \right\}
$$

where $x_{i0} = 1$ for $i = 1, \ldots, I$. Denote, $\theta = \{a_j, b_j; j = 0, \ldots, J\}$ for the model parameters to be estimated.

The likelihood function on the basis of given observed data can be obtained as

$$
L(\theta) \propto \prod_{i=1}^{I} F(\tau_i; x_{ij}, \theta)^{n_i} (1 - F(\tau_i; x_{ij}, \theta))^{k_i - n_i}
$$

$$
= \prod_{i=1}^{I} \left[ 1 - \left\{ 1 + \left( e^{\lambda_i \tau_i} - 1 \right)^{\alpha_i} \right\}^{-1} \right]^{n_i} \left[ \left\{ 1 + \left( e^{\lambda_i \tau_i} - 1 \right)^{\alpha_i} \right\}^{-1} \right]^{k_i - n_i}.
$$

Therefore, the log-likelihood function without the normalized constant can be obtained as,

$$
\ln L(\theta) \propto \sum_{i=1}^{I} n_i \ln \left[ \frac{(e^{\lambda_i \tau_i} - 1)^{\alpha_i}}{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}} \right] - (k_i - n_i) \ln \left[ 1 + \left( e^{\lambda_i \tau_i} - 1 \right)^{\alpha_i} \right]
$$

$$
= \sum_{i=1}^{I} \left[ n_i \alpha_i \ln \left( e^{\lambda_i \tau_i} - 1 \right) - k_i \ln \left\{ 1 + \left( e^{\lambda_i \tau_i} - 1 \right)^{\alpha_i} \right\} \right]. \quad (2)
$$

Hence the MLE of $\theta$, say $\hat{\theta} = \{\hat{a}_j, \hat{b}_j; j = 0, \ldots, J\}$ would be derived as

$$
\hat{\theta} = \arg \max_{\theta} \ln L(\theta)
$$

provided $\sum_{i=1}^{I} n_i > 0$.

**Theorem 2.1.** To obtain the MLE, the set of estimating equations are given as follows,

for the parameter $a_j$,

$$
\sum_{i=1}^{I} \alpha_i x_{ij} \ln(e^{\lambda_i \tau_i} - 1) \left[ n_i - \frac{k_i (e^{\lambda_i \tau_i} - 1)^{\alpha_i}}{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}} \right] = 0 \quad (4)
$$
and for the parameter $b_j$,

$$
\sum_{i=1}^I \frac{\hat{\alpha}_i \lambda_i \tau_i x_{ij} e^\lambda \tau_i}{(e^\lambda \tau_i - 1)} \left[ n_i - \frac{k_i (e^\lambda \tau_i - 1) \hat{\alpha}_i}{1 + (e^\lambda \tau_i - 1) \hat{\alpha}_i} \right] = 0.
$$

(5)

**Proof.** Given in the appendix.

3. Density Power Divergence (DPD) Measure

Minimum divergence estimation method namely Density power divergence for robust parameter estimation was developed by Basu et al. (1998). Tuning parameter $\beta$ is the special feature of DPD measure. DPD measure between any two probability distributions say, $F$ and $G$ on the same variable with density respectively, $f$ and $g$ can be obtained as:

$$
D_\beta(g,f) = \int \left\{ f^{\beta+1}(x) - \frac{\beta + 1}{\beta} g(x)f^\beta(x) + \frac{1}{\beta} g^{\beta+1}(x) \right\} dx
$$

(6)

where $(0 \leq \beta \leq 1)$ is the tuning parameter which strikes a balance between efficiency and robustness.

Also, it is simple to show that,

$$
\lim_{\beta \to 0^+} D_\beta(g,f) = D_{KL}(g|f)
$$

where $D_{KL}(g|f)$ is the Kullback-Leibler (KL) divergence measure which is the measure of the information lost when a probability distribution $F$ is used to approximate another probability distribution $G$ on the same variable. Based on the dichotomous data for one-shot devices, the DPD measure between empirical probability distribution, $\pi_i = \left( \frac{n_i}{k_i}, 1 - \frac{n_i}{k_i} \right)$ and theoretical probability distribution, $p_i = (1 - \{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^{-1}, \{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^{-1})$ for the given $i$th group (for $i = 1, \ldots, I$) can be obtained as,

$$
D_\beta(p_i, \pi_i) = \left[ 1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i(\beta+1)} \right] \frac{1}{\{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^{\beta+1}} - \frac{\beta + 1}{\beta} \left[ \frac{n_i}{k_i} \right] \left( \frac{\{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^\beta}{\{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^\beta} \right)
$$

$$+ \left( 1 - \frac{n_i}{k_i} \right) \left( \frac{1}{\{1 + (e^{\lambda_i \tau_i} - 1)^{\alpha_i}\}^{\beta+1}} \right)
$$

$$+ \frac{1}{\beta} \left[ \left( \frac{n_i}{k_i} \right)^{\beta+1} + \left( 1 - \frac{n_i}{k_i} \right)^{\beta+1} \right]
$$

(7)

and the KL divergence is obtained as,

$$
D_{KL}(\pi_i|p_i) = \left[ \frac{n_i}{k_i} \left\{ \ln \left( \frac{n_i}{k_i - n_i} \right) - \alpha_i \ln \left( e^{\lambda_i(\tau_i)} - 1 \right) \right\} \right]
$$

$$+ \ln \left\{ \left( \frac{k_i - n_i}{k_i} \right) \left( 1 + (e^{\lambda_i(\tau_i)} - 1)^{\alpha_i} \right) \right\}
$$

(8)
Considering all the I groups and imposing weights proportional to the group size, the weighted DPD measure with the weights $w_i = \frac{k_i}{K}$ for $i = 1, \ldots, I$ where $K = k_1 + k_2 + \ldots + k_I$ is given as:

$$D_{\beta}^w(\theta) = \sum_{i=1}^{I} \frac{k_i}{K} D_{\beta}(p_i, \pi_i)$$

$$= \sum_{i=1}^{I} \frac{k_i}{K} \left\{ \frac{1}{1 + (e^{\lambda (\tau_i)} - 1)^{a_i}} \right\}^{\beta} \left[ \frac{1 + \left( e^{\lambda (\tau_i)} - 1 \right)^{\alpha_i}}{1 + \left( e^{\lambda (\tau_i)} - 1 \right)^{a_i}} \right]$$

$$- \frac{\beta + 1}{\beta} \left\{ \frac{n_i}{k_i} \left( e^{\lambda (\tau_i)} - 1 \right)^{\alpha_i} \alpha + \left( 1 - \frac{n_i}{k_i} \right) \right\}$$

$$+ \frac{1}{\beta} \left\{ \left( \frac{n_i}{k_i} \right)^{\beta + 1} + \left( 1 - \frac{n_i}{k_i} \right)^{\beta + 1} \right\}$$

(9)

The relationship between likelihood function and weighted DPD measure can be established as follows:

$$\lim_{\beta \to 0^+} D_{\beta}^w(\theta) = C - \frac{1}{K} \ln L(\theta)$$

where $C$ being a constant which is $\theta$ independent.

Weighted Minimum density power divergence estimators (WMDPDE) for $\theta$ can be defined as follows:

$$\hat{\theta}_{\beta} = \text{argmin}_{\theta} D_{\beta}^w(\theta)$$

(10)

**Theorem 3.1.** The set of estimating equations to minimize WMDPD measure is given as follows,

$$\sum_{i=1}^{I} k_i \left\{ \left( 1 - (1 + (e^{\lambda (\tau_i)} - 1)^{a_i})^{-1} \right)^{\beta - 1} + (1 + (e^{\lambda (\tau_i)} - 1)^{a_i})^{-(\beta - 1)} \right\}$$

$$\left\{ \left( 1 - (1 + (e^{\lambda (\tau_i)} - 1)^{a_i})^{-1} \right) - \frac{n_i}{k_i} \right\} \left[ \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} \right] = 0_{2J}$$

(11)

with $\theta = (a_1, \ldots, a_j, a_J, b_1, \ldots, b_j, b_J)'$ and where,

$$\frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial a_j} = \frac{\alpha_i x_{ij} (e^{\lambda (\tau_i)} - 1)^{a_i} \ln (e^{\lambda (\tau_i)} - 1)}{\{1 + (e^{\lambda (\tau_i)} - 1)^{a_i}\}^2}$$

(12)

$$\frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial b_j} = \frac{\alpha_i \lambda_i \tau_i x_{ij} e^{\lambda (\tau_i)} (e^{\lambda (\tau_i)} - 1)^{a_i - 1}}{\{1 + (e^{\lambda (\tau_i)} - 1)^{a_i}\}^2}$$

(13)

For $\beta = 0$, WMDPDE reduces to MLE.

**Proof.** : Given in the appendix.  

\[\square\]
3.1. Asymptotic Property

Here, we present the asymptotic distribution of the weighted minimum density power divergence estimator under the logistic exponential distribution based on the failure count data.

Theorem 3.2. Suppose $\theta_0$ is the true value of the parameter $\theta$, when $k_i \to \infty$ for $i = 1, \ldots, I$ and $\lim_{k_i \to \infty} \frac{k_i}{K}$ finite for $i = 1, \ldots, I$.

$$\sqrt{K}(\hat{\theta}_\beta - \theta_0) \xrightarrow{L} N\left(0, J^{-1}_\beta(\theta_0)KJ^{-1}_\beta(\theta_0)\right)$$

(14)

Proof. : Given in the appendix.

3.2. Testing of hypothesis based on weighted Density Power Divergence (WDPD)

Testing statistical hypothesis is of fundamental importance for the inferential analysis. Due to lack of robustness of MLE, tests based on MLE may not be able to perform well in presence of data contamination. Here, Robust WDPD based test statistic for the testing of simple hypothesis based on the idea of Basu et al. (2013) is presented. The null and alternative hypothesis is given as follows:

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0$$

where $\theta = \{a_j, b_j ; j = 1, 2, \ldots, J\}$.

The robust WDPD based test statistic is defined as,

$$\Lambda_\beta(\hat{\theta}_\beta, \theta_0) = 2KD_w(\hat{\theta}_\beta, \theta_0)$$

(15)

where,

$$D_w(\hat{\theta}_\beta, \theta_0) = \sum_{i=1}^{I} \frac{k_i}{K} \left[ \{F_{\beta+1}^i(\theta_0) + \bar{F}_{\beta+1}^i(\theta_0)\} - \frac{\beta + 1}{\beta} \{F_{\beta}^i(\hat{\theta}_\beta)F_{\beta}^i(\theta_0) \right.$$  

$$+ \bar{F}_{\beta}^i(\hat{\theta}_\beta)\bar{F}_{\beta}^i(\theta_0)\right] + \frac{1}{\beta} \left\{F_{\beta+1}^i(\hat{\theta}_\beta) + \bar{F}_{\beta+1}^i(\hat{\theta}_\beta)\right\}$$

Applying Taylor series expansion of second order around $\theta = \theta_0$ at $\theta = \hat{\theta}_\beta$

$$\Lambda_\beta(\hat{\theta}_\beta, \theta_0) = 2K \left[ D_w^u(\theta_0, \theta_0) + (\hat{\theta}_\beta - \theta_0)^T \nabla D_w^u(\theta_0, \theta_0) \right.$$  

$$\left. + \frac{1}{2}(\hat{\theta}_\beta - \theta_0)^T \nabla^2 D_w^u(\theta_0, \theta_0)(\hat{\theta}_\beta - \theta_0)\right]$$

When $H_0$ is true,

$$\Lambda_\beta(\hat{\theta}_\beta, \theta_0) = \sqrt{K}(\hat{\theta}_\beta - \theta_0)^T \nabla^2 D_w^u(\theta_0, \theta_0)\sqrt{K}(\hat{\theta}_\beta - \theta_0)$$

(16)

(since $D_w^u(\theta_0, \theta_0) = 0$, $\nabla D_w^u(\theta_0, \theta_0) = 0$)
The establishment of asymptotic distribution of $\Lambda_{\beta}$ values of $X$ can be expressed as a matrix of order $p$, $W$. $W$ is a diagonal matrix with diagonal elements $\lambda_i$ which are eigen values of $\Sigma^{1/2} A \Sigma^{1/2}$ and $W = P' \Sigma^{-1/2} X$.

Result 1: Let $X \sim N_p(0, \Sigma)$ and $A$ be a $p \times p$ real symmetric matrix then $X'AX$ can be expressed as

$$X'AX = X' \Sigma^{-1/2} \Sigma^{1/2} A \Sigma^{1/2} \Sigma^{-1/2} X = X' \Sigma^{-1/2} P \Lambda P' \Sigma^{-1/2} X = W' \Lambda W.$$ 

Here, $P$ is an orthogonal matrix containing the eigen vectors of $\Sigma^{1/2} A \Sigma^{1/2}$ and $\Lambda$ is a diagonal matrix with diagonal elements $\lambda_i$ which are eigen values of $\Sigma^{1/2} A \Sigma^{1/2}$ and $W = P' \Sigma^{-1/2} X$.

As $W \sim N(0, I)$, therefore, the distribution of $X'AX$ is same as the distribution of $\sum_{i=1}^{r} \lambda_i W_i^2$, where $W_1, \ldots, W_r$ are the independent standard normal variables. Here, $r = \text{rank}(\Sigma^{1/2} A \Sigma^{1/2}) = \text{rank}(A \Sigma)$ and $\lambda_i$ are the non-zero eigenvalues of $A \Sigma$.

Using the above result, the asymptotic distribution of test statistic $\Lambda_{\beta}(\hat{\theta}_\beta, \theta_0)$ can be described by $\sum_{i=1}^{r} \lambda_i^{(\beta)} W_i^2$, where $\lambda_i^{(\beta)}$s are the non-zero eigen values of $(\nabla^2 D^w_\beta(\theta_0, \Sigma \beta(\theta_0)))$ and $\Sigma \beta(\theta_0) = J_{\beta}^{-1}(\theta_0) K_{\beta}(\theta_0) J_{\beta}^{-1}(\theta_0)$ with $r = \text{rank}(\nabla^2 D^w_\beta(\theta_0, \theta_0) \Sigma \beta(\theta_0))$.

Further, let us define,

$$\Lambda^*_{\beta}(\hat{\theta}_\beta, \theta_0) = \frac{\Lambda_{\beta}(\hat{\theta}_\beta, \theta_0)}{\lambda_{\max}} \leq \sum_{i=1}^{r} W_i^2$$

where $\lambda_{\max} = \max(\lambda_i^{(\beta)}; 1, 2, \ldots, r)$. As, $\sum_{i=1}^{r} W_i^2 \sim \chi^2_r$, $H_0$ is rejected when $\Lambda^*_{\beta}(\hat{\theta}_\beta, \theta_0) \geq \chi^2_{(r, 1-\alpha)}$ where $\chi^2_{(r, 1-\alpha)}$ is the upper $(1-\alpha)$ quantile point of $\chi^2_r$.

3.3. Power Function of WDPA Based Test

In this section we have studied the power function of WDPA based test. The alternative hypothesis is set as $H_{1,K}: \theta_K = \theta_0 + K^{-1/2}d$, where $\theta_K \in \theta \subset \mathbb{R}^{2J}$ and $d$ is a fixed vector in $\mathbb{R}^{2J}$. The expression $\sqrt{K}(\hat{\theta}_\beta - \theta_0)$ can be written as,

$$\sqrt{K}(\hat{\theta}_\beta - \theta_0) = \sqrt{K}(\hat{\theta}_\beta - \theta_K) + d$$

Under $H_{1,K},$

$$\sqrt{K}(\hat{\theta}_\beta - \theta_K) \xrightarrow{L_{K\rightarrow\infty}} N(0_{2J}, \Sigma_{\beta}(\theta_K))$$

and therefore

$$\sqrt{K}(\hat{\theta}_\beta - \theta_0) \xrightarrow{L_{K\rightarrow\infty}} N(d, \Sigma_{\beta}(\theta_K))$$

The establishment of asymptotic distribution of $\Lambda_{\beta}(\hat{\theta}_\beta, \theta_0)$ under $H_{1,K}$ is based on the Corollary 2.2 of Dik and De Gunst (1985) given as follows.

Result 2: Let $X \sim N_p(\mu, \Sigma)$, $A$ be a real-symmetric non-negative definite matrix of order $p$, $r = \text{rank}(\Sigma A \Sigma)$, $r \geq 1$ and $\lambda_1, \ldots, \lambda_r$ be the positive eigen values of $A \Sigma$. Then the distribution of $X'AX$ is equivalent to the distribution of $\sum_{i=1}^{r} \lambda_i(W_i + v_i)^2 + \Psi$, where $W_1, \ldots, W_r$ are independent standard normal variables.
and $v = \Lambda^{-1}P'S'A\mu$, $\Psi = \mu'A\mu - v'\Lambda v$. Here, $S$ be any square-root of $\Sigma$ and $S'AS = P\Lambda P'$ where $\Lambda$ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_r$ which are positive eigen values of $S'AS$ and $P$ is an orthogonal matrix with column vectors being the eigen vectors of the corresponding eigen values.

Using the above result, the asymptotic distribution of $\Lambda_\beta(\hat{\theta}_\beta, \theta_0)$ under $H_{1,K}$ is equivalent with the distribution of $\sum_{i=1}^r \lambda_i^{(\beta)}(\theta_0)(W_i + v_i)^2 + \Psi$ where $W_i$s are independent standard normal variables, $\lambda_1^{(\beta)}(\theta_0), \ldots, \lambda_r^{(\beta)}(\theta_0)$ are the positive eigen values of $\left(\nabla^2 D_{\beta}^w(\theta_0, \theta_0)\Sigma_{\beta}(\theta_0)\right)$. The values $v = (v_1, \ldots, v_r)$ are given as $v = \Lambda^{-1}P'S'\nabla^2 D_{\beta}^w(\theta_0, \theta_0)d$ and $\Psi = d'\nabla^2 D_{\beta}^w(\theta_0, \theta_0)d - v'\Lambda v$ where $S$ is any square-root of $\Sigma_{\beta}(\theta_0)$, $\Lambda = \text{diag}(\lambda_1^{(\beta)}(\theta_0), \ldots, \lambda_r^{(\beta)}(\theta_0))$ which are positive eigen values of $\left(S'\nabla^2 D_{\beta}^w(\theta_0, \theta_0)S\right)$ and $P$ is an orthogonal matrix with column vectors being the eigen vectors of the corresponding eigen values.

Though the asymptotic distribution of $\Lambda_\beta(\hat{\theta}_\beta, \theta_0)$ under $H_{1,K}$ using above method is very informative yet it is not useful for determining the power function due to its complex nature. The alternative approach for determination of approximation of power function is given as follows:

The first order Taylor expansion of $D_{\beta}^{w}(\hat{\theta}_\beta, \theta_0)$ under $\theta^*, \theta^* \neq \theta_0$ is given as,

$$D_{\beta}^{w}(\hat{\theta}_\beta, \theta_0) = D_{\beta}^{w}(\theta^*, \theta_0) + A_\beta'(\hat{\theta}_\beta - \theta^*), \text{ where } A_\beta = \nabla D_{\beta}^{w}(\theta^*, \theta_0)$$

As,

$$\sqrt{K}(\hat{\theta}_\beta - \theta^*) \xrightarrow{L} \frac{K \rightarrow \infty} {N(0_{2J}, \Sigma_{\beta}(\theta^*))}$$

Then, $\sqrt{K}(D_{\beta}^{w}(\hat{\theta}_\beta, \theta_0) - D_{\beta}^{w}(\theta^*, \theta_0))$ and $A_\beta'\sqrt{K}(\hat{\theta}_\beta - \theta^*)$ have the same asymptotic distribution and

$$\sqrt{K}(D_{\beta}^{w}(\hat{\theta}_\beta, \theta_0) - D_{\beta}^{w}(\theta^*, \theta_0)) \xrightarrow{L} \frac{K \rightarrow \infty} {N(0_{2J}, \Sigma_{\beta}^*(\theta^*))}$$

where, $\Sigma_{\beta}^*(\theta^*) = A_\beta'\Sigma_{\beta}(\theta^*)A_\beta$.

Therefore the power function can be obtained as

$$\pi_{K,\alpha}^{(\beta)}(\theta^*) = Pr[2KD_{\beta}^{w}(\hat{\theta}_\beta, \theta_0) > c_{\alpha}^{(\beta)}]$$

$$= Pr \left[ \frac{\sqrt{K}(D_{\beta}^{w}(\hat{\theta}_\beta, \theta_0) - D_{\beta}^{w}(\theta^*, \theta_0))}{\Sigma_{\beta}^*(\theta^*)} > \frac{\sqrt{K}}{\Sigma_{\beta}^*(\theta^*)} \left( \frac{c_{\alpha}^{(\beta)}}{2K} - D_{\beta}^{w}(\theta^*, \theta_0) \right) \right]$$

$$\pi_{K,\alpha}^{(\beta)}(\theta^*) = 1 - \Phi \left[ \frac{\sqrt{K}}{\Sigma_{\beta}^*(\theta^*)} \left( \frac{c_{\alpha}^{(\beta)}}{2K} - D_{\beta}^{w}(\theta^*, \theta_0) \right) \right] \quad (17)$$

where, $\Phi(x)$ is the standard normal distribution function and $c_{\alpha}^{(\beta)}$ is $(1 - \alpha)$ percentile
of the distribution of $\Lambda_\beta(\hat{\theta}, \theta_0)$ under $H_0$.

4. Simulation Experiment

In this section, a simulated environment is created using the Markov Chain Monte Carlo Simulations based on 1000 generations and performances of MLE and WMDPDEs are observed. The lifetimes of the one-shot devices are considered to follow the Logistic-Exponential distribution. An accelerated life test is conducted under 3 testing groups with an inspection time for each of the groups. Different number of devices are put to test in each group which are subjected to two types of stress factors. The layout summaries are given in Table (2).

**Table 2.** Layout of one-shot device ALT design for simulation

| Groups | Inspection Times | Devices | Failures | Stress 1 | Stress 2 |
|--------|------------------|---------|----------|----------|----------|
| 1      | 1.00             | 9       | $n_1$    | 0.2      | 0.4      |
| 2      | 1.00             | 12      | $n_2$    | 0.3      | 0.6      |
| 3      | 1.00             | 15      | $n_3$    | 0.4      | 0.8      |

The three different sets of model parameters are taken so that the results can be analysed with at least three different view points and the performances of the estimators can be assessed. To study the robustness of the WMDPDEs, the generated data has been contaminated in three different ways and results are analysed. Different sets of model parameters for pure ALT data are given in the Table (3) and the contamination schemes are given in the Table (4).

**Table 3.** True model parameters to generate pure data

| S.No. | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
|-------|-------|-------|-------|-------|
| $\theta_1$ | 0.2   | -0.6  | -0.2  | 0.4   |
| $\theta_2$ | 0.4   | -0.3  | 0.1   | 0.2   |
| $\theta_3$ | 0.1   | 0.1   | 0.1   | 0.1   |

**Table 4.** Model parameters to generate contaminated data

| S.No. | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
|-------|-------|-------|-------|-------|
| $\theta_1$ | $a_1 + 0.05$ | $a_2 + 0.03$ | $b_1 + 0.02$ | $b_2 + 0.09$ |
| $\theta_2$ | $a_1 + 0.02$ | $a_2 + 0.03$ | $b_1 - 0.03$ | $b_2 + 0.02$ |
| $\theta_3$ | $a_1 + 0.03$ | $a_2 + 0.03$ | $b_1 + 0.02$ | $b_2 + 0.02$ |

The algorithm to obtain the MLEs and WMDPDEs is given as follows:

- Generate a Logistic-Exponential random samples of size $k_i$ with the parameters $\alpha_i$ and $\lambda_i$ using the command “rlogis.exp($k_i, \alpha_i, \lambda_i$)” for $i = 1, 2, 3$ from “library(reliaR)” in R programming and sort it.
- At the given inspection times, obtain the number of failures.
- Use the Coordinate-Descent method to obtain the MLEs and WMDPDEs. Coordinate descent is an optimization scheme which successively minimizes along
with the coordinate directions to find minimum value of a function. The algorithm is as follows:
i) Choose initial value of $\theta = (a_1, a_2, b_1, b_2)$ say $\theta_0 = (a_1^{(0)}, a_2^{(0)}, b_1^{(0)}, b_2^{(0)})$.
ii) Let at $t$th iteration for $t = 0, 1, 2, \ldots$, the estimate of $\theta$ be $\theta^t = (a_1^{(t)}, a_2^{(t)}, b_1^{(t)}, b_2^{(t)})$. Then at $t + 1$th iteration, the estimate can be derived as,
\[
\begin{align*}
a_1^{(t+1)} &= a_1^{(t)} - h \frac{\partial H(a_1^{(t)}, a_2^{(t)}, b_1^{(t)}, b_2^{(t)})}{\partial a_1}, \\
a_2^{(t+1)} &= a_2^{(t)} - h \frac{\partial H(a_1^{(t)}, a_2^{(t)}, b_1^{(t)}, b_2^{(t)})}{\partial a_2}, \\
b_1^{(t+1)} &= b_1^{(t)} - h \frac{\partial H(a_1^{(t)}, a_2^{(t)}, b_1^{(t)}, b_2^{(t)})}{\partial b_1}, \\
b_2^{(t+1)} &= b_2^{(t)} - h \frac{\partial H(a_1^{(t)}, a_2^{(t)}, b_1^{(t)}, b_2^{(t)})}{\partial b_2},
\end{align*}
\]
where $H = -\ln L(\theta)$ in case of MLE and $H = D^w_j(\theta)$ in case of WMDPDE where $h$ is the learning rate. Value of learning rate is chosen as $h = 0.01$.
iii) The process continues until $max(|a_1^{(t+1)} - a_1^{(t)}|, |a_2^{(t+1)} - a_2^{(t)}|, |b_1^{(t+1)} - b_1^{(t)}|, |b_2^{(t+1)} - b_2^{(t)}|)$ is less than some pre-specified threshold value and if it is satisfied the final estimate is obtained as $\theta^{t+1} = (a_1^{(t+1)}, a_2^{(t+1)}, b_1^{(t+1)}, b_2^{(t+1)})$.

### 4.1. Result and Interpretation

Bias and root-mean-square errors (RMSE) of the MLE and WMDPDE in case of pure data and contaminated data settings are given in the Tables 5-8. It can be observed that bias of WMDPDEs are lesser than the bias of MLEs in the pure data as well as the contaminated data settings. It is discerned from the Tables 5 and 6 that biases of MLEs are affected in the contaminated data setting as compared to pure data setting. It is observed from Tables 7 and 8 that as the value of $\beta$ increases, bias decreases. Also, it is observed from these tables that there is not much difference in the bias of WMDPDEs for pure data and contaminated data schemes. Therefore, it may be concluded that WMDPDEs are robust estimators without its efficiency being compromised. Due to superiority of WMDPDEs numerically proven here, these estimators are preferable choice over MLEs.

| Table 5. MLE outcomes in pure data setting |
|-------------------------------------------|
| a1 | Bias  | RMSE |
| a2 | Bias  | RMSE |
| b1 | Bias  | RMSE |
| b2 | Bias  | RMSE |
| $\theta_1$ | 0.00234 | 0.02251 |
| $\theta_2$ | 0.00203 | 0.02053 |
| $\theta_3$ | 0.00242 | 0.02184 |

| Table 6. MLE outcomes in contaminated data setting |
|------------------------------------------|
| a1 | Bias  | RMSE |
| a2 | Bias  | RMSE |
| b1 | Bias  | RMSE |
| b2 | Bias  | RMSE |
| $\theta_1$ | 0.00673 | 0.02644 |
| $\theta_2$ | 0.00660 | 0.02672 |
| $\theta_3$ | 0.00504 | 0.02123 |
One-shot devices stay in torpid state until used, hence behaviour of such devices are revealed only when these are put to test. As optimization is often required at the highest cost efficiency (Altarazi & Allaf, 2017), here, it is essential to design a set of inspection times which will optimize a scientifically defined cost function. In this section, a cost function is defined which is based on the asymptotic co-variance matrix of the WMDPDE and the expected number of failures. The aim is to find a set of inspection times, which will give a good precision of the estimator as well as try to
reduce the experimental cost due to destruction of experimental units. Therefore, the cost function is defined as follows,

$$Cost = C_1 \times |V| + C_2 \times E\left(\sum_{i=1}^{I} n_i\right)$$  \hspace{1cm} (18)$$

where, $|V|$ is the determinant value of covariance matrix based on asymptotic distribution of WMMDPDE, $E(\sum_{i=1}^{I} n_i)$ is the expected total number failures from all I different groups. Cost $C_1$ is imposed on $|V|$ which reflects the precision cost of the estimator and $C_2$ is the cost per failure of experimental units. Given the values of group sizes, the optimal inspection times will minimize the cost function defined in (18). This cost function brings a trade-off between precision of the estimation and experimental cost.

5.1. Search Algorithm

Here, the objective is to find the inspection times which will minimize the above defined cost. Population-based heuristic Natural optimization method namely Genetic Algorithm (GA) is applied which was developed by Holland (1975) and popularized by Goldberg (1989). GA is an optimization method which produces generations of population by the processes of selection, crossover and mutation (Ghosh, Iquebal & Prajneshu, 2011). It is applicable to discrete as well as continuous variables and can deal with large number of variables. It can perform optimization with extremely complex cost surfaces and is capable of discovering global optimum avoiding trap of local optima. GA is inherently parallel, modifiable, easily distributed and adaptable to different problems (Ou S.L., Liu & Ou, 2014). GA works well with numerically generated data, experimental data, or analytical functions (Haupt R.L. & Haupt, 2004). To search the optimal inspection times through GA, the strategy of keeping the best or the elite observations of the current population in the new population is followed, readers may see Chakraborty and Chaudhuri (2003) for detailed discussion. Thus traits of the observations with the best cost i.e. minimum cost is retained from generation to generation. The procedure of GA using elitism is as follows.

i) Set the size of initial population $N_{pop}$ where each element in the population is a set of inspection times with dimension $N_{var} = I$.

ii) Obtain cost for each set of inspection times and select the parents for pairing.

iii) The parents are selected using a specified selection procedure based on cost. Among the various selection methods, random rank selection procedure and tournament selection procedure are used which are explained below.

iv) Each set of parents would produce two off springs. Odd numbered element is chosen as mother and even numbered element is chosen as father. The off springs produced are:

Offspring 1 = $b \times$ mother + $(1-b) \times$ father

Offspring 2 = $(1-b) \times$ mother + $b \times$ father

where, $b$ is a random number. Thus a new population is formed containing the
parents and off springs.

v) Now the population is mutated with the mutation rate \( mr \). So the number of mutations are \( mn = mr \times N_{pop} \times N_{var} \). The values in randomly selected \( mn \) positions out of \( N_{pop} \times N_{var} \) positions are replaced with the randomly generated new values. Hence a mutated population is obtained which would function as initial population for the next generation.

vi) The process is repeated up to say, \( B \) generations. For each generation, a local minimum is obtained which is the set of inspection times with least cost. Among these \( B \) local minima, the global minimum is obtained which is the set of inspection times with minimum cost.

vii) This global minimum is the set of optimum inspection times.

5.1.1. Random Rank Selection Procedure

- Rearrange the observations with respect to cost in increasing order.
- Define \( N_{keep} = 0.5 \times N_{pop} \).
- Keep \( N_{keep} \) observations and rank them while discarding rest of the observations.
- Use a permutation of ranks and rearrange the \( N_{keep} \) size data with respect to this permutation.
- The resultant observations are chosen as the parents which are to be paired.

5.1.2. Tournament Selection Procedure

- Randomly pick a pair of observations such that this pair is not selected again.
- The observation having lesser cost in the pair is kept as a parent and other one is discarded. Though the same pair can not be chosen again, yet the discarded observation or the chosen observation in the pair, can be chosen again for pairing.
- Repeat this process up to \( N_{keep} \) times.
- The resultant observations are chosen as the parents which are to be paired.

Here, a numerical experiment is conducted to derive optimal inspection times implementing GA under the following set-ups. The chosen model parameters are given in Table 9 with tuning parameter \( \beta = 0.1 \) and the layout is given in Table 10. Costs are set as \( C_1 = C_2 = 0.5 \) where \( N_{pop} = 12 \), \( N_{var} = 5 \), GA continues till \( B = 250 \) generations. In crossover, random number \( b \) follows \( U(0, 1) \), where in mutation stage, mutation rate is set as \( mr = 0.20 \), and the value in the selected position is replaced by a new random number from \( U(0, 10) \).

Table 9. Model parameters

| S.No. | \( a_1 \) | \( a_2 \) | \( b_1 \) | \( b_2 \) |
|-------|--------|--------|--------|--------|
| \( \theta_1 \) | 0.2    | -0.6   | -0.2   | 0.4    |
| \( \theta_2 \) | -0.3   | -0.1   | -0.2   | 0.1    |

5.2. Result and Interpretation

The Optimum inspection times \( \tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \) by the random rank selection procedure and tournament selection procedure are given in the Tables 11 and 12. Though the
two selection procedures yield very different set of inspection times yet the resultant costs for both of them are very similar. Thus these two selection procedures can be used alternatively for Genetic Algorithm.

Table 11. Optimum Inspection Times (For $\theta_1$)

| Selection Method          | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\tau_4$ | $\tau_5$ | Cost  |
|---------------------------|----------|----------|----------|----------|----------|-------|
| Random rank selection     | 6.13382  | 0.98725  | 2.69014  | 8.90694  | 4.05132  | 2.46078|
| Tournament selection      | 8.34024  | 1.56001  | 7.02301  | 0.98418  | 0.21274  | 2.32247|

Table 12. Optimum Inspection Times (For $\theta_2$)

| Selection Method          | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\tau_4$ | $\tau_5$ | Cost  |
|---------------------------|----------|----------|----------|----------|----------|-------|
| Random rank selection     | 9.18581  | 2.63731  | 9.62269  | 0.05032  | 4.53493  | 1.80780|
| Tournament selection      | 4.29600  | 9.97924  | 0.67811  | 8.40763  | 0.02557  | 1.95393|

6. Application to the murine model with Melioidosis data

A real data is adopted from the study conducted by West et al. [2012] where they have developed an experimental murine model of pneumonic Melioidosis induced by a gram-negative bacterium Burkholderia pseudomallei which causes potentially fatal tropical disease Melioidosis. Various dilutions of Burkholderia pseudomallei were given to the mice with different CFU/lung deposition doses. Between the two strain of mice chosen for experiment, one of the strain showed more resistance than the other one. Clearly the doses of B. Pseudomallei ($x_{i1}$) and starin of mice ($x_{i2}$) are considered as two stress factors/covariates. The strains of mice are coded as: BALB/c-strain (value 1), C57BL/6-strain (value 0). The deposition doses are divided by 100 for computational conveniences of the analysis. Those mice were observed over different number of days and number of failure were reported within follow-up times where death due to Melioidosis indicates failure. The description of data is given in the Table 13. $a_1$, $a_2$, $b_1$ and $b_2$ are the model parameters of interest. To check if the logistic-exponential distribution can be fitted to the data, a bootstrap testing is conducted which is based on the distance-based test statistic $D = \max_i |n_i - e_i|$, where $n_i$ is the observed failure and $e_i$ is the expected failure for $i = 1, \ldots, 10$. The MLEs are used to compute estimated failures as $e_i = k_i \times F(\tau_i; x_{ij}, \theta)$ and the observed value of $D = 2.5$. To calculate the approximate p-value [Balakrishna & Ling, 2012], 10,000 bootstrap samples are generated. In this case, the p value is coming as 0.1122 which suggests the suitability of logistic-exponential model to the given data. MLEs and WMDPDEs of the model parameters are obtained and bootstrap estimates of bias and root-mean-square-error generated from 1000 bootstrap samples are computed. In
Table 13. Details of the murine model with Melioidosis data (West et al., 2012)

| Covariates | Groups | Observation Days | Number of Mice under observation | Mice deaths | Deposition dose | Mouse strain |
|------------|--------|------------------|----------------------------------|-------------|----------------|--------------|
|            | 1      | 14               | 4                                | 0           | 0.56           | 1            |
|            | 2      | 5                | 5                                | 1           | 1.32           | 1            |
|            | 3      | 61               | 5                                | 4           | 2.92           | 1            |
|            | 4      | 3                | 4                                | 4           | 10.29          | 1            |
|            | 5      | 2                | 5                                | 5           | 16.37          | 1            |
|            | 6      | 14               | 4                                | 0           | 0.56           | 0            |
|            | 7      | 5                | 5                                | 0           | 1.32           | 0            |
|            | 8      | 61               | 5                                | 0           | 2.92           | 0            |
|            | 9      | 5                | 4                                | 4           | 3.75           | 0            |
|            | 10     | 2                | 5                                | 5           | 16.37          | 0            |

The coordinate descent algorithm, the initial parameter value is set as (-5, -4, -2, 1), which is chosen through a grid search process. The MLE and WMDPDE outcomes are reported in the Tables (14) and (15). The hazard functions for the mice groups are shown in Figure (2).

Table 14. MLE outcomes for the murine model with Melioidosis data

| a1   | a2   | b1   | b2   |
|------|------|------|------|
| Estimate | -5.003640 | -4.000270 | -2.003780 | 0.999720 |
| BT Bias   | -0.119668 | 0.005523  | -0.123156 | 0.005516  |
| BT RMSE   | 0.137218  | 0.008015  | 0.140787  | 0.008010  |

Table 15. WMDPDE outcomes for the murine model with Melioidosis data

| a1   | b1   | b2   |
|------|------|------|
| Estimate | -5.003640 | -4.000270 | -2.003780 | 0.999720 |
| BT Bias   | -0.119668 | 0.005523  | -0.123156 | 0.005516  |
| BT RMSE   | 0.137218  | 0.008015  | 0.140787  | 0.008010  |

For robust WDPD based testing of hypothesis, the null and alternative hypothesis is taken as follows:

\[ H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0 \]

where, \( \theta_0 = (-4, -3, -2, 1)' \). The values of WDPD based test statistic is given in
Figure 2. Hazard rates of the mice for 10 different groups: Murine model with Melioidosis data
Table 16. The rank\((\nabla^2 D^w_\beta(\theta_0, \theta_0) \Sigma_\beta(\theta_0))\) = 4 and \(\chi^2_{(4,0.95)} = 0.711\). From the values of test statistic in the Table 16, it is seen that in all cases \(\chi^2_{(4,0.95)}\) value is greater than these calculated values. Hence the null hypothesis is failed to be rejected.

| \(\beta\)  | 0.2  | 0.4  | 0.6  | 0.8  | 1.0  |
|-------------|------|------|------|------|------|
| Value       | 0.000026478 | 0.000026357 | 0.000026257 | 0.000026176 | 0.000026116 |

7. Discussion and Conclusion

In this work, the robust weighted minimum density power divergence estimators (WMDPDE) are derived for point estimation of a Logistic-Exponential lifetime distribution under one-shot device testing data and the asymptotic property along with the robust WDPP based testing of hypothesis are studied. Through the Markov Chain Monte Carlo simulation of size 1000, the behaviour of conventional maximum likelihood estimator (MLE) and WMDPDEs under the ALT design are observed. Since the estimating equations are not in closed form, coordinate-descent method is used to derive the estimates. In the study of the robustness of WMDPDE, numerically it is found that WMDPDEs are the preferable choice in the pure data as well as the contaminated data settings over MLEs. Finally, those theoretical results are applied in the reliability analysis of the murine model with Melioidosis data. Apart from those accomplishments, a search for optimum inspection times is also conducted applying Genetic Algorithm. This optimization is performed with respect to a defined cost function which incorporates a trade-off between the precision of the estimation and the experimental cost.

The reliability analysis of one-shot devices in a more complicated situation such as missing information on stress factors/covariates can also be conducted. Instead of one inspection time per group, several inspection times per group with the competing causes of failure can also be studied. Works in those directions are in pipeline and we are optimistic about reporting those findings soon.

Disclosure statement

The authors report no conflict of interest.

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Appendix A. Proof of Theorems

A.1. Proof of Theorem (2.1).

The set of estimating equations for MLE are,

$$\frac{\partial}{\partial a_j} \left( \sum_{i=1}^{I} \left[ n_i \hat{\alpha}_i \ln(e^{\lambda_i(\tau_i)} - 1) - k_i \ln \left\{ 1 + (e^{\lambda_i(\tau_i)} - 1)^{\alpha_i} \right\} \right] \right) = 0_J$$

and,

$$\frac{\partial}{\partial b_j} \left( \sum_{i=1}^{I} \left[ n_i \hat{\alpha}_i \ln(e^{\lambda_i(\tau_i)} - 1) - k_i \ln \left\{ 1 + (e^{\lambda_i(\tau_i)} - 1)^{\alpha_i} \right\} \right] \right) = 0_J$$

where

$$\hat{\alpha}_i = \hat{\tau}_i x_{ij} e^{\lambda_i(\tau_i)} \left( e^{\lambda_i(\tau_i)} - 1 \right)^{\alpha_i} \left\{ 1 + (e^{\lambda_i(\tau_i)} - 1)^{\alpha_i} \right\}^{-1}.$$
A.2. Proof of Theorem (3.1).

The set of estimating equations for WMDPDEs are,

$$\Rightarrow \frac{\partial D_{\beta}^w(\theta)}{\partial \theta} = 0_{2J}$$

$$\Rightarrow \sum_{i=1}^{I} \frac{k_i}{K} \left[ F^\beta(\tau_i; x_{ij}, \theta) \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} - \tilde{F}^\beta(\tau_i; x_{ij}, \theta) \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} \right. \right.$$  

$$\left. - \left\{ \left( \frac{n_i}{k_i} \right) F^{\beta-1}(\tau_i; x_{ij}, \theta) \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} \right) - \left( \frac{k_i - n_i}{k_i} \right) \tilde{F}^{\beta-1}(\tau_i; x_{ij}, \theta) \right\} = 0_{2J}$$

$$\Rightarrow \sum_{i=1}^{I} \frac{k_i}{K} \left[ F^\beta(\tau_i; x_{ij}, \theta) - \tilde{F}^\beta(\tau_i; x_{ij}, \theta) - \left( \frac{n_i}{k_i} \right) \tilde{F}^{\beta-1}(\tau_i; x_{ij}, \theta) \right. \right.$$  

$$\left. + \left( \frac{k_i - n_i}{k_i} \right) \tilde{F}^{\beta-1}(\tau_i; x_{ij}, \theta) \right] \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} = 0_{2J}$$

$$\Rightarrow \sum_{i=1}^{I} \frac{k_i}{K} \left\{ F^{\beta-1}(\tau_i; x_{ij}, \theta) + \tilde{F}^{\beta-1}(\tau_i; x_{ij}, \theta) \right\} \right.$$  

$$\left[ F^{\beta-1}(\tau_i; x_{ij}, \theta) - \frac{n_i}{k_i} \right] \frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial \theta} = 0_{2J}$$

where,

$$F(\tau_i; x_{ij}, \theta) = \left( 1 - (1 + (e^{\lambda \tau_i} - 1)^{\alpha_i})^{-1} \right)$$

$$\frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial a_j} = \alpha_i x_{ij} (e^{\lambda \tau_i} - 1)^{\alpha_i} \ln(e^{\lambda \tau_i} - 1) \right.$$  

$$\frac{\partial F(\tau_i; x_{ij}, \theta)}{\partial b_j} = \frac{\alpha_i \lambda_i \tau_i x_{ij} e^{\lambda \tau_i} (e^{\lambda \tau_i} - 1)^{\alpha_i}}{1 + (e^{\lambda \tau_i} - 1)^{\alpha_i}}$$

A.3. Proof of Theorem (3.2).

Based on [Calvino, Martin & Pardo] (2021), the proof of the theorem carries on as follows.

WDPD measure keeping only the terms involving parameters can be written as,

$$D_K = \sum_{i=1}^{I} \frac{k_i}{K} \left[ P_{i1}^{\beta+1} + \tilde{P}_{i1}^{\beta+1} - \frac{\beta + 1}{\beta} \left\{ \frac{n_i}{k_i} P_{i1}^{\beta} + \left( \frac{k_i - n_i}{k_i} \right) \tilde{P}_{i1}^{\beta} \right\} \right]$$

where, $P_{i1} = F(\tau_i; x_{ij}, \theta), \tilde{P}_{i1} = 1 - P_{i1} = \tilde{F}(\tau_i; x_{ij}, \theta)$
Let us define, $X_{ui} \sim Bin(1, P_{i1})$, then, $\sum_{ui=1}^{K_i} X_{ui} = n_i$. Therefore,

$$D_K = \sum_{i=1}^{I} k_i \left[ \frac{1}{k_i} \sum_{u_i=1}^{k_i} (P_{i1}^{\beta+1} + \bar{P}_{i1}^{\beta+1}) - \frac{\beta + 1}{\beta} \left( \sum_{u_i=1}^{k_i} X_{ui} P_{i1}^{\beta} + \sum_{u_i=1}^{k_i} (1 - X_{ui}) \bar{P}_{i1}^{\beta} \right) \right]$$

$$= \sum_{i=1}^{I} \frac{k_i}{K} \left[ \frac{1}{k_i} \sum_{u_i=1}^{k_i} \left\{ (P_{i1}^{\beta+1} + \bar{P}_{i1}^{\beta+1}) - \frac{\beta + 1}{\beta} \left( X_{ui} P_{i1}^{\beta} + (1 - X_{ui}) \bar{P}_{i1}^{\beta} \right) \right\} \right]$$

$$= \sum_{i=1}^{I} \frac{k_i}{K} \left[ \frac{1}{k_i} \sum_{u_i=1}^{k_i} \left\{ (P_{i1}^{\beta+1} + \bar{P}_{i1}^{\beta+1}) - \frac{\beta + 1}{\beta} \left( X_{ui} (P_{i1}^{\beta} - \bar{P}_{i1}^{\beta}) + \bar{P}_{i1}^{\beta} \right) \right\} \right]$$

$$\implies D_K = \sum_{i=1}^{I} \frac{k_i}{K} \left[ \frac{1}{k_i} \sum_{u_i=1}^{k_i} H_i^{\beta}(\theta) \right]$$

where,

$$H_i^{\beta}(\theta) = \left\{ P_{i1}^{\beta+1} + \bar{P}_{i1}^{\beta+1} - \frac{\beta + 1}{\beta} \left( X_{ui} (P_{i1}^{\beta} - \bar{P}_{i1}^{\beta}) + \bar{P}_{i1}^{\beta} \right) \right\}$$

For the $i^{th}$ group,

$$Y_i = \frac{\partial}{\partial \theta} H_i^{\beta}(\theta) = (\beta + 1) P_{i1}^{\beta} \frac{\partial}{\partial \theta} P_{i1} - (\beta + 1) \bar{P}_{i1}^{\beta} \frac{\partial}{\partial \theta} P_{i1} - \frac{\beta + 1}{\beta} [X_{ui} \left( \beta P_{i1}^{\beta-1} \frac{\partial}{\partial \theta} P_{i1} + \beta \bar{P}_{i1}^{\beta-1} \frac{\partial}{\partial \theta} P_{i1} \right) - \beta \bar{P}_{i1}^{\beta-1} \frac{\partial}{\partial \theta} P_{i1}]$$

$$= (\beta + 1) \left[ P_{i1}^{\beta} - \bar{P}_{i1}^{\beta} - X_{ui} \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial}{\partial \theta} P_{i1} \right]$$

$$= (\beta + 1) \left[ P_{i1}^{\beta} - \bar{P}_{i1}^{\beta} - X_{ui} \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial}{\partial \theta} P_{i1} \right]$$

$$= (\beta + 1) \left[ P_{i1}(P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1}) - X_{ui} \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial}{\partial \theta} P_{i1} \right]$$

$$= (\beta + 1) \left[ (P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1})(P_{i1} - X_{ui}) \right] \frac{\partial}{\partial \theta} P_{i1}.$$

Therefore,

$$Y_{is} = (\beta + 1) \left[ (P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1})(P_{i1} - X_{ui}) \right] \frac{\partial}{\partial \theta} P_{i1}$$

where $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{is}, \ldots, Y_{2J})'$. 

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Now, \( E(Y_i) = 0 \),
\[
Var(Y_{is}) = (\beta + 1)^2 \left[ (P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1}) \frac{\partial(P_{i1})}{\partial \theta_s} \right]^2 Var(X_{u1})
\]
\[
= (\beta + 1)^2 \left[ (P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1}) \frac{\partial(P_{i1})}{\partial \theta_s} \right]^2 P_{i1} \bar{P}_{i1}
\]

and,
\[
Cov(Y_{is1}, Y_{is2}) = (\beta + 1)\left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right)^2 \frac{\partial(P_{i1})}{\partial \theta_{s1}} \frac{\partial(P_{i1})}{\partial \theta_{s2}} Var(X_{u1})
\]
\[
= (\beta + 1)\left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right)^2 \frac{\partial(P_{i1})}{\partial \theta_{s1}} \frac{\partial(P_{i1})}{\partial \theta_{s2}} P_{i1} \bar{P}_{i1}
\]

Therefore,
\[
Cov \left( \frac{\partial}{\partial \theta} H_{i\beta} (\theta) \right) = (\beta + 1)^2 K_{i\beta} (\theta)
\]
\[
K_{i\beta} (\theta)_{ss} = \frac{Var(Y_{is})}{(\beta + 1)^2} \quad \text{(variance term)}
\]
\[
K_{i\beta} (\theta)_{s1s2} = \frac{Cov(Y_{is1}, Y_{is2})}{(\beta + 1)^2} \quad \text{(Covariance term)}
\]

Applying Central Limit Theorem (CLT), when \( k_i \to \infty \) for \( i = 1, \ldots, I \),
\[
\sqrt{k_i} \left( \frac{1}{k_i} \sum_{u_i=1}^{k_i} \frac{\partial}{\partial \theta} H_{i\beta} (\theta) \right) \sim N \left( 0, (\beta + 1)^2 K_{i\beta} (\theta) \right)
\]

We denote,
\[
\nabla D_K (\theta) = (D_{K,1}(\theta), \ldots, D_{K,s}(\theta), \ldots, D_{K,2J}(\theta))' \quad \text{where,} \quad D_{K,s}(\theta) = \frac{\partial}{\partial \theta_s} D_K
\]

define, \( T_\beta = -\sqrt{K} \nabla D_K (\theta) = -\sqrt{K} \sum_{i=1}^{I} k_i \left( \frac{1}{k_i} \sum_{u_i=1}^{k_i} \frac{\partial}{\partial \theta} H_{i\beta} (\theta) \right) \)
\[
\Rightarrow T_\beta \sim N \left( 0, (\beta + 1)^2 \sum_{i=1}^{I} k_i \frac{K_{i\beta} (\theta)}{K} \right), \quad \text{where} \quad \frac{k_i}{K} \text{ is finite when } k_i \to \infty \quad \text{(A1)}
\]

Now,
\[
\frac{\partial(D_{K,s}(\theta))}{\partial \theta_{s2}} = \sum_{i=1}^{I} \frac{k_i}{K} \left( \frac{1}{k_i} \sum_{u_i=1}^{k_i} \frac{\partial^2 H_{i\beta}(\theta)}{\partial \theta_{s2} \partial \theta_{s1}} \right)
\]
Applying CLT, when \( k_i \to \infty \) for \( i = 1, \ldots, I \), \( \frac{1}{k_i} \sum_{u_i=1}^{k_i} X_{ui} \xrightarrow{p} P_{i1} \).

Therefore, 
\[
\frac{1}{k_i} \sum_{u_i=1}^{k_i} \frac{\partial^2 H_{i\beta}(\theta)}{\partial \theta_{s_2} \partial \theta_{s_1}} \xrightarrow{p} (\beta + 1) \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial (P_{i1})}{\partial \theta_{s_2}} \frac{\partial (P_{i1})}{\partial \theta_{s_1}}
\]
and,
\[
\frac{\partial (D_{K,s}(\theta))}{\partial \theta_{s_2}} \xrightarrow{p} \frac{1}{K} \sum_{i=1}^{l} k_i (\beta + 1) \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial (P_{i1})}{\partial \theta_{s_2}} \frac{\partial (P_{i1})}{\partial \theta_{s_1}}
\]

Consider \( \theta_0 \) to be the true value of parameters, then applying the Taylor series expansion \( D_{K,s}(\theta) \) around \( \theta_0 \)
\[
D_{K,s}(\theta) = D_{K,s}(\theta_0) + \sum_{m=1}^{2J} \left. \frac{\partial (D_{K,s}(\theta))}{\partial \theta_m} \right|_{\theta=\theta_0} (\hat{\theta}_m - \theta_{0m})
+ \frac{1}{2} \sum_{l=1}^{2J} \sum_{m=1}^{2J} \left. \frac{\partial^2 (D_{K,s}(\theta))}{\partial \theta_l \partial \theta_m} \right|_{\theta=\theta_0} (\hat{\theta}_l - \theta_{0l}) (\hat{\theta}_m - \theta_{0m}) \tag{A2}
\]

Since, \( D_{K,s}(\hat{\theta}_\beta) = 0 \) it can be written that,
\[
-\sqrt{K} D_{K,s}(\theta_0) = \sqrt{K} \sum_{m=1}^{2J} \left. \frac{\partial (D_{K,s}(\theta))}{\partial \theta_m} \right|_{\theta=\theta_0} (\hat{\theta}_m - \theta_{0m})
+ \frac{1}{2} \sum_{l=1}^{2J} \sum_{m=1}^{2J} \left. \frac{\partial^2 (D_{K,s}(\theta))}{\partial \theta_l \partial \theta_m} \right|_{\theta=\theta_0} (\hat{\theta}_l - \theta_{0l}) (\hat{\theta}_m - \theta_{0m}) \tag{A3}
\]

Denoting,
\[
A_{s,m} = \left. \frac{\partial (D_{K,s}(\theta))}{\partial \theta_m} \right|_{\theta=\theta_0} + \frac{1}{2} \sum_{l=1}^{2J} \left. \frac{\partial^2 (D_{K,s}(\theta))}{\partial \theta_l \partial \theta_m} \right|_{\theta=\theta_0} (\hat{\theta}_l - \theta_{0l})
\]

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Then,

$$A_{s,m} = \sum_{i=1}^{I} \frac{k_i}{K} (\beta + 1) \left( P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1} \right) \frac{\partial(P_{i1})}{\partial \theta_m} \frac{\partial(P_{i1})}{\partial \theta_s}$$

where \( \frac{k_i}{K} \) is finite when \( k_i \to \infty \).

\[ \Rightarrow A_{\beta} \to (\beta + 1)J_{\beta}(\theta_0) \quad \text{(A4)} \]

where, \( A_{\beta} \) is the \( 2J \times 2J \) matrix with \( (s,m) \)th element as \( A_{s,m} \) and

\[ J_{\beta}(\theta_0) = \left[ \left( \sum_{i=1}^{I} \frac{k_i}{K} (P_{i1}^{\beta-1} + \bar{P}_{i1}^{\beta-1}) \frac{\partial(P_{i1})}{\partial \theta_m} \frac{\partial(P_{i1})}{\partial \theta_s} \right)_{s,m} \right] \]

We can write,

\[ -\sqrt{K} D_{K,s}(\theta_0) = \sqrt{K} \sum_{s=1}^{2J} (\hat{\theta}_s - \theta_{0s}) A_{s,m} \]

Therefore, it can be expressed as

\[ T_{\beta} = \sqrt{K} (\hat{\theta}_\beta - \theta_0) A_{\beta} \]

\[ \Rightarrow \sqrt{K} (\hat{\theta}_\beta - \theta_0) = A_{\beta}^{-1} T_{\beta} \]

\[ \Rightarrow \sqrt{K} (\hat{\theta}_\beta - \theta_0) = A_{\beta}^{-1} T_{\beta} \sim N \left( 0, J_{\beta}(\theta_0)^{-1} K_{\beta}(\theta_0)^{-1} \right) \]

(by equation (A4))

where,

\[ K_{\beta}(\theta_0) = \sum_{i=1}^{I} \frac{k_i}{K} K_{i\beta}(\theta) ; \quad \frac{k_i}{K} \text{ is finite, when } k_i \to \infty \]