HOMOLOGICAL ALGEBRA OF MODULES OVER POSETS

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Abstract. Homological algebra of modules over posets is developed, as closely parallel as possible to that of finitely generated modules over noetherian commutative rings, in the direction of finite presentations and resolutions. Centrally at issue is how to define finiteness to replace the noetherian hypothesis which fails. The tameness condition introduced for this purpose captures finiteness for variation in families of vector spaces indexed by posets in a way that is characterized equivalently by distinct topological, algebraic, combinatorial, and homological manifestations. Tameness serves both theoretical and computational purposes: it guarantees finite presentations and resolutions of various sorts, all related by a syzygy theorem, amenable to algorithmic manipulation. Tameness and its homological theory are new even in the finitely generated discrete setting of \( \mathbb{N}^n \)-gradings, where tame is materially weaker than noetherian. In the context of persistent homology of filtered topological spaces, especially with multiple real parameters, the algebraic theory of tameness yields topologically interpretable data structures in terms of birth and death of homology classes.

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Overview. A module over a poset is a family of vector spaces indexed by the poset elements with a homomorphism for each poset relation. The setup is inherently commutative: the homomorphism for a poset relation $p \preceq q$ is the composite of homomorphisms for the relations $p \preceq r$ and $r \preceq q$ whenever $r$ lies between $p$ and $q$. This paper lays the foundation for an extensive theory of modules over arbitrary posets, with a view toward abstract mathematical theory, algorithmic challenges, and statistical implications. The mathematics includes commutative and homological algebra as they interact with topological, analytic, algebraic, or polyhedral geometric structure on the poset, if any is given. The algorithmic challenges involve effectively encoding and manipulating arbitrary poset modules. The statistical considerations stem from applied topology, where modules over posets arise from persistent homology.

This installment covers initial homological aspects: the extent to which modules over posets behave like multigraded modules over polynomial rings when it comes to finite presentations and resolutions. The long-term investigation tests the frontier of multigraded algebra regarding how far one can get without a ring and with no hypotheses on the multigrading other than a partial order. The syzygy theorem for poset modules here vastly generalizes the one for finitely generated modules over polynomial rings, along the way introducing finite data structures to enable algorithmic computation.

The poset of utmost interest is the real vector space $\mathbb{R}^n$, with its usual componentwise partial order. A module over $\mathbb{R}^n$ is equivalently an $\mathbb{R}^n$-graded module over the polynomial ring whose exponents are allowed to be nonnegative real numbers instead of integers. In this setting, the noetherian hypothesis fails spectacularly, and essentially nothing is known about homological behavior of its category of modules.
The infrastructure developed here meets the lack of noetherian hypotheses head on, to open the possibility of working directly with modules over $\mathbb{R}^n$ and, with no additional difficulty, arbitrary posets.

The focus, and the most subtle point, is the nature of a suitable finiteness condition to replace the noetherian hypothesis. The tame condition introduced here is the natural candidate because it captures equivalent topological, algebraic, combinatorial, and homological manifestations of finiteness for variation of vector spaces parametrized by a poset. Tameness serves both theoretical and computational purposes: it guarantees various finite presentations and resolutions all related by a syzygy theorem, and the data structures thus produced are amenable to algorithmic manipulation. Tameness, its syzygy theorem, and its data structures are new and theoretically as well as computationally valuable even in the discrete setting over the poset $\mathbb{Z}^n$, which is ordinary commutative algebra of polynomial rings, where tame is much weaker than noetherian.

No restriction on the underlying poset is required. For example, the lack of local finiteness of $\mathbb{R}^n$ is immaterial. Moreover, in that particular setting, if the partial orderings and the modules possess supplementary geometry, be it subanalytic, semialgebraic, or piecewise-linear, for instance, then the data structures and transitions between the topological, algebraic, combinatorial, and homological perspectives take advantage of and preserve the geometry.

Beyond the abstract route to graded module theory over real-exponent polynomial rings and arbitrary posets, one impetus for these developments lies in data science applications, where the poset consists of “parameters” indexing a family of topological subspaces of a fixed topological space. Taking homology of the subspaces in this topological filtration yields a poset module, called the persistent homology of the filtration, referring to how homology classes are born, persist for a while, and then die as the parameter moves up in the poset. In ordinary persistent homology, the poset is totally ordered—usually the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, or a subset $\{1, \ldots, m\}$. This case is well studied (see [EH10], for example), and the algebra is correspondingly simple [Cra13]. Persistence with multiple totally ordered parameters, introduced by Carlsson and Zomorodian [CZ09], has been developed in various ways, often assuming that the poset is $\mathbb{N}^n$. That discrete framework has been preferred in part because it arises frequently when filtering finite simplicial complexes, but also because settings involving continuous parameters unavoidably produce modules that fail to be finitely presented in several fundamental ways. Tameness, with its data structures and syzygy theorem, circumvent these limitations.

Multigraded algebra can be expressed equivalently in terms of modules, or sheaves, or functors, or derived categories, and the literature exhibits all of these. The exposition throughout this paper is intentionally kept at the most elementary level, with posets instead of thin skeletal categories, for instance, and with modules instead of sheaves or functors on posets. At the risk of masking the depth of the content in these enriched contexts, this choice of elementary language makes the exposition accessible
to a wide audience, including statisticians applying persistent homology in addition to topologists, combinatorialists, algebraists, geometers, and programmers.

The power of the foundations here is demonstrated by [Mil20b], for example, which proves conjectures made by Kashiwara and Schapira concerning the relationship between subanalytic and piecewise-linear stratifications of vector spaces and constructibility of sheaves on real vector spaces in the derived category with microsupport restricted to a cone; see [KS17, Conjecture 3.17] and [KS19, Conjecture 3.20]. The theory here as well as in [Mil20a, Mil20c] was developed simultaneously and independently from that of Kashiwara and Schapira [KS18], cf. [Mil17]. The conical-microsupport theory is roughly equivalent to the subanalytic special case of poset module theory for partially ordered real vector spaces, and similarly for the later PL theory [KS19]; this is essentially the content of [Mil20b]. A detailed comparison of the two viewpoints, including key differences, is left to [Mil20b], where the derived sheaf background is reviewed.

Acknowledgements. First, a special acknowledgement goes to Ashleigh Thomas, who has been a long-term collaborator on this project. She was listed as an author on earlier drafts of [Mil17] (of which this is roughly the first quarter), but her contributions lie more properly beyond these preliminaries (see [MT20], for example), so she declined in the end to be named as an author on this installment. Early in the development of the ideas here, Thomas put her finger on the continuous rather than discrete nature of multiparameter persistence modules for fly wings. She computed the first examples explicitly, namely those in Example 1.2, and produced the biparameter persistence diagrams there as well as some of the figures in Example 3.21.

Justin Curry pointed out connections from the combinatorial viewpoint taken here, in terms of modules over posets, to higher notions in algebra and category theory, particularly those involving constructible sheaves, which are in the same vein as Curry’s proposed uses of them in persistence [Cur14]; see Remarks 2.4, 3.2, 4.26, and 6.11.

The author is indebted to David Houle, whose contribution to this project was seminal and remains ongoing; in particular, he and his lab produced the fruit fly wing images [Hou03]. Paul Bendich and Joshua Cruz took part in the genesis of this project, including early discussions concerning ways to tweak persistent (intersection [BH11]) homology for investigations of fly wings. Ville Puuska discovered several errors in an early version of Section 4, resulting in substantial correction and alteration; see Examples 2.7 and 4.16. Banff International Research Station provided an opportunity for valuable feedback and suggestions at the workshop there on Topological Data Analysis (August, 2017) as parts of this research were being completed; many participants, especially the organizers, Uli Bauer and Anthea Monod, as well as Michael Lesnick, shared important perspectives and insight. Thomas Kahle requested that Proposition 5.7 be an equivalence instead of merely the one implication it had stated. Hal Schenck gave

1Bibliographic note: this conjecture appears in v3 (the version cited here) and earlier versions of the cited arXiv preprint. It does not appear in the published version [KS18], which is v6 on the arXiv.
helpful comments on an earlier version of the Introduction. Passages in Examples 1.1 and 1.2 are based on or taken verbatim from [Mil15]. Portions of this work were funded by NSF grant DMS-1702395.

1.1. Modules over posets. There are many essentially equivalent ways to think of a poset module. The definition in the first line of this Introduction is among the more elementary formulations; see Definition 2.1 for additional precision. Others include a

- representation of a poset [NR72];
- functor from a poset to the category of vector spaces (e.g., see [Cur19]);
- vector-space valued sheaf on a poset (e.g., see [Cur14, §4.2] or [Mil20b, §3.3]);
- representation of a quiver with (commutative) relations (e.g., see [Oud15, §A.6]);
- representation of the incidence algebra of a poset [DRS72]; or
- module over a directed acyclic graph [CL18].

The premise here is that commutative algebra provides an elemental framework out of which flows corresponding structure in these other contexts, in which the reader is encouraged to interpret all of the results. [Mil20b] provides an example of how that can look, in that case from sheaf perspectives. Expressing the foundations via commutative algebra is natural for its infrastructure surrounding resolutions. And as the objects are merely graded vector spaces with linear maps among them—there are no rings to act—it is also the most elementary language available.

Some of the formulations of poset modules are only valid when the poset is assumed to be locally finite (see [DRS72], for instance), or when the object being acted upon satisfies a finitary hypothesis [KN09] in which the algebraic information is nonzero on only finitely many points in any interval. This is not a failing of any particular formulation, but rather a signal that the theory has a different focus. Combinatorial formulations are built for enumeration. Representation theories are built for decomposition into and classification of irreducibles. While commutative algebra appreciates a direct sum decomposition when one is available, such as over a noetherian ring of dimension 0, its initial impulse is to relate arbitrary modules to simpler ones by less restrictive decomposition, such as primary decomposition, or by resolution, such as by projective or injective modules. That is the tack taken here.

1.2. Topological tameness. The tame condition (Definitions 2.6 and 2.11) on a module $M$ stipulates that the poset admit a partition into finitely many domains of constancy for $M$. This finiteness generalizes topological tameness for persistent homology in a single parameter (see [CdS+16, §3.8], for example), reflecting the intuitive notion that given a filtration of a topological space from data, only finitely many topologies should appear. Tameness is thus a topological concept, designed to control the size and variation of homology groups of subspaces in a fixed topological space.

Example 1.1. Let $Q = \mathbb{R}_- \times \mathbb{R}_+$ with the coordinatewise partial order, so $(r, s) \in Q$ for any nonnegative real numbers $-r$ and $s$. Let $X = \mathbb{R}^2$ be the plane containing an embedded planar graph. Define $X_{rs} \subseteq X$ to be the set of points at distance at least $-r$
from every vertex and within $s$ of some edge. Thus $X_{rs}$ is obtained by removing the union of the balls of radius $r$ around the vertices from the union of $s$-neighborhoods of the edges. In the following portion of an embedded graph, $-r$ is approximately twice $s$:

The biparameter persistent homology module $M_{rs} = H_0(X_{rs})$ summarizes the geometry of the embedded planar graph.

Relevant properties of these modules are best highlighted in a simplified setting.

**Example 1.2.** Using the setup from Example 1.1, the zeroth persistent homology for the toy-model embedded graph at left in Figure 1 is the $\mathbb{R}^2$-module $M$ shown at center.

Each point of $\mathbb{R}^2$ is colored according to the dimension of its associated vector space in $M$, namely 3, 2, or 1 proceeding up (increasing $s$) and to the right (increasing $r$). The structure homomorphisms $M_{rs} \to M_{r's'}$ are all surjective.

This $\mathbb{R}^2$-module fails to be finitely presented for three fundamental reasons. First, the three generators sit infinitely far back along the $r$-axis. (Fiddling with the sign on $r$ does not help: the natural maps on homology proceed from infinitely large radius to 0 regardless of how the picture is drawn.) Second, the relations that specify the transition from vector spaces of dimension 3 to those of dimension 2 or 1 lie along a real algebraic curve, as do those specifying the transition from dimension 2 to dimension 1. These curves have uncountably many points. Third, even if the relations are discretized—restrict $M$ to a lattice $\mathbb{Z}^2$ superimposed on $\mathbb{R}^2$, say—the relations march off to infinity roughly diagonally away from the origin. (See Example 1.3 for the right-hand image.)

Nonetheless, the $\mathbb{R}^2$-module here is tame, with four constant regions: over the bottom-left region (yellow) the vector space is $k^3$; over the middle (olive) region the vector space is $k^2$; over the upper-right (blue) region the vector space is $k$; and over the remainder of $\mathbb{R}^2$ the vector space is 0. The homomorphisms between these vector spaces do not depend on which points in the regions are selected to represent them. For instance, $k^3 \to k^2$ always identifies the two basis vectors corresponding to the connected components that are the left and right halves of the horizontally infinite red strip.
In principle, tameness can be reworked to serve directly as a data structure for algorithmic computation, especially in the presence of an auxiliary hypothesis to regulate the geometry of the constant regions—when they are semialgebraic or piecewise linear (Definition 2.15.1 or 2.15.2), for example. The algorithms would generalize those for polyhedral “sectors” in the discrete case [HM05] (or see [MS05, Chapter 13]).

1.3. Combinatorial tameness: finite encoding. Whereas the topological notion of tameness requires little more than an arbitrary subdivision of the poset into regions of constancy (Definition 2.6), the combinatorial incarnation imposes additional structure on the constant regions, namely that they should be partially ordered in a natural way. More precisely, it stipulates that the module $M$ should be pulled back from a $P$-module along a poset morphism $Q \to P$ in which $P$ is a finite poset and the $P$-module has finite dimension as a vector space over the field $k$ (Definition 4.1).

Example 1.3. The right-hand image in Example 1.2 is a finite encoding of $M$ by a three-element poset $P$ and the $P$-module $H = k^3 \oplus k^2 \oplus k$ with each arrow in the image corresponding to a full-rank map between summands of $H$. Technically, this is only an encoding of $M$ as a module over $Q = \mathbb{R}_+ \times \mathbb{R}_-$. The poset morphism $Q \to P$ takes all of the yellow rank 3 points to the bottom element of $P$, the olive rank 2 points to the middle element of $P$, and the blue rank 1 points to the top element of $P$. (To make this work over all of $\mathbb{R}^2$, the region with vector space dimension 0 would have to be subdivided, for instance by introducing an antidiagonal extending downward from the origin, thus yielding a morphism from $\mathbb{R}^2$ to a five-element poset.) This encoding is semialgebraic (Definition 2.15): its fibers are real semialgebraic sets.

In general, constant regions need not be situated in a manner that makes them the fibers of a poset morphism (Example 4.4). Nonetheless, over arbitrary posets, modules that are tame by virtue of admitting a finite constant subdivision (Definition 2.11) always admit finite encodings (Theorem 4.22), although given constant regions are typically subdivided by the constructed encoding poset morphism. This implication is what demands precision in the definition of tame via constant subdivision; it makes subtle use of the no-monodromy condition in Definition 2.6. In the case where the poset is a real vector space, if the constant regions have additional geometry (Definition 2.15), then a similarly geometric finite encoding is possible (Theorem 4.22.3).

Remark 1.4. Filtrations of finite simplicial complexes by products of intervals yield persistent homology modules that are not naturally modules over a polynomial ring in $n$ (or any finite number of) variables. This is for the same reason that single-parameter persistent homology is not naturally a module over a polynomial ring in one variable: though there might only be finitely many topological transitions, they can (and often do) occur at incommensurable real numbers. That said, filtering a finite simplicial complex automatically induces a finite encoding. Indeed, the parameter space maps to the poset of simplicial subcomplexes of the original simplicial complex by sending a parameter to the simplicial subcomplex it indexes.
Remark 1.5. The framework of poset modules arising from filtrations of topological spaces is more or less an instance of MacPherson’s exit path category [Tre09, §1.1]. In that context, Lurie defined a notion of constructibility in the Alexandrov topology [Lur17, Definitions A.5.1 and A.5.2], independently of the developments here and for different purposes. It would be reasonable to speculate that tameness should correspond to Alexandrov constructibility, given that encoding of a poset module is defined by pulling back along a poset morphism (in Lurie’s language, a continuous morphism of posets), but it does not; see Remark 4.26. The difference between constant in the sense of tameness via constant subdivision (Section 2.2) and locally constant in the sheaf-theoretic sense with the Alexandrov topology makes tameness—in the equivalent finitely encoded formulation—rather than Alexandrov constructibility the right notion of finiteness for the syzygy theorem as well as for algorithmic computation and data analysis applications of persistent homology. That contrasts with the comparison between tameness and subanalytic constructibility in the usual topology on real vector spaces, which are essentially the same notion for the relevant sheaves; see [Mil20b, §4].

1.4. Algebraic tameness: fringe presentation. To compute with poset modules algebraically, in theoretical as well as algorithmic senses, requires presentations. When the poset is $\mathbb{Z}^n$, the modules are multigraded over the usual polynomial ring in $n$ variables, free presentations are available. But over arbitrary posets, there are no free modules, and even when there are, requiring finite presentation is unreasonably restrictive, cf. Example 1.2. Furthermore, there is nothing special about generators (in topological language, “births”) as opposed to cogenerators (“deaths”). These issues are all resolved by (i) using arbitrary upsets instead the right-angled principal upsets that give rise to free modules and (ii) symmetrically involving downsets. The resulting notion of fringe presentation (Definition 3.16) is a homomorphisms from a direct sum of interval modules for upsets to a direct sum of interval modules for downsets.

Fringe presentation is expressed by a monomial matrix (Definition 3.17): an array of scalars with rows labeled by upsets and columns labeled by downsets.

Example 1.6. Over the poset $\mathbb{R}^2$, the monomial matrix

\[
\begin{bmatrix}
\phi_{11}
\end{bmatrix}
\]

represents a fringe presentation of $M = k$ as long as $\phi_{11} \in k$ is nonzero. That is, the monomial matrix specifies a homomorphism $k[\square] \to k[\square]$ with image $M$, which has $M_\mathbf{a} = k$ over the yellow parameters $\mathbf{a}$ and 0 elsewhere. The blue upset specifies the generators (births) at the lower boundary of $M$;
unchecked, these persist all the way up and to the right. But the red downset specifies
the cogenerators (deaths) along the upper boundary of $M$. This example illustrates
how fringe presentations are topologically interpretable in terms of birth and death of
homology classes, when the modules in question are persistent homology.

When birth upsets and death downsets are semialgebraic, or piecewise linear, or
otherwise manageable algorithmically, monomial matrices render fringe presentations
effective data structures for multiparameter persistent homology (multipersistence).

It is evidence for the naturality of the definitions that the algebraic condition of
admitting a finite fringe presentation is equivalent to the topological and combinatorial
notions of tameness; this equivalence is part of the syzygy theorem (Theorem 6.12).

Although the data structure of fringe presentation is aimed at $\mathbb{R}^n$-modules, it is new
and lends insight already for finitely generated $\mathbb{N}^n$-modules (even when $n = 2$), where
monomial matrices have their origins [Mil00, Section 3]. The context there is more or
less that of finitely determined modules; see Definition 5.14 in particular, which is
really just the special case of fringe presentation in which the upsets are localizations
of $\mathbb{N}^n$ and the downsets are duals—that is, negatives—of those.

It may be helpful to understand the relaxation from free presentation to fringe pre-
sentation step by step over $\mathbb{Z}^n$ or $\mathbb{R}^n$. First, tame modules can have generators that
sit infinitely far back along various axes, as in Example 1.2. This issue is solved by
allowing flat modules instead of free ones. Over $\mathbb{Z}^n$, for instance, this means that
(multigraded translates of) localizations of the polynomial ring by inverting variables
should be used instead of only (translates of) the ring itself; see Remark 5.11.

The next relaxation concerns cogenerators (deaths) as opposed to generators (births).
Switching these means injective hulls and copresentations instead of flat covers and
presentations. Commutative algebra has considered multigraded injectives for decades
[GW78] (see [MS05, Chapter 11] for an exposition), even algorithmically [Mil02, HM05].

The goal, however, is to place flat presentations and injective copresentations on
equal footing, so as to incorporate births and deaths simultaneously. These Matlis
dual concepts (see Remark 5.11) are combined by composing a flat cover $F \to M$ with
an injective hull $M \rightarrow E$ to get a homomorphism $F \to E$ whose image is $M$. This
homomorphism is a flange presentation of $M$ (Definition 5.12), which splices a flat
resolution to an injective one in the same way that Tate resolutions (see [Coa03], for
example) transition from a free resolution to an injective one over a Gorenstein local
ring of dimension 0. Flange presentation is the most direct generalization to multiple
parameters of the presentation corresponding to a bar code or persistence diagram. The
key realization is that with multiple parameters, while births still correspond to gen-
erators, deaths correspond to cogenerators rather than to relations among generators.

The final relaxation, from summands that are flat or injective to arbitrary upset
or downset modules, provides finite data structures for tame modules even when they
have infinite numbers of generators or cogenerators.
**Example 1.7.** The module $M$ in Example 1.6 is tame but has uncountably many generators, uncountably many cogenerators, and an even worse set of relations. The fringe presentation in Example 1.6 gathers the lower boundary points into a single *upset module* and all upper boundary points into a single *downset module* (Definition 3.1). In contrast, a free $\mathbb{R}^n$-module of rank 1 has its nonzero components on a principal upset, which has exactly one lower corner. Thus fringe presentation sacrifices flatness and injectivity for finiteness and flexibility to serve over arbitrary posets.

**Remark 1.8.** Any $\mathbb{R}^n$-module $M$ can be approximated by a $\mathbb{Z}^n$-module, the result of restricting $M$ to, say, the rescaled lattice $\varepsilon\mathbb{Z}^n$. Suppose, for the sake of argument, that $M$ is bounded, in the sense of being zero at parameters outside of a bounded subset of $\mathbb{R}^n$; think of Example 1.2, ignoring those parts of the module there that lie outside of the depicted square. Ever better approximations, by smaller $\varepsilon \to 0$, yield sets of lattice points ever more closely hugging an algebraic curve. Neglecting the difficulty of computing where those lattice points lie, how is a computer to store or manipulate such a set? Listing the points individually is an option, and perhaps efficient for particularly coarse approximations, but in $n$ parameters the dimension of this storage problem is $n - 1$. As the approximations improve, the most efficient way to record such sets of points is surely to describe them as the allowable ones on one side of the given algebraic curve. And once the computer has the curve in memory, no approximation is required: just use the (points on the) curve itself. In this way, even in cases of multipersistence where the entire topological filtration setup can be approximated by finite simplicial complexes, understanding the continuous nature of the un-approximated setup can be at the same time more transparent and more efficient.

**Remark 1.9.** $\mathbb{Z}^n$-graded commutative algebra is decades old [GW78], but the perspective arising from their equivalence with multipersistence is relatively new [CZ09]. Initial steps have included descriptions of the set of isomorphism classes [CZ09], presentations [CSV17] and algorithms for computing [CSZ09, CSV12] or visualizing [LW15] them, as well as interactions with homological algebra of modules, such as persistence invariants [Kmu08] and certain notions of multiparameter noise [SCL+16]. That multipersistence modules can fail to be finitely generated (cf. Example 1.2) in situations reflecting reasonably realistic data analysis was observed by Patriarca, Scalamiero, and Vaccarino [PSV12, Section 2]. Their algorithm for discrete parameters keeps track of generators not individually but gathered together as generators of monomial ideals. Huge numbers of predictable syzygies among generators are swallowed and hence are
present only implicitly. And that is good, as nothing topologically new about persistence of homology classes is taught by the well known syzygies of monomial ideals (see [EMO20], for example), which in this setting are merely an interference pattern from the merging of separate birth points of the same class.

1.5. **Homological tameness: the syzygy theorem.** Just as upsets and downsets can be used to present poset modules, they can be used to resolve them. As in polynomial algebra, this line of thinking culminates in a syzygy theorem (Theorem 6.12 for modules; Theorem 6.17 for complexes) to the effect that, remarkably, the topological, algebraic, combinatorial, and homological notions of tameness available respectively via

- constant subdivision (Definition 2.11),
- fringe presentation (Definition 3.16),
- poset encoding (Definition 4.1), and
- indicator resolution (Definition 6.1)

are equivalent. The moral is that the tame condition over arbitrary posets appears to be the right notion to stand in lieu of the noetherian hypothesis over \( \mathbb{Z}^n \): the tame condition is robust, has multiple characterizations from different mathematical perspectives, and enables algorithmic computation in principle. The syzygy theorem is the main take-away from the paper. It engulfs the statements of its stepping stones, most notably Theorems 4.19 and 4.22, whose proofs isolate crucial ideas.

The syzygy theorem directly reflects the more usual syzygy theorem for finitely determined \( \mathbb{Z}^n \)-modules (Theorem 5.19), with upset and downset resolutions being the arbitrary-poset analogues of free and injective resolutions, respectively, and fringe presentation being the arbitrary-poset analogue of flange presentation. Indeed, the proof of the syzygy theorem works by reducing to the finitely determined case (Section 5) over \( \mathbb{Z}^n \). The main point is that given a finite encoding of a module over an arbitrary poset \( Q \), the encoding poset can be embedded in \( \mathbb{Z}^n \). The proof is completed by pushing the data forward to \( \mathbb{Z}^n \), applying the more usual syzygy theorem to finitely determined modules there, and pulling back to \( Q \).

It bears mentioning that even if one is interested in ring-theoretic situations where the poset is \( \mathbb{Z}^n \) or \( \mathbb{R}^n \), one can and should do homological algebra of tame modules over a finite encoding poset rather than (only) over the original parameter space.

**Remark 1.10.** Topological tameness via constant subdivision is a priori weaker (that is, more inclusive) than combinatorial tameness via finite encoding, and algebraic tameness via fringe presentation is a priori weaker than homological tameness via upset or downset resolution. Thus the syzygy theorem leverages relatively weak topological structure into powerful homological structure. In particular, it provides concrete, computable, combinatorially describable representatives for objects in the derived category. The proof [Mil20b] of two conjectures due to Kashiwara and Schapira ([KS17, Conjecture 3.17] and [KS19, Conjecture 3.20]) relies on the fact that, although the tameness characterizations require no additional structure on the underlying poset, any
additional structure that is present—subanalytic, semialgebraic, or piecewise-linear—is preserved by the transitions among tameness characterizations in the syzygy theorem.

1.6. Bar codes and further developments. Tame modules over the totally ordered set of integers or real numbers are, up to isomorphism, the same as “bar codes”: finite multisets of intervals. The most general form of this bijection between algebraic objects and essentially combinatorial objects over totally ordered sets is due to Crawley-Boevey [Cra13]. At its root this bijection is a manifestation of the tame representation theory of the type $A$ quiver; that is the context in which bar codes were invented by Abeasis and Del Fra, who called them “diagrams of boxes” [AD80, ADK81]. Subsequent terminology for objects equivalent to these diagrams of boxes include bar codes themselves (see [Ghr08]) and planar depictions discovered effectively simultaneously in topological data analysis, where they are called persistence diagrams [ELZ02] (see [CEH07] for attribution) and combinatorial algebraic geometry, where they are called lace arrays [KMS06].

No combinatorial analogue of the bar code can classify modules over an arbitrary poset because there are too many indecomposable modules, even over seemingly well behaved posets like $\mathbb{Z}^n$ [CZ09]: the indecomposables come in families of positive dimension. Over arbitrary posets, every tame module does still admit a decomposition of the Krull–Schmidt sort, namely as a direct sum of indecomposables [BC19], but again, there are too many indecomposables for this to be useful in general. Instead of decomposing modules as direct sums of elemental pieces, which be arbitrarily complicated [BE20], the commutative algebra view advocates expressing poset modules in terms of intervals, especially indicator modules for upsets and downsets, by way of less rigid constructions like fringe presentation (Section 3), primary decomposition [Mil20a, MT20], or resolution (Section 6). This relaxes the direct sum in a $K$-theoretic way, allowing arbitrary complexes instead of split short exact sequences.

Various aspects of bar codes are reflected in the equivalent concepts of tameness. The finitely many regions of constancy are seen in topological tameness by constant subdivision. The matching between left and right endpoints is seen in algebraic tameness by fringe presentation, where the left endpoints form lower borders of birth upsets and the right endpoints form upper borders of death downsets. The expressions of modules in terms of bars is seen, in its relaxed form, in homological tameness, where modules become “virtual” sums, in the sense of being formal alternating combinations rather than direct sums of intervals. Primary decomposition [Mil20a] isolates elements that would, in a bar code, lie in bars unbounded in fixed sets of directions (see also [HOST19]).

Bar codes rely on some concept of minimality: left endpoints must correspond to minimal generators, and right endpoints to minimal cogenerators. These are not available over arbitrary posets and are subtle to define and handle properly even for partially ordered real vector spaces [Mil20c]. When minimality is available, instead of a bijection (perfect matching) from a multiset of births to a multiset of deaths, the best one can settle for is a linear map from a filtration of the birth multiset to a filtration of the death multiset [Mil20d].
2. Tame poset modules

2.1. Modules over posets.

Definition 2.1. Let $Q$ be a partially ordered set (poset) and $\preceq$ its partial order. A module over $Q$ (or a $Q$-module) is

- a $Q$-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
- a homomorphism $M_q \to M_{q'}$ whenever $q \preceq q'$ in $Q$ such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \preceq q' \preceq q''$.

A homomorphism $M \to N$ of $Q$-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms $M_q \to N_q$, that commute with the structure homomorphisms $M_q \to M_{q'}$ and $N_q \to N_{q'}$.

The last bulleted item is commutativity. In important instances (e.g., Example 2.3), it reflects that inclusions of topological subspaces induce functorial maps on homology.

Example 2.2. A module over the poset $\mathbb{R}^n$ whose partial order is componentwise comparison is the same thing as an $\mathbb{R}^n$-graded module over the monoid algebra $k[\mathbb{R}^n_+]$, where $\mathbb{R}^n_+ = \{ r \in \mathbb{R} \mid r \geq 0 \}$ is the additive monoid of nonnegative real numbers. (This is immediate from the definitions, see [Les15, §2.1], for instance.) This case generalizes that of $\mathbb{Z}^n$-modules, which are $\mathbb{Z}^n$-graded modules over polynomial rings $k[\mathbb{N}^n]$: elements of $k[\mathbb{R}^n_+]$ are polynomials with real exponents.

Example 2.3. Let $X$ be a topological space and $Q$ a poset.

1. A filtration of $X$ indexed by $Q$ is a choice of subspace $X_q \subseteq X$ for each $q \in Q$ such that $X_q \subseteq X_{q'}$ whenever $q \preceq q'$.
2. The $i$th persistent homology of the filtered space $X$ is the associated homology module, meaning the $Q$-module $\bigoplus_{q \in Q} H_i X_q$.

Remark 2.4. There are a number of abstract, equivalent ways to phrase Example 2.3. For example, a filtration is a functor from $Q$ to the category $\mathcal{S}$ of subspaces of $X$ or an $\mathcal{S}$-valued sheaf on $Q$ with its Alexandrov topology, whose base is the set of principal upsets (dual order ideals with unique minimal element). For background on and applications of many of these perspectives, see Curry’s dissertation [Cur14], particularly §4.2 there. See also [Mil20b], which details the transition from modules to constructible sheaves.

Example 2.5. A real multifiltration of $X$ is a filtration indexed by $\mathbb{R}^n$, with its partial order by coordinatewise comparison. Example 1.1 is a real multifiltration of $X = \mathbb{R}^2$ with $n = 2$. The persistent homology of a real $n$-filtered space $X$ is a multipersistence module, which is an $\mathbb{R}^n$-module.
2.2. Constant subdivisions.

**Definition 2.6.** Fix a $Q$-module $M$. A **constant subdivision** of $Q$ subordinate to $M$ is a partition of $Q$ into constant regions such that for each constant region $I$ there is a single vector space $M_I$ with an isomorphism $M_I 	o M_i$ for all $i \in I$ that has no monodromy: if $J$ is some (perhaps different) constant region, then all comparable pairs $i \preceq j$ with $i \in I$ and $j \in J$ induce the same composite homomorphism $M_I 	o M_i 	o M_J$.

**Example 2.7.** Consider the poset module (kindly provided by Ville Puuska [Puu18])

in which the structure morphisms $M_a \to M_b$ are all identity maps on $k$, except for the rightmost one. This example demonstrates that module structures need not be recoverable from their **isotypic subdivision**, in which elements of $Q$ lie in the same region when their vector spaces are isomorphic via a poset relation. In cases like this, refining the isotypic subdivision appropriately yields a constant subdivision. Here, the two minimum elements must lie in different constant regions and the two maximum elements must lie in different constant regions. Any partition accomplishing these separations—that is, any refinement of a partition that has a region consisting of precisely one maximum and one minimum—is a constant subdivision. Of course, a finite poset always admits a constant subdivision with finitely many regions, since the partition into singletons works.

**Example 2.8.** Constant subdivisions need not refine the isotypic subdivision in Example 2.7, one reason being that a single constant region can contain two or more incomparable isotypic regions. For a concrete instance with a single constant region comprised of uncountably many incomparable isotypic regions, let $M$ be the $\mathbb{R}^2$-module that has $M_a = 0$ for all $a \in \mathbb{R}^2$ except for those on the antidiagonal line spanned by $[1, -1] \in \mathbb{R}^2$, where $M_a = k$. There is only one such $\mathbb{R}^2$-module because all of the degrees of nonzero graded pieces of $M$ are incomparable, so all of the structure homomorphisms $M_a \to M_b$ with $a \neq b$ are zero. Every point on the line is a singleton isotypic region. This conclusion reverses entirely when the line is thickened to a strip of positive width, where the single isotypic region comprising the support yields a constant subdivision.

The direction of the line in Example 2.8 is important: an antidiagonal line, whose points form an antichain in $\mathbb{R}^2$, behaves radically differently than diagonal lines.

**Example 2.9.** Let $M$ be an $\mathbb{R}^2$-module with $M_a = k$ whenever $a$ lies in the closed diagonal strip between the lines of slope 1 passing through any pair of points. The structure homomorphisms $M_a \to M_b$ could all be zero, for instance, or some of them
could be nonzero. But the length $|a - b|$ of any nonzero such homomorphism must in any case be bounded above by the Manhattan (i.e., $\ell^\infty$) distance between the two points, since every longer structure homomorphism factors through a sequence that exits and re-enters the strip.

In particular, the structure homomorphism between any pair of points on the upper boundary line of the strip is zero because it factors through a homomorphism that points upward first; therefore such pairs of points lie in distinct regions of any constant subdivision. The same conclusion holds for pairs of points on the lower boundary line of the strip. When the strip has width 0, so the upper and lower boundary coincide, the module is supported along a diagonal line whose uncountably many points must all lie in distinct constant regions.

The reference to “no monodromy” in Definition 2.6 agrees with the usual notion.

**Lemma 2.10.** Fix a constant region $I$ subordinate to a poset module $M$. The composite isomorphism $M_I \to M_i \to \cdots \to M_{i'} \to M_I$ is independent of the path from $i$ to $i'$ through $I$, if one exists. In particular, when $i = i'$ the composite is the identity on $M_I$.

**Proof.** The second claim follows from the first. When the path has length 0, the claim is that $M_I \to M_i \to M_I$ is the identity on $M_I$, which follows by definition. For longer paths the result is proved by induction on path length. \qed

Constant subdivision is the subtle part of the central finiteness concept of the paper.

**Definition 2.11.** Fix a poset $Q$ and a $Q$-module $M$.

1. A constant subdivision of $Q$ is *finite* if it has finitely many constant regions.
2. The $Q$-module $M$ is *$Q$-finite* if its components $M_q$ have finite dimension over $k$.
3. The $Q$-module $M$ is *tame* if it is $Q$-finite and $Q$ admits a finite constant subdivision subordinate to $M$.

**Remark 2.12.**

1. In ordinary totally ordered persistent homology, tameness means simply that the bar code (see Section 1.6) has finitely many bars, or equivalently, the persistence
diagram has finitely many off-diagonal dots: finiteness of the set of constant regions precludes infinitely many non-overlapping bars (the bar code can’t be “too long”), while the vector space having finite dimension precludes a parameter value over which lie infinitely many bars (the bar code can’t be “too wide”).

2. The tameness condition here includes but is much less rigid than the compact tameness condition in [SCL+16], the latter meaning more or less that the module is finitely generated over a scalar multiple of $\mathbb{Z}^n$ in $\mathbb{Q}^n$.

3. Some literature calls Definition 2.11.2 pointwise finite dimensional (PFD). The terminology here agrees with that in [Mil00], on which Section 5 here is based.

**Lemma 2.13.** Any refinement of a constant subdivision subordinate to a $Q$-module $M$ is a constant subdivision subordinate to $M$.

*Proof.* Choosing the same vector space $M_I$ for every region of the refinement contained in the constant region $I$, the lemma is immediate from Definition 2.6. $\square$

2.3. **Auxiliary hypotheses.**

Effectively computing with real multifiltered spaces (Example 2.5) requires keeping track of the shapes of various regions, such as constant regions. (In later sections, other regions along these lines include upsets, downsets, and fibers of poset morphisms.) The fact that applications of persistent homology often arise from metric considerations, which are semialgebraic in nature, or are approximated by piecewise linear structures suggests the following auxiliary hypotheses for algorithmic developments. The subanalytic hypothesis is singled out for theoretical purposes surrounding conjectures of Kashiwara and Schapira ([KS17, Conjecture 3.17], [KS19, Conjecture 3.20]; cf. [Mil20b]).

**Definition 2.14.** An abelian group $Q$ is *partially ordered* if it is generated by a submonoid $Q_+$, called the *positive cone*, that has trivial unit group. The partial order is:

\[ q \preceq q' \iff q' - q \in Q_+. \]

A partially ordered group is

1. *real* if the underlying abelian group is a real vector space of finite dimension;
2. *discrete* if the underlying abelian group is free of finite rank.

**Definition 2.15.** Fix a subposet $Q$ of a real partially ordered group. A partition of $Q$ into subsets is

1. *semialgebraic* if the subsets are real semialgebraic varieties;
2. *piecewise linear (PL)* if the subsets are finite unions of convex polyhedra, where a *convex polyhedron* is an intersection of finitely many closed or open half-spaces;
3. *subanalytic* if the subsets are subanalytic varieties;
4. *of class $\mathcal{X}$* if the subsets lie in a family $\mathcal{X}$ of subsets of $Q$ closed under complement, finite intersection, negation, and Minkowski sum with the positive cone $Q_+$.

A module over $Q$ is *semialgebraic*, or *PL, subanalytic*, or *of class $\mathcal{X}$* if $Q_+$ is and the module is tamed by a subordinate finite constant subdivision of the corresponding type.
Remark 2.16. Subposets of real partially ordered groups are allowed in Definition 2.15 to be able to speak of, for example, piecewise linear sets in rational vector spaces, or semialgebraic subsets of $\mathbb{Z}^n$, such as the set of lattice points in a right circular cone (e.g. [Mil20a, Example 5.9]). When $Q$ is properly contained in the ambient real vector space, subsets of $Q$ are semialgebraic, PL, or subanalytic when they are intersections with $Q$ of the corresponding type of subset of the ambient real vector space.

Proposition 2.17. Fix a partially ordered real vector space $Q$.

1. The classes of semialgebraic, PL, and subanalytic subsets of $Q$ are each closed under complements, finite intersections, and negation.
2. The Minkowski sum $S + Q_+$ of a semialgebraic set $S$ with the positive cone is semialgebraic if $Q_+$ is semialgebraic.
3. The Minkowski sum $S + Q_+$ of a PL set with the positive cone is semialgebraic if $Q_+$ is polyhedral.
4. The Minkowski sum $S + Q_+$ of a bounded subanalytic set $S$ with the positive cone is subanalytic if $Q_+$ is subanalytic.

Proof. See [Shi97] (for example) to treat the semialgebraic and subanalytic cases of item 1. The PL case reduces easily to checking that the complement of a single polyhedron is PL, which in turn follows because a real vector space is the union of the (relatively open) faces in any finite hyperplane arrangement, so removing a single one of these faces leaves a PL set remaining.

For item 2, use that the image of a semialgebraic set under linear projection is a semialgebraic set, and then express $S + Q_+$ as the image of $S \times Q_+$ under the projection $Q \times Q \to Q$ that acts by $(q, q') \mapsto q + q'$. The same argument works for item 3. The same argument also works for item 4 but requires that the restriction of the projection to the closure of $S \times Q_+$ be a proper map, which always occurs when $S$ is bounded. \qed

3. Fringe presentation by upsets and downsets

To define the concept of fringe presentation precisely requires some elementary background on posets. That includes upsets and downsets and the modules constructed from them (Definition 3.1). Less obviously, notions of connectedness (Definition 3.5) play a key role, especially in computing vector spaces of homomorphisms between upset and downset modules (Proposition 3.10). Situations where these Hom sets have dimension 1 (Corollary 3.11) are particularly key, leading to the notion of connected homomorphisms of interval modules (Definition 3.14). In general, the basic poset material in Section 3.1 should be useful as a reference more widely than for the application to fringe presentation here. Section 3.2 goes on to introduce fringe presentation (Definition 3.16) and monomial matrix (Definition 3.17), along with some simple examples.
3.1. Upsets and downsets.

**Definition 3.1.** The vector space $k[Q] = \bigoplus_{q \in Q} k$ that assigns $k$ to every point of the poset $Q$ is a $Q$-module with identity maps on $k$. More generally,

1. an *upset* (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in $Q$ (so $U + \mathbb{R}^n_+ = U$, when $Q = \mathbb{R}^n$) determines an indicator submodule or upset module $k[U] \subseteq k[Q]$; and

2. dually, a *downset* (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in $Q$ (so $D - \mathbb{R}^n_+ = D$, when $Q = \mathbb{R}^n$) determines an indicator quotient module or downset module $k[Q] \twoheadrightarrow k[D]$.

When $Q$ is a subposet of a real partially ordered group (Definition 2.14), an indicator module of either sort is semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ if the corresponding upset or downset is of the same type (Definition 2.15).

**Remark 3.2.** Indicator submodules $k[U]$ and quotient modules $k[D]$ are $Q$-modules, not merely $U$-modules or $D$-modules, by setting the graded components indexed by elements outside of the ideals to 0. It is only by viewing indicator modules as $Q$-modules that they are forced to be submodules or quotients, respectively. For relations between these notions and those in Remark 2.4, again see Curry’s thesis [Cur14]. For example, upsets form the open sets in the topology from Remark 2.4.

**Example 3.3.** Ising crystals at zero temperature, with polygonal boundary conditions and fixed mesh size, are semialgebraic upsets in $\mathbb{R}^n$. That much is by definition: fixing a mesh size means that the crystals in question are (staircase surfaces of finitely generated) monomial ideals in $n$ variables. Remarkably, such crystals remain semialgebraic in the limit of zero mesh size; see [Oko16] for an exposition and references.

**Example 3.4.** Monomial ideals in polynomial rings with real exponents, which correspond to upsets in $\mathbb{R}^n_+$, are considered in [ASW15], including aspects of primality, irreducible decomposition, and Krull dimension. Upsets in $\mathbb{R}^n$ are also considered in [MMc15], where the combinatorics of their lower boundaries, and topology of related simplicial complexes, are investigated in cases with locally finite generating sets.

**Definition 3.5.** A poset $Q$ is

1. *connected* if every pair of elements $q, q' \in Q$ is joined by a path in $Q$: a sequence $q = q_0 \preceq q'_0 \preceq q_1 \preceq q'_1 \preceq \cdots \preceq q_k \preceq q'_k = q'$ in $Q$;

2. *upper-connected* if every pair of elements in $Q$ has an upper bound in $Q$;

3. *lower-connected* if every pair of elements in $Q$ has a lower bound in $Q$; and

4. *strongly connected* if $Q$ is upper-connected and lower-connected.

**Example 3.6.** $\mathbb{R}^n$ is strongly connected. The same is true of any partially ordered abelian group. (See [Mil20a] for additional basic theory of those posets.)
Example 3.7. A poset \( Q \) is upper-connected if (but not only if, cf. Example 3.6) it has a maximum element—one that is preceded by every element of \( Q \). Similarly, \( Q \) is lower-connected if it has a minimum element—one that precedes every element of \( Q \).

Remark 3.8. The relation \( q \sim q' \) defined by the existence of a path joining \( q \) to \( q' \) as in Definition 3.5.1 is an equivalence relation.

Definition 3.9. Fix a poset \( Q \). For any subset \( S \subseteq Q \), write \( \pi_0 S \) for the set of connected components of \( S \): the maximal connected subsets of \( S \), or equivalently the classes under the relation from Remark 3.8.

Proposition 3.10. Fix a poset \( Q \).

1. For an upset \( U \) and a downset \( D \),
\[
\text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}^{\pi_0(U \cap D)},
\]
a product of copies of \( \mathbb{k} \), one for each connected component of \( U \cap D \).

2. For upsets \( U \) and \( U' \),
\[
\text{Hom}_Q(\mathbb{k}[U'], \mathbb{k}[U]) = \mathbb{k}^{\{ \pi \subseteq \pi_0 U' \mid \pi \subseteq U \}},
\]
a product of copies of \( \mathbb{k} \), one for each connected component of \( U' \) contained in \( U \).

3. For downsets \( D \) and \( D' \),
\[
\text{Hom}_Q(\mathbb{k}[D], \mathbb{k}[D']) = \mathbb{k}^{\{ \pi \subseteq \pi_0 D' \mid \pi \subseteq D \}},
\]
a product of copies of \( \mathbb{k} \), one for each connected component of \( D' \) contained in \( D \).

Proof. For the first claim, the action \( \varphi_q \) of \( \varphi : \mathbb{k}[U] \to \mathbb{k}[D] \) on the copy of \( \mathbb{k} \) in any degree \( q \in U \cap D \) is 0 because \( \mathbb{k}[D]_q = 0 \), so assume \( q \in U \cap D \). Then \( \varphi_q = \varphi' : \mathbb{k} \to \mathbb{k} \) if \( q \preceq q' \in U \cap D \) because \( \mathbb{k}[U]_{q'} \to \mathbb{k}[U]_q \) and \( \mathbb{k}[D]_q \to \mathbb{k}[D]_{q'} \) are identity maps on \( \mathbb{k} \). Similarly, \( \varphi_q = \varphi' \) if \( q \succeq q' \in U \cap D \). Induction on the length of the path in Definition 3.5.1 shows that \( \varphi_q = \varphi' \) if \( q \) and \( q' \) lie in the same connected component of \( U \cap D \). Thus \( \text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) \subseteq \mathbb{k}^{\pi_0(U \cap D)} \). On the other hand, specifying for each component \( S \in \pi_0(U \cap D) \) a scalar \( \alpha_S \in \mathbb{k} \) yields a homomorphism \( \varphi : \mathbb{k}[U] \to \mathbb{k}[D] \), if \( \varphi_q \) is defined to be multiplication by \( \alpha_S \) on the copies of \( \mathbb{k} = \mathbb{k}[U]_q \) indexed by \( q \in S \) and 0 for \( q \in U \setminus D \); that \( \varphi \) is indeed a \( Q \)-module homomorphism follows because \( \mathbb{k}[D]_{q'} = 0 \) (that is, \( q' \not\in D \)) whenever \( q' \preceq q \not\in D \) but \( q' \) does not lie in the connected component of \( U \cap D \) containing \( q \). Said another way, pairs of elements of \( U \cap D \) either lie in the same connected component of \( U \cap D \) or they are incomparable.

The proofs of the last two claims are similar (and dual to one another), particularly when it comes to showing that a homomorphism of indicator modules of the same type—that is, source and target both upset or both downset—is constant on the relevant connected components. The only point not already covered is that if \( U' \) is a connected upset and \( U' \not\subseteq U \) then every homomorphism \( \mathbb{k}[U'] \to \mathbb{k}[U] \) is 0 because \( q' \in U' \setminus U \) implies \( \mathbb{k}[U']_{q'} \to 0 = \mathbb{k}[U]_{q'} \). \( \square \)
The cases of interest in this paper and its sequels [Mil20c, Mil20d], particularly real and discrete partially ordered groups (Definition 2.14) such as \( \mathbb{R}^n \) and \( \mathbb{Z}^n \), have strong connectivity properties, thereby simplifying the conclusions of Proposition 3.10. First, here is a convenient notation.

**Corollary 3.11.** Fix a poset \( Q \) with upsets \( U, U' \) and downsets \( D, D' \).

1. \( \text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k} \) if \( U \cap D \neq \emptyset \) and either \( U \) is lower-connected as a subposet of \( Q \) or \( D \) is upper-connected as a subposet of \( Q \).
2. If \( U \) and \( U' \) are upsets and \( Q \) is upper-connected, then \( \text{Hom}_Q(\mathbb{k}[U'], \mathbb{k}[U]) = \mathbb{k} \) if \( U' \subseteq U \) and 0 otherwise.
3. If \( D \) and \( D' \) are downsets and \( Q \) is lower-connected, then \( \text{Hom}_Q(\mathbb{k}[D], \mathbb{k}[D']) = \mathbb{k} \) if \( D \supseteq D' \) and 0 otherwise. \( \square \)

**Example 3.12.** Consider the poset \( \mathbb{N}^2 \), the upset \( U = \mathbb{N}^2 \setminus \{0\} \), and the downset \( D \) consisting of the origin and the two standard basis vectors. Then \( \mathbb{k}[U] = \mathbb{m} = \langle x, y \rangle \) is the graded maximal ideal of \( \mathbb{k}[\mathbb{N}^2] = \mathbb{k}[x, y] \) and \( \mathbb{k}[D] = \mathbb{k}[\mathbb{N}^2]/\mathbb{m}^2 \). Now calculate

\[
\text{Hom}_{\mathbb{N}^2}(\mathbb{k}[U], \mathbb{k}[D]) = \text{Hom}_{\mathbb{N}^2}(\mathbb{m}, \mathbb{k}[\mathbb{N}^2]/\mathbb{m}^2) = \mathbb{k}^2,
\]

a vector space of dimension 2: one basis vector preserves the monomial \( x \) while killing the monomial \( y \), and the other basis vector preserves \( y \) while killing \( x \).

**Example 3.13.** For an extreme case, consider the poset \( Q = \mathbb{R}^2 \) with \( U \) the closed half-plane above the antidiagonal line \( y = -x \) and \( D = -U \), so that \( U \cap D \) is totally disconnected: \( \pi_0(U \cap D) = U \cap D \). Then \( \text{Hom}_Q(\mathbb{k}[U], \mathbb{k}[D]) = \mathbb{k}^{\mathbb{R}} \) is a vector space of beyond continuum dimension, the copy of \( \mathbb{R} \) in the exponent being the antidiagonal line.

The proliferation of homomorphisms in Examples 3.12 and 3.13 is undesirable for both computational and theoretical purposes; it motivates the following concept.

**Definition 3.14.** Let each of \( S \) and \( S' \) be a nonempty intersection of an upset in a poset \( Q \) with a downset in \( Q \), so \( \mathbb{k}[S] \) and \( \mathbb{k}[S'] \) are subquotients of \( \mathbb{k}[Q] \). A homomorphism \( \varphi : \mathbb{k}[S] \to \mathbb{k}[S'] \) is **connected** if there is a scalar \( \lambda \in \mathbb{k} \) such that \( \varphi \) acts as multiplication by \( \lambda \) on the copy of \( \mathbb{k} \) in degree \( q \) for all \( q \in S \cap S' \).

The cases of interest in the rest of this paper concern three situations: both \( S \) and \( S' \) are upsets, or both are downsets, or \( S = U \) is an upset and \( S' = D \) is downset with \( U \cap D \neq \emptyset \). However, the full generality of Definition 3.14 is required in the sequel to this work [Mil20c].

**Remark 3.15.** Corollary 3.11 says that homomorphisms among indicator modules are automatically connected in the presence of appropriate upper- or lower-connectedness.
3.2. Fringe presentations.

**Definition 3.16.** Fix any poset \( Q \). A fringe presentation of a \( Q \)-module \( M \) is
- a direct sum \( F \) of upset modules \( \k[U] \),
- a direct sum \( E \) of downset modules \( \k[D] \), and
- a homomorphism \( F \rightarrow E \) of \( Q \)-modules with
  - image isomorphic to \( M \) and
  - components \( \k[U] \rightarrow \k[D] \) that are connected (Definition 3.14).

A fringe presentation
1. is finite if the direct sums are finite;
2. dominates a constant subdivision of \( M \) if the subdivision is subordinate to each summand \( \k[U] \) of \( F \) and \( \k[D] \) of \( E \); and
3. is semialgebraic, PL, subanalytic, or of class \( \mathcal{X} \) if \( Q \) is a subposet of a partially ordered real vector space of finite dimension and the fringe presentation dominates a constant subdivision of the corresponding type (Definition 2.15).

Fringe presentations are effective data structures via the following notational trick. Topologically, it highlights that births occur along the lower boundaries of the upsets and deaths occur along the upper boundaries of the downsets, with a linear map over the ground field to relate them.

**Definition 3.17.** Fix a finite fringe presentation \( \varphi : \bigoplus_p \k[U_p] = F \rightarrow E = \bigoplus_q \k[D_q] \).
A **monomial matrix** for \( \varphi \) is an array of scalar entries \( \varphi_{pq} \) whose columns are labeled by the birth upsets \( U_p \) and whose rows are labeled by the death downsets \( D_q \):

\[
\begin{bmatrix}
U_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\
\vdots & \ddots & \ddots & \vdots \\
U_k & \varphi_{k1} & \cdots & \varphi_{k\ell}
\end{bmatrix}
\]

\( \k[U_1] \oplus \cdots \oplus \k[U_k] = F \quad \rightarrow \quad E = \k[D_1] \oplus \cdots \oplus \k[D_\ell]. \)

**Proposition 3.18.** With notation as in Definition 3.17, \( \varphi_{pq} = 0 \) unless \( U_p \cap D_q \neq \emptyset \). Conversely, if an array of scalars \( \varphi_{pq} \in \k \) with rows labeled by upsets and columns labeled by downsets has \( \varphi_{pq} = 0 \) unless \( U_p \cap D_q \neq \emptyset \), then it represents a fringe presentation.

**Proof.** Proposition 3.10.1 and Definition 3.14. \( \square \)

**Example 3.19.** Fringe presentation in one parameter reflects the usual matching between left endpoints and right endpoints of a module, once it has been decomposed as
a direct sum of bars. A single bar, say an interval \([a, b]\) that is closed on the left and open on the right, has fringe presentation

\[
\begin{array}{cc}
\text{with image} & \\
\bullet & a \\
\downarrow & \Rightarrow \\
\circ & b
\end{array}
\]

in which a subset \(S \subseteq \mathbb{R}\) is drawn instead of writing \(k[S]\). With multiple bars, the bijection from left to right endpoints yields a monomial matrix whose scalar entries form the identity, with rows labeled by positive rays having the specified left endpoints (the ray is the whole real line when the left endpoint is \(-\infty\)) and columns labeled by negative rays having the corresponding right endpoints (again, the whole line when the right endpoint is \(+\infty\)). In practical terms, the rows and columns can be labeled simply by the endpoints themselves, with (say) a bar over a closed endpoint and a circle over an open one. Thus a standard bar code, in monomial matrix notation, has the form

\[
\begin{bmatrix}
b_1 \\
\vdots \\
b_k
\end{bmatrix}
\begin{bmatrix}
a_1 & \cdots & a_k \\
1 & \ddots & \\
& & 1
\end{bmatrix}
\]

**Example 3.20.** Although there are many opinions about what the multiparameter analogue of a bar code should be, the analogue of a single bar is generally accepted to be some kind of interval in the parameter poset—that is, \(k[U \cap D]\), where \(U\) is an upset and \(D\) is a downset—sometimes with restrictions on the shape of the interval, depending on context. This case of a single bar explains the terminology “birth upset” and “death downset”. For instance, a fringe presentation of the yellow interval

locates the births along the lower boundary of the blue upset and the deaths along the upper boundary of the red downset. The scalar entries relate the births to the deaths. In this special case of one bar, the monomial matrix is \(1 \times 1\) with a single nonzero scalar entry; choosing bases appropriately, this nonzero entry might as well be 1.
**Example 3.21.** Consider an $\mathbb{N}^2$-filtration of the following simplicial complex.

Each simplex is present above the correspondingly colored rectangular curve in the following diagram, which theoretically should extend infinitely far up and to the right.

Each little square depicts the simplicial complex that is present at the parameter occupying its lower-left corner. Taking zeroth homology yields an $\mathbb{N}^2$-module that replaces the simplicial complex in each box with the vector space spanned by its connected components. A fringe presentation for this $\mathbb{N}^2$-module is

\[
\begin{pmatrix}
1 & 0 & 1 \\
-1 & 1 & 1 \\
0 & -1 & 1 \\
\end{pmatrix},
\]

where the grey square atop the third column represents the downset that is all of $\mathbb{N}^2$. This fringe presentation means that, for example, the connected component that is the blue endpoint of the simplicial complex is born along the union of the axes with the origin removed but the point $[1]$ appended. The purple downset, corresponding to the left edge, records the death—along the upper purple boundary—of the homology class represented by the difference of the blue (left) and gold (middle) vertices. Computations and figures for this example were kindly provided by Ashleigh Thomas.
Remark 3.22. The term “fringe” is a portmanteau of “free” and “injective” (that is, “frinj”), the point being that it combines aspects of free and injective resolutions while also conveying that the data structure captures trailing topological features at both the birth and death ends.

4. Encoding poset modules

Sections 2 and 3 introduce two finiteness conditions: a topological one (tameness, Definition 2.11), which is the intuitive control of homological variation in a filtration of a topological space, and an algebraic one (fringe presentation, Definition 3.16), which provides effective data structures. To interpolate between them, a third finiteness condition, this one combinatorial in nature (finite encoding, Definition 4.1), serves as a theoretical tool whose functorial essence supports much of the development in this paper; the category of tame modules (Section 4.5) is best dealt with using this language, for instance. The main result of Section 4, namely Theorem 4.22, says that tame $Q$-modules can be encoded in the manner of Definition 4.1. Theorems 4.19 and 4.22 are a substantial portion of the main result of the paper (Theorem 6.12), and their proofs contribute key arguments not repeated there although their statements are largely subsumed.

4.1. Finite encoding.

Definition 4.1. Fix a poset $Q$. An encoding of a $Q$-module $M$ by a poset $P$ is a poset morphism $\pi : Q \to P$ together with a $P$-module $H$ such that $M \cong \pi^*H = \bigoplus_{q \in Q} H_{\pi(q)}$, the pullback of $H$ along $\pi$, which is naturally a $Q$-module. The encoding is finite if
1. the poset $P$ is finite, and
2. the vector space $H_p$ has finite dimension for all $p \in P$.

Example 4.2. Example 1.2 shows a constant isotypic subdivision of $\mathbb{R}^2$ which happens to form a poset and therefore produces an encoding.

Example 4.3. A finite encoding of the module in Example 3.21 is as follows.
Example 4.4. There is no natural way to impose a poset structure on the set of regions in a constant subdivision. Take, for example, $Q = \mathbb{R}^2$ and $M = \mathbb{k}_0 \oplus \mathbb{k}[\mathbb{R}^2]$, where $\mathbb{k}_0$ is the $\mathbb{R}^2$-module whose only nonzero component is at the origin, where it is a vector space of dimension 1. This module $M$ induces only two isotypic regions, namely the origin and its complement, and they constitute a constant subdivision.

Neither of the two regions has a stronger claim to precede the other, but at the same time it would be difficult to justify forcing the regions to be incomparable.

Example 4.5. Take $Q = \mathbb{Z}^n$ and $P = \mathbb{N}^n$. The convex projection $\mathbb{Z}^n \to \mathbb{N}^n$ sets to 0 every negative coordinate. The pullback under convex projection is the Čech hull [Mil00, Definition 2.7]. More generally, suppose $a \preceq b$ in $\mathbb{Z}^n$. The interval $[a, b] \subseteq \mathbb{Z}^n$ is a box (rectangular parallelepiped) with lower corner at $a$ and upper corner at $b$. The convex projection $\pi : \mathbb{Z}^n \to [a, b]$ takes every point in $\mathbb{Z}^n$ to its closest point in the box. A $\mathbb{Z}^n$-module is finitely determined if it is finitely encoded by $\pi$.

Example 4.6. The indicator module $\mathbb{k}[Q]$ is encoded by the morphism from $Q$ to the one-point poset with vector space $H = \mathbb{k}$. This generalizes to other indicator modules.

1. Any upset module $\mathbb{k}[U] \subseteq \mathbb{k}[Q]$ is encoded by a morphism from $Q$ to the chain $P$ of length 1, consisting of two points $0 < 1$, that sends $U$ to 1 and the complement $\overline{U}$ to 0. The $P$-module $H$ that pulls back to $\mathbb{k}[U]$ has $H_0 = 0$ and $H_1 = \mathbb{k}$.

2. Dually, any downset module $\mathbb{k}[D]$ is also encoded by a morphism from $Q$ to the chain $P$ of length 1, but this one sends $D$ to 0 and the complement $\overline{D}$ to 1, and the $P$-module $H$ that pulls back to $\mathbb{k}[D]$ has $H_0 = \mathbb{k}$ and $H_1 = 0$.

Definition 4.7. Fix a poset $Q$ and a $Q$-module $M$.

1. A poset morphism $\pi : Q \to P$ or an encoding of a $Q$-module (perhaps different from $M$) is subordinate to $M$ if there is a $P$-module $H$ such that $M \cong \pi^*H$.

2. When $Q$ is a subposet of a partially ordered real vector space, an encoding of $M$ is semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ if the partition of $Q$ formed by the fibers of $\pi$ is of the corresponding type (Definition 2.15).

Example 4.8. The “antidiagonal” $\mathbb{R}^2$-module $M$ in Example 2.8 has a semialgebraic poset encoding by the chain with three elements, where the fiber over the middle element is the antidiagonal line, and the fibers over the top and bottom elements are the open half-spaces above and below the line, respectively. In contrast, using the diagonal line spanned by $[1, 1] \in \mathbb{R}^2$ instead of the antidiagonal line yields a module with no finite encoding; see Example 2.9.

Lemma 4.9. An indicator module is constant on every fiber of a poset morphism $\pi : Q \to P$ if and only if the module is the pullback along $\pi$ of an indicator $P$-module.
Proof. The "if" direction is by definition. For the "only if" direction, observe that if $U \subseteq Q$ is an upset that is a union of fibers of $P$, then the image $\pi(U) \subseteq P$ is an upset whose preimage equals $U$. The same argument works for downsets. \qed

Example 4.10 (Pullbacks of flat and injective modules). An indecomposable flat $\mathbb{Z}^n$-module $k[b + Z\tau + \mathbb{N}^n]$ is an upset module for the poset $\mathbb{Z}^n$. Pulling back to any poset under a poset map to $\mathbb{Z}^n$ therefore yields an upset module for the given poset. The dual statement holds for any indecomposable injective module $k[b + Z\tau - \mathbb{N}^n]$: its pullback is a downset module.

Pullbacks have particularly transparent monomial matrix interpretations.

Proposition 4.11. Fix a poset $Q$ and an encoding of a $Q$-module $M$ via a poset morphism $\pi : Q \to P$ and $P$-module $H$. Any monomial matrix for a fringe presentation of $H$ pulls back to a monomial matrix for a fringe presentation that dominates the encoding by replacing the row labels $U_1, \ldots, U_k$ and column labels $D_1, \ldots, D_\ell$ with their preimages, namely $\pi^{-1}(U_1), \ldots, \pi^{-1}(U_k)$ and $\pi^{-1}(D_1), \ldots, \pi^{-1}(D_\ell)$. \qed

4.2. Uptight posets.

Constructing encodings from constant subdivisions uses general poset combinatorics.

Definition 4.12. Fix a poset $Q$ and a set $\Upsilon$ of upsets. For each poset element $a \in Q$, let $\Upsilon_a \subseteq \Upsilon$ be the set of upsets from $\Upsilon$ that contain $a$. Two poset elements $a, b \in Q$ lie in the same uptight region if $\Upsilon_a = \Upsilon_b$.

Remark 4.13. The partition of $Q$ into uptight regions in Definition 4.12 is the common refinement of the partitions $Q = U \cup (Q \setminus U)$ for $U \in \Upsilon$.

Remark 4.14. Every uptight region is the intersection of a single upset (not necessarily one of the ones in $\Upsilon$) with a single downset. Indeed, the intersection of any family of upsets is an upset, the complement of an upset is a downset, and the intersection of any family of downsets is a downset. Hence the uptight region containing $a$ equals $(\bigcap_{U \in \Upsilon_a} U) \cap (\bigcap_{U \not\in \Upsilon_a} \overline{U})$, the first intersection being an upset and the second a downset.

Proposition 4.15. In the situation of Definition 4.12, the relation on uptight regions given by $A \preceq B$ whenever $a \preceq b$ for some $a \in A$ and $b \in B$ is reflexive and acyclic.

Proof. The stipulated relation on the set of uptight regions is

- reflexive because $a \preceq a$ for any element $a$ in any uptight region $A$; and
- acyclic because going up from $a \in Q$ causes the set $\Upsilon_a$ in Definition 4.12 to (weakly) increase, so a directed cycle can only occur with a constant sequence of sets $\Upsilon_a$. \qed
Example 4.16. The relation in Proposition 4.15 makes the set of uptight regions into a directed acyclic graph, but the relation need not be transitive. An example in the poset $Q = \mathbb{N}^2$, kindly provided by Ville Punska [Puu18], is as follows. Notationally, it is easier to work with monomial ideals in $k[x, y] = k[\mathbb{N}^2]$, which correspond to upsets (see Example 3.12). Let $\Upsilon = \{U_1, \ldots, U_4\}$ consist of the upsets with indicator modules

$$\begin{align*}
k[U_1] &= \langle x^2, y \rangle, & k[U_2] &= \langle x^3, y \rangle, & k[U_3] &= \langle xy \rangle, & k[U_4] &= \langle x^2y \rangle.
\end{align*}$$

Identifying each monomial $x^ay^b$ with the corresponding pair $(a, b) \in \mathbb{N}^2$, it follows that $\Upsilon_{x^2} = \{U_1\}$, and $\Upsilon_{x^3} = \Upsilon_y = \{U_1, U_2\}$, and $\Upsilon_{xy} = \{U_1, U_2, U_3\}$ represent three distinct uptight regions; call them $A$, $B$, and $C$. They satisfy $A \prec B \prec C$ because $x^2 \prec x^3$ and $y \prec xy$. However, $A \not\prec C$ because $A = \{x^2\}$ while $U_4$ forces $C = xyk[y]$ to consist of all lattice points in a vertical ray starting at $xy$.

Definition 4.17. In the situation of Definition 4.12, the **uptight poset** is the transitive closure $P_\Upsilon$ of the directed acyclic graph of uptight regions in Proposition 4.15.

4.3. Constant upsets.

Definition 4.18. Fix a constant subdivision of $Q$ subordinate to $M$. A **constant upset** of $Q$ is either

1. an upset $U_I$ generated by a constant region $I$ or
2. the complement of a downset $D_I$ cogenerated by a constant region $I$.

Theorem 4.19. Let $\Upsilon$ be the set of constant upsets from a constant subdivision of $Q$ subordinate to $M$. The partition of $Q$ into uptight regions for $\Upsilon$ forms another constant subdivision subordinate to $M$.

Proof. Suppose that $A$ is an uptight region that contains points from constant regions $I$ and $J$. Any point in $I \cap A$ witnesses the containments $A \subseteq D_I$ and $A \subseteq U_I$ of $A$ inside the constant upset and downset generated and cogenerated by $I$. Any point $j \in J \cap A$ is therefore sandwiched between elements $i, i' \in I$, so $i \preceq j \preceq i'$, because $j \in U_I$ (for $i$) and $j \in D_I$ (for $i'$). By symmetry, switching $I$ and $J$, there exists $j' \in J$ with $i' \preceq j'$. The sequence

$$M_I \to M_i \to M_j \to M_Y \to M_I \to M_J,$$

where the first and last isomorphisms come from Definition 2.6 and the homomorphisms in between are $Q$-module structure homomorphisms, induces isomorphisms $M_I \to M_Y$ and $M_J \to M_Y$ by definition of constant region. Elementary homological algebra implies that $M_I \to M_J$ is an isomorphism. The induced isomorphism $M_I \to M_J$ is independent of the choices of $i, j, i'$, and $j'$ (in fact, merely considering independence of the choices of $i$ and $j'$ would suffice) because constant subdivisions have no monodromy.

The previous paragraph need not imply that $I = J$, but it does imply that all of the vector spaces $M_J$ for constant regions $J$ that intersect $A$ are—viewing the data of the original constant subdivision as given—canonically isomorphic to $M_I$, thereby allowing...
the choice of $M_A = M_I$. This, plus the lack of monodromy in constant subdivisions, ensures that $M_A \to M_a \to M_b \to M_B$ is independent of the choices of $a \in A$ and $b \in B$ with $a \preceq b$. Thus the uptight subdivision is also constant subordinate to $M$. □

Example 4.20. Theorem 4.19 does not claim that $I = U_I \cap D_I$, and in fact that claim is often not true, even if the isotypic subdivision (Example 2.7) is already constant. Consider $Q = \mathbb{R}^2$ and $M = k_0 \oplus k[\mathbb{R}^2]$, as in Example 4.4, and take $I = \mathbb{R}^2 \smallsetminus \{0\}$. Then $U_I = D_I = \mathbb{R}^2$, so $U_I \cap D_I$ contains the other isotypic region $J = \{0\}$. The uptight poset $P_M$ has precisely four elements:
1. the origin $\{0\} = U_J \cap D_J$;
2. the complement $U_J \smallsetminus \{0\}$ of the origin in $U_J$;
3. the complement $D_J \smallsetminus \{0\}$ of the origin in $D_J$; and
4. the points $\mathbb{R}^2 \smallsetminus (U_J \cup D_J)$ lying only in $I$ and in neither $U_J$ nor $D_J$.

Oddly, uptight region 4 has two connected components, the second and fourth quadrants $A$ and $B$, that are incomparable: any chain of relations from Definition 2.6 that realizes the equivalence $a \sim b$ for $a \in A$ and $b \in B$ must pass through the positive quadrant or the negative quadrant, each of which accidentally becomes comparable to the other isotypic region $J$ and hence lies in a different uptight region.

4.4. Finite encoding from constant subdivisions.

Definition 4.21. If $Q$ is a subposet of a partially ordered real vector space, then a $Q$-module $M$ has compact support if $M$ has nonzero components $M_q$ only in a bounded set of degrees $q \in Q$. A constant subdivision subordinate to such a module is compact if it has exactly one unbounded constant region (namely those $q \in Q$ for which $M_q = 0$).

Theorem 4.22. Fix a $Q$-finite module $M$ over a poset $Q$.
1. $M$ admits a finite encoding if and only if there exists a finite constant subdivision of $Q$ subordinate to $M$. More precisely,
2. the uptight poset of the set of constant upsets from any constant subdivision yields an uptight encoding of $M$ that is finite if the constant subdivision is finite.
3. If $Q$ is a subposet of a partially ordered real vector space and the constant subdivision in the previous item is
   - semialgebraic, with $Q_+$ also semialgebraic; or
   - PL, with $Q_+$ also polyhedral; or
   - compact and subanalytic, with $Q_+$ also subanalytic; or
   - of class $X$,
then the relevant uptight encoding is semialgebraic, PL, subanalytic, or class $X$.

Proof. One direction of item 1 is easy: a finite encoding induces a constant subdivision almost by definition. Indeed, if $\pi : Q \to P$ is a poset encoding of $M$ by a $P$-module $H$, then each fiber $I$ of $\pi$ is a constant region with $M_I = H_{\pi(I)}$. If $i \preceq j$ with $i \in I$ and $j \in J$, then the composite homomorphism $M_I \to M_i \to M_j \to M_J$ is merely the structure morphism $H_{\pi(I)} \to H_{\pi(J)}$ of the $P$-module $H$. 


The hard direction is producing a finite encoding from a constant subdivision. For that, it suffices to prove item 2. Let $\Upsilon$ be the set of constant upsets from a constant subdivision of $Q$ subordinate to $M$. Consider the quotient map $Q \to P_\Upsilon$ of sets that sends each element of $Q$ to the uptight region containing it. Proposition 4.15 and Definition 4.17 imply that this map of sets is a morphism of posets. By Definition 2.6 the vector spaces $M_A$ indexed by the uptight regions $A \in P_\Upsilon$ constitute a $P_\Upsilon$-module $H$ that is well defined by Theorem 4.19. The pullback of $H$ to $Q$ is isomorphic to $M$ by construction. The claim about finiteness follows because the number of uptight regions is bounded above by $2^{2^r}$, where $r$ is the number of constant regions in the original constant subdivision: every element of $Q$ lies inside or outside of each constant upset and inside or outside of each constant downset.

For claim 3, every constant upset is a Minkowski sum $I + Q_+$ or the complement of $I - Q_+ = -(-I + Q_+)$ by Definition 4.18. These are semialgebraic, PL, subanalytic, or of class $\mathcal{X}$, respectively, by Proposition 2.17 (or Definition 2.15 for class $\mathcal{X}$). Note that in the compact subanalytic case, the unique unbounded constant region $I$ afforded by Definition 4.21 has $I + Q_+ = I - Q_+ = Q$, which is subanalytic. □

Example 4.23. For the “antidiagonal” $\mathbb{R}^2$-module $M$ in Examples 2.8 and 4.8, every point on the line is a singleton isotypic region, but these uncountably many isotypic regions can be gathered together: the finite encoding there is the uptight poset for the two upsets that are the closed and open half-spaces bounded below by the antidiagonal.

Example 4.24. In any encoding of the “diagonal strip” $\mathbb{R}^2$-module $M$ in Example 4.4, the poset must be uncountable by Theorem 4.22.

4.5. The category of tame modules.

Example 4.25. The kernel of a homomorphism of tame modules need not be tame. The upset $U \subseteq \mathbb{R}^2$ that is the closed half-space above the antidiagonal line $L$ given by $a + b = 1$ has interior $U^o$, also an upset. The quotient module $N = k[U]/k[U^o]$ is the translate by one unit (up or to the right) of the antidiagonal module in Examples 2.8, 4.8, and 4.23. Both $M = k[U] \oplus k[U]$ and $N$ are tame. The surjection $\varphi : M \to N$ that acts in every degree $a = \begin{bmatrix} a \\ b \end{bmatrix}$ along $L$ by sending the basis vectors of $M_a = k^2$ to $b$ and $-a$ in $N_a = k$ has kernel $K = \ker \varphi$ that is the submodule of $M$ with

- $k^2$ in every degree from $U^o$, and
- the line in $k^2$ through $\begin{bmatrix} 0 \\ b \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ in every degree from $L$. 

0 $\to$ \begin{align*} &\end{align*} $\to$ \begin{align*} &\end{align*} $\to$ 0
That is, $K_a$ agrees with $M_a$ for degrees $a$ outside of $L$, and $K_a$ is the line in $M_a$ of slope $b/a$ through the origin when $a$ lies on $L$. This kernel $K$ is not tame. Indeed, if $a$ and $a'$ are distinct points on $L$, then the homomorphisms $K_a \to K_{a+a'}$ and $K_{a'} \to K_{a+a'}$ have different images, so $a$ and $a'$ are forced to lie in different constant regions in every constant subdivision of $\mathbb{R}^2$ subordinate to $K$. (Note the relation between this example and Proposition 3.10 for $Q = U \subset \mathbb{R}^2$ and $D = L \subset Q$.)

Remark 4.26. Encoding of a $Q$-module $M$ by a poset morphism is related to viewing $M$ as a sheaf on $Q$ with its Alexandrov topology that is constructible in the sense of Lurie [Lur17, Definitions A.5.1 and A.5.2]. The difference is that poset encoding requires constancy (in the sense of Definition 2.6) on fibers of the encoding morphism, whereas Alexandrov constructibility requires only local constancy in the sense of sheaf theory. This distinction is decisive for Example 4.25, where the kernel $K$ is constructible but not finitely encoded.

Because of Remark 4.26, allowing arbitrary homomorphisms between tame modules would step outside of the tame class. More formally, inside the category of $Q$-modules, the full subcategory generated by the tame modules contains modules that are not tame. To preserve tameness, it is thus necessary to restrict the allowable morphisms.

Definition 4.27. A homomorphism $\varphi : M \to N$ of $Q$-modules is tame if $Q$ admits a finite constant subdivision subordinate to both $M$ and $N$ such that for each constant region $I$ the composite isomorphism $M_I \to M_i \to N_i \to N_I$ does not depend on $i \in I$. The map $\varphi$ is semialgebraic, PL, subanalytic, or class $\mathcal{X}$ if this constant subdivision is.

Lemma 4.28. The kernel and cokernel of any tame homomorphism of $Q$-modules are tame morphisms of tame modules. The same is true when tameness is replaced by semialgebraic, PL, subanalytic, or class $\mathcal{X}$.

Proof. Any constant subdivision as in Definition 4.27 is subordinate to both the kernel and cokernel of $M \to N$, with the vector spaces associated to any constant region $I$ being $\ker(M_I \to N_I)$ and $\operatorname{coker}(M_I \to N_I)$. \hfill $\Box$

Definition 4.29. The category of tame modules is the subcategory of $Q$-modules whose objects are the tame modules and whose morphisms are the tame homomorphisms.

Remark 4.30. To be precise with language, a morphism of tame modules is required to be tame, whereas a homomorphism of tame modules is not. That is, morphisms in the category of tame modules are called morphisms, whereas morphisms in the category of $Q$-modules are called homomorphisms. To avoid confusion, the set of tame morphisms from a tame module $M$ to another tame module $N$ is denoted $\operatorname{Mor}(M, N)$ instead of $\operatorname{Hom}(M, N)$.

Proposition 4.31. Over any poset $Q$, the category of tame $Q$-modules is abelian. If $Q$ is a subposet of a partially ordered real vector space of finite dimension, then the category of semialgebraic, PL, subanalytic, or class $\mathcal{X}$ modules is abelian.
Proof. Over any poset, the category in question is a subcategory of the category of $Q$-modules, which is abelian. The subcategory is not full, but $\text{Mor}(M, N)$ is an abelian subgroup of $\text{Hom}(M, N)$; this is most easily seen via Theorem 4.22, for if $\varphi : M \to N$ and $\varphi' : M \to N'$ are finitely encoded by $\pi : Q \to P$ and $\pi' : Q \to P'$, respectively, then $\varphi + \varphi'$ is finitely encoded by $\pi \times \pi' : Q \to P \times P'$. The same construction, but with the source of $\pi'$ being a new module $M'$ instead of $M$, shows that the ordinary product and direct sum of a pair of finitely encoded modules serves as a product and coproduct in the tame category. Kernels and cokernels of morphisms in the tame category exist by Lemma 4.28, which also implies that every monomorphism is a kernel (it is the kernel of its cokernel in the category of $Q$-modules) and every epimorphism is a cokernel (it is the cokernel of its kernel in the category of $Q$-modules).

The semialgebraic, PL, and class $\mathcal{X}$ cases have the same proof, noting that $\pi \times \pi'$ has fibers of the desired type if $\pi$ and $\pi'$ both do. The subanalytic case only follows from this argument when restricted to the category of modules whose nonzero graded pieces lie in a bounded subset of $Q$ (the subset is allowed to depend on the module). However, the argument in the previous paragraph can be done directly with common refinements of pairs of constant subdivisions, so reducing to Theorem 4.22 is not necessary. □

5. Finitely determined $\mathbb{Z}^n$-modules

Unless otherwise stated, this section is presented over the discrete partially ordered group $Q = \mathbb{Z}^n$ with $Q_+ = \mathbb{N}^n$. It begins by reviewing the structure of finitely determined $\mathbb{Z}^n$-modules (Section 5.1), including (minimal) injective and flat resolutions (Sections 5.2 and 5.3), before getting to flange presentations (Section 5.4) and the syzygy theorem (Section 5.5). These latter two underlie the general syzygy theorem (Section 6.2), including existence of fringe presentations. They also serve as models for the concepts of socle, cogenerator, and downset hull over real polyhedral groups, covered in the sequel [Mil20c], as well as dual notions of top, generator, and upset cover.

The main references for $\mathbb{Z}^n$-modules used here are [Mil00, MS05]. The development of homological theory for injective and flat resolutions in the context of finitely determined modules is functorially equivalent to that for finitely generated modules, by [Mil00, Theorem 2.11], but it is convenient to have on hand statements in the finitely determined case. Flange presentation (Section 5.4) and the characterization of finitely determined modules in Proposition 5.7 and (hence) Theorem 5.19 are apparently new.

5.1. Definitions.

The essence of the finiteness here is that all of the relevant information about the relevant modules should be recoverable from what happens in a bounded box in $\mathbb{Z}^n$.

Definition 5.1. A $\mathbb{Z}^n$-finite module $N$ is finitely determined if for each $i = 1, \ldots, n$ the multiplication map $\cdot x_i : N_b \to N_{b+e_i}$ is an isomorphism whenever $b_i$ lies outside of some bounded interval. For notation, $k[N^n] = k[x]$, where $x = x_1, \ldots, x_n$ is a sequence of variables, and $e_i$ is the standard basis vector whose only nonzero entry is 1 in slot $i$. 
Remark 5.2. This notion of finitely determined is the same notion as in Example 4.5. A module is finitely determined if and only if, after perhaps translating its \( Z^n \)-grading, it is \( a \)-determined for some \( a \in \mathbb{N}^n \), as defined in [Mil00, Definition 2.1].

Remark 5.3. For \( Z^n \)-modules, the finitely determined condition is weaker—that is, more inclusive—than finitely generated, but it is much stronger than tame or (equivalently, by Theorem 4.22) finitely encoded. The reason is essentially Example 4.5, where the encoding has a very special nature. For a generic sort of example, the restriction to \( Z^n \) of any \( R^n \)-finite \( R^n \)-module with finitely many constant regions of sufficient width is a tame \( Z^n \)-module, and there is simply no reason why the constant regions should be commensurable with the coordinate directions in \( Z^n \). Already the toy-model fly wing modules in Examples 1.2 and 1.3 yield infinitely generated but tame \( Z^n \)-modules, and this remains true when the discretization \( Z^n \) of \( R^n \) is rescaled by any factor.

Example 5.4. The local cohomology of an affine semigroup ring is tame but usually not finitely determined; see [HM05] and [MS05, Chapter 13], particularly Theorem 13.20, Example 13.17, and Example 13.4 in the latter.

5.2. Injective hulls and resolutions.

Remark 5.5. Every \( Z^n \)-finite module that is injective in the category of \( Z^n \)-modules is, by [MS05, Theorem 11.30], a direct sum of downset modules \( k[D] \) for coprincipal downsets \( D = a + \tau - \mathbb{N}^n \), said to be \( a \)-generated by \( a \) along the face \( \tau \) of \( \mathbb{N}^n \). Note that faces of \( \mathbb{N}^n \) correspond to subsets of \( [n] = \{1, \ldots, n\} \) via \( \tau \leftrightarrow \{ i \in [n] \mid e_i \in \tau \} \).

Minimal injective resolutions work for finitely determined modules just as they do for finitely generated modules. The standard definitions are as follows.

Definition 5.6. Fix a \( Z^n \)-module \( N \).

1. An injective hull of \( N \) is an injective homomorphism \( N \rightarrow E \) in which \( E \) is an injective \( Z^n \)-module (see Remark 5.5). This injective hull is
   - finite if \( E \) has finitely many indecomposable summands and
   - minimal if the number of such summands is minimal.

2. An injective resolution of \( N \) is a complex \( E^\bullet \) of injective \( Z^n \)-modules whose differential \( E^i \rightarrow E^{i+1} \) for \( i \geq 0 \) has only one nonzero homology \( H^0(E^\bullet) \cong N \) (so \( N \hookrightarrow E^0 \) and \( \text{coker}(E^{i-1} \rightarrow E^i) \hookrightarrow E^{i+1} \) are injective hulls for all \( i \geq 1 \)). \( E^\bullet \) is
   - has length \( \ell \) if \( E^i = 0 \) for \( i > \ell \) and \( E^\ell \neq 0 \);
   - is finite if \( E^\bullet = \bigoplus_i E^i \) has finitely many indecomposable summands; and
   - is minimal if \( N \hookrightarrow E^0 \) and \( \text{coker}(E^{i-1} \rightarrow E^i) \hookrightarrow E^{i+1} \) are minimal injective hulls for all \( i \geq 1 \).

Proposition 5.7. The following are equivalent for a \( Z^n \)-module \( N \).

1. \( N \) is finitely determined.
2. \( N \) admits a finite injective resolution.
3. \( N \) admits a finite minimal injective resolution.
Any finite minimal resolution is unique up to isomorphism and has length at most \( n \).

**Proof.** The proof is based on existence of finite minimal injective hulls and resolutions for finitely generated \( \mathbb{Z}^n \)-modules, along with uniqueness and length \( n \) given minimality, as proved by Goto and Watanabe [GW78].

First assume \( N \) is finitely determined. Translating the \( \mathbb{Z}^n \)-grading affects nothing about existence of a finite injective resolution. Therefore, using Remark 5.2, assume that \( N \) is \( a \)-determined. Truncate by taking the \( N_{\geq 0} \)-graded part of \( N \) to get a positively \( a \)-determined—and hence finitely generated—module \( N_{\geq 0} \); see [Mil00, Definition 2.1]. Take any minimal injective resolution \( N_{\geq 0} \to E^\bullet \). Extend backward using the Čech hull [Mil00, Definition 2.7], which is exact [Mil00, Lemma 2.9], to get a finite minimal injective resolution \( \hat{\mathcal{C}}(N_{\geq 0} \to E^\bullet) = (N \to \hat{\mathcal{C}}E^\bullet) \), noting that \( \hat{\mathcal{C}} \) fixes indecomposable injective modules whose \( N_{\geq 0} \)-graded parts are nonzero and is zero on all other indecomposable injective modules [Mil00, Lemma 4.25]. This proves \( 1 \Rightarrow 3 \).

That \( 3 \Rightarrow 2 \) is trivial. The remaining implication, \( 2 \Rightarrow 1 \), follows because every indecomposable injective is finitely determined and the category of finitely determined modules is abelian. (The category of \( \mathbb{Z}^n \)-modules each of which is nonzero only in a bounded set of degrees is abelian, and constructions such as kernels, cokernels, or direct sums in the category of finitely determined modules are pulled back from there.) □

5.3. **Flat covers and resolutions.**

Minimal flat resolutions are not commonplace, but the notion is Matlis dual to that of minimal injective resolution. In the context of finitely determined modules, flat resolutions work as well as injective resolutions. The definitions are as follows.

**Definition 5.8.** Fix a \( \mathbb{Z}^n \)-module \( N \).

1. A **flat cover** of \( N \) is a surjective homomorphism \( F \to N \) in which \( F \) is a flat \( \mathbb{Z}^n \)-module (see Remark 5.11). This flat cover is
   - **finite** if \( F \) has finitely many indecomposable summands and
   - **minimal** if the number of such summands is minimal.

2. A **flat resolution** of \( N \) is a complex \( F^\bullet \) of flat \( \mathbb{Z}^n \)-modules whose differential \( F_{i+1} \to F_i \) for \( i \geq 0 \) has only one nonzero homology \( H_0(F_i) \simeq N \) (so \( F_0 \to N \) and \( F_{i+1} \to \ker(F_i \to F_{i-1}) \) are flat covers for all \( i \geq 1 \)). The flat resolution \( F^\bullet \)
   - has **length** \( \ell \) if \( F_i = 0 \) for \( i > \ell \) and \( F_\ell \neq 0 \);
   - is **finite** if \( F^\bullet = \bigoplus_i F_i \) has finitely many indecomposable summands; and
   - is **minimal** if \( F_0 \to N \) and \( F_{i+1} \to \ker(F_i \to F_{i-1}) \) are minimal flat covers for all \( i \geq 1 \).

**Definition 5.9.** The **Matlis dual** of a \( \mathbb{Z}^n \)-module \( M \) is the \( \mathbb{Z}^n \)-module \( M^\vee \) defined by

\[
(M^\vee)_a = \text{Hom}_k(M_{-a}, k),
\]

so the homomorphism \( (M^\vee)_a \to (M^\vee)_b \) is transpose to \( M_{-b} \to M_{-a} \).

**Lemma 5.10.** \( (M^\vee)^\vee \) is canonically isomorphic to \( M \) for any \( \mathbb{Z}^n \)-finite module \( M \). □
Remark 5.11. By the adjunction between $\text{Hom}$ and $\otimes$, a module is flat if and only its Matlis dual is injective (see [Mil00, §1.2], for example). The Matlis dual of Remark 5.5 therefore says that every $\mathbb{Z}^n$-finite flat $\mathbb{Z}^n$-module is isomorphic to a direct sum of upset modules $k[U]$ for upsets of the form $U = b - \tau + \mathbb{N}^n = b + \mathbb{N}^n + \mathbb{Z}\tau$. These upset modules are the graded translates of localizations of $k[\mathbb{N}^n]$ along faces.

5.4. Flange presentations.

Definition 5.12. Fix a $\mathbb{Z}^n$-module $N$.

1. A flange presentation of $N$ is a $\mathbb{Z}^n$-module morphism $\varphi : F \to E$, with image isomorphic to $N$, where $F$ is flat and $E$ is injective in the category of $\mathbb{Z}^n$-modules.
2. If $F$ and $E$ are expressed as direct sums of indecomposables, then $\varphi$ is based.
3. If $F$ and $E$ are finite direct sums of indecomposables, then $\varphi$ is finite.
4. If the number of indecomposable summands of $F$ and $E$ are simultaneously minimized then $\varphi$ is minimal.

Remark 5.13. The term flange is a portmanteau of flat and injective (i.e., “flainj”) because a flange presentation is the composite of a flat cover and an injective hull.

The same notational trick to make fringe presentations effective data structures (Definition 3.17) works on flange presentations.

Definition 5.14. Fix a based finite flange presentation $\varphi : \bigoplus_p F_p = F \to E = \bigoplus_q E_q$. A monomial matrix for $\varphi$ is an array of scalar entries $\varphi_{qp}$ whose columns are labeled by the indecomposable flat summands $F_p$ and whose rows are labeled by the indecomposable injective summands $E_q$:

$$
\begin{bmatrix}
E_1 & \cdots & E_\ell \\
F_1 & \varphi_{11} & \cdots & \varphi_{1\ell} \\
\vdots & \vdots & \ddots & \vdots \\
F_k & \varphi_{k1} & \cdots & \varphi_{k\ell}
\end{bmatrix}
$$

$F_1 \oplus \cdots \oplus F_k = F \xrightarrow{\varphi} E = E_1 \oplus \cdots \oplus E_\ell$.

The entries of the matrix $\varphi_{\cdot, \cdot}$ correspond to homomorphisms $F_p \to E_q$.

Lemma 5.15. If $F = k[a + \mathbb{Z}\tau' + \mathbb{N}^n]$ is an indecomposable flat $\mathbb{Z}^n$-module and $E = k[b + \mathbb{Z}\tau - \mathbb{N}^n]$ is an indecomposable injective $\mathbb{Z}^n$-module, then $\text{Hom}_{\mathbb{Z}^n}(F, E) = 0$ unless $(a + \mathbb{Z}\tau' + \mathbb{N}^n) \cap (b + \mathbb{Z}\tau - \mathbb{N}^n) \neq \emptyset$, in which case $\text{Hom}_{\mathbb{Z}^n}(F, E) = k$.

Proof. Corollary 3.11.1. \qed

Definition 5.16. In the situation of Lemma 5.15, write $F \preceq E$ if their degree sets have nonempty intersection: $(a + \mathbb{Z}\tau' + \mathbb{N}^n) \cap (b + \mathbb{Z}\tau - \mathbb{N}^n) \neq \emptyset$. 
**Proposition 5.17.** With notation as in Definition 5.14, \( \varphi_{pq} = 0 \) unless \( F_p \preceq E_q \). Conversely, if an array of scalars \( \varphi_{qp} \in \mathbb{k} \) with rows labeled by indecomposable flat modules and columns labeled by indecomposable injectives has \( \varphi_{pq} = 0 \) unless \( F_q \preceq E_q \), then it represents a flange presentation.

*Proof.* Lemma 5.15 and Definition 5.16.

The unnatural hypothesis that a persistence module be finitely generated results in data types and structure theory that are asymmetric regarding births as opposed to deaths. In contrast, the notion of flange presentation is self-dual: their duality interchanges the roles of births \((F)\) and deaths \((E)\).

**Proposition 5.18.** A \( \mathbb{Z}^n \)-module \( N \) has a finite flange presentation \( F \to E \) if and only if the Matlis dual \( E^\vee \to F^\vee \) is a finite flange presentation of the Matlis dual \( N^\vee \).

*Proof.* Matlis duality is an exact, contravariant functor on \( \mathbb{Z}^n \)-modules that takes the subcategory of finitely determined \( \mathbb{Z}^n \)-modules to itself (these properties are immediate from the definitions), interchanges flat and injective objects therein, and has the property that the natural map \((N^\vee)^\vee \to N\) is an isomorphism for finitely determined \( N \) (Lemma 5.10); see [Mil00, §1.2] for a discussion of these properties.

5.5. **Syzygy theorem for \( \mathbb{Z}^n \)-modules.**

**Theorem 5.19.** A \( \mathbb{Z}^n \)-module is finitely determined if and only if it admits one, and hence all, of the following:

1. a finite flange presentation; or
2. a finite flat presentation; or
3. a finite injective copresentation; or
4. a finite flat resolution; or
5. a finite injective resolution; or
6. a minimal one of any of the above.

Any minimal one of these objects is unique up to noncanonical isomorphism, and the resolutions have length at most \( n \).

*Proof.* The hard work is done by Proposition 5.7. It implies that \( N \) is finitely determined \( \Leftrightarrow \) \( N^\vee \) has a minimal injective resolution \( \Leftrightarrow \) \( N \) has a minimal flat resolution of length at most \( n \), since the Matlis dual of any finitely determined module \( N \) is finitely determined. Having both a minimal injective resolution and a minimal flat resolution is stronger than having any of items 1–3, minimal or otherwise, so it suffices to show that \( N \) is finitely determined if \( N \) has any of items 1–3. This follows, using that the category of finitely determined modules is abelian as in the proof of Proposition 5.7, from the fact that every indecomposable injective or flat \( \mathbb{Z}^n \)-module is finitely determined.
Remark 5.20. Conditions 1–6 in Theorem 5.19 remain equivalent for \( \mathbb{R}^n \)-modules, with the standard positive cone \( \mathbb{R}^n_+ \), assuming that the finite flat and injective modules in question are finite direct sums of localizations of \( \mathbb{R}^n \) along faces and their Matlis duals. (The equivalence, including minimality, is a consequence of the generator and cogenerator theory over real polyhedral groups [Mil20c].) The equivalent conditions do not characterize \( \mathbb{R}^n \)-modules that are pulled back under convex projection from arbitrary modules over an interval in \( \mathbb{R}^n \), though, because all sorts of infinite things can happen inside of a box, such as having generators along a curve.

6. Homological algebra of poset modules

6.1. Indicator resolutions.

**Definition 6.1.** Fix any poset \( Q \) and a \( Q \)-module \( M \).

1. An **upset resolution** of \( M \) is a complex \( F_\bullet \) of \( Q \)-modules, each a direct sum of upset submodules of \( \mathbb{k}[Q] \), whose differential \( F_i \to F_{i-1} \) decreases homological degrees, has components \( \mathbb{k}[U] \to \mathbb{k}[U'] \) that are connected (Definition 3.14), and has only one nonzero homology \( H_0(F_\bullet) \cong M \).

2. A **downset resolution** of \( M \) is a complex \( E^\bullet \) of \( Q \)-modules, each a direct sum of downset quotient modules of \( \mathbb{k}[Q] \), whose differential \( E^i \to E^{i+1} \) increases cohomological degrees, has components \( \mathbb{k}[D'] \to \mathbb{k}[D] \) that are connected, and has only one nonzero homology \( H^0(E^\bullet) \cong M \).

An upset or downset resolution is called an **indicator resolution** if the “up-” or “down-” nature is unspecified. The **length** of an indicator resolution is the largest (co)homological degree in which the complex is nonzero. An indicator resolution

3. is **finite** if the number of indicator module summands is finite,

4. dominates a constant subdivision or encoding of \( M \) if the subdivision or encoding is subordinate to each indicator summand, and

5. is **semialgebraic**, PL, subanalytic, or of class \( \mathcal{X} \) if \( Q \) is a subposet of a real partially ordered group and the resolution dominates a constant subdivision or encoding of the corresponding type.

**Definition 6.2.** Monomial matrices for indicator resolutions are defined similarly to those for fringe presentations in Definition 3.17, except that for the cohomological case the row and column labels are source and target upsets, respectively, while in the homological case the row and column labels are target and source downsets, respectively:
As in Proposition 4.11, pullbacks have transparent monomial matrix interpretations.

**Proposition 6.3.** Fix a poset $Q$ and an encoding of a $Q$-module $M$ by a poset morphism $\pi : Q \to P$ and $P$-module $H$. Monomial matrices for any indicator resolution of $H$ pull back to monomial matrices for an indicator resolution of $M$ that dominates the encoding by replacing the row and column labels with their preimages under $\pi$. \hfill \square

**Definition 6.4.** Fix any poset $Q$ and a $Q$-module $M$.

1. An **upset presentation** of $M$ is an expression of $M$ as the cokernel of a homomorphism $F_1 \to F_0$ such that each $F_i$ is a direct sum of upset modules and every component $k[U'] \to k[U]$ of the homomorphism is connected (Definition 3.14).

2. A **downset copresentation** of $M$ is an expression of $M$ as the kernel of a homomorphism $E^0 \to E^1$ such that each $E^i$ is a direct sum of downset modules and every component $k[D'] \to k[D]$ of the homomorphism is connected.

These indicator presentations are finite, or dominate a constant subdivision or encoding of $M$, or are semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ as in Definition 6.1.

**Example 6.5.** In one parameter, the bar $[a, b)$ in Example 3.19, has upset presentation

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

with cokernel isomorphic to the single bar. When there are multiple bars, the bijection from left to right endpoints yields a monomial matrix whose scalar entries again form an identity matrix, with rows labeled by positive rays having the specified left endpoints (the ray is the whole real line when the left endpoint is $-\infty$) and columns labeled by positive rays having the corresponding right endpoints—but with their open or closed nature reversed—as left endpoints (the ray is empty when the specified right endpoint is $+\infty$).

**Example 6.6.**

**Lemma 6.7.** The homomorphisms in indicator presentations and resolutions are tame, so their kernels and cokernels are tame. If the indicator modules in question are semialgebraic, PL, subanalytic, or of class $\mathcal{X}$ then the morphisms are, as well.
Proof. Any connected homomorphism among indicator modules is tame—and satisfies one of the auxiliary hypotheses, if the source and target do—by Definition 4.27, so the conclusion follows from Proposition 4.31.

Example 6.8. The poset module in Example 2.7 has an upset presentation

\[
\begin{pmatrix}
T & B \\
L & 2 & 1 \\
R & -1 & -1 \\
\end{pmatrix}
\]

in which the monomial matrix has row and column labels

- \(L\), the upset generated by the leftmost element;
- \(R\), the upset generated by the rightmost element;
- \(T\), the upset consisting solely of the maximal element depicted on top; and
- \(B\), the upset consisting solely of the maximal element depicted on the bottom.

Although the disjoint union of \(T\) and \(B\) is an upset, and there is a homomorphism \(\varphi : \mathbb{k}[T \cup B] \to \mathbb{k}[L \oplus \mathbb{k}[R]]\) whose cokernel is the desired poset module, there is no way to arrange for the homomorphism \(\varphi\) to be connected.

Remark 6.9. It is tempting to think that a fringe presentation is nothing more than the concatenation of the augmentation map of an upset resolution (that is, the surjection at the end) with the augmentation map of a downset resolution (that is, the injection at the beginning), but there is no guarantee that the components \(F_i \to E_j\) of the homomorphism thus produced are connected (Definition 3.14). In contrast, a flange presentation (Definition 5.12) is in fact nothing more than the concatenation of the augmentation maps of a flat resolution and an injective resolution, since connected homomorphisms are forced by Lemma 5.15.

6.2. Syzygy theorem for modules over posets.

Proposition 6.10. For any inclusion \(\iota : P \to Z\) of posets and \(P\)-module \(H\) there is a \(Z\)-module \(\iota_* H\), the pushforward to \(Z\), whose restriction to \(\iota(P)\) is \(H\) and is universally repelling: \(\iota_* H\) has a canonical map to every \(Z\)-module whose restriction to \(\iota(P)\) is \(H\).

Proof. At \(z \in Z\) the pushforward \(\iota_* H\) places the colimit \(\lim_{\to} \mathbb{H}_{\leq z}\) of the diagram of vector spaces indexed by the elements of \(P\) whose images precede \(z\). The universal property of colimits implies that \(\iota_* H\) is a \(Z\)-module with the desired universal property.

Remark 6.11. With perspectives as in Remark 2.4, the pushforward is a left Kan extension [Cur14, Remark 4.2.9]. This instance is a special case of [Cur19, Example 4.4].
Theorem 6.12 (Syzygy theorem). A module $M$ over a poset $Q$ is tame if and only if it admits one, and hence all, of the following:

1. a finite constant subdivision of $Q$ subordinate to $M$; or
2. a finite poset encoding subordinate to $M$; or
3. a finite fringe presentation; or
4. a finite upset presentation; or
5. a finite downset copresentation; or
6. a finite upset resolution; or
7. a finite downset resolution; or
8. any of the above dominating any given finite encoding; or
9. a finite encoding subordinate to any given one of items 1–7; or
10. a finite constant subdivision subordinate to any given one of items 1–7.

The statement remains true over any subposet of a real partially ordered group if “tame” and all occurrences of “finite” are replaced by “semialgebraic”, “PL”, or “class $X$”. Moreover, any tame or semialgebraic, PL, or class $X$ morphism $M \to M'$ lifts to a similarly well behaved morphism of presentations or resolutions as in parts 3–7. All of these results except item 9 hold in the subanalytic case if $M$ has compact support.

Proof. Tame is equivalent to item 1 without auxiliary hypotheses by Definition 2.11 and with auxiliary hypotheses by Definition 2.15. Tame is equivalent to item 2 by Theorem 4.22. With auxiliary hypotheses, $1 \Rightarrow 2$ by Theorem 4.22.3; to apply that result in the subanalytic case starting from an arbitrary subanalytic finite constant subdivision subordinate to $M$, construct a compact such subdivision by keeping the bounded constant regions as they are and taking the union of all unbounded constant regions to get a single unbounded one. The implication $2 \Rightarrow 1$ holds because the fibers of the encoding poset morphism form a constant subdivision of the relevant type.

The necessity to construct an auxiliary compact subdivision from the given one is the reason to exclude item 9 from the subanalytic case, as the upcoming argument produces constant subdivisions, not directly encodings. For all of the other cases, item 9 proceeds via item 10, given the uptight constructions in the previous paragraph. For item 10, to produce a subordinate finite constant subdivision given a finite fringe presentation, take the common refinement of the canonical constant subdivision subordinate to each of its indicator summands. The same construction works if indicator presentations or resolutions are given, and it preserves auxiliary hypotheses by Proposition 2.17.1.

What remains is item 8: a finitely encoded $Q$-module $M$ has finite upset and downset resolutions and (co)presentations, as well as a finite fringe presentation, all dominating the given encoding. (As noted in the first paragraph, the fibers of the encoding morphism are already a constant subdivision of the relevant type.) The domination takes care of the cases with auxiliary hypotheses by Definitions 3.16.3, 4.7.2, 6.1.5, and 6.4.

Fix a $Q$-module $M$ finitely encoded by a poset morphism $\pi : Q \to P$ and $P$-module $H$. The finite poset $P$ has order dimension $n$ for some positive integer $n$; as such $P$ has an
embedding \( \iota: P \hookrightarrow \mathbb{Z}^n \). The pushforward \( \iota_* H \) (Proposition 6.10) is finitely determined (Definition 5.1; see also Example 4.5) as it is pulled back from any box containing \( \iota(P) \).

The desired presentation or resolution is pulled back to \( Q \) (via \( \iota \circ \pi: Q \to \mathbb{Z}^n \)) from the corresponding flange, flat, or injective presentation or resolution of \( \iota_* H \) afforded by Theorem 5.19. These pullbacks are finite indicator resolutions of \( M \) dominating \( \pi \) by Example 4.10 and Lemma 4.9. The component homomorphisms are connected because, by Corollary 3.11 and Example 3.6 (see Definition 3.5), components of flange presentations, flat resolutions, and injective resolutions over \( \mathbb{Z}^n \) are automatically connected.

The preceding argument proves the claim about a morphism \( M \to M' \), as well, since

- only one poset morphism is required to encode the morphism \( M \to M' \);
- the push-pull constructions are functorial; and
- morphisms of finitely determined modules can be lifted to the relevant presentations and resolutions, since the relevant covers, presentations, and resolutions are free or injective in the category of finitely determined modules. \( \square \)

**Remark 6.13.** Comparing Theorems 6.12 and 5.19, what happened to minimality? It is not clear in what generality minimality can be characterized. The sequel [Mil20c] to this paper can be seen as a case study for posets arising from abelian groups that are either finitely generated and free or real vector spaces of finite dimension. The answer is much more nuanced in the real case, obscuring how minimality might generalize beyond these cases.

**Remark 6.14.** In the situation of the proof of Theorem 6.12, composing two applications of Proposition 4.11—one for the encoding \( \pi: Q \to P \) and one for the embedding \( \iota: P \hookrightarrow \mathbb{Z}^n \)—yields a monomial matrix for a fringe presentation of \( M \) directly from a monomial matrix for a flange presentation.

**Remark 6.15.** Lesnick and Wright consider \( \mathbb{R}^n \)-modules [LW15, §2] in finitely presented cases. As they indicate, homological algebra of such \( \mathbb{R}^n \)-modules is no different than finitely generated \( \mathbb{Z}^n \)-modules. This can be seen by finite encoding: any finite poset in \( \mathbb{R}^n \) is embeddable in \( \mathbb{Z}^n \), because a product of finite chains is all that is needed.

6.3. **Syzygy theorem for complexes of modules.**

Theorem 6.12 is stated for individual modules, but the proof works just as well for complexes, in a sense recorded here for reference in the proof of a version in the language of derived categories of constructible sheaves [Mil20b, Theorem 4.5].

**Definition 6.16.** Fix a complex \( M^* \) of modules over a poset \( Q \).

1. \( M^* \) is *tame* if its modules and morphisms are tame (Definitions 2.11 and 4.27).
2. A constant subdivision or poset encoding is *subordinate* to \( M^* \) if it is subordinate to all of the modules and morphisms therein, and in that case \( M^* \) is said to *dominate* the subdivision or encoding.
3. An upset resolution of $M^\bullet$ is a complex of $Q$-modules in which each $F_i$ is a direct sum of upset modules and the components $k[U] \to k[U']$ are connected, with a homomorphism $F^\bullet \to M^\bullet$ of complexes inducing an isomorphism on homology.

4. A downset resolution of $M^\bullet$ is a complex of $Q$-modules in which each $E_i$ is a direct sum of downset modules and the components $k[D] \to k[D']$ are connected, with a homomorphism $M^\bullet \to E^\bullet$ of complexes inducing an isomorphism on homology.

These resolutions are finite, or dominate a constant subdivision or encoding, or are semialgebraic, PL, subanalytic, or of class $\mathfrak{X}$ as in Definition 6.1.

**Theorem 6.17** (Syzgy theorem for complexes). Theorem 6.12 holds verbatim for a bounded complex $M^\bullet$ in place of the module $M$ as long as items 3, 4, and 5 are ignored.

**Proof.** As already noted, the proof is the same. It bears mentioning that finite injective and flat resolutions of complexes exist in the category of finitely determined $\mathbb{Z}^n$-modules because finite injective resolutions do (Proposition 5.7): any of the standard constructions that produce injective resolutions of complexes given that modules have injective resolutions works in this setting, and then Matlis duality (Definition 5.9) produces finite flat resolutions (see Remark 5.11). □

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