Pattern formation and firing synchronization in networks of map neurons

Qing Yun Wang\textsuperscript{1,2,5}, Zhi Sheng Duan\textsuperscript{1}, Lin Huang\textsuperscript{1}, Guan Rong Chen\textsuperscript{3} and Qi Shao Lu\textsuperscript{1}

\textsuperscript{1} State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University, Beijing 100871, People’s Republic of China
\textsuperscript{2} School of Statistics and Mathematics, Inner Mongolia Finance and Economics College, Huhhot 010051, People’s Republic of China
\textsuperscript{3} Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, People’s Republic of China
\textsuperscript{4} School of Science, Beijing University of Aeronautics and Astronautics, Beijing 100083, People’s Republic of China

E-mail: nmqingyun@163.com and eegchen@cityu.edu.hk

New Journal of Physics 9 (2007) 383
Received 17 July 2007
Published 26 October 2007
Online at http://www.njp.org/
doi:10.1088/1367-2630/9/10/383

Abstract. Patterns and collective phenomena such as firing synchronization are studied in networks of nonhomogeneous oscillatory neurons and mixtures of oscillatory and excitable neurons, with dynamics of each neuron described by a two-dimensional (2D) Rulkov map neuron. It is shown that as the coupling strength is increased, typical patterns emerge spatially, which propagate through the networks in the form of beautiful target waves or parallel ones depending on the size of networks. Furthermore, we investigate the transitions of firing synchronization characterized by the rate of firing when the coupling strength is increased. It is found that there exists an intermediate coupling strength; firing synchronization is minimal simultaneously irrespective of the size of networks. For further increasing the coupling strength, synchronization is enhanced. Since noise is inevitable in real neurons, we also investigate the effects of white noise on firing synchronization for different networks. For the networks of oscillatory neurons, it is shown that firing synchronization decreases when the noise level
increases. For the missed networks, firing synchronization is robust under the noise conditions considered in this paper. Results presented in this paper should prove to be valuable for understanding the properties of collective dynamics in real neuronal networks.

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1. Introduction

Networks, in a range of disciplines from biology to physics, social sciences and informatics, have received significant attention in recent years [1]–[4]. The human brain is a large system, with more than a hundred specialized modules with different functions. At the smallest grain, the cerebral cortex consists of about 10^10 neurons that are connected to each other by synapses, together forming neural networks [5, 6]. However, what can an investigation of network structure and dynamics contribute to our understanding of the brain and cognitive function?

It is well known that processes of generation and propagation of neuron excitation waves and patterns in neural networks are a matter of topical interest because of their importance for understanding normal and pathological types of activity in neuronal systems [7]. An important goal in neuroscience is to understand these spatio-temporal patterns of brain activity. Hence, spatial patterns and their propagation in neuronal systems have been extensively studied, both experimentally and by means of numerical simulations. It was shown in [8] that complex wave propagation exists in a ring neuronal network when the noise level was changed. The spatio-temporal dynamics of spatially extended neuronal networks have been investigated [9]–[12]. Results show that there exists an optimal noise level, at which beautiful circular waves can appear, which implies that noise can induce spatial order in spatially extended neuronal networks. Effects of small-world connectivity on noise-induced spatial patterns in neural media have also been investigated [13, 14]. Kanakov et al have studied spatio-temporal dynamics in networks of oscillatory and excitable Luo–Rudy cells [7]. They found rich spatio-temporal patterns including pacemakers, spiral waves and complicated irregular activity. Volman et al [15], have investigated the possible causes underlying the appearance of structured spatio-temporal patterns in the activity of neuronal networks by function-follow-form.

Synchronization of a set of interacting individuals or units has been intensively studied because of its ubiquity in the natural world. The dependence of the emergent collective phenomena on the coupling strength and on the topology was unveiled for homogeneous and heterogeneous complex networks [16]. It was concluded that different sources of diversity, such as those represented by quenched disorder or noise, could induce a resonant collective behavior.
in an ensemble of coupled bistable or excitable systems [17]. In particular, mammalian nervous systems exhibit a diversity of synchronized behaviors [18]–[21]. It has been supposed that theoretical studies of such synchronized behaviors in neuronal assemblies play an important role in our understanding of information processing in nervous systems. Complete synchronizations of three types of regular neuronal networks have been theoretically investigated and it was proved that different coupling styles had different critical values, which depended on specific coupling styles [22]. Sato and Shiino [23] found that the effects of the width of an action potential played an important role in synchronization phenomena of the coupled neurons by means of return map analysis. The spatial synchronization in chaotic modified Hodgkin–Huxley (MHH) thermosensitive neurons in complex networks has been studied by Gong et al [24]. They found that the synchronization which is absent in the regular network can be greatly enhanced by random shortcuts between the neurons. Phase synchronization effects of chaotic oscillators with a type-I intermittency behavior were studied [25]. As an example, phase synchronization in a chain of coupled nonidentical model maps replicating neural spiking activity was investigated [26]. It was shown that an increase of the coupling strength could lead to desynchronization phenomena, i.e. global or cluster synchronization is changed to a regime where synchronization is intermittent with the incoherent state. Then a regime of a fully incoherent nonsynchronous state, spatio-temporal intermittency, appears. The relationship between structural and functional connectivity at different levels of synchronization was explored by means of a realistic network of cat cortical connectivity [27].

Map neuronal networks have been regarded as alternative models for investigating the dynamics of neuronal systems. The analysis of complex large-scale neurobiological networks is computationally efficient by means of a map neuron model. Interconnected with synaptic currents, map neurons demonstrated responses very similar to those found with Hodgkin–Huxley models and in experiments [28]. Perc [11, 12], studied the effects of spatio-temporal additive noise on the spatial dynamics of excitable neuronal media that is locally modeled by a two-dimensional (2D) map. Subthreshold stimulus-aided temporal order and synchronization in a square lattice noisy neuronal network have been investigated by means of the discrete-time Rulkov map neuron [29]. Recently, Wei and Luo [30], studied ordering of spatio-temporal chaos in discrete neural networks with small-world connections. They found that the discrete neural network with small-world connections could capture well the maximal spatio-temporal order, which has been observed in continuous neuronal networks. However, the spatio-temporal dynamics of map neuronal networks is still one of the most flourishing and fascinating topics of research in the field of nonlinear science.

To the best of our knowledge, dynamics of spatially extended map neuronal networks with changing coupling strength have never been reported and thus remains a challenging problem to be solved in the field of neuronal dynamics. In this paper, we will study the pattern formation and firing synchronization in two types of 2D square lattice map neuronal networks. The results show that as the coupling strength is increased, typical patterns can transit from random portraits to target waves or parallel ones depending on the size of networks. Furthermore, the synchronization firstly decreases, and then increases for any type of networks. At the same time, it is shown in the networks of oscillatory neurons that firing synchronization decreases when the noise level increases; while for mixed networks, firing synchronization is robust for the noise conditions considered in this paper. Hopefully, our results may provide a useful tip for understanding the properties of collective dynamics in coupled real neurons.
2. Networks of map neurons

The neuronal networks studied in this paper are described by the following coupled 2D Rulkov map neurons,

\[
\begin{align*}
\alpha_{i,j} = x_{i,j}(n+1) & = \alpha_{i,j} f(x_{i,j}(n)) + y_{i,j}(n) - D(4x_{i,j}(n) - x_{i+1,j}(n) - x_{i-1,j}(n) - x_{i,j+1}(n) - x_{i,j-1}(n)) \\
\gamma & = y_{i,j}(n) - \beta x_{i,j}(n) - \gamma, \quad i, j = 1, 2, \ldots, N,
\end{align*}
\]

where \(x_{i,j}(n)\) is the membrane potential of the neuron labeled \((i, j)\), and \(y_{i,j}(n)\) is the variation of ion concentration. These represent the fast and slow dynamics, respectively. The slow evolution of \(y_{i,j}(n)\) is due to the small values of the positive parameters \(\beta\) and \(\gamma\), say, \(\beta = \gamma = 0.004\). \(n\) is the discrete time series, while \(\alpha_{i,j}\) are control parameters of the networks. \(f(x) = 1/(1+x^2)\) is a nonlinear function. \(D\) is the coupling strength. In what follows, periodic boundary conditions of the coupling are used such that the networks studied are spatially extended square lattices.

Nonlinear dynamics of an individual 2D Rulkov map neuron, which depends on the control parameter \(\alpha\) has been extensively investigated [31]. Results showed that when the parameter \(\alpha\) is changed, the neuron exhibits rich firing behaviors, such as silent, periodic and/or chaotic spiking, spiking-bursting behaviors. For \(\alpha < 2.0\), the map exhibits a single excitable steady state. It is noted that for \(\alpha > 2.0\) the excitable steady state loses its stability via a Hopf bifurcation, and then the neuron begins to oscillate. In this paper, we investigate the pattern formation and firing synchronization in networks of oscillatory neurons, and mixtures of oscillatory and excitable neurons, respectively. In the following calculations, we use the control parameter \(\alpha_{i,j}\), uniformly distributed in the interval \(\alpha_{i,j} \in [2.1; 2.3]\) for the former case; while for the latter case, the control parameters \(\alpha_{i,j}\) are uniformly distributed in the interval \(\alpha_{i,j} \in [1.95; 2.35]\). We average the runs of 50 networks with different realizations of these uniform distributions for any simulation. The initial conditions are chosen to be identical in each neuron, so that all neurons in the network initially stay in the quiescent state simultaneously.

3. Typical patterns in networks of map neurons

Firstly, some typical patterns are depicted in the networks of oscillatory neurons as the coupling strength \(D\) varies. As shown in figure 1, the characteristic spatial snapshots of \(x\) are presented for different coupling strengths with a network of size \(128 \times 128\) after a waiting time of \(8 \times 10^4\) units. It is seen in figure 1(a) that neurons fire disordered random-like portraits in spatial structures for a small coupling strength, which are similar to the random portraits induced by large spatial white noise [28]. As the coupling strength is increased, neurons begin to swarm, thus exhibiting several big clusters of firing patterns as illustrated in figures 1(b) and (c) as compared with figure 1(a). Furthermore, it is shown that fired clusters become knitted together as the coupling strength is increased. With the coupling strength increasing further, ordered patterns emerge spatially, which propagate through the neurons in the form of circular waves as shown in figure 1(d). The corresponding space-time evolution in the latter case is an almost regular target wave structure, but it contains defects in the form of additional pacemakers. In fact, since a larger \(D\) leads to a faster diffusive spread, local excitations can propagate further through space in a given amount of time. Hence, it is understandable that as the coupling
Figure 1. Snapshots of membrane voltage after a waiting time of $8 \times 10^4$ units in a 2D lattice of $128 \times 128$ oscillatory neurons at different values of the coupling coefficient (a) $D = 0.001$; (b) $D = 0.004$; (c) $D = 0.015$ and (d) $D = 0.02$. The control parameters $\alpha_{i,j}$ are distributed uniformly on the interval $[2.1; 2.3]$.

In what follows, we will study the pattern formation of networks of mixtures of oscillatory and excitable neurons. To obtain a mixed network, we set the interval of uniform distribution $\alpha_{i,j}$ to $[1.95; 2.35]$. We perform the same computational analysis of the model as in the
previous setting. It is shown in figures 3 and 4 that target and parallel waves depending on the size of networks can also appear when the coupling strength is beyond a critical value. Evolutions of the typical waves with the coupling strength increasing are similar to those of the networks of oscillatory neurons. Since there exists a small fraction of excitable neurons, compared to the networks of oscillatory neurons, larger couplings are needed to form the target and parallel waves for the mixed networks. Meanwhile, typical target and parallel waves of the mixed networks are not so regular as those of oscillatory networks since there is a fraction of excitable neurons in the mixed networks, which hinder orderly activities of neurons in the network.

4. Firing synchronization

It is well known that it is impossible to present the above several snapshots in continuous time. Consequently, it is not enough to understand spatio-temporal dynamics of a 2D lattice neuronal

**Figure 2.** Snapshots of membrane voltage after waiting time $8 \times 10^4$ units in a 2D lattice of $256 \times 256$ oscillatory neurons at different values of the coupling coefficient. (a) $D = 0.004$; (b) $D = 0.012$ and (c) $D = 0.02$. The control parameters $\alpha_{i,j}$ are distributed uniformly on the interval $[2.1; 2.3]$.

**Figure 3.** Snapshots of membrane voltage after waiting time $8 \times 10^4$ units in a 2D lattice of $128 \times 128$ mixture of oscillatory and excitable neurons at different values of the coupling coefficient (a) $D = 0.004$; (b) $D = 0.02$ and (c) $D = 0.035$. The control parameters $\alpha_{i,j}$ are distributed uniformly on the interval $[1.95; 2.35]$.
network by means of some typical patterns. However, we can capture the essence of spatio-temporal dynamics by presenting time traces of a firing rate function $\Pi(m)$, which simply measures the fraction of spatial neurons when the membrane potential $x$ reaches its threshold value $x_{th} = -0.1$ from below at a certain time. It can be described as follows: suppose that $m$ neurons fire at the given discrete time $n$, then we define a function $\Pi(m) = \frac{m}{N_T}$. Clearly, if $\Pi(m) = 0$, this means that none of the neurons is perturbed to fire. On the other hand, $\Pi(m) = 1$ indicates that all neurons are in the firing state simultaneously at the given time $n$, which constitutes global synchrony corresponding to the most ordered temporal dynamics in neuronal networks.

In what follows, temporal evolutions of the firing rate in neuronal networks are shown in figures 5 and 6 for networks of different sizes and coupling strengths, respectively. It is found in figures 5 and 6 that the rates of firing can almost vary periodically for the large coupling strength,
which correspond to the target and parallel waves as shown in figures 1–4, respectively. For small coupling, it can be observed that the rate of firing is a bit disordered as shown in the dotted lines of figures 5 and 6. Furthermore, it can be found that the average rate of firing becomes high when the coupling strength is increased. Hence, this implies that the firing synchronization can be enhanced by the coupling strength.

From the above analysis, it can be seen that although evolutions of typical patterns are drastically different for different networks, the firing rates have similar characteristics of variation as the coupling strength is increased. Hence, based on the firing rate, it is reasonable to study the firing synchronization for different networks. For this purpose, we introduce the following quantity. It is the average $\langle \Pi_{m} \rangle$ of the rate of firing $\Pi_{m}$, which can show the number of fired neurons in the given time interval. We call it the synchronization index, and denote it as $\sigma = \langle \Pi_{m} \rangle$. It is obvious that the larger $\sigma$ is, the more neurons can synchronize in the given time interval.

Synchronization index is shown in figures 7(a) and (b) as the coupling strength is increased for different networks. One sees that the synchronization index initially becomes small, which indicates that the small coupling destroys synchronization. This demonstrates the fact that the small coupling cannot make the neurons drive their nearest neighborhood ones to synchronize their behaviors. On the contrary, neurons can exhibit random-like fluctuation due to the effects of small couplings. As a result, this lowers the degree of firing synchronization. When the coupling increases further, and then goes beyond a threshold, the synchronization index begins to increase, which is a well-known result in diffusively coupled systems. Hence, the variation of synchronization index can exhibit a ‘U’-like shape structure as shown in figures 7(a) and (b) when the coupling strength is varied. It can also be found that degree of synchronization in the mixed networks is lower than that of the networks of oscillatory neurons, as illustrated in figures 7(a) and (b).
5. Effects of noise on firing synchronization

Actually, in neural systems, noise is inevitable, which arises from intrinsic and extrinsic sources, such as the random switching of ion channels, the quasi-random release of neurotransmitter by the synapse, and most importantly random synaptic input from other neurons. Hence, it is necessary to explore the effects of noise on firing synchronization. To do this, Gaussian white noise is added to the following equations,

\[
\begin{align*}
    x^{(i,j)}(n+1) &= \alpha^{(i,j)} f(x^{(i,j)}(n)) + y^{(i,j)}(n) \\
    &\quad - D(4x^{(i,j)}(n) - x^{(i+1,j)}(n) - x^{(i-1,j)}(n) - x^{(i,j+1)}(n)) \\
    &\quad - x^{(i,j-1)}(n)) + W\xi^{(i,j)}(n) \\
    y^{(i,j)}(n+1) &= y^{(i,j)}(n) - \beta x^{(i,j)}(n) - \gamma, \quad i, j = 1, 2, \ldots, N,
\end{align*}
\]

where \(W\) denotes the level of noise and \(\xi^{(i,j)}\) the Gaussian white noise for the \((i, j)\)th space neuron.

In what follows, firing synchronizations in the networks of map neurons with white noise will be investigated. Similar to the above study, the synchronization index \(\sigma\) is shown in figure 8 for the size 128 \times 128 network with noise as the coupling strength is increased. It is found that the synchronization index in the network of oscillatory neurons with noise decreases when noise appears, which implies a lowering of the firing synchronization. Meanwhile the synchronization index of the mixed networks with noise is almost the same as that without noise, which shows the robustness of firing synchronization to the noise considered in this case. For other networks, similar investigations have been conducted and similar results to the above can be found.
Figure 8. For different noise levels, variations of the synchronization index $\sigma$ are shown for the networks of oscillatory neurons and a mixture of oscillatory and excitable neurons in (a) and (b), respectively, when the coupling strength $D$ changes.

6. Conclusion and discussion

In conclusion, collective phenomena were investigated numerically in networks of oscillatory neurons, and a mixture of oscillatory and excitable neurons, respectively. We have observed various spatio-temporal patterns of activity, including complicated irregular behaviors, target and parallel waves depending on the size of networks when the coupling strength was increased. Furthermore, it was shown that firing synchronizations initially decrease and then increase for two types of networks when the coupling strength was increased. Hence, their variations can exhibit the ‘U’-like shape structures. It was also found that synchronization of oscillatory networks was stronger than that of mixed networks. Moreover, the effects of noise on firing synchronization have also been investigated. It is clear that the coupling constant $D$ plays an important role in the collective dynamics of the neurons. In real situations, this $D$ corresponds probably to the gap-junction connectivity between neurons. Consequently, the results in this paper have potential implications for understanding synchronization.

Nonlinear wave propagation is ubiquitous in nature, appearing in cardiac tissue dynamics, cortical spreading depression and slow wave sleep [32]. It is well known that cardiac cells exhibit properties of either excitable or oscillatory systems. The former case is observed in working myocardium, and the latter is found in natural cardiac pacemakers. Normal heart activity is controlled by waves of excitation generated in the sinoatrial node and propagating through the conducting system and working myocardium. Deviations from the normal regime are often associated with pathological types of wave dynamics in the cardiac tissue. Also, some experiments in vitro were reported [33] that after approximately 24 h of culture time, irregular spontaneous activity arises in the culture, and it further organizes itself into several pacemakers emitting target waves. Significant scientific effort has been made to understand these waves and develop a way of controlling them to recover normal function. The application of dynamical modeling has provided valuable insights into the mechanisms underlying such nonlinear wave phenomena in several domains. Hence, the obtained waves depending on the size of networks and coupling strength in this paper can provide important guidance for controlling pathological types of wave dynamics and understanding the evolution of waves in neuronal systems.
Acknowledgments

We thank the referee for valuable comments. This work was supported by the National Natural Science Foundation of China (nos. 10702023 and 60674093).

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