Draft: Volume Integral of the Pressure Gradient and Archimedean’s
Principle

Galen T. Pickett

Department of Physics and Astronomy,
California State University Long Beach,
1240 Bellflower Blvd., Long Beach, CA 90840, USA
(Dated: July 30, 2014)

Abstract

The theorems of vector analysis (divergence theorem, etc.) are typically first applied in the undergraduate physics curriculum in the context of the electromagnetic field and the differential forms of Maxwell’s equations. However, these tools are analyzed in depth several courses later in the junior-senior level. I discuss here a “bridge” problem, using the language of vector calculus in a mechanics setting to understand Archimedes’ principle as a consequence of hydrostatic equilibrium and the superposition of the external forces. It is my hope that this treatment will help students better integrate and understand these and similar vector analysis results in contexts beyond electromagnetism.
I. INTRODUCTION

There is a qualitative transition in the sophistication of the mathematical tools required to describe mechanical systems of a few free bodies to those dominated by the behavior of continuous fields. Second-order ordinary differential equations are indispensable in the study of point-system mechanics, particularly in a first year course, and many curricula are organized around first understanding the properties of these equations (particularly in the case in which the equation of motion has the form $\ddot{x} = \text{const}$). In a first course in electromagnetism, the fundamental quantities to understand are the distributed electromagnetic fields, $\vec{E}$, $\vec{B}$, with dynamics given by Maxwell’s Equations using the language of partial differential equations with appropriate boundary conditions. Even very good companion works can give the impression to undergraduates that the ideas of “divergence, gradient, and curl” have their primary (perhaps only) role in describing electricity and magnetism. Thus, beginning students get the impression that these mathematical tools are somehow wedded to the separate subjects of the physics canon, rather than being independent, widely applicable tools.

My aim here is to present a mechanics problem which is usually dealt with in the algebra-based mechanics curriculum and recast it using the mathematical language students expect in electricity and magnetism. I have two aims. First, I will connect Archimedes’ principle (usually developed as a “just-so” statement of how a “buoyant force” works) to real forces acting on real bodies. The ideas of buoyancy, density, and volume are known areas of difficulty for science teaching candidates, K-12 students, and even quite advanced students majoring in physics. As we shall see, the buoyant force is merely a consequence of the superposition of many forces acting on the outer edge of an object, and thus is no more mysterious than any other force in mechanics. Secondly, I wish to give an undergraduate a context in which vector analysis has a definite application in the most familiar, concrete physics a student knows, namely mechanics.

Thus, the integration of these mathematical tools and the “net forces” ideas in this nexus of problems and placing that integrated understanding firmly within the grasp of an intermediate-level student of physics is the goal. It is not my intention to merely display yet another derivation of the principle, although I will certainly do that. Rather, it is my intention to integrate this basic idea into our curriculum in a fundamental way. If this work is
used to merely guide a new set of derivations in a lecture hall, its purpose will be unfulfilled. There is a growing consensus across the academy that the value of integrating methods and content, and creating a unified, flexible intellectual framework supporting lifelong learning is the eventual goal of a baccalaureate course of study — even in professional and STEM fields, perhaps particularly so. Too often a university education can be experienced as a list of topics or courses logically disconnected from each other. The work of integrating the education into a seamless whole can be left to a student, and may or may not occur. Physics is particularly blessed in this regard, in that our discipline celebrates the unification of disparate phenomena in a way that few disciplines have a history of doing. There are a number of excellent introductory texts that take this integrated point of view as opposed to a historical development. The “right” ideas are developed at the beginning of study, and the special cases amenable to analysis (exact or approximate) are labeled as such.

In the physics program at CSU Long Beach where I am a faculty member, we have for a number of years been intentional with this very issue, stemming from our activity as a PhysTEC supported site in 2010. One element of this program was to develop a Learning Assistant Program on the Colorado model, along with a course in physics pedagogy we use to train physics majors to teach in our lower-level laboratories as Learning Assistants. A part of that course, and the key to explaining to physics majors why this low-level material is so hard to learn (and teach) is discussing the excellent article by Reddish on mental models in physics learning. One of the most striking features of this treatment is the realization that mental structures of physics in a student’s head are incomplete, disorganized, and even for very well-trained students (and faculty) there exists “holes” in even very basic concepts.

In taking this course, students are required to be present in a science tutoring center, and as is often the case, have duties to help students with courses they themselves had never taken. Algebra-based mechanics and K-6 level physical sciences are never taken by physics students, and we assume that advanced students (with advanced mathematical and physics tools) should be able to handle the more elementary ideas in those “precursor” courses.

It is a relatively simple problem that stumped my stable of learning assistants that prompted the development below — a problem using Archimedes’ principle. It is my experience that anything involving fluids is inordinately tricky to explain. The basic functioning of a centrifuge can be understood at an engineering level and qualitatively well enough for the devices to work correctly in the hands of research-caliber scientists and engineers. Yet, the
explanation in terms of hydrostatic equilibrium in a non-inertial frame is subtle with tendrils reaching up into very sophisticated mechanics.\textsuperscript{10,11} Of the many treatments of the principle (starting with a good exposition of Archimedes’ words themselves\textsuperscript{12} and even quite interesting derivations centering on energy ideas\textsuperscript{13}) not many cement the principle into the basic foundation of mechanics many students understand and can already use quite well. Even in a static case, in which Archimedes’ principle and buoyancy should be sufficient to solve many problems, my group of junior level physics students “knew” Archimedes’ principle, but were unable to apply it. And, embarrassingly enough, likewise for myself. Only when I had put the problem into the language of vector analysis did Archimedes’ principle actually make sense to me (in the sense that I had connected it to my own ideas of mechanics). The point of this paper is to elucidate the conditions that lead me to “reconstruct” a theorem of vector calculus Archimedes himself must have discovered: \textsuperscript{14}

$$\iiint d\vec{A}p = \iiint dv\nabla p$$

which some will recognize as the three-dimensional version of the “mean-value theorem,” but which I will simply regard as “Archimedes’ Theorem” — a fitting companion to Gauss’ law.

I will first describe the elementary problem that revealed to us all that we had a buoyancy gap in our understanding. Then I will describe hydrostatic equilibrium and the new vector identity, and in the discussion describe some applications and “clicker” style questions.

\section{II. ELEMENTARY PROBLEM}

The problem that send my physics major Learning Assistants off into a cycle of recursive algebra is a relatively straightforward problem encountered in the algebra-based mechanics sequence. Here is a variation:

A hot air balloon of volume $V$ is in equilibrium surrounded by air of density $\rho_a$. At $t = 0$ ballast of mass $m_o$ is released from the balloon, and it is observed to accelerate upwards with an acceleration $a$. What is the final mass of the balloon (in terms of $m_o, \rho_a, g, V$)?

These were students, by and large, who had just taken our junior-level course in mechanics, or had completed a course in introductory modern physics and differential equations, so there was no issue with their preparation. Yet, they reported that they spent an inordinate
amount of time circling around the problem without coming to a resolution, much to the
cornsternation of the the first-year students who were hoping to get some help on the problem
in our tutoring center.

What my students knew, and what they had connected to the stock of problems they had
personally solved, were two different things. It was only the presence of our Master Teacher
in Residence, Ms. Kathryn Beck of Bolsa Grande High School that saved our collective
bacon by remarking that the buoyant force is just $\rho_a V g$ both before the ballast had been
dropped and after.\textsuperscript{15} Given that this force is the same before and after leads immediately to
the observation that

$$M = m_0 g/a$$  \hfill (2)

In short, my students had “heard” of Archimedes’ principle (as I had myself), but in
never having had to use it in a physical context (this sort of problem has fallen out of
our “calculus”-based mechanics curriculum). They were quite confident in their use of free
body diagrams, but the nature of the buoyant force was not connected from the beginning
to the end of the problem.

And, once I had the benefit of Ms. Beck’s observation (she was clearly very, very familiar
with this sort of problem, having taught it many years in many guises and having charged
generations of students to construct and predict the displacement of cardboard boats) I was
left with a conundrum. There is in reality no \textbf{buoyant} force, there is only the effect of the
surrounding air pressure on the balloon. As I had drilled into my own mind in the course of
working through my own Ph.D. thesis\textsuperscript{16}, only gradients of the pressure field give forces. The
presence of a pressure gradient is the \textbf{only} way the air can affect the balloon, not through
an unsystematic “watch me pull a rabbit out of a hat” buoyant force.

\section{Pressure}

And, here is how I connect the buoyant force and the presence of a non-uniform pressure
field. The air around the balloon is in hydrostatic equilibrium. Thus, a small cube of air
with edge lengths $dx, dy, dz$ is in equilibrium when:

$$(\rho_a dx dy dz)g = P(x, y, z) dx dy - P(x, y, z + dz) dx dy$$  \hfill (3)
FIG. 1. A small cube of the ambient air at density $\rho$ is subjected to an inward force on its upper face, an upward force on its lower face, and its weight. An abstract “net force” or “free body” diagram is at right.

Here, the pressure on the top face of the box produces a downward force, and the pressure on the bottom face produces an upward force. Rearranging a bit gives a differential equation for the pressure:

$$\rho ag = -\partial_z P$$

(4)

with the familiar solution:

$$P = P_o - \rho az,$$

(5)

where $P_o$ is the pressure at $z = 0$. This treatment is elementary enough, certainly not worth writing a manuscript to share.

So, now we know the pressure field around the balloon. What is the microscopic mechanism of transferring force to the balloon from this pressure field? Again, mirroring the constructions in Ref. 1 I consider an infinitesimal element of the surface area (with normal directed outward) of the balloon, $d\vec{A}$. As in Figure 2 the air puts a force on this area element, $d\vec{F} = -Pd\vec{A}$. Note that there is a pressure field inside the balloon as well, but if we are concerned with the balloon as the system, all such internal forces cancel in pairs — a distinct advantage of the Matter and Interactions curriculum 5 is the clear demarcation between system and surroundings in every physical situation.

Thus, the total force from the air is a sum over all of these small air forces:

$$\vec{F} = \iiint -d\vec{A}P$$

(6)

This looks suspiciously like the “flux” expression in Gauss’ Law, except $\vec{F}_a$ is a vector quantity and not a scalar formed by the flux of a vector field through a closed surface.
Here is the clever bit, which generates a new (to me) theorem of vector calculus. Let us dot bot sides of Eq. 6 with \( \hat{z} \):

\[
F_z = \iiint -d\vec{A} \cdot (P\hat{z}) = \iiint dv - \nabla \cdot (P\hat{z}).
\]  

(7)

The second equality follows form the divergence theorem applied to the vector field \( P\hat{z} \).

Thus, we have

\[
F_z = - \iiint dv \partial_z P = -(\partial_z P) \iiint dv\rho_a = gM_a,
\]

(8)

where the pressure gradient in Eq. 4 is used in the second equality.

Eureka.

This gives just the \( z \) component of the total force, and in this case, as the pressure has no gradients in either the \( x \) nor \( y \) directions, this is all the analysis can tell us, although if \( P \) depends on each of \( x, y, z \) nontrivially, we have derived the the full version of Archimedes’ theorem, Eq. 1.

IV. DISCUSSION

The development of Archimedes’ Principle as a consequence of hydrostatic equilibrium, Eq. 5 and a generalization of the divergence theorem, Eq. 1 is clearly within the ability of an advanced undergraduate to use, and, in my opinion, relaxes the appearance of the Archimedes’ Principle from an ad hoc statement and grounds it solidly upon the development of general equilibrium, forces, and superposition of forces. It is altogether more clear to an advanced student why the principle is true which allows it to be used in different contexts.
The design of numerous hand-made ‘‘bob’’ accelerometers is a case in point. Here, an object less dense than water is submerged completely, yet tethered to the bottom of a clear, plastic bottle. The bob reacts quickly to an applied acceleration ... accelerate forward, and the bob immediately leans forward, and the magnitude of the acceleration can be measured by the angle of the tilt. I have seen many students fumbling with why this should occur. There is some loose talk of ‘‘water rushing backward, the bob getting out of the way,’’ and yet this is clearly an unsystematic treatment. Suppose the bottle is accelerated along the x-axis. A pressure field is immediately set up in the fluid (on the timescale it would take a shockwave to traverse the bottle) with a large pressure at the rear of the bottle, and a smaller pressure at the front. This component of the gradient of the pressure field has to match the hydrostatic equilibrium condition, Eq. 5, with an acceleration \( \vec{a} \):

\[
P = P_o + (\vec{g} - \vec{a}) \cdot \vec{r}. \tag{9}
\]

Then, an application of Eq. 1 yields

\[
\vec{F}_{\text{fluid}} = (\vec{a} - \vec{g})M_{\text{fluid}}. \tag{10}
\]

Thus, for a quiescent bottle, there is a net pressure force in the direction opposite \( \vec{g} \), but given the equivalence principle, we should be able to replace \( \vec{g} \) with an accelerating noninertial frame with acceleration equal to \(-\vec{g}\) with no observable effects. And, if we moved the whole discussion off into the International Space Station, we would have \( \vec{g} = 0 \), and to get the bob to point away form the bottom of the bottle, we would have to accelerate the bottle in the \( \hat{z} \) direction with the magnitude of the terrestrial \( g \) to get the same behavior.

A suitable probe for the reasonableness of this whole discussion is the following ‘‘clicker’’-style question. A helium-filled balloon is suspended from a thread one meter from the floor of an enclosed elevator. At the instant you cut the thread, the cable supporting the elevator is severed. While you are hurtling toward certain severe injury, you observe the balloon

- A) Rise toward the ceiling of the elevator.
- B) Remain one meter from the floor of the elevator.
- C) Fall toward the floor of the elevator.

The correct answer is B), of course, for a very physical reason that is opaque in the usual reasoning of the principle. In the freely falling elevator, the pressure field is uniform, and
hence the net force from the air on the helium balloon is zero. The balloon thus has only its weight force acting on it, and it freely falls along with the elevator. An unsophisticated reading of the problem will assume that the buoyant force is always acting, so the balloon will not only rise upward, it will rise upward faster and faster until it hits the roof of the elevator. A suitable demonstration (in a very special case) is found in dropping a water-filled container with a neutrally buoyant object floating in its midsection. If the buoyant force always acts, the ball will remain in equilibrium while the container falls, and he top of the container will fall toward the ball. Yet, in the freely falling system, the water, ball, and container all should share the same acceleration.

And further, we have a solid explanation in terms of real forces in an inertial frame for the functioning of a centrifuge. For, as the centrifuge spins, there is an acceleration field of the form $-\omega^2\mathbf{r}_\perp$. Thus, the equivalent of the hydrostatic condition is (in cylindrical coordinates)

$$\partial_r P = -\omega^2 r \rho_l$$

(11)

with the analog of Eq. (11) is then

$$F_{\text{fluid}} = \omega^2 \rho_l \int \int \int dv = \omega^2 \rho_l M R$$

(12)

where $R$ radial position of the geometric center of the displaced volume. This is exactly the position of the center of mass of the fluid that would exists had the object not been submerged in the fluid. The dependence on $R$ on $F_{\text{fluid}}$ provides an enhancement of the density-discriminating effect, crucial for the operation of a gas-centrifuge. But again, this is solely a function of adding up the pressure on each surface of an object, and determining the net force. From there, we can connect to a student’s early training in free body diagrams (and the Momentum Principle, in the language of Ref. 5).

What is going on here is extremely clear from a computational point of view. As in Figure 2 we attempt to calculate the effect of the pressure forces on the finite elements in the entire body under consideration. As the pressures are all equal in the $x$ and $y$ directions, it is just the $z$ direction that I will concentrate upon here. Each element of volume has two faces, and the pressures at neighboring surfaces give canceling forces. It is just the pressure at the top, and the pressure at the bottom of the overall column that is important, and this gradient is just adding up the increments in the weights of the fluid elements. What we have here is little more than

$$\Delta P = dxdy(P_{\text{top}} - P_{n-1} + P_{n-1} - P_{n-2} + \cdots P_1 - P_{\text{bot}})$$

(13)
with
\[(P_i - P_{i-1})dx dy = (dx dy dz) \rho g.\]  \hspace{1cm} (14)

A student asked to compute the effect of the pressure gradient will find Eq 1 as a trivial restatement of a massive internal cancellation of terms. Exactly this sort of exercise is well within the capability of students completing a computation-enriched curriculum.\(^5,10,11\)

V. CONCLUSION

Here, I have related an episode revealing a “hole” in the preparation of undergraduate students in physics at CSU Long Beach (and perhaps in many more institutions beside), and proposed a solution. The solution brings the mathematics of vector analysis back into the physics curriculum between the hiatus in electrodynamics courses (just when it would be very useful e.g. in an introduction to quantum mechanics). And, the argument provides a clear insight into how deeply Archimedes himself understood the theorems of vector calculus.\(^14\)

---

\(^4\) [Galen.Pickett@csulb.edu]

1. Harry Moritz Schey, *Div, Grad, Curl, and All That*, 4th edition (W. W. Norton, 2005).
2. Michael E. Loverude1, Christian H. Kautz1 and Paula R. L. Heron, Am. J. Phys. 71, 1178 (2003).
3. Paula R. L. Heron1, Michael E. Loverude1, P. S. Shaffer1 and L. C. McDermott, Am. J. Phys. 71, 1188 (2003);
4. George D. Kuh, *High-Impact Educational Practices: What They Are, Who Has Access to Them, and Why They Matter*, (AAC and U, 2008).
5. Ruth W. Chabay, Bruce A. Sherwood, *Matter and Interactions, vol. 1*, 3rd edition (John Wiley and Sons, 2011).
6. David E. Meltzer and Peter S. Shaffer, eds. *Teacher Education in Physics*, (American Physical Society, 2011).
7. PHYS 390 “Exploring Physics Teaching” is taken by approximately 30 physics majors annually at CSU Long Beach, and counts as upper-division elective credit in the physics major.
8. V. Otero, S. Pollock, N. Finkelstein, Am. J. of Phys. 78, 1218 (2010).
Edward F. Reddish, Am. J. Phys. 62(9), 796 (1994).

John Robert Taylor, Classical Mechanics, (University Science Books, 2005).

Herbert Goldstein, Charles P. Poole Jr., John L. Safko Classical Mechanics, 3rd edition (Pearson, 2001).

Erlend H. Graf, The Phys. Teach. 42, 296 (2004).

V. M. Rudiak Phys. Teach. 2, 293 (1964).

While there is evidence that Archimedes had developed ideas of infinitesimals in various geometric problems, he almost certainly had no concept of a vector field or vector calculus. It is a testament to his powers of insight that his principle has such deep roots in vector analysis.

We have had four TIR’s at CSU Long Beach: Rod Ziolkowski of Whitney High School in Cerritos, CA; Kathryn Beck of Bolsa Grande High School in Garden Grove, CA; Kevin Dwyer of Cypress High School in Cypress, CA; and, Meredith Ashbran of Da Vinci High School in Hawthorne, CA. Interacting with these physicists has been extremely valuable for both our students and our faculty. Generally, high school physicists have an enormous wealth of techniques and a depth of pedagogical experience that I, for one, envy.

Galen T. Pickett, J. Chem. Phys. 104, 1657 (1996);