A criterion for the properness of the $K$-energy in a general Kähler class

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Abstract In this paper, we give a criterion for the properness of the $K$-energy in a general Kähler class of a compact Kähler manifold by using Song–Weinkove’s result in (Commun Pure Appl Math 61(2):210–229, 2008). As applications, we give some Kähler classes on $\mathbb{CP}^2 \# 3\mathbb{CP}^2$ and $\mathbb{CP}^2 \# 8\mathbb{CP}^2$ in which the $K$-energy is proper. Finally, we prove Song-Weinkove’s result on the existence of critical points of $\hat{J}$ functional by the continuity method.
The behavior of the $K$-energy plays an important role in Kähler geometry. It is conjectured by Tian [27] that there exists a constant scalar curvature Kähler (cscK) metric in a Kähler class $\Omega$ if and only if the $K$-energy is proper on $\Omega$. For the Kähler-Einstein case, this was proved by Tian when $M$ has no nontrivial holomorphic vector fields. For the general case, Chen–Tian [4] showed that the $K$-energy is bounded from below if $M$ has a cscK metric. On toric manifolds, using Donaldson’s idea in [8] Zhou–Zhu [33] gave a sufficient condition on the properness of the (modified) $K$-energy on the space of invariant potentials. In a series of papers [17–19], S. Paul gave a sufficient and necessary condition on the lower boundedness and properness of the $K$-energy on the finite dimensional spaces of Bergman metrics. However, it is still difficult to analyze the behavior of the $K$-energy on general Kähler manifolds. In this paper, we give a sufficient condition for the properness of the $K$-energy in a general Kähler class on a compact Kähler manifold by using the $J$-flow.

The $J$-flow was introduced by Donaldson [7] and Chen [2] independently, and it was used to obtain the properness or the lower bound of the $K$-energy on a compact Kähler manifold with negative first Chern class. As pointed out by Chen [2], the $J$-flow is a gradient flow of the functional $\hat{J}_{\omega, \chi_0}$, which is strictly convex along any $C^{1,1}$ geodesics. Thus, if there is a critical point of $\hat{J}_{\omega, \chi_0}$ in a Kähler class, then $\hat{J}_{\omega, \chi_0}$ is bounded from below and the $K$-energy is proper when the first Chern class is negative by the formula of $K$-energy relating $\hat{J}$. Therefore, to obtain the properness of the $K$-energy it suffices to study the existence of the critical point of $\hat{J}_{\omega, \chi_0}$. In [24] Song–Weinkove gave a sufficient and necessary condition to this problem, and their result directly implies that the $K$-energy is proper on a Kähler class $[\chi_0]$ on a $n$-dimensional Kähler manifold $X$ of $c_1(X) < 0$ with the property that there is a Kähler metric $\chi' \in [\chi_0]$ such that

$$\left(-n \frac{c_1(X) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} \chi' + (n-1)c_1(X)\right) \wedge \chi^{n-2} > 0. \quad (1.1)$$

Moreover, Song–Weinkove asked whether the conclusion still holds if the inequality (1.1) is not strict. Fang et al. [10] studied the $J$-flow on the boundary of the Kähler cone.
and gave an affirmative answer in complex dimension 2. Later, Song–Weinkove [25] gave a result on the properness of the $K$-energy on a minimal surface with general type. Besides, Lejmi-Székelyhidi [16] studied the relation of the convergence of $J$-flow to a notion of stability.

To state our main results, we recall Tian’s $\alpha$-invariant for a Kähler class $[\chi_0]$:

\[
\alpha_X([\chi_0]) = \sup \left\{ \alpha > 0 \mid \exists C > 0, \int_X e^{-\alpha(\varphi - \sup \varphi)} \chi_0^n \leq C, \forall \varphi \in \mathcal{H}(X, \chi_0) \right\},
\]

where $\mathcal{H}(X, \chi_0)$ denotes the space of Kähler potentials with respect to the metric $\chi_0$. For any compact subgroup $G$ of Aut($X$), and a $G$-invariant Kähler class $[\chi_0]$, we can similarly define the $\alpha_{X,G}$ invariant by using $G$-invariant potentials in the definition.

**Theorem 1.1** Let $X$ be a $n$-dimensional compact Kähler manifold. If the Kähler class $[\chi_0]$ satisfies the following conditions for some constant $\epsilon$:

1. $0 \leq \epsilon < \frac{n+1}{n} \alpha_X([\chi_0])$,
2. $\pi c_1(X) < \epsilon [\chi_0]$,
3. \[
\left( -n \frac{\pi c_1(X) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} + \epsilon \right) [\chi_0] + (n - 1)\pi c_1(X) > 0,
\]

then the $K$-energy is proper on $\mathcal{H}(X, \chi_0)$. If instead of (1), we assume $[\chi_0]$ is $G$-invariant for a compact subgroup $G$ of Aut($X$), and $0 \leq \epsilon < \frac{n+1}{n} \alpha_{X,G}([\chi_0])$, then the $K$-energy is proper on the space of $G$-invariant potentials.

In the polarized algebraic case, our theorem translates into the following

**Corollary 1.2** Let $X$ be a $n$-dimensional compact Kähler manifold and $L$ an ample holomorphic line bundle on $X$. If there is a positive number $\epsilon > 0$ such that the following conditions hold:

1. $\alpha_X(\epsilon \pi c_1(L)) > \frac{n}{n+1}$, (or equivalently, $\alpha_X(\pi c_1(L)) > \frac{\epsilon n}{n+1}$),
2. $K_X + \epsilon L > 0$,
3. \[
\frac{nK_X \cdot L^{n-1} + \epsilon L^n}{L^n} L - (n - 1)K_X > 0,
\]

then the $K$-energy is proper on the Kähler class $\pi c_1(L)$.

A direct corollary of Theorem 1.1 is the following result, which gives a partial answer to the question of Song–Weinkove in [24] and generalize a result of Fang et al. [10] to higher dimensions.
Corollary 1.3 Let $X$ be a compact Kähler manifold with $c_1(X) < 0$. If the Kähler class $[\chi_0]$ satisfies
\[
-n \frac{c_1(X) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} - (n-1)c_1(X) \geq 0
\]
then the $K$-energy is proper on $H(X, \chi_0)$.

Note that when the strict inequality holds, our condition (1.2) is stronger than (1.1). For some technical reasons, we cannot weaken (1.2) to Song–Weinkove’s original condition. Moreover, in Theorem 1.1 we don’t require the condition that $c(X) < 0$. In the case of $c_1(X) > 0$, we have the following result. Although this result is very simple and its original proof is very direct (see, for example, Tian’s book [27], page 95), we feel it is still interesting to write it as a corollary of Theorem 1.1.

Corollary 1.4 Let $X$ be a compact Kähler manifold with $c_1(X) > 0$. If the $\alpha$-invariant $\alpha_X(\pi c_1(X)) > \frac{n}{n+1}$, then the $K$-energy is proper on the Kähler class $\pi c_1(X)$.

An application of Corollary 1.3 is the following result, which says that the properness of the $K$-energy is not a sufficient condition of the smooth convergence of the $J$-flow. This answers a question of Ross [21]. This result is also implied by Corollary 1.2 of [10].

Corollary 1.5 There exists a compact Kähler surface with a Kähler class $\Omega_1$ such that the $K$-energy is proper on $\Omega_1$ but the $J$-flow doesn’t converge smoothly.

One might ask whether the conditions of Theorem 1.1 are optimal. We calculate two concrete examples here and apply Theorem 1.1 to determine the Kähler classes on which the $K$ energy is proper. Let $X$ be the blowup of $\mathbb{CP}^2$ at three general points and $E_1, \ldots, E_3$ the exceptional divisors of the blowing up map. Denote by $F_i$, $i = 1, 2, 3$ the strict transforms of lines through two of the three blowing up centers. Consider the class $L_\lambda = (E_1 + E_2 + E_3) + \lambda (F_1 + F_2 + F_3)$ for a positive rational number $\lambda$. Then applying Theorem 1.1, if $\lambda$ satisfies
\[
\frac{5}{6} < \lambda < \frac{6}{5},
\]
then the $K$-energy is proper on $\pi c_1(L_\lambda)$. The details are contained in Sect. 4. This example was also studied by Zhou–Zhu [33]. They analyzed the expression of the $K$-energy carefully on toric manifolds and showed that the $K$-energy is is proper on $G$-invariant metrics if
\[
0.61 \approx \frac{1}{1 + \frac{\sqrt{10}}{5}} < \lambda < 1 + \frac{\sqrt{10}}{5} \approx 1.63.
\]

We see that the conditions (1.4) is less restrictive than (1.3). Therefore, the conditions in Theorem 1.1 is not sharp.
A criterion for the properness of the $K$-energy

Theorem 1.1 shows that the $K$-energy and $\alpha$-invariants are closely related. Dervan [6] proved that the $\alpha$ invariant also closely relates the $K$-stability. Dervan [6] studied the $\alpha$-invariant and $K$-stability for general polarizations on Fano manifolds. An example of [6] is the blowup of $\mathbb{CP}^2$ at eight points in general positions. Let $E_i$ be the exceptional divisors, and $L_\lambda = 3H - \sum_{i=1}^7 E_i - \lambda E_8$, where $\lambda$ is a positive rational number. Dervan proved that when

\[ 0.76 \approx \frac{1}{9} (10 - \sqrt{10}) < \lambda < \sqrt{10} - 2 \approx 1.16, \] (1.5)

$(X, L_\lambda)$ is $K$-stable. Applying Theorem 1.1 to this example, we know that the $K$-energy is proper on $\pi c_1(L_\lambda)$ if

\[ \frac{4}{5} < \lambda < \frac{10}{9}. \] (1.6)

Note that the interval (1.6) is strictly contained in (1.5). According to Tian’s conjecture and general Yau et al. conjecture, this example hints that the conditions in Theorem 1.1 is not optimal. There are some overlap regions between our results and Dervan’s for the properness of the $K$-energy and $K$-stability. All these results provide some support to general Yau et al. conjecture for general cscK metrics.

Finally, we reprove the existence of the critical point of $\hat{J}$ functional under the condition (2.4) by the continuity method. Fang et al. [9] discussed a class of fully nonlinear flows in Kähler geometry, which includes the $J$-flow as a special case. In Remark 1.3 of [9], they asked whether the critical points of those fully nonlinear flows can be solved by using the elliptic method instead of the geometric flow method. Later, in a series of papers [13–15] Guan and his collaborators gave some $C^2$ estimates for these critical points on Hermitian manifolds. Then Sun proved Song–Weinkove’s result by the elliptic method on general Hermitian manifolds [26]. Here we give a different proof only using the estimates in [24,30,31].

**Theorem 1.6** (cf. [24]) *If there is a metric $\chi' \in [\chi_0]$ satisfying*

\[ (nc\chi' - (n - 1)\omega) \wedge \chi'^{n-2} > 0, \]

*then there is a smooth Kähler metric $\chi \in [\chi_0]$ satisfying the equation*

\[ \omega \wedge \chi^{n-1} = c\chi^n. \]

In a forthcoming paper, we will generalize the results in this paper to the properness of the log $K$-energy and discuss the existence of critical points of the $J$ flow with conical singularities. Our method can also be applied to study the modified $K$-energy for extremal Kähler metrics and we will discuss this elsewhere.
2 Preliminaries

In this section, we recall some basic facts on \( J \)-flow. We follow the notations in [24]. Let \( (X, \omega) \) be a \( n \)-dimensional compact Kähler manifold with a Kähler form \( \omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \), and \( \chi_0 \) another Kähler form on \( X \). We denote by \( \mathcal{H}(X, \chi_0) \) the space of Kähler potentials

\[
\mathcal{H}(X, \chi_0) = \left\{ \varphi \in C^\infty(X, \mathbb{R}) \mid \chi_\varphi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0 \right\}.
\]

The \( J \)-flow is defined by

\[
\frac{\partial \varphi}{\partial t} = c - \frac{\omega \wedge \chi_\varphi^{n-1}}{\chi_\varphi^n}, \quad \varphi|_{t=0} = \varphi_0 \in \mathcal{H},
\]

where \( c \) is a constant defined by

\[
c = \frac{[\omega] \cdot [\chi_0]^{n-1}}{[\chi_0]^n}.
\]

A critical point of \( J \)-flow is a Kähler form \( \chi \) satisfying

\[
\omega \wedge \chi^{n-1} = c \chi^n.
\]

Donaldson [7] showed that a necessary condition for the existence of the critical metrics (2.3) is \( [c \chi_0 - \omega] > 0 \), and Chen [2] showed that it is also sufficient in complex dimension 2. In a series of papers [24,30] Weinkove and Song–Weinkove obtain a sufficient and necessary condition for any dimension:

**Theorem 2.1** (cf. [24]) *The following conditions are equivalent:*

1. *There is a metric \( \chi' \in [\chi_0] \) satisfying*

\[
(nc \chi' - (n-1)\omega) \wedge \chi^{n-2} > 0.
\]

2. *For any initial data \( \varphi_0 \in \mathcal{H} \), the \( J \)-flow (2.1) converges smoothly to \( \varphi_\infty \in \mathcal{H} \) with the limit metric \( \chi_\infty \) satisfying (2.3).*

3. *There is a smooth Kähler metric \( \chi \in [\chi_0] \) satisfying the Eq. (2.3).*

The convergence of the \( J \)-flow can be used to determine in which Kähler class the \( K \)-energy is proper or bounded from below. Note that the \( J \)-flow is the gradient flow of the functional

\[
\hat{J}_{\omega, \chi_0}(\varphi) = \int_0^1 \int_X \frac{\partial \varphi}{\partial t} \left( \omega \wedge \chi_\varphi^{n-1} - c \chi_\varphi^n \right) \frac{dt}{(n-1)!}.
\]
When $\omega$ is a positive $(1, 1)$ form, the functional $\hat{J}_{\omega, \chi_0}$ is strictly convex along any $C^{1,1}$ geodesics (cf. [2]). Therefore, under the Assumption (1) of Theorem 2.1, the critical point of $\hat{J}_{\omega, \chi_0}$ exists and $\hat{J}_{\omega, \chi_0}$ is bounded from below. When $c_1(X) < 0$, we can choose $\omega = -\text{Ric}(\chi_0) > 0$, then the $K$-energy can be written as

$$\mu_{\chi_0}(\varphi) = \int_X \log \frac{\chi^n_{\varphi}}{\chi^n_{\varphi_0}} + \hat{J}_{\omega, \chi_0}(\varphi). \quad (2.6)$$

where $\omega = -\text{Ric}(\chi_0) > 0$. Since the first term of this formula is always proper (see Lemma 4.1 of [24]), one can conclude that the $K$-energy is proper on $[\chi_0]$, provided (2.4) holds for $[\omega] = -\pi c_1(X)$.

3 Proofs of Theorem 1.1 and Corollary 1.3–1.5

In this section, we prove Theorem 1.1 and Corollary 1.3–1.5.

**Proof of Theorem 1.1** We focus on the $G = \{1\}$ case, the proof in the general case is identical. Recall the Aubin–Yau functionals

$$I_{\chi_0}(\varphi) = \int_X \varphi \left( \frac{\chi^n_\varphi}{n!} - \frac{\chi^n_{\varphi_0}}{n!} \right),$$

$$J_{\chi_0}(\varphi) = \int_0^1 dt \int_X \frac{\partial \varphi}{\partial t} \left( \frac{\chi^n_\varphi}{n!} - \frac{\chi^n_{\varphi_0}}{n!} \right).$$

Direct calculation shows that

$$I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) = -\int_0^1 dt \int_X \frac{\partial \varphi}{\partial t} \Delta_{\chi_\varphi} \varphi \frac{\chi^n_\varphi}{n!}$$

$$= -\int_0^1 \int_X \frac{\partial \varphi}{\partial t} \left( \chi^n_\varphi - \chi_0 \wedge \chi_{\varphi}^{n-1} \right) \frac{dt}{(n-1)!}.$$ 

By the assumption (2), there exists $\chi_1 \in [\chi_0]$ such that $\omega_1 := -\text{Ric}(\chi_1)$ satisfies

$$\omega_1 + \epsilon \chi_0 > 0. \quad (3.1)$$

By the Assumption (3), there exists $\tilde{\chi}', \tilde{\chi} \in [\chi_0]$ such that

$$(nc + \epsilon) \tilde{\chi}' - (n - 1) \tilde{\omega} > 0,$$
\[ \hat{\omega} := -Ric(\tilde{\chi}) \] and
\[ c = -\pi c_1(X) \cdot [\chi_0]^{n-1} / [\chi_0]^n. \]

Note that \( c + \epsilon > 0 \) by assumption (2). Set
\[ \chi' := \frac{1}{n(c + \epsilon)} \left( (nc + \epsilon) \tilde{\chi}' + (n - 1)\epsilon \chi_0 + (n - 1) \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \chi_1^n \right). \]

Then \( \chi' \in [\chi_0] \) and \( \chi' \) satisfies
\[ n(c + \epsilon) \chi' - (n - 1)(\omega_1 + \epsilon \chi_0) = (nc + \epsilon) \tilde{\chi}' - (n - 1)\hat{\omega} > 0. \] (3.2)

In particular, \( \chi' > 0 \). We define the modified \( \hat{J} \) functional by
\[ \hat{J}_{\omega_1, \chi_0}^\epsilon(\varphi) = \hat{J}_{\omega_1, \chi_0}(\varphi) + \epsilon \left( I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) \right) \]
\[ = \int_0^1 \int_X \frac{\partial \varphi}{\partial t} \left( (\omega_1 + \epsilon \chi_0) \wedge \chi_0^{n-1} - (c + \epsilon) \chi^n \right) \frac{dt}{(n - 1)!}. \]

which is exactly the functional \( \hat{J}_{\omega_1 + \epsilon \chi_0, \chi_0} \) defined by (2.5). Thus, by Chen’s result in [2] if there is a Kähler metric \( \chi \) satisfying
\[ (\omega_1 + \epsilon \chi_0) \wedge \chi^{n-1} = (c + \epsilon) \chi^n, \] (3.3)

then \( \hat{J}_{\omega_1, \chi_0}^\epsilon \) is bounded from below on \( \mathcal{H}(X, \chi_0) \). By Theorem 2.1 the critical metric (3.3) exists if there exists a Kähler metric \( \chi' \in [\chi_0] \) such that
\[ (n(c + \epsilon) \chi' - (n - 1)(\omega_1 + \epsilon \chi_0)) \wedge \chi'^{n-2} > 0. \] (3.4)

Clearly, (3.4) can be implied by (3.2). Therefore, if (3.1) and (3.2) hold, then \( \hat{J}_{\omega_1, \chi_0}^\epsilon \) is bounded from below on \( \mathcal{H}(X, \chi_0) \) and we have
\[ \hat{J}_{\omega_1, \chi_0}(\varphi) \geq -\epsilon \left( I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi) \right) - C, \quad \forall \varphi \in \mathcal{H}(X, \chi_0). \] (3.5)

Next, we claim that for \( \omega := -Ric(\chi_0) \), there is a constant \( C(\chi_1, \chi_0) \) such that for any \( \varphi \in \mathcal{H}(X, \chi_0) \)
\[ |\hat{J}_{\omega, \chi_0}(\varphi) - \hat{J}_{\omega_1, \chi_0}(\varphi)| \leq C(\chi_1, \chi_0). \] (3.6)
In fact, by the explicit expression of the $\hat{J}$ functional from [2] we have

$$\hat{J}_{\omega_0, \chi_0}(\varphi) = \sum_{p=0}^{n-1} \frac{1}{(p+1)!(n-p-1)!} \int_X \varphi \omega \wedge \chi_0^{n-p-1} \wedge (\sqrt{-1} \partial \bar{\partial} \varphi)^p$$

$$- nc \int_0^1 dt \int_X \frac{\partial \varphi_t}{\partial t} \frac{\chi_{\varphi_t}^{n}}{n!}$$

$$= \sum_{p=0}^{n-1} c_p \int_X \varphi \omega \wedge \chi_0^{n-p-1} \wedge (\chi_{\varphi} - \chi_0)^p - nc \int_0^1 dt \int_X \frac{\partial \varphi_t}{\partial t} \chi_{\varphi_t}^{n}/n!$$

$$= \sum_{p=0}^{n-1} c'_p \int_X \varphi \omega \wedge \chi_0^{n-p-1} \wedge \chi_{\varphi}^p - nc \int_0^1 dt \int_X \frac{\partial \varphi_t}{\partial t} \chi_{\varphi_t}^{n}/n!,$$

where $c_p, c'_p$ are universal constants. Since $\omega_1 - \omega = -\sqrt{-1} \partial \bar{\partial} f$ where $f = \log \frac{\chi_{\varphi}^{n}}{\chi_0^{n}},$

we have

$$|\hat{J}_{\omega_0, \chi_0}(\varphi) - \hat{J}_{\omega_1, \chi_0}(\varphi)| \leq \left| \sum_{p=0}^{n-1} c'_p \int_X \varphi \sqrt{-1} \partial \bar{\partial} f \wedge \chi_0^{n-p-1} \wedge \chi_{\varphi}^p \right|$$

$$= \sum_{p=0}^{n-1} c'_p \int_X f (\chi_{\varphi} - \chi_0) \wedge \chi_0^{n-p-1} \wedge \chi_{\varphi}^p \leq C \|f\|_{C^0}.$$

Therefore, (3.6) is proved.

Now using Tian’s $\alpha$-invariant we have (see Lemma 4.1 of [24])

$$\int_X \log \frac{\chi_{\varphi}^{n}}{\chi_0^{n}} \frac{\chi_{\varphi}^{n}}{n!} \geq \alpha I_{\chi_0}(\varphi) - C$$

$$\geq \frac{n+1}{n} \alpha \cdot (I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi)) - C, \quad \forall \varphi \in \mathcal{H} \quad (3.7)$$

for any $\alpha \in (0, \alpha_{[\chi_0]}(X))$. Combining the inequalities (3.5)–(3.7) we have

$$\mu_{\chi_0}(\varphi) = \int_X \log \frac{\chi_{\varphi}^{n}}{\chi_0^{n}} \frac{\chi_{\varphi}^{n}}{n!} + \hat{J}_{\omega_0, \chi_0}(\varphi)$$

$$\geq \int_X \log \frac{\chi_{\varphi}^{n}}{\chi_0^{n}} \frac{\chi_{\varphi}^{n}}{n!} + \hat{J}_{\omega_1, \chi_0}(\varphi) - C(\chi_0, \chi_1)$$

$$\geq \left( \frac{n+1}{n} \alpha - \epsilon \right) (I_{\chi_0}(\varphi) - J_{\chi_0}(\varphi)) - C.$$
Therefore, if $\alpha_X(\{\chi_0\}) > \frac{n}{n+1}\epsilon$ then the $K$ energy is proper. \hfill \Box

**Remark 3.1** We see from the above proof that the $K$-energy is proper if (3.1) and (3.4) hold. However, we are unable to show that (3.4) holds if there exists a Kähler metric $\chi' \in [\chi_0]$ such that

$$\left(-n\pi c_1(X) \cdot [\chi_0]^{n-1} \chi' - (n - 1)\omega + \epsilon \chi' \right) \wedge \chi'^{n-2} > 0.$$ 

If it were true, then Song-Weinkove’s question would be answered as a corollary.

**Proof of Corollary 1.3** Let $\epsilon \in (0, \frac{n+1}{n}\alpha_X(\{\chi_0\}))$. Since the manifold $X$ has negative first Chern class, the condition (2) in Theorem 1.1 automatically holds. The third condition in Theorem 1.1 follows directly from (1.2). Thus, the corollary is proved. \hfill \Box

**Proof of Corollary 1.4** Let $[\chi_0] = \pi c_1(X)$. By the assumption $\alpha_X(\{\chi_0\}) > \frac{n}{n+1}$, we can choose $\epsilon \in (1, \frac{n+1}{n}\alpha_X(\{\chi_0\}))$. Therefore, the three conditions of Theorem 1.1 hold and the corollary is proved. \hfill \Box

**Proof of Corollary 1.5** In complex dimension 2, the $J$ flow converges smoothly if and only if the inequality

$$-2\frac{c_1(X) \cdot [\chi_0]^{n-1}}{[\chi_0]^n} \chi' + c_1(X) > 0 \quad (3.8)$$

holds for some $\chi' \in [\chi_0]$. By Corollary 1.4, the $K$-energy is still proper on $\mathcal{H}(X, \chi_0)$ if the inequality (3.8) is not strictly. Therefore, on any Kähler class lying on the boundary of the cone defined by (3.8), the $K$-energy is proper but the $J$-flow doesn’t converge smoothly. \hfill \Box

%4 Toric varieties

In this section, we apply Theorem 1.1 to projective toric manifolds. First we recall some basic facts on toric varieties from Fulton’s book [11]. Let $N$ be a lattice of rank $n$, $M = \text{Hom}(N, \mathbb{Z})$ is its dual. The complex torus group $T$ is defined to be $T = N \otimes \mathbb{Z} \mathbb{C}^* = \text{Hom}(M, \mathbb{C}^*)$, and we have the real torus group $T_R = N \otimes \mathbb{R} S^1 = \text{Hom}(M, S^1)$. A complete toric variety $X_{\Delta}$ is defined by a fan $\Delta$ in $N_R = N \otimes \mathbb{R}$, consists of strongly convex rational polyhedral cones, such that the union of these cones is the whole of $N_R$. We assume each cone is generated by a subset of a $\mathbb{Z}$-basis of $N$, then $X_{\Delta}$ is smooth. Each 1-dimensional cone $\rho_i$ of $\Delta$ corresponds to an irreducible divisor $D_i$ and it is well known that the Picard group of $X_{\Delta}$ is generated by these $D_i$’s.

Let $u_i \in N$ be the primitive generator of the 1-dimensional cone $\rho_i$. For any $T$-invariant divisor $D = \sum_i a_i D_i, a_i \in \mathbb{Z}$, we associate to it a piecewise linear function $\phi_D$ on $N_R$, defined by

\[ \phi_D(x) = \sum_i a_i \langle x, u_i \rangle \]
The divisor $D$ is ample if and only if $\phi_D$ is strictly concave\(^1\) in the sense that for any $n$-dimensional cone $\sigma$ of $\Delta$, the graph of $\phi_D$ on the compliment of $\sigma$ is strictly under the graph of the linear function $u_\sigma$ whose restriction on $\sigma$ equals $\phi_D$. For such a $D$, we can also associate to it a polytope $P_D$ in $M_\mathbb{R}$, defined by

$$P_D := \{ m \in M_\mathbb{R} \mid \langle m, v \rangle \geq \phi_D(v), \forall v \in N_\mathbb{R}\}.$$

We have

$$H^0(X_\Delta, \mathcal{O}(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{C}\chi^m,$$

where $\chi^m$ is the rational function on $X_\Delta$ defined by $m \in M$. If we restrict $\chi^m$ to $\mathbb{C}^n$, the $\chi^m$ has the form $z_1^{m_1} \cdots z_n^{m_n}$.

Besides the natural $T$-action, the toric manifold $X_\Delta$ also has some discrete symmetries from the fan. Let

$$\mathcal{W} = \{ g \in GL(n, \mathbb{Z}) \mid g \text{ preserves } \Delta \}.$$

Since each $g \in \mathcal{W}$ is decided by a permutation of $\{u_i\}$, so $\mathcal{W}$ is finite. Every $g$ induces a $\tilde{g} \in \text{Aut}(X)$. For a given ample divisor $D$, we consider the subgroup of $\mathcal{W}$ preserving the class of $D$:

$$K_D := \{ g \in \mathcal{W} \mid \tilde{g}^* D \sim D \},$$

where “$\sim$” means “linearly equivalent”. Note that $\tilde{g}^* D$ corresponds to the function $\phi_D \circ g$, so we have a combinatorial characterization of $K_D$:

$$K_D = \{ g \in \mathcal{W} \mid \exists m \in M, \text{ s.t. } \phi_D \circ g = \phi_D + \langle m, \cdot \rangle \}.$$

Let $G$ be the compact subgroup of $\text{Aut}(X)$ generated by $T_\mathbb{R}$ and $K_D$, we compute the $\alpha$-invariant $\alpha_G(\pi[D])$, extending previous works of Song [23] and Cheltsov–Shramov [1] in the toric Fano case.

First, we make a normalization: assume the barycenter of $P_D$ is the origin of $M_\mathbb{R}$. This is equivalent to taking power of the line bundle $\mathcal{O}(D)$ and change the divisor in its linear equivalent class. Since we are computing the $\alpha$-invariant, this does not lose any generality. Under this assumption, for any $g \in K_D$, the induced linear action $g^*$ on $M_\mathbb{R}$ preserves $P_D$. Actually, by definition of $K_D$, $g^* P_D$ is a translation of $P_D$. However, since $g^*$ is linear, the barycenter of $g^* P_D$ is also the origin. This implies $g^* P_D = P_D$. Now the result is:

\(^1\) In Fulton’s terminology, it is called “convex”. We choose this name according to the usual notation.
Theorem 4.1 The $\alpha_G$ invariant equals
\[
\alpha_G(\pi[D]) = \min_{i} \min_{y \in P_D^K} \frac{1}{y \cdot u_i + a_i},
\]
where $P_D^K$ is the set of $K_D$ (induced action on $M_\mathbb{R}$) fixed points in $P_D$.

Proof To prove this, we use the result of J.-P. Demailly [5], saying that
\[
\alpha_G(\pi[D]) = \inf_{k \in \mathbb{Z}^+} \inf_{|\Sigma| \subset |kD|} \lct \left( \frac{1}{k} |\Sigma| \right)
\]
where the second infimum is taken for all $G$ invariant linear systems. If $|\Sigma_1| \subset |\Sigma_2| \subset |kD|$, obviously $\lct \left( \frac{1}{k} |\Sigma_1| \right) \leq \lct \left( \frac{1}{k} |\Sigma_2| \right)$, so we only need take all $G$ invariant and irreducible linear systems $|\Sigma|$.

Note that $H^0(X, kD) = \text{span}\{s_m | m \in kP_D \cap M\}$ is just the decomposition into one dimensional invariant subspaces of the $T$-action, and the torus acts on these lines with different characters. Take a $G$ invariant linear system $|\Sigma|$ and $m \in kP_D \cap M$, then $s = \bigotimes_{m \in \Gamma} s_m \in H^0(X, kND)$, and it spans a one dimensional $G$-invariant linear system. Moreover, we have $\lct \left( \frac{1}{kN} (s) \right) \leq \lct \left( \frac{1}{k} |\Sigma| \right)$ by Hölder inequality. So without loss of generality, we can take $\Sigma$ to be one dimensional in the following.

Let $s \in H^0(X, kD)$. Assume $s$ corresponds to the lattice point $m \in kP_D \cap M$, and $m$ is fixed by $K_D$. Now the divisor of $s$ is given by
\[
(s) = \sum_i (m \cdot u_i + ka_i)D_i.
\]
Since $X$ is smooth, $D_i$'s have simple normal crossing intersections with each other. So we have
\[
\lct \left( \frac{1}{k} (s) \right) = \min_i \frac{k}{m \cdot u_i + ka_i}.
\]
From this and the above discussion, we have
\[
\alpha_G(\pi[D]) = \inf_{k \in \mathbb{Z}^+} \inf_{\frac{m}{k} \in P_D \cap \frac{1}{k} M} \min_i \frac{1}{\frac{m}{k} \cdot u_i + a_i} = \min_{i} \min_{y \in P_D^K} \frac{1}{y \cdot u_i + a_i}.
\]

Now we consider a concrete example: the blowup of $\mathbb{C}P^2$ at three general points.

The fan of $M$ is generated by the following primitive vectors:
\[
u_1 = (1, 0), \ nu_2 = (1, 1), \ nu_3 = (0, 1), \ nu_4 = (0, -1), \ nu_5 = (-1, -1), \ nu_6 = (0, -1).
\]
They correspond to all the \((-1)\) - curves \(D_1, \ldots, D_6\) on \(M\). The intersection numbers satisfy:

\[ D_i \cdot D_i = -1, \quad D_i \cdot D_{i+1} = 1, \quad i = 1, \ldots, 6, \]

where \(D_7 = D_1\) is understood. Note also that the anti-canonical divisor is given by \(K^{-1} = D_1 + \cdots + D_6\). The bundle we choose is \(L_{\lambda} = (D_1 + D_3 + D_5) + \lambda(D_2 + D_4 + D_6)\), where \(\lambda \in \mathbb{Q}\). This is the same class as we mentioned in the introduction. Actually if we blow down \(D_1, D_3\) and \(D_5\), we will get \(\mathbb{C}P^2\), and the image of \(D_2, D_4, D_6\) are lines.

**Proposition 4.2** Under the above assumptions, if \(\frac{5}{6} < \lambda < \frac{6}{5}\) then the \(K\)-energy is proper on the space of \(G\)-invariant potentials for the class \(\pi c_1(L_{\lambda})\).

**Proof** For this class to be ample, we need the function \(\phi_{L_{\lambda}}\) to be strictly concave, and this is easily seen to be

\[ \frac{1}{2} < \lambda < 2. \]

Now we apply our theorem to this case. Write \(D = aL_{\lambda}\) for some positive \(a\). The \(\alpha\)-invariant can be computed using our Theorem 4.1: Since the elements of \(\mathcal{K}_D\) can be enumerated, one can check easily that \(P^K_D = \{ 0 \in M_{\mathbb{R}} \}\). So

\[ \alpha_G(\pi[D]) = \min \left\{ \frac{1}{a}, \frac{1}{\lambda a} \right\}. \]

So the condition \(\alpha_G > \frac{2}{3}\) translates to \(0 < a < \frac{3}{2}, 0 < a < \frac{3}{2\lambda}\). Similarly, by considering the concexity of \(\phi_{K+D}\), we can translate the condition \(K + D > 0\) into

\[ (2 - \lambda)a > 1, \quad (2\lambda - 1)a > 1. \]

The last condition says that

\[ (1 - R)D + K^{-1} > 0, \]

where \(R\) is the mean value of the scalar curvature of the class \(\pi[D]\). Direct computation\(^2\) shows that

\[ R = \frac{2(1 + \lambda)}{a\lambda(4\lambda - 1 - \lambda^2)}. \]

\(^2\) Just compute the intersection number, or use the fact that \(R = \frac{Vol(\partial P_L)}{Vol(Y_L)}\), where \(Vol(\partial P_L)\) is computed using Donaldson’s special boundary measure, see [8] and [33].
Again, use the piecewise linear function $\phi\left((1-R)D-K\right)$, we get the condition

$$\frac{2(\lambda + 1)}{4\lambda - 1 - \lambda^2} + \frac{1}{\lambda - 2} < a, \quad \frac{2(\lambda + 1)}{4\lambda - 1 - \lambda^2} + \frac{1}{1 - 2\lambda} < a.$$ 

In conclusion, we have

$$\frac{1}{2} < \lambda < 2,$$

and

$$\max\left\{ \frac{1}{2-\lambda}, \frac{1}{2\lambda - 1}, \frac{2(\lambda + 1)}{4\lambda - 1 - \lambda^2} + \frac{1}{\lambda - 2}, \frac{2(\lambda + 1)}{4\lambda - 1 - \lambda^2} + \frac{1}{1 - 2\lambda} \right\} < a$$

$$< \min\left\{ \frac{3}{2}, \frac{3}{2\lambda} \right\}.$$ 

Then for any $\frac{5}{6} < \lambda < \frac{6}{5}$, we can find a suitable $a$ satisfying these conditions. So we can apply Theorem 1.1 (or Corollary 1.2) to conclude that the $K$-energy is proper on the space of $G$-invariant potentials for the class $\pi c_1(aL_\lambda)$ and hence on $\pi c_1(L_\lambda)$. □

### 5 An example of Dervan

In this section, we would like to compare our result with that of Dervan [6] on the $\alpha$-invariant and K-stability for general polarizations on Fano manifolds. His sufficient condition involves the quantity

$$\mu(X, L) := \frac{-K_X \cdot L^{n-1}}{L^n}.$$ 

And he proved that if $\alpha_X(L) > \frac{n}{n+1} \mu(X, L)$ and $-K_X \geq \frac{n}{n+1} \mu(X, L)L$, then $(X, L)$ is $K$-stable. This condition is quite similar to ours, but at present we don’t know whose condition is stronger.

We know compare our result with that of Dervan on a concrete example, the Del Pezzo surface $X$ of index one. It is the blowup of $\mathbb{C}P^2$ at eight points in general positions. Let $E_i$ be the exceptional divisors, and $L_\lambda = 3H - \sum_{i=1}^{7} E_i - \lambda E_8$, then $L_1 = -K_X$ and we set $L = aL_\lambda$, where $a$ and $\lambda$ are both positive and $\lambda \in \mathbb{Q}$. We know the exceptional curves on $X$ are $E_i$’s and the strictly transforms of following curves in $\mathbb{C}P^2$:

- lines through 2 points,
- conics through 5 points,
- cubics through 7 points, vanishing doubly at 1 of them,
- quartics through 8 points, vanishing doubly at 3 points,
- quintics through 8 points, vanishing doubly at 6 points,
- sextics through 8 points, vanishing doubly at 7 points and triply at another point.
**Proposition 5.1** Under the above assumptions, if \( \frac{4}{5} < \lambda < \frac{10}{9} \) the K-energy is proper on the Kähler class \( \pi c_1(L_\lambda) \).

**Proof** By Kleiman’s ampleness criterion, we know that \( L_\lambda \) is ample when \( \lambda < \frac{4}{3} \). According to Dervan [6], we have

\[
\alpha(\pi[L_\lambda]) \geq \min \left\{ 1, \frac{1}{2-\lambda} \right\}.
\]

Now we look for the \( \lambda \) for which the class \( L = aL_\lambda \) satisfies our conditions for some \( a > 0 \). The condition on the \( \alpha \)-invariant gives

\[
\frac{1}{a} \min \left\{ 1, \frac{1}{2-\lambda} \right\} > \frac{2}{3}.
\]

For the ampleness of \( L + K \), by computing the intersection number with the above exceptional curves we have

\[
a > \frac{1}{4 - 3\lambda}, \text{ when } \lambda \geq 1,
\]

and

\[
a > \frac{1}{\lambda}, \text{ when } \lambda < 1.
\]

For the last condition, denote \( b = \frac{4-2\lambda}{2-\lambda^2} - a \), we require the divisor \( 3(1 - b)H - \sum_{i=1}^{7} (1 - b)E_i - (1 - \lambda b)E_8 \) to be ample. This is equivalent to the conditions

\[
b < \frac{1}{\lambda}, \text{ when } \lambda \geq 1,
\]

and

\[
b < \frac{1}{4 - 3\lambda}, \text{ when } \lambda < 1.
\]

From these inequalities, now we can conclude that when \( \frac{4}{5} < \lambda < \frac{10}{9} \), the K-energy is proper on the Kähler class \( 3H - \sum_{i=1}^{7} E_i - \lambda E_8 \). \( \square \)

**6 Existence of critical points by the continuity method**

In this section, we prove Theorem 1.6 by the continuity method by using the estimates of [24,30,31].

\[\text{Springer}\]
Proof of Theorem 1.6  By the assumption, we can choose the reference metric $\chi_0$ as $\chi'$ in (2.4). Consider the continuity equation

$$\omega_t \wedge \chi_t^{n-1} = c_t \chi_t^n,$$  \hspace{1cm} (6.1)

where

$$\chi_t = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi_t, \quad \omega_t = (1 - t) \chi_0 + t \omega.$$  

Clearly, $\omega_0 = \chi_0$ and $\omega_1 = \omega$. The constant $c_t$ is given by

$$c_t = \frac{[\omega_t] \cdot [\chi_0]^{n-1}}{[\chi_0]^n} = (1 - t) + c \cdot t,$$  \hspace{1cm} (6.2)

where the constant $c$ is given by (2.2). We denote by $g_t$ the corresponding metric tensor of the Kähler form $\omega_t$. Clearly, $\varphi = 0$ is a solution to the Eq. (6.1) when $t = 0$. Define the set

$$S = \{ s \in [0, 1] \mid \text{the Eq. (6.1) has a solution when } t = s \}.$$  

We first prove the openness of $S$. Consider the operator $L_t(\varphi) = \chi_\varphi^{i\bar{j}} g^{i\bar{j}} : \mathcal{U} \to C^\alpha_0$, where

$$\mathcal{U} := \left\{ \varphi \in C^{2,\alpha}(X; \mathbb{R}) \mid \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0 \right\} / \mathbb{R},$$

and

$$C^\alpha_0 := C^\alpha(X; \mathbb{R}) / \mathbb{R}.$$  

Here we take the quotient space norm, and the Hölder semi-norms are taken with respect to a fixed metric, say, $\chi_0$. For any $t_0 \in S$, the linearization of the operator $L_t(\varphi)$ at $(t_0, \varphi_{t_0})$, $DL|_{(t_0, \varphi_{t_0})} : C^{2,\alpha}_0 \to C^\alpha_0$ is given by

$$DL|_{(t_0, \varphi_{t_0})}(f) = -h^{p \bar{q}}_{t_0} f_{p \bar{q}}, \quad h^{p \bar{q}}_t = \chi_t^{i \bar{q}} \chi_t^{p \bar{j}} g_t^{i \bar{j}}.$$  

Here $C^{2,\alpha}_0 := C^{2,\alpha}(X; \mathbb{R}) / \mathbb{R}$. It is obvious to see that $DL|_{(t_0, \varphi_{t_0})}$ (We write $DL$ for short.) is elliptic. By strong maximum principle, its kernel in $C^{2,\alpha}_0$ is trivial. We claim

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that $DL$ is a self-adjoint operator in $L^2$. Actually, for any real-valued smooth function $\eta$, we have

$$ (DL(f), \eta)_{L^2} = \int_X \eta DL(f) \frac{\chi^n}{n!} = -\int_X \eta \chi^{i\overline{j}} DL f_{i\overline{j}} \frac{\chi^n}{n!} $$

$$ = \int_X \eta_p \chi^{i\overline{j}} f_{i\overline{j}} \frac{\chi^n}{n!} + \int_X \eta \chi^{i\overline{j}} \chi^{p\overline{q}} f_{i\overline{j}} \frac{\chi^n}{n!} $$

Here all the covariant derivatives are taken with respect to $\chi_0$. Observe that $\chi^{i\overline{j}} f_{i\overline{j}} = c_{t_0}$, so $\chi^{i\overline{j}} g_{i\overline{j},p} = 0$. Since both $\omega_{t_0}$ and $\chi_0$ are Kähler, we have $g_{i\overline{j},p} = g_{i\overline{j},p}$ by comparing the expressions in local coordinates. So $\chi^{p\overline{q}} f_{i\overline{j}} = 0$. We have

$$ (DL(f), \eta)_{L^2} = \int_X \eta_p \chi^{i\overline{j}} f_{i\overline{j}} \frac{\chi^n}{n!} $$

Then another integration by parts gives the result: $(DL(f), \eta)_{L^2} = (f, DL(\eta))_{L^2}$.

Now we conclude that the kernel of the adjoint of $DL$ is also trivial (modulo constant functions), and hence $DL|_{(t_0, \varphi_{t_0})}$ is invertible. By standard inverse function theorem, the set $S$ is open.

Since $0 \in S$, by the openness there exists $t_1 > 0$ such that (6.1) has a smooth solution for $t \in [0, t_1)$. By Lemmas 6.1 and 6.2 below, for any $t \in S$ the metric $\chi_t$ is uniformly equivalent to $\chi_0$. By the elliptic estimates in [12] we can get $C^{2,\alpha}(X, \omega)$ estimates of $\varphi_t$.

This together with the classical Schauder estimates can improve the regularity to $C^\infty$. Thus, $S$ is closed and the theorem is proved. \hfill \Box

**Lemma 6.1** Fix some $t_2 \in (0, t_1)$. Let $\varphi_t$ be a solution of (6.1) with $t \geq t_2 > 0$. If the inequality (2.4) holds, then there exist constants $A, \lambda, C > 0$ independent of $t$ such that

$$ \Lambda_\omega \chi_t \leq \lambda \Lambda_{\omega_0} \chi_t \leq \lambda C e^{A(\varphi_t - \inf_X \varphi_t)}, \quad \forall \ t \geq t_2. $$

**Proof** Here we follow the estimates in [24] and [30]. Sometimes we omit the subscript $t$ for simplicity. We choose $\chi_0$ to be the $\chi'$ in (2.4). Write

$$ 3 \text{ We can also prove this by observing that } (DL(f), \eta)_{L^2} = -\int_X \eta < \sqrt{-1} \partial \overline{\partial} f, \omega_t > \chi_0 \frac{\chi^n}{n!} = -\int_X \eta \sqrt{-1} \partial \overline{\partial} f \wedge \ast \omega_t. \text{ Here } \ast \text{ is the Hodge star operator associated with } \chi_0. \text{ Since } \partial \text{ and } \overline{\partial} \text{ commute with } \ast \text{ it is obvious that this equals } -\int_X f \sqrt{-1} \partial \overline{\partial} \eta \wedge \ast \omega_t = (f, DL(\eta))_{L^2}. \text{ A good reference for Hodge star operator on Kähler manifold is } [32]. $$

4 The adaptation of the classical Evans–Krylov theorem to the complex case has been carried out by Siu [22] P100–107.
\[ \tilde{f} = \frac{1}{n} \chi^{k\bar{j}} \chi^{i\bar{i}} g_{t,i\bar{j}} \partial_k \partial_{\bar{i}} f = \frac{1}{n} h^{k\bar{i}} \partial_k \partial_{\bar{i}} f. \]

We calculate

\[ \tilde{\Delta}(\Lambda_{\omega_t} \chi) = \frac{1}{n} h^{k\bar{i}} R_{k\bar{i}}^{i\bar{j}} (g_t) \chi_{i\bar{j}} + \frac{1}{n} h^{k\bar{i}} g_{t,i\bar{j}} \partial_k \partial_{\bar{i}} \chi_{i\bar{j}}, \quad (6.3) \]

where \( R_{k\bar{i}}^{i\bar{j}} (g_t) \) denotes the curvature tensor of \( g_t \). Note that by Eq. (6.1),

\[ 0 = -g_{t,i\bar{j}} \partial_i \partial_{\bar{j}} (\chi^{k\bar{l}} g_{t,k\bar{l}}) = g_{t,i\bar{j}} h^{p\bar{q}} \partial_i \partial_{\bar{j}} \chi_{p\bar{q}} - g_{t,i\bar{j}} h^{r\bar{q}} \chi^{p\bar{r}} \partial_i \partial_{\bar{r}} \partial_{\bar{j}} \chi_{p\bar{q}} - g_{t,i\bar{j}} h^{p\bar{q}} \chi^{r\bar{q}} \partial_i \partial_{\bar{r}} \partial_{\bar{j}} \chi_{p\bar{q}} + \chi^{k\bar{l}} R_{k\bar{l}} (g_t). \]

Therefore, we have

\[ \tilde{\Delta} \log(\Lambda_{\omega_t} \chi) = \frac{\tilde{\Delta}(\Lambda_{\omega_t} \chi)}{\Lambda_{\omega_t} \chi} - \frac{|\tilde{\nabla}(\Lambda_{\omega_t} \chi)|^2}{(\Lambda_{\omega_t} \chi)^2} \]

\[ \geq \frac{1}{n \Lambda_{\omega_t} \chi} \left( h^{k\bar{i}} R_{k\bar{i}}^{i\bar{j}} (g_t) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}} (g_t) \right), \]

where we used the inequality by Lemma 3.2 in [30]

\[ n |\tilde{\nabla}(\Lambda_{\omega_t} \chi)|^2 \leq (\Lambda_{\omega_t} \chi) g_{t,i\bar{j}} h^{p\bar{q}} \chi^{r\bar{q}} \partial_i \partial_{\bar{r}} \partial_{\bar{j}} \chi_{p\bar{q}}. \quad (6.4) \]

For any \( A \), we have

\[ \tilde{\Delta} \left( \log(\Lambda_{\omega_t} \chi) - A \varphi \right) \geq \frac{1}{n \Lambda_{\omega_t} \chi} \left( h^{k\bar{i}} R_{k\bar{i}}^{i\bar{j}} (g_t) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}} (g_t) \right) \]

\[ - \frac{1}{n} (n c_t A - Ah^{k\bar{i}} \chi_{0,k\bar{i}}). \]

Fix \( t_0 \in [0,1] \). We choose \( A \) large enough such that

\[ - \frac{1}{A \Lambda_{\omega_t} \chi} \left( h^{k\bar{i}} R_{k\bar{i}}^{i\bar{j}} (g_t) \chi_{i\bar{j}} - \chi^{k\bar{l}} R_{k\bar{l}} (g_t) \right) \leq \epsilon, \quad (6.5) \]

where we used the fact that

\[ \Lambda_{\omega_t} \chi \geq C > 0 \]
for some constant $C$ independent of $t$. We assume that $\log(\Lambda_{o_t} \chi) - A \varphi$ achieves its maximum at the point $(x_t, t)$. Then at the point $(x_t, t)$ we have

$$0 \geq \tilde{\Delta} \left( \log(\Lambda_{o_t} \chi) - A \varphi \right)$$

$$\geq \frac{1}{n \Lambda_{o_t} \chi} \left( h^{k \bar{i}} R^i_{k \bar{j}} (g_t) \chi_{i \bar{j}} - \chi^{k \bar{i}} R^i_{k \bar{j}} (g_t) \right) - \frac{1}{n} (n c_t A - A h^{k \bar{i}} \chi_{0, k \bar{i}})$$

$$\geq \frac{A}{n} \left( -\epsilon - n c_t + h^{k \bar{i}} \chi_{0, k \bar{i}} \right).$$

Therefore, at the point $(x_t, t)$ we have

$$h^{k \bar{i}} \chi_{0, k \bar{i}} - n c_t \leq \epsilon.$$

We choose normal coordinates for the metric $\chi_0$ so that the metric $\chi_t$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. We denote the diagonal entries of $g_t$ by $\mu_1, \ldots, \mu_n$. Thus, we have

$$\sum_{i=1}^{n} \frac{\mu_i}{\lambda_i^2} - n c_t \leq \epsilon,$$

which implies the inequality

$$\epsilon \geq \sum_{i=1, i \neq k}^{n} \mu_i \left( \frac{1}{\lambda_i} - 1 \right)^2 - \sum_{i=1, i \neq k}^{n} \mu_i + \frac{\mu_k}{\lambda_k^2} - 2 \frac{\mu_k}{\lambda_k} + n c_t$$

$$\geq n c_t - \sum_{i=1, i \neq k}^{n} \mu_i - 2 \frac{\mu_k}{\lambda_k}. \quad (6.6)$$

On the other hand, by the assumption and the choice of the metric $\chi_0$ we have

$$(n c_t \chi_0 - (n - 1) \omega) \wedge \chi_0^{n-2} \wedge \beta_k > B \epsilon \chi_0^{n-1} \wedge \beta_k$$ \quad (6.7)

for sufficiently small $\epsilon > 0$. Here $\beta_k$ denotes the $(1, 1)$ form $\sqrt{-1} dz^k \wedge d \bar{z}^k$ and we choose the constant $B$ such that $B t_2 > 1$. Combining (6.7) with (6.2), we have

$$(n c_t \chi_0 - (n - 1) \omega_t) \wedge \chi_0^{n-2} \wedge \beta_k$$

$$= (1 - t) \chi_0^{n-1} \wedge \beta_k + t \left( n c_t \chi_0 - (n - 1) \omega \right) \wedge \chi_0^{n-2} \wedge \beta_k$$

$$\geq B \epsilon t \chi_0^{n-1} \wedge \beta_k. \quad (6.8)$$

By the argument of [24], the inequality (6.8) implies that

$$\sum_{i=1, i \neq k}^{n} \mu_i < n c_t - B t \epsilon. \quad (6.9)$$
Combining (6.9) with (6.6), we have
\[
\frac{\lambda_k}{\mu_k} < \frac{2}{(Bt - 1)\epsilon} \leq \frac{2}{(Bt_2 - 1)\epsilon}, \quad t \geq t_2
\]
for \( k = 1, \ldots, n \). Hence, at the point \((x_t, t)\) we have the estimate
\[
\Lambda_{\omega_t} \chi \leq \frac{2n}{(Bt_2 - 1)\epsilon}, \quad t \geq t_2.
\]
Thus, for any \( x \in X \) we have
\[
\Lambda_{\omega_t} \chi \leq \frac{2n}{(Bt_2 - 1)\epsilon} e^{A(\varphi - \inf_X \varphi)}.
\]
Since \( \omega_t \leq \chi_0 + \omega \leq \lambda \omega \) and \( \chi \) is a positive \((1, 1)\) form, we have
\[
\Lambda_{\omega_t} \chi \leq \frac{2n\lambda}{(Bt_2 - 1)\epsilon} e^{A(\varphi - \inf_X \varphi)}.
\]
The lemma is proved. \( \square \)

**Lemma 6.2** Under the assumptions of Lemma 6.1, there exists a uniform constant \( C \) such that
\[
\text{osc}_X \varphi_t \leq C, \quad \forall \ t \geq t_2.
\]

**Proof** The argument follows directly from [25] and [24] and we sketch the details here. We normalize \( \varphi_t \) by
\[
\int_X \varphi_t \omega^n_t = 0.
\]
Therefore, we have \( \sup_X \varphi_t \geq 0 \) and \( \inf_X \varphi_t \leq 0 \). Define
\[
u = e^{-N\hat{\varphi}}, \quad \hat{\varphi} = \varphi - \sup_X \varphi,
\]
where \( N = \frac{A}{1-\delta} \). Here \( A \) is the constant in Lemma 6.1 and \( \delta \) is a small positive constant to be determined later. Using the argument in [30], there is a constant \( C \) independent of \( t \) such that for all \( p \geq 1 \) and \( t \geq t_2 \) we have
\[\text{osc}_X \varphi_t \leq C, \quad \forall \ t \geq t_2.\]
A criterion for the properness of the $K$-energy

\[
\int_X |\nabla u|^p \frac{\omega^n_t}{n!} \leq C p \|u\|_{C^{0,\delta}}^{1-\delta} \int_X u^{p-(1-\delta)} \frac{\omega^n_t}{n!}.
\]

Note that the Sobolev constant of $\omega_t$ is uniformly bounded for all $t \in [0, 1]$. Thus, following the argument of [31] we can prove that $u$ is bounded and hence $\inf_X \hat{\phi}_t \geq -C$ for some constant $C$. In other words, we have

\[
0 \geq \inf_X \phi_t \geq \sup_X \phi_t - C \geq -C,
\]

which implies that $\text{osc}_X \phi_t \leq C$. The lemma is proved.

\[\square\]

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