ANALYTIC EXTENSION OF EXCEPTIONAL CONSTANT MEAN CURVATURE ONE CATENOIDs IN DE SITTER 3-SPACE

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Abstract. Catenoids in de Sitter 3-space $S^3_1$ belong to a certain class of space-like constant mean curvature one surfaces. In a previous work, the authors classified such catenoids, and found that two different classes of countably many exceptional elliptic catenoids are not realized as closed subsets in $S^3_1$. Here we show that such exceptional catenoids have closed analytic extensions in $S^3_1$ with interesting properties.

1. Introduction.

We denote by $S^3_1$ the de Sitter 3-space, which is a simply-connected Lorentzian 3-manifold with constant sectional curvature 1. Let $R^4_1$ be the Lorentz-Minkowski 4-space with the metric $(\cdot, \cdot)$ of signature $(-+++)$. Then

$$S^3_1 = \{X \in R^4_1; \langle X, X \rangle = 1\}$$

with metric induced from $R^4_1$. We identify $R^4_1$ with the $2 \times 2$ Hermitian matrices $\text{Herm}(2)$ by

$$(t, x, y, z) \leftrightarrow \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix},$$

where $i = \sqrt{-1}$. Then $S^3_1$ is represented as

$$S^3_1 = \{X \in \text{Herm}(2); \det X = -1\} = \{ae_3a^*; a \in \text{SL}(2, \mathbb{C})\},$$

where $a^* := \overline{a}$ is the conjugate transpose of $a$, and

$$e_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

To draw surfaces in $S^3_1$, we use the stereographic hollow ball model given in [4] as follows:

(1) $\Pi : S^3_1 \ni (t, x, y, z) \mapsto \frac{1}{\delta}(x, y, z) \in R^3$

$$\delta := t + \sqrt{t^2 + x^2 + y^2 + z^2} = t + \sqrt{2t^2 + 1}.$$
This projection $\Pi$ is the composition of central projection of $S^3_1$ to the unit sphere $S^3$ centered at the origin in $\mathbb{R}^4$ and usual stereographic projection of $S^3$ into $\mathbb{R}^3$ from $(0, 0, 0, -1)$. The image of $\Pi$ is the set
\[ \mathcal{D}^3 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{2} - 1 < |\xi| < \sqrt{2} + 1 \right\}, \]
where $|\xi| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ for $\xi = (\xi_1, \xi_2, \xi_3)$.

In [1], the authors classified all catenoids in $S^3_1$ (i.e. weakly complete constant mean curvature one surfaces in $S^3_1$ of genus zero with two regular ends whose hyperbolic Gauss map is of degree one). There are three types of catenoids:

- elliptic catenoids,
- the parabolic catenoid, and
- hyperbolic catenoids.

Parabolic catenoids have only one congruence class, whose secondary Gauss map is given by
\[ g = \frac{1 + \log z}{-1 + \log z}, \]
and they are rotationally symmetric surfaces with one cone-like singular point and two embedded ends. On the other hand, the secondary Gauss maps of hyperbolic catenoids are of the form
\[ g = \frac{g_0 - i}{g_0 + 1}, \quad g_0 := \exp((m + i\tau) \log z) = z^{m+i\tau}, \]
where $m$ is a non-negative integer, and $\tau$ is a non-zero real number. When $m \neq 0$ (resp. $m = 0$), hyperbolic catenoids admit only cuspidal edge singularities (resp. cone-like singular points), see [1, Page 36]. Recently, in a joint work with Seong-Deog Yang, the authors [2] proved that all hyperbolic catenoids do not admit any analytic extension.

On the other hand, there are many subclasses of elliptic catenoids, whose secondary Gauss maps $g$ are given by

(i) $g = z^\alpha$ \quad ($0 < \alpha < 1$),
(ii) $g = z^\alpha$ \quad ($\alpha > 1$),
(iii) $g = z^m + c$ \quad ($m = 2, 3, \ldots$) with $c \in (0, \infty) \setminus \{1\}$,
(iv) $g = z^m + 1$ \quad ($m = 2, 3, \ldots$),
(v) $g = (z^m - 1)/(z^m + 1)$ \quad ($m = 2, 3, \ldots$).

Except for the two cases (iv) and (v), all elliptic catenoids are closed subsets of $S^3_1$, since the singular sets of catenoids of type (i)–(iii) are compact. In this paper, we call the catenoids in the class (iv) (resp. (v)) exceptional catenoids of type I (resp. exceptional catenoids of type II) and we study these two classes.
For each \( m = 2, 3, \ldots \), we set

\[
F^I_m := \frac{z^{m+1}}{2\sqrt{m}} \begin{pmatrix}
(m + 1)z \\
(m - 1)z^m - m - 1
\end{pmatrix}
\]

and

\[
F^II_m := \frac{z^{m+1}}{2\sqrt{m}} \begin{pmatrix}
z((1 - m)z^m + m + 1) \\
-(m + 1)z^m + m - 1
\end{pmatrix}
\]

The maps \( f^I_m : C \setminus \{0\} \to S^3_1 \) defined by

\[
f^I_m := F^I_m \epsilon_3 (F^I_m)^* \quad (J = I, II)
\]

give the exceptional catenoids. These expressions are obtained by shifting \( m \) to \( m - 1 \) in \cite{1}, Prop. 4.9. We will show that the image of each \( f^I_m \) \( (J = I, II) \) has an analytic extension \( C^J_m \) which is a closed set in \( S^3_1 \).

A subset \( \mathcal{A} \) of a manifold \( M^n \) is called almost embedded (resp. almost immersed) if there is a discrete subset \( D \) of \( \mathcal{A} \) such that \( \mathcal{A} \setminus D \) is the image of an embedding (resp. an immersion) of a manifold into \( M^n \). For example (cf. \cite{1}),

- catenoids of class (iii) are not almost immersed,
- catenoids of class (i) are almost immersed, but not almost embedded,
- catenoids of class (ii) are almost embedded.

In Section 2, we investigate the geometric properties of \( f^I_m \), and show that the image of each \( f^I_m \) has an analytic extension whose image is immersed outside of a compact set. See Figures 1, 2, 3, where \( f^I_2, C^I_2 \) and \( C^I_3 \) are drawn in the stereographic hollow ball model (1). In Section 3, we show that each \( C^II_m \) can be realized as a warped product of a certain trochoid and hyperbola. In particular, \( C^II_2 \) and \( C^II_3 \) are almost embedded, and \( C^II_m \) \( (m \geq 4) \) are almost immersed (cf. Section 3). See Figures 4, 5, where \( f^II_2, C^II_2 \) and \( C^II_3 \) are drawn in the stereographic hollow ball model (1) as well.
It is well-known that the de Sitter space $S^3_1$ can be compactified by including two spheres $\partial \pm S^3_1$. These two sets $\partial \pm S^3_1$ are called the ideal boundaries. In the stereographic hollow ball model, the relations

\begin{equation}
\partial \pm S^3_1 = \{ \xi \in \mathbb{R}^3 ; |\xi| = \sqrt{2} \pm 1 \}
\end{equation}
Figure 5. The set $C^I_3$ and half of it.

hold (cf. (2)). If a subset $A$ of $S^3_1$ is closed, then each element of the set

$$\overline{\Pi(A)} \cap \partial S^3_1 (\subset \partial_- S^3_1 \cup \partial_+ S^3_1)$$

is called an endpoint, where $\overline{\Pi(A)}$ is the closure of $\Pi(A)$ in $R^3$. Then the set $\overline{\Pi(C^I_m)} \cap \partial S^3_1$ consists of one (resp. two) point(s) if $J = I$ and $m$ is odd (resp. if $J = I$ and $m$ is even, or $J = I I$). On the other hand, $\overline{\Pi(C^I_m)} \cap \partial S^3_1$ always consists of two points, that is, the number of the endpoints of $C^I_m$ ($J = I, I I$) is three or four (cf. Theorems 4 and 6). This is a remarkable phenomenon, since other elliptic catenoids in $S^3_1$ do not have any analytic extensions and have exactly two endpoints.

2. Exceptional catenoids of type I.

In this section, we show that $f^I_m$ has an analytic extension. For each integer $m \geq 2$, we set

$$f^I_m(r, \theta) = (x_0(r, \theta), x_1(r, \theta), x_2(r, \theta), x_3(r, \theta)),$$

with $z = re^{i\theta}$ ($r > 0, \theta \in S^1 := R/2\pi Z$). Then

$$x_0 = \frac{m^2 - 1}{4m} r^{m+1} \left(2 \cos m\theta - \frac{m \mp 1}{m \pm 1} r^m\right),$$

$$x_1 + ix_2 = \frac{(m - 1)^2}{4m} e^{i(m+1)\theta} + \frac{(m + 1)^2}{4m} e^{-i(m-1)\theta} - e^{i\theta m^2 - 1} r^m.$$

We know that $f^I_m(r, \theta)$ has self-intersections, since it contains swallowtail singularities (cf. Proposition A.1 in Appendix A). The limit curve

$$\gamma_m(\theta) := \lim_{r \to 0} (x_1, x_2)$$

gives a closed regular planar curve.

A hypo-trochoid is a roulette traced by a point attached to a disk of radius $r_c$ rolling along the inside of a fixed circle of radius $r_m$, where the point is
Figure 6. The trochoids for $m = 2, 3, 4$.

a distance $d$ from the center of the interior circle. The parametrization of a hypo-trochoid is given by

$$
x(s) = (r_c - r_m) \cos s + d \cos \left( \frac{r_c - r_m}{r_m} s \right),
$$

$$
y(s) = (r_c - r_m) \sin s - d \sin \left( \frac{r_c - r_m}{r_m} s \right).
$$

We prove the following:

**Proposition 1.** The plane curve $\gamma_m(\theta)$ has the following properties:

(a) $\gamma_m(\theta + \pi) = (-1)^{m+1} \gamma_m(\theta)$ for $\theta \in \mathbb{R}$,

(b) the image of $\gamma_m$ is a convex curve if $m = 2, 3$,

(c) $\gamma_m$ is a hypo-trochoid with (cf. Figure 6)

$$
r_c = \frac{m - 1}{2}, \quad r_m = \frac{m^2 - 1}{4m}, \quad d = \frac{(m + 1)^2}{4m}.
$$

**Proof.** The first two assertions follow immediately. The last assertion follows from the expressions

$$
x_1 = \frac{(m - 1)^2 \cos(m + 1)\theta + (m + 1)^2 \cos(m - 1)\theta}{4m},
$$

$$
x_2 = \frac{(m - 1)^2 \sin(m + 1)\theta - (m + 1)^2 \sin(m - 1)\theta}{4m}.
$$

\[\square\]

We set $\Omega := \Omega^+ \cup \Omega^-$, where

$$
\Omega^\pm := \{(r, \theta) \in \mathbb{R} \times S^1; \pm r > 0\} \quad (S^1 := \mathbb{R}/2\pi \mathbb{Z}).
$$

The expressions (6) and (7) are meaningful for $r < 0$ as well, and $f_m^1$ can be extended to $\Omega$. We denote this extension by $\vec{f}_m^1: \Omega \to S^3_+$. If $m$ is odd, then

$$
\vec{f}_m^1(-r, \theta + \pi) = \vec{f}_m^1(r, \theta).
$$
In particular, if \( m \) is odd, the image of \( f_m^1 \) coincides with that of \( \tilde{f}_m^1 \). On the other hand, if \( m \) is even,
\[
\tilde{f}_m^1(-r, \theta) = \iota \circ \tilde{f}_m^1(r, \theta),
\]
where \( \iota \) is the isometric involution given by
\[
\iota: S^3_1 \ni (t, x, y, z) \mapsto (-t, x, y, -z) \in S^3_1.
\]
Thus, if \( m \) is even, \( f_m^1(\Omega^+) \) and \( f_m^1(\Omega^-) \) are congruent, but do not coincide with each other. The singular set of \( \tilde{f}_m^1 \) is \( \Sigma := \Sigma^+_m \cup \Sigma^-_m \), where
\[
\Sigma^+_m := \{(r, \theta) \in \Omega^+ : r^m + 2 \cos m\theta = 0\},
\]
each of which consists of \( m \) components. The image of each component of the singular set is a curve with singularities which is bounded in \( S^3_1 \), whose endpoints are
\[
P_k := (0, \gamma_m(\alpha_k), 0), \quad \alpha_k := \frac{2k + 1}{2m} \pi \quad (k = 0, \ldots, 2m - 1).
\]
We denote by \( A^+_m \) the domain in \( \Omega^+ \) containing a neighborhood of \( r = \pm \infty \), and \( B_m^\pm := \Omega \setminus A_m^\pm \). Then we have the expressions
\[
A^+_m = \{(r, \theta) \in \Omega^+ : \epsilon^m(r^m + 2 \cos m\theta) > 0\},
\]
\[
B^+_m = \{(r, \theta) \in \Omega^+ : \epsilon^m(r^m + 2 \cos m\theta) < 0\},
\]
where \( \epsilon \) is the sign of \( r \) (cf. Figure 7). We next consider the light-like lines
\[
L_k := \{(t, \gamma_m(\alpha_k), -t) : t \in \mathbb{R} \} \subset S^3_1
\]
passing through \( P_k \) for \( k = 0, 1, \ldots, 2m - 1 \), and set
\[
C^1_m := \tilde{f}_m^1(\Omega) \cup L_0 \cup \cdots \cup L_{2m-1}.
\]
Then \( C^1_m \) is the analytic extension of \( f_m^1 \). In fact,

**Theorem 2.** For each integer \( m \geq 2 \),
(i) $C^1_m$ is a closed set of $S^3$. In particular, if $m$ is odd, then $C^1_m$ is the closure of the image of $f^1_m$. On the other hand, if $m$ is even, then the closure of the image of $f^1_m$ is just half of $C^1_m$. The other half can be obtained by the isometric involution $\iota$ of $S^3$ given in (10).

(ii) Moreover, $C^1_m$ is analytically immersed outside the compact set consisting of the image of $\Sigma_m$, and the points $\{(0, \gamma_m(\alpha_k), 0) ; k = 0, \ldots, 2m - 1\}$.

Proof. By (6), $x_0(r, \theta)$ diverges for $r \to \pm \infty$. Take a sequence $\{\zeta_j = (r_j, \theta_j)\}_{j=1,2,\ldots}$ on $\Omega$ such that $\lim_{j \to \infty} r_j = 0$. Taking a subsequence if necessary, we may assume $\{\zeta_j\}$ is included in $\Omega^+$ or $\Omega^-$, and $\lim_{j \to \infty} \theta_j = \beta$. If $\cos m \beta \neq 0$, (6) implies that $\lim_{j \to \infty} x_0(r_j, \theta_j)$ diverges. On the other hand, if $\cos m \beta = 0$, that is, $\beta = \alpha_k$ for some $k$, then $\lim_{j \to \infty} (x_0(\zeta_j) + x_3(\zeta_j))$ tends to 0, that is, $\tilde{f}^1_m(\zeta_j)$ is asymptotic to the line $L_k$. Conversely, for each point $Q_{k,t} := (t, \gamma_m(\alpha_k), -t) \in L_k$, we set

$$\zeta_j := \left(\frac{1}{j}, \frac{1}{m} \cos^{-1} \left(\frac{4mt}{j(m^2 - 1)}\right)\right) \quad (j = 1, 2, \ldots),$$

where $\cos^{-1}$ is the inverse function of cos as a map

$$\cos^{-1}: (-1, 1) \to \left(m\alpha_k - \frac{\pi}{2}, m\alpha_k + \frac{\pi}{2}\right).$$

Then $\lim_{j \to \infty} \zeta_j = (0, \alpha_k)$ and $\lim_{j \to \infty} \tilde{f}^1_m(\zeta_j) = Q_{k,t}$, where $\alpha_k$ is as in (11).

Hence $C^1_m$ is the closure of the image of $\tilde{f}^1_m$, proving the first part of (i). The second part of (i) is already proven. We next prove (ii). Since $\tilde{f}^1_m$ is an analytic immersion on $\Omega \setminus \Sigma_m$, it is sufficient to show that $C^1_m$ is parametrized analytically on a neighborhood of $L_k$, which gives an immersion on $L_k \setminus \{P_k\}$.

For this purpose, we set $s := (\cos m\theta)/r$. Then the $x_j$ ($j = 0, 1, 2, 3$) have the following expressions:

$$x_0 \pm x_3 = \frac{m^2 - 1}{4m} r^{\pm 1} \left(2rs - \frac{m \pm 1}{m \pm 1} r^m\right),$$

$$x_1 + ix_2 = \frac{e^{i\cos^{-1}(sr)/m}}{4m} \left((m - 1)^2 e^{i\cos^{-1}(sr)} + (m + 1)^2 e^{-i\cos^{-1}(sr)(m^2 - 1)r^m}\right).$$

Since $\left.\partial(x_1 + ix_2)/\partial r\right|_{(0,s)} \neq 0$ if $s \neq 0$, one can easily check that $\tilde{f}^1_m(r, s)$ is an immersion at $(0, s)$ for each $s \in \mathbb{R} \setminus \{0\}$, which proves the assertion. □
Remark 3. For the parametrization \((r, s)\) as in the proof of Theorem 2, the origin \((0, 0)\) is a singular point for each \(k = 0, 1, \ldots, 2m - 1\), whose image is the point \(P_k\) given in (11). One can show that this parametrization gives a wave front on a neighborhood of \((0, 0)\), and the origin is a cuspidal edge (resp. swallowtail) when \(m = 2\) (resp. \(m = 3\)), see Appendix A.

Next, we consider the endpoints of \(C_m^1\). Let
\[
(15) \quad \begin{align*}
    p_{\pm} := (0, 0, \pm (\sqrt{2} - 1)) &\in \partial_+ S^3_1, \\
    n_{\pm} := (0, 0, \pm (\sqrt{2} + 1)) &\in \partial_- S^3_1,
\end{align*}
\]
where \(\partial_{\pm} S^3_1\) are the ideal boundaries given in (5). We set
\[
(16) \quad y := (y_1, y_2, y_3) := f_1^m \circ \delta(x_1, x_2, x_3),
\]
where \(\delta = x_0 + \sqrt{2x_0^2 + 1}\) (cf. (1)).

Theorem 4. If \(m\) is even (resp. odd), the set of endpoints of \(C_m^1\) is \(\{p_{\pm}, n_{\pm}\}\) (resp. \(\{p_{-}, n_{+}\}\)). More precisely, let \(\{\zeta_j = (r_j, \theta_j)\}\) be a sequence in \(\Omega\) whose image under \(\sim f_1^m\) is unbounded. Then the following cases occur:

1. \(\lim_{j \to \infty} y(\zeta_j) = n_-\) holds when \(\lim_{j \to \infty} r_j = +\infty\) (that is, \(\{\zeta_j\}\) lies in \(\Omega^+\) and diverges).
2. When \(\lim_{j \to \infty} r_j = -\infty\), that is, if \(\{\zeta_j\}\) lies in \(\Omega^-\) and diverges, then \(\lim_{j \to \infty} y(\zeta_j)\) is \(p_+\) (resp. \(n_-\)) if \(m\) is even (resp. odd).
3. When \(r_j \to 0\) and \(\{\zeta_j\}\) is contained in \(A_m^+\) (resp. \(A_m^-, B_m^+, B_m^-\)), the limit of \(y(\zeta_j)\) is obtained as in the following table:

| The domain containing \(\{\zeta_j\}\) | \(A_m^+\) | \(A_m^-\) | \(B_m^+\) | \(B_m^-\) |
|----------------------------------------|----------|----------|----------|----------|
| \(\lim_{j \to \infty} y(\zeta_j)\) for even \(m\) | \(p_-\) | \(n_+\) | \(n_-\) | \(p_-\) |
| \(\lim_{j \to \infty} y(\zeta_j)\) for odd \(m\) | \(p_-\) | \(p_-\) | \(n_+\) | \(n_-\) |

Proof. We rewrite (16) as
\[
(17) \quad y_l = \frac{x_l/x_0}{1 + \text{sgn}(x_0)\sqrt{2 + 1/(x_0)^2}} \quad (l = 1, 2, 3),
\]
where \(\text{sgn}(x_0)\) denotes the sign of \(x_0\). By (6) and (7),
\[
\lim_{r \to \pm \infty} \frac{x_3}{x_0} = 1, \quad \lim_{r \to \pm \infty} \frac{x_l}{x_0} = 0 \quad (l = 1, 2),
\]
\[
\lim_{r \to + \infty} x_0 = -\infty, \quad \lim_{r \to - \infty} (-1)^m x_0 = \infty,
\]
proving (1) and (2).
We prove (3) for the case that \( \{ \zeta_j \} \subset B_m \). Noticing that \( r_j < 0 \), (13) implies that
\[
(-1)^m \left( \frac{r_j^{m-1} \cos m\theta_j}{r_j} \right) > 0
\]
holds for each \( j \). Since \( \{ x_0(\zeta_j) \} \) is unbounded, so is \( (\cos m\theta_j)/r_j \). Then the sign of \( (\cos m\theta_j)/r_j \) is equal to \( (-1)^m \) for sufficiently large \( j \) because \( r_j \) tends to 0. Therefore by (6), \( \text{sgn}(x_3(\zeta_j)) = (-1)^m \). On the other hand, (6) implies that \( \lim_{j \to \infty} x_3(\zeta_j)/x_0(\zeta_j) = -1 \). Thus, we have
\[
\lim_{j \to \infty} y_3(\zeta_j) = \frac{-1}{1 + (-1)^m \sqrt{2}} = 1 - (-1)^m \sqrt{2}.
\]
Since \( x_1 \) and \( x_2 \) are bounded near \( r = 0 \), \( y_l(\zeta_j) \) tends to 0 for \( l = 1, 2 \). Thus we have the conclusion. The other cases can be proved similarly.

3. Exceptional catenoids of type II.

Here we show that the image of the exceptional catenoid \( f^\Pi \) in \( S^3 \) has an analytic extension. For each integer \( m \geq 2 \), we set
\[
f^\Pi_m(r, \theta) = (x_0(r, \theta), x_1(r, \theta), x_2(r, \theta), x_3(r, \theta)),
\]
with \( z = re^{i\theta} \). By (4), \( f^\Pi_m \)'s components are
\[
\begin{align*}
x_0 &= \frac{1 - m^2}{4m} \left( \frac{r + 1}{r} \right) \cos m\theta, \\
x_3 &= \frac{1 - m^2}{4m} \left( \frac{r - 1}{r} \right) \cos m\theta, \\
x_1 &= -\frac{(m^2 + 1) \cos m\theta \cos \theta + 2m \sin m\theta \sin \theta}{2m}, \\
x_2 &= -\frac{(m^2 + 1) \cos m\theta \sin \theta - 2m \sin m\theta \cos \theta}{2m},
\end{align*}
\]
where \( z = re^{i\theta} \). The secondary Gauss map \( g_m \) of \( f^\Pi_m \) is a meromorphic function on \( C \cup \{ \infty \} \) given by (cf. [1, (39)])
\[
g_m = (z^m - 1)/(z^m + 1).
\]
Since the singular set \( \Sigma_m \) of the map \( f^\Pi_m \) is
\[
\Sigma_m = \{ z \in C \setminus \{ 0 \} ; |g_m(z)| = 1 \} = \{ re^{i\theta} \in C \setminus \{ 0 \} ; \cos m\theta = 0 \},
\]
we have \( \Sigma_m = \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_{2m-1} \), where
\[
\sigma_k := \{ z = re^{ia_k} ; r > 0 \} \quad (a_k := \frac{(2k+1)}{2m} \pi)
\]
for \( k = 0, \ldots, 2m - 1 \). In particular, if we set

\[
\Omega_k := \left\{ re^{i\theta} : \frac{(2k-1)\pi}{2m} < \theta < \frac{(2k+1)\pi}{2m}, \ r > 0 \right\},
\]

then the union of the \( \Omega_k \ (k = 0, \ldots, 2m - 1) \) is the regular set of \( f^m \), that is, the regular set consists of a disjoint union of \( 2m \) sectors.

**Proposition 5.** The map \( f^m \) satisfies:

1. For each \( m \geq 2 \), the image \( f^m(\sigma_k) \) consists of a point. More precisely,

\[
f^m(\sigma_k) = (-1)^k (0, -\sin \alpha_k, \cos \alpha_k, 0),
\]

where \( \alpha_k \) is as in (19) \((k = 0, \ldots, 2m - 1)\).

2. The endpoints of the image of \( f^m \) are the four points \( p_\pm \) and \( n_\pm \) as in (15).

**Proof.** Substituting \( \theta = \alpha_k \) into (18) and using that \( \cos m\theta = 0 \) and \( \sin m\theta = (-1)^k \) on \( \sigma_k \), we get the first assertion.

To prove the second assertion, we remark that

\[
\text{sgn}(x_0) = (-1)^{k+1} \quad \text{(on } \Omega_k \text{)},
\]

for each \( k \), since \( \text{sgn}(\cos m\theta) = (-1)^k \). Take a sequence \( \{z_j\} \) on \( C \setminus \{0\} \) such that \( \Pi \circ f^m(z_j) \) converges to one of the points in the ideal boundary. By (i), we may assume that each \( z_j \notin \Sigma_m \). With finitely many sectors, we may also assume \( \{z_j\} \subset \Omega_k \) for some \( k \). Then \( x_0(z_j) \) diverges to \( \infty \) or \( -\infty \) as \( j \to \infty \), that is, \( \{r_j + r_j^{-1}\}_{j=1,2,\ldots} \) is unbounded, where \( r_j := |z_j| \). Taking a subsequence, we may assume

\[
\lim_{j \to \infty} r_j = 0 \quad \text{or} \quad \lim_{j \to \infty} r_j = \infty.
\]

We set \( y := \Pi \circ f^m \). Since \( x_1 \) and \( x_2 \) are bounded (cf. (18)), \( y_l(z_j) \to 0 \) for \( l = 1, 2 \), where \( y = (y_1, y_2, y_3) \). On the other hand, by (18) and (22), we have \( \lim_{j \to \infty} \frac{x_3(z_j)}{x_0(z_j)} = \pm 1 \). Thus, we have \( \lim_{j \to \infty} y_3(z_j) = \pm \sqrt{2} \pm 1 \), which proves (ii). \( \square \)

It should be remarked that \( x_1, x_2 \) depend only on the variable \( \theta \), and

\[
(x_1(\theta), x_2(\theta)) = -2\gamma_m(\theta)
\]

holds. Here, \( \gamma_m \) is exactly the same hypo-trochoid as given in Proposition 1. For fixed \( \theta \), the image of the curve defined by \( r \mapsto (x_0(r, \theta), x_3(r, \theta)) \) coincides with

\[
\left\{ (t, z) \in R^2_1 : t^2 - z^2 = \frac{(m^2 - 1)^2}{(2m)^2} \cos^2 m\theta, \ \text{sgn} (\cos m\theta)t < 0 \right\}.
\]
In particular, it is half of a hyperbola when $\cos m \theta \neq 0$. If $\cos m \theta = 0$, the image reduces to a point. So we can conclude that the real analytic extension of the image of $f_m$ coincides with the set

$$C_m^\| := \left\{ (t, x, y, z) \in \mathbb{R}^4; (x, y) = -2\gamma_m(\theta),
\quad t^2 - z^2 = \frac{(m^2 - 1)^2}{(2m)^2} \cos^2 m \theta, \ \theta \in [0, 2\pi) \right\}.$$

For each $k = 0, \ldots, 2m - 1$, the analytic extension $C_m^\|$ contains a union of two light-like lines

$$L_k^\pm := \left\{ (t, -\sin \alpha_k, \cos \alpha_k, \pm t) ; t \in \mathbb{R} \right\},$$

where $\alpha_k$ is as in (19). Moreover, $C_m^\|$ is symmetric with respect to the isometric involution

$$S_3^1 \ni (t, x, y, z) \longmapsto (t, \cos(2\alpha_k)x + \sin(2\alpha_k)y, \sin(2\alpha_k)x - \cos(2\alpha_k)y, z) \in S_3^1.$$

This involution fixes the two lines $L_k^+$ and $L_k^-$. Suppose that $m$ is an odd integer. By (a) of Proposition 1, $\gamma_m$ is $\pi$-periodic. In this case, one half of the hyperbola at $\theta + \pi$ is just the other half of the hyperbola (23) at $\theta$, and $C_m^\|$ coincides with the closure of the image of $f_m^\|$.

In the case $m$ is even, $C_m^\|$ does not coincide with the closure of the image of $f_m^\|$. Moreover, $C_m^\|$ contains the image of the map $\iota \circ f_m^\|$, which is congruent to $f_m^\|$, where $\iota$ is the involution as in (10), and $C_m^\|$ is just the closure of the union of the images of $f_m^\|$ and $f_m^\|$. Figure 4 shows $C_m^\|$ and the image of $f_m^\|$ for $m = 2$.

Summarizing the above, we get the following:

**Theorem 6.** For each $m = 2, 3, \ldots$, the set $C_m^\|$ gives the real analytic extension of the exceptional catenoid $f_m^\|$, and has the following properties:

(i) The projection of $C_m^\|$ into the $xy$-plane in $\mathbb{R}_1^4$ is the hypo-trochoid $-2\gamma_m$. Furthermore, the section of $C_m^\|$ by a plane containing a point of the hypo-trochoid and perpendicular to the $xy$-plane is a hyperbola unless the plane passes through the cone-like singularity of $C_m^\|$.

(ii) $C_m^\|$ is almost immersed and has four endpoints. Two of them lie in $\partial_+ S_3^1$ and the others lie in $\partial_- S_3^1$. Moreover, $C_m^\|$ is almost embedded if $m = 2, 3$.

(iii) If $m$ is odd, then $C_m^\|$ is the closure of the image of $f_m^\|$. On the other hand, if $m$ is even, then the closure of the image of $f_m^\|$ is just half of $C_m^\|$. The other half can be obtained by the isometric involution of $S_3^1$ given in (10).
When $m \geq 4$, $C^m_{II}$ has self-intersections. It should be remarked that similar phenomena occur for parabolic or hyperbolic catenoids in the class of space-like maximal surfaces in $\mathbb{R}^4_1$ (see [3]).

**Remark 7.** As shown in [2], $C^m_{II}$ is analytically complete, that is, $C^m_{II}$ admits no analytic extension.

To end this paper, we remark that the replacement
\[ s \mapsto \text{(} r = e^s \text{)} \]
of the parameter of $f^m_{II}$ induces constant mean curvature surfaces in anti-de Sitter space. This induces a family of surfaces
\[ \tilde{f}_m := (x_0(s, \theta), x_1(\theta), x_2(\theta), x_3(s, \theta)) \]
given by $(x_0, x_3) = \frac{1-m^2}{2m^2} \cos m \theta (\cos s, \sin s)$, and $x_1, x_2$ as in (18), where $m = 2, 3, 4, \ldots$. For each $m$, the corresponding surface lies in the space form
\[ H^3_1(-1) := \{(t, x, y, z) : t^2 - x^2 - y^2 + z^2 = -1\} \]
of constant curvature $-1$ realized in $(\mathbb{R}^4_2, + - - +)$. The image of $\tilde{f}_m$ gives a compact almost immersed time-like surface of constant mean curvature one having a finite number of cone-like singularities. Moreover, if $m$ equals 2 or 3, the surface is almost embedded in the sense given in the introduction. To draw the surfaces, we use the ‘solid torus model’ of $H^3_1(-1)$, that is, we define the following projection
\[ \hat{H} : H^3_1(-1) \ni (t, x, y, z) \mapsto \frac{1}{\rho} \left( \left(1 + \frac{t}{\rho}\right)x, \left(1 + \frac{t}{\rho}\right)y, z\right) \in \mathbb{R}^3, \]
where $\rho := \sqrt{x^2 + y^2}$. The image of $\hat{H}$ is the interior of the solid torus obtained by rotating the unit disk with center $(1, 0, 0)$ about the third axis in $\mathbb{R}^3$. The images of $\hat{H} \circ \tilde{f}_m$ for $m = 2, 3$ are given in Figure 8.
APPENDIX A. SINGULARITIES OF EXCEPTIONAL CATENOIDS OF TYPE I

In this appendix, we discuss properties of singularities of the exceptional catenoids of type I. By the criteria in [5, Theorem 3.4], we have the following:

**Proposition A.1.** The singular set of $f^I_m$ is

$$
\Sigma_m := \{ z = re^{i\theta} \in \mathbb{C} \setminus \{0\} : r^m + 2m\cos \theta = 0 \}.
$$

The $m$ points

$$
z = re^{i\theta}, \quad (r, \theta) = \left(2^{1/m}, \frac{1}{m}(2j + 1)\pi\right), \quad (j = 0, \ldots, m - 1)
$$

are swallowtails, and the $2m$ points

$$
z = re^{i\theta}, \quad (r, \theta) = \left\{ \begin{array}{ll}
(2^{1/(2m)}, \frac{1}{m}(\frac{3}{4} + 2j)\pi) \\
(2^{1/(2m)}, \frac{1}{m}(\frac{5}{4} + 2j)\pi),
\end{array} \right. \quad (j = 0, \ldots, m - 1)
$$

are cuspidal cross caps. Other points in $\Sigma_m$ are cuspidal edges.

Next, we discuss singularities of the parametrization of $C^I_m$ near the line $L_k$ ($k = 0, \ldots, 2m - 1$), as in the proof of Theorem 2. Without loss of generality, we may assume $k = 0$. Then the parametrization is expressed as

$$
\tilde{f}^I_m(r, s) := (x_0(r, s), x_1(r, s), x_2(r, s), x_3(r, s)),
$$

where

$$
x_0 + x_3 := \frac{m^2 - 1}{4m} \left(2r^2 - s - \frac{m - 1}{m + 1}r^{m+1}\right),
$$

(A.24) $x_0 - x_3 := \frac{m^2 - 1}{4m} \left(2s - \frac{m + 1}{m - 1}r^{m-1}\right),$

$$
x_1 + ix_2 := \frac{(m-1)^2}{4m} e^{i(m+1)\theta} + \frac{(m+1)^2}{4m} e^{-i(m-1)\theta} - \frac{m^2 - 1}{4m} r^m e^{i\theta}
$$

and

(A.25) $\theta := \theta(r, s) = \frac{1}{m} \cos^{-1}(rs),$

where we consider $\cos^{-1}(rs) \in [0, \pi]$. As shown in Theorem 2, the map $\tilde{f}^I_m$ is an immersion at $(0, s)$ if $s \neq 0$. We show the following:

**Proposition A.2.** The map $\tilde{f}^I_m$ is a wave front near the origin, and the origin $(0, 0)$ is a cuspidal edge (resp. swallowtail) when $m = 2$ (resp. $m = 3$).

**Proof.** Since

$$
x_1(0, 0) + ix_2(0, 0) = -ie^{i\pi/(2m)} = \sin \frac{\pi}{2m} - i \cos \frac{\pi}{2m},
$$
$x_1(0,0) \neq 0$. Then
\[
\pi : S^3_1 \ni (x_0, x_1, x_2, x_3) \mapsto (x_0 + x_3, x_0 - x_3, x_1) \in \mathbb{R}^3
\]
gives a local coordinate system of $S^3_1$ around $\tilde{f}_m^1(0,0)$. So it is sufficient to show the conclusion for the map
\[
F(r,s) := \pi \circ \tilde{f}_m^1(r,s) = (X(r,s), Y(r,s), Z(r,s)),
\]
where
\[
X(r,s) := 2(m+1)r^2s - (m-1)r^{m+1},
\]
\[
Y(r,s) := 2(m-1)s - (m+1)r^{m-1},
\]
\[
Z(r,s) := \frac{m-1}{m+1}\cos(m+1)\theta + \frac{m+1}{m-1}\cos(m-1)\theta - r^m \cos \theta.
\]
By (A.25),
\[
\theta_r = -\delta s, \quad \theta_s = -\delta r \quad \left( \delta(r,s) := \frac{1}{m \sin m\theta(r,s)} \right)
\]
hold. Then we have
\[
F_r = ((m+1)r(4s - (m-1)r^{m-1}), -(m^2 - 1)r^{m-2},
\]
\[
(2s - mr^{m-1}) \cos \theta - sr \delta \lambda \sin \theta),
\]
\[
F_s = (2(m+1)r^2, 2(m-1), 2r \cos \theta - r^2 \delta \lambda \sin \theta),
\]
where
\[
\lambda := 2s + r^{m-1}.
\]
By a direct computation, we have $F_r \times F_s = \lambda \nu$, where
\[
\nu := (\nu_1, \nu_2, \nu_3)
\]
\[
\nu_1 := -(m-1)(2 \cos \theta - r \delta(2s + (m+1)r^{m-1}) \sin \theta),
\]
\[
\nu_2 := -(m+1)r^2(2 \cos \theta - r \delta(2s - (m-1)r^{m-1}) \sin \theta),
\]
\[
\nu_3 := 4(m^2 - 1)r.
\]
Since
\[
\nu(0,0) = (-2(m-1) \cos(\pi/(2m)), 0, 0) \neq 0,
\]
$\nu$ is the normal vector field of $F$, and $\lambda$ in (A.27) is an identifier of singularities, that is, $\{(r,s) : \lambda(r,s) = 0\}$ is the singular set. Since $d\lambda \neq 0$, the singular points of $F$ are non-degenerate. Thus, the singular direction is
\[
\xi := -2 \frac{\partial}{\partial r} + (m-1)r^{m-2} \frac{\partial}{\partial s}.
\]
On the other hand, since \(2F_r + (m + 1)r^{m-2}F_s = 0\) when \(\lambda(r, s) = 0\),

\[
\eta := 2 \frac{\partial}{\partial r} + (m + 1)r^{-2} \frac{\partial}{\partial s}
\]

is the null direction. Moreover,

\[
d\nu(\eta)(0, 0) = (0, 0, 4(m^2 - 1)),
\]

which is not proportional to \(\nu(0, 0)\). Thus, the map \(F\) gives a wave front near the origin.

When \(m = 2\), the singular direction and the null direction are linearly independent at the origin. Hence, by the criterion in [6, Proposition 1.3], the origin is a cuspidal edge.

Finally, when \(m = 3\), the singular direction and the null direction are

\[
\xi = -2 \frac{\partial}{\partial r} + 2r \frac{\partial}{\partial s}, \quad \eta = 2 \frac{\partial}{\partial r} + 4r \frac{\partial}{\partial s},
\]

which are proportional at the origin. Moreover, \(\det(\xi, \eta) = -12r\), where \(\det\) is the determinant function on the \((r, s)\)-plane. Hence by the criterion in [6, Proposition 1.3], the origin is a swallowtail. \(\square\)

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