Heterotic – Type II duality in the hypermultiplet sector

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ABSTRACT

We revisit the duality between heterotic string theory compactified on $K^3 \times T^2$ and type IIA compactified on a Calabi-Yau threefold $X$ in the hypermultiplet sector. We derive an explicit map between the field variables of the respective moduli spaces at the level of the classical effective actions. We determine the parametrization of the $K^3$ moduli space consistent with the Ferrara-Sabharwal form. From the expression of the holomorphic prepotential we are led to conjecture that both $X$ and its mirror must be $K^3$ fibrations in order for the type IIA theory to have an heterotic dual. We then focus on the region of the moduli space where the metric is expressed in terms of a prepotential on both sides of the duality. Applying the duality we derive the heterotic hypermultiplet metric for a gauge bundle which is reduced to 24 point-like instantons. This result is confirmed by using the duality between the heterotic theory on $T^3$ and M-theory on $K^3$. We finally study the hyper-Kähler metric on the moduli space of an $SU(2)$ bundle on $K^3$.

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1 Introduction

Among the backgrounds of different string theories, duality relations are conjectured to hold [1, 2]. (For a review see, for example, [3–5].) One distinguishes perturbative from non-perturbative dualities where the former hold at weak coupling and can be tested in string perturbation theory. T-duality and mirror symmetry are prominent examples of perturbative dualities. Non-perturbative dualities on the other hand involve the string coupling in a non-trivial way and generically relate weak to strong coupling physics. The example we want to focus on in this paper is the heterotic/type IIA duality, or more specifically the duality between heterotic strings compactified on $K^3 \times T^2$ and type IIA theories compactified on Calabi-Yau threefolds $X$. The backgrounds of this duality have four infinitely extended Lorentzian ($d = 4$) and six compact dimensions. Eight supercharges are unbroken or in other words the effective $d = 4$ theory has $N = 2$ supersymmetry.

In $N = 2$ supergravity the scalar field space is a direct product of the form

$$M = M_h \times M_v,$$

where $M_h$ is a $4n_h$-dimensional quaternionic-Kähler manifold spanned by the scalars of $n_h$ hypermultiplets, while $M_v$ is a $2n_v$-dimensional special-Kähler manifold spanned by the scalars of $n_v$ vector multiplets [6–9]. The heterotic dilaton $S$ is part of a vector multiplet while the type II dilaton $\phi$ resides in a hypermultiplet. As a consequence $M_v$ is quantum corrected on the heterotic side but ‘exact’ in type II backgrounds. Conversely $M_h$ is exact on the heterotic side (if one includes $\alpha'$-corrections) but quantum corrected in type II backgrounds.

The heterotic-type IIA duality has been perturbatively well tested for the vector multiplets [10–15]. (For a review and further references see, for example, [5, 16].) This component is a special Kähler manifold described by a holomorphic prepotential $F$. Under the duality the heterotic dilaton $S$ is mapped to a geometrical modulus of the Calabi-Yau manifold in the type II background. More precisely, the Calabi-Yau is constrained to be a $K^3$-fibration where the size of the $\mathbb{P}^1$-base is the type II dual of the heterotic dilaton. Thus, weak heterotic couplings corresponds to a large $\mathbb{P}^1$-base – a limit which can be systematically studied in type II backgrounds.

On the other hand, much less is known for the duality among the hypermultiplets of $M_h$ [17, 18]. One of the reasons is that $M_h$ being a quaternionic-Kähler manifold is a more complicated and cannot be characterized as succinctly as $M_v$ by a holomorphic prepotential. However, recently there has been considerable progress in the understanding of $M_h$ on the type II side of the duality [19–27] and therefore it seems worthwhile to revisit the hypermultiplet sector in the heterotic-type IIA duality [28].

The goal of this paper is to assemble further properties of $M_h$ on both sides of the duality. On the type IIA side $M_h$ is spanned by the axio-dilaton, the complex structure moduli of $X$ and the deformations of the RR three-form $C_3$. On the heterotic side, the hypermultiplet scalars are the $K^3$ geometric and $B$-field moduli and the deformations of the gauge bundle over $K^3$. First we give the map between the two classical moduli spaces of the dual backgrounds at the level of the effective action. To do this, it is convenient to consider an elliptically fibered $K^3$ on the heterotic side [17, 18]. In order to check the
heterotic/type IIA duality we go to a region of the hypermultiplet moduli space where both heterotic and type IIA theories are in the perturbative regime. This is realized when both \( \alpha' \) corrections on the heterotic side and \( g_s \) corrections on the type IIA side are negligible. This limit is described in [17] and we will call it the ‘double classical limit’. On the heterotic side, it corresponds to taking the elliptically fibred \( K3 \) with large base and large fibre. On the type IIA side, it is realized by taking the dilaton to be large (corresponding to small \( g_s \)) and by making the Calabi-Yau manifold undergo a specific stable degeneration [17, 29].

At small type IIA string coupling, the quaternionic metric has a special form in that it is in the image of the c-map [30,31]. In this limit the dilaton and universal axion together with the deformations of the RR three-form \( C_3 \) are fibred over a special-Kähler base spanned by the complex structure moduli of the Calabi-Yau threefold \( X \). As a consequence the entire quaternionic metric is determined by the holomorphic prepotential \( G \) of the special-Kähler base. The heterotic hypermultiplet moduli space at leading order in \( \alpha' \) has a different structure: First, it is not in the image of the c-map. Second, it has a fibration structure, where the base is spanned by the \( K3 \) moduli, while the fibre by the bundle moduli. The duality implies that in the double classical limit one recovers both structures on the hypermultiplet moduli space.

Even though the heterotic moduli space is not in the image of the c-map, this is true for the \( K3 \) moduli space \( SO(4,20)/SO(4) \times SO(20) \) [30]. In section 3 we derive the explicit parametrization of this space consistent with the Ferrara-Sabharwal form. This allows us to compare the heterotic metric with the type IIA metric at the level of the prepotential and extract the explicit map of the moduli that gives the match. Furthermore, given the form of the prepotential, we are led to conjecture that:

*If type IIA compactified on the CY \( X \) has an heterotic dual, then \( X \) and its mirror \( \tilde{X} \) both have to be a \( K3 \) fibration.*

With the results just described at hand we are able to obtain further information about the heterotic hypermultiplet metric from the type IIA side. The type IIA prepotential depends on all the complex structure moduli of \( X \) spanning the special-Kähler base. The leading contribution in the degeneration parameter (corresponding to large elliptic fibre of \( K3 \)) is the piece that matches with the \( K3 \) prepotential. Among the subleading terms we identify the ones related to \( \alpha' \) corrections on the heterotic side. Moreover, there are terms in the prepotential that depend on the complex structure moduli corresponding to the heterotic bundle moduli. We consider in detail the case when the bundle is reduced to 24 point-like instantons [32–37]. Using the conjecture on the mirror \( \tilde{X} \), we derive the explicit prepotential describing the classical metric on the heterotic side, in the limit of large base and large fibre of the heterotic \( K3 \). In particular we compute how the \( K3 \) moduli couple to the bundle moduli.

Finally, we check our results using a different duality which relates the heterotic string compactified on \( T^3 \) with M-theory compactified on \( K3 \) [2]. Applying twice the duality map, we are able to confirm the conjecture stated above and to rederive the point-like instanton prepotential. Furthermore, taking the rigid limit of the hypermultiplet moduli space, we find the metric on the moduli space of a smooth \( SU(2) \) bundle. To do this we use the fact that in this limit, the wanted prepotential is given by Seiberg-Witten prepotential \( \mathcal{F}_{SW} \) [38].
The paper is organized as follows. Section 2 is introductory in that we recall some basic facts about $N = 2$ supergravity (section 2.1), type II compactified on Calabi-threefolds (section 2.2), heterotic compactified on $K3 \times T^2$ (section 2.3) and the heterotic-type IIA duality (section 2.4). In section 3 we study the $K3$ moduli space and give the $K3$ metric in the c-map form. In section 4 we focus on the heterotic–type IIA duality. In section 4.1 we use the duality with type IIA to derive the structure of heterotic metric, in section 4.2 we consider the situation of 24 point-like instantons while in section 4.3 we studying the heterotic hypermultiplet space using duality with M-theory. Finally, in Appendix A we present the detailed derivation of the result given in section 3, while in B we analyze the two different classical limits.

2 Preliminaries

2.1 $N = 2$ in $d = 4$

To set the stage let us first briefly recall some properties of four-dimensional $N = 2$ supergravity (for a review see, for example, [9]). The theory consists of a gravitational multiplet, $n_v$ vector multiplets and $n_h$ hypermultiplets. The bosonic components of the gravitational multiplet are the spacetime metric $g_{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$ and the graviphoton $A_0^\mu$. A vector multiplet contains a vector $A^\mu$ and a complex scalar $t$ as bosonic components. Finally, a hypermultiplet features four real scalars $q^u$. For $n_v$ vector- and $n_h$ hypermultiplets there are thus a total of $2n_v + 4n_h$ real scalar fields in the spectrum. For an ungauged theory the bosonic matter Lagrangian is given by

$$L = -iN_{IJ} F_{\mu}^{I+} F_{\mu}^{J+} + iN_{IJ} F_{\mu}^{I-} F_{\mu}^{J-} + g_{ij}(t, \bar{t}) \partial_\mu t^i \partial^\mu \bar{t}^j + h_{uv}(q) \partial_\mu q^u \partial^\mu q^v, \quad (2.1)$$

where $g_{ij}$, $i, j = 1, \ldots, n_v$, is the metric defined on the $2n_v$-dimensional space $M_v$, which $N = 2$ supersymmetry constrains to be a special-Kähler manifold [7]. This implies

$$g_{ij} = \partial_i \partial_j K^\nu, \quad \text{for} \quad K^\nu = -\ln i \left( \bar{X}^I \mathcal{F}_I - X^I \bar{\mathcal{F}}_I \right). \quad (2.2)$$

Both $X^I(t)$ and $\mathcal{F}_I(t)$, $I = 0, 1, \ldots, n_v$, are holomorphic functions of the scalars $t^i$ and $\mathcal{F}_I = \partial \mathcal{F} / \partial X^I$ is the derivative of a holomorphic prepotential $\mathcal{F}(X)$ which is homogeneous of degree two. Furthermore, it is possible to go to a system of ‘special coordinates’ where $X^I = (1, t^i)$.

The $F_{\mu \nu}^{I\pm}$ that appear in the Lagrangian (2.1) are the self-dual and anti-self-dual parts of the field strengths of the gauge bosons in the vector multiplets together with the graviphoton. Their kinetic matrix $N_{IJ}$ is a function of the $t^i$ given by

$$N_{IJ} = \bar{\mathcal{F}}_{IJ} + 2i \frac{\text{Im} \mathcal{F}_{IK} \text{Im} \mathcal{F}_{JL} X^K X^L}{\text{Im} \mathcal{F}_{KL} X^K X^L}, \quad (2.3)$$

where $\mathcal{F}_{IJ} = \partial_I \mathcal{F}_J$.

The metric on the $4n_h$-dimensional space $M_h$ is denoted by $h_{uv}(q)$, $u, v = 1, \ldots, 4n_h$, in the Lagrangian (2.1). It is constrained by $N = 2$ supersymmetry to be a quaternionic-Kähler manifold [6,8]. The holonomy group of such manifolds is given by $Sp(1) \times Sp(n_h)$.
and in addition they admit a triplet of complex structures $J^x$, $x = 1, 2, 3$, which satisfy the quaternionic algebra
\[ J^x J^y = -\delta^{xy} 1 + \epsilon^{xyz} J^z. \] (2.4)

A special class of quaternionic-Kähler manifolds which will play a role in the following are the so called ‘special quaternionic-Kähler manifolds’. They can always be constructed from any special Kähler manifold $M_{SK}^{2n-2}$ (of dimension $2n - 2$) by the c-map [30]
\[ c : \frac{SU(1,1)}{U(1)} \times M_{SK}^{2n-2} \to M_{QK}^{4n}. \] (2.5)

An explicit form of the metric on $M_{QK}^{4n}$ is known as the Ferrara-Sabharwal metric which reads [31]
\[ L = -(\partial \phi)^2 - e^{4\phi}(\partial \bar{\phi} + \bar{\xi}_A \partial \xi^A - \xi^A \partial \bar{\xi}_A)^2 + g_{ab} \partial \bar{z}^a \partial z^b + e^{2\phi} \text{Im}N^{AB}(\partial \bar{\xi}_A - N \partial \xi^A)(\partial \xi^B - N \partial \bar{\xi}_B), \] (2.6)

where the coordinates of the first factor are $(\phi, \bar{\phi})$, while $z^a, a = 1, \ldots, n - 1$ are the complex coordinates of $M_{SK}^{2n-2}$ with $g_{ab}$ being its metric. This metric is determined in terms of a holomorphic prepotential $G$ by the relation (2.2) with $F$ replaced by $G$. $\xi^A, \bar{\xi}_A, A = 0, \ldots, 2n - 2$ are the $2n$ real coordinates of a complex torus which is fibred over $M_{SK}^{2n-2}$. Their couplings $N^{AB}$ are determined in terms of the holomorphic prepotential $G$ of $M_{SK}^{2n-2}$ via
\[ N^{AB} = \mathcal{G}^{AB} + 2i \frac{\text{Im}(G_{AC})Z^C \text{Im}(G_{BD})Z^D}{\text{Im}(G_{CD})Z^C Z^D}, \] (2.7)

where $Z^A = (1, z^a)$ are the homogeneous coordinates on $M_{SK}^{2n-2}$.

### 2.2 Type II compactified on Calabi-threefolds

Calabi-Yau manifolds $X$ have $h^{1,1}(X)$ Kähler moduli and $h^{1,2}(X)$ complex structure moduli which can be varied independently. Hence the total moduli space is a product
\[ M(X) = M_{(1,1)}(X) \times M_{(1,2)}(X), \] (2.8)

with each component being a special Kähler manifold. In type IIA one has $M_{(1,1)} = M_v$ which, as discussed in the previous section, is characterized by a holomorphic $\mathcal{F}_\Pi$ of the generic form
\[ \mathcal{F}_\Pi = i d_{ijk} t^i t^j t^k + c + \mathcal{O}(e^{-2\pi i t^1}). \] (2.9)

The $t^i$ now denote the Kähler moduli of $X$ with $d_{ijk}$ being the classical intersection numbers of their dual two-cycles. $c$ is a constant proportional to the Euler number of $X$ while the last term denotes the worldsheet instanton corrections.

On the other hand $M_{(1,2)} \subset M_s$ where the full $M_s$ also features the dilaton and axion plus $2(h^{1,2} + 1)$ scalars from the RR-sector. At the string tree level $M_s$ is a special quaternionic-Kähler manifold in the image of the c-map (2.5), i.e. the metric is of the form (2.6). Now $(\phi, \bar{\phi})$ are the dilaton and axion respectively while the $z^a, a = 1, \ldots, h^{1,2}$, are the complex structure moduli parametrizing the special Kähler manifold $M_{(1,2)}$ with metric $g_{ab}$. In full generality this metric is only known for Calabi-Yaus with $h^{1,2}$ small.
The real \((\xi^A, \tilde{\xi}^A), A = 0, \ldots, h^{1,2}\), arise from expanding the RR three-form \(C_3\) and are the fibre coordinates (coordinates of the intermediate Jacobian). So altogether there are indeed \(4(h^{1,2} + 1)\) scalars \((\phi, \tilde{\phi}, z^a, s^A, \tilde{\xi}^A)\).

In type IIB the assignment is reversed with \(M_{(1,2)} = M_v\) while \(M_{(1,1)} \subset M_h\). The full \(M_h\) is again completed by the axio-dilaton together with \(2(h^{1,1} + 1)\) scalars from the RR-sector. Mirror symmetry states that type IIA compactified on \(X\) is equivalent to type IIB compactified on the mirror manifold \(\tilde{X}\). This implies in particular

\[
M_{(1,1)}(X) = M_{(1,2)}(\tilde{X}) , \quad M_{(1,2)}(X) = M_{(1,1)}(\tilde{X}) .
\]

(2.10)

or equivalently

\[
F_{\text{IIA}}(X) = G_{\text{IIB}}(\tilde{X}) , \quad G_{\text{IIA}}(X) = F_{\text{IIB}}(\tilde{X}) .
\]

(2.11)

Since the dilaton is in the hypermultiplet sector the metric (2.6) receives quantum correction which generically cannot be written in the form (2.6). In other words the quantum corrected hypermultiplet sector is no longer a special quaternionic-Kähler manifolds and the quantum corrections cannot be viewed as correction of the holomorphic prepotential \(G_{\text{II}}\) [19, 39–42].

### 2.3 Heterotic compactified on \(K3 \times T^2\)

On the heterotic side the vector multiplets arise from the original ten-dimensional gauge group \(E_8 \times E_8\) or \(SO(32)\) together with three Kaluza-Klein vector fields of \(T^2\).\(^1\) Their couplings are described by a prepotential of the form [43–45]

\[
F_{\text{het}} = iS_{\eta_{\alpha \beta}} T^\alpha T^\beta + f(T) + g(T, W) + \mathcal{O}(e^{-2\pi i S}) ,
\]

(2.12)

where \(S\) denotes the heterotic dilaton and \(\eta_{\alpha \beta}\) is an \(SO(2, n-2)\) metric \((\alpha, \beta = 1, \ldots, n)\). The \(T^\alpha\) consist of two moduli parametrizing Kähler and complex structure deformations of the \(T^2\) together with Wilson-line moduli in the Cartan subalgebra of the \(E_8 \times E_8\) or \(SO(32)\) which parametrize the Coulomb branch of this gauge bundle. \(f(T)\) are the one-loop threshold corrections (computed for example in [43, 44, 46]) while \(g(T, W)\) denotes the couplings of additional gauge multiplets \(W\) which are light due to non-perturbative effects [45]. Finally, \(\mathcal{O}(e^{-2\pi i S})\) stands for non-perturbative corrections which are governed by the dilaton. The string tree level corresponds to large \(S\) in which case only the first term in (2.12) survives and the vector multiplets parametrize the space

\[
M_v = \frac{\text{SU}(1,1)}{U(1)} \times \frac{\text{SO}(2, n-2)}{\text{SO}(n-2) \times \text{SO}(2)} .
\]

(2.13)

Among the one-loop and non-perturbative corrections one finds the Seiberg-Witten \(F_{\text{SW}}\) in an appropriate double scaling limit where all gravitational corrections are turned off [47, 48]. (We return to this point section 2.4.)

The hypermultiplet sector sensitively depends on choice of gauge bundle and is constrained by the Bianchi identity

\[
\int_{K3} \text{tr} F^2 + n_5 = n + n_5 = \int_{K3} \text{tr} R^2 = 24 ,
\]

(2.14)

\(^1\)The fourth vector field is the graviphoton.
where \( n_5 \) counts the number of background five-branes and \( n = \int_{K3} \text{tr} F^2 \) is the instanton number. In perturbative compactifications \( n_5 = 0 \) holds. The hypermultiplet scalars are the \( K3 \) moduli (or rather a subset thereof), the moduli arising from the NS \( B \)-field and the moduli of instanton solutions on \( K3 \), i.e. the gauge bundle deformations. In fact the space of gauge bundle moduli is fibred over the moduli space of a \( K3 \) base.\(^2\)

At specific loci of its moduli space the \( K3 \) becomes singular. These singularities follow an ADE-classification. Near such a singularity one can replace the \( K3 \) by an ALE space which is asymptotic to \( \mathbb{C}^4/\Gamma \) where the finite group \( \Gamma \) depends on the choice of the ADE-singularity. In [18, 50–52] it was argued that in this limit the hypermultiplet moduli space coincides with the moduli space of a three-dimensional gauge theory with the corresponding ADE-gauge group.

On the heterotic side the dilaton is part of a vector multiplet and hence \( M_h \) receives \( \alpha' \)-corrections but is “exact” in string \((g_s)\) perturbation theory. However, this argument is too naive since non-perturbative effects which do not have the canonical \( e^{-g_s^2} \) dependence can contribute to \( M_h \). In fact the moduli space of instanton solutions is generically singular when the size \( \rho \) of an instanton shrinks to zero, i.e. \( \rho \to 0 \) [53]. This singularity can be seen in string theory already at the tree level and in particular in the heterotic hypermultiplet moduli space. In ref. [32, 33] the singularity was interpreted as a non-perturbative effect – either as an enhanced gauge symmetry for the \( SO(32) \) heterotic string or as a wrapped 5-brane becoming tensionless for the \( E_8 \times E_8 \) heterotic string.

For an arbitrary gauge group \( G \) the moduli space of instantons on \( K3 \) is a hyper-Kähler manifold with (quaternionic) dimension

\[
N_n = nh(G) - \text{dim}(G) \, ,
\]

(2.15)

where \( n \) is the instanton number and \( h(G) \) the dual Coxeter number of \( G \). For a small \( E_8 \) instanton \( n \) changes by one and thus \( h(E_8) - 1 = 29 \) hypermultiplets are fixed where the extra modulus parametrizes the location of the small instanton. In \( d = 6 \) this modulus is part of an additional tensor multiplet which in \( d = 4 \) can be dualized to an additional \( U(1) \) vector multiplet.\(^3\) These \( U(1) \) vector multiplets are denoted by \( W \) in (2.12) and some of their couplings have been computed, for example in ref. [45].

For a generic instanton background with structure group \( SU(n) \), we can describe its moduli by means of the spectral cover [29]. In fact, \( SU(n) \) bundles over an elliptically fibred manifold are described by a spectral cover \( C_H \) (i.e. an \( n \)-cover of the base \( \pi : C_H \to B \)) and a line bundle \( \mathcal{N} \) over \( C_H \) [29]. The line bundle over \( C_H \) is given in terms of its first Chern class \( c_1(\mathcal{N}) \) and a twist by a flat bundle on \( C_H \) (if it is not simply-connected). When the elliptic fibration is a \( K3 \) surface, the spectral cover is a curve that in general has non-trivial fundamental group and the first Chern class of \( \mathcal{N} \) is completely determined by the spectral curve. In fact [29],

\[
\mathcal{N} = K_{C_H}^{3/2} \otimes K_B^{-1/2} \otimes F \, ,
\]

\(^2\)Strictly speaking there also is the possibility of charged matter multiplets. However, for most of our considerations we go to the Higgs- or Coulomb branch of these theories where they become massive or can be viewed as additional gauge neutral bundle moduli. Some couplings of the charged matter fields have been determined recently in ref. [49].

\(^3\)This multiplet is know as the \( N = 2 \) vector-tensor multiplet [43] which is dual to vector multiplet.
where \( K_C \) and \( K_B \) are the canonical bundles of \( C_H \) and \( B \) and \( \mathcal{F} \) is a flat bundle over \( C_H \). Such flat bundles are classified by the Jacobian \( J(C_H) \) of \( C_H \).

2.4 Heterotic-type IIA duality

In [1, 2] it was conjectured that apart from perturbative dualities also non-perturbative dualities hold among string backgrounds. In these cases the dualities involve the string coupling in a non-trivial way. One of the prominent example is the duality of the heterotic string compactified on a four-torus \( T^4 \) and type IIA string theory compactified on \( K3 \). These backgrounds have six Lorentzian space-time dimension and 16 unbroken supercharges. The respective dilatons are mapped inversely to each other mapping the strong coupling region of one theory to the weak coupling regime of the dual theory.

A close “cousin” of this duality is conjectured to hold in four space-time dimensions with eight unbroken supercharges and can be viewed as a fibred version of the six-dimensional duality. More precisely, one fibres both sides over a \( \mathbb{P}^1 \) base such that on the heterotic side the compactification manifold is \( K3 \times T^2 \) with the \( K3 \) being elliptically fibred (with a section) over the \( \mathbb{P}^1 \). (We denoted the \( \mathbb{P}^1 \) on the heterotic side by \( B_H \) and the \( K3 \) by \( S_H \) in the following.) On the type II side the \( K3 \) is fibred over the \( \mathbb{P}^1 \) base (which we denote by \( B_{II} \) in the following) in such a way that the resulting threefold \( X \) is Calabi-Yau. Due to the elliptic fibration of the \( K3 \) the threefold \( X \) is also elliptically fibred over a ruled surface \( \Sigma \). Thus the conjectured duality relates the heterotic string compactified on \( K3 \times T^2 \) with an elliptic \( K3 \) to type IIA string theory compactified on a \( K3 \)-fibred Calabi-Yau threefolds [10–12, 15]. The heterotic dilaton, which is part of a vector multiplet, is mapped to the type IIA Kähler modulus which controls the size of the \( B_{II} \) and thus it is not a strong-weak duality but rather relates the string coupling to a geometrical modulus. This picture has been confirmed in the vector multiplet sector by organizing the type II vector multiplet couplings in a form corresponding to the heterotic perturbation theory and then comparing both sides for a (large) number of dual backgrounds.\(^4\) The success of these checks is partly related to the fact that the \( N = 2 \) vector multiplet couplings are encoded in terms of the holomorphic prepotential \( \mathcal{F} \) which can be easily compared or constrained.

Concretely one finds in the large dilaton/large base volume limit

\[
\mathcal{F}_{II} = \mathcal{F}_{het} = iS\eta_{\alpha\beta}T^\alpha T^\beta + id_{\alpha\beta\gamma}T^\alpha T^\beta T^\gamma + \ldots, \tag{2.16}
\]

where for simplicity we have adopted the heterotic notation. On the type IIA side, \( S \) denotes the volume modulus of the base \( B_{II} \), while the \( T^\alpha \) denote the Kähler moduli of the generic \( K3 \) fibre. \( \eta_{\alpha\beta} \) is an \( SO(2, n - 2) \) metric while \( d_{\alpha\beta\gamma} \) are the (classical) intersection numbers of the Kähler moduli on the type II side which are related to one-loop threshold corrections on the heterotic side. The ellipsis stand for exponentially suppressed terms and moduli parametrizing singular \( K3 \) fibres (see [12, 13, 15] for more details).

At special points in this moduli space non-Abelian gauge bosons occur. On the heterotic side this is a standard Higgs mechanism where all (or part) of the original \( E_8 \times E_8 \) gauge symmetry becomes visible. On the type II side the non-Abelian gauge enhancement occurs in the vicinity of the ADE-singularities of the \( K3 \) fibres. In an

\(^4\)See, for example, [10–15] or the reviews [16, 54].
appropriate “double scaling limit” which corresponds to turning off gravity and zooming
in on one specific singularity one can replace the $K3$ fibre by an ALE space and recover
the Seiberg-Witten $F_{SW}$ from $F_{II}$ [47, 48].

A similar analysis for the hypermultiplet sector was pioneered in [17] where on the
heterotic side the dual type II dilaton was identified. However, due to the more compli-
cated geometry of the hypermultiplets the duality has not been established at a similar
quantitative level and one of the goals of this paper is to improve this situation. In order
to do so we need however to first recall the results of [17]. For simplicity we confine our
attention to $E_8 \times E_8$ backgrounds henceforth.

As a first step to repeat the analysis in the hypermultiplet sector ref. [17] identified
the volume of the heterotic base $B_H$ with the type IIA dilaton. With this identification
it is in principle possible to formally organize the heterotic hypermultiplet couplings in
a type IIA perturbation theory, or in other words in powers of the type IIA dilaton. The
problem is that on the heterotic side very little is explicitly known about $M_h$.

In the duality, type IIA tree-level contributions correspond to $B_H$ being large. In this
limit one can still have $\alpha'$ corrections on the heterotic side, as they depend generically on
the volume of $K3$. Therefore one can achieve a further simplification by considering $K3$s
with large volume and large $B_H$. A sufficient condition for a generic $K3$ (i.e. without
ADE singularities) with large base to have large volume is to also take the elliptic fibre
to be large. The modulus parametrizing the (complexified) fibre volume corresponds to
a specific complex structure modulus of the Calabi-Yau threefold $X$ that we will call
the “smoothing parameter” $\sigma$. Taking the volume of the fibre to be infinitely large then
corresponds to a particular limit (let us call it $\sigma \to \infty$) in the complex structure moduli
space of $X$ which is known as the stable degeneration limit $X \to X^\sharp$. In this limit $X^\sharp$ is
still a $K3$ fibration over $B_H$, but now the $K3$ is degenerate over each point: It is the union
of two rational elliptic surfaces intersecting along an elliptic curve. Correspondingly, $X^\sharp$
is the union of two three-folds $X_1$ and $X_2$, intersecting along an elliptically fibred $K3$ surface
$S_*$. The type IIA complex structure moduli describing the 18 complex deformations of
$S_*$ are mapped by duality to the complex structure moduli of the heterotic $S_H$. On the
other side, the type IIA moduli describing the complex structure deformation of $X_1$ and
$X_2$ are mapped to the complex deformations of the two spectral curves describing the
bundle moduli of the two $E_8$ factors.

Let us now consider the third cohomology group of $X^\sharp$. It splits into three pieces [17]:

$$H_3(X^\sharp) \to H_3(X_1) + H_3(X_2) + M_{20}.$$  \hspace{1cm} (2.17)

The first and the second subspaces are the spaces of three-cycles of $X_1$ and of $X_2$ respec-
tively; they are disjoint as there are no three-cycles in $S_*$. The third subspace in (2.17) is
the set of three-cycles constructed by two-cycles of $S_*$. Since for generic complex structure
the only $(1,1)$-cycles of $S_*$ are the fibre and the base of the elliptic fibration, there
are 20 two-cycles of $S_*$ that are homologically trivial in $X^\sharp$ and that are the boundary
of a three-chain in $X_1$ and a three-chain in $X_2$. By these chains one can construct closed
three-cycles in $X^\sharp$.

Going away from the degenerate limit, a set of three-cycles blow up which we will call
$M^\sharp_{20}$. In fact, as $\sigma$ becomes finite, a circle fibred over $S_*$ appears (it was shrunk to zero
in the degenerate limit). The fibration of this circle over the 20 two-cycles orthogonal to
base and fibre of $S_*$ gives the three-cycles in $M_{20}^*$. Together with the three-cycles of $M_{20}$, they form a symplectic subspace of $H_3(X^2)$.

On the type IIA side $H_3(X)$ determines the complex structure deformations $z^a$ but also the $2(h^{1,2} + 2)$ real moduli $\xi^A, \tilde{\xi}^A, A = 0, \ldots, h^{1,2}$ which arise as zero modes of the RR three-form $C_3$. In the dual heterotic vacuum the $C_3$-moduli arising from $H_3(X_1)$ correspond to bundle moduli of the first $E_8$, while the $C_3$-moduli arising from $H_3(X_2)$ correspond to bundle moduli of the second $E_8$. In particular, they correspond to the Jacobians of the two spectral curves related to the two $E_8$ factors. Furthermore, the $C_3$-moduli arising from $M_{20}$ are mapped to the heterotic $B$-field along two-cycles of $S_H$. Away from the degenerate limit there are also the moduli given by $C_3$ along $M_{20}$. These correspond to the Kähler moduli (with exclusion of fibre and base volumes) of the heterotic $S_H$. They are frozen in the limit of large fibre and base in heterotic theory.

Let us summarize the map of the spectrum in table 2.1. In particular, note that we can give a type IIA counterpart for all heterotic $K3$ moduli.

| IIA | Het | #\(\mathbb{R}\) |
|-----|-----|-----------|
| dilaton/axion | $\phi, \tilde{\phi}$ | vol($\mathbb{P}^1$) + $B$-field | $\Phi$ | 2 |
| smoothing parameter | $\sigma$ | vol(fibre) + $B$-field | $s$ | 2 |
| $cs(S_*)$ | $\tau$ | $cs(S_H)$ | $t$ | 36 |
| $C_3(M_{20})$ | $\xi$ | $20$ $B$-fields | $\tilde{c}$ | 20 |
| $C_3(M_{20}')$ | $\xi$ | $20$ $J$ moduli | $c$ | 20 |
| $cs(X_1)$ | $E_8^1$-bdl moduli: spectral cover $C_H^1$ | $c$ | model dep. |
| $C_3(X_1)$ | $E_8^1$-bdl moduli: Jacobian $J(C_H^1)$ | model dep. |
| $cs(X_2)$ | $E_8^2$-bdl moduli: spectral cover $C_H^2$ | model dep. |
| $C_3(X_2)$ | $E_8^2$-bdl moduli: Jacobian $J(C_H^2)$ | model dep. |

Table 2.1: Heterotic-type II map at the tree-level.

### 3 K3 moduli space in Ferrara-Sabharwal form

One of our goals is to make (part of) the table 2.1 explicit at the level of the effective action. In order to do so we start at the type IIA tree level and take the stable degeneration limit which allows us to neglect $\alpha'$ corrections in the dual heterotic background. Furthermore, in this section we freeze all gauge bundle moduli for simplicity and only study the ‘geometrical’ moduli space arising from deformations of the $K3$ metric together with the moduli arising from the NS $B$-field. For this subset of fields we determine the precise map between heterotic and type II field variables.

It is convenient to introduce the notation

$$\text{Gr}_{n,m} \equiv \frac{\text{SO}(n,m)}{\text{SO}(n) \times \text{SO}(m)}, \quad (3.1)$$

which has real dimension $\text{dim}_R(\text{Gr}_{n,m}) = nm$. The 58 moduli of the $K3$ metric form the space

$$\text{Gr}_{3,19} \times R^+, \quad (3.2)$$

9
where the $R^+$ corresponds to the volume. Including the 22 $B$-fields the moduli space is locally given by \[55\]

$$M_{K3} = \text{Gr}_{4,20} .$$

However, it can happen that a specific solution of (2.14) fixes some of the $K3$ moduli. Since $N = 2$ constrains $M_h$ to be quaternionic-Kähler one typically obtains the $4n$-dimensional subspace

$$\mathcal{M}_{K3} = \text{Gr}_{4,n} \subset M_{K3} ,$$

where $n \leq 20$.\(^5\) If we additionally freeze the $B$-fields $\mathcal{M}_{K3}$ is reduced to $\text{Gr}_{3,n-1} \times R^+$. Fixing also the Kähler class of $K3$ a further reduction to $\text{Gr}_{2,n-1}$ occurs. Finally, if the $K3$ is also elliptically fibred the complex structure moduli preserving the elliptic fibration span $\text{Gr}_{2,n-2}$. This can be summarized by the chain

$$\mathcal{M}_{K3} = \text{Gr}_{4,n} \rightarrow \text{Gr}_{3,n-1} \times R^+ \rightarrow \text{Gr}_{2,n-1} \rightarrow \text{Gr}_{2,n-2} .$$

The heterotic action for the $K3$ moduli can be written as \[56, 57\]

$$\mathcal{L} = -\frac{1}{8}(\partial \rho)^2 + \frac{1}{16}(\partial \hat{M})^2 - \frac{1}{4} e^{-\rho} \hat{M}^{IJ} \partial B_I \partial B_J ,$$

where $e^{-\rho}$ is the $K3$ volume, $\hat{M}$ is a $\text{SO}(3, n-1)$ matrix parametrizing $\text{Gr}_{3,n-1}$ and $B_I$ are $n + 2$ $B$-fields. In order to prepare the comparison with type IIA we need to rewrite this action in the Ferrara-Sabharwal form (2.6). Indeed it is known [30] that $\text{Gr}_{4,n}$ is a special quaternionic manifold, i.e. this space is in the image of the c-map with a special Kähler base $M_{SK}^{2(n-1)} = \frac{SU(1,1)}{U(1)} \times \text{Gr}_{2,n-2}$. So we only need to identify the base inside $\text{Gr}_{4,n}$ and then put (3.6) into the form (2.6). However, this turns out to be surprisingly involved.

As we reviewed in section 2.1 the Ferrara-Sabharwal metric (2.6) in general is a metric of a special quaternionic-Kähler manifold $M_{SK}^{4n}$ which is in the image of the c-map (2.5). In other words it has a base $\frac{SU(1,1)}{U(1)} \times M_{SK}^{2(n-2)}$ where the first factor is parametrized by the coordinates $(\phi, \tilde{\phi})$ (corresponding to dilaton and axion on the type II side), while the second factor is parametrized by the complex variables $z^a$. Our first task therefore is to identify this base in the heterotic metric (3.6) in the stable degeneration limit.

In order to do so we only keep the two Kähler moduli $\tilde{\Phi}, s$ corresponding to the volume of the base $B_H$ and the elliptic fibre (first two lines in table 2.1) and for the moment freeze the other $n$ Kähler moduli. As noted in (3.5) the $2(n-2)$ complex structure deformations $t^i$ which preserve this elliptic fibration span the moduli space $\text{Gr}_{2,n-2}$. (They correspond to the third line in table 2.1.) Thus altogether we have in this limit

$$\mathcal{M}_{K3} \rightarrow \frac{SU(1,1)}{U(1)} \times M_{SK}^{2(n-1)} ,$$

with

$$M_{SK}^{2(n-1)} = \frac{SU(1,1)}{U(1)} \times \text{Gr}_{2,n-2} ,$$

where the two $\frac{SU(1,1)}{U(1)}$ factors are spanned by $\tilde{\Phi}$ and $s$. From (2.5) we now see that we can reconstruct the entire $\mathcal{M}_{K3} = \text{Gr}_{4,n}$ as a c-map of the special-Kähler base $M_{SK}^{2(n-1)}$.

\(^5\)For the standard embedding and the case of point-like instantons [32, 36], all $K3$ moduli can be varied freely and one has $n = 20$. In the following we always discuss the case of arbitrary $n$ however.
The 2n-dimensional space which is fibred over \( M_{SK}^{2(n-1)} \) is parametrized by the \( n \) Kähler moduli which were frozen so far and \( n \) B-fields.

In appendix A we perform a number of field redefinitions which at the end put (3.6) into the form (2.6). A complication arises from the fact that the metric of \( M_{SK}^{2(n-1)} \) (denoted by \( g_{ab} \) in (2.6)) is not immediately in the canonical form as given in (2.2). Rather it is in field variables where the holomorphic prepotential does not exist and only after a symplectic rotation can be expressed as the derivative of a prepotential \( G \) [58]. After this rotation we find (as expected)

\[
G_{\text{het}} = is \eta_{ij} t^i t^j, \quad i, j = 2, \ldots, n - 1,
\]

where the \( t^i \) are the \( (n - 2) \) complex structure deformations preserving the elliptic K3 and \( \eta_{ij} \) is an \( \text{SO}(2, n - 2) \) metric.

### 4 Heterotic - Type IIA duality in the hypermultiplet sector

#### 4.1 General considerations

Let us now see what in turn the heterotic results imply on the type IIA side. From (3.8) we conclude that in the stable degeneration limit the complex structure moduli space of \( X \) has to take a particular form. There has to be a limit where the holomorphic prepotential \( G_{II} \) which appears in the Ferrara-Sabharwal metric and which describes the complex structure moduli space of \( X \) takes a form identical to (3.8), i.e.

\[
G_{II} = i\sigma \eta_{ij} \tau^i \tau^j + \ldots, \quad i, j = 2, \ldots, n - 1,
\]

where \( \sigma, \tau^i \) are now complex structure moduli of the dual type IIA background. However, the form (4.1) generically does not appear in the complex structure moduli space of \( X \). In fact comparing (4.1) with (2.16) we see that \( G_{II} \) coincides with \( F_{II} \) in the large volume (or more precisely large \( B_{II} \) limit) if we identify \( \sigma \leftrightarrow S, \tau^i \leftrightarrow T^\alpha \). This implies that in the complex structure moduli space there is a limit where the prepotential \( G_{II}(X) \) agrees with \( F_{II}(\tilde{X}) \) in the Kähler moduli space of some other (mirror) manifold \( \tilde{X} \). Of course this is precisely the statement of mirror symmetry and the necessary agreement of (2.16) with (4.1) suggests that the mirror threefold \( \tilde{X} \) also has to be a K3 fibration. We are thus lead to the conjecture:

**Type II compactified on a Calabi-Yau threefold \( X \) can be dual to a heterotic string vacuum only if \( X \) and its mirror threefold \( \tilde{X} \) are simultaneously K3 fibrations.**

The modulus \( \sigma \) is than the mirror dual of the volume \( S \) of the base \( B_{II} \) of \( \tilde{X} \) and the \( \tau^i \) are the mirror duals of the K3 fibre Kähler moduli \( T^\alpha \) of \( \tilde{X} \).

Once we assume the validity of this conjecture we can draw further conclusions in that the structure of the vector multiplet couplings of the K3-fibred \( \tilde{X} \) has additional properties. For example, the Coulomb branch has a non-Abelian gauge enhancement at

---

6Note that the set of Calabi-Yau three-folds studied in [18] do satisfy this conjecture.
the ADE-singulites of $K3$ as we briefly discussed in section 2.3. Thus in an appropriate double scaling limit one has $G_{\text{II}} = G_{\text{SW}}(ADE)$ \[18, 50–52\].

At a finite number of points on $B_{\text{II}}$ the $K3$ fibre can degenerate and the Kähler moduli associated with the components of such degenerate fibres have no intersection with the $B_{\text{II}}$ volume $S$ \[15\]. Translated to the complex structure moduli space of $X$ this says that $G_{\text{II}}$ can be parametrized analogous to (2.12) by

$$G_{\text{II}} = i\sigma \eta_{ij} \tau^i \tau^j + f(\tau) + g(\tau, \omega) + \ldots , \quad (4.2)$$

where the ellipsis stand for terms which are exponentially suppressed in the large $\sigma$ limit. $f(\tau)$ is a function of the $\tau^i$ which in the large complex structure limit is cubic plus a constant $\rho$ proportional to the Euler number plus exponentially suppressed terms

$$f(\tau) = id_{ijk} \tau^i \tau^j \tau^k + \rho + \mathcal{O}(e^{-2\pi i \tau}) \quad (4.3)$$

This in turn further determines the structure of $G_{\text{het}}$ of which we only computed the leading contribution in the large $s$-limit in (3.8). The couplings of $f(\tau)$ given in (4.3) we can understand on the heterotic side as $\alpha'$ corrections. Recall that the heterotic $t^i$ (which are dual to the $\tau^i$) correspond to complex structure deformations of $K3$ that preserve the elliptic fibration. Furthermore, they control the volume of the transcendental two-cycles $C_2$ which is approximately $\text{vol}(C_2) \sim e^{-\rho/2} t$. In the limit of large $K3$ volume (i.e. $e^{-\rho/2}$ large) and for generic values of $t^i$, these cycles are also large. However, when some $t^i$ go to zero, a two-cycle shrinks and a worldsheet instanton wrapping $C_2$ contributes to the prepotential. Thus we can identify the last term in (4.3) as a contribution from worldsheet instantons on the heterotic side.\footnote{Note that this term appears already at the classical level on the type IIA side.}

The cubic term in (4.3) is a bit more difficult to understand since it appears as a perturbative $\alpha'$ correction. On the type II side it becomes relevant when $\sigma$ is finite, i.e. when we go away from the degeneration limit in type IIA. Departing from this limit on the heterotic side means that the $\alpha'$ corrections become relevant. However, the cubic term cannot be interpreted as an instanton correction, as it becomes subleading when the volumes of the wrapped cycles become small. It would be nice to identify the origin of this term in more detail on the heterotic side.

The function $g(\tau, \omega)$ in (4.2) summarizes on the type II side the contribution of the degenerate fibres whose moduli we denote by $\omega$. As already stated above it cannot have any $\sigma$-dependence. Furthermore, they are related to vector multiplet couplings of the type IIA theory on the mirror $\tilde{X}$. Thus using the results of \[45\] we can give generically

$$g(\tau, \omega) = c_{ij} \tau^i \tau^j \omega + c_i \tau^i \omega^2 + c \omega^3 + \ldots , \quad (4.4)$$

where $c_{ij}$, $c_i$ and $c$ are model dependent constants.

### 4.2 Point-like instantons

We now consider the case in which the heterotic gauge bundle on $K3$ is reduced to 24 point-like instantons \[32, 36\]. A point-like instanton is a limit of a smooth bundle with instanton number 1, in which all the curvature is concentrated into a single point.\footnote{A point-like instantons can be thought as an NS5 brane wrapping the four-dimensional space-time times the torus $E_{\text{II}}$.} If
the bundle is given only by point-like instantons, we need 24 of them in order to satisfy the Bianchi identity (2.14). The holonomy around the location of a point-like instanton is trivial. Hence, the full $E_8 \times E_8$ gauge group is unbroken. The classical hypermultiplet moduli space is given by the 44-dimensional space

$$\text{Gr}_{4,20} \ltimes \text{Sym}^2(S_H),$$

where the first factor is the $K3$ moduli space, while the second factor denotes the location of the 24 instantons on $S_H$. Since the shape of $S_H$ depends on the moduli of the first factor, the product is a warped product [59]. On the vector multiplet side each point-like instanton introduces one new $d = 4$ vector multiplet [32] so that on the 16-dimensional Coulomb branch of $E_8 \times E_8$ we have altogether 43 vector multiplets which includes the three vector multiplets of the heterotic compactification on $T^2$.

The dual type IIA Calabi-Yau manifold is a singular elliptic fibration over the Hirzebruch surface $\mathbb{F}_n$ [34,35]. $\mathbb{F}_n$ is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. It is described by the homogeneous coordinates $(z, w, u, v)$ identified under the transformations

$$\mu : (z, w, u, v) \rightarrow (z, w, \mu u, \mu v) \quad \lambda : (z, w, u, v) \rightarrow (\lambda z, \lambda w, \lambda^n u, v).$$

An elliptic fibration over this space can be described by the Weierstrass model

$$y^2 = x^3 + f(z, w, u, v) x \zeta^4 + g(z, w, u, v) \zeta^6.$$  \hspace{1cm} (4.5)

For constant $f$ and $g$, this equation describes a torus embedded into the weighted projective space $\mathbb{P}^2_{1,2,3}$. When $f$ and $g$ are functions of the base coordinates, equation (4.7) describes a torus fibration. The elliptic fibre degenerates over points where the discriminant $\Delta = 4f^3 + 27g^2$ vanishes. If the degeneration is sufficiently hard, this produces a singularity on the Calabi-Yau. The elliptic fibration over $\mathbb{F}_n$ is also a $K3$ fibration over the $\mathbb{P}^1_{z,w}$ base spanned by the $(z, w)$-coordinates.

In order to make contact with the point-like instanton case, we need to constrain the complex structure of the Calabi-Yau. In fact, the corresponding bundle must leave the full $E_8 \times E_8$ gauge group unbroken. On the type IIA side, this is realized by a Calabi-Yau with two $E_8$ singularities. The corresponding Weierstrass model takes the form [34,35]

$$y^2 = x^3 + f_8(z, w) v^4 u^4 x + g_{12-n}(z, w) v^5 u^7 + g_{12}(z, w) v^6 u^6 + g_{12+n}(z, w) v^7 u^5,$$

with discriminant

$$\Delta = v^{10} u^{10} \left( 4f_8^3 v^2 u^2 + 27(g_{12+n} v^2 + g_{12} v u + g_{12-n} u^2)^2 \right).$$

This space has one $E_8$ singularity at $u = 0$ and one $E_8$ singularity at $v = 0$. The rest of the discriminant locus intersects these singularities in double points: It intersects $\{ v = 0 \}$ at $12 - n$ points (zeroes of $g_{12-n}$), while it intersects $\{ u = 0 \}$ at $12 + n$ points (zeroes of $g_{12+n}$). At these points the singularity becomes worse than an $E_8$ singularity. The locations of such singularities on the $\mathbb{P}^1_{z,w}$ base of the $K3$ fibration are controlled by the complex structure of $X$ (related to the coefficients of $g_{12-n}$ and $g_{12+n}$). They correspond to the positions of the heterotic point-like instantons on the base $B_H$. Their positions on the elliptic fibre of $S_H$ are dual to $C_3$ deformations along three cycles of $X_1$ and $X_2$, respectively.
as explained in Section 2.4. Looking at the $K3$ fibration structure, the $K3$ fibre has two $E_8$ singularities over each point of the $\mathbb{P}^1_{z,w}$ base.

We can go to the Coulomb branch by resolving the singularities. Each $E_8$ singularity is resolved by blowing up eight $\mathbb{P}^1$s along the elliptic fibre. They sit at $u = 0$ (or $v = 0$). Fibering them along the $\mathbb{P}^1_{z,w}$ base of the $K3$ fibration, we get eight exceptional divisors $D_{i,\sigma}^{E_8}$ ($i = 1, \ldots, 8$ runs over the eight $E_8$ two-cycles and $\sigma = 1, 2$ runs over the two $E_8$s). On top of the 24 zeros of $g_{12-n}$ and $g_{12+n}$, the singularity is worse than an $E_8$ one. The resolution is made by blowing up the points $\{v = 0, \ g_{12-n} = 0\}$ and $\{u = 0, \ g_{12+n} = 0\}$ of the $\mathbb{F}_n$ base. The corresponding exceptional divisors $D^\omega_a$ ($a = 1, \ldots, 24$) are elliptic fibrations over the blown-up two-cycles in $\mathbb{F}_n$. Before resolving, the only large divisors were the $K3$ fibre $D^\sigma$, the exceptional divisors $D^{E_8}_{k,\sigma}$, and the elliptic fibration $D^\tau_u$ of the base of the $K3$ fibration. After resolving we have in addition $5 + 8 + 24 = 40$ exceptional divisors. The smooth Calabi-Yau $X$ has then $h^{2,1}(X) = h^{1,1}(X) = 43$, i.e. Euler characteristic $\chi(X) = 0$.

$X$ is self-mirror [59] and then it trivially respects the conjecture that the mirror of a $K3$ fibration with heterotic dual is also a $K3$ fibration. Under mirror symmetry, the complex structure moduli related to the point-like instanton positions $\omega_a$ are mapped to the coefficients of the Kähler form along the (Poincaré dual of the) exceptional divisors $D^\omega_a$ located at those points. On the other hand, the complex structure moduli $\tau^i$ related to the complex structure of the heterotic $K3$ are mapped under mirror symmetry to the coefficients along $D^{\tau_u}$, $D^{\tau_{\sigma}}$ and $D^{E_8}_{k,\sigma}$ ($k = 1, \ldots, 8$ and $\sigma = 1, 2$). This allows us to give the leading behavior of $g(\tau, \omega)$ anticipated in (4.4), using the intersection numbers of the divisors $D^\tau$ and $D^\omega$. The elliptic fibration structure of $D^\omega$ implies that their triple self-intersection is zero. Moreover, as they are blown up at different points on the $\mathbb{F}_n$ base, they do not intersect each other. Their double self-intersection is an elliptic fibration over points of the blown-up $\mathbb{F}_n$ base. These points in general miss the $D^\tau$ divisor, except for $D^{\tau_u}$ that is the class $\mathbb{F}_n$. Hence, the only non-zero triple intersections with two $D^\omega$ divisors are $D^{\tau_{\sigma}} \cdot D^\omega_a \cdot D^\omega_b$. The $D^\tau$ divisors are fibrations of the $K3$ two-cycles over the $\mathbb{P}^1_{z,w}$ base. If we intersect two of them, we obtain a curve that is the fibration of the intersection points on the $K3$ fibre over the $\mathbb{P}^1_{z,w}$ base. These points in general miss the locations of the $D^\omega$ divisors. This does not happen for the exceptional $E_8$ divisors $D^{E_8}_{k,\sigma}$, as they are located at $u = 0$ and $v = 0$, where the $D^\omega$ divisors sit. In particular, the $12 - n$ $D^\omega$ divisors located at $v = 0$ intersect the $D^{E_8}_{k,\sigma}$ divisors with $\sigma = 1$, while the $12 + n$ ones located at $u = 0$ intersect the $D^{E_8}_{k,\sigma}$ divisors with $\sigma = 2$.

We are now able to give $g(\tau, \omega)$ more explicitly:

$$g(\tau, \omega) = i \tau^u \sum_{a=1}^{24} \omega^a \omega^a + \sum_{a=1}^{12-n} i \omega^a \sum_{k,\ell} \eta_{k,\ell}^{(1)} T_{E_8}^{k,1} T_{E_8}^{\ell,1} + \sum_{a=13-n}^{24} i \omega^a \sum_{k,\ell} \eta_{k,\ell}^{(2)} T_{E_8}^{k,2} T_{E_8}^{\ell,2} + \ldots \ (4.10)$$

$\eta_{k,\ell}^{(\sigma)}$ is the intersection matrix of the $E_8$ two-cycles of the $K3$ fibre and $\sigma = 1, 2$ refers to the two $E_8$ singularities. We will independently confirm (4.10) in the next section.
4.3 Hypermultiplet moduli space from three-dimensional vector multiplets

The hypermultiplet moduli space is not affected by compactification on a further circle and in the resulting three-dimensional theory we have an additional duality at our disposal. It descends from a duality in seven space-time dimensions which relates M-theory compactified on $K3$ to the heterotic string compactified on $T^3$ [2]. Compactifying this duality on another $K3$ to three dimensions one obtains a relation between two different heterotic theories and two different type IIA theories:

$$
\text{Het}/(S_H \times \tilde{T}^2 \times \tilde{S}^1) \leftrightarrow \text{M-th}/(S_H \times \tilde{S}_H) \leftrightarrow \text{Het}/(S^1 \times T^2 \times \tilde{S}_H) \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
\text{IIA}/(X \times \tilde{S}^1) \hspace{1cm} \text{IIA}/(S^1 \times \tilde{X})
$$

where we take the $K3$ surfaces $S_H$ and $\tilde{S}_H$ to be elliptically fibred. Starting from M-theory in the center, the duality on the left first uses the above mentioned seven-dimensional duality between M-theory and the heterotic string for $\tilde{S}_H$ and then in a second step the heterotic–type IIA duality in four space-time dimensions. On the right the seven-dimensional duality is used for $S_H$ and as a consequence in the type IIA background the mirror Calabi-Yau $\tilde{X}$ appears. In all cases the moduli space of the effective three-dimensional theories is the product of two quaternionic-Kähler spaces. Decompressing on a circle one of the two spaces is projected to a special Kähler manifold and we recover the product of the four-dimensional hypermultiplet and vector multiplet moduli spaces (1.1). Going through the correspondence in detail reveals that the hypermultiplet moduli space of the heterotic theory on the left is mapped to the vector multiplet moduli space of the heterotic theory on the right and vice versa via the c-map [28, 59]. Including the type IIA theories yields the following relations between the respective moduli spaces:

$$
M_v(X) = M_v(S_H \times \tilde{T}^2) \leftrightarrow M_h(\tilde{S}_H \times T^2) = M_h(\tilde{X}), \\
M_h(X) = M_h(S_H \times \tilde{T}^2) \leftrightarrow M_v(\tilde{S}_H \times T^2) = M_v(\tilde{X}).
$$

Hence, one can derive the hypermultiplet moduli space of the heterotic theory in the background $S_H \times \tilde{T}^2$ by considering the vector multiplet moduli space of the heterotic string in the background $T^2 \times \tilde{S}_H$ and compactifying on a circle $S^1$. This is the ‘heterotic version’ of deriving the IIA hypermultiplet moduli space on $X$ by considering the vector multiplet moduli space of type IIA on $\tilde{X}$, compactifying on an circle and adding the quantum corrections. This also implies that if the dual of the left heterotic theory is type IIA compactified on the CY $X$ than the dual of the right heterotic theory is type IIA compactified on the mirror $\tilde{X}$ (see [59]). Therefore the duality (4.11) supplies another argument in support of the conjecture which we stated in section 4: In order that the dualities on both sides hold $X$ and $\tilde{X}$ have to be simultaneously $K3$ fibrations.

From the duality (4.11) we can also obtain the prepotential as an expansion in the smoothing parameter $\sigma$ (or the fibre size $s$). As we reviewed in section 2.4 the vector multiplet moduli space of the heterotic string on $T^2 \times \tilde{S}_H$ is given as an expansion in the dilaton $S$ (see (2.16)). Under the duality (4.11), this modulus is mapped to the fibre size $s$ of $S_H$ and then to the smoothing parameter $\sigma$ of type IIA on $X$. Hence,
the prepotential of the vector multiplet moduli space of the heterotic theory on $T^2 \times \tilde{S}_H$ gives the prepotential of the type IIA hypermultiplet moduli space at leading order in the parameter $\sigma$ which controls the stable degeneration.

**Point-like instantons**

Let us apply these considerations to the situation where the bundle is given by 24 point-like instantons on the smooth $K^3 \tilde{S}_H$. In this case, the dual heterotic background is the same, i.e. 24 point-like instantons on a smooth $K^3 \tilde{S}_H$. Hence, both are dual to type IIA on a CY three-fold which is self-mirror and has $h^{2,1} = h^{1,1} = 43$.

The prepotential on the heterotic side is given by the expression (2.12). When the gauge bundle consists of 24 pointlike instantons, we can be more explicit about the functions $f(T)$ and $g(T, W)$. First of all, following the duality (4.11), one sees that while $S$ corresponds to the complexified volume $s$ of the $S_H$ fibre, the $T$ moduli (that on the vector multiplet side are the complex structure $T_{18}$ of $T^2$, the complexified volume $T_{17}$ of $T^2$ and the 16 Wilson lines) correspond to the complex structure of $S_H$ that preserves the elliptic fibration, and $W$ to the positions of the pointlike instantons.

Ref. [45] determined certain properties of $f(T)$ and $g(T, W)$. For example, $f(T)$ is shown to be a cubic polynomial in the ‘large’ $T$ limit. This corresponds to keeping the complex structure of $S_H$ generic in order to avoid shrinking two-cycles, that may introduce $\alpha'$ corrections. In the large $S$ limit $g(T, W)$ also is a cubic polynomial, that in our case reduces to:

$$g(T, W) = -\frac{1}{2} U \sum_i W_i^2 + \sum_i \sum_{m=1}^{16} \gamma_i^m W_i T_m^2 .$$

(4.13)

When $\mathrm{Re} T_{17} > \mathrm{Re} T_{18}$ holds we have to identify $U = T_{18}$ while for $\mathrm{Re} T_{18} > \mathrm{Re} T_{17}$ we have $U = T_{17}$. By translating this result in terms of the complex structure of $S_H$ and the positions of the pointlike instantons, we obtain the same expression as in (4.10). In particular, the moduli $U$ and $T_m$ can be identified with specific complex structure deformations of $S_H$ (see [60] for details).

**SU(2) bundle from 3d Seiberg-Witten**

In heterotic string theory compactified on $K^3 \times T^2$, the hypermultiplet moduli space has a fibration structure, where the base is spanned by the $K^3$ moduli and the fibre by the bundle moduli. The fibre is known to be an hyper-Kähler space (see for example [5]). In this section we consider the situation with an $SU(2)$ bundle on the smooth $K^3 \tilde{S}_H$ and determine the metric on the fibre at generic values of the $K^3$ moduli. To do so, we freeze the geometric moduli at a generic point by taking the volume of $K^3$ to infinity.

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9Heterotic string theory on $K^3 \times T^2$ with point-like instantons is dual to type IIB on $K^3 \times T^2/\mathbb{Z}_2$ orientifold with 16 D7-branes wrapping $K^3$ and 24 D3-branes. The vector multiplet moduli space prepotential has been worked out in [61] at the classical level. Applying the duality map (worked out in detail, for example, in [62]), one recognizes the first term in (4.13), where in type IIB $W_i$ are the positions of the D3-branes on $T^2$ and $U$ the complex structure of the torus. The second term should describe a possible coupling between the D3 and the D7 position moduli, arising at quantum level in type IIB.
The unbroken gauge group (before switching on Wilson lines) is $E_7 \times E_8$. The dual type IIA CY three-fold $X$ has $h^{1,1}(X) = 18$ and $h^{1,2}(X) = 64$ resulting in a 65-dimensional hypermultiplet moduli space $[10, 63]$. Applying the duality (4.11) and decompactifying along $S^1$, one gets a vector multiplet moduli space of (complex) dimension 64. Whenever the $K3$ manifold develops and ADE singularity on one side of the duality, the dual heterotic theory will have a non-abelian gauge bundle with the corresponding structure group. Since there is an $SU(2)$ bundle on $S_H$, the dual surface $\tilde{S}_H$ has an $A_1$-singularity [59]. On the other hand, since $S_H$ is smooth, the dual heterotic theory has unbroken $E_8 \times E_8$ (perturbative) gauge group (before switching on Wilson lines) and the Bianchi identity is saturated by 24 point-like instantons. Each point-like instanton gives rise to a tensor multiplet in six space-time dimensions, that becomes a vector multiplet after compactifying on $T^2$. When the point-like instanton sits at a singular point of the $K3$ manifold, new (non-perturbative) vector multiplets arise. In [36] the gauge enhancement when $k$ point-like instantons sit on an $A_1$-singularity is determined to be $SU(2)^{k-3}$. In our situation, the resulting gauge group (excluding the $U(1)$ factor coming from the graviphoton) therefore is

$$ (U(1)^3 \times E_8 \times E_8) \times SU(2)^{k-3} \times U(1)^{24}. \quad (4.14) $$

The first factor is the perturbative unbroken gauge group; the second is the enhancement due to $k$ instantons on the $A_1$-singularity; the last factor corresponds to 24 vector multiplets associated with the 24 point-like instantons. Thus the dimension of the Coulomb branch is $3 + 8 + 8 + (k - 3) + 24 = 40 + k$. The expected dimension (i.e. 64) is obtained when $k = 24$ which means that all the point-like instantons have to be on top of the $A_1$ singularity.

If we freeze the vector multiplet moduli dual to the $S_H$ moduli, we are left with the non-perturbative factor, that is mapped to the bundle moduli space. Hence, we claim that the bundle moduli space (at $\alpha' \to 0$) can be reconstructed by the moduli space of the three-dimensional Seiberg-Witten theory [38] with gauge group $SU(2)^{21} \times U(1)^{24}$.

There is more: The bundle moduli space of the heterotic theory compactified on $S_H \times \tilde{T}^2$ is described by the deformations of a Riemann surface $\mathcal{C}_H$ (the spectral curve) together with its Jacobian $J(\mathcal{C}_H)$ [29]. When the bundle has structure group $SU(2)$ with integrated second Chern class $c_2 = 24$, the genus of this curve is $g_{\mathcal{C}_H} = 45$ and the curve is a double cover of the $\mathbb{P}^1$ base of $S_H$. On the other hand, we have just seen that the same moduli space in the rigid limit is dual to the moduli space of a three-dimensional supersymmetric Yang-Mills theory. This space is also described by a Riemann surface (the Seiberg-Witten curve) together with its Jacobian. This Riemann surface is a double cover of the complex plane. When the gauge group is $SU(N)$, the genus of the Seiberg-Witten curve is $g_{SW} = N - 1$. A match at the level of the curves arises when $g_{SW} = 45$, i.e. for the gauge group $SU(46)$. Note that the group $SU(2)^{21} \times U(1)^{24}$ that we found above, is a subgroup of $SU(46)$ with same rank. One then would expect points in the vector multiplet moduli space of the heterotic theory compactified on $T^2 \times \tilde{S}_H$, where this enhancement is realized. These points should be in a region of the moduli space away from the studied regime. It would be nice to give an interpretation of this enhancement in terms of the dual hypermultiplet moduli space.
5 Conclusions

In this paper we considered the duality between heterotic string theory compactified on $K3 \times T^2$ with the $K3$ being elliptically fibred and type IIA compactified on a Calabi-Yau three-fold $X$ with $X$ being both an elliptic and a $K3$ fibration. We focused on the hypermultiplet sector of the duality which has not been studied extensively so far. We analyzed a region of the moduli space where the metric is given by classical computations on both sides of the duality. More precisely, we neglected perturbative and non-perturbative quantum corrections in the string coupling $g_s$ on the type IIA side. In this limit the hypermultiplet metric is of the Ferrara-Sabharwal form (2.6) and described by a holomorphic prepotential $G$. On the heterotic side the hypermultiplet moduli space is exact in string perturbation theory but does receive $\alpha'$ corrections. They can be neglected for elliptically fibred $K3$'s with both the base of the fibration and the elliptic fibre large. On the type II side this corresponds to taking a specific complex structure modulus (which we denoted as the smoothing parameter $\sigma$) large. In this region of the moduli space we were able to match the field variables, the holomorphic prepotentials and thus the respective effective actions of the two dual theories. In order to do so we first had to give the hypermultiplet metric of the $K3$ moduli space in the Ferrara-Sabharwal form and then, with the resulting heterotic prepotential at hand, we were able to explicitly perform the match. Furthermore, from the form of the prepotential we were led to the conjecture that type IIA theory compactified on the Calabi-Yau threefold $X$ has a heterotic dual if not only $X$ but simultaneously its mirror $\tilde{X}$ is a $K3$ fibration. Mirror symmetry then gives the leading terms of the prepotential $G$ in terms of the four-cycle intersection numbers of the mirror $\tilde{X}$.

Apart from the $K3$ moduli the heterotic prepotential also depends on gauge bundle moduli. Using properties of the type IIA theory we identified additional terms on the heterotic side, some of which correspond to the leading $\alpha'$ corrections. We also considered in detail the case when the heterotic gauge bundle on $K3$ reduces to 24 point-like instantons. The dual type IIA is compactified on a self-mirror Calabi-Yau manifold with Hodge numbers $h^{1,1} = h^{1,2} = 43$. In this case we have been able to compute the structure of the intersection numbers that enter in the prepotential. This gave us the explicit dependence of $G$ on the bundle moduli.

We confirmed our results by using the duality between M-theory compactified on $K3$ and the heterotic theory compactified on $T^3$. Applying this duality twice we obtained information about the hypermultiplet moduli space of the heterotic theory compactified on $K3 \times T^2$ from the vector multiplet moduli space of the heterotic theory also compactified on $K3 \times T^2$ – albeit with a different geometric and gauge background. Using this correspondence we were also able to derive the hyper-Kähler metric on the moduli space of an $SU(2)$ bundle on $K3$.

From our analysis it seems feasible to understand in detail how the heterotic/type IIA duality has to be modified once $\alpha'$ and $g_s$ corrections are included. We can go away from the ‘double classical’ limit in two ways (see also appendix B). First of all we can consider subleading corrections in $g_s$ on the type II side keeping the heterotic $\alpha'$ corrections suppressed. On the type II side this corresponds to allow $g_s$ to be $\mathcal{O}(1)$ while keeping the smoothing parameter $\sigma$ large. On the heterotic side this corresponds to having the base volume $\Phi$ of the elliptic $K3$ small and the fibre volume $s$ large. In
this case the hypermultiplet metric is no longer of the Ferrara-Sabharwal form and thus cannot be described by a holomorphic \( G \). On the other hand, since we are suppressing the \( \alpha' \) corrections, we obtain the heterotic classical metric at large \( K3 \) volume. The second option is to consider subleading corrections in \( \alpha' \) on the heterotic side keeping the \( g_s \) corrections on the type II side suppressed. This corresponds to \( \phi \) large but taking subleading corrections in \( \sigma \) or correspondingly \( \tilde{\Phi} \) large and \( s \) small on the heterotic side. In this case the hypermultiplet metric is of the Ferrara-Sabharwal form and a holomorphic \( G \) exists. Hence, there are regions in the moduli space \( M_h \) where only the classical heterotic metric is corrected by quantum effects, and regions where only the type IIA metric is corrected. This would allow us to understand quantum corrections on one side using the classical results valid on the other side, and vice versa. In particular, quantum corrections to \( M_h \) have been better understood on the type II side of the duality [19–26]. It would be interesting to identify the heterotic counterpart of these corrections. They will give information both on the classical heterotic metric and on some heterotic quantum corrections. Moreover, considering the degeneration limit of \( X \) on the type IIA side one obtains information on the classical heterotic metric. Including the \( g_s \) corrections allows to explore regions of the moduli space where the \( K3 \) base is small, even though the volume is large and then the \( \alpha' \) corrections are negligible. In this respect, the point-like instanton case that we analyzed in detail in this paper is promising. In fact the dual type IIA Calabi-Yau has \( \chi(X) = 0 \). Since the perturbative \( g_s \) corrections are proportional to the Euler characteristic, the only \( g_s \) corrections are the non-perturbative ones. This makes it easier to include the quantum corrections on the type IIA side (in the large \( \sigma \) limit) and obtain the full classical heterotic metric.

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Appendix

A    \textit{K3 moduli space metric in Ferrara-Sabharwal form}

In this section we supply the details (i.e. the field redefinitions) which put the metric (3.6) into the Ferrara-Sabharwal form (2.6). For convenience we repeat the action (3.6)

\[ \mathcal{L} = -\frac{1}{8}(\partial \rho)^2 + \frac{1}{16}(\partial \hat{M})^2 - \frac{1}{4}e^\rho \hat{M}^{I\bar{J}} \partial B_I \partial B_{\bar{J}}, \quad I, J = 0, \ldots, 21, \]  

where \( e^{-\rho} \) is the K3 volume, \( \hat{M} \) is a SO(3,19) matrix parametrizing Gr\(_{3,19} \) and \( B_I \) are 22 \( B \)-fields.\(^\text{10}\)

As a first step we recall a more explicit representation of \( \hat{M} \) in terms of scalar fields that was determined in [64]. Since K3 is a hyper-Kähler manifold there exist three complex structures \( J^x, \ x = 1, 2, 3 \). One can define their deformation \( \zeta^x \) via

\[ J^x = e^{-\frac{1}{2}\rho} \zeta^x \omega^I, \]  

where \( \omega^I \) are the 22 two-forms of K3. The 66 \( \zeta^x \) are constrained by the six equations

\[ J^x \wedge J^y = 2\delta^{xy} \text{vol}(K3) \Rightarrow \eta^{I\bar{J}} \zeta^x \zeta^y = 2\delta^{xy}, \]  

where the intersection matrix

\[ \eta^{I\bar{J}} = \int_{K3} \omega^I \wedge \omega^{\bar{J}} \]  

has signature (3,19). Furthermore the SO(3) rotation among the \( J^x \) substracts additionally 3 angles so that there are \( 66 - 6 - 3 = 57 \) independent parameters which can be viewed as the coordinates of Gr\(_{3,19} \). In terms of the \( \zeta^x \) the matrix \( \hat{M} \) takes the form [64]

\[ \hat{M}_{IJ} = -\eta_{IJ} + \zeta_I \zeta_{\bar{J}}. \]  

For our purpose it is convenient to choose a specific basis where we can easily identify the Kähler moduli corresponding to the base and fibre of the elliptic K3. Furthermore we also want to choose a parametrization of Gr\(_{3,19} \) which treats the Kähler moduli as fibred over the base Gr\(_{2,18} \) which is spanned by the complex structure deformations preserving the elliptic fibration. To do so we split the basis \( \{ \omega^I \} \) into two sets of elements which are orthogonal to each other with respect to the metric (A.4). The first set consists of the two elements \( \omega^0, \omega^1 \) corresponding to the \( \mathbb{P}^1 \)-base and the elliptic fibre of the K3. The second set contains the others 19 elements \( \omega^A \). In this basis the intersection matrix (A.4) splits into two blocks:

\[ \eta^{0\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta^{AB} = \int_{K3} \omega^A \wedge \omega^B, \quad A = 2, \ldots, 21. \]  

\(^\text{10}\)In this appendix we do the computation for all K3 moduli but one could easily freeze some of them to obtain the results for a subspace.
Furthermore let us fix the $SO(3)$ redundancy by choosing $\Omega = J^1 + iJ^2$ for the holomorphic two-form $\Omega$ and $J^3$ for the real Kähler form. Now the constraints (A.3) can be explicitly solved by introducing the following parametrization

$$\zeta_0 = -\eta^{AB}\zeta_A v_B,$$
$$\zeta_1 = 0,$$
$$\zeta_A = \frac{X_A}{\sqrt{\eta^{BC}X_BX_C}},$$

(A.7)

where $\eta^{BC}X_BX_C = 0$ and we defined $\zeta_I \equiv \frac{1}{2}(\zeta^I + i\zeta^I)$. Inserted into $\hat{M}$ of (A.5) we arrive at

$$\hat{M}_{I,J} = \begin{pmatrix}
  e^{-R} + \frac{1}{4}e^R + v_A M^{AB} v_B & -\frac{1}{2}e^R - v_C M_{CB} - \frac{1}{2}e^R v_B \\
  -\frac{1}{4}e^R & v_A M_{AB} + e^R v_B
\end{pmatrix}$$

(A.8)

with $I = \{\alpha, A\}$ and the indices $A, B$ raised by $\eta^{AB}$. The matrix $M_{AB}$ takes the form

$$M_{AB} = -\eta_{AB} + 2\frac{X_A \bar{X}_B + \bar{X}_A X_B}{\eta^{CD}X_CX_D}.\tag{A.9}$$

Note that the field redefinitions (A.7) express the 66 $\zeta^I$ in terms of 20 complex $X^A$, 20 real $v_A$ plus one real $R$. However, the constraint $\eta^{BC}X_BX_C = 0$ removes two degrees of freedom and we see from (A.9) that the overall complex scale of the $X^A$ does not appear in $\hat{M}$. Thus effectively the $X_A$ depend on 18 complex structure moduli $t^i, i = 1, \ldots, 18$ which preserve the elliptic fibration. In addition there are the 22 (real) Kähler moduli $v_A, R, \rho$ adding up to 58 $K3$ moduli altogether.

Inserting (A.8) into (A.1) we obtain

$$\mathcal{L} = -\frac{1}{8}(\partial \rho)^2 - \frac{1}{8}(\partial R)^2 + \frac{1}{16}(\partial M)^2 - \frac{1}{4}e^{R} M^{AB} \partial v_A \partial v_B - \frac{1}{4}e^R M_{AB} \partial B_A \partial B_B$$

$$- \frac{1}{4}e^R \left( (\partial b - \frac{1}{2} \partial b_s + v \cdot \partial B)^2 + \partial b_s ((v \cdot M \cdot v) \partial b_s - 2v \cdot M \cdot \partial B) \right) - \frac{1}{4}e^{R}(\partial b_s)^2,$$

(A.10)

where we used the notation $B_1 = b$ and $B_2 = b_s$. As a next step let us define

$$\chi = \frac{1}{4}(\rho + R), \quad a = -\frac{1}{2}(b + \frac{1}{2}v^C B_C), \quad s = \nu_F + ib_s,$$
$$\bar{c}_A = \frac{1}{\sqrt{2}}(B_A - b_s v_A), \quad c^A = \frac{1}{2\sqrt{2}} v^A,$$

(A.11)

where $\nu_F = e^{-\frac{1}{2}(\rho - R)}$ is the volume of the fibre. These redefinitions put the metric (A.10) into the form:

$$\mathcal{L} = -(\partial \chi)^2 - \frac{1}{4}e^{4\chi}(\partial a + \partial c^A \partial \bar{c}_A - c^A \partial \bar{c}_A)^2 - \frac{\partial s \partial \bar{s}}{(s + \bar{s})^2} + \frac{1}{16}(\partial M)^2$$

$$- e^{2\chi} M^{AB} \frac{\partial \bar{c}_A - \mathcal{N}_{AD} \partial c^D}{\partial \bar{c}_B - \mathcal{N}_{BC} \partial c^C},$$

(A.12)

where

$$\mathcal{N}_{AD} = -i \left[(s - \bar{s})\eta_{AD} + (s + \bar{s})M_{AD}\right].$$

(A.13)
This is almost of the form (2.6). The only subtlety left is the fact that the $X^A$ are not all independent (they obey $X_A X^A = 0$) and therefore they cannot yet be viewed as homogeneous coordinates of a special Kähler manifold. In fact the parametrization (A.12), (A.13) corresponds to a field basis of the special Kähler base where the holomorphic prepotential $\mathcal{G}$ does not exist [58]. This basis is frequently encountered in heterotic compactifications [43, 65] and can be related to the standard basis as given in (2.2), (2.3) by a symplectic rotation.

In order to rotate to the standard basis, let us first note that the constraint $X_A X^A = 0$ can be solved by the choice

$$X_A = \{ X^0, X^0 \eta^{ij} t_i t_j, iX^0 t_i \} \quad , \quad i, j = 1, ..., 18 , \quad (A.14)$$

where we splitted

$$\eta_{AB} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & \eta_{ij} \end{pmatrix} . \quad (A.15)$$

Inserted into (A.9) one obtains [65]

$$\frac{1}{16} (\partial M)^2 = \frac{\partial}{\partial t_i} \frac{\partial}{\partial \bar{t}_j} K(t, \bar{t}) , \quad (A.16)$$

with

$$K = - \log \left( \frac{2 X_A \eta^{AB} \bar{X}_B}{|X^0|^2} \right) = - \log \left( \eta_{ij} (t^i + \bar{t}^i)(t^j + \bar{t}^j) \right) . \quad (A.17)$$

Together with the metric for $s$ from (A.12) we have

$$K = - \log (s + \bar{s}) - \log \left( \eta_{ij} (t^i + \bar{t}^i)(t^j + \bar{t}^j) \right) , \quad (A.18)$$

which indeed is the Kähler potential of the special Kähler manifold

$$M_{SK} = \frac{SU(1, 1)}{U(1)} \times \text{Gr}_{2,18} , \quad (A.19)$$

with a prepotential (3.8), i.e. $\mathcal{G}_{\text{het}} = is \eta_{ij} t^i t^j$. The symplectic rotation

$$(Z^s, \mathcal{G}_s) \mapsto (\mathcal{G}_s, -Z^s) \quad (A.20)$$

precisely maps to the field basis (A.14) and furthermore transforms the matrix $\mathcal{N}$ defined in (2.7) into the form (A.13) [43, 58, 65].
Thus we finally succeeded in putting the heterotic metric (A.1) into the Ferrara-Sabharwal form (2.6). Therefore we can now give the precise (classical) map between type IIA and heterotic moduli which is summarized in table A.1.  

**B  Refined classical limits**

Let us first determine the geometrical meaning of some of the moduli introduced in the previous appendix. To do so we consider the Kähler-form $J$ evaluated in the basis of the previous appendix. That is we insert (A.7) into (A.2) to arrive at

\[ J = J^3 = e^{-\frac{1}{2} \rho} \zeta^3 \omega^0 = (e^{-2 \chi} - \frac{1}{2} \nu_F v_A v_A) \omega^0 + \nu_F \omega^1 + \nu_F v_A \omega^A. \]  

(B.1)

In the basis (A.6) the elliptic fibre of $S_H$ is dual to $\omega^0$, while the $\mathbb{P}^1$-base is dual to $\omega^1 - \omega^0$. This can be checked by computing the intersection matrix for fibre and base to be

\[ \eta = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \]  

(B.2)

We see that the fibre has no self-intersection but it intersects once with the base. The base itself has self-intersection $-2$ which confirms that the base is a $\mathbb{P}^1$ while the fibre is elliptic. Now we compute

\[ \text{vol}(S_H) = \frac{1}{2} \int_{K3} J \wedge J = e^{-\rho}, \quad \text{vol}(F_H) = \int_{K3} J \wedge \omega^0 = \nu_F = e^{-\frac{1}{2}(\rho+R)}, \]

\[ \text{vol}(B_H) = \int_{K3} J \wedge (\omega^1 - \omega^0) = e^{-\frac{1}{2}(\rho+R)} - \frac{1}{2} e^{-\frac{1}{2}(\rho-R)}(v^A v_A + 2). \]

(B.3)

We see that $e^{-\rho}$ is the $K3$ volume, $\nu_F$ is the volume of the fibre while the volume of the base is a more complicated linear combination.

Let us now discuss various limits we can take. First of all the $K3$ volume becomes large in the limit $\rho \to -\infty$ and $R$ generic. From (B.3) we see that in this limit also the volume of base and fibre are both large which implies that the $\alpha'$ corrections on the heterotic side vanish. To see this more explicitly we note that $\alpha'$ corrections arise from worldsheet instantons wrapping a minimal two-sphere in $K3$ (two-cycles with self-intersection equal to $-2$) with the strength of the correction depending on the area of the wrapped two-sphere. For general complex structure of $S_H$, generic two-spheres are orthogonal to the Piccard lattice and then their area is given by

\[ \text{vol}(S^2) = (\text{vol}(S_H))^{1/2} \left| \int_{S^2} \Omega \right| \sim e^{-\rho/2}. \]  

(B.4)

We see that the area is going to infinity when the volume of $S_H$ is large and the complex structure is generic. Thus the $\alpha'$ corrections vanish on the heterotic side in the limit

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11 In the last two rows of the table A.1 we have been schematic: Because of the symplectic transformation (A.20), the pair of fields $(\psi^s, \tilde{c}_s)$, which correspond to the index $A = s$, should be swapped.

12 For generic $v^A$, the volume of the base $B_H$ gets a non-zero contribution also from $\int_{K3} \Omega \wedge (\omega^1 - \omega^0)$. We are neglecting these contributions here, as they are irrelevant for our conclusions.
\[ \rho \to -\infty \text{ and } R \text{ generic.} \] From table A.1 we further conclude that in the dual type II theory \( \Phi \sim g_s^{-2} \) becomes large in that limit and therefore the type II \( g_s \) corrections vanish correspondingly. Or in other words in this limit \( \alpha' \) corrections on the heterotic side and \( g_s \) corrections on the type II side vanish simultaneously.

However, since the \( K3 \) is elliptically fibred one can also take two more refined limits where either one of these corrections is kept. In the limit \( \rho \text{ finite and } R \to -\infty \) the volume of \( S_H \) is finite while the volume of the base \( B_H \) is still infinite. Therefore heterotic \( \alpha' \) corrections are kept while the \( g_s \) corrections continue to vanish. Note that in this limit, the IIA CY three-fold \( X \) goes away from the degenerate limit \( X^\sharp \) which has been discussed in [18].

The other possible limit is \( \rho \to -\infty, R \to \infty, e^{-\frac{1}{2}(\rho+R)} \) finite and \( v^Av_A = -2 \). The volume of \( S_H \) and \( F_H \) is infinite while the volume of \( B_H \) is finite. This corresponds to negligible heterotic \( \alpha' \) corrections but keeping the type II \( g_s \)-corrections. Note that in this limit the IIA CY is the degenerate \( X^\sharp \).

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