FEEDBACK STABILIZATION OF BILINEAR COUPLED HYPERBOLIC SYSTEMS

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Abstract. This paper studies the problem of stabilization of some coupled hyperbolic systems using nonlinear feedback. We give a sufficient condition for exponential stabilization by bilinear feedback control. The specificity of the control used is that it acts on only one equation. The results obtained are illustrated by some examples where a theorem of Mehrenberger has been used for the observability of compactly perturbed systems [18].

1. Introduction. Second order infinite dimensional coupled systems are frequently encountered in practical engineering problems, for example in civil engineering structures (suspension bridges models [24]) and in many problems coming from elasticity (see for instance Lions [17] and references therein). In recent years, the study of this kind of systems has become fairly common and is now an established area of research with an extensive and long list of publications and conference communications (see [25], [10], [1], [2], [3], [4], [16], [12], [5] and the references therein). In this paper we are interested in the problem of stabilization of a class of coupled systems where the control acts only on the first equation of these systems, by using a technique that links the stabilization with a property of controllability for the undamped system. This method started with Haraux [15] for the bounded input operator and after by Tucsnak and Ammari for the unbounded feedback [6]. This method is also used in semi-linear and bilinear problems (see Berrahmoune [11], Ouzahra [19], Barbu [9], El Harraki [14] and references therein).

The systems we consider in this paper are described by:

\[
\begin{cases}
\ddot{w}_1(t) + A_1w_1(t) + u(t)B\dot{w}_1(t) + C\dot{w}_2(t) = 0, \\
\ddot{w}_2(t) + A_2w_2(t) - C\dot{w}_1(t) = 0, \\
w_1(0) = w_1^0, \quad \dot{w}_i(0) = w_i^1, \quad 1 \leq i \leq 2,
\end{cases}
\]

(1)

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The difference between these two systems is that input operator $B$ in system (1) acts on the speed variable $\dot{w}_1$ and in system (2) acts on state variable $w_1$. We note also that in system (2) the operator $D$ “damping operator” appears. As a model for the “damping operator” which has been studied in the literature, we mention the one defined by $Dv(x) = a(x)v(x), \forall v \in L^2(\Omega)$ (see for details Zuazua [26]). More precisely, in these systems, the operators $A_1,A_2$ are unbounded positive and self-adjoint in a Hilbert space $H$. The control operator $B$ is assumed here to be bounded from $H$ to $H$.

The coupling operator $C$ is supposed bounded from $H$ to $H$. Generally $C$ is not necessarily bounded, indeed we refer to the paper by E. M. Ait Benhassi et.al (see [10]) where the authors consider an output feedback for the closed loop system given by

\[
\begin{align*}
\dot{w}_1(t) + A_1w_1(t) + u(t)Bw_1(t) + D\dot{w}_1(t) + C\dot{w}_2(t) &= 0, \\
\dot{w}_2(t) + A_2w_2(t) - C\dot{w}_1(t) &= 0, \\
w_i(0) = w_i^0, \quad \dot{w}_i(0) = w_i^1, &\quad 1 \leq i \leq 2,
\end{align*}
\]

(2)

The authors use unbounded coupling operator to show the stabilization. We also point out that for the coupled linear systems there are in literature many results in different cases (see [2], [25], [5], [3] and references therein). In [1] the authors have considered coupled systems in the case of bounded (even compact) coupling operators $C$, they showed that, under a condition on the operators of each equation and on the boundary feedback operator, the energy of smooth solutions of this system decays polynomially at $\infty$. We note that when $C = 0$, the exponential stability does not hold, since the second equation in the systems like (1) is conservative. Recently, Ammari and Nicaise [3] have characterized the polynomial and analytic stabilization of a wave equation coupled with an Euler–Bernoulli beam, augmented with the output $u(t) = -B^*\dot{w}_1(t)$. They prove that in the case where the control zone does not satisfy the geometric control condition, they have a polynomial stability result for all regular initial data. Moreover, they give a precise estimate on the analyticity of reachable functions where they have an exponential stability. In [22], the authors proves that the uniform stability of semigroups associated to displacement coupled dissipator systems is equivalent to the uniform stability of velocity coupled dissipator systems. Unbounded control operator is used in [10] where the authors combined boundedness property of the transfer function of the associated open loop system with an observability estimate (see also [10], [1], [2]).

We mention that in the existing literature, there are no results concerning the stabilization of bilinear coupled hyperbolic systems. Such characterization is the main goal of this paper, indeed we will show the exponential stabilization under an observability inequality. For the weak stabilization we refer to Ball and Slemrod [8], where it has been shown that if $B$ is compact and $(S(t))_{t \geq 0}$ is a semigroup of contractions such that

\[ \langle BS(t)w, S(t)w \rangle = 0 \ \forall t \geq 0 \Rightarrow w = 0, \]

then the feedback

\[ u(t) = -\langle w(t), Bw(t) \rangle, \]
weakly stabilizes the following system
\[
\begin{align*}
\dot{w}(t) &= Aw(t) + v(t)Bw(t), \\
\dot{w}(0) &= u_0.
\end{align*}
\]

We extend this result to the coupled system (1). Compactness of \(C\) is used in the applications to deduce an observability estimate.

Our paper is organized as follows, in section 2, we prove the exponential stabilization for a coupled system without damping operator, in section 3, we show the stabilization of system with damping operator and we characterize the weak stabilization when the exponential one doesn’t holds. Finally, section 4 deals with applications.

2. System without damping operator. In this section we show the exponential stabilization of a bilinear coupled system in the absence of the damping operator using the strong observability inequality.

Let \(H\) be a Hilbert space equipped with the norm \(\|\cdot\|_H\). The operators \(A_1, A_2\) are unbounded, positive and self-adjoint in \(H\), \(C\) is bounded self-adjoint operator defined from \(H\) to \(H\) and \(B\) is a bounded operators defined from \(H\) to \(H\). Consider the following system
\[
\begin{align*}
\ddot{w}_1(t) + A_1 w_1(t) + u(t)B\dot{w}_1(t) + C\dot{w}_2(t) &= 0, \\
\ddot{w}_2(t) + A_2 w_2(t) - C\dot{w}_1(t) &= 0,
\end{align*}
\]
with the following initial conditions
\[
\begin{align*}
w_1(0) &= w_0^1, \quad \dot{w}_1(0) = \dot{w}_1^1, \\
w_2(0) &= w_0^2, \quad \dot{w}_2(0) = \dot{w}_2^1,
\end{align*}
\]
and the corresponding homogeneous system
\[
\begin{align*}
\ddot{\varphi}_1(t) + A_1 \varphi_1(t) + C\dot{\varphi}_2(t) &= 0, \\
\ddot{\varphi}_2(t) + A_2 \varphi_2(t) - C\dot{\varphi}_1(t) &= 0, \\
\varphi_i(0) &= w_0^i, \quad \dot{\varphi}_i(0) = \dot{w}_i^1, \quad 1 \leq i \leq 2.
\end{align*}
\]
In the following, we use the following feedback
\[
\begin{align*}
u(t) &= U(\dot{w}_1(t)),
\end{align*}
\]
where
\[
U(z) = \begin{cases} 
\frac{\langle Bz, z \rangle}{\|z\|^2}, & z \neq 0, \\
0, & z = 0.
\end{cases}
\]
Let \( E := D(A_1^\frac{1}{2}) \times H \times D(A_2^\frac{1}{2}) \times H, \quad \|\cdot\|_E = (\|\cdot\|^2_{D(A_1^\frac{1}{2})} + \|\cdot\|^2_H + \|\cdot\|^2_{D(A_2^\frac{1}{2})} + \|\cdot\|^2_H)^{\frac{1}{2}}. \)

The following theorem ensures the exponential stabilization of system (3)-(4).

**Theorem 2.1.** Suppose that there exists \( T > 0 \) and \( \delta > 0 \) such that every solution of system (5) satisfies:
\[
\begin{align*}
\|w_1^0\|^2_{D(A_1^\frac{1}{2})} + \|w_1^1\|^2_H + \|w_2^0\|^2_{D(A_2^\frac{1}{2})} + \|w_2^1\|^2_H &\leq \delta \int_0^T \langle B\dot{\varphi}_1(t), \dot{\varphi}_1(t) \rangle dt.
\end{align*}
\]
Then the feedback (6) exponentially stabilizes (3)-(4).
Proof. Let the following energy function
\[ E(t) = \frac{1}{2} \| u_1(t) \|_{D(A_1^\frac{1}{2})}^2 + \| \dot{u}_1(t) \|_H^2 + \| u_2(t) \|_{D(A_2^\frac{1}{2})}^2 + \| \dot{u}_2(t) \|_H^2. \]
By using system (3) we have
\[
\dot{E}(t) = \langle \dot{u}_1(t), \ddot{u}_1(t) \rangle_{D(A_1^\frac{1}{2})} + \langle \dot{u}_2(t), \ddot{u}_2(t) \rangle_{D(A_2^\frac{1}{2})} \\
= \langle \dot{u}_1(t), \ddot{u}_1(t) \rangle_{D(A_1^\frac{1}{2})} + \langle A_1 u_1(t), \ddot{u}_1(t) \rangle_{D(A_1^\frac{1}{2})} + \langle \dot{u}_2(t), \ddot{u}_2(t) \rangle_{D(A_2^\frac{1}{2})}+ \langle A_2 u_2(t), \ddot{u}_2(t) \rangle_{D(A_2^\frac{1}{2})} \\
= -u(t) \langle B \dot{u}_1(t), \dot{u}_1(t) \rangle.
\]
For \( u(t) = \frac{\langle B \dot{u}_1(t), \dot{u}_1(t) \rangle}{\| \dot{u}_1(t) \|_H^2} \), we obtain:
\[
\dot{E}(t) = -\frac{\| \dot{u}_1(t) \|_H^2}{\| \dot{u}_1(t) \|_H^2},
\]
hence
\[
E(T) - E(0) = \int_0^T -\frac{\| \dot{u}_1(t) \|_H^2}{\| \dot{u}_1(t) \|_H^2} dt.
\]
Now we decompose the system (3) in the following form
\[
\begin{align*}
\psi_1 &= w_1 - \varphi_1, \\
\psi_2 &= w_2 - \varphi_2,
\end{align*}
\]
where \((\psi_1, \psi_2)\) is the solution of the following system:
\[
\begin{align*}
\dot{\psi}_1(t) + A_1 \psi_1(t) + C \dot{\psi}_2(t) &= -u(t) B \dot{u}_1(t), \\
\dot{\psi}_2(t) + A_2 \psi_2(t) - C \dot{\psi}_1(t) &= 0, \\
\psi_1(0) &= 0, \\
\psi_2(0) &= 0,
\end{align*}
\]
where \((\varphi_1, \varphi_2)\) is the solution of the following system:
\[
\begin{align*}
\dot{\varphi}_1(t) + A_1 \varphi_1(t) + C \varphi_2(t) &= -u(t) B \varphi_1(t), \\
\dot{\varphi}_2(t) + A_2 \varphi_2(t) - C \varphi_1(t) &= 0, \\
\varphi_1(0) &= 0, \\
\varphi_2(0) &= 0,
\end{align*}
\]
We have
\[
\langle B \dot{\varphi}_1, \dot{\varphi}_2 \rangle_{H} = \langle B \dot{u}_1(t), \dot{u}_1(t) \rangle - \langle B \dot{u}_1(t), \dot{u}_1(t) \rangle - \langle B \dot{\varphi}_1(t), \dot{\varphi}_2(t) \rangle,
\]
by boundedness of \(B\) we deduce that
\[
\|B \dot{\varphi}_1, \dot{\varphi}_2\| \leq \|B \dot{u}_1(t), \dot{u}_1(t)\| + K \| \dot{u}_1(t) \| \| \dot{\varphi}_1(t) \| + K \| \dot{\varphi}_1(t) \| \| \dot{\varphi}_1(t) \|,
\]
where \((K := \|B\|_{L^2(H)})\)
On the other hand system (5) can be written in the following form
\[
\begin{align*}
\dot{y} &= Ay, \\
y(0) &= y_0.
\end{align*}
\]
where
\[
y = \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}
\quad \text{and} \quad
A = \begin{pmatrix}
0 & I_d & 0 & 0 \\
-A_1 & 0 & 0 & -C \\
0 & 0 & 0 & I_d \\
0 & C & -A_2 & 0
\end{pmatrix}.
\]
According to the hypotheses on the operators \(A_1, A_2 \) and \(C, A\) is skew-adjoint operator then it generates a semigroup of contractions \((S(t))_{t \geq 0}\) (by using Lumer-Phillip’s theorem and Proposition 3.7.2 in [23]), then the solution of (5) is given by
\[
y(t) = S(t)y_0.
\]
then
\[ \|y(t)\|_E \leq \|S(t)\|_E \|y_0\|_E \leq \|y_0\|_E, \]
thus
\[ \|\hat{\varphi}_1(t)\|_H \leq \|y(t)\|_E \leq \|y_0\|_E := \left(\|w_1^0\|_{D(A_{1/2}^2)}^2 + \|w_1^0\|_{H_1}^2 + \|w_2^0\|_{D(A_{1/2}^2)}^2 + \|w_2^0\|_{H_2}^2\right)^{1/2}. \]
Therefore
\[ \left|\langle B\hat{\varphi}_1, \hat{\varphi}_1 \rangle \right| \leq \left|\langle B\hat{w}_1(t), \hat{w}_1(t) \rangle \right| + K\|\hat{w}_1(t)\|\|\hat{\varphi}_1(t)\| + K\|y_0\|_E\|\hat{\varphi}_1(t)\|. \]
Thus the system (8) can be written in the form
\[
\begin{align*}
\dot{x} &= Ax + f(x), \\
x(0) &= x_0, \tag{9}
\end{align*}
\]
where
\[
x = \begin{pmatrix}
\psi_1 \\
\hat{\varphi}_1 \\
\psi_2 \\
\hat{\varphi}_2
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & I_d & 0 & 0 \\
-A_1 & 0 & 0 & -C \\
0 & 0 & 0 & I_d \\
0 & C & -A_2 & 0
\end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix}
0 \\
-u(t)B\hat{w}_1(t) \\
0 \\
0
\end{pmatrix}.
\]
Our assumptions imply that \(A\) generates a \(C_0\) semi group of contractions \((S(t))_{t \geq 0}\) and that \(f\) is locally Lipschitz, then the solution of (9) is given by
\[ x(t) = S(t)x_0 + \int_0^t S(t - \tau)f(x(\tau))d\tau. \]
Hence \((S(t))_{t \geq 0}\) is a semi group of contractions, then
\[ \|x(t)\|_E \leq \int_0^t \|f(x(\tau))\|_E d\tau, \]
thus
\[ \|\hat{\varphi}_1(t)\|_H^2 \leq \left(\int_0^T \|u(t)B\hat{w}_1(t)\|_H dt\right)^2, \]
therefore
\[ \|\hat{\varphi}_1(t)\|_H \leq K(T)\int_0^T |U(\hat{w}_1(t))|^2\|\hat{w}_1(t)\|_H^2 dt)^{1/2}. \]
Furthermore
\[ \|\hat{w}_1(t)\| \leq 2E(t) \leq 2E(0) \]
\[ \|\hat{w}_1(t)\| \leq \sqrt{2} \sqrt{E(0)} \leq \|y_0\|_E. \]
Then
\[ \left|\langle B\hat{\varphi}_1, \hat{\varphi}_1 \rangle \right| \leq \left|\langle B\hat{w}_1(t), \hat{w}_1(t) \rangle \right| + K\|\hat{w}_1(t)\|\|\hat{\varphi}_1(t)\| + K\|\hat{\varphi}_1(t)\|\|\hat{\varphi}_1\|
\leq \left|\langle B\hat{w}_1(t), \hat{w}_1(t) \rangle \right| + 2K\|y_0\|_E\|\hat{\varphi}_1\|
\leq \left|\langle B\hat{w}_1(t), \hat{w}_1(t) \rangle \right| + 2K\|y_0\|_E\|T\int_0^T |U(\hat{w}_1(t))|^2\|\hat{w}_1(t)\|_H^2 dt\right)^{1/2}. \]
By using the Gronwall’s inequality, we deduce that
\[
\left| \langle B\dot{w}_1(t), \dot{w}_1(t) \rangle \right| \leq \frac{\left| \langle B\ddot{w}_1(t), \ddot{w}_1(t) \rangle \right|}{\| \ddot{w}_1(t) \|^2} \| y_0 \|_E \| \dot{w}_1(t) \|.
\]
\[
\int_0^T \left| \langle B\ddot{w}_1(t), \ddot{w}_1(t) \rangle \right| dt \leq \| y_0 \|_E (T \int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}}.
\]
According to (10) we obtain
\[
\int_0^T \left| \langle B\ddot{w}_1(t), \ddot{w}_1(t) \rangle \right| dt \leq \| y_0 \|_E (T \int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}}
\]
\[
+ 2K \| y_0 \|_E (T^3 \int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}}
\]
\[
\leq \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \| y_0 \|_E (\int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}},
\]
then
\[
\int_0^T \left| \langle B\ddot{w}_1(t), \ddot{w}_1(t) \rangle \right| dt \leq \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \| y_0 \|_E (\int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}}.
\]
By using (7) we have
\[
\| y_0 \|^2_E = \| w_1^0 \|^2_{D(A^1_1)} + \| w_1^0 \|^2_{D(A^2_2)} + \| w_2^0 \|^2_{D(A^3_3)} + \| w_2^0 \|^2_{D(A^2_2)} \leq \delta \int_0^T \langle B\ddot{\varphi}_1(t), \ddot{\varphi}_1(t) \rangle dt,
\]
hence
\[
\| y_0 \|^2_E \leq \delta \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \| y_0 \|_E (\int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt)^{\frac{1}{2}}
\]
thus
\[
\| y_0 \|_E \leq \delta \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \left( \int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt \right)^{\frac{1}{2}}
\]
therefore
\[
E(t) \leq \left( \frac{\delta}{\sqrt{2}} \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \right)^2 \left( \int_0^T |U(\ddot{w}_1(t))|^2 \| \dot{w}_1(t) \|^2 dt \right)
\]
\[
\leq \left( \frac{\delta}{\sqrt{2}} \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \right)^2 (E(0) - E(T))
\]
then
\[
E(T) \leq E(0) \leq \left( \frac{\delta}{\sqrt{2}} \left( T^{\frac{3}{2}} + 2KT^{\frac{3}{2}} \right) \right)^2 (E(0) - E(T)).
\]
We deduce that
\[
(1 + K')E(T) \leq E(0),
\]
where \( K' = \frac{1}{\left( \frac{\delta}{\sqrt{2}} (T^{\frac{3}{2}} + 2KT^{\frac{3}{2}}) \right)^2} \). Moreover, in view to the semi group property we deduce that
\[
E(t) \leq M e^{-\gamma t} E(0),
\]
where
\[
M = \frac{1 + K'}{K'} \text{ and } \gamma = \frac{1}{T} \log \left( \frac{1 + K'}{K'} \right).
\]
\[ \square \]
3. System with damping operator. In this section we prove the exponential stabilization of a kind of coupled systems using a damping operator and a weak inequality of observability. Under the same assumptions on $A_1$, $A_2$ and $C$, we make appear the damping operator $'D'$ in system (3)-(4) as follows:

\[
\begin{aligned}
\dot{w}_1(t) + A_1 w_1(t) + D\dot{w}_1(t) + u(t)Bw_1(t) + C\dot{w}_2(t) &= 0, \\
\dot{w}_2(t) + A_2 w_2(t) - C\dot{w}_1(t) &= 0,
\end{aligned}
\tag{11}
\]

with the following initial conditions

\[
w_1(0) = w_1^0, \quad \dot{w}_1(0) = w_1^1, \quad w_2(0) = w_2^0, \quad \dot{w}_2(0) = w_2^1,
\tag{12}
\]

where $D$ is a bounded operator defined from $H$ to $H$. Let the corresponding homogeneous system

\[
\begin{aligned}
\dot{\varphi}_1(t) + A_1 \varphi_1(t) + C\dot{\varphi}_2(t) &= 0, \\
\dot{\varphi}_2(t) + A_2 \varphi_2(t) - C\dot{\varphi}_1(t) &= 0, \\
\varphi_1(0) &= w_1^0, \quad \dot{\varphi}_1(0) = w_1^1, \quad \varphi_2(0) = w_2^0, \quad \dot{\varphi}_2(0) = w_2^1.
\end{aligned}
\tag{13}
\]

We use the feedback defined by

\[u(t) = f((Bw_1(t), w_1(t))),\]

then the system (11)-(12) is equivalent to

\[
\begin{aligned}
\dot{w}_1(t) + A_1 w_1(t) + D\dot{w}_1(t) + f((Bw_1, w_1))Bw_1(t) + C\dot{w}_2(t) &= 0, \\
\dot{w}_2(t) + A_2 w_2(t) - C\dot{w}_1(t) &= 0, \\
w_1(0) = w_1^0, \quad \dot{w}_1(0) = w_1^1, \quad w_2(0) = w_2^0, \quad \dot{w}_2(0) = w_2^1.
\end{aligned}
\tag{14}
\]

3.1. Well posedness and stabilization. Let $A_1$, $A_2$, $B$, $C$ and $D$ satisfies the hypotheses above. Let $f$ be a locally lipschitz function satisfying the assumptions:

- $H_1$- $s f(s) \geq 0$, $\forall s \in \mathbb{R}$,
- $H_2$- $f \in L^\infty(\mathbb{R})$,
- $H_3$- $f(\infty) = f(0) = 0$.

**Theorem 3.1.** Under these assumptions, for all $(w_1^0, w_1^1, w_2^0, w_2^1) \in D(A_1^\frac{1}{2}) \times H \times D(A_2^\frac{1}{2}) \times H$, there exists a unique weak solution to system (14) such that

\[(w_1, w_2) \in C([0, \infty[; D(A_1^\frac{1}{2}) \times D(A_2^\frac{1}{2})) \cap C^1([0, \infty[; H \times H)).\]

**Proof.** By classical semi group (see [20]).

**Theorem 3.2.** Suppose that $B$ and $D$ are self-adjoint and satisfies:

- $H_4$- $\langle Dv, v \rangle > 0$, $\forall v \in H$, $v \neq 0$,
- $H_5$- $\|Dv\|^2 \leq \text{const} \langle Dv, v \rangle$, $\forall v \in H$, $v \neq 0$,
- $H_6$- $\langle Bv, v \rangle \neq 0$, $\forall v \in H$, $v \neq 0$,

and that every solution $(\varphi_1, \varphi_2)$ of (13) obeys the following inequality

\[
\|w_1^0\|_{D(A_1^\frac{1}{2})}^2 + \|w_1^1\|^2_H + \|w_2^0\|_{D(A_2^\frac{1}{2})}^2 + \|w_2^1\|^2_H \
\leq \text{const} \int_0^{T_0} \langle D\varphi_1, \varphi_1 \rangle + \|\varphi_1\|^2 dt \tag{15}
\]

for some positive number $T_0 > 0$. Then the system (14) is exponentially stabilizable.
Proof. Let us consider the following energy functional:

\[ E(t) = \frac{1}{2} \left( \left\| A_1^{\frac{1}{2}} w_1 \right\|_H^2 + \left\| \dot{w}_1 \right\|_H^2 + \left\| A_2^{\frac{1}{2}} w_2 \right\|_H^2 + \left\| \dot{w}_2 \right\|_H^2 \right) \]

and let

\[ V(t) = \frac{1}{2} \left( \left\| A_1^{\frac{1}{2}} w_1 \right\|_H^2 + \left\| \dot{w}_1 \right\|_H^2 + \left\| A_2^{\frac{1}{2}} w_2 \right\|_H^2 + \left\| \dot{w}_2 \right\|_H^2 + F(\langle Bw_1(t), w_1(t) \rangle) \right), \]

where

\[ F(s) := \int_0^s f(\tau) d\tau. \]

We have

\[
\dot{V}(t) = \langle A_1 w_1, \dot{w}_1 \rangle + \langle \ddot{w}_1, \dot{w}_1 \rangle + \langle A_2 w_1, \dot{w}_2 \rangle + \langle Bw_1, \dot{w}_1 \rangle f(\langle Bw_1, w_1 \rangle)
= -\langle D\dot{w}_1, \dot{w}_1 \rangle,
\]

then

\[ V(t_2) - V(t_1) = -\int_{t_1}^{t_2} \langle D\dot{w}_1, \dot{w}_1 \rangle dt, \]

hence the decrease of \( V \) according to the condition \( (H_4) \).

\begin{itemize}
  \item We show that there exist \( C_0, T_0 \) such that
  \[ V(T_0) \leq C_0 \int_0^{T_0} \langle D\dot{w}_1(t), \dot{w}_1(t) \rangle dt \]

indeed, the solution of (14) may be decomposed as:

\[
\begin{cases}
  w_1 = \varphi_1 + \varphi_1, \\
  w_2 = \varphi_2 + \varphi_2,
\end{cases}
\]

where \((\varphi_1, \varphi_2)\) solves

\[
\begin{cases}
  \ddot{\varphi}_1(t) + A_1 \varphi_1(t) + D\dot{\varphi}_1(t) + C\dot{\varphi}_2(t) = -f(\langle Bw_1(t), w_1(t) \rangle)Bw_1(t), \\
  \ddot{\varphi}_2(t) + A_2 \varphi_2(t) - C\dot{\varphi}_1(t) = 0, \\
  \varphi_1(0) = 0, \dot{\varphi}_1(0) = 0, \varphi_2(0) = 0, \dot{\varphi}_2(0) = 0.
\end{cases}
\]

\begin{itemize}
  \item Since \( V \) is non-increasing then:
  \[ V(T_0) \leq V(0) \leq \text{Const} \left( \left\| w_1^0 \right\|_{D(A_1^{\frac{1}{2}})}^2 + \left\| w_1^0 \right\|_H^2 + \left\| w_2^0 \right\|_{D(A_2^{\frac{1}{2}})}^2 + \left\| w_2^0 \right\|_H^2 \right) \]

\[ \leq \text{Const} \int_0^{T_0} \left( \langle D\varphi_1, \dot{\varphi}_1 \rangle + \left\| \varphi_1(t) \right\|_H^2 dt \right) \]

\[ \leq \text{Const} \int_0^{T_0} \left( \langle D\dot{w}_1, \dot{w}_1 \rangle + \langle D\dot{\varphi}_1, \dot{\varphi}_1 \rangle + \left\| w_1(t) \right\|_H^2 + \left\| \varphi_1(t) \right\|_H^2 \right) dt \]

\begin{itemize}
  \item Furthermore
  \[ \int_0^{T_0} \left( \langle D\dot{\varphi}_1, \dot{\varphi}_1 \rangle + \left\| \varphi_1(t) \right\|_H^2 \right) dt \leq \text{const} \int_0^{T_0} \left( \langle D\dot{w}_1, \dot{w}_1 \rangle + \left\| w_1(t) \right\|_H^2 \right) dt \]
\end{itemize}
\end{itemize}
By using standard continuity arguments, one can deduce from (16) that
\[
\int_0^{T_0} \left( (D\dot{w}_1, \dot{w}_1) + \|w_1(t)\|^2 \right) dt \leq \int_0^{T_0} \left( (D\dot{w}_1, \dot{w}_1) + \|w_1(t)\|^2 + \|w_2(t)\|^2 \right) dt
\]
\[
\leq \int_0^{T_0} \left( \| - \dot{w}_1(t) - f(Bw_1(t), w_1(t))Bw_1(t) \|^2 \right) dt
\]
\[
\leq \int_0^{T_0} \left( \|D\dot{w}_1(t)\|^2 + \|f(Bw_1(t), w_1(t))Bw_1\|^2 \right) dt
\]
\[
\leq \text{const} \int_0^{T_0} \left( (D\dot{w}_1, \dot{w}_1) + \|w_1\|^2 \right) dt.
\]
From where by combining with the above we have
\[
V(T_0) \leq \text{const} \int_0^{T_0} \left( (D\dot{w}_1(t), \dot{w}_1(t)) + \|w_1\|^2 \right) dt.
\]
It remains to show that
\[
\int_0^{T_0} \|w_1(t)\|^2 \leq \text{const} \int_0^{T_0} (D\dot{w}_1(t), \dot{w}_1(t)).
\]
By contradiction, suppose that there exists two sequences \((w_{1n})_n\) and \((w_{2n})_n\) satisfying:
\[
\lim_{n \to \infty} \frac{\int_0^{T_0} \|w_{1n}\|^2 dt}{\int_0^{T_0} (D\dot{w}_{1n}(t), \dot{w}_{1n}(t)) dt} = \infty
\]

(17)
and \(w_{2n} = 0, \quad \forall n \geq 1.\)

Let us consider \(\lambda_n := \left( \int_0^{T_0} \|w_{1n}(t)\|^2_H dt \right)^{\frac{1}{2}}\) and \(v_n = \frac{w_{1n}}{\lambda_n}.\)

By replacing in the first equation of system (14) we obtain:
\[
\ddot{v}_n(t) + Av_n(t) + D\dot{v}_n(t) + f(\lambda_n^2(Bv_n, v_n))Bv_n = 0.
\]
(18)

On the other hand, we have
\[
\int_0^{T_0} (D\dot{v}_n(t), \dot{v}_n(t)) dt \to 0 \quad \text{quand} \quad n \to +\infty,
\]
(19)
\[
\|v_n(t)\|_{L^2(0,T;H)} = 1.
\]
Then \((v_n)_n\) is bounded in \(L^{\infty}(0, \infty; D(A_1^\frac{1}{2})) \cap W^{1,\infty}(0, \infty; H).\) Moreover, considers that injection \(D(A_1^\frac{1}{2}) \subset H\) is compact. We deduce via a standard compactness argument that there exists a subsequence, still denoted by \((v_n)_n\) such that\([21]:\)
\[
v_n \rightharpoonup v \quad \text{weakly in} \quad L^2(0, T_0; D(A_1^\frac{1}{2}))\]
(20)
\[
\dot{v}_n \rightharpoonup \dot{v} \quad \text{weakly in} \quad L^2(0, T_0; H)\]
(21)
\[
v_n \to v \quad \text{strongly in} \quad L^2(0, T_0; H)\]
(22)
\[
v_n \to v \quad \text{a.e in} \quad (0, T_0) \times \Omega.\]
(23)

From (22) we get
\[
\|v(t)\| = 1.
\]
On the other hand, the conditions \((H_5)-(19)-(21)\) ensure that \((D\dot{v}(t), \dot{v}(t)) = 0 \ a.e \ in \ (0, T_0).\) It follows by taking into account \(H_4\) that
\[
\dot{v} = 0 \quad \text{ae in} \quad (0, T_0) \times \Omega.
We conclude our proof by distinguishing the following three situations:

(a)- There exists a subsequence, still denoted by \( (\lambda_n)_n \), such that \( \lambda_n \to \infty \). From \((H_5), (19)\) and \((21)\) we have
\[
D\dot{v}_n \to 0 \quad \text{ae in} \quad (0, T_0) \times \Omega.
\]
Moreover, \( v \) does not depend on the time variable \( t \) and
\[
\langle Bv_n, v_n \rangle \to \langle Bv, v \rangle, \quad n \to \infty.
\]
Finally the hypotheses \((H_3)-(H_6)\) show
\[
f(\lambda_n^2 \langle Bv_n, v_n \rangle) \to 0, \quad n \to \infty. \quad (24)
\]
Thus
\[
A_1v = 0 \Rightarrow v = 0. \quad (A_1 \text{ positive})
\]
Absurd with
\[
\|v(t)\| = 1.
\]

(b)- There exists a subsequence, still denoted by \( \lambda_n \), such that \( \lambda_n \to 0 \). This possibility leads to \((24)\) by virtue of the last equality in \((H_3)\). As in the preceding case, this contradicts \( \|v(t)\| = 1 \).

(c)- There exists a subsequence, still denoted by \( (\lambda_n)_n \), such that \( \lambda_n \to \lambda \in (0, \infty) \). Then \( v \) satisfies this equation
\[
Av + f(\lambda^2 \langle Bv, v \rangle)Bv = 0,
\]
multiplying by \( \lambda^2 v \) we obtain
\[
\|\lambda v\|^2 + f(\langle B\lambda v, \lambda v \rangle) \langle B\lambda v, \lambda v \rangle = 0.
\]
We have \( \lambda v = 0 \) then \( v = 0 \), also this contradicts
\[
\|v(t)\| = 1.
\]
Therefore we conclude that
\[
V(T_0) \leq C_0 \int_0^{T_0} \{\langle D\dot{y}(t), \dot{y}(t) \rangle\} dt
\]
then
\[
V(T_0) \leq \frac{C_0}{1 + C_0} V(0).
\]
As a result we conclude that
\[
V(t) \leq \delta e^{-\omega t} V(0), \quad \forall t \geq 0, \quad (25)
\]
where \( \delta = 1 + \frac{1}{C_0} \) and \( \omega = \frac{1}{C_0} \log(1 + \frac{1}{C_0}) \). Finlay from \((25)\) we deduce that
\[
E(t) \leq V(t) \leq \delta e^{-\omega t} V(0) = \delta e^{-\omega t} E(0).
\]
Hence the result.
Remark. Note that the condition (7) does not holds when $B$ is compact (see [8]), which implies that we do not have the exponential stabilization. In this case the following proposition characterize the weak stabilization.

**Proposition 1.** Under the hypotheses above on $A_1$, $A_2$, $C$ and let $B$ be compact such that

$$\langle BS(t)y, S(t)y \rangle_E = 0 \Rightarrow y = 0, \quad \forall y \in E, \quad t \geq 0$$

then the feedback (6) weakly stabilizes (1).

Where $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $(S(t))_{t \geq 0}$ is the semigroup of contractions generated by $A$ on $E$.

**Proof.** System (1) can be written in the form:

\[
\begin{cases}
\dot{y}(t) = Ay(t) + u(t)By(t), \\
y(0) = y_0
\end{cases}
\]

where

\[
y = \begin{pmatrix} w_1 \\ \dot{w}_1 \\ w_2 \\ \dot{w}_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I_d & 0 & 0 \\ -A_1 & 0 & 0 & -C \\ 0 & 0 & 0 & I_d \\ 0 & C & -A_2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

and then we use the same technique as in Theorem 3.1 in [19].

\[\square\]

4. Applications. In this section, we present two examples of coupled wave equations. In both examples we show the desired observability estimates using a compact coupling in order to establish the exponential stabilization.

**Example 1: System without damping:**

Consider the following strongly coupled wave system:

\[
\begin{align*}
\dot{u}_1(t, x) - \alpha \frac{\partial^2 u_1}{\partial x^2}(t, x) - \alpha \frac{\partial^2}{\partial x^2}^{-1} \dot{u}_2(t, x) + v(t) \dot{u}_1(t) &= 0, \quad (t, x) \in (0, \infty) \times (0, 1), \\
\dot{u}_2(t, x) - \alpha \frac{\partial^2 u_2}{\partial x^2}(t, x) + \alpha \frac{\partial^2}{\partial x^2}^{-1} \dot{u}_1(t, x) &= 0, \\
u_i(t, 0) &= u_i(t, 1) = 0, \\
u_i(0) &= u_i^0, \quad \dot{u}_i(0) = u_i^1, \quad 1 \leq i \leq 2, \quad t \in (0, \infty)
\end{align*}
\]

where $\alpha \geq 0$. To put this control system into the form (3), consider the spaces $H = L^2(0, 1)$ and $E = \mathcal{H}^1(0, 1) \times L^2(0, 1) \times \mathcal{H}^1(0, 1) \times L^2(0, 1)$ and the operators $A_1 = A_2 = -\frac{\partial^2}{\partial x^2}$, with the domain $D(A_1) = D(A_2) = \mathcal{H}^2(0, 1) \cap \mathcal{H}^1(0, 1)$ which are obviously self-adjoint positive operators. In this case, the domains of the fractional power operators are given by

\[D(A_1^{\frac{1}{2}}) = D(A_2^{\frac{1}{2}}) = \mathcal{H}^1(0, 1).\]

The operator $B = I_d$ and finally

\[C = \alpha \left(-\frac{\partial^2}{\partial x^2}\right)^{-1} : L^2(0, 1) \rightarrow L^2(0, 1)\]
Proof of Lemma 4.2.

The eigenvalues of $A$ are $\lambda_n = \text{sgn}(n)i\pi n, n \in \mathbb{Z}^*$. The corresponding eigenfunctions are given by

$$\Phi^n = \left( \begin{array}{c} 1 \\ -1 \\ \lambda_n \\ -1 \lambda_n \end{array} \right) \frac{\sin(n\pi x)}{\sqrt{2}}, \quad n \in \mathbb{Z}^*$$

and form an orthonormal basis in $E$.

Proof of Lemma 4.2. • $(\Phi^n)_{n \in \mathbb{Z}^*}$ is an orthonormal sequence in

$$E = H^1_0(0, 1) \times L^2(0, 1) \times H^1_0(0, 1) \times L^2(0, 1),$$

where $H^1_0(0, 1)$ is equipped with the norm $\|\nabla\|_{L^2(0, 1)}$. 

C is self-adjoint, positive and compact operator (see Proposition 8.2.1 [7]). The homogeneous system corresponding to (28) is given by

$$\begin{cases}
\ddot{\varphi}_1(t, x) - \frac{\partial^2 \varphi_1(t, x)}{\partial x^2} = 0, \\
\ddot{\varphi}_2(t, x) - \frac{\partial^2 \varphi_2(t, x)}{\partial x^2} = 0, \\
\varphi_1(t, 0) = \varphi_2(t, 0) = 0, \\
\varphi_1(t, 1) = \varphi_2(t, 1) = 0.
\end{cases}$$

We have the following theorem:

**Theorem 4.1.** For all $T \geq 2$. There exist two positive constants $C_1$ and $C_2$ such that

$$\| \langle \varphi_1, \varphi_2 \rangle \|_E \simeq \int_0^T \int_0^1 |\dot{\varphi}_2|^2 dx dt,$$

holds for any $(\varphi_1^0, \varphi_2^0, \varphi_1^1, \varphi_2^1) \in E$ and $(\varphi_1, \varphi_2)$ solution of system (29).

System (29) is a compact perturbation of this system

$$\begin{cases}
\ddot{\varphi}_1(t, x) - \frac{\partial^2 \varphi_1(t, x)}{\partial x^2} = 0, \\
\ddot{\varphi}_2(t, x) - \frac{\partial^2 \varphi_2(t, x)}{\partial x^2} = 0, \\
\varphi_1(t, 0) = \varphi_1(t, 1) = 0, \\
\varphi_1(t, 0) = \varphi_1(t, 1) = 0.
\end{cases}$$
In fact we have
\[
\|\Phi^n\|_H^2 = \left\| \frac{1}{\lambda_n} \sin(n\pi x) \right\|_{H^1_0(0,1)}^2 + \left\| \frac{1}{\lambda_n} \sin(n\pi x) \right\|_{L^2(0,1)}^2 + \frac{\lambda_n}{\sqrt{2}} \left\| \frac{\sin(n\pi x)}{\sqrt{2}} \right\|_{L^2(0,1)}^2 \\
= 2 \left\| \frac{1}{\lambda_n} \sin(n\pi x) \right\|_{H^1_0(0,1)}^2 + 2 \left\| \frac{\sin(n\pi x)}{\sqrt{2}} \right\|_{L^2(0,1)}^2 \\
= \int_0^1 \left( \cos^2(n\pi x) + \sin^2(n\pi x) \right) dx = 1.
\]

and
\[
\langle \Phi^n, \Phi^m \rangle_E = 2\left( \frac{1}{\lambda_n} \sin(n\pi x), \frac{1}{\lambda_m} \sin(m\pi x) \right)_{H^1_0(0,1)} + 2\left( \frac{\sin(n\pi x)}{\sqrt{2}}, \frac{\sin(m\pi x)}{\sqrt{2}} \right)_{L^2(0,1)} \\
= \frac{1}{nm\pi^2} \int_0^1 n\pi \cos(n\pi x)\cos(m\pi x)dx + \int_0^1 \sin(n\pi x)\sin(m\pi x)dx \\
= \int_0^1 \cos ((n-m)\pi x) dx = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\]

Hence \((\Phi^n)_{n \in \mathbb{Z}^*}\) is an orthonormal sequence.

\[\bullet\] The completeness of \((\Phi^n)_{n \in \mathbb{Z}^*}\) in \(E\) is a consequence of the fact that these are all the eigenfunctions of the compact skew-adjoint operator \(A^{-1}\). It follows that \((\Phi^n)_{n \in \mathbb{Z}^*}\) is an orthonormal basis in \(E\).

**Remark 1.**
1. \(\sum_{n \in \mathbb{Z}^*} a_n \Phi^n \in H \iff \sum_{n \in \mathbb{Z}^*} |a_n|^2 < \infty\).
2. The solution of (32) with the initial data
\[
y(0) = \sum_{n \in \mathbb{Z}^*} a_n \Phi^n \in H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1)
\]
is given by
\[
y(t) = \sum_{n \in \mathbb{Z}^*} a_n e^{\lambda_n t} \Phi^n.
\]

We look for the internal observability estimates
\[
\| (u^0, u^1, u^0_2, u^1_2) \|_E^2 \simeq \int_0^T \int_0^1 |\varphi_1|^2 dx dt, \tag{33}
\]
where \((\varphi_1, \varphi_2)\) is the solution of (31).

Firstly we recall the Ingham’s theorem:

**Theorem 4.3.**
1. Let \(\lambda_n\) be a sequence of real numbers and \(\gamma > 0\) such that
\[
\lambda_{n+1} - \lambda_n \geq \gamma > 0,
\]
for all \(n\) and \(T > \frac{\pi}{\gamma}\). Then \(\exists C = C(T, \gamma) > 0\) such that for any finite sequence \((a_n)\), we have
\[
\sum_{n \in \mathbb{Z}^*} |a_n|^2 \leq C \int_{-T}^T \left| \sum_{n \in \mathbb{Z}^*} a_n e^{\lambda_n t} \right|^2 dt.
\]
2. Let \( \gamma_\infty := \lim \lambda_{n+1} - \lambda_n \). Then for \( T > \frac{\pi}{\gamma_\infty} \), \( \exists c_1, c_2 > 0 \), such that
\[
 c_1 \sum |a_n|^2 \leq \int_{-T}^{T} \left| \sum a_n e^{i\lambda_n t} \right|^2 dt \leq c_2 \sum |a_n|^2.
\]

**Theorem 4.4.** Let \( T \geq 2 \), there exist two positive constants \( C_1 \) and \( C_2 \) such that (33) holds for any \( (u_0^0, u_1^0, u_2^0, u_1^1) \in E \) and \( (\varphi_1, \varphi_2) \) solution of the unperturbed system (31).

**Proof.** Firstly, we have that
\[
\left\| (u_1^0, u_1^1, u_2^0, u_1^1) \right\|_E = \sum_{n \in \mathbb{Z}^*} |a_n|^2.
\]

Let calculate:
\[
\int_{0}^{T} \int_{0}^{1} |\varphi_1(t, x)|^2 dx dt.
\]

By semi-group theory, we check that the solutions of (31) obeys the following regularity
\[
(\varphi_1, \varphi_2) \in C([0, T], L^2(0, 1) \times L^2(0, 1)) \cap C^1([0, T], H_0^1(0, 1) \times H_0^1(0, 1)).
\]

We obtain from Fubini’s Theorem that
\[
\int_{0}^{T} \int_{0}^{1} |\varphi_1(t, x)|^2 dx dt = \int_{0}^{1} \int_{0}^{T} \sum_{n \in \mathbb{Z}^*} |a_n e^{i\pi n \pi t} \sin(n\pi x)|^2 dt dx.
\]

Applying Ingham’s inequality for the sequences \( \{n\pi \} \) and \( \{a_n \sin(n\pi x)\} \),
\[
((n+1)\pi - n\pi = \pi = \gamma), \quad \gamma_\infty = \lim \left( (n+1)\pi - n\pi = \pi \right).
\]

By Theorem 4.3 we obtain
\[
\int_{0}^{T} \int_{0}^{1} |\varphi_1(t, x)|^2 dx dt \geq \int_{0}^{1} \sum_{n \in \mathbb{Z}^*} |a_n \sin(n\pi x)|^2 dx
\]
\[
\geq \sum_{n \in \mathbb{Z}^*} |a_n|^2 \int_{0}^{1} \sin^2(n\pi x) dx, \quad (T > \frac{2\pi}{\gamma} = 2).
\]

Let \( C = \inf_{n \in \mathbb{Z}^*} \int_{0}^{1} \sin^2(n\pi x) dx, \) \( C > 0 \), in fact,
\[
C_n = \int_{0}^{1} \sin^2(n\pi x) dx \geq \frac{1}{2} - \frac{1}{2|n\pi|}
\]
\[
\lim_{n \to +\infty} C_n > 0 \implies \exists n_0 \in \mathbb{N} \text{ such that } C > 0. \text{ Hence (33) occurs for the system (31).} \]

**Proof of Theorem 4.4.** System (29) can be rewrite as \( y'(t) = (A + B)y(t) \). Where \( A \) is skew-adjoint, have a resolvent compact and generates a semi-group. Moreover \( A + B \) has a pseudo basis of generalized eigenvectors whose eigenvalues are of finite type (see [18], for definition).

In fact, let \( \{z_k\} \) be an orthonormal basis in \( L^2(0, 1) \), satisfying
\[
\begin{cases}
-\Delta z_k = \gamma_z^2 z_k, & z_k \text{ in } (0, 1), \\
z_k = 0 & \text{for } x = 0 \text{ and } x = 1.
\end{cases}
\]

Since \( Z_k := \{\beta \cdot z_k, \beta \in \mathbb{C}^2\} \), is stable by \( A + B \), we obtain a Riesz basis of subspaces generated by generalized eigenvectors for \( A + B \).
Thus we apply Theorem (2.6) in [18] with \( p_1(x) =: \| \dot{\phi}_1 \|_{L^2(0,1)} \) to deduce the inequality (30) for the system (29).

We give the following proposition:

**Proposition 2.** According to Theorem 2.1 we deduce that the system (28) is exponentially stabilizable by the corresponding feedback given in (6).

**Example 2: System with damping**

We consider the following strongly coupled wave system:

\[
\begin{cases}
\ddot{u}_1(t, x) - \frac{\partial^2 u_1}{\partial x^2}(t, x) + a(x) \dot{u}_1(t, x) + v(t) u_1(t, x) + C \ddot{u}_2(t, x) = 0, \\
\ddot{u}_2(t, x) - \frac{\partial^2 u_2}{\partial x^2}(t, x) - C \ddot{u}_1(t, x) = 0, \\
u_i(t, -1) = u_i(t, 1) = 0, & 1 \leq i \leq 2, \\
u_i(0, x) = u_i^0, & \dot{u}_i(0, x) = u_i^1, \quad 1 \leq i \leq 2, \quad x \in (-1, 1),
\end{cases}
\]  

(36)

where the function \( a \) defined by

\[
a \in L^\infty(-1, 1), \quad a(x) > 1, \quad \forall x \in (-1, 1).
\]  

(37)

The coupling operator \( C \) is defined as follows

\[
C : L^2(-1, 1) \longrightarrow L^2(-1, 1)
\]

\[
f \longrightarrow \int_{-1}^{1} k(x, u) f(u) du
\]

where \( k \in L^2((-1, 1) \times (-1, 1)) \) and it is chosen such that \( C \) is self-adjoint and compact, for example one can consider \( k(x, u) = (x^2 + \frac{2}{3} xu + u^2) \) (see p 150 [13]).

The system (36) has the form (11) if we set

\[
A_1 = A_2 = -\frac{\partial^2}{\partial x^2}, \quad B = I_d, \quad D(A_1^\frac{1}{2}) = D(A_2^\frac{1}{2}) = H_0^1(-1, 1)
\]

and \( D \) defined by

\[
(Dy)(x) := a(x) y(x) \quad \forall y \in L^2(-1, 1).
\]

The homogeneous system corresponding to (36) is given by

\[
\begin{cases}
\ddot{\varphi}_1(t, x) - \frac{\partial^2 \varphi_1}{\partial x^2}(t, x) + C \ddot{\varphi}_2(t, x) = 0, & (t, x) \in (0, \infty) \times (-1, 1), \\
\ddot{\varphi}_2(t, x) - \frac{\partial^2 \varphi_2}{\partial x^2}(t, x) - C \ddot{\varphi}_1(t, x) = 0, & (t, x) \in (0, \infty) \times (-1, 1) \\
\varphi_i(t, -1) = \varphi_i(t, 1) = 0, & 1 \leq i \leq 2, \\
\varphi_i(0, x) = \varphi_i^0, & \varphi_i(0, x) = \varphi_i^1, \quad 1 \leq i \leq 2, \quad x \in (-1, 1),
\end{cases}
\]  

(38)

System (38) is a perturbation of the following system

\[
\begin{cases}
\ddot{\varphi}_1(t, x) - \frac{\partial^2 \varphi_1}{\partial x^2}(t, x) = 0, & (t, x) \in (0, \infty) \times (-1, 1), \\
\ddot{\varphi}_2(t, x) - \frac{\partial^2 \varphi_2}{\partial x^2}(t, x) = 0 & (t, x) \in (0, \infty) \times (-1, 1) \\
\varphi_i(t, -1) = \varphi_i(t, 1) = 0, & 1 \leq i \leq 2, \\
\varphi_i(0, x) = \varphi_i^0, & \varphi_i(0, x) = \varphi_i^1, \quad 1 \leq i \leq 2, \quad x \in (-1, 1),
\end{cases}
\]  

(39)
then by similar arguments of example 1 we deduce that for all $T \geq 2$ there exists two positive constants $C_1$ and $C_2$ such that the following estimation holds
\[
\|(\varphi_1^0, \varphi_1^1, \varphi_2^0, \varphi_2^1)\|^2_E \approx \int_0^T \int_{-1}^1 |\dot{\varphi}_1|^2 \, dx \, dt, \tag{40}
\]
for all solution of (38). In other hand using (37) we have
\[
\int_0^T \int_{-1}^1 a(x)|\dot{\varphi}_1(t, x)|^2 \, dx \, dt \geq \int_0^T \int_{-1}^1 |\varphi_1(t, x)|^2 \, dx \, dt \tag{41}
\]
by combining (41) with (40) we deduce that for all $T \geq 2$ there exist a positive constant $C_1$ such that
\[
\|(u_0^1, u_1^1, u_0^2, u_2^1)\|_E \leq C_1 \int_0^T \int_{-1}^1 a(x)|\dot{\varphi}_1(t, x)|^2 \, dx \, dt + \int_0^T \int_{-1}^1 |\varphi_1(t, x)|^2 \, dx \, dt, \tag{42}
\]
holds for all solution of system (38). Therefore Theorem 3.2 ensures the exponential stabilization of system (36) using the appropriate feedback
\[
v(t) = f \left( \int_{-1}^1 |y(t, x)|^2 \, dx \right) \tag{43}
\]
where $f$ is a locally lipschitz function which satisfies the hypotheses $H_1$, $H_2$ and $H_3$.

**Conclusion.** In this work, a non linear feedback has been proposed to study the stabilization of a class of coupled hyperbolic systems. The stabilising controls acts only on one equation and guarantees the exponential decay of those systems. This article leaves the open question of the study of the stabilization when the control operator $B$ is unbounded and when the coupling operator $C$ is nonlinear.

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