Betti numbers of transversal monomial ideals

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Abstract

In this paper, by a modification of a previously constructed minimal free resolution for a transversal monomial ideal, the Betti numbers of this ideal is explicitly computed. For convenient characteristics of the ground field, up to a change of coordinates, the ideal of \( t \)-minors of a generic pluri-circulant matrix is a transversal monomial ideal. Using a Gröbner basis for this ideal, it is shown that the initial ideal of a generic pluri-circulant matrix is a stable monomial ideal when the matrix has two square blocks. By means of the Eliahou-Kervair resolution, the Betti numbers of this initial ideal is computed and it is proved that, for some significant values of \( t \), this ideal has the same Betti numbers as the corresponding transversal monomial ideal. The ideals treated in this paper, naturally arise in the study of generic singularities of algebraic varieties.

Key Words: Betti numbers; Pluri-circulant matrix; Stable monomial ideal; Transversal monomial ideal.

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1. Introduction

Let \( S = k[y_{i,j(i)} : 1 \leq i \leq n, 1 \leq j(i) \leq b_i] \) be the polynomial ring in \( m = b_1 + \cdots + b_n \) indeterminates over a field \( k \). Let \( D = D(b_1, b_2, \ldots, b_n) \) be the matrix

\[
\begin{bmatrix}
y_{11} & \cdots & y_{1b_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & y_{21} & \cdots & y_{2b_2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & y_{n1} & \cdots & y_{nb_n}
\end{bmatrix}.
\]

Let \( I_t(D) \in S \) be the ideal generated by \( t \)-minors of \( D \). This is a square-free monomial ideal and is called a transversal monomial ideal. In \((9), \S 3\) the Hilbert series of \( S/I_t(D) \) has been computed by means of the simplicial complex associated to \( I_t(D) \) and its minimal free resolution has been constructed. In this paper we outline a modification of this resolution and compute the Betti numbers of \( I_t(D) \).

Let \( R = k[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq b] \) be the polynomial ring in \( nb \) indeterminates over \( k \) and let \( P = \begin{bmatrix} M_1 & M_2 & \cdots & M_b \end{bmatrix} \) be a generic pluri-circulant matrix where \( M_j \) is the generic circulant matrix with the first row \((x_{1j} \ x_{2j} \ \ldots \ x_{nj})\). The ideal generated by \( t \)-minors of \( P \) has been considered in \([8]\). Under some hypothesis on \( k \), this ideal is closely related to a certain transversal monomial ideal. In fact, if \( k \) possesses the \( n \)th roots of unity and \( \text{char}(k) \nmid n \) then over such ground field, up to a linear change of coordinates, the matrix \( P \) is equivalent to the matrix \( D \) with \( b_1 = b_2 = \cdots = b_n = b \) \((9), \S 4\). On the other hand, with the same assumptions on the ground field, it has been proved that for a suitable monomial order on \( R \), certain set of \( t \)-minors of the first \( t \) rows of \( P \) forms a Gröbner basis for \( I_t(P) \) and its initial ideal \( J_t \) has been computed (see \([6], \S 3\)). For a filed of arbitrary characteristic, such a result is known only for \( t = n, n - 1 \) \((8), \S 5\). However, the monomial ideal \( J_t \) can be studied in its own. We show that for \( b = 2 \), \( J_t \) is a stable monomial ideal. This class of monomial ideals have been introduced and studied by Eliashou and Kervaire \([2]\). Using the Eliashou-Kervaire resolution for stable monomial ideals, we compute the Betti numbers of \( J_t \). For \( t = n, n - 1 \) and \( n - 2 \), we prove that the Betti numbers of \( J_t \) are equal to the corresponding Betti numbers of \( I_t(D) \). These equalities are not immediate and require some unexpected combinatorics.

The ideals treated here, naturally arise in the study of the local equations of generic singularities of algebraic varieties (see \([6], [7]\)).
2. The minimal free resolution and Betti numbers of \( I_t(D) \)

The notation employed for description of the minimal free resolution of \( I_t(D) \) in (§3) can be modified to make this resolution more accessible. This is the first task of the section. Using this setting, we compute the Betti numbers of \( I_t(D) \) explicitly. In the special case \( m = n \), the modification allows one to define a structure of a graded differential algebra on the resolution.

Let \( V \) be a free \( S \)-module of rank \( m = b_1 + \cdots + b_n \) generated by symbols \( e_{i,j(i)} \) in one-to-one correspondence with the indeterminates \( y_{i,j(i)} \). Let \( W \) be the free \( S \)-module of rank \( n \) generated by symbols \( \varepsilon_1, \ldots, \varepsilon_n \) in one-to-one correspondence with rows of \( D \). For \( q = 0, 1, \ldots, m - t - 1 \), let \( E_q = \bigoplus_{p=1}^{q+1} \wedge^p W \) and let \( C_q \subset (\wedge^{t+q} V) \otimes E_q \) be the free \( S \)-module generated by the basis elements

\[
e_{i_1,j_1} \wedge \cdots \wedge e_{i_r,j_r} \wedge \cdots \wedge e_{i_s,j_s} \otimes \varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_{s-t+1}}
\]

where

\[
t \leq s \leq n, \quad r_1, \ldots, r_s \geq 1, \quad \text{and,} \quad 1 \leq k_1 < \cdots < k_{s-t+1} \leq s.
\]

As for the elements of the wedge product, we adopt the usual convention on the order of vectors \( e_{i,j(i)} \) to appear in the lexicographic order of their indices, i.e.,

\[
1 \leq i_1 < \cdots < i_s \leq n,
\]

and

\[
1 \leq j_u(i_v) < j_{u+1}(i_v) \leq b_{i_v}, \forall u,v.
\]

Clearly, \( r_1 + \cdots + r_s = t + q \), and \( t \leq s \leq \min\{n,t+q\} \).

For simplicity, we may drop the first subscripts in each \( e_{i,j(i)} \) and denote the basis elements of \( C_q \) by

\[
e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge \cdots \wedge e_{j_s} \otimes \varepsilon_{k_1} \wedge \cdots \wedge \varepsilon_{k_{s-t+1}}.
\]

To keep a reference of the above basis elements, it may be helpful to replace \( y_{i,j(i)} \) with \( e_{j(i)} \) in the matrix \( D \). Then the above basis elements are obtained by selecting the rows \( i_1, \ldots, i_s \) in the resulting matrix and choosing \( r_1 \) nonzero entries on the \( i_1 \)st row, ... , and, \( r_s \) nonzero entries on the \( i_s \)th row, and finally, specifying further rows
For \( q \geq 0 \) the boundary map \( d : C_{q+1} \rightarrow C_q \) can now be defined by

\[
d(\xi \otimes \delta) = \sum_{v^*} \Delta_v(\xi) \otimes \delta + \Lambda(\xi \otimes \delta),
\]
where the asterisk sign over \( v \) in the summation means that we sum only over those values of \( v \) for which \( r_v \geq 2 \).

By (2.1), (2.2) and some straightforward computation, it follows that \( C \) is indeed a complex. However, in general, the complex \( C \) does not lead to a free resolution for \( I_t(D) \). We construct a subcomplex \( K \subset C \) such that an augmentation of the quotient complex \( C/K \) is the minimal free resolution for \( I_t(D) \).

For \( q \geq 0 \), let \( K_q \subset C_q \) be the submodule generated by all expressions

\[
\xi \otimes \left( \sum_{p=1}^{s-t+2} (-1)^p \epsilon_{k_1} \wedge \cdots \wedge \epsilon_{k_p} \wedge \cdots \wedge \epsilon_{k_{s-t+2}} \right)
\]

for some \( k_1, \ldots, k_{s-t+2} \) with \( 1 \leq k_1 < \cdots < k_{s-t+1} < k_{s-t+2} = s \), where as above,

\[
\xi = e_{j_1(i_1)} \wedge \cdots \wedge e_{j_1(i_t)} \wedge \cdots \wedge e_{j_1(i_s)} \wedge \cdots \wedge e_{j_1(i_s)}.
\]

It can be checked that \( K \) is a sub-complex of \( C \), i.e., \( d(K_{q+1}) \subset d(K_q) \). Let \( L_q = C_q/K_q \) for \( q = 0, \ldots, m-t-1 \). While among the summands of any of the above expressions there is only one \( \epsilon \) without an index equal to \( i_s \), we will consider the representatives

\[
e_{j_1(i_1)} \wedge \cdots \wedge e_{j_1(i_1)} \wedge \cdots \wedge e_{j_1(i_s)} \wedge \cdots \wedge e_{j_1(i_s)} \otimes \epsilon_{k_1} \wedge \cdots \wedge \epsilon_{k_{s-t}} \wedge \epsilon_{i_t},
\]

as basis elements of \( L_q \). In particular \( L_q \) is a free \( S \)-module. Although one may ignore writing the index \( i_s \), it may be kept to signify the action of the differentiations by diagonal entries.

Finally, the augmentation map \( d : L_0 \longrightarrow I_t(D) \) is defined as the determinant map, i.e.,

\[
d(e_{j_1(i_1)} \wedge \cdots \wedge e_{j_1(i_t)} \otimes \epsilon_{i_t}) = y_{i_1 j_1(i_1)} \cdots y_{i_t j_1(i_t)}.
\]

The main result of [9] may now be stated. For the proof we refer to ([9], Theorem 3.1).

2.1. Theorem. The complex \( L \) with the induced boundary maps is the minimal free resolution for \( I_t(D) \).
2.2. Remark. The complex $L_*$ is clearly linear. However, the linearity also follows via properties of the simplicial complex associated to $I_t(D)$ (see [7], Proposition 2.1). In fact, $I_t(D)$ is weakly polymatroidal, and hence, it has linear quotients and in particular, it has linear resolution (see [5], Theorem 1.4).

The natural multiplication

$$(\xi \otimes \delta)(\zeta \otimes \tau) = (\xi \wedge \zeta) \otimes (\delta \wedge \tau),$$

is not well-defined on the complex $C_*$ to turn it into a differential algebra unless $t = n$, or, $b_1 = \cdots = b_n = 1$. Even for $t = n$, the Leibnitz formula fails. However, in the latter case, $C_*$ is a graded differential algebra under the above multiplication. Thus we may state the following which should have been well-known.

2.3. Corollary. For $m = n$ the complex $C_*$ is a graded differential algebra under the multiplication (2.4) and the complex $K_*$ is a homogenous ideal in $C_*$ and hence $L_*$ inherits the structure of a graded differential algebra as the minimal free resolution of the ideal generated by all square-free monomials of degree $t$ in $n$ indeterminates. In this case, with the exception of the augmentation map, the boundary maps descend to $\Lambda$.

We now return to the general case. Although when the minimal free resolution is known, the Betti numbers are theoretically available, explicit formulation of these numbers provide finer information. We now pursue on the computation of the Betti numbers of $I_t(D)$.

2.4. Proposition. With the notations as the above,

$$\beta_q(I_t(D)) = \sum_{s=t}^{\text{Min}\{t+q, n\}} \binom{s-1}{t-1} \sum_{1 \leq i_1 < \cdots < i_s \leq n} \sum_{r_1 + \cdots + r_s = t+q, r_1, \ldots, r_s \geq 1} \binom{b_{i_1}}{r_1} \cdots \binom{b_{i_s}}{r_s}.$$

For the case $b_1 = \cdots = b_n = b$,

$$\beta_q(I_t(D)) = \sum_{s=t}^{t+q} \binom{s-1}{t-1} \binom{n}{s} \sum_{r_1 + \cdots + r_s = t+q, r_1, \ldots, r_s \geq 1} \binom{b}{r_1} \cdots \binom{b}{r_s}.$$
For \( b_1 = \cdots = b_n = 2 \),

\[
\beta_q(L(D)) = \sum_{s=t}^{t+q} \binom{s-1}{t-1} \binom{n}{s} \binom{s}{t+q-s} 2^{2s-t-q}.
\]

The usual conventions \( \binom{\alpha}{\beta} = 0 \) for \( \alpha < \beta \), and \( \binom{\alpha}{0} = 1 \) for \( \alpha \geq 0 \), are to be adopted in these formulas. Thus, for example, the precise lower bound and upper bound of the last summation would be \( s = \text{Max}\{t, \lceil \frac{t+q}{2} \rceil \} \) and \( s = \text{Min}\{t+q, n\} \), respectively.

**Proof.** We use the expressions (2.3) for the basis of \( L_q \). In the first formula, \( \binom{s-1}{t-1} \) is the number of choices for \( k_1, \ldots, k_{s-t} \), the second summation is for the choices of \( i_1, \ldots, i_s \) and the last summation is for the number of choices of \( r_1, \ldots, r_s \). The second equality is immediate. For the last equality, while for each \( v \in \{1, \ldots, s\} \), \( r_v = 1 \) or 2, the binomial coefficient counts the number of cases and the power of 2 is the number of \( r_v \)'s with \( r_v = 1 \). \( \Box \)

**2.5. Remark.** It is important to emphasize the condition \( r_1, \ldots, r_s \geq 1 \) in the first and the second formulas in the above proposition. Otherwise, the last summation would simplify using the following so-called generalized Vandermonde convolution \[4]:

\[
\sum_{r_1+\cdots+r_s=t+q} \binom{b_{i_1}}{r_1} \cdots \binom{b_{i_s}}{r_s} = \binom{b_{i_1} + \cdots + b_{i_s}}{r_1 + \cdots + r_s}.
\]

1. **Betti numbers of the initial ideal of the ideal of \( t \)-minors of generic pluri-circulant matrices**

Generic pluri-circulant matrices and their ideals of \( t \)-minors arise naturally in the study of generic projections in algebraic geometry [6]. These ideals are closely related to transversal monomial ideals. In this section we recall some results on the ideals of \( t \)-minors of generic pluri-circulant matrices and then we determine Betti numbers of the initial ideals of these ideals of minors and compare them with the Betti numbers computed in the previous section.

Let \( R = k[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq b] \) be the polynomial ring in \( nb \) indeterminates
over $k$ and let $P = \begin{bmatrix} M_1 & M_2 & \cdots & M_b \end{bmatrix}$ be a generic pluri-circulant matrix where

$$M_j = \begin{bmatrix} x_{1j} & x_{2j} & \cdots & x_{nj} \\ x_{nj} & x_{1j} & \cdots & x_{n-1,j} \\ \vdots & \vdots & \ddots & \vdots \\ x_{2j} & x_{3j} & \cdots & x_{1j} \end{bmatrix}$$

is a generic circulant matrix. Let $k$ possess the $n$th roots of unity and $\text{char}(k) \nmid n$. Then, under a linear change of variables in $R$, $I_t(P)$ converts to the ideal $I_t(D)$ in $S$ considered in the previous section with $b_1 = b_2 = \cdots = b_n = b$ (see [9], § 4). Let

$$T = \begin{bmatrix} T_1 & T_2 & \cdots & T_b \end{bmatrix},$$

where

$$T_j = \begin{bmatrix} x_{1j} & x_{2j} & \cdots & \cdots & x_{nj} \\ 0 & x_{1j} & \cdots & \cdots & x_{n-1,j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{1j} & \cdots & x_{n-t+1,j} \end{bmatrix}, \ j = 1, 2, \cdots, b.$$ 

Let $J_t$ be the ideal in $R$ generated by products of the entries of the main diagonals of $T$. The ideal $J_t$ is $Q$-primary where $Q$ is the prime ideal generated by the indeterminates on the last row of $T$. It is known that, under the above assumptions on the ground field, for a suitable monomial order on $R$, the set of $t$-minors of the first $t$ rows of $P$ whose main diagonals correspond to the main non-zero diagonals of $T$, forms a Gröbner basis for $I_t(P)$ and $J_t$ is its initial ideal ([6], Theorem 3.3). For arbitrary filed $k$, this result is only known for $b = 2, t = n, n - 1$ ([8], Theorem 5.4). We show that for $b = 2$, $J_t$ is a stable monomial ideal. Using the Eliahou-Kervaire resolution for stable monomial ideals [2], we prove that all Betti numbers of $J_t$ and $I_t(D)$ are equal at least for $t = n, n - 1, n - 2$. In general, Betti numbers of the initial ideal of a given ideal, only give upper bounds for Betti numbers of the the original ideal.

Recall that a monomial ideal $I \subset k[z_1, \cdots, z_n]$ is said to be stable if for every monomial $w \in I$ and index $i < m = \text{max}(w)$, the monomial $z_iw/z_m$ again belongs to $I$, where $\text{max}(w)$ denotes the largest index of the variables dividing $w$. Let $G(I)$ be the unique minimal generating set of $I$ consisting of monomials. Note that $I$ is stable if and only if the above condition holds for every $w \in G(I)$. Clearly, no nontrivial square-free monomial ideal is stable. In particular, $I_t(D)$ is not a stable monomial ideal. For $b = 2$, to
simplify the notation we use $d = n - t + 1$, and we consider the re-indexing of indeterminates in the ring $R$ such that the matrices $T_1$ and $T_2$ turn to the following forms, respectively:

$$T'_1 = \begin{bmatrix}
z_1 & z_2 & \cdots & z_d & z_{2d+1} & z_{2d+2} & \cdots & z_{2d+t-1} \\
0 & z_1 & \cdots & z_{d-1} & z_d & z_{2d+1} & \cdots & z_{2d+t-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & z_1 & z_2 & \cdots & z_{d-1} & z_d
\end{bmatrix},$$

$$T'_2 = \begin{bmatrix}
z_{d+1} & z_{d+2} & \cdots & z_{2d} & * & * & \cdots & * \\
0 & z_{d+1} & \cdots & z_{2d-1} & z_{2d} & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & z_{d+1} & z_{d+2} & \cdots & z_{2d-1} & z_{2d}
\end{bmatrix}.$$

In other words, we first re-index the indeterminates on the last row of $[T_1 \ T_2]$ so that the nonzero entries on the last row are linearly ordered. Then the remaining indeterminates on the first row of this matrix are re-indexed linearly. Since the last $t - 1$ indeterminates on the first row do not appear in $G(J_t)$, they are replaced by *’s. With this new indexing, the ideal $J_t$ is stable.

3.1. Lemma. For $b = 2$, with the above re-indexing of the indeterminates, $J_t$ is a stable monomial ideal. In particular, the associated Eliahou-Kervaire complex provides a linear minimal free resolution for $J_t$ equipped with a structure of graded differential algebra.

Proof. Recall that $G(J_t)$ consists of the products of entries of the main diagonals of $[T'_1 \ T'_2]$. Observe that every monomial in $G(J_t)$ has a unique representation in the form

$$z_{i_1} \cdots z_{i_q} z_{j_1} \cdots z_{j_r} z_{k_1} \cdots z_{k_s}$$

with

$$1 \leq i_1 \leq \cdots \leq i_q \leq d,$$

$$2d + 1 \leq j_1 \leq \cdots \leq j_r \leq 2d + t - 1 - q,$$

$$d + 1 \leq k_1 \leq \cdots \leq k_s \leq 2d,$$

$$q + r + s = t.$$
Conversely, any such representation identifies a unique monomial in $G(J_t)$. Let $w \in G(J_t)$ and let $w = z_{i_1} \cdots z_{i_q} z_{j_1} \cdots z_{j_r} z_{k_1} \cdots z_{k_\nu}$ be its unique representation. Since all monomials of degree $t$ in $z_k$ with $1 \leq \ell \leq 2d$ belong to $J$, we need to check the stability condition for the case $r \geq 1$. Then the maximum index of $w$ is $j_r$. We need to show that $w' = \frac{w}{z_{j_r}} \in G(J_t)$ for all $1 \leq \ell < j_r$. This can be checked directly. In fact, if $i_\tau \leq \ell \leq i_{\tau + 1}$, then $w' = z_{i_1} \cdots z_{i_\tau} z_{w'_{\tau + 1}} \cdots z_{i_q} z_{j_1} \cdots z_{j_{\tau - 1}} z_{k_1} \cdots z_{k_\nu} \in G(J_t)$. If $j_\tau \leq \ell \leq j_{\tau + 1}$, then $w' = z_{i_1} \cdots z_{i_q} z_{j_1} \cdots z_{j_{\tau - 1}} z_{k_1} \cdots z_{k_\nu} \in G(J_t)$. If $i_\tau \leq \ell < j_{\tau + 1}$, then $w' = z_{i_1} \cdots z_{i_q} z_{j_1} \cdots z_{j_{\tau - 1}} z_{k_1} \cdots z_{k_\nu} \in G(J_t)$. The last claim follows from (2), §2 Theorem 2.1 and Remark 1).

3.2. Remark. For $b > 2$, the ideal $J_t$ is not a stable monomial ideal as it can be inspected for $b = 3$, $n = t = 3$. On the other hand, even for $b = 2$, the ideal $J_t$ is not Borel fixed (see the definition in [11] or [2]), as it can be checked for $n = t = 5$.

Recall that by (2), §3, the Betti numbers of a stable monomial ideal $I$ is given by

$$\beta_q(I) = \sum_{w \in G(I)} \left( \max(w) - 1 \right),$$

where $G(I)$ is the minimal generating set of $I$. We will use this result to compute the Betti numbers of $J_t$ explicitly.

3.3. Lemma. For $b = 2$, let $v_\ell$ be the number of monomials in $G(J_t)$ with largest index $\ell$ and let $d = n - t + 1$. Then

(a) For $1 \leq \ell \leq 2d$,

$$v_\ell = \binom{t + \ell - 2}{t - 1}.$$

(b) For $2d + 1 \leq \ell \leq 2d + t - 1$,

$$v_\ell = \sum_{k=0}^{t-j-1} \binom{n-t+j+k}{k} \binom{n-k-1}{n-t} = \sum_{\tau=1}^{n-t+1} \binom{n+\tau-1}{t-j-1} \binom{n-t+j-\tau}{j-1} = \sum_{\tau=1}^{n-t+1} \binom{n}{t-j-\tau} \binom{n-t+j}{j+\tau-1},$$

where $j = \ell - 2d$.

(c) For $q = 0, 1, \cdots, 2n - t + 1$,

$$\beta_q(J_t) = \sum_{\ell=q+1}^{2n-t+1} \binom{\ell - 1}{q} v_\ell,$$

where $v_\ell$ is given in (a) and (b).
3.4. Remark. Observe that the first equality for \( v_\ell \) in (b) also makes sense for \( d + 1 \leq \ell \leq 2d \) and it reduces to (a). This follows from the well-known identity
\[
\sum_{k=a-m}^{c-n} \binom{a+k}{m} \binom{c-k}{n} = \binom{a+c+1}{m+n+1}
\]
for all non-negative integers \( a, m, c, n \) with \( a \geq m \) and \( c \geq n \). However, the second and third equality in (b) is valid only for \( 2d + 1 \leq \ell \leq 2d + t - 1 \).

Proof of Lemma 3.3. The ideal \( J_t \) contains all monomials of degree \( t \) in \( z_1, \ldots, z_{2d} \). Thus for \( 1 \leq \ell \leq 2d \), a monomials in \( G(J_t) \) with largest index \( \ell \) is of the form \( wz_\ell \) where \( w \) is any monomial of degree \( t - 1 \) in \( z_1, \ldots, z_\ell \). This settles (a). For \( 2d + 1 \leq \ell \leq 2d + t - 1 \) a monomial \( w \in G(J_t) \) with largest index \( \ell = 2d + j \) can be uniquely written in the form \( w = w_1z_\ell w_2 \), where \( w_1 \) varies in the set of all monomials of degree \( k \) in \( z_1, \ldots, z_d, z_{d+1}, \ldots, z_\ell \) and \( w_2 \) ranges over all monomials of degree \( t - k - 1 \) in \( z_{d+1}, \ldots, z_{2d} \), for \( k = 0, \ldots, t - j - 1 \). This proves the first equality in (b). To prove the second equality, we employ another method to count the same monomials. Using the index configuration in \( [T'_1 \quad T'_2] \), for fixed \( j, 1 \leq j = \ell - 2d \leq t - 1 \), any such monomial can be uniquely written in the form \( w = u_1z_\ell z_{d+j+\tau}u_2 \), where \( u_1 \) varies over the set of all monomials of degree \( t - j - 1 \) in \( z_1, \ldots, z_d, z_{d+1}, \ldots, z_\ell, z_{d+1}, \ldots, z_{d+j+\tau} \) and \( u_2 \) ranges over all monomials of degree \( j - 1 \) in \( z_{d+j+\tau}, \ldots, z_{2d} \), for \( \tau = 1, \ldots, d \). In fact, \( u_1 \) is a product any diagonal entries of
\[
\begin{bmatrix}
z_1 & \cdots & z_d & z_{2d+1} & \cdots & z_\ell & z_{d+1} & \cdots & z_{d+j+\tau} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & z_1 & \cdots & z_d & z_{2d+1} & \cdots & z_\ell & z_{d+1} & \cdots & z_{d+j+\tau}
\end{bmatrix}
\]
as a submatrix of \( [T'_1 \quad T'_2] \) with \( t - j - 1 \) rows, and \( u_2 \) is a product of any diagonal entries of
\[
\begin{bmatrix}
z_\tau & \cdots & z_{2d} & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & z_\tau & \cdots & z_{2d}
\end{bmatrix}
\]
a submatrix of \( T'_2 \) wit \( j - 1 \) rows. This clarifies the second equality in (b). The third equality in (b) follows from the second equality as a combinatorial identity, or, by using the identity (3.2) below. The assertion (c) is just the formula for the Betti numbers of a stable monomial ideal. \( \square \)
3.5. Remark. While for \( b = 2 \) the minimal free resolution of \( J_t \) has a natural structure of graded differential algebra, the minimal free resolution of \( I_t(D) \) and hence that of \( I_t(P) \) has no such natural structure as explained prior to Corollary 2.3. In particular, the minimal free resolution of \( I_t(P) \) is not a “natural lifting” of the Eliahou-Kervaire resolution of \( J_t \). More importantly, if \( \text{char}(k) | n \) and no minimal free resolution for \( I_t(P) \) is known, the minimal free resolution of \( J_t \) does not naturally lift to the minimal free resolution of \( I_t(P) \). This is contrary to what one could have hoped, since, at least for \( t = n, n - 1, J_t = \text{in}(I_t(P)) \). However, regardless of characteristic of the ground field, these three ideals have the same Betti numbers.

The following result should be true for all \( 1 \leq t \leq n \). We only prove it for \( t = n, n - 1, n - 2 \). Although the same procedure works for any specific value of \( t \), we are not able to provide a unified proof for arbitrary \( t \). We will replace \( d \) with \( n - t + 1 \).

3.6. Theorem. For \( b = 2, t = n, n - 1, n - 2 \), the ideals \( J_t, I_t(D) \) and \( I_t(P) \) have equal Betti numbers, i.e.,

\[
\beta_q(J_t) = \beta_q(I_t(D)) = \beta_q(I_t(P))
\]

for all \( q = 0, \cdots, 2n - t \).

Proof. The last equality is clear due to the explanations at the beginning of this section. Thus we prove the first equality.

For \( t = n \), the claim is rather straightforward. Indeed, by Lemma 3.3,

\[
\beta_q(J_n) = \sum_{\ell = q + 1}^{2} \binom{\ell - 1}{q} \binom{n + \ell - 2}{n - 1} + \sum_{j = q - 1}^{n - 1} \binom{j + 1}{q} \binom{n}{j + 1}.
\]

For \( q \geq 2 \), the first sum is zero. By Proposition 2.4,

\[
\beta_q(I_t(D)) = \binom{n}{q} 2^{n - q}.
\]

Thus for \( q \geq 2 \) the equality \( \beta_q(J_n) = \beta_q(I_t(D)) \) is just a well-known combinatorial identities. For \( q = 0, 1 \), the proof is similar.

For \( t = n - 1, n - 2 \) to settle the equality \( \beta_q(J_t) = \beta_q(I_t(D)) \) we try to write both sides as \( \mathbb{Z}[q] \)-linear combinations of \( \binom{n}{q - \alpha} 2^{n - q + \alpha} \) for \( \alpha = 0, \pm 1, \pm 2 \). The main combinatorial identities to be employed are

\[
\binom{i}{\tau} \binom{i - \tau}{q - \tau} = \binom{q}{\tau} \binom{i}{q},
\]  
(3.1)
\[
\binom{n + \rho}{q} = \sum_{i=0}^{\rho} \binom{\rho}{i} \binom{n}{q - i} \tag{3.2}
\]

and
\[
\sum_{i=q}^{n} \binom{i}{q} \binom{n}{i} = \left( \frac{n}{q} \right) 2^{n-q} \tag{3.3}
\]

For \( t = n - 1 \), by (c), (a) and the last equality of (b) in Lemma 3.3,
\[
\beta_q(J_{n-1}) = \sum_{\ell=q+1}^{4} \binom{\ell - 1}{q} \binom{n + \ell - 3}{n - 2} + \sum_{j=q+3}^{n-2} \binom{j + 3}{q} \sum_{\tau=1}^{2} \binom{n}{j + \tau + 1} \binom{j + 1}{2 - \tau}.
\]

We treat the case \( q \geq 4 \) so that the first sum is zero. For \( 0 \leq q \leq 3 \), similar computation works where the first sum recovers the missing quantity expected for the required equality. For \( q \geq 4 \), using \( i = j + 3 \) we get
\[
\beta_q(J_{n-1}) = \sum_{i=q}^{n+1} (i - 2) \binom{i}{q} \binom{n}{i-1} + \sum_{i=q}^{n} \binom{i}{q} \binom{n}{i}.
\]

By (3.1) the first sum in \( \beta_q(J_{n-1}) \) reduces to
\[
\sum_{i=q}^{n+1} (i + 1) \binom{i}{q} \binom{n}{i-1} = 3 \sum_{i=q}^{n+1} \binom{i}{q} \binom{n}{i-1} - 3 \sum_{i=q}^{n+1} \binom{i-1}{q} \binom{n}{i-1} - 3 \sum_{i=q}^{n+1} \binom{i-1}{q} \binom{n}{i-1}.
\]

Using (3.2) and (3.3) we have
\[
\sum_{i=q}^{n+1} (i + 1) \binom{i}{q+1} \binom{n}{i-1} = (q + 1) \sum_{i=q}^{n+1} \left[ \binom{i-1}{q+1} + \binom{i-1}{q} + \binom{i-1}{q-1} \right] \binom{n}{i-1}
\]
\[
= (q + 1) \left[ \binom{n}{q+1} 2^{n-q-1} + 2 \binom{n}{q} 2^{n-q} + \binom{n}{q-1} 2^{n-q+1} \right].
\]

Hence by (3.3) we get
\[
\beta_q(J_{n-1}) = (q + 1) \binom{n}{q+1} 2^{n-q-1} - 2q \binom{n}{q} 2^{n-q} + (q - 2) \binom{n}{q-1} 2^{n-q+1}.
\]

On the other hand, by Proposition 2.3,
\[
\beta_q(I_{n-1}(D)) = n \binom{n-1}{q} 2^{n-q-1} + (n - 1) \binom{n}{q-1} 2^{n-q+1}.
\]
Using (3.1) this reduces to
\[ \beta_q(I_{n-1}(D)) = (q+1) \binom{n}{q+1} 2^{n-q-1} + q \binom{n+1}{q} 2^{n-q+1} - 2 \binom{n}{q-1} 2^{n-q+1}. \]

By (3.2) this is equal to \( \beta_q(J_{n-1}) \) computed above.

For \( t = n-2 \), the proof is almost similar. More specifically, by Lemma 3.3,
\[ \beta_q(J_{n-2}) = \sum_{\ell=q+1}^6 \binom{\ell-1}{q} \binom{n+\ell-4}{n-3} + \sum_{j=q-5}^{n-3} \binom{j+5}{q} \sum_{\tau=1}^3 \binom{n}{j+\tau+2} \binom{j+2}{3-\tau}. \]

Again we treat the case \( q \geq 6 \) so that the first sum is zero. Using \( i = j + 3 \) we get
\[ \beta_q(J_{n-2}) = \sum_{i=q-2}^n \binom{i-1}{2} \binom{i+2}{q} \binom{n}{i} + \sum_{i=q-2}^{n-1} \binom{i-1}{2} \binom{i+2}{q} \binom{n}{i+1} + \]
\[ \sum_{i=q-2}^{n-2} \binom{i+2}{q} \binom{n}{i+2} = \sum_i \left( \binom{i-1}{2} \binom{i+2}{q} \binom{n}{i} + (i-2) \binom{i+1}{q} \binom{n}{i} \right). \]

Finally, with the same method as the case \( t = n-1 \) we arrive to
\[ \beta_q(J_{n-2}) = \binom{q+2}{2} \binom{n}{q+2} 2^{n-q-2} + 4 \binom{q+1}{2} \binom{n}{q+1} 2^{n-q-1} + \]
\[ q(3q-4) \binom{n}{q} 2^{n-q} + 2(q-1)(q-3) \binom{n}{q-1} 2^{n-q+1} + \binom{q-3}{2} \binom{n}{q-2} 2^{n-q+2}. \]

By Proposition 2.3 and manipulations as the previous case \( \beta_q(I_{n-2}(D)) \) amounts to the same quantity. \( \square \)

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