Electron-photon interaction in a quantum point contact coupled to a microwave resonator

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(Dated: April 14, 2016)

We study a single-mode cavity weakly coupled to a voltage-biased quantum point contact. In a perturbative analysis, the lowest order predicts a thermal state for the cavity photons, driven by the emission noise of the conductor. The cavity is thus emptied as all transmission probabilities of the quantum point contact approach one or zero. Two-photon processes are identified at higher coupling, and pair absorption dominates over pair emission for all bias voltages. As a result, the number of cavity photons, the cavity damping rate and the second order coherence $g^{(2)}$ are all reduced and exhibit less bunching than the thermal state. These results are obtained with a Keldysh path integral formulation and reproduced with rate equations. They can be seen as a backaction of the cavity measuring the electronic noise. Extending the standard $P(E)$ theory to a steady-state situation, we compute the modified noise properties of the conductor and find quantitative agreement with the perturbative calculation.

PACS numbers: 84.40.Az, 42.50.Ar, 42.50.Ct, 72.70.+m

I. INTRODUCTION

In the last years the research on circuit quantum electrodynamics (cQED) has been extended to mesoscopic devices, e.g., quantum dots (QDs)1–10, tunnel junctions11–19 or dc-biased Josephson junctions20–24. The high sensitivity of the cavity field offers a powerful way to probe the electronic conductor in a non-invasive way,25–27 thus acting as an excellent tool to investigate correlated electronic systems such as Majorana fermions28–30 and Kondo physics2,31. Furthermore, many interesting phenomena due to the interplay between electrons and photons have been demonstrated in this hybrid device: quadrature squeezing3,32, spin-photon coupling3,32, coupling between distant QDs33–36, and photon lasing37–40. These demonstrations open a huge avenue of possibilities in quantum information processing and quantum state engineering41 with mesoscopic devices coupled to transmission lines. Also the photons emitted by a quantum conductor carry information on the dynamics of electrons42–44. They have been used to characterize the photonic side of the dynamical Coulomb blockade effect (DCB)45–47 in which photons are radiated in inelastic electron scattering.

Even with the rapid progress on cQED with mesoscopic devices the physics of a quantum point contact (QPC) coupled to a superconductor microwave resonator has been practically unexplored. The QPC is a coherent conductor formed by two metallic gates on the top of a two-dimensional electron gas. A voltage applied on the gates creates a one-dimension constriction connecting the two sides of the electron gas. The gate voltage controls both number of channels $n$ connecting the two metallic reservoirs and their transmission probabilities $T_n$. As microwave photons are created by the scattering of electrons tunneling from one reservoir to the other, the QPC is an excellent device to investigate photon statistics with controlled electron scattering. In the case of a coherent conductor QPC coupled to an open transmission line, it has been predicted that the photons emitted by the QPC present sub- or super-Poissonian statistics, depending on the transmission probability and source-drain voltage48–52. However, for a microwave cavity exchanging photons with the QPC, less is known about photon statistics. The cases of a tunnel junction16 and a quantum dot3 have been investigated with a quantum master equation.

In this article, we discuss a dc-biased coherent QPC coupled to a microwave resonator formed with a transmission line cavity (TLC). A single mode is considered in the cavity, with frequency $\omega_0 = 2\pi\omega_n$, such that our model equivalently describes a lumped LC circuit coupled to the QPC. When the dc-bias $V$ is smaller than the cavity frequency $\hbar\omega_n/e$ in reduced units, electrons coherently traversing the QPC do not carry enough energy to emit photons to the cavity and the cavity remains in the vacuum state at zero temperature. Despite this absence of photons, vacuum fluctuations in the cavity still affect and suppress electron tunneling with Frank-Condon factors5. This suppression is alternatively captured by the equilibrium $P(E)$ theory53 of the DCB34,55.

For $V > \hbar\omega_0/e$, the electrons scattered by the QPC emit and absorb photons to/from the cavity. We want to characterize the distribution of photons by studying the mean number of photon, the damping rate of the cavity and the second-order coherence $g^{(2)}(0)$ at vanishing time. For a weak electron-photon (e-p) coupling, the field radiated in an open transmission line exhibits a negative-binomial form for the photon distribution, similar to blackbody radiation, approaching a Poisson distribution with $g^{(2)}(0) = 1$ when the number of bosonic modes is infinite. In our case of a single-mode cavity, we consistently find a thermal distribution with $g^{(2)}(0) = 2$ as with other conductors5,16, independently of the transmission coefficients of the QPC. Proceeding with the next-to-leading order at weak light-matter coupling, we obtain analytically a decrease in $g^{(2)}(0)$ controlled by the bias voltage $V$. This decrease is caused by a two-photon absorption process whereas two-photon emission is energetically forbidden for $V < 2\hbar\omega_0/e$. The balance is reestablished at large bias where $g^{(2)}(0) = 2$ is recovered.

Interestingly, our next-to-leading order calculation can also
be understood as a backaction effect. In the presence of electron transport accompanied by photon emission, the cavity reaches a non-equilibrium stationary state characterized by a Bose-Einstein photon distribution. This cavity state generates an out-of-equilibrium DCB affecting transport which in turn modifies the photon distribution. Here we show that the conventional $P(E)$ theory extended to the stationary state recovers quantitatively the results of the straightforward perturbative calculation, revealing further the physics behind our analytical results.

We also discuss the connections between different theoretical approaches to describe the coupled QPC-cavity system. We first consider a capacitive model initially devised in Ref. 27 and included here in the Keldysh path integral framework. By inspecting the leading electron-photon order, we show its equivalence with a Keldysh path integral method introduced by Kindermann and Nazarov in which electronic variables have already been integrated out. We also apply a rate equation approach, obtained by neglecting off-diagonal elements in the quantum master equation, and show that it coincides with results from the path integral methods in the rotating-wave approximation (RWA). Denoting the damping rate of the cavity, we focus below on the case \( |eV - \hbar \omega_0| \gg \kappa \) where RWA is applicable, leaving aside the regime \( eV \approx \hbar \omega_0 \) where the antibunching inherited from electrons plays a crucial role in the photon distribution.

The article is organized as follows. Section II discusses different architectures coupling the QPC to the TLC. In Sec. III we introduce the capacitive model and derive an effective action for the photons at weak e-p coupling. We compute the non-quadratic corrections to the effective action and provide analytical results.

Here we discuss how the QPC can be coupled to a TLC. Fig. 1(a) depicts the TLC-QPC hybrid system. The TLC is defined by the gate voltage \( V \). A voltage \( V \) applied to the TLC voltage node controls the electronic transport in the QPC. (b) and (c) Circuit representation of a galvanic and capacitive coupling between the TLC and the QPC, respectively.

\[
\lambda_g = \sqrt{\frac{\pi Z_c}{R_K}} \quad (1)
\]

where \( Z_c = \sqrt{L_c/C_c} \) is the cavity impedance and \( R_K = \hbar/e^2 \) is the quantum resistance. A coupling strength of \( \lambda_g^2 \approx 0.3 \) has been reached and even larger values are expected in the near future.

In the capacitive coupling scheme, Fig. 1(c), the QPC is inside the cavity and both left (l) and right (r) electronic reservoirs are coupled to the central and ground lines via capacitances \( C_\alpha \) and \( C_{\alpha g} \), where \( \alpha = l, r \). The reservoirs are also coupled between themselves via capacitance \( C_{qpc} \). In this case, the dimensionless coupling constant between the lead \( \alpha \) and the cavity field is

\[
\lambda_\alpha = \frac{C_\alpha C_{\alpha' s} + C_{qpc} C_s}{C_{ls} C_{rs} + C_{qpc} C_s \sqrt{2\hbar \omega_0 C_c}} \quad (2)
\]

where \( \alpha' = l, r \) with \( \alpha \neq \alpha' \), \( e \) is the electron charge, \( f(x_0) \) is the wavefunction of the cavity field at the coupling position \( x_0 \), \( C_{\alpha s} = C_\alpha + C_{\alpha g} \) and \( C_s = C_{ls} + C_{rs} \). The coupling constant was obtained via circuit theory. A detailed description of the circuit theory applied to a quantum dot coupled to the TLC is given in Ref. [5].

In the next section, we show that the photonic properties are characterized by the difference between the e-p coupling of each lead with the TLC, i.e., the relevant quantity defining...
the coupling is
\[ \lambda_c = \lambda_l - \lambda_r = \frac{C_l C_{rg} - C_r C_q g_e f(x_0)}{C_l s C_{rs} + C_{qpc} C_q \sqrt{2} i \omega \lambda_c^2}. \]

This equation reveals that if \( C_l(g) = C_r(g) \) the QPC is completely decoupled from the TLC. It also shows that the maximum coupling occurs when one reservoir is coupled to the central line and the other one to ground line. Thus, in this geometry, placing the QPC where the cavity field is maximum coupling occurs when one reservoir is coupled to the forward (+) and backward (−) branches of the Keldysh time-ordered contour. Since we are interested in the cavity field properties the specific formula of the QPC action is not necessary and, hence, we keep it as general as possible. The QPC is only characterized by its transmission probability \( T_{\text{eff}} \).

Integrating over the fermionic degrees of freedom obtains an effective action \( \langle S_{\text{eff}} \rangle \) describing the photons. Assuming a small cavity-QPC coupling constant we derive \( S_{\text{eff}} \) using the cumulant expansion\(^56\). To second-order in the e-p coupling we obtain after averaging over electrons
\[ (e^{i S_{\text{eff}}/\hbar}) \simeq e^{i \langle S_{\text{eff}} \rangle + i (\delta S_{\text{eff}}^2)_{\text{e-p}}/2 \hbar/\hbar}, \]
with \( \langle \ldots \rangle_e = \int [D_c, e^\dagger] \langle \ldots \rangle e^{it S_{\text{eff}}/\hbar}, \) and \( \delta S_{\text{e-p}} = S_{\text{e-p}} - \langle S_{\text{e-p}} \rangle_e \). Thus, the photon effective action takes the form
\[ S_{\text{eff}} = S_{\text{cav}} + \langle S_{\text{e-p}} \rangle_e + \frac{i}{2 \hbar} (\delta S_{\text{e-p}}^2)_{\text{e-p}}. \]

The first term describes the uncoupled TLC, the second and third terms are the linear and quadratic contribution resulting from the e-p coupling. The linear term is
\[ \langle S_{\text{e-p}} \rangle_e = -\sqrt{2} \int_\omega [a_q(\omega) + a_q(\omega)] [g_r(\dot{n}_r(\omega)) + g_l(\dot{n}_l(\omega))]. \]

The quadratic action
\[ \langle \delta S_{\text{e-p}}^2 \rangle_e = -\frac{2 \hbar}{i} \int_\omega [a_q^\dagger(\omega) a_q(\omega)] \left( \begin{array}{c} \Sigma^R(\omega) \end{array} \right) \left( \begin{array}{c} a_q^\dagger \\ a_q \end{array} \right)_\omega - S_a \]
introduces the advanced \( \Sigma^A(\omega) \), retarded \( \Sigma^R(\omega) \) and Keldysh \( \Sigma^K(\omega) \) self-energies, while
\[ S_a = \frac{2 \hbar}{i} \int_\omega [a_q(\omega)] [g_r(\dot{n}_r(\omega)) + g_l(\dot{n}_l(\omega))] \left( \begin{array}{c} \Sigma^K(\omega) \end{array} \right) \left( \begin{array}{c} \Sigma^R(\omega) \end{array} \right) \left( \begin{array}{c} a_q^\dagger(\omega) \\ a_q(\omega) \end{array} \right)_\omega + c.c., \]
where we use the notation \( \lambda_{\alpha} = \lambda_{\alpha l} - \lambda_{\alpha r} \), where the cavity-QPC action is
\[ S_{\text{cav}} = \int_\omega \left( \begin{array}{c} a_q^\dagger(\omega) a_q(\omega) \end{array} \right) \left( \begin{array}{c} 0 \\ [G^A]_{\omega}^{-1}(\omega) \end{array} \right) \left( \begin{array}{c} a_q^\dagger(\omega) \\ a_q(\omega) \end{array} \right)_\omega, \]
is the anomalous action. This term is responsible for produc-
ing quadrature squeezing when the quantum conductor is ac-
biased\(^5\). In the time-domain the retarded and Keldysh self-
energies are expressed as

\[
\Sigma^R(t_2 - t_1) = -\frac{i}{\hbar} \Theta(t_2 - t_1) \langle [\delta \hat{\eta}(t_2), \delta \hat{\eta}(t_1)] \rangle \quad (12a)
\]

\[
\Sigma^K(t_2 - t_1) = -\frac{i}{\hbar} \langle [\delta \hat{\eta}(t_1), \delta \hat{\eta}(t_2)] \rangle . \quad (12b)
\]

with the notation \(\delta \hat{A}(t) \equiv \hat{A}(t) - \langle \hat{A} \rangle\), thus in terms of the
density-density correlators

\[
\langle \delta \hat{n}_\alpha(t_1) \delta \hat{n}_\beta(t_2) \rangle = \sum_{\alpha, \beta = l, r} \int_\omega \frac{i}{\hbar} I_\alpha(\omega) \omega e^{-i\omega t}, \quad (14)
\]

where \(I_\alpha\) is the charge current towards the lead \(\alpha = l/r\). For a time-independent bias applied to the QPC, we intro-
duce the noise power spectrum function \(\langle \delta \hat{\eta}_\alpha(\omega_1) \delta \hat{\eta}_\beta(\omega_2) \rangle = 2\pi S_{\alpha\beta}(\omega_1)\delta(\omega_1 + \omega_2)\) in its non-symmetrized form. Using the above relations, the self-energies of Eqs. (12) decompose in frequency space as

\[
\Sigma^R(\omega) = \Delta(\omega) - i\Gamma(\omega)/2, \quad \Sigma^K(\omega) = \Sigma^R(\omega)\quad (15)
\]

with the real part

\[
\Delta(\omega) = -\sum_{\alpha, \beta} g_{\alpha} g_{\beta} \int_{\omega_1} \frac{S_{\alpha\beta}(\omega_1) - S_{\alpha\beta}(-\omega_1)}{\omega^2} \omega \quad (16)
\]

and the imaginary part

\[
\Gamma(\omega) = \sum_{\alpha, \beta} \frac{g_{\alpha} g_{\beta} S_{\alpha\beta}(\omega) - S_{\alpha\beta}(-\omega)}{\omega^2},
\]

while the Keldysh component is

\[
\Sigma^K(\omega) = -i \sum_{\alpha, \beta} \frac{g_{\alpha} g_{\beta} S_{\alpha\beta}(\omega) + S_{\alpha\beta}(-\omega)}{\omega^2}.
\]

The photon effective action, given by Eq. (6) and the first term of Eq. (11), is

\[
S_{\text{eff}} = \int_\omega \left( a_c^* \omega, a_q^* \omega \right) \begin{pmatrix} 0 & G^{-1}_K(\omega) \\ G_K(\omega) & -\Sigma^K(\omega) \end{pmatrix} \begin{pmatrix} a_c \\ a_q \end{pmatrix} \omega (18)
\]

where \(G^{-1}_R(\omega) = h\omega - \omega\bar{\epsilon}_0 - \Sigma^R\) is the inverse of the retarded (advanced) GF. The Keldysh GF is \(G^K(\omega) = G_K(\omega)\Sigma^K(\omega)G_A(\omega)\). So far the photon effective action has been derived assuming only weak electron-photon coupling but arbitrary transmission or electron-electron interactions.

To further characterize the photon effective action, we assume a non-interacting QPC, i.e., in the absence of both electron-electron interaction and e-p coupling, which implies in energy-independent transmission probabilities\(^42\). We compute the nonsymmetrized noise via the scattering formalism\(^42\). Charge conservation which imposes

\[
S_{rr}(\omega) = S_{tt}(\omega) = -S_{rt}(\omega) = -S_{tr}(\omega).
\]

As the self-energies are proportional to the square of the e-p coupling constant the pole of the GFs are weakly modi-
fied by them. Therefore, we approximate the\(\bar{\epsilon}\)-\((\omega) \approx \Sigma^R\) with \(i = R, A, \text{ and } K\). This approximation is equivalent to the rotating-wave approximation (RWA) performed in the time-domain consisting in averaging to zero all the fast oscillating terms.

The anomalous action, \(S_a\), is neglected within the RWA, since its contribution oscillates with frequency \(2\omega_0\). The final expression for the effective action is obtained by shifting the classical field to absorb the linear action \(\langle S_{\text{e-p}} \rangle\), namely

\[
a_c(\omega) \to a_c(\omega) + \sqrt{2} \sum_{\alpha} g_\alpha \langle \hat{n}_\alpha(\omega) \rangle G_R(\omega), \quad (17)
\]

thereby producing a correction to the photons correlators. For the number of photons the correction is proportional to \(\lambda \sum_n T_n/R^K\), which is higher-order in the e-p coupling and can be neglected, see below. The third and fourth orders in the e-p coupling discussed in Sec. IV are also not altered by this shift in the classical field, as discussed in Appendix A.

B. Results from scattering theory

The photon effective action, given by Eq. (6) and the first term of Eq. (11), is

\[
S_{\text{eff}} = \int_\omega \left( a_c^* \omega, a_q^* \omega \right) \begin{pmatrix} 0 & G^{-1}_K(\omega) \\ G_K(\omega) & -\Sigma^K(\omega) \end{pmatrix} \begin{pmatrix} a_c \\ a_q \end{pmatrix} \omega (18)
\]

where \(G^{-1}_R(\omega) = h\omega - \omega\bar{\epsilon}_0 - \Sigma^R\) is the inverse of the retarded (advanced) GF. The Keldysh GF is \(G^K(\omega) = G_K(\omega)\Sigma^K(\omega)G_A(\omega)\). So far the photon effective action has been derived assuming only weak electron-photon coupling but arbitrary transmission or electron-electron interactions.

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\[
S_{rr}(\omega) = S_{tt}(\omega) = -S_{rt}(\omega) = -S_{tr}(\omega).
\]

with

\[
S_{rr}(\omega) = \frac{1}{R^K} \left[ 2h\omega \Theta(h\omega) \sum_n T_n^2 + \sum_n T_n R_n \bar{S}(\omega) \right],
\]

where we introduce the notation

\[
\bar{S}(\omega) = (h\omega + eV)\Theta(h\omega + eV) + (h\omega - eV)\Theta(h\omega - eV),
\]

\(\Theta(\omega)\) is the Heaviside step function and \(R_n = 1 - T_n\).

We are now in a position to compute the the real and imagi-
ary parts of the retarded self-energy. The real part

\[
\Delta(\omega) = -\frac{(g_r - g_l)^2}{h\pi} \sum_n T_n P \int_{\omega_1} \frac{1}{\omega_1(\omega_1 + \omega)} d\omega = 0, \quad (22)
\]
is proportional to the difference between the e-p couplings and the transmission probability. However, the integral is zero for any value of \( \omega \), meaning that the cavity frequency is not shifted by the e-p coupling. This result extends the absence of a cavity frequency pull obtained for a tunnel junction\(^{15,27}\) to leading order in the e-p coupling. Higher orders in the coupling or an energy dependence of the transmission provide the non-linearities needed for a shift in the cavity resonant frequency.

The imaginary part of the self-energy is related to the exchange of photons between the cavity and the QPC. It determines the cavity damping rate \( \kappa \), i.e., the photon losses to the electronic environment. To leading order, \( \kappa \approx \kappa_0 = \Gamma(\omega_0)/\hbar \), with

\[
\kappa_0 = \frac{\lambda^2}{e^2} [S_{rr}(\omega_0) - S_{rr}(-\omega_0)] = \frac{\omega_0 \lambda^2}{2\pi} \sum_n T_n, \tag{23}
\]

where \( \lambda = (g_r - g_l)/h\omega_0 \). The cavity damping rate has a simple dependence on the transmission probability \( T_n \), and it increases with the number of electron channels. Considering a single channel QPC and \( T_1 = 1 \), one can access the dimensionless e-p constant by measuring the cavity peak broadening\(^{36,73,91}\).

Finally, the Keldysh self-energy

\[
\Sigma^K(\omega_0) = -i\frac{\lambda^2 R_K}{2\pi} [S_{rr}(\omega_0) + S_{rr}(-\omega_0)],
\]

also depends on the QPC transmissions. We note that the different self-energies are non-zero only if \( g_r \) differs from \( g_l \) corresponding to an inhomogeneous coupling between the leads and the cavity field.

C. Cavity field properties

We now present results for the cavity photons. The number of photons is related to the Keldysh GF via the formula \( 2\langle n \rangle + 1 = iG^K(t = 0) \). From the Gaussian action (18), we easily find

\[
\langle n \rangle = \frac{S_{rr}(-\omega_0)}{S_{rr}(\omega_0) - S_{rr}(-\omega_0)} = F(eV - h\omega_0) \Theta(eV - h\omega_0), \tag{24}
\]

where \( F = \sum_n T_n (1 - T_n) / \sum_n T_n \) is the QPC Fano factor. The number of photons is determined by the emission noise\(^{90,93}\) over the rate at which the cavity loses photons to the QPC. At zero temperature and \( V \leq h\omega_0/e \) the number of photons is zero. In this case, electrons traversing the QPC must have an energy in a window of size \( eV \) and are therefore not able to emit a photon with energy \( h\omega_0 \) to the cavity. Remarkably, the number of photons decreases as the transmission probabilities increase and even vanish in the limit of perfectly transmitting channels. The physical reason is that photon emission, similarly to quantum noise, is related to charge discreteness in electron transport. At perfect transmission, there is a continuous flow of charges with no noise and no photon emission. In this case however, the damping rate of the cavity \( \kappa_0 \) remains finite as the bath of electrons can still absorb photons from the cavity. For weak transmissions \( T_n \ll 1 \), the result (24) coincides with previous studies for a tunnel junction\(^{15,16}\) and metallic QDs\(^{35}\).

Next we compute the photon second-order coherence

\[
g^{(2)}(\tau) = \langle \hat{a}^\dagger(0)\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}(0) \rangle / \langle n \rangle^2. \tag{25}
\]

At zero time, \( g^{(2)}(0) = \langle n(n - 1) \rangle / \langle n \rangle^2 \) indicates whether the cavity field presents super-Poissonian (\( g^{(2)}(0) > 1 \)) or sub-Poissonian (\( g^{(2)}(0) < 1 \)) photon statistics. Also, its time-evolution is a direct measurement of photon bunching (\( g^{(2)}(\tau) < g^{(2)}(0) \)) or antibunching (\( g^{(2)}(\tau) > g^{(2)}(0) \)). The quadratic action (18) implies a Gaussian field distribution, similar to a thermal state, for which the calculation of correlation functions is straightforward by using Wick’s theorem.

\[
g^{(2)}(\tau) = 1 + |g_t(\tau)|^2 = 1 + e^{-\kappa_0 \tau},
\]

exhibits photon bunching and super-Poissonian statistics \( g^{(2)}(0) = 2 \). These results are in fact a direct consequence of the Gaussian field distribution\(^{15,16}\). They are valid at weak coupling \( \lambda \ll 1 \) where non- quadratic terms in the effective action can be discarded. The case of a dc-bias Josephson junction shows that increasing e-p interaction may lead to strongly antibunched photons\(^{24}\). The next section is devoted to the study of non-quadratic terms in the action to investigate in particular how they modify \( g^{(2)}(0) \) as the coupling \( \lambda \) increases.

IV. BEYOND THE THERMAL DISTRIBUTION

A. Alternative gauge

We showed above that the cavity field follows a thermal distribution at weak electron-photon coupling. Deviations to this statistical description emerge by expanding the action (7) beyond second order. Here, instead of expanding further Eq. (7), we use an alternative gauge for which the action integrated over electronic variables has been derived exactly\(^{90,91}\). This model was recently used to study a quantum tunneling detector coupled to a coherent conductor\(^{90}\) and the light emitted by the tunneling of electrons from a STM tip to a metallic surface\(^{84}\).

The corresponding action is \( S_{cav} + S_{e-p} \) where the unperturbed cavity action \( S_{cav} \) is still given by Eq. (6). The average over electrons produces the non-linear photon action

\[
S_{e-p} = -\frac{i\hbar}{2} \sum_n \text{Tr} \ln \left[ 1 + \frac{T_n}{4} (\{\hat{G}_t, \hat{G}_r\} - 2) \right], \tag{26}
\]

where the trace is over the Keldysh indices (±) and time. \( \hat{G}_\alpha \) is the Keldysh GF of the electrons in the reservoir \( \alpha \), and they are defined in terms of the equilibrium GF

\[
\hat{G}_\text{eq}^\alpha(\epsilon) = \left( \frac{1 - 2f(\epsilon_\alpha)}{2 - 2f(\epsilon_\alpha)} \frac{2f(\epsilon_\alpha)}{2f(\epsilon_\alpha) - 1} \right), \tag{27}
\]
where \( f(\epsilon) \) is the Fermi function. \( \hat{G}_r(t, t') = \hat{G}_q(t - t') \) and \( \hat{G}_l(t, t') = \hat{U}^\dagger(t)\hat{G}_q(t - t')\hat{U}(t') \) with

\[
\hat{U}(t) = \begin{pmatrix} e^{\lambda\varphi_+(t)} & 0 \\ 0 & e^{\lambda\varphi_-(t)} \end{pmatrix}
\]  

(28)

where \( \varphi_\pm(t) = a^\dagger_\pm(t) - a^\pm_\pm(t) \) and \( a^\pm \) is the complex photon field defined in the \( \pm \) Keldysh contour. The matrix \( \hat{U}(t) \) describes the photons. For details on the derivation of \( S_{\text{ep}} \) see Ref. [59].

The link between this description and the model from Sec. III is best seen in the tunneling limit where scattering in the QPC is accounted for by the tunnel Hamiltonian \( \hat{H}_T = \hat{T} + \hat{T}^\dagger \), where \( \hat{T} \) describes the tunneling of one electron from the left to the right reservoir and \( \hat{T}^\dagger \) the reversed process. Applying the gauge transformation via the unitary operator \( \hat{U}_0 = \exp[i(\hat{a}^\dagger - \hat{a})]/\hbar\omega_0 \) cancels the electron-photon coupling term \( (\hat{a}^\dagger + \hat{a}) \eta \) in Eq. (4) while dressing the tunneling part \( \hat{H}_T = \hat{U}_0^\dagger\hat{H}_T\hat{U}_0 = \hat{T}e^{-\lambda(\hat{a}^\dagger - \hat{a})} + \hat{T}^\dagger e^{\lambda(\hat{a}^\dagger - \hat{a})} \) with \( \lambda = (g_r - g_l)/\hbar\omega_0 \). This new form of the tunnel Hamiltonian implies that each tunneling event is accompanied by the coherence excitation of the cavity with the displacement operators \( e^{\pm\lambda(\hat{a}^\dagger - \hat{a})} \). The same prescription was used in deriving \(^{59}\) Eq. (26). For completeness, we also show in Appendix A that the expansion of Eq. (26) to second order in \( \lambda \) agrees with the results of the previous section with the same identification \( \lambda = (g_r - g_l)/\hbar\omega_0 \).

### B. Non-quadratic effects

We assume for simplicity weak transmission probabilities \( T_n \ll 1 \) and expand the action (26) as

\[
x\lambda_{\text{ep}} \approx -\frac{ig_r}{8} \sum_n \text{Tr} \left[ \{ [\hat{G}_l, \hat{G}_r] - 2 \} \right],
\]  

(29)

where we introduce the dimensionless conductance \( g_r = \sum_n T_n \). In this limit, the QPC description is equivalent to a tunnel junction.

Equation (29) is further expanded to fourth order in \( \lambda \). The effective action is

\[
S_{\text{eff}} = S_q + S_{\text{mq}}^{(3)} + S_{\text{mq}}^{(4)}.
\]  

(30)

The second order \( S_q \) (from now on, the subscript \( q \) stands for the quadratic approximation) has been derived in the previous section, see Eq. (18), and is rederived in Appendix A for the present model. Assuming weak transmission implies a Fano factor \( F = 1 \) in Eq. (24) for the average number of photons \( \langle n \rangle_q \), or

\[
\langle n \rangle_q = \frac{\lambda^2 S_{\text{tr}}(-\omega_0)}{e^2\kappa_0} = \frac{(eV - \hbar\omega_0)}{2\hbar\omega_0} \Theta(eV - \hbar\omega_0).
\]  

(31)

In this limit, the non-symmetrized noise power can be expressed as \( S_{\text{tr}}(\omega) = (g_r / R_K) S(\omega) \) where \( S(\omega) \) is given in Eq. (21).

The next terms in Eq. (30) are derived in Appendix A. The third-order expansion in the e-p coupling is

\[
S_{\text{mq}}^{(3)} = C_0 \int dt_1 dt_2 dt_3 dt_4 e^{i\omega_0(t_1 - t_2)} \times f(\epsilon) \delta(\epsilon(2[\varphi_+(t_2) - \varphi_-(t_1)])^2 - \sum_\sigma [\varphi_\sigma(t_2) - \varphi_\sigma(t_1)]^2)
\]  

(32)

where \( C_0 = -\lambda^3 g_r / 24\hbar^2 \). As the non-quadratic terms are small for \( \lambda \ll 1 \), we compute their contributions to the number of photons, retarded self-energy and \( g_r^{(3)}(0) \) perturbatively. \( S_{\text{mq}}^{(3)} \) being odd in the number of bosons, it must be expanded at least to second order to contribute. The corresponding term is of order six in \( \lambda \) and is negligible compared to \( S_{\text{mq}}^{(4)} \). It is discarded in what follows.

The perturbation scheme employed here consists in expanding to first-order \( e^{i\omega_0(t)/h} \approx 1 + i\omega_0(t)/h \). Any photon expectation value is obtained by computing

\[
\langle \ldots \rangle_\q = \frac{i}{\hbar}\langle \ldots S^{(4)}_\q \rangle_{q},
\]  

(34)

where the first term is the quadratic contribution and the second originates from \( S_{\text{mq}}^{(4)} \). The averages \( \langle \ldots \rangle_q \) are taken with respect to the quadratic action \( S_q \) defined in Eq. (18) at weak transmission.

In the Keldysh field theory formalism the number of photons is defined on the \( \pm \) contour by \( \langle n \rangle = \langle a^\dagger_+ a_+ \rangle \). Therefore, the non-quadratic contribution, \( \langle n \rangle_{\text{mq}} = i \langle a^\dagger_+ a_+ S^{(4)}_\q \rangle / h \), is

\[
\langle n \rangle_{\text{mq}} = \frac{i}{\hbar} C \int dt_1 dt_2 dt_3 dt_4 e^{i\omega_0(t_1 - t_2)} \times f(\epsilon) \delta(\epsilon)(2[\varphi_+(t_2) - \varphi_-(t_1)])^4_q
\]  

(35)

\[
\times \sum_\sigma (a^\dagger_+ a_+ [\varphi_\sigma(t_2) - \varphi_\sigma(t_1)])^4_q \rangle_{q}.
\]

This expectation value is evaluated using Wick’s theorem. Details of this calculation are presented in the Appendix B. The number of photons is

\[
\langle n \rangle = \frac{\bar{S}(\omega_0)}{2\hbar\omega_0} - \frac{\lambda^2}{8\hbar^2(\omega_0)^3} \left[ S^2(-\omega_0)\bar{S}(2\omega_0) - \bar{S}(\omega_0)\bar{S}(-2\omega_0) \right].
\]  

(36)

The first term \( \langle n \rangle_q \) given by Eq. (31). The second and smaller term originates from the non-quadratic part of the action and describe departure from the thermal-like state. It can be given a physically more transparent form by using
\[ \langle n \rangle_q = \bar{S}(\omega_0)/2\hbar \omega_0 \] and \[ \langle n \rangle_q + 1 = \bar{S}(\omega_0)/2\hbar \omega_0 \] such that the second term of Eq. (36) reads
\[
\frac{\bar{S}^2(-\omega_0)}{(2\hbar \omega_0)^2} \bar{S}(2\omega_0) - \frac{\bar{S}^2(\omega_0)}{(2\hbar \omega_0)^2} \bar{S}(-2\omega_0)
= \langle n \rangle_q^2 \bar{S}(2\omega_0) - (\langle n \rangle_q + 1)^2 \bar{S}(-2\omega_0)
\] (37)

To interpret this decomposition, we have to recall that \( \bar{S} \) is proportional to the absorption/emission noise for the QPC. The first term in Eq. (37) therefore corresponds to the absorption of a pair of photons from the cavity conditioned by their presence \( \propto \langle n \rangle_q^2 \), while the second term describes photon-pair stimulated emission \( \propto (\langle n \rangle_q + 1)^2 \).

The cavity damping rate is obtained from the retarded self-energy computed with the correction \( S_{\text{mq}}^{(4)} \). The result assumes the form
\[
\kappa = \kappa_0 \left[ 1 - \frac{\lambda^2}{2\hbar \omega_0} (\bar{S}(\omega_0) + \bar{S}(-\omega_0)) + \frac{\lambda^4}{e^2} S_{rr}(0) \right] + \frac{\lambda^4}{2\hbar \omega_0 e^2} \left[ S_{rr}(-\omega_0) \bar{S}(2\omega_0) - S_{rr}(\omega_0) \bar{S}(-2\omega_0) \right],
\] (40)

where the last term recovers the competition between two-photon emission and absorption. The net effect of this competition is to increase \( \kappa \) as two-photon absorption dominates over emission. In addition, there is a Franck-Condon reduction of the leading damping rate \( \kappa_0 \). The second term \( \propto S_{rr}(0) \) describes a process in which a single photon is absorbed and reemitted with no energy cost for electrons. Since absorption is \( \propto \langle n \rangle_q \) and emission \( \propto \langle n \rangle_q + 1 \), the net effect is positive for the damping rate.

The bias voltage dependence of the damping rate is shown in Fig. 2 for \( \lambda^2 = 0.15 \). Interestingly, the different non-quadratic corrections compensate each other for \( eV > 2\hbar \omega_0 \) and we obtain \( \kappa = \kappa_0 \) in this case. In the range \( 0 \leq eV \leq 2\hbar \omega_0/e \), \( \kappa \) is smaller than \( \kappa_0 \) but increases linearly with the dc-bias. For \( eV \leq \hbar \omega_0 \), the cavity is in the vacuum state. The expression of the damping rate simplifies as
\[
\kappa = \frac{\lambda^2}{e^2} [(1 - \lambda^2) S_{rr}(\omega_0) + \lambda^2 S_{rr}(0)]
\] (41)

showing that the absorption of photons with energy \( \hbar \omega_0 \) is reduced by the Franck-Condon factor \( 1 - \lambda^2/2 \) affecting the electronic transmissions. The linear voltage dependence comes here from the zero-energy photon emission/absorption \( \propto S_{rr}(0) \).

We finally compute \( g^{(2)}(0) = \langle a^*_z a^*_x \rangle / \langle n \rangle^2 \). Using Eq. (34) we write its numerator as
\[
\langle a^*_z a^*_x \rangle = 2 \langle n \rangle^2 + \frac{i}{\hbar} \langle a^*_z S_{mq}^{(4)} a_x \rangle_{q,fc}.
\] (42)

The first term comes from pairing each \( a^*_z \) with \( a_x \), \( S_{mq}^{(4)} \) being contracted with one pair only. The second term is averaged with the quadratic part of the action \( S_q \) and only the contraction pairing each field \( a^*_z \) or \( a_x \) to a field from \( S_{mq}^{(4)} \) is kept (fully connected diagram), with the result
\[
\frac{i}{\hbar} \langle a^*_z S_{mq}^{(4)} a_x \rangle_{q,fc} = -\frac{\lambda^2}{32\hbar \omega_0^3} \left[ \bar{S}(\omega_0) + \bar{S}(-\omega_0) \right] \times [\bar{S}^2(-\omega_0) \bar{S}(2\omega_0) - \bar{S}^2(\omega_0) \bar{S}(-2\omega_0)].
\] (43)

Combining this expression with Eq. (36) for the number of photons, we find
\[
g^{(2)}(0) = 2 - \frac{\lambda^2}{8\hbar \omega_0} \left[ \bar{S}(\omega_0) + \bar{S}(-\omega_0) \right] \times [\bar{S}^2(-\omega_0) \bar{S}(2\omega_0) - \bar{S}^2(\omega_0) \bar{S}(-2\omega_0)].
\] (44)
The deviation from the quadratic prediction \( g^{(2)}(0) = 2 \) involves again a balance between two-photon emission and absorption. For a low number of photons, \( g^{(2)}(0) \approx 2P_2/P_1^2 \), where \( P_n \) is the probability to host \( n \) photons in the cavity. Two-photon absorption reduces \( P_2 \) in comparison to \( P_1^2 \). Since it is more efficient than two-photon emission, \( g^{(2)}(0) \) is found to be smaller than 2 for all voltages. This is shown in Fig. 3. For \( eV < \hbar \omega_0 \), no photon are present and \( g^{(2)}(0) \) vanishes. In the range \( \hbar \omega_0 < eV \leq 2\hbar \omega_0 \), two-photon emission is prohibited, Eq. (44) simplifies to

\[
g^{(2)}(0) = 2 - \frac{\lambda^2}{(2\hbar \omega_0)^2} eV \bar{S}(2\omega_0),
\]

and \( g^{(2)}(0) \) decreases with voltage as the cavity population increases and two-photon absorption processes \( \langle n \rangle_q^2 \) are reinforced. For a voltage larger than \( 2\hbar \omega_0/e \), two-photon emission sets in and Eq. (44) takes the form

\[
g^{(2)}(0) = 2 - \frac{\lambda^2}{(2\hbar \omega_0)^2} eV \bar{S}(2\omega_0),
\]

increasing with the voltage. At large voltage \( eV \gg \hbar \omega_0, E_c \), the quadratic result \( g^{(2)}(0) = 2 \) is finally recovered.

**V. RATE EQUATIONS**

The RWA used so far averages to zero terms that do not conserve energy. It removes most off-diagonal elements of the density matrix. In this section, we apply a rate equation approach to the QPC-TLC system, corresponding to a quantum master equation approach in which off-diagonal elements are disregarded, and indeed recover most results from the previous section. In this way the physical picture of two-photon processes is further justified.

\( P_n \) is the probability to have \( n \) photons in the cavity. Its time evolution is fixed by

\[
\dot{P}_n = - \left( \Gamma_{n \rightarrow n+1} + \Gamma_{n \rightarrow n-1} + \Gamma_{n \rightarrow n+2} + \Gamma_{n \rightarrow n-2} \right) P_n \\
+ \Gamma_{n+1 \rightarrow n} P_{n+1} + \Gamma_{n-1 \rightarrow n} P_{n-1} + \Gamma_{n+2 \rightarrow n} P_{n+2} \\
+ \Gamma_{n-2 \rightarrow n} P_{n-2}.
\]

where \( \Gamma_{i \rightarrow j} \) denotes the rate from \(|i\rangle \) to \(|j\rangle \) photons. \( \Gamma_{n \rightarrow n \pm 1} \) corresponds to single-photon and \( \Gamma_{n \rightarrow n \pm 2} \) to two-photon emission/absorption. These rates are calculated in Appendix C via Fermi-Golden rule in the limit of weak transmissions \( T_n \ll 1 \). Fig. 4 illustrates the ladder of transitions processes.

**FIG. 4. Schematic representation of all allowed emission/absorption processes.** The cavity emission and absorption processes are defined by the QPC absorption \([S_{rr}(\omega)]\) and emission \([S_{rr}(-\omega)]\) noise.

The leading (second in \( \lambda \)) order in the e-p coupling involves only single-photon exchange. Setting \( \Gamma_{n \rightarrow n \pm 2} = 0 \), the steady state solution of Eq. (47) is the Bose-Einstein distribution

\[
P_n^0 = \left( 1 - \frac{S_{rr}(-\omega_0)}{S_{rr}(\omega_0)} \right) \left( \frac{S_{rr}(-\omega_0)}{S_{rr}(\omega_0)} \right)^n = \left( \langle n \rangle_q \right)^n, \tag{48}
\]

corresponding to a thermal Gaussian state. The mean number of photons \( \sum_n n P_n^0 \) recovers \( \langle n \rangle_q \) given in Eq. (31).

Two-photon rates are higher orders in \( \lambda \) and can be treated perturbatively with respect to the distribution \( P_n^0 \). Writing \( P_n = P_n^0 + p_n \), we look for the steady-state solution \( \dot{P}_n = 0 \) of Eq. (47). Expanding to lowest non-vanishing order in \( \lambda \) gives

\[
- \left( [(n+1)S_{rr}(-\omega_0) + nS_{rr}(\omega_0)]p_n + (n+1)S_{rr}(\omega_0)p_{n+1} \right) \\
+ nS_{rr}(-\omega_0)p_{n-1} = \frac{\lambda^2}{4} \left( (n+1)(n+2)S_{rr}(2\omega_0)P_n^{0^2} \right. \\
+ \left. [(n+1)(n+2)S_{rr}(2\omega_0) + n(n-1)S_{rr}(2\omega_0)]P_n^0 \right) \\
- \left( n(n-1)S_{rr}(-2\omega_0)P_{n-2}^0 \right).
\]

The correction to the mean number of photon is determined by multiplying this expression by \( n \) and summing over \( n \). We finally obtain that \( \langle n \rangle = \sum_n n(P_n^0 + p_n) \), calculated perturbatively, coincides with Eq. (36) from Sec. IV.

We proceed in two steps in order to compute \( g^{(2)}(0) \). We...
multiply Eq. (49) by $n^2$ and sum over $n$ to obtain
\[
\sum_n n(n - 1) p_n = -\frac{\lambda^2}{32(\hbar \omega_0)^4} [9 \tilde{S}(-\omega_0) + \tilde{S}(\omega_0)] \\
\times [\tilde{S}^2(-\omega_0) \tilde{S}(2\omega_0) - \tilde{S}^2(\omega_0) \tilde{S}(-2\omega_0)]
\]
(50)
corresponding to the two-photon correction to the average $\langle a^\dagger a^2 \rangle$ while the leading order is simply $\sum_n n(n - 1) P_n^0 = 2(\sum_n n P_n^0)^2 = 2(n^2)$. We then include the denominator $(n^2)$ expanded in $\lambda$ and recover exactly Eq. (44) from Sec. IV.

These results not only reinforce our physical interpretation for the formulas derived in Sec. IV but also shows that the cavity properties in the weak coupling regime are well described by a diagonal density matrix.

VI. DYNAMICAL BACKACTION

The QPC-TLC hybrid system (with weak transmissions) can also be discussed within an enlightening approach which emphasizes backaction. The cavity provides a readout of the noise power spectrum of the QPC or tunnel junction. In return, there is backaction from the cavity with a DCB effect which reduces transport. The modified noise properties of electrons are imprinted in the state of the cavity. This effect is captured by the fourth order in $\lambda$ calculation of the previous sections but it can be made more explicit by extending the $P(E)$ theory to a non-equilibrium steady state situation.

We begin by assuming that the cavity is in the thermal state characterized by the photon distribution (48) and the mean number of photons is $\langle n \rangle_q$, see Eq. (31). For weak transmissions $T_n \ll 1$, we use the tunnel Hamiltonian $H_T = \mathcal{T} e^{-\lambda(\hat{a}^\dagger - \hat{a})} + \mathcal{T} e^{\lambda(\hat{a}^\dagger - \hat{a})}$, where the cavity field $\hat{a}$ plays the role of the environment, and proceed with the $P(E)$ approach by computing the current noise correlator. The tunneling limit allows for a factorization of the electron and environment variables such that the noise takes the convoluted form
\[
S_{\text{DCB}}(\omega) = \int_{-\infty}^{\infty} S_{\text{rr}}(\hbar \omega - E) P(E) dE.
\]
(51)
$S_{\text{rr}} (S_{\text{DCB}})$ is the noise in the absence (presence) of the cavity. The $P(E)$ function,
\[
P(E) = \frac{1}{2\pi \hbar} \int dt e^{iEt/\hbar} \langle \hat{X}^\dagger(t) \hat{X}(0) \rangle,
\]
(52)
where $\hat{X}(t) = \exp[-\lambda(\hat{a}^\dagger(t) - \hat{a}(t))]$ characterizes the environment. For $E > 0$, it gives the probability for the QPC/tunnel junction to emit a photon of energy $E$ in the environment during a tunneling event. The $E < 0$ part describes photon absorption.

$P(E)$ can be evaluated exactly in a Gaussian state using Wick’s theorem. We nevertheless expand in $\lambda$ for consistency with our weak coupling scheme, and find
\[
P(E) = (1 - \lambda^2(\langle n \rangle_q + n_1)) \delta(E) + \lambda^2 \langle n \rangle_q \delta(E + \hbar \omega_0) \\
+ \lambda^2 n_1 \delta(E - \hbar \omega_0)
\]
(53)
where $n_1 = \langle n \rangle_q + 1$. It can be check that $P(E)$ is normalized, $\int dE P(E) = 1$. This expression is valid at zero temperature. The presence of non-zero values for $E < 0$ therefore indicates that the cavity is in an out-of-equilibrium state and is able to provide energy to the quantum conductor. For $eV < \hbar \omega_0$, the cavity is empty and $\langle n \rangle_q = 0$, $n_1 = 1$; the elastic peak $\delta(E)$ is renormalized by vacuum fluctuations and inelastic photon emission occurs for $E = \hbar \omega_0$.

We consider the noise properties of the QPC. The absorption noise is
\[
S_{\text{DCB}}(\omega_0) = (1 - \lambda^2(\langle n \rangle_q + n_1)) S_{\text{rr}}(\omega_0) + \lambda^2 n_1 S_{\text{rr}}(0) \\
+ \lambda^2 \langle n \rangle_q S_{\text{rr}}(2\omega_0).
\]
(54)
In addition to the renormalized single-photon absorption, the second term describes the correlated emission and reabsorption of a photon by the QPC, present also when the cavity is in a vacuum state, and the third term, pair absorption, requires an occupied cavity.

A similar analysis for the emission noise
\[
S_{\text{DCB}}(-\omega_0) = (1 - \lambda^2(\langle n \rangle_q + n_1)) S_{\text{rr}}(-\omega_0) \\
+ \lambda^2 \langle n \rangle_q S_{\text{rr}}(0) + \lambda^2 n_1 S_{\text{rr}}(-2\omega_0)
\]
(55)
reveals a renormalized single-photon emission, a correlated photon absorption and reemission, and pair emission. The first and second terms are non-zero only for $eV > \hbar \omega_0$, the third term for $eV > 2\hbar \omega_0$.

To summarize, the cavity back-action provides new absorption and emission mechanisms. Their effect on the noise properties of a tunnel junction have been studied in Ref. 16. We focus here on the effect on the cavity state. To lowest order in the e-p coupling, the damping rate and number of photons of the cavity
\[
\kappa_{\text{DCB}} = \frac{\lambda^2}{e^2} [S_{\text{DCB}}(\omega_0) - S_{\text{DCB}}(-\omega_0)]
\]
(56a)
\[
\langle n \rangle_{\text{DCB}} = \frac{\lambda^2 S_{\text{DCB}}(-\omega_0)}{e^2 \kappa_{\text{DCB}}}.
\]
(56b)
can be computed using Eqs. (54) and (55) and the relations $\langle n \rangle_q = \tilde{S}(\omega_0)/2\hbar \omega_0$ and $n_1 = \tilde{S}(\omega_0)/2\hbar \omega_0$. We obtain
\[
\kappa_{\text{DCB}} = \kappa_0 \left[ 1 - \frac{\lambda^2}{2\hbar \omega_0} (\tilde{S}(\omega_0) + \tilde{S}(-\omega_0)) \right] + \lambda^4 \frac{S_{\text{rr}}(0)}{e^2}
\]
\[
+ \frac{\lambda^4}{2\hbar \omega_0 e^2} [S_{\text{rr}}(-\omega_0) \tilde{S}(2\omega_0) - S_{\text{rr}}(\omega_0) \tilde{S}(-2\omega_0)]
\]
(57)
\[
\langle n \rangle_{\text{DCB}} = \frac{\tilde{S}(\omega_0) - \tilde{S}(-\omega_0)}{2\hbar \omega_0} - \frac{\lambda^2}{8(\hbar \omega_0)^2} \left[ \tilde{S}^2(-\omega_0) \tilde{S}(2\omega_0) - \tilde{S}^2(\omega_0) \tilde{S}(-2\omega_0) \right].
\]
(58)
These expressions coincide exactly with those obtained in Sec. IV B, see Eqs. (40) and (36).

VII. SUMMARY AND CONCLUSION

We investigated the properties of a single-mode cavity field coupled to a quantum point contact or tunnel junction. We
first used a Keldysh path integral framework using two formulations related by a unitary gauge transformation: a coupling between the quantum voltage operator of the cavity and the lead densities, and an inductive coupling where each scattering event is accompanied by the excitation of the cavity. Expanding for weak electron-photon coupling to a quadratic form, we found a Gaussian thermal field distribution with \( g^{(2)}(0) = 2 \), and the photon self-energies are fully characterized by the emission and absorption noise of the quantum point contact. The damping rate is constant in this limit, independent of the bias voltage.

Proceeding with the next order in the electron-photon coupling, we identified two-photon processes: pair emission or absorption as well as correlated photon emission and absorption. We recovered these effects using a rate equation approach and a \( P(E) \) calculation adapted to our non-equilibrium situation. We obtained a reduced number of photons relatively to the thermal state prediction, a suppressed \( g^{(2)}(0) < 2 \) for a not too large bias voltage, and a reduced cavity damping rate for \( eV < 2\hbar\omega_0 \), which are explained by noting the preeminence of two-photon absorption over emission.

It is tempting to speculate about the extension of these findings to higher orders in the electron-photon coupling \( \lambda \). Increasing powers of \( \lambda \) will introduce emission and absorption processes involving three, four and higher number of photons, in which photon absorption is always energetically more favourable than emission. This should modify the photon distribution by adding more weight to small numbers of photons compared to larger numbers as the numerical results of Ref. 5 seem also to indicate. Interestingly, the balance between emission and absorption depends on the bias voltage such that high voltages are expected to bring the system back to a Gaussian thermal distribution.

We emphasize a strong difference with dc-biased Josephson junctions where the physics of strong electron-photon coupling is essentially described by generalized Frank-Condon factors and only processes conserving energy occur as there is no electronic dissipation. In tunnel junctions, the non-conserved part of the energy can be dissipated in the leads which allows for more processes. The question is still open whether it is possible to realize a non-classical state with \( g^{(2)}(0) \) < 1 apart from the region \( eV \approx \hbar\omega_0 \). As a prospect of future investigation, we mention the extension of our approach by including fourth-order current correlations. They appear to the order considered in this work when the QPC transmission probabilities are no longer small.

### VIII. ACKNOWLEDGMENTS

We thank C. Altimiras, A. Clerk, P. Joyez, F. Portier and P. Simon for fruitful discussions. U.C.M. acknowledges the support from CNPq-Brazil (Project No. 229659/2013-6).

### Appendix A: Alternative gauge formulation

In this appendix we detail the Keldysh path integral framework presented in Sec. IV.

#### 1. Quadratic effective action

We discuss the equivalence between the Keldysh path integral formulations presented in Sec. III and IV. To show the correspondence we redefine the quadratic action Eq. (18) starting form the action

\[
\mathcal{S}_{e-p} = -\frac{i\hbar}{2} \sum_n \text{Tr} \ln \left[ 1 + \frac{T_n}{4} \{ \tilde{G}_t, \tilde{G}_r \} - 2 \right].
\]  

(A1)

Using \( \tilde{G}_t(t, t') = \tilde{U}(t) \tilde{G}_{eq}(t - t') \tilde{U}(t') \) and \( \tilde{G}_r(t, t') = \tilde{G}_{eq}(t - t') \), the anticommutator can be written as

\[
\{ \tilde{G}_t(t, t'), \tilde{G}_r(t', t) \} = 2 + 2 \tilde{M}(t, t'),
\]  

(A2)

where

\[
\tilde{M}(t, t') = \int \frac{d\epsilon d\epsilon'}{(2\pi \hbar)^2} e^{-i\omega_{lr}(t-t')} \left( m_1(x) m_2(x) \right) m_3(x) m_4(x),
\]  

(A3)

\[\omega_{lr} = (\epsilon_l - \epsilon_r)/\hbar, \quad x = \{ t, t', \epsilon_l, \epsilon_r \} \] and

\[
m_1(x) = 2f(\epsilon_l)f(\epsilon_r)[\epsilon^{M(\varphi_-(t') - \varphi_-(t))} - \epsilon^{M(\varphi_-(t) - \varphi_+(t))}]
\]

\[+ 2f(\epsilon_r)f(\epsilon_l)[\epsilon^{M(\varphi_+(t') - \varphi_-(t))} - \epsilon^{M(\varphi_+(t) - \varphi_+(t))}]
\]  

(A4a)

\[
m_2(x) = f(\epsilon_r)[1 - 2f(\epsilon_l)] \sum_{\sigma = \pm} \sigma e^{M(\varphi_{\sigma}(t') - \varphi_{\sigma}(t))}
\]  

(A4b)

\[
m_3(x) = \tilde{f}(\epsilon_r)[1 - 2f(\epsilon_l)] \sum_{\sigma = \pm} \sigma e^{M(\varphi_{\sigma}(t') - \varphi_{\sigma}(t))}
\]  

(A4c)

\[
m_4(x) = 2f(\epsilon_l)f(\epsilon_r)[\epsilon^{M(\varphi_+(t') - \varphi_+(t))} - \epsilon^{M(\varphi_-(t) - \varphi_-(t))}]
\]

\[+ 2f(\epsilon_r)f(\epsilon_l)[\epsilon^{M(\varphi_+(t') - \varphi_-(t))} - \epsilon^{M(\varphi_-(t) - \varphi_+(t))}]
\]  

(A4d)

where \( f(\epsilon) \) is the Fermi function, \( \tilde{f}(\epsilon) = 1 - f(\epsilon) \) and \( \varphi_{\pm} = a_{\pm}^\dagger - a_{\pm} \) is the photon complex field defined in the \( \pm \) Keldysh time-ordered contour. We use the following expansion strategy: (i) one notices that the matrix \( \tilde{M} \) vanishes for \( \lambda = 0 \) and is linear in \( \lambda \) at small coupling, (ii) therefore expanding in \( T_n \tilde{M} \) is consistent with an expansion in \( \lambda \), (iii) the second order in \( T_n \tilde{M} \),

\[
\mathcal{S}_{e-p} = \frac{i\hbar}{4} \sum_n T_n \text{Tr}[\tilde{M}] + \frac{i\hbar}{8} \sum_n T_n^2 \text{Tr}[\tilde{M}^2],
\]  

(A5)

thus exhausts all possible contributions up to second order in \( \lambda \). Orders in \( T_n \) higher than two must all vanish if we restrict the action to second order in \( \lambda \). Expanding directly the action Eq. (A1) up to \( T_n^2 \) gives

\[
\mathcal{S}_{e-p} = -\frac{i\hbar}{8} \sum_n T_n \left( 1 + \frac{T_n}{2} \right) \text{Tr}[\{ \tilde{G}_t, \tilde{G}_r \}]
\]  

(A6)

\[+ \frac{i\hbar}{32} \sum_n T_n^2 \left( \text{Tr}[\tilde{G}_t \tilde{G}_r \tilde{G}_t \tilde{G}_r] + \text{Tr}[\tilde{G}_t \tilde{G}_r \tilde{G}_t \tilde{G}_t] \right).
\]
where \( \tau_0 \) is the identity matrix, it can be shown that the third term of Eq. (A6) is equal to the identity. Thus, we define the quadratic e-p action as \( S_{e-p} = S_1 + S_2 \), where

\[
S_1 = \frac{-i\hbar}{8} \sum_n T_n \left(1 + \frac{T_n}{2}\right) \text{Tr}[\{\hat{G}_t, \hat{G}_r\}] \quad (A8a)
\]

\[
S_2 = \frac{i\hbar}{32} \sum_n T_n^2 \text{Tr}[\hat{G}_t \hat{G}_r \hat{G}_t \hat{G}_r]. \quad (A8b)
\]

Before proceeding with the derivation of \( S_{e-p} \) we rotate the equilibrium GF and \( \tilde{U} \) matrices. The rotating matrix is

\[
\hat{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \hat{L}^\dagger.
\]

The rotated equilibrium matrix, defined as \( \hat{G}_t^{\alpha} = \hat{L}^\dagger \hat{G}_t^{\alpha} \hat{L}, \) is

\[
\hat{G}_t^{\alpha}(\epsilon_\alpha) = \begin{pmatrix} 0 & 2[1 - 2f(\epsilon_\alpha)] \\ 1 & -1 \end{pmatrix}.
\]

and the rotated \( \tilde{U}_i = \hat{L}^\dagger \tilde{U} \hat{L} \) is conveniently written as

\[
\tilde{U}_i(t) = A(t) \tau_0 + B(t) \tau_z,
\]

where \( \tau_z \) is the first Pauli matrix, and

\[
A(t) = \frac{1}{2} \left[ e^{i \lambda \varphi_+(t)} + e^{-i \lambda \varphi_-(t)} \right]
\]

\[
\approx 1 + \frac{\lambda}{\sqrt{2}} \varphi_c(t) + \frac{\lambda^2}{4} \left[ \varphi_c^2(t) + \varphi_\ell^2(t) \right] \quad (A11a)
\]

\[
B(t) = \frac{1}{2} \left[ e^{i \lambda \varphi_+(t)} - e^{-i \lambda \varphi_-(t)} \right]
\]

\[
\approx \frac{\lambda}{\sqrt{2}} \varphi_\ell(t) \left[1 + \frac{\lambda}{\sqrt{2}} \varphi_c(t) \right], \quad (A11b)
\]

the above equations were obtained by expanding the exponentials to second-order in \( \lambda \) and performing the Keldysh rotation on the photonic complex fields \( \varphi \pm = (\varphi_c \pm \varphi_\ell) / \sqrt{2} \), with \( i \varphi_c(q) = a^*_c(q) - a_c(q) \).

Using the above relations and Fourier transforming the equilibrium GF \( \hat{G}_t^{\alpha}(t) = \frac{1}{2\pi \hbar} \int d\epsilon_\alpha \hat{G}_t^{\alpha}(\epsilon_\alpha) e^{-i\omega_\alpha t/\hbar} \), we rewrite \( S_1 \) as

\[
S_1 = g_1 \int dt_1 dt_2 dt_3 dt_4 \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \left[ A^\dagger(t_1) A(t_2) A^\dagger(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)]
\]

\[
+ 2 A_1(t_1) A(t_2) \left( A^\dagger(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B^\dagger(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right)
\]

\[
+ 2 A^\dagger(t_1) B(t_2) \left( A(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right)
\]

\[
+ A^\dagger(t_1) B(t_2) A^\dagger(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B^\dagger(t_1) A(t_2) B(t_3) A(t_4)
\]

\[
\times \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right] e^{-i(\omega_1 t_1 + \omega_2 t_2) t_3} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_4} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_5} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_6}, \quad (A12)
\]

where \( g_1 = -i \sum_n T_n / 16 \pi^2 \hbar \). To write the above equation we have shifted the energies \( \epsilon_\alpha \rightarrow \epsilon_\alpha + eV_\alpha \), \( V_\alpha \) is the voltage applied to the lead \( \alpha \), and defined \( V = V_1 - V_2 \). The trace is only over the matrix structure. Analogously, the second term is written as

\[
S_2 = g_2 \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 dt_1 dt_2 dt_3 dt_4 \left[ A^\dagger(t_1) A(t_2) A^\dagger(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)]
\]

\[
+ 2 A^\dagger(t_1) A(t_2) \left( A(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right)
\]

\[
+ 2 A^\dagger(t_1) B(t_2) \left( A(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B(t_3) A(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right)
\]

\[
+ A^\dagger(t_1) B(t_2) A^\dagger(t_3) B(t_4) \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] + B^\dagger(t_1) A(t_2) B(t_3) A(t_4)
\]

\[
\times \text{Tr}[\hat{G}_t^{\alpha}(\epsilon_1) \hat{G}_t^{\alpha}(\epsilon_2) \hat{G}_t^{\alpha}(\epsilon_3) \hat{G}_t^{\alpha}(\epsilon_4)] \right] e^{-i(\omega_1 t_1 + \omega_2 t_2) t_3} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_4} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_5} e^{-i(\omega_1 t_1 + \omega_2 t_2) t_6}, \quad (A13)
\]

Eq. (A12) as

\[
S_1 = 2g_1 \int dt_1 dt_2 dt_3 dt_4 e^{-i(\omega_1 t_1 + \omega_2 t_2) t_3} \left( A^\dagger(t_1) A(t_2)
\]

\[
+ B^\dagger(t_1) B(t_2) - 4 [f(\epsilon_1) \bar{f}(\epsilon_1) + f(\epsilon_1) \bar{f}(\epsilon_1)] B^\dagger(t_1) B(t_2)
\]

\[
+ 2 [f(\epsilon_1) \bar{f}(\epsilon_1) - f(\epsilon_1) \bar{f}(\epsilon_1)][A^\dagger(t_1) B(t_2) - B^\dagger(t_1) A(t_2)] \right) \quad (A14)
\]

Using Eqs. (A11) one can show that the integrals over the two first terms do not depend on the cavity field variables and...
can be disregarded. The remaining terms are written as
\[ S_1 = \frac{4g_1 \lambda}{\sqrt{2}} \int dt_1 dt_2 \langle \varphi(q) \rangle \langle \varphi(q) \rangle e^{-i(\omega_1 + eV/t)} (\varphi(q) (t_1) + \varphi(q) (t_2)) [f(\epsilon_2) \bar{f}(\epsilon_1) - f(\epsilon_1) \bar{f}(\epsilon_2)] + \frac{g_2}{\lambda^2} \int dt_1 dt_2 \langle \varphi(q) \rangle e^{-i(\omega_1 + eV/t)} (\varphi(q) (t_1) + \varphi(q) (t_2)) [f(\epsilon_2) \bar{f}(\epsilon_1) - f(\epsilon_1) \bar{f}(\epsilon_2)] + 2 \langle \varphi(q) \rangle \langle \varphi(q) \rangle [f(\epsilon_2) \bar{f}(\epsilon_1) + f(\epsilon_1) \bar{f}(\epsilon_2)] \}
\]

Next we Fourier transform \( \varphi(q) (t) = \int \omega \varphi(q) (\omega) e^{-i\omega t}, \) obtaining
\[ S_1 = -\frac{i\lambda^2}{4\pi} g_0 \int \omega \langle \varphi(q) \rangle - \omega \left( \frac{0}{\omega} \right) (\varphi(q) \rangle \omega + \sqrt{2} \lambda \epsilon V g_0 \int \omega \langle \varphi(q) \rangle \delta(\omega),\]
\]
where we defined \( g_0 = \sum_n T_n (1 + \alpha T_n) / 2, \) \( \tilde{\varphi}(\omega) = (\varphi(q) \rangle + \varphi(q) \langle \omega) \) is defined in Eq. (21).

Following the same steps we derive the action
\[ S_2 = \frac{i\lambda^2}{8\pi} \sum_n T_n \int \omega \langle \varphi(q) \rangle - \omega \left( \frac{0}{\omega} \right) (\varphi(q) \rangle \omega - \frac{\hbar \omega}{\omega} \tilde{Y}(\omega) \rangle (\varphi(q) \rangle \omega) - \frac{\sqrt{2} \lambda \epsilon V}{2} \sum_n T_n \int \omega \langle \varphi(q) \rangle \delta(\omega),\]
\]
where
\[ \tilde{Y}(\omega) = 2 \left[ \tilde{\varphi}(\omega) + \tilde{S}(\omega) + \hbar \omega - 4 \hbar \omega T \right].\]

Adding Eqs. (A16) and (A17) and changing variables \( \varphi(q) = a^+ (q) \) we obtain the same quadratic e-p action (11) as in Sec. III. The quadratic action (18) is finally exactly recovered (with the same self-energies) by applying the same RWA as in Sec. III.

A difference occurs nevertheless in the linear term in the action. In this case, it assumes the form
\[ S_L = i2\sqrt{2} \pi \hbar \lambda (I) \int \omega [a^+ (\omega) - a(q) (\omega)] \delta(\omega),\]
\]
where \( (I) = eV \sum_n T_n / h \) is the QPC current, in contrast with the linear term (10) obtained in Sec. III for the other gauge. This difference is however expected since a change of gauge, via a unitary transformation, also modifies observables. With the operator \( U_0, \) it merely shifts field operators. The linear term (A18) can be removed by shifting the classical field
\[ a_c = a_c - 2\sqrt{2} \pi \hbar \lambda (I) \delta(\omega) G_R(\omega).\]

2. Non-quadratic action

Here we show how to obtain the non-quadratic actions \( S_{1q}^{(3)} \) and \( S_{1q}^{(4)} \) used in Sec. IV B. It is convenient to consider the first term of the action defined in Eq. (A5). Considering \( \lambda \ll 1 \) we first expand the exponentials in the matrix elements [Eqs. (A4)] of the matrix \( \tilde{M} \) to fourth-order, then we take the trace over the matrix structure and, finally, we shift the energies \( \epsilon, \) by the voltage applied to the lead \( \alpha, \) i.e., \( \epsilon_{(r)} \rightarrow \epsilon_{(r)} \pm V/2. \) Thus, obtaining
\[ S_{\text{eff}} = -\frac{i g_c}{8\pi^2 \hbar} \sum_{p=1}^{4} \lambda \int dt_1 dt_2 \langle \varphi(q) \rangle \delta(\omega),\]
\]
where \( g_c = \sum_n T_n. \) The terms \( p = 1 \) and \( 2 \) will produce the linear [Eq. (A18)] and quadratic [Eq. (11)] actions. \( S_{1q}^{(3)} \) [Eq. (32)] and \( S_{1q}^{(4)} \) [Eq. (33)] are given by terms \( p = 3 \) and \( 4, \) respectively. As they are defined by the difference between the complex fields, \( \varphi \), the shift of classical fields, Eq. (A19), does not alter the form of \( S_{1q}^{(3)} \) and \( S_{1q}^{(4)} \).

Appendix B: Correction to the number of photons

We present here the main steps used in the derivation of the non-quadratic contribution to mean number of photons \( \langle n \rangle_{\text{q}} = \langle a^+ a \rangle_{\text{q}} + S_{1q}^{(4)}. \) For simplicity we set \( \hbar = 1 \) and \( \epsilon = 1. \) Our starting point is Eq. (35), we rewrite it as
\[ \langle n \rangle_{\text{q}} = C_1 \sum_{\beta = \pm} \int dt_1 dt_2 \langle \varphi(q) \rangle \delta(\omega),\]
\]
where \( C_1 = -\lambda^4 g_c / 192 \pi^2, g_c = \sum_n T_n, a_{\beta} = \omega_{1\beta} + \beta V, \) and \( \omega_{1\beta} = \epsilon_1 - \epsilon_\beta. \) Using Wick’s theorem we write \( \langle n \rangle_{\text{q}} = 12 C_1 (n_1 - 2 n_2), \) where
\[ n_1 = \sum_{\beta = \pm} \int dt_1 dt_2 \langle a^+ [\varphi_\beta (t_2) - \varphi_\beta (t_1)] [\varphi_\beta (t_2) - \varphi_\beta (t_1)] \rangle \delta(\omega) + 2 \langle [a^+ a_\beta] \rangle \delta(\omega),\]
\]
and
\[ n_2 = \sum_{\beta} \int dt_1 dt_2 \langle a^+ [\varphi_\beta (t_2) - \varphi_\beta (t_1)] \rangle \delta(\omega),\]
\]

A detailed derivation of \( n_1 \) is presented. Before computing the diagram we define the correlators in the frequency-domain. Using the definition \( \varphi_\beta (t) = a^+ (t) - a_\beta (t) \) we have
\[
\langle [\varphi(t_2) - \varphi(t_1)]^2 \rangle = -2i \int_0^\omega D_{\sigma\varphi}(\omega)[2 - e^{i\omega(t_1-t_2)} - e^{-i\omega(t_1-t_2)}] \tag{B4a}
\]

\[
\langle a^*_+ [\varphi(t_2) - \varphi(t_1)] \rangle = -i \int_0^\omega D_{\sigma\varphi}(\omega)[e^{-i\omega t_2} - e^{-i\omega t_1}] \tag{B4b}
\]

\[
\langle a_+ [\varphi(t_2) - \varphi(t_1)] \rangle = i \int_0^\omega D_{\sigma\varphi}(\omega)[e^{i\omega t_2} - e^{i\omega t_1}] \tag{B4c}
\]

where we used
\[
\langle a_+(\omega)\varphi(\omega) \rangle = 2\pi i D_{\sigma\varphi}(\omega)\delta(\omega_1 + \omega_2), \tag{B5}
\]

with \(D_{\sigma\varphi}(\omega) = -i\langle a_+(\omega)a^*_\sigma(\omega) \rangle\). Using Eqs. (B4) we rewrite Eq. (B2) as
\[
n_1 = -\frac{2i}{(2\pi)^3} \sum_\beta \int dt_1 dt_2 dc dt dc dt_3 dc dw dw_1 dw_2 D_{\sigma\varphi}(\omega) D_{\sigma}(\omega_1) \times D_{\sigma}(\omega_2) f(\epsilon_r)f(\epsilon_t)e^{-i\sigma\beta}(t_1-t_2)[e^{-i\omega t_1} - e^{-i\omega t_1}] \times [2 - e^{i\omega_1(t_1-t_2)} - e^{-i\omega_1(t_1-t_2)}][e^{i\omega_2 t_2} - e^{-i\omega_1 t_1}]. \tag{B6}
\]

The integrals are performed in the following sequence. First change variables \(t_1 = \tau + t_2\), then integrate over \(t_2\) and \(\omega_2\), respectively. Next, we integrate over \(\tau\) obtaining \(\delta\)-functions that depend on \(\omega\) and \(\omega_1\). At this step we replace \(\omega\) and \(\omega_1\) in the \(\delta\)-functions by \(\omega_0\). The smallness of \(\kappa_0\) in comparison with \(\omega_0\) validates this approximation. Finally, we integrate over the energies and use the definition
\[
\sum_\beta \int dt dz f(\epsilon_r)f(\epsilon_t)\delta(\alpha \beta - \omega) = \bar{S}(\omega) \tag{B7}
\]

Equation (B6) is now written as
\[
n_1 = -\frac{i}{\pi}[2\bar{S}(0) - 4\bar{Y}(\omega_0) + \bar{Y}(2\omega_0)] \times \sum_\sigma \int d\omega_1 D_{\sigma\varphi}(\omega) D_{\sigma}(\omega_1) D_{\sigma}(\omega_1) \tag{B8}
\]

where we defined \(\bar{Y}(\omega_0) = \bar{S}(\omega_0) + \bar{S}(\omega_0)\). The final step is to perform the integrals over \(\omega\) and \(\omega_1\). With the help of the relation \(a_\pm = (a_\pm \pm a_\pm) / \sqrt{2}\) we rewrite the GFs \(D_{\sigma\varphi}(\omega)\) in terms of the Keldysh, retarded and advanced GFs defined in Sec. III B considering \(T_n \ll 1\). The result of these integrals is
\[
\int d\omega D_{\sigma\varphi}(\omega) = -\frac{i\pi}{2\omega_0} \bar{Y}(\omega_0) \tag{B9a}
\]

\[
\sum_\sigma \int d\omega_1 D_{\sigma}(\omega_1) D_{\sigma}(\omega_1) = -\frac{\pi^2 \bar{S}(\omega_0)}{2\lambda^2 g_\omega^2} \bar{Y}(\omega_0) \tag{B9b}
\]

Replacing Eqs. (B9) into Eq. (B8) we obtain
\[
n_1 = \frac{\pi^2}{2\lambda^2 g_\omega^2} \bar{Y}(\omega_0)[6\bar{S}(0) - 4\bar{Y}(\omega_0) + \bar{Y}(2\omega_0)] \times \bar{S}(\omega_0) \tag{B10}
\]

Following the same steps presented above we compute \(n_2\), resulting in
\[
n_2 = \frac{\pi^2}{2\lambda^2 g_\omega^2} \{2\bar{S}(\omega_0)\bar{Y}(\omega_0)[\bar{Y}(\omega_0)\bar{S}(0) \tag{B11}
\]

\[-2\bar{S}(\omega_0)\bar{S}(\omega_0)] - 2\bar{S}(\omega_0)\bar{S}(\omega_0) - \bar{S}(\omega_0)\bar{S}(\omega_0)\bar{S}(0) - [\bar{S}^2(\omega_0) + 2\bar{S}(\omega_0)] \times [\bar{Y}(\omega_0)\bar{S}(\omega_0) - \bar{S}(\omega_0)\bar{S}(0) - \bar{S}(\omega_0)\bar{S}(\omega_0)] \}. \]

Appendix C: Transition rate

In this section we derive the coefficients \(\Gamma_{i \rightarrow j}\) of the rate equation model presented in Sec. V. They are obtained via Fermi golden rule
\[
\Gamma_{i \rightarrow j} = \frac{2\pi}{\hbar} \langle f \mid \hat{H}_T \mid i \rangle^2 \delta(E_i - E_f). \tag{C1}
\]

This equation describes the transition from the state \(|i\rangle\) to \(|f\rangle\). \(E_i(f)\) is the energy of the state \(|i\rangle\) \((|f\rangle\), and the interacting Hamiltonian is
\[
\hat{H}_T(\lambda) = \hat{T}e^{-\lambda(a^\dagger - \bar{a})} + \bar{T}e^{\lambda(a^\dagger - \bar{a})}. \tag{C2}
\]

As we are considering QPC with tunneling probabilities \(T_n \ll 1\), the tunneling operator is well describe by \(\hat{T} = \gamma \sum_k c^\dagger_{\sigma,k} c_{\sigma,k} \), where \(\gamma\) is the tunneling amplitude and \(c^\dagger_{\sigma,k}\) is the electron creation operator in the reservoir \(\sigma\). Considering \(\lambda \ll 1\), we expand the exponential to second-order and rewrite Eq. (C2) as
\[
\hat{H}_T = \left(1 - \frac{\lambda^2}{2}(2\bar{n} + 1)\right)\hat{H}_T(0) + \lambda(a^\dagger - a)\hat{I} + \frac{\lambda^2}{2}(a^\dagger a + a a^\dagger)\hat{H}_T(0), \tag{C3}
\]

with \(\bar{n}\) the photon number operator, and \(\hat{I} = \gamma \sum_{k,q} c^\dagger_{\sigma,q} c_{\sigma,q,k}\). The first term does not produce any transition, and hence, it is neglected. The second and third terms give rise to transitions from the state \(|n\rangle\) to \(|n \pm 1\rangle\) and \(|n\pm 2\rangle\) states, respectively. As expected, the expansion in the e-p coupling shows that the transitions to the state \(|n\pm 1\rangle\) and \(|n\pm 2\rangle\) will be proportional to \(\lambda^2\) and \(\lambda^4\), respectively.

Considering \(|i\rangle = |n\rangle \otimes |E_g\rangle\) and \(|f\rangle = |m\rangle \otimes |E_e\rangle\), in which \(|n\rangle\) is the Fock state of \(n\)-photons with energy \(E_n = n\hbar \omega_1\), \(|E_g\rangle\) and \(|E_e\rangle\) are the electron states with respective energies \(E_g\) and \(E_e\).
The coefficient $\Gamma_{n \rightarrow n \pm 1}$ is obtained replacing $\hat{H}_T$ in Eq. (C1) by the second term of Eq. (C3). It is equal to

$$\Gamma_{n \rightarrow n \pm 1} = \frac{2\pi \lambda^2}{h} |\langle n \pm 1 | (a_1 - a) | n \rangle|^2 |\langle E_c | \hat{f} | E_g \rangle|^2 \times \delta(E_g - E_c - \pm \hbar \omega_0)$$

$$= \frac{2\pi \lambda^2}{h} \left( n + \frac{1}{2} \pm \frac{1}{2} \right) |\langle E_c | \hat{f} | E_g \rangle|^2 \delta(E_g - E_c - \pm \hbar \omega_0).$$

(C4)

We need to evaluate the matrix element $\langle E_c | \hat{f} | E_g \rangle$. For that let us consider the matrix element $\langle E_c | \gamma_{\epsilon,\epsilon'} c_{\gamma}^\dagger k | E_g \rangle$. It is different of zero only if

$$|E_\gamma\rangle = |0, \ldots, 0_{\gamma,q_1}, \ldots, 1_{l,k}, \ldots\rangle,$$

(C5a)

$$|E_\epsilon\rangle = |0, \ldots, 1_{\epsilon,q_1}, \ldots, 0_{l,k}, \ldots\rangle,$$

(C5b)

i.e., the initial state $|E_\gamma\rangle$ the state $k$ in the left lead must be occupied, while the state $q$ in the right lead must be empty. This implies that $\langle E_\gamma | \gamma_{\epsilon,\epsilon'} c_{\gamma}^\dagger k | E_g \rangle = f(\epsilon_k) f(\epsilon_q)$ with $E_g = \epsilon_k$, $E_\gamma = \epsilon_q$, $f(\epsilon_k)$ the Fermi function and $f(\epsilon_q) = 1 - f(\epsilon_q)$. A similar expression is found for second term of the $\hat{f}$. Therefore, we write

$$|\langle E_c | \hat{f} | E_g \rangle|^2 \delta(E_g - E_c - \pm \hbar \omega_0) = \rho_0^2 \gamma^2 \int \rho_0 d\epsilon_k d\epsilon_q [f(\epsilon_k) f(\epsilon_q)$$

$$\times \delta(\epsilon_k - \epsilon_q - \pm \hbar \omega_0 + V) + f(\epsilon_q) f(\epsilon_k) \delta(\epsilon_k - \epsilon_q - \pm \hbar \omega_0 - V)]$$

$$= \rho_0^2 \gamma^2 \bar{S}(\mp 2\omega_0).$$

(C6)

where $\rho_0$ is the electronic density of the state, which we assume to be the same for both leads. We considered that the chemical potential in the left (right) reservoir is $\mu_l = V/2$ ($\mu_r = -V/2$). Thus, to obtain Eq. (C6) we shift $\epsilon_k(q) \rightarrow \epsilon_k(q) \pm V/2$. The integrals over the energies were preformed with the help of Eq. (B7). Therefore, the transition rate from the Fock state $|n\rangle$ to $|n \pm 1\rangle$, Eq. (C4), is

$$\Gamma_{n \rightarrow n \pm 1} = \frac{2\pi \lambda^2 \rho_0^2 \gamma^2}{h} \left( n + \frac{1}{2} \pm \frac{1}{2} \right) \bar{S}(\mp 2\omega_0).$$

(C7)

As expected the transition rate from $|n\rangle$ to $|n + 1\rangle$ $|n - 1\rangle$ is proportional to the QPC emission (absorption) noise power of one-photon.

Analogously, we obtained the transition coefficients $\Gamma_{n \rightarrow n \pm 2}$, which are

$$\Gamma_{n \rightarrow n \pm 2} = \frac{\pi \lambda^4 \rho_0^2 \gamma^2}{2h} \left( n + \frac{1}{2} \pm \frac{1}{2} \right) \left( n + \frac{1}{2} \pm \frac{3}{2} \right) \bar{S}(\mp 2\omega_0).$$

(C8)

The transition rate from $|n\rangle$ to $|n + 2\rangle$ $|n - 2\rangle$ is characterized the QPC emission (absorption) of two-photon. These transitions coefficients are next-to-leading order correction to the cavity absorption and emissions processes.

Finally, the rate equation, Eq. (47), is fully characterized by the transitions rates defined in Eqs. (C7) and (C8).

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