ON GRAND SOBOLEV SPACES AND POINTWISE DESCRIPTION OF BANACH FUNCTION SPACES

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Abstract. We study weighted grand Sobolev spaces, defined on arbitrary open sets $\Omega$ (of finite or infinite measure) in $\mathbb{R}^n$. We give a new characterization of grand Sobolev spaces in terms of pointwise inequality. The same description is valid as well for Banach function spaces provided that the Hardy–Littlewood maximal operator is bounded.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open connected set having finite or infinite Lebesgue measure, which we will denote as $|\Omega|$. The Lebesgue space $L^q(\Omega), 1 \leq q < \infty$, is defined to be the space of all measurable functions $f$ on $\Omega$ such that

$$\|f\|_{L^q(\Omega)} := \left( \int_{\Omega} |f(x)|^q \, dx \right)^{1/q} < \infty,$$

and the Sobolev space $W^{1,q}(\Omega)$ is defined to be the space of all $f \in L^q(\Omega)$ such that the weak gradient $\nabla f$ belongs to $L^q(\Omega)$ with a finite norm

$$\|f\|_{W^{1,q}(\Omega)} := \|f\|_{L^q(\Omega)} + \|
abla f\|_{L^q(\Omega)}.$$

If $w$ is a weight, i.e. a measurable, positive, and finite almost everywhere (a.e.) function, the weighted Lebesgue space $L^q(\Omega, w)$ and weighted Sobolev space $W^{1,q}(\Omega, w)$ are defined similarly with norms

$$\|f\|_{L^q(\Omega, w)} := \left( \int_{\Omega} |f(x)|^q w(x) \, dx \right)^{1/q} < \infty$$

and

$$\|f\|_{W^{1,q}(\Omega, w)} := \|f\|_{L^q(\Omega, w)} + \|\nabla f\|_{L^q(\Omega, w)} < \infty.$$
Sobolev functions are known to have a pointwise characterization, which is a classic nowadays.

**Theorem A1.** Let $1 < q < \infty$. A function $f$ belongs to $W^{1,q}(\mathbb{R}^n)$ if and only if there exists a non-negative $g \in L^q(\mathbb{R}^n)$ such that the inequality

$$|f(x) - f(y)| \leq |x - y|(g(x) + g(y))$$

(1.1)

holds for all $x, y$ outside of some set $\Sigma \subset \mathbb{R}^n$ of measure zero.

The necessity part of Theorem A1 has been firstly obtained in [3], while the sufficiency has been proved independently in [25] and [49]. Afterwards, this description turned out to be useful in various aspects of analysis such as Sobolev spaces with higher-order derivatives [4], Hardy–Sobolev spaces [38], Sobolev spaces on Carnot groups [49], the theory of Mappings with Bounded Distortion on Carnot Groups [50], and many others. Moreover, since the formula (1.1) does not involve the notion of derivative, it turned out to be a starting point to define a counterpart of Sobolev spaces on metric structures and to investigate its properties [25], for comprehensive coverage of the theory of Sobolev spaces on the metric measure spaces the reader is referred to [28] and references therein.

Another natural direction of developing the topic is to consider Sobolev spaces based on other functional classes than those of Lebesgue functions. Thus, in [37] a pointwise description in terms of the Young function was obtained for Orlicz–Sobolev mappings.

In this paper we ask if the characterization (1.1) can be obtained for grand Sobolev spaces, spaces that are slightly larger than Sobolev ones. A grand Lebesgue space $L^q(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, was first introduced in [30], to study the question of the integrability of the Jacobian of an orientation-preserving mapping belonging to the Sobolev space $W^{1,n}(\Omega)$. Grand Lebesgue spaces have been thoroughly studied in the 1D-case when $\Omega = I = (0,1)$, some recent results can be found in [1, 9, 13, 15, 32, 33] (see Section 2 for definitions and properties). The grand version of Sobolev spaces $W^{1,q}(\Omega)$ was defined and studied in [44] (see Section 3). For further discussion of grand Lebesgue and Sobolev spaces we refer the reader to [5, 7, 11, 14, 16, 18, 21, 23, 24, 31, 36, 37, 39, 45] and [8, §7.2]. Note, that the situation when $|\Omega| = \infty$ is not easy, neither for $L^q(\Omega)$ nor for $W^{1,q}(\Omega)$. The space $L^q(\Omega)$ over the unbounded domain $\Omega$ was introduced and developed in [33, 45].

In the present paper, we define the weighted grand Sobolev space $W^{1,q}_a(\Omega, w)$, when $\Omega$ may be unbounded. The main result of the paper is a pointwise estimate in the spirit of (1.1) stated in Theorem 4.5 in Section 4. Moreover, a generalization in the case of Banach function spaces, provided that the Hardy–Littlewood maximal operator is bounded, is given in Theorem 5.3, and a number of new pointwise characterizations for various spaces is also available, see Corollaries 5.4–5.8.
2. Grand Lebesgue spaces

Let us recall a definition of grand Lebesgue spaces, given in [30].

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for $n \geq 2$. For $1 < q < \infty$, the grand Lebesgue space $L^q(\Omega)$ consists of all measurable functions $g$, $g \in \bigcap_{1 < p < q} L^p(\Omega)$, such that

$$\|g\|_{L^q(\Omega)} := \sup_{0 < \varepsilon < q - 1} \left( \varepsilon \int_{\Omega} |g(x)|^{q-\varepsilon} \, dx \right)^{\frac{1}{q-\varepsilon}} < \infty. \quad (2.1)$$

**Remark 2.2.** In some cases it is more convenient to consider a multiplier $\frac{1}{|\Omega|}$ in front of the integral in (2.1).

The spaces $L^q(\Omega)$ are rearrangement invariant Banach function spaces and the following continuous embeddings hold:

$$L^q(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{q-\varepsilon}(\Omega) \quad \text{for} \quad 0 < \varepsilon < q - 1.$$

For any given $1 < q < \infty$, the inclusion $L^q(\Omega) \subset L^q(\Omega)$ is strict and, moreover, the spaces $L^q(\Omega)$ are not reflexive [13].

To define and deal with grand Lebesgue space $L^q(\Omega)$ in case of unbounded domains $\Omega \subset \mathbb{R}^n$, i.e. $|\Omega| \leq \infty$, one needs to introduce a class of weights.

**Definition 2.3.** A weight $w$ is said to be in the Muckenhoupt class $A_q(\mathbb{R}^n)$, $1 < q < \infty$, if

$$[w]_{A_q(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{q-1} \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{q-1}} \, dx \right) < \infty,$$

where $Q \subset \mathbb{R}^n$ is a cube with edges parallel to the coordinate axes and $\frac{1}{|Q|} \int_Q f(x) \, dx$ denotes integral average of $f$ over $Q$, i.e.

$$f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx := \frac{1}{|Q|} \int_Q f(x) \, dx. \quad (2.2)$$

The class $A_q(\mathbb{R}^n)$, $1 < q < \infty$, possesses the following properties which are consequences of the Hölder inequality and the reverse Hölder inequality, see for instance [9 Theorem IV].

**Proposition A2.** Let $1 < q < \infty$.

(i) If $q < p$, then $A_q(\mathbb{R}^n) \subseteq A_p(\mathbb{R}^n)$ and $[w]_{A_q(\mathbb{R}^n)} \leq [w]_{A_p(\mathbb{R}^n)}$;

(ii) $[w]_{A_q(\mathbb{R}^n)} \geq 1$;

(iii) If $w \in A_q(\mathbb{R}^n)$ and $0 \leq \alpha \leq 1$, then $w^\alpha \in A_q(\mathbb{R}^n)$ and $[w^\alpha]_{A_q(\mathbb{R}^n)} \leq [w]^\alpha_{A_q(\mathbb{R}^n)}$;

(iv) If $w \in A_q(\mathbb{R}^n)$, then there exists $0 < \sigma < q - 1$ such that $w \in A_{q-\sigma}(\mathbb{R}^n)$;

(v) If $w \in A_q(\mathbb{R}^n)$, then there exists $\alpha > 1$ such that $w^\alpha \in A_q(\mathbb{R}^n)$. 

Now we are ready to define a generalized version of grand Lebesgue spaces, following [43, 48].

Definition 2.4. Let \( 1 < q < \infty \) and \( w, a \) be weights such that \( wa^\varepsilon \in L^1_{\text{loc}}(\Omega) \), for all \( \varepsilon \in (0, q-1) \). The generalized grand Lebesgue space \( L^q_a(\Omega, w) \) consists of all measurable functions \( g \) defined on \( \Omega \) such that

\[
\|g\|_{L^q_a(\Omega, w)} := \sup_{0 < \varepsilon < q-1} \left( \varepsilon \int_\Omega |g(x)|^{q-\varepsilon} w(x)a^\varepsilon(x) \, dx \right)^{\frac{1}{q-\varepsilon}} < \infty.
\]

Remark 2.5. The function \( a \) in the Definition 2.4 of grand Lebesgue space \( L^q_a(\Omega, w) \) is called a grandisator in [43, 48]. It is introduced for the close control of the behavior of \( g \in L^q_a(\Omega, w) \) at infinity.

To work with weights in the context of grand spaces the following lemma (see [48, Lemma 5]) appears to be useful.

Lemma A3. If \( w \in A_q(\mathbb{R}^n) \) and \( a^\delta \in A_q(\mathbb{R}^n) \) for some \( \delta > 0 \), then there exists \( 0 < \varepsilon < \delta \) such that \( wa^\varepsilon \in A_{q-\varepsilon}(\mathbb{R}^n) \).

Note that if \( \delta \geq q - 1 \), then it may happen that \( q - \varepsilon < 1 \). In this case one should consider \( \alpha < \frac{q-1}{\delta} < 1 \) and \( \delta_0 = \alpha \delta < q - 1 \). Then by Proposition A2(iii) we can apply Lemma A3 to \( a^{\delta_0} \in A_q(\mathbb{R}^n) \).

It has been proved in [48, Lemma 3] that the following chain of embeddings holds

\[
L^q(\Omega, w) \hookrightarrow L^q_a(\Omega, w) \hookrightarrow L^{q-\varepsilon}(\Omega, wa^\varepsilon), \quad 0 < \varepsilon < q - 1, \quad (2.3)
\]

if and only if \( a \in L^q(\Omega, w) \).

Some of the properties possessed by the space \( L^q_a(\Omega, w) \) are proved below:

Proposition 2.6. Let \( wa^\varepsilon \in L^1_{\text{loc}}(\Omega) \) for all \( \varepsilon \in (0, q-1) \). Then

(i) if \( a \in L^q(\Omega, w) \), then the generalized grand Lebesgue space \( L^q_a(\Omega, w) \) is a Banach space;
(ii) if \( |f| \leq |g| \) a.e. on \( \Omega \) then \( \|f\|_{L^q_a(\Omega, w)} \leq \|g\|_{L^q_a(\Omega, w)} \);
(iii) if \( 0 \leq f \not\rightarrow f \) a.e. in \( \Omega \) then \( \|f_n\|_{L^q_a(\Omega, w)} \not\rightarrow \|f\|_{L^q_a(\Omega, w)} \);
(iv) if \( E \subset \Omega \), \( |E| < \infty \) and \( a \in L^q(\Omega, w) \), then \( \chi_E \in L^q_a(\Omega, w) \);
(v) if \( w \in A_q(\Omega) \) and \( a^\delta \in A_q(\Omega) \) for some \( \delta > 0 \), then \( L^q_a(\Omega, w) \subset L^1_{\text{loc}}(\Omega) \).

Proof. Properties (i)–(iii) can be easily checked and we leave them as an exercise for the reader.
Consider $E \subset \Omega$, $|E| < \infty$, for $0 < \varepsilon < q - 1$, using Hölder’s inequality with the exponents $\frac{q}{q-\varepsilon}$ and $\frac{q}{\varepsilon}$, we get

$$\|\chi_E\|_{L^q_0(\Omega,w)} = \sup_{0 < \varepsilon < q - 1} \varepsilon^{\frac{1}{q-\varepsilon}} \left( \int_{\Omega} |\chi_E(x)|^q w(x) \frac{1}{a^\varepsilon(x)} dx \right)^{\frac{1}{q}}$$

$$\leq \sup_{0 < \varepsilon < q - 1} \varepsilon^{\frac{1}{q-\varepsilon}} \left( \int_{\Omega} |\chi_E(x)|^q w(x) dx \right)^{\frac{1}{q}} \left( \int_{\Omega} a^\varepsilon(x) w(x) dx \right)^{\frac{1}{q}} \|a\|_{L^q(\Omega,w)}^{-\frac{1}{q}}$$

$$= \|\chi_E\|_{L^q(\Omega,w)} \sup_{0 < \varepsilon < q - 1} \varepsilon^{\frac{1}{q-\varepsilon}} \left( \int_{\Omega} a^\varepsilon(x) w(x) dx \right)^{\frac{1}{q}} \|a\|_{L^q(\Omega,w)}^{-\frac{1}{q}}.$$ (2.4)

Then the point (iv) follows from (2.3) and (2.4).

To prove (v) let us consider a compact $K \subset \Omega$ and $f \in L^q_0(\Omega, w)$. By Lemma [A3] we know that there exists $0 < \varepsilon_0 < \delta$ such that $wa^{\varepsilon_0} \in A_{q-\varepsilon_0}(\Omega)$. Then by Hölder inequality, we obtain

$$\int_{K} |f(x)| dx = \int_{K} |f(x)| (wa^{\varepsilon_0})^{\frac{1}{q-\varepsilon_0}} (wa^{\varepsilon_0})^{-\frac{1}{q-\varepsilon_0}} dx$$

$$\leq C_K |K| \varepsilon_0^{\frac{1}{q-\varepsilon_0}} \varepsilon_0^{\frac{1}{q-\varepsilon_0}} \|f\|_{L^{q-\varepsilon_0}(\Omega,wa^{\varepsilon_0})} [wa^{\varepsilon_0}]_{A_{q-\varepsilon_0}(\Omega)} \left( \int_{K} wa^{\varepsilon_0} \right)^{-\frac{1}{q-\varepsilon_0}}$$

$$\leq C_K |K| \|f\|_{L^q_0(\Omega,w)} \left( \varepsilon_0 \int_{K} wa^{\varepsilon_0} \right)^{-\frac{1}{q-\varepsilon_0}} [wa^{\varepsilon_0}]_{A_{q-\varepsilon_0}(\Omega)} < \infty.$$ (2.5)

**Remark 2.7.** In view of Proposition 2.6 the space $L^q_0(\Omega, w)$ is “very close” to become a Banach function space (refer to Definition 5.1 for the notion of Banach function space).

3. Grand Sobolev spaces

We begin with the following definition.

**Definition 3.1.** Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain. The grand Sobolev space, denoted by $W^{1,q}(\Omega)$, consists of all $f \in \bigcap_{0 < \varepsilon < q - 1} W^{1,q-\varepsilon}(\Omega)$ such that

$$\|f\|_{W^{1,q}(\Omega)} := \sup_{0 < \varepsilon < q - 1} \varepsilon^{\frac{1}{q-\varepsilon}} \|f\|_{W^{1,q-\varepsilon}(\Omega)} < \infty.$$ (3.1)

In [11] the grand Sobolev space was defined by the means of the norm

$$\|f\|_{W^{1,q}(\Omega)} := \|f\|_{L^q(\Omega)} + \|
abla f\|_{L^q(\Omega)} < \infty.$$ (3.2)
However, the norms (3.1) and (3.2) are equivalent.

**Proposition 3.2.** The following estimates hold

\[
\frac{1}{4} \| f \|_{W^{1,q}}(\Omega) \leq \| f \|_{W^{1,q}}(\Omega) \leq \| f \|_{W^{1,q}}(\Omega).
\]

**Proof.** We use Hölder’s inequality for sums and obtain

\[
\| f \|_{W^{1,q}}(\Omega) = \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \| f \|_{L^{q-\varepsilon}}(\Omega) + \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \| \nabla f \|_{L^{q-\varepsilon}}(\Omega)
\]

\[
\leq 2 \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} (\| f \|_{L^{q-\varepsilon}}(\Omega) + \| \nabla f \|_{L^{q-\varepsilon}}(\Omega))
\]

\[
\leq 2 \sup_{0<\varepsilon<q-1} 2^{\frac{q-1}{q-\varepsilon}} \varepsilon^{\frac{1}{q-\varepsilon}} (\| f \|_{L^{q-\varepsilon}}(\Omega) + \| \nabla f \|_{L^{q-\varepsilon}}(\Omega))^{\frac{1}{q-\varepsilon}}
\]

\[
\leq 2^\alpha \| f \|_{W^{1,q}}(\Omega).
\]

Also, we have

\[
\| f \|_{W^{1,q}}(\Omega) = \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} (\| f \|_{L^{q-\varepsilon}}(\Omega) + \| \nabla f \|_{L^{q-\varepsilon}}(\Omega))
\]

\[
\leq \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} (\| f \|_{L^{q-\varepsilon}}(\Omega) + \| \nabla f \|_{L^{q-\varepsilon}}(\Omega))
\]

\[
\leq \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \| f \|_{L^{q-\varepsilon}}(\Omega) + \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \| \nabla f \|_{L^{q-\varepsilon}}(\Omega)
\]

\[
= \| f \|_{W^{1,q}}(\Omega)
\]

and the assertion follows. \(\square\)

Now, we define the generalized grand Sobolev space on \(\Omega \subset \mathbb{R}^n, |\Omega| \leq \infty\), as follows.

**Definition 3.3.** Let \(1 < q < \infty\) and \(w, a\) be weight functions on \(\Omega\) such that \(wa^\varepsilon \in L^1_{\text{loc}}(\Omega)\) for all \(\varepsilon \in (0, q-1)\). The generalized grand Sobolev space \(W^{1,q}_{a}(\Omega, w)\) is defined as the collection of all \(f \in L^1_a(\Omega, w)\) having a weak gradient \(\nabla f\) in \(L^1_a(\Omega, w)\), equipped with the norm

\[
\| f \|_{W^{1,q}_a(\Omega, w)} := \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \| f \|_{W^{1,q-\varepsilon}(\Omega, wa^\varepsilon)} < \infty.
\]

**Remark 3.4.** By the definition of the space \(W^{1,q}_a(\Omega, w)\) and Proposition 3.2, it can be shown that

\[
\frac{1}{4} (\| f \|_{L^1_a(\Omega, w)} + \| \nabla f \|_{L^1_a(\Omega, w)}) \leq \| f \|_{W^{1,q}_a(\Omega, w)} \leq \| f \|_{L^1_a(\Omega, w)} + \| \nabla f \|_{L^1_a(\Omega, w)}.
\]

**Remark 3.5.** If the weight \(w \in A_q(\Omega)\) then the weighted Sobolev space \(W^{1,q}(\Omega, w)\) is a Banach space and functions \(f \in W^{1,q}(\Omega, w)\) belong also \(W^{1,1}_{\text{loc}}(\Omega)\), see [34, Proposition 2.1]. Similar properties are valid for grand Sobolev spaces \(W^{1,q}_a(\Omega, w)\).
**Proposition 3.6.** Let $1 < q < \infty$, $wa^\varepsilon \in L^1_{\text{loc}}(\Omega)$ for all $\varepsilon \in (0, q - 1)$, and $a \in L^q(\Omega, w)$. Let also $w \in A_q(\Omega)$ and $a^\delta \in A_q(\Omega)$ for some $\delta > 0$. Then $W^{1,q}_a(\Omega, w) \subset W^{1,1}_{\text{loc}}(\Omega)$, and a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset W^{1,q}_a(\Omega, w)$ is also Cauchy in $W^{1,1}_{\text{loc}}(\Omega)$.

*Proof.* Let $K \subset \Omega$ be a compact. From the proof of Proposition 2.6(v) we obtain

$$
\int_K |f(x)| + |\nabla f(x)| \, dx \leq 4C_K |K| \left( \varepsilon_0 \int_K wa^{\varepsilon_0} \right)^{\frac{1}{q-\delta}} \left[ wa^{\varepsilon_0} \right]_{A_q-\varepsilon_0(\Omega)} \|f\|_{W^{1,q}_a(\Omega,w)}.
$$

Hence, $f \in W^{1,1}_{\text{loc}}(\Omega)$. Moreover, for any Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset W^{1,q}_a(\Omega, w)$ we have $\|f_n - f_m\|_{W^{1,1}_{\text{loc}}(\Omega)} \leq C\|f_n - f_m\|_{W^{1,q}_a(\Omega,w)}$, i.e. $\{f_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in $W^{1,1}_{\text{loc}}(\Omega)$. 

**Proposition 3.7.** Let $1 < q < \infty$, $wa^\varepsilon \in L^1_{\text{loc}}(\Omega)$ for all $\varepsilon \in (0, q - 1)$, and $a \in L^q(\Omega, w)$. Then the grand Sobolev space $W^{1,q}_a(\Omega, w)$ is a Banach space if $w \in A_q(\Omega)$ and $a^\delta \in A_q(\Omega)$ for some $\delta > 0$.

*Proof.* It is enough to prove completeness. Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $W^{1,q}_a(\Omega, w)$, by Proposition 3.6 it is so in $W^{1,1}_{\text{loc}}(\Omega)$, denote the limit by $f$. On the other hand, $\{f_n\}_{n \in \mathbb{N}}$ and $\{\partial_\alpha f_n\}_{n \in \mathbb{N}}$ are Cauchy in Banach spaces $L^q_{\alpha}(\Omega, w)$ for all multi-indexes $\alpha$, $|\alpha| = 1$. Let $h$ and $h_\alpha$ be corresponding limits. Then it is easy to see that $h = f$ and $h_\alpha = \partial_\alpha f$ and $\|f_n - f\|_{W^{1,q}_a(\Omega,w)} \to 0$, i.e. $W^{1,q}_a(\Omega, w)$ is complete. 

**Theorem 3.8.** The embedding

$$
W^{1,q}_a(\Omega, w) \hookrightarrow W^{1,q}_a(\Omega, w)
$$

holds if $a \in L^q(\Omega, w)$.

*Proof.* Let $a \in L^q(\Omega, w)$ and $f \in W^{1,q}_a(\Omega, w)$. Similarly to (2.4) we can obtain

$$
\|f\|_{L^q(\Omega,w)} \leq \|f\|_{L^q(\Omega,w)} \|a\|_{L^q_a(\Omega,w)} \|a\|_{L^q(\Omega,w)}^{-1} \tag{3.4}
$$

and

$$
\|\nabla f\|_{L^q(\Omega,w)} \leq \|\nabla f\|_{L^q(\Omega,w)} \|a\|_{L^q_a(\Omega,w)} \|a\|_{L^q(\Omega,w)}^{-1} \tag{3.5}
$$

Since (3.3) holds, using (3.4) and (3.5), we obtain

$$
\|f\|_{W^{1,q}_a(\Omega,w)} \leq \|a\|_{L^q(\Omega,w)} \|a\|_{L^q_a(\Omega,w)}^{-1} \left( \|f\|_{L^q(\Omega,w)} + \|\nabla f\|_{L^q(\Omega,w)} \right)
\leq 4\|a\|_{L^q(\Omega,w)} \|a\|_{L^q_a(\Omega,w)}^{-1} \|f\|_{W^{1,q}(\Omega,w)}
= K_a \|f\|_{W^{1,q}_a(\Omega,w)},
$$

where $K_a = 4\|a\|_{L^q(\Omega,w)} \|a\|_{L^q_a(\Omega,w)}^{-1}$. Now since $a \in L^q(\Omega, w)$ and in view of (2.3), the embedding $L^q(\Omega, w) \hookrightarrow L^q_a(\Omega, w)$ holds, it follows that $K_a < \infty$. 


In view of Theorem 3.8 and embeddings (2.3), if \( a \in L^q(\Omega, w) \) we have that the following embeddings hold for all \( 0 < \varepsilon < q - 1 \)

\[
W^{1,q}(\Omega, w) \hookrightarrow W^{1,q-\varepsilon}(\Omega, wa^\varepsilon).
\]

4. Maximal operator and the pointwise estimate for generalized grand Sobolev functions

**Definition 4.1.** Let \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( 0 < t \leq \infty \). We define the Hardy–Littlewood maximal operator or simply the maximal operator by

\[
M_t g(x) := \sup_{0 < r \leq t} \int_{B(x, r)} |g(y)| \, dy,
\]

where \( B(x, r) \) is a ball with radius \( r > 0 \) and center \( x \in \mathbb{R}^n \). In the case \( t = \infty \), we write \( Mg(x) \) instead of \( M_{\infty} g(x) \).

In the celebrated paper [40] it was proved that the maximal operator \( M : L^q(\mathbb{R}^n, w) \to L^q(\mathbb{R}^n, w) \) is bounded for \( 1 < q < \infty \) if and only if \( w \in A_q(\mathbb{R}^n) \). Later, the class \( A_q(\mathbb{R}^n) \) turned out to be very useful since it also characterizes the \( L^q \)-boundedness of several other operators such as the Hilbert transform [29]. In fact, the same class characterizes also the boundedness of the maximal operator on grand Lebesgue spaces. In particular, the following theorem was proved in [45].

**Theorem A4.** Let \( 1 < q < \infty \) and \( I = (0, 1) \). The maximal operator

\[
Mg(x) := \sup_{I \ni x} \int_I |g(y)| \, dy,
\]

where the supremum is taken over all nondegenerate intervals \( J \) contained in \( I \), is bounded between grand Lebesgue spaces \( L^q(I, w) \) if and only if

\[
\sup \left( \int_J w(x) \, dx \right)^{q-1} < \infty,
\]

where supremum is taken over all intervals \( J \subset I \).

The following theorem, proved in [45], gives the sufficient condition for the maximal operator \( M \) to be bounded in the space \( L^q_a(\mathbb{R}^n, w) \).

**Theorem A5.** Let \( 1 < q < \infty \) and \( \delta > 0 \) be such that \( a^\delta \in A_q(\mathbb{R}^n) \) for some \( a \in L^q(\mathbb{R}^n, w) \). If \( w \in A_q(\mathbb{R}^n) \), then the maximal operator is bounded on \( L^q_a(\mathbb{R}^n, w) \), i.e.

\[
\|Mg\|_{L^q_a(\mathbb{R}^n, w)} \leq K_q \|g\|_{L^q_a(\mathbb{R}^n, w)}
\]

(4.1)

for all \( g \in L^q_a(\mathbb{R}^n, w) \), where \( K_q \) is a positive constant.

**Remark 4.2.** It is unknown if the condition \( w \in A_q(\mathbb{R}^n) \) is also necessary for the boundedness of \( M \) on \( L^q_a(\mathbb{R}^n, w) \).
As a consequence of Theorem A5 we obtain the boundedness of the maximal operator on grand Lebesgue spaces in the case of a bounded domain \( \Omega \subset \mathbb{R}^n \).

**Proposition 4.3.** Let \( 1 < q < \infty, \Omega \subset \mathbb{R}^n \), be a bounded domain and \( t \in (0, \infty) \) be a fixed number. Then the inequality
\[
\| M_t g \|_{L^q(\Omega)} \leq K_q \| g \|_{L^q(\Omega)} \tag{4.2}
\]
holds for all \( g \in L^q(\Omega) \) with some constant \( K_q \) independent of \( g \).

**Proof.** We take a weight \( a \equiv 1 \) in \( \mathbb{R}^n \) and a weight \( w: \mathbb{R}^n \to (0, \infty) \) such that \( w \in A_q(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) and \( w \equiv 1 \) in some cube \( Q \supset \Omega \) big enough such that \( B(x, t) \subset Q \) for any \( x \in \Omega \). Then we have
\[
wa^\varepsilon = w \in L^1_{\text{loc}}(\Omega) \quad \text{for all } \varepsilon \in (0, q - 1).
\]
We extend \( g \in L^q(\Omega) \) by 0 outside \( \Omega \). By (4.1) we get
\[
\| M_t g \|_{L^q(\Omega)} \leq \| M_t g \|_{L^q_1(\mathbb{R}^n, w)} \leq K_q \| g \|_{L^q_0(\mathbb{R}^n, w)} = K_q \| g \|_{L^q(\Omega)}
\]
and (4.2) is proved. \( \square \)

The following estimate was obtained in [22, Lemma 7.16]. For convenience of the reader we provide here the statement and a new proof of it.

**Proposition 4.4.** If \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \) then, for every ball \( B := B(\xi, r) \subset \mathbb{R}^n \) centered at \( \xi \in \mathbb{R}^n \) of radius \( r \), the inequality
\[
| f(x) - f_B | \leq C \int_B \frac{| \nabla f(y) |}{| x - y |^{n-1}} dy \tag{4.3}
\]
holds for every \( x \in B \setminus S_B \), where \( S_B \) is measurable subset of \( B \) of zero Lebesgue measure: \( | S_B | = 0 \), and \( f_B \) is the integral average defined by (2.2).

Before proving Proposition 4.4 we recall the following lemma.

**Lemma A6 ([12, 4.5.2, Lemma 1]).** There exists a constant \( C = C(n) \) such that, for any ball \( B(\xi, r) \subset \mathbb{R}^n \) and function \( f \in C^1(B(\xi, r)) \), the following inequality
\[
 \int_{B(\xi, r)} | f(y) - f(z) | dz \leq Cr^n \int_{B(\xi, r)} \frac{\nabla f(z)}{| y - z |^{n-1}} dz
\]
holds for any point \( y \in B(\xi, r) \).

**Proof of Proposition 4.4.** By Lemma A6, for a ball \( B := B(\xi, r) \subset \mathbb{R}^n, f \in C^1(B), x \in B \) and \( \rho \) small enough (\( B(x, \rho) \subset B \)), we come to the following
inequalities

\[
\int_{B(x, \rho)} |f(y) - f_B| \, dy = \int_{B(x, \rho)} \left| \int_{B} (f(y) - f(z)) \, dz \right| \, dy \\
\leq \int_{B(x, \rho)} \int_{B} |f(y) - f(z)| \, dz \, dy \\
\leq C r^n \int_{B(x, \rho)} \int_{B} \frac{|
abla f(z)|}{|y - z|^{n-1}} \, dz \, dy.
\]

Now the Lebesgue Differentiation Theorem, with \( \rho \to 0 \), gives just (4.3) for \( C^1 \)-functions.

In view of the estimate

\[
\int_{B} |f(x) - f_B| \, dx \leq C \int_{B} \frac{|
abla f(y)|}{|x - y|^{n-1}} \, dy \, dx \leq C_1 r \int_{B} |
abla f(y)| \, dy
\]

we can apply standard approximation arguments and to justify inequality (4.3) for any \( f \in W^{1, q}_{\text{loc}}(\mathbb{R}^n) \).

Theorem 4.5. Let \( 1 < q < \infty \) and \( \delta > 0 \) be such that \( a^\delta \in A_q(\mathbb{R}^n) \) for some \( a \in L^q(\mathbb{R}^n, w) \), and \( w \in A_q(\mathbb{R}^n) \). Then \( f \in W^{1, q}_a(\mathbb{R}^n, w) \) if and only if \( f \in L^q_a(\mathbb{R}^n, w) \) and there exists a non negative function \( g \in L^q_a(\mathbb{R}^n, w) \) such that the inequality

\[
|f(x) - f(y)| \leq |x - y|(g(x) + g(y)) \tag{4.4}
\]

holds for \( x, y \in \mathbb{R}^n \) a.e.

Proof. Proposition 3.6 ensures that (4.3) holds for \( f \in W^{1, q}_a(\mathbb{R}^n, w) \). Now, for \( x, y \in \mathbb{R}^n \), consider a ball \( B \) of the least radius with center \( \xi = \frac{1}{2}(x + y) \) such that \( x, y \in B \) and \( |x - \xi| = |y - \xi| \). Applying (4.3), we obtain

\[
|f(x) - f(y)| \leq |f(x) - f_B| + |f(y) - f_B| \\
\leq C \left( \int_{2B} \frac{|
abla f(z)|}{|x - z|^{n-1}} \, dz + \int_{2B} \frac{|
abla f(z)|}{|y - z|^{n-1}} \, dz \right) \\
:= C(I_1 + I_2)
\]

for all \( x, y \in B \setminus S_B \).

Let \( f \in W^{1, q}_a(\mathbb{R}^n, w) \). Then by Proposition 4.4, the estimate

\[
|f(x) - f_B| \leq C \int_{2B} \frac{|
abla f(y)|}{|x - y|^{n-1}} \, dy
\]

holds for every \( x \in B \setminus S_B \) where \( |S_B| = 0 \).
Applying the Kirszbraun extension theorem [12, 3.1, Theorem 1], we obtain

$$I_1 \leq \int_{B_1} \frac{\|f(z)\|}{|x-z|^{n-1}} \, dz$$

$$= \sum_{j=1}^{\infty} \int_{\frac{1}{2j}B_1 \setminus \frac{1}{2j}B_1} \frac{\|f(z)\|}{|x-z|^{n-1}} \, dz$$

$$\leq \omega_n t \sum_{j=1}^{\infty} \frac{2^n}{2j} \int_{\frac{1}{2j}B_1} |\nabla f(z)| \, dz$$

$$\leq 2^n \omega_n t M_\ell (|\nabla f|)(x)$$

where $\omega_n$ is the volume of the unit ball. Similarly, $I_2$ can be estimated. Using these estimates, (4.4) gives

$$|f(x) - f(y)| \leq 3C \cdot 2^{n-1} \omega_n |x-y| (M_\ell (|\nabla f|)(x) + M_\ell (|\nabla f|)(y))$$

(4.5)

for all $x, y \in B \setminus S_B$ with $|S_B| = 0$. Now, take $g = 3C \cdot 2^{n-1} \omega_n M_\ell (|\nabla f|)$. Since $|\nabla f| \in L^2_0(\mathbb{R}^n, w)$, by Theorem A.6, $g \in L^2_0(\mathbb{R}^n, w)$. Hence, by (4.5) we obtain (4.4).

Conversely, by conditions of the theorem the inequality (4.4) holds with some functions $f \in L^2_0(\mathbb{R}^n, w)$ and $g \in L^2_0(\mathbb{R}^n, w)$ for almost all $x, y \in \mathbb{R}^n \setminus S$ where $S$ is some set of zero measure. By Proposition 4.6 we can assume that $f, g \in L^1_{loc}(\mathbb{R}^n)$.

For any $k \in \mathbb{N}$ we define a set

$$A_k = \{ x \in \mathbb{R}^n \setminus S : g(x) \leq k \}.$$

Then for all points $x, y \in A_k$ we have

$$|f(x) - f(y)| \leq 2k |x-y|.$$

Therefore $f$ is a Lipschitz function on the set $A_k$ in the conventional sense. Applying the Kirszbraun extension theorem [12, 3.1, Theorem 1], we obtain an extension $\tilde{f}_k : \mathbb{R}^n \to \mathbb{R}$ of $f : A_k \to \mathbb{R}$ to a Lipschitz function on $\mathbb{R}^n$ with the same Lipschitz constant. In particular, for all points $x, y \in \mathbb{R}^n \setminus A_k$ we have

$$|\tilde{f}(x) - \tilde{f}(y)| \leq 2k |x-y|.$$

(4.6)

Take an arbitrary vector $e_i$ of the standard basis in $\mathbb{R}^n$, $i = 1, 2, \ldots, n$. We will write points $x \in \mathbb{R}^n$ as $x = (\tilde{x}, x_i) = \tilde{x} + x_i e_i$ where $\tilde{x} \in \mathbb{R}^{n-1}$ and $x_i \in \mathbb{R}$. For any $\tilde{x} \in \mathbb{R}^{n-1}$ the restriction $\mathbb{R} \ni x_i \mapsto \tilde{f}_k(\tilde{x} + x_i e_i)$ is a Lipschitz function with respect to $x_i \in \mathbb{R}$. Hence it is absolutely continuous on every line parallel to $i$-coordinate axis and therefore it has the partial derivative $\frac{d\tilde{f}_k}{dx_i}(\tilde{x} + x_i e_i)$ for almost all $x_i \in \mathbb{R}$. Thus, by Tonelli’s theorem (see, for example, [10, Theorem 13.8]) the partial derivative $\frac{d\tilde{f}_k}{dx_i}(x)$ exists for almost all $x \in \mathbb{R}^n, i = 1, 2, \ldots, n$.

Again by Tonelli’s theorem the intersection

$$\{ \tilde{x} + x_i e_i : x_i \in \mathbb{R} \} \cap A_k$$
is measurable for almost all $\bar{x} \in \mathbb{R}^{n-1}$. Take $\bar{x} \in \mathbb{R}^{n-1}$ such that

1. the intersection $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\} \cap A_k$ is measurable and has a positive Lebesgue measure;
2. the restriction of $g$ to the line $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\}$ belongs to the class $L^1_{\text{loc}}$.

Now let $t \in \mathbb{R}$ be a value such that

3. $x + te_i \in A_k$;
4. $x + te_i$ is a density point with respect to $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\} \cap A_k$;
5. there exists the partial derivative

$$\frac{\partial}{\partial x_i} \tilde{f}(\bar{x} + te_i) = \frac{d}{dt} \tilde{f}(\bar{x} + te_i);$$

6. $\bar{x} + te_i$ is a Lebesgue point of the restriction $g$: $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\} \to \mathbb{R}$.

Properties (1)–(2) are fulfilled for almost all $\bar{x} \in \mathbb{R}^{n-1}$ such that the intersection is $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\} \cap A_k$ is not empty. For fixed $\bar{x} \in \mathbb{R}^{n-1}$ properties (3)–(6) hold for almost all $t$ such that $\bar{x} + te_i \in A_k$. Therefore, by Tonelli’s theorem, properties (1)–(6) are satisfied for almost all $x \in A_k$.

Our immediate goal is to evaluate a derivative:

$$\left| \frac{\partial \tilde{f}_k}{\partial x_i}(x) \right| \leq \begin{cases} 2g(x) & \text{for almost all } x \in A_k, \ i = 1, 2, \ldots, n, \\ 2k & \text{for almost all } x \in \mathbb{R}^n \setminus A_k. \end{cases}$$ (4.7)

The second line is a consequence of (4.6) and the Rademacher differentiability theorem.

Let $x = \bar{x} + te_i \in A_k$ be a point meeting all the above-mentioned differentiability properties (1)–(6). In view of (4.3) for the function

$$\mathbb{R} \ni \tau \to h(\tau) = \tilde{f}(\bar{x} + (t + \tau)e_i) = \tilde{f}(x + \tau e_i)$$

we have an estimate of the difference relation provided $x + \tau e_i \in A_k$:

$$\left| \frac{h(\tau) - h(0)}{\tau} \right| = \left| \frac{\tilde{f}(x + \tau e_i) - \tilde{f}(x)}{\tau} \right| = \left| \frac{f(x + \tau e_i) - f(x)}{\tau} \right| \leq g(x) + g(x + \tau e_i)$$

$$= 2g(x) + g(x + \tau e_i) - g(x).$$ (4.8)

The last row of the relations (4.8) shows that the estimate for the derivative $h'(0)$ depends on the behavior of the difference $g(x + \tau e_i) - g(x)$. Since $x$ is the Lebesgue point of the restriction $g$: $\{\bar{x} + x_ie_i : x_i \in \mathbb{R}\} \to \mathbb{R}$ then

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} |g(x + \tau e_i) - g(x)| d\tau = o(1) \quad \text{as } \delta \to 0.$$
We fix an arbitrary number $\varepsilon > 0$. Using the Chebyshev inequality, we deduce

$$\frac{|\{\tau \in (-\delta, \delta) : |g(x + \tau e_i) - g(x)| \geq \varepsilon\}|}{\delta} \leq \frac{1}{\varepsilon} \int_{-\delta}^{\delta} |g(x + \tau e_i) - g(x)| \, d\tau = \frac{o(\delta)}{\varepsilon}$$

as $\delta \to 0$. Therefore, we obtain

$$1 \geq \frac{|\{\tau \in [-\delta, \delta] : |g(x + \tau e_i) - g(x)| < \varepsilon\}|}{2\delta} \geq \frac{2\delta - o(\delta)}{2\delta} \to 1 \quad (4.9)$$

as $\delta \to 0$.

As soon as the point $x$ is a density point with respect to the intersection \{\(x + x_ie_i : x_i \in \mathbb{R}\} \cap A_k\), we come to

$$\frac{|\{x - \delta e_i, x + \delta e_i \cap A_k\}|}{2\delta} \to 1 \quad \text{as} \quad \delta \to 0. \quad (4.10)$$

We introduce the notation

$$T = \{\tau \in [-1, 1] : |g(x + \tau e_i) - g(x)| < \varepsilon\}.$$

By virtue of (4.9), the point 0 is a density point with respect to the set $T$. Similarly to the previous one, due to (4.10), the point 0 is also a density point with respect to the set

$$P = \{\tau \in [-1, 1] : |x - \tau e_i, x + \tau e_i \cap A_k\}.$$

From the definition of a density point, we conclude that 0 is the density point of the intersection $T \cap P$. From here, (4.9) and (4.10), we derive the relations

$$\left|\frac{h(\tau) - h(0)}{\tau}\right| = \left|\frac{\hat{f}(x + \tau e_i) - \hat{f}(x)}{\tau}\right| = 2g(x) + g(x + \tau e_i) - g(x) \leq 2g(x) + \sup_{\tau \in [-\delta, \delta] \cap (T \cap P)} |g(x + \tau e_i) - g(x)| \leq 2g(x) + \varepsilon \quad (4.11)$$

for all points $\tau \in [-\delta, \delta] \cap (T \cap P)$. Passing to the limit in (4.11) as $\tau \to 0$, $\tau \in [-\delta, \delta] \cap (T \cap P)$, we get

$$|h'(0)| = \left|\frac{\partial \hat{f}_k}{\partial x_i}(x)\right| \leq 2g(x) + \varepsilon.$$

Since here $\varepsilon > 0$ is an arbitrary positive number, the inequality (4.7) is proven.

As long as $A_k \subset A_{k+1}$, $k = 1, 2, \ldots$, and $|\mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} A_k| = 0$, for almost all $x \in \mathbb{R}^n$ there exist limits

$$\lim_{k \to \infty} \tilde{f}_k(x) = f(x),$$

and, for $i = 1, 2, \ldots, n$, we define

$$w_i(x) := \lim_{k \to \infty} \frac{\partial f_k}{\partial x_i}(x) = \begin{cases} \frac{\partial f_1}{\partial x_i}(x), & \text{if} \ x \in A_1, \\ \frac{\partial f_l}{\partial x_i}(x), & \text{if} \ x \in A_l \setminus A_{l-1}, \ l = 2, 3, \ldots. \end{cases}$$
Fixing an arbitrary point \( x_0 \in A_k \) and taking into account the inequality \( k \leq g(x) \) at \( x \in R^n \setminus A_k \) we have the following estimates: \(|\tilde{f}_k(x) - \tilde{f}_k(x_0)| \leq 2k|x - x_0| \leq 2g(x)|x - x_0|\) at \( x \in R^n \setminus (A_k \cup S) \). Therefore

\[
|\tilde{f}_k(x)| \leq \begin{cases} 
|f(x)|, & \text{if } x \in A_k, \\
2g(x)|x - x_0| + |f(x_0)|, & \text{if } x \in R^n \setminus (A_k \cup S).
\end{cases}
\] (4.12)

Moreover \(|w_i(x)| \leq 2g(x)\) for almost all \( x \in R^n \) by (4.7) and the inequality \( k < g(x) \) in \( x \in R^n \setminus A_k \).

Take an arbitrary test function \( \varphi \in C^\infty(\Omega) \). By the conditions of Theorem 4.5 the function \( f \) is integrable on \( \text{supp} \varphi \). Since \( \tilde{f}_k \) has the first generalized derivatives, we have

\[
\int_{\{x: \varphi(x) \neq 0\}} \varphi(x) \frac{\partial \tilde{f}_k}{\partial x_i}(x) \, dx = - \int_{\{x: \varphi(x) \neq 0\}} \tilde{f}_k(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx.
\]

On the compact set \( \{x : \varphi(x) \neq 0\} \) the sequences \( \tilde{f}_k(x) \) and \( \frac{\partial \tilde{f}_k}{\partial x_i}(x) \) have the majorants \( \max_{\{x: \varphi(x) \neq 0\}} (|f(x)|, 2g(x)|x - x_0| + |f(x_0)|) \) and \( 2g(x) \), respectively.

Therefore, by the Lebesgue dominated convergence theorem we obtain

\[
\int_{\{x: \varphi(x) \neq 0\}} \varphi(x)w_i(x) \, dx = - \int_{\{x: \varphi(x) \neq 0\}} f(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx
\]

for all \( \varphi \in C^\infty(\Omega) \). Consequently, \( w_i(x) \) is the generalized derivative \( \frac{\partial f}{\partial x_i}(x) \), \( i = 1, \ldots, n \), of the function \( f \) and, by virtue of (4.7) and Proposition 2.6(ii), we have the estimate

\[
\|\nabla f\|_{L^q_0(R^n, w)} \leq 2\sqrt{n} \|g\|_{L^q_0(R^n, w)}.
\]

Therefore, \( f \in L^q_a(R^n, w) \). Thus, it is proven that \( f \in W^{1,q}_a(R^n, w) \). \( \square \)

**Remark 4.6.** The proof of Theorem 4.5 is based on arguments of the paper [49] where the similar assertion for Sobolev functions, defined on Carnot groups, was given. See also [25] for an independent proof for Sobolev functions defined on Euclidean spaces.

The following result is the version of Theorem 4.5 for bounded domains.

**Theorem 4.7.** Let \( 1 < q < \infty \) and \( \Omega \subset R^n \) be a bounded domain. A function \( f \) belongs to \( W^{1,q}(\Omega) \) if and only if \( f \in L^q(\Omega) \) and there exist a non negative function \( g \in L^q(\Omega) \) and a set \( S \subset \Omega \) of measure zero such that the inequality

\[
|f(x) - f(y)| \leq |x - y|(g(x) + g(y))
\]

holds for all points \( x, y \in \Omega \setminus S \) with \( B(x, 3|x - y|) \subset \Omega \).

In view of Proposition 4.3 the case \( \Omega \subset R^n \), where \( |\Omega| < \infty \) can be proved with similar arguments making evident changes.
Remark 4.8. If $f \in W^{1,q}(\Omega)$ then $|f| \in W^{1,q}(\Omega)$ because by Theorem 4.4 there exist a non negative function $g \in L^{q}(\Omega)$ and a set $S \subset \Omega$ of measure zero such that

$$
|f(x)| \leq |f(y)| \leq |x - y|(g(x) + g(y))
$$

holds for all points $x, y \in \Omega \setminus S$ with $B(x, 3|x - y|) \subset \Omega$.

5. Pointwise estimates for Banach function spaces

Note, that the proof of Theorem 4.5 is based on the following facts:

1. $f, \nabla f, g \in L_{\text{loc}}^{1}(\mathbb{R}^{n})$;
2. the maximal operator $M_{g}$ is bounded in $L^{q}(\mathbb{R}^{n}, w)$;
3. the space $L_{\text{loc}}^{q}(\mathbb{R}^{n}, w)$ satisfies the lattice property, i.e. if $|f| \leq |g|$ a.e. then $\|f\| \leq \|g\|$.

Hence, this proof can be applied almost verbatim for proving a similar result for function spaces, which meet conditions (1)–(4). It includes many and various spaces, for example, grand Lesbegue, Musielak–Orlicz, Lorentz and Marcinkiewicz spaces, as well as Lebesgue spaces with variable exponents. In particular, it includes the general concept, that covers many different spaces at once, namely, the theory of Banach function spaces. We refer the reader to [2] and [12] for details.

Definition 5.1. Let $(R, \mu)$ be a $\sigma$-finite measure space and let $\mathcal{M}$ denote the set of all measurable functions on $(R, \mu)$. We say that a function $\| \cdot \| : \mathcal{M} \to [0, \infty]$ is a Banach function norm if for all $f_{n}, f, g \in \mathcal{M}$ and $\alpha \in \mathbb{R}$:

1. $\|f\| = 0$ if and only if $f = 0$ a.e., $\|\alpha f\| = |\alpha|\|f\|$ and $\|f + g\| \leq \|f\| + \|g\|$;
2. if $|f| \leq |g|$ a.e. then $\|f\| \leq \|g\|$;
3. if $0 \leq f_{n} \nrightarrow f$ then $\|f_{n}\| \nrightarrow \|f\|$
4. for every measurable $E \subset R$, $\mu(E) < \infty$: $\|\chi_{E}\| < \infty$;
5. for every measurable $E \subset R$, there exists a constant $C_{E} > 0$ (independent of $f$), such that $\int_{E} |f| d\mu \leq C_{E}\|f\|$.

The space $X(R) = \{f \in \mathcal{M} : \|f\| < \infty\}$ with norm $\| \cdot \|$ is called a Banach function space.

Remark 5.2. The property (1) is a local version of (5) in the sense that it considers only compact sets $E \subset R$.

Let $\Omega \subset \mathbb{R}^{n}$ be an open set and $X(\Omega)$ be a Banach function space w.r.t. the Lebesgue measure. The Sobolev space $W^{1,q}(\Omega)$ denotes the space of weakly differentiable mappings $f$ with $f, \nabla f \in X(\Omega)$. This space is equipped with a norm

$$
\|f\|_{W^{1,q}(\Omega)} := \|f\|_{X(\Omega)} + \|\nabla f\|_{X(\Omega)}.
$$

It is easy to see that conditions (1) and (2) are fulfilled for any Banach function space $X$. Hence, we can formulate the theorem in a general way as follows.
Theorem 5.3. Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( X(\Omega) \) be a Banach function space such that the Hardy–Littlewood maximal operator is bounded in \( X(\Omega) \). Then a function \( f \) belongs to \( W^X(\Omega) \) if and only if \( f \in X(\Omega) \) and there exist a non negative function \( g \in X(\Omega) \) such that the inequality
\[
|f(x) - f(y)| \leq |x - y|(g(x) + g(y))
\]
holds for almost all \( x, y \in \Omega \) with \( B(x, 3|x - y|) \subset \Omega \).

Let us consider several specific examples.

Corollary 5.4. A function \( f \) belong to Lorentz–Sobolev space \( W^{p,q}(\mathbb{R}^n) \) with \( p > 1 \) and \( q \geq 1 \) if and only if \( f \in L^{p,q}(\mathbb{R}^n) \) and there exist a non negative function \( g \in L^{p,q}(\mathbb{R}^n) \) such that the inequality \( 5.1 \) is fulfilled for all \( x, y \in \mathbb{R}^n \setminus \sigma \) where \( \sigma \) is a set of measure zero.

Properties of Lorentz spaces can be found in [2, Chapter 4.4].

Corollary 5.5. Consider a Young function \( A \) such that there exists a positive constant \( c \), for which
\[
\int_0^t \frac{A(s)}{s^2} ds \leq \frac{A(ct)}{t}
\]
holds for all \( t > 0 \). Then \( f \) belongs to Orlicz–Sobolev space \( W^A(\mathbb{R}^n) \) if and only if \( f \in L^A(\mathbb{R}^n) \) and there exists a non negative function \( g \in L^A(\mathbb{R}^n) \) such that the inequality \( 5.1 \) is fulfilled for all \( x, y \in \mathbb{R}^n \setminus \sigma \) where \( \sigma \) is a set of measure zero.

For the research on the maximal operator in Orlicz spaces the reader is referred to [35] and [11] Theorem 3.3.

Remark 5.6. In [47] Theorem 1.2] the characterization in spirit of [26] is given for the Orlicz–Sobolev function \( W^A(\mathbb{R}^n) \) with the Young function \( A \) and a complimentary \( A^* \) satisfying the \( \Delta_2 \)-condition. The right hand side of \( 5.1 \) is expressed in terms of the Young function:
\[
|f(x) - f(y)| \leq C|x - y|(A^{-1}(M_{\beta|y|}(A(g)(x))) + A^{-1}(M_{\beta|y|}(A(g)(y))))
\]
for some \( C > 0 \) and \( \sigma \geq 1 \). Here \( A^{-1} \) is a generalized inverse of \( A \).

Corollary 5.7. Let \( p: \Omega \to [1, \infty] \) be a measurable function with \( p^- = \text{ess inf}_{y \in \Omega} p(y) > 1 \) and \( \frac{1}{p} \) is globally log-Hölder continuous. Then \( f \) belongs to Sobolev space with variable exponent \( W^{1,p(\cdot)}(\Omega) \) if and only if \( f \in L^{p(\cdot)}(\Omega) \) and there exist a non negative function \( g \in L^{p(\cdot)}(\Omega) \) such that the inequality \( 5.1 \) holds for all \( x, y \in \Omega \setminus \sigma \) with \( B(x, 3|x - y|) \subset \Omega \), where \( \sigma \subset \Omega \) is a set of measure zero.

\( \alpha: \Omega \to \mathbb{R} \) is globally log-Hölder continuous if there exist \( c_1, c_2 > 0 \) and \( \alpha_\infty \in \mathbb{R} \) s.t.
\[
|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(\epsilon + |x - y|/|y|)} \quad \text{for all } x, y \in \Omega, \quad \text{and} \quad |\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(\epsilon + |x|)} \quad \text{for all } x \in \Omega.
\]
The theory of Lebesgue and Sobolev spaces with variable exponent and, in particular, conditions about boundedness of maximal operator can be found in [10, Section 4.3]).

**Corollary 5.8.** Let \( X \) be a rearrangement invariant Banach function space with the upper Boyd index \( \beta_X < 1 \). A function \( f \) belongs to \( W^X(\Omega) \) if and only if \( f \in X(\Omega) \) and there exist a nonnegative function \( g \in X(\Omega) \) such that the inequality \((5.1)\) holds for all \( x, y \in \Omega \setminus S \) with \( B(x, 3|x - y|) \subset \Omega \), where \( S \subset \Omega \) is a set of measure zero.

**Remark 5.9.** If \( X \) is a rearrangement invariant Banach function space then the maximal operator is bounded if and only if the upper Boyd index \( \beta_X < 1 \), see [2, Chapter 3, Definition 5.12 and Theorem 5.17]. The formulas for calculating the Boyd indices of classical function spaces may be found in, for example, [17].

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