Normalized ground states for 3D dipolar Bose-Einstein condensate with attractive three-body interactions

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Abstract

We study the existence of normalized ground states for the 3D dipolar Bose-Einstein condensate equation with attractive three-body interactions:

$$-\Delta u + \beta u + \lambda_1 |u|^2 u + \lambda_2 (K \ast |u|^2) u - |u|^4 u = 0.$$  \hspace{1cm} (DBEC)

When $\lambda_2 = 0$ or $u$ is radial, (DBEC) reduces to the cubic-quintic NLS

$$-\Delta u + \beta u + \lambda_1 |u|^2 u - |u|^4 u = 0,$$  \hspace{1cm} (CQNLS)

which has been recently studied by Soave in [31]. In particular, it was shown that for any $\lambda_1 < 0$ and $c > 0$, (CQNLS) possesses a radially symmetric ground state solution with mass $c$ and for $\lambda_1 \geq 0$, (CQNLS) has no non-trivial solution. We show that by adding a dipole-dipole interaction to (CQNLS), the geometric nature of (CQNLS) changes dramatically and techniques as the ones from [31] cannot be used anymore to obtain similar results. More precisely, due to the axisymmetric nature of the dipole-dipole interaction potential, the energy corresponding to (DBEC) is not stable under symmetric rearrangements, hence conventional arguments based on the radial symmetry of solutions are inapplicable. We will overcome this difficulty by appealing to subtle variational and perturbative methods and prove the following:

(i) If the pair $(\lambda_1, \lambda_2)$ is unstable and $\lambda_1 < 0$, then for any $c > 0$, (DBEC) has a ground state solution with mass $c$.

(ii) If the pair $(\lambda_1, \lambda_2)$ is unstable and $\lambda_1 \geq 0$, then there exists some $c^* = c^*(\lambda_1, \lambda_2) \geq 0$ such that for all $c > c^*$, (DBEC) has a ground state solution with mass $c$. Moreover, any non-trivial solution of (DBEC) in this case must be non-radial.

(iii) If the pair $(\lambda_1, \lambda_2)$ is stable, then (DBEC) has no non-trivial solutions.

1 Introduction and main results

In this paper, we prove existence of solitary waves for the equation modeling 3D dipolar Bose-Einstein condensates (DBEC) with attractive three-body interactions:

$$i \partial_t \phi = -\Delta \phi + \beta |\phi|^2 \phi + \lambda_1 |\phi|^2 \phi + \lambda_2 (K \ast |\phi|^2) \phi - |\phi|^4 \phi,$$  \hspace{1cm} (1.1)

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants and the dipole-dipole kernel $K$ is defined by

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^4}.$$ 

To be more precise, we will be seeking solitary wave solutions $u$ of (1.1) which satisfy the stationary DBEC equation:

$$-\Delta u + \beta u + \lambda_1 |u|^2 u + \lambda_2 (K \ast |u|^2) u - |u|^4 u = 0$$  \hspace{1cm} (1.2)

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and possess a prescribed mass $\|u\|_2^2 = c$ for a given $c \in (0, \infty)$. It then follows directly that the function $\phi(t, x) = e^{i\lambda t}u(x)$ is a solution to (1.1), for any solution $u$ of (1.2). Equations (1.1) and (1.2) can be generalized to

$$i\partial_t \phi = -\Delta \phi + \lambda_1 |\phi|^2 \phi + \lambda_2 (K * |\phi|^2) \phi + \lambda_3 |\phi|^p \phi$$

(1.3)

and

$$-\Delta u + \beta u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u + \lambda_3 |u|^p u = 0$$

(1.4)

with $\lambda_3 \in \mathbb{C}$ and $p \in (0, \infty)$ respectively\(^1\). The potentials $|u|^2 u$ and $(K * |u|^2) u$ describe the two-body and long range dipole-dipole interactions respectively. For the potential $|u|^p u$, the cases $p = 5$ and $p = 6$ correspond to the Lee-Huang-Yang-correction (LHY-correction) and three-body interaction respectively. When in a physical experiment the parameters $\lambda_1, \lambda_2$ are tuned so that they lie in the unstable regime (see Definition 1.1), the classical Gross-Pitaevskii theory would predict a collapse of the gas, which was not seen during the experiment. In order to stabilize the Gross-Pitaevskii equation, it has been suggested to incorporate a higher order repulsive term such as the LHY-correction or three-body interactions; we refer to the papers [11, 19, 22, 24, 30, 34] and the references therein for a more comprehensive introduction on the physical background of (1.3). There has also been an ongoing investigation of models for collapsing Bose-Einstein condensates with attractive three-body interaction (see for instance [1, 23, 29, 32]). The subject of this paper is to study this aspect of the DBEC, which itself presents a mathematically challenging problem.

The first mathematically rigorous analysis for (1.3) dates back to the work of Carles, Markowich and Sparber in [14], where (1.3) was considered without any higher order term ($\lambda_3 = 0$). The authors proved local and global well-posedness and finite time blow-up results for (1.3). Particularly, 1D and 2D DBEC-models were also derived from the 3D model via dimension reduction. Later, Antonelli and Sparber [2] proved existence of ground states for (1.4) (again in the case $\lambda_3 = 0$) in the unstable regime using the so-called Weinstein functional method. Further regularity and symmetry results of the ground states were also established. The existence of normalized ground states for (1.3) in the unstable regime without higher order term was later proved by Bellazzini and Jeanjean [9] using mountain pass arguments. In particular, it was shown that ground states obtained by mountain pass are automatically the Weinstein optimizers found in [2]. Further existence, stability and well-posedness results for (1.3) and (1.4) with or without an external trapping potential were also obtained in [9]. In [7], Bellazzini and Forcella were able to utilize the ground states given in [2] and [9] to formulate a sharp scattering threshold for (1.3) without higher order term. The first results for (1.3) and (1.4) with higher order term were given by the Authors [27, 28], where the cases $\lambda_3 < 0, p = 5$ and $\lambda_3 > 0, p \in \{4, 6\}$ were studied. We also refer to [3, 4, 5, 6, 8, 13, 15, 16, 17, 18, 25, 33] and the references therein for recent analytical and numerical progress on (1.3) and (1.4).

From now on we focus on the DBEC (1.2) with attractive three-body interactions. A solitary wave solution of (1.2) is of fundamental importance for studying (1.1), since it might be the only observable quantity in physical experiments and can be seen as a balance point between linear and nonlinear effects. Here, we will be looking for ground state solutions with prescribed mass, i.e., the total number of particles in the gas.

Before we state the main results of the paper, we firstly fix some definitions and notation. The Hamiltonian $E(u)$ corresponding to (1.2) is defined by

$$E(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \lambda_1 \|u\|_4^4 + \lambda_2 \int_{\mathbb{R}^3} (K * |u|^2)|u|^2 \, dx - \frac{1}{6} \|u\|_6^6.$$  

For $c > 0$, the manifold $S(c)$ is defined by

$$S(c) = \{ u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c \}.$$  

The definition of unstable and stable regimes is given as follows:

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\(^1\)More generally, an external trapping potential $V_{ext}$ should also be contained in (1.3). We consider in this paper the case $V_{ext} = 0$, which corresponds to the so-called self-bounded model.

\(^2\)In the case where the three-body interaction is under consideration, the parameter $\lambda_3$ might also have non-trivial imaginary part, indicating a three-body loss effect.
Definition 1.1 (Unstable and stable regimes). We define the unstable and stable regimes as follows:

(i) The pair \((\lambda_1, \lambda_2)\) is said to be in the **unstable** regime if

\[
\lambda_2 > 0 \quad \text{and} \quad \lambda_1 - \frac{4\pi}{3}\lambda_2 < 0
\]

or

\[
\lambda_2 < 0 \quad \text{and} \quad \lambda_1 + \frac{8\pi}{3}\lambda_2 < 0;
\]

(ii) The pair \((\lambda_1, \lambda_2)\) is said to be in the **stable** regime if

\[
\lambda_2 > 0 \quad \text{and} \quad \lambda_1 - \frac{4\pi}{3}\lambda_2 > 0
\]

or

\[
\lambda_2 < 0 \quad \text{and} \quad \lambda_1 + \frac{8\pi}{3}\lambda_2 > 0;
\]

We use the following form of the Fourier transform:

\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-i\xi \cdot x} \, dx.
\]

Due to [14], the Fourier transform \(\hat{K}\) of \(K\) is given by

\[
\hat{K}(\xi) = \frac{4\pi}{3} \frac{\xi_1^2 - \xi_2^2 - \xi_2^2}{|\xi|^3} \in \left[ -\frac{4\pi}{3}, \frac{8\pi}{3} \right].
\]

The following quantities will also be used throughout the paper:

\[
A(u) := \|\nabla u\|_2^2,
\]

\[
B(u) := \lambda_1 \|u\|_4^4 + \lambda_2 \int_{\mathbb{R}^3} (K * |u|^2)|u|^2 \, dx,
\]

\[
C(u) := \|u\|_6^6,
\]

\[
Q(u) := A(u) + \frac{3}{4}B(u) - C(u).
\]

Next, we define the set \(V(c)\) by

\[
V(c) = \{u \in S(c) : Q(u) = 0\}
\]

and we define the variational problem \(\gamma(c)\) by

\[
\gamma(c) := \inf\{E(u) : u \in V(c)\}.
\]

Finally, for \(u \in H^1(\mathbb{R}^3)\) and \(t > 0\) we define the function \(u^t\) by

\[
u^t(x) = t^\frac{\beta}{2}u(tx).
\]

It is a direct calculation to check that the rescaling \((1.6)\) leaves the \(L^2\)-norm invariant.

The main result of the present paper is the following:

**Theorem 1.2** (Existence of ground states). The following statements hold true:

(i) If \((\lambda_1, \lambda_2)\) is an unstable pair and \(\lambda_1 < 0\), then for any \(c \in (0, \infty)\) the variational problem \((1.5)\) has a positive optimizer \(u_c \in S(c)\).

(ii) If \((\lambda_1, \lambda_2)\) is an unstable pair and \(\lambda_1 \geq 0\), then there exists some \(c^* = c^*(\lambda_1, \lambda_2) \in [0, \infty)\) such that \(c \mapsto \gamma(c)\) is constantly equal to \(\frac{\beta}{2}\) on \((0, c^*)\) and strictly less than \(\frac{\beta}{2}\) on \((c^*, \infty)\). Furthermore, for any \(c \in (c^*, \infty)\) the variational problem \((1.5)\) has a positive optimizer \(u_c \in S(c)\).
Moreover, the optimizers given by (i) and (ii) are solutions of (1.2) with \(\beta = \beta_c > 0\).

(iii) Let \(u_c\) be the solution of (1.2) given by (ii). Then for any \(x \in \mathbb{R}^3\), \(u_c(\cdot + x)\) is not radially symmetric.

(iv) If \((\lambda_1, \lambda_2)\) is a stable pair, then (1.2) has no non-trivial solution in \(H^1(\mathbb{R}^3)\).

We make a couple of comments on Theorem 1.2: the quintic potential \(|u|^4u\) is energy-critical in 3D. Moreover, since \(K\) has vanishing sphere integral, it is in fact a Calderon-Zygmund kernel and therefore a bounded mapping from \(L^p\) to \(L^p\) for all \(p \in (1, \infty)\). Hence (1.2) can be seen as the focusing energy-critical NLS perturbed by a cubic-like lower order term. The similar cubic-quintic model

\[-\Delta u + \beta u + \lambda_1 |u|^2 u - |u|^4 u = 0\]  

(1.7)

has been recently studied by Soave in [31] (in fact, a general class of focusing energy-critical NLS with combined powers including (1.7) was studied therein). In particular, Soave proved that for any \(\lambda_1 < 0\) and \(c > 0\), \(\gamma(c)\) defined by (1.5) (corresponding to (1.7)) has a normalized positive, radially symmetric ground state \(u_c \in S_c\) and \(\gamma(c) \in (0, \frac{S^3}{3})\), where \(S\) is the best constant for the Sobolev inequality:

\[S = \inf_{u \in D^{1,2}(\mathbb{R}^3)} \frac{\|\nabla u\|^2_2}{\|u\|^6_6}.\]

On the contrary, for \(\lambda_1 \geq 0\), Soave showed that (1.7) has no non-trivial solution\(^3\).

It remains an open problem whether there exist normalized ground states of (1.2) for \(c \in (0, c^*)\) in the case where \((\lambda_1, \lambda_2)\) is unstable and \(\lambda_1 \geq 0\). As it will become clear from the proof of Theorem 1.2, the existence of ground states is, loosely speaking, equivalent to showing that \(\gamma(c)\) is strictly smaller than \(\frac{S^3}{3}\). In the case of (1.7), Soave showed this directly by constructing a sequence of test functions that asymptotically meets the assumption, where the construction of the test functions is based on certain delicate cut-off refinement of the radial Aubin-Talenti ground states. However, the radial symmetry is principally incompatible with the dipolar potential and Soave’s arguments can not be utilized for finding a reasonable minimizing sequence in the case \(\lambda_1 \geq 0\). In view of such consideration, we conjecture that in the case of Theorem 1.2 (ii), the number \(c^*\) is strictly positive and there is no ground states of \(\gamma(c)\) for all \(c \in (0, c^*)\). We note, however, that this is not contradicting Theorem 1.2 (ii), since for large mass we can find a minimizing sequence without invoking the radial symmetry at all, while simultaneously the gradient and quintic potential energies remain controllable.

The rest of the paper is organized as follows: In Section 2 we provide some auxiliary lemmas which will be useful for proving the main results. In Section 3 we prove our main results.

## 2 Preliminaries

In this section we collect some useful auxiliary results which will be later used in the proof of Theorem 1.2.

### 2.1 Pohozaev identity

We begin with the well-known Pohozaev identity.

**Lemma 2.1.** Let \(u\) be a solution of (1.2). Then

\[A(u) + 3\beta\|u\|^2_2 + \frac{3}{2}B(u) - C(u) = 0.\]  

(2.1)

Consequently, we have

\[4\beta\|u\|^2_2 = -B(u),\]  

(2.2)

\[Q(u) = 0.\]  

(2.3)

\(^3\)This was in fact originally shown in the case \(\lambda_1 > 0\), but the proof extends verbatim to \(\lambda_1 \geq 0\).
Proof. (2.1) follows generally by multiplying (1.2) with \( x \cdot \nabla u \) and then integrating by parts (for more details see [27]). On the other hand, by multiplying (1.2) with \( \bar{u} \) and integrating by parts we obtain
\[
A(u) + \beta\|u\|_2^2 + B(u) - C(u) = 0. \tag{2.4}
\]
Eliminating \( A(u) \) and \( C(u) \) in (2.1) and (2.4) yields (2.2); Eliminating \( \beta\|u\|_2^2 \) in (2.1) and (2.4) yields (2.3). \( \square \)

2.2 Characterization of optimizers of \( \gamma(c) \)

In the following, we show that any optimizer of \( \gamma \) is automatically a solution of (1.2).

Lemma 2.2. Let \( c > 0 \). If \( \gamma(c) \) is attained at some \( u \in V(c) \), then \( u \) solves (1.2).

Proof. Suppose that \( \gamma(c) \) is attained at \( u \in V(c) \). Then in view of the Lagrange multiplier theorem, there exist \( \mu_1, \mu_2 \in \mathbb{C} \) such that
\[
E'(u) - \mu_1 Q'(u) - 2\mu_2 u = 0,
\]
or equivalently
\[
(1 - 2\mu_1)(-\Delta u) + (1 - 3\mu_1)(\lambda_1|u|^2 + \lambda_2(K + |u|^2))u + (6\mu_1 - 1)|u|^4 u - 2\mu_2 u = 0. \tag{2.5}
\]
The Pohozaev identity corresponding to (2.5) is given by
\[
0 = (1 - 2\mu_1)A(u) + \frac{3}{4}(1 - 3\mu_1)B(u) + (6\mu_1 - 1)C(u). \tag{2.6}
\]
Eliminating \( B(u) \) in (2.5) and (2.6) and using the fact that \( Q(u) = 0 \), we infer that
\[
\mu_1(A(u) + 3C(u)) = 0.
\]
Since \( A(u) + 3C(u) > 0 \), we know that \( \mu_1 = 0 \) and the proof is complete. \( \square \)

2.3 The energy landscape along the \( L^2 \)-invariant scaling

For any function \( u \in H^1(\mathbb{R}^3) \), we recall that \( u^t \) is the \( L^2 \)-invariant scaling of \( u \) defined by (1.6). The following lemma shows that we can always find some \( t^* > 0 \) such that \( u^{t^*} \in V(c) \) is a local maximum of the mapping \( t \mapsto E(u^t) \). Thus \( u^{t^*} \) can be viewed as the peak of a mountain pass.

Lemma 2.3. Let \( c > 0 \) and \( u \in S(c) \). Then:

(i) \( \frac{\partial}{\partial t} E(u^t) = \frac{Q(u^t)}{t} \), for all \( t > 0 \).

(ii) There exists a \( t^* > 0 \) such that \( u^{t^*} \in V(c) \).

(iii) We have \( t^*(u) < 1 \) if and only if \( Q(u) < 0 \). Moreover, \( t^*(u) = 1 \) if and only if \( Q(u) = 0 \).

(iv) The following inequalities hold:
\[
Q(u^t) \begin{cases} 
> 0, & t \in (0, t^*(u)), \\
< 0, & t \in (t^*(u), \infty).
\end{cases}
\]

(v) \( E(u^t) < E(u^{t^*}) \) for all \( t > 0 \) with \( t \neq t^* \).

Proof. (i) follows from direct calculation. Next, define \( y(t) := \frac{\partial}{\partial t} E(u^t) \). Then
\[
y(t) = tA(u) + \frac{3}{4}t^2B(u) - t^5C(u),
\]
\[
y'(t) = A(u) + \frac{3}{4}B(u) - 5t^4C(u),
\]
\[
y''(t) = \frac{3}{2}B(u) - 20t^3C(u).
\]
If $B(u) \leq 0$, then $g''(t)$ is negative on $(0, \infty)$; if $B(u) > 0$, then $g''(t)$ is positive on $(0, \left(\frac{3B}{4u^3}\right)^{1/4})$ and negative on $(-\left(\frac{3B}{4u^3}\right)^{1/4}, \infty)$. Since $y'(0) = A(u) > 0$ and $y'(t) \to -\infty$ as $t \to \infty$, we conclude simultaneously from both cases that there exists a $t_0 > 0$ such that $y'(t)$ is positive on $(0, t_0)$ and negative on $(t_0, \infty)$. From the expression for $y(t)$ we obtain that $\lim_{t \to 0^+} y(t) = 0$ and $\lim_{t \to \infty} y(t) = -\infty$. Thus $y(t)$ has a zero at $t^* > t_0$, $y(t)$ is positive on $(0, t^*)$ and $y(t)$ is negative on $(t^*, \infty)$. Since $y(t) = \frac{\partial E(u^t)}{\partial t} = \frac{Q(u^t)}{t}$, (ii) and (iv) are shown. For (iii), we first let $Q(u) < 0$. Then
\[ 0 > Q(u) = \frac{Q(u^1)}{1} = y(1), \]
which is only possible if $t^* < 1$. Conversely, let $t^* < 1$. Then
\[ Q(u) = y(1) < y(t^*) < 0. \]

This completes the proof of (iii). To see (v), we use that
\[ E(u^t) = E(u^*) + \int_t^{t^*} y(s) \, ds. \]
Then (v) follows from the fact that $y(t)$ is positive on $(0, t^*)$ and negative on $(t^*, \infty)$.

\[ \square \]

2.4 Palais-Smale sequences with vanishing virial

In this subsection we prove the existence of a bounded Palais-Smale (PS) sequence. Since $E(u)$ is unbounded below on $S(c)$ (which can be easily verified using Hölder’s inequality), the boundedness of a PS-sequence does not directly follow from the mountain pass geometry. Nevertheless, by Lemma 2.2 we will be seeking optimizers on the manifold $V(c)$ containing functions $u$ with vanishing virial $Q(u)$, from which the boundedness of the PS-sequence follows. Hence, the problem reduces to finding a PS-sequence with vanishing virial. To show this, we firstly introduce the following definition of homotopy-stable family:

Definition 2.4 (Homotopy-stable family, [21, Def. 3.1]). Let $B$ be a closed subset of a metric space $X$. We say that a class $\mathcal{F}$ of compact subsets of $X$ is a homotopy-stable family with closed boundary $B$ if

(i) $B$ is contained in every set in $\mathcal{F}$ contains $B$;

(ii) For any $A \in \mathcal{F}$ and any $\eta \in C([0,1] \times X, X)$ satisfying $\eta(t, x) = x$ for all $(t, x) \in \{(0) \times X \cup ([0,1] \times B)$, we have $\eta(1) \times A) \in \mathcal{F}$. If $B$ is empty, we call $\mathcal{F}$ a homotopy-stable family without boundary.

Lemma 2.5 (Existence of PS-sequence with vanishing virial). For each $c > 0$ there exists $s$ PS-sequence $(u_n)_n \subset S(c)$ with vanishing virial, in other words $(u_n)_n$ satisfies
\[ E(u_n) = \gamma(c) + o(1), \]
\[ E'(u_n) = o(1), \]
\[ \text{dist}(u_n, V(c)) = o(1) \]
with $o(1) = o_n(1)$ as $n \to \infty$.

Proof. We define
\[ X := S(c), \]
\[ \mathcal{F} := \{\{u\} : u \in S(c)\}, \]
\[ B := \emptyset. \]
It follows directly that $\mathcal{F}$ is a homotopy-stable family without boundary. Next, we define
\[ \varphi(u) = E(u^t). \]
Then
\[
\inf_{A \in \mathcal{F}} \max_{u \in A} \varphi(u) = \inf_{u \in S(c)} E(u^*) =: \gamma_2(c).
\]
We show that \(\gamma(c) = \gamma_2(c)\). Since \(u^* \in V(c)\), it follows that \(\gamma_2(c) \geq \gamma(c)\). On the other hand, if \(u \in V(c)\), then
\[
E(u) \geq \inf_{v \in V(c)} E(v) = \inf_{v \in V(c)} E(v^*) \geq \inf_{u \in S(c)} E(v^*) = \gamma_2(c),
\]
which implies that \(\gamma(c) \geq \gamma_2(c)\). It is also standard to check that \(S(c)\) is a \(C^1\)-manifold and \(\varphi\) is a \(C^1\) functional on \(S(c)\). Thus the claim follows from [21, Thm. 3.2] by setting \(A_n = \{u_n\} \in \mathcal{F}\) therein, where \((u_n)_n \subset V(c)\) is a minimizing sequence, i.e. \(E(u_n) = \gamma(c) + o(1)\).

The following corollary is an immediate consequence of Lemma 2.5. The proof is standard and we refer to [10, Prop. 4.1] for related arguments.

**Corollary 2.6.** Let \(c > 0\) and \((u_n)_n \subset S(c)\) be the bounded Palais-Smale sequence constructed in Lemma 2.5. Then there exist \(u \in H^1(\mathbb{R}^3, \mathbb{C}), \beta \in \mathbb{R}\), a (not relabeled) subsequence \((u_n)_n\) and a sequence \((\beta_n)_n \subset \mathbb{R}\) such that:

1. \(u_n \rightharpoonup u\) in \(H^1(\mathbb{R}^3)\).
2. \(\beta_n \to \beta\) in \(\mathbb{R}\).
3. \(-\Delta u_n + \beta_n u_n + \lambda_1 |u_n|^2 u_n + \lambda_2 (K * |u_n|^2) u_n - |u_n|^4 u_n \to 0\) in \(H^{-1}(\mathbb{R}^3)\).
4. \(-\Delta u + \beta u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u - |u|^4 u = 0\) in \(H^{-1}(\mathbb{R}^3)\).

### 2.5 Nonvanishing weak limit

We recall that from Corollary 2.6 we have found a function \(u\) that will solve (1.2) for some \(\beta \in \mathbb{R}\). However, it is *a priori* unclear whether \(u\) is non-vanishing. We show that this is indeed the case when \(\gamma(c)\) is strictly smaller than \(\frac{S^2}{3}\). The original proof of Soave [31] relied on radial symmetry arguments\(^4\) and is not applicable in our case. We will remove this restriction using the \(pqr\)-lemma from [20, Lem. 2.1].

**Lemma 2.7.** Let \(c > 0\). Suppose that the mountain pass level \(\gamma(c)\) of the PS-sequence \((u_n)\) given by Lemma 2.5 satisfies

\[
\gamma(c) < \frac{S^2}{3} \quad \text{and} \quad \gamma(c) \neq 0.
\]

Then \((u_n)_n\) has a nonzero weak limit \(u\) in \(H^1(\mathbb{R}^3)\).

**Proof.** By Corollary 2.6 we know that \((u_n)_n\) is a bounded sequence in \(H^1(\mathbb{R}^3)\), hence also in \(L^p(\mathbb{R}^3)\) for all \(p \in [2, 6]\). We can then distinguish between two cases:

(i) \(\|u_n\|_4 = o(1)\). Then the claim follows from [31, Lem. 3.3].

(ii) \(\|u_n\|_4 \neq o(1)\). Then the claim follows from the \(pqr\)-lemma ([20, Lem. 2.1]) and the Lieb-translation ([26, Lem. 6]).

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\(^4\)In the original proof of Soave, the condition \(\|u_n\|_4 = o(1)\) was in fact a consequence of the Strauss’ compactness lemma for radial functions.
2.6 A qualitative description of the mapping $c \mapsto \gamma(c)$

We first show that the number $\gamma(c)$ is always positive and will never exceed $\frac{S^2}{3}$.

**Lemma 2.8.** For all $c > 0$ we have $\gamma(c) \in (0, \frac{S^2}{3}]$.

**Proof.** Let $u \in V(c)$. Using the Gagliardo-Nirenberg and Sobolev inequalities we obtain that

$$\|\nabla u\|_2^2 = A(u) = -\frac{3}{4} B(u) + C(u) \leq C(c^*\|\nabla u\|_2^3 + \|\nabla u\|_6^6).$$

Since $c \neq 0$ we can divide by $\|\nabla u\|_2^2$ and modify the constants to obtain that

$$\inf_{u \in V(c)} \|\nabla u\|_2 > 0.$$ 

Since

$$E(u) = E(u) - \frac{1}{3}Q(u) = \frac{1}{6} (A(u) + C(u)) \geq \frac{1}{6} A(u) = \frac{1}{6} \|\nabla u\|_2^2$$

for $u \in V(c)$, the lower bound follows by taking infimum over $V(c)$ on both sides. It is left to show the upper bound. We define

$$\tilde{E}(u) := \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{4} \lambda_1 \|u\|_4^4 - \frac{1}{6} \|u\|_6^6,$$

$$\tilde{Q}(u) := \|\nabla u\|_2^2 + \frac{3}{4} \lambda_1 \|u\|_4^4 - \|u\|_6^6.$$ 

Then $\tilde{E}(u)$ and $\tilde{Q}(u)$ are the energy and virial corresponding to the equation

$$-\Delta u + \beta u + \lambda_1 |u|^2 u - |u|^4 u = 0 \quad (2.7)$$

respectively. For a radially symmetric $u$ we have

$$\tilde{E}(u) := E(u),$$

$$\tilde{Q}(u) := Q(u).$$

Now we define

$$u_\varepsilon(x) := \varphi(x) \cdot \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{3}{4}}, \quad (2.8)$$

$$v_\varepsilon := c^{\frac{3}{2}} \frac{u_\varepsilon}{\|u_\varepsilon\|_2}. \quad (2.9)$$

where $\varphi \in C^\infty_c(\mathbb{R}^3)$ is a real radial cut-off function with $c \equiv 1$ in $B_1$, $\phi \equiv 0$ in $B_2$ and $\phi$ is radially decreasing. Denote by $I^*$ the number introduced in Lemma 2.3, but corresponding to (2.7). Due to [31, Lem. 6.4] there exist $C_1, C_2 > 0$ such that

$$\tilde{E}(v_\varepsilon^*) \leq \frac{S^2}{3} + C_1 \varepsilon^{\frac{3}{2}} + C_2 \varepsilon = \frac{S^2}{3} + O(\varepsilon) \quad (2.10)$$

for all sufficiently small $\varepsilon$. Since $v_\varepsilon^*$ is radially symmetric, using (2.8) and (2.9) we obtain that $v_\varepsilon^* \in V(c)$ and $E(v_\varepsilon^*) = \tilde{E}(v_\varepsilon^*)$. The upper bound follows by the definition of $\gamma(c)$. 

**Lemma 2.9.** The curve $c \mapsto \gamma(c)$ is non-increasing and continuous on $(0, \infty)$.

**Proof.** The proof is standard, see for instance [10].

The following lemma is fundamental for the proof of Theorem 1.2 (ii).

**Lemma 2.10.** Let $(\lambda_1, \lambda_2)$ be unstable. Then $\gamma(c) \to 0$ as $c \to \infty$. 

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Proof. By [9, Remark 4.1] we have that the set

\[ \{ u \in H^1(\mathbb{R}^3) : B(u) < 0 \} \]

is not empty. Assuming \( B(u) < 0 \) and using the Fourier transform and the Gagliardo-Nierenberg inequality, we estimate

\[ -B(u) \leq \frac{\|u\|^4}{(2\pi)^3} \max \left\{ \left| \lambda_1 - \lambda_2 \frac{4\pi}{3} \right|, \left| \lambda_1 + \lambda_2 \frac{8\pi}{3} \right| \right\} \leq \frac{\|u\|_2 \|\nabla u\|_3}{(2\pi)^3} \max \left\{ \left| \lambda_1 - \lambda_2 \frac{4\pi}{3} \right|, \left| \lambda_1 + \lambda_2 \frac{8\pi}{3} \right| \right\}, \]

so that

\[ C_{GN} := \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}, B(u) < 0} \frac{\|u\|_2 \|\nabla u\|_2^3}{-B(u)} > 0. \tag{2.11} \]

Let \((u_n)_n\) be a minimizing sequence for (2.11). The rescaling \( u(x) \to q u(s x) \) leaves the minimized quantity in (2.11) invariant, so we may assume that

\[ \|u_n\|_2 = n, \quad -B(u_n) = 1, \]

and, therefore,

\[ \|\nabla u_n\|_2 = n^{-\frac{1}{4}} (C_{GN} + o(1))^{-\frac{1}{4}}. \]

Using Lemma 2.3 we find \( \theta_n \in (0, \infty) \) such that \( Q(u_n^{\theta_n}) = 0 \). This is equivalent to

\[ 0 = \theta_n^2 \|\nabla u_n\|_2^2 + \frac{3}{4} \theta_n^4 B(u_n) - \theta_n^6 \|u_n\|_6^6 \]
\[ = \theta_n^2 n^{-\frac{2}{3}} (C_{GN} + o(1))^{-\frac{2}{3}} - \frac{3}{4} \theta_n^4 n^{-\frac{2}{3}} - \theta_n^6 \|u_n\|_6^6, \]

which implies

\[ \theta_n^2 \|u_n\|_6^6 = n^{-\frac{2}{3}} (C_{GN} + o(1))^{-\frac{2}{3}} - \frac{3}{4} \theta_n^4 n^{-\frac{2}{3}} - \theta_n^6 \|u_n\|_6^6. \tag{2.12} \]

If either \( \liminf \theta_n > 0 \) or \( \limsup \theta_n > 0 \), then for sufficiently large \( n \), the right hand side of (2.12) will be negative, while the left hand side is always positive, which is a contradiction. Thus \( \lim \theta_n = 0 \). Consequently,

\[ \gamma_n^2 \leq E(u_n^{\theta_n}) \]
\[ = \frac{1}{2} \theta_n^2 n^{-\frac{2}{3}} (C_{GN} + o(1))^{-\frac{2}{3}} - \frac{1}{4} \theta_n^4 n^{-\frac{2}{3}} - \frac{1}{6} \theta_n^6 \|u_n\|_6^6 \]
\[ = \frac{1}{2} \theta_n^2 n^{-\frac{2}{3}} (C_{GN} + o(1))^{-\frac{2}{3}} - \frac{1}{4} \theta_n^4 n^{-\frac{2}{3}} - \frac{1}{6} \theta_n^2 n^{-\frac{2}{3}} (C_{GN} + o(1))^{-\frac{2}{3}} = o(1). \]

Combining that with the non-increasing monotonicity of \( c \mapsto \gamma(c) \), proved in Lemma 2.9, completes the proof.

\[ \square \]

3 Proof of Theorem 1.2

Having all the preliminaries we are in the position to prove Theorem 1.2. We will firstly show the existence of ground states as long as \( \gamma(c) \) is strictly smaller than \( \frac{\sqrt{2}}{3} \) (Lemma 3.1). Afterwards, we show that this assumption is guaranteed by the conditions given in Theorem 1.2. It is worth noting that the proof of Lemma 3.1 provides with an alternative for the ones from [31] without any use of radial symmetry arguments, which may be of independent interest.

**Lemma 3.1.** Let \( (\lambda_1, \lambda_2) \) be unstable and let \( c \in (0, \infty) \). If \( \gamma(c) \in \left(0, \frac{\sqrt{2}}{3}\right) \), then (1.5) has at least an optimizer \( u_c \in \mathcal{V}_c \).
Proof. Let \((u_n, \beta_n)_n\) and \((u, \beta)\) be the PS-sequence and its limit deduced from Corollary 2.6. Using the splitting properties given by [12, Theorem 1] for \(L^p\)-norms and by [2] for \(B(u)\) we obtain that
\[
\begin{align*}
A(u_n - u) + A(u) &= A(u_n) + o(1), \\
B(u_n - u) + B(u) &= B(u_n) + o(1), \\
C(u_n - u) + C(u) &= C(u_n) + o(1), \\
D(u_n - u) + D(u) &= D(u_n) + o(1),
\end{align*}
\]
where \(D(v) := \|v\|_2^2\). Since \(E(u) = \frac{1}{2}A(u) + \frac{1}{4}B(u) - \frac{1}{6}C(u)\), we see that
\[
E(u_n - u) + E(u) = E(u_n) + o(1).
\]
From the lower semicontinuity of the \(L^2\)-norm we obtain that
\[
\|u\|_2^2 \leq \liminf_{n \to \infty} \|u_n\|_2^2 = c.
\]
By Lemma 2.7, we know that \(u \neq 0\). Consequently, we infer that \(u \in V(c_1)\) for some \(c_1 \in (0, c]\). We also have \(\gamma(c) = E(u_n) + o(1)\) from Lemma 2.6. Hence
\[
E(u_n - u) + \gamma(c_1) \leq E(u_n) + E(u) = E(u_n) + o(1) = \gamma(c) + o(1).
\]
On the other hand, direct calculation results in
\[
-Q(v) + 3E(v) = \frac{1}{2}A(v) + \frac{1}{2}C(v)
\]
for all \(v \in H^1(\mathbb{R}^3)\). Using \(Q(u) = 0\) and \(Q(u_n) = o(1)\) from Lemma 2.6 we obtain that
\[
Q(u_n - u) = Q(u_n) = Q(u) = Q(u_n) + o(1) = o(1).
\]
Inserting this into (3.2), we conclude that \(E(u_n - u) \geq o(1)\), since the right-hand side of (3.2) is always nonnegative. From Lemma 2.9 we know that \(\gamma(c_1) \geq \gamma(c)\), therefore it follows from (3.1) that \(E(u_n - u) \leq o(1)\). Thus \(E(u_n - u) = o(1)\). Since \(A(\cdot)\) and \(C(\cdot)\) are nonnegative, we obtain from \(E(u_n - u) = o(1)\), \(Q(u_n - u) = o(1)\) and (3.2) that
\[
\begin{align*}
A(u_n - u) &= o(1), \\
C(u_n - u) &= o(1).
\end{align*}
\]
But \(E(u_n - u)\) is a linear combination of \(A(u_n - u), B(u_n - u)\) and \(C(u_n - u)\), it follows immediately that \(B(u_n - u) = o(1)\). Now Corollary 2.6 implies
\[
\frac{1}{2}A(u_n) + \beta D(u_n) + B(u_n) + C(u_n) = \frac{1}{2}A(u) + \beta D(u) + B(u) + C(u) + o(1).
\]
Using the previous splitting properties one has then
\[
\beta D(u_n) = \beta D(u) + o(1) = \beta(D(u_n) - D(u_n - u) + o(1)).
\]
From this we infer that \(\beta D(u_n - u) = o(1)\). Let us now show that \(\beta > 0\). Due to (2.2) it is equivalently to show that \(B(u) < 0\). Suppose in contrast that \(B(u) \geq 0\). Since
\[
0 = Q(u) = A(u) + \frac{3}{4}B(u) - C(u)
\]
it follows that \(A(u) \leq C(u)\), or in other words \(\|\nabla u\|_2^2 \leq \|u\|_6^6\). Together with the Sobolev inequality \(S\|u\|_6^6 \leq \|\nabla u\|_2^2\) we obtain that
\[
S^2 \leq \|\nabla u\|_2^2.
\]
Thus
\[ E(u) = E(u) - \frac{1}{6} Q(u) = \frac{1}{3} A(u) + \frac{1}{8} B(u) \geq \frac{1}{3} A(u) \geq \frac{S^2}{3}. \]

Since \( E(u_n - u) = o(1) \) and \( E(u_n) = \gamma(c) + o(1) \), from the splitting property it follows that \( \gamma(c) = E(u) \geq \frac{S^2}{3} \). But \( \gamma(c) \in (0, \frac{S^2}{3}) \) according to our assumption, hence we obtain a contradiction and \( B(u) < 0 \).

Therefore \( D(u_n - u) = o(1) \). Together with \( A(u_n - u) = o(1) \) we infer that \( u_n \to u \) in \( H^1(\mathbb{R}^3) \) and \( u \in S(c) \). Using now Lemma 2.2 the proof is complete.

**Proof of Theorem 1.2.** We begin with the proof of (iv). In fact, by the definition of a stable pair, we immediately see that \( B(u) \) is always positive for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \). Hence the proof of (iv) follows the one for [31, Thm. 1.2. 2] verbatim, and we omit the details here. For (i), we note that according to [31, Lem. 6.4], (2.10) can in fact be modified to the version that there exist \( C_1, C_2 > 0 \) such that

\[ \tilde{E}(u_n^\epsilon) \leq \frac{S^2}{3} - C_1 \epsilon^2 + C_2 \epsilon < \frac{S^2}{3} \]

for all sufficiently small \( \epsilon \). Now (i) follows from Lemma 3.1.

(ii) is a direct consequence of Lemma 2.10 and Lemma 3.1 by defining

\[ c_* := \inf \left\{ c > 0 : \gamma(c) < \frac{S^2}{3} \right\}. \]

That the ground states are in fact solutions of (1.2) with positive \( \beta \) follows from Lemma 2.2 and the proof of Lemma 3.1; that the solutions of (1.2) are positive follows directly from the strong maximum principle. Finally, we note that in the case of (iii), if a solution of (1.2) is radial, then it must be a solution of (1.7) with some \( \lambda_1 \geq 0 \). However, this is impossible due to [31, Thm. 1.2. 2]]. This finishes the desired proof.

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