GLOBAL SMOOTH SOLUTION FOR THE SIPN-POLARIZED TRANSPORT EQUATION WITH LANDAU-LIFSHITZ-BLOCH EQUATION

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Abstract. The Landau-Lifshitz-Bloch equation is often used to describe micromagnetic phenomenon under high temperature. In this paper, we establish the existence and uniqueness of global smooth solution for the initial problem of the spin-polarized transport equation with Landau-Lifshitz-Bloch equation in dimension two.

1. Introduction. It is well known that the Landau-Lifshitz equation has a fundamental importance in understanding the ferromagnetism in materials, which describes the magnetization dynamics of ferromagnets at low temperature[7]. The Landau-Lifshitz equation is investigated widely. Many important and interesting results have been obtained, see [5]. At temperatures below the critical (so-called Curie) temperature, the magnetization vector $Z(x,t) = (Z_1(x,t), Z_2(x,t), Z_3(x,t)) \in S^2$ satisfies the following Landau-Lifshitz-Gilbert equation

$$Z_t = Z \times \Delta Z - \lambda Z \times (Z \times \Delta Z),$$

where $\lambda > 0$ is the Gilbert coefficients. “$\times$” denotes the vector outer product. However, for high temperatures the model must be replaced by a more thermodynamically consistent approach, so in order to describe the dynamics of magnetization vector $Z$ in a ferromagnetic body for a wide range of temperatures, Garanin derived the Landau-Lifshitz-Bloch equation from statistical mechanics with the mean field approximation in [3, 4]. The Landau-Lifshitz-Bloch equation essentially interpolates between the Landau-Lifshitz-Gilbert equation at low temperatures and the Ginzburg-Landau theory of phase transitions. It is valid not only below but also above the Curie temperature.

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The Landau-Lifshitz-Bloch equation is given as follows

\[ u_t = -\gamma \times H_{\text{eff}} + \frac{L_1}{|u|^2} (u \cdot H_{\text{eff}}) u - \frac{L_2}{|u|^2} u \times (u \times H_{\text{eff}}), \]

where \( u = (u_1, u_2, u_3) \), \( \gamma \), \( L_1 \), \( L_2 \) are constants, \( H_{\text{eff}} \) is the effective field. We can also rewrite 2 as follows

\[ u_t = -\gamma \times H_{\text{eff}} + \gamma a_\parallel \frac{|u|^2}{|u|^2} (H_{\text{eff}} \cdot u) u - \gamma a_\perp \frac{|u|^2}{|u|^2} u \times (u \times H_{\text{eff}}) \]

with \( \gamma a_\parallel = L_1 \), \( \gamma a_\perp = L_2 \). Here \( a_\parallel \) and \( a_\perp \) are dimensionless damping parameters that depend on the temperatures and are defined as follows [1]

\[ a_\parallel(\theta) = \frac{2\theta}{3\theta_c} \lambda, \quad a_\perp(\theta) = \begin{cases} \lambda \left(1 - \frac{\theta}{3\theta_c}\right), & \text{if } \theta < \theta_c, \\ a_\parallel(\theta), & \text{if } \theta \geq \theta_c, \end{cases} \]

where \( \lambda > 0 \) is a constant.

When the effective field \( H_{\text{eff}} \) is given by

\[ H_{\text{eff}} = \Delta u - \frac{1}{\chi_\parallel} \left(1 + \frac{3}{5} \frac{T}{T_c} |u|^2\right) u, \]

where \( \chi_\parallel \) is the longitudinal susceptibility, if \( L_1 = L_2 \), 2 can be reduced as follows [8]

\[ u_t = k_1 \Delta u + \gamma u \times \Delta u - k_2 (1 + \mu |u|^2) u, \]

where the coefficients \( k_1, k_2, \gamma, \mu > 0 \) and the existence of global weak solution for the equation 2 has been obtained.

In 2007, Garcia-Cervera and Wang [2] consider the weak solutions of the following spin-polarized transport equations in materials

\[ \frac{\partial s}{\partial t} = -\text{div} J_s - D_0(x) s - D_0(x) s \times m, \]
\[ \frac{\partial m}{\partial t} = -m \times (h + s) + \alpha m \times \frac{\partial m}{\partial t}, \]
\[ s(x, 0) = s_0(x), \quad m_0(x, 0) = m_0(x), \]

where \((s, m)\) is the unknown, \( s = (s_1, s_2, s_3) : \Omega \rightarrow \mathbb{R}^3 \) denotes the spin accumulation and \( m = (m_1, m_2, m_3) : \Omega \rightarrow \mathbb{R}^3 \) is the magnetization field, \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( J_s \) is the spin current,

\[ J_s = m \otimes J_c - D_0(x) [\nabla s - \beta m \otimes (\nabla s \cdot m)], \]

\( J_c \) is the applied electric current, \( 0 < \beta < 1 \) is the spin polarization parameter and \( D_0(x) \) is the diffusion parameter depending on the material. In the second equation of 5, \( h = -\nabla_m \Phi + h_d + \Delta m \) denotes the anisotropy, exchange and self-induced energy. In [6], the authors considered the periodic boundary conditions, assumed that diffusion material \( D_0(x) \equiv 1 \), and ignored the anisotropic energy \( \nabla_m \Phi \) and self-induced \( h_d \). Under these assumptions and simplifications, they established the existence of global smooth solutions of the spin polarized transport equation of problem 5.
In this paper, we consider the following spin polarized transport equation with Landau-Lifshitz-Bloch equation

\[ u_t = k_1 \Delta u + \gamma u \times (\Delta u + s) - k_2 (1 + \mu |u|^2)u, \]  
\[ \frac{\partial s}{\partial t} = -\text{div} J_s - s - s \times u, \]  
\[ u_0(x, 0) = u_0(x), \quad s(x, 0) = s_0(x), \]  

where \( J_s = u \otimes J_e - [\nabla s - \beta u \otimes (\nabla s \cdot u)] \) and \( J_e \) is a known function of variable \( x \), the constants \( k_1, k_2, \gamma, \mu, \beta > 0 \). Without loss of generality, we assume \( k_1 = \gamma = 1, k_2 = k \) in the systems. In the sequel, we will concentrate on the existence and uniqueness of global smooth solution for the initial problem (7)-(9).

The main result of this paper is as follows:

**Theorem 1.1.** Let \( u_0(x) \in H^m(\mathbb{R}^2), s_0 \in H^m(\mathbb{R}^2), J_e(x) \in H^m(\mathbb{R}^2) (m \geq 2) \) and \( \beta \| u_0 \|^2_\infty \ll 1 \), then for any \( T > 0 \), there exists a unique smooth solution \((u, s)\) of problem 7-9 satisfying

\[ \partial_t^j \partial_x^\alpha u \in L^\infty([0, T]; L^2(\mathbb{R}^2)), \quad \partial_t^j \partial_x^\alpha u \in L^2([0, T]; L^2(\mathbb{R}^2)), \]
\[ \partial_t^j \partial_x^\alpha s \in L^\infty([0, T]; L^2(\mathbb{R}^2)), \quad \partial_t^j \partial_x^\alpha s \in L^2([0, T]; L^2(\mathbb{R}^2)), \]

where \( 2j + |\alpha| \leq m \) and \( 2k + \beta \leq m + 1 \).

The rest of this paper is as follows. In section 2, we first give the proof of smooth local solution of 7-9 in \( \mathbb{R}^2 \), and then give a priori uniform estimates of the global smooth solution of 7-9, finally, the existence and uniqueness of the global smooth solution are proved.

2. **The Proof of Theorem 1.1.** From [9] it can be shown that there exists \( T > 0 \) and a smooth solution of problem 7-9 in \([0, T]\). Indeed, it is easy to check that \( e^{t\Delta} \) is the analytic semigroup generated by \( \Delta \) in \( L^2(\mathbb{R}^2) \), let

\[ X = \{ w|w \in C([0, T]; H^m(\mathbb{R}^2)), \quad t^\alpha w \in C^\alpha([0, T]; H^m(\mathbb{R}^2)), \quad w(0) = w_0 \} \]
and

\[ Y = \{ w|w \in X, \quad \|w\|_{C([0, T]; H^m(\mathbb{R}^2))} + \|t^\alpha w\|_{C^\alpha([0, T]; H^m(\mathbb{R}^2))} \leq \delta \} \]

with \( 0 < \alpha < 1 \) and \( m \geq 2 \), \( w = (u, s)^T \). Define a nonlinear operator \( \Gamma \) on \( Y \) by \( \Gamma(w) = v \), where \( v = (v_1, v_2)^T \) is the solution of

\[
\begin{align*}
\begin{cases}
v_{1t} = \Delta v_1 + u \times (\Delta u + s) - k(1 + \mu |u|^2)u, \quad v_1(0) = u_0, \\
v_{2t} = \Delta v_2 - \text{div}(u \otimes J_e) - \text{div}\beta[u \otimes (\nabla s \cdot u)], \quad v_2(0) = s_0.
\end{cases}
\end{align*}
\]

By Theorem 4.3.5 of reference [9] (pp. 137-139), for every \( w \in Y, \Gamma(w) \in C([0, T]; H^m(\mathbb{R}^2)) \) and \( t^\alpha \Gamma(w) \in C^\alpha([0, T]; H^m(\mathbb{R}^2)) \), then, using the same arguments as in the proof of Theorem 8.1.1 (pp. 290-294), there exists \( T > 0 \) and \( \delta > 0 \) such that \( \Gamma: Y \rightarrow Y \) is a contraction i.e., there exists a unique smooth local solution of problem 7-9. In order to prove Theorem 1.1, it suffices to give a priori estimates for the smooth solution of problem 7-9.

The following Gagliardo-Nirenberg inequality will be used many times.
Lemma 2.1. (Gagliardo-Nirenberg Inequality) Assume that $u \in L^q(\Omega), D^m u \in L^r(\Omega), \Omega \subset \mathbb{R}^n, 1 \leq q, r \leq \infty, 0 \leq j \leq m$. Then
\[ \|D^j u\|_{L^p(\Omega)} \leq C(j, m; p, r, q) \|u\|_{W^{m, p}(\Omega)}^{a} \|u\|_{L^r(\Omega)}^{1-a}, \]
where $C(j, m; p, r, q)$ is a positive constant, and
\[ \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a)\frac{1}{q}, \quad \frac{j}{m} \leq a \leq 1. \]

Lemma 2.2. Let the initial data $u_0 \in L^\infty$, then for the smooth solution of problem 7-9, we have
\[ \|u(\cdot, t)\|_{L^\infty} \leq \|u_0(x)\|_{L^\infty}, \quad \forall t \geq 0. \]

Proof. Taking the scalar product of $|u|^{p-2} u$ with equation 7, and integrating the result over $\mathbb{R}^2$, we get
\[
\int_{\mathbb{R}^2} |u|^{p-2} u \cdot u_t dx = \int_{\mathbb{R}^2} |u|^{p-2} u \cdot \Delta u dx + \int_{\mathbb{R}^2} |u|^{p-2} u \cdot (\Delta u + s) dx
- k \int_{\mathbb{R}^2} |u|^{p-2} u \cdot (1 + \mu |u|^2) u dx
\leq - \int_{\mathbb{R}^2} |u|^{p-2} \nabla u \cdot \nabla u dx - (p - 2) \int_{\mathbb{R}^2} |u|^{p-4} (u \cdot \nabla u)^2 dx
\leq 0,
\]
so we have
\[ \frac{1}{p} \frac{d}{dt} \|u(\cdot, t)\|_{L^p}^{p} \leq 0, \]
which implies
\[ \|u(\cdot, t)\|_{L^p} \leq \|u_0(x)\|_{L^p}, \quad \forall t \geq 0, \]
let $p \to \infty$, estimate 13 is obtained. \qed

For simplicity, we denote $\| \cdot \|_{L^p} = \| \cdot \|_{p}, \quad p \geq 2$.

Lemma 2.3. Let $d = 2, u_0(x) \in L^2, s_0(x) \in L^2, j_0(x) \in H^m (m \geq 2)$ and $\beta \|u_0\|^2 \ll 1$, then for the smooth solution of problem 7-9, we have the following estimate
\[ \|u(\cdot, t)\|^2 + \|s(\cdot, t)\|^2 + \int_0^t (\|\nabla u\|^2 + \|\nabla s\|^2) dx \leq C, \quad \forall t \geq 0, \]
where the constant $C$ depends on $k, \mu, \|u_0\|^2$ and $\|s_0\|^2$.

Proof. Taking the scalar product of $u$ with equation 7, and then integrating the result over $\mathbb{R}^2$, we have
\[ \frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \|\nabla u(\cdot, t)\|^2 + k \int_{\mathbb{R}^2} (1 + \mu |u|^2) |u|^2 dx = 0, \]
where $k > 0, \mu > 0$. \qed
Lemma 2.4. Let \( u_0(x) \in H^m, s_0(x) \in H^m, j_c(x) \in H^m, (m \geq 2) \) and \( \beta \|u_0\|_\infty \ll 1 \), then for the smooth solution of problem 7-9, we have

\[
\|s(\cdot, t)\|_2^2 + \|s(\cdot, t)\|_2^2 + \|s(\cdot, t)\|_2^2 \leq C,
\]

where the constant \( C \) depends on \( \|u_0(x)\|_2^2 \), and we have used H"older’s inequality and \( \beta \|u_0\|_\infty \ll 1 \).

Summing 16 with 17, we get

\[
\frac{d}{dt} \left\{ \|u(\cdot, t)\|_2^2 + \|s(\cdot, t)\|_2^2 \right\} + \|\nabla u(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 \leq C,
\]

then by Gronwall’s inequality, we find the estimate 15, then the proof of Lemma 2.3 is completed. \( \square \)

Lemma 2.4. Let \( u_0(x) \in H^m, s_0(x) \in H^m, j_c(x) \in H^m, (m \geq 2) \) and \( \beta \|u_0\|_\infty \ll 1 \), then for the smooth solution of problem 7-9, we have

\[
\|\nabla u(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 + \int_0^t \left\{ \|\Delta u(\cdot, t)\|_2^2 + \|\Delta s(\cdot, t)\|_2^2 \right\} dt \leq C(\|u_0(x)\|_{H^1}, \|s_0(x)\|_{H^1}).
\]

and

\[
\|\Delta u(\cdot, t)\|_2^2 + \|\Delta s(\cdot, t)\|_2^2 + \int_0^t \left\{ \|\nabla \Delta u(\cdot, t)\|_2^2 + \|\nabla \Delta s(\cdot, t)\|_2^2 \right\} dt \leq C(\|u_0(x)\|_{H^2}, \|s_0(x)\|_{H^2}).
\]

Proof. Taking the scalar product of \( \Delta u \) with equation 7 and then integrating the result over \( \mathbb{R}^2 \), we have

\[
\int_{\mathbb{R}^2} u_t \cdot \Delta u dx = \int_{\mathbb{R}^2} \Delta u \cdot \Delta u dx + \int_{\mathbb{R}^2} u \times s \cdot \Delta u dx
\]

\[
- k \int_{\mathbb{R}^2} (1 + \mu|u|^2) u \cdot \Delta u dx,
\]

where

\[
-k \int_{\mathbb{R}^2} (1 + \mu|u|^2) u \cdot \Delta u dx = k\|\nabla u\|_2^2 + \int_{\mathbb{R}^2} k\mu|u|^2 \nabla u \cdot \nabla u dx
\]

\[
+ \int_{\mathbb{R}^2} k\mu \nabla u^2 \cdot \nabla u^2 dx.
\]
By Hölder inequality, we have
\[ \int \mathbb{R}^2 u \times s \cdot \Delta u \, dx \leq \|u\|_\infty \|s\|_2 \|\Delta u\|_2, \]
then by Gagliardo-Nirenberg inequality, we get
\[ \frac{1}{2} \frac{d}{dt} \|\nabla u(\cdot, t)\|_2^2 + \|\Delta u(\cdot, t)\|_2^2 + k \|\nabla u(\cdot, t)\|_2^2 \]
\[ \leq \|u\|_\infty \|s\|_2 \|\Delta u\|_2 \leq \|s\|_2 \|\nabla \|_2 \|\Delta u\|_2^2 \]
\[ \leq \frac{1}{2} \|\Delta u(\cdot, t)\|_2^2 + C(\|s_0\|_2^2 + \|u_0\|_2^2) \|\Delta u\|_2^2, \]
where we have used the estimate 15. Thus we get
\[ \frac{d}{dt} \|\nabla u(\cdot, t)\|_2^2 + \|\Delta u(\cdot, t)\|_2^2 + k \|\nabla u(\cdot, t)\|_2^2 \leq C(\|s_0\|_2^2 + \|u_0\|_2^2) \|\nabla u\|_2^2, \]  \tag{21}
then by Gronwall’s inequality, we have
\[ \|\nabla u(\cdot, t)\|_2^2 + \int_0^t (\|\Delta u(\cdot, t)\|_2^2 + k \|\nabla u(\cdot, t)\|_2^2) \, dt \leq C. \tag{22} \]
Taking the scalar product of \(\Delta^2 u\) with equation 7, and then integrating the result over \(\mathbb{R}^2\), we have
\[ \int \mathbb{R}^2 u_t \cdot \Delta^2 u \, dx = \int \mathbb{R}^2 \Delta u \cdot \Delta^2 u \, dx + \int \mathbb{R}^2 u \times (\Delta u + s) \cdot \Delta^2 u \, dx \]
\[ - k \int \mathbb{R}^2 (1 + \mu |u|^2) u \cdot \Delta^2 u \, dx. \tag{23} \]
By Gagliardo-Nirenberg inequality, we get
\[ \|\nabla u\|_\infty \leq C \|\nabla u\|_1 \mathbb{R}^2 \|\nabla u\|_2^\frac{1}{2}. \]
Thus, by Hölder inequality, the estimate 15 and 22, we have
\[ \int \mathbb{R}^2 u \times (\Delta u + s) \cdot \Delta^2 u \, dx \]
\[ = - \int \mathbb{R}^2 \nabla u \times (\Delta u + s) \cdot \nabla \Delta u \, dx - \int \mathbb{R}^2 u \times \nabla s \cdot \nabla \Delta u \, dx \]
\[ \leq \|\nabla u\|_\infty \left( \|\Delta u\|_2 + \|s\|_2 \right) \|\nabla \Delta u\|_2 + \|\Delta u\|_\infty \|\nabla s\|_2 \|\nabla \Delta u\|_2 \]
\[ \leq \frac{1}{6} \|\nabla \Delta u\|_2^2 + C(1 + \|\Delta u\|_2^2) \]
\[ \leq \frac{1}{3} \|\nabla \Delta u\|_2^2 + C + \|\Delta u\|_2^2 \]
\[ \leq \frac{1}{3} \|\nabla \Delta u\|_2^2 + C(1 + \|\Delta u\|_2^2) \]  \tag{24}
and
\[ \left| k \int \mathbb{R}^2 (1 + \mu |u|^2) u \cdot \Delta^2 u \, dx \right| = \left| k \int \mathbb{R}^2 (\nabla \Delta u \cdot \nabla u + \mu \nabla (|u|^2 u) \cdot \nabla \Delta u) \, dx \right| \]
\[ \leq k \|\nabla \Delta u\|_2 \|\nabla u\|_2 \left( 1 + 3\mu \|u\|_\infty^2 \right) \]
\[ \leq \frac{1}{6} \|\nabla \Delta u\|_2^2 + C(1 + \|\Delta u\|_2^2). \]  \tag{25}
where we have used Sobolev imbedding theorem $H^2 \subset L^\infty$. Combining 23, 24 and 25, we have

$$\frac{d}{dt}\|\Delta u\|_2^2 + \|\nabla \Delta u\|_2^2 \leq C(1 + \|\Delta u\|_2^2)(1 + \|\Delta u\|_2^2 + \|\nabla s\|_2^2).$$  \hfill (26)

The generalized Gronwall’s inequality says that if $f' = C(f \cdot g) + C$, $f \leq C \exp \left( \int_0^t g \, dt \right) + C$, by replacing $f$ and $g$ by $\|\Delta u\|_2^2$ and $\|\nabla s\|_2^2$ respectively, and the boundedness of $\int_0^t g \, dt$ from 15, we have

$$\|\Delta u\|_2^2 + \int_0^t \|\nabla \Delta u\|_2^2 \, dx \leq C.$$  \hfill (27)

Taking the scalar product of $\Delta s$ with equation 8, and then integrating the result over $\mathbb{R}^2$ for the space variable $x$, we have

$$\int_{\mathbb{R}^2} s_1 \cdot \Delta s \, dx = \int_{\mathbb{R}^2} \Delta s \cdot \Delta s \, dx - \int_{\mathbb{R}^2} \text{div}(u \otimes J_e(x)) \cdot \Delta s \, dx - \int_{\mathbb{R}^2} s \cdot \Delta s \, dx + \beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta s \, dx - \int_{\mathbb{R}^2} s \times u \cdot \Delta s \, dx.$$  \hfill (28)

By Hölder inequality, we have

$$- \int_{\mathbb{R}^2} \text{div}(u \times J_e(x)) \cdot \Delta s \, dx$$

$$\leq \|J_e(x)\|_\infty \|\nabla u\|_2 \|\Delta s\|_2 + \|u\|_4 \|\nabla J_e(x)\|_4 \|\Delta s\|_2$$

$$\leq \|\nabla u\|_2 \|\Delta s\|_2 \|J_e(x)\|_{H^2} + C \|u\|_{H^1} \|J_e(x)\|_{H^2} \|\Delta s\|_2$$

$$\leq C\|J_e(x)\|_{H^2}(\|\nabla u\|_2 + \|u\|_{H^1})\|\Delta s\|_2$$

$$\leq \frac{1}{6} \|\Delta s\|_2^2 + C(\|u_0\|_{H^1}),$$

where we have used the Lamma 2.3, the estimate 22 and Sobolev imbedding theorems $H^1 \subset L^4$, $H^2 \subset L^\infty$.

By using Gagliardo-Nirenberg inequality and estimates 22, 27, we obtain

$$\beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta s \, dx$$

$$= \beta \int_{\mathbb{R}^2} (\Delta s \cdot u) u \cdot \Delta s \, dx + \beta \int_{\mathbb{R}^2} (\nabla u \cdot (\nabla s \cdot u)) \cdot \Delta s \, dx$$

$$+ \beta \int_{\mathbb{R}^2} (\nabla s \cdot \nabla u) u \cdot \Delta s \, dx$$

$$\leq \beta \|u\|_{\infty}^2 \|\Delta s\|_2^2 + 2\beta \|u\|_\infty \|\nabla u\|_4 \|\nabla s\|_4 \|\Delta s\|_2$$

$$\leq \frac{1}{12} \|\Delta s\|_2^2 + C(\|\nabla u\|_{H^1} \|\Delta s\|_2) \|\Delta s\|_2$$

$$\leq \frac{1}{12} \|\Delta s\|_2^2 + C(\|\nabla \Delta s\|_2)$$

$$\leq \frac{1}{6} \|\Delta s\|_2^2 + C\|\nabla s\|_2^2,$$

where we have used the condition $\beta \|u\|_\infty \ll 1$ and Sobolev imbedding theorems $H^1 \subset L^4$. 

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Similarly, we get
\[- \int_{\mathbb{R}^2} s \times u \cdot \Delta s dx \leq \|u\|_\infty \|s\|_2 \|\Delta s\|_2 \leq C \|\Delta u\|_2 \|s\|_2 \|\Delta s\|_2 \]
\[\leq \frac{1}{6} \|\Delta s\|_2^2 + 12 \|\Delta u\|_2^2 + C. \quad (31)\]

Thus inserting estimates 29, 30, 31 into 28, we have
\[\frac{1}{2} \frac{d}{dt} \left( \|\nabla s(\cdot, t)\|_2^2 + \|\Delta s(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 \right) \]
\[\leq \int_{\mathbb{R}^2} \text{div}(u \otimes J_r(x)) \cdot \Delta s dx - \beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta s dx \]
\[- \int_{\mathbb{R}^2} s \times u \cdot \Delta s dx \]
\[\leq \frac{1}{2} \|\Delta s\|_2^2 + 12 \|\Delta u\|_2^2 + C(1 + \|\nabla s\|_2^2). \quad (32)\]

Combining 22 and 32 we have
\[\frac{d}{dt} \left( \|\nabla s(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 \right) + \|\Delta u(\cdot, t)\|_2^2 + \|\Delta s(\cdot, t)\|_2^2 \]
\[+ k \|\nabla s(\cdot, t)\|_2^2 + \|\nabla s(\cdot, t)\|_2^2 \]
\[\leq C(1 + \|\nabla s\|_2^2). \quad (33)\]

Using estimates 15, 22 and applying Gronwall's inequality, we obtain the estimate 19.

Taking the scalar product of \( \Delta^2 s \) with equation 8, and then integrating the result over \( \mathbb{R}^2 \), we have
\[\int_{\mathbb{R}^2} s_t \cdot \Delta^2 s dx = \int_{\mathbb{R}^2} \Delta s \cdot \Delta^2 s dx - \int_{\mathbb{R}^2} \text{div}(u \otimes J_r(x)) \cdot \Delta^2 s dx - \int_{\mathbb{R}^2} s \cdot \Delta^2 s dx \]
\[+ \beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta^2 s dx - \int_{\mathbb{R}^2} s \times u \cdot \Delta^2 s dx, \quad (34)\]

where
\[\int_{\mathbb{R}^2} \text{div}(u \otimes J_r(x)) \cdot \Delta^2 s dx \]
\[\leq \|J_r(x)\|_\infty \|\Delta u\|_2 \|\nabla s\|_2 + 2 \|\nabla J_r(x)\|_\infty \|\nabla u\|_2 \|\Delta s\|_2 \]
\[+ \|u\|_\infty \|\Delta J_r(x)\|_2 \|\nabla s\|_2 \]
\[\leq \frac{1}{6} \|\nabla \Delta s\|_2^2 + C(\|J_r(x)\|_2^2 \|\nabla s\|_2 + \|\nabla J_r(x)\|_2 \|\nabla s\|_2 \]
\[+ \|\Delta s\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_2^2 \]
\[\leq \frac{1}{6} \|\nabla \Delta s\|_2^2 + C(1 + \|\Delta u\|_2^2), \quad (35)\]

By Gagliardo-Nirenberg inequality, we get
\[\|\nabla u\|_4 \leq C \|\nabla u\|_H^\frac{1}{2}, \|\nabla u\|_4 \leq C \|\Delta u\|_H^\frac{1}{2}, \|\Delta u\|_4 \leq C \|\Delta u\|_H^\frac{1}{2}.\]
By Hölder inequality, we have
\[
\beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta^2 s dx \\
\leq \beta (\|u\|_\infty^2 \|\nabla \Delta s\|_2 + \|u\|_\infty \|\Delta s\|_4 \|\nabla u\|_4 + \|u\|_\infty \|\Delta u\|_4 \|\nabla s\|_4 \\
+ \|\nabla u\|_2 \|\nabla s\|_4) \|\nabla \Delta s\|_2 \\
\leq C (\|\nabla \Delta s\|_2 + \|\Delta s\|_4 \|\nabla u\|_2 \|\nabla s\|_4 + \|\Delta u\|_4 \|\nabla s\|_4 + \|\nabla \Delta s\|_2 \\
+ \|\Delta u\|_4 \|\nabla \Delta s\|_2 + \|\nabla \Delta s\|_2 \|\nabla s\|_4 \|\nabla u\|_2) \\
\leq \frac{1}{6} \|\nabla \Delta s\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + C (1 + \|\Delta s\|_2^2 + \|\Delta s\|_4^2 + \|\Delta u\|_2^2 + \|\Delta u\|_4^2).
\]

(36)

Thus inserting estimates 35, 36, 37 into 34, we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta s(\cdot, t)\|_2^2 + \|\Delta u(\cdot, t)\|_2^2 + \|\Delta s(\cdot, t)\|_2^2 \\
\leq \int_{\mathbb{R}^2} \text{div}(u \otimes J_\tau(x)) \cdot \Delta s dx - \beta \int_{\mathbb{R}^2} \text{div}(u \otimes (\nabla s \cdot u)) \cdot \Delta s dx \\
- \int_{\mathbb{R}^2} s \times u \cdot \Delta^2 s dx \\
\leq \frac{1}{2} \|\nabla \Delta s\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + C (1 + \|\Delta u\|_2^2 + \|\Delta u\|_4^2 + \|\Delta s\|_2^2 + \|\Delta s\|_4^2).
\]

(37)

Summing 26 with 38, we have
\[
\frac{d}{dt} \left( \|\Delta u\|_2^2 + \|\Delta s\|_2^2 \right) + \|\nabla \Delta u\|_2^2 + \|\nabla \Delta s\|_2^2 \\
\leq C (1 + \|\Delta u\|_2^2 + \|\Delta u\|_4^2 + \|\Delta s\|_2^2 + \|\Delta s\|_4^2).
\]

(39)

By Gronwall’s inequality, we can prove estimate 20. □

Lemma 2.5. Let \( u_0(x) \in H^m \), \( s_0(x) \in H^m \), \( j_\tau(x) \in H^m \) \((m \geq 2)\) and \( \beta \|u_0\|_\infty \ll 1 \), then the solutions of the problem 7–9 satisfy the following estimate
\[
\sup_{t \in [0, T]} \left( \|D^m u(\cdot, t)\|_2^2 + \|D^m s(\cdot, t)\|_2^2 \right) \\
+ \int_0^T \left( \|D^{m+1} u(\cdot, t)\|_2^2 + \|D^{m+1} s(\cdot, t)\|_2^2 \right) dt \leq C_{m+1},
\]
\[
\forall T > 0, t \in [0, T], \text{ where the constant } C \text{ depends on } T, \|u_0\|_{H^m}, \text{ and } \|s_0\|_{H^m}.
\]

Proof. It is easy to prove this lemma by using induction argument. In fact, by Lemma 2.3 and Lemma 2.4, the estimate 40 is verified for \( m = 0, 1, 2 \).
Taking the scalar product of $\Delta^3 u$ with equation 7 and then integrating the result over $\mathbb{R}^2$, we have
\begin{equation}
\int_{\mathbb{R}^2} u_t \cdot \Delta^3 u \, dx = \int_{\mathbb{R}^2} \Delta u \cdot \Delta^3 u \, dx + \int_{\mathbb{R}^2} u \times (\Delta u + s) \cdot \Delta^3 u \, dx
- k \int_{\mathbb{R}^2} (1 + \mu |u|^2) u \cdot \Delta^3 u dx.
\end{equation}

By Hölder inequality, Sobolev imbedding theorems and the estimates 19, 20, we have
\begin{equation}
\int_{\mathbb{R}^2} u \times (\Delta u + s) \cdot \Delta^3 u \, dx = \int_{\mathbb{R}^2} \Delta \left( u \times (\Delta u + s) \right) \cdot \Delta^2 u \, dx
\leq \int_{\mathbb{R}^2} \left\{ 2 \nabla u \times \nabla \Delta u + \sum_{j=0}^{2} \left( \begin{array}{c} 2 \\ j \end{array} \right) \nabla^j u \cdot \nabla^{2-j} s \right\} \cdot \Delta^2 u \, dx
\leq C \left( \| \nabla u \|_\infty \| \nabla \Delta u \|_2 + \sum_{j=0}^{1} \| \nabla^j u \|_\infty \| \nabla^{2-j} s \|_2 \right) \| \Delta^2 u \|_2^2 dx
\leq \frac{1}{4} \| \Delta^2 u \|_2^2 + C \left( 1 + \| \nabla \Delta u \|_2^2 + \| \nabla \Delta u \|_2^4 \right).
\end{equation}

Similarly, we get
\begin{equation}
\left| k \int_{\mathbb{R}^2} (1 + \mu |u|^2) u \cdot \Delta^3 u dx \right|
= \left| k \int_{\mathbb{R}^2} \Delta (u + \mu |u|^2 u) \cdot \Delta^2 u dx \right|
\leq C \left( \| \Delta u \|_2 + \| u \|_\infty \| \nabla u \|_4^2 + \| u \|_\infty \| \Delta u \|_2 \right) \| \Delta^2 u \|_2^2
\leq \frac{1}{4} \| \nabla \Delta u \|_2^2 + C,
\end{equation}

where we have used Sobolev imbedding theorems $H^2 \subset L^\infty, H^1 \subset L^4$ and estimates 19, 20.

Combining 41, 42 and 43, we have
\begin{equation}
\frac{d}{dt} \| \nabla \Delta u \|_2^2 + \| \Delta^2 u \|_2^2 \leq C \left( 1 + \| \nabla \Delta u \|_2^2 + \| \nabla \Delta u \|_2^4 \right).
\end{equation}

Taking the scalar product of $\Delta^3 s$ with 8, and then integrating the result over $\mathbb{R}^2$, we have
\begin{equation}
\int_{\mathbb{R}^2} s_t \cdot \Delta^3 s dx = \int_{\mathbb{R}^2} \Delta s \cdot \Delta^3 s dx - \int_{\mathbb{R}^2} \nabla \cdot (u \otimes J_e(x)) \cdot \Delta^3 s dx - \int_{\mathbb{R}^2} s \cdot \Delta^3 s dx
+ \beta \int_{\mathbb{R}^2} \nabla \cdot (u \otimes (\nabla s \cdot u)) \cdot \Delta^3 s dx - \int_{\mathbb{R}^2} s \times u \cdot \Delta^3 s dx,
\end{equation}
where
\[
\int_{\mathbb{R}^2} \nabla \cdot (u \otimes J_c(x)) \cdot \Delta^3 sdx = \int_{\mathbb{R}^2} \Delta \nabla \cdot (u \otimes J_c(x)) \cdot \Delta^2 sdx
\]
\[
\leq C \left( ||J_c(x)||_\infty ||\nabla \Delta u||_2 + 3 ||\nabla J_c(x)||_\infty ||\Delta u||_2 + 3 ||\nabla u||_4 ||\Delta J_c(x)||_4 + ||u||_\infty ||\nabla \Delta J_c(x)||_2 \right) ||\Delta^2 s||_2
\]
\[
\leq \frac{1}{6} ||\Delta^2 s||_2^2 + C(1 + ||\nabla \Delta u||_2^2)
\]
and
\[
\beta \int_{\mathbb{R}^2} \nabla \cdot (u \otimes (\nabla s \cdot u)) \cdot \Delta^3 sdx = \beta \int_{\mathbb{R}^2} \Delta \nabla \cdot (u \otimes (\nabla s \cdot u)) \cdot \Delta^2 sdx
\]
\[
\leq C \left( ||u||_\infty ||\nabla \Delta u||_4 ||\nabla s||_4 + ||u||_\infty ||\Delta u||_4 ||\Delta s||_4 + ||\nabla u||_\infty ||\nabla \Delta s||_4 + ||u||_\infty ||\nabla u||_2 ||\Delta s||_2 + ||u||_\infty ||\Delta s||_2 + ||u||_\infty ||\Delta^2 s||_2 \right) ||\Delta^2 s||_2
\]
\[
\leq C \left( ||\nabla \Delta u||_2 \frac{1}{2} ||\nabla \Delta u||_2 + ||\nabla u||_2 \frac{1}{2} ||\nabla s||_2^2 ||\nabla s||_2^2 + ||\nabla u||_2 \frac{1}{2} ||\nabla \Delta s||_2 + ||\nabla u||_2 ||\Delta u||_2 \right) ||\Delta^2 s||_2
\]
\[
\leq \frac{1}{6} ||\Delta^2 s||_2^2 + \frac{1}{2} ||\Delta^2 u||_2^2 + C(1 + ||\nabla \Delta s||_2^2 + ||\nabla \Delta s||_2^2 + ||\nabla \Delta u||_2^2)
\]
and
\[
-\int_{\mathbb{R}^2} s \times u \cdot \Delta^3 sdx = \int_{\mathbb{R}^2} \Delta(s \times u) \cdot \Delta^2 sdx
\]
\[
\leq ||u||_\infty ||\Delta s||_2 ||\Delta^2 s||_2 + 2 ||\nabla u||_4 ||\nabla s||_4 ||\Delta^2 s||_2 + ||s||_\infty ||\Delta u||_2 ||\Delta^2 s||_2
\]
\[
\leq C \left( ||\Delta s||_2 + ||\nabla u||_2 \frac{1}{2} ||\nabla u||_2 \frac{1}{2} ||\nabla s||_2 + ||\Delta s||_2 ||\Delta u||_2 \right) ||\nabla \Delta s||_2
\]
\[
\leq \frac{1}{6} ||\Delta^2 s||_2^2 + C.
\]
Combining 45, 46, 47 and 48, we have
\[
\frac{1}{2} \frac{d}{dt} ||\nabla \Delta s(\cdot, t)||_2^2 + ||\Delta^2 s(\cdot, t)||_2^2 + ||\nabla \Delta s(\cdot, t)||_2^2
\]
\[
\leq \int_{\mathbb{R}^2} \nabla \cdot (u \otimes J_c(x)) \cdot \Delta^3 sdx - \beta \int_{\mathbb{R}^2} \nabla \cdot (u \otimes (\nabla s \cdot u)) \cdot \Delta^3 sdx
\]
\[-\int_{\mathbb{R}^2} s \times u \cdot \Delta^3 sdx
\]
\[
\leq \frac{1}{2} ||\Delta^2 s||_2^2 + \frac{1}{2} ||\Delta^2 u||_2^2 + C(1 + ||\nabla \Delta u||_2^2 + ||\nabla \Delta s||_2^2 + ||\nabla \Delta s||_2^2).
\]
Summing 44 with 49, we have
\[
\frac{d}{dt} \left( ||\nabla \Delta u||_2^2 + ||\nabla \Delta s||_2^2 \right) + ||\Delta^2 u||_2^2 + ||\Delta^2 s||_2^2
\]
\[
\leq C \left( 1 + ||\nabla \Delta u||_2^2 + ||\nabla \Delta u||_2^2 + ||\nabla \Delta s||_2^2 + ||\nabla \Delta s||_2^2 \right).
\]
By Gronwall’s inequality, we get
\[
\sup_{t \in [0, T]} \left( \| D^3 u(\cdot, t) \|_2^2 + \| D^3 s(\cdot, t) \|_2^2 \right)
+ \int_0^T \left( \| D^4 u(\cdot, t) \|_2^2 + \| D^4 s(\cdot, t) \|_2^2 \right) dt \leq C_4.
\]
(51)

Now assume that the estimate 40 is valid for \( m = K \geq 3 \), i.e., we have
\[
\sup_{t \in [0, T]} \left( \| D^K u(\cdot, t) \|_2^2 + \| D^K s(\cdot, t) \|_2^2 \right)
+ \int_0^T \left( \| D^{K+1} u(\cdot, t) \|_2^2 + \| D^{K+1} s(\cdot, t) \|_2^2 \right) dt \leq C_{K+1},
\]
(52)
where constant \( C_{M+1} \) depends on \( T \) and \( \| u_0 \|_{H^K}, \| s_0 \|_{H^K} \).

We should to prove the estimate 40 with \( m = K + 1 \). Taking the scalar product of \( \Delta^{K+1} u \) with equation 7, and then integrating the result over \( \mathbb{R}^2 \), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+2} u \|_2^2 = - \int_{\mathbb{R}^2} \{ \nabla^K [u \times (\Delta u + s)] \} \cdot \nabla^{K+2} u dx
- k \int_{\mathbb{R}^2} \{ \nabla^K (u + \mu |u|^2 u) \} \cdot \nabla^{K+2} u dx.
\]
(53)

By Hölder inequality, we have
\[
\left| \int_{\mathbb{R}^2} \{ \nabla^K [u \times (\Delta u + s)] \} \cdot \nabla^{K+2} u dx \right|
\leq \sum_{j=1}^K \binom{K}{j} \int_{\mathbb{R}^2} |\nabla^j u \times \nabla^{K+2-j} u| \cdot \nabla^{K+2} u dx
\]
\[
+ \sum_{j=0}^K \binom{K}{j} \int_{\mathbb{R}^2} |\nabla^j u \times \nabla^{K-j} s| \cdot \nabla^{K+2} u dx
\]
\[
\leq C \left\{ \sum_{j=1}^2 \| \nabla^j u \|_\infty \| \nabla^{K+2-j} u \|_2 + \sum_{j=0}^2 \| \nabla^j u \|_\infty \| \nabla^{K-j} s \|_2
+ \chi(K \geq 2) \sum_{j=3}^{K-1} \| \nabla^j u \|_4 (\| \nabla^{K+2-j} u \|_4 + \| \nabla^{K-j} s \|_4)
+ \| \nabla^K u \|_2 \| s \|_\infty \right\} \| \nabla^{K+2} u \|_2,
\]
where the characteristic function
\[
\chi(K \geq 2) = \begin{cases} 1, & K \geq 2, \\ 0, & 0 \leq K \leq 1. \end{cases}
\]

Using Sobolev imbedding theorems and the estimate 52, we get
\[
\left| \int_{\mathbb{R}^2} \{ \nabla^K [u \times (\Delta u + s)] \} \cdot \nabla^{K+2} u dx \right|
\leq \frac{1}{4} \| \nabla^{K+2} u \|_2^2 + C \left( 1 + \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+1} u \|_2^2 \right),
\]
(54)
where we have used Sobolev imbedding theorems
\( H^{K+1} \subset W^{2, \infty}, \ H^K \subset W^{j, \infty}, \ j = 0, 1 \).
and
\[ H^{j+1} \subset W^{j,4}, \ j = 0, 1, \cdots, K. \]

Similarly, we have
\[
\left| k \int_{\mathbb{R}^2} \{ \nabla^K (u + \mu|u|^2 u) \} \cdot \nabla^{K+2} u \, dx \right|
\leq k \| \nabla^{K+2} u \|_2 \| \nabla^K u \|_2
+ C \sum_{j_1 + j_2 + j_3 = K} \| \nabla^{j_1} u \|_6 \| \nabla^{j_2} u \|_6 \| \nabla^{j_3} u \|_6 \| \nabla^{K+2} u \|_2
\leq \frac{1}{4} \| \nabla^{K+2} u \|_2^2 + C(1 + \| \nabla^{K+1} u \|_2^2),
\] (55)

where we have used the estimate 52 and Sobolev imbedding theorems
\[ H^{j+1} u \subset W^{j,6}, \ j = 0, 1, \cdots, K. \]

Thus inserting estimates 54, 55 into 53, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+2} u \|_2^2 \leq C(1 + \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+1} u \|_2^2).
\] (56)

Making the scalar product of \( \Delta^{K+1} s \) with equation 8 and then integrating the result over \( \mathbb{R}^2 \), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^{K+1} s \|_2^2 + \| \nabla^{K+2} s \|_2^2 + \| \nabla^{K+1} s \|_2^2
\leq - \int_{\mathbb{R}^2} \{ \nabla^{K+1} \cdot (u \otimes J_{x}(x)) \} \cdot \nabla^{K+2} s \, dx
+ \beta \int_{\mathbb{R}^2} \nabla^{K+1} \cdot (u \otimes (\nabla s \cdot u)) \cdot \nabla^{K+2} s \, dx
- \int_{\mathbb{R}^2} \nabla^K (s \times u) \cdot \nabla^{K+2} s \, dx.
\] (57)

By Hölder inequality, we have
\[
\left| - \int_{\mathbb{R}^2} \nabla^K (s \times u) \cdot \nabla^{K+2} s \, dx \right|
\leq \sum_{j=0}^{K} \left( \begin{array}{c} K \\ j \end{array} \right) \int_{\mathbb{R}^2} |(\nabla^j s \times \nabla^{K-j} u) \cdot \nabla^{K+2} u| \, dx
\leq C \left\{ \sum_{j=0}^{2} \| \nabla^j s \|_\infty \| \nabla^{K-j} u \|_2 + \chi(K \geq 2) \sum_{j=3}^{K-1} \| \nabla^j s \|_4 \| \nabla^{K-j} u \|_4 ight. \\
+ \left. \| \nabla^K s \|_2 \| u \|_\infty \right\} \| \nabla^{K+2} s \|_2.
\] (58)

Using Sobolev imbedding theorems and the estimate 52, we get
\[
\left| \int_{\mathbb{R}^2} \nabla^K (s \times u) \cdot \nabla^{K+2} s \, dx \right| \leq \frac{1}{6} \| \nabla^{K+2} s \|_2^2 + C,
\] (59)
Similarly, we have
\[
\left| \int_{\mathbb{R}^2} \nabla^{K+1} \cdot (u \otimes J_e(x)) \cdot \nabla^{K+2} s \, dx \right|
\]
\[
\leq \sum_{j=0}^{K+1} \binom{K+1}{j} \int_{\mathbb{R}^2} |(\nabla^j u \times \nabla^{K+1-j} J_e(x)) \cdot \nabla^{K+2} s| \, dx
\]
\[
\leq C \left\{ \sum_{j=0}^{2} \| \nabla^j u \|_\infty \| \nabla^{K+1-j} J_e(x) \|_2 + \| \nabla^{K+1} u \|_2 \| J_e(x) \|_\infty \right\} \| \nabla^{K+2} s \|_2
\]
\[
+ \chi(K \geq 2) \sum_{j=3}^{K} \| \nabla^j u \|_4 \| \nabla^{K+1-j} J_e(x) \|_4 \right\} \| \nabla^{K+2} s \|_2
\]
\[
\leq \frac{1}{6} \| \nabla^{K+2} s \|_2^2 + C(1 + \| \nabla^{K+1} u \|_2^2)
\]
and
\[
\left| \beta \int_{\mathbb{R}^2} \nabla^{K+1} \cdot (u \otimes (\nabla s \cdot u)) \cdot \nabla^{K+2} s \, dx \right|
\]
\[
\leq \sum_{j_1+j_2+j_3=K+1} \| \nabla^{j_1} u \|_6 \| \nabla^{j_2+1} u \|_6 \| \nabla^{j_3} u \|_6 \| \nabla^{K+2} s \|_2
\]
\[
\leq \frac{1}{6} \| \nabla^{K+2} s \|_2^2 + \frac{1}{2} \| \nabla^{K+2} u \|_2^2 C(1 + \| \nabla^{K+1} s \|_2^2 + \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+1} u \|_4^2).
\]

Thus inserting estimates 58, 59, 60 into 57, we obtain
\[
\frac{d}{dt} \| \nabla^{K+1} s \|_2^2 + \| \nabla^{K+2} s \|_2^2
\]
\[
\leq C(1 + \| \nabla^{K+1} s \|_2^2 + \| \nabla^{K+1} u \|_2^2 + \| \nabla^{K+1} u \|_4^2).
\]

Using estimates 56, 62 and applying Gronwall’s inequality, we establish the estimate 40 with \( m = K + 1 \). This lemma is proved.

Now we devote to prove the uniqueness. Assume that there exist two solutions \((u_j, s_j) \ (j = 1, 2)\). Let \((\phi, \psi) = (u_1 - u_2, s_1 - s_2)\), then \((\phi, \psi)\) satisfies the following equations
\[
\frac{\partial \phi}{\partial t} - \Delta \phi - \phi \times (\Delta u_1 + s_1) - u_2 \times (\Delta \phi + \psi)
\]
\[
= - k(1 + \mu |u_1|^2) \phi - k\mu (u_1 + u_2) \cdot \phi u_2,
\]
\[
\frac{\partial \psi}{\partial t} - \Delta \psi + \text{div} (\psi \otimes J_e(x)) + \psi + \psi \times u_1 + s_2 \times \phi
\]
\[
= - \text{div} \left( \phi \otimes (\nabla s_1 \cdot u_1) + u_2 \otimes (\nabla \psi \cdot u_1) + u_2 \otimes (\nabla s_2 \cdot \phi) \right)
\]
\[
\phi(x, 0) = 0, \ \psi(x, 0) = 0.
\]

Making the scalar product of equation 63 with \( \phi - \Delta \phi \) and then integrating the result over \( \mathbb{R}^2 \), we have
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \phi(\cdot, t) \|_2^2 + \| \nabla \phi(\cdot, t) \|_2^2 \right\} + \{ \| \nabla \phi(\cdot, t) \|_2^2 + \| \Delta \phi(\cdot, t) \|_2^2 \}
\]

By using Gagliardo-Nirenberg inequality, one has
\[ \|\phi\|_\infty \leq C \|\phi\|_{L^2}^{\frac{1}{2}} \|\phi\|_{H^2}^{\frac{1}{2}}. \] (66)

Applying the estimate 40 with \( m \geq 1 \) and inequality 66, we get
\[ \frac{1}{2} \frac{d}{dt} \left\{ \|\psi\|_2^2 + \|\nabla \psi\|_2^2 \right\} + \{ \|\psi\|_2^2 + 2\|\nabla \psi\|_2^2 + \|\Delta \psi\|_2^2 \} \]
\[ \leq \frac{1}{4} \|\Delta \psi\|_2^2 + C(\|\phi\|_2^2 + \|\psi\|_2^2). \] (67)

Taking the scalar product of equation 64 with \( \psi - \Delta \psi \) and then integrating the result over \( \mathbb{R}^2 \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \left\{ \|\psi\|_2^2 + \|\nabla \psi\|_2^2 \right\} + \{ \|\psi\|_2^2 + 2\|\nabla \psi\|_2^2 + \|\Delta \psi\|_2^2 \} \]
\[ = - \int_{\mathbb{R}^2} \left\{ \psi \times u_1 + s_2 \times \phi \right\} \cdot (\psi - \Delta \psi) dx - \int_{\mathbb{R}^2} \text{div}(\psi \otimes J_c(x)) \cdot (\psi - \Delta \psi) dx \]
\[ - \int_{\mathbb{R}^2} \text{div} \left( \phi \otimes (\nabla s_1 \cdot u_1) + u_2 \otimes (\nabla \psi \cdot u_1) + u_2 \otimes (\nabla s_2 \cdot \phi) \right) \cdot (\psi - \Delta \psi) dx \]
\[ \leq C \left\{ \|u_1\|_\infty \|\nabla \psi\|_2 \Delta \psi_2 + \|s_2\|_\infty \|\phi\|_2 (\|\psi\|_2 + \|\Delta \psi\|_2) \right\} \]
\[ + \left\{ \|J_c(x)\|_\infty \|\nabla \psi\|_2 \Delta \psi_2 + \|\nabla J_c(x)\|_\infty \|\nabla \psi\|_2 (\|\psi\|_2 + \|\Delta \psi\|_2) \right\} \]
\[ + \left\{ \|u_1\|_\infty \|\nabla \psi_4\| \|\nabla s_1\|_4 + \|u_1\|_\infty \|\phi\|_\infty \|\Delta s_1\|_2 \right\} \]
\[ + \left\{ \|\nabla \psi_4\|_\infty \|\nabla s_1\|_4 \|\nabla u_1\|_4 (\|\psi_2\| + \|\Delta \psi\|_2) \right\} \]
\[ + \left\{ \|u_2\|_\infty \|\nabla \psi_4\| \|\nabla u_1\|_4 (\|\psi_2\| + \|\Delta \psi\|_2) \right\} \]
\[ + \left\{ \|\phi\|_\infty \|\nabla u_2\|_4 \|\nabla s_2\|_4 + \|u_2\|_\infty \|\phi\|_\infty \|\Delta s_2\|_2 \right\} \]
\[ + \left\{ \|u_2\|_\infty \|\nabla s_2\|_4 \|\nabla \psi_4\| (\|\psi_2\| + \|\Delta \psi\|_2) \right\}. \]

By using Gagliardo-Nirenberg inequality, one has
\[ \|\nabla \psi\|_4 \leq C \|\psi\|_2^{\frac{1}{2}} \|\phi\|_{H^2}^{\frac{1}{2}}, \|\nabla \psi\|_4 \leq C \|\psi\|^{\frac{3}{2}} \|\psi\|_{H^2}, \|\psi\|_\infty \leq C \|\psi\|_2^{\frac{1}{2}} \|\psi\|_{H^2}^{\frac{1}{2}}. \] (68)

Applying the estimate 40 with \( m \geq 1 \) and inequality 66 and 68, we get
\[ \frac{1}{2} \frac{d}{dt} \left\{ \|\psi\|_2^2 + \|\nabla \psi\|_2^2 \right\} + \{ \|\nabla \psi\|_2^2 + \|\Delta \psi\|_2^2 \} \]
\[ \leq \frac{1}{2} \|\Delta \psi\|_2^2 + \frac{1}{4} \|\Delta \phi\|_2^2 \]
\[ + C(\|\phi\|_2^2 + \|\psi\|_2^2 + \|\nabla \phi\|_2^2 + \|\nabla \psi\|_2^2). \] (69)

Summing 67 with 69 and applying Gronwall’s inequality, we get
\[ \|\phi\|_2^2 + \|\nabla \phi\|_2^2 + \|\psi\|_2^2 + \|\nabla \psi\|_2^2 = 0. \] (70)
Therefore the global solution \((u, s)\) is unique for \(m \geq 2\). This completes the proof of Theorem 1.1.

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