SEIDEL MINOR, PERMUTATION GRAPHS
AND COMBINATORIAL PROPERTIES

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Abstract. A permutation graph is an intersection graph of segments lying between two parallel lines. A Seidel complementation of a finite graph at a vertex $v$ consists in complementing the edges between the neighborhood and the non-neighborhood of $v$. Two graphs are Seidel complement equivalent if one can be obtained from the other by a sequence of Seidel complementations.

In this paper we introduce the new concept of Seidel complementation and Seidel minor. We show that this operation preserves cographs and the structure of modular decomposition. The main contribution of this paper is to provide a new and succinct characterization of permutation graphs namely, a graph is a permutation graph if and only if it does not contain any of the following graphs: $C_5$, $C_7$, $XF_6^2$, $XF_5^{2n+3}$, $C_{2r}$, $r \geq 6$ and their complements as a Seidel minor. This characterization is in a sense similar to Kuratowski’s characterization [15] of planar graphs by forbidden topological minors.

Keywords: Graph; Permutation graph; Seidel complementation; Seidel minor; Modular decomposition; Cograph; Local complementation; Well Quasi Order.

1. Introduction

A lot of graph classes are frequently characterized by a list of forbidden induced subgraphs. For instance such characterization is known for cographs, interval graphs, chordal graphs... However, it is not always convenient to deal with this kind of characterizations and the list of forbidden subgraphs can be quite large. Some characterizations rely on the use of local operators such as minors, local complementation or Seidel switch.

Certainly, Kuratowski’s characterization of planar graphs by forbidden topological minors is one of the most famous [15].

A nice characterization of circle graphs, i.e. the intersection graphs of chords in a circle, was given by Bouchet [2], using an operation called local complementation. This operation consists in complementing the graph induced by the neighborhood of a vertex. His characterization states that a graph is a circle graph if and only if it does not contain $W_5$, $W_7$ and $BW_3$ as vertex minor. This operation has strong connections with a graph decomposition called rank-width, this relationship is presented in the work of Oum [19, 20].

Another example of local operator is the Seidel switch. The Seidel switch is a graph operator introduced by Seidel in his seminal paper [23]. A Seidel switch in a graph consists in complementing the edges between a subset of vertices $S$ and its complement $V \setminus S$.

Seidel switch has been intensively studied since its introduction: Colbourn et al. [6] proved that deciding whether two graphs are Seidel switch equivalent is $\text{ISO}$-Complete. The Seidel switch has also applications in graph coloring [14]. Other interesting applications of Seidel

$^1$\(W_5\) (resp. \(W_7\)) is the wheel on five (resp. seven) vertices, i.e. a chordless cycle vertices plus a dominating vertex, and \(BW_3\) is a wheel on three vertices where the cycle is subdivided.
switch concerns structural graph properties \cite{12,13}. It has also been used by Rotem and Urrutia \cite{22} to show that the recognition of circular permutation graphs (CPG for short) can be polynomially reduced to the recognition of permutation graphs. Years later, Sritharan \cite{24} presented a nice and efficient algorithm to recognize CPGs in linear time. Once again it is a reduction to permutation graph recognition, and it relies on the use of a Seidel switch. Montgolfier et al. \cite{17,18} used it to characterize graphs completely decomposable w.r.t. Bi-join decomposition. Seidel switch is not only relevant to the study of graphs. Ehrenfeucht et al. \cite{8} showed the interest of this operation for the study of 2-structures and recently, Bui-Xuan et al. extended these results to broader structures called Homogeneous relations \cite{3,5,4}.

We present in this paper a novel characterization of the well known class of permutation graphs, i.e., the intersection graphs of segments lying between two parallel lines. Permutation graphs were introduced by Even, Lempel and Pnueli \cite{21,9}. They established that a graph is a permutation graph if and only if the graph and its complement are transitively orientable. They also gave a polynomial time procedure to find a transitive orientation when it is possible. A linear time algorithm recognition algorithm is presented in \cite{16}.

This results constitutes, in a sense, an improvement compared to Gallai’s characterization of permutation graphs by forbidden induced subgraphs which counts no less than 18 finite graphs, and 14 infinite families \cite{10}.

For that we introduce a new local operator called Seidel complementation. In few words, the Seidel complementation on an undirected graph at a vertex \(v\) consists in complementing the edges between the neighborhood and the non-neighborhood of \(v\). A schema of Seidel complementation is depicted in Figure 2. Thanks to this operator and the corresponding minor, we obtain a compact list of Seidel minor obstructions for permutation graphs.

The main result of this paper is a new characterization of permutation graphs. We show that a graph is a permutation graph if and only if it does not contain any of the following graphs \(C_5\), \(C_7\), \(XF_{6,2}\), \(XF_{5,n+3}\), \(C_{2n}, n \geq 6\) or their complements as Seidel minors.

The proof is based on a study of the relationships between Seidel complementation and modular decomposition. We show that any Seidel complementation of a prime graph w.r.t. modular decomposition is a prime graph. As a consequence we get that cographs are stable under Seidel complementation. We also present a complete characterization of equivalent cographs, which leads to a linear time algorithm for verifying Seidel complement equivalence of cographs.

Our notion of Seidel complementation is a combination of local complementation and Seidel switch. The use of a vertex as pivot comes from local complementation, and the transformation from Seidel switch.

The paper is organized as follows. In section 2 we present the definitions of Seidel complementation and Seidel minor. Then we show some structural properties of Seidel complementation and introduce the definitions and notations used in the sequel of the paper. In section 3 we show the relationships between Seidel complementation and modular decomposition, namely we prove that Seidel complementation preserves the structure of modular decomposition of a graph. Finally we show that cographs are closed under this relation. Section 4 is devoted to prove the main theorem, namely a graph is a permutation graph if and only if it does not contain any of the forbidden Seidel minors. We also prove that permutation graphs are not well quasi ordered under the Seidel minor relation. In section 5 we show that any Seidel complement equivalent graphs are at distance at most one from each other, we show that to decide when two graphs are equivalent under the Seidel complement relation is
ISO-Complete and we provide two polynomial algorithms to solve this problem on cographs and on permutation graphs. And finally in section 6 we propose a definition of Seidel complementation for tournaments, and we show, with that definition, we have same property w.r.t. modular decomposition as for undirected graphs.

2. Definitions and notations

In this paper only undirected, finite, loop-less and simple graphs are considered. We present here some notations used in the paper. The graph induced by a subset of vertices $X$ is noted $G[X]$. For a vertex $v$, $N(v)$ denotes the neighborhood of $v$, and $\overline{N}(v)$ represents the non-neighborhood. Sometimes we need to use a refinement of the neighborhood on a subset of vertices $X$, noted $N_X(v) = N(v) \cap X$. Let $A$ and $B$ be two disjoint subsets of $V$, and let $E[A, B] = \{ab \in E : a \in A \text{ and } b \in B\}$ be the set of edges between $A$ and $B$. For two sets $A$ and $B$, let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

**Definition 2.1** (Seidel complement). Let $G = (V, E)$ be a graph, and let $v$ be a vertex of $V$, and the Seidel complement at $v$ on $G$, denoted $G * v$ is defined as follows: Swap the edges and the non-edges between $G[N(v)]$ and $G[\overline{N}(v)]$, namely

$$G * v = (V, E\Delta\{xy : vx \in E, vy \notin E\})$$

From the previous definition it is straightforward to notice that $G * v * v = G$.

**Proposition 2.2.** Let $G$ be a graph. If $vw$ is an edge of $G$, then $G * v * w * v = G * w * v * w$. This operation is denoted $G * v w$.

**Proof.** Let us consider the neighborhood of $v$ and $w$. Let $N(v) = A \cup B$ and let $N(w) = B \cup C$, where $B$ is obviously $N(v) \cap N(w)$, and let $D$ be $\overline{N}(v) \cap \overline{N}(w)$. We know that $v$ is connected to $A \cup B \cup \{w\}$ and that $w$ is connected to $B \cup C \cup \{v\}$. But we do not know how the sets $A$, $B$, $C$ and $D$ are connected. We just say there are mixed edges between each set. See Figure 1.

**Remark 2.3.** One can remark from Figure 1 that the $*$ operation merely exchanges the vertices $v$ and $w$ without modifying the graph $G[V \setminus \{v\} \cup \{w\}]$.

**Remark 2.4.** Proposition 2.2 remains true even if $v, w$ is not an edge of $G$. The proof is similar to the proof of the Proposition 2.2.

**Definition 2.5** (Seidel Minor). Let $G = (V, E)$ and $H = (V', E')$ be two graphs. $H$ is a Seidel minor of $G$ (noted $H \preceq S G$) if $H$ can be obtained from $G$ by a sequence of the following operations:

- Perform a Seidel complementation at a vertex $v$ of $G$,
- Delete a vertex of $G$.

**Definition 2.6** (Seidel Equivalent Graphs). Let $G = (V, E)$ and $H = (V, F)$ be two graphs. $G$ and $H$ are said to be Seidel equivalent if and only if there exists a word $\omega$ defined on $V^*$ such that $G * \omega \cong H$.

At first glance Seidel complementation seems to be just a particular case of Seidel switch, but after a careful examination, one can see that they are not comparable.

Actually two graphs $G$ and $H$ that belong to the same Seidel switch equivalence class share a common combinatorial structure called a 2-graph [23] [8]. A 2-graph $\Omega = (V, D)$ is 3-regular
Figure 1. \( G * w * v * w = G * v * w * v = G * vw \)

Figure 2. An illustration of the Seidel complement concept.

hypergraph where \( V \) is the ground set and \( D \) is the set of hyperedges, and for each subset \( S \)
of V of size 4, we have $|D \cap S| \equiv 0 \mod 2$. A 2-graph can be obtained from a graph by taking in D all the triples of vertices with an odd number of edges. And from that definition we can see that the Seidel complementation applied on a graph does not preserve the underlying 2-graphs. We can also see that with the Seidel complementation and with the Seidel switch starting from a same graph, the graphs we obtain with each operator are different. The reader can convince himself by looking at the house, i.e. a cycle C on five vertices plus a short chord connecting two vertices at distance two in C.

3. Modular decomposition and cographs

In this section we investigate the relationships between Seidel complementation and modular decomposition. This study is relevant in order to prove the main result. Actually a permutation graph is uniquely representable if and only if it is prime w.r.t. modular decomposition. And one of the results of this section is to prove that if a graph is prime w.r.t. modular decomposition this property is preserved by Seidel complementation. As a consequence for permutation graphs, it means that if the graph is uniquely representable so are their Seidel complement equivalent graphs.

Let us now briefly recall the definition of module. A module in a graph is subset of vertices M such that any vertex outside M is either completely connected to M or is completely disjoint from M. Modular decomposition is a decomposition of graph introduced by Gallai [10]. The modular decomposition of a graph G is the decomposition of G into its modules. Without going too deeply into the details, there exists for each graph a unique modular decomposition tree, and it is possible to compute it in linear time (cf. [25]).

In the sequel of this section we show that if G is prime, i.e. not decomposable, w.r.t. modular decomposition, then applying a Seidel complementation at any vertex of the graph preserves this property. Then we prove that the family of cographs is closed under Seidel minor. And finally show how the modular decomposition tree of a graph is modified by a Seidel complementation.

3.1. Modular decomposition.

**Theorem 3.1.** Let $G = (V, E)$ be graph, and let $v$ be an arbitrary vertex of G. G is prime w.r.t. modular decomposition if and only if $G*v$ is prime w.r.t. to modular decomposition.

**Proof.** Let us proceed by contradiction. Let us assume that G is prime and $G*v$ has a module M.

We have to consider two cases: (1) $v \in M$ and (2) $v \notin M$.

(1) Since M is not trivial: we have $|M| \geq 2$ and $|\overline{M}| \geq 1$.

We can identify four representative vertices of G: let A be a vertex of $\overline{N}(v) \cap M$, let B be a vertex of $\overline{N}(v) \cap \overline{M}$, let C be a vertex of $N(v) \cap M$ and let D be a vertex of $N(v) \cap \overline{M}$. Since M is a module we have the following edges: CA and CD and the following non-edges: BA and BD (cf. Figure 3(a)).

By definition of Seidel complementation at a vertex, it is equivalent to swap the edges and non-edges between the neighborhood and the non-neighborhood of v. We obtain the result depicted in Figure 3(b). Now we can clearly see that $\overline{M} \cup \{v\}$ is a module in G, and since $|\overline{M}| \geq 1$ we obtain a non-trivial module. Thus a contradiction.

(2) Let us consider the case where v does not belong to M. We can assume, w.l.o.g., that $M \subseteq N(v)$. We can partition $N(v)$ into $A_1, A_2$ such that $N_M(A_1) = M$ and $N_M(A_2) = \emptyset$. 
And similarly we can partition $\overline{N}(v)$ into $B_1, B_2$ such that $N_M(B_1) = M$ and $N_M(B_2) = \emptyset$. (cf. Figure 3(c)-(d))

Since we have proceeded to a Seidel complement on $v$, the original configuration in $G$ is such that $N_M(B_1) = \emptyset$ and $N_M(B_2) = M$. This is the only modification w.r.t. $M$. Thus $M$ is also a module in $G$. Contradiction.

From the second case of the proof of Theorem 3.1 we can deduce the following corollary:

**Corollary 3.2.** Let $G = (V, E)$ and let $v$ be a vertex. And let $M$ be a module of $G$ such that $v$ does not belong to $M$, then $M$ is also a module in $G^* v$.

![Diagram](attachment:image.png)

**Figure 3.** Details of theorem 3.1. The Figures (a)-(b) correspond to the case where $v$ belongs to $M$. And the Figures (c)-(d) correspond to the other case.

### 3.2. Cographs.

Cographs are the graphs which are completely decomposable w.r.t. modular decomposition. There exist several characterizations of cographs (see [7]), one of them is given by a forbidden induced subgraph, i.e. cographs are the graphs without $P_4$ - a chordless path on four vertices – as induced subgraph. Another fundamental property of cograph is the fact that its modular decomposition tree – called its co-tree – has only series (1) and parallel (0) nodes as internal nodes. An example of a cograph and its associated co-tree is given in Figure 4(a). A co-tree is a rooted tree, where the leaves represent the vertices of the graph, and the internal nodes of the co-tree encode the adjacency of the vertices of the graph. Two vertices are adjacent iff their Least Common Ancestor\(^2\) (LCA) is a series node (1). Conversely two vertices are disconnected iff their LCA is a parallel node (0). The following theorem shows that the class of cographs is closed under Seidel complementation.

**Theorem 3.3.** Let $G = (V, E)$ be a cograph, and $v$ a vertex of $G$, then $G^* v$ is also a cograph.

**Proof.** Let $T$ be the co-tree of $G$. The Seidel complementation at a vertex $v$ is obtained as follows: Let $T'$ be the tree obtained by $T^* v$. $P(v)$, the former parent node of $v$, becomes the new root of $T'$, and now the parent of $v$ in $T'$ is the former root, namely $R(T)$. In other words by performing a Seidel complementation we have reversed the path from $P(v)$ to $R(T)$.

It is easy to see that $G[N(v)]$ and $G[\overline{N}(v)]$ are not modified. Now to see that the adjacency between $G[N(v)]$ and $G[\overline{N}(v)]$ is reversed, it is sufficient to remark that for two vertices $u$ and $y$ is first node in common on the paths from the leaves to the root.

\(^2\)The LCA of two leaves $x$ and $y$ is first node in common on the paths from the leaves to the root.
and \( w \), \( u \) belonging to the neighborhood of \( v \) and \( w \) belonging to the non-neighborhood of \( v \). If \( u \) and \( w \) are adjacent in \( G \) it means that their LCA is a series node. We note that this node lies on the path from \( v \) to \( R(T) \). After proceeding to a Seidel complementation their LCA is modified and it is now a parallel node, consequently reversing the adjacency between the neighborhood and the non-neighborhood.  

An example of the Seidel complement of the co-tree is given in Figure 4(b).

**Remark 3.4 (Exchange property).** Actually a Seidel complementation on a cograph, or more precisely on its co-tree is equivalent of exchanging the root of the co-tree with the vertex \( v \) used to proceed to the Seidel complement, i.e. the vertex \( v \) is attached to the former root of the co-tree and the new root is the former parent of the vertex \( v \).

Except for this transformation, the other parts of the co-tree remain unchanged, i.e. the number and the types of internal nodes are preserved, and no internal nodes are merged.

**Figure 4.** (a) An example of a cograph on 5 vertices and its respective co-tree. (b) A schema of a Seidel complement at a vertex \( v \) on a co-tree.

**Proposition 3.5.** The Seidel complementation of a cograph on its co-tree can be performed in \( O(1) \)-time.

**Proof.** It suffices to consider the co-tree of \( G \). As noticed in remark 3.4 to perform a Seidel complementation at a vertex \( v \) is equivalent to exchange a vertex – i.e. a leaf – with the root of the tree. We need to store, in a lookup table, for each vertex its parent node in the tree and the root of the tree. Updating the structure can easily be done in constant time. □

### 3.3. Modular decomposition tree

In this section we will show how the modular decomposition tree of a graphs is modified. Using Theorems 3.1, 3.3 and 3.2.

Let \( G = (V, E) \) be a graph, and let \( T(G) \) (\( T \) for short) be its modular decomposition tree.

Modular decomposition tree is a generalization of the co-tree for cographs. The only difference with co-tree is that the modular decomposition tree can contain prime nodes. Prime nodes corresponds to graphs that are not decomposable w.r.t. modular decomposition.

We generalize the operation on the co-tree, described in Theorem 3.3, to arbitrary modular decomposition tree.

**Theorem 3.6.** Let \( G = (V, E) \) be a graph, and let \( T \) be its modular decomposition tree. Let \( v \) be a vertex of \( G \). By applying a Seidel complement at \( v \) the modular decomposition tree of \( T \ast v \) of \( G \ast v \) is obtained by:

- performing a Seidel complement in every prime node lying on the path from \( v \) to \( R(T) \).
• making \( P(v) \) the root of \( T \ast v \).
• Reverse the path from \( P(v) \) to \( R(T) \): if \( \alpha \) and \( \beta \) are prime node in \( T \) with \( \beta = P(\alpha) \) then \( \alpha = P(\beta) \) and \( \beta \) is connected in place of the subtree coming from \( v \).

Proof. If \( G \) is prime w.r.t. modular decomposition, its modular decomposition tree has only one internal node labeled prime, and the leaves represent the vertices. And we know by theorem 3.1 that \( G \ast v \) is also prime.

When the graph is not prime and is not a cograph, it admits a modular decomposition tree with more than one internal node. We have seen in Theorem 3.3 that when all the internal nodes are of type parallel or series, the statement holds.

It remains to deal with the case of prime nodes. We have to notice, as a consequence of Corollary 3.2, that any module that do not contain \( v \) are not impacted by the Seidel complement at \( v \). It means that only the modules that contain \( v \) are modified by the Seidel complement. And all the module that contain \( v \) are precisely the nodes lying on the path from \( v \) to \( R(T) \).

The next part of the proof is illustrated in Figure 5. Let us consider the case when the path from \( v \) to the root of \( T \) is constituted of two prime nodes \( \alpha \) and \( \beta \), with \( \beta = P(\alpha) \) and \( \alpha \) is connected to \( \beta \) on the vertex \( b \) and \( v \) is connected to \( \alpha \) on the vertex \( a \). Since \( \alpha \) is a module, all the vertices connected to \( \alpha \) are also connected to \( N_{\alpha}(b) \) the neighbors of \( b \) in \( \beta \). By performing a Seidel complementation at \( v \) we must remove the edges between \( N(v) \) and \( \overline{N(v)} \), in particular we must disconnect \( N_{\alpha}(a) \) from \( N_{\beta}(b) \) and we must connect \( \overline{N_{\beta}(b)} \) to \( N_{\alpha}(a) \).

By performing the Seidel complement in \( \alpha \) (resp. \( \beta \)) at \( a \) (resp. \( b \)), we satisfy the condition of Seidel complementation in each prime node. Now we need to realize the conditions above mentioned. By making now \( \alpha \) the root and by connecting \( \beta \) to \( a \) and connecting \( v \) to \( b \). The condition is realized. Now since \( \beta \) is a module attached under \( \alpha \) every vertex contained in \( \beta \) is connected to the the neighbors in \( \alpha \). And every vertex connected of \( N_{\alpha}(a) \) are no longer connected to \( N_{\beta}(b) \). We still have \( N_{\alpha}(a) \) completely connected to \( N_{\beta}(b) \) and \( \overline{N_{\alpha}(a)} \) disconnected from \( \overline{N_{\beta}(b)} \).

We can easily generalize this to paths of height greater than 2. \( \square \)

\[ \begin{array}{c}
\text{Figure 5. Effects of Seidel complementation on a modular decomposition tree.}
\end{array} \]
4. Permutation graphs

In this section we show that the class of permutation graphs is closed under Seidel minor, and we prove the main theorem that states that a graph is a permutation graph if and only if it does not contain any of the following graphs: $C_5$, $C_7$, $XF_6^2$, $XF_2^{2n+3}$, $C_{2n}$, $n \geq 6$ or their complements as Seidel minor.

**Definition 4.1** (Permutation graph). A graph $G = (V, E)$ is a permutation graph if there exist two permutations $\sigma_1, \sigma_2$ on $V = \{1, \ldots , n\}$, such that two vertices $u, v$ of $V$ are adjacent iff $\sigma_1(u) < \sigma_1(v)$ and $\sigma_2(v) < \sigma_2(u)$. $R = \{\sigma_1, \sigma_2\}$ is called a representation of $G$, and $G(R)$ is the permutation graph represented by $R$.

More properties of permutation can be found in [1]. An example of a permutation graph is presented in Figure 6.

![Figure 6. A permutation graph and its representation.](http://wwwteo.informatik.uni-rostock.de/isgci/classes/AUTO_3080.html)

**Theorem 4.2** (Gallai’67 [10]). A permutation graph is uniquely representable iff it is prime w.r.t. modular decomposition.

**Theorem 4.3** ([10]). A graph is a permutation graph if and only if it does not contain one of the finite graphs as induced subgraphs $T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}$ nor their complements and does not contain the graphs given by the infinite families: $XF_1^{2n+3}$, $XF_5^{2n+3}$, $XF_6^{2n+2}$, $XF_2^{n+1}$, $XF_3^n$, $XF_4^n$, the Holes, and their complements.

**Operation S**: Let $\sigma = A \cdot v \cdot B$ be a permutations on $[n]$. Let $v$ be an element of $[n]$. The operation $S$ at an element $v$ of $[n]$ noted $\sigma * v$ is done. Let $\sigma * v = B \cdot v \cdot A$.

**Remark 4.4.** Let $R = \{\sigma_1, \sigma_2\}$ be a permutation representation of a permutation graph $G$ then $R * v = \{\sigma_1 * v, \sigma_2 * v\}$ is a permutation representation of a graph $H$.

**Theorem 4.5.** Let $G = (V, E)$ be a permutation graph, and let $v$ be a vertex of $G$, and let $R = \{\sigma_1, \sigma_2\}$ be the permutation representation of $G$. We have $G(R * v) = G * v$.

**Proof.** Let $G = (V, E)$ be a permutation graph and $v$ a vertex of $G$. Let us prove that $G * v$ remains a permutation graph. Operation $S$ applied simultaneously on $R = \{\sigma_1, \sigma_2\}$ is depicted in Figure 9(a). Let $\sigma_1$ be $A \cdot v \cdot B$ and $\sigma_2$ be $C \cdot v \cdot D$.

The operation $S$ on $R = \{\sigma_1, \sigma_2\}$ corresponds to a Seidel complementation at $v$. We have to prove that the graphs induced by the neighborhood $G[N(v)]$ and $G[N(v)]$ are unchanged.

Let us begin with the non-neighborhood of $v$. It is easy to notice on Figure 9(a) that the non-neighborhood of $v$ is contained in the two vertical rectangles, one on the left of $v$ and the other one on their right, $(A, C)$ and $(B, D)$. By proceeding to the transformation described above, and by keeping the order of the words, it is easy to notice that first of all, these
vertices remain disconnected from \( v \) and since the order of vertices in the words are preserved then this subgraph remain unchanged. In a similar manner for the subgraph induced by the neighborhood of \( v \), now the vertices of their neighborhood are contained in the gray crosses \((A, D)\) and \((B, C)\) and for the same reason as for the non-neighborhood, the subgraph remains unchanged and it is still connected to \( v \).

Now let us consider the less obvious part which is to swap the adjacency between \( G[N(v)] \) and \( G[N(v)] \). Let \( w \) be a neighbor of \( v \) and let \( u \) be a non-neighbor of \( v \). Let us assume, \( w.l.o.g. \), that \( w \) and \( u \) are connected. Let us consider the case where \( u \) belongs to the \((A, C)\) rectangle and \( w \in (A, D) \), if \( uw \in E \) it means that \( \sigma_1(w) < \sigma_1(u) \) and \( \sigma_2(u) < \sigma_2(w) \), after proceeding to a Seidel complement at \( v \) we obtain \( \sigma'_1 = \sigma_1 \ast v \) and \( \sigma'_2 = \sigma_2 \ast v \) but now according to the transformation we have \( \sigma'_1(w) < \sigma'_1(u) \) and \( \sigma'_2(w) < \sigma'_2(u) \). And according to the definition 4.1 now \( u \) and \( w \) are no longer connected. The proof is similar for the other cases. \( \square \)

Corollary 4.6. The Seidel complementation at a vertex \( v \) of a permutation graph can be achieved in \( O(1) \)-time.
Proof. It is sufficient to consider the permutation representation of $G$ as two doubly linked lists. Then the Seidel complementation consists of applying the pattern described in the proof of Theorem 4.5. It consists w.l.o.g. on $\sigma_1$ to exchange $A$ and $B$: $A \cdot v \cdot B$ becomes $B \cdot v \cdot A$. So it suffices to change the successor of $v$ in the list as the first element of $A$ and the predecessor of $v$ as the last element of $B$. Then update the first and last element of the new list. We proceed similarly for $\sigma_2$. All these operations can obviously be done in constant time. □

An arbitrary remark. To perform a Seidel complementation at a vertex on a graph can require in the worst case $O(n^2)$-time. It suffices to consider the graph consisting of a star $K_{1,n}$ and a stable $S_n$, whose size is $2n + 1$ with $n + 1$ connected components. Applying a Seidel complementation on the vertex of degree $n$ results in a connected graph with $O(n^2)$ edges.

4.1. Finite Families. In this section we show that it is possible to reduce the list of forbidden induced subgraphs by using Seidel Complementation. Actually a lot of forbidden subgraphs are Seidel equivalent. The graphs that are Seidel complement equivalent are in the same box in Figure 7. Thus, the list of finite forbidden graphs is reduced from 18 induced subgraphs to only 6 finite Seidel minors. The forbidden Seidel minors are $C_5$, $C_7$, $X_5F_2^4$ and their complements.

Proposition 4.7. The graphs $X_3$, $X_2$, $X_36$ (cf. Figure 7(c)-(e)) are Seidel complement equivalent.

Proposition 4.8. The graphs $X_{30}$, $X_{32}$, $X_{33}$ and $X_{34}$ (cf. Figure 7(f)-(i)) are Seidel complement equivalent.

Proofs of propositions 4.7 and 4.8 are not presented here, they essentially consist for each graph to find which vertex allows us to transform one graph into another.

The following proposition show that two forbidden finite graphs contain actually an instance of a member of an infinite family as Seidel minor. Thus it is no longer necessary to keep them in the list of forbidden Seidel minors.

Proposition 4.9. The graph $X_5F_0^4$ is a Seidel minor of $T_2$ and $X_{31}$.

Proof. $X_5F_0^4 \leq_S T_2$
and $XF^0_4 <_S X_{31}$

**Proposition 4.10.** The graph $C_6$ is a Seidel minor of $XF^0_4$.

**Proof.** Applying a Seidel complementation on the degree 2 vertex of the $C_4$ in $XF^0_4$ we obtain $C_6$. □

4.2. **Infinite Families.** We show in this section that actually forbidden infinite families under the relation on induced subgraphs are redundant when the Seidel minor operation is considered. Consequently the following propositions allows us to reduce from 14 infinite families with the induced subgraph relation to only 4 infinite families under Seidel minor relation. The forbidden families are $XF^{2n+3}_5$ and $C_{2n}, n \geq 6$ and their complements.

**Proposition 4.11.** The Hole is a Seidel minor of $XF^3_4$, $XF^n_4$ and $XF^{n+1}_2$.

**Proof.**

$XF^{2n+1}_5$ is a Seidel minor of $XF^{2n+2}_6$.

**Proposition 4.13.** $XF^{2n+1}_5$ is a Seidel minor of $XF^{2n+3}_1$.

**Proposition 4.14.** $XF^{2n+1}_5$ is a Seidel minor of $C_{2n+3}$. 
4.3. Main Theorem.

**Definition 4.15 (Seidel Complement Stable).** A graph $G = (V, E)$ is said to be Seidel complement stable if: \( \forall v \in V : G \cong G * v \)

Few small graphs are Seidel complement stable, for instance, $P_4$, $C_5$, and more trivially $K_n$ the clique on $n$ vertices and $S_n$ the stable on $n$ vertices.

**Lemma 4.16.** The graph $XF_5^3$ is Seidel complement stable.

**Proof.** $XF_5^3$ is a path of length $n$ dominated by two non-adjacent vertices $C$ and $D$. In addition to that, a vertex $A$ is connected to $D$ and 1, and a vertex $B$ is connected to $C$ and $n + 1$. This graph is represented in Figure 8(f).

The degree sequence for this graph for $n \geq 1$ is $[2; 2; 4 \times n; n + 2; n + 2]$. Except for $n = 3$ the degree sequence allows us to “identify” the vertices. $A$ and $B$ are the vertices of degree 2, $C$ and $D$ are the vertices of degree $n + 1$ and the vertices of the path $[1, n + 1]$ are the vertices of degree 4.

Now let us formulate two easy observations. Since the graph presents of lot of symmetries, i.e. $A$ is equivalent to $B$; $C$ is equivalent to $D$. It suffices to check that the graph obtained after a Seidel complement on the following vertices will preserve the desired properties. So the set of vertices to consider is $\{A, D, 1, \ldots, [n + 1]\}$.

Now two easy observations: $G$ denotes $XF_5^3 \ : G \cong G * D$. Since $D$ is connected to $\{A, 1, \ldots, n + 1\}$. After the Seidel complement it means that $C$ is now connected to only $B$ and $A$. And it also means that $B$ is connected to $C$, and since $B$ was only connected to $n + 1$ in the original graph, it is now connected to $\{A, 1, \ldots, n\}$. So now the path consists of the vertices $\{A, 1, \ldots, n\}, B$ and $D$ dominate this path and $C$ and $n + 1$ constitute the extremities. The function $\varphi$ is given by this permutation.

$$\sigma = \begin{pmatrix} A & B & C & D & 1 & 2 & \ldots & n + 1 \\ 1 & D & A & C & 2 & 3 & \ldots & B \end{pmatrix}$$

Let us show now that $G \cong G * A$. By definition, the subgraph induced by $\{B, C, 2, \ldots, n + 1\}$ remains unchanged. The vertex 1 is now connected to $\{3, \ldots, n + 1, B\}$, and is still connected to $A$ and $D$. Concerning $D$, it is now only connected to $B$ and $C$ in $G[N(A)]$. So the bijection $\varphi$ is given by the following permutation:

$$\sigma = \begin{pmatrix} A & B & C & D & 1 & 2 & \ldots & n + 1 \\ A & n & C & n + 1 & D & B & \ldots & n - 1 \end{pmatrix}$$

It is easy to see that $G \cong G * 1$. The path is $3, 4, \ldots, n + 1, B, D, 1, C$. The vertex $A$ is connected to $\{3, 4, \ldots, n + 1, B, D, 1\}$ and the vertex $2$ is connected to $\{4, \ldots, n + 1, B, D, 1, C\}$. Let us consider the case for the vertex 2. Actually $G \cong G * 2$. The path is $4, 5, \ldots, n + 1, B, D, 2, C$. And the vertex 1 is connected to $\{4, 5, \ldots, n + 1, B, D, 2, C\}$. And the vertex 3 is connected to $\{5, \ldots, n + 1, B, D, 2, C, A\}$.

Concerning the vertices on the path, let us consider the case of their vertex $k$ such that $k \in [3, n - 1]$. It is clear that the graph $G[\{C, D, k - 1, k, k + 1\}]$ remains unchanged as for the graph $G[V \setminus \{C, D, k - 1, k, k + 1\}]$. The vertex $C$ is now connected to $A, k - 1, k$ and $k + 1$. So
it is 4. A similar thing happens for D. It is now connected to B, k−1, k and k+1. Concerning k−1 and k+1, k−1 is connected to every vertex except k−2 and k+1, so its degree is n + 2. And k+1 is connected to every vertex except k−1 and k+2. Concerning A and B their degrees are now equal to 4 (because of k−1 and k+1). And concerning the vertices k−2 and k+2 their degrees equal 2 because they are no longer connected to C, D, k±1 but are now connected to k±1 (i.e. k−1 and k+1 swap roles).

Now the extremities of the path are k−2 and k+2. The path is of the form: k−2, . . . , 1, A, C, k, D, B, n + 1, n, . . . , k + 2
Consequently the graph XF5n is Seidel complement stable.

Lemma 4.17. The Seidel stable class of the hole Cn consists of Cn, XF4n−6.

Due to lack of space the proof is omitted, but in a few words, it relies on the “regular” structure of XF5n and Lemma 4.16.

Theorem 4.18 (Main Theorem). A graph is a permutation graph if and only if it does not contain as finite graphs C5, C7 and XF5 and their complements and as infinite families XF2n+3 and C2n, n ⩾ 6 and their complements as Seidel minor.

Sketch of Proof. This theorem relies on Gallai’s result (cf. Theorem 4.3). If G is not a permutation graph then it contains one of the graphs listed in Theorem 4.3 as an induced subgraph. Thanks to previous propositions 4.7, 4.10 concerning the finite families, and propositions 4.11, 4.14 concerning the infinite families. We are able to reduce these induced subgraphs into a smaller set of graphs which are now forbidden Seidel minor. It remains to prove that this list is minimal. Concerning the infinite families, Lemma 4.16 proves that it is not possible to get rid of this families since it is Seidel stable. Concerning even holes (since odd holes are dismissed because they contain XF5n−1 as Seidel minors) Lemma 4.17 says that it is not possible to get rid of them. The same kind of argument holds for the finite graphs.

Corollary 4.19. The class of permutation graphs is not well quasi ordered under Seidel minor relation.

Proof. XF2n+3 constitutes an obstruction for permutation graphs. But for even values XF2n is a permutation graph. Furthermore, it is easy to check that for k and l two positive integers such that k < l, XF2k is not an induced subgraph of XF2l. Consequently the family XF2n is an infinite family of finite permutation graphs. Since XF5 is Seidel stable by Lemma 4.16, these graphs are not comparable each other with the Seidel minor relation. It is thus an infinite anti-chain for the Seidel minor relation and consequently permutation graphs are not well quasi ordered under Seidel minor relation.

5. Distance between Seidel complement equivalent graphs

In this section we show that if two graphs G and H are Seidel complement equivalent, they are at distance at most 1 from each other.

5.1. General remarks.

Lemma 5.1. Let G and H be two graphs, G and H are Seidel complement equivalent if and only if they are at distance at most 1.

Proof. If G and H are isomorphic they are Seidel complement equivalent by Definition 2.6. Let us assume that G and H are not isomorphic. As an observation of the proof of Lemma
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2.2 and Remark 2.4 we can notice that for any graph $G$ and any pair of vertices $v$ and $w$ of $G$ we have the following:

$$G_1 = G * v * w \cong G * w = G_2$$

This equality allows us to reduce, when given a sequence of Seidel complementation, by one. It is not longer reducible when the sequence is only of length 1 and by hypothesis since $G$ and $H$ are not isomorphic it is the best we can do.

**Corollary 5.2.** The number of graphs that are Seidel complement equivalent to a graph $G$ is at most $n + 1$.

**Corollary 5.3.** Seidel complement equivalence is polynomially reducible to Graph Isomorphism problem.

**Lemma 5.4.** Let $G$ and $H$ be two prime graphs w.r.t. modular decomposition, to decide if $G$ is isomorphic to $H$ is polynomially reducible to Seidel complement equivalence problem.

**Proof.** The reduction is as follows: $G'$ (resp. $H'$) is obtained from $G$ (resp. $H$) by adding a universal vertex $x$ (resp. $y$). As a consequence $G'$ has only one non-trivial module: $G$. The modular decomposition tree of $G'$ is composed of two internal nodes, the root is a series node with two children: the universal vertex and the prime node labeled by $G$.

**Claim:** $G$ is isomorphic to $H$ iff $G'$ is Seidel complement equivalent to $H'$.

**Proof.**

$(\Rightarrow)$ is obvious.

$(\Leftarrow)$ Any graph that is Seidel complement equivalent to $G'$ is actually isomorphic to $G'$. Since $G$ is a prime graph, its modular decomposition tree $T(G)$ is only a prime node labeled $G$. Since $G'$ is obtained from $G$ by adding a universal vertex to $G$, its modular decomposition tree is simply a series node as the root, his first child is the universal vertex and the second child is the modular decomposition tree of $G$. Thanks to Theorem 3.6 and 5.1, it is easy to realize that only one graph in the Seidel complement equivalence class of $G'$ possess a universal vertex. Actually performing a Seidel complement at any vertex attached to the prime node of $T(G')$ will result in graph without a universal vertex since the root of modular decomposition tree obtained will be a prime node. And since all the graph that are Seidel equivalent to a given graph are at distance at most 1, $G'$ and $H'$ are Seidel equivalent iff $G'$ and $H'$ are isomorphic if $G$ and $H$ are isomorphic.

It is enough to conclude.

5.2. The case of cographs and permutation graphs. In this section we show that for the class of cographs and permutation graphs. We can decide, in linear time for cographs and in quadratic time for permutation graphs if the graphs are given with their co-tree for cographs or with their intersection model for permutation graphs.

**Lemma 5.5.** To decide if two cographs $G$ and $H$ are Seidel complement equivalent can be computed in linear time $O(n)$.

**Proof.** Let us consider the co-trees $T(G)$ and $T(H)$. We modify $T(G)$ and $T(H)$ as follows: Let $T'(G)$ be the co-tree of $G$ on which we add a dummy vertex attached to the root of $T(G)$. We proceed in a similar manner for $T'(H)$.

$G$ and $H$ are Seidel complement equivalent if and only if $T'(G)$ and $T'(H)$ are isomorphic.

$(\Rightarrow)$ This direction is easy, since according to Remark 3.4 Theorem 3.3 and Lemma 5.1, if $G$ and $H$ are Seidel complement equivalent then $T'(G)$ and $T'(H)$ are isomorphic.

$\square$
Let us assume now that $T'(G)$ and $T'(H)$ are isomorphic and let $\varphi : V(T'(G)) \rightarrow V(T'(H))$ be the mapping function. The isomorphism considered here is the labeled isomorphism, i.e. labels of the internal nodes, 0 or 1, are preserved.

Using the result of Theorem 5.1, we know that cographs are at distance at most 1. It is thus sufficient to find the actual vertex to transform one co-tree into another. Let us call the dummy vertices added to turn $T(G)$ (resp. $T(H)$) into $T'(G)$ and $T'(H)$ $d_{u_G}$ and $d_{u_H}$. Now since we want to transform $T(H)$ into $T(G)$ it suffices to pick a vertex $f$ in $T(H)$ such that it is the image by $\varphi$ of $d_{u_G}$ i.e. $f = \varphi(d_{u_G})$. Once we have obtained this vertex in $T(G)$ it is sufficient to proceed to a Seidel complement on $f$, $H* f$, so now $P(f)$ is the root of $T(H* f)$ as requested since $f$ was an image of $d_{u_G}$ and $f$ is now attached to the former root $R(H)$. Consequently we have shown that when $T'(G)$ and $T'(H)$ are isomorphic we can find a vertex permitting us to transform $T(H)$ into $T(G)$ and hence proving that they are Seidel complement equivalent.

This procedure can be achieved in linear time, since deciding if two given trees are isomorphic is well known to be linear [1], and finding the actual vertex and performing the Seidel complementation is done in constant time.

Lemma 5.6. Let $G$ and $H$ be two permutation graphs given with their representation then we can decide in $O(n^2)$ if $G$ is Seidel complement equivalent to $H$.

Proof. From Lemma 5.1 we know that every Seidel complement equivalent graphs are at distance at most one from each other. It suffices, w.l.o.g. to apply on $H$ every possible Seidel complementation, and check if one the graph obtained is isomorphic to $G$. We can decide in $O(n)$ time if two permutation diagrams (and hence the corresponding permutation graphs). Finally, in the worst case we must try on all the vertices of $H$ if $G \sim H* v$. Seidel complementation on the permutation diagram can be performed in constant time, the final complexity is $O(n^2)$.

6. TOURNAMENTS

We present in this section a notion of Seidel complementation applied to tournaments. Let $T = (V, A)$ be a tournament. And let $v$ a vertex of $T$.

Definition 6.1. The Seidel complementation on a tournament $T$ applied on a vertex $v$ is defined as follows:

- reverse the direction of all the arcs lying between $N^+(v)$ and $N^-(v)$.
- reverse all the arcs incident to $v$, i.e. $N^+(v)$ becomes $N^-(v)$ and conversely.

Lemma 6.2. Let $T$ be a tournament and let $v$ be a vertex of $T$. $T$ is prime w.r.t. modular decomposition iff $T* v$ is prime.

Proof. The proof is almost the same as the proof of Theorem 3.1 it suffices to replace the edges and non edges by the arcs.

The modular decomposition tree of a tournament is modified in the same way as described in Theorem 3.6 for undirected graphs. It suffices to use Definition 6.1 instead of 2.1

7. CONCLUSION AND PERSPECTIVES

We have shown that the new paradigm of Seidel minor provides a nice and compact characterization of permutation graphs.
A lot of questions remain open. A natural question lies in the fact that Theorem 4.18 is obtained using Gallai’s result on forbidden induced subgraphs. Is it possible to give a direct proof of Theorem 4.18 without using Gallai’s result?

Another direction concerns graph decomposition. Oum [19] has shown that local complementation preserves rank-width. Is there a graph decomposition that is preserved by Seidel complementation?

Finally, it could be interesting to generalize the Seidel complement operator to directed graphs, and possibly to hypergraphs.

We hope that this Seidel minor will be relevant in the future as a tool to study graph decomposition and to provide similar characterizations, as the one presented for permutation graphs, to other graph classes.

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