Probability distribution of constrained Random Walks

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Abstract

In this paper we consider a sequence of $n$ coin tosses, whose outcome depends on the previous $n-1$ tosses. In particular, their distribution is not i.i.d. We compute the limiting distribution of this sequence using the method of images.

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1 Introduction

We study the following question - suppose a particle is taking a random walk on a finite rod. What is the chance that it will be at a particular position? How much will it deviate on an average? We answer this question using the method of images. Intuitively, one would expect that in the limiting case, the answer should be the fundamental solution of the Heat Equation on a finite rod. That is indeed the case. The idea we employ is very similar to the idea used in [1], where the authors solve the Schrödinger Equation for the particle in a box by explicitly computing the propagator.

To motivate our results, let us first recall the Central Limit Theorem (C.L.T), which is for i.i.d random variables.

Theorem 1.1. (Central Limit Theorem, c.f. [2]) Let $x$ be any real number and $t$ a positive real number. Toss a fair coin $n$ times and denote $n_H$ and $n_T$ to be the number of heads and tails respectively. Define

$$S_n^{x,t} := x + \left( \frac{n_H - n_T}{\sqrt{n}} \right) \sqrt{t}.$$ 

Then for any extended real numbers $a$ and $b$

$$\lim_{n \to \infty} \text{Prob}(a \leq S_n^{x,t} \leq b) = \int_a^b G(t, x, y) dy, \quad \text{where} \quad G(t, x, y) := \frac{1}{\sqrt{2\pi t}} \exp \left( - \frac{(y - x)^2}{2t} \right).$$
2 Main Results

We will now modify the setup of the C.L.T and consider coin tosses that are not i.i.d. The two main theorems we prove in this paper are:

**Theorem 2.1.** Let $x$ and $t$ be positive real numbers and $n \in \mathbb{N}$ a positive integer. Toss a coin exactly $n$ times. Let $n_H^k$ and $n_T^k$ denote the number of heads and tails respectively after $k$ tosses. Define

$$S_{x,t}^k := x + \left(\frac{n_H^k - n_T^k}{\sqrt{n}}\right)\sqrt{t}. \tag{1}$$

The coin is a “fair” coin except for one small difference - for each $k$, the quantity $S_{x,t}^k$ has to be strictly greater than zero. Then for any extended real numbers $a$ and $b$ such that $(a,b) \subset [0,\infty)$

$$\lim_{n \to \infty} \operatorname{Prob}(a \leq S_{n,t}^x \leq b) = \frac{\int_a^b \mathcal{H}(t,x,y)dy}{\int_0^\infty \mathcal{H}(t,x,y)dy}, \quad \text{where} \quad \mathcal{H}(t,x,y) := \mathcal{G}(t,x,y) - \mathcal{G}(t,x,-y).$$

**Theorem 2.2.** Let $L$ be a positive real numbers and $x$ a real number strictly between $0$ and $L$. Let $t$ be positive real numbers and $n \in \mathbb{N}$ a positive integer. Toss a coin exactly $n$ times. Denote $n_H^k$ and $n_T^k$ to be the number of heads and tails respectively after $k$ tosses and define

$$S_{x,t}^k := x + \left(\frac{n_H^k - n_T^k}{\sqrt{n}}\right)\sqrt{t}. \tag{2}$$

The coin is a “fair” coin except for one small difference - for each $k$, $S_{x,t}^k$ has to be strictly between $0$ and $L$. Then for any real numbers $a$ and $b$ such that $(a,b) \subset [0,L]$

$$\lim_{n \to \infty} \operatorname{Prob}(a \leq S_{n,t}^x \leq b) = \frac{\int_a^b \mathcal{K}(t,x,y)dy}{\int_0^L \mathcal{K}(t,x,y)dy}, \quad \text{where} \quad \mathcal{K}(t,x,y) := \frac{1}{L} \sum_{m=0}^{\infty} \sin \left(\frac{m\pi x}{L}\right) \sin \left(\frac{m\pi y}{L}\right) \exp \left(-\frac{m^2\pi^2 t}{L^2}\right).$$

**Remark 2.3.** The reader should observe that $\mathcal{G}(t,x,y)$, $\mathcal{H}(t,x,y)$ and $\mathcal{K}(t,x,y)$ are exactly the same as the fundamental solution for the Heat equation on an infinite rode, semi-infinite rod and finite rod respectively.

**Remark 2.4.** The reader should note that in equations (1) and (2), the denominator is $\sqrt{n}$, not $\sqrt{k}$. The $n$ is chosen before performing the sequence of tosses, and then we toss the coin exactly $n$ times. For example, let us consider the setup of Theorem 2.1. Suppose $n=9$, $x=3.01$, $t=1$ and we have tossed a coin three times, getting consecutive tails. On the fourth toss we will get a head with probability one; if we got a tail then $S_{x,t}^{x,3}$ would be negative. Secondly, once we have chosen $n=9$, we have decided that we will toss exactly $9$ times. A similar thing holds for the setup of Theorem 2.2, where $S_{x,t}^k$ has to stay strictly between $0$ and $L$. 


3 Correspondence between forbidden tosses and allowed tosses

The basic idea in proving Theorem 2.1 and 2.2 is that we establish a one to one correspondence between forbidden tosses and allowed tosses, after we make a suitable reflection. After that, the distribution function (in the case of Theorem 2.2) appears as an infinite sum. In order to evaluate the infinite sum, we write the summand as a Fourier Transform, and use the Poisson Summation formula. That produces the Heat Kernel for a finite rod. The author learned of this idea from [1].

First, let us set up some notation. We will identify a sequence of \( n \) tosses with an element

\[
\omega := \omega_1 \omega_2 \ldots \omega_n \in \{-1, 1\}^n,
\]

with 1 denoting heads and -1 denoting tails. For each \( k \leq n \), define

\[
S_k(\omega) := x + \frac{\sum_{i=1}^{k} \omega_i \sqrt{t}}{\sqrt{n}}.
\]

Next, given a set \( U \subset \mathbb{R} \) we define \( P_n(U) \) to be the set of all \( n \)-tosses \( \omega \) such that \( S_n(\omega) \) belongs to \( U \), i.e.

\[
P_n(U) := \{ \omega \in \{-1, 1\}^n : S_n(\omega) \in U \}.
\] (3)

Next, given a positive integer \( m \leq n \), we define the following two subsets of \( P_n(U) \):

\[
F_{n,m}^0(U) := \{ \omega \in P_n(U) : \exists i_1 < i_2 < \ldots < i_m \in \{0, 1, 2, \ldots n\},
\]

such that \( S_{i_2k-1}(\omega) \leq 0 \), \( S_{i_2k}(\omega) \geq L \) \( \forall k \in \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor + 1\} \},
\]

\[
F_{n,m}^L(U) := \{ \omega \in P_n(U) : \exists i_1 < i_2 < \ldots < i_m \in \{0, 1, 2, \ldots n\},
\]

such that \( S_{i_2k-1}(\omega) \geq L \), \( S_{i_2k}(\omega) \leq 0 \) \( \forall k \in \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor + 1\} \}.
\]

Observe that

\[
F_{n,m}^0(U) \cap F_{n,m}^L(U) = F_{n,m+1}^0(U) \cup F_{n,m+1}^L(U) \quad \forall m \in \{1, 2, \ldots, n-1\}. \] (4)

Finally, given a collection of real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \), we define

\[
U_{\lambda_1} := \{ x \in \mathbb{R} : 2\lambda_1 - x \in U \} \quad \text{and} \quad U_{\lambda_1, \lambda_2, \ldots, \lambda_k} := (U_{\lambda_1, \lambda_2, \ldots, \lambda_{k-1}})\lambda_k.
\]

The set \( U_{\lambda_1} \) is simply the reflection of the set \( U \) through the point \( x = \lambda_1 \). Similarly, \( U_{\lambda_1, \lambda_2, \ldots, \lambda_k} \) is obtained by first reflecting \( U \) through \( x = \lambda_1 \), then through \( x = \lambda_2 \) and so on until \( x = \lambda_k \).

We are now ready to establish a one to one correspondence between forbidden tosses and allowed tosses, after we take a suitable reflection.

**Proposition 3.1.** Let \( U \subset [0, \infty) \) be an open set and \( x \in (0, \infty) \). Then there exists a bijection between \( F_{n,1}^0(U) \) and \( P_n(U_0) \) if \( n \) is sufficiently large.
Proposition 3.2. Let $U \subset [0,L]$ be an open set and $x \in (0,L)$. If $n$ is sufficiently large, there exists a bijection

$$F^0_{n,m}(U) \iff P_n(U_{0,-L,-2L,\ldots, -(m-1)L})$$

and

$$F^L_{n,m}(U) \iff P_n(U_{L,2L,\ldots,mL})$$

for every $m \in \{1,2,\ldots,n\}$.

Proof of Proposition 3.1: We will explicitly show the bijection. Let $\omega \in F^0_{n,1}(U)$ and $i_1$ be the smallest integer such that $S_{i_1}(\omega) \leq 0$. Define the map

$$\Phi : F^0_{n,1}(U) \to P_n(U_0),$$

given by $\Phi(\omega) := \omega_1 \ldots \omega_{i_1}^* \omega_{i_1+1}^* \ldots \omega_{n}^*$,

where $1^* := -1$ and $-1^* := 1$. It is easy to see that $\Phi$ maps to $P_n(U_0)$ if $n$ is large and $U$ is open. Next, suppose $\omega \in P_n(U_0)$ and $i_1$ be the smallest integer such that $S_{i_1}(\omega) \leq 0$. Define the map

$$\Psi : P_n(U_0) \to F^0_{n,1}(U),$$

given by $\Psi(\omega) := \omega_1 \ldots \omega_{i_1}^* \omega_{i_1+1}^* \ldots \omega_{n}^*$.

It is easy to see that $\Psi$ maps to $F^0_{n,1}(U)$ if $n$ is sufficiently large and $U$ is open. It is also immediate that $\Phi$ and $\Psi$ are inverses of each other, since

$$\Phi \circ \Psi = \text{Id} \quad \text{and} \quad \Psi \circ \Phi = \text{Id}.$$ This proves Proposition 3.1.

Proof of Proposition 3.2: We will prove (5); the proof of (6) is identical. Let $\omega \in F^0_{n,m}(U)$ and $i_1, i_2, i_3, \ldots, i_m$ be the unique collection of $m$ integers such that $i_1$ is the smallest integer such that $S_{i_1}(\omega) \leq 0$ and for all $r \in \{2, \ldots, m\}$, $i_r$ is the smallest integer such that $i_r > i_{r-1}$ and $S_{i_r}(\omega)$ is either less than or equal to zero or greater than or equal to $L$, depending on whether $r$ is odd or even respectively. Now we define the map

$$\Phi : F^0_{n,m}(U) \to P_n(U_{0,-L,-2L,\ldots, -(m-1)L}),$$

given by $\Phi(\omega) := \omega_1 \ldots \omega_{i_1}^* \omega_{i_1+1}^* \ldots \omega_{i_2}^* \omega_{i_2+1} \ldots \omega_{i_3} \ldots$.

It is easy to see that $\Phi$ maps to $P_n(U_{0,-L,-2L,\ldots, -(m-1)L})$ if $n$ is sufficiently large and $U$ is open.

Next, suppose $\omega \in P_n(U_{0,-L,-2L,\ldots, -(m-1)L})$ and $i_1, i_2, i_3, \ldots, i_m$ be the unique collection of $m$ integers such that $i_1$ is the smallest integer such that $S_{i_1}(\omega) \leq 0$ and for all $r \in \{2, \ldots, m\}$, $i_r$ is the smallest integer such that $i_r > i_{r-1}$ and $S_{i_r}(\omega)$ is less than or equal to $-(r-1)L$. Now we define the map

$$\Psi : P_n(U_{0,-L,-2L,\ldots, -(m-1)L}) \to F^0_{n,m}(U),$$

given by $\Psi(\omega) := \omega_1 \ldots \omega_{i_1}^* \omega_{i_1+1}^* \ldots \omega_{i_2}^* \omega_{i_2+1} \ldots \omega_{i_3} \ldots$.

It is easy to see that $\Psi$ maps to $F^0_{n,m}(U)$ if $n$ is sufficiently large and $U$ is open. It is also immediate that $\Phi$ and $\Psi$ are inverses of each other, since

$$\Phi \circ \Psi = \text{Id} \quad \text{and} \quad \Psi \circ \Phi = \text{Id}.$$ This proves (5). The proof of (6) is identical.
4 Proofs of the main results

We are now ready to prove Theorem 2.1 and 2.2.

Proof of Theorem 2.1: First, we observe that for sufficiently large $n$

\[
\text{Prob}\left(a \leq S_{n}^{x,t} \leq b\right) = \frac{|P_n((a,b))| - |F_{n,1}^0((a,b))|}{|P_n((0,\infty))| - |F_{n,1}^0((0,\infty))|} = \frac{|P_n((a,b))| - |P_n((-b,-a))|}{|P_n((0,\infty))| - |P_n((-\infty,0))|}
\]

by Proposition 3.1

\[
\lim_{n \to \infty} \text{Prob}\left(a \leq S_{n}^{x,t} \leq b\right) = \frac{\int_{a}^{b} G(t,x,y)dy - \int_{-\infty}^{0} G(t,x,y)dy}{\int_{-\infty}^{0} G(t,x,y)dy - \int_{0}^{\infty} G(t,x,y)dy}
\]

by C.L.T

\[
\frac{b}{\int_{-\infty}^{\infty} H(t,x,y)dy} = \frac{a}{\int_{0}^{\infty} H(t,x,y)dy}
\]

Proof of Theorem 2.2: Let $F_n((a,b))$ denote the set of all forbidden $n$-tosses in the setup of this Theorem. Note that

\[
F_n((a,b)) = F_{n,1}^0((a,b)) \cup F_{n,1}^L((a,b)).
\]

Using equations (7), (4), Proposition 3.2 and the Inclusion Exclusion Principle, we conclude that

\[
|P_n((a,b))| - |F_n((a,b))| = \sum_{m=-\infty}^{\infty} P_n((2mL + a, 2mL + b)) - P_n((2mL - b, 2mL - a))
\]

Note that the above expression is actually a finite sum; when $m$ is sufficiently large the terms will
become zero. Hence, we conclude that for large \( n \),
\[
\text{Prob}(a \leq S_n^{x,t} \leq b) = \frac{|P_n((a, b))| - |F_n((0, L))|}{|P_n((0, L))| - |F_n((0, L))|} \\
= \sum_{m=-\infty}^{\infty} P_n((2mL + a, 2mL + b)) - P_n((2mL - b, 2mL - a))
\]
\[
= \sum_{m=-\infty}^{\infty} P_n((2mL + a, 2mL + L)) - P_n((2mL - L, 2mL))
\]
\[
= \sum_{m=-\infty}^{\infty} \left( P_n((2mL + a, 2mL + b)) - P_n((2mL - b, 2mL - a)) \right) \times 2^{-n}
\]
\[
= \sum_{m=-\infty}^{\infty} \left( P_n((2mL + a, 2mL + L)) - P_n((2mL - L, 2mL)) \right) \times 2^{-n}
\]
\[
\Rightarrow \lim_{n \to \infty} \text{Prob}(a \leq S_n^{x,t} \leq b) = \frac{\int K(t, x, y)dy}{\int K(t, x, y)dy},
\] where
\[
K(t, x, y) := \sum_{m=-\infty}^{\infty} \mathcal{G}(t, x, y + 2mL) - \mathcal{G}(t, x, -y + 2mL).
\]

To go from (8) to (9), we interchanged the order of limit and summation; this is justified in section 5. Finally, we evaluate \( K \) by writing \( \mathcal{G} \) as a Fourier Transform and using the Poisson Summation Formula. Let
\[
f(k) := \exp(-2\pi^2tk^2 + 2\pi k \sqrt{-1}(x - y)) - \exp(-2\pi^2tk^2 + 2\pi k \sqrt{-1}(x + y)).
\]

Then we get that
\[
K(t, x, y) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(k) \exp(-4\pi \sqrt{-1}mkL)dk
\]
\[
= \frac{1}{2L} \sum_{m=-\infty}^{\infty} f\left(\frac{m}{2L}\right) \quad \text{(by the Poisson Summation Formula, c.f. [3])}
\]
\[
= \frac{1}{L} \sum_{m=-\infty}^{\infty} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \exp\left(-\frac{m^2\pi^2t}{L^2}\right).
\]

5 Justifying the interchange of limit and summation

We now justify the interchange of limit and summation that was used in going from (8) to (9). First, we state a couple of well known inequalities.

**Theorem 5.1. (Stirling’s Inequality)** For all \( n \in \mathbb{N} \), we have
\[
\sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n} \leq n! \leq en^{n+\frac{1}{2}} e^{-n}.
\]

**Theorem 5.2. (Bernoulli’s Inequality)** Let \( x, r \in \mathbb{R} \), such that \( x > -1 \) and \( r \geq 1 \). Then
\[
\frac{1}{(1 + x)^r} \leq \frac{1}{1 + rx}.
\]
Proposition 5.3. Let $U$ be a subset of $\mathbb{R}$ and $P_n(U)$ be as defined in (3). Define

$$\alpha_{m,n} := \frac{|P_n((2mL + a, 2mL + b))|}{2^n} \quad \text{and} \quad \beta_{m,n} := \frac{|P_n((2mL - b, 2mL - a))|}{2^n}.$$ 

Then there exists a constant $C$ (independent of $m$ and $n$), such that

$$\alpha_{m,n} \leq \frac{C}{\left(1 - \frac{(2mL + a - x)^2}{2t}\right)^2 \left(1 - \frac{(2mL + a - x)^2}{t}\right)} \quad (10)$$

$$\beta_{m,n} \leq \frac{C}{\left(1 - \frac{(2mL - b - x)^2}{2t}\right)^2 \left(1 - \frac{(2mL - b - x)^2}{t}\right)} \quad (11)$$

Furthermore,

$$\lim_{n \to \infty} \sum_{m=-\infty}^{\infty} (\alpha_{m,n} - \beta_{m,n}) = \sum_{m=-\infty}^{\infty} \lim_{n \to \infty} (\alpha_{m,n} - \beta_{m,n}). \quad (12)$$

Proof: We note that

$$\alpha_{m,n} \leq \sum_{i=0}^{b-a+1} \left(\frac{n}{2} - \frac{n}{2\sqrt{t}} \sqrt{n} \right) \leq (b-a+1) \left(\frac{n}{2} - \frac{n}{2\sqrt{t}} \sqrt{n} \right) \leq \frac{C'}{\sqrt{n} \left(1 - \frac{(2mL + a - x)^2}{t}\right)^{\frac{3}{2}} \left(1 - \frac{(2mL + a - x)^2}{2t}\right)^{\frac{1}{4}} \sqrt{n}}$$

(\text{using Theorem 5.1})

$$\leq \frac{C'}{\left(1 - \frac{(2mL + a - x)^2}{2t}\right)^2 \left(1 - \frac{(2mL + a - x)^2}{t}\right)} \quad (\text{using Theorem 5.2}),$$

where $C'$ is some constant. Inequality (11) follows from (10) by replacing $a$ by $-b$ and $b$ by $-a$. Finally, (12) follows from (10), (11), the triangle inequality and the Dominated Convergence Theorem (since the rhs of (10) and (11) are infinite summable). Hence, (8) implies (9).

\[\square\]

6 Further remarks

We end this paper with a few natural questions that one can investigate in the future.

Question 6.1. Can this idea of “method of images” be used to find the limiting distribution of constrained random walks defined on a free group (or a finitely generated group)? After all, it seems quite natural to extend the idea of taking a suitable reflection on any group.

Question 6.2. One can consider higher dimensional analogues of Theorem 2.1 and 2.2. Is the distribution function the same as the fundamental solution of the corresponding higher dimensional Heat Equation?
**Question 6.3.** Do the random variables considered in Theorem 2.1 and 2.2 satisfy a Large Deviation Principle? If yes, is it possible to compute the rate function explicitly?

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