A SURVEY ON TINGLEY’S PROBLEM FOR OPERATOR ALGEBRAS

ANTONIO M. PERALTA

ABSTRACT. We survey the most recent results on extension of isometries between special subsets of the unit spheres of \( C^* \)-algebras, von Neumann algebras, trace class operators, preduals of von Neumann algebras, and \( p \)-Schatten-von Neumann spaces, with special interest on Tingley’s problem.

1. Introduction

The problem of extending a surjective isometry between two subsets of the unit spheres of two operator algebras was treated in several talks during the conference on preserver problems held in Szeged in June 2017. The conference “Preservers Everywhere” gathered a substantial group of world experts on preservers problems. It became clear that the problems regarding the extension of this type of surjective isometries constitute an intensively studied line in recent times. Let us try to unify all these problems in the following statement.

Problem 1.1. Let \( X \) and \( Y \) be two Banach spaces whose unit spheres are denoted by \( S(X) \) and \( S(Y) \), respectively. Let \( S_1 \) and \( S_2 \) be two subsets of \( S(X) \) and \( S(Y) \), respectively. Suppose \( \Delta : S_1 \to S_2 \) is a surjective isometry. Does \( \Delta \) extend to a real linear isometry from \( X \) onto \( Y \)?

Henceforth, we shall write \( \mathbb{T} \) for the unit sphere of \( \mathbb{C} \). The complex conjugation on \( \mathbb{T} \) cannot be extended to a complex linear isometry on \( \mathbb{C} \). So, in the case of complex Banach spaces, a complex linear extension is simply hopeless for all cases. Similar constrains will appear in subsequent results.

These problems, whose origins are in geometry, are nowadays a central topic for those researchers working on preservers. If in Problem 1.1 we consider \( S_1 = S(X) \) and \( S_2 = S(Y) \) we meet the so-called Tingley’s problem. This problem was named after the contribution of D. Tingley, who established that for any two finite dimensional Banach spaces \( X \) and \( Y \), every surjective isometry \( \Delta : S(X) \to S(Y) \) preserves antipodal points, that is, \( \Delta(-x) = -\Delta(x) \), for every \( x \) in \( S(X) \) (see [53, THEOREM in page 377]). Tingley’s problem remains open even in the case of two dimensional Banach spaces.

Let us observe that, given a surjective isometry \( \Delta : S(X) \to S(Y) \), where \( X \) and \( Y \) are Banach spaces, we can always consider the natural (positively) homogeneous extension \( F_\Delta : X \to Y \) given by \( F_\Delta(0) = 0 \), and \( F_\Delta(x) = \|x\| \Delta \left( \frac{x}{\|x\|} \right) \) for \( x \neq 0 \).

2010 Mathematics Subject Classification. Primary 47B49, Secondary 46A22, 46B20, 46B04, 46A16, 46E40.

Key words and phrases. Tingley’s problem; extension of isometries; von Neumann algebra; \( p \)-Schatten-von Neumann, trace class operators.
Clearly, $F_\Delta$ is a bijection, however it is a hard question to decide whether $F_\Delta$ is an isometry. Actually, the Mazur-Ulam theorem implies that $F_\Delta$ is real linear as soon as it is an isometry.

We have already found our first connection with the Mazur-Ulam theorem. Tingley’s problem and Problem 1.1 can be considered as generalization of this pioneering result in Functional Analysis. P. Mankiewicz established in 1972 an intermediate result which provides an useful tool for our purposes.

**Theorem 1.2.** [31, Theorem 5 and Remark 7] *Every bijective isometry between convex sets in normed linear spaces with nonempty interiors admits a unique extension to a bijective affine isometry between the corresponding spaces.*

During the thirty years elapsed after Tingley’s paper, a lot of hard efforts from many authors, especially many Chinese mathematicians, and the elite Chinese group leaded by G.G. Ding, have been conducted in the seeking of a solution to Tingley’s problem in concrete spaces. The huge contribution due to mathematicians like R.S. Wang, G.G. Ding, D. Tan, L. Cheng, Y. Dong, X.N. Fang, J.H. Wang, and R. Liu, among others, have been overview in full detail in the excellent surveys published by G.G. Ding [12] and X. Yang and X. Zhao [55].

A reborn interest on the problems concerning extension of isometries between subsets of the unit spheres of two operator algebras has been materialized in a fruitful series of recent papers dealing with Tingley’s problem and related questions for certain operator algebras, which have been published during the short interval determined by the last three years. The abundance of new results for operator algebras motivates and justifies the writing of this survey with the aim of completing and updating the surveys [12, 55], and providing a recent state of the art of these problems. The real “avalanche” of recent achievements provides enough material to write a new and detailed survey on this topic.

We strive for conciseness and for restrict the results to the setting of operator algebras, despite that some of the results have been already extended to the strictly wider setting of JB$^*$-triples (compare [24, 26]). So, few or none proofs are explicitly included. The main tools and results are reviewed with a full bibliographic information. We shall also insert some new arguments to establish some additional statements.

In section 2 we gather some of the key tools applied in many of the proofs given to solve Tingley’s problem. Most of the studies make use of a result, which was originally established by L. Cheng, Y. Dong in [5], and proves that a surjective isometry \( \Delta : S(X) \to S(Y) \) between the unit spheres of two Banach spaces, maps maximal proper faces of the closed unit ball of \( X \) to maximal proper faces of the closed unit ball of \( Y \) (see Theorem 2.2). The section also contains a recent generalization of this result due to F.J. Fenández-Polo, J. Garcés, I. Villamaya and the author of this note, which assures the following: Let \( \Delta : S(X) \to S(Y) \) be a surjective isometry between the unit spheres of two Banach spaces, and suppose that these spaces satisfy the following two properties:

(h.1) Every norm closed face of \( B_X \) (respectively, of \( B_Y \)) is norm-semi-exposed;
(h.2) Every weak$^*$ closed proper face of \( B_{X^*} \) (respectively, of \( B_{Y^*} \)) is weak$^*$-semi-exposed.

Then the following statements hold:
(a) Let $\mathcal{F}$ be a convex set in $S(X)$. Then $\mathcal{F}$ is a norm closed face of $B_X$ if and only if $\Delta(\mathcal{F})$ is a norm closed face of $B_Y$;
(b) Let $e \in S(X)$. Then $e \in \partial_e(B_X)$ if and only if $\Delta(e) \in \partial_e(B_Y)$
(see Corollary 2.3).

It should be remarked that hypotheses (h.1) and (h.2) above hold whenever $X$ and $Y$ are $C^*$-algebras, hermitian parts of $C^*$-algebras, von Neumann algebra preduals, preduals of the hermitian part of a von Neumann algebra, $JB^*$-triples, and $JBW^*$-triple preduals (see [21] and the comments after Corollary 2.3).

In section 2 we shall also survey the main results on the facial structure of the closed unit ball of a $C^*$-algebra due to C.A. Akemann and G.K. Pedersen [1] and C.M. Edwards and G.T. Rüttimann [17].

Section 3 is completely devoted to present the most recent achievements on Tingley’s problem in the setting of $C^*$-algebras. In all sections we shall insert an introductory paragraph with the equivalent results in the commutative setting. We begin from the results by R. Tanaka, which assure that every surjective isometry from the unit sphere of a finite dimensional $C^*$-algebra $A$ onto the unit sphere of another $C^*$-algebra $B$ extends to a unique surjective real linear isometry from $A$ onto $B$, and the same conclusion holds when $A$ and $B$ are finite von Neumann algebras (see Theorem 3.4). In Theorem 3.8 we revisit the solution to Tingley’s problem for surjective isometries between the unit spheres of two compact $C^*$-algebras found by R. Tanaka and the author of this survey in [43]. This solution also covers the case of a surjective isometry between the unit spheres of two $K(H)$ spaces. In this note $K(H)$ and $B(H)$ will denote the spaces of compact and bounded linear operators on a complex Hilbert space $H$, respectively.

Accordingly to the chronological order, the next step in the study of Tingley’s problem on $C^*$-algebras is a result by F.J. Fernández-Polo and the author of this note, which shows that for any two complex Hilbert spaces $H_1$ and $H_2$, every surjective isometry $\Delta : S(B(H_1)) \rightarrow S(B(H_2))$ admits a unique extension to a surjective complex linear or conjugate linear surjective isometry $T$ from $B(H_1)$ onto $B(H_2)$ satisfying $\Delta(x) = T(x)$, for every $x \in S(B(K))$ (see Theorem 3.9). The most conclusive result on Tingley’s problem has been also obtained by the same authors in a result showing that every surjective isometry $\Delta : S(M) \rightarrow S(N)$ between the unit spheres of two von Neumann algebras admits a unique extension to a surjective real linear isometry $T : M \rightarrow N$. Furthermore, under these hypotheses, there exist a central projection $p$ in $N$ and a Jordan $*$-isomorphism $J : M \rightarrow N$ such that defining $T : M \rightarrow N$ by $T(x) = \Delta(1)(pJ(x) + (1 - p)J(x)^*)$ ($x \in M$), then $T$ is a surjective real linear isometry and $T|_{S(M)} = \Delta$ (see Theorem 3.15).

Section 4 is devoted to survey the results on Tingley’s problem for surjective isometries between the unit spheres of von Neumann algebra preduals. In [21], F.J. Fernández-Polo, J. Garcés, I. Villanueva and the author of this survey gave a complete solution to Tingley’s problem for surjective isometries on the unit sphere of the space $C_1(H)$ of trace class operators on an arbitrary complex Hilbert space $H$ (see Theorem 4.5).

It is well known that the space $C_1(H)$ identifies with the dual of the space $K(H)$ and with the predual of $B(H)$. It seems a natural question whether the previous positive solution to Tingley’s problem in the setting of trace class operators remains true for preduals of general von Neumann algebras.
When the writing of this survey was being completed (precisely, on December 27th, 2017), an alert message came to this author from arxiv. This alert was about a very recent preprint by M. Mori (see [33]), which has been an impressive discovering, and made this author change the original project to insert some nice achievements, one of them is a positive solution to Tingley’s problem for preduals of general von Neumann algebras (see Theorem 4.6).

Henceforth, the hermitian part of a C*-algebra $A$ will be denoted by $A_{sa}$. As we commented before, when in Problem 1.1 the subsets $S_1$ and $S_2$ are the unit spheres of two Banach spaces, we find the so-called Tingley’s problem. Another interesting variant of Problem 1.1 is obtained when $X$ and $Y$ are von Neumann algebras or C*-algebras and $S_1$ and $S_2$ are the unit spheres of their respective hermitian parts. In Section 5, we shall study the problem of extending a surjective isometry $\Delta : S(M_{sa}) \to S(N_{sa})$, where $M$ and $N$ are von Neumann algebras. In this section we shall show that the same tools given by F.J. Fernández-Polo and the author of this survey in [27] can be, almost literarily, applied to find a surjective complex linear isometry $T : M \to N$ satisfying $T(a^*) = T(a)^*$ for all $a$ in $M$ and $T(x) = \Delta(x)$ for all $x$ in $S(M_{sa})$ (see Theorem 5.8).

It should be remarked here that, after completing the writing of this chapter, the preprint by M. Mori [33] became available in arxiv. Section 5 in [33] is devoted to the study of Tingley’s problem for surjective isometries between the unit spheres of the hermitian parts of two von Neumann algebras, and our Theorem 5.8 is also established by M. Mori with a alternative proof.

The sixth and final section of this paper is devoted to review the main result on a topic which had its own protagonism in the meeting held in Szeged. We are talking about the problem of extending a surjective isometry between the sets of positive norm-one operators of two type I von Neumann factors $B(H_1)$ and $B(H_2)$. During the talk presented by G. Nagy in this conference, he presented a recent achievement which shows that for a finite dimensional complex Hilbert space $H$, every isometry $\Delta : S(B(H)^+) \to S(B(H)^+)$ admits a (unique) extension to a surjective complex linear isometry $T : B(H) \to B(H)$ satisfying $T(x) = \Delta(x)$ for all $x \in S(B(H)^+)$ (see Theorem 6.5), where for a C*-algebra $A$, the symbol $A^+$ will denote the cone of positive elements in $A$, and $S(A^+)$ will stand for the sphere of positive norm-one operators. It was conjectured by Nagy that the same conclusion holds for every complex Hilbert space $H$.

We culminate this section, and the results in this note, by surveying a recent work where we provide a proof to Nagy’s conjecture. The main result is treated in Theorem 6.10, where it is shown that every surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$, where $H_1$ and $H_2$ are complex Hilbert spaces, admits an extension to a surjective complex linear isometry (actually, a *-isomorphism or a *-anti-automorphism) $T : B(H_1) \to B(H_2)$.

We shall revisit one of the main tools employed to establish the above result. This tool is a geometric characterization of projections in atomic von Neumann algebras. Let us recall some notation first. Suppose that $E$ and $P$ are non-empty subsets of a Banach space $X$. Following the notation employed in the recent paper [42], the unit sphere around $E$ in $P$ is the set

$$Sph(E; P) := \{x \in P : \|x - b\| = 1 \text{ for all } b \in E\}.$$
To simplify the notation, given a C*-algebra $A$, and a subset $E \subset A$, we shall write $\text{Sph}^+(E)$ or $\text{Sph}^+(A)$ for the set $\text{Sph}(E; S(A^+))$. The geometric characterization of projections reads as follows: let $M$ be an atomic von Neumann algebra, and let $a$ be a positive norm-one element in $M$. Then the following statements are equivalent:

(a) $a$ is a projection;
(b) $\text{Sph}^+_M(\text{Sph}^+_M(a)) = \{a\}$.

(see Theorem 6.6). This characterization also holds when $M$ is replaced by $K(H_3)$, where $H_3$ is a separable complex Hilbert space (Theorem 6.8). Moreover, if $a$ is a positive norm-one element in an arbitrary C*-algebra $A$ satisfying $\text{Sph}^+_A(\text{Sph}^+_A(a)) = \{a\}$, then $a$ is a projection (see [42, Proposition 2.2]).

This geometric characterization has been also applied to prove that if $H_3$ and $H_4$ are separable complex Hilbert spaces, then every surjective isometry $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$ admits a unique extension to a surjective complex linear isometry $T$ from $K(H_3)$ onto $K(H_4)$ (see Theorem 6.9).

2. Geometric background

In this section we survey the basic geometric tools which are frequently applied in most of the studies extending isometries. The results gathered in this section are established in the general setting of Banach spaces.

A non-empty convex subset $F$ of a convex set $C$ is said to be a face of $C$ if $\alpha x + (1 - \alpha)y \in F$ with $x, y \in C$ and $0 < \alpha < 1$ implies $x, y \in F$. An element $x$ in the unit sphere of a Banach space $X$ is an extreme point of $B_X$ precisely when the set $\{x\}$ is a face of $B_X$. Accordingly to the standard notation, from now on, the extreme points of a convex set $C$ will be denoted by $\partial e(C)$. The Krein-Milman theorem is a fantastic tool to assure the existence and abundance of extreme points in any non-empty compact convex subset of a locally convex, Hausdorff, topological vector space.

Up to now, most of the studies on Tingley’s problem are based on a good and appropriate knowledge of the geometric properties of the involved spaces. This is because the most general geometric conclusion which can be derived from the existence of a surjective isometry between the unit spheres of two Banach spaces is the following result, which was originally established by L. Cheng and Y. Dong [5], and later rediscovered by R. Tanaka [50, 49]. From now on, given a normed space $X$, the symbol $B_X$ will stand for the closed unit ball of $X$.

**Theorem 2.1.** ([5, Lemma 5.1], [50, Lemma 3.3], [49, Lemma 3.5]) Let $\Delta : S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and let $M$ be a convex subset of $S(X)$. Then $M$ is a maximal proper face of $B_X$ if and only if $\Delta(M)$ is a maximal proper (closed) face of $B_Y$.

As we commented at the introduction, Tingley’s problem remains open even in the case of two dimensional Banach spaces, the reason, probably, being the lacking of a concrete description of the maximal convex subsets of the unit sphere of a general Banach space.
All strategies based on Theorem 2.1 above require a concrete description of the maximal proper norm-closed faces of $B_X$ in terms of the algebraic or geometric properties of $X$. This is the point where the results of C.A. Akemann and G.K. Pedersen [1], C.M. Edwards and G.T. Rüttimann [17], C.M. Edwards, F.J. Fernández-Polo, C. Hoskin and A.M. Peralta [14], and F.J. Fernández-Polo and A.M. Peralta [23], describing the facial structure of the closed unit ball of $C^*$-algebras, von Neumann algebra preduals, $JB^*$-triples and their dual spaces, and $JBW^*$-triples and their preduals, become an useful tool.

We recall now the “facear” and “pre-facear” operations introduced in [17]. For each $F \subseteq B_X$ and $G \subseteq B_{X^*}$, we define

$$F' = \{ a \in B_{X^*} : a(x) = 1 \ \forall x \in F \}, \ G_i = \{ x \in B_X : a(x) = 1 \ \forall a \in G \}.$$  

Then, $F'$ is a weak$^*$ closed face of $B_{X^*}$ and $G_i$ is a norm closed face of $B_X$. The subset $F$ is said to be a norm-semi-exposed face of $B_X$ if $F = (F')'$, while the subset $G$ is called a weak$^*$-semi-exposed face of $B_{X^*}$ if $G = (G_i)'$. The mappings $F \mapsto F'$ and $G \mapsto G_i$ are anti-order isomorphisms between the complete lattices $S_i(B_X)$ of norm-semi-exposed faces of $B_X$, and $S_{i^*}(B_{X^*})$ of weak$^*$-semi-exposed faces of $B_{X^*}$ and are inverses of each other.

If in Theorem 2.1 we assume a richer geometric structure on the spaces $X$ and $Y$, then the conclusion of this result was improved in a recent paper by F.J. Fernández-Polo, J. Garcés, I. Villanueva and the author of this note in [21].

**Theorem 2.2.** [21, Proposition 2.4] Let $\Delta : S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and let $C$ be a convex subset of $S(X)$. Suppose that for every extreme point $\phi_0$ in $\partial_c(B_{X^*})$, the set $\{ \phi_0 \}$ is a weak$^*$-semi-exposed face of $B_{X^*}$. Then $C$ is a norm-semi-exposed face of $B_X$ if and only if $\Delta(C)$ is a norm-semi-exposed face of $B_Y$.

The real interest of the previous theorem is the following corollary.

**Corollary 2.3.** [21, Corollary 2.5] Let $X$ and $Y$ be Banach spaces satisfying the following two properties:

1. Every norm closed face of $B_X$ (respectively, of $B_Y$) is norm-semi-exposed;
2. Every weak$^*$ closed proper face of $B_{X^*}$ (respectively, of $B_{Y^*}$) is weak$^*$-semi-exposed.

Let $\Delta : S(X) \to S(Y)$ be a surjective isometry. The following statements hold:

(a) Let $F$ be a convex set in $S(X)$. Then $F$ is a norm closed face of $B_X$ if and only if $\Delta(F)$ is a norm closed face of $B_Y$;

(b) Let $e \in S(X)$. Then $e \in \partial_c(B_X)$ if and only if $\Delta(e) \in \partial_c(B_Y)$.

As it is observed in [21], the hypotheses of the above corollary hold whenever $X$ and $Y$ are $C^*$-algebras [1, Theorems 4.10 and 4.11], hermitian parts of $C^*$-algebras (see [16, Corollary 5.1] and [1, Theorem 3.11]), von Neumann algebra preduals [17, Theorems 5.3 and 5.4], preduals of the hermitian part of a von Neumann algebra (see [15, Theorem 4.4] and [17, Theorem 4.1]), or more generally, $JB^*$-triples (cf. [14, Corollary 3.11] and [23, Corollary 1]), or $JBW^*$-triple preduals [17, Corollaries 4.5 and 4.7].

By extending a result of D. Tingley [53, §4], M. Mori has recently added in [33, Proposition 2.3] more information to the conclusion of the above Corollary 2.3. Actually with similar arguments we can deduce the following result.
Proposition 2.4. Let $\Delta : S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces. Then the following statements hold:

(a) If $\mathcal{M}$ is a maximal proper face of $\mathcal{B}_X$, then $\Delta(-\mathcal{M}) = -\Delta(\mathcal{M})$;

(b) If $X$ and $Y$ satisfy the hypotheses of Corollary 2.3, then $\Delta(-F) = -\Delta(F)$ for every proper norm closed face of $\mathcal{B}_X$.

Elements $a, b$ in a $C^*$-algebra $A$ are said to be orthogonal if $ab^* = b^*a = 0$. The set of partial isometries in $A$ can be equipped with a partial order defined by $e \leq v$ if $v - e$ is a partial isometry orthogonal to $e$, equivalently, $v = e + (1 - ee^*)v(1 - v^*v)$.

This seems to be an optimal moment to recall the facial structure of the closed unit ball of a $C^*$-algebra. We recall first some basic notions required to understand the results. Let $A$ be a $C^*$-algebra. It was shown by Akemann and Pedersen in [1] that norm closed faces of $\mathcal{B}_A$ are in one-to-one correspondence with the compact partial isometries in $A^{**}$. Let us recall that a projection $p$ in $A^{**}$ is said to be open if $A \cap (pA^{**}p)$ is weak* dense in $pA^{**}p$, equivalently, there exists an increasing net of positive elements in $A$, all of them bounded by $p$, converging to $p$ in the strong* topology of $A^{**}$ (see [41, §3.11], [46, §III.6 and Corollary III.6.20]). A projection $p \in A^{**}$ is called closed if $1 - p$ is open. A closed projection $p$ in $A^{**}$ is called compact if $p \leq x$ for some norm-one positive element $x \in A$.

Compact partial isometries in the bidual of a $C^*-$algebra were studied by C.M. Edwards and G.T. Rüttimann in [18, §5] as an application of the more general notion of compact tripotent in the bidual of a JB$^*$-triple. C.A. Akemann and G.K. Pedersen consider an alternative term for the same notion. A partial isometry $v \in A^{**}$ belongs locally to $A$ if $v^*v$ is a compact projection and there exists a norm-one element $x$ in $A$ satisfying $v = xv^*v$ (compare [1, Remark 4.7]). It was shown by C.A. Akemann and G.K. Pedersen that a partial isometry $v$ in $A^{**}$ belongs locally to $A$ if and only if $v^*$ belongs locally to $A$ (see [1, Lemma 4.8]). We know from [18, Theorem 5.1] that a partial isometry $v$ in $A^{**}$ belongs locally to $A$ if and only if it is compact in the sense introduced in [18].

Akemann and Pedersen gave in [1, Lemma 4.8 and Remark 4.11] an interesting procedure to understand well those partial isometries in $A^{**}$ belonging locally to $A$. Borrowing a paragraph from the just quoted paper we recall that "the partial isometries $v$ in $A^{**}$ that belong locally to $A$ are obtained by taking an element $x$ in $A$ with norm 1 and polar decomposition $x = u|x|$ (in $A^{**}$), and then letting $v = ue$ for some compact projection $e$ contained in the spectral projection $\chi_{(1)}(|x|)$ of $|x|$ corresponding to the eigenvalue 1." Accordingly to most of the basic references, for each element $x$ in $A$ we set $|x| = (x^*x)^{1/2}$.

We are now in position to revisit the results by C.A. Akemann and G.K. Pedersen.

Theorem 2.5. [1, Theorems 4.10 and 4.11] Let $A$ be a $C^*$-algebra. The following statements hold:

(a) For each norm closed face $F$ of $\mathcal{B}_A$ there exists a unique partial isometry $v$ in $A^{**}$ belonging locally to $A$ such that

$$F = F_v = \{v\}_v = (v + (1 - vv^*)B_A^**(1 - v^*v)) \cap B_A = \{x \in B_A : xv^* = vv^*\}.$$
Furthermore, the mapping \( v \mapsto F_v \) is an anti-order isomorphism from the complete lattice of partial isometries in \( A^{**} \) belonging locally to \( A \) onto the complete lattice of norm closed faces of \( \mathcal{B}_A \);

(b) For each weak* closed face \( \mathcal{G} \) of \( \mathcal{B}_A \) there exists a unique partial isometry \( v \) in \( A^{**} \) belonging locally to \( A \) such that \( \mathcal{G} = \{ v \} \), and the mapping \( v \mapsto \{ v \} \), is an order isomorphism from the complete lattice of partial isometries in \( A^{**} \) belonging locally to \( A \) onto the complete lattice of weak* closed faces of \( \mathcal{B}_A \).

A non-zero projection \( p \) in a C*-algebra \( A \) is called minimal if \( pAp = \mathbb{C}p \). A non-zero partial isometry \( e \) in a C*-algebra \( A \) is minimal if \( ee^* \) (equivalently, \( e^*e \)) is a minimal projection in \( A \). By Kadison’s transitivity theorem minimal partial isometries in \( A^{**} \) belong locally to \( A \), and hence every maximal proper face of the unit ball of a C*-algebra \( A \) is of the form

\[
(v + (1 - vv^*)B_{A^{**}}(1 - v^*v)) \cap \mathcal{B}_A
\]

for a unique minimal partial isometry \( v \) in \( A^{**} \) (compare [1, Remark 5.4 and Corollary 5.5]).

Another technical result of geometric nature, which is frequently applied in the study of Tingley’s problem and should be considered in any survey on this topic, was established by X.N. Fang, J.H. Wang and G.G. Ding in [20] and [11], respectively.

**Theorem 2.6.** ([20, Corollary 2.2], [11, Corollary 1]) Let \( X \) and \( Y \) be normed spaces and let \( \Delta : S(X) \to S(Y) \) be a surjective isometry. Then, for any \( x, y \) in \( S(X) \), we have \( \| x + y \| = 2 \) if and only if \( \| \Delta(x) + \Delta(y) \| = 2 \).

This result plays a role, for example, in some of the proofs in [21, 42].

### 2.1. A taste of Jordan structures.

Many recent advances on Tingley’s problem and it’s variants on C*-algebras make an explicit use of the Jordan theory of JB*-triples (see, for example, the proofs in [43, 25, 26, 27] and [21]). Although we are not going to enter in the deep details of the proofs, it seems convenient to recall the basic notions and connections with this theory.

We recall that, accordingly to the definition introduced in [30], a JB*-triple is a complex Banach space \( E \) admitting a continuous triple product \( \{ a, b, c \} \) which is conjugate linear in \( b \) and linear and symmetric in \( a \) and \( c \), and satisfies the following axioms:

\[\begin{align*}
(\text{JB}^1) & \quad L(a, b)L(c, d) - L(c, d)L(a, b) = L(L(a, b)(c), d) - L(c, L(b, a)(d)), \quad \text{for every} \quad a, b, c, d \in E, \quad \text{where} \quad L(a, b) \quad \text{is the operator on} \quad E \quad \text{defined by} \quad L(a, b)(x) = \{ a, b, x \}; \\
(\text{JB}^2) & \quad L(a, a) \quad \text{is a hermitian operator on} \quad E \quad \text{with non-negative spectrum}; \\
(\text{JB}^3) & \quad \|\{ a, a, a\}\| = \| a \|^3, \quad \text{for every} \quad a \in E.
\end{align*}\]

Examples of JB*-triples include the spaces \( B(H, H') \) of bounded linear operators and the spaces \( K(H, H') \) of all compact operators between two complex Hilbert spaces, complex Hilbert spaces, and all C*-algebras when equipped with the triple product defined by \( \{ x, y, z \} := \frac{1}{2} (xy^*z + zy^*x) \). JB*-triples constitute a category which produces a Jordan model valid to generalize C*-algebras. Every JB*-algebra is a JB*-triple under the triple product

\[\{ a, b, c \} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^* \]

For the basic notions and results on JB*-triples the reader is referred to the monograph [6].
A linear mapping between JB*-triples is called a triple homomorphism if it preserves triple products. Surjective real linear isometries between C*-algebras and JB*-triples are deeply connected to triple isomorphisms (see [8, 7] and [22]). Many of the results in this survey can be complemented with a good description of the real triple isomorphisms between von Neumann algebras. Let us add that real linear triple isomorphisms play a fundamental role in the original proofs of the main results in [43, 25, 26, 27].

3. Tingley’s problem on C*-algebras

Tingley’s problem for surjective isometries between the unit spheres of two commutative C*-algebras is completely solved by arguments, solves Tingley’s problem for commutative C* ∆ : S(ℓ∞(Γ)) → S(ℓ∞(Ω)) and ∆ : S(C0(Γ1)) → S(C0(Γ2)) is a surjective isometry, then we can always find an extension to a surjective real linear isometry between the corresponding spaces, where C0(Γ), c(Γ1), and ℓ∞(Γ) denote the spaces of all complex null, convergent, and bounded functions on Γ, respectively. A similar conclusion holds for a surjective isometry Δ : S(L∞(Ω, Σ, μ)) → S(L∞(Ω, Σ, μ)).

The previous result reveals the importance of considering real linear surjective isometries between C0(L) spaces. A generalization of the Banach-Stone theorem to real linear surjective isometries (see [19] and [32]) assures that for each surjective real linear isometry T : C0(L1) → C0(L2) there exist a homeomorphism ϕ : L2 → L1, a clopen subset K2 of L2, and a unitary continuous function u : L2 → C such that

\[ T(f)(s) = u(s) f(ϕ(s)), \quad ∀f ∈ C_0(L_1), s ∈ K_2, \]

and

\[ T(f)(s) = u(s) \overline{f(ϕ(s))}, \quad ∀f ∈ C_0(L_1), s ∈ L_2 \setminus K_2. \]

Having this theorem in mind, the conclusion in [9] can be explicitly obtained as a consequence of the above Theorem 3.1.
In 2014, 2016, and 2017, R. Tanaka publishes the first achievements on Tingley’s problem for surjective isometries between the unit spheres of two non-commutative $C^*$-algebras; his results focus on finite dimensional $C^*$-algebras, and more generally on finite von Neumann algebras (see [49, 50, 51, 52]). From now on, we shall write $M_n(\mathbb{C})$ for the space of all $n \times n$ matrices with complex entries.

**Theorem 3.2.** [50, Theorem 6.1] Let $\Delta : S(M_n(\mathbb{C})) \to S(M_n(\mathbb{C}))$ be a surjective isometry. Then $\Delta$ admits a (unique) extension to a complex linear or to a conjugate linear surjective isometry on $M_n(\mathbb{C})$. Furthermore, there exist a complex linear or conjugate linear $^*$-automorphism $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and a unitary matrix $u$ in $M_n(\mathbb{C})$ such that one of the next statements hold:

1. $\Delta(x) = u\Phi(x)$, for all $x \in S(M_n(\mathbb{C}))$;
2. $\Delta(x) = u\Phi(x)^*$, for all $x \in S(M_n(\mathbb{C}))$.

Again surjective real linear isometries seem to be behind the results. The proof of the above Theorem 3.2 is based on the following well known fact: The extreme points of the closed unit of $M_n(\mathbb{C})$ are precisely the unitary matrices in $M_n(\mathbb{C})$. Let $U_n$ denote the set of all unitary matrices in $M_n(\mathbb{C})$. It follows from the above fact and from Corollary 2.3 that a surjective isometry $\Delta : S(M_n(\mathbb{C})) \to S(M_n(\mathbb{C}))$ maps $U_n$ onto itself, and thus the restriction $\Delta|_{U_n} : U_n \to U_n$ gives a surjective isometry too. Similar conclusions also hold when $M_n(\mathbb{C})$ is replaced by a finite dimensional $C^*$-algebra, or more generally, by a finite von Neumann algebra. We are naturally lead to the outstanding theorem of O. Hatori and L. Molnár.

**Theorem 3.3.** [28, Corollary 3] Every surjective isometry between the unitary groups of two von Neumann algebras extends to a surjective real linear isometry between the von Neumann algebras. More concretely, let $M_1$ and $M_2$ be von Neumann algebras whose unitary groups are denoted by $U_1$ and $U_2$. Let $\Upsilon : U_1 \to U_2$ be a bijection. Then $\Upsilon$ is a surjective isometry if and only if there exist a central projection $p \in M_2$ and a Jordan $^*$-isomorphism $\Phi : M_1 \to M_2$ such that

$$\Upsilon(u) = \Upsilon(1)(p \Phi(u) + (1 - p) \Phi(u)^*),$$

for all $u \in U_1$.

R.V. Kadison and G.K. Pedersen showed in [29] that every element in a finite von Neumann algebra $M$ can be expressed as the convex combination (actually as the midpoint) of two unitary elements in $M$. Tanaka’s arguments rely on the facial structure of von Neumann algebras and the property of preservation of midpoints between unitary elements. By this arguments the above Theorem 3.2 was generalized by R. Tanaka in the following form:

**Theorem 3.4.** ([52, Theorem 4.2] and [51]) Let $\Delta : S(M_1) \to S(M_2)$ be a surjective isometry, where $M_1$ and $M_2$ are finite von Neumann algebras. There exists a surjective real linear isometry $T : M_1 \to M_2$ satisfying $\Delta(a) = T(a)$ for all $a \in S(M_1)$. More concretely, we can find a central projection $p \in M_2$ and a Jordan $^*$-isomorphism $\Phi : M_1 \to M_2$ such that

$$\Delta(a) = \Delta(1)(p \Phi(a) + (1 - p) \Phi(a)^*),$$

for all $a \in S(M_1)$. The same conclusion holds when $\Delta : S(A) \to S(B)$ is a surjective isometry from the unit sphere of a finite dimensional $C^*$-algebra onto the unit sphere of another $C^*$-algebra.
The Hatori-Molnár theorem is applied by Tanaka to synthesize a surjective real linear isometry $T : M_1 \to M_2$.

The first results on Tingley’s problem for (non-necessarily commutative) operator algebras opened the exploration of this problem for more general classes of operator algebras.

The next natural steps are perhaps, the C*-algebras $K(H)$ and $B(H)$ of all compact and bounded linear operators on an infinite dimensional complex Hilbert space $H$, respectively. There is a clear obstruction in the case of $K(H)$ because $\partial_c(B_{K(H)}) = \emptyset$, even more, $K(H)$ contains no unitary elements, and hence Theorem 3.3 is meaningless to synthesize a surjective real linear isometry in this setting. Surprisingly, we shall get back to Hatori-Molnár theorem (Theorem 3.3) when we survey the recent solution to Tingley’s problem for general von Neumann algebras obtained in [27].

3.1. Tingley’s problem for compact C*-algebras.

Along the paper, given a vector $x_0$ in a Banach space $X$, the translation with respect to $x_0$ will be denoted by $T_{x_0}$.

Let us consider the C*-algebra $K(H)$ of all compact operators on an arbitrary complex Hilbert spaces $H$. It is well known that $K(H)^{**} = B(H)$. There is a clear advantage in this case because minimal partial isometries in $K(H)^{**} = B(H)$ are precisely the rank-one partial isometries which clearly belong to $K(H)$. Furthermore, compact partial isometries in $K(H)^{**}$ are all finite rank partial isometries in $K(H)$.

A C*-algebra $A$ is called compact if it can be written as a $c_0$-sum of the form $A = \bigoplus_j K(H_j)$, where each $H_j$ is a complex Hilbert space (compare [2, 56]). In this case $A^{**} = \bigoplus_j B(H_j)^{**}$, and every minimal partial isometry in $A^{**}$ is a rank-one partial isometry in one of the factors, and hence belongs to $A$. Actually compact partial isometries in $A^{**}$ are finite rank partial isometries, and hence they all belong to $A$. The following proposition was derived in [43] by combining these facts with Corollary 2.3, the Akemann-Pedersen theorem (see Theorem 2.5), the comments in (1), and Mankiewicz’ theorem (see Theorem 1.2).

Proposition 3.5. [43, Proposition 3.2] Let $A$ and $B$ be compact C*-algebras, and suppose that $\Delta : S(A) \to S(B)$ is a surjective isometry. Then the following statements hold:

(a) $\Delta$ maps norm closed proper faces of $B_A$ to norm closed proper faces of $B_B$;
(b) For each (minimal) partial isometry $e_1$ in $A$ there exists a unique (minimal) partial isometry $u_1$ in $B$ such that $\Delta((e_1 + (1 - e_1 e_1^*) B_A \cdots (1 - e_1^* e_1)) \cap B_A) = (u_1 + (1 - u_1 u_1^*) B_B \cdots (1 - u_1^* u_1)) \cap B_B$. Moreover, there exists a surjective real linear isometry $T_{e_1} : (1 - e_1 e_1^*) A (1 - e_1^* e_1) \to (1 - u_1 u_1^*) B (1 - u_1^* u_1)$ such that $\Delta(e_1 + x) = u_1 + T_{e_1}(x)$, for every $x \in B_{(1 - e_1 e_1^*) A (1 - e_1^* e_1)}$;
(c) The restriction of $\Delta$ to each norm closed proper face of $B_A$ is an affine function;
(d) For each partial isometry $e_1$ in $A$ there exists a unique partial isometry $u_1$ in $B$ such that $\Delta(e_1) = u_1$. Moreover, the rank of $e_1$ coincides with the rank of $u_1$ and both are finite.
The proof of the above result can be outlined and guessed by the reader from the previously commented results.

A result determining when a partial isometry is at distance two from another minimal partial isometry in a compact C*-algebra was first considered in [43].

**Lemma 3.6. [43, Lemma 3.5]** Let e and w be partial isometries in a compact C*-algebra A. Suppose that e is minimal and \( \|e - w\| = 2 \). Then

\[
w = -e + (1 - ee^*)w(1 - e^*e).
\]

Let \( \Delta : S(A) \to S(B) \) be a surjective isometry between the unit spheres of two compact C*-algebras. Let us pick a minimal partial isometry e in A. Proposition 3.5 implies that \( \Delta(e) \) and \( \Delta(-e) \) are minimal partial isometries in B. Since \( \|\Delta(e) - \Delta(-e)\| = \|e + e\| = 2 \), Lemma 3.6 assures that

\[
\Delta(-e) = -\Delta(e) + (1 - \Delta(e)\Delta(e)^*)\Delta(-e)(1 - \Delta(e)^*\Delta(e)),
\]

and we derive from the minimality of \( \Delta(-e) \) that \( \Delta(-e) = -\Delta(e) \). A more elaborated argument was applied in [43], via similar arguments, to establish a version of the original theorem of Tingley [53] for finite rank partial isometries.

**Theorem 3.7. [43, Theorem 3.7]** Let \( \Delta : S(A) \to S(B) \) be a surjective isometry between the unit spheres of two compact C*-algebras. The following statements hold:

(a) If e is a partial isometry in A, then \( \Delta(-e) = -\Delta(e) \);
(b) If \( e_1, \ldots, e_m \) are mutually orthogonal partial isometries in A, then \( \Delta(e_1, \ldots, e_m) \) are mutually orthogonal partial isometries in B and

\[
\Delta(e_1 + \ldots + e_m) = \Delta(e_1) + \ldots + \Delta(e_m).
\]

If we take a projection p in a C*-algebra A, the subspace \( (1 - p)A(1 - p) \) is a C*-subalgebra of A. However, if we take a partial isometry e in A, the subspace \( (1 - ee^*)A(1 - e^*e) \) need not be, in general, a C*-subalgebra of A. However, \( (1 - ee^*)A(1 - e^*e) \) is a norm closed subspace of A which is also closed under the triple product given by

\[
(1) \quad \{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).
\]

This is equivalent to say that \( (1 - ee^*)A(1 - e^*e) \) is a JB*-subtriple of A is the sense defined in [30] (see subsection 2.1).

Suppose that e is a partial isometry in a compact C*-algebra A, and let B be another compact C*-algebra. Suppose \( \Delta : S(A) \to S(B) \) is a surjective isometry. Let us consider the surjective real linear isometry

\[
T_e : (1 - ee^*)A(1 - e^*e) \to (1 - \Delta(e)\Delta(e)^*)A(1 - \Delta(e)^*\Delta(e))
\]
given by Proposition 3.5(b). Let \( e_1 \) be any partial isometry in \( (1 - p)A(1 - p) \). By Propositions 3.5 and 3.7 we have

\[
\Delta(e) + T_e(e_1) = \Delta(e + e_1) = \Delta(e) + \Delta(e_1),
\]

we get \( T_e(e_1) = \Delta(e_1) \). Furthermore, let \( e_1, \ldots, e_m \) be mutually orthogonal partial isometries in \( (1 - ee^*)A(1 - e^*e) \), and let \( \alpha_1, \ldots, \alpha_m \) be positive real numbers with
max\{\alpha_1, \ldots, \alpha_m\} \leq 1. By the same results above we deduce that
\[
\Delta \left( e + \sum_{j=1}^{\infty} \alpha_j e_j \right) = \Delta(e) + T_{e} \left( \sum_{j=1}^{\infty} \alpha_j e_j \right) = \Delta(e) + \sum_{j=1}^{\infty} \alpha_j T_{e}(e_j)
\]
\[
= \Delta(e) + \sum_{j=1}^{\infty} \alpha_j \Delta(e_j).
\]
Furthermore, if \( w \) is a partial isometry in \( A \) such that \( e, e_j \in (1 - w w^*)A(1 - w^* w) \) for all \( j \), we also have
\[
\Delta \left( e + \sum_{j=1}^{\infty} \alpha_j e_j \right) = \Delta(e) + \sum_{j=1}^{\infty} \alpha_j \Delta(e_j)
\]
\[
= T_{w}(e) + \sum_{j=1}^{\infty} \alpha_j T_{w}(e_j) = T_{w} \left( e + \sum_{j=1}^{\infty} \alpha_j e_j \right)
\]
A triple spectral resolution assures that every compact operator can be approximated in norm by finite linear combinations of mutually orthogonal minimal partial isometries, and the same statement holds for every element in a compact \( \mathbb{C}^* \)-algebra. Therefore, under the above hypotheses, we deduce from the continuity of \( T_{w} \) and \( \Delta \) that for each non-zero partial isometry \( w \in A \) we have
\[
\Delta(x) = T_{w}(x), \text{ for all } x \in S((1 - w w^*)A(1 - w^* w)).
\]
A straight consequence of (4) gives the following: if \( w_1 \) and \( w_2 \) are non-zero partial isometries we have
\[
T_{w_2}(x) = \Delta(x) = T_{w_1}(x),
\]
for all \( x \in S((1 - w_1 w_1^*)A(1 - w_1^* w_1)) \cap S((1 - w_2 w_2^*)A(1 - w_2^* w_2)) \).

The lacking of possibility to apply the Hatori-Molnár theorem to synthesize a surjective real linear isometry between \( A \) and \( B \) forces us to apply a different strategy in [43]. This different approach is worth to be, at least, outlined here.

In a first step we assume that we can find a non-zero subfactor \( K(H_1) \) of \( A \) such that \( A \) is the orthogonal sum of \( K(H_1) \) and its orthogonal complement \( J = K(H_1)^\perp \) and the latter is non-zero. Let us take two non-zero projections \( p_1 \) in \( K(H_1) \) and \( p_2 \in J \), and define a mapping \( T : A \rightarrow K(H_1) \oplus^+ J \rightarrow B \) given by
\[
T(x) = T_{p_1}(\pi_2(x)) + T_{p_2}(\pi_1(x))
\]
where \( \pi_1 \) and \( \pi_2 \) denote the canonical projections of \( A \) onto \( K(H_1) \) and \( J \), respectively, and \( T_{p_1} \) and \( T_{p_2} \) are defined by Proposition 3.5. The mapping \( T \) is real linear because \( T_{p_1} \) and \( T_{p_2} \) are. Clearly \( T \) is bounded with \( \|T\| \leq 2 \). A minimal partial isometry in \( A \) either lies in \( K(H_1) \) or in \( J \). Let us pick an element \( x \) in \( S(A) \) which can be written in the form \( x = e + \sum_{j=1}^{\infty} \alpha_j e_j + \sum_{k=1}^{\infty} \beta_k e_k \), where \( e, e_j, e_k \) are mutually orthogonal minimal partial isometries in \( A \), \( \alpha_j, \beta_k \in \mathbb{R}^+ \), \( e_j \in B(H_1) \) and \( e_k \in J \) for all \( j, k \), and \( e \) either lies in \( B(H_1) \) or in \( J \). If \( e \in K(H_1) \) (respectively, \( e \in J \)), by (4), we have \( \Delta(e) = T_{p_1}(e) = T(e) \) (respectively, \( \Delta(e) = T_{p_2}(e) = T(e) \)). Now, by (3) and (4) we have
\[
\Delta(x) = \Delta(e) + \sum_{j=1}^{\infty} \alpha_j \Delta(e_j) + \sum_{k=1}^{\infty} \beta_k \Delta(e_k) = \Delta(e) + \sum_{j=1}^{\infty} \alpha_j T_{p_2}(e_j) + \sum_{k=1}^{\infty} \beta_k T_{p_1}(e_k)
\]
\[ T(e) = T(x) = \sum_{j=1}^{\infty} \alpha_j T(e_j) + \sum_{k=1}^{\infty} \beta_k T(e_k) = T(x). \]

The norm density of this kind of elements \( x \) in \( S(A) \) together with the norm continuity of \( T \) and \( \Delta \) proves that \( T(x) = \Delta(x) \) for all \( x \in S(A) \).

In the second case we assume that \( A = K(H) \) for some complex Hilbert space \( H \). If \( H \) is finite dimensional Theorem 3.4 proves that our mapping \( \Delta : S(A) \to S(B) \) admits a unique extension to a surjective real linear isometry from \( A \) onto \( B \). We can therefore assume that \( H \) is infinite dimensional.

Let us take three mutually orthogonal minimal projections \( p_1, p_2 \) and \( p_3 \) in \( A \), and the corresponding surjective real linear isometries \( T_{p_1}, T_{p_2}, \) and \( T_{p_3} \) given by Proposition 3.5. We can decompose \( A \) in the form
\[
A = C_p + (p_1 A p_2 \oplus p_2 A p_1) + ((1 - p_2) A p_1 \oplus p_1 A (1 - p_2)) + (1 - p_1) A (1 - p_1),
\]
where \( C_p = C_p + (p_1 A p_2 \oplus p_2 A p_1) \subseteq (1 - p_3) A (1 - p_3), \) and \((1 - p_2) A p_1 \oplus p_1 A (1 - p_2)) \subseteq (1 - p_2) A (1 - p_2). \) Let \( \pi_1, \pi_2, \) and \( \pi_3 \) denote the corresponding projections of \( A \) onto \( C_p \), \( (p_1 A p_2 \oplus p_2 A p_1) \), \((1 - p_2) A p_1 \oplus p_1 A (1 - p_2)) \) and \((1 - p_1) A (1 - p_1), \) respectively. We synthesize a mapping \( T : A \to B \) given by
\[
T(x) = T_{p_1}(\pi_1(x)) + T_{p_2}(\pi_2(x)) + T_{p_3}(\pi_3(x)).
\]
The mapping \( T \) is continuous and real linear because \( T_{p_1}, T_{p_2}, \) and \( T_{p_3} \) are.

If we prove that
\[
T(e) = \Delta(e),
\]
then a similar argument to that given in the first step above, based on (3) and (4), the norm density in \( S(A) \) of elements which can be written as finite positive combinations of mutually orthogonal projections, and the continuity of \( T \) and \( \Delta \), shows that \( T(x) = \Delta(x) \) for all \( x \in S(A) \).

Let \( e \) be a minimal partial isometry in \( A \). Since \( H \) is infinite dimensional, we can find another minimal projection \( p_4 \) which is orthogonal to \( p_1, p_2, p_3, e \).

Since \( e \in (1 - p_4) A (1 - p_4) \), the statement in (4) implies that \( \Delta(e) = T_{p_4}(e) \).

Let us write \( e = p_1 e p_1 + p_1 e p_2 + p_2 e p_1 + p_1 e (1 - p_2) + (1 - p_2) e p_1 + (1 - p_1) e (1 - p_1) \). Clearly, \( p_1 e p_1, p_1 e p_2, p_2 e p_1 \in (1 - p_4) A (1 - p_4) \). Since \( p_1, p_2, e \in (1 - p_4) A (1 - p_4) \), we also deduce that \( p_1 e (1 - p_2), (1 - p_2) e p_1, (1 - p_1) e (1 - p_1) \in (1 - p_4) A (1 - p_4). \) By applying (5) to \( T_{p_4} \) and \( T_{p_5} \) (respectively, to \( T_{p_4} \) and \( T_{p_2} \), and \( T_{p_4} \) and \( T_{p_3} \)) we get
\[
T(e) = T_{p_4}(p_1 e p_1 + p_1 e p_2 + p_2 e p_1) + T_{p_2}(p_1 e (1 - p_2) + (1 - p_2) e p_1) + T_{p_3}((1 - p_1) e (1 - p_1))
\]
\[
= T_{p_4}(p_1 e p_1 + p_1 e p_2 + p_2 e p_1) + T_{p_2}(p_1 e (1 - p_2) + (1 - p_2) e p_1) + T_{p_3}((1 - p_1) e (1 - p_1))
\]
\[
= T_{e_4}(e) = \Delta(e).
\]

We have sketched the main arguments leading to one of the main achievements in [43].

**Theorem 3.8.** [43, Theorem 3.14] Let \( \Delta : S(A) \to S(B) \) be a surjective isometry between the unit spheres of two compact \( C^* \)-algebras. Then there exists a (unique) surjective real linear isometry \( T : A \to B \) such that \( T(x) = \Delta(x) \), for every \( x \) in \( S(A) \). In particular, the same conclusion holds when \( A = K(H_1) \) and \( B = K(H_2) \), where \( H_1 \) and \( H_2 \) are arbitrary complex Hilbert spaces.
Surjective real linear isometries between (real) C*-algebras were studied in deep by Ch.H. Chu, T. Dang, B. Russo, B. Ventura in [7]. Theorem 6.4 in [7] proves that every surjective real linear isometry between (real) C*-algebras is a triple isomorphism with respect to the triple product defined in (2). Studies on surjective real linear isometries on JB*-triples and real JB*-triples have been considered by T. Dang [8] and F.J. Fernández-Polo, J. Martínez and the author of this survey [22].

3.2. Tingley’s problem for $B(H)$.

After having revisited the solution to Tingley’s problem for compact C*-algebras published in [43], the next natural challenge is to consider a surjective isometry $\Delta : S(B(H_1)) \to S(B(H_2))$, where $H_1, H_2$ are arbitrary complex Hilbert spaces. Let us observe that if $H_1$ or $H_2$ is finite dimensional, then the extension of $\Delta$ to a surjective real linear isometry is guaranteed by Tanaka’s theorem (see Theorem 3.4).

The problem in the setting of $B(H)$ spaces has been recently solved in a contribution by F.J. Fernández-Polo and the author of this survey in [25]. The main conclusion gives a positive solution to Tingley’s problem in the setting just commented.

**Theorem 3.9.** [25, Theorem 3.2] Let $H_1$ and $H_2$ be complex Hilbert spaces. Suppose that $\Delta : S(B(H_1)) \to S(B(H_2))$ is a surjective isometry. Then there exists a surjective complex linear or conjugate linear surjective isometry $T$ from $B(H_1)$ onto $B(H_2)$ satisfying $\Delta(x) = T(x)$, for every $x \in S(B(K))$.

Actually, a stronger conclusion has been achieved.

**Theorem 3.10.** [25, Theorem 3.2] Let $(H_i)_{i \in I}$ and $(K_j)_{j \in J}$ be two families of complex Hilbert spaces. Suppose $\Delta : S \left( \bigoplus_{j}^\ell \infty B(K_j) \right) \to S \left( \bigoplus_{i}^\ell \infty B(H_i) \right)$ is a surjective isometry. Then there exists a surjective real linear isometry $T : S \left( \bigoplus_{j}^\ell \infty B(K_j) \right) \to S \left( \bigoplus_{i}^\ell \infty B(H_i) \right)$ satisfying $T|_{S(E)} = \Delta$.

The strategy to obtain the previous two theorems also begins with results based on the facial structure of the closed unit ball of $B(H)$, Theorem 2.1, Corollary 2.3 and the Akemann-Pedersen theorem (Theorem 2.5). The latter result forces us to face a serious additional obstacle which requires a completely new strategy. More concretely, we have already seen in the previous subsection that, for a compact C*-algebra $A$, the norm closed faces of $B_A$ are determined by finite rank partial isometries in $A$. However, for a general C*-algebra $A$ the maximal proper faces of $B_A$ are determined by minimal partial isometries in $A^{**}$. This is a serious obstacle which makes invalid the arguments in previous subsections and in [43, 24] to the case of a surjective isometry $\Delta : S(B(H_1)) \to S(B(H_2))$, because, in principle, $\Delta$ cannot be applied to every minimal projection in $B(H_1)^{**}$. The novelities in [25] are based on certain technical results which provides an antidote to avoid these difficulties.

Two results from [25] deserve to be highlighted by their own right.
Theorem 3.11. [25, Theorem 2.3] Let $A$ and $B$ be $C^*$-algebras, and suppose that $\Delta : S(A) \to S(B)$ is a surjective isometry. Let $e$ be a minimal partial isometry in $A$. Then 1 is isolated in the spectrum of $|\Delta(e)|$.

The consequences of the previous result reveal to be stronger after the next additional theorem.

Theorem 3.12. [25, Theorem 2.5] Let $A$ be a $C^*$-algebra, and let $H$ be a complex Hilbert space. Suppose that $\Delta : S(A) \to S(B(H))$ is a surjective isometry. Let $e$ be a minimal partial isometry in $A$. Then $\Delta(e)$ is a minimal partial isometry in $B(H)$. Moreover, there exists a surjective real linear isometry $T_e : (1 - ee^*)A(1 - e^*e) \to (1 - \Delta(e)\Delta(e)^*)B(H)(1 - \Delta(e)^*\Delta(e))$
such that $\Delta(e + x) = \Delta(e) + T_e(x)$, for all $x$ in $B(1 - ee^*)A(1 - e^*e)$.

In particular, the restriction of $\Delta$ to the face $F_e = e + (1 - ee^*)B_A(1 - e^*e)$ is a real affine function.

Technical algebraic and geometric manipulations combined with the previous theorem determine a precise control of a surjective isometry $\Delta : S(B(K)) \to S(B(H))$ on algebraic elements in the sphere which can be expressed as finite positive linear combinations of mutually orthogonal minimal partial isometries. It should be remarked here that a traditional spectral resolution with finite linear combinations of mutually orthogonal projections is only valid to approximate hermitian elements in the sphere.

Theorem 3.13. [25, Theorem 2.7] Let $\Delta : S(B(H_1)) \to S(B(H_2))$ be a surjective isometry where $H_1$ and $H_2$ are complex Hilbert spaces with dimension greater than or equal to 3. Then the following statements hold:

(a) For each minimal partial isometry $v$ in $B(H_1)$, the mapping

$$T_v : (1 - vv^*)B(H_1)(1 - vv^*) \to (1 - \Delta(v)\Delta(v)^*)B(H_2)(1 - \Delta(v)^*\Delta(v))$$
given by Theorem 3.12 is complex linear or conjugate linear;

(b) For each minimal partial isometry $v$ in $B(H_1)$ we have $\Delta(-v) = -\Delta(v)$ and $T_v = T_{-v}$. Furthermore, $T_v$ is weak*-continuous and $\Delta(e) = T_v(e)$ for every minimal partial isometry $e \in (1 - vv^*)B(H_1)(1 - v^*v)$;

(c) For each minimal partial isometry $v$ in $B(H_1)$ the equality $\Delta(w) = T_v(w)$ holds for every partial isometry $w \in (1 - vv^*)B(H_1)(1 - v^*v) \setminus \{0\}$;

(d) Let $w_1, \ldots, w_n$ be mutually orthogonal non-zero partial isometries in $B(H_1)$, and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers with $\lambda_1 = 1$, and $\lambda_j \leq 1$ for all $j$. Then

$$\Delta \left( \sum_{j=1}^{n} \lambda_j w_j \right) = \sum_{j=1}^{n} \lambda_j \Delta \left( w_j \right);$$

(e) For each minimal partial isometry $v$ in $B(H_1)$ we have $\Delta(x) = T_v(x)$ for every $x \in S(\mathcal{B}(1 - vv^*)B(H_1)(1 - v^*v))$;

(f) For each partial isometry $w$ in $B(H_1)$ the element $\Delta(w)$ is a partial isometry;

(g) Suppose $v_1, v_2$ are mutually orthogonal minimal partial isometries in $B(H_1)$ then $T_{v_1}(x) = T_{v_2}(x)$ for every $x$ in the intersection

$$(1 - v_1v_1^*)B(H_1)(1 - v_1v_1^*) \cap ((1 - v_2v_2^*)B(H_1)(1 - v_2v_2^*));$$
(h) Suppose \( v_1, v_2 \) are mutually orthogonal minimal partial isometries in \( B(H_1) \) then exactly one of the following statements holds:
(1) The mappings \( T_{v_1} \) and \( T_{v_2} \) are complex linear;
(2) The mappings \( T_{v_1} \) and \( T_{v_2} \) are conjugate linear.

The synthesis of a surjective real linear isometry in the proof of Theorem 3.9 (see [25, Theorem 3.2]) is given with similar arguments to those we sketched in page 14 with the obvious modifications and the new tools developed in Theorems 3.12 and 3.13. That is, assuming that \( H \) is infinite dimensional, we pick three mutually orthogonal minimal projections \( p_1, p_2 \) and \( p_3 \) in \( A \), and the corresponding surjective real linear isometries \( T_{p_1}, T_{p_2}, \) and \( T_{p_3} \) given by Theorem 3.12. By decomposing \( B(H_1) \) in the form

\[
B(H_1) = \mathbb{C}p_1 \oplus (p_1B(H_1)p_2 \oplus p_2B(H_1)p_1)
\]

\[
\oplus((1 - p_2)B(H_1)p_1 \oplus p_1B(H_1)(1 - p_2)) \oplus (1 - p_1)B(H_1)(1 - p_1),
\]

with

\[
\mathbb{C}p_1 \oplus (p_1B(H_1)p_2 \oplus p_2B(H_1)p_1) \subset (1 - p_3)B(H_1)(1 - p_3),
\]

and

\[
((1 - p_2)B(H_1)p_1 \oplus p_1B(H_1)(1 - p_2)) \subset (1 - p_2)B(H_1)(1 - p_2),
\]

and denoting by \( \pi_1, \pi_2, \) and \( \pi_3 \) the corresponding projections of \( B(H_1) \) onto

\[
\mathbb{C}p_1 \oplus (p_1B(H_1)p_2 \oplus p_2B(H_1)p_1),
\]

\[
((1 - p_2)B(H_1)p_1 \oplus p_1B(H_1)(1 - p_2)) \text{ and } (1 - p_1)B(H_1)(1 - p_1),
\]

respectively. We synthesize a mapping \( T : B(H_1) \to B(H_2) \) given by

\[
T(x) = T_{p_3}(\pi_1(x)) + T_{p_2}(\pi_2(x)) + T_{p_1}(\pi_3(x)).
\]

The mapping \( T \) is weak* continuous and real linear because \( T_{p_1}, T_{p_2}, \) and \( T_{p_3} \) are. By the new tools given by Theorem 3.13 it is shown in the proof of [25, Theorem 3.2] that \( \Delta(e) = T(e) \) for every minimal partial isometry \( e \) in \( B(H_1) \).

Contrary to the case of \( K(H) \) spaces and compact \( C^* \)-algebras, where every element in the sphere can be approximated in norm by norm-one elements which are finite linear combination of mutually orthogonal minimal partial isometries, elements in the sphere of \( B(H) \) can be approximated only in the weak* topology by these kind of algebraic elements. To solve this additional obstacle, it is established in [25] an identity principle in the following terms.

**Proposition 3.14.** [25, Proposition 3.1] Let \( H_1 \) and \( H_2 \) be complex Hilbert spaces. Suppose that \( \Delta : S(B(H_1)) \to S(B(H_2)) \) is a surjective isometry, and there exists a weak*-continuous real linear operator \( T : B(H_1) \to B(H_2) \) such that \( \Delta(v) = T(v) \), for every minimal partial isometry \( v \) in \( B(H_1) \). Then \( T \) and \( \Delta \) coincide on the whole \( S(B(H_1)) \).

The above proposition, Theorem 3.13 and the above observation are, in essence, all the arguments required to prove Theorem 3.9. The proof of Theorem 3.10 required additional technical adaptations which can be found in [25].

### 3.3. Tingley’s problem for von Neumann algebras.

The most recent, and for the moment, the most general conclusion on Tingley’s problem is an affirmative solution to this problem for surjective isometries between the unit spheres of two arbitrary von Neumann algebras, which has been recently obtained by F.J. Fernández-Polo and the author of this survey in [27]. The result reads as follows:
Theorem 3.15. [27, Theorem 3.3] Let $\Delta : S(M) \to S(N)$ be a surjective isometry between the unit spheres of two von Neumann algebras. Then there exists a surjective real linear isometry $T : M \to N$ whose restriction to $S(M)$ is $\Delta$. More precisely, there exist a central projection $p$ in $N$ and a Jordan $^*$-isomorphism $J : M \to N$ such that, defining $T : M \to N$ by $T(x) = \Delta(1)(pJ(x) + (1-p)J(x)^*)$ ($x \in M$), then $T$ is a surjective real linear isometry and $T|_{S(M)} = \Delta$.

The mathematical difficulties of the problem in this general setting are considerable. The techniques, procedures and strategies applied in previous case to synthesize a surjective real linear isometry and to apply the facial structure are no longer valid under the new hypotheses.

Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two $C^*$-algebras. A combination of Theorem 2.5 and Corollary 2.3 (see also the subsequent comments) gives a one-to-one correspondence between compact partial isometries in the corresponding second duals.

Theorem 3.16. Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two $C^*$-algebras. Then the following statements hold:

(a) For each non-zero compact partial isometry $e \in A^{**}$ there exists a unique (non-zero) compact partial isometry $\phi_\Delta(e) \in B^{**}$ such that $\Delta(F_e) = F_{\phi_\Delta(e)}$, where $F_e = (e + (1-e^*)B_{A^{**}}(1-e^*)B_A);$

(b) The mapping $e \mapsto \phi_\Delta(e)$ defines an order preserving bijection between the sets of non-zero compact partial isometries in $A^{**}$ and the set of non-zero compact partial isometries in $B^{**}$;

(c) $\phi_\Delta$ maps minimal partial isometries in $A^{**}$ to minimal partial isometries in $B^{**}$.

The above result produces no alternative to our obstacles because compact partial isometries in the second dual cannot be transformed under $\Delta$. Technical arguments based on ultraproducts techniques and a subtle uniform generalization of Lemma 3.6 are appropriately applied in [27] to obtain generalizations of the above Theorems 3.11 and 3.12.

Theorem 3.17. [27, Theorem 2.7] Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two $C^*$-algebras. Let $e$ be a non-zero partial isometry in $A$. Then 1 is isolated in the spectrum of $|\Delta(e)|$.

Mankiewicz’s theorem (Theorem 1.2) plays a fundamental role in the second part of the statement of the next theorem.

Theorem 3.18. [27, Theorem 2.8] Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two $C^*$-algebras. Then $\Delta$ maps non-zero partial isometries in $A$ to non-zero partial isometries in $B$. Moreover, for each non-zero partial isometry $e$ in $A$, we have $\phi_\Delta(e) = \Delta(e)$, and there exists a surjective real linear isometry

$$T_e : (1-e^*)A(1-e^*) \to (1-\Delta(e)\Delta(e)^*)B(1-\Delta(e)^*\Delta(e))$$

such that

$$\Delta(e + x) = \Delta(e) + T_e(x), \text{ for all } x \in B_{(1-e^*)A(1-e^*)}.$$ 

In particular the restriction of $\Delta$ to the face $F_e = e + (1-e^*)B_A(1-e^*)$ is a real affine function.
Another crucial step in the study of Tingley’s problem on von Neumann algebras asserts that the mapping $\phi_\Delta$ given by Theorem 3.16 preserves antipodal points.

**Theorem 3.19.** [27, Theorem 2.11] Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two C*-algebras. Then, for each non-zero compact partial isometry $e$ in $A^{**}$ we have $\phi_\Delta(-e) = -\phi_\Delta(e)$, where $\phi_\Delta$ is the mapping given by Theorem 3.16. Consequently, for each non-zero partial isometry $e \in A$ we have $\Delta(-e) = -\Delta(e)$.

The orthogonal complement of a subset $S$ in a C*-algebra $A$ is defined by

$$S^\perp := \{ x \in A : x \perp a, \text{ for all } a \in S \}.$$

The previous theorems provide the key tools to extend Theorem 3.13 to the setting of von Neumann algebras.

**Proposition 3.20.** [27, Proposition 2.12] Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two C*-algebras. Then the following statements hold:

(a) For each non-zero partial isometry $v$ in $A$, the surjective real linear isometry

$$T_v : (1 - vv^*)A(1 - vv^*) \to (1 - \Delta(v)\Delta(v)^*)B(1 - \Delta(v)^*\Delta(v))$$

given by Theorem 3.18 satisfies $\Delta(e) = T_v(e)$, for every non-zero partial isometry $e \in (1 - vv^*)A(1 - v^*v)$;

(b) Let $w_1, \ldots, w_n$ be mutually orthogonal non-zero partial isometries in $A$, and let $\lambda_1, \ldots, \lambda_n$ be real numbers with $1 = |\lambda_1| \geq \max(|\lambda_j|)$. Then

$$\Delta \left( \sum_{j=1}^n \lambda_j w_j \right) = \sum_{j=1}^n \lambda_j \Delta(w_j);$$

(c) Suppose $v, w$ are mutually orthogonal non-zero partial isometries in $A$ then $T_v(x) = T_w(x)$ for every $x \in \{v\}^\perp \cap \{w\}^\perp$;

(d) If $A$ is a von Neumann algebra, then for each non-zero partial isometry $v$ in $A$ we have $\Delta(x) = T_v(x)$ for every $x \in S((1 - vv^*)A(1 - v^*v))$.

Given a surjective isometry $\Delta : S(M) \to S(N)$ between the unit spheres of two von Neumann algebras, the synthesis of a surjective real linear extension to a surjective real linear isometry $T : M \to N$ follows completely different arguments than those in the cases of compact C*-algebras and $B(H)$. The technique in this case relies again in the Hatori-Molnár theorem (Theorem 3.3). R. Tanaka proves in [52] that a surjective isometry between the unit spheres of two finite von Neumann algebras maps unitary elements to unitary elements. This result has been extended to general von Neumann algebras in [27, Theorem 3.2].

**Theorem 3.21.** [27, Theorem 3.2] Let $\Delta : S(M) \to S(N)$ be a surjective isometry between the unit spheres of two von Neumann algebras. Then $\Delta$ maps unitaries in $M$ to unitaries in $N$.

From now on, let the symbol $U(A)$ denote the unitary group of a C*-algebra $A$. The above Theorem 3.21 opens the door to apply the Hatori-Molnár theorem (Theorem 3.3) to synthesize a surjective real linear isometry $T : M \to N$ satisfying that $T(u) = \Delta(u)$ for all $u \in U(M)$. The difficulties to finish the proof of Theorem 3.15 reside in proving that $\Delta(x) = T(x)$ for all $x \in S(M)$. This is solved in [27] with
a convenient application of the theory of convex combinations of unitary operators in von Neumann algebras developed by C.L. Olsen and G.K. Pedersen in [38] and [39]. These are the main lines in the proof of Theorem 3.15.

It is worth to make a stop to comment the first connection with a very recent contribution of M. Mori. In the preprint [33], M. Mori establishes a generalization of the above Theorem 3.21.

**Theorem 3.22.** [33, Theorem 3.2] Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two unital $C^*$-algebras. Then $\Delta$ maps unitaries in $A$ to unitaries in $B$.

The proof presented by M. Mori in [33] is based on the following geometric result, which is a nice discovering by itself, and might be useful in some other contexts.

**Lemma 3.23.** Let $A$ be a unital $C^*$-algebra, and let $x$ be an element in $\partial_e(B_A)$. Then $x$ is a unitary if and only if the set $A_x := \{y \in \partial_e(B_A) : \|x \pm y\| = \sqrt{2}\}$ has an isolated point as metric space.

We finish this section with a couple of open problems.

**Open Problem 1.** (Tingley’s problem for $C^*$-algebras) Let $\Delta : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two $C^*$-algebras. Does $\Delta$ admits an extension to a surjective real linear isometry from $A$ onto $B$?

When $A$ is unital $C^*$-algebra M. Mori shows in [33, Proposition 3.4 and Problem 6.1] that a particular version of the above problem can be restated in the following terms:

**Open Problem 2.** Let $A$ be a unital $C^*$-algebra and let $\Delta : S(A) \to S(A)$ be a surjective isometry. Suppose that $\Delta(x) = x$ for every invertible element in the unit sphere of $A$. Is $\Delta$ equal to the identity mapping on $S(A)$?

4. TINGLEY’S PROBLEM ON VON NEUMANN ALGEBRA PREDUALS

Let us begin this section with another result due to G.G. Ding. Let $\Gamma$ be a index set, we denote by $\ell^1_R(\Gamma)$ the Banach space of all absolutely summable families of real numbers equipped with the norm $\| (\xi_j)_{j \in \Gamma} \|_1 = \sum_{j \in \Gamma} |\xi_j|$.

**Theorem 4.1.** [10, Theorem 1] Let $\Delta : S(\ell^1_R(\Gamma_1)) \to S(\ell^1_R(\Gamma_2))$ be a surjective isometry. Then there exists a one-to-one bijection $\sigma : \Gamma_1 \to \Gamma_2$ and a family of real numbers $\{\theta_j : j \in \Gamma_1\} \subseteq \mathbb{T}$ such that

$$\Delta \left( \sum_{j \in \Gamma_1} \xi_j e_j \right) = \sum_{j \in \Gamma_2} \theta_j \xi_{\sigma(j)} \hat{e}_j,$$

where $\{e_j : j \in \Gamma_1\}$ and $\{\hat{e}_j : j \in \Gamma_2\}$ are the canonical basis of $\ell^1_R(\Gamma_1)$ and $\ell^1_R(\Gamma_2)$, respectively. In particular, there exists a surjective real linear isometry $T : \ell^1_R(\Gamma_1) \to \ell^1_R(\Gamma_2)$ whose restriction to $S(\ell^1_R(\Gamma_1))$ coincides with $\Delta$.

Given a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, the symbol $L^1_R(\Omega, \Sigma, \mu)$ will denote the Banach space of real valued measurable functions $f : \Omega \to \mathbb{R}$ satisfying $\int_\Omega |f| d\mu < \infty$. 


and admits non-commutative counterparts. The \( \lambda \) elements satisfying \( \Delta(\ell) \) will be denoted by \( L^\infty_\lambda(\Omega, \Sigma, \mu) \).

The previous result of Ding is complemented by the following result due to D. Tan.

**Theorem 4.2.** [48, Theorem 3.4] Let \( (\Omega, \Sigma, \mu) \) be a \( \sigma \)-finite measure space and let \( Y \) be a real Banach space. Then every surjective isometry \( \Delta : S(L^1_\mu(\Omega, \Sigma, \mu)) \to S(Y) \) can be uniquely extended to a surjective real linear isometry from \( L^1_\lambda((\Omega, \Sigma, \mu)) \) onto \( Y \).

Regarding \( \ell^1_\lambda(\Gamma_1) \) and \( L^1_\lambda(\Omega, \Sigma, \mu) \) as predual spaces of the hermitian parts of the von Neumann algebras \( \ell^\infty_\mu(\Gamma_1) \) and \( L^\infty_\mu(\Omega, \Sigma, \mu) \), respectively, it seems natural to ask whether Theorems 4.1 and 4.2 admits non-commutative counterparts. The duality \( c_0^* = \ell^1 \) and \( (\ell^1)^* = \ell^\infty \) admits a non-commutative alter ego in the form \( K(H)^* = C_1(H) \) and \( C_1(H) = B(H) \), where \( C_1(H) \) is the space of trace class operators on a complex Hilbert space \( H \). This will be treated in the next subsection.

4.1. Tingley’s problem on trace class operators.

Tingley’s problem for surjective isometries between unit spheres of spaces of trace class operators has been approached by F.J. Fernández-Polo, J.J. Garcés, I. Villanueva and the author of this note in [21]. We shall review here the main achievements in this line.

When the space \( C_1(H) \) is regarded as the predual of the von Neumann algebra \( B(H) \), or as the dual space of the \( C^* \)-algebra \( K(H) \), we can get back to Corollary 2.3 and subsequent comments whose consequences were already observed in [21].

**Proposition 4.3.** [21, Proposition 2.6] Let \( \Delta : S(C_1(H)) \to S(C_1(H')) \) be a surjective isometry, where \( H \) and \( H' \) are complex Hilbert spaces. Then the following statements hold:

(a) A subset \( F \subset S(C_1(H)) \) is a proper norm-closed face of \( B_{C_1(H)} \) if and only if \( \Delta(F) \) is.
(b) \( \Delta \) maps \( \partial_v(B_{C_1(H)}) \) into \( \partial_v(B_{C_1(H')}) \);
(c) \( \dim(H) = \dim(H') \);
(d) For each \( e_0 \in \partial_v(B_{C_1(H)}) \) we have \( \Delta(ie_0) = i\Delta(e_0) \) or \( \Delta(ie_0) = -i\Delta(e_0) \);
(e) For each \( e_0 \in \partial_v(B_{C_1(H)}) \) if \( \Delta(ie_0) = i\Delta(e_0) \) (respectively, \( \Delta(ie_0) = -i\Delta(e_0) \)) then \( \Delta(\lambda e_0) = \lambda \Delta(e_0) \) (respectively, \( \Delta(\lambda e_0) = \overline{\lambda} \Delta(e_0) \)) for every \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \).

The strategy to solve Tingley’s problem on \( C_1(H) \) is based on techniques of linear algebra and geometry to obtain first a solution in the case of finite dimensional spaces.

**Theorem 4.4.** [21, Theorem 3.7] Let \( \Delta : S(C_1(H)) \to S(C_1(H)) \) be a surjective isometry, where \( H \) is a finite dimensional complex Hilbert space. Then there exists a surjective complex linear or conjugate linear isometry \( T : C_1(H) \to C_1(H) \) satisfying \( \Delta(x) = T(x) \) for every \( x \in S(C_1(H)) \). More concretely, there exist unitary elements \( u, v \in M_{\dim(C)} = B(H) \) such that one of the following statements holds:

(a) \( \Delta(x) = uxv \), for every \( x \in S(C_1(H)) \);
(b) \( \Delta(x) = u^*xv \), for every \( x \in S(C_1(H)) \);
(c) \( \Delta(x) = u\overline{\tau}v \), for every \( x \in S(C_1(H)) \);
(d) \( \Delta(x) = ux^*v \), for every \( x \in S(C_1(H)) \), where \( \overline{(x_{ij})} = (\overline{x_{ij}}) \).

Surprisingly, the solution in the finite dimensional case is applied, in a very technical argument, to derive a solution to Tingley’s problem for surjective isometries between the unit spheres of two spaces of trace class operators.

**Theorem 4.5.** [21, Theorem 4.1] Let \( \Delta : S(C_1(H)) \to S(C_1(H)) \) be a surjective isometry, where \( H \) is an arbitrary complex Hilbert space. Then there exists a surjective complex linear or conjugate linear isometry \( T : C_1(H) \to C_1(H) \) satisfying \( \Delta(x) = T(x) \), for every \( x \in S(C_1(H)) \).

4.2. **Tingley’s problem on von Neumann preduals.**

According to what is commented at the introduction, a very recent contribution by M. Mori has changed the original plans and the structure of this survey. The preprint [33] contains, among other interesting results, a complete positive solution to Tingley’s problem for surjective isometries between the unit spheres of von Neumann algebra preduals.

**Theorem 4.6.** [33, Theorem 4.3] Let \( M \) and \( N \) be von Neumann algebras, and let \( \Delta : S(M_*) \to S(N_*) \) be a surjective isometry. Then there exists a (unique) surjective real linear isometry \( T : M_* \to N_* \) satisfying \( T(x) = \Delta(x) \), for every \( x \in S(M_*) \).

It is perhaps interesting to take a brief look at the method applied by M. Mori to synthesize the surjective real linear isometry \( T \). Let \( \Delta : S(M_*) \to S(N_*) \) be a surjective isometry, where \( M \) and \( N \) are von Neumann algebras. When Corollary 2.3 and the subsequent comments is combined with the Akemann-Pedersen theorem (see Theorem 2.5), we can conclude that for each maximal partial isometry \( u \in \partial_e(B_M) \) there exists a unique maximal partial isometry \( T_1(u) \in \partial_e(B_N) \) satisfying \( \Delta(\{u\}) = \{T_1(u)\} \). This gives a bijection \( T_1 : \partial_e(B_M) \to \partial_e(B_N) \).

Let \( (E, d) \) be a metric space. The **Hausdorff distance** between two sets \( S_1, S_2 \subseteq E \) is defined by

\[
d_H(S_1, S_2) := \max\{ \sup_{x \in S_1} \inf_{y \in S_2} d(x, y), \sup_{y \in S_2} \inf_{x \in S_1} d(x, y) \}.
\]

The lattice of partial isometries can be equipped with a distance defined by

\[
\delta_H(v, w) := d_H(\{v\}, \{w\}).
\]

It is shown by M. Mori that this distance enjoys the following properties:

**Proposition 4.7.** [33, Lemmas 4.1 and 4.2] Let \( M \) be a von Neumann algebra. Then the following statements hold:

(a) \( \delta_H(u, v) = \|u - v\| \), for every \( u \in \mathcal{U}(M) \) and every \( v \in \partial_e(B_M) \);

(b) An element \( u \in \partial_e(B_M) \) is a unitary if and only if the set

\[
\hat{M}_u := \{ e \in \partial_e(B_M) : \delta_H(u, e) \leq \sqrt{2} \}
\]

has an isolated point with respect to the metric \( \delta_H \).

Applying Proposition 2.4(a) and Proposition 4.7(b), M. Mori concludes that \( T_1(\mathcal{U}(M)) = \mathcal{U}(N) \), and by Proposition 4.7(a), \( T_1|_{\mathcal{U}(M)} : \mathcal{U}(M) \to \mathcal{U}(N) \) is a surjective isometry. The mapping \( T_1 \) fulfills the hypothesis of the Hatori-Molnár
theorem (see Theorem 3.3), and thus there exists a surjective real linear (weak*-continuous) isometry \( \tilde{T}_1 : M \to N \) whose restriction to \( \mathcal{U}(M) \) is \( T_1 \). The technical arguments developed by M. Mori in the proof of [33, Theorem 4.3] finally show that the mapping \( T_2 : N^* \to M^* \) defined by

\[
T_2(\varphi)(x) := \Re \varphi(\tilde{T}_1(x)) - i \Im \varphi(\tilde{T}_1(ix)), \quad \varphi \in N^*, x \in M,
\]

is a real linear isometry whose restriction to \( N_* \) gives a surjective real linear isometry \( T_2|_{N_*} : N_* \to M_* \) and \( (T_2|_{M_*})^{-1}(\phi) = \Delta(\phi) \) for all \( \phi \) in \( M_* \).

5. ISOMETRIES BETWEEN THE SPHERES OF HERMITIAN OPERATORS

A second and interesting variant of Problem 1.1 is obtained when \( X \) and \( Y \) are von Neumann algebras or \( C^* \)-algebras and \( S_1 \) and \( S_2 \) are the unit spheres of their respective hermitian parts. In this section we consider two von Neumann algebras \( M, N \) and a surjective isometry \( \Delta : S(M_{sa}) \to S(N_{sa}) \). Our goal will consist in showing that the same tools in [27] can be, almost literally, applied to find a surjective complex linear isometry \( T : M \to N \) satisfying \( T(a^*) = T(a)^* \) for all \( a \in M \) and \( T(x) = \Delta(x) \) for all \( x \in S(M_{sa}) \).

Given a \( C^* \)-algebra \( A \), its hermitian part \( A_{sa} \) is not, in general, a \( C^* \)-subalgebra of \( A \). However, \( A_{sa} \) is a real closed subspace of \( A \) which satisfies the hypotheses of Corollary 2.3 (see the comments after this corollary). After applying this corollary, we find the necessity of describing the facial structure of \( B_{A_{sa}} \). Fortunately for us, the Akemann-Pedersen theorem (Theorem 2.5) has a forerunner in [16, Corollary 5.1] where C.M. Edwards and G.T. Rüttimann described the facial structure of the closed unit ball of the hermitian part of every \( C^* \)-algebra. We recall that partial isometries in \( A_{sa} \) are all elements of the form \( e = p - q \), where \( p \) and \( q \) are orthogonal projections in \( A \).

**Theorem 5.1.** [16, Corollary 5.1] Let \( A \) be a \( C^* \)-algebra. Then for each norm-closed face \( F \) of \( B_{A_{sa}} \), there exists a unique pair of orthogonal compact projections \( p, q \) in \( A^{**} \) such that

\[
F = \{ x \in B_{A_{sa}} : x(p - q) = p + q \} = \{ p - q \},
\]

\[
= \{ x \in B_{A_{sa}} : x = (p - q) + (1 - p - q)x(1 - p - q) \}.
\]

Combining this theorem of Edwards and Rüttimann with the above Corollary 2.3 we easily get the following version of Theorem 3.16.

**Theorem 5.2.** Let \( \Delta : S(A_{sa}) \to S(B_{sa}) \) be a surjective isometry, where \( A \) and \( B \) are \( C^* \)-algebras. Then the following statements hold:

(a) For each non-zero compact partial isometry \( e \in A_{sa}^{**} \) there exists a unique (non-zero) compact partial isometry \( \phi_\Delta(e) \in B_{sa}^{**} \) such that \( \Delta(F_e) = F_{\phi_\Delta(e)} \), where

\[ F_e = (e + (1 - e^2)B_{A_{sa}}(1 - e^2)) \cap B_{A_{sa}}; \]

(b) The mapping \( e \mapsto \phi_\Delta(e) \) defines an order preserving bijection between the sets of non-zero compact partial isometries in \( A_{sa}^{**} \) and the set of non-zero compact partial isometries in \( B_{sa}^{**} \);

(c) \( \phi_\Delta \) maps minimal partial isometries in \( A_{sa}^{**} \) to minimal partial isometries in \( B_{sa}^{**} \).

The arguments in the proofs of [27, Theorems 2.7, 2.8 and 2.11 and Proposition 2.12] literally works to obtain the following four results.
Theorem 5.3. [27, Theorem 2.7] Let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where $A$ and $B$ are $C^*$-algebras. Let $e$ be a non-zero partial isometry in $A_{sa}$. Then $1$ is isolated in the spectrum of $|\Delta(e)|$.

Theorem 5.4. [27, Theorem 2.8] Let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where $A$ and $B$ are $C^*$-algebras. Then $\Delta$ maps non-zero partial isometries in $A_{sa}$ into non-zero partial isometries in $B_{sa}$. Moreover, for each non-zero partial isometry $e$ in $A_{sa}$, we have $\phi_{\Delta}^*(e) = \Delta(e)$, where $\phi_{\Delta}^*$ is the mapping given by Theorem 5.2, and there exists a surjective (real) linear isometry

$$T_e : (1 - e^2)A_{sa}(1 - e^2) \to (1 - \Delta(e)^2)B_{sa}(1 - \Delta(e)^2)$$

such that

$$\Delta(e + x) = \Delta(e) + T_e(x), \text{ for all } x \in \mathcal{B}(1 - e^2)A_{sa}(1 - e^2).$$

In particular the restriction of $\Delta$ to the face $F_e = e + (1 - e^2)B_{sa}(1 - e^2)$ is a real affine function.

Theorem 5.5. [27, Theorem 2.11] Let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where $A$ and $B$ are $C^*$-algebras. Then, for each non-zero compact partial isometry $e$ in $A_{sa}^*$ we have $\phi_{\Delta}^*(-e) = -\phi_{\Delta}^*(e)$, where $\phi_{\Delta}^*$ is the mapping given by Theorem 5.2. Consequently, for each non-zero partial isometry $e \in A_{sa}$ we have $\Delta(-e) = -\Delta(e)$.

Proposition 5.6. [27, Proposition 2.12] Let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where $A$ and $B$ are $C^*$-algebras. Then the following statements hold:

(a) For each non-zero partial isometry $v$ in $A_{sa}$, the surjective real linear isometry

$$T_v : (1 - v^2)A_{sa}(1 - v^2) \to (1 - \Delta(v)^2)B_{sa}(1 - \Delta(v)^2)$$

given by Theorem 5.4 satisfies $\Delta(e) = T_v(e)$, for every non-zero partial isometry $e \in (1 - v^2)A_{sa}(1 - v^2)$;

(b) Let $w_1, \ldots, w_n$ be mutually orthogonal non-zero partial isometries in $A_{sa}$, and let $\lambda_1, \ldots, \lambda_n$ be real numbers with $1 = |\lambda_1| \geq \max\{|\lambda_j|\}$. Then

$$\Delta \left( \sum_{j=1}^{n} \lambda_j w_j \right) = \sum_{j=1}^{n} \lambda_j \Delta (w_j);$$

(c) Suppose $v, w$ are mutually orthogonal non-zero partial isometries in $A_{sa}$ then $T_v(x) = T_w(x)$ for every $x \in \{v\}^\perp \cap \{w\}^\perp$;

(d) If $A$ is a von Neumann algebra, then for each non-zero partial isometry $v$ in $A_{sa}$ we have $\Delta(x) = T_v(x)$ for every $x \in S(1 - vv^*)A_{sa}(1 - v^*v)$.

Back to our goal, we observe that the case of $M_2(\mathbb{C})$ of all $2 \times 2$ matrices with complex entries must be treated independently.

Proposition 5.7. Let $A = M_2(\mathbb{C})$, $B$ a $C^*$-algebra, and let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry. Then there exists a surjective complex linear isometry $T : A \to B$ satisfying $T(a^*) = T(a)^*$, for all $a \in A$, and $T(x) = \Delta(x)$, for all $x \in S(A_{sa})$.

Proof. Since $A$ is finite dimensional, it follows from the hypotheses that $S(B_{sa})$ is compact, and hence $B$ is finite dimensional. Having in mind that the rank of a von Neumann algebra $M$ is the cardinal of a maximal set of mutually orthogonal
projects, Proposition 5.6 assures that $B$ must have rank 2. Therefore $B = \mathbb{C} \oplus \mathbb{C}$ or $B = M_2(\mathbb{C})$. We shall show that the first case is impossible.

Suppose $B = \mathbb{C} \oplus \mathbb{C}$. We pick two orthogonal minimal projections $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and a symmetry $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $A$.

By Theorem 5.2(b) and Proposition 5.6, $\Delta(p_1)$ and $\Delta(p_2)$ are orthogonal minimal partial isometries in $B_s$, and $\Delta(s)$ is a symmetry in $B$. We can assume, without loss of generality, that $\Delta(p_1) = (\pm 1, 0)$, $\Delta(p_2) = (0, \pm 1)$, and $\Delta(s) = (\sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \{\pm 1\}$. By hypotheses,

$$\frac{1 + \sqrt{5}}{2} = \| \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \| = \| p_1 - s \| = \| \Delta(p_1) - \Delta(s) \|
$$

$$= \| (\pm 1, 0) - (\sigma_1, \sigma_2) \| \in \{1, 2\},$$

which is impossible. Therefore, $B = M_2(\mathbb{C})$.

Let us take a surjective complex linear and symmetric isometry $T_1 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ mapping $\Delta(p_1)$ and $\Delta(p_2)$ to $p_1$ and $p_2$, respectively. We set $\Delta_1 = T_1 \circ \Delta$.

Then $\Delta_1 : S(A_{sa}) \to S(B_{sa})$ is a surjective isometry with $\Delta_1(p_i) = p_i$ for $i = 1, 2$.

An arbitrary pair of orthogonal minimal projections in $A_{sa}$ writes in the form

$q_1 = \begin{pmatrix} s_0 \\ \sqrt{s_0(1 - s_0)} \lambda \sqrt{s_0(1 - s_0)} \\ 1 - s_0 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 - s_0 \\ -s_0(1 - s_0) \\ s_0 \end{pmatrix}$

for a unique $s_0 \in (0, 1)$ and a unique $\lambda \in \mathbb{T}$ (the cases $s_0 = 0, 1$ give $p_1$ and $p_2$). By Theorem 5.2(b) and Proposition 5.6, $\Delta_1(q_1)$ and $\Delta_1(q_2)$ are orthogonal minimal partial isometries in $B_{sa}$. It is well known that $\Delta_1(q_1) = \pm \begin{pmatrix} t_0 \\ \mu \sqrt{t_0(1 - t_0)} \\ 1 - t_0 \end{pmatrix}$

for a unique $t_0 \in [0, 1]$ and a unique $\mu \in \mathbb{T}$ (compare [44, Theorems 1.3] or [40, §3]).

If $\Delta_1(q_1) = - \begin{pmatrix} t_0 \\ \mu \sqrt{t_0(1 - t_0)} \\ 1 - t_0 \end{pmatrix}$, then by hypothesis,

$$1 + \sqrt{t_0} = \| \begin{pmatrix} t_0 + 1 \\ \mu \sqrt{t_0(1 - t_0)} \\ 1 - t_0 \end{pmatrix} \| = \| - \Delta_1(q_1) + \Delta_1(p_1) \|
$$

$$= \| - q_1 + p_1 \| = \| q_1 - p_1 \| = \| \begin{pmatrix} s_0 - 1 \\ \lambda \sqrt{s_0(1 - s_0)} \\ 1 - s_0 \end{pmatrix} \| = \sqrt{(1 - s_0)},$$

which is impossible.

If $\Delta_1(q_1) = \begin{pmatrix} t_0 \\ \mu \sqrt{t_0(1 - t_0)} \\ 1 - t_0 \end{pmatrix}$, then by hypothesis,

$$\sqrt{(1 - t_0)} = \| \begin{pmatrix} t_0 - 1 \\ \mu \sqrt{t_0(1 - t_0)} \\ 1 - t_0 \end{pmatrix} \| = \| \Delta_1(q_1) - \Delta_1(p_1) \|
$$

$$= \| q_1 - p_1 \| = \| \begin{pmatrix} s_0 - 1 \\ \lambda \sqrt{s_0(1 - s_0)} \\ 1 - s_0 \end{pmatrix} \| = \sqrt{(1 - s_0)},$$

which implies that $t_0 = s_0$. That is, for each $s_0 \in [0, 1]$ and $\lambda \in \mathbb{T}$, there exists a unique $\mu \in \mathbb{T}$ such that

(6) $\Delta_1 \left( \begin{pmatrix} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \\ 1 - s_0 \end{pmatrix} \right) = \begin{pmatrix} s_0 \\ \mu \sqrt{s_0(1 - s_0)} \\ 1 - s_0 \end{pmatrix}$.
In particular, $\Delta_2\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{\mu_0}{\sqrt{2}}\right)$, for certain $\mu_0 \in \mathbb{T}$.

Let us take a surjective complex linear symmetric isometry $T_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ satisfying $T_2(p_j) = p_j$ for every $j = 1, 2$ and $T_2\Delta_2\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

We set $\Delta_2 = T_2 \circ \Delta_1 : S(A_{sa}) \to S(B_{sa})$. Proposition 5.6(b) applied to $\Delta_2$ gives

$$1 = \Delta_2(p_1) + \Delta_2(p_2) = \Delta_2(1) = \Delta_2\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) + \Delta_2\left(\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right),$$

which assures that $\Delta_2\left(\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Let us denote $r_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $r_2 = 1 - r_1$. A new application of Proposition 5.6(b) gives

$$\Delta_2(r_1 - r_2) = \Delta_2(r_1) - \Delta_2(r_2) = r_1 - r_2.$$

Take an arbitrary projection $q_1 = \left(\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}$ with $s_0 \in (0, 1)$ and $\lambda \in \mathbb{T}$. We deduce from the hypothesis and (6) (applied to $\Delta_2$) that

$$\frac{1 + \sqrt{5 - 8\Re(\lambda)s_0(1 - s_0)}}{2} = \left\|\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right\| - \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} - \Delta_2\left(\begin{array}{c} 0 \\ 1 \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0},$$

which assures that the scalar $\mu$ in (6) for $\Delta_2$ must satisfy $\mu = \lambda$ or $\mu = \bar{\lambda}$. Consequently, for each $s_0 \in (0, 1)$ and $\lambda \in \mathbb{T}$ we have

$$\Delta_2\left(\left(\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}\right) = \left(\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}$$

or

$$\Delta_2\left(\left(\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}\right) = \left(\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}. $$

We can also deduced above and Proposition 5.6(b) that

$$\Delta_2\left(\left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}\right) = \left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0},$$

or

$$\Delta_2\left(\left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}\right) = \left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}. $$

Suppose first that $\Delta_2\left(\left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}\right) = \left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}. $ Given $s_0 \in (0, 1)$ and $\lambda \in \mathbb{T}$, we have

$$\frac{1 + \sqrt{5 + 8\Im(\lambda)s_0(1 - s_0)}}{2} = \left\|\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right\| - \left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0},$$

and

$$\frac{1 + \sqrt{5 - 8\Im(\lambda)s_0(1 - s_0)}}{2} = \left\|\begin{array}{c} s_0 \\ \lambda \sqrt{s_0(1 - s_0)} \end{array}\right\| - \left(\begin{array}{c} 0 \\ i \end{array}\right) \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0} \frac{\lambda\sqrt{s_0(1 - s_0)}}{1 - s_0}. $$
and thus, (7) and the hypothesis prove that
\[
\Delta_2(q_1) = \Delta_2\left(\begin{pmatrix} s_0 & \lambda \sqrt{s_0(1-s_0)} \\ -\lambda \sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = \left(\begin{pmatrix} s_0 & \lambda \sqrt{s_0(1-s_0)} \\ -\lambda \sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = q_1,
\]
for every \(q_1\) as above. Let \(q_2 = 1 - q_1\). By Proposition 5.6(b) we also have
\[
1 = \Delta_2(p_1) + \Delta_2(p_2) = \Delta_2(1) = \Delta_2(q_1) + \Delta_2(q_2),
\]
which assures that \(\Delta_2(q_2) = q_2\). We have therefore proved that \(\Delta_2(q_i) = q_i\), for every pair of orthogonal minimal projections \(q_1, q_2\) in \(A_{sa}\). Since every element \(x\) in \(S(A_{sa})\) can be written as a linear combination of the form \(x = \sum_{j=1}^{2} \mu_j q_j\), where \(q_1\) and \(q_2\) are orthogonal minimal projections in \(A_{sa}\), \(\mu_j \in \mathbb{R}\) and \(\max\{|\mu_j|\} = 1\), a new application of Proposition 5.6(b) gives
\[
\Delta_2(x) = \Delta_2\left(\sum_{j=1}^{2} \mu_j q_j\right) = \sum_{j=1}^{2} \mu_j \Delta_2(q_j) = \sum_{j=1}^{2} \mu_j q_j = x.
\]
This shows that \(\Delta_2(x) = x\), for every \(x\) in \(S(A_{sa})\), and hence \(\Delta(x) = T_2^{-1}T_1^{-1}(x)\), for every \(x\) in \(S(A_{sa})\).

Assume now that \(\Delta_2\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\). Similar arguments to those given above show that, in this case, we have
\[
\Delta_2(q_1) = \Delta_2\left(\begin{pmatrix} s_0 & \lambda \sqrt{s_0(1-s_0)} \\ -\lambda \sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = \left(\begin{pmatrix} s_0 & \lambda \sqrt{s_0(1-s_0)} \\ -\lambda \sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = q_1,
\]
for every minimal projection \(q_1\) as above, where \((x_{ij}) = (x_{i,j})\), and \(\Delta_2(x) = \overline{x}\), for every \(x\) in \(S(A_{sa})\). Therefore, \(\Delta(x) = T_2^{-1}T_1^{-1}(\overline{x})\), for every \(x\) in \(S(A_{sa})\). Denoting \(S = T_2^{-1}T_1^{-1}\) we have a complex linear and symmetric isometry \(S : M_2(\mathbb{C}) \to M_2(\mathbb{C})\). We define \(T : M_2(\mathbb{C}) \to M_2(\mathbb{C})\) by \(T(h + ik) := S(h) + iS(k) = S(h + ik) = S((h + ik)^*) = S((h + ik)^t)\), for every \(h, k \in A_{sa}\), which provides the mapping \(T\) in the statement of the proposition. \(\square\)

We can state now the desired result and its proof, where we show that the synthesis of a surjective isometry is even easier in this setting.

**Theorem 5.8.** Let \(\Delta : S(M_{sa}) \to S(N_{sa})\) be a surjective isometry, where \(M\) and \(N\) are von Neumann algebras. Then there exists a surjective complex linear isometry \(T : M \to N\) satisfying \(T(a^*) = T(a)^*\), for all \(a \in M\), and \(T(x) = \Delta(x)\), for all \(x \in S(M_{sa})\).

**Proof.** We shall distinguish the following three cases,

1. \(M\) contains no type \(I_2\) von Neumann factors;
2. \(M\) contains a type \(I_2\) von Neumann factor but \(M\) is not a type \(I_2\) von Neumann factor;
3. \(M\) is a type \(I_2\) von Neumann factor.
Case (3) is solved by Proposition 5.7.

Case (2). We can assume that $M = J_1 \oplus J_2$, where $J_1$ and $J_2$ are non-zero orthogonal weak$^*$ closed ideals of $M$ and $J_1 = M_2(\mathbb{C})$. We can now mimic the arguments we gave in the solution to Tingley's problem for compact operators in page 13. Let us take two non-zero projections $p_1$ in $J_1$ and $p_2$ in $J_2$, and define a mapping $T : M \to N$ given by

$$T(x) = T_{p_1}(\pi_2(x)) + T_{p_2}(\pi_1(x))$$

where $\pi_1$ and $\pi_2$ stand for the canonical projections of $M$ onto $J_1$ and $J_2$, respectively, and $T_{p_1}$ and $T_{p_2}$ are the surjective weak$^*$ continuous complex linear and symmetric isometries given by Theorem 5.4. The mapping $T$ is complex linear and weak$^*$ continuous because $T_{p_1}$ and $T_{p_2}$ are. Any projection $p$ in $M$ can be written in the form $p = p_1 + p_2$ where $p_j$ is a projection in $J_1$. Let us pick an algebraic element $x \in S(M_{sa})$ which can be written in the form $x = \sum \alpha_j p_j + \sum \beta_k q_k$, where $p_j, q_k$ are mutually orthogonal non-zero projections in $M_{sa}$, $\alpha_j, \beta_k \in \mathbb{R}\setminus\{0\}$, $\max\{\alpha_j, |\beta_k|\} = 1$, $p_j \in J_1$ and $q_k \in J_2$ for all $j, k$. By definition of $T$ and Proposition 5.6(b) we have

$$\Delta(x) = \sum_{j=1}^{\infty} \alpha_j \Delta(p_j) + \sum_{k=1}^{\infty} \beta_k \Delta(q_k) = \sum_{j=1}^{\infty} \alpha_j T_{p_2}(p_j) + \sum_{k=1}^{\infty} \beta_k T_{p_1}(q_k)$$

$$= \sum_{j=1}^{\infty} \alpha_j T(p_j) + \sum_{k=1}^{\infty} \beta_k T(q_k) = T(x).$$

The norm density of this kind of algebraic elements $x \in S(M_{sa})$ together with the norm continuity of $T$ and $\Delta$ prove that $T(x) = \Delta(x)$ for all $x \in S(M)$.

Case (1). $M$ contains no type $I_2$ von Neumann factors. Let us define a vector measure on the lattice $\mathcal{P}roj(M)$ of all projections of $M$ defined by $\mu : \mathcal{P}roj(M) \to N$, $\mu(p) = \Delta(p)$ if $p \in S(M)$ and $\mu(0) = 0$. Proposition 5.6(b) assure that $\mu$ is finitely additive, that is

$$\mu\left(\sum_{j=1}^{m} p_j\right) = \sum_{j=1}^{m} \mu(p_j),$$

whenever $p_1, \ldots, p_m$ are mutually orthogonal projections in $M$. We further have $\|\mu(p)\| \leq 1$ for every $p \in \mathcal{P}roj(M)$. By the Bunce-Wright-Mackey-Gleason theorem (see [3, Theorem A] or [4, Theorem A]) there exists a bounded (complex) linear operator $T : M \to N$ satisfying $T(p) = \mu(p) = \Delta(p)$, for every $p \in \mathcal{P}roj(M) \setminus \{0\}$. By definition $T(p) \in N_{sa}$ for every projection $p$ in $M$. Therefore $T$ is a symmetric map, that is, $T(a^*) = T(a)^*$ for all $a \in M$.

Finally, Proposition 5.6(b) also guarantees that $\Delta$ and $T$ coincide on algebraic elements in $S(M_{sa})$ which can be written as finite real linear combinations of mutually orthogonal projections. Since this kind of algebraic elements are norm dense in $S(M_{sa})$, we deduce from the norm continuity of $\Delta$ and $T$ that $T(x) = \Delta(x)$ for all $x \in S(M_{sa})$. \hfill \Box

Remark 5.9. After completing the writing of this chapter, the preprint by M. Mori [33] became available in arxiv. Section 5 in the just quoted paper is devoted to study Theorem 5.8 with a different proof based on a theorem of Dye on orthoisomorphisms (see [33, §5] and [13]). So, Theorem 5.8 should be also credited to M. Mori. It is
surprising that the arguments developed by Mori find a similar obstacle with type $I_2$ von Neumann factors when applying Dye’s theorem. To solve the difficulties, Mori build a analogue to our Proposition 5.7 in [33, Proposition 5.2 and its proof]. The proof of Proposition 5.7 is a bit simpler with pure geometry-linear algebra arguments.

**Open Problem 3.** Let $\Delta : S(A_{sa}) \to S(B_{sa})$ be a surjective isometry between the unit spheres of the hermitian parts of two $C^*$-algebras. Does $\Delta$ admit an extension to a surjective complex linear isometry from $A$ onto $B$?

6. Isometries between the spheres of positive operators

Contrary to the results revised in previous sections, in the third variant of Problem 1.1 treated in this survey the theory on the facial structure of a $C^*$-algebra revised in section 2 will not play any role. Let us estate the concrete statement. Given a subset $B$ of a Banach space $X$, the symbol $S(B)$ will stand for the intersection of $B$ and $S(X)$. Given a $C^*$-algebra $A$, the symbol $A^+$ will denote the cone of positive elements in $A$, while $S(A^+)$ will stand for the sphere of positive norm-one operators. The concrete variant of Problem 1.1 reads as follows.

**Problem 6.1.** Let $\Delta : S(X^+) \to S(Y^+)$ be a surjective isometry, where $X$ and $Y$ are Banach spaces which can be regarded as linear subspaces two $C^*$-algebras $A$ and $B$, $S(X^+) = S(X) \cap A^+$ and $S(Y^+) = S(Y) \cap B^+$. Does $\Delta$ admit an extension to a surjective complex linear isometry $T : X \to Y$?

Problem 6.1 is too general. We can easily find non isomorphic Banach spaces $X$ and $Y$ which are linear subspaces of two $C^*$-algebras $A$ and $B$, for which $S(X^+)$ and $S(Y^+)$ reduce to a single point.

Before dealing with the historical background and forerunners, we shall make some observations. If we have a surjective isometry $\Delta : S(A^+) \to S(B^+)$ between the spheres of positive elements in two arbitrary $C^*$-algebras the application of Theorems 2.1 and 2.2 is non-viable because $A^+$ and $B^+$ are not Banach spaces.

Another comment, the hypotheses in Problem 6.1 are strictly weaker than those in Theorems 3.4, 3.8, 3.10, 3.15, 4.5, 4.6, and 5.8. However, the conclusion is also weaker because we need to find a surjective isometry $T : A \to B$ whose restriction to $S(A^+)$ coincides with $\Delta$, we do not have to show that $T$ and $\Delta$ coincide on the whole $S(A)$ nor on $S(A_{sa})$. That is, the synthesis of the mapping $T$ is, a priori, easier at the cost of loosing the main geometric tools.

We can now go survey the main achievements in this line. Let us recall some terminology. According to the notation in previous sections, we shall denote by $(C_p(H), \| \cdot \|_p)$ the Banach space of all $p$-Schatten-von Neumann operators on a complex Hilbert space $H$, where $1 \leq p \leq \infty$. For $p = 1$ we find the space of trace class operators. By an standard abuse of notation we identify $C_\infty(H)$ with $B(H)$. Let the symbol $C_p(H)^+$ denote the set of all positive operators in $C_p(H)$. The elements in the set $S(C_p(H)^+) = S(C_p(H)) \cap C_p(H)^+$ are usually called *density operators*.

Our first result, which was obtained by L. Molnár and W. Timmermann in [35], provides a complete positive solution to Problem 6.1 for the space $C_1(H)$ of trace class operators on an arbitrary complex Hilbert space $H$. 


Theorem 6.2. [35, Theorem 4] Let $H$ be an arbitrary complex Hilbert space. Then every surjective isometry $\Delta : S(C_n(H)^+) \to S(C_n(H)^+)$ admits a unique extension to a surjective complex linear isometry on $C_n(H)$.

In 2012, G. Nagy and L. Molnár solve Problem 6.1 in the finite dimensional case for every $1 \leq p$.

Theorem 6.3. [34, Theorem 1] Let $H$ be a finite dimensional complex Hilbert space, and let $\infty > p > 1$. Then every isometry $\Delta : S(C_p(H)^+) \to S(C_p(H)^+)$ admits a unique extension to a surjective complex linear isometry on $C_p(H)$.

Let us observe that the mapping $\Delta$ in the above theorem is not assumed to be surjective a priori. However, as a consequence of the result $\Delta$ is surjective.

Theorem 6.3 was extended by G. Nagy to arbitrary complex Hilbert spaces in [36].

Theorem 6.4. [36, Theorem 1] Let $H$ be an arbitrary complex Hilbert space, and let $p \in (1, \infty)$. Then every isometry $\Delta : S(C_p(H)^+) \to S(C_p(H)^+)$ admits a unique extension to a surjective complex linear isometry on $C_p(H)$.

Problem 6.1 has been explored, in a very recent paper due G. Nagy, for surjective isometries $\Delta : S(B(H)^+) \to S(B(H)^+)$ under the hypothesis of $H$ being finite dimensional. In the paper [37] we can find the following result.

Theorem 6.5. [37, Theorem] Let $H$ be a finite dimensional complex Hilbert space, and let $\Delta : S(B(H)^+) \to S(B(H)^+)$ be an isometry. Then $\Delta$ is surjective and there exists a (unique) surjective complex linear isometry $T : B(H) \to B(H)$ satisfying $T(x) = \Delta(x)$, for all $x \in S(B(H)^+)$. The arguments developed by Nagy in the paper [37] develop some interesting tools and results in the finite dimensional setting. Some of them have been successfully extended to arbitrary dimensions. Let $E$ and $P$ be subsets of a Banach space $X$. Following the notation employed in the recent paper [42], the unit sphere around $E$ in $P$ is defined as the set

$$Sph(E; P) := \{ x \in P : \| x - b \| = 1 \text{ for all } b \in E \}.$$  

To simplify the notation, given a $C^*$-algebra $A$, and a subset $E \subset A$ we shall write $Sph^+(E)$ or $Sph^+_A(E)$ for the set $Sph(E; S(A^+))$.

In [37, Proof of Claim 1] G. Nagy proves that if $H$ is a finite dimensional complex Hilbert space, and $a$ is a positive norm-one element in $B(H) = M_n(C)$, then

$$a \text{ is a projection if, and only if, } Sph^+_{M_n(C)} \left( Sph^+_{M_n(C)}(a) \right) = \{ a \}.$$

We have recently generalized Nagy’s result to the setting of atomic von Neumann algebras. We recall that a von Neumann algebra $M$ is called atomic if it coincides with the weak* closure of the linear span of its minimal projections. It is known that every atomic von Neumann algebra $M$ can be written in the form $M = \bigoplus_j B(H_j)$, where each $H_j$ is a complex Hilbert space (compare [46, §V.1] or [45, §2.2]).

Theorem 6.6. [42, Theorem 2.3] Let $M$ be an atomic von Neumann algebra, and let $a$ be a positive norm-one element in $M$. Then the following statements are equivalent:
In the setting of atomic von Neumann algebras. For brevity we shall
consider the setting of compact operators. It can be concluded that given two atomic
operators, the conditions hold when

\[ Sph_{\lambda}^+(\{a\}) = \{a\}. \]

Actually, if \( a \) is a positive norm-one element in an arbitrary \( C^* \)-algebra \( A \) satisfying \( Sph_{\lambda}^+(\{a\}) = \{a\} \), then \( a \) is a projection (see [42, Proposition 2.2]).

**Open Problem 4.** Does the equivalence in Theorem 6.6 hold when \( M \) is a general von Neumann algebra or a \( C^* \)-algebra?

For a separable infinite dimensional complex Hilbert space \( H_3 \) and the \( C^* \)-algebra \( K(H_3) \), of compact operators on \( H_3 \), we have actually established a more general result, whose finite dimensional version was given by G. Nagy in [37, Proof of Claim 1].

**Theorem 6.7.** [42, Theorem 3.3] Let \( H_3 \) be a separable infinite dimensional complex Hilbert space. Then the identity

\[ Sph_{K(H_3)}^+(\{a\}) = \left\{ b \in S(K(H_3)^+) : s_{K(H_3)}(a) \leq s_{K(H_3)}(b), \text{ and } 1 - r_{B(H_3)}(a) \leq 1 - r_{B(H_3)}(b) \right\}, \]

holds for every \( a \) in the unit sphere of \( K(H_3)^+ \).

A consequence of the above theorem gives an appropriate version of Theorem 6.6 for \( K(H_3) \).

**Theorem 6.8.** [42, Theorem 2.5] Let \( a \) be a positive norm-one element in \( K(H_3) \), where \( H_3 \) is a separable complex Hilbert space. Then the following statements are equivalent:

(a) \( a \) is a projection;
(b) \( Sph_{K(H_3)}^+(\{a\}) = \{a\} \).

Thanks to Theorems 6.6 and 6.8 it can be concluded that given two atomic von Neumann algebras \( M \) and \( N \) (respectively, separable complex Hilbert spaces \( H_3 \) and \( H_4 \)), and a surjective isometry \( \Delta : S(M^+) \to S(N^+) \) (respectively, \( \Delta : S(K(H_3)^+) \to S(K(H_4)^+) \)), then \( \Delta \) maps \( \text{Proj}(M) \setminus \{0\} \) onto \( \text{Proj}(N) \setminus \{0\} \) (respectively, \( \text{Proj}(K(H_3)) \setminus \{0\} \) onto \( \text{Proj}(K(H_4)) \setminus \{0\} \)), and the restriction

\[ \Delta|_{\text{Proj}(M)\setminus\{0\}} : \text{Proj}(M)\setminus\{0\} \to \text{Proj}(N)\setminus\{0\} \]

(respectively, \( \Delta|_{\text{Proj}(K(H_3))\setminus\{0\}} : \text{Proj}(K(H_3))\setminus\{0\} \to \text{Proj}(K(H_4))\setminus\{0\} \)) is a surjective isometry.

These are some of the tools that combined with many other technical arguments are applied to give a partial solution to Problem 6.1 in the setting of compact operators.

**Theorem 6.9.** [42, Theorem 3.7] Let \( H_3 \) and \( H_4 \) be separable complex Hilbert spaces. Let us assume that \( H_3 \) is infinite dimensional. We suppose that \( \Delta : S(K(H_3)^+) \to S(K(H_4)^+) \) is a surjective isometry. Then there exists a surjective complex linear isometry \( T : K(H_3) \to K(H_4) \) satisfying \( T(x) = \Delta(x) \), for all \( x \in S(K(H_3)^+) \). We can further conclude that \( T \) is a *-isomorphism or a *-anti-isomorphism.

Additional technical results are given in [42, §4] to give a complete solution to Problem 6.1 in the setting of atomic von Neumann algebras. For brevity we shall not comment some of the deep technical results required to establish this solution. The final result reads as follows:
Suppose $\Delta$:

**Open Problem 5.** Let $\Delta : S(A^+ ) \to S(B^+ )$ be a surjective isometry, where $A$ and $B$ are $C^*$-algebras. Does $\Delta$ admit an extension to a surjective complex linear isometry from $A$ onto $B$?

**Open Problem 6.** Let $H$ be an arbitrary complex Hilbert space, and let $p \in (1, \infty)$. Suppose $\Delta : S(C_p(H)^+ ) \to S(C_p(H)^+ )$ is a surjective isometry. Does $\Delta$ admit a unique extension to a surjective real linear isometry on $C_p(H)$.

A more general version has been also posed by M. Mori in [33, Problem 6.3].

**Open Problem 7.** Let $1 < p < \infty$, $p \neq 2$, let $M$, $N$ be von Neumann algebras and $\Delta : S(L^p(M)) \to S(L^p(N))$ be a surjective isometry between the unit spheres of two noncommutative $L^p$-spaces (with respect to fixed normal semifinite faithful weights). Does $\Delta$ admit an extension to a real linear surjective isometry $T : L^p(M) \to L^p(N)$?

**Acknowledgements** Authors partially supported by the Spanish Ministry of Economy and Competitiveness (MINECO) and European Regional Development Fund project no. MTM2014-58984-P and Junta de Andalucía grant FQM375.

I thank the organizers of the meeting “Preservers Everywhere, Szeged-2017” for a successful and fruitful initiative.

**References**

1. C.A. Akemann, G.K. Pedersen, Facial structure in operator algebra theory, *Proc. Lond. Math. Soc.* 64, 418-448 (1992).
2. J. C. Alexander, Compact Banach algebras, *Proc. London Math. Soc.* (3) 18, 1-18 (1968).
3. L.J. Bunce, J.D.M. Wright, The Mackey-Gleason problem, *Bull. Amer. Math. Soc.* 26, 288-293 (1992).
4. L.J. Bunce, J.D.M. Wright, The Mackey-Gleason problem for vector measures on projections in von Neumann algebras, *J. London Math. Soc.* 49, 133-149 (1994).
5. L. Cheng, Y. Dong, On a generalized Mazur-Ulam question: extension of isometries between unit spheres of Banach spaces, *J. Math. Anal. Appl.* 377, 464-470 (2011).
6. Ch.-H. Chu. Jordan structures in geometry and analysis., Cambridge, Cambridge University Press, 2012.
7. Ch.H. Chu, T. Dang, B. Russo, B. Ventura, Surjective isometries of real $C^*$-algebras, *J. London Math. Soc.* 47, 97-118 (1993).
8. T. Dang, Real isometries between JB*-triples, *Proc. Amer. Math. Soc.* 114, 971-980 (1992).
9. G.G. Ding, The representation theorem of onto isometric mappings between two unit spheres of $l^\infty$-type spaces and the application on isometric extension problem, *Sci. China Ser. A* 47, 722-729 (2004).
10. G.G. Ding, The representation theorem of onto isometric mappings between two unit spheres of $l^1(\Gamma)$ type spaces and the application to the isometric extension problem, *Acta. Math. Sin. (Engl. Ser.)* 20, 1089-1094 (2004).
11. G.G. Ding, The isometric extension of the into mapping from a $L^\infty(\Gamma)$-type space to some Banach space, *Illinois J. Math.* 51(2), 445-453 (2007).
12. G.G. Ding, On isometric extension problem between two unit spheres, *Sci. China Ser. A* 52, 2069-2083 (2009).
13. H.A. Dye, On the geometry of projections in certain operator algebras, *Ann. of Math.* (2) 61, 73-89 (1955).
A SURVEY ON TINGLEY’S PROBLEM FOR OPERATOR ALGEBRAS

[14] C.M. Edwards, F.J. Fernández-Polo, C.S. Hoskin, A.M. Peralta, On the facial structure of the unit ball in a JB*-triple, *J. Reine Angew. Math.* **641**, 123-144 (2010).

[15] C.M. Edwards, G.T. Rüttimann, On the facial structure of the unit balls in a GL-space and its dual, *Math. Proc. Cambridge Philos. Soc.* **98**, 305-322 (1985).

[16] C.M. Edwards, G.T. Rüttimann, On the facial structure of the unit balls in a GM-space and its dual, *Math. Z.* **193**, 597-611 (1986).

[17] C.M. Edwards, G.T. Rüttimann, On the facial structure of the unit balls in a JBW*-triple and its predual, *J. Lond. Math. Soc.* **38**, 317-332 (1988).

[18] C.M. Edwards, G.T. Rüttimann, Compact tripotents in bidual JB*-triples, *Math. Proc. Camb. Phil. Soc.* **120**, 155-173 (1996).

[19] A.J. Ellis, Real characterizations of function algebras amongst function spaces, *Bull. London Math. Soc.* **22**, 381-385 (1990).

[20] X.N. Fang, J.H. Wang, Extension of isometries between the unit spheres of normed space $E$ and $C(\Omega)$, *Acta Math. Sinica* (Engl. Ser.) **22**, 1819-1824 (2006).

[21] F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, I. Villanueva, Tingley’s problem for spaces of trace class operators, *Linear Algebra Appl.* **529**, 294-323 (2017).

[22] F.J. Fernández-Polo, J. Martínez, A.M. Peralta, Surjective isometries between real JB*-triples, *Math. Proc. Cambridge Phil. Soc.*, **137**, 709-723 (2004).

[23] F.J. Fernández-Polo, A.M. Peralta, On the facial structure of the unit ball in the dual space of a JB*-triple, *Math. Ann.* **348**, 1019-1032 (2010).

[24] F.J. Fernández-Polo, A.M. Peralta, Low rank compact operators and Tingley’s problem, preprint 2016. arXiv:1611.0218v1

[25] F.J. Fernández-Polo, A.M. Peralta, On the extension of isometries between the unit spheres of von Neumann algebras, preprint 2017. arXiv:1709.08529v1

[26] F.J. Fernández-Polo, A.M. Peralta, Tingley’s problem through the facial structure of an atomic JBW*-triple, *J. Math. Anal. Appl.* **455**, 750-760 (2017).

[27] F.J. Fernández-Polo, A.M. Peralta, On the extension of isometries between the unit spheres of von Neumann algebras, to appear in *Trans. Amer. Math. Soc.*

[28] O. Hatori and L. Molnár, Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in $C^*$-algebras, *J. Math. Anal. Appl.* **409**, 158-167 (2014).

[29] R.V. Kadison, G.K. Pedersen, Means and convex combinations of unitary operators, *Math. Scand.* **57**, 249-266 (1985).

[30] W. Kaup, A Riemann Mapping Theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183**, 503-529 (1983).

[31] P. Mankiewicz, On extension of isometries in normed linear spaces, *Bull. Acad. Pol. Sci.*, Sér. Sci. Math. Astron. Phys. **20**, 367-371 (1972).

[32] T. Miura, Real-linear isometries between function algebras, *Publ. Math. Debrecen* **82**, 183-192 (2013).

[33] G. Nagy, Isometries on positive operators of unit norm, *Publ. Math. Debrecen* **82**, 183-192 (2013).

[34] C.L. Olsen, Unitary approximation, *J. Funct. Anal.* **85**, no. 2, 392-419 (1989).

[35] C.L. Olsen, G.K. Pedersen, Convex combinations of unitary operators in von Neumann algebras, *J. Funct. Anal.* **66**, no. 3, 365-380 (1986).

[36] G.K. Pedersen, Measure theory for $C^*$ algebras, II, *Math. Scand.* **22**, 63-74 (1968).

[37] G.K. Pedersen, $C^*$-algebras and their automorphism groups, London Mathematical Society Monographs Vol. 14, Academic Press, London, 1979.

[38] A.M. Peralta, On the unit sphere of positive operators, preprint 2017. arXiv:1711.05652v1

[39] A.M. Peralta, R. Tanaka, A solution to Tingley’s problem for isometries between the unit spheres of compact $C^*$-algebras and JB*-triples, to appear in *Sci. China Math.* arXiv:1608.06327v1.
[44] I. Raeburn, A.M. Sinclair, The C*-algebra generated by two projections, *Math. Scand.* **65**, no. 2, 278-290 (1989).

[45] S. Sakai, *C*-algebras and W*-algebras. Springer Verlag, Berlin 1971.

[46] M. Takesaki, *Theory of operator algebras I*, Springer, New York, 2003.

[47] D. Tan, Extension of isometries on unit sphere of $L^\infty$, *Taiwanese J. Math.* **15**, 819-827 (2011).

[48] D. Tan, On extension of isometries on the unit spheres of $L^p$-spaces for $0 < p \leq 1$, *Nonlinear Anal.* **74**, 6981-6987 (2011).

[49] R. Tanaka, A further property of spherical isometries, *Bull. Aust. Math. Soc.* **90**, 304-310 (2014).

[50] R. Tanaka, The solution of Tingley’s problem for the operator norm unit sphere of complex $n \times n$ matrices, *Linear Algebra Appl.* **494**, 274-285 (2016).

[51] R. Tanaka, Spherical isometries of finite dimensional C*-algebras, *J. Math. Anal. Appl.* **445**, no. 1, 337-341 (2017).

[52] R. Tanaka, Tingley’s problem on finite von Neumann algebras, *J. Math. Anal. Appt.* **451**, 319-326 (2017).

[53] D. Tingley, Isometries of the unit sphere, *Geom. Dedicata* **22**, 371-378 (1987).

[54] R.S. Wang, Isometries between the unit spheres of $C_0(\Omega)$ type spaces, *Acta Math. Sci.* (English Ed.) **14**, no. 1, 82-89 (1994).

[55] X. Yang, X. Zhao, On the extension problems of isometric and nonexpansive mappings. In: *Mathematics without boundaries*. Edited by Themistocles M. Rassias and Panos M. Pardalos. 725-748, Springer, New York, 2014.

[56] K. Ylinen, *Compact and finite-dimensional elements of normed algebras*, Ann. Acad. Sci. Fenn. Ser. A I, No. **428**, 1968.

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

E-mail address: aperalta@ugr.es