On complete hypersurfaces with constant scalar curvature $n(n-1)$ in the unit sphere

Jinchuan Bai, Yong Luo

Abstract

Let $M^n$ be an $n$-dimensional complete and locally conformally flat hypersurface in the unit sphere $S^{n+1}$ with constant scalar curvature $n(n-1)$. We show that if the total curvature $(\int_M |H|^n \, dv)^{\frac{1}{n}}$ of $M$ is sufficiently small, then $M^n$ is totally geodesic.

Keywords Hypersurfaces; Constant scalar curvature; Gap theorem

MSC2020 53C24, 53C42

1 Introduction

Inspired by the famous Bernstein theorem that a minimal entire graph $M^n(n \leq 7)$ immersed in $\mathbb{R}^{n+1}$ must be a hyperplane [12], there are many results characterizing the hyperplane as the unique minimal hypersurface in a Euclidean space under certain geometrical or analytical assumptions [2][3][4][5][11][10]. In particular, Ni [8] and Yun [15] proved that a complete minimal hypersurface in a Euclidean space is a hyperplane if its total scalar curvature is sufficiently small.

A hypersurface is minimal if its mean curvature is zero and the mean curvature is, in our definition, $\frac{1}{n}$ of the first elementary symmetric function of the second fundamental form. Except mean curvature, scalar curvature is the most important curvature for hypersurfaces and for hypersurfaces in a Euclidean space scalar curvature is the second elementary symmetric function of the second fundamental form.

Recently, Li, Xu and Zhou studied complete hypersurfaces in a Euclidean space with zero scalar curvature and they obtained a Ni and Yun’s type Gap theorem.

Theorem 1.1 ([6]). Let $M^n(n \geq 3)$ be a locally conformally flat complete hypersurface in $\mathbb{R}^{n+1}$ with zero scalar curvature. Then there exists a positive
constant $C(n)$ depending only on $n$ such that $M$ is a hyperplane if

$$\int_M |H|^n \, dv < C(n).$$

It is very natural to ask if one could obtain similar result for hypersurfaces in the unit sphere with zero second elementary symmetric function of the second fundamental form. In this paper we answer this question affirmatively.

**Theorem 1.2.** Let $M^n(n \geq 3)$ be a complete locally conformally flat hypersurface in unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature $n(n-1)$. There exists a sufficiently small number $\alpha$ which depends only on dimension $n$ such that if

$$\left(\int_M |H|^n \, dv\right)^{\frac{1}{n}} < \alpha,$$

then $M$ is totally geodesic.

Theorem 1.2 is proved by analyzing the following Simons’ type inequality (where to derive this inequality the assumption of locally conformally flatness is used)

$$|r|\Delta |r| \geq \frac{n}{n-2} \text{tr}(r^2) + n |r|^2,$$

where $|r|^2 = \sum_{i,j} r_{ij}^2$ and $r_{ij} := R_{ij} - (n-1)\delta_{ij}$. If $M$ is complete with sufficiently small total curvature, from the above inequality we obtain that $|r| = 0$, which implies by the Gauss equations that $nHh_{ij} = \sum_k h_{ik}h_{kj}$ for any $i, j = 1, \ldots, n$. Then we can prove that $M$ has at least $n-1$ zero principle curvatures. At last we prove the following fact to finish the proof of Theorem 1.2.

**Proposition 1.1.** Assume that $M$ is a complete hypersurface in unit sphere $\mathbb{S}^{n+1}$. If $M$ has at least $n-1$ zero principle curvatures everywhere, then $M$ is totally geodesic.

Proposition 1.1 was pointed out by Wu that it can be proved by similar argument with that used in the proof of Theorem 2.2 in [13]. For convenience of the reader we will give the details of the proof of Proposition 1.1.

The rest of the article is arranged as follows. In section 2 we will list and prove several useful Lemmas which will be used in the rest of our paper. Theorem 1.2 and Proposition 1.1 are proved in section 3.

## 2 Notations and Lemmas

Assume that $M^n$ is a hypersurface in unit sphere $\mathbb{S}^{n+1}$. We choose a local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}\}$ along $M$, where $\{e_i\}_{i=1, \ldots, n}$ are tangent to $M$ and $e_{n+1}$ are normal to $M$. Let $\{\omega_A\}_{A=1, \ldots, n+1}$ be the corresponding
dual coframe, and \( \{ \omega_{AB} \} \) the connection 1-forms on \( S^{n+1} \). We make the convention on the range of indices that \( 1 \leq A, B, ... \leq n + 1 \) and \( 1 \leq i, j, ... \leq n \). The structure equation of \( S^{n+1} \) are

\[
d\omega_A = - \sum_B \omega_{AB} \wedge \omega_B + \omega_{BA} = 0,
\]

\[
d\omega_{AB} = - \sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,
\]

where \( K_{ABCD} \) is the curvature tensor of \( S^{n+1} \). When restricted on \( M \) we have \( \omega_{n+1} = 0 \). Hence \( 0 = d\omega_{n+1} = - \sum_i \omega_{n+1i} \wedge \omega_i \). By Cartan’s lemma, there exists local functions \( h_{ij} \) such that

\[
\omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\]

The second fundamental form is \( h = \sum_{ij} h_{ij} \omega_i \otimes \omega_j \). We also write \( h = (h_{ij}) \) as a matrix and call the eigenvalues of \( h \) the principle curvatures of \( M \). The mean curvature of \( M \) in \( S^{n+1} \) is defined by

\[
H := \frac{1}{n} \text{tr} h.
\]

The structure equations of \( M \) are

\[
d\omega_i = - \sum_j \omega_{ij} \wedge \omega_j + \omega_{ji} = 0,
\]

\[
d\omega_{ij} = - \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l,
\]

and the Gauss equations and Codazzi equations are given by

\[
R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + h_{ik} h_{jl} - h_{il} h_{jk} \quad (2.1)
\]

\[
h_{ij,k} = h_{ik,j} \quad (2.2)
\]

where the covariant derivative of \( h_{ij} \) is defined by

\[
\sum_{k=1}^n h_{ij,k} \omega_k = dh_{ij} - \sum_{k=1}^n h_{kj} \omega_{ki} - \sum_{k=1}^n h_{ik} \omega_{kj}
\]

In particular, components of the Ricci tensor are given by

\[
R_{ij} = (n-1)\delta_{ij} + nH h_{ij} - \sum_k h_{ik} h_{kj} \quad (2.3)
\]

Assume that \( R \) is the scalar curvature of \( M \), then by (2.3) we have
where $|h|^2 = \sum_{i,j} (h_{ij})^2$ is the squared norm of the second fundamental form.

We have

Lemma 2.1. Let $M^n (n \geq 3)$ be a hypersurface in $\mathbb{S}^{n+1}$ with constant scalar curvature $R = n(n−1)$. If $M$ is locally conformally flat, we have

$$R_{ijk} = -\varphi_{ij}\delta_{lk} + \varphi_{ik}\delta_{lj} - \delta_{ij}\varphi_{lk} + \delta_{ik}\varphi_{lj}$$

where $\varphi_{ij} = \frac{R_{ij} - \frac{n}{n-2}H_{ij}}{n-2}$.

Proof. Recall that the Riemannian curvature tensor has the following decomposition

$$R_{ijk} = W_{ijk} + \frac{1}{n-2}(R_{ik}\delta_{lj} - R_{ij}\delta_{lk} + R_{ij}\delta_{ik} - R_{lk}\delta_{ij}) - \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{lj} - \delta_{ij}\delta_{lk}).$$

Since $M$ is locally conformally flat, i.e. $W_{ijk} = 0$, and $R = n(n−1)$, we get

$$R_{ijk} = -\varphi_{ij}\delta_{lk} + \varphi_{ik}\delta_{lj} - \delta_{ij}\varphi_{lk} + \delta_{ik}\varphi_{lj}$$

by direct computation. 

Lemma 2.2 ([9]). Let $a_i, i = 1, 2, \ldots, n$ be real numbers satisfying

$$\sum_{i=1}^{n} a_i = 0, \sum_{i=1}^{n} a_i^2 = |\mu|^2,$$

then

$$\left|\sum_{i=1}^{n} a_i^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}}|\mu|^3.$$

The equality holds if and only if $n-1$ terms of $\{a_i\}_{i=1}^{n}$ are equal.

Following the argument in [9], we have

Lemma 2.3. Let $M^n (n \geq 3)$ be a hypersurface in $\mathbb{S}^{n+1}$ with constant scalar curvature $R = n(n−1)$. Then at least $n-1$ eigenvalues of $h_{ij}$ are zero if

$$\sum_{j} h_{ij} h_{jk} = nH h_{ik}.$$

Proof. Since $\sum_{j} h_{ij} h_{jk} = nH h_{ik}$ and $R = n(n−1)$, by (2.4) we have

$$\text{tr} (h^3) = \sum_{i,j,k} h_{ij} h_{jk} h_{ik} = nH \sum_{i,j,k} (h_{ik})^2 = n^3 H^3$$

Set $\mu_{ij} := h_{ij} - H \delta_{ij}$. Then we have

$$|\mu|^2 = (n-1)nH^2.$$
and
\[ \text{tr}(h^3) = \text{tr}(\mu^3) + 3H\text{tr}(\mu^2) + nH^3 = \text{tr}(\mu^3) + 3H|h|^2 - 2nH^3. \]

Therefore, we have
\[ |\text{tr}(\mu^3)| = n(n-1)(n-2)|H^3| = \frac{n-2}{\sqrt{n(n-1)}}|\mu|^3. \]

Let \( p \in M \) and assume that \( \mu_{ij} \) is diagonal at \( p \), then by Lemma 2.2 we may assume that \( (\mu_{ij}) = \text{diag}\{v_1, \ldots, v_1, v_2\} \). Thus at \( p \),
\[ h_{ij} = \text{diag}\{v_1 + H, \ldots, v_1 + H, v_2 + H\} \]

Since \( \sum_j h_{ij}h_{jk} = nHh_{ik} \), we have
\[ (v_1 + H)^2 = ((n-1)(v_1 + H) + (v_2 + H))(v_1 + H), \]

and
\[ (v_2 + H)^2 = ((n-1)(v_1 + H) + (v_2 + H))(v_2 + H). \]

Therefore we have \( (v_1 + H) = 0. \)

Recall that the following Michael-Simon’s inequality holds true.

**Lemma 2.4** ([7]). Let \( M^n \) be a sub-manifold in \( \mathbb{R}^{n+p} \). Then for any function \( \varphi \in C^1_0(M) \), we have
\[
\left( \int_M |\varphi|^{\frac{n}{n-1}} \, dM \right)^{\frac{n-1}{n}} \leq C_1 \left( \int_M |\nabla \varphi| \, dM + \int_M |H\varphi| \, dM \right)
\]

Following the argument of [6] and [14] (see also [1]) we get the following variant of Michael-Simon’s inequality.

**Lemma 2.5.** Let \( M^n (n \geq 3) \) be a hypersurface in \( S^{n+1} \). Suppose that \( \|H\|_n C_1 < 1 \) where \( C_1 \) is a constant in Lemma 2.4. Then for any \( f \in C^1_0(M) \) we have
\[
\left( \int_M |f|^{\frac{2n}{n-2}} \, dM \right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla f|^2 + f^2 \, dM,
\]
where \( C_s = 2(C_1^2 - \frac{2n-2}{2n-2})^2 \)

**Proof.** Seem \( M^n \) has mean curvature \( \bar{H} \) as a submanifold in \( \mathbb{R}^{n+2} \). Then it is easy to see that
\[ |\bar{H}|^2 = 1 + |H|^2. \]
By Lemma 2.4 we have
\[
\left( \int_M |\varphi|^{\frac{n}{n-1}} dM \right)^{\frac{n-1}{n}} \leq C_1 \left( \int_M |\nabla \varphi| + \int_M |H||\varphi| dM \right) \\
\leq C_1 \left( \int_M |\nabla \varphi| + \int_M |\varphi| dM + \int_M |H||\varphi| dM \right)
\]

Let \( \varphi = f^{\frac{2(n-1)}{n}} \). Then by Hölder’s inequality, we have
\[
\left( \int_M |f|^{\frac{2n}{n-2}} dM \right)^{\frac{n-1}{n}} \leq C_1 \left( \int_M \frac{2(n-1)}{n-2} |f|^{\frac{2n}{n-2}} dM \right)^{\frac{1}{2}} \left( \int_M \frac{1}{n-2} |\nabla f|^{2} dM \right)^{\frac{1}{2}} \\
+ \left( \int_M f^{2} dM \right)^{\frac{1}{2}} \left( \int_M |f|^{\frac{2n}{n-2}} dM \right)^{\frac{1}{2}} + \|H\|_n \left( \int_M |f|^{\frac{2n}{n-2}} dM \right)^{\frac{n-1}{n}}
\]
and the conclusion follows. \( \square \)

### 3 Proof of Theorem 1.2

**Proof.** Let
\[
\sum_{i,j} r_{ij} = R_{ij} - (n-1) \delta_{ij},
\]
then
\[
\sum_{i,j} r_{ij,k} = r_{ik,j},
\]
since \( M \) is locally conformally flat. By Ricci formula and Lemma 2.1, we have
\[
\sum_{i,j} r_{ij,ll} = \sum_{i,j} r_{il,jl} + \sum_{i,j} r_{ijkl} R_{kijl} + \sum_{i,j} r_{ik} R_{kijl} \\
= \frac{1}{n-2} \left( 2 \sum_{k} r_{kl} r_{kj} + (n-2) r_{ij} - \sum_{l,k} r_{kl}^2 \delta_{ij} \right) + \sum_{k} r_{ik} r_{kj} + (n-1) \sum_{k} r_{ik} \delta_{kj} \\
= \frac{n}{n-2} \sum_{k} r_{ik} r_{kj} - \frac{1}{n-2} |r|^2 \delta_{ij} + n r_{ij}.
\]
Multiplying \( r_{ij} \) to both sides of above identity and summing up for \( i,j \) from 1 to \( n \) we obtain
\[
\sum_{i,j} r_{ij} \Delta r_{ij} = \frac{n}{n-2} \text{tr}(r^3) + n |r|^2,
\]
(3.1)
where \( r \) is the matrix of \( (r_{ij}) \) and \( |r|^2 = \sum_{i,j} r_{ij}^2 \). By
\[
\sum_{i,j} r_{ij} \Delta r_{ij} + \sum_{i,j} |\nabla r_{ij}|^2 = |r| \Delta |r| + |\nabla |r||^2,
\]

we get
\[ |r| \Delta |r| \geq \frac{n}{n - 2} \text{tr}(r^3) + n|r|^2, \quad (3.2) \]
since \( \sum_{i,j} |\nabla r_{ij}|^2 \geq |\nabla |r||^2 \).

Let \( f \) be a cut-off function supported in a geodesic ball \( B(o, R) \) with \( o \in M \) such that
\[ |\nabla f| \leq \frac{C}{R}, \quad |\Delta f| \leq \frac{C}{R^2}. \]

Multiplying both sides of (3.2) by \( f^2 |r|^q \) \((q > -1)\), and by integral by parts we have (remark: actually since \( |r| \) may have zero points, we need to choose the test function \( f^2\left(|r| + \epsilon\right)^q \)(\(q > -1\)) and then let \( \epsilon \to 0^+\))
\[
0 \geq 2 \int_M f^2 |r|^{q+1} \langle \nabla f, \nabla |r| \rangle + (q + 1) \int_M f^2 |r|^q |\nabla |r||^2 \\
+ \frac{n}{n - 2} \int_M f^2 |r|^q \text{tr}(r^3) + n \int_M f^2 |r|^{q+2}.
\]
Therefore
\[
n \int_M f^2 |r|^{q+2} + (q + 1) \int_M f^2 |r|^q |\nabla |r||^2 \leq \left| \frac{n}{n - 2} \int_M f^2 |r|^q \text{tr}(r^3) \right| \\
+ \frac{2}{q + 2} \int_M \text{div}(f \nabla f) |r|^{q+2}. \quad (3.3)
\]

Let \( q = \frac{n-4}{2} \). Using the Gauss formula,
\[
\sum_{i,j} r_{ij}^2 = \sum_{i,j} (nH h_{ij} - \sum_k h_{ik} h_{kj})^2 \\
= n^4 H^4 - 2nH \text{tr}(h^3) + \sum_{i,j} \left( \sum_k h_{ik} h_{kj} \right)^2 \\
\leq 4n^4 H^4.
\]
It follows that
\[
\int_M |r|^{q+2} = \int_M |r|^{n/2} \\
\leq (4n^4)^{\frac{n}{2-n}} \int_M |H|^n < (4n^4)^{\frac{n}{2-n}} \alpha < +\infty.
\]

By Hölder’s inequality and Lemma 2.5, we have
\[ \left| \int_M f^2 |r|^q \text{tr}(r^3) \right| \leq \int_M f^2 |r|^{q+3} \]
\[ \leq \left( \int_M (f |r|^{q+1}) \frac{2}{q+2} \right) \right)^{\frac{n-2}{n}} \left( \int_M |r|^\frac{q}{2} \right) \]
\[ \leq C_s \left( \int_M |r|^\frac{q}{2} \right) \frac{\frac{q}{2}}{q+2} \left( \int_M |\nabla (f |r|^{q+1})|^2 + f^2 |r|^{q+2} \right) \]
\[ \leq 2C_s \left( \int_M |r|^\frac{q}{2} \right) \frac{\frac{q}{2}}{q+2} \left[ \int_M |\nabla f|^2 |r|^{q+2} + (\frac{q}{2} + 1)^2 \int_M f^2 |r|^{q+2} \right]. \]
\[ + \left( \int_M f^2 |r|^{q+2} \right) \]

Combining this estimate with (3.3) we get

\[ E \int_M f^2 |r|^{q+2} + F \int_M f^2 |r|^{q} |\nabla r|^2 \]
\[ \leq 2C_s \frac{n}{n-2} \left( \int_M |r|^\frac{q}{2} \right) \frac{\frac{q}{2}}{q+2} \int_M |\nabla f|^2 |r|^{q+2} + \frac{2}{q+2} \int_M \text{div}(f \nabla r)|r|^{q+2}, \]

where \( E = n - 2C_s(\int_M |r|^\frac{q}{2})^{\frac{q}{2}} \) and \( F = q + 1 - 2C_s \frac{n}{n-2} (\frac{q}{2} + 1)^2 (\int_M |r|^\frac{q}{2})^{\frac{q}{2}} \).

Assume that \( \alpha \) is sufficiently small such that \( E, F \) are greater than zero. Letting \( R \to +\infty \), we obtain that \( |r| = 0 \). Therefore,

\[ nHh_{ij} = \sum_k h_{ik} h_{kj}. \]

By Lemma 2.3, at least \( n - 1 \) eigenvalues of \( h_{ij} \) are zero.

At last by following the argument used in the proof of Theorem 2.2 in [13], we can prove that all eigenvalues of \( (h_{ij}) \) are zero, i.e. \( M \) is totally geodesic. For convenience of the reader we give the details here.

**Proof of Proposition 1.1.** Since \( n - 1 \) eigenvalues of \( h \) are zero, by the Gauss equation we see that the sectional curvature of \( M \) equals to one everywhere. Then by the Bonnet-Myers’ theorem we see that \( M \) is compact. If \( M \) is not totally geodesic, then there is a point \( p \) such that the biggest eigenvalue of \( h \), say \( \lambda_1 = \mu \) attains its maximum value \( \mu(p) > 0 \) at \( p \). Then there is a neighborhood of \( p \in U \subset M \) such that on \( U \), \( \lambda_1 = \mu > 0 \) and \( \lambda_2 = \ldots = \lambda_n = 0 \).

Next we restrict our analysis on \( U \).

Assume that \( 2 \leq r, s \leq n \), then

\[ \sum_i h_{1r,i} \omega_i = dh_{1r} - \sum_k (h_{k1} \omega_{kr} + h_{kr} \omega_{k1}) = -\mu \omega_{1r}. \]
Note that
\[ \sum_i h_{1r,i} \omega_i = h_{1r,1} \omega_1 + \sum_{i>1} h_{1r,i} \omega_i = h_{11} \omega_1 + \sum_{i>1} h_{ir,1} \omega_i = \mu \omega_1. \]

Therefore
\[ \mu \omega_1 = -\mu \omega_1 = \mu \omega_1, \]
which implies that
\[ \omega_1 = (\log \mu)_r \omega_1. \]

Take the exterior differentiation of the above equation and by the structure equation on \( M \) we obtain
\[ d\omega_1 = -\sum_s \omega_s \wedge \omega_1 + \omega_r \wedge \omega_1, \]
\[ d\omega_r = d((\log \mu)_r \omega_1) = -((\log \mu)_r \sum_s \omega_s \wedge \omega_1 + \sum_s ((\log \mu)_r \omega_s + (\log \mu)_s \omega_s) \wedge \omega_1 \wedge \omega_1 \]
\[ = -\sum_s (\log \mu)_r \omega_s \wedge \omega_1 + \sum_s (\log \mu)_r \omega_s \wedge \omega_1 - \sum_s (\log \mu)_s \omega_s \wedge \omega_1. \]

Comparing the above formulas we get on \( U \)
\[ (\log \mu)_r = (\log \mu)_s + \delta_{rs}. \]

Since \( p \) is a maximum point of \( \mu \) on \( M \), we see that at \( p \),
\[ 0 \geq (\log \mu)_{rr} = (\log \mu)_r^2 + \delta_{rr} = 1, \]
a contradiction. Hence we get the conclusion of Proposition 1.1.

This completes the proof of Theorem 1.2.

**Declarations**

**Funding** The second author is supported by the NSF of China (Grant No. 12271069) and Chongqing NSF (Grant No. cstc2021jcjy-nsxmX0443).

**Conflict of interest** The authors hereby state that there are no conflicts of interest regarding the presented results.

**Data availability statements** No data sets were generated or analysed during the current study.
References

[1] J. C. Bai and Y. Luo, Remarks on gap theorems for complete hypersurfaces with constant scalar curvature, to appear in J. Math. Study, doi:10.4208/jms.v56n2.23.01.

[2] O. Chodosh and C. Li, Stable minimal hypersurfaces in $\mathbb{R}^4$, arXiv:2108.11462, preprint.

[3] M. P. do Carmo and C. K. Peng, Stable compete minimal surfaces in $\mathbb{R}^3$ are planes, Bull. Amer. Math. Soc. 1(1979), 903-906.

[4] M. P. do Carmo and C. K. Peng, Stable complete minimal hypersurfaces, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), Sci. Press Beijing, Beijing, 1982, 1349-1358.

[5] D. Fisher-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative curvature, Comm. Pure Appl. Math. 33(1980), no. 2, 199-211.

[6] Y. W. Li, X. W. Xu and J. R. Zhou, The complete hyper-surfaces with zero scalar curvature in $\mathbb{R}^{n+1}$, Ann. Global Anal. Geom. 44(2013), no. 4, 401-416.

[7] J. H. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$, Comm. Pure Appl. Math. 26(1973), 361-379.

[8] L. Ni, Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$, Comm. Anal. Geom. 9(2001), no. 3, 641-656.

[9] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, Amer. J. Math. 96(1974), 207-213.

[10] R. Schoen, L. Simon and S. T. Yau, Curvature estimates for minimal hypersurfaces, Acta Math. 134(1975), 275-288.

[11] Y. B. Shen and X. H. Zhu, On stable minimal hypersurfaces in $\mathbb{R}^{n+1}$, Amer. J. Math. 120(1998), no. 1, 103-116.

[12] J. Simons, Minimal varieties in riemannian manifolds, Ann. of Math. (2)88(1968), 62-105.

[13] B. Y. Wu, On hypersurfaces with two distinct principal curvatures in a unit sphere, Differential Geom. Appl. 27(2009), no. 5, 623-634.

[14] H. W. Xu, $L_{a\pi}$-pinching theorems for submanifolds with parallel mean curvature in a sphere, J. Math. Soc. Japan 46(1994), no. 3, 503-515.
[15] G. Yun, Total scalar curvature and $L^2$ harmonic 1-forms on a minimal hypersurface in Euclidean space, Geom. Dedicata 89(2002), 135-141.

JINCHUAN BAI, YONG LUO
MATHEMATICAL SCIENCE RESEARCH CENTER OF MATHEMATICS,
CHONGQING UNIVERSITY OF TECHNOLOGY,
CHONGQING, 400054, CHINA
baijinchuan23@163.com, yongluo-math@cqut.edu.cn