Domain Wall and Periodic Solutions of a Coupled $\phi^6$ Model

Avinash Khare
Institute for Physics, Bhubaneswar, Orissa 751005, India

Avadh Saxena
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Lab, Los Alamos, NM 87545, USA

Abstract:
Coupled triple well ($\phi^6$) one-dimensional potentials occur in both condensed matter physics and field theory. Here we provide a set of exact periodic solutions in terms of elliptic functions (domain wall arrays) and obtain single domain wall solutions in specific limits. Both topological, nontopological (e.g. some pulse-like solutions) as well as mixed domain walls are obtained. We relate these solutions to structural phase transitions in materials with polarization, shuffle modes and strain. We calculate the energy and the asymptotic interaction between solitons for various solutions. We also consider the discrete analog of these coupled models and obtain several single and periodic domain wall exact solutions.
1 Introduction

Many physical problems of interest often have a coupling between two order parameters. In a recent paper we obtained exact single and periodic domain wall solutions of coupled double well ($\phi^4$) models [1] (both continuum and discrete) which arise in many second order transitions [2, 3, 4]. Similarly, coupled triple well or $\phi^6$ models [5] arise in the context of first order phase transitions. There exist analogous coupled models in field theoretical contexts [6, 7, 8]. Specifically, when a first order transition is driven by two primary order parameters, the free energy should be expanded to sixth order in both order parameters with a biquadratic (or possibly other symmetry allowed) coupling. An example of such a transition is the cubic to monoclinic structural transformation in shape memory alloys NiTi and AuCd in which two phonon (or shuffle) modes are simultaneously active [9]. A simpler example is the square to oblique lattice transition driven simultaneously by two different strain components with a biquadratic coupling [10]. Here our motivation is to obtain all possible single and periodic domain wall solutions of these models and then connect to experimental observations wherever possible.

The paper is organized as follows. In Sec. II we provide twenty distinct periodic solutions of a coupled continuum $\phi^6$ model. In doing so, we obtain previously unknown periodic solutions of the standard (uncoupled) $\phi^6$ model [11, 12, 13]. We also calculate the total energy and the interaction energy between solitons for all the solutions. Section III contains six different solutions for the corresponding discrete $\phi^6$ case. The latter arises in the context of first order phase transitions on a lattice. Finally, we conclude in Sec. IV with remarks on related coupled models.

2 Solutions of a Coupled $\phi^6$ Continuum Model

In the literature, the examples of a coupled $\phi^6$ model are scarce. The reason is presumably that one field (or order parameter) is usually dominant and leads to a first order transition. One does not need nonlinearity in the secondary order parameter (or the field) to drive the transition. However, in crystalline phase transitions this is not necessarily true and within the context of Landau theory, in certain transitions symmetry allows one to go to the sixth order nonlinearity in both fields (as primary order parameters)
with a biquadratic coupling \[5, 9, 10\]. In this paper, we adopt this viewpoint and consider the solutions of a coupled \(\phi^6\) model with the potential

\[
V(\phi, \psi) = \left( a_1 \frac{1}{2} \phi^2 - b_1 \frac{1}{4} \phi^4 + c_1 \frac{1}{6} \phi^6 \right) + \left( a_2 \frac{1}{2} \psi^2 - b_2 \frac{4}{4} \psi^4 + c_2 \frac{6}{6} \psi^6 \right) + \frac{d}{2} \phi^2 \psi^2,\]

(1)

where \(a_{1,2}, b_{1,2}, c_{1,2}\) and \(d\) are material (or system) dependent parameters; \(\phi\) and \(\psi\) are scalar fields. A comprehensive analysis of this coupled continuum \(\phi^6\) model shows that it has \textit{twenty} possible periodic solutions, i.e. eleven “bright-bright”, three “bright-dark” and six “dark-dark” solutions. In particular, we obtain six solutions below the transition temperature \(T_c\), ten at \(T_c\), one above \(T_c\) and three in mixed phase in the sense that while one of the field is above \(T_c\), the other one is below \(T_c\). The latter situation is akin to the one found in multiferroic materials where one transition (e.g. antiferromagnetic) takes place at a higher temperature than the other transition (e.g. ferroelectric) or vice versa \[14, 15\]. However, in the corresponding discrete case we are able to obtain only \textit{six} periodic solutions.

The (static) equations of motion which follow from Eq. (1) are

\[
\begin{align*}
\frac{d^2 \phi}{dx^2} &= a_1 \phi - b_1 \phi^3 + c_1 \phi^5 + d \phi \psi^2, \\
\frac{d^2 \psi}{dx^2} &= a_2 \psi - b_2 \psi^3 + c_2 \psi^5 + d \psi \phi^2.
\end{align*}
\]

(2)

These coupled equations have twenty distinct periodic (elliptic function) solutions which we discuss one by one systematically. It is worth pointing out that in turn in the single soliton limit, these lead to \textit{nine} distinct (hyperbolic) soliton solutions. In particular, one obtains three solutions below the transition temperature \(T_c\), three at \(T_c\), one above \(T_c\) and two in the mixed phase in the sense that while one of the field is above \(T_c\), the other one is below \(T_c\).

For static solutions the energy is given by

\[
E = \int \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{d\psi}{dx} \right)^2 + V(\phi, \psi) \right] dx,
\]

(3)

where the limits of integration are from \(-\infty\) to \(\infty\) for hyperbolic solutions (single solitons) on the full line. On the other hand, for the periodic solutions (i.e. soliton lattices), the limits are from \(-K(k)\) to \(K(k)\). Here \(K(k)\) (and \(E(k)\) below) denote complete elliptic integral of the first (and second) kind \[16\].
Using the equations of motion, one can show that for all of our solutions

$$V(\phi, \psi) = \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{d\psi}{dx} \right)^2 \right] + C,$$

(4)

where the constant $C$ in general varies from solution to solution. Hence the energy measured with respect to the reference energy (e.g. the local or global minimum) $\hat{E} = E - \int C \, dx$ is given by

$$\hat{E} \equiv E - \int C \, dx = \int \left[ \left( \frac{d\phi}{dx} \right)^2 + \left( \frac{d\psi}{dx} \right)^2 \right] \, dx.$$

(5)

Below we give an explicit expression for the energy in the case of all twenty periodic solutions (and hence the corresponding nine hyperbolic solutions). In each case we also give an expression for the constant $C$. To begin with, we first obtain ten bright-bright solutions at $T_c$.

2.1 Solutions I and II

We look for the most general solutions to the coupled Eqs. (2) in terms of Jacobi elliptic functions $\text{sn}(x, m)$, $\text{cn}(x, m)$ and $\text{dn}(x, m)$ [16] where the modulus $m \equiv k^2$. It is easily shown that

$$\phi = A \sqrt{1 \pm \text{sn}(B x + x_0, m)}, \quad \psi = F \sqrt{1 \pm \text{sn}(B x + x_0, m)},$$

(6)

are solutions to the coupled Eqs. (2) provided the following six coupled equations are satisfied

$$a_1 - b_1 A^2 + dF^2 + c_1 A^4 = -\frac{B^2}{4},$$

(7)

$$- b_1 A^2 + dF^2 + 2c_1 A^4 = -\frac{mB^2}{2},$$

(8)

$$c_1 A^4 = \frac{3mB^2}{4},$$

(9)

$$a_2 - b_2 F^2 + dA^2 + c_2 F^4 = -\frac{B^2}{4},$$

(10)

$$- b_2 F^2 + dA^2 + 2c_2 F^4 = -\frac{mB^2}{2},$$

(11)

$$c_2 F^4 = \frac{3mB^2}{4}.$$

(12)

Here $+(-)$ sign in $\phi$ goes with $+(-)$ sign in $\psi$, $A$ and $F$ denote the amplitudes of the kink lattice, $B$ is an inverse characteristic length while $x_0$ is the (arbitrary) location of the kink. Three of these equations
determine the three unknowns $A, F, B$ while the other three equations give three constraints between the seven parameters $a_{1,2}, b_{1,2}, c_{1,2}, d$. In particular, $A, F, B$ are given by

$$B^2 = \frac{4a_1}{(5m - 1)}, \quad A^2 = \frac{8m(d + b_2)a_1}{(5m - 1)(b_1b_2 - d^2)}, \quad F^2 = \frac{8m(d + b_1)a_1}{(5m - 1)(b_1b_2 - d^2)},$$

while the three constraints are

$$a_1 = a_2, \quad c_1(d + b_2)^2 = c_2(d + b_1)^2, \quad (b_1b_2 - d^2)^2 = \frac{64ma_1c_1(d + b_2)^2}{3(5m - 1)}.$$

For $d = 0$ the last constraint reduces to the constraint (for the uncoupled $\phi^6$ \cite{11,12,13} model)

$$b_1^2 = \frac{64ma_1c_1}{3(5m - 1)}.$$  \hspace{1cm} (15)

So far as we are aware of, the uncoupled (i.e. $d = 0$) kink lattice solution $\phi = A \sqrt{1 \pm \text{sn}(Bx + x_0, m)}$, satisfying

$$B^2 = \frac{4a_1}{(5m - 1)}, \quad A^2 = \frac{8ma_1}{(5m - 1)b_1},$$

and the constraint \cite{15} have never been explicitly written down before in the literature.

At $m = 1$ these solutions go over to the topological (bright-bright) solutions

$$\phi = A \sqrt{1 \pm \text{tanh}(Bx + x_0)}, \quad \psi = F \sqrt{1 \pm \text{tanh}(Bx + x_0)},$$

provided

$$B^2 = a_1, \quad A^2 = \frac{2(d + b_2)a_1}{(b_1b_2 - d^2)}, \quad F^2 = \frac{2(d + b_1)a_1}{(b_1b_2 - d^2)}.$$  \hspace{1cm} (18)

Further, two of the constraints remain the same as above [Eq. \cite{13}] while the third one is now given by

$$(b_1b_2 - d^2)^2 = \frac{(16/3)a_1c_1(d + b_2)^2}{(16/3)a_1c_1}.$$  \hspace{1cm} (19)

which for $d = 0$ reduces to

$$b_1^2 = \frac{(16/3)a_1c_1}{(16/3)a_1c_1}.$$  \hspace{1cm} (20)

This corresponds to the point (i.e. temperature) where there are three degenerate minima (or the point where a discontinuous transition takes place) of the usual (uncoupled) $\phi^6$ field theory \cite{11,12,13}.
Energy: Corresponding to the periodic solutions (6), the energy $\hat{E}$ and the constant $C$ are given by

$$
\hat{E} = \frac{(A^2 + F^2)B}{4} E(k),
C = -\frac{1}{4} (1 - k^2)(A^2 + F^2)B^2.
$$

(21)

On using the expansion formulas for $E(k)$ [and $K(k)$ to be used later in the paper] around $k = 1$ (note $m = k^2$) as given in [16]

$$
E(k) = 1 + \frac{k'^2}{2} \left( \ln \left( \frac{4}{k'} \right) - \frac{1}{2} \right) + ..., \quad K(k) = \ln \left( \frac{4}{k'} \right) + \frac{k'^2}{4} \left( \ln \left( \frac{4}{k'} \right) - 1 \right) + ..., \quad k'^2 = 1 - k^2,
$$

(22)

(23)

the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (17)] plus the asymptotic (i.e. widely separated solitons) interaction energy [13]. We find

$$
\hat{E} = E_{\text{kink}} + E_{\text{int}} = (A^2 + F^2)B \left( \frac{1}{4} + \frac{k'^2}{8} \left[ \ln \left( \frac{4}{k'} \right) - \frac{1}{2} \right] \right).
$$

(24)

Note that these solutions exist only when $b_1b_2 > d^2$. The interaction energy vanishes at $k^2 = 1$, as it should.

2.2 Solution III

It turns out that the coupled Eqs. (2) also admit a novel mixed solution at $T = T_c$ given by

$$
\phi = A\sqrt{1 + \text{sn}(Bx + x_0, m)}, \quad \psi = F\sqrt{1 - \text{sn}(Bx + x_0, m)}, \quad (25)
$$

provided the following six coupled equations are satisfied

$$
a_1 - b_1 A^2 + dF^2 + c_1 A^4 = \frac{-B^2}{4},
$$

(26)

$$
-b_1 A^2 - dF^2 + 2c_1 A^4 = \frac{-mB^2}{2},
$$

(27)

$$
c_1 A^4 = \frac{3mB^2}{4},
$$

(28)

$$
a_2 - b_2 F^2 + dA^2 + c_2 F^4 = \frac{-B^2}{4},
$$

(29)

$$
-b_2 F^2 - dA^2 + 2c_2 F^4 = \frac{-mB^2}{2},
$$

(30)
\[ c_2 F^4 = \frac{3mB^2}{4}. \] (31)

Three of these equations determine the three unknowns \( A, F, B \) while the other three equations give three constraints between the seven parameters \( a_{1,2}, b_{1,2}, c_{1,2}, d \). In particular, \( A, F, B \) are given by

\[
B^2 = \frac{4a_1(b_1b_2 - d^2)}{((5m - 1)b_1b_2 - 16mb_1d + (11m + 1)d^2)},
\]
\[
A^2 = \frac{8ma_1(b_2 - d)}{((5m - 1)b_1b_2 - 16mb_1d + (11m + 1)d^2)},
\]
\[
F^2 = \frac{8ma_1(b_1 - d)}{((5m - 1)b_1b_2 - 16mb_1d + (11m + 1)d^2)} \] (32)
while the three constraints are

\[
[(5m - 1)b_1b_2 + (11m + 1)d^2 - 16mb_2d]a_1 = [(5m - 1)b_1b_2 + (11m + 1)d^2 - 16mb_1d]a_2,
\]
\[
c_1(b_2 - d)^2 = c_2(b_1 - d)^2, \quad (b_1b_2 - d^2) = \frac{64ma_1c_1(b_2 - d)^2}{3[(5m - 1)b_1b_2 + (11m + 1)d^2 - 16mb_1d]}.
\] (33)

For \( d = 0 \) the last constraint reduces to the constraint (for the uncoupled \( \phi^6 \) model) as given by Eq. (20). Note that no new solution is obtained by having the \(-\text{sn}\) term inside the square root in \( \phi \) and the \(+\text{sn}\) term inside the square root in \( \psi \) in Eq. (25) since such a solution is trivially obtained from Eq. (25) by merely interchanging \( A \) and \( F \). This is so because the potential \( V(\phi, \psi) \) is completely symmetric in \( \phi \) and \( \psi \). Similar comments apply to many of the other asymmetric solutions given below.

At \( m = 1 \) this solution goes over to the topological (bright-bright) solution

\[
\phi = A \sqrt{1 + \tanh(Bx + x_0)}, \quad \psi = F \sqrt{1 - \tanh(Bx + x_0)},
\] (34)
provided

\[
B^2 = \frac{a_1(b_1b_2 - d^2)}{(b_1b_2 - b_1d + d^2)}, \quad A^2 = \frac{a_1(b_2 - d)}{2(b_1b_2 - b_1d + 3d^2)}, \quad F^2 = \frac{a_1(b_1 - d)}{2(b_1b_2 - 4b_1d + 3d^2)},
\] (35)
while the three constraints are

\[
(b_1b_2 + 3d^2 - 4b_2d)a_1 = (b_1b_2 + 3d^2 - 4b_1d)a_2,
\]
\[
c_1(b_2 - d)^2 = c_2(b_1 - d)^2, \quad (b_1b_2 - d^2) = \frac{16a_1c_1(b_2 - d)^2}{3(b_1b_2 + 3d^2 - 4b_1d)}.
\] (36)

For \( d = 0 \), the last constraint reduces to the constraint (20) which is the point where there are three degenerate minima (i.e. the point where a discontinuous transition takes place) of the usual (uncoupled) \( \phi^6 \) field theory [11, 12, 13].
Energy: Remarkably, corresponding to the periodic solution (25), the energy $\hat{E}$ is again the same as that for solutions I and II and is given by Eq. (21). Thus the interaction energy too remains unchanged and is again given by Eq. (24). However, $C$ is different for this solution than those for solutions I and II and is given by

$$C = \frac{(a_1 A^2 + a_2 F^2)}{4} - \frac{B^2}{16} (A^2 + F^2)(3 + m).$$

(37)

Note that this solution also exists only when $b_1 b_2 > d^2$.

2.3 Solutions IV and V

It is easily shown that

$$\phi = A \sqrt{1 \pm \sqrt{\text{sn}(Bx + x_0, m)}}, \quad \psi = F \sqrt{1 \pm \sqrt{\text{sn}(Bx + x_0, m)}},$$

(38)

are also solutions to the coupled Eqs. (2) provided the following six coupled equations are satisfied

$$a_1 - b_1 A^2 + dF^2 + c_1 A^4 = \frac{-mB^2}{4},$$

(39)

$$- b_1 A^2 + dF^2 + 2c_1 A^4 = \frac{-B^2}{2},$$

(40)

$$c_1 A^4 = \frac{3B^2}{4},$$

(41)

$$a_2 - b_2 F^2 + dA^2 + c_2 F^4 = \frac{-mB^2}{4},$$

(42)

$$- b_2 F^2 + dA^2 + 2c_2 F^4 = \frac{-B^2}{2},$$

(43)

$$c_2 F^4 = \frac{3B^2}{4}.$$  

(44)

Three of these equations determine the three unknowns $A, F, B$ while the other three equations give three constraints between the seven parameters $a_{1,2}, b_{1,2}, c_{1,2}, d$. In particular, $A, F, B$ are given by

$$B^2 = \frac{4a_1}{(5 - m)}, \quad A^2 = \frac{8(d + b_2)a_1}{(5 - m)(b_1 b_2 - d^2)}, \quad F^2 = \frac{8(d + b_1)a_1}{(5 - m)(b_1 b_2 - d^2)},$$

(45)

while the three constraints are

$$a_1 = a_2, \quad c_1(d + b_2)^2 = c_2(d + b_1)^2, \quad (b_1 b_2 - d^2)^2 = \frac{64a_1 c_1(d + b_2)^2}{3(5 - m)}.$$  

(46)
For \( d = 0 \) the last one reduces to the constraint (for the uncoupled \( \phi^6 \) model) \[11, 12, 13\]

\[ b_1^2 = \frac{64a_1c_1}{3(5 - m)}. \] (47)

So far as we are aware of, the uncoupled (i.e. \( d = 0 \)) kink lattice solution \( \phi = A\sqrt{1 \pm \sqrt{m}\text{sn}(Bx + x_0, m)} \), satisfying

\[ B^2 = \frac{4a_1}{(5 - m)}, \quad A^2 = \frac{8a_1}{(5 - m)b_1}, \] (48)

and the constraint \[17\] have never been explicitly written down before in the literature.

At \( m = 1 \) these solutions go over to the topological (bright-bright) solutions \[17\] satisfying the constraints \[18\] to \[20\].

**Energy:** Corresponding to the periodic solutions \[85\], the energy \( \hat{E} \) and the constant \( C \) are given by

\[ \hat{E} = \frac{(A^2 + F^2)B}{4} \left[ E(k) - k'^2K(k) \right], \]
\[ C = \frac{1}{4}(1 - k'^2)(A^2 + F^2)B^2. \] (49)

On using the expansion formulas for \( E(k) \) and \( K(k) \) around \( k = 1 \) (note \( m = k^2 \)) as given by Eqs. \[22\] and \[23\], the energy of the periodic solutions can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solutions [Eq. \[17\]] plus the interaction energy. We find

\[ \hat{E} = E_{kink} + E_{int} = (A^2 + F^2)B \left( \frac{1}{4} - \frac{k'^2}{8} \left[ \ln \left( \frac{4}{k'} \right) + \frac{1}{2} \right] \right). \] (50)

Note that these solutions also exist only when \( b_1b_2 > d^2 \). The interaction energy vanishes at \( k = 1 \).

### 2.4 Solution VI

It turns out that the coupled Eqs. \[2\] also admit a novel mixed solution at \( T = T_c \) given by

\[ \phi = A\sqrt{1 + \sqrt{m}\text{sn}(Bx + x_0, m)} , \quad \psi = F\sqrt{1 - \sqrt{m}\text{sn}(Bx + x_0, m)}, \] (51)

provided the following six coupled equations are satisfied

\[ a_1 - b_1A^2 + dF^2 + c_1A^4 = \frac{-mB^2}{4}, \] (52)
\[ -b_1A^2 - dF^2 + 2c_1A^4 = \frac{B^2}{2}, \] (53)
\[ c_1 A^4 = \frac{3B^2}{4}, \]  
(54)  
\[ a_2 - b_2 F^2 + c_2 F^4 = \frac{-mB^2}{4}, \]  
(55)  
\[ -b_2 F^2 - dA^2 + 2c_2 F^4 = \frac{-B^2}{2}, \]  
(56)  
\[ c_2 F^4 = \frac{3B^2}{4}. \]  
(57)

Three of these equations determine the three unknowns \( A, F, B \) while the other three equations give three constraints between the seven parameters \( a_{1,2}, b_{1,2}, c_{1,2}, d \). In particular, \( A, F, B \) are given by

\[
 B^2 = \frac{4a_1(b_1b_2 - d^2)}{[(5 - m)b_1b_2 - 16b_1d + (11 + m)d^2]},
\]
\[
 A^2 = \frac{8ma_1(b_2 - d)}{[(5 - m)b_1b_2 - 16b_1d + (11 + m)d^2]},
\]
\[
 F^2 = \frac{8ma_1(b_1 - d)}{[(5 - m)b_1b_2 - 16b_1d + (11 + m)d^2]},
\]
(58)

while the three constraints are

\[
 [(5 - m)b_1b_2 + (11 + m)d^2 - 16bd)|a_1 = [(5 - m)b_1b_2 + (11 + m)d^2 - 16bd)|a_2,
\]
\[
 c_1(b_2 - d)^2 = c_2(b_1 - d)^2,
\]
\[
 (b_1 - d) = \frac{64a_1 c_1(b_2 - d)^2}{3[(5 - m)b_1b_2 + (11 + m)d^2 - 16bd].
\]
(59)

For \( d = 0 \) the last one reduces to the constraint (57).

At \( m = 1 \) this solution goes over to the topological (bright-bright) solution (34) satisfying the same constraints as given by Eqs. (35) and (36).

**Energy**: Remarkably, corresponding to the periodic solution (51), the energy \( \hat{E} \) is again the same as that for solutions IV and V and is given by Eq. (49). Thus the interaction energy too remains unchanged and is again given by Eq. (50). However, \( C \) is different for this solution than those for solutions IV and V and is given by

\[
 C = \frac{(a_1 A^2 + a_2 F^2)}{4} - \frac{B^2}{16}(A^2 + F^2)(1 + 3m).
\]
(60)

Note that this solution also exists only when \( b_1 b_2 > d^2 \).

### 2.5 Solutions VII and VIII

Apart from the above six solutions, at \( T = T_c \), the field equations also admit mixed type of solutions. In particular, it is easily shown that

\[
 \phi = A\sqrt{1 \pm \text{sn}(Bx + x_0, m)}, \quad \psi = F\sqrt{1 \pm \text{msn}(Bx + x_0, m)},
\]
(61)
are solutions to the coupled Eqs. provided the following six coupled equations are satisfied
\[ a_1 - b_1 A^2 + dF^2 + c_1 A^4 = \frac{-B^2}{4}, \]  
\[ -b_1 A^2 + \sqrt{md} F^2 + 2c_1 A^4 = \frac{-m B^2}{2}, \]  
\[ c_1 A^4 = \frac{3m B^2}{4}, \]  
\[ a_2 - b_2 F^2 + dA^2 + c_2 F^4 = \frac{-m B^2}{4}, \]  
\[ -\sqrt{mb_2} F^2 + dA^2 + 2\sqrt{mc_2} F^4 = \frac{-\sqrt{m} B^2}{2}, \]  
\[ c_2 F^4 = \frac{3B^2}{4}. \]  

Three of these equations determine the three unknowns \( A, F, B \) while the other three equations give three constraints between the seven parameters \( a_{1,2}, b_{1,2}, c_{1,2}, d \). In particular, \( A, F, B \) are given by
\[
B^2 = \frac{4a_1 (b_1 b_2 - d^2)}{[5m - 1)b_1 b_2 - (8\sqrt{m} - 3m - 1)d^2 - 8db_1 (1 - \sqrt{m})]}, \\
A^2 = \frac{8\sqrt{m} (d + \sqrt{mb_2}) a_1}{[5m - 1)b_1 b_2 - (8\sqrt{m} - 3m - 1)d^2 - 8db_1 (1 - \sqrt{m})]}, \\
F^2 = \frac{8(b_1 + \sqrt{mb_2}) a_1}{[5m - 1)b_1 b_2 - (8\sqrt{m} - 3m - 1)d^2 - 8db_1 (1 - \sqrt{m})]},
\]
while the three constraints are
\[
[5m - 1)b_1 b_2 - (8\sqrt{m} - 3m - 1)d^2 - 8db_1 (1 - \sqrt{m})]a_2 \\
= [(5 - m)b_1 b_2 - (8\sqrt{m} - 3 - m)d^2 + 8db_1 (1 - \sqrt{m})]a_1, \quad c_1 (d + \sqrt{mb_2})^2 = c_2 (b_1 + \sqrt{md})^2, \\
(b_1 b_2 - d^2) = \frac{64a_1 c_1 (d + \sqrt{mb_2})^2}{3[(5m - 1)b_1 b_2 - (8\sqrt{m} - 3m - 1)d^2 - 8db_1 (1 - \sqrt{m})]}.
\]

For \( d = 0 \) the last constraint reduces to the constraint \( 65 \).

At \( m = 1 \) these solutions go over to the topological (bright-bright) solutions \( 17 \) satisfying the constraints \( 15 \) to \( 20 \).

**Energy:** Corresponding to the periodic solutions \( 61 \), the energy \( \hat{E} \) and the constant \( C \) are given by
\[
\hat{E} = \frac{(A^2 + F^2) B}{4} E(k) - \frac{BF^2}{4} k^2 K(k), \\
C = \frac{1}{4} [a_1 A^2 + a_2 F^2] - \frac{B^2}{16} [(3 + k^2) A^2 + (1 + 3k^2) F^2].
\]
On using the expansion formulas (22) and (23), the energy of the periodic solutions can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solutions [Eq. (17)] plus the interaction energy. We find

\[ \hat{E} = E_{\text{kink}} + E_{\text{int}} = (A^2 + F^2)B \left( \frac{1}{4} + \frac{k'^2}{8} \left[ \ln \left( \frac{4}{k'} \right) - \frac{1}{2} \right] \right) - \frac{BF^2}{4} k'^2 \ln \left( \frac{4}{k'} \right). \]

(71)

Note that these solutions exist only when \( b_1 b_2 > d^2 \). The interaction energy vanishes at \( k = 1 \), as it should.

### 2.6 Solutions IX and X

Finally, there are two more mixed type of solutions at \( T = T_c \) given by

\[ \phi = A \sqrt{1 \pm \text{sn}(Bx + x_0, m)}, \quad \psi = F \sqrt{1 \pm \sqrt{m} \text{sn}(Bx + x_0, m)}, \]

(72)

provided the following six coupled equations are satisfied

\[ a_1 - b_1 A^2 + dF^2 + c_1 A^4 = -\frac{B^2}{4}, \]

(73)

\[ -b_1 A^2 - \sqrt{md} F^2 + 2c_1 A^4 = -\frac{m B^2}{2}, \]

(74)

\[ c_1 A^4 = \frac{3 m B^2}{4}, \]

(75)

\[ a_2 - b_2 F^2 + dA^2 + c_2 F^4 = -\frac{m B^2}{4}, \]

(76)

\[ -\sqrt{m} b_2 F^2 - dA^2 + 2 \sqrt{m} c_2 F^4 = -\frac{\sqrt{m} B^2}{2}, \]

(77)

\[ c_2 F^4 = \frac{3 B^2}{4}. \]

(78)

Three of these equations determine the three unknowns \( A, F, B \) while the other three equations give three constraints between the seven parameters \( a_{1,2}, b_{1,2}, c_{1,2}, d \). In particular, \( A, F, B \) are given by

\[ B^2 = \frac{4 a_1 (b_1 b_2 - d^2)}{[(5m - 1)b_1 b_2 + (11m + 1)d^2 - 16 \sqrt{mdb_1}]}, \]

\[ A^2 = \frac{8 \sqrt{m} (\sqrt{mb_2} - d) a_1}{[(5m - 1)b_1 b_2 + (11m + 1)d^2 - 16 \sqrt{mdb_1}]}, \]

\[ F^2 = \frac{8 \sqrt{m} (b_1 - \sqrt{m} d) a_1}{[(5m - 1)b_1 b_2 + (11m + 1)d^2 - 16 \sqrt{mdb_1}]}, \]

(79)
while the three constraints are

\[
[(5m - 1)b_1b_2 + (11m + 1)d^2 - 16d\sqrt{mb_1}]a_2 =
\]

\[
[(8\sqrt{m} - 3 - m)b_1b_2 + (8\sqrt{m} + 3 + m)d^2 - 16mdb_2]a_1,
\]

\[c_1(\sqrt{mb_2} - d)^2 = mc_2(b_1 - \sqrt{md})^2, \quad (b_1b_2 - d^2) = \frac{64a_1c_1(\sqrt{mb_2} - d)^2}{3(5m - 1)b_1b_2 + (11m + 1)d^2 - 16d\sqrt{mb_1}}.\]  

(80)

For \(d = 0\) the last constraint reduces to the constraint (16).

At \(m = 1\) these solutions go over to the topological (bright-bright) solutions (34) satisfying the constraints (35) and (36).

**Energy**: The energy \(\hat{E}\) and the constant \(C\) for the solutions (72) are the same as for solutions VII and VIII and are given by Eq. (70) and hence the interaction energy is again as given by Eq. (71). Note that these solutions also exist only when \(b_1b_2 > d^2\).

### 2.7 Solution XI

We shall now present six periodic solutions which exist in the case \(T < T_c\). Out of these one is bright-bright, three dark-dark and two bright-dark solutions. It is easily shown that

\[
\phi = \frac{A\sin(Bx + x_0, m)}{\sqrt{1 - D\sin^2(Bx + x_0, m)}}, \quad \psi = \frac{F\sin(Bx + x_0, m)}{\sqrt{1 - D\sin^2(Bx + x_0, m)}},
\]

(81)

is a periodic variant of the bright-bright solution provided the following six equations are satisfied

\[
a_1 = (3D - 1 - m)B^2, \quad (82)
\]

\[
dF^2 - b_1A^2 - 2a_1D = 2(m - D - mD)B^2, \quad (83)
\]

\[-dDF^2 + b_1DA^2 + a_1D^2 + c_1A^4 = mDB^2, \quad (84)\]

\[a_2 = (3D - 1 - m)B^2, \quad (85)\]

\[
dA^2 - b_2F^2 - 2a_2D = 2(m - D - mD)B^2, \quad (86)\]

\[-dDA^2 + b_2DF^2 + a_2D^2 + c_2F^4 = mDB^2. \quad (87)\]

13
From here one can determine the four unknowns $A, F, B, D$ and further one obtains two constraints between the seven parameters $a_{1,2}, b_{1,2}, c_{1,2}, d$. We obtain

$$B^2 = \frac{a_1}{(3D - 1 - m)}, \quad A^2 = \frac{2(d + b_2)B^2[2(1 + m)D - m - 3D^2]}{(b_1b_2 - d^2)};$$
$$F^2 = \frac{2(d + b_1)B^2[2(1 + m)D - m - 3D^2]}{(b_1b_2 - d^2)};$$
$$\frac{[2(1 + m)D - m - 3D^2]^2}{D(1 - D)(m - D)(3D - 1 - m)} = \frac{3(b_1b_2 - d^2)^2}{4a_1c_1(d + b_2)^2},$$

(88)

while the two constraints are

$$a_1 = a_2, \quad c_2(d + b_1)^2 = c_1(d + b_2)^2.$$  

(89)

Using $A^2 > 0$ as well as $D < m$, it follows that

$$\frac{1 + m - \sqrt{1 - m + m^2}}{3} < D < m.$$  

(90)

As a result, $a_1, a_2$ are positive so long as

$$\frac{1 + m - \sqrt{1 - m + m^2}}{3} < D < \frac{1 + m}{3},$$  

(91)

while they are negative if

$$\frac{1 + m}{3} < D < m.$$  

(92)

In the limit $d = 0$, one obtains the uncoupled kink lattice solution [11, 12, 13]

$$\phi = \frac{Asn(Bx + x_0, m)}{\sqrt{1 - Dsn^2(Bx + x_0, m)}},$$  

(93)

satisfying

$$A^2 = \frac{2B^2[2(1 + m) - m - 3D^2]}{b_1}, \quad B^2 = \frac{a_1}{(3D - 1 - m)},$$
$$\frac{[2D(1 + m) - m - 3D^2]^2}{D(1 - D)(m - D)(3D - 1 - m)} = \frac{3b_1^2}{4a_1c_1}.$$  

(94)

At $m = 1$ the solution reduces to the topological (bright-bright) solution

$$\phi = \frac{Atanh(Bx + x_0)}{\sqrt{1 - Dtanh^2(Bx + x_0)}}, \quad \psi = \frac{Ftanh(Bx + x_0)}{\sqrt{1 - Dtanh^2(Bx + x_0)}},$$  

(95)
provided

\[
B^2 = \frac{a_1}{(3D - 2)}, \quad A^2 = \frac{2(d + b_2)B^2(1 - D)(3D - 1)}{(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2(d + b_1)B^2(1 - D)(3D - 1)}{(b_1b_2 - d^2)},
\]

\[
D = \frac{1}{3}\left[1 + \frac{(b_1b_2 - d^2)}{\sqrt{(b_1b_2 - d^2)^2 - 4a_1c_1(d + b_2)^2}}\right],
\]

(96)

while the two constraints are the same as given by Eq. (89). Since \( A^2 > 0 \), it then follows that \( 1/3 < D < 1 \).

Further, \( a_1, a_2 > 0 \) if \( 2/3 < D < 1 \) while \( a_1, a_2 < 0 \) in case \( 1/3 < D < 2/3 \). The constraint \( D < 1 \) implies the inequality

\[
(b_1b_2 - d^2)^2 > (16/3)a_1c_1(d + b_2)^2,
\]

(97)

which for \( d = 0 \) reduces to

\[
b_1^2 > (16/3)a_1c_1.
\]

(98)

This corresponds to the situation when there are two degenerate absolute minima and a local minimum or maximum at \( \phi = 0 \) depending on if \( a_1 > 0 \) or \( a_1 < 0 \), respectively [11, 12, 13].

**Special case of** \( b_1b_2 = d^2 \)

One can show that the solution (81) exists even in the case \( b_1b_2 = d^2 \). It turns out that such a solution exists only if

\[
b_1A^4 = b_2F^4,
\]

(99)

and further

\[
a_1 = a_2 < 0, \quad b_1c_2 = b_2c_1, \quad B^2 = \frac{-a_1}{\sqrt{1 - m + m^2}},
\]

\[
D = \frac{(1 + m) - \sqrt{1 - m + m^2}}{3}, \quad c_1A^4 = B^2(1 - D)[m(1 + D) - 2D].
\]

(100)

In the limit \( m \to 1 \), relation (100) takes the simpler form

\[
B^2 = -a_1, \quad D = \frac{1}{3}, \quad c_1A^4 = -\frac{4a_1}{9}.
\]

(101)

**Energy:** Corresponding to the periodic solution (81), the energy \( \hat{E} \) and the constant \( C \) (using the appropriate integrals given in [17]) are given by

\[
\hat{E} = (A^2 + F^2)BI_1,
\]

15
\[ C = -\frac{1}{2} (A^2 + F^2)B^2, \]  

(102)

where

\[ I_1 = \frac{[A_1 K(k) + A_2 E(k) + A_3 \Pi(D, k)]}{4D^2(1 - D)(k^2 - D)}, \]  

(103)

with

\[ A_1 = (k^2 - D)(2D + k^2 - 3D^2), \quad A_2 = D[2D(1 + k^2) - k^2 - 3D^2], \]
\[ A_3 = 4(1 - D)(k^2 - D)[(1 + k^2)D - k^2] - [D^2 - 2(1 + k^2)D + 3k^2][2(1 + k^2)D - k^2 - 3D^2]. \]  

(104)

Here \( \Pi(D, k) \) denotes the complete elliptic integral of the third kind \[16, 17\].

In order to obtain the interaction energy corresponding to this solution, one needs the expansion formula for \( \Pi(D, k) \) around \( k = 1 \). Unfortunately, only the leading term in the expansion is given in various books that we have encountered, but the subleading term is not mentioned. In particular, the result well known in the literature is \[17\]

\[ K(k = 1) - (1 - \alpha^2)\Pi(\alpha^2, k = 1) = \frac{\alpha}{2} \ln \left[ \frac{1 + \alpha}{1 - \alpha} \right]. \]  

(105)

We now show that starting from the basic definitions of \( K(k) \) and \( \Pi(\alpha^2, k) \), the subleading term in the expansion is easily derived.

Around \( k = 1 \), we have (note \( k'^2 = 1 - k^2 \))

\[ K(k) - (1 - \alpha^2)\Pi(\alpha^2, k) = K(1) - (1 - \alpha^2)\Pi(\alpha^2, 1) + (k^2 - 1) \left[ \frac{dK}{dk^2} - (1 - \alpha^2) \frac{d\Pi}{dk^2} \right]_{k^2 = 1} + O(k'^4), \]  

(106)

where \( K(k) \) and \( \Pi(\alpha^2, k) \) are defined by \[16, 17\]

\[ K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \quad \Pi(\alpha^2, k) = \int_0^1 \frac{dx}{(1 - \alpha^2x^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}. \]  

(107)

The well known result (105) is easily obtained by using basic definitions (107) of \( K(k) \) and \( \Pi(\alpha^2, k) \), i.e.

\[ K(1) - (1 - \alpha^2)\Pi(\alpha^2, 1) = \alpha^2 \int_0^1 \frac{dx}{(1 - \alpha^2x^2)} = \frac{\alpha}{2} \ln \left[ \frac{1 + \alpha}{1 - \alpha} \right]. \]  

(108)

Similarly, using the basic definitions (107) it is straightforward to show that

\[ \left[ \frac{dK}{dk^2} - (1 - \alpha^2) \frac{d\Pi}{dk^2} \right]_{k^2 = 1} = \frac{\alpha^2}{2(1 - \alpha^2)} \left[ \int_0^1 \frac{dx}{1 - x^2} - \int_0^1 \frac{dx}{1 - \alpha^2x^2} \right]. \]  

(109)
These integrals are easily evaluated and thus we find that

\[
K(k) - (1 - \alpha^2)\Pi(\alpha^2, k) = \frac{\alpha}{2} \ln \left[ \frac{1 + \alpha}{1 - \alpha} \right]
\]

\[-\frac{k^2 \alpha}{2(1 - \alpha^2)} \left( \alpha \ln \left[ \frac{4}{k^2} \right] - \frac{1}{2} \ln \left[ \frac{1 + \alpha}{1 - \alpha} \right] \right) + O(k^4).
\]

(110)

On using the expansions of \(E(k), K(k),\) and \(\Pi(D, k)\) around \(k = 1\) as given by Eqs. (22), (23) and (110), the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (95)] plus the interaction energy. We find

\[
\dot{E} = E_{\text{kink}} + E_{\text{int}} = (A^2 + F^2)B[I_1^{(0)} + k^2 I_1^{(1)}],
\]

(111)

where \(I_1^{(0)}\) and \(I_1^{(1)}\) are given by

\[
I_1^{(0)} = \frac{(3D - 1)}{4D(1 - D)} + \frac{(1 + 3D)}{8D^{3/2}} \ln \left[ \frac{1 + \sqrt{D}}{1 - \sqrt{D}} \right],
\]

(112)

\[
I_1^{(1)} = \frac{(1 + 3D^2)}{16D(1 - D)^2} - \frac{1}{16D^{3/2}} \ln \left[ \frac{1 + \sqrt{D}}{1 - \sqrt{D}} \right].
\]

(113)

Note that this solution exists only when \(b_1b_2 \geq d^2\). The interaction energy, as expected, vanishes at \(k = 1\).

2.8 Solution XII

There are actually three periodic solutions, all of which at \(m = 1\) reduce to the same hyperbolic nontopological (dark-dark) soliton solution. However, for \(m < 1\), all three (periodic) solutions are distinct which we now discuss one by one. One of the solution is

\[
\phi = \frac{A \text{cn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{cn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}},
\]

(114)

provided the following six equations are satisfied

\[
a_1 - b_1 A^2 + dF^2 + c_1 A^4 = -(1 - D)B^2, \]

(115)

\[(1 + D)b_1 A^2 - (1 + D)dF^2 - 2a_1 D - 2c_1 A^4 = 2(m - D)(1 - D)B^2, \]

(116)

\[dDF^2 - b_1 DA^2 + a_1 D^2 + c_1 A^4 = mD(1 - D)B^2, \]

(117)
\[ a_2 - b_2 F^2 + dA^2 + c_2 F^4 = -(1 - D)B^2, \]  
\[ (1 + D) b_2 F^2 - (1 + D) dA^2 - 2a_2 D - 2c_2 F^4 = 2(m - D)(1 - D)B^2, \]  
\[ dDA^2 - b_2 DF^2 + a_2 D^2 + c_2 F^4 = mD(1 - D)B^2. \]

From here one can determine the four unknowns \(A, F, B, D\) and further one obtains two constraints between the seven parameters \(a_{1,2}, b_{1,2}, c_{1,2}, d\). We get

\[ A^2 = \frac{2(d + b_2)B^2[(1 + 2D)m - (D + 2D)]}{(1 - D)(b_1b_2 - d^2)}, \quad F^2 = \frac{2(d + b_1)B^2[2(1 + 2D)m - (D + 2D)]}{(1 - D)(b_1b_2 - d^2)}, \]

\[ B^2 = \frac{(1 - D)a_1}{[2m - 1 - D(2 - m)]}, \quad \frac{[(1 + 2D)m - (2 + D)D]^2}{D(m - D)[2m - 1 - D(2 - m)]} = \frac{3(b_1b_2 - d^2)^2}{4a_1c_1(d + b_2)^2}, \]

while the two constraints are

\[ a_1 = a_2, \quad c_2(d + b_1)^2 = c_1(d + b_2)^2. \]

Using \(A^2 > 0\), it follows that

\[ 0 < D < \sqrt{1 - m + m^2} - (1 - m). \]

It is worth noting that for \(m < 1/2\), \(a_1, a_2 < 0\) irrespective of the value of \(D\).

In the limit \(d = 0\), one obtains the uncoupled pulse lattice solution

\[ \phi = \frac{A \text{cn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \]

satisfying

\[ A^2 = \frac{2B^2[(1 + 2D)m - (D + 2D)]}{(1 - D)b_1}, \quad B^2 = \frac{(1 - D)a_1}{[2m - 1 - D(2 - m)]}, \]

\[ \frac{[(1 + 2D)m - (2 + D)D]^2}{D(m - D)[2m - 1 - D(2 - m)]} = \frac{3b_1^2}{4a_1c_1}. \]

So far as we are aware of, this solution has never been explicitly written down before in the literature.

In the limit \(m = 1\), the solution \(\text{[111]}\) goes over to the hyperbolic, nontopological, dark-dark soliton solution

\[ \phi = \frac{A \text{sech}(Bx + x_0)}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \quad \psi = \frac{F \text{sech}(Bx + x_0)}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \]
provided

\[ B^2 = a_1, \quad A^2 = \frac{2(d + b_2)B^2(1 + D)}{(b_1b_2 - d^2)}, \]
\[ F^2 = \frac{2(d + b_1)B^2(1 + D)}{(b_1b_2 - d^2)}, \quad \frac{(1 + D)^2}{D} = \frac{3(b_1b_2 - d^2)^2}{4a_1c_1(d + b_2)^2}, \quad (127) \]

while the two constraints are the same as given by Eq. (122). Note that \( a_1, a_2 \) are now positive. The condition \( D < 1 \) gives the same constraint as given by Eq. (127).

Special case of \( b_1b_2 = d^2 \)

One can show that the solution \((114)\) exists even in the case \( b_1b_2 = d^2 \). It turns out that such a solution exists only if

\[ b_1A^4 = b_2F^4, \quad (128) \]

and further if

\[ a_1 = a_2 < 0, \quad b_1c_2 = b_2c_1, \quad B^2 = \frac{-a_1}{\sqrt{1 - m + m^2}}, \]
\[ D = \sqrt{1 - m + m^2} - (1 - m), \quad c_1A^4 = -a_1 \left[ 2 - \frac{(2 - m)}{\sqrt{1 - m + m^2}} \right]. \quad (129) \]

In the limit \( m \to 1 \), the constraint \((129)\) gives

\[ B^2 = -a_1, \quad D = 1, \quad c_1A^4 = -a_1. \quad (130) \]

Thus, for the case \( b_1b_2 = d^2 \) the hyperbolic dark-dark soliton solution \((126)\) reduces to a constant solution

\[ \phi = A, \quad \psi = F. \quad (131) \]

Energy: Corresponding to the periodic solution \((114)\), the energy \( \hat{E} \) and the constant \( C \) are given by

\[ \hat{E} = (A^2 + F^2)BI_2, \]
\[ C = -\frac{(1 - k^2)(A^2 + F^2)B^2}{2(1 - D)}. \quad (132) \]

Here

\[ I_2 = \frac{(1 - D)[B_1K(k) + B_2E(k) + B_3\Pi(D,k)]}{4D^2(k^2 - D)}. \quad (133) \]
with
\[ B_1 = \frac{-(k^2 - D)[(1 - D)^2 - (1 - k^2)(1 - 4D)]}{(1 - D)}, \quad B_2 = \frac{D[(1 - D^2 - (1 - k^2)(1 + 2D)]}{(1 - D)}, \]
\[ B_3 = 4(k^2 - D)[(1 + D)k^2 - D] - \frac{[D^2 - 2(1 + k^2)D + 3k^2][k^2 - 2(1 - k^2)D - D^2]}{(1 - D)}. \] (134)

On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D, k) \) around \( k = 1 \) as given above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (126)] plus the interaction energy. We find
\[ \hat{E} = E_{kink} + E_{int} = (A^2 + F^2)B[I_2^{(0)} + k^2I_2^{(1)}], \] (135)
where \( I_2^{(0)} \) and \( I_2^{(1)} \) are given by
\[ I_2^{(0)} = \frac{(1 + D)}{4D} - \frac{(1 - D)^2}{8D^{3/2}} \ln \left[ \frac{1 + \sqrt{D}}{1 - \sqrt{D}} \right], \] (136)
\[ I_2^{(1)} = \frac{(D^2 - 4D - 1)}{16D(1 - D)} - \frac{(3D^2 + 6D - 1)}{16D^{3/2}(1 - D)} \ln \left[ \frac{1 + \sqrt{D}}{1 - \sqrt{D}} \right] + \frac{1}{(1 - D)} \ln \left( \frac{4}{k'} \right). \] (137)

Note that this solution exists only when \( b_1b_2 \geq d^2 \). The interaction energy, as expected, vanishes at \( k = 1 \).

### 2.9 Solution XIII

Another solution is given by
\[ \phi = \frac{A \text{dn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{dn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \] (138)
provided the following six equations are satisfied
\[ a_1 - b_1A^2 + dF^2 + c_1A^4 = -(m - D)B^2, \] (139)
\[ (m + D)b_1A^2 - (m + D)dF^2 - 2a_1D - 2mc_1A^4 = 2(m - D)(1 - D)B^2, \] (140)
\[ dmDF^2 - b_1mA^2 + a_1D^2 + c_1m^2A^4 = D(m - D)B^2, \] (141)
\[ a_2 - b_2F^2 + dA^2 + c_2F^4 = -(m - D)B^2, \] (142)
satisfying solution (126) satisfying the same constraint as given by Eq. (127).

From here one can determine the four unknowns $a, b, c, d$. We obtain

$$A^2 = \frac{2(d + b_2)B^2[2D(1-m) + m - D^2]}{(m-D)(b_1b_2 - d^2)}, \quad F^2 = \frac{2(d + b_1)B^2[2D(1-m) + m - D^2]}{(m-D)(b_1b_2 - d^2)},$$

$$B^2 = \frac{(m-D)a_1}{[2m(1-D) - m^2 + D]}, \quad \frac{[2D(1-m) + m - D^2]}{D(1-D)[2m(1-D) - m^2 + D]} = \frac{3(b_1b_2 - d^2)^2}{4a_1c_1(d + b_2)^2}, \quad (145)$$

while the two constraints are

$$a_1 = a_2, \quad c_2(d + b_1)^2 = c_1(d + b_2)^2. \quad (146)$$

The best bound on $D$ is given by $0 < D < m$ which ensures $A^2 > 0$. Further, unlike two of the above solutions (i.e. solutions XI and XII), $a_1, a_2$ are always positive.

In the limit $d = 0$, one obtains the uncoupled pulse lattice solution

$$\phi = \frac{A\ln(Bx + x_0, m)}{\sqrt{1 - D\text{sn}^2(Bx + x_0, m)}}, \quad (147)$$

satisfying

$$A^2 = \frac{2B^2[2D(1-m) + m - D^2]}{(m-D)b_1}, \quad B^2 = \frac{(m-D)a_1}{[2m(1-D) + D - m^2]}, \quad$$

$$\frac{[2D(1-m) + m - D^2]}{D(1-D)[2m(1-D) + D - m^2]} = \frac{3b_1^2}{4a_1c_1}. \quad (148)$$

In the limit $m = 1$, the solution $[138]$ also goes over to the hyperbolic, nontopological, dark-dark soliton solution $[129]$ satisfying the same constraint as given by Eq. $[127]$.

**Special case of** $b_1b_2 = d^2$

One can show that the solution $[138]$ exists even in the case $b_1b_2 = d^2$. It turns out that such a solution exists only if

$$b_1A^4 = b_2F^4, \quad (149)$$

and further if

$$a_1 = a_2, \quad b_1c_2 = b_2c_1, \quad B^2 = \frac{-a_1}{\sqrt{1 - m + m^2}},$$

$$D = \sqrt{1 - m + m^2 + (1-m)}, \quad c_1A^4 = \frac{3D(1-D)B^2}{(m-D)}. \quad (150)$$
In the limit $m \to 1$, the constraint (150), as expected, reduces to Eq. (130) and the solution (138) reduces to the constant solution (131).

**Energy:** Corresponding to the periodic solution (138), the energy $\hat{E}$ and the constant $C$ are given by

$$\hat{E} = (A^2 + F^2)BI_3,$$

$$C = \frac{k^2(1-k^2)(A^2 + F^2)B^2}{2(k^2 - D)}.$$  \hspace{1cm} (151)

Here

$$I_3 = \frac{(k^2 - D)[D_1 K(k) + D_2 E(k) + D_3 \Pi(D,k)]}{4D^2(1-D)},$$ \hspace{1cm} (152)

with

$$D_1 = -[(1-D)^2 - (1-k^2)], \quad D_2 = \frac{D[(1-D^2 + (1-k^2)(2D - 1)]}{(k^2 - D)},$$

$$D_3 = 4(1-D)[(1-D)k^2 + D] - \frac{[D^2 - 2(1+k^2)D + 3k^2][k^2 + 2(1-k^2)D - D^2]}{(k^2 - D)}. \hspace{1cm} (153)$$

On using the expansion formulas for $E(k), K(k)$ and $\Pi(D,k)$ around $k = 1$ obtained above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (126)] plus the interaction energy. We find

$$\hat{E} = E_{kink} + E_{int} = (A^2 + F^2)B[I_3^{(0)} + k^2I_3^{(1)}],$$ \hspace{1cm} (154)

where $I_3^{(0)}$ is in fact the same as $I_2^{(0)}$, and given by Eq. (136) while $I_3^{(1)}$ is given by

$$I_3^{(1)} = \frac{(D^2 + 8D - 5)}{16D(1-D)} + \frac{(7D^2 - 2D + 3)}{16D^{3/2}(1-D)} \ln \left[ \frac{1 + \sqrt{D}}{1 - \sqrt{D}} \right] - \frac{1}{(1-D)} \ln \left( \frac{4}{k^2} \right).$$ \hspace{1cm} (155)

Note that this solution exists only when $b_1b_2 \geq d^2$. As expected, the interaction energy vanishes at $k = 1$.

### 2.10 Solution XIV

Yet another solution is

$$\phi = \frac{A \cn(Bx + x_0, m)}{\sqrt{1 - D \sn^2(Bx + x_0, m)}}, \quad \psi = \frac{F \dn(Bx + x_0, m)}{\sqrt{1 - D \sn^2(Bx + x_0, m)}},$$ \hspace{1cm} (156)
provided the following six equations are satisfied

\[ a_1 - b_1 A^2 + d F^2 + c_1 A^4 = -(1 - D) B^2, \]  
\[ (1 + D) b_1 A^2 - (m + D) d F^2 - 2 a_1 D - 2 c_1 A^4 = 2(m - D)(1 - D) B^2, \]  
\[ d m D F^2 - b_1 D A^2 + a_1 D^2 + c_1 A^4 = m D (1 - D) B^2, \]  
\[ a_2 - b_2 F^2 + d A^2 + c_2 F^4 = -(m - D) B^2, \]  
\[ (m + D) b_2 F^2 - (1 + D) d A^2 - 2 a_2 D - 2 m c_2 F^4 = 2(m - D)(1 - D) B^2, \]  
\[ d D A^2 - b_2 m D F^2 - a_2 D^2 + c_2 m^2 F^4 = D(m - D) B^2. \] 

Solving these equations, we determine the four unknowns \( A, F, B, D \)

\[ B^2 = \frac{3D(1 - D)(m - D)(b_1 b_2 - d^2)^2}{4c_1[(m - D^2)(d + b_2) - 2D(1 - m)(b_2 - d)]^2}, \]
\[ A^2 = \frac{2B^2[(m - D^2)(d + b_2) - 2D(1 - m)(b_2 - d) + 2D(1 - m)(b_2 - d)]}{(b_1 b_2 - d^2)(m - D)}, \]
\[ F^2 = \frac{2B^2[(m - D^2)(d + b_1) + 2D(1 - m)(b_1 - d)]}{(b_1 b_2 - d^2)(m - D)} \]
\[ (1 - D)a_1 + (1 - m)d F^2 = [2m - 1 - (2 - m)D]B^2, \]  

while the two constraints between the seven parameters are

\[ (1 - D)^2 c_1 A^4 = (m - D)^2 c_2 F^4, \]  

and

\[ (1 - D)a_2 = (1 - m)b_2 F^2 + \frac{[D^2 + 4mD - 3D(1 + m^2) + m^2]B^2}{(m - D)}. \] 

In this case too the best bound on \( D \) is given by \( 0 < D < m \). Further, \( a_1 < 0 \) in case \( m < 1/2 \).

In the limit \( m = 1 \), the solution \( \text{[156]} \) also goes over to the hyperbolic, nontopological, dark-dark soliton solution \( \text{[126]} \) satisfying the same constraints as given by Eq. \( \text{[127]} \).

Unlike the solutions XI, XII and XIII, the solution \( \text{[156]} \) does not exist in the case \( b_1 b_2 = d^2 \). This is because, in order that such a solution exist, \( m \) and \( D \) must satisfy

\[ \sqrt{b_2}[m - D^2 - 2(1 - m)D] = -\sqrt{b_1}[m - D^2 + 2(1 - m)D], \]  

23
which is impossible.

**Energy**: Corresponding to the periodic solution (156), the energy $\hat{E}$ and the constant $C$ are given by

$$
\hat{E} = B (A^2 I_2 + F^2 I_3),
$$

$$
C = -\frac{1}{2D^3} \left[ A^2 (a_1 D^2 + b_1^2 DA^2 - c_1 A^4 - dmDF^2) + F^2 (-ma_2 D^2 + \frac{b_2}{2} Dm^2 F^2 - \frac{c_2}{3} m^3 F^4) \right],
$$

(167)

where $I_2, I_3$ are as given by Eqs. (133), (134), (152) and (153). On using the expansion formulas for $E(k), K(k)$ and $\Pi(D,k)$ around $k = 1$ as derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (126)] plus the interaction energy. We find

$$
\hat{E} = E_{kink} + E_{int} = [(A^2 + F^2) I_2^{(0)} + k^2 (A^2 I_2^{(1)} + F^2 I_3^{(1)})] B,
$$

(168)

where $I_2^{(0)}, I_2^{(1)}, I_3^{(1)}$ are given by Eqs. (136), (137) and (155) respectively.

Note that this solution exists only when $b_1 b_2 > d^2$. As expected, the interaction energy vanishes at $k = 1$.

### 2.11 Solution XV

Finally, we obtain two periodic solutions which at $m = 1$ go over to the same bright-dark soliton solution. The first solution is given by

$$
\phi = \frac{A \text{sn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{cn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}},
$$

(169)

provided the following six equations are satisfied

$$
a_1 + dF^2 = (3D - 1 - m) B^2,
$$

(170)

$$
b_1 A^2 + d(1 + D) F^2 + 2a_1 D = 2(D + Dm - m) B^2,
$$

(171)

$$
dDF^2 + b_1 DA^2 + a_1 D^2 + c_1 A^4 = m DB^2,
$$

(172)

$$
a_2 - b_2 F^2 + c_2 F^4 = -(1 - D) B^2,
$$

(173)

$$
(1 + D)b_2 F^2 + dA^2 - 2a_2 D - 2c_2 F^4 = 2(m - D)(1 - D) B^2,
$$

(174)
\[-dDA^2 - b_2DF^2 + a_2D^2 + c_2F^4 = mD(1 - D)B^2. \quad (175)\]

On solving these equations we determine the four parameters $A, F, B, D$ as well as obtain two constraints between the seven parameters. We find

\[
A^2 = \frac{2B^2[b_2(2mD - m + 2D - 3D^2) - d(2mD + m - 2D - D^2)]}{(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2[b_1(2mD + m - 2D - D^2) - d(2mD - m + 2D - 3D^2)]}{(1 - D)(b_1b_2 - d^2)}, \quad (176)
\]

\[
B^2 = \frac{3D(1 - D)(m - D)(b_1b_2 - d^2)^2}{4c_1[b_2(2mD - m + 2D - 3D^2) - d(2mD + m - 2D - D^2)]^2},
\]

\[
a_1 + dF^2 = [3D - 1 - m]B^2, \quad (177)
\]

while the two constraints are

\[
c_1A^4 = (1 - D)^2c_2F^4, \quad (178)
\]

\[
a_2 - b_2F^2 + \frac{B^2[(1 - D)^2 + 3D(m - D)]}{(1 - D)} = 0. \quad (179)
\]

From positivity considerations, one can show from here that

\[
\frac{(1 + m) - \sqrt{1 - m + m^2}}{3} < D < \sqrt{1 - m + m^2} - (1 - m). \quad (180)
\]

Further, it follows that $a_1 < 0$ if $D < (1 + m)/3$.

At $m=1$, the solution (169) goes over to the hyperbolic, bright-dark soliton solution

\[
\phi = \frac{A \tanh(Bx + x_0)}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \quad \psi = \frac{F \text{sech}(Bx + x_0)}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \quad (181)
\]

and the relations (176) to (179) take the simpler form

\[
A^2 = \frac{2B^2(1 - D)[b_2(3D - 1) - d(1 + D)]}{(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2(1 - D)[b_1(1 + D) - d(3D - 1)]}{(b_1b_2 - d^2)}, \quad (182)
\]

\[
B^2 = \frac{3D(b_1b_2 - d^2)^2}{4c_1[b_2(3D - 1) - d(1 + D)]^2},
\]

\[
a_1 + dF^2 = (3D - 2)B^2, \quad (183)
\]

25
while the two constraints are

\[ c_1 A^4 = (1 - D)^2 c_2 F^4, \]  
\[ a_2 - b_2 F^2 + B^2(1 + 2D) = 0. \]  

From here one can show that \( 1/3 < D < 1 \) and further \( a_1 < 0 \) if \( D > 2/3 \) while \( a_2 < 0 \) if \( D > \frac{\sqrt{3} - 1}{2} \).

**Special case of \( b_1b_2 = d^2 \)**

One can show that the solution \( 169 \) exists even in the case \( b_1b_2 = d^2 \). It turns out that such a solution exists only if

\[ \sqrt{b_2}[2(1 + m)D - m - 3D^2] = \sqrt{b_1}[m - D^2 - 2(1 - m)D], \]  

and further while relations \( 177 \) to \( 179 \) are still valid, the relations \( 176 \) are no longer valid. Instead of separate expressions for \( A^2, F^2 \), one now only has a constraint

\[ b_1 A^2 + \sqrt{b_1b_2}(1 - D)F^2 = 2B^2[2(1 + m)D - 3D^2 - m]. \]  

In the limit \( m \to 1 \), the hyperbolic bright-dark solution \( 181 \) is still valid provided \( D \) has the value

\[ D = \frac{\sqrt{b_2} + \sqrt{b_1}}{3\sqrt{b_2} - \sqrt{b_1}}, \]  

while the constraint \( 187 \) takes the simpler form

\[ b_1 A^2 + \sqrt{b_1b_2}(1 - D)F^2 = 2B^2(1 - D)(3D - 1). \]  

The relations \( 183 \) to \( 185 \) are still valid.

**Energy**: Corresponding to the periodic solution \( 169 \), the energy \( \dot{E} \) and the constant \( C \) are given by

\[ \dot{E} = B(A^2 I_1 + F^2 I_2), \]
\[ C = -\frac{1}{2D^3} \left[ A^2(a_1 D^2 + \frac{b_1}{2} DA^2 + \frac{c_1}{3} A^4 + dDF^2) + F^2(-a_2 D^2 + \frac{b_2}{2} DF^2 - \frac{c_2}{3} F^4) \right], \]

where \( I_1, I_2 \) are as given by Eqs. \( 103, 104, 133 \) and \( 134 \). On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D, k) \) around \( k = 1 \) as derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution [Eq. \( 181 \)] plus the interaction energy. We find

\[ \dot{E} = E_{kink} + E_{int} = B A^2[I_1^{(0)} + k^2 I_1^{(1)}] + B F^2[I_2^{(0)} + k^2 I_2^{(1)}], \]
where $I_{1,2}^{(0,1)}$ are given by Eqs. (112), (113), (136) and (137).

Note that this solution exists only when $b_1 b_2 \geq d^2$. As usual, the interaction energy vanishes at $k = 1$.

### 2.12 Solution XVI

Finally, the last solution below $T_c$ is given by

$$
\phi = \frac{A \text{sn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{dn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}},
$$

(192)

provided the following six equations are satisfied

$$
a_1 + dF^2 = (3D - 1 - m)B^2, \quad \text{(193)}
$$

$$
b_1 A^2 + d(m + D)F^2 + 2a_1 D = 2(D + Dm - m)B^2, \quad \text{(194)}
$$

$$
dmDF^2 + b_1 DA^2 + a_1 D^2 + c_1 A^4 = mDB^2, \quad \text{(195)}
$$

$$
a_2 - b_2 F^2 + c_2 F^4 = -(m - D)B^2, \quad \text{(196)}
$$

$$
(m + D)b_2 F^2 + a_2 D^2 - 2a_2 D - 2mc_2 F^4 = 2(m - D)(1 - D)B^2, \quad \text{(197)}
$$

$$
- dDA^2 - mb_2 DF^2 + a_2 D^2 + m^2 c_2 F^4 = D(1 - D)B^2. \quad \text{(198)}
$$

On solving these equations we obtain the four unknowns $A, F, B, D$ and further also obtain two constraints between the seven parameters. We find

$$
A^2 = \frac{2B^2[b_2(2mD - m + 2D - 3D^2) - d(m + 2D - 2mD - D^2)]}{(b_1 b_2 - d^2)},
$$

$$
F^2 = \frac{2B^2[b_1(2D + m - 2Dm - D^2) - d(2mD - m + 2D - 3D^2)]}{(m - D)(b_1 b_2 - d^2)}, \quad \text{(199)}
$$

$$
B^2 = \frac{3D(1 - D)(m - D)}{4c_1[b_2(2mD - m + 2D - 3D^2) - d(m + 2D - 2mD - D^2)]^2}, \quad a_1 + dF^2 = [3D - 1 - m]B^2, \quad \text{(200)}
$$

while the two constraints are

$$
c_1 A^4 = (m - D)^2 c_2 F^4, \quad \text{(201)}
$$

$$
a_2 - b_2 F^2 + \frac{B^2[(m - D)^2 + 3D(1 - D)]}{(m - D)} = 0. \quad \text{(202)}
$$
From positivity considerations, one can show from here that
\[
\frac{(1 + m) - \sqrt{(1 + m)(m - 1/2)}}{3} < D < m.
\] (203)

Further, it follows that \( a_1 < 0 \) if \( D < (1 + m)/3 \).

At \( m=1 \), the solution (192) goes over to the hyperbolic, bright-dark soliton solution (181) and the constraints (199) to (202) take the simpler form as given by Eqs. (182) to (185).

**Special case of** \( b_1b_2 = d^2 \)

One can show that the solution (192) exists even in the case \( b_1b_2 = d^2 \). It turns out that such a solution exists only if
\[
\sqrt{b_2}[2(1 + m)D - m - 3D^2] = \sqrt{b_1}[m - D^2 + 2(1 - m)D],
\] (204)
and further while relations (200) to (202) are still valid, the relation (199) is no longer valid. Instead of separate expressions for \( A^2, F^2 \), one now only has a constraint
\[
b_1A^2 + \sqrt{b_1b_2}(1 - D)F^2 = 2B^2[2(1 + m)D - 3D^2 - m].
\] (205)

In the limit \( m \to 1 \), the hyperbolic bright-dark solution (181) is still valid provided the constraints (188) and (189) as well as Eqs. (183) to (185) are satisfied.

**Energy:** Corresponding to the periodic solution (192), the energy \( \hat{E} \) and the constant \( C \) are given by
\[
\hat{E} = B(A^2I_1 + F^2I_3),
\]
\[
C = -\frac{1}{2D^3} \left[ A^2(a_1D^2 + \frac{b_1}{2}DA^2 + \frac{C_1}{3}A^4 + dmDF^2) + F^2(-ma_2D^2 + \frac{b_2}{2}Dm^2F^2 - \frac{C_2}{3}m^3F^4) \right],
\] (206)
where \( I_1, I_3 \) are as given by Eqs. (112), (113), (152) and (153). On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D,k) \) around \( k = 1 \) as derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution [Eq. (181)] plus the interaction energy. We find
\[
\hat{E} = E_{kink} + E_{int} = BA^2[I_1^{(0)} + k^2I_1^{(1)}] + BF^2[I_3^{(0)} + k^2I_3^{(1)}],
\] (207)
where \( I_1^{(0,1)} \) are given by Eqs. (112) and (113) while \( I_3^{(0)} = I_2^{(0)} \) and \( I_3^{(1)} \) are given by Eqs. (136) and (155), respectively.

Note that this solution exists only when \( b_1b_2 \geq d^2 \). The interaction energy vanishes at \( k = 1 \).
2.13 Solution XVII

So far, we have presented ten solutions at \( T = T_c \) and six solutions for \( T < T_c \). We now present one solution which exists when \( T_c < T < T_p \) where \( T_p \) denotes the point of inflection where \( \phi^6 \) potential (at \( d = 0 \)) has an absolute minimum at \( \phi = 0 \) and two points of inflection. Note that the point of inflection occurs when \( b^2 = 4a_1c_1 \).

It is easily shown that

\[
\phi = \frac{A}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}},
\]

is a solution to the field Eqs. \( \text{(2)} \) provided the following six equations are satisfied

\[
a_1 - b_1A^2 + c_1A^4 + dF^2 = DB^2, \tag{209}
\]

\[-dF^2 + b_1A^2 - 2a_1 = 2(D - 1 - m)B^2, \tag{210}
\]

\[a_1D = (3m - D - Dm)B^2, \tag{211}
\]

\[a_2 - b_2F^2 + c_2F^4 + dA^2 = DB^2, \tag{212}
\]

\[-dA^2 + b_2F^2 - 2a_2 = 2(D - 1 - m)B^2, \tag{213}
\]

\[a_2D = (3m - D - Dm)B^2. \tag{214}
\]

From here one can determine the four unknowns \( A, F, B, D \) and further one obtains two constraints between the seven parameters \( a_{1,2}, b_{1,2}, c_{1,2}, d \). We get

\[
B^2 = \frac{a_1D}{[3m - D(1 + m)]}, \quad A^2 = \frac{2(d + b_2)B^2[D^2 - 2(1 + m)D + 3m]}{D(b_1b_2 - d^2)}, \tag{215}
\]

\[
F^2 = \frac{2(d + b_1)B^2[D^2 - 2(1 + m)D + 3m]}{D(b_1b_2 - d^2)}, \quad \frac{|D^2 - 2(1 + m)D + 3m|^2}{(1 - D)(m - D)[3m - D(1 + m)]} = \frac{3(b_1b_2 - d^2)^2}{4a_1c_1(d + b_2)^2},
\]

while the two constraints are

\[a_1 = a_2, \quad c_2(d + b_1)^2 = c_1(d + b_2)^2. \tag{216}\]

Note that \( a_1, a_2 \) are always positive.
In the limit \( d = 0 \), one obtains the uncoupled pulse lattice solution \[\phi = A \sqrt{1 - D \text{sn}^2(Bx + x_0, m)}, \tag{217}\]
satisfying
\[
A^2 = \frac{2B^2[D^2 - 2D(1 + m) + 3m]}{D b_1}, \quad B^2 = \frac{(1 - D)a_1}{[2m - 1 - D(2 - m)]}, \\
\frac{[D^2 - 2D(1 + m) + 3m]^2}{(1 - D)(m - D)[3m - (1 + m)D]} = \frac{3b_1^2}{4a_1 c_1}. \tag{218}
\]

At \( m = 1 \) the solution reduces to the nontopological (dark-dark) solution
\[
\phi = A \sqrt{1 - D \tanh^2(Bx + x_0)}, \quad \psi = F \sqrt{1 - D \tanh^2(Bx + x_0)}, \tag{219}
\]
provided
\[
B^2 = \frac{a_1 D}{(3 - 2D)}, \quad A^2 = \frac{2(d + b_2)B^2(1 - D)(3 - D)}{D(b_1b_2 - d^2)}, \\
F^2 = \frac{2(d + b_1)B^2(1 - D)(3 - D)}{D(b_1b_2 - d^2)}, \quad \frac{(3 - D)^2}{(3 - 2D)} = \frac{3(b_1b_2 - d^2)^2}{4a_1 c_1(d + b_2)^2}, \tag{220}
\]
while the two constraints are the same as given by Eq. \[\text{(216)}.\] The constraint \( 0 < D < 1 \) implies the relation
\[
4a_1 c_1(d + b_2)^2 < (b_1b_2 - d^2)^2 < (16/3)a_1 c_1(d + b_2)^2, \tag{221}
\]
which for \( d = 0 \) reduces to the constraint
\[
4a_1 c_1 < b_1^2 < (16/3)a_1 c_1, \tag{222}
\]
i.e. \( T_c < T < T_p \). This corresponds to the situation when there is an absolute minimum at \( \phi = 0 \) and there are two degenerate local minima \[\text{[11][12][13]}\].

**Special case of** \( b_1b_2 = d^2 \)

One can show that the solution \[\text{[208]}\] exists even in the case \( b_1b_2 = d^2 \) provided \( m < 1 \). It turns out that such a solution exists only if
\[
b_1 A^4 = b_2 F^4, \quad c_2 b_1 = c_1 b_2, \tag{223}
\]
and further
\[ D = (1 + m) - \sqrt{1 - m + m^2}, \quad c_1 A^4 = \frac{3B^2(1 - D)(m - D)}{D}, \quad a_1 = a_2 = \frac{B^2[3m - D(1 + m)]}{D}. \quad (224) \]

**Energy:** Corresponding to the periodic solution (208), the energy \( \hat{E} \) and the constant \( C \) are given by (using appropriate integrals in [17])

\[ \hat{E} = (A^2 + F^2)BI_4, \]
\[ C = \frac{m}{2}(A^2 + F^2)B^2, \quad (225) \]

where
\[ I_4 = \frac{[G_1 K(k) + G_2 E(k) + G_3 \Pi(D, k)]}{4D^2(1 - D)(k^2 - D)}, \quad (226) \]

with
\[ G_1 = D(k^2 - D)(D^2 - 2D + 4Dk^2 - 3k^2), \quad G_2 = D^2[D^2 - 2D(1 + k^2) + 3k^2], \]
\[ G_3 = 4D(1 - D)(k^2 - D)[3k^2 - (1 + k^2)D] - D[D^2 - 2(1 + k^2)D + 3k^2]^2. \quad (227) \]

On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D, k) \) around \( k = 1 \) derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (219)] plus the interaction energy. We find

\[ \hat{E} = E_{kink} + E_{int} = (A^2 + F^2)B[I_4^{(0)} + k^2I_4^{(1)}], \quad (228) \]

where \( I_4^{(0)} \) and \( I_4^{(1)} \) are given by

\[ I_4^{(0)} = \frac{(3 - D)}{4(1 - D)} - \frac{(D + 3)}{8D^{1/2}} \ln \frac{1 + \sqrt{D}}{1 - \sqrt{D}}, \quad (229) \]
\[ I_4^{(1)} = \frac{-(D^2 - 8D + 3)}{16(1 - D)^2} + \frac{3(D + 4)}{16D^{3/2}} \ln \frac{1 + \sqrt{D}}{1 - \sqrt{D}} - \frac{3}{2D^2} \ln \left( \frac{4}{k^2} \right). \quad (230) \]

Note that this solution exists only when \( b_1b_2 \geq d^2 \). The interaction energy vanishes at \( k = 1 \).
2.14 Solution XVIII

Finally, we present three novel mixed phase solutions, i.e. in which one of the field exists for \( T > T_c \) while the other one exists for \( T < T_c \).

It is easily shown that

\[
\phi = \frac{A}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{sn}(Bx + x_0, m)}{\sqrt{1 - D \text{sn}^2(Bx + x_0, m)}},
\]

is a solution to the field Eqs. (2) provided the following six equations are satisfied

\[
a_1 - b_1 A^2 + c_1 A^4 = DB^2, \tag{232}
\]

\[
dF^2 + b_1 DA^2 - 2a_1 D = 2D(D - 1 - m)B^2, \tag{233}
\]

\[
a_1 D - dF^2 = (3m - D - Dm)B^2, \tag{234}
\]

\[
a_2 + dA^2 = (3D - 1 - m)B^2, \tag{235}
\]

\[
-dDA^2 - b_2 F^2 - 2a_2 D = 2(m - D - mD)B^2, \tag{236}
\]

\[
a_2 D^2 + b_2 DF^2 + c_2 F^4 = mDB^2. \tag{237}
\]

On solving these equations, we obtain the four unknowns \( A, F, B, D \) as well as two constraints between the seven parameters. We find

\[
A^2 = \frac{2B^2 [b_2 (D^2 - 2D - 2mD + 3m) + d(2D + 2mD - m - 3D^2)]}{(b_1 b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2 [d(D^2 - 2D - 2mD + 3m) + b_1 (2D + 2mD - m - 3D^2)]}{(b_1 b_2 - d^2)},
\]

\[
B^2 = \frac{3(1 - D)(m - D)(b_1 b_2 - d^2)^2}{4Dc_1 [b_2 (D^2 - 2D - 2mD + 3m) + d(2D + 2mD - m - 3D^2)]^2},
\]

\[
a_1 D - dF^2 = [3m - (1 + m)D]B^2, \tag{238}
\]

while the two constraints are

\[
D^2 c_1 A^4 = c_2 F^4, \quad a_2 + dA^2 = (3D - 1 - m)B^2. \tag{239}
\]

At \( m = 1 \) the solution reduces to the dark-bright solution

\[
\phi = \frac{A}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \quad \psi = \frac{F \tanh(Bx + x_0)}{\sqrt{1 - D \tanh^2(Bx + x_0)}}, \tag{240}
\]
provided

\[ A^2 = \frac{2B^2(1-D)[b_2(3-D) + d(3D-1)]}{(b_1b_2 - d^2)} \]

\[ F^2 = \frac{2B^2(1-D)[d(3-D) + b_1(3D-1)]}{(b_1b_2 - d^2)} \]

\[ B^2 = \frac{3(b_1b_2 - d^2)^2}{4Dc_1[b_2(3-D) + d(3D-1)]^2} \]

\[ a_1D - dF^2 = [3 - 2D]B^2 \] (241)

while the two constraints are

\[ D^2c_1A^4 = c_2F^4, \quad a_2 + dA^2 = (3D - 2)B^2. \] (242)

**Energy:** Corresponding to the periodic solution (231), the energy \( \hat{E} \) and the constant \( C \) are given by (using appropriate integrals in [17])

\[ \hat{E} = (A^2I_4 + F^2I_1)B, \]

\[ C = \frac{m}{2}A^2B^2 - \frac{F^2}{4D}[B^2(1 + m - D) + a_2], \] (243)

where \( I_1 \) and \( I_4 \) are given by Eqs. (103), (104), (226) and (227).

On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D,k) \) around \( k = 1 \) as derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-bright) soliton solution [Eq. (240)] plus the interaction energy. We find

\[ \hat{E} = E_{\text{kink}} + E_{\text{int}} = BA^2[I_4^{(0)} + k^2I_4^{(1)}] + BF^2[I_1^{(0)} + k^2I_1^{(1)}], \] (244)

where \( I_{1,4}^{(0,1)} \) are given by Eqs. (112), (113), (229) and (230).

Note that this solution exists only when \( b_1b_2 > d^2 \). The interaction energy vanishes at \( k = 1 \).

### 2.15 Solution XIX

Another novel mixed phase solution is given by

\[ \phi = \frac{A}{\sqrt{1 - D\text{sn}^2(Bx + x_0, m)}}, \quad \psi = \frac{F\text{cn}(Bx + x_0, m)}{\sqrt{1 - D\text{sn}^2(Bx + x_0, m)}}. \] (245)

This is a solution to the field Eqs. (2) provided the following six equations are satisfied

\[ a_1 - b_1A^2 + c_1A^4 + dF^2 = DB^2 \] (246)
\[ -d(1+D)F^2 + b_1 DA^2 - 2a_1 D = 2D(D - 1 - m)B^2, \]  
\[ a_1 D + dF^2 = (3m - D - Dm)B^2, \]  
\[ a_2 + dA^2 - b_2 F^2 + c_2 F^4 = -(1-D)B^2, \]  
\[ -dDA^2 + b_2(1+D)F^2 - 2c_2 F^4 - 2a_2 D = 2(1-D)(m - D)B^2, \]  
\[ a_2 D^2 - b_2 DF^2 + c_2 F^4 = mD(1-D)B^2. \]  

On solving these equations, we obtain the four unknowns \( A, F, B, D \) as well as two constraints between the seven parameters. We find

\[
A^2 = \frac{2B^2[b_2(D^2 - 2D - 2mD + 3m) - d(m + 2mD - D^2 - D^2)]}{D(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2[b_1(m + 2mD - 2D - D^2) - d(D^2 - 2D - 2mD + 3m)]}{(1-D)(b_1b_2 - d^2)},
\]

\[
B^2 = \frac{3D(1-D)(m - D)(b_1b_2 - d^2)^2}{4c_1[b_2(D^2 - 2D - 2mD + 3m) - d(m + 2mD - 2D - D^2)]^2},
\]

\[a_1 D + dF^2 = [3m - (1 + m)D]B^2,\]  

while the two constraints are

\[ D^2 c_1 A^4 = (1-D)^2 c_2 F^4, \quad a_2 D - b_2 F^2 + (2 + D)B^2 = 0. \]

At \( m = 1 \) the solution reduces to the dark-dark soliton solution

\[
\phi = \frac{A}{\sqrt{1-D \tanh^2(Bx + x_0)}}, \quad \psi = \frac{F \text{sech}(Bx + x_0)}{\sqrt{1-D \tanh^2(Bx + x_0)}},
\]

provided

\[
A^2 = \frac{2B^2(1-D)[b_2(3-D) - d(3D - 1)]}{D(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2[b_1(1+D) - d(3-D)]}{(b_1b_2 - d^2)},
\]

\[
B^2 = \frac{3D(b_1b_2 - d^2)^2}{4c_1[b_2(3-D) - d(1+D)]^2}, \quad a_1 D + dF^2 = [3m - (1 + m)D]B^2,
\]

while the two constraints are

\[ D^2 c_1 A^4 = (1-D)^2 c_2 F^4, \quad a_2 D - b_2 F^2 + (2 + D)B^2 = 0. \]
**Special case of** \( b_1 b_2 = d^2 \)

One can show that the solution \((245)\) exists even in the case \( b_1 b_2 = d^2 \). It turns out that such a solution exists only if

\[
\sqrt{b_2}[D^2 - 2(1 + m)D + 3m] = \sqrt{b_1}[m(1 + 2D) - D(2 + D)].
\]  

(257)

At \( m = 1 \) this implies

\[
\sqrt{b_2}(3-D) = \sqrt{b_1}(1+D).
\]  

(258)

**Energy:** Corresponding to the periodic solution \((245)\), the energy \( \hat{E} \) and the constant \( C \) are given by (using appropriate integrals in [17])

\[
\hat{E} = (A^2 I_4 + F^2 I_2)B,
\]

\[
C = \frac{m}{2} A^2 B^2 - \frac{F^2}{4D} \left[ \frac{B^2(1-mD)}{(1-D)} - a_2 \right],
\]

(259)

where \( I_2 \) and \( I_4 \) are given by Eqs. \([133], [134], [220] \) and \([221] \).

On using the expansion formulas for \( E(k), K(k) \) and \( \Pi(D, k) \) around \( k = 1 \) as derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution \([\text{Eq. (254)}]\) plus the interaction energy. We find

\[
\hat{E} = E_{\text{kink}} + E_{\text{int}} = BA^2[I_4^{(0)} + k^2 I_4^{(1)}] + BF^2[I_2^{(0)} + k^2 I_2^{(1)}],
\]

(260)

where \( I_{2,4}^{(0,1)} \) are given by Eqs. \([136], [137], [220] \) and \([221] \).

Note that this solution exists only when \( b_1 b_2 \geq d^2 \). The interaction energy vanishes at \( k = 1 \).

### 2.16 Solution XX

Finally, yet another novel mixed phase solution is given by

\[
\phi = \frac{A}{\sqrt{1 - D \sin^2(Bx + x_0, m)}}, \quad \psi = \frac{F \text{dn}(Bx + x_0, m)}{\sqrt{1 - D \sin^2(Bx + x_0, m)}}.
\]  

(261)

This is a solution to the field Eqs. \([2]\) provided the following six equations are satisfied

\[
a_1 - b_1 A^2 + c_1 A^4 + dF^2 = DB^2,
\]

(262)

\[
-d(m + D)F^2 + b_1 DA^2 - 2a_1 D = 2D(D - 1 - m)B^2,
\]

(263)
\[ a_1D + dmF^2 = (3m - D - Dm)B^2, \]  
(264)

\[ a_2 + dA^2 - b_2F^2 + c_2F^4 = -(m - D)B^2, \]  
(265)

\[- dDA^2 + b_2(m + D)F^2 - 2mc_2F^4 - 2a_2D = 2(1 - D)(m - D)B^2, \]  
(266)

\[ a_2D^2 - b_2mDF^2 + c_2m^2F^4 = D(m - D)B^2. \]  
(267)

On solving these equations, we obtain the four unknowns \(A, F, B, D\) as well as two constraints between the seven parameters. We find

\[
A^2 = \frac{2B^2[b_2(D^2 - 2D - 2mD + 3m) - d(m + 2D - 2mD - D^2)]}{D(b_1b_2 - d^2)},
\]

\[
F^2 = \frac{2B^2[b_1(m + 2D - 2mD - D^2) - d(D^2 - 2D - 2mD + 3m)]}{(m - D)(b_1b_2 - d^2)},
\]

\[
B^2 = \frac{3D(1 - D)(m - D)(b_1b_2 - d^2)}{(D^2 - 2D - 2mD + 3m) - d(m + 2D - 2mD - D^2))^2},
\]

\[ a_1D + mdF^2 = [3m - (1 + m)D]B^2, \]  
(268)

while the two constraints are

\[
D^2c_1A^4 = (m - D)^2c_2F^4, \quad a_2D - mb_2F^2 = \frac{B^2[D^2 + mD(3m - 2) - 2m^2]}{B^2}. \]  
(269)

At \(m = 1\) the solution reduces to the dark-dark soliton solution (2.54) satisfying the relations (2.55) and (2.56).

**Special case of** \(b_1b_2 = d^2\)

One can show that the solution (2.61) exists even in the case \(b_1b_2 = d^2\). It turns out that such a solution exists only if

\[
\sqrt{b_2}[D^2 - 2(1 + m)D + 3m] = \sqrt{b_1}[m + 2D - 2mD - D^2]. \]  
(270)

At \(m = 1\) this implies the constraint (2.65).

**Energy:** Corresponding to the periodic solution (2.61), the energy \(\hat{E}\) and the constant \(C\) are given by (using appropriate integrals in [17])

\[
\hat{E} = (A^2I_4 + F^2I_5)B, \]

\[
C = \frac{m}{2} A^2B^2 - \frac{F^2}{4D} \left[ \frac{mB^2(2m - m^2 - D)}{(m - D)} - a_2 \right], \]  
(271)

36
where $I_3$ and $I_4$ are given by Eqs. (152) and (226), respectively.

On using the expansion formulas for $E(k), K(k)$ and $\Pi(D, k)$ around $k = 1$ derived above, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (254)] plus the interaction energy. We find

$$\hat{E} = E_{\text{kin}} + E_{\text{int}} = BA^2[(I_3^{(0)} + k'^2I_4^{(1)})] + BF^2[I_3^{(0)} + k'^2I_4^{(1)}],$$

(272)

where $I_3^{(0)} = I_2^{(0)}, I_3^{(1)}, I_4^{(0,1)}$ are given by Eqs. (136), (155), (229) and (240).

Note that this solution exists only when $b_1b_2 \geq d^2$. The interaction energy vanishes at $k = 1$, as it should.

**Interaction energy:** Summarizing, we thus have obtained twenty solutions of the coupled $\phi^6$ model. We have calculated the corresponding soliton and interaction energy in each case. If we carefully look at the interaction energy expression in each case, we find that except for the solution XI, for all other solutions, the soliton interaction energy (in the dilute gas or asymptotic limit [13]) has the leading term of the form $k'^2 \ln(4/k')$. Out of these nineteen solutions, solutions I, II, III, XII and XV have repulsive interaction energy while the solutions IV, V, VI, XIII, XVI and XIX have attractive interaction energy. However, for the solutions VII, VIII, IX, X, XIV and XX, the interaction energy can be attractive or repulsive, depending on the values of the various parameters of the model. Finally, for solution XI, the interaction energy (in the dilute gas or asymptotic limit) has the leading term of the form $k'^2$ and is repulsive. Since the lattice periodicity $r = 2nK(k)/B$ with $n = 1$ for dn based solutions, and $n = 2$ for sn and cn based solutions, using $K(k) \sim \ln(4/k')$ we find that $k'^2 = 4n^2 \exp(-rB/n)$. Thus the interaction is purely exponential for solution XI whereas it is of the form $k'^2 \ln(4/k') = (2rnB) \exp(-rB/n)$ for all other solutions.

### 3 Solutions of a Coupled $\phi^6$ Discrete Model

We shall now consider a discrete variant of the above continuum coupled $\phi^6$ model and obtain its static solutions. We start from the coupled static field Eqs. (2). The discrete analogue of these field equations...
has the form

\[
\begin{align*}
(1/h^2)[\phi_{n+1} + \phi_{n-1} - 2\phi_n] &= a_1 \phi_n - b_1 \phi_n^3 + c_1 \phi_n^5 + d\psi_n^2 \phi_n, \\
(1/h^2)[\psi_{n+1} + \psi_{n-1} - 2\psi_n] &= a_2 \psi_n - b_2 \psi_n^3 + c_1 \psi_n^5 + d\phi_n^2 \psi_n.
\end{align*}
\]

(273)

where \( h \) is the lattice spacing. We are unable to find any solution of these coupled equations. However, as in the Ablowitz-Ladik discretization of the nonlinear Schrödinger equation [18], if we replace \( \phi, \psi \) in the \( \phi^5, \psi^5 \) terms by the corresponding average, then we can find the exact solutions. Thus, we consider the coupled equations

\[
\begin{align*}
(1/h^2)[\phi_{n+1} + \phi_{n-1} - 2\phi_n] &= a_1 \phi_n - b_1 \phi_n^3 + \frac{c_1}{2} \phi_n^4[\phi_{n+1} + \phi_{n-1}] + d\psi_n^2 \phi_n, \\
(1/h^2)[\psi_{n+1} + \psi_{n-1} - 2\psi_n] &= a_2 \psi_n - b_2 \psi_n^3 + \frac{c_2}{2} \psi_n^4[\psi_{n+1} + \psi_{n-1}] + d\phi_n^2 \psi_n.
\end{align*}
\]

(274)

We shall show that while in the continuum case there are twenty periodic solutions, we are able to obtain only six periodic solutions in the discrete case. We shall also see that none of the discrete solutions has a smooth continuum limit, i.e. in the limit \( h \to 0 \), none of them goes over to the continuum solutions obtained in the last section.

### 3.1 Solution I

It is easy to show that

\[
\phi_n = A \sqrt{\text{sn}(hB[n+c], m)}, \quad \psi_n = F \sqrt{\text{sn}(hB[n+c], m)},
\]

(275)

is a solution to the above equations provided

\[
\begin{align*}
a_1 &= a_2 = (2/h^2)[\text{cn}(hB, m) \text{dn}(hB, m) - 1] < 0, \quad (d + b_1)A^2 = (d + b_2)F^2, \\
A^2 &= \frac{2(d + b_2)\text{sn}^2(hB, m)\text{cn}(hB, m)\text{dn}(hB, m)}{h^2(d^2 - b_1b_2)}, \\
c_1A^4 &= c_2F^4, \quad c_1a_1 = \frac{\text{cn}(hB, m)\text{dn}(hB, m) - 1 |(d^2 - b_1b_2)|^2}{\text{cn}^2(hB, m)\text{dn}^2(hB, m)(b_2 + d)^2}. \quad (276)
\end{align*}
\]
It is worth noting the completely different form of the solutions in the continuum and the discrete cases. We shall see that this is true for all the six solutions that we obtain in the discrete case. Also note that while in the continuum case \( b_1 b_2 > d^2 \), in the discrete case \( d^2 > b_1 b_2 \).

\[ m = 1 \text{ limit} \]

In the \( m = 1 \) limit, this solution goes over to the bright-bright topological solution

\[
\phi = A \tanh(hB[n + c]) , \quad \psi = F \tanh(hB[n + c]) ,
\]

provided

\[
a_1 = a_2 = -(2/h^2) \tanh^2(hB) < 0 , \quad A^2 = \frac{2(d + b_2) \tanh^2(hB) \sech^2(hB)}{h^2(d^2 - b_1 b_2)} ,
\]

\[
c_1 a_1 = - \frac{\tanh^2(hB)(d^2 - b_1 b_2)^2}{(b_2 + d)^2 \sech^4(hB, m)} ,
\]

while the other two relations are as given by Eq. (276). We note that this solution exists only if \( d^2 > b_1 b_2 \).

In the limit \( d = 0 \) (i.e. uncoupled case), the solution is given by

\[
\phi_n = A \sqrt{m} \text{sn}(hB[n + c], m) ,
\]

provided

\[
a_1 = (2/h^2)[\text{cn}(hB, m) \text{dn}(hB, m) - 1] < 0 , \quad A^2 = - \frac{2 \text{sn}^2(hB, m) \text{cn}(hB, m) \text{dn}(hB, m)}{h^2 b_1} ,
\]

\[
c_1 a_1 = \frac{[\text{cn}(hB, m) \text{dn}(hB, m) - 1]}{\text{cn}^2(hB, m) \text{dn}^2(hB, m)} .
\]

In the limit \( m = 1 \) the uncoupled solution is given by

\[
\phi_n = A \tanh(hB[n + c]) ,
\]

provided

\[
a_1 = - \frac{2 \tanh^2(hB)}{h^2} , \quad A^2 = - \frac{2 \tanh^2(hB) \sech^2(hB)}{h^2 b_1} , \quad c_1 a_1 = - \frac{\tanh^2(hB) \sech^2(hB)}{\sech^4(hB)} .
\]

Thus, such a solution exists provided \( a_1, b_1 < 0 \).
## 3.2 Solution II

We shall now obtain three periodic solutions, all of which in the limit $m \to 1$ go over to the same dark-dark soliton solution. The first one of these, i.e.

$$
\phi = A \sqrt{m \text{cn}(hB[n + c], m)}, \quad \psi = F \sqrt{m \text{cn}(hB[n + c], m)},
$$

is a solution to the above equations provided

$$
a_1 = a_2 = \left(\frac{2}{h^2}\right) \left[ \frac{\text{cn}(hB, m)}{\text{dn}^2(hB, m)} - 1 \right], \quad A^2 = \frac{2(d + b_2) \text{sn}^2(hB, m) \text{cn}(hB, m)}{h^2(b_1b_2 - d^2) \text{dn}^4(hB, m)},
$$

$$(d + b_1)A^2 = (d + b_2)F^2, \quad c_1A^4 = c_2F^4,
$$

$$
c_1a_1 = \left[ \frac{\text{cn}(hB, m)}{\text{dn}^2(hB, m)} - 1 \right] \frac{\text{dn}^4(hB, m)(b_1b_2 - d^2)^2}{\text{cn}^2(hB, m)(b_2 + d)^2}.
$$

Again notice the completely different form of the solutions in the continuum and the discrete cases. One can show that $a_1, a_2 > (\ <) 0$ provided $m > (\ <) 1/2$. Note that this solution exists only if $b_1b_2 > d^2$.

In the $m = 1$ limit, three of the solutions (i.e. solutions II, as well as solutions III and IV to be discussed below) go over to the nontopological soliton solution

$$
\phi_n = A \text{sech}(hB[n + c]), \quad \psi_n = F \text{sech}(hB[n + c]),
$$

provided

$$
a_1 = a_2 = \left(\frac{2}{h^2}\right) \left[ \text{cosh}(hB) - 1 \right] > 0, \quad A^2 = \frac{2(d + b_2) \text{sinh}^2(hB) \cosh(hB)}{h^2(d^2 - b_1b_2)},
$$

$$(d + b_1)A^2 = (d + b_2)F^2, \quad c_1A^4 = c_2F^4,
$$

$$
c_1a_1 = \left[ \text{cosh}(hB) - 1 \right] \frac{\text{sech}^2(hB)(b_1b_2 - d^2)^2}{(b_2 + d)^2}.
$$

In the limit $d = 0$ (i.e. the uncoupled case), the solution is given by

$$
\phi_n = A \sqrt{m \text{cn}(hB[n + c], m)},
$$

provided

$$
a_1 = \left(\frac{2}{h^2}\right) \left[ \frac{\text{cn}(hB, m)}{\text{dn}^2(hB, m)} - 1 \right], \quad A^2 = \frac{2\text{sn}^2(hB, m) \text{cn}(hB, m)}{h^2b_1 \text{dn}^4(hB, m)},
$$

$$
c_1a_1 = \left[ \frac{\text{cn}(hB, m)}{\text{dn}^2(hB, m)} - 1 \right] \frac{\text{dn}^4(hB, m)}{\text{cn}^2(hB, m)}.
$$
In the limit $m = 1$, the uncoupled solution takes the form

$$
\phi_n = \operatorname{Asech}(hB[n + c]), \quad (289)
$$

provided

$$
a_1 = (2/h^2) \left[ \cosh(hB) - 1 \right], \quad A^2 = \frac{2 \tanh^2(hB)}{h^2 b_1 \operatorname{sech}^3(hB)},
$$

$$
\frac{c_1 a_1}{b_1^2} = ([1 - \operatorname{sech}(hB)] \operatorname{sech}(hB)). \quad (290)
$$

Thus, the uncoupled solution exists only if $a_1, b_1 > 0$.

### 3.3 Solution III

It is easy to show that another periodic solution is given by

$$
\phi = A \operatorname{dn}(hB[n + c], m), \quad \psi = F \operatorname{dn}(hF[n + c], m), \quad (291)
$$

provided

$$
a_1 = a_2 = (2/h^2) \left[ \frac{\operatorname{dn}(hB, m)}{\operatorname{cn}^2(hB, m)} - 1 \right] > 0, \quad A^2 = \frac{2(d + b_2) \operatorname{sn}^2(hB, m) \operatorname{dn}(hB, m)}{h^2(b_1 b_2 - d^2) \operatorname{cn}^4(hB, m)},
$$

$$(d + b_1)A^2 = (d + b_2)F^2, \quad c_1 A^4 = c_2 F^4,
$$

$$
c_1 a_1 = \left[ \frac{\operatorname{dn}(hB, m)}{\operatorname{cn}^2(hB, m)} - 1 \right] \frac{\operatorname{cn}^4(hB, m)(b_1 b_2 - d^2)^2}{\operatorname{dn}^2(hB, m)(b_2 + d)^2}. \quad (292)
$$

Note that as in the continuum case, here too $a_1, a_2 > 0$. Further, this solution exists only if $b_1 b_2 > d^2$.

Finally, in the limit $m = 1$, this solution goes over to the solution (285) satisfying the constraints (286).

In the limit $d = 0$ (i.e. the uncoupled case), the solution is given by

$$
\phi = A \operatorname{dn}(hB[n + c], m), \quad (293)
$$

provided

$$
a_1 = (2/h^2) \left[ \frac{\operatorname{dn}(hB, m)}{\operatorname{cn}^2(hB, m)} - 1 \right], \quad A^2 = \frac{2 \operatorname{sn}^2(hB, m) \operatorname{dn}(hB, m)}{h^2 b_1 \operatorname{cn}^3(hB, m)},
$$

$$
\frac{c_1 a_1}{b_1^2} = \left[ \frac{\operatorname{dn}(hB, m)}{\operatorname{cn}^2(hB, m)} - 1 \right] \frac{\operatorname{cn}^4(hB, m)}{\operatorname{dn}^2(hB, m)}. \quad (294)
$$

In the limit $m = 1$, the uncoupled solution goes over to the solution (289) satisfying relations (290).
3.4 Solution IV

It is easy to show that yet another periodic solution is given by

\[ \phi = A \sqrt{m} \mathrm{cn}(hB[n + c], m), \quad \psi = F \mathrm{dn}(hB[n + c], m), \quad (295) \]

provided

\[ a_1 = \left( \frac{2}{h^2} \right) \left[ \frac{\mathrm{cn}(hB, m)}{\mathrm{dn}^2(hB, m)} - 1 \right], \quad a_2 = \left( \frac{2}{h^2} \right) \left[ \frac{\mathrm{dn}(hB, m)}{\mathrm{cn}^2(hB, m)} - 1 \right] > 0, \]

\[ A^2 = \frac{2[ddn^5(hB, m) + b_2 \mathrm{cn}^5(hB, m)] \mathrm{sn}^2(hB, m)}{h^2(b_1 b_2 - d^2) \mathrm{cn}^4(hB, m) \mathrm{dn}^4(hB, m)}, \]

\[ F^2 = \frac{2[b_1 \mathrm{dn}^5(hB, m) + d \mathrm{cn}^5(hB, m)] \mathrm{sn}^2(hB, m)}{h^2(b_1 b_2 - d^2) \mathrm{cn}^4(hB, m) \mathrm{dn}^4(hB, m)}, \]

\[ c_1 A^4 = \frac{2\mathrm{sn}^4(hB, m)}{h^2 \mathrm{dn}^4(hB, m)}, \quad c_2 F^4 = \frac{2\mathrm{sn}^4(hB, m)}{h^2 \mathrm{cn}^4(hB, m)} . \quad (296) \]

Note that \( a_2 > 0 \) while \( a_1 > (\langle \rangle) 0 \) depending on if \( m > (\langle \rangle) 1/2. \)

Again note that like the previous two solutions, this one also exists only if \( b_1 b_2 > d^2. \) As mentioned above, in the limit \( m = 1, \) this solution also goes over to the solution (285) satisfying the constraints (286). Further, in the decoupled case (i.e. \( d = 0, \)) \( \phi_n \) and \( \psi_n \) satisfy solutions (287) and (293), respectively, with the appropriate constraints.

3.5 Solution V

Finally, there are two periodic solutions, both of which at \( m = 1 \) go over to the (same) bright-dark soliton solution. The first one is given by

\[ \phi = A \sqrt{m} \mathrm{sn}(hB[n + c], m), \quad \psi = F \sqrt{m} \mathrm{cn}(hB[n + c], m), \quad (297) \]

provided \( d < 0 \) and further

\[ a_1 - m|d|F^2 = \left( \frac{2}{h^2} \right) [\mathrm{cn}(hB, m) \mathrm{dn}(hB, m) - 1] < 0, \]

\[ a_2 - m|d|A^2 = \left( \frac{2}{h^2} \right) \left[ \frac{\mathrm{cn}(hB, m)}{\mathrm{dn}^2(hB, m)} - 1 \right] , \]

\[ A^2 = \frac{2[|d| - b_2 \mathrm{dn}^5(hB, m)] \mathrm{sn}^2(hB, m) \mathrm{cn}(hB, m)}{h^2(b_1 b_2 - d^2) \mathrm{dn}^4(hB, m)}, \]
\[ F^2 = \frac{2[b_1 - |d[dn^5(h\beta, m)]|sn^2(h\beta, m)cn(h\beta, m)]}{h^2(b_1b_2 - d^2)dn^4(hB, m)}, \]
\[ c_1A^4 = \frac{2sn^4(hB, m)}{h^2}, \quad c_2F^4 = \frac{2sn^4(hB, m)}{h^2dn^4(hB, m)}. \] (298)

Note that this solution exists only if \( b_1b_2 > d^2 \). In the decoupled limit (i.e. \( d = 0 \)), the \( \phi_n \) and \( \psi_n \) fields go over to the solutions (279) and (287), respectively, satisfying appropriate constraints.

### 3.6 Solution VI

Finally, the second solution is given by
\[ \phi = A\sqrt{msn(hB[n + c], m)}, \quad \psi = Fdn(hB[n + c], m), \] (299)
provided \( d < 0 \) and further
\[ a_1 - |d|F^2 = \frac{(2/h^2)[cn(hB, m)dn(hB, m) - 1]}{0}, \]
\[ a_2 - |d|A^2 = \frac{(2/h^2)[dn(hB, m)]}{0}, \]
\[ A^2 = \frac{2[|d| - b_2cn^5(hB, m)|sn^2(hB, m)dn(hB, m)]}{h^2(b_1b_2 - d^2)cn^4(hB, m)}, \]
\[ F^2 = \frac{2[b_1 - |d|cn^5(hB, m)]sn^2(hB, m)dn(hB, m)}{h^2(b_1b_2 - d^2)cn^4(hB, m)}, \]
\[ c_1A^4 = \frac{2sn^4(hB, m)}{h^2}, \quad c_2F^4 = \frac{2sn^4(hB, m)}{h^2cn^4(hB, m)}. \] (300)

Note that \( a_2 > 0 \).

In the limit \( m = 1 \), both of these solutions (i.e. solutions V and VI) go over to the bright-dark soliton solution
\[ \phi = A\tanh(hB[n + c]), \quad \psi = Fsech(hB[n + c]), \] (301)
provided \( d < 0 \) and further
\[ a_1 - |d|F^2 = \frac{-(2/h^2)\tanh^2(hB)}{0}, \quad a_2 - |d|A^2 = \frac{(2/h^2)[cosh(hB) - 1]}{0}, \]
\[ A^2 = \frac{2[|d| - b_2\sech^5(hB)]\tanh^2(hB)}{h^2(b_1b_2 - d^2)\sech^3(hB, m)}, \]
\[ F^2 = \frac{2[b_1 - |d|\sech^5(hB)]\tanh^2(hB)}{h^2(b_1b_2 - d^2)\sech^3(hB)}, \]
\[ c_1A^4 = \frac{2\tanh^4(hB)}{h^2}, \quad c_2F^4 = \frac{2\sin^4(hB)}{h^2}. \] (302)
Note that this solution exists only if \( b_1 b_2 > d^2 \). In the decoupled limit (i.e. \( d = 0 \)), the \( \phi_n \) and the \( \psi_n \) fields go over to the solutions (279) and (293), respectively, satisfying appropriate constraints.

As in the continuum case, can one say if these discrete solutions are valid below, above or at \( T_c \)? There is a note of caution here. Since we are considering a discrete model here, and since the corresponding Hamiltonian from which the coupled discrete field Eqs. (274) can be derived is not known, strictly speaking we cannot directly associate \( a_1, b_1 \) and other parameters of the discrete problem and draw conclusions about the nature of the transition by using the corresponding continuum model potential (1).

However, as a first guess, it might be reasonable to assume that for the same values of the ratio of the parameters (i.e. \( a_1 c_1/b_1^2 \)) as in the continuum case, one can classify if the solution is below, above or at \( T_c \). For example, since solution I exists if \( a_1, b_1 < 0 \) (\( c_1 \) is always assumed to be positive), this implies that this solution exists with \( T < T_{II}^c \). It is worth reminding that in case \( b_1 < 0 \), then the uncoupled \( \phi^6 \) model characterized by the potential (1) corresponds to a second order (and not a first order) transition. Similarly, since for solutions II, III and IV, \( b_1, a_1 > 0 \) and \( b_1^2 \geq 4a_1 c_1 \), this implies that \( T < T_{II}^p \). Finally, solutions V and VI correspond to mixed-type solutions with \( \phi_n \) field corresponds to \( T < T_{II}^c \) while \( \psi_n \) field corresponds to \( T < T_{II}^p \).

Since we do not know the Hamiltonian associated with Eqs. (274), we are unable to find the soliton and interaction energy explicitly for these discrete solutions and hence cannot comment whether the Peierls-Nabarro barrier is zero or non-zero.

4 Conclusion

We have provided a set of twenty distinct exact, periodic domain wall solutions for a coupled \( \phi^6 \) model with biquadratic coupling. These in turn lead to nine distinct (hyperbolic) soliton solutions both at, above and below \( T_c \). The corresponding discrete case was also considered and we found six different periodic solutions which in turn lead to three distinct (hyperbolic) soliton solutions. The soliton and interaction energy were obtained for all twenty continuum solutions. For the six solutions of the discrete model, the calculation of the Peierls-Nabarro barrier [20, 21, 22] and soliton scattering [23] are important topics of further study. Similarly, scattering of solitons for the twenty solutions in the coupled \( \phi^6 \) continuum model is an interesting
issue with these static solutions boosted with a certain velocity. We also discovered previously unknown periodic solutions of the uncoupled $\phi^6$ model \cite{11, 12, 13}.

It would be useful to explore whether the different solutions are completely disjoint or if there are any possible bifurcations linking them via, for instance, analytical continuation. We have not carried out an explicit stability analysis of various periodic solutions \cite{24}. However, the energy calculations and interaction energy between solitons (for $m \sim 1$) could provide useful insight in this direction. Similarly, it would be worth exploring the problem of a coupled $\phi^6$ model in the presence of an external field. Our results are relevant for domain walls in structural phase transitions in ferroelectrics and elastic materials \cite{5, 9, 10, 12, 13, 19} and possibly in field theoretic contexts \cite{6, 7, 8}. These ideas and solutions can be generalized to other coupled models for first order transitions such as coupled asymmetric double well potentials ($\phi^2-\phi^3-\phi^4$) and will be discussed elsewhere.

5 Acknowledgment

A.K. acknowledges the hospitality of the Center for Nonlinear studies at LANL. This work was supported in part by the U.S. Department of Energy.

References

[1] A. Khare and A. Saxena, J. Math. Phys. 47, (2006); nlin.SI/0608047

[2] S. Aubry and R. Pick, Ferroelectrics 8, 471 (1973).

[3] T. Abel and R. Siems, Ferroelectrics 153, 177 (1994).

[4] A. A. Kornyshev, D. A. Kossakowski, and S. Leikin, J. Chem. Phys. 97, 6809 (1992).

[5] A. Saxena, G. R. Barsch, and D. M. Hatch, Phase Trans. 46, 89 (1994).

[6] D. Bazeia, M. J. dos Santos, and R. F. Ribeiro, Phys. Lett. A 208, 84 (1995).

[7] D. Bazeia, R. F. Ribeiro, and M. M. Santos, Phys. Rev. E 54, 2943 (1996).
[8] S.-Y. Lou, J. Phys. A 32, 4521 (1999).

[9] D. M. Hatch and H. T. Stokes, Phys. Rev. B 65, 014113 (2001).

[10] D. M. Hatch, T. Lookman, A. Saxena, and S. R. Shenoy, Phys. Rev. B 68, 104105 (2003).

[11] S. N. Behera and A. Khare, Pramana 15, 245 (1980).

[12] F. Falk, Z. Phys. B 51, 177 (1983).

[13] M. Sanati and A. Saxena, J. Phys. A 32, 4311 (1999).

[14] T. Kimura, T. Goto, H. Shintani, K. Ishizaka, T. Arima, and Y. Tokura, Nature 426, 55 (2003).

[15] M. Fiebig, Th. Lottermoser, and R. V. Pisarev, J. Appl. Phys. 93, 8194 (2003).

[16] I. S. Gradshteyn and I. M. Ryzhyk, Tables of Integrals, Series and Products (Academic, San Diego, 1994).

[17] P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists (Springer-Verlag, Berlin, 1954).

[18] M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 16, 598 (1975); ibid 17, 1011 (1976).

[19] A. N. Das and B. Ghosh, J. Phys. C 16, 1803 (1983).

[20] F. R. N. Nabarro, Theory of Crystal Dislocations, (Dover, New York, 1987).

[21] O. M. Braun and Yu. S. Kivshar, Phys. Rev. B 43, 1060 (1991).

[22] P. G. Kevrekidis, Physica D 183, 68 (2003).

[23] S. V. Dmitriev, P. G. Kevrekidis, B. A. Malomed, and D. J. Frantzeskakis, Phys. Rev. E 68, 056603 (2003).

[24] F. Zimmerschied and H. J. W. Müller-Kirsten, Phys. Rev. D 49, 5387 (1994).