MORDELL-WEIL LATTICE OF INOSE’S ELLIPTIC K3 SURFACE ARISING FROM THE PRODUCT OF 3-ISOGENOUS ELLIPTIC CURVES

MASATO KUWATA AND KAZUKI UTSUMI

Abstract. From the product of two elliptic curves, Shioda and Inose [6] constructed an elliptic K3 surface having two II* fibers. Its Mordell-Weil lattice structure depends on the morphisms between the two elliptic curves. In this paper, we give a method of writing down generators of the Mordell-Weil lattice of such elliptic surfaces when two elliptic curves are 3-isogenous. In particular, we obtain a basis of the Mordell-Weil lattice for the singular K3 surfaces $X_{[3,3,3]}$, $X_{[3,2,3]}$ and $X_{[3,0,3]}$.

1. Introduction

In the study of the geometry, arithmetic and moduli of K3 surfaces, elliptic K3 surfaces with large Picard number play a vital role. In 1977 Shioda and Inose [6] gave a classification of singular K3 surfaces, that is, K3 surfaces with maximum Picard number. For this purpose, they constructed elliptic K3 surfaces $E$ with two singular fibers of type II* starting from the Kummer surface $\text{Km}(E_1 \times E_2)$ with the product of two elliptic curves $E_1$ and $E_2$. They constructed $E$ as a double cover of $\text{Km}(E_1 \times E_2)$ with certain properties (now called a Shioda-Inose structure). Later, Inose [1] gave an explicit model of such an elliptic K3 surface as a quartic surface in $\mathbb{P}^3$, and remarked that it is the quotient of $\text{Km}(E_1 \times E_2)$ by an involution. We call the Kodaira-Néron model of $E$ the Inose surface associated with $E_1$ and $E_2$, and denote it by $\text{Ino}(E_1, E_2)$. We thus have a “Kummer sandwich” diagram:

$\begin{align*}
\text{Km}(E_1 \times E_2) & \xrightarrow{\pi_2} \text{Ino}(E_1, E_2) \xrightarrow{\pi_1} \text{Km}(E_1 \times E_2)
\end{align*}$

(cf. [7]). Also, $E$ as an elliptic surface with two II* fibers is denoted by $F^{(1)}_{E_1, E_2}$. This notation reflects that it is a part of the construction of elliptic K3 surfaces of high rank by the first named author [4], where he constructed $F^{(n)}_{E_1, E_2}$, $n = 1, \ldots, 6$, which has various Mordell-Weil rank up to 18.

The structure of the Mordell-Weil lattice of $F^{(1)}_{E_1, E_2}$ is known to be isomorphic to $\text{Hom}(E_1, E_2)(2)$ if $E_1$ and $E_2$ are nonisomorphic (see [8]). Here, for a lattice $L$, we denote by $L(n)$ the lattice structure on $L$ with the pairing multiplied by $n$. However, given an isogeny $\varphi \in \text{Hom}(E_1, E_2)$ and the Weierstrass equation of $F^{(1)}_{E_1, E_2}$, it is quite difficult to write down the coordinates of the section corresponding to $\varphi$, and it has been worked out only in limited cases (cf. [2, 3]). Most known examples fall into the case where the degree of isogeny $\deg \varphi$ equals 2, in which case the calculations are straightforward. One particular example of the case $\deg \varphi = 4$ is dealt in [2, Example 9.2]. In this paper we consider a family of the pairs of elliptic curves $E_1$ and $E_2$ with an isogeny $\varphi : E_1 \rightarrow E_2$ of degree 3 defined over $k$. We write down a formula of the section of $F^{(1)}_{E_1, E_2}$ coming from $\varphi$ defined over the base field $k$. To do so, we first work with the surface $F^{(6)}_{E_1, E_2}$, which has a simple affine model that can

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be viewed as a family of cubic curves with a rational point over \( k \). We modify the method in [2] to find sections of \( F_{E_1,E_2}^{(1)} \). We also give a section of \( F_{E_1,E_2}^{(2)} \) coming from the isogeny \( \varphi \), and give a basis defined over the field \( k(E_1[2],E_2[2]) \) when \( E_1 \) and \( E_2 \) do not have a complex multiplication.

In [7] we study some examples of singular \( K3 \) surfaces in detail. In particular, we determine a basis of the MWL of the Inose surface \( F_{E_1,E_2}^{(1)} \) and that of \( F_{E_1,E_2}^{(2)} \) for the singular \( K3 \) surfaces \( X_{[3,3,3]} \), \( X_{[3,2,3]} \) and \( X_{[3,0,3]} \) which correspond to the quadratic forms \( 3x^2 + 3xy + 3y^2, 3x^2 + 2xy + 3y^2, \) and \( 3x^2 + 3y^2 \) respectively.

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### 2. Inose Surface

Throughout this paper the base field \( k \) of algebraic varieties is assumed to be a number field.

Let \( \text{Km}(E_1 \times E_2) \) be the Kummer surface associated with the product of elliptic curves \( E_1 \) and \( E_2 \), that is, the minimal resolution of the quotient surface \( E_1 \times E_2 / \{ \pm 1 \} \). If \( E_1 \) and \( E_2 \) are defined by the equations

\[
E_1 : y_1^2 = f_1(x_1), \quad E_2 : y_2^2 = f_2(x_2),
\]

where \( f_1(x_1) \) and \( f_2(x_2) \) are cubic polynomials, an affine singular model of \( \text{Km}(E_1 \times E_2) \) is given as the hypersurface in \( \mathbb{A}^3 \) defined by the equation

\[
f_2(x_2) = t^2f_1(x_1),
\]

where \( t = y_2/y_1 \). Then, the map \( \text{Km}(E_1 \times E_2) \to \mathbb{P}^1 \) induced by \( (x_1,x_2,t) \mapsto t \) is a Jacobian fibration, which is sometimes called Inose’s pencil (cf. [3]).

Take a parameter \( u \) such that \( t = u^3 \), and consider (2.1) as a family of cubic curves in \( \mathbb{P}^2 = \{(x_1 : x_2 : z)\} \), or a cubic curve over \( k(u) \). Then, we see that it has a rational point \( (x_1 : x_2 : z) = (1 : u^2 : 0) \). Using this point as the origin, we consider it as an elliptic curve over \( k(u) \). In particular, if \( f_1(x_1) \) and \( f_2(x_2) \) are given by

\[
f_1(x_1) = x_1^3 + a_2x_1^2 + a_4x_1 + a_6, \quad \text{and} \quad f_2(x_2) = x_2^3 + a_2'x_2^2 + a_4'x_2 + a_6',
\]

then we can convert (2.1) to the Weierstrass form

\[
Y^2 = X^3 - \frac{1}{3}AX + \frac{1}{64}\left( \Delta_{E_1}u^6 + B + \frac{\Delta_{E_2}}{u^6} \right),
\]

where

\[
\left\{
\begin{aligned}
A &= (a_2^2 - 3a_4)(a_2'^2 - 3a_4'), \\
B &= \frac{32}{27}(2a_2^3 - 9a_2a_4 + 27a_6)(2a_2'^3 - 9a_2'a_4' + 27a_6'), \\
\Delta_{E_1} &= 16(a_2^2a_4^2 - 4a_2a_4a_6 + 18a_2a_4a_6 - 4a_4^3 - 27a_6^2), \\
\Delta_{E_2} &= 16(a_2'^2a_4'^2 - 4a_2'a_4'a_6' + 18a_2'a_4'a_6' - 4a_4'^3 - 27a_6'^2).
\end{aligned}
\right.
\]
Let \( s = t^2 = u^6 \). Define \( F_{E_1E_2}^{(1)} \) to be
\[
(2.2) \quad F_{E_1E_2}^{(1)} : Y^2 = X^3 - \frac{1}{3} A X + \frac{1}{64} \left( \Delta_{E_1}s + B + \frac{\Delta_{E_2}}{s} \right).
\]
This Jacobian fibration has two reducible fibers of type \( \Pi^* \) at \( s = 0 \) and \( s = \infty \).

**Definition 2.1** (cf. [2]). The Kodaira-Néron model of the elliptic surface \( F_{E_1E_2}^{(1)} \) over \( k \) defined by (2.2) is called the Inose surface associated with \( E_1 \) and \( E_2 \), and it is denoted by \( \text{Ino}(E_1, E_2) \).

**Remark 2.2.** Over a certain extension of \( k \), the equation of \( F_{E_1E_2}^{(1)} \) may be given by
\[
F_{E_1E_2}^{(1)} : Y^2 = X^3 - 3\sqrt{J_1J_2} X + s + \frac{1}{s} - 2\sqrt{(1-J_1)(1-J_2)},
\]
where \( J_i = j(E_i)/1728 \) (\( i = 1,2 \)) with \( j(E_i) \) the \( j \)-invariant of \( E_i \) (cf. [1], [8]).

**Definition 2.3.** For \( n \geq 1 \), the elliptic surface \( F_{E_1E_2}^{(n)} \) over \( k \) is defined by
\[
(2.3) \quad F_{E_1E_2}^{(n)} : Y^2 = X^3 - \frac{1}{3} A X + \frac{1}{64} \left( \Delta_{E_1}s^n + B + \frac{\Delta_{E_2}}{s^n} \right).
\]

**Remark 2.4.** (1) The Kodaira-Néron model of \( F_{E_1E_2}^{(n)} \) is a \( K3 \) surface for \( n = 1, \ldots, 6 \), but not for \( n \geq 7 \) ([4]).

(2) Since \( u^6 = t^2 \), Inose’s pencil on \( \text{Km}(E_1 \times E_2) \) is isomorphic to \( F_{E_1E_2}^{(2)} \).

However, the isomorphism between (2.1) and (2.3) for \( n = 2 \) may not be defined over \( k \) itself.

3. Mordell-Weil Lattice of the Inose Surface

In this section we give a summary of known facts on the Mordell-Weil lattice of \( F_{E_1E_2}^{(1)} \) over \( k(s) \).

**Theorem 3.1.** ([8] Theorem 6.3) The Jacobian fibration \( F_{E_1E_2}^{(1)} \) on \( \text{Ino}(E_1, E_2) \) has two singular fibers of type \( \Pi^* \) at \( s = 0 \) and \( \infty \), and the other singular fibers and its Mordell-Weil rank are given in the table below. Here, \( j_i = j(E_i), \) \( i = 1,2, \) are the \( j \)-invariants, and \( h \) is the rank of \( \text{Hom}_k(E_1, E_2) \).

| \( j \)-invariants | singular fibers | Mordell-Weil rank |
|---------------------|-----------------|------------------|
| \( j_1 \neq j_2, j_1j_2 \neq 0 \) | \( 4I_1 \) | \( h \) |
| \( j_1 \neq j_2, j_1j_2 = 0 \) | \( 2\Pi \) | \( h \) |
| \( j_1 = j_2 \neq 0, 1728 \) | \( 4I_1 \) | \( h - 1 \) |
| \( j_1 = j_2 = 1728 \) | \( 2I_2 \) | \( 0 \) |
| \( j_1 = j_2 = 0 \) | \( 4I_2 \) | \( 0 \) |

Assume \( j_1 \neq j_2 \). Then, the Mordell-Weil group \( F_{E_1E_2}^{(1)}(\overline{\mathbb{Q}}(s)) \) is torsion-free, and isomorphic to the lattice \( \text{Hom}_k(E_1, E_2)(2) \), where the paring of \( \text{Hom}_k(E_1, E_2) \) is given by
\[
(\varphi, \psi) = \frac{1}{2} (\deg(\varphi + \psi) - \deg \varphi - \deg \psi) \quad \varphi, \psi \in \text{Hom}_k(E_1, E_2).
\]

**Remark 3.2.** The notation \( \langle n \rangle \) means that the paring of the lattice is multiplied by \( n \).
4. 3-ISOGENIES OF ELLIPTIC CURVES

We recall some general facts on 3-isogenies between elliptic curves following J. Top [10, §3].

Lemma 4.1. Let $E$ be an elliptic curve over a number field $k$, and $G \subset E(\bar{k})$ a subgroup of order three that is stable under the action of $\text{Gal}(\bar{k}/k)$. Then, the pair of $E$ and $G$ is one of the following:

(i) $E$ is given by $y^2 = x^3 + d$ and $G$ is generated by $P = (0, \sqrt{d})$.
(ii) $E$ is given by $y^2 = x^3 + a(x-b)^2$ and $G$ is generated by $P = (0, b\sqrt{a})$.

Proof. $E$ can be given by an equation of the form $y^2 = f(x)$ with $\deg f = 3$. Then, $G$ consists of three points $G = \{0 : 1 : 0\}$, $P = (\alpha, \beta)$, and $2P = -P = (\alpha, -\beta)$ with $\beta \neq 0$. Since $G$ is Galois invariant, we have $\alpha \in k$ and $\beta^2 \in k$. Replacing $x$ by $x + \alpha$ if necessary, we may assume $\alpha = 0$. The curve is now given by an equation $y^2 = x^3 + ax^2 + cx + d$, and $P = (0, \sqrt{d})$. The tangent line at $P$ is given by $y = cx/(2\sqrt{d}) + \sqrt{d}$, and this tangent line intersects with $E$ at $P$ with multiplicity $3$ if and only if $c^2 = 4ad$. If $c = 0$, our equation is $y^2 = x^3 + d$. If $c \neq 0$, the equation can be written as $y^2 = x^3 + a(x-b)^2$, and $P = (0, b\sqrt{a})$. \hfill \Box

If $E$ is given by $y^2 = x^3 + d$ and $P = (0, \sqrt{d})$, then the quotient $E/G$ is given by the equation

$$E/G : y^2 = x^3 - 27d,$$

and the quotient map $\varphi : E \to E/G$ is given by

$$\varphi : (x_1, y_1) \mapsto (x_2, y_2) = \left( \frac{x_1^3 + 4d}{x_1^2}, \frac{x_1^3 - 8d}{x_1^2} \right).$$

In this case $E$ and $E/G$ are isomorphic over $\bar{k}$, and the $j$-invariants are equal to 0.

If $E$ is given by $E : y^2 = x^3 + a(x-b)^2$, and $P = (0, b\sqrt{a})$, then the quotient $E/G$ is given by

$$E/G : y^2 = x^3 - 3a(x_2 - (4a + 27b)/9)^2.$$

The isogeny $\varphi : E \to E/G$ is given by $\varphi(x_1, y_1) = (\varphi_x(x_1), \varphi_y(x_1)y_1)$, where

$$\varphi(x_1) = \frac{3x_1^4 + 4ax_1^2 - 12abx_1 + 12ab^2}{3x_1^4}, \quad \varphi_y(x_1) = \frac{x_1^3 + 4abx_1 - 8ab^2}{x_1^3}.$$

5. THE RATIONAL SECTION OF $F^{(1)}_{E_1, E_2}$ ARISING FROM A 3-ISORENY

In this section, we assume $E_1$ and $E_2$ are 3-isogenous over $k$, and we find an explicit section of $F^{(1)}_{E_1, E_2}$.

In the case where the $j$-invariants of both $E_1$ and $E_2$ are equal to 0, the Mordell-Weil group $F^{(1)}_{E_1, E_2}(k(s))$ is trivial by Theorem 4.1 and we have nothing to do.

As for the second case in §4, suppose that $E_1$ and $E_2$ are given by

$$E_1 : y^2 = x^3 + a(x_1 - b)^2,$$
$$E_2 : y^2 = x^3 + a'(x_2 - b')^2,$$

where $a, b \in k, a' = -3a$, and $b' = (4a + 27b)/9$. We work with the cubic curve over $k(u)$ in $\mathbf{P}^2 = \{(x_1 : x_2 : z)\}$ given by

$$C_u : x_2^3 + a'(x_2 - b'z)^2 z = u^6(x_1^3 + a(x_1 - bz)^2z).$$
which is isomorphic over \( k(u) \) to \( F_{E_1, E_2}^{(0)} \) with the choice of origin \( O = (1 : u^2 : 0) \). Its Weierstrass equation is given by

\[
F_{E_1, E_2}^{(0)} : Y^2 = X^3 - \frac{1}{3}aa'(a + 6b)(a' + 6b')X - \frac{1}{4}(a^2b^3(4a + 27b)u^6 + a'^2b'^3(4a' + 27b'))u^6 + \frac{1}{54}aa'(3(a + 3b)^2 - a^2)(3(a' + 3b')^2 - a'^2).
\]

The change of coordinates are given by

\[
\begin{align*}
X &= \frac{c_6 u^6 + c_4 u^4 + c_2 u^2 + c_0}{3a^2((3x_1 + az)^2 - (3x_2 + a'z))^2}, \\
Y &= \frac{d_{10} u^{10} + d_6 u^6 + d_4 u^4 + d_0}{6a^3((3x_1 + az)^2 - (3x_2 + a'z))^2},
\end{align*}
\]

where

\[
\begin{align*}
c_6 &= 2a(a + 6b)(3x_1 + az) - a(3(a + 3b)^2 - a^2)z, \\
c_4 &= a(a + 6b)(3x_2 + a'z), \\
c_2 &= -a'(a' + 6b')(3x_1 + az), \\
c_0 &= 2a'(a' + 6b')(3x_2 + a'z) + a'(3(a' + 3b')^2 - a'^2)z, \\
d_{10} &= -a(3(a + 3b)^2 - a^2)((3x_1 + az)^2 + 2a(a + 6b)z^2) + 6a^2(a + 6b)^2(3x_1 + az)z, \\
d_6 &= a(3(a + 3b)^2 - a^2)((3x_2 + a'z)^2 + 2a'(a' + 6b')z^2) - 6aa'(a + 6b)(a' + 6b')(3x_1 + az)z, \\
d_4 &= a'(3(a' + 3b')^2 - a'^2)((3x_1 + az)^2 + 2a(a + 6b)z^2) - 6aa'(a + 6b)(a' + 6b')(3x_2 + a'z)z, \\
d_0 &= -a'(3(a' + 3b')^2 - a'^2)((3x_2 + a'z)^2 + 2a'(a' + 6b')z^2) + 6a^2(a' + 6b')^2(3x_2 + a'z)z.
\end{align*}
\]

**Remark 5.1.** The origin \( O \) in our case is not an inflection point of the cubic curve \( C_u \). Thus, three collinear points \( P, Q, R \in C_u \) do not satisfy the equation \( P + Q + R = O \) under the group law. Instead, we have \( P + Q + R = \overline{O} \), where \( \overline{O} \) is the third point of intersection between \( C_u \) and the tangent line at \( O \).

**Remark 5.2.** By definition, the surface \( F_{E_1, E_2}^{(1)} \) is obtained as the quotient of \( 5.3 \) by the automorphism \( (X, Y, u) \mapsto (X, Y, -\omega u) \), where \( \omega \) is a third root of unity. It should be noted that the automorphism \( (x_1 : x_2 : z, u) \mapsto (x_1 : x_2 : z, -\omega u) \) on \( C_u \) does not correspond to this automorphism since the quotient by the latter gives a rational surface.

Replacing \( x_2 \) in \( 5.2 \) by \( \varphi(x_1) \) given by \( 4.1 \), we obtain

\[
\frac{1}{x_1^4}(x_1^3 + a(x_1z - bz)^2z)((x_1^3 + 4abx_1^2z^2 - 8ab^2z^3)^2 - (u^3x_1^3)^2) = 0.
\]

Define two homogeneous polynomials in \( x_1, z \) by

\[
p_x^+(x_1, z) = x_1^3 + 4abx_1z^2 - 8ab^2z^3 - u^3x_1^3,
\]
\[
p_x^-(x_1, z) = x_1^3 + 4abx_1z^2 - 8ab^2z^3 + u^3x_1^3.
\]
Let $Q$, $O$ are the three roots of $p$ which passes through $Q$ conic $Q$ (see Figure 1).

The condition that $l$ and $\ell$ divided by the coefficients of $q$ if $m$ homogeneous linear equations in $Q$ counting multiplicity. It is not difficult to find the sixth point $p$ by the equation $q$ if $m$.

Proposition 5.3. Let $D^+_\varphi$ (resp. $D^-_\varphi$) be the divisor on the cubic curve $C_u$ defined by the equation $p^\pm_\varphi(x_1, z) = 0$ (resp. $p^-_\varphi(x_1, z) = 0$).
(i) The divisor $D_\varphi^+$ (resp. $D_\varphi^-$) determines a $k$-rational point $P_\varphi^+$ (resp. $P_\varphi^-$) in $F_{E_1,E_2}^{(6)}(k(u))$.

(ii) $P_\varphi^+ - P_\varphi^-$ is in the image of $F_{E_1,E_2}^{(6)}(k(s)) \to F_{E_1,E_2}^{(6)}(k(u))$ induced by $s \mapsto u^6$.

The height of its pre-image in $F_{E_1,E_2}^{(6)}(k(s))$ is 6.

Proof. Since $D_\varphi^+$ and $D_\varphi^-$ are both defined over $k(u)$, $Q_4^+$ and $Q_\bar{4}$ are $k(u)$-rational points on $C_u$. Let $\Psi_u : C_u \to F_{E_1,E_2}^{(6)}$ be the isomorphism over $k(u)$ defined by the formula \([5,4]\), and $P_\varphi^+$ (resp. $P_\varphi^-$) be the point in $F_{E_1,E_2}^{(6)}(k(u))$ given by $\Psi_u(Q_4^+)$ (resp. $\Psi_u(Q_\bar{4})$). Let $\sigma$ be the automorphism of $k(u)$ defined by $u \mapsto -\omega u$. It induces an automorphism of $C_u(k(u))$ and that of $F_{E_1,E_2}^{(6)}(k(u))$. We show that the $(k(u)$-rational point $P_\varphi^+ - P_\varphi^- \in F_{E_1,E_2}^{(6)}(k(u))$ is invariant under $\sigma$. This proves that $P_\varphi^+ - P_\varphi^-$ belongs to the image of $F_{E_1,E_2}^{(6)}(k(s))$ under the map $s \mapsto u^6$.

First, consider the automorphism $\sigma^4 : u \mapsto u$. The explicit conversion formula \([5,4]\) shows that $\sigma^4(\Psi_u(Q)) = \Psi_{-u}(\sigma^4(Q)) = -\Psi_u(\sigma^4(Q))$ for $Q \in C_u(k(u))$. Since $\sigma^4$ exchanges $Q_4^+$ and $Q_\bar{4}$ by definition, we have $\sigma^4(P_\varphi^+) = -P_\varphi^+$, and $\sigma^4(P_\varphi^-) = -P_\varphi^-$. This implies that $\sigma^3$ leaves $P_\varphi^+ - P_\varphi^-$ invariant.

Next, consider the automorphism $\sigma^4 : u \mapsto \omega u$. Since $\sigma^4$ leaves $p_{\varphi^+}^M(x_1,z)$ (resp. $p_{\varphi^-}^M(x_1,z)$) invariant, $D_\varphi^+$ (resp. $D_\varphi^-$) is invariant under $\sigma^4$. By construction, the divisors $D_\varphi^+ + Q_{\varphi}^+$ and $D_\varphi^- + Q_{\varphi}^-$ are linearly equivalent by the function $q^+(x_1,x_2,z)/q^-(x_1,x_2,z)$. This implies that the divisor class $[Q_4^+ - Q_\bar{4}] = [D_\varphi^- - D_\varphi^+] \in \text{Pic}_k^0(C_u)$ is invariant under $\sigma^4$. The Mordell-Weil group $F_{E_1,E_2}^{(6)}(k(u))$ can be identified with $\text{Pic}_k^0(F_{E_1,E_2}^{(6)})$ by the map $P \mapsto [P - O]$, where $[\ ]$ stands for the divisor class. Since $\Psi_u$ induces an isomorphism of groups $\text{Pic}_k^0(C_u) \to \text{Pic}_k^0(F_{E_1,E_2}^{(6)})$, we can identify $F_{E_1,E_2}^{(6)}(k(u))$ with $\text{Pic}_k^0(F_{E_1,E_2}^{(6)})$, and $P_\varphi^+ - P_\varphi^- = [\Psi_u^{-1}(P_\varphi^+) - O] = [\Psi_u^{-1}(P_\varphi^-) - O] = [Q_4^+ - Q_\bar{4}]$, we see that $P_\varphi^+ - P_\varphi^-$ is invariant under $\sigma^4$.

We thus conclude that $P_\varphi^+ - P_\varphi^-$ is invariant under the automorphism $\sigma = \sigma^3 \circ \sigma^4$.

The calculation of the height is the same as in \([8]\). □

We denote by $P_{\varphi}^{(1)}$ the section $P_\varphi^+ - P_\varphi^-$ in $F_{E_1,E_2}^{(6)}(k(s))$. Carrying out calculations according to the above recipe, we obtain the coordinates of $P_{\varphi}^{(1)}$ as follows:

$$P_{\varphi}^{(1)} = \left( -\frac{3S^2}{16aa'(S-9bb')} + \frac{c_1S + c_0}{288a^3(b-9b')^3}, \right)$$

where

$$S = \frac{9}{2}\left( \frac{b^2s}{9} + \frac{9b'^2}{s} \right),$$

$$c_2 = 8aa' + 81bb', \quad c_1 = \frac{16}{9}a^2a'^2 - 144aa'bb' + 729b^2b'^2,$$

$$c_0 = bb'(80a^2a'^2 - 194aa'bb' + 2187b^2b'^2),$$

$$d_3 = 36(aa' + 9bb'), \quad d_2 = 2(16a^2a'^2 - 162aa'bb' + 2187b^2b'^2),$$

$$d_1 = -108bb'(8a^2a'^2 + 135aa'bb' - 243b^2b'^2),$$

$$d_0 = -3bb'(128a^2a'^2 - 6912a^2b^2b' + 26244aa'bb'^2 - 19683b^4b'^3).$$
6. Rational sections of $F_{E_1, E_2}^{(2)}$

Next, we consider the Mordell-Weil lattice $F_{E_1, E_2}^{(2)}$. We use the same notation as the previous section.

Let $\alpha_1, \alpha_2, \alpha_3$ be the three roots of $x^3 + a(x_1 - b)^2 = 0$. For $i = 1, 2, 3$, define $\beta_i = \varphi_i(\alpha_i)$. Since $\varphi$ is an isogeny of odd degree, $(\beta_i, 0)$ are the 2-torsion points of $E_2$. Thus, we have $k(E_1[2], E_2[2]) = k(\alpha_1, \alpha_2, \alpha_3)$, which we denote by $k_2$ for short.

Let $R_{ij}$ be the point $(\alpha_i : \beta_j : 1)$ in $C_u$. The quotient of $C_u$ by the action $u \mapsto \omega u$ is nothing but the model of Kummer surface $[2, 1]$.

**Theorem 6.1** (cf. [8, Theorem 1.2]). Suppose $j(E_1) \neq j(E_2)$, and let $h$ be the rank of $\text{Hom}_k(E_1, E_2)$. Then, the Mordell-Weil lattice $F_{E_1, E_2}^{(2)}(k(t))$ contains a sublattice of index $2^h$ naturally isomorphic to

$$\text{Hom}_k(E_1, E_2)(4) \oplus A_2^*(2)^{\oplus 2},$$

where $A_2^*$ denotes the dual lattice of the root lattice $A_2$, and $(n)$ denotes the lattice with the pairing multiplied by $n$. In particular, the determinant of the height matrix of $F_{E_1, E_2}^{(2)}(k(t))$ equals $2^4/3^2$ times the determinant of $\text{Hom}_k(E_1, E_2)$.

**Proof.** Except for the last statement, it is Theorem 1.2 of [8]. The last statement follows from the fact that if $L' \subset L$ is a sublattice of finite index in $L$, then we have $L' = \det L \times [L : L']^2$. \hfill $\square$

In case $\text{Hom}_k(E_1, E_2) = 0$, it is known (cf. [2, Prop. 3.3]) that, by taking $R_{11}$ as the origin, $F_{E_1, E_2}^{(2)}(k(t))$ is generated by $R_{22}, R_{33}, R_{23}, R_{32}$, and is isomorphic to the lattice $A_2^*(2)$.

Recall that the origin $O$ in $C_u$ is $(x_1 : x_2 : z) = (1 : u^2 : 0)$. Let $O_\omega$ and $O_{\omega z}$ be the points in $C_u$ defined by $(1 : (\omega u)^2 : 0)$ and $(1 : (\omega^2 u)^2 : 0)$ respectively. Clearly, the action $u \mapsto \omega u$ induces a cyclic permutation $O \mapsto O_\omega \mapsto O_{\omega z}$. Thus, the divisor $O + O_\omega + O_{\omega z}$ is invariant under this action. Let $q(x_1, x_2, z)$ be the quadratic form in $[\varphi(x, z)]$ and $l(x_1, x_2, z)$ the linear form in $[x_1, x_2]$. Consider the function $f = \frac{q(x_1, x_2, z)}{l(x_1, x_2, z)}$. The divisor of this function is

$$\text{div}(f) = D_4^+ + Q_4^+ + 2O - (O + O_\omega + O_{\omega z} + 2O + \overline{O}).$$

Thus, we have

$$Q_4^+ - \overline{O} \sim (O + O_\omega + O_{\omega z}) - D_4^+,$$

where $\overline{O}$ is the third point of intersection between $C_u$ and the tangent line at $O$. Let $P_{\overline{O}}$ be the section in $F_{E_1, E_2}^{(6)}(k(u))$ corresponding to $\overline{O} \in C_u(k(u))$. Then, since $(O + O_\omega + O_{\omega z}) - D_4^+$ is invariant under the action $u \mapsto \omega u$, the section $P_{\overline{O}} - P_{\overline{O}}$ is also invariant under this action. This implies that $P_{\overline{O}} - P_{\overline{O}}$ is a $(k)$-rational section where $t = u^4$.

**Theorem 6.2.** Suppose $E_1$ and $E_2$ are isogenous by an isogeny $\varphi$ of degree 3 over $k$, and $E_1$ and $E_2$ do not have complex multiplication. Let $P_{\varphi}^{(2)} = P_\varphi^+ - P_{\overline{O}}$ be the section of $F_{E_1, E_2}^{(2)}$ defined as above. Then, $P_{\varphi}^{(2)}$, $R_{22}, R_{33}, R_{23}, R_{32}$ are linearly independent and generate the Mordell-Weil group $F_{E_1, E_2}^{(2)}(k(t))$.

**Proof.** Since $E_1 \sim E_2$, but they do not have complex multiplication, we have $\text{rank} \text{Hom}_k(E_1, E_2) = 1$, and we are in case (ii) of Lemma 1.1. By straightforward calculations, we have

$$P_{\varphi}^{(2)} = \left( T^2 + 4aT + \frac{4}{3}(a^2 - 3bb') , \left( bt + \frac{b'}{T} \right)(T^2 + 6aT + 4(2a^2 + bb')) \right),$$
where \( b' = 4a + 27b \) and \( T = bt - b'/t \). It is easy to show that height of \( P_{\varphi}^{(2)} \) equals 4. Since the coordinates of \( R_{ij} \) involve the roots of the cubic equation \( x_1^3 + a(x_1 - b)^2 = 0 \), we do not write down the explicit coordinates here, but calculations are straightforward. (We will show them in the numerical examples.)

The height matrix with respect to \( P_{\varphi}^{(2)}, R_{22}, R_{33}, R_{23}, R_{32} \) is given by

\[
\begin{pmatrix}
12 & 0 & 0 & -3 & -3 \\
0 & 4 & 2 & 0 & 0 \\
0 & 2 & 4 & 0 & 0 \\
-3 & 0 & 0 & 4 & 2 \\
-3 & 0 & 0 & 2 & 4
\end{pmatrix}.
\]

Its determinant is \( 2^4/3 \), which equals \( 2^4/3^2 \) times \( \det \text{Hom}_k(E_1, E_2) = 3 \). Thus, \( P_{\varphi}^{(2)}, R_{22}, R_{33}, R_{23}, R_{32} \) are the generators of \( F_{E_1, E_2}^{(2)}(k(t)) \).

Remark 6.3. The height paring of given two rational points is easily computed by the height formula in [3].

### 7. Singular K3 surfaces

A complex K3 surface whose Picard number equals the maximum possible number 20 is called a singular K3 surface. Shioda and Inose showed that a complex singular K3 surface is isomorphic to an Inose surface \( \text{Ino}(E_1, E_2) \) for some elliptic curves \( E_1 \) and \( E_2 \) that have complex multiplication and are isogenous to each other.

**Theorem 7.1** ([3]). There is a one-to-one correspondence between the set of isomorphism classes of complex singular K3 surfaces and the set of equivalence classes of positive-definite even integral lattices of rank 2 with respect to \( SL_2(\mathbb{Z}) \):

\[
\{ \text{singular K3 surfaces over } \overline{\mathbb{Q}} \} / \text{isom.}
\]

\[
\{ \left( \begin{array}{cc} 2a & b \\ b & 2c \end{array} \right) \mid a, b, c \in \mathbb{Z}, \ a, c > 0, \ b^2 - 4ac < 0 \} / SL_2(\mathbb{Z}),
\]

which associates a singular K3 surface \( X \) with its transcendental lattice \( T_X \).

In fact, a singular K3 surface corresponding to the lattice \( Q = \left( \begin{array}{cc} 2a & b \\ b & 2c \end{array} \right) \) is constructed as follows. Let \( E_1 \) and \( E_2 \) be the complex elliptic curves \( \mathbb{C}/\mathbb{Z} \oplus \tau_1 \mathbb{Z} \) and \( \mathbb{C}/\mathbb{Z} \oplus \tau_2 \mathbb{Z} \), respectively, where

\[
\tau_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \tau_2 = \frac{b + \sqrt{b^2 - 4ac}}{2}.
\]

Then, the Inose surface \( \text{Ino}(E_1, E_2) \) is a singular K3 surface corresponding to \( Q \), which is often denoted by \( X_{[a,b,c]} \).

In the following, we study in detail the Mordell-Weil group of \( F^{(1)} \) and \( F^{(2)} \) for \( X_{[3,3,3]}, X_{[3,2,3]}, \) and \( X_{[3,0,3]} \)

#### 7.1. The singular K3 surface \( X_{[3,3,3]} \)

The transcendental lattice of the singular K3 surface \( X_{[3,3,3]} \) is given by the matrix

\[
\begin{pmatrix}
6 & 3 \\
3 & 6
\end{pmatrix}.
\]

Then, elliptic curves \( E_1 \) and \( E_2 \) are given by

\[
E_1 = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\omega, \quad E_2 = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}(-3\omega^2),
\]

where \( \omega = \frac{1 + \sqrt{-3}}{2} \). Their \( j \)-invariants are given by

\[
j(E_1) = 0, \quad j(E_2) = -1288000.
\]
and we take the following Weierstrass forms of them:
\[E_1 : y_1^2 = x_1^3 + 6(x_1 + 1)^2,\]
\[E_2 : y_2^2 = x_2^3 - 2(3x_2 + 1)^2.\]
This is nothing but the case \(a = 6, b = -1\) in (5.1). Then, \(F_{E_1,E_2}^{(1)}\) is given by
\[F_{E_1,E_2}^{(1)} : Y^2 = X^3 - 27\left(s - 506 + \frac{9t}{s}\right),\]
after rescaling \(X\) and \(Y\). In this case, \(F_{E_1,E_2}^{(1)}\) itself has complex multiplication by \(\omega\).

There are two 3-isogenies from \(E_1\) to \(E_2\). One is the 3-isogeny \(\varphi_1\) in §4. We denote it by \(\varphi_1\). The other, denoted by \(\varphi_2\), is a composite of \(\varphi_1\) and multiplication by \(\omega\) on \(E_2\). They are given by
\[\varphi_1(x, y) = (\varphi_2(x), \varphi(y)z), \quad \varphi_2(x, y) = (\omega\varphi_2(x), \varphi(y)z),\]
where
\[\varphi(x) = \frac{x^3 + 8x^2 + 24x + 24}{x^2}, \quad \varphi(y) = -\frac{x^3 - 24x - 48}{x^2}.\]
These isogenies form a basis of the lattice \(\text{Hom}_E(E_1,E_2)\) whose Gram matrix is given by
\[\begin{pmatrix}
6 & 3 \\
3 & 6
\end{pmatrix}.\]
The isogeny \(\varphi_1\) yields a \(\mathbb{Q}\)-rational section \(P^{(1)}_\varphi = P^+_\varphi - P^-_\varphi\) of \(F_{E_1,E_2}^{(1)}\) (see §5). If we let \(S = \frac{1}{2}(\frac{7}{6} + \frac{1}{2})\), its coordinates are given by
\[P^{(1)}_\varphi = \left(\frac{s^3 - 93s^2 + 963s + 4129}{64(S - 1)}, \frac{3s(s + 3)(s^4 - 140S^3 + 4758S^2 - 13100S + 258481)}{256(s - 3)^4}\right).\]
The height matrix with respect to sections \(P^{(1)}_\varphi\) and \([-\omega]P^{(1)}_\varphi = (\omega X, -Y)\) is given by
\[\begin{pmatrix}
6 & 3 \\
3 & 6
\end{pmatrix}.\]
Thus, by Theorem 5.1 these sections form a basis of the Mordell-Weil group \(F_{E_1,E_2}^{(1)}(\mathbb{Q}(s)) = F_{E_1,E_2}^{(1)}(\mathbb{Q}(\omega)(s)).\)

Next, we consider \(F_{E_1,E_2}^{(2)}(\mathbb{Q}(t))\). The explicit formula for the sections \(P^{(2)}_\varphi, R_{22}, R_{33}, R_{32,3}, R_{32}\) described in Theorem 6.2 are as follows:
\[P^{(2)}_\varphi = \left(\frac{1}{4}T_2^2 - 6T_+ + 15, \frac{1}{8}T_+(T_2^2 - 36T_+ + 300)\right),\]
\[R_{22} = (-15 - 2t\omega, -3\sqrt{3}T_+), \quad R_{32} = [-\omega]R_{22},\]
\[R_{33} = (-24\omega, -3\sqrt{3}T_+), \quad R_{32} = [-\omega]R_{23},\]
where \(T_- = t - 3/t\), and \(T_+ = t + 3/t\).

The height matrix with respect to \(P^{(2)}_\varphi\), \([-\omega]P^{(2)}_\varphi, R_{22}, R_{33}, R_{33,2}, R_{32}\) is given by
\[\begin{pmatrix}
12 & 6 & 0 & 0 & -3 & -3 \\
6 & 12 & 0 & 0 & 0 & -3 \\
0 & 0 & 4 & 2 & 0 & 0 \\
0 & 0 & 2 & 4 & 0 & 0 \\
-3 & 0 & 0 & 0 & 4 & 2 \\
-3 & -3 & 0 & 0 & 2 & 4
\end{pmatrix}.\]
Its determinant is \(2^2 \cdot 3\). Since \(2^2/3^2 \cdot \det \text{Hom}_E(E_1, E_2) = 2^4/3^2 \cdot \det \left(\frac{1}{2}, \begin{pmatrix}
6 & 3 \\
3 & 6
\end{pmatrix}\right) = 2^2 \cdot 3\), the above sections generate \(F_{E_1,E_2}^{(2)}(\mathbb{Q}(t))\) by Theorem 6.1.
7.2. The singular $K3$ surface $X_{[3,2,3]}$. The transcendental lattice of the singular $K3$ surface $X_{[3,2,3]}$ is given by the matrix

$$
\begin{pmatrix}
6 & 2 \\
2 & 6
\end{pmatrix}.
$$

Then, elliptic curves $E_1$ and $E_2$ are given by $C/Z \oplus \tau_1 Z$ and $C/Z \oplus \tau_2 Z$, where

$$
\tau_1 = \frac{-1 + 2\sqrt{-2}}{3}, \quad \tau_2 = 1 + 2\sqrt{-2}.
$$

The $j$-invariants of $E_1$ and $E_2$ are given by

$$
j(E_1) = 26125000 - 18473000\sqrt{2}, \quad j(E_2) = 26125000 + 18473000\sqrt{2}.
$$

We thus work on the base field $k = \mathbb{Q}(\sqrt{2})$. We choose $E_1$ and $E_2$ such that their Weierstrass forms are given by

$E_1 : y_1^2 = x_1^3 + 6(3 - \sqrt{2})x_1^2 + 9(3 + 2\sqrt{2} + 3)x_1$, 

$E_2 : y_2^2 = x_2^3 + 6(3 + \sqrt{2})x_2^2 + 9(3 - 2\sqrt{2} + 3)x_2$.

They are so-called $\mathbb{Q}$-curves; they are Galois conjugates and isogenous to each other.

Both $E_1$ and $E_2$ have complex multiplication by $\mathbb{Z}[1 + 2\sqrt{-2}]$. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{-2})$ be the Hilbert class field of $K$. They admit two isogenies $\varphi_1, \varphi_2 : E_1 \to E_2$ of degree 3 defined over $H$ given by

$$
\varphi_1(x_1, y_1) = (\varphi_{1,x}(x_1), \varphi_{1,y}(x_1), y_1), \quad \varphi_2(x_1, y_1) = (\varphi_{2,x}(x_1), \varphi_{2,y}(x_1), y_1),
$$

where

$$
\varphi_{1,x}(x_1) = \frac{(1 - 2\sqrt{-2})x_1(x_1 + 3 - 3\sqrt{-2})^2}{9(x_1 + 3 \pm 3\sqrt{-2} + 2\sqrt{2} + 4\sqrt{-1})},
$$

$$
\varphi_{2,x}(x_1) = \frac{(1 + 2\sqrt{-2})x_1(x_1 + 3 + 3\sqrt{-2})^2}{9(x_1 + 3 - 3\sqrt{-2} + 2\sqrt{2} - 4\sqrt{-1})^2}.
$$

The isogeny $\varphi_i$ yields a $H$-rational section $P_{\varphi_i}^+$ of $F_{E_1, E_2}^{(6)}$ for $i = 1, 2$.

Let $\tilde{\varphi}_i : E_2 \to E_1$ be the dual isogeny of $\varphi_i$ for $i = 1, 2$. The endomorphisms $\tilde{\varphi}_2 \circ \varphi_1$ and $\varphi_1 \circ \tilde{\varphi}_2$ correspond to the complex multiplication of $E_1$ by $1 \pm 2\sqrt{-2}$.

The surface $F_{E_1, E_2}^{(6)}$ is defined over $k$ and given by

$$
F_{E_1, E_2}^{(6)} : Y^2 = X^3 + \frac{575}{12}X + \left(\frac{u^6}{(1 - \sqrt{2})^3} - \frac{34937}{108} - \frac{(1 - \sqrt{2})^3}{u^6}\right).
$$

By letting $s' = s/(1 - \sqrt{2})^3 = u^{1/6}/(1 - \sqrt{2})^3$, the Weierstrass equation of the surface $F_{E_1, E_2}^{(1)}$ is given by

$$
F_{E_1, E_2}^{(1)} : Y^2 = X^3 + \frac{575}{12}X + \left(s' - \frac{34937}{108} - \frac{1}{s'}\right).
$$

With this model, the $X$-coordinate of the section $P_{\varphi_1}^{(1)} = P_{\varphi_1}^+ - P_{\varphi_1}^-$ of $F_{E_1, E_2}^{(1)}$ (see §5) is given by

$$
X(P_{\varphi_1}^{(1)}) = \frac{-\sqrt{-1}}{12(2 - \sqrt{-1})^8(S + 2\sqrt{-1})}(3S'^3 + c_2S'^2 + c_1S' + c_0),
$$

where

$$
S' = s' - \frac{1}{s'}, \quad c_2 = -42(23 - 10\sqrt{-1}), \quad c_1 = 2(9402 - 13685\sqrt{-1}), \quad c_0 = -4(61663 + 50160\sqrt{-1}).
$$
The $Y$-coordinate can be obtained easily, but it is rather complicated and we do not include here. The section $F_{\varphi_2}^{(1)}$ is the image of $F_{\varphi_1}^{(1)}$ under the complex conjugate $\sqrt{-1} \mapsto -\sqrt{-1}$. As in the previous example, the height matrix with respect to the sections $P_{\varphi_1}^{(1)}$ and $P_{\varphi_2}^{(1)}$ is given by $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$, and they form a basis of the Mordell-Weil lattice $F_{E_1}^{(1)}F_{E_2}^{(1)}(\bar{k}(s)) = F_{E_1}^{(1)}F_{E_2}^{(1)}(H(s))$.

Let $L = H(\sqrt{1 - \sqrt{2}}) = Q(\sqrt{1 - \sqrt{2}}, \sqrt{-1})$. Then, all the 2-torsion points of $E_1$ and $E_2$ are defined over $L$, and thus, the field of definition of $F_{E_1,E_2}^{(2)}(\bar{k}(t))$ is $L$. If we let $t' = u^{1/3}/(1 - \sqrt{2})^{3/2}$, the Weierstrass equation of $F_{E_1,E_2}^{(2)}$ is given by

$$F_{E_1,E_2}^{(2)} : Y^2 = X^3 + \frac{575}{12} X + \left(t'^2 - \frac{34937}{108} - \frac{1}{t'^2}\right).$$

Let $P_{\varphi_i}^{(2)} = P_{\varphi_i}^{+} - P_\infty$ for $i = 1, 2$ (see [6]). The $X$-coordinate of the section $P_{\varphi_1}^{(2)}$ is given by

$$X(P_{\varphi_1}^{(2)}) = \frac{-1 + \sqrt{-1}}{2(1 + 2\sqrt{-1})^2} \left(T^2 + c_1 T + c_0\right),$$

where

$$T = t + \sqrt{-1}/t, \quad c_1 = \sqrt{1 - \sqrt{2}}(9 + 13\sqrt{-1} - 2\sqrt{2} + 11\sqrt{-2}), \quad c_0 = \frac{1}{6}(161 - 97\sqrt{-1}).$$

Since $P_{\varphi_1}^{(2)}$ is defined over $k = Q(\sqrt{2})$, $P_{\varphi_2}^{(2)}$ is also the image of $P_{\varphi_1}^{(2)}$ by the complex conjugate $\sqrt{-1} \mapsto -\sqrt{-1}$.

The explicit formula for the sections $R_{22}$ described in Theorem 6.2 is

$$R_{22} = \left(\left(1 - 2\sqrt{-1}\right)(2 + \sqrt{2} - \sqrt{-2})\sqrt{1 - \sqrt{2}} - \frac{1}{6}(1 - 2\sqrt{-1})^4, T\right).$$

Let $\sigma \in \text{Gal}(L/Q)$ be the automorphism given by $\sqrt{1 - \sqrt{2}} \mapsto -\sqrt{1 - \sqrt{2}}$, and $\gamma$ the complex conjugate $\sqrt{-1} \mapsto -\sqrt{-1}$. Then, we have

$$R_{33} = -\sigma(R_{22}), \quad R_{23} = \gamma(R_{22}), \quad R_{32} = \gamma(R_{33}).$$

The height matrix with respect to $F_{\varphi_1}^{(2)}$, $F_{\varphi_2}^{(2)}$, $R_{22}$, $R_{33}$, $R_{23}$ and $R_{32}$ is given by

$$\begin{pmatrix} 12 & 3 & 0 & 0 & -3 & -3 \\ 3 & 12 & -3 & -3 & 0 & 0 \\ 0 & -3 & 4 & 2 & 0 & 0 \\ -3 & 0 & 0 & 4 & 2 & -3 \\ -3 & 0 & 0 & 2 & 4 & -3 \end{pmatrix}.$$

Its determinant is $2^7/3^2 = 2^4/3^2 \cdot \det \text{Hom}_k(E_1, E_2)$, and we can see that the above sections generate $F_{E_1,E_2}^{(2)}(\bar{k}(t))$ as in the previous example.

7.3. The singular $K3$ surface $X_{[3,0,3]}$. The transcendental lattice of the singular $K3$ surface $X_{[3,0,3]}$ is given by the matrix

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

In this case, the complex elliptic curves $E_1/C$ and $E_2/C$ are given by

$$E_1 : C/\mathbb{Z} \oplus \mathbb{Z}(3\sqrt{-1}), \quad E_2 : C/\mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}.$$  

Their $j$-invariants are given by

$$j(E_1) = 76771008 + 44330496\sqrt{3}, \quad j(E_2) = 1728.$$
We work with the base field \( k = \mathbb{Q}(\sqrt{3}) \), and we take the following Weierstrass forms, which corresponds to the formula (4.1) with \( a = 9(2 + \sqrt{3}), b = (1 - \sqrt{3})/3 \).

(\text{The equation of } E_2 \text{ is scaled differently from } (5.1).)

\[
\begin{align*}
E_1 : y_1^2 &= x_1^3 + (2 + \sqrt{3})(3x_1 - 1 + \sqrt{3})^2 \\
E_2 : y_2^2 &= x_2^3 - 27(2 + \sqrt{3})(x_2 - 9 - 3\sqrt{3})^2
\end{align*}
\]

The curves \( E_1 \) and \( E_2 \) are isogenous with complex multiplication by \( \mathbb{Q}(\sqrt{-1}) \), and there are two 3-isogenies from \( E_1 \) to \( E_2 \). One is obtained by a coordinate changes of the 3-isogeny \( \varphi \) in (4.1). We denote it by \( \varphi_1 \). The other, denoted by \( \varphi_2 \), is a composite of \( \varphi_1 \) and the complex multiplication \( [\sqrt{-1}] \) on \( E_2 \). With our equation of \( E_2 \), the multiplication-by-\( \sqrt{-1} \) map on \( E_2 \) is given by \((x_2, y_2) \mapsto (-x_2 + 36 + 18\sqrt{3}, \sqrt{-1}y_2) \).

The 3-isogenies \( \varphi_1, \varphi_2 \) are given by

\[
\begin{align*}
\varphi_1(x_1, y_1) &= (\varphi_x(x_1), \varphi_y(x_1)y_1), \\
\varphi_2(x_1, y_1) &= (-\varphi_x(x_1) + 36 + 18\sqrt{3}, \sqrt{-1}\varphi_y(x_1)y_1),
\end{align*}
\]

where

\[
\begin{align*}
\varphi_x(x_1) &= \frac{x_1^3 + 12(2 + \sqrt{3})x_1^2 + 12(1 + \sqrt{3})x_1 + 8}{x_1} \\
\varphi_y(x_1) &= -\frac{x_1^3 - 12(1 + \sqrt{3})x_1 - 16x_1^3}{x_1}
\end{align*}
\]

Over the base field \( k \), the equation of \( F_{E_1, E_2}^{(6)} \) is given by

\[
F_{E_1, E_2}^{(6)} : Y^2 = X^3 - (387 + 224\sqrt{3})X + 3(3 + 2\sqrt{3})u^6 + \frac{45 + 26\sqrt{3}}{9u^6},
\]

after replacing \( u \) by \( 3u \). By letting \( s' = s(3 + 2\sqrt{3})/9 = u^6(3 + 2\sqrt{3})/9 \), the equation of \( F_{E_1, E_2}^{(1)} \) is given by

\[
F_{E_1, E_2}^{(1)} : Y^2 = X^3 - (387 + 224\sqrt{3})X + (7 + 4\sqrt{3})(s' + \frac{1}{s'}).
\]

Using the method described in (4.1) the section \( P_{\varphi_1}^{(1)} \) of \( F_{E_1, E_2}^{(1)} \) is given by

\[
P_{\varphi_1}^{(1)} = \left( \frac{c_3S'^3 + c_2S'^2 + c_1S' + c_0}{144(S' + 2)}, \frac{s'(s' - 1)(d_4S'^4 + d_3S'^3 + d_2S'^2 + d_1S' + d_0)}{1728(s' + 1)^3} \right),
\]

where

\[
S' = s' + \frac{1}{s'},
\]

\[
c_3 = (2 - \sqrt{3})^2, \quad c_2 = -42, \quad c_1 = 12(91 + 36\sqrt{3}), \quad c_0 = -8(1267 + 680\sqrt{3}),
\]

\[
d_4 = (2 - \sqrt{3})^4, \quad d_3 = -4(25 - 12\sqrt{3}), \quad d_2 = 24(107 + 15\sqrt{3}),
\]

\[
d_1 = 16(461 + 444\sqrt{3}), \quad d_0 = -16(54676 + 32091\sqrt{3}).
\]

If we write \( P_{\varphi_1}^{(1)} = (x(s'), y(s')) \), then we have \( P_{\varphi_2}^{(1)} = (-x(-s'), \sqrt{-1}y(-s')) \). The height matrix with respect to the sections \( P_{\varphi_1}^{(1)} \) and \( P_{\varphi_2}^{(1)} \) is given by \(
\begin{pmatrix}
6 & 0 \\
0 & 6
\end{pmatrix}
\)

and they form a basis of \( F_{E_1, E_2}^{(1)}(k(s)) = F_{E_1, E_2}^{(1)}(\mathbb{Q}(\sqrt{3}, \sqrt{-1})(s)) \).

Next, we consider \( F_{E_1, E_2}^{(2)}(\bar{k}(s)) \). The field of definition of 2-torsion subgroups \( E_1[2] \) and \( E_2[2] \) is \( \mathbb{Q}(\sqrt{3} + 2\sqrt{3}) \). In order to compute generators of \( F_{E_1, E_2}^{(2)}(\bar{k}(t)) \), we need to work with the field \( L = \mathbb{Q}(\sqrt{3} + 2\sqrt{3}, \sqrt{-1}) \).
The explicit formulas for the sections $P_{\varphi_1}^{(2)}$, $P_{\varphi_2}^{(2)}$, $R_{22}$, $R_{33}$, $R_{23}$, $R_{32}$ described in Theorem 6.2 are as follows:

\[
P_{\varphi_1}^{(2)} = \left( \frac{1}{6} (\sqrt{3} T_+^2 + 6(1 + \sqrt{3}) T_+ + 42 + 20\sqrt{3}) \right),
\]

\[
P_{\varphi_2}^{(2)} = \left( \frac{1}{6} (\sqrt{3} T_-^2 + 6(1 + \sqrt{3}) T_- - 42 + 20\sqrt{3}) \right),
\]

\[R_{22} = \left( -7 - 4\sqrt{3} + 2(1 + \sqrt{3}) \alpha, (2 + \sqrt{3}) T_+ \right),
\]

\[R_{33} = \left( -7 - 4\sqrt{3} - 2(1 + \sqrt{3}) \alpha, -(2 + \sqrt{3}) T_+ \right),
\]

\[R_{23} = \left( 7 + 4\sqrt{3} - 2(1 + \sqrt{3}) \alpha, (2 + \sqrt{3}) T_- \right),
\]

\[R_{32} = \left( 7 + 4\sqrt{3} + 2(1 + \sqrt{3}) \alpha, -(2 + \sqrt{3}) T_- \right),
\]

where

\[
\alpha = \sqrt{3 + 2\sqrt{3}}, \quad T_+ = \frac{3t}{\alpha} + \frac{\alpha}{3t}, \quad T_- = \frac{3t}{\alpha} - \frac{\alpha}{3t}.
\]

The height matrix with respect to $P_{\varphi_1}^{(2)}$, $P_{\varphi_2}^{(2)}$, $R_{22}$, $R_{33}$, $R_{23}$, $R_{32}$ is given by

\[
\begin{pmatrix}
12 & 0 & 0 & 0 & -3 & -3 \\
0 & 12 & -3 & -3 & 0 & 0 \\
1 & 0 & -3 & 4 & 2 & 0 \\
0 & -3 & 2 & 4 & 0 & 0 \\
-3 & 0 & 0 & 0 & 4 & 2 \\
-3 & 0 & 0 & 0 & 2 & 4
\end{pmatrix}
\]

Its determinant is $2^4 = 4^2 \cdot \det \text{Hom}(E_1, E_2)$ as expected, and we can see that the above sections generate $F_{E_1, E_2}^{1,2}(\tilde{k}(t))$ as before.
[10] J. Top, Descent by 3-isogeny and 3-rank of quadratic fields, Advances in number theory (Kingston, ON, 1991), Oxford Sci. Publ., Oxford Univ. Press, New York, 1993, pp. 303–317. MR 1368429

Faculty of Economics, Chuo University
742-1 Hachioji-shi, Tokyo 192-0393 Japan
E-mail address: kuwata@tamacc.chuo-u.ac.jp

College of Science and Engineering, Ritsumeikan University
1-1-1 Noji-higashi, Kusatsu Shiga 525-8577 Japan
E-mail address: kutsumi@fc.ritsumei.ac.jp