The switching element for a Leonard pair

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Abstract
Let $V$ denote a vector space with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

We call such a pair a Leonard pair on $V$. Let \{\textit{v}$_i$\}$^d_{i=0}$ (resp. \{\textit{w}$_i$\}$^d_{i=0}$) denote a basis for $V$ referred to in (i) (resp. (ii)). We show that there exists a unique linear transformation $S : V \to V$ that sends \textit{v}$_0$ to a scalar multiple of \textit{v}$_d$ (resp. \textit{w}$_0$) fixes \textit{w}$_0$ (resp. \textit{v}$_d$), and sends \textit{w}$_i$ to a scalar multiple of \textit{w}$_i$ for $1 \leq i \leq d$. We call $S$ the switching element. We describe $S$ from many points of view.

1 Leonard pairs

We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix $X$ is said to be tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume $X$ is tridiagonal. Then $X$ is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 \cite{29} Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Leonard pair on $V$ we mean an ordered pair $A, A^*$ where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy (i) and (ii) below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

Note 1.2 It is a common notational convention to use $A^*$ to represent the conjugate-transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i) and (ii) above.
denote the $K$ matrices that are the 

Leonard system in terms of the switching element. We fin ish the paper with some 

of the earlier remarks there exists a unique linear transformatio n on $V$ that sends 

$D$, $v_i$ for 

$d$, $v_i$ to a scalar multiple of $w_i$ for. We show that up to multiplication 

by a nonzero scalar, $S$ is the unique linear transformation on $V$ that fixes each of $[0]$, $[D]$ and sends $[0^*]$ to $[D^*]$. A decomposition of $V$ is a sequence of one-dimensional subspaces whose direct sum is $V$. Let $x, y$ denote an ordered pair of distinct elements from the set 

$D, 0^*, D^*$. By Theorem 8.3 there exists a decomposition $[xy]$ of $V$ such that for 

$d$, $v_i$ to a scalar multiple of $v_0$, and sends $v_i$ to a scalar multiple of $v_i$ for 

$d$. We show that each component of $[0^*D]$ (resp. $[D^*D]$, $[0^*0]$, $[D^*0]$) is an eigenspace for $S^*S^{-1}S^{-1}S$ (resp. $S^*S$). We find the corresponding eigenvalues. We consider a certain basis for $V$ whose $i$th component is contained in the $i$th component of $[0^*D]$ for $0 \leq i \leq d$. With respect to this basis the matrix representing $A$ (resp. $A^*$) is lower bidiagonal (resp. upper bidiagonal) [29 Lemma 3.9]. We display the matrices that represent $S$ and $S^*$ with respect to this basis. In a related result we characterize the Leonard pair concept in terms of the switching element. We finish the paper with some open problems.

2 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a Leonard system. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let $d$ denote a nonnegative integer and let $\text{Mat}_{d+1}(K)$ denote the $K$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $K$. We index the rows and columns by $0, 1, \ldots, d$. We let $K^{d+1}$ denote the $K$-vector space of all $d+1$ by $d$ matrices that have entries in $K$. We index the rows by $0, 1, \ldots, d$. We view $K^{d+1}$ as a left module for $\text{Mat}_{d+1}(K)$. We observe this module is irreducible. For the rest of this
paper, let \( \mathcal{A} \) denote a \( \mathbb{K} \)-algebra isomorphic to \( \text{Mat}_{d+1}(\mathbb{K}) \) and let \( V \) denote a simple left \( \mathcal{A} \)-module. We remark that \( V \) is unique up to isomorphism of \( \mathcal{A} \)-modules, and that \( V \) has dimension \( d + 1 \). Let \( \{v_i\}_{i=0}^d \) denote a basis for \( V \). For \( X \in \mathcal{A} \) and \( Y \in \text{Mat}_{d+1}(\mathbb{K}) \), we say \( Y \) represents \( X \) with respect to \( \{v_i\}_{i=0}^d \) whenever \( Xv_j = \sum_{i=0}^d Y_{ij}v_i \) for \( 0 \leq j \leq d \). For \( A \in \mathcal{A} \) we say \( A \) is multiplicity-free whenever it has \( d + 1 \) mutually distinct eigenvalues in \( \mathbb{K} \). Assume \( A \) is multiplicity-free. Let \( \{\theta_i\}_{i=0}^d \) denote an ordering of the primitive idempotents of \( A \), and for \( 0 \leq i \leq d \) put

\[
E_i = \prod_{0 \leq j \leq d, j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j},
\]

where \( I \) denotes the identity of \( \mathcal{A} \). We observe (i) \( AE_i = \theta_i E_i \) (\( 0 \leq i \leq d \)); (ii) \( E_i E_j = \delta_{i,j} E_i \) (\( 0 \leq i, j \leq d \)); (iii) \( \sum_{i=0}^d E_i = I \); (iv) \( A = \sum_{i=0}^d \theta_i E_i \). Let \( \mathcal{D} \) denote the subalgebra of \( \mathcal{A} \) generated by \( A \). Using (i)–(iv) we find the sequence \( \{E_i\}_{i=0}^d \) is a basis for the \( \mathbb{K} \)-vector space \( \mathcal{D} \). We call \( E_i \) the primitive idempotent of \( A \) associated with \( \theta_i \). It is helpful to think of these primitive idempotents as follows. Observe \( V = E_0 V + E_1 V + \cdots + E_d V \) (direct sum).

For \( 0 \leq i \leq d \), \( E_i V \) is the (one-dimensional) eigenspace of \( A \) in \( V \) associated with the eigenvalue \( \theta_i \), and \( E_i \) acts on \( V \) as the projection onto this eigenspace. We note that for \( X \in \mathcal{A} \) the following are equivalent: (i) \( X \in \mathcal{D} \); (ii) \( XA = AX \); (iii) \( XE_i V \subseteq E_i V \) for \( 0 \leq i \leq d \).

By a Leonard pair in \( \mathcal{A} \) we mean an ordered pair of elements taken from \( \mathcal{A} \) that act on \( V \) as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

**Definition 2.1** [29] By a Leonard system in \( \mathcal{A} \) we mean a sequence

\[
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
\]

that satisfies (i)–(v) below.

(i) Each of \( A, A^* \) is a multiplicity-free element in \( \mathcal{A} \).

(ii) \( \{E_i\}_{i=0}^d \) is an ordering of the primitive idempotents of \( A \).

(iii) \( \{E_i^*\}_{i=0}^d \) is an ordering of the primitive idempotents of \( A^* \).

(iv) For \( 0 \leq i, j \leq d \),

\[
E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases}
\]

(v) For \( 0 \leq i, j \leq d \),

\[
E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases}
\]

We say \( \Phi \) is over \( \mathbb{K} \).

Leonard systems are related to Leonard pairs as follows. Let \( (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a Leonard system in \( \mathcal{A} \). Then \( A, A^* \) is a Leonard pair in \( \mathcal{A} \) [37, Section 3]. Conversely, suppose \( A, A^* \) is a Leonard pair in \( \mathcal{A} \). Then each of \( A, A^* \) is multiplicity-free [29].
The relatives of \( \Phi \) are as follows: The group generated by symbols \( \ast \) Leonard systems will be called \( D \).

We will use the following notational convention.

For a given Leonard system \( \Phi = (A; \{ E_i \}_{i=0}^d; A^* ; \{ E_i^* \}_{i=0}^d) \) in \( \mathcal{A} \), each of the following is a Leonard system in \( \mathcal{A} \):

\[
\Phi^* := (A^*; \{ E_i^* \}_{i=0}^d; A; \{ E_i^d \}_{i=0}^d),
\]

\[
\Phi^↓ := (A; \{ E_i \}_{i=0}^d; A^* ; \{ E_{d-i}^* \}_{i=0}^d),
\]

\[
\Phi^↓ := (A; \{ E_{d-i} \}_{i=0}^d; A^* ; \{ E_i^* \}_{i=0}^d).
\]

Viewing \( \ast \), \( ↓ \), \( \downarrow \) as permutations on the set of all Leonard systems in \( \mathcal{A} \),

\[
s^2 = \downarrow^2 = \downarrow\downarrow = 1, \tag{4}
\]

\[
\downarrow\ast = \ast \downarrow \quad \downarrow\ast = \ast \downarrow \downarrow \quad \downarrow\downarrow = \downarrow\downarrow. \tag{5}
\]

The group generated by symbols \( \ast \), \( \downarrow \), \( \downarrow \) subject to the relations \((4), (5)\) is the dihedral group \( D_4 \). We recall that \( D_4 \) is the group of symmetries of a square and has 8 elements.

Two Leonard systems will be called relatives whenever they are in the same orbit of this \( D_4 \) action. The relatives of \( \Phi \) are as follows:

| name          | relative                                                                 |
|---------------|---------------------------------------------------------------------------|
| \( \Phi \)    | \( (A; \{ E_i \}_{i=0}^d; A^* ; \{ E_i^* \}_{i=0}^d) \)                        |
| \( \Phi^\downarrow \) | \( (A; \{ E_i \}_{i=0}^d; A^* ; \{ E_{d-i}^* \}_{i=0}^d) \)                           |
| \( \Phi^\downarrow\downarrow \) | \( (A; \{ E_{d-i} \}_{i=0}^d; A^* ; \{ E_i^* \}_{i=0}^d) \)                        |
| \( \Phi^* \)   | \( (A^*; \{ E_i^* \}_{i=0}^d; A; \{ E_i^d \}_{i=0}^d) \)                        |
| \( \Phi^{\downarrow\ast} \) | \( (A^*; \{ E_{d-i}^* \}_{i=0}^d; A; \{ E_i^d \}_{i=0}^d) \)                       |
| \( \Phi^{\downarrow\downarrow\ast} \) | \( (A^*; \{ E_i^* \}_{i=0}^d; A; \{ E_{d-i}^d \}_{i=0}^d) \)                      |
| \( \Phi^{\downarrow\ast\downarrow\ast} \) | \( (A^*; \{ E_{d-i}^* \}_{i=0}^d; A; \{ E_{d-i}^d \}_{i=0}^d) \)                  |

We will use the following notational convention.

**Definition 3.1** For \( g \in D_4 \) and for an object \( f \) associated with \( \Phi \) we let \( f^g \) denote the corresponding object associated with \( \Phi^g^{-1} \).

4 The parameter array

In this section we recall some parameters.
Definition 4.1 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$. For $0 \leq i \leq d$ we let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp. $E_i^*$). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{K}$.

We will use the following notation. Let $\lambda$ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in $\lambda$ that have coefficients in $\mathbb{K}$.

Definition 4.2 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. For $0 \leq i \leq d$ we define polynomials $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ in $\mathbb{K}[\lambda]$ as follows:

$$
\begin{align*}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}), \\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{i-1}^*), \\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*)\cdots(\lambda - \theta_{d-i+1}^*).
\end{align*}
$$

Note that each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic with degree $i$ for $0 \leq i \leq d$.

Definition 4.3 [20] Theorem 4.6] Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$. Referring to Definition 4.2 we define scalars

$$
\begin{align*}
\varphi_i &= (\theta_0^* - \theta_i^*) \frac{\text{tr}(\tau_i(A)E_i^*)}{\text{tr}(\tau_{i-1}(A)E_0^*)} & (1 \leq i \leq d), \\
\phi_i &= (\theta_0^* - \theta_i^*) \frac{\text{tr}(\eta_i(A)E_i^*)}{\text{tr}(\eta_{i-1}(A)E_0^*)} & (1 \leq i \leq d),
\end{align*}
$$

where $\text{tr}$ means trace. We note that in [9], [7] the denominators are nonzero by [20] Corollary 4.5. The sequence $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is called the first split sequence (resp. second split sequence) of $\Phi$.

Definition 4.4 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$. By the parameter array of $\Phi$ we mean the sequence $\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d$, where the $\theta_i, \theta_i^*$ are from Definition 4.1 and the $\varphi_i, \phi_i$ are from Definition 4.3.

Theorem 4.5 [20] Theorem 1.9] Let $d$ denote a nonnegative integer and let

$$
\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d
$$

(8)

denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a Leonard system $\Phi$ over $\mathbb{K}$ with parameter array [X] if and only if (PA1)–(PA5) hold below.

(PA1) $\varphi_i \neq 0, \phi_i \neq 0$ (1 $\leq i \leq d$).
(PA2) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$).

(PA3) For $1 \leq i \leq d$,

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d).$$

(PA4) For $1 \leq i \leq d$,

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0).$$

(PA5) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \frac{\theta_i^* - \theta_{i+1}}{\theta_i^* - \theta_i}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$.

Suppose (PA1)–(PA5) hold. Then $\Phi$ is unique up to isomorphism of Leonard systems.

The $D_4$ action affects the parameter array as follows.

Lemma 4.6 [29, Theorem 1.11] Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let $({\theta_i}_i^d_{i=0}; {\theta_i^*}_i^d_{i=0}; {\varphi_i}_i^d_{i=1}; {\phi_i}_i^d_{i=1})$ denote the parameter array of $\Phi$. Then the following (i)–(iii) hold.

(i) The parameter array of $\Phi^*$ is

$$(\{\theta_i^*_i\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

(ii) The parameter array of $\Phi^\downarrow$ is

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_{d-1}\}_{i=1}^d; \{\phi_{d-1}\}_{i=1}^d).$$

(iii) The parameter array of $\Phi^\uparrow$ is

$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d).$$

We finish this section with a comment.

Lemma 4.7 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$. Let $D$ denote the subalgebra of $A$ generated by $A$, and let $X$ denote an element of $D$ such that $XE_0^* = 0$. Then $X = 0$.

Proof. Immediate from [37, Lemma 5.9]. □
5 The switching element \( S \)

Definition 5.1 For a Leonard system \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) in \( A \) we define

\[
S = \sum_{r=0}^{d} \frac{\phi d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r,
\]

where \( \{\varphi_i\}_{i=1}^d \) (resp. \( \{\phi_i\}_{i=1}^d \)) denotes the first (resp. second) split sequence of \( \Phi \). We call \( S \) the switching element for \( \Phi \).

Note 5.2 Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a Leonard system. In what follows we will often make use of the switching element of \( \Phi^* \). By (10) and Lemma 4.6(i),

\[
S^* = \sum_{r=0}^{d} \frac{\phi_1 \phi_2 \cdots \phi_r}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r^*.
\]

We call \( S^* \) the dual switching element for \( \Phi \).

Lemma 5.3 The switching element (10) and the dual switching element (11) are invertible with

\[
S^{-1} = \sum_{r=0}^{d} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi d \phi_{d-1} \cdots \phi_{d-r+1}} E_r,
\]

\[
S^{*-1} = \sum_{r=0}^{d} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_1 \phi_2 \cdots \phi_r} E_r^*.
\]

Proof. To obtain (12) we note that the sum on the right in (10) times the sum on the right in (12) is equal to the identity; this is verified using equations (ii), (iii) below (11). Line (13) is similarly obtained.

Theorem 5.4 Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a Leonard system with switching element \( S \) and dual switching element \( S^* \). Then the switching element and the dual switching element for the relatives of \( \Phi \) are given in the following table:

| relative \( \Phi \) | \( \Phi^i \) | \( \Phi^i \downarrow \) | \( \Phi^i \downarrow \uparrow \) | \( \Phi^* \) | \( \Phi^* \downarrow \) | \( \Phi^* \downarrow \uparrow \) |
|---------------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| switching element \( S \) | \( S^{-1} \) | \( \varphi \phi^{-1} S \) | \( \varphi^{-1} \phi S^{-1} \) | \( S^* \) | \( S^{*-1} \) | \( \varphi^{-1} \phi S^{*-1} \) |
| dual switching element \( S^* \) | \( \varphi \phi^{-1} S^* \) | \( S^{*-1} \) | \( \varphi^{-1} \phi S^{*-1} \) | \( S \) | \( S^{-1} \) | \( \varphi^{-1} \phi S^{-1} \) |

In the above table we abbreviate

\[
\varphi = \varphi_1 \varphi_2 \cdots \varphi_d, \quad \phi = \phi_1 \phi_2 \cdots \phi_d,
\]

where \( \{\varphi_i\}_{i=1}^d \) (resp. \( \{\phi_i\}_{i=1}^d \)) is the first (resp. second) split sequence of \( \Phi \).

Proof. Apply \( D_4 \) to (10) and use Lemma 4.6.

We now describe the switching element from various points of view.
6 Representing S as a polynomial

Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system in \(A\) with switching element \(S\). Let \(D\) denote the subalgebra of \(A\) generated by \(A\), and recall \(\{E_i\}_{i=0}^d\) is a basis for \(D\). By this and \((10)\) we find \(S \in D\), so \(S\) is a polynomial in \(A\). In the present section we find this polynomial.

Lemma 6.1 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system in \(A\). Then for \(0 \leq i \leq d\) there exists a unique monic polynomial \(p_i\) in \(\mathbb{K}[\lambda]\) with degree \(i\) such that

\[
p_i(A)E_0^*V = E_i^*V.
\]

Proof. The existence of \(p_i\) is established in [37, Theorem 8.3]. Concerning uniqueness, suppose we are given a monic polynomial \(p'_i\) in \(\mathbb{K}[\lambda]\) of degree \(i\) such that \(p'_i(A)E_0^*V = E_i^*V\). We show \(p_i = p'_i\). To this end we define \(f = p_i - p'_i\) and show \(f = 0\). By construction \(f(A)E_0^*V \subseteq E_i^*V\). Each of \(p_i, p'_i\) is monic of degree \(i\) so the degree of \(f\) is at most \(i - 1\). By this and \((10)\) we find \(f(A)E_0^*V\) is included in \(\sum_{k=0}^{i-1} E_k^*V\). By these comments we find \(f(A)E_0^*V = 0\) so \(f(A)E_0^* = 0\). Now \(f(A) = 0\) in view of Lemma 4.7. This implies \(f = 0\) since \(I, A, A^2, \ldots, A^d\) are linearly independent. \(\square\)

Lemma 6.2 [37, Lemma 17.5] Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system and let \((\{\theta_i\}_{i=0}^d; \{\tau_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\psi_i\}_{i=1}^d)\) denote the corresponding parameter array. Let the polynomials \(\{p_i\}_{i=0}^d\) be from Lemma 6.1. Then

\[
p_i(\theta_0) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\tau_i^*(\delta_i^*)} \quad (0 \leq i \leq d).
\]

Moreover \(p_i(\theta_0) \neq 0\).

Definition 6.3 [37, Definition 14.1] Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system over \(\mathbb{K}\) and let the polynomials \(\{p_i\}_{i=0}^d\) be as in Lemma 6.1. For \(0 \leq i \leq d\) we define

\[
u_i = \frac{p_i}{p_i(\theta_0)}.
\]

where \(\theta_0\) is from Definition 4.1.

Lemma 6.4 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system in \(A\) and let the polynomials \(\{u_i\}_{i=0}^d\) be from Definition 6.3. Then

\[
u_i(A)E_0^*V = E_i^*V \quad (0 \leq i \leq d).
\]

Proof. Combine Lemma 6.1 and (15). \(\square\)
Lemma 6.5 [37, Theorem 14.7] Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^\ast\}_{i=0}^d)\) denote a Leonard system with eigenvalue sequence \(\{\theta_i\}_{i=0}^d\) and dual eigenvalue sequence \(\{\theta_i^\ast\}_{i=0}^d\). Let the polynomials \(\{u_i\}_{i=0}^d\) be as in Definition 6.3 and recall \(\{u_i^\ast\}_{i=0}^d\) are the corresponding polynomials for \(\Phi^\ast\). Then for \(0 \leq i, j \leq d\),

\[ u_i(\theta_j) = u_j^\ast(\theta_i^\ast). \] (16)

Theorem 6.6 Let \(\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^\ast\}_{i=0}^d)\) denote a Leonard system and let \(S\) denote the switching element for \(\Phi\). Then

\[ S = u_d(A), \] (17)

where the polynomial \(u_d\) is from Definition 6.3.

Proof. By \(D_4\) symmetry it suffices to show \(S^\ast = u_d^\ast(A^\ast)\). Using the comments below (11) we find

\[ u_d^\ast(A^\ast) = \sum_{i=0}^d u_d^\ast(\theta_i^\ast)E_i^\ast. \] (18)

For \(0 \leq i \leq d\) we compute \(u_d^\ast(\theta_i^\ast)\) as follows. By Lemma 6.1 the polynomial \(p_i\) is invariant under \(\downarrow\); that is \(p_i^\downarrow = p_i\). We apply \(\downarrow\) to (14) using this and Lemma 4.6 to get

\[ p_i(\theta_d) = \frac{\phi_1\phi_2\cdots\phi_i}{\tau_i(\theta_d^\ast)}. \]

Combining this with (14), (15) we find

\[ u_i(\theta_d) = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i} \]

By this and Lemma 6.5 we get

\[ u_d^\ast(\theta_i^\ast) = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i} \] (19)

Evaluating (18) using (19) and comparing the result with (11) we find \(S^\ast = u_d^\ast(A^\ast)\). The result follows.

The switching element is characterized as follows.

Theorem 6.7 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^\ast\}_{i=0}^d)\) denote a Leonard system in \(A\) with switching element \(S\). Let \(D\) denote the subalgebra of \(A\) generated by \(A\). Then for all nonzero \(X \in A\) the following (i), (ii) are equivalent.

(i) \(X\) is a scalar multiple of \(S\).

(ii) \(X \in D\) and \(XE_0^\ast V \subseteq E_d^\ast V\).

Suppose (i), (ii) hold. Then \(XE_0^\ast V = E_d^\ast V\).

Proof. (i)⇒(ii): We mentioned in the first paragraph of this section that \(S \in D\). We have \(SE_0^\ast V = E_d^\ast V\) by Lemma 6.4 and Theorem 6.6.
(ii)⇒(i): For \(0 \leq i \leq d\) we define \(D_i = \text{Span}\{u_i(A)\}\). Observe that \(D = \sum_{i=0}^{d} D_i\) (direct sum). Also observe by Lemma 6.4 that \(D_iE_0^*V = E_0^*V\) for \(0 \leq i \leq d\). We assume \(XE_0^*V \subseteq E_0^*V\) so \(X \in D_d\) and in other words \(X\) is a scalar multiple of \(u_d(A)\). By this and Theorem 6.6 we find \(X\) is a scalar multiple of \(S\).

Now suppose (i), (ii) hold. We mentioned in the proof of (i)⇒(ii) that \(SE_0^*V = E_0^*V\). But \(X\) is nonzero and a scalar multiple of \(S\) so \(XE_0^*V = E_0^*V\). \(\square\)

7 Decompositions and flags

In this section we recall the notion of a decomposition and a flag.

By a decomposition of \(V\) we mean a sequence \(\{V_i\}_{i=0}^{d}\) of subspaces of \(V\) such that \(V_i\) has dimension 1 for \(0 \leq i \leq d\) and \(\sum_{i=0}^{d} V_i = V\) (direct sum). Let \(\{V_i\}_{i=0}^{d}\) denote a decomposition of \(V\). By the inversion of this decomposition we mean the decomposition \(\{V_{d-i}\}_{i=0}^{d}\).

By a flag on \(V\) we mean a sequence \(\{F_i\}_{i=0}^{d}\) of subspaces of \(V\) such that \(F_i\) has dimension \(i + 1\) for \(0 \leq i \leq d\) and \(F_{i-1} \subseteq F_i\) for \(1 \leq i \leq d\). The following construction yields a flag on \(V\). Let \(\{V_i\}_{i=0}^{d}\) denote a decomposition of \(V\). Define

\[
F_i = V_0 + V_1 + \cdots + V_i \quad (0 \leq i \leq d).
\]

Then \(\{F_i\}_{i=0}^{d}\) is a flag on \(V\). We say this flag is induced by the decomposition \(\{V_i\}_{i=0}^{d}\).

We recall what it means for two flags on \(V\) to be opposite. Suppose we are given two flags on \(V\): \(\{F_i\}_{i=0}^{d}\) and \(\{F'_i\}_{i=0}^{d}\). We say these flags are opposite whenever there exists a decomposition \(\{V_i\}_{i=0}^{d}\) of \(V\) such that

\[
F_i = V_0 + \cdots + V_i, \quad F'_i = V_{d-i} + \cdots + V_{d-1}
\]

for \(0 \leq i \leq d\). In this case

\[
F_i \cap F'_j = 0 \quad \text{if} \quad i + j < d \quad (0 \leq i, j \leq d)
\]

and

\[
V_i = F_i \cap F'_{d-i} \quad (0 \leq i \leq d).
\]

In particular the decomposition \(\{V_i\}_{i=0}^{d}\) is uniquely determined by the given flags. We say this decomposition is induced by the given flags.

We end this section with some notation.

**Notation 7.1** Let \(F = \{F_i\}_{i=0}^{d}\) denote a sequence of subspaces of \(V\). Then for \(X \in A\) we write \(XF\) to denote the sequence \(\{XF_i\}_{i=0}^{d}\). We say \(X\) fixes \(F\) whenever \(XF = F\). Let \(F' = \{F'_i\}_{i=0}^{d}\) denote a second sequence of subspaces of \(V\). We write \(F \subseteq F'\) whenever \(F_i \subseteq F'_i\) for \(0 \leq i \leq d\).
8 Some decompositions and flags associated with a Leonard system

We now return our attention to Leonard systems. Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d) \) denote a Leonard system in \( A \). Using \( \Phi \) we construct four mutually opposite flags and consider the decompositions that they induce. We start with a definition.

**Definition 8.1** For notational convenience let \( \Omega \) denote the set consisting of four symbols \( 0, D, 0^*, D^* \).

**Definition 8.2** Let \( (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d) \) denote a Leonard system in \( A \). For \( z \in \Omega \) we define a flag on \( V \) which we denote by \([z]\). To define this flag we display the \( i \)th component for \( 0 \leq i \leq d \).

| \( z \) | \( i \)th component of \([z]\) |
|------|-------------------------------|
| 0    | \( E_0V + E_1V + \cdots + E_iV \) |
| \( D \) | \( E_dV + E_{d-i}V + \cdots + E_dV \) |
| \( 0^* \) | \( E_0^*V + E_1^*V + \cdots + E_i^*V \) |
| \( D^* \) | \( E_d^*V + E_{d-i}^*V + \cdots + E_d^*V \) |

**Lemma 8.3** Referring to Definition 8.2, the following (i)–(iv) hold for \( 0 \leq i \leq d \).

(i) The \( i \)th component of \([0]\) is equal to \( \eta_{d-i}(A)V \).

(ii) The \( i \)th component of \([D]\) is equal to \( \tau_{d-i}(A)V \).

(iii) The \( i \)th component of \([0^*]\) is equal to \( \eta_{d-i}^*(A^*)V \).

(iv) The \( i \)th component of \([D^*]\) is equal to \( \tau_{d-i}^*(A^*)V \).

**Proof.** (i): Recall that \( V = \sum_{j=0}^d E_jV \) (direct sum). Further recall that for \( 0 \leq j \leq d \), \( E_jV \) is an eigenspace for \( A \) with eigenvalue \( \theta_j \). This implies that for \( 0 \leq j, k \leq d \), \((A - \theta_k I)E_jV \) equals 0 if \( j = k \) and \( E_jV \) if \( j \neq k \). By these comments and Definition 4.2 we have \( \eta_{d-i}(A)E_jV = 0 \) for \( i + 1 \leq j \leq d \) and \( \eta_{d-i}(A)E_jV = E_jV \) for \( 0 \leq j \leq i \). Therefore \( \eta_{d-i}(A)V = \sum_{j=0}^i E_jV \) and this is the \( i \)th component of \([0]\).

(ii)–(iv): Similar. \( \square \)

**Lemma 8.4** [31, Theorem 7.3] The four flags in Definition 8.2 are mutually opposite.

**Definition 8.5** Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d) \) denote a Leonard system in \( A \). Let \( z, w \) denote an ordered pair of distinct elements of \( \Omega \). By Lemma 8.4 the flags \([z]\), \([w]\) are opposite. Let \([zw]\) denote the decomposition of \( V \) induced by \([z]\), \([w]\).

We mention a few basic properties of the decompositions from Definition 8.5.
Lemma 8.6 Referring to Definition 8.3, for distinct \( z, w \in \Omega \) the following (i)–(iii) hold.

(i) The decomposition \([zw]\) is the inversion of \([wz]\).

(ii) For \( 0 \leq i \leq d \) the \( i^{th} \) component of \([zw]\) is the intersection of the \( i^{th} \) component of \([z]\) and the \( (d-i)^{th} \) component of \([w]\).

(iii) The decomposition \([zw]\) induces \([z]\) and the inversion of \([zw]\) induces \([w]\).

Proof. Routine using Section 7 and Definition 8.5. \( \square \)

Example 8.7 We display some of the decompositions from Definition 8.5. For each decomposition in the table below we give the \( i^{th} \) component for \( 0 \leq i \leq d \).

| decomposition | \( i^{th} \) component |
|--------------|-----------------|
| \([0^*D]\)   | \((E^*_0V + \cdots + E^*_iV) \cap (E_iV + \cdots + E_dV)\) |
| \([D^*D]\)   | \((E^*_dV + \cdots + E^*_{d-i}V) \cap (E_iV + \cdots + E_dV)\) |
| \([0^*0]\)   | \((E^*_0V + \cdots + E^*_iV) \cap (E_{d-i}V + \cdots + E_0V)\) |
| \([D^*0]\)   | \((E^*_dV + \cdots + E^*_{d-i}V) \cap (E_{d-i}V + \cdots + E_0V)\) |
| \([0D]\)     | \(E_iV\) |
| \([0^*D^*]\) | \(E^*_iV\) |

Lemma 8.8 Referring to Definition 8.3, the following (i)–(iv) hold for \( 0 \leq i \leq d \).

(i) The \( i^{th} \) component of \([0^*D]\) is equal to \( \tau_i(A)E^*_0V \) and \( \eta^*_0(A^*)E_dV \).

(ii) The \( i^{th} \) component of \([D^*D]\) is equal to \( \tau_i(A)E^*_dV \) and \( \tau^*_d(A^*)E_0V \).

(iii) The \( i^{th} \) component of \([0^*0]\) is equal to \( \eta_i(A)E^*_0V \) and \( \eta^*_0(A^*)E_dV \).

(iv) The \( i^{th} \) component of \([D^*0]\) is equal to \( \eta_i(A)E^*_dV \) and \( \tau^*_d(A^*)E_0V \).

Proof. (i): We first show that \( \tau_i(A)E^*_0V \) is equal to the \( i^{th} \) component of \([0^*D]\). Denote this \( i^{th} \) component by \( U_i \). By Lemma 8.3(ii) \( \tau_i(A)V \) is equal to \( \sum_{j=i}^d E_jV \), so \( \tau_i(A)E^*_0V \) is contained in \( \sum_{j=0}^d E^*_jV \). By these comments and the definition of \( U_i \) we find \( \tau_i(A)E^*_0V \) is contained in \( \sum_{k=0}^i E^*_kV \). By Lemma 8.2 we have \( \tau_i(A)E^*_0V \neq 0 \) so \( \tau_i(A)E^*_0V \neq 0 \). By this and since \( U_i \) has dimension 1 we find \( \tau_i(A)E^*_0V = U_i \). We have now shown that \( \tau_i(A)E^*_0V \) is equal to the \( i^{th} \) component of \([0^*D]\). In a similar way we find that \( \eta^*_0(A^*)E_dV \) is equal to the \( i^{th} \) component of \([0^*D]\).

(ii)–(iv): Apply (i) to the relatives of \( \Phi \). \( \square \)
9 The action of $S$ on the flags

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$ and let $S$ denote the corresponding switching element. In this section we characterize $S$ via its action on the flags from Definition 5.2.

**Theorem 9.1** Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$ and let $S$ denote the corresponding switching element. Then for all nonzero $X \in \mathcal{A}$ the following (i), (ii) are equivalent.

(i) $X$ is a scalar multiple of $S$.

(ii) $X[0] \subseteq [0]$, $X[D] \subseteq [D]$, and $X[0^*] \subseteq [D^*]$.

Suppose (i), (ii) hold. Then equality is attained everywhere in (ii).

**Proof.** (i)$\Rightarrow$(ii): We show $S[0] = [0]$, $S[D] = [D]$, and $S[0^*] = [D^*]$. For $0 \leq i \leq d$ we find $SE_i V \subseteq E_i V$ since $S$ is a polynomial in $A$, and $SE_i V = E_i V$ since $S^{-1}$ exists. Therefore $S[0] = [0]$ and $S[D] = [D]$. We now show that $S[0^*] = [D^*]$. To this end we fix an integer $i$ ($0 \leq i \leq d$) and show

$$S(E_0^* V + E_1^* V + \cdots + E_i^* V) = E_d^* V + E_{d-1}^* V + \cdots + E_{d-i}^* V.$$  \tag{20}$$

Using Lemma 6.3 and Theorem 6.6 we find that for $0 \leq j \leq i$,

$$SE_j^* V = u_d(A)u_j(A)E_0^* V = u_j(A)u_d(A)E_0^* V = u_j(A)E_d^* V.$$

By (3) and since the polynomial $u_j$ has degree $j$,

$$u_j(A)E_d^* V \subseteq E_{d-j}^* V + E_{d-j+1}^* V + \cdots + E_d^* V.$$

Combining these comments we obtain

$$SE_j^* V \subseteq E_{d-j}^* V + E_{d-j+1}^* V + \cdots + E_d^* V,$$

and it follows that

$$S(E_0^* V + E_1^* V + \cdots + E_i^* V) \subseteq E_d^* V + E_{d-1}^* V + \cdots + E_{d-i}^* V.$$

In the above inclusion each side has the same dimension since $S^{-1}$ exists, so the inclusion becomes equality and (20) holds.

(ii)$\Rightarrow$(i): By Theorem 6.7 it suffices to show that $X \in \mathcal{D}$ and $XE_0^* V \subseteq E_d^* V$. We first show $X \in \mathcal{D}$. Recall that the flags $[0]$, $[D]$ induce the decomposition $[0D]$. We assume $X[0] \subseteq [0]$ and $X[D] \subseteq [D]$ so $X[0D] \subseteq [0D]$. This means that $XE_i V \subseteq E_i V$ for $0 \leq i \leq d$, so $X \in \mathcal{D}$. To get $XE_0^* V \subseteq E_d^* V$, consider the $0^\text{th}$ component in the inclusion $X[0^*] \subseteq [D^*]$.

Suppose (i), (ii) hold. We mentioned in the proof of (i)$\Rightarrow$(ii) that $S[0] = [0]$, $S[D] = [D]$, and $S[0^*] = [D^*]$. But $X$ is nonzero and a scalar multiple of $S$ so $X[0] = [0]$, $X[D] = [D]$, and $X[0^*] = [D^*]$. \qed
10 The action of $S$ on the decompositions

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let $S$ denote the corresponding switching element. In this section we characterize $S$ via its action on the decompositions from Definition 8.5.

**Theorem 10.1** Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let $S$ denote the corresponding switching element. Then for all nonzero $X \in A$ the following (i), (ii) are equivalent.

(i) $X$ is a scalar multiple of $S$.

(ii) $X[0^*0] \subseteq [D^*0]$ and $X[0^*D] \subseteq [D^*D]$.

Suppose (i), (ii) hold. Then equality holds everywhere in (ii).

**Proof.** (i)⇒(ii): By Theorem 9.1 we have $S[0^*0] \subseteq [D^*0]$ and $S[0] \subseteq [0]$ so $S[0^*0] \subseteq [D^*0]$. Since each component of a decomposition has dimension 1 and since $S^{-1}$ exists, we find $S[0^*0] = [D^*0]$. In a similar way we obtain $S[0^*D] = [D^*D]$.

(ii)⇒(i): By Theorem 9.1 it suffices to show $X[0] \subseteq [0]$, $X[D] \subseteq [D]$, and $X[0^*] \subseteq [D^*]$. We first show $X[0] \subseteq [0]$. By Lemma 8.6(i) and since $X[0^*0] \subseteq [D^*0]$ we find $X[00^*] \subseteq [0D^*]$. The decompositions $00^*$ and $0D^*$ each induce the flag $[0]$ by Lemma 8.6(iii) so $X[0] \subseteq [0]$. Next we show $X[D] \subseteq [D]$. By Lemma 8.6(i) and since $X[0^*D] \subseteq [D^*D]$ we find $X[0D^*] \subseteq [DD^*]$. The decompositions $0D^*$ and $DD^*$ each induce the flag $[D]$ by Lemma 8.6(iii) so $X[D] \subseteq [D]$. Finally we show $X[0^*] \subseteq [D^*]$. By Lemma 8.6(iii) we find that $0^*D$ induces $[0^*]$ and $D^*D$ induces $[D^*]$. By this and since $X[0^*D] \subseteq [D^*D]$ we find $X[0^*] \subseteq [D^*]$. Suppose (i), (ii) hold. We mentioned in the proof of (i)⇒(ii) that $S[0^*0] = [D^*0]$ and $S[0^*D] = [D^*D]$. But $X$ is nonzero and a scalar multiple of $S$ so $X[0^*0] = [D^*0]$ and $X[0^*D] = [D^*D]$. □

11 Some group commutators

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system with switching element $S$ and dual switching element $S^*$. In this section we consider linear transformations such as $S^*S^{-1}S^{*-1}S$. As we will see, these maps are closely related to the decompositions from Definition 8.5. We start with a lemma.
Lemma 11.1 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)\) denote a Leonard system in \(A\), with switching element \(S\) and dual switching element \(S^*\). Then referring to Definition 8.2 the following (i)–(iv) hold.

(i) \(S^*S^{-1}S^{-1}S\) fixes each of \([0^*], [D]\).
(ii) \(S^*SS^{-1}S^{-1}\) fixes each of \([D^*], [D]\).
(iii) \(S^*-1S^*S\) fixes each of \([0^*], [0]\).
(iv) \(S^*-1SS^*S^{-1}\) fixes each of \([D^*], [0]\).

Proof. By Theorem 9.1 we find \(S[0] = [0], S[D] = [D], \) and \(S[0^*] = [D^*]\). Applying this to \(\Phi^*\) we find \(S^*[0^*] = [0^*], S^*[D^*] = [D^*], \) and \(S^*[0] = [D]\). Combining these comments we routinely obtain the result. □

Corollary 11.2 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)\) denote a Leonard system in \(A\), with switching element \(S\) and dual switching element \(S^*\). Then referring to Definition 8.5 the following (i)–(iv) hold.

(i) \(S^*S^{-1}S^{-1}S^*\) fixes \([0^*D]\).
(ii) \(S^*SS^{-1}S^{-1}\) fixes \([D^*D]\).
(iii) \(S^*-1S^*S\) fixes \([0^*0]\).
(iv) \(S^*-1SS^*S^{-1}\) fixes \([D^*0]\).

Proof. (i): For notational convenience abbreviate \(T = S^*S^{-1}S^{-1}S\). By Lemma 11.1 we have \(T[0^*] = [0^*]\) and \(T[D] = [D]\) so \(T[0^*D] \subseteq [0^*D]\). By this and since \(T^{-1}\) exists we find \(T[0^*D] = [0^*D]\).

(ii)–(iv) Apply (i) to the relatives of \(\Phi\) and use Theorem 5.4 □

Referring to Corollary 11.2, each part (i)–(iv) is asserting that for \(0 \leq i \leq d\), the \(i^{th}\) component of the given decomposition is an eigenspace for the given operator. We now find the corresponding eigenvalue. We will focus on case (i); the eigenvalues for the remaining cases will be found using the \(D_4\) action.
Lemma 11.3 [22, Theorem 5.2] Let \( (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a Leonard system and let \( (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \) denote the corresponding parameter array. Then for \( 0 \leq i \leq d \),

\[
\begin{align*}
\eta_i(A)E_0^*E_0 &= \frac{\varphi_1\varphi_2 \cdots \varphi_i}{\eta_d(\theta_0^*)} \eta_{d-i}(A^*)E_0, \\
\eta_i(A)E_d^*E_0 &= \frac{\varphi_1\varphi_2 \cdots \varphi_{d-i+1}}{\tau_d^*(\theta_0^*)} \tau_{d-i}(A^*)E_0, \\
\tau_i(A)E_0^*E_d &= \frac{\varphi_1\varphi_2 \cdots \varphi_i}{\eta_d(\theta_0)} \eta_{d-i}(A)E_d, \\
\tau_i(A)E_d^*E_d &= \frac{\varphi_1\varphi_2 \cdots \varphi_i}{\eta_d(\theta_d)} \eta_{d-i}(A)E_d^*, \\
\eta_i^*(A^*)E_0^*E_d^* &= \frac{\varphi_1\varphi_2 \cdots \varphi_i}{\eta_d(\theta_0)} \eta_{d-i}(A)E_0^*, \\
\eta_i^*(A^*)E_d^*E_0 &= \frac{\varphi_1\varphi_2 \cdots \varphi_i}{\eta_d(\theta_0)} \eta_{d-i}(A)E_0^*.
\end{align*}
\]

Lemma 11.4 [22, Theorem 5.6] Let \( (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a Leonard system and let \( (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \) denote the corresponding parameter array. Then

\[
E_0^*E_d^*E_0 = \frac{\varphi_1\varphi_2 \cdots \varphi_d}{\tau_d(\theta_d)\tau_d^*(\theta_d)} E_0^*. \tag{29}
\]
Lemma 11.5 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system with switching element \(S\) and dual switching element \(S^*\). Then the following (i)–(iv) hold.

(i) \(SE_0^*\) is equal to each of
\[
\frac{\tau_d(\theta_d)^e(\theta_d^*)}{\varphi_1\varphi_2\cdots\varphi_d} E_d^*E_dE_0^*, \quad \frac{\eta_d(\theta_d)^e(\theta_d^*)}{\varphi_1\varphi_2\cdots\varphi_d} E_d^*E_dE_0^*.
\]

(ii) \(S^{-1}E_d^*\) is equal to each of
\[
\frac{\tau_d(\theta_d)^e(\theta_d^*)}{\phi_1\phi_2\cdots\phi_d} E_d^*E_dE_0^*, \quad \frac{\eta_d(\theta_d)^e(\theta_d^*)}{\phi_1\phi_2\cdots\phi_d} E_d^*E_dE_0^*.
\]

(iii) \(S^*E_0\) is equal to each of
\[
\frac{\tau_d(\theta_d)^e(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_d} E_d^*E_dE_0^*, \quad \frac{\eta_d(\theta_d)^e(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_d} E_d^*E_dE_0^*.
\]

(iv) \(S^{*-1}E_d\) is equal to each of
\[
\frac{\tau_d(\theta_d)^e(\theta_d^*)}{\phi_1\phi_2\cdots\phi_d} E_d^*E_dE_0^*, \quad \frac{\eta_d(\theta_d)^e(\theta_d^*)}{\phi_1\phi_2\cdots\phi_d} E_d^*E_dE_0^*.
\]

Proof. We first show that \(SE_0^*\) is equal to the expression on the left in (30). By Theorem 5.4 we have \(SE_0^*V = E_d^*V\) so \(SE_0^* = E_d^*SE_0^*\). The element \(E_d^*E_dE_0^*\) is nonzero by (29) and Lemma 4.7 so it forms a basis for \(E_d^*AE_0^*\). This space contains \(SE_0^*\) so there exists \(\alpha \in \mathbb{K}\) such that \(SE_0^* = \alpha E_d^*E_dE_0^*\). To find \(\alpha\), note that \(E_0S = E_0\) by (10) so \(E_0E_0^* = \alpha E_0E_d^*E_dE_0^*\). Comparing this with (29) we find
\[
\alpha = \frac{\tau_d(\theta_d)^e(\theta_d^*)}{\varphi_1\varphi_2\cdots\varphi_d}.
\]

We have now shown that \(SE_0^*\) is equal to the expression on the left in (30). To obtain the remaining assertions, apply \(D_4\) and use Lemma 4.6 and Theorem 5.4. \(\square\)

Lemma 11.6 Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system with switching element \(S\) and dual switching element \(S^*\). Then for \(0 \leq i \leq d\),
\[
S^*S^{-1}S^{*-1}S\tau_i(A)E_0^* = \frac{\phi_1\phi_2\cdots\phi_i\varphi_1\varphi_2\cdots\varphi_{d-i}}{\varphi_1\varphi_2\cdots\varphi_i\phi_1\phi_2\cdots\phi_{d-i}} \tau_i(A)E_0^*.
\]

Proof. We evaluate the expression on the left in (34). Recall \(S\), \(A\) commute; pull \(S\) to the right past \(\tau_i(A)\). Now evaluate \(SE_0^*\) using the expression on the left in (30) and in the resulting expression evaluate \(\tau_i(A)E_d^*E_d\) using (29); we find the left-hand side of (34) is a scalar multiple of
\[
S^*S^{-1}S^{*-1}\tau_{d-i}(A^*)E_dE_0^*.
\]
In line (35) pull $S^{*-1}$ to the right past $\tau_{d-i}(A^*)$. Now evaluate $S^{*-1}E_d$ using the expression on the left in (33) and in the resulting expression evaluate $\tau_{d-i}(A^*)E_0E_d^*$ using (27); this shows (35) is a scalar multiple of
\[ S^*S^{-1}\eta_i(A)E_d^*E_dE_0^*. \tag{36} \]

In line (36) pull $S^{-1}$ to the right past $\eta_i(A)$. Now evaluate $S^{-1}E_d^*$ using the expression on the right in (31) and in the resulting expression evaluate $\eta_i(A)E_0E_d^*$ using (21); this shows (36) is a scalar multiple of
\[ S^*\eta_{d-i}(A^*)E_0E_d^*E_0^*. \tag{37} \]

In line (37) pull $S^*$ to the right past $\eta_{d-i}(A^*)$. Now evaluate $S^*E_0E_d^*E_d$ using the expression on the left in (31) and in the resulting expression evaluate $\eta_{d-i}(A^*)E_0E_0^*$ using (20); this shows (37) is a scalar multiple of $\tau_i(A)E_0^*$. By the above comments we find that the left-hand side of (34) is a scalar multiple of $\tau_i(A)E_0^*$. Keeping track of the scalar we routinely verify (34). \hfill $\square$

\textbf{Theorem 11.7} Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\varphi_i^*\}_{i=1}^d)$, switching element $S$ and dual switching element $S^*$. Then the following (i)--(iv) hold for $0 \leq i \leq d$.

(i) The eigenvalue of $S^*S^{-1}S^{*-1}S$ on the $i^{th}$ component of $[0^*D]$ is
\[ \frac{\phi_1\phi_2\cdots\phi_i\varphi_1\varphi_2\cdots\varphi_{d-i}}{\varphi_1\varphi_2\cdots\varphi_i\phi_1\phi_2\cdots\phi_{d-i}}. \tag{38} \]

(ii) The eigenvalue of $S^*S^{*-1}S$ on the $i^{th}$ component of $[D^*D]$ is
\[ \frac{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}\phi_d\phi_{d-1}\cdots\phi_{i+1}}{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}\varphi_d\varphi_{d-1}\cdots\varphi_{i+1}}. \tag{39} \]

(iii) The eigenvalue of $S^*S^{-1}S^*S$ on the $i^{th}$ component of $[0^*0]$ is
\[ \frac{\varphi_1\varphi_2\cdots\varphi_i\phi_1\phi_2\cdots\phi_{d-i}}{\phi_1\phi_2\cdots\phi_i\varphi_1\varphi_2\cdots\varphi_{d-i}}. \tag{40} \]

(iv) The eigenvalue of $S^*S^{*-1}S^*S^{-1}$ on the $i^{th}$ component of $[D^*0]$ is
\[ \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}\varphi_d\varphi_{d-1}\cdots\varphi_{i+1}}{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}\phi_d\phi_{d-1}\cdots\phi_{i+1}}. \tag{41} \]

\textbf{Proof.} (i): Let $\varepsilon_i$ denote the expression in (38). We show $S^*S^{-1}S^{*-1}S - \varepsilon_iI$ is zero on the $i^{th}$ component of $[0^*D]$. But this is immediate from Lemma 11.6 and since this $i^{th}$ component equals $\tau_i(A)E_0^*V$ by Lemma 11.6.(i).

(ii)--(iv): Apply the $D_i$ action and use Lemma 4.6 Theorem 5.3. \hfill $\square$
12 Representing the elements $S$, $S^*$, $S^{-1}$, $S^{*-1}$ by matrices

**Definition 12.1** Let $(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and fix a nonzero $v_0^* \in E_0^* V$. By Lemma 8.8(i) the vectors $\tau_i(A)v_0^*$ $(0 \leq i \leq d)$ form a basis for $V$. For $X \in A$ let $X^2$ denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents $X$ with respect to this basis. We observe $\frac{\tau_i(A)}{v_0^*} : A \rightarrow \text{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras.

**Example 12.2** [38] Section 21] Let $\Phi = (A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let the isomorphism $\frac{\tau_i(A)}{v_0^*} : A \rightarrow \text{Mat}_{d+1}(\mathbb{K})$ be as in Definition 12.1. Then

$$ A^2 = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & 0 \\ 1 & \theta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \theta_d \end{pmatrix}, \quad A^*\tau = \begin{pmatrix} \theta_0^* & \varphi_1 & \varphi_2 & \cdots & 0 \\ \theta_1^* & \theta_1 & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \varphi_d & \theta_d & \vdots \end{pmatrix}, $$

where $(\{\theta_i\}_{i=0}^d;\{\theta_i^*\}_{i=0}^d;\{\varphi_i\}_{i=0}^d;\{\phi_i\}_{i=1}^d)$ denotes the parameter array of $\Phi$.

Let $\Phi = (A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$, with switching element $S$ and dual switching element $S^*$. Our goal for this section is to find $S^2$, $(S^{-1})^2$, $S^{*2}$, $(S^{*-1})^2$.

**Lemma 12.3** Let $(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d)$ denote a Leonard system with switching element $S$ and dual switching element $S^*$. Then for $0 \leq i \leq d$,

$$ S^*\eta_{d-i}(A^*)E_d = \frac{\phi_1^* \phi_2 \cdots \phi_i}{\varphi_1^* \varphi_2 \cdots \varphi_i} \tau_{d-i}(A^*)E_d, \quad (42) $$

$$ S^*\eta_i(A^*)E_0 = \frac{\phi_i^* \phi_{d-1} \cdots \phi_{d-i+1}}{\varphi_i^* \varphi_{d-1} \cdots \varphi_{d-i+1}} \tau_i(A^*)E_0, \quad (43) $$

$$ S^*\eta_{d-i}(A)E^*_d = \frac{\phi_i^* \phi_2 \cdots \phi_i}{\varphi_i^* \varphi_2 \cdots \varphi_i} \tau_{d-i}(A)E^*_d, \quad (44) $$

$$ S^*\eta_i(A)E^*_0 = \frac{\phi_i^* \phi_2 \cdots \phi_i}{\varphi_i^* \varphi_2 \cdots \varphi_i} \tau_i(A)E^*_0. \quad (45) $$

**Proof.** We first show (42). Using (23) we find that the left-hand side of (42) is a scalar multiple of

$$ S\tau_i(A)E^*_0 E_d. \quad (46) $$

In (10) pull $S$ to the right past $\tau_i(A)$. Now evaluate $SE^*_0$ using the expression on the left in (38) and in the resulting expression evaluate $\tau_i(A)E^*_d E_d$ using (24); this shows that (46) is a scalar multiple of

$$ \tau_{d-i}(A^*)E_d E^*_0 E_d. \quad (47) $$

By [38] Theorem 23.8],

$$ E_0 E^*_0 E_0 = \frac{\phi_1^* \phi_2 \cdots \phi_d}{\eta_d(\theta_0)\eta_d^*(\theta_0^*)} E_0.$$
Applying \(\downarrow\) to this and using Lemma 1.6 we find
\[
E_d E_0^* E_d = \frac{\varphi_1 \varphi_2 \cdots \varphi_d}{\tau_d(\theta_d) \eta_d^*} E_d.
\]
Using this we find that (17) is a scalar multiple of \(\tau_{d-i}^*(A^*) E_d\). By the above comments the left-hand side of (12) is a scalar multiple of \(\tau_{d-i}^*(A^*) E_d\). Keeping track of the scalar we routinely verify (12). To obtain (43)–(45) apply \(D_4\) to (12) using Lemma 1.6 and Theorem 5.4. □

Before we proceed we recall some scalars. Given a Leonard system \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) and given nonnegative integers \(r, s, t\) such that \(r + s + t \leq d\), in \([31]\) Definition 13.1 we defined a scalar \([r, s, t]_q \in \mathbb{K}\), where \(q + q^{-1} + 1\) is the common value of (9). For example, if \(q \neq 1\) and \(q \neq -1\) then
\[
[r, s, t]_q = \frac{(q; q)_{r+s}(q; q)_{r+t}(q; q)_{s+t}}{(q; q)_r(q; q)_s(q; q)_t(q; q)_{r+s+t}},
\]
where
\[
(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).
\]
We mention some features of \([r, s, t]_q\) that we will use. By \([31]\) Lemma 13.2 we find \([r, s, t]_q\) is symmetric in \(r, s, t\). We also have the following.

**Lemma 12.4** Referring to Definition 4.2, for \(0 \leq j \leq d\) we have
\[
\tau_j = \sum_{i=0}^j [i, j-i, d-j]_q \tau_{j-i}(\theta_d) \eta_i,
\]
\[
\eta_j = \sum_{i=0}^j [i, j-i, d-j]_q \eta_{j-i}(\theta_0) \tau_i,
\]
\[
\tau_j^* = \sum_{i=0}^j [i, j-i, d-j]_q \tau_{j-i}^*(\theta_d^*) \eta_i^*,
\]
\[
\eta_j^* = \sum_{i=0}^j [i, j-i, d-j]_q \eta_{j-i}^*(\theta_0^*) \tau_i^*.
\]

**Proof.** This is a routine consequence of \([31]\) Theorem 15.2. □

**Lemma 12.5** Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a Leonard system with switching element \(S\). Then for \(0 \leq j \leq d\),
\[
S \tau_j(A) = \sum_{i=j}^d [j, i-j, d-i]_q \frac{\phi_d \phi_{d-1} \cdots \phi_{d-j+1} \tau_{i-j}^*(\theta_d^*)}{\varphi_1 \varphi_2 \cdots \varphi_i} \tau_i(A),
\]
\[
S^{-1} \tau_j(A) = \sum_{i=j}^d [j, i-j, d-i]_q \frac{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}{\phi_1 \varphi_2 \cdots \varphi_j \eta_{i-j}^*(\theta_0^*)} \tau_i(A).
\]

**Proof.** Concerning (52), let \(L\) (resp. \(R\)) denote the expression on the left (resp. right). We show \(L = R\). To do this we first show that \(L E_0^* = R E_0^*\). We evaluate \(L E_0^*\) using (26)

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and in the resulting expression evaluate $S\eta_{d-j}^* (A^*) E_0$ using (12); this shows $LE_0^*$ is a scalar multiple of

$$\tau_{d-j}^* (A^*) E_0 E_0^*.$$  

(54)

Now in (53) evaluate $\tau_{d-j}^* (A^*)$ using (50) and in the resulting expression evaluate $\eta_i^* (A^*) E_0 E_0^*$ using (20); this shows that (54) is a scalar multiple of

$$\sum_{i=0}^{d-j} [i, d - j - i, j] q \varphi_a \varphi_{d-1} \cdots \varphi_{d-i+1} \tau_{d-j-i}^* (\theta_d^*) \tau_{d-i} (A) E_0^*.$$  

In this expression we replace $i$ by $d - i$ and find it is equal to

$$\sum_{i=d-j}^{d} [j, i - j, d - i] q \varphi_a \varphi_{d-1} \cdots \varphi_{d-i+1} \tau_{i-j}^* (\theta_d^*) \tau_i (A) E_0^*.$$  

(55)

So far we have shown that $LE_0^*$ is a scalar multiple of (55). Keeping track of the scalar we routinely find $LE_0^* = RE_0^*$. Now $L = R$ by Lemma 4.7. To obtain (53), apply $\downarrow$ to (52) and recall $S^\downarrow = S^{-1}$ from Theorem 5.2.

\[ \square \]

**Theorem 12.6** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=0}^d; \{\phi_i\}_{i=0}^d)$ denote the corresponding parameter array. Let $S$ denote the switching element for $\Phi$ and let the isomorphism $\xi : A \to \text{Mat}_{d+1}(\mathbb{K})$ be from Definition 12.7. Then each of $S^2$, $(S^{-1})^2$ is lower triangular. Moreover for $0 \leq j \leq i \leq d$ their $(i, j)$ entries are given as follows.

$$S^2_{i,j} = \frac{[j, i - j, d - i] q \phi_a \phi_{d-1} \cdots \phi_{d-j+1} \tau_{i-j}^* (\theta_d^*)}{\varphi_1 \varphi_2 \cdots \varphi_i},$$

$$(S^{-1})^2_{i,j} = \frac{[j, i - j, d - i] q \varphi_a \varphi_{d-1} \cdots \varphi_{d-i+1} \tau_{i-j} (\theta_d^*)}{\phi_1 \phi_2 \cdots \phi_i}.$$  

**Proof.** Follows from Lemma 12.5 and Definition 12.1. \[ \square \]

**Lemma 12.7** Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system with dual switching element $S^*$. Then for $0 \leq j \leq d$,

$$S^* \tau_j (A) E_0^* = \sum_{i=0}^{j} \frac{[i, j - i, d - j] q \phi_1 \phi_2 \cdots \phi_i \tau_{j-i} (\theta_d)}{\varphi_1 \varphi_2 \cdots \varphi_i} \tau_i (A) E_0^*,$$  

(56)

$$S^{*-1} \tau_j (A) E_0^* = \sum_{i=0}^{j} \frac{[i, j - i, d - j] q \varphi_1 \varphi_2 \cdots \varphi_i \eta_{j-i} (\theta_0)}{\phi_1 \phi_2 \cdots \phi_j} \tau_i (A) E_0^*.$$  

(57)

**Proof.** First we show (56). In the left-hand side of (56) we evaluate $\tau_j (A)$ using (48) to find

$$S^* \tau_j (A) E_0^* = \sum_{i=0}^{j} [i, j - i, d - j] q \tau_{j-i} (\theta_d) S^* \eta_i (A) E_0^*.$$  

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In this equation we evaluate $S^* \eta_i(A)E_0^*$ using (45) and get (56).

Next we show (57). By (45),

$$S^{*-1} \tau_j(A)E_0^* = \frac{\varphi_1 \varphi_2 \cdots \varphi_j \eta_j(A)E_0^*}{\phi_1 \phi_2 \cdots \phi_j}.$$ 

In this equation we evaluate $\eta_j(A)$ using (49) and get (57). □

Theorem 12.8 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $A$ and let $\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d$ denote the corresponding parameter array. Let $S^*$ denote the dual switching element for $\Phi$ and let the isomorphism $\# : A \to \text{Mat}_{d+1}(K)$ be from Definition 12.1. Then each of $S^*$, $(S^*)^2$ is upper triangular. Moreover for $0 \leq i \leq j \leq d$ their $(i,j)$ entries are given as follows.

$$S^*_{i,j} = \frac{[i, j - i, d - j] \varphi_1 \varphi_2 \cdots \varphi_i \tau_j - i(\theta_d)}{\phi_1 \phi_2 \cdots \phi_j},$$

$$(S^*)^2_{i,j} = \frac{[i, j - i, d - j] \varphi_1 \varphi_2 \cdots \varphi_j \eta_j - i(\theta_0)}{\phi_1 \phi_2 \cdots \phi_j}.$$ 

Proof. Follows from Lemma 12.7 and Definition 12.1. □

13 Leonard pairs in matrix form

In this section we restate Theorem 12.6 and Theorem 12.8 in more concrete terms. Let us consider the following situation.

Definition 13.1 Let $d$ denote a nonnegative integer. Let $A$ and $A^*$ denote matrices in $\text{Mat}_{d+1}(K)$ of the form

$$A = \begin{pmatrix}
\theta_0 & 0 & & \\
1 & \theta_1 & 0 & \\
& 1 & \theta_2 & \\
& & & \ddots \\
0 & & & 1 & \theta_d
\end{pmatrix}, \quad A^* = \begin{pmatrix}
\theta_0^* & \varphi_1 & \varphi_2 & \cdots & 0 \\
\theta_1^* & \varphi_2 & \varphi_3 & \cdots & 0 \\
\theta_2^* & \varphi_3 & \varphi_4 & \cdots & 0 \\
& & & \ddots & \\
0 & \cdots & \varphi_d & \cdots & \theta_d^*
\end{pmatrix},$$

(58)

where

$\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad (0 \leq i, j \leq d),$

$\varphi_i \neq 0 \quad (1 \leq i \leq d).$

Observe $A$ (resp. $A^*$) is multiplicity-free, with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). For $0 \leq i \leq d$ let $E_i$ (resp. $E_i^*$) denote the primitive idempotent of $A$ (resp. $A^*$) associated with $\theta_i$ (resp. $\theta_i^*$).
Lemma 13.2 Referring to Definition 13.1, the following (i), (ii) are equivalent.

(i) The pair $A, A^*$ is a Leonard pair.

(ii) The sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system.

Suppose (i), (ii) hold. Then the sequence $\{\varphi_i\}_{i=1}^d$ is the first split sequence of $\Phi$. Moreover the isomorphism $\natural$ from Definition 12.1 is the identity map.

Proof. The equivalence of (i), (ii) is established in [35, Lemma 6.2]. By [32, Theorem 14.3], the sequence $\{\varphi_i\}_{i=1}^d$ is the first split sequence of $\Phi$. Comparing Example 12.2 with (58) we find $A^\natural = A$ and $A^{\natural*} = A^*$. Now $\natural$ is the identity map since $A, A^*$ generate $\text{Mat}_{d+1}(K)$ by [37, Corollary 5.5]. □

Theorem 13.3 Referring to Lemma 13.2, assume the equivalent conditions (i), (ii) hold, and let $S$ denote the switching element element for $\Phi$. Then each of $S, S^{-1}$ is lower triangular. Moreover for $0 \leq j \leq i \leq d$ their $(i, j)$ entries are given as follows.

$$S_{i,j} = \left[ j, i-j, d-j \right] q \varphi_1 \varphi_2 \cdots \varphi_i \phi_{d-1} \cdots \phi_{d-i+1} (\theta_d^*)^{i-j} / \varphi_1 \varphi_2 \cdots \varphi_i ;$$

$$S^{-1}_{i,j} = \left[ j, i-j, d-j \right] q \varphi_1 \varphi_2 \cdots \varphi_j \eta_{i-j}^* (\theta_0^*) / \phi_1 \phi_2 \cdots \phi_j .$$

In the above lines $\{\varphi_i\}_{i=1}^d$ is the second split sequence of $\Phi$.

Proof. Combine Theorem 12.6 with Lemma 13.2. □

Theorem 13.4 Referring to Lemma 13.2, assume the equivalent conditions (i), (ii) hold, and let $S^*$ denote the dual switching element element for $\Phi$. Then each of $S^*, S^{*-1}$ is upper triangular. Moreover for $0 \leq i \leq j \leq d$ their $(i, j)$ entries are given as follows.

$$S^*_{i,j} = \left[ i, j-i, d-j \right] q \varphi_1 \varphi_2 \cdots \varphi_i \phi_{j-i-1} (\theta_d) / \varphi_1 \varphi_2 \cdots \varphi_i ;$$

$$S^{*-1}_{i,j} = \left[ i, j-i, d-j \right] q \varphi_1 \varphi_2 \cdots \varphi_j \eta_{j-i} (\theta_0) / \phi_1 \phi_2 \cdots \phi_j .$$

In the above lines $\{\varphi_i\}_{i=1}^d$ is the second split sequence of $\Phi$.

Proof. Combine Theorem 12.8 with Lemma 13.2. □
14 A characterization of a Leonard system in terms of the switching element

In this section we give a characterization of a Leonard system in terms of its switching element. This characterization is a variation on [35, Theorem 6.3] and is stated as follows.

**Theorem 14.1** Referring to Definition [33.1] the following (i), (ii) are equivalent.

(i) The pair $A, A^*$ is a Leonard pair.

(ii) There exists an invertible $X \in \text{Mat}_{d+1}(K)$ and there exist nonzero scalars $\phi_i \in K$ $(1 \leq i \leq d)$ such that $X^{-1}AX = A$ and

\[
X^{-1}A^*X = \begin{pmatrix}
\theta^*_d & \phi_d & \phi_{d-1} & 0 \\
\theta^*_{d-1} & \phi_{d-1} & \phi_{d-2} & \\
& \ddots & \ddots & \ddots \\
0 & & \phi_1 & \theta_0^*
\end{pmatrix}.
\] (59)

Suppose (i), (ii) hold. Then $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ is a Leonard system, $\phi_i^d$ is the second split sequence of $\Phi$, and $X$ is a scalar multiple of the switching element for $\Phi$.

**Proof.** For notational convenience we abbreviate $V = K^{d+1}$. For $0 \leq i \leq d$ let $e_i$ denote the vector in $V$ with $i^{th}$ coordinate 1 and all other coordinates 0. Observe that $\{e_i\}_{i=0}^d$ is a basis for $V$. From the form of $A$ in (58) we find

\[
(A - \theta_iI)e_i = e_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_dI)e_d = 0.
\] (60)

From the form of $A^*$ in (58) we find

\[
(A^* - \theta^*_iI)e_i = \varphi_i e_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta^*_0I)e_0 = 0.
\] (61)

(i)⇒(ii): By Lemma [13.2] the sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ is a Leonard system in $\text{Mat}_{d+1}(K)$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$, and first split sequence $\{\varphi_i\}_{i=0}^d$. Let $\{\phi_i\}_{i=0}^d$ denote the second split sequence of $\Phi$ and let $S$ denote the switching element for $\Phi$. Then $S^{-1}$ exists by Lemma [5.3]. Also $S$ commutes with $A$ by Theorem [6.7] so $S^{-1}AS = A$. We now show that (59) holds with $X = S$. From (60) we find

\[
e_i = \tau_i(A)e_0 \quad (0 \leq i \leq d).
\] (62)

From the equation on the right in (61) we find $A^*e_0 = \theta^*_0e_0$; using this and $e_0 \neq 0$ we find $e_0$ is a basis for $E_0^*V$. By this and (61), (62) we have

\[
(A^* - \theta^*_iI)e_i = \varphi_i \tau_{i-1}(A)e_0^* \quad (1 \leq i \leq d), \quad (A^* - \theta^*_0I)e_0^* = 0.
\] (63)

We apply $\downarrow$ to (63) and use Lemma [14.6] to find

\[
(A^* - \theta^*_{d-i}I)e_d = \phi_{d-i+1} \tau_{d-i-1}(A)e_d^* \quad (1 \leq i \leq d), \quad (A^* - \theta^*_dI)e_d^* = 0.
\] (64)

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By Theorem 6.1 and since $e_0 \in E_0^* V$ we find $Se_0 \in E_{0}^* V$. Combining this with (64) we have

$$(A^* - \theta_{d-i}^* I)\tau_i(A)Se_0 = \phi_{d-i+1}\tau_i(A)Se_0 \quad (1 \leq i \leq d), \quad (A^* - \theta_{d}^* I)Se_0 = 0. \tag{65}$$

Evaluating (65) using $S^{-1}AS = A$ and (62) we routinely find

$$(S^{-1}A^*S - \theta_{d-i}^* I)e_i = \phi_{d-i+1}e_{i-1} \quad (1 \leq i \leq d), \quad (S^{-1}A^*S - \theta_{d}^* I)e_0 = 0.$$

By this we find (60) holds with $X = S$, as desired.

(ii)⇒(i): We show $\Phi = (A; \{E_i^d\}_{i=0}^{d}; A^*; \{E_i^d\}_{i=0}^{d})$ is a Leonard system in Mat$_{d+1}(\mathbb{K})$. To do this we invoke [35, Theorem 5.1]. According to that theorem it suffices to display a decomposition $\{U_i\}_{i=0}^{d}$ of $V$ such that

$$(A - \theta_{d-i} I)U_i = U_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_{d} I)U_d = 0, \tag{66}$$

$$(A^* - \theta_{d-i}^* I)U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta_{d}^* I)U_0 = 0, \tag{67}$$

and a decomposition $\{V_i\}_{i=0}^{d}$ of $V$ such that

$$(A - \theta_{d-i} I)V_i = V_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_{d} I)V_d = 0, \tag{68}$$

$$(A^* - \theta_{d-i}^* I)V_i = V_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta_{d}^* I)V_0 = 0. \tag{69}$$

Define $U_i = \text{Span}\{e_i\}$ for $0 \leq i \leq d$. The sequence $\{U_i\}_{i=0}^{d}$ is a decomposition of $V$ since $\{e_i\}_{i=0}^{d}$ is a basis for $V$. The decomposition $\{U_i\}_{i=0}^{d}$ satisfies (66) by (60). The decomposition $\{V_i\}_{i=0}^{d}$ satisfies (67) by (61) and since $\varphi_i \neq 0$ for $1 \leq i \leq d$. Now define $V_i = \text{Span}\{Xe_i\}$ for $0 \leq i \leq d$. The sequence $\{V_i\}_{i=0}^{d}$ is a decomposition of $V$ since $X^{-1}$ exists, and since $\{e_i\}_{i=0}^{d}$ is a basis for $V$. The decomposition $\{V_i\}_{i=0}^{d}$ satisfies (68) by (60) and since $X^{-1}AX = A$. The decomposition $\{V_i\}_{i=0}^{d}$ satisfies (69) by (69) and since $\phi_i \neq 0$ for $1 \leq i \leq d$. We have now verified (66)–(69) so [35, Theorem 5.1] applies; by that theorem $\Phi$ is a Leonard system in Mat$_{d+1}(\mathbb{K})$. Now the pair $A, A^*$ is a Leonard pair by Lemma 13.2.

Now suppose (i), (ii) hold. We mentioned in the proof of (i)⇒(ii) that $\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^d\}_{i=0}^{d})$ is a Leonard system in Mat$_{d+1}(\mathbb{K})$. Next we show that $X$ is a scalar multiple of the switching element $S$ for $\Phi$. To do this we invoke Theorem 6.7. Let $\mathcal{D}$ denote the subalgebra of Mat$_{d+1}(\mathbb{K})$ generated by $A$. The element $X$ commutes with $A$ so $X \in \mathcal{D}$. By the left-most column in (62) we find $X^{-1}A^*Xe_0 = \theta_{d}^* e_0$ so $Xe_0 \in E_{0}^* V$. But $e_0$ is a basis for $E_0^* V$ so $XE_0^* V \subseteq E_{0}^* V$. Now $X$ is a scalar multiple of $S$ by Theorem 6.7. We saw in the proof of (i)⇒(ii) that the sequence $\{\phi_i\}_{i=1}^{d}$ is the second split sequence of $\Phi$. \qed

The following is a variation on [36, Theorem 3.2].
Theorem 14.2 Let $d$ denote a nonnegative integer and let
\begin{equation}
(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \tag{70}
\end{equation}
denote a sequence of scalars taken from $\mathbb{K}$. Assume this sequence satisfies the conditions (PA1) and (PA2) in Theorem 4.5. Then the following (i), (ii) are equivalent.

(i) The sequence \( (70) \) satisfies (PA3)--(PA5) in Theorem 4.5.

(ii) There exists an invertible $X \in \text{Mat}_{d+1}(\mathbb{K})$ such that
\[
X^{-1} \begin{pmatrix} \theta_0 & 1 & \theta_1 & 1 & \theta_2 & \cdots & 0 \\ 0 & 1 & \theta_{d-1} & \cdots & \theta_d \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_d & \theta_d & \varphi_d & \cdots & 0 \\ \end{pmatrix} \begin{pmatrix} \theta_0 & 1 & \theta_1 & 1 & \theta_2 & \cdots & 0 \\ 0 & 1 & \theta_{d-1} & \cdots & \theta_d \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_d & \theta_d & \varphi_d & \cdots & 0 \\ \end{pmatrix} = \begin{pmatrix} \theta_0 & 1 & \theta_1 & 1 & \theta_2 & \cdots & 0 \\ 0 & 1 & \theta_{d-1} & \cdots & \theta_d \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \theta_0^* & \varphi_1 & \theta_1^* & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_d & \theta_d & \varphi_d & \cdots & 0 \\ \end{pmatrix}
\]

Proof. (i)$\Rightarrow$(ii): By Theorem 4.5 there exists a Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ over $\mathbb{K}$ that has parameter array \((70)\). Let $\natural$ denote the corresponding isomorphism from Definition 12.1. Applying $\natural$ to each term in $\Phi$ if necessary, we may assume $\Phi$ is in $\text{Mat}_{d+1}(\mathbb{K})$, and that $\natural$ is the identity map. Now $A, A^*$ are of the form \((58)\). Now (ii) holds by Theorem 14.1.

(ii)$\Rightarrow$(i): Follows from Theorem 14.1 and Theorem 4.5 $\square$

Note 14.3 We comment on how the switching element is related to the matrix $G$ that appears in [35, Theorem 6.3] and [36, Theorem 3.2]. In [35, Theorem 6.3] reference is made to a Leonard pair $A, A^*$ of the form \((58)\); let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote the corresponding Leonard system from Lemma 13.2. Then $G$ is a nonzero scalar multiple of $S^*^{-1}Y$, where $S^*$ denotes the dual switching element for $\Phi$ and $Y$ denotes the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$ whose \((i, i)\) entry is
\[
\frac{\phi_1 \phi_2 \cdots \phi_i}{\varphi_1 \varphi_2 \cdots \varphi_i}
\]
for $0 \leq i \leq d$.

Proof. From the data given in [35, Theorem 6.3] one readily verifies that $YG^{-1}$ is invertible and commutes with $A^*$. Moreover $YG^{-1}E_0V \subseteq E_dV$ where $V = \mathbb{K}^{d+1}$. Now applying Theorem 6.7 to $\Phi^*$ we find $YG^{-1}$ is a nonzero scalar multiple of $S^*$. So $G$ is a nonzero scalar multiple of $S^{-1}Y$. $\square$
15 Open problems

In this section $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denotes a Leonard system in $A$ with switching element $S$ and dual switching element $S^*$. We will be discussing the flags $[z] \quad (z \in \Omega)$ (71)

from Definition 8.2

Definition 15.1 For distinct $x, y \in \Omega$, by a switching element of type $(x, y)$ we mean an element $X$ in $A$ that sends $[x]$ to $[y]$ and fixes the remaining two flags in (71).

Example 15.2 By Theorem 9.1 we find that for nonzero $X \in A$,

(i) $X$ is a switching element of type $(0^*, D^*)$ if and only if $X$ is a scalar multiple of $S$.
(ii) $X$ is a switching element of type $(D^*, 0^*)$ if and only if $X$ is a scalar multiple of $S^{-1}$.
(iii) $X$ is a switching element of type $(0, D)$ if and only if $X$ is a scalar multiple of $S^*$.
(iv) $X$ is a switching element of type $(D, 0)$ if and only if $X$ is a scalar multiple of $S^*-1$.

Problem 15.3 Find a necessary and sufficient condition on the parameter array of $\Phi$ for there to exist a switching element of type $(0^*, D)$.

Problem 15.4 Find a necessary and sufficient condition on the parameter array of $\Phi$ for there to exist a switching element of type $(0^*, D)$ and a switching element of type $(0, D^*)$.

Note 15.5 For certain $\Phi$ there exists a second Leonard system $\Phi' = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ in $A$ such that the decomposition $\{F_iV\}_{i=0}^d$ coincides with $[0D^*]$ and the decomposition $\{F_i^*V\}_{i=0}^d$ coincides with $[0^*D]$. See for example 5. In this case the switching element for $\Phi'$ is a switching element for $\Phi$ of type $(0^*, D)$, and the dual switching element for $\Phi'$ is a switching element for $\Phi$ of type $(0, D^*)$. 

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