ODE/IM correspondence for modified $B_2^{(1)}$ affine Toda field equation

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Abstract

We study the massive ODE/IM correspondence for modified $B_2^{(1)}$ affine Toda field equation. Based on the $\psi$-system for the solutions of the associated linear problem, we obtain the Bethe ansatz equations. We also discuss the T–Q relations, the T-system and the Y-system, which are shown to be related to those of the $A_3/\mathbb{Z}_2$ integrable system. We consider the case that the solution of the linear problem has a monodromy around the origin, which imposes nontrivial boundary conditions for the T-/Y-system. The high-temperature limit of the T- and Y-system and their monodromy dependence are studied numerically.
1 Introduction

It has been recognized that the relation between classical and quantum integrable systems is useful for studying non-perturbative properties of supersymmetric gauge theories and the AdS/CFT correspondence [1, 2, 3]. The ODE/IM correspondence [1, 5, 6] provides an interesting example of this classical/quantum correspondence, which relates the spectral determinants of certain ordinary differential equations (ODE) to the Bethe ansatz equations in the massless limit of certain integrable models (IM). It is an interesting problem to make a complete list of this ODE/IM correspondence. The ordinary differential equations for the integrable models related to classical Lie algebras have been proposed in [7]. The Wronskian of the solutions obeys the functional relations called the $\psi$-system, which leads to the Bethe ansatz equations of the related quantum integrable system. The $\psi$-system for classical Lie algebra has been reformulated in the form of the matrix valued linear differential equations [8], where the Bethe ansatz equations of the integrable models associated with the untwisted affine Lie algebra $X^{(1)}$ of a classical Lie algebra $X$ are related to the linear differential equations associated with the Langlands dual ($X^{(1)}$)$^\vee$.

The ODE/IM correspondence has been generalized to massive integrable models. It was found that for the classical sinh-Gordon equation modified by a conformal transformation, the spectral problem for the associated linear problem leads to the functional relations of the quantum sine-Gordon model [9]. By taking the conformal limit, it reduces to the ODE/IM correspondence for the Schrödinger type differential equation [4, 5].

Recently the massive ODE/IM correspondence has been generalized to a class of modified affine Toda field equations [10, 11, 12, 13, 14, 15]. In particular, Locke and one of the present authors studied the modified affine Toda equations for affine Lie algebra $\hat{g}$, where $\hat{g}$ is an untwisted affine Lie algebra including exceptional type [13]. It has been shown that from their associated linear problems one obtains the $\psi$-system which leads to the Bethe ansatz equations for the affine Lie algebra $\hat{g}$.

It would be an interesting problem to explore the modified affine Toda field equation with an affine Lie algebra $\hat{g}$ which is not of the form of the Langlands dual of an untwisted one, where the corresponding integrable models are not identified yet. In this paper we will work with the modified affine Toda field equation associated with the affine Lie algebra $B^{(1)}_2$ (or $C^{(1)}_2$), which provides the simplest and nontrivial example. This equation also appears in the study of the area of minimal surface with a null-polygonal boundary in $AdS_4$ spacetime [16, 17, 18, 19], which is dual to the gluon scattering amplitudes with specific momentum configurations. The equation of motion of strings is described by the $B^{(1)}_2$ affine Toda field equation modified by the conformal transformation. The Stokes problem of the associated linear system determines the functional equations for the cross-ratios of external momenta. These functional relations are known to be the same as the $Y$-system of the homogeneous sine-Gordon model [20, 21] and the free energy of the $Y$-system determines the area of the minimal surface.

The purpose of this paper is to apply the massive ODE/IM correspondence to the modified $B^{(1)}_2$ affine Toda field equation and to investigate the functional relations for the
Stokes coefficients of the linear problem, which include the Bethe ansatz equations, the T–Q relations, the T-system and the Y-system. We study the boundary condition of the T-system arising from the nontrivial monodromy of the linear problem solution around the origin. This monodromy condition also appears in the study of the form factors via the AdS/CFT correspondence [22, 23].

This paper is organized as follows: In sect. 2 we introduce modified $B_2^{(1)}$ affine Toda equation and the associated linear problem. In sect. 3 we discuss the $\psi$-system and derive the Bethe ansatz equations. In sect. 4 we discuss the spinor representation of Eq. (2.1) can be written in the form of the compatibility condition.

In sect. 5 we argue the T-system and Y-system and their boundary conditions which come from the monodromy of the solution of the linear system around the origin. In sect. 6 we investigate the high-temperature limit of the Y-system in the presence of monodromy. Sect. 7 is devoted for conclusions and discussion. In the appendix, we summarize the auxiliary T-functions and their functional relations used in this paper.

2 Modified $B_2^{(1)}$ affine Toda field equation

The Lie algebra $B_2 = so(5)$ has simple roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$, where $e_i$ ($i = 1, 2$) is an orthonormal basis of $\mathbb{R}^2$. We denote the highest root by $\theta$, which is given by $\theta = \alpha_1 + 2\alpha_2$, and define the extended root $\alpha_0 = -\theta$. $\omega_1 = e_1$ and $\omega_2 = \frac{1}{2}(e_1 + e_2)$ are the fundamental weights satisfying $2\omega_i \cdot \alpha_j/\alpha_j^2 = \delta_{ij}$. Let $\{H^i, E^\alpha\} (i = 1, 2, \alpha \in \Delta)$ be the Chevalley basis of $B_2$, where $\Delta$ is the set of roots.

Let $\phi = (\phi^1, \phi^2)$ be the two-component scalar field on the complex plane with coordinates $(z, \bar{z})$. We define the modified affine Toda field equation for $B_2^{(1)}$ by

$$\partial \bar{\partial} \phi - \frac{m^2}{\beta} (\alpha_1 e^{\beta \alpha_1 \cdot \phi} + 2 \alpha_2 e^{\beta \alpha_2 \cdot \phi} + p(z) \bar{p}(\bar{z}) e^{\beta \alpha_0 \cdot \phi}) = 0.$$  \hspace{1cm} (2.1)

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, $m$ is a mass parameter and $\beta$ is a coupling parameter. $p(z)$ is a holomorphic function of $z$ and is chosen as

$$p(z) = z^{4M} - s^{4M},$$  \hspace{1cm} (2.2)

with $M > \frac{1}{4}$ and $s$ is a complex parameter. This equation is obtained by the conformal transformation $z \rightarrow w$ with $\frac{\partial w}{\partial z} = p^{\frac{1}{4}}$ and the field redefinition $\phi \rightarrow \phi - \frac{1}{4\beta} \rho^\nu \log(p \bar{p})$, where $\rho^\nu = \omega_1 + 2\omega_2$ is the co-Weyl vector. Note that the Coxeter number of $B_2$ is 4. Eq. (2.1) can be written in the form of the compatibility condition $[\partial + A_z, \bar{\partial} + A_{\bar{z}}] = 0$ of the linear differential equations defined in a $B_2$-module:

$$(\partial + A_z) \Psi = 0, \hspace{0.5cm} (\bar{\partial} + A_{\bar{z}}) \Psi = 0,$$  \hspace{1cm} (2.3)

Note that the modified equation in [17] is $\partial \bar{\partial} \phi - \frac{m^2}{\beta} (\alpha_1 \sqrt{p \bar{p}} e^{\beta \alpha_1 \cdot \phi} + 2 \alpha_2 e^{\beta \alpha_2 \cdot \phi} + \sqrt{p \bar{p}} e^{\beta \alpha_0 \cdot \phi}) = 0$, which is obtained by the same conformal transformation but a different field redefinition $\phi \rightarrow \phi - \frac{1}{4\beta} \rho^\nu \log(p \bar{p})$. This modified equation is related to (2.1) by a field redefinition.
where the connections are defined by
\[
A_z = \frac{\beta}{2} \partial \phi \cdot H + m e^\lambda \left( e^{\beta \alpha_1 \phi/2} E_{\alpha_1} + e^{\beta \alpha_2 \phi/2} E_{\alpha_2} + p(z) e^{\beta \alpha_0 \phi/2} E_{\alpha_0} \right),
\]
\[
\bar{A}_z = -\frac{\beta}{2} \partial \phi \cdot H + m e^{-\lambda} \left( e^{\beta \alpha_1 \phi/2} E_{-\alpha_1} + e^{\beta \alpha_2 \phi/2} E_{-\alpha_2} + \bar{p}(z) e^{\beta \alpha_0 \phi/2} E_{-\alpha_0} \right). \tag{2.4}
\]
Here \( \lambda \) is the spectral parameter. We are interested in the special class of solutions of \([2.1]\), which satisfy the periodicity condition \( \phi(\rho, \theta + \frac{\pi}{4M}) = \phi(\rho, \theta) \) and the boundary conditions at infinity and the origin of the complex plane:
\[
\phi(\rho, \theta) = \frac{2M \rho^\nu}{\beta} \log \rho + \cdots, \quad (\rho \to \infty) \tag{2.5}
\]
\[
\phi(\rho, \theta) = 2g \log \rho + \cdots, \quad (\rho \to 0) \tag{2.6}
\]
where we have introduced the polar coordinate \((\rho, \theta)\) by \( z = \rho e^{i\theta} \) and \( g \) is a 2-vector satisfying \( \beta \alpha_a \cdot g + 1 > 0 \) \((a = 0, 1, 2)\). Due to the special form \((2.2)\) of \( p(z) \), \((2.1)\) and the linear problem are invariant under the Symanzik rotation
\[
\hat{\Omega}_k : (z, s, \lambda) \to (ze^{\frac{2\pi i k}{4M}}, se^{\frac{2\pi i k}{4M}}, \lambda - \frac{2\pi i k}{M}), \tag{2.7}
\]
for an integer \(k\). This also acts on the solution \( \Psi(z, \bar{z}) \), which is denoted as \( \Psi_k(z, \bar{z}) := \hat{\Omega}_k \Psi(z, \bar{z}) \). The linear problem is also invariant under the transformation:
\[
\hat{\Pi} : (\lambda, A_z, A_{\bar{z}}, \Psi) \to (\lambda - \frac{2\pi i}{4M}, SA_z S^{-1}, SA_{\bar{z}} S^{-1}, S\Psi) \tag{2.8}
\]
where \( S = \exp(\frac{2\pi i}{4M} \rho^\nu \cdot H) \).

We now consider the solutions of the linear differential equations \([2.3]\) in the basic \( B_2^{(1)} \)-module \( V^{(a)} \) \((a = 1, 2)\) associated with the highest weight \( \omega_a \). Let \( e_j^{(a)} \) be the orthonormal basis of \( V^{(a)} \) with \( H^i \) eigenvalue \( (h_j^{(a)})^i \), where \( i, j = 1, \cdots, \dim V^{(a)} \). For the Lie algebra \( B_2, V^{(1)} \) is 5-dimensional vector representation, whose matrix representation is given by
\[
E_{\alpha_1} = e_{1,2} + e_{4,5}, \quad E_{\alpha_2} = \sqrt{2}(e_{2,3} + e_{3,4}), \quad E_{\alpha_0} = -(e_{1,2} + e_{5,2}) \tag{2.9}
\]
and \( E_{-\alpha_i} = E^{T}_{\alpha_i} \). Here \( e_{ab} \) denotes the matrix whose \((i, j)\)-element is \( \delta_{ia} \delta_{jb} \). Similarly, \( V^{(2)} \) is a 4-dimensional spinor representation. Its matrix representation is given by
\[
E_{\alpha_1} = e_{23}, \quad E_{\alpha_2} = e_{12} + e_{34}, \quad E_{\alpha_0} = e_{41} \tag{2.10}
\]
and \( E_{-\alpha_i} = E^{T}_{\alpha_i} \).

We are interested in the small (or subdominant) solution \( \Psi^{(a)} \), which decays fastest along the positive real axis. This was studied in \([13]\) for \( g^\nu \) for an untwisted affine Lie algebra \( g \). In general, the small solution \( \Psi^{(a)} \) at large \( \rho \) is given by
\[
\Psi^{(a)}(z, \bar{z}, \lambda, g) = C^{(a)} \exp \left( -2\mu^{(a)} \frac{\rho^{M+1}}{M+1} m \cosh(\lambda + i\theta(M+1)) \right) e^{-i\theta M \rho^\nu \cdot H} \mu^{(a)}, \tag{2.11}
\]
with $C^{(a)}$ being a normalization constant. Here $\mu^{(a)}$ and $\mu^{(a)}$ denote the eigenvector and its eigenvalue of the matrix $\Lambda_+ = E_{a_0} + E_{a_1} + E_{a_2}$ with the eigenvalue of the largest real part. Applying the Symanzik rotation $\hat{\Omega}_k$ ($k \in \mathbb{Z}$), one obtains the small solution $\Psi_k^{(a)}$ in the Stokes sector

$$s_{-k} : \left| \theta + \frac{2\pi k}{4(M + 1)} \right| < \frac{\pi}{4(M + 1)} \quad (2.12)$$

For the vector representation (2.9), the eigenvalues of $\Lambda_+$ are $\sqrt{2}e^{i\pi(2k+1)}$ ($k = 0, 1, 2, 3$) and 0. For the spinor representation (2.10), they are $\pm 1$ and $\pm i$. For $V^{(1)}$, one has two eigenvalues with the largest real part and the corresponding solutions in $V^{(1)}$ are not subdominant along the real axis. So we introduce the $\frac{1}{2}$-rotated Symanzik solution $\Psi^{(1)}_{\frac{1}{2}}$. This is a solution of the linear problem with the $\frac{1}{2}$-rotated connection $(A_{\frac{1}{2}})_z$ which is obtained by replacing $E_{\pm a_0} \rightarrow -E_{\pm a_0}$ in (2.4). Then the $\Psi^{(1)}_{\frac{1}{2}}$ behaves along the real positive axis as (2.11) with $\mu^{(1)}_1 = \sqrt{2}$ and $\mu^{(1)}_2 = (1, \sqrt{2}, \sqrt{2}, \sqrt{2}, 1)^T$.

We define the basis of the solutions around $\rho = 0$ behaves as $\rho \rightarrow 0$:

$$\chi_i^{(a)}(z, \bar{z}|\lambda, g) = e^{-(\lambda+i\theta)g h_i^{(a)}} e_i^{(a)} + O(\rho), \quad i = 1, \ldots, \dim V^{(a)} \quad (2.13)$$

which are invariant under $\hat{\Omega}_k$. The small solution $\Psi^{(1)}_{\frac{1}{2}}$ and $\Psi^{(2)}$ can be expanded in this basis as

$$\Psi^{(1)}_{\frac{1}{2}}(z, \bar{z}|\lambda, g) = \sum_{i=1}^5 Q_i^{(1)}(\lambda, g) \chi_i^{(1)}(z, \bar{z}|\lambda, g),$$

$$\Psi^{(2)}(z, \bar{z}|\lambda, g) = \sum_{i=1}^4 Q_i^{(2)}(\lambda, g) \chi_i^{(2)}(z, \bar{z}|\lambda, g). \quad (2.14)$$

We call $Q_i^{(a)}(\lambda, g)$ the Q-functions. From the relation $\hat{\Omega}_1 \hat{\Pi} \Psi^{(a)} = \Psi^{(a)}$, the coefficients $Q_i^{(a)}(\lambda, g)$ satisfy the quasi-periodicity condition:

$$Q_i^{(a)}(\lambda - 2\pi i M + 1), g) = \exp\left(-\frac{2\pi i}{4}(\rho^g + \beta g) \cdot h_i^{(a)}\right) Q_i^{(a)}(\lambda, g). \quad (2.15)$$

Note that we can rescale $z$ and $\bar{z}$ such that the mass parameter $m$ is fixed to be an arbitrary non-zero constant. Then the Q-functions depend on the mass parameter through $s/m$.

3 \textbf{ψ-System and the Bethe ansatz equations}

The linear problem in the basic $B_2^{(1)}$-modules $V^{(a)}$ can be also defined in other $B_2$-modules corresponding to the (anti-)symmetrized tensor product of $V^{(a)}$’s. The inclusion maps
between the modules induce the relation between the small solutions, which is called the \( \psi \)-system \( ^{[7]} \).
For example, we consider the inclusion map
\begin{align*}
\iota_1 &: V^{(1)} \land V^{(1)} \hookrightarrow V^{(2)} \otimes V^{(2)}, \quad (3.1) \\
\iota_2 &: V^{(2)} \land V^{(2)} \hookrightarrow V^{(1)}.
\end{align*}
By these maps the highest weight state \( \psi_{(1)} \) is mapped to \( \sqrt{2} \psi_{(2)} \otimes \psi_{(2)} \) and \( \psi_{(1)}^2 \land \psi_{(2)}^2 \) to \( \psi_{(1)}^2 \). We use this map to relate the solutions of the linear problem defined on the different modules. \( \Psi_{(1)}^{(1)} \land \Psi_{(0)}^{(1)} \) is a solution of the linear problem \( ^{[23]} \) on \( V^{(1)} \land V^{(1)} \) due to invariance of \( ^{[23]} \) under the Symanzik rotation \( \Omega_1 \). This solution is mapped into the module \( V^{(2)} \otimes V^{(2)} \) by \( \iota_1 \). Now \( \Psi_{(2)}^{(2)} \otimes \Psi_{(2)}^{(2)} \) is the unique solution in \( V^{(2)} \otimes V^{(2)} \) with the same asymptotic behavior at large \( \rho \). In a similar way we can identify \( \Psi_{\frac{1}{2}}{(2)} \land \Psi_{\frac{1}{2}}{(2)} \) with \( \Psi_{(1)}^{(1)} \). Thus we obtain the \( \psi \)-system:
\begin{align*}
\iota_1(\Psi_{(1)}^{(1)} \land \Psi_{(0)}^{(1)}) &= \Psi_{(2)}^{(2)} \otimes \Psi_{(2)}^{(2)}, \quad (3.3) \\
\iota_2(\Psi_{\frac{1}{2}}^{(2)} \land \Psi_{\frac{1}{2}}^{(2)}) &= \Psi_{\frac{1}{2}}^{(1)}.
\end{align*}
Expanding the small solutions in the basis \( \{ \lambda_i^{(a)} \} \) and substituting them into the \( \psi \)-system, one obtains the functional relation for the Q-functions \( Q_1^{(a)} \) and \( Q_2^{(a)} \):
\begin{align*}
Q_1^{(1)}(\lambda - \frac{2\pi i}{8M})Q_2^{(1)}(\lambda + \frac{2\pi i}{8M}) - Q_2^{(1)}(\lambda - \frac{2\pi i}{8M})Q_1^{(1)}(\lambda + \frac{2\pi i}{8M}) &= 2Q_1^{(2)}(\lambda)Q_2^{(2)}(\lambda), \quad (3.5) \\
Q_1^{(2)}(\lambda - \frac{2\pi i}{8M})Q_2^{(2)}(\lambda + \frac{2\pi i}{8M}) - Q_2^{(2)}(\lambda - \frac{2\pi i}{8M})Q_1^{(2)}(\lambda + \frac{2\pi i}{8M}) &= Q_1^{(1)}(\lambda). \quad (3.6)
\end{align*}
Denoting the zeros of the Q-functions \( Q_1^{(a)}(\lambda) \) by \( \lambda_i^{(a)}(n = 1, 2, \ldots) \), one obtains the Bethe ansatz equations
\begin{align*}
\frac{Q_1^{(2)}(\lambda_i^{(1)} - \frac{\pi i}{4M})^2}{Q_1^{(2)}(\lambda_i^{(1)} + \frac{\pi i}{4M})^2} \frac{Q_1^{(1)}(\lambda_i^{(1)} + \frac{\pi i}{2M})}{Q_1^{(1)}(\lambda_i^{(1)} - \frac{\pi i}{2M})} &= -1, \quad (3.7) \\
\frac{Q_1^{(2)}(\lambda_i^{(2)} - \frac{\pi i}{2M})^2}{Q_1^{(2)}(\lambda_i^{(2)} + \frac{\pi i}{2M})^2} \frac{Q_1^{(1)}(\lambda_i^{(2)} + \frac{\pi i}{4M})}{Q_1^{(1)}(\lambda_i^{(2)} - \frac{\pi i}{4M})} &= -1.
\end{align*}
Note that these differ from those of the integrable model based on the \( U_q(A_3^{(2)}) \) \( ^{[24]} \), which is expected from the Langlands duality between \( A_3^{(2)} \) and \( B_2^{(1)} \). The Bethe ansatz equations for \( U_q(A_3^{(2)}) \) do not include the squared Q-functions. It would be interesting to study the solutions of the Bethe ansatz equations \( ^{[3.7]} \) and \( ^{[3.8]} \) in the conformal limit and explore the corresponding integrable model.
4 Quantum Wronskian and T–Q relations

4.1 Spinor representation and discrete symmetries

The $\psi$-system in the previous section has been obtained by investigating the asymptotic solution in a single Stokes sector, $S_0$ for example. Now we consider the solutions of the linear problem in the whole complex plane. We focus on $V^{(2)}$ because this is the minimal dimensional representation and the solution in the vector representation can be constructed via the inclusion map $\iota_2$.

Since we are considering a $SO(5)$ spinor, it is natural to introduce the charge conjugation. Associated with the linear problem (2.3) in the spinor representation, we define the transposed linear problem:

$$
(\partial - A_T^T) \bar{\Psi} = 0, \quad (\bar{\partial} - A_T^T) \bar{\Psi} = 0.
$$

(4.1)

The solution $\bar{\Psi}(z, \bar{z}|\lambda, g)$ of these equations are related to $\Psi(z, \bar{z}|\lambda, g)$ by the charge conjugation:

$$
\bar{\Psi}(z, \bar{z}|\lambda, g) = F \Psi(z, \bar{z}|\lambda, g),
$$

(4.2)

where

$$
F = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
$$

(4.3)

This is a $\mathbb{Z}_2$ symmetry of the linear problem. Note that $\bar{\Psi} = -\Psi$.

One can define the inner product $\langle \bar{\Psi}, \Psi \rangle := \sum_{\alpha=1}^4 \bar{\Psi}^\alpha \Psi^\alpha$ between $\Psi = (\Psi^\alpha)$ and $\bar{\Psi} = (\bar{\Psi}^\alpha)$. The inner product is independent of $z$ and $\bar{z}$ when $\Psi (\bar{\Psi})$ is a solution of the (transposed) linear problem. The Wronskian of any four linearly independent solutions $\Psi_i (i = 1, 2, 3, 4)$

$$
\langle \Psi_1, \Psi_2, \Psi_3, \Psi_4 \rangle := \det(\Psi_1, \Psi_2, \Psi_3, \Psi_4),
$$

(4.4)

is also independent of $z$ and $\bar{z}$.

We define the $(-k)$-rotated solution $s_k := \Psi^{(2)}_{-k}$ in the module $V^{(2)}$. This is the sub-dominant solution in the Stokes sector $S_k$ but it gives a divergent solution in the sectors $S_{k-2}$ and $S_{k+2}$. One can choose $\{s_{k-1}, s_k, s_{k+1}, s_{k+2}\}$ as a basis of the solutions. We normalize the solution $s_k$ such that

$$
\langle s_{k-1}, s_k, s_{k+1}, s_{k+2} \rangle = 1,
$$

(4.5)

by choosing the normalization constant $C^{(2)}$ in (2.11) as $(-16)^{-\frac{1}{2}}$. From the asymptotic behavior of $s_k$ and $\bar{s}_k$ at large $\rho$, we find $\langle \bar{s}_k, s_k \rangle = \langle s_k, s_{k+1} \rangle = 0$ and $\langle \bar{s}_k, s_{k+2} \rangle = \frac{1}{16}$.
Then from the condition (4.5) we find

\[ s_k^{(2)} = -\frac{1}{16} s_{k-1} s_k s_{k+1}. \]  

(4.6)

We write it in the form \( s_k = -\frac{1}{16} s_{k-1} \wedge s_k \wedge s_{k+1}. \) Since the basis \( e_i^{(2)} \) is orthonormal, we can fix the normalization of \( \mathcal{X}_1^{(2)} \) as

\[ \text{det}(\mathcal{X}_1^{(2)}, \mathcal{X}_2^{(2)}, \mathcal{X}_3^{(2)}, \mathcal{X}_4^{(2)}) = 1, \]  

(4.7)

which simplifies the functional relations described below.

### 4.2 \( T-Q \) relation

Now we take \( \{s_{-2}, s_{-1}, s_0, s_1\} \) as the basis of the solutions of the linear system. We introduce a set of functions \( \mathcal{T}_{a,m}(\lambda) (a = 1, 2, 3, m \in \mathbb{Z}) \) by

\[ \mathcal{T}_{1,m}(\lambda) = \langle s_{-2}, s_{-1}, s_0, s_{m+1}\rangle^{[-m]}, \]

(4.8)

\[ \mathcal{T}_{2,m}(\lambda) = \langle s_{-2}, s_{-1}, s_1, s_{m+1}\rangle^{[-m]}, \]

(4.9)

\[ \mathcal{T}_{3,m}(\lambda) = \langle s_{-2}, s_0, s_1, s_{m+1}\rangle^{[-m]}, \]  

(4.10)

where \( f^{[m]}(\lambda) \equiv f(\lambda + \frac{m 2\pi i}{24M}). \) A solution \( s_k (k \in \mathbb{Z}) \) is expanded in terms of this basis as

\[ s_k = -\mathcal{T}_{1,k-2}^{[k]} s_{-2} + \mathcal{T}_{3,k-1}^{[k]} s_{-1} - \mathcal{T}_{2,k-1}^{[k]} s_0 + \mathcal{T}_{1,k-1}^{[k]} s_1. \]  

(4.11)

The coefficients of \( s_{-1}, s_0 \) and \( s_1 \) follow from the definition of \( \mathcal{T}_{a,m} \) directly. The coefficient of \( s_{-2} \) is evaluated as \( \langle s_k, s_{-1}, s_0, s_1 \rangle. \) Using the identity:

\[ \langle s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4}\rangle^{[2]} = \langle s_{i_1+1}, s_{i_2+1}, s_{i_3+1}, s_{i_4+1}\rangle, \]  

(4.12)

which follows from the Symanzik rotation, it is shown to be equal to \( \langle s_{-2}, s_{-1}, s_0, s_{k-1}\rangle^{[2]} = -\mathcal{T}_{1,k-2}^{[k]} \).

We expand \( s_{-k} \) in terms of the basis \( \mathcal{X}_i^{(2)} \):

\[ s_{-k}(z, \bar{z}) = \sum_{i=1}^{4} Q_i(\lambda - k \frac{2\pi i}{4M}, g) \mathcal{X}_i^{(2)}(z, \bar{z} | \lambda, g), \]  

(4.13)

where \( Q_i := Q_i^{(2)} \). The exterior product \( s_{-i_1} \wedge s_{-i_2} \cdots \wedge s_{-i_p} \) in \( \wedge^p V^{(2)} \) is also expanded in the basis \( \mathcal{X}_i^{(2)} \). The coefficient of the highest weight vector is evaluated as

\[ s_{-i_1} \wedge s_{-i_2} \cdots \wedge s_{-i_p} = \mathcal{W}_{i_1i_2 \cdots i_p}^{(p)} \mathcal{X}_1^{(2)} \wedge \cdots \wedge \mathcal{X}_p^{(2)} + \cdots, \]  

(4.14)
where we introduce the determinant

\[
W_{i_1 i_2 \cdots i_p}^{(p)} := \det \begin{pmatrix}
Q_1(\lambda - i_1 \frac{2\pi i}{4M}) & Q_1(\lambda - i_2 \frac{2\pi i}{4M}) & \cdots & Q_1(\lambda - i_p \frac{2\pi i}{4M}) \\
Q_2(\lambda - i_1 \frac{2\pi i}{4M}) & Q_2(\lambda - i_2 \frac{2\pi i}{4M}) & \cdots & Q_2(\lambda - i_p \frac{2\pi i}{4M}) \\
\vdots & \vdots & & \vdots \\
Q_p(\lambda - i_1 \frac{2\pi i}{4M}) & Q_p(\lambda - i_2 \frac{2\pi i}{4M}) & \cdots & Q_p(\lambda - i_p \frac{2\pi i}{4M})
\end{pmatrix}.
\] (4.15)

For \( p = 1 \) we have \( W_k^{(1)} = Q_k^{(2)}[0,-k] \). For \( p = 4 \), we obtain \( W_{i_1 i_2 i_3 i_4}^{(4)} = \langle s_{-i_1}, s_{-i_2}, s_{-i_3}, s_{-i_4} \rangle \).

In particular, from the normalization condition (4.7) we find that

\[
W_{k-1,k,k+1,k+2}^{(4)} = 1.
\] (4.16)

The relation (4.16) can be regarded as the quantum Wronskian relation [9]. Let us consider two more examples. For \( p = 2 \) with \( i_1 = -k \) and \( i_2 = -k + 1 \), using the \( \psi \)-system (3.4), we find

\[
W_{k,k-1}^{(2)} = Q_1^{(1)}(\lambda - k \frac{2\pi i}{4M}).
\] (4.17)

For \( W_{k+1,k+1, k-1}^{(3)} \), using (4.6), we have

\[
\langle \bar{s}_{-k}, \mathcal{X}_4^{(2)} \rangle = -\frac{1}{16} W_{k+1,k,k-1}^{(3)},
\] (4.18)

which becomes \( \langle s_{-k}, F^T \mathcal{X}_4^{(2)} \rangle \) by the formula (4.2). We then get

\[
W_{k+1,k,k-1}^{(3)} = 16Q_1^{(2)}(\lambda - k \frac{2\pi i}{4M}).
\] (4.19)

We note that the determinants (4.15) satisfy the Plücker relations

\[
W_{i_1 i_2 \cdots i_{p-1}}^{(p-1)} W_{i_1 i_2 \cdots i_p}^{(p)} - W_{i_1 i_2 \cdots i_{p-1}}^{(p-1)} W_{i_2 \cdots i_{p-1} i_p}^{(p)} + W_{i_2 \cdots i_{p-1} i_p}^{(p-1)} W_{i_1 \cdots i_{p-1}}^{(p)} = 0.
\] (4.20)

In particular one finds

\[
0 = W_0^{(1)} W_{12}^{(2)} - W_1^{(1)} W_{02}^{(2)} + W_2^{(1)} W_{01}^{(2)},
\]

\[
0 = W_0^{(2)} W_{123}^{(3)} - W_1^{(2)} W_{023}^{(3)} - W_2^{(2)} W_{012}^{(3)},
\]

\[
0 = W_0^{(3)} W_{1234}^{(4)} - W_1^{(3)} W_{023}^{(4)} + W_2^{(3)} W_{0123}^{(4)}.
\] (4.21)

From these equations, we can solve \( W_{0234}^{(4)} \) as

\[
W_{0234}^{(4)} = \frac{W_0^{(1)}}{W_1^{(1)}} + \frac{W_1^{(1)} W_{01}^{(2)} W_2^{(1)} W_{02}^{(2)}}{W_1^{(1)} W_{12}^{(2)}} + \frac{W_2^{(2)} W_{012}^{(3)}}{W_{123}^{(3)} W_{12}^{(2)}} + \frac{W_2^{(3)} W_{0123}^{(4)}}{W_{12}^{(3)}}.
\] (4.22)
This equation is the T–Q relation of the $A_3$-type quantum integrable models. Now using (4.17) and (4.19), (4.22) becomes

$$
\mathcal{T}_{1,1}^{[-1]} Q_1^{(2)} Q_1^{(1)[-1]} Q_1^{(2)[-2]} = Q_1^{(2)[2]} Q_1^{(1)[-1]} Q_1^{(2)[-2]} + Q_1^{(2)[-4]} Q_1^{(1)[-1]} Q_1^{(2)[-2]}
$$

$$
+ Q_1^{(1)[-3]} Q_1^{(2)} Q_1^{(2)} + Q_1^{(2)[-4]} Q_1^{(1)[-1]}.
$$

This is the T–Q relation for the $A_3/\mathbb{Z}_2$-type. From this relation we obtain the Bethe equations, which was also derived in the previous section by using the $\psi$-system.

One can also derive a set of the relations:

$$
0 = W_{013}^{(3)} W_{234}^{(4)} - W_{213}^{(3)} W_{013}^{(4)} + W_{134}^{(2)} W_{0213},
$$

$$
0 = -W_{12}^{(2)} W_{013}^{(3)} + W_{01}^{(2)} W_{123}^{(3)} + W_{13}^{(2)} W_{013}^{(3)},
$$

$$
0 = W_{23}^{(3)} W_{134}^{(4)} - W_{13}^{(2)} W_{234}^{(3)} - W_{34}^{(2)} W_{123}^{(3)},
$$

$$
0 = W_{01}^{(2)} W_{13}^{(2)} - W_{1}^{(1)} W_{23}^{(2)} - W_{3}^{(1)} W_{12}^{(2)},
$$

$$
0 = W_{01}^{(2)} W_{23}^{(2)} - W_{2}^{(1)} W_{13}^{(2)} + W_{3}^{(1)} W_{12}^{(2)}.
$$

From these equations $W_{013}^{(4)}$ is solved as

$$
W_{013}^{(4)} = \frac{W_{01}^{(2)} W_{12}^{(3)}}{W_{12}^{(2)}} + \frac{W_{01}^{(2)} W_{23}^{(2)}}{W_{23}^{(2)}} + \frac{W_{01}^{(2)} W_{12}^{(4)}}{W_{12}^{(2)}} + \frac{W_{01}^{(2)} W_{23}^{(4)}}{W_{23}^{(2)}} + \frac{W_{01}^{(2)} W_{12}^{(3)}}{W_{12}^{(2)}} + \frac{W_{01}^{(2)} W_{23}^{(3)}}{W_{23}^{(2)}}.
$$

From (4.17), (4.19), (4.25) and $W_{013}^{(4)} = \mathcal{T}_{2,1}^{[-3]}$, we get the T–Q relation for $\mathcal{T}_{2,1}$, $Q_1^{(1)}$ and $Q_1^{(2)}$:

$$
\mathcal{T}_{2,1}^{[1]} Q_1^{(1)[-1]} Q_1^{(1)[-1]} (Q_1^{(2)})^2 = (Q_1^{(2)})^2 Q_1^{(1)[-1]} Q_1^{(1)[-1]} + (Q_1^{(2)})^2 Q_1^{(1)[-1]} Q_1^{(1)[-1]} + (Q_1^{(2)})^2 Q_1^{(1)[-1]} Q_1^{(1)[-1]}
$$

At the zeros $\lambda_{1n}^{(2)}$ of $Q_1^{(2)}(\lambda)$, $\mathcal{T}_{2,1}^{[1]}$ might have a double pole. Absence of the double pole in $\mathcal{T}_{2,1}^{[1]}$ leads to eq. (3.8). By the shift of $\lambda$ and evaluating (4.26) at zeros of $Q_1^{(1)}(\lambda)$, we obtain eq. (3.7). Thus one obtains the Bethe ansatz equations again.

## 5 T-system and Y-system

Now we study the functional relations which are satisfied by $\mathcal{T}_{a,m}$. First we calculate the product of $\mathcal{T}_{a,1}$ and $\mathcal{T}_{1,m}$. From the Plücker relation

$$
\langle s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4} \rangle \langle s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4} \rangle = \langle s_{i_1}, s_{j_2}, s_{j_3}, s_{j_4} \rangle \langle s_{j_1}, s_{i_2}, s_{i_3}, s_{i_4} \rangle + \langle s_{i_4}, s_{j_2}, s_{j_3}, s_{j_4} \rangle \langle s_{j_1}, s_{i_2}, s_{i_3}, s_{i_4} \rangle = 0.
$$

(5.1)
we get the identities

\[ T_{1,1}^{[+1]} T_{1,m-1}^{[-1]} = T_{1,m}^{[m]} + T_{2,m-1}^{[m+1]}, \]
\[ T_{2,1}^{[+1]} T_{1,m-1}^{[-1]} = T_{2,m}^{[m]} + T_{3,m-1}^{[m+1]}, \]
\[ T_{3,1}^{[+1]} T_{1,m-1}^{[-1]} = T_{3,m}^{[m]} + T_{1,m-2}^{[m+2]}. \]  
(5.2)

These relations are a generalization of the fusion relation of the modified sinh-Gordon equation \(^9\) to the modified \(B_{2}^{(1)}\) affine Toda field equation. However, one finds

\[ T_{1,m}^{[+1]} T_{1,m}^{[-1]} = T_{1,m+1} T_{1,m-1} + \langle s_{-1}, s_{0}, s_{m+1}, s_{m+2}\rangle^{[-m-1]}, \]  
(5.3)

where the second term in the r.h.s. is not the form of the \(T_{a,m}\) functions. We add this function to a member of the T-functions and define

\[ T_{1,m}(\lambda) = T_{1,m} = \langle s_{-2}, s_{-1}, s_{0}, s_{m+1}\rangle^{[-m]}, \]  
(5.4)
\[ T_{2,m}(\lambda) = \langle s_{-1}, s_{0}, s_{m+1}, s_{m+2}\rangle^{[-m-1]}, \]  
(5.5)

for \(m \in \mathbb{Z}\). The new function \(T_{2,m}\) satisfies the identity

\[ T_{2,m}^{[+1]} T_{2,m}^{[-1]} = T_{2,m-1} T_{2,m+1} + \langle s_{-1}, s_{m}, s_{m+1}, s_{m+2}\rangle^{[-m]} T_{1,m}, \]  
(5.6)

We then introduce

\[ T_{3,m}(\lambda) = \langle s_{-1}, s_{m}, s_{m+1}, s_{m+2}\rangle^{[-m]}. \]  
(5.7)

But this is not new. Using \(4.12\) and \(4.6\), we can show that \(T_{3,m} = T_{1,m}\). Finally we obtain the T-system of \(A_{3}/\mathbb{Z}_{2}\) type:

\[ T_{1,m}^{[+1]} T_{1,m}^{[-1]} = T_{1,m-1} T_{1,m+1} + T_{2,m} \]
\[ T_{2,m}^{[+1]} T_{2,m}^{[-1]} = T_{2,m+1} T_{2,m-1} + T_{1,m} T_{1,m}, \]  
(5.8)

which is obtained by the reduction of \(A_{3}\) T-system with the identification \(T_{1,m} = T_{3,m}\). Other functions \(T_{2,m}, T_{3,m}\) can be expressed in terms of \(T_{a,m}\) by using \(T_{2,1} = T_{2,1}^{[-1]}, T_{3,1} = T_{3,1}^{[-2]}\) and \(5.2\). They also satisfy the identities:

\[ T_{3,m+1} T_{1,m-1} = T_{3,m}^{[-1]} T_{1,m}^{[2]} - T_{2,m}, \]
\[ T_{2,m+1} T_{1,m} = T_{1,m+1} T_{2,m} + T_{2,m+1}. \]  
(5.9)

We next introduce the Y-functions by

\[ Y_{1,m} = \frac{T_{1,m+1} T_{1,m-1}}{T_{2,m}}, \quad Y_{2,m} = \frac{T_{2,m+1} T_{2,m-1}}{T_{1,m} T_{1,m}}. \]  
(5.10)
They satisfy the Y-system of $A_3/Z_2$ type

\[
\frac{Y_{a,m}^{[+1]} Y_{a,m}^{-[1]}}{Y_{a+1,m} Y_{a-1,m}} = \frac{(1 + Y_{a,m+1})(1 + Y_{a,m-1})}{(1 + Y_{a+1,m})(1 + Y_{a-1,m})},
\]

where $a = 1, 2$ and $Y_{3,m} = Y_{1,m}$. The T-system (5.8) and the Y-system (5.11) imply that the Langlands duality between the modified $B_2^{(1)}$ affine Toda equation and the functional equations of the $A_3/Z_2$ quantum integrable system.

We now discuss the boundary condition of the T-system and the Y-system. It is easy to see that $T_{a,-1} = 0$ and $T_{a,0} = 1$. In order to determine the boundary conditions $T_{1,m}$ for large $m$, we need to study the small solutions $s_m$ in the whole complex plane. When $4(M + 1)$ is not a rational number, the Stokes sectors cover the complex plane infinitely many times. So the T-functions $T_{a,m}$ are defined independently for arbitrary positive integer $m$.

In this paper we will consider the case $4(M + 1) = n$ with $n \geq 6$ being a positive integer in detail. In this case there are $n$ Stokes sectors in the complex plane. When we go around the origin, the solution $s_k(ze^{-2\pi i})$ is defined in the sector $S_{k+n}$, which is the same as $S_k$. Then the small solution $s_{k+n}(z)$ is proportional to $s_k(ze^{-2\pi i})$:  

\[
s_{k+n}(z) \propto s_k(ze^{-2\pi i}).
\]

For $g = 0$, the linear system has no simple pole at the origin. The solution has no monodromy around it. Then we have $s_k(ze^{-2\pi i}) = s_k(z)$, which implies 

\[
s_{k+n}(z, \lambda) \propto s_k(z, \lambda).
\]

The condition (5.13) leads to the boundary conditions for the T-/Y-functions: $T_{a,n-3} = 0$ and $Y_{a,n-4} = 0$. The truncated T-/Y-system becomes the same as the one for the $n$-point gluon scattering amplitudes in AdS$_4$ at strong coupling [19].

For $g \neq 0$, the solutions of the linear system have monodromy around the origin. We introduce a monodromy matrix $\Omega(\lambda)$ by

\[
\begin{pmatrix}
  s_1 \\
  s_0 \\
  s_{-1} \\
  s_{-2}
\end{pmatrix}
(z e^{-2\pi i}, \lambda) = \Omega(\lambda)
\begin{pmatrix}
  s_1 \\
  s_0 \\
  s_{-1} \\
  s_{-2}
\end{pmatrix}
(z, \lambda).
\]

From the normalization condition (4.5) we find $\det \Omega(\lambda) = 1$. We also introduce the proportionality factor $B(\lambda)$ in (5.13) for $k = 1$ by

\[
s_{n+1}(z, \lambda) = B(\lambda)s_1(z e^{-2\pi i}, \lambda).
\]

Let us expand the solution $s_0(z, \lambda)$ in the basis $X_i(z, \bar{z} | \lambda, g)$ whose coefficient has been defined as $Q_i(\lambda, g)$. Then we substitute its Symanzik rotation into (5.15). In the basis $X_i$, the

---

2When $n$ is a rational number, we can do similar arguments. But it is not discussed in this paper.
monodromy matrix becomes diagonal and takes the form \( \text{diag}(e^{2\pi i \beta g_{h_1}^{(2)}}, \ldots, e^{2\pi i \beta g_{h_4}^{(2)}}) \). Moreover from the quasi-periodicity condition \( (2.15) \) one finds that \( B(\lambda) = -1 \). Plugging \((5.15)\) into \((5.14)\), we get the relation

\[
\begin{pmatrix}
  s_{n+1} \\
  s_n \\
  s_{n-1} \\
  s_{n-2}
\end{pmatrix}
(z, \lambda) = -\Omega(\lambda)
\begin{pmatrix}
  s_1 \\
  s_0 \\
  s_{-1} \\
  s_{-2}
\end{pmatrix}
(z, \lambda),
\]

which generalizes the condition \((5.13)\) and determines the boundary condition for the T-system. It is convenient to use the (multi-)trace of the monodromy matrix \( \Omega \): \( \text{tr} \Omega \) and \( \text{tr}^{(2)} \Omega \equiv \frac{1}{2}((\text{tr} \Omega)^2 - \text{tr} \Omega^2) \), which are basis independent quantities. These traces can be also expressed using the Wronskians:

\[
\text{tr} \Omega = -\langle s_{-2}, s_{-1}, s_0, s_{n+1} \rangle + \langle s_{-2}, s_{-1}, s_1, s_n \rangle - \langle s_{-2}, s_0, s_1, s_{n-1} \rangle + \langle s_{-1}, s_0, s_1, s_{n-2} \rangle,
\]

\[
\text{tr}^{(2)} \Omega = \langle s_{-2}, s_{-1}, s_n, s_{n+1} \rangle + \langle s_{-2}, s_{-1}, s_0, s_{n-1} \rangle + \langle s_{-2}, s_{n-1}, s_0, s_{n+1} \rangle + \langle s_{n-2}, s_{-1}, s_0, s_{n+1} \rangle + \langle s_{n-2}, s_{n-1}, s_0, s_{n-1} \rangle + \langle s_{n-2}, s_{n-1}, s_1, s_{n-1} \rangle + \langle s_{n-2}, s_{-1}, s_n, s_1 \rangle.
\]

Here the r.h.s. of these equations are expressed by \( T_{2,m}, T_{3,m} \) and the auxiliary T-functions \( W_{1,m}, W_{2,m}, \bar{W}_{2,m} \) defined in \([23]\), in addition to the T-functions \((5.5)\). In Appendix A, we will summarize these auxiliary T-functions and their recursion relations. In the diagonal basis, they are evaluated as

\[
\text{tr} \Omega = 4 \cos(\beta g_1 \pi) \cos(\beta g_2 \pi)
\]

\[
\text{tr}^{(2)} \Omega = 2 + 4 \cos[\beta(g_1 - g_2) \pi] \cos[\beta(g_1 + g_2) \pi]
\]

where \( g_i \equiv g \cdot e_i \) \((i = 1, 2)\). For \( g = 0 \), one finds that \( \text{tr} \Omega = 4 \) and \( \text{tr}^{(2)} \Omega = 6 \). The monodromy conditions \((5.17)\) determine \( T_{a,n} \) \((a = 1, 2)\). Then the T-system extends up to \( m = n + 1 \) and the T-functions \( T_{a,m} \) for \( m \geq n + 1 \) are determined by the T-system and the monodromy conditions. Concerning the Y-system \((5.11)\), it also extends up to \( m = n - 2 \). It is convenient to introduce new Y-functions \( \bar{Y}_a \) \((a = 1, 2, 3)\) by

\[
\bar{Y}_1 = \bar{Y}_3 = \frac{T_{1,n-2}}{T_{2,n-1}}, \quad \bar{Y}_2 = \frac{T_{2,n-2}}{T_{1,n-1}T_{1,n-1}}.
\]

whose functional relations are given by

\[
\frac{\bar{Y}_a^{[+1]}\bar{Y}_a^{-[-1]}}{\bar{Y}_a^{[+1]}\bar{Y}_a^{[-1]}} = \frac{1 + Y_{a,n-2}}{(1 + Y_{a+1,n-1})(1 + Y_{a-1,n-1})}.
\]

The Y-system \((5.11)\) for \( m = n - 2 \) and \((5.21)\) contains \( Y_{a,n-1} \) in the r.h.s. of the equations. \( Y_{a,n-1} \) are expressed as

\[
Y_{1,n-1} = -T_{1,n}\bar{Y}_1, \quad Y_{2,n-1} = T_{2,n}\bar{Y}_2,
\]
and $T_{a, n}$ are expressed in terms of the lower $T$-functions. For the $n \neq 4\ell$ $(\ell = 1, 2, \cdots)$ case, they are also expressed in terms of the lower $Y$-functions by solving (5.10). Then (5.11) and (5.21) with (5.22) become the closed functional relations. Note that the present $T$- and $Y$-systems are the same as those of form factors in AdS$_4$ [23]. However the function $p(z)$ has different pole structure from the present one.

In the case of even $n$ and $g_1 = 0$ (or $g_2 = 0$), one can consider the limit to the modified sinh-Gordon equation [9] (or gluon scattering amplitudes in AdS$_3$ [19]), where in this limit the $SO(5)$ spinor is decomposed into left and right-handed spinors. In this reduction the $T$-functions $T_{a, m}$ reduce to the functions $T_{k}$ $(k = 1, \cdots, \frac{n}{2} - 2)$, which are defined by the inner product of the left-handed spinors. They satisfy

$$T_{1, 2k+1} = 0, \quad T_{1, 2k} = -T_{k}^{[2]}, \quad T_{2, 2k} = T_{k}^{[3]}T_{k}^{[+1]}, \quad T_{2, 2k+1} = -T_{k}^{[2]}T_{k+1}^{[2]},$$

$$\langle s_{-2}, s_{0}, s_{1}, s_{n-1} \rangle = T_{\frac{n+2}{2}}, \quad \langle s_{-2}, s_{-1}, s_{1}, s_{n} \rangle = T_{\frac{n}{2}}.$$

(5.23)

Here $T_{k}$ obey the functional relations

$$T_{k}^{[2]}T_{k}^{[-2]} = 1 + T_{k-1}T_{k+1}.$$  

(5.24)

Using (5.23), we can rewrite tr$\Omega$ in terms of the left-handed part and right-handed part. Decomposing these two parts we obtain $T_{\frac{n}{2}} - T_{\frac{n}{2} - 2} = 2 \cos \pi \beta g_2$, which is the trace of monodromy in left-handed part. The $Y$-functions $Y_{a, m}$ and $\bar{Y}_{a}$ reduce to $Y_{k} = T_{k+1}T_{k-1}$ $(k = 1, \cdots, n/2 - 2)$ and $\bar{Y} = -T_{\frac{n}{2} - 2}$ as

$$Y_{1, 2k} = 0, \quad Y_{1, 2k+1} = -1, \quad Y_{2, 2k+1} = \infty, \quad Y_{2, 2k} = Y_{k},$$

$$\bar{Y}_{1}Y_{2, n-2} = \bar{Y}_{k}^{[2]}, \quad \bar{Y}_{2} = \infty.$$  

(5.25)

Here the $Y$-functions $Y_{k}$ and $\bar{Y}$ satisfy the $D_{n/2}$-type $Y$-system [9, 22]

$$Y_{k}^{[2]}Y_{k}^{[-2]} = (1 + Y_{k-1})(1 + Y_{k+2}), \quad (k = 1, \cdots, \frac{n}{2} - 3),$$

$$\bar{Y}_{k}^{[2]}\bar{Y}_{k}^{[-2]} = 1 + Y_{\frac{n}{2} - 2}^{2},$$

$$Y_{\frac{n}{2} - 2}^{[2]}\bar{Y}_{\frac{n}{2} - 2}^{[-2]} = (1 + Y_{\frac{n}{2} - 3})(1 - 2 \cos \pi \beta g_2 \bar{Y} + \bar{Y}^2).$$

(5.26)

6 High-temperature limit of the Y-system

In the previous section we have seen that the $T$-/Y-system becomes the extended one in the presence of monodromy. The standard approach to analyze the (extended) $Y$-system is to derive the Thermodynamic Bethe ansatz (TBA) equations and investigate their free energy. The IR (or low-temperature) limit of the TBA system are characterized by the WKB approximation, whereas in the UV (or high-temperature) limit is characterized by the spectral parameter independent $Y$-functions and the free energy is determined by the dilog formulas [27, 28, 29]. Since the present $Y$-system is very complicated, we
leave the detailed TBA analysis to the subsequent paper. Instead we will study the high-temperature limit of the Y-system and their solutions explicitly for the simplest case $n = 6$.

For $n = 6$. Using the T-system, $T_{a,m}$ ($a = 1, 2, m = 1, \ldots, 6$) are solved in terms of $T_{1,1} = x$ and $T_{2,1} = y$ using (5.8). Substituting them into the monodromy conditions we obtain two equations for $x$ and $y$

$$
x^6 - 6x^4(-1 + y) - 2y(-3 + y^2) + 3x^2(-1 - 4y + 3y^2) + 4\cos[\beta g_1\pi]\cos[\beta g_2\pi] = 0,
$$
$$- 2x^6 + y^2(-3 + y^2)^2 + 3x^4(8 - 8y + 3y^2) - 6x^2(3 - 2y^3 + y^4)
$$
$$- 4\cos[\beta(g_1 - g_2)\pi]\cos[\beta(g_1 + g_2)\pi] = 0. \tag{6.1}
$$

For $g_1 = g_2 = 0$, i.e. when there is no monodromy around the origin, we find the solutions of the above algebraic equations are given by $(x, y) = (0, -1), (0, 2), (\pm2\sqrt{3}, 5)$ and $(\pm\sqrt{3}, 2)$. For $(x, y) = (0, -1)$, we get $Y_{1,2k-1} = -1, Y_{1,2k} = 0, Y_{2,2k-1} = \infty \quad (k \geq 1)$, $Y_{2,2} = 0$ and $Y_{2,4} = -1$. This solution corresponds to the $AdS_3$ limit of the Y-system. For $(x, y) = (\pm\sqrt{3}, 2)$, we get $Y_{1,1} = \frac{1}{2}, Y_{2,2} = \frac{1}{2}$ and $Y_{1,2} = Y_{2,2} = 0$, which corresponds to the constant Y-system of the 6-point amplitudes [19]. For other solutions, we do not find any corresponding physical quantities.

Now we turn on $g_2$ with keeping $g_1 = 0$. We find the solutions of (6.1) with $x = 0$ are given by

$$y = \pm \sqrt{2 + 2\cos\frac{2\pi}{3}(1 - \beta g_2)}. \tag{6.2}
$$

This gives

$$Y_{2,2} = 1 + 2\cos\frac{2\pi}{3}(1 - \beta g_2), \quad \bar{Y}_{1}Y_{2,4} = -y, \tag{6.3}
$$

which turns out to be a constant solution. This corresponds to a constant solution of the $AdS_3$ form factor with $n = 6$ gluons [22]. Note that $\bar{Y}^{[2]} = \bar{Y}_{1}Y_{2,4}$ in (5.23) for $n = 6$. As for the solutions starting from $(\sqrt{3}, 2)$, we solve eqs. (6.1) numerically. The graphs of $T_{1,1} = x(g_2)$ and $T_{2,1} = y(g_2)$ are shown in Fig. 1 and the corresponding Y-functions $Y_{1,1}$ and $Y_{2,1}$ are shown in Fig. 2. There arise four branches from the point $(\sqrt{3}, 2)$, which arrive at the solutions $(x, y) = (3, 4), (2, 3), (1, 0)$ and $(0, 1)$ at $g_2 = 1$. These solutions provide a deformation of the constant T-/Y-systems by the monodromy parameter $g_2$.

### 7 Conclusions and discussion

In this paper, we studied the massive ODE/IM correspondence for modified $B^{(1)}_2$ affine Toda field equation. By investigating the solutions of the linear problem associated with
the modified affine Toda equation, we derived the $\psi$-system. This leads to the Bethe ansatz equations corresponding to the integrable model which is not identified yet. We also derived the same Bethe ansatz equations from the T–Q relations. We constructed the T-system and Y-system from the Wronskians of the solutions of the linear problem. These systems have non-trivial boundary conditions due to the presence of monodromy around the origin. It would be interesting to generalize the present approach to modified affine Toda field equations associated with other affine Lie algebras which are not of the Langlands dual of an untwisted affine Lie algebra [30]. It is also interesting to study the massless limit of this linear problem and investigate the description by using the free field realization of conformal field theory [31, 32, 33, 34].

For the linear system associated the null-polygonal minimal surface in AdS$_4$, we have seen that the corresponding integrable system is the homogeneous sine-Gordon model [20, 21]. When the solution has monodromy around the origin, we have seen the T-system and Y-system are extended and they take the form that appears in the strong-coupling limit of the form factor in $\mathcal{N} = 4$ super Yang–Mills theory. For a general polynomial $p(z)$ and the appropriate boundary conditions for the solutions of the linear problem, one can describe the minimal surface problem using the massive ODE/IM correspondence. In particular it is interesting to explore the ODE/IM correspondence for the minimal surface in AdS$_5$, where the corresponding quantum integrable model is not known yet.

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Auxiliary T-functions In this appendix we summarize the auxiliary T-functions and their recursion relations [23]. From these relations we express \( \text{tr}\Omega \) and \( \text{tr}^{(2)}\Omega \) in terms of \( T_{a,m} (m \leq n) \). Furthermore we can express \( Y_{a,n-1} \) in the lower Y-functions and get a closed Y-system. We define the functions \( U_{1,m}, V_{1,m}, W_{1,m}, W_{2,m} \) \( (m \in \mathbb{Z}) \) by

\[
U_{1,m} = \langle s_{-2}, s_{-1}, s_m, s_{m+2} \rangle^{-m}, \quad V_{1,m} = \langle s_{-2}, s_0, s_{m+1}, s_{m+2} \rangle^{-m},
\]

\[
W_{1,m} = \langle s_{-2}, s_0, s_{m+1}, s_{m+3} \rangle^{-m-1},
\]

\[
W_{2,m} = \langle s_{-1}, s_0, s_{m+1}, s_{m+4} \rangle^{-m-2}, \quad \tilde{W}_{2,m} = \langle s_{-2}, s_1, s_{m+2}, s_{m+3} \rangle^{-m-2}.
\]

From the Plücker relation \((5.1)\), we can show that these auxiliary T-functions satisfy

\[
U_{1,m}T_{1,m} = T_{1,m-1}^{[-1]}T_{2,m+1} + T_{1,m+1}^{[+1]}T_{2,m}^{-1},
\]

\[
V_{1,m}T_{3,m} = T_{1,m-1}^{[+1]}T_{2,m+1} + T_{3,m+1}^{[-1]}T_{2,m}^{[+1]},
\]

\[
W_{1,m}T_{2,m} = V_{1,m}^{[-1]}T_{1,m}^{[+1]} - T_{1,m}^{-1}T_{3,m}^{[+1]},
\]

\[
W_{2,m}T_{2,m+1} = U_{1,m+1}U_{1,m} - T_{2,m}^{[-1]}T_{2,m+2}^{[+1]},
\]

\[
\tilde{W}_{2,m}T_{2,m+1} = V_{1,m+1}V_{1,m} - T_{2,m}^{[-1]}T_{2,m+2}^{[-1]}.
\]

Then \( U_{1,m}, V_{1,m}, W_{1,m}, W_{2,m} \) and \( \tilde{W}_{2,m} \) are expressed in terms of \( T_{a,s} \). The (symmetrized) trace of the monodromy matrix becomes

\[
\text{tr}\Omega = - \left( T_{1,n}^{[n]} - T_{3,n-2}^{[n-1]} + T_{3,n-2}^{[n-2]} - T_{1,n-4}^{[n-2]} \right),
\]

\[
\text{tr}^{(2)}\Omega = T_{2,n}^{[n-1]} + T_{2,n-4}^{[n-1]} - W_{2,n-4}^{[n-1]} + W_{2,n-3}^{[n-1]} - W_{2,n-3}^{[n-1]} - W_{1,n-4}^{[n-1]}.
\]

Using \( T_{3,1} = T_{1,1}^{[-2]} \), \( T_{2,1} = T_{2,1}^{[-1]} \) and the identities \((5.2)\), we get

\[
T_{3,n-2} = T_{1,1}^{[-n+1]}T_{1,n-3} + T_{1,n-4}^{[n+2]},
\]

\[
T_{2,n-1} = T_{2,1}^{[-n+1]}T_{1,n-2} + \{ T_{1,1}^{[-n+1]}T_{1,n-3} - T_{1,n-4}^{[n+2]} \}.
\]
The (symmetrized) traces are expressed in terms of $T_{a,s}$ as

$$
\text{tr} \Omega = - \left( T_{1,n}^{[n]} - T_{1,n-4}^{[n-2]} - \left( T_{1,1}^{[-n+1]} T_{1,n-2}^{[+1]} \right) + \left( T_{1,1}^{[-n]} T_{1,n-3}^{[n]} - T_{1,n-4}^{[n+1]} \right) \right)
$$

$$
\text{tr}^{(2)} \Omega = T_{2,n}^{[n-1]} + T_{2,n}^{[1]}
$$

$$
\left( - \frac{T_{3,n-2}^{[n]} T_{1,n-2}^{[n-2]}}{T_{2,n-2}^{[n-1]}} + \frac{1}{T_{2,n-2}^{[n-1]}} T_{1,n-1}^{[n]} T_{2,n-2}^{[n-1]} + T_{1,n-1}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} - \frac{1}{T_{2,n-3}^{[n-1]}} T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} + \frac{1}{T_{2,n-1}^{[n-1]}} T_{1,n-1}^{[n]} T_{2,n-1}^{[n-2]} + T_{1,n-1}^{[n]} T_{2,n-1}^{[n-2]} - \frac{1}{T_{2,n-2}^{[n-1]}} T_{1,n-2}^{[n]} T_{2,n-2}^{[n-1]} + T_{1,n-2}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} - \frac{1}{T_{2,n-3}^{[n-1]}} T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} + \frac{1}{T_{2,n-1}^{[n-1]}} T_{1,n-1}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-1}^{[n]} T_{2,n-1}^{[n-3]} - \frac{1}{T_{2,n-3}^{[n-1]}} T_{1,n-3}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-1}^{[n-3]} + T_{1,n-5}^{[n]} T_{2,n-3}^{[n-2]} - \frac{1}{T_{2,n-4}^{[n-1]}} T_{1,n-4}^{[n]} T_{2,n-3}^{[n-2]} + T_{1,n-4}^{[n]} T_{2,n-3}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-2}^{[n-2]} - \frac{1}{T_{2,n-4}^{[n-1]}} T_{1,n-4}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-4}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} - \frac{1}{T_{3,n-4}^{[n-1]}} T_{1,n-4}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-4}^{[n]} T_{2,n-2}^{[n-3]} + T_{1,n-3}^{[n]} T_{2,n-1}^{[n-2]} \right) .
$$

From these equations we can write $T_{1,n}$ and $T_{2,n}$ in terms of lower $T$-functions. In the case of $n \neq 4\ell$ with $\ell = 1, 2, \ldots$, the $T$-functions can be further expressed in terms of the Y-functions by solving (5.10). We then obtain a closed Y-system.

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