ABSTRACT. We study dynamical properties of direction foliations on the complex plane pulled back from direction foliations on a half-translation torus $T$, i.e. a torus equipped with a strict and integrable quadratic differential. If the torus $T$ admits a pseudo-Anosov map we give a homological criterion for the appearance of dense leaves and leaves with bounded deviation on the universal covering of $T$, called Panov plane. Our result generalizes Dmitri Panov’s explicit construction of dense leaves for certain arithmetic half-translation tori [P10]. Certain Panov planes are related to the polygonal table of the periodic wind-tree model. In fact, we show that the dynamics on periodic wind-tree billiards can be converted to the dynamics on a pair of singular planes.

Possible strategies to generalize our main dynamical result to larger sets of directions are discussed. Particularly we include recent results of Frączek and Ulcigrai [FU11, FU12] and Delecroix [D13] for the wind-tree model. Implicitly Panov planes appear in Frączek and Schmoll [FS13], where the authors consider Eaton Lens distributions.

1. DYNAMICS ON PANOV PLANES

For fixed $\theta \in S^1$ we denote by $\mathcal{F}_\theta(C)$ the foliation of $C$ by oriented lines in direction $\theta \in S^1$. The foliation by lines without orientation is denoted by $\mathcal{F}_{\pm\theta}(C)$. We write $f_p \in \mathcal{F}_{\pm\theta}(C)$ for the line through $p \in C$ and use the same notation in the oriented case. Using polygonal subsets of $C$ we define a class of (flat) surfaces which generally admit un-oriented foliations and in some cases also the oriented foliation. The subject of polygonal surfaces itself is not new and it plays a major role in the study of polygonal billiards. A sufficient background can be found in the textbooks [MT02] and [Z06]. For convenience of the reader, we briefly recall some essential terms and constructions. Consider a (not necessarily finite) disjoint collection $\bigcup_{i \in \mathbb{N}} P_i$ of polygons $P_i \subset C$ with boundaries $\partial P_i$ consistently oriented, say counterclockwise. The polygons define a connected surface without boundary, denoted by $X$, if each edge in $\bigcup_{i \in \mathbb{N}} P_i$ is identified with a parallel edge of the same length and opposite orientation.

![Figure 1](image-url)

**Figure 1.** Three types of folded tori, i.e. tori with attached flat spheres

---

*Date: March 13, 2014.*

*2000 Mathematics Subject Classification.* 14H15, 14H52, 30F30, 30F60, 37A60, 37C35, 37N20, 58D15, 58D27.
The surface \( X \) is called \textit{translation surface}, if all the edge identifications are \textit{translations}. If some edge identifications are translations combined with a half-turn, then \( X \) is called \textit{half-translation surface}. Figure 1 (A), (B) and (C) shows three half-translation tori, i.e. half-translation surfaces of genus one, represented by a single L-shaped polygon. Since all edge identifications are holomorphic \((z \mapsto \pm z + c)\) the surface \( X \) is a Riemann surface with a flat metric defined away from a discrete set of vertex points. Because the edge identifications are by half-turns any line foliation \( \mathcal{F}_{\pm \theta}(C) \) of \( C \) induces a line foliation \( \mathcal{F}_{\pm \theta}(X) \) on \( X \). If \( X \) is a translation surface it admits directed line foliations \( \mathcal{F}_{\theta}(X) \) and each of those define a flow in direction \( \theta \). However, if \( X \) is a \textit{strict} half-translation surface, that is one which is not already a translation surface, the leaves are not consistently orientable. For an example, look at Figure 1 (B). The red (oriented) leaves, have two legs each and define a parallel family (red shaded area) bordered by the single blue leaf terminating in a singular point. No orientation consistent with the orientation of the parallel leaves in the (red) family can be given to the blue leaf.

We study the behavior of leaves in direction foliations on the universal cover \( C^T \rightarrow T \) of a \textit{half-translation torus} \( T \). We call those covers \textit{Panov planes} after Dmitri Panov, who studied them in [P10]. In terms of complex analysis a Panov plane is defined as a pair \((C, \pi^*q)\) where \( q \in \mathcal{Q}(C/\Lambda) \) is an integrable, strict, meromorphic quadratic differential on a torus \( C/\Lambda \) and \( \pi : C \rightarrow C/\Lambda \) a \( \mathbb{Z}^2 \) cover. Recall a quadratic differential is called integrable, if it has at most simple poles (subsequently called singularities). Coherent with earlier use of the term we call a quadratic differential \textit{strict}, if it is not the square of an Abelian differential. Because the Euler characteristic of a torus is 0, any integrable and strict quadratic differential on a torus has cone points and therefore it must have simple singularities.

**Classes of affine maps.** In order to study leaves on half-translation surfaces \( X \) we need to assume the existence of a class of maps stabilizing certain foliations. Namely an orientation preserving homeomorphism \( \phi : X \rightarrow X \) of a half-translation surface \( X \) is called \textit{affine}, if it acts affine linear on the set of polygons \( P_i \subset C \) defining \( X \). The affine maps of \( X \) form a group denoted by \( \text{Aff}^+(X) \). On a connected compact surface \( X \) an affine map \( \phi \in \text{Aff}^+(X) \) has constant derivative \( D\phi \in \text{PGL}_2^+(\mathbb{R}) \) which is area preserving, i.e. \( D\phi \in \text{PSL}_2(\mathbb{R}) \). Using the trace on \( \text{SL}_2(\mathbb{R}) \) (or \( \text{PSL}_2(\mathbb{R}) \)) we can define three classes of affine maps: \( \phi \in \text{Aff}^+(X) \) is \textit{elliptic} if \(|\text{tr}(D\phi)| < 2\), \textit{parabolic} if \(|\text{tr}(D\phi)| = 2\), and \textit{hyperbolic} if \(|\text{tr}(D\phi)| > 2\).

**Affine pseudo-Anosov maps.** An affine homeomorphism \( \phi : X \rightarrow X \) with \textit{hyperbolic} derivative is called \textit{pseudo-Anosov}. A hyperbolic map \( A \in \text{PSL}_2(\mathbb{R}) \) has two real eigendirections: The \textit{stable direction} in which \( A \) is contracting and the \textit{unstable direction} in which \( A \) is expanding. The foliations in the respective eigendirections of \( \phi \) are the stable foliation \( \mathcal{F}^s = \mathcal{F}^s(X) \) and the unstable foliation \( \mathcal{F}^u = \mathcal{F}^u(X) \). Affine pseudo-Anosovs belong to the more general class of pseudo-Anosov maps on surfaces for which the following is known:

**Theorem 1.1.** (See Thurston [Th88].) A pseudo-Anosov homeomorphism \( f : X \rightarrow X \) on a compact surface \( X \) satisfies the following:

i. the stable foliation \( \mathcal{F}^s \) for \( f : X \rightarrow X \) is uniquely ergodic (i.e., there exists a unique transverse measure \( \mu^s \), up to multiplication by a non-zero scalar);

ii. the transverse measure satisfies \( f_*\mu^s = e^{h(f)}\mu^s \), where \( h(f) > 0 \) denotes the topological entropy of \( f \).

The same statements apply to the unstable foliation, because it is the stable foliation of \( f^{-1} \). Any pseudo-Anosov map \( f : X \rightarrow X \) induces an invertible linear map in homology \( f_* : H_1(X;\mathbb{Z}) \rightarrow H_1(X;\mathbb{Z}) \). If \( X \) is a half-translation torus, say \( T \), then \( H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2 \), so \( f_* \in \text{SL}_2(\mathbb{Z}) \) and \( f_* \) is
there is a constant $0 < \varepsilon < 1$ on $C^\infty$. Theorem 1.3. Come back to those results. Through covering construction. In the section on the geodesic flow in moduli space (pg. 8–11) we model and strip-billiards [FU12]. The strip billiards in [FU12] are related to certain Panov planes of ergodic directions of Hausdorff dimension larger than $1/2$. The method applies to the wind-tree directionally bounded in $\mathbb{C}$. In fact in [FS13] the authors utilize techniques from Frâczek and Ulcigrai [FU11, FU12] to show that directional boundedness in $\mathbb{C}$ is confined in a strip of width $2\varepsilon$ in direction of the line $\mathbb{R}v \subset \mathbb{C}$. The line and strip are placed into of $C^\infty$ with respect the lattice $\mathbb{Z}^2 \cong D_p \subset C^\infty$. Our result for dynamics in eigendirections of pseudo-Anosovs is:

**Theorem 1.2.** Let $T$ be a half-translation torus and $C^\infty \to T$ be the associated Panov plane. Assume $\phi \in \text{Aff}^+(T)$ is a pseudo-Anosov and let $\phi_* \in SL_2(\mathbb{Z}) \cong \text{Aut}(H_1(T;\mathbb{Z}))$ denote its induced action on homology.

- The stable eigenleaf $f_p^s$ emanating from any singular point $p \in C^\infty$ is dense in $C^\infty$, if and only if it is convex and $\phi_*^n = \text{id}$ for some $n \in \mathbb{N}$.
- If $\phi_*$ is hyperbolic, all leaves of the stable foliation of $\phi$ on $C^\infty$ are directionally bounded with respect to the stable direction of $\phi_*$.  

**Remark 1.** The proof of Theorem 1.2 in [JS13b] uses standard tools such as geometry, the action of the pseudo-Anosov and its induced action on homology, we do not need to invoke general theorems on foliations. In [JS13b] we also show generalizations of Theorem 1.2 to larger sets of directions. To state the result in such a form would take a significant amount of additional technical background.

Related results for foliations in eigendirections of pseudo-Anosov maps on universal covers were also published by Boyland [B09] (dense orbits) and Pollicott and Sharp [PS07] (ergodicity). We add their results applied to Panov planes below. Their methods fall into part of dynamical systems called topological dynamics.

**Remark 2.** With the help of Teichmüller theory, i.e. deformations of surfaces in moduli space, one can prove dynamical statements for uncountable sets of directions on the windtree-model, see [FU11, FU12] and [D13]. The results in those articles are related to the statements of Theorem 1.2. In fact in [FS13] the authors utilize techniques from Frâczek and Ulcigrai [FU11, FU12] to show directional boundedness in almost every direction on particular Panov planes.

In [Ho10] Hooper presents a method, essentially using Teichmüller theory, which produces sets of ergodic directions of Hausdorff dimension larger than $1/2$. The method applies to the wind-tree model and strip-billiards [FU12]. The strip billiards in [FU12] are related to certain Panov planes through covering construction. In the section on the geodesic flow in moduli space (pg. 8–11) we come back to those results.

A direct application of Panov’s argument [P10] allows us to explicitly construct dense leaves:

**Theorem 1.3.** [JS13b] Under the assumptions of Theorem 1.2, if the singular point $p \in C^\infty$ is a fixed point of the (lifted) pseudo-Anosov $\phi$ and the action of $\phi_* : H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z})$ is elliptic, then $f_p^s$ is dense on $C^\infty$. 

either hyperbolic, parabolic, or elliptic. Note that the type of $f_*$ is well defined, since the trace is a conjugation invariant.
To show this theorem one needs that elliptic elements in $\text{SL}_2(\mathbb{Z})$ have bounded order ($\leq 6$) and therefore certain powers of elliptic maps are trivial. Then homology convexity of $f_p^n$ for a pseudo-Anosov acting elliptic on homology is not hard to establish, see [JS13b]. In [JS13b] we also provide some conditions for convexity of (un-)stable leafs emanating from singular points. The topology of the base torus of any Panov plane implies the existence of singular points and those are metrically cone points of total angle $\pi$.

Recall that every half-translation surface is doubly covered by a translation surface, called the orientation cover. In the complex analytic language, the pull back of a strict quadratic differential (with at most simple singularities) to the orientation cover is the square of an Abelian differential with branching exactly over the cone points having an odd multiple of $\pi$ as total angle.

**Theorem 1.4.** [JS13b] Let $O^T \rightarrow C^T$ be the orientation cover of the Panov plane $C^T$. Assume that there is an affine pseudo-Anosov on $C^T$ acting elliptic on homology and fixing a singular point $p \in C^T$. Then the lifts of $f_p^n$ to $O^T$ are dense. In particular the stable direction of $\phi$ is flow-transitive on $O^T$.

Recall the spectral radius of an affine pseudo-Anosov $\phi : T \rightarrow T$ is the absolute value of its leading eigenvalue $\lambda_1$ in homology $\phi_* : H_1(T; \mathbb{Z}) \rightarrow H_1(T; \mathbb{Z})$. In particular the spectral radius of a pseudo-Anosov acting elliptic on homology is one. In fact, if $\phi$ is an affine pseudo-Anosov on a translation surface then $h(\phi) = \log \lambda = \log \lambda_1$, where $\lambda$ denotes the leading eigenvalue of $D\phi$.

The (abstract) existence of dense orbits in eigendirections of pseudo-Anosov homeomorphisms on universal abelian covers has been observed by Boyland [B09, Thm. 1.2]. Boyland’s result for half-translation tori states:

**Theorem 1.5** (Boyland). Let $T$ be a half-translation torus and $\phi \in \text{Aff}^+(T)$ be a pseudo-Anosov with spectral radius one. Then the (un)stable eigenfoliation of $\phi$ on the Panov plane $C^T$ contains dense leaves.

**Ergodicity.** We also like to mention an ergodicity result of Pollicott and Sharp [PS07]. In order to state this result, let $\mu$ be a transverse measure of a direction foliation on a half-translation torus $T$ and $\tilde{\mu}$ its lift to $C^T$. To bring the main theorem of [PS07] in the context of Panov planes, recall that an affine pseudo-Anosov on a half-translation torus is isotopic to the identity, if it acts trivially on homology $H_1(T; \mathbb{Z})$ (for details see [JS13b], or [FM11]).

**Theorem 1.6** (Pollicott-Sharp [PS07]). Let $C^T \rightarrow T$ be a Panov plane. Assume $\tilde{\phi} \in \text{Aff}^+(C^T)$ is a lift of a pseudo-Anosov $\phi \in \text{Aff}^+(T)$ acting elliptic on homology and $\tilde{\phi}$ fixes a (non-)singular point $p \in C^T$. Let $\mu^u$ be the transverse measure of the unstable eigenfoliation of $\phi$. Then the lift $\tilde{\mu}^u$ of $\mu^u$ to $C^T$ is ergodic and satisfies

$$\tilde{\phi}_* \tilde{\mu}^u = e^{h(\phi)} \tilde{\mu}^u.$$

Eigenfoliations of pseudo-Anosov maps have been studied by Thurston [FM11, FLP11].

We now show a connection of Panov planes to the periodic wind-tree model.

**THE PERIODIC EHRENFEST WIND-TREE MODEL AND PANOV PLANES**

Certain Panov planes can be used to study dynamical systems appearing in physics, such as the wind-tree model. Recall that a billiard on a polygonal table with (polygonal) obstacles is the dynamics defined by a point particle moving on lines with unit speed. At obstacles the particle is reflected according to the light reflection laws: The angle of incidence is the angle of reflection. In case the particle hits a corner point its trajectory will not be continued.

Figure 2 shows three doubly periodic surfaces, each hosts particular dynamics. Using the $\mathbb{Z}^2$ translation symmetry one easily checks, that the two tables shown in Figure 2 (B) and Figure 2 (C) are universal covers, i.e. $\mathbb{Z}^2$-covers, of the half-translation tori as shown in Figures 1 (B) and 1 (C) respectively. In particular both are Panov planes. Either of the two half-translation tori can be realized as translation torus with attached flat sphere(s), for that reason we call them folded.
tori. We will also use the term pillow-case for flat spheres with four singular points. A pillow-case is the quotient of a torus $\mathbb{R}^2/\Lambda$ ($\Lambda \subset \mathbb{C}$ a lattice) with respect to the action of the idempotent $-\text{id}_\Lambda: \mathbb{R}^2/\Lambda \to \mathbb{R}^2/\Lambda$ induced from $-\text{id}_{\mathbb{R}^2}: \mathbb{R}^2 \to \mathbb{R}^2$. For $a, b \in (0, 1)$ let

$$R_{a,b} := (-a/2, a/2) \times (-b/2, b/2) \subset \mathbb{R}^2.$$ 

The periodic Ehrenfest wind-tree model $E_{a,b}$, or simply the wind-tree model (Figure 2 (A)) is the billiard on the infinite table $T_{a,b}$. The state space $E_{a,b}$ of the billiard is the full measure subset of $T_{a,b} \times S^1$, with the restriction that at edges the angle has to point inwards. Since a billiard particle on $T_{a,b}$ adopts at most four different directions any billiard trajectory is confined to a flow invariant subspace of the shape $T_{a,b} \times \{\pm \theta, \pi \pm \theta\}$ for some $\theta \in S^1$. Let us denote the billiard flow restricted to the invariant subspace $T_{a,b} \times \{\pm \theta, \pi \pm \theta\}$ by $e^\theta_t(z)$. More precisely $e^\theta_t(z)$ is the state $(z, \theta) \in T_{a,b} \times \{\pm \theta, \pi \pm \theta\}$ a billiard particle with state $e^\theta_0(z) := (z, \theta)$ reaches at (time) $t \in \mathbb{R}$ while traveling with unit speed. Let us further denote the projection of $e^\theta_t(z)$ to $T_{a,b}$ by $b^\theta_t(z)$.

There is a connection between the wind-tree dynamics (Figure 2 (A)) and the L-shaped surface dynamics shown in Figure 1 (A), it is indirect though: One needs to consider the phase space $T_{a,b} \times \{\pm \theta, \pi \pm \theta\}$. This is a translation surface with a particular $\mathbb{Z}^2$ translation symmetry and its quotient with respect to that symmetry is a four-fold translation cover of a certain L-shaped translation surface, see [DHL11] and [FU11].

The wind-tree model: History and recent progress. The wind-tree model was introduced about one hundred years ago by Paul and Tatyana Ehrenfest [EE] and has been considered an important model in statistical physics ever since. However, very few results on it are rigorous. Based on new mathematical tools and broader interest in infinite billiards, several rigorous and interesting statements on the periodic wind-tree model appeared recently, particularly [HLT10, DHL11, D13] and [FU11], but also [CG10]. Recurrence for a full measure set of directions was shown by Hubert, Lelièvre and Troubetzkoy for rational tables [HLT10] and for general tables by Avila and Hubert [AH11] using Teichmüller flow techniques. Given that many dynamical properties, for example recurrence and ergodicity on $\mathbb{Z}$-covers are understood (see [HW11] and [HuW12]), the periodic wind-tree model serves as a stepping stone to investigate dynamical phenomena on $\mathbb{Z}^d$ covers.
Dmitri Panov [P10] located dense leaves in the eigenfoliation of certain pseudo-Anosov maps on the planes covering rational tori with one fold. Panov planes and wind-tree models are connected, we show that the wind-tree dynamics is “contained” in the dynamics on a pair of Panov planes.

**Converting wind-tree dynamics to Panov plane dynamics.** For the argument below recall that a point \((x_0, y_0) \in \mathcal{T}_{a,b} \subset \mathbb{C}\) defines a point on the three Panov planes \(C_{\hat{a},b}, C_{a,\hat{b}}\) and \(C_{\hat{a},\hat{b}}\) which we also denote by \((x_0, y_0)\).

**Proposition 1.7.** Fix \(\theta_0 \in S^1, a, b \in (0, 1)\) and \((x_0, y_0) \in \mathcal{T}_{a,b}\). Let \((x(t), y(t)) \in \mathcal{T}_{a,b}\) be the billiard trajectory of a particle starting at \((x_0, y_0)\) in direction \(\theta_0\). Further let \((x_a(t), y_a(t)) \in C_{a,b}\) and \((x_b(t), y_b(t)) \in C_{\hat{a},\hat{b}}\) be the leaves starting in direction \(\theta\) at \((x_0, y_0)\). Then

\[
x(t) = x_b(t) \quad \text{and} \quad y(t) = y_a(t)
\]

for any \(t \in \mathbb{R}\).

The direct geometric argument below was observed by Michael Burr (Clemson). An alternative and less elementary way is to consider certain quotients of the unfolded billiard followed by coverings.

**Proof.** Because \(e^{\theta_0}(x_0, y_0) = e^{-\theta_0}(x_0, y_0)\), we only need to consider the case \(t \geq 0\). Without restrictions of generality consider a billiard trajectory \(\gamma \subset E_{a,b}\) which hits the obstacles, since otherwise \(\gamma_a = \gamma\) on \(C_{\hat{a},b}\) and \(C_{a,\hat{b}}\). The (projection of the) billiard trajectory on \(\mathcal{T}_{a,b}\) decomposes into a sequence of segments \(\gamma = s_1s_2 \cdots\). Besides the first and last segment each \(s_i\) is the straight line segment between two reflections. We define the corresponding trajectory \(\gamma_a = s_1s_2 \cdots \subset C_{\hat{a},\hat{b}}\) segment by segment. To start, put \(s_1^0 = s_1\). To continue reflect \(\gamma_2 := s_2s_3 \cdots\) at the vertical line through the center of the rectangle where \(s_2\) emanates. Denote this billiard trajectory by \(\gamma'_2 := r_1(s_2s_3 \cdots)\). The second segment \(s_2^0 \subset C_{\hat{a},\hat{b}}\) of \(\gamma_a\) is the first segment of \(\gamma'_2\). Note, that this is the dynamically correct continuation of \(s_1^0\) on the Panov plane \(C_{\hat{a},\hat{b}}\). To proceed, take away the first segment of \(\gamma'_2\) and repeat the previous construction with that trajectory, to obtain \(s_3^0 \subset C_{\hat{a},\hat{b}}\) and so forth. Each new segment produced by a vertical reflection defines a dynamical continuation of the already constructed path on \(C_{\hat{a},\hat{b}}\). Since \(y\)-coordinates are invariant under reflections at verticals, the claim follows.

The trajectory \(\gamma_b \subset C_{a,b}\) is constructed from \(\gamma\) the same way using horizontal reflections. \(\square\)

Successive reflections about the middle points of all edges along the lines of the previous proof convert any billiard trajectory on \(\mathcal{T}_{a,b}\) into a trajectory on the Panov plane \(C_{\hat{a},\hat{b}}\). In this case there are no invariant coordinates, though.

**DEHN TWISTS, LATTICE SURFACES AND TORUS TWISTS**

We consider examples of Panov planes and half-translation tori having many affine pseudo-Anosov maps. In this section \(C_{a,b}\) and \(\mathcal{T}_{a,b}\) denotes any of the three possible folded planes and associated folded tori, \(C_{a,b}\) for example stands for \(C_{\hat{a},\hat{b}}, C_{a,\hat{b}}\), and \(C_{\hat{a},\hat{b}}\).

Consider a half-translation surface \(X\) having a non-trivial parabolic affine homeomorphism. A parabolic map stabilizes its single eigen-direction which is completely periodic, i.e. every leaf in the stabilized direction is compact (if \(X\) is compact). In polygonal local coordinates a parabolic map appears as a linear *Dehn twist*. If for instance \(\phi_h \in \text{Aff}^+(X)\) and \(\phi_v \in \text{Aff}^+(X)\) are Dehn twists stabilizing the horizontal foliation of \(X\) and vertical foliation of \(X\), respectively, their derivatives have the shape \(D\phi_h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: P_h\) and \(D\phi_v = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} =: P_v\) for some \(\alpha, \beta \in \mathbb{R}\). Taking combinations
of (multi-)twists, like \( \phi^{-1} \phi_v \), defines pseudo-Anosov homeomorphisms on \( X \). Using pairs of non-commuting parabolic maps to construct pseudo-Anosovs is known as Thurston construction (see [FLP11, FM11]).

A half-translation surface \( X \) is called lattice surface, if \( \text{PSL}_2(\mathbb{R}) / \text{DAff}^+ (X) \) has finite (hyperbolic) volume, i.e. \( \text{DAff}^+ (X) \subset \text{PSL}_2(\mathbb{R}) \) is a lattice. L-shaped surfaces, like the lower left polygon in Figure 3, or the polygons in Figure 5, having horizontal and vertical Dehn twists are already lattice surfaces. That follows from McMullen’s [McM03], or Calta’s [C04] characterization of lattice surfaces in genus two.

**Proposition 1.8.** The following are equivalent for the folded tori \( T_{a,b} \), L-shaped surfaces \( L_{a,b} \) and folded planes \( C_{a,b} \):

1. The horizontal and vertical directions in \( L_{a,b} \) have parabolic stabilizers
2. The horizontal and vertical directions in \( T_{a,b} \) have parabolic stabilizers
3. The horizontal and vertical directions in \( C_{a,b} \) have parabolic stabilizers
4. \( C_{a,b} \) is a lattice surface
5. \( T_{a,b} \) is a lattice surface
6. \( L_{a,b} \) is a lattice surface
7. \( \frac{1}{a} = x + y \sqrt{d} \) and \( \frac{1}{b} = 1 - x + y \sqrt{d} \) for \( x, y \in \mathbb{Q} \) and \( d \in \mathbb{N} \) squarefree.

We call a Panov plane with the lattice property lattice Panov plane. The statement in the proposition holds in greater generality: Tori obtained from genus two lattice surfaces by so called twists are lattice surfaces, see Figure 3 on page 8. This is shown in [McM06] where the twist construction has been introduced and applied to a different class of problems.

Let \( O_{a,b} \rightarrow \hat{T}_{\hat{a},\hat{b}} \) and \( O_{\pi_{\hat{a},\hat{b}}} \rightarrow \hat{T}_{\hat{a},\hat{b}} \) be the orientation covers. In this case the orientation covers have genus 3, see Figure 3. The lattice property is stable under quotients by involutions [McM06] and covers branched only over periodic points under the action of the affine group [GJ00]. In our examples all branching appears over (periodic) Weierstrass points. Thus the statements in Proposition 1.8 are a consequence of the “zigzag”-diagram of covering maps. Note that only the surfaces in the lower two rows of diagram 1.1 are of finite genus.

\[
\begin{array}{c}
\hat{T}_{\hat{a},\hat{b}} & \xrightarrow{O_{\pi_{\hat{a},\hat{b}}}} & L_{a,b} & \xrightarrow{O_{\hat{a},\hat{b}}} & T_{\hat{a},\hat{b}} \\
\downarrow{\pi_{\hat{a},\hat{b}}} & & \downarrow{\pi_{a,b}} & & \downarrow{\pi_{a,b}} \\
C_{\hat{a},\hat{b}} & & C_{\tilde{a},\tilde{b}}
\end{array}
\]

Figure 3 shows that the orientation cover of the twice folded \( L \) is branched over (periodic) Weierstrass points. The quotient \( \hat{T}_{\hat{a},\hat{b}} \) is obtained by rotating the two L-shaped polygons in the upper right of Figure 3 onto each other. That rotation will twist the edges connecting the two decks onto itself. The maps indicated in Figure 3 are maps of (half)-translation surfaces: The left arrow denotes a two-fold translation cover, the right arrow denotes a two-fold half-translation cover. The top map is a translation isomorphism defined by cutting and pasting of polygons. The half-translation surface on the right is called a twist of the genus two surface on the lower left.

**Twists of one-forms.** The construction in Diagram 1.1 and Figure 3 is just one way to twist the genus 2 base surface (to the left) into a half-translation torus. It is natural to ask how many (half-translation) tori can be obtained by twists. In fact the construction appears also in Vasilyev [V05] and as already mentioned in McMullen [McM06].
To state the problem more precisely consider a translation surface \((X, \sigma)\) together with an affine involution \(\sigma \in \text{Aff}^+(X)\) and assume \(\pi : (\tilde{X}, \tilde{\sigma}) \to (X, \sigma)\) is a double translation cover admitting a lifted involution \(\tilde{\sigma} \in \text{Aff}^+(\tilde{X})\), i.e. \(\pi \circ \tilde{\sigma} = \sigma \circ \pi\). The sheet exchange map \(\delta \in \text{Aff}^+(\tilde{X})\), \(\pi \circ \delta = \pi\) commutes with \(\tilde{\sigma}\) and defines two types of quotients: \(\tilde{X} \to \tilde{X}/\tilde{\sigma}\) and \(\tilde{X} \to \tilde{X}/\delta \circ \tilde{\sigma}\). Both quotients carry an induced half-translation structure, say \(q\), which is in general not a translation structure. Following McMullen [McM06], we call \((\tilde{X}/\tilde{\sigma}, q)\) a twist of \((X, \sigma)\) and a torus twist, if \(X/\sigma\) is a torus. The twist in Figure 3 is the one obtained by using the involution \(\delta \circ \tilde{\sigma}\).

Because a genus two surface is hyperelliptic, there are fifteen possible torus twists which are genuine half-translation tori:

**Theorem 1.9.** [JS13a] Let \((X, \sigma)\) be a hyperelliptic translation surface of genus \(g\) and \(\sigma \in \text{Aff}^+(X)\) its hyperelliptic involution. Then there are \((\frac{2g+2}{2g+2})\) half-translation twists \(\tilde{X}\) of genus \(\tilde{g}\). In particular there are \(\left(\frac{2g+2}{4}\right) = \frac{1}{6} (g+1)(4g^2-1)\) strict half-translation tori among those twists. The twists are the branched double covers \(\tilde{X} \to X/\sigma \cong \mathbb{C}P^1\) branched over a subset of the image of the Weierstrass points in \(X/\sigma\) with respect to the natural map \(X \to X/\sigma\). In particular any twist of \((X, \sigma)\) is a lattice surface if and only if \((X, \sigma)\) is.

**Geodesics in moduli space, recurrence and generalizations of Theorem 1.2**

We now discuss some results which provide ideas how to show versions of Theorem 1.2 for larger sets of directions. Particularly relevant are the techniques used in [DHL11, FU11] and [D13] to study the wind-tree model. The method of Delecroix [D13] and Frączek’s and Ulcigrai’s [FU12] analysis of strip billiards are depend on Hooper [Ho10]. Hoopers method identifies sets of directions of Hausdorff Dimension larger than \(1/2\) having uniform dynamical properties, such as ergodicity [Ho10, FU12], or escaping orbits [D13]. Note, that the statements of Theorem 1.2 hold for a countable set of pseudo-Anosov eigendirections. Teichmüller methods can be used to show more general statements for certain Panov planes. In [FS13] for instance, the authors show directional boundedness of trajectories in almost every direction on Panov planes covering tori with two singular points using those methods.
The general idea is to study geodesics in moduli space, the orbits of the Teichmüller geodesic flow defined by applying the one-parameter group \((g_t)_{t \in \mathbb{R}} \subset \text{SL}_2(\mathbb{R})\), where \(g_t \equiv \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}\) to a half-translation surface. To consider geodesics is motivated by the fact that a pseudo-Anosov on a half-translation surface defines a periodic geodesic in moduli space. The pseudo-Anosov is recovered as a return map of the family of surfaces parameterized by the geodesic loop in moduli space. In particular pseudo-Anosov maps define recurrent orbits of the Teichmüller flow.

Directions on a half-translation surface for which the dynamical conclusions of Theorem 1.2 hold share the following properties:

- **Uniquely ergodic foliations.** Eigenfoliations of pseudo-Anosov maps (on compact surfaces) are uniquely ergodic, with respect to Lebesgue measure.
- **Elliptic, or hyperbolic homology.** The map on homology induced by a pseudo-Anosov should be elliptic, or hyperbolic.

Let us briefly describe why those properties have interpretations for the Teichmüller flow. The geodesic flow interpretation of unique ergodicity for direction foliations is a widely applied success of the Teichmüller picture. In fact Masur’s criterion [M92] states that unique ergodicity of the vertical foliation on a half-translation surface, say \(X\), is a consequence of the recurrence of the geodesic \((g_tX)_{t \in \mathbb{R}}\) defined by \(X\) in moduli space.

To come to the second property, consider the homology class of a loop obtained from closing up a “very long” segment of a (vertical) leaf. This segment can be written with respect to a particular homology basis, itself being constructed by closing vertical segments, for details see [Z06, 480-482]. Unique ergodicity now implies that all very long vertical leaves essentially define the same homology class, in the sense that each basis class appears with nearly the same frequency. This uniformity motivates to study the limiting behavior of homology along a geodesic. To do that one needs to generalize objects defined over a particular surface, such as the first homology group (or vector space) to objects that make sense over moduli space. The (homological) Hodge bundle \(\mathcal{H}\) over moduli space (of genus \(g\) curves) is the bundle with fiber the first homology group of the surface defining the respective point in moduli space. The changes of homology along a geodesic is recorded by the Kontsevich-Zorich cocycle \(G_{KZ}^t : \mathcal{H} \to \mathcal{H}\). Given an ergodic measure for the geodesic flow then Oseledet’s Theorem characterizes the asymptotic behavior of the Kontsevich-Zorich cocycle in terms of Lyapunov exponents. By deep recent results of Eskin and Mirzakhani [EM13] there are natural flow invariant ergodic measures on all \(\text{SL}_2(\mathbb{R})\)-invariant loci in strata of Abelian differentials. Hyperbolicity and ellipticity are properties which can be read off the Lyapunov exponents. If any one of the two properties holds, it can be used to study the dynamical behavior of vertical leaves on surfaces in invariant loci [FS13, FS14]. Despite recent results in [EKZ12] relating the so called Siegel-Veech constants to the sum of the Lyapunov exponents, there are no general methods to calculate particular Lyapunov exponents. There are also challenges to work out this strategy for concrete examples, because we generally do not know what the smallest \(\text{SL}_2(\mathbb{R})\)-invariant locus containing a geodesic is. Moreover the Kontsevich-Zorich cocycle approach applies to Abelian differentials, and since Panov planes (and their quotient tori) are quadratic differentials one needs to consider their orientation covers. But the genus of those orientation covers can be arbitrarily large and for that reason the calculation of Lyapunov exponents is eventually difficult.

Despite all these general problems the geodesic flow method has been effectively applied by Frączek and Ulcigrai [FU11] to study ergodicity of the wind-tree model:

**Theorem 1.10** (Frączek-Ulcigrai). There exists a set \(\mathcal{P} \subset [0, 1]^2\) of full Lebesgue measure, such if

1. \(a, b \in (0, 1)\) are rational numbers, or
2. \(a, b \in (0, 1)\) of the shape \(\frac{1}{1-a} = x + y\sqrt{d}\) and \(\frac{1}{1-b} = 1 - x + y\sqrt{d}\) for \(x, y \in \mathbb{Q}\) and \(d\) a positive square-free integer, or
(3) \((a, b) \in \mathcal{P}\),
then for almost every \(\theta \in S^1\) the directional billiard flow \((e^\theta_t)_{t \in \mathbb{R}}\) on \(\mathcal{E}_{a,b}\) is recurrent, not ergodic, not transitive and there are uncountably many ergodic components.

In this theorem non-trivial Lyapunov exponents, as related to the hyperbolic case, play a key role. Grivaux and Hubert [GH13] on the other hand constructed affine manifolds of quadratic differentials (of arbitrary genus) with vanishing Lyapunov exponents.

In [FS13] the authors apply study the geodesics in moduli space to provide an extension of the bounded leaf statement in Theorem 1.2. As a direct consequence of [FS13] there is a full measure set of directions on almost every Panov plane of the kind \(C_{0,\beta}\) and \(C_{\alpha,0}\), i.e. the Panov planes with one fold, for which the leaves are directionally bounded. The locus in moduli space for which needs to be studied is this case are the genus two, degree two covers of (unimodular) tori branched over two points. We now come to Hoopers approach [Ho10] to generate uncountably many directions with certain dynamical properties. In fact, we explain a manifestation of Hooper’s strategy for the wind-tree model by Delecroix [D13] and for strip billiards by Frączek and Ulcigrai [FU12].

\begin{figure}[h]
\centering
\subfigure[100 iterations]{
\includegraphics[width=0.4\textwidth]{fig_a}
}\hspace{1cm}
\subfigure[1000 iterations]{
\includegraphics[width=0.4\textwidth]{fig_b}
}
\subfigure[100,000 iterations]{
\includegraphics[width=0.4\textwidth]{fig_c}
}\hspace{1cm}
\subfigure[500,000 iterations]{
\includegraphics[width=0.4\textwidth]{fig_d}
}
\caption{A recurring orbit on the table with parameters \(2\alpha - 1 = \sqrt{17} = 2\beta + 3\)}
\end{figure}

**Divergent directions for the wind-tree model.** To state Delecroix’s result [D13] for the wind-tree model, we need some conventions.

**Definition 1.11.** We call a direction \(\theta \in S^1\) on \(\mathcal{T}_{a,b}\) self-avoiding, if for all \(z \in \mathcal{T}_{a,b}\) (for which the forward orbit is defined) \(t \mapsto b^\theta_t(z) \in \mathcal{T}_{a,b}\) is a topological embedding.

Informally this means the billiard ball eventually leaves any compact set without self-crossings.
Theorem 1.12 (Delecroix [D13]). Let $T_{a,b}$ be the Ehrenfest wind-tree model with rational parameters $a, b \in (0, 1)$, or parameters of the shape
\[
\frac{1}{1-a} = x + z\sqrt{d} \quad \text{and} \quad \frac{1}{1-b} = 1 - x + z\sqrt{d}
\]
where $d \in \mathbb{N}$ is square free. Then there is a dense set of directions $\Lambda(a, b) \subseteq S^1$ of Hausdorff dimension $HD(\Lambda(a, b)) > 1/2$, such that any direction $\theta \in \Lambda$ is self-avoiding.

In [D13] and [Ho10] those directions are studied using renormalization of (cocycles over) interval exchange maps capturing the dynamics of the direction flow. Instead of the traditional Rauzy induction Delecroix uses Ferenczi-Zamboni induction [FZ10, FZ11]. In fact, the directions in Delecroix's theorem are explicit and can be identified with the eigendirections of an (infinite) Rauzy induction. Delecroix uses Ferenczi-Zamboni induction to construct a dense orbit on the Panov plane which is a definition of the Panov plane in Delecroix's theorem.

\[\prod i \in S \subset \mathbb{N} \phi^m_{\alpha_i} \phi^2_{\beta_i}, \quad m_i, n_i \in \mathbb{N}\]

where $\phi_{\alpha}, \phi_{\beta} \in \text{Aff}^+(L)$ are (positively twisting) parabolic stabilizers of the horizontal and vertical foliation on an $L$-shaped lattice surface. The limiting eigendirections have particular continued fraction expansions, see [D13, Prop. 14].

Wind-tree trajectories which are limits of pseudo-Anosov eigendirections, may help to construct recurrent, or dense trajectories. Figure 4 shows an orbit in the eigendirections of the pseudo-Anosov defined by the Dehn-twists $P^{-1}_{h} P_{v}$ on $T_{a, b}$ with parameters $2\alpha - 1 = \sqrt{17} = 2\beta + 3$. For a definition of $(\alpha, \beta)$ and $(a, b)$ parameters for $L$-shaped surfaces see Figure 5. Their relations are given in equation 1.3. The plotted orbit is constructed from a dense orbit on the Panov plane $C_{\alpha, \beta}$ by means of Proposition 1.7 preserving the $y$-coordinate. For a detailed description of that orbit see Lemma 1.19 and below.

At this point we formulate the question: Given a dense direction on one of the two Panov planes corresponding to a specific wind-tree model. If this direction is homology elliptic (in any possible interpretation discussed so far), can we make conclusions on the dynamics of the corresponding wind-tree orbit? More generally: If we know something about the dynamics on one model (wind-tree), what can we say about the dynamics in the same direction on the other (Panov)?

Fračzek’s and Ulcigrai’s analysis of strip billiards. To contrast their non-ergodicity result [FU11] Fračzek and Ulcigrai were studying ergodic directions on strip billiards. In fact, in [FU12] Fračzek and Ulcigrai describe a set of ergodic directions of Hausdorff dimension larger than 1/2 on certain infinite strip billiards. Strip billiards on the other hand are related to both, the periodic wind-tree model and Panov planes of the type $C_{a, b}$ through unfolding and covering constructions. To see that one considers the orientation cover $O^Z_{a} \to C_{a, b}/\mathbb{Z}$ of the quotient surface in $C_{a, b} \to C_{a, b}/\mathbb{Z}$. The translation surface $O^Z_{a}$ corresponds to the translation surface of the (unfolded) table $T(1/2, 1, a/2)$ in [FU12, Figure 1]. The set of ergodic directions in [FU12] have a certain elliptic action on homology paralleling use of the ellipticity in Theorem 1.2. The results of Fračzek and Ulcigrai do not directly require the setting of lattice surfaces, therefore it would be interesting to see to what extend those techniques apply to Panov planes.

Besides extensions to large sets of directions a result of Eskin and Chaika [CE13] may apply to extend our results beyond surfaces with the lattice property, or spaces of lattice surface covers.

Lattice tori with non-trivial affine Torelli group

The affine Torelli group $\mathcal{I}_A(X) \ltimes \text{Aff}^+(X)$ of a half-translation surface $X$ is the group of affine maps that act trivially on $H_1(X; \mathbb{Z})$. Let us call the group of derivatives $\mathcal{I}_V(X) := D \mathcal{I}_A(X) \subset \text{SL}_2(\mathbb{R})$ the Torelli-Veech group.
A result of Veech [V89] implies, that the affine Torelli group on a translation surface is trivial. This is not true for half-translation surfaces, in fact there are half-translation tori with nontrivial affine Torelli group:

**Lemma 1.13.** Let \( \mathbb{T}_{\hat{a}, \hat{b}} \) be a folded torus with (positive) parameters of the shape

\[
a = \frac{\sqrt{D} - 1}{2} + k, \quad b = \frac{\sqrt{D} - 1}{2} - k
\]

for positive \( D \equiv 1 \mod 8 \) and \( k \in \mathbb{Z} \). Then \( I_A(\mathbb{T}_{\hat{a}, \hat{b}}) \) contains a free group.

Note that for fixed \( D \) the above describes a one parameter \( (k \in \mathbb{Z}) \) family in the two parameter family (over \( \mathbb{Q} \)) of lattice tori, as in part seven of Proposition 1.8, constructed by twisting genus 2 lattice surfaces. It is an interesting question whether the set of lattice tori with non trivial Torelli group in this two parameter family is actually larger.

Below we describe elements of the affine Torelli group, that appear as powers of pseudo-Anosovs acting elliptic on homology.

**Corollary 1.14.** For any pair of parameters \( (a, b) \) as in the previous Lemma there is an uncountable, dense set of directions for which the direction foliation on \( \mathbb{C}_{\hat{a}, \hat{b}} \) has dense orbits starting at singular points. Moreover the elements of the affine Torelli group \( I_A(\mathbb{T}_{\hat{a}, \hat{b}}) \) whose lifts to \( \text{Aff}^+(\mathbb{C}_{\hat{a}, \hat{b}}) \) fix all the singular points on \( \mathbb{C}_{\hat{a}, \hat{b}} \) has finite index in \( I_A(\mathbb{T}_{\hat{a}, \hat{b}}) \).

**Half-translation tori with non-trivial affine Torelli group.** We describe a class of lattice half-translation tori having homology elliptic pseudo-Anosovs. The universal covers of those tori admit dense orbits. The results below are numerically easier if we rescale the L-shaped polygons to have a core square of size 1 with a rectangle of width \( \alpha \) attached to the (right) side and topped by another rectangle of height \( \beta \). The relation between the \( (a, b) \)-parameters we have used so far and the new \( (\alpha, \beta) \)-parameters is

\[
\alpha = \frac{a}{1-a}, \quad \beta = \frac{b}{1-b}
\]

By Proposition 1.8, any L-shaped polygon, of total width \( 1 + \alpha \) and total height \( 1 + \beta \), defines surfaces admitting horizontal and vertical Dehn twists, if

\[
a = x + y \sqrt{d} \quad \text{and} \quad \beta = z + y \sqrt{d}
\]

for \( x, y, z \in \mathbb{Q}, x + z = -1 \) and \( d \in \mathbb{N} \). The original condition \( x + z = 1 \) on \( x \) and \( z \) as in [McM03], is obtained from this by replacing the parameters \( a \) and \( \beta \) with \( p = 1 + \alpha \) and \( q = 1 + \beta \).

**Simple twist surfaces.** Let us call a lattice surface modeled on an L-shaped polygon *simple*, if its horizontal and vertical foliation have parabolic stabilizers (affine Dehn twists) twisting the respective long cylinder exactly once. The “long cylinder” (in either direction) is the one containing the one-by-one square. Let us now determine the parameters \( a, \beta \in \mathbb{R}_+ \), for which \( \mathbb{T}_{\hat{a}, \hat{b}} \) is simple.
The minimal parabolic stabilizers of bottom and top horizontal cylinders in $T_{\alpha,\beta}$ are given by the linear Dehn twists:

$$P_b = \begin{bmatrix} 1 & 1 + \alpha \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_t = \begin{bmatrix} 1 & \frac{2}{\beta} \\ 0 & 1 \end{bmatrix}$$

Note, that the (minimal possible) horizontal twist of the top cylinder maps $(0, \beta/2)^T$ to $(1, \beta/2)^T$. The vertical cylinder decomposition delivers an analogous set of parabolic matrices.

For the existence of a global parabolic map fixing the horizontal foliation of $L_{\alpha,\beta}$, the one parameter groups $(P_b)$ and $(P_t)$ generated by powers of the respective maps, have to have an infinite (cyclic) intersection. That is, there exist numbers $b_h, k_h \in \mathbb{Z}$, such that

$$b_h(1 + \alpha) = k_h \frac{2}{\beta}.$$

Simplicity of the twists implies $b_h = 1$. For the horizontal and vertical direction on $L_{\alpha,\beta}$ the compatibility conditions imply: $1 + \alpha = k_h \frac{2}{\beta}$ and $1 + \beta = k_v \frac{2}{\alpha}$, or:

$$\beta(1 + \alpha) = 2k_h, \quad \text{and} \quad \alpha(1 + \beta) = 2k_v.$$

The solution $\alpha$ fulfills an integral, quadratic equation:

$$0 = \alpha^2 - \alpha(\alpha - \beta - 1) - (\alpha + \alpha \beta) = \alpha^2 - (2(k_v - k_h) - 1)\alpha - 2k_v.$$

Hence

$$\alpha = k_v - k_h - \frac{1}{2} \pm \frac{\sqrt{4(k_v - k_h)^2 + 4(k_v + k_h) + 1}}{2}$$

where

$$4(k_v - k_h)^2 + 4(k_v + k_h) = 4(k_v(k_v + 1) + k_v(k_v + 1) - 2k_v) \equiv 0 \mod 8.$$

Using further $\alpha - \beta = 2(k_h - k_v)$ one finds:

$$\alpha = \frac{\pm \sqrt{D} - 1}{2} + k \quad \text{and} \quad \beta = \frac{\pm \sqrt{D} - 1}{2} - k,$$

where $D \equiv 1 \mod 8$ and $k \in \mathbb{Z}$ are such that both $\alpha, \beta \geq 0$. The negative roots do not contribute non-negative solutions, so we drop those. The expression for $\alpha$ and $\beta$ in equation 1.5 defines the heights of pillowcases, if the associated (top cylinder) twist multiplicities $k_h$ and $k_v$ are positive. If the height of the top pillowcase is zero, the compatibility condition of moduli in the horizontal direction disappears. We call a pair of numbers $(\alpha, \beta)$ given by equation (1.5) simple twist pair, if both are non-negative and at least one is strictly positive.

**Thurston construction of pseudo-Anosovs.** Besides minor deviations we follow Panov’s construction of homology elliptic pseudo-Anosov maps [P10] on half-translation tori.

Suppose $\alpha, \beta \in \mathbb{Q}_{\geq 0}[\sqrt{D}]$ is a simple twist pair. Using the presentation of $T_{\alpha,\beta}$ by an $L$-shaped polygon in the complex plane (with boundary identifications) one confirms that the horizontal twist $P_h$ and the vertical twist maps $P_v$ are given by

$$P_h = \begin{bmatrix} 1 & \alpha + 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_v = \begin{bmatrix} 1 & 0 \\ \beta + 1 & 1 \end{bmatrix}.$$

Both twists induce global affine maps on $T_{\alpha,\beta}$, $T_{\alpha,\beta}$ and $L_{\alpha,\beta}$. The maps also twist the respective small cylinder of the translation surface $L_{\alpha,\beta}$ an even number of times, which is essential for the escaping orbit construction in [D13].

For trim the following expressions put $\alpha_1 := 1 + \alpha, \beta_1 := 1 + \beta$ and consider

$$P_h^{-1} \cdot P_v = \begin{bmatrix} 1 - \alpha_1 \beta_1 & -\alpha_1 \\ \beta_1 & 1 \end{bmatrix}.$$
The eigenvalues of \( P^{-1} \) are

\[
\lambda_{\pm} = \frac{1}{2} \left( 2 - \alpha_1 \beta_1 \pm \sqrt{(2 - \alpha_1 \beta_1)^2 - 4} \right) = \frac{1}{2} \left( 2 - \alpha_1 \beta_1 \pm \alpha_1 \beta_1 \sqrt{1 - \frac{4}{\alpha_1 \beta_1}} \right),
\]

with eigenvectors

\[
\mathbf{v}_{\lambda_{\pm}} = \left[ -\frac{\beta_1}{2} \left( 1 \pm \frac{1}{\sqrt{1 - \frac{4}{\alpha_1 \beta_1}}} \right) \right].
\]

In particular \( P^{-1} \circ P \) is hyperbolic. As a side note, one easily verifies the following interesting algebraic identities for the slopes \( m_{\pm} \) of the eigendirections

\[
m_+ \cdot m_- = \frac{1 + \beta}{1 + \alpha} \quad \text{and} \quad 1 + \beta + m_+ + m_- = 0.
\]

Without restrictions assume \( k \geq 0 \) in the representation of \( \alpha \) and \( \beta \), otherwise we switch the roles of \( \alpha \) and \( \beta \). If \( D \equiv 1 \mod 8 \) and \( k \in \mathbb{Z} \), then \( \beta = \sqrt{\frac{D-1}{2}} - k \geq 0 \) implies \( D \geq (2k+1)^2 \). Let us denote the non-negative integers by \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

**Corollary 1.15.** We have

\[
\mathbb{N}_0 \subset \left\{ \frac{\sqrt{D}-1}{2} : D \equiv 1 \mod 8 \right\}.
\]

**Proof.** Taking \( D = (2j+1)^2 = 4j(j+1) + 1 \equiv 1 \mod 8 \) for \( j \in \mathbb{N}_0 \) gives \( \sqrt{D} = 2j + 1 \), or \( j = \frac{\sqrt{D}-1}{2} \).

In particular \( \sqrt{D} = 2j + 1 \) can be any odd number. Moreover

\[
\alpha = j + k \quad \text{and} \quad \beta = j - k \quad \text{for} \quad j \geq |k| \in \mathbb{N}_0.
\]

From the general expression for \( \alpha \) and \( \beta \), it is clear that only pillowcases of integer, or half-integer height (given by \( \alpha/2 \) and \( \beta/2 \)), fulfill the simple twist condition, moreover:

**Corollary 1.16.** The arithmetic surfaces \( \mathbb{T}_{\alpha, \beta} \), i.e. the ones with parameters \( \alpha, \beta \in \mathbb{Q}_+ \), with simple vertical and horizontal twists, fulfill the condition \( \alpha + \beta \in 2\mathbb{N} \).

**Proof.** Since \( \sqrt{D} \) is odd, all rational simple twist pairs \( (\alpha, \beta) \) are of the shape \( \alpha = j + k \) and \( \beta = j - k \) with \( j \in \mathbb{N} \) and \( j \geq k \).

In the rational case, both \( \alpha \) and \( \beta \) are either even, or both are odd. For \( \beta = 0 \) we have the examples considered by Panov [P10]. The next Lemma shows, that there is a reasonable set of lattice surfaces with simple twists.

**Lemma 1.17.** Both components of tuples in

\[
\left\{ (\alpha, \beta) = \left( \frac{\sqrt{D}-1}{2} + k, \frac{\sqrt{D}-1}{2} - k \right) : D \equiv 1 \mod 8, k \in \mathbb{Z}, \text{ such that } \alpha, \beta \in \mathbb{R}_+ \right\}
\]

are dense in \( \mathbb{R}_+ \).

**Proof.** We have already noticed, \( n^2 \equiv 1 \mod 8 \), if and only if \( n \in \mathbb{N} \) is odd. Therefore for any square-free number \( m \equiv 1 \mod 8 \), for example 17, 33 and 57, we have \( n^2m = 8l + 1 \) whenever \( n \) is odd. In that case

\[
\alpha = \frac{(2n+1)^2m-1}{2} + k = \frac{(2n+1)\sqrt{m}-1}{2} + k = n\sqrt{m} + \frac{\sqrt{m}-1}{2} + k.
\]
Consider that expression modulo $\mathbb{Z}$ as a function of $n \in \mathbb{N}$. Density of the orbit generated by an irrational circle rotation implies: $(\mathbb{N}, \sqrt{m} + x)/\mathbb{Z} = \mathbb{R}/\mathbb{Z}$ for any $x \in \mathbb{R}$, if and only if $\sqrt{m} \notin \mathbb{Q}$. Thus if $m \in \mathbb{N}$ is square-free, we have density on $\mathbb{R}/\mathbb{Z}$. Because the parameter $k \in \mathbb{Z}$ is free, we have

$$R = \left\{ \frac{(2n+1)\sqrt{m} - 1}{2} + k : n \in \mathbb{N} \text{ and } k \in \mathbb{Z} \right\} \subset \left\{ \frac{\sqrt{D} - 1}{2} + k : l \in \mathbb{N}_0 \text{ and } k \in \mathbb{Z} \right\}$$

for any square-free $m$. Since the root terms of such $\alpha$ are always positive (if $D > 1$), we see particularly

$$R_+ \subset \{ \alpha \geq 0 \} \quad \text{and} \quad R_+ \subset \{ \beta \geq 0 \}.$$

We shall show that there are admissible pairs with the above property. There are infinitely many square-free numbers which are congruent to 1 modulo 8. In fact, by Dirichlet's Theorem there are already infinitely many primes among those. Take one of those primes, say $m$, then equation 1.8 implies those $k \in \mathbb{Z}$ for which

$$0 < \alpha = \frac{(2n+1)\sqrt{m} - 1}{2} + k = n\sqrt{m} + \frac{\sqrt{m} - 1}{2} + k < \frac{\sqrt{m} - 1}{2}$$

are negative (we assume $n \in \mathbb{N}$). For such an $\alpha$ the corresponding $\beta = \frac{(2n+1)\sqrt{m} - 1}{2} - k$ is positive.

By the argument in the beginning the $\alpha$ in the open interval $(0, \frac{\sqrt{m} - 1}{2})$ are dense. As a function of $m$ (prime), the interval becomes arbitrarily large, so the set of simple twist pairs $(\alpha, \beta)$, contains a dense set of $\alpha$ and by symmetry a dense set of $\beta$ as well. \hfill \Box

Finally the appearance of homology elliptic pseudo-Anosovs as needed in Theorem 1.3 to conclude the existence of dense orbits follows.

**Lemma 1.18.** For $D \equiv 1 \mod 8$ pick $k \in \mathbb{Z}$, such that

$$\alpha = \frac{\sqrt{D} - 1}{2} + k \quad \text{and} \quad \beta = \frac{\sqrt{D} - 1}{2} - k$$

are both positive. Then there are affine pseudo-Anosovs $\phi_{\tilde{a}, \tilde{b}} \in \text{Aff}^+ (\mathbb{T}_{\tilde{a}, \tilde{b}})$ and $\phi_{\tilde{a}, \tilde{b}} \in \text{Aff}^+ (\mathbb{T}_{\tilde{a}, \tilde{b}})$ with derivatives $D\phi_{\tilde{a}, \tilde{b}} = D\phi_{\tilde{a}, \tilde{b}} = P_{\tilde{h}}^{-1} P_{\tilde{v}}$. Moreover on the twice folded torus $\mathbb{T}_{\tilde{a}, \tilde{b}}$ the induced map in homology $\phi_{\tilde{a}, \tilde{b}} \in \text{Aut}(H_1(\mathbb{T}_{\tilde{a}, \tilde{b}}; \mathbb{Z})) \cong \text{SL}_2(\mathbb{Z})$ is elliptic.

The proof is completely straightforward. As for the last claim, the vertical and horizontal Dehn twists on $\mathbb{T}_{\tilde{a}, \tilde{b}}$ induce the following maps in homology

$$P_{h*} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{v*} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

so $\text{tr}(\phi_{\tilde{a}, \tilde{b}}) = \text{tr}(P_{h*}^{-1} P_{v*}) = 1$ and $\phi_{\tilde{a}, \tilde{b}}$ is elliptic. For once folded tori homology elliptic pseudo-Anosovs of the above kind are less common:

**Lemma 1.19.** On $\mathbb{T}_{\tilde{a}, \tilde{b}}$ the induced map $\phi_{\tilde{a}, \tilde{b}} \in \text{Aut}(H_1(\mathbb{T}_{\tilde{a}, \tilde{b}}; \mathbb{Z})) \cong \text{SL}_2(\mathbb{Z})$ is elliptic, if $D = 9 + 4k(k+1)$ (some $k \in \mathbb{N}$). Otherwise it is hyperbolic with negative eigenvalues.

**Proof.** Let $P_h$ and $P_v$ be the respective stabilizers of the horizontal and vertical directions on the simple twist folded torus $\mathbb{T}_{\tilde{a}, \tilde{b}}$. Their action in homology action is given by

$$P_{h*} = \begin{bmatrix} 1 & 1 + \beta_k(1 + \alpha_k) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{v*} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

So elliptic in homology means $|\text{tr}(P_{h*}^{-1} P_{v*})| = |1 - \beta(1 + \alpha)| < 2$. A short calculation shows $D = 9 + 4k(k+1), k \in \mathbb{N}$. Assuming that $D = 9 + 4k(k+1)$, then $9 + 4k(k+1) > 1 + 4k + 4k^2 = \ldots$
\((2k + 1)^2\) and so \(\beta > \frac{(2k+1)-1}{2} - k = 0\), so \((\alpha, \beta)\) is an admissible pair. Further for such a pair \(\beta(1 + \alpha) = 2\) and \(\alpha_k = \beta_k + 2k > 2k\) for all \(k \in \mathbb{N}\). On the other hand equation 1.6 on page 13 shows \(P^{-1}_h P_v\) is hyperbolic: From \(\alpha > 2\) follows
\[
|\text{tr}(P^{-1}_h P_v)| = |2 - (1 + \beta)(1 + \alpha)| = 1 + \alpha > 3.
\]

As a consequence of Theorem 1.3 on page 3 we have:

**Corollary 1.20.** The orbits starting at a pillow-case singularity in eigendirections of \(P^{-1}_h P_v\) on \(C_{\tilde{\alpha}, \tilde{\beta}}\) are dense.

Some iterates of the corresponding orbit on the wind-tree model are shown in Figure 4. Since elliptic elements in \(\text{SL}_2(\mathbb{Z})\) have order at most 6, particular powers of the above homology elliptic pseudo-Anosov maps define elements in the Torelli group of the respective Panov plane. If the underlying surface is a lattice surface we may conjugate any homology elliptic pseudo-Anosov by another affine map. Standard results on the Torelli group [FM11] say, it contains a free group in this case.

**Corollary 1.21.** If \(T_{\tilde{\alpha}, \tilde{\beta}}\) is a simple twist surface, the affine Torelli group \(\mathcal{I}_A(T_{\tilde{\alpha}, \tilde{\beta}})\) contains a free sub-group of order 2 generated by powers of two elliptic pseudo-Anosovs. The same is true for the once-folded simple twist tori \(T_{\tilde{\alpha}, \tilde{\beta}}\) and \(T_{\tilde{\alpha}, \tilde{\beta}}\), if the respective surface has homology elliptic pseudo-Anosovs.

We stress that in the eyes of a geometric topologist our tori should be marked in the singular points. The tori with one fold for example are in reality two marked tori and the ones with two folds have four marked points.

**Panov’s construction.** We recall the key observation of Panov to construct dense orbits.

Recall that strict half-translation torus \(T\) has singular points of total angle \(\pi\). Suppose that \(\phi \in \text{Aff}^+(T)\) is a homology elliptic pseudo-Anosov with a singular fixed point, say \(p \in T\). By the density of eigenleaves on \(T\), any unstable eigenleaf \(f^u_p\) emanating from \(p\) intersects a stable eigenleaf \(f_s^p\) starting at the same point. Fix an intersection point \(e \in f^u_p \cap f^s_p\) and construct the loop \(\gamma_e\) by going from \(p\) to \(e\) along \(f^u_p\) and then back to \(p\) along \(f^s_p\).

Denote lifts of \(p\), of the (un)stable-leaves and of loops to \(C^T\) by adding a tilde. For example \(\tilde{p} \in C^T\) denotes a lift of \(p\). Lifts of curves and leaves, such as \(\tilde{\gamma}_e\), are assumed to start at \(\tilde{p}\). Now consider a lift \(\tilde{\phi}\) of \(\phi\) fixing \(\tilde{p}\). If \(p\) is singular, then so is \(\tilde{p}\) and \(\tilde{\phi}\) fixes each of the lifted leaves \(\tilde{f}^{(s)}_p\). If the induced map in homology \(\phi_\ast\) is elliptic of order \(k\), then \(\phi_\ast^{mk}\) is the identity for all \(m \in \mathbb{Z}\). So \(\tilde{\phi}^{mk}\) fixes all points in \(\pi^{-1}(p) \cong \mathbb{Z}^2\) and therefore all possible lifts of \(f^{u(s)}_p\), in particular
\[
\lim_{m \to \infty} \tilde{\phi}^{mk}(\tilde{e}) = \tilde{q} \in \pi^{-1}(p) \setminus \{\tilde{p}\}, \quad \text{where} \ \tilde{e} \in \pi^{-1}(e) \ \text{and} \ e \in f^s_p \cap f^u_p
\]
Here \(\tilde{q}\) denotes the endpoint of \(\tilde{\gamma}_p\). We have the easy consequence:

**Corollary 1.22.** If the orbit \(O_{\tilde{q}} = \{\tilde{\phi}^m(\tilde{q}) : m \in \mathbb{Z}\}\) is infinite, the leaf \(\tilde{f}^u_p\) is unbounded. If \(O_{\tilde{q}}\) is finite, the leaf \(\tilde{f}^u_p\) has all points of \(O_{\tilde{q}}\) as accumulation points.

To conclude density in the finite orbit case, we show in [JS13a], that any point in \(\pi^{-1}(p)\), i.e. any deck, has uniformly bounded distance from the accumulation points of \(\tilde{f}^u_p\) in \(\pi^{-1}(p)\).
Acknowledgements. This research was initiated at the trimester program on geometry and dynamics of Teichmüller spaces at the Hausdorff Institute Bonn in 2010. We thank the organizers for the invitation and Ferran Valdez for a stimulating introduction to Panov’s results on dense orbits. Further we wish to thank Vincent Delecroix, Pat Hooper, Pascal Hubert and Richard Evan Schwartz for explaining some of their results on the wind-tree model and related dynamical systems to us. Several parts of this research rely on the geometry of moduli space and the Teichmüller geodesic flow. We owe credit to Benson Farb, Giovanni Forni and Erwan Lanneau, who freely shared their insights in these subjects with us. We particularly thank Corinna Ulcigrai for mentioning Boyland’s result [B09], Krzysztof Frączek for bringing the paper of Pollicot and Sharp [PS07] to our attention and both for discussions about their deep (non-)ergodicity results for the wind-tree model [FU11] and strip billiards [FU12]. Finally we thank the referee for a careful reading and for valuable suggestions.

REFERENCES

[AH11] Avila, Arthur; Hubert Pascal. Recurrence for the wind-tree model. Preprint 2011/2012.
[B09] Boyland, Philip. Transitivity of Surface Dynamics Lifted to Abelian Covers. Ergodic Theory and Dynamical Systems 29, no. 5 (2009), 1417–1449.
[C04] Calta, Kariane. Veech surfaces and complete periodicity in genus two. J. Amer. Math. Soc. 17 (2004), no. 4, 871–908.
[CE13] Chaika, Jon; Eskin, Alex. Every flat surface is Birkhoff and Osceledets generic in almost every direction. arXiv:1305.1104 (2013)
[CG10] Conze, Jean-Pierre; Gutkin, Eugene. On recurrence and ergodicity for geodesic flows on noncompact periodic polygonal surfaces. Ergodic Theory Dyn. Systems 32 (2012), no. 2, 491–515.
[D13] Delecroix, Vincent. Divergent directions in some periodic wind-tree models. Journal of Modern Dynamics, Volume 7, No. 1, 2013, 1–29.
[DHL11] Delecroix, Vincent; Hubert, Pascal; Lelièvre, Samuel. Diffusion for the periodic wind-tree model. Annales Scientifiques de l’Ecole Normale Supérieure Volume 47, fascicule 3 (2014), 28 pgs. ArXiv:1107.1810v3.
[EE] Ehrenfest, Paul; Ehrenfest, Tatiana. The conceptual foundations of the statistical approach in mechanics, Translated from the German by Michael J. Moravcsik, Reprint of the 1959 English edition, Dover Publications, Inc., New York, 1990.
[EM13] Eskin, Alex; Mirzakhani Mariam; Invariant and stationary measures for the SL₂(R) action on moduli space. arXiv:1302.3320.
[EKZ12] Eskin, Alex; Kontsevich, Maxim; Zorich, Anton. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. Publications mathématiques de l’IHÉS, November 2013, 1–127.
[FM11] Farb, Benson; Margalit, Dan. A primer on mapping class groups. Princeton University Press 2011.
[FLP11] Fathi, Albert; Laudenbach, François; Poénaru, Valentin. Thurston’s Work on surfaces. Translation form french by Djun M. Kim and Dan Margalit. Princeton University Press 2012.
[FZ10] Ferenczi Sébastien; Zamboni Luca Q. Structure of K-interval-exchange transformations: Induction trajectories, and distance theorems, J. Anal. Math., 112 (2010), 289–328.
[FZ11] Ferenczi Sébastien; Zamboni Luca Q. Eigenvalues and simplicity of interval-exchange transformations, Ann. Sci. Éc. Norm. Sup. (4), 44 (2011), 361–392.
[FS13] Frączek, Krzysztof; Schmoll, Martin. Directional localization of light rays in a periodic array of retro-reflector lenses. To appear in Nonlinearity.
[FS14] Frączek, Krzysztof; Schmoll, Martin. In preparation.
[FU11] Frączek, Krzysztof; Ulcigrai, Corinna. Non-ergodic Z-periodic billiards and infinite translation surfaces. To appear in Inventiones Math.
[FU12] Frączek, Krzysztof; Ulcigrai, Corinna. Ergodic directions for billiards in a strip with periodically located obstacles. To appear in Communications in Mathematical Physics. arXiv:1208.5212.
[GH13] Grivaux, Julien; Hubert, Pascal. Loci in strata of meromorphic differentials with fully degenerate Lyapunov spectrum. arXiv:1307.3481v1
[GI00] Gutkin, Eugene; Judge, Chris. Affine mappings of translation surfaces: geometry and arithmetic. Duke Math. J. 103 (2000), no. 2, 191–213.
[HaWe80] Hardy, J.; Weber, J. Diffusion in a periodic wind-tree model. J. Math. Phys. 21 (1980), no. 7, 1802–1808.
[Ho10] Hooper, W. Patrick. The invariant measures of some infinite interval exchange maps. arXiv:1005.1902
[HW11] Hooper, W. Patrick; Weiss, Barack. Generalized Staircases: recurrence and symmetry. arXiv:0905.3736v1
[H11] Hubert, Pascal. Oral communication.
[HLT10] Hubert, Pascal; Lelièvre, Samuel; Troubetzkoy, Serge. The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion. J. Reine Angew. Math. 656 (2011), 223–244.

[HuW12] Hubert, Pascal; Weiss, Barak. Ergodicity for infinite periodic translation surfaces. Compos. Math. 149(2013), no. 8, 1364–1380. 57S25 (37A25)

[JS13a] Johnson, Chris; Schmoll, Martin. Hyperelliptic translation surfaces and folded tori. Topology and its Applications Vol 161 (2014), 73–94.

[JS13b] Johnson, Chris; Schmoll, Martin. In final preparation.

[JS13c] Johnson, Chris; Schmoll, Martin. In preparation.

[M92] Masur, Howard. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential, Duke Math. J. 66 (1992), 387–442.

[MT02] Masur, Howard; Tabachnikov, Sergei. Rational billiards and flat structures. Handbook of dynamical systems, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.

[McM03] McMullen, Curtis T. Teichmüller curves on Hilbert modular surfaces. J. Amer. Math. Soc. 16 (2003), no. 4, 857–885 (electronic).

[McM06] McMullen, Curtis T. Prym varieties and Teichmüller curves. Duke Math. J. 133 (2006), no. 3, 569–590.

[P10] Panov, Dmitri. Foliations with unbounded deviation on $T^2$. J. Mod. Dyn. 3 (2009), no. 4, 589–594.

[PS07] Pollicott, Mark; Sharp Richard. Pseudo-Anosov foliations on periodic surfaces. Topology and its Applications 154 (2007) 2365–2375.

[Th88] Thurston, William. On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.) 19 (1988) 417–431.

[V05] Vasilyev, Sergey. Genus two Veech surfaces arising from general quadratic differentials. arXiv:math/0504180v1 2005

[V89] Veech, William. Teichmüller curves in the moduli space, Eisenstein series and an application to triangular billiards, Invent. Math. 97 (1989), pp. 533–683.

[Z06] Zorich, Anton. Flat surfaces. Frontiers in number theory, physics, and geometry. I, 437–583, Springer, Berlin, 2006.