Stochastic evolutions in superspace and superconformal field theory

Jørgen Rasmussen
Centre de recherches mathématiques, Université de Montréal
Case postale 6128, succursale centre-ville, Montréal, Qc, Canada H3C 3J7
rasmusse@crm.umontreal.ca

Abstract

Some stochastic evolutions of conformal maps can be described by SLE and may be linked to conformal field theory via stochastic differential equations and singular vectors in highest-weight modules of the Virasoro algebra. Here we discuss how this may be extended to superconformal maps of $N = 1$ superspace with links to superconformal field theory and singular vectors of the $N = 1$ superconformal algebra in the Neveu-Schwarz sector.

Keywords: Stochastic evolutions, superconformal field theory, superspace.
1 Introduction

Stochastic Löwner evolutions (SLEs) have been introduced by Schramm [1] and further developed in [2, 3] as a mathematically rigorous way of describing certain two-dimensional systems at criticality. The method involves the study of stochastic evolutions of conformal maps.

An intriguing link to conformal field theory (CFT) has been examined by Bauer and Bernard [4] (see also [5]) in which the SLE differential equation is associated to a particular random walk on the Virasoro group. The relationship can be made more direct by establishing a connection between the representation theory of CFT and quantities conserved in mean under the stochastic process. This is based on the existence of level-two singular vectors in the associated Virasoro highest-weight modules.

In [6] it is discussed how SLE appears naturally at grade one in a hierarchy of SLE-type growth processes. Extending the approach of Bauer and Bernard, each of these may be linked to CFT, and a direct relationship to the Yang-Lee singularity is found at grade two through the construction of a level-four null vector.

The objective of the present work is to extend this to a link between stochastic evolutions of superconformal maps in $\mathcal{N} = 1$ superspace and $\mathcal{N} = 1$ superconformal field theory (SCFT). It can be made more direct by relating some quantities conserved in mean under the process to a level-3/2 singular vector in the Neveu-Schwarz sector of the SCFT. These results rely on an Ito calculus, not just for commuting and anti-commuting variables as discussed in [7] and references therein, but for general non-commuting objects (e.g. group- or algebra-valued entities) on superspace. The generalization is straightforward, though.

After a brief review on SLE and the approach of Bauer and Bernard, we summarize some basics on $\mathcal{N} = 1$ superspace and $\mathcal{N} = 1$ SCFT in the superfield formalism. We then discuss how graded stochastic evolutions may be linked to SCFT and establish relationships through two different ways of obtaining level-3/2 singular vectors. Some coupled stochastic differential equations are solved in the process. In one case, the solution suggests the introduction of an SLE-type trace and may be interpreted as describing the evolution of hulls in the complex plane. The final section contains some concluding remarks.

2 SLE and CFT

Geometrically, (chordal) SLE [1, 2, 3] describes the evolution of boundaries of simply-connected domains in the complex plane. One may think of this as Brownian motion on the set of conformal maps $\{g_t\}$ satisfying the Löwner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z$$ (1)
$B_t$ is one-dimensional Brownian motion with $B_0 = 0$ and $\kappa \geq 0$. This is the only adjustable parameter thus characterizing the process which is often denoted SLE$_\kappa$.

Introducing the function
\[ f_t(z) := g_t(z) - \sqrt{\kappa} B_t \]
one has the stochastic differential equation
\[ df_t(z) = \frac{2}{f_t(z)} dt - \sqrt{\kappa} dB_t, \quad f_0(z) = z \]
based on which we may establish contact with CFT. One defines the SLE trace as
\[ \gamma(t) := \lim_{z \to 0} f_t^{-1}(z) \]
As discussed in [3], its nature depends radically on $\kappa$.

To unravel the link [4] one considers Ito differentials of Virasoro group elements $\mathcal{G}_t$:
\[ \mathcal{G}_t^{-1} d\mathcal{G}_t = \left( -2L_{-2} + \frac{\kappa}{2} L_{-1}^2 \right) dt + \sqrt{\kappa} L_{-1} dB_t, \quad \mathcal{G}_0 = 1 \]
Such group elements are obtained by exponentiating generators of the Virasoro algebra
\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \]
where $c$ is the central charge. The conformal transformation generated by a Virasoro group element has a simple action on a primary field of weight $\Delta$. In particular, the transformation generated by $\mathcal{G}_t$ reads
\[ \mathcal{G}_t^{-1} \phi_\Delta(z) \mathcal{G}_t = (\partial_z f_t(z))^\Delta \phi_\Delta(f_t(z)) \]
for some conformal map $f_t$. Using that the Virasoro generators act as
\[ [L_n, \phi_\Delta(z)] = (z^{n+1} \partial_z + \Delta(n+1)z^n) \phi_\Delta(z) \]
one finds that the conformal map $f_t$ associated to the random process $\mathcal{G}_t$ [5] must be a solution to the stochastic differential equation (3).

This link does not teach us to which CFT with central charge $c$ a given SLE$_\kappa$ may be associated. A refinement could be reached, though, by determining for which $\kappa$ the stochastic process (5) would allow us to relate conformal correlation functions on one side to probabilities of stopping-time events on the other side. A goal is thus to find the evolution of the expectation values of the observables of the stochastic process (5) where the observables are thought of as functions on the Virasoro group. The direct link is established by relating the representation theory of the Virasoro algebra to quantities conserved in mean under the random process.

---

1 For simplicity, we do not distinguish explicitly between boundary and bulk primary fields, nor do we write the anti-holomorphic part.
To achieve this, let $|\Delta\rangle$ denote the highest-weight vector of weight $\Delta$ in the Verma module $V_\Delta$, and consider the time evolution of the expectation value of $G_t|\Delta\rangle$:

$$\partial_t E[G_t|\Delta\rangle] = E[G_t \left( -2L_{-2} + \frac{\kappa}{2}L_{-1}^2 \right) |\Delta\rangle] \quad (9)$$

For some values of $\kappa$ (in relation to the central charge $c$), the linear combination $-2L_{-2} + \frac{\kappa}{2}L_{-1}^2$ will produce a singular vector when acting on the highest-weight vector. This happens provided

$$c_\kappa = 1 - \frac{3(4 - \kappa)^2}{2\kappa}, \quad \Delta_\kappa = \frac{6 - \kappa}{2\kappa} \quad (10)$$

In this case the expectation value $\langle 11 \rangle$ vanishes in the module obtained by factoring out the singular vector from the reducible Verma module $V_\Delta$. The representation theory of the factor module is thereby linked to the description of quantities conserved in mean.

We see that this direct relationship between SLE$_\kappa$ and CFT is through the existence of a level-two singular vector in a highest-weight module. This has been extended in [6] where a link between elements of a hierarchy of SLE-type growth processes and CFT has been established. At grade two, a direct relationship between an SLE-type stochastic differential equation and the Yang-Lee singularity is based on a level-four null vector. Ordinary SLE appears at grade one in this hierarchy.

In the following we shall discuss how the scenario outlined above may be extended to stochastic evolutions in superspace with links to SCFT.

3 Superspace and SCFT

Here we shall give a very brief summary of results from the theory of $N = 1$ superspace and $N = 1$ SCFT required in the following. We refer to [8, 9] and references therein for recent accounts on the subject. As we only consider $N = 1$ superspace and $N = 1$ SCFT we shall simply refer to them as superspace and SCFT, respectively.

3.1 Superconformal maps

Let

$$(z, \theta) \mapsto (z', \theta'), \quad \begin{cases} z' = g(z) + \theta \gamma(z) \\ \theta' = \tau(z) + \theta s(z) \end{cases} \quad (11)$$

denote a general superspace coordinate transformation. The associated superderivative reads

$$D = \partial_\theta + \theta \partial_z \quad (12)$$

Here $\theta, \theta', \gamma$ and $\tau$ are anti-commuting or (Grassmann) odd entities, while $z, z', g$ and $s$ are even. A superconformal transformation may be characterized by

$$Dz' = \theta'D\theta' \quad (13)$$

3
in which case we have
\[ \gamma(z) = \tau(z)s(z), \quad \partial_z g(z) = s^2(z) - \tau(z)\partial_z \tau(z) \] (14)
and
\[ D = (D\theta')D', \quad D' = \partial_\theta + \theta'\partial_{\theta'} \] (15)
We shall be interested in locally invertible superconformal maps implying that the complex part of \( z' \) is non-vanishing if the complex part of \( z \) is non-vanishing. We assume this is the case for \( z \neq 0 \).

The superconformal transformations are generated by \( T \) and \( G \) with modes \( L_n \) and \( G_r \) satisfying the superconformal algebra
\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \\
[L_n, G_r] = \left(\frac{n}{2} - r\right)G_{n+r} \\
\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}
\] (16)
The algebra is said to be in the Neveu-Schwarz (NS) sector when the supercurrent \( G \) has half-integer modes, and in the Ramond sector when they are integer. We shall consider the NS sector only.

### 3.2 Primary superfields and singular vectors

A superfield depends on the superspace coordinates and thus enjoys an almost trivial Taylor expansion in \( \theta \):
\[ \Phi(z, \theta) = \phi(z) + \theta\varphi(z) \] (17)
Primary superfields are defined by having simple transformation properties with respect to the superconformal generators (16). With \( \Delta \) denoting the (super)conformal dimension, a primary superfield may be characterized by
\[
[L_n, \Phi_{\Delta}(z, \theta)] = (z^{n+1}\partial_z + \frac{1}{2}(n + 1)z^n\partial_\theta + \Delta(n + 1)z^n)\Phi_{\Delta}(z, \theta) \\
[G_r, \Phi_{\Delta}(z, \theta)] = (z^{r+1/2}(\partial_\theta - \theta\partial_z) - \Delta(2r + 1)\theta z^{r-1/2})\Phi_{\Delta}(z, \theta)
\] (18)
As indicated, we are considering even primary superfields only.

The superconformal transformations generated by Virasoro supergroup elements \( \mathcal{G} \) extend the ordinary conformal case (7):
\[
\mathcal{G}^{-1}\Phi_{\Delta}(z, \theta)\mathcal{G} = (D\theta')^{2\Delta} \Phi_{\Delta}(z', \theta')
\] (19)
Supergroup elements may be constructed by exponentiating the generators of the superconformal algebra. Modes of the supercurrent must in that case be accompanied by Grassmann odd parameters.
A highest-weight module of the superconformal algebra is reducible if it contains a submodule generated from a singular vector. The simplest non-trivial singular vector in a highest-weight module with highest weight $\Delta$ appears at level $3/2$ and is given by

$$|\chi; 3/2\rangle = \left( (\Delta + \frac{1}{2}) G_{-3/2} - L_{-1} G_{-1/2} \right) |\Delta\rangle$$

(20)

provided

$$12\Delta = (2\Delta + 1)(3\Delta + c)$$

(21)

4 Graded stochastic evolutions and SCFT

4.1 General link

Since we already have experience with stochastic calculus of generically non-commutative objects from the Virasoro case in section 2, for example, the inclusion of anti-commuting variables is straightforward. An important feature is the vanishing of $\partial^2_\eta$ (\eta odd) appearing in the second-order term in the extended or ‘graded’ Ito formula. The Ito formula for commuting as well as anti-commuting variables is discussed in [7] and references therein. The techniques will be illustrated as we go.

We consider first the stochastic differential

$$G_t^{-1}dG_t = \alpha dt + \beta dB_t, \quad G_0 = 1$$

which we can think of as a random walk on the Virasoro supergroup. $\alpha$ and $\beta$ are even and generically non-commutative expressions in the generators of the superconformal algebra. It follows from $d(G_t^{-1}G_t) = 0$ that the Ito differential of the inverse element is given by

$$d(G_t^{-1})G_t = (-\alpha + \beta^2) dt - \beta dB_t$$

(23)

The superconformal transformation generated by $G_t$ acts on a primary superfield as [19]:

$$G_t^{-1}\Phi_\Delta(z, \theta)G_t = (D\theta'_t)^{2\Delta}\Phi_\Delta(z'_t, \theta'_t)$$

(24)

The superspace coordinate $(z'_t, \theta'_t)$ is now a stochastic function of $(z, \theta)$. To relax the notation, the subscript $t$ will often be suppressed below.

A goal is to compute the Ito differential of both sides of (24) and thereby relate the stochastic differential equations of $G_t$ and $(z', \theta')$. Let’s first consider the Ito differential of the left hand side of (24):

$$d\left(G_t^{-1}\Phi_\Delta G_t\right) = d(G_t^{-1})\Phi_\Delta G_t + G_t^{-1}\Phi_\Delta dG_t + d(G_t^{-1})\Phi_\Delta dG_t$$

$$= \left( -\left[ \alpha - \frac{1}{2}\beta^2, G_t^{-1}\Phi_\Delta G_t \right] + \frac{1}{2} \left[ \beta, \left[ \beta, G_t^{-1}\Phi_\Delta G_t \right] \right] \right) dt$$

$$- \left[ \beta, G_t^{-1}\Phi_\Delta G_t \right] dB_t$$

(25)
To facilitate the comparison we want to express the differentials in the *adjoint* representation of the algebra generators only. We should thus choose $\beta$ and $\alpha_0$ linear in the generators, with $\alpha_0$ defined through $\alpha = \alpha_0 + \frac{1}{2} \beta^2$. Extending this to a $b$-dimensional Brownian motion, $\bar{B}_t = (B_t^{(1)}, ..., B_t^{(b)})$ with $B_0^{(i)} = 0$, we are led to consider

$$G_t^{-1} dG_t = \left( \alpha_0 + \frac{1}{2} \sum_{i=1}^{b} \beta_i^2 \right) dt + \sum_{i=1}^{b} \beta_i dB_t^{(i)}, \quad G_0 = 1 \tag{26}$$

The associated Ito calculus treats the basic differentials according to the rules

$$(dt)^2 = dt dB_t^{(i)} = 0, \quad dB_t^{(i)} dB_t^{(j)} = \delta_{ij} dt \tag{27}$$

and we find

$$d \left( G_t^{-1} \Phi \Delta G_t \right) = \left( -[\alpha_0, G_t^{-1} \Phi \Delta G_t] + \frac{1}{2} \sum_{i=1}^{b} [\beta_i, [\beta_i, G_t^{-1} \Phi \Delta G_t]] \right) dt$$

$$- \sum_{i=1}^{b} [\beta_i, G_t^{-1} \Phi \Delta G_t] dB_t^{(i)}$$

$$= (D\theta')^{2\Delta} \left( -[\alpha_0, \Phi \Delta (z', \theta')] + \frac{1}{2} \sum_{i=1}^{b} [\beta_i, [\beta_i, \Phi \Delta (z', \theta')]] \right) dt$$

$$- (D\theta')^{2\Delta} \sum_{i=1}^{b} [\beta_i, \Phi \Delta (z', \theta')] dB_t^{(i)} \tag{28}$$

With the subscript $t$ suppressed, we can express the Ito differentials of the superspace coordinate as

$$dz' = z'_0 dt + \sum_{i=1}^{b} z'_i dB_t^{(i)}, \quad z'|_{t=0} = z$$

$$d\theta' = \theta'_0 dt + \sum_{i=1}^{b} \theta'_i dB_t^{(i)}, \quad \theta'|_{t=0} = \theta$$

$$d(D\theta') = (D\theta'_0) dt + \sum_{i=1}^{b} (D\theta'_i) dB_t^{(i)} \tag{29}$$

where we have added the Ito differential of $D\theta'$. The initial conditions are required to match the initial condition on $G_t$ in (26). With (29) at hand, we compute the Ito differential of the right hand side of (24):

$$d \left( (D\theta')^{2\Delta} \Phi \Delta (z', \theta') \right)$$

$$= (D\theta')^{2\Delta} \partial_{z'} \Phi \Delta (z', \theta') dz' + (D\theta')^{2\Delta} d\theta' \partial_{\theta'} \Phi \Delta (z', \theta')$$

$$+ 2\Delta (D\theta')^{2\Delta-1} \Phi \Delta (z', \theta') d(D\theta') + \frac{1}{2} (D\theta')^{2\Delta} \partial^2_{z'} \Phi \Delta (z', \theta')(dz')^2$$
\[
+ \Delta (2\Delta - 1) \left( D\theta' \right)^{2\Delta - 2} \Phi_\Delta (z', \theta') (d(D\theta'))^2 + (D\theta')^{2\Delta} d\theta' \partial_z \partial_{y} \Phi_\Delta (z', \theta') dz' \\
+ 2\Delta \left( D\theta' \right)^{2\Delta - 1} \partial_z \Phi_\Delta (z', \theta') dz' d(D\theta') + 2\Delta \left( D\theta' \right)^{2\Delta - 1} d\theta' \partial_{\theta} \Phi_\Delta (z', \theta') d(D\theta') \\
= (D\theta')^{2\Delta} \left\{ \sum_{i=1}^{b} \left( z'_i \right)^2 \partial^2_{z_i} + \sum_{i=1}^{b} z'_i \theta'_i \partial_{y_i} \partial_{z_i} + \left( z'_0 + 2\Delta \sum_{i=1}^{b} (D'\theta'_i) z'_i \right) \partial_{z'_0} \\
+ \left( \theta'_0 + 2\Delta \sum_{i=1}^{b} (D'\theta'_i) \theta'_i \right) \partial_{y_i} + 2\Delta \left( (D'\theta'_0) + (\Delta - \frac{1}{2}) \sum_{i=1}^{b} (D'\theta'_i)^2 \right) \right\} \Phi_\Delta (z', \theta') dt \\
+ (D\theta')^{2\Delta} \sum_{i=1}^{b} \left\{ z'_i \partial_{z'_i} + \theta'_i \partial_{y_i} + 2\Delta (D'\theta'_i) \right\} \Phi_\Delta (z', \theta') dB_t^{(i)} \\
\tag{30}
\]

Note the positions of the odd differential \( d\theta' \). We have used the transformation rule \((15)\) in the rewriting. A comparison of the \( dB_t^{(i)} \) terms in \((28)\) and \((30)\) yields

\[
[\beta_i, \Phi_\Delta (z', \theta')] = - (z'_i \partial_{z'_i} + \theta'_i \partial_{y_i} + 2\Delta (D'\theta'_i)) \Phi_\Delta (z', \theta') \\
\tag{31}
\]

With \( \alpha_0 \) and \( \beta_i \) linear in the algebra generators

\[
\alpha_0 = \sum_{n \in \mathbb{Z}} \left( y_{0,n} L_n + \eta_{0,n} G_{n+1/2} \right) \\
\beta_i = \sum_{n \in \mathbb{Z}} \left( y_{i,n} L_n + \eta_{i,n} G_{n+1/2} \right) \\
\tag{32}
\]

we find that

\[
z'_i = - \sum_{n \in \mathbb{Z}} (y_{i,n} + \theta' \eta_{i,n}) (z')^{n+1} \\
\theta'_i = - \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} (n+1) \theta' y_{i,n} + z' \eta_{i,n} \right) (z')^n \\
\tag{33}
\]

A comparison of the \( dt \) terms allows us to extract information on \([\alpha_0, \Phi_\Delta (z', \theta')]\) from which we deduce that

\[
z'_0 = - \sum_{n \in \mathbb{Z}} (y_{0,n} + \theta' \eta_{0,n}) (z')^{n+1} + \frac{1}{2} \sum_{i=1}^{b} (z'_i \partial_{z'_i} + \theta'_i \partial_{y_i}) z'_i \\
\theta'_0 = - \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} (n+1) \theta' y_{0,n} + z' \eta_{0,n} \right) (z')^n + \frac{1}{2} \sum_{i=1}^{b} (z'_i \partial_{z'_i} + \theta'_i \partial_{y_i}) \theta'_i \\
\tag{34}
\]

Using \((33)\), one may of course eliminate \( z'_i \) and \( \theta'_i \) from these expressions, but the result is then less compact than \((34)\). Only finitely many of the expansion coefficients \( y \) and \( \eta \) will be chosen non-vanishing rendering the sums in the solution \((33)\) and \((34)\) finite.

In conclusion, the construction above establishes a general link between a class of stochastic evolutions in superspace and SCFT: the stochastic differentials \((29)\) describing the evolution of the superconformal maps are expressed in terms of the parameters of the random walk on the Virasoro supergroup \((26)\), and this has been achieved via the definition of primary superfields in SCFT \((24)\).
4.2 Expectation values and singular vectors

To obtain a more direct relationship along the lines of section 2, we should link the representation theory of the superconformal algebra through the construction of singular vectors to quantities conserved in mean under the stochastic process. From the supergroup differential it follows that the time evolution of the expectation value of $G_t|\Delta\rangle$ is given by

$$\partial_t E[G_t|\Delta\rangle] = E[G_t \left( \alpha_0 + \frac{1}{2} \sum_{i=1}^{b} \beta_i^2 \right)|\Delta\rangle]$$  \hspace{1cm} (35)

We should thus look for processes allowing us to put

$$\left( \alpha_0 + \frac{1}{2} \sum_{i=1}^{b} \beta_i^2 \right)|\Delta\rangle \approx 0$$  \hspace{1cm} (36)

in the representation theory. $G_t|\Delta\rangle$ is then a so-called martingale of the stochastic process $G_t$.

Before doing that, let us indicate how time evolutions of much more general expectation values may be evaluated. This extends one of the main results in [4] on ordinary SLE. In this regard, observables of the process $G_t$ are thought of as functions of $G_t$. We introduce the graded vector fields

$$(\nabla_n F)(G_t) = \frac{d}{du} F(G_t e^{uL_n})|_{u=0}$$

$$(\nabla_{n+1/2} F)(G_t) = \frac{d}{d\nu} F(G_t e^{\nu G_{n+1/2}})|_{\nu=0}$$  \hspace{1cm} (37)

associated to the generators $L_n$ and $G_{n+1/2}$. Here $\nabla_{n+1/2}$ and $\nu$ (and of course $G_{n+1/2}$) are odd, and $F$ is a function admitting a 'sufficiently convergent' Laurent expansion. Referring to the notation in [32], we then find that

$$\partial_t E[F(G_t)] = E \left[ \left( \alpha_0(\nabla) + \frac{1}{2} \sum_{i=1}^{b} \beta_i^2(\nabla) \right) F(G_t) \right]$$  \hspace{1cm} (38)

with

$$\alpha_0(\nabla) = \sum_{n \in \mathbb{Z}} \left( y_{0,n} \nabla_n + \eta_{0,n} \nabla_{n+1/2} \right)$$

$$\beta_i(\nabla) = \sum_{n \in \mathbb{Z}} \left( y_{i,n} \nabla_n + \eta_{i,n} \nabla_{n+1/2} \right)$$  \hspace{1cm} (39)

This follows from [22], [23], $\alpha = \alpha_0 + \frac{1}{2} \sum_{i=1}^{b} \beta_i^2$ and

$$\nabla_n (G_t^{-N}) = \frac{d}{du} \left( e^{-uL_n} G_t^{-1} \right)^N |_{u=0}$$

$$\nabla_{n+1/2} (G_t^{-N}) = \frac{d}{d\nu} \left( e^{-\nu G_{n+1/2}} G_t^{-1} \right)^N |_{\nu=0}$$  \hspace{1cm} (40)
It is seen that (38) reduces to (35) as it should when \( F(\mathcal{G}_t) = \mathcal{G}_t|\Delta) \).

We now return to (36) and shall discuss the example where it corresponds to the level-3/2 singular vector (20). We consider two different ways of obtaining this vector and examine the associated stochastic differential equations in superspace.

It turns out that a one-dimensional Brownian motion suffices if we choose
\[
\alpha_0 = -y\eta G_{-3/2} , \quad \beta = (yL_{-1} + \eta G_{-1/2})\sqrt{\kappa} , \quad y^2 = 0 \tag{41}
\]
The condition \( y^2 = 0 \) ensures that the term with \( L^2_{-1} \) in \( \beta^2 \) vanishes, and is obtained by writing \( y \) as an even monomial in Grassmann odd parameters. \( \eta \) is odd. With this choice of normalization (one can rescale \( \alpha_0 \) by any non-vanishing real number) and by comparing with (20) and (21), we find
\[
c_\kappa = \frac{15}{2} - 3\left(\kappa + \frac{1}{\kappa}\right) , \quad \Delta_\kappa = \frac{2 - \kappa}{2\kappa} \tag{42}
\]
This is in accordance with the ordinary labeling of reducible highest-weight modules of the superconformal algebra with \( \Delta = h_{1,3} \) or \( \Delta = h_{3,1} \). We see that the stochastic models with \( \kappa \) and \( 1/\kappa \) may be linked to the same SCFT, albeit via the two different primary superfields \( \Phi_{h_{1,3}} \) and \( \Phi_{h_{3,1}} \).

The stochastic differential equations associated to (41) read
\[
dz' = \frac{y\theta'\eta}{z'} dt - (y + \theta'\eta)\sqrt{\kappa} dB_t , \quad z'|_{t=0} = z
\]
\[
d\theta' = \frac{y\eta}{z'} dt - \eta\sqrt{\kappa} dB_t , \quad \theta'|_{t=0} = \theta \tag{43}
\]
Unlike the similar equation in ordinary SLE (3), these can actually be solved explicitly. To see this, we introduce the functions
\[
w_t = z'_t + (y + \theta'\eta)\sqrt{\kappa} B_t , \quad w_0 = z
\]
\[
\mu_t = \theta'_t + \eta\sqrt{\kappa} B_t , \quad \mu_0 = \theta \tag{44}
\]
which are seen to satisfy the ordinary (but coupled) differential equations
\[
\partial_t w_t = \frac{y\mu\eta}{w_t} , \quad w_0 = z
\]
\[
\partial_t \mu_t = \frac{y\eta}{w_t} , \quad \mu_0 = \theta \tag{45}
\]
From \( \partial_t(\mu_t w_t) \) it follows that \( \mu_t w_t = \theta z + y\eta t \) and in particular
\[
w_t = z + \frac{\theta y\eta}{z} t , \quad \mu_t = \theta + \frac{y\eta}{z} t \tag{46}
\]
Returning to the original stochastic functions we find that they are given by
\[
z'_t = z + \frac{\theta y\eta}{z} t - (y + \theta\eta)\sqrt{\kappa} B_t
\]
\[
\theta'_t = \theta + \frac{y\eta}{z} t - \eta\sqrt{\kappa} B_t \tag{47}
\]
It is easily verified that \((z, \theta) \mapsto (z', \theta')\) indeed is a superconformal transformation. It is observed that the complex part of \(z\) is fixed under this map.

A different scenario, but in some sense closer in spirit to ordinary SLE, emerges if we work with \textit{two-dimensional} Brownian motion (even though SLE is based on \textit{one-dimensional} Brownian motion only) and choose

\[
\alpha_0 = -y\eta G_{-3/2}, \quad \beta_1 = (yL_{-1} + \eta G_{-1/2})\sqrt{\kappa}, \quad \beta_2 = iyL_{-1}\sqrt{\kappa} \quad (48)
\]

Note that \(y^2\) is \textit{not} required to vanish in this case. Nevertheless, the expression (36) corresponds to the same level-3/2 singular vector as in the previous case, and we again recover the parameterization (42). The associated stochastic differentials now read

\[
dz' = \frac{y\theta'\eta}{z'} dz - (y + \theta'\eta)\sqrt{\kappa}dB_t^{(1)} + iy\sqrt{\kappa}dB_t^{(2)}, \quad z'|_{t=0} = z
\]

\[
d\theta' = \frac{\eta\theta'}{z'} dt - \eta\sqrt{\kappa}dB_t^{(1)}, \quad \theta'|_{t=0} = \theta \quad (49)
\]

These can be expressed as ordinary differential equations by introducing

\[
w_t = z' + (y + \theta'\eta)\sqrt{\kappa}B_t^{(1)} + iy\sqrt{\kappa}B_t^{(2)}, \quad w_0 = z
\]

\[
\mu_t = \theta' + \eta\sqrt{\kappa}B_t^{(1)}, \quad \mu_0 = \theta \quad (50)
\]

as we find that

\[
\partial_t w_t = \frac{y\mu_t\eta}{w_t - y\sqrt{\kappa}B_t^{(1)}}, \quad w_0 = z
\]

\[
\partial_t \mu_t = \frac{\eta\theta'}{w_t - y\sqrt{\kappa}B_t^{(1)}}, \quad \mu_0 = \theta \quad (51)
\]

Here we have defined the complex Brownian motion

\[
B_t^+ := B_t^{(1)} + iB_t^{(2)}, \quad B_0^+ = 0 \quad (52)
\]

and it has been used that the complex part of \(z'\) is non-vanishing for a locally invertible superconformal map. We note that if \(y^2 = 0\) the set of equations (51) reduces to (45).

Despite the resemblance to Löwner’s equation (1), the stochastic differential equations (51) may be solved explicitly. To see this, we use the expansion (11) of \(z'\) and \(\theta'\) in (19) with \(y\) chosen as \(y = 1\) for simplicity. The four functions \(g, \gamma, \tau\) and \(s\) are then expanded with respect to the odd parameter \(\eta\) and we end up with eight coupled differentials. They are easily solved, though, and we find the solution to (19) to be

\[
z' = z - \sqrt{\kappa}B_t^+ + \theta\eta \left( \int_0^t \frac{1}{z - \sqrt{\kappa}B_s^+} ds - \sqrt{\kappa}B_t^{(1)} \right)
\]

\[
\theta' = \theta + \eta \left( \int_0^t \frac{1}{z - \sqrt{\kappa}B_s^+} ds - \sqrt{\kappa}B_t^{(1)} \right) \quad (53)
\]
The map \((z, \theta) \mapsto (z', \theta')\) is seen to be superconformal.

This solution is only well-defined when the denominator of the integrands has a non-vanishing complex part: \(z^c \neq \sqrt{\kappa B_t^+}, 0 \leq s \leq t\), with \(z^c\) denoting the complex part of \(z\). With reference to the definition of the SLE trace in ordinary SLE \([4]\), it thus seems natural to define the 'supertrace' \(\Gamma(t)\) to be

\[
\Gamma(t) := \sqrt{\kappa B_t^+}
\]  

and for \(t > 0\) to introduce the hulls \(K_t\) as the union of \(\Gamma[0, t]\) and the bounded connected components of \(\mathbb{C} \setminus \Gamma[0, t]\). The (stochastic) superconformal maps may be interpreted as describing the evolution of these hulls, and are taken as maps from \((z, \theta)\) with \(z^c \in \mathbb{C} \setminus K_t\). Note that this also ensures that they are locally invertible.

5 Conclusion

Motivated by the success of SLE in the description of critical systems, we have extended the study in \([4]\) (and \([6]\)) of the link between SLE and CFT to a link between stochastic evolutions in superspace and SCFT. Our analysis has led to very general results and has been illustrated by establishing a direct relationship based on the evaluation of level-3/2 singular vectors in highest-weight modules of the superconformal algebra. Two different scenarios were outlined. A set of coupled stochastic differential equations were solved in each of them. In one case, the solution suggested the introduction of a supertrace and could be interpreted as describing the evolution of hulls in the complex plane.

Our approach may be adapted to stochastic evolutions in 'ordinary' space and CFT. This would extend the results of \([4]\) and \([6]\), and will be discussed elsewhere.

Acknowledgements

The author is grateful to J. Nagi for numerous helpful discussions on superspace and SCFT. He also thanks C. Cummins, P. Jacob, P. Mathieu, and Y. Saint-Aubin for comments.

References

[1] O. Schramm, Israel J. Math. 118 (2000) 221.

[2] G.F. Lawler, O. Schramm, W. Werner, Acta Math. 187 (2001) 237; Acta Math. 187 (2001) 275; Ann. Inst. Henri Poincaré PR 38 (2002) 109; Conformal restriction: the cordal case, math.PR/0209343.

[3] S. Rohde, O. Schramm, Basic properties of SLE, math.PR/0106036.

[4] M. Bauer, D. Bernard, Phys. Lett. B 543 (2002) 135; Commun. Math. Phys. 239 (2003) 493.
[5] R. Friederich, W. Werner, *Conformal restriction, highest-weight representations and SLE*, math-ph/0301018.

[6] F. Lesage, J. Rasmussen, *SLE-type growth processes and the Yang-Lee singularity*, math-ph/0307058.

[7] A. Rogers, *Supersymmetry and Brownian motion on supermanifolds*, quant-ph/0201006.

[8] M. Dörrozapf, Rev. Math. Phys. 11 (1999) 137.

[9] J. Nagi, *Superconformal primary fields on a graded Riemann sphere*, hep-th/0309243.