On the associative Nijenhuis Relation

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Abstract

In this brief note we would like to give the construction of a free commutative unital associative Nijenhuis algebra on a commutative unital associative algebra based on an augmented modified quasi-shuffle product.

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1 Introduction

The associative analog of the Nijenhuis relation \(^1\) may be regarded as the homogeneous version of the Rota-Baxter relation \(^2 3 4 5 6 7 8 9\). Some of its algebraic aspects especially with regard to the notion of quantum bihamiltonian systems were investigated by Cariñena \(\textit{et al.}\) in \(^1\). The Lie algebraic version of the associative Nijenhuis relation is investigated in \(^10 11\) in the context of the classical Yang-Baxter equation which is closely related to the Lie algebraic version of the Rota-Baxter relation (see especially \(^12\)). Likewise in \(^13 14\) the deformation of Lie brackets defined by Nijenhuis operators and the connection to the (modified) classical Yang-Baxter relation are studied respectively. In Kreimer’s work \(^15 16\) the connection of the Rota-Baxter relation to the Riemann-Hilbert problem in the realm of perturbative Quantum Field Theory is reviewed.

The algebraic properties of the associative notion of the Nijenhuis relation, respectively Nijenhuis algebras, provide interesting insights into associative analogs of Lie algebraic structures. Giving the definition of a commutative unital associative Nijenhuis algebra we use an augmented modified quasi-shuffle product to give explicitly the construction of the free commutative unital associative Nijenhuis algebra generated by a commutative unital associative \(K\)-algebra. We follow thereby closely the inspiring work of Guo \(\textit{et al.}\) \(^17 18\). The main aspect of this construction of the free object is the use of a “homogenized” notion of Hoffman’s quasi-shuffle product giving an associative, unital, and commutative composition. This natural ansatz relies on the close resemblance between the Rota-Baxter and the associative Nijenhuis relation. The free construction of the former is essentially given by the quasi-shuffle product.

Let us remark here that in recent papers \(^9 19 20\) it was shown that non-commutative\(^2\) Rota-Baxter algebras always give Loday-type algebras, i.e. dendriform di- and trialgebra structures \(^21 22 23\).

The paper is organized as follows. In the first section we give the definition of a commutative, unital and associative Nijenhuis \(K\)-algebra respectively the associative Nijenhuis relation. We introduce the notion of a Nijenhuis homomorphism and a free Nijenhuis algebra. Section two contains the definition of the modified and augmented modified quasi-shuffle product being commutative, having a unit and of which we prove the associativity property explicitly. In the following section the augmented modified quasi-shuffle product algebra is identified as the free commutative associative unital Nijenhuis algebra, thereby giving an explicit construction of it. This section closes with a couple of remarks especially concerning the relation to Loday-type algebras. We end this work with a short summary and outlook in the final section.

Throughout this paper, we will consider \(K\) to be a commutative field of characteristic zero. The term \(K\)-algebra always means if not stated otherwise associative commutative unital \(K\)-algebra.

\(^2\)not necessarily commutative
The associative Nijenhuis Relation

Let $\mathcal{A}$ be a $\mathbb{K}$-algebra (see last remark above). On $\mathcal{A}$ we have the $\mathbb{K}$-linear map $N : \mathcal{A} \to \mathcal{A}$. We call $\mathcal{A}$ a Nijenhuis $\mathbb{K}$-algebra in case the operator $N$ holds the following so called associative Nijenhuis relation [1]:

$$N(x)N(y) + N^2(xy) = N(N(x)y + xN(y)), \quad x, y \in \mathcal{A}. \quad (1)$$

The map $N$ might be called associative Nijenhuis operator or just Nijenhuis map for short. In this setting ”associative” refers to the relation (1) to distinguish it clearly form its Lie algebraic version [1, 10, 11, 13, 14]:

$$[N(x), N(y)] + N^2([x, y]) = N([N(x), y] + [x, N(y)]). \quad (2)$$

A Nijenhuis map on a $\mathbb{K}$-algebra $\mathcal{A}$ gives also a Nijenhuis map for the associated Lie algebra $(\mathcal{A}, [\cdot, \cdot]), [\cdot, \cdot]$ being the commutator. 

Equation (1) may be interpreted as the homogeneous version of the standard form of the Rota-Baxter relation of weight $\lambda$ [9]:

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)). \quad (3)$$

A simple transformation $R \to \lambda^{-1}R$ gives the standard form of (3). The homogeneity of (1) destroys this freedom to renormalize the operator $N$ so as to allow for either sign in front of the second term on the lefthand side of (1) for instance.

Let us give two examples of operators fulfilling the associative Nijenhuis relation. Left or right multiplication $L_a b := ab, \ R_a b := ba, \ a, b \in \mathcal{A}$ both hold relation (1).

Another class of associative Nijenhuis operators comes from idempotent Rota-Baxter operators. Let $\hat{R}$ be such an idempotent ($\hat{R}^2 = \hat{R}$) Rota-Baxter operator (3). The operator $\tilde{R} := 1 - \hat{R}$ being idempotent, too, also holds equation (3). Define the following operator:

$$N_\tau := R - \tau \tilde{R}, \ \tau \in \mathbb{K}. \quad (4)$$

The map $N_\tau$ is an associative Nijenhuis operator. One finds similar examples and further algebraic aspects related to equation (1) in [1].

Nijenhuis Homomorphism and the Universal Property

We call an algebra homomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ between Nijenhuis (Rota-Baxter) $\mathbb{K}$-algebras $\mathcal{A}_1$ respectively $\mathcal{A}_2$ a Nijenhuis (Rota-Baxter) homomorphism if and only if $\phi$ intertwines with the Nijenhuis (Rota-Baxter) operators $N_1$, $N_2$ ($R_1$, $R_2$):

$$\phi \circ X_i = X_2 \circ \phi, \ X_i = N_i (R_i), \ i = 1, 2. \quad (5)$$

A Nijenhuis (Rota-Baxter) $\mathbb{K}$-algebra $F(\mathcal{A})$ generated by a $\mathbb{K}$-algebra $\mathcal{A}$ together with an associative algebra homomorphism $j_\mathcal{A} : \mathcal{A} \to F(\mathcal{A})$ is called a free Nijenhuis (Rota-Baxter of weight $\lambda \in \mathbb{K}$) $\mathbb{K}$-algebra if the following universal property holds: for any associative Nijenhuis (Rota-Baxter) $\mathbb{K}$-algebra $\mathcal{X}$ and $\mathbb{K}$-algebra homomorphism $\phi : \mathcal{A} \to \mathcal{X}$, there exists a unique Nijenhuis
(Rota-Baxter) homomorphism $\tilde{\phi} : F(\mathcal{A}) \to \mathcal{X}$ such that the diagram:

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{j} & F(\mathcal{A}) \\
\downarrow{\phi} & & \downarrow{\tilde{\phi}} \\
\mathcal{X} & & \mathcal{X}
\end{array}
$$

commutes.

In the next section we introduce Hoffman’s quasi-shuffle product and an augmented respectively augmented modified version of it to give the construction of the free Nijenhuis algebra.

### 3 Modified Quasi-Shuffle Product

Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. We denote the product on $\mathcal{A}$ by $[ab] \in \mathcal{A}$, $a, b \in \mathcal{A}$ and the algebra unit by $e \in \mathcal{A}$, $[ea] = a$, $a \in \mathcal{A}$.

Consider the tensor module of $\mathcal{A}$:

$$
\mathcal{T}(\mathcal{A}) := \bigoplus_{n \geq 0} \mathcal{A}^\otimes n.
$$

We denote generators $a_1 \otimes \cdots \otimes a_n$ by concatenation, i.e. words $a_1 \ldots a_n$. Generally we use capitals $U \in \mathcal{A}^\otimes n$, $n > 0$ for words and lower case letters $a \in \mathcal{A}$ for letters. The natural grading on $\mathcal{T}(\mathcal{A})$ is defined by the length of words, $l(U + V) := l(U) + l(V) = n + m$, $U \in \mathcal{A}^\otimes n$, $V \in \mathcal{A}^\otimes m$, $n, m > 1$, and $l(a) = 1$.

The empty word is denoted by $1$ and $l(1) = 0$.

We denote the associative commutative quasi-shuffle product \[26\] on $\mathcal{T}(\mathcal{A})$ by:

$$
aU \ast bV := a(U \ast bV) + b(aU \ast V) - \lambda [ab](U \ast V)
$$

whereby the unit of this product is given by the algebra unit $e \in \mathcal{A}$, $e \ast U = U$, $U \in \mathcal{T}(\mathcal{A})$.

In \[17\] Guo et al. gave a non-recursive definition of the quasi-shuffle product by introducing the notion of mixed shuffles (see later in the section 4).

For the parameter $\lambda$ being zero we get the ordinary shuffle product.

The augmented quasi-shuffle product on the augmented tensor module:

$$
\bar{\mathcal{T}}(\mathcal{A}) := \bigoplus_{n > 0} \mathcal{A}^\otimes n
$$

is defined in the following way:

$$
aU \boxdot bV := [ab](U \ast V)
$$

whereby the unit of this product is given by the algebra unit $e \in \mathcal{A}$, $e \boxdot U = U \boxdot e = U$, $U \in \mathcal{T}(\mathcal{A})$.

The associativity and commutativity of $\boxdot$ for arbitrary $\lambda \in \mathbb{K}$ follows from the associativity and commutativity of the quasi-shuffle product and the product $[\ ]$ in $\mathcal{A}$. 
We define now a modified quasi-shuffle product on \( T(A) \) using the algebra unit \( e \in A \):

\[
a U \otimes b V := a(U \otimes b V) + b(aU \otimes V) - \lambda e[ab](U \otimes V). \tag{10}
\]

For the empty word we have \( 1 \otimes U = U \otimes 1 := U, U \in T(A) \).

We prove the associativity property (commutativity being obvious) for this product by induction on the length of words. The linearity allows us to reduce the proof to generators in \( T(A) \). Choose \( a, b, c \in T(A) \), \( l(a + b + c) = 3 \):

\[
a \otimes (b \otimes c) = a \otimes ((bc + cb - \lambda e[bc])
\]

\[
= abc + b(ac + ca - \lambda e[ac]) - \lambda e[ab]c
\]

\[
+ acb + c(ab + ba - \lambda e[ab]) - \lambda e[ac]b
\]

\[
- \lambda(\lambda e[bc] + e(a[bc] + [bc]a - \lambda e[a(bc)])) + \lambda^2(e[e[bc]])
\]

\[
= abc + bca + bac + cab + cba
\]

\[
- \lambda(\lambda e[bc] + e[ac]c + e[ac]b + e[bc]a + e[ab]c + e[bc]a + e[bc]c + e[bc]a)
\]

\[
+ \lambda^2(e[e[bc]] + e[ac]c + e[bc]a)
\]

\[
(a \otimes b) \otimes c = (ab + ba - \lambda e[ab]) \otimes c
\]

\[
= a(bc + cb - \lambda e[bc]) + cab - \lambda e[ac]b
\]

\[
+ b(ac + ca - \lambda e[ac]) + cba - \lambda e[bc]a
\]

\[
- \lambda(e([ab]c + c[ab] - \lambda e[[ab][c]]) + ce[ab]) + \lambda^2(e[ec][ab])
\]

\[
= abc + acb + cab + bac + cba + cba
\]

\[
- \lambda(\lambda e[bc] + e[ac]c + e[ac]b + e[bc]a + e[ab]c + e[bc]a + e[bc]a)
\]

\[
+ \lambda^2(e[[ab][c]] + e[ac]c + e[bc]a)
\]

We see already here that this product is associative if and only if \( \lambda = 1 \), eliminating the two unwanted terms in the \( \lambda^2 \)-part, i.e. \( ec[ab] \) and \( ea[bc] \)!

Now we assume that associativity is true for \( l(U + V + W) = k - 1 > 3 \) and choose \( aX, bY, cZ \in T(A), l(aX + bY + cZ) = k \) (we use now \( \lambda = 1 \)):

\[
a X \otimes (bY \otimes cZ) = a X \otimes (b(Y \otimes cZ) + c(bY \otimes Z) - e[bc](Y \otimes Z))
\]

\[
= a X \otimes (bY \otimes cZ) + a X \otimes (c(bY \otimes Z))
\]

\[
- a X \otimes e[bc](Y \otimes Z)
\]

\[
= a(X \otimes b(Y \otimes cZ)) + b(aX \otimes (Y \otimes cZ))
\]

\[
- e[ab](X \otimes (Y \otimes cZ))
\]

\[
+ a(X \otimes c(bY \otimes Z)) + c(aX \otimes (bY \otimes Z))
\]

\[
- e[ac](X \otimes (bY \otimes Z))
\]

\[
- a(X \otimes e[bc](Y \otimes Z)) + e(aX \otimes [bc](Y \otimes Z))
\]

\[
+ e[ae](X \otimes [bc](Y \otimes Z))
\]

\[
= a((X \otimes bY) \otimes cZ) + b((aX \otimes Y) \otimes cZ) -
\]

\[
c((aX \otimes bY) \otimes Z) + ee[[ab][c]]( (X \otimes Y) \otimes Z) -
\]

\[
e[ac]((X \otimes bY) \otimes Z) - e[bc]((aX \otimes Y) \otimes Z)
\]

\[
- e[ab]((X \otimes Y) \otimes cZ)
\]

\[
= (aX \otimes bY) \otimes cZ
\]
The modified quasi-shuffle is homogeneous in contrast to the ordinary quasi-shuffle product (7). Anticipating the result in the next section we mention here that this fact reflects the homogeneity difference between the Rota-Baxter relation (3) and equation (1).

The augmented modified quasi-shuffle product on the augmented tensor module \( \bar{T}(A) \) is defined as follows:

\[
aU \boxtimes bV := [ab](U \otimes V)
\]

(11)

\[
e \boxtimes V = V \boxtimes e = V.
\]

(12)

We define now the following two linear operators \( B^+_e, B^- : \bar{T}(A) \to \bar{T}(A) \) on the augmented tensor module \( \bar{T}(A) \):

\[
B^+_e(a_1 \ldots a_n) := ea_1 \ldots a_n
\]

(13)

\[
B^-(a_1 \ldots a_n) := a_2 \ldots a_n.
\]

(14)

The first one is later to be identified as the Nijenhuis operator with respect to the augmented modified quasi-shuffle product (11), i.e. in the following section we will show that the triple \( (\bar{T}(A), \boxtimes, B^+_e) \) defines a Nijenhuis algebra, moreover we will see that it fulfills the universal property.

4 Free Associative Nijenhuis Algebra

We will use the augmented modified quasi-shuffle (11) product to give the construction of the free Nijenhuis \( \mathbb{K} \)-algebra. The existence of such an object follows from general arguments in the theory of universal algebras [5, 27], since the category of Nijenhuis \( \mathbb{K} \)-algebras, defined through the identity (1) forms a variety (in the sense of universal algebras, see especially section 6 of [5] for a concise summary of the main ideas).

Let us remark here that the construction of the free Rota-Baxter algebra (of weight \( \lambda \neq 0 \)) uses the augmented quasi-shuffle product (9) and works analogously to what follows. The quasi-shuffle essentially embodies the structure of relation (3). The case \( \lambda = 0 \) gives the ”trivial” Rota-Baxter algebra, i.e. relation (3) without the second term on the lefthand side. This construction was essentially given in [17] using a non-recursive notion of a so called mixed shuffle product. However the formulation using Hoffman’s quasi-shuffle is new.

The essential point (for both constructions of the free objects) lies in the (augmented) quasi-shuffle product (or mixed shuffle in [17]) and its ”merging” of letters \( \lambda[ab], a, b \in A \) in the third term on the righthand side of (9), whereby \( \lambda \) is arbitrary. It represents the second term on the lefthand side of (3). The modified quasi-shuffle product (11) reflects relation (1), in particular, the third term on the righthand side of it is related to the second term on the lefthand side of the associative Nijenhuis relation. It has to be underlined that the associativity of (11) relies on the minus sign on the righthand side. This is due to the fact that the following bilinear map on a \( \mathbb{K} \)-algebra \( A \), \( N \) being an arbitrary \( \mathbb{K} \)-linear map:

\[
\mu_N(a, b) := N(a)b + aN(b) - N(ab)
\]

(15)
gives an associative product if and only if the following map, called $\mu$-Nijenhuis torsion in $[1]$

\[ T_{\mu,N}(a,b) := N(\mu_N(a,b)) - N(a)N(b) \]  

is a 2-Hochschild cocycle. This is especially true for $N$ being a Nijenhuis operator holding exactly relation (1).

We now show that $B^+_e$ holds the associative Nijenhuis relation:

\[ B^+_e(U) \otimes B^+_e(V) + (B^+_e)^2(U \otimes V) = [ee](U \otimes V) + ee\left(\left[u_1v_1\right](B^{-}(U) \otimes B^{+}(V))\right) \]

\[= e\left(u_1(B^{-}(U) \otimes V) + v_1(U \otimes B^{-}(V))\right) - e\left[u_1v_1\right](B^{-}(U) \otimes B^{+}(V)) \]

\[ + ee\left(u_1v_1\right)(B^{-}(U) \otimes B^{-}(V)) \]

\[= B^+_e(U \otimes B^+_e(V) + B^+_e(U \otimes V) \]  

(17)

For the triple $(\tilde{T}(A), \boxtimes, B^+_e)$ to be the free Nijenhuis algebra over $A$ we have to show that the universal property holds. Let $\mathcal{U}$ be an arbitrary Nijenhuis $\mathbb{K}$-algebra with Nijenhuis operator $N$ and $\phi : A \rightarrow \mathcal{U}$ an $\mathbb{K}$-algebra homomorphism.

We have to extend the $\mathbb{K}$-algebra map $\phi$ to a Nijenhuis homomorphism $\tilde{\phi} : \tilde{T}(A) \rightarrow \mathcal{U}$ using the following important fact: every generator $a_1 \ldots a_n \in \tilde{T}(A)$ can be written using the modified augmented quasi-shuffle product:

\[ a_1 \ldots a_n = a_1 \boxtimes B^+_e(a_2 \boxtimes B^+_e(a_3 \boxtimes B^+_e(\ldots B(a_{n-1} \boxtimes B^+_e(a_n)) \ldots))) \]  

(18)

Since the extension $\tilde{\phi}$ is supposed to be a Nijenhuis algebra homomorphism we have the unique possible extension:

\[ \tilde{\phi}(a_1 \ldots a_n) := \phi(a_1)N(\phi(a_2)N(\ldots N(\phi(a_{n-1})N(\phi(a_n)) \ldots)). \]  

(19)

Defining the map $N_w : \mathcal{U} \rightarrow \mathcal{U}, w \in \mathcal{U}, \ N_w(x) := N(wx), \ x \in \mathcal{U}$ we can write $[19] \text{ like } [17]$. 

\[ \tilde{\phi}(a_1 \ldots a_n) = \phi(a_1) \{ \circ_{i=2}^{n} N_{\phi(a_i)}(1_{\mathcal{U}}) \}. \]  

(20)

So we are left to show that $\tilde{\phi}$ is a Nijenhuis homomorphism. By construction it is a well defined $\mathbb{K}$-linear map. The Nijenhuis property $[5]$ follows by $[17]$

\[ \tilde{\phi}(B^+_e(a_1 \ldots a_n)) = \phi(ea_1 \ldots a_n) \]

\[= \phi(a_1 \ldots a_n) \]

\[= \circ_{i=1}^{n} N_{\phi(a_i)}(1_{\mathcal{U}}) \]

\[= N(\phi(a_1) \circ_{i=2}^{n} N_{\phi(a_i)}(1_{\mathcal{U}})) \]

\[= N(\tilde{\phi}(a_1 \ldots a_n)). \]  

(21)

Finally we would like to show that $\tilde{\phi}$ is a $\mathbb{K}$-algebra homomorphism, i.e. preserves multiplication. This we proof by induction on the length of words,
\( I(X + Y) = m + n = k \). For \( k = 2 \) we have, \( a, b \in \mathcal{T}(A) \):

\[
\tilde{\phi}(a \boxtimes b) = \tilde{\phi}([ab])
\]

\[
= \phi([ab])
\]

\[
= \phi(a)\phi(b) = \tilde{\phi}(a)\tilde{\phi}(b).
\]

Let \( X := u_1 \ldots u_n \in A^\otimes n \), \( Y := v_1 \ldots v_m \in A^\otimes m \), of length \( m + n > 2 \):

\[
\tilde{\phi}(X \boxtimes Y) = \tilde{\phi}([u_1v_1] \{ B^-(X) \otimes B^-(Y) \})
\]

\[
= \tilde{\phi}([u_1v_1] \{ B^+(B^-(X)) \boxtimes B^-(Y) + B^-(X) \boxtimes B^+(B^-(Y))
\]

\[
- B^+(B^-(X) \boxtimes B^-(Y)) \})
\]

\[
= \phi(u_1)\phi(v_1) N \{ \tilde{\phi}(\tilde{\phi}(B^+(B^-(X))) \boxtimes \tilde{\phi}(B^-(Y))
\]

\[
+ \tilde{\phi}(B^-(X)) \tilde{\phi}(B^+(B^-(Y))) - N\{\tilde{\phi}(B^-(X)) \tilde{\phi}(B^-(Y))\} \}
\]

\[
= \phi(u_1)\phi(v_1) N \{ \tilde{\phi}(B^-(X)) \tilde{\phi}(B^-(Y))
\]

\[
+ \tilde{\phi}(B^-(X)) N\{\tilde{\phi}(B^-(Y))\} - N\{\tilde{\phi}(B^-(X)) \tilde{\phi}(B^-(Y))\} \}
\]

\[
= \phi(u_1)\phi(v_1) N\{\tilde{\phi}(B^-(X))\} \tilde{\phi}(Y)
\]

In the third line we used the same trick as in (21). Apparently by going from line (\#) to the next we see that the use of the augmented modified quasi-shuffle product in the above construction of the free Nijenhuis algebra is limited to the commutative case.

Using the augmented quasi-shuffle product \((\#)\) the proof of the universal property for the triple \((\mathcal{T}(A), \boxtimes, \lambda, B^+_e)\) goes analogously, giving the free Rota-Baxter \(K\)-algebra of weight \(\lambda\) over \(A\).

The algebras \((\mathcal{T}(A), *, \lambda, B^+_e)\) and \((\mathcal{T}(A), \otimes, B^+_e)\) define a Rota-Baxter algebra and a Nijenhuis algebra, respectively.

The above construction may also be used to give the free Nijenhuis algebra on a set \(S\) by working with the polynomial algebra \(K[S]\).

**Mixed Shuffle Product**

Guo’s et al mixed shuffle product in [17] may be described the following way. Let \( Z := z_1 \ldots z_n \), \( X := x_{n+1} \ldots x_{n+m} \in \mathcal{T}(A) \) be two words. The mixed shuffle relies on so called admissible pairs \((k, k+1)\), \(k \in \mathbb{N}\), defined as follows. First take the ordinary shuffle product of \(X, Z \in \mathcal{T}(A)\), \(I(X + Z) = m + n\), i.e. relation \((7)\) with \(\lambda = 0\):

\[
X * Z := x_1(B^-(X) * Z) + z_1(X * B^-(Z))
\]

\[
= \sum_{\tau = 1}^{\Gamma(m,n)} W^{\tau}(X,Z) \in A^\otimes m+n.
\]
\[ \Gamma_{m,n} := \binom{m+n}{m}. \] Every pair of consecutive letters in one of the words \( W_{\tau}^{(X,Z)} \) in the above shuffle sum \((22)\) of the form \( \ldots x_i z_j \ldots \), \( x_i \) being the \( k \)-th letter in this word gives an admissible pair \( (k,k+1) \). We denote the set of admissible pairs by \( P_{W_{\tau}^{(X,Z)}} := \{(k,k+1) \mid 1 \leq k < m+n\} \). A word may contain no, one or several admissible pairs. If it contains any there is no change and we just keep the word \( W_{\tau}^{(X,Z)} \). If it contains admissible pairs, we add the following sum to the above shuffle product \((22)\):

\[ \sum_{p \in P_{W_{\tau}^{(X,Z)}}} (-\lambda)^{|p|} W_{\tau}^{((X,Z))}_p \in \bigoplus_{i=1}^{m+n-1} \mathcal{A}^{\otimes i}. \] (23)

Keeping the word \( W_{\tau}^{(X,Z)} \) in \((22)\), the empty set must be excluded \( p \neq \emptyset \) in \((23)\). The word \( W_{\tau}^{((X,Z))}_p \) denotes the original \("shuffled") string but with consecutive letters at position \( k \), \( (k,k+1) \in p \in P_{W_{\tau}^{(X,Z)}} \) being \"merged\" (in \( \mathcal{A} \)) to one letter, i.e. \( \ldots [x_i z_j] \ldots \).

**Dendriform Di- and Trialgebra Structures**

We mentioned in the beginning the relation between Rota-Baxter algebras and Loday-type algebras \([9, 19, 20]\), i.e. dendriform di- and trialgebra structures (see \([21, 22]\) for definitions).

In this final part we would like to relate the above given constructions of the free Rota-Baxter algebra, i.e. especially the augmented and augmented modified quasi-shuffle products to these Loday-type structures.

In \([24]\) Ronco defined a dendriform dialgebra structure on the tensor coalgebra \( \tilde{T}(V) \) over a vector space \( V \) in connection to the ordinary shuffle product \([22]\).

The two compositions \( \prec, \succ \) are defined using the fact, that the shuffle sum \([22]\) may be characterized by words either beginning with \( x_1 \) or \( z_1 \):

\[ X \prec Z := x_1(B^-(X) \ast Z) \]
\[ X \succ Z := z_1(X \ast B^-(Z)). \]

In the context of \( \tilde{T}(\mathcal{A}) \) over the \( \mathbb{K}\)-algebra \( \mathcal{A} \) we see how this relates to the augmented shuffle product \([9]\), respectively the \"trivial\" Rota-Baxter map \( B^+_e \) of weight \( \lambda = 0 \):

\[ X \prec Z := X \square B^+_e(Z) \]
\[ X \succ Z := B^+_e(X) \square Z, \]

which gives the dendriform dialgebra structure related to the Rota-Baxter relation of weight zero found by Aguiar \([19]\). Using \( B^+_e \) and \( \lambda \text{id}_{\tilde{T}(\mathcal{A})} - B^+_e =: \hat{B}^+_e \) the dendriform compositions \( X \prec Z := X \square B^+_e(Z) \), \( X \succ Z := -\hat{B}^+_e(X) \square Z \) relate to the augmented quasi-shuffle product, i.e. to the Rota-Baxter relation of weight \( \lambda \neq 0 \) \([9]\).

Loday and Ronco related in \([22]\) the dendriform trialgebra structure to Hoffman’s quasi-shuffle product:

\[ X \prec Z := x_1(B^-(X) \ast Z) = X \square B^+_e(Z) \] (24)
\[ X \succ Z := z_1(X \ast B^-(Z)) = B^+_e(X) \square Z \] (25)
\[ X \bullet Z := [x_1 z_1](B^-(X) \ast B^-(Z)) = X \square Z. \] (26)
Including the $\lambda$-term in the $\square$-product in \cite{24} we might describe this structure using the free Rota-Baxter algebra $(\mathcal{T}(\mathcal{A}), \boxtimes, \lambda, B^+_e)$ of weight $\lambda \neq 0$. The last term is just the product $X \boxtimes Z$ \cite{20}. Let us remark here an observation with respect to the work \cite{24} of Ronco. From the previous results it follows that Rota-Baxter algebras always give dendriform algebra structures. We used in the definition of the extension of an algebra homomorphism to a Rota-Baxter homomorphism the fact \cite{18}, that the elements in $\mathcal{T}(\mathcal{A})$ may be written using the Rota-Baxter map $B^+_e$ instead of $\otimes$:

\[
a_1 \ldots a_n = a_1 \boxtimes B^+_e(a_2 \boxtimes B^+_e(a_3 \boxtimes B^+_e(\ldots B^+_e(a_{n-1} \boxtimes B^+_e(a_n)) \ldots)))
= B^+_e(\ldots B^+_e(B^+_e(B^+_e(a_n) \boxtimes a_{n-1}) \boxtimes a_{n-2}) \boxtimes a_{n-3} \ldots) \boxtimes a_1.
\]

With respect to the dendriform (trialgebra) structures this may be written using either of the compositions $\prec$, $\succ$ and introducing the ”identity” maps $\omega_{\prec, \succ}$ on $\mathcal{T}(\mathcal{A})$ ($a_i \in \mathcal{A}$ being letters):

\[
\omega_{\prec}(a_1 \ldots a_n) = a_1 \prec (a_2 \prec (a_3 \prec (\cdots (a_{n-1} \prec a_n) \ldots))) \quad (27)
\]

These maps are of importance with respect to the primitive elements and brace-algebras \cite{24,25} in a dendriform Hopf algebra. The connection to Loday’s recent work \cite{23} has to be clarified in a future work.

\section{Summary and Outlook}

We have shown that Hoffman’s (augmented) quasi-shuffle product essentially gives the free commutative associative unital Rota-Baxter algebra on a commutative associative unital $\mathbb{K}$-algebra. As the associative Nijenhuis relation may be regarded as the homogeneous version of the Rota-Baxter relation the natural way to construct a free commutative associative Nijenhuis algebra is to ”homogenize” the (augmented) quasi-shuffle product. Commutativity following by construction, we showed explicitly the associativity. Due to the homogeneity of the associative Nijenhuis relation the sign, i.e. free parameter, in front of the ”merging” part in the modified quasi-shuffle product has to be fixed to be 1. We then showed that this new product effectively gives the free associative Nijenhuis structure generated by a commutative associative unital $\mathbb{K}$-algebra. The ”shift” operator $B^+_e$ acts in the former case as the Rota-Baxter map and as the Nijenhuis operator in the latter.

Finally we related these free objects to the fascinating dendriform di- and trialgebra structures given in the work of Loday and Ronco. Further work concerning this link and the algebraic properties of the augmented modified quasi-shuffle product has to be done. Following the work of Guo \textit{et al.} on the free Rota-Baxter algebra further investigations into the structure of associative Nijenhuis algebras need to be done.

Both the Rota-Baxter and the associative Nijenhuis relation are of interest with respect to the Hopf algebraic formulation of the theory of renormalization in perturbative Quantum Field Theory and especially to aspects of integrable systems.

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