A Proximal Alternating Direction Method of Multiplier for Linearly Constrained Nonconvex Minimization

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Abstract

Consider the minimization of a nonconvex differentiable function over a polyhedron. A popular primal-dual first-order method for this problem is to perform a gradient projection iteration for the augmented Lagrangian function and then update the dual multiplier vector using the constraint residual. However, numerical examples show that this approach can exhibit “oscillation” and may not converge. In this paper, we propose a proximal alternating direction method of multipliers for the multi-block version of this problem. A distinctive feature of this method is the introduction of a “smoothed” (i.e., exponentially weighted) sequence of primal iterates, and the inclusion, at each iteration, to the augmented Lagrangian function a quadratic proximal term centered at the current smoothed primal iterate. The resulting proximal augmented Lagrangian function is inexactly minimized (via a gradient projection step) at each iteration while the dual multiplier vector is updated using the residual of the linear constraints. When the primal and dual stepsizes are chosen sufficiently small, we show that suitable “smoothing” can stabilize the “oscillation”, and the iterates of the new proximal ADMM algorithm converge to a stationary point under some mild regularity conditions. Furthermore, when the objective function is quadratic, we establish the linear convergence of the algorithm. Our proof is based on a new potential function and a novel use of error bounds.
1 Introduction

Consider the following linearly constrained optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad Ax = b, \ x \in P,
\end{align*}
\]

where \( f \) is differentiable (not necessarily convex) and

\[
P := \{x \in \mathbb{R}^n \mid \ell_i \leq x_i \leq u_i, \ i = 1, 2, ..., n\}
\]

is a bounded box with \( \ell_i < u_i \) for all \( i \), and the matrix \( A \) has dimension \( \mathbb{R}^{m \times n} \).

Problems of the form (1.1) arise in many applications involving big data, including nonnegative matrix factorization [9, 20, 23], phase retrieval [22], distributed matrix factorization [16], polynomial optimization [14], asset allocation [21], zero variance discriminant analysis [1], to name just a few.

A popular approach to solve problem (1.1) is to dualize the linear equality constraint and apply a primal-dual type algorithm to the resulting augmented Lagrangian function. Such approach is particularly attractive when the objective function \( f(x) \) has a separable structure since in this case the corresponding primal minimization problem can be decomposed and often times solvable efficiently in parallel, while the dual update can be carried out in closed form.

The local convergence analysis of the classical augmented Lagrangian method was for smooth objective function with smooth equality constraint [2]. Global convergence of a primal-dual method for smooth objective and smooth equality constraint was given recently in [10]. When the decision variable \( x \) consists of many small variable blocks, the popular alternating direction method of multipliers (ADMM) is often the preferred algorithm to solve (1.1), see [4] for a detailed coverage of the method and many applications from a set of diverse fields. It is well known [7] that for the two-block (strongly) convex case ADMM can be viewed as a variant of proximal-point method or operator-splitting method, from which one can derive linear convergence of the method. The paper [5] proves that the direct extension of ADMM to three-block case may not converges and state some sufficient conditions for the extended algorithm to converge. The reference [11] uses the error bound analysis [17] to establish the linear convergence rate of ADMM for a family of convex programming problem with any number of variable blocks with a reduced dual stepsize. However, for nonconvex problems, the convergence of the augmented Lagrangian method or ADMM has not been well understood, despite the fact they have been widely used in applications. In [12], the convergence of ADMM was established for some special nonconvex problems such as consensus-based sharing problems by using the augmented Lagrangian function as the potential function. This approach was further extended [6] to a larger family of nonconvex-nonsmooth problems under some technical assumption such as the prox-regularity of the objective function. The papers [3, 13, 15] proved the convergence of some inexact ADMM for certain nonsmooth, nonconvex problems. However, these references all require at least one block of the variable to be unconstrained and a strong feasibility assumption holds, namely, suppose the linear equality constraint is

\[
A_1x_1 + A_2x_2 = b,
\]

then the image of \( A_1 \) is contained in the image of \( A_2 \) and the variable block \( x_2 \) does not have any other constraint. Recently, [8] established the convergence of multi-block ADMM algorithm for the so called multi-affine constraints which are linear in each variable block but otherwise nonconvex.
This paper also has some technical assumptions, including the similar strong feasibility constraint and the objective function for some block must be strongly convex. We see that the studies of these papers do not cover the general problem (1.1).

The contribution of this paper is as follows. We propose a proximal alternating direction method of multipliers (ADMM) to solve a linearly constrained nonconvex differentiable minimization problem. A distinctive feature of the algorithm is to introduce a “smoothed” (i.e., exponentially weighted) sequence of primal iterates, and at each iteration add to the augmented Lagrangian function an extra quadratic proximal term centered at the smoothed primal iterate. The resulting proximal augmented Lagrangian function is inexactly minimized at each iteration while the dual multiplier vector is updated using the residual of the linear constraints. The algorithm is well suited for large scale optimization involving big data, and easily extends to the multiple variable block case, resulting in a variant of the well-known ADMM algorithm. When the primal and dual stepsizes are chosen sufficiently small, we show that the iterates of the proximal ADMM algorithm converge to a stationary point of the nonconvex problem under some mild regularity conditions. Moreover, we present a numerical example showing that the “smoothing” step is necessary for the convergence of the proximal ADMM when the objective function is nonconvex. Furthermore, for a quadratic objective function, we establish the linear convergence of the algorithm.

2 Preliminaries

2.1 The set of stationary solutions

We first define the solution of the problem (1.1) in this subsection. Due to the linearity of the constraints, there exists a set of Lagrangian multipliers for each stationary point of (1.1) such that the KKT condition holds. We denote the set of stationary points of (1.1) by $X^*$. Writing down the KKT condition, letting $y, \mu, \nu$ be the multipliers, we have:

\[
\nabla f(x^*) + A^T y^* - \mu^* + \nu^* = 0 \\
Ax^* = b \\
\ell_i \leq x_i^* \leq u_i, \text{ for all } i \\
\mu^* \geq 0, \\
\nu^* \geq 0, \\
\mu_i^*(\ell_i - x_i^*) = 0, \text{ for all } i \\
\nu_i^*(x_i^* - u_i) = 0, \text{ for all } i, 
\]

where $\mu_i^*$ and $\nu_i^*$ denote the $i$-th component of $\mu^*$ and $\nu^*$ respectively. Let $X^*, Y^*$ be the sets of all $x^*$ and $y^*$ satisfying the KKT condition. Note that (2.1) and (2.2) are the complementarity conditions. It means that either ($\ell_i - x_i^*$) or $\mu_i^*$ must be zero for all $i$ and similarly for $\nu^*$. A stronger condition, , which holds generically, is called “strict complementarity condition”.

**Definition 2.1** If for all solutions $(x^*, y^*, \mu^*, \nu^*)$ of the KKT system, for any $i$, exactly one of $\mu_i^*$ and $(\ell_i - x_i^*)$ is zero and exact one of $\nu_i^*$ and $(x_i^* - u_i)$ is zero, then we say the original problem satisfies the strict complementarity condition.
2.2 Assumptions

In this subsection, we give our main assumptions, which are valid in many practical problems.

Assumption 2.2

(a) The origin is in the relative interior of the set \(AP - b = \{Ax - b \mid x \in P\}\).

(b) The strict complementarity condition holds for (1.1).

(c) The objective function \(f\) is a differentiable function with Lipschitz continuous gradient

\[
\|\nabla f(x) - \nabla f(x')\| \leq L\|x - x'\|, \quad \text{for some } L > 0 \text{ and } \forall x, x' \in P.
\]

Note that Assumption 2.2(a) is equivalent to the feasibility of (1.1) for all slightly perturbed \(b\) from the range space of \(A\); in particular it does not require the full row rank of \(A\). Assumption 2.2(c) implies the existence of a constant \(\gamma\) (possibly negative) such that

\[
\langle \nabla f(x) - \nabla f(x'), x - x' \rangle \geq \gamma\|x - x'\|^2, \quad \text{for all } x, x' \in P. \tag{2.3}
\]

Assumption 2.2(b) is reasonable since the strict complementarity is valid generically, as we argue in the proposition below.

Proposition 2.3 Suppose \(f(x) = g(x) + q^T x\), \(g\) is Lipschitz-differentiable and \(v\) is a constant vector. If the data vector \((v, b)\) is generated from a continuous measure, then with probability 1, the strict complementarity condition holds for (1.1).

Proof We will use the fact that Lipschitz continuous functions map a zero-measure set to a zero-measure set. For active sets \(S_1\) and \(S_2\) the KKT condition with respect to \(\{1, 2, \ldots, n\}\) is

\[
\nabla g(x) + A^T y - \mu + \nu = -q
\]

\[
Ax = b,
\]

\[
x_i = \ell_i, \mu_i \geq 0, \quad i \in S_1
\]

\[
x_i = u_i, \nu_i \geq 0, \quad i \in S_2
\]

\[
x_i \in [\ell_i, u_i], \quad \text{for all } i
\]

\[
\mu_i = 0, \quad i \notin S_1
\]

\[
\nu_i = 0, \quad i \notin S_2
\]

We prove that for any \(S_1, S_2 \subseteq \{1, 2, \ldots, n\}\), with probability 0, the strictly complementary condition does not hold. Since \(S_1, S_2\) have only finitely many choices, we only need to consider fixed \(S_1\) and \(S_2\). Suppose the assumption does not hold, without loss of generality, assume that \(x_1 = \ell_1, \mu_1 = 0\) and \(1 \notin S_1\). Consider the Lipschitz continuous map \(\Phi\)

\[
\Phi(x, y, \mu, \nu) = (\nabla g(x) + A^T y - \mu + \nu, Ax)
\]

from the set

\[
T = \{(x, y, \mu, \nu) \mid x_i = \ell_i, i \in S_1, \quad x_i = u_i, i \in S_2,
\]

\[
\ell_i \leq x_i \leq u_i, \mu_i = 0, \quad i \notin S_1, \quad \nu_i = 0, \quad i \notin S_2,
\]

\[
\mu \geq 0, \nu \geq 0, x_1 = \ell_1\}
\]
to $\mathbb{R}^{n+m}$. Clearly $\Phi$ maps from a $n + m - 1$ dimension subset to $n + m$-dimension space. Hence, the image is zero-measure in $\mathbb{R}^{n+m}$. Consequently, the choice for $(q,b)$ such that the solution exists is of measure zero.

2.3 A Proximal Inexact Augmented Lagrangian Multiplier Method

We will state our algorithm in this subsection based on the augmented Lagrangian function. The Augmented Lagrangian function for (1.1) is given by

$$L(x; y) = f(x) + y^T(Ax - b) + \frac{\rho}{2}\|Ax - b\|^2,$$

where $\rho > 0$ is a constant. The classical augmented Lagrangian multiplier method minimizes $L(x; y)$ for a fixed $y$ over the box constraint $P$, and then updates $y$ using the residual of the primal equality constraint $Ax = b$. Unfortunately, due to the nonconvexity of $f$, the exact minimization of $L(x; y)$ with respect to $x$ can be difficult. Thus, it is often more practical to minimize $L(x; y)$ inexactly with respect to $x$. In particular, we recall the following simple inexact augmented Lagrangian multiplier method (which also corresponds to the linearized ADMM algorithm when there is only one primal variable block).

Algorithm 1 An Inexact Augmented Lagrangian Multiplier Method

1: Let $\alpha > 0$ and $c > 0$;
2: Initialize $x^0, y^0$;
3: for $t = 0, 1, 2, \ldots, \text{do}$
4: \hspace{1em} $y^{t+1} = y^t + \alpha(Ax^t - b)$;
5: \hspace{1em} $x^{t+1} = [x^t - c\nabla_xL(x^t; y^{t+1})]_+$.
6: end for

Though easy to implement numerically, the above inexact augmented Lagrangian multiplier method can behave erratically or even diverge for nonconvex problems (see Figure 1 in Section 6 for a numerical example). To stabilize the convergence behaviour of the inexact augmented Lagrangian multiplier method, we propose a proximal version of the augmented Lagrangian multiplier method. In this new method, we introduce an exponential averaging (or smoothing) scheme to generate an extra sequence $\{z^t\}$ and insert an extra quadratic proximal term centered at $z^t$ to the augmented Lagrangian function so that the next primal iterate $x^{t+1}$ does not deviate too much from the stabilized iterate $z^t$. More specifically, let

$$K(x, z; y) = L(x; y) + \frac{p}{2}\|x - z\|^2,$$

where $p$ is a positive parameter. Note that the function $K$ is Lipschitz differentiable with modulus

$$L_K = L + p + \rho\sigma^2,$$

where $\sigma$ is the spectral norm of the matrix $A$, and can be made strongly convex in $x$ with modulus

$$\gamma_K = p + \gamma > 0$$

if $p$ is chosen to be larger than $-\gamma$. We consider the following proximal inexact augmented Lagrangian multiplier method.
Algorithm 2 A Proximal Inexact Augmented Lagrangian Multiplier Method

1: Let $\rho > 0$, $\alpha > 0$, $0 < \beta \leq 1$ and $\frac{1}{L_K} > c > 0$;
2: Initialize $x^0 \in P$, $z^0 \in P$, $y^0 \in \mathbb{R}^m$;
3: for $t = 0, 1, 2, \ldots$ do
4: $y^{t+1} = y^t + \alpha (Ax^t - b)$;
5: $x^{t+1} = [x^t - c\nabla_x K(x^t, z^t; y^{t+1})]_+;
6: z^{t+1} = z^t + \beta (x^{t+1} - z^t)$.
7: end for

Let
\[
\begin{align*}
    d(y, z) &= \min_{x \in P} K(x, z; y), \quad (2.5) \\
    x(y, z) &= \arg\min_{x \in P} K(x, z; y), \quad (2.6) \\
    P(z) &= \min_{x \in P, Ax=b} (f(x) + \frac{p}{2}\|x - z\|^2), \quad (2.7) \\
    x(z) &= \arg\min_{x \in P, Ax=b} \left( f(x) + \frac{p}{2}\|x - z\|^2 \right). \quad (2.8)
\end{align*}
\]

It should be noted that if $p > -\gamma$, then $f(x) + \frac{p}{2}\|x - z\|^2$ is strongly convex, so there holds
\[
K(x, z; y) \geq d(y, z), \quad P(z) \geq d(y, z), \quad \forall \ y, z, \quad (2.9)
\]
where the first inequality is due to (2.5), while the second inequality follows from the strong duality
\[
P(z) = \max_y d(y, z).
\]

Note that the subproblem of the Algorithm 2 is easy since it involves only projection to the box $P$. Compared to Algorithm 1, the proximal inexact augmented Lagrangian method (Algorithm 2) constructs an auxiliary sequence $\{z^t\}$ which is a recursive average of the primal sequence $\{x^t\}$, and uses it to build a quadratic proximal term in the augmented Lagrangian function $L(x^t; y^t)$. Notice that the recursive averaging step is computationally simple, so Algorithm 2 has a similar per-iteration complexity to Algorithm 1. It should be noted that this new quadratic proximal term in the augmented Lagrangian function introduces an extra term $p(x^t - z^t)$ in the gradient of $L(x^t; y^{t+1})$. This extra term is an exponentially weighted average of all the previously generated primal iterates $\{x^0, x^1, \ldots, x^t\}$. As such, it is different from the well known “momentum term” in the backpropagation training algorithm which is equal to the difference of the previous two primal iterates. Also, this extra term is different from the Nesterov’s acceleration scheme which adds to the gradient descent direction a specific (iteration dependent) average of previous two primal iterates.

In the rest of the paper, we fix parameters $c < 1/L_K$ and $p > -\gamma$. Our main claim is that the introduction of the proximal term can ensure the global convergence of Algorithm 2 for the nonconvex problem (1.1).

Theorem 2.4 Suppose Assumption 2.2 holds. Moreover, suppose the parameters $c$ and $p$ are selected to satisfy
\[
\frac{1}{L_K} > c > 0, \quad p > -\gamma
\]
and that the primal and dual stepsizes $\beta$ and $\alpha$ to be sufficiently small. Then the dual iterates $\{y^t\}$ are bounded. Moreover, there holds
\[
\lim_{t \to \infty} \|x^{t+1} - x^t\| = 0,
\lim_{t \to \infty} \text{dist}(x^t, X^*) = 0,
\lim_{t \to \infty} \text{dist}(z^t, X^*) = 0,
\]
and every limit point of the sequence $\{(x^t, y^t)\}$ generated by Algorithm 2 is a primal-dual stationary point of (1.1).

We will prove this theorem in the next section.

3 Convergence Analysis

3.1 Key Lemmas

In this subsection, we will give some key lemmas that are needed to establish the main theorem.

3.1.1 Error Bounds

First, we develop some error bounds which establish the relation between the primal and dual residual and the distance to the solution set.

Lemma 3.1 Suppose Assumption 2.2(a) holds. Let $y_k = \tilde{y}_0 + \tilde{y}_k$ with $\tilde{y}_k \in \text{Range}(A)$ and some fixed $\tilde{y}_0 \perp \tilde{y}_k$, and let $x(y_k, z_k)$ be given by (2.6). If
\[
\|Ax(y_k, z_k) - b\| \to 0,
\]
then $y_k$ is bounded.

Proof According to Assumption 2.2(a), there exists a positive $r > 0$, such that for any direction $d \in \text{Range}(A)$, we can find an $x \in P$ satisfying $\|Ax - b\| = r$ and $Ax - b$ has the same direction as $d$. We claim that if $\|y_k\|$ goes to infinity, then $\|Ax(y_k, z_k) - b\|$ must be bounded away from 0. We prove this by contradiction. Assume that $\|y_k\| \to \infty$ and $\|Ax(y_k, z_k) - b\| \to 0$. Since $\tilde{y}_0$ is fixed and $\tilde{y}_0 \perp \tilde{y}_k$, it follows that $\|\tilde{y}_k\| \to \infty$. By Assumption 2.2(a), there exists a $x_k \in P$ such that $Ax_k - b$ is of the same direction as $-\tilde{y}_k$ and $\|Ax_k - b\| = r$. Let
\[
M = \max_{x,z \in P} \left\{ |f(x)| + \frac{\rho}{2} \|x - z\|^2 + \frac{\rho}{2} \|Ax - b\|^2 + \langle y_0, Ax - b \rangle \right\},
\]
then if $\|\tilde{y}_k\| > 4M/r$ and $\|Ax(y_k, z_k) - b\| < r/2$, we have
\[
\tilde{y}_k^T (Ax_k - b) = -\|\tilde{y}_k\| \|Ax_k - b\| = -\|\tilde{y}_k\| r < -4M
\]
and

\[
\tilde{y}_k^T (Ax_k - b) = -r \|\tilde{y}_k\| \\
\leq -r \frac{|\tilde{y}_k^T (Ax(y_k, z_k) - b)|}{\|Ax(y_k, z_k) - b\|} \\
\leq -r \frac{|\tilde{y}_k^T (Ax(y_k, z_k) - b)|}{r/2} \\
\leq 2\tilde{y}_k^T (Ax(y_k, z_k) - b),
\]

where the second step follows from Cauchy-Schwartz inequality. Hence, we have

\[
K(x_k, z_k; y_k) - K(x(y_k, z_k), z_k; y_k) \leq 2M + \tilde{y}_k^T (Ax_k - b) - \tilde{y}_k^T (Ax(y_k, z_k) - b) \\
\leq 2M + \tilde{y}_k^T (Ax_k - b) - \frac{1}{2} \tilde{y}_k^T (Ax_k - b) \\
= 2M + \frac{1}{2} \tilde{y}_k^T (Ax_k - b) \\
< 0
\]

where the last step is due to (3.1). This further implies

\[
K(x_k, z_k; y_k) < K(x(y_k, z_k), z_k; y_k)
\]

which is a contradiction. ■

The boundedness of \(y_k\) will be used to establish the dual error bound later in Lemma 3.6. Next result shows that \(x(z)\) is continuous in \(z\) and \(x(y, z)\) is continuous in \((y, z)\).

**Lemma 3.2** Suppose \(p > -\gamma\). If the sequences \(\{y_k\}, \{z_k\}\) satisfy \(y_k \to y\) and \(z_k \to z\), then we have

\[
\|x(z) - x(z')\| \leq \frac{p}{p + \gamma} \|z - z'\|
\]

and

\[
x(y_k, z_k) \to x(y, z).
\]

**Proof** We only prove the Lipschitz continuity of \(x(z)\), the other claim can be proved similarly. Let \(f(x; z) = f(x) + \frac{x}{2} \|x - z\|^2\). For \(z, z' \in P\), we have

\[
\begin{align*}
f(x(z); z') - f(x(z'); z') &= (f(x(z); z) - f(x(z'); z)) - (f(x(z'); z') - f(x(z'); z)) + (f(x(z); z') - f(x(z); z)) \\
&= (f(x(z); z) - f(x(z'); z)) - \frac{P}{2} (-2(z' - z)^T x(z') + \|z'\|^2 - \|z\|^2) \\
&\quad + \frac{P}{2} (-2(z' - z)^T x(z) + \|z'\|^2 - \|z\|^2) \\
&= (f(x(z); z) - f(x(z'; z)) + p(z' - z)^T (x(z') - x(z)) \\
&\leq -\frac{P + \gamma}{2} \|x(z) - x(z')\|^2 + p(z' - z)^T (x(z') - x(z)),
\end{align*}
\]

where the last inequality is due to the strong convexity of \(f(x; z)\) in variable \(x\). On the other hand, again by the strong convexity, we have

\[
f(x(z); z') - f(x(z'); z') \geq \frac{P + \gamma}{2} \|x(z) - x(z')\|^2. \tag{3.2}
\]
Hence, we have
\[-(p + \gamma)\|x(z) - x(z')\|^2 + p(z' - z)^T(x(z') - x(z)) \geq 0,
\]
which by Cauchy-Schwartz inequality further implies
\[
\|x(z) - x(z')\| \leq \frac{p}{p + \gamma} \|z - z'\|.
\]
Hence, \(x(\cdot)\) is Lipschitz continuous with modulus \(p/(p + \gamma)\).

A couple of corollaries are in order. We denote \(x^+, y^+, z^+\) be the updated variables of \(x, y, z\) by Algorithm 2.

**Corollary 3.3** Suppose \(p > -\gamma\). Then for any \(\epsilon > 0\), there exists a \(\delta(\epsilon) > 0\), such that for vectors \(x, z \in P\) and any \(y\) satisfying
\[
\max\{\|x - x^+\|, \|z - x^+\|, \|Ax(y^+, z) - b\|\} < \delta(\epsilon),
\]
we have
\[
\max\{\text{dist}(x, X^*), \text{dist}(z, X^*)\} < \epsilon.
\]

**Proof** We prove by contradiction. Suppose that
\[
\lim_{k \to \infty} \|x_k - x_k^+\| = \lim_{k \to \infty} \|z_k - x_k^+\| = \lim_{k \to \infty} \|Ax(y_k^+, z_k) - b\| = 0
\]
while
\[
\text{either } \lim_{k \to \infty} \text{dist}(x_k, X^*) > 0, \text{ or } \lim_{k \to \infty} \text{dist}(z_k, X^*) > 0. \tag{3.3}
\]
Since \(\{y_k\}\) is bounded (cf. Lemma 3.1), by passing to a subsequence if necessary, we can assume that
\[
x_k \to \bar{x}, \quad z_k \to \bar{z}, \quad y_k^+ \to \bar{y}.
\]
Then, it follows from (3.3) that
\[
\text{either } \bar{x} \notin X^* \text{ or } \bar{z} \notin X^*.
\]
Moreover, we have
\[
\bar{x} - x^+ = 0, \quad \bar{z} - z^+ = 0,
\]
and by Lemma 3.2, we have
\[
Ax(\bar{y}, \bar{z}) - b = 0.
\]
These imply that \(\bar{x} \in X^*, \ \bar{z} \in X^*\), which is a contradiction. ■

**Corollary 3.4** Suppose \(p > -\gamma\). Then for any \(\epsilon > 0\), there exists a \(\delta(\epsilon) > 0\) such that if
\[
\|Ax(y^+, z) - b\| < \delta(\epsilon)
\]
then
\[
\|x(y^+, z) - x(z)\| < \epsilon, \text{ for any } z \in P.
\]
Proposition 3.5 Let

to make use of the Hoffman bound, which is given below.

implying

\[ x(\bar{x}, \bar{z}) - x(\bar{z}) > 0. \]

Notice that \( x(\bar{y}, \bar{z}) \) is defined by \( (2.6) \) and we have

\[ Ax(\bar{y}, \bar{z}) - b = 0, \]

which is because of the continuity of \( x(\cdot, \cdot) \). It follows from the KKT condition for \( (2.6) \) at \( z_k \) that \( \bar{y} \) (the limit of \( y_k^+ \)) is the optimal dual multiplier for the problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) + \frac{p}{2} \| x - \bar{z} \|^2 \\
\text{subject to} & \quad Ax = b, \; x \in P,
\end{align*}
\]

implying \( x(\bar{y}, \bar{z}) = x(\bar{z}) \). This is a contradiction.

\[ \]

Proof Again we prove by contradiction. Suppose the contrary so that there exists a sequence \((y_k^+, z_k) \to (\bar{y}, \bar{z})\) such that

\[ \|Ax(y_k^+, z_k) - b\| \to 0 \]

and

\[ \|x(\bar{y}, \bar{z}) - x(\bar{z})\| > 0. \]

Next we develop some primal and dual error bounds. To prove the dual error bound, we need to make use of the Hoffman bound, which is given below.

**Proposition 3.5** Let \( A \in \mathbb{R}^{m \times n} \), \( C \in \mathbb{R}^{k \times n} \) and \( b \in \mathbb{R}^m \), \( d \in \mathbb{R}^k \), then the distance from a point \( \bar{x} \in \mathbb{R}^n \) to the set:

\[ S = \{ x \mid Ax \leq b, Cx = d \} \]

is bounded by:

\[ \text{dist}(\bar{x}, S)^2 \leq \theta^2 (\| (Ax - b)_+ \|^2 + \| C\bar{x} - d \|^2), \]

where \( (\cdot)_+ \) means the projection to the nonegative orthant and \( \theta \) is a positive constant depending on \( A \) and \( C \) only.

**Lemma 3.6 (Error Bounds)** Suppose \( p > -\gamma \), \( \rho > 0 \) are fixed. Then there exist positive constants \( \sigma_1, \ldots, \sigma_5, \sigma_4 > 0 \) (independent of \( y_t^{t+1} \) and \( z_t \)) such that the following error bounds hold:

\[
\begin{align*}
\| x^{t+1} - x^t \| & \geq \sigma_1 \| x^t - x(y_t^{t+1}, z_t^t) \|, \\
\| x^{t+1} - x^t \| & \geq \sigma_2 \| x^{t+1} - x(y_t^{t+1}, z_t^t) \|, \\
\| y - y^t \| & \geq \sigma_3 \| x(y, z) - x(y^t, z^t) \|, \\
\| z^t - z^{t+1} \| & \geq \sigma_4 \| x(z^t) - x(z_t^{t+1}) \|, \\
\| z^t - z^{t+1} \| & \geq \sigma_4 \| x(y_t^{t+1}, z^t) - x(y_t^{t+1}, z_t^{t+1}) \|,
\end{align*}
\]

where \( \sigma_1 = c \gamma_K = c(p + \gamma) \), \( \sigma_2 = \sigma_1/(1 + \sigma_1) \), \( \sigma_3 = \gamma_K / \sigma = (\gamma + p + \rho \sigma^2) / p \), \( \sigma_4 = \gamma_K / p = (\gamma + p + \rho \sigma^2) / p \) and \( \sigma_4 = (\gamma + p) / p \). Furthermore, suppose Assumption \( (2.7) \) holds, then there exist positive scalars \( \Delta, \sigma_5 \) such that

\[ \text{dist}(y, Y^*(z)) \leq \sigma_5 \| Ax(y, z) - b \|, \]

if \( \| Ax(y, z) - b \| \leq \Delta \) and \( \text{dist}(z, X^*) \leq \Delta \), where \( Y^*(z) \) denotes the solution set of dual multipliers for \( (2.7) \).
Proof We first prove (3.4). By the definition of $K(x,y;z)$ (cf. (2.4)) and Assumption 2.2(b), $\nabla K(x,z^t;y^{t+1})$ is Lipschitz continuous in $x$

$$\|\nabla_x K(x^t,z^t,y^{t+1}) - \nabla_x K(x(y^{t+1},z^t), y^{t+1})\| \leq (p + L + \rho \sigma^2)\|x^t - x(y^{t+1},z^t)\|.$$ 

So the Lipschitz constant for $c\nabla_x K$ is $c(L + p + \rho \sigma^2)$, where $\sigma$ is the spectral norm of $A$. Since $\gamma_K = p + \gamma > 0$, it follows that $cK(x,z^t; y^{t+1})$ is strongly convex in $x$ with modulus $c(p + \gamma)$, so from [18] that the following global error bound holds

$$\|x - [x - c\nabla K(x,z^t; y^{t+1})]_+\| \geq \sigma_1\|x - x(y^{t+1}, z^t)\|, \quad \forall x \in \mathbb{R}^n,$$  

(3.10)

where $\sigma_1 = c\gamma_K$. Specializing (3.10) at $x = x^t$ and noticing that $[x^t - c\nabla K(x^t,z^t; y^{t+1})]_+ = x^{t+1}$, we obtain (3.4). By the triangular inequality, we further obtain

$$\|x^{t+1} - x(y^{t+1},z^t)\| \leq \|x^t - x^{t+1}\| + \|x^t - x(y^{t+1},z^t)\|$$

$$\leq \|x^t - x^{t+1}\| + \frac{1}{\sigma_1}\|x^t - x^{t+1}\|$$

$$\leq \left(1 + \frac{1}{\sigma_1}\right)\|x^t - x^{t+1}\|.$$ 

This shows that (3.5) holds with $\sigma_2 = \sigma_1/(1 + \sigma_1)$.

Note that we have proved (3.7) in Lemma 3.2 and here we provide another proof based on (3.4). Define

$$f(x;z) = f(x) + \frac{p}{2}\|x - z\|^2, \quad \psi(x;z) = P_F [x - (L + p)^{-1}\nabla f(x;z)],$$

where $F := \{x \mid Ax = b, x \in P\}$ denotes the feasible set of (1.1) and $P_F$ denotes the projection onto $F$. Notice that $\nabla f(x;z)$ is Lipschitz continuous with modulus $(L + p)$. Then we have

$$\|x(z^{t+1}) - x(z^t)\| \stackrel{(i)}{\leq} \frac{L + p}{\gamma + p}\|\psi(x(z^t); z^{t+1}) - x(z^t)\|$$

$$\stackrel{(ii)}{=} \frac{L + p}{\gamma + p}\|P_F [x(z^t) - (p + L)^{-1}\nabla f(x(z^t); z^{t+1})]

- P_F [x(z^t) - (p + L)^{-1}\nabla f(x(z^t); z^t)]\|$$

$$\leq \frac{L + p}{\gamma + p}\frac{1}{p + L}\|\nabla f(x(z^t); z^{t+1}) - \nabla f(x(z^t); z^t)\|$$

$$= \frac{L + p}{\gamma + p}\|z^t - z^{t+1}\|$$

$$= \frac{p}{\gamma + p}\|z^t - z^{t+1}\|,$$

where step (i) follows from a similar argument for (3.4) with the following correspondences

$$x(z^t) \leftrightarrow x^t, \quad x^{t+1} \leftrightarrow \psi(x(z^t); z^{t+1}), \quad x(z^{t+1}) \leftrightarrow x(y^{t+1}, z^t),$$

$$c \leftrightarrow (L + p)^{-1}, \quad K(x,z;y) \leftrightarrow f(x;z), \quad \sigma_1 \leftrightarrow \frac{\gamma + p}{L + p};$$

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The optimal solution of (2.7) is given by $x(z^*)$ where $y(z^*)$ holds. Now for any $y,z \in I$, let

$$x(z^*) = P_F \left[ x(z^*) - (p + L)^{-1} \nabla f(x(z^*); z^*) \right];$$

and step (iii) follows from the nonexpansiveness of the projection operator. The last two equality steps follow direct calculation. This proves (3.7).

The proof of (3.6) and (3.8) is almost the same as that for (3.7), and is therefore skipped.

It remains to prove the dual error bound (3.9). To this end, we first write down, for any $z$, the optimality conditions for the strongly convex proximal optimization problem (2.7) (note that $\gamma_K = p + \gamma > 0$) as follows:

$$\nabla f(x(z)) + p(x(z) - z) + A^Ty^*(z) - \mu(z) + \nu(z) = 0,$$

$$Ax(z) = b,$$

$$\mu_i(z)(x_i - \ell_i) = 0, \quad i = 1, \cdots, n,$$

$$\nu_i(z)(x_i - u_i) = 0, \quad i = 1, \cdots, n,$$

$$\mu_i(z), \nu_i(z) \geq 0, \quad i = 1, \cdots, n.$$ (3.11)

where $y^*(z)$, $\mu(z)$, $\nu(z)$ are the Lagrangian multipliers. If $z \in X^*$, by strong convexity, the unique optimal solution of (2.7) is given by $x(z) = z$. Moreover, for $z \in X^*$, the strict complementarity condition

$$\mu_i(z) = 0 \implies x_i(z) = z_i > \ell_i, \quad i = 1, \cdots, n,$$

$$\nu_i(z) = 0 \implies x_i(z) = z_i < u_i, \quad i = 1, \cdots, n,$$

holds. Now for any $y, z$, let $I(y, z)$, $I(z)$ denote the set of inactive inequality constraints (the inequality constraint holds strictly) in problem (2.6) at $x(y, z)$ and (2.7) at $x(z)$ respectively. We prove that there exists a $\Delta > 0$, such that

$$I(y, z) = I(z), \text{ if } \|Ax(y, z) - b\| \leq \Delta \text{ and dist}(z, X^*) \leq \Delta.$$ We prove it by contradiction. Suppose the contrary, then there is a sequence $\Delta_k \rightarrow 0$ and sequences $\{y_k\}, \{z_k\}$ with

$$\|Ax(y_k, z_k) - b\| \leq \Delta_k, \quad \text{dist}(z_k, X^*) \leq \Delta_k,$$

such that $I(y_k, z_k) \neq I(z_k)$, for all $k$. Note that $y^*(z_k)$ is the optimal dual solution for the problem (2.7) with $z = z_k$, so we have $x(y^*(z_k), z_k) = x(z_k)$ and $\|Ax(y^*(z_k), z_k) - b\| = 0$, for all $k$. By Lemma 3.1, we know that $\{y_k\}, \{y^*(z_k)\}$ are bounded. So we can assume that (passing to a subsequence if necessary)

$$y_k \rightarrow y^*, \quad \text{for some } y^* \in \mathbb{R}^m,$$

$$y^*(z_k) \rightarrow \bar{y}, \quad \text{for some } \bar{y} \in \mathbb{R}^m$$

and

$$z_k \rightarrow z^*, \quad \text{for some } z^* \in X^*.$$
According to Lemma 3.2, we have \( x(z^k) \to x(z^*) \) and \( x(y_k, z_k) \to x(y^*, z^*) \). Since for any \( i \) at most one of \( \mu_i \) and \( \nu_i \) can be nonzero, \( \{\mu_i(z_k)\}_{k=1}^{\infty}, \{\nu_i(z_k)\}_{k=1}^{\infty} \) are bounded by (3.11). Hence, passing to a subsequence if necessary, we can assume

\[
\mu_i(z_k) \to \mu^*, \quad \nu_i(z_k) \to \nu^*, \quad \text{for all } i.
\]

Hence, \((x^*(z^*), y^*, \mu^*, \nu^*)\) is a solution to the KKT system (3.11). Using the strict complementarity property at \( z^* \in X^* \), we have \( \mu_i, \nu_i \neq 0, i \in I(z^*) \). When \( k \to \infty \), we have

\[
x_i(z_k) \in (\ell_i, u_i), \quad i \in I(z^*), \\
\mu_i > 0, \quad i \notin I(z^*) \quad \text{and} \quad x_i(z_k) = \ell_i, \\
\nu_i > 0, \quad i \notin I(z^*) \quad \text{and} \quad x_i(z_k) = u_i.
\]

This means that \( I(z_k) = I(z^*) \). Similarly, we can consider the KKT conditions for (2.6) and can show that for large enough \( k \) there holds \( x(y_k, z_k) \to x(y^*, z^*) = x(z^*) \), and \( I(y_k, z_k) = I(y^*, z^*) = I(z^*) \). This implies \( I(y_k, z_k) = I(z^*) = I(z_k) \) for large \( k \), a contradiction. So next we assume that \( I(y, z) = I(z) \). Also define \( A_1 = \{i \mid x_i(z) = x_i(y, z) = \ell_i\} \) and \( A_2 = \{i \mid x_i(z) = x_i(y, z) = u_i\} \). Recall the optimization problem (2.6) for function \( K \):

\[
\begin{align*}
\text{minimize} & \quad f(x) + \frac{\rho}{2} \|x - y\|^2 + y^T(Ax - b) + \frac{\rho}{2} \|Ax - b\|^2 \\
\text{subject to} & \quad \ell_i \leq x_i \leq u_i, \quad i = 1, \ldots, n,
\end{align*}
\]

and its KKT conditions:

\[
\begin{align*}
\nabla f(x(y, z)) + p(x(y, z) - z) + A^T y + \rho A^T(Ax(y, z) - b) - \mu(y, z) + \nu(y, z) &= 0, \quad \mu_i(y, z) \geq 0, \quad \ell_i - x_i(y, z) = 0, \quad i \in A_1, \\
\nu_i(y, z) \geq 0, \quad x_i(y, z) - u_i = 0, \quad i \in A_2, \quad (3.13) \\
\mu_i(y, z) &= 0, \quad \nu_i(y, z) = 0, \quad i \notin A_1 \cup A_2, \\
\mu_i(z), \quad \nu_i(z) &\geq 0, \quad x_i(z) \in [\ell_i, u_i], \quad i \in I(z).
\end{align*}
\]

Notice that the optimality conditions (3.11) for the convex proximal problem (2.7) can be rewritten (using the information about the active and inactive sets) as:

\[
\begin{align*}
\nabla f(x(z)) + p(x(z) - z) + A^T y^*(z) - \mu(z) + \nu(z) &= 0, \quad Ax(z) - b = 0, \\
\mu_i(z) \geq 0, \quad \ell_i - x_i(z) = 0, \quad i \in A_1, \\
\nu_i(z) \geq 0, \quad (x_i(z) - u_i) = 0, \quad i \in A_2, \\
\mu_i(z), \quad \nu_i(z) &\geq 0, \quad x_i(z) \in [\ell_i, u_i], \quad i \in I(z).
\end{align*}
\]

The KKT system (3.14) is linear in the variables \((x(z), y(z), \mu(z), \nu(z))\). By (3.13), the vector \((x(y, z), y, \mu(y, z), \nu(y, z))\) satisfies (3.11) approximately. Using Hoffman bound (Proposition 3.5) to (3.14) at the point \((x(y, z), y, \mu(y, z), \nu(y, z))\), we obtain

\[
\begin{align*}
dist(y, Y^*(z))^2 &\leq \min_{(x(z), y(z), \mu(z), \nu(z)) \text{ satisfying } (3.14)} \|y - y(z)\|^2 + \|(x(y, z), \mu(y, z), \nu(y, z)) - (x(z), \mu(z), \nu(z))\|^2 \\
&\leq 2\theta^2 \left( \|\nabla f(x(y, z)) - \nabla f(x(z)) + p(x(y, z) - x(z))\|^2 \\
&\quad + \rho^2 \|A^T(Ax(y, z) - b)\|^2 + \|Ax(y, z) - b\|^2 \right) \\
&\leq 2\theta^2 ((L + p)^2 \|x(y, z) - x(z)\|^2 + (1 + \rho^2 \sigma^2) \|Ax(y, z) - b\|^2),
\end{align*}
\]

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Note that (3.16) is a quadratic inequality in the form $u\sigma$ for some $f$ for any $t$, if $c < 1/L_K$,

$$K(x^t, z^t; y^t) - K(x^{t+1}, z^{t+1}; y^{t+1}) \geq \frac{1}{2c} \|x^t - x^{t+1}\|^2 + \frac{p}{2\beta} \|z^t - z^{t+1}\|^2 - \alpha\|Ax^t - b\|^2.$$ 

**Proof** First, we have the trivial equality:

$$K(x^t, z^t; y^t) - K(x^t, z^t; y^{t+1}) = -\alpha\|Ax^t - b\|^2.$$
Next, notice that updating $x$ is a standard gradient projection, hence,

$$K(x^t, z^t; y^{t+1}) - K(x^{t+1}, z^t; y^{t+1}) \geq \frac{1}{2c}||x^t - x^{t+1}||^2.$$  \hspace{1cm} (3.17)

Moreover, recall that in Algorithm 2, $z^{t+1} = z^t + \beta(x^{t+1} - z^t)$, we have

$$K(x^{t+1}, z^t; y^{t+1}) - K(x^{t+1}, z^{t+1}; y^{t+1}) = \frac{p}{2} ||x^{t+1} - z^t||^2 - ||x^{t+1} - z^{t+1}||^2$$

$$= \frac{p}{2}(z^{t+1} - z^t)^T((x^{t+1} - z^t) + (x^{t+1} - z^{t+1}))$$

$$= \frac{p}{2}(2/\beta - 1)||z^t - z^{t+1}||^2$$

$$\geq \frac{p}{2\beta}||z^t - z^{t+1}||^2,$$  \hspace{1cm} (3.18)

for $\beta \leq 1$. Combining the above three inequalities yields the desired result.

\[\square\]

**Lemma 3.8 (Dual Ascent)** For any $t$, we have

$$d(y^{t+1}, z^{t+1}) - d(y^t, z^t) \geq \alpha(Ax^t - b)^T(Ax(y^{t+1}, z^t) - b) + \frac{p}{2}(z^{t+1} - z^t)^T(z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})).$$

**Proof** First, recall that

$$K(x, z; y) = f(x) + \langle y, Ax - b \rangle + \frac{p}{2}||Ax - b||^2 + \frac{p}{2}||x - z||^2$$

so we have

$$d(y^{t+1}, z^t) - d(y^t, z^t) = K(x(y^{t+1}, z^t), z^t; y^{t+1}) - K(x(y^t, z^t), z^t; y^t)$$

$$\geq K(x(y^{t+1}, z^t), z^t; y^{t+1}) - K(x(y^t, z^t), z^t; y^t)$$

$$= \langle y^{t+1} - y^t, Ax(y^{t+1}, z^t) - b \rangle,$$

$$\geq \alpha(Ax^t - b)^T(Ax(y^{t+1}, z^t) - b),$$

where the inequality is because $x(y^t, z^t) = \arg \min_x K(x, z^t; y^t)$. Next, using the same technique, we have

$$d(y^{t+1}, z^{t+1}) - d(y^{t+1}, z^t) = K(x(y^{t+1}, z^{t+1}), z^{t+1}; y^{t+1}) - K(x(y^{t+1}, z^t), z^t; y^{t+1})$$

$$\geq K(x(y^{t+1}, z^{t+1}), z^{t+1}; y^{t+1}) - K(x(y^{t+1}, z^{t+1}), z^t; y^{t+1})$$

$$= \frac{p}{2}||x(y^{t+1}, z^{t+1} - z^t||^2 - ||x(y^{t+1}, z^{t+1} - z^t||^2$$

$$= \frac{p}{2}(z^{t+1} - z^t)^T(z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})).$$ \hspace{1cm} (3.19)

Combining these, we get the desired result.

\[\square\]

**Lemma 3.9 (Proximal Descent)** For any $t \geq 0$, there holds

$$P(z^{t+1}) - P(z^t) \leq p(z^{t+1} - z^t)^T(z^t - x(z^t)) + \frac{p\tilde{L}}{2}||z^t - z^{t+1}||^2,$$ \hspace{1cm} (3.20)

where $\tilde{L} = \tilde{\sigma}_4^{-1} + 1$.
Proof First, using Danskin’s theorem in convex analysis [19], we have:
\[ \nabla P(z^t) = p(z^t - x(z^t)), \]
where \( P(z) \) is defined by (1.1). So it suffices to prove that
\[ \|\nabla P(z^t) - \nabla P(z^{t+1})\| \leq (\hat{\sigma}_4^{-1} + 1)\|z^t - z^{t+1}\|. \]
But this is a direct corollary of the error bound (3.6) in Lemma 3.6. The proof is complete.

The three terms \( K(x^t, z^t; y^t), -d(y^t, z^t) \) and \( P(z^t) \) individually do not need to decrease after each iteration; however, some weighted sum of them does! This is the main idea of the proof of Theorem 2.4. We will see the details in the next subsection.

3.2 Proof of Theorem 2.4

Now we are ready to prove the main theorem, i.e. the global convergence of the Algorithm 2. To establish the convergence of Algorithm 2, we construct a potential function which decreases sufficiently after each iteration. This potential function is a linear combination of the primal, dual and proximal terms considered in the previous subsection. Specifically, we will prove that the potential function
\[ \phi^t = \phi(x^t, z^t; y^t) = K(x^t, z^t; y^t) - 2d(y^t, z^t) + 2P(z^t) \]
decreases sufficiently after each iteration for sufficiently small \( \alpha \) and \( \beta \), where the functions \( K, d \) and \( P \) are defined in (2.4)-(2.7). Note that \( x^t \in P \) and \( z^t \in P \) for all \( t \) (see the definition of the Algorithm 2), so \( P(z^t) \) is bounded from below. Moreover, it follows from (2.9) that
\[ \phi^t = (K(x^t, z^t; y^t) - d(y^t, z^t)) + (P(z^t) - d(y^t, z^t)) + P(z^t) \tag{3.21} \]
is also bounded below.

Proof Using the three descent lemmas in Subsection 3.1.2, we get
\[
\begin{align*}
\phi^t - \phi^{t+1} & \geq \left( \frac{1}{2c} \|x^{t+1} - x^t\|^2 - \alpha \|Ax^t - b\|^2 + \frac{p}{2\beta} \|z^t - z^{t+1}\|^2 \right) \\
& \quad + 2 \left( \alpha (Ax^t - b)^T (Ax(y^{t+1}, z^t) - b) + p(z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})) \right) \\
& \quad + 2 \left( p(z^{t+1} - z^t)^T (x(z^t) - z^t) - \frac{p\bar{L}}{2} \|z^{t+1} - z^t\|^2 \right) \\
& = \left( \frac{1}{2c} \|x^{t+1} - x^t\|^2 - \alpha \|Ax^t - b\|^2 + \frac{p}{2\beta} \|z^t - z^{t+1}\|^2 \right) + 2\alpha (Ax^t - b)^T (Ax(y^{t+1}, z^t) - b) \\
& \quad + p(z^{t+1} - z^t)^T ((z^{t+1} - z^t) - 2(x(y^{t+1}, z^{t+1}) - x(z^t))) - \frac{p\bar{L}}{2} \|z^t - z^{t+1}\|^2 \\
& = \left( \frac{1}{2c} \|x^{t+1} - x^t\|^2 - \alpha \|Ax^t - b\|^2 + \frac{p}{2\beta} \|z^t - z^{t+1}\|^2 \right) + 2\alpha (Ax^t - b)^T (Ax(y^{t+1}, z^t) - b) \\
& \quad + p(z^{t+1} - z^t)^T ((z^{t+1} - z^t) - 2(x(y^{t+1}, z^{t+1}) - x(y^t, z^t))) - \frac{p\bar{L}}{2} \|z^t - z^{t+1}\|^2.
\end{align*}
\tag{3.22}
Let $\lambda$ be an arbitrary positive scalar, and by the fact that
$$\|(z^{t+1} - z^t)/\lambda - \lambda(x(y^{t+1}, z^t) - x(z^t))\|^2 \geq 0,$$
we have
$$-2(z^{t+1} - z^t)^T(x(y^{t+1}, z^t) - x(z^t)) \geq -\|z^t - z^{t+1}\|^2/\lambda - \lambda\|x(y^{t+1}, z^t) - x(z^t)\|^2.$$

Using Cauchy-Schwarz inequality and the error bound (3.8) in Lemma 3.6, we have
$$-2(z^{t+1} - z^t)^T(x(y^{t+1}, z^{t+1}) - x(y^{t+1}, z^t)) \geq -\|z^t - z^{t+1}\|\|x(y^{t+1}, z^{t+1}) - x(y^{t+1}, z^t)\|
\geq -\frac{1}{\sigma_4}\|z^t - z^{t+1}\|^2.$$

Substituting these two inequalities into (3.22), we have
$$\phi^t - \phi^{t+1} \geq \frac{1}{2c}\|x^{t+1} - x^t\|^2 - (\alpha\|Ax^t - b\|^2 - 2\alpha(Ax^t - b)^T(Ax(y^{t+1}, z^t) - b) + \alpha\|Ax(y^{t+1}, z^t) - b\|^2)
+ \alpha\|Ax(y^{t+1}, z^t) - b\|^2 + \left(\frac{p}{2\beta} + p - \frac{p}{\sigma_4} - \frac{p}{\lambda} - \frac{pL}{2}\right)\|z^t - z^{t+1}\|^2
-p\lambda\|x(y^{t+1}, z^t) - x(z^t)\|^2.$$

By completing the square, we further obtain
$$\phi^t - \phi^{t+1} \geq \frac{1}{2c}\|x^{t+1} - x^t\|^2 - \alpha\|A(x(y^{t+1}, z^t) - x^t)\|^2 + \alpha\|Ax(y^{t+1}, z^t) - b\|^2
+ \left(\frac{p}{2\beta} + p - \frac{p}{\sigma_4} - \frac{p}{\lambda} - \frac{pL}{2}\right)\|z^t - z^{t+1}\|^2
-p\lambda\|x(y^{t+1}, z^t) - x(z^t)\|^2$$
$$\geq \left(\frac{1}{2c} - \frac{\alpha\sigma_4^2}{\sigma_4^2}\right)\|x^t - x^{t+1}\|^2 + \alpha\|Ax(y^{t+1}, z^t) - b\|^2
+ \left(\frac{p}{2\beta} + p - \frac{p}{\sigma_4} - \frac{p}{\lambda} - \frac{pL}{2}\right)\|z^t - z^{t+1}\|^2
-p\lambda\|x(y^{t+1}, z^t) - x(z^t)\|^2, \quad (3.23)$$

where $\sigma$ is the spectral norm of $A$, $\lambda$ is any positive scalar, and the last step is due to the error bound (3.4) in Lemma 3.6. Let $\lambda = D\beta$ for some sufficiently large $D$ (for example, $D > 6$), and set $\beta$ sufficiently small ($\beta' \leq \beta$) for some constant $\beta' > 0$, such that
$$\frac{p}{2\beta} + p - \frac{p}{\sigma_4} - \frac{p}{\lambda} - \frac{pL}{2} \geq \frac{p}{3\beta}.$$

Therefore, if we choose $\alpha < \frac{\sigma_4^2}{4c\sigma^2}$, then it follows from (3.23), that
$$\phi^t - \phi^{t+1} \geq \frac{1}{4c}\|x^t - x^{t+1}\|^2 + \alpha\|Ax(y^{t+1}, z^t) - b\|^2
+ \left(\frac{p}{3\beta} + \frac{p}{3\beta} - \frac{p}{\sigma_4} - \frac{p}{\lambda} - \frac{pL}{2}\right)\|z^t - z^{t+1}\|^2
-p\lambda\|x(y^{t+1}, z^t) - x(z^t)\|^2. \quad (3.24)$$
It remains to bound the term \( pD\beta \|x(y^{t+1}, z^t) - x(z^t)\|^2 \) in the above expression. The main proof idea is as follows. In view of the dual error bound (3.9), when the residuals are sufficiently small, we can use the dual residual \( Ax(y^{t+1}, z^t) - b \) to bound dist\((y^{t+1}, Y^*(z^t))\) and then further use the error bound (3.6) to bound \( ||x(y^{t+1}, z^t) - x(z^t)||\). When some residual is not too small, we will make use of the compactness of the feasible set to bound the term \( ||x(y^{t+1}, z^t) - x(z^t)||^2 \).

Let us define

\[
M = \max_{x_1, x_2 \in P} \|x_1 - x_2\|, \quad \zeta = \min\{\Delta, \delta(\Delta/\sqrt{6D})\}
\]

and set

\[
\beta < \min\left\{\beta', \frac{\zeta^2}{8c\rho D M^2}, \frac{\zeta^2 \alpha}{2p\rho D M^2}, \frac{\alpha \sigma_3^2}{2pD\sigma_3^2}\right\},
\]

where \( \Delta \) is defined in Lemma 3.6 and \( \delta(\cdot) \) is defined in Corollary 3.3 and Corollary 3.4. We also define the following three conditions:

\[
\|x^t - x^{t+1}\|^2 \leq 8c\rho D M^2 \beta, \quad (3.27)
\]

\[
\|Ax(y^{t+1}, z^t) - b\|^2 \leq \frac{2p\rho D M^2}{\alpha} \beta, \quad (3.28)
\]

\[
\|x^{t+1} - z^t\|^2 = ||(z^t - z^{t+1})/\beta||^2 \leq 6D\|x(y^{t+1}, z^t) - x(z^t)\|^2. \quad (3.29)
\]

We now consider two cases.

**Case 1.** Conditions (3.27)-(3.29) hold. In this case, it follows from (3.25)-(3.26) that

\[
\|x^t - x^{t+1}\| \leq \zeta = \min\{\Delta, \delta(\Delta/\sqrt{6D})\} \leq \Delta,
\]

\[
\|Ax(y^{t+1}, z^t) - b\| \leq \zeta = \min\{\Delta, \delta(\Delta/\sqrt{6D})\} \leq \Delta,
\]

which further implies

\[
\|x^{t+1} - z^t\| = ||(z^t - z^{t+1})/\beta|| \leq \sqrt{6D}\|x(y^{t+1}, z^t) - x(z^t)\| \leq \sqrt{6D}\frac{\Delta}{\sqrt{6D}} = \Delta,
\]

where the last inequality follows from Corollary 3.4. Therefore, the error bounds (3.6) and (3.9) in Lemma 3.6 hold and we have

\[
pD\beta \|x(y^{t+1}, z^t) - x(z^t)\|^2 \leq pD\beta \sigma_3^{-2} \cdot \text{dist}(y^{t+1}, Y^*(z^t))^2 \leq \frac{pD\beta \sigma_3^{-2} \sigma_5^2}{\alpha} \|Ax(y^{t+1}, z^t) - b\|^2 \leq \frac{\alpha}{2} \|Ax(y^{t+1}, z^t) - b\|^2,
\]

where the last step follows from (3.26). It then follows from (3.24) that

\[
\phi^t - \phi^{t+1} \geq \frac{1}{4c} \|x^t - x^{t+1}\|^2 + \frac{\alpha}{2} \|Ax(y^{t+1}, z^t) - b\|^2 + \frac{p}{3\beta} \|z^t - z^{t+1}\|^2. \quad (3.30)
\]

**Case 2.** One of the conditions (3.27)-(3.29) is violated. Then we consider three subcases:
Case 2.1. \( \|x^t - x^{t+1}\|^2 \geq 8cpD\beta M^2 \). In this case, we have
\[
\frac{1}{4c}\|x^t - x^{t+1}\|^2 - pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
\geq \frac{1}{8c}\|x^t - x^{t+1}\|^2 + \frac{1}{8c}8cpD\beta M^2 - pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
\geq \frac{1}{8c}\|x^t - x^{t+1}\|^2 + pD\beta M^2 - pD\beta M^2 \\
= \frac{1}{8c}\|x^t - x^{t+1}\|^2,
\]
where the second step is due to (3.25).

Case 2.2. \( \|Ax(y^{t+1}, z^t) - b\|^2 \geq \frac{2pD\beta}{\alpha} M^2 \). In this case, we use this condition and (3.25) to obtain
\[
\alpha\|Ax(y^{t+1}, z^t) - b\|^2 - pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
\geq \frac{\alpha}{2}\|Ax(y^{t+1}, z^t) - b\|^2 + \frac{\alpha}{2} \cdot \frac{2pD\beta}{\alpha} M^2 - pD\beta M^2 \\
= \frac{\alpha}{2}\|Ax(y^{t+1}, z^t) - b\|^2.
\]

Case 2.3. \( \|(z^t - z^{t+1})/\beta\|^2 \geq 6D\|x(x^{t+1}, z^t) - x(z^t)\|^2 \). In this case, we have
\[
\frac{p}{3\beta}\|z^t - z^{t+1}\|^2 - pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
\geq \frac{p}{3\beta}\|z^t - z^{t+1}\|^2 - pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
\geq \frac{p}{6\beta}\|z^t - z^{t+1}\|^2 + \frac{p}{6\beta}6\beta D\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
- pD\beta\|x(x^{t+1}, z^t) - x(z^t)\|^2 \\
= \frac{p}{6\beta}\|z^t - z^{t+1}\|^2.
\]

Considering (3.24), we have in all three subcases:
\[
\phi^t - \phi^{t+1} \geq \frac{1}{8c}\|x^t - x^{t+1}\|^2 + \frac{\alpha}{2}\|Ax(y^{t+1}, z^t) - b\|^2 + \frac{p}{6\beta}\|z^t - z^{t+1}\|^2.
\]
Combining this with (3.30) yields
\[
\phi^t - \phi^{t+1} \geq \frac{1}{8c}\|x^t - x^{t+1}\|^2 + \frac{\alpha}{2}\|Ax(y^{t+1}, z^t) - b\|^2 + \frac{p}{6\beta}\|z^t - z^{t+1}\|^2, \quad \forall t \geq 0. \tag{3.31}
\]

Since \( \phi^t \) is bounded below, we must have
\[
\max\{\|x^{t+1} - x^t\|, \|Ax(y^{t+1}, z^t) - b\|, \|z^t - x(y^{t+1}, z^t)\|\} \to 0.
\]
This together with Corollary 3.3 shows that the KKT condition for (1.1) is satisfied in the limit. This completes the proof.

Theorem 2.4 establishes the global convergence of Algorithm 2 to a stationary solution. However, it does not address the question of convergence rate. The latter is considered in the next section for the special case when the objective function is (nonconvex) quadratic.
4 Linear Convergence for Quadratic Programming

In this section, we consider a nonconvex quadratic program (QP), which is a special case of (1.1) with \( f(x) \) being a quadratic function

\[
    f(x) = \frac{1}{2} x^T Q x + r^T x. \tag{4.1}
\]

We will strengthen the convergence result of Theorem 2.4 in this case by showing that Algorithm 2 converges linearly to a stationary point of the nonconvex QP problem.

By Theorem 2.4, we have \( \text{dist}(x^t, X^*) \to 0 \), \( \text{dist}(z^t, X^*) \to 0 \) and \( \|x^{t+1} - x^t\| \to 0 \) as \( t \to \infty \). Since \( X^* \) is the union a finite number of polyhedral sets, it follows that the connected components of \( X^* \) are properly separated in the sense that there is a positive distance between each pair of distinct connected components of \( X^* \). As a result, the sequences \( \{x^t\} \) and \( \{z^t\} \) will both converge to one unique connected component of \( X^* \). Moreover, it is known [17] that for a quadratic programming problem, the objective function value \( f(x) \) is constant on each of the connected component of \( X^* \). Let \( f^* \) denote the value of \( f(x) \) over the connected component of \( X^* \) to which \( x^t \) (and \( z^t \)) converges. Then \( f(x^t) \to f^* \) as \( t \to \infty \). We summarize the above analysis as follows.

Claim 4.1 Assume the parameters of Algorithm 2 are chosen to guarantee its convergence (cf. Theorem 2.4). We have the following.

1. The quadratic cost function \( f \) is constant on each of the connected components of \( X^* \).
2. The two sequences \( \{x^t\} \), \( \{z^t\} \) converge to a same connected component of \( X^* \).
3. Let \( z^t = \arg\min_{z \in X^*} \text{dist}(z^t, X^*) \) and \( x^* \) is any limit point of \( \{x^t\} \), then \( f(x^*) = f(z^t) = f^* \), where \( f^* \) is a constant, for all sufficiently large \( t \).

Denote

\[
    \Delta_p(z) = K(x^t, z; y^t) - d(y^t, z),
    \Delta_d(z) = P(z) - d(y^t, z),
    \Delta_{prx}(z) = P(z) - f^*,
    z^t = \arg\min_{x \in X^*} \|x - z^t\|. \tag{4.2}
\]

To prove the linear convergence, we make use of some “cost-to-go” estimates from [11].

Lemma 4.2 There exist constants \( \tau_1, \tau_2, \tau_3 > 0 \), such that

\[
    K(x^{t+1}, z^t; y^{t+1}) - d(y^{t+1}, z^t) \leq \tau_1 \|x^t - x^{t+1}\|^2, \tag{4.3}
    P(z^t) - d(y^{t+1}, z^t) \leq \tau_2 \|Ax(y^{t+1}, z^t) - b\|^2, \tag{4.4}
    P(z^t) - f^* \leq \tau_3 \|z^t - x(z^t)\|^2. \tag{4.5}
\]

Proof The proof of (4.3) and (4.4) is simply to combine Lemma 3.1 of [11] with (3.4), (3.9) in Lemma 3.6. Specifically, we only need to replace \( L(x, y), d(y), p^* \) of [11], Lemma 3.1] by \( K(x, z; y), d(y, z), P(z) \) respectively with \( z \) fixed, since the error estimates of [11] are independent of the linear term in \( K(x, z; y) \). To prove the estimate (4.5), we first notice that \( f \) has a Lipschitz continuous
gradient and that the classic proximal algorithm belongs to the class of approximate gradient projection algorithm (see [17] Theorem 3.3), namely, we have

\[ x(z^t) = [z^t - \nabla P(z^t) + \Theta(t)]_+ , \]

where \( \Theta(t) \) satisfies \( \| \Theta(t) \| \leq \eta \| x(z^t) - z^t \| \) for some \( \eta > 0 \). Note that the above are similar to the inequalities (3.5) and (3.7) in the proof of the second part of Lemma 3.1 of [11]. Therefore, similar to [11] Lemma 3.1, there exists a constant \( \tau > 0 \) such that

\[ f(x(z^t)) - f(\tilde{z}^t) \leq \tau'(\| z^t - x(z^t) \|^2 + \| z^t - \tilde{z}^t \|^2), \]

where \( \tilde{z}^t \) is the projection of \( z^t \) to stationary solution set of problem (1.1). According to Claim 4.1, \( \tilde{z}^t \) is in the connected component that \( \{ x^t \}, \{ z^t \} \) converge to and \( f(\tilde{z}^t) = f^* \). From Theorem 2.1 in [17], there exists a constant \( \bar{\tau} > 0 \), such that

\[ \| z^t - \tilde{z}^t \| \leq \bar{\tau}\| z^t - x(z^t) \|. \]

Combining the above two inequalities and using the definition of \( P(z) \), we have

\[ P(z^t) - f^* = f(x(z^t)) + \frac{p}{2} \| x(z^t) - z^t \|^2 - f(\tilde{z}^t) \leq \left( (\tilde{z}^t)^2 + 1 \right) \tau' + \frac{p}{2} \| z^t - x(z^t) \|^2. \]

This completes the proof of (4.5) by setting \( \tau_3 = ((\tilde{z}^t)^2 + 1) \tau' + \frac{p}{2} \).

Since the term \( \| z^t - z^{t+1} \|^2 \) in (3.31) is not directly related to Lemma 4.2, we split it to get the term \( \| z - x(z) \|^2 \) in order to make use of the estimate (4.5). We prove the following lemma, which is a corollary of (3.31).

**Lemma 4.3** For \( t > 0 \) and \( \beta \) sufficiently small, we have

\[ \phi^t - \phi^{t+1} \geq C_1(\beta)\| x^t - x^{t+1} \|^2 + C_2(\beta)\| Ax(y^{t+1}, z^t) - b \|^2 + C_3(\beta)\| z^t - x(z^t) \|^2 + C_4(\beta)\| z^t - z^{t+1} \|^2, \]

where \( C_i(\beta) > 0, i = 1, 2, 3, 4 \) are constants depending on \( \beta \).

**Proof** We assume that \( t \) is sufficiently large such that the dual error bound (3.9) holds. According to equation (3.31), we have

\[ \phi^t - \phi^{t+1} \geq \frac{1}{8c} \| x^t - x^{t+1} \|^2 + \frac{\alpha}{2} \| Ax(y^{t+1}, z^t) - b \|^2 + \frac{p}{6} \| z^t - z^{t+1} \|^2 + \frac{p}{12} \| z^t - z^{t+1} \|^2, \]

where the last step is due to \( z^{t+1} - z^t = \beta(x^{t+1} - z^t) \). We now bound the last term in (4.6). To this end, let us split the term \( z^t - x(z^t) \) as

\[ z^t - x(z^t) = (z^t - x^{t+1}) + (x^{t+1} - x(y^{t+1}, z^t)) + (x(y^{t+1}, z^t) - x(z^t)). \]
By the convexity of the norm squared function $\| \cdot \|_2$, it follows that
\[
3(\|u\|^2 + \|v\|^2 + \|w\|^2) \geq \|u + v + w\|^2, \quad \forall \ u, v, w \in \mathbb{R}^n.
\]

Applying this inequality to the above equation, we obtain
\[
3(\|z^t - x^{t+1}\|^2 + \|x^{t+1} - x(y^{t+1}, z^t)\|^2 + \|x(y^{t+1}, z^t) - x(z^t)\|^2) \geq \|z^t - x(z^t)\|^2.
\]

It follows that
\[
\|z^t - x^{t+1}\|^2 \geq \frac{1}{3} \left( \|x(z^t) - z^t\|^2 - 3\|x(y^{t+1}, z^t) - x(z^t)\|^2 - 3\|x^{t+1} - x(y^{t+1}, z^t)\|^2 \right)
\]
\[
\geq \frac{1}{3} \left( \|x(z^t) - z^t\|^2 - 3\frac{\sigma^2}{2\sigma_3} \|Ax(y^{t+1}, z^t) - b\|^2 - \frac{3}{\sigma_2^2} \|x^t - x^{t+1}\|^2 \right),
\]

where the last step is due to the error bounds (3.5), (3.8) and (3.9) in Lemma 3.6. Substituting this into the last term of (4.6), we have
\[
\phi^t - \phi^{t+1} \geq \frac{1}{8c} \|x^t - x^{t+1}\|^2 + \frac{\alpha}{2} \|Ax(y^{t+1}, z^t) - b\|^2 + \frac{p}{12\beta} \|z^t - z^{t+1}\|^2
\]
\[
+ \frac{p^2}{12} \frac{1}{3} \left( \|x(z^t) - z^t\|^2 - 3\frac{\sigma^2}{2\sigma_3} \|Ax(y^{t+1}, z^t) - b\|^2 - \frac{3}{\sigma_2^2} \|x^t - x^{t+1}\|^2 \right)
\]
\[
\geq \left( \frac{1}{8c} - \frac{p^2}{12\sigma_2^2} \right) \|x^t - x^{t+1}\|^2 + \left( \frac{\alpha}{2} - \frac{p^2}{12\sigma_2^2} \frac{\sigma^2}{2\sigma_3} \|Ax(y^{t+1}, z^t) - b\|^2 \right)
\]
\[
+ \frac{p}{12\beta} \|z^t - z^{t+1}\|^2 + \frac{p^2}{36} \|z^t - x(z^t)\|^2.
\]

This completes the proof.

Now we can prove the linear convergence Algorithm 2 for quadratic programming.

**Theorem 4.4** After finitely many iterations of Algorithm 2, we have
\[
\Delta^t(z^t) - \Delta^{t+1}(z^{t+1}) \geq \kappa \Delta^{t+1}(z^{t+1}),
\]
where $\kappa > 0$ is a constant. Hence, $\Delta^{t+1}(z^{t+1}) < \frac{1}{1+\kappa} \Delta^t(z^t)$.

**Proof** Assume $t$ is sufficiently large such that the dual error bound (3.9) holds. To prove the linear convergence, we need to relate the gap $\Delta^t(z^t) = \Delta^t_p(z^t) + \Delta^t_d(z^t) + \Delta^t_{prx}(z^t) + \Delta^t_{prx}(z^t)$ to the estimates in Lemma 4.2.

Then using Lemmas 3.7 and 3.9 and Lemma 4.2, we have
\[
\Delta^{t+1}_p(z^t) \leq \tau_1 \|x^t - x^{t+1}\|^2, \quad (4.7)
\]
\[
\Delta^{t+1}_d(z^t) \leq \tau_2 \|Ax(y^{t+1}, z^t) - b\|^2, \quad (4.8)
\]
\[
\Delta^{t+1}_{prx}(z^t) \leq \tau_3 \|z^t - x(z^t)\|^2. \quad (4.9)
\]

Recall the inequality (3.18) in the proof of Lemma 3.7
\[
K(x^{t+1}, z^{t+1}, y^{t+1}) - K(x^{t+1}, z^t, y^{t+1}) \leq -\frac{p}{2\beta} \|z^t - z^{t+1}\|^2,
\]

and the inequality (3.19)
\[ d(y^{t+1}, z^t) - d(y^{t+1}, z^{t+1}) \leq -\frac{p}{2} (z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})). \]

Then for \( \Delta_{p}^{t+1}(z^t) \), we have
\[
\Delta_{p}^{t+1}(z^{t+1}) - \Delta_{p}^{t+1}(z^t) = (K(z^{t+1}, z^{t+1}, y^{t+1}) - K(x^{t+1}, z^t; y^{t+1})) + (d(y^{t+1}, z^t) - d(y^{t+1}, z^{t+1})) \\
\leq -\frac{p}{2\beta} \|z^t - z^{t+1}\|^2 - \frac{p}{2} (z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})).
\]

For \( \Delta_{d}^{t+1}(z^t) \), we have
\[
\Delta_{d}^{t+1}(z^{t+1}) - \Delta_{d}^{t+1}(z^t) = d(y^{t+1}, z^t) - d(y^{t+1}, z^{t+1}) + P(z^{t+1}) - P(z^t) \\
\leq -\frac{p}{2} (z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})) + P(z^{t+1}) - P(z^t).
\]

Also, it follows from the definition (4.2) that
\[
\Delta_{prx}^{t+1}(z^t) = \Delta_{prx}^{t+1}(z^t) + P(z^{t+1}) - P(z^t).
\]

Combining the above three inequalities, we have
\[
\Delta_{p}^{t+1}(z^{t+1}) + \Delta_{d}^{t+1}(z^{t+1}) + \Delta_{prx}^{t+1}(z^{t+1}) \\
\leq \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
-\frac{p}{2\beta} \|z^t - z^{t+1}\|^2 - \frac{p}{2} (z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1}))) \\
-\frac{p}{2} (z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1}))) + 2 (P(z^{t+1}) - P(z^t)) \\
= \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
-\frac{p}{2\beta} \|z^t - z^{t+1}\|^2 - p(z^{t+1} - z^t)^T (z^{t+1} + z^t - 2x(y^{t+1}, z^{t+1})) \\
2p(z^{t+1} - z^t)^T (z^t - x(z^t)) + \frac{p}{\sigma_4} \|z^t - z^{t+1}\|^2 \\
= \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
\left(-\frac{p}{2\beta} + \frac{p}{\sigma_4} - p\right) \|z^t - z^{t+1}\|^2 + 2p(z^{t+1} - z^t)^T (x(y^{t+1}, z^{t+1}) - x(z^t)),
\]

where the second step follows from (3.20). Using the inequality \( 2a^T b \leq ||a||^2 + ||b||^2 \) for any \( a, b \in \mathbb{R}^n \), we have
\[
\Delta_{p}^{t+1}(z^{t+1}) + \Delta_{d}^{t+1}(z^{t+1}) + \Delta_{prx}^{t+1}(z^{t+1}) \\
\leq \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
\left(-\frac{p}{2\beta} + \frac{p}{\sigma_4} - p\right) \|z^t - z^{t+1}\|^2 + p||x(y^{t+1}, z^{t+1}) - x(z^t)||^2 \\
= \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
\left(-\frac{p}{2\beta} + \frac{p}{\sigma_4} - p\right) \|z^t - z^{t+1}\|^2 + p||x(y^{t+1}, z^{t+1}) - x(z^t)||^2 \\
\leq \Delta_{p}^{t+1}(z^t) + \Delta_{d}^{t+1}(z^t) + \Delta_{prx}^{t+1}(z^t) \\
\left(-\frac{p}{2\beta} + \frac{p}{\sigma_4} - p\right) \|z^t - z^{t+1}\|^2 + \frac{p\sigma_4^2}{\sigma_3^2} \|Ax(y^{t+1}, z^t) - b\|^2,
\]
where the last step is due to the error bounds \([3.6], (3.9)\) in Lemma \([3.6] \). Let

\[
D_1(\beta) = \tau_1, \quad D_2(\beta) = \tau_2 + \frac{p\sigma_2^2}{\sigma_3}, \quad D_3(\beta) = \tau_3, \quad D_4(\beta) = -\frac{p}{2\beta} + \frac{p}{\sigma_4},
\]

then we have

\[
\Delta^{t+1}(z^{t+1}) = \Delta_{p}^{t+1}(z^{t+1}) + \Delta_{d}^{t+1}(z^{t+1}) + \Delta_{prx}^{t+1}(z^{t+1})
\]

\[
\leq \Delta_{p}^{t+1}(z^{t}) + \Delta_{d}^{t+1}(z^{t}) + \Delta_{prx}^{t+1}(z^{t})
\]

\[
\left(-\frac{p}{2\beta} + \frac{p}{\sigma_4}\right)\|z^{t} - z^{t+1}\|^2 + \frac{p\sigma_2^2}{\sigma_3}\|Ax(y^{t+1}, z^{t}) - b\|^2
\]

\[
\leq D_1(\beta)\|x^{t} - x^{t+1}\|^2 + D_2(\beta)\|Ax(y^{t+1}, z^{t}) - b\|^2
\]

\[
+ D_3(\beta)\|z^{t} - x(z^{t})\|^2 + D_4(\beta)\|z^{t} - z^{t+1}\|^2,
\]

where the last step follows from \([4.7]-[4.9] \).

Recall the definition of potential function \(\phi\) (cf. \([3.21]\) ). It follows that

\[
\phi^{t} - \phi^{t+1} = (\Delta_{p}^{t+1}(z^{t}) + \Delta_{d}^{t+1}(z^{t}) + \Delta_{prx}^{t+1}(z^{t})) - (\Delta_{p}^{t+1}(z^{t+1}) + \Delta_{d}^{t+1}(z^{t+1}) + \Delta_{prx}^{t+1}(z^{t+1}))
\]

\[
= \Delta^{t}(z^{t}) - \Delta^{t+1}(z^{t+1})
\]

\[
\geq C_1(\beta)\|x^{t} - x^{t+1}\|^2 + C_2(\beta)\|Ax(y^{t+1}, z^{t}) - b\|^2
\]

\[
+ C_3(\beta)\|z^{t} - x(z^{t})\|^2 + C_4(\beta)\|z^{t} - z^{t+1}\|^2
\]

\[
\geq \kappa (D_1(\beta)\|x^{t} - x^{t+1}\|^2 + D_2(\beta)\|Ax(y^{t+1}, z^{t}) - b\|^2
\]

\[
+ D_3(\beta)\|z^{t} - x(z^{t})\|^2 + D_4(\beta)\|z^{t} - z^{t+1}\|^2)
\]

\[
\geq \kappa \Delta^{t+1}(z^{t+1}),
\]

where \(\kappa := \min_{i:D_i(\beta)>0}\{\frac{C_i(\beta)}{D_i(\beta)}\}\) and the first inequality follows from Lemma \([4.3] \). This completes the proof. 

\[\Box\]

5 Multi-block Case and a Linearized Proximal ADMM

In this section we consider the multi-block case. Consider the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f(x_1, x_2, \cdots, x_k) \\
\text{subject to} & \quad \sum_{j=1}^{k} A_j x_j = b, \quad x \in P,
\end{align*}
\]

\[(5.1)\]

where \(x_j \in \mathbb{R}^n\) for all \(1 \leq j \leq k\), \(x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^n\) and \(P = \prod_{i=1}^{k} [\ell_i, u_i]\). We also denote \(A = [A_1, A_2, \cdots, A_k]\) and \(\sigma = \max_{1 \leq i \leq k} \|A_i\|_2\), where \(\| \cdot \|_2\) is the spectral norm of a matrix.

Similar to Assumption \([2.2]\) we make the following assumptions.

Assumption 5.1

(a) The point 0 is in the relative interior of the set \(AP - b = \{Ax - b \mid x \in P\}\).

(b) The strict complementarity condition holds for \([5.1]\).
The objective function \( f \) is a differentiable function with Lipschitz continuous partial derivatives, namely, for all \( 1 \leq i \leq k \),

\[
\| \nabla_{x_i} f(x) - \nabla_{x_i} f(x') \| \leq \bar{L} \| x - x' \|, \quad \text{for some } \bar{L} > 0 \text{ and } \forall x, x' \in \mathbb{R}^n.
\]

Similar to (2.3), we can still define a constant \( \gamma \) as in Section 2. To solve this problem, we use the following linearized proximal ADMM 3, which updates the primal variables blockwise. Let us denote

\[
x^t(j) = (x^t_{j+1}, x^t_{j+1}, \ldots, x^t_{j-1}, x^t_j, \ldots, x^t_k).
\]

Here \( [.] \) means the projection to the set \( \prod_{i \in \mathcal{N}_j}[\ell^i, u^i] \), where \( \mathcal{N}_j \) is the set of indexes for the \( j \)-th variable block \( x_j \). Note that here we take the stepsize \( c < \frac{1}{L + p + \rho \bar{s}^2} \).

To prove the convergence of Algorithm 3, we can follow the same line as that for the proof of Theorem 2.4. The only differences are the proof for (3.4)-(3.8) in Lemma 3.6 and the proof of Lemma 3.7.

For the primal error bounds (3.4)-(3.8), we only give the proof of the first one and the others can be proved using the same techniques as that in the proof of Lemma 3.6.

**Lemma 5.2** For any \( t \), there exists a constant \( \bar{s}_1 \), such that

\[
\| x^t - x^{t+1} \| \geq \bar{s}_1 \| x^t - x(y^{t+1}, z^t) \|.
\]

**Proof** Since this lemma is not related to the update of \( y, z \), for notation simplicity, we denote \( K(x) = K(x, z^t, y^{t+1}) \). The proof consists of two parts:

1. When the primal variables are updated via the block coordinate gradient descent scheme, it is a type of approximate gradient projection algorithm, namely,

\[
x^t+1 = [x^t - c \nabla K(x^t) + \Theta(t)]_+,
\]

where \( \Theta(t) \) satisfies

\[
\| \Theta(t) \| < \eta \| x^t - x^{t+1} \| \quad (5.2)
\]

for some positive constant \( \eta > 0 \).
2. Prove the approximate gradient projection algorithm has the primal error bound.

We consider the first part. Notice that

\[
x_j^{t+1} = [x_j^t - c \nabla x_j K(x^t(j))]_+^2, \\
= [x_j^t - c \nabla x_j K(x^t) + c(\nabla x_j K(x^t(j)) - \nabla x_j K(x^t))]_+^2, \\
= [x^t - c \nabla x_j K(x^t) + \Theta_j(t)]_+^2,
\]

where \( \Theta_j(t) = c(\nabla x_j K(x^t(j)) - \nabla x_j K(x^t)) \). Due to the Lipschitz continuity of the partial gradient of \( K \), we have

\[
\|\Theta_j(t)\| \leq c(\bar{L} + p + \rho \sigma^2)\|x^t(j) - x^t\|
\leq c(\bar{L} + p + \rho \sigma^2)\sqrt{\sum_{i=1}^{j-1} \|x_i^{t+1} - x_i^t\|^2}
\leq c(\bar{L} + p + \rho \sigma^2)\sum_{i=1}^{j-1} \|x_i^t - x_i^{t+1}\|, \quad (5.3)
\]

where the last inequality is proved by just squaring both sides of the inequality. Since \( \sum_{i=1}^{j-1} \|x_i^t - x_i^{t+1}\| \leq \sum_{i=1}^k \|x_i^t - x_i^{t+1}\| \), we have

\[
\|\Theta(t)\| = \|(\Theta_1(t), \ldots, \Theta_k(t))\|
\leq c(\bar{L} + p + \rho \sigma^2)k \sum_{i=1}^k \|x_i^t - x_i^{t+1}\|
\leq c(\bar{L} + p + \rho \sigma^2)k^{3/2}\|x^t - x^{t+1}\|,
\]

where the last inequality is due to Cauchy-Schwartz inequality. This finishes the proof of (5.2) with \( \eta = c(\bar{L} + p + \rho \sigma^2)k^{3/2}/2. \)

For the second part, we have

\[
\|x^t - x^{t+1}\| = \|x^t - [x^t - c\nabla K(x^t) + \Theta(t)]_+\|
\geq (i) \|x^t - [x^t - c\nabla K(x^t)]_+\| - \|[x^t - c\nabla K(x^t)]_+ - [x^t - c\nabla K(x^t) + \Theta(t)]_+\|
\geq (ii) \sigma_1\|x^t - x(y^{t+1}, z^t)\| - \|[x^t - c\nabla K(x^t)]_+ - [x^t - c\nabla K(x^t) + \Theta(t)]_+\|
\geq (iii) \sigma_1\|x^t - x(y^{t+1}, z^t)\| - \|\Theta(t)\|
\geq (iv) \sigma_1\|x^t - x(y^{t+1}, z^t)\| - \eta\|x^t - x^{t+1}\|,
\]

where (i) is because of the triangular inequality, (ii) is due to the error bound (3.4) in Lemma 3.6, (iii) is due to the nonexpansiveness of the projection operator and (iv) is because of (5.2). Hence setting \( \sigma_1 = \sigma_1/(1 + \eta) \) completes the proof.

Next we establish a simple lemma to ensure that the sufficient decrease result Lemma 3.7 holds true for the multi-block case.
Lemma 5.3 For any \( t \), we have
\[
K(x^t, z^t; y^{t+1}) - K(x^{t+1}, z^t; y^{t+1}) \geq \frac{1}{2c} \|x^t - x^{t+1}\|^2.
\]

Proof Notice that, by Assumption 5.1(c), the partial gradient of \( K \) with respect to any block is \( c^{-1} \)-Lipschitz continuous, so we have
\[
K(x^t(j), z^t; y^{t+1}) - K(x^{t}(j + 1), z^t; y^{t+1}) \geq \frac{1}{2c} \|x^t_j - x^{t+1}_j\|^2, \text{ for all } 1 \leq j \leq k.
\]
Here \( x^t(0) = x^t \). Summing this from 0 to \( k - 1 \) and using the fact that
\[
\sum_{j=1}^{k} \|x^t_j - x^{t+1}_j\|^2 = \|x^t - x^{t+1}\|^2
\]
yields the desired result.

This shows that the descent condition (3.17) holds true for the multi-block case, which further implies Lemma 3.7 remains valid. Equipped with these, we conclude that the Algorithm 3 converges globally.

Theorem 5.4 Suppose Assumption 5.1 holds and the parameters \( c \) and \( p \) satisfy
\[
\frac{1}{L + p + \rho \sigma^2} > c > 0, \quad p > -\gamma
\]
and that the primal and dual stepsizes \( \beta \) and \( \alpha \) are sufficiently small. Then the dual iterates \( \{y^t\} \) are bounded. Moreover, there holds
\[
\lim_{t \to \infty} \|x^{t+1} - x^t\| = 0,
\]
\[
\lim_{t \to \infty} \text{dist}(x^t, X^*) = 0,
\]
\[
\lim_{t \to \infty} \text{dist}(z^t, X^*) = 0,
\]
and every limit point of the sequence \( \{(x^t, y^t)\} \) generated by Algorithm 3 is a primal-dual stationary point of (5.1).

6 Numerical Results

A natural question is whether we can set \( \beta = 1 \) in Algorithm 2 and thus eliminating the sequence \( \{z^t\} \) from the iterations. In this section, we give some numerical results showing that \( \beta < 1 \) is needed for convergence and hence the sequence \( \{z^t\} \) is necessary in Algorithm 2. We consider the case of quadratic programming where the cost function is \( f(x) = \frac{1}{2} x^T Qx + r^T x \). We use the following distributions to randomly generate the data matrices:
1. $Q$ is a $500 \times 500$ matrix and $Q = -U^T U$, where $U$ is a $500 \times 500$ matrix with each entry following the normal distribution $N(0, 1)$.

2. $r \in \mathbb{R}^{500}$, with each entry generated from $N(0, 1)$.

3. $A \in \mathbb{R}^{10 \times 500}$ with each entry following $N(0, 1)$.

4. $b = Ax_0$, with $x_0 \in \mathbb{R}^{500}$ uniformly distributed over $[0, 5]^{500}$.

5. $\ell_i = 0$ and $u_i = 1000$ for all $1 \leq i \leq 500$.

In our experiment, we set $\rho = 1000$. We will see that if we set $\beta = 1$ (i.e., no stabilization step), Algorithm 1 oscillates after $2 \times 10^6$ iterations with $\alpha = 1000$, $\alpha = 50$ and $\alpha = 1$. However, for $\alpha = 50$, we find that Algorithm 2 converges with $\beta = 0.01$ or $\beta = 0.02$. The following are the figures for these cases.

Case 1. We use $\alpha = 1000, 50, 1$ and plot the curves for $\|Ax_t - b\|$ and $\frac{1}{c}\|x_t - x_{t+1}\|$.

![Figure 1](image)

Figure 1: $\|Ax_t - b\|$ v.s. iterations with $\beta = 1$

We see that Algorithm 1 oscillates after $2 \times 10^6$ iterations for these choices of $\alpha$.

Case 2. We choose $\alpha = 50$, $\beta = 0.01$ and $\beta = 0.02$ and plot $f(x^t)$, $\|Ax^t - b\|$, $\|x^{t+1} - z^t\|$ and $\frac{1}{c}\|x^t - x^{t+1}\|$.
Figure 2: $f(x^t)$ v.s. iterations

Figure 3: $\|Ax^t - b\|$ v.s. iterations
We see that Algorithm 2 converges with $\beta = 0.01$ and $\beta = 0.02$. The algorithm with $\beta = 0.02$ is faster, which suggests that we can try larger $\beta$ to achieve higher convergence speed.

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