On the Relevance of Disorder for Dirac Fermions with Imaginary Vector Potentials

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We consider the effects of disorder in a Dirac-like Hamiltonian. In order to use conformal field theory techniques, we argue that one should consider disorder in an imaginary vector potential. This affects significantly the signs of the lowest order beta functions. We present evidence for the existence of two distinct universality classes, depending on the relative strengths of the gauge field verses impurity disorder strengths. In one class all disorder is driven irrelevant by the gauge field disorder.

I. INTRODUCTION

In order to explain the main qualitative features of the Quantum Hall effect one needs gauge invariance and impurities. From the gauge invariance one can obtain the quantization condition of the plateau, at least for the integer case \[ \nu = 20/9 \]. The impurities are necessary to localize the states being filled on a plateau where \( \sigma_{xy} \) remains constant.

From the gauge arguments alone, it seems possible to infer that extended states exist at the center of impurity broadened Landau bands \[ \nu \geq 2 \]. This is rather striking, since the issue of whether states are localized or extended is normally a difficult problem in Anderson localization, involving the study of Quantum Mechanics in disordered potentials, at least in more than two dimensions. In two dimensions, with no magnetic field, states are in principle always localized, no matter what the strength of the disorder \[ \nu \geq 2 \]. It seems clear from these observations that there exists a competition between localization due to impurity disorder and the consequences of gauge invariance. One expects then that the presence of the magnetic field can drive impurity disorder irrelevant.

We recently presented a computation of the correlation length exponent \( \nu \) in a certain model with no disorder and obtained \( \nu = 20/9 \), in good agreement with experiments and numerical simulations \[ \nu \geq 2 \]. This exponent essentially followed from gauge invariance. In this paper we consider the effects of disorder in the above model. We argue that to use conformal field theory techniques, one must view the gauge potential as imaginary, whereas the usual scalar potential is real. We present evidence that there may actually be two distinct universality classes depending on the relative strengths of the gauge field verses impurity disorder. In one universality class all disorder is driven irrelevant due to the disorder in the gauge potential. The gauge field is not normally thought of as disordered, since the magnetic field is usually considered uniform. The disordered component of the gauge field can be thought of as arising from the local electro-magnetic field due to the random impurities as a source. Alternatively, conduction electrons find their way along the most conductive paths, and the shapes of these paths can be rather complicated in the presence of impurity disorder. Magnetic flux enclosed by such paths is random, and this may perhaps effectively lead to disorder in the gauge field, as in the network model \[ \nu \geq 2 \].

II. RENORMALIZATION GROUP ANALYSIS

We consider fermions in two dimensions with hamiltonian \( H \) and second quantized action

\[
S_{2+1} = \int dt d^2x \, \Psi^\dagger (i \partial_t - H) \Psi
\]  

(2.1)

For the purposes of studying the consequences of disorder it is convenient to Fourier transform in time

\[
\Psi(x, t) = \int d\varepsilon \, e^{i\varepsilon t} \, \Psi_\varepsilon(x)
\]  

(2.2)

such that

\[
S_{2+1} = \int d\varepsilon \int d^2x \, \Psi_\varepsilon^\dagger (\varepsilon - H) \Psi_\varepsilon
\]  

(2.3)

The functional integral is defined by \( e^{iS_{2+1}} \). For a fixed \( \varepsilon \), one has a euclidean field theory with functional integral defined by \( e^{-S} \) where

\[
S = i \int d^2x \, \psi_\varepsilon^\dagger (H - \varepsilon) \psi_\varepsilon
\]  

(2.4)

With the hermiticity properties inherited from \( 2 + 1 \) dimensions, one has that \( S \) is anti-hermitian \( S^\dagger = -S \).

We will study the hamiltonian

\[
H = \frac{1}{\sqrt{2}} (-i \partial_x - A_x) \sigma_x + \frac{1}{\sqrt{2}} (-i \partial_y - A_y) \sigma_y + V(x, y)
\]  

(2.5)

where \( A_\mu \) is the electro-magnetic vector potential and \( V(x, y) \) is an impurity potential. This kind of Dirac
hamiltonian has been considered before in the context of Quantum Hall transitions, with real \( V \) and \( A_\mu \), however its meaning in [1] is rather different. Letting 
\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]
and introducing the complex coordinates
\[
z = (x + iy)/\sqrt{2}, \quad \bar{z} = (x - iy)/\sqrt{2}
\]
and gauge fields
\[
A_z = (A_x - iA_y)/\sqrt{2}, \quad A_\bar{z} = (A_x + iA_y)/\sqrt{2},
\]
on the action is
\[
S = \int \frac{d^2 x}{2\pi} \left[ \psi_1^\dagger (\partial_z - iA_z) \psi_2 + \psi_2^\dagger (\partial_{\bar{z}} - iA_{\bar{z}}) \psi_1 - iV(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \right] \tag{2.6}
\]
where we have dropped the \( \varepsilon \) term and suppressed the \( \varepsilon \) subscripts, since it was sufficient to study \( \varepsilon = 0 \) in [1]. Using the hermiticity properties \( \partial_\mu^\dagger = -\partial_\mu \), \( A_\mu^\dagger = A_{\bar{\mu}} \) one verifies \( S^\dagger = -S \) when \( V \) is real.

We wish to study disorder in the above model using techniques from conformal field theory. When \( V = 0 \), the model is conformally invariant and the action is usually expressed in terms of left (L) and right (R) moving fermions. Making the identifications:
\[
\psi_1^\dagger = \psi_R^\dagger, \quad \psi_2 = \psi_R, \quad \psi_2^\dagger = \psi_L^\dagger, \quad \psi_1 = \psi_L \tag{2.7}
\]
the action becomes the one appropriate to conformal field theory:
\[
S = \int \frac{d^2 x}{2\pi} \left( \psi_R^\dagger (\partial_z - iA_z) \psi_R + \psi_L^\dagger (\partial_{\bar{z}} - iA_{\bar{z}}) \psi_L - iV(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L) \right) \tag{2.8}
\]
The L, R designations come from the equations of motion when \( V = 0 \): \( \psi_L = \psi_L(z) \), \( \psi_R = \psi_R(\bar{z}) \).

The conformal field theory is defined by functional integrals over \( \psi_L, \psi_R \), which is not evidently the same as the functional integral over \( \psi_{1,2}, \psi_{1,2}^\dagger \) since, as can be seen from Eq. (2.7), the hermiticity properties are not compatible. This is most apparent in the conventional bosonization for the fermions, which reads:
\[
\psi_L = e^{-i\phi_L}, \quad \psi_R = e^{-i\phi_R}, \quad \psi_R^\dagger = e^{i\phi_L}, \quad \psi_L^\dagger = e^{i\phi_R} \tag{2.9}
\]
where \( \phi = \phi_L(z) + \phi_R(\bar{z}) \) is a free scalar field. The current coupled to the gauge field then has the following bosonized expressions:
\[
j_z = \frac{1}{2\pi} \psi_L^\dagger \psi_L = \frac{i}{2\pi} \partial_z \phi \tag{2.10}
\]
\[
j_{\bar{z}} = \frac{1}{2\pi} \psi_R^\dagger \psi_R = -\frac{i}{2\pi} \partial_{\bar{z}} \phi \tag{2.10}
\]
which implies \( j_\mu = \epsilon_{\mu\nu} \partial_\nu \phi/2\pi \), where \( \epsilon_{12} = -\epsilon_{21} = 1 \).

The bosonized action then takes the form
\[
S = \int d^2 x \left[ \frac{1}{8\pi} \partial_\mu \phi \partial_\mu \phi - \frac{i}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi A_\mu \right] \tag{2.11}
\]
The important point is that whereas the kinetic term for the real scalar field is real, the term which couples to the gauge field is imaginary if \( A_\mu \) is real. Indeed, in [1], it was shown that the Hall conductivity \( \sigma_{xy} \) computed from \( S \) is real only if one makes an analytic continuation which effectively makes \( A_\mu \) imaginary and thus renders \( S \) real.

Based on the above considerations, we consider the effects of disorder in the real potential \( V \) and an imaginary vector potential \( A_\mu \). The role of imaginary vector potentials in localization problems has been previously recognized by Hatano and Nelson [11] and in [12]. Let \( A_\mu \rightarrow A_\mu + iA^d_\mu \) where \( A^d_\mu \) is a disordered component of the gauge field, and \( A_\mu = \partial_\mu \chi \) is the background value with the properties assumed in [1]. The disordered potentials are taken to have gaussian probability distributions:
\[
P[A^d] = \exp \left( -\frac{1}{g_A} \int d^2 x \frac{A_\mu^d A^d_\mu}{2} \right) \tag{2.12}
\]
P[\( V \)] = \( \exp \left( -\frac{1}{2g_V} \int d^2 x (V(x) - V_0)^2 \right) \)

where \( V_0 \) is the mean value of \( V(x) \) and \( g_V, g_A \) represent positive variances of the distribution.

An effective action which incorporates the effects of the disorder can be obtained using the supersymmetric method [13]. For our particular problem this was carried out by Bernstein [14]. Let us outline the main features. Introducing bosonic ghosts \( \beta_{L,R}, \beta_{L,R}^\dagger \) that couple in the same way as the fermions, the functional integral over \( V, A^d \) leads to the effective action:
\[
S_{\text{eff}} = S_{\text{cft}} + \int \frac{d^2 x}{2\pi} \left( -iV_0 \partial_\nu \phi_0 + g_V \mathcal{O}_V + g_A \mathcal{O}_A \right) \tag{2.13}
\]
where \( S_{\text{cft}} \) is the conformal field theory action for the fermions plus a ghost action obtained from the fermionic one with the replacement \( \psi_{L,R} \rightarrow \beta_{L,R} \), and \( \mathcal{O}_0 \) is the operator the \( V \) couples to in (2.8) plus ghosts. The operators \( \mathcal{O}_{V,A} \) are:
\[
\mathcal{O}_V = \frac{1}{2} \left( \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L + \beta_L^\dagger \beta_R + \beta_R^\dagger \beta_L \right)^2 \tag{2.14}
\]
\[
\mathcal{O}_A = \left( \psi_R^\dagger \psi_R + \beta_R^\dagger \beta_R \right) \left( \psi_L^\dagger \psi_L + \beta_L^\dagger \beta_L \right) \tag{2.15}
\]
The interactions can be written as a current-current perturbation of the \( \text{OSP}(2|2) \) super-current algebra [14].

The operators \( \mathcal{O}_V, \mathcal{O}_A \) do not form a closed operator algebra and another term \( g_M \mathcal{O}_M \) in the lagrangian is generated under renormalization, with
\[
\mathcal{O}_M = \frac{1}{2} \left( \psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L + \beta_L^\dagger \beta_R - \beta_R^\dagger \beta_L \right)^2 \tag{2.15}
\]

\(^1\text{In [1], this corresponded to letting the topological angle } \theta \rightarrow -i\theta.\)
Using the operator product expansions
\[ \psi_1^\dagger(z) \psi_L(0) \sim \psi_L(z) \psi_1^\dagger(0) \sim 1/z \] (2.16)
and similarly for \( \beta_R, \beta_R \), one obtains the operator product expansions
\[
\begin{align*}
\mathcal{O}_V(z, \bar{z}) \mathcal{O}_V(0) & \sim -\frac{4}{z^\nu} \mathcal{O}_V(0) \\
\mathcal{O}_M(z, \bar{z}) \mathcal{O}_M(0) & \sim \frac{4}{z^\nu} \mathcal{O}_M(0) \\
\mathcal{O}_A(z, \bar{z}) \mathcal{O}_M(0) & \sim 2 \frac{2}{z^\nu} (\mathcal{O}_V(0) + \mathcal{O}_M(0)) \\
\mathcal{O}_A(z, \bar{z}) \mathcal{O}_V(0) & \sim 2 \frac{2}{z^\nu} (\mathcal{O}_V(0) + \mathcal{O}_M(0)) \\
\mathcal{O}_M(z, \bar{z}) \mathcal{O}_V(0) & \sim \frac{1}{z^\nu} (4 \mathcal{O}_A(0) - 2 \mathcal{O}_V(0) + 2 \mathcal{O}_M(0))
\end{align*}
\] (2.17)
From these one can determine the lowest order (1-loop) beta functions,
\[ \beta_g = \frac{dg}{d \log R} \] (2.18)
where \( R \) is a length scale [13]. One finds
\[ \begin{align*}
\beta_{g_V} & = 4g_V^2 + 4g_Mg_V - 4g_A(g_M + g_V) \\
\beta_{g_M} & = -4g_M^2 - 4g_Mg_V - 4g_A(g_M + g_V) \\
\beta_{g_A} & = -8g_Mg_V
\end{align*} \] (2.19)
In [14] the analogous result was obtained for a real vector potential, which differs by signs \( g_A \rightarrow -g_A \). The special case \( g_M + g_V = 0 \) has some very interesting properties, namely \( \beta_{g_M + g_V} = 0 \) when \( g_M + g_V = 0 \), and the only non-zero beta function is \( \beta_{g_A} \). For the case of a real vector potential, one can check numerically that renormalization group flows are drawn to the line \( g_M + g_V = 0 \) in the UV. This nearly conformal situation was studied in [16], and is exactly soluble using \( gl(N|N) \) supercurrent algebra.

The change in sign for an imaginary vector potential has some important consequences as far as the relevance or irrelevance of disorder. The \( \mathcal{O}_M \) operator which is generated corresponds to a term in the hamiltonian \( iM(x, y) \sigma_z \), which leads to a term in the lagrangian \( iM(\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2) \). This is anti-hermitian, as it should be in the original \( \psi_{1,2} \) formulation if \( M \) is real. Thus a positive \( g_M \) corresponds a variance in a gaussian distribution of \( M \). Since the couplings \( g \) represent variances of the potentials, we consider the renormalization group flow when all couplings are initially positive. The beta function \( \beta_{g_A} \), then shows that \( g_M \) is a marginally irrelevant coupling, i.e. it decreases at larger distances. At large enough distance it is eventually driven to zero. It will eventually be driven negative using the 1-loop beta functions, but since this is unphysical let us assume that the flow is cut-off when \( g_M = 0 \). To get a qualitative understanding of the renormalization group flow, let us set \( g_M = 0 \) since it is marginally irrelevant. One sees that \( g_A \) is then exactly marginal, i.e. it’s beta function vanishes. What is interesting is that \( g_V \) can be marginally relevant or irrelevant depending on the initial strengths of \( g_V \) verses \( g_A \). The two classes are:

**Class A (Gauge Dominated)** Here \( g_A > g_V \). In this class \( g_A \) drives \( g_V \) to be irrelevant. Integrating the beta function, one finds
\[ g_V(R) = \frac{g_A}{1 + R^4g_A} \] (2.20)
i.e. \( g_V(R) \) decreases at large distances. \( g_V \) is more than marginally irrelevant: it decreases much faster than logarithmically because of the linear term in the beta function.

**Class B (Impurity Dominated)** Here \( g_A < g_V \), and \( g_V \) is relevant
\[ g_V(R) = \frac{g_A}{1 - R^4g_A} \] (2.21)

To verify that this picture is not spoiled by a non-zero \( g_M \), we integrated the beta functions numerically for an initial value of \( g_M = 1/4 \). Indeed one sees two classes roughly separated by \( g_A = g_V \). In Class A all couplings decrease with increasing \( R \), whereas in Class B \( g_A, g_M \) decrease and \( g_V \) increases.

**III. DISCUSSION**

If disorder is indeed irrelevant in Class A, there isn’t much left for the critical exponents of a transition to depend on. What remains is the constant mean value of the potential \( V_0 \), which is relevant, but clearly the exponent is independent of the strength of \( V_0 \). In [6] the impurity region was shrunk to a circular defect of uniform strength and gauge arguments led to the exponent \( \nu = 20/9 \). This is a reasonable proposal for the exponent governing Class A. For Class B the impurity disorder is relevant and so it is possible that one flows to a new fixed point with a different exponent. However the constant potential \( V_0 \) corresponds to a dimension 1 operator and is more relevant. If the exponent in this class is indeed different, then the percolation type exponent of \( 7/3 \) is a possibility, since the impurities dominate and the electrons must somehow percolate through islands of impurities [17].

There is a small amount of evidence for two universality classes in both numerical simulations and experiments. Huckestein’s work gave \( \nu = 2.35 \pm 0.03 \) which is consistent with \( 7/3 \) [8]. On the other hand Aoki-Ando’s simulation gave \( 2.2 \pm .1 \), closer to \( 20/9 \) [19] [20]. It isn’t
clear yet whether there is any statistical significance to this. Furthermore Ando apparently doesn’t incorporate randomness in the gauge field directly. It would be very interesting to perform a simulation where one can tune the strength of the gauge verses impurity disorder.

IV. ACKNOWLEDGMENTS

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**FIG. 1.** Renormalization group flow with initial values $g_A = g_M = .25$, for various values of $g_V$. Increasing length scale is from top to bottom.