A refinement of the A-polynomial of quivers

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1 Introduction

For a finite quiver $Q = (Q_0, Q_1)$ Kac proved that the number of absolutely irreducible representations of $Q$ of dimension vector $n \in \mathbb{Z}_{\geq 0}^{Q_0}$ over $\mathbb{F}_q$ is given by a polynomial $A_n(q)$. He conjectured that $A_n$ has non-negative integer coefficients.

In this preliminary note we propose an refinement of this polynomial. For simplicity we will concentrate on $S_g$, the quiver consisting of one vertex and $g$ loops, though we expect many of the results to extend to a general quiver. We only give a brief sketch of proofs, fuller details will appear elsewhere.

We define a priori rational functions $A_{\lambda}(q)$ indexed by partitions $\lambda$ which give a decomposition

$$A_n(q) = \sum_{|\lambda| = n} A_{\lambda}(q)$$

of the $A$-polynomial of $S_g$. (We drop $g$ from the notation if there is no risk of confusion.)

Computations suggest that for $g > 0$, which we assume from now on, $A_{\lambda}(q)$ is in fact a polynomial in $q$ with non-negative integer coefficients. For example, for $g = 2$ and $n = 3$ we obtain

$$A_{(1,1,1)}(q) = q^{10} + q^8 + q^7, \quad A_{(2,1)} = q^6 + q^5, \quad A_{(3)} = q^4$$

with sum

$$A_3(q) = q^{10} + q^8 + q^7 + q^6 + q^5 + q^4$$

It would be quite interesting to understand this decomposition directly in terms of absolutely indecomposable representations of the quiver.

We sketch below a proof of the following formula for $A_{\lambda}(1)$ (the case $\lambda = (1^n)$ was previously proved by Reineke [9] by different methods). By the conjectures of [2] the number $A_n(1) = \sum_{|\lambda| = n} A_{\lambda}(1)$ should equal the dimension of the middle dimensional cohomology group of the character variety $\mathcal{M}_n$ studied there. A refined version of this conjecture states that $A_{\lambda}(1)$ is the number of connected components of type $\lambda$ of a natural $\mathbb{C}^*$ action on the moduli space of Higgs bundles, which is diffeomorphic to $\mathcal{M}_n$. A proof of this conjecture for $\lambda = (1^n)$ was recently given by Reineke [10, Theorem 7.1]. The refined conjecture originates in [2, Remark 4.4.6] and was in fact the motivation to construct the truncated polynomials $A_{\lambda}(q)$ studied in this note.

**Theorem 1.0.1.** For any non-zero partition $\lambda$ we have

$$A_{\lambda}(1) = \frac{1}{\rho} \sum_{d|m} \frac{\mu(d)}{d^2} \frac{1}{P_1(m/d)P_N(m/d)} \prod_{l \geq 1} \left( \rho P_l(m/d) - 1 + m_l/d \right)$$

(1.0.1)

where $\lambda = (1^{m_1}2^{m_2} \cdots N^{m_N})$ with $N = l(\lambda)$, the length of $\lambda$,

$$P_l(m) := \sum_{j \geq 1} \min(i, j) m_j, \quad m := (m_1, m_2, \ldots), \quad \rho := 2g - 2,$$

and $\mu$ is the Möbius function of number theory.
Note that if the entries of $m$ have no common factor then the right hand side of (1.0.1) consists of only one term.

**Corollary 1.0.2.** As a function of $g$, the quantity $A_{\lambda}(1)$ is a polynomial of degree $l(\lambda) - 1$; its leading coefficient in $\rho := 2g - 2$ is

$$
\frac{1}{P_1(m)P_{\lambda}(m)} \prod_{i \geq 1} \frac{P_i(m)^{m_i}}{m_i!}.
$$

(1.0.2)

**Remark 1.0.3.** In particular, we recover the fact (noticed numerically in [2] and proved in greater generality in [3]) that $A_{\alpha}(1)$ is a polynomial in $\rho$ of degree $n - 1$ and leading coefficient $n^{n-2}/n!$. The appearance of the term $n^{-2}$, the number of spanning trees on $n$ labelled vertices, is not a coincidence, see Remark 2.0.5.

We also note the following important property (here we write $A_{\rho \lambda}$ with $\rho := 2g - 2$ for $A_{\lambda}$ to indicate the dependence on $g$), which was inspired by the interpretation of $A_{\lambda}(1)$ in terms of the moduli space of Higgs bundles mentioned above.

**Proposition 1.0.4.** Let $n$ be a positive integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ a non-zero partition. Define $n_{\lambda} := (n\lambda_1, n\lambda_2, \ldots)$. Then

$$
A_{\rho \lambda}(q) = A_{n\lambda}(q), \quad \rho := 2g - 2.
$$

In particular,

$$
A_{(0)}(q) = q^{n(n-1)/2}.
$$

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## 2 Definition of $A_{\lambda}(q)$

Hua’s formula expresses the $A$-polynomial of a quiver in terms of generating series. Let

$$
\text{Hua}(T; q) := \sum_{A} q^{q(q-1)(A,A)} b_A(T^{-1}) T^{A}. \quad (2.0.3)
$$

Then for $S_g$ we have

$$(q - 1) \text{Log} \left( \text{Hua}(T; q) \right) = \sum_{n \geq 0} A_n(q) T^n. \quad (2.0.4)
$$

Here Log is the plethystic equivalent of the usual log (see for example [2] for a discussion). The main use of Log is as a convenient tool to manipulate the conversion of series to infinite products. It takes a factor of the form $(1 - w)^{-1}$, where $w$ is a monomial in some set of variables, to $w$.

We are interested in the values $A_n(1)$. Because the individual terms in the generating function $\text{Hua}(T; q)$ have high order poles at $q = 1$ it is not easy to recover these numbers from (2.0.4).

Consider Hua’s formula as the limit as $N \rightarrow \infty$ of the truncated series

$$
\text{Hua}_N(T; q) := \sum_{A_1 \leq N} q^{q(q-1)(A,A)} b_A(T^{-1}) T^{A}. \quad (2.0.5)
$$
Using multiplicities \( m = (m_1, \ldots, m_N) \) to parametrize partitions we can write this series as the specialization 
\( x_i = T^i \) for \( i = 1, 2, \ldots, N \) of the following series in the variables \( x := (x_1, x_2, \ldots, x_N) \) and 
\[
\text{Hu}_{N}(x; q) := \sum_m \frac{q^{(x-1)^mN}H_{m}}{(q^{-1})_m} x^m
\]
where 
\[
x^m := x_1^{m_1}x_2^{m_2} \cdots x_N^{m_N} \quad (q)_m := \prod_{i=1}^{N}(q)_{m_i} \quad \mathcal{H}_N := (\min(i,j)), \quad i, j = 1, 2, \ldots, N.
\]
It is remarkable that the inverse of \( \mathcal{H}_N \) has a very simple and sparse structure, namely,
\[
\mathcal{H}_N^{-1} = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2 & -1 \\
\vdots & & \ddots & & \ddots & & \ddots & & -1 \\
0 & \cdots & -1 & 2 & -1 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}.
\]
This is the Cartan matrix of the tadpole \( T_N \) (obtained by folding the \( A_{2N} \) diagram in the middle), a positive definite symmetric matrix of determinant 1.

We define refinements of the \( A \)-polynomial by replacing \( \text{Hua}(T; q) \) by \( \text{H}_{\lambda}(x; q) \). More precisely, define \( A_{\lambda}(q) \) for \( \lambda \) a partition with \( l(\lambda) \leq N \) as \( A_{\lambda}(q) \), where 
\[
(q - 1) \log (\text{H}_{\lambda}(x; q)) = \sum_m A_{\lambda}(q) x^m, \tag{2.0.6}
\]
and \( m = (m_1, \ldots, m_N) \) are the multiplicities of \( \lambda \), so \( \lambda = (1^m_12^m_2 \cdots N^m_N) \). It is straightforward to check that the definition is independent of \( N \) as long as \( l(\lambda) \leq N \).

As mentioned, \( A_{\lambda}(q) \) is a priori a rational function of \( q \) but we expect it to actually be a polynomial. We have in any case 
\[
A_n = \sum_{|\lambda|=n} A_{\lambda}.
\]

**Remark 2.0.5.** Appropriately scaled, \( \text{H}_{\lambda}(x; q) \) converges as \( g \to \infty \) to 
\[
G(x; q) := \sum_{m} q^{m\mathcal{H}_m} \frac{x^m}{m!}, \quad m := m_1! \cdots m_N!.
\]
By the exponential formula of combinatorics
\[
(q - 1) \log G(x; q) \bigg|_{q=1}
\]
is the exponential generating function for certain weighted trees. This gives an interpretation of the leading term of \( A_{\lambda}(1) \) as \( g \to \infty \) in terms of weighted trees in the spirit of [3]. See Remark 3.0.7

**Proof of Proposition 1.0.4.** Let \( m = (m_1, \ldots, m_N) \) be the multiplicities of \( \lambda \). Then the multiplicities of \( n\lambda \) are
\[
m[n] := (0, \ldots, m_1, 0, \ldots, m_2, 0, \ldots, m_N),
\]
where \( m_i \) is located at the spot \( ni \). Note that \( m \) and \( m[n] \) have exactly the same non-zero entries. Hence \( (m)_m = (m)_{m} \). Also, it is easy to check that 
\[
\lambda^{m[n]}\mathcal{H}_{m[n]}m[n] = n^{m}\mathcal{H}_{m[m]}
\]
Now the first claim follows from the definition of \( A_{\lambda} \). The second claim follows from the first since \( A_{1}(q) = q^2 \). \( \square \)
3 Truncation to order $N = 1$

We will first consider the case $N = 1$ in detail as it contains all of the main ingredients of the general case. When $N = 1$ we can interpret $\text{Hua}_N$ as follows. Consider the action by conjugation of $\text{GL}_n(\mathbb{F}_q)$ on $g$-tuples of $n \times n$ matrices $X := (X_1, \ldots, X_g)$ with coefficients in $\mathbb{F}_q$. The number $N_n(q)$ of such $g$-tuples, each weighed by $1/|\text{Stab}(X)|$, where $\text{Stab}(X)$ is the stabilizer of $X$ in $\text{GL}_n(\mathbb{F}_q)$, is

$$N_n(q) = \frac{q^{n^2}}{|\text{GL}_n(\mathbb{F}_q)|} = \frac{q^{(a-1)w^2}}{(q-1)_n},$$

since

$$|\text{GL}_n(\mathbb{F}_q)| = (-1)^n q^{\frac{1}{2}n(n-1)}(q)_n = q^{\frac{1}{2}n^2}(q-1)_n$$

Hence

$$\text{Hua}_1(x_1; q) = \sum_{n \geq 0} N_n(q) x_1^n.$$ 

We want to get an expression for $\text{Hua}_N(x; q)$ as $q \to 1$. The general question of describing the asymptotics of series like (2.0.3) as $q \to 1$ has a long history. Already Ramanujan used such an asymptotics to test his famous $q$-series identities now known as the Rogers–Ramanujan formulas. More recently, the question arose in conformal field theory and there is a beautiful conjecture of Nahm [7] that relates the modular behaviour of $q$-series of this type with torsion elements in the Bloch group.

To conform with the standard format in the literature we will change $q$ to $q^{-1}$. The basic result is the following (see [5],[13]).

**Proposition 3.0.6.** Let $a$ and $T$ be fixed positive real numbers and let $z$ be the positive real root of

$$Tz^2 + z - 1 = 0.$$  \hspace{1cm} (3.0.7)

Then with $q = e^{z}$ and $t \searrow 0$ we have

$$\log \sum_{n \geq 0} \frac{q^{\frac{1}{2}w^2}}{(q)_n} T^n = c_{-1}t^{-1} + c_0 + c_1t + \cdots,$$

where

$$c_{-1} = \text{Li}_2(1-z) + \frac{1}{2}a \log^2 z, \quad c_0 = - \frac{1}{2} \log(z + a(1 - z))$$

It is a classical fact going back to Lambert that the trinomial equation (3.0.7) can be solved in terms of hypergeometric functions. Concretely, we can expand $z$ as a power series in $T$ and obtain

$$z = z(T) = \sum_{n \geq 0} \frac{1}{(a-1)n+1} \binom{an}{n} (-T)^n.$$  \hspace{1cm} (3.0.8)

This result follows easily from Lagrange’s formula [12, p. 133] (see also [8, p. 43]). For example, if $a = 2$ then

$$z = -1 + \frac{\sqrt{1+4T}}{2T} = 1 - T + 2T^2 - 5T^3 + 14T^4 - 42T^5 + 132T^6 + \cdots,$$

where the coefficients are, up to a sign, the Catalan numbers.

We would now like to combine (3.0.8) with Proposition 3.0.6. To find an expression for $c_{-1}$ as a power series in $T$ we differentiate (3.0.7) with respect to $T$ to find

$$\frac{\partial z}{\partial T} = -\frac{z^a}{1 + az^{a-1}T} = -\frac{(1-z)z}{T(z + a(1 - z))}.$$
On the other hand
\[ \frac{\partial c_{-1}}{\partial T} = \log z \left( \frac{1}{1-z} + \frac{a}{z} \right) \frac{\partial z}{\partial T} \]
and these combine to give
\[ T \frac{\partial c_{-1}}{\partial T} = -\log z \]
As it happens, \( \log z \) also has an explicit hypergeometric power series expansion \[12, p. 134, Ex. 3\]. In fact, all powers \( z^s \) have such an expansion, namely
\[ z^s = 1 + \sum_{n \geq 1} \frac{1}{a^n} \binom{n}{s} \left( -T \right)^n n! \]
from which we obtain by differentiation with respect to \( s \) and setting \( s = 0 \)
\[ \log z = \sum_{n \geq 1} \frac{1}{a^n} \binom{n}{1} \left( -T \right)^n n! \]
It follows that
\[ c_{-1} = \sum_{n \geq 1} (-1)^{n-1} \frac{1}{a^n} \binom{n}{1} \left( -T \right)^n n^2 \]
and therefore
\[ A_{\{1\}}(1) = \frac{1}{n^2} \sum_{d|n} \mu \left( \frac{n}{d} \right) \frac{1}{\rho} \binom{\rho d + d - 1}{d} \]
\[ \rho = 2g - 2. \]

**Remark 3.0.7.** As observed by Polya \[8, p. 44\] if we set \( z = 1 + w/a, T = -U/a \) and let \( a \) go to infinity then \( \text{3.0.7} \) becomes
\[ U e^w = w \]
and \( z(T) \) becomes \( w(U) = \sum_{n \geq 1} n^{w-1} U^n / n! \), the exponential generating function for labelled rooted trees.

## 4 General case

### 4.1 Lagrange’s inversion formula

Consider
\[ (q - 1) \log [\text{Hua}_N(x; q)] \bigg|_{q=1} \]
The asymptotic expansion now involves \( N \) saddle points \( z_1, \ldots, z_N \), which are solutions to the system of equations
\[ 1 - z_i = x_i \prod_{j=1}^{N} z_j^{a_{i,j}}, \quad i = 1, 2, \ldots, N, \quad (4.1.1) \]
where \( a_{i,j} = -\rho \min(i, j) \). These equations determine \( z_i = z_i(x) \) implicitly.

We will carry on the discussion of solving \( (4.1.1) \) for an generic symmetric matrix \( A = (a_{i,j}) \in \mathbb{Z}^{N \times N} \) as far as we can before specializing to our situation. As in the one variable case of the previous section we can expand \( z_i \) as power series in the \( x_j \)'s. We obtain expressions for the corresponding coefficients by applying a multi-variable version of Lagrange’s formula due to Stieltjes (see \[11\] and \[11\] for other multi-variable versions and some history on the matter). Note that \( z_i(0) = 1 \).

**Remark 4.1.1.** The kind of analysis done in this section appears prominently in the physics literature under the heading of \( Q \)-systems, originating from the work of Kirillov–Reshetikhin on representation theory and the combinatorics of the Bethe Ansatz. There is a substantial literature on the subject. The basic application of Lagrange’s inversion can be found for example in \[4\]; see also Nahm’s paper \[7\] already mentioned. We preferred to rederive the results we needed from scratch.
Theorem 4.1.2. Let \( z_1, \ldots, z_N \) be implicitly given by

\[
    z_i = y_i + x_i f_i(z_1, \ldots, z_N), \quad i = 1, 2, \ldots, N,
\]

where \( f_1, \ldots, f_N \) are analytic. Then for \( g \) analytic we have

\[
    g(z) = \frac{1}{D} \sum_m a_m(y) x^m/m!,
\]

where \( m = (m_1, \ldots, m_N), x = (x_1, \ldots, x_N), \) etc., \( x^m := x_1^{m_1} \cdots x_N^{m_N}, \) \( m! := m_1! \cdots m_N! \), and

\[
    a_m(y) := \frac{\partial^{m_1}}{\partial y_1^{m_1}} \cdots \frac{\partial^{m_N}}{\partial y_N^{m_N}} \left[ g(y) f_1(y)^{m_1} \cdots f_N(y)^{m_N} \right],
\]

and

\[
    D := \det \left( \frac{\partial z_i}{\partial y_j} \right).
\]

By differentiating (4.1.2) with respect to the \( y \)'s we see that

\[
    \left( I_N - \left( x_i \frac{\partial f_i}{\partial z_j} \right) \right) \left( \frac{\partial z_i}{\partial y_j} \right) = I_N,
\]

where \( I_N \) is the \( N \times N \) identity matrix and hence

\[
    D^{-1} = \det \left( I_N - \left( x_i \frac{\partial f_i}{\partial z_j} \right) \right)
\]

We apply this theorem to \( g(z_1, \ldots, z_N) = z_1^{a_1} \cdots z_N^{a_N} \) and \( f_i(z_1, \ldots, z_N) = -\prod_{j=1}^N z_j^{a_j} \). Note that the Jacobian matrix \( \partial z_i/\partial x_j \) at \( x = 0 \) is the identity matrix. Setting \( y_i = 1 \) we obtain

\[
    z_1^{a_1} \cdots z_N^{a_N} = \frac{1}{D} \sum_m (-1)^{m_1+\ldots+m_N} \prod_{i=1}^N \left( s_i + P_i(m) \right) x^m,
\]

where \( P_i(m) := \sum_{j=1}^N a_{ij} m_j \) and \( z_1, \ldots, z_N \) satisfy (4.1.1). In particular, taking \( s_1 = \cdots = s_N = 0 \) we find

\[
    D = \sum_m (-1)^{m_1+\ldots+m_N} \prod_{i=1}^N \left( P_i(m) \right) x^m.
\]

Remark 4.1.3. Note that (4.1.6) implies that the power series on the right hand side is an algebraic function of \( x_1, \ldots, x_N \) for any choice \( a_{i,j} \in \mathbb{Q} \). This generalizes an observation of Hurwitz in the one variable case (see [8] footnote 1, p. 43).

By (4.1.4)

\[
    D^{-1} = \det \left( I_N - FAZ^{-1} \right),
\]

where \( F \) and \( Z \) are the diagonal matrices

\[
    F := \text{diag}(x_1 f_1, \ldots, x_N f_N), \quad Z := \text{diag}(z_1, \ldots, z_N).
\]

In particular, \( D^{-1} \) is a Laurent polynomial in the \( z'_i \).

For example, in the dimension \( N = 1 \) case of [13] with \( z = z(T) = 1 + O(T) \) a solution to \( 1 - z = T z^a \) for a generic \( a \in \mathbb{Z} \), we find \( D^{-1} = 1 + x a z^{a-1} \) and hence

\[
    D = \frac{z}{1 + a(1-z)} = \sum_{n \geq 0} \left( \frac{an}{n} \right) (-T)^n.
\]
Lemma 4.1.4. Let $A = (a_{i,j})$ be an $N \times N$ matrix and $x_1, \ldots, x_N$ independent variables. Then
\[
\det(\text{diag}(x_1, \ldots, x_N) - A) = \sum_I (-1)^{|I|} D_I x^I,
\]
where $I$ runs over all subsets of $\{1, 2, \ldots, N\}$, $x^I := \prod_{i \in I} x_i$ and $D_I$ is the determinant of the matrix obtained from $A$ by striking the rows and columns whose indices are in $I$.

Combining the lemma with (4.1.7) we get
\[
D^{-1} = \sum_I (-1)^{|I|} D_I \prod_{i \in I} \frac{x_i f_i}{z_i}, \tag{4.1.8}
\]
where $I$ is the complement of $I$.

We would like an explicit form for the series expansion of $z_1^{s_1} \cdots z_N^{s_N}$. To do this combine (4.1.5), which already yields $z_1^{s_1} \cdots z_N^{s_N}$ as a ratio of two power series, with (4.1.8) to obtain
\[
z_1^{s_1} \cdots z_N^{s_N} = \sum_I D_I \prod_{m} (-1)^{m_1+\cdots+m_N} \prod_{j=1}^{N} \left( \frac{s_j + \sum_{k=1}^{N} a_{k,j} m_k + \sum_{i \in I} (a_{i,j} - \delta_{1,i})}{m_j} \right) x^m,
\]
where $\delta_{1,i} := 1$ if $i \in I$ and is 0 otherwise and
\[
R(m; s) := \sum_I (-1)^{|I|} D_I \prod_{i \in I} \frac{m_i}{s_i + P_i(m)}.
\]
Note that it follows from the above identity that $R(m; 0) = 0$ for $m$ non-zero. We can write $R$ back as a determinant using Lemma 4.1.4. By the generalized matrix-tree theorem we can interpret this determinant in terms of weighted spanning trees on $N$ labelled vertices. For a generic symmetric matrix $A$ we have
\[
\frac{\partial}{\partial s_i} R(m; s) \bigg|_{s=0} = \frac{m_i \tau(m)}{\prod_{i=1}^{N} P_i(m)}, \tag{4.1.9}
\]
where
\[
\tau(m) := \sum_T \prod_{i=1}^{N} m_i^{d_i-1} \prod_{e} a_{i,j}.
\]
Here the second product runs over all edges $i \rightarrow j$ of the tree $T$ and $d_i$ is the number of edges connecting to the vertex $i$ in $T$.

It turns out that $\tau(m)$ has a particularly simple form when $A = \mathcal{H}_N$; this is not true in general.

Lemma 4.1.5. For $A = \mathcal{H}_N$ we have
\[
\tau(m) = P_2(m) \cdots P_{N-1}(m).
\]

4.2 Schl"afli’s differential

For a generic symmetric matrix $A = (a_{i,j}) \in \mathbb{Z}^{N \times N}$ define
\[
V(x) = \lim_{q \to 1} \left\{ (q-1) \log \left[ \sum_m q^m \prod_{i=1}^{N} x_i^{a_{i,j}} \right] \right\}
\]
and let $z_i = z_i(x)$ be the power series solutions to the saddle point equations (4.1.1) of the previous section. Then $V$ and $z_i$ are related as follows.
Proposition 4.2.1. We have
\[ dV(x) = \sum_{i=1}^{N} \log z_i(x) \frac{dx_i}{x_i}. \]

Remark 4.2.2. We call \( dV \) Schlafli’s differential because of its analogy to the situation in hyperbolic geometry (see [6]). There \( V \) is the volume of a polyhedron in hyperbolic 3-space, \( \log z_i \) correspond to the length of an edge and \( \frac{dx_i}{x_i} \) to \( d\theta_i \), where \( \theta_i \) is the dihedral angle of the polytope at that edge. Although we will not emphasize this aspect, \( V \) can be expressed as a sum of dilogarithms of the \( z_i \)'s. This is what we see in Proposition 3.0.6.

4.3 Proof of Theorem 1.0.1

We return to the case of interest, where \( A = -\rho H_N \). It follows from Proposition 4.2.1 that
\[ z_i = \prod_m (1 - x^m)^{m_i A_m(1)} \tag{4.3.1} \]

Given a non-zero \( m \) chose \( i \) such that \( m_i \neq 0 \). Formula (1.0.1) now follows from (4.1.9) and Lemma 4.1.5 by taking the logarithm of (4.3.1). The M\"obius inversion in (1.0.1) accounts for the difference between \( \log \) and \( \text{Log} \).

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