FRACTAL DIMENSION OF RANDOM ATTRACTOR FOR
STOCHASTIC NON-AUTONOMOUS DAMPED WAVE
EQUATION WITH LINEAR MULTIPLICATIVE WHITE NOISE

SHENGFAN ZHOU* AND MIN ZHAO
Department of Mathematics
Zhejiang Normal University
Jinhua, 321004, China

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Abstract. In this paper, we first present some conditions for bounding the
fractal dimension of a random invariant set of a non-autonomous random dy-
namical system on a separable Banach space. Then we apply these conditions
to prove the finiteness of fractal dimension of random attractor for stochastic
damped wave equation with linear multiplicative white noise.

1. Introduction. It is well known that the finite dimensionality of attractor is one
of most important topics in studying the asymptotic behavior of infinite-dimensional
dynamical systems. Recently, the random attractors for autonomous and non-
autonomous random dynamical systems have been studied widely since Crauel and
Flandoli [8] established a criterion for the existence of a random attractor for an au-
tonomous random dynamical systems, see [3-5, 7, 9, 10, 16, 24-27] and the references
wherin. As we know, there are several approaches to estimate the upper bound
of Hausdorff and fractal dimension of random attractors, see [9, 11, 12, 19, 20, 29].
Of those works, Crauel and Flandoli [9], Debussche [11, 12] developed some tech-
nique for bounding the Hausdorff dimension of random attractors for autonomous
random dynamical systems; and Langa [20] generalized the method of [12] to the
fractal dimension but requiring differentiability of random dynamical system. Wang
and Tang [29] gave a method to bound the fractal dimension of random attractor
similar to [20], but requiring some “strong” conditions that the Lipshitz constant
of system and the “contraction” coefficient of the infinite-dimensional part of sys-
tem are deterministic constants independent of the sample points which are only
satisfied by some special random systems with uniform bounded derivative of the
nonlinearity. In fact, the difficulty to a random system is that a random attractor
is not uniformly bounded along the sample path of the sample points.

The finite fractal dimension of attractor plays a very important role in the finite-
dimensional reduction theory of infinite dimensional dynamical systems basing on

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* Corresponding author: Shengfan Zhou.
the fact that a compact set \( A \) in a metric space with fractal dimension \( \dim_f(A) \) less than \( m/2 \) for some \( m \in \mathbb{N} \) can be placed in the graph of Hölder continuous mapping which maps a compact subset of \( \mathbb{R}^m \) onto \( A \). But no such a finite parametrization is available for a set if just knowing the boundedness of its Hausdorff dimension (see [20, 22]).

Motivated by the ideas of [11, 20, 26, 29], in this article, we first give some conditions for bounding the fractal dimension of a random invariant set for a non-autonomous random dynamical system originated from Debussche [11]. Our conditions don’t require the differentiability of random dynamical system and just need the boundedness of expectation of some random variables. And then we consider the following non-autonomous stochastic damped wave equation with multiplicative white noise

\[
\begin{align*}
    d u_t + a d u_t + ( - \Delta u + f(u, x) ) d t &= g(x, t) d t + a u \circ d W(t) \quad \text{in} \ U \times (\tau, +\infty), \tau \in \mathbb{R}, \\
    u(x, t)|_{x \in \partial U} &= 0, \quad t \geq \tau, \\
    u(x, \tau) &= u_\tau(x), \quad u_t(x, \tau) = u_1\tau(x), \quad x \in U,
\end{align*}
\]

(1)

where \( u = u(x, t) \) is a real-valued function on \( U \times [\tau, +\infty), \tau \in \mathbb{R}, U \) is an open bounded set of \( \mathbb{R}^n \) (\( n \leq 3 \)) with a smooth boundary \( \partial U, g(x, \cdot) \in C_b(\mathbb{R}, H^1_0(U)) \), \( \alpha > 0, a \in \mathbb{R}, W(t) \) is a one-dimensional two-sided Wiener process, the random term “\( a u \circ d W(t) \)” is in the Stratonovich sense: \( f(u, x) \) satisfies the following condition:

(H) there exist constants \( c_1, c_2, c_3 > 0 \) and functions \( \beta_i(x) \in L^1(U), i = 1, 2, \) such that

\[
f(\cdot, x) \in C^1(\mathbb{R}^n; \mathbb{R}), \quad f(0, x) = 0, \quad |f_\beta(u, x)| \leq c_1(1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}, x \in U,
\]

(2)

\[
c_2 |u|^{p+1} - \beta_1(x) \leq G(u, x) \leq c_3 u f(u, x) + \beta_2(x), \quad \forall u \in \mathbb{R}, x \in U,
\]

(3)

where \( G(u, x) = \int_0^u f(r, x) dr \) and

\[
\begin{align*}
    1 \leq p < \infty, & \quad \text{when} \quad n = 1, 2, \\
    1 \leq p < 3, & \quad \text{when} \quad n = 3.
\end{align*}
\]

(4)

An example of such nonlinearity is \( f(u, x) = |u|^{p-1}u \) arising in the relativistic quantum mechanics equation [23]. The attractor for damped wave equation of type (1) has been studied widely. For the global attractors, pullback attractors (or kernel sections) and the bounds of their Hausdorff and fractal dimensions for the deterministic autonomous and non-autonomous wave equations without noise (i.e., \( a = 0 \)), we can see [2, 6, 17, 18, 23]. For the random attractor and the bounds of its Hausdorff dimensions for the stochastic wave equations (1) with additive noise (i.e., the random term is “\( a d W(t) \)” independent of \( u \)), we can see [9, 10, 11, 13, 21, 27, 30, 32]. For the stochastic system (1) with linear multiplicative noise “\( a u \circ d W(t) \)” and sufficient small coefficient \( a \), when the nonlinear function \( f \) and its derivative \( f' \) are both uniformly bounded, Fan [14] proved the existence of random attractor and obtained an upper bound of the Hausdorff and fractal dimension of the random attractor by using the method of [12]. Later, Wang [25] proved the existence of random attractor under condition (H). We notice that in [14], the obtained upper bound of the dimension of random attractor is independent of the random term and also that of the dimension of attractor for the corresponding deterministic system without noise. However, in general, the random attractor should depend on the noise term for nonlinear stochastic equations.

Notice that equation (1) is a non-autonomous stochastic equation that the external term \( g \) is time-dependent. For this case, Wang have established an efficacious
theory about the existence and upper semi-continuity of random attractors for non-autonomous stochastic systems by introducing two parametric spaces, see [24-27]. Here we first prove that when $f$ satisfies (H) and the coefficient $a$ is small enough, the system (1) has a random attractor in $H^1_0(U) \times L^2(U)$ which is bounded in $[H^2(U) \cap H^1_0(U)] \times H^1_0(U)$ by “iteration”, then we apply our criteria to obtain an upper bound of fractal dimension of random attractor which implies that the random attractor of (1) can be embedded in a finite dimensional Euclidean space. We will see that our obtained upper bound is influenced by the noise term. It is worth mentioning that we need to prove the higher regularity of random attractor by a recurrence method, for this aim, we establish some new argument basing on the idea of [31].

This paper is organized as follows. In Section 2, we first present some concepts related to non-autonomous random dynamical system and random attractor, then we give some sufficient conditions to obtain an upper bound of fractal dimension of a random invariant set. In section 3, we apply our method to get an upper bound of fractal dimension of random attractor of system (1).

2. Fractal dimension of random invariant sets. In this section, we give some sufficient conditions to bound the fractal dimension of a random invariant set for a non-autonomous random dynamical system.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X$ be a separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(X)$. First we present some basic concepts related with non-autonomous random dynamical system and random attractor (see [1, 26] for details).

**Definition 2.1.** The space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system, if $\{\theta_t : \Omega \rightarrow \Omega, t \in \mathbb{R}\}$ is a family of measure preserving transformations such that $(t, \omega) \rightarrow \theta_t \omega$ is $\mathbb{P}$-measurable, $\theta_0$ is the identity on $\Omega$, $\theta_{s+t} = \theta_s \theta_t$ for all $s, t \in \mathbb{R}$. In addition, if for any $F \in \mathcal{F}$, provided $\mathbb{P}(\theta_t^{-1}F \Delta F) = 0$, it holds that $\mathbb{P}(F) = 0$ or 1 for all $t \in \mathbb{R}$, then called the space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ or $(\theta_t)_{t \in \mathbb{R}}$ an ergodic metric dynamical system.

**Definition 2.2.** Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ be a metric dynamical system. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \rightarrow X$ is called a continuous (non-autonomous) random dynamical system (RDS) or cocycle on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the mapping $\Phi$ satisfies

(i) $\Phi(t, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \rightarrow X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on $X$;

(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \Phi(s, \tau, \omega, \cdot))$;

(iv) $\Phi(t, \tau, \omega, \cdot) : X \rightarrow X$ is continuous.

Let $B = \{B(\tau, \omega) \subseteq X : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of some subsets of $X$ which is parameterized by $(\tau, \omega) \in \mathbb{R} \times \Omega$. The family $B$ is measurable with respect to $\mathcal{F}$ in $\Omega$ if the set $B(\tau, \omega)$ is a closed nonempty subset of $X$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, and the mapping $\omega \in \Omega \rightarrow d(x, B(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$.

**Definition 2.3.** A family $B = \{B(\tau, \omega) \subseteq X : \tau \in \mathbb{R}, \omega \in \Omega\}$ is invariant under $\Phi$ if $\Phi(t, \tau, \omega)B(\tau, \omega) = \Phi(t, \tau, \omega, B(\tau, \omega)) = B(t + \tau, \theta_t \omega)$ for all $t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega$.

**Definition 2.4.** A family $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for every $\epsilon > 0$, $\lim_{t \rightarrow \infty} e^{-\epsilon |t|}\|B(t + \tau, \theta_t \omega)\|_X = 0$ for $\omega \in \Omega$, where $\|B(\tau, \omega)\|_X = \sup_{x \in B(\tau, \omega)} \|x\|$.
Let $\mathcal{D} = \mathcal{D}(X)$ be the collection of the tempered families of nonempty subsets of $X$.

**Definition 2.5.** A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a measurable $\mathcal{D}$-pullback attracting (or absorbing) set for $\Phi$ if

(i) $K$ is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$ in $\Omega$;

(ii) for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $B \in \mathcal{D}$, it holds: $\lim_{t \to +\infty} d_H(\Phi(t, \tau - t, \theta_{-t}, B(\tau - t, \theta_{-t})), K(\tau, \omega)) = 0$ (or there exists $T(\tau, \omega) > 0$ such that $\Phi(t, \tau - t, \theta_{-t}, B(\tau - t, \theta_{-t})) \subseteq K(\tau, \omega)$, $\forall t \geq T(\tau, \omega)$), where $d_H(\cdot, \cdot)$ denotes the Hausdorff semi-distance between two subsets of $X$.

**Definition 2.6.** A family $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback random attractor for $\Phi$ if

(i) $A(\tau, \omega)$ is measurable in $\omega$ with respect to $\mathcal{F}$ and compact in $X$ for all $\tau \in \mathbb{R}$, $\omega \in \Omega$;

(ii) $A$ is invariant, i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\Phi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta_t, \omega), \forall t \geq 0$;

(iii) $A$ attracts every member of $\mathcal{D}$, i.e., for every $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{t \to +\infty} d_H(\Phi(t, \tau - t, \theta_{-t}, B(\tau - t, \theta_{-t})), A(\tau, \omega)) = 0$.

For the existence of random attractor, as a direct consequence of [7, 8, 10, 26], we have the following Theorem.

**Theorem 2.7.** Let $\Phi$ be a continuous RDS on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. If $\Phi$ has a compact measurable (w.r.t. $\mathcal{F}$) $\mathcal{D}$-pullback attracting set $K$ in $\mathcal{D}$, then $\Phi$ has a unique $\mathcal{D}$-pullback random attractor $A$ in $\mathcal{D}$ given by: for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$A(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t}, K(\tau - t, \theta_{-t})).$$

For the finiteness of fractal dimension of a random invariant set for a non-autonomous random dynamical system, we have the following result originated from the idea of Theorem 1.4 of [11].

**Theorem 2.8.** Let $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ be a RDS on a separable Banach space $X$ over $\mathbb{R}$ and ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. Assume that there exists a family of bounded closed random subsets $\{\chi(\tau,\omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of $X$ satisfying the following conditions: for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(H1) there exists a tempered random variable $R_\omega$ (independent of $\tau$) such that the diameter $\|\chi(\tau,\omega)\|_X$ of $\chi(\tau,\omega)$ is bounded by $R_\omega$, i.e., $\sup_{\tau \in \mathbb{R}} \sup_{\omega \in \chi(\tau,\omega)} \|u\|_X \leq R_\omega < \infty$ and $R_{\theta_\tau \omega}$ is continuous in $t$ for all $t \in \mathbb{R}$;

(H2) invariance: $\chi(\tau + \tau, \theta_{\tau} \omega) = \Phi(t, \tau, \omega) \chi(\tau, \omega)$ for all $t \geq 0$;

(H3) there exist positive numbers $\delta$, $t_0$, random variable $C_0(\omega) \geq 0$, $\mathbb{C}_1(\omega)$ and $m$-dimensional projector $P_m: X \to P_mX$ ($\dim(P_mX) = m$) such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any $u, v \in \chi(\tau, \omega)$, it holds that

$$\|P_m \Phi(t_0, \tau, \omega)u - P_m \Phi(t_0, \tau, \omega)v\|_X \leq e^{t_0 \delta} C_0(\theta_t \omega) \|u - v\|_X$$

and

$$\|(I - P_m) \Phi(t_0, \tau, \omega)u - (I - P_m) \Phi(t_0, \tau, \omega)v\|_X \leq e^{t_0 \delta} C_0(\theta_t \omega) \|u - v\|_X,$$

where $\delta$, $t_0$, $m$ are independent of $\tau$ and $\omega$. 


(H4) $t_0$, $\delta$, $C_0(\omega)$, $C_1(\omega)$ satisfy conditions:

\[
\begin{align*}
-\infty &< E[C_1(\omega)] < 0, \quad t_0 \geq \frac{\ln \frac{1}{m}}{2E[C_1(\omega)]} > 0, \quad 3E[C_1^2(\omega)] + E[C_1^4(\omega)] < \infty, \\
0 &< \delta \leq \min \left\{ \frac{1}{2m}, \frac{1}{m} \frac{1}{2E[C_1^2(\omega)] + E[C_1^4(\omega)]} \right\}, \tag{7}\end{align*}
\]

where “$E$” denotes the expectation.

Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, the fractal dimension of $\chi(\tau, \omega)$ has an upper bound:

\[
\dim_f \chi(\tau, \omega) = \lim_{\varepsilon \to 0^+} \sup \frac{\ln N_\varepsilon(\chi(\tau, \omega))}{-\ln \varepsilon} \leq \frac{2m \ln \left( \frac{\sqrt{m}}{\delta} + 1 \right)}{\ln \frac{4}{3}} < \infty, \tag{8}\]

where $N_\varepsilon(\chi(\tau, \omega))$ is the minimal number of balls with radius $\varepsilon > 0$ covering $\chi(\tau, \omega)$ in $X$.

Proof. Let $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by (H1), taking $u_0 \in \chi(\tau, \omega)$, then

\[
\chi(\tau, \omega) \subseteq B(u_0, R_\omega). \tag{9}\]

For any $u \in \chi(\tau, \omega) = \chi(\tau, \omega) \cap B(u_0, R_\omega)$, by (5)-(6), we have

\[
||P_m \Phi(t_0, \tau, \omega)u - P_m \Phi(t_0, \tau, \omega)u_0||_X \leq e^{\int_{0}^{t_0} C_0(\theta, \omega)ds} R_\omega, \tag{10}\]

and

\[
||(I - P_m) \Phi(t_0, \tau, \omega)u - (I - P_m) \Phi(t_0, \tau, \omega)u_0||_X \leq (e^{\int_{0}^{t_0} C_1(\tau, \omega)ds} + 2\delta e^{\int_{0}^{t_0} C_0(\theta, \omega)ds}) R_\omega. \tag{11}\]

By Lemma 1.2 in [11], there exist $v_{01}, \ldots, v_{01}^{n_1} \in P_m X$ such that

\[
B_{P_m X}(P_m \Phi(t_0, \tau, \omega)u_0, e^{\int_{0}^{t_0} C_0(\theta, \omega)ds} R_\omega) \subseteq \bigcup_{j=1}^{n_1} B_{P_m X}(v_{01}^j, \delta e^{\int_{0}^{t_0} C_0(\theta, \omega)ds} R_\omega), \tag{12}\]

where $B_{P_m X}(v, r)$ denotes the ball in $P_m X$ of radius $r$ and center $v$, and

\[
n_1 \leq \left( \frac{\sqrt{m}}{\delta} + 1 \right)^m. \tag{13}\]

Take

\[
u_{01}^j = v_{01}^j + (I - P_m) \Phi(t_0, \tau, \omega)u_0 \in X, \quad j = 1, \ldots, n_1, \tag{14}\]

then there exists a $j \in \{1, \ldots, n_1\}$ such that

\[
||\Phi(t_0, \tau, \omega)u - u_{01}^j||_X \leq ||P_m \Phi(t_0, \tau, \omega)u - v_{01}^j||_X + ||(I - P_m) \Phi(t_0, \tau, \omega)u - (I - P_m) \Phi(t_0, \tau, \omega)u_0||_X \leq (e^{\int_{0}^{t_0} C_1(\tau, \omega)ds} + 2\delta e^{\int_{0}^{t_0} C_0(\theta, \omega)ds}) R_\omega. \tag{15}\]

Thus, by (15) and (H2), it follows that

\[
\chi(t_0 + \tau, \theta_t \omega) = \Phi(t_0, \tau, \omega)(\chi(\tau, \omega) \cap B(u_0, R_\omega)) \subseteq \bigcup_{j=1}^{n_1} B(u_{01}^j, (e^{\int_{0}^{t_0} C_1(\theta, \omega)ds} + 2\delta e^{\int_{0}^{t_0} C_0(\theta, \omega)ds}) R_\omega). \tag{16}\]
Making the recursion to the inclusion (16), we have
\[
\chi(kt_0 + \tau, \theta_{kt_0} \omega) = \Phi(kt_0, \tau, \omega) \chi(\tau, \omega) \subseteq \bigcup_{j=1}^{n_1 \cdots n_k} B(u_{ij}^j, \prod_{l=1}^k \sigma_l R_\omega), \quad k \geq 1, \quad (17)
\]
where
\[
n_l \leq \left( \frac{\sqrt{m}}{\delta} + 1 \right)^m, \quad \sigma_l = e^{\int_{t_0}^{t_0} C_1(\theta_s \omega) ds} + 2\delta e^{\int_{t_0}^{t_0} C_0(\theta_s \omega) ds}, \quad l = 1, \ldots, k.
\]
Thus, by (18), the minimal number \( N_{r_k}(\chi(kt_0 + \tau, \theta_{kt_0} \omega)) \) of balls with radius \( r_k = (\prod_{l=1}^k \sigma_l R_\omega \) covering \( \chi(kt_0 + \tau, \theta_{kt_0} \omega) \) in \( X \) satisfies
\[
N_{r_k}(\chi(kt_0 + \tau, \theta_{kt_0} \omega)) \leq n_1 \cdots n_k \leq \left( \frac{\sqrt{m}}{\delta} + 1 \right)^{km}.
\]
Set
\[
J = \left\{ \omega \in \Omega : \int_0^{t_0} [C_0(\theta_s \omega) - C_1(\theta_s \omega)] ds > \ln \frac{1}{8\delta} \right\}. \quad (20)
\]
(a) If \( \theta_{(l-1)t_0} \omega \in J \), then we have
\[
\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds - \int_{(l-1)t_0}^{lt_0} C_1(\theta_s \omega) ds = \int_0^{t_0} [C_0(\theta_s(\theta_{(l-1)t_0} \omega)) - C_1(\theta_s(\theta_{(l-1)t_0} \omega))] ds > \ln \frac{1}{8\delta},
\]
that is,
\[
8\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds} > e^{\int_{(l-1)t_0}^{lt_0} C_1(\theta_s \omega) ds},
\]
thus, by (7), (18) and (22),
\[
\sigma_l = e^{\int_{(l-1)t_0}^{lt_0} C_1(\theta_s \omega) ds} + 2\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds} < 10\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds} \leq \frac{1}{2} e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds}, \quad l \geq 1
\]
and
\[
\prod_{l=1}^k \sigma_l \leq \frac{1}{2^k} \sum_{l=1}^k \zeta_j(\theta_{(l-1)t_0} \omega) \cdot e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds},
\]
where
\[
\zeta_j(\omega) = \begin{cases} 1, & \omega \in J, \\ 0, & \omega \notin J. \end{cases}
\]
Since \( (\theta_t)_{t \in \mathbb{R}} \) is measure-preserving and ergodic on \( (\Omega, \mathcal{F}, \mathbb{P}) \), by Birkhoff ergodic Theorem [24], we have that
\[
\mathbb{E}[C_1(\theta_s \omega)] = \mathbb{E}[C_1(\omega)], \quad \mathbb{E}[C_1^s(\theta_s \omega)] = \mathbb{E}[C_1^s(\omega)], \quad \forall s \in \mathbb{R}, \quad i = 0, 1 \quad (26)
\]
and for all \( \omega \in \Omega \) (in fact, for a.e. \( \omega \in \Omega \),
\[
\frac{1}{k} \sum_{l=1}^k \zeta_j(\theta_{(l-1)t_0} \omega) \cdot \int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds \rightarrow \mathbb{E} \left( \zeta_j(\omega) \cdot \int_0^{t_0} C_0(\theta_s \omega) ds \right) \text{ as } k \rightarrow \infty.
\]
Thus, for every \( \omega \in \Omega \), there exists a large integer \( k_0(\omega) \in \mathbb{N} \) such that
\[
\sum_{l=1}^k \zeta_j(\theta_{(l-1)t_0} \omega) \cdot \int_{(l-1)t_0}^{lt_0} C_0(\theta_s \omega) ds \leq 2k \mathbb{E} \left( \zeta_j(\omega) \cdot \int_0^{t_0} C_0(\theta_s \omega) ds \right), \quad \forall k \geq k_0(\omega).
\]
By (20), (25) and Hölder inequality, we have

\[ E \left( \zeta_j(\omega) \int_0^{t_0} C_0(\theta_s, \omega) ds \right) = E \left( \zeta_j(\omega) \left( \frac{\int_0^{t_0} C_0(\theta_s, \omega) ds \int_0^{t_0} |C_0(\theta_s, \omega) - C_1(\theta_s, \omega)| ds}{\int_0^{t_0} |C_0(\theta_s, \omega) - C_1(\theta_s, \omega)| ds} \right) \right) \leq E \left( \zeta_j(\omega) \left[ \left( \int_0^{t_0} C_0(\theta_s, \omega) ds \right)^2 - \int_0^{t_0} C_0(\theta_s, \omega) ds \int_0^{t_0} C_1(\theta_s, \omega) ds \right] \right) \leq \frac{1}{2 \ln \frac{1}{\varepsilon_0}} E \left( \zeta_j(\omega) \left[ 3 \left( \int_0^{t_0} C_0(\theta_s, \omega) ds \right)^2 + \left( \int_0^{t_0} C_1(\theta_s, \omega) ds \right)^2 \right] \right) \leq \frac{1}{2 \ln \frac{1}{\varepsilon_0}} t_0^2 \left( 3E[C_0^2(\omega)] + E[C_1^2(\omega)] \right). \] (29)

Thus, by (7), (24) and (29), we have

\[ \prod_{l=1}^k \sigma_l \leq \frac{1}{2^k} e^{k \frac{1}{4} t_0^2 \left( 3E[C_0^2(\omega)] + E[C_1^2(\omega)] \right)} \leq \frac{1}{2^k} e^{k \ln \frac{3}{4}} = \left( \frac{3}{4} \right)^k. \] (30)

(b) If \( \theta_{(i-1)t_0} \omega \notin J \), by (20), we have

\[ \int_{(i-1)t_0}^{it_0} |C_0(\theta_s, \omega) - C_1(\theta_s, \omega)| ds \leq \ln \frac{1}{\varepsilon_0}, \]

i.e.,

\[ 2\delta e^{\int_{(i-1)t_0}^{it_0} C_0(\theta_s, \omega) ds} \leq \frac{1}{2} e^{\int_{(i-1)t_0}^{it_0} C_1(\theta_s, \omega) ds}, \] (31)

then by (18) and (31),

\[ \sigma_l \leq \frac{5}{4} e^{\int_{(i-1)t_0}^{it_0} C_1(\theta_s, \omega) ds}, \quad \prod_{l=1}^k \sigma_l \leq \left( \frac{5}{4} \right)^k e^{\sum_{l=1}^k \int_{(i-1)t_0}^{it_0} C_1(\theta_s, \omega) ds}. \] (32)

By the ergodic Theorem and (26), for any \( \omega \in \Omega \),

\[ \frac{1}{k} \sum_{l=1}^k \int_{(i-1)t_0}^{it_0} C_1(\theta_s, \omega) ds \to E \left( \int_0^{t_0} C_1(\theta_s, \omega) ds \right) = t_0 E[C_1(\omega)] \quad \text{as} \quad k \to \infty. \] (33)

It follows that for every \( \omega \in \Omega \), there exists \( k_1(\omega) \) such that for \( k \geq k_1(\omega) \),

\[ \sum_{l=1}^k \int_{(i-1)t_0}^{it_0} C_1(\theta_s, \omega) ds \leq 2kt_0 E[C_1(\omega)]. \] (34)

Thus, by (7), (32) and (34), for all \( k \geq k_1(\omega) \),

\[ \prod_{l=1}^k \sigma_l \leq \left( \frac{5}{4} \right)^k \left( e^{2t_0 E[C_1(\omega)]} \right)^k \leq \left( \frac{3}{4} \right)^k. \] (35)
Therefore, combining (30) of (a) and (35) of (b), for every \( \omega \in \Omega \) and \( k \geq k_2(\omega) = \max\{k_0(\omega), k_1(\omega)\} \), we have
\[
\prod_{l=1}^{k} \sigma_l \leq \left( \frac{3}{4} \right)^{k-\infty} 0. \tag{36}
\]
Note that the right side of (17) is independent of \( \tau \), replacing \( \omega \) by \( \theta_{-kt_0} \omega \) in (17), we have, for \( k \geq 1 \)
\[
\chi(\tau, \omega) = \Phi(kt_0, \tau - kt_0, \theta_{-kt_0} \omega) \chi(\tau - kt_0, \theta_{-kt_0} \omega) \subseteq \bigcup_{j=1}^{n_1 \ldots n_k} B(w_{0j}, \prod_{l=1}^{k} \sigma_l) R_{\theta_{-kt_0} \omega}). \tag{37}
\]
By (H1) and [1], there exists a tempered random variable \( \tilde{b}_\omega (>0) \) such that
\[
R_{\theta_{-kt_0} \omega} \subseteq \tilde{b}_\omega e^{\frac{1}{2} \ln \frac{4}{3}} \, \forall k \geq k_2(\omega). \tag{38}
\]
By (36) and (38), we have
\[
0 < r_k(\omega) = \left( \prod_{l=1}^{k} \sigma_l \right) R_{\theta_{-kt_0} \omega} \leq \left( \frac{3}{4} \right)^{k-\infty} \tilde{b}_\omega e^{\frac{1}{2} \ln \frac{4}{3}} = \tilde{b}_\omega e^{\frac{1}{2} \ln \frac{4}{3} - \frac{\ln 3}{2}} 0. \tag{39}
\]
Now let us estimate the upper bound of fractal dimension of \( \chi(\tau, \omega) \): \( \text{dim}_f \chi(\tau, \omega) = \limsup_{\varepsilon \to 0^+} \frac{\ln N_\varepsilon(\chi(\tau, \omega))}{\ln \varepsilon} \).
\]
Let \( 0 < \varepsilon < 1 \) and \( n_\varepsilon = n_\varepsilon(\omega) \in \mathbb{N} \) be an integer such that \( r_{n_\varepsilon}(\omega) \leq \varepsilon < r_{n_\varepsilon-1}(\omega) \).
Then \( n_\varepsilon \to +\infty \) as \( \varepsilon \to 0^+ \). Taking \( \varepsilon \) small such that \( n_\varepsilon - 1 \geq k_2(\omega) \), then by (39),
\[
\frac{1}{-\ln r_{n_\varepsilon}(\omega)} \leq \frac{1}{-\ln r_{n_\varepsilon-1}(\omega)} \leq \frac{n_\varepsilon - 1}{\frac{n_\varepsilon}{2} \ln \frac{4}{3} - \ln \tilde{b}_\omega}, \tag{40}
\]
and by (19),
\[
N_\varepsilon(\chi(\tau, \omega)) \leq N_{r_{n_\varepsilon-1}(\omega)}(\chi(\tau, \omega)) \leq n_1 \cdot \ldots \cdot n_{n_\varepsilon - 1} \leq \left( \frac{\sqrt{M}}{\delta} + 1 \right)^{(n_\varepsilon - 1)m}. \tag{41}
\]
Then it follows from (40)-(41) that
\[
\text{dim}_f \chi(\tau, \omega) = \limsup_{\varepsilon \to 0^+} \frac{\ln N_\varepsilon(\chi(\tau, \omega))}{-\ln \varepsilon} \leq \limsup_{n_\varepsilon \to +\infty} \frac{\ln N_{r_{n_\varepsilon-1}(\omega)}(\chi(\tau, \omega))}{-\ln r_{n_\varepsilon}(\omega)} \leq \limsup_{n_\varepsilon \to +\infty} \frac{(n_\varepsilon - 1)m \ln \left( \frac{\sqrt{M}}{\delta} + 1 \right)}{\frac{n_\varepsilon}{2} \ln \frac{4}{3} - \ln \tilde{b}_\omega} = \frac{2m \ln \left( \frac{\sqrt{M}}{\delta} + 1 \right)}{\ln \frac{4}{3}} < \infty. \tag{42}
\]
\[
\square
\]
3. Stochastic damped nonlinear wave equation with linear multiplicative white noise. In this section, we devote to prove the boundedness of fractal dimension of random attractor by Theorem 2.8 for the following initial boundary value problem of non-autonomous stochastic damped nonlinear wave equation with linear multiplicative white noise

\[
\begin{align*}
\{ & du_t + adu_t + (-\Delta u + f(u, x))dt = g(x, t)dt + au \circ dW(t) \text{ in } U \times (\sigma, +\infty), \tau \in \mathbb{R}, \\
& u(x, t)|_{x \in \partial U} = 0, \quad t \geq \tau, \\
& u(x, \tau) = u_\tau(x), \quad u_t(x, \tau) = u_{1\tau}(x), \quad x \in U,
\end{align*}
\]  
(43)

where \( \alpha > 0, a \in \mathbb{R}, u = u(x, t) \) is a real-valued function on \( U \times [\sigma, +\infty) \), \( \tau \in \mathbb{R}, U \) is an open bounded set of \( \mathbb{R}^n \) with a smooth boundary \( \partial U \), \( g(x, \cdot) \in C_0(\mathbb{R}, H^1_0(U)) \), \( f(\cdot, \cdot) \) satisfies the condition (H) in section 1, \( W(t) \) is a one-dimensional two-sided Wiener process on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \), the Borel \( \sigma \)-algebra \( \mathcal{F} \) on \( \Omega \) being generated by the compact open topology, and \( \mathbb{P} \) being the corresponding Wiener measure on \( \mathcal{F} \). For any \( t \in \mathbb{R} \), define \( \theta_t \) on \( \Omega : (t, \omega) \mapsto \omega(t + \cdot) - \omega(t) \), then \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) is an ergodic metric dynamical system [1].

Let \( A = -\Delta \), \( D(A) = H^2(U) \cap H_0^1(U) \), then \( A \) is a self-adjoint positive linear operator with eigenvalues \( \{ \lambda_i \}_{i \in \mathbb{N}} : 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \lambda_m \to +\infty \) as \( m \to +\infty \). For \( r \in \mathbb{R} \), let \( V_r = D(A^r) \) be a Hilbert space, where the inner product is \( (u, v)_r = (A^r u, A^r v) \) and \( V_0 = L^2(U), V_1 = H_0^1(U), (A^{1/2} u, A^{1/2} v) = (\nabla u, \nabla v) \). The injection \( V_1 \hookrightarrow V_r \) is compact if \( r_1 > r_2 \). Write \( E_r = D(A^{r + \frac{1}{2}}) \times D(A^r) \) for \( r \in \mathbb{R} \). For simplicity, denote the inner and norm of \( L^2(U) \) as \( (\cdot, \cdot) \) and \( \| \cdot \| \), and write \( E_0 = H_0^1(U) \times L^2(U) = E \).

For our purpose, we first transfer (43) into a (deterministic) random system without noise terms. Write an Ornstein-Uhlenbeck stationary process as
\[
z(\theta_t \omega) := -\alpha \int_{-\infty}^t e^{\alpha s}(\theta_t \omega)(s)ds, \quad \forall t \in \mathbb{R}, \omega \in \Omega,
\]
(44)
which solves the Itô equation
\[
dz(\theta_t \omega) + \alpha z(\theta_t \omega)dt = dW(t).
\]
(45)

It is known from [1, 7, 15] that there is a \( \theta_t \)-invariant set \( \tilde{\Omega} \subset \Omega \) of full \( \mathbb{P} \) measure such that for every \( \omega \in \tilde{\Omega}, t \mapsto z(\theta_t \omega) \) is continuous in \( t \) and
\[
\begin{align*}
& \mathbb{E}\left[ e^{\int_{t}^{t+\epsilon} z(\theta_s \omega)^2 ds} \right] \leq e^{\frac{\alpha^2 t}{2}}, \quad \text{for } \alpha^2 \geq 2\epsilon \geq 0, \tau \in \mathbb{R}, t \geq 0; \\
& \mathbb{E}\left[ e^{\int_{t}^{t+\epsilon} z(\theta_s \omega)ds} \right] \leq e^{\frac{\alpha^3 t}{2}}, \quad \text{for } \alpha^3 \geq \epsilon^2 \geq 0, \tau \in \mathbb{R}, t \geq 0;
\end{align*}
\]
(46)
\[
\lim_{t \to +\infty} e^{-\epsilon t} |z(\theta_{-t} \omega)| = 0, \quad \forall \epsilon > 0; \quad \mathbb{E}[|z(\theta_t \omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi \alpha^r}}, \quad \forall r > 0, s \in \mathbb{R},
\]
(47)
where \( \Gamma(\cdot) \) is the Gamma function.

Introducing a variable transformation
\[
v(t, \tau, x) = u_t(t, \tau, x) + \varepsilon u(t, \tau, x) - auz(\theta_t \omega), \quad \varepsilon = \frac{\lambda_1 \alpha}{\alpha^2 + 3\lambda_1},
\]
(48)
then the problem (43) can be written as the following equivalent deterministic random system in \( E \):
\[
\begin{align*}
\varphi + \Lambda \varphi &= F(\varphi, \theta_t \omega, t), \quad \varphi_t(\omega) = (u_{\tau}, u_{1\tau} + \varepsilon u_{\tau} - auz(\theta_t \omega))^T, \quad t \geq \tau, \tau \in \mathbb{R},
\end{align*}
\]
(49)
where
\[
\begin{align*}
\varphi(t, \tau, x) &= \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t + \varepsilon u - auz(\theta_t \omega) \\ u \end{pmatrix}, \\
\Lambda &= \begin{pmatrix} \varepsilon I & 0 \\ A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I \end{pmatrix},
\end{align*}
\]
(50)
\[ F(\varphi, \theta_t \omega, t) = \left( \frac{ax(\theta_t \omega)u}{2} + az(\theta_t \omega)u - az(\theta_t \omega)v - f(u, x) + g(x, t) \right). \] (51)

From now on, we will consider (49) for \( \omega \in \Omega \) and still write \( \Omega \) as \( \Omega \). It follows from assumption (H) and [25] that the solution \( \varphi(\cdot, \tau, \omega) \) of (49) exists globally for \( t \in [\tau, +\infty) \) and \( \varphi(\cdot, \tau, \omega) \in C([\tau, +\infty); E) \), which defines a continuous random dynamical system

\[ \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E, \quad (t, \tau, \omega, \varphi) \to \Phi(t, \tau, \omega, \varphi) = \Phi(t, \tau, \omega) \varphi \] (52)

over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) by

\[ \Phi(t, \tau, \omega) \varphi = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi(\theta_{-\tau} \omega)) \]
\[ = \left( u_t(t + \tau, \tau, \theta_{-\tau} \omega, \varphi(\theta_{-\tau} \omega)) + \varepsilon u(t + \tau, \tau, \theta_{-\tau} \omega, \varphi(\theta_{-\tau} \omega)) - au(t, \theta_{-\tau} \omega) \right), \]

where

\[ \Phi(0, \tau, \omega) \varphi = \varphi(\theta_{-\tau} \omega), \]
\[ \Phi(t, \tau - t, \theta_{-\tau} \omega) \varphi = \varphi(t, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau - t}(\theta_{-\tau} \omega)). \] (54)

It is easy to see that \( \Psi(t, \tau, \omega) = R_{-\tau}^{-1} \Phi(t, \tau, \omega) R_{\tau} \) is a continuous random dynamical system over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) associated with (43), where for \( t \geq \tau \)

\[ \Psi(t, \tau, \omega) : \psi(\tau) = (u_T, u_{1T})^T \mapsto \Psi(t, \tau, \omega) \psi(\tau) = (u(t, \tau, \omega, \psi(\tau)), u_T(t, \tau, \omega, \psi(\tau)))^T, \] (55)

and \( R_{\epsilon, \theta_t} : (u, v)^T \to (u, v + \varepsilon u - au(t, \theta_{-\tau} \omega))^T \) is an isomorphism of \( E \). Thus, \( \Phi \) and \( \Psi \) have the same dynamics.

In the following, we first prove the existence of random attractor \( \mathcal{A}(\tau, \omega) \) of \( \Phi \) in \( E \) based on Theorem 2.7 and study the boundness of \( \mathcal{A}(\tau, \omega) \) in \( E^1 \) by “iteration”, then we prove the finiteness of the fractal dimension of \( \mathcal{A}(\tau, \omega) \) by Theorem 2.8.

**Remark.** In the proof of following Lemmas and Theorems, we just do them for \( n = 3 \) only, the proof is easier for the cases of \( n = 1, 2 \).

### 3.1. Boundedness of solutions

First we have the following estimate of solutions for (49).

**Lemma 3.1.** Suppose the assumption (H) holds. Then for any \( \tau \in \mathbb{R}, \omega \in \Omega, t \geq 0 \) and any set \( B(\tau, \omega) \in B \in \mathcal{D}(E) \), the solution \( \varphi(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau - t}(\theta_{-\tau} \omega)) \in \mathcal{E} \) of (49) with \( \varphi_{\tau - t}(\theta_{-\tau} \omega)) \in B(\tau - t, \theta_{-\tau} \omega) \) satisfies:

\[
\| \varphi(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau - t}(\theta_{-\tau} \omega)) \|^2 \leq \left( \| \varphi_{\tau - t}(\theta_{-\tau} \omega) \|^2 + 2c_3 c_1 (|u_{\tau - t}|^2 + |u_{\tau - t}|^2)^{p+1} + c_6 \right) \times \\
e^{-\int_{\tau - t}^{\tau} (\rho - c_4 |a||z(\theta_{-\tau} \omega)| + |z(\theta_{-\tau} \omega)|^2) \, ds} \\
+ c_5 \int_{\tau - t}^{\tau} e^{-\int_{\tau - t}^{\tau} (\rho - c_4 |a||z(\theta_{-\tau} \omega)| + |z(\theta_{-\tau} \omega)|^2) \, ds} \, ds, \quad \forall \tau \geq \tau - t,
\] (56)

where

\[
\begin{align*}
    & c_4 = \max \left\{ \frac{2c_1}{\lambda^2(\tau)}, \frac{c_1}{\lambda(\tau)}, \frac{1}{\lambda^2(\tau)} \right\}, \quad \rho = \min \left\{ \frac{1}{\tau}, \frac{\varepsilon}{\lambda^2(\tau)} \right\}, \\
    & c_5 = \frac{1}{\lambda^2(\tau)} \left( f_1 \beta_{1}(x) \, dx + 2f_1 \beta_{2}(x) \, dx \right), \\
    & c_6 = 4 \int_{U} \beta_{2}(x) \, dx + 2 \int_{U} \beta_{1}(x) \, dx.
\end{align*}
\] (57)
Thus, by (60), we have

\[ \frac{1}{2} \frac{d}{dt} |||\varphi(r)|||^2 \geq 2 \bar{G}(u, x) + (\Lambda \varphi, \varphi)_E + \varepsilon (f(u, x), u) \]

\[ = (f(u, x), azu(\theta_{r-\tau}) + (azu(\theta_{r-\tau}), u)_1) + ((2\varepsilon - az)azu - azv + g(x, r), v), \]

where \( \bar{G}(u, x) = \int_U G(u, x)dx \). By [14, 32], we have

\[ (\Lambda \varphi, \varphi)_E \geq \frac{\varepsilon}{2} (||u||^2 + ||v||^2) + \frac{\alpha}{2} ||v||^2. \]

By (H), we have

\[ \left\{ \begin{array}{l}
(f(u, x), u) \geq \frac{1}{c_3} \left( \bar{G}(u, x) - 2 \int_U \beta_2(x)dx \right), \\
||u||^{p+1} \leq \frac{1}{c_3} (G(u, x) + \beta_1(x)), \quad |f(u, x)| \leq c_1 (|u| + |u|^p).
\end{array} \right. \]  

Thus, by (60), we have

\[ (f(u, x), az(\theta_{r-\tau})u) \leq \frac{1}{c_2} \frac{|a| \cdot |z(\theta_{r-\tau})|}{\sqrt{\lambda_1}} ||u||^2 + \frac{c_1}{c_2} |a| \cdot |z(\theta_{r-\tau})| \left( \bar{G}(u, x) + \int_U \beta_1(x)dx \right). \]  

By \( \frac{1}{\sqrt{\lambda_1}} \leq 1 \), we have

\[ \frac{\varepsilon}{2} (||u||^2 + ||v||^2) \geq \left( |a| ||z(\theta_{r-\tau})| + \frac{|a|^2 ||z(\theta_{r-\tau})|^2}{2\sqrt{\lambda_1}} \right) (||u||^2 + ||v||^2) - az||v||^2, \]

\[ (azu(\theta_{r-\tau})u, u) \leq |a| \cdot |z(\theta_{r-\tau})| \cdot ||u||^2, \quad (g(x, r), v) \leq \frac{1}{2\alpha} ||g||^2 + \frac{\alpha}{2} ||v||^2. \]

where \( ||g||^2 = \sup_{r \in \mathbb{R}} ||g(r)||^2 \) is finite. By putting (59)-(63) into (58), we have that for all \( r \geq \tau - t, \)

\[ \frac{d}{dt} |||\varphi(r)|||^2 \geq 2 \bar{G}(u, x) + 2 \int_U \beta_1(x)dx + \varepsilon (||u||^2 + ||v||^2) + c_3 \left( \frac{1}{c_3} \bar{G}(u, x) + \frac{1}{c_3} \int_U \beta_1(x)dx \right) \]

\[ \leq \frac{2}{c_3} |a| \cdot |z(\theta_{r-\tau})| \left( \frac{1}{\sqrt{\lambda_1}} + 2 \right) \left( \frac{c_1}{c_2} |a| ||z(\theta_{r-\tau})||^2 \right) \left( ||u||^2 + ||v||^2 \right) + \frac{c_1}{c_2} |a| \cdot |z(\theta_{r-\tau})| \left( 2 \bar{G}(u, x) + 2 \int_U \beta_1(x)dx \right) + \frac{1}{\alpha} ||g||^2 \]

\[ + c_1 \left( \int_U \beta_1(x)dx + 2 \int_U \beta_2(x)dx \right) \leq c_4 \left( |a| \cdot |z(\theta_{r-\tau})| + |a|^2 ||z(\theta_{r-\tau})||^2 \right) \left( ||u||^2 + ||v||^2 + 2 \bar{G}(u, x) + 2 \int_U \beta_1(x)dx \right) + c_5. \]

Set

\[ y(r) = |||\varphi(r)|||^2 \geq 2 \bar{G}(u, x) + \int_U \beta_1(x)dx \geq |||\varphi(r)|||^2, \quad \forall r \geq \tau - t, \]

Then by (49) and (65),

\[ y(\tau - t, \tau - t, \varphi_{r-\ell}) \leq |||\varphi_{r-\ell}(\theta_{r-\tau})|||^2 \geq 2c_3 c_1 (||u_{r-\ell}||^2 + ||u_{r-\ell}||^{p+1}) + c_6, \]

and by (64),

\[ \frac{d}{dt} y(r) + \left( \rho - c_4 |a| \cdot |z(\theta_{r-\tau})| + |a|^2 ||z(\theta_{r-\tau})||^2 \right) y(r) \leq c_5, \quad \forall r \geq \tau - t. \]
By Gronwall’s inequality to (67) on \([\tau-t,r] \ (r \geq \tau-t)\), we have

\[
||\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))||_E^2 \\
\leq \varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega)) \\
\leq \varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega)) \\
e^{-\int_{\tau-t}^{r} \varphi(t,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))} \left(\rho-c_4||\varphi||_{\mathcal{D}(E)}+\beta||\varphi||_{\mathcal{D}(E)}\right)dt
\]

(68)

Thus, we have the following boundedness of solutions for (49).

**Lemma 3.2.** Suppose (H) and

\[
\rho_0 = \rho - c_4 \left(\frac{|a|}{\sqrt{\pi\alpha}} + \frac{|a|^2}{2\alpha}\right) > 0
\]

(69)

hold. Then for any \(\tau \in \mathbb{R}, \omega \in \Omega\) and any set \(B \in \mathcal{D}(E)\), there exist \(T(\tau,\omega,B) \geq 0\) and a tempered random variable (independent of \(\tau\))

\[
M_0(\omega) = \left(2c_5 \int_{-\infty}^{0} e^{-\int_{0}^{t} \varphi(t,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))} \left(\rho-c_4||\varphi||_{\mathcal{D}(E)}+\beta||\varphi||_{\mathcal{D}(E)}\right)dt\right)^{1/2} > 0
\]

(70)

such that the solution \(\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega)) \in E\) of (49) with \(\varphi_{-t}(\theta_{-\tau}\omega) \in B(\tau-t,\theta_{-\tau}\omega)\) satisfies:

\[
||\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))||_E \leq M_0(\omega), \quad \forall t \geq T(\tau,\omega,B),
\]

(71)

that is, the tempered family

\[
B_0 = \{B_0(\omega) = \{\varphi \in E : ||\varphi||_E \leq M_0(\omega)\} : \omega \in \Omega\}, \quad \text{(independent of } \tau \text{)}
\]

(72)

is a measurable \(\mathcal{D}(E)\)-pullback absorbing set for \(\Phi\). In particular,

\[
\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_0(\theta_{-\tau}\omega)) \in B_0(\omega), \quad \forall t \geq T(\tau,\omega,B_0), \quad r \geq \tau-t.
\]

(73)

Moreover, in addition, for any \(k \geq 1\) with

\[
|a|^2 \leq \min \left\{\frac{\alpha^3}{(k\alpha)^2}, \frac{\alpha^2}{2k}, 1\right\},
\]

(74)

then the expectation \(E(M_0^2(\omega)) < \infty\).

**Proof.** By (68), we have

\[
||\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))||_E^2 \\
\leq \left(||\varphi_{-t}(\theta_{-\tau}\omega)||_E^2 + 2c_3c_1(||u_{t-\tau}\|_2^2 + ||u_{t-\tau}\|_1^{p+1}) + c_6\right) \times \\
e^{-\int_{-\infty}^{0} \varphi(t,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))} \left(\rho-c_4||\varphi||_{\mathcal{D}(E)}+\beta||\varphi||_{\mathcal{D}(E)}\right)dt + \frac{1}{2}M_0^2(\omega).
\]

(75)

For the initial data \(\varphi_{-t}(\theta_{-\tau}\omega) \in B(\tau-t,\theta_{-\tau}\omega) \in \mathcal{D}(E)\), by (69), we have

\[
\sup \left(||\varphi_{-t}(\theta_{-\tau}\omega)||_E^2 + 2c_3c_1(||u_{t-\tau}\|_2^2 + ||u_{t-\tau}\|_1^{p+1}) + c_6\right) \times \\
e^{-\int_{-\infty}^{0} \varphi(t,\tau-t,\theta_{-\tau}\omega,\varphi_{-t}(\theta_{-\tau}\omega))} \left(\rho-c_4||\varphi||_{\mathcal{D}(E)}+\beta||\varphi||_{\mathcal{D}(E)}\right)dt \rightarrow 0\text{ as } t \rightarrow +\infty.
\]

(76)

Then from (74), there exists \(T(\tau,\omega,B) \geq 0\) such that (71) holds. By (47) and ergodic Theorem, for any \(\omega \in \Omega\),

\[
\lim_{t \rightarrow +\infty} \frac{1}{t} \int_{-\infty}^{0} c_4(||z(\theta_{-\tau}\omega)|| + ||a||^2||z(\theta_{-\tau}\omega)||^2)dt \rightarrow c_4 \left(\frac{|a|}{\sqrt{\pi\alpha}} + \frac{|a|^2}{2\alpha}\right),
\]

(77)
which implies that there exists a $t_0(\omega) > 0$ such that
\[
\int_{-t}^{0} c_4 ||a||z(\theta_t(\omega))| + |a|^2|z(\theta_t(\omega))|^2|d\leq (c_4 \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right) + \rho_0) t, \quad \forall t \geq t_0(\omega)
\]
and
\[
\int_{-\infty}^{0} e^{-s} f_s^a(\rho - c_4 ||a||z(\theta_s(\omega))| + |a|^2|z(\theta_s(\omega))|^2)ds \\
\leq \int_{-t_0(\omega)}^{0} e^{-\rho s + f_s^a c_4 ||a||z(\theta_s(\omega))| + |a|^2|z(\theta_s(\omega))|^2}d\omega + \frac{2}{\rho_0} e^{-\frac{1}{2}t_0(\omega)}.
\]
Thus, for any $\epsilon > 0$,
\[
e^{-\epsilon t} M_0(\theta_{-\omega}) \\
= \sqrt{2c_5} \left( e^{-2\epsilon t} \int_{-t_0(\omega)}^{0} e^{-\rho s + f_s^a c_4 ||a||z(\theta_s(\omega))| + |a|^2|z(\theta_s(\omega))|^2}d\omega + \frac{2}{\rho_0} e^{-2\epsilon t - \frac{1}{2}t_0(\omega)} \right)^{1/2} \\
\to 0 \quad \text{as} \quad t \to +\infty,
\]
that is, $M_0(\omega)$ is a tempered random variable. By (46), (69) and (70), for any $k \geq 1$ satisfying (74), then
\[
E(M_0^{2k}(\omega)) = 2^{k} \sum_{\ell} E \left( \int_{0}^{+\infty} e^{-\rho s} f_s^a c_4 ||a||z(\theta_s(\omega))| + |a|^2|z(\theta_s(\omega))|^2}ds \right)^k \\
\leq 2^{k} \sum_{\ell} \left( \int_{0}^{+\infty} e^{-\frac{s}{2\rho_0 - k\rho_0}} E e^{k s} f_s^a c_4 ||a||z(\theta_s(\omega))| + |a|^2|z(\theta_s(\omega))|^2}ds \right)^k \\
\leq 2^{k} \sum_{\ell} \left( \int_{0}^{+\infty} e^{-\frac{s}{2\rho_0}} k^{k-1} \left( \int_{0}^{+\infty} e^{-\frac{s}{2\rho_0}} ds \right) \right)^k \\
= 2^{k} \sum_{\ell} \left( \frac{2(k-1)}{k\rho_0} \right)^k < \infty.
\]

3.2. Decomposition of solutions. In this subsection, we assume that (H) and (69) hold. Next we decompose the solution of (49) into a sum of two components, one component decays exponentially and another one is bounded in a “higher regular” space.

For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, set
\[
B_1(\tau, \omega) = \bigcup_{t \geq T(\tau, \omega, B_0)} \varphi(\tau, \tau - t, \theta_{-\omega}, B_0(\theta_{-\omega})) \subseteq B_0(\omega),
\]
then by (73) and the cocycle property of $\varphi$, it holds that for $r \geq \tau - t$ and $t \geq 0$,
\[
\varphi(r, \tau - t, \theta_{-\omega}, B_1(\tau - t, \theta_{-\omega})) \subseteq B_1(r, \theta_{-\omega}) \subseteq B_0(\theta_{-\omega}).
\]
Let $\varphi(r) = \varphi(r, \tau - t, \theta_{-\omega}, \varphi_{\tau-t}(\theta_{-\omega}))$ ($r \geq \tau - t$) be a solution of (49) with $\varphi_{\tau-t}(\theta_{-\omega}) \in B_1(\tau - t, \theta_{-\omega}) \subseteq B_0(\theta_{-\omega})$, then it follows from (83) that
\[
||\varphi(r, \tau - t, \theta_{-\omega}, \varphi_{\tau-t}(\theta_{-\omega}))||_E \leq M_0(\theta_{-\omega}), \quad \forall r \geq \tau - t.
\]
Decompose $\varphi(r)$ into a sum $\varphi(r) = \varphi_L(r) + \varphi_N(r)$, where $\varphi_L(r) = (u_L, v_L)^T$ and $\varphi_N(r) = (u_N, v_N)^T$ satisfy, respectively,
\[
\begin{cases}
\varphi_L + A\varphi_L = F_L(\varphi_L, \theta_{\omega}), \quad r > \tau - t, \\
\varphi_L(\tau - t, t - t, \theta_{-\omega}, \varphi_{\tau-t}(\theta_{-\omega})) = \varphi_{\tau-t}(\theta_{-\omega}),
\end{cases}
\]
(85)
and
\[
\begin{cases}
\phi_N + \Lambda \phi_N = F_N(\phi_N, u, \theta_t, t), & r > t - t, \\
\phi_N(r, \tau - t, \theta_{\tau t}, \varphi_{\tau t}(\theta_{\tau t})) = (0, 0)^T,
\end{cases}
\]
where
\[
F_L(\phi_L, \theta_t) = \left(\frac{az(\theta_t)u_L}{(2\varepsilon - az(\theta_t))az(\theta_t)u_L - az(\theta_t)v_L}, \frac{az(\theta_t)u_N}{(2\varepsilon - az(\theta_t))az(\theta_t)u_N - az(\theta_t)v_N - f(u, x) + g(x, t)}\right).
\]
(86)

(87)

(88)

First, we show the component $\phi_L$ decays exponentially.

**Lemma 3.3.** For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $0 \leq \mu \leq 1$ and $t \geq 0$,
\[
\|\phi_L(\tau, \tau - t, \theta_{\tau t}, \varphi_{\tau t}(\theta_{\tau t}))\|_{E^\mu}^2 \\
\leq \|\phi_{\tau t}(\theta_{\tau t})\|_{E^\mu, e}^2 - \int_0^t \left(\varepsilon - 4|a| |z(\theta_t)| - \frac{1}{\sqrt{\lambda_1}} |a|^2 |z(\theta_t)|^2\right) ds.
\]
(89)

**Proof.** Taking the inner product of (85) in $E^\mu$ with $\phi_L = (u_L, v_L)^T$, we have, for all $r \geq \tau - t$,\[
\frac{1}{2} \frac{d}{dt} \|\phi_N\|_{E^\mu}^2 + (A_\mu, \phi_L)_{E^\mu} \\
= (az(\theta_{\tau t})u_L, u_L)_{2\mu + 1} + (2\varepsilon - az(\theta_{\tau t}))az(\theta_{\tau t})u_L - az(\theta_{\tau t})v_L, v_L)_{2\mu}.
\]
(90)

Similar to (59), (62)-(63), we have
\[
(A_\mu, \phi_L)_{E^\mu} \geq \frac{\varepsilon}{2} \|\phi_L\|_{E^\mu}^2 + \frac{\alpha}{2} \|A_\mu v_L\|^2,
\]
(91)

\[
\begin{split}
(az(\theta_{\tau t})u_L, u_L)_{2\mu + 1} - (azv_L, v_L)_{2\mu} \leq |a| \cdot |z(\theta_{\tau t})| \|\phi_L\|_{E^\mu}^2.
\end{split}
\]
(92)

Thus, for all $r \geq \tau - t$,\[
\frac{d}{dr} \|\phi_L(r)\|_{E^\mu}^2 \leq \left(-\varepsilon + 4|a| |z(\theta_{\tau t})| + \frac{1}{\sqrt{\lambda_1}} |a|^2 |z(\theta_{\tau t})|^2\right) \|\phi_L(r)\|_{E^\mu}^2.
\]
(93)

we have that for $r \geq \tau - t$,
\[
\begin{split}
\|\phi_L(r, \tau - t, \theta_{\tau t}, \varphi_{\tau t}(\theta_{\tau t}))\|_{E^\mu}^2 \\
\leq \|\phi_{\tau t}(\theta_{\tau t})\|_{E^\mu, e}^2 - \int_0^t \left(\varepsilon - 4|a| |z(\theta_t)| - \frac{1}{\sqrt{\lambda_1}} |a|^2 |z(\theta_t)|^2\right) ds.
\end{split}
\]
(94)

(95)

(96)

For the second component $\phi_N$, we have the following boundedness of $\phi_N$ in a “higher regular” space.

**Lemma 3.4.** Assume that
\[
\rho_1 = \frac{\varepsilon}{2} - \frac{8|a|}{\sqrt{\pi\alpha}} - \frac{2|a|^2}{\alpha\sqrt{\lambda_1}} > 0.
\]
(97)
Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, there exists $M_{1\nu}(\omega) > 0$ (independent of $\tau$ and $t$) such that the solution $\varphi(t, \tau - t, \theta_{\tau}\omega, \varphi_{\tau}(\theta_{\tau}\omega))$ of (86) satisfies

$$\|A^{\nu+1/2}u_N(t, \tau - t, \theta_{\tau}\omega, \varphi_{\tau}(\theta_{\tau}\omega))\|^2 + \|A^{\nu}u_{N,t}(t, \tau - t, \theta_{\tau}\omega, \varphi_{\tau}(\theta_{\tau}\omega))\|^2 \leq M_{1\nu}^2(\omega),$$  

(98)

where

$$0 < \nu \leq \frac{1}{4}.$$  

(99)

**Proof.** Taking the inner product of (86) in $E$ with $A^{2\nu}\varphi_N = (A^{2\nu}u_N, A^{2\nu}v_N)^T$, we have

$$\frac{1}{2} \frac{d}{dt} \left( \|A^{\nu+1/2}u_N\|^2 + \|A^{\nu}v_N\|^2 \right) + 2 \int_{\mathbb{R}} f(u(x)) \cdot A^{2\nu}u_N \, dx + (A_{\varphi}N, A^{2\nu}\varphi) \nu = (az(\theta_{\tau}\omega, u_N), A^{2\nu}u_N) - (az(\theta_{\tau}\omega) v_N, A^{2\nu}v_N) + (f(u, x), A^{2\nu}az(\theta_{\tau}\omega)u_N)

+ (g(x, r) + (2\nu - az(\theta_{\tau}\omega))az(\theta_{\tau}\omega)u_N, A^{2\nu}v_N), \quad \forall \tau \geq t.$$  

(100)

Similar to (59), (61)-(63), we have

$$\|A\varphi_N, A^{2\nu}\varphi ) \nu \geq \frac{\varepsilon}{2} \left( \|A^{\nu+1/2}u_N\|^2 + \|A^{\nu}v_N\|^2 \right) + \frac{\alpha}{2} \|A^{\nu}v_N\|^2,$$  

(101)

$$= (az(\theta_{\tau}\omega)u_N, A^{2\nu}u_N) - (az(\theta_{\tau}\omega) v_N, A^{2\nu}v_N)

\leq |a| \cdot |z(\theta_{\tau}\omega)| \cdot (\|A^{\nu+1/2}u_N\|^2 + \|A^{\nu}v_N\|^2),$$  

(102)

$$\|g(x, r), A^{2\nu}v_N\| \leq \frac{1}{2\alpha} \|g\|^2 + \frac{\alpha}{2} \|A^{\nu}v_N\|^2,$$  

(103)

$$\leq \left(|a| \cdot |z(\theta_{\tau}\omega)| + \frac{1}{2\sqrt{\lambda_1}} |a|^2 |z(\theta_{\tau}\omega)|^2 \right) \left( \|A^{\nu+1/2}u_N\|^2 + \|A^{\nu}v_N\|^2 \right),$$  

(104)

where $\|g\|^2 = \sup_{r \in \mathbb{R}} \|g(\cdot, r)\|_1^2 < \infty$. For $0 \leq \nu \leq \frac{1}{4}$, we have $H_0^2(U) = D(A^{2\nu}) = H^{\nu}(U)$ and

$$H^{(1)}(U) \subset H^{\nu}(U) \quad \text{if} \quad r_1 \geq r_2 \quad \text{and} \quad H^{\nu}(U) \subset L^q(U), \quad \text{where} \quad \frac{1}{q} = \frac{1}{2} - \nu.$$  

(105)

Thus, by (60), (105), (84) and (95), we have

$$\int f(x, x), A^{2\nu}u_N = \int \int f(x, x) \cdot A^{2\nu}u_N \, dx

\leq \|f(x, x)\| \cdot \|A^{2\nu}u_N\|

\leq c_1(1 + \|u_N\|^2) \cdot (\|A^{2\nu}u\| + \|A^{2\nu}v\|)

\leq c_1 \left( M_0^2(\theta_{\tau}\omega) + M_0^2(\theta_{\nu}\omega) e^{-\frac{1}{4} \int_{t-t}^{t-t} (-4|a||z(\theta_{\tau}\omega)| - \frac{\alpha}{2\sqrt{\lambda_1}} |a|^2 |z(\theta_{\tau}\omega)|^2) \, ds} \right)

\leq c_8 \left( M_0^2(\theta_{\tau}\omega) + M_0^2(\theta_{\nu}\omega) e^{-\frac{1}{4} \int_{t-t}^{t-t} (-4|a||z(\theta_{\tau}\omega)| - \frac{\alpha}{2\sqrt{\lambda_1}} |a|^2 |z(\theta_{\tau}\omega)|^2) \, ds} \right)

\leq R_1(\theta_{\tau}\omega, \theta_{\nu}\omega), \quad \forall \tau \geq t,$$  

where $

R_1(\theta_{\tau}\omega, \theta_{\nu}\omega)$ is a constant independent of $\tau$ and $t$.\]
and by (99),

\[
(f(u, x), A^{2\nu} az(\theta_{r-\tau}\omega)u_N)
\]

\[
\leq |a||z(\theta_{r-\tau}\omega)|| \int_U f(u, x) \cdot A^{2\nu} u_N dx
\]

\[
\leq |a||z(\theta_{r-\tau}\omega)| c_1 \int_U (|u| + |u|^p) \cdot |A^{2\nu} u_N| dx
\]

\[
\leq |a||z(\theta_{r-\tau}\omega)| c_1 \left( \int_U (|u| + |u|^p) dx \right)^{\frac{\alpha}{\nu}} \left( \int_U |A^{2\nu} u_N| dx \right)^{\frac{\nu}{\nu + \alpha}}
\]

\[
\leq |a||z(\theta_{r-\tau}\omega)| c_9 \left( 1 + ||u||_p^p \right) \cdot |A^{2\nu} u_N|
\]

\[
\leq c_{10} |A^{2\nu} u_N| + \frac{|a|^2 |z(\theta_{r-\tau}\omega)|^2}{2}\|A^{\nu+\frac{1}{2}} u_N\|^2, \quad \forall r \geq \tau - t,
\] (106)

and

\[
\int_U f_u^r(u, x) u_t \cdot A^{2\nu} u_N dx
\]

\[
\leq c_{11} \int_U |u_t| \cdot (1 + |u|^{p-1}) \cdot |A^{2\nu} u_N| dx
\]

\[
\leq c_{11} \left( \int_U |u_t|^2 dx \right)^{\frac{1}{2}} \left( \int_U (1 + |u|^{p-1}) dx \right)^{\frac{1}{2}} \left( \int_U |A^{2\nu} u_N|^6 dx \right)^{\frac{1}{6}}
\]

\[
\leq c_{12} |u_t|_{L_0^2} \cdot (1 + ||u||_p^{p-1}) \cdot |A^{\nu+\frac{1}{2}} u_N|
\]

\[
\leq c_{13} |A^{2\nu} u_N| + \frac{\varepsilon}{4} \|A^{\nu+\frac{1}{2}} u_N\|^2, \quad \forall r \geq \tau - t.
\] (107)

Thus, by putting (101)-(107) into (100), we have

\[
\frac{d}{dt} \left( |A^{\nu+1/2} u_N|^2 + \|A^\nu v_N\|^2 + 2 \int_U f(u, x) \cdot A^{2\nu} u_N dx \right)
\]

\[
+ \left( \frac{\varepsilon}{2} - 4|a||z(\theta_{r-\tau}\omega)|^2 \right) \times \left( \int_U f(u, x) \cdot A^{2\nu} u_N dx \right)^{\frac{\nu}{\nu + \alpha}}
\]

\[
\leq c_9 \left( 1 + |z(\theta_{r-\tau}\omega)| + \frac{|a|^2 |z(\theta_{r-\tau}\omega)|}{\sqrt{\lambda}} \right) R_1(\theta_{r-\tau}\omega, \theta) + c_{14} M_{R_0}^3(\theta_{r-\nu}\omega) e^{-2 \int_{r-\tau}^r \left( \varepsilon - 4|a||z(\theta_{r-\omega})| + \frac{|a|^2 |z(\theta_{r-\omega})|^2}{\sqrt{\lambda}} \right) ds}
\]

\[
\leq q_1(\theta_{r-\tau}\omega), \quad \forall r \geq \tau - t,
\]

where \( c_{14} \) is a positive constant independent of \((r, \tau, \omega)\). Set

\[
y_N = \|A^{\nu+1/2} u_N\|^2 + \|A^\nu v_N\|^2 + 2 \int_U f(u, x) \cdot A^{2\nu} u_N dx
\]

\[
\geq |A^{\nu+1/2} u_N|^2 + \|A^\nu v_N\|^2 - 2R_1(\theta_{r-\tau}\omega, \theta_{r-\omega}).
\] (109)

We have, for \( r \geq \tau - t \),

\[
\frac{d}{dt} y_N(r) \leq - \left( \frac{\varepsilon}{2} - 4|a||z(\theta_{r-\tau}\omega)| - \frac{2|a|^2 |z(\theta_{r-\tau}\omega)|^2}{\sqrt{\lambda}} \right) y_N(r) + q_1(\theta_{r-\tau}\omega).
\] (110)
By
g_{\tau-t} q_{1}(\theta_{\xi-\tau})e^{-T_{\tau-t}} \left( \frac{2}{\sqrt{\lambda}} \left( 4|\varphi(\theta_{\xi-\tau})| + \frac{2|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} \right) ds
\leq c_{14} \int_{0}^{\infty} \left( 1 + |\varphi(\theta_{\xi-\tau})|^{4} + M_{0}^{4p}(\theta-\xi) \right) e^{-\frac{2}{\sqrt{\lambda}} t_{\xi-\tau}} \left( 4|\varphi(\theta_{\xi-\tau})| + \frac{2|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} ds
d\xi
+c_{14} M_{0}^{p}(\theta-\xi) e^{-\frac{2}{\sqrt{\lambda}} t_{\xi-\tau}} \left( 8|\varphi(\theta_{\xi-\tau})| + \frac{8|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} ds
d\xi
\leq c_{14} \int_{0}^{\infty} \left( 1 + |\varphi(\theta_{\xi-\tau})|^{4} + M_{0}^{4p}(\theta-\xi) \right) e^{-\frac{2}{\sqrt{\lambda}} t_{\xi-\tau}} \left( 4|\varphi(\theta_{\xi-\tau})| + \frac{2|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} ds
d\xi
+c_{14} M_{0}^{p}(\theta-\xi) e^{-\frac{2}{\sqrt{\lambda}} t_{\xi-\tau}} \left( 8|\varphi(\theta_{\xi-\tau})| + \frac{8|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} ds
\leq M_{1}\nu(\omega) < \infty, \quad \forall t \geq 0,
(113)
where
\begin{equation}
R_{1}(\omega, \theta-\xi) = c_{8} \left( 1 + M_{0}^{2\nu}(\omega) + M_{0}^{p}(\theta-\xi) e^{-f_{\nu}(\epsilon-4|\varphi(\theta_{\xi-\tau})| - \frac{2|\theta_{\xi-\tau}|}{\sqrt{\lambda}} \right)^{2} ds \right).
\end{equation}
(114)

By (47), (97) and (80), $M_{1}(\omega)$ is tempered.

**Lemma 3.5.** Suppose (97) holds. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, assume that $B_{\nu}(\tau, \omega) \subseteq B_{1}(\tau, \omega)$ and $B_{\nu}(\tau, \omega) \in B_{\nu}(\mathcal{D}(E^{\nu}))$, where $\nu$ is as in (99). Then there exist a random variable $t_{1}(\omega) > 0$ and a tempered random variable $M_{1}(\omega) > 0$ (independent of $\tau$) such that for any $t \geq 0$, the solution $\varphi(\tau, \tau-t, \theta-\tau, \varphi(\theta-\tau))$ of (49) with $\varphi(\theta-\tau) \in B_{\nu}(\tau-t, \theta-\xi)$ satisfies

$$
||\varphi(\tau, \tau-t, \theta-\tau, \varphi(\theta-\tau))||_{E^{\nu}}^{2}
= ||A^{1/2}u(\tau, \tau-t, \theta-\tau, \varphi(\theta-\tau))||_{E^{\nu}}^{2} + ||A^{1/2}u(\tau, \tau-t, \theta-\tau, \varphi(\theta-\tau))||_{E^{\nu}}^{2}
\leq M_{1}(\omega), \quad \forall t \geq t_{1}(\omega).
(115)
$$

**Proof.** Taking the inner product of (49) in $E$ with $A^{2\nu} \varphi = (A^{2\nu} u, A^{2\nu} v)^{T}$, similar to (110), we have that for $r \geq \tau-t$,

$$
\frac{d}{dr} \tilde{y}(r) \leq -\left( \frac{2}{\sqrt{\lambda}} - 4|\varphi(\theta-\tau)| - \frac{2|\theta-\tau|}{\sqrt{\lambda}} \right) \tilde{y}(r)
+c_{15} \left( 1 + |\varphi(\theta-\tau)|^{2} + M_{0}^{2\nu}(\theta-\tau) \right),
(116)
$$
where

\[ \tilde{y} = ||A^{\nu+1/2}u||^2 + ||A^\nu v||^2 - 2\int_U f(u, x) \cdot A^{2\nu} v dx \geq ||\varphi(\tau)||_{L^2}^2 - 2c_{16}[1 + M_0^{2p}(\omega)]. \]  

(117)

Thus, we have that for \( t \geq 0 \),

\[ \begin{aligned}
|\varphi(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))|_{L^2}^2 & \leq \tilde{y}(\tau, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \omega)) + 2c_{16}[1 + M_0^{2p}(\omega)] \\
& \leq \tilde{y}(\tau - t, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \omega))e^{-\int_{\tau-t}^{\tau} \left( \frac{\xi - 4|a||z(\theta - \omega)|}{\sqrt{\varsigma_1}} \right) ds} \\
& + \kappa c(A(\theta, \omega)) \int_{\tau-t}^{\tau} \left( \frac{\xi - 4|a||z(\theta - \omega)|}{\sqrt{\varsigma_1}} \right) ds \\
& \leq \left( M_0^2(\theta, \omega) + 2c_{16}[1 + M_0^{2p}(\theta, \omega)] \right) e^{-\int_{\tau-t}^{\tau} \left( \frac{\xi - 4|a||z(\theta - \omega)|}{\sqrt{\varsigma_1}} \right) ds} \\
& + \frac{1}{2} M_{1, \nu}^2(\omega). \end{aligned} \]

(118)

where

\[ M_{1, \nu}^2(\omega) = 4c_{16}[1 + M_0^{2p}(\omega)] \\
+ 2c_{15} \int_{t_0}^{+\infty} [1 + M_0^{2p}(\theta, \omega) + z^2(\theta, \omega)] e^{-\int_{t_0}^{\tau} \left( \frac{\xi - 4|a||z(\theta - \omega)|}{\sqrt{\varsigma_1}} \right) ds} d\xi. \]

(119)

Since \( M_0^2(\omega) + 2c_{16}[1 + M_0^{2p}(\omega)] \) is tempered and \( \rho_1 = \frac{\pi}{2} - \frac{8|a|}{\sqrt{\varsigma_1}} - \frac{2|a|^2}{\sqrt{\varsigma_1}} > 0 \), we have, as \( t \to +\infty \)

\[ \left( M_0^2(\theta, \omega) + 2c_{16}[1 + M_0^{2p}(\theta, \omega)] \right) e^{-\int_{t_0}^{\tau} \left( \frac{\xi - 4|a||z(\theta - \omega)|}{\sqrt{\varsigma_1}} \right) ds} \to 0. \]

(120)

By (47), (97) and (80), \( \tilde{M}_{1, \nu}^2(\omega) \) is tempered.

Basing on the proof of Lemmas 3.3-3.5 and recursion method, we have the following Lemma.

**Lemma 3.6.** Assume that (H), (69), (97) hold, and for any \( \tau \in \mathbb{R}, \omega \in \Omega, \ t \geq 0 \), let \( B_\epsilon(\tau, \omega) \subseteq B_1(\tau, \omega), \ B_\kappa(\tau, \omega) \subseteq B_\kappa(\Omega) \) and \( \varphi_{t - \tau}(\theta, \omega) \in B_\kappa(\tau - t, \theta, \omega) \). Then

(i) for \( \nu \leq \kappa - 1 \), there exist \( t_\kappa(\omega) > 0 \) and a tempered random variable \( M_\kappa(\omega) \geq 0 \) (independent of \( \tau \) and \( t \)) such that the solution \( \varphi(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega)) \) of (49) satisfies

\[ \begin{aligned}
||\varphi(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||_{L^2}^2 & = ||A^{\nu+1/2}u(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||^2 + ||A^\nu v(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||^2 \\
& \leq M_\kappa^2(\omega), \quad \forall t \geq t_\kappa(\omega); \end{aligned} \]

(121)

(ii) for \( \nu \leq \kappa - 1 - \nu \), there exists \( M_\kappa(\omega) > 0 \) (independent of \( \tau \) and \( t \)) such that the solution \( \varphi_N(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega)) \) of (86) satisfies

\[ \begin{aligned}
||\varphi_N(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||_{L^2}^2 & = ||A^{\nu+1/2}u_N(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||^2 \\
& + ||A^\nu v_N(\tau, \tau - t, \theta, \tau - \omega, \varphi_{\tau - t}(\theta - \omega))||^2 \\
& \leq M_\kappa^2(\omega), \quad \forall t \geq t_\kappa(\omega); \end{aligned} \]

(122)
where $\nu$ is as in (99).

### 3.3. Existence of random attractor.

In this subsection, we assume that (H), (69) and (97) hold. We will prove the existence of a random attractor $A(\tau, \omega)$ for $\Phi$ by Theorem 2.7 and prove that $A(\tau, \omega)$ is included in the bounded ball of $E^1$ basing on Lemma 3.6 and the “iteration” method similar to [31]. To construct a non-empty compact measurable tempered attracting set for $\Phi$ in $E^1$, we need the following result which generalizes Lemma 1.3 of [31] to random dynamical system.

**Lemma 3.7.** Let $(X, d)$ be a Polish space, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ be an ergodic metric dynamical system and $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ be a continuous RDS on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. Assume that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$,

(i) there exist a constant $C_1 > 0$ and a random variable $Q_1(\omega)$ (independent of $\tau$) such that

$$d(\Phi(t, \tau, \omega)\varphi_1, \Phi(t, \tau, \omega)\varphi_2) \leq C_1 e^{\int_0^t Q_1(\theta, \omega)ds} d(\varphi_1, \varphi_2), \quad \forall \varphi_1, \varphi_2 \in X;$$  (123)

(ii) there exist random variables $C_2(t, \omega)$, $C_3(t, \omega)$, $Q_2(\omega)$, $Q_3(\omega)$ (independent of $\tau$) and three subsets $K_1$, $K_2$, $K_3 \subset X$ such that

$$d_X(\Phi(t, \tau, \omega)K_1, K_2) \leq C_2(t, \omega) e^{\int_0^t Q_2(\theta, \omega)ds},$$  
$$d_X(\Phi(t, \tau, \omega)K_2, K_3) \leq C_3(t, \omega) e^{\int_0^t Q_3(\theta, \omega)ds};$$  (124)

(iii) the expectations of $Q_i(\omega)$ ($i = 1, 2, 3$) satisfy:

$$|E[Q_1(\omega)]| < \infty, \quad -\infty < E[Q_2(\omega)], E[Q_3(\omega)] < 0.$$  (125)

Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists $T_0(\omega) > 0$ (independent of $\tau$) such that for positive constant $\sigma$ with

$$0 < \sigma \leq \frac{-E[Q_2(\omega)]}{3|E[Q_1(\omega)]| - E[Q_2(\omega)] - 3E[Q_3(\omega)]},$$  (126)

and $\min\{\sigma t, (1 - \sigma)t\} \geq T_0(\omega)$, $t > 0$, it holds that

$$d_X(\Phi(t, \tau, \omega)K_1, K_3) \leq (C_1 C_2(\sigma t, \omega) + C_3((1 - \sigma)t, \omega)) e^{\frac{\sigma}{3} E[Q_3(\omega)]t}.$$  (127)

**Proof.** For any fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, taking $\varphi_1 \in K_1$ and let $t = t_1 + t_2$ with $t_1, t_2 \geq 0$. By (124), there exists $\varphi_2 \in K_2$ such that

$$d(\Phi(t_1, \tau, \omega)\varphi_1, \varphi_2) \leq C_2(t_1, \omega) e^{\int_0^{t_1} Q_2(\theta, \omega)ds},$$  (128)

By the cocycle property of $\Phi$, (123), and (128), we have

$$d(\Phi(t_2 + t_1, \tau, \omega)\varphi_1, \Phi(t_2, t_1 + \tau, \theta_t, \omega)\varphi_2) = d(\Phi(t_2, t_1 + \tau, \theta_t, \omega)\Phi(t_1, \tau, \omega)\varphi_1, \Phi(t_2, t_1 + \tau, \theta_t, \omega)\varphi_2)$$
$$\leq C_1 e^{\int_0^{t_1} Q_1(\theta, \omega)ds} d(\Phi(t_1, \tau, \omega)\varphi_1, \varphi_2)$$
$$\leq C_1 e^{\int_0^{t_1} Q_1(\theta, \omega)ds} C_2(t_1, \omega) e^{\int_0^{t_1} Q_2(\theta, \omega)ds}$$
$$\leq C_1 C_2(t_1, \omega) e^{\int_0^{t_2} Q_1(\theta, \omega)ds + \int_0^{t_1} Q_2(\theta, \omega)ds}.$$  (129)

By (124), there exists $\varphi_3 \in K_3$ such that

$$d(\Phi(t_2, t_1 + \tau, \theta_t, \omega)\varphi_2, \varphi_3) \leq C_3(t_2, \omega) e^{\int_0^{t_2} Q_3(\theta, \omega)ds}.$$  (130)
Combining (129)-(130), we have
\[
dX(\Phi(t_2 + t_1, \tau, \omega)K_1, K_3)
\leq \inf_{t_1, t_2 \in [0, t], t_1 + t_2 = t}
\left(C_1 C_2 (t_1, \omega) e^{\int_0^{t_1} Q_1 (\theta_{s+t} \omega) ds + \int_0^{t_1} Q_2 (\theta_s \omega) ds} + C_3 (t_2, \omega) e^{\int_0^{t_2} Q_3 (\theta_{s+t} \omega) ds}\right).
\] (131)

By the property of measure-preserving and ergodicity of \( \{\theta_t\}_{t \in \mathbb{R}} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and the Birkhoff ergodic Theorem, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t Q_i (\theta_{s+t} \omega) ds = \mathbb{E}[Q_i (\omega)], \quad \forall t \in \mathbb{R}, i = 1, 2, 3,
\] (132)

which implies that there exists a large \( T_0 (\omega) > 0 \) such that
\[
\left\{ \begin{array}{ll}
\frac{1}{t} \int_0^t Q_1 (\theta_{s+t} \omega) ds \leq \frac{3}{2} |\mathbb{E}[Q_1 (\omega)]|, \\
-\infty < \frac{1}{t} \int_0^t Q_2 (\theta_s \omega) ds \leq \frac{3}{2} \mathbb{E}[Q_2 (\omega)] < 0, \\
-\infty < \frac{3}{2} \mathbb{E}[Q_3 (\omega)] \leq \frac{1}{t} \int_0^t Q_3 (\theta_{s+t} \omega) ds \leq \frac{3}{2} \mathbb{E}[Q_3 (\omega)] < 0.
\end{array} \right. \quad \forall t \geq T_0 (\omega). \] (133)

Let \( t_2 = \sigma t \) and \( t_1 = (1 - \sigma) t, t > 0 \), where \( \sigma \) satisfies \( (126) \), then for \( \min \{ \sigma t, (1 - \sigma) t \} \geq T_0 (\omega) \),
\[
\int_0^t Q_1 (\theta_{s+t} \omega) ds + \int_0^t Q_2 (\theta_s \omega) ds = \int_0^t \sigma Q_1 (\theta_{s+t} \omega) ds + \int_0^t (1 - \sigma) Q_2 (\theta_s \omega) ds
\leq \frac{3}{2} \sigma \mathbb{E}[Q_1 (\omega)] + (1 - \sigma) \mathbb{E}[Q_2 (\omega)] \quad \text{(by (133))}
\leq \frac{3}{2} \sigma \mathbb{E}[Q_3 (\omega)] \quad \text{(by (126))}
\leq \sigma \cdot \frac{3}{2} \int_0^t Q_3 (\theta_{s+t} \omega) ds \quad \text{(by (133))}
\leq \frac{3}{2} \mathbb{E}[Q_3 (\omega)] t. \quad \text{(by (133))}
\] (134)

By (131) and (134), we have, for all \( \min \{ \sigma t, (1 - \sigma) t \} \geq T_0 (\omega) \)
\[
dX(\Phi(t, \tau, \omega)K_1, K_3) \leq (C_1 C_2 ((1 - \sigma) t, \omega) + C_3 (\sigma t, \omega)) e^{\frac{3}{2} \mathbb{E}[Q_3 (\omega)] t}.
\] (135)

Now let us show the Lipschitz property of \( \varphi(t + \tau, \tau, \omega, \varphi_r (\omega)) \) on the bounded random set \( B_1 (\tau, \omega) \) for any \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t \geq 0 \).

Let \( \varphi_r (\omega) = (u_r (\tau, \omega), v_r (\tau, \omega)) \in B_1 (\tau, \omega) \), \( \varphi_j (r) = \varphi_j (r, \tau, \theta_r \omega, \varphi_j (\theta_r \omega)) = (u_j (r), v_j (r)) \), \( r \geq \tau, j = 1, 2 \), and let
\[
\psi (r) = \varphi_1 (r) - \varphi_2 (r) = (u_1 (r) - u_2 (r), v_1 (r) - v_2 (r)) = (\xi (r), \eta (r)),
\] (136)
then
\[
\left\{ \begin{array}{l}
\dot{\psi} + \Delta \psi = F(\varphi_1, \theta_r \omega, r) - F(\varphi_2, \theta_r \omega, r), \\
\psi (r) = (\xi (r), \eta (r)) = (u_1 - u_2, v_1 - v_2, r \geq \tau,
\end{array} \right.
\] (137)
where
\[
F(\varphi_1, \theta_r \omega, r) - F(\varphi_2, \theta_r \omega, r) = \left( \begin{array}{c}
a z \theta_r \omega \xi \\
(2 \varepsilon - a z) a z \xi - a z \eta - f(u_1, x) + f(u_2, x)
\end{array} \right).
\] (138)

By the definition of \( B_1 (\tau, \omega) \) and Lemma 3.2, we have
\[
||\varphi_1 (r)||_{E} \leq M_0 (\theta_r \omega), \quad ||\varphi_2 (r)||_{E} \leq M_0 (\theta_r \omega), \quad \forall r \geq \tau.
\] (139)

**Lemma 3.8.** Suppose that
\[
|a|^2 \leq \min \left\{ \frac{\alpha^3}{((p - 1) c_4)^2}, \frac{\alpha^2}{2(p - 1)c_4} \right\}.
\] (140)
Then there exists a tempered random variable \( L(\omega) > 0 \) with \( 0 < \mathbb{E}[L(\omega)] < \infty \) such that for any \( r \in \mathbb{R}, \omega \in \Omega, t \geq 0 \), it holds that
\[
\|\varphi(t + \tau, \theta_{-\tau}\omega, \varphi_1(\theta_{-\tau}\omega)) - \varphi(t + \tau, \theta_{-\tau}\omega, \varphi_2(\theta_{-\tau}\omega))\|_{\mathcal{E}} \leq e^{\int_0^t L(\omega)ds} \|\varphi_1 - \varphi_2\|_{\mathcal{E}}. \tag{141}
\]

**Proof.** Taking the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{E}} \) of (137) with \( \psi(r) \), we find that for \( r \geq \tau \),
\[
\frac{d}{dt} \|\psi(r)\|_{\mathcal{E}}^2 \leq \left( -\varepsilon + 4|a||z(\theta_{-\tau}\omega)| + \frac{|a|^2|z(\theta_{-\tau}\omega)|^2}{\sqrt{\lambda_1}} \right) \|\psi(r)\|_{\mathcal{E}}^2 + \frac{1}{\alpha} \int_0^r \|f(u_2(r), x) - f(u_1(r), x)\|^2 dx. \tag{142}
\]
By Hölder inequality and (139),
\[
\|f(u_2(r), x) - f(u_1(r), x)\|^2 \leq c_{17} \int_0^r (1 + |u_1|^2(p-1) + |u_2|^{2(p-1)})\|\xi\|^2 dx \leq c_{18}(1 + M_0^{2(p-1)}(\theta_{-\tau}\omega))\|\xi(r)\|^2. \tag{143}
\]
Thus, for \( r \geq \tau \),
\[
\frac{d}{dt} \|\psi(r)\|_{\mathcal{E}}^2 \leq \left( -\varepsilon + 4|a||z(\theta_{-\tau}\omega)| + \frac{|a|^2|z(\theta_{-\tau}\omega)|^2}{\sqrt{\lambda_1}} + \frac{c_{18}}{\alpha} (1 + M_0^{2(p-1)}(\theta_{-\tau}\omega)) \right) \|\psi(r)\|_{\mathcal{E}}^2. \tag{144}
\]
Then by Gronwall inequality to (144), it follows that for \( r \geq \tau \),
\[
\|\varphi_1(r) - \varphi_2(r)\|_{\mathcal{E}}^2 \leq \|\psi(r)\|_{\mathcal{E}}^2 \leq \left( -\varepsilon + 4|a||z(\theta_{-\tau}\omega)| + \frac{|a|^2|z(\theta_{-\tau}\omega)|^2}{\sqrt{\lambda_1}} + \frac{c_{18}}{\alpha} (1 + M_0^{2(p-1)}(\theta_{-\tau}\omega)) \right) \|\psi(r)\|_{\mathcal{E}}^2 ds. \tag{145}
\]
Set \( r = t + \tau \), we obtain (141), where
\[
L(\omega) = 2|a||z(\omega)| + \frac{|a|^2|z(\omega)|^2}{\sqrt{\lambda_1}} + \frac{c_{18}}{2\alpha} (1 + M_0^{2(p-1)}(\omega)). \tag{146}
\]
By (74), (81) and (140), we have
\[
0 < \mathbb{E}[L(\omega)] \leq \begin{cases} \frac{2|a|}{\sqrt{\lambda}} + \frac{|a|^2}{\alpha \sqrt{\lambda}}, & 1 \leq p \leq 2, \\ \frac{2|a|}{\sqrt{\lambda}} + \frac{|a|^2}{\alpha \sqrt{\lambda}} + \frac{c_{18}}{2\alpha} \left( 1 + \frac{2p-2}{\rho_0}(2(p-2))^{p-2} \right), & 2 \leq p < 3. \end{cases} \tag{147}
\]

**Lemma 3.9.** For any \( \tau \in \mathbb{R}, \omega \in \Omega \), there exist a \( T_{\nu}(\omega) \geq 0 \), a random bounded ball \( B_1(\omega) \) of \( E^1 \) with radius \( M_1(\omega) \) (defined by (113)), a positive number \( \bar{\sigma} \) and a tempered random variable \( \tilde{Q}(\omega) > 0 \) such that the solution \( \varphi(\tau, t, \theta_{-\tau}\omega) \) of (49) with initial value \( \varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau-t, \theta_{-\tau}\omega) \), it holds that for \( t \geq T_{\nu}(\omega) \),
\[
d_{E}(\varphi(\tau, t, \theta_{-\tau}\omega), B_1(\tau-t, \theta_{-\tau}\omega), \tilde{B}_1(\omega)) \leq \tilde{Q}(\theta_{-\tau}\omega)e^{-\frac{1}{2p_0}(\frac{c_{18}}{\alpha \sqrt{\lambda}} + \frac{|a|^2}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_1}} |a|^2|z(\omega)|^2)} ds. \tag{148}
\]

**Proof.** Let \( \varphi_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\tau-t, \theta_{-\tau}\omega) \). Let \( K_{\nu}(\omega) \subset E^\nu \subset E \) be the random ball of \( E^\nu \) with radius \( M_{1\nu}(\omega) \) defined by (113), where \( \nu \) is as in (99). By Lemma 3.3, we have that for \( t \geq 0 \),
\[
d_{E}(\varphi(\tau, t, \theta_{-\tau}\omega), B_1(\tau-t, \theta_{-\tau}\omega), K_{\nu}(\omega)) \leq M_{1\nu}^2(\theta_{-\tau}\omega)e^{-\int_0^r \left( c_{18}(1 + M_0^{2(p-1)}(\theta_{-\tau}\omega)) \right) \|\xi(r)\|^2 ds}. \tag{149}
\]
By Lemma 3.3, Lemma 3.5 and Lemma 3.6 to \( \varphi_{\tau-t}(\theta_{-t}\omega) \in K_{\nu}(\theta_{-t}\omega) \), it follows that there exist \( t_{1\nu}(\omega) \geq 0 \) and a random ball \( K_{2\nu}(\omega) \) of \( E_{2\nu}^1 \) with radius \( M_{2\nu}(\omega) \) (defined by (122)) such that

\[
d_{E}(\varphi(\tau, \tau-t, \theta_{-t}\omega, K_{\nu}(\theta_{-t}\omega)), K_{2\nu}(\omega)) \\
\leq cl_{\nu}d_{E}(\varphi(\tau, \tau-t, \theta_{-t}\omega, K_{\nu}(\theta_{-t}\omega)), K_{2\nu}(\omega)) \\
\leq cl_{\nu}\tilde{M}_{1\nu}^2(\theta_{-t}\omega)e^{-\int_{t}^{0}\left(-\frac{4|a|}{|a|^2} + \frac{|a|^2}{2|a|^2}\right)ds} \quad (t \geq t_{1\nu}(\omega)),
\]

where \( c_{1\nu} \) is a constant, \( \tilde{M}_{1\nu}^2(\omega) \) is defined by (119). By Lemma 3.7, Lemma 3.8 and (149)-(150), there exists \( T_{1\nu}(\omega) > 0 \) (independent of \( \tau \)) such that for \( t \geq T_{1\nu}(\omega) \),

\[
d_{E}(\varphi(\tau, \tau-t, \theta_{-t}\omega, B_{1}(\tau-t, \theta_{-t}\omega)), K_{2\nu}(\omega)) \\
\leq \tilde{M}_{2,1}^2(\theta_{-t}\omega)e^{-\frac{4|a|}{|a|^2} + \frac{|a|^2}{2|a|^2}t} \quad (151),
\]

where \( \tilde{M}_{2,1}^2(\theta_{-t}\omega) = M_{0}^2(\theta_{-\sigma_{t}}\omega) + c_{1\nu}\tilde{M}_{1\nu}^2(\theta_{-1-\sigma_{t}}\omega) \) is tempered and

\[
0 < \sigma_{1} \leq \frac{\varepsilon - \frac{4|a|}{\sqrt{\alpha}} - \frac{|a|^2}{2\sqrt{\alpha}}}{3E[L(\omega)] + 4\varepsilon - \frac{16|a|}{\sqrt{\alpha}} - \frac{3|a|^2}{\alpha \sqrt{\alpha}}}. \quad (152)
\]

Since \( \nu > 0 \) is fixed, there exists an integer \( \tilde{k} \geq 1 \) such that \( 1 - \nu \leq (\tilde{k}-1)\nu < 1 \). By above recursion of \( k \) \( (\leq \lfloor \frac{k}{2} \rfloor + 2) \) steps, there exist \( T_{k\nu}(\omega) > 0 \) (independent of \( \tau \)) and a random ball \( \hat{B}_{1}(\omega) \) of \( E_{1}^1 \) with radius \( M_{1}(\omega) \) (defined by (122)) such that for \( t \geq T_{k\nu}(\omega) \),

\[
d_{E}(\varphi(\tau, \tau-t, \theta_{-t}\omega, B_{1}(\tau-t, \theta_{-t}\omega)), \hat{B}_{1}(\omega)) \\
\leq \tilde{M}_{k,1}^2(\theta_{-t}\omega)e^{-\frac{4|a|}{|a|^2} + \frac{|a|^2}{2|a|^2}t} \quad (153),
\]

where

\[
0 < \sigma_{j} \leq \frac{\varepsilon - \frac{4|a|}{\sqrt{\alpha}} - \frac{|a|^2}{2\sqrt{\alpha}}}{3E[L(\omega)] + \frac{4|a|}{\sqrt{\alpha}} - \frac{|a|^2}{2\sqrt{\alpha}}} \quad (154),
\]

and

\[
\tilde{M}_{k\nu}^2(\theta_{-t}\omega) = M_{0}^2(\theta_{-\sigma_{t-1}}\omega) + c_{1\nu}\tilde{M}_{1\nu}^2(\theta_{-1-\sigma_{t-1}}\omega) \\
+ \ldots + c_{(\tilde{k}-1)\nu}\tilde{M}_{1\nu}^2(\theta_{-1-\sigma_{(\tilde{k}-1)}}\omega) \quad (155)
\]

is tempered.

To this end, combining Lemmas 3.1, 3.9 and Theorem 2.7, our main result about the existence of a random attractor for the RDS \( \Phi \) is as follows.

**Theorem 3.10.** Suppose \( (H), (69), (97) \) and (140) hold. Then for any \( \tau \in \mathbb{R}, \omega \in \Omega, \) the RDS \( \Phi \) associated with (49) possesses a \( D(E) \)-pullback random attractor \( \mathcal{A}(\tau, \omega) \subseteq \hat{B}_{1}(\omega) \cap B_{0}(\omega) \in D(E) \). Moreover,

\[
\|\mathcal{A}(\tau, \omega)\| \leq M_{1}(\omega),
\]

where \( M_{1}(\omega) \) is the radius of the bounded ball \( \hat{B}_{1}(\omega) \subset E_{1}^1 \) defined by (122).
Proof. For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by (153), (154), (155) and the compactness of embedding $E^1 \hookrightarrow E$, $B_1(\omega)$ is a compact measurable $D(E)$-pullback attracting ball in $E$. By Theorem 2.7 and Lemma 3.1, the RDS $\Phi$ possesses a $D(E)$-pullback random attractor $A(\tau, \omega) \subseteq B_1(\omega)$. From (113), (122) and Lemma 3.6, the radius $M_1(\omega)$ of $B_1(\omega) \subseteq E^1$ can be written as the following form

$$M_1^2(\omega) = c_{18} + c_{20}M_0^{2p_k}(\omega) + c_{21}M_0^{4k}(\theta_{-t}\omega)e^{-\frac{1}{2}t} + f_1^0 \left(8\alpha|z(\eta, \omega)| + 8\alpha|z(\xi, \omega)|^2 \right) ds$$

$$+ c_{22} \int_0^\infty \left(1 + |z(\theta_{-t}\omega)|^4 + M_0^{4p_k}(\theta_{-t}\omega)\right) e^{-\frac{1}{2}t} + f_1^0 \left(4\alpha|z(\eta, \omega)| + 4\alpha|z(\xi, \omega)|^2 \right) ds$$

$$+ c_{23}M_0^{2k}(\theta_{-t}\omega)e^{-2k} \int_0^\infty \left(\varepsilon - 4\alpha|z(\theta, \omega)| - \frac{|\varepsilon|^2(\theta, \omega)}{\sqrt{\lambda_1}} \right) ds.$$  \hspace{1cm} (157)

3.4. Fractal dimension of random attractor. Assume that (H), (69), (97) hold. This subsection focuses to estimate an upper bound of the fractal dimension of $A(\tau, \omega)$ for $\Phi$ defined by Theorem 3.10. Now let us check that $A(\tau, \omega)$ satisfies the conditions of Theorem 2.8 one by one.

It is easy to see from Lemma 3.2, Theorem 3.10 and Definition 2.6 that $A(\tau, \omega)$ satisfies conditions (H1) and (H2). By $A(\tau, \omega) \subseteq B_1(\tau, \omega)$ and Lemma 3.8, we have the Lipschitz property (141) of $\Phi$ on $A(\tau, \omega)$.

Let $\{e_j\}_{j \in \mathbb{N}} \subset D(A)$ be the eigenvectors of operator $A$ corresponding to the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ with $Ae_j = \lambda_j e_j$ for $j \in \mathbb{N}$, then $\{e_j\}_{j \in \mathbb{N}}$ form an orthonormal base of $L^2(U)$ and $H_1^1(U)$. Let

$$H_n(U) = \text{span}\{e_1, e_2, \ldots, e_n\}, \quad H_n^\perp(U) = \text{span}\{e_{n+1}, e_{n+2}, \ldots\}, \quad n \in \mathbb{N},$$

then $H_n(U) \times H_n^\perp(U)$ is a 2n-dimensional subspace of $E$. Let

$$P_n : E \rightarrow H_n(U) \times H_n^\perp(U), \quad Q_n = I - P_n : E \rightarrow H_n^\perp(U) \times H_n^\perp(U)$$

be the orthonormal projectors. For $\varphi = (u, v) \in E$, let $\varphi_{nq} = Q_n \varphi = (u_{nq}, v_{nq}) \in H_n^\perp(U) \times H_n^\perp(U)$, then

$$\lambda_{n+1}||u_{nq}||^2 \leq ||u_{nq}||^2, \quad \lambda_{n+1}||u_{nq}||^2 \leq ||u_{nq}||^2.$$  \hspace{1cm} (160)

Lemma 3.11. Suppose (H), (69), (97) hold and the number $\alpha$ is small enough such that

$$|a|^2 \leq \min \left\{ \frac{\alpha^3}{64}, \frac{\sqrt{|1|}a^2}{8}, \frac{\alpha^3}{16(p-1)kC_4}, \frac{32(p-1)kC_4}{32(p-1)kC_4} \right\}.$$  \hspace{1cm} (161)

Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, there exist random variables $C_2(\omega)$, $C_3(\omega) > 0$ and a 2n-dimensional finite dimensional projector $P_n : E \rightarrow P_n E = H_n(U) \times H_n^\perp(U)$ ($\dim(P_n E) = 2n$ (independent of $\tau$ and $\omega$)) such that for any $\varphi_{j\tau}(\omega) = (u_{j\tau}\omega), v_{j\tau}(\omega) \in A(\tau, \omega)$, $j = 1, 2$, it holds that

$$||(I - P_n)\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{1\tau}(\theta_{-\tau}\omega) - (I - P_n)\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{2\tau}(\theta_{-\tau}\omega)||_E$$

$$\leq e^{\tilde{C}_3(\theta_{-\tau}\omega)ds} + c_{24}e^{\gamma_{n+1}}e^{\tilde{C}_3(\theta_{-\tau}\omega)ds} \cdot ||\varphi_{1\tau} - \varphi_{2\tau}||_E$$

and

$$||P_n\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{1\tau}(\theta_{-\tau}\omega) - P_n\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{2\tau}(\theta_{-\tau}\omega)||_E$$

$$\leq e^{\tilde{C}_3(\theta_{-\tau}\omega)ds} ||\varphi_{1\tau} - \varphi_{2\tau}||_E,$$  \hspace{1cm} (162)

where

$$E[C_2(\omega)] < \infty, \quad 0 \leq E[C_3(\omega)], E[C_3(\omega)] < \infty.$$  \hspace{1cm} (164)
and
\[ \gamma_{n+1} = \frac{1}{\sqrt{\lambda_{n+1}}} \to 0 \text{ as } n \to +\infty. \] (165)

Proof. Taking the inner product of (137) with \( \psi_{nq} = Q_n\psi \) in \( E \), we have that for \( r \geq \tau \),
\[
\frac{d}{dr} ||\psi_{nq}(r)||^2_E \leq -\varepsilon + 4|a||z(\theta_{r-\tau})| + \frac{|a|^2|z(\theta_{r-\tau})|^2}{\sqrt{\lambda_1}} ||\psi_{nq}(r)||^2_E \]
\[ + \frac{1}{\alpha} ||f(u_2(r), x) - f(u_1(r), x)||^2. \] (166)

By the invariance of \( A(\tau, \omega) \) and Theorem 3.10, for \( r \geq \tau \), \( \varphi_1(r), \varphi_2(r) \in A(r, \theta, \omega) \subseteq \tilde{B}_1(\theta, \omega) \subset E \) and
\[
||\varphi_1(r)||_{E^1} \leq M_1(\theta, \omega), \quad ||\varphi_2(r)||_{E^1} \leq M_1(\theta, \omega), \quad E(M^2(\theta, \omega)) < \infty, \quad \forall r \geq \tau, \] (167)
where \( M_1(\omega) > 0 \) is defined by (157), which implies that
\[
|u_j(r)| \leq \tilde{c}(\Omega)|A u_j(r)|_0 \leq c_{25}M_1(\theta, \omega), \quad r \geq \tau, \quad j = 1, 2, \] (168)
where \( c_{25} > 0 \) is a constant independent of \( \tau, r \) and \( \omega \). Thus, by (160), (168) and (141), we have
\[
2(f(u_2(r), x) - f(u_1(r), x), \eta_{nq})
= 2 \int_{\Omega} [(f(u_2(r), x) - f(u_1(r), x)] \eta_{nq} dx
\leq 2\sqrt{c_{26}}(1 + M_1^{(p-1)}(\theta, \omega))||\xi(r)||_1 \cdot ||\eta_{nq}(r)||_{-1}
\leq 2\sqrt{c_{26}} \frac{1}{\sqrt{\lambda_{n+1}}}(1 + M_1^{(p-1)}(\theta, \omega))||\xi(r)||_1 \cdot ||\eta_{nq}(r)||
\leq 2\sqrt{\lambda_{n+1}}(1 + M_1^{(p-1)}(\theta, \omega))||\xi(r)||^2_1 + \alpha ||\eta_{nq}(r)||^2. \] (169)

Putting (169) into (166), we have that for \( r \geq \tau \),
\[
\frac{d}{dr} ||\psi_{nq}(r, \tau, \theta_r-\tau, \varphi_{1r}(\theta_r-\tau))||^2_E
\leq -\varepsilon + 4|a||z(\theta_{r-\tau})| + \frac{|a|^2|z(\theta_{r-\tau})|^2}{\sqrt{\lambda_1}} ||\psi_{nq}(r)||^2_E
\[ + \gamma_{n+1}^2 \frac{2c_{26}}{\alpha}(1 + M_1^{(p-1)}(\theta, \omega))||\xi(r)||^2_1 + \alpha ||\eta_{nq}(r)||^2. \] (170)
Since
\[
\sqrt{x} \leq e^x, \quad \forall x \geq 0, \] (171)

it follows that
\[
\int_{\tau}^{\tau+t} (1 + M_1^{(p-1)}(\theta, \omega))e^{-\varepsilon(\tau+t-r)} dr
\leq \left( \int_{\tau}^{\tau+t} (1 + M_1^{(p-1)}(\theta, \omega))^2 dr \right)^{\frac{1}{2}} \left( \int_{\tau}^{\tau+t} e^{-2\varepsilon(\tau+t-r)} dr \right)^{\frac{1}{2}}
\leq \frac{1}{2\varepsilon} e^{\frac{1}{2}(1 + M_1^{(p-1)}(\theta, \omega))^2} ds. \] (172)
By the Gronwall inequality to (170) on \([\tau, \tau + t] \ (t \geq 0)\), we have that for \(t \geq 0\),

\[
||\psi_{n+1}(\tau + t, \tau, \theta_{\tau+\omega}, \omega, \varphi_{1}(\theta_{\tau+\omega}))||_{E}^2 \\
\leq ||\psi_{n+1}(\tau)||_{E}^2 e^{\int_{\tau}^{\tau+t}(\epsilon + 4|a||z(\theta_{\tau+\omega})| + \frac{|a|^2|z(\theta_{\tau+\omega})|^2}{\sqrt{\lambda_1}})ds} \\
+ \frac{2c_{26}}{\alpha} \gamma_n^2 \int_{\tau}^{\tau+t}||\varphi_{1}(\theta_{\tau+\omega})||_{E}^2 ds \\
\leq ||\varphi_{1}(\theta_{\tau+\omega})||_{E}^2 e^{\int_{\tau}^{\tau+t}(-\epsilon + 4|a||z(\theta_{\tau+\omega})| + \frac{|a|^2|z(\theta_{\tau+\omega})|^2}{\sqrt{\lambda_1}})ds} \\
+ c_{27} \gamma_n^2 \int_{\tau}^{\tau+t}||\varphi_{1}(\theta_{\tau+\omega})||_{E}^2 ds \\
\leq e^{\int_{\tau}^{\tau+t}2L(\theta_{\tau+\omega})ds + c_{24} \gamma_n^2 \int_{\tau}^{\tau+t}C_2(\theta_{\tau+\omega})ds} \int_{\tau}^{\tau+t}||\varphi_{1}(\theta_{\tau+\omega})||_{E}^2 ds,
\]

(173)

where

\[
C_2(\omega) = \frac{-\epsilon}{2} + 2|a||z(\omega)| + \frac{|a|^2|z(\omega)|^2}{2\sqrt{\lambda_1}},
\]

(174)

\[
C_3(\omega) = L(\omega) + 2|a||z(\omega)| + \frac{|a|^2|z(\omega)|^2}{2\sqrt{\lambda_1}} + 1 + M_4^{(p-1)}(\omega).
\]

(175)

By (47), (81) and (146), it follows that

\[
E[C_2(\omega)] = \frac{-\epsilon}{2} + \frac{2|a|}{\sqrt{\pi}\alpha} + \frac{|a|^2}{4\alpha\sqrt{\lambda_1}} < 0,
\]

(176)

\[
E[C_2^2(\omega)] \leq \frac{3}{4} \left(\epsilon^2 + \frac{8|a|^2}{\alpha} + \frac{\Gamma(\frac{q}{2})|a|^4}{\sqrt{\pi}\lambda_1\alpha^2}\right) < \infty.
\]

(177)

\[
E[L^2(\omega)] \\
\leq 3E \left(\frac{4|a|^2|z(\omega)|^2}{4\lambda_1} + \frac{|a|^4|z(\omega)|^4}{4\lambda_1} + \frac{c_{18}^2}{2\alpha^2}(1 + M_4^{(p-1)}(\omega))\right) \\
\leq 3 \left(\frac{2|a|^2}{\alpha} + \frac{\Gamma(\frac{1}{2})|a|^4}{4\sqrt{\pi}\lambda_1\alpha^2} + \frac{c_{18}^2}{2\alpha^2}(1 + E[M_4^{(p-1)}(\omega)])\right) < \infty.
\]

(178)

Obviously,

\[
E[M_4^{(p-1)}(\omega)] \leq c_{28}(1 + E[M_4^2(\omega)]) < \infty, \quad \text{for} \ p \leq \frac{5}{4}.
\]

For \(p > \frac{5}{4}\), by (157), we have

\[
M_1^{(p-1)}(\omega) \leq c_{29} + c_{30}M_0^{8(p-1)\frac{1}{4}}(\omega) \\
+ c_{31}M_0^{16(p-1)\frac{1}{4}}(\omega)e^{4(p-1)\left(-\frac{1}{2}t + \int_{\tau}^{\tau+\xi} \left(8|a||z(\theta_{\tau+\omega})| + \frac{2|a|^2|z(\theta_{\tau+\omega})|^2}{\sqrt{\lambda_1}}\right)ds\right)} \\
+ c_{32}M_0^{8(p-1)\frac{1}{4}}(\omega)e^{-8(p-1)\frac{1}{4} \int_{\tau}^{\tau+\xi} \left(-\frac{1}{2}t + \int_{\tau}^{\tau+\xi} \left(8|a||z(\theta_{\tau+\omega})| + \frac{2|a|^2|z(\theta_{\tau+\omega})|^2}{\sqrt{\lambda_1}}\right)ds\right)} \\
+ c_{33} \int_{\tau}^{\tau+\xi} \left(1 + |z(\theta_{\tau+\omega})|^4\right)K \int_{\tau}^{\tau+\xi} \left(1 + |z(\theta_{\tau+\omega})|^4\right)M_0^{4p\frac{1}{4}}(\omega) \\
\cdot e^{-\frac{1}{2}t + \int_{\tau}^{\tau+\xi} \left(8|a||z(\theta_{\tau+\omega})| + \frac{2|a|^2|z(\theta_{\tau+\omega})|^2}{\sqrt{\lambda_1}}\right)ds} d\xi.
\]

(179)
By (161) and (108),
\[
E \left( \int_0^\infty M_0^{4p_k} (\theta - \xi \omega)e^{-\frac{\sqrt{2}\xi + f_0^\infty \left( \frac{|\omega||z(\theta, \omega)| + \frac{|2\omega|^2|z(\theta, \omega)|^2}{\sqrt{\lambda_1}} \right)}{\sqrt{\lambda_1}}} \, ds \right)^{4(p-1)} \\
\leq E \left( \int_0^\infty M_0^{8p_k} (\theta - \xi \omega)e^{-\rho_1 \xi} \, ds \right)^{4(p-1)} \\
+ E \left( \int_0^\infty e^{\rho_1 \xi - \frac{\sqrt{2}\xi + f_0^\infty \left( \frac{|\omega||z(\theta, \omega)| + \frac{|2\omega|^2|z(\theta, \omega)|^2}{\sqrt{\lambda_1}} \right)}{\sqrt{\lambda_1}}} \, ds \right)^{4(p-1)} \\
\leq E \left( \int_0^\infty M_0^{32(p-1)} p_k (\theta - \xi \omega)e^{-2(p-1)\rho_1 \xi} \, ds \right)^{4(p-1)} \\
+ E \left( \int_0^\infty e^{(p-1) \left( \rho_1 \xi - \frac{3}{2} \xi + f_0^\infty \left( \frac{|\omega||z(\theta, \omega)| + \frac{|2\omega|^2|z(\theta, \omega)|^2}{\sqrt{\lambda_1}} \right) \right)} \, ds \right)^{4(p-1)} \\
\leq \left( \frac{4^{p-5}}{2(p-1)\rho_1} \right)^{4(p-5)} \int_0^\infty e^{-2(p-1)\rho_1 \xi} E \left[ M_0^{32(p-1)} p_k (\theta - \xi \omega) \right] \, ds \\
+ \left( \frac{4^{p-5}}{2(p-1)\rho_1} \right)^{4(p-5)} \int_0^\infty e^{(p-1) \left( \rho_1 - \frac{3}{2} \xi + f_0^\infty \left( \frac{|\omega||z(\theta, \omega)| + \frac{|2\omega|^2|z(\theta, \omega)|^2}{\sqrt{\lambda_1}} \right) \right)} \, ds \\
\leq \left( \frac{4^{p-5}}{2(p-1)\rho_1} \right)^{4(p-5)} \frac{1}{2(p-1)\rho_1} E \left[ M_0^{32(p-1)} p_k (\theta - \xi \omega) \right] + \frac{1}{(p-1)\rho_1} \\
< \infty.
\]
Similarly, the expectation of other terms of (179) are all finite, so, \( E[M_1^{8(p-1)}(\omega)] < \infty \), thus,
\[
0 \leq E[C_3^2(\omega) = \tilde{C}_3 \leq 5 \left( E[L^2(\omega)] + \frac{2|\omega|^2}{\alpha} + \frac{1}{2\sqrt{\lambda_1}} \frac{|\omega|^4}{\alpha} + 1 + E[M_1^{8(p-1)}(\omega)] \right) < \infty.
\]
From (141) and \( C_3(\omega) > L(\omega) \geq 0 \), (163) holds. \( \square \)

As a consequence of Theorem 2.8 and Lemma 3.11, we have the following main result of this section.

**Theorem 3.12.** Suppose \((H), (69), (97) and (161) hold. Then for any \( \tau \in \mathbb{R}, \omega \in \Omega \), the fractal dimension of \( \mathcal{A}(\tau, \omega) \) has finite upper bound:
\[
\dim_f \mathcal{A}(\tau, \omega) \leq \frac{4n_0 \ln \left( \frac{\sqrt{2\lambda_1\lambda_2 n_0 + 1}}{c_{24}} + 1 \right)}{\ln \frac{3}{\beta}} < \infty,
\]
where
\[
n_0 = \min \left\{ n : c_{24} \gamma_n + 1 \leq \frac{1}{20} \left( 1 - \frac{1}{8} e^{-\frac{1}{\ln 2} \left( 3E[C_3^2(\omega)] + E[C_3^2(\omega)] \right)} \right) \right\} < \infty.
\]
**Proof.** From (176), (177) and (181),
\[
-\infty < E[C_3^2(\omega)] < 0, \quad 0 < 3E[C_3^2(\omega)] + E[C_3^2(\omega)] < \infty.
\]
Take \( t_0 > 0 \) satisfying
\[
t_0 \geq \frac{\ln \frac{3}{\beta}}{-\varepsilon + \frac{4|\omega|}{\sqrt{\alpha\lambda_1}} + \frac{|\omega|^2}{2\alpha \sqrt{\lambda_1}}} > 0.
\]
Comparing (6) and (162), we see that
\[
0 < \delta_n = c_{24} \gamma_{n+1} = \frac{c_{24}}{\sqrt{\lambda_{n+1}}} \xrightarrow{n \to \infty} 0.
\] (186)

Thus there exists a finite integer \( n_0 \in \mathbb{N} \) such that
\[
0 < \delta_{n_0} = \min \left\{ \frac{1}{20}, \frac{1}{8} e^{-\ln 1.8 \frac{1}{\delta_{n_0}} (\mathbb{E}[C_2^2(\omega)] + \mathbb{E}[C_2^2(\omega)])} \right\}.
\] (187)

Then by Theorem 2.8, for any \( \tau \in \mathbb{R}, \omega \in \Omega \), we have
\[
\dim_f A(\tau, \omega) \leq \frac{4n_0 \ln \left( \frac{\mathbb{E}[C_2^2(\omega)]}{\delta_{n_0}} + 1 \right)}{\ln 4/3} < \infty.
\] (188)

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*E-mail address*: zhoushengfan@yahoo.com
*E-mail address*: zhaomin1223@126.com