Explicit $Q$ expressions for inhomogeneous P- and SV-waves in isotropic viscoelastic media

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Abstract
We derive explicit expressions for the dissipation factors of inhomogeneous P and SV-waves in isotropic viscoelastic media. The $Q^{-1}$ values are given as concise and simple functions of material parameters and the wave inhomogeneity parameter using two different definitions. Unlike homogenous waves, inhomogeneous waves may have significant differences in the values of dissipation factors because of different definitions. For example, under one of the three dissipation factor definitions that $Q^{-1}$ is equal to the time-averaged dissipated-energy density divided by twice the time-averaged strain-energy density, it is found and proved that the dissipation factor of SV-waves is totally independent of the inhomogeneity parameter. For materials in which P-waves are normally more dissipative than S-waves (e.g. a porous reservoir), the dissipation factors of P-waves tend to decrease with increasing degree of inhomogeneity. Based on Buchan’s classic real value energy balance equation, a parallel investigation is conducted for each step similar to that based on the Carcione equations, including derivation of explicit formulas (with inhomogeneity angle representing the degree of inhomogeneity of a plane wave), and dissipation curves calculations. We also obtain an inhomogeneity independent formula of $Q_{SV}^{-1}$, and exactly the same phase velocity and attenuation dispersion results for the example material.

Keywords: attenuation, seismic waves, viscoelasticity, wave propagation

1. Introduction
Natural materials are normally dissipative, and viscoelastic waves in such media are generally inhomogeneous waves in which the complex propagation vector is not parallel to the attenuation vector. The dissipative properties are commonly described by the dissipation factor $Q^{-1}$ or inverse quality factor, which is generally defined as the ratio of the time-averaged dissipated-energy density to the stored energy. The stored energy has been defined in three different ways: (i) the peak potential (strain) energy density normalized by $2\pi$ and denoted as $Q_{B}^{-1}$ (Borcherdt 1977), (ii) the time-averaged strain-energy density and denoted as $Q_{V}^{-1}$ (Carcione 2001) or (iii) the time-averaged energy density and denoted as $Q_{T}^{-1}$ (Buchen 1971). For homogeneous waves in isotropic and weakly dissipative solids, the dissipation factor under the three different definitions tends to be the same (Borcherdt & Wennerberg 1985; Carcione 2001; Wang & Guo 2004; Wang 2008). In such situations, and even in high loss media, the dissipation factors are easily and explicitly expressed by concise and simple formulas using the wave parameters, e.g. complex wave-numbers; or materials parameters, e.g. complex modulus. Therefore, these equations are actually viewed as the definition of $Q^{-1}$ in many publications (Borcherdt 1977; Carcione 2001; Cerveny & Psencik 2008).
In the present context, the word ‘explicit’ is taken to mean that the dissipation factor formulas are expressed as functions of the known material parameters and the inhomogeneity parameter(s). For instance, Wang (2019) defines the dissipation factor in terms of the viscoelastic parameter. Very recently, inhomogeneous waves have been considered in the investigation of the behavior of reflection and refraction at an interface of very complicated materials, such as a porous solid with two immiscible viscous fluids (Sharma & Kumar 2011), double porosity and dual-permeability (DPDP) materials (Kumar et al. 2018, 2019). But Sharma & Kumar (2011) use the definition of $Q^{-1}$ of homogeneous waves, and Kumar et al. (2018, 2019) did not consider $Q^{-1}$ for their waves.

However, the applicability of some concise and simple dissipation formulas (e.g. the ratio of the imaginary part of the squared complex wavenumber over its real part) as the definition is problematic for anisotropic media, especially for inhomogeneous plane waves (Krebes & Le 1994; Cerveny & Psencik 2008). Berryman & Wang (2000) also encountered a similar problem when it was applied to slow Biot P-waves. Therefore, this explicit formula developed for homogeneous waves is not suitable for describing wave dissipation behavior of inhomogeneous waves, which should have their own explicit formula.

Borcherdt & Wennerberg (1985) derived explicit dissipation formulas for inhomogeneous P- and S-waves under the definition of $Q_{v}^{-1}$ and showed that $Q^{-1}$ for inhomogeneous waves is greater than the corresponding characteristics for homogeneous waves, with the deviation increasing as the degree of inhomogeneity is increased. However, there is no guarantee that this result and the explicit formula under the definition $Q_{b}^{-1}$ will be the same for cases under the definitions of $Q_{v}^{-1}$ or $Q_{T}^{-1}$. Using Carcione’s definition of $Q_{v}^{-1}$ and solving an algebraic equation of the sixth degree, Cerveny & Psencik (2006, 2008) derived an exact formula in which $Q_{v}^{-1}$ is expressed in terms of the viscoelastic material parameters and polarization vectors of homogeneous or inhomogeneous waves. Isotropic media are a special limiting case of anisotropic media. But the polarization vectors are not the intrinsic parameters of waves nor is the inhomogeneity parameter. Therefore, these equations are not explicit formula for the dissipation factors.

This article aims to develop the explicit formulas under the definitions of $Q_{v}^{-1}$ or $Q_{T}^{-1}$ and systematically investigate their dependence on the inhomogeneity parameters. We will use both the complex form of the energy balance equation (Carcione & Cavallini 1993; Carcione 2001) and the classic real form of the energy balance equation (Buchen 1971) to derive the explicit formulas. The degree of a wave's inhomogeneity will be chosen as the inhomogeneity parameter, $D$ (Cerveny & Psencik 2005), and the inhomogeneity angle, $\gamma$ (Buchen 1971), respectively, in the two sets of derivations. Both sets of formulas will be applied to calculate the phase velocity and attenuation dispersion curves of an example material. The example material is chosen such that its P-wave is more dissipated than its S-wave, which is a normal observation in a sedimentary reservoir.

It is worth pointing out that the complex form of the energy balance equation was developed for anisotropic viscoelastic media and poro-viscoelastic media. As a special case, the application of this new equation to isotropic viscoelastic media can not only check the effectiveness and consistency with the classic real form of the energy balance equation, but it can also provide the reference for application of the equation in anisotropic and porous media. The origin of this paper lies in definitive and explicit $Q^{-1}$ expressions for inhomogeneous P- and SV-waves in isotropic viscoelastic media under the definitions of $Q_{v}^{-1}$ or $Q_{T}^{-1}$ that explicitly show a result previously implicit, where the dissipation factors of SV-waves are independent of the degree of inhomogeneity and that significant differences in attenuation arise from the three different common definitions of $Q^{-1}$. Moreover, we show how the dissipation of P-waves depends on the degree of inhomogeneity of the wave.

2. The commonly defined wave inhomogeneity and dissipation factors

The degree of a wave's inhomogeneity is normally expressed by either the inhomogeneity parameter $D$ (Cerveny & Psencik 2005, 2006, 2008) or the inhomogeneity angle $\gamma$ (Buchen 1971; Borcherdt 1977; Carcione 2001). The particle displacement component $u_{i}$ of an inhomogeneous harmonic plane wave in this article is assumed to be expressed as

\[ u_{i} = A\hat{u}_{i} \exp [i\omega (\mathbf{p} \cdot \mathbf{r} - t)]. \tag{1} \]

Here, $A$ is the complex amplitude, and the over-hat ‘$\hat{}$’ on $u$ refers to a complex unit vector $\hat{u} \cdot \hat{u} = 1$. Quantity $\mathbf{p}$ is the complex slowness vector, and $\mathbf{r}$ is the spatial coordinate (position) vector. With the mixed specification of the slowness vector $\mathbf{p}$ (Cerveny & Psencik 2005), in equation (1), there are two parts to consider as shown in equation (2): $\sigma$ (complex slowness in direction of wave) and $D$ (the real valued inhomogeneity parameter in the perpendicular direction) as given by the expression (Cerveny & Psencik 2005):

\[ \mathbf{p} = \sigma\hat{n} + iD\hat{m} = \hat{n}\text{Re}(\sigma) + i\hat{n}\text{Im}(\sigma) + iD\hat{m}. \tag{2} \]

Here, $\hat{n}$ is the propagation direction and $\hat{n} \cdot \hat{m} = 0$. $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote taking the real and imaginary parts, respectively.
Note that the time dependence $\exp(-i\omega t)$ in equation (1) will result in a negative sign ‘−’ for the first order time derivative. Alternatively, $u_i$ can also expressed in terms of the complex wave vector $k$.

$$u_i = A \hat{u}_i \exp \left[ i (k \cdot \mathbf{r} - \omega t) \right],$$

(3)

where vector $k$ is expressed as (Buchen 1971)

$$k = \omega \mathbf{p} = \mathbf{k} + i \alpha.$$  

(4)

The degree of a wave’s inhomogeneity in Buchen’s (1971) paper is expressed with the inhomogeneity angle $\gamma$, which refers to the angle between the two real vectors $\mathbf{k}$ and $\alpha$.

Inserting equation (2) into (4) gives

$$\mathbf{k} = \omega \text{Re}(\sigma) \hat{n}, \quad \alpha = \omega \text{Im}(\sigma) \hat{n} + \omega D \hat{m}.$$  

(5)

The inhomogeneous wave angle $\gamma$ is given by

$$\cos \gamma = \frac{\text{Im}(\sigma)}{\sqrt{\text{Im}^2(\sigma) + D^2}}.$$  

(6)

Cerveny & Psencik (2005) mentioned that the combination of $(\sigma, D)$ to represent the degree of inhomogeneity of a wave has the advantages of simplicity and always being physically meaningful, whereas the combination of $(\mathbf{k}, \gamma)$ may yield a nonphysical solution. We agree with these comments but contend that the combination of $(\mathbf{k}, \gamma)$ has its own advantages, namely that $\gamma$ is a dimensionless quantity always within the range $(-\pi / 2, \pi / 2)$, whereas $D$ has the units of slowness. Therefore, we normalize $D$ with the relaxed slowness (inverse phase velocity at $\omega = 0$):

$$D = \omega \text{Re}(\sigma [\omega = 0]).$$  

(7)

Thus, $D$ has no physical units and is dimensionless, just like $\gamma$. Theoretically speaking, the range of $D$ is $(-\infty, +\infty)$. But it can be normally set in the range of $(-1, 1)$. It deserves mentioning that $\gamma$ may be frequency dependent for a given $D$ since $\text{Im}(\sigma)$ may be frequency dependent (equation 6).

Similar to Buchen (1971), the complex quantities are expressed as

$$k^2 = \mathbf{k} \cdot \mathbf{k} = \omega^2 p^2_{\text{p}, \sigma} = \Omega_1 + i \Omega_2, \quad \Omega_1 = k^2 - \alpha^2, \quad \Omega_2 = 2k\alpha \cos \gamma.$$  

(8)

The phase velocity $v_{\text{phy}}$ is simply defined as

$$v_{\text{phy}} = \omega / k = 1 / \text{Re}(\sigma).$$  

(9)

It was proved (Borcherdt & Wennerberg 1985; Carcione 2001) that an increase in the inhomogeneity of the wave field causes the velocity to decrease.

However, the dissipation factor has been commonly defined in the following three different ways that we denote by $Q_{\text{B}}^{-1}$, $Q_{\text{V}}^{-1}$ and $Q_{\text{T}}^{-1}$, respectively.

$Q_{\text{B}}^{-1}$: the ratio of the time-averaged (denoted with symbol $\langle \cdot \rangle$) dissipated-energy density $\langle \varepsilon_D \rangle$ to the peak potential (strain) energy density and normalized by $2\pi$ (Borcherdt 1977, 1982; Borcherdt & Wennerberg 1985).

$Q_{\text{V}}^{-1}$: the time-averaged dissipated-energy density $\langle \varepsilon_D \rangle$ divided by twice the time-averaged strain-energy density $\langle \varepsilon_S \rangle$ (Carcione 2001).

$$Q_{\text{V}}^{-1} = \langle \varepsilon_D \rangle / (2 \langle \varepsilon_S \rangle).$$  

(10)

$Q_{\text{T}}^{-1}$: the time-averaged dissipated-energy density $\langle \varepsilon_D \rangle$ divided by the time-averaged energy density $\langle \varepsilon_T \rangle$ (normally the sum of $\langle \varepsilon_S \rangle$ and the time-averaged kinetic-energy density $\langle \varepsilon_K \rangle$) (Buchen 1971).

$$Q_{\text{T}}^{-1} = \langle \varepsilon_D \rangle / \langle \varepsilon_T \rangle, \quad \langle \varepsilon_T \rangle = \langle \varepsilon_K \rangle + \langle \varepsilon_S \rangle.$$  

(11)

For homogenous waves, without approximations, all factors $Q_{\text{B}}^{-1}$, $Q_{\text{V}}^{-1}$ and $Q_{\text{T}}^{-1}$ can be expressed by the following concise and simple formula (Borcherdt & Wennerberg 1985; Carcione 2001), $Q_{\text{BH}}^{-1}$, $Q_{\text{TV}}^{-1}$ and $Q_{\text{TH}}^{-1}$ as

$$Q_{\text{BH}}^{-1} = Q_{\text{TV}}^{-1} = \text{Im}(k^2) / \text{Re}(k^2),$$  

(12)

and

$$Q_{\text{TH}}^{-1} = 2 \alpha v_{\text{phy}} / \omega.$$  

(13)
For homogeneous waves propagating in an isotropic and weakly dissipative material, the three \(Q^{-1}\) values are approximately the same, \(Q^{-1}_0 = Q^{-1}_v = Q^{-1}_s\) (Carcione 2001). Therefore, equations (12) and (13) are actually viewed as the definition of \(Q^{-1}\) in many publications (Borcherdt 1977; Carcione 2001; Cerveny & Psencik 2008). However, the applicability of the definition of equation (12) is problematic for anisotropic media, especially for inhomogeneous plane waves (Krebes & Le 1994; Cerveny & Psencik 2008).

On the other hand, for inhomogeneous waves we will find that the three \(Q^{-1}\) values could be significantly different. The dependence of \(Q^{-1}_s\) on the wave inhomogeneity parameter in isotropic viscoelastic material has been extensively investigated by Borcherdt (1977, 1982, 2009) and Borcherdt & Wennerberg (1985).

### 3. The complex form of the energy balance equation

The wave equation of a viscoelastic material is well-known and can be written as

\[
[\lambda + \mu] \nabla \nabla \mathbf{u} + (\mu \nabla^2 + \omega^2 \rho) \mathbf{I} \cdot \mathbf{u} = 0, \tag{14}
\]

where, \(\rho\) is density, \(\lambda\) and \(\mu\) are the complex Lame (modulus) coefficients. Substituting equation (1) into (14) results in two slowness values, \(p_P\), \(p_S\) and the corresponding complex amplitude vectors, \(\hat{u}_P\), \(\hat{u}_S\) for P- and S-waves, respectively,

\[
\begin{align*}
\hat{u}_P \cdot \mathbf{p}_P &= p_P^2 = \rho / (\lambda + 2\mu), \\
\hat{u}_S \cdot \mathbf{p}_S &= p_S^2 = \rho / \mu, \\
\end{align*}
\tag{15}
\]

Here, \(\epsilon\) is an arbitrary scalar.

According to Carcione & Cavallini (1993) and Carcione (2001), the complex form of the energy balance equation without body force terms, is written as

\[
-\frac{1}{2} \text{div} (\Sigma \cdot \mathbf{v}^*) = 2i\omega \left[ \frac{1}{4} \rho \mathbf{v}^* \cdot \mathbf{v} - \frac{1}{4} \text{Re}(\mathbf{S}^T \cdot \mathbf{M} \cdot \mathbf{S}^*) \right] + \omega \frac{1}{2} \text{Im}(\mathbf{S}^T \cdot \mathbf{M} \cdot \mathbf{S}^*), \tag{16}
\]

Here, \(\mathbf{v}\) is the particle velocity vector \(\mathbf{v} = \hat{\mathbf{u}} = -i\omega \mathbf{u}\); \(\mathbf{S}\) is a shortened form of the symmetric strain tensor

\[
\mathbf{S} = [u_{xx}, u_{yy}, u_{zz}, 2u_{xy}, 2u_{xz}, 2u_{yz}], \tag{17}
\]

\(\Sigma\) is the stress tensor \(\Sigma = [\sigma_{ij}] (i, j = x, y, z)\); and \(\mathbf{M}\) is the complex 6 \times 6 stiffness tensor. For isotropic media,

\[
\mathbf{M} =
\begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}. \tag{18}
\]

On a time-averaged basis, each term in equation (16) has precise physical meaning (Carcione & Cavallini 1993 for details). For the sake of completeness, we list the following time-averaged energy densities.

The time-averaged stored kinetic-energy density \(\langle \epsilon_k \rangle\) is given by:

\[
\langle \epsilon_k \rangle = \frac{1}{4} \rho \mathbf{v}^* \mathbf{v} = \frac{1}{2} \rho \langle \text{Re}(\mathbf{v}^T) \cdot \text{Re}(\mathbf{v}) \rangle. \tag{19}
\]

The time-averaged stored strain-energy density \(\langle \epsilon_s \rangle\) is defined as

\[
\langle \epsilon_s \rangle = \frac{1}{4} \text{Re}(\mathbf{S}^T \cdot \mathbf{M} \cdot \mathbf{S}^*) = \frac{1}{2} \langle \text{Re}[\mathbf{S}^T] \cdot \text{Re}[\mathbf{M}] \cdot \text{Re}[\mathbf{S}] \rangle. \tag{20}
\]

The time-averaged dissipated-energy density \(\langle \epsilon_d \rangle\) is given by

\[
\langle \epsilon_d \rangle = \frac{1}{2} \text{Im}(\mathbf{S}^T \cdot \mathbf{M} \cdot \mathbf{S}^*) = \langle \text{Re}[\mathbf{S}^T] \cdot \text{Im}[\mathbf{M}] \cdot \text{Re}[\mathbf{S}] \rangle. \tag{21}
\]

The time-averaged product of real wavefields implies the consistency between the complex energy balance equation (equation (16)) and the classic real energy balance equation (Buchen 1971), although this was not claimed by Carcione & Cavallini (1993). The complex energy balance equation was applied by Cerveny & Psencik (2006) for viscoelastic, anisotropic media.
Carcione & Cavallini (1993) defined the complex power flux vector \( \mathbf{P} = -\Sigma \cdot \mathbf{v}^* / 2 \). This can be written as \( P_j = -\sigma_j \mathbf{u}^c / 2 \). Thus, \( \text{Re}(P_j) = -\left(\text{Re}(\sigma_j) \text{Re}(\mathbf{u}^c)\right) \), which is same as the energy flux defined in Buchen (1971).

The investigation of the consistency between the two equations draws our attention to the divergence operation on the mean energy flux in the wave. With the approach based on the complex energy balance equation, equation (16) shows that \( \langle \varepsilon_D \rangle \) is obtained as \( \frac{1}{2} \text{Re}[\text{div}(\Sigma \cdot \mathbf{v}^*)] \). The corresponding \( \langle \varepsilon_D \rangle \) in Buchen (1971) was actually obtained by the formula

\[
\text{div} [\left(\text{Re}(\Sigma) \cdot \text{Re}(\mathbf{v})\right)] = \frac{1}{2} \text{div}[\text{Re}(\Sigma \cdot \mathbf{v}^*)].
\]

The consistency between the two operations: \( \text{Re}[\text{div}(\mathbf{F})] \) and \( \text{div}[\text{Re}(\mathbf{F})] \) seems not straightforward; so for the sake of completeness, this is given in Appendix A.

Inserting equation (2) into equation (1) we obtain the displacement expression as

\[
u_i = A \hat{u}_i \exp \left[-\omega (\hat{n} \cdot \mathbf{r} \text{Im}(\sigma) + D \hat{m} \cdot \mathbf{r})\right] \exp \left[i \omega (\hat{n} \cdot \mathbf{r} \text{Re}(\sigma) - t)\right].
\]

Since equation (2) must satisfy the solutions (15), we have

\[
p \cdot p = p^2 = \left(\sigma \hat{n} + iD \hat{m}\right) \cdot \left(\sigma \hat{n} + iD \hat{m}\right) = \sigma^2 - D^2, \quad \text{or} \quad \sigma^2 = p^2 + D^2.
\]

Note that \( p^2 \) only depends on material properties for a given wave type, i.e. \( p^2 \) only depend on the degree of the wave inhomogeneity \( D \).

Without loss of generality for isotropic media, we consider a 2D inhomogeneous wave propagating along the \( z \)-direction in the \( xy \)-plane. Thus, \( \hat{n} = \hat{z} \) and \( \hat{m} = \hat{x} \), and the complex slowness vector can be written as

\[
p = iD \hat{x} + \sigma \hat{z} \quad \text{or} \quad p_x = iD, \quad p_z = \sigma.
\]

### 4. Explicit dissipation factor for inhomogeneous waves based on the complex energy balance equations

#### 4.1. For the P-wave

With equations (15) and (23), the particle displacement of a P-wave is written as

\[
u = A \left(p_x, 0, p_z\right) \exp \left[-\omega (D \hat{x} + \text{Im}(\sigma) \hat{z}) \cdot \mathbf{r}\right] \exp \left[i \omega (\text{Re}(\sigma) z - t)\right]
\]

Substituting equation (26) into equations (19)–(21) gives

\[
\langle \varepsilon_K \rangle = \frac{\alpha^2|A|^2}{4} \rho \left( |p_x^2 + D^2| + D^2 \right) \exp \left[-2\omega (D \hat{x} + \text{Im}(\sigma) \hat{z}) \cdot \mathbf{r}\right],
\]

\[
\langle \varepsilon_S \rangle = \frac{\alpha^2|A|^2}{4} ES_R \exp \left[-2\omega (D \hat{x} + \text{Im}(\sigma) \hat{z}) \cdot \mathbf{r}\right]
\]

\[
ES_R = \left(D^4 + |p_x^2 + D^2|^2\right) \text{Re}(\lambda + 2\mu) - D^2 \left(\text{Re}(p_x^2 + D^2) \text{Re}(2\lambda) + 4D^2 |p_x^2 + D^2| \text{Re}(\mu)\right)
\]

and

\[
\langle \varepsilon_D \rangle = \frac{\alpha^2|A|^2}{2} ES_I \exp \left[-2\omega (D \hat{x} + \text{Im}(\sigma) \hat{z}) \cdot \mathbf{r}\right]
\]

\[
ES_I = -\left(D^4 + |p_x^2 + D^2|^2\right) \text{Im}(\lambda + 2\mu) + D^2 \left(\text{Re}(p_x^2 + D^2) \text{Im}(2\lambda) - 4D^2 |p_x^2 + D^2| \text{Im}(\mu)\right)
\]

According to the \( Q^{-1} \) definition (10)

\[
Q_{PV}^{-1} = \frac{ES_I}{ES_R}
\]

According to the \( Q^{-1} \) definition (11)

\[
Q_{PT}^{-1} = \frac{2ES_I}{\rho \left( |p_x^2 + D^2| + D^2 \right) + ES_R}
\]
Setting $D = 0$ gives the dissipation factors of homogeneous P-waves

$$Q_{PVH}^{-1} = Q_{PV}^{-1} (D = 0) = -\frac{\text{Im}(\lambda + 2\mu)}{\text{Re}(\lambda + 2\mu)}. \tag{32}$$

With equations (15) and (4), one may easily find that $Q_{PVH}^{-1}$ satisfies equation (12).

Similarly, we have

$$Q_{PVH}^{-1} = Q_{PV}^{-1} (D = 0) = -2 \frac{|p_p|^2 \text{Im}(\lambda + 2\mu)}{\rho + |p_p|^2 \text{Re}(\lambda + 2\mu)} = 2 \frac{\text{Im}(p_p)}{\text{Re}(p_p)}. \tag{33}$$

The proof of equation (33) is given in Appendix B. Using equations (5) and (9), one may easily find that equation (33) satisfies (13).

### 4.2. For the S-wave

With equations (15) and (23), the particle displacement of an S-wave is written as

$$u = A (p_p, 0, -p_p) \exp [-\omega (D\hat{x} + \text{Im}(\sigma) \hat{z}) \cdot r] \exp [i\omega (\text{Re} (\sigma) z - t)]. \tag{34}$$

Substituting equation (34) into the equations (19)–(21) gives

$$\langle \epsilon_K \rangle = \frac{\omega^2 |A|^2}{4} \rho \left( \left| \sqrt{p_S^2 + D^2} \right|^2 + D^2 \right) \exp [-2\omega (D\hat{x} + \text{Im}(\sigma) \hat{z}) \cdot r], \tag{35}$$

$$\langle \epsilon_S \rangle = \frac{\omega^2 |A|^2}{4} \left( 4|p_S^2 + D^2|D^2 + |p_S^2 + 2D^2|^2 \right) \text{Re}(\mu) \exp [-2\omega (D\hat{x} + \text{Im}(\sigma) \hat{z}) \cdot r], \tag{36}$$

and

$$\langle \epsilon_D \rangle = -\frac{\omega^2 |A|^2}{2} \left( 4|p_S^2 + D^2|D^2 + |p_S^2 + 2D^2|^2 \right) \text{Im}(\mu) \exp [-2\omega (D\hat{x} + \text{Im}(\sigma) \hat{z}) \cdot r]. \tag{37}$$

Using the $Q^{-1}$ definition of equation (10), we have

$$Q_{SV}^{-1} = -\frac{\text{Im}(\mu)}{\text{Re}(\mu)}. \tag{38}$$

Using the $Q^{-1}$ definition of equation (11), we have

$$Q_{ST}^{-1} = -2 \frac{\left( 4|p_S^2 + D^2|D^2 + |p_S^2 + 2D^2|^2 \right) \text{Im}(\mu)}{\rho \left( \left| \sqrt{p_S^2 + D^2} \right|^2 + D^2 \right) + \left( 4|p_S^2 + D^2|D^2 + |p_S^2 + 2D^2|^2 \right) \text{Re}(\mu)}. \tag{39}$$

The significant difference between the two dissipation factors from the different definitions can be immediately identified by comparing $Q_{SV}^{-1}$ and $Q_{ST}^{-1}$. It seems surprising that $Q_{SV}^{-1}$ is independent of the degree of wave inhomogeneity. But the physical meaning behind this result has already been reported by Borcherdt & Wennerberg (1985, p. 1738) who mentioned that for inhomogeneous SV and SH waves, the ratios of the mean energy dissipated per unit volume to the mean potential energy density are equal and independent of the degree of inhomogeneity of the waves. This statement is definitely implied by the result of equation (38). Unlike the definitions of $Q_{SV}^{-1}$ and $Q_{ST}^{-1}$, Borcherdt & Wennerberg (1985) defined the dissipation $Q^{-1}$ as the ratio of the mean energy dissipated per unit volume to the peak energy density stored (not mean strain energy) and normalized by $2\pi$. Their $Q_{SV}^{-1}$ is inhomogeneity dependent. Thus, it seems the result of equation (38) has not been well-known. For example, equation (38) should be a special case of the qSV wave for anisotropic media in Cerveny & Psencik (2006). But this has not been mentioned before.

In a similar way to equations (32) and (33) for the P-wave, the dissipation factors of homogeneous S-waves are obtained by setting $D = 0$ in equation (39)

$$Q_{STH}^{-1} = -2 \frac{|p_S|^2 \text{Im}(\mu)}{\rho + |p_S|^2 \text{Re}(\mu)} = 2 \frac{\text{Im}(p_S)}{\text{Re}(p_S)}. \tag{40}$$
Since $Q_{SV}^{-1}$ is independent of the degree of the wave inhomogeneity, we can directly obtain $Q_{SVH}^{-1} = Q_{SV}^{-1}$.

It is very important to note that for an infinite wave inhomogeneity parameter (i.e. $D = \infty$), the P-wave dissipation factors depend on the dissipation factors of homogeneous S-waves. By equations (28)–(30) together with (31) and (39), we have the following concise and simple limiting dissipation factors:

$$Q_{PV}^{-1}(D = \infty) = Q_{SV}^{-1} = -\frac{\text{Im}(\mu)}{\text{Re}(\mu)},$$

$$Q_{PT}^{-1}(D = \infty) = Q_{ST}^{-1}(D = \infty) = -2\frac{\text{Im}(\mu)}{\text{Re}(\mu)}.$$

(41) 

5. Explicit dissipation factor expressions for inhomogeneous waves based on Buchen’s energy balance equations

Unlike the complex energy balance equation, Buchen (1971) developed a real valued energy balance equation for isotropic viscoelastic materials for which the two approaches should be consistent and give the same dissipation factors.

The complex shear modulus is written as $\mu = \text{Re}(\mu) + i\text{Im}(\mu)$, $\text{Im}(\mu) \leq 0$ for a time dependence of $\exp(-iot)$ assumed in this paper. But in Buchen (1971), it was written as $\mu = \text{Re}(\mu) - i\text{Im}(\mu)$ with the implicit assumption that $\text{Im}(\mu) \geq 0$. Therefore, the $\text{Im}(\mu)$ in Buchen (1971) will be replaced with $-\text{Im}(\mu)$ for the following time-averaged energy terms for both P- and SV-waves as

$$\langle \varepsilon_k \rangle = \frac{1}{4} \rho \omega |A|^2 \sqrt{\Omega_1^2 + \Omega_2^2 \sec^2 \gamma} \exp(-2\alpha \cdot r),$$

(43) 

$$\langle \varepsilon_s \rangle = \frac{1}{4} |A|^2 \left\{ \rho \omega^2 \Omega_1 + 2\text{Re}(\mu) \Omega_2 \tan^2 \gamma \right\} \exp(-2\alpha \cdot x),$$

(44) 

and

$$\langle \varepsilon_D \rangle = \frac{1}{2} \rho \omega \Omega_1 \langle |A|^2 \rangle \exp(-2\alpha \cdot x) \left\{ \rho \omega^2 - 2\text{Im}(\mu) \Omega_2 \tan^2 \gamma \right\},$$

$$\langle \varepsilon_D \rangle = \langle \varepsilon_D \rangle / \omega.$$ 

(45) 

To investigate the consistency of the two energy balance equations, we can also provide the explicit $Q^{-1}$ formulas in terms of the inhomogeneity angle $\gamma$.

Using the definitions of $Q_V^{-1}$ and $Q_T^{-1}$, and Buchen’s equations (43)–(45) gives

$$Q_V^{-1} = -\frac{\Omega_1 \left\{ \rho \omega^2 - 2\text{Im}(\mu) \Omega_2 \tan^2 \gamma \right\}}{\rho \omega^2 \Omega_1 + 2\text{Re}(\mu) \Omega_2 \tan^2 \gamma},$$

(46) 

$$Q_T^{-1} = 2 \frac{\Omega_2 \left\{ \rho \omega^2 - 2\text{Im}(\mu) \Omega_2 \tan^2 \gamma \right\}}{\rho \omega^2 \sqrt{\Omega_1^2 + \Omega_2^2 \sec^2 \gamma} + \left\{ \rho \omega^2 \Omega_1 + 2\text{Re}(\mu) \Omega_2 \tan^2 \gamma \right\}}.$$ 

(47) 

We can temporarily view these two equations are for the P-wave and the $\Omega_1$ and $\Omega_2$ quantities are assumed to be calculated from $\omega^2 p_{SV}^2$ (equation (8)). For the SV-wave, where we write $Q_{SV}^{-1}$ and $Q_{ST}^{-1}$, the $\Omega_1$ and $\Omega_2$ should be calculated from $\omega^2 p_{ST}^2$. An immediate question that arises is equation (38) gives the inhomogeneity factor independent of $Q_{SV}^{-1}$. But equation (46) seems not right at first sight. Actually, this can be easily proved.

For the SV-wave, we have

$$\frac{\rho}{\mu} = \frac{\rho}{\text{Re}(\mu) + i\text{Im}(\mu)} = \frac{k^2}{\omega^2} = \frac{\Omega_1 + i\Omega_2}{\omega^2}.$$ 

(48) 

Then equation (48) gives

$$\Omega_1 = \rho \omega^2 \frac{\text{Re}(\mu)}{\text{Re}^2(\mu) + \text{Im}^2(\mu)} \text{ and } \Omega_2 = -\rho \omega^2 \frac{\text{Im}(\mu)}{\text{Re}^2(\mu) + \text{Im}^2(\mu)}.$$ 

(49) 

Equation (48) can also be written as

$$\rho \omega^2 = \left( \Omega_1 + i\Omega_2 \right) \cdot \left( \text{Re}(\mu) + i\text{Im}(\mu) \right) = \Omega_1 \text{Re}(\mu) - \Omega_2 \text{Im}(\mu).$$ 

(50)
and

\[ \Omega_2 \text{Re} (\mu) + \Omega_1 \text{Im} (\mu) = 0. \]  (51)

Equation (51) can be understood by the real value of \( \rho \omega^2 \) or by using equation (49).

Substituting equation (51) into equation (46) to replace \( \text{Re} (\mu) \Omega_2^2 \) with \( -\Omega_1 \text{Im} (\mu) \Omega_2 \), and using equations (48) and (49) directly gives

\[ Q_{SV}^{-1} = \frac{\Omega_2 \left\{ \rho \omega^2 - 2 \text{Im} (\mu) \Omega_2 \tan^2 \gamma \right\}}{\rho \omega^2 \Omega_1 - 2 \Omega_1 \text{Im} (\mu) \Omega_2 \tan^2 \gamma} = \frac{\Omega_2}{\Omega_1} = \frac{\text{Im} (k^2)}{\text{Re} (k^2)} = \frac{\text{Im} (\mu)}{\text{Re} (\mu)}. \]  (52)

Equation (52) is exactly same as equation (38). But the consistency of \( Q_{P\&S}^{-1} \) and \( Q_{PV}^{-1} \) between the two energy balance equations are not apparent straight away and will be illustrated with the following examples.

6. Examples

The dissipative properties of the examples will be described with the Cole–Cole model (Liu et al. 2018) with the complex relaxation function, \( \Omega_z (\omega) \), of the viscoelastic modulus given by

\[ \Omega_z (\omega) = \left[ 1 + \left( -i \omega \tau_z^\sigma \right)^\alpha \right] / \left[ 1 + \left( -i \omega \tau_z^\sigma \right)^\alpha \right]. \]  (53)

Here, \( \tau_z^\sigma \) and \( \tau_z^\gamma \) are stress and strain relaxation times of the modulus \( Z \) and \( \alpha \) is the fractional derivative order. Thus, a complex modulus \( Z(\omega) \) (representing the shear modulus \( \mu (\omega) \) and bulk modulus \( K(\omega) = \lambda (\omega) + 2 \mu (\omega) / 3 \), etc.) can be written as \( Z(\omega) = Z(0) \Omega_z (\omega) \) where \( Z(0) \) is the relaxed modulus that is assumed to be known (Liu et al. 2018).
The sample material has material density $\rho = 2304 \text{ kg/m}^3$, bulk modulus $K(0) = 7.8 \times 10^9 \text{ N/m}^2$ and shear modulus $\mu(0) = 6.0 \times 10^9 \text{ N/m}^2$. The relaxed velocities are obtained as $V_{p0} = 2612 \text{ m s}^{-1}$, $V_{s0} = 1606 \text{ m s}^{-1}$. The parameters of the Cole–Cole model are $\tau_\mu = 2\pi \times 4.44 \times 10^{-5} \text{ s}$, $\tau_\varepsilon = 2\pi \times 4.77 \times 10^{-5} \text{ s}$, $\alpha_\mu = 0.527$ for $\mu(\omega)$; and $\tau_K = 2\pi \times 5.1 \times 10^{-5} \text{ s}$, $\tau_\varepsilon = 2\pi \times 1.0 \times 10^{-4} \text{ s}$, $\alpha_K = 0.505$ for $K(\omega)$.

Figure 1 shows the dispersion curves of homogeneous waves in the example material. Figure 1a is the P-wave phase velocity, $v_{pH}$. Figure 1b shows the P-wave dissipation factors, $Q_{V}^{-1}$ and $Q_{T}^{-1}$, corresponding to the definitions $Q_V^{-1}$ and $Q_T^{-1}$. In similar fashion, figure 1c is the S-wave phase velocity, $v_{SH}$, and figure 1d shows the S-wave dissipation factors, $Q_{SV}^{-1}$ and $Q_{ST}^{-1}$. In this example material symbolizing a near surface porous reservoir, the P-wave is more dissipative than the S-wave and non-attenuating at zero frequency and infinite frequency. But there is no distinguishing difference between $Q_V^{-1}$ and $Q_T^{-1}$ for homogeneous P- or S-waves.

We now investigate the phase velocity dispersion and dissipation factor dependence on the degree of inhomogeneity $D$ of the plane waves. We set $D$ equal to $[0.00, 0.01, 0.15, 0.30, 0.50]$ normalized with $V_{p0} = 2612 \text{ m s}^{-1}$ or $V_{s0} = 1606 \text{ m s}^{-1}$ (relaxed velocity of the homogeneous P-wave and S-wave, respectively). It is not necessary to set a negative $D$ value since the equations of the phase velocity and dissipation factor as developed are even functions of $D$.

Figure 2 shows the dispersion curves of inhomogeneous P-waves. Phase velocities (upper) increase with increasing frequency, and decrease with an increasing inhomogeneity parameter $D$, as reported by, for example, Borcherdt & Wennerberg (1985) and Carcione (2001).

However, the middle diagrams indicate that the dissipation factors under the definition of $Q_T^{-1}$ decrease with increasing $D$. This seems contradictory to the Borcherdt & Wennerberg (1985) conclusion that $Q_T^{-1}$ for inhomogeneous waves is greater than the corresponding characteristics for homogeneous waves. But we must appreciate that Borcherdt & Wennerberg (1985) assumed a more dissipative S-wave than P-wave and defined the dissipative factor as $Q_V^{-1}$. Our example is different to their case. On the other hand, equations (41) and (42) strongly support our results. Since our example material has much larger dissipation factors of homogenous P-waves than S-waves (at $D = 0$), the dissipation factors of P-waves must decrease with
Figure 3. Comparison between $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ of P-waves at different inhomogeneity parameters $D$. (a) $D = 0.01/Vp0$; (b) $D = 0.15/Vp0$; (c) $D = 0.3/Vp0$ and (d) $D = 1000.0/Vp0$.

an increasing $D$ to finally satisfy the limiting dissipation factors which only depend on the complex shear modulus (equations (41) and (42)). By the same reasoning, $Q_{SV}^{-1}$ itself is already equal to the limiting dissipation factor (equations (38), (41) and (52)) and is thus independent of $D$.

The lower diagrams show the corresponding inhomogeneity angle (lower) for a given $D$. The inhomogeneity angle is calculated by equation (6) and will be used to calculate the dissipation factor with equations (46) and (47), which are developed from Buchen’s (1971) energy balance equations (43)–(45). The Buchen equations give the exact same results as those from Carcione’s (2001) complex energy balance equations. Because of this, we only show the result based on the Carcione equations. It is interesting to see that at zero or infinite frequency (non-attenuating), the inhomogeneity angle $\gamma$ can only be $0$ or $90^\circ$.

Figure 3 is the comparison between $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ of P-waves denoted as $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ at different inhomogeneity parameter $D$ values. Both $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ decrease with increasing $D$ as depicted in figure 2. In figure 3d, we set $D = 1000/Vp0$, which can be approximately viewed as ‘infinite’ for this numerical calculation. Thus, the $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ curves in figure 3d may be viewed as their limiting dissipation factors. Please notice here that $Q_{PT}^{-1}$ is double $Q_{PV}^{-1}$, as indicated by equations (41) and (42), which can also explain the reason that the difference between $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ becomes significant with increasing $D$ (see diagrams 3c and 3d). On the other hand, because $Q_{PV}^{-1}(D = \infty) = Q_{SV}^{-1}$, equation (41), and $Q_{SV}^{-1}$ is $D$ independent, the $Q_{SV}^{-1}$ in figure 3d can be viewed as the $Q_{SV}^{-1}$ curve with no necessity to draw it separately.

In a similar way to the calculations for the P-wave above, figure 4 shows the dispersion curves for inhomogeneous S-waves. Phase velocities (upper) increase with increasing frequency, and decrease with increasing inhomogeneity parameter $D$ as expected. The middle diagrams show increasing $Q_{ST}^{-1}$ with increasing $D$, which is the opposite trend to its P-wave counterpart, e.g. $Q_{PV}^{-1}$ shown in the middle of figure 2. The lower diagrams show the corresponding inhomogeneity angle $\gamma$. To provide sufficient illustration on the dependence of $\gamma$ on $D$, we calculate extra dissipation curves with $D = 0.001/Vs0$ and $0.03/Vs0$. But only their $\gamma$ curves are shown in the lower diagram. Again, at zero or infinite frequency (non-attenuating) the inhomogeneity angle $\gamma$ can only be $0$ or $90^\circ$. 


Figure 4. Dispersion curves of inhomogeneous S-waves for phase velocities (upper plots), dissipation factors (middle plots) under the definition of $Q_{T}^{-1}$, and the corresponding inhomogeneity angle (lower plots) for a given $D$.

7. Conclusions

Based on Carcione’s (2001) complex energy balance equations and the mixed specification of slowness vector (Cerveny & Psencik 2005), explicit formulas for the dissipation factors of P- and SV-waves are developed under the two different definitions, $Q_{V}^{-1}$ and $Q_{T}^{-1}$, respectively. The dissipation factor of the SV-wave $Q_{SV}^{-1}$ (under the definition of $Q_{V}^{-1}$) is found to be independent of the degree of inhomogeneity of the wave. The example viscoelastic material is chosen to represent the dissipative features of a reservoir for which P-waves are normally more dissipative than S-waves. The calculated dissipation factors of P-waves under the definitions, $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ (i.e. $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$) decrease with an increasing degree of inhomogeneity $D$. For their counterpart S-waves, $Q_{SV}^{-1}$ is independent of $D$ and $Q_{ST}^{-1}$ shows the opposite trend, increasing with increasing $D$.

It is very important to notice that the values of the dissipation factors could have significant differences due to the different definitions for inhomogeneous plane P- and SV- waves. To ensure the correctness, we repeated each step of the investigation mentioned above in a parallel way based on Buchen (1971) classic real value energy balance equation, including derivation of explicit formulas for $Q_{PV}^{-1}$ and $Q_{PT}^{-1}$ (with inhomogeneity angle $\gamma$ representing the degree of inhomogeneity of the plane wave), and dissipation curve calculations. We also obtain the inhomogeneity independent formula for $Q_{SV}^{-1}$ and exactly the same phase velocity and dissipation factor dispersion results for the example material.

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Appendix A. Consistency check for two divergence operations on a harmonic wave

We assume an inhomogeneous harmonic wave is expressed as:
\[
F = (\Re(F) + i \Im(F)) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad \text{and} \quad \mathbf{k} = \omega \mathbf{p} = \kappa + i\alpha, \tag{A.1}
\]
or
\[
F = [\Re(F) C - \Im(F)] + i(\Re(F) S - \Im(F) C) \exp (-\alpha \cdot r). \tag{A.2}
\]

Here
\[
S = \sin (\kappa \cdot r - \omega t), \quad C = \cos (\kappa \cdot r - \omega t). \tag{A.3}
\]

The operation \(\text{div} \Re(F)\) is used by Buchen (1971) to obtain the time-averaged dissipated-energy density ratio
\[
\text{div} \Re(F) = \text{div} \left[ (\Re(F) C - \Im(F)) - \kappa \cdot [\Re(F) S + \Im(F) C] \exp (-\alpha \cdot r) \right]. \tag{A.4}
\]

Using the identity
\[
\text{div} (f \mathbf{A}) = (\nabla f) \cdot \mathbf{A} + (\text{div} \mathbf{A}) f \tag{A.5}
\]
results in
\[
\text{div} \Re(F) = -\alpha \cdot [\Re(F) C - \Im(F)] - \kappa \cdot [\Re(F) S + \Im(F) C] \exp (-\alpha \cdot r). \tag{A.6}
\]

The operation \(\Re(\text{div}(F))\) is used by Carcione & Cavallini (1993) to obtain the time-averaged dissipated-energy density.

Using the identity (A.5) results in
\[
\Re(\text{div}(F)) = \Re \left\{ -\alpha \cdot [\Re(F) C - \Im(F)] + \exp (-\alpha \cdot r) [\kappa \cdot \Re(F) S + \Im(F) C + i (\kappa \cdot \Re(F) C + \kappa \cdot \Im(F) S) - \alpha \cdot r] \right\}. \tag{A.7}
\]

Then, we have
\[
\Re(\text{div}(F)) = -\alpha \cdot [\Re(F) C - \Im(F)] - \kappa \cdot [\Im(F) C + \Re(F) S] \exp (-\alpha \cdot r). \tag{A.8}
\]
Comparing with (A.6), we have
\[
\text{div} \Re(F) = \Re(\text{div}(F)). \tag{A.9}
\]

Appendix B. Proof of equation (33)

Equation (15) gives
\[
\lambda + 2\mu = \frac{\rho}{p^2_p} = \rho \frac{\Re(p^2_p) - i \Im(p^3_p)}{\Re^2(p^2_p) + \Im^2(p^2_p)} = \rho \frac{\Re(p^2_p) - i \Im(p^3_p)}{|p^3_p|^2}. \tag{B.1}
\]

Substituting the real and imaginary parts of \(\lambda + 2\mu\) into equation (33) leads to
\[
Q^{-1}_{\text{FTH}} = \frac{2\Im(p^2_p)}{|p^2_p| + \Re(p^2_p)}. \tag{B.2}
\]
Expressing \(p^2_p\) with its the real and imaginary parts gives
\[
p^2_p = \Re(p^2_p) - i \Im(p^2_p) + 2 \Re(p_p) \Im(p_p). \tag{B.3}
\]
Then,
\[
|p^2_p| = \sqrt{\left(\Re^2(p_p) - \Im^2(p_p)\right)^2 + \left[2 \Re(p_p) \Im(p_p)\right]^2} = \Re^2(p_p) + \Im^2(p_p) \tag{B.4}
\]
Substituting equations (B.3) and (B.4) into equation (B.2) results in equation (33)
\[
Q^{-1}_{\text{FTH}} = \frac{4 \Re(p_p) \Im(p_p)}{\Re^2(p_p) + \Im^2(p_p) + \Re^2(p_p) - \Im^2(p_p)} = 2 \frac{\Im(p_p)}{\Re(p_p)}. \tag{B.5}
\]
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