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To cite this article: M Blasone et al 2017 J. Phys.: Conf. Ser. 804 012006

View the article online for updates and enhancements.
Inequivalent representations in the functional integral framework

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Abstract. An important feature of Quantum Field Theory is the existence of unitarily inequivalent representations of canonical commutation relations. When one works with the functional integral formalism, it is not clear, however, how this feature emerges. By following the seminal work of M. Swanson on canonical transformations in phase-space path integral, we generalize his approach to coherent-state functional integrals which in turn will lead to a simplified formalism which makes the appearance of the inequivalent representations more transparent.

1. Introduction
Since the early days of its formulation, it was realized that Quantum Field Theory (QFT) is not simply the extension of quantum mechanics (QM) to relativistic systems \cite{1}. There is indeed more: QFT deals with systems with an infinite number of degrees of freedom and thus the Stone–von Neumann uniqueness theorem \cite{2}, stating the unitary equivalence of the unitary irreducible representations of the canonical commutation relations (CCR), does not apply. Consequently there exist in QFT infinitely many unitarily inequivalent representations of the field algebra or, in other words, for a given dynamics, there is an infinite number of (inequivalent) Hilbert spaces.

A related, and often not sufficiently appreciated, feature of QFT is that the interacting (Heisenberg) fields do not have a unique representation in terms of the asymptotic fields, i.e. fields which directly act on the Fock space. In fact, the functional relation between the asymptotic fields and the Heisenberg fields is known as Haag’s map and represents the so called weak operatorial relation, i.e. a relation valid only for expectation values over states belonging to the Hilbert space of the asymptotic fields. All this is of course well known and indeed important phenomena like the spontaneous symmetry breaking mechanism \cite{3, 4} or the Hawking black hole radiation \cite{5, 6} and quantization of dissipative systems \cite{7}, are possible in QFT only because of the existence of inequivalent representations.

In the last two decades the fundamental role of inequivalent representations has been recognized in the problem of quantization of mixed particles: in such case, a simple canonical transformation (rotation) of fields with different masses has a dramatic effect on the structure of the Hilbert space, leading to the orthogonality of the vacua for fields with definite flavor.
and those with definite mass. This effect has been shown to be valid for both bosonic and fermionic fields [8, 9] and lead to phenomenologically relevant corrections in flavor oscillation formulas [10]. Further recent developments include the study of dynamical generation of fermionic mixing [11, 12] with extensions to curved backgrounds [13].

Aforementioned progress has been achieved in the context of canonical (i.e., operator) formalism, where there is a clear distinction between of dynamics (i.e., the operatorial Heisenberg equations) and that of the representation which embodies the boundary conditions for the Heisenberg equations and sets up the Hilbert space. On the other hand, one should be able to achieve the same result by means of the functional integral formalism, where one deals with a generating functional for Green’s functions.

Here we show what role the inequivalent representations play in the study of linear canonical transformations in functional integrals. To this end, we generalize a treatment developed by M. Swanson in Ref. [14] for path integrals, to the framework of coherent states functional integrals which are better suited for the field-theory considerations.

Plan of the paper is as follows; in the following section we present the van Hove model [15] which is a simple toy-model system exhibiting inequivalent representations [4]. In Section 3 we present a brief review of the theory of canonical transformations in phase-space path integral, following Ref. [14]. In Section 4 we present our approach to the canonical transformations in the context of coherent-state functional integrals. In Section 5 we use the general framework obtain to study van Hove model in more detail.

2. van Hove model and inequivalent representations in QFT

The van Hove model is described by the Hamiltonian [4, 15]

$$\hat{H} = \int d^3k \left[ \omega_k \hat{a}_k^\dagger \hat{a}_k + \nu_k \left( \hat{a}_k^\dagger + \hat{a}_k \right) \right].$$

This can be diagonalized by the so-called dynamical map

$$\hat{\alpha}_k = \hat{a}_k + g_k \ , \ \hat{\alpha}_k^\dagger = \hat{a}_k^\dagger + g_k,$$

with $g_k = \frac{\nu_k}{\omega_k}$. The energy spectrum is defined by the dispersion relation

$$E_k = \omega_k .$$

The physical vacuum is defined by

$$\hat{\alpha}_k |0(g)\rangle = 0 .$$

This is related with the “interacting” vacuum $|0\rangle$ satisfying

$$\hat{a}_k |0\rangle = 0$$

by the relation

$$|0(g)\rangle = \hat{G}^{-1} |0\rangle .$$

Here $\hat{G}$ is the generator of the canonical transformation

$$\hat{G} = \exp \left( \int d^3k \left[ g_k \hat{a}_k^\dagger - g_k^* \hat{a}_k \right] \right).$$

The “scalar product” between the two vacua is

$$\langle 0 | 0(g) \rangle = \exp \left( \int d^3k |g_k|^2 \right) .$$
If we now consider the case of a translationally invariant system, i.e. \( \nu_k = \nu \delta(k) \) and then \( g_k = c \delta(k) \), the foregoing product can be explicitly written as

\[
\langle 0|0(g) \rangle = \exp \left( -\frac{1}{2} \frac{V \nu^2}{(2\pi)^3} \right).
\] (9)

In the long-wave (i.e., \( V \to \infty \)) limit this product goes to zero, i.e. the Hilbert spaces constructed on the two vacua are orthogonal. From the second Schur’s lemma [16] then follows the unitary inequivalence of the CCR representations constructed over ensuing Hilbert spaces. Actually, the Haag’s map (2) can be seen as a weak relation and it is still defined. This is an example of the celebrated Haag’s theorem [17].

3. Canonical transformations in phase space path integral

Let us now first briefly review the concept of canonical transformations in the framework of phase-space PIs. This subject has been treated literature by several authors [18, 19, 20, 21, 22]. In the following we will closely follow the approach introduced by M. Swanson in Ref. [14]. There the key object of interest is the evolution kernel in mixed phase-space representation, namely

\[
W_{fi} = \langle p_f | \exp \left[ -iH(p,q;t)(t_f - t_i) \right] | q_i \rangle.
\] (10)

The latter can be rewritten in the time-sliced PI form

\[
W_{fi} = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{N} \int \frac{dp_j dq_j}{2\pi} \exp \left\{ i \sum_{j=1}^{N} \left[ -q_j(p_{j+1} - p_j) - \Delta t H(p_j, q_j) - iq_j p_1 \right] \right\},
\] (11)

where \( \Delta t \equiv (t_f - t_i)/N, t_i \equiv t_0, t_f \equiv t_{N+1}, p(t_0 + (N + 1)\Delta t) \equiv p_{N+1} = p_f, q(t_0) \equiv q_0 = q_i \) and the limit \( \Delta t \to 0 \) or equivalently \( N \to \infty \) is understood to be taken at the appropriate stage of calculations.

To define a QM form of the canonical transformation, one starts from the completeness relation

\[
\langle p_f | q_i \rangle = \int \frac{dP_i dQ_i}{\sqrt{2\pi}} \langle p_f | Q_N \rangle e^{iP_i Q_i} \langle P_i | q_i \rangle,
\] (12)

and assumes the following correspondence relations [14]

\[
\langle p_f | Q_N \rangle = \frac{1}{\sqrt{2\pi}} \exp \left\{ i \left[ P_f (Q_f - Q_N) + F(p_f, Q_f) \right] \right\},
\] (13)

\[
\langle P_i | q_i \rangle = \frac{1}{\sqrt{2\pi}} \exp \left\{ -i \left[ P_i Q_i + F(p_i, Q_i) \right] \right\}.
\] (14)

In order to identify \( F(p,Q) \), one substitutes these relations in Eq. (12). To save the consistence of this equation, one has to make the identifications

\[
-P_f (Q_f - Q_i) = F(p_f, Q_f) - F(p_f, Q_i),
\] (15)

\[
-q_i (p_f - p_i) = F(p_f, Q_i) - F(p_i, Q_i),
\] (16)

which are nothing but ansatz proposed by Fukutaka and Kashiwa [21]. However, these seem to lead to the correct result only for \( p_i = 0 \). This condition implies an over-specification of boundary conditions. In Ref. [14] Swanson deals with this problem by assuming the boundary condition \( q_i = 0 \), and then shifts the solution via Galilean boost.
In any case, the relations (15)-(16) define the new variables. To recover the classical result we can expand these relations:

$$P_j = \frac{\partial F(p_j, Q_j)}{\partial Q_j} + \frac{1}{2} \frac{\partial^2 F(p_j, Q_j)}{\partial Q_j^2} (Q_f - Q_i) + \ldots ,$$

(17)

where the classical canonical transformation correspond to the first order result. Therefore, we can identify the function $F$ with the third type classical generating function of the canonical transformation [23]. Writing (15)-(16) in the infinitesimal form

$$P_j = -\frac{F(p_j, Q_j) - F(p_j, Q_{j-1})}{\Delta Q_j} ,$$

(18)

$$q_j = -\frac{F(p_{j+1}, Q_j) - F(p_j, Q_j)}{\Delta p_j} ,$$

(19)

where $\Delta Q_j = Q_j - Q_{j-1}$, $\Delta p_j = p_{j+1} - p_j$ we get

$$-q_j(p_{j+1} - p_j) = P_{j+1}(Q_{j+1} - Q_j) + F(p_{j+1}, Q_{j+1}) - F(p_j, Q_j) .$$

(20)

This relation permit us to rewrite the argument of the PI as

$$\sum_{j=0}^{N} [-q_j(p_{j+1} - p_j) - \epsilon H(q_j, p_j)]
= F(p_f, Q_f) - F(p_i, Q_i) + \sum_{j=0}^{N} [P_{j+1}(Q_{j+1} - Q_j) - \epsilon H(P_j, Q_j, \Delta P_j, \Delta Q_j)] .$$

Now the Jacobian could give a finite contribution to the PI action because of the appearance of anomalous corrections (Liouville anomaly) [14, 19, 20, 21]:

$$J^{-1} = \prod_{j=1}^{N} \left[ 1 + \frac{1}{2} \left( \frac{\partial^3 F}{\partial p_j^2 \partial Q_j} \frac{\partial p_j}{\partial q_j} + \frac{\partial^3 F}{\partial Q_j^2 \partial p_j} \frac{\partial Q_j}{\partial q_j} \right) \Delta Q_j + \frac{1}{2} \frac{\partial^4 F}{\partial p_j^2 \partial Q_j^2} \frac{\partial p_j}{\partial q_j} \frac{\partial Q_j}{\partial q_j} \Delta P_j \Delta Q_j \right] .$$

(22)

When $\Delta Q_j = O(\Delta t)$, $\Delta P_j = O(\Delta t)$, exponentiating the expression (22), we get

$$J^{-1} = e^{\sum_{j=1}^{N} \log(1 + A_j \Delta Q_j + B_j \Delta P_j)} \sim e^{\sum_{j=1}^{N} (A_j \Delta Q_j + B_j \Delta P_j)} ,$$

(23)

where, in the last passage we have expanded the logarithm and following Ref.[14] we have introduced the short notation

$$A_j = \frac{1}{2} \left( \frac{\partial^3 F}{\partial p_j^2 \partial Q_j} \frac{\partial p_j}{\partial q_j} + \frac{\partial^3 F}{\partial Q_j^2 \partial p_j} \frac{\partial Q_j}{\partial q_j} \right) ,$$

(24)

$$B_j = \frac{1}{2} \frac{\partial^3 F}{\partial p_j^2 \partial Q_j} \frac{\partial p_j}{\partial q_j} \frac{\partial Q_j}{\partial q_j} .$$

(25)

Eq. (23) can be rewritten as

$$J^{-1} = e^{\sum_{j=1}^{N} \Delta \left( A_j \frac{\Delta Q_j}{\Delta t} + B_j \frac{\Delta P_j}{\Delta t} \right)} .$$

(26)

A naive approach to this subject could thus lead to various inconsistencies as discusses by many authors (cf. e.g., [22] and citations therein).
4. Canonical transformations and inequivalent representations
In this section we present our analysis of canonical transformation in the framework of coherent-state (or holomorphic-representation) functional integrals: indeed such a formalism seems to offer, in a natural way, a correspondence between canonical transformations in PIs and a change in the CCR representations.

As shown by Klauder [24], coherent states PI can be expressed in terms of a true Gaussian measure, solving convergence problems. Moreover, this measure is naturally symmetric in the integration variables, avoiding problems due to the asymmetry of the phase space path integral in the definition of canonical transformations [22]. Moreover in the holomorphic PI integral the extension to the field theoretical case is really natural. Actually, this is a context where the holomorphic functional integral is widely used [25].

4.1. QM and finite degrees of freedom systems
We are going to treat the case of quantum mechanical many body systems in the coherent states PI framework. In the next subsection we shall pass to the field theoretical case.

Let us start with some basic definitions. Let \( \{ z_k \} \) be a sequence of complex numbers so that
\[
\sum_k z_k^* z_k < \infty .
\] (27)
A coherent states in Schrödinger picture are defined as
\[
|\{ z \} \rangle = e^{-\sum_k z_k \hat{a}_k^\dagger} |0 \rangle .
\] (28)
These are eigenstates of annihilation operators
\[
\hat{a}_k |\{ z \} \rangle = z_k |\{ z \} \rangle ,
\] (29)
where the corresponding bra vectors are defined as
\[
\langle \{ z^\ast \} | = \langle 0 | e^{\sum_k z_k^\ast \hat{a}_k},
\] (30)
and are eigenstates of creation operators
\[
\langle \{ z^\ast \} | \hat{a}_k^\dagger = \langle \{ z^\ast \} | z_k^\ast .
\] (31)

Moving base (or Heisenberg-picture base) vectors are defined as
\[
|\{ z \}, t \rangle = e^{-iH(\{ \hat{a} \}, \{ \hat{a}^\dagger \}) t} |\{ z \} \rangle .
\] (32)
A resolution of the identity can be written in terms of these base states as
\[
\int [d\mu(z)] |\{ z \}, t \rangle \langle \{ z^\ast \}, t | = \mathbb{I} ,
\] (33)
where
\[
[d\mu(z)] = \prod_k d\mu(z_k) = \prod_k \int \frac{dz_k dz_k^*}{2\pi i} e^{-\sum_k z_k^* z_k} .
\] (34)
The scalar product between two coherent states is done by the relation
\[
\langle \{ \tilde{z}^\ast \} | \{ z \} \rangle = e^{\sum_k \tilde{z}^* z_k} .
\] (35)
With these ingredients we are able to construct a functional-integral form for the evolution kernel:

\[
\langle \{z^*\} f, t f | \{z\} i, t_i \rangle = \int_{\{z\}(t_i) = \{z\}} \prod_k \mathcal{D}z_k^* \mathcal{D}z_k e^{\sum_k z^*_k z_k f_k} e^{i \sum_k f_k \int_{i}^{f} dt (iz_k^*(t) z_k(t) - H(z_k^*, z_k))},
\]

(36)

that has to be interpreted as a formal way to indicate the time sliced expression

\[
\langle \{z^*\} f, t f | \{z\} i, t_i \rangle = \lim_{N \to \infty} \prod_{j=1}^{N} \int [d\mu(z_j^*)] d\mu(z_j) e^{z_j^* f z_j f} e^{-\sum_{j=1}^{N+1} \sum_k z_j^* k_k z_j k_k} \Delta t.
\]

Here

\[
H(z^*_{j+1 k}, z_{j k}) \equiv \frac{\langle \{z^*\}_{j+1}, t_{j+1} | \hat{H}(\hat{a}_k, \hat{a}_k^\dagger) | \{z\}_{j}, t_j \rangle}{\langle \{z^*\}_{j+1}, t_{j+1} | \{z\}_{j}, t_j \rangle}.
\]

(37)

Now we consider another set of coherent states \(|\{\zeta\}\rangle\), constructed on the Hilbert space \(\hat{\mathcal{H}}\). Here it is very important to remark that when we consider these new states, we are dealing with another representation of the CCR. These states are defined as

\[
|\{\zeta\}\rangle = e^{\sum_k \xi_k \hat{a}_k^\dagger} |\emptyset\rangle,
\]

(39)

\[
\langle \{\zeta^*\} | = \langle \{\zeta\} | e^{\sum_k \xi_k \hat{a}_k},
\]

(40)

\[
\hat{a}_k |\{\zeta\}\rangle = \xi_k |\{\zeta\}\rangle, \quad \langle \{\zeta^*\} | \hat{a}_k^\dagger = \xi_k^* \langle \{\zeta^*\} |,
\]

(41)

where

\[
\hat{a}_k = \exp(-i\hat{K}) \hat{a}_k \exp(i\hat{K}), \quad \hat{a}_k^\dagger = \exp(-i\hat{K}) \hat{a}_k^\dagger \exp(i\hat{K}).
\]

(42)

where \(\hat{K} \equiv \sum_k \hat{K} (\hat{a}_k, \hat{a}_k^\dagger) \theta_k\) is a Hermitian operator. In Ref. [18, 26] it was noticed that in the phase-space framework the correspondence relations might be written as formal PIs

\[
\langle \{\zeta^*\} |\{\zeta\}\rangle = \int_{\{\xi(0)\} = \{\zeta\}} \prod_k \mathcal{D}\xi_k^* \mathcal{D}\xi_k e^{\sum_k \xi_k^* (\theta_k(0)) \xi_k(0) e^{\sum_k \int_0^{\theta_k} d\theta'_k [\xi_k^* (\theta'_k) \xi_k (\theta'_k) - K (\xi_k^*, \xi_k)]}} \mathcal{N}(\xi^*, \xi),
\]

(43)

\[
\langle \{\zeta^*\} |\{z\}\rangle = \int_{\{\xi(0)\} = \{z\}} \prod_k \mathcal{D}\xi_k^* \mathcal{D}\xi_k e^{\sum_k \xi_k^* (\theta_k(0)) \xi_k(0) e^{\sum_k \int_0^{\theta_k} d\theta'_k [\xi_k^* (\theta'_k) \xi_k (\theta'_k) - K (\xi_k^*, \xi_k)]}} \mathcal{N}(\xi^*, \xi).
\]

(44)

These are impossible to solve in the general and we have to resort to the following ansatz

\[
\langle \{z^*\} |\{\zeta\}\rangle = A(\{\zeta^*\}, \{z\}) \exp[F(\{\zeta^*\}, \{z\})],
\]

(45)

\[
\langle \{\zeta^*\} |\{z\}\rangle = A(\{\zeta^*\}, \{z\}) \exp[F(\{\zeta^*\}, \{z\})].
\]

(46)

As shown in [27, 28] these functions can be determined, solving the formal path integrals in the stationary-phase approximation, obtaining

\[
A(\{\zeta^*\}, \{z\}) = \prod_k \left( \frac{\partial^2 F_2(\{\zeta^*\}, \{z\})}{\partial \xi_k^* \partial \xi_k^*} \right)^{1/2},
\]

(47)

\[
F(\{\zeta^*\}, \{z\}) = F_2(\{\zeta^*\}, \{z\}) + \sum_k \frac{i}{2} \int_0^{\theta_k} d\theta'_k \frac{\partial^2 K(\xi_k^*, \xi_k^* \xi_k, \xi_k)}{\partial \xi_k \partial \xi_k^*}.
\]

(48)
where $\xi^c_k(\theta)$ and $\xi^k(\theta)$ are stationary phase solutions and
\begin{equation}
F_2(\{\xi\}^*, \{z\}) = \sum_k F_{2k}(\xi_k^*, z_k),
\end{equation}
is the classical second-type generating function of the canonical transformation. Let us now introduce the function $G(\{\xi\}^*, \{z\}) = F(\{\xi\}^*, \{z\}) + \log A(\{\xi\}^*, \{z\})$. In QM these two representations are always unitarily equivalent, because of the Stone–von Neumann theorem. Therefore overcompletness ensures that the expression
\begin{equation}
\frac{\langle\{\xi\}^*_{j+1}\{z\}_{j+1}}{\langle\{\xi\}^*_{j}\{z\}_{j}\rangle} = e^{G(\{\xi\}^*_{j+1}\{z\}_{j+1})-G(\{\xi\}^*_{j}\{z\}_{j})},
\end{equation}
is well defined. Multiplying both sides by $\langle\{z\}^*_{j+1}\{\xi\}^*_{j+1}\rangle$ and integrating over [d$\mu(\xi_{j+1})$] we get
\begin{equation}
\langle\{z\}^*_{j+1}\{z\}_{j+1}\rangle = \int [d\mu(\xi_{j+1})] \left[ e^{G(\{\xi\}^*_{j+1}\{z\}_{j+1})-G(\{\xi\}^*_{j}\{z\}_{j})} \langle\{z\}^*_{j+1}\{\xi\}^*_{j+1}\{z\}_{j+1}\rangle \right].
\end{equation}
With this we arrive on the consistency relation
\begin{equation}
G(\{\xi\}^*_{j+1}, \{z\}_{j+1}) - G(\{\xi\}^*_{j+1}, \{z\}_{j}) = \sum_k z^*_j(z_{j+1} - z_j).
\end{equation}
Similarly we find
\begin{equation}
G(\{\xi\}^*_{j+1}, \{z\}_{j}) - G(\{\xi\}^*_{j}, \{z\}_{j}) = \sum_k \xi^*_j(\xi^{*+1}_j - \xi^*_j).
\end{equation}
These two relations are formally analogous to Eqs. (18)-(19). Because the formalism developed by Fukutaka and Kashiwa and by Swanson works well, we could make the identification $G(\{\xi\}^*, \{z\}) = F_2(\{\xi\}^*, \{z\})$. As it is clear from Eq. (47), this identification is exact for linear transformations.

We find then, that the argument of the functional integral, in the new variables, reads
\begin{align*}
&- \sum_{j=1}^{N+1} \sum_k z^*_j(z_{j+1} - z_j) - i \sum_{j=0}^{N} H(z_j, z^*_j) \Delta t

&= F_2(\{\xi\}^*_i, \{z\}_i) - F_2(\{\xi\}^*_f, \{z\}_f) + \sum_{j=1}^{N+1} \sum_k \xi^*_j(z_{j+1} - z_j) \\
&- i \sum_{j=0}^{N} H(\xi_j, \xi^*_j) \Delta \xi_j \Delta \xi^*_j \Delta t.
\end{align*}
The inverse Jacobian, with anomalous correction takes the form
\begin{equation}
J^{-1} = \prod_{j=1}^{N} \prod_{k} \left[ 1 + \frac{1}{2} \frac{\partial^2 F_2(\xi^*_k, z_j)}{\partial z^*_k \partial \xi^*_k} \frac{\partial z^*_j}{\partial \xi^*_k} \Delta \xi^*_j \right],
\end{equation}
which very similar to the Swanson result. However, here we have not imposed particular boundary conditions for the consistency of our equations.

1 A formulation of classical mechanics in terms of complex phase space variable was introduced by Strocchi [29]. However here we use a slightly different notation that generalizes the notation of Goldstein [23].
4.2. Field theoretical case and inequivalent representations
As seen in Section 2, in QFT the Stone–von Neumann theorem does not apply and we must deal with many unitarily inequivalent representations of the CCR. In this section, we restrict our attention to linear transformations. In this case the following correspondence relations are exact

\[ \langle z^* | \zeta \rangle = \exp \left\{ \int d^3k \ F_2^* [\zeta^*(k), z(k)] \right\}, \quad (56) \]

\[ \langle \zeta^* | z \rangle = \exp \left\{ \int d^3k \ F_2 [\zeta^*(k), z(k)] \right\}. \quad (57) \]

However, \( F_2^{\zeta^*, z} \equiv \int d^3k \ F_2 [\zeta^*(k), z(k)] \) is now a functional, defined on an infinite dimensional functional space, and as such it might be a divergent quantity. To see this we decompose \( F_2 \) as

\[ F_2 = \int d^3k \ F_2 [\zeta^*(k), z(k)] + C, \quad (58) \]

where the arbitrary constant \( C \) was singled out. But now this constant is determined by the vacuum structure. In fact it follows easily that

\[ \langle 0 | 0 \rangle = \exp(C). \quad (59) \]

If these two representations are unitarily inequivalent \( C \) diverges and so the generating functional as well. Therefore we cannot derive a definition of new variables as we did above in deriving the consistency relations (52)-(53). We limit ourselves to formally define the relations

\[ F_2 [\zeta^*_{j+1} (k), z_{j+1} (k)] - F_2 [\zeta^*_{j+1} (k), z_{j} (k)] = z^*_{j+1} (k)[z_{j+1} (k) - z_{j} (k)], \quad (60) \]

\[ F_2 [\zeta^*_{j+1} (k), z_{j} (k)] - F_2 [\zeta^*_{j} (k), z_{j} (k)] = \zeta^*_{j+1} (k)[\zeta^*_{j+1} (k) - \zeta_{j} (k)], \quad (61) \]

that there not affected by the divergence seen above. The functional integral can be now *formally* expanded in a generic representation as

\[ \langle z^*_f, t_f | z_i, t_i \rangle = \prod_k \int_{\tilde{z}(t_i) = z_i}^{\tilde{z}(t_f) = z^*_f} D\zeta^*_k (k) D\zeta^*_k (k) e^{\int d^3k \ \tilde{z}_f \tilde{z}_f} \]

\[ \times e^{i \int d^3k \ \left\{ F_2 [\zeta^*_k (k), \tilde{z}_f - F_2 [\zeta^*_k (k), \tilde{z}_i] \right\} + \int d t \left\{ -i \zeta^*_k (k,t) \dot{\zeta}^*_k (k,t) + H (\zeta (k,t), \zeta^* (k,t)) \right\} \}, \quad (62) \]

where we denoted \( \tilde{z} = z [\zeta (k,t), \zeta^* (k,t)] \). Let us stress that, in general, only a combination of these new variables can be fixed by boundary conditions because of the uncertainty principle.

As shown by Torre [30] in the case of quadratic Hamiltonians, because of the existence of unitarily inequivalent representations, the unitary time evolution operator may not exist so that expressions such as (62) lose sense. In the following section we show a physical view of these mathematical considerations, using a close connection between QFT and statistical physics.

5. Functional integral analysis of the van Hove model
We start our analysis of the van Hove model by working in the Minkowski space. We will perform the Wick rotation to the Euclidean regime at the end of our reasoning. For the sake of simplicity we start with a single-mode van Hove model, the extension to many modes being trivial. The Feynman kernel, after an integration by parts, can be written as

\[ \langle z^*_f, t_f | z_i, t_i \rangle = \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{d\zeta^*_j d\zeta^* \Delta t}{2 \pi i} \]

\[ \times e^{-\sum_{j=1}^{N+1} \zeta_j (\zeta_j - \zeta_{j-1}) - \int \sum_{j=0}^{N} \omega (\zeta_{j+1} - g^* (\zeta_j - g^*) (\zeta_j - g) + (\zeta_j + g^* (\zeta_j + g - g^*) (\zeta_j - g) - g^*) (\zeta_j - g) \Delta t. \quad (63) \]
If we take \( g = \nu/\omega \), we find

\[
(z^*_f, t_f | z_i, t_i) = \lim_{N \to \infty} \prod_{j=1}^N \int \frac{d\zeta_j d\zeta_i}{2\pi i} e^{i\zeta_j \zeta_i - \frac{\nu}{2}(\zeta_j + \zeta_i) + (\bar{\zeta})^2} \times e^{-\sum_{j=1}^{N+1}(\omega \zeta_j - \frac{\nu^2}{2}) \Delta t}.
\] (64)

Passing to the continuous limit we get

\[
(z^*_f, t_f | z_i, t_i) = e^{-\frac{\nu}{2}(\zeta_j + \zeta_i) + (\bar{\zeta})^2} \int D\zeta e^{i\zeta_j \zeta_i - \frac{\nu}{2}(\zeta_j + \zeta_i) + (\bar{\zeta})^2} dt. \] (65)

Apart from the constant pre-factor, fixed by the mixed boundary conditions, this is the harmonic oscillator kernel and as such it can be exactly solved in the saddle-point approximation. We thus find

\[
(z^*_f, t_f | z_i, t_i) = e^{\zeta^*_j \zeta_i - \frac{\nu}{2}(\zeta_j + \zeta_i) + (\bar{\zeta})^2 - i(t_f - t_i)\frac{\nu^2}{2}}. \] (66)

With the help of Wick’s rotation we derive the Euclidean kernel

\[
\langle z^*_f, \beta | z_i, 0 \rangle = e^{\zeta^*_j \zeta_i - \frac{\nu}{2}(\zeta_j + \zeta_i) + (\bar{\zeta})^2 - \beta \bar{\zeta}}. \] (67)

The ensuing partition function is then easily obtained in the form

\[
Z_{vH} = Z_{ho} e^{-\beta \bar{\zeta}}. \] (68)

Here \( Z_{ho} \) indicates the harmonic-oscillator partition function (cf. e.g. [1, 22]). The extension of these results to systems with many degrees of freedom is now formally straightforward. The Euclidean kernel is now

\[
\langle \{ z^*_f \}, \beta | \{ z_i \}, 0 \rangle = e^{\sum_k \zeta^*_f \zeta_k e^{-\beta \nu_k} - \frac{\nu_k}{\omega_k}(\zeta^*_f \zeta_k + \zeta_k) + (\bar{\zeta})^2 - \beta \bar{\zeta}^2}, \] (69)

and the Euclidean partition function reads

\[
Z_{vH} = Z_{ho} e^{-\beta \sum \frac{\nu^2}{\omega_k}}. \] (70)

Reading this expression as the partition function of an interacting field-theoretical system in a box of a finite volume \( V \) we can interpret \( Z_{ho} \) as a partition function of quasi-particles. The energy gap can be obtained from the thermodynamical relation

\[
F_{vH} - F_{ho} = \sum_k \frac{\nu_k^2}{\omega_k}, \] (71)

where \( F = \beta \ln Z \) is the Helmholtz free energy. Taking the large volume limit, in the case of the translationally invariant van Hove model, for which \( \nu(k)/\omega(k) = c\delta(k) \), we find

\[
\Delta F = F_{vH} - F_{ho} = \int \frac{d^3k}{(2\pi)^3} c^2 \omega(k) \delta^2(k) = \frac{V c^2 \omega(0)}{(2\pi)^3}, \] (72)

If we now perform the thermodynamic limit we see that energy gap \( \Delta F \) diverges. This is the physical meaning of the inequivalence between these two representations: an infinite energy is required to pass from one vacuum to another one.
6. Conclusions
In this paper we have shown how a careful study of canonical transformations in the coherent-state functional integrals makes clearer the connection between classical and quantum theory of canonical transformations. This, on one hand side, helps to bring into a new light previous results on this subject found by other authors in the context of phase-space path integrals. On the other hand, we have demonstrated how the existence of unitarily inequivalent representations of the CCR in QFT affects this study: although the (classical) generator of the canonical transformations between two representations typically do not exist, the transformation can still be (at least formally) well defined. This fact is directly reflected in the existence of an infinite energy gap among inequivalent vacuum states.

Acknowledgments
It is pleasure to acknowledge helpful conversations with G. Vitiello and H. Kleinert. PJ was supported by the GAČR Grant No. GA14-07983S.

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