1. Introduction

Let $\mathcal{N}$ be the class of closed simply connected, smooth $n$–manifolds admitting nonnegative sectional curvature and $\mathcal{P} \subset \mathcal{N}$ the corresponding class for positive curvature. Known examples suggest that $\mathcal{N}$ ought to be much larger than $\mathcal{P}$. On the other hand, there is no known obstruction that distinguishes between the two classes. So it is actually possible that $\mathcal{N} = \mathcal{P}$.

In [PetWilh2] we will give a deformation of the nonnegatively curved metric on the Gromoll-Meyer sphere [GromMey] to a positively curved metric. The purpose of this note is to elucidate a few abstract principles that will be used in this deformation, and possibly could be helpful for other deformations to positive curvature.

Besides a few exceptions, [Cheeg], [Dear2], [GrovVerdZil], [GrovZil1], [GrovZil2], [Gui] all known examples of compact nonnegatively curved manifolds are constructed as Riemannian submersions of compact Lie groups. A result in [Tapp2] then implies that the zero curvature planes of the nonexceptional examples are
contained in totally geodesic 2-dimensional flats. As far as we are aware, the exceptional examples also have totally geodesic flats, provided of course that they have any zero curvature planes at all ([Dear2] and [GrovVerdZil]).

All known examples with nonnegative curvature, some zero curvatures, and positive curvature at a point, are the images of Riemannian submersions of compact Lie groups and hence have all zero planes contained in totally geodesic flats ([EschKer], [Esch], [GromMey], [Ker1], [Ker2], [PetWilh1], [Tapp1], [Wilh], and [Wilk].) So in most cases, any attempt to put positive curvature on a known nonnegatively curved example must confront the issue of how to put positive curvature on a neighborhood of a totally geodesic flat torus.

More than 20 years ago Strake observed that the presence of a totally geodesic flat torus in a nonnegatively curved manifold means that there can be no deformation that is positive to first order. In principle, a first order deformation should be much easier to construct and verify than a higher order one. In fact, if \( \{g_t\}_{t \in \mathbb{R}} \) is \( C^\infty \) family of metrics with \( g_0 \) a metric of nonnegative curvature, and if

\[
\frac{\partial}{\partial t} \sec_{g_t} P \bigg|_{t=0} > 0
\]

for all planes \( P \) so that \( \sec_{g_0} P = 0 \), then \( g_t \) has positive curvature for all sufficiently small \( t > 0 \).

On the other hand, if for all planes \( P \) with \( \sec_{g_0} P = 0 \) we have

\[
\frac{\partial}{\partial t} \sec_{g_t} P \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \sec_{g_t} P \bigg|_{t=0} > 0,
\]

then, without more information, we can not make any conclusion about obtaining positive curvature. For instance, if \( \Phi_t \) is a flow that moves the zero planes to positive curvature, then the variation \( (\Phi_t)^* g \) can satisfy the conditions above, yet clearly each of the metrics \( (\Phi_t)^* g \) are isometric to \( g \).

The obvious problem with such a gauge transformation is that it only moves zero planes to new places. Unfortunately, the discussion above illustrates that any attempt to put positive curvature on a (generic) known nonnegatively curved example must confront this issue. It is not enough to consider the effect of a deformation on the set, \( Z \), of zero planes of the original metric. Instead we to have check that the curvature becomes positive in an entire neighborhood of \( Z \).

To this bleak reality we offer the following ray of hope–

The very rigidity of totally geodesic flats can be exploited in attempts to deform them.

The rigidity of a totally geodesic flat within a fixed nonnegatively curved manifold is of course well known and well understood. Here we have in mind a different sort of rigidity. We will look at certain types of deformations that preserve totally geodesic flats, and other types of deformations that preserve aspects of the rigidity of totally geodesic flats. The tremendous advantage of this rigidity is that it will allow us to change one component of the curvature tensor while controlling the change in other components. Since the problem of prescribing the curvature tensor is highly over determined, in general, this is an entirely unreasonable thing
to expect; nevertheless, the rigidity of totally geodesic flats will allow us to do this in certain narrowly constrained situations.

Besides Cheeger deformations, the metric changes that we use to go from the Gromoll-Meyer metric to our positively curved metric are

- a deformation that we call the **Orthogonal Partial Conformal Change**
- scaling of the fibers of the Riemannian submersion $Sp(2) \to S^4$, to create integrally positive curvature, and
- another deformation that we call the **Tangential Partial Conformal Change**

To describe a general **Partial Conformal Change** we start with a distribution $D \subset TM$, and decompose our original metric as

$$g = g_D + g_{D^\perp}.$$  

We then conformally change $g_D$ while fixing $g_{D^\perp}$.

Our use of the terms “**Orthogonal**” and “**Tangential**” is meant to reflect the relationship of the original zero curvature planes to the distribution $D$.

An abstraction of the orthogonal partial conformal change is discussed in Sections 2 and 3. It preserves nonnegative curvature, the zero curvature locus, and has the effect of redistributing certain positive curvatures along the initial zero curvature locus. The idea that such a change is possible goes back at least to [Wals].

Having a broader class of nonnegatively curved metrics could certainly be an advantage. In fact, if we were to perform our other deformations without doing the orthogonal partial conformal change we could make the old zero planes positively curved, but as far as we can tell would not get positive curvature.

The fiber scaling is the central idea of the deformation to positive curvature on the Gromoll–Meyer sphere, $\Sigma^7$. In section 4, we prove an abstract theorem about fiber scaling. This result implies that if we start with the metric from [Wilh] and scale the fibers of $Sp(2) \to S^4$, then we get integrally positive curvature over the sections that have zero curvature in [Wilh]. More precisely, the zero locus in [Wilh] consists of a (large) family of totally geodesic 2–dimensional tori. We will show that after scaling the fibers of $Sp(2) \to S^4$, the integral of the curvature over any of these tori becomes positive. The computation is fairly abstract, and the argument is made in these abstract terms, so no knowledge of the metric of [Wilh] is required.

In addition to proving that fiber scaling creates integrally positive curvature, our argument in section 4 will provide a precise formula for what happens to the curvature of each of the old zero curvature planes. The leading order term has both signs, so the metric with the fibers scaled has curvatures of both signs. On the other hand, the leading order term is also the Hessian of a function and along any one of our originally flat tori it can be canceled by a conformal change of metric. The details are carried out in section 5. Thus by reading sections 4 and 5 the reader can get a quick impression of what the entire deformation does to the curvature of a single torus that is initially totally geodesic and flat.

Unfortunately, the conformal factor required to cancel the Hessian term from fiber scaling varies from torus to torus. To achieve positive curvature on all of the initially flat tori our actual deformation substitutes another partial conformal change for the conformal change described in section 5. This is our Tangential Partial Conformal Change. We have deferred our discussion of the Tangential Partial Conformal Change to [PetWilh2], where we also exploit the rigidity of totally
geodesic flats to show that the important curvatures change as though we had performed an actual conformal change.

Section 6 is the first place in the paper where totally geodesic flats do not play a prominent role. Instead we detail an observation that Cheeger deformations can be used to create positive curvature even when the initial metric has curvatures of both signs. Modulo the so-called “Cheeger Reparametrization” of the Grassmannian, Cheeger deformations preserve positive curvatures. In addition, any plane whose projection to the orbits “corresponds” to a positively curved plane will become positively curved provided the deformation is carried out for a sufficiently long period.

We do not imagine that we are the first to make this observation, and in fact, took for granted that this idea was well understood when we wrote the first draft of [PetWilh2]. We have subsequently become aware that these ideas are not as well known as we originally assumed, so we have included them for the sake of completeness.

The curvatures of the zero planes of [Wilh] are not affected by Cheeger deformations, but most nearby planes feel the effect. Part of the role of long term Cheeger deformations is to simplify the problem of estimating the curvatures in a neighborhood of the original zero curvature locus.

Sections 7 and 8 are also part of our strategy to solve this problem, and are the sections that are most dependent on the others. While this paper is an attempt to divide some of our deformation of the Gromoll–Meyer sphere into digestible, abstract pieces, the reader should be aware that in at least one respect the argument is an intertwined whole.

In Section 7, we analyze the effect of certain Cheeger deformations on our formula for the curvatures of our tori after fiber scaling. We will show that Cheeger deformations have the effect of compressing the bulk of these curvatures into a small set, $T_0$. Because $T_0$ is small the orthogonal partial conformal change will allow us to make certain curvatures much larger on $T_0$, and “pay” with only a small decrease in curvature outside of $T_0$. This synergy makes the problem of verifying positive curvature more tractable, is crucial to our whole argument, and explained in greater detail in Section 8.

It is natural to speculate on the extent to which some (or all) of these ideas might be useful in other deformations to positive curvature. For example there are non simply connected examples with nonnegative curvature that according to Synge’s Theorem can not admit positive curvature, so it is natural to ask where our methods break down in these examples. While we have not made an exhaustive study of this question, we can point out that if a totally geodesic flat is vertical for the submersion whose fibers are scaled, then our curvature formula shows that it will continue to be flat. This is the case for the metrics on $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $S^3 \times S^2$ in [Wilk], with respect to the isometric $SO(3)$–action of that paper. Since our total argument in [PetWilh2] is very long, there are many obstructions to using it in general. It seems more likely that individual pieces will find other applications.

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2. Technical Introduction

In this section we state our four main results, Theorem 2.1, Theorem 2.5, Theorem 2.6, and Lemma 2.10. Additionally, we formulate abstract settings for which these results are valid. All of the structure that we assume here actually occurs on the Gromoll–Meyer sphere. All of these abstract features are stated explicitly in bullet points in this section. In [PetWilh2] we will verify that these axioms are actually valid on the Gromoll–Meyer sphere.

These four results provide a detailed, technical overview of the strategy to put positive curvature on the Gromoll–Meyer sphere, but even after they are proven and the axioms are verified, a fairly substantial portion of the overall argument will remain to be completed in [PetWilh2].

We have divided this section into four subsections. The results stated in the first two subsections, Theorems 2.1, 2.5, and 2.6 only depend on the hypotheses stated in their respective subsections. The result of the third subsection, Lemma 2.10, requires all of the the axioms of this section.

Theorem 2.1 is proven in sections 5 and 6. Theorems 2.5 and 2.6 are proven in sections 3 and 4, and Lemma 2.10 is proven in section 9. Some of the individual results of these sections are valid under hypotheses that are more general than our overall abstract framework. Occasionally, we state them at this greater level of generality. We have done this only when it seems to simplify the exposition, and have indicated the places where this occurs.

To help elucidate certain key ideas, we have stated and proven Theorem 2.1 and Lemma 2.10 for a single flat torus, whereas, the Gromoll–Meyer sphere with the metric of [Wilh] contains two large families of flat tori. In the fourth subsection, we give a brief preview of how we will overcome this difficulty in [PetWilh2].

2.1. Set up for fiber scaling and conformal change.

- Let

\[ M \xrightarrow{p_{BM}} B \]

be Riemannian submersions of nonnegatively curved manifolds with \( M \) compact and \( B \) a twisted sphere.

Let \( g_s \) be the metric obtained from \( g_0 \) by scaling the lengths of the fibers of \( p_{BM} \) by

\[ \sqrt{1 - s^2}. \]

Further assume:

- There is an isometric action by a Lie group \( G_1 \) on \( M \) that is by symmetries of \( p_{BM} \).
- The intrinsic metrics on the principal orbits of \( G_1 \) in \( B \) are homotheties of each other.
- The normal distribution to the principal orbits of \( G_1 \) on \( B \) is integrable.

In addition we have a totally geodesic torus \( T \subset M \) spanned by commuting, orthogonal, geodesic fields \( X \) and \( W \) with the properties that

- \( D_{p_{BM}} (W^H) \) is a Killing field for the \( G_1 \)–action on \( B \).
- \( X \) is horizontal, invariant under \( G_1 \), and orthogonal to the orbits of \( G_1 \).
- \( D_W [W^H] = 0 \).
- the integral curves of \( X \) on \( T \) are periodic pass through the principle orbits of \( G_1 \) except possibly for finitely many points.
We use the term “geodesic field” for any field $X$ so that $\nabla_X X = 0$.

**Theorem 2.1.** There is a conformal change, $\tilde{g} = e^{2f} g$, so that $\text{curv}_{\tilde{g}} (X, W) > 0$.

We use the superscripts $^H$ and $^V$ to denote the horizontal and vertical parts of vectors.

In section 5 we also provide formulas for how $R(W, X) X$ and $(R(X, W) W)^H$ change under the fiber scaling deformation.

### 2.2. Set up for Orthogonal Partial conformal change.

Inspite of Theorem 2.1, there appear to be flat tori in the Gromoll-Meyer sphere where a combination of Cheeger deformations, fiber scaling and a conformal is not sufficient to obtain positive curvature in a neighborhood of the tori. The problem is that these deformations only produce positive curvature to higher order on the initially flat tori. In principle, such a deformation could produce positive curvature, but much more needs to be verified. As far as we can tell this verification must fail for the Gromoll–Mey er sphere.

We describe in sections 3 and 4 a method, called orthogonal partial conformal change, that will allow us in [PetWilh2] to change the metric on the Gromoll–Meyer sphere to one that has nonnegative curvature, the same zero curvatures, and to which we will be able to apply a combination of Cheeger deformations, fiber scaling, and (partial) conformal changes and get positive curvature.

To get a quick idea of why our orthogonal partial conformal change is possible we offer the following example.

**Example 2.2.** Let $\{g_t\}_{t \geq 0}$ be a family of rotationally symmetric positively curved metrics on $S^n$ with $g_0$ round. Let $(M, g)$ be nonnegatively curved. Then

$$\{(M \times S^n, g + g_t)\}_{t \geq 0}$$

is a deformation of $(M \times S^n, g + g_0)$ preserving nonnegative curvature.

A less trivial example is given in Theorem 2.1 in [Wals].

The standing hypotheses for sections 3 and 4 are as follows.

- Let $\{g_t\}_{t \geq 0}$ be a family of rotationally symmetric positively curved metrics on $S^n$ with $g_0$ round. Let $(M, g)$ be nonnegatively curved. Then

$$\{(M \times S^n, g + g_t)\}_{t \geq 0}$$

is a deformation of $(M \times S^n, g + g_0)$ preserving nonnegative curvature.

- Suppose that every zero curvature plane of $M$ that contains $X$ is tangent to a totally geodesic flat 2-dimensional torus, $S$. Let $S$ be the collection of these flat tori $S$, and suppose that $\bigcup_{S \in S} (S \cap O) = O$.

- Suppose $K_0 \subset p_{BM}^{-1} (B \setminus \{-\ast\})$ be a compact neighborhood of $p_{BM}^{-1} (\{\ast\})$ so that the union

$$\bigcup_{S \in S} p_{BM} (S)$$

contains a neighborhood of all of the points of $p_{BM} (K_0)$ that have zero curvatures.

- Suppose the horizontal lift of every zero curvature plane of $p_{BM} (K_0)$ is contained in some horizontal $S \in S$. 

Throughout the paper we set
\[
\text{curv} (X, W) \equiv R (X, W, W, X).
\]

Given \(X, W, Z, V \in T_pM\) we can describe a two parameter variation of the plane span \(\{X, W\}\) by
\[
\text{span} \{X + \sigma Z, W + \tau V\}, \text{ for } \sigma, \tau \in \mathbb{R}.
\]
Clearly any plane sufficiently close to span \(\{X, W\}\) has such a representation with \(\sigma, \tau\) being small. Corresponding to this variation we have a quartic curvature polynomial
\[
P (\sigma, \tau) = \text{curv} (X + \sigma Z, W + \tau V).
\]
When \((M, g)\) is nonnegatively curved and span \(\{X, W\}\) is a zero curvature plane this polynomial has a minimum at \((\sigma, \tau) = (0, 0)\). By expanding \(P (\sigma, \tau)\) we see that this minimum is nondegenerate and isolated provided
\[
\sigma^2 \text{curv} (Z, W) + 2\sigma \tau (R (X, Z, V, W) + R (X, V, Z, W)) + \tau^2 \text{curv} (X, V) > 0
\]
for all \((\sigma, \tau)\) so that \(\sigma^2 + \tau^2 = 1\).

**Definition 2.3.** Let \((M, g)\) be a nonnegatively curved manifold, \(p \in M, \ H_p\) a subspace of \(T_pM\), and \(Z_p\) a family of planes of \(T_pM\).

We say that \(Z_p\) is \(\varepsilon\)-nondegenerate with respect to \(H_p\) if and only if there is an open subset \(N_p\) of \(2\)-planes in \(H_p\) so that
1: sec \((P) > \varepsilon > 0\) for all planes \(P \subset H_p\) that are not in \(N_p\).
2: every zero curvature plane in \(N_p\) \(\subset H_p\) is in \(Z_p\).
3: every plane in \(N_p\) \(\setminus (Z_p \cap H_p)\) can be written \(\text{span} \{X + \sigma_0 Z, W + \tau_0 V\}\) where \(|X| = |W| = |Z| = |V| = 1\),
\[
\text{span} \{X, W\} \in Z_p, \text{ and}
\]
\[
\sigma^2 \text{curv} (Z, W) + 2\sigma \tau (R (X, Z, V, W) + R (X, V, Z, W)) + \tau^2 \text{curv} (X, V) > \varepsilon
\]
for all \((\sigma, \tau)\) \(\in \mathbb{R}^2\) with \(\sigma^2 + \tau^2 = 1\).

If \(H_p = T_pM\) and \(Z_p\) consists of all zero planes at \(p\), then we simply say that the zero locus at \(p\) is \(\varepsilon\)-nondegenerate.

**Remark 2.4.** We do not require that \(Z_p \subset H_p\). We also do not require that \(Z_p\) consist entirely of zero planes.

The standing hypotheses for sections 3 and 4 also include the following.
- For any compact subset \(K \subset O\), there is an \(\varepsilon > 0\) so that the zero planes curvature planes \(\cup_{S \in \mathcal{S}} TS\) are \(\varepsilon\)-nondegenerate with respect to \(H_{pBM}\).
- Suppose there is a distribution \(\mathcal{P}\) that is perpendicular to each \(S \in \mathcal{S}\), and let
\[
\varphi \equiv \kappa \circ \text{dist} (\ast, \cdot) \circ p_{BM}
\]
where \(\kappa : \mathbb{R} \rightarrow \mathbb{R}\) is chosen so that \(\varphi\) is smooth.

Let \(\hat{g}\) be obtained from \(g\) by multiplying the lengths of all vectors in \(\mathcal{P}\) by \(\varphi\), while keeping the orthogonal complement and the metric on the orthogonal complement of \(\mathcal{P}\) fixed. That is, \(\hat{g}\) is obtained from \(g\) by doing a partial conformal change with distribution \(\mathcal{D} = \mathcal{P}\) and conformal factor \(\varphi^2\). We call this deformation an orthogonal partial conformal change, and we assume that
- \(\pi : (M, \hat{g}) \rightarrow \Sigma\) is a Riemannian submersion.
Theorem 2.5. Let $K \subset O \cap K_0$ be a fixed compact set. For all $\delta > 0$ with
\[ \delta R^g (V, X, X, V) > (D_X D_X \varphi) |V|^2 |X|^2_g \text{ for all } V \in \mathcal{P}, \]
there is a $\delta' > 0$ so that if
\[ |\varphi - 1|_{C^1} < \delta', \]
then $(\Sigma, \tilde{g})$ is nonnegatively curved on $K$. Moreover $(\Sigma, \tilde{g})$ has precisely the same 0 curvature planes on $K$ as $(\Sigma, g)$.

In particular, if $\varphi$ is equal to 1 outside of $K$, and is otherwise as above, then $(\Sigma, \tilde{g})$ is nonnegatively curved. Moreover $(\Sigma, \tilde{g})$ has precisely the same 0 curvature planes as $(\Sigma, g)$.

Although the Orthogonal Partial Conformal Change of [PetWilh2] will be performed with a function $\varphi$ as above that is 1 outside of a compact subset $K$ of $O$, we will need to choose this set $K$ to be very near $\{*, -*\}$ and as such we will not be able to exploit the $\varepsilon$–nondegeneracy of the zero locus. This is because there is no $\varepsilon$ for which the zero locus of $\Sigma^T$ is $\varepsilon$–nondegenerate.

As a substitute we add the following hypotheses.

- There is an $\varepsilon > 0$ and an extension $TS$ of $\cup_{S \in S} TS|_{p^{-1}_{BM}((*, -*))}$ to a neighborhood, $U$, of $p^{-1}_{BM}((*, -*))$ so that $TS$ is $\varepsilon$–nondegenerate with respect to $H_{pBM}$.

We further assume

- There is a $\delta > 0$ so that the planes in $TS$ have the form
  \[ \text{span } \{X \cos \sigma + Z \sin \sigma, W \cos \sigma + V \sin \sigma\} \]
  with $\sigma \in \mathbb{R}$, span $\{X, W\}$ tangent to an $S \in S$, $|X| = |Z| = |W| = |V| = 1$, $Z \perp X$, $V \perp W$, $Z \in H_{BM}^p$, and $|V^p| > \delta$.
- The distribution $\mathcal{P}$ is vertical over $\{*, -*\}$.

With these additional hypotheses and some further constraints on $\varphi$ we have

Theorem 2.6. For all $\delta > 0$ with
\[ \delta R^g (V, X, X, V) > (D_X D_X \varphi) |V|^2 |X|^2_g \text{ for all } V \in \mathcal{P}, \]
there is a $\delta' > 0$ so that if for all $p \in M$ and $E$ horizontal with $Dp_{BM}(E) \perp X$
\[ |\varphi - 1|_{C^1} < \delta', \]
\[ |1 - \varphi(p)| < \delta \text{ dist} (p^{-1}_{BM}(*), p) \]
\[ -\delta' g(\nabla E X, E) (D_X \varphi)|_U > \delta \left( |D_X \varphi| + \frac{|1 - \varphi(p)|}{\text{dist} (p^{-1}_{BM}(*), p)} \right), \]
\[ -\delta' D_X D_X \varphi|_U > \delta \left( |D_X \varphi| + \frac{|1 - \varphi(p)|}{\text{dist} (p^{-1}_{BM}(*), p)} \right), \]
\[ \varphi = 1 \text{ on } \Sigma - K_0, \]
then $(\Sigma, \tilde{g})$ is nonnegatively curved. Moreover $(\Sigma, \tilde{g})$ has precisely the same 0 curvature planes as $(\Sigma, g)$.
2.3. Set up for Curvature Compression. The proof of Theorem 2.1, will provide us with a precise formula for \( \text{curv} (X, W) \) after our fiber scaling and conformal change. It is given in Proposition 6.3, and says

\[
e^{-2f} \text{curv} (X, W) = s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 |W|^2 D_X D_X I + O (s^6)
\]

where \( \psi \equiv |W^\mathcal{H}|_{g_0} \) is the length of the horizontal part of \( W \). The function \( I \) is part of the definition of the conformal factor, \( f \). We are free to choose \( I \) subject to the constraints that the integrals of \( D_X I \) and \( D_X D_X I \) over a closed integral curve of \( X \) are 0, and pointwise \( D_X I = O (1) \) and \( D_X D_X I = O (1) \).

On the Gromoll–Meyer sphere Cheeger deformations have a huge quantitative impact on the above formula for \( \text{curv} (X, W) \). In fact, by running one of our Cheeger deformations for a long time, we shall see that the vast bulk of the first three of these terms is compressed into a small neighborhood of the poles \( \{ *, -s \} \) of \( X \).

What happens to these curvatures is a lot like what happens to \( R^2 \) under the long term Cheeger deformation by the standard \( SO (2) \)–action. The metric becomes a positively curved paraboloid that is very nearly flat except near the fixed point, \((0, 0)\), where there is a lot of curvature. In this example, the radial field, \( \partial_r \), plays the role of our field, \( X \), and the lengths of the circles centered about the origin play the role of our function \( \psi \).

We describe here an abstraction of what happens on the Gromoll–Meyer sphere. Let \( (M, g_\infty) \) be a Riemannian submersion.

- Let \( G = G_1 \times G_2 \) act isometrically on \( M \) and by symmetries of \( \pi \).

Let \( g_{\nu, l} \) be the metric on \( M \) obtained by doing the Cheeger deformation with \( G = G_1 \times G_2 \) on \( M \), where the scale on the \( G_1 \) factor in \((G_1 \times G_2) \times M \) is \( \nu \) and the scale on the \( G_2 \) factor in \((G_1 \times G_2) \times M \) is \( l \).

Because of our curvature formula, we are interested in how the length of the \( \pi \)–horizontal part, \( W^\mathcal{H}; g_{\nu, l} \), of a \((G_1 \times G_2)\)–Killing field \( W \) is affected by Cheeger deforming with \((G_1 \times G_2)\). In the Gromoll–Meyer sphere we will consider the case when \( \nu \) is very small and

\[
l = O \left( \nu^{1/3} \right).
\]

So we adopt these hypotheses for our abstract framework here. We set

\[
\psi_\infty = |W^\mathcal{H}; g_\infty|_{g_\infty} \quad \text{and} \quad \psi_{\nu, l} = |W^\mathcal{H}; g_{\nu, l}|_{g_{\nu, l}}.
\]

In Section 8 we prove the following formula for \( \psi_{\nu, l} \) in terms of \( \psi_\infty \).

**Lemma 2.7.** Let \( K^1_W \) be the Killing field on \( G_1 \) that corresponds to \( W \). Suppose \( W \) lies in the direction of the projection of \( W^\mathcal{H}; g_\infty \) onto the orbits of \( G_1 \), and that

\[
\rho = \frac{1}{|K^1_W|_{\mathcal{H}_1}}.
\]

Let \( K^2_{W, M} \) be a vector in the direction of the projection of \( W^\mathcal{H}; g_\infty \) onto the orbit of \( G_2 \). We normalize \( K^2_{W, M} \) so that \( |K^2_{W, M}|^2 = 1 \), where \( K^2_W \) is the corresponding
Killing field on $G_2$. Then
\[
\psi_{t,l}^2 = \frac{\psi_{\infty}^2}{\rho^2 \frac{\psi_{\infty}'}{\psi_{\infty}} + \frac{\psi_{\infty}'}{\psi_{\infty}} + 1}
\]
where
\[
\psi_{\infty}^4 \equiv \left( K_{W}^{2} , W^{M, g_{\infty}} \right)^{2}.
\]
Now we restrict our attention to a curve $\gamma : [0, b] \to M$ so that $\gamma(0) = \ast$,
- $\left| \frac{\psi_{\infty}}{\psi_{\infty}'} \right|$ is bounded,
- $\psi_{\infty}'' \leq 0$,
- $\psi_{\infty}(0) = \psi_{\infty}'(0) = 0$,
- $\left( \frac{\psi_{\infty}}{\psi_{\infty}'} \right)$ is bounded, and
- $\psi_{\infty}'(0) = \psi_{\infty}(0) = 1$.

In section 8 we prove the following quantitative description of how $\left( \psi_{t,l}^2 \right)$ is compressed as $\nu \to 0$.

**Proposition 2.8.** For $\nu$ sufficiently small and $l = O(\nu^{1/3})$, we have
\[
\left( \psi_{t,l}' \right)^2 \bigg|_{[0, \nu]} \geq \frac{97}{100} \frac{1}{\rho^2 + 1},
\]
\[
\left( \psi_{t,l}' \right)^2 \bigg|_{[\nu^{1/3}, \nu]} \leq O\left( \nu^{\frac{1+\beta}{3}} \right),
\]
for any fixed $\beta < \frac{7}{5}$.

**Remark 2.9.** Keeping in mind that $\rho$ is a fixed “background constant”, $\rho = \frac{1}{|K_{W}|_{10}}$, that is independent of $\nu$, these formulas tell us that $\left( \psi_{t,l}^2 \right)$ is large on the small interval $[0, \nu]$, and then rapidly becomes very small, its generic order being $\nu^{\frac{1+\beta}{3}}$.

2.4. **Set up for Synergy.** In section 9, we discuss in an abstract setting, how the orthogonal partial conformal change of Sections 2 and 3 will play a role in making the problem of verifying positive curvature more tractible. This will involve a synergy between the curvature compression principle, fiber scaling, and the orthogonal partial conformal change.

The addition of the orthogonal partial conformal change will aid us in verifying the positivity of the curvatures of planes of the form
\[
\text{span} \{X, W + \tau V\},
\]
where $V \in \mathcal{P}$ is perpendicular to $X$, $W$, and $W^{M}$, and $\tau \in \mathbb{R}$. It is necessary that such planes have positive curvature, but of course it is not sufficient. To the axioms of the previous two subsections we add,
- Cheeger deforming $M$ by $G_1 \times G_2$ preserves the Riemannian submersions
\[
M \xrightarrow{\text{PSM}} \sum \xrightarrow{\text{PSM}} B.
\]

Let $g_{t,l}$ be the metric on $M$ obtained by doing the Cheeger deformation with $G = G_1 \times G_2$ on $M$, where the scale on the $G_1$ factor in $(G_1 \times G_2) \times M$ is $\nu$ and the scale on the $G_2$ factor in $(G_1 \times G_2) \times M$ is $l$. We assume that
\[
l = O\left( \nu^{1/3} \right).
\]
We now carry out metric deformations in the following order.

1: Cheeger deform with $G_1 \times G_2 = G$ and with the Cheeger parameter $\nu$ being small.
2: Perform the orthogonal partial conformal change with $\varphi$ as in the proof of Theorem 2.6.
3: Scale the fibers of the Riemannian submersion $\pi : (M, g_0) \to B$, as in Section 4, and
4: perform a conformal change with the conformal factor $e^{2f}$ that we have asserted to exist in Theorem 2.1.

As usual we call the initial metric $g$ and the final metric $\tilde{g}$.

In section 9 we will only concern ourselves with the curvatures along a single torus $T$ in our family $\mathcal{S}$. As we did above, we suppose that $T$ is spanned by commuting, orthogonal, geodesic fields $X$ and $W$.

Let $\text{Fix}_{G_1, B}$ be the fixed point set of the $G_1$-action on $B$. To all of the axioms stated above we add

- For unit $V \in \mathcal{P}$,
  $$\frac{\text{Hess}^{g_\nu} (f) (W, V)^2}{\text{curv} (X, V)} = o \left( s^4 \right) \left( \text{grad} |W^\nu| \right).$$

- The ratio
  $$\frac{g \left( A_X^\tau W^\nu, V \right)^2}{\text{curv}^\varphi (X, V)} \leq C$$
  for all $\nu$.

**Lemma 2.10.** There is a function $\xi : (0, 1) \to \mathbb{R}_+$ with $\lim_{t \to 0} \xi (t) = 0$ so that for all $\tau \in \mathbb{R}$ and all $V \in \mathcal{P}$ with $V \perp W^\nu$

$$Q (\tau) \equiv \text{curv}^\varphi (X, W + \tau V) \geq (1 - \xi (\nu)) \text{curv}^\varphi (X, W) > 0,$$

provided that the $\varphi$ used in the orthogonal partial conformal change is chosen appropriately.

**2.5. The Role of the Tangential Partial Conformal Change.** Unfortunately we do not know a means to apply a conformal change as part of a deformation to positive curvature on the Gromoll–Meyer sphere, $\Sigma^7$. The problem is that $\Sigma^7$ contains two families of initially flat tori, $\mathcal{F}_\zeta$ and $\mathcal{F}_\xi$ whose intersection is non-empty. The conformal factor required to make $\mathcal{F}_\xi$ positively curved is different from the conformal factors required to make $\mathcal{F}_\zeta$ positively curved.

Despite this problem we will still be able to apply Theorem 2.1 and Lemma 2.10 in [PetWilh2]. The conformal change will be replaced with a deformation that we call the tangential partial conformal change. We will exploit the rigidity of totally geodesic flats to show that the tangential partial conformal change has the same effect on the curvatures of the initially flat tori as an actual conformal change, thereby making Theorem 2.1 and Lemma 2.10 applicable.
3. Deformations Preserving Totally Geodesic Families

In the next two sections we prove Theorems 2.5 and 2.6. Conceptually, the proofs have two parts.

1: Show that the $S_2S$ are also totally geodesic ats with respect to $\tilde{g}$, and
2: show that the rest of the curvature tensor of $\tilde{g}$ is close enough to the curvature tensor of $g$ so that $\tilde{g}$ remains nonegatively curved.

The first issue is handled in this section, while the second is treated in the next section.

For the first two results of this section we can get by with something weaker than the standing hypotheses of subsection 2.2. For the next two results, let $M$ be complete and nonnegatively curved. Let $O$ be open and precompact. Let $S$ be a family of totally geodesic ats in $M$ so that $\cup_{S \in S} (S \cap O) = O$, and in addition let all of the zero curvature planes of $(M, g)$ be tangent to an $S \in S$.

**Lemma 3.1.** (Converse Gauss Lemma) Let $\tilde{g}$ be another metric on $M$ satisfying $g(Z, \cdot) = \tilde{g}(Z, \cdot) : TM \to \mathbb{R}$ for all vectors $Z$ tangent to a submanifold in $S$. Then $S \cap O \subset (M, \tilde{g})$ is totally geodesic and flat for all $S \in S$.

**Proof.** Let $Z$ be a unit vector field on $O$ that at every point is tangent to an $S \in S$. Let $V$ a field on $O$ that is normal to every $S \in S$. Our assumptions imply that $Z$ is also a unit field for $\tilde{g}$ and $V$ is also normal to every $S \in S$ with respect to $\tilde{g}$. Using that all $S \in S$ are totally geodesic for $g$, we have

\[
0 = g(\nabla_Z V, Z) \\
= g([Z, V], Z) \\
= \tilde{g}([Z, V], Z) \\
= \tilde{g}(\tilde{\nabla}_Z V, Z)
\]

showing that they are also totally geodesic for $\tilde{g}$.

Since the intrinsic metric on members of $S$ does not change, each $S \in S$ remains flat. \hfill \Box

Note the proof that each $S \in S$ is totally geodesic does not use the hypothesis that the $S$’s are flat.

To explain the name of the lemma, let $(M, g)$ be an arbitrary Riemannian manifold. Let $S$ be the collection of minimal geodesics emanating from a fixed point $p \in M$, and let $O$ be the open dense set on which $\text{dist}(p, \cdot) : M \to \mathbb{R}$ is smooth. If $\tilde{g}$ agrees with $g$ on the radial $g$–geodesics from $p$, then they are also $\tilde{g}$–geodesics.

As an application of the Converse Gauss Lemma we have the following abstraction of Theorem 2.1 in [Wals].

**Theorem 3.2.** Let $M$ be complete and nonnegatively curved. Let $O \subset M$ be open and precompact. Let $S$ be a family of totally geodesic ats in $M$ so that $\cup_{S \in S} (S \cap O) = O$, and in addition let all of the zero curvature planes of $(M, g)$ be tangent to an $S \in S$. Suppose that there is an $\varepsilon > 0$ so that the zero planes of $M \cap O$ are $\varepsilon$-nondegenerate at all points.

Let $\tilde{g}$ be another metric so that $g(Z, \cdot) = \tilde{g}(Z, \cdot) : TM \to \mathbb{R}$
for all vectors $Z$ tangent to a submanifold in $S$.

Then $(M, \tilde{g})$ is nonnegatively curved on $O$ with precisely the same zero curvature planes as $g$, provided $\tilde{g}$ is sufficiently close to $g$ in the $C^2$–topology.

Proof. Since $O$ is precompact and satisfies the $\varepsilon$–nondegeneracy condition, we can find $\delta > 0$ and a neighborhood $\mathcal{N}$ of the zero curvature planes such that any plane in $\mathcal{N}$ has the form span $\{X + \sigma_0 Z, W + \tau_0 V\}$ where

- $\text{curv} (X, W) = 0$,
- $|X| = |W| = |Z| = |V| = 1$
- $(\sigma_0, \tau_0) \in [-\delta, \delta]^2$,
- $\sigma^2 \text{curv} (Z, W) + 2\sigma \tau (R(X, Z, V, W) + R(X, V, Z, W) + \tau^2 \text{curv} (X, V) > \varepsilon$, for all $\sigma^2 + \tau^2 = 1$
- $(0, 0)$ is the only zero for $P(\sigma, \tau) = \text{curv} (X + \sigma Z, W + \tau V)$ on $[-\delta, \delta]^2$, and
- $\text{sec} (P) > \varepsilon > 0$ for all planes $P$ not in $\mathcal{N}$.

Any nonpositively curved planes of $\tilde{g}$ must lie in $\mathcal{N}$ if $g$ and $\tilde{g}$ are sufficiently close in the $C^2$–topology. Thus the theorem will follow if we can show that for all choices of $\{X, Z, W, V\}$ as above

$\tilde{P} (\sigma, \tau) = \text{curv} \tilde{g} (X + \sigma Z, W + \tau V)$

is nonnegative on $[-\delta, \delta]^2$ and only vanishes when $\sigma = \tau = 0$, provided $\tilde{g}$ is sufficiently $C^2$ close to $g$.

Because $X$ and $W$ are tangent to a 0–curvature plane in a nonnegatively curved manifold

$R^g (X, W) W = R^g (W, X) X = 0.$

Since they are also tangent to a totally geodesic flat that is preserved under our deformation we have

$R^g (X, W) W = R^g (W, X) X = 0.$

So the constant and linear terms of $P (\sigma, \tau)$ and $\tilde{P} (\sigma, \tau)$ vanish. If $\tilde{g}$ is sufficiently close to $g$, then we also have that

$\sigma^2 \text{curv} \tilde{g} (Z, W) + 2\sigma \tau (R^g (X, Z, V, W) + R^g (X, V, Z, W)) + \tau^2 \text{curv} \tilde{g} (X, V) > \varepsilon.$

It follows that $\tilde{P} (\sigma, \tau)$ has an isolated minimum at $(0, 0)$ and that this is the only minimum on $[-\delta, \delta]^2$, provided $\tilde{g}$ is sufficiently close to $g$ in the $C^2$–topology.

The previous result can be viewed as a modified version of Theorem 2.5. The submersion $M^{PM} \Sigma$ is dropped, and the hypothesis

$\delta R^g (V, X, X, V) > (D_X D_X \varphi) |V|^2_g |X|^2_g$

is replaced with the hypothesis that $\tilde{g}$ is $C^2$–close to $g$.

Therefore by combining the proof of the previous result with O’Neill’s horizontal curvature equation we get the following.

**Corollary 3.3.** Theorem 2.5 holds if we replace the hypothesis

$\delta R^g (V, X, X, V) > (D_X D_X \varphi) |V|^2_g |X|^2_g,$

with the hypothesis that $\tilde{g}$ is $C^2$–close to $g$. 
Proof. By hypothesis there is an open subset $\mathcal{N}$ of the horizontal distribution along $K$ so that

- $\sec(P) > \varepsilon > 0$ for all horizontal planes $P$ that are not in $\mathcal{N}$,
- every positively curved plane in $\mathcal{N}$ can be written span $\{X + \sigma_0 Z, W + \tau_0 V\}$ where $|X| = |W| = |Z| = |V| = 1$,
- $\sigma^2 \text{curv}(Z, W) + 2\sigma \tau (R(X, Z, V, W) + R(X, V, Z, W)) + \tau^2 \text{curv}(X, V) > \varepsilon$.

Let $\tilde{H}$ be the horizontal distribution of $\pi$ with respect to $\tilde{g}$. The horizontal zero curvature planes of $g$ are also in $\tilde{H}$. This is because they are all tangent to some $S \in \mathcal{S}$.

Let $\tilde{\mathcal{N}}$ be the $\tilde{g}$-orthogonal projection of $\mathcal{N}$ onto $\tilde{H}$. Then $\tilde{\mathcal{N}}$ is close to $\mathcal{N}$ if $\tilde{g}$ is close to $g$.

Arguing as in the proof of Theorem 3.2 with $\tilde{\mathcal{N}}$ playing the role of $\mathcal{N}$ we see that the horizontal distribution with respect to $\tilde{g}$ is nonnegatively curved on $\pi(K)$ with precisely the same zero curvature planes as the horizontal distribution of $g$. It follows from O’Neil’s horizontal curvature equation that $(\Sigma, \tilde{g})$ is nonnegatively curved on $\pi(K)$ and that every zero curvature plane of $(\Sigma, \tilde{g})$ on $\pi(K)$ is a zero plane of $(\Sigma, g)$ on $\pi(K)$. Since the zero planes of $(\Sigma, g)$ lift to horizontal flats, that are preserved under our deformation, it follows that zero planes of $(\Sigma, g)$ on $\pi(K)$ are also zero planes of $(\Sigma, \tilde{g})$ on $\pi(K)$.

4. ORTHOGONAL PARTIAL CONFORMAL CHANGE

Although Theorem 3.2 and Corollary 3.3 give us a means to deform nonnegative curvature, it is impossible for them to have a large quantitative effect on the curvature tensor. This is because the deformations discussed are $C^2$-small.

On the other hand, Theorem 2.5, which we prove in this section, holds for certain deformations that are only $C^1$-small deformations.

The standing hypotheses for this section are stated in subsection 2.2, but the main lemma that we prove here applies to a more general type of deformation. To state it we assume the following.

- $M$ is a compact, smooth $n$-manifold with a Riemannian metric $g$ and a family of Riemannian metrics $\{g_t\}_{t>0}$
- $\{E_i\}$ is an orthonormal frame for $g$ with dual coframe $\{\theta^i\}$.
- $\{E_i^t\}$ is an orthonormal frame for $g_t$ with dual coframe $\{\theta^t_i\}$.
- $R_{i,j,k,l}^t \equiv R(E_i, E_j, E_k, E_l)$ and $R_{i,j,k,l}^t \equiv R^t(E_i^t, E_j^t, E_k^t, E_l^t)$.
- There are smooth functions $\{\phi_t^i\}_{i=1}^n, \{\psi_t^i\}_{i=1}^n$, and $\{\lambda_t^i\}_{i=1}^n$ on $M$ so that
  \[ \theta_t^i = \phi_t^i \theta^i, \]
  \[ d\phi_t^i = \psi_t^i \theta^i \]
  \[ d\psi_t^i = \lambda_t^i \theta^i, \]
  \[ \phi_t^i \rightarrow 1, \text{ uniformly as } t \rightarrow 0, \text{ and } \]
  \[ \psi_t^i \rightarrow 0, \text{ uniformly as } t \rightarrow 0. \]

In the presence of our other hypotheses, the uniform convergence of $\phi_t^i$ to 1 and $\psi_t^i$ to 0, as $t \rightarrow 0$ is equivalent to the convergence of $g_t$ to $g$ in the $C^1$-topology.
Lemma 4.1. Except for the quadruples of indices that up to symmetries of the curvature can be reduced to $\{1, i, i, 1\}$ we have

$$\lim_{t \to 0} R_{i,j,k,l}^t = R_{i,j,k,l}.$$

Proof. To simplify the notation we let $\tilde{g}$ equal $g_t$ for $t$ sufficiently small. We use the symbols $\tilde{E}_i, \tilde{\theta}_i, \tilde{R},$ etc. for concepts pertaining to $\tilde{g}$.

Our proof makes heavy use of the Cartan formalism ([Spiv], Chap 7).

Following ([Spiv], Chap 7) we define $\tilde{b}_{jk}^i$, $\tilde{a}_{jk}^i$, and $\tilde{\omega}_j^i$ by

\[
\begin{align*}
    d\theta^i & = \frac{1}{2} \sum_{j,k=1}^n \tilde{b}_{jk}^i \theta^j \wedge \theta^k, \\
    a_{jk}^i & = \frac{1}{2} \left( \tilde{b}_{jk}^i + \tilde{b}_{kj}^i - \tilde{b}_{ij}^i \right), \\
    \omega_j^i & = \sum_{k=1}^n a_{jk}^i \theta^k.
\end{align*}
\]

It then follows ([Spiv], Chap 7) that

\[
\begin{align*}
    \tilde{b}_{jk}^i & = -\tilde{b}_{kj}^i \quad \text{and} \\
    \tilde{a}_{jk}^i & = -\tilde{a}_{kj}^i
\end{align*}
\]

The forms $\Omega_j^i$

\[
\Omega_j^i \equiv d\omega_j^i + \sum_{k=1}^n \omega_k^j \wedge \omega_j^k
\]

are then curvatures. Specifically

\[
g(R(X, Y)E_j, E_i) = \Omega_j^i (X, Y).
\]

We now check how these functions get changed for the new frame.

\[
\begin{align*}
    d\theta^i & = d\phi^i \theta^i + \frac{1}{2} \sum_{j,k=1}^n \phi^i \tilde{b}_{jk}^i \theta^j \wedge \theta^k \\
    & = \psi^i \theta^1 \wedge \theta^i + \frac{1}{2} \sum_{j,k=1}^n \phi^i \tilde{b}_{jk}^i \theta^j \wedge \theta^k \\
    & = \frac{1}{2} \psi^i \tilde{\theta}^1 \wedge \tilde{\theta}^i - \frac{1}{2} \phi^i \phi^j \theta^1 \wedge \theta^j \\
    & \quad + \frac{1}{2} \sum_{j,k=1}^n \phi^i \phi^j \phi^k \tilde{b}_{jk}^i \tilde{\theta}^j \wedge \tilde{\theta}^k.
\end{align*}
\]

(4.1)

So the only $\tilde{b}_{jk}^i$'s that depend on the $\psi$'s are

\[
\tilde{b}_{1i}^i = -\tilde{b}_{ii}^i = \frac{\psi^i}{\phi^1 \phi^i} + \frac{\phi^i}{\phi^1 \phi^i} \tilde{b}_{ii}^i.
\]

So among the

\[
a_{jk}^i = \frac{1}{2} \left( \tilde{b}_{jk}^i + \tilde{b}_{kj}^i - \tilde{b}_{ij}^i \right)
\]
the ones potentially affected by the $\psi$'s are $\tilde{a}_{i,1}^i$, $\tilde{a}_{1,i}^i$, and $\tilde{a}_1^i$. However, the antisymmetry $\tilde{a}_{jk}^i = -\tilde{a}_{kj}^i$ implies that $\tilde{a}_{i,1}^i = 0$, and that $\tilde{a}_{1,i}^i = -\tilde{a}_{i,i}^i$. The antisymmetry of the $b$s then gives us

$$\tilde{a}_{i,i}^i = -\tilde{b}_{i,i}^i.$$ 

So in fact, the $\tilde{a}$s that depend on the $\psi$'s are

$$\tilde{a}_{i,i}^i = -\tilde{a}_{i,i}^i = \frac{1}{2} \left( \tilde{b}_{i,i}^i + \tilde{b}_{i,i}^i - \tilde{b}_{i,i}^i \right) = \frac{\psi^i}{\phi^1 \phi^i} \tilde{b}_{i,i}^i = \tilde{a}_{i,i}^i + \phi^i \phi^j \phi^k \tilde{a}_{i,j}^i$$

Thus the only $\omega^i$'s that depend on the $\psi$'s are

$$\tilde{\omega}_1^i = -\tilde{\omega}_1^1 = \tilde{a}_{1,i}^i \tilde{\theta} + \sum_{k \neq i} \tilde{a}_{i,k}^i \tilde{\theta}^k$$

$$= \frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta} + \sum_{k} a_{i,k}^i \theta^k + O \left( C^0 \right)$$

where by $O \left( C^0 \right)$ we mean $O \left( \max \{1 - \phi^i\} \right) \sum_k \theta^k$.

It follows that the only $\Omega_1^i$'s that depend on the $\lambda$'s are

$$\Omega_1^i = -\Omega_1^i = d\tilde{\omega}_1^i + \sum_{k=1}^n \omega_1^k \wedge \omega_1^k.$$ 

Only the first term depends on the $\lambda$'s. It is

$$d\tilde{\omega}_1^i = d \left( \sum_{k=1}^n \tilde{a}_{1,k}^i \tilde{\theta}^k \right)$$

$$= \sum_{k=1}^n \left( \tilde{a}_{1,k}^i \wedge \tilde{\theta}^k - \tilde{a}_{1,k}^i \tilde{\theta} \right)$$

$$= d \left( \frac{\psi^i}{\phi^1 \phi^i} \tilde{\theta} \right) + d\omega_1^i + O \left( C^1 \right)$$

where by $O \left( C^1 \right)$ we mean

$$d \left( \max \{1 - \phi^i\} \sum_k \theta^k \right) + O \left( \max \{1 - \phi^i, \psi^i\} \sum_k \theta^k \right).$$

We conclude that

$$d\tilde{\omega}_1^i = \lambda^i \tilde{\theta} + \tilde{\theta}^i + d\omega_1^i + O \left( C^1 \right).$$

So we note that the only curvatures affected by the $C^2$ change are the sectional curvature spanned by $E_1$ and $E_i$. □
Corollary 4.2.

\[ \tilde{a}^i_{1i} = \left( \frac{\psi^i}{\phi^i} + \frac{\phi^i}{\phi^i} a^i_{1i} \right) \]

For \( i \neq 1 \neq j \neq 1 \neq k \) we have

\[ \left| \tilde{a}^k_{ij} - a^k_{ij} \right| = \max_{l \in \{i,j,k\}} O \left( 1 - \phi^j \right) \max_{l \in \{i,j,k\}} \{ b^k_{ij} \} \]

\[ \tilde{a}^k_{ii} = \frac{1}{\phi^k} b^k_{ij} = \frac{1}{\phi^k} a^k_{ii} \]

Proof. The first equation is derived explicitly in our proof above.

The other equations follow by combining equation 4.1 with the relation \( a^i_{jk} = \frac{1}{2} \left( b^i_{jk} + b^i_{kj} - b^i_{ij} \right) \).

In our applications we will also need to know something more specific about how the other components of the curvature tensor change with such a deformation.

Corollary 4.3. Except for the quadruples of indices that up to to symmetries of the curvature can be reduced to \( (1, k, k, 1) \)

\[ \left| R^q \left( \tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l \right) - R \left( E_i, E_j, E_k, E_l \right) \right| \]

\[ \leq O \left( \max_q \{ \psi^q, 1 - \phi^q \} \max_p \left\{ \left\{ a^i_{p,i}, a^i_{p,k}, a^p_{j,k}, a^p_{j,k} \right\}^2, |da^i_{jp}|, a^i_{jp} b^i_{p,k} \right\} \right) \]

\[ \leq O \left( \max_q \{ \psi^q, 1 - \phi^q \} \max_{i,j,k} \left\{ |\omega^j_i|^2, |da^i_{jk}| \right\} \right) \]

Proof. We have

\[ \left| \sum_{p=1}^n \tilde{\omega}^i_{p} \wedge \tilde{\omega}^p_{j} \left( \tilde{E}_l, \tilde{E}_k \right) - \sum_{p=1}^n \omega^i_{p} \wedge \omega^p_{j} \left( E_l, E_k \right) \right| \]

\[ \leq \left( O \left( \max_q \{ 1 - \phi^q, \psi^q \} \max_p \left\{ \left\{ a^i_{p,i}, a^i_{p,k}, a^p_{j,k}, a^p_{j,k} \right\}^2, |da^i_{jp}|, a^i_{jp} b^i_{p,k} \right\} \right) \right) , \]

and

\[ d\tilde{\omega}^i_{j} \left( \tilde{E}_l, \tilde{E}_k \right) = d \left[ \sum_{p=1}^n \tilde{a}^i_{jp} \tilde{\theta}^p \right] \left( \tilde{E}_l, \tilde{E}_k \right) \]

\[ = \left[ \sum_{p=1}^n \tilde{d}a^i_{jp} \wedge \tilde{\theta}^p + \tilde{a}^i_{jp} d\tilde{\theta}^p \right] \left( \tilde{E}_l, \tilde{E}_k \right) \]

\[ = d\omega^i_{j} \left( E_l, E_k \right) + O \left( \max_q \{ \psi^q, 1 - \phi^q \} \sum_{p=1}^n \omega^i_{jp} \wedge \theta^p \left( E_i, E_k \right) + O \left( \max_q \{ \psi^q, 1 - \phi^q \} \sum_{p=1}^n a^i_{jp} d\theta^p \left( E_i, E_k \right) \right) \]

Combining the displays gives the first inequality. Since \( O \left( \max_{i,j,k} \left\{ \left| b^i_{jk} \right| \right\} \right) \leq O \left( \max_{i,j} |\omega^i_{j}| \right) = O \left( \max_{i,j,k} a^i_{jk} \right) \) the second inequality follows from the first. \( \square \)
We now return to our discussion of an orthogonal partial conformal change, defined in subsection 2.2.

Let \( \{ E_i \} \) be an orthonormal frame for \( g \) with \( X = E_1 \), and \( \text{span} \{ E_2, \ldots, E_p \} = \mathcal{P} \). Set \( \phi^i = \varphi \) for \( i = 2, \ldots, p \), and \( \phi^1 \equiv 1 \) otherwise. This shows that the orthogonal partial conformal change, defined in subsection 2.2 is an example of the above lemma.

Applying the last formula in the proof of the preceding lemma to our orthogonal partial conformal change yields

**Corollary 4.4.** For \( V \in \mathcal{P} \)

\[
R^g (V, X, X, V) = R^g (V, X, X, V) - D_X (D_X \varphi) |V|^2_g |X|^2_g + O(C^1) .
\]

**Proof of Theorem 2.5.** Combine Corollary 4.4, and Lemma 4.1 with the proof of Corollary 3.3.

4.1. Orthogonal Partial Conformal Change Near the Singularity. In this subsection we prove Theorem 2.6. We start with some lemmas.

**Lemma 4.5.** There is a single constant \( C > 0 \) so that any point of \( M - p_{BM}^{-1} \{ *, -* \} \) has a neighborhood \( B \) and an orthonormal frame

\[
\{ X, E_2, \ldots, E_b \}
\]

of the horizontal distribution of \( p_{BM} \) so that, \( \{ X, E_2, \ldots, E_b \} \) extends to an orthonormal frame

\[
\{ X, E_2, \ldots, E_b, V_1, \ldots, V_k \} \equiv \{ F_1, \ldots, F_n \}
\]

that satisfies

\[
|a^i_{j,k}| = |\omega^j_i (F_k)| \leq C \\
|\text{da}^i_{j,k}| \leq C
\]

for all indices that do not correspond to \( \omega^X_i (E_i) = -\omega^{E_i}_X (E_i) \), \( D_X (a^i_{1,i}) \), and \( da^{E_i}_{E_k, E_i} \).

In the exceptional cases we have

\[
\text{da}^{E_i}_{E_k, E_i} (E_k) = \frac{1}{t^2} + O(1) , \\
\text{da}^{E_i}_{E_k, E_i} (F_k) \leq C \text{ for } F_k \neq E_k , \\
\omega^{E_i}_X (E_i) = -\omega^X_E (E_i) = a^i_{1,i} = \frac{1}{t} + O(t) , \text{ and } \\
|D_{F_j} \left( \omega^{E_i}_X (E_i) \right)| = |D_{F_j} (a^i_{1,i})| \leq C
\]

except the case when \( F_j = X \). When \( F_j = X \)

\[
|D_X \left( \omega^{E_i}_X (E_i) \right)| = |D_X (a^i_{1,i})| = -\frac{1}{t^2} + O(1)
\]

**Remark 4.6.** Typically there are topological obstructions to defining a frame of \( p_{BM}^{-1} \{ *, -* \} \). We emphasize however, that this lemma asserts the existence of a single constant \( C \) and local framings through out \( p_{BM}^{-1} \{ *, -* \} \) with the above properties.
Proof. Parameterize the integral curves $c_X$ of $X$ in $B$ on $[0, \cdot)$ with $c_X(0) = \ast$. Let $E$ be a normal parallel field along $c_X$ with $|E| = 1$. If $J$ is a Jacobi field along $c_X$ with $J(0) = 0$, $J'(0) = E$, then we have,

$$\nabla_X \nabla_X J|_0 = -R(J, X) X|_0 = 0.$$ 

So on $B$ we have

$$\nabla JX = \nabla X J = E + O(t^2).$$

So the lifted fields on $M$ satisfy

$$\nabla_E X = \frac{1}{t} E + A_E X + O(t).$$

This gives us

$$\omega^E_X (E) = -\omega^X_E (E) = \frac{1}{t} + O(t),$$

$$\left| D_X \left( \omega^E_X (E_i) \right) \right| = \left| D_X (a^i_{1,i}) \right| = \frac{1}{t^2} + O(1),$$

and if $V_j$ is vertical with $|V_j| = 1$,

$$\left| \omega^V_X (E_i) \right| \leq C.$$ 

Small enough metric spheres in $B$ around $\ast$ are almost round. The intrinsic sectional curvature of these metric spheres at distance $t$ from $\ast$ is nearly $\frac{1}{t^2}$. We shall see that around any point we can choose the frame $\{E_i\}$ so that

$$\left| \omega^E_{E_j} (E_k) \right| \leq C,$$

$$d \omega^E_{E_k} (E_i, E_k) = \frac{1}{t^2} + O(1),$$

$$d a^E_{E_k, E_i} (E_k) = \frac{1}{t^2} + O(1),$$

and in all other cases

$$\left| a^E_{E_j, E_k} \right| \leq C$$

$$\left| d a^E_{E_j, E_k} \right| \leq C,$$

and

$$\left| d a^E_{E_i, E_j} \right| \leq C.$$

Indeed the euclidean model for such a frame $\{E_i\}$ of the metric spheres around the origin is one that that is intrinsically parallel at a point. Exponentiation of such a frame, and then applying Gram-Schmidt gives us the desired frame $\{E_i\}$.

The remaining $a$s are either components of the $A$ or $T$ tensors of our submersion, or components of the 1-form

$$\langle \nabla V_j, V_k \rangle.$$ 

Since $p_{BM}^1(B)$ is precompact all of these quantities (and their differentials) are bounded, for any fixed vertical frame $\{V_j\}_{j=1}^k$. \hfill \Box

Combining the previous lemma with the relations

$$a^i_{jk} = \frac{1}{2} \left( b^i_{jk} + b^i_{kj} - b^i_{lj} \right),$$

and
\[ a^i_{1,i} = \frac{1}{2} (b^i_{1i} + b^1_{1i} - b^i_{11}) = b^i_{1i} \]

we get

**Corollary 4.7.** There is a \( C > 0 \) so that with respect to the frame chosen above we have

\[ b^i_{1i} = -b^i_{11} = \frac{1}{t} + O(t) \]

for \( i = 2, \ldots, b \), and for all other triples of indices

\[ |b^i_{jk}| \leq C. \]

As a consequence of Corollary 4.3 and the previous two results we have

**Lemma 4.8.** Except for the quadruples of indices that up to to symmetries of the curvature tensor can be reduced to \((i, p, p, l)\) with \( p = 1, \ldots, b \)

\[ \hat{R}_{i,j,k,l} \text{ is close to } R_{i,j,k,l} \]

provided \( \varphi \) is sufficiently close to 1 in the \( C^1 \)-topology and \( \frac{1 - \varphi}{t} \) is sufficiently small.

*Proof.* From Corollary 4.3 we see that we only need to worry about components of the curvature tensor for which some of the quantities \( a^1_{p,l}, a^1_{p,k}, a^p_{j,l}, a^p_{j,k}, da^i_{j,p}, a^i_{p,j}b^p_{l,k} \) blow up.

In Lemma 4.5 we saw that the only ones of these quantities that are unbounded are

\[ da^i_{1,i} (X), da^i_{1,j} (E_k), b^i_{1,i}, \text{ and } a^i_{1,i} \]

where \( i, k = 1, \ldots, b \).

The quantities, \( da^i_{1,i} (X) \) and \( da^i_{1,j} (E_k) \) only appear in the sectional curvatures, \( R(X, E_i, E_i, X) \) and \( R(E_k, E_i, E_i, E_k) \).

Since we are assuming that \( \frac{1 - \varphi}{t} \) is small, the only times when \( b^i_{1i} \) and \( a^1_{1,i} \) can make a substantial difference to the change in the curvature tensor is when they are multiplied by an \( \tilde{a}^i_{p,l} \) for which \( |\tilde{a}^i_{p,l} - a^1_{p,l}| \) is as large as \( O(\varphi') \). That is an \( \tilde{a}^i_{p,l} \) of the form \( \tilde{a}^i_{1,j} \). The only components of the curvature tensor where these pairs occur in the Cartan expansion,

\[ R^\tilde{g} \left( \tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l \right) = d\tilde{\omega}^i_{j} \left( \tilde{E}_l, \tilde{E}_k \right) + \sum_{p=1}^{n} \tilde{\omega}^i_{p} \wedge \tilde{\omega}^j_{p} \left( \tilde{E}_l, \tilde{E}_k \right) \]

are sectional curvatures where the indices have the form \((l, p, p, l)\) with \( p = 1, \ldots, b \). \( \square \)

**Lemma 4.9.** For \( V \in \mathcal{P} \) with \( |V|^2 = 1 \)

\[ \hat{R} (V, X, X, V) = R (V, X, X, V) - D_X (D_X \varphi) + O \left( C^1 \right), \]

For arbitrary \( U \) and \( V \) perpendicular to \( X \)

\[ \hat{R} \left( U, \tilde{E}_i, \tilde{E}_i, V \right) = R (U, E_i, E_i, V) + \omega_{E_i}^X (E_i) \varphi' \left( U^P, V^P \right) + O \left( \varphi' \right) + \frac{O(1 - \varphi)}{t} \]

\[ \hat{R} \left( \tilde{E}_k, \tilde{E}_i, \tilde{E}_i, \tilde{E}_k \right) = R (E_k, E_i, E_i, E_k) + \frac{O(1 - \varphi)}{t} + O \left( \varphi' \right). \]

\[ \hat{R} \left( X, \tilde{E}, \tilde{E}, X \right) = R (X, E, E, X) + O \left( \frac{1 - \varphi}{t} \right) + O \left( \varphi' \right) + O \left( \varphi'' \right) |E^P|. \]
Proof: The first statement is a repeat of Corollary 4.4.

To prove the second statement we note that

\[ R^2 \left( U, \bar{E}, \bar{E}_i, V \right) = d\tilde{\omega}_E^{V} (U, E_i) + \sum_k \tilde{\omega}_k^V (U) \tilde{\omega}_E^k (\bar{E}_i) - \tilde{\omega}_E^k (\bar{E}_i) \tilde{\omega}_E^k (U). \]

By anti-symmetry of \( R \), we may assume that both \( U \) and \( V \) are perpendicular to \( \bar{E}_i \). It follows from Lemma 4.5 and Corollary 4.3 that

\[
\left| d\tilde{\omega}_E^{V} (U, \bar{E}_i) + \sum_k -\tilde{\omega}_k^V (\bar{E}_i) \tilde{\omega}_E^k (U) - \left[ d\omega^V_{E_i} (U, E_i) + \sum_k -\omega^V_k (E_i) \omega^k_{E_i} (U) \right] \right|
\leq (O(\varphi') + O(1 - \varphi^2)).
\]

Similarly

\[
\left| \sum_{k \neq 1} \tilde{\omega}_k^V (U) \tilde{\omega}_E^k (E_i) - \sum_{k \neq 1} \omega_k^V (U) \omega^k_{E_i} (E_i) \right| \leq (O(\varphi') + O(1 - \varphi^2)).
\]

Finally, Corollary 4.2 give us

\[
\tilde{\omega}_1^V (U) \tilde{\omega}_E^1 (\bar{E}_i) = \tilde{\omega}_1^V a_{1U}^V a_{1E_i}^V a_{1E_i}^V = (a_{1U}^V + a_{1E_i}^V (1 - \varphi) + \varphi' \langle U^P, V^P \rangle) \left( a_{1E_i}^V + a_{1E_i}^V O(1 - \varphi) + O(\varphi') |E_i^P|^2 \right)
\]

\[
= a_{1U}^V a_{1E_i}^V (1 + O(1 - \varphi))^2 + a_{1E_i}^V O(\varphi') |E_i^P|^2 + a_{1E_i}^V \varphi' \langle U^P, V^P \rangle
\]

\[
= \omega_1^V (U) \omega_{E_i}^1 (E_i) + \omega_{E_i}^X (E_i) \varphi' \langle U^P, V^P \rangle + O(\varphi') + \frac{O(1 - \varphi)}{t},
\]

where we have used the hypothesis that \( P \) is vertical over \{*, *\} to drop the term \( a_{1U}^V O(\varphi') |E_i^P|^2 \).

Combining displays gives us the second equation.

\[
\tilde{R} (E_k, E_i, E_i, E_k) = d\tilde{\omega}_E^{E_k} (E_k, E_i) + \sum_j \tilde{\omega}_E^{E_k} (E_k) \tilde{\omega}_E^j (E_i) - \tilde{\omega}_E^{E_k} (E_i) \tilde{\omega}_E^j (E_k)
\]

Set \( \tilde{a} = \tilde{a} - a \), \( \tilde{b} = \tilde{b} - b \)

\[
d\tilde{\omega}_E^{E_k} (\bar{E}_k, \bar{E}_i) = d \left[ \sum_{p=1}^n \tilde{a}_p \tilde{\theta}_p \right] (\bar{E}_k, \bar{E}_i)
\]

\[
= \sum_{p=1}^n \left[ d\tilde{a}_p \tilde{\theta}_p (\bar{E}_k, \bar{E}_i) + \tilde{a}_p d\tilde{\theta}_p (\bar{E}_k, \bar{E}_i) \right]
\]

\[
= d\omega_E^{E_k} (E_k, E_i) + \tilde{a}_i \tilde{\theta}_i (\bar{E}_k, \bar{E}_i) + \tilde{a}_k \tilde{\theta}_k (\bar{E}_k, \bar{E}_i) + O(1 - \varphi, \varphi')
\]

\[
= d\omega_E^{E_k} (E_k, E_i) + \tilde{a}_k (\bar{E}_k) + \tilde{a}_i (\bar{E}_i) + O(1 - \varphi, \varphi')
\]

From Corollary 4.2 we have

\[
\tilde{a}_i = O(1 - \varphi) a_i |E^P_k|
\]
So
\[
\begin{align*}
d\omega^{E_k}_{E_1} (\tilde{E}_k, \tilde{E}_i) &= d\omega^{E_k}_{E_1} (E_k, E_i) + D_{E_k} [a_{k,i} (O (1-\varphi) |E^P_k|)] + D_{E_i} [a_{i,k} (O (1-\varphi) |E^P_i|)] + O (1-\varphi, \varphi') \\
&= d\omega^{E_k}_{E_1} (E_k, E_i) + O (1-\varphi) |E^P_k| D_{E_k} (a_{k,i}) + O (1-\varphi) (a_{i,k} D_{E_k} |E^P_i|) \\
&\quad + O (1-\varphi) |E^P_i| D_{E_i} a_{i,k} + O (1-\varphi) a_{i,k} D_{E_k} |E^P_i| + O (1-\varphi, \varphi') \\
&= d\omega^{E_k}_{E_1} (E_k, E_i) + O (1-\varphi) (a_{k,i} D_{E_k} |E^P_k|) + O (1-\varphi) a_{i,k} D_{E_i} |E^P_i| + O \left( \frac{1-\varphi}{t}, \varphi' \right)
\end{align*}
\]

Let \{P_i\} be a smooth orthonormal frame for \( P \). We have
\[
D_{E_k} |E^P_k| = D_{E_k} \left( \sum_i \langle E_k, P_i \rangle^2 \right)^{1/2}
\]
\[
= \frac{1}{2 |E^P_k|} \sum_i 2 \langle E_k, P_i \rangle D_{E_k} \langle E_k, P_i \rangle
\]
\[
= \sum_i \frac{\langle E_k, P_i \rangle}{|E^P_k|} (\langle E_k, \nabla E_k P_i \rangle + (\nabla E_k E_k, P_i))
\]
\[
\langle \cdot, \nabla P_i \rangle \text{ is a } C^\infty \text{-tensor so its values are uniformly bounded and } \nabla E_k E_k \text{ is a combination } X \text{ and a uniformly bounded combination of the } E' \text{'s so we conclude that}
\]
\[
|D_{E_k} |E^P_k| | \leq O (1).
\]

So
\[
\begin{align*}
d\omega^{E_k}_{E_1} (\tilde{E}_k, \tilde{E}_i) &= d\omega^{E_k}_{E_1} (E_k, E_i) + O \left( \frac{1-\varphi}{t}, \varphi' \right).
\end{align*}
\]

From Lemma 4.5 and the proof of Lemma 4.1 we see that among the terms in the sum \( \sum_j \bar{\omega}^j_{E_k} (\tilde{E}_k) \bar{\omega}^j_{E_1} (\tilde{E}_i) - \bar{\omega}^j_{E_k} (\tilde{E}_i) \bar{\omega}^j_{E_1} (\tilde{E}_k) \) the only ones that could change a lot are the first term when \( j = 1 \). From Corollary 4.2 we have
\[
\bar{\omega}^1_{E_k} (\tilde{E}_k) \bar{\omega}^1_{E_1} (\tilde{E}_i) = (a_{i,k} + a_{k,i} O (1-\varphi) |E^P_k| + O (\varphi') |E^P_k|) (a_{i,k} + a_{k,i} O (1-\varphi) |E^P_i| + O (\varphi') |E^P_i|)
\]

We have \( a_{i,k} = \frac{1}{t} + O (t), a_{i,i} = \frac{1}{t} + O (t), |E^P_k| = O (t), \) and \( |E^P_i| = O (t) \). Combining these with the display above gives
\[
\bar{\omega}^{E_k}_{E_1} (\tilde{E}_k) \bar{\omega}^{E_1}_{E_i} (E_i) = \omega^{E_k}_{E_1} (E_k) \omega^{E_i}_{E_1} (E_i) + O \left( \frac{1-\varphi}{t} \right) + O (\varphi').
\]

Thus
\[
\hat{R} (E_k, E_i, E_i, E_k) = R (E_k, E_i, E_i, E_k) + O \left( \frac{1-\varphi}{t} \right) + O (\varphi'),
\]
as claimed.

\[
\hat{R} (X, E, E, X) = d\bar{\omega}^{X}_{E} (X, E) + \sum_j \bar{\omega}^j_{E} (X) \bar{\omega}^j_{E} (E) = \hat{R} (X, E, E, X) - \bar{\omega}^j_{E} (E) \bar{\omega}^j_{E} (X)
\]

Since \( \bar{\omega}^X = 0 = \bar{\omega}^E, \) we have
\[
\sum_j \bar{\omega}^j_{E} (X) \bar{\omega}^j_{E} (E) = \sum_j \bar{\omega}^j_{E} (X) \omega^j_{E} (E) = \omega^j_{E} (E) \omega^j_{E} (X) + O (1-\varphi) + O (\varphi')
\]
\[
d\bar{\omega}^X (X, E) = D_X \bar{\omega}^X (E) - D_E \bar{\omega}^X (X) - \bar{\omega}^X [X, E]
\]
We have
\[ \tilde{\omega}^X_{\tilde{E}} \left( \tilde{E} \right) = a^{X,E} + \frac{\varphi'}{\varphi} |E^p| + a^{X,E,O} \left( 1 - \varphi \right) |E^p| \]
and
\[ D_X \tilde{\omega}^X_{\tilde{E}} \left( \tilde{E} \right) = D_X a^{X,E} + |E^p| D_X \frac{\varphi'}{\varphi} + \frac{\varphi'}{\varphi} D_X |E^p| + O \left( 1 - \varphi \right) |E^p| D_X a^{X,E,E} + O \left( \varphi' \right) a^{X,E,E} |E^p| + a^{X,E,E,O} \left( 1 - \varphi \right) \]

Let \( \{ P_i \} \) be a smooth orthonormal frame for \( \mathcal{P} \).

\[ D_X |E^p| = D_X \left( \sum_i \langle E, P_i \rangle^2 \right)^{1/2} \]
\[ = \frac{1}{|E^p|} \sum_i \langle E, P_i \rangle D_X \langle E, P_i \rangle \]
\[ = \sum \frac{\langle E, P_i \rangle}{|E^p|} \left( \langle E, \nabla_X P_i \rangle + \langle \nabla_X E_k, P_i \rangle \right) \]
\( \langle \cdot, \nabla_P \rangle \) is a \( C^\infty \)-tensor so its values are uniformly bounded and
\[ \nabla_X E_k = \frac{1}{t} E_k + \text{a bounded vector}. \]

So we get a uniform bound on \( D_X |E^p| \), and it follows that
\[ D_X \tilde{\omega}^X_{\tilde{E}} \left( \tilde{E} \right) = D_X \tilde{\omega}^X_{\tilde{E}} \left( \tilde{E} \right) + O \left( \frac{1 - \varphi}{t} \right) + O \left( \varphi' \right) + O \left( \varphi'' \right) |E^p| \]

We write
\[ \tilde{\omega}^X_{\tilde{E}} \left[ X, \tilde{E} \right] = \tilde{\omega}^X_{\tilde{E}} \left( \tilde{\nabla}_X \tilde{E} - \tilde{\nabla}_E X \right) = \tilde{\omega}^X_{\tilde{E}} \left( \tilde{\nabla}_E X \right) \]
and
\[ \omega^X_{\tilde{E}} \left[ X, E \right] = \omega^X_{\tilde{E}} \left( \tilde{\nabla}_E X \right) \]

We have
\[ \tilde{\nabla}_E X = \frac{1}{t} E + A_XX + O \left( t \right) \]
\[ \tilde{\nabla}_E X = \tilde{\nabla}_E X + O \left( \varphi' \right) \frac{1}{t} |E^p|^2 E + O \left( 1 - \varphi \right) \frac{1}{t} |E^p| \]
So
\[ \tilde{\omega}^X_{\tilde{E}} \left[ X, \tilde{E} \right] = \omega^X_{\tilde{E}} \left[ X, E \right] + \omega^X_{\tilde{E}} \left( O \left( \varphi' \right) \frac{1}{t} |E^p|^2 E + O \left( 1 - \varphi \right) \frac{1}{t} |E^p| \right) \]
\[ = \omega^X_{\tilde{E}} \left[ X, E \right] + O \left( \varphi' \right) \frac{1}{t} |E^p|^2 E + O \left( 1 - \varphi \right) \frac{1}{t} |E^p| \]
\[ + \frac{1}{t} O \left( \varphi'' \right) |E^p|^2 + O \left( 1 - \varphi \right) + \frac{1}{t} O \left( \varphi' \right)^2 |E^p|^2 \]
\[ = \omega^X_{\tilde{E}} \left[ X, E \right] + \frac{1}{t^2} O \left( \varphi' \right) |E^p|^2 + O \left( \varphi' \right) + O \left( 1 - \varphi \right) \]
\[ = \omega^X_{\tilde{E}} \left[ X, E \right] + O \left( \varphi' \right) + O \left( 1 - \varphi \right) \]
Combining displays we have
\[ \tilde{R} \left( X, \tilde{E}, \tilde{E}, X \right) = R \left( X, E, E, X \right) + O \left( \frac{1 - \varphi}{t} \right) + O \left( \varphi' \right) + O \left( \varphi'' \right) |E^p|. \]
Proof of Theorem 2.6: Let $K$ be a compact subset of $O \cap K_0$ so that $K_0 = K \cup U$. By Theorem 2.5 we only need to verify our curvature condition on $U$.

Let $\{X, W\} \in S$. A plane that is not perpendicular to either $X$ or $W$ can be written as $\text{span}\{X + \sigma z, W + \tau V\}$ with $\sigma, \tau \in \mathbb{R}$. By the Converse Gauss Lemma, $R(X, W) W = R(W, X) X = 0$. Therefore

\[
\text{curv} (X + \sigma z, W + \tau V) = \\
\sigma^2 \text{curv} (z, W) + 2\sigma \tau \left( \tilde{R}(X, W, V, z) + \tilde{R}(X, V, W, z) \right) + \tau^2 \text{curv} (X, V) + \\
2\sigma^2 \tau \left( \tilde{R}(z, W, V, z) + 2\sigma \tau^2 \tilde{R}(X, V, W, z) \right) + \sigma^2 \tau^2 \text{curv} (z, V)
\]

Combining Lemmas 4.8 and 4.9 we have

\[
\text{curv} (X + \sigma z, W + \tau V) = \text{curv} (X + \sigma z, W + \tau V) - \tau^2 D_X (D_X \varphi) |V^p|^2_g + \\
+ \sigma^2 \tau^2 \omega_{ZE} (Z^E) \varphi' |V^p|^2 + \left( \frac{O(1 - \varphi)}{\text{dist} \left( p_{BM}^1 (*), p \right)} + O(\varphi') \right) (\sigma^2 + 2\sigma \tau + \tau^2 + 2\sigma^2 \tau + 2\sigma^2 \tau^2),
\]

where $Z^E$ is the component of $Z$ that is tangent to (the horizontal lift of) a metric spheres around $\{*, -*\}$ in $B$.

For the moment assume we are on $U$ and that our deformed plane $\text{span}\{X + \sigma z, W + \tau V\}$ is in the extension $T \mathcal{S}$ of $U \in S T S_{|_{p_{BM}^1 (*, -*)}}$.

Our hypotheses about the form of $T \mathcal{S}$ and the fact that

\[
-\delta' D_X D_X \varphi |U| > \delta \left( |D_X \varphi| + \frac{|1 - \varphi (p)|}{\text{dist} \left( p_{BM}^1 (*), p \right)} \right)
\]

gives us that the second term satisfies

\[
- \frac{\delta'}{\delta} |V^p|^2_g D_X D_X \varphi |U| > \delta' D_X D_X \varphi |U| > \delta \left( |D_X \varphi| + \frac{|1 - \varphi (p)|}{\text{dist} \left( p_{BM}^1 (*), p \right)} \right).
\]

Similarly our hypotheses about the form of $T \mathcal{S}$ give us that the third term satisfies

\[
\frac{\delta'}{\delta^2} \omega_{ZE} (Z^E) |V^p|^2 \varphi' \geq -\delta' g \left( \nabla_{ZE} X, Z^E \right) (D_X \varphi) |U| > \delta \left( |D_X \varphi| + \frac{|1 - \varphi (p)|}{\text{dist} \left( p_{BM}^1 (*), p \right)} \right).
\]

Since $\delta$ is fixed and we can choose $\delta'$ to be as close to $0$ as we please, it follows that $T \mathcal{S} \setminus U \in S T S_{|_{p_{BM}^1 (*, -*)}}$ is positively curved under our deformation.

On the other hand, we have assumed that $T \mathcal{S}$ is $\varepsilon$–nondegenerate with respect to $\mathcal{H}_{p_{BM}}$, so we can argue as in the proof of Theorem 2.5 to conclude that $(\Sigma, \tilde{g})$ is nonnegatively curved with precisely the same zero curvatures $(\Sigma, g)$.

Planes that are perpendicular to $X$ have the form $\text{span} \{z, W + \tau V\}$ and

\[
\text{curv} (z, W + \tau V) = \text{curv} (z, W) + 2\tau R(W, z, z, V) + \tau^2 \text{curv} (z, V).
\]
Planes that are perpendicular to \( W \) have the form \( \text{span} \{ X + \sigma z, V \} \) and
\[
\text{curv}(X + \sigma z, V) = \text{curv}(X, V) + 2\sigma R(X, V, V, z) + \sigma^2 \text{curv}(z, V)
\]
Similar arguments show that these curvatures are positive.

5. **Integrally Positive Curvature**

Here we give abstract criteria that are sufficient to create integrally positive curvature on a totally geodesic flat, when the fibers of a Riemannian submersion are scaled.

Throughout this section we use the set-up of section 2.1. Thus we have a Riemannian submersion

\[
p_{BM}: (M, g_0) \to B
\]

where \((M, g_0)\) has nonnegative sectional curvature. Let \( g_s \) be the metric obtained from \( g_0 \) by scaling the lengths of the fibers of \( p_{BM} \) by

\[
\sqrt{1 - s^2}.
\]

As usual we use the superscripts \( \mathcal{H} \) and \( \mathcal{V} \) to denote the horizontal and vertical parts of the vectors, \( R \) and \( A \) are the curvature and \( A \)-tensors for the unperturbed metric \( g \), \( R^{g_s} \) denotes the new curvature tensor of \( g_s \), and \( R^B \) is the curvature tensor of the base.

**Theorem 5.1.** Let \( T \subset M \) be a totally geodesic, flat torus spanned by commuting, orthogonal, geodesic fields \( X \) and \( W \) such that \( X \) is horizontal for \( p_{BM} \) and \( Dp_{BM}(W) \) is a Jacobi field along an integral curve \( \gamma \) of \( Dp_{BM}(X) \).

Then
\[
R^{g_s}(X, W, W, X) = -\frac{s^2}{2} \left( D_X \left( D_X |W^H|^2 \right) \right) + s^4 |A_X W^V|^2.
\]

In particular, if \( \gamma \) is a closed integral curve of \( X \), then
\[
\int_{\gamma} \text{curv}_{g_s}(X, W) = s^4 \int_{\gamma} |A_X W^V|^2.
\]

So the curvature of \( \text{span} \{ X, W \} \) is integrally positive along \( \gamma \), provided \( |A_X W^V|^2 \) is not identically 0 along \( \gamma \).

**Remark 5.2.** With additional hypotheses, listed as bullet items on page 27, we also obtain formulas for \( R^{g_s}(W, X, X) \) and \( (R^{g_s}(X, W) W)^H \) in Lemma 5.9 at the end of this section.

The reader should note that the above curvature formula is as important as the fact that the integral is positive. Since \( X \) is a geodesic field, the larger term
\[
-\frac{s^2}{2} \left( D_X \left( D_X |W^H|^2 \right) \right)
\]

is the Hessian, \( \text{Hess} f(X, X) \), of the function
\[
f = -\frac{s^2}{2} |W^H|^2.
\]

Therefore, we can cancel it with a conformal change involving \( f \). Such a conformal change will create other terms of order \( s^4 \) in our expression for \( \text{curv}_{g_s}(X, W) \). To compare these terms with \( s^4 |A_X W^V|^2 \), we will evaluate \( A_X W^V \) in the presence of the additional hypotheses listed as bullet items on page 27, after we prove the theorem above. These additional hypotheses will also allow us to obtain formulas
for the (1, 3)-tensor, \( R^g_\nu (W, X) X \) and the horizontal part of the (1, 3)-tensor, \( R^g_\nu (X, W) W \).

After refining our formula for \( \text{curv}_g (X, W) \), we will explain in the next section precisely how to combine fiber scaling and a conformal change to put positive curvature on a single initially flat torus, subject to the additional hypotheses mentioned in 2.1.

Scaling the fibers of a Riemannian submersion was dubbed the “canonical variation” in [Bes]. One can find formulas for how curvature changes under the canonical variation in any of [Bes], [Dear1], [GromDur], or [GromWals]. Since the particular “W” that we have in mind is neither horizontal nor vertical for \( p_{BM} \), we need multiple formulas just to find \( \text{curv}(X, W) \).

Given vertical vectors \( U, V \in \mathcal{V} \) and horizontal vectors \( X, Y, Z \in \mathcal{H} \), for \( p_{BM} : M \to B \) we have

\[
\begin{align*}
R^g_\nu (X, V) U & = (1 - s^2) (R(X, V) U) + (1 - s^2) s^2 A X U V \\
R^g_\nu (V, X) Y & = (1 - s^2) R(V, X) Y + s^2 (R(V, X) Y) V + s^2 A_X A Y V \\
R^g_\nu (X, Y) Z & = (1 - s^2) R(X, Y) Z + s^2 (R(X, Y) Z) V + s^2 R^B(X, Y) Z
\end{align*}
\]

Let \( \text{Lemma 5.4} \). And vertical parts and applying the formulas above we obtain the following.

**Lemma 5.4.** Let \( X \) be a horizontal vector for \( p_{BM} \) and let \( W \) be an arbitrary vector in \( TM \). Splitting \( W \) into horizontal and vertical parts and applying the formulas above we obtain the following.

\[
\begin{align*}
R^g_\nu (W, X) X & = (1 - s^2) R(W, X) X + s^2 (R(W, X) X) V \\
& \quad + s^2 R^B(W^H, X) X + s^2 A_X A_X W V \\
(R^g_\nu (X, W) W)^H & = (1 - s^2) (R(X, W) W)^H \\
& \quad + (1 - s^2) s^2 A_X W V + s^2 R^B(X, W^H) W^H
\end{align*}
\]

**Remark 5.5.** Notice that the first curvature terms vanish in both formulas on the totally geodesic flat tori.

**Proof of Theorem 5.1:** Using the fact that \( \text{curv}^g_\nu (X, W) = 0 \) and either of the formulas for \( R^g_\nu (W, X) X \) or \( R^g_\nu (X, W) W \) we have

\[
(5.5) \quad \text{curv}^g_\nu (X, W) = s^2 \text{curv}^B(X, W^H) - (1 - s^2) s^2 |A_X W V|^2.
\]

Since \( D_{p_{BM}} (W^H) \) is a Jacobi field along \( \gamma \), and writing \( W^H \) for \( D_{p_{BM}} (W^H) \) we have

\[
\begin{align*}
\text{curv}^B(X, W^H) & = - \langle \nabla^B_X \nabla^B_X W^H, W^H \rangle \\
& = - D_X \langle \nabla^B_X W^H, W^H \rangle + \langle \nabla^B_X W^H, \nabla^B_X W^H \rangle \\
& = - \frac{1}{2} D_X D_X \langle W^H, W^H \rangle + \langle \nabla^B_X W^H, \nabla^B_X W^H \rangle
\end{align*}
\]

Since \( \nabla_X W \equiv 0 \) we have

\[
\begin{align*}
0 & \equiv \nabla_X W \\
& = \nabla_X W^H + \nabla_X W^V.
\end{align*}
\]
The horizontal part of this equation gives us
\[ A_X W^V = -\left( \nabla_X W^\mathcal{H} \right)^\mathcal{H}. \]
Identifying \((\nabla_X W^\mathcal{H})^\mathcal{H}\) with \(\nabla_X^B W^\mathcal{H}\) and substituting into the formula for \(\text{curv}^B (X, W^\mathcal{H})\) we obtain
\[ \text{curv}^B (X, W^\mathcal{H}) = -\frac{1}{2} D_X D_X |W^\mathcal{H}|^2 + |A_X W^V|^2 \]
Substituting this into our formula for \(\text{curv}^g (X;W^\mathcal{H})\) yields
\[ \text{curv}^g (X;W^\mathcal{H}) = -s^2 \frac{1}{2} D_X D_X |W^\mathcal{H}|^2 + s^2 |A_X W^V|^2 - \left( 1 - s^2 \right) s^2 |A_X W^V|^2 \]
proving Theorem 5.1. \(\square\)

The additional assumptions from 2.1 that we use to help evaluate \(A_X W^V\) are the following.

- There is an isometric action by \(G_1\) on \(M\) that is by symmetries of \(p_{BM}\).
- The intrinsic metrics on the principal orbits of \(G_1\) in \(B\) are homotheties of each other.
- The normal distribution to the principle orbits of \(G_1\) on \(B\) is integrable.
- \(W^\mathcal{H}\) is a Killing field for the \(G_1\)-action on \(B\).
- \(Dp_{BM} (X)\) is invariant under the action that \(G_1\) induces on \(B\).
- \(Dp_{BM} (X)\) is orthogonal to the orbits of \(G_1\).

Lemma 5.6. With the additional hypotheses mentioned above
\[ A_X W^V = -\frac{D_X |W^\mathcal{H}|}{|W^\mathcal{H}|} W^\mathcal{H}. \]
and
\[ \text{curv}^g (X, W) = -s^2 \frac{1}{2} D_X D_X |W^\mathcal{H}|^2 + s^4 |D_X |W^\mathcal{H}||^2. \]

Proof. Let \(Z\) be a \(G_1\)-invariant field on \(B\) that is normal to the principal orbits of \(G_1\). Writing \(X\) for \(Dp_{BM} (X)\) and using the fact that normal distribution to the principal orbits of \(G_1\) on \(B\) is integrable it then follows that all terms in the Koszul formula for
\[ \langle \nabla^B_{W^\mathcal{H}X} Z \rangle \]
vanish. In particular, \(\nabla^B_{W^\mathcal{H}X}\) is tangent to the orbits of \(G_1\).

If \(K\) is another Killing field for \(G_1\), then \(X\) commutes with \(K\) as well as \(W^\mathcal{H}\), thus \([K, W^\mathcal{H}]\) is perpendicular to \(X\) as it is again a Killing field. Combining this with our hypothesis that the intrinsic metrics on the principal orbits of \(G_1\) in \(B\) are homotheties of each other, we see from Koszul’s formula that \(\nabla_{W^\mathcal{H}X}\) is proportional to \(W^\mathcal{H}\) and can be calculated by
\[ \langle \nabla_{W^\mathcal{H}X} W^\mathcal{H} \rangle = \langle \nabla_X W^\mathcal{H}, W^\mathcal{H} \rangle = \frac{1}{2} D_X |W^\mathcal{H}|^2 = |W^\mathcal{H}| D_X |W^\mathcal{H}|, \text{ so} \]
\[ \nabla_{W^\mathcal{H}X} = \frac{D_X |W^\mathcal{H}|}{|W^\mathcal{H}|} W^\mathcal{H}. \]
Since
\[ A_X W^V = - (\nabla_X W^H)^H \]
we conclude that with the additional hypotheses mentioned above
\[ A_X W^V = - \frac{D_X |W^H|}{|W^H|} W^H. \]
Plugging this into our curvature formula we get
\[ \text{curv}^{g_\ast} (X, W) = - s^2 \frac{1}{2} D_X D_X |W^H|^2 + s^4 |D_X |W^H||^2. \]
\[ \square \]

As we’ve mentioned, these hypotheses also give us certain other components of the (1,3) curvature tensor.

Lemma 5.7. Using \( W^H \) for \( D_{pBM} (W) \) and \( X \) for \( D_{pBM} (X) \)
\[ R^B (W^H, X) X = - \left( \frac{D_X D_X |W^H|}{|W^H|} \right) W^H \]
Proof. Since \( X \) is a geodesic field and \( W^H \) is a Jacobi field along the integral curves of \( X \)
\[ R^B (W^H, X) X = - \nabla_X \nabla_X W^H. \]
We discovered above that
\[ \nabla_X W^H = \nabla_{W^H} X = \frac{D_X |W^H|}{|W^H|} W^H. \]
Thus
\[
R^B (W^H, X) X = - \nabla_X \left( \frac{D_X |W^H|}{|W^H|} W^H \right) \\
= - D_X \left( \frac{D_X |W^H|}{|W^H|} \right) W^H - \left( \frac{D_X |W^H|}{|W^H|} \right) \nabla_X W^H \\
= - \left( \frac{|W^H| D_X D_X |W^H| - (D_X |W^H|)^2}{|W^H|^2} \right) W^H - \left( \frac{D_X |W^H|}{|W^H|} \right)^2 W^H \\
= - \left( \frac{D_X D_X |W^H|}{|W^H|} \right) W^H. \\
\square \]

Lemma 5.8. Using \( W^H \) for \( D_{pBM} (W) \) and \( X \) for \( D_{pBM} (X) \)
\[ R^B (X, W^H) W^H = - |W^H| \nabla_X (\text{grad} |W^H|). \]
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Proof. Let $Z$ be any vector field. Using that $W^H$ is a Killing field we get

$$\langle \nabla_{W^H} W^H, Z \rangle = -\langle \nabla_Z W^H, W^H \rangle$$

$$= -\frac{1}{2} D_Z \langle W^H, W^H \rangle$$

$$= -\frac{1}{2} D_Z |W^H|^2$$

$$= -|W^H| D_Z |W^H|$$

$$= -\langle |W^H| \text{ grad } |W^H|, Z \rangle$$

showing that

$$\nabla_{W^H} W^H = -|W^H| \text{ grad } |W^H|.$$ 

Thus

$$R^B (X, W^H) W^H = \nabla_X \nabla_{W^H} W^H - \nabla_{W^H} \nabla_X W^H$$

$$= -\nabla_X (|W^H| \text{ grad } |W^H|) - \nabla_{W^H} \left( \frac{D_X |W^H|}{|W^H|^2} W^H \right)$$

$$= -\left( D_X |W^H| \right) \text{ grad } |W^H| - \left( |W^H| \nabla_X \text{ grad } |W^H| \right) - \frac{D_X |W^H|}{|W^H|^2} \nabla_{W^H} W^H$$

$$= -\left( D_X |W^H| \right) \text{ grad } |W^H| - \left( |W^H| \nabla_X \text{ grad } |W^H| \right) + \frac{D_X |W^H|}{|W^H|^2} |W^H| \text{ grad } |W^H|$$

$$= -\left( |W^H| \nabla_X \text{ grad } |W^H| \right)$$

Combining the calculations above we have

Lemma 5.9. Let $X$ and $W$ be as in Theorem 5.1. Then

$$R^{B^2} (X, W) X = -s^2 \left( \frac{D_X D_X |W^H|}{|W^H|^2} \right) W^H - s^2 \frac{D_X |W^H|}{|W^H|^2} A_X W^H$$

$$R^{B^2} (X, W) W^H = -\left( 1 - s^2 \right) s^2 \frac{D_X |W^H|}{|W^H|^2} A_{W^H} W^V - s^2 |W^H| \nabla_X \left( \text{ grad } |W^H| \right).$$

Remark 5.10. The two $A$-tensors $A_X W^H$ and $A_{W^H} W^V$ involve derivatives of vectors that are not tangent or normal to the totally geodesic tori. They cannot be determined abstractly, and are in fact dependent on the particular geometry. We give estimates for them in the case of the Gromoll-Meyer sphere in the section of [PetWilh2] called “Concrete $A$-tensor estimates”.

6. Positive Curvature on a Single Initially Flat Torus

In this section we complete the proof of Theorem 2.1. To simplify notation we set

$$\psi = |W^H|,$$

and extend $\psi$ arbitrarily to $M$.

After scaling the fibers of $p_{BM}$ by $\sqrt{1 - s^2}$ we have from 5.6

$$(6.0) \text{ curv}_{B^2} (X, W) = -s^2 (D_X (\psi D_X \psi)) + s^4 (D_X \psi)^2.$$
We remind the reader that after the conformal change \( \tilde{g} = e^{2f} g_s \) we will have
\[
e^{-2f} \text{curv}_{\tilde{g}} (X, W) = \text{curv}_{g_s} (X, W) - |W|^2_{g_s} \text{Hess} f (X, X) - \text{Hess} f (W, W) + (D_X f)^2 |W|^2_{g_s},
\]
provided \( X \) is unit and \( W \) is perpendicular to \( \text{grad} f \) (cf [Pet] Exercise 3.5)

Our choice of conformal factor will look like
\[
f = \frac{s^2}{\psi^2} + \text{a much smaller term.}
\]

The first conformal term \( -|W|^2_{g_s} \text{Hess} f (X, X) \) will nearly cancel with the leading term \( -s^2 (D_X (\psi D_X \psi)) \) in \( \text{curv}_{g_s} (X, W) \). For our initial metric \( \nabla_W W = 0 \), so \( \text{Hess} f (W, W) \) has order \( s^4 \), as do the other two conformal terms, \( (D_X f)^2 |W|^2_{g_s} \)
and \( |\nabla f|^2 |W|^2_{g_s} \). In the remainder of this section we will see more precisely what these terms actually are.

To do this we name the “much smaller term”, \( E \). Along our torus \( T \), the function \( E \) and its gradient have size \( O (s^4) \). The gradient is proportional to \( X \). We call the proportionality factor \( s^4 I' \). For \( E \) to be well defined on \( T \) it is necessary that \( I' \) integrate to zero along closed integral curves for \( X \). We will put further constraint on the restriction of \( I \) to \( T \) as we proceed. Since we are only concerned with \( \text{curv}(X, W) \) we can extend \( I \) to \( M \) in any way that we like.

Thus
\[
\text{grad} f = -\frac{s^2}{(1-s^2) |W|^2} \psi \text{grad} \psi + s^4 I' X
\]

To understand the effect that this conformal change has on our curvatures we will need to know the Hessian of \( f \), and hence a covariant derivative that we have yet to compute.

**Proposition 6.1.**
\[
\nabla^g_W W = -s^2 \psi \text{grad} \psi,
\]

**Proof.** Before the fiber scaling \( \nabla_W W = 0 \). Breaking \( W \) into horizontal and vertical parts and using the Koszul formula we get
\[
\nabla^g_W W = \nabla^g_{W^\nu} W^\nu + \nabla^g_{W^\kappa} W^\kappa + \nabla^g_{W^\nu \kappa} W^\nu + \nabla^g_{W^\nu \kappa} W^\kappa
\]
\[
= (\nabla_{W^\nu} W^\nu)^\nu + (1-s^2) (\nabla_{W^\nu} W^\nu)^\kappa + (\nabla_{W^\kappa} W^\nu)^\nu + (1-s^2) (\nabla_{W^\kappa} W^\nu)^\kappa + (\nabla_{W^\nu \kappa} W^\nu)^\nu + (1-s^2) (\nabla_{W^\nu \kappa} W^\nu)^\kappa + \nabla_{W^\nu \kappa} W^\nu
\]

Rearranging terms and using the fact that \( \nabla_W W = 0 \) yields
\[
\nabla^g_W W = -s^2 \left( (\nabla_{W^\nu} W^\nu) + \nabla_{W^\nu \kappa} W^\kappa + \nabla_{W^\nu \kappa} W^\nu \right)^\kappa.
\]

We also have \( (\nabla_W W)^\kappa = 0 \), so
\[
\left( \nabla_{W^\nu} W^\nu + \nabla_{W^\nu \kappa} W^\kappa + \nabla_{W^\nu \kappa} W^\nu + \nabla_{W^\nu \kappa} W^\kappa \right)^\kappa = 0.
\]
Thus

\[ \nabla^g_w W = -s^2 \left[ (\nabla^g W^\nabla) + \nabla^g W^\nabla + \nabla^g W^\nabla \right] \]

\[ = s^2 \nabla^g W^\nabla \]

\[ = -s^2 |W^\nabla| \text{grad} |W^\nabla| \]

\[ = -s^2 \psi \text{grad} \psi, \]

where we have used an equation in the proof of Lemma 5.8 for the next to last inequality.

\[ \square \]

**Proposition 6.2.**

\[
\text{Hess} f (X, X) = - \frac{s^2}{(1-s^2)|W|^2} D_X (\psi D_X \psi) + s^4 I''
\]

\[
\text{Hess} f (W, W) = - \frac{s^4}{(1-s^2)|W|^2} \psi^2 |\text{grad} \psi|^2 + O (s^6)
\]

**Proof.** Since

\[ \text{grad} f = - \frac{s^2}{(1-s^2)|W|^2} \psi \text{grad} \psi + s^4 I' X \]

we have

\[
\text{Hess} f (X, X) = - \frac{s^2}{(1-s^2)|W|^2} \langle \nabla_X (\psi \text{grad} \psi), X \rangle + s^4 \langle \nabla_X (I' X), X \rangle
\]

\[ = - \frac{s^2}{(1-s^2)|W|^2} \left( (D_X \psi)^2 + \psi \langle \nabla_X (\text{grad} \psi), X \rangle \right) + s^4 I''
\]

\[ = - \frac{s^2}{(1-s^2)|W|^2} \left( (D_X \psi)^2 + \psi D_X D_X \psi \right) + s^4 I''
\]

\[ = - \frac{s^2}{(1-s^2)|W|^2} D_X (\psi D_X \psi) + s^4 I''
\]

Since \( W \) is perpendicular to \( \text{grad} f \) we have

\[
\text{Hess} f (W, W) = \langle \nabla_W \text{grad} f, W \rangle
\]

\[ = - \langle \text{grad} f, \nabla_W W \rangle
\]

Using the previous proposition this gives us

\[
\text{Hess} f (W, W) = - \left( - \frac{s^2}{(1-s^2)|W|^2} \psi \text{grad} \psi, -s^2 \psi \text{grad} \psi \right) - s^4 \langle I' X, -s^2 \psi \text{grad} \psi \rangle
\]

\[ = - \frac{s^4}{(1-s^2)|W|^2} \psi^2 |\text{grad} \psi|^2 + O (s^6)
\]

\[ \square \]

**Proposition 6.3.** After fiber scaling and the conformal change we have

\[
e^{-2f \text{curv} (X, W)} = s^4 \left( D_X \psi \right)^2 + s^4 \frac{\psi^2}{|W|^2} \left( D_X \psi \right)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 \left( I'' \right) |W|^2 + O (s^6)
\]
Remark 6.4. We pick \( I'' \) so that the first four terms are \( O(s^4) \). The third can have either sign, and since the integral of \( I'' \) over an integral curve of \( X \) is 0, the term \( s^4 I'' |W|_g^2 \) also has both signs. After proving the proposition we will argue that the integral \( e^{-2f} \text{curv} (X, W) \) is positive, and hence that an appropriate choice of \( I'' \) will give us pointwise positive curvature.

Proof. Combining \( |X| = 1 \), equation 6.0, the formula for the curvature of a conformal change ([Pet], exercise 3.5), and the fact that \( W \) is perpendicular to \( \text{grad} f \) we have

\[
e^{-2f} \text{curv}_g (X, W) = -s^2 (D_X (\psi D_X \psi)) + s^4 (D_X \psi)^2 - |W|_g^2 \text{Hess} f (X, X) - \text{Hess} f (W, W) + (D_X f)^2 |W|_g^2 - |\text{grad} f|^2 |W|_g^2 \cdot
\]

To evaluate this we will need

\[
|W|_g^2 = (1 - s^2) |W'|^2 + |W'\gamma|^2 = |W|^2 - s^2 |W'|^2 = |W|^2 - s^2 (|W|^2 - |W'\gamma|^2)
\]

\[
= (1 - s^2) |W|^2 + s^2 |W'\gamma|^2 = (1 - s^2) |W|^2 + s^2 \psi^2.
\]

Combining this with the previous proposition we see that the sum of the first and third term is

\[
-s^2 (D_X (\psi D_X \psi)) - |W|_g^2 \text{Hess} f (X, X)
\]

\[
= -s^2 (D_X (\psi D_X \psi)) + (1 - s^2) |W|^2 \frac{s^2}{(1 - s^2) |W|^2} D_X (\psi D_X \psi)
\]

\[
+ s^2 \psi^2 \frac{s^2}{(1 - s^2) |W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|_g^2
\]

\[
= s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6).
\]

The sum of the fourth and last terms is

\[
-\text{Hess} f (W, W) - |\text{grad} f|^2 |W|_g^2
\]

\[
= \frac{s^4}{(1 - s^2) |W|^2} \psi^2 |\text{grad} \psi|^2 - \frac{s^4}{(1 - s^2) |W|^4} |\text{grad} \psi|^2 |W|^2 + O(s^6)
\]

\[
= O(s^6).
\]

The fifth term of our curvature formula is

\[
(D_X f)^2 |W|_g^2 = s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + O(s^6).
\]

Combining equations we obtain

\[
e^{-2f} \text{curv} (X, W) = s^4 (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O(s^6)
\]

as desired. \( \square \)
To understand the sign of the above formula we will need to understand some relationships between the integrals of the first three terms.

**Proposition 6.5.** Let \( \gamma : [0, l] \to M \) be a closed integral curve of \( X \). Then using \( \psi' \) for \( DX \psi \)

\[
\int_\gamma \psi^2 (\psi')^2 \, dt = -\frac{1}{3} \int_\gamma \psi^2 (\psi\psi'') \, dt
\]

\[
\int_\gamma \psi^2 (\psi\psi')' \, dt = -2 \int_\gamma \psi^2 (\psi')^2 \, dt
\]

**Proof.** The first equation follows from integration by parts

\[
\int_\gamma \psi^2 (\psi')^2 \, dt = \int_\gamma \psi' (\psi^2 \psi') \, dt
\]

\[
= \psi' \frac{1}{3} \psi^3 \bigg|_0^l - \int_\gamma \psi'' \frac{1}{3} \psi^3 \, dt
\]

\[
= -\frac{1}{3} \int_\gamma \psi'' \psi^3 \, dt
\]

So

\[
\int_\gamma \psi^2 (\psi\psi')' \, dt = \int_\gamma \psi^2 \left\{ (\psi')^2 + \psi\psi'' \right\} \, dt
\]

\[
= \int_\gamma \psi^2 \left\{ (\psi')^2 - 3 (\psi')^2 \right\} \, dt
\]

\[
= -2 \int_\gamma \psi^2 (\psi')^2 \, dt
\]

\[
\square
\]

**Proof of Theorem 2.1:** Using the second equation of the previous proposition we can re-write the integral of our curvature over \( \gamma \) as

\[
\int_\gamma e^{-I} \text{curv} (X, W) = \int_\gamma s^4 (DX \psi)^2 + s^4 \frac{\psi^2}{|W|^2} (DX \psi)^2 + s^4 \frac{\psi^2}{|W|^2} DX (\psi DX \psi) - s^4 I'' |W|^2 + O(s^6)
\]

\[
= \int_\gamma s^4 (DX \psi)^2 - s^4 \frac{\psi^2}{|W|^2} (DX \psi)^2 - s^4 I'' |W|^2 + O(s^6).
\]

Since \( \psi^2 = |W|^2 \) we always have

\[
\frac{\psi^2}{|W|^2} \leq 1
\]

and we assumed that it is not always 1. It follows that the integral

\[
\int_\gamma s^4 (DX \psi)^2 - s^4 \frac{\psi^2}{|W|^2} (DX \psi)^2 \geq O(s^4) > 0
\]

Since \( \gamma \) is closed and \( I \) is smooth

\[
\int_\gamma I'' = 0.
\]
so we also have
\[ \int_{\gamma} e^{-2f} \text{curv}(X,W) > O\left(s^4\right) > 0. \]

However, the quantity
\[ s^4(D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X(\psi D_X\psi) \]
can have some negative values, but by choosing \( I'' \) to be sufficiently negative in the region where
\[ s^4(D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X(\psi D_X\psi) < 0 \]
we can make \( e^{-2f} \text{curv}(X,W) \) positive in this region. We will have to pay for this by having \( I'' \) be nonnegative on the rest of \([0, l] \). Since
\[ \int_{\gamma} s^4(D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X(\psi D_X\psi) > 0 \]
this can be achieved while keeping \( e^{-2f} \text{curv}(X,W) > 0 \) point wise. □

7. Long Term Cheeger Principle

In the presence of a group of isometries, \( G \), a method for perturbing the metric on a manifold, \( M \), of nonnegative sectional curvature is proposed in [Cheeg]. Various special cases of this method were first studied in [Berg3] and [BourDesSent]. An exposition can be found in [Muet]. Although this technique has been used repeatedly in the literature, our impression is that it is not widely understood.

To understand the effect of a Cheeger deformation on the curvature of a nonnegatively curved manifold, in our view, it is crucial to exploit the “Cheeger reparametrization” of the Grassmannian. We will review the definition of the Cheeger reparametrization below. For now we recall (see e.g. [PetWilh1])

**Proposition 7.1.** Let \((M, g_{\text{Cheeg}})\) be a Cheeger deformation by \( G \) of the nonnegatively curved manifold \((M,g)\). Then modulo the Cheeger reparametrization,

1: If a plane \( P \) is positively curved with respect to \( g \), then it is positively curved with respect to \( g_{\text{Cheeg}} \).

2: If a plane \( P \) has a nondegenerate projection onto the orbits of \( G \) and “corresponds” to a positively curved plane in \( G \), then \( P \) is positively curved with respect to a Cheeger deformed metric, provided the Cheeger deformation is “run for a sufficiently long time”.

**Proposition 7.2.** Let \((M, g_{\text{Cheeg}})\) be a Cheeger deformation by \( G \) of \((M,g)\). Then modulo the Cheeger reparametrization,

1: If a plane \( P \) is positively curved with respect to \( g \), then it is positively curved with respect to \( g_{\text{Cheeg}} \).

2: If a plane \( P \) has a nondegenerate projection onto the orbits of \( G \) and “corresponds” to a positively curved plane in \( G \), then \( P \) is positively curved with respect to a Cheeger deformed metric, provided the Cheeger deformation is “run for a sufficiently long time”.
The meaning of “run for a sufficiently long time” will also be explained below.

To explain these results we offer a review that is sufficient for our purposes. None of this review is original, and in fact some of it is copied verbatim from [PetWilh1], whose main contribution to the theory of Cheeger deformations is expository.

If \( G \) is a compact group of isometries of \( M \), then we let \( G \) act on \( G \times M \) by

\[
(7.2) \quad g(p, m) = (pg^{-1}, gm).
\]

If we endow \( G \) with a biinvariant metric and \( G \times M \) with the product metric, then the quotient of (7.2) is a new metric on \( M \). It was observed in [Cheeg], that in a certain sense we may expect the new metric to have more curvature and less symmetry than the original metric. The “sense” in which this is true is modulo the Cheeger reparametrization.

The quotient map for the action (7.2) is

\[
q_{G \times M} : (g, m) \mapsto gm.
\]

The vertical space for \( q_{G \times M} \) at \( (g, m) \) is

\[
V_{q_{G \times M}} = \{ (-k, k) \mid k \in \mathfrak{g} \}
\]

where the \(-k\) in the first factor stands for the value at \( g \) of the Killing field on \( G \) given by the circle action

\[
(\exp(ak), g) \mapsto g \exp(-kt)
\]

and the \( k \) in the second factor is the value of the Killing field

\[
m \mapsto \frac{d}{dt} \exp(\tau k)m
\]
on \( M \) at \( m \).

We recall from [Cheeg], [PetWilh1] that there is a reparametrization of the tangent space, that we will call the Cheeger reparameterization. It is given by

\[
v \mapsto Dq_{G \times M} (\dot{v})
\]

where

\[
\dot{v} \equiv (k_v, v)
\]
is the vector tangent to \( G \times M \) that is horizontal for \( q_{G \times M} : G \times M \rightarrow M \), and projects to \( v \) under \( \pi_2 : G \times M \rightarrow M \).

From now on we will assume that the metric on the \( G \)-factor in \( G \times M \) is biinvariant. This means that we have only a one parameter family \( (M, g_l)_{l \in \mathbb{R}} \) of Cheeger deformed metrics, where \( l \) denotes the scale of the biinvariant metric in \( G \times M \). As \( l \rightarrow \infty \), \( (M, g_l) \) converges to the metric on the \( M \)-factor in \( G \times M \), so we will often call the original metric \( g_{\infty} \) [Pet].

With an understanding of the Cheeger re-parameterization the proof of Proposition 7.1 is now clear. \( Dq_{G \times M} (\tilde{P}) \) is positively curved if \( \tilde{P} \) is positively curved, and \( \tilde{P} \) is positively curved if its projection onto either \( M \) or \( G \) is positively curved. Since the projection onto \( M \) is \( P \), we get the conclusion of Proposition 7.1.

The proof of Proposition 7.2 is only a little harder. If \( P \) happens to be positively curved, then so is \( \tilde{P} \) and hence also \( Dq_{G \times M} (\tilde{P}) \).
On the other hand, if
\[ \dot{P} = \text{span} \{ \dot{v}, \dot{w} \} \]
when \( l = 1 \), then for arbitrary \( l \),
\[ \dot{P} = \text{span} \left\{ \left( \frac{k_v}{l^2}, v \right), \left( \frac{k_w}{l^2}, w \right) \right\} . \]
So
\[
\text{curv}_{(M,g_l)} \left( D_{g_M} \left( \frac{k_v}{l^2}, v \right), D_{g_M} \left( \frac{k_w}{l^2}, w \right) \right) \geq \text{curv}_{G,l} \left( \frac{k_v}{l^2}, \frac{k_w}{l^2} \right) + \text{curv}_M (v, w)
\]
where \( \text{curv}_{G,l} \) stands for the curvature with respect to the biinvariant metric with scale \( l \), and \( \text{curv}_{G,1} \) stands for the curvature with respect to the biinvariant metric with scale 1. Thus if \( \text{curv}_{G,1} (k_v, k_w) \) happens to be positive, then the term \( \frac{1}{l^6} \text{curv}_{G,1} (k_v, k_w) \) will dominate the term \( \text{curv}_M (v, w) \), when \( l \) is sufficiently small, and we conclude that
\[
\text{curv}_{(M,g_l)} \left( D_{g_M} \left( \frac{k_v}{l^2}, v \right), D_{g_M} \left( \frac{k_w}{l^2}, w \right) \right) > 0.
\]

The utility of using the Cheeger reparametrization is undeniable. As we have seen, it provides a simple way to track changes of curvature. It also preserves horizontal spaces of Riemannian submersions, [PetWilh1].

**Proposition 7.3.** Let \( A_H : H \times M \rightarrow M \) be an action that is by isometries with respect to both \( g_\infty \) and \( g_l \). Let \( \mathcal{H}_{A_H} \) denote the distribution of vectors that are perpendicular to the orbits of \( A_H \).

Then \( u \) is in \( \mathcal{H}_{A_H} \) with respect to \( g_\infty \) if and only if \( D_{g_M} (\dot{u}) \) is in \( \mathcal{H}_{A_H} \) with respect to \( g_1 \). In fact,
\[
g_\infty (u, w) = g_1 (u, D_{g_M} (\dot{w}))
\]
for all \( u, w \in TM \).

**Proof.** Starting with the left and side we take the horizontal lifts to \( G \times M \)
\[
g_1 (u, D_{g_M} (\dot{w})) = g_{G \times M} \left( (0, u) - (0, u)^V, \dot{w} \right)
\]
Since \( \dot{w} \) is horizontal this becomes
\[
g_1 (u, D_{g_M} (\dot{w})) = g_{G \times M} ((0, u), \dot{w}) = g_\infty (u, w).
\]

\[ \square \]

7.1. **Quadratic Nondegeneracy and Cheeger Deformations.** Cheeger deformations and the Cheeger reparametrization also play a role in verifying the Quadratic Nondegeneracy Condition.

**Definition 7.4.** We say that a four tuple of vectors \( \{ X, Z, W, V \} \) is \( \varepsilon \)-nondegenerate provided for all \( (\sigma, \tau) \) with \( \sigma^2 + \tau^2 = 1 \) the total quadratic term of \( P(\sigma, \tau) = \text{curv} (X + \sigma Z, W + \tau V) \) is \( > \varepsilon \). That is
\[
\frac{\sigma^2}{|Z|^2} \text{curv} (Z, W) + 2 \frac{\sigma \tau}{|X||W||V||Z|} (R(X, W, V, Z) + R(X, V, W, Z)) + \frac{\tau^2}{|X|^2|V|^2} \text{curv} (X, V) > \varepsilon.
\]
for all \((\sigma, \tau)\) with \(\sigma^2 + \tau^2 = 1\).

We use \(X_{W_{(\theta)}}^{\theta}, W_{(\theta)}, V_{(\theta)}, \) and \(Z_{(\theta)}^{\theta}\) for \(D_{G \times M}(\hat{X})\), \(D_{G \times M}(\hat{Z})\), \(D_{G \times M}(\hat{V})\), and \(D_{G \times M}(\hat{W})\).

Proposition 7.5. Let \((M, g_{\infty})\) be nonnegatively curved and \(q_{G \times M} : G \times (M, g_{\infty}) \to (M, g_1)\) a be Cheeger submersion.

Suppose that \(X\) is orthogonal to the orbits of \(G\) on \((M, g_{\infty})\), and

\[
\text{span} \{X_{\text{Ch}}, W_{\text{Ch}}\}
\]

is one of the zero curvature planes of \((M, g_{\infty})\). Then for \(l \in (0, 1)\) the four tuple \(\{W_{\text{Ch}}, Z_{\text{Ch}}, W_{\text{Ch}}, V_{\text{Ch}}\}\) is \(\varepsilon\)-nondegenerate with respect to \(g_{\infty}\) if either

1. \(\{X, Z, W, V\}\) is \(\frac{\varepsilon}{l}\)-nondegenerate with respect to \(g_{\infty}\), where \(l \equiv \frac{1}{\sqrt{1 + \tau^2}}\), or
2. When \(l = 1\), \(\text{curv}^G(Z_{\text{Ch}}, W_{\text{Ch}}) > \varepsilon\).

Proof: As usual we compute \(R^g(x, y, z, w)\) by lifting the four vectors to \(G \times M\) and symbolically think of \(R^g\) as being decomposed into

\[
R^g = R^{g_{\infty}} + R^G + R^A,
\]

where \(R^M\) and \(R^G\) are the contributions that come from projecting the lifted vectors onto the \(M\) and \(G\) factors and \(R^A\) is the contribution that comes from the \(A\)-tensor via O’Neill’s Horizontal Curvature Equation.

It turns out that each piece of the decomposition, \(R^g = R^{g_{\infty}} + R^G + R^A\), is nonnegative on our total quadratic term. To see this just combine the facts that

- **A:** \((M, g_1)\) is nonnegatively curved,
  - \(\text{curv}^G\left\{D_{G \times M}(\hat{X}), D_{G \times M}(\hat{W})\right\} = 0\)

- **B:** \((M, g_{\infty})\) is nonnegatively curved
  - \(\text{curv}^{g_{\infty}}\{X, W\} = 0\)

- **C:** \(G\) is nonnegatively curved
  - \(\text{curv}^G\left\{dp_G(\hat{X}), dp_G(\hat{W})\right\} = 0\).

Note that \(A, B, C\), and \(i\) hold by hypothesis, and \(ii\) and \(iii\) are consequences of \(i\).

Combining \(B\) and \(ii\) shows that \(R^{g_{\infty}}\) is nonnegative on our total quadratic term. Indeed, if it were not, then the fact that \(R^{g_{\infty}}(X, W) W = R^{g_{\infty}}(W, X) X = 0\) would imply that \((M, g_{\infty})\) is not nonnegatively curved. Similarly, combining \(C\) and \(iii\) give us that \(R^G\) is nonnegative on our total quadratic term.

To see that \(R^A\) is nonnegative on the total quadratic term, we use \(i\) to conclude that \(A^{g_{\infty}}(W) = 0\). Therefore, writing \(A\) for \(A^{g_{\infty}}\) and omitting the hats,

\[
0 \leq \langle A_{X + \sigma Z}(W + \tau V), A_{X + \sigma Z}(W + \tau V) \rangle
= \tau^2 \langle A_X V, A_X V \rangle + \sigma^2 \langle A_X V, A_Z W \rangle + \sigma \tau \langle A_X V, A_Z V \rangle + \sigma \tau \langle A_Z W, A_X V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle
+ \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle
= \tau^2 \langle A_X V, A_X V \rangle + 2\sigma \tau \langle A_X V, A_Z W \rangle + \sigma^2 \langle A_Z W, A_Z W \rangle
+ 2\sigma \tau \langle A_X V, A_Z W \rangle + 2\sigma \tau \langle A_X V, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle
= |\tau A_X V + \sigma A_Z W|^2 + 2\sigma \tau \langle A_X V, A_Z W \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle + \sigma^2 \tau \langle A_Z W, A_Z V \rangle
\]
So if all four vectors are unit, then $R^A$ is $\left| \tau A_X \vec{V} + \sigma A_Z \vec{W} \right|^2$ on the total quadratic term and hence is also nonnegative.

Since each of $R^g$, $R^G$, $R^A$ is nonnegative on our total quadratic term and $R^g = R^g + R^G + R^A$, we have $\varepsilon$–nondegeneracy on $\{X_{Ch}, Z_{Ch}, W_{Ch}, V_{Ch}\}$ if any one of $R^g$, $R^G$, $R^A$ is $\varepsilon$–nondegenerate on our 4–tuple.

We have

$$|W_{Ch}|^2 \leq |W|^2 \left( \frac{1}{l^2} + 1 \right) = |W|^2 \left( \frac{1 + l^2}{l^2} \right)$$

$$|W_{Ch}| \leq |W| \left( \frac{\sqrt{1 + l^2}}{l} \right), \text{ so}$$

$$\frac{1}{|W_{Ch}|} \geq \frac{1}{|W|} \tilde{l}, \text{ where } \tilde{l} = \frac{l}{\sqrt{1 + l^2}},$$

and a similar inequality holds for $Z$ and $V$. So if 1 holds, then on the four tuple $\{X_{Ch}, W_{Ch}, V_{Ch}, Z_{Ch}\}$,

$$\sigma^2 \text{curv}^g (Z_{Ch}, W_{Ch}) + 2\sigma \tau (R^g (X_{Ch}, W_{Ch}, V_{Ch}, Z_{Ch}) + R^g (X_{Ch}, V_{Ch}, W_{Ch}, Z_{Ch})) + \tau^2 \text{curv}^g (X_{Ch}, V_{Ch})$$

$$\geq \tilde{l}^2 (\sigma^2 \text{curv}^g (Z, W) + 2\sigma \tau (R^g (X, W, V, Z) + R^g (X, V, W, Z)) + \tau^2 \text{curv}^g (X, V)) > \varepsilon.$$

So $\{X_{Ch}, W_{Ch}, V_{Ch}, Z_{Ch}\}$ is $\varepsilon$–nondegenerate if 1 holds.

Since $X$ is perpendicular to the orbits, the portion of the total quadratic term that comes from $R^G$ is $\text{curv}^G (Z_{Ch}, W_{Ch})$, which is assumed to be $> \varepsilon$ when $l = 1$. This implies that in general

$$\text{curv}^G (Z_{Ch}, W_{Ch}) \geq \frac{\varepsilon}{\tilde{l}^6}$$

and hence we have $\varepsilon$–nondegeneracy if 2 holds and $l \in (0, 1)$.

8. Curvature Compression Principle

Proof of Lemma 2.7

We have

$$\psi_\infty = \left| W^{\mathcal{H}, g_{\infty}} \right|_{g_{\infty}} = \frac{\langle W, W^{\mathcal{H}, g_{\infty}} \rangle}{|W^{\mathcal{H}, g_{\infty}}|} = \psi_\infty |W^{\mathcal{H}, g_{\infty}}| = \psi_\infty^2$$

Because of the formula

$$g_{\infty} (u, w) = g_{\nu, l} (u, Dq_{G \times M} (\tilde{w})), $$
$Dq_{G\times M}$ is horizontal with respect to $g_{\nu,l}$. So setting $G_{\nu,l} \equiv (G_1, \nu_{bi}) \times (G_2, \nu_{bi})$

$$\psi_{\nu,l} = \left| \frac{W_{\nu,1}}{G_{\nu,l}\times M} \right|$$

$$= \left| \frac{\langle W, Dq_{G\times M} \left( \frac{\hat{W}_{\nu,1}}{G_{\nu,l}\times M} \right) \rangle_{g_{\nu,l}}}{\hat{W}_{\nu,1} / G_{\nu,l}\times M} \right|$$

$$= \frac{1}{\hat{W}_{\nu,1} / G_{\nu,l}\times M} \langle W, \hat{W} / G_{\nu,l}\times M \rangle_{g_{\nu,l}}$$

where we use the general formula

$$g_{\nu,l}(u, w) = g_{\nu,l}(u, Dq_{G\times M} (\hat{w}))$$

for the next to last equation.

We have

$$\hat{W} = \left( \frac{\langle W, W_{\nu,1} \rangle}{\nu^2}, \frac{\langle W, K_{W_{\nu,1}} \rangle}{\nu^2}, \frac{\langle W, K_{W_{\nu,1}}^2 \rangle}{\nu^2}, \frac{\langle W, W_{\nu,1} \rangle}{\nu^2}, \frac{\langle W, K_{W_{\nu,1}} \rangle}{\nu^2}, \frac{\langle W, K_{W_{\nu,1}}^2 \rangle}{\nu^2} \right)$$

This gives us

$$\left| \hat{W} \right|^2 = \rho^2 \langle W, W_{\nu,1} \rangle^2 + \frac{\langle K_{W_{\nu,1}} \rangle^2}{\nu^2} + \frac{\langle K_{W_{\nu,1}}^2 \rangle^2}{\nu^2} + \left| W_{\nu,1} \right|^2$$

and hence

$$\psi_{\nu,l}^2 = \frac{\psi_{\nu,l}^4}{\rho^2 \psi_{\nu,l}^4 + \psi_{\nu,l}^2 + \psi_{\nu,l}^2}$$

Straightforward calculation gives us formal derivatives of $\psi_{\nu,l}$ in some unspecified direction

**Proposition 8.1.**

$$\psi_{\nu,l} = \left( \psi_{\nu,l} - \frac{2\nu^3}{\rho^2} \left( \frac{\psi_{\nu,l}}{\psi_{\nu,l}} \right)^3 \right) \frac{\psi_{\nu,l}^3}{\psi_{\nu,l}^3}$$
\[
\psi_{\nu,l}'' = \left( \psi_{\nu,l}' - \frac{6\varphi_{\infty}^2 \varphi_{\infty}'}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' - \frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)'' \right) \psi_{\nu,l}^3
\]

\[
-3 \frac{\psi_{\nu,l}'}{t^2} \left( \psi_{\infty}' - \frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' \right) \psi_{\nu,l}^3
\]

\[
+3 \left( \psi_{\nu,l}' - \frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' \right)^2 \psi_{\nu,l}^5
\]

**Proof of Proposition 2.8** Setting \( D^2 = \rho^2 \frac{\psi_{\infty}^2}{t^2} + \frac{\varphi_{\infty}^4}{t^2 \psi_{\infty}} + 1 \) we have

\[
\frac{\psi_{\nu,l}'}{\psi_{\infty}^2} = \frac{1}{D^2}
\]

\[
= \frac{\nu^2}{\rho^2 \psi_{\infty}^2 + \frac{\varphi_{\infty}^4}{t^2 \psi_{\infty}} + \nu^2}
\]

So

\[
(\psi_{\nu,l}')^2 = \left( \psi_{\infty}' - \frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' \right)^2 \frac{1}{D^6}
\]

On \([0, \nu]\) our hypotheses imply that \( \frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' \) is much smaller than \( \psi_{\infty}' \); so on \([0, \nu]\) we have

\[
(\psi_{\nu,l}')^2 \geq \frac{99}{100} (\psi_{\infty}')^2 \frac{1}{D^6}
\]

\[
\geq \frac{98}{100} \frac{\nu^2}{\rho^2 \psi_{\infty}^2 + 1 + \nu^2}
\]

\[
\geq \frac{97}{100} \frac{1}{\rho^2 \psi_{\infty}^2 + \nu^2 + 1}
\]

\[
\geq \frac{97}{100} \frac{\nu^2}{\rho^2 \psi_{\infty}^2 + \nu^2 + 1}
\]

\[
\geq \frac{97}{100} \frac{1}{\rho^2 + 1}
\]

We get the upper estimate on \([\nu^2, \nu]\) by using

\[
\psi_{\nu,l}' \leq 1,
\]

\[
\frac{2\varphi_{\infty}^3}{t^2} \left( \frac{\varphi_{\infty}}{\psi_{\infty}} \right)' = O \left( \frac{\psi_{\infty}^3}{t^2} \right)
\]
and

\[
\frac{1}{D^6} = \left( \frac{\nu^2}{\rho^2 \psi_\infty^2 + \frac{\nu^2}{\psi_\infty^2} + \nu^2} \right)^3 \\
\leq O \left( \frac{\nu^2}{\rho^2 \psi_\infty^2 + \psi_\infty^2 + \nu^2} \right)^3 \\
\leq O \left( \frac{\nu^2}{\rho^2 \nu^2 + \nu^2} \right)^3 \\
\leq \left( \frac{\nu^2}{\rho^2 \nu^2} \right)^3 \\
\leq \frac{1}{\rho^2} \nu^{6(1-\beta)}
\]

So

\[
\left( \psi_{\nu,1} \right)^2 \bigg|_{[\nu^3, \frac{\nu^3}{4}]} \leq O \left( \frac{1}{\rho^2} \nu^{6(1-\beta)} \right) \\
= O \left( \frac{1}{\rho^2} \nu^{6(1-\beta)-\frac{3}{2}} \right) \\
= O \left( \nu^{\frac{3}{4} - 6\beta} \right) \square
\]

The results in the remainder of this section will be used in [PetWilh2], but not in this paper.

**Lemma 8.2.** In addition to the assumptions above suppose that

\[
\text{curv}^M (X, W^{H, g_\infty}) \geq C_1 \psi_\infty^2
\]

for some positive constant \(C_1\).

Let \(\gamma : [0, \pi/4] \rightarrow M\) be as above, then for any \(\beta > 0\)

\[-(D_X (\psi_{\nu,1} D_X \psi_{\nu,1})) > 0,
\]

provided \(t \geq \frac{\nu}{\sqrt{3}\rho} + \beta \nu\), and \(\nu\) is sufficiently small.

**Proof.** Because the second derivative of \(\psi_{\nu,1}\) is so complicated, we divide the proof of the first inequality into the case where \(t \geq O (\nu^{1/2})\) and the case where \(t \leq \)
Since $\psi_\infty \equiv \left| W^{g,\infty} \right|_{g_\infty}$ and $\psi_{\nu,l} \equiv \left| W^{g,\nu,l} \right|_{g_{\nu,l}}$ we have using Proposition 5.7

\[
-\psi_{\nu,l} \psi''_{\nu,l} = \operatorname{curv}^B (X, W^{g,\nu,l}) \\
\geq \operatorname{curv}^M (X, W^{g,\nu,l}) \\
\geq \psi_{\nu,l}^2 \operatorname{curv}^M \left( X, \frac{W^{g,\nu,l}}{W^{g,\nu,l}} \right) \\
\geq \frac{\psi_{\nu,l}^2}{\left| \frac{W^{g,\nu,l}}{W^{g,\nu,l}} \right|_{g_{\nu,l} \times M}} C_1 \psi_\infty^2 \\

\left| \frac{W^{g,\nu,l}}{W^{g,\nu,l}} \right|^2 = \psi_\infty^2 D^2 \quad \text{and} \quad \psi_{\nu,l}^2 = \frac{\psi_\infty^2}{D^2} \quad \text{we have}
\]

\[
-\psi_{\nu,l} \psi''_{\nu,l} \geq \frac{\psi_\infty^2}{D^2} \frac{1}{\psi_\infty^2 D^2} C_1 \psi_\infty^2 \\
\geq \frac{\psi_\infty^2}{D^4}
\]

and

\[
\left( \psi_{\nu,l}' \right)^2 = \left( \psi_\infty' - \frac{2 \varphi_\infty^3}{t^2} \left( \frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \\
\]

So it would be enough to prove

\[
\left( \psi_\infty' - \frac{2 \varphi_\infty^3}{t^2} \left( \frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \leq C_1 \psi_\infty^2 D^2 \\

\]

For $t \geq O \left( \nu^{1/2} \right)$

\[
\left( \psi_\infty' - \frac{2 \varphi_\infty^3}{t^2} \left( \frac{\varphi_\infty}{\psi_\infty} \right)' \right)^2 \leq \left( \psi_\infty' \right)^2 + O \left( \frac{t^3}{t^2} \right) + O \left( \frac{t^6}{t^2} \right)
\]

and since $D^2 = \rho^2 \frac{\psi_\infty^2}{t^2} + \frac{\varphi_\infty^4}{t^2 \psi_\infty^2} + 1,$

\[
C_2 t^2 \left( 1 + \rho^2 \frac{t^2}{\nu^2} \right) \leq C_1 \psi_\infty^2 D^2, \\
\]

for another positive constant $C_2.$ So the desired inequality would follow from

\[
\left( \psi_\infty' \right)^2 \leq C_2 t^2 \left( 1 + \rho^2 \frac{t^2}{\nu^2} \right)
\]
or

\[ 1 \leq O \left( \frac{t^4}{\nu^2} \right) \]

or

\[ t \geq O \left( \nu^{1/2} \right) . \]

For \( t \leq O \left( \nu^{1/2} \right) \),

\[
(\psi'_{\nu,l})^2 = \left( \psi'_\infty - \frac{2\nu_\infty^3}{\nu^3} \begin{pmatrix} \varphi_\infty \\ \psi_\infty \end{pmatrix} \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \\
\leq \left( \psi'_\infty - O \left( \frac{\nu^{3/2}}{\nu^3} \right) \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} ,
\]

\[
|\psi_{\nu,l}''| \geq \left| \begin{pmatrix} \frac{\psi_{\nu,l}}{\psi_\infty} - 6\frac{\nu_\infty^2 \varphi'_\infty}{\nu^3} \begin{pmatrix} \varphi_\infty \\ \psi_\infty \end{pmatrix} \right)^2 \frac{\psi_{\nu,l}^4}{\psi_\infty^4} \\
-3 \frac{\psi'_\infty}{\psi_\infty} \left( \psi'_\infty - \frac{2\nu_\infty^3}{\nu^3} \begin{pmatrix} \varphi_\infty \\ \psi_\infty \end{pmatrix} \right)^2 \frac{\psi_{\nu,l}^4}{\psi_\infty^4} \\
+3 \left( \psi'_\infty - 2\frac{\nu_\infty^3}{\nu^3} \begin{pmatrix} \varphi_\infty \\ \psi_\infty \end{pmatrix} \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right|
\]

\[
\geq \left| \begin{pmatrix} \psi''_\infty - O \left( \frac{t^2}{\nu^3} \right) - O \left( \frac{t^3}{\nu^4} \right) \right)^2 \frac{\psi_{\nu,l}^4}{\psi_\infty^4} \\
-3 \frac{\psi'_\infty}{\psi_\infty} \left( \psi'_\infty - O \left( \frac{t^3}{\nu^4} \right) \right) \frac{\psi_{\nu,l}^4}{\psi_\infty^4} \\
+3 \left( \psi'_\infty - O \left( \frac{t^3}{\nu^4} \right) \right)^2 \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right|
\]

Since \( t \leq O \left( \nu^{1/2} \right) \) we conclude that

\[
|\psi_{\nu,l}''| \geq \left| \begin{pmatrix} \psi''_\infty \frac{\psi_{\nu,l}^4}{\psi_\infty^4} - 3 \left( \psi'_\infty \right)^2 \left( \frac{\psi_{\nu,l}^4}{\psi_\infty^4} - \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right) \right| + O,
\]

where “O” stands for a quantity that is too small to matter.

Recalling that \( \frac{\psi_{\nu,l}^2}{\psi_\infty} = D^2 = \rho^2 \frac{\psi_{\nu,l}^2}{\psi_\infty} + \frac{\rho^4}{\nu^2 D^6} + 1 \) we have

\[
\left( \frac{\psi_{\nu,l}^4}{\psi_\infty} - \frac{\psi_{\nu,l}^6}{\psi_\infty^6} \right) = \frac{1}{D^4} - \frac{1}{D^6} \\
= \frac{D^2 - 1}{D^6} \\
= \frac{\rho^2 \psi_{\nu,l}^2}{D^6} + O \\
= \frac{\rho^2 \psi_{\nu,l}^2}{\nu^2 D^6} + O
\]
Since \( \psi''(0) = 0 \), it follows that for \( t \leq O(\nu^{1/2}) \),
\[
|\psi_{\nu,l}''| \geq 3 (\psi'')^2 \frac{\rho \psi^2}{\nu^2 D^6} + O.
\]

Thus our total derivative is positive when
\[
(\psi')^2 \left( \frac{\psi^6}{\psi^6_\infty} \right) \leq 3 (\psi')^2 \frac{\rho \psi^2}{\nu^2 D^6} + O, \quad \text{or}
\]
\[
\frac{1}{D^6} \leq 3 \rho^2 \frac{\psi^2}{\nu^2 D^6} + O, \quad \text{or}
\]
\[
\nu \leq \sqrt{3} \rho \psi_\infty + O, \quad \text{or}
\]

Since \( \psi'(0) = 1, \psi''(0) = 0 \), this is equivalent to
\[
t \geq \frac{\nu}{\sqrt{3} \rho} + O
\]

\[\Box\]

**Lemma 8.3.**

(8.3) \[\left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}'} \left[ \psi_{\nu,l} \psi_{\nu,l}' \right]' \right| \leq \max \left\{ \psi_{\nu,l}^2, \frac{\nu^2}{3\rho^2} \right\}.\]

**Proof.** From the previous result we have that for \( t \geq \frac{\nu}{\sqrt{3} \rho} + O, \left| \left[ \psi_{\nu,l} \psi_{\nu,l}' \right]' \right| \leq \left| \psi_{\nu,l} \psi_{\nu,l}' \right| \), so

\[
\left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}'} \left[ \psi_{\nu,l} \psi_{\nu,l}' \right]' \right| \leq \psi_{\nu,l}^2
\]

For some \( t \leq \frac{\nu}{\sqrt{3} \rho} + O \) the above inequality fails, but then we have

\[
\left| \frac{\psi_{\nu,l}}{\psi_{\nu,l}'} \left[ \psi_{\nu,l} \psi_{\nu,l}' \right]' \right| \leq \frac{\psi_{\nu,l}^2}{\psi_{\nu,l}} \left[ \psi_{\nu,l}' \right]^2
\]

Estimating as in Proposition 2.8 we have that for \( t \leq \frac{\nu}{\sqrt{3} \rho} + O \)

\[
\left[ \psi_{\nu,l}' \right]^2 \leq \left( \psi_{\nu,l}' \right)^2 + O
\]

and

\[
\psi_{\nu,l}'' = \psi_{\nu,l} \psi_{\nu,l}' - 3 (\psi')^2 \left( \psi_{\nu,l}^3 \psi_{\nu,l}'^4 - \psi_{\nu,l}^5 \psi_{\nu,l}'^5 \right) + O
\]

\[
= -3 (\psi')^2 \left( \psi_{\nu,l}^3 \psi_{\nu,l}'^4 - \frac{1}{\psi_{\nu,l}^5} \right) + O
\]
Using $D^2 = \rho^2 \frac{\psi^2}{\psi^2} + \frac{\psi^5}{\psi^5} + 1$ we have

\[
\left( \frac{\psi^3}{\psi^3} - \frac{\psi^5}{\psi^5} \right) = \frac{1}{\psi^3} \left( \frac{1}{D^3} - \frac{1}{D^5} \right) = \frac{1}{\psi^5} \left( \frac{D^2 - 1}{D^5} \right) = \rho^2 \left( \frac{\psi^5}{D^5} \right) + O
\]

So

\[
\frac{\psi_{\nu,d}}{\psi_{\nu,d}} \left[ \psi_{\nu,d} \psi_{\nu,d} \right]' \leq \frac{\psi_{\nu,d}}{\psi_{\nu,d}} \left[ \psi_{\nu,d} \right]^2
\]

\[
\leq \frac{1}{3} \left( \psi_{\nu,d}' \right)^2 \frac{\rho^2}{\psi^2} \left( \frac{\psi_{\nu,d}}{\psi^5} \right)^2 = \frac{\rho^2 D^5}{3 \rho^2} \left( \frac{1}{D^6} \right) \psi_{\nu,d}
\]

\[
= \frac{\rho^2 D^5}{3 \rho^2} \left( \frac{1}{D^6} \right) \psi_{\nu,d}
\]

\[
= \frac{\rho^2}{3 \rho^2} \left[ \frac{1}{D^2} \right]
\]

Since $D^2 \geq 1$, we get the desired inequality.

\[\square\]

9. Synergy

In this section we prove Lemma 2.10. All of the hypotheses stated in the technical introduction are in force.

The curvature of span $\{X, W + \tau V\}$ is a quadratic polynomial in $\tau$

\[Q(\tau) = \text{curv} (X, W) + 2 \tau R(W, X, X, V) + \tau^2 \text{curv} (X, V)\]

whose minimum value is

\[\text{curv} (X, W) - \frac{R(W, X, X, V)^2}{\text{curv} (X, V)}.\]

Let $g$ the metric after the indicated Cheeger Deformations are performed. Let $\tilde{g}$ be the metric after all of the deformations, and let $\bar{g}$ be the metric obtained by omitting the orthogonal partial conformal change. That is $\bar{g} = \tilde{g}$ when then Orthogonal Partial Conformal Factor $\phi \equiv 1$.

To illustrate the role of the Orthogonal Partial Conformal Change we first study the situation where it is not performed. That is we compute the above minimum for the metric $\bar{g}$. 

Proposition 9.1.

\[ \text{curv}^{\bar{g}}(X,W) - \frac{R^g(W,X,X,V)^2}{\text{curv}^{\bar{g}}(X,V)} = \left( s^4(D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} (D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 \right) \]

\[ - s^4 (D_X\psi)^2 \frac{g\left(\frac{A_X W}{|W|^2},V\right)^2}{\text{curv}^{\bar{g}}(X,V)} + o \left( s^4 (D_X\psi)^2 \right). \]

Proof. According to Proposition 6.3, the first four terms are just \( \text{curv}^{\bar{g}}(X,W) \).

Because \( X \) and \( W \) are initially tangent to a totally geodesic flat in a nonnegatively curved manifold our initial curvature, \( R^g \), satisfies

\[ R^g(W,X) = 0. \]

In particular,

\[ R^g(W,X,X,V) = 0. \]

Our hypotheses on \( V \) combined with Lemma 5.9 give us that after fiber scaling

\[ R^{g_\nu}(W,X,X,V)^2 = s^4(D_X\psi)^2 g_\nu \left( A_X \frac{W^{\nu}}{|W|^2},V \right)^2. \]

It remains to verify that this formula continues to hold after our conformal change. After the conformal change we have

\[ e^{-2f} R^g(W,X,X,V) = R^{g_\nu}(W,X,X,V) \]

\[ - g_\nu(W,V) \text{Hess}^{g_\nu}(f)(X,X) - g_\nu(X,X) \text{Hess}^{g_\nu}(f)(W,V) \]

\[ + g_\nu(X,V) \text{Hess}^{g_\nu}(f)(W,X) \]

\[ + g_\nu(W,V) D_X f D_X f - g_\nu(X,X) g_\nu(W,V) |\text{grad} f|^2 \]

Our hypotheses about \( V \) immediately simplifies this to

\[ e^{-2f} R^g(W,X,X,V) = R^{g_\nu}(W,X,X,V) + o \left( s^4(D_X\psi)^2 \right). \]

We also have

\[ \text{curv}^g(X,V) = \text{curv}^g(X,V) + O \left( s^2 |V| \right). \]

The result follows by combining Proposition 6.3 and equation 9.1 with the previous two equations.

\[ \Box \]

Proof of the Synergy Lemma 2.10

Combining the previous result with the hypothesis that

\[ \frac{g\left(\frac{W^{\nu}}{|W|^2},A_X V\right)^2}{\text{curv}^{\bar{g}}(X,V)} \leq C \]

for all \( \nu \), we conclude that for the metric \( \bar{g} \) the minimum of \( Q(\tau) \) satisfies

\[ \text{curv}^{\bar{g}}(X,W) - \frac{R^g(W,X,X,V)^2}{\text{curv}^{\bar{g}}(X,V)} \]

\[ \geq s^4(D_X\psi)^2 (1 - C) + s^4 \frac{\psi^2}{|W|^2} (D_X\psi)^2 + s^4 \frac{\psi^2}{|W|^2} D_X (\psi D_X \psi) - s^4 I'' |W|^2 + O \left( s^6 \right). \]
Let \( \gamma \) be a closed integral curve of \( X \). At the end of section 6 we saw that
\[
\int_\gamma c^{-2} \operatorname{curv} (X, W) = \int_\gamma s^4 (D_X \psi)^2 - s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 - s^4 I'' |W|^2 + O (s^6).
\]
Repeating that argument gives us the the integral of the right hand side of inequality 9.1 is
\[
\int_\gamma s^4 (D_X \psi)^2 (1 - C) - s^4 \frac{\psi^2}{|W|^2} (D_X \psi)^2 - s^4 I'' |W|^2 + O (s^6).
\]
Comparing these quantities shows that the metric \( \bar{g} \) can fail to satisfy our conclusion. Depending on the precise value of \( C \) we may even get that the minimum of \( Q (\tau) \) is negative somewhere along \( \gamma \) for all choices of \( I'' \). In any event, our conclusion is false without the orthogonal partial conformal change.

It follows from Theorem 2.6 that the orthogonal partial conformal change does not \( \operatorname{curv} (X, W) \) and \( R (W, X, X, V) \). Its effect on \( \operatorname{curv} (X, V) \) is given in Corollary 4.4 and is
\[
\operatorname{curv}^\varphi (X, V) = \operatorname{curv}^\varphi (X, V) - \varphi'' |V|^2 |X|^2 + O (C^1).
\]
where we use \( \varphi'' \) for \( D_X D_X (\varphi) \). The goal will now be to select \( \varphi'' \) appropriately so as to adjust our estimate for
\[
(D_X \psi)^2 \bar{g} \left( \frac{W^n}{|W|^2}, X \right)^2 \operatorname{curv}^\varphi (X, V)
\]
Recall that
\[
\varphi \equiv \kappa \circ \operatorname{dist} (\cdot, \cdot) \circ \operatorname{p}_{BM},
\]
and \( \varphi \equiv 1 \) outside of the compact neighborhood \( K_0 \) of \( \operatorname{p}_{BM}^{-1} (\cdot) \). We also require that \( \varphi \equiv 1 \) in a (very small) neighborhood of \( \operatorname{p}_{BM}^{-1} (\cdot) \). Together these hypotheses imply that \( \kappa : \mathbb{R} \to \mathbb{R} \), and is 1 outside of a compact interval, \([a, b]\) with \( a \) very close to 0. It follows that
\[
\int_{[a, b]} \kappa'' = 0.
\]
Equation 9.1, therefore, implies that, on average, to leading order, our Orthogonal Partial Conformal Change will neither increase nor decrease \( \operatorname{curv} (X, V) \) along the integral curves of \( X \). On the other hand, it gives us a way to redistribute \( \operatorname{curv} (X, V) \) along the integral curves of \( X \).

Our curvature compression result, Proposition 2.8 and inequality 9.1 together suggest an appealing choice for \( \varphi'' \). Indeed Proposition 2.8 says, for example, that
\[(D_X \psi)^2 \leq \nu^{4.6} \text{ outside of an interval of the form } [0, \kappa (\nu)] \]. We choose \( \varphi'' \) to be negative (and relatively large in absolute value) on an interval like \([0, O (\nu)] \). Since \( \int_{[a, b]} \varphi'' = 0 \) we must “pay for this” by having \( \varphi'' \) be positive (but relatively small) on \([\kappa (\nu), b]\). With such a choice of \( \varphi \) we can make the integral over any integral curve of \( X \) satisfy
\[
\kappa (\nu) \int \operatorname{curv}^\varphi (X, W) > \int \frac{R^\varphi (W, X, X, V)^2}{\operatorname{curv}^\varphi (X, V)}
\]
for the appropriately chosen function \( \kappa (\nu) \) with \( \lim_{\nu \to 0} \kappa (\nu) = 0 \). Indeed we have made the denominator \( \operatorname{curv}^\varphi (X, V) \) larger on the region \([a, O (\nu)] \) where \( R^\varphi (W, X, X, V)^2 \) is relatively large. We have done this at the expense of making it very slightly smaller on \([\kappa (\nu), b]\), but on this region \( R^\varphi (W, X, X, V)^2 \) is very
small. So our redistribution of \( \text{curv}(X,V) \) can in fact give us the desired inequality in an integral sense.

We obtain the point wise inequality by combining the integral inequality with a judicious choice of \( I^\nu \). Namely that it be sufficiently negative on the the complement of \( [\alpha, O(\nu)] \).

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