Optimal Control Problems with Time Delays (Preliminary Version)

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Abstract

This paper provides necessary conditions of optimality for optimal control problems with time delays in both state and control variables. Different versions of the necessary conditions cover fixed end-time problems and, under additional hypotheses, free end-time problems. The conditions improve on previous available conditions in a number of respects. They can be regarded as the first generalized Pontryagin Maximum Principle for fully non-smooth optimal control problems, involving delays in state and control variables, only special cases of which have previously been derived. Even when the data is smooth, the conditions advance the existing theory. For example, we provide a new ‘two-sided’ generalized transversality condition, associated with the optimal end-time, which gives more information about the optimal end-time than the ‘one-sided’ condition in the earlier literature. But there are improvements in other respects, relating to the treatment of initial data, specifying past histories of the state and control, and to the unrestrictive nature of the hypotheses under which the necessary conditions are derived.

Key Words: Optimal Control, Maximum Principle, Time Delay Systems, Nonsmooth Analysis.

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1 Introduction

This paper concerns optimal control problems, in which we seek to minimize a cost

\[ J(x(.), u(.)) = g(x(S), x(T)) + \int_{[S,T]} L(t, \{x(t-h_k)\}_{k=0}^N, \{u(t-h_k)\}_{k=0}^N) dt, \]

over control functions \( u(.) \) such that \( u(t) \in U(t) \), a.e., and state trajectories \( x(.) \) satisfying an end-point constraint \( (x(S), x(T)) \in C \) and a dynamic constraint, formulated as a controlled delay differential equation:

\[ \dot{x}(t) = f(t, \{x(t-h_k)\}_{k=0}^N, \{u(t-h_k)\}_{k=0}^N), \text{ a.e. } t \in [S,T]. \] (1.1)

Here, \([S,T]\) is a given time interval, \( h_0 < h_1 < \ldots < h_N \) are given numbers such that \( h_0 = 0 \), \( f(\ldots) : [S,T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R}^n \) and \( L(\ldots) : [S,T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R} \) are given functions and \( U(t) \), \( S \leq t \leq T \), and \( C \) are given sets. We write \( h := h_N \). Notice that,
according to this formulation, delays may occur in both \( x \) and \( u \) variables.

Under suitable hypotheses on the function \( f(\cdot) \), we can unambiguously associate a state trajectory \( x(\cdot) : [S, T] \rightarrow \mathbb{R}^n \) with a given control function \( u(\cdot) : [S, T] \rightarrow \mathbb{R}^m \) (in some appropriate function class) and initial data in the form of (a.e.) specified values \( d^x(s) \) and \( d^u(s) \), \( S - h \leq s < S \), of the \( x \) variable and the \( u \) variable, respectively, on the ‘delay interval’ \([S - h, S]\), and the initial value \( x_0 \) of the \( x \) variable. The state trajectory \( x(\cdot) \) is the absolutely continuous solution to (1.1), consistent with the initial data, in the sense that, for each \( k \):

\[
 x(t) = x_0 + \int_{[S, t]} f(s, t, \{x(s - h_k)\}, \{u(s - h_k)\}; \{d(t - h_k)\})ds.
\]

Here, and throughout the paper, \( \{x(s - h_k)\}_{k=0}^N \) is written simply as \( \{x(s - h_k)\} \), etc. The function \( f(\cdot; \{d(s - h_k)\}) \) appearing in (1.2) is

\[
f(t, x_0, \ldots, x_N, u_0, \ldots, u_N; d_0, \ldots, d_N)
:= f \left( t, \left\{ \begin{array}{ll} x_k & \text{if } t - h_k \geq S \\ d^x_k & \text{if } t - h_k < S \end{array} \right\}_{k=0}^N, \left\{ \begin{array}{ll} u_k & \text{if } t - h_k \geq S \\ d^u_k & \text{if } t - h_k < S \end{array} \right\}_{k=0}^N \right),
\]

describes how the initial segments of the state and control variables, gathered together as a single function \( d(\cdot) = (d^x(\cdot), d^u(\cdot)) \) on the time interval \([S - h, S]\), affect the evolution of the state trajectory \( x(\cdot) \). Note that the right side of (1.2) makes sense because \( x(t - h_k) \) and \( u(t - h_k) \) need to be evaluated only when \( t - h_k \in [S, T] \) and the vector \( d(t - h_k) \) needs to be evaluated only when \( t - h_k \in [S - h, S] \). This formulation of the dynamic constraint and cost covers, as special cases, situations in which there are only time delays in the states, only time delays in the controls, or when the delay times for controls and states differ, since, if the delay times differ, we can take \( \{h_1, \ldots, h_N\} \) to comprise all the time delays (in states and controls).

This paper provides necessary conditions of optimality for a ‘feasible process’ \((\bar{x}(\cdot), \bar{u}(\cdot))\) (i.e. a state trajectory/control policy pair satisfying the constraints of the problem) and accompanying initial data to be a minimizer, in the form of a generalized Pontryagin Maximum Principle (PMP). Necessary conditions for optimal control problems with time delays, of this nature, go back to the beginnings of optimal control theory (see, e.g., [1]). Early derivations of necessary conditions (see, e.g., [1], [3], [13] and the extensive references in [1] and [14]) were typically based on the application of abstract multiplier rules (due to Hestenes, Neustadt, Warga, Gamkrelidze and others), which are specially adapted to the structure of optimal control problems interpreted as optimization problems over function spaces, and which take account of density theorems relating to ‘original’ and ‘relaxed’ state trajectories, through consideration of Gamkrelidze’s ‘quasiconvex families of functions’ (or by other means). In common with the classical (delay-free conditions), these necessary conditions assert the existence of a ‘co-state’ trajectory \( p(\cdot) \) satisfying a co-state equation and transversality conditions, and Weierstrass condition telling us that a Hamiltonian-type function, evaluated along \((\bar{x}(\cdot), p(\cdot))\) is maximized over possible values of the control variable at \( \bar{u}(\cdot) \). A distinctive feature of these conditions is that the co-state equation is an ‘advance functional differential equation’, namely

\[
 -\dot{p}(t) = \sum_{k=0}^N p(t + h_k) \cdot \nabla_{x_k} f(t + h_k, \{\bar{x}(t - h_j + h_k)\}_{j=0}^N, \{\bar{u}(t - h_j + h_k)\}_{j=0}^N; \{\bar{d}(t - h_j + h_k)\}_{j=0}^N) \quad \text{a.e.}
\]

(\( \nabla_{x_k} \) refers to partial differentiation with respect to the \( k \)’th delayed state argument.)
We derive necessary conditions of this nature, for general, possibly non-commensurate, delays in both state and control variables. We also provide generalizations in which the initial data (specifying the past histories of $x(.)$ and $u(.)$), are included in the cost, and in which the terminal time $T$ is a choice variable (‘free-time’ problems). They reduce to Clarke’s nonsmooth PMP when when there are no time delays. Some special cases of these results were announced in [2]. The novel aspects of our work are as follows:

Nonsmoothness: We provide the first set of necessary conditions, in the form of a generalized Pontryagin Maximum Principle, for ‘fully’ nonsmooth problems (i.e. problems in which the only regularity hypothesis on the data w.r.t. the state variable is ‘Lipschitz continuity’) involving delays in states and controls. They resemble the classical necessary conditions for ‘smooth’ problems except that, in the costate relation classical derivatives are replaced by set-valued subdifferentials of nonsmooth analysis. Two earlier papers [6], [7] provide necessary conditions for nonsmooth optimal control problems with time delays in the state alone. The difference is that, in these papers, the dynamic constraint is modelled as a differential inclusion, and the relation for the costate arc (combined with the Weierstrass condition) is a generalization of Clarke’s Hamiltonian inclusion condition with ‘advanced’ arguments. As observed in [7 Section 1], necessary conditions expressed in terms of the Hamiltonian inclusion imply the nonsmooth Maximum Principle only for problems having special structure and not ‘fully’ nonsmooth problems, as in this paper. Furthermore, the methods of [6] and [7] cannot be adapted to cover problems with time delays in the control, because it is not possible to express a controlled delay differential equation (with delays in the control) as a delay differential inclusion (which can take account only of delays in the state). [7] allows both distributed and discrete delays (in the state variable), whereas we allow only discrete delays (in both state and control variables). Necessary conditions for optimal control problems involving differential inclusions are also provided in [3] and [15] for fixed time optimal control problems involving a single time delay in the state. We mention that Warga [19] showed that a broad class of optimal control problems involving delays and/or functional differential equations, distinct from the problems considered in this paper, can be fitted to an abstract framework within which nonsmooth necessary conditions can be derived; [19] requires a special ‘additively-coupled’ structure for the control delay dependence.

Free End-Time: This paper treats free end-time optimal control problems with time delays. In a delay-free context, optimal control problems with free end-time can be reformulated as standard optimal control problems on a fixed time interval, as a result of a transformation of the time variable. Optimality conditions for free end-time problems can be obtained from those for fixed end-time problems by applying fixed time conditions to the reformulated problem. For no time delays then, the derivation of free end-time conditions is straightforward. For optimal control problems with time delays, the reduction of free end-time problems to fixed end-time problems, in order to derive optimality conditions, cannot generally be achieved. This is because the time transformation, whose object is to fix the end-time, also generates a non-standard optimal control problem with time delays, since the time delays now depend on the state variable. We follow a different approach, which is new in a time delays context, based on a perturbation of the end-time. We thereby derive a modified transversality condition, which supplies additional information about the optimal end-time, expressed in terms of the ‘essential value’ of a maximized Hamiltonian-like function, introduced in [5]. Our free-time transversality condition, which is ‘two-sided’, is stronger than the ‘one-sided’ condition in [14].

The Weierstrass Condition: A significant feature of the necessary conditions provided in this paper is that they incorporate an ‘integral’ form of the Weierstrass condition for problems involving general, non-commensurate time delays in state and control. For optimal control
problems involving delays only in the state, the integral and pointwise forms of the Weierstrass condition are equivalent. But when we allow non-commensurate control delays, the integral form of the condition (appearing in this paper) is stronger than the pointwise form. While integral forms have been proved in special cases (‘additively-coupled’ non-commensurate time delays in the control (see, e.g. [19]) or commensurate control delays[14], only pointwise forms (or weak ‘differentiated’ forms), of the Weierstrass condition are provided for general time delays in the control, elsewhere in the literature. An exception is the important, but apparently overlooked, work of Warga and Zhu [20]. For controlled functional differential equations with non-additively-coupled, non-commensurate control delays, these authors establish the requisite ‘quasi-convexity’ properties required for the derivation of the integral form of the condition, though they explore their implications to the theory of necessary conditions only in a special case. Ideas in [20] play a key role in the derivation of the integral condition in this paper.

Initial Data: In this paper, the ‘initial data’ function \( d(t) = (d^x(t), d^u(t)), S - h \leq t \leq S \), specifying the past history of state variable (the \( d^x(.) \) component) and control variable (the \( d^u(.) \) component) is regarded as a choice variable, which is required to satisfy

\[
d(t) \in D(t) \text{ a.e. } t \in [S - h, S]
\]

and is taken account of in the cost by the integral cost term \( + \int_{S-h}^S \Lambda(t, d(t)) \, dt \).

The multifunction \( D(.) \) and the integrand \( \Lambda(.,.) \) are required to satisfy merely weak measurability hypotheses and the component of the necessary conditions relating the optimal choice of initial data takes the form of a ‘strong’ Maximum Principle. Optimality conditions relating to the initial data to be found in earlier work provide less information (in the case of [6]) and are derived under much stronger hypotheses. In [7], which concerns only state delays, it is assumed that the integral cost term is a Lipschitz function of \( d^x(.) \) (w.r.t. the sup norm) and \( D(t) \) is required to be closed for each \( t \). The relevant component of the necessary conditions is a less informative ‘weak’ Weierstrass condition governing the initial data. [14] provides a ‘strong’ Weierstrass condition (in integrated form) for the initial data, but under stronger hypotheses: \( D(t) \) is must be a closed, convex product set and the control delays are assumed commensurate.

Consider the important special case of a single delay or, more generally, commensurate delays (i.e. all delays are integer multiples of a single positive number). For fixed time problems, all the necessary conditions of this paper can be simply derived, using a transformation technique widely attributed to Guinn ([10], [9]), but which is, in fact, due to Warga [17]. This transformation converts an optimal control problem with commensurate delays (in state and control) to a delay-free problem, to which the standard PMP is applicable. (Note that free end-time problems with commensurate delays cannot be reduced to delay-free problems in this way because, when the end-time is free, the transformed problem does not have a suitable structure for application of delay-free necessary conditions.)

Notation: The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is \( |x| \). \( \mathbb{B} \) indicates the closed unit ball in \( \mathbb{R}^n \). The distance function \( d_A(.) : \mathbb{R}^n \rightarrow \mathbb{R} \) of a non-empty set \( A \subset \mathbb{R}^n \) is defined as

\[
d_A(x) := \inf\{|x - y| : y \in A\}, \quad \text{for } x \in \mathbb{R}^n.
\]

The convex hull of the set \( A \) is written \( \text{co} A \). Let \( I \subset \mathbb{R} \). The indicator function of the set \( I \) is written \( \chi_I(t) := \{1 \text{ if } t \in I \text{ and } 0 \text{ otherwise} \} \). Given a multifunction \( F(.) : \mathbb{R}^n \to \mathbb{R}^k \), we denote by \( \text{Gr} F(.) \) the graph of \( F(.) \), namely the set \( \{(x, v) \in \mathbb{R}^{n+k} | v \in F(x)\} \). Given real numbers \( a \) and \( b \), \( a \vee b := \max\{a, b\} \) and \( a \wedge b := \min\{a, b\} \).
$W^{1,1}( [a, b]; \mathbb{R}^n )$ denotes the space of absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$, with norm
\[
\|x\|_{W^{1,1}} := |x(a)| + \int_a^b |\dot{x}(t)| \, dt.
\]

We make use of several constructs from nonsmooth analysis, described in detail, for example, in \[16\] or \[8\]: given a closed set $E \subset \mathbb{R}^n$ and $x \in E$, the proximal normal cone of $E$ at $x$ is
\[
N^P_E(x) := \{ \zeta \in \mathbb{R}^n : \exists \epsilon > 0 \text{ and } M > 0 \text{ s.t. } \zeta \cdot (y - x) \leq M|y - x|^2 \text{ for all } y \in x + \epsilon B \}.
\]
The limiting normal cone at $x$ is
\[
N_E(x) := \{ \lim_{i \to \infty} \zeta_i : \zeta_i \in N^P_E(x_i) \text{ and } x_i \in E \text{ for all } i, \text{ and } x_i \to x \}.
\]
If $E$ is convex, these two normal cones coincide with the normal of cone of convex analysis.

Given a lower semicontinuous function $f(\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom } f(\cdot) := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$, the proximal subdifferential of $f(\cdot)$ at $x$ is the set
\[
\partial_P f(x) := \left\{ \zeta \in \mathbb{R}^n : \exists \sigma > 0 \text{ and } \epsilon > 0 \text{ such that, for all } y \in x + \epsilon B, \right\}.
\]
The limiting subdifferential of $f(\cdot)$ at $x$ is
\[
\partial f(x) := \{ \lim_{i \to \infty} \zeta_i : \zeta_i \in \partial_P f(x_i), x_i \to x \text{ and } f(x_i) \to f(x) \}.
\]
The partial limiting subdifferential $\partial_{x_i} f(\bar{x})$ w.r.t. $x_i$ at $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_N)$ is the limiting subdif-
erential with respect to the $x_i$ variable at $\bar{x}_i$ when the other variables are fixed. The projected limiting subdifferential w.r.t. $x_i$ of $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_N)$, written
\[
\tilde{\partial}_{x_i} f(\bar{x}) := \Pi_{x_i} \partial f(\bar{x}),
\]
is the projection of the limiting subdifferential of $f(\cdot)$ at $\bar{x}$ onto the $i$'th coordinate. The partial and projected limiting subdifferentials coincide with the classical partial derivative, when $f(\cdot)$ is continuously differentiable near $\bar{x}$, but can differ for Lipschitz functions.

Given an essentially bounded function $h(\cdot) : (a, b) \to \mathbb{R}$ and a point $T \in (a, b)$, the essential value of $h(\cdot)$ at $T$ is the closed interval
\[
\text{ess} \inf_{t \to T} h(t) := \left[ \lim_{\epsilon \downarrow 0} \text{ess inf}_{T-\epsilon \leq t \leq T+\epsilon} h(t), \lim_{\epsilon \downarrow 0} \text{ess sup}_{T-\epsilon \leq t \leq T+\epsilon} h(t) \right].
\]
2 Necessary Conditions for Fixed End-Time Problems

We consider the following optimal control problem:

\[
\begin{align*}
\text{(P)} & \quad \text{Minimize } J(x(\cdot), u(\cdot), d(\cdot)) := g(x(S), x(T)) + \int_{[S-h,S]} \Lambda(t, d(t)) \, dt \\
& \quad + \int_{[S,T]} L(t, \{x(s-h_k)\}, \{u(s-h_k)\}; \{d(t-h_k)\}) \, dt \\
& \quad \text{over } x(\cdot) \in W^{1,1}([S,T]; \mathbb{R}^n) \text{ and measurable functions } \\
& \quad u(\cdot) : [S, T] \to \mathbb{R}^m, \ d(\cdot) = (d^x, d^\mu)(\cdot) : [S-h, S] \to \mathbb{R}^n \times \mathbb{R}^m,
\end{align*}
\]

such that

\[
\begin{align*}
\dot{x}(t) &= f(t, \{x(s-h_k)\}, \{u(s-h_k)\}; \{d(t-h_k)\}) \text{ a.e. } t \in [S, T], \\
u(t) &\in U(t) \text{ a.e. } t \in [S, T], \\
d(t) &\in D(t) \text{ a.e. } t \in [S-h, S], \\
(x(S), x(T)) &\in C.
\end{align*}
\]

Here, and below, expressions such as \(\{x(t-h_k)\}\) should always be interpreted as \(\{x(t-h_k)\}_{k=0}^{N}\). The index \(k\) will be reserved for such expressions, and the values of \(k\) will always run from 0 to \(N\). We write \(h := h_N\).

The data comprises an interval \([S, T]\), real numbers \(h_0, \ldots, h_N\) such that \(h_0 = 0 < \ldots < h_N\), functions \(g(\ldots) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ \Lambda(\ldots, \ldots) : [S-h, S] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \ f(\ldots, \ldots) : [S, T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R}\), and \(L(\ldots, \ldots) : [S, T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R}\), multfunctions \(U(\cdot) : [S, T] \rightharpoonup \mathbb{R}^m\) and \(D(\cdot) : [S-h, S] \rightharpoonup \mathbb{R}^{n+m}\) and a set \(C \subset \mathbb{R}^n \times \mathbb{R}^m\).

A feasible process is a 3-tuple \((x(\cdot), u(\cdot), d(\cdot))\), in which \(x(\cdot) : [S, T] \to \mathbb{R}^n\), \(u(\cdot) : [S, T] \to \mathbb{R}^m\) and \(d(\cdot) : [S-h, S] \to \mathbb{R}^{n+m}\) are functions satisfying the constraints in \((P)\), and for which \(t \to \Lambda(t, d(t))\) is integrable on \([S-h, S]\) and

\[
t \to L(t, \{x(s-h_k)\}, \{u(s-h_k)\}; \{d(t-h_k)\})
\]

is integrable on \([S, T]\). We say that a feasible process \((\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot))\) is a \(L^\infty\) local minimizer if there exists \(\epsilon > 0\) such that

\[
J(x(\cdot), u(\cdot), d(\cdot)) \geq J(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot))
\]

for any feasible process \((x(\cdot), u(\cdot), d(\cdot))\) satisfying \(\|x(-) - \tilde{x(-)\|}_{L^\infty([S,T] ; \mathbb{R}^n)} \leq \epsilon\).

We shall invoke the following hypotheses, in which \(\tilde{f}(t, x_k, d_k) := (f, L)(t, \{x_k\}, \{u_k\})\) and \((\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot))\) is a given feasible process. For some \(\epsilon > 0\):

(H1): \(g(\ldots)\) is Lipschitz continuous on \((\tilde{x}(S), \tilde{x}(T)) + \epsilon \mathbb{B}\) and \(C\) is a closed subset of \(\mathbb{R}^{2n}\).

(H2): For every \(z \in \mathbb{R}^{(1+N)\times n}\), the function \(\tilde{f}(\cdot, z, \cdot)\) is \(L([S, T]) \times \mathcal{B}\) measurable, the set \(\text{Gr } U(\cdot)\) is \(L([S, T]) \times \mathcal{B}\) measurable and the set \(\text{Gr } D(\cdot)\) is \(L([S-h, S]) \times \mathcal{B}\) measurable. Here, for a given interval \(I\), \(L(I)\) denotes Lebesgue subsets of \(\mathbb{R}\), \(\mathcal{B}\) denotes the Borel sets of a Euclidean space, and \(L(I) \times \mathcal{B}\) denotes the product \(\sigma\)-algebra.
(H3): There exists a function a Borel measurable function \( k(.,.,.,.) : [S,T] \times \mathbb{R}^{n \times (N+1)} \times \mathbb{R}^{m \times (N+1)} \) such that \( t \rightarrow k(t,\{\bar{u}(t-h_k)\},\{\bar{d}(t-h_k)\}) \) is integrable and

\[
\hat{f}(t,\{u_k\};\{d_k\} \equiv k(t,\{u_k\},\{d_k\}) \text{-Lipschitz on} \{\bar{x}(t-h_k)\} + \varepsilon \mathbb{B}^{n \times (N+1)}
\]

for all \( \{u_k\} \in U(t-h_0) \times \ldots \times U(t-h_N) \), \( \{d_k\} \in D(t-h_0) \times \ldots \times D(t-h_N) \) a.e. \( t \in [S,T] \).

Fix \( x(.) : [S,T] \rightarrow \mathbb{R}^n, u(.) : [S,T] \rightarrow \mathbb{R}^m, d(.) : [S-h, S] \rightarrow \mathbb{R}^{n+m}, \lambda \geq 0 \) and \( p(.) : [S,T] \rightarrow \mathbb{R} \).

Define, for \( t \in [S,T] \), \( u \in \mathbb{R}^m \) and \( d \in \mathbb{R}^m \)

\[\mathcal{H}_\lambda(t,u,d;x(\cdot),u(\cdot),d(\cdot),p(\cdot)) := (p(t+h_0) \cdot f - \lambda L)
\]

\[
(t + h_0, \{x(t + h_0 - h_k)\}, u(t + h_0 - h_1), \ldots,
\]

\[
u(t + h_0 - h_N); d, d(t + h_0 - h_1), \ldots, d(t + h_0 - h_N) \chi_{[S,T]}(t + h_0)
\]

\[
+ \ldots +
\]

\[
+ (p(t + h_N) \cdot f - \lambda L)
\]

\[
(t + h_N, \{x(t + h_N - h_k)\}, u(t + h_N - h_0), \ldots,
\]

\[
u(t + h_N - h_{N-1}); u, d(t + h_N - h_0), \ldots, d(t + h_N - h_{N-1}), d \chi_{[S,T]}(t + h_N).
\]

**Theorem 2.1.** Let \((\bar{x}(\cdot), \bar{u}(\cdot), \bar{d}(\cdot)) \) be an \( L^\infty \) local minimizer for (P). Assume hypotheses (H1)-(H3) are satisfied for some \( \varepsilon > 0 \).

Then there exist \( p_k(.) \in W^{1,1}([S-h_k, T]; \mathbb{R}^n), k = 0, \ldots, N, \) and \( \lambda \geq 0 \) such that

\[
\begin{cases}
\dot{p}_k(t) = 0 & \text{for } t \in [S-h_k, S] \\
p_k(t) = 0 & \text{for } t \in [(T-h_k) \cap S, T],
\end{cases}
\]

for \( k = 1, \ldots, N, \) with the following properties, in which \( p(.) \in W^{1,1}([S,T]; \mathbb{R}^n) \) is the function

\[
p(t) := \sum_{k=0}^{N} p_k(t) \text{ for } t \in [S,T].
\]

(a): (nontriviality) \( (\lambda, p(\cdot)) \neq 0, \)

(b): (adjoint inclusion)

\[\{-\dot{p}_0(t-h_k)\} \in \co \partial_{\{x_k\}} (p \cdot f - \lambda(L + \Lambda))(t, \{x_k\}, \{\bar{u}(t-h_k)\}; \{\bar{d}(t-h_k)\})
\]

\[\{x_k\} = \{\bar{x}(t-h_k)\}, \text{ a.e. } t \in [S-h, T].\]

(c): (integral Weierstrass condition)

\[
\int_{[S-h,T]} (p \cdot f - \lambda(L + \Lambda))(t, \{\bar{x}(t-h_k)\}, \{u(t-h_k)\}; \{d(t-h_k)\})dt
\]

\[\leq \int_{[S-h,T]} (p \cdot f - \lambda(L + \Lambda))(t, \{\bar{x}(t-h_k)\}, \{\bar{u}(t-h_k)\}; \{\bar{d}(t-h_k)\})dt
\]

for any selectors \( u(.) \) of \( U(.) \) and \( d(.) \) of \( D(.) \) such that the integrand on the left side of (4.7) is integrable.

(d): \( (p(S), -p(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T)). \)

It follows from the definition (2.2) and (2.3) of \( p(.) \) and from condition (b) that \( p(.) \) satisfies the ‘advance’ functional differential inclusion:
(b*): \(-\tilde{p}(t) \in \sum_{j=0}^{N} \chi_{[S,T-h_k]}(t) \co \tilde{\partial}_{x_j} (p \cdot f - \lambda L)\)

\((t+h_j, \{x_k\}, \{\tilde{u}(t-h_k+h_j)\}, \{\tilde{d}(t-h_k+h_j)\})\) \in \tilde{\partial}_{x_k} (\{\tilde{x}(t-h_k+h_j)\}) \quad \text{a.e. } t \in [S,T]

in which \(\tilde{\partial}_{x_i}\) denotes the projected limiting subdifferential onto the \(i\)’th delayed state coordinate.

**Condition (c) implies**

\((c^*)\): *(Pointwise Maximum Principle)*

\[\mathcal{H}_\lambda(t, \tilde{u}(t), \tilde{d}(t); \tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot), p(\cdot)) = \max_{u \in U(t), d \in D(t)} \mathcal{H}_\lambda(t, u, d; \tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot), p(\cdot))\]

\(a.e. t \in [S-h,T],\)

A proof of Thm. 2.4 is given in a later section.

**Comments**

(a): The key conditions in Thm. 2.4 are (a), (b*), (c)-(e) involving the function \(p(\cdot)\). When \((\{x_k\}) \rightarrow (f,L)(t, \{x_k\}, \{u_k\}; \{d_k\})\) is \(C^1\) near \(\{\tilde{x}(t-h_k)\}\), they reduce to standard necessary conditions expressed in terms of a costate function \(p(\cdot)\) satisfying the ‘advance’ functional differential equation (1.3) of the Introduction. The condition (b), expressed in terms of the collection of \(p_k(.)\)’s in the sum decomposition (2.2) of \(p(\cdot)\) is more precise condition than (b*) in a non-smooth setting because the subdifferential \(\partial_{x_k}(p \cdot f)\) is a subset, and in some cases a strict subset, of the product of projected partial subdifferentials \(\tilde{\partial}_{x_0}(p \cdot f) \times \ldots \times \tilde{\partial}_{x_N}(p \cdot f)\).

(b): The integral Weierstrass condition (c) (for control functions and initial data functions), which allows simultaneous variation of the entries in all the control delay slots (provided they are all associated with some control function) is a stronger condition (when there are time delays in the control), than the pointwise condition (c)*, expressed in terms of the Hamiltonian function (2.1), which involves variations in the control slots only one at a time. Note that, elsewhere in the literature (see, e.g., [19], integral forms of the Weierstrass condition are given only under the addition hypothesis that the control delays are additively coupled. Apparently the only exception is [20], where, in a special setting, Warga and Zhu derive an integral form of the condition for non-additively coupled control delays.

(c): The initial data segments \(d^x(s)\) and \(d^u(s)\), \(s \in [S-h,S]\), for the \(x\) and \(u\) variables are treated in a more general way than in the previous literature. Here they are regarded as choice variables that are required to satisfy \(d(t) = (d^x, d^u)(t) \in D(t)\) a.e.. \(D(t)\) need not be closed or bounded, and is not necessarily a product set that captures, separately, constraints on initial state and initial control segments. The optimal initial segment \(d(.)\) is characterized by two versions of ‘strong’ Weierstrass condition (the pointwise condition \(d\) and integral condition \(d^*\)) that provide more information about \(d(.)\) than the ‘weak’ condition in [7] (when it is applicable) expressed in terms of normal cones of \(D(t)\). An integrated version of the ‘strong’ Weierstrass condition on the initial data for the optimal state variable is included in the necessary conditions of [14], but \(D(t)\) is required to be a closed convex set.

(d): Our nonsmooth necessary conditions allow time delays in the control. They improve on earlier nonsmooth necessary conditions for time delay problems, which allow delays only
in the state [6], [7], or require a separable structure for the control delay dependence [19]. In common with [20], they improve on available necessary conditions in [13] for smooth problems with delays in both state and control, because they do not require the control delays to be commensurate.

3 Necessary Conditions for a Free End-Time Problem

Consider next a related problem to (P) above, in which the end-time $T$ is free, and included in the choice variables, and in which there are no control delays.

$$(P_{FT}) \begin{cases} 
\text{Minimize} & J(x(\cdot), u(\cdot), d(\cdot), T) := \tilde{g}(x(S), x(T), T) + \int_{[S-h, S]} \Lambda(t, d(t)) dt \\
+ \int_{[S, T]} L(t, \{x(t-h_k)\}, u(t); \{d(t-h_k)\}) dt, \\
\text{over} & T \geq S, x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n) \text{ and measurable functions} \\
u(\cdot) : [S, T] \to \mathbb{R}^m, d(\cdot) : [S-h, S] \to \mathbb{R}^n \\
such that} & \dot{x}(t) = f(t, \{x(t-h_k)\}, u(t); \{d(t-h_k)\}) \quad \text{a.e. } t \in [S, T], \\
u(t) \in U(t) \text{ a.e. } t \in [S, T], \\
d(t) \in D(t) \text{ a.e. } t \in [S-h, S] \\
x(S), x(T), T \in \tilde{C}.
\end{cases}$$

Here, $h := h_N$. The data for $(P_{FT})$ comprises a real number $S$, real numbers $h_0, \ldots, h_N$ such that $0 = h_0 < \ldots < h_N$, functions $\tilde{g}(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $\Lambda(\cdot, \cdot) : [S-h, S] \times \mathbb{R}^n \to \mathbb{R}$, $f(\cdot, \cdot, \cdot) : [S, \infty) \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^n \to \mathbb{R}^n$ and $L(\cdot, \cdot, \cdot) : [S, \infty) \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^m \to \mathbb{R}$, multifunctions $U(\cdot) : [S, \infty) \rightrightarrows \mathbb{R}^m$ and $D(\cdot) : [S-h, S] \rightrightarrows \mathbb{R}^n$ and a set $\tilde{C} \subset \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$.

A feasible process for $(P_{FT})$ is a 4-tuple $(x(\cdot), u(\cdot), d(\cdot), T)$, where $T$ is a number ($T \geq S$), $x(\cdot) : [S, T] \to \mathbb{R}^n$, $u(\cdot) : [S, T] \to \mathbb{R}^m$ and $d(\cdot) : [S-h, S] \to \mathbb{R}^n$ are functions in the specified spaces, satisfying the constraints in $(P_{FT})$ and such that $t \to \Lambda(t, d(t))$ is integrable on $[S-h, S]$, and $t \to L(t, \{x(t-h_k)\}; \{d(t-h_k)\})$ is integrable on $[S, T]$.

A feasible process $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot), \tilde{T})$ is an $L^\infty$ local minimizer if there exists $\epsilon > 0$ such that

$$J(x(\cdot), u(\cdot), d(\cdot), T) \geq J(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot), \tilde{T})$$

for any feasible process $(x(\cdot), u(\cdot), d(\cdot), T)$ satisfying $\|x(\cdot) - \tilde{x}(\cdot)\|_{L^\infty([S, T\cap T], \mathbb{R}^n)} + |T - \tilde{T}| \leq \epsilon$.

We shall invoke the following hypotheses, in which $\tilde{f}(\cdot, \cdot, \cdot) := (f, L)(\cdot, \cdot, \cdot)$ and $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot), \tilde{T})$ is a given feasible process. For some $\epsilon > 0$:

(HFT1): $g(\cdot, \cdot, \cdot)$ is Lipschitz continuous on $(\tilde{x}(S), \tilde{x}(T), \tilde{T}) + \epsilon \mathbb{B}^{2n+1}$. $\tilde{C}$ is a closed subset of $\mathbb{R}^{2n+1}$.

(HFT2): For every $z \in \mathbb{R}^{(1+N)\times n}$, the function $\tilde{f}(\cdot, z, \cdot)$ is $\mathcal{L}([S, \tilde{T} + \epsilon]) \times \mathcal{B}$ measurable, the set $\text{Gr} \ U(\cdot)$ is $\mathcal{L}([S, \infty]) \times \mathcal{B}$ measurable and the set $\text{Gr} \ D(\cdot)$ is $\mathcal{L}([S-h, S]) \times \mathcal{B}$ measurable.
(HFT3): There exists a function a Borel measurable function \( k(.,.) : [S,T] \times \mathbb{R}^m \times \mathbb{R}^{n \times (N+1)} \) and numbers \( \tilde{k} > 0 \) and \( \tilde{c} > 0 \) such that \( t \to k(t, \bar{u}(t), \{\bar{d}(t-h_k)\}) \) is integrable on \([S,T]\) and 

\[
\tilde{f}(t,.,u;\{d_k\}) = k(t,u,\{d_k\})\text{-Lipschitz on } \{\bar{x}(t-h_k)\} + \varepsilon \mathbb{R}^{n \times (N+1)}
\]

for all \( u \in U(t) \), \( \{d_k\} \in D(t-h_0) \times \ldots \times D(t-h_N) \) a.e. \( t \in [S,T] \). Moreover 

\[
\tilde{f}(t,.,u;\{d_k\}) \text{ is } \tilde{k}\text{-Lipschitz and } \tilde{c}\text{-bounded on } \{\bar{x}(t-h_k)\} + \varepsilon \mathbb{R}^{n \times (N+1)}
\]

for all \( u \in U(t) \), \( \{d_k\} \in D(t-h_0) \times \ldots \times D(t-h_N) \) a.e. \( t \in [\bar{T} - \epsilon, \bar{T} + \epsilon] \).

There follows a set of necessary conditions for \((\bar{x}(.),\bar{u}(.),\bar{d}(.),\bar{T})\) to be an \( L^\infty\)-local minimizer for the free end-time problem \((P_{FT})\).

**Theorem 3.1.** Let \((\bar{x}(.),\bar{u}(.),\bar{d}(.),\bar{T})\) be an \( L^\infty\) local minimizer for \((P_{FT})\). Assume hypotheses (HFT1)-(HFT3). Assume also that \( \bar{T} - S > \theta \).

Then there exists \( p_k(.) \in W^{1,1}([S-h_k,\bar{T}]; \mathbb{R}^n) \) such that

\[
\begin{cases}
\dot{p}_k(t) = 0 & \text{for } t \in [S-h_k, S] \\
p_k(t) = 0 & \text{for } t \in [\bar{T} - h_k, \bar{T}] ,
\end{cases}
\]

\( k = 1, \ldots, N, \lambda \geq 0 \) and \( \xi \in \mathbb{R} \), with the following properties, in which \( p(.) \in W^{1,1}([S,\bar{T}]; \mathbb{R}^n) \) is the function:

\[
p(t) = \sum_{k=0}^{N} p_k(t) \quad \text{for } t \in [S, \bar{T}] . \tag{3.1}
\]

Conditions (a)-(d), (b*) and (c*) of Thm. 2.1 (in which \( f,L \) does not depend on the ‘delayed’ control variables \( u(t-h_1), \ldots, u(t-h_N) \) and \([S,T]\) is replaced by \([S,\bar{T}]\)). Furthermore, in place of (d), the following ‘free end-time’ transversality condition is satisfied:

\[(d'):\quad \langle p(S), -p(\bar{T}), \xi \rangle = \lambda \partial \bar{g}(\bar{x}(S), \bar{x}(\bar{T}), \bar{T}) + N_{\bar{C}}(\bar{x}(S), \bar{x}(\bar{T}), \bar{T}), \]

in which \( \xi \) is some number that satisfies

\[
\xi \in \text{ess} \left\{ \max_{u \in U(t)} (p(\bar{T}) \cdot f - \lambda L)(t, \{\bar{x}(\bar{T} - h_k)\}, u) \right\} .
\]

**Comments**

(a): Condition (d’) is apparently the first generalized tranversality condition for free end-time optimal control problems with delays, when the data is assumed to be merely measurable w.r.t. to the time variable. But it provides new information even in the continuous case. Indeed, suppose that \( \bar{C} = C \times \mathbb{R} \), i.e. the free end-time \( T \) is unconstrained and \( g \) is a \( C^1 \) function, \((f,L)\) are continuous w.r.t. the time variable and \( U(t) \) is a constant compact set \( U \). Then the ‘essential value’ is single valued and the transversality condition provides the following information about the optimal end-time \( \bar{T} \):

\[
\max_{u \in U} \left( p(\bar{T}) \cdot f - \lambda L \right)(\bar{T}, \{\bar{x}(\bar{T} - h_k)\}, u) = \nabla_T g(\bar{x}(S), \{\bar{x}(\bar{T}), \bar{T}) . \tag{3.2}
\]

Note that this is an equality (or ‘two sided’) relation. The necessary conditions in [14] include an inequality (or ‘one sided’) version of this relation which conveys less information.

(b): The transversality condition (3.2) is used in [2], to derive sensitivity relations and construct algorithms for the computation of solutions to free end-time optimal control problems with delays.
4 Proof of Theorem 2.1

We shall make extensive use of the following existence theorem for delay differential equations with accompanying estimates, which is can be regarded as a generalization of Filippov’s Existence Theorem [16 Thm. 2.4.3.] for (delay free) differential inclusions.

**Theorem 4.1.** Take $0 =: h_0 < \ldots < h_{N-1} < h_N$, $\bar{\epsilon} \in (0, \infty) \cup \{+\infty\}$, a function $f(\cdot) : [S, T] \times \mathbb{R}^{(N+1) \times n} \to \mathbb{R}^n$ and a measurable function $d(\cdot) : [S - h, S] \to \mathbb{R}^n$. Take also $y(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$ (the ‘reference trajectory’) and $\xi \in \mathbb{R}^n$ (the ‘initial state’). Assume

(a): $f(\cdot, \{x_k\})$ is measurable for each $\{x_k\} \in \mathbb{R}^{(N+1) \times n}$.
(b): there exists $k(\cdot) \in L^1$ such that
\[
\{x_k\} \to f(t, \{x_k\}; \{d(t - h_k)\})
\]
is $k(t)$ Lipschitz continuous on $(y((t - h_0) \lor S), \ldots, y((t - h_N) \lor S)) + \bar{\epsilon}B$, a.e.

Assume also that
\[
|y(S) - \xi| + \int_S^T |\dot{y}(t) - f(t, \{y(t - h_k)\}); \{d(t - h_k)\})|dt
\]
\[
\leq \bar{\epsilon} \times \left(\exp\{(N + 1) \int_S^T k(t) dt\}\right)^{-1}.
\]

(In the case $\bar{\epsilon} = +\infty$, this last condition is automatically satisfied and the function (4.1) is required to be $k(t)$ Lipschitz on $\mathbb{R}^n$.) Then:

(A): there exists a solution $x(\cdot) \in W^{1,1}([S, T]; \mathbb{R}^n)$ to the equation
\[
\dot{x}(t) = f(t, \{x(t - h_k)\}); \{d(t - h_k)\}) \ a.e. \ t \in [S, T]
\]
with initial state $x(S) = \xi$ and satisfying
\[
||y(. ) - x(\cdot)||_{L^\infty(S, t)} \leq |y(S) - x(S)| + \int_S^t |y(s) - \dot{x}(s)| ds \leq e^{(N + 1) \int_S^t k(s) ds} \left(\sum_{k=1}^n \left|\xi - y(S)\right| + \int_S^t |\dot{y}(s) - k(s); \{y(s - h_k)\}); \{d(s - h_k)\})|ds\right),
\]
for all $t \in [S, T]$.

(B): if $\bar{\epsilon} = +\infty$, (4.2) has a unique solution $x(\cdot)$ in $W^{1,1}$ (for given $x(S)$, $u(\cdot)$ and $d(\cdot)$).

Proof. (A) is proved by means of a straightforward adaptation of the proof of the well-known related existence theorem for differential inclusions, based on ‘Picard iteration’ (see, e.g., [16 Proof of Thm. 2.4.3.]). To prove (B) note that, if the hypotheses are satisfied with $\bar{\epsilon} = +\infty$, then (4.2) has a solution, by part (A). If there are two solutions $x(\cdot)$ and $y(\cdot)$, we deduce from hypothesis (b) that the continuous function $\xi(t) := ||x(\cdot) - y(\cdot)||_{L^\infty([0, t])}$ satisfies
\[
|\xi(t)| \leq \int_{[S, t]} (N + 1) \times k(s)|\xi(s)|ds \quad \text{for } t \in [S, T].
\]
We conclude from Gronwall’s inequality that $\xi(\cdot) \equiv 0$, whence $x(\cdot) = y(\cdot)$. ∎
We first validate the assertions of Thm. 2.1 when hypotheses (H1)-(H3) are supplemented by several additional hypotheses. We then show that the assertions remain true when the additional hypotheses are removed. The additional hypotheses (which make reference to the initial state \( \bar{x}(S) \) of the process \((\bar{x}(\cdot), \bar{u}(\cdot)) \) under consideration) are as follows:

(A0): \((f, L)(t, \{x_k\}, \{u_k\}; \{d_k\}) \) does not depend on the initial data \(\{d_k\} \) and \(\Lambda(\cdot, \cdot) \equiv 0\).

(When (A0) is satisfied, we write \(f(t, \{x_k\}, \{u_k\})\) in place of \(f(t, \{x_k\}, \{u_k\}; \{d_k\})\).

(A1): There exists a bounded, \( \mathcal{L} \times \mathcal{B} \) measurable function \( \tilde{L}(\cdot, \cdot) : [S, T] \times \mathbb{R}^m \to \mathbb{R} \) such that \( L(t, \{x_k\}, \{u_k\}) = \tilde{L}(t, u_0) \).

(A2): There exist integrable functions \(c_0(\cdot) : [S, T] \to \mathbb{R} \) and \( k_0(\cdot) : [S, T] \to \mathbb{R} \) such that, for all selectors \(u(\cdot) \in U(\cdot)\) and a.e. \(t \in [S, T]\), the mapping

(i): \(\{x_k\} \to f(t, \{x_k\}, \{u(t-h_k)\})\) is \(c_0(t)\) bounded on \(\mathbb{R}^{n \times (N+1)}\),

(ii): \(\{x_k\} \to f(t, \{x_k\}, \{u(t-h_k)\})\) is \(k_0(t)\)-Lipschitz continuous on \(\mathbb{R}^{n \times (N+1)}\).

(A3): \(C = \mathbb{R}^n \times \mathbb{R}^m\).

(A4): There exist continuously differentiable functions \(l_0(\cdot) : \mathbb{R}^n \to \mathbb{R}\) and \(l_1(\cdot) : \mathbb{R}^n \to \mathbb{R}\) and \(\alpha \geq 0\), such that

\[
g(x_0, x_1) = l_0(x_0) + \alpha|x_0 - \bar{x}(S)| + l_1(x_1) \quad \text{for all } (x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.3}
\]

**Step 1:** We confirm the assertions of Thm. 2.1 (with \( \lambda = 1 \)) under (H1)-(H3), (A0)-(A4).

Assume that \((\bar{x}(\cdot), \bar{u}(\cdot))\) is an \(L^\infty\) local minimizer for problem \((P)\), under hypotheses (H0)-(H3) and (A1)-(A4). Then, for some \(\epsilon > 0\), \((\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{d}(\cdot))\) is a minimizer for

\[
\begin{aligned}
\text{(P')} & \quad \text{Minimize } J(x(\cdot), u(\cdot)) := l_0(x(S)) + \alpha|x(S) - \bar{x}(S)| + l_1(x(T)) + \int_{[S, T]} \tilde{L}(t, u(t)) \, dt, \\
& \quad \text{subject to } \\
& \quad \tilde{x}(t) = f(t, \{x(t-h_k)\}, \{u(t-h_k)\}), \\
& \quad u(t) \in U(t) \text{ a.e. } t \in [S, T], \\
& \quad ||x(\cdot) - \tilde{x}(\cdot)||_{L^\infty} \leq \epsilon.
\end{aligned}
\]

For each positive integer \(i\), consider the related problem:

\[
\begin{aligned}
\text{(P$_i$)} & \quad \text{Minimize } J_i(x(\cdot), \{y_k(\cdot)\}, u(\cdot)) := l_0(x(S)) + \alpha|x(S) - \bar{x}(S)| + l_1(x(T)) \\
& \quad + \int_{[S, T]} \tilde{L}(t, u(t)) \, dt + i \times \left( \sum_{k=0}^{N} \int_{[(S+h_k) \wedge T, T]} k_0(t)|y_k(t) - x(t-h_k)|^2 \, dt \right) \\
& \quad \text{over } \{y_k(\cdot) \in L^1_{k_0(\cdot)}(\{(S+h_k) \wedge T, T\}; \mathbb{R}^n)\} \text{ and selectors } u(\cdot) \text{ of } U(\cdot) \text{ s.t.} \\
& \quad \tilde{x}(t) = f(t, \{y_k(t)\}, \{u(t-h_k)\}), \text{ a.e.} \\
& \quad u(t) \in U(t) \text{ a.e. } t \in [S, T], \\
& \quad ||x(\cdot) - \tilde{x}(\cdot)||_{L^\infty} \leq \epsilon.
\end{aligned}
\]

Here, \(k_0(\cdot)\) is the integrable bound of hypothesis (A2) and \(L^1_{k_0(\cdot)}([a, b])\) denotes measurable functions \(\phi : [a, b] \to \mathbb{R}^n\) such that \(k_0(t)\phi(t) \in L^1\). Observe that the cost can be infinite because the \(y_k(\cdot)\)’s are allowed to be \(L^1\) functions and the cost involves \(L^2\) norms. Write the infimum costs of \((P_i)\) and \((P')\) as \(\inf(P_i)\) and \(\inf(P')\), respectively.

**Lemma 4.2.**

\[
\lim_{i \to \infty} \inf(P_i) = \inf(P').
\]
Proof. We deduce from the special structure of the \((P_i)\)'s and \((P^c)\) that

\[
-\infty < \inf(P_i) \leq \inf(P_j) \leq \inf(P^c) \quad \text{for any index values } i < j. \tag{4.4}
\]

Fix \(i > 0\) and take any feasible process \((x(.), \{y_k(.)\}, u(.))\) for \((P_i)\) such that \(J_i(x(.), \{y_k(.)\}, u(.)) < \infty\). (Such a feasible process, namely \((\bar{x}(.), [\bar{x}(.-h_k)], \bar{u}(.), \bar{d}(.))\), exists.) By Filippov’s Thm., applied with \(y(.) = x(.)\) and initial state \(\xi = x(S)\), there exists a feasible process \((x_i(.), u(.)\)) for \((P^c)\) (with the same \(u(.)\) and \(d(.)\)) such that \(x_i(S) = x(S)\) and

\[
||x_i(.) - x(.)||_{L^\infty} \leq K \times \sum_{k=0}^{N} \int_{[(S+h_k)\wedge T,T]} k_0(t)|y_k(t) - x(t-h_k)|dt.
\]

\((K\) is a number that does not depend on our choice of \((x(.), \{y_k(.)\}, u(.)\)).) With the help of Hölder’s inequality, we can show that

\[
J_i(x(.), \{y_k(.)\}, u(.)) - J_i(\{x_i(.), x_i(.-h_k)\}, u(.)) \\
\geq \sum_{k=0}^{N} \int_{[(S+h_k)\wedge T,T]} k_0(t) (i \times |y_k(t) - x(t-h_k)|^2 - K k_{l_1}|y_k(t) - x(t-h_k)|) dt, \\
\geq \sum_{k=0}^{N} \left( \int_{[S,T]} k_0(t) dt \right)^{-1} \times i \times z_k^2 - K k_{l_1} z_k, \\
\geq \sum_{k=0}^{N} \min_{z \in \mathbb{R}} \left( \int_{[S,T]} k_0(t) dt \right)^{-1} \times i \times z^2 - K k_{l_1} z \geq -\gamma \times i^{-1},
\]

in which \(\gamma := \frac{1}{2} \times (N+1) \times (K k_{l_1})^2 \times ||k_0(.)||_{L^1} \) and \(k_{l_1}\) is a Lipschitz constant for \(l_1(.)\). Since \((x(.), \{y_k(.)\}, u(.)\)) was chosen arbitrarity, we have shown that

\[
\inf(P_i) \geq \inf(P^c) - \gamma \times i^{-1}.
\]

Combining this inequality with (4.4) gives the desired relation. \(\square\)

Now write problem \((P_i)\) as

Minimize \(\{J_i(x(.), \{y_k(.)\}, u(.)) \mid (x(.), \{y_k(.)\}, u(.)) \in A_c\}\),

in which

\[
A_c := \{(x(.), \{y_k(.)\}, u(.)) \mid (x(.), \{y_k(.)\}, u(.)) \in W^{1,1}, \ y_k(.) \in L^1_{k_0(.)}([(S+h_k)\wedge T,T]) \text{ for } k = 0, \ldots, N, \ u(.) \text{ is a selector of } U(.), \ \dot{x} = f \text{ a.e., } ||x(.) - \bar{x}(.)||_{L^\infty} \leq \epsilon\).
\]

Equip \(A_c\) with the metric

\[
d_c((x(.), \{y_k(.)\}, u(.)), (x(.), \{y_k(.)\}, u(.))) = \\
||x'(S) - x(S)|| + \sum_{k=0}^{N} \int_{[(S+h_k)\wedge T,T]} k_0(t)|y_k'(t) - y_k(t)| dt + \text{ meas } \{t \in [S,T] \mid u'(t) \neq u(t)\}
\]

It can be shown that, w.r.t. this metric, \(A_c\) is complete and \(J_i(., ., .)\) is lower semicontinuous on \(A_c\). (The lower semicontinuity can be deduced from Fatou’s Lemma, as demonstrated in [16] page...
Define the functions for each $i$.

Take a minimizer.

Lemma 4.3. By Lemma 4.2, there exists $\gamma_i > 0$ such that, for each $i$, $(\bar{x}(\cdot), y_0(t) = \bar{x}(t-h), \ldots, y_N(t) = \bar{x}(t-h_N), \bar{u}(\cdot), \bar{d}(\cdot))$ is a $\gamma_i$ minimizer for $(P_i)$. According to Ekeland’s Theorem, there exists, for each $i$, $(x_i(\cdot), y_0^i(\cdot), \ldots, y_N^i(\cdot), x_i(\cdot)) \in A_i$ which is a minimizer for the optimization problem:

\[(P_i) \quad \text{Minimize } J_i((x(\cdot), y_k(\cdot), u(\cdot))) + \gamma_i^\frac{1}{2} d_e((x(\cdot), y_k(\cdot), u(\cdot)), (x_i(\cdot), y_k^i(\cdot), u_i(\cdot))) \quad \text{for } (x(\cdot), y_k(\cdot), u(\cdot)) \in A_i,
\]

and

\[d_e((x_i(\cdot), y_k^i(\cdot), u_i(\cdot)), (x(\cdot), \bar{x}(\cdot-h_k)), \bar{u}(\cdot))) \leq \gamma_i^\frac{1}{2}.
\]

It can be deduced from (4.5), with the help of Thm. 4.1, that $||x_i(\cdot) - \bar{x}(t)||_{L^\infty} \to 0$, as $i \to \infty$. We have then, for $i$ sufficiently large,

\[||x_i(\cdot) - \bar{x}(\cdot)||_{L^\infty} \leq \varepsilon/2.
\]

Define $m_i(\cdot, \cdot) : [S - h, S] \times \mathbb{R}^{n+m} \to \mathbb{R}$ as

\[m_i(t, u) := \begin{cases} 0 & \text{if } u = u_i(t), \\ 1 & \text{if } u \neq u_i(t). \end{cases}
\]

The cost function for $(P_i)$ can be written

\[\tilde{J}_i((x(\cdot), y_k(\cdot), u(\cdot))) := i \times \left( \sum_{k=0}^{N} \int_{[(S+h_k) \land T, T]} k_0(t)||y_k(t) - x(t-h_k)||^2 dt \right)
\]

\[\quad + \int_{[S, T]} (\tilde{L}(t, u(t)) + \gamma_i^\frac{1}{2} m_i(t, u(t))) dt
\]

\[\quad + l_0(x(S)) + \alpha|x(S) - \bar{x}(S)| + \gamma_i^\frac{1}{2} |x(S) - x_i(S)| + l_1(x(T)).
\]

**Lemma 4.3.** Take a minimizer $(x^*(\cdot), y_k^*(\cdot), u^*(\cdot))$ for $(P_i)$ such that $||x^*(\cdot) - \bar{x}(\cdot)||_{L^\infty} \leq \varepsilon/2$. Define the functions $p_k(\cdot) : [S - h_k, T] \to \mathbb{R}$, $k = 0, \ldots, N$ according to

\[
\begin{aligned}
-\dot{p}_k(t - h_k) &= 2 \times i \times k_0(t)(y_k^*(t) - x^*(t-h_k)) \quad \text{a.e. } t \in [(S+h_k) \land T, T], \\
p_k(t) &= 0 \quad \text{for } t \in [(T-h_k) \lor S, T] \quad \text{if } k > 0, \\
\dot{p}_k(t - h_k) &= 0 \quad \text{a.e. } t \in [S, (S+h_k) \land T], \\
-p_0(T) &= \nabla l_1(x^*(T)),
\end{aligned}
\]

and let $p(\cdot) : [S, T] \to \mathbb{R}^n$ be the $W^{1,1}$ function $p(t) := \sum_{k=0}^{N} p_k(t)$ for $t \in [S, T]$. Then

(b)' : \{\{-\dot{p}_k(t - h_k)\}\} in co\partial_{x^*}(p \cdot f - L)(t, \{y_k^*(t)\}, \{u^*(t-h_k)\}) \quad \text{a.e. } t \in [S, T],

(c)' : For any selector $u(\cdot)$ of $U(\cdot)$,

\[\int_{[S, T]} (p \cdot f - \lambda L)(t, y_k^*(t), u(t-h_k)) dt \leq \int_{[S, T]} (p \cdot f - \lambda L)(t, y_k^*(t), \bar{u}(t-h_k)) dt,
\]

(d)' : $p(S) \in \nabla l_0(x^*(S)) + (\gamma_i^\frac{1}{2} + \alpha) \mathcal{B}$, $-p(T) = \nabla l_1(x^*(T)).$

**Proof.** Take $\delta > 0$, $\xi \in \mathbb{R}^n$, functions $y_k(\cdot) \in L^p_{\text{loc}}(\{(S+h_k) \land T, T\})$, $k = 0, \ldots, N$, and a selector $u(\cdot)$ of $U(\cdot)$. Let $x(\cdot)$ be the corresponding state trajectory, with initial condition $x(S) = x^*(S) + \delta \xi$. Assume that $x(t) \in \bar{x}(t) + \varepsilon/2$ for all $t \in [S, T].$
By ‘optimality’ of \((x^*(.), \{y^*_k(\cdot)\}, u^*(\cdot))\) and since \(x(\cdot)\) and \(x^*(\cdot)\) satisfy the dynamic constraint,

\[
\delta^{-1} \left( \sum_{k=0}^{N} \int_{[S+h_k] \cap T,T} i \times k_0(t) \left( |y_k(t) - x(t-h_k)|^2 - |y^*_k(t) - x^*(t-h_k)|^2 \right) dt \right) \\
+ \delta^{-1} \left( l_0(x^*(S) + \delta \xi) - l_0(x^*(S)) + \gamma_\delta^2 \delta \xi \right) \right) + l_1(x(T)) - l_1(x^*(T)) \\
+ \alpha \left( |x^*(S) + \delta \xi - x^*(S)| - |x^*(S) - x(S)| \right) \\
+ \delta^{-1} \left( \int_{[S,T]} (\tilde{L}(t,u(t)) - \tilde{L}(t,u^*(t))) + \gamma_{\delta}^2 (m_i(t,u(t)) - m_i(t,u^*(t))) dt \right) \\
+ \delta^{-1} \left( \int_{[S,T]} p(t) \cdot (\dot{x}(t) - \dot{x}^*(t)) - f(t,\{y_k(t)\},\{u(t-h_k)\}) - f(t,\{y^*_k(t)\},\{u^*(t-h_k)\}) \right) dt \geq 0. \\
\] (4.8)

Performing an integration by parts yields the identity

\[
\int_{[S,T]} p(t) \cdot (\dot{x}(t) - \dot{x}^*(t)) dt = - \int_{[S,T]} \dot{p}(t) \cdot (x(t) - x^*(t)) dt \\
+ p(T) \cdot (x(T) - x^*(T)) - p(S) \cdot (x(S) - x^*(S)). \\
\]

Substituting this expression into (4.8), employing the expansion

\[
|y_k(t) - x(t-h_k)|^2 = |y_k(t) - x^*(t-h_k)|^2 + |x(t-h_k) - x^*(t-h_k)|^2 \\
- 2(y_k(t) - x^*(t-h_k)) \cdot (x(t-h_k) - x^*(t-h_k)) - 2(y_k(t) - y^*_k(t)) \cdot (x(t-h_k) - x^*(t-h_k))
\]

and using the estimates

\[
l_1(x(T)) - l_1(x^*(T)) - \nabla l_1(x^*(T)) \cdot (x(T) - x^*(T)) \leq \theta(|x(T) - x^*(T)|) \\
l_0(x(S)) - l_0(x^*(S)) - \delta \nabla l_0(x^*(S)) \cdot \xi \leq \theta(\delta |\xi|) \\
|x(S) - x(S)| - |x^*(S) - x(S)| \leq \delta |\xi|,
\]

(for some function \(\theta(\cdot) : [0, \infty) \to [0, \infty)\) such that \(\lim_{\alpha \to 0} \alpha^{-1} \theta(\alpha) = 0\), we arrive at

\[
\delta^{-1} \left( \sum_{k=0}^{N} \int_{[S,T]} i \times k_0(t) \left( |y_k(t) - x^*(t-h_k)|^2 - |y^*_k(t) - x^*(t-h_k)|^2 \right) \chi_{[S,T]}(t-h_k) dt \right) \\
+ (\nabla l_0(x^*(S)) - p(S)) \cdot \xi + (\gamma_{\delta}^2 + \alpha) |\xi| \\
+ \delta^{-1} \left( \int_{[S,T]} (\tilde{L}(t,u(t)) - \tilde{L}(t,u^*(t))) + \gamma_{\delta}^2 (m_i(t,u(t)) - m_i(t,u^*(t))) dt \right) \\
+ \delta^{-1} \int_{[S,T]} -p(t) \cdot [f(t,\{y_k(t)\},\{u(t-h_k)\}) - f(t,\{y^*_k(t)\},\{u^*(t-h_k)\})] dt \\
+ E_1(\{y_k(\cdot)\},x(\cdot),\{y^*_k(\cdot)\},x^*(\cdot)) + E_2(\{y_k(\cdot)\},x(\cdot),\{y^*_k(\cdot)\},x^*(\cdot)) \geq 0. \quad (4.9)
\]

in which the first ‘error’ term \(E_1(\cdot)\) is

\[
E_1(\{y_k(\cdot)\},x(\cdot),\{y^*_k(\cdot)\},x^*(\cdot)) := \\
\delta^{-1} \sum_{k=0}^{N} \int_{[S,T]} \left( -2i k_0(t)(y_k(t) - x^*(t-h_k)) - \dot{p}_k(t-h_k) \right) \cdot (x(t-h_k) - x^*(t-h_k)) \\
+ \delta^{-1} \left( \nabla l_1(x^*(T)) + p(T) \right) \cdot (x(T) - x^*(T)),
\]

\[
\chi_{[S,T]}(t-h_k) dt
\]}
and the second ‘error’ term $E_2(.)$ is some function that satisfies

$$E_2(\{y_k(.)\}, x(.), \{y'_k(.)\}, x^*(.) \leq \delta^{-1} \left( \theta(\delta|x|) + \theta(|x(T) - x^*(T)|) \right) + \delta^{-1} K(i) \times \left( \sum_{k=0}^{N} \int_{[S+h_k,T,T]} k_0(t)|y_k(t) - y'_k(t)|dt \times ||x(.) - x^*(.)||_{L^\infty} + \int_{[S,T]} k_0(t)|x(t) - x^*(t)|^2dt \right).$$

(4.10)

($K(i)$ is some number that depends on $i$, but not on the choice of $\xi, u(.)$ and $\{y_k(.)\}$)

Note that $E_1(.)\equiv 0$, because of the defining relations of the $p_k(.)$’s.

We now confirm the assertions of the lemma by examining inequality (4.9), for various choices of $\delta > 0$, $\xi$, the $y_k(.)$’s, $u(.)$.

**Confirmation of (d)’**: Notice that $-p(T) = \nabla l_1(x^*(T))$, by definition of the $p_k(.)$’s. To verify the other transversality condition, take any $\xi \in \mathbb{R}^n$ and sequence $\delta_j \downarrow 0$. For each $j$ let $x_j(.)$ be the state trajectory corresponding to $y^*_k(\cdot)$, $k = 0, \ldots, N, u(.)$ and $d^*(.)$ and with initial value $x_j(S) = x^*(S) + \delta_j \xi$. It is easy to show that

$$||x_j(.) - x^*(.)||_{L^\infty} \to 0, \text{ as } j \to \infty$$

(4.11)

and there exist a number $C$ such that, for $j = 1, 2, \ldots$,

$$\delta_j^{-1} ||x_j(.) - x^*(.)||_{L^\infty} \leq C.$$  

(4.12)

Now consider (4.9) when $\delta = \delta_j$, $y_k(.) = y^*_k(.)$ for $k = 0, \ldots, N$, $u(.) = u^*(.)$, $d(.) = d^*(.)$ and $x(.) = x_j(.)$. From (4.11) and (4.12) we see that

$$E_2(\{y_k(.)\}, x_j(.)\{y'_k(.)\}, x^*(.) \to 0 \text{ as } j \to \infty.$$  

We may pass to the limit as $j \to \infty$, to obtain

$$(\nabla l_0(x^*(S)) - p(S)) : \xi + (\gamma_i^{-\frac{1}{2}} + \alpha)|\xi| \geq 0.$$  

Since this inequality is valid for every $\xi \in \mathbb{R}^n$, we conclude that $p(S) \in \nabla l_0(x^*(S)) + (\gamma_i^{-\frac{1}{2}} + \alpha)\mathbb{B}$.

**Confirmation of (b)’**: Choose $y_k(t) \in L^2_{\mathcal{K}o(.)}(\{(S + h_k) \land T, T\})$ such that

$$y_k(t) \in y^*_k(t) + (1 + k_0(t))^{-1}\mathbb{B} \text{ a.e. } t, \text{ for } k = 0, \ldots, N.$$  

(4.13)

Define

$$\mathcal{S} := \{\tilde{t} \in (S, T) \mid \tilde{t} \text{ is a Lebesgue point of } t \to k_0(t)(|y_k(t) - x^*(t - h_k)|^2 - |y^*_k(t) - x^*(t - h_k)|^2)$$  

and $t \to f(t, \{y_k(t)\}, \{u^*(t - h_k)\}) - f(t, \{y^*_k(t)\}, \{u^*(t - h_k)\})$)

Take $\delta_j \downarrow 0$ and $\tilde{t} \in \mathcal{S}$. For each $j$, $t \in [(S + h_k) \land T, T]$, let

$$y^j_k(t) = \begin{cases} y_k(t) & \text{ if } t \in [\tilde{t}, \tilde{t} + \delta_j] \\ y^*_k(t) & \text{ if } t \notin [\tilde{t}, \tilde{t} + \delta_j] \end{cases}$$  

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for \( k = 0, \ldots, N \). Write \( x_j(.) \) for the state trajectory corresponding to \( \{y_{k}^j(.)\} \) and \( u^*(.) \), with initial value \( x^*(S) \). For each \( j \), consider (4.9) with \( \delta = \delta_j, \xi = 0, \{y_{k}^j(.)\} = \{y_{k}^j(.)\} \) and \( u(.) = u^*(.) \). Making use of (4.13), we can show that (4.11) and (4.12) are satisfied, and \( \int k_0(t)|y_{k}^j(t) - y^*(t)|dt \to 0 \) as \( j \to \infty \). It follows that

\[
E_2(\{y_{k}^j(.)\}, x_j(.)\{y_{k}^*(.)\}, x^*(.) \to 0 \) as \( j \to \infty \).
\]

Since \( t \in S \), we can pass to the limit in (4.9) as \( j \to \infty \), to obtain

\[
\phi(\bar{t}, \{y_{k}(\bar{t})\}) \geq 0,
\]

where

\[
\phi(t, \{y_{k}\}) := \sum_{k} k_0(t)i \left( |y_{k} - x^*(t - h_k)|^2 - |y_{k}^*(t) - x^*(t - h_k)|^2 \right) \chi_{[S,T]}(t - h_k) - p(t) \cdot (f(t, \{y_{k}\}, \{u^*(t - h_k)\}) - f(t, \{y_{k}^*\}, \{u^*(t - h_k)\})
\]

Since \( S \) has full measure

\[
\int_{[S,T]} \phi(t, \{y_{k}(t)\}) dt \geq 0.
\]

But the \( y_{k}(.)\)'s are arbitrary measurable functions satisfying \( y_{k}(t) \in y_{k}^*(t) + (1 + k_0(t))^{-1}B \) for a.e. \( t \in ([S + h_k] \cap T, T] \). Invoking a measurable selection theorem, we can deduce that

\[
\phi(t, \{y_{k}^*\}) = \min \{\phi(t, \{y_{k}\}) | y_{k} \in y_{k}^*(t) + (1 + k_0(t))B, \text{ for } k = 0, \ldots, N \}.
\]

But then

\[
-2ik_0(t) \left( (y_{k}^*(t) - x^*(t - h_0))\chi_{[S,T]}(t - h_0), \ldots, (y_{k}^*(t) - x^*(t - h_N))\chi_{[S,T]}(t - h_N) \right) \in \partial_{\{x_k\}} (-p \cdot f(t, \{y_{k}^*\}, \{u^*(t - h_k)\})
\]

From the defining relations for the \( p_k(.)\)'s we deduce

\[
\{p_0(t - h_k)\} \in \partial_{\{x_k\}} (-p \cdot f(t, \{y_{k}^*\}, \{u^*(t - h_k)\})) \text{ a.e } t \in [S, T].
\]

This relation implies (b)'.

**Confirmation of (c)'**: See Appendix.

To complete Step 1, we observe that Lemma 4.3 provides perturbed versions of the desired conditions, with cost multiplier \( \lambda = 1 \), in terms of costate functions that we now write \( p_k(.) \), \( k = 0, \ldots, N \) to emphasize the \( \bar{i} \) dependence, under the stated hypotheses. Condition (4.5) ensures that, along a subsequence, \( y_{k}^j(.) \) converges in \( L^1 \) and a.e. to \( x(t - h_k) \) on \([S + h_k] \cap T, T] \) (for each \( k \)), \( x_i(.) \) converges uniformly to \( x(.) \), \( u_i(.) \) converges a.e. to \( u(.) \), as \( \bar{i} \to \infty \). The \( p_k(.)\)'s converge uniformly to \( W^{1,1} \) functions \( p_k(.) \), \( k = 0, \ldots, N \) and their time derivatives converge weakly in \( L^1 \) to the time derivatives of the \( p_k(.)\)'s. A standard convergence analysis can be used to justify passage to the limit in conditions (b)'-(d)' as \( \bar{i} \to \infty \), to recover the required necessary conditions (b)-(d). Notice, in particular, that the limiting \( p(.) \) function satisfies

\[
p(S) \in \nabla l_0(\bar{x}(S)) + \alpha B = \bar{\partial}l(\bar{x}(S)),
\]

where \( \bar{l}(.) \) is the function \( x_0 \to l(x_0) + \alpha |x_0 - \bar{x}(S)| \), which is the left transversality condition. Of course (a) is automatically satisfied because \( \lambda = 1 \).
Step 2: We show that, if the assertions of Thm. 2.1 are valid under (H1)-(H3) and (A0)-(A4), then they are also valid under (H1)-(H3), (A0)-(A3).

Assume that Thm. 2.1 is valid under (H1)-(H3) and (A0)-(A4). Suppose \((\bar{x}(\cdot), \bar{u}(\cdot))\) is an \(L^\infty\) local minimizer for \((P)\) when we impose hypotheses (H1)-(H3) and (A0)-(A3). By (H1), \(l(\cdot)\) is Lipschitz continuous on a ball about \((\bar{x}(S), \bar{x}(T))\). By redefining this function outside the ball (a change which does not affect \(L^\infty\) local minimizers), we can arrange that \(l(\cdot)\) is Lipschitz continuous on \(\mathbb{R}^n \times \mathbb{R}^n\); write the Lipschitz constant \(k_l\).

For \(i = 1, 2, \ldots\), let \(l^i(\cdot)\) be the \('i - quadratic inf convolution’ of \(l(\cdot)\):

\[
l^i(z) := \inf_{y \in \mathbb{R}^n \times \mathbb{R}^n} \{l(y) + i \times |y - z|^2\}.
\]

(4.14)

The key ‘quadratic inf convolution’ properties of \(l^i(\cdot)\) (see [S]) are: take any \(z \in \mathbb{R}^n \times \mathbb{R}^n\) and let \(y \in \mathbb{R}^n \times \mathbb{R}^n\) be any vector achieving the infimum in (4.14) (one such vector exists). Let

\[
\eta^i(z) := 2i(y - z).
\]

(i): \(l^i(\cdot)\) is locally Lipschitz continuous with Lipschitz constant \(k_l\).

(ii): \(l^i(z) \geq l_j(z) - k_l^2 \times i^{-1}\).

(iii): \(l^i(z') - l^i(z) \leq \eta^i(z) \cdot (z' - z) + i \times |z' - z|^2\) for all \(z' \in \mathbb{R}^n \times \mathbb{R}^n\)

(iv): \(\eta^i(z) \in \partial_p l(y)\).

(v): \(|y - z| \leq k_l \times i^{-1}\).

Since \((\bar{x}(\cdot), \bar{u}(\cdot))\) is an \(L^\infty\) local minimizer for \((P)\), \((\bar{x}(\cdot), \bar{u}(\cdot))\) is a minimizer for

\[
(Q) : \quad \text{Minimize } \{J((x(\cdot), u(\cdot)) | (x(\cdot), u(\cdot)) \in \mathcal{B}_\epsilon\},
\]

for some \(\epsilon > 0\), where

\[
J((x(\cdot), u(\cdot)) := \int_{[S,T]} \tilde{L}(t, u(t)) dt + l(x(S), x(T)),
\]

and \(\mathcal{B}_\epsilon := \{\text{feasible processes } (x(\cdot), u(\cdot)) \text{ for } (P) | ||x(\cdot) - \bar{x}(\cdot)||_{L^\infty} \leq \epsilon\}\).

For each \(i\), consider the problem

\[
(Q^i) : \quad \text{Minimize } \{J^i((x(\cdot), u(\cdot)) | (x(\cdot), u(\cdot)) \in \mathcal{B}_\epsilon\}
\]

in which

\[
J^i((x(\cdot), u(\cdot)) := \int_{[S,T]} \tilde{L}(t, u(t)) dt + l^i(x(S), x(T))\).
\]

Equip \(\mathcal{B}_\epsilon\) with the metric

\[
d_\epsilon((x'(\cdot), u'(\cdot)), (x(\cdot), u(\cdot))) = \text{meas } \{t \in [S,T] | u'(t) \neq u(t)\} + |x'(S) - x(S)|.
\]

It can be shown that, w.r.t. this metric, \(\mathcal{B}_\epsilon\) is complete and \(J^i(\cdot, \cdot)\) is continuous on \(\mathcal{B}_\epsilon\).

Now note that, in view of property (ii) of the quadratic inf convolution operation, \((\bar{x}(\cdot), \bar{u}(\cdot))\) is a \(\gamma_k\)-minimizer for \((Q^i)\), where \(\gamma_k = k_l^2 i^{-1}\). In consequence of Ekeland’s Theorem, there exists \((x_i(\cdot), u_i(\cdot)) \in \mathcal{B}_\epsilon\) which is a minimizer for \((\tilde{Q}^i)\):

\[
(\tilde{Q}^i) \text{ Min } \{J^i((x(\cdot), u(\cdot)) + \gamma_k t d_\epsilon((x(\cdot), u(\cdot)), (x_i(\cdot), u_i(\cdot))) | (x(\cdot), u(\cdot)) \in \mathcal{B}_\epsilon\}.
\]

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and
\[ d_c((x_i(.), u_i(.)), (\bar{x}(.), \bar{u}(.))) \leq \frac{1}{\gamma_i} \, . \] (4.15)

The cost function for \((\tilde{Q}^i)\) can be written
\[ J_i(x(\cdot), u(\cdot)) := \int_{[S, T]} (L(t, u(t)) + \gamma_i^{\frac{1}{2}} m_i(t, u(t))) dt + i^0(x(S), x(T)) + \gamma_i^{\frac{3}{2}} |x(S) - x_i(S)|. \]

\((m_i(t, u))\) was defined in \((4.6)\). But by property (iii) of quadratic inf convolutions (see above),
\[
(l^i(x_0, x_1) - (l^i(x_i(S), x_i(T))) \\
\leq \eta_0^i \cdot (x_0 - x_i(S)) + \eta_1^i \cdot (x_1 - x_i(T)) + i \times (|x_0 - x_i(S)|^2 + |x_1 - x_i(T)|^2)
\]
for all \((x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n\), with equality when \((x_0, x_1) = (x_i(S), x_i(T))\). Here
\[
(\eta_0^i, \eta_1^i) \in \partial p_i^0(y_0, y_1)
\]
for some \((y_0, y_1) \in (x_i(S), x_i(T)) + k_l i^{-1}(\mathbb{B} \times \mathbb{B})\). So \((x_i(.), u_i(.))\) is also a minimizer for
\[
(\tilde{Q}_i^0) : \text{ Minimize } \{J_i^0((x(.), u(.)) | (x(.), u(.)) \in A_k\}
\]
in which
\[
J_i^0((x(.), u(.)) := \int_{[S, T]} (\bar{L}(t, u(t)) + \gamma_i^{\frac{1}{2}} m_i(t, u(t))) dt + \eta_0^i \cdot (x(S) - x_i(S)) + \eta_1^i \cdot (x(T) - x_i(T)) + i \times (|x(S) - x_i(S)|^2 + |x(T) - x_i(T)|^2) + \gamma_i^{\frac{3}{2}} |x(S) - x_i(S)|. \]

The data for Problem \((\tilde{Q}^i)\) satisfies (H1), (H2), (A0)-(A3) and (A4). We may therefore apply the Maximum Principle (with \(\lambda = 1\)), which we know to be valid under these hypotheses. We conclude the existence of \(p_k(\cdot) : [S, T] \rightarrow \mathbb{R}^n, k = 0, \ldots, N, \) and \(p^i(\cdot) : [S, T] \rightarrow \mathbb{R}^n\) such that
\[(2.2) \text{ and } (2.3), \text{ as well as conditions (a) and (b) of the theorem statement, are satisfied, when }\]
\((x_i(.), u_i(.))\) replaces \((\bar{x}(.), \bar{u}(.))\) and \(p_k^i(.)\) replaces \(p_k(.)\), etc., and when \(\lambda = 1\). In addition, we have, for any selector \(u(.)\) of \(U(.)\),
\[
(c)' : \int_{[S, T]} (p \cdot f - \lambda \bar{L})(t, \{x_i(t - h_k)\}, \{u(t - h_k)\}) dt \leq \int_{[S, T]} (p^{\prime i} - \lambda \bar{L})(t, \{x_i(t - h_k)\}, \{u_i(t - h_k)\}) dt
\]
\[
(d)' : (p^i(S), -p^{\prime i}(T)) \in \partial l((x_i(S) + x_i(T)) + k_l i^{-1}\mathbb{B}) + \gamma_i^{\frac{3}{2}} \mathbb{B} \times \{0\}. \]

We deduce from \((4.15)\) that, along some subsequence, \(u_i(t) \rightarrow \bar{u}(t)\), a.e. \(t \in [S, T]\) and \(x_i(t) \rightarrow \bar{x}(t)\) uniformly over \(t \in [S, T]\). On the other hand, we can show from the conditions that \(p_k^i(.)\), 
\(k = 1, \ldots, N\) and \(p^i(.)\), \(i = 1, 2, \ldots\), are uniformly bounded on their domains and their derivatives are uniformly integrably bounded. We may deduce from Ascoli’s theorem that, after a further subsequence extraction, for each \(k, p_k^i(.)\) converge uniformly to some \(W^{1,1}\) function \(p_k(.)\) as \(i \rightarrow \infty\), and the derivatives \(p_k^i(.)\) converge weakly in \(L^1\) to \(p_k(.)\), for each \(k\). The function \(p^i(.)\) converges likewise to some \(p(.)\). A standard analysis permits us to pass to the limit in conditions (a) and (b) (modified as indicated above) and \((c)' \cdot (d)'\). We thereby achieve confirmation of all assertions of Thm. \([2.1]\) (with \(\lambda = 1\)), for the special case of when the additional hypotheses (A1)-(A4) and (A5)' are satisfied.

**Step 3:** Assume that the assertions of Thm. \([2.1]\) are valid under hypotheses (H1)-(H3), (A0) - (A2) and (A3). Then the assertions remain valid (with \(\lambda = 1\)) when \((\bar{x}(.), \bar{u}(.))\) is an \(L^\infty\) local solution under the weaker set of hypotheses, in which (A3) is replaced by (A3)', where
(A3)': There exists a closed set $C_0 \subset \mathbb{R}^n$ such that $C = C_0 \times \mathbb{R}^n$.

A simple contradiction argument (c.f. the proof of the ‘Exact Penalization Thm.’ [16, p.48]), based on Thm. 4.1, permits us to conclude that $(\bar{x}(.), \bar{u}(.) )$ remains an $L^\infty$ local minimizer for a modified problem in which the endpoint constraint set $C_0 \times \mathbb{R}^n$ is replaced by $\mathbb{R}^n \times \mathbb{R}^n$ and the endpoint cost function $l(.)$ in (P) is replaced by the Lipschitz continuous function $\bar{l}(.)$:

$$\bar{l}(x_0, x_1) := l(x_0, x_1) + Kd_{c_0}(x_0),$$

for any

$$K > k_l \times \exp\{(N + 1) \times \int_S^T k_0(t)dt\},$$

in which $k_l$ is the Lipschitz constant of $l_i(.)$. Applying Thm. 2.1 to the modified problem, which is permissible since $C = \mathbb{R}^n \times \mathbb{R}^n$ and $l_0(.) = \bar{l}_0(\cdot)$, yields the desired necessary conditions for the original problem. Note that the transversality condition for the modified problem implies

$$(p(S), -p(T) \in \partial \bar{l}(\bar{x}(S), \bar{x}(T)) \subset \partial l(\bar{x}(S), \bar{x}(T)) + K \partial d_{c_0}(\bar{x}(S)) \times \{0\}$$

$$(\bar{x}(S), \bar{x}(T)) \subset \partial l(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T)),$$

which is the appropriate left transversality condition for the original problem.

**Step 4:** Assume the assertions of Thm. 2.1 are valid under (H1)-(H3), (A0)-(A2), (A3)’. Then they are valid under (H0)-(H3), (A0)-(A2).

Assume that the assertions of Thm. 2.1 are valid under hypotheses (H1)-(H3), (A0) - (A2) and (A3)’. Suppose that $(\bar{x}(.), \bar{u}(.) )$ is an $L^\infty$ local solution to (P) under (H1)-(H3), (A1) and (A2) alone. We must show that $(\bar{x}(.), \bar{u}(.) )$ satisfies the Maximum Principle.

Take $\gamma_i \downarrow 0$. For $i = 1, 2, \ldots$, consider the problem with $(n + n)$-dimensional state vector $(z, x)$:

$$\left\{ \begin{array}{l}
\text{Minimize } J_i^1(z(\cdot), x(\cdot), u(\cdot)) \text{ subject to } \\
(\dot{z}(t), \dot{x}(t)) = (0, f(t, \{x(t - h_k)\}, \{u(t - h_k)\}))) \\
u(t) \in U(t) \text{ a.e. } t \in [S, T], \\
(z(S), x(S)) \in \tilde{C} := \{(z, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid z = x\}
\end{array} \right.$$
In consequence of Ekeland’s Theorem, there exists \((z_i(\cdot), x_i(\cdot), u_i(\cdot)) \in A_i^1\), for which
\[
d_{C}((z_i(\cdot), x_i(\cdot), u_i(\cdot)), (\tilde{z}(\cdot), \tilde{x}(\cdot), \tilde{u}(\cdot)) \leq \gamma^\frac{1}{2}_i,
\] (4.16)
and minimizes \(\tilde{J}_1(\cdot)\) over \(A_i^1\). Here,
\[
\tilde{J}_1(z(\cdot), x(\cdot), u(\cdot)) = J_1(z(\cdot), x(\cdot), u(\cdot)) + \gamma^\frac{1}{2}_i \left( \int_{[S,T]} m_i(t, u(t)) \, dt + |x(S) - x_i(S)| + |z(S) - z_i(S)| \right).
\]
From (4.16) it can be deduced that \(||(z_i(\cdot), x_i(\cdot)) - (\tilde{x}(S), \tilde{x}(\cdot))||_{L^\infty} \to 0\) as \(i \to \infty\) and therefore that, for \(i\) sufficiently large, \((z_i, x_i(\cdot))\) is an \(L^\infty\) local minimizer for \((P_i^\gamma)\). The data for this last problem satisfies \((H1)-(H3), (A0)-(A3), (A4)'\) and \((A5)'\), and we may therefore apply the Maximum Principle (with \(\lambda = 1\)). We deduce the existence of a costate trajectory \(p(\cdot)\) with associated decomposition \(p(\cdot) = \sum_{k=0}^N p_k(\cdot)\) (arising from the \(x\)-state) and a costate trajectory \(q(\cdot)\) (arising from the \(z\)-state), and with properties given by Thm. 2.1. Conditions \((b)\) and \((c)\) imply that \(q(\cdot)\) is a constant, which we write \(q\),

\[(b)' : \{-\hat{p}_k(t - h_k)\} \in co \partial x_0,\ldots,x_N p(t) \cdot f(t, \{x_i(t - h_k)\}, \{u_i(t - h_k)\}, \text{ a.e. } t \in [S,T].
\]

\[(c)' : \text{ for any selector } u(\cdot) \text{ of } U(\cdot)
\]
\[
\int_{[S,T]} (p(t) \cdot f - \lambda \bar{L})(t, \{x_i(-h_k)\}, \{u(t - h_k)\}) \, dt \leq \int_{[S,T]} (p(t) \cdot f - \lambda \bar{L})(t, \{x_i(-h_k)\}, \{u_i(t - h_k)\}) \, dt + \gamma^\frac{1}{2}_i |T - S|
\]
Let us examine the implications of the transversality condition \((e)\). In this connection, we make use of the fact that
\[
\max\{g(z_i(T), x_i(T)) - g(\tilde{x}(T), \tilde{x}(\cdot)) + \gamma_i, d_C(z_i(T), x_i(T))\} > 0, \text{ for } i \text{ sufficiently large}.
\]
Indeed if this were not the case then, since \(z_i(T) = x_i(S)\), we would have \(g(x_i(S), x_i(T)) - g(\tilde{x}(S), \tilde{x}(\cdot)) \leq -\gamma_i \) and \(d_C(x_i(S), x_i(T)) = 0\). We could also arrange, by choosing \(i\) sufficiently large, that \(||x_i(\cdot) - \tilde{x}(\cdot)||_{L^\infty} \leq \epsilon\) for \(\epsilon\) arbitrarily small. This contradicts the \(L^\infty\) local optimality of \((x_i(\cdot), u_i(\cdot))\) for \((P)\).

Note that \(N_C(x, x) = \{(e_0, e_1)|e_0 + e_1 = 0\}\) which, in combination with the transversality condition \((d)\), yields the information
\[
q \in -p(S) + 2\gamma^\frac{1}{2}_i B.
\] (4.17)
In consequence of the max rule for limiting subdifferentials, we know that
\[
\partial \max\{g(z, x) - g(\tilde{z}(T), \tilde{x}(T)) + \gamma_i, d_C(z, x)\} \subset \lambda \partial g(z, x) + (1 - \lambda) \partial d_C(z, x),
\]
for some \(\lambda \in [0,1]\). Moreover, \(\lambda > 0\) implies
\[
'dC(z, x) = \max\{g(z, x) - g(\tilde{z}(T), \tilde{x}(T)) + \gamma_i, d_C(z, x)\}.
\]
Since \(z_i(S) = z_i(T) = x_i(S)\), and in view of (4.17), we deduce from condition \((d)\) that
We claim that, for \( i \) sufficiently large,

\[
\lambda + \|p(.)\|_{L^\infty} > 0.
\]

Indeed if this were not true, we would have \( \lambda = 0 \) and \( (p(S), p(T)) = 0 \). It would follow that

\[
(q, 0) \in \partial d_C(x_i(S), x_i(T)) \quad \text{and} \quad |q| \leq 2\gamma_i.
\]

But \( \lambda = 0 \) also implies \((1 - \lambda) > 0\), whence \( d_C(x_i(S), x_i(T)) > 0 \). This last condition is known to implies that all elements \( \xi \) in the set \( \partial d_C(x_i(S), x_i(T)) \) have Euclidean length (see [16]). This is not possible because \((q, 0)\) is such an element, and has Euclidean length \( 2\gamma_i \) (which can be made arbitrarily small). We can therefore arrange, by positive scaling of the Lagrange multipliers \((\{p_k(.)\}, p(.), \lambda)\), that

\[
\lambda + \|p(.)\|_{L^\infty} = 1.
\]

Bearing in mind that \( \partial d_C(z) \subset N_C(z) \) (for \( z \in C \)), we have arrived at a set of relations \((a)' - (d)'\) that are approximate version of those asserted in Thm. 2.1 involving the multiplier set \( \{p_k(.)\}, \lambda \), with reference to \((x_i(.), u_i(.))\). To emphasize the fact that these relations depend on \( i \), we rewrite the multiplier set \( \{p_k^0(.)\}, p^i(\cdot), \lambda^i\). Condition (4.16) ensures that, along a subsequence \( u_i(t) \to \bar{u}(t) \) a.e., We deduce from Thm. 4.1 that \( x_i(.) \to \bar{x}(.) \) uniformly. For each \( k, p_k^i(\cdot), i = 1, 2, \ldots \), is a uniformly bounded sequence of absolutely continuous functions with uniformly integrably bounded derivatives. It follows that, for each \( k \), \( \{p_k^i(.)\} \) converges to an absolutely continuous function \( p_k(.) \), \( k = 0, \ldots, N \), and \( \{p_k^i(.)\} \) converges to \( p_k(.) \) weakly in \( L^1 \), along a subsequence. We can also arrange that \( \lambda_i \to \lambda \) for some \( \lambda \in [0, 1] \). A standard convergence analysis permits us to pass to the limit in relations \((a)' - (d)'\), and thereby arrive at the assertions of Thm. 2.1.

\textbf{Step 5:} Suppose the assertions of Thm. 2.1 are valid under \((H1)-(H3), (A0)-(A2)\). Then they are valid under \((H1)-(H3), (A0)\) and \((A1)\).

Assume that the assertions of Thm. 2.1 are valid under \((H1)-(H3), (A0)-(A2)\). Suppose \((\bar{x}(.), \bar{u}(.)\)) is an \( L^\infty \) local minimizer when the hypotheses are satisfied, with the possible exception of \((A2)\). For \( i = 1, 2, \ldots \), define family of functions

\[
U_i := \{u(.)\} \text{ is a selector of } U(.) \mid k(t, \{u(t - h_k)\}) + f(t, \{\bar{x}(t - h_k)\}, \{u(t - h_k)\}) \leq k(t, \{\bar{u}(t - h_k)\}) + |\dot{x}(t)| + i \quad (4.18)
\]

For each \( i \), let \( U_i(.) : [S, T] \to \mathbb{R}^m \) be the multifunction defined, \( \mathcal{L} \), a.e. by the condition

\[
\text{Gr } U_i(.) = U_i. \quad (4.19)
\]

We note that

\[
\bar{u}(t) \in U_1(t) \subset U_2(t) \subset \ldots \quad \text{and} \quad U_i(t) = U(t), \text{ a.e.} \quad (4.20)
\]

For each \( i \), \((\bar{x}(.), \bar{u}(.)\)) continues to be an \( L^\infty \) local minimizer for \((P_i)\) which is the modification of \((P)\) in which \( U_i(.) \) replaces \( U(.) \). Because the data for \((P_i)\) satisfies \((A2)\), the assertions of Thm. 2.1 are available to us: they yield (for each \( i \)) a cost multiplier \( \lambda_i \geq 0 \) and a costate arc \( p^i(.) \), with decomposition \( p^i(.) = \sum_k p_k^i(.) \), such that \( \lambda_i + \|p^i(.)\|_{L^\infty} = 1 \) and \((b)-(d)\) of Thm. 2.1 are satisfied (when \((\lambda_i, \{p_k^i(.)\}, p^i(.)\)) replaces \((\lambda, \{p_k(.)\}, p(.)\))). Extracting subsequences we can arrange that \( p_j^i(.) \to p_j(.) \), for each \( j \) and \( p^i(.) \to p(.) \), uniformly, as \( i \to \infty \). A
where \( \bar{\phi} \). Passing to the limit as \( \ell \rightarrow \infty \), the underlying time interval is now \( \int_{[S,T]} \), by 'integrability', measurable and possibly violated. Then (\( \bar{\phi} \)), integrable. For this purpose, define, for each integer \( \ell \),

\[
\bar{m}(t, \{u_k\}) := (p \cdot f - \lambda \bar{\phi})(t, \{\bar{x}(t-h_k)\}, \{u_k\}).
\]

We must validate (c) when \( u(.) \) is an arbitrary selector of \( U(.) \) such that \( m(t, \{u(t-h_k)\}) \) is integrable. For this purpose, define, for each integer \( \ell \),

\[
S_{\ell} := \{ t \in [S,T] \mid m(t, \{u(t-h_k)\}) - m(t, \{\bar{u}(t-h_k)\}) \geq \ell \}
\]

By 'integrability', meas\{\( S_{\ell} \)\} \( \rightarrow 0 \) as \( \ell \rightarrow \infty \). For each \( \ell \), define the subsets \( A^\ell_j \subset [S,T] \), \( j = 0, \ldots, K \), in which \( K \) is the integer \( K := \text{int} (T-S)/h_1 \), recursively: \( A^\ell_0 = S_{\ell} \) and \( A^\ell_j = ((A^\ell_{j-1} + h_1) \cap [S,T]) \cup \cdots \cup ((A^\ell_{j-1} + h_N) \cap [S,T]), j = 1, \ldots, N \). Now write \( A^\ell := \cup_{j=0}^K A^\ell_j \). Define

\[
\bar{u}(t) := \bar{u}(t)\chi_{A^\ell}(t) + u(t)\chi_{[S,T] \setminus A^\ell}(t).
\]

We can deduce from the special structure of \( A^\ell \) that \( \bar{u}(.) \) is a selector of \( U(.) \) for \( i \) sufficiently large, and

\[
' \in A^\ell \implies 'u(t-h_k) \in A^\ell, k = 0, \ldots, N' \quad \text{and} \quad 't \notin A^\ell \implies 'u(t-h_k) \notin A^\ell, k = 0, \ldots, N'.
\]

It follows that, for each \( \ell \),

\[
\int_{[S,T] \cap A^\ell} (m(t, \{u(t-h_k)\}) - m(t, \{\bar{u}(t-h_k)\})) dt \leq 0.
\]

Passing to the limit as \( \ell \rightarrow \infty \), using the fact that meas\{\( A^\ell \)\} \( \leq N^K \rightarrow 0 \), we arrive at

\[
\int_{[S,T]} (m(t, \{u(t-h_k)\}) - m(t, \{\bar{u}(t-h_k)\})) dt \leq 0.
\]

We have shown that the assertions of Thm. \( \mathbb{1} \) are valid under (H1)-(H3), (A0)-(A1).

Now suppose \( (\bar{x}(.), \bar{u}(.) ) \) is an \( L^\infty \) local minimizer under (H1)-(H3), but when either (A0) or (A1) are possibly violated. Then \( (\bar{y}(.), \bar{z}(.) ) : [S-h,T] \rightarrow \mathbb{R}^{n+1}, \bar{v}(.) : [S-h,T] \rightarrow \mathbb{R}^{2m+n} \) is an \( L^\infty \) local minimizer for the reformulated problem

\[
(P_r)
\]

\[
\begin{align*}
&\text{Minimize} \quad g(y(S-h), y(T)) + z(T) \\
&\text{over} \quad (y(.), z(.)) \in W^{1,1}([S-h,T]; \mathbb{R}^{n+1}) \text{ and measurable } u(.) : [S-h,T] \rightarrow \mathbb{R}^{2m+n}, \\
&\text{such that} \\
&(\bar{y}(t), \bar{z}(t)) = (\phi, M)(t, \{y(s-h_k)\}, \{v(s-h_k)\}) \text{ a.e. } t \in [S-h,T], \\
&(v(t) \in V(t) \text{ a.e. } t \in [S-h,T], \\
&((y(S-h), y(T)) \in \mathbb{R}^+ \times C, z(S-h) \geq 0).
\end{align*}
\]

where \( \bar{y}(t) := \bar{x}(S)\chi_{[S-h,S]}(t) + \bar{x}(t)\chi_{[S,T]}(t), \bar{z}(t) := \int_{[S,S+h]} M((s, \{\bar{x}(s-h_k)\}, \{v(s-h_k)\})) ds \\
\text{and} \quad \bar{v}(t) := (0, \bar{d}(t))\chi_{[S-h,S]}(t) + (\bar{u}, 0)\chi_{[S,T]}(t).
\]

The underlying time interval is now \([S-h,T]\) and \((\phi, M)(.)\) and \( V(.) \) are:

\[
\begin{align*}
\phi(t, \{y_k\}, \{v_k = (u_k, d_k)\}) := 0\chi_{[S-h,S]}(t) + f(t, \{y_k\}, \{u_k\}; \{d_k\})\chi_{[S,T]}(t) \\
M(t, \{y_k\}, \{v_k = (u_k, d_k)\}) := \Lambda(t, d_0)\chi_{[S-h,S]}(t) + L(t, \{y_k\}, \{u_k\}; \{d_k\})\chi_{[S,T]}(t) \\
V(t) := ((0) \times D(t))\chi_{[S-h,S]}(t) + (U(t) \times \{0\})\chi_{[S,T]}(t).
\end{align*}
\]

\[23\]
Notice that the cost integrands have been eliminated by ‘state augmentation’ and the ‘old’ initial data \(d(.)\) has been absorbed into the ‘new’ control \(v(.)\). There is no need for ‘new’ initial data because \(((f, L)(t, \ldots))\chi_{[S, T]}(t) = (0, 0)\) for \(t \in [S-h, S]\). The data for the reformulated problem continues to satisfy \((\text{H}1)-(\text{H}3)\), now with reference to \(((\tilde{y}(.), \tilde{z}(.)), \tilde{v}(.))\), but also the additional conditions. Application the special case of Thm. 2.1 in which \((\text{A}0)\) and \((\text{A}1)\) are satisfied, yields the conditions listed in Thm. 2.1 for \(L^\infty\) local minimizer \((\tilde{x}(.), \tilde{u}(.), \tilde{d}(.))\), in the original problem.

Note that condition \((b)\) in the theorem statement and the defining relations for the \(p_k(.)\)’s and \(p(.)\) do indeed imply the ‘advance’ differential inclusion condition \((b^*)\) for \(p(.)\). This is because each \(p_k(.)\) satisfies the relation \(-\dot{p}_k(.) \in \partial \tilde{v}_k\lambda \tilde{L}(p \cdot f - \lambda L)(.)\), involving the projected limiting subdifferential \(\partial \tilde{v}_k\lambda \tilde{L}(p \cdot f - \lambda L)\). \((b^*)\) now follows from the definition (2.3) of \(p(.)\) and condition (2.2), which implies \(-\dot{p}(t) = -\sum_{k=0}^N \tilde{p}_k(t)\) a.e.

Finally, we justify including the ‘pointwise’ version of the Weierstrass condition \((b^*)\) in the assertions of Thm. 2.1. By consideration of the reformulated problem \((P_r)\) of Step 5, we may restrict attention to the case when \((\text{A}0)\) holds and \(L(.)\) ≡ 0. A simple analysis using ‘needle variations’ permits us to deduce from the integral Weierstrass condition \((b)\) that, for each \(i\),

\[
\mathcal{H}_\lambda(t, \bar{u}(t); \bar{x}(.), \bar{u}(.), p(.)) = \max_{u \in U_i(t)} \mathcal{H}_\lambda(t, u; \bar{x}(.), \bar{u}(.), p(.)) \text{ a.e. } t \in [S, T].
\]

(4.21)

Here \(U_i(.)\) is the multifunction defined by (4.18) and (4.19). \((\mathcal{H}_\lambda(t, u; .)\) is the function (2.1); we have suppressed reference to \(d(.)\) in the notation for \(\mathcal{H}_\lambda(t, u; .)\), since \((\text{H}0)\) is assumed to be satisfied.) But we may use (4.20) to show that (4.21) remains valid, when we substitute \(U(t)\) for \(U^i(t)\). We have confirmed the validity of the pointwise Weierstrass condition.

5 Proof of Thm. 3.1

To prove the PMP for free end-time problems, we initially assume that the following additional hypotheses are satisfied

(A0): \(f(t, \{x_k\}, \{u_k\}; \{d_k\})\) does not depend on the initial data \(\{d_k\}\) and \(\Lambda(., .) \equiv 0\).

(When (A0) is satisfied, we write \(f(t, \{x_k\}, \{u_k\})\) in place of \(f(t, \{x_k\}, \{u_k\}; \{d_k\})\).)

(A1): \(L(., .) \equiv 0\).

(A2): There exist integrable functions \(c_0(.) : [S, T+\epsilon] \rightarrow \mathbb{R}\) and \(k_0(.) : [S, T+\epsilon] \rightarrow \mathbb{R}\) such that, for all selectors \(u(.)\) \(U(.)\) and a.e. \(t \in [S, T+\epsilon]\), the mapping \(\{x_k\} \rightarrow f(t, \{x_k\}, \{u(t-h_k)\})\) is \(c_0(t)\) bounded and \(k_0(t)\)-Lipschitz continuous on \(\mathbb{R}^{n \times (N+1)}\).

The proof (under the additional hypotheses) proceeds in two steps. In the first step we consider the case when only the left endpoint of state trajectories is constrained. In the second, we show the PMP for problems involving general endpoint constraints can be derived by applying it to a sequence of perturbed problems with free right endpoints (the case treated in Step 1), and passage to the limit.
Step 1: Let \((x^*(\cdot), u^*(\cdot), T^*)\) be an \(L^\infty\) be local minimizer the following problem which, when regarded as a special case of \((P_{FT})\), has data satisfying (H1)-(H3), (A0) and (A2):

\[
\begin{align*}
(Q_{FT}) \quad \begin{cases}
\text{Minimize} & g'(x(T), T) + \alpha \left( \int_{[S, T]} m(t, u(t)) dt + |x(S) - x^*(S)| + |T - T^*| \right) \\
\text{subject to} & \dot{x}(t) = f(t, \{x(t - h_k)\}, u(t)), \ a.e. \ t \in [S, T^* + \epsilon] \\
& u(t) \in U(t) \ a.e. \ t \in [S, T^* + \epsilon], \\
& x(S) \in C_0,
\end{cases}
\end{align*}
\]

Here, \(\alpha \geq 0\) is a given number, \(m(\cdot, \cdot)\) is a given bounded, \(\mathcal{L} \times \mathcal{B}\) measurable function and \(g'(. \cdot)\) is a given Lipschitz continuous function. It is assumed that \(T^* - S > h\). Our goal in this step is to prove the following necessary conditions:

There exist \(p_k(\cdot) \in W^{1,1}([S - h_k, T^*]; \mathbb{R}^n), k = 0, \ldots, N,\) satisfying (2.2) and \(p(\cdot)\) given by (2.3) (when \(T = T^*\)), such that

\[
\begin{align*}
(b') & : \{-\dot{p}_k(t - h_k)\} \in \text{co} \partial x_0, \ldots, x_N p(t) \cdot f(t, \{x^*(t - h_k)\}, u(t)), \ a.e. \ t \in [S, T^*], \\
(c') & : (p \cdot f - \lambda m)(t, \{x^*(t - h_k)\}, u^*(t))) = \max_{u \in U(t)} (p \cdot f - \lambda m)(t, \{\bar{x}_i(t - h_k)\}, u) \ a.e. \ t \in [S, T^*], \\
(d') & : (-p(T^*), \xi) \in \partial g'(x^*(T), T^*) + 2\alpha \mathbb{B}, \ p(S) \in \alpha \mathbb{B} + NC_0(x^*(S)), \text{ for some } \xi \in \mathbb{R} \ s.t. \\
& \xi \in \text{ess sup} \left\{ \max_{t \in U(t)} p(T^* - t) \cdot f(t, x^*(T^* - h_0), \ldots, x^*(T^* - h_N), u) \right\}.
\end{align*}
\]

We may assume, without loss of generality, that \(g'(. \cdot)\) is continuously differentiable, not merely Lipschitz continuous. This is because, if \(g'(. \cdot)\) were Lipschitz continuous, we could replace it by its \(i\)-quadratic inf convolution \(g'_i(\cdot \cdot)\), for \(i = 1, 2, \ldots\). For each \(i\), \((x^*(\cdot), u^*(\cdot), T^*)\) is a \(\gamma_i\)-minimizer of the perturbed problem, for some sequence \(\gamma_i \downarrow 0\). We may then apply Ekeland’s theorem with the following metric on the space of processes:

\[
d_E((x'(.), u'(.), T'), (x(.), u(.), T)) = |x'(S) - x(S)| + |T' - T| + \text{meas } \{t \in [S, T \land T'] | u'(t) \neq u(t)\}. \tag{5.1}
\]

We thereby arrive at a minimizer \((x_i(\cdot), u_i(\cdot), T_i)\) for a perturbed version of \((Q_{FT})\), in which \(g(\cdot \cdot)\) is replaced by \(g'_i(\cdot \cdot)\). The element \((x_i(\cdot), u_i(\cdot), T_i)\) remains a minimizer when \(g'_i(\cdot \cdot)\) is replaced by a quadratic function (plus a perturbation term) that majorizes \(g_{i+1}(\cdot \cdot)\) and coincides at \((T_i, x_i(T_i))\). We have arrived at in this way a problem again with the structure of \((Q_{FT})\), but in which \(g(\cdot \cdot)\) has been replaced by a continuously differentiable function. (It is precisely in anticipation of stage of the analysis that the cost in \((Q_{FT})\) is furnished with the ‘perturbation’ term \(\alpha (\cdot \cdot)\). The special case of the Maximum Principle (with smooth terminal state and time cost) can be applied, with reference to \((x_i(\cdot), u_i(\cdot), T_i)\). We obtain the asserted necessary conditions (for Lipschitz continuous \(g'_i(\cdot \cdot)\)) in the limit as \(i \to \infty\). (The details are very similar to those followed in Step 2 of the proof of Thm. 2.1.)

So we assume \(g'(\cdot \cdot)\) is continuously differentiable. For \(T\) fixed at \(T = T^*\), \((x^*(\cdot), u^*(\cdot))\) is an \(L^\infty\) local minimizer for the corresponding fixed time problem. We then deduce from Thm. 2.1 existence of functions \(\{p_k(\cdot)\}\) and \(p(\cdot)\) satisfying (2.2) and (2.3), and conditions (b)'-(d)'. We also know that \(p(S) \in \alpha \mathbb{B} + NC_0(x^*(S))\). It remains to validate the transversality condition involving the optimal end-time. Take \(\delta \in (0, h_1)\) such that \(T^* - \delta \geq S\). \((h_1\) is the shortest time
delays period.) Take also any \( \gamma > 0 \) and let \( v^*(\cdot) \) be a measurable selector on \( [T^*, T^* + \delta] \) of the multifunction

\[
t \mapsto \{ u \in U(t) \mid p(T^*) \cdot f(t, x(T^*-h_0), \ldots, x(T^*-h_N), u) \geq \max_{u' \in U(t)} p(T^*) \cdot f(t, x(T^*-h_0), \ldots, x(T^*-h_N), u') - \gamma \}.
\]

Now consider an extension of \( u^*(\cdot) : [S, T^*] \rightarrow \mathbb{R}^m \) to \( [S, T^* + \delta] \), obtained by setting \( u^*(t) = v^*(t) \) for \( t \in (T^*, T^* + \delta) \). Extend also \( x^*(\cdot) \) to \( [S, T^* + \delta] \), as the state trajectory corresponding to the extended control function \( u^*(\cdot) \) and the original initial state \( x(S) \). Since \( (x^*(\cdot), u^*(\cdot), T^*) \) is an \( L^\infty \) local minimizer, we have

\[
\tilde{g}(x^*(T^* - \delta), T^* - \delta) + 2\alpha \delta \geq \tilde{g}(x^*(T^*), T^*) \quad \text{and} \quad \tilde{g}(x^*(T^* + \delta), T^* + \delta) + 2\alpha \delta \geq \tilde{g}(x^*(T^*), T^*) \quad (5.2)
\]

for all \( \delta > 0 \) sufficiently small. Since \( \tilde{g}(. , .) \) is continuously differentiable and \( \nabla_x \tilde{g}(x^*(T^*), T^*) = -p(T^*) \), we can deduce from the second inequality in (5.2) that

\[
-2\alpha \leq \nabla_T \tilde{g}(x^*(T^*), T^*) - \delta^{-1} \left[ \int_{[T^*, T^* + \delta]} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u^*(s)) \, ds \right] + o(\delta)
\]

\[
\leq \nabla_T \tilde{g}(x^*(T^*), T^*) - \delta \left[ \sup_{s \in [T^* - \delta, T^* + \delta]} \left[ \sup_{u \in U(s)} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u) \right] \right] + \gamma + o(\delta).
\]

(in which \( o(.) \) is some increasing function such that \( o(\delta') \rightarrow 0 \) as \( \delta' \downarrow 0 \)). (We make use of the fact that \( T^* \geq S + h \) and the essential boundedness of \( k_f(\cdot) \) and \( c_f(\cdot) \) on a neighbourhood of \( T^* \) to justify these relationships.)

Exploiting the first inequality in (5.2), we arrive at

\[
-2\alpha \leq -\nabla_T \tilde{g}(x^*(T^*), T^*) + \delta^{-1} \left[ \int_{[T^*, T^* + \delta]} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u^*(s)) \, ds \right] + o_1(\delta)
\]

\[
\leq -\nabla_T \tilde{g}(x^*(T^*), T^*) \left[ \sup_{s \in [T^* - \delta, T^* + \delta]} \left[ \sup_{u \in U(s)} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u) \right] \right] + o_1(\delta),
\]

(for some increasing \( o_1(.) \) such that \( \lim_{s \uparrow 0} o_1(s) = 0 \)). It follows

\[
\sup_{s \in [T^* - \delta, T^* + \delta]} \left[ \sup_{u \in U(s)} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u) \right] - \gamma - o(\delta) - 2\alpha
\]

\[
\leq \nabla_T \tilde{g}(x^*(T^*), T^*) \leq \sup_{s \in [T^* - \delta, T^* + \delta]} \left[ \sup_{u \in U(s)} p(T^*) \cdot f(s, \{x^*(T^* - h_k)\}, u) \right] + o_1(\delta) + 2\alpha.
\]

Since \( \gamma \) and \( \delta \) are arbitrary positive numbers we conclude,

\[
\nabla_T \tilde{g}(T^*, x^*(T^*)) \in \operatorname{ess} \min_{t \rightarrow T^*} \left\{ \max_{u \in U(t)} p(T^*) \cdot f(t, x^*(T^* - h_0), \ldots, x^*(T^* - h_N), u) \right\} + 2\alpha \mathbb{B}.
\]
as required to complete the derivation of the necessary conditions for the free end-time problem of step 1.

**Step 2:** Let \((\bar{x}(\cdot), \bar{u}(\cdot), \bar{T})\) be a minimizer for \((P_{FT})\). Assume that (H1)-(H3) and (A0)-(A2) are satisfied. We show that the assertions of Thm. 3.1 are valid. (Now the endpoint cost function is \(g(x(S), x(T), T)\) and the endpoint constraint is \((x(S), x(T), T) \in C\).

Take \(\gamma_i \downarrow 0\). For \(i = 1, 2, \ldots\), consider the problem with \((n + n)\)-dimensional state vector \((z, x)\):

\[
\begin{cases}
\text{Minimize} & \max\{g(z(T), x(T), T) - g(\bar{z}(\bar{T}), \bar{x}(\bar{T}), \bar{T}) + \gamma_i, d_C(z(T), x(T), T)\} \text{ s.t.} \\
\dot{z}(t) = 0, \dot{x}(t) = f(t, \{x(t - h_k)\}, u(t)) \\
(z(S), x(S)) \in \tilde{C} := \{(z, x) \in \mathbb{R}^n \times \mathbb{R}^n | z = x\}.
\end{cases}
\]

We see that, for \(\tilde{c}\) sufficiently small, \((\bar{z}(\cdot) \equiv \bar{x}(S), \bar{x}(\cdot), \bar{u}(\cdot), \bar{T})\) is an \(\gamma_i\)-minimizer for each \(i\).

From this point the analysis follows the same path as that in Step 4 of the proof of the necessary conditions for the fixed time problem. That is to say, we use Ekeland’s Theorem to establish the existence of a new process \((z_i, x_i(\cdot), u_i(\cdot), T_i)\), ‘close’ to \((\bar{z}(\cdot), \bar{x}(\cdot), \bar{u}(\cdot), \bar{T})\) for large \(i\), that is a minimizer for a perturbed problem. The perturbed problem has the special structure for which Step 1 provides necessary conditions. We apply the earlier derived necessary conditions, and obtain the assertions of the theorem in the limit as \(i \to \infty\). The difference with the earlier ‘fixed time’ analysis is that we now use the metric \((5.1)\) on free end-time processes, in place of the earlier metric on fixed time processes.

So far, our proof of the Thm. 3.1 covers only the special case when the extra hypotheses (A0)-(A2). To show that the assertions of the Thm. remain valid when we remove (A0)-(A2) by techniques essentially the same as those employed in Step 5 of the proof of the fixed time PMP, based on the state augmentation and domain extension.

## 6 Appendix

The purpose of this Appendix is to prove condition (c)’ in Lemma 4.3, namely the ‘integral Weierstrass condition’ for problem \((P^1_i)\). Take an arbitrary selector \(u(\cdot)\) of \(U(\cdot)\). We must show

\[
(c)^\prime: \int_{[S,T]} Q_i(t, \{u(t - h_k)\}) dt \leq \int_{[S,T]} Q_i(t, \{u^*(t - h_k)\}) dt
\]

where \(Q_i(t, \{u_k\}) := p(t) \cdot f(t, \{y^*_k(t)\}, \{u_k\}) - \tilde{L}(t, u_0) - \gamma_i^2 m_i(t, u_0)\). (Note that the integrands are integrable in consequence of the supplementary hypothesis (A2).)

We state for future use the following generalization of Hurwitz’s theorem, concerning the simultaneous approximation of a finite collection of positive real numbers by rational numbers. A proof is appears in [20, Lemma 4.2] of this classical result [12, Thm. 200]. \((\mathbb{Z}^+\) denotes the positive integers.)

**Lemma 6.1.** Take positive numbers \(h_1, \ldots, h_N\). Then there exist a sequence \(n_j \to \infty\) in \(\mathbb{Z}^+\) and a sequence of \(\epsilon_j \downarrow 0\) in \(\mathbb{R}\) such that

\[
\max_{k=1, \ldots, N} \left\{ \min_{m \in \mathbb{Z}^+} |n_j h_k - m| \right\} \leq \epsilon_j, \text{ for } j = 1, 2, \ldots.
\]
Take any $K > 0$ and define the set
\[ A^K := \{ t \in [S, T] \mid |c_0(t)| \leq K \} . \]

Here $c_0(.) \in L^1$ is as in supplementary hypothesis (A2). Define $\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [S, T] \setminus A^K, \\ u^*(t) & \text{if } t \in A^K. \end{cases}$

Take $\lambda \in (0, 1)$. For $j = 1, 2, \ldots$ let $n_j$ and $\epsilon_j$ be as in Lemma 6.1 and

\[ A_{j, \lambda} := \bigcap_{k=0}^{n_j-1} \left( S + \frac{k(T - S)}{n_j}, S + \frac{(k + \lambda)(T - S)}{n_j} \right) . \]

Now define the selector $u_{j, \lambda}(.) : [S, T] \to \mathbb{R}^m$ be

\[ u_{j, \lambda}(.) = \begin{cases} \tilde{u}(t) & \text{if } t \in A_{j, \lambda} , \\ u^*(t) & \text{if } t \in [S, T] \setminus A_{j, \lambda} . \end{cases} \]

In view of Lemma 6.1,

\[ \{u_{j, \lambda}(t - h_k)\} = \{\tilde{u}(t - h_k)\} \chi_{A_{j, \lambda}}(t) + (\{u^*(t - h_k)\}) \chi_{[0, T] \setminus A_{j, \lambda}}(t) \]

for all $t \in [S, T] \setminus B_{j, \lambda}$, where
\[
B_{j, \lambda} := \left\{ S, S + \frac{1}{n_j}(T - S), \ldots, S + \frac{n_j}{n_j}(T - S), \right. \\
\left. S + \frac{\lambda}{n_j}(T - S), S + \frac{1+\lambda}{n_j}(T - S), \ldots, S + \frac{(n_j-1)+\lambda}{n_j}(T - S) \right\} + 2\epsilon_j(T - S)(-1, +1) \cap [S, T].
\]

Note that $\sigma_j := \text{meas} \{B_{j, \lambda}\} \leq 4\epsilon_j \times \frac{2n_j+1}{n_j} |T - S| \to 0$, as $j \to 0$.

Now write $x_{j, \lambda}(.)$ for the solution to the dynamical equation in $(\hat{P}_{\lambda})$, when $u(.) = u_{j, \lambda}(.)$, \{y_k(.)\} = \{y_k^*(.)\}, with initial value $x_{j, \lambda}(S) = x^*(S)$. Since $\text{meas} \{A_{j, \lambda}\} = \lambda$ and since $c_0(.)$ is essentially bounded on $[S, T] \setminus A^K$, we have

\[ ||x_{j, \lambda}(.) - x^*(.)||_{L^\infty} \leq \gamma(K) \left( \text{meas} \{A_{j, \lambda}\} + \text{meas} \{B_{j, \lambda}\} \right) \leq \gamma(K)(\lambda + \sigma_j) \]

for some number $\gamma(K)$ that depends on $K$ but is independent of $j$. According to (4.8),

\[ \int_{[S, T]} [Q_i(t, \{u_{j, \lambda}(t - h_k)\}) - Q_i(t, u^*\{t - h_k\})] dt \leq \theta(|||x_{j, \lambda}(.) - x^*(.)||_{L^\infty}) + ||k_0(.)||_{L^1} ||x_{j, \lambda}(.) - x^*(.)||_{L^\infty}^2 . \]

Noting that $\tilde{u}(.)$ coincides with $u^*(.)$ on $A^K$, we see that

\[ \int_{[S, T]} Q_i(t, \{u_{j, \lambda}(t - h_k)\}) dt \leq \int_{[S, T] \setminus A^K} [Q_i(t, \{u(t - h_k)\}), Q_i(t, \{u^*(t - h_k)\})] q_{j, \lambda}(t) dt + \gamma'(K)\sigma_j . \]

for some number $\gamma'(K) > 0$ independent of $j$, in which

\[ q_{j, \lambda}(t) := \begin{bmatrix} 1 & 0 \end{bmatrix}^T \chi_{A_{j, \lambda}}(t) + \begin{bmatrix} 0 & 1 \end{bmatrix}^T \chi_{[0, T] \setminus A_{j, \lambda}}(t) . \]

The sequence \{q_{j, \lambda}(.)\} is equi-integrable and uniformly bounded and therefore has a weak limit in $L^1$. By consideration of $C^1$ test functions, we can easily show that $q_{j, \lambda}(.) \to \begin{bmatrix} \lambda & 1 - \lambda \end{bmatrix}^T$ weakly* in $L^\infty$ as $j \to \infty$, and so

\[
\int_{[S, T] \setminus A^K} [Q_i(t, \{u(t - h_k)\}), Q_i(t, \{u^*(t - h_k)\})] q_{j, \lambda}(t) dt \\
\to \int_{[S, T] \setminus A^K} \left[ \lambda Q_i(t, \{u(t - h_k)\}) + (1 - \lambda)Q_i(t, \{u^*(t - h_k)\}) \right] dt .
\]
as \( j \to \infty \). We deduce from the preceding relations that

\[
\lambda \int_{[S,T] \setminus A^K} [Q_i(t, \{u(t-h_k)\}) - Q_i(t, \{u^*(t-h_k)\})] \, dt \\
\leq \limsup_{j \to \infty} \left[ \tilde{k} \sigma_j + \theta(\tilde{k}(\lambda + \sigma_j)) - (\tilde{k}(\lambda + \sigma_j)^2 ||k_0(\cdot)||_{L^1}) \right] = \theta(\tilde{k}\lambda) - (\tilde{k}\lambda)^2 ||k_0(\cdot)||_{L^1},
\]

which is valid for any \( \lambda \in (0, 1] \). Dividing across by \( \lambda \) and passing to the limit as \( \lambda \downarrow 0 \) gives

\[
\int_{[S,T] \setminus A^K} [Q_i(t, \{u(t-h_k)\}) - Q_i(t, \{u^*(t-h_k)\})] \, dt \leq 0.
\]

But the integrand here is an integrable function. Since \( \text{meas \{A^K\}} \to 0 \) as \( K \to \infty \), the inequality is valid when \([S,T]\) replaces \([S,T]\setminus A^K\). We have confirmed condition \((c)'\).

7 Proof of Thm. 3.1

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