DOUBLE-NORMAL PAIRS IN SPACE

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Abstract. A double-normal pair of a finite set $S$ of points that spans $\mathbb{R}^d$ is a pair of points $\{p, q\}$ from $S$ such that $S$ lies in the closed strip bounded by the hyperplanes through $p$ and $q$ perpendicular to $pq$. A double-normal pair $\{p, q\}$ is strict if $S \setminus \{p, q\}$ lies in the open strip. The problem of estimating the maximum number $N_d(n)$ of double-normal pairs in a set of $n$ points in $\mathbb{R}^d$ was initiated by Martini and Soltan [Discrete Math. 290 (2005), 221–228]. It was shown in a companion paper that in the plane, this maximum is $3\lfloor n/2 \rfloor$, for every $n > 2$. For $d \geq 3$, it follows from the Erdős–Stone theorem in extremal graph theory that $N_d(n) = \frac{1}{2}(1 - 1/k)n^2 + o(n^2)$ for a suitable positive integer $k = k(d)$. Here we prove that $k(3) = 2$ and, in general, $[d/2] \leq k(d) \leq d - 1$. Moreover, asymptotically we have $\lim_{n \to \infty} k(d)/d = 1$. The same bounds hold for the maximum number of strict double-normal pairs.

§1. Introduction. Let $V$ be a set of $n$ points that spans $\mathbb{R}^d$. A double-normal pair of $V$ is a pair of distinct points $\{p, q\}$ in $V$ such that $V$ lies in the closed strip bounded by the hyperplanes $H_p$ and $H_q$ through $p$ and $q$, respectively, that are perpendicular to the line $pq$. A double-normal pair $\{p, q\}$ is strict if $V \setminus \{p, q\}$ is disjoint from the hyperplanes $H_p$ and $H_q$. Define the double-normal graph of $V$ as the graph on the vertex set $V$ in which two vertices $p$ and $q$ are joined by an edge if and only if $\{p, q\}$ is a double-normal pair. The number of edges of this graph, that is, the number of double-normal pairs induced by $V$ is denoted by $N(V)$.

We define the strict double-normal graph of $V$ analogously and denote its number of edges by $N'(V)$.

Martini and Soltan [10, Problems 3 and 4] asked for the maximum numbers $N_d(n)$ and $N'_d(n)$ of double-normal pairs and strict double-normal pairs of a set of $n$ points in $\mathbb{R}^d$:

$$N_d(n) := \max_{V \subset \mathbb{R}^d \mid |V| = n} N(V)$$

and

$$N'_d(n) := \max_{V \subset \mathbb{R}^d \mid |V| = n} N'(V).$$

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Clearly, we have $N(V) \geq N'(V)$ and $N_d(n) \geq N'_d(n)$. It is not difficult to see that $N'_2(n) = n$. In another paper [12] we show that $N_2(n) = 3[n/2]$. Here we only consider the case $d \geq 3$.

**Theorem 1.** The maximum number of double-normal and strict double-normal pairs in a set of $n$ points in $\mathbb{R}^3$ satisfy $N_3(n) = n^2/4 + o(n^2)$ and $N'_3(n) = n^2/4 + o(n^2)$.

In fact, since the collection of double-normal graphs in Euclidean space is closed under the taking of induced subgraphs, the Erdős–Stone theorem [3] implies that for each $d \in \mathbb{N}$, there exist unique $k(d), k'(d) \in \mathbb{N}$ such that $N_d(n) = \frac{1}{2} \left(1 - 1/k(d)\right)n^2 + o(n^2)$ and $N'_d(n) = \frac{1}{2} \left(1 - 1/k'(d)\right)n^2 + o(n^2)$. The number $k(d)$ (respectively $k'(d)$) can also be characterized as the largest $k$ such that complete $k$-partite graphs with arbitrarily many points in each class occur as subgraphs of double-normal (respectively strictly double-normal) graphs in $\mathbb{R}^d$.

Theorem 1 states that $k(3) = k'(3) = 2$ and is a special case of the next theorem.

**Theorem 2.** For each $d$, there exist unique integers $k(d), k'(d) \geq 1$ such that $N_d(n)$, the maximum number of double-normal pairs, and $N'_d(n)$, the maximum number of strict double-normal pairs in a set of $n$ points in $\mathbb{R}^d$, satisfy

\[ N_d(n) = \frac{1}{2} \left(1 - \frac{1}{k(d)}\right)n^2 + o(n^2) \]

and

\[ N'_d(n) = \frac{1}{2} \left(1 - \frac{1}{k'(d)}\right)n^2 + o(n^2). \]

For any $d \geq 3$, we have

\[ \lceil d/2 \rceil \leq k'(d) \leq k(d) \leq d - 1. \]

Asymptotically, as $d \to \infty$, we have

\[ k(d) \geq k'(d) \geq d - O(\log d). \]

Although this theorem gives the exact values $k(3) = k'(3) = 2$, we do not know whether $k(4)$ or $k'(4)$ equals 2 or 3.

Two notions related to double-normal pairs have been studied before. We define a diameter pair of $S$ to be a pair of points $\{p, q\}$ in $S$ such that $|pq| = \text{diam}(S)$. Note that a diameter pair is also a strictly double-normal pair. The maximum number of diameter pairs in a set of $n$ points is known for all $d \geq 2$, and in the case of $d \geq 4$, if $n$ is sufficiently large [1, 4–6, 13, 14]. We call a pair $pq$ of a set $S \subset \mathbb{R}^d$ antipodal if there exist parallel hyperplanes $H_1$ and $H_2$ through $p$ and $q$, respectively, such that $S$ lies in the closed strip bounded by the hyperplanes. The pair is called strictly antipodal if there exist parallel hyperplanes through $p$ and $q$ such that $S \setminus \{p, q\}$ lies in the open strip bounded by the hyperplanes. Clearly, a (strictly) double-normal pair of a set is also a
(strictly) antipodal pair. The problem of determining the asymptotic behaviour of the maximum number of antipodal or strictly antipodal pairs in a set of \( n \) points is open already in \( \mathbb{R}^3 \). For a thorough discussion of antipodal pairs, see the series of papers [7–9].

The paper is structured as follows. In §2, we collect some geometric lemmas on double-normal pairs. They are applied in §3 together with a Ramsey-type argument to derive the upper bound of Theorem 2 (Theorem 7). Finally, in §4 we show the two lower bounds of Theorem 2 (Corollaries 10 and 16). The asymptotic lower bound follows from a random construction closely related to the construction by Erdős and Füredi [2] of strictly antipodal sets of size exponential in the dimension.

We use the following notation. The inner product of \( x, y \in \mathbb{R}^d \) is denoted by \( \langle x, y \rangle \), the linear span of \( S \subset \mathbb{R}^d \) by \( \text{lin} S \), the convex hull of \( S \) by \( \text{conv} S \), the diameter of \( S \) by \( \text{diam}(S) \), the cardinality of a finite set \( S \) by \( |S| \), and the complete \( k \)-partite graph with \( N \) vertices in each class by \( K_k(N) \). An angle with vertex \( b \) and sides \( ba \) and \( bc \) is denoted by \( \angle abc \), which we also use to denote its angular measure. All angles in this paper have angular measure in the range \((0, \pi)\). The Euclidean distance between \( p \) and \( q \) is denoted \( |pq| \).

§2. Geometric properties of the double-normal relation. Here we collect some elementary geometric properties of double-normal pairs. They will be used in the next section where we find upper bounds to \( k(d) \).

If a unit vector \( u \) is almost orthogonal to two given unit vectors \( u_1 \) and \( u_2 \), then \( u \) is still almost orthogonal to any unit vector in the span of \( u_1 \) and \( u_2 \), with an error that becomes worse the closer \( u_1 \) and \( u_2 \) are to each other. The next lemma quantifies this observation.

**Lemma 3.** Let \( u, u_1, u_2 \) be unit vectors with \( u_1 \neq \pm u_2 \), such that for some \( \varepsilon_1, \varepsilon_2 > 0 \), \(|\langle u, u_1 \rangle| \leq \varepsilon_1 \) and \(|\langle u, u_2 \rangle| \leq \varepsilon_2 \). Then for any unit vector \( v \in \text{lin} \{u_1, u_2\} \) we have \(|\langle u, v \rangle| < (\varepsilon_1 + \varepsilon_2)/\sin \theta \), where \( \theta \in (0, \pi) \) satisfies \( \cos \theta = \langle u_1, u_2 \rangle \).

**Proof.** Let \( u' \) be the orthogonal projection of \( u \) onto the plane \( \text{lin} \{u_1, u_2\} \). Then the quantity \( \langle u, v \rangle = \langle u', v \rangle \) is maximized when \( v \) is a positive multiple of \( u' \), and then \(|\langle u, v \rangle| = |ou'| \). It follows from the hypotheses that \( u' \) lies in the parallelogram \( P \) symmetric around \( o \) with sides perpendicular to \( u_1 \) and \( u_2 \), respectively, and with the sides perpendicular to \( u_i \) at distance \( 2\varepsilon_i, i = 1, 2 \). The sides of \( P \) form an angle of \( \theta \), and their lengths are \( 2\varepsilon_1/\sin \theta \) and \( 2\varepsilon_2/\sin \theta \). The maximum value of \(|ou'|\) is attained at a vertex of the parallelogram \( P \), that is, \(|ou'|\) is at most half the largest diagonal of \( P \). By the law of cosines, half a diagonal of \( P \) has length

\[
\sqrt{\frac{\varepsilon_1^2}{\sin^2 \theta} + \frac{\varepsilon_2^2}{\sin^2 \theta} \pm 2 \frac{\varepsilon_1 \varepsilon_2}{\sin^2 \theta} \cos \theta} < \sqrt{\frac{\varepsilon_1^2}{\sin^2 \theta} + \frac{\varepsilon_2^2}{\sin^2 \theta} + 2 \frac{\varepsilon_1 \varepsilon_2}{\sin^2 \theta} = \frac{\varepsilon_1 + \varepsilon_2}{\sin \theta}. \]


Suppose that \( y_1, y_2, y_3 \) are collinear, with \( y_2 \) between \( y_1 \) and \( y_3 \), and that \( xy_2 \) is a double-normal pair in some set that contains \( x, y_1, y_2, y_3 \). Then, since the segment \( y_1y_3 \) has to lie in the half-space through \( y_2 \) with normal \( y_2x \), it follows that \( y_1y_3 \) lies in the boundary of this half-space. That is, \( xy_2 \perp y_1y_2 \). If \( y_1, y_2, y_3 \) are close to collinear, then intuitively \( y_1y_2 \) will still be close to orthogonal to \( xy_2 \). This is the content of the next lemma.

**Lemma 4.** Let \( x, y_1, y_2, y_3 \) be different points from \( V \subset \mathbb{R}^d \), with \( xy_2 \) a double-normal pair in \( V \). Let \( \varepsilon > 0 \) and suppose that \( \angle y_1y_2y_3 > \pi - \varepsilon \). Let \( u \) be a unit vector parallel to \( y_1y_2 \) and \( v \) a unit vector parallel to \( xy_2 \). Then \( |\langle u, v \rangle| < \varepsilon \).

**Proof.** Without loss of generality, \( \varepsilon < \pi/2 \). Note that \( \angle xy_2y_1, \angle xy_2y_3 \leq \pi/2 \). Since also
\[
\pi - \varepsilon < \angle y_1y_2y_3 \leq \angle y_1y_2x + \angle xy_2y_3 \leq \angle y_1y_2x + \pi/2,
\]
we obtain
\[
\pi/2 - \varepsilon < \angle y_1y_2x \leq \pi/2,
\]
and it follows that
\[
|\langle u, v \rangle| = \cos \angle y_1y_2x < \cos(\pi/2 - \varepsilon) = \sin \varepsilon < \varepsilon.
\]

Consider the situation where \( y_1, y_2, y_3 \) are “almost” collinear with \( y_2 \) the “middle” point, but now there are two double-normal pairs \( x_1y_2 \) and \( x_2y_2 \) in a set that contains \( x_1, x_2, y_1, y_2, y_3 \). Then \( y_1, y_2, y_3 \) all lie inside the wedge \( W \) formed by the intersection of the half-spaces \( H_1 \) and \( H_2 \) through \( y_2 \) with normals \( x_1 - y_2 \) and \( x_2 - y_2 \), respectively. If \( y_1, y_2, y_3 \) are collinear with \( y_2 \) between \( y_1 \) and \( y_3 \), then necessarily \( y_1, y_2, y_3 \) all lie on the “ridge” bd \( H_1 \cap \text{bd}H_2 \) of the wedge \( W \), and \( y_1y_2 \) is orthogonal to the plane \( \Pi \) through \( x_1, x_2, y_2 \). If \( y_1, y_2, y_3 \) are close to collinear, then intuitively \( y_1y_2 \) will still be close to orthogonal to \( \Pi \). The next lemma quantifies this intuition. It is an immediate consequence of Lemmas 3 and 4.

**Lemma 5.** Let \( x_1, x_2, y_1, y_2, y_3 \) be different points in \( V \subset \mathbb{R}^d \), with \( x_1y_2 \) and \( x_2y_2 \) double-normal pairs in \( V \). Let \( \varepsilon > 0 \). Suppose that \( \angle y_1y_2y_3 > \pi - \varepsilon \). Then for any unit vector \( u \) parallel to the line \( y_1y_2 \) and any unit vector \( v \) parallel to the plane \( x_1x_2y_2 \) we have \( |\langle u, v \rangle| < 2\varepsilon/\sin \angle x_1y_2x_2 \).

If the angle \( \angle x_1y_2x_2 \) in the previous lemma is small, then the bound obtained may be too large to be useful. In the next lemma, we show that we can still obtain a small upper bound if \( |y_1y_2| \) is much smaller than \( |x_1x_2| \). We need four double-normal pairs instead of the two required by Lemma 5, but we do not need \( y_3 \).

**Lemma 6.** Let \( x_iy_j, i, j = 1, 2, \) be four double-normal pairs in a set \( V \subset \mathbb{R}^d \) that contains \( x_1, x_2, y_1, y_2 \). Let \( u \) be a unit vector parallel to \( y_1y_2 \) and \( v \) a unit vector parallel to the plane \( x_1x_2y_2 \). Then
\[
|\langle u, v \rangle| \leq \frac{\sqrt{2}}{\cos^2 \angle x_1y_2x_2} \frac{|y_1y_2|}{|x_1x_2|}.
\]
Proof. Let \( u := \|y_1 y_2\|^{-1} (y_1 - y_2) \), \( u_1 := \|x_1 y_2\|^{-1} (x_1 - y_2) \) and \( u_2 := \|x_1 x_2\|^{-1} (x_1 - x_2) \). Then \( \langle u_1, u_2 \rangle = \cos \theta \) where \( \theta := \angle_{x_2 x_1} y_2 \).

Since the angles \( \angle_{x_1 y_1} y_2, \angle_{x_1 y_2} y_1, \angle_{x_2 y_2} y_1 \) are non-obtuse, we obtain

\[
\langle x_1 - y_1, y_2 - y_1 \rangle \geq 0, \quad \langle x_1 - y_2, y_1 - y_2 \rangle \geq 0,
\]

and

\[
\langle y_2 - x_2, y_2 - y_1 \rangle \geq 0.
\]

From (1) we obtain \( \langle x_1 - y_2, y_2 - y_1 \rangle \geq -\|y_1 y_2\|^2 \), that is,

\[
\langle u, u_1 \rangle \leq \frac{\|y_2 y_1\|}{\|x_1 y_2\|} =: \varepsilon_1.
\]

From (2), \( \langle u, u_1 \rangle \geq 0 \). Next, add (1) and (3) to obtain \( \langle x_2 - x_1, y_2 - y_1 \rangle \leq \|y_1 y_2\|^2 \), that is,

\[
\langle u, u_2 \rangle \leq \frac{\|y_1 y_2\|}{\|x_1 x_2\|} =: \varepsilon_2.
\]

The analogues of (1) and (3) with \( x_1 \) and \( x_2 \) interchanged similarly give \(-\langle u, u_2 \rangle \leq \varepsilon_2 \). By Lemma 3, for any unit vector \( v \) parallel to the plane \( \Pi \) through \( x_1, x_2, y_2 \), that is, with \( v \in \mathrm{lin}\{u_1, u_2\} \), we have

\[
|\langle u, v \rangle| \leq \frac{\varepsilon_1 + \varepsilon_2}{\sin \theta}.
\]

By the law of sines in \( \triangle x_1 x_2 y_2 \),

\[
\frac{\varepsilon_1}{\varepsilon_2} = \frac{|x_1 x_2|}{|x_1 y_2|} = \frac{\sin \alpha}{\sin \varphi},
\]

where \( \varphi := \angle_{x_1 x_2} y_2 \) and \( \alpha := \angle_{x_1 y_2} x_2 \). It follows from (4) that

\[
|\langle u, v \rangle| \leq \frac{\varepsilon_2}{\sin \theta} \left( 1 + \frac{\sin \alpha}{\sin \varphi} \right).
\]

Since \( \alpha, \theta, \varphi \leq \pi/2 \) and \( \alpha + \theta + \varphi = \pi \), we have

\[
\sin \theta, \sin \varphi \geq \sin(\pi/2 - \alpha) = \cos \alpha,
\]

hence

\[
|\langle u, v \rangle| \leq \frac{\varepsilon_2}{\cos \alpha} \left( 1 + \frac{\sin \alpha}{\cos \alpha} \right) = \frac{\varepsilon_2}{\cos^2 \alpha}(\cos \alpha + \sin \alpha)
\]

\[
\leq \frac{\varepsilon_2}{\cos^2 \alpha} \sqrt{2} = \frac{\sqrt{2}}{\|x_1 y_2\|^2} |\|y_1 y_2\|^2|.
\]

\( \square \)
§3. Upper bound on the number of double-normal pairs. Recall that $k(d)$ denotes the largest $k$ such that for each $N \in \mathbb{N}$, the complete $k$-partite graph with $N$ vertices in each class, $K_k(N)$, is a subgraph of some double-normal graph in $\mathbb{R}^d$.

**Theorem 7.** For all $d \geq 3$, we have $k(d) \leq d - 1$.

This theorem is a straightforward consequence of the following technical result.

**Proposition 8.** There exist a family of $k = k(d)$ not necessarily distinct points $\{p_1, \ldots, p_k\}$ and a family of $k^2$ not necessarily distinct unit vectors $\{u_{i,j} : 1 \leq i, j \leq k\}$, all in $\mathbb{R}^d$, such that the following hold:

1. $\{p_1, p_2, \ldots, p_k\}$ has at least two distinct points and no obtuse angles;  
2. $\{u_{1,1}, u_{2,2}, \ldots, u_{k,k}\}$ is an orthogonal set;  
3. If $i \neq j$, then $u_{i,j} = -u_{j,i}$;  
4. If $p_i \neq p_j$, then $u_{i,j} = |p_j p_i|^{-1}(p_j - p_i)$;  
5. For any distinct $i, j$, $u_{i,i}$ is orthogonal to $u_{i,j}$;  
6. Each $u_{i,j}$ is orthogonal to the subspace $\text{lin}\{p_j - p_1 : j = 2, \ldots, k\}$;  
7. If $p_i = p_j$, then $u_{i,i}$ is orthogonal to $u_{i,j} = u_{j,i}$.

**Proof.** The proof consists of three steps.

**Step 1.** We will use a geometric Ramsey-type result from [11] and the pigeonhole principle to show that for any $\epsilon > 0$ there exists $N$ such that for any $K_k(N)$ with classes $V_1, \ldots, V_k$ contained in some double-normal graph in $\mathbb{R}^d$, there exist points $a_i, b_i, c_i \in V_i$ ($i = 1, \ldots, k$) such that

$$
\langle a_i b_i c_i \rangle > \pi - \epsilon, \quad i = 1, \ldots, k,  
$$

$$
|a_{i+1} c_{i+1}| \leq \epsilon |a_i c_i|, \quad i = 1, \ldots, k - 1,  
$$

$$
|a_i b_i| \geq \frac{1}{2} |a_i c_i|, \quad i = 1, \ldots, k.  
$$

**Step 2.** We use the results from §2 to show that if we set $u_{i,i} := |a_i b_i|^{-1}(a_i - b_i)$ and $u_{i,j} := |b_j b_i|^{-1}(b_j - b_i)$, then

$$
|\langle u_{i,i}, u_{i,j} \rangle| < \epsilon, \quad i, j = 1, \ldots, k, \ i \neq j,  
$$

$$
|\langle u_{i,i}, u_{i,j} \rangle| < 4\epsilon, \quad i, j = 1, \ldots, k, \ i \neq j.  
$$

**Step 3.** The proposition will follow by setting $\epsilon := 1/n$ and taking subsequences of the sequences $a_i^{(n)}, b_i^{(n)}, c_i^{(n)}$, $i = 1, \ldots, k$, such that $b_i^{(n)}$ converges to $p_i$, and each $u_{i,j}^{(n)}$ converges as $n \to \infty$. The details follow.

Let $\epsilon > 0$ be given. Write $t := \lceil (\epsilon \cos \epsilon)^{-1} \rceil$. In Step 1, applying [11, Theorem 4] we first choose a sufficiently large $N$ depending only on $\epsilon$ and $d$ such that each class $V_i$ of any $K_k(N)$ contained in a double-normal graph in $\mathbb{R}^d$...
has a subset \( V'_i \) of size \( 2^{k-1} + 1 \) such that for any \( a, b, c, d \) from the same \( V'_i \) with \( a \neq b \) and \( c \neq d \), the angle between the lines \( ab \) and \( cd \) is less than \( \varepsilon \). We now replace the original \( V_i \) by \( V'_i \). If we assume \( \varepsilon < \pi/3 \), we obtain a natural linear ordering (more precisely, a betweenness relation) on the points of each \( V_i \), by defining for each \( x, y, z \in V_i \) that \( y \) is between \( x \) and \( z \) if \( \angle xyz > \pi - \varepsilon \). Then \( |yx| < |zx| \) whenever \( y \) is between \( x \) and \( z \).

Next we run Algorithm 1 on \( V_1, \ldots, V_k \). Note that at the start of the outer \textbf{for} loop, \(|V_j| = 2^{k-i} + 1\) for all \( j = i, \ldots, k \). That we can find a \( V'_j \) as required inside the inner \textbf{for} loop, is seen as follows. Write \( V_j := \{p_1, \ldots, p_{2^{2k-i} + 1}\} \) with the points in their natural order (where \( p_j \) is between \( p_i \) and \( p_k \) if \( \angle p_ip_jp_k > \pi - \varepsilon \)). Let \( p'_j \) be the orthogonal projection of \( p_i \) onto the line \( \ell \) through \( p_1 \) and \( p_{2^{2k-i} + 1} \). Since \( \varepsilon < \pi/2 \), the points \( p'_j \) are in order on \( \ell \), and

\[
|p_1p_{2^{2k-i}+1}| = |p'_1p'_{2^{2k-i}+1}|
= \sum_{s=1}^{t} |p'_{2^{k-i-1}(s-1)+1}p'_{2^{k-i-1}s+1}|
> \cos \varepsilon \sum_{s=1}^{t} |p_{2^{k-i-1}(s-1)+1}p_{2^{k-i-1}s+1}|,
\]

where the last inequality holds, because the angle between \( \ell \) and the line through any two \( p_i \) is less than \( \varepsilon \). Thus,

\[
\frac{1}{t} \sum_{s=1}^{t} |p_{2^{k-i-1}(s-1)+1}p_{2^{k-i-1}s+1}| < \frac{1}{t \cos \varepsilon} |p_1p_{2^{2k-i}+1}| < \varepsilon |p_1p_{2^{2k-i}+1}|.
\]

It follows that for some \( s \in \{1, \ldots, t\} \),

\[
|p_{2^{k-i-1}(s-1)+1}p_{2^{k-i-1}s+1}| < \varepsilon |p_1p_{2^{2k-i}+1}|.
\]

Let \( V'_j := \{p_{2^{k-i-1}(s-1)+1}, \ldots, p_{2^{k-i-1}s+1}\} \). Then \( |V'_j| = 2^{k-i-1} + 1 \) and

\[
\text{diam}(V'_j) < \varepsilon |p_1p_{2^{2k-i}+1}| = \varepsilon \text{diam}(V_j).
\]
When the algorithm is done, we have sets $V_1, \ldots, V_k$ such that $\text{diam}(V_{i+1}) \geq \varepsilon \text{diam}(V_i)$ for each $i = 1, \ldots, k - 1$, and $|V_i| = 2^i k^{-i} + 1 \geq 3$ for each $i = 1, \ldots, k$. Let $a_i, c_i$ be a diameter of $V_i$ and choose any $b_i \in V_i \setminus \{a_i, c_i\}$. Then (12) and (13) hold. To ensure (14), exchange $a_i$ and $c_i$ if necessary such that $|a_i, b_i| \geq |c_i, b_i|$. Then (14) follows from the triangle inequality.

In Step 2 we show (15) and (16). Let $1 \leq i, j \leq k$, $i \neq j$. Without loss of generality, $i < j$. Then (15) follows upon applying Lemma 4 with $x := b_i$, $y_1 := a_j$, $y_2 := b_j$, $y_3 := c_j$.

If $\angle a_i, b_j, b_i \geq \pi/6$, then by Lemma 5 with $x_1 := a_i$, $x_2 := b_i$, $y_1 := a_j$, $y_2 := b_j$, $y_3 := c_j$,

$$|\langle u_{i,i}, u_{j,j} \rangle| < \frac{2 \varepsilon}{\sin \angle a_i, b_j, b_i} \leq \frac{2 \varepsilon}{\sin \pi/6} = 4 \varepsilon.$$  

If $\angle a_i, b_j, b_i < \pi/6$, then by Lemma 6 with $x_1 := a_i$, $x_2 := b_i$, $y_1 := a_j$, $y_2 := b_j$,

$$|\langle u_{i,i}, u_{j,j} \rangle| < \frac{\sqrt{2}}{\cos^2 \frac{\angle a_i, b_j, b_i}{|a_i, b_i|}} = \frac{\sqrt{2}}{\cos^2 (\pi/6)} \frac{|a_i c_j|}{|a_i, c_i|} < \frac{(8 \sqrt{2}/3) \varepsilon}{4 \varepsilon},$$  

which shows (16).

In Step 3, we let $n \in \mathbb{N}$ be arbitrary, set $\varepsilon := 1/n$, and choose $a_i^{(n)}, b_i^{(n)}, c_i^{(n)}$, $i = 1, \ldots, k$, as in the first stage of the proof. We may assume, after translating and scaling each $\bigcup_{i=1}^k V_i^{(n)}$ if necessary, that $\{b_1^{(n)}, \ldots, b_k^{(n)}\}$ has diameter 1 and is contained in the unit ball. Thus, we may pass to subsequences to assume that for each $i$, $b_i^{(n)}$ converges to $p_i$, say,

$$u_{i,i}^{(n)} := |a_i^{(n)} b_i^{(n)}|^{-1} (a_i^{(n)} - b_i^{(n)})$$  

converges to $u_{i,i}$, say, and

$$u_{i,j}^{(n)} := |b_j^{(n)} b_i^{(n)}|^{-1} (b_j^{(n)} - b_i^{(n)})$$  

converges to $u_{i,j}$, say. Then $\text{diam}\{p_1, \ldots, p_k\} = 1$, and since there are no obtuse angles in $\{b_1^{(n)}, \ldots, b_k^{(n)}\}$, there will still be no obtuse angles between distinct elements of $\{p_1, \ldots, p_k\}$. Thus, (5) holds. Also, (6) follows from (16), (7) from the definition of $u_{i,j}^{(n)}$, (8) from the definitions of $u_{i,j}^{(n)}$ and $p_i$, and (9) from (15). Properties (8) and (9) immediately imply that $u_{i,i}$ is orthogonal to $p_i - p_j$ for all $j \neq i$. Since the subspace $\text{lin}\{p_i - p_j : j \neq i\}$ is the same for all $i$, we obtain (10).

Finally, suppose $p_i = p_i' \neq p_j$. Since $\angle b_i^{(n)} b_j^{(n)} b_i^{(n)} \to 0$ as $n \to \infty$ and $\triangle b_i, b_i', b_j$ is not obtuse, we obtain that $\angle b_i^{(n)} b_i' b_j^{(n)} \to \pi/2$ and $\angle b_i^{(n)} b_i' b_j^{(n)} \to \pi/2$ as $n \to \infty$, giving $u_{i,i'} \perp u_{i,j}$. This shows (11).
Proof of Theorem 7. Let $k = k(d)$. Consider the points $p_1, \ldots, p_k$ and vectors $u_{i,j}, 1 \leq i, j \leq k$, given by Proposition 8. There exist distinct $i$ and $j$ such that $p_i \neq p_j$. By (6), the $k$ unit vectors $u_{1,1}, \ldots, u_{k,k}$ are pairwise orthogonal. By (10), they are also orthogonal to $p_i - p_j$, which is a multiple of $u_{i,j}$ by (8). Thus, we have found $k + 1$ pairwise orthogonal vectors. That is, $k(d) + 1 \leq d$. 

§4. Constructions with many strict double-normal pairs.

Theorem 9. Let $m \geq 2$. Suppose that there exist $m$ points $p_1, \ldots, p_m \in \mathbb{R}^d$ and $m$ unit vectors $u_1, \ldots, u_m \in \mathbb{R}^d$ such that, for all triples of distinct $i, j, k$, the angle $\angle p_i p_j p_k$ is acute, and

$$\langle u_i, p_i - p_j \rangle < \langle u_i, p_k - p_j \rangle < \langle u_i, p_j - p_i \rangle. \quad (17)$$

Then, for any $N \in \mathbb{N}$, there exists a strict double-normal graph in $\mathbb{R}^{d+m}$ containing a complete $m$-partite $K_m(N)$. In particular, $k'(d + m) \geq m$.

Geometrically, (17) means that if we project the points $p_1, \ldots, p_m$ orthogonally onto the line through $p_i$ parallel to $u_i$, then the projected points are on the ray from $p_i$ in the direction of $u_i$, and the furthest is at less than twice the distance from $p_i$ than the closest (other than $p_i$).

Proof. Identify $\mathbb{R}^d$ with the first $d$ coordinates of $\mathbb{R}^{d+m}$, and let $v_1, \ldots, v_m \in \mathbb{R}^{d+m}$ be pairwise orthogonal unit vectors that are also orthogonal to $\mathbb{R}^d$. We will construct countably infinite sets $V_1, \ldots, V_m \subset \mathbb{R}^{d+m}$, with each $V_i$ on a circular arc through $p_i$ in the plane $\Pi_i := p_i + \text{lin}(u_i, v_i)$. Then we will verify that for any distinct $i, j$ and any $x \in V_i$ and $y \in V_j$, $xy$ is a strict double-normal pair of $\bigcup_i V_i$.

We will use a small $\varepsilon > 0$ that will depend only on the given points $p_1, \ldots, p_m$ and vectors $u_1, \ldots, u_m$. As the proof progresses, we will put finitely many constraints on $\varepsilon$, all depending only on the points $p_i$ and vectors $u_i$.

Let $\alpha_i := \min_{j \neq i} \langle u_i, p_j \rangle$ and $\beta_i := \max_{j} \langle u_i, p_j \rangle$. By condition (17),

$$\langle u_i, p_i - p_j \rangle - \alpha_i < \beta_i - \alpha_i < \alpha_i - \langle u_i, p_i \rangle,$$

hence $\langle u_i, p_i \rangle < \frac{1}{2} (\beta_i - \langle u_i, p_i \rangle) < \alpha_i$.

We choose $\varepsilon > 0$ small enough so that $\frac{1}{2} (\beta_i + \varepsilon + \langle u_i, p_i \rangle) < \alpha_i - \varepsilon$ for all $i$. Choose any $r_i \in (\frac{1}{2} (\beta_i + \varepsilon + \langle u_i, p_i \rangle), \alpha_i - \varepsilon)$, and set $c_i := p_i + r_i u_i$, $a_i := p_i + (\alpha_i - \varepsilon) u_i$, $b_i := p_i + (\beta_i + \varepsilon) u_i$, $q_i := p_i + 2r_i u_i$ (Figure 1). Denote the circle with centre $c_i$ and radius $r_i$ in the plane $\Pi_i$ by $C_i$. Then $p_i q_i$ is a diameter of $C_i$ parallel to $u_i$, and $a_i$ and $b_i$ are strictly between $c_i$ and $q_i$. Choose any $x_1 \in C_i \setminus \{p_i\}$ such that $\angle p_1 x_1 c_i$ is acute. We will now recursively choose $x_2, x_3, \ldots$ on the minor arc $\gamma_i$ of $C_i$ between $x_1$ and $p_i$ such that for any $z$ on the segment $a_i b_i$, the angle $\angle x_1 x_2 x_3$ is acute for all distinct $s, t \in \mathbb{N}$.

Assume that for some $t \in \mathbb{N}$ we have already chosen $x_1, \ldots, x_t \in \gamma_i$ with $x_{s+1}$ between $x_s$ and $p_i$ for each $s = 1, \ldots, t - 1$, and such that $\angle x_1 x_j x_k$ is acute for all $1 \leq j, k \leq t, j \neq k$, and for all $z$ on the segment $a_i b_i$. Since $p_i x_t q_i$ is a right angle, $\angle p_i x_t b_i$ is acute, and the line in $\Pi_i$ through $x_t$ and perpendicular to $b_i x_t$ intersects $C_i$ in a point $y_i$ between $x_t$ and $p_i$. Let $x_{t+1}$ be any point on $\gamma_i$.
between \( y \) and \( p_i \). Now consider any \( z \) on the segment \( a_i b_i \). We have to show that \( \angle z x_{t+1} x_s \) and \( z x_s x_{t+1} \) are acute for all \( s = 1, \ldots, t \). This can be simply seen as follows:

\[
\angle z x_{t+1} x_s \leq \angle z x_{t+1} x_t \leq \angle c_i x_{t+1} x_t < \pi/2
\]

and

\[
\angle z x_s x_{t+1} \leq \angle z x_t x_{t+1} \leq \angle b_i x_t x_{t+1} < \angle b_i x_t y = \pi/2.
\]

Finally, let \( V_i := \{ x_t : t \in \mathbb{N} \} \). Then \( \text{diam}(V_i) = |p_i x_1| \), which can be made arbitrarily small by choosing \( x_1 \) close enough to \( p_i \). We can assume that all \( \text{diam}(V_i) < \varepsilon \). This finishes the construction.

Let \( 1 \leq i < j \leq m \), \( x \in V_i \) and \( y \in V_j \). We have to show that all \( z \in \bigcup V_i \setminus \{ x, y \} \) are in the open slab bounded by the hyperplanes through \( x \) and \( y \) orthogonal to \( x y \). First consider the case where \( z \in V_k \), \( k \neq i, j \). Since \( \langle p_i p_j p_k \rangle \) and \( \angle p_j p_i p_k \) are acute, \( \langle p_i - p_j, p_k - p_j \rangle > 0 \) and \( \langle p_j - p_i, p_k - p_i \rangle > 0 \).

Noting that \( |x p_i|, |y p_j|, |z p_k| < \varepsilon \), it follows that \( \langle x - y, z - y \rangle > 0 \) and \( \langle y - x, z - x \rangle > 0 \) if \( \varepsilon \) is sufficiently small, depending only on the given points. That is, \( z \) is in the open slab determined by \( x y \).

Next consider the case where \( z \in V_i \cup V_j \). Without loss of generality, \( z \in V_i \).

Then

\[
\langle x - y, z - y \rangle = \langle x - y, z - x \rangle + |x y|^2 \geq -\varepsilon |x y| + |x y|^2 > 0,
\]

as long as \( \varepsilon < |x y| \). It remains to verify that \( \langle y - x, z - x \rangle > 0 \). Denote the orthogonal projection of a point \( p \in \mathbb{R}^{d+m} \) onto the plane \( \Pi_i \) by \( p' \). Since \( V_j \subset \Pi_j \subset \mathbb{R}^d + \text{lin}(v_j) \), it follows that \( p'_j, y' \in p_i + \text{lin}(u_i) \). In particular, \( p'_j \) is also the orthogonal projection of \( p_j \) onto the line \( p_i + \text{lin}(u_i) \). By hypothesis,
\[ p_j' = p_i + \lambda u_i \text{ for some } \lambda \in [\alpha_i, \beta_i]. \]

Since \(|p_j'y'| \leq |p_jy| < \varepsilon\), it follows that \(y' = p_i + \mu u_i\) where \(\mu \in [\alpha_i - \varepsilon, \beta_i + \varepsilon]\), that is, \(y'\) is on the segment \(a_ib_i\).

By construction, the angle \(\angle y'xz\) is acute, hence \(\langle y - x, z - x \rangle = \langle y' - x, z - x \rangle > 0.\)

\[ \square \]

**Corollary 10.** We have \(k'(d) \geq \lceil d/2 \rceil.\)

**Proof.** Let \(m := \lceil d/2 \rceil.\) Let \(p_1, \ldots, p_m\) be the vertices of a regular simplex in \(\mathbb{R}^{m-1}\) inscribed in the unit sphere. Then \(p_i\) and \(u_i := -p_i\) satisfy the conditions of Theorem 9. It follows that \(k'(d) \geq k'(2m - 1) \geq m.\)

**Theorem 11.** There exist \(m := \lfloor \frac{1}{4} e^{d/20} \rfloor\) distinct points \(p_1, \ldots, p_m \in \mathbb{R}^d\) and unit vectors \(u_1, \ldots, u_m \in \mathbb{R}^d\) such that for all distinct \(1 \leq i, j, k \leq m\), the angle \(\angle p_i p_j p_k\) is acute, and condition (17) is satisfied.

The proof of Theorem 11 is probabilistic, and is a modification of an argument of Erdős and Füredi [2]. Write \([d]\) for the set \(\{1, 2, \ldots, d\}\) of all integers from 1 to \(d\). For any \(A \subseteq [d]\), let \(\chi(A) \in \{0, 1\}^d\) denote its characteristic vector. The routine proofs of the following three lemmas are omitted.

**Lemma 12 [2, Lemma 2.3].** Let \(A, B,\) and \(C\) be distinct subsets of \([d]\). Then we have \(\angle \chi(A) \chi(C) \chi(B) \leq \pi/2\), and equality holds if and only if \(A \cap B \subseteq C \subseteq A \cup B\).

**Lemma 13 [2].** If \(A, B,\) and \(C\) are subsets of \([d]\) chosen independently and uniformly, then we have \(\Pr[A \cap B \subseteq C \subseteq A \cup B] = (3/4)^d.\)

**Lemma 14.** Let \(A, B, C \subseteq [d]\) and consider the unit vector

\[ u := (1/\sqrt{d})(\chi([d]) - 2\chi(A)). \]

Then we have \(\langle u, \chi(A) \rangle \leq \langle u, \chi(B) \rangle\), with equality if and only if \(A = B\). Also,

\[ \langle u, \chi(B) - \chi(C) \rangle \geq \langle u, \chi(C) - \chi(A) \rangle \]

if and only if

\[ 4|A \cap C| + |B| \geq 2|A \cap B| + |A| + 2|C|. \]

**Lemma 15.** If \(A, B,\) and \(C\) are subsets of \([d]\) chosen independently and uniformly, then we have

\[ \Pr[4|A \cap C| + |B| \geq 2|A \cap B| + |A| + 2|C|] \leq \left( \frac{65}{72} \right)^d < e^{-d/10}. \]

**Proof.** Let \(X\) be the random variable

\[ X := 4|A \cap C| + |B| - 2|A \cap B| - |A| - 2|C| = \sum_{i=1}^{d} X_i, \]

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where $X_i$ is the contribution of the element $i \in [d]$ to $X$, that is,

$$X_i := \begin{cases} 
1 & \text{if } i \in B \setminus (A \cup C) \text{ or } i \in (A \cap C) \setminus B, \\
0 & \text{if } i \in A \cap B \cap C \text{ or } i \notin A \cup B \cup C, \\
-1 & \text{if } i \in A \setminus (B \cup C) \text{ or } i \in (B \cap C) \setminus A, \\
-2 & \text{if } i \in C \setminus (A \cup B) \text{ or } i \in (A \cap B) \setminus C.
\end{cases}$$

Note that

$$\Pr[X_i = 1] = \Pr[X_i = 0] = \Pr[X_i = -1] = \Pr[X_i = -2] = 1/4.$$

We now bound $\Pr[X \geq 0]$ from above. For any $\lambda \geq 1$,

$$\Pr[X \geq 0] = \Pr[\lambda^X \geq 1] \leq \mathbb{E}[\lambda^X] = \prod_{i=1}^{d} \mathbb{E}[\lambda^{X_i}] = \left(\frac{\lambda + 1 + \lambda^{-1} + \lambda^{-2}}{4}\right)^d,$$

where we used Markov’s inequality and independence. Set $\lambda := 3/2$, which is close to minimizing the right-hand side. This gives $\Pr[X \geq 0] \leq (65/72)^d$. \qed

**Proof of Theorem 11.** Let $m := [(1/4)e^{d/20}]$. Choose subsets $A_1, \ldots, A_{2m}$ randomly and independently from the set $[d]$. For $i \in [d]$, define $p_i := \chi(A_i)$ and $u_i := (1/\sqrt{d})(\chi([d]) - 2\chi(A_i))$. Let $i, j, k \in [d]$ be distinct.

Assume that $A_i, A_j, A_k$ are distinct sets. Then by Lemma 12, $\angle p_i p_k p_j$ fails to be acute if and only if

$$A_i \cap A_j \subseteq A_k \subseteq A_i \cup A_j,$$  \hspace{1cm} (18)

and condition (17) is violated if and only if

$$\langle u_i, \chi(A_i) - \chi(A_j) \rangle \geq \langle u_i, \chi(A_k) - \chi(A_j) \rangle$$  \hspace{1cm} (19)

or

$$\langle u_i, \chi(A_k) - \chi(A_j) \rangle \geq \langle u_i, \chi(A_j) - \chi(A_i) \rangle.$$  \hspace{1cm} (20)

Condition (19) is equivalent to $\langle u_i, \chi(A_i) \rangle \geq \langle u_i, \chi(A_k) \rangle$. This, in turn, is equivalent to $A_i = A_k$, by the first statement of Lemma 14, contradicting our assumption that $A_i, A_j, A_k$ are distinct. By the second statement of Lemma 14, (20) is equivalent to

$$4|A_i \cap A_j| + |A_k| \geq 2|A_i \cap A_k| + |A_i| + 2|A_j|.$$  \hspace{1cm} (21)

Thus, for distinct points $p_i, p_j, p_k$, at least one of the conditions (18) and (21) holds if and only if $\angle p_i p_k p_j$ is a right angle or condition (17) is violated.

Note that if some two of the sets coincide, say $A_i = A_k$, then (18) also holds. Let us call a triple of distinct numbers $(i, j, k)$ bad if at least one of (18) and (21) holds. It follows that if no triple $(i, j, k)$ is bad, then all points $p_i$ are distinct,
all angles $\angle p_i p_j p_k$ are acute, and condition (17) is also satisfied. We will show that with positive probability, some $m$ of the $A_1, \ldots, A_{2m}$ will be without bad triples, which will prove the theorem.

By Lemmas 13 and 15 and the union bound, we obtain that

$$\Pr[(i,j,k) \text{ is bad}] \leq (3/4)^d + e^{-d/10} < 2e^{-d/10}.$$ 

By linearity of expectation, the expected number of bad triples is at most

$$2m(2m-1)(2m-2)2e^{-d/10} < 16m^3e^{-d/10}.$$ 

In particular, there exists a choice of subsets $A_1, \ldots, A_{2m} \subseteq [d]$ with less than $16m^3e^{-d/10}$ bad triples. For each bad triple $(i,j,k)$, remove $A_i$ from $\{A_1, \ldots, A_{2m}\}$. We are left with more than $2m - 16m^3e^{-d/10}$ sets without any bad triple. Since $m \leq (1/4)e^{d/20}$ implies that $2m - 16m^3e^{-d/10} \geq m$, we obtain $m$ points $p_i$ with unit vectors $u_i$ satisfying the theorem. \hfill \Box

COROLLARY 16. We have $k'(d) \geq d - O(\log d)$.

Proof. Let $n$ be the unique integer such that 

$$[(1/4)e^{n/20}] + n < d < [(1/4)e^{(n+1)/20}] + n + 1.$$ 

By Theorems 11 and 9, $k'(m+n+1) \geq m$ for any $m = 2, \ldots, [(1/4)e^{(n+1)/20}]$. In particular, if we take $m = d - n - 1$, we obtain

$$k'(d) \geq d - n - 1 > d - 20\log(4d) - 1. \hfill \Box$$

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