Stability Criterion of 1+1 Dilatonic Black Hole

Ramón Becar$^1$, Samuel Lepe$^2$ and Joel Saavedra $^2$

$^1$Departamento de Física, Universidad de Concepción, Casilla 160 C, Concepción, Chile
$^2$Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile
E-mail: joel.saavedra@ucv.cl

Abstract. We study the stability of the black hole that is a solution of the 2d dilaton gravity derived from string-theoretical models. We do this by means of the Quasinormal Modes (QNM) approach. In order to find the QNM of this geometry, we consider perturbations described by a massive scalar field non-minimally coupled to gravity. We find that the QNM frequencies turn out to be pure imaginary, consequently leading to purely damped modes, in agreement with the literature of dilatonic black holes. Our result exhibits a large range of stability for this geometry against the scalar perturbations that excite the overtone with $n > \mu$, $\mu$ being a dimensionless parameter related to the mass of the black hole. We consider both the minimal coupling case, i.e. for which the coupling parameter $\zeta$ vanishes, and the case $\zeta = \frac{1}{4}$.

1. Introduction
Two-dimensional theories of gravity have recently attracted much attention [1][2][3], and the black hole solutions have played an important role in revealing various aspects of black hole from string theory [4][5]. For this reason, it is important to study the dynamics of these fields and the metric perturbations in such black hole backgrounds in order to find stable solutions. One of this issues are the so-called quasinormal modes (QNMs), known as the "ringing" of black holes, that play an important role in classical aspects of black holes physics. In this work we are interested in the stability of the 1+1-dilatonic black hole through the study of the QNMs. Determination of QNMs for a specific geometry implies solving the field equations for different types of perturbation (scalar, fermionic, vectorial, etc...) with suitable boundary conditions that reflect the fact that the geometry describe one black hole. Quasinormal modes for a scalar classical perturbation of black holes are defined as the solutions of the Klein-Gordon equation characterized by purely ingoing waves at the horizon $\Phi \sim e^{-i\omega(t+r)}$, since at least a classically outgoing flux is not allowed at the horizon. In addition, one has to impose boundary conditions on the solutions at the asymptotic region (infinity) and it is crucial to use the asymptotic geometry of the space time under study. In the case of an asymptotically flat space time, the condition we need to impose over the wave function is a purely outgoing waves $\Phi \sim e^{i\omega(t-r)}$ at the infinity [6]. In general the QNMs are given by $\omega_{QNM} = \omega_R + i\omega_I$ where $\omega_R$ and $\omega_I$ are the real and the imaginary part of the frequency. Therefore the study of QNMs can be implemented as one simple test for studying the stability of the system. In this sense any imaginary frequency with the wrong sign would mean an exponentially growing mode, rather than damping of it.

In this work we analytically compute the QNMs of 1+1-dilatonic black hole and we use this fact as a test for the stability of the system. The organization of this article is as follows: In
Sec. II we specify the 1+1-dilatonic black hole. In Sec. III we determine the QNMs and we establish a criterion for the stability. Finally, we conclude in Sec. IV.

2. 1 + 1-Dilatonic Black Hole
In order to have a theory with a dynamical degree of freedom we consider a two dimensional gravity coupled to a dilatonic field, described by the action

$$S_g = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right).$$

This theory presents a black hole solution whose general static metric can be written as,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)},$$

where $f(r) = 1 - e^{-\phi}$ and $\phi = r/r_0$. If we perform the change of coordinates $x = \frac{r-r_0}{r_0}$, then $f(x) = 1 - e^{-x}$ and the horizon of this black hole is located in $x = 0$. This solution described the well known string-theoretic black hole [4][5] [7].

3. Quasinormal Modes
In order to study the QNMs, we consider a scalar field with no-minimal coupling to gravity propagating in the background of the dilatonic black hole. The field equations are obtained from the action [7]

$$S[\varphi] = -\frac{1}{2} \int d^2x \sqrt{-g} \left( (\nabla \varphi)^2 + \left( m^2 + \zeta R \right) \varphi^2 \right),$$

where $\zeta$ is one parameter from the no-minimal coupling. The field equations reads,

$$\left( -\mu^2 - \zeta R \right) \varphi = 0,$$

where $\mu = r_0m$, and in terms of the $x$ coordinate and by using a solution of the form $\varphi = e^{-\omega t}R(x)$, we can be writte the equation

$$f \partial_x^2 R(x) + e^{-x} \partial_x R(x) - \left( \frac{\omega^2}{f} - \mu^2 - \zeta e^{-x} \right) R(x) = 0.$$

If we perform the changes $z = 1 - e^{-x}$ and $R(z) = z^\alpha (1 - z)^\beta F(z)$, the last equation reads as the hypergeometric equation for the function $F(z)$, that is

$$z(1 - z)F'''(z) + (c - (a + b + 1)z)F'(z) - abF(z) = 0,$$

where $\zeta' = \zeta/r_0^2$ and the coefficients $a$, $b$ and $c$ are given through the relations

$$c = 2\alpha + 1, \quad a + b = 2(\alpha + \beta) + 1, \quad ab = (\alpha + \beta)(\alpha + \beta + 1) - \zeta',$$

being $\alpha$ and $\beta$

$$\alpha = \pm i\omega, \quad \beta = \pm \sqrt{\omega^2 - \mu^2},$$
and in addition without loss of generality, we put \( \alpha = -i\omega \) and \( \beta = \sqrt{\omega^2 - \mu^2} \). It is well known that the hypergeometric equation has three regular singular point at \( z = 0 \), \( z = 1 \) and \( z = \infty \), and it has two independent solutions in the neighborhood of each point \([8]\). The solutions of the radial equation reads as follows

\[
F(z) = C_1 F_1(a, b, c; z) + C_2 z^{1-c} F_1(a - c + 1, b - c + 1, 2 - c; z) \tag{9}
\]

so that the solution for \( R(z) \) is

\[
R(z) = C_1 z^{-i\omega} (1 - z)^{-i\sqrt{\omega^2 - \mu^2}} F_1(a, b, c; z) + C_2 z^{i\omega} (1 - z)^{-i\sqrt{\omega^2 - \mu^2}} * F_1(a - c + 1, b - c + 1, 2 - c; z), \tag{10}
\]

In the neighborhood of the horizon \( (z = 0) \)

\[
R(z) = C_1 e^{-i\omega \ln z} + C_2 e^{i\omega \ln z}, \tag{11}
\]

and \( \varphi \) can be written in the following way

\[
\varphi \sim C_1 e^{-i\omega (t + \ln z)} + C_2 e^{-i\omega (t - \ln z)}, \tag{12}
\]

where it is easy to see that the first term corresponds to an ingoing wave and the second term corresponds to an outgoing wave in the black hole. For computing the QNMs we must to impose that in the horizon we must have only ingoing waves, and in order to satisfy this condition we put \( C_2 = 0 \). Then the radial solution at the horizon is

\[
R(z) = C_1 z^{-i\omega} (1 - z)^{-i\sqrt{\omega^2 - \mu^2}} F_1(a, b, c; z), \tag{13}
\]

In order to implement boundary conditions at infinity \( (z = 1) \), we use the linear transformation \( z \to 1 - z \), Kummer’s formula \([8]\) for the hypergeometric function, and we obtain,

\[
R(z) = C_1 z^{-i\omega} (1 - z)^{-i\sqrt{\omega^2 - \mu^2}} \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} * F_1(a, b, a + b - c + 1; 1 - z) + C_1 z^{-i\omega} (1 - z)^{i\sqrt{\omega^2 - \mu^2}} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} * F_1(c - a, c - b, c - a - b + 1; 1 - z), \tag{14}
\]

and this solution near infinity \( (z = 1) \) takes on the form

\[
R(z) = C_1 (1 - z)^{-i\sqrt{\omega^2 - \mu^2}} \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} + C_1 (1 - z)^{i\sqrt{\omega^2 - \mu^2}} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}, \tag{15}
\]

and the solution for the field near infinity takes the following form

\[
\varphi \sim C_1 e^{-i\sqrt{\omega^2 - \mu^2} (t + \ln(1 - z))} \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} + C_1 e^{i\sqrt{\omega^2 - \mu^2} (t - \ln(1 - z))} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}, \tag{16}
\]
In order to compute the QNMs, we need to impose the boundary conditions upon the solution of the radial equation at infinity, meaning that only purely outgoing waves are allowed there. This implies

\[ a = -n, \quad \text{or} \quad b = -n, \]

where \( n = 0, 1, 2, \ldots \). These conditions lead directly to an exact determination of the quasinormal modes

\[
\omega = -\frac{i}{4} \left( 1 - \sqrt{1 - 4\zeta} - \frac{(1 + \sqrt{1 - 4\zeta})\mu^2}{n + n^2 + \zeta'} \right) + n \left( 2 - \frac{2\mu^2}{n + n^2 + \zeta'} \right) \tag{18}
\]

Note that the quasinormal modes are purely imaginary as in the 2+1-dilatonic case[13]. Besides this, (18) shows this kind of instabilities of black hole under scalar perturbations could imply an exponentially growing mode if the wrong sign of the pure imaginary frequency if chosen (positive). This fact can be explicitly shown in the following cases where \( \zeta' = 0 \) (minimal coupling)

\[
\omega = -i \left( \frac{n^2 - \mu^2}{2n} \right) \tag{19}
\]

where we see that the overtones \( n > \mu \) guarantees the stability under scalar perturbations, and a similar situation occurs in the conformal case \( \zeta' = 1/4 \)

\[
\omega = -i \left( \frac{(1 + 2n)^2 - 4\mu^2}{4 + 8n} \right) \tag{20}
\]

if \( n > \mu - 1/2 \).

4. Final Remark

In this paper we have computed the exact values of the quasinormal modes of dilatonic black holes which are purely imaginary (this kind of QNM's was also reported in Refs. [9][10][11][12][13]). This geometry is unstable under scalar perturbations that excite the zero mode. In the opposite direction this result shows the large stabilities of the dilatonic black hole for perturbations that excite the overtone with \( n > \mu \) in the minimal case and \( n > \mu - 1/2 \) in the conformal case, for all overtones in higher damping limit. Finally we want to note that this result allows to compute the QNMs modes of the five dimensional case [4] [5], where the metric is the product of an asymptotically flat two-dimensional geometry and a three-sphere with a constant radius. These two parts are completely decoupled from each other. We left this issue for a future work.

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