On limiting behavior of stationary measures for stochastic evolution systems with small noise intensity

Lifeng Chen¹, Zhao Dong², Jifa Jiang¹,*, & Jianliang Zhai³

¹Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China;
²Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China;
³Wu Wen-Tsun Key Laboratory of Mathematics, University of Science and Technology of China, Hefei 230026, China

Email: lfchenchina@163.com, dzhao@amt.ac.cn, jiangjf@shnu.edu.cn, zhaijl@ustc.edu.cn

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Abstract The limiting behavior of stochastic evolution processes with small noise intensity $\epsilon$ is investigated in distribution-based approaches. Let $\mu_\epsilon$ be a stationary measure for stochastic process $X_\epsilon$ with small $\epsilon$ and $X^0$ be a semiflow on a Polish space. Assume that \{\mu_\epsilon : 0 < \epsilon \leq \epsilon_0\} is tight. Then all their limits in the weak sense are $X^0$-invariant and their supports are contained in the Birkhoff center of $X^0$. Applications are made to various stochastic evolution systems, including stochastic ordinary differential equations, stochastic partial differential equations, and stochastic functional differential equations driven by Brownian motion or Lévy processes.

Keywords stationary measure, Lyapunov function, limit measure, support, Birkhoff center, stochastic evolution system

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1 Introduction

Mumford [44] addressed that “Stochastic differential equations are more fundamental and relevant to modeling the world than deterministic equations · · ·. A major step in making the equation more relevant is to add a small stochastic term. Even if the size of the stochastic term goes to 0, its asymptotic effects need not. It seems fair to say that all differential equations are better models of the world when a stochastic term is added and that their classical analysis is useful only if it is stable in an appropriate sense to such perturbations.” This shows that it is important to check the asymptotic stability of stochastic systems with small noise. For this purpose, a basic method is to study the stationary measures and their limit measures. The latters are called zero-noise limits by Young [49] and Cowieson and Young [11], where they proved Sinai, Ruelle and Bowen (SRB) measures can be realized as zero-noise limits. Huang et al. [27,30] have investigated stochastic ordinary differential equations with small white noise where the drift vector

* Corresponding author
field is dissipative. They have shown that limiting measures are invariant for the flow generated by the drift vector field and their supports are in its global attractor. For non-degenerate noise, Freidlin and Wentzell [18] estimated the concentration of limiting measures for stationary measures via the large-deviation technique and proved that the stationary measure value $\mu^\epsilon(P)$ tends to zero for any subset $P$ not intersecting with any attractor for the drift vector field, which implies that any limiting measure will be concentrated on the global attractor of the drift vector field; Li and Yi [36, 37] have presented a more precise estimation for stationary measures near the global attractor or outside of the global attractor via the Fokker-Planck equation and the level set method developed in [27], which are applied by them to study systematic measures of biological networks including degeneracy, complexity, and robustness. Hwang [31] proved the limiting probability measure of Gibbs measures for gradient systems with additive noise concentrates on the minimal energy states. Huang et al. [29] have explored the stochastic stability of invariant sets and measures for gradient systems with noise.

This paper is intended to establish the close connection between deterministic dynamical systems and their stochastic perturbations by considering the limiting behavior of stationary measures for stochastic evolution systems with small random perturbations. These stochastic evolution systems $X^\epsilon(t,x)$ may be solutions of various stochastic differential equations driven by white or Lévy noise with the intensity $\epsilon$. The corresponding solution of deterministic equations is denoted by $X^0(t,x)$. Let $\mu^\epsilon$ be the stationary probability measure of $X^\epsilon(t,x)$. We prove that all their weak limits of stationary measures $\mu^\epsilon$ of $X^\epsilon$ are $X^0$-invariant and their supports are contained in the Birkhoff center of $X^0$ as $\epsilon$ tends to zero (see Theorem 2.1). For various stochastic differential equations with small noise intensity, we prove the probability convergence property and provide the existence of stationary measures and their tightness and applications to all corresponding stochastic systems (see Sections 3–5). Usually, a global attractor for the finite dimensional system has positive Lebesgue measure if it is not a globally stable equilibrium; however, the Birkhoff center always has zero Lebesgue measure for the dissipative system. Compared with the existing results, which mostly focus on stochastic ordinary differential equations (SODEs) with non-degenerate noise, ours gives much more precise positions for limiting measures to be concentrated on. We note that our result is the best if we do not put any restriction on types of noise because we can construct a diffusion term such that a sequence of stationary measures converges weakly to a given invariant measure of the drift vector field (see Proposition 3.19 and Remark 4.8). As far as we know, among all existing examples (see, for example, [18, 30, 31]), the limiting measures are concentrated on stable orbits, such as, stable equilibria or closed orbits. A natural question arises: when a dissipative drift vector field has no stable motion in its global attractor, where is any limiting measure concentrated? Utilizing our result, we construct Bernoulli’s lemniscate with non-degenerate noise such that stationary measures converge weakly to a delta measure at a saddle of the drift vector field; however, the global attractor in this case is the closed domain surrounded by the lemniscate of Bernoulli (see Example 3.20).

In the May-Leonard system perturbed by one-dimensional white noise (see Example 3.24), we have proved that the limiting measures will be concentrated on the three saddles when the deterministic May-Leonard system admits a heteroclinic cycle. Also, from this example, the limiting measures can be distinguished by different initial values because of the various kinds of asymptotic behavior for deterministic equations. In a word, limiting measures are always concentrated on “most relatively stable positions”.

The rest of this paper is organized as follows. In Section 2, we present the framework to study the limiting measures of stationary measures for stochastic evolution processes and their supports. From Sections 3–5, we prove the probability convergence, the existence of stationary measures and their tightness for various stochastic differential equations. Specially, in Section 3, we deal with all these problems of stochastic ordinary differential equations (SODEs). In Section 4, we investigate stochastic reaction-diffusion equations, stochastic 2D Navier-Stokes equations and 1D stochastic Burgers type equations driven by Brownian motion or Lévy process. In Section 5, we consider a class of stochastic functional differential equations (SFDEs). Appendix A collects the basic properties with respect to invariant measures of deterministic flows.

Here and throughout this article, we will use the same symbol $|\cdot|$ to denote the Euclidean norm of a vector or the operator norm of a matrix. Sometimes we will write $X^\epsilon(t,x)$, $X^0(t,x)$ as $X^\epsilon_t(x)$, $X^\epsilon_0(x)$,
respectively, unless noted otherwise.

2 The general framework to study limiting measures

In this section, we will give a general criterion on studying limiting measures of stationary measures for stochastic evolution processes and describe their concentration.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(M, \rho)$ be a Polish space (i.e., a separable complete metric space) and $\mathcal{B}(M)$ be the Borel $\sigma$-algebra on $M$. Throughout this paper, we will denote by $\mathcal{P}(M)$ and $C_b(M)$ the sets of all probability measures and real-valued bounded continuous functions on $M$, respectively. Let $\mu \in \mathcal{P}(M)$. We say that a sequence $\{\mu_n\} \subset \mathcal{P}(M)$ converges weakly to $\mu$, denoted by $\mu_n \xrightarrow{w} \mu$, if

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu, \quad \forall f \in C_b(M).$$

We say that $\Pi \subset \mathcal{P}(M)$ is tight if for every $\eta > 0$, there exists a compact set $K \subset M$ such that $\mu(K) > 1 - \eta$ for every $\mu \in \Pi$. By Prokhorov’s theorem, since $M$ is a Polish space, $\Pi \subset \mathcal{P}(M)$ is tight if and only if it is relatively compact in the weakly convergent topology. The support of $\mu \in \mathcal{P}(M)$, denoted by $\text{supp}(\mu)$, is the smallest closed set whose complement has measure zero under $\mu$.

Assume that $\Phi_t(x) := X^0_t(x)$ is a deterministic semi-dynamical system (semiflow) on $(M, \rho)$ and for $0 < \epsilon \ll 1$, $X^\epsilon_t(x)$ is a noise driven process on $(M, \rho)$ with noise intensity $\epsilon$.

Throughout this article we assume that $\Phi : \mathbb{R}^+ \times M \to M$ is a mapping with the following properties:

(i) $\Phi(x)$ is continuous for all $x \in M$,

(ii) $\Phi_t(\cdot)$ is Borel measurable for all $t \in \mathbb{R}^+$,

(iii) $\Phi_0 = \text{id}$, $\Phi_t \circ \Phi_s(x) = \Phi_{t+s}(x)$ for all $t, s \in \mathbb{R}^+$, $x \in M$. Here, $\circ$ denotes composition of mappings.

Let $\{X^\epsilon_t(x)\}_{t \geq 0}$ be a family of processes with the initial value $x$ on the state space $M$, $\epsilon \in (0, 1]$. The transition probability function is defined as

$$P^\epsilon_t(x, A) := \mathbb{P}(X^\epsilon_t(x) \in A), \quad t \geq 0, \quad x \in M, \quad A \in \mathcal{B}(M).$$

A probability measure $\nu^\epsilon \in \mathcal{P}(M)$ is called stationary (or invariant) with respect to the transition semigroup $\{P^\epsilon_t\}_{t \geq 0}$ if

$$P^\epsilon_t \nu^\epsilon = \nu^\epsilon \quad \text{for any} \ t \geq 0.$$

Let $\mathcal{S}^\epsilon$ denote the set of all stationary measures of the process $\{X^\epsilon_t\}_{t \geq 0}$. For technical reasons, it is convenient to have the following properties.

**Hypothesis (Probability convergence).** For any given compact set $K \subset M$, $T > 0$ and $\delta > 0$,

$$\lim_{\epsilon \to 0} \sup_{x \in K} \mathbb{P}(\rho(X^\epsilon(T, x), \Phi(T, x)) \geq \delta) = 0. \quad (2.1)$$

**Theorem 2.1.** Assume the hypothesis (2.1) holds. If $\nu^\epsilon_i \in \mathcal{S}^\epsilon_i$, and $\nu^\epsilon_i \xrightarrow{w} \nu$ as $\epsilon_i \to 0$, then $\nu$ is an invariant measure of $\Phi$, i.e., $\nu \circ \Phi_t^{-1} = \nu$ for every $t \geq 0$. Moreover, this invariant probability measure $\nu$ is concentrated on $B(\Phi)$, where $B(\Phi) := \{x \in M : x \in \omega(x)\}$ denotes the Birkhoff center of $\Phi$ (see the definition in Appendix A).

**Proof.** Let $\nu^\epsilon_i \xrightarrow{w} \nu$ as $\epsilon_i \to 0$. It suffices to prove that for any nonzero $g \in C_b(M)$ and $T > 0$,

$$\int g(x) \nu \circ \Phi_T^{-1}(dx) = \int g(x) \nu(dx), \quad (2.2)$$

or equivalently,

$$\int g(\Phi(T, x)) \nu(dx) = \int g(x) \nu(dx).$$

Since $\{\nu^\epsilon_i\}$ is relatively compact, it is tight. For every $\eta > 0$, there exists a compact set $K \subset M$ such that $\inf_{\epsilon_i} \nu^\epsilon_i(K) \geq 1 - \frac{\eta}{\|g\|}$. It holds that

$$\left| \int g(x) \nu^\epsilon_i \circ \Phi(T, \cdot)^{-1}(dx) - \int g(x) \nu^\epsilon_i(dx) \right| = \left| \int g(\Phi(T, x)) \nu^\epsilon_i(dx) - \int g(X^\epsilon_i(T, x)) \nu^\epsilon_i(dx) \right|$$
\[
\leq \int E|g(\Phi(T, x)) - g(X_{\epsilon_i}^\tau(T, x))|\mu^{\epsilon_i}(dx) \\
\leq \int E[I_K(x)|g(\Phi(T, x)) - g(X_{\epsilon_i}^\tau(T, x))|\mu^{\epsilon_i}(dx) + 2\eta.
\]

\(\bar{K} := \Phi(T \times K) \subset M\) is a compact set since \(\Phi(T, x)\) is continuous on \(x\). We claim that there exists \(\delta > 0\) such that \(\forall y, z \in M\) with \(z \in \bar{K}\) and \(\rho(y, z) < \delta\), one has \(|g(y) - g(z)| < \eta\). If not, then there exist \(\eta_0 > 0\) and \(y_n \in M\) and \(z_n \in \bar{K}\) with \(\rho(y_n, z_n) < \frac{\delta}{n}\) such that \(|g(y_n) - g(z_n)| \geq \eta_0\), \(n = 1, 2, \ldots\) The compactness of \(\bar{K}\) and \(\{z_n\} \subset \bar{K}\) imply that, without loss of generality, \(z_n \to z_0 \in \bar{K}\) as \(n \to \infty\). Therefore, it follows from \(\rho(y_n, z_n) < \frac{\delta}{n}\) that \(y_n \to z_0\). By the continuity of \(g\), letting \(n \to \infty\), we have

\[
0 = |g(z_0) - g(z_0)| \geq \eta_0,
\]

which leads to a contradiction.

Hence one can derive that

\[
\int E[I_K(x)|g(\Phi(T, x)) - g(X_{\epsilon_i}^\tau(T, x))|\mu^{\epsilon_i}(dx)
\leq \int E[I_{\rho(\Phi(T, x), X_{\epsilon_i}^\tau(T, x)} \geq \delta}(\omega)|g(\Phi(T, x)) - g(X_{\epsilon_i}^\tau(T, x))|\mu^{\epsilon_i}(dx)
+ \int E[I_{\rho(\Phi(T, x), X_{\epsilon_i}^\tau(T, x)} < \delta}(\omega)|g(\Phi(T, x)) - g(X_{\epsilon_i}^\tau(T, x))|\mu^{\epsilon_i}(dx)
\leq 2\|g\| \sup_{x \in \bar{K}} P(\rho(X_{\epsilon_i}^\tau(T, x), \Phi(T, x)) \geq \delta) + \eta.
\]

Therefore, by the hypothesis (2.1), one can show that

\[
\limsup_{\epsilon_i \to 0} \left| \int g(x)\mu^{\epsilon_i} \Phi(T, \cdot)^{-1}(dx) - \int g(x)\mu^{\epsilon_i}(dx) \right|
\leq 2\|g\| \limsup_{\epsilon_i \to 0} P\{\rho(X_{\epsilon_i}^\tau(T, x), \Phi(T, x)) \geq \delta\} + \eta + 2\eta
= 3\eta.
\]

Since \(\eta > 0\) is arbitrary and \(\mu^{\epsilon_i} \rightharpoonup \mu\), (2.2) holds. This shows that \(\mu\) is an invariant probability measure of the semiflow \(\Phi\).

It remains to prove that \(\mu(B(\Phi)) = 1\). Indeed, the result of this fact relies on the following well-known lemma, the Poincaré recurrence theorem.

\[\textbf{Lemma 2.2.} \text{ The support of semiflow } \Phi\text{-invariant probability measure } \mu \text{ is contained in } B(\Phi). \text{ Consequently this implies that } \mu(B(\Phi)) = 1.\]

The above result is a slightly variant version of the Poincaré recurrence theorem (see, e.g., [40, Theorem 2.3, p. 29]) to obtain the concentration of invariant measures. For readers’ convenience, we also give a self-contained proof of Lemma 2.2 which is postponed to Appendix A.

\[\textbf{Remark 2.3.} \text{ Observing the proof of Theorem 2.1, we only need to prove the probability convergence property for a compact set } K \text{ satisfying the definition of tightness. This remark will be used in stochastic partial differential equations (SPDEs) of Section 4.}\]

In order to apply Theorem 2.1 to various stochastic differential equations, the hypothesis (2.1) and the existence of stationary measures for \(X_{\epsilon_i}^\tau(x)\) and their tightness are needed to be proved. In the rest of this paper, we will check them for various stochastic evolution systems.

\section{ODEs driven by Lévy noise}

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a filtration \(\{\mathcal{F}_t, t \geq 0\}\) satisfying the usual conditions, \(W = \{W_t, t \geq 0\}\) be a \(k\)-dimensional Wiener process and \(N\) be a Poisson random measure on \(\mathbb{R}_+\).
V a non-negative function stationary measures of (3.1) for a given \( \epsilon \) where \( N(dt, dy) = N(dt, dy) - \nu(dy)dt \). Denote by \( (L_2(\mathbb{R}^k, \mathbb{R}^m), \| \cdot \|_2) \) the Hilbert space of all Hilbert-Schmidt operators from \( \mathbb{R}^k \) to \( \mathbb{R}^m \). Actually, \( L_2(\mathbb{R}^k, \mathbb{R}^m) \) is an \( m \times m \) matrix set.

Consider the following SODEs driven by a Lévy process:

\[
\dot{X}^{\epsilon,x}(t) = b(X^{\epsilon,x}(t))dt + \sigma(X^{\epsilon,x}(t))dW_t + \epsilon \int_{|y|_M < \epsilon} F(X^{\epsilon,x}(t-), y)\tilde{N}(dt, dy)
\]

with the initial condition \( X^{\epsilon,x}(0) = x \in \mathbb{R}^m \) and \( \epsilon, c > 0 \). The mappings \( b : \mathbb{R}^m \to \mathbb{R}^m \) and \( \sigma : \mathbb{R}^m \to L_2(\mathbb{R}^k, \mathbb{R}^m) \) are \( \mathcal{B}(\mathbb{R}^m) \) measurable functions, \( F : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m \) is a \( \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^l) \) measurable function.

\( b, \sigma \) and \( F \) are called to satisfy the local Lipschitz condition, respectively, if for every integer \( n \geq 1 \), there is a positive constant \( L_1(n) \) such that for all \( x, y \in \mathbb{R}^m \) with \( |x| \leq n \) and \( |y| \leq n \),

\[
|b(x) - b(y)|^2 \leq L_1(n)|x - y|^2, \\
\|\sigma(x) - \sigma(y)\|^2 \leq L_1(n)|x - y|^2, \\
\int_{\|z\|_M < \epsilon} |F(x, z) - F(y, z)|^2 \nu(dz) \leq L_1(n)|x - y|^2,
\]

respectively. In addition, we say that \( F \) satisfies the local growth condition, if for every integer \( n \geq 1 \), there is a positive constant \( L_2(n) \) such that for all \( x \in \mathbb{R}^m \) with \( |x| \leq n \),

\[
\int_{\|z\|_M < \epsilon} |F(x, z)|^2 \nu(dz) \leq L_2(n)(1 + |x|^2).
\]

If \( L_i(n), i = 1, 2 \) are independent of \( n \), we say that the coefficient functions admit global Lipschitz and linear growth conditions.

For a \( C^2 \) scalar function \( V \), and \( \epsilon \geq 0 \), we define

\[
\mathcal{L}^\epsilon V(x) := \langle \nabla V(x), b(x) \rangle + \frac{\epsilon^2}{2} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \\
+ \int_{|y|_M < \epsilon} (V(x + \epsilon F(x, y)) - V(x) - \langle \nabla V(x), \epsilon F(x, y) \rangle)\nu(dy),
\]

where \( A(x) = (a_{ij}(x)) := \sigma(x)\sigma^T(x) \) is the diffusion matrix. Let us recall that \( \mathcal{F}^\epsilon \) is the set of all stationary measures of (3.1) for a given \( \epsilon \). The following is the main result of this section.

**Theorem 3.1** (Support on limiting measures). Let \( b, \sigma \) and \( F \) in (3.1) be locally Lipschitz continuous and locally linear growth, and \( F(x, y) \) be locally bounded with respect to \( (x, y) \). Suppose that there exists a non-negative function \( V \in C^2(\mathbb{R}^m) \) and \( \epsilon_0 > 0 \) such that

\[
\inf_{|x| > R} V(x) \to +\infty \quad \text{as} \quad R \to \infty,
\]

\[
\sup_{|x| > R} \mathcal{L}^\epsilon V(x) \leq -A_R \to -\infty \quad \text{as} \quad R \to \infty
\]

uniformly in \( \epsilon \in (0, \epsilon_0] \). Then \( \mathcal{F}^\epsilon \) is nonempty for every \( \epsilon \in (0, \epsilon_0] \), and the family \( \mathcal{F} = \bigcup_{0 < \epsilon \leq \epsilon_0} \mathcal{F}^\epsilon \) is tight. Furthermore, if \( \mu_{\epsilon_i}^x \in \mathcal{F}^\epsilon_i \), and \( \mu_{\epsilon_i}^x \overset{w}{\to} \mu \) as \( \epsilon_i \to 0 \), then \( \mu \) is an invariant measure of \( X^0(t) \), which is concentrated on the Birkhoff center \( B(X^0) \).

The proof of Theorem 3.1 follows from Subsections 3.1 and 3.2 and Theorem 2.1.

### 3.1 The criterion for probability convergence

By standard arguments, we have the following theorem.
Lemma 3.2. Suppose that the coefficient functions $b$, $\sigma$ and $F$ admit global Lipschitz and linear growth conditions with positive constant $L$. Then the system (3.1) admits a unique strong solution $X^{\epsilon,x} = \{X^{\epsilon,x}(t) : t \geq 0\}$, which is adapted and has càdlàg sample paths. Moreover, for every fixed $T > 0$, there is a constant $D_{L,T} > 0$ such that for each $x \in \mathbb{R}^m$, one has

$$\sup_{\epsilon \in [0,1]} \sup_{t \in [0,T]} E|X^{\epsilon,x}(t)|^2 \leq D_{L,T}(1 + |x|^2).$$

(3.8)

Denote by $X^{0,x}(t)$ the solution for (3.1) as $\epsilon = 0$. Then we also have the following proposition.

Proposition 3.3. Suppose that the coefficient functions $b$, $\sigma$ and $F$ admit global Lipschitz and linear growth conditions with positive constant $L$. Then there exists a constant $D_{L,T}^*$ such that for every $\epsilon \in (0,1]$,

$$E \left[ \sup_{0 \leq t \leq T} |X^{\epsilon,x}(t) - X^{0,x}(t)|^2 \right] \leq D_{L,T}^* \epsilon^2(1 + |x|^2)$$

(3.9)

for all $x \in \mathbb{R}^m$.

Theorem 3.4. Let $b$, $\sigma$ and $F$ be locally Lipschitz continuous and locally linear growth. If there exist a function $V \in C^2(\mathbb{R}^m, \mathbb{R}_+)$, $\epsilon_0 > 0$ and a constant $c^* \in \mathbb{R}$, such that (3.6) and

$$\mathcal{L}V(x) \leq c^*V(x)$$

(3.10)

uniformly in $\epsilon \in [0,\epsilon_0]$ hold, then there exists a global unique solution $X^{\epsilon,x}(t)$ to (3.1) for all $x \in \mathbb{R}^m$ and $\epsilon \in [0,\epsilon_0]$. Moreover, the hypothesis (2.1) holds, i.e., for any given compact set $K \subset \mathbb{R}^m$, $T > 0$ and $\delta > 0$,

$$\lim_{\epsilon \to 0} \sup_{x \in K} P(|X^{\epsilon,x}(T) - X^{0,x}(T)| \geq \delta) = 0.$$  

(3.11)

Proof. For the global existence and uniqueness of the solution to (3.1) we refer to a similar proof in [34, Theorems 1.1.3 and 3.3.5], for example. Without loss of generality, we assume $c^* > 0$. Let $\tau_n^{\epsilon,x} = \inf\{t : |X^{\epsilon,x}(t)| > n\}$ and $\tau_n^{0,x} = \inf\{t : |X^{0,x}(t)| > n\}$. It is easy to see that $\tau_n^{\epsilon,x} \uparrow +\infty$ and $\tau_n^{0,x} \uparrow +\infty$ as $n \to \infty$.

For each $n \in \mathbb{N}^*$, let $S^n(r)$ be a non-increasing $C^\infty$ function with values in $[0,1]$ such that

$$S^n(r) = \begin{cases} 1, & \text{if } r \in [0,n], \\ n + \frac{1}{r}, & \text{if } r \in [n+1, +\infty). \end{cases}$$

Construct the functions

$$b_n(x) = b(xS^n(|x|)), \quad \sigma_n(x) = \sigma(xS^n(|x|)), \quad F_n(x,y) = F(xS^n(|x|), y).$$

(3.12)

Then $b_n$, $\sigma_n$ and $F_n$ clearly satisfy global Lipschitz and linear growth conditions. Let $X_n^{\epsilon,x}(t)$ be the solution associated with the functions $b_n$, $\sigma_n$ and $F_n$. It is easy to see that

$$X_n^{\epsilon,x}(t) = X_n^{\epsilon,x}(t) \quad \text{for } t \leq \tau_n^{\epsilon,x}.$$  

(3.13)

Repeating the proof in [34, Theorem 3.3.5], we know that

$$P(\tau_n^{\epsilon,x} \leq T) \leq \exp(c^*T)V(x) \inf_{|y| > n} V(y),$$

which implies that $P(\tau_n^{\epsilon,x} \leq T) \to 0$ as $n \to \infty$ uniformly for $x \in K$. This shows that $\forall \eta > 0, \exists N_0 \in \mathbb{N}^*$ such that $\forall n \geq N_0$, we have

$$\sup_{x \in K} P(\tau_n^{\epsilon,x} \leq T) < \eta.$$  

(3.14)

The compactness of $K$ and continuity for the solution $X^{0,x}(t)$ with respect to the initial point ensure that there exists $N_1 \in \mathbb{N}^*$, such that for all $n \geq N_1$,

$$\inf_{x \in K} \tau_n^{0,x} > T.$$  

(3.15)
Now choosing \( n \geq N_0 \lor N_1 \), and using (3.13)–(3.15) and (3.9), we have
\[
\sup_{x \in K} P\{|X^{t,x}(T) - X_n^{0,x}(T)| \geq \delta\} \leq \sup_{x \in K} P\{|X^{t,x}(T) - X^{0,x}(T)| \geq \delta, T < \tau_n^{t,x} \}
+ \sup_{x \in K} P(\tau_n^{t,x} \land \tau_n^{0,x} \leq T)
= \sup_{x \in K} P\{|X^{t,x}(T) - X_n^{0,x}(T)| \geq \delta, T < \tau_n^{t,x} \}
+ \sup_{x \in K} P(\tau_n^{t,x} \leq T)
\leq \frac{1}{\delta^2} \sup_{x \in K} E|X^{t,x}(T) - X_n^{0,x}(T)|^2 + \eta
\leq \sup_{x \in K} \frac{D_{\epsilon_n,T} \epsilon^2}{\delta^2} + \eta
\leq \eta, \quad \text{as } \epsilon \to 0.
\]
The assertion (3.11) follows immediately from that \( \eta \) is arbitrarily small. \( \square \)

**Remark 3.5.** The results of Theorem 3.4 are valid if we replace merely (3.10) by a wider condition
\[
\mathcal{L}^*V(x) \leq cV(x) + K
\tag{3.16}
\]
for some constants \( c \) and \( K \).

In fact, let \( \tilde{V}(x) = V(x) + \frac{K^+}{a^+} \), where \( a^+ = \max\{a, 0\} \). Then \( \tilde{V} \in \mathcal{C}^2(\mathbb{R}^m, \mathbb{R}_+) \), \( \lim_{R \to \infty} \inf_{|x| > R} \tilde{V}(x) = +\infty \) and
\[
\mathcal{L}^*\tilde{V}(x) = \mathcal{L}^*V(x) \leq cV(x) + K \leq (a^+ + 1)V(x) + K^+ = (a^+ + 1)\tilde{V}(x).
\]

### 3.2 The criteria on the existence of stationary measures and their tightness

Following the arguments as in [33,34], we obtain the criterion on the tightness of a family of stationary measures for (3.1).

**Theorem 3.6 (Tightness criterion).** Suppose that \( h, \sigma \) and \( F \) in (3.1) are locally Lipschitz continuous and locally linear growth, and \( F(x,y) \) is locally bounded with respect to \( (x,y) \), and that there exists a scalar function \( V \in \mathcal{C}^2(\mathbb{R}^m, \mathbb{R}_+) \) such that (3.6) and (3.7) hold. Then for any \( x \in \mathbb{R}^m \), there exists at least a stationary measure \( \mu_x^\epsilon \in \mathcal{I}^\epsilon \) for every \( \epsilon \), and the set of stationary measures \( \mathcal{I} = \bigcup_{0 < \epsilon \leq \epsilon_0} \mathcal{I}^\epsilon \) is tight.

**Proof.** For any fixed \( x \in \mathbb{R}^m \), it follows from Theorem 3.4 that the solution \( X^\epsilon(t,x) \) is globally defined on \([0, +\infty)\). For any \( n \in \mathbb{N}^* \), we define stopping time \( \tau_n^\epsilon = \inf\{t : |X^\epsilon(t,x)| > n\} \). Then Itô’s formula and Doob’s optional sampling theorem (see [1,32]) imply that
\[
EV(X^\epsilon(t \land \tau_n^\epsilon, x)) - V(x) = E \int_0^{t \land \tau_n^\epsilon} \mathcal{L}^*V(X^\epsilon(s,x))ds.
\]
Since \( V \in \mathcal{C}^2(\mathbb{R}^m) \) and \( F(x,y) \) is locally bounded, applying the Taylor expansion and (3.5), we obtain
\[
\sup_{\epsilon \in [0, \epsilon_0]} \int_{|y| < c} (V(x + \epsilon F(x,y)) - V(x) - \langle \nabla V(x), \epsilon F(x,y) \rangle) \nu(dy) < \infty.
\]
By \( V \in \mathcal{C}^2(\mathbb{R}^m) \) again and (3.7),
\[
C := \sup_{\epsilon \in [0, \epsilon_0]} \sup_{x \in \mathbb{R}^m} \mathcal{L}^*V(x) < \infty.
\]
Hence we have
\[
\mathcal{L}^*V(X^\epsilon(s,x)) \leq -I_{\{|X^\epsilon(s,x)| > R\}} A_R + C.
\]
It is easy to get
\[ A_R E \int_0^{t \wedge \tau_n^F} I_{\{|X^F(s,x)|>R\}} ds \leq V(x) + Ct, \]
where we have used the condition (3.7). Since \( t \wedge \tau_n^F \to t \) a.s. as \( n \to \infty \), letting \( n \to \infty \) and then changing the order of integration in the last inequality, we have for \( t > 0 \),
\[ \frac{1}{t} \int_0^t P^e(s,x,U_R^c) ds \leq \frac{1}{A_R} \left( \frac{V(x)}{t} + C \right), \tag{3.17} \]
where \( U_R^c = \{ x \in \mathbb{R}^m : |x| > R \} \). This implies that
\[ \lim_{R \to \infty} \liminf_{t \to \infty} \frac{1}{t} \int_0^t P^e(s,x,U_R^c) ds = 0. \]
Applying [34, Theorem 3.3.1], we know there exists at least a stationary measure \( \mu_x^e \), which is produced by the Krylov-Bogoliubov procedure, i.e., \( \mu_x^e \) is a weak limit of a subsequence of probability measures on \( \mathbb{R}^m \) defined by
\[ P^e, t(x,B) = \frac{1}{t} \int_0^t P^e(s,x,B) ds. \]
For any \( \mu \in \mathcal{I} \), say \( \mu \in \mathcal{I}^e \). Using the invariance of \( \mu \), Fubini’s theorem, Fatou’s lemma and (3.17) we obtain
\[ \mu(U_R^c) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}^m} P^e(s,x,U_R^c) \mu(dx) ds \]
\[ = \lim_{t \to \infty} \int_{\mathbb{R}^m} \frac{1}{t} \int_0^t P^e(s,x,U_R^c) ds \mu(dx) \]
\[ \leq \int_{\mathbb{R}^m} \limsup_{t \to \infty} \frac{1}{t} \int_0^t P^e(s,x,U_R^c) ds \mu(dx) \]
\[ \leq \frac{C}{A_R}. \]
It follows that \( \sup_{\mu \in \mathcal{I}} \mu(U_R^c) \leq \frac{C}{A_R} \to 0 \) as \( R \to \infty \). This completes the proof. \( \square \)

**Remark 3.7.** From our proof, the conclusions still hold if \( C \leq 0 \) and there is a constant \( \gamma > 0 \) such that \( A_R \geq \gamma > 0 \) for \( R \) sufficiently large.

**Remark 3.8.** Huang et al. [27] gave the estimate of stationary measures in the essential domain of a Lyapunov-like function in the case \( F \equiv 0 \), which provides the criterion for the tightness of stationary measures.

Our results allow the nonlinear terms in (3.1) to be polynomial growth, which is stated as the following corollary.

**Corollary 3.9.** Suppose that \( b, \sigma \) and \( F \) in (3.1) are locally Lipschitz continuous and locally linear growth, and that \( F(x,y) \) is locally bounded with respect to \( (x,y) \). If there are positive constants \( c_1, c_2 \) and \( q \geq 2 \) such that for \( |x| \) sufficiently large, one has
\[ \langle b(x), x \rangle \leq -c_1|x|^q, \]
\[ \frac{1}{2} \|\sigma(x)\|^2 + \int_{|y|<c} |F(x,y)|^2 \nu(dy) \leq c_2|x|^q, \]
then the conclusions of Theorems 3.4 and 3.6 hold.

**Proof.** Define \( V : \mathbb{R}^m \to \mathbb{R}_+ \) by
\[ V(x) := \frac{1}{2} \sum_{i=1}^m (x_i)^2. \]
Then
\[ \mathcal{L}^\epsilon V(x) = \langle b(x), x \rangle + \frac{\epsilon^2}{2} \| \sigma (x) \|_2^2 + \epsilon^2 \int_{|y|_d < \epsilon} |F(x, y)|^2 \nu (dy) \]
\[ \leq -(c_1 - \epsilon^2 c_2) |x|^q \]
\[ \leq -\frac{c_1}{2} |x|^q \]
for $|x|$ sufficiently large and $\epsilon$ sufficiently small. It is easy to see that (3.6) and (3.7) and all other conditions in Theorems 3.4 and 3.6 hold. The proof is completed.

It is easy to see that Theorem 3.1 follows immediately from Theorems 3.4, 3.6 and 2.1.

3.3 On the uniqueness and ergodicity of stationary measures

In this subsection, we will provide a result for the uniqueness and ergodicity of stationary measures with respect to $\{P_t\}_{t \geq 0}$. Here, $\epsilon$ is dropped when there is no risk of confusion. To achieve this goal, we will give the sufficient conditions for $\{P_t\}_{t \geq 0}$ to be irreducible and strong Feller.

Lemma 3.10. Suppose that the assumptions of Lemma 3.2 hold. If the non-degeneracy
\[ \sup_{x \in \mathbb{R}^m} |\sigma^T (x) (\sigma (x) \sigma^T (x))^{-1}| =: \tilde{K} < \infty \quad (3.18) \]
holds, then the semigroup $\{P_t\}_{t \geq 0}$ of the solution for (3.1) is irreducible.

We furthermore assume the following:
(a) $b, \sigma \in C_b^1 (\mathbb{R}^m)$,
(b) there exists a non-negative function $\tilde{c} \in L^2 (\mathbb{R}^l, \mathcal{B} (\mathbb{R}^l), \nu)$ such that
\[ |F(x, y)| \leq \tilde{c}(y), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^l, \]
(c) there exists a constant $C > 0$ such that
\[ \int_{|y|_d < \epsilon} \| D_x F (0, y) \|_2^2 \nu (dy) \leq C, \]
\[ \int_{|y|_d < \epsilon} \| D_x F (x_1, y) - D_x F (x_2, y) \|_2^2 \nu (dy) \leq C |x_1 - x_2|^2, \quad x_1, x_2 \in \mathbb{R}^m. \]

Then the semigroup $\{P_t\}_{t \geq 0}$ has the strong Feller property.

Proof. For the diffusion case, i.e., $F \equiv 0$, it is well known that the semigroup $\{P_t\}_{t \geq 0}$ of the solution for (3.1) is strong Feller and irreducible (see, e.g., [50]). We will prove the jump diffusion case. In the following, we denote by $X(t)$ the solution to (3.1) and $X^d(t)$ to the diffusion case.

(1) Irreducibility:

Step 1. Suppose $\nu (\{|y|_d < c\}) < \infty$.

Let $\{\tau_i\}_{i \geq 1}$ be the interarrival times of the Poisson random measure $N$. Then $\{\tau_i\}_{i \geq 1}$ is a point process associated with the Poisson random measure $N$ which satisfies
(i) $\{y_{\tau_i}\} \subset U := \{y : |y|_d < c\}$,
(ii) $\{(\tau_i, y_{\tau_i})\}_{i \geq 1}$ is independent and for the measurable set $\widetilde{O} \subset U (\subset \mathbb{R}^l)$, $t > 0$,
\[ P(\tau_i > t, y_{\tau_i} \in \widetilde{O}) = e^{-\nu (U)} \nu (\widetilde{O}). \]

On $[0, \tau_1)$, $X(t) = X^d(t)$ and $X_{\tau_1} = X_{\tau_1-} + F (X_{\tau_1}, y_{\tau_1})$. Since $\{\tau_i, y_{\tau_i}\}_{i \geq 1}$ is independent of the solution $X^d(t)$, as proved in [13, 14], we have the relationship for $x \in \mathbb{R}^m$, $t > 0$, $\Gamma \in \mathcal{B} (\mathbb{R}^m)$,
\[ P_t (x, \Gamma) = e^{-\nu (U)} P_t^0 (x, \Gamma) + \int_0^t \int_{\mathbb{R}^m} e^{-\nu (U)} P_{t-s} (x + F(z, y), \Gamma) \nu (dy) ds P_t^0 (x, dz), \]
where $P_t^0$ is the semigroup of the solution for (3.1) with $F \equiv 0$, which is irreducible. Therefore, we have that $P_t$ is irreducible.

**Step 2.** Suppose $\nu(\{|y|_{\mathbb{R}^d} < c\}) = \infty$. The irreducibility of $X(t)$ can be proved by using the arguments in [15]. So we omit it.

(2) The strong Feller property:

Denote by $X(t,x)$ the solution of (3.1) ($\epsilon = 1$) with the initial value $x$.

For any $\phi \in C^1(\mathbb{R}^m)$, $t \geq 0$, and $h \in \mathbb{R}^m$,

$$D_x E\phi(X(t,x)) h = E[D_x \phi(X(t,x)) h],$$

where $h = D_x(X(t,x)) h$ is the solution of the equation

$$d h = D_x b(X(t,x)) h dt + D_x \sigma(X(t,x)) h dW_t$$

$$+ \int_{|y|_{\mathbb{R}^d} < c} D_x F(X(t-, x), y) h \tilde{N}(dt, dy), \quad h_0 = h. \quad (3.19)$$

From (a)–(c) and (3.18), by the standard method, we know that there exists some constant $C_T > 0$, independent of $h$ such that

$$E|\eta_t^h|^2 \leq C_T |h|^2 \quad \text{for all } t \in [0, T]. \quad (3.20)$$

We can prove that $V(t,x) := E\phi(X(t,x))$ is a solution of the equation

$$\left\{ \begin{array}{l}
\frac{dV(t,x)}{dt} = LV(t,x), \\
V(0,x) = \phi(x)
\end{array} \right. \quad (3.21)$$

(see [19, Theorem 3.1, p. 89]). Using Itô’s formula on $V(t-s,x)$ with respect to $s \in [0,t]$ and $x \in \mathbb{R}^m$ of $X(t,x)$, noting that $V$ is the solution of (3.21), we have

$$\phi(X(t,x)) = V(t,x) + \int_0^t \left[ \frac{\partial}{\partial s} V(t-s, X(s,x)) + L V(t-s, X(s,x)) \right] ds$$

$$+ \int_0^t D_x V(t-s, X(s,x)) \sigma(X(s,x)) dW_s$$

$$+ \int_0^t \int_{U} [V(t-s, X(s-,x) + F(X(s-, x), y)) - V(t-s, X(s-, x))] \tilde{N}(ds, dy)$$

$$= V(t,x) + \int_0^t D_x V(t-s, X(s,x)) \sigma(X(s,x)) dW_s$$

$$+ \int_0^t \int_{U} [V(t-s, X(s-,x) + F(X(s-, x), y)) - V(t-s, X(s-, x))] \tilde{N}(ds, dy). \quad (3.22)$$

Multiplying both sides of (3.22) by $\int_0^t [\sigma^T(X(s,x))(\sigma(X(s,x))\sigma^T(X(s,x)))^{-1} \eta_s^h, dW_s]_{\mathbb{R}^k}$ and taking expectations, we get the following Bismut-Elworthy-Li formula for (3.1) (see, e.g., [12, Lemma 7.13]):

$$D_x E\phi(X(t,x)) h = \frac{1}{t} E \left[ \phi(X(t,x)) \int_0^t [\sigma^T(X(s,x))(\sigma(X(s,x))\sigma^T(X(s,x)))^{-1} \eta_s^h, dW_s]_{\mathbb{R}^k} \right]. \quad (3.23)$$

Indeed,

$$E \left[ \phi(X(t,x)) \int_0^t [\sigma^T(X(s,x))(\sigma(X(s,x))\sigma^T(X(s,x)))^{-1} \eta_s^h, dW_s]_{\mathbb{R}^k} \right]$$

$$= E \int_0^t \left\langle [D_x V(t-s, X(s,x))\sigma(X(s,x))]^T, \sigma^T(X(s,x))(\sigma(X(s,x))\sigma^T(X(s,x)))^{-1} \eta_s^h \right\rangle_{\mathbb{R}^m} ds$$

$$= E \int_0^t \left\langle [D_x V(t-s, X(s,x))]^T, \sigma(X(s,x))\sigma^T(X(s,x))(\sigma(X(s,x))\sigma^T(X(s,x)))^{-1} \eta_s^h \right\rangle_{\mathbb{R}^m} ds.$$
= E \int_0^t D_x V(t - s, X(s, x)) \eta_s^h ds \\
= E \int_0^t D_x P_{t-s}(\phi(X(s, x))) h ds \\
= \int_0^t D_x EP_{t-s}(\phi(X(s, x))) h ds \\
= t D_x \phi(X(t, x)) h.

For any \( \phi \in C^1_b(\mathbb{R}^m) \), from (3.18), (3.20) and (3.23), it follows

\[
|D_x \phi(X(t, x)) h|^2 \leq \frac{\|\phi\|^2}{t^2} E \left( \int_0^t \langle \sigma^T(X(s, x)) \sigma(X(s, x))^{-1} \eta_s^h, dW_s \rangle \right)^2 \\
\leq \frac{\|\phi\|^2}{t^2} E \left( \int_0^t \| \sigma^T(X(s, x)) \sigma(X(s, x))^{-1} \eta_s^h \|^2 ds \right) \\
\leq \frac{\|\phi\|^2}{t} \tilde{K}C_T |h|^2.
\]

Then there exists a positive constant \( \tilde{C}_T \) such that

\[
|D_x \phi(X(t, x))| \leq \frac{\|\phi\|}{\sqrt{t}} \tilde{C}_T.
\]

Therefore, we have

\[
|P_t \phi(x) - P_t \phi(y)| \leq \frac{\tilde{C}_T}{\sqrt{t}} \|\phi\| |x - y| \quad \text{for all } t \in (0, T].
\]

By [12, Lemma 7.15], \( \{ P_t \}_{t \geq 0} \) has the strong Feller property.

**Remark 3.11.** The uniformly elliptic property of the diffusion matrix \( \sigma \sigma^T \), i.e., there is a constant \( \lambda > 0 \) such that \( \xi^T \sigma(x) \sigma(x)^T \xi \geq \lambda |\xi|^2 \) for all \( x \in \mathbb{R}^m \) and \( \xi \in \mathbb{R}^m \), implies (3.18). Indeed, the boundedness of \( \bar{\sigma} = \sigma \sigma^T \sigma^{-1} \) follows from the fact that

\[
|\bar{\sigma}(x)\xi|^2 = \xi^T \bar{\sigma}(x)^T \bar{\sigma}(x) \xi = \xi^T (\sigma(x) \sigma^T(x))^{-1} \xi \leq \lambda^{-1} |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^m.
\]

**Theorem 3.12.** Suppose the assumptions of Theorem 3.4 are satisfied. If the non-degeneracy

\[
|\sigma^T(x) (\sigma(x) \sigma^T(x))^{-1}| < \infty \tag{3.24}
\]

holds, then the semigroup \( \{ P_t \}_{t \geq 0} \) is irreducible.

Furthermore, if

(a) \( b, \sigma \in C^1(\mathbb{R}^m) \);

(b) for any \( n \in \mathbb{N}^* \), there exists a non-negative function \( c_n \in L^2(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l), \nu) \) such that

\[
\sup_{|x| \leq n} |F(x, y)| \leq c_n(y), \quad y \in \mathbb{R}^l;
\]

(c) there exist positive constants \( C \) and \( C_r \) for any \( r > 0 \) such that

\[
\int_{|y| < r} \| D_x F(0, y) \|_2^2 \nu(dy) \leq C, \\
\int_{|y| < r} \| D_x F(x_1, y) - D_x F(x_2, y) \|_2^2 \nu(dy) \leq C_r |x_1 - x_2|^2, \quad |x_1| \vee |x_2| \leq r,
\]

then the semigroup \( \{ P_t \}_{t \geq 0} \) has the strong Feller property.
Proof. It can be readily checked that for each \( n \geq 1 \) the coefficients \( b_n, \sigma_n \) and \( F_n \) as in the proof of Theorem 3.4 satisfy the assumptions of Lemma 3.10. Therefore, the transition semigroup \( P^n_t \) corresponding to \( X_n(t) \) enjoys the strong Feller property and irreducibility.

Thus, for any \( t > 0 \) and \( f \in \mathcal{B}_b(\mathbb{R}^m) \),

\[
|Ef(X^z(t)) - Ef(X^{x_0}(t))| = |E(f(X^z(t)); t < \tau_n^z) + E(f(X^z(t)); t \geq \tau_n^z) - E(f(X^{x_0}(t)); t < \tau_n^{x_0}) - E(f(X^{x_0}(t)); t \geq \tau_n^{x_0})|
\]

\[
\leq |E(f(X^z(t)); t < \tau_n^z) - E(f(X^{x_0}(t)); t < \tau_n^{x_0})| + |E(f(X^{x_0}(t)); t \geq \tau_n^{x_0}) - E(f(X^z(t)); t \geq \tau_n^z)|
\]

Since \( P(\tau_n^z \leq T) \to 0 \) as \( n \to \infty \) uniformly for \( x \) in a compact subset of \( \mathbb{R}^m \), we have, \( \forall \eta > 0 \), there is a sufficient large \( n \in \mathbb{N}^* \) such that

\[
P(\tau_n^z \leq T) \leq \eta
\]

for all \( z \in B_{\frac{1}{2}}(x_0) \). Note that \( P^n_t \) is strong Feller, which implies

\[
\lim_{x \to x_0} |Ef(X^z_n(t)) - Ef(X^{x_0}_n(t))| = 0.
\]

Consequently,

\[
\lim_{x \to x_0} |Ef(X^z(t)) - Ef(X^{x_0}(t))| \leq 4\eta.
\]

Since \( \eta \) is arbitrary, the strong Feller property of \( P_t \) holds.

Now, we prove that the semigroup \( P_t \) is irreducible. In fact, for any open ball \( B_\delta(z) \subset \mathbb{R}^m \), choose \( n \in \mathbb{N}^* \) sufficiently large such that \( \overline{B_\delta(z)} \subset B_n(0) \). Then for each \( t > 0 \) and \( x \in \mathbb{R}^m \), we have

\[
P(X^z(t) \in B_\delta(z)) = P(X^z(t) \in B_\delta(z), t < \tau_n^z) + P(X^z(t) \in B_\delta(z), t \geq \tau_n^z)
\]

\[
\geq P(X^z(t) \in B_\delta(z), t < \tau_n^z)
\]

\[
= P(X^z(t \land \tau_n^z) \in B_\delta(z), t < \tau_n^z)
\]

\[
= P(X^z(t \land \tau_n^z) \in B_\delta(z)) - P(X^z(t \land \tau_n^z) \in \overline{B_\delta(z)}, t \geq \tau_n^z)
\]

Since \( |X^z(\tau_n^z)| \geq n \), we get

\[
P(X^z(\tau_n^z) \in B_\delta(z)) = 0.
\]

Therefore, the irreducibility of \( P_t \) follows from the fact that \( P^n_t \) is irreducible. \( \square \)

**Corollary 3.13.** Suppose all the assumptions of Theorems 3.6 and 3.12 are satisfied. Then there exists an \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \), (3.1) has a unique stationary measure \( \mu^\epsilon \) and the family \( \{\mu^\epsilon\}_{0 < \epsilon \leq \epsilon_0} \) is tight. If \( \mu^\epsilon \rightharpoonup^\star \mu \) as \( \epsilon \to 0 \), then \( \mu \) is an invariant measure of \( X^0(t) \), which is concentrated on the Birkhoff center \( B(X^0) \).

### 3.4 Examples

**Example 3.14** (The monotone cyclic feedback systems with noise). A typical monotone cyclic feedback system is given by the \( N + 1 \) equations

\[
\dot{x}_i(t) = -b_i x_i(t) + f_i(x_{i+1}(t)), \quad 0 \leq i \leq N,
\]

(3.25)
where each $b_i$ is positive, $N \geq 0$, the indices are taken mod $N + 1$ and each $f^i$ enjoys the monotonicity property

$$\frac{df^i(s)}{ds} \neq 0 \quad \text{for all } s \in \mathbb{R}, \quad 0 \leq i \leq N. \tag{3.26}$$

After a sequence of normalizing transformations fully described in [38], we may assume that

$$\frac{df^i(s)}{ds} > 0, \quad \delta \frac{df^N(s)}{ds} > 0 \quad \text{for all } s \in \mathbb{R}, \quad 0 \leq i \leq N - 1, \tag{3.27}$$

where $\delta \in \{-1,1\}$. The monotone cyclic feedback systems (3.25) and (3.27) arise in versions of the classical Goodwin model of enzyme synthesis and in the theory of neural networks. In application, the functions $f^i$, $i = 0, 1, \ldots, N$, are often assumed to have sigmoidal shapes. Hence, we always assume that $f^i$, $i = 0, 1, \ldots, N$, are bounded and continuously differentiable with bounded derivatives. Then Mallet-Paret and Smith [39] proved the following Poincaré-Bendixson theorem.

**Theorem 3.15 (The Poincaré-Bendixson theorem).** Consider the system (3.25) with each $f^i$ satisfying the above assumptions. Let $x(t)$ be a solution of (3.25) on $[0, \infty)$. Let $\omega(x)$ denote the $\omega$-limit set of this solution in the phase space $\mathbb{R}^{N+1}$. Then either

(a) $\omega(x)$ is a single non-constant periodic orbit; or else

(b) for solutions with $u(t) \in \omega(x)$ for all $t \in \mathbb{R}$, we have

$$\alpha(u) \cup \omega(u) \subset \mathcal{E},$$

where $\alpha(u)$ and $\omega(u)$ denote the $\alpha$- and $\omega$-limit sets, respectively, of this solution, and where $\mathcal{E} \subset \mathbb{R}^{N+1}$ denotes the set of equilibrium points of (3.25).

Now we consider the system driven by a Lévy process

$$dx^i(t) = [-b_i x^i(t) + f^i(x^{i+1}(t))] dt + \epsilon \sum_{j=1}^{k} \sigma_{ij}(x(t)) dW_j(t) + \epsilon \int_{|y| < c} F^i(x(t), y) \tilde{N}(dt, dy) \tag{3.28}$$

for $0 \leq i \leq N$, where the $((N + 1) \times k)$-dispersion matrix $\sigma(x) := (\sigma_{ij}(x))$ and $F$ have global Lipschitz continuous and linear growth properties.

Define $V : \mathbb{R}^{N+1} \to \mathbb{R}_+$ by

$$V(x) := \frac{1}{2} \sum_{i=0}^{N} (x^i)^2.$$

Then

$$\mathcal{L} V(x) = -\sum_{i=0}^{N} b_i (x^i)^2 + \sum_{i=0}^{N} x^i f^i(x^{i+1}) + \frac{\epsilon^2}{2} \sum_{i=0}^{N} a_{ii}(x)$$

$$+ \int_{|y| < c} (V(x + \epsilon F(x, y)) - V(x) - \langle \nabla V(x), \epsilon F(x, y) \rangle)_{\mathbb{R}^{N+1}, \mathbb{R}^{N+1}} \nu(dy).$$

It follows from the assumptions that all $f^i$, $i = 0, 1, \ldots, N$, are bounded and the dispersion matrix $\sigma(x)$ and $F$ have linear growth that there is a positive constant $\tilde{L}$ such that

$$\mathcal{L} V(x) \leq -b \sum_{i=0}^{N} (x^i)^2 + \tilde{L} (|x| + \epsilon^2 (|x|^2 + 1)),$$

where $b = \min_{0 \leq i \leq N} b_i$. This shows that there are $\epsilon_0 > 0$ and $R > 0$ such that as $\epsilon \in (0, \epsilon_0]$ one enjoys

$$\mathcal{L} V(x) \leq -\frac{b}{2} \sum_{i=0}^{N} (x^i)^2 \quad \text{for } |x| > R.$$

By Theorem 3.6, the set of all stationary measures for (3.25) $(0 < \epsilon \leq \epsilon_0)$ is tight.

From the Poincaré-Bendixson theorem, we know that the Birkhoff center $B(\Phi) = \mathcal{E} \cup \mathcal{P}$, where $\Phi$ is the flow generated by (3.25) and $\mathcal{P}$ denotes the set of nontrivial periodic orbits.

Applying Theorem 3.1, we conclude the following theorem.
**Theorem 3.16.** Let $\mu = \lim_{\epsilon \to 0} \mu^\epsilon$ be a weak limit point of $\{\mu^\epsilon\}$. Then $\mu$ is an invariant measure of the flow $\Phi$ and the $\text{supp}(\mu)$ is contained in $E \cup P$.

**Remark 3.17.** Theorem 3.16 is still valid for those systems if the Poincaré-Bendixson theorem holds for the unperturbed systems, for example, planar systems and Morse-Smale higher dimensional systems. Remark 3.17. Theorem 3.16 is still valid for those systems if the Poincaré-Bendixson theorem holds for the unperturbed systems, for example, planar systems and Morse-Smale higher dimensional systems,

Consider the system to have polynomial growth.

**Example 3.14.** We give an example to show our result can be used for the system with drift terms and diffusion terms on $(O, \sigma^O)$.

**Example 3.15.** We give an example to show our result can be used for the system with drift terms and diffusion terms on $(O, \sigma^O)$.

Let $\mathcal{I}_{x_0}$ (resp. $\mathcal{M}_{x_0}^\epsilon$) the set of invariant measures (resp. stationary measures) generated by the family of probability measures

$$P^{(0,t)}(x_0,B) = \frac{1}{T} \int_0^T \delta_{X^0,x_0(t)}(B)ds$$

(resp. $P^{(\epsilon,t)}(x_0,B) = \frac{1}{T} \int_0^T P(s,x_0,B)ds$) via the Krylov-Bogoliubov procedure. Then we have the following proposition.

**Proposition 3.19.** Suppose that $b$ is globally Lipschitz continuous. Then $X^{\epsilon,x_0}(t) = X^{0,x_0}(t)$ for all $t \geq 0$ and $\mathcal{M}_{x_0}^\epsilon = \mathcal{I}_{x_0}$ for all $\epsilon$. In particular, for any $\mu \in \mathcal{I}_{x_0}$, $\mu_{x_0}^\epsilon \equiv \mu_{x_0}^\epsilon$ $\mu$ as $\epsilon \to 0$.

This proposition illustrates that under a mild regular condition on drift terms, for any invariant measure $\mu$ of $\dot{x} = b(x)$, there exists a diffusion term $\sigma$ with small noise intensity $\epsilon$ such that there is a sequence of stationary measures for (3.30) converging to $\mu$ weakly as $\epsilon \to 0$.

Example 3.18 is completed.

We have observed from examples that the limiting measures of stationary measures will be concentrated on stable orbits of the deterministic system decided by drift terms. However, the following two examples show that the limit measure can be concentrated on saddles for the deterministic system. In summary, limiting measures are always concentrated on “most relatively stable positions”.

$$\begin{align*}
\begin{cases}
\frac{dx}{dt} &= [x - y - x(x^2 + y^2)]dt + \epsilon(x^2 + y^2)dW_1^t, \\
\frac{dy}{dt} &= [x + y - y(x^2 + y^2)]dt + \epsilon(x^2 + y^2)dW_2^t.
\end{cases}
\end{align*}$$ (3.29)

Let $V(x,y) = x^2 + y^2$. Then for $0 < \epsilon < \frac{1}{\sqrt{2}}$, for $x^2 + y^2$ sufficiently large. This shows that all conditions of Theorem 3.1 hold. It is easy to see that the Birkhoff center for the corresponding deterministic system in (3.29) with $\epsilon = 0$ is $\{O, \mathbf{S}^1\}$ where $\mathbf{S}^1$ denotes the unit cycle. Employing Theorem 3.1, we have that $\text{supp}(\mu) \subset \{O, \mathbf{S}^1\}$ for any stationary measures $\{\mu^\epsilon\}$ of (3.29) such that $\mu = \lim_{\epsilon \to 0} \mu^\epsilon$ in the sense of a weak limit.

In particular, $O(0,0)$ is a solution of (3.29), which implies that $\mu^\epsilon = \delta_O$ is a stationary measure of (3.29) and concentrates at the origin. If we replace $x^2 + y^2$ in the diffusion terms by $x^2 + y^2 - 1$, then $\mathbf{S}^1$ is invariant for (3.29) in this case. Therefore, the Haar measure on $\mathbf{S}^1$ is a stationary measure for any $\epsilon$.

Which invariant measure for the deterministic system $\dot{x} = b(x)$ can be the limiting measure for a sequence of stationary measures for (3.1)? Such a problem strongly depends on the type of noise, which is shown as follows.

Suppose that $X^{0,x_0}(t)$ is a bounded solution of $\dot{x} = b(x)$. Let $r > 0$ such that $X^{0,x_0}(t) \in B_r(O)$ for all $t \geq 0$. We can construct a $C^\infty$ diffusion term $\sigma$ satisfying $\sigma = 0$ on $B_r(O)$ and $\sigma = \text{constant matrix} \mathbf{M}$ on $(B_{r+1}(O))^c$. Consider SODEs

$$dX^{\epsilon,x}(t) = b(X^{\epsilon,x}(t))dt + \epsilon \sigma(X^{\epsilon,x}(t))dW_t.$$

(3.30)
**Example 3.20** (The lemniscate of Bernoulli system with noise). Let \( I(x, y) = (x^2 + y^2)^2 - 4(x^2 - y^2) \). Define
\[
V(I) := \frac{I^2}{2(1 + I^2)^{\frac{3}{2}}} \quad \text{and} \quad H(I) := \frac{I}{(1 + I^2)^{\frac{1}{2}}}.
\]
Consider the vector field
\[
b(x, y) := -\left( \nabla V(I) + \left( \frac{\partial H(I)}{\partial y}, -\frac{\partial H(I)}{\partial x} \right) \right) =: -[\nabla V(I) + \Theta(x, y)],
\]
where
\[
\nabla V(I) = \frac{dV(I)}{dI} \left( \frac{\partial I}{\partial x}, \frac{\partial I}{\partial y} \right)^T, \quad \Theta(x, y) = \frac{dH(I)}{dI} \left( \frac{\partial I}{\partial y}, -\frac{\partial I}{\partial x} \right)^T.
\]
By a calculation,
\[
\frac{\partial I}{\partial x} = 4x(x^2 + y^2) - 8x, \quad \frac{\partial I}{\partial y} = 4y(x^2 + y^2) + 8y.
\]
Consider the unperturbed system of ordinary differential equations
\[
\begin{align*}
\frac{dx}{dt} &= -f(I)(4x(x^2 + y^2) - 8x) - g(I)(4y(x^2 + y^2) + 8y), \\
\frac{dy}{dt} &= -f(I)(4y(x^2 + y^2) + 8y) - g(I)(-4x(x^2 + y^2) + 8x).
\end{align*}
\]
Here,
\[
f(I) = \frac{dV(I)}{dI} = \frac{I(I^2 + 4)}{4(1 + I^2)^{\frac{3}{2}}} \quad \text{and} \quad g(I) = \frac{dH(I)}{dI} = \frac{I^2 + 4}{4(1 + I^2)^{\frac{1}{2}}}.
\]
We will summarize the global behavior of (3.32) in the following proposition.

**Proposition 3.21.** The system (3.32) has a global Lipschitz constant and the equilibria \((0, 0), P^+(\sqrt{2}, 0)\) and \(P^-(-\sqrt{2}, 0)\). \(V(I)\) is its Lyapunov function. When the initial point \(p\) locates outside of the Bernoulli lemniscate
\[
L : (x^2 + y^2)^2 = 4(x^2 - y^2),
\]
its \(\omega\)-limit set \(\omega(p) = L\), which is a red curve in Figure 1; when the initial point \(p \neq P^-\) (resp. \(p \neq P^+\)) locates left (resp. right) inside the Bernoulli lemniscate, its \(\omega\)-limit set the left (resp. right) branch of \(L\). However, the Birkhoff center \(B(\Phi)\) for this solution flow \(\Phi\) is \(\{O, P^+, P^-\}\).

![Figure 1](Color online) The phase portrait of (3.32) with \(b(x, y) = -\nabla V(x, y) - \Theta(x, y)\)
Proof. It is easy to see that
\[
\lim_{|x,y| \to \infty} V(I(x, y)) = \infty. \tag{3.34}
\]
Since \(\nabla V(I)\) and \(\Theta(x, y)\) are orthogonal, the derivative of the function \(V(I(x, y))\) along a solution is
\[
\dot{V} = -|\nabla V(I)|^2. \tag{3.35}
\]
(3.34) and (3.35) imply that all positive trajectories for (3.32) are bounded. LaSalle’s invariance principle deduces that for any \(p \in \mathbb{R}^2\),
\[
\omega(p) \subset \{(x, y) : \nabla V(I) = 0\} = L \cup \{P^+, P^-\}.
\]
In particular, the equilibria for (3.32) are contained in \(L \cup \{P^+, P^-\}\). It is easy to calculate that there uniquely exists an equilibrium on \(L\), which is the origin \(O\), and that the other equilibria are \(P^+\) and \(P^-\).

It is not hard to get that
\[
Db(0, 0) = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix}, \quad Db(\sqrt{2}, 0) = Db(-\sqrt{2}, 0) = 16 \begin{pmatrix} \frac{20}{17} & -\frac{5}{17} \\ \frac{5}{17} & \frac{20}{17} \end{pmatrix}.
\]
This implies that \((0, 0)\) is a saddle point and \((\pm \sqrt{2}, 0)\) are unstable foci. Combining LaSalle’s invariance principle and the Poincaré-Bendixson theorem, we can obtain the \(\omega\)-limit set of each trajectory for (3.32), as shown in Figure 1.

By estimation, we can obtain the following inequalities:
\[
|\frac{\partial b_i(x, y)}{\partial x}|, \quad |\frac{\partial b_i(x, y)}{\partial y}| \leq 130 \sqrt{8}, \quad r \geq 4, \quad i = 1, 2,
\]
where \(r = \sqrt{x^2 + y^2}\). Therefore, \(b(x, y)\) is a globally Lipschitz function. This completes the proof. \(\square\)

Now we consider the perturbed system of (3.32) driven by Brownian motion:
\[
\begin{aligned}
&dx = b_1(x, y)dt + \epsilon[\sigma_{11}(x, y)dW^1_t + \sigma_{12}(x, y)dW^2_t], \\
&dy = b_2(x, y)dt + \epsilon[\sigma_{21}(x, y)dW^1_t + \sigma_{22}(x, y)dW^2_t],
\end{aligned} \tag{3.36}
\]
where \(\sigma_{ij} (i, j \in \{1, 2\})\) satisfies the global Lipschitz condition, which implies that there exist non-negative constants \(C_1\) and \(C_2\) such that
\[
|a_{ij}(x, y)| \leq C_1|x(y)|^2 + C_2 \leq C_1r^2 + C_2 \quad \text{for} \quad i, j = 1, 2,
\]
where
\[
a_{ij}(x, y) = \sum_{k=1}^2 \sigma_{ik}(x, y)\sigma_{jk}(x, y).
\]

**Theorem 3.22.** Suppose that \(\sigma_{ij} (i, j \in \{1, 2\})\) satisfy the following global Lipschitz conditions:
(i) if \(C_1 = 0\), then for any \(\epsilon\), the system (3.36) admits at least one stationary measure \(\mu^\epsilon\);
(ii) if \(C_1 > 0\), then the system (3.36) possesses at least one stationary measure \(\mu^\epsilon\) for \(0 < \epsilon < \frac{1}{8\sqrt{2}C_1} \).

If, in addition, the diffusion matrix \(a(x, y)\) is positively definite everywhere, then for a given \(\epsilon\) as above, the stationary measure \(\mu^\epsilon\) is unique, and \(\mu^\epsilon \Rightarrow \delta_0(\cdot)\) as \(\epsilon \to 0\), where \(\delta_0(\cdot)\) denotes the Dirac measure at the saddle \(O\).

**Proof.** In fact, from the above inequalities one can see that for \(r > 4\),
\[
\left|\frac{\partial^2 V}{\partial x^2}\right|, \quad \left|\frac{\partial^2 V}{\partial y^2}\right|, \quad \left|\frac{\partial^2 V}{\partial y \partial x}\right| \leq 104\sqrt{2}, \quad |\nabla V(x, y)|^2 \geq \frac{r^2}{4\sqrt{2}}.
\]
and
\[
\mathcal{L}V(x, y) = - \left[\frac{1}{4\sqrt{2}} - 208\sqrt{2}C_1\epsilon^2\right]r^2 + 208\sqrt{2}C_2\epsilon^2 \to -\infty \quad \text{as} \quad r \to \infty.
\]
Applying the Khasminskii Theorem 3.6, we conclude that there is at least one stationary measure for (3.36) if $0 < \epsilon < \frac{1}{\sqrt{2}}$ with $C_1 > 0$ or all $\sigma_i(x, y)$ are bounded on the plane, and that this stationary measure is unique if $a(x, y)$ is positively definite everywhere.

In (i) and (ii), from the tightness criterion it follows that the set of stationary measures $\{\mu^\epsilon : \epsilon \in (0, \epsilon_0]\}$ is tight. Thus Prokhorov’s theorem implies that any sequence $\{\mu^\epsilon_i\}$ of stationary measures with $\epsilon_i \to 0$ contains a weakly convergent subsequence. Let $\mu$ be any weak limit measure. Then Theorem 2.1 deduces that $\text{supp}(\mu) \subset B(\Phi)$. However, in view of Proposition 3.21, $B(\Phi) = \{O, P^+, P^-, \}$, which implies that $\text{supp}(\mu) \subset \{O, P^+, P^-, \}$.

Finally, we show that $\mu(\{P^+, P^-, \}) = 0$. Since matrix $Db(P^+)$ has all eigenvalues with positive real parts. Thus there exists a positive definite matrix $B$ satisfying

$$
(Db(P^+))^T B + B(Db(P^+)) = I.
$$

Let $\hat{V}^+(z) = (z - P^+)^T B(z - P^+)$, where $z = (x, y)$. It is easy to see that there exists a neighborhood $\mathcal{U}$ of $P^+$ such that $\langle \nabla \hat{V}^+, b \rangle > 0$ on $\mathcal{U} \setminus \{P^+\}$ (see, e.g., [26]). We denote

$$
\rho_M := \sup_{(x, y) \in \mathcal{U}} \hat{V}^+(x, y)
$$

(called the essential upper bound of $\hat{V}^+$). Then

$$
\mathcal{L}' \hat{V}^+(x, y) = (b(x, y), \nabla \hat{V}^+(x, y))
$$

$$
+ \frac{\epsilon^2}{2} \left[a_{11}(x, y) \frac{\partial^2 V}{\partial x^2} + 2a_{12}(x, y) \frac{\partial^2 V}{\partial y \partial x} + a_{22}(x, y) \frac{\partial^2 V}{\partial y^2}\right]
$$

$$
\geq \frac{\epsilon^2}{2} \left[a_{11}(x, y) b_{11} + 2a_{12}(x, y) b_{12} + a_{22}(x, y) b_{22}\right]
$$

$$
\geq \tilde{m} \epsilon^2 =: \gamma > 0, \quad \forall (x, y) \in \mathcal{U}.
$$

(3.37)

We used here the fact that $B$ is positively definite and $A(x, y)$ is positively definite on $B_{\delta}(P^+)$. It follows from (3.37) that $\hat{V}^+$ is an anti-Lyapunov function with respect to (3.36) in $B_{\delta}(P^+)$ with the anti-Lyapunov constant $\tilde{m}\epsilon^2$ and essential lower bound $\rho_m = 0$ (see, e.g., [28, Definition 2.2]). It is obvious that

$$
\nabla \hat{V}^+(x, y) = (2b_{11}(x - \sqrt{2}) + 2b_{12}y, 2b_{12}(x - \sqrt{2}) + 2b_{22}y)
$$

$$
\neq 0, \quad \forall (x, y) \in (\hat{V}^+)^{-1}(\rho) \text{ for a.e. } \rho \in [0, \rho_M],
$$

(3.38)

where $(\hat{V}^+)^{-1}(\rho) = \{(x, y) \in \mathcal{U} : \hat{V}^+(x, y) = \rho\}$. Note that

$$
\frac{\epsilon^2}{2} \left[a_{11}(x, y) \left(\frac{\partial \hat{V}^+}{\partial x}\right)^2 + 2a_{12}(x, y) \frac{\partial \hat{V}^+}{\partial x} \frac{\partial \hat{V}^+}{\partial y} + a_{22}(x, y) \left(\frac{\partial \hat{V}^+}{\partial y}\right)^2\right]
$$

$$
\leq \frac{\epsilon^2}{2} ||A(x, y)|| \left(\frac{\partial \hat{V}^+}{\partial x}(x, y), \frac{\partial \hat{V}^+}{\partial y}(x, y)\right)^2
$$

$$
\leq \frac{\epsilon^2}{2} \sup_{(x, y) \in B_{\delta}(P^+)} ||A(x, y)|| \hat{M}_0 ||V^+(x, y)|| \epsilon^2
$$

$$
=: \tilde{M} \epsilon^2 =: H(\rho), \quad (x, y) \in (\hat{V}^+)^{-1}(\rho) \text{ for } \rho \in [0, \rho_M].
$$

(3.39)

Without loss of generality, we may assume $\mu^\epsilon(\mathcal{U}) > 0$ for each $\epsilon > 0$. It is easy to verify that $\hat{\mu}^\epsilon(\cdot) = \frac{\mu^\epsilon(\cdot || A_\epsilon(\cdot))}{\mu^\epsilon(\mathcal{U})}$ is a stationary measure in $\mathcal{U}$. By a regularity result on stationary measures in [6], we have known that $\hat{\mu}^\epsilon$ admits a positive density function $\hat{u} \in W^{1, p}_{loc}(\mathcal{U})$. Let

$$
\Omega_{\rho} = \{(x, y) \in \mathcal{U} : \hat{V}^+(x, y) < \rho\}, \quad \Omega_{\rho}^* = \Omega_{\rho} \cup (\hat{V}^+)^{-1}(\rho)
$$
for each \( \rho \in [0, \rho_M] \). The regularity implies that \( \mu^\epsilon(\Omega^*_\rho_{\rho_0}) = \mu^\epsilon(\{P^+\}) = 0 \). The measure estimate theorem in [30, Theorem B(a)] asserts that for any \( \rho_0 \in (0, \rho_M) \),

\[
\mu^\epsilon(\Omega^*_\rho_{\rho_{\rho_0}}) = \mu^\epsilon(\Omega_{\rho}) e^{\frac{\epsilon}{\rho} \int_{\rho_0}^\rho \frac{M}{2} d\rho} \leq e^{\frac{\epsilon}{\rho} \int_{\rho_0}^\rho \frac{M}{2} d\rho}, \quad \rho \in (\rho_0, \rho_M),
\]

i.e.,

\[
\mu^\epsilon(\Omega_{\rho_{\rho_0}}) \leq \mu^\epsilon(\Omega_{\rho}) e^{-\frac{\epsilon}{\rho} \int_{\rho_0}^\rho \frac{M}{2} d\rho} \leq e^{-\frac{\epsilon}{\rho} \int_{\rho_0}^\rho \frac{M}{2} d\rho}, \quad \rho \in (\rho_0, \rho_M).
\]

Since \( \mu^\epsilon \xrightarrow{\rho \to 0} \mu \) as \( \epsilon \to 0 \), and \( \Omega_{\rho_{\rho_0}} \) is an open set, we have \( \mu(\Omega_{\rho_{\rho_0}}) \leq e^{-\frac{\epsilon}{\rho} \int_{\rho_0}^\rho \frac{M}{2} d\rho} \). Finally, letting \( \rho_0 \to 0 \), we obtain \( \mu(\{P^+\}) = 0 \). Analogously, we can verify that \( \mu(\{P^-\}) = 0 \). We conclude that \( \mu = \delta_O \), i.e., \( \mu^\epsilon \xrightarrow{\rho \to 0} \delta_O \) as \( \epsilon \to 0 \).

**Remark 3.23.** From the above arguments, we have obtained that if the system (3.32) is driven by Brownian motion and the diffusion matrix is positively definite everywhere, then any limiting measure is \( \delta_O \). However, if we get rid of the non-degenerate condition for the diffusion matrix, then it is possible for limiting measures to be either \( \delta_{P^+} \) or \( \delta_{P^-} \) from Proposition 3.19. The problem is whether or not such result still holds if it is driven by Lévy processes. We can only get that the limiting measure is supported in \( \{O, P^+, P^-\} \).

Example 3.20 is completed.

**Example 3.24** (The May-Leonard system with a noise perturbation). Consider the May-Leonard system with a white noise perturbation

\[
\begin{align*}
dy_1 &= y_1(1 - y_1 - \beta y_2 - \gamma y_3) dt + \epsilon y_1 \circ dW_t, \\
dy_2 &= y_2(1 - y_2 - \beta y_1 - \gamma y_3) dt + \epsilon y_2 \circ dW_t, \\
dy_3 &= y_3(1 - y_3 - \beta y_1 - \gamma y_2) dt + \epsilon y_3 \circ dW_t,
\end{align*}
\]

where \( \circ \) denotes the Stratonovich stochastic integral, \( \beta, \gamma > 0 \) and \( \epsilon \) denotes noise intensity.

Recalling [8], we have the following stochastic decomposition formula:

\[
\Phi^\epsilon(t, \omega, y) = g^\epsilon(t, \omega, y_0) \Phi^0 \left( \int_0^t g^\epsilon(s, \omega, y_0) ds, \frac{y}{y_0} \right),
\]

where \( \Phi^\epsilon \) and \( \Phi^0 \) are the solutions of (3.40) and the corresponding deterministic system without noise (i.e., \( \epsilon = 0 \)) respectively, and \( g^\epsilon \) is the solution of the stochastic logistic equation

\[
dg = g(1 - g) dt + \epsilon g \circ dW_t.
\]

In order to obtain the stationary properties for (3.40) in detail, we need the asymptotic properties for the deterministic flow \( \Phi^0 \). It is well known from Hirsch [24] that the flow \( \Phi^0 \) admits an invariant surface \( \Sigma \) (called carrying simplex), homeomorphic to the closed unit simplex \( \Delta^2 = \{ y \in \mathbb{R}^3_+ : y_1 + y_2 + y_3 = 1 \} \) by radial projection, such that every trajectory in \( \mathbb{R}^3_+ \setminus \{O\} \) is asymptotic to one in \( \Sigma \). So we will draw phase portraits on \( \Sigma \) (see Table 1).

It is easy to see that \( \Phi^0 \) always possesses equilibria: the origin \( O(0,0,0) \), three axial equilibria \( R_1(1,0,0), R_2(0,1,0), R_3(0,0,1) \) and the unique positive equilibrium \( P = \left( \frac{1}{1 + \beta + \gamma}, \frac{1}{1 + \beta + \gamma}, \frac{1}{1 + \beta + \gamma} \right) \). \( \Phi^0 \) has planar equilibria

\[
R_{12} = \frac{1}{1 - \beta \gamma}(1 - \beta, 1 - \gamma, 0), \quad R_{23} = \frac{1}{1 - \beta \gamma}(0,1 - \beta, 1 - \gamma), \quad R_{31} = \frac{1}{1 - \beta \gamma}(1 - \gamma, 0, 1 - \beta)
\]

if and only if \( (1 - \beta)(1 - \beta \gamma) > 0, (1 - \gamma)(1 - \beta \gamma) > 0 \). The classification for the flow \( \Phi^0 \) on the carrying simplex \( \Sigma \) is summarized in Table 1.
Table 1  The classification for the flow $\Phi^0$ on $\Sigma$

| Parameter conditions | Equilibria | Phase portraits |
|----------------------|------------|-----------------|
| a: $0 < \beta, \gamma < 1$ | $O, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, P$ | ![Diagram] |
| b: (i) $\beta + \gamma < 2$ | $O, R_1, R_2, R_3, P$ | ![Diagram] |
| or $\gamma \geq 1, \beta < 1$ | | |
| (ii) $\beta \geq 1, \gamma < 1$ | $O, R_1, R_2, R_3, P$ | ![Diagram] |
| or $\gamma \geq 1, \beta < 1$ | | |
| c: (i) $\beta + \gamma = 2$ | $O, R_1, R_2, R_3, P$ | ![Diagram] |
| (ii) $\beta, \gamma \neq 1$ | | |
| d: (i) $\beta + \gamma > 2$ | $O, R_1, R_2, R_3, P$ | ![Diagram] |
| (ii) $\gamma > 1, \beta \leq 1$ | | |
| or $\gamma < 1, \beta > 1$ | | |
| e: $\beta, \gamma > 1$ | $O, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, P$ | ![Diagram] |
| f: $\beta = \gamma = 1$ | $\forall x \in \Sigma \cup \{O\}$ | ![Diagram] |

Set by $\omega(z)$ the $\omega$-limit set of the trajectory $\Phi^0(t, z)$. Then it follows from Table 1 that any $\omega(z)$ is either an equilibrium, or a closed orbit, or a heteroclinic cycle. We define

$$A(\omega(z)) := \left\{ y \in \mathbb{R}^3_+ : \lim_{t \to \infty} \text{dist}(\Phi^0(t, y), \omega(z)) = 0 \right\}$$

to be the attracting domain of $\omega(z)$. Let $A_\Sigma(\omega(z))$ denote the attracting domain of $\omega(z)$ on $\Sigma$, which can be derived from Table 1 for each case. It follows from [9, Proposition 4.13] that any pair of points on $L(y) := \{ \lambda y : \lambda > 0 \}$ has the same $\omega$-limit set. We can obtain the attracting domain of $\omega(z)$ as follows:

$$A(\omega(z)) = \bigcup \{ L(y) : y \in A_\Sigma(\omega(z)) \}. \tag{3.43}$$

The relation (3.43) together with Table 1 gives the attracting domains of an equilibrium, a closed orbit, or a heteroclinic cycle, respectively.

Using the stochastic decomposition formula (3.41), we have shown that the convergence in the probability hypothesis (2.1) holds, i.e., for every $t > 0$ and compact set $K \subset \mathbb{R}^3_+$, $\Phi^\epsilon(t, x)$ converges in probability to $\Phi^0(t, x)$ uniformly in $x \in K$ as $\epsilon \to 0$ (see [8, Proposition 6.1]). Let $\mathcal{I}^\epsilon$ denote the set of all stationary measures of (3.40) and $\mathcal{I} = \bigcup_{0 < \epsilon \leq \epsilon_0} \mathcal{I}^\epsilon$. Then $\mathcal{I}$ is tight (see [8, Theorem 6.1]). Moreover, any weak limit of a sequence $\{ \mu^\epsilon \} \subset \mathcal{I}$ is an invariant measure of $\Phi^0$ (see [8, Proposition 6.2]).

It is not difficult to see that the stochastic logistic equation (3.42) has a unique nontrivial stationary solution $u^\epsilon(\theta_t \omega)$, where $\{ \theta_t \}_{t \in \mathbb{R}}$ is the metric dynamical system generated by Brownian motion. It follows
from the stochastic decomposition formula (3.41) that \( u^\epsilon(\theta_t \omega) Q \) is a stationary solution of (3.40) for any equilibrium \( Q \in \mathcal{E} \), whose law, denoted by \( \mu_Q^\epsilon \), defines an ergodic stationary measure of (3.40) (see [8, Theorem 5.3]). In addition, for each \( y \in \mathcal{A}(Q) \), \( P^\epsilon(t,y,\cdot) \overset{w}{\to} \mu_Q^\epsilon \) as \( \epsilon \to 0 \), and
\[
\mu_Q^\epsilon \overset{w}{\to} \delta_Q \quad \text{as} \quad \epsilon \to 0.
\]

These results are available for the cases a, b, e and f in Table 1 and as well as any equilibrium version cases c and d in Table 1.

In the case c in Table 1, the carrying simplex is full of periodic orbits \( \Gamma(h) \) (0 < \( h \leq \frac{1}{27} \)):
\[
\begin{align*}
&y_1 + y_2 + y_3 = 1, \\
y_1 y_2 y_3 = h,
\end{align*}
\]
whose attracting domain is the invariant cone surface \( \Lambda(h) \):
\[
\frac{y_1 y_2 y_3}{(y_1 + y_2 + y_3)^3} = h, \quad 0 < h \leq \frac{1}{27}.
\]
Then there exists a unique ergodic nontrivial stationary measure \( \nu_h^\epsilon \) supported on the cone surface \( \Lambda(h) \) (see [8, Theorem 7.8]).

We can obtain a uniform characteristic: the expected occupation measure converges weakly to an ergodic stationary measure for (3.40) and all stationary measures are obtained for each case (see [8, Theorems 7.6–7.9]) except the case d. Here, the expected occupation measure is defined by
\[
\mu^\epsilon_{c,T}(y, A) := \frac{1}{T} \int_0^T P^\epsilon(s,y,A) ds, \quad y \in \mathbb{R}_+^3, \quad A \in \mathcal{B}(\mathbb{R}_+^3).
\]

But the case d is quite different. If \( y \in \text{Int} \mathbb{R}_+^3 \setminus \overline{\mathcal{L}(P)} \), the corresponding expected occupation measure has infinite weak limit points, which are not ergodic (see [8, Appendix B]).

Applying Theorem 2.1, we conclude the following theorem.

**Theorem 3.25.** Consider the system (3.40) with \( \beta, \gamma > 0 \). Then
(i) for any equilibrium \( Q \in \mathcal{E} \), \( \mu_Q^\epsilon \overset{w}{\to} \delta_Q \) as \( \epsilon \to 0 \), which is valid to the cases a, b, e and f;
(ii) for the case c, \( \nu_h^\epsilon \) converges weakly to the Haar measure on the closed orbit \( \Gamma(h) \) as \( \epsilon \to 0 \), 0 < \( h \leq \frac{1}{27} \);
(iii) for the case d, if \( \mu^i := \nu_{y^i}^\epsilon \in \mathcal{M}_y^\epsilon, y \neq 0, i = 1,2, \ldots, \) satisfying \( \epsilon^i \to 0 \) and \( \mu^i \overset{w}{\to} \mu \) as \( i \to \infty \), then \( \mu(\{R_1,R_2,R_3\}) = 1 \). Here, \( \mathcal{M}_y^\epsilon \) denotes the set of all weak limits for \( P^\epsilon_{c,T}(y,\cdot) \) as \( T \to \infty \).

**Remark 3.26.** The Birkhoff center in Example 3.20 consists of the origin \( O \) (saddle) and strongly unstable foci \( \{P^+, P^-\} \). Relatively, the origin \( O \) is more stable than \( \{P^+, P^-\} \). So all limiting measures are concentrated on the origin in the case that the diffusion matrix is non-degenerate. The Birkhoff center for the flow \( \Phi^0 \) on \( \Sigma \) is composed of \( P \) (which is strongly repelling on \( \Sigma \)) and three saddles \( \{R_1,R_2,R_3\} \), which are relatively more stable than \( P \) and supported by all limiting measures determined by those solutions with initial points on \( \Sigma \). Nevertheless, if the drift system is Morse-Smale and the diffusion matrix is positively definite everywhere, then we conjecture that all limiting measures will be concentrated on either a stable equilibrium or a stable periodic orbit.

Example 3.24 is completed.

### 4 Applications to SPDEs

Although our main result can be applied to many SPDEs, we prefer to apply it to stochastic reaction-diffusion equations, stochastic 2D Navier-Stokes equations and stochastic Burgers type equations driven by Brownian motion or Lévy processes.
4.1 The stochastic reaction diffusion equation with a polynomial nonlinearity

Let $\Lambda \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Lambda$ and let $g$ be a polynomial of odd degree with the negative leading coefficient

$$g(u) = \sum_{i=0}^{2k-1} a_i u^i, \quad a_{2k-1} < 0. \quad (4.1)$$

Consider the following initial boundary value problem:

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + g(u), & x \in \Lambda, \quad t > 0, \\
u(x, t) = 0, & x \in \partial \Lambda, \quad t \geq 0, \\
u(x, 0) = \varphi \in L^2(\Lambda).
\end{cases} \quad (4.2)$$

Then by Temam [47, p. 84, Theorem III.1.1], Equation (4.2) has a unique solution $u(x, t; \varphi) \in L^2(\Lambda)$. Therefore we can define a semigroup $T(t)$ on $L^2(\Lambda)$: $T(t) : \varphi \in L^2(\Lambda) \mapsto u(t; \varphi) \in H_0^1(\Lambda)$. Thanks to [47, p. 88, Theorem III.1.2], (4.2) has a global attractor $A_{1,2}(\Lambda)$ which is bounded in $H_0^1(\Lambda)$, compact and connected in $L^2(\Lambda)$. We have that for any $\varphi \in H_0^1(\Lambda)$, $\{u(x, t; \varphi)\}_{t \geq 0}$ is bounded in $H_0^1(\Lambda)$. Furthermore, since $-\Delta$ has compact resolvent, trajectory $\{u(x, t; \varphi)\}_{t \geq 0}$ has a compact closure. Let

$$V(\varphi) = \int_\Lambda \left( \frac{1}{2} |\nabla \varphi|^2 - G(\varphi) \right) dx, \quad G(u) = \int_0^u g(\xi)d\xi, \quad \varphi \in H_0^1(\Lambda).$$

Then

$$\frac{d}{dt} V(u(x, t; \varphi)) = \frac{d}{dt} \int_\Lambda \left( \frac{1}{2} |\nabla u|^2 - G(u) \right) dx$$

$$= \int_\Lambda \left( \frac{1}{2} \nabla u \cdot \nabla \frac{\partial u}{\partial t} - g(u) \frac{\partial u}{\partial t} \right) dx$$

$$= - \int_\Lambda \left( (\Delta u + g(u)) \frac{\partial u}{\partial t} \right) dx$$

$$= - \int_\Lambda \left( \frac{\partial u}{\partial t} \right)^2 dx \leq 0,$$

where we have used the Green formula and the boundary condition in the third equality. Finally, by LaSalle’s invariance principle, the $\omega$-limit set $\omega(\varphi)$ is contained in the equilibrium set of $T(t)$ for any $\varphi \in H_0^1(\Lambda)$, i.e., $u(x, t; \varphi) = \varphi$ satisfies the following boundary value problem:

$$\begin{cases}
\Delta \varphi + g(\varphi) = 0, & x \in \Lambda, \\
\varphi(x) = 0, & x \in \partial \Lambda.
\end{cases}$$

This implies that all solutions for (4.2) are convergent to equilibrium points.

**Proposition 4.1.** The Birkhoff center for (4.2) is the equilibrium set $E$.

Now let us consider the noise perturbation system of the reaction diffusion equation, such as (4.2). Consider the following stochastic reaction diffusion equation in $\Lambda$ with Dirichlet boundary conditions:

$$\begin{cases}
dX(t, x) = \nu \Delta X(t, x)dt + g(x, X(t, x))dt + \epsilon \sigma(x, X(t, x))dW(t), \\
X(t, x) = 0, & x \in \partial \Lambda, \quad t > 0, \\
X(0) = h \in L^2(\Lambda),
\end{cases} \quad (4.3)$$

Here, $\nu > 0$, $g : \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \Lambda \times \mathbb{R} \to L^2$ are two measurable functions. $W(t) = \{W_k(t)\}_{k \in \mathbb{N}}$ is a sequence of independent one-dimensional standard Brownian motion on some filtered probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$. 
For $p \geq 1$, let $L^p(\Lambda)$ be the usual $L^p$-space over $\Lambda$ with the standard norm $\| \cdot \|_p$. For $m \in \mathbb{N}$, let $\mathbb{H}_0^m(\Lambda)$ be the usual $m$-order Sobolev space over $\Lambda$ with Dirichlet boundary conditions, and its norm is denoted by $\| \cdot \|_{2, m}$. Denote $\mathbb{H}^{-m}(\Lambda)$ to be the dual space of $\mathbb{H}_0^m(\Lambda)$. Notice that the following Poincaré inequality holds: for some $\lambda_\Lambda > 0$,

$$\lambda_\Lambda \| u \|^2_2 \leq \| \nabla u \|^2_2, \quad u \in \mathbb{H}_0^1(\Lambda).$$

Let $V := \mathbb{H}_0^1(\Lambda)$ and denote $\| \nabla u \|_2$ by $\| u \|_V$.

Now we identify the Hilbert space $H := L^2(\Lambda)$ with itself by the Riesz representation, and set for $q \geq 2$,

$$V_q := \mathbb{H}_0^1(\Lambda) \cap L^q(\Lambda), \quad V_q^* = \mathbb{H}^{-1}(\Lambda) + L^{q'}(\Lambda),$$

where $q' := q/(q - 1)$. For any $u \in V_q$ and $w = w_1 + w_2 \in \mathbb{H}^{-1}(\Lambda) + L^{q'}(\Lambda)$, we have

$$\langle u, w \rangle_{V_q, V_q^*} = \langle u, w_1 \rangle_{\mathbb{H}^{-1}, \mathbb{H}^{-1}} + \langle u, w_2 \rangle_{L^{q'}(\Lambda), L^{q'}(\Lambda)}.$$

In what follows, we consider the evolution triple

$$V_q \subset H \subset V_q^*.$$ 

Assume that

(C1) there exist $q \geq 2$, $c_i > 0$, $i = 1, 2, 3, 4$ and $h_1 \in L^1(\Lambda)$, $h_2 \in L^{q'}(\Lambda)$ such that for all $u, u' \in \mathbb{R}$ and $x \in \Lambda$,

$$(u - u')(g(x, u) - g(x, u')) \leq c_1 |u - u'|^2,$$

$$ug(x, u) \leq -c_2 |u|^q + c_3 |u|^2 + h_1(x),$$

$$|g(x, u)| \leq c_4 |u|^{q-1} + h_2(x),$$

and the mapping $u \mapsto g(x, u)$ is continuous;

(C2) there exist $c_5, c_6 > 0$ and $h_3 \in L^1(\Lambda)$ such that for all $u, u' \in \mathbb{R}$ and $x \in \Lambda$,

$$\| \sigma(x, u) - \sigma(x, u') \|^2_{L^2} \leq c_5 |u - u'|^2$$

and

$$\| \sigma(x, u) \|^2_{L^2} \leq c_6 |u|^2 + h_3(x),$$

where $l^2$ is the Hilbert space of all square summable sequences of real numbers. By [35, Theorem 3.2] or [51, Theorem 3.6], under (C1) and (C2), for any $p \geq 1$ and $h \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\Lambda))$, there exists a unique $L^2(\mathbb{R}^+)$-valued $\mathcal{F}_t$-adapted process $X^{\epsilon, h}$ with

$$X^{\epsilon, h} \in C_{\text{loc}}([0, \infty), L^2(\Lambda)) \cap L^2_{\text{loc}}([0, \infty), V) \cap L^q_{\text{loc}}([0, \infty), L^q(\Lambda)), \quad \mathbb{P}\text{-a.s.},$$

and (4.3) holds in $V_q^*$. Moreover, we can obtain the following lemma.

**Lemma 4.2.** Assume (C1)–(C2) hold, and $q > 2$ or $q = 2, \nu = \frac{q - 1}{q} > 0$. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$E\left( \sup_{t \in [0, T]} \| X^{\epsilon, h}(t) \|_2^2 \right) + E\left( \int_0^T \| X^{\epsilon, h}(t) \|_{V}^2 dt \right) \leq C(E(\| h \|_2^2) + T). \quad (4.4)$$

**Remark 4.3.** By Fubini’s theorem and (4.4), we have

$$\int_0^T E(\| X^{\epsilon, h}(t) \|_{V}^2) dt < \infty.$$ 

This implies that there exists $\mathcal{T}_0 \in \mathcal{B}([0, T])$ with zero Lebesgue measure such that

$$E(\| X^{\epsilon, h}(t) \|_{V}^2) < \infty, \quad t \in [0, T] \setminus \mathcal{T}_0.$$ 

Hence for any $t \in [0, T] \setminus \mathcal{T}_0$, there exists $\Omega_t \in \mathcal{F}$ with $\mathbb{P}(\Omega_t) = 1$ such that

$$X^{\epsilon, h}(t, \omega) \in V, \quad \omega \in \Omega_t.$$
Proof. Denote by \((L^2(H), \| \cdot \|_{L^2, H})\) the Hilbert space of all Hilbert-Schmidt operators for \(L^2\) to \(H\). By Itô’s formula,
\[
\| X^{\epsilon, h}(t) \|_2^2 + 2\nu \int_0^t \| X^{\epsilon, h}(s) \|_2^2 ds = \| h \|_2^2 + 2\epsilon \int_0^t \langle X^{\epsilon, h}(s), g(x, X^{\epsilon, h}(s)) \rangle_{H, H} ds
\]
\[
+ 2\epsilon \int_0^t \langle X^{\epsilon, h}(s), \sigma(x, X^{\epsilon, h}(s)) dW(s) \rangle_{H, H}
\]
\[
+ \epsilon^2 \int_0^t \| \sigma(x, X^{\epsilon, h}(s)) \|_{L^2, H}^2 ds
\]
\[
=: I_1 + I_2(t) + I_3(t) + I_4(t).
\]

For \(I_3\), by (C2),
\[
E \left( \sup_{t \in [0, T]} |I_3'(t)| \right) \leq 2\epsilon E \left( \int_0^T \| X^{\epsilon, h}(t) \|_2^2 \| \sigma(x, X^{\epsilon, h}(t)) \|_{L^2, H}^2 dt \right)^{1/2}
\]
\[
\leq 2\epsilon E \left( \sup_{t \in [0, T]} \| X^{\epsilon, h}(t) \|_2^2 \right) + 2\epsilon E \left( \int_0^T \| \sigma(x, X^{\epsilon, h}(t)) \|_{L^2, H}^2 dt \right)
\]
\[
\leq 2\epsilon E \left( \sup_{t \in [0, T]} \| X^{\epsilon, h}(t) \|_2^2 \right) + 2\epsilon c_6 \lambda \left( \int_0^T \| X^{\epsilon, h}(t) \|_2^2 dt \right) + 2\epsilon T \| h_3 \|_1
\]
\[
\leq 2\epsilon E \left( \sup_{t \in [0, T]} \| X^{\epsilon, h}(t) \|_2^2 \right) + 2\epsilon c_6 \lambda \left( \int_0^T \| X^{\epsilon, h}(t) \|_2^2 dt \right) + 2\epsilon T \| h_3 \|_1. \tag{4.5}
\]

For \(I_4\), applying (C2) again, we have
\[
E(I_4'(T)) \leq \epsilon^2 c_6 \lambda \left( \int_0^T \| X^{\epsilon, h}(s) \|_{L^1} ds \right) + \epsilon^2 T \| h_3 \|_1. \tag{4.6}
\]

\(I_2\) will be estimated according to two cases.

(1) The case \(q > 2\) or \(q = 2, c_2 - c_3 \geq 0\).
By (C1), it is easy to see that there exist \(\kappa \geq 0\) and \(\tilde{h}_1 \in L^1(\Lambda)\) such that
\[
u g(x, u) \leq -\kappa |u|^q + \tilde{h}_1(x).
\]
Hence
\[
I_2(t) \leq 2\| \tilde{h}_1 \|_1^2 t. \tag{4.7}
\]

(2) The case \(q = 2, c_2 - c_3 < 0, \nu + \frac{c_2 - c_3}{\lambda} > 0\).
By (C1),
\[
I_2(t) \leq 2(c_3 - c_2) \int_0^t \| X^{\epsilon, h}(s) \|_2^2 ds + 2t \| h_1 \|_1
\]
\[
\leq \frac{2(c_3 - c_2)}{\lambda} \int_0^t \| X^{\epsilon, h}(s) \|_1^2 ds + 2t \| h_1 \|_1. \tag{4.8}
\]
Combining (4.5)–(4.8), we conclude that there exist \(\epsilon_0, \eta_1, \eta_2 > 0\) such that for \(0 < \epsilon \leq \epsilon_0\),
\[
\frac{1}{2} E \left( \sup_{t \in [0, T]} \| X^{\epsilon, h}(t) \|_2^2 \right) + \eta_1 E \left( \int_0^T \| X^{\epsilon, h}(t) \|_{L^1}^2 dt \right) \leq E(\| h \|_2^2) + \eta_2 T.
\]
This completes the proof. \(\square\)

Denote by \(X^{0, h}\) the solution for (4.3) when \(\epsilon = 0\). Applying the results in Lemma 4.2 and Itô’s formula to \(\| X^{\epsilon, h}(t) - X^{0, h}(t) \|_2^2\), we can obtain the following theorem.
Theorem 4.4. If the assumptions of Lemma 4.2 hold, then
(1) for any $\bar{M} > 0$, $\delta > 0$ and $t \geq 0$,
$$\lim_{\epsilon \to 0} \sup_{\|h\|_V^2 \leq \bar{M}} \mathbb{P}(\|X^{\epsilon,h}_t - X^{0,h}_t\|_2^2 \geq \delta) = 0;$$

(2) there exists at least one stationary measure $\mu^{\epsilon,h}$ for $X^{\epsilon,h}$;
(3) for $\epsilon \in (0, \epsilon_0]$, denote by $\{\mu^{\epsilon}_i, i \in I\}$ all stationary measures for the semigroup $\{P^\epsilon_t\}_{t \geq 0}$. Then $\{\mu^{\epsilon}_i, i \in I, \epsilon \in (0, \epsilon_0]\}$ is tight.

Proof. Repeating the proof of Lemma 4.2, we can get that there is a positive constant $C$ such that
$$\sup_{t \in [0,T]} \|X^{0,h}(t)\|_2^2 + \int_0^T \|X^{0,h}(t)\|_2^2 \, dt \leq C(\|h\|_2^2 + T), \quad (4.9)$$
where $C$ can be chosen to be the same as (4.4).

Define the stopping time $\tau_N = \inf\{s \in [0,T], \|X^\epsilon_s\|_2^2 \geq N\} \wedge T$. Applying Itô’s formula to $\|X^\epsilon(t) - X^{0,h}(t)\|_2^2$, we have
$$\|X^\epsilon(t \wedge \tau_N) - X^{0,h}(t \wedge \tau_N)\|_2^2 + 2\nu \int_0^{t \wedge \tau_N} \|X^\epsilon(s) - X^{0,h}(s)\|_V^2 \, ds$$
$$= 2 \int_0^{t \wedge \tau_N} \langle X^\epsilon(s) - X^{0,h}(s), g(x, X^\epsilon(s)) - g(x, X^{0,h}(s)) \rangle_{H,H} \, ds$$
$$+ 2\epsilon \int_0^{t \wedge \tau_N} \langle X^\epsilon(s) - X^{0,h}(s), \sigma(x, X^\epsilon(s)) \rangle_{H,H} \, dW(s)$$
$$+ \epsilon^2 \int_0^{t \wedge \tau_N} \|\sigma(x, X^\epsilon(s))\|_{2,H}^2 \, ds. \quad (4.10)$$

By the definition of $\tau_N$ and (4.9), we know that
$$\left\{2\epsilon \int_0^{t \wedge \tau_N} \langle X^\epsilon(s) - X^{0,h}(s), \sigma(x, X^\epsilon(s)) \rangle_{H,H} \, dW(s) \right\}_{t \in [0,T]}$$
is a martingale. Taking expectations of (4.10) and by (C1), (4.6) and (4.4), we have
$$\mathbb{E}\|X^\epsilon(t \wedge \tau_N) - X^{0,h}(t \wedge \tau_N)\|_2^2 + 2\nu \mathbb{E} \int_0^{t \wedge \tau_N} \|X^\epsilon(s) - X^{0,h}(s)\|_V^2 \, ds$$
$$\leq 2c_1 \mathbb{E} \int_0^{t \wedge \tau_N} \|X^\epsilon(s) - X^{0,h}(s)\|_2^2 \, ds + \epsilon^2 \mathbb{E} \int_0^{t \wedge \tau_N} \|\sigma(x, X^\epsilon(s))\|_{2,H}^2 \, ds$$
$$\leq 2c_1 \int_0^t \mathbb{E}\|X^\epsilon(s \wedge \tau_N) - X^{0,h}(s \wedge \tau_N)\|_2^2 \, ds + \epsilon^2 \mathbb{E} \int_0^T \|\sigma(x, X^\epsilon(s))\|_{2,H}^2 \, ds$$
$$\leq 2c_1 \int_0^t \mathbb{E}\|X^\epsilon(s \wedge \tau_N) - X^{0,h}(s \wedge \tau_N)\|_2^2 \, ds + \epsilon^2 \frac{c_6}{\lambda_3} \mathbb{E} \left(\int_0^T \|X^\epsilon(s)\|_V^2 \, ds\right) + \epsilon^2 T \|h_3\|_1$$
$$\leq 2c_1 \int_0^t \mathbb{E}\|X^\epsilon(s \wedge \tau_N) - X^{0,h}(s \wedge \tau_N)\|_2^2 \, ds + \epsilon^2 \frac{c_6}{\lambda_3} \mathbb{E} \left(\|h\|_2^2 + T\right) + \epsilon^2 T \|h_3\|_1.$$

By Gronwall’s lemma, for any $t \in [0,T]$ and integer $N$,
$$\mathbb{E}\|X^\epsilon(t \wedge \tau_N) - X^{0,h}(t \wedge \tau_N)\|_2^2 \leq \epsilon^2 \left(\frac{c_6}{\lambda_3} \mathbb{E} \left(\|h\|_2^2 + T\right) + T \|h_3\|_1\right) \cdot e^{2c_1 T}.$$

Letting $N \to \infty$ and using Fatou’s lemma, we get that for any $t \in [0,T]$,
$$\mathbb{E}\|X^\epsilon(t) - X^{0,h}(t)\|_2^2 \leq \epsilon^2 \left(\frac{c_6}{\lambda_3} \mathbb{E} \left(\|h\|_2^2 + T\right) + T \|h_3\|_1\right) \cdot e^{2c_1 T}.$$
Hence, for any $t \in [0, T]$,
\[
\lim_{\epsilon \to 0} \sup_{\|h\|_V^2 \leq \delta} \mathbb{P}(\|X^{\epsilon,h}_t - X^{0,h}_0\|_V^2 \geq \delta) \leq \lim_{\epsilon \to 0} \frac{C^2}{\delta} \left( \frac{c_0 C}{\lambda}(\overline{M} + T) + T\|h_3\|_1 \right) \cdot e^{2\epsilon T} = 0.
\]
We have obtained (1).

(2) Utilizing Lemma 4.2, we deduce that for any $L > 0$,
\[
\lim_{t \to \infty} \sup_{\epsilon \to 0} \frac{1}{t} \int_0^t \mathbb{P}(\|X^{\epsilon,h}_s\|_V \geq L)ds \leq \frac{1}{L^2} \lim_{t \to \infty} \sup_{\epsilon \to 0} \frac{1}{t} \int_0^t \mathbb{E}(\|X^{\epsilon,h}_s\|_V^2)ds \leq \frac{C}{L^2}.
\]
(4.11)
Notice that the embedding $V \subset H$ is compact. Then there exists at least one stationary measure $\mu^{\epsilon,h}$ for $X^{\epsilon,h}$ by Prokhorov’s theorem.

(3) For $\epsilon \in (0, \epsilon_0]$, choosing $\mu^{\epsilon} \in \{\mu^{\epsilon,i} : i \in I_\epsilon\}$, by the definition of stationary measures, we have
\[
\mu^{\epsilon}(\|h\|_V \geq L) = \int_H P(\|X^{\epsilon,h}_s\|_V \geq L)\mu^{\epsilon}(dh), \quad \forall \epsilon \geq 0,
\]
and hence, by Fubini’s theorem, Fatou’s lemma and (4.11),
\[
\mu^{\epsilon}(\|h\|_V \geq L) \leq \lim_{t \to \infty} \sup_{\epsilon \to 0} \frac{1}{t} \int_0^t \int_H P(\|X^{\epsilon,h}_s\|_V \geq L)\mu^{\epsilon}(dh)ds
\]
\[
\leq \lim_{t \to \infty} \sup_{\epsilon \to 0} \frac{1}{t} \int_0^t \int_H P(\|X^{\epsilon,h}_s\|_V \geq L)ds\mu^{\epsilon}(dh)
\]
\[
\leq \frac{C}{L^2}.
\]
It follows from the fact that \( \{h \in V : \|h\|_V \leq L\} \) is compact in $H$ that \( \{\mu^{\epsilon,i} : i \in I_\epsilon, \epsilon \in (0, \epsilon_0]\} \) is tight.

Using Young’s inequality, we can get that the polynomial $g$ in (4.1) satisfies the condition (C1). This fact, together with Theorems 4.4 and 2.1, implies the main result in this subsection.

**Theorem 4.5.** Assume that $\nu = 1$, $g$ is given in (4.1) and $\sigma$ satisfies the condition (C2). Then any limiting measures of stationary measures for (4.3) are supported in the set of equilibrium points $\mathcal{E}$.

Specially, we consider the one-dimensional cubic reaction-diffusion equation:
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \lambda^2 u(1 - u^2), \quad 0 < x < 1, \quad t > 0, \\
u(0, t) &= u(1, t) = 0,
\end{aligned}
\]
(4.12)
where $\lambda$ is a positive parameter.

**Proposition 4.6** (See [22, Theorem 4.3.12]). If $\lambda \in (n\pi, (n + 1)\pi]$ where $n$ is an integer, then there are $2n + 1$ equilibrium points $\phi_0 = 0, \phi_j^+, \phi_j^-, j = 1, 2, \ldots, n$ of (4.12), and the points $\phi_j^+$ are hyperbolic with $\text{dim}W^u(\phi_j^+) = j - 1, j = 1, 2, \ldots, n$. If $\lambda \in (n\pi, (n + 1)\pi)$, then $\phi_0 = 0$ is hyperbolic, $\text{dim}W^u(0) = n$, and the attractor $A_\lambda$ is given by
\[
A_\lambda = W^u(0) \cup \left( \bigcup_{j=1}^n W^u(\phi_j^+) \right).
\]
Here, $W^u(\phi)$ denotes the unstable manifold for the equilibrium $\phi$. Hence the Birkhoff center for the semiflow $\Phi$ is given by
\[
B(\Phi) = \{0\} \cup \{\phi_j^+, \phi_j^- : j = 1, 2, \ldots, n\}.
\]
Now we consider the perturbed equation driven by Brownian motion:
\[
\begin{aligned}
dX(t, x) &= \Delta X(t, x)dt + \lambda X(t, x)(1 - (X(t, x))^2)dt + \epsilon \sigma(x, X(t, x))dW(t), \\
X(t, 0) &= X(t, 1) = 0, \quad t > 0, \\
X(0) &= h \in L^2([0, 1]).
\end{aligned}
\]
(4.13)
It is easy to see that the cubic polynomial \( g(u) = u(1 - u^2) \) satisfies the condition (C1). From this together with Theorems 4.4 and 2.1 and Proposition 4.6, we have the following theorem.

**Theorem 4.7.** Assume that \( \sigma \) in (4.13) satisfies the condition (C2) and \( \lambda \in (n\pi, (n+1)\pi] \). If \( \mu \) is any weak limit point of \( \{\mu^*\} \) as \( \epsilon_i \to 0 \), then \( \mu \) is supported in the set of equilibrium points

\[
\{0\} \cup \{\phi_j^+, \phi_j^- : j = 1, 2, \ldots, n\}.
\]

**Remark 4.8.** Let \( \lambda \in (n\pi, (n+1)\pi] \) and \( \phi \), for example, \( \phi_j^+ \), be an equilibrium of (4.12). Then denote by

\[
m_j = \min\{\phi_j^+(x) : x \in [0, 1]\} \quad \text{and} \quad M_j = \max\{\phi_j^+(x) : x \in [0, 1]\}
\]

the minimal and maximal values of \( \phi_j^+ \), respectively. As constructed in Proposition 3.19, one can construct the diffusion term \( \sigma \) such that \( \sigma(u) = 0 \) on \([m_j, M_j]\) and (C2) is satisfied. Thus, (4.13) has a sequence of stationary measures whose weak limit is concentrated on \( \phi_j^+ \). This shows that the result of Theorem 4.7 is the best as a general result.

### 4.2 The stochastic 2D Navier-Stokes equation driven by Lévy noise

Let \( D \) be an open bounded domain with smooth boundary \( \partial D \) in \( \mathbb{R}^2 \). Denote by \( u \) and \( p \) the velocity and the pressure fields, respectively. The Navier-Stokes equation is given as follows:

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= h \quad \text{in} \quad D \times [0, T], \\
\nabla \cdot u &= 0 \quad \text{in} \quad D \times [0, T]
\end{align*}
\]

(4.14)

with the conditions

\[
\begin{align*}
\{0\} \cup \{\phi_j^+, \phi_j^- : j = 1, 2, \ldots, n\}.
\end{align*}
\]

(4.14)

where \( \nu > 0 \) is the viscosity, and \( h \) stands for the external force acting on the fluid.

Define

\[
V = \{v \in W^{1,2}_0(D, \mathbb{R}^2) : \nabla \cdot v = 0 \ \text{a.e. in} \ D\}, \quad \|v\|_V := \left( \int_D (|\nabla v_1|^2 + |\nabla v_2|^2) \text{dx} \right)^{1/2},
\]

and let \( H \) be the closure of \( V \) in the following norm:

\[
\|v\|_H := \left( \int_D |v|^2 \text{dx} \right)^{1/2}.
\]

By the Poincaré inequality, we have the Gelfand triples \( V \subset H \equiv H^* \subset V^* \).

Define the Stokes operator \( A \) in \( H \) by

\[
Au = P_H \Delta u, \quad \forall u \in D(A) = W^{2,2}(D, \mathbb{R}^2) \cap V,
\]

where the linear operator \( P_H \) (Helmholtz-Hodge projection) is the projection operator from \( L^2(D, \mathbb{R}^2) \) into \( H \). Since \( V \) coincides with \( D(A^{1/2}) \), we can endow \( V \) with the norm \( \|u\|_V = \|A^{1/2}u\|_H \). Because \( A \) is a positive self-adjoint operator with compact resolvent, there is a complete orthonormal system \( \{e_1, e_2, \ldots\} \) of eigenvectors of \( A \) in \( V \), with corresponding eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \) (\( Ae_i = \lambda_i e_i \)), and

\[
\|u\|^2_V \geq \lambda_1 \|u\|^2_H, \quad u \in V.
\]

It is well known that the Navier-Stokes equation can be reformulated as follows:

\[
u' = \nu A u + F(u) + h_0(u), \quad u(0) = u_0 \in H,
\]

(4.16)

and

\[
F : D_F \subset H \times V \to H, \quad F(u, v) = -P_H[(u \cdot \nabla)v], \quad F(u) = F(u, u), \quad h_0 = P_H h.
\]
One can show that the following mappings:

\[ A : V \to V^*, \quad F : V \times V \to V^* \]

are well defined, and

\[ \langle F(u, v), w \rangle_{V^*} = -\langle F(u, w), v \rangle_{V^*}, \quad \langle F(u, v), v \rangle_{V^*} = 0, \quad u, v, w \in V. \]

**Assumption 1.** For the mapping \( h_0 : V \to V^* \) there exists \( \vartheta_0 \in (0, \nu) \) such that

\[ ||h_0(v) - h_0(w)||_{V^*} \leq \vartheta_0 ||v - w||_V, \quad v, w \in V. \]

Define

\[ A(v) = \nu Av + F(v) + h_0(v). \]

Since \( A \) is linear, \( F \) is bilinear and Assumption 1 holds, we can easily get that

(H1) the map \( s \mapsto \langle A(v_1 + s v_2), v \rangle_{V^*} \) is continuous on \( \mathbb{R} \) for all \( v, v_1, v_2 \in V \).

As estimated in [42, Lemma 2.3] or [7, p. 293], we have that for any \( \eta > 0 \),

\[ \langle F(u) - F(v), u - v \rangle_{V^*} \leq 2||u - v||_V^{3/2}||u - v||_H^{1/2} ||v||_{L^4(D, \mathbb{R}^2)}^4 \leq \eta||u - v||_V^4 + \frac{27}{16\eta^2} ||v||_{L^4(D, \mathbb{R}^2)}^4 ||u - v||_H^2, \quad u, v \in V. \]

Take \( \eta = \frac{\nu - \vartheta_0}{2} \). Then

\[ 2\langle A(u) - A(v), u - v \rangle_{V^*} \leq -2\nu||u - v||_V^2 + 2\eta||u - v||_V^4 + \frac{27\nu}{8\eta^2} ||v||_{L^4(D, \mathbb{R}^2)}^4 ||u - v||_H^2 \]

\[ + 2\|
\|h_0(u) - h_0(v)\|_{V^*}||u - v||_V \leq -(\nu - \vartheta_0)||u - v||_V^2 + \frac{27}{(\nu - \vartheta_0)^2} ||v||_{L^4(D, \mathbb{R}^2)}^4 ||u - v||_H^2. \]

Thus we get that

(H2) \( 2\langle A(u) - A(v), u - v \rangle_{V^*} \leq \frac{27}{(\nu - \vartheta_0)^2} ||v||_{L^4(D, \mathbb{R}^2)}^4 ||u - v||_H^2 \) for all \( u, v \in V \).

Similarly, we can prove that

(H3) \( 2\langle A(v), v \rangle_{V^*} + (\nu - \vartheta_0)||v||_V^2 \leq (\nu - \vartheta_0)^{-1} ||h_0(0)||_{V^*}^2 \) for all \( v \in V \).

It follows from (2.91) in [7, p. 291] that for all \( u, v \in V \), \( ||F(u), v \rangle_{V^*} \leq 2||v||_{L^4(D, \mathbb{R}^2)} ||u||_V \). An easy calculation deduces that

\[ ||A(v)||_{V^*} \leq \nu||v||_V + 2||v||_{L^4(D, \mathbb{R}^2)} + \vartheta_0 ||v||_V + ||h_0(0)||_{V^*}. \]

Applying [42, Lemma 2.1], we have

\[ ||v||_{L^4(D, \mathbb{R}^2)}^4 \leq 2||v||_H^2 ||v||_V^2, \quad v \in V. \] (4.17)

Therefore, we obtain that there exists a positive constant \( C \) such that

(H4) \( ||A(v)||_{V^*} \leq C(1 + ||v||_V)(1 + ||v||_H^2), \quad v \in V. \)

Now we present an attractor result for the deterministic system (4.16).

**Proposition 4.9.** If Assumption 1 holds, then the deterministic system (4.16) generates a dynamical system \( \{\Phi(t)\}_{t \geq 0} \) which possesses a connected global attractor \( \mathcal{A}_H \). Besides, \( \Phi(t) |_{\mathcal{A}_H} \) is a group.

The proof is contained in [47, Theorems IV.8.2 and III.2.2] or [22, Theorem 4.4.5].

Consider the stochastic 2D Navier-Stokes equation driven by Lévy noise:

\[ dX_t^{c,h} = (\nu A X_t^{c,h} + F(X_t^{c,h}) + h_0(X_t^{c,h}))dt + \epsilon B(X_t^{c,h})dW_t + \epsilon \int_Z f(X_t^{c,h}, z)\tilde{N}(dt, dz) \] (4.18)

with a deterministic initial value \( X_0^{c,h} = h \in H \).
Here, \( \{W_t\}_{t \geq 0} \) is a \( U \)-valued cylindrical Wiener process on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) with \( U \) a separable Hilbert space, \( (Z, \mathcal{Z}) \) is a locally compact Polish space, and \( \nu \) is a \( \sigma \)-finite measure on \( Z \). Set \( N \) being a Poisson random measure on \([0, \infty) \times Z\) with the \( \sigma \)-finite intensity measure \( \lambda \otimes \nu \), where \( \lambda \) is the Lebesgue measure on \([0, \infty)\). \( \bar{N}([0, t] \times O) = N([0, t] \times O) - t\nu(O), O \in \mathcal{Z} \) with \( \nu(O) < \infty \), is the compensated Poisson random measure. Denote by \((L_2(U, H), \| \cdot \|_2)\) the Hilbert space of all Hilbert-Schmidt operators from \( U \) to \( H \).

**Assumption 2.** Suppose that \( B : V \to L_2(U, H) \) and \( f : V \times Z \to H \) satisfy the following conditions: there is a positive constant \( L \) such that for any \( v, w \in V \),

\[
\|B(v) - B(w)\|_2^2 + \int_Z \|f(v, z) - f(w, z)\|^2_H \nu(dz) \leq L\|v - w\|_H^2,
\]

\[
\int_Z \|f(v, z)\|^2_H \nu(dz) \leq L(1 + \|v\|_H^4),
\]

\[
\int_Z \|f(v, z)\|^2_H \nu(dz) \leq L(1 + \|v\|_H^2).
\]

This implies that

\[
\|B(v)\|_2^2 + \int_Z \|f(v, z)\|^2_H \nu(dz) \leq F + C\|v\|_H^2 \leq K(1 + \|v\|_H^2),
\]

where \( F \) and \( C \) depend on \( L \) and \( \|B(0)\|_2 \) and \( K := \max\{(\nu - \bar{\nu})^{-1}\|h_0\|_H^2, F, C\} \).

By Assumptions 1 and 2 with (H1)–(H4), all hypotheses in [7] are satisfied. So we have the following proposition.

**Proposition 4.10** (See [7, Theorem 1.2]). Suppose that Assumptions 1 and 2 hold. Then

(i) for any \( h \in L^2(\Omega, \mathcal{F}_0, P; H) \), \( T > 0 \), (4.18) has a unique solution \( \{X^\tau_{\epsilon, h}\}_{t \in [0, T]} \),

(ii) there exists a constant \( C_T \) independent of \( \epsilon \) and \( h \) such that, for \( \epsilon \in (0, 1) \),

\[
\sup_{t \in [0, T]} E\|X^\tau_{\epsilon, h}\|_{H}^4 + E \int_0^T \|X^\tau_{\epsilon, h}\|_H^2 \|X^\tau_{\epsilon, h}\|_V^2 dt \leq C_T(E\|h\|_H^4 + T)
\]

and

\[
E \int_0^T \|X^\tau_{\epsilon, h}\|_V^2 dt < \infty.
\]

**Lemma 4.11.** There exist \( \eta > 0 \) and \( \epsilon_\eta > 0 \) such that for any \( h \in H \),

\[
\sup_{s \in [0, T]} E\|X^{\tau, h}_s\|_H^2 + \eta E \int_0^T \|X^{\tau, h}_s\|_V^2 ds \leq \|h\|_H^2 + (1 + \epsilon_\eta^2)KT, \quad \forall \epsilon \in (0, \epsilon_\eta].
\]

In particular,

\[
\sup_{s \in [0, T]} \|X^{0, h}_s\|_H^2 + \eta \int_0^T \|X^{0, h}_s\|_V^2 ds \leq \|h\|_H^2 + 2KT.
\]

**Remark 4.12.** As in Remark 4.3, there exists \( T_0 \in \mathcal{B}([0, T]) \) with zero Lebesgue measure and for any \( t \in [0, T) \setminus T_0 \), there exists \( \Omega_t \in \mathcal{F} \) with \( P(\Omega_t) = 1 \) such that

\[
X^{\tau, h}(t, \omega) \in V, \quad \omega \in \Omega_t.
\]

**Proof.** Using a similar argument to [7, (4.18)], we have

\[
\|X^{\tau, h}_t\|_H^2 = \|h\|_H^2 + 2 \int_0^t \langle A(X^{\tau, h}_s), X^{\tau, h}_s \rangle_{V^*, V} ds
\]

\[
+ 2\epsilon \int_0^t \langle B(X^{\tau, h}_s) dW_s, X^{\tau, h}_s \rangle_{H, H} ds.
\]
where $\bar{X}^{\epsilon,h}$ is any $V$-valued progressively measurable $dt \times P$ version of $X^{\epsilon,h}$.

Set $\sigma_N := \inf \{s \geq 0 : \|X^{\epsilon,h}_s\|_H \geq N \} \wedge T$. By (4.20) and (4.21),

$$L_N(t) = 2 \epsilon \int_0^t \langle B(\bar{X}^{\epsilon,h}_s) dW_s, \bar{X}^{\epsilon,h}_s \rangle_{H,H} + 2 \epsilon \int_0^t \int_Z \langle f(\bar{X}^{\epsilon,h}_s, z) \bar{X}^{\epsilon,h}_s \rangle_{H,H} \bar{N}(ds, dz),$$

(4.23)

where $\bar{X}^{\epsilon,h}$ is a square integrable martingale. Combining (H3) and (4.19), we obtain

We obtain

$$\sup_{s \in [0,t]} \mathbb{P}(\|X^{\epsilon,h}_s\|^2_H \geq 0) = 0;$$

which implies

$$\sup_{s \in [0,t]} \mathbb{P}(\|X^{\epsilon,h}_s\|^2_H \geq 0) = 0;$$

This completes the proof.}

**Theorem 4.13.** Suppose Assumptions 1 and 2 hold. Then

(1) for any $\tilde{M} > 0, \delta > 0$ and $T \geq 0$,

$$\lim_{\epsilon \to 0} \sup_{\|h\|_V \leq \tilde{M}} \mathbb{P}(\|X^{\epsilon,h}_T - X^{0,h}_T\|^2_H \geq \delta) = 0;$$

(2) there exists at least one stationary measure $\mu^{\epsilon,h}$ for $X^{\epsilon,h}$;

(3) for $\epsilon \in (0, \epsilon_0]$, denote by $\{\mu^{\epsilon,h}_i, i \in I_\epsilon\}$ all stationary measures for the semigroup $\{P_t\}_{t \geq 0}$. Then $\{\mu^{\epsilon,h}_i, i \in I_\epsilon, \epsilon \in (0, \epsilon_0]\}$ is tight.

**Proof.** (1) Set $\sigma_N := \inf \{s \geq 0 : \|X^{\epsilon,h}_s\|_H \geq N \} \wedge T$ and $\rho(v) := \frac{27}{16 \lambda^2 \delta} \|v\|^2_{L^4(D, R^2)}$. By Itô’s formula, we have

$$\exp \left( - \int_0^{\epsilon N} \rho(X^{0,h}_t)dt \right) \|X^{\epsilon,h}_t - X^{0,h}_t\|^2_H$$

$$= \int_0^{\epsilon N} \exp \left( - \int_0^s \rho(X^{0,h}_t)dt \right) (\langle A(\bar{X}^{\epsilon,h}_s) - A(X^{0,h}_s), \bar{X}^{\epsilon,h}_s - X^{0,h}_s \rangle_{V', V}) ds$$

$$= \int_0^{\epsilon N} \exp \left( - \int_0^s \rho(X^{0,h}_t)dt \right) \left( \langle A(\bar{X}^{\epsilon,h}_s) - A(X^{0,h}_s), \bar{X}^{\epsilon,h}_s - X^{0,h}_s \rangle_{V', V} \right) ds$$

$$= \int_0^{\epsilon N} \exp \left( - \int_0^s \rho(X^{0,h}_t)dt \right) \left( \langle A(\bar{X}^{\epsilon,h}_s) - A(X^{0,h}_s), \bar{X}^{\epsilon,h}_s - X^{0,h}_s \rangle_{V', V} \right) ds.$$
\[- \rho(X^{0,h}_s)\|\bar{X}^{e,h}_s - X^{0,h}_s\|^2_H ds \\
+ 2\epsilon \int_0^{t\wedge \sigma_N} \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \langle B(\bar{X}^{e,h}_s) dW_s, \bar{X}^{e,h}_s - X^{0,h}_s \rangle_{H,H} \]
\[+ \epsilon^2 \int_0^{t\wedge \sigma_N} \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \|B(\bar{X}^{e,h}_s)\|^2 ds \\
+ 2\epsilon \int_0^{t\wedge \sigma_N} \int_Z \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \langle f(\bar{X}^{e,h}_r, z), \bar{X}^{e,h}_s - X^{0,h}_s \rangle_{H,H} \bar{N}(ds,dz) \]
\[+ \epsilon^2 \int_0^{t\wedge \sigma_N} \int_Z \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \|f(\bar{X}^{e,h}_r, z)\|^2 \bar{N}(ds,dz), \]

where $\bar{X}^{e,x}$ is any $V$-valued progressively measurable $dt \times P$ version of $X^{e,x}$.

From (4.20), (4.21) and Lemma 4.11, we know that

\[L_N(t) = 2\epsilon \int_0^{t\wedge \sigma_N} \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \langle B(\bar{X}^{e,h}_s) dW_s, \bar{X}^{e,h}_s - X^{0,h}_s \rangle_{H,H} \]
\[+ 2\epsilon \int_0^{t\wedge \sigma_N} \int_Z \exp \left( - \int_0^s \rho(X^{0,h}_r) dr \right) \langle f(\bar{X}^{e,h}_r, z), \bar{X}^{e,h}_s - X^{0,h}_s \rangle_{H,H} \bar{N}(ds,dz) \]
is a square integrable martingale. Therefore from (H2), (4.19) and Lemma 4.11,

\[E \left( \exp \left( - \int_0^t \rho(X^{0,h}_r) dr \right) \|X^{e,h}_{t\wedge \sigma_N} - X^{0,h}_{t\wedge \sigma_N}\|^2_H \right) \leq \epsilon^2 E \int_0^{t\wedge \sigma_N} \|B(\bar{X}^{e,h}_r)\|^2 ds + \epsilon^2 E \int_0^{t\wedge \sigma_N} \int_Z \|f(\bar{X}^{e,h}_r, z)\|^2 \bar{N}(dz) ds \]
\[\leq \epsilon^2 E \int_0^t K(1 + \|\bar{X}^{e,h}_s\|^2_H) ds \]
\[\leq K\epsilon^2 t(1 + \sup_{s \in [0,t]} E \|X^{e,h}_s\|^2_H) ds \]
\[\leq \epsilon^2 KT(1 + \|h\|^2_H + (1 + \epsilon_0^2)KT), \]

which implies

\[E \left( \exp \left( - \int_0^t \rho(X^{0,h}_r) dr \right) \|X^{e,h}_{t\wedge \sigma_N} - X^{0,h}_{t\wedge \sigma_N}\|^2_H \right) \leq \epsilon^2 KT(1 + \|h\|^2_H + (1 + \epsilon_0^2)KT). \]

By (4.17) and Lemma 4.11,

\[\int_0^t \rho(X^{0,h}_r) dr \leq \frac{54}{(\nu - \vartheta_0)^3} \int_0^t \|X^{0,h}_r\|^2_H \|X^{0,h}_r\|^2_V ds \]
\[\leq \frac{54}{(\nu - \vartheta_0)^3} C_\eta(\|h\|^2_H + 2KT)^2, \]

where $\eta$ comes from Lemma 4.11, we obtain

\[E(\|X^{e,h}_{t\wedge \sigma_N} - X^{0,h}_{t\wedge \sigma_N}\|^2_H) \leq \epsilon^2 KT \exp \left( \frac{54}{(\nu - \vartheta_0)^3} C_\eta(\|h\|^2_H + 2KT)^2 \right)(1 + \|h\|^2_H + (1 + \epsilon_0^2)KT). \]

Letting $N \to \infty$, we have

\[E(\|X^{e,h}_t - X^{0,h}_t\|^2_H) \leq \epsilon^2 KT \exp \left( \frac{54}{(\nu - \vartheta_0)^3} C_\eta(\|h\|^2_H + 2KT)^2 \right)(1 + \|h\|^2_H + (1 + \epsilon_0^2)KT), \]

which implies the first part of this theorem by Chebyshev’s inequality.

Notice that the embedding $V \subset H$ is compact. Then the proofs for (2) and (3) are exactly the same as those in Theorem 4.4, so we omit them. \qed
Theorem 4.14. Assume that (4.18) satisfies Assumptions 1 and 2. Let \( \{ \mu^\varepsilon \} \) be a sequence of stationary measures for (4.18) such that \( \mu^\varepsilon \xrightarrow{\varepsilon \to 0} \mu \). Then \( \mu \) is an invariant measure of \( \Phi \) and its support is contained in the Birkhoff center for \( \Phi(t)|_{A_H} \).

Proof. It follows directly from Theorems 4.13 and 2.1. \( \square \)

4.3 Stochastic Burgers type equations

Consider the classic Burgers equation (see [12, pp. 257–258])

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \lambda u \frac{\partial u}{\partial x}, \quad 0 < x < 1, \quad t > 0, \\
u(0, t) &= u(1, t) = 0, \\
u(x, 0) &= \varphi(x).
\end{aligned}
\] (4.24)

The solution generates a strongly monotone flow in a suitable function space \( V = W^{1,2}_0(0, 1) \) (see [20, 23, 45]). It is easy to obtain that the trivial solution is the unique stationary solution for (4.24). Set \( H = L^2([0, 1]) \) and let \( A \) denote the Laplace operator. Since \( V \) coincides with \( D(A^{1/2}) \), we endow \( V \) with the norm \( \|u\|_V = \|A^{1/2}u\|_H \), which is equivalent to the usual norm in the Sobolev space \( V \). Also we can prove that \( \|u^\varphi(t)\|_V^2 \) is bounded on \([0, \infty)\) for every initial value \( \varphi \in V \).

In fact, if \( u(0) = \varphi \in V \), then there exists a unique solution \( u^\varphi \in C([0, \infty), V) \cap L^2([0, \infty), D(A)) \) of (4.24). This fact can be obtained by a fixed-point theorem, and the proof is omitted. We have

\[
\|u^\varphi(t)\|^2_V + 2\int_0^t \|u^\varphi(s)\|^2_V ds = \|\varphi\|^2_V,
\] (4.25)

where we have used the fact that \( \langle u, Au \rangle_H \) is bounded.

Since \( \langle u, Du \rangle_H \) is bounded, we have

\[
\frac{\partial^2 u}{\partial x^2} + \lambda u \frac{\partial u}{\partial x} \in C([0, 1], D(A)) \quad \text{and} \quad u(0, t) = u(1, t) = 0,
\]

and hence

\[
\|u^\varphi(t)\|^2_V + \int_0^t \|u^\varphi(s)\|^2_{D(A)} ds \leq \|\varphi\|^2_V + (\lambda c)^2 \int_0^t \|u^\varphi(s)\|^2_V ds,
\]

by Gronwall’s inequality and (4.25).

Consider the one-dimensional stochastic Burgers equation driven by Lévy noise:

\[
dX_t^{\varepsilon,h} = (\Delta X_t^{\varepsilon,h} + X_t^{\varepsilon,h} \cdot \nabla X_t^{\varepsilon,h}) dt + \varepsilon B(X_t^{\varepsilon,h}) dW_t + \varepsilon \int_Z f(X_t^{\varepsilon,h}, z) \Delta t dz
\] (4.26)

with a deterministic initial value \( X_0^{\varepsilon,h} = h \).

Theorem 4.15. Suppose (4.26) satisfies Assumption 2 in Subsection 4.2 and \( h \in H \). Then the results of Theorem 4.13 hold. Moreover, any limiting measures for its stationary measures are the Dirac measure \( \delta_0 \).
Proof. Itô’s formula deduces that
\[
\|X_t^{\epsilon,h}\|_H^2 + 2 \int_0^t \|X_s^{\epsilon,h}\|_V^2 \, ds = \|h\|_H^2 + 2\epsilon \int_0^t \langle B(X_{s-}^{\epsilon,h})dW_s, X_s^{\epsilon,h} \rangle_{H,H} \\
+ 2\epsilon \int_0^t \int_Z \langle f(X_{s-}^{\epsilon,h}, z), X_s^{\epsilon,h} \rangle_{H,H} \tilde{N}(ds, dz) \\
+ \epsilon^2 \int_0^t \|B(X_s^{\epsilon,h})\|_2^2 \, ds + \epsilon^2 \int_0^t \|f(X_s^{\epsilon,h}, z)\|_H^2 \, N(ds, dz). 
\]  
(4.27)
By the same argument of Lemma 4.11, there exist \( \eta > 0, \epsilon_\eta > 0 \) and \( F > 0 \) such that
\[
\sup_{s \in [0,T]} \mathbb{E}\|X_s^{\epsilon,h}\|_H^2 + \eta \mathbb{E} \int_0^T \|X_s^{\epsilon,h}\|_V^2 \, ds \leq \|h\|_H^2 + (1 + \epsilon_\eta^2)FT, \ \forall \epsilon \in (0, \epsilon_\eta]. 
\]  
(4.28)
Define \( A : V \rightarrow V^* \) by
\[
A(v) := v_{xx} + vv_x.
\]
According to [7, Lemma 2.1(2)], for any \( u, v \in V \), there is a positive constant \( C \) such that
\[
2\langle A(u) - A(v), u - v \rangle_{V^*, V} \leq -\|u - v\|_V^2 + C(1 + \|v\|_V^2)\|u - v\|_H^2. 
\]  
(4.29)
Combining (4.28) and (4.29), similar to the proof of Theorem 4.13, we can obtain the probability convergence, existence of stationary measures and their tightness. Thus the last conclusion follows immediately from Theorem 2.1.

5 FDEs driven by white noise

Consider the \( m \)-dimensional stochastic functional differential equations (SFDEs)
\[
dX^{\epsilon,\phi}(t) = b(X_t^{\epsilon,\phi})dt + c\sigma(X_t^{\epsilon,\phi})dW(t), \\
X_0^{\epsilon,\phi} = \phi \in C := C([-\tau, 0], \mathbb{R}^m), 
\]  
(5.1)
where \( W = \{W_t = (W_t^1, \ldots, W_t^k), t \geq 0\} \) is a \( k \)-dimensional Wiener process, \( b(\cdot) : C \rightarrow \mathbb{R}^m \) and \( \sigma(\cdot) : C \rightarrow \mathbb{R}^{m \times 1} \) satisfy the global Lipschitz condition and linear growth condition, i.e., there exists a positive constant \( L \) such that \( \forall \phi, \psi \in C \),
\[
(b) \ |b(\phi)|^2 + \|\sigma(\phi)\|^2 \leq L^2(1 + \|\phi\|^2). 
\]
Here, \( C \) denotes the set of continuous functions \( \phi(s) \) from \([-\tau, 0]\) into \( \mathbb{R}^m \) with the uniform norm
\[
||\phi|| = \sup_{-\tau \leq s \leq 0} |\phi(s)|.
\]
It is known that the hypotheses (a) and (b) are sufficient to ensure the global existence and uniqueness of a strong solution to (5.1). Let \( X^{\epsilon,\phi}(t) \) denote the solution to (5.1) with the initial data \( X_0^{\epsilon,\phi} = \phi \). Then the segment process of \( X^{\epsilon,\phi}(t) \) is given by
\[
X_t^{\epsilon,\phi}(\theta) = X^{\epsilon,\phi}(t + \theta), \ \theta \in [-\tau, 0], 
\]
i.e., \( \{X_t^{\epsilon,\phi}\}_{t \geq 0} \) is a process on \( C \). Furthermore, the segment process \( \{X_t^{\epsilon,\phi}\}_{t \geq 0} \) to (5.1) is immediately a Feller process on the path space \( C \) (see, e.g., [43, Theorem III.3.1, pp. 67–68]), where the associated Markov semigroup
\[
P_t^{\phi}g(\phi) = \mathbb{E}_g(X_t^{\epsilon,\phi}), \ \ t \geq 0, \ \phi \in C, \ \ g \in B_b(C). 
\]  
(5.2)
Consider the corresponding \( m \)-dimensional deterministic functional differential equations (FDEs)
\[
dX^{\phi}(t) = b(X_t^{\phi})dt, \ \ X_0^{\phi} = \phi \in C. 
\]  
(5.3)
Under the hypothesis (a), it is easy to see that (5.3) generates a semiflow \( \Phi_t(\phi) = \Phi_t^{\phi}, \ t \geq 0 \) on \( C \).

The following result reveals a close connection between (5.1) and (5.3).
Lemma 5.1. Suppose (a) and (b) hold. Let $K \subset \mathcal{C}$ be a compact (bounded) set and $T > 0$. Then for sufficiently small $\epsilon > 0$,
\[
\sup_{\phi \in K} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| X^t_{\epsilon, \phi} - \Phi_t(\phi) \|^2 \right] \leq C \epsilon^2,
\]
where $C = C(K, L, T)$ is a positive constant, depending only on $K$, $L$ and $T$.

The proof is easy, so we omit it.

The following probability convergence can be obtained by Chebyshev’s inequality.

Corollary 5.2. Let $K \subset \mathcal{C}$ be a compact set. Then for any $T > 0$ and $\delta > 0$, we have
\[
\lim_{\epsilon \to 0} \sup_{\phi \in K} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| X^t_{\epsilon, \phi} - \Phi_t(\phi) \| \geq \delta \right\} = 0.
\]

In order to apply Theorem 2.1, we need to present a criterion on the existence of stationary measures for (5.1) and their tightness. For this purpose, from now on, we focus on the following stochastic functional differential equations:
\[
dX^t = [ -BX^t + Ag(X^t) ]dt + c\sigma(X^t)dW(t)
\]
with the initial data $X^0_t = \phi \in \mathcal{C}$, where $B = (b_{ij})_{m \times m}$, $A = (a_{ij})_{m \times m}$ are two matrices, $\sigma(\phi) = (\sigma_{ij}(\phi))_{m \times k}$ is an $m \times k$ matrix valued function defined on $\mathcal{C}$, and $g : \mathcal{C} \to \mathcal{C}$ is a measurable function.

We will suppose the following assumptions on $g$ and $\sigma$:

(A1) There exists a positive constant $\tilde{L}$ such that for all $\phi, \psi \in \mathcal{C}$,
\[
|g(\phi) - g(\psi)| \leq \tilde{L} \| \phi - \psi \|.
\]

(A2) There exists a positive constant $L$ such that for all $\phi, \psi \in \mathcal{C}$,
\[
\| \sigma(\phi) - \sigma(\psi) \|_2 \leq L \| \phi - \psi \|.
\]

To show the existence of a stationary measure, it is sufficient to prove the tightness of the segments by using the Arzelà-Ascoli tightness characterization and the Krylov-Bogoliubov theorem.

Using the idea presented in [17, Proposition 2.1], we have the following theorem.

Theorem 5.3. Assume (A1), (A2) and
\[
\langle x, Bx \rangle \geq b|x|^2 \quad \text{for any} \quad x \in \mathbb{R}^m.
\]

If $b$ satisfies the following:
\[
b > \frac{\gamma^2 e^{6\gamma} (16\tilde{L}^3 |A|^3)^2}{(1 - \kappa e^{-3\gamma})^2},
\]
where $\kappa \in (1, e^{3\gamma})$ is arbitrary, and $\gamma = \gamma(\epsilon) = 9\left(\frac{2\sqrt{\pi} - 1}{\sqrt{\pi} - 1}\right) + 1$, then there exists $\epsilon_0 > 0$ such that
\[
\sup_{0 < \epsilon < \epsilon_0} \sup_{t \geq 0} \mathbb{E}[\| X^t_{\epsilon, \phi} \|^6 ] \leq 2e^{3\gamma}(\| \phi \|^6 + \tilde{M}),
\]
where $\tilde{M}$ is a constant independent of $\epsilon$ in $(0, \epsilon_0)$. Furthermore, for each $\phi \in \mathcal{C}$, there exists at least a stationary measure $\mu^{\epsilon, \phi}$ for (5.4) corresponding to the segment process $\{ X^t_{\epsilon, \phi} \}_{t \geq 0}$.

Proof. Fixing $\epsilon$, to simplify notation, we let $X(t) = X^t_{\epsilon, \phi}(t)$ and set $Z(t) = |X(t)|^2$, $t \geq 0$. By Itô’s formula, (5.5), (A1) and (A2), we have
\[
dZ(t) \leq -2bZ(t)dt + 2|A||X(t)||g(X_t)|dt + 2\epsilon^2(2L^2\|X_t\|^2 + \|\sigma(0)\|^2_2)dt
\]
\[
+ 2\epsilon\langle X(t), \sigma(X_t) dW(t) \rangle
\]
\[
\leq -2bZ(t)dt + 2|A||X(t)||g(X_t)|dt + 2\epsilon^2(2L^2\|X_t\|^2 + \|\sigma(0)\|^2_2)dt
\]
\[
+ 2\epsilon\langle X(t), \sigma(X_t) dW(t) \rangle
\]
\[ \begin{align*}
\leq -2bZ(t)dt + 2|A|\left( \underline{L}\|X_t\|^2 + \overline{L}\|X_t\|^2 + \frac{|g(0)|^2}{L} \right) dt + 2\epsilon \langle X(t), \sigma(X_t)dW(t) \rangle \\
=:-2bZ(t)dt + C\|X_t\|^2 dt + Ddt + 2\epsilon \langle X(t), \sigma(X_t)dW(t) \rangle, \quad t \geq 0,
\end{align*} \]
where \( C = C(\epsilon) = 4\overline{L}|A| + 2\epsilon^2L^2 \) and \( D = D(\epsilon) = \frac{2|A||g(0)|^2}{L} + 2\epsilon^2\|\sigma(0)\|^2_2 \). Then the stochastic variation of the constant formulas yields
\[ Z(t) \leq e^{-2bt}Z(0) + \int_0^t e^{-2b(t-s)}(C\|X_s\|^2 + D)ds + 2\epsilon \int_0^t e^{-2b(t-s)}\langle X(s), \sigma(X_s)dW(s) \rangle \]
\[ \leq e^{-2bt}Z(0) + \frac{C}{2b} \sup_{0 \leq s \leq t} \|X_s\|^2 + \frac{D}{2b} \epsilon^2 + 2\epsilon \sup_{0 \leq s \leq \tau} \int_0^t e^{-2b(t-s)}\langle X(s), \sigma(X_s)dW(s) \rangle, \quad \forall t \geq 0. \]
Hence, for \( 0 \leq t \leq \tau \) we obtain
\[ \sup_{0 \leq t \leq \tau} e\gamma'Z(t) \leq Z(0) + \frac{C}{2b} \epsilon^2 \sup_{0 \leq s \leq \tau} \|X_s\|^2 + \frac{D}{2b} \epsilon^2 + 2\epsilon \sup_{0 \leq t \leq \tau} \int_0^t e^{-2b(t-s)}\langle X(s), \sigma(X_s)dW(s) \rangle, \]
where we have used the fact that \( b > \frac{1}{2} \). It is easy to see that for any \( \kappa \in (1, e^{3\tau}) \), there exists \( \gamma = \gamma(\kappa) > 1 \) such that
\[ (x_1 + x_2 + x_3 + x_4)^3 \leq \kappa x_1^3 + \gamma(x_2^3 + x_3^3 + x_4^3) \]
for all \( x_1, x_2, x_3, x_4 \geq 0 \). (5.8)
Combining the above inequality (5.8) and taking expectations, we obtain
\[ E\left[ \sup_{0 \leq t \leq \tau} e^{3t}|Z(t)|^3 \right] \leq \kappa E(|Z(0)|^3) + \gamma \left( \frac{C^3}{8b^3} \epsilon^3 E\left[ \sup_{0 \leq s \leq \tau} \|X_s\|^6 \right] + \frac{D^3}{8b^3} \epsilon^3 \right) + 8\epsilon^3 \gamma^3 \left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t e^{-2b(t-s)}\langle X(s), \sigma(X_s)dW(s) \rangle \right|^3 \right]. \] (5.9)
From Lévy’s celebrated martingale characterization of Brownian motion (see [32, Theorem 3.16, p. 157]), we know that there exists a one-dimensional Brownian motion \( B \) with respect to the same filtration such that
\[ \langle X(s), \sigma(X_s)dW(s) \rangle = \beta(s, \omega)dB(s), \]
where
\[ \beta(s, \omega) = \left( \sum_{j=1}^k \left( \sum_{i=1}^m X_i(s)\sigma_{ij}(X_s) \right)^2 \right)^{\frac{1}{2}}. \]
By the technical result [17, Lemma 2.2] and (A2), we get
\[ E\left[ \sup_{0 \leq t \leq \tau} \left| \int_0^t e^{-2b(t-s)}\langle X(s), \sigma(X_s)dW(s) \rangle \right|^3 \right] \]
\[ \leq 2\epsilon a_{3,2b}(2L^3 + \|\sigma(0)\|^3_2) E\|X_t\|^6 + L^3 E\|X_0\|^6 + \|\sigma(0)\|^3_2, \]
where
\[ a_{3,2b} = C_3 \left( \frac{3 - 1}{6b} \right)^{3\alpha-1} \Gamma \left( \frac{3\alpha - 1}{3 - 1} \right)^{3-1} \left[ \left( \frac{1}{4b} \right)^{1-2\alpha} \Gamma(1 - 2\alpha) \right]^{\frac{1}{2}} \]
\[ =: \Lambda(\alpha) \left( \frac{1}{b} \right)^{\frac{1}{2}}, \]
(5.10)
where \( C_3 = \left( \frac{3\alpha - 1}{6b} \right)^{\frac{1}{2}} \) is the universal positive constant in the Burkholder-Davis-Gundy inequality (see [41, Theorem 7.3, p. 40]) and \( \Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t} \) is a Gamma function and \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right). \)

\footnote{For arbitrary \( \kappa \in (1, e^{3\tau}) \), choose \( \gamma = \gamma(\kappa) = 9\left( \frac{4\pi}{(\sqrt{\kappa-1})^2} + 1 \right) \) such that (5.8) holds.}
Continuing from (5.9) and using the above inequality, we have

\[
E \left[ \sup_{0 \leq t \leq \tau} e^{3s} |Z(t)|^3 \right] \leq \kappa E(|Z(0)|^3 + \gamma \frac{C^3}{8\delta^3} e^{3\tau} E[\|X_0\|^6 + \|X_\tau\|^6] + \gamma \frac{D^3}{8\delta^3} e^{3\tau} \\
+ \gamma \delta e^{3\tau} 2\tau a_{3,2b} [(2L^3 + \|\sigma(0)\|_2^2) E\|X_\tau\|^6 + L^3 E\|X_0\|^6 + \|\sigma(0)\|_2^2] \\
\leq \gamma \frac{D^3}{8\delta^3} e^{3\tau} + 16e^3 \gamma a_{3,2b} e^{3\tau} \|\sigma(0)\|_2^3 \\
+ \kappa E|Z(0)|^3 + \left( \gamma \frac{C^3}{8\delta^3} e^{3\tau} + 16e^3 \gamma a_{3,2b} e^{3\tau} L^3 \right) E\|X_\tau\|^6 \\
+ \left[ \gamma \frac{C^3}{8\delta^3} e^{3\tau} + 16e^3 \gamma a_{3,2b} e^{3\tau} (2L^3 + \|\sigma(0)\|_2^2) \right] E\|X_\tau\|^6. \tag{5.11}
\]

Define a Lyapunov function \( V : C([-\tau, 0], \mathbb{R}) \rightarrow \mathbb{R}_+ \) by

\[
V(\zeta) = \sup_{-\tau \leq s \leq 0} e^{3s} |\zeta(s)|^3.
\]

Let \( \psi(s) = |\phi(s)|^2, s \in [-\tau, 0] \). Therefore, (5.11) along with the fact that \( E|Z(0)|^3 \leq EV(\psi), E\|X_0\|^6 \leq e^{3\tau} EV(\psi), E[\sup_{0 \leq s \leq \tau} e^{3s} |Z(s)|^3] = e^{3\tau} EV(\tau) \) and \( E\|X_\tau\|^6 \leq e^{3\tau} EV(\tau) \) imply that

\[
\left\{ 1 - \gamma e^{3\tau} \left[ \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} (2L^3 + \|\sigma(0)\|_2^2) \right] \right\} EV(\tau) \\
\leq \left[ \kappa e^{-3\tau} + \gamma e^{3\tau} \left( \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} L^3 \right) \right] EV(\psi) + \gamma \left( \frac{D^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} \|\sigma(0)\|_2^3 \right) \tag{5.12}
\]

We assume that

\[
\left\{ 1 - \gamma e^{3\tau} \left[ \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} (2L^3 + \|\sigma(0)\|_2^2) \right] \right\} > 0,
\]

\[
\delta = \delta(\epsilon) := \frac{\kappa e^{-3\tau} + \gamma e^{3\tau} \left( \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} L^3 \right)}{1 - \gamma e^{3\tau} \left[ \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} (2L^3 + \|\sigma(0)\|_2^2) \right]} < 1,
\]

which is equivalent to

\[
\left\{ \frac{C^3}{8\delta^3} + 16e^3 \gamma (2L^3 + \|\sigma(0)\|_2^2) a_{3,2b} < \frac{1}{\gamma e^{3\tau}}, \right\} \\
\left\{ \frac{C^3}{4\delta^3} + 16e^3 \gamma (3L^3 + \|\sigma(0)\|_2^3) a_{3,2b} < \frac{1}{\gamma e^{3\tau}}. \right\} \tag{5.13}
\]

By (5.10) and the fact that \( b > 1 \), it suffices to show that

\[
\left\{ b > \gamma^2 e^{6\tau} \left[ \frac{C^3}{8} + 16e^3 \gamma (2L^3 + \|\sigma(0)\|_2^2) \Lambda \right]^2, \right\} \\
\left\{ b > \gamma^2 e^{6\tau} \left[ \frac{C^3}{4} + 16e^3 \gamma (3L^3 + \|\sigma(0)\|_2^3) \Lambda \right]^2 \right\} \tag{5.14}
\]

This shows that we can find \( \epsilon_0 > 0 \) such that for each \( \epsilon \leq \epsilon_0 \), (5.14) holds as long as (5.6) is satisfied. Let

\[
\rho := \frac{\gamma \left( \frac{D^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} \|\sigma(0)\|_2^3 \right)}{1 - \gamma e^{3\tau} \left[ \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} (2L^3 + \|\sigma(0)\|_2^2) \right]}.
\]

Then for every \( \epsilon \leq \epsilon_0 \),

\[
\frac{\rho}{1 - \delta} = \frac{\gamma \left( \frac{D^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} \|\sigma(0)\|_2^3 \right)}{1 - \kappa e^{-3\tau} - \gamma e^{3\tau} \left[ \frac{C^3}{8\delta^3} + 16e^3 \gamma a_{3,2b} (3L^3 + \|\sigma(0)\|_2^3) \right]} \leq \frac{\rho(\epsilon_0)}{1 - \delta(\epsilon_0)}.
\]

If (5.14) holds, then from (5.13) we have

\[
EV(Z_\tau) \leq \delta EV(\psi) + \rho. \tag{5.15}
\]
Iterating (5.15), we get

$$\text{EV}(Z_{kT}) \leq \delta^k \text{EV}(\psi) + \rho \left( \frac{1}{1 - \delta} \right) \leq \text{EV}(\psi) + \frac{\rho}{1 - \delta} \quad \text{for all} \ k \in \mathbb{N}^*.$$  

(5.16)

This implies

$$\sup_{k \in \mathbb{N}^*} \text{EV}(Z_{kT})^3 \leq e^{3\tau} \left( \text{EV}(\psi) + \frac{\rho}{1 - \delta} \right).$$

Note that for \( t \in [k\tau, (k+1)\tau], \|Z_t\|^3 \leq \|Z_{k\tau}\|^3 + \|Z_{(k+1)\tau}\|^3, \forall k \in \mathbb{N}. \) In term of the original process \( X, \) we conclude that for all \( 0 < \epsilon \leq \epsilon_0, \)

$$\sup_{t \geq 0} \text{EV}(X_t,\phi)^6 \leq 2e^{3\tau} \left( \text{EV}(\psi) + \frac{\rho}{1 - \delta} \right) \leq 2e^{3\tau} \left( \|\phi\|^6 + \frac{\rho(\epsilon_0)}{1 - \delta(\epsilon_0)} \right).$$

By adopting the Arzelà-Ascoli tightness characterization, we can show that the law \{\( P_t(\phi, \cdot) \)\}_{t \geq 0} of the segment process \( \{X_t^\phi\}_{t \geq 0} \) is tight in \((C, B(C))\) (see [17, Theorem 2.3]), which implies that \( \{Q_t(\cdot) := \frac{1}{t} \int_0^t P_s(\phi, \cdot) ds\}_{t \geq 0} \) is tight in \((C, B(C))\). Then applying the Krylov-Bogoliubov theorem, we can conclude that \( \{Q_t(\cdot)\}_{t \geq 0} \) has at least a weakly convergent limit \( \mu^{e,\phi} \) which is stationary for the segment process \( \{X_t^\phi\}_{t \geq 0} \). We omit the details and refer the readers to the proof of [17, Theorems 2.3 and 3.2] and [12, Theorem 3.1.1, p. 21].

For each \( 0 < \epsilon \leq \epsilon_0, \) the following assumption is a sufficient condition to guarantee the uniqueness of a stationary measure.

(A3) The diffusion matrix \( \sigma \sigma^T \) is uniformly elliptic in \( C, \) i.e., there is a constant \( \lambda > 0 \) such that \( x^T \sigma(\phi)(\sigma(\phi))^T x \geq \lambda |x|^2 \) for all \( \phi \in C \) and \( x \in \mathbb{R}^m. \)

The next result is implicitly proved in [21, Theorem 3.1].

**Lemma 5.4.** Under the assumptions (A1)–(A3), there exists a unique stationary measure (5.4), i.e., \( \mu^{e,\phi} \equiv \mu^e \) is independent of \( \phi \) for each \( 0 < \epsilon \leq \epsilon_0. \)

**Remark 5.5.** If \( \sigma \) satisfies (A2) and (A3), then \( \sigma \) admits a continuous bounded right inverse, i.e., there exists a continuous function \( \delta : C \to \mathbb{R}^{k \times m} \) such that for all \( \phi \in C, \sigma(\phi)\delta(\phi) = I_m, \) and \( \sup_{\phi \in C} |\delta(\phi)| < \infty. \)

Actually, it is easy to see that \( \delta = \sigma^T(\sigma \sigma^T)^{-1}. \)

Let \( B_R(0) := \{ \phi \in C : \|\phi\| \leq R \} \) for a given \( R > 0. \) The following result gives the tightness for the family of stationary measures \( \{\mu^{e,\phi}\}_{0 < \epsilon \leq \epsilon_0, \phi \in B_R(0)}. \)

**Theorem 5.6.** Suppose the assumptions of Theorem 5.3 hold. Then the set of stationary measures \( \{\mu^{e,\phi}\}_{0 < \epsilon \leq \epsilon_0, \phi \in B_R(0)} \) is tight. If we additionally assume that (A3) holds, then the set of stationary measures \( \{\mu^e\}_{0 < \epsilon \leq \epsilon_0} \) is tight.

**Proof.** Fix \( R > 0. \) For given \( 0 < \epsilon \leq \epsilon_0 \) and \( \phi \in B_R(0), \) from the proof of Theorem 5.3, we know that there exists a sequence \( \{T_n\} \to +\infty, \) depending on \( \epsilon \) and \( \phi, \) such that

$$\frac{1}{T_n} \int_0^{T_n} P(X_{s+\phi}^\epsilon \in \cdot) ds \overset{w}{\to} \mu^{e,\phi}(), \quad \text{as} \ n \to \infty.$$  

(5.17)

Then

$$\mu^{e,\phi}(\varphi \in C : |\varphi(0)| > \lambda)$$

$$\leq \liminf_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} P(|X_{s+\phi}^\epsilon(0)| > \lambda) ds$$

$$= \liminf_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} P(|X_{s+\phi}^\epsilon(s)| > \lambda) ds$$

$$\leq \sup_{0 < \epsilon \leq \epsilon_0} \sup_{t \geq 0} E|X_t,\phi(t)|^6$$

$$\leq \frac{2e^{3\tau}(R^6 + M)}{\lambda^6} \to 0 \quad \text{uniformly in} \ 0 < \epsilon \leq \epsilon_0 \quad \text{and} \ \phi \in B_R(0) \quad \text{as} \ \lambda \to \infty.$$
From the Kolmogorov’s tightness argument (see [32, Problem 2.4.11, p. 64]) to obtain the first inequality, Chebyshev’s inequality to the second inequality, and (5.7) to the last inequality. This means

\[
\lim_{\lambda \to \infty} \sup_{0 < c \leq \epsilon_0, \phi \in B_R(0)} \mu^{c, \phi} = 0.
\]

(5.18)

For every \( \gamma > 0 \), by a similar argument as used in the above first inequality, we have

\[
\mu^{c, \phi} \{ \varphi \in \mathcal{C} : \sup_{-t \leq u < v \leq 0} |\varphi(v) - \varphi(u)| > \gamma \}
\]

\[
\leq \liminf_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{P} \left( \sup_{-t \leq u < v \leq 0} |X^{c, \phi}_t(v) - X^{c, \phi}_t(u)| > \gamma \right) dt
\]

\[
= \liminf_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{P} \left( \sup_{-t \leq u < v \leq 0} |X^{c, \phi}(t + v) - X^{c, \phi}(t + u)| > \gamma \right) dt
\]

\[
\leq \sup_{t \geq \tau} \mathbb{P} \left( \sup_{-t \leq u < v \leq 0} |X^{c, \phi}(t + v) - X^{c, \phi}(t + u)| > \gamma \right)
\]

\[
= \sup_{t \geq \tau} \left( \sup_{-t \leq u < v \leq 0} \int_u^v |b(X^{c, \phi}_s)| ds > \frac{\gamma}{2} \right)
\]

\[
\leq \sup_{t \geq \tau} \left( \sup_{-t \leq u < v \leq 0} \int_u^v \left| \frac{\sigma}{\epsilon} \right| ds \right) + \sup_{t \geq \tau} \left( \sup_{-t \leq u < v \leq 0} \left| \int_u^v \sigma(X^{c, \phi}_s) dW(s) \right| > \frac{\gamma}{2} \right).
\]

Here, \( b(\phi) := -B(0) + Ag(\phi) \), which maps bounded sets in \( \mathcal{C} \) into bounded sets in \( \mathbb{R}^m \). From this fact and (5.7), it is easy to yield that

\[
\lim_{\lambda \to \infty} \sup_{0 < c \leq \epsilon_0, \phi \in B_R(0)} \sup_{t \geq 0} \mathbb{P} \left( \sup_{-t \leq u < v \leq 0} \int_u^v |b(X^{c, \phi}_s)| ds > \frac{\gamma}{2} \right) = 0.
\]

Let \( J^v(v) := \epsilon \int_0^v \sigma(X^{c, \phi}_r) dW(s), v \geq 0 \). The continuity of \( \sigma \) implies that \( \{J^v(v), v \geq 0\} \) is a continuous \( m \)-dimensional local martingale. Then by the Burkholder-Davis-Gundy inequality, Hölder’s inequality, (A2), \( C_r \)-inequality and Fubini’s theorem, we have for any \( t > s \geq 0 \),

\[
\mathbb{E}[J^v(t) - J^v(s)]^6 = \mathbb{E} \left[ \epsilon \int_s^t \sigma(X^{c, \phi}_r) dW(r) \right]^6
\]

\[
\leq \tilde{C}_6 (2e^{3\tau} (R^6 + \tilde{M}) + 1 |t - s|^3),
\]

where \( \tilde{C}_6 = 2^{5/6} C_6 (L^6 + \|\sigma(0)\|_2^6) \) is a constant independent of \( \epsilon \). This means that there exists some positive constant \( c \) such that

\[
\sup_{0 < c \leq \epsilon_0, \phi \in B_R(0)} \sup_{t \geq 0} \mathbb{E}[J^v_t(v) - J^v_t(u)]^6 \leq c |v - u|^3 \quad \text{for all } u, v \in [0, \tau].
\]

From the Kolmogorov’s tightness argument (see [32, Problem 2.4.11, p. 64]), we can deduce

\[
\lim_{\delta \to 0} \sup_{0 < c \leq \epsilon_0, \phi \in B_R(0)} \sup_{t \geq 0} \mathbb{P} \left( \sup_{-t \leq u < v \leq 0} |J^v(v) - J^v(u)| > \frac{\gamma}{2} \right) = 0.
\]

In other words,

\[
\lim_{\delta \to 0} \sup_{0 < c \leq \epsilon_0, \phi \in B_R(0)} \sup_{t \geq 0} \mathbb{P} \left( \epsilon \sup_{-t \leq u < v \leq 0} \left| \int_u^v \sigma(X^{c, \phi}_s) dW(s) \right| > \frac{\gamma}{2} \right) = 0.
\]
Therefore we obtain
\[ \lim_{\delta \to 0} \sup_{0 < \epsilon \leq \epsilon_0, \phi \in B_R(0)} \mu^\epsilon,\phi \{ \phi \in \mathcal{C} : \sup_{-\tau \leq x, y \leq 0} |\varphi(v) - \varphi(u)| > \gamma \} = 0. \] (5.19)

Consequently, the conclusion follows immediately from (5.18) and (5.19).

\[ \] \[ \]

**Example 5.7** (Hopfield neural network models with noise). We consider the stochastic delayed Hopfield equations
\[ dX^\epsilon(t) = [-BX^\epsilon(t) + Ag(X^\epsilon(t - \tau))] dt + \epsilon\sigma(X^\epsilon) dW(t), \] (5.20)
where \( B = \text{diag}(b_1, \ldots, b_m) \), \( A = (a_{ij})_{m \times m} \), \( g(x) = (g_1(x_1), \ldots, g_m(x_m))^T \), and \( \sigma(\phi) = (\sigma_{ij}(\phi))_{m \times m} \) is an \( m \times m \) matrix defined on \( \mathcal{C} \).

Hopfield-type neutral networks have many applications to parallel computation and signal processing involving the solution of optimization problems. It is often required that the network should have a unique stationary solution that is globally attractive. For this purpose, we present the following theorem.

**Theorem 5.8.** Assume (A1)-(A4), and (A4) there exists some constant \( M > 0 \) such that \( |g(x)| \leq M \) for all \( x \in \mathbb{R}^m \). If \( b \) satisfies the condition (5.6), where \( b = \min_{1 \leq i \leq m} b_i \), then for each \( \epsilon \in (0, \epsilon_0] \), the system (5.20) has a unique invariant measure \( \mu^\epsilon \) for the segment process \( \{X^\epsilon_t\}_{t \geq 0} \). Furthermore, \( \mu^\epsilon \) converges weakly to \( \delta_p \) as \( \epsilon \to 0 \), where \( p \) is a globally asymptotically stable equilibrium for the differential equations (5.20) with \( \epsilon = 0 \).

**Proof.** Let \( \tilde{b}(\phi) = -B\phi(0) + Ag(\phi(-\tau)) \), \( \phi \in \mathcal{C} \). It is easy to see that \( \tilde{b} \) is globally Lipschitz continuous in \( \mathcal{C} \). The existence and uniqueness of invariant measures, the tightness of the set \( \{\mu^\epsilon, 0 < \epsilon \leq \epsilon_0\} \) follow from Lemma 5.4 and Theorem 5.6, respectively. The probability convergence condition holds by Corollary 5.2. Combining Assumption (A4), we get that the unperturbed system has a unique equilibrium \( \hat{p} \) which is globally asymptotically stable (see [48, Theorem 2.4]), where we have used the fact that the operator norm is less than the trace norm. The final assertion follows from Theorem 2.1.

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Appendix A  The Poincaré recurrence theorem for the continuous dynamical system

In this appendix, we give a full proof of the Poincaré recurrence theorem for a semiflow (or flow) on the separable metric space \((M, \rho)\). The original idea is borrowed from Mañé [40] and Hirsch [25].

Throughout this appendix we assume that \(\Phi : \mathbb{R}_+ \times M \to M\) is a mapping with the following properties:

(i) \(\Phi_t(x)\) is continuous, for all \(x \in M\);
(ii) \(\Phi_t(\cdot)\) is Borel measurable, for all \(t \in \mathbb{R}_+\);
(iii) \(\Phi_0 = \text{id}, \Phi_t \circ \Phi_s(x) = \Phi_{t+s}(x)\), for all \(t, s \in \mathbb{R}_+\). Here, \(\circ\) denotes composition of mappings.

Let \(x \in M\). Then the \(\omega\)-limit set of \(x\) is defined by \(\omega(x) = \{y \in M : \text{for every neighborhood } U \text{ of } y, \text{ and for every } k \in \mathbb{N}, \text{ there exists } s > k \text{ such that } \Phi_s(x) \in U\}\).

We note that \(\Phi_t(x)\) is continuous, thus \(\omega(x) = \{y \in M : \text{for every neighborhood } U \text{ of } y, \text{ and for every } k \in \mathbb{N}, \text{ there exists } s \geq k \text{ and } s \in \mathbb{Q} \text{ such that } \Phi_s(x) \in U\}\).

By using the semigroup properties of \(\Phi\), it is easy to see that \(\omega(x) = \omega(\Phi_t(x))\) for every \(t \in \mathbb{R}_+\).

Let \((M, \mathcal{B}(M), \mu)\) be a probability space and \(\mu\) be a \(\Phi\)-invariant probability measure, i.e., \(\mu \circ \Phi_t^{-1} = \mu\) for all \(t \in \mathbb{R}_+\), where \(\mathcal{B}(M)\) is the \(\sigma\)-algebra of Borel sets in \(M\).

The proof of the following Poincaré recurrence theorem follows the lines of argument for the discrete time measurable mapping case (see, e.g., [40, pp. 28–29]).

**Theorem A.1.** Let \(M\) be a separable metric space and \(\mu\) be \(\Phi\)-invariant. Then \(\mu(\mathcal{B}(\Phi)) = 1\), where \(\mathcal{B}(\Phi) = \{x \in M : x \in \omega(x)\}\) denotes the Birkhoff center of \(\Phi\).

**Proof.** For given \(t \in \mathbb{R}_+\), let \(A\) be an open set in \(M\) and

\[A_0 = \{x \in A : \forall k \in \mathbb{N}, \exists s \geq k \text{ and } s \in \mathbb{Q} \text{ such that } \Phi_s \circ \Phi_t(x) \in A\}.
\]

We claim that \(A_0 \in \mathcal{B}(M)\) and \(\mu(A) = \mu(A_0)\). In fact, for every \(k \in \mathbb{N}\), let

\[C_k = \{x \in A : \Phi_s \circ \Phi_t(x) \notin A, \forall s \geq k \text{ and } s \in \mathbb{Q}\}.
\]

It is easy to see that \(A_0 = A \setminus \bigcup_{k=1}^{\infty} C_k\). Let \(x \in A_0\) if and only if \(x \in A\) and \(\forall k \in \mathbb{N}, \exists s \geq k \text{ and } s \in \mathbb{Q}\) such that \(\Phi_s \circ \Phi_t(x) \in A\) if and only if \(x \in A\) and \(x \notin \bigcup_{k=1}^{\infty} C_k\). Note that \(C_k = A \setminus \bigcup_{s \in k, s \in \mathbb{Q}} (\Phi_s \circ \Phi_t)^{-1}(A)\) which shows that \(C_k \in \mathcal{B}(M)\) and implies that

\[C_k \subset \bigcup_{s \geq 0} \Phi_s^{-1}(A) \setminus \bigcup \Phi_{s+t}^{-1}(A) = \bigcup s \geq 0, s \in \mathbb{Q} \Phi_s^{-1}(A) \setminus \bigcup s \geq k, s \in \mathbb{Q} \Phi_{s+t}^{-1}(A),
\]

where we have used the fact that \(\Phi_t(x)\) is continuous and \(A\) is an open set. From (A.1), we have

\[
\mu(C_k) \leq \mu \left( \bigcup_{s \geq 0, s \in \mathbb{Q}} \Phi_s^{-1}(A) \right) - \mu \left( \bigcup_{s \geq k, s \in \mathbb{Q}} \Phi_{s+t}^{-1}(A) \right)
\]
\[
\leq \mu \left( \bigcup_{s \geq 0, s \in \mathbb{Q}} \Phi_s^{-1}(A) \right) - \mu \circ \Phi_t^{-1} \left( \bigcup_{s \geq k, s \in \mathbb{Q}} \Phi_s^{-1}(A) \right)
\]
\[
= \mu \left( \bigcup_{s \geq 0, s \in \mathbb{Q}} \Phi_s^{-1}(A) \right) - \mu \circ \Phi_k^{-1} \left( \bigcup_{s \geq 0, s \in \mathbb{Q}} \Phi_s^{-1}(A) \right)
\]
\[
= 0.
\]

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Therefore, we get $A_0 \in \mathcal{B}(M)$ and $\mu(A_0) \geq \mu(A) - \sum_{k=1}^{\infty} \mu(C_k) = \mu(A)$. This completes the proof of the claim.

Since, furthermore, $M$ is a separable metric space, we can find the countable basis $\{U_n\}_{n \in \mathbb{N}}$ of $M$ such that $\lim_{n \to \infty} \text{diam}(U_n) = 0$ and $\bigcup_{n=1}^{\infty} U_n = M$ for every $k \in \mathbb{N}$. Let

$$\hat{U}_n = \{x \in U_n : \forall k \in \mathbb{N}, \exists s \geq k \text{ and } s \in \mathbb{Q} \text{ such that } \Phi_s \circ \Phi_t(x) \in U_n\}$$

for every $n \in \mathbb{N}$. From the above claim, we have $\hat{U}_n \in \mathcal{B}(M)$ and $\mu(U_n \setminus \hat{U}_n) = 0$. Let

$$\hat{M} = \limsup_{n \to \infty} \hat{U}_n = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{\infty} \hat{U}_n.$$

Then we have

$$\mu(M \setminus \hat{M}) = \mu \left( \bigcup_{k=0}^{\infty} \left( M \setminus \bigcup_{n=k}^{\infty} \hat{U}_n \right) \right)$$

$$= \mu \left( \bigcup_{k=0}^{\infty} \left( \bigcup_{n=k}^{\infty} U_n \setminus \hat{U}_n \right) \right)$$

$$\leq \mu \left( \bigcup_{k=0}^{\infty} \left( U_n \setminus \hat{U}_n \right) \right)$$

$$= 0.$$

This means $\mu(\hat{M}) = 1$. Due to this fact it is sufficient to prove that $\hat{M} \subset \{x \in M : x \in \omega(x)\}$ which we now prove. Let $x \in \hat{M}$. For any $r > 0$, since $\lim_{n \to \infty} \text{diam}(U_n) = 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ $\text{diam}(U_n) < \frac{r}{2}$. Note that $x \in \bigcup_{n=1}^{\infty} U_n$. Hence there exists $n \geq N$ such that $x \in U_n \subset U_n$. It follows that $U_n \subset B(x, r)$. This implies that $\forall k \in \mathbb{N}, \exists s \geq k$ and $s \in \mathbb{Q}$ such that $\Phi_s \circ \Phi_t(x) \in U_n \subset B(x, r)$, i.e., $x \in \omega(\Phi_t(x)) = \omega(x)$.

This theorem immediately implies the following assertion.

**Remark A.2.** Let $\text{supp}(\mu)$ denote the support of $\mu$, where $\mu$ is $\Phi$-invariant. Then $\text{supp}(\mu) \subset B(\Phi)$.

If we additionally assume that $\Phi_t(\cdot)$ is continuous for every $t \in \mathbb{R}_+$, i.e., $\Phi$ is a semiflow (if we can replace $\mathbb{R}_+$ by $\mathbb{R}$, then $\Phi$ defines a flow), then we can prove the following assertion.

**Proposition A.3.** $\text{supp}(\mu)$ is forward invariant. If, in addition, $\text{supp}(\mu)$ is a compact set and $\Phi_t(\cdot): M \rightarrow M$ is an injective mapping (or homeomorphism), then $\text{supp}(\mu)$ is invariant.

**Proof.** Let $H := \text{supp}(\mu)$. The continuity of $\Phi_t(\cdot)$ implies that $\Phi_t^{-1}(H)$ is a closed set. By the invariance of $\mu$, $\mu(\Phi_t^{-1}(H)) = \mu(H) = 1$. This implies $H \subset \Phi_t^{-1}(H)$. Therefore, $\Phi_t(H) \subset H$.

The injectivity of $\Phi_t$ implies that $\Phi_t^{-1}(\Phi_t(H)) = H$. Thus

$$1 = \mu(H) = \mu(\Phi_t^{-1}(\Phi_t(H))) = \mu(\Phi_t(H)).$$

Note that $\Phi_t(H)$ is a closed set (more precisely, a compact set) from the fact that $H$ is a compact set and $\Phi_t(\cdot)$ is continuous. Therefore, we have $H \subset \Phi_t(H)$.

In the following we assume that $H := \text{supp}(\mu)$ is a closed invariant set. Let $\Phi|_H$ denote the restricted semiflow. Then by the Poincaré recurrence theorem, we obviously have the following corollary.

**Corollary A.4.** It holds that $H = B(\Phi|_H)$. This means that every point of $H$ is recurrent for $\Phi|_H$.

**Proof.** The fact that $H \subset B(\Phi|_H) \subset H$ follows directly from the Poincaré recurrence theorem and the definition of the support of the invariant measure $\mu$.

If, in addition, $H$ is a compact set, we refer to Hirsch [25] for additional properties of the support of $\mu$ in the case that $\mu$ is ergodic. Recall that the $\Phi$-invariant probability measure $\mu$ is said to be ergodic if for any $A \in \mathcal{B}(M)$ with the property $\Phi_t(A) = A$ for all $t \in \mathbb{R}_+$, we have either $\mu(A) = 0$ or $\mu(A) = 1$.

A subset $A \subset M$ is an attractor if $A$ is compact and invariant ($\Phi_t(A) = A$) and contained in an open set $N \subset M$ such that

$$\lim_{t \to \infty} \text{dist}(\Phi_t(x), A) = 0$$

uniformly in $x \in N$. 

Furthermore, if there is an attractor that contains all $\omega$-limit points, then we call that $\Phi$ is *dissipative*.

A nonempty compact invariant set $A \subset M$ is called *attractor-free* if the restricted flow $\Phi|_A$ has no attractor other than $A$ itself. By a result of Conley [10] $A$ is attractor-free which is equivalent to that $A$ is connected and every point of $A$ is chain recurrent for $\Phi|_A$. The definition of chain recurrent set we refer the reader to [2] since this notion will not be used here. Meanwhile, for a detailed discussion about this relation we refer to Benaïm [3, p. 23].

Furthermore, for this compact invariant set $H := \text{supp}(\mu)$, the following result is proved in [4] and [25], respectively.

**Proposition A.5.** Each component of $H$ is attractor-free. In addition, if $\mu$ is ergodic, then $H$ itself is attractor-free.

If $\Phi$ is a strongly monotone semiflow in an ordered Banach space $M$, and let $A$ denote attractor-free, then by more detailed structural analysis of $A$, Hirsch [25] pointed out that either $A$ is unordered, or $A$ is contained in a totally ordered, compact arc of equilibria.