Unique additive information measures -
Boltzmann-Gibbs-Shannon, Fisher and beyond

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Abstract

It is proved that the only additive and isotropic information measure that can depend on the probability distribution and also on its first derivative is a linear combination of the Boltzmann-Gibbs-Shannon and Fisher information measures. Power law equilibrium distributions are found as a result of the interaction of the two terms. The case of second order derivative dependence is investigated and a corresponding additive information measure is given.

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I. INTRODUCTION

In 1957 in his famous paper on information theory and statistical mechanics, Jaynes suggested to look statistical mechanics as a form of statistical inference. He argued that the usual rules of statistical mechanics are justified independently of experimental verification and additional physical arguments, because "they still represent the best estimates that could have been made on the basis of information available" \[1\]. He recognized the physical importance of the train of thought of Shannon, where it was proved that the logarithmic form of the information measure is a consequence of some simple properties that any information measure should have. Based on this observation, he suggested to start statistical physics from a maximum entropy principle.

Dealing with a discrete probability space where a variable \(x\) can assume the discrete values \((x_1, ..., x_n)\) with the corresponding probabilities \((p_1, ..., p_n)\) there exist a function \(H(p_1, ..., p_n)\), which is

- continuous,

- with equal probabilities is monotone increasing with \(n\),

- satisfies the composition law.

These conditions are those that we would expect from a measure of information. With the above properties the function \(H(p_1, ..., p_n) = -k \sum_{i=1}^{n} p_i \ln p_i\) is unique up to the positive multiplier \(k\).

Later these conditions were investigated, generalized and clarified extensively both from mathematical and from physical points of view. It turned out that continuity is not so important. Monotonicity can be replaced by concavity and determines the sign of the function only. The most important assumption from a physical point of view is the composition law. Later the composition law was reformulated in a more convenient way as additivity. Rényi recognized that the only possibility to find a different additive measure of information is to generalize also the method of averaging \[2\].

The uniqueness is the key feature that connect the information measure to the physical entropy and ensures that the consequences of the macroscopic Second Law (first of all the existence of the universal, absolute temperature) are valid for the quantities of the statistical physical approach. Because of the uniqueness, the arising statistics is the same, independently of the microscopic dynamics.
As additivity gives the connection between the statistical and thermodynamic theories, its investigation is particularly interesting if one would like to explain phenomena that is seemingly out of the framework of traditional methods of statistical physics [3, 4, 5, 6].

In all previously mentioned researches it was assumed explicitly that the entropy is a local function of the variables. In this paper we weaken this assumption and look for additive entropy functions interpreted on a continuous state space, that depend not only on the probability distribution but also on the derivatives of the mentioned probability distribution. In the following we will prove generalizations of the statement of Shannon for derivative dependent information measures. In the next section we prove, that the unique additive, continuously differentiable entropy/information measure that depends only on the first derivatives of the probability distribution is a linear combination of the Boltzmann-Gibbs-Shannon and the Fisher information measures. In the third section we investigate the physical meaning of the constructed unique information measure. Then we construct an additive information measure that contains second order derivatives. Finally there are some conclusions.

II. WEAKLY NONLOCAL INFORMATION MEASURES: FIRST ORDER

Let us consider an $n$ dimensional continuous probability space $X \subset \mathbb{R}^n$, where the probability measure can be given by a continuously differentiable nonnegative function, the probability density $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$, which is normalized,

$$\int_X f(x)dx = 1. \tag{II.1}$$

A first order weakly nonlocal information measure is a function $s(f, Df)$ of the probability density $f$ and its derivative $Df$ with some expected properties. An information measure is positive, increases with increasing uncertainty, and is additive for independent sources of uncertainty. In case of derivative dependent information measures it is convenient to require the isotropy of $s$, too. These conditions can be formulated as follows

1. Isotropy. An isotropic function $s$ of $f$ and $Df$ has the following form

$$s(f, Df) = \hat{s}(f, (Df)^2). \tag{II.2}$$
2. Additivity. For the sake of simplicity we restrict ourself for two independent distribution functions $f_1(x_1)$ and $f_2(x_2)$ defined on $X = X_1 \times X_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad n_1 + n_2 = n$. The generalization to finite number of distributions is straightforward. Then additivity requires

$$s_n(f_1f_2, D(f_1f_2)) = s_{n_1}(f_1, Df_1) + s_{n_2}(f_2, Df_2), \quad (\text{II.3})$$

where the subscripts denote the different dimensions of the domains.

Without isotropy additivity cannot be formulated easily because the domain of the function $\hat{s}$ is the same on both sides of the above formula. Although most probability distributions in physics are defined on spaces that are highly anisotropic, here we restrict ourselves to isotropic information measures on isotropic state spaces and leave that problem for further investigations. Fortunately, in the simplest situation, when the state space is the Descartes product of isotropic subspaces (position-momentum) one can keep the simple formulation of additivity with some straightforward assumption.

For independent probability distributions the unified probability density $f(x_1, x_2)$ is the product of the probability densities $f_1(x_1)$ and $f_2(x_2)$. Thus, we have $Df(x_1, x_2) = (f_2(x_2)Dx_1, f_1(x_1))$ and omitting the variables $x_1$ and $x_2$ $(Df)^2 = (f_2Df_1)^2 + (f_1Df_2)^2$. As a consequence, for isotropic information measures the additivity requirement can be written as

$$\hat{s}(f_1f_2, (f_2Df_1)^2 + (f_1Df_2)^2) = \hat{s}(f_1, (Df_1)^2) + \hat{s}(f_2, (Df_2)^2). \quad (\text{II.4})$$

Differentiating the above equality by $(Df_1)^2$ and $(Df_2)^2$, respectively we have that

$$f_2^2 \partial_2 \hat{s}(f_1f_2, (f_2Df_1)^2 + (f_1Df_2)^2) = \partial_2 \hat{s}(f_1, (Df_1)^2),$$

$$f_1^2 \partial_2 \hat{s}(f_1f_2, (f_2Df_1)^2 + (f_1Df_2)^2) = \partial_2 \hat{s}(f_2, (Df_2)^2).$$

Here $\partial_2$ denotes the partial derivative of $\hat{s}$ by its second argument. Therefore

$$f^2 \partial_2 \hat{s}(f, (Df)^2) = -\kappa_1 = \text{const.},$$

hence

$$\hat{s}(f, (Df)^2) = -\kappa_1 \frac{(Df)^2}{f^2} + \hat{s}(f). \quad (\text{II.5})$$

Here $\hat{s}$ is an arbitrary function (the local part of the entropy). Repeating the above train of thought with the derivatives by the first argument of $\hat{s}$, one finds that

$$f \partial_1 \hat{s}(f) = -\kappa = \text{const.}$$
Consequently, \( \bar{s}(f) = -\kappa \ln f + s_0 \), where \( s_0 = 0 \) by further applying additivity. Therefore, the most general isotropic and additive first order weakly nonlocal information measure is
\[
s(f, Df) = -\kappa_1 \frac{(Df)^2}{f^2} - \kappa \ln f.
\] (II.6)

The first term has the form of a Fisher information \[7, 8\] and the second term has the form of a Shannon information measure. It is clear from the previous calculations that (II.6) is unique with the above requirements (isotropy and additivity).

### III. POWER LAW TAILS IN MICROCANONICAL AND CANONICAL ENSEMBLES

There are several attempts to find the physical significance of Fisher information (see e.g. \[9, 10\]). The observation in the previous section puts these investigations into a new light. Accepting Jaynes approach in suggesting the central role of information in statistical physics one should require the extremum of the two terms together. However, even if we accept the idea of Jaynes, that the unique information measures are important from a physical point of view there are several important questions to be answered. E. g. What is the physics behind the second term? What could be the value of the constant \( \kappa_1 \)?

Let us consider a classical one dimensional ideal gas where the Hamiltonian is given as \( H(p) = \frac{p^2}{2m} \), where \( p \) is the momentum and \( m \) is the mass of the particles. In this case, according to the maximum entropy principle one should find the maximum of the average of the entropy density (II.6) subject to the constraints of fixed average energy and normality. Therefore we face to the following variational problem:
\[
\int \left( -\kappa_1 \frac{(Df)^2}{f} - \kappa f \ln f \right) dp - \beta \left( \int f \frac{p^2}{2m} dp - E \right) - \alpha \left( \int f dp - 1 \right) = \text{extremum}
\] (III.1)

In this case the partition function formalism does no help, the above variational problem leads to the following Euler-Lagrange equation for \( R = \sqrt{f} \):
\[
4\kappa_1 R'' - 2\kappa R^2 \ln R + \frac{\beta}{2m} p^2 R + (\alpha - \kappa)R = 0.
\] (III.2)

Here the dash denotes the derivation \( R' = \frac{dR}{dp} \). The corresponding natural boundary condition can be interpreted as an entropy current \( J_s = R' \delta R \) [11].
For any positive $\kappa_1$ the solutions of the above equation are different of the classical Maxwell-Boltzmann distribution, but the properties of the distribution are similar. Let us choose $\frac{2}{\kappa_1} = 0.1$, $\frac{\beta}{m\kappa_1} = 1$ and $\frac{2-\kappa}{4\kappa_1} =: \Lambda$. The symmetric solutions of the above equation with the condition $R'(0) = 0$ have finite support if $\alpha > 0.9897$, the value where we can get back a Maxwell-Boltzmann like Gaussian distribution. Above that value the function have power law tail of the form $R_{\text{tail}}(p) = D \pm Cp^{\gamma}$ as one can see from Figure 1.

![Figure 1](image)

**FIG. 1:** The square root of the distribution function $f(p)$ for different values of $\Lambda = (1, 1.1, 1.2, 1.3, 1.5, 2, 5)$.

We have got a one parameter family of power law tail distributions parameterized by the Lagrange multiplier $\alpha (\Lambda)$. The parameter can be regarded to different nonzero entropy currents at the boundary as one can see from the boundary conditions defined above. The arising family of power law tail distributions are different form the distributions of nonadditive (nonextensive) statistics of Tsallis, Beck-Cohen or Kaniadakis $[3, 6, 12]$ and were defined by a unique and additive information measure.

A related answer to the above questions emerges considering quantum mechanics as a particular application and focusing on the concavity properties of the Fisher term. With a suitable reinterpretation of the terms one can recognize a time independent Schrödinger equation of a harmonic oscillator in $[11, 12]$ if $\kappa = 0$. The connection
of quantum mechanics to Fisher information was pointed out by several independent researches \[13, 14\]. We may determine the \( \kappa_1 \) constant in these systems. E.g. for a single particle system one can get, that \( \kappa_1 = h^2 \).

The understanding the role of Fisher information in quantum mechanics can give some clues to the further understanding the physics behind (II.6). In quantum mechanics only the Fisher term appears and one should explain the missing Boltzmann-Gibbs-Shannon term in an information theoretical approach. In this respect Hall and Regginato argued with strengthening some basic laws of quantum mechanics and introduced the exact uncertainty principle \[15, 16\]. Ván and Fülöp suggested mass scale invariance, requiring the possibility of particle interpretation for the probability distribution \[17\].

On the other hand Białynicki-Birula and Mycielski gives an example that that the quantum potential could be supplemented by a Boltzmann-Gibbs-Shannon term \[18\]. The solution of the supplemented Schrödinger equation gives non dispersive free solutions, the so-called ”Gaussons”, as a result of the additional logarithmic term.

IV. WEAKLY NONLOCAL INFORMATION MEASURES: SECOND ORDER

One can ask about properties of information measures depending on higher order derivatives of the distribution function. Here we investigate the second order case. The requirements are similar as previously

1. Isotropy. According to the representation theorems of isotropic functions that depend on a vector \((Df)\) and a symmetric second order tensor \((D^2f)\) we can write \[19\]:

\[
s(f, Df, D^2f) = \hat{s}(f, (Df)^2, Df \cdot D^2f \cdot Df, Df \cdot D^2f \cdot D^2f \cdot Df, ..., Df \cdot (D^2f)^n \cdot Df, \quad Tr(D^2f), Tr(D^2f \cdot D^2f), ..., Tr((D^2f)^n)).
\]

2. Additivity. Here we require that

\[
s(f_1f_2, D(f_1f_2), D^2(f_1f_2)) = s(f_1, Df_1, D^2f_1) + s(f_2, Df_2, D^2f_2)
\]
As previously, we need to consider isotropy in the formulation of additivity. Let us observe, that the above form (IV.1) is restricted very much by the requirement of additivity. Simple calculations show that the second order nonlocal information measure is more difficult than the first order one. Its form depend on the dimension of the phase space. As an example I give the general additive version of the following isotropic function (this is the unique general form of weakly nonlocal information measure in three dimension)

\[
s_3(f, Df, D^2f) = s_3(f, (Df)^2, Df \cdot D^2f \cdot Df, Df \cdot (D^2f)^2 \cdot Df, Tr(D^2f),
\]

\[
Tr(D^2f)^2, Tr(D^2f)^3). \quad (IV.1)
\]

One can derive that

\[
s_3(f, Df, D^2f) = -\kappa \ln f - \kappa_1 \frac{(Df)^2}{f^2} - (\kappa_2 + \kappa_5) \frac{(Df)^4}{f^4} + (\kappa_3 + \kappa_6) \frac{(Df)^6}{f^6} - \\
\kappa_2 \frac{1}{f^3} Df \cdot D^2f \cdot Df + (2\kappa_3 + 3\kappa_6) \frac{(Df)^2}{f^5} Df \cdot D^2f \cdot Df - \kappa_3 \frac{1}{f^4} Df \cdot (D^2f)^2 \cdot Df - \kappa_4 \frac{1}{f} Tr(D^2f) - \\
\kappa_5 \frac{1}{f^2} Tr(D^2f)^2 - \kappa_6 \frac{1}{f^3} Tr(D^2f)^3. \quad (IV.2)
\]

The concavity properties of the above function are not straightforward.

V. CONCLUSIONS

There are several attempts to understand the reason of the appearance of Fisher information in different disciplines of physics [9, 13, 20, 21]. The usual justification and interpretation is based on estimation theory. The above proof of uniqueness explains, why any dynamical background that preserves the additivity - gives the same Fisher like form, independently of the estimation theoretical background.

We have seen that for information measures with second order derivatives the number of additive terms depends on the dimension of the probability (phase) space. Therefore the physical significance of information measures containing higher than first order derivatives is dubious.

There are more questions than answers in this work. However, in the relationship of thermodynamics and statistical physics the idea of Jaynes is the key of understanding that deserves further investigations.
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