ACCESSIBLE HYPERBOLIC COMPONENTS IN
ANTI-HOLOMORPHIC DYNAMICS

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ABSTRACT. The tricorn, the connectedness locus of anti-holomorphic qua-
dratic family is known to be non-locally connected. The boundary of every
hyperbolic components of odd periods has inaccessible arcs from the comple-
ment of the tricorn. As the period increases, the decorations become more and
more complicated and it seems natural to think that every hyperbolic compo-
nent of sufficiently large and odd period is inaccessible. Contrary to such an
anticipation, we show that there exists infinitely many accessible hyperbolic
components for the tricorn.

1. INTRODUCTION

The Mandelbrot set \( M \) is the connectedness locus of the (holomorphic) quadratic
family \( Q_c(z) = z^2 + c \), i.e., the set of parameters \( c \) for which the filled Julia set
\[ K(Q_c) = \{ z \in \mathbb{C} | \{Q^n_c(z)\}_{n \geq 0}: \text{bounded} \} \]
is connected. For the anti-holomorphic quadratic family \( f_c(z) = \overline{z}^2 + c \), the con-
nectedness locus \( M^* = \{ c \in \mathbb{C} | K(f_c): \text{connected} \} \) is called the tricorn [MI92] or
the Mandelbar set [CHRSC89].

It is well-known that \( M \) is connected and full [DHS14], and the famous MLC
conjecture asserts that \( M \) is locally connected. If the MLC conjecture holds, then
by Carathéodory’s theorem, \( \partial M \) is a quotient of the circle and hence every point
in the boundary is the landing point of some parameter ray. In particular, every
point in \( \partial M \) is accessible from the complement of \( M \); that is, there exists a path
in \( \mathbb{C} \setminus M \) which converges to that point.

On the contrary, it is known that \( M^* \) is not locally connected. Indeed, there are
real-analytic arcs of positive length in \( \partial M^* \) which consists of inaccessible points
from the complement [HS14]. Such arcs are contained in the boundary of hyperbolic
components of odd period. Moreover, it is also known that every parameter ray for
\( M^* \) accumulating to a hyperbolic component of odd period at least three does not
converge to a point [MI16].

Decorations of \( M^* \) accumulating to a hyperbolic component of odd period be-
come more and more complicated as the period increases, so it seems natural to
think that such a hyperbolic component has no accessible boundary point if the
period is sufficiently large. In fact, the accessibility of a boundary point \( c \) is related
to the condition that the basin of infinity, when projected into the repelling Ecalle
cylinder of the periodic point for \( f_c \), contains a horizontal circle (we give a sufficient
condition in Theorem 2.8). However, since the projection of the basin of infinity
into the Ecalle cylinder is an annulus, whose modulus is inversely reciprocal to the
period of the hyperbolic component, it is unlikely to contain a horizontal circle
when the period is large.
In this article, we prove that this speculation is in fact incorrect:

**Theorem 1.1.** There exist infinitely many hyperbolic components of odd periods in $\text{Int } \mathcal{M}^*$ which are accessible from $\mathbb{C} \setminus \mathcal{M}^*$.

Since the second iterate $f_2^2(z) = (z^2 + \bar{c})^2 + c$ is holomorphic in $z$, the situation depends on the parity of the period. For example, if a periodic point is of odd period, then its multiplier (as a periodic point for $f_2^2$) is always non-negative real. Thus the boundary of a hyperbolic component of odd period consists of maps with parabolic periodic point. In fact, the boundary arcs consists of three cusps and three arcs connecting them, which are called parabolic arcs (see [CHRSC89], [NS03] and [MNS14] for more details). When the period is even, the situation looks similar to the case of the Mandelbrot set, except possibly the root is blown-up to an arc with cusp.

The existence of such arcs consisting parabolic parameters is the source of complicated topological structure such as non-path connectivity [HS14], [IM21], non-landing parameter rays [IM16] and failure of self-similarity [IM21] (see also [IM20]). Non-path connectivity and failure of self-similarity are proved by the existence of “wiggly umbilical cords”; that is, the continuum connecting the origin to a given hyperbolic component of odd period (“the umbilical cord”) does not converge to a point. The only landing umbilical cords are those on the real line up to symmetry, and this shows that the natural dynamical correspondence (the *straightening map*) between the unions of hyperbolic components and parts of the decorations is not continuous. Accessibility and inaccessibility of hyperbolic components also shows that decorations of hyperbolic components of odd periods are topologically different in general (see Figure 1 and Figure 2).

![Figure 1. Accessible hyperbolic component of period 3 (centered at the airplane).](image)

The accessible hyperbolic components in Theorem 1.1 are centered at real parameters and accumulates to the anti-Chebyshev map $f_{-2}$. We can numerically see the decorations of the “baby tricorn-like set” centered at such a hyperbolic component converge to truncated dyadic parameter rays of the tricorn. Also in the phase space for a parameter in the baby tricorn-like set, the decorations attached to the small Julia set are also converge to truncated dyadic dynamical rays (see Figure 3). We in fact use such a convergence in the phase space to prove the accessibility.
The structure of the paper is as follows: In Section 2, we recall basic facts on the dynamics of anti-holomorphic quadratic polynomials, especially on parabolic maps and their bifurcations, and prove a sufficient condition for accessibility (Theorem 2.8). In Section 3, we prove that there exists a sequence of critically finite parameters \( c_n \to \hat{c} = -2 \), such that up to scaling, the family \( \{ f_{c_n + \epsilon t}(z) \} \) converges to the original family \( \{ f_t(z) \} \) locally uniformly (Theorem 3.3). This is a simple modification of the classical fact for the holomorphic case (see [EE85], [DH85] and [McM00]).

To show the exponential convergence of decorations to truncated dyadic rays, we need a concrete construction for hybrid conjugacies of renormalizations; we give precise construction of polynomial-like restrictions in Section 4 and hybrid conjugacies in Section 5. In Section 6, we show the hybrid conjugacies just constructed converge to the identity as \( n \to \infty \) exponentially fast. In Section 7, we define a “quasiconformal Böttcher coordinate” for renormalizations. In Section 8, we apply a result by Rivera-Letelier [RL01] on convergence of filled Julia sets at semi-hyperbolic map to show that the decorations of the small filled Julia set converge to (truncated) dyadic dynamical rays exponentially fast. In Section 9, we show the exponential convergence of the Fatou coordinate for a specific parameter. Finally in Section 10, we gather all the estimates to show the accessibility. The key fact is that while we have exponential convergence for everything, the decorations (which
Figure 3. Decorations of baby tricorn-like sets (rescaled to 5:9) and corresponding Julia sets at parabolic parameters for period 15 (top), 55 (middle), 105 (bottom).

are exponentially close to truncated dyadic rays) grow only linearly in terms of the number of iteration (or equivalently the Green potential).

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2. Preliminaries

2.1. The tricorn. We denote \( K_c = K(f_c) \) be the filled Julia set of \( f_c \) and \( J_c = \partial K_c \) be the Julia set of \( f_c \).

Let \( \text{B"ott}_c \) be the B"ottcher coordinate for \( f_c \); i.e., the conformal isomorphism defined near infinity such that

\[
\text{B"ott}_c(f_c(z)) = \frac{\text{B"ott}_c(z)^2}{z}, \quad \lim_{z \to \infty} \frac{\text{B"ott}_c(z)}{z} = 1.
\]

Just as the same as the holomorphic case, we can extend the domain of definition of the B"ottcher coordinate by the functional equation above, and we may assume the following:

- \( \text{B"ott}_c : (\mathbb{C} \setminus K_c) \to (\mathbb{C} \setminus \mathbb{D}) \) is a well-defined conformal isomorphism when \( c \in \mathcal{M}^* \), where \( \mathbb{D} \) is the unit disk;
- \( \text{B"ott}_c(c) \) is well-defined when \( c \notin \mathcal{M}^* \).

**Theorem 2.1** (Nakane [Nak93]). The map

\[
\Phi^* : \mathbb{C} \setminus \mathcal{M}^* \to \mathbb{C} \setminus \mathbb{D}
\]

\[
c \mapsto \text{B"ott}_{f_c}(c)
\]

is a real-analytic diffeomorphism. In particular, \( \mathcal{M}^* \) is connected and full.

**Definition 2.2.** A map \( f_c \) is hyperbolic if \( c \notin \mathcal{M}^* \) or \( f_c \) has an attracting cycle.

A hyperbolic component is a maximal connected open set of \( c \in \mathbb{C} \) for which \( f_c \) has an attracting cycle. The period of a hyperbolic component is defined by that of the attracting cycle of \( f_c \) for any (hence all) \( c \) in the hyperbolic component.

For the holomorphic quadratic family, a hyperbolic component is a component of the interior of \( \mathcal{M} \). On the other hand, an interior component of \( \mathcal{M}^* \) can contain several hyperbolic components [CHRSC89].

2.2. External rays.

**Definition 2.3** (Dynamical rays and parameter rays). For \( \theta \in \mathbb{R}/\mathbb{Z} \), let

\[
R_c(\theta) := (\text{B"ott}_c)^{-1}(\{re^{2\pi i \theta} \mid r > 1\}), \quad \mathcal{R}(\theta) := (\Phi^*)^{-1}(\{re^{2\pi i \theta} \mid r > 1\}).
\]

We call \( R_c(\theta) \) the dynamical ray of angle \( \theta \) for \( f_c \) and \( \mathcal{R}(\theta) \) the parameter ray of angle \( \theta \).

Since \( \text{B"ott}_c \) is the B"ottcher coordinate for the holomorphic polynomial \( f_c^2 \), every dynamical ray of rational angle lands at a point unless it bifurcates [DH84]. Furthermore, every parameter ray of rational angle for the Mandelbrot set lands at a point. On the contrary, there are many parameter rays of rational angles for the tricorn which do not land:

**Theorem 2.4** (Inou-Mukherjee [IM16]). Any parameter ray for \( \mathcal{M}^* \) accumulating to a hyperbolic component of odd period greater than one does not land at a point. The angle of every such a ray is rational.
2.3. Accessibility.

Definition 2.5. A point \( c_0 \in \partial \mathcal{M}^* \) is accessible if there exists a continuous arc \( c = c(s) : [0, \delta] \to \mathbb{C} \) for some \( \delta > 0 \) such that \( c(0) = c_0 \) and \( c(s) \not\in \mathcal{M}^* \) for \( 0 < s \leq 1 \). We call such \( c(s) \) a path to \( c \). A hyperbolic component is accessible if it has an accessible boundary point.

A point or a hyperbolic component is inaccessible if it is not accessible.

Every hyperbolic component \( H \) of even period is accessible, since there are external rays landing on the boundary of \( H \) [MNS14, Lemma 7.3, Lemma 7.4]. Hence we need only consider hyperbolic components of odd period.

By Theorem 2.4, we cannot prove accessibility for a hyperbolic component of odd period greater than one by showing a parameter ray land at a boundary point. However, it is not difficult to see that the hyperbolic components of period three is still accessible:

**Theorem 2.6** (Inou-Mukherjee [IM16]). Every hyperbolic components of period one or three contains undecorated sub-arcs. In particular, it is accessible.

2.4. Parabolic arcs. Here we recall some basic facts on hyperbolic components of odd period. For further detail, see [NS03] and [MNS14].

The multiplicity of a periodic point \( z \) for \( f_c \) is, by definition, that of \( z \) for \( f_{2c} \).

More precisely, let \( p \) be the period of \( z \) for \( f_c \). Then the multiplier \( \lambda \) is defined by

\[
\lambda = \begin{cases} 
(f^p)'(z) & \text{for } p \text{ even,} \\
(f^{2p})'(z) & \text{for } p \text{ odd.}
\end{cases}
\]

A simple but important fact is that when \( p \) is odd, \( \lambda = |\frac{\partial}{\partial z}(f^p)(z)|^2 \) is a non-negative real number.

Let \( H \) be a hyperbolic component of odd period \( p \geq 3 \). Then by the above fact, for any \( c \in \partial H \), there exists a periodic point \( z \) such that \( (f^{2p})'(z) = 1 \). Moreover, one can prove \( z \) has the exact period \( p \).

We say a parameter \( c \in \partial H \) is a cusp if \( z \) is a double parabolic point, i.e., \( z \) has two attracting petals. Otherwise, \( z \) has only one attracting petal and we say \( c \) is non-cusp. The boundary \( \partial H \) consists of three cusps and three arcs connecting them. These three arcs are called parabolic arcs. There exists a unique parabolic arc \( \gamma \subset \partial H \) such that the parabolic periodic points disconnect the filled Julia set \( K_c \) for \( c \in \gamma \). We call \( \gamma \) the root arc; the others are called the co-root arcs.

Roughly speaking, the root arc is the arc which the “umbilical cord” accumulates (see Figure 4). Cusps are contained in the interior of the tricorn, hence \( H \) can be accessible only at points in (the middle of) the parabolic arcs.

We are mainly interested in accessibility on co-root arcs, since there are less decorations that accumulate to them (precisely speaking, there is no umbilical cord).

2.5. Fatou coordinates. Let \( \gamma \subset \partial H \) be a parabolic arc of a hyperbolic component \( H \) of odd period \( p \geq 3 \). For \( c \in \gamma \), let \( x = x_c \) be the parabolic periodic point for \( f_c \) whose immediate basin contains the critical value \( c \). Let \( \text{Fat}_{c, \text{attr}} \) and \( \text{Fat}_{c, \text{rep}} \) be attracting and repelling Fatou coordinates; i.e., univalent maps defined on domains whose boundaries contain \( x_c \) satisfying the “anti-holomorphic Abel equation”

\[
\text{Fat}_{c, *}(f_c(z)) = \text{Fat}_{c, *}(z) + \frac{1}{2} \quad (* = \text{attr, rep})
\]
where both sides are defined, and the range of $\text{Fat}_{c,\text{attr}}$ (resp. $\text{Fat}_{c,\text{rep}}$) contains a right (resp. left) half plane. Let us denote by $V_{c,*}$ the domain of definition of $\text{Fat}_{c,*}$.

Fatou coordinates are unique up to real translation. Hence it follows that the imaginary part $E_{c,*}(z) = \text{Im} \text{Fat}_{c,*}(z)$ is well-defined. We call $E_{c,*}(z)$ the \textit{Eckalle height} of a point $z$ in the domain of $\text{Fat}_{c,*}$.

We can extend $\text{Fat}_{c,\text{attr}}$ holomorphically to the basin of attraction of $x_c$, and $\Psi_c := \text{Fat}_{c,\text{rep}}^{-1}$ can also be extended holomorphically to the whole complex plane. In particular, the \textit{critical Eckalle height} $E_{c,\text{attr}}(c)$ is well-defined.

The following simple fact is used later:

\textbf{Lemma 2.7.} For $c = \frac{1}{4}$, the external ray $R_{1/4}(0)$ is equal to $(\frac{1}{2}, +\infty)$, and

$$\Psi_{1/4}(R_{1/4}(0)) \subset \mathbb{R}.$$ 

Let $c_0 \in \gamma$ and consider a perturbation $c$ of $c_0$ with $c \notin \overline{H}$. There still exist attracting and repelling Fatou coordinates $\text{Fat}_{c,*} : V_{c,*} \to \mathbb{C}$. They are again unique up to real translation; so we normalize them so that $\text{Fat}_{c,*} \to \text{Fat}_{c_0,*}$ as $c \to c_0$. Moreover, the forward orbit of any point $z \in \text{Fat}_{c,\text{attr}}$ eventually escapes from $V_{c,\text{attr}}$ and enters in $V_{c,\text{rep}}$. This induces an isomorphism between the \textit{Eckalle cylinders}, i.e., an isomorphism from $V_{c,\text{attr}}/f_c$ to $V_{c,\text{rep}}/f_c$. Since Eckalle cylinders are isomorphic to $\mathbb{C}/\mathbb{Z}$ by $\text{Fat}_{c,*}$, such an isomorphism is in fact a translation; in other words, we have

$$\text{Fat}_{c,\text{rep}}(f_c^{2n}(z)) - \text{Fat}_{c,\text{attr}}(z) = n + C \quad \text{if} \quad z \in V_{c,\text{attr}} \text{ and } f_c^{2n}(z) \in V_{c,\text{rep}},$$

for some constant $C$. The “anti-holomorphic Abel equation” \cite{1} implies $C \in \mathbb{R}$. This constant is called the \textit{lifted phase}.

\textbf{2.6. Accessible parameters and Fatou coordinates.} The following theorem gives a sufficient condition for a parameter $c_0$ in a parabolic arc to lie in an “open beach”, or an undecorated arc:

\textbf{Theorem 2.8.} Let $H$ be a hyperbolic component of odd period and $c_0 \in \partial H$ be a non-cusp point. If there exists $\varepsilon > 0$ such that

$$\{z \in V_{c,\text{rep}} | |E_{c_0,\text{rep}}(z) - E_{c_0,\text{attr}}(c_0)| < \varepsilon\}$$

is contained in the basin of infinity, then there exists a neighborhood $U$ of $c_0$ such that $U \cap M^* = U \cap \overline{H}$.
In particular, $c_0$ is accessible.

Proof. Take a small neighborhood $U$ such that for any $c \in U$,

1. \(\{|E_{c,\text{rep}}(z) - E_{c_0,\text{attr}}(c_0)| < \varepsilon/2\}\) is contained in the basin of infinity for $f_c$.
2. \(|E_{c,\text{attr}}(c) - E_{c_0,\text{attr}}(c_0)| < \varepsilon/2\).

If further $c \notin \mathcal{H}$, there exists some $n > 0$ such that $f_c^n(c) \in V_{c,\text{rep}}$. Since the lifted phase is real, $E_{c,\text{rep}}(f_c^n(c)) = E_{c,\text{attr}}(c)$, hence $c$ is in the basin of infinity and $c \notin \mathcal{M}^*$. \(\square\)

3. Scaling limit

In this section, we generalize a traditional result on convergence of some scaling limits to the original quadratic family (see [DH85], [EE85] and [McM00]). For simplicity, we consider the following specific situation:

Let $n$ be an integer and set $N = 2^n + 3$. Define a sequence of parameters $\{c_n\}$ as follows:

\[ c_n = \min\{c \in \mathbb{R} | Q_{c_n}^{-1}(0) = 0\}. \]

Namely, $c_n$ is the unique real parameter such that

\[ c_n = Q_{c_n}(0) < 0 = Q_{c_n}(0) < Q_{c_n}^{-1}(0) < \cdots < Q_{c_n}^2(0). \]

Note that (2) and (3) also hold for $f_c$ instead of $Q_c$ since everything is real (Figure 5).

![Figure 5. The critical orbit of $f_{c_n}$ (or $Q_{c_n}$)](image)

In the following, we consider the case $n$ large and $c \in \mathbb{C}$ is close to $c_n$. In particular, $c$ is close to $\hat{c} := \lim c_n = -2$. Let $\beta_c$ be the $\beta$-fixed point for $f_c$, i.e., the landing point of $R_c(0)$ and let $\lambda_c = (f_c^2)'(\beta_c)$ be its multiplier. Let $u = \text{Koe}_c(z)$ be a linearizing (Koenigs) coordinate at $\beta_c$ for $f_c$, i.e., a holomorphic map defined near $\beta_c$ such that

\[ \text{Koe}_c(f_c^2(w)) = \lambda_c \text{Koe}_c(w), \quad \text{Koe}_c(\beta_c) = 0. \]

We may assume $\text{Koe}_c(0)$ is defined; we normalize $\text{Koe}_c$ so that $\text{Koe}_c(0) = 1$, hence

\[ u = \text{Koe}_c(z) = 1 + a_c z + O(z^2) \quad \text{near } z = 0 \]

with $a_c \neq 0$. The Poincaré map $\text{Poi}_c := \text{Koe}_c^{-1}$ satisfies

\[ z = \text{Poi}_c(u) = a_c^{-1}(u - 1) + O((u - 1)^2) \quad \text{near } u = 1. \]

Now we consider the first return map $f_c^N$ near 0. We factor $f_c^N = (f_c^2)^n \circ f_c^3$; that is, first $f_c^3$ anti-holomorphically maps a small neighborhood of 0 into a neighborhood
of \( \beta_c \), which is easy to estimate because the number of iteration is small, and then \((f_c^2)^N\) holomorphically maps it to a neighborhood of 0, which can be estimated in terms of the linearizing coordinate.

Let us denote
\[
\text{Koe}_c \circ f_c^3(z) = \sum_{i=0}^{\infty} A_i(c)z^i
\]
(5)
\[
= A_0(c) + A_2(\hat{c})z^2(1 + O(|z| + |c - \hat{c}|)).
\]

Then \( A_1(c) \) is a real-analytic function on \( c \) (\( A_1(c) \) contains terms in both \( c \) and \( \hat{c} \)) and \( A_2(\hat{c}) \neq 0 \) (indeed, we have \( A_2(\hat{c}) = \frac{64}{\pi^2} > 0 \)).

**Lemma 3.1.** There exist real-analytic functions \( B(c) \) and \( B^*(c) \) defined near \( \hat{c} \) such that
\[
A_0(c) = (c - \hat{c})B(c) + (\overline{c - \hat{c}})B^*(c).
\]
Moreover,
\[
b_0 := B(\hat{c}) = \frac{27 \cdot 7}{15\pi^2}, \quad b_0^* := B^*(\hat{c}) = -\frac{28}{15\pi^2}.
\]
In particular, \(|b_0| \neq |b_0^*|\).

**Proof.** Let \( v_c := f_c^3(0) \) be such that \( A_0(c) = \text{Koe}_c \circ f_c^3(0) = \text{Koe}_c(v_c) \). Note that \( v_c \) is real analytic with \( v_c \to \hat{c} = 2 \) as \( c \to \hat{c} = -2 \). Since \( \text{Koe}_c(\beta_c) = 0 \) and \( \text{Koe}_c^*(\beta_c) \neq 0 \) for any \( c \) near \( \hat{c} \), we have

\[
\text{Koe}_c(z) = \text{Koe}_c^*(\beta_c)(z - \beta_c) + O((z - \beta_c)^2)
\]
near \( \beta_c \) and thus
\[
A_0(c) = \text{Koe}_c(v_c) = \text{Koe}_c^*(\beta_c)(v_c - \beta_c) + O((v_c - \beta_c)^2).
\]

Since \( v_c = (c^2 + \hat{c})^2 + c, \beta_c = f_c(\beta_c) = \overline{\beta_c^2} + c \) and \( \beta_c = f_c^2(\beta_c) = (\beta_c^2 + \hat{c})^2 + c \), we have
\[
\frac{\partial v_c}{\partial c} = 4(c^2 + \hat{c}) + 1, \quad \frac{\partial v_c}{\partial \beta_c} = 2(c^2 + \hat{c}),
\]
\[
\frac{\partial \beta_c}{\partial c} = 4\beta_c \frac{\partial \beta_c}{\partial \beta_c} + 1, \quad \frac{\partial \beta_c}{\partial \beta_c} = 2(2\beta_c \frac{\partial \beta_c}{\partial \beta_c} + 1)\overline{\beta_c},
\]
Thus
\[
\frac{\partial \beta_c}{\partial c} = -\frac{1}{4|\beta_c|^2 - 1}, \quad \frac{\partial \beta_c}{\partial \beta_c} = -\frac{2\overline{\beta_c}}{4|\beta_c|^2 - 1},
\]
and
\[
\frac{\partial}{\partial c}(v_c - \beta_c) \bigg|_{c=\hat{c}} = -15 + \frac{1}{15} = -\frac{224}{15},
\]
\[
\frac{\partial}{\partial \beta_c}(v_c - \beta_c) \bigg|_{c=\hat{c}} = 4 + \frac{4}{15} = \frac{64}{15}.
\]

Hence we have
\[
v_c - \beta_c = \frac{32}{15}(-7(c - \hat{c}) + 2\overline{(c - \hat{c})}) + O((c - \hat{c})^2).
\]

Since \( \text{Koe}_c^*(\beta_c) \) is a real-analytic function of the form \( \text{Koe}_c^*(\beta_c) = \text{Koe}_c^*(\beta_c) + O(|c - \hat{c}|) \) near \( c = \hat{c} \), we conclude that the functions \( B(c) \) and \( B^*(c) \) in the statement exist.
Moreover, since $f_2^3(z) = Q^2_2(z)$ is the Chebyshev map of degree 4, we have
\begin{equation}
\text{Poiz}_c(w) = 2 \cos \left( \frac{\pi}{2} \sqrt{w} \right).
\end{equation}
Hence
\[
A_0(c) = \text{Koe}_c'(\beta)(v_c - \beta_c) + O((v_c - \beta_c)^2) + O(c - \hat{c})(v_c - \beta_c)
= \frac{32}{15 \text{Poiz}_c(0)}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2)
= -\frac{128}{15 \pi^2}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2).
\]
\[
\frac{\partial}{\partial c} A_0(c) = \frac{32}{15 \text{Poiz}_c(0)}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2)
= -\frac{128}{15 \pi^2}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2).
\]
\[
\frac{\partial^2}{\partial c^2} A_0(c) = \frac{32}{15 \text{Poiz}_c(0)}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2)
= -\frac{128}{15 \pi^2}(-7(c - \hat{c}) + 2(c - \hat{c})) + O((c - \hat{c})^2).
\]

By definition, $c_n$ satisfies
\[
\lambda_c^n A_0(c_n) = \lambda_c^n \text{Koe}_{c_n} \circ f_3^n(0) = \text{Koe}_{c_n}(f_2^{n+3}(0)) = 1.
\]
Hence
\[
\lambda_c^{-n} = A_0(c_n) = (c_n - \hat{c})B(c_n) + (c_n - \hat{c})B^*(c_n)
= b_0(c_n - \hat{c}) + b_0^*(c_n - \hat{c}) + O(|c_n - \hat{c}|^2).
\]

Lemma 3.2. \[
c_n = \hat{c} + \frac{b_0 \lambda_c^{-n} - b_0^* \lambda_c^{-n}}{|b_0|^2 - |b_0^*|^2} + o(\lambda_c^{-n}).
\]
In particular, $c_n - \hat{c} = O(\lambda_c^{-n})$.

Proof. It just follows by solving the following system of equations:
\[
\begin{cases}
\lambda_c^{-n} = b_0(c_n - \hat{c}) + b_0^*(c_n - \hat{c}) + O(|c_n - \hat{c}|^2), \\
\lambda_c^n = b_0^*(c_n - \hat{c}) + b_0(c_n - \hat{c}) + O(|c_n - \hat{c}|^2).
\end{cases}
\]

Let $\Lambda = |\lambda_c|(= 16 > 1)$. In this section, we prove the following:

Theorem 3.3. Let $\frac{3}{4} \leq \delta < 1$ and $\frac{1}{2} < \gamma < 1$. Then for $|Z| = O(\Lambda^{1-\delta} n)$ and $t = O(\Lambda^{1-\gamma} n)$,
\[
g_c^n(Z + \rho_n(t)) \sim \frac{1}{\alpha_n} \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}} \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}},
\]
and the convergence is exponentially fast, where
\[
g_c^n(Z) := \alpha_n f_c(Z/\alpha_n), \quad \rho_n(t) := k_n b_0 t - k_n^* b_0^* t, \quad k_n := \frac{a_{\hat{c}}}{\alpha_n \lambda_c^n},
\]
and $\alpha_n$ is the constant satisfying\[1\]
\[
|\alpha_n| = \left| \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}} \right|, \quad \arg \alpha_n = -\frac{1}{3} \arg \left( \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}} \right).
\]

\[1\] Since $b_0$ and $b_0^*$ are real, we may omit to take complex conjugates here. However, we keep taking complex conjugate in order to make the difference between the arguments for holomorphic and anti-holomorphic cases clear.

\[2\] We actually use the condition $\frac{d}{dt} |\alpha_n|^2 \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}} = 1$. Hence we can just let $\alpha_n = \frac{\lambda_c^n A_2(\hat{c})}{a_{\hat{c}}} (< 0)$ because all constants are real.
Hence it follows from (7) that the vertical and horizontal scaling factor is computed as follows:

\[ k_n = O(\Lambda^{-2n}). \]

**Remark 3.4.** By the theorem, the hyperbolic component \( \mathcal{H}_n \) centered at \( c_n \) converges to \( \mathcal{H}_0 \) up to scaling. The asymptotic aspect ratio (the limit of the ratio of the vertical and horizontal scaling factor) is computed as follows:

By Lemma 3.1 and the fact \( k_n > 0 \), for \( t = \zeta + \xi i \),

\[
\rho_n(t) = k_n \frac{2^7}{15\pi^2} (7(\zeta + \xi i) - 2(\zeta - \xi i)) = \frac{2^7 k_n}{15\pi^2} (5\zeta + 9\xi i).
\]

Therefore, the asymptotic aspect ratio is 9:5.

Let \( \gamma \) and \( \delta \) as in the theorem, and consider \( c = c_n + s \) with \( |\frac{s}{c_n - c}| = O(\Lambda^{-\gamma n}) \) and \( z = O(\Lambda^{-\delta n}) \). Then by (5),

\[
Koe \circ f^N_c(z) = \lambda^n_c A_0(c) + \lambda^n_c A_2(\hat{c}) \hat{z}^2 (1 + O(|z| + |c - \hat{c}|)).
\]

**Lemma 3.5.**

\[
\lambda^n_c = \lambda^n_c (1 + O(n|c - \hat{c}|)) = \lambda^n_{c_n} (1 + O(ns)).
\]

In particular, \( \lambda^n_c = O(\Lambda^{-n}) \) and \( c_n - \hat{c} = O(\Lambda^{-n}) \).

**Proof.** Since \( \lambda_c = \lambda_{\hat{c}} + O(|c - \hat{c}|) \), we have

\[
n \log \lambda_c = n \log \lambda_{\hat{c}} + n \log(1 + O(|c - \hat{c}|)) = n \log \lambda_{\hat{c}} + O(n|c - \hat{c}|)
\]

The second equality follows similarly. \(\square\)

**Proof of Theorem 3.3.** By the above lemma,

\[
\lambda^n_c A_0(c) = \lambda^n_c (c - \hat{c}) B(c) + \lambda^n_c (c - \hat{c}) B^*(c)
\]

\[
= \lambda^n_{c_n} (1 + O(ns)) \left\{ (c_n - \hat{c} + s) B(c_n + s) + (c_n - \hat{c} + s) B^*(c_n + s) \right\}
\]

\[
= \lambda^n_{c_n} (1 + O(ns)) \times \left\{ (c_n - \hat{c} + s) B(c_n)(1 + O(s)) + (c_n - \hat{c} + s) B^*(c_n)(1 + O(s)) \right\}
\]

\[
= \lambda^n_{c_n} (1 + O(ns)) \times \left\{ (c_n - \hat{c}) B(c_n) + (c_n - \hat{c}) B^*(c_n) + B(c_n) s + B^*(c_n) \hat{s} + O(|c_n - \hat{c}|s) \right\}
\]

\[
= \lambda^n_{c_n} A_0(c_n)(1 + O(ns)) + \lambda^n_{c_n} B(c_n)s + B^*(c_n)\hat{s} + O(|c_n - \hat{c}|s)
\]

\[
+ \lambda^n_{c_n} O(ns^2) + \lambda^n_{c_n} (1 + O(ns)) O(|c - \hat{c}|s)
\]

\[
= 1 + \lambda^n_{c_n} (B(c_n)s + B^*(c_n)\hat{s}) + O(ns) + O(ns\Lambda^{-\gamma n}) + O(s)
\]

\[
= 1 + \lambda^n_{c_n} (B(c_n)s + B^*(c_n)\hat{s}) + O(ns).
\]

Hence it follows from (7) that

\[
Koe \circ f^N_c(z) - 1 = \lambda^n_c (A_0(c) + A_2(\hat{c}) \hat{z}^2 (1 + O(|z| + |c - \hat{c}|))) - 1
\]

\[
= \lambda^n_c A_2(\hat{c}) \hat{z}^2 (1 + O(|z| + |c - \hat{c}|)) + \lambda^n_{c_n} (B(c_n)s + B^*(c_n)\hat{s} + O(ns))
\]

\[
= \lambda^n_c A_2(\hat{c}) \hat{z}^2 (1 + O(|z| + n|c - \hat{c}|)) + \lambda^n_{c_n} (B(c_n)s + B^*(c_n)\hat{s} + O(ns)).
\]
Since we assume $z = O(\Lambda^{-\delta n})$ and $\delta < 1$, we have
\[
\lambda_c^n z^2 \left( |z| + n|c - \hat{c}| \right) = O(\Lambda^n \Lambda^{-2\delta n}(\Lambda^{-\delta n} + n\Lambda^{-n})) \\
= O(\Lambda^{(1-3\delta)n}).
\]
By the assumption, $s = O(|c - \hat{c}| \Lambda^{-\gamma n}) = O(\Lambda^{-(1+\gamma)n})$ and $\gamma < 1$, it follows that
\[
\text{Koe} \circ f_c^N(z) - 1 = \lambda_c^n A_2(\hat{c}) \bar{z}^2 + \lambda_c^n B(c_n)s + \lambda_c^n B^*(c_n)\bar{s} + O(ns) + O(\Lambda^{(1-3\delta)n}) \\
= O(\Lambda^n \Lambda^{-2\delta n} + O(\Lambda^n \Lambda^{-(1+\gamma)n}) + O(n\Lambda^{-(1+\gamma)n}) + O(\Lambda^{(1-3\delta)n}) \\
= O(\Lambda^{(1-2\delta)n}) + O(\Lambda^{-\gamma n}).
\]
Since $a_c = a_\varepsilon + O(|c - \hat{c}|) = a_\varepsilon + O(\Lambda^{-n})$ and $a_\varepsilon \neq 0$, we have
\[
f_c^N(z) = \text{Poi}_c \circ \text{Koe}_c \circ f_c^N(z) \\
= \frac{1}{a_c} \left\{ \lambda_c^n A_2(\hat{c}) \bar{z}^2 + \lambda_c^n B(c_n)s + \lambda_c^n B^*(c_n)\bar{s} \right\} \\
+ O(\Lambda^{(1-\gamma)n} + O(\Lambda^{(1-3\delta)n}) + O(\Lambda^{-2\gamma n}) \\
= \lambda_c^n A_2(\hat{c}) \bar{z}^2 + \lambda_c^n \frac{B(c_n)s + B^*(c_n)\bar{s}}{a_\varepsilon} \\
+ O(\Lambda^{-(2\gamma n)} + O(\Lambda^{(1-3\delta)n}) + O(\Lambda^{(2-4\delta)n}) + O(\Lambda^{-(1+\gamma)n}) \\
= \lambda_c^n A_2(\hat{c}) \bar{z}^2 + \lambda_c^n \frac{B(c_n)s + B^*(c_n)\bar{s}}{a_\varepsilon} + O(\Lambda^{-2\gamma n}) + O(\Lambda^{(2-4\delta)n}).
\]
Let $\alpha_n$ be as in the theorem and let $Z = \alpha_n z$. Then $\alpha_n = O(\Lambda^n)$ and $f_c^N$ is conjugate to
\[
g_c(Z) = \alpha_n f_c^N(Z/\alpha_n) \\
= \hat{Z}^2 + \alpha_n \left\{ \lambda_c^n (B(c_n)s + B^*(c_n)\bar{s}) + O(\Lambda^{-2\gamma n} + O(\Lambda^{(2-4\delta)n}) \right\} \\
= \hat{Z}^2 + \xi_n(s) + O(\Lambda^{(1-\gamma)n}) + O(\Lambda^{(3-4\delta)n})
\]
where
\[
\xi_n(s) := \frac{\alpha_n \lambda_c^n}{a_\varepsilon} (B(c_n)s + B^*(c_n)\bar{s}).
\]
Now recall that $k_n = \frac{a_\varepsilon}{\alpha_n \lambda_c^n}$. Let
\[
s = \rho_n(t) := k_n \frac{B(c_n)t - B^*(c_n)\bar{t}}{k_n B(c_n)\bar{t} - k_n B^*(c_n)\bar{t}}.
\]
Then for $t = O(\Lambda^{(1-\gamma)n})$, we have
\[
\frac{s}{c - \hat{c}} = O(\Lambda^{-\gamma n}),
\]
and
\[
\xi_n(s) = \frac{\alpha_n \lambda_c^n}{a_\varepsilon} (B(c_n)(k_n \frac{B(c_n)t - B^*(c_n)\bar{t}}{k_n B(c_n)\bar{t} - k_n B^*(c_n)\bar{t}} + B^*(c_n)(k_n B(c_n)\bar{t} - k_n B^*(c_n)\bar{t})) \\
= (|B(c_n)|^2 - |B^*(c_n)|^2) t = (|b_0|^2 - |b_0^*|^2) t - O(\Lambda^{-\gamma n}).
\]
Therefore, we have
\[
(8) \ g_{c_n + \rho_n(t)}(z) - \frac{1}{2} \bar{z}^2 + (|b_0|^2 - |b_0^*|^2) t \right\} = O(\Lambda^{(3-4\delta)n}) + O(\Lambda^{(1-2\gamma)n}) + O(\Lambda^{-\gamma n}).
\]
Since $\gamma \in (\frac{3}{2}, 1)$ and $\delta \in (\frac{3}{4}, 1)$, $1 - 2\gamma$ and $3 - 4\delta$ are negative. Hence $g_{c_n + \rho_n(t)}$ converges exponentially to $\bar{z}^2 + (|b_0|^2 - |b_0^*|^2) t$. \qed
4. Construction of polynomial-like restrictions

In the following, we consider the case \( c = c_n + \rho_n(t) \) with

\[
t = O(1) (= O(\Lambda^{(1-\gamma)n}) \text{ for any } \gamma \in (\frac{1}{2}, 1)).
\]

For \( f_{-2} \), the external rays of angles 1/3 and 2/3 land at the “\( \alpha \)-fixed point” \(-1\). Since \(-1\) is repelling, \( R_c(1/3) \) and \( R_c(2/3) \) land at the same point when \( n \) is sufficiently large.

We construct Yoccoz puzzles from these rays. Let \( D^0 = D^0_c \) be the puzzle piece of depth 1 containing 0 for \( f_c \). Namely, let

\[
\Gamma_0 = \text{B"{o}tt}^{-1}([\{z\} = 2]) \cup R_c(1/3) \cup R_c(2/3)
\]

and let \( D^0 \) be the closure of the bounded component of \( \mathbb{C} \setminus f_c^{-1}(\Gamma_0) \) containing 0.

Let

\[
D^1 = D^1_c := f_c^{-1}\left(-\text{Koe}^{-1}_c(\lambda_c^{-n-1}\text{Koe}_c(D_0))\right).
\]

In other words, \( D^1 \) is the puzzle piece of depth \( N = 2n + 3 \) containing 0. Hence we have the following:

**Lemma 4.1.** The set \( D^1 \) is the component of \( f_c^{-N}(D^0) \) containing 0 and \( f_c^N : \text{Int} D^1 \to \text{Int} D^0 \) is a quadratic-like mapping.

Let \( D^k_c := (f_c^k|_{D^1_c})^{-k}(D_0) \) for \( k > 1 \). It is easy to see from the definition that \( \text{diam } D^1 = O(\lambda^{-\frac{n+1}{2}}) \). Therefore, \( f_c^N : D^k_c \to D^{k-1}_c \) is close to a composition of a linear map and \( z^2 \) for \( k \geq 2 \). More precisely,

**Lemma 4.2.** Fix \( k \geq 1 \). Then \( \text{diam } D^k_c \approx \Lambda^{-\left(1-2^{-\gamma}\right)n} \), and for \( z \in D^k_c \), we have

\[
f_c^N(z) + \frac{\lambda_c^{n+1}\text{Koe}'_c(-c_n)}{\text{Koe}_c(0)} \cdot z^2 = O(\Lambda^{-n}).
\]

Here, \( A \asymp B \) stands for \( A = O(B) \) and \( B = O(A) \). Note that all constants can depend on \( k \) (and \( m \) which appears later), but uniform on \( n \), since we will fix \( m \) and consider \( k \leq m \), then we let \( n \) tends to infinity.

**Proof.** First observe that \( c - c_n = \rho_n(t) = O(\Lambda^{-2n}) \). Take \( \frac{1}{2} \leq a < 1 \) and let \( z = O(\Lambda^{-an}) \). Recall that \( \lambda_c^m = \lambda_c^{m_n}(1 + O(m(c - c_n))) \) by Lemma \[3.5\]. Hence it follows that

\[
\text{Koe}_c(-\bar{z}^2 + c) = \text{Koe}_c(-c_n) - \text{Koe}'_c(-c_n)(\bar{z}^2 + c - c_n) + O((\bar{z}^2 + c - c_n)^2)
\]

\[
= \text{Koe}_c(-c_n) - \text{Koe}'_c(-c_n)(\bar{z}^2 + \rho_n(t)) + O(\Lambda^{-4an}),
\]

and

\[
\lambda_c^{n+1}\text{Koe}_c(-c_n) = \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}(\text{Koe}_c(-c_n) - \text{Koe}_c(-c_n))
\]

\[
+ (\lambda_c^{n+1} - \lambda_c^{n+1})\text{Koe}_c(-c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}O(c - c_n) + \lambda_c^{n+1}O((n + 1)(c - c_n))O(\beta_c + c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}) + O(n\Lambda^{-2n})
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}),
\]

\[
\lambda_c^{n+1}\text{Koe}_c(-c_n) = \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}(\text{Koe}_c(-c_n) - \text{Koe}_c(-c_n))
\]

\[
+ (\lambda_c^{n+1} - \lambda_c^{n+1})\text{Koe}_c(-c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}O(c - c_n) + \lambda_c^{n+1}O((n + 1)(c - c_n))O(\beta_c + c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}) + O(n\Lambda^{-2n})
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}),
\]

\[
\lambda_c^{n+1}\text{Koe}_c(-c_n) = \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}(\text{Koe}_c(-c_n) - \text{Koe}_c(-c_n))
\]

\[
+ (\lambda_c^{n+1} - \lambda_c^{n+1})\text{Koe}_c(-c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + \lambda_c^{n+1}O(c - c_n) + \lambda_c^{n+1}O((n + 1)(c - c_n))O(\beta_c + c_n)
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}) + O(n\Lambda^{-2n})
\]

\[
= \lambda_c^{n+1}\text{Koe}_c(-c_n) + O(\Lambda^{-n}),
\]
so

\[
\lambda_{c}^{n+1} \text{Koe}_{c}(-\bar{z}^2 + \bar{c}) = \lambda_{c}^{n+1} \text{Koe}_{c}(-c) + O(\Lambda^{-n}) - \lambda_{c}^{n+1} \text{Koe}_{c}'(-c)(\bar{z}^2 + \rho_n(t)) \\
+ (\lambda_{c}^{n+1} \text{Koe}_{c}'(-c) - \lambda_{c}^{n+1} \text{Koe}_{c}'(\bar{c}))(O(\Lambda^{-2n}) + O(\Lambda^{-(4a-1)n})) \\
= \lambda_{c}^{n+1} \text{Koe}_{c}(-c) - \lambda_{c}^{n+1} \text{Koe}_{c}'(-c)(\bar{z}^2 + \rho_n(t)) \\
+ O(\Lambda^{-n}) + O(n\Lambda^{-n}) + O(\Lambda^{-2n}) + O(\Lambda^{-(4a-1)n}) \\
= \lambda_{c}^{n+1} \text{Koe}_{c}(-c) - \lambda_{c}^{n+1} \text{Koe}_{c}'(-c)(\bar{z}^2 + \rho_n(t)) + O(\Lambda^{-n}),
\]

and since \(\text{Poic}_{c}((\lambda_{c}^{n+1} \text{Koe}_{c}(-c))) = f_{c}^{N}(0) = 0\),

\[
f_{c}^{N}(z) = \text{Poic}_{c}(\lambda_{c}^{n+1} \text{Koe}_{c}(-\bar{z}^2 + \bar{c})) \\
= \text{Poic}_{c}(\lambda_{c}^{n+1} \text{Koe}_{c}(-\bar{z}^2 + \bar{c})) + O(\Lambda^{-2n}) \\
= \frac{1}{\text{Koe}_{c}(0)}(-\lambda_{c}^{n+1} \text{Koe}_{c}'(-c)(\bar{z}^2 + \rho_n(t)) + O(\Lambda^{-n})) + O(\Lambda^{-2n}) \\
= -\frac{\lambda_{c}^{n+1} \text{Koe}_{c}'(-c)}{\text{Koe}_{c}(0)}\bar{z}^2 + O(\Lambda^{-n}) = O(\Lambda^{-(2a-1)n}).
\]

Therefore, if \(z \in D_{c}^{1}\) and \(f_{c}^{N}(z) = O(\Lambda^{-(2a-1)n})\), then \(z = O(\Lambda^{-n})\). Hence \(\text{diam}
D_{c}^{k} \asymp \Lambda^{-(1-2^{-k})n}\) follows by induction. \(\square\)

5. Straightening

Fix \(m \geq 0\). Consider a polynomial-like restriction

\[f_{c}^{N} : D_{c}^{m+1} \rightarrow D_{c}^{m}.\]

The straightening is a quadratic anti-holomorphic polynomial hybrid equivalent to it. We follow the construction by Douady and Hubbard and give a precise construction of such a polynomial and a hybrid conjugacy.

First we rescale as before, so let \(g_{c}(z) := \alpha_{n}f_{c}^{N}(\frac{z}{\alpha_{n}})\) and \(\tilde{D}_{c}^{m} := \alpha_{n}D_{c}^{m}\). Then \(g_{c} : \tilde{D}_{c}^{m+1} \rightarrow \tilde{D}_{c}^{m}\) is quadratic-like.

Since we consider the case \(|\frac{c-n}{c+n}| = O(\Lambda^{-2n})/O(\Lambda^{-n}) = O(\Lambda^{-n})\) and \(|z| < \text{diam} D_{c}^{m+1} = O(\Lambda^{-(1-2^{-m+1})n})\), we can apply Theorem 3.3 and with \(\gamma < 1\) sufficiently close to 1 and \(\delta = 1 - 2^{-m-1}(m \geq 2)\). Namely, we have

\[g_{c}(z) = \bar{z}^2 + s + O(\Lambda^{-(1-2^{-m+1})n})\]

where \(s = (|b_{0}|^2 - |b_{0}^2|^2)t\), and \(g_{c}\) is close to \(f_{s}(z) = \bar{z}^2 + s\). (As mentioned before, \(m\) will be fixed, so constants can depend on \(m\).)

Next we enlarge the domain of \(g_{c} : \tilde{D}_{c}^{m+1} \rightarrow \tilde{D}_{c}^{m}\).

**Lemma 5.1.** For sufficiently large \(n\), there exist \(r_{m}\) and \(R_{m}\) with \(0 < r_{m} < R_{m}\) such that

1. \(r_{m} \asymp R_{m} \asymp \Lambda^{-(1-2^{-m})n}\), and
2. for any \(c = c_{n} + \rho_{n}(t)\) with \(t = O(1)\),
\[D_{c}^{m+1} \subset \mathbb{D}(r_{m}) \subset D_{c}^{m} \subset \mathbb{D}(R_{m}),\]

where \(\mathbb{D}(r) := \{z \in \mathbb{C} \mid |z| < r\}\).
Proof. There are $r_0$ and $R_0$ with $0 < r_0 < R_0$ independent of $n$ such that $\partial D_0^c \cap \{z \in \mathbb{C} \mid r_0 \leq |z| \leq R_0\}$ for any $c = c_n + \rho_n(t)$ with $t = O(1)$. By Lemma 4.2, there exists a constant $K$ such that for sufficiently large $n$ and for $c = c_n + \rho_n(t)$ with $t = O(1)$, we have

$$K^{-1} \Lambda^{-n/2} |f_c^N(z)|^{1/2} \leq |z| \leq K \Lambda^{-n/2} |f_c^N(z)|^{1/2}$$

if $f_c^N(z) \in \overline{D(R_0)} \setminus D_c^{n+1}$. Now the existence of $r_m$ and $R_m$ simply follows by induction. 

Let $R := |a_n| R_m \sim \Lambda^{-m}$ such that $\bar{D}_c^m \subset \mathbb{D}(R)$. Let 

$$U = \mathbb{D}(R), \quad U' = U_1 := g_c^{-1}(U), \quad \text{and} \quad U_2 := f_c^{-1}(U).$$

Then $g_c : U_1' \to U$ and $f_c : U_2' \to U$ are quadratic-like mappings. In particular, $g_c : U_1' \to U$ still satisfies (9).

Let 

$$\gamma_1 : \mathbb{R}/\mathbb{Z} \to \partial U_1', \quad \gamma_2 : \mathbb{R}/\mathbb{Z} \to \partial U_2'$$

be such that $g_c(\gamma_1(\theta)) = f_c(\gamma_2(\theta)) = R e^{-4\pi \theta}$ and arg $\gamma_2(\theta)$ ($j = 1, 2$) are close to $2\pi \theta$. Let $A_0 := \{z \in \mathbb{C} \mid \sqrt{R} \leq |z| \leq R\}$ and $A_j := U \setminus U_j'$. Define diffeomorphisms 

$$h_j : A_0 \to A_j$$

by 

$$h_j(re^{2\pi i \theta}) = \frac{R - r}{R - \sqrt{R}} \gamma_j(\theta) + \frac{r - \sqrt{R}}{R - \sqrt{R}} R e^{2\pi i \theta} = r e^{2\pi i \theta} - \frac{R - r}{R - \sqrt{R}} \gamma_j(\theta),$$

where $\gamma_j(\theta) := \gamma_j(\theta) - \sqrt{R} e^{2\pi i \theta}$. Let $h := h_2 \circ h_1^{-1} : A_1 \to A_2$. Then 

$$h(h_1(re^{2\pi i \theta})) = h_1(re^{2\pi i \theta}) + \frac{R - r}{R - \sqrt{R}} (\gamma_1(\theta) - \gamma_2(\theta)).$$

Let $h(z) = z$ on $\{|z| > R\}$. Then $h_j : \{|z| \geq \sqrt{R}\} \to \mathbb{C} \setminus U_j'$ is a piecewise smooth homeomorphism, hence quasiconformal. Thus $h = h_2 \circ h_1^{-1} : (\mathbb{C} \setminus U_1') \to (\mathbb{C} \setminus U_2')$ is also a quasiconformal homeomorphism.

Now define an anti-(i.e., orientation reversing) quasiregular mapping $F_c : \mathbb{C} \to \mathbb{C}$ as follows: 

$$F_c(z) = \begin{cases} 
  g_c(z) & \text{if } z \in U_1', \\
  h^{-1}(f_c(h(z))) & \text{if } z \notin U_1'. 
\end{cases}$$

Define an almost complex structure $\sigma$ as follows: Let $\sigma := \sigma_0$ (the standard complex structure) on $\mathbb{C} \setminus U$ and on $K(g_c)$ for $z \in U \setminus K(g_c)$, let $n > 0$ be the smallest integer satisfying $F_c^n(z) \notin U$. Then define 

$$\sigma := (F_c^n)^* \sigma_0$$

at $z$. Then $\sigma$ is $F_c$-invariant. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\eta = \eta_c : \mathbb{C} \to \mathbb{C}$ fixing $0$ and $\infty$ such that $\eta^* \sigma_0 = \sigma$ and $\eta$ is tangent to the identity at $\infty$ (note that $\eta$ is holomorphic near $\infty$).

Then $P := \eta \circ F_c \circ \eta^{-1}$ is an anti-holomorphic map and because of the normalization of $\eta$, hence $P = f_{\tilde{c}}$ for some $\tilde{c}$. We denote $\tilde{c} = \chi_n(c)$. Note that $s \neq \chi_n(c)$ in general (see Lemma 6.2 below).
6. Estimate of quasiconformal homeomorphisms

Here we give the following estimate of the hybrid conjugacy $\eta = \eta_c$ defined in the previous section:

**Lemma 6.1.** The hybrid conjugacy $\eta$ satisfies $\eta(z) - z = O(\Lambda^{-(1-2^{-m+1})n})$ on $U$.

**Proof.** Let $u = u(c, z) := g_c(z) - f_s(z)$, which is anti-holomorphic in variable $z$ and $u = O(\Lambda^{-(1-2^{-m+1})n})$ by (9). To simplify the notation, we assume $|u| \leq \ell = O(\Lambda^{-(1-2^{-m+1})n})$ on $U'$.

Let $\tau := 2\pi \theta$ with $0 \leq \theta \leq 1$ and $w(\tau) := Re^{-2\tau i}$. By replacing $R$ by $R/2$ if necessary, we can find an anti-holomorphic branch $z = G_c(w)$ of $g_c^{-1}$ on the disk $\Delta_\tau := \{w \in \mathbb{C} | |w - w(\tau)| < R/2\}$ such that

$$\tilde{z} = G_c(w) = (w - s - u)^{1/2} = \sqrt{w} \left(1 - \frac{s + u}{w}\right)^{1/2} = \sqrt{w} + O\left(\frac{1}{\sqrt{w}}\right),$$

where we choose the branch of $\sqrt{w}$ such that $\sqrt{w(\tau)} = \sqrt{R}e^{-\tau i}$. Let $V_1(w) := G_c(w) - \sqrt{w}$ such that

$$V_1(w(\tau)) = \gamma_1^{1}\left(\frac{\tau}{2\pi}\right) = \gamma_1^{1}(\theta).$$

Since $V_1(w)$ is holomorphic and $V_1(w) = O(1/R^{1/2})$ on $\Delta_\tau$, we have

$$V_1'(w(\tau)) = O(1/R^{3/2}),$$

hence

$$\frac{d}{d\tau} V_1(w(\tau)) = V_1'(w(\tau)) \cdot w'(\tau) = O(1/R^{1/2}).$$

Similarly, we define a branch $z = F_s(w)$ of $f_s^{-1}$ on $\Delta_\tau$ such that

$$\tilde{z} = F_s(w) = \sqrt{w} \left(1 - \frac{s}{w}\right)^{1/2}.$$

Let $V_2(w) := F_s(w) - \sqrt{w}$ such that

$$V_2(w(\tau)) = \gamma_2^{1}\left(\frac{\tau}{2\pi}\right) = \gamma_2^{1}(\theta)$$

with derivative $\frac{d}{d\tau} V_2(w(\tau)) = O(1/R^{1/2})$. Hence we have

$$\Gamma(\tau) := \gamma_2^{1}(\theta) - \gamma_1^{1}(\theta) = V_2(w(\tau)) - V_1(w(\tau)).$$

Since

$$V_2(w) - V_1(w) = F_s(w) - G_c(w)$$

$$= \sqrt{w} \left\{ \left(1 - \frac{s}{w}\right)^{1/2} - \left(1 - \frac{s + u}{w}\right)^{1/2} \right\}$$

$$= \sqrt{w} O(u/w),$$

we have

$$\Gamma(\tau) = \sqrt{R} O(\ell/R) = O(\ell/R^{1/2}).$$

By the Schwarz lemma,

$$\frac{d}{dw} (V_2(w) - V_1(w)) = O(\ell/R^{3/2})$$
at \( w = w(\tau) \), hence it follows that

\[
\Gamma'(\tau) = \left. \frac{d}{dw} (V_2(w) - V_1(w)) \right|_{w=w(\tau)} \cdot w'(\tau)
\]

\[
= O(\ell/R^{3/2}) \cdot O(R) = O(\ell/R^{1/2}).
\]

Let \( H(z) := h(z) - z \). By [10], it follows that for \( z = h_1(re^{\tau i}) \),

\[
H(z) = \frac{R - r}{R - \sqrt{R}} \Gamma(\tau).
\]

Then

\[
H_r = -\frac{\Gamma(\tau)}{R - \sqrt{R}} = O(\ell/R^{3/2}),
\]

\[
H_r = \frac{R - r}{R - \sqrt{R}} \cdot \Gamma'(\tau) = O(\ell/R^{1/2}),
\]

\[
z_r = e^{\tau i} - \frac{\gamma_1(\theta)}{R - \sqrt{R}} = O(1),
\]

\[
z_{\bar{r}} = ire^{r i} + \frac{R - r}{R - \sqrt{R}} \frac{d}{d\tau} V_1(w(\tau)) = O(R),
\]

and similarly \( \bar{z}_r = e^{-\tau i} + O(1/R^{3/2}) = O(1) \) and \( \bar{z}_{\bar{r}} = -ire^{-\tau i} + O(1/R^{1/2}) = O(R) \).

In particular, \( z_r \bar{z}_r - z_{\bar{r}} z_r = -2ir + O(1/R^{1/2}) \) where \( r \) ranges from \( \sqrt{R} \) to \( R \).

Therefore

\[
(H_z \quad H_{\bar{z}}) = (H_r \quad H_{\bar{r}}) \begin{pmatrix} z_r & z_{\bar{r}} \\ \bar{z}_r & \bar{z}_{\bar{r}} \end{pmatrix}^{-1}
\]

\[
= (O(\ell/R^{3/2}) \quad O(\ell/R^{1/2})) \cdot O(1/R^{1/2}) \begin{pmatrix} O(R) & O(R) \\ O(1) & O(1) \end{pmatrix}
\]

\[
= (O(\ell/R) \quad O(\ell/R))
\]

on \( A_1 \). Thus the complex dilatation \( \mu_h \) of \( h \) satisfies

\[
\mu_h = \frac{H_{\bar{z}}}{1 + H_z} = O(\ell/R)
\]

as well. This implies that

\[
\|\mu_\eta\|_{\infty} = \|\mu_h\|_{\infty} = O(\ell/R) (= O(A^{-1/2})�)
\]

where \( \cdot \) is the norm for \( L^p(\mathbb{C}) \). Since \( \mu_\eta \) is supported on \( U \setminus K(q_\ell) \), it is of compact support. Hence by [1192, Theorem 4.24] and the proof of its Corollary 2, the following holds: For given \( m > 0 \) and \( p > 2 \), if \( n \) is sufficiently large, then we have

\[
|\eta(z) - z| \leq K_p \|\mu_\eta\|_{p} |z|^{1-2/p}
\]

for any \( z \in \mathbb{C} \), where \( K_p > 0 \) is a constant depending only on \( p \). Since \( \mu_\eta \) is supported on \( U \), we have

\[
\|\mu_\eta\|_{p} \leq \|\mu_\eta\|_{\infty} \cdot (\text{Area } U)^{1/p}
\]

\[
= \|\mu_\eta\|_{\infty} (\pi R^2)^{1/p}
\]

\[
= O(\ell R^{-1+2/p}).
\]
Hence if $z \in U$, we obtain
\[ |\eta(z) - z| \leq O(\ell R^{-1+2/\nu}) \cdot R^{1-2/\nu} = O(\ell) = O(\Lambda^{-1-2^{-m+1}}n). \]

\[ \square \]

**Lemma 6.2.** Let $\tilde{c} \in \mathcal{M}^*$ and assume a sequence $\{b_n = c_n + \rho_n(t_n)\}$ satisfies $\chi_n(b_n) = \tilde{c}$ for sufficiently large $n$. Then $s_n = ((|b_0|^2 - |b_0'|^2)t_n \to \tilde{c}$ and $g_{n,b_n}$ converges locally uniformly to $f_{\tilde{c}}$.

**Proof.** Let $\eta_n$ be the hybrid conjugacy between $g_{b_n} : U' \to U$ and $f_{\tilde{c}}$ constructed in the previous section.

Since the critical value of $g_{b_n}$ is $s_n + O(\Lambda^{-1-2^{-m+1}}n)$ and it is mapped to $\tilde{c}$ by $\eta_n$, it follows by Lemma 6.1 that
\[ \tilde{c} - \{s_n + O(\Lambda^{-1-2^{-m+1}}n)\} = O(\Lambda^{-1-2^{-m+1}}n). \]
Hence $s_n \to \tilde{c}$ as $n \to \infty$ and local uniform convergence follows by [8]. \[ \square \]

7. **Estimate of Böttcher coordinates**

Recall that Böttc is the Böttcher coordinate for $f_c$. For $c = c_n + \rho_n(t)$ with $t = O(1)$, we define a “quasiconformal Böttcher coordinate” $\text{Bott}_{g_c} : (\overline{U} \setminus K(g_c)) \to \mathbb{C}$ for $g_c$ as follows: First for $z \in A_1 = \overline{U} \setminus U'$, let
\[ \text{Bott}_{g_c}(z) = \text{Bott}_{\chi_n(c)} \circ \eta_c(z). \]
Then extend it to $\overline{U} \setminus K(g_c)$ continuously by the functional equation
\[ (12) \quad \text{Bott}_{g_c}(g_c(z)) = \text{Bott}_{g_c}(z)^2. \]
By construction, the map $\text{Bott}_{g_c} : U' \setminus K(g_c) \to \mathbb{C}$ is a quasiconformal homeomorphism onto its image.

**Lemma 7.1.** Fix $m > 1$. On $\tilde{D}_c^{m} \setminus \tilde{D}_c^{m+1}$, $\text{Bott}_{g_c}$ satisfies
\[ |\text{Bott}_{g_c}(z) - z| = O(\Lambda^{-2^{-m-1}n}). \]

**Proof.** Note that any $z \in \tilde{D}_c^{m} \setminus \tilde{D}_c^{m+1}$ satisfies $|z| \geq |a_n|r_{m+1} \approx \Lambda^{n/2^{m+1}}$ by Lemma 5.1 and that the functional equation (12) implies $\text{Bott}_{c}(z) = z + O(1/z)$ near $\infty$ for any $c \in C$. Hence by Lemma 6.1 we have
\[ |\text{Bott}_{g_c}(z) - z| \leq |\text{Bott}_{\chi_n(c)}(\eta_c(z)) - \eta_c(z)| + |\eta_c(z) - z|
\begin{align*}
&= O(1/\eta_c(z)) + |\eta_c(z) - z| \\
&= O(1/z) + |\eta_c(z) - z| \\
&= O(\Lambda^{-2^{-m-1}n}) + O(\Lambda^{-1-2^{-m+1}}n) = O(\Lambda^{-2^{-m-1}n})
\end{align*} \]
since $m > 1$. \[ \square \]
The following theorem is proved by Rivera-Letelier [RL01] Theorem C] for holomorphic quadratic polynomials:

**Theorem 8.1 (Rivera-Letelier).** Assume $Q_{c_0}$ is semihyperbolic for $c_0 \in \partial \mathcal{M}$. Then there exists $C > 0$ such that if $c \in \mathbb{C}$ is close to $c_0$,

$$d_H(J(Q_{c_0}), K(Q_c)) \leq C|c - c_0|^{1/2},$$

where $d_H$ is the Hausdorff distance.

Indeed, we can apply this theorem for $f_c(z) = \bar{z}^2 + c$ with $c_0 = \hat{c} = -2$. Namely, there exists $C > 0$ such that

$$K(f_c) \subset \{|\text{Im } z| < C|c - \hat{c}|^{1/2}\}.$$

**Outline of proof of Theorem 8.1.** Since $f_\hat{c}^2 = Q^2_{\hat{c}}$, the dynamical properties in [RL01 §2, 3] hold for $f_\hat{c}$.

Then we can apply the proof of [RL01 Proposition 4.2] to the biquadratic family $(z^2 + a)^2 + b$. Note that we need the $\lambda$-lemma [MBSSS94] in order to extend a holomorphic motion only to the closure of the domain of definition, hence we can apply the proof to a two-dimensional parameter space. □

Let $c = c_n + \rho_n(t)$ with $t = O(1)$ as before. Recall that $\text{Poi}_c = \text{Koe}_{c^{-1}}$ can be extended to an entire function, normalized so that $\text{Koe}_c(0) = 1$. Let $\hat{K}_c := \text{Poi}_{c^{-1}}(K(f_c))$. Recall that for $c = \hat{c} = -\frac{5}{2}$, we have

$$\text{Poi}_{-2}(w) = 2 \cos \left( \frac{\pi}{2} \sqrt{w} \right).$$

Hence it follows that $\hat{K}_{-2} = [0, \infty]$, $\text{Poi}_{-2}([0, 4]) = [-2, 2] = K_{-2}$ and, since $D^0_{-2} \cap K_{-2} = [-1, 1]$, it follows that

$$\text{Koe}_{-2}(D^0_{-2} \cap K(f_{-2})) = \left[ \frac{4}{9}, \frac{16}{9} \right].$$

By Theorem 8.1, $\text{Koe}_c(D^0_c \cap K(f_c))$ is a compact set contained in a $O(\Lambda^{-\frac{1}{2} n})$-neighborhood of $\hat{K}_c^0 = [\frac{4}{9}, \frac{16}{9}]$. Thus we can take compact sets $\hat{L}_c \subset \hat{K}_c \subset \hat{K}_c$ contained in a $O(\Lambda^{-\frac{1}{2} n})$-neighborhood of $[0, 4]$ such that

$$\text{Poi}_c(\hat{K}_c^0) = K(f_c), \quad \hat{L}_c = \hat{K}_c \setminus \text{Koe}_c(\text{Int } D^0_c).$$

(See Figures 5 and 7.) Then we have the following:

**Lemma 8.2.** For $z \in \hat{L}_c$,

$$\arg(1 - z) = l\pi + O(\Lambda^{-\frac{1}{2} n})$$

for some $l \in \mathbb{Z}$.

**Proof.** This simply follows from the fact that $\hat{L}_c$ is contained in a $O(\Lambda^{-\frac{1}{2} n})$-neighborhood of $[0, 4] \setminus \left( \frac{4}{9}, \frac{16}{9} \right)$. □

Now we estimate the argument of points outside $D^1_c$. Let $L_c := K(f_c) \setminus f_c(D^1_c)$.\footnote{In this section we use $-2$ rather than $\hat{c}$ because the argument here does not work for other Misiurewicz parameter $\hat{c}$.}
Figure 6. The filled Julia set of $f_c$ and $D^0_c, D^1_c$.

Figure 7. Near the critical value and the linearizing coordinate.

**Lemma 8.3.** If $z \in L_c$, then

$$\arg(c - z) = l\pi + O(\Lambda^{-\frac{1}{2}n})$$

for some $l \in \mathbb{Z}$.

**Proof.** First observe that

$$\text{Int } f_c(D^1_c) = -\text{Poi}_c(\lambda^{-n-1}_c \text{Koe}_c(\text{Int } D^0_c)).$$

Since $c$ is $O(\Lambda^{-2n})$-close to $c_n = -\text{Poi}_c(\lambda^{-n-1}_c)$, it is enough to show that for any $z \in \hat{K}^0_c \setminus \lambda^{-n-1}_c \text{Koe}_c(\text{Int } D^0_c)$, we have

$$\arg(z - \lambda^{-n-1}_c) = l\pi + O(\Lambda^{-\frac{1}{2}n})$$

for some $l \in \mathbb{Z}$.

Observe that $\hat{L}_c$ is contained in a $O(\Lambda^{-\frac{1}{2}n})$-component of $[0, \frac{2}{3}] \cup [\frac{10}{9}, 4]$. Let $\hat{L}'_c$ be the intersection of $\hat{L}_c$ with a small neighborhood of $[0, \frac{2}{3}]$ and $\hat{L}''_c$ be the other part; i.e., the intersection of $\hat{L}_c$ with a small neighborhood of $[\frac{16}{9}, 4]$ (if $c \in \mathcal{M}^*$, they are simply the connected components). Then $z \in \hat{K}^0_c \setminus \lambda^{-n-1}_c \text{Koe}_c(\text{Int } D^0_c)$ satisfies either

1. $z \in \lambda^{-n-1}_c(\hat{L}'_c)$,
(2) $z \in \lambda_c^{-k}(\tilde{L}_c)$ but not in $\lambda_c^{-k-1}(\tilde{L}_c)$ for some $0 \leq k \leq n+1$.

For the first case, the lemma follows immediately by the previous lemma and $\lambda^n_c = 16^n + O(n\Lambda^{-n})$. So let us consider the second case. Similarly by the previous lemma, we have

$$\arg(z - \lambda_c^{-k}) = l\pi + P(\Lambda^{-\frac{1}{2}n}).$$

Moreover, since $\tilde{L}_c$ is $O(\Lambda^{-\frac{1}{2}n})$-close to $\left[\frac{16}{\pi^2}, 4\right]$, $z$ is also close to $16^{-k}\left[\frac{16}{\pi^2}, 4\right]$ (in particular we may take $l = 0$). Hence the lemma follows because $\lambda_c^{-k}$ is close to $16^{-k}$ and $\lambda_c^{-n-1}$ is close to $16^{-n-1} < 16^{-k}$ for large $n$. □

By taking inverse images by $f_c(z) = \bar{z}^2 + c$, we have the following:

**Corollary 8.4.** For $z \in (D^0_c \setminus D^1_c) \cap K(f_c)$,

$$\arg z = \frac{l\pi}{2} + O(\Lambda^{-\frac{1}{2}n})$$

for some $l \in \mathbb{Z}$.

In fact the corollary holds for $z \in K(f_c) \setminus D^1_c$.

By applying Lemma 4.2 repeatedly, we have the following:

**Lemma 8.5.** Fix $m > 0$. Then for $z \in (D^m_c \setminus D^{m+1}_c) \cap K(f_c)$,

$$\arg z = \frac{l\pi}{2m} + O(\Lambda^{-\frac{1}{2}n}).$$

### 9. Estimate of Fatou coordinates

Let $c = c_n + \rho_c(t)$ be such that $\chi_n(c) = \omega/4$, i.e., $g_c$ is hybrid equivalent to $f_{\omega/4}(z) = \bar{z}^2 + \frac{\omega}{4}$, where $\omega = \frac{-1 + \sqrt{3}i}{2}$ is a cubic root of unity. The existence of such $c$ is guaranteed by the compactness of the renormalizable parameters [IM21, Corollary 7.1] (see also [IK12, Theorem D]), the surjectivity onto hyperbolic maps [IM21, Theorem 5.9] (see also [IK12, Theorem C]) and a standard limiting argument (see [IK12, Lemma 9.2, Lemma 9.5]). See also [SW21].

We describe Fatou coordinates for $g_c$ in terms of those of $f_{\omega/4}$. Let $\eta_c : \mathbb{C} \to \mathbb{C}$ be the quasiconformal map constructed in Section 4. Recall that $\eta_c$ is a hybrid conjugacy from $F_c : U_1' \to \mathbb{C}$ to $f_{\omega/4}$. Since a hybrid conjugacy is holomorphic in a parabolic basin, the attracting Fatou coordinate satisfies

$$\Fat_{c, \text{attr}} = \Fat_{\omega/4, \text{attr}} \circ \eta_c.$$

Therefore, the attracting Ecalle height is invariant by taking hybrid conjugacy. In particular, the critical Ecalle height satisfies

$$E_{c, \text{attr}}(c) = E_{\omega/4, \text{attr}}(\omega/4) = 0$$

by Lemma 2.7

Let

$$\Fat_{\omega/4, \text{rep}} : V_0(:= V_{\text{rep}}) \to \mathbb{C}$$

be a repelling Fatou coordinate for $f_{\omega/4}$. Since the critical Ecalle height is zero, we may assume $\Fat_{\omega/4, \text{rep}}(\omega/4) = 0$. We describe a repelling Fatou coordinate for $g_c$ in terms of $\Fat_{\omega/4, \text{rep}}$. Take $n$ sufficiently large so that $\eta_c^{-1}(V_0)$ is contained in $U_1'$ and $\Fat_{\omega/4, \text{rep}} \circ \eta_c$ is a quasiconformal conjugacy between $g_c$ and $z \mapsto \bar{z} + \frac{\omega}{4}$.

By pushing forward the standard complex structure $\sigma_0$ by $\Fat_{\omega/4, \text{rep}} \circ \eta_c$, we can define an almost complex structure $\sigma$ on $\mathbb{C}/\mathbb{Z}$. Since both of the ends correspond to the filled Julia set, $\sigma = \sigma_0$ on neighborhoods of the ends. More precisely,
the complex dilatation of $\sigma$ is supported on a compact set independent of $n$ by Lemma 6.2.

By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\zeta : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ fixing 0 such that $\zeta^* \sigma_0 = \sigma$. Let $\hat{\zeta} : \mathbb{C} \to \mathbb{C}$ be the lift of $\zeta$ fixing 0. Then Fat$_{g_e} := \hat{\zeta} \circ$ Fat$_{\omega/4,\text{rep}} \circ \eta_e$ is a repelling Fatou coordinate for $g_e$ sending the critical value to zero.

**Lemma 9.1.** For $z \in \mathbb{R}$, the repelling Fatou coordinate Fat$_{g_e}$ converges to Fat$_{\omega/4,\text{rep}}(z)$ uniformly and exponentially.

**Proof.** By Lemma 6.1 it suffices to show that $\zeta(z) - z$ converges to zero exponentially for $z \in \mathbb{R}/\mathbb{Z}$.

Let $\xi : \mathbb{C}^* \to \mathbb{C}^*$ be the quasiconformal mapping satisfying $\xi(\exp(2\pi i z)) = \exp(2\pi i \zeta(z))$ and $\xi(1) = 1$. The support of the Beltrami coefficient $\mu_\xi$ of $\xi$ is contained in a compact annulus $\{ z \in \mathbb{C} : 1/r^* \leq |z| \leq r^* \}$ for some $r^* > 1$ that is independent of $c = c_n + \rho_n(t)$ with $t = O(1)$. Moreover, by definition, the maximal dilatations of $\zeta$ and $\xi$ are bounded by that of $\eta_e$, which is $O(\ell/R) = O(\Lambda^{-1 - 2^{-m}n})$ by (11). Hence a similar argument to the proof of Lemma 6.1 yields

$$\| \mu_\xi \|_p = O(\| \mu_\xi \|_\infty) = O(\ell/R)$$

for some $p > 2$ that is independent of both $c$ and sufficiently large $n$, and thus the normal solution $\hat{\xi}$ of the Beltrami equation for $\mu_\xi$ (i.e., $\hat{\xi}_z = \mu_\xi \hat{\xi}_z$ a.e., $\hat{\xi}(0) = 0$, and $\hat{\xi}$ is tangent to the identity at $\infty$) satisfies

$$|\hat{\xi}(z) - z| \leq K_p \| \mu_\xi \|_p \cdot |z|^{1 - 2/p} = O(\ell/R)$$

when $|z| \leq r^*$. Since $\xi(z) = \hat{\xi}(z)/\hat{\xi}(1) = z + O(\ell/R)$ for $|z| = 1$, we obtain

$$|\zeta(z) - z| \approx |e^{2\pi i \hat{\xi}(z)} - e^{2\pi i z}|$$

$$= |\xi(e^{2\pi i z}) - e^{2\pi i z}|$$

$$= O(\ell/R) = O(\Lambda^{-1 - 2^{-m}n})$$

for $z \in \mathbb{R}/\mathbb{Z}$. \hfill $\square$

### 10. Proof of Theorem 1.1

We use the notations in Section 9. Recall that $\Psi_\ell = \text{Fat}_{\omega/4,\text{rep}}^{-1}$ can be extended holomorphically on $\mathbb{C}$. Since the critical Ecalle height for $f_e$ is zero, the hypothesis of Theorem 2.8 for $c_0 = c = c_n + \rho_n(t)$ is equivalent that $\Psi_\ell^{-1}(K(f_e))$ does not intersect $\mathbb{R}$.

Let $\Psi : \mathbb{C} \to \mathbb{C}$ be the holomorphic extension of $\Psi_{\omega/4}$. Recall that

$$R_{\omega/4}(1/3) = \omega(1/2, \infty) = \hat{\Psi}(\mathbb{R})$$

is a half line by Lemma 2.7.

Take $x_0 \in R_{\omega/4}(1/3)$ and consider a closed subarc $\gamma_0 \subset R_{\omega/4}(1/3)$ between $x_0$ and $f_{\omega/4}(x_0)$. Let $r_0 := |\text{B"{o}tt}_{\omega/4}(x_0)|$ and $s_0 = \Phi_{\omega/4}(x_0) \in \mathbb{R}$.

**Lemma 10.1.** Both $\Psi_{\mu}([s_0, s_0 + 1/2])$ and $\text{B"{o}tt}_{g_e}^{-1}(e^{2\pi i 3/\mu}[r_0, r_0^2])$ are contained in a $O(\Lambda^{-m})$-neighborhood of $\gamma_0$ for some $\mu > 0$.

**Proof.** This is an immediate consequence of Lemma 7.1 and Lemma 9.1 \hfill $\square$
Therefore to prove that the hyperbolic component containing $c_n$ in its boundary is accessible, it is enough to show that $\hat{K}_c := \alpha_n K(f_c)$ does not intersect this neighborhood, say $W$.

**Lemma 10.2.** $2^{j_n} = O(n)$.

**Proof.** Recall that if $z \in \tilde{D}_m \setminus \tilde{D}_m^+ + d$, then $\Lambda^{\frac{1}{2m}+\frac{1}{n}} \lesssim |z| \lesssim \Lambda^{\frac{1}{2m}}$ for large $n$ by Lemma 4.2, where $\Lambda \lesssim B$ stands for $A = O(B)$. Hence $\text{B"{o}t}_{g_c}(z)$ also satisfies this asymptotic inequality by Lemma 7.1.

On the other hand, we have $\log |\text{B"{o}t}_{g_c}(g_j^{c_n}(x_0))| = 2^{j_n} \log |\text{B"{o}t}_{g_c}(x_0)|$.

Since $x_0$, $m$ and $d$ are fixed, we have $2^{j_n} = O(n)$. \hfill \Box

Hence, by Theorem 3.3 we have the following:

**Corollary 10.3.** $g_j^{c_n}(W)$ is contained in an $O(n\Lambda^{-\mu n})$-neighborhood of $R_{\omega/4}(1/3)$.

Therefore by Lemma 8.5 it follows that $g_j^{c_n}(W)$ does not intersect $\hat{K}_c$ for sufficiently large $n$. Hence neither does $W$ and we have proved Theorem 1.1.

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