On finite termination of the generalized Newton method for solving absolute value equations

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Abstract
Motivated by the framework constructed by Brugnano and Casulli (SIAM J. Sci. Comput. 30: 463–472, 2008), we analyze the finite termination property of the generalized Newton method (GNM) for solving the absolute value equation (AVE). More precisely, for some special matrices, GNM is terminated in at most $2n + 2$ iterations. A new result for the unique solvability and unsolvability of the AVE is obtained. Numerical experiments are given to demonstrate the theoretical analysis.

Keywords Absolute value equation · The generalized Newton method · Finite termination · Unique solvability · Unsolvability

Mathematics Subject Classification 90C33 · 90C30 · 65H10

1 Introduction

Given $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, we consider the system of absolute value equations (AVE)

$$Ax - |x| - b = 0, \quad (1.1)$$

where $|x| = (|x_1|, \cdots, |x_n|)^T$ represents the componentwise absolute value of the unknown vector $x \in \mathbb{R}^n$. AVE (1.1) has attracted considerable attention in the optimization community since its relevance to many mathematical programming problems, including the linear...
complementarity problem, the bimatrix game and others (see, e.g., Mangasarian 2007a; Mangasarian and Meyer 2006; Prokopyev 2009 and the references therein). Recently, the non-Lipschitz generalization of AVE (1.1) was defined for the first time in Yilmaz and Sahiner (2023).

In general, solving AVE (1.1) is NP-hard (Mangasarian 2007a). In addition, if AVE (1.1) is solvable, checking whether AVE (1.1) has a unique solution or multiple solutions is NP-complete (Prokopyev 2009). Nevertheless, many necessary or sufficient conditions, which guarantee the AVE (1.1) having unique solution for any \( b \in \mathbb{R}^n \), have been constructed, see (Mezzadri 2020; Mangasarian and Meyer 2006; Rohn et al. 2014; Zhang and Wei 2009; Wu and Li 2018; Wu and Shen 2021; Hladík and Moosaei 2023; Wu and Guo 2016; Hladík 2023) and the references therein. Particularly, one of the known sufficient conditions is described in Lemma 1.1.

**Lemma 1.1** (Mangasarian and Meyer 2006) AVE (1.1) is uniquely solvable for any \( b \in \mathbb{R}^n \) if \( \|A^{-1}\| < 1 \).

A number of algorithms have been proposed during the past several years for finding the unique solution of AVE (1.1). For example, the generalized Newton method (GNM) (Mangasarian 2009) and its extensions (Caccetta et al. 2011; Bello Cruz et al. 2016; Wang et al. 2019), the SOR-like iterative methods (Ke and Ma 2017; Guo et al. 2019), the concave minimization methods (Zamani and Hladík 2021; Mangasarian 2007a,b), the Levenberg-Marquardt method (Iqbal et al. 2015), the generalized Gauss-Seidel iterative method (Edalatpour et al. 2017) and its improvements (Ali and Pan 2023; Ali et al. 2022), the exact and inexact Douglas-Rachford splitting methods (Chen et al. 2023) and others, see, e.g., (Rahpeymaii et al. 2019; Gu et al. 2017; Yu et al. 2022; Ke 2020; Saheya et al. 2018; Mansoori and Erfanian 2018; Chen et al. 2021; Abdallah et al. 2018; Khan et al. 2023; Alcantara et al. 2023) and the references therein. Most of the methods mentioned above are proved to be convergent under the condition that \( \|A^{-1}\| < 1 \) (In this situation, it follows from Lemma 1.1 that AVE (1.1) is uniquely solvable). Among these, GNM often obtains the exact solution of AVE (1.1) in just a few iterations, which makes it a competitive method. Theoretically, however, GNM can only be applied to the case with more restrictions. Concretely, Mangasarian in Mangasarian (2009) showed that the sequence generated by GNM is well defined and linearly convergent if \( \|A^{-1}\| < 1/4 \). Thereafter, the researchers in Lian et al. (2018), Bello Cruz et al. (2016), Zamani and Hladík (2023) relaxed the condition to \( \|A^{-1}\| < 1/3 \). Numerically, however, we find that GNM may still work under the less stringent condition that \( \|A^{-1}\| > 0 \) (Chen et al. 2023). Thus, it would be very useful to establish convergence of GNM under this assumption. This paper is devoted to solve this problem to some extend. Indeed, under mild conditions, we conclude that GNM is terminated in at most \( 2n + 2 \) iterations. Our work here is based on a property of GNM proposed by Mangasarian (2009) (see Lemma 2.3 below) and inspired by the work of Brugnano and Casulli (2008).

The rest of this paper is organized as follows. In the next section, some preliminaries are given. The finite termination property of GNM is further discussed in Sect. 3. Numerical experiments are reported in Sect. 4. Conclusions are made in Sect. 5.

**Notation.** We use \( \mathbb{R}^{n \times n} \) to denote the set of all \( n \times n \) real matrices and \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \). \( I \) is the identity matrix with suitable dimension. \( \|A\| \) denotes the spectral norm of the matrix \( A \). For a matrix \( X \), \( \mathcal{N}(X) \) is the null space of \( X \). Let \( \rho(A) \) denote the spectral radius of the square matrix \( A \). Throughout this paper, \( D(x) \equiv \text{diag}(\text{sign}(x)) \) with \( \text{sign}(x) \) denoting a vector with components equal to \(-1, 0, \) or \(1\), respectively, depending on whether the corresponding element in the vector \( x \) is negative, zero, or positive; and for \( x \in \mathbb{R}^n \), \( \text{diag}(x) \) represents a
diagonal matrix with $x_i$ as its diagonal entries for every $i = 1, 2, \cdots, n$. For $X \in \mathbb{R}^{m \times n}$, $X_{(i,j)}$ refers to its $(i, j)$th entry, $|X|$ is in $\mathbb{R}^{n \times n}$ with its $(i, j)$th entry $|X_{(i,j)}|$. Inequality $X \geq Y$ means $X_{(i,j)} \geq Y_{(i,j)}$ for all $(i, j)$. In particular, $X \geq 0$ means that $X$ is a nonnegative matrix. $A^\top$ denotes the transpose of $A$.

2 Preliminaries

In this section, we briefly introduce some preliminaries, which lay the foundation for our later arguments.

A matrix $A \in \mathbb{R}^{n \times n}$ is called a $Z$-matrix if $A_{(i,j)} \leq 0$ for all $i \neq j$. A $Z$-matrix $A$ is called an $M$-matrix if $A^{-1} \geq 0$ (Berman and Plemmons 1994). The following property can be found in Berman and Plemmons (1994).

Lemma 2.1 Let $A \in \mathbb{R}^{n \times n}$ be an $M$-matrix. Let $B \in \mathbb{R}^{n \times n}$ be a $Z$-matrix. If $B \geq A$, then $B$ is also an $M$-matrix.

We assume, throughout the rest of this paper, that either\(^1\)

$$A - I \text{ is an } M - \text{ matrix} \tag{2.1a}$$

or

$$\exists \ 0 < v \in \mathcal{N}(A^\top - I), \text{ and } A - I + D \text{ is an } M\text{-matrix for all diagonal matrices } D = \text{diag}(d) \text{ such that } d \geq 0 \text{ and } d \neq 0. \tag{2.1b}$$

Remark 2.1 For (2.1a), it is well known that there are various characterizations of $M$-matrix and some of them are easy to check (Plemmons 1977). However, it seems difficult to verify (2.1b), especially for large $n$. Here, we explore (2.1b) for $n = 1$ and $n = 2$, and it requires further study to show how to verify (2.1b) for $n \geq 3$.

If $n = 1$, (2.1b) is satisfied if and only if $A = 1$. For a $2 \times 2$ $Z$-matrix $A = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$, it satisfies (2.1b) if and only if

$$a_{11} > 1, \quad a_{22} > 1, \quad a_{12} > 0, \quad a_{21} > 0 \quad \text{and} \quad (a_{11} - 1)(a_{22} - 1) = a_{12}a_{21}. \tag{2.2}$$

The proof is as follows.

If (2.1b) holds, then $A - I$ is singular and thus $(a_{11} - 1)(a_{22} - 1) = a_{12}a_{21}$. In order to prove $a_{12} > 0$ and $a_{21} > 0$, it suffices to prove $a_{12}a_{21} \neq 0$. If $a_{12} = 0$, then $a_{11} = 1$ or $a_{22} = 1$. Suppose $a_{11} = 1$, then $A - I + D$ with $d_1 = 0$ and $d_2 > 0$ is singular, which leads to a contradiction. The same argument goes to $a_{22} = 1$. Hence, we have $a_{12} > 0$ and $a_{21} > 0$. We finally remark that we have $a_{11} > 1$ and $a_{22} > 1$ because $A - I + D$ is an $M$-matrix for the diagonal matrix $D = \text{diag}(d)$ with $d_1 = 0$ and $d_2 > 0$ or $d_1 > 0$ and $d_2 = 0$.

Conversely, if (2.2) holds, then it is easy to check $A - I$ is singular and $A - I + D$ is an $M$-matrix for all diagonal matrices $D = \text{diag}(d)$ such that $d \geq 0$ and $d \neq 0$. In addition, $(a_{11} - 1)(a_{22} - 1) = a_{12}a_{21}$ implies that $(A^\top - I)v = 0$ holds for $v_2 > 0$ and $v_1 = \frac{a_{21}v_2}{a_{11} - 1} > 0$. In conclusion, (2.2) implies (2.1b).

\(^1\) In the original version of this paper, $\forall \ 0 < v \in \mathcal{N}(A^\top - I)$ was used in (2.1b), and an anonymous referee suggested the new presentation.
**Remark 2.2** In (Brugnano and Casulli 2008; Armijo et al. 2023; Barrios et al. 2016), the piecewise linear system

\[
\max\{0, x\} + Tx = c \tag{2.3}
\]

is considered, which is equivalent to

\[
(I + 2T)x + |x| = 2c. \tag{2.4}
\]

Obviously, AVE (2.4) can be rewritten as

\[
Ax + |x| = b \tag{2.5}
\]

with \( A = I + 2T \) and \( b = 2c \). Note that AVE (2.5) can be reformulated as

\[
A(-x) - | -x| = -b, \tag{2.6}
\]

which is \( Ay - |y| = -b \) with \( y = -x \). Similarly, AVE (1.1) can be reformulated as

\[
Ay + |y| = -b
\]

with \( y = -x \). In this sense, we do not distinguish between AVE (1.1) and AVE (2.5) and we focus on AVE (1.1) in this paper.

Clearly, (2.4) is equivalent to

\[
\hat{A}x - |x| = -2c
\]

with \( \hat{A} = -2T - I \), which is mentioned in Armijo et al. (2023). Although all papers (Brugnano and Casulli 2008; Armijo et al. 2023; Barrios et al. 2016) explore the finite termination property of the semi-smooth Newton method for the piecewise linear system (2.3), different conditions are used. The assumption (2.1) is based on (2.5) and inspired by Brugnano and Casulli (2008).

For the convenience of reference, we formally present GNM for AVE (1.1) in Algorithm 2.1.

**Algorithm 2.1** (Mangasarian 2009) Assume that \( x^0 \in \mathbb{R}^n \) is an arbitrary initial guess. For \( k = 0, 1, 2, \cdots \) until the iterative sequence \( \{x^k\}^\infty_{k=0} \) is convergent, compute \( x^{k+1} \) by

\[
x^{k+1} = \left[ A - D(x^k) \right]^{-1} b. \tag{2.7}
\]

The following lemma implies that GNM (2.7) is well defined when \( \|A^{-1}\| < 1 \).

**Lemma 2.2** (Caccetta et al. 2011) Suppose that \( \|A^{-1}\| < 1 \) and \( D = \text{diag}(d) \) with \( d_i \in [-1, 1] \) \((i = 1, 2, \cdots, n)\). Then, \( A - D \) is nonsingular.

The GNM iteration (2.7) has an attractive property presented in the following lemma. However, the upper bound of \( k \) is unknown.

**Lemma 2.3** (Mangasarian 2009) Let \( \|A^{-1}\| < 1 \). If \( D(x^{k+1}) = D(x^k) \) for some \( k \) for the well defined GNM iteration (2.7), then \( x^{k+1} \) solves AVE (1.1).
3 Main results

In this section, based on (2.1), we will discuss the finite termination property of the GNM iteration (2.7). In our analysis we assume exact arithmetic.

**Lemma 3.1** Let the matrix A satisfy either (2.1a) or (2.1b). If A satisfies (2.1b), assume also that \( D(x^0) \neq I \) and \( v^T b < 0 \). Then \( A - D(x^k) \) is an M-matrix for \( k = 0, 1, 2, \cdots \), and the GNM iteration (2.7) is well defined.

**Proof** If A satisfies (2.1a), it follows from Lemma 2.1 that \( A - D(x^0) \) is an M-matrix with \( D(x^0) \in [-I, I] \) and thus the GNM iteration (2.7) is well defined.

If A satisfies (2.1b) and \( D(x^0) \neq I \), then \( A - D(x^0) = A - I + [I - D(x^0)] \) is an M-matrix. Since \( v^T b < 0 \), one has

\[
v^T [A - D(x^0)] x^1 = v^T [(A - I) + [I - D(x^0)]] x^1 = v^T [I - D(x^0)] x^1 = v^T b < 0,
\]

which implies that at least one entry of \( x^1 \) is strictly negative. Thus \( D(x^1) \neq I \) and \( A - D(x^1) = A - I + [I - D(x^1)] \) is an M-matrix. In the same manner, we can recursively prove that \( A - D(x^k) \) is an M-matrix with \( k \geq 2 \) and thus the GNM iteration (2.7) is also well defined.

We are now in the position to present the finite termination property of the GNM iteration (2.7).

**Theorem 3.1** Let the matrix A satisfy either (2.1a) or (2.1b). If A satisfies (2.1b), assume also that \( D(x^0) \neq I \) and \( v^T b < 0 \). Then, the GNM iteration (2.7) converges to an exact solution of AVE (1.1) in at most \( 2n + 2 \) iterations.

**Proof** The proof is inspired by (Brugnano and Casulli 2008, Theorem 2). It follows from Lemma 3.1 that (2.7) is well defined. By iterative scheme (2.7), we have

\[
[A - D(x^k)] x^{k+1} = [A - D(x^{k-1})] x^k = b, \quad k = 1, 2, \cdots,
\]

which implies

\[
[A - D(x^k)] x^{k+1} = [A - D(x^k)] x^k + \xi^k, \quad k = 1, 2, \cdots, \tag{3.1}
\]

where \( \xi^k = [D(x^k) - D(x^{k-1})] x^k \). Next we prove \( \xi^k \geq 0 \) (\( k = 1, 2, \cdots \)). Denote \( d^k_i \) as the i-th diagonal entry of \( D(x^k) \), we have

\[
\begin{cases}
   x^k_i > 0 \iff d^k_i = 1 \Rightarrow d^k_i - d^{k-1}_i \geq 0 \Rightarrow (d^k_i - d^{k-1}_i)x^k_i \geq 0 \Rightarrow \xi^k_i \geq 0, \\
   x^k_i = 0 \Rightarrow (d^k_i - d^{k-1}_i)x^k_i = 0 \Rightarrow \xi^k_i = 0, \\
   x^k_i < 0 \iff d^k_i = -1 \Rightarrow d^k_i - d^{k-1}_i \leq 0 \Rightarrow (d^k_i - d^{k-1}_i)x^k_i \geq 0 \Rightarrow \xi^k_i \geq 0.
\end{cases}
\]

Since \( [A - D(x^k)]^{-1} \geq 0 \) and \( \xi^k \geq 0 \), it follows from (3.1) that \( x^{k+1} \geq x^k \). Hence, \( D(x^{k+1}) \geq D(x^k) \) (\( k = 1, 2, \cdots \)). From the proof of Lemma 2.3, if \( D(x^{k+1}) = D(x^k) \) for some \( k \geq 0 \), then \( x^{k+1} \) is an exact solution of AVE (1.1). If \( D(x^{k+1}) \neq D(x^k) \), namely, \( d^{k+1}_i > d^k_i \) for some \( i \in \{1, 2, \cdots, n\} \) and \( k \geq 1 \), then \( D(x^{k+1}) \neq D(x^k) \) may occur at most \( n - m + 1 \) times, where

\[
m = \sum_{i=1}^{n} d^1_i.
\]
In the hypothetical case that \( D(x^0) \neq D(x^1) \), \( m = -n \) and \( \sum_{i=1}^{n} d_i^k = k - n - 1 \) (1 \( \leq k \leq 2n + 1 \)), \( D(x^{k+1}) \neq D(x^k) \) would occur \( 2n + 1 \) times. Therefore, the sequence \( \{x^k\} \) generated by (2.7) converges to an exact solution of AVE (1.1) in at most \( 2n + 2 \) iterations. \( \Box \)

As by-products of Theorem 3.1, we have the following results.

**Theorem 3.2** Let the matrix \( A \) satisfy either (2.1a) or (2.1b) with \( v^\top b < 0 \). Then the solution of AVE (1.1) exists and is unique.

**Proof** The existence of a solution is proved constructively by Theorem 3.1. In the following, we prove the uniqueness.

For any two vectors \( x, y \in \mathbb{R}^n \), one has

\[
D(x)x - D(y)y = Q(x - y),
\]

where \( Q = \text{diag}(q) \in \mathbb{R}^{n \times n} \) is a diagonal matrix whose diagonal entries \( q_i \in [-1, 1] \) (\( i = 1, 2, \cdots, n \)). Indeed, we have

1. \( x_i, y_i \geq 0 \) \( \Rightarrow \) \( \text{sign}(x_i) = 0 \) or 1, \( \text{sign}(y_i) = 0 \) or 1 \( \Rightarrow \) \( q_i = 1 \);
2. \( x_i, y_i < 0 \) \( \Rightarrow \) \( \text{sign}(x_i) = \text{sign}(y_i) = -1 \) \( \Rightarrow \) \( q_i = -1 \);
3. \( x_i \geq 0 > y_i \) \( \Rightarrow \) \( \text{sign}(x_i) = 1 \) or 0, \( \text{sign}(y_i) = -1 \) \( \Rightarrow \) \( -1 \leq q_i < 1 \);
4. \( x_i < 0 \leq y_i \) \( \Rightarrow \) \( \text{sign}(x_i) = -1, \text{sign}(y_i) = 1 \) or 0 \( \Rightarrow \) \( -1 \leq q_i < 1 \).

Now, we assume that \( x \) and \( y \) are both solutions of AVE (1.1), i.e.,

\[
[A - D(x)]x = b, \quad [A - D(y)]y = b.
\]

If \( A \) satisfies (2.1a), it is obvious that \( A - Q \) is an M-matrix. Therefore,

\[
[A - D(x)]x - [A - D(y)]y = (A - Q)(x - y) = 0,
\]

which implies \( x = y \) according to the nonsingularity of \( A - Q \). If \( A \) satisfies (2.1b) and \( v^\top b < 0 \), we have

\[
v^\top [A - D(x)]x = v^\top [I - D(x)]x = v^\top b < 0,
\]

\[
v^\top [A - D(y)]y = v^\top [I - D(y)]y = v^\top b < 0.
\]

Therefore, there are \( i \) and \( j \) such that \( x_i < 0 \) and \( y_j < 0 \), which implies that at least one of the diagonal entries of \( Q \) is strictly less than 1. Thus, \( A - Q = A - I + (I - Q) \) is an M-matrix and it follows from (3.2) that \( x = y \). \( \Box \)

**Remark 3.1** The unique solvability condition (2.1a) of AVE (1.1) is also studied by Wu and Guo (2016) and we give a new proof in Theorem 3.2. In addition, Hladík and Moosaei in Hladík and Moosaei (2023) show that (2.1a) implies \( \rho(|A^{-1}|) < 1 \), a sufficient condition due to Rohn et al. (2014). If (2.1a) holds, \( A \) is an M-matrix and \( ||A^{-1}|| < 1 \) implies \( \rho(|A^{-1}|) = \rho(A^{-1}) \leq ||A^{-1}|| < 1 \). In general, however, \( ||A^{-1}|| < 1 \) does not imply \( \rho(|A^{-1}|) < 1 \) (Zhang and Wei 2009). Moreover, (2.1a) does not imply \( ||A^{-1}|| < 1 \) and vice versa. For example, let (Zhang and Wei 2009)

\[
A = \begin{pmatrix} 1.5 & -3 \\ 0 & 1.5 \end{pmatrix},
\]

then \( A - I \) is an M-matrix but \( ||A^{-1}|| \approx 1.6095 > 1 \). Let

\[
A = \begin{pmatrix} 1.5 & -1.25 \\ 0 & 1.5 \end{pmatrix}.
\]
we have \( \|A^{-1}\| = 1 \) and \( A - I \) is an \( M \)-matrix. On the other hand, for the matrix satisfied \( \|A^{-1}\| < 1 \), \( A - I \) may not be a \( Z \)-matrix not to mention satisfying (2.1a). For example (Zhang and Wei 2009),

\[
A = \begin{pmatrix}
1 & -0.01 \\
0.01 & 1
\end{pmatrix}.
\]

However, when \( A \) is symmetric and \( \|A^{-1}\| \geq 1 \), it can be proved that (2.1a) does not hold. Indeed, if \( A \) is symmetric and \( A - I \) is an \( M \)-matrix, then all eigenvalues of \( A - I \) are positive (Berman and Plemmons 1994, P. 141), which implies that all eigenvalues of \( A \) is larger than one. Thus, all eigenvalues of \( A^{-1} \) lay in \((0, 1)\), which implies that \( \|A^{-1}\| < 1 \). This leads to a contradiction.

It follows from Lemma 2.2 that (2.1b) does not hold whenever \( \|A^{-1}\| < 1 \). When \( \|A^{-1}\| \geq 1 \), (2.1b) can hold. For instance (see Examples 4.5-4.6 for more detail), the matrix

\[
A = \begin{pmatrix}
3 & -2 \\
-2 & 3
\end{pmatrix}
\]

satisfies (2.1b) with \( \|A^{-1}\| = 1 \), and the matrix

\[
A = \begin{pmatrix}
3 & -1 \\
-4 & 3
\end{pmatrix}
\]

satisfies (2.1b) with \( \|A^{-1}\| \approx 1.1708 > 1 \).

Finally, we have the following corollary, which is different from the existing results in Mangasarian and Meyer (2006), Prokopyev (2009).

**Corollary 3.1** Let the matrix \( A \) satisfy (2.1b). Then we have the following claims.

1. If \( v^T b = 0 \) and \( A \) is symmetric, then a solution of AVE (1.1) exists but is not unique;
2. If \( v^T b > 0 \), then AVE (1.1) has no solutions.

**Proof** If (2.1b) and \( v^T b = 0 \) hold, then \( b \) is in the range of \( A - I \). Consequently, there exists a vector \( u \) such that \( (A - I)u = b \). Let \( \alpha < \min_i \frac{u_i}{v_i} \) and denote

\[
x(\alpha) = u - \alpha v,
\]

then \( x(\alpha) > 0 \), from which and the symmetry of \( A \) we have

\[
(A - I)x(\alpha) = (A - I)u - \alpha(A - I)v = (A - I)u - \alpha(A^T - I)v = (A - I)u = b.
\]

(3.3)

Hence, \( x(\alpha) \) is solution of AVE (1.1).

If (2.1b) and \( v^T b > 0 \) hold, let \( x \) be a solution of AVE (1.1), then we have

\[
v^T [A - D(x)]x = v^T [A - I + I - D(x)]x = v^T [I - D(x)]x = v^T b > 0,
\]

which is impossible since \( x - |x| \leq 0 \) and \( v > 0 \).

**Remark 3.2** In (3.3), we use the symmetry of \( A \). However, we find that the symmetry seems redundant for \( n = 2 \). Indeed, let \( A = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \), we have

\[
\begin{cases}
a_{11}x_1 - a_{12}x_2 - |x_1| = b_1, \\
a_{21}x_1 + a_{22}x_2 - |x_2| = b_2.
\end{cases}
\]

(3.4)
If \( x_1, x_2 \geq 0 \), it follows from (3.4) that
\[
\begin{align*}
(a_{11} - 1)x_1 - a_{12}x_2 &= b_1, \\
-a_{21}x_1 + (a_{22} - 1)x_2 &= b_2.
\end{align*}
\tag{3.5}
\]

If (2.1b) holds, it follows from (2.2) that \( a_{11} - 1 > 0 \). Multiplying the first equation of (3.5) by \( -\frac{a_{21}}{a_{11} - 1} \) follows
\[
-a_{21}x_1 + \frac{a_{12}a_{21}}{a_{11} - 1}x_2 = -\frac{a_{21}}{a_{11} - 1}b_1,
\]
which is equivalent to the second equation of (3.5) since \( a_{22} - 1 = \frac{a_{12}a_{21}}{a_{11} - 1} \) and \( b_2 = -\frac{a_{21}}{a_{11} - 1}b_1 \). \(^3\) Thus, any \( x \) with \( x_2 \geq 0 \) and \( x_1 = \frac{a_{12}x_2 + b_1}{a_{11} - 1} \geq 0 \) are solutions of (3.4). For instance, if
\[
A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 3 \end{bmatrix},
\]
then (2.1b) is satisfied with \( v = (1, \frac{1}{3}) \) and \( v^\top b = 0 \). It is easy to check that any \( x \in \mathbb{R}^2 \) with \( x_1 = x_2 - 1 \) and \( x_2 \geq 1 \) are solutions of AVE
\[
\begin{align*}
2x_1 - x_2 - |x_1| &= -1, \\
-3x_1 + 4x_2 - |x_2| &= 3.
\end{align*}
\]

When \( n \geq 3 \), it is an open question whether the first assertion of Corollary 3.1 is valid without the symmetry of \( A \).

### 4 Numerical experiments

In this section, we will give six numerical examples to illustrate the theoretical analysis presented in this paper. The purpose here is to show that GNM is applicable under the conditions in Theorem 3.1, no matter \( \|A^{-1}\| < 1 \), \( \|A^{-1}\| = 1 \) or \( \|A^{-1}\| > 1 \). In our implementations, GNM is terminated if \( \text{RES}(x^k) = \|Ax^k - |x^k| - b\| \leq 10^{-7} \). All experiments are implemented in MATLAB R2018b. In all tables, the required number of iterations \( (K) \) are listed with \( \text{RES} \), \( \delta_k \) and \( \Delta_k \), where \( \delta_k \) and \( \Delta_k \) are defined by (in MATLAB expressions)
\[
\delta_k = \sum (d^k - d^{k-1}), \quad k = 1, \cdots, K
\]
and
\[
\Delta_k = \sum (d^k - d^{k-1} \geq 0), \quad k = 1, \cdots, K
\]
with \( D(x^k) = \text{diag}(d^k) \).

\(^2\) It follows from the fact that there is a positive vector \( v \) such that \( (A^\top - I)v = 0 \).

\(^3\) It follows from the fact that there is a positive vector \( v \) such that \( (A^\top - I)v = 0 \) and \( v^\top b = 0 \).
Table 1  Numerical results for Example 4.1

| n    | K | δ₁ | δ₂ | δ₃ | δ₄ | Δ₁  | Δ₂  | Δ₃  | Δ₄  | RES       |
|------|---|----|----|----|----|-----|-----|-----|-----|----------|
| 2000 | 3 | -98| 170| 0  | -  | 1560| 2000| 2000| -   | 1.3229 × 10⁻¹³ |
| 4000 | 3 | -348| 342| 0  | -  | 3100| 4000| 4000| -   | 1.8705 × 10⁻¹³ |
| 6000 | 3 | -552| 506| 0  | -  | 4642| 6000| 6000| -   | 2.2644 × 10⁻¹³ |
| 8000 | 3 | -756| 658| 0  | -  | 6182| 8000| 8000| -   | 2.5992 × 10⁻¹³ |
| 10000| 4 | -1016| 848| 2  | 0  | 7706| 10000| 10000| 10000| 2.8946 × 10⁻¹³ |

Example 4.1  The first example is inspired by Brugnano and Casulli (2008), in which we have \( \|A^{-1}\| < 1 \) and (2.1a) is satisfied. Consider AVE (1.1) with

\[
A = \begin{pmatrix}
7 & -2 & 0 & \cdots & 0 & 0 \\
-2 & 7 & -2 & \cdots & 0 & 0 \\
0 & -2 & 7 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 7 & -2 \\
0 & 0 & \cdots & -2 & 7
\end{pmatrix} \in \mathbb{R}^{n \times n}
\]  
and  
\( b = Ax^* - |x^*| \),

where \( x^* = -1 + 2 \times \text{rand}(n, 1) \) with the random seed \text{rng}(168).\(^4\) For this example, we choose \( x^0 = -1 + 2 \times \text{rand}(n, 1) \) with the random seed \text{rng}(256) as the initial point. Numerical results are reported in Table 1. It follows from Table 1 that GNM converges to the exact solution in at most four iterations for different values of \( n \). In addition, it follows from \( \delta_k \geq 0 \) (\( 2 \leq k \leq K \)) and \( \Delta_k = n(2 \leq k \leq K) \) that \( x_k \geq x_{k-1} \) (\( 2 \leq k \leq K \)). However, \( \delta_1 < 0 \) or \( \Delta_1 < n \) implies that \( x_1 \geq x^0 \) does not hold.

Example 4.2  Let the matrix \( A \) be produced by the following MATLAB algorithm. According to (Plemmons 1977, N39) the matrix \( A \) satisfies (2.1a). In addition, we always have \( \|A^{-1}\| < 1 \) for the values of \( n \) used below. For this example, we set \( x^* = -1 + 2 \times \text{rand}(n, 1) \), \( b = Ax^* - |x^*| \) and \( x^0 = -2 + 4 \times \text{rand}(n, 1) \).

\[
\text{rng}(256); \\
D = 1 + \text{rand}(n, 1); \\
A = -5 + 4 \times \text{rand}(n, n); \\
\text{for } i = 1:n \\
\quad \text{while } A(i, i) * D(i) \leq (\text{abs}(A(i, 1:i-1)) * D(1:i-1) + \text{abs}(A(i, i+1:n)) * D(i+1:n)) \\
\quad \quad A(i, i) = A(i, i) + 10; \\
\text{end} \\
\text{end} \\
A = A + \text{eye}(n, n); 
\]

In contrast with Example 4.1, the matrix \( A \) of this example is dense and it is neither symmetric nor diagonally dominant. However, numerical results reported in Table 2 also demonstrate that GNM converges to the exact solution in at most four iterations, and it has \( x_k \geq x_{k-1} \) (\( 2 \leq k \leq K \)) while \( x^1 \geq x^0 \) does not hold.

\(^4\) Here and in the sequel, a random seed is used for the sake of reproducibility.
When the matrix $A$ satisfies $(2.1a)$ and $\|A^{-1}\| \geq 1$, we construct the following two examples.

**Example 4.3** Consider AVE (1.1) with

$$A = \begin{pmatrix} 1.5 & -1.25 \\ 0 & 1.5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4 \\ 16 \end{pmatrix}. $$

Note that $\|A^{-1}\| = 1$. Moreover, the unique solution of the AVE (1.1) is $x^* = (88, 32)^\top$. For this example, we set $x^0 = -1 + 2 * \text{randn}(2, 1)$ with the random seed rng(168). Numerical results are reported in Table 3. It follows from Table 3 that GNM converges to the exact solution at the second iteration with $x^2 \geq x^1 \geq x^0$ for this example.

**Example 4.4** Consider AVE (1.1) with

$$A = \begin{pmatrix} 1.5 & -3 \\ 0 & 1.5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -2 \\ -3 \end{pmatrix}. $$

Note that $\|A^{-1}\| \approx 1.6095 > 1$. Moreover, the unique solution of the AVE (1.1) is $x^* = (-2.24, -1.2)^\top$. For this example, we set $x^0 = -0.5 + 2 * \text{randn}(2, 1)$ with the random seed rng(256). Numerical results are reported in Table 4. It follows from Table 4 that GNM converges to the exact solution at the second iteration with $x^2 \geq x^1$.

Before concluding this section we give two examples with the matrix $A$ satisfying $(2.1b)$ and $v^\top b < 0$. 

| Table 2 | Numerical results for Example 4.2 |
|---------|---------------------------------|
| $n$     | $K$  | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\Delta_4$ | RES      |
|---------|------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|----------|
| 2000    | 3    | -176        | 150         | 0           | -1449       | 2000        | 2000        | -           | 3.5299 $\times 10^{-10}$ |
| 4000    | 3    | -406        | 340         | 0           | -2902       | 4000        | 4000        | -           | 1.4446 $\times 10^{-9}$  |
| 6000    | 4    | -304        | 488         | 6           | 4452        | 6000        | 6000        | 6000        | 3.8508 $\times 10^{-9}$  |
| 8000    | 4    | -644        | 648         | 6           | 5833        | 8000        | 8000        | 8000        | 6.7734 $\times 10^{-9}$  |
| 10000   | 4    | -958        | 814         | 4           | 7248        | 10000       | 10000       | 10000       | 1.0707 $\times 10^{-8}$  |

| Table 3 | Numerical results for Example 4.3 |
|---------|---------------------------------|
| $n$     | $K$  | $\delta_1$ | $\delta_2$ | $\Delta_1$ | $\Delta_2$ | RES |
|---------|------|-------------|-------------|-------------|-------------|-----|
| 2       | 2    | 4           | 0           | 2           | 2           | 0   |

| Table 4 | Numerical results for Example 4.4 |
|---------|---------------------------------|
| $n$     | $K$  | $\delta_1$ | $\delta_2$ | $\Delta_1$ | $\Delta_2$ | RES      |
|---------|------|-------------|-------------|-------------|-------------|----------|
| 2       | 2    | -2          | 0           | 1           | 2           | 2.2204 $\times 10^{-16}$ |
Example 4.5 Consider AVE (1.1) with
\[
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 3 & -1 & \cdots & 0 & 0 \\
0 & -1 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 3 & -1 \\
0 & 0 & \cdots & -1 & 2
\end{pmatrix} \in \mathbb{R}^{n \times n}
\]
and \( b = Ax^* - |x^*| \), where \( x^*_i = (-1)^i \) for \( i = 1, 2, \ldots, n \). For this example, (2.1b) is satisfied with \( v = (1, 1, \cdots, 1)^\top \). In addition, we have \( \|A^{-1}\| = 1 \) and \( v^\top b = -n < 0 \). Thus, GNM with \( D(x^0) \neq I \) will converge within finite iterations. Here we set \( x^0 = -1 + 2 \times \text{randn}(n, 1) \) with the random seed \( \text{rng}(168) \). Numerical results are reported in Table 5. It follows from Table 5 that GNM converges to the exact solution at the third iteration for different values of \( n \). In addition, it follows from \( \delta_k \geq 0 (2 \leq k \leq K) \) and \( \Delta_k = n(2 \leq k \leq K) \) that \( x^k \geq x^{k-1} (2 \leq k \leq K) \). However, \( \delta_1 < 0 \) or \( \Delta_1 < n \) implies that \( x^1 \geq x^0 \) does not hold.

Example 4.6 Consider AVE (1.1) with
\[
A = \begin{pmatrix}
3 & 0 \\
-4 & 3
\end{pmatrix}
\]
and \( b = \begin{pmatrix}
-5 \\
-4
\end{pmatrix} \).

For this example, \( \|A^{-1}\| \approx 1.1708 > 1 \), \( v = (2, 1)^\top \), and \( v^\top b = -14 < 0 \). The exact solution is \( x^* = (-2, -3)^\top \). Here, we set \( x^0 = -0.1 + 2 \times \text{randn}(2, 1) \) with the random seed \( \text{rng}(168) \). Numerical results are reported in Table 6, which are similar to those of Example 4.4.

In conclusion, under the conditions in Theorem 3.1, the above mentioned examples show that GNM converges to the unique solution of AVE (1.1) within \( 2n + 2 \) iterations.

5 Conclusions

For two special classes of matrices, the finite termination property of the generalized Newton method (GNM) for the absolute value equation is further considered in this paper. Under the conditions in Theorem 3.1, GNM is finitely convergent for \( \|A^{-1}\| > 0 \), which is weaker.
than $\|A^{-1}\| < \frac{1}{3}$ (commonly used in the literature). In addition, a new proof of an existing unique solvability condition is given and a new unique solvability condition as well as a new unsolvability condition is developed. Numerical results demonstrate our claims.

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**Declarations**

**Conflicts of interest** The authors declare that there is no conflict of interest to this work.

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