Rigged configurations approach for the spin-1/2 isotropic Heisenberg model

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Abstract

We continue the rigged configurations analysis of the solutions to the Bethe ansatz equations for the spin-1/2 isotropic Heisenberg model. We analyze the non self-conjugate strings of Deguchi–Giri and the counter example for the string hypothesis discovered by Essler–Korepin–Schoutens. In both cases clear discrete structures appear.

Keywords: Bethe ansatz, rigged configurations, Heisenberg model

1. Introduction

Bethe’s famous solution [B] to the spin-1/2 isotropic Heisenberg model under the periodic boundary condition is one of the prototypical examples of the quantum integrable models. In this problem, our first goal is to find all eigenvectors and eigenvalues of the Hamiltonian so that we can describe the complete set of solutions for the corresponding Schrödinger equation. Basically Bethe’s method is comprised of two steps. First we assume a specific form of the eigenstates which depend on parameters $\lambda_j$. We call them the Bethe vectors. Then we determine the parameters $\lambda_j$ by solving the so-called Bethe ansatz equations (see equation (8) for the precise form). Later Bethe’s method was reformulated in a more algebraic form by Faddeev’s school (see [F, KBI]).

Despite the huge amount of works that have appeared over several decades, there are still aspects that remain unclear about the Bethe ansatz theory. When we try to construct the eigenvectors according to the Bethe ansatz, we first have to solve the Bethe ansatz equations. Here we usually assume that it is enough to consider pairwise distinct solutions to the equations. Then one immediately notices that there are too many solutions to the Bethe ansatz equations and only some of them correspond to non-zero Bethe vectors (see, for example, [HNS] and [KS14a, section 4.2]). We call the solutions corresponding to non-zero Bethe vectors regular solutions and the remaining solutions singular solutions.
Unfortunately, the set of regular solutions is too small to construct all the eigenvectors. Therefore, it is natural to look for a method to construct the eigenvectors from a subset of the singular solutions. A particularly convenient method is to introduce certain regularization (see equation (11)) to the solutions to the Bethe ansatz equations [AV, BSMZ]. Recently, Nepomechie–Wang [NW13] worked out the procedure in general and proposed a criterion to select the subset of the singular solutions which generate the remaining eigenvectors after the regularization (see also [NW14] for an alternative derivation). This proposal is checked by an extensive numerical computation [HNS]. Also the corresponding eigenvalues are obtained in [KS14b]. See section 3 and section appendix A for a somewhat extensive review of these developments. Therefore, we now have a conjectural characterization of the solutions to the Bethe ansatz equations which shall generate all the eigenvectors.

Despite these developments, not much is known about the structure of the solutions to the Bethe ansatz equations. Let us call the union of the regular solutions and Nepomechie–Wang’s physical singular solutions physical solutions to the Bethe ansatz equations. A popular method for the analysis of the physical solutions is the so-called string hypothesis [B, T]. In this hypothesis, we assume that the roots of physical solutions appear in the form of strings

\[ a + bi, a + (b - 1)i, a + (b - 2)i, \ldots, a - bi, \quad (a \in \mathbb{R}, b \in \mathbb{Z}_{\geq 0}/2). \quad (1) \]

Based on this assumption, we derive (half) integers called the Bethe–Takahashi quantum numbers (see [T, equation (2.11)]), which label the strings in a solution (see, for example, [HC, SD] for detailed explanations). However, Essler–Korepin–Schoutens [EKS] discovered a counter example to this scenario. Namely, for the 2-down spin sector of length more than 21.86 chains, they discovered a pair of real roots which have the same Bethe–Takahashi quantum number, thus we cannot apply the Bethe–Takahashi quantum number for classification of them. Therefore, we think it is worthwhile to consider a possible method to provide well-defined quantum numbers to classify strings in the solutions to the Bethe ansatz equations.

In this paper, we elaborate the rigged configurations approach for this problem [KS14a] by analyzing much more complicated solutions. The rigged configurations approach is based on the following observation. We introduce an order within the set of physical solutions of the same strings type based on the real parts of the longest strings. Then, from the relative positions of the strings, we construct a bijection with combinatorial objects called the rigged configurations. We apply this method for two classes of solutions. One is the non self-conjugate strings due to Deguchi–Giri [DG] (see section 5). The non self-conjugate string is a set of roots which is not a single string, nevertheless we cannot split it into smaller strings, see equation (20) for an example. Based on the rigged configurations approach, we provide a clear demonstration that the non self-conjugate strings are obtained by a fusion of smaller strings even if they look irregular.

Another solution which we analyze in the present paper is the counter example to the string hypothesis at 2-down spin sector of length 25 chains discovered by Essler–Korepin–Schoutens [EKS] (see section 6). We analyze 255 real solutions to the corresponding Bethe ansatz equations by the rigged configurations approach. Then we naturally find two exceptional real solutions which violate the rigged configurations structure. Next we make correspondence between the set of a combination of two exceptional real solutions and 20 physical complex solutions and the set of the rigged configurations. All physical solutions in this case naturally fit into the rigged configurations scheme. As a result, we can determine the quantum
numbers (we call riggings) of all physical solutions in the present case. Our result agrees with the interpretation by [EKS] except for the exceptional real solutions.

In both cases, we observe clear discrete structures within the solutions to the Bethe ansatz equations. Underlying our philosophy is that the Bethe ansatz method for the spin-1/2 isotropic Heisenberg model is a well-defined mathematical procedure.

This paper is organized as follows. In section 2, we briefly review standard results from the algebraic Bethe ansatz analysis of the spin-1/2 isotropic Heisenberg model. In section 3, we review recent developments about the theory of the singular solutions. In section 4 we review necessary facts about the rigged configurations. In section 5 we analyze the non self-conjugate strings. In section 6 we analyze the counter example to the string hypothesis. In section appendix A we provide complete proofs for the assertions in section 3 according to the line described in [KS14b]. In appendix B we provide an update for our previous paper [KS14a].

All numerical results in this work are due to Mathematica.

2. Algebraic Bethe ansatz

In this section, we briefly review standard facts from the algebraic Bethe ansatz analysis for the spin-1/2 isotropic Heisenberg model with the periodic boundary condition (see [F, KBI] for the details). In the length $N$ chain case, the Hamiltonian is

$$H_N = J \frac{N}{4} \sum_{k=1}^{N} \left( \sigma^1_k \sigma^1_{k+1} + \sigma^2_k \sigma^2_{k+1} + \sigma^3_k \sigma^3_{k+1} - I_N \right), \quad \sigma^a_{N+1} = \sigma^a_1 \quad (a = 1, 2, 3)$$

where $\sigma^a (a = 1, 2, 3)$ are the Pauli matrices and $I_N = I \otimes I$ with the $2 \times 2$ identity matrix $I$. The space of states of our systems is $H_N \cong V \otimes \cdots \otimes V$ where $V_j \cong \mathbb{C}^2$. Then $\sigma^a_k$ acts as $\sigma^a$ on the $k$-th site $V_k$ and as the identity operator on the rest of sites. Below we aim to construct the solutions to the Schrödinger equation $H_N \Psi = \lambda \Psi$ where $\lambda$ is the energy eigenvalue.

A basic apparatus of the algebraic Bethe ansatz is the transfer matrices. For this purpose, we introduce the so-called Lax operators

$$L_k (\lambda) = \begin{pmatrix} \lambda \sigma_k^3 + i \sigma_k^2 & -i \sigma_k^- \\ i \sigma_k^2 & \lambda \sigma_k^- - i \sigma_k^3 \end{pmatrix}, \quad \left( k = 1, 2, \ldots, N, \sigma_k^\pm = \sigma_k^1 \pm i \sigma_k^2 \right)$$

which act on the space $\mathbb{C}_0^2 \otimes \delta_N$. Here the first $\mathbb{C}_0^2$ is the auxiliary space $\mathbb{C}^2$ which provides the above $2 \times 2$ matrix presentation of $L_k (\lambda)$. The matrix entries of $L_k (\lambda)$ act on $\delta_N$. Then the transfer matrix is

$$T_N (\lambda) = L_N (\lambda) L_{N-1} (\lambda) \cdots L_1 (\lambda). \quad (4)$$

From the $2 \times 2$ matrix presentation of $T_N (\lambda)$

$$T_N (\lambda) = \begin{pmatrix} A_N (\lambda) & B_N (\lambda) \\ C_N (\lambda) & D_N (\lambda) \end{pmatrix}$$

we define the operators $A_N (\lambda), B_N (\lambda), C_N (\lambda)$ and $D_N (\lambda)$ which act on the space of states $\delta_N$. Let $T_N (\lambda) = \text{tr} \left[ \lambda \delta_N \right] T_N (\lambda) = A_N (\lambda) + D_N (\lambda)$. Then the fundamental formula in the algebraic Bethe ansatz is
\[ H_N = \frac{iJ}{2} \frac{d}{d\lambda} \log \tau_N(\lambda) \bigg|_{\lambda=\frac{iJ}{2N}} - \frac{NJ}{2} \delta_N. \] (6)

Therefore, in order to obtain the eigenvectors and the eigenvalues of the Hamiltonian \( H_N \), it is enough to diagonalize \( \tau_N(\lambda) \). Let us start from the spacial vector \(|0\rangle_N = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}^{\otimes N} \in \mathcal{F}_N \). Note that the vector \(|0\rangle_N \) is an eigenvector of \( \tau_N(\lambda) \). On this vector, we apply \( B_N(\lambda) \) as the creation operator. We call the following vectors the Bethe vectors.

\[ \Psi_\lambda(\lambda_1, \cdots, \lambda_\ell) = B_N(\lambda_1) \cdots B_N(\lambda_\ell) |0\rangle_N. \] (7)

Since the operators \( B_N \) are commutative (see (22)), we see that the order of the parameters \( \lambda_1, \cdots, \lambda_\ell \) is not important. It is known that \( \ell \) is the number of down spins in the state \( \Psi_\lambda \). Then the main construction is as follows.

**Theorem 1.** The Bethe vector \( \Psi_\lambda(\lambda_1, \cdots, \lambda_\ell) \) is an eigenvector of the transfer matrix \( \tau_N(\lambda) \) if and only if the numbers \( \lambda_1, \cdots, \lambda_\ell \) satisfy the following system of algebraic equations

\[ \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right)^N = \prod_{j=1}^{\ell} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad (k = 1, \cdots, \ell). \] (8)

These equations are called the Bethe ansatz equations. □

By a direct computation, we see that non-zero Bethe vector \( \Psi_\lambda(\lambda_1, \cdots, \lambda_\ell) \) is the eigenvector of the Hamiltonian \( H_N \) with the energy eigenvalue \( \mathcal{E} = -\frac{J}{2} \sum_{j=1}^{\ell} \frac{1}{\lambda_j^2 + \frac{1}{4}} \). If a pairwise distinct solution \( (\lambda_1, \cdots, \lambda_\ell) \) to the Bethe ansatz equations corresponds to a non-zero Bethe vector, we call it a **regular solution**.

Note that we can introduce a natural action of the Lie algebra \( \mathfrak{sl}_2 \) on the space \( \mathcal{F}_N \). Then it is known that the Bethe vectors thus obtained correspond to the highest weight vectors. The other eigenvectors of the Hamiltonian are obtained by the lowering operators of \( \mathfrak{sl}_2 \). Nevertheless, it is known for a long time that the regular solutions to the Bethe ansatz equations do not cover all the vectors of \( \mathcal{F}_N \). This will be the subject of the next section.

### 3. Singular solutions to the Bethe ansatz equations

As we see in the previous section, if \( \{\lambda_1, \cdots, \lambda_\ell\} \) is a solution to the Bethe ansatz equations, we have

\[ H_N \Psi_\lambda(\lambda_1, \cdots, \lambda_\ell) = \mathcal{E}_{\lambda_1, \cdots, \lambda_\ell} \Psi_\lambda(\lambda_1, \cdots, \lambda_\ell), \quad \mathcal{E}_{\lambda_1, \cdots, \lambda_\ell} = -\frac{J}{2} \sum_{j=1}^{\ell} \frac{1}{\lambda_j^2 + \frac{1}{4}}. \] (9)

Therefore, the following type of solutions correspond to divergent energy eigenvalues.

**Definition 2.** The pairwise distinct solutions to the Bethe ansatz equations of the form

\[ \left\{ \frac{i}{2}, -\frac{i}{2}, \lambda_3, \cdots, \lambda_\ell \right\}. \] (10)

are called **singular**. □
The Bethe vectors corresponding to singular solutions are the null vector. However, in the following arguments based on the rigged configurations, it is necessarily to analyze all physically relevant solutions to the Bethe ansatz equations. Therefore, we need to introduce a certain regularization scheme corresponding to the singular solutions. We remark that this problem was already noticed in Bethe’s original paper [B].

Following Nepomechie–Wang [NW13], let us consider the following regularization

\[ \lambda_1 = \frac{i}{2} + \epsilon + c e^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon. \]  

As we shall see in the sequel, this regularization leads to finite Bethe vectors. Moreover, Kirillov–Sakamoto [KS14b] derived the corresponding energy eigenvalues under the regularization. Note that this type of regularization was also considered by Avdeev–Vladimirov [AV] in 1987 and Beisert et al [BSMZ, equation (3.4)] in 2003. In the case of \( \ell = 2 \), the resulting eigenvector coincides with the explicit expression of [EKS, equation (26)]. See [GD1] where the authors derived several explicit expressions for the singular eigenvectors, including the \( \ell = 2 \) case, under the same regularization considered here based on the result of [D].

In general, our regularization scheme proceeds as follows. We start from the general singular solution (10). If a singular solution satisfies the following criterion due to Nepomechie–Wang

\[ \left( \prod_{j=3}^{\ell} \frac{\lambda_j + i \frac{3}{2}}{\lambda_j - \frac{i}{2}} \right)^N = 1, \]  

we call it a physical singular solution. We set the number \( c \) of (11) as follows

\[ c = 2i^{N+1} \prod_{j=3}^{\ell} \frac{\lambda_j + i \frac{3}{2}}{\lambda_j - \frac{i}{2}}. \]  

According to [NW13],

\[ \lim_{\epsilon \to 0} \frac{1}{e^{N\epsilon}} B_N \left( \frac{i}{2} + \epsilon + c e^N \right) B_N \left( -\frac{i}{2} + \epsilon \right) B_N (\lambda_3) \cdots B_N (\lambda_{\ell}) |0\rangle_N \]  

gives a well-defined eigenvector of the Hamiltonian with the eigenvalue [KS14b]

\[ E_{\frac{i}{2},-\frac{i}{2}, \ldots, -\frac{i}{2}} = -J - \frac{\ell}{2} \sum_{j=3}^{\ell} \frac{1}{\lambda_j^2 + \frac{1}{4}}. \]  

Then the main conjecture of [NW13] states that the Bethe vectors corresponding to regular solutions and physical singular solutions provide the complete set of the highest weight vectors of the model. Hao–Nepomechie–Sommese [HNS] computed all pairwise distinct solutions to the Bethe ansatz equations up to \( N = 14 \) and confirmed this conjecture. Therefore, we believe this conjecture is very likely to be true.

**Example 3.** When \( N = 4 \), we have the singular solution \( \{ i/2, -i/2 \} \). By an explicit computation, we obtain
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^4} B_i \left( \frac{i}{2} + \epsilon + \epsilon^4 \right) B_i \left( -\frac{i}{2} + \epsilon \right) |0\rangle_i = (0, 0, 0, -2, 0, 0, 2, 0, 0, 0, 0). \tag{16}
\]

If we put \( \epsilon = 2i \), we obtain the correct eigenvector with the energy eigenvalue \(-J\). \( \square \)

Note that the above example forms the initial step of the inductive proof for the assertions in the present section, which is given in the appendix at the end of the present paper.

4. Rigged configurations

Rigged configurations are combinatorial objects introduced by [KKR, KR]. In [KS14a], it was pointed out that the set of regular and physical singular solutions to the Bethe ansatz equations have natural correspondence with the rigged configurations. In the next sections we aim to elaborate this claim by analyzing more complicated solutions. For this purpose we review basic definitions about the rigged configurations following [KS14a].

The rigged configurations are comprised of a set of data \( \mu \) and \( \nu \). \( \mu \) is a sequence of positive integers which specifies the shape of the space of states. If we consider the spin-1/2-Heisenberg model, the space of states is \( \mathcal{H} \equiv \bigotimes_{j=1}^N \mathbb{C}^2 \). Then we set \( \nu_{\mu}(\nu) = \nu_j - \nu_j \) for \( j = 1, \ldots, g \) where \( g \) is the length of the partition \( \nu \). In the definition of the rigged configurations we do not make distinction about the order within the multiset \( \nu \). For \( k \in \mathbb{Z}_{\geq 0} \), define the vacancy numbers \( R(\nu) \) by

\[
R(\nu) = \sum_{j=1}^N \min(k, \nu_j) - 2 \sum_{j=1}^g \min(k, \nu_j). \tag{17}
\]

Note that although we do not explicitly write \( \mu \) in \( R(\nu) \), the vacancy numbers do depend on \( \mu \). The condition that the data \( \mu \) and \( \nu \) to be a rigged configuration is as follows:

\[
0 \leq J_i \leq P_i(\nu), \quad (0 \leq i \leq g). \tag{18}
\]

This condition implicitly requires \( 0 \leq P(\nu) \) as the condition on \( \nu \), in which case the partition \( \nu \) is called admissible. We call \( \nu \) the configuration and \( J_i \) \( (1 \leq i \leq g) \) the rigging.

In the spin-1/2 case, the situation becomes simple. In this case, we have \( \mu = (1^N) \), thus the vacancy number is

\[
R(\nu) = N - 2 \sum_{j=1}^g \min(k, \nu_j). \tag{19}
\]

Here \( \sum_{j=1}^g \min(k, \nu_j) \) is the number of the boxes within the first \( k \) columns of the Young diagram \( \nu \). Then the partition \( \nu \) is admissible when the number of the boxes \( |\nu| \) satisfies \( |\nu| \leq N/2 \). In the application to the Bethe ansatz, \( |\nu| \) is the number of down spins \( \ell \). For example, when \( N = 14 \), \( \ell = 7 \), \( (\nu, J) = \{(3, 0), (2, 1), (1, 4), (1, 2)\} \) is a rigged configuration. We use the following diagram to represent this rigged configuration:
Here we depict $\nu$ as the Young diagram and we put the riggings (resp. vacancy numbers) on the right (resp. left) of the corresponding rows $\nu_i$.

For higher spin cases, the situation becomes more subtle. For the spin-1 case, we have $\mu = (2^N)$. Let $N = 2$. Then $\nu = (1)$ and $(2)$ are admissible, though $\nu = (1, 1)$ is not admissible since we have $R((1, 1)) = \{\min(1, 1) + \min(1, 1)\} - 2\min(1, 1) + \min(1, 1) = -2$.

The basic idea of the correspondence between the solutions to the Bethe ansatz equations and the rigged configurations is as follows. In many cases, roots of the solutions to the Bethe ansatz equations take the following forms:

$$a + bi, a + (b - 1)i, a + (b - 2)i, \ldots, a - bi, \quad (a \in \mathbb{R}, b \in \mathbb{Z}_{\geq 0}/2).$$

We call this a $(2b + 1)$-string. Then we correspond a length $(2b + 1)$ row of $\nu$. The riggings specify the relative positions of the $(2b + 1)$-strings. Below we explain how to realize this correspondence based on concrete examples.

5. Non self-conjugate strings

Deguchi and Giri [DG] reported a new type of solutions which they call non self-conjugate strings. This class of solutions first appears when $N = 12$ and its typical structure is as follows:

$$\{a + i, a + \delta_1 + i\delta_2, a + \delta_1 - i\delta_2, a - i\} \quad (20)$$

where $a \in \mathbb{R}$ and the two real numbers $\delta_1$ and $\delta_2$ are supposed to be small. A complicated feature of this type of solutions is that it is difficult to split the string of solutions into distinct strings of solutions. In this section, we use the rigged configurations to analyze the non self-conjugate strings$^1$.

Following [KS14a] we construct a bijection between the rigged configurations and the solutions to the Bethe ansatz equations for the $N = 12$ and $\ell = 6$ case. Here we concentrate on the physical solutions which contain three strings of lengths 3, 2 and 1. This case corresponds to the rigged configurations of the following type:

For the sake of brevity, we call the above rigged configurations by $(n_1, n_2)$. Since $0 \leq n_1 \leq 2$ and $0 \leq n_2 \leq 6$, we have 21 such rigged configurations. Now we introduce the order for 21 solutions with length 3, 2 and 1 strings according to the real parts of the 3-strings. Then we have a clear classification as follows (asterisk for the solutions number means non self-conjugate strings). We remark that the solution #11* is a physical singular solution. In the following diagrams, the solutions are depicted on the complex plane and the spacing between the dotted lines is 0.5.

$^1$ For the numerical computations of the case $N = 12$ and $\ell = 6$, we greatly benefited from the supplementary tables of [HNS].
Our basic principle is to assign larger riggings for right strings and smaller riggings for left strings. Then if we compare the positions of 2-strings in Groups 1, 2 and 3, we can assign $r_1 = 0$ for Group 1, $r_1 = 1$ for Group 2 and $r_1 = 2$ for Group 3 without ambiguity. Then, within each group, we can assign $r_0, r_1, \ldots, r_{62}$ from a smaller solutions number to a larger solutions number. For example, solution #6* corresponds to (0, 5), solution #11* corresponds to (1, 3) and solution #16* corresponds to (2, 1), respectively. This is in agreement with the results of [DG] based on the Bethe–Takahashi quantum number.
We observe that within each group, the positions of the 2-string and 3-string are almost unchanged whereas the 1-string moves from left to right. In this interpretation, we see that there are collisions of 1-string and 3-string at the solutions #6*, #11* and #16*. To summarize, we have demonstrated that the non self-conjugate strings of the form (20) is generated by a fusion of a 3-string and a 1-string even if they look irregular. As we can see, they fit naturally with the rigged configurations picture.

Finally we remark that there are small deviations for the real parts of three roots within each 3-string. However, as we can see in the following numerical tables, there are always clear gaps between the real parts of neighboring solutions. Therefore, in this case, there is no ambiguity to introduce an order based on the real parts of the 3-strings. In the following tables, the first column indicates the solutions number. Within each solution, the first line contains a 3-string and the second line contains a 2-string and a 1-string.

**Group 1.**

|   |   |   |   |
|---|---|---|---|
| 1 | 0.542 419 27 | 0.544 556 99 + 0.996 391 65i | 0.544 556 99 − 0.996 391 65i |
|   | −0.458 105 68 + 0.500 177 85i | −0.458 105 68 − 0.500 177 85i | −0.715 321 88 |
| 2 | 0.527 080 58 | 0.529 578 75 + 0.996 604 93i | 0.529 578 75 − 0.996 604 93i |
|   | −0.641 278 30 + 0.503 350 13i | −0.641 278 30 − 0.503 350 13i | −0.303 681 49 |
| 3 | 0.498 935 78 | 0.502 001 96 + 0.997 249 69i | 0.502 001 96 − 0.997 249 69i |
|   | −0.692 843 88 + 0.505 152 34i | −0.692 843 88 − 0.505 152 34i | −0.117 251 96 |
| 4 | 0.465 646 65 | 0.469 417 36 + 0.998 097 39i | 0.469 417 36 − 0.998 097 39i |
|   | −0.719 762 99 + 0.506 144 40i | −0.719 762 99 − 0.506 144 40i | 0.035 044 06 |
| 5 | 0.424 300 10 | 0.429 606 41 + 0.999 413 30i | 0.429 606 41 − 0.999 413 30i |
|   | −0.738 693 44 + 0.506 831 56i | −0.738 693 44 − 0.506 831 56i | 0.193 873 95 |
| 6* | 0.384 905 22 + 0.019 061 27i | 0.367 308 04 + 0.991 797 19i | 0.367 308 04− 0.991 797 19i |
|   | −0.752 213 26 + 0.507 293 83i | −0.752 213 26 − 0.507 293 83i | 0.384 905 22 − 0.019 061 27i |
| 7 | 0.230 566 69 | 0.230 832 74 + 0.999 670 59i | 0.230 832 74 − 0.999 670 59i |
|   | −0.760 561 74 + 0.507 453 13i | −0.760 561 74 − 0.507 453 13i | 0.828 891 33 |

**Group 2.**

|   |   |   |   |
|---|---|---|---|
| 8 | 0.206 695 77 | 0.205 975 72 + 1.000 386 08i | 0.205 975 72 − 1.000 386 08i |
|   | 0.105 784 35 + 0.500 000 00i | 0.105 784 35 − 0.500 000 00i | −0.830 215 90 |
| 9 | 0.059 726 272 | 0.060 070 63 + 0.999 273 37i | 0.060 070 63 − 0.999 273 37i |
|   | 0.108 473 10 + 0.500 000 00i | 0.108 473 10 − 0.500 000 00i | −0.396 813 73 |
| 10 | 0.010 757 119 | 0.012 499 79 + 0.999 589 01i | 0.012 499 79 − 0.999 589 01i |
|   | 0.069 413 54 + 0.500 000 00i | 0.069 413 54 − 0.500 000 00i | −0.174 583 78 |
| 11* | 0.018 539 900i | 0.993 775 01i | −0.993 775 01i |
|   | 0.500 000 000i | −0.500 000 000i | −0.018 539 900i |
| 12 | −0.010 757 119 | −0.012 499 79 + 0.999 589 01i | −0.012 499 79 − 0.999 589 01i |
|   | −0.069 413 54 + 0.500 000 00i | −0.069 413 54 − 0.500 000 00i | 0.174 583 78 |
| 13 | −0.059 726 272 | −0.060 070 63 + 0.999 273 37i | −0.060 070 63 − 0.999 273 37i |
|   | −0.108 473 10 + 0.500 000 00i | −0.108 473 10 − 0.500 000 00i | 0.396 813 73 |
In [DG] another set of non self-conjugate strings are reported at $N = 12$ and $\nu = (3, 1, 1)$. The total number of the corresponding solutions is 84. Let us denote the solutions in the following way:

$$\{\lambda_1, \lambda_2, \eta^{(i)}, \eta^{(0)}, \eta^{(-i)}\}.$$ 

Here $\lambda_1$ and $\lambda_2$ are 1-strings ($\lambda_1 < \lambda_2$) and $\eta$ corresponds to the 3-string ($\text{Im}(\eta^{(i)}) > \text{Im}(\eta^{(0)}) > \text{Im}(\eta^{(-i)})$). Below we denote the corresponding rigged configurations $\{3, \eta, (1, r_2), (1, r_2)\}$ by $(\eta, r_2, r_3)$. Note that it is enough to consider the case $\text{Re}(\eta) \geq 0$ since we can derive the remaining rigged configurations by the flip operation (see KS14a, section 3). We arrange the corresponding 84 solutions according to the descending order of $\eta^{(0)}(0)$. Then we recognize a regular structure related with the rigged configurations$^3$. To be more specific, we start from the first solution and proceed through the list one by one. We can identify (2, 0, 0), (2, 0, 1), ..., (2, 0, 6) by choosing the solutions such that $\lambda_1$ is decreasing and $\lambda_2$ is increasing. After removing the selected seven solutions from the list, we determine the solutions corresponding to (2, 1, 1), (2, 1, 2), ..., (2, 1, 6) similarly. We repeat the procedure until (2, 6, 6). Note that $\lambda_1$ for (2, $i$, $i$) should be increasing. Next we start from (1, 0, 0) and proceed similarly. The resulting rigged configurations coincide with the identification of [GD2] in terms of the Bethe–Takahashi quantum number. As a result, we can see that the non self-conjugate strings in this case is also obtained by a fusion of a 1-string and a 3-string, as in the previous case.

$^2$ Alternatively we may define the order according to $\text{Re}(\eta^{(0)})$. This choice does not affect the final result. However we prefer to use $\text{Re}(\eta^{(0)})$ since our interest here is a special behavior of $\eta^{(0)}$.

$^3$ The readers should refer to supplementary tables included in the arXiv submission of the present paper. For the numerical values in the present case, we greatly benefited from the work [GD2].
6. Exceptional real solutions

In this section, we analyze the famous counter example to the string hypothesis discovered by Essler, Korepin and Schoutens [EKS] in 1992. For this purpose, we consider the case $N = 25$ and $\ell = 2$ where the corresponding Bethe ansatz equations are

\[
\left( \frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - \frac{i}{2}} \right)^{25} = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}, \quad \left( \frac{\lambda_2 + \frac{i}{2}}{\lambda_2 - \frac{i}{2}} \right)^{25} = \frac{\lambda_2 - \lambda_1 + i}{\lambda_2 - \lambda_1 - i}. \tag{21}
\]

We consider the solutions which satisfy $\lambda_1 \neq \lambda_2$. Moreover, we identify the solutions which are connected by the permutation such as $\{\lambda_1, \lambda_2\}$ and $\{\lambda_2, \lambda_1\}$. Then we have 255 real solutions and 21 complex solutions.

We start from the analysis of the real solutions. In the rigged configurations picture, they should correspond to the following rigged configurations:

\[
\begin{align*}
\begin{array}{c}
21 \\
21 \\
\end{array}
\end{align*}
\]

During this section, we refer to the above rigged configuration as $(\eta, \rho_1)$. Then we have $21 \geq \eta \geq \rho_2 \geq 0$. Now we introduce an order for the set of the real solutions $\{\lambda_1, \lambda_2\}$. Without loss of generality we can suppose that $\lambda_1 < \lambda_2$. Then we arrange 255 real solutions in descending order of the larger roots $\lambda_2$. We call them $\{\lambda_1^{(1)}, \lambda_2^{(1)}\}$, $\{\lambda_1^{(2)}, \lambda_2^{(2)}\}$, $\{\lambda_1^{(255)}, \lambda_2^{(255)}\}$. By definition, we have $\lambda_2^{(r)} > \lambda_1^{(r+1)}$. Remarkably, this simple procedure clarifies a natural relationship between the real solutions and the rigged configurations and eventually two exceptional real solutions appear as the elements which violate the rigged configurations structure.

Let us analyze the real solutions starting from $\{\lambda_1^{(1)}, \lambda_2^{(1)}\}$, $\{\lambda_1^{(2)}, \lambda_2^{(2)}\}$, $\ldots$, $\{\lambda_1^{(255)}, \lambda_2^{(255)}\}$. In both diagrams, the rightward arrows represent the real axis on which we locate each solution $\{\lambda_1^{(r)}, \lambda_2^{(r)}\}$. Here the spacing of the dotted lines is 0.5 and the vertical arrows indicate the position of the origin 0.
As we can see in the left one of the above diagrams, we can confirm the following property:

$$\lambda_2^{(1)} \succ \lambda_2^{(2)} \succ \cdots \succ \lambda_2^{(22)},$$
$$\lambda_1^{(1)} \prec \lambda_1^{(2)} \prec \cdots \prec \lambda_1^{(22)}.$$

Again, we correspond the larger riggings with the right roots and smaller riggings with the left roots. Therefore we have the following correspondence between the solutions $$\{\lambda_1^{(i)}, \lambda_2^{(i)}\}$$ and the rigged configurations $$(n_1, n_2)$$:

$$\{\lambda_1^{(1)}, \lambda_2^{(1)}\} \longrightarrow (21, 0), \quad \{\lambda_1^{(2)}, \lambda_2^{(2)}\} \longrightarrow (21, 1), \ldots, \quad \{\lambda_1^{(22)}, \lambda_2^{(22)}\} \longrightarrow (21, 21).$$

Let us postpone the analysis of the solution $$\{\lambda_2^{(23)}, \lambda_2^{(23)}\}$$ for a while and consider the analysis of the following two groups of solutions:

$$\{\lambda_1^{(24)}, \lambda_2^{(24)}\}, \{\lambda_1^{(25)}, \lambda_2^{(25)}\}, \ldots, \{\lambda_1^{(44)}, \lambda_2^{(44)}\},$$
$$\{\lambda_1^{(45)}, \lambda_2^{(45)}\}, \{\lambda_1^{(46)}, \lambda_2^{(46)}\}, \ldots, \{\lambda_1^{(64)}, \lambda_2^{(64)}\}.$$

The first group of solutions are presented in the left one of the following two diagrams and the second group in the right diagram.

As we can confirm from the left one of the above diagrams, we have

$$\lambda_2^{(24)} \succ \lambda_2^{(25)} \succ \cdots \succ \lambda_2^{(44)},$$
$$\lambda_1^{(24)} \prec \lambda_1^{(25)} \prec \cdots \prec \lambda_1^{(44)},$$

and from the right one of the above diagrams, we have

$$\lambda_2^{(45)} \succ \lambda_2^{(46)} \succ \cdots \succ \lambda_2^{(64)},$$
$$\lambda_1^{(45)} \prec \lambda_1^{(46)} \prec \cdots \prec \lambda_1^{(64)}.$$

Moreover, we can recognize clear gaps between the three groups of values:

$$\{\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(22)}\}, \{\lambda_1^{(24)}, \lambda_1^{(25)}, \ldots, \lambda_1^{(44)}\} \text{ and } \{\lambda_1^{(45)}, \lambda_1^{(46)}, \ldots, \lambda_1^{(64)}\}.$$

Therefore, we conclude the following correspondences between the solutions $$\{\lambda_1^{(i)}, \lambda_2^{(i)}\}$$ and the rigged configurations $$(n_1, n_2)$$:
Let us return to the solution \( \{ \lambda_1^{(23)}, \lambda_2^{(23)} \} \). From the above analysis, we see that this solution is isolated and does not fit into the rigged configurations picture. Therefore, the solution \( \{ \lambda_1^{(23)}, \lambda_2^{(23)} \} \) is one of the exceptional real solutions.

Let us continue in a similar way. The left one of the following diagrams shows the solutions \( \{ \lambda_1^{(65)}, \lambda_2^{(65)} \}, \ldots, \{ \lambda_1^{(83)}, \lambda_2^{(83)} \} \) and the right diagram shows the solutions \( \{ \lambda_1^{(84)}, \lambda_2^{(84)} \}, \ldots, \{ \lambda_1^{(101)}, \lambda_2^{(101)} \} \). The first group corresponds to the rigged configurations \((18, 0), \ldots, (18, 18)\) and the second group corresponds to \((17, 0), \ldots, (17, 17)\), respectively.

The left one of the following diagrams shows the solutions \( \{ \lambda_1^{(102)}, \lambda_2^{(102)} \}, \ldots, \{ \lambda_1^{(118)}, \lambda_2^{(118)} \} \) and the right diagram shows the solutions \( \{ \lambda_1^{(119)}, \lambda_2^{(119)} \}, \ldots, \{ \lambda_1^{(134)}, \lambda_2^{(134)} \} \). The first group corresponds to the rigged configurations \((16, 0), \ldots, (16, 16)\) and the second group corresponds to \((15, 0), \ldots, (15, 15)\), respectively.

The left one of the following diagrams shows the solutions \( \{ \lambda_1^{(135)}, \lambda_2^{(135)} \}, \ldots, \{ \lambda_1^{(149)}, \lambda_2^{(149)} \} \) and the right diagram shows the solutions \( \{ \lambda_1^{(150)}, \lambda_2^{(150)} \}, \ldots, \{ \lambda_1^{(163)}, \lambda_2^{(163)} \} \).
The first group corresponds to the rigged configurations $(14, 0), \ldots, (14, 14)$ and the second group corresponds to $(13, 0), \ldots, (13, 13)$, respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(164)}, \lambda_2^{(164)}\}, \ldots, \{\lambda_1^{(176)}, \lambda_2^{(176)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(177)}, \lambda_2^{(177)}\}, \ldots, \{\lambda_1^{(188)}, \lambda_2^{(188)}\}$. The first group corresponds to the rigged configurations $(12, 0), \ldots, (12, 12)$ and the second group corresponds to $(11, 0), \ldots, (11, 11)$, respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(189)}, \lambda_2^{(189)}\}, \ldots, \{\lambda_1^{(199)}, \lambda_2^{(199)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(200)}, \lambda_2^{(200)}\}, \ldots, \{\lambda_1^{(209)}, \lambda_2^{(209)}\}$. The first group corresponds to the rigged configurations $(10, 0), \ldots, (10, 10)$ and the second group corresponds to $(9, 0), \ldots, (9, 9)$, respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(210)}, \lambda_2^{(210)}\}, \ldots, \{\lambda_1^{(218)}, \lambda_2^{(218)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(219)}, \lambda_2^{(219)}\}, \ldots, \{\lambda_1^{(226)}, \lambda_2^{(226)}\}$. 

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The first group corresponds to the rigged configurations (8, 0), ⋯, (8, 8) and the second group corresponds to (7, 0), ⋯, (7, 7), respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(227)}, \lambda_2^{(227)}\}$, ⋯, $\{\lambda_1^{(233)}, \lambda_2^{(233)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(234)}, \lambda_2^{(234)}\}$, ⋯, $\{\lambda_1^{(239)}, \lambda_2^{(239)}\}$. 

The first group corresponds to the rigged configurations (6, 0), ⋯, (6, 6) and the second group corresponds to (5, 0), ⋯, (5, 5), respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(240)}, \lambda_2^{(240)}\}$, ⋯, $\{\lambda_1^{(244)}, \lambda_2^{(244)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(245)}, \lambda_2^{(245)}\}$, ⋯, $\{\lambda_1^{(248)}, \lambda_2^{(248)}\}$. 

The first group corresponds to the rigged configurations (4, 0), ⋯, (4, 4) and the second group corresponds to (3, 0), ⋯, (3, 3), respectively.

The left one of the following diagrams shows the solutions $\{\lambda_1^{(249)}, \lambda_2^{(249)}\}$, $\{\lambda_1^{(250)}, \lambda_2^{(250)}\}$ and $\{\lambda_1^{(251)}, \lambda_2^{(251)}\}$ and the right diagram shows the solutions $\{\lambda_1^{(252)}, \lambda_2^{(252)}\}$ and $\{\lambda_1^{(253)}, \lambda_2^{(253)}\}$. The first group corresponds to the rigged configurations (2, 0), (2, 1) and (2, 2) and the second group corresponds to (1, 0) and (1, 1), respectively.

The left one of the following diagrams shows the solution $\{\lambda_1^{(254)}, \lambda_2^{(254)}\}$ and the right diagram shows the solution $\{\lambda_1^{(255)}, \lambda_2^{(255)}\}$. The first solution corresponds to the rigged configuration (0, 0) and the second solution corresponds to the exceptional real solution.
Some readers might feel that the identification of the solution \( \lambda_1^{(255)}, \lambda_2^{(255)} \) with the exceptional real solution is somewhat tricky. In this case, we introduce another order based on the values of the smaller roots \( \lambda_i^{(0)} \). Then the solution \( \{\lambda_1^{(255)}, \lambda_2^{(255)}\} \) naturally appears as the exceptional real solution.

Next, let us consider the complex solutions. Among 21 such solutions, there is the solution \( i/2, -i/2 \). Since this solution does not satisfy the condition (12), this is unphysical. Therefore, we have to consider the remaining 20 complex solutions together with the two exceptional real solutions detected in the previous step. We show these 22 solutions on the complex plain as follows (the spacing of the dotted lines is 0.5).

Here the complex solutions take the form of 2-strings:

\[
\lambda_1 = a + \left( \frac{1}{2} + \delta \right)i, \quad \lambda_2 = a - \left( \frac{1}{2} + \delta \right)i
\]

where \( a, \delta \in \mathbb{R} \). These solutions should correspond to the rigged configurations

\[
21 \square \square r
\]

where \( 0 \leq r \leq 21 \). Again, we can construct a correspondence based on the real parts of \( \{\lambda_1, \lambda_2\} \) without any ambiguity. In particular, the exceptional real solutions correspond to the following rigged configurations:

\[
21 \square \square 1 \quad 21 \square \square 20
\]

Let us compare the above analysis with appendix of \([EKS]\). For the comparison we have to divide their \( \Lambda_1 \) and \( \Lambda_2 \) by 2. In their table 2, they consider real solutions. In our analysis, the two row rigged configurations \( (n_1, n_2) \) corresponding to their solutions are \( (11, 10), (12, 9), (13, 8), (14, 7), (15, 6), (16, 5), (17, 4), (18, 3), (19, 2), (20, 1) \) and \( (21, 0) \) from top to bottom of their table. In their table 3, they consider the complex solutions. According to our analysis, their solutions correspond to the one row rigged configurations of the riggings \( 21, 19, 18, 17...4, 3, 2, 0 \) from top to bottom of their table. In table 4, they consider real solutions. According to our analysis, their solutions correspond to the two row rigged configurations with the riggings \( (21, 21), (21, 20), (21, 19), (21, 18), (21, 17), (21, 16), (21, 15), (21, 14), (21, 13), (21, 12), (21, 11), (21, 10), (21, 9), (21, 8), (21, 7), (21, 6), (21, 5), (21, 4), (21, 3), (21, 2), (21, 1), (21, 0) \) from top to bottom of their table. Therefore, under a suitable transformation, our analysis agrees with their tables 2–4 based on the Bethe–Takahashi quantum number. However, our interpretation of the exceptional real solutions are different from their interpretation given in table 1 of their paper. A possible reasoning for this discrepancy might be the following. In the derivation of the Bethe–Takahashi quantum number
[T], we assume that the solutions take the form of a \((2b + 1)\)-string:

\[
a + bi, a + (b - 1)i, a + (b - 2)i, \ldots, a - bi, \quad (a \in \mathbb{R}, b \in \mathbb{Z}_{\geq 0}/2).
\]

However, the exceptional real solutions take a very different form from the standard 2-strings.

According to [EKS] we have additional exceptional real solutions when \(N > 61.34\). We expect we can analyze such cases in a similar way. However, currently we are unable to construct complete solutions for such cases.

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Appendix A. Proofs for the assertions in section 3

Since the proofs for the assertions in section 3 are scattered around several places, we believe it is worthwhile to provide proofs for the relevant assertions in a systematic way.

Recall that from the relation of the transfer matrix \(T_N(\lambda)\) with the \(R\)-matrices (known as the \(RTT = TTR\) relation), one can derive the commutation relations between the fundamental operators \(A_N, B_N\) and \(D_N\) (see [F, KBI]):

\[
\begin{align*}
[ B_N (\lambda), B_N (\mu) ] &= 0, \quad (22) \\
A_N (\lambda) B_N (\mu) &= \frac{\lambda - \mu - i}{\lambda - \mu} B_N (\mu) A_N (\lambda) + \frac{i}{\lambda - \mu} B_N (\lambda) A_N (\mu), \quad (23) \\
D_N (\lambda) B_N (\mu) &= \frac{\lambda - \mu + i}{\lambda - \mu} B_N (\mu) D_N (\lambda) - \frac{i}{\lambda - \mu} B_N (\lambda) D_N (\mu). \quad (24)
\end{align*}
\]

Let

\[
\Psi^{(c)}_\lambda := \frac{1}{e^N} B_N \left( \frac{i}{2} + c + e^{N} \right) B_N \left( - \frac{i}{2} + c \right) |0\rangle_N. \quad (25)
\]

Lemma 4. The vector \(\lim_{N \to 0} \Psi^{(c)}_\lambda\) is well-defined.

Proof. Since \(B_N\) operators commute with each other, it is enough to prove

\[
B_N \left( \frac{i}{2} + c + e^{N} \right) B_N \left( - \frac{i}{2} + c \right) |0\rangle_N \sim e^N \quad (26)
\]

where \(\sim\) means the order of the leading term in the expansion of \(e\). Our argument is similar to that given in [NW13, appendix A].

The proof is by induction on \(N\). As the starting point, the \(N = 4\) case is checked explicitly as in (16). Suppose that we have checked (26) for the \(N - 1\) case. Let us denote

\[
|0\rangle_N = |0\rangle_{N-1} \otimes \left( \frac{1}{0} \right)_N.
\]
From the definition of the transfer matrix at (4), we see that $T_N$ and $T_{N-1}$ are related as

$$T_N (\lambda) = L_N (\lambda) T_{N-1} (\lambda).$$

By definition of $L_N (\lambda)$ at (3), we have

$$\begin{pmatrix} A_N (\lambda) & B_N (\lambda) \\ C_N (\lambda) & D_N (\lambda) \end{pmatrix} = \begin{pmatrix} a_N (\lambda) & b_N (\lambda) \\ c_N (\lambda) & d_N (\lambda) \end{pmatrix} \begin{pmatrix} A_{N-1} (\lambda) & B_{N-1} (\lambda) \\ C_{N-1} (\lambda) & D_{N-1} (\lambda) \end{pmatrix},$$

where the operators $a_N (\lambda)$, $b_N (\lambda)$, $c_N (\lambda)$ and $d_N (\lambda)$ act on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as

$$a_N (\lambda) = \begin{pmatrix} \lambda + \frac{i}{2} & 0 \\ 0 & \lambda - \frac{i}{2} \end{pmatrix}, \quad b_N (\lambda) = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix},$$

$$c_N (\lambda) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad d_N (\lambda) = \begin{pmatrix} \lambda - \frac{i}{2} & 0 \\ 0 & \lambda + \frac{i}{2} \end{pmatrix},$$

and the operators $A_{N-1} (\lambda)$, $B_{N-1} (\lambda)$, $C_{N-1} (\lambda)$ and $D_{N-1} (\lambda)$ act on $|0\rangle_{N-1}$. In particular,

$$B_N (\lambda) = a_N (\lambda) B_{N-1} (\lambda) + b_N (\lambda) D_{N-1} (\lambda).$$

It follows that

$$B_N (\lambda_1) B_N (\lambda_2) |0\rangle_N = \left\{ a_N (\lambda_1) B_{N-1} (\lambda_1) + b_N (\lambda_1) D_{N-1} (\lambda_1) \right\} \times \left\{ a_N (\lambda_2) B_{N-1} (\lambda_2) + b_N (\lambda_2) D_{N-1} (\lambda_2) \right\} |0\rangle_N$$

$$= \left\{ a_N (\lambda_1) a_N (\lambda_2) B_{N-1} (\lambda_1) B_{N-1} (\lambda_2) \\ + a_N (\lambda_1) b_N (\lambda_2) B_{N-1} (\lambda_1) D_{N-1} (\lambda_2) \\ + b_N (\lambda_1) a_N (\lambda_2) D_{N-1} (\lambda_1) B_{N-1} (\lambda_2) \\ + b_N (\lambda_1) b_N (\lambda_2) D_{N-1} (\lambda_1) D_{N-1} (\lambda_2) \right\} |0\rangle_N \quad (27)$$

for $\lambda_1$, $\lambda_2$ arbitrary.

We now set $\lambda_1 = \frac{i}{2} + \epsilon + c e^N$ and $\lambda_2 = -\frac{i}{2} + \epsilon$, and we consider the four terms of the right hand side of (27) respectively. In the first term, we have

$$B_{N-1} (\lambda_1) B_{N-1} (\lambda_2) |0\rangle_{N-1} \sim e^{N-1}$$

by the induction hypothesis, and we have

$$a_N (\lambda_1) a_N (\lambda_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N = \begin{pmatrix} \epsilon (i + \epsilon + c e^N) & 0 \\ 0 & \epsilon (i + \epsilon + c e^N) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N \sim \epsilon.$$

Hence the first term on the right hand side of (27) is of order $e^N$.

The fourth term on the right hand side of (27) is zero since we have

$$b_N (\lambda_1) b_N (\lambda_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N = 0.$$
From (24), we have
\[ D_{N-1}(\lambda_1)B_{N-1}(\lambda_2) = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2} B_{N-1}(\lambda_1)D_{N-1}(\lambda_2) - \frac{i}{\lambda_1 - \lambda_2} B_{N-1}(\lambda_1)D_{N-1}(\lambda_2) \]
\[ = \frac{2i + c e^N}{i + c e^N} B_{N-1}(\lambda_2)D_{N-1}(\lambda_1) - \frac{i}{i + c e^N} B_{N-1}(\lambda_1)D_{N-1}(\lambda_2). \]

We apply this relation to the third term on the right hand side of (27). Then the combination of the second term and the third term is
\[ \left\{ a_N(\lambda_1)b_N(\lambda_2) - \frac{i}{i + c e^N} b_N(\lambda_1)a_N(\lambda_2) \right\} B_{N-1}(\lambda_1)D_{N-1}(\lambda_2) |0\rangle_N \]
\[ + \frac{2i + c e^N}{i + c e^N} b_N(\lambda_1)a_N(\lambda_2)B_{N-1}(\lambda_2)D_{N-1}(\lambda_1) |0\rangle_N. \]

The first term of (28) is
\[ \left\{ a_N(\lambda_1)b_N(\lambda_2) - \frac{i}{i + c e^N} b_N(\lambda_1)a_N(\lambda_2) \right\} \left( \frac{1}{i e^{N+1}} \right) = \left( \frac{ic e^{N+1}}{i + c e^N} + ic e^N \right) \left( \frac{1}{i e^N} \right) \sim e^N. \]

Also there is no divergence in \( B_{N-1}(\lambda_1)D_{N-1}(\lambda_2) |0\rangle_{N-1} \). Thus the first line of (28) is of order \( e^N \). In the second line of (28), we have
\[ D_{N-1}(\lambda_i) |0\rangle_{N-1} = \left( \lambda_i - \frac{i}{2} \right)^{N-1} |0\rangle_{N-1} \sim e^{N-1} \]
(the first equality follows from the definition of the operator \( D_N \), see [F, KBI]), and we have
\[ \frac{2i + c e^N}{i + c e^N} b_N(\lambda_1)a_N(\lambda_2) \left( \frac{1}{i e^N} \right) = \frac{2i + c e^N}{i + c e^N} \left( \frac{0}{0} \right) \left( \frac{1}{0} \right) \sim e. \]
Thus the second line of (28) is also of order \( e^N \).

Summing up, we obtain the relation (26). \( \square \)

Now we need to show that the vector \( \lim_{\epsilon \to 0} \Psi^\epsilon \) is an eigenvector of the Hamiltonian \( H_N \). Recall the following standard relations (see [F, KBI])
\[ \left\{ A_N(\lambda) + D_N(\lambda) \right\} B_N(\lambda_1) \cdots B_N(\lambda_\ell) |0\rangle_N \]
\[ = A(\lambda; \lambda_1, \cdots, \lambda_\ell) \prod_{j=1}^\ell B_N(\lambda_j) |0\rangle_N \]
\[ + \sum_{k=1}^{\ell} \left\{ A_k(\lambda; \lambda_1, \cdots, \lambda_\ell) B_N(\lambda_j) \prod_{j=1}^\ell B_N(\lambda_j) |0\rangle_N \right\}, \]

where
\[ A(\lambda; \lambda_1, \cdots, \lambda_\ell) = \left( \lambda + i \right) \prod_{j=1}^\ell \left( \lambda - \lambda_j - i \right) + \left( \lambda - i \right) \prod_{j=1}^\ell \left( \lambda - \lambda_j + i \right). \]

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and for $k = 1, 2, \ldots, k$ we have

$$
A_k (\lambda; \lambda_1, \ldots, \lambda_k) = \frac{i}{\lambda - \lambda_k} \left\{ \left( \lambda_k + \frac{i}{2} \right) \prod_{j=1}^{\ell} \frac{\lambda_j - \lambda_k - i}{\lambda_k - \lambda_j} - \left( \lambda_k - \frac{i}{2} \right) \prod_{j=1}^{\ell} \frac{\lambda_j - \lambda_k - i}{\lambda_k - \lambda_j} \right\}.
$$

(31)

As in the standard arguments, we can show $A_k (\lambda; \lambda_1, \ldots, \lambda_k) = 0$ for $k = 3, \ldots, \ell$ by the Bethe ansatz equations. Therefore we need to check the following cases.

**Lemma 5.** Let $\lambda_1 = \frac{i}{2} + c + c e^N$ and $\lambda_2 = -\frac{i}{2} + c$.

(a) If we take

$$
c = -\frac{2}{i^{N+1}} \prod_{j=3}^{\ell} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j - \frac{i}{2}},
$$

then we have $A_1 (\lambda; \lambda_1, \ldots, \lambda_{\ell}) = 0$.

(b) If we take

$$
c = 2i^{N+1} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}},
$$

then we have $A_2 (\lambda; \lambda_1, \ldots, \lambda_{\ell}) = 0$.

**Proof.** (a) We have

$$
A_1 = \frac{i}{\lambda - \lambda_1} \left\{ \left( \lambda_1 + \frac{i}{2} \right) \prod_{j=2}^{\ell} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j - \frac{i}{2}} - \left( \lambda_1 - \frac{i}{2} \right) \prod_{j=2}^{\ell} \frac{\lambda_j - \frac{i}{2}}{\lambda_j - \frac{3i}{2}} \right\}
$$

$$
= \frac{i}{\lambda - \lambda_1} \left\{ i^N \cdot c e^N \prod_{j=3}^{\ell} \frac{\lambda_j - \frac{i}{2}}{\lambda_j - \frac{3i}{2}} - e^N \cdot \frac{-2i}{i^N} \prod_{j=3}^{\ell} \frac{\lambda_j - \frac{i}{2}}{\lambda_j - \frac{3i}{2}} \right\}
$$

$$
= \frac{i e^N}{\lambda - \lambda_1} \left\{ c \cdot i^{N-1} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} - 2 \prod_{j=3}^{\ell} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j - \frac{i}{2}} \right\}.
$$

Therefore, if we take $c$ as in (32), we see that $A_1 = 0$.

(b) We have

$$
A_2 = \frac{i}{\lambda - \lambda_2} \left\{ \left( \lambda_2 + \frac{i}{2} \right) \prod_{j=1}^{\ell} \frac{\lambda_j - \frac{i}{2}}{\lambda_j - \frac{3i}{2}} - \left( \lambda_2 - \frac{i}{2} \right) \prod_{j=1}^{\ell} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j - \frac{i}{2}} \right\}
$$
\[
\frac{i}{\lambda - \lambda_2} \left\{ \epsilon^N \left( \frac{-2i}{\lambda - \lambda_2} \prod_{j=3}^{\ell} \frac{-\frac{i}{2} - \lambda_j - i}{i - \lambda_j} - (-i)^N \frac{\epsilon^N}{i} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2} - i}{\lambda_j + \frac{i}{2}} \right) \right\} \\
= \frac{i \epsilon^N}{\lambda - \lambda_2} \left\{ 2 \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{3i}{2} - \frac{i}{2} - \lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}} - \frac{\epsilon}{i^{N+1}} \prod_{j=3}^{\ell} \frac{\lambda_j - \frac{i}{2} + i}{\lambda_j + \frac{i}{2}} \right\}. 
\]

Therefore, if we take \( c \) as in (33), we see that \( \Lambda_2 = 0 \). \( \square \)

Now we can prove the theorem of Nepomechie–Wang [NW13].

**Theorem 6.** Suppose that the singular solution \( \lambda = \{ \lambda_1 = \frac{i}{2}, \lambda_2 = -\frac{i}{2}, \lambda_3, \ldots, \lambda_\ell \} \) satisfies the condition

\[
\left\{ -\prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2} - \frac{i}{2} - \lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}} \right\} = 1. 
\]  

(34)

Then the vector \( \lim_{\epsilon \to 0} \Psi_\ell^{(c)} \) is a well-defined eigenvector of the Hamiltonian \( \mathcal{H}_N \).

**Proof.** The remaining thing to check is that the compatibility condition of (32) and (33) is the condition (34). By equating the right hand sides of (32) and (33), we obtain

\[
\prod_{j=3}^{\ell} \left( \frac{\lambda_j - \frac{i}{2} + \frac{i}{2} - \lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}} \right) = (-1)^N. 
\]

On the other hand, if we take the product of the Bethe ansatz equations, we obtain

\[
\prod_{j=3}^{\ell} \left( \frac{\lambda_k + \frac{i}{2} - \frac{i}{2} - \lambda_k - \frac{3i}{2}}{\lambda_k + \frac{i}{2}} \right) = \prod_{k \neq j} \left( \frac{\lambda_k - \lambda_j + i}{\lambda_k + \lambda_j} \right). 
\]

Combining two expressions, we obtain (34). \( \square \)

Finally, we have to compute the energy eigenvalue for the state \( \lim_{\epsilon \to 0} \Psi_\ell^{(c)} \).

**Proposition 7 [KS14b].** For the singular solution \( \lambda = \{ \lambda_1 = \frac{i}{2}, \lambda_2 = -\frac{i}{2}, \lambda_3, \ldots, \lambda_\ell \} \), the energy eigenvalue of the corresponding vector \( \lim_{\epsilon \to 0} \Psi_\ell^{(c)} \) is

\[
\mathcal{E} = -J - \frac{\ell}{2} \sum_{j=3}^{\ell} \frac{1}{\lambda_j + \frac{i}{2}}. 
\]

**Proof.** Let \( \epsilon = \frac{2}{\ell} \mathcal{E} + N. \) Then

\[
\epsilon = \left. \frac{d}{d\lambda} \log A \right|_{\lambda = \frac{i}{2}} = \left. \frac{i \delta \lambda}{2 \lambda} \right|_{\lambda = \frac{i}{2}}, 
\]  

(35)
where

\[
\Lambda(\lambda; \lambda_1, \ldots, \lambda_\ell) = \left(\lambda + \frac{i}{2}\right)^N \prod_{j=1}^\ell \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + \left(\lambda - \frac{i}{2}\right)^N \prod_{j=1}^\ell \frac{\lambda_j - \lambda - i}{\lambda_j - \lambda}.
\]

(36)

See (6). On this expression, we apply the regularization

\[
\lambda_1 = \frac{i}{2} + \epsilon + c e^{N_1}, \quad \lambda_2 = -\frac{i}{2} + \epsilon.
\]

The denominator of \(\epsilon\) is

\[
\epsilon_{\text{denom}} \equiv \Lambda \left(\frac{i}{2}; \lambda_1, \ldots, \lambda_\ell\right) = \frac{\lambda_j + \frac{i}{2}}{1 - \frac{i}{2}} = i^N \cdot e + c e^{N_1} \cdot \frac{\epsilon}{e - i} \cdot \prod_{j=3}^\ell \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}
\]

\[
= i^N \cdot \frac{i + e + c e^{N_1}}{1 + e^{N_1-1} (e - i)} \prod_{j=3}^\ell \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}.
\]

On the other hand, the numerator is

\[
\frac{i}{d\lambda} \frac{d\Lambda}{d\lambda} = A_0(\lambda) + \sum_{j=1}^\ell A_j(\lambda) + \text{terms containing at least one } \left(\lambda - \frac{i}{2}\right)
\]

where

\[
A_0(\lambda) = i^N \left(\lambda + \frac{i}{2}\right)^{N-1} \prod_{j=1}^\ell \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j}
\]

and for \(j = 1, 2, \ldots, \ell\),

\[
A_j(\lambda) = i \left(\lambda + \frac{i}{2}\right)^{N-1} \prod_{k=1}^{j-1} \frac{\lambda - \lambda_k - i}{\lambda - \lambda_k} \cdot \frac{1}{\prod_{k=1}^{j+1} \frac{\lambda - \lambda_k - i}{\lambda - \lambda_k}} \cdot \frac{\lambda - \lambda_{j+1} - i}{\lambda - \lambda_{j+1}} \cdot \frac{\lambda - \lambda_{j+1} - i}{\lambda - \lambda_{j+1}}.
\]

Below we compute the contribution from each term one by one.
Let us consider $A_0(\lambda)$:

$$A_0\left(\frac{i}{2}\right) = i^{N+}\cdot \frac{i - \frac{i}{2} + \epsilon + c e^{\epsilon}}{\left\{\frac{i}{2} - \left(\frac{i}{2} + \epsilon + c e^{\epsilon}\right)\right\}^2} \cdot \frac{i - \left(\frac{i}{2} + \epsilon\right) - i}{\prod_{j=3}^{\ell} \lambda_j - \frac{i}{2}} = i^{N+} \cdot \frac{i + \epsilon + c e^{N}}{(1 + c e^{N-1}) (e - i)} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}.$$ 

Therefore, we obtain

$$\frac{1}{\varepsilon_{\text{deno}}} \cdot A_0\left(\frac{i}{2}\right) = N.$$

Let us consider $A_1(\lambda)$ and $A_2(\lambda)$.

$$A_1\left(\frac{i}{2}\right) = i^{N+1} \cdot \frac{i}{\left\{\frac{i}{2} - \left(\frac{i}{2} + \epsilon + c e^{\epsilon}\right)\right\}^2} \cdot \frac{i - \left(\frac{i}{2} + \epsilon\right) - i}{\prod_{j=3}^{\ell} \lambda_j - \frac{i}{2}} = -i^{N} \frac{1}{\varepsilon (1 + c e^{N-1}) (e - i)^2} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}.$$ 

On the other hand, we have

$$A_2\left(\frac{i}{2}\right) = i^{N+1} \cdot \frac{i - \left(\frac{i}{2} + \epsilon + c e^{\epsilon}\right) - i}{\left\{\frac{i}{2} - \left(\frac{i}{2} + \epsilon + c e^{\epsilon}\right)\right\}^2} \cdot \frac{i}{\prod_{j=3}^{\ell} \lambda_j - \frac{i}{2}} = -i^{N} \frac{i + \epsilon + c e^{N}}{\varepsilon (1 + c e^{N-1}) (e - i)^2} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}.$$ 

Thus we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_{\text{deno}}} \left\{ A_1\left(\frac{i}{2}\right) + A_2\left(\frac{i}{2}\right) \right\} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon_{\text{deno}}} \times \left( -i^{N} \frac{(e - i) + (i + \epsilon + c e^{N})(1 + c e^{N-1})}{\varepsilon (1 + c e^{N-1})^2 (e - i)^2} \prod_{j=3}^{\ell} \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)$$

$$= -\lim_{\varepsilon \to 0} \frac{(1 + c e^{N-1}) (e - i)}{i + \epsilon + c e^{N}} \times \frac{2e + i c e^{N-1} + 2c e^{N} + c^2 e^{2N-1}}{\varepsilon (1 + c e^{N-1})^2 (e - i)^2}.$$
\[= \lim_{\epsilon \to 0} \frac{2e + \epsilon c e^{N-1} + 2c e^N + c^2 e^{2N-1}}{e(1 + c e^{N-1})(e - i)(i + e + c e^N)} = -2.\]

Finally, for \(j = 3, 4, \ldots, \ell\), we have

\[
A_j\left(\frac{i}{2}\right) = \frac{i^{N+1} \epsilon i + e + c e^N}{e + c e^N} : \frac{\epsilon}{e - i} : \frac{\lambda_3 + \frac{i}{2}}{\lambda_3 - \frac{i}{2}} : \frac{\lambda_{j-1} + \frac{i}{2}}{\lambda_{j-1} - \frac{i}{2}} : \frac{i}{(\lambda_j - \frac{i}{2})^2}
\]

Thus we have

\[
\frac{1}{\epsilon_{\text{deno}}} A_j\left(\frac{i}{2}\right) = -\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}} \cdot \frac{1}{(\lambda_j - \frac{i}{2})^2} = -\frac{1}{\lambda_j^2 + \frac{1}{4}}.
\]

To summarize, we have

\[\epsilon = N - 2 = \sum_{j=3}^{\ell} \frac{1}{\lambda_j^2 + \frac{1}{4}}.\]

Therefore, we obtain the result. \(\square\)

**Appendix B. Update for our previous paper**

Here we would like to provide an update for our previous paper [KS14a]. In the \(N = 12, \ell = 5\) case, we have the following solution:

\[
\begin{align*}
\lambda_1 &= 0 \\
\lambda_2 &\approx 0.1789782217190060052976927892101 \\
\lambda_3 &\approx -0.1789782217190060052976927892101 \\
\lambda_4 &= \frac{i}{2} \\
\lambda_5 &= -\frac{i}{2}
\end{align*}
\]

Here, \(\xi = \lambda_2^2 = \lambda_3^2\) is the smallest positive real solution of the equation

\[5120\xi^5 + 11520\xi^4 - 4992\xi^3 - 9312\xi^2 + 2020\xi - 55 = 0.\]  

(37)

This solution corresponds to the following rigged configuration:

\[
\begin{array}{c|c|c}
2 & 1 \\
4 & 2 \\
4 & 2 \\
4 & 2 \\
\end{array}
\]

In ‘tableN12M5.pdf’ of [HNS], this solution appears as \#235 which is counted as a regular solution. Based on this result, we noted in [KS14a, Conjecture 14(C)] such that we need one
more restriction on the riggings when \( \ell \) is odd. However, as Deguchi–Giri [DG] pointed out, the above solution is singular. Therefore, we come to the conclusion that we no longer need restriction [KS14a, equation (26)] when \( \ell \) is odd.

We remark that equation (37) admits two more positive real solutions and both of them provide physical singular solutions. In the rigged configurations language, they are \{ (2, 1), (1, 3), (1, 2), (1, 1) \} for the smaller solution and \{ (2, 1), (1, 4), (1, 2), (1, 0) \} for the larger solution. Similarly the equation (37) admits two negative real solutions which provide physical singular solutions. In the rigged configurations language, they are \{ (4, 1), (1, 4) \} for the smaller solution and \{ (3, 1), (2, 2) \} for the larger solution. Thus the five roots of equation (37) determine all the physical singular solutions for the case \( N = 12 \) and \( \ell = 5 \).

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