Vibration control of two-degree-of-freedom exponentially damped oscillators

Guozhong Xiu¹*, Shi Bao¹, Wenfei Zhao¹ and Jian Yuan¹

¹Institute of System Science and Mathematics, Naval Aeronautical University, Yantai 264001, P.R.China

*E-mail: xuguozhong2013@163.com

Abstract. Vibration control of two-degree-of-freedom (2DOF) exponentially damped oscillators is considered. Firstly, the equations of motion for 2DOF exponentially damped oscillators are established by using the nonviscous damping models that depend on the past history of motion. Secondly, the equations of motion for 2DOF exponentially damped oscillators are transformed into state equations with six dimensions by using a set of internal variables. Thirdly, Sliding control design is proposed. Furthermore, this control method can also be used in the multi-degree-of-freedom (MDOF) exponentially damped oscillators. Finally, numerical simulations are provided to illustrate the above control designs.

1. Introduction

Viscoelastic materials are widely used to control or reduce vibrations and sound radiation in aerospace structures, industrial machines, civil engineering structures, etc. Of many nonviscously damping models, the convolution integral model is possibly the most general model within the scope of linear analysis. The integral constitutive models are derived based on the materials properties of stress relaxation and creep applying Boltzmann’s superposition principle. The stress relaxation functions and creep functions are memory and hereditary kernels in the integral constitutive equations. They can be expressed by a series of exponential functions, power-law functions, Mittage-Leffler functions, or other types of functions. Equations of motion of such systems are a set of coupled second-order integro-differential equations. The presence of the “integral” term makes the vibration analysis and control design more complicated than the classical ones. The integral type damping models may be also called nonviscously damping models and the corresponding oscillators are called nonviscously damped oscillators.

Researches on nonviscously damped oscillators are mainly concentrated on two types: one is the exponentially damped oscillators where the damping force are expressed by exponentially fading memory kernel; and the other is the fractional-order oscillators where the viscoelastic relaxation functions are characterized by power-law functions or Mittage-Leffler functions. Adhikari and his colleagues have systematically investigated the structural dynamics with exponentially damped models, including dynamics of exponentially damped single-degree-of-freedom and multi-degree-of-freedom systems, identification and quantification of damping in [1,2].

In recent years, Various fractional control techniques have been proposed, such as CRONE control, fractional PID control, fractional sliding control [3,4], fractional adaptive control [5,6], fractional optimal control, etc. However, control designs for exponentially damped oscillators are very limited. This paper is going to propose sliding control design for 2DOF exponentially damped oscillators.
2. The equation of motion for 2DOF exponentially damped oscillators

We consider two-degree-of-freedom nonviscously damped system shown in Fig.1. The spring-mass system with nonviscous damping is composed of $M$ and $m$. The two masses are connected by a rod made of some viscoelastic material, with the cross-sectional area $A$ and length $L$. The viscoelastic rod can be seen as consisting of spring of stiffness $k$ and nonviscous damper with damping function given by $G(t) = c_t \mu_1 e^{-\mu_1 t}$. $X(t)$ is the displacement of $M$, $x(t)$ is the displacement of $m$.

Through the force analysis of mass $M$, we can get the result

$$M \ddot{x} (t) - f_d (t) + kX (t) = f(t) \quad (1)$$

where, $f(t)$ is the external force acted on $M$, $f_d(t)$ is the force induced by the viscoelastic rod:

$$f_d (t) = A \sigma (t) \quad (2)$$

The creep of the viscoelastic rod is

$$\varepsilon(t) = \frac{x(t) - X(t)}{L} \quad (3)$$

The following nonviscous damping model that depend on the past history of motion via convolution integral over kernel function is adopted

$$\sigma(t) = \int_0^t G(t - \tau) \varepsilon(\tau) d\tau \quad (4)$$

Here, we will use a damping model for which the kernel function has the special form

$$G(t) = c_t \mu_1 e^{-\mu_1 t} \quad (5)$$

Substituting Eq.(3) and (4) into (2) yields

$$f_d (t) = \frac{Ac}{L} \int_0^t \mu_1 e^{-\mu_1 (t - \tau)} \dot{x}(\tau) d\tau + \frac{Ac}{L} \int_0^t \mu_1 e^{-\mu_1 (t - \tau)} \dot{X}(\tau) d\tau \quad (6)$$

In Eq. (6), we denote that $k = \frac{A}{L}$.

Substituting Eq.(6) into Eq.(1), we obtain the equations of motion for $M$

$$M \ddot{X} (t) + c \int_0^t \mu_1 e^{-\mu_1 (t - \tau)} \dot{X}(\tau) d\tau - c \int_0^t \mu_1 e^{-\mu_1 (t - \tau)} \dot{x}(\tau) d\tau + (K + k) X(t) - kx(t) = f(t) \quad (7)$$

Through the force analysis of mass $m$, we can get the result

$$m \ddot{x} (t) + f_d (t) = 0 \quad (8)$$

Substituting Eq. (6) into Eq.(8), we obtain the equations of motion for $m$
\[ m\ddot{x}(t) + c\int_0^t \mu e^{-\mu(t-\tau)} \dot{x} d\tau - c\int_0^t \mu e^{-\mu(t-\tau)} \dot{x} d\tau + kx(t) - kX(t) = 0 \]  \hspace{1cm} (9)

Eq. (7) and Eq. (9) constitute the following equations of motion for 2DOF exponentially damped oscillators:
\[
\begin{align*}
(M\ddot{X}(t) + c\int_0^t \mu e^{-\mu(t-\tau)} \dot{X} d\tau - c\int_0^t \mu e^{-\mu(t-\tau)} \dot{X} d\tau + (K + k)X(t) - kX(t) &= f(t) \\
m\ddot{x}(t) + c\int_0^t \mu e^{-\mu(t-\tau)} \dot{x} d\tau - c\int_0^t \mu e^{-\mu(t-\tau)} \dot{x} d\tau + kx(t) - kX(t) &= 0 
\end{align*}
\]  \hspace{1cm} (10)

3. State-space formalism[7]

First, we introduce the internal variable \( y_1(t), y_2(t) \) through following relationship:
\[ y_1(t) = \int_0^t \mu e^{-\mu(t-\tau)} \dot{X} d\tau, \quad y_2(t) = \int_0^t \mu e^{-\mu(t-\tau)} \dot{x} d\tau \]  \hspace{1cm} (11)

Applying Leibniz’s rule for differentiation of an integral to Eq.(11), one obtains
\[ \dot{y}_1(t) = \int_0^t -\mu^2 e^{-\mu(t-\tau)} \ddot{X}(\tau) d\tau + \mu \dot{X}(t) \]  \hspace{1cm} (12)

Multiplying Eq.(11) by the relaxation parameter \( \mu_1 \), then adding it to Eq.(12)
\[ \dot{y}_1(t) + \mu_1 y_1(t) = \mu_1 \dot{X}(t) \]  \hspace{1cm} (13)

Using the same way, we can get
\[ \dot{y}_2(t) + \mu_2 y_2(t) = \mu_2 \dot{x}(t) \]  \hspace{1cm} (14)

The control input \( u(t) \) is acted on \( M \), then the dynamics of the controlled oscillators is described by
\[
\begin{align*}
M\ddot{X} + cy_1 - cy_2 + (K + k)X - kx &= f + u \\
m\ddot{x} + cy_2 - cy_1 + kx - kX &= 0
\end{align*}
\]  \hspace{1cm} (15)

Defining the following state vector:
\[ \bar{x} = \begin{bmatrix} \dot{X} & \dot{x} & y_1 & y_2 & X & x \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T \]  \hspace{1cm} (16)

Then the equation of motion (15) is transformed into the following equations in the state space
\[
\begin{align*}
\dot{x}_1 &= -\frac{c}{M} x_3 + \frac{c}{M} x_4 - \frac{K + k}{M} x_5 + \frac{k}{M} x_6 + \frac{f}{M} + \frac{u}{M} \\
\dot{x}_2 &= \frac{c}{m} x_1 - \frac{c}{m} x_4 + \frac{k}{m} x_5 - \frac{k}{m} x_6 \\
\dot{x}_3 &= -\mu_1 x_3 + \mu_1 x_1 \\
\dot{x}_4 &= -\mu_2 x_4 + \mu_2 x_2 \\
\dot{x}_5 &= x_1 \\
\dot{x}_6 &= x_2
\end{align*}
\]  \hspace{1cm} (17)

Eqs.(17) can be represented in the first-order form as
\[ \dot{z}(t) = Az(t) + ru + df \]  \hspace{1cm} (18)
4. Sliding mode control of vibration in the 2DOF exponentially damped oscillators

In the following we propose sliding control design for 2DOF exponentially damped oscillators

4.1. Sliding surface design

We constructed the sliding surface as

\[ s = x_1 + \int_0^t \left[ \frac{c}{M} (x_3 - x_4) + \frac{K+k}{m} x_5 - k \frac{1}{M} x_6 \right] d\tau \]  \hspace{1cm} (19)

Differentiating Eq.(19) leads to

\[ \dot{s} = \dot{x}_1 + \frac{c}{M} x_3 - \frac{c}{M} x_4 + \frac{K+k}{M} x_5 - \frac{k}{M} x_6 \]  \hspace{1cm} (20)

Letting \( \dot{s}(t) = 0 \), we can get the following sliding mode dynamics:

\[
\begin{align*}
\dot{x}_1 &= -\frac{c}{M} x_3 + \frac{c}{M} x_4 - \frac{K+k}{M} x_5 + \frac{k}{M} x_6 \\
\dot{x}_2 &= -\frac{c}{m} x_3 + \frac{c}{m} x_4 + \frac{k}{m} x_5 - \frac{k}{m} x_6 \\
\dot{x}_3 &= -\mu_1 x_3 + \mu_1 x_1 \\
\dot{x}_4 &= -\mu_1 x_4 + \mu_1 x_2 \\
\dot{x}_5 &= x_1 \\
\dot{x}_6 &= x_2 
\end{align*}
\]  \hspace{1cm} (21)

Through analysis, we found the following Lyapnov candidate

\[ V_1(t) = \frac{1}{2} M x_1^2 + \frac{1}{2} K x_2^2 + \frac{1}{2} m x_3^2 + \frac{1}{2} k \left( x_5 - x_6 \right)^2 + \frac{c}{2 \mu_1} \left( x_4 - x_5 \right)^2 \]  \hspace{1cm} (22)

Derivation of \( V_1(t) \), we get a very good conclusion.
\[ \dot{V}(t) = Mx_1 \dot{x}_1 + Kx_2 \dot{x}_2 + mx_1 \dot{x}_2 + kx_5 x_2 - kx_6 x_1 - kx_3 x_2 + kx_5 x_1 + \frac{C_1}{\mu_t} (x_4 - x_3) (\dot{x}_4 - \dot{x}_3) \]
\[ = x_1 \left[ M \dot{x}_1 + c x_3 - c x_4 + (K + k) x_3 - k x_6 \right] + x_2 \left[ m \ddot{x}_2 - c x_3 + c x_4 - k x_4 + k x_6 \right] - (x_4 - x_3)^2 \]
\[ = - (x_4 - x_3)^2 \]

If \( \dot{V}(t) = 0 \), it means the equation of motion is not nonviscously damped system. So \( \dot{V}(t) < 0 \), the sliding mode dynamics (21) is asymptotically stable.

4.2. Sliding mode control law design

For the sake of sliding mode control law, we define the second Lyapunov candidate \( V_2(t) = \frac{1}{2} s^2 \). Derivation of \( V_2(t) \) leads to

\[ \dot{V}_2(t) = s \ddot{s} = s \left[ \dot{x}_1 + \frac{c}{M} x_3 - \frac{c}{M} x_4 + \frac{K + k}{M} x_3 - \frac{k}{M} x_6 \right] \]

(23)

Substituting the state equation (17) into (23)

\[ \dot{V}_2(t) = s \left[ \frac{f}{m} + \frac{u}{m} \right] \]

(24)

The external exciting force \( f(t) \) of the equation of motion is not exactly known, but the extent of the imprecision is upper bounded by a known constant \( F \). In this case, Eq.(24) is calculated as

\[ \dot{V}_2(t) \leq \frac{1}{m} \left[ s |F + su| \right] \]

(25)

From (25) we thus construct the following control law

\[ u(t) = - \left( F + \frac{\rho_2}{\sqrt{2}} \right) s \text{sign}(s) - \rho_1 s \]

(26)

Where, \( \rho_1 \) and \( \rho_2 \) are positive constants, \( \text{sign}(\cdot) \) is sign function.

Substituting (26) into (25), we get

\[ \dot{V}_2(t) \leq \frac{1}{m} \left[ s |F - |s| \left( F + \frac{\rho_2}{\sqrt{2}} \right) - \rho_2 s^2 \right] \]
\[ = - \frac{\rho_1}{m} \sqrt{v_2 - 2 \rho_2 V_2} \]

In terms of Lyapunov stability theorem, we derive that \( s \to 0 \) and \( X(t) \to 0, x(t) \to 0 \) as \( t \to \infty \).
5. Numerical simulation
In this section, we present numerical simulations in MATLAB to evaluate the performance of the sliding mode control for two-degree-of-freedom exponentially damped oscillators. We take the parameters in the Eq. (10) respectively as \( M = 1, K = 2, \mu_1 = 0.56, m = 0.2, c = 0.4, k = 0.5 \). The external force is assumed to be \( f(t) = 30\cos 6t \). In Fig.2 shows the forced vibration response of the equations of motion.

![Figure 2](image1.png)

In the sliding control law (26), we take the parameters as \( F + \frac{\rho_1}{\sqrt{2}} = 31, \rho_2 = 1 \). In Fig.3 shows the state response of the sliding mode control system.

![Figure 3](image2.png)

6. Conclusions
Sliding control designs for 2DOF exponentially damped oscillators are presented. We have transformed the equation of motion into state equations by using a set of internal variables. Through analysis, We find the perfect Lyapunov candidate to make the sliding mode dynamics asymptotically stable. Sliding control laws has been designed respectively for 2DOF exponentially damped oscillator. Furthermore, this control method can also be applied to MDOF exponentially damped oscillators. Finally, numerical simulations are provided to illustrate the above control designs.

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