Baryons as Solitons in Three Dimensional Quantum Chromodynamics

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Abstract

We show that baryons of three dimensional Quantum Chromodynamics can be understood as solitons of its effective lagrangian. In the parity preserving phase we study, these baryons are fermions for odd $N_c$ and bosons for even $N_c$, never anyons. We quantize the collective variables of the solitons and there by calculate the flavor quantum numbers, magnetic moments and mass splittings of the baryon. The flavor quantum numbers are in agreement with naive quark model for the low lying states. The magnetic moments and mass splittings are smaller in the soliton model by a factor of $\log \frac{F_N}{N_c m_\pi}$. We also show that there is a dibaryon solution that is an analogue of the deuteron. These solitons can describe defects in a quantum anti–ferromagnet.
1. Introduction

In accompanying paper [1], we constructed the low energy effective lagrangian of the ‘mesons’ of three dimensional QCD, with \( N_c \) colors and \( 2n \) flavors. It is a nonlinear sigma model with the Grassmannian \( Gr_n = SU(2n)/S(U(n) \times U(n)) \) as the target space. The field variables \( \chi \) and \( A \) take values in the group \( SU(2n) \) and the Lie algebra \( G = SU(n) \oplus SU(n) \oplus R \) respectively. The effective action is

\[
S = S_1 + S_k + S_m = \frac{F_\pi}{2} \int \text{tr} \nabla_\mu \chi \nabla^\mu \chi d^3x + \frac{k}{4\pi} \int \text{tr} \left[ A_1 dA_1 + \frac{2}{3} A_1^3 \right] - \frac{k}{4\pi} \int \text{tr} \left[ A_2 dA_2 + \frac{2}{3} A_2^3 \right] + \frac{F_\pi}{2} m_\pi^2 \int \text{tr} \epsilon \epsilon^\dagger \chi d^3x.
\]

Here, \( A = A_1 + A_2 + A_3 \) is a gauge field valued in \( SU(n) \oplus SU(n) \oplus R \) Lie algebra. The covariant derivative is

\[
\nabla_\mu \chi = \partial_\mu \chi - i\chi A_\mu.
\]

Also, \( \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the matrix that commutes with \( G \).

The Chern–Simons terms are necessary to realize the discrete symmetries of 3DQCD correctly. Comparison with 3DQCD shows that the level number \( k \) of the Chern–Simons theory is \( N_c \), the number of colors. This lagrangian describes pseudo–scalars (‘pions’) of mass \( m_\pi \) and vector mesons of mass \( \frac{2\pi F_\pi}{k} \).

We will show in this paper that this effective lagrangian also describes the baryons: they are the topological soliton solutions. The ideas are very similar to those in four dimensions. (For a review, see Ref. [2]). The Chern–Simons terms provide the short repulsion necessary for the stability of these solitons [3] [4]. We argue that these solitons are the baryons of 3DQCD. We then study the low energy properties (mass splittings, magnetic moments, flavor quantum numbers) by using an effective lagrangian
for the collective motion. This effective lagrangian (for the case of four flavors, $n = 2$), is a $0 + 1$ dimensional nonlinear sigma model on the coset space $S(U(2) \times U(2))/U(1) \times U(1)$.

We find that the quantum numbers of the solitons are the same, (at low energies) as those of the baryons in the quark model. However, the mass splittings are smaller than in the naive quark model. This is because the size of the soliton is bigger than that of the baryon in the quark model.

A summary of the behaviour of our effective action under discrete symmetries is perhaps useful. There are three discrete transformations $P_0, \sigma$ and $P_2$ of interest.

$P_0 : \chi(x_1, x_2, t) \rightarrow \chi(-x_1, x_2, t) \quad A_{1,2,3}(x, t) \rightarrow A_{1,2,3}(-x_1, x_2, t)$

$\sigma : \chi(x, t) \rightarrow \chi(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_1(x, t) \leftrightarrow A_2(x, t) \quad A_3(x, t) \rightarrow -A_3(x, t)$

$P_2 : \chi(x, t) \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi(x, t) \quad A_{1,2,3}(x, t) \rightarrow A_{1,2,3}(x, t) .$

Under $P_0$, the terms $S_1$ and $S_m$ are invariant; under $\sigma$, only $S_1$ is invariant. Also, $P_2$ leaves $S_1$ and $S_k$ invariant. Thus the only symmetry of the total effective action is the product $P_0\sigma P_2 = P_1 P_2 = P$. It is this product that we should identify with physical parity.

2. Baryons of 3DQCD as Solitons

As in the four dimensional Skyrme model, we should expect baryons to arise as solitons of this effective lagrangian. In fact we will see now that there are such solitons. Furthermore we will show that they are fermions when $k$ is odd (bosons when $k$ is even) and that their wavefunctions transform under the flavor symmetry $G$ as expected from the quark model.

We will consider only the special case $n = 2$ in detail. (This is analogous to the Skyrme model with $SU(3)$ symmetry [2].) The cases $n > 2$ are exactly analogous and nothing much is learned by being more general. The case $n = 1$ has some special features because the third homotopy group of the coset space (which is $CP^1$) is also non-trivial; we will comment on it later.

The Grassmannian has nontrivial second homotopy group, which allows for the exis-
ence of topological solitons. The topological current is, in our choice of variables,

\[ j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho_3. \]  

(2)

The soliton number is just the vorticity of \( A_3 \) at infinity:

\[ Q = \frac{1}{2\pi} \int d\theta A_3. \]  

(3)

This current is to be identified with baryon number current of 3DQCD. For, it is equal to the expectation value \( \frac{1}{N_c} < \sum_i \bar{q}^i \gamma_\mu q_i > \) coupled to an external \( A_3 \) gauge field. (To be precise, this vacuum expectation value is equal to the above topological current up terms that do not contribute to the total charge) \([5],[6],[7]\).

A rotationally symmetric ansatz for the static solution is

\[ \chi_0(r, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi(r)e^{iQ\theta} & \sin \phi(r) & 0 \\ 0 & -\sin \phi(r) & \cos \phi(r)e^{-iQ\theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

(4)

and

\[ A_\mu^0 dx^\mu = \frac{1}{2} V(r)dt \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{2} A(r)d\theta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ + \frac{1}{2} [\tilde{V}(r)dt + \tilde{A}(r)d\theta] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\( V, \tilde{V}, \tilde{A} \) are Lagrange multipliers which can be eliminated by their equations of motion:

\[ V(r) = \frac{k}{2\pi F_\pi r} A'(r), \quad \tilde{V} = 0, \quad \tilde{A}(r) = Q \cos^2 \phi(r). \]  

(5)

In the above ansatz, \( Q \) is the soliton number of the solution. (We will first study \( Q = 1 \), but later we will need \( Q = 2 \) for the dibaryon.) The boundary conditions on \( \phi \) and \( A \) are

\[ \phi(0) = \frac{\pi}{2}, A(0) = 0 \quad \text{and} \quad \phi(r) \to 0, A(r) \to Q \quad \text{as} \quad r \to \infty \]  

(6)
After substituting into the lagrangian and eliminating \( V, \tilde{V}, \tilde{A} \), we get the energy integral

\[
E(\phi, A) = \pi F_\pi \int_0^\infty \left[ 2r\phi'^2 + \frac{A^2 \sin^2 \phi + (Q - A)^2 \cos^2 \phi + \frac{Q^2}{4} \sin^2 2\phi}{r} \right. \\
\left. + \left( \frac{k}{2\pi F_\pi} \right)^2 \frac{A'^2}{r} + 4m^2_\pi r \sin^2 \phi \right] dr
\]

It is clear that under scaling

\[
\phi(r) \to \phi(\lambda r) \quad A(r) \to A(\lambda r)
\] (7)

the first two terms are invariant, the third goes like \( \lambda^{-2} \) and the last term like \( \lambda^2 \). This shows that the solution is stable under scaling (“Derrick’s Theorem”). The mass term tends to shrink the soliton while the vector mesons provide a short range repulsive force which tends to expand it.

The Euler–Lagrange equations for the cases \( Q = 1, 2 \) can be solved numerically, by relaxation methods. In Fig. 1 we plot the solution for \( Q = 1 \). The energy of the baryon (Fig. 2) is almost a linear function \( E \sim a\frac{2\pi F_\pi}{N_c} + bm_\pi \), with a numerical fit to the constants, \( a \sim 1.058, b \sim 1.167 \). This almost linear dependence can be understood by a variational argument (an approximate ‘virial theorem’). We will comment on the \( Q = 2 \) solution later.

In the limit \( m \neq 0, k = 0 \), the soliton will shrink to a point. In the other limit \( m \to 0 \) keeping \( k \) fixed, it will expand to infinite size. It is a surprising fact that, in the absence of a current quark mass, the soliton will expand to infinity and disappear. In general, the size of the soliton is of the same order as the pion Compton wavelength, which is very different from the situation in four dimensions.

If both \( m \) and \( k \) are zero the solution tends to the well–known soliton of the \( CP^1 \) model,

\[
\phi = \frac{\pi}{2} - \arctan \frac{r}{a}, \quad A(r) = \frac{r^2}{a^2 + r^2}.
\] (8)
This case is scale invariant, so that there is a soliton of every possible size $a$. However, in this limit, (which is not related to 3DQCD since $k = 0$) the moment of inertia of the soliton is infinite \[8\], leading to a spontaneous breaking of rotation invariance.

One can argue on purely topological grounds that these solitons are fermions when $N_c$ is odd and bosons when $N_c$ is even. First one shows that they are of spin $\frac{N_c}{2}$ mod $Z$ by following an argument analogous to Witten \[7\]. One consider a closed path in the configuration space which corresponds to creating a soliton anti-soliton pair, separating them to a large distance, then rotating the soliton through $2\pi$ and then annihilating them. This process has a probability amplitude $(-1)^{N_c}$. (We don’t give the details since the following collective co-ordinate method will show quite explicitly that the spin is $N_c/2$ mod $Z$). Then one can use the general spin statistics theorems \[9\] of soliton theories to show that they are fermions (bosons) for odd $N_c$ (even $N_c$).

3. **Collective Coordinate Quantization**

The three dimensional sigma model has a global invariance under $G = S(U(n) \times U(n))$. Therefore given any static solution $\chi_0(x)$, we can find another one, $X\chi_0(x)$ for $X \in G$, of the same energy. By allowing $X$ to be a slowly varying function of time, we will excite the lowest energy states of the soliton. However, not all such rotations produce a physically distinct soliton: there is a subgroup $H$ of $G$ that changes $\chi_0$ only by a gauge transformation. Thus the configuration space of the collective motion of the soliton is a coset space $G/H$. In fact, the effective action for collective motion is a **one dimensional** nonlinear sigma model on $G/H$. This can be described again by a variable $X$ valued in $G$ and a one dimensional gauge field valued in $H$. We will see that the Chern–Simons term of the three dimensional sigma model induces a Chern–Simons term for the one dimensional theory as well. This term will then dominate the low energy properties of the soliton. (In particular, how the soliton wave function transforms under $G$).
Let us first determine $H$, the subgroup of $G$ that only changes $\chi_0$ by a gauge transformation. Recall that the gauge transformations of the three dimensional theory are right multiplications by $G$ while the global symmetry is a left multiplication. Thus $h$ is in $H$ if $\chi_0^\dagger(x) h \chi_0(x)$ is in $G$ for all $x$. A short calculation will show that such elements are of the form

$$h = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & e^{i\alpha} & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & h_2 \end{pmatrix}$$

(9)

where $h_{1,2}$ are in $U(n-1)$. (It is useful to consider first the special case $r = 0$ which will already require that $h$ be block diagonal.) The dimension of $G/H$ is then $4n-3$. (There are also two translational collective modes which we are ignoring. They can be taken care of trivially [10]. Our soliton has one less degree of freedom than the instanton of the two dimensional Grassmannian sigma model [11], because scale invariance is not symmetry. In the limit $m = k = 0$ we recover this collective mode).

For a more detailed study, we will restrict to the case $n = 2$. Then $H$ is the abelian group $U(1) \times U(1)$, generated by

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(10)

The collective variables $X, a, \tilde{a}$ describe a deformation of the soliton configuration,

$$\chi(x, t) = X(t) \chi_0(x)$$

(11)

and

$$A(x, t) = A_0^\mu dx^\mu + [a(t)y + \tilde{a}(t)\tilde{y}] dt - iP_G \left( \chi_0^\dagger X^\dagger \dot{X} \chi_0 \right) dt,$$

(12)

where $A_0^\mu$ is the static solution.

The collective action will be a nonlinear sigma model on $G/H$ with $a, \tilde{a}$ playing the role of one dimensional gauge fields valued in $H$. 
The general one dimensional nonlinear sigma model on $G/H$ has action

$$S_{\text{coll}} = \frac{1}{2} I_1 \int \text{tr} D_t X_1^\dagger D_t X_1 dt + \frac{1}{2} I_2 \int \text{tr} D_t X_2^\dagger D_t X_2 dt +$$

$$\frac{1}{2} I_3 \int (\dot{\xi} + \ddot{a})^2 dt + \mu \int adt + \tilde{\mu} \int \tilde{a} dt$$

up to higher derivatives in $t$. Here,

$$X = \begin{pmatrix} e^{i\frac{\xi}{2} X_1} & 0 \\ 0 & e^{-i\frac{\xi}{2} X_2} \end{pmatrix}, \text{ for } X_{1,2} \in SU(2)$$

(13)

and

$$D_t X_1 = \dot{X}_1 + X_1 (i a \tau_3 + \frac{1}{2} i \ddot{a} \tau_3)$$

$$D_t X_2 = \dot{X}_2 + X_2 (-i a \tau_3 + \frac{1}{2} i \ddot{a} \tau_3)$$

$$D_t \xi = \dot{\xi} + \ddot{a}.$$ 

The $I_{1,2,3}$ are ‘moments of inertia’ determined by the microscopic theory.

This action has a gauge invariance (up to boundary terms) under the right action of $H$,

$$X_1 \rightarrow X_1 e^{i(\lambda + \frac{1}{2} \dot{\lambda}) \tau_3},$$

$$X_2 \rightarrow X_2 e^{i(-\lambda + \frac{1}{2} \dot{\lambda}) \tau_3},$$

$$\xi \rightarrow \xi + \lambda$$

$$a \rightarrow a - \dot{\lambda}, \quad \ddot{a} \rightarrow \ddot{a} - \dot{\lambda}.$$ 

The Chern–Simons terms of the one dimensional gauge fields are linear. In order for $e^{iS_{\text{coll}}}$ to be gauge invariant, $\mu, \tilde{\mu}$ have to be integers. Of course, there is no curvature for such a gauge field. The Chern–Simons term will contribute a phase to the wave function which will determine its transformation properties under the global symmetry.

Parity invariance imposes certain relations among the constants of the effective lagrangian. Under parity,

$$P : \chi(r, \theta, t) \rightarrow \sigma \chi(r, -\theta, t) \sigma.$$ 

(14)
One can check that
\[ \sigma \chi_0(r, -\theta) \sigma = \tau_2 \chi_0(r, \theta) \tau_2 \] (15)
where
\[ \tau_2 = \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
\end{pmatrix}. \] (16)

Then, we have
\[ P : X \chi_0(r, \theta) \rightarrow \sigma X(t) \sigma \tau_2 \chi_0(r, \theta) \tau_2. \] (17)

The last factor is an element of \( G \), so is a gauge transformation. Hence parity acts on the collective variable as follows:
\[ P : X \rightarrow \sigma X \sigma \tau_2. \] (18)

Therefore, in the collective action,
\[ P : X_1 \leftrightarrow X_2 \begin{pmatrix}
0 & -i \\
i & 0 \\
\end{pmatrix}, \quad \xi \rightarrow -\xi, \quad a \rightarrow a, \quad \tilde{a} \rightarrow -\tilde{a} \] (19)
so that we get
\[ I_1 = I_2 \quad \tilde{\mu} = 0. \] (20)

If it weren’t for the Chern–Simons terms, there would have been an additional discrete symmetry,
\[ X_1 \leftrightarrow X_2 \quad a \rightarrow -a \quad \tilde{a} \rightarrow \tilde{a}. \] (21)

Thus by arguments entirely analogous to the ones that led to the three dimensional effective action, we get
\[ S_{\text{coll}} = \frac{1}{2} I_1 \int \text{tr} D_t X_1^\dagger D_t X_1 dt + \frac{1}{2} I_1 \int \text{tr} D_t X_2^\dagger D_t X_2 dt + \]
\[ \frac{1}{2} I_3 \int (\dot{\xi} + \tilde{a})^2 dt + \mu \int a dt \]
Now we can determine the constants $I_{1,3}$ and $\mu$ from our ‘microscopic’ theory, which is our three dimensional effective action. For $I_{1,3}$ we get the integrals:

\[ I_1 = I_3 = \pi \int_0^\infty \sin^2 2\phi \ rdr \quad (22) \]

To calculate the Chern–Simons term we will need the projections

\[
A_1 = \left[ \left( a + \frac{1}{2} \tilde{a} - \frac{1}{2} V \right) dt - \frac{1}{2} A d\theta \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
A_2 = \left[ \left( -a + \frac{1}{2} \tilde{a} - \frac{1}{2} V \right) dt - \frac{1}{2} A d\theta \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Since $A$ lies in an abelian subgroup of $G$, the $A^3$ terms in the Chern–Simons term will not contribute. Also, we can drop the terms of order zero in $a, \tilde{a}$ since they are part of the energy of the static soliton. Thus we get

\[
\int \text{tr} A_1 da_1 - \int \text{tr} A_2 da_2 = 2 \int a dt \int d\theta A' = 4\pi Q \int a dt \quad (23)
\]

since $\int d\theta \frac{\partial A}{\partial r} = 2\pi Q.$ ($Q$ is the soliton number). Thus we see that $\mu = kQ, \tilde{\mu} = 0$ and, the collective action is

\[
S_{\text{coll}} = \frac{1}{2} I_1 \int \text{tr} D_1 X_1^\dagger D_1 X_1 dt + \frac{1}{2} I_1 \int \text{tr} D_1 X_2^\dagger D_1 X_2 dt +
\frac{1}{2} I_3 \int (\dot{\xi} + \tilde{\alpha})^2 dt + kQ \int a dt
\]

Let us now consider the wave functions of the soliton. They can be thought of as functions of $X$ that satisfy a constraint coming from the $H$ gauge invariance. The wave function of the soliton would have been invariant under the right action of $H$ if this collective action were gauge invariant. But, $S_{\text{coll}}$ is invariant only up to a boundary term so that the wave functions are invariant only up to a phase:

\[
\psi(X e^{(i\lambda y + i\tilde{\lambda} \tilde{y})}) = e^{-ikQ\lambda} \psi(X).
\]

\[\dagger\] These will diverge in the limit $m = k = 0$. This is the problem noted in [8]. It appears to be coincidence that $I_1 = I_3$. 

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They are sections of a nontrivial line bundle on $G/H$.

Left multiplication by $G$ will leave this constraint invariant, so that the states can be classified by representations of this flavor symmetry. Furthermore, the right multiplication

$$X \to X \exp \left[ i \frac{\alpha}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

(25)

will leave the constraint invariant. This symmetry, corresponds to spatial rotations of the soliton. For,under spatial rotations,

$$X(t)\chi_{0}(r, \theta) \to X(t)\chi_{0}(r, \theta + \alpha) =$$

$$X(t) \exp \left[ i \frac{\alpha}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \chi_{0}(r, \theta) \exp \left[ i \frac{\alpha}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right].$$

The last factor is just a gauge transformation in $G$, so that the net effect is a change of $X$. However, there is an ambiguity in the definition of the spin operator, since any linear combination of $y, \tilde{y}$ can be added without changing its physical meaning. A convenient choice will be one that changes sign under parity (spin is a pseudo–scalar). If we define the generator of rotations to be a right multiplication by

$$s = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(26)

it will change sign under parity. (Also, left and right multiplications by $s$ are equivalent). This matrix differs from the naive generator of rotations by a linear combination of $y, \tilde{y}$:

$$s = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \tilde{y}.$$

(27)

We can now solve the constraint and determine the quantum numbers of the different states of the soliton. Any function on $G = S(U(2) \times U(2))$ can be expanded in terms of
the matrices of its irreducible representations. Recall that

\[ (g_1, g_2, e^{i\alpha}) \rightarrow \begin{pmatrix} g_1 e^{i\alpha} & 0 \\ 0 & g_2 e^{-i\alpha} \end{pmatrix} \]  

(28)
gives a homomorphism $SU(2) \times SU(2) \times U(1) \rightarrow S(U(2) \times U(2))$. The kernel of this homomorphism is $(-1, -1, -1)$. So representations of $G$ can be labelled by $(j, j', n)$ for $j, j' = 0, \frac{1}{2}, 1 \ldots$ and $n = \cdots, -1, 0, 1, \cdots$. In order that $(-1, -1, -1)$ be represented by the identity, $2(j + j') + n$ must be even. It is useful to note that $n = 2s$, $s$ being the spin variable defined earlier.

In terms of the generators of $SU(2) \times SU(2) \times U(1)$, a basis for wavefunctions is labelled by $|j, j', n; j_3L, j'_3L; j_3R, j'_3R \rangle$. The constraint on the wave function becomes (for $B = 1$)

\[ \tilde{y}_R = (j_3R + j'_3R) + \frac{n}{2} = 0 \quad y_R = 2(j_3R - j'_3R) = -k. \]  

(29)

These conditions imply that the spin $s$ of any baryon state is integer (or half–integer) according to whether $k$ is integer (or half–integer). For, the spin is $s = \frac{n}{2}$, which from the constraint is $(j + j')$; however, $(j + j') = (j_3R + j'_3R) \mod Z$. On the other hand $j_3R + j'_3R = j_3R - j'_3R \mod Z$ which in turn is $\frac{k}{2} \mod Z$ from the second constraint. Thus $s = \frac{k}{2} \mod Z$.

One representation that contains such a state is $(j, j', n) = (\frac{k}{2}, 0, k)$. Then $j_3R = -\frac{k}{2}, j'_3R = 0, n = k$ is the only choice that satisfies the conditions. The different left indices will then describe a state that transforms under the representation $(\frac{k}{2}, 0, k)$ of the flavour group $G$. This is part of a more general solution, $(\frac{k-r}{2}, \frac{r}{2}, k-2r)$ for $r = 0, 1, \cdots k$. There is exactly one state, $j_3R = -\frac{k-r}{2}, j'_3R = \frac{r}{2}$ in each of these representations that satisfies the constraint. So there is one multiplet of the left action of $G$ for each $r$. All these representations can be grouped into a symmetric tensor representation of order $k$ of the group $SU(4)$ that contains $G$. These are therefore precisely the baryon wave functions predicted by the naive quark model. The soliton model predicts many more states than
the naive quark model. Only in the limit \( k = N_c \to \infty \) will the two agree. This is exactly the analogue of what happens in the Skyrme model in 3 + 1 dimensions.

The hamiltonian operator that we get from the collective action is the Laplace operator on \( G/H \). Its eigenvalues will give the mass splittings of the different baryon multiplets. The above wavefunctions are eigenfunctions with eigenvalues

\[
H | j, j', n; j_3L, j'_3L; j_3R, j'_3R > = \left[ \frac{1}{2I_1}[j(j+1) + j'(j'+1)] + \frac{(n/2)^2}{2I_3} \right] | j, j', n; j_3L, j'_3L; j_3R, j'_3R >.
\]

The dependence of mass splittings on \( j, j', n \) agree with the prediction of the static quark model. There they arise from spin–spin coupling of the quarks due to gluon exchange. However, the magnitude of the splittings is much smaller in the soliton model. In the limit that the pion mass (or current quark mass) goes to zero, the soliton becomes very large and its moment of inertia diverges:

\[
I_{1,3} \sim \frac{1}{F_\pi} \log \frac{m_\pi N_c}{F_\pi}.
\]

Then the mass splittings go to zero as \( m_\pi \to 0 \). In the static quark model on the other hand, even in this limit, the mass splittings are non–zero. We believe that in 2 + 1 dimensions, the soliton model is a more reliable description of the size, and hence, the mass splittings of the baryon.

4. The dibaryon solution

In the Skyrme model with \( SU(2) \) symmetry, it is known that there is a dibaryon solution [12] with cylindrical rather than spherical symmetry. Roughly speaking, it describes a pair of baryons rotating around each other. The baryon number density is concentrated on a toroid. There is an analogous dibaryon solution in 3DQCD as well. This is a bound state of two baryons formed by balancing their long range attraction (due to exchange of massive pions) by a short range repulsion (due to vector meson exchange).
The baryon number density again has a maximum in a ring. However, in our case it has the same rotational symmetry as the one baryon solution. Therefore it is a solution with the same static ansatz as before, but with $Q = 2$. In Fig. 3 we give a comparison of the baryon number density $A'/r$ for the baryon and dibaryon solution. The numerical solution shows that the baryon number density indeed has a maximum at a ring of radius about $\frac{4\pi}{m_\pi}$. The dibaryon configuration has a positive binding energy (a few percent of rest mass, Fig. 4). This shows that this is stable against decay into a pair or baryons. For higher values of $Q$, there is a static solution, but it is probably unstable with respect to non-spherically symmetric perturbations.

The wave–functions of the dibaryon can be determined by the same collective co–ordinate method as before. The only difference is that $N_c$ is replaced by $2N_c$. Thus the low lying states of this dibaryon form a symmetric tensor representation of $SU(4)$ or order $2N_c$. This is a little surprising from the quark model point of view. The only way the quark wavefunctions can be completely symmetric in flavor variables is for a rotational degree of freedom to be excited. ($2N_c$ quarks cannot be put into a completely anti–symmetric representation of color $SU(N_c)$). The natural possibility is that $N_c$ of the quarks are in a state with orbital angular momentum $+1$ and the others in a state with angular momentum $-1$. This agrees with the picture that the dibaryon consists of a pair of baryons orbiting each other.

If the ‘baryons’ can be identified as the charge carriers in a model of superconductivity, the dibaryon bound state we are describing can be thought of as a Cooper pair. It will have charge $2e$ and a small binding energy. If the pions are identified with the spin waves of anti–ferromagnetism, the attractive force is ultimately anti–ferromagnetic in origin.

5. **Soliton model with $n = 1$**

The special case of $n = 1$ is somewhat different because there are no non–abelian
Chern-Simons terms that can be added to the action to recover the correct parity properties. In this case the coset space $SU(2)/U(1)$ is just $CP^1$, or equivalently, $S^2$. Clearly $H_4(CP^1) = 0$ so that there are no Wess–Zumino terms that can be added to the action. This case is analogous to the Skyrme model with $SU(2)$ symmetry. As in that case, we can study this case by embedding it into the higher flavor case. (For example, we can imagine that one parity doublet of quarks in the case $n = 2$ is much heavier than the other.)

We can also study this case directly; $\pi_3(CP^1) = Z$ so that there is another kind of topological term (Hopf term) that can be added. The coefficient of the Hopf term can be determined by comparison to QCD. This term then reduces to a model of Wilczek and Zee [13]. (Except that the coefficient of the Hopf term corresponds to bosons for even $N_c$ and fermions for odd $N_c$).

The effective action then becomes

$$S = \frac{F_\pi}{2} \int \text{tr} \nabla_\mu \chi \nabla^\mu \chi d^3x + \frac{\theta}{4\pi^2} \int \bar{A} dA + \frac{1}{2} F_\pi m_\pi^2 \int \text{tr} \chi \epsilon \chi^\dagger \epsilon d^3x + \cdots \quad (31)$$

Here, $\bar{A} = \text{Im} \text{tr} \partial_\mu \chi \nabla^\mu \epsilon$. The second term is the Hopf term, which is quantized. Hence $\theta$ must be periodic with period $2\pi$. In order for parity to be a symmetry, $\theta$ must be a multiple of $\pi$. Whether it is an even or odd multiple of $\pi$ is determined by comparing the global anomaly of the flavor $SU(2)$ symmetry with QCD. We will get $\theta = \pi N_c \text{ mod } 2\pi$.

At this level, this lagrangian does not contain any solitons. The Chern–Simons terms that stabilized the soliton against collapsing to a point have disappeared. However at sufficiently short distances, higher derivative terms will become important. The first such term is a Maxwell term for the gauge field:

$$S = \frac{F_\pi}{2} \int \text{tr} \nabla_\mu \chi \nabla^\mu \chi d^3x + \frac{\theta}{4\pi^2} \int \bar{A} dA + \frac{1}{2} F_\pi m_\pi^2 \int \text{tr} \chi \epsilon \chi^\dagger \epsilon d^3x - \frac{1}{4e^2} \int F_{\mu\nu} F^\mu\nu d^3x + \cdots \quad (32)$$
Lagrangians of this type have been studied in the literature in a cosmological context [14]. Except for the Hopf term, our effective lagrangian describes an abelian Higgs model (with an infinite mass for the Higgs field) with a global $SU(2)$ symmetry, broken by the pion mass. There are solitons in this theory of mass about $F_\pi(a + b \frac{m_\pi}{c})$, $a$ and $b$ being constants of order one. If the $SU(2)$ symmetry is exact, (the pion mass is zero) the soliton is infinitely large. (These are called ‘semi–local strings’ in the cosmological context). Otherwise they will have a size of order $\frac{1}{m_\pi}$. The same arguments that we made earlier will show that these are fermions or bosons depending on whether $N_c$ is even or odd. The collective variable $X$ is valued in $U(1) \times U(1)$; the wave functions of the solitons will be spanned by the basis $|j_3, j'_3>$ for $j_3, j'_3 = 0, \frac{1}{2}, \frac{3}{2}, \cdots$. However, the Hopf term will require that $j_3 + j'_3 = \frac{N_c}{2}$ mod $Z$. Thus we have integer spin for even $N_c$ and half–integer spin for odd $N_c$.

6. Quark Model of Baryons

The naive quark model of baryons assumes that the flavor symmetry breaking results in constituent masses to the quarks. Massive quarks carry spin. In order to preserve parity, we assume $n$ quarks are spin up and the other $n$ are spin down. The flavor symmetry is now $U(n) \times U(n) = U(1) \times G$. The first $U(1)$ is baryon number symmetry. The $U(1)$ factor within $G$ is interpreted as spin.

We first discuss the constituent quark mass. In 3+1 dimensions, the constituent quark mass $M$ is roughly $M + m = M_B/N_c$, where $M_B$ is the mass of some baryon and $m$ is the current quark mass. It agrees with the Skyrme model. In our case, the soliton model predicts something different: $M_B/N_c \sim \tilde{F}_\pi + \sqrt{(m\tilde{F}_\pi)}$. (In this section we denote $F_\pi/N_c$ by $\tilde{F}_\pi$; this is the quantity that is kept finite in the large $N_c$ limit.) Thus the dependence of the constituent quark mass on the current quark mass has an exponent $\frac{1}{2}$ instead of 1 as one would naively expect. This is reminiscent of anomalous critical exponents in the
theory of phase transitions.

The baryon wave functions from the nonrelativistic quark model agree with the low lying modes of soliton model. As in 3+1 dimensions, we assume that baryons form completely symmetric representation under $G$ since the quarks are antisymmetric in color. It is more convenient to look for the symmetric representation of $SU(2n)$ which combines the flavor and spin symmetries. The Young tableau consists $N_c$ boxes in the same row. Each box represents $2n$-dimensional representation of $SU(2n)$, corresponding to $2n$ quark states. We can then decompose it to irreducible representations of $G$. Consider $n = 2$, $G = S(U(2) \times U(2))$. We can label the $SU(4)$ symmetric representation with $N_c$ boxes by a sum of $|j, j', n>$, where $j, j'$ denote spin $j$ and $j'$ representations of the two $SU(2)$, respectively, and $n$ labels the remaining $U(1)$ group. Since there are $N_c$ boxes,

$$j + j' = \frac{N_c}{2}. \quad (33)$$

The spin is

$$s = \frac{n}{2} = j - j'. \quad (34)$$

These agree with the solutions to the constraints found in the soliton model. Of course, the soliton model has more states and there is complete agreement only in the limit $N_c \to \infty$.

One of the interesting predictions of the naive quark model in 3+1 dimensions is the spin-dependent mass splitting. It comes from the residual gluon exchange. If we simply take over it to 2+1 dimensions, the Hamiltonian is

$$\frac{\mu}{M^2} \sum s_is_j \sim \frac{\mu}{M^2} S^2 + \text{const.}, \quad (35)$$

where $\mu$ is some constant of dimension 3, $S$ is the spin of the baryon. This is, however, different from the soliton prediction. The moment of inertia is $\ln(\bar{F}_\pi/m_\pi)/\bar{F}_\pi$ there. The soliton model predicts a spin–splitting that is smaller by a factor $\ln \bar{F}_\pi/m_\pi$. 

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One can also work out the prediction of the magnetic moments. Let us consider electromagnetic $U(1)$ charge to be the baryon number. (For simplicity we consider only the iso–singlet part of electric charge). In the quark model, one simply writes the interacting Hamiltonian as

$$H_{\text{int}} = \sum \frac{s_i}{M} B = B \frac{S}{M}. \quad (36)$$

In the soliton model, the $U(1)$ current is just the topological current (19). The gauge coupling is then a Chern–Simons term,

$$S_{\text{int}} = \frac{1}{2\pi} \int d^3 x \epsilon^{\mu\nu\rho} A_{\mu}^{em} \partial_{\nu} A_{3}^{\rho}. \quad (37)$$

Since $A_3$ is pseudo-vector, this is parity invariant. The interaction is gauge invariant provided $\partial_{\nu} A_3^{\rho}$ is well-behaved at infinity. We are interested in $B_i$ non-zero, i.e., magnetic coupling. We have to compute the current $j_i$. For $\chi = X(t)\chi_0$, the equation of motion tells us

$$A_{3i} = \text{tr} \chi_0^\dagger \partial_i \chi_0, \ A_{30} = \text{tr} \chi_0^\dagger X^\dagger \dot{X} \chi_0 \epsilon. \quad (38)$$

Since $A_{3i}$ is time independent, only $A_{30}$ contributes to the current. One can show that

$$j_i = \partial_i A_{30} = (\sin^2 \phi)' \text{tr} X^\dagger \dot{X} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \end{array} \right). \quad (39)$$

The interaction Lagrangian is

$$S_{\text{int}} = \frac{1}{2\pi} \int dt d^2 x A_{i}^{em} \epsilon^{ij} \partial_j (\sin^2 \phi) \text{tr} X^\dagger \dot{X} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \end{array} \right). \quad (40)$$

One can integrate by parts, since $\sin \phi$ is well behaved when $m_{\pi}$ is not zero. For constant magnetic field, the result is

$$S_{\text{int}} \sim \int dt \frac{\ln \left( \frac{F_{\pi}}{m_{\pi}} \right)}{F_{\pi}^2 \lambda} B \text{tr} X^\dagger \dot{X} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \end{array} \right). \quad (41)$$
Recall the definition of the angular momentum $S$, we have the interacting Hamiltonian

$$H_{\text{int}} \sim BS/\tilde{F}_\pi,$$

which qualitatively agrees with the quark model prediction. Thus the soliton model predicts that the color and magnetic moment of the constituent quark is smaller by a factor of $\log \frac{\tilde{F}_\pi}{m_\pi}$ than one would naively expect.

At long distances, the force between two constituent quarks has an additional piece mediated by pion exchange. If it is attractive, one has an explanation of how dibaryon bound state is formed (equivalently one can check the long range force between two baryons). For simplicity we consider $N = 2$ and $N_c = 3$ case. There are two pions $\pi^\pm$. We write down the effective Lagrangian of the pions and the constituent quarks $\psi_u$ and $\psi_d$,

$$S = \int d^3x (\bar{\Psi} \gamma^\mu \partial_\mu \Psi + M \bar{\Psi} \pi \cdot \tau \Psi + F_\pi \partial_\mu \pi \cdot \partial_\mu \pi + F_\pi m_\pi^2 \pi^+ \pi^-),$$

where

$$\Psi = \left( \begin{array}{c} \psi_u \\ \psi_d \end{array} \right), \pi = (\pi^+, \pi^-, \pi_3), \pi^2_3 + \pi^+ \pi^- = 1,$$

and $\tau$’s are Pauli matrices. We have omitted residual color force. One can easily compute the force between the two quarks. We first do non-relativistic reduction of $\Psi$, then integrate over $\pi$’s, it turns out that the only force is an attractive one between $u$-quark and $d$-quark,

$$U(r) \sim -\frac{m_\pi^2}{F_\pi} e^{-m_\pi r}.$$  

This attractive force provides a mechanism for the formation of dibaryons.

In summary, the soliton model predicts a larger contribution of current quark mass to the baryon mass, and moment of inertia is much larger than the naive quark model. The soliton model arises in a systematic $1/N_c$ expansion, whereas the naive quark model doesn’t seem to become exact in any limit. We can conclude that in $2+1$ dimensions (or lower), soliton model describes baryons more accurately than the naive quark model, or
its refinement, the analog of chiral quark model in 3+1 dimensions. The underlying reason for this is the infrared behavior of pions: the pion condensate in $d \geq 2 + 1$ is negligible at long distance, while it is more important in $d \leq 2 + 1$. $d = 2 + 1$ is in some sense a critical dimension, with a logarithmic divergence in the moments of inertia.

7. Acknowledgements

S.G.R. thanks P. Wiegmann for rekindling his interest in soliton models. We also thank A.P. Balachandran and K. Gupta for discussions. This work is supported in part by the US Department of Energy Contract No. DE-AC02-76ER13065.
Figure Captions

**Fig. 1** One baryon solution for $2\pi F_{\pi}/N_c m_\pi = 7$. The solid line represents the function $\phi$ and the dashed line represent the function $A$.

**Fig. 2** Baryon energy in units of $m_\pi$ as function of the ratio $2\pi F_{\pi}/N_c m_\pi$.

**Fig. 3** Baryon number density for the single baryon (solid line) and dibaryon (dashed line) solutions. The choice of parameters is as before: $2\pi F_{\pi}/N_c m_\pi = 7$.

**Fig. 4** Binding energy of the dibaryon as function of the ratio $2\pi F_{\pi}/N_c m_\pi$. Here, $\Delta E = 2E_{Q=1} - E_{Q=2}$
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