COBORDISMS OF MAPS WITHOUT PRESCRIBED SINGULARITIES

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Abstract. Let $N$ and $P$ be smooth closed manifolds of dimensions $n$ and $p$ respectively. Given a Thom-Boardman symbol $I$, a smooth map $f : N \to P$ is called an $\Omega^I$-regular map if and only if the Thom-Boardman symbol of each singular point of $f$ is not greater than $I$ in the lexicographic order. We will represent the group of all cobordism classes of $\Omega^I$-regular maps of $n$-dimensional closed manifolds into $P$ in terms of certain stable homotopy groups. As an application we will study the relationship among the stable homotopy groups of spheres, the above cobordism group and higher singularities.

1. Introduction

Let $N$ and $P$ be smooth ($C^\infty$) manifolds of dimensions $n$ and $p$ respectively. Let $k \gg n, p$. Let $J^k(N, P)$ denote the $k$-jet bundle of the manifolds $N$ and $P$ with the canonical projection $\pi^k_N \times \pi^k_P$ onto $N \times P$, whose fiber is denoted by $J^k(n, p)$. Here, $\pi^k_N$ and $\pi^k_P$ map a $k$-jet to its source and target respectively. Let $I = (i_1, i_2, \ldots, i_k)$ be a Thom-Boardman symbol (simply symbol) where $i_1, i_2, \ldots, i_k$ are a finite number of integers with $i_1 \geq i_2 \geq \cdots \geq i_k \geq 0$. In [13] there have been defined what is called the Boardman manifold $\Sigma^I(N, P)$ in $J^k(N, P)$. A smooth map germ $f : (N, x) \to (P, y)$ has $x$ as a singularity of the symbol $I$ if and only if $j^k f \in \Sigma^I(N, P)$. Let $\Omega^I(N, P)$ denote the open subset of $J^k(N, P)$ which consists of all Boardman manifolds $\Sigma^I(N, P)$ with symbols $I'$ of length $k$ and $I' \leq I$. It is known that $\Omega^I(N, P)$ is an open subbundle of $J^k(N, P)$ over $N \times P$, whose fiber is denoted by $\Omega^I(n, p)$. A smooth map $f : N \to P$ is called an $\Omega^I$-regular map if and only if $j^k f(N) \subset \Omega^I(N, P).

Let $J$ be another Thom-Boardman symbol with $I \leq J$. In this paper we will represent the set of all cobordism classes as $\Omega^J$-regular maps of $\Omega^I$-regular maps of $n$-dimensional closed manifolds into $P$ in terms of certain stable homotopy groups.

Let $P$ be a connected (resp. an oriented) smooth manifold of dimension $p$. We define the notion of (resp. oriented) $\Omega^I$-cobordisms of $\Omega^J$-regular maps. Let $f_i : N_i \to P$ ($i = 0, 1$) be two $\Omega^J$-regular maps, where $N_i$ are closed (resp. oriented) smooth $n$-dimensional manifolds. We say that they are (resp. oriented) $\Omega^I$-cobordant when there exists an $\Omega^I$-regular map, say $\Omega^I$-cobordism $F : (W, \partial W) \to (P \times [0, 1], P \times 0 \cup P \times 1)$ such that, for a sufficiently small positive number $\varepsilon$,

(i) $W$ is a (resp. an oriented) smooth manifold of dimension $n + 1$ with $\partial W = N_0 \cup (-N_1)$ and the collar of $\partial W$ is identified with $N_0 \times [0, \varepsilon] \cup N_1 \times [1 - \varepsilon, 1],$

(ii) $F|N_0 \times [0, \varepsilon] = f_0 \times id_{[0, \varepsilon]}$ and $F|N_1 \times [1 - \varepsilon, 1] = f_1 \times id_{[1 - \varepsilon, 1]}.$

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Let $\mathfrak{MCob}_{n,P}^{(\Omega^I, \Omega^I_{n+1})}$ (resp. $\mathfrak{DCob}_{n,P}^{(\Omega^I, \Omega^I_{n+1})}$) denote the set of all (resp. oriented) $\Omega^I$-cobordism classes of $\Omega^I$-regular maps into $P$. We will provide them the structures of modules in Section 4.

We need some notion to represent them by using stable homotopy groups. Let $E \to X$ and $F \to Y$ be smooth vector bundles of dimensions $n$ and $p$ over smooth manifolds, and let $\pi_X$ and $\pi_Y$ be the projections of $X \times Y$ onto $X$ and $Y$ respectively. Define the vector bundle $J^k(E, F)$ over $X \times Y$ by

$$J^k(E, F) = \bigoplus_{i=1}^k \text{Hom}(S^i(\pi_X^*E), \pi_Y^*F).$$

Here, $S^i(E)$ is the vector bundle $\cup_{x \in X} S^i(E_x)$ over $X$, where $S^i(E_x)$ denotes the $i$-fold symmetric product of $E_x$. The canonical fiber $\bigoplus_{i=1}^k \text{Hom}(S^i(\mathbb{R}^n), \mathbb{R}^p)$ is canonically identified with $J^k(n, p)$. If we provide $N$ and $P$ with Riemannian metrics, then $J^k(TN, TP)$ is identified with $J^k(N, P)$ over $N \times P$ (see Section 2). Let $\Omega^I(E, F)$ denote the open subbundle of $J^k(E, F)$ associated to $\Omega^I(n, p)$.

Let $G_m$ refer to the grassmann manifold $G_{m,\ell}$ (resp. oriented grassmann manifold $G_{m,\ell}^+$) of all (resp. oriented) $m$-subspaces of $\mathbb{R}^{m+\ell}$. Let $\gamma^m_{G_m}$ and $\gamma^\ell_{G_m}$ denote the canonical vector bundles of dimensions $m$ and $\ell$ over the space $G_m$, respectively such that $\gamma^m_{G_m} = \gamma^\ell_{G_m}$ is the trivial bundle $\mathbb{R}^{m+\ell}$. Let $T(\gamma^m_{G_m})$ denote the Thom spaces of $\gamma^m_{G_m}$. Let $i^G : G_n \to G_{n+1}$ denote the injection mapping an $n$-plane $a$ to the $(n+1)$-plane generated by $a$ and the $(n+\ell+1)$-th unit vector $e_{n+\ell+1}$ in $\mathbb{R}^{n+\ell+1}$.

Let $i_{+1}^{(\Omega^I, \Omega^I_{n+1})} : \Omega^I(n, p) \to \Omega^I(n+1, p+1)$ denote the map which sends a $k$-jet $z = j^k f$ to $j^k f \cdot \text{id}_p$, where $f : \mathbb{R}^n \to \mathbb{R}^p$ is a map germ and $\text{id}_p$ is the identity of $\mathbb{R}$. Set $\Omega^I_n = \Omega^I(\gamma^m_{G_m}, TP)$, $\Omega^I_{n+1} = \Omega^I(\gamma^m_{G_{n+1}}, TP \oplus \varepsilon^I_P)$, $\gamma^I_{n+1} = (\pi_{G_{n+1}}^k)^* (\gamma^I_n) | \Omega^I_n$ and $\gamma^I_{n+1} = (\pi_{G_{n+1}}^k)^* (\gamma^I_n) | \Omega^I_{n+1}$, where $\varepsilon^I_P = P \times \mathbb{R}$. There exists a fiberwise map $j_{+1}^{(\Omega^I, \Omega^I_{n+1})} : \Omega^I_n \to \Omega^I_{n+1}$ associated to $i_{+1}^{(\Omega^I, \Omega^I_{n+1})}$ covering $i^G \times \text{id}_P$. Then $j_{+1}^{(\Omega^I, \Omega^I_{n+1})}$ induces the bundle map

$$b(\gamma)^{(\Omega^I, \Omega^I_{n+1})} : \gamma^I_n \to \gamma^I_{n+1}$$

covering $j_{+1}^{(\Omega^I, \Omega^I_{n+1})}$ and the Thom map $T(b(\gamma)^{(\Omega^I, \Omega^I_{n+1})})$. We denote the image of

$$T(\gamma^I_n) = (\pi_{G_{n+1}}^k)^* (\gamma^I_n) | \Omega^I_n$$

by $\text{Im}^\mathfrak{M} \left( T(b(\gamma)^{(\Omega^I, \Omega^I_{n+1})}) \right)$ (resp. $\text{Im}^\mathfrak{D} \left( T(b(\gamma)^{(\Omega^I, \Omega^I_{n+1})}) \right)$) in the nonoriented (resp. oriented) case. We are ready to state the main result of this paper. The following theorem will be proved by applying the homotopy principles in [15], [16] and [11].

**Theorem 1.1.** Let $n$ and $p$ be natural numbers with $p > 1$ and $\ell \gg n, p$. Let $P$ be a connected $p$-dimensional manifold and be oriented in the oriented case. Let $I$ and $J$ be Thom-Boardman symbols with $I \leq J$ such that if $n \geq p$, then $I \geq (n-p+1, 0)$. Then there exist the isomorphisms

$$\omega^I_{\Omega^I_n} : \mathfrak{MCob}_{n,P}^{(\Omega^I, \Omega^I_{n+1})} \to \text{Im}^\mathfrak{M} \left( T(b(\gamma)^{(\Omega^I, \Omega^I_{n+1})}) \right),$$

$$\omega^\Omega_{\Omega^I_n} : \mathfrak{DCob}_{n,P}^{(\Omega^I, \Omega^I_{n+1})} \to \text{Im}^\mathfrak{D} \left( T(b(\gamma)^{(\Omega^I, \Omega^I_{n+1})}) \right).$$
As for the image of $T(b(\gamma^1(\Omega_{n,P}^{1,0}))))$, we will prove the following theorem.

**Theorem 1.2.** If either (i) $n < p$ or (ii) $n = p \geq 1$ and $I = (1,0)$, then the homomorphism $T(b(\gamma^1(\Omega_{n,P}^{1,0}))))$ in (1.2) is surjective.

These theorem show the importance of the homotopy type of $\Omega^I(n,p)$. In [5], [7], [8] and [9] we have determined the homotopy type of $\Omega^{I,0}(n,p)$ for $i = \max\{n-p+1,1\}$, and studied $\mathcal{C}ob_{n,P}(\Omega^{I,0}_n,\Omega^{I,0}_{n+1})$ and $\mathcal{C}ob_{n,P}(\Omega^{I,0}_n,\Omega^{I,0}_{n+1})$ in the case $n \geq p$.

When $n = p \geq 2$, $I = J = (1,0)$ and $P$ is closed, we will prove in Section 6 that $\pi_{n+\ell}(\hat{\gamma}_{\Omega_{n+1}^{I,0}})$ is, in the oriented case, isomorphic to $\pi_{n+\ell}(T(\nu^I))$ by using the results in [6]. Let $F_m$ denote the space of all base point preserving maps of the $m$-sphere $S^m$ and let $F = \lim_{m \to \infty} F_m$. By using $S$-dual spaces and duality maps in the suspension category in [35] and [39], we can prove that $\pi_{n+\ell}(T(\nu^I))$ is isomorphic to the set of homotopy classes $[P,F]$. Consequently, we have the following theorem.

**Theorem 1.3.** If $n = p \geq 2$ and $P$ is a closed, connected, oriented and $n$-dimensional manifold, then there exists the isomorphism $\mathcal{C}ob_{n,P}(\Omega^{1,0}_n,\Omega^{1,0}_{n+1}) \to [P,F]$.

We have constructed the surjection of $\mathcal{C}ob_{n,P}(\Omega^{1,0}_n,\Omega^{1,0}_{n+1})$ onto $[P,F]$ from a different point of view ([7, Corollary 2]). This surjection turns out to be a bijection. Namely, $F$ is the classifying space of this cobordism group. Theorem 1.3 suggests that the stable homotopy groups representing these kind of cobordism groups yield many invariants related to the singularities of $\Omega^I$-regular maps.

In [35] the group of the cobordism classes of smooth maps of $n$-dimensional manifolds into $P$ having only a given class of $C^\infty$ simple singularities has been represented by the homotopy classes of $P$ to a certain space in the case $n < p$. The approach leading to their results is quite different from ours using homotopy principles in this paper. This should be compared with our Corollary 5.9.

We will actually work in a more general situation than above. We will generalize the definition of the maps in Theorem 1.1 in Section 3, and will prove the generalized forms of Theorems 1.1 and 1.2 in Sections 4 and 5 respectively (see Theorem 5.6). Consequently we can apply these theorems to the groups of cobordism classes of smooth maps having only singularities of certain $C^\infty$ simple types by using [10] and [17]. In Section 6 we will prove Theorem 1.3. We will argue by applying Theorem 1.3 that stable maps of spheres are detected by higher singularities through $\mathcal{C}ob_{n,S^n}(\Omega^{1,0}_n,\Omega^{1,0}_{n+1})$ in low dimensions.

2. Preliminaries

Throughout the paper all manifolds are smooth of class $C^\infty$. Maps are basically continuous, but may be smooth (of class $C^\infty$) if necessary. Given a fiber bundle $\pi^E : E \to X$ and a subset $C$ in $X$, we denote $\pi^{-1}(C)$ by $E|_C$. Let $\pi^F : F \to Y$ be another fiber bundle. A map $\tilde{b} : X \to F$ is called a fiber map over a map $b : X \to Y$ if $\pi^F \circ \tilde{b} = b \circ \pi^E$ holds. The restriction $\tilde{b}|(E|_C) : E|_C \to F$ (or $F|_b(C)$) is denoted by $\tilde{b}|_C$. In particular, for a point $x \in X$, $E|_x$ and $\tilde{b}|_x$ are simply denoted by $E_x$ and $\tilde{b}_x : E_x \to F_{\tilde{b}(x)}$, respectively. When $E$ and $F$ are vector bundles, a fiberwise homomorphism, epimorphism and monomorphism $E \to F$ are simply
called homomorphism, epimorphism and monomorphism respectively. The trivial bundle $X \times \mathbb{R}^k$ is denoted by $\varepsilon_X$.

Let $E \to X$ (resp. $F \to Y$) be $n$-dimensional (resp. $p$-dimensional) vector bundle. Let us recall the $k$-jet bundle $J^k(E, F)$ with fiber $J^k(n, p)$ in Introduction, where $k$ may be $\infty$. The origin of $\mathbb{R}^m$ is simply denoted by $0$. Let $L^k(m)$ denote the group of all $k$-jets of local diffeomorphisms of $(\mathbb{R}^m, 0)$. We define the action of $L^k(p) \times L^k(n)$ on $J^k(n, p)$ by $(j^k_0 h_1, j^k_0 h_2) \cdot j^k_0 f = j^k_0 (h_1 \circ f \circ h_2^{-1})$. In particular, $O(p) \times O(n)$ acts on $J^k(n, p)$. Let $\Omega(n, p)$ be an open subset of $J^k(n, p)$, which is invariant with respect to the action of $L^k(p) \times L^k(n)$. Let $\Omega(E, F)$ be the open subbundle of $J^k(E, F)$ associated to $\Omega(n, p)$. Let $E_1 \to X_1$ and $F_1 \to Y_1$ be other $n$-dimensional vector bundles, and let $b_1 : E \to E_1$ and $b_2 : F \to F_1$ be bundle maps covering $b_1 : X \to X_1$ and $b_2 : Y \to Y_1$ respectively. Then $b_1$ and $b_2$ yield the isomorphisms $S^i(E_x) \to S^i(E_1, b_1(x))$ and $S^i(F_y) \to S^i(F_1, b_2(y))$ for any $x \in X$ and $y \in Y$ for $1 \leq i \leq k$ and hence, we have the bundle map

$$(2.1) \quad j : (b_1, b_2) : J^k(E, F) \to J^k(E_1, F_1)$$

covering $b_1 \times b_2$. Then $j(b_1, b_2)$ induces the bundle map $j : (b_1, b_2)_\Omega : \Omega(E, F) \to \Omega(E_1, F_1)$.

If we provide $N$ and $P$ with Riemannian metrics, then the Levi-Civita connections induce the exponential maps $\exp_{N,x} : T_x N \to N$ and $\exp_{P,y} : T_y P \to P$. In dealing with the exponential maps we always consider the convex neighborhoods (28). We define the smooth bundle map

$$(2.2) \quad J^k(N, P) \to J^k(TN, TP) \quad \text{over } N \times P$$

by sending $z = j^k_x f \in (\pi_N^k \times \pi_P^k)^{-1}(x, y)$ to the $k$-jet of $(\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}$ at $0 \in T_x N$, which is regarded as an element of $J^k(T_x N, T_y P) = (J^k_x(TN, TP))$ (see Proposition 8.1 for the smoothness of exponential maps). More strictly, (2.2) gives a smooth equivalence of the fiber bundles under the structure group $L^k(p) \times L^k(n)$. Namely, it gives a smooth reduction of the structure group $L^k(p) \times L^k(n)$ of $J^k(N, P)$ to $O(p) \times O(n)$, which is the structure group of $J^k(TN, TP)$. Let us recall Boardman submanifolds (see 13, 28). The Boardman submanifold $\Sigma^k(N, P)$ of $J^k(N, P)$ is identified with $\Omega^{k}(TN, TP)$ under (2.2). The same is true for $\Omega^{k}(N, P)$ and $\Omega^{k}(TN, TP)$.

3. MAPS $\omega_{\Omega}^{(\Omega_{n}^{k}, \Omega_{n+1}^{k})}$ AND $\omega_{\Omega}^{(\Omega_{n}^{k}, \Omega_{n+1}^{k})}$

Let $i_{k+1} : J^k(n, p) \to J^k(n+1, p+1)$ be the map defined by $i_{k+1}(j^k_0 f) = j^k_0 (f \times id_\partial)$. Let $\Omega(n, p)$ be given as in Section 2. Let $\Omega^*(n+1, p+1)$ be an open subset of $J^k(n+1, p+1)$ which is invariant with respect to the action of $L^k(p+1) \times L^k(n+1)$ such that $i_{k+1}(\Omega(n, p)) \subset \Omega^*(n+1, p+1)$. Let $\Omega = (\gamma^n_{G_n}, TP)$ and $\Omega^* = (\gamma^n_{G_n} + \varepsilon^1_{G_n}, TP + \varepsilon^1_P)$ be the open subbundles of $J^k(\gamma^n_{G_n}, TP)$ and $J^k(\gamma^n_{G_n} + \varepsilon^1_{G_n}, TP + \varepsilon^1_P)$ associated to $\Omega(n, p)$ and $\Omega^*(n+1, p+1)$ respectively. Then we have the fiber map $i_{k+1}^* : \Omega = (\gamma^n_{G_n}, TP) \to \Omega^* = (\gamma^n_{G_n} + \varepsilon^1_{G_n}, TP + \varepsilon^1_P)$ over $G_n \times P$ associated to $i_{k+1}(\Omega(n, p)) : \Omega(n, p) \to \Omega^*(n+1, p+1)$. Let $c_{\gamma^n_{G_n} + \varepsilon^n_{G_n}} : \gamma^n_{G_n} + \varepsilon^n_{G_n} \to \gamma^{n+1}_{G_n+1}$ be the bundle map, which is induced from, and is covered by $i^G : G_n \to G_{n+1}$. Let us define

$$(\Omega^i_{\Omega}) : \Omega = (\gamma^n_{G_n}, TP) \to \Omega^* = (\gamma^{n+1}_{G_{n+1}}, TP + \varepsilon^1_P)$$
to be the composite of $i^{(1,1)}$ and

$$j(c_{\gamma^n_G} \oplus e_{G_n}, \mathbb{id}_{TP} \oplus e_p): \Omega^r(\gamma^n_G \oplus e_{G_n}, TP \oplus e_p) \to \Omega^r(\gamma^n_{G+1} \oplus e_{G_n} \oplus e_p).$$

We set $\Omega = \Omega(\gamma^n_G, TP)$, $\Omega^* = \Omega^r(\gamma^n_{G+1}, TP \oplus e_p)$, $\gamma'^{G} = (\pi^n_G)^*(\gamma'^{G})|_{\Omega}$ and $\gamma'^{G} = (\pi^n_{G+1})^*(\gamma'^{G})|_{\Omega}$ for simplicity. Since $(i^G)^*(\gamma^n_{G+1}) = \gamma^n_G \oplus e_{G_n}$ and $(i^G)^*(\gamma'^{G}) = \gamma'^{G}$, we have the bundle maps

$$b(\gamma \oplus e^1): (\pi^n_G)^*(\gamma^n_G \oplus e_{G_n})|_{\Omega} \to (\pi^n_{G+1})^*(\gamma^n_{G+1})|_{\Omega},$$

$$b(\gamma): \gamma'^{G} \to \gamma'^{G},$$

and the Thom map $T(b(\gamma)(\Omega^r))$. Thus we have the homomorphism

$$T(b(\gamma)(\Omega^r))_*: \pi_{n+\ell}(T(\gamma'^{G})) \to \pi_{n+\ell}(T(\gamma'^{G})).$$

We denote the image of $T(b(\gamma)(\Omega^r))_*$ by $\text{Im}^n(T(b(\gamma)(\Omega^r)))$ in the case of $G_m = G_{m,\ell}$ or by $\text{Im}^n(T(b(\gamma)(\Omega^r)))$ in the case of $G_m = G_{m,\ell}$ and $P$ being oriented respectively.

**Definition 3.1.** We define $\mathcal{NCob}^{(n,n')}_{n,P}$ (resp. $\mathcal{NCob}^{(n,n')}_{n,P}$) to be the set of all (resp. oriented) $\Omega^r$-cobordism classes of $\Omega$-regular maps of (resp. oriented) $n$-dimensional manifolds into a connected (resp. oriented) manifold $P$ by following the definition of $\mathcal{NCob}^{(n,n')}_{n,P}$ (resp. $\mathcal{NCob}^{(n,n')}_{n,P}$) in Introduction and by replacing $\Omega^l_n$ and $\Omega^l_{n+1}$ by $\Omega$ and $\Omega^r_{n+1}$ respectively.

Let $\mathcal{NCob}^{(n,n')}$ refer to $\mathcal{NCob}^{(n,n')}_{n,P}$ or $\mathcal{NCob}^{(n,n')}$ and let $\text{Im}(T(b(\gamma)(\Omega^r)))$ refer to $\text{Im}^n(T(b(\gamma)(\Omega^r)))$ or $\text{Im}^n(T(b(\gamma)(\Omega^r)))$ for simplicity, depending on whether we work in the nonoriented case or oriented case respectively.

Let $M$ be an $m$-dimensional compact manifold such that $M$ should be oriented in the oriented case. Take an embedding $e_M: M \to S^{m+\ell}$ and identify $M$ with $e_M(M)$. Let $c_M: M \to G_m$ be the classifying map defined by sending a point $x \in M$ to the $m$-plane $T_xM \in G_m$. Let $\nu^M_M$ be the orthogonal normal bundle of $M$ in $S^{m+\ell}$. Let $c^M_M: TM \to \gamma^m_m$ (resp. $c^M_M: \nu^M_M \to \gamma^m_m$) be the bundle map covering the classifying map $c_M: M \to G_m$, which is defined by sending a vector $v$ of $T_xM$ (resp. $w \in \nu^M_M$) to $(T_xM, v)$ (resp. $(T_xM, w)$). Then we have the canonical trivializations $t^M_M: TM \oplus \nu^M_M \to \varepsilon^m_M$ and $t^M_M: \gamma^m_m \oplus \varepsilon^m_M$ with $t^M_M = (\pi^m_m)^* \oplus s_M$ is homotopic (resp. equal) to $c_M$, then $c_M$ and $\nu^M_M$ and the projection $\pi^m_M: \Omega(\gamma^m_m, TP) \to G_m$ induce the bundle maps $C^M_M: TM \to (\pi^m_M)^* \varepsilon^m_M|_{\Omega}$ and $C^M_M: \nu^M_M \to (\pi^m_M)^* \varepsilon^m_M|_{\Omega}$ covering $s_M$ such that $t^M_M \circ (C^M_M \oplus C^M_M) \circ t^M_M = c_M \times s_M$. Furthermore, if there is a map $\nu^M_M: \Omega(\gamma^m_m, TP) \to G_m$ covering $s_M$ such that $t^M_M \circ (C^M_M \oplus C^M_M) \circ t^M_M = c_M \times s_M$, then $\nu^M_M$ is the trivialization induced from $t^M_M$.

Take an embedding $e_N: N \to S^{n+\ell}$ and apply the above notation. Then we have the bundle map

$$j(c_{TN}, \mathbb{id}_{TP}): \Omega(TN, TP) \to \Omega(\gamma^n_G, TP).$$
Let $f : N \to P$ be an $\Omega$-regular map and $j^k f : N \to \Omega(TN, TP)$ be the jet extension of $f$. Then we have the composites
\[ j^{(\Omega, \Omega^*)} \circ j(c_{TN}, id_{TP})_\Omega \circ j^k f : N \to \Omega^* = \Omega^*(\gamma_{G_{n+1}}, TP \oplus \varepsilon^1_P). \]
and
\[ b(\tilde{\gamma})(\Omega, \Omega^*) \circ C_{\nu_N} : \nu_N^f \to \tilde{\gamma}^f. \]
covering $j^{(\Omega, \Omega^*)} \circ j(c_{TX}, id_{TP})_\Omega \circ j^k f$. Let $a_N : S^{n+\ell} \to T(\nu_N^f)$ be the Pontrjagin-Thom construction. We now define the map
\[ \omega : \mathcal{C}ob^b((\Omega, \Omega^*)) \to \text{Im}(T(b(\tilde{\gamma})(\Omega, \Omega^*))) \]
by mapping the cobordism class $[f]$ to the homotopy class of $T(b(\tilde{\gamma})(\Omega, \Omega^*)) \circ T(C_{\nu_N}) \circ a_N$. Here, $\omega$ refers to $\omega^b_{\Omega}((\Omega, \Omega^*))$ or $\omega^b_{\Omega}((\Omega, \Omega^*))$ depending on whether we work in the nonoriented case or oriented case. In the rest of the paper we often deal with the nonoriented case and oriented case at the same time. We have to prove that $\omega([f])$ does not depend on the choice of a representative $f$.

**Lemma 3.2.** Let two $\Omega$-regular maps $f_i : N_i \to P$ ($i = 0, 1$) are $\Omega^*$-cobordant. Then we have $\omega^b_{\Omega}((\Omega, \Omega^*))([f_0]) = \omega^b_{\Omega}((\Omega, \Omega^*))([f_1])$. If $N_i$, $P$ are oriented and $f_i$ ($i = 0, 1$) are oriented $\Omega^*$-cobordant, then we have $\omega^b_{\Omega}((\Omega, \Omega^*))([f_0]) = \omega^b_{\Omega}((\Omega, \Omega^*))([f_1])$.

**Proof.** Let $F : (W, \partial W) \to (P \times [0, 1], P \times 0 \cup P \times 1)$ be an $\Omega^*$-cobordism of $f_0$ and $f_1$ as given in Definition 3.1. Take embeddings $e_{N_i} : N_i \to S^{n+\ell}$ and $e_W : W \to S^{n+\ell} \times [0, 1]$. Let us identify $TW|_{N_i} = TN_i \oplus \varepsilon^1_{N_i}$. Then we may assume that the trivializations $t_{N_i} : TN_i \oplus \nu_N^f \to \varepsilon^1_{N_i}$ and $t_W : TW \oplus \nu_W^f \to \varepsilon^1_{W}$ satisfy $t_W|_{N_i} = (t_{N_i} \oplus id_{\varepsilon^1_{N_i}}) \circ (id_{TN_i} \oplus k^T_{N_i})$, where $k^T_{N_i} : \varepsilon^1_{N_i} \oplus \nu_N^f \to \nu_N^f \oplus \varepsilon^1_{N_i}$ is the map exchanging the components of $\varepsilon^1_{N_i}$ and $\nu_N^f$ and $\varepsilon^1_{N_i} \oplus \nu_N^f = \varepsilon^1_{N_i} \oplus \varepsilon^1_{N_i} = \varepsilon^1_{N_i} \oplus \varepsilon^1_{N_i}$. Therefore, we have that
\[ c^T_{\nu_N^f} \circ (c_{TN_i} \circ ((\iota^T \circ c_{N_i}) \times id_\mathbb{R})) = c_{TW}|N_i \oplus \varepsilon^1_{N_i}, \]
\[ C_{\nu_W^f}|_{N_i} = b(\tilde{\gamma})(\Omega, \Omega^*) \circ C_{\nu_N^f}. \]
Let $a_W : S^{n+k+1} \times [0, 1] \to T(\nu_W^f)$ be the Pontrjagin-Thom construction for $e_W$. Then the composite $T(C_{\nu_W^f}) \circ a_W$ gives a homotopy between $T(b(\tilde{\gamma})(\Omega, \Omega^*)) \circ T(C_{\nu_{N_0}}) \circ a_{N_0}$ and $T(b(\tilde{\gamma})(\Omega, \Omega^*)) \circ T(C_{\nu_{N_1}}) \circ a_{N_1}$. This proves $\omega([f_0]) = \omega([f_1])$. In the oriented case we only need to provide manifolds, which appear in the proof, with the orientations.

4. $\omega^b_{\Omega}((\Omega, \Omega^*))$ and $\omega^b_{\Omega}((\Omega, \Omega^*))$ are isomorphisms

Let $O(m, q)$ be an open subset of $J^k(m, q)$, which is invariant with respect to the action $L^k(q) \times L^k(m)$. Given an $m$-dimensional manifold $M$ with $\partial M$ and an $q$-dimensional manifold $Q$, we can define the open subbundle $O(M, Q)$ in $J^k(M, Q)$. Let $C^\infty(M, Q)$ denote the space consisting of all $O$-regular maps equipped with the $C^\infty$-topology. Let $\Gamma_O(M, Q)$ denote the space consisting of all continuous sections
of the fiber bundle $\pi_{M}^{Q} \colon O(M, Q) : O(M, Q) \rightarrow M$ equipped with the compact-open topology. Then there exists the continuous map

$$j_{O} : C^{\infty}_{O}(M, Q) \rightarrow \Gamma_{O}(M, Q)$$

defined by $j_{O}(f) = j^{k}f$.

**Definition 4.1.** In this paper we say that $O(m, q)$ satisfies the relative homotopy principle in the existence level if the following property holds. Let $M$ and $Q$ be any manifolds as above. Let $C$ be a closed subset of $M$ such that if $\partial M \neq \emptyset$, then $\partial M \subset C$. Let $s$ be a section of $\Gamma_{O}(M, Q)$ which has an $O$-regular map $g$ defined on a neighborhood of $C$ into $Q$. Then there exists an $O$-regular map $f : M \rightarrow Q$ such that $f^{k}f$ is homotopic to $s$ relative to a neighborhood of $C$ by a homotopy $s_{\lambda}$ in $\Gamma_{O}(M, Q)$ with $s_{0} = s$ and $s_{1} = j^{k}f$.

Let $\Omega(n, p)$ and $\Omega^{*}(n + 1, p + 1)$ be an open subset of $J^{k}(n, p)$ and $J^{k}(n + 1, p + 1)$ respectively. We say that the pair $(\Omega(n, p), \Omega^{*}(n + 1, p + 1))$ is admissible to the $h$-Principle if the following properties are satisfied:

(i) $\Omega(n, p)$ (resp. $\Omega^{*}(n + 1, p + 1)$) is invariant with respect to the action of $L^{k}(p) \times L^{k}(n)$ (resp. $L^{k}(p + 1) \times L^{k}(n + 1)$).

(ii) $i_{i+1}(\Omega(n, p)) \subset \Omega^{*}(n + 1, p + 1)$.

(iii) $\Omega(n, p)$ and $\Omega^{*}(n + 1, p + 1)$ satisfy the relative homotopy principle in the existence level in Definition 4.1 respectively.

**Theorem 4.2.** Let $P$ be a connected $p$-dimensional manifold and be oriented in the oriented case. Assume that the pair $(\Omega(n, p), \Omega^{*}(n + 1, p + 1))$ is admissible to the $h$-Principle. Then the maps

$$\omega^{(\Omega, \Omega^{*})}_{n,p} : \Omega \text{Cob}^{(\Omega, \Omega^{*})}_{n,p} \rightarrow \text{Im} \left\{ T(\mathbf{b}(\gamma)^{(\Omega, \Omega^{*})}) \right\}$$

$$\omega^{(\Omega, \Omega^{*})}_{\Omega} : \text{Cob}^{(\Omega, \Omega^{*})}_{n,p} \rightarrow \text{Im} \left\{ T(\mathbf{b}(\gamma)^{(\Omega, \Omega^{*})}) \right\}$$

are bijective.

**Proof.** In the oriented case we only need to provide manifolds which appear in the proof with the orientations.

We first prove that $\omega$ is injective. For this, take two $\Omega$-regular maps $f_{i} : N_{i} \rightarrow P$ ($i = 0, 1$) such that $\omega([f_{0}]) = \omega([f_{1}])$. Recall the map $T(\mathbf{b}(\gamma)^{(\Omega, \Omega^{*})}) \circ T(\mathbf{c}_{\nu_{N_{0}}}) \circ a_{N_{0}}$, which represents $\omega([f_{i}])$. Let $pr_{X} : X \times [0, 1] \rightarrow X$ be the canonical projection for a space $X$. There is a homotopy $H : S^{n+\ell} \times [0, 1] \rightarrow T(\hat{\gamma}_{\Omega^{*}})$ such that if $\varepsilon$ is sufficiently small, then

(i) $H[S^{n+\ell} \times [0, \varepsilon]] = T(\mathbf{b}(\gamma)^{(\Omega, \Omega^{*})}) \circ T(\mathbf{c}_{\nu_{N_{0}}}) \circ a_{N_{0}} \circ (pr_{S^{n+\ell}} \times [0, \varepsilon]),$ 

(ii) $H[S^{n+\ell} \times [1-\varepsilon, 1]] = T(\mathbf{b}(\gamma)^{(\Omega^{*}, \Omega^{*})}) \circ T(\mathbf{c}_{\nu_{N_{1}}}) \circ a_{N_{1}} \circ (pr_{S^{n+\ell}} \times [1-\varepsilon, 1]),$ 

(iii) $H$ is smooth around $H^{-1}(\Omega^{*})$ and is transverse to $\Omega^{*}$.

We set $W = H^{-1}(\Omega^{*})$. Then we have

(iv) $W \cap S^{n+\ell} \times [0, \varepsilon] = N_{0} \times [0, \varepsilon]$ and $H|N_{0} \times [0, \varepsilon] = j^{(\Omega, \Omega^{*})} \circ j(\mathbf{c}_{\nu_{N_{0}}, \text{id}_{TP}}) \circ j^{k}f_{0} \circ (pr_{N_{0}})[N_{0} \times [0, \varepsilon]),$

(v) $W \cap S^{n+\ell} \times [1-\varepsilon, 1] = N_{1} \times [1-\varepsilon, 1]$ and $H|N_{1} \times [1-\varepsilon, 1] = j^{(\Omega^{*}, \Omega^{*})} \circ j(\mathbf{c}_{\nu_{N_{1}}, \text{id}_{TP}}) \circ j^{k}f_{1} \circ (pr_{N_{1}})[N_{1} \times [1-\varepsilon, 1]),$

(vi) $TW|N_{0} \times [0, \varepsilon] = (T\nu_{0} \oplus \varepsilon N_{0}) \times [0, \varepsilon]$ and $TW|N_{1} \times [1-\varepsilon, 1] = (T\nu_{1} \oplus \varepsilon N_{1}) \times [1-\varepsilon, 1],$

(vii) $\nu_{W}|N_{0} \times [0, \varepsilon] = \nu_{N_{0}} \times [0, \varepsilon]$ and $\nu_{W}|N_{1} \times [1-\varepsilon, 1] = \nu_{N_{1}} \times [1-\varepsilon, 1].$
By (iii) we have the bundle map \( C'_{\nu W} : \nu W \to T_{\Omega}^{C} \) such that
\[
\begin{align*}
(\text{vii}) \quad C'_{\nu W} |_{N_0 \times [0, \varepsilon]} &= C'_{\nu N_0} \circ (pr_{\nu N_0} |_{N_0} \times [0, \varepsilon]) \text{ by (i) and (ii),} \\
(\text{ix}) \quad C'_{\nu W} |_{N_1 \times [1 - \varepsilon, 1]} &= C'_{\nu N_1} \circ (pr_{\nu N_1} |_{N_1} \times [1 - \varepsilon, 1]) \text{ by (ii) and (vii).}
\end{align*}
\]
It follows from [3, Proposition 3.3] that there exists a bundle map
\[
C'_{TW} : TW \to (\pi^k_{G_{n+1}})^{*}(\gamma^{n+1}_{G_{n+1}})|_{\Omega},
\]
covering \( H|W : W \to \Omega^{*} \) such that \( t_{\Omega} \circ (C'_{TW} \oplus C'_{\nu W}) \circ t_{W}^{-1} \) is homotopic to \( (H|W) \times id_{\Omega^{n+1}} \). Since \( \gamma^{n+1}_{G_{n+1}} \) is the universal bundle \( \ell \gg n \), \( C'_{TW} \) is homotopic to \( C_{TW} \). Furthermore, we may assume by (iv), (v) and (vi) that
\[
\begin{align*}
(\text{x}) \quad C'_{\nu W} |_{N_0 \times t} &= b(\gamma \oplus \varepsilon^{1})^{(\Omega, \Omega^{*})} \circ (C_{TW}(N_0 \times t) \oplus (c_{\Omega_{N_0 \times t}} \times id_{k})) \text{ for } 0 \leq t \leq \varepsilon, \\
(\text{xii}) \quad C'_{\nu W} |_{N_1 \times t} &= b(\gamma \oplus \varepsilon^{1})^{(\Omega, \Omega^{*})} \circ (C_{TW}(N_1 \times t) \oplus (c_{\Omega_{N_1 \times t}} \times id_{k})) \text{ for } 1 - \varepsilon \leq t \leq 1.
\end{align*}
\]
Hence, \( c_{\nu W} \) is homotopic to \( \pi^k_{G_{n+1}} \circ H|W \) relative to \( N_0 \times [0, \varepsilon] \cup N_1 \times [1 - \varepsilon, 1] \).

Let \( pr_{TW_{\hat{\nu}^{\gamma^{1}_{g_n}}}} : T(P \times [0, 1]) \to TP \oplus \nu_{P}^{1} \) be the canonical bundle map covering \( pr_{P} : P \times [0, 1] \to P \). Then we have the bundle map
\[
j(c_{TW}, pr_{TW_{\hat{\nu}^{\gamma^{1}_{g_n}}}}_{\nu W}) : \Omega^{*}(TW, T(P \times [0, 1])) \to \Omega^{*}(\gamma^{n+1}_{G_{n+1}}, TP \oplus \nu_{P}^{1})
\]
covering \( c_{W} \times pr_{P} \). Therefore, since \([0, 1]\) is contractible, there is the section \( s_{W} : W \to \Omega^{*}(TW, T(P \times [0, 1])) \) such that
\[
\begin{align*}
\pi^k_{P \times [0, 1]} \circ s_{W} |_{N_0 \times [0, \varepsilon]} &= f_{0} \times id_{[0, \varepsilon]}, \\
\pi^k_{P \times [0, 1]} \circ s_{W} |_{N_1 \times [1 - \varepsilon, 1]} &= f_{1} \times id_{[1 - \varepsilon, 1]}, \\
pr_{P} \circ \pi^k_{P \times [0, 1]} \circ s_{W} &= \pi^k_{P} \circ (H|W),
\end{align*}
\]
and that \( j(c_{TW}, pr_{TW_{\hat{\nu}^{\gamma^{1}_{g_n}}}}_{\nu W}) \circ s_{W} \) is homotopic to \( H|W \) relative to \( N_0 \times [0, \varepsilon] \cup N_1 \times [1 - \varepsilon, 1] \).

Since \( \Omega^{*}(TW, T(P \times [0, 1])) \) satisfies the relative homotopy principle in the existence level, there exists an \( \Omega^{*}\)-regular map \( F : W \to P \times [0, 1] \) such that \( F(x, t) = f_{0}(x) \times t \) for \( 0 \leq t \leq \varepsilon \), \( F(x, t) = f_{1}(x) \times t \) for \( 1 - \varepsilon \leq t \leq 1 \) and that \( j^{\gamma}F \) is homotopic to \( s_{W} \) relative to \( N_0 \times [0, \varepsilon/2] \cup N_1 \times [1 - \varepsilon/2, 1] \). This implies that the \( \Omega^{*}\)-regular maps \( f_{0} \) and \( f_{1} \) are \( \Omega^{*}\)-cobordant. This proves that \( \omega \) is injective.

We next prove that \( \omega \) is surjective. Let an element \( \tilde{\alpha} \) of \( \text{Im} (T(b(\gamma^{1}_{g_n}))^{(\Omega, \Omega^{*})}) \) be represented by a map \( \alpha : S^{n+1} \to \tilde{T}_{\Omega}^{C} \) such that \( T(b(\gamma^{1}_{g_n}))^{(\Omega, \Omega^{*})}) ([\alpha]) = \tilde{\alpha} \). We may suppose that \( \alpha \) is smooth around \( \alpha^{-1}(\Omega) \) and is transverse to \( \Omega \). We set \( N = \alpha^{-1}(\Omega) \). If \( N = \emptyset \), then \( [\alpha] \) must be a null element, while we can deform \( \alpha \) so that \( N \neq \emptyset \) even in this case. Since \( \alpha \) is transverse to \( \Omega \), we have the bundle map \( C'_{\nu N} : \nu N \to \tilde{T}_{\Omega}^{C} \) covering \( \alpha|N \). It follows from [3, Proposition 3.3] that there exists a bundle map
\[
C'_{TN_{\hat{\nu}^{\gamma^{1}_{g_n}}}} : TN \oplus \varepsilon^{1}_{N} \to (\gamma^{n}_{G_{n}} \oplus \varepsilon^{1}_{G_{n}})|_{\Omega}
\]
covering \( \alpha|N : N \to \Omega \) such that the composite
\[
(t_{\Omega} \oplus id_{\varepsilon^{1}_{N}}) \circ (id_{\gamma^{n}_{G_{n}}} \oplus k_{G_{n}}) \circ (C'_{TN_{\hat{\nu}^{\gamma^{1}_{g_n}}}}) \circ (id_{TN} \oplus k_{N}^{n}) \circ (t_{N}^{-1} \oplus id_{\varepsilon^{1}_{N}})
\]
is homotopic to \( \alpha|N \times id_{\Omega^{n+1}} \). Since \( \gamma^{n}_{G_{n}} \) is the universal bundle \( \ell \gg n \), \( C'_{TN_{\hat{\nu}^{\gamma^{1}_{g_n}}}} \) is homotopic to \( C_{TN} \), \( (\alpha|N \times id_{\Omega^{n+1}}) \), and \( t_{\Omega} \circ (C_{TN} \oplus C'_{\nu N}) \circ t_{N}^{-1} \) is homotopic to \( \alpha|N \times id_{\Omega^{n+1}} \). Hence, \( c_{N} \) is homotopic to \( \pi^{k}_{G_{n}} \circ \alpha|N \). By [3, Proposition 3.3] again, \( C'_{\nu N} \) and \( C_{\nu N} \) are homotopic as bundle maps \( \nu N \to \tilde{T}_{\Omega}^{C} \). Since we
have the bundle map
\[ j(c_{TN}, id_{TP})\Omega : \Omega(TN, TP) \rightarrow \Omega(\gamma^s_{c_n}, TP) \]
covering \( c_n \), there is the section \( s_N : N \rightarrow \Omega(TN, TP) \) such that \( \pi^N_h \circ s_N = \pi^N_h \circ \alpha|N \) and \( j(c_{TN}, id_{TP})\Omega \circ s_N \) is homotopic to \( \alpha|N \). Since \( \Omega(TN, TP) \) satisfies the relative homotopy principle in the existence level, there exists an \( \Omega \)-regular map \( j \) such that \( j \circ s_N \) is homotopic to \( s_N \). This proves that \( j(c_{TN}, id_{TP})\Omega \circ j^k f \) and \( \alpha|N \) are homotopic. This proves that
\[ \omega([f]) = [T(b(\tilde{\gamma})^{(\Omega,\Omega')}) \circ T(C_{\nu_N}) \circ a_N] \]
\[ = [T(b(\tilde{\gamma})^{(\Omega,\Omega')}) \circ T(C'_{\nu_N}) \circ a_N] \]
\[ = [T(b(\tilde{\gamma})^{(\Omega,\Omega')}) \circ a] \]
\[ = \tilde{\alpha}. \]
This is what we want. \( \square \)

Let \( \mathcal{O}(m, q) \) be an open subset of \( J^k(m, q) \), which is invariant with respect to the action \( L^k(q) \times L^k(m) \). In [10] du Plessis has called \( \mathcal{O}(m, q) \) extensible when there exists an open subset \( \mathcal{O}'(m + 1, q) \) of \( J^k(m + 1, q) \), which is invariant with respect to the action \( L^k(q) \times L^k(m + 1) \), such that \( \tilde{i}(\mathcal{O}'(m + 1, q)) = \mathcal{O}(m, q) \). Here, \( \tilde{i} \) is the map which is canonically induced by the inclusion \( i^m : \mathbb{R}^m = \mathbb{R}^m \times 0 \rightarrow \mathbb{R}^{m+1} \). This extensibility yields that not only \( J_\mathcal{O} \) is a weak homotopy equivalence, but also \( \mathcal{O}(m, q) \) satisfies the relative homotopy principle in the existence level. However, the last assertion is not stated explicitly. So we explain an outline of the proof.

**Lemma 4.3.** Let \( \mathcal{O}(m, q) \) be an extensible open subset as given above. Then \( \mathcal{O}(m, q) \) satisfies the relative homotopy principle in the existence level in Definition 4.1.

**Proof.** For the \( \mathcal{O} \)-regular map \( g \) and the closed subset \( C \) in Definition 4.1, we take a closed neighborhood \( V(C) \) of \( C \) such that \( V(C) \) is an \( n \)-dimensional submanifold with boundary and that \( g \) is defined on a neighborhood of \( V(C) \), where \( J^k g = s \). Without loss of generality we may assume that \( N \setminus \text{Int}V(C) \) is nonempty. Take a smooth function \( h_C : N \rightarrow [0, 1] \) such that
\[ \begin{cases} 
    h_C(x) = 1 & \text{for } x \in C, \\
    h_C(x) = 0 & \text{for } x \in N \setminus \text{Int}V(C), \\
    0 < h_C(x) < 1 & \text{for } x \in \text{Int}V(C) \setminus C.
\end{cases} \]
(4.1)
By the Sard Theorem ([24]) there is a regular value \( r \) of \( h_C \) with \( 0 < r < 1 \). Then \( h_C^{-1}(r) \) is a submanifold and we set \( L_0 = h_C^{-1}([r, 1]) \). We may represent \( N \) as the union of an increasing finite sequence
\[ L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset L_\sigma = N \]
of compact \( n \)-manifolds with boundary such that \( L_{i+1} = L_i \cup \partial L_i \cup [0, 1]) \cup H_{i+1} \), where \( H_{i+1} \) is the \( j \)-handle \( D^j \times D^{n-j} \) with \( \partial L_i \times 1 \cap H_{i+1} = \partial D^j \times D^{n-j} \). For a sufficiently small \( \varepsilon \), we set \( L_{i+1}^\varepsilon = L_i \cup \partial L_i \cup [0, \varepsilon]) \).
Let $\rho^{i+1}_{C^\infty}$ and $\rho^{i+1}_{\Gamma^0}$ in the following diagram denote the maps, which are canonically induced from the inclusion $L^*_i \to L^*_{i+1}$.

\[
\begin{array}{ccc}
C^\infty_{\Gamma^0}(L^*_i, P) & \xrightarrow{j_o} & \Gamma_{\Omega}(L^*_i, P) \\
\rho^{i+1}_{C^\infty} & \downarrow & \downarrow \rho^{i+1}_{\Gamma^0} \\
C^\infty_{\Gamma^0}(L^*_i, P) & \xrightarrow{j_o} & \Gamma_{\Omega}(L^*_i, P).
\end{array}
\]

It has been proved in [22] and [16] that $\rho^{i+1}_{C^\infty}$ and $\rho^{i+1}_{\Gamma^0}$ are Serre fibrations. By the induction on $i \geq 0$ we construct an $\mathcal{O}$-regular map $g^{i+1} \in C^\infty(L^*_{i+1}, P)$, a homotopy $u^{i+1}_\lambda \in \Gamma_{\Omega}(L^*_{i+1}, P)$ relative to $L_0$ and a homotopy $s^{i+1}_\lambda \in \Gamma_{\Omega}(L^*_{i+1}, P)$ relative to $L_i$ such that $u^{i+1}_0 = s|L^*_{i+1}, u^{i+1}_i = j^k g^i, g^{i+1}|L_i = g^i, s^{i+1}_0 = s^{i+1}_1, s^{i+1}_i = j^k g^{i+1}$. Let $u^{i+1}_\lambda \in \Gamma_{\Omega}(L^*_{i+1}, P)$ be the homotopy defined by $v^{i+1}_\lambda = u^{i+1}_{2\lambda} (0 \leq \lambda \leq 1/2)$ and $v^{i+1}_\lambda = u^{i+1}_{2\lambda-1} (1/2 \leq \lambda \leq 1)$.

In fact, we start with $g^0 = g|L_0$ and $u^{i+1}_\lambda = s^{i+1}_\lambda = s|L_0$ for any $\lambda$, and next assume that $g^i, u^i_\lambda$ and $s^i_\lambda$ are already constructed. By applying the homotopy extension property to $s|L^*_{i+1}$ and $v^{i}_\lambda$, we have the homotopy $u^{i+1}_\lambda \in \Gamma_{\Omega}(L^*_{i+1}, P)$ relative to $L_0$ such that $u^{i+1}_0 = s|L^*_{i+1}, u^{i+1}_i = v^{i}_\lambda|L^*_i$. Let $F^i_{C^\infty}$ and $F^i_{\Gamma^0}$ denote the fibers of $\rho^{i+1}_{C^\infty}$ and $\rho^{i+1}_{\Gamma^0}$ over $g^i$ and $j^k g^i$ respectively. Since $j_o$ induces the weakly homotopy equivalence $F^i_{C^\infty} \to F^i_{\Gamma^0}$, we have an $\mathcal{O}$-regular map $g^{i+1}$ and a homotopy $s^{i+1}_\lambda$ relative to $L_i$ such that $g^{i+1}|L_i = g^i, s^{i+1}_0 = u^{i+1}_0, s^{i+1}_i = j^\infty g^{i+1}$. This is what we want.

Define $s_\lambda \in \Gamma_{\Omega}(L_\lambda, P)$ to be the homotopy $v^{i}_\lambda$. Then we have $s_0 = s$ and the required $\mathcal{O}$-regular map $f = g^{\infty}$ with $s_1 = j^k g^{\infty}$. \hfill \Box

**Proof of Theorem 1.1.** Let $I = (i_1, i_2, \ldots, i_k)$. It has been proved in [15] that if $i_k > n-p-d'$, where $d'$ is the sum of $\alpha_1, \cdots, \alpha_{k-1}$ with $\alpha_\ell$ being 1 or 0 depending on $i_\ell - i_{\ell+1} > 1$ or otherwise, then $\Omega^I(n, p)$ is extensible.

Let $n < p$. By Lemma 4.3, $\Omega^I(n, p)$ and $\Omega^I(n+p+1)$ satisfy the relative homotopy principle in the existence level (see also [24] and [21]).

In [11] Theorem 0.1 it has been proved that if $n \geq p \geq 2$ and $I \geq (n-p+1, 0)$, then $\Omega^I(n, p)$ and $\Omega^I(n+p+1)$ satisfy the relative homotopy principle in the existence level. Here we remark the following. In [11], Theorem 0.1] $V$ is assumed to be $\partial V = \emptyset$. However, this does not matter, because we only need to consider the manifold $V - \partial V$.

Therefore, the pair $(\Omega^I(n, p), \Omega^I(n+p+1))$ is admissible to the h-Principle if (i) $n < p$ or (ii) $n \geq p$ and $I \geq (n-p+1, 0)$. This proves the theorem. \hfill \Box

In [17] Section 0, Theorem 1 there have been given extensible open subsets $\Omega(n, p)$, which are associated to smooth maps having only singularities of certain $C^\infty$ simple type. In [10] we have constructed the submanifolds $\sum D_i(n, p) (i \geq 4)$ and $\sum E_i(n, p)$ in $J^k(n, p)$, which play the similar role for the singularities of types $D_i$ and $E_i$ respectively as $\sum_i (n-p+1, \cdots, 1, 0)(n, p)$ does for the singularities of type $A_i$ (see [12]). Consider a subset $\Omega(n, p)$ of $J^k(n, p)$, which consists of all regular jets and a number of prescribed submanifolds $\sum A_i(n, p), \sum D_i(n, p)$ and $\sum E_i(n, p)$. We can find when $\Omega(n, p)$ becomes open by using the adjacency relations of these singularities given in [12, Corollary 8.7]. We can apply Theorem 4.2 to these open subsets $\Omega(n, p)$ (see also Corollary 5.9).
Let us provide $\mathfrak{Ceb}^{(0,\Omega^+)}_{\mathbb{R}_{n,P}}$ with the structure of a module so that $\omega$ is an isomorphism. This is a standard argument and the details are left to the reader. Given two $\Omega$-regular maps $f_i : N_i \to P$ ($i = 0, 1$), we define the sum $[f_0] + [f_1]$ to be the cobordism class of the $\Omega$-regular maps $f : N_0 \cup N_1 \to P$ defined by $f|N_i = f_i$. The null element is defined to be represented by an $\Omega$-regular maps $f : N \to P$, which has an $\Omega^*$-cobordism $F : (W, \partial W) \to (P \times [0,1], P \times 0)$ with $\partial W = N$ such that $F|N = f$ under the identification $P \times 0 = P$.

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Let $C_m$ denote the set consisting of all smooth map germs $(\mathbb{R}^m, 0) \to \mathbb{R}$ and let $m_m$ denote the ideal in $C_m$ which consists of all smooth map germs vanishing at the origin. For a $k$-jet $z = j^k f \in J^k(n, q)$ we define the $\mathbb{R}$-algebra $Q(z) = C_m\big(/(f'(m_n) + \mathfrak{m}_m^{k+1})\big)$. If two $\mathbb{R}$-algebras $A$ and $B$ are isomorphic, then we write $A \approx B$. Let $\Omega(n, p)$ and $\Omega(n+1, p+1)$ be subsets of $J^k(n, p)$ and $J^k(n+1, p+1)$ respectively satisfying the conditions:

(C1) $\Omega(n, p)$ and $\Omega(n+1, p+1)$ are open and $i_{n+1}(\Omega(n, p)) \subset \Omega(n+1, p+1)$.

(C2) If a $k$-jet $z \in J^k(n, p)$ (resp. $z \in J^k(n+1, p+1)$) has a $k$-jet $w \in \Omega(n, p)$ (resp. $w \in \Omega(n+1, p+1)$) such that $Q(z) \approx Q(w)$, then $z \in \Omega(n, p)$ (resp. $z \in \Omega(n+1, p+1)$).

(C3) If a $k$-jet $z \in \Omega(n+1, p+1)$ does not lie in $\Sigma^{n+1}(n+1, p+1)$, then there exists a $k$-jet $w \in \Omega(n, p)$ such that $Q(z) \approx Q(w)$.

By (C2), $\Omega(n, p)$ and $\Omega(n+1, p+1)$ are invariant with respect to the actions of $L^k(p) \times L^k(n)$ and $L^k(p+1) \times L^k(n+1)$ respectively.

**Lemma 5.1.** The open subsets $\Omega^I(n, p)$ and $\Omega^I(n+1, p+1)$ for the symbol $I = (i_1, \cdots, i_k)$ satisfy the conditions (C1), (C2) and (C3).

**Proof.** The condition (C1) and (C2) are satisfied by [33, 2, Corollary(Morin)].

We consider the usual coordinates $x = (x_1, x_2, \cdots, x_{n+1})$ of $\mathbb{R}^{n+1}$ and $y = (y_1, y_2, \cdots, y_{p+1})$ of $\mathbb{R}^{p+1}$. Since the given $k$-jet $z \in \Omega(n+1, p+1)$ does not lie in $\Sigma^{n+1}(n+1, p+1)$, $z$ is represented as $z = j^k f$ with

$$(y_1^t \circ f(x^t), \cdots, y_{p+1}^t \circ f(x^t)) = (g^1(x^t), \cdots, g^{p-n+1}(x^t), x_{i_1+1}, \cdots, x_{n+1}),$$

where $g^t \in m_{n+1}^2$ under suitable coordinates $x' = (x'_1, x'_2, \cdots, x'_{n+1})$ of $\mathbb{R}^{n+1}$ and $y' = (y_1^t, y_2^t, \cdots, y_{p+1}^t)$ of $\mathbb{R}^{p+1}$. Let $V$ and $W$ be the subspaces of $\mathbb{R}^{n+1}$ and $\mathbb{R}^{p+1}$ defined by the equations $x'_{n+1} = 0$ and $y'_{p+1} = 0$, and let $i_V : V \to \mathbb{R}^{n+1}$ and $\pi_W : \mathbb{R}^{p+1} \to W$ be the inclusion and the projection defined by $\pi_W(y_1^t, y_2^t, \cdots, y_{p+1}^t) = (y_1^t, y_2^t, \cdots, y_{p+1}^t)$ respectively. Let $\hat{x} = (x_1, x_2, \cdots, x_n)$ and $\hat{x}' = (x'_1, x'_2, \cdots, x'_{n+1})$. Define the map $\overline{f} : V \to W$ by $\overline{f} = \pi_W \circ f \circ i_V$ and the functions $\overline{g}$ on $V$ by $\overline{g}(\hat{x}') = y_j^t \circ \overline{g}(\hat{x}')$ ($1 \leq j \leq p$). Replacing the coordinates $x'_j$ and $y'_j$ by $x_j$ and $y_j$ we define the germ $\overline{f} : \mathbb{R}^n \to \mathbb{R}^p$ by

$$\overline{f}(\hat{x}) = \left\{ \begin{array}{ll}
(\overline{g}^1(\hat{x}), \overline{g}^2(\hat{x}), \cdots, \overline{g}^{p-n+1}(\hat{x}), x_{i_1+1}, \cdots, x_n) & \text{for } i_1 < n, \\
(\overline{g}^1(\hat{x}), \overline{g}^2(\hat{x}), \cdots, \overline{g}^{p}(\hat{x})) & \text{for } i_1 = n.
\end{array} \right.$$
If we write \( g(x') \) for \( f(x') \) to avoid the confusion, then
\[
Q(z) = C_{n+1}/(f^*(m_{p+1}) + \mathbf{m}_{n+1}^{k+1}) \\
= C_{n+1}/(g^*(m_{p+1}) + \mathbf{m}_{n+1}^{k+1}) \\
\approx C_{n}/(\mathcal{T}(m_p) + \mathbf{m}_n^{k+1}) \\
\approx C_{n}/(\mathcal{F}(m_p) + \mathbf{m}_n^{k+1}) \\
= Q(j_0^k \mathcal{F}).
\]

By Corollary(Morin), the Thom-Boardman symbol of \( \mathcal{F} \) is equal to \( I \). This proves the assertion.

A point of \( \mathbb{R}^m \) is expressed as \( x_1 e_1 + x_2 e_2 + \cdots + x_m e_m \) or \( (x_1, x_2, \ldots, x_m) \), where \( e_1, e_2, \ldots, e_m \) are the canonical orthonormal basis. Let \( pr_{p+1} : \mathbb{R}^{p+1} \to \mathbb{R} \) be the projection mapping \( y_1, y_2, \ldots, y_{p+1} \) to \( y_{p+1} \). Let \( j(pr_{p+1}) : J^1(n+1, p+1) \to J^1(n+1, 1) \) denote the map defined by mapping a 1-jet \( j_0^1 f \) to the 1-jet \( j_0^1 (pr_{p+1} \circ f) \).

Let \( K \) be a finite simplicial complex of dimension \( i \) and \( L \) be its subcomplex of dimension less than \( i \) such that \( K \setminus L \) is a manifold.

**Lemma 5.2.** Let \( \Omega(n, p) \) and \( \Omega(n+1, p+1) \) satisfy (C1), (C2) and (C3) as above. Let \( i \leq n < p \) and \( (K, L) \) be given as above. Let \( \psi : (K, L) \to (\Omega(n+1, p+1), i+1(\Omega(n, p))) \) be a map such that \( \psi|(K \setminus L) \) is smooth. Then there exists a homotopy \( \psi_L : (K, L) \to (\Omega(n+1, p+1), i+1(\Omega(n, p))) \) such that

(i) \( \psi_0 = \psi \),
(ii) \( \psi_L | L = \psi | L \),
(iii) \( j(pr_{p+1}) \circ \pi^k_1 \circ \psi(u)(e_i) = \begin{cases} 0 & \text{for } i < n + 1, \\ e_{p+1} & \text{for } i = n + 1. \end{cases} \)

**Proof.** Let us define \( \kappa : K \to \mathbb{R}^{p+1} \) by \( \kappa(u) = (\pi^k_1 \circ \psi(u))(e_{n+1}) \). Since \( \psi(L) \subset i+1(\Omega(n, p)) \), we have that, for any \( u \in L \), \( \kappa(u) = e_{p+1} \). Since \( \dim K \leq n \), \( K \setminus L \) is a manifold and since \( \psi|(K \setminus L) \) is smooth, there exists a deformation \( \kappa_\lambda \) of \( \kappa \) relative to a neighborhood of \( L \) with \( \kappa_0 = \kappa \) such that the set of \( \lambda \)'s for which \( \kappa_\lambda \) does not take the value \( 0 \in \mathbb{R}^{p+1} \) is dense in \([0, 1] \).

Consider the fiber bundle \( q_{p+1} : J^p(n+1, p+1) \to \mathbb{R}^{p+1} \) defined by \( q_{p+1}(j_0^p f) = j_0^p f(e_{n+1}) \). By applying the covering homotopy property to \( \psi \) and \( \kappa_\lambda \), there exists a homotopy \( \varphi_\lambda \) relative to \( L \) such that \( \varphi_0 = \psi \) and \( q_{p+1} \circ \varphi_\lambda = \kappa_\lambda \). Since \( \Omega(n+1, p+1) \) is an open subset and \( K \) is compact, there exists a \( r \in [0, 1] \) such that if \( \lambda \leq r \), then \( \varphi_\lambda(K) \subset \Omega(n+1, p+1) \) and \( \kappa_\lambda \) does not take the value \( 0 \in \mathbb{R}^{p+1} \).

In the proof an element of \( GL(m) \) is regarded as a linear isomorphism of \( \mathbb{R}^m \). Let \( h_1^1 : (K, L) \to (GL(p+1), E_{p+1}) \) be the homotopy defined by \( h_1^1(u) = ((1 - \lambda) + \lambda/||\kappa_r(u)||)E_{p+1} \). Then we have \( h_1^1(u)(\kappa_r(u)) \in S^p \), and we may assume without loss of generality that \( h_1^1(u)(\kappa_r(u)) \neq e_{p+1} \) for any \( u \in K \). By considering the rotation which is the identity on all points orthogonal to both \( \kappa_r(u) \) and \( e_{p+1} \) and rotate the great circle through \( \kappa_r(u) \) and \( e_{p+1} \) so as to carry \( \kappa_r(u) \) to \( e_{p+1} \) (when \( \kappa_r(u) = e_{p+1} \), we consider \( E_{p+1} \)), we have the homotopy \( h_2^1 : (K, L) \to (SO(p+1), E_{p+1}) \) relative to \( L \) such that \( h_0^2(u) = E_{p+1} \) and \( h_2^1(u)(h_1^1(u)(\kappa_r(u)) = e_{p+1} \) for any \( u \in K \).

Let \( h_1 : (K, L) \to (GL(p+1), E_{p+1}) \) be the homotopy defined by \( h_1 = h_1^1 \) for \( 0 \leq \lambda \leq 1/2 \) and \( h_1 = h_2^1 \) for \( 1/2 \leq \lambda \leq 1 \). Define \( \kappa^J : K \to J^1(n+1, 1) \) by
\[
\kappa_\lambda^J(u) = j(pr_{p+1}) \circ \pi^k_1 \circ (h_1(u) \circ \varphi_r(u)).
\]
Since \( \mathcal{K}^J(u) \) is of rank 1 for any \( u \in K \), we have the unique vector \( v(u) \) of length 1 such that \( v(u) \) is perpendicular to \( \text{Ker}(\mathcal{K}^J(u)) \) and that \( \mathcal{K}^J(u)(v(u)) \) is directed to the same orientation of \( e_{p+1} \). Since \( h_1(u)(\kappa_r(u)) = e_{p+1}, v(u) \) cannot be equal to \(-e_{n+1}\). We set \( v(u) = \|\mathcal{K}^J(u)(v(u))\| \).

Note that \( v(u) \neq -e_{n+1} \) for any \( u \in K \). By considering the rotation which is the identity on all points orthogonal to both \( v(u) \) and \( e_{n+1} \) so as to carry \( e_{n+1} \) to \( v(u) \), we again have the homotopy \( H^1_\lambda: (K, L) \to (SO(n + 1), E_{n+1}) \) relative to \( L \) such that \( H^1_\lambda(u) = E_{n+1} \) and \( H^1_\lambda(e_{n+1}) = v(u) \) for any \( u \in K \). Let \( H^2_\lambda: (K, L) \to (GL(n + 1), E_{n+1}) \) be the homotopy relative to \( L \) defined by \( H^2_\lambda(u) = ((1 - \lambda) + \lambda/v(u))E_{n+1} \). Let \( H_\lambda: (K, L) \to (GL(n + 1), E_{n+1}) \) be the homotopy defined by \( H_\lambda(u) = H^2_\lambda(u) \) for \( 0 \leq \lambda \leq 1/2 \) and \( H_\lambda(u) = H^2_{2\lambda-1}(u) \circ H^1_1(u) \) for \( 1/2 \leq \lambda \leq 1 \). Then we have that, for any \( u \in K \),

\[
\mathcal{K}^J(u) \circ H^1_1(u)(e_{n+1}) = \mathcal{K}^J(u) \circ H^2_1(u) \circ H^1_1(u)(e_{n+1})
\]

\[
= \mathcal{K}^J(u) \circ H^2_1(u)(v(u))
\]

\[
= \mathcal{K}^J(u)(v(u))/v(u)
\]

\[
= e_{p+1}.
\]

Since \( H^1_1(u) \in SO(n + 1) \) and \( e_i \) is orthogonal to \( e_{n+1} \) for \( i < n + 1 \), \( H^1_1(u)(e_i) \) is orthogonal to \( H^1_1(u)(e_{n+1}) = v(u) \). Namely, \( H_1(u)(e_i) \) lies in \( \text{Ker}(\mathcal{K}^J(u)) \). Hence, we have

\[
\mathcal{K}^J(u) \circ H_1(u)(e_i) = 0 \quad \text{for } i < n + 1.
\]

Define the homotopy \( \psi_\lambda: (K, L) \to (\Omega(n + 1, p + 1), i_{n+1}(\Omega(n, p))) \) relative to \( L \) by

\[
\psi_\lambda(u) = \begin{cases} 
\varphi_{3\lambda}(u) & \text{for } 0 \leq \lambda \leq 1/3, \\
h_{3\lambda-1}(u) \circ \varphi_3(u) & \text{for } 1/3 \leq \lambda \leq 2/3, \\
h_1(u) \circ \varphi_3(u) \circ H_{3\lambda-2}(u) & \text{for } 2/3 \leq \lambda \leq 1.
\end{cases}
\]

By the definition we have

\[
j(p_{r+1}) \circ \pi_k \circ \psi_1(u)(e_i) = \begin{cases} 
0 & \text{for } i < n + 1, \\
e_{p+1} & \text{for } i = n + 1.
\end{cases}
\]

This is what we want. \( \square \)

**Proposition 5.3.** Under the same assumption of Lemma 5.2, we have a homotopy \( \Psi_\lambda: (K, L) \to (\Omega(n + 1, p + 1), i_{n+1}(\Omega(n, p))) \) such that

(i) \( \Psi_0 = \psi \),

(ii) \( \Psi_\lambda|L = \psi|L \),

(iii) \( \Psi_1(K) \subset i_{n+1}(\Omega(n, p)) \).

**Proof.** Let \( \psi_\lambda \) be the homotopy given in Lemma 5.2. Let us express \( \psi_\lambda(u) = (f^1(u), f^2(u), \ldots, f^{p+1}(u)) \) using the coordinates of \( \mathbb{R}^{p+1} \), where \( f^1(u) \) is regarded as a polynomial of degree \( k \) with constant 0. We note that

\[
f^{p+1}(u)(x_1, \ldots, x_{n+1}) = x_{n+1} + \text{higher term}.
\]

Let \( C^\infty_0(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \) denote the set of all germs of local diffeomorphisms of \( (\mathbb{R}^{n+1}, 0) \). Let us define a homotopy of maps \( \Phi_\lambda: (K, L) \to C^\infty_0(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \) by

\[
\Phi_\lambda(u)(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1} + \lambda(f^{p+1}(u) - x_{n+1})).
\]
It is obvious that $\Phi(u)$ is a germ of a diffeomorphism of $(\mathbb{R}^{n+1}, 0)$. Then we have the inverse $\Phi(u)^{-1}$ such that
\[(5.1)\quad pr_{p+1} \circ \psi_1 \circ \Phi(u)^{-1}(x_1, \ldots, x_{n+1}) = x_{n+1}.
\]
We now define $\phi_\lambda : (K, L) \to (\Omega(n + 1, p + 1), i_{+1}(\Omega(n, p)))$ by
\[\phi_\lambda(u) = \psi_1 \circ J_0^k(\Phi(u)^{-1}).\]

Next we exclude the terms containing $x_{n+1}$ from $y_j \circ \phi_\lambda$ $(1 \leq j \leq p)$. Let $\eta_\lambda : (K, L) \to (J^k(n + 1, p + 1), J^k(n, p))$ be the homotopy defined by
\[\eta_\lambda(u)(x) = (1 - \lambda)\phi_1(u)((x_1, \ldots, x_{n+1})
+ \lambda(\phi_1(u)(x_1, \ldots, x_n, 0) + (0, \ldots, 0, x_{n+1})).\]
It is obvious that $\eta_\lambda(K) \subset i_{+1}(\Omega(n, p))$ and that $\eta_\lambda(L) = \psi_1(L)$. It remains to prove that $\eta_\lambda$ is a homotopy into $\Omega(n + 1, p + 1)$. It follows from (5.1) and (5.2) that
\[pr_{p+1} \circ \eta_\lambda(u)(x) = (1 - \lambda)(x_{n+1}) + \lambda x_{n+1} = x_{n+1}.
\]
Let us express $\eta_\lambda(u) = (g_1^\lambda(u), g_2^\lambda(u), \ldots, g_{p+1}^\lambda(u))$, where $g_i^\lambda(u)$ is regarded as a polynomial of degree $k$ with constant 0. Consider the ideal $\mathfrak{I}_\lambda(u)$ generated by
\[g_1^\lambda(u), g_2^\lambda(u), \ldots, g_{p+1}^\lambda(u) \text{ in } m_{n+1}/m_{n+1}^k.\]
Then $\mathfrak{I}_\lambda(u)$ is constantly equal to $\mathfrak{I}_0(u)$, and hence $Q(\eta_\lambda(u)) \approx Q(\psi_1(u))$. Since $\psi_1(u) \in \Omega(n + 1, p + 1)$, we have $\eta_\lambda(u) \in \Omega(n + 1, p + 1)$ by (C2). Then the required homotopy $\Psi_\lambda$ is defined by $\Psi_\lambda = \psi_3\phi_\lambda (0 \leq \lambda \leq 1/3)$, $\Psi_\lambda = \phi_{3\lambda^{-1}} (1/3 \leq \lambda \leq 2/3)$ and $\Psi_\lambda = \eta_3\lambda^{-2} (2/3 \leq \lambda \leq 1)$. □

**Proposition 5.4.** Let $n < p$. Let $\Omega(n, p)$ and $\Omega(n + 1, p + 1)$ denote the open subspaces, which satisfy (C1), (C2) and (C3). Then $(i_{+1})_* : \pi_i(\Omega(n, p)) \to \pi_i(\Omega(n + 1, p + 1))$ is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$.

**Proof.** Let $\iota_n : \mathbb{R}^n \to \mathbb{R}^p$ and $\iota_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$ denote the map defined by
\[\iota_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0),
\iota_{n+1}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, 0, \ldots, 0, x_{n+1}).\]
We first prove that $(i_{+1})_*$ is surjective for $0 \leq i \leq n$. Indeed, let $[a] \in \pi_i(\Omega(n + 1, p + 1))$ be represented by $a : (S^i, e_1) \to (\Omega(n + 1, p + 1), \iota_{n+1})$. Then by Proposition 5.3 we have a homotopy $\varphi : (S^i, e_1) \to (\Omega(n + 1, p + 1), \iota_{n+1})$ such that $\varphi(1) \subset i_{+1}(\Omega(n, p))$.

Next let $[b] \in \pi_i(\Omega(n, p))$ be represented by $b : (S^i, e_1) \to (\Omega(n, p), \iota_n)$ such that $(i_{+1})_*([b]) = 0$. Then we have a homotopy $\tilde{\varphi} : S^i \times I \to (\Omega(n + 1, p + 1), \iota_{n+1})$ such that $\tilde{\varphi}|_{S^i \times 0} = i_{+1} \circ b$ under the identification $S^i \times 0 = i_{+1} \circ b$. Then $\Phi_\lambda : (S^i \times I, S^i \times 0 \cup e_1 \times I \cup S^i \times 1) \to \Omega(1, 0) (n + 1, n + 1)$ is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$.

**Proposition 5.5.** Let $(i) n < p$, or $(ii) n = p > 1$ and $I = (1, 0)$. Then $(i_{+1}^*)_* : \pi_i(\Omega^2(n, p)) \to \pi_i(\Omega^2(n + 1, p + 1))$ is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$.

**Proof.** The case (i) follows from Lemma 5.1 and Proposition 5.4. The case (ii) is proved as follows. Let $d : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the diffeomorphism defined by $d(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ and let $d^2 : \Omega(1, 0)(n + 1, n + 1) \to \Omega(1, 0)(n + 1, n + 1)$ be the diffeomorphism, which maps a $k$-jet $j_k^0 f$ to $j_k^0 (d \circ f \circ d^{-1})$. □
Let \( j : SO(n+1) \to SO(n+2) \) be the map defined by \( j(A) = (1) + A \) for \( A \in SO(n+1) \). Let us recall the topological embedding \( i^S_{SO} : SO(m+1) \to \Omega^{(1,0)}(m,m) \) defined in \([7]\), Proposition 2.4, which is equivariant with respect to the action of \( SO(m) \times SO(m) \) and that \( i^S_{SO}(SO(m+1)) \) is a deformation retract of \( \Omega^{(1,0)}(m,m) \).

Then we have the following commutative diagram. The proof of commutativity is a rather wearisome calculation using the definition of \( i^S_{SO} \), and so it is left to the reader.

\[
\begin{array}{ccc}
SO(n+1) & \xrightarrow{j} & SO(n+2) \\
\downarrow i^S_{n+1} & & \downarrow i^S_{n+2} \\
\Omega^{(1,0)}(n,n) & \xrightarrow{d^0} & \Omega^{(1,0)}(n+1,n+1)
\end{array}
\]

Then the assertion of the corollary follows from the corresponding assertion for the map \( j \).

Recall that \( \Omega_n = \Omega(\gamma^n_{G_n}, TP) \) and \( \Omega_{n+1} = \Omega(\gamma^{n+1}_{G_{n+1}}, TP \oplus \varepsilon_{P}^1) \). Then Theorem 1.2 follows from Lemma 5.1, Proposition 5.5 and the following theorem.

**Theorem 5.6.** Let \( \Omega(n,p) \) and \( \Omega(n+1,p+1) \) denote the open subspaces, which satisfy (C1), (C2) and (C3) as in Proposition 5.4 when \( n < p \), or \( \Omega^{(1,0)}(n,n) \) and \( \Omega^{(1,0)}(n+1,n+1) \) when \( n = p \) respectively. Let \( G_n \) and \( G_{n+1} \) refers to \( G_n, \ell \) and \( G_{n+1, \ell} \) in the nonoriented case (resp. \( \bar{G}_n, \ell \) and \( \bar{G}_{n+1, \ell} \) in the oriented case). Then the homomorphism

\[
T(b(\gamma^{(1)}_{\Omega_n}))(\gamma_{\Omega_{n+1}})) \to T(\gamma_{\Omega_{n+1}})
\]

is an isomorphism for \( 0 \leq i < n \) and an epimorphism for \( i = n \).

**Proof.** We first show that \( (i^G)_* : \pi_i(G_n) \to \pi_i(G_{n+1}) \) is an isomorphism for \( 0 \leq i < n \) and an epimorphism for \( i = n \). We give a proof only in the nonoriented case, since the proof in the oriented case is analogous. Let us consider the canonical maps

\[
\begin{align*}
q_1 : O(n+\ell+1)/O(\ell) \times O(n) \times E_1 \to G_{n+1, \ell}, \\
q_2 : O(n+\ell+1)/O(\ell) \times O(n) \times E_1 \to O(n+\ell+1)/O(n+\ell) \times E_1 = S^{n+\ell}, \\
j : G_{n, \ell} \to O(n+\ell+1)/O(\ell) \times O(n) \times E_1.
\end{align*}
\]

It is obvious that \( q_1 \) and \( q_2 \) yield the structures of the fiber bundles with the fibers \( S^n \) and \( G_{n, \ell} \), which is included by \( j \), respectively. Since \( j_* : \pi_i(G_{n, \ell}) \to \pi_i(O(n+\ell+1)/O(\ell) \times O(n) \times E_1) \) is an isomorphism for \( 0 \leq i < n+\ell-1 \) and since \( (q_1)_* : \pi_i(O(n+\ell+1)/O(\ell) \times O(n) \times E_1) \to \pi_i(G_{n+1, \ell}) \) is an isomorphism for \( 0 \leq i < n \) and an epimorphism for \( i = n \), the assertion follows.

Let us recall the fiber map \( j^{(1)}_{(\Omega_n, \Omega_{n+1})} \) covering \( i^G \times id_P \). We next prove that

\[
(j^{(1)}_{(\Omega_n, \Omega_{n+1})})_* : \pi_i(\Omega_n) \to \pi_i(\Omega_{n+1})
\]

is an isomorphism for \( 0 \leq i < n \) and an epimorphism for \( i = n \). Let us consider the diagram which is induced from the homomorphisms \( (j^{(1)}_{(\Omega_n, \Omega_{n+1})})_* \) of the exact sequence of the homotopy groups for the fiber bundle \( \Omega_n \) over \( G_{n, \ell} \times P \) to the exact sequence of the fiber bundle \( \Omega_{n+1} \) over \( G_{n+1, \ell} \times P \). Then the second assertion about (5.3) follows from \([13]\) Lemma 3.2, the first assertion about \( (i^G)_* \) and Proposition 5.4. Then it follows from \([10]\) Section 5.9 Theorem 1.2.8 that

\[
(j^{(1)}_{(\Omega_n, \Omega_{n+1})})_* : H_i(\Omega_n) \to H_i(\Omega_{n+1})
\]
is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$. By virtue of the Thom Isomorphism Theorem, we have that

$$T(b(\tilde{\gamma})(\Omega_0, \Omega_{n+1}))_* : H_{i+\ell}(T(\tilde{\gamma}_* \Omega_n)) \longrightarrow H_{i+\ell}(T(\tilde{\gamma}_* \Omega_{n+1}))$$

is an isomorphism for $0 \leq i < n$ and an epimorphism for $i = n$. Since both of Thom spaces are simply connected, the assertion of theorem follows from the Hurewicz Isomorphism Theorem.

Let $K^k$ denote the group of $k$-jets of contact transformations, which acts on $J^k(m, q)$, defined in [30, (2.6)]. The $K^k$-orbit of $z \in J^k(m, q)$ is denoted by $K^k z$. It is known [31, Theorem 2.1] that $z'$ lies in $K^k z$ if and only if $Q(z') \approx Q(z)$. Hence, given a $\mathbb{R}$-algebra $Q$ which has a $k$-jet $z$ with $Q \approx Q(z)$, we write $\Sigma_Q = K^k z$. In general, $Q(z)$ is written as $\mathbb{R}[x_1, \ldots, x_m]/(f_1, \ldots, f_q) + m^k_{m+1}$, and hence let $b$ denote the minimal number of generators for the ideal $(f_1, \ldots, f_q)$ modulo $m^k_{m+1}$. If we define the integer $\iota(Q) = m - b$, then it is easy to see that $\iota(Q)$ is an invariant of the isomorphism class of $Q$. Du Plessis [17] has proved the following theorem (see also several examples which are concerned with simple singularities).

**Theorem 5.7** (17). Let $\Omega(m, q)$ be an open set of $J^k(m, q)$, which is invariant with respect to the action of $K^k$. Assume that for each $K^k$-orbit $\Sigma_Q \subset \Omega(m, q)$, there exists a $K^k$-orbit $\Sigma_{Q'} \subset \Omega(m, q)$ for another $\mathbb{R}$-algebra $Q'$ such that $\Sigma_{Q'} \subset \text{Closure}(\Sigma_Q)$ and $-\iota(Q') < q - m$. Then $\Omega(m, q)$ is extensible.

**Lemma 5.8.** Assume that $\Omega(n, p)$ and $\Omega(n+1, p+1)$ satisfy (C1), (C2) and (C3) and that $\Omega(n+1, p+1) \cap \Sigma^{n+1}(n+1, p+1) = \emptyset$. Assume that $\Omega(n, p)$ satisfies the assumption of Theorem 5.7 for $(m, q) = (n, p)$. Then $\Omega(n+1, p+1)$ also satisfies the assumption of Theorem 5.7 for $(m, q) = (n+1, p+1)$. Namely, $\Omega(n, p)$ and $\Omega(n+1, p+1)$ are extensible.

In particular, the pair $(\Omega(n, p), \Omega(n+1, p+1))$ is admissible to the $h$-Principle.

**Proof.** Take a $K^k$-orbit $\Sigma_Q \subset \Omega(n+1, p+1)$. Then there exists $k$-jets $z_1 \in \Sigma_Q$ and $z_2 \in \Omega(n, p)$ with $Q \approx Q(z_1) \approx Q(z_2)$ by (C3). Hence, $K^k z_2 \subset \Omega(n, p)$. By assumption, there exists a $K^k$-orbit $\Sigma_{Q'}$ for another $\mathbb{R}$-algebra $Q'$ such that $\Sigma_{Q'} \subset \text{Closure}(K^k z_2)$ and $-\iota(Q') < p - n$. By (C1), $i_{+1}(K^k z_2) \subset \Omega(n+1, p+1)$ and $i_{+1}(\Sigma_{Q'}) \subset i_{+1}(\text{Closure}(K^k z_2))$. Since $Q(z_2) \approx Q$, we have $K^k(i_{+1}(z_2)) = \Sigma_Q$, and hence we have

$$i_{+1}(\text{Closure}(K^k z_2)) \subset \text{Closure}(K^k(i_{+1}(z_2))) = \text{Closure}(\Sigma_Q).$$

Since the $\mathbb{R}$-algebras associated to $i_{+1}(\Sigma_{Q'})$ are all isomorphic to $Q'$, the $K^k$-orbit of $i_{+1}(\Sigma_{Q'})$ in $J^k(n+1, p+1)$ is contained in $\text{Closure}(\Sigma_Q)$. This is what we want to prove.

If we note $\text{codim}\Sigma^{n+1}(n+1, p+1) = (n+1)(p+1)$, we have the following corollary of Theorems 4.2, 5.6 and Lemma 5.8.

**Corollary 5.9.** Let $n < p$. Assume that $\Omega(n, p)$ and $\Omega(n+1, p+1)$ satisfy (C1), (C2) and (C3). Assume that $\Omega(n, p)$ satisfies the assumption of Theorem 5.7 for
$$(m, q) = (n, p)$$. Then the homomorphisms
\[
\omega_{\mathcal{N}}^{(\Omega_p, \Omega_{n+1})} : \mathcal{OCob}_{n, p}^{(\Omega_p, \Omega_{n+1})} \to \pi_{n+\ell} \left( T(\gamma_{n+1, \ell}^{(1, 0)} \mathcal{N}_{n+1, \ell, p} \oplus \mathcal{N}_{p, n+1}) \right)
\]
\[
\omega_{\mathcal{O}}^{(\Omega_p, \Omega_{n+1})} : \mathcal{OCob}_{n, p}^{(\Omega_p, \Omega_{n+1})} \to \pi_{n+\ell} \left( T(\gamma_{n+1, \ell}^{(1, 0)} \mathcal{O}_{n+1, \ell, p} \oplus \mathcal{O}_{p, n+1}) \right)
\]
are isomorphisms.

6. Cobordisms of fold-maps

In this section we study $\mathcal{OCob}_{n, P}^{(\Omega^{(1, 0)}, \Omega^{(1, 0)})}$ in the equidimension $n = p$, where $G_{n+1}$ refers to $\mathcal{G}_{n+1, \ell}$. For this we study $\pi_{n+\ell}(\gamma_{n+1, \ell}^{(1, 0)})$ in place by Theorems 1.1 and 1.2 and prove Theorem 1.3. Take a Riemannian metric on $P$.

Let $SO_{n+2}(\gamma_{G_{n+1}}^{n+1, \ell} \mathcal{N}_{G_{n+1}}^{1, \ell} \mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1})$ denote the total space of the open subbundle of $\text{Hom}(\gamma_{G_{n+1}}^{n+1, \ell} \mathcal{N}_{G_{n+1}}^{1, \ell} \mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1})$, which is associated to $SO(n + 2)$. Namely, it consists of all isomorphisms which preserve orientations and norms of vectors of $G_{n+1}$. Since the topological embedding $i_{SO}^{(1, 0)} : SO(n + 2) \to \Omega^{(1, 0)}(n + 1, n + 1)$ in [13 Proposition 2.4] is equivariant with respect to the action of $SO(n + 1) \times SO(n + 1)$ and since $i_{SO}^{(1, 0)}(SO(n + 2))$ is a deformation retract of $\Omega^{(1, 0)}(n + 1, n + 1)$, there exists the homotopy equivalent fiber map
\[
i_{SO} : SO_{n+2}(\gamma_{G_{n+1}}^{n+1, \ell} \mathcal{N}_{G_{n+1}}^{1, \ell} \mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1}) \to \Omega^{(1, 0)}_{n+1} = \Omega^{(1, 0)} \mathcal{N}_{G_{n+1}}^{1, \ell} \mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1}.
\]

Let $SO(\mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1})$ denote the total space of the principal bundle over $P$ associated to $TP \oplus \mathcal{N}_{p, n+1}$, whose fiber is $SO(n + 1)$. Let $(SO(\ell) \times E_{n+1}) \setminus SO(n + \ell) = SO(n + 1)$ be the Stiefel manifold. Consider the natural actions of $SO(n + 1)$ on $SO(\mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1})$ from the right-hand side and on $SO(n + 2)$ through $SO(n + 1) \times E_{1}$ from the left-hand side, and the natural actions of $SO(n + 1)$ on $SO(n + 2)$ through $SO(n + 1) \times E_{1}$ from the right-hand side and on $(SO(\ell) \times E_{n+1}) \setminus SO(n + \ell) = SO(n + 1)$ from the left-hand side respectively. Then we can express as
\[
SO_{n+2} = SO(\mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1}) \times SO(n + 2) \\
\times ((SO(\ell) \times E_{n+1}) \setminus SO(n + \ell + 1)).
\]

Identify the quotient space $SO(\mathcal{P}_{n+1} \oplus \mathcal{N}_{p, n+1})/SO(n + 1)$ with $P$. Then we have the projection $pr_{SO}^{SO} : SO_{n+2} \to P$ by forgetting the component $SO(n + 2) \times SO(n + 1)$.
\((SO(\ell) \times E_{n+1}) \backslash SO(n+\ell+1))\), denoted by \(f\), which is the canonical fiber of \(pr^F_P\).

Let \(pr_{S^{n+1}}^f : f \to SO(n+2)/SO(n+1) = S^{n+1}\) be the projection forgetting the component \((SO(\ell) \times E_{n+1}) \backslash SO(n+\ell+1)\). Since the last space is \((\ell - 1)\)-connected, 
\[
(pr_{S^{n+1}}^f)_* : \pi_i(f) \to \pi_i(S^{n+1})
\]
is an isomorphism for \(i < \ell\). Hence, we have the following lemma.

**Lemma 6.2.** The homomorphism \((pr^F_P)_* : \pi_i(SO_{n+2}) \to \pi_i(P)\) is an isomorphism for \(i < n + 1\) and is an epimorphism for \(i = n + 1\).

Let us define a bundle map
\[
b_\gamma : (\kappa^h_{G_{n+1}})^*(\gamma_{G_{n+1}})_{SO_{n+2}} \to TP \oplus \varepsilon_P^1
\]
as follows. We denote an element of \(SO_{n+2}\) and \((\pi^h_{G_{n+1}})^*(\gamma_{G_{n+1}})_{SO_{n+2}}\) by \([a, y, h]\) and by \([a, y, h, v]\) respectively, where \(a \in G_{n+1}, y \in P, v \in a\) and \( h : a \to T_yP \oplus \mathbb{R}\) is an isomorphism preserving orientations and norms. Then we set \(b_\gamma([a, y, h, v]) = h(v)\). Let us consider the trivialization \((t_P \oplus id_P \times \mathbb{R}) \circ (id_TP \oplus k_P) : TP \oplus \varepsilon_P \to \varepsilon_{P+1}\). By \([3, Proposition 3.3]\), there exists a bundle map
\[
b_\delta : \gamma_{\Omega(1,0)}^{(1)}(\kappa_{\Omega(1,0)})_{SO_{n+2}} \to \nu_P^\ell
\]
such that \((t_P \oplus id_P \times \mathbb{R}) \circ (id_TP \oplus k_P) \circ (b_\gamma \oplus \nu_P) \circ (\nu_{\Omega(1,0)}^{(1)} \times id_P \times \mathbb{R})\) is homotopically equivalent to \(pr^F_P \oplus id_{P+1}\).

According to \([33]\), \(\nu_P^\ell \equiv \nu_{\Omega(1,0)}^{(1)}\) and \(\nu_P^\ell \equiv \nu_{\Omega(1,0)}^{(1)}\). Since \(\nu_P^\ell \equiv \nu_{\Omega(1,0)}^{(1)}\), \(\nu_P^\ell \equiv \nu_{\Omega(1,0)}^{(1)}\). Hence, we have the following lemma.

**Lemma 6.3.** The homomorphism induced from the Thom map of \(b_\gamma\)
\[
T(b_\gamma)_* : \pi_{i+\ell}(T(\gamma_{\Omega(1,0)}^{(1)}_{\kappa_{\Omega(1)}})) \to \pi_{i+\ell}(T(\nu_P^\ell))
\]
is an isomorphism for \(i < n + 1\) and is an epimorphism for \(i = n + 1\).

**Proof.** By Lemma 6.2, (6.2), \([33, Section 5, 9 Theorem]\) and the Thom Isomorphism Theorem,
\[
T(b_\delta)_* : H_{i+\ell}(T(\gamma_{\Omega(1,0)}^{(1)}_{\kappa_{\Omega(1)}})) \to H_{i+\ell}(T(\nu_P^\ell))
\]
is an isomorphism for \(i < n + 1\) and is an epimorphism for \(i = n + 1\). Since \(T((\kappa^h_{G_{n+1}})^*(\gamma_{G_{n+1}}))_{SO_{n+2}}\) and \(T(\nu_P^\ell)\) are simply connected, it follows from the Hurewicz Isomorphism Theorem that
\[
T(b_\gamma)_* : \pi_{i+\ell}(T(\gamma_{\Omega(1,0)}^{(1)}_{\kappa_{\Omega(1)}})) \to \pi_{i+\ell}(T(\nu_P^\ell))
\]
is an isomorphism for \(i < n + 1\) and is an epimorphism for \(i = n + 1\). \(\Box\)
consists of all S-maps of degree \( d \). By Theorem 1.1, Proposition 6.1 and Lemma 6.3, \( \mathcal{OC}ob_{n,P}(\Omega^{(1,0)},\Omega^{(1,0)})(d) \) is mapped bijectively onto \( \{S^{n+\ell}, T(\nu_{\ell})\}_d \).

We prove the following refined form of Theorem 1.3.

**Theorem 6.4.** Let \( n = p \geq 2 \) and \( P \) be a closed, connected, oriented \( n \)-dimensional manifold. Then there exist a bijection \( \omega_d : \mathcal{OC}ob_{n,P}(\Omega^{(1,0)},\Omega^{(1,0)})(d) \to [P, F^d] \).

**Proof.** Let us define the map \( c_F : \{S^0 P^0, S^\ell\} \to [P, F] \). Let \( \{\beta\} \in \{S^0 P^0, S^\ell\} \) be represented by \( \beta : S^0 P^0 \to S^\ell \). For a point \( x \in P \) we define \( \beta(x) : S^\ell = S^0 \cup S^\ell \to S^\ell \) by \( \beta(\{x \cup x\} \cup S^\ell) \circ (t_x \times id_{S^\ell}) \), where \( t_x : S^0 \to \{x \cup x\} \) is the canonical identification. Then we set \( c_F(\{\beta\})(x) = \{\beta(x)\} \). It is easy to check that \( c_F \) is bijective. Furthermore, we have proved in [17] Lemma 2.4 that \( c_F \) maps \( \{S^{n+\ell}, T(\nu_{\ell})\}_d \) to the subset of \( \{S^0 P^0, S^\ell\} \) which consists of all \( \{\beta\} \) such that \( \beta(x) \) is of degree \( d \), namely \( c_F(\{\beta\})(x) \in F_d \) for any \( x \in P \). This proves the assertion. \( \square \)

Let \( \pi_n^S \) denote the \( n \)-th stable homotopy group of spheres \( \lim_{\ell \to \infty} \pi_n(S^\ell) \). It follows from [2] that \( [S^n, F^0] \) is canonically isomorphic to \( \pi_n^S \). So identifying \( [S^n, F^0] \) with \( \pi_n^S \), we have the following corollary.

**Corollary 6.5.** The map \( \omega_0 : \mathcal{OC}ob_{n,S^n}(\Omega^{(1,0)},\Omega^{(1,0)})(0) \to \pi_n^S \) is an isomorphism.

7. **Stable maps of spheres**

Let \( \mathcal{OC}ob_{n,S^n}(\Omega^{(1,0)},\Omega^{(1,0)})(0) \to \mathcal{OC}ob_{n,S^n}(\Omega^{(1,0)},\Omega^{(1,0)})(0) \) denote the homomorphism which maps an \( \Omega^{(1,0)} \)-cobordism class \( [f] \) to the \( \Omega^{(1,0)} \)-cobordism class of \( f \). Let \( I([f]) \) denote the smallest symbol \( f \) such that \( \mathcal{OC}ob_{n,S^n}(\Omega^{(1,0)},\Omega^{(1,0)})(0) \) is a null element. Then there exists an \( \Omega^{(1,0)} \)-regular cobordism \( F : (V, \partial V) \to (S^n \times I, S^n \times 0) \) such that \( \partial V = N, \) the collar of \( \partial V \) is identified with \( N \times [0, \varepsilon], \) and \( F|N \times [0, \varepsilon] = f \times id_{[0, \varepsilon]} \).

In this section we show that the singularities of symbol \( I([f]) \) of \( F \) detect the stable homotopy class \( \omega_0([f]) \in \pi_n^S \) in low dimensions. We have to prepare some machinery for this purpose, although the dimensions are low.

Let \( D \) and \( P \) denote the total tangent bundle defined on \( J^\infty(V, Y) \) and \( (\pi_Y^*)^*(TY) \) respectively. Let us recall the fundamental property of \( D \) over \( J^\infty(V,Y) \). Let \( f : (V, x) \to (Y, y) \) be a map defined on a neighborhood \( U_x \) of \( x \) with coordinates \( (x_1, \cdots, x_{n+1}) \) and \( f \) be a smooth function in the sense of [13] Definition 1.4] defined on a neighborhood of \( j^\infty f \). We have the local vector fields \( D_i \) defined around \( z \) with the property

\[
D_i f \circ j^\infty f = \frac{\partial}{\partial x_i} (f \circ j^\infty f) \quad (1 \leq i \leq n + 1),
\]

which span \( D \). It follows that \( d(j^\infty f)(\partial/\partial x_i)(f) = D_i f (j^\infty f) \), where \( d(j^\infty f) : TV \to T(J^\infty(V,Y)) \) around \( x \). This implies \( d(j^\infty f)(\partial/\partial x_i) = D_i \). Hence, we have \( D \cong (\pi_Y^*)^*(TV) \). There have been defined the homomorphism \( d_1 : D \to P \) over \( J^\infty(V,Y) \). If \( z = j^\infty f \), then \( d_{1,z}(D_i) = (z, dz_f(\partial/\partial x_i)) \). The manifold \( \Sigma^1(V,Y) \) is defined to be the submanifold of \( J^\infty(V,Y) \) which consists of all jets \( z \) such that the kernel rank of \( d_{1,z} \) is 1. Since \( d_1|_{\Sigma^1(V,Y)} \) is of constant rank \( n \), we set \( K_1 = \text{Ker}(d_1) \) and \( P_1 = \text{Cok}(d_1) \), which are vector bundles over \( \Sigma^1(V,Y) \). Let \( 1_r \)

denote \( (1, \cdots, 1) \). The Boardman manifold \( \Sigma^{1,r}(V,Y) \ (r \geq 1) \) has the following properties (13).
(7-i) There exists the $(r + 1)$-th intrinsic derivative
\[ d_{r+1} : T(\Sigma^1, (V, Y))|_{\Sigma^r} \rightarrow \text{Hom}(S^rK_1, P_1)|_{\Sigma^r} \rightarrow 0, \]
so that \( \text{Ker}(d_{r+1}) = T(\Sigma^1r, (V, Y)) \). Namely, \( d_{k+1} \) induces the isomorphism of the normal bundle \( T(\Sigma^1r, (V, Y))|_{\Sigma^r} \rightarrow T(\Sigma^1, (V, Y)) \) of \( \Sigma^1r \) in \( \Sigma^1r \) on \( \text{Hom}(S^kK_1, P_1)|_{\Sigma^r} \).

(7-ii) \( \Sigma^1r+1, (V, Y) \) is defined to be the submanifold of \( \Sigma^1r, (V, Y) \) which consists of all jets \( s \) such that \( d_{r+1}s |_{K_1} \) vanishes.

(7-iii) The \( (r + 2) \)-th intrinsic derivative \( d_{r+2} \) is defined to be the intrinsic derivative
\[ d(d_{r+1}(K_1)) : T(\Sigma^1, (V, Y))|_{\Sigma^{r+1}} \rightarrow \text{Hom}(K_1, P_1)|_{\Sigma^{r+1}}. \]

(7-iv) The submanifold \( \Sigma^1r, (V, Y) \) is actually defined so that it coincides with the inverse image of the submanifold \( \Sigma^1r, (V, Y) \) in \( J^r(V, Y) \) by \( \pi^\infty \). The codimension of \( \Sigma^1r, (V, Y) \) in \( J^r(V, Y) \) is \( r \).

**Theorem 7.1.** Let \( V \) be an oriented \((n + 1)\)-manifold with \( \partial V \), which may be empty, \( Y \) be an oriented \((n + 1)\)-manifold and let \( C \) be a closed subset of \( V \). Let \( s \) be a section of \( \Gamma_{\Omega^1}(V, Y) \) which has a fold-map \( g \) defined on a neighborhood of \( C \) into \( Y \), where \( j^\infty g = s \). Then there exists an \( \Omega^{(1, 1)} \)-regular mapping \( f : V \rightarrow Y \) and a homotopy \( s_\lambda \in \Gamma_{\Omega^1}(V, Y) \) relative to a neighborhood of \( C \) such that \( s_0 = s \) and \( s_1 = j^\infty f \).

**Proof.** In the proof we use the notation introduced in [13]. By (2.2) we always identify \( J^r(TV, TY) \) with \( J^r(TV, TY) \), where \( r \) may be \( \infty \). We may assume that \( s \) is transverse to \( \Sigma^n, (V, Y) \) and set \( S^1r(s) = s^{-1}(\Sigma^1r, (V, Y)) \). It follows that \( (\pi^\infty_s \circ s)(V \setminus (S^1r(s))) \subset \Omega^{1-1}(V \setminus S^1r(s), Y) \).

We find a section \( s \) of \( \Omega^{(1, 1, 0)}(V, Y) \) such that \( \pi_2^\infty \circ s = \pi_2^\infty \circ s \). We set \( (s|S^1r(s)) \circ K_1 = K_1 \) and \( (s|S^1r(s)) \circ P_1 = P_1 \). Since \( V \) and \( Y \) are oriented and since \( K_1 \) and \( P_1 \) are line bundles, we have that \( K_1 \) and \( P_1 \) are isomorphic. In particular, we have the isomorphism \( K_1|_{S^1r(s)} = P_1|_{S^1r(s)} \). Consider the homomorphism
\[ r^3 : \text{Hom}(S^3(TV), TY)|_{S^1r(s)} \rightarrow \text{Hom}(S^3K_1, P_1)|_{S^1r(s)} \]
which is induced from the inclusion \( S^3K_1|_{S^1r(s)} \rightarrow S^3(TV, TY)|_{S^1r(s)} \) and the projection \( TY|_{S^1r(s)} \rightarrow P_1|_{S^1r(s)} \). Since \( S^3K_1 \approx K_1 \), there exists the isomorphism \( r^3 : S^3K_1|_{S^1r(s)} \rightarrow P_1|_{S^1r(s)} \), which induces the isomorphism \( K_1|_{S^1r(s)} \rightarrow \text{Hom}(S^3K_1, P_1)|_{S^1r(s)} \). Since \( S^3K_1 \) is a closed submanifold of \( V \) such that \( S^3(s) \cap C = \emptyset \) in \( V \), there exists a homomorphism \( h^3 : S^3(TV)|_{S^1r(s)} \rightarrow (\pi^\infty_s \circ s)^*(TY)|_{S^1r(s)} \) such that \( r^3 \circ h^3 = r^3 \). We extend \( h^3 \) to the homomorphism \( H^3 : S^3(TV) \rightarrow (\pi^\infty_s \circ s)^*(TY) \). If \( (\pi^\infty_s \circ s)_{TY} : (\pi^\infty_s \circ s)^*(TY) \rightarrow TY \) denote the canonical bundle map covering \( \pi^\infty_s \circ s \), then we define the section \( s : V \rightarrow J^\infty(TV, TY) \) so that
\[ \pi^\infty_s \circ s(x) = \pi^\infty_s \circ s(x) \oplus (\pi^\infty_s \circ s)_{TY} \circ H^3|_x \]
and that \( s \) is the composite of \( \pi^\infty_s \circ s \) and the canonical inclusion \( J^3(TV, TY) \rightarrow J^\infty(TV, TY) \).

We now show that \( s(V) \subset \Omega^{(1, 1, 0)}(V, Y) \). In fact, it is obvious that \( s(V) \subset \Omega^2(V, Y) \). It remains to prove that if \( x \in S^1r(s) \), then
\[ (7.1) \quad d_{3,s(x)} : K_1,s(x) \rightarrow \text{Hom}(S^3K_1,s(x), P_1,s(x)) \]
is an isomorphism. In other words the homomorphism \( S^3 \mathbf{K}_{1,s(x)} \to \mathbf{P}_{1,s(x)} \) induced from \( \partial_3 \mathbf{s}(x) \) is an isomorphism. For any point \( x \in S^{12}(s) \), let \( y = \pi_2^{\infty} \circ \mathfrak{s}(x) \), \( U_y \) and \( V_y \) be convex neighborhoods of \( x \) and \( y \) respectively. Let \( t \) and \( u \) be the coordinates of \( \exp_{Y,t}(K_{1,x}) \) and \( \exp_{Y,u}((\pi_2^{\infty} \circ \mathfrak{s})_{TY}(P_1,y)) \) respectively, where \( P_1 \) is regarded as a line subbundle of \( (\pi_2^{\infty} \circ \mathfrak{s})(TY)|_{S^{12}(s)} \) by virtue of the metric of \( Y \). It follows from (7.1), (2.2) and the definition of \( i^3 \) that

\[
(\partial^3 D_t)u_{|s(x)} = \partial^3 u/\partial t^3(x) \neq 0 \quad \text{for } x \in S^{12}(s).
\]

Hence, we have that \( s(S^{12}(s)) \subseteq \Sigma^{(1,1,0)}(V,Y) \).

By (13) Theorem 0.1 and (22) there exists an \( \Omega^{(1,1,0)} \)-regular map \( G : V \to Y \) such that \( j^\infty G \) and \( s \) are homotopic relative to a neighborhood of \( C \) as sections of \( \Omega^1(V,Y) \) over \( V \). Here, we again note the remark which has been given at the end of the proof of Theorem 1.1.

Let \( \ell \) be a natural number with \( \ell \gg n \). Let \( V \) be an \((n + 1)\)-manifold with \( \partial V = N \), and let \( \tau_V \) be the stable \( \ell \)-dimensional tangent bundle of \( V \). Given a fold map \( f : N \to S^n \) of degree 0, we have the bundle map \( T(f) : TN \oplus \varepsilon^n_{\ell} \to \varepsilon_{S^n+1} \) by (38) Corollary 2. Let us consider the obstruction for \( T(f) \) to be extended to the trivialization of \( \tau_V \). For this purpose, we have the primary obstruction \( o(\tau_V, T(f)) \) defined in \( H^{i+1}(V,N;\pi_i(SO(\ell))) \) for some \( i \). Let \( \hat{V} = V \cup N CN \), which is obtained by pasting \( V \) and the cone of \( N \). Let \( \tau(\hat{V}, T(f)) \) be the \( \ell \)-dimensional vector bundle, which is obtained by pasting \( \tau_V \) and \( \varepsilon_{CN}^\ell \) by using \( T(f) \). We have the primary obstruction \( o(\tau(\hat{V}, T(f))) \in H^{i+1}(\hat{V}; \pi_i(SO(\ell))) \approx H^{i+1}(V,N;\pi_i(SO(\ell))) \) for \( \tau(\hat{V}, T(f)) \) to be trivial. It is not difficult to see that \( o(\tau_V, T(f)) = \pm o(\tau(\hat{V}, T(f))) \) under the isomorphism.

**Remark 7.2.** In this case we may take \( \ell = n + 2 \) and consider the subbundle \( SO(\tau(\hat{V}, T(f)), \varepsilon_{\hat{V}}^{n+2}) \) of \( Hom(\tau(\hat{V}, T(f)), \varepsilon_{\hat{V}}^{n+2}) \) associated to \( SO(n + 2) \). Since \( i_{n+2} : SO(n + 2) \to \Omega^{(1,0)}(n + 1, n + 1) \) is a homotopy equivalence, \( o(\tau(\hat{V}, T(f))) \) coincides with the obstruction to find a section of \( \Omega^{(1,0)}(\tau(\hat{V}, T(f)), \varepsilon_{\hat{V}}^{n+2}) \), namely the Thom polynomial of the closure \( C(\Sigma^{(1,1)}(\tau(\hat{V}, T(f)), \varepsilon_{\hat{V}}^{n+2})) \) (see, for example, (33) Proposition 3.1). This Thom polynomial is equal to the second Stiefel-Whitney class \( w_2(\tau(\hat{V}, T(f))) \) by (33).

If \( n + 1 = 4m \) and \( \omega_0([f]) \) lies in what is called the \( J \)-image \( J(\pi_{4m-1}(SO(\ell))) \) of order \( j_m \) in (14), then we can choose a fold-map \( f \) such that \( N = S^n \) by (24) Proposition 5.1. This is also true for the case \( n = 1 \). Furthermore, we can take \( V \) to be a parallelizable manifold. Hence, the above dimension \( i \) is equal to \( n \).

We have the following lemma due to (33) Lemma 2]. Let \( a_m \) denote 2 for \( m \) odd and 1 for \( m \) even.

**Lemma 7.3** (32). Let \( n + 1 = 4m \). Let \( V \) be a parallelizable manifold. Then \( o(\tau_V, T(f)) \) is related to the \( m \)-th Pontryagin class \( P_m(\tau(\hat{V}, T(f))) \) by the identity \( P_m(\tau(\hat{V}, T(f))) = \pm a_m(2m - 1)! \omega(\tau(\hat{V}, T(f))) \).

We next see how \( o(\tau_V, T(f)) \) varies depending on the choice of \( V \) and \( f \) (the following argument is available for the case \( n = 1 \)). Let two fold maps \( f_i : S^n \to S^n \) of degree 0 (\( i = 0, 1 \)) be \( \Omega^{(1,0)} \)-cobordant by a cobordism \( F : (W, \partial W) \to (S^n \times I, S^n \times 0 \cup S^n \times 1) \) of degree 0 as in Definition 3.1. Assume that there exists a
parallelizable \((n + 1)\)-manifold \(V_i\) with \(\partial V_i = S^n \times i\). Then we have the bundle maps \(\mathcal{T}(f_i) : T(S^n \times i) \oplus \varepsilon_{S^n \times i}^1 \to \varepsilon_{S^n \times i}^{n+1}\) and \(\mathcal{T}(F) : TW \oplus \varepsilon_{W}^1 \to \varepsilon_{S^n \times i}^{n+1}\) by [3] Corollary 2 such that \(TW|_{S^n \times i} = T(S^n \times i) \oplus \varepsilon_{S^n \times i}^1\) and the stabilizations of \(\mathcal{T}(F)|_{N_i}\) and \(\mathcal{T}(f_i)\) are equal. Consider the almost parallelizable manifold \(\hat{W} = V_0 \cup S^n \times W \cup V_1(-V_1)\), which is obtained by pasting \(V_0\), \(W\) and \(V_1\) with orientation reversed. Let 
\[
o(\tau(\hat{W})) \in H^{n+1}(\hat{W}; \pi_n(SO(\ell))) \approx \pi_n(SO(\ell))
\]
be the unique primary obstruction for \(\tau(\hat{W})\) to be trivial. Then it is obvious that 
\[
o(\tau_{v_0}, \mathcal{T}(f_0)) - o(\tau_{v_1}, \mathcal{T}(f_1)) = \pm o(\tau(\hat{W})) \quad \text{in} \quad \pi_n(SO(\ell)).
\]
Define the integer \(m(n)\) for \(n > 1\) to be the minimal nonnegative number such that there exists an \((n + 1)\)-dimensional almost parallelizable closed manifold \(\hat{W}'\) such that \(o(\tau(\hat{W}')) = m(n)\). We will see \(m(1) = 0\) later. We have the following theorem due to [44] Theorems 1 and 2.

**Theorem 7.4** (32). Let \(n + 1 = 4m\). Then we have

(i) The Pontrjagin class \(P_m(\hat{W}')\) of an almost parallelizable closed manifold \(\hat{W}'\) is divisible by \(\pm jm_a(m(2m - 1))!\).

(ii) There exists an almost parallelizable closed manifold \(\hat{W}_0\) with \(P_m(\hat{W}_0) = \pm jm_a(m(2m - 1))!\).

Consequently, we have \(m(n) = j_m\).

Let \(\mathbb{P}^n\) denote the real projective space of dimension \(n\). Let \(C_+\mathbb{P}^n\) and \(C_-\mathbb{P}^n\) denote the cone of \(\mathbb{P}^n \times [0, 1]/\mathbb{P}^n \times 1\) and \(\mathbb{P}^n \times [-1, 0]/\mathbb{P}^n \times (-1)\) respectively and let \(S\mathbb{P}^n\) denote the suspension \(C_+\mathbb{P}^n \cup_{\mathbb{P}^n} C_-\mathbb{P}^n\). For the projection of the double covering \(p_{\mathbb{P}^n} : S^n \to \mathbb{P}^n\), let \(S(p_{\mathbb{P}^n}) : S^{n+1} \to S\mathbb{P}^n\) be its suspension.

**Lemma 7.5.** Let \(n + 1 = 4m\). Then we have

(i) \(S(p_{\mathbb{P}^n})_* : \pi_i(S^{n+1}) \to \pi_i(S\mathbb{P}^n)\) is an isomorphism modulo 2-torsion for \(0 \leq i \leq n + 1\).

(ii) \(S(p_{\mathbb{P}^n})_* : \pi_{n+1}(S^{n+1}) \to \pi_{n+1}(S\mathbb{P}^n)\) is injective.

**Proof.** Since \(S\mathbb{P}^n\) is simply connected and \(H_i(S\mathbb{P}^n)\) is a 2-torsion for \(0 < i < n + 1\), the assertion (i) follows from [41] Section 6, 21 Theorem.

Let \(q : S\mathbb{P}^n \to S(S\mathbb{P}^n) = S\mathbb{P}^{n-1}\) be the collapsing map. Then it is obvious that the degree of \(q \circ S(p_{\mathbb{P}^n})\) is equal to 2. This shows (ii). \(\square\)

Let us define the map \(g : \mathbb{P}^n \to SO(n + 1)\) as follows. Let \(I_+\) be the \((n + 1)\)-matrix \(E_n + 1\). For an element \(v \in \mathbb{P}^n\), take a column vector \((s_1, \ldots, s_{n+1})\) of length 1 representing \(v\). Let \(G(v)\) denote the \((n+1)\)-matrix whose \((i, j)\) component is \(\delta_{ij} - 2s_is_j\), where \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) if \(i \neq j\). Then we set \(g(v) = I_+G(v)\). We note that \(g\) is well-known as the characteristic map of the tangent bundle of \(S^{n+1}\) (see [41] Section 23.4). We define the space \(M(n + 2)\) to be the union \(C_+\mathbb{P}^n \times SO(n + 1) \cup_{p \times SO(n+1)} C_-\mathbb{P}^n \times SO(n + 1)\), which is pasted by the diffeomorphism \((v, S) \to (v, g(v)S)\) for \(S \in SO(n + 1)\). We have the map \(p_{\mathbb{P}^n} : M(n + 2) \to S\mathbb{P}^n\), which induces the structure of the fiber bundle with fiber \(S(n + 1)\).

Let \(p_{S^{n+1}} : SO(n + 2) \to SO(n + 2)/SO(n + 1) \times E_1 = S^{n+1}\) be the map defined by \(p_{S^{n+1}}(S) = S\mathbb{e}_{n+2}\) for \(S \in SO(n + 2)\). Let \(D_{n+1}^+\) and \(D_{n+1}^-\) denote the hemispheres of \(S^{n+1}\) which consist of all points \((x_1, \ldots, x_{n+2})\) such that \(x_{n+2} \geq 0\) and \(x_{n+2} \leq 0\).
respectively. Let us recall that \(SO(n+2)\) is identified with the union \(D^{n+1} \times SO(n+1) \cup p_x \times SO(n+1) D^{n+1} \times SO(n+1)\), which is pasted by the diffeomorphism \((s, S) \rightarrow (s, g(s))S\). By the construction we have the natural bundle map \(\pi_{S}^{SO} : SO(n+2) \rightarrow \mathcal{M}(n+2)\) covering \(S(p_{n+1})\).

\[
\begin{array}{ccc}
SO(n+2) & \xrightarrow{\pi_{S}^{SO}} & \mathcal{M}(n+2) \\
p_{S,n+1} \downarrow & & \downarrow i_{n+1} \\
S_{n+1} & \xrightarrow{S(p_{n+1})} & \mathcal{S}^{n+1}
\end{array}
\]

According to [5, Theorem 3.5] and [6, Section 2], we have the topological embeddings \(i_{M} : \mathcal{M}(n+2) \rightarrow \Omega^{1}(n+1, n+1)\) and \(i_{SO}^{n+1} : SO(n+2) \rightarrow \Omega^{1,0}(n+1, n+1)\) such that \(i_{M}(\mathcal{M}(n+2))\) and \(i_{SO}^{n+1}(SO(n+2))\) are the deformation retracts of \(\Omega^{1}(n+1, n+1)\) and \(\Omega^{1,0}(n+1, n+1)\) respectively and that the following diagram commutes

\[
\begin{array}{ccc}
SO(n+2) & \xrightarrow{\pi_{M}^{SO}} & \mathcal{M}(n+2) \\
i_{n+1}^{SO} \downarrow & & \downarrow i_{M} \\
\Omega^{1,0}(n+1, n+1) & \xrightarrow{i_{n+1}^{(1,0)}} & \Omega^{1}(n+1, n+1),
\end{array}
\]

where \(i_{n+1}^{(1,0)}\) denotes \(\pi_{T}^{1}(\Omega^{1,0})(n+1, n+1) : \Omega^{1,0}(n+1, n+1) \rightarrow \Omega^{1}(n+1, n+1)\).

**Lemma 7.6.** Let \(n+1 = 4m\). Then \((\pi_{M}^{SO})_{*} : \pi_{n}(SO(n+2)) \rightarrow \pi_{n}(\mathcal{M}(n+2))\) is injective.

**Proof.** Consider the following diagram, which is induced from the homomorphism \((\pi_{M}^{SO})_{*}\), between the exact sequences for the fiber bundles \(p_{S,n+1}\) and \(p_{S}^{n+1}\) in (7.2).

\[
\begin{array}{ccc}
\pi_{n+1}(S_{n+1}) & \xrightarrow{\partial} & \pi_{n}(SO(n+1)) \\
\downarrow & & \downarrow \\
\pi_{n+1}(S^{n+1}) & \xrightarrow{\partial} & \pi_{n}(SO(n+2))
\end{array}
\]

It is known that \(\pi_{n}(SO(n+1)) \approx \mathbb{Z} \oplus \mathbb{Z}, \pi_{n}(SO(n+2)) \approx \mathbb{Z}\) and \(\partial : \pi_{n+1}(S^{n+1}) \rightarrow \pi_{n}(SO(n+1))\) is injective. Then the assertion follows from Lemma 7.5.

Consider the homomorphism

\[
(\pi_{n+1}^{SO})_{*} : H^{n+1}(\hat{V}; \pi_{n}(\Omega^{1,0}(n+1, n+1))) \rightarrow H^{n+1}(\hat{V}; \pi_{n}(\Omega^{1}(n+1, n+1))),
\]

which is injective by (7.3) and Lemma 6. Therefore, we have the following proposition.

**Proposition 7.7.** Let \(n+1 = 4m\). Let \(V\) be parallelizable. Then, for the obstruction \(\sigma(\tau(\hat{V}, \mathcal{T}(f)))\), \((\pi_{n+1}^{SO})_{*}(\sigma(\tau(\hat{V}, \mathcal{T}(f))))\) becomes the unique obstruction to find a section of \(\Omega^{1}(\tau(\hat{V}, \mathcal{T}(f)), \varepsilon_{n+2}^{n+2})\).

We note that \((\pi_{n+1}^{SO})_{*}(\sigma(\tau(\hat{V}, \mathcal{T}(f))))\) is not necessarily a primary obstruction.

In the rest of the paper we are only concerned with the case \(n < 8\). We have chosen \(V\) to be parallelizable. This is justified by the following lemma and \(\text{codim} \Sigma^{3}(n+1, n+1) = 9\). By the definition of \(\tau(\hat{V}, \mathcal{T}(f))\) in the case \(\partial V = S^{n}\), \(\mathcal{T}(f)\) yields the section of \(\Omega^{1}(\tau(\hat{V}, \mathcal{T}(f)), \varepsilon_{n+2}^{n+2})|_{CS^{n}}\), which we denote by \(s_{T(f)}\).
Lemma 7.8. Let $n < 8$ and $\partial V = S^n$. Then we have the following.

(i) If $P_1(\tau(\hat{V}, T(f)))$ does not vanish, then any extension $F : V \to S^n \times [0, 1]$ such that $F|S^n \times 0 = f$ has the singularities of codimension 4 and of type (2).

(ii) If $P_1(\tau(\hat{V}, T(f)))$ vanishes, then we have 

(ii-a) $s_{\tau(f)}$ is extendable to a section of $\Omega^1(\tau(\hat{V}, T(f)), \varepsilon^{n+2})$ over $\hat{V}$ except for a single point in the interior of $V$. In particular, $V$ is parallelizable.

(ii-b) $P_2(\tau(\hat{V}, T(f)))$ vanishes if and only if $s_{\tau(f)}$ is extendable to a section of $\Omega^1(\tau(\hat{V}, T(f)), \varepsilon^{n+2})$ over $\hat{V}$.

Proof. (i) is clear.

(ii-a) We triangulate $\hat{V}$ so that $CV$ is a subcomplex. Since $\text{codim} \Sigma^2(n + 1, n + 1) = 4$, $\Omega^1(n + 1, n + 1)$ is 2-connected. By considering the fiber bundle $p_{\text{SP}^n} : M(n + 2) \to \text{SP}^n$ it follows that $TV$ is trivial on $CV$ and the 3-skeleton of $V$. Since $P_1(\tau(\hat{V}, T(f))) = 0$, it follows from [34, Proof of Lemma 2] that $TV$ is trivial on $CV$ and the 4-skeleton of $V$. Then the assertion follows from $\pi_i(\text{SO}(8)) = 0$ for $i = 4, 5, 6$ by applying the obstruction theory for $p_{\text{SP}^n} : M(n + 2) \to \text{SP}^n$ with fiber $\text{SO}(8)$.

(ii-b) We can define $\sigma(\tau(\hat{V}, T(f)))$ by (ii-a), and hence the assertion follows from Lemma 7.3 and Proposition 7.7.

Let us see that the singularities of symbol $I([f])$ of $F$ actually detect the stable homotopy class $\omega_0([f]) \in \pi^S_6$ for $\pi^S_2 \approx \pi^S_4 \approx \mathbb{Z}(2)$, $\pi^S_3 \approx \mathbb{Z}(24)$ and $\pi^S_5 \approx \mathbb{Z}(240)$. We use the above notation. Note that in dimensions $n = 1, 2$, stable tangent bundles $\tau_V$ and $\tau_W$ are trivial (note that an orientable 3-manifold is parallelizable).

Case: $n = 1$ We may take $N = S^1$. We have by [31, Section 38] that $\sigma(\tau_V, T(f)) = \sigma(\tau(\hat{V}, T(f)))$ is equal to the second Stiefel Whitney class $w_2(\hat{V})$. This is, as an invariant in $\mathbb{Z}(2)$, coincides with the number of the singularities of the symbol $(1, 1, 0)$ of $F$ modulo 2, since the Thom polynomial is the dual class of $S^1(F)$. Hence, we have $m(1) = 0$.

Case: $n = 2$ It follows from Theorem 7.1 that we can choose an $\Omega^{[1,1,0]}$-regular map $F$ for a fold-map $f$. Hence, $\sigma(\tau_V, T(f))$, namely, $\sigma(\tau(\hat{V}, T(f)))$ lies in $H^2(\hat{V}; \pi_1(\text{SO}(f)))$ and coincides with $w_2(\tau(\hat{V}, T(f)))$ by Remark 7.2. Suppose that $w_2(\tau(\hat{V}, T(f)))$ vanishes. Since $\pi_2(\text{SO}(3)) \approx \{0\}$, the second obstruction in $H^3(\hat{V}; \pi_2(\text{SO}(3)))$, for $\tau(\hat{V}, T(f))$ to be trivial, always vanishes. This implies that $\omega_0([f]) \neq 0$ if and only if $w_2(\tau(\hat{V}, T(f)))$ does not vanish for any choice of $V$ and $F$ (consequently, $I([f]) = (1, 1, 0)$).

Case: $n = 3$ By Theorem 7.4 we have that $m(3) = j_1 = 24$ and $a_1 = 2$. By Lemmas 7.3, 7.8 and Proposition 7.7, we have that $\sigma(\tau(\hat{V}, T(f)))$ is equal to $\pm P_1(\tau(\hat{V}, T(f)))/2$. By [34] the Thom polynomial of $\Sigma^2(\tau(\hat{V}, T(f)), \varepsilon^3)$ is equal to $P_1(\tau(\hat{V}, T(f)))$. Consequently, an $\Omega^2$-regular map $F$, for an element $\omega_0([f]) \in \pi^S_3 \approx \mathbb{Z}(24)$, has the corresponding algebraic number of the singular points of the symbol $(2, 0)$ modulo 24. We remark that this number coincides with the $e$-invariant introduced in [1] and [32].

Case: $n = 7$ By [27, Section 7, Discussions and computations], an element of $\pi^S_7 \approx \mathbb{Z}(240)$ is detected by $P_2(\tau(\hat{V}, T(f)))/6$ modulo 240. We note codim $\Sigma^3(n + 1, n + 1) = 9$. By Theorem 7.4 we have that $m(7) = j_2 = 240$ and $a_1 = 1$. Let $f : S^7 \to S^7$ be a fold map with $\omega_0([f]) \neq 0$. If $P_1(\tau(\hat{V}, T(f)))$ does not vanish, then
we have that \( I([f]) = (2, 0) \) or \((2, 1)\) by Lemma 7.8 (i). If \( P_1(\tau \hat{V}, T(f)) \) vanishes, then \( V \) becomes parallelizable by Lemma 7.8 (ii). Since \( P_2(\tau \hat{V}, T(f)) \) does not vanish for any cobordism \( F : (V, S^7) \to (S^7 \times [0, 1], S^7 \times 0) \) with \( F|S^7 = f \), the secondary obstruction \( (\Omega_G^{1,0}, \phi(\tau \hat{V}, T(f))) \) does not vanish by Lemmas 7.3, 7.8 and Proposition 7.7. Therefore, we have \( I([f]) = (2, 0) \) or \((2, 1)\).

Let \( IV_4 = (x^2 + y^2, x^4) \) and \((x^2 + y^3, xy^2)\) stand for the orbit of the \( k \)-jets of the \( C^\infty \)-stable germs \((\mathbb{R}^8, 0) \to (\mathbb{R}^8, 0)\) of the symbols \((2, 0)\) and \((2, 1)\), which are characterized by the local algebras \( \mathbb{R}[[x, y]/(x^2 + y^2, x^4) \) and \( \mathbb{R}[[x, y]/(x^2 + y^3, xy^2)\), by the group action of \( \text{Diff}(\mathbb{R}^8, 0) \times \text{Diff}(\mathbb{R}^8, 0) \) respectively. They have been defined in \( \text{[32]} \). If we apply an elaborated result in \( \text{[20]} \) to the jet bundle \( J^k(\tau \hat{V}, T(f)), \varepsilon^k_\hat{V}, T^k \), then we obtain the cycle \( \langle x^2 + y^3, xy^2 \rangle - 2IV_4 \) under the integer coefficients of the Vassiliev complex \( \text{[20]} \) Theorem 2.7) and the Thom polynomial of \( \langle x^2 + y^3, xy^2 \rangle - 2IV_4 \) is equal to \( 9P_2(\tau \hat{V}, T(f)) \) \( \text{[20]} \) Section 3). We denote the algebraic numbers of the singular points of types \((x^2 + y^3, xy^2)\) and \( IV_4 \) by \( A \) and \( B \) respectively. Then \( A - 2B \) is divisible by \( 6 \cdot 9 = 54 \) and \((A - 2B)/54 \) corresponds to the stable homotopy class \( \omega_0([f]) \).

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