A BRIANÇON-SKODA TYPE RESULT FOR A NON-REDUCED ANALYTIC SPACE

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Abstract. We present here an analogue of the Briançon-Skoda theorem for a germ of an analytic space $Z$ at $x$, such that $O_{Z,x}$ is Cohen-Macaulay, but not necessarily reduced. More precisely, we find a sufficient condition for membership of a function in a power of an arbitrary ideal $a^l \subset O_{Z,x}$ in terms of size conditions of Noetherian differential operators applied to that function. This result generalizes a theorem by Huneke in the reduced case.

1. Introduction

The Briançon-Skoda theorem, [12], states that for any ideal $a \subset O_{\mathbb{C}^n,0}$ generated by $m$ germs, we have the inclusion $a^{\min(m,n)+l-1} \subset a^l$, where $\overline{I}$ denotes the integral closure of $I$. The generalization to an arbitrary regular Noetherian ring was proven algebraically in [17], and for rational singularities in [1].

Huneke, [14], showed that for a quite general Noetherian reduced local ring $S$, there is an integer $N$ such that $a^{N+l-1} \subset a^l$, for all ideals $a$ and $l \geq 1$. In particular this applies when $S = O_{V,x}$, the local ring of holomorphic functions of a germ of a reduced analytic space $V$. This case of Huneke’s theorem was recently reproven analytically, [7]. Let $|a|^2 = \sum_{i=1}^{m} |a_i|^2$, which up to constants does not depend on the choice of the generators $a_i$. Since a function $\phi$ in $a^M$ is characterized by the property that $|\phi| \leq C|a|^M$, [10], an equivalent formulation of the theorem is that $\phi$ belongs to $a^l$ whenever $|\phi| \leq C|a|^{N+l-1}$ (on $V$).

We will consider a germ of an analytic space, i.e. a pair $(Z, O_{Z,x})$, or just $Z$ for brevity, of a germ of an analytic variety $Z$ at a point $x$ and its local ring $O_{Z,x} = O_{\mathbb{C}^n,0}/\mathcal{J}$ for some $\mathcal{J} \subset O_{\mathbb{C}^n,0}$ such that $Z(\mathcal{J}) = Z$. The reduced space $Z_{\text{red}}$ has the same underlying variety $Z$, but the structure ring is $O_{Z,x}/\sqrt{0} = O_{\mathbb{C}^n,0}/\sqrt{\mathcal{J}}$. We assume throughout this paper that $O_{Z,x}$ is Cohen-Macaulay, which in particular gives that $\mathcal{J}$ has pure dimension.

The aim of this paper is to find an appropriate generalization of the Briançon-Skoda theorem to this setting – when $S = O_{Z,x}$. Now that we have dropped the assumption that $Z$ is reduced, the situation becomes different; the integral closure of any ideal contains the nilradical $\sqrt{0}$ by

\begin{flushleft}
\textit{Date:} January 2, 2010.
\textit{2000 Mathematics Subject Classification.} 32C30, 13B22, 32A26.
\end{flushleft}
definition, so $\overline{a^n} \subset a$ can only hold if $\sqrt{0} \subset a$, or equivalently, if each element of $\mathcal{O}_{Z,x}$ that vanishes on $Z$ belongs to $a$. Clearly this does not hold for any $a$. In the following example we consider the most simple non-reduced space. It will nevertheless help illustrate some general notions, and also our main result, Theorem 1.2.

Example 1.1. Consider the analytic space $Z$ whose underlying space $Z_{red}$ is \{\(w = 0\)\} = $\mathbb{C}^{n-1} \subset \mathbb{C}^n$, such that $\mathcal{O}_Z = \mathbb{C}[[z_1, \ldots, z_{n-1}, w]]/w^k$, $k > 2$. The nilradical is $(w)$, and is not contained in $a = (w^2)$. It may be helpful to think of the space $Z$ as $\mathbb{C}^{n-1}$ with an extra infinitesimal direction transversal to $Z$, and its structure sheaf being the $k$:th order Taylor expansions in that direction. For each $f \in \mathcal{O}_{Z,x}$ we have
\[
f(z, w) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial w^i}(z, 0)w^i,
\]and
\[\mathcal{O}_{Z,x} \simeq \bigoplus_{0}^{k-1} \mathcal{O}_{Z_{red},x}.
\]
Although the function $w$ is identically zero on $Z$ (and therefore it most definitely satisfies $|w| \leq C|a|^M$ for any $M$), but the element $w$ does not belong to $a = (w^2)$. Since $Z$ is non-reduced, evaluating $w$, or any other element, as a function on $Z$ does not give enough information to determine ideal membership. We also have to take into account the transversal derivatives.

A germ of a holomorphic differential operator $L$ is called Noetherian with respect to an ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ if $L\phi \in \sqrt{J}$ for all $\phi \in J$. We say that $L_1, \ldots, L_M$ is a defining set of Noetherian operators for $J$, if $\phi \in J$ if and only if $L_1 \phi, \ldots, L_M \phi \in \sqrt{J}$. The existence of a defining set for any ideal $J$ is due to Ehrenpreis [13] and Palamodov [19], see also [10], [15] and [18]. In the example above, a defining set is $1, \partial/\partial w, \ldots, \partial^{k-1}/\partial w^{k-1}$.

If $L$ is Noetherian with respect to $J$, then $L\psi$ is a well-defined function on $Z_{red}$ for any $\psi \in \mathcal{O}_{Z,x}$, and $L$ induces an intrinsic mapping $L : \mathcal{O}_{Z,x} \rightarrow \mathcal{O}_{Z_{red},x}$. Let $\mathcal{N}(Z)$ be the set of all such mappings; this set does not depend on the choice of local embedding of $Z$. If $(L_i)$ is a defining set for $J$, then by definition any element $\phi \in \mathcal{O}_{Z,x}$ is determined uniquely by the tuple of functions $(L_i \phi)$ on $Z_{red}$, cf. Example 1.1. This fact indicates that it is natural to impose size conditions on the whole set $(L_i \phi)$ to generalize the Briançon-Skoda theorem:

**Theorem 1.2.** Let $Z$ be a germ of an analytic space such that $\mathcal{O}_{Z,x}$ is Cohen-Macaulay. Then there exists an integer $N$ and operators $L_1, \ldots, L_M \in \mathcal{N}(Z)$ such that for all ideals $a \subset \mathcal{O}_{Z,x}$ and all $l \geq 1$,
\[
|L_j \phi| \leq C|a|^{N+l-1}, \quad 1 \leq j \leq M,
\]
implies that $\phi \in a^l$. 

Remark 1.3. Let $\mu_0$ be the maximal order of $L_j$ for $1 \leq j \leq M$. If $\phi \in a^{l+\mu_0}$, then $L_j \phi \in a^l$, so for $N = -\mu_0 + 1$ and $l >> 0$, the inequalities (1) are necessary conditions for $\phi$ belonging to $a^l$. Note also that the special case $a = (0)$ of the theorem means that $L_1, \ldots, L_M$ is a defining set of Noetherian operators for $J$, if $L_j$ is any representative for $L_j$ in the ambient space.

We will now prove Theorem 1.2 by elementary means in the case of Example 1.1:

Example 1.4. As we saw in the previous example, a set of defining differential operators for the ideal $(w^k)$ consists of $L_j = \partial^j / \partial w^j$, $0 \leq j \leq k - 1$. We assume that $|L_j \phi| \leq |a|^{r+(k-1)-j}$, (2) for all $z \in \mathbb{C}^{n-1}$, $w = 0$, where $r = \min(n-1,m)$. We will allow ourselves to abuse notation; for example, we will write simply $a$ when we actually are referring to some element that belongs to $a$. Using Briançon-Skoda for $\mathbb{C}^{n-1}$ we get

$$\frac{\partial^j \phi}{\partial w^j} = a^{k-j} + k_j w, \quad k_j \in \mathcal{O}_{\mathbb{C}^n,0}. \tag{3}$$

We will show that

$$\phi = \sum_{i=0}^p w^i a^{k-i} + g_p w^{p+1}, \quad g_p \in \mathcal{O}_{\mathbb{C}^n,0} \tag{4}$$

holds for $p \leq k - 1$. For $p = k - 1$ it implies that $\phi \in a$ and for $p = 0$ it reduces to (3) with $j = 0$. Assume that (4) holds for some $p < k - 1$. Let us differentiate (4) $p + 1$ times with respect to $w$, and compare the result with (3) for $j = p + 1$. This gives

$$g_p \in \sum_{i=0}^p w^i a^{k-i-p-1} + (w).$$

Now we substitute this back into (4), and get

$$\phi \in \sum_{i=0}^{p+1} w^i a^{k-i} + (w^{p+2}).$$

By induction (4) holds also for $p = k - 1$. This proves the theorem for $l = 1$, and the same argument works for all $l$. Moreover we get that $N = r + k - 1$ works in Theorem 1.2. This is optimal as the following example shows, i.e. $N = r + k - 1$ is the Briançon-Skoda number for this particular analytic space. We get of course back the usual Briançon-Skoda number $N = r$ in the reduced case $k = 1$.

Example 1.5. To show that the example above is optimal, we need to find $\phi_p$, for each $0 \leq p \leq k - 1$, such that $\phi_p \notin a$ and $|\partial^j \phi_p| \leq |a|^{r+(k-1)-j}$ holds for $j \neq p$ and $|\partial^p \phi_p| \leq |a|^{r+(k-1)-p-1}$. Take $n = 2$
and \( a = (z + w) \). Then \( r = 1 \), since \( n - 1 = m = 1 \). A suitable choice is now \( \phi_p = w^p z^{k-1-p} \). This does not belong to \( a \), because if it did we would have \( w^p z^{k-1-p} = (z + w)(a_0(z) + \cdots + a_{k-1}(z) w^{k-1}) \), which would give \( a_0 = \cdots = a_{p-1} = 0 \) and \( za_p = z^{k-1-p} \), \( za_{p+1} = -a_p \), \( za_{p+2} = -a_{p+1} \), etc, so \( a_{k-1} = \pm 1/z \). This is a contradiction since \( a_{k-1} \) is holomorphic at \( 0 \in \mathbb{C}^n \).

In Section \([5]\) we will apply the proof of the main theorem, with a minor modification, to the case where \( Z_{\text{red}} \) is smooth. We then recover the optimal result as in Examples \([1.4]\) and \([1.5]\) when \( J = (w^k) \).

Although the formulation of Theorem \([1.2]\) is intrinsic, we will choose an embedding and work in the ambient space exclusively. If \( \phi \in \mathcal{O}_{\mathbb{C}^n,0} \) annihilates a certain vector-valued residue current, \( R^{a^\vee} \land R^Z \), then it turns out that it belongs to \( a^\vee + J \) (i.e. the image of \( \phi \) in \( \mathcal{O}_{Z,x} \) belongs to \( a^\vee \)); this follows by solving a certain sequence of \( \overline{\partial} \)-equations. Alternatively, one can also use a division formula to obtain an explicit integral representation of the membership. This way of proving ideal membership is described in \([7]\) and goes back to \([2]\) and \([4]\). We conclude that the proof is reduced to showing that \( \phi R^{a^\vee} \land R^Z = 0 \) whenever \( \phi \) satisfies \((\text{II})\). The vanishing of this current is proved in Section \([4]\).

### 2. Coleff-Herrera currents

Assume that \( Z \) is a germ of an analytic variety of pure codimension \( p \) in \( \mathbb{C}^n \).

**Definition 2.1.** A Coleff-Herrera current on \( Z \) is a \( \overline{\partial} \)-closed current \( \mu \) of bidegree \((0,p)\) with support on \( Z \) that is annihilated by \( \phi \) if \( \phi \in \mathcal{O}_{\mathbb{C}^n,0} \) vanishes on \( Z \). One also requires that \( \mu \) satisfies the standard extension property (SEP), that is, \( \mu = \lim_{\varepsilon \to 0} \chi(|h|^2/\varepsilon^2)\mu \), if \( h \in \mathcal{O}_{\mathbb{C}^n,0} \) does not vanish identically on any component of \( Z \), and \( \chi \) is a smooth cut-off function approximating \( 1_{[1,\infty)} \).

The set of all Coleff-Herrera currents on \( Z \) is an \( \mathcal{O} \)-module which we denote by \( \mathcal{CH}_Z \). It is easy to see for any \( \mu \in \mathcal{CH}_{Z,x} \), that \( \text{ann} \mu \) is a pure-dimensional ideal whose associated primes correspond to components of \( Z \). There is a direct way, due to J-E. Björk \([10]\), to obtain a defining set \( L_1 \ldots L_\nu \) for the ideal \( J = \text{ann} \mu \). Furthermore, the operators \( L_j \) are related to \( \mu \) by the formula \((\text{III})\). To obtain the formula \((\text{III})\), we need to consider

*Björk’s proof.* Let \( \zeta \) denote the first \( n - p \) coordinates in \( \mathbb{C}^n \) and \( \eta \) the \( p \) last. It follows from the local parametrization theorem once one can find holomorphic functions \( f_1 \ldots f_p \), forming a complete intersection, such that \( Z \) is a union of a number of irreducible components of \( V_f = \{ f_1 = 0 \} \ldots = \{ f_p = 0 \} \).
\[ \cdots = f_p = 0 \}. \] Moreover,
\[ z = \zeta, \quad w = f(\zeta, \eta) \]
are local holomorphic coordinates outside the hypersurface \( W \) defined by
\[ h := \det \frac{\partial f}{\partial \eta}. \]
By possibly rotating the coordinates \((\zeta, \eta)\), we can make sure \( h \) will not vanish identically on any component of \( Z \). Since \( \mu \) is a Coleff-Herrera current on the complete intersection \( V_f \), we get by Theorem 4.2 in [5],
\[ \mu = A \left[ \frac{\partial}{\partial f_1^{1+M_1}} \wedge \cdots \wedge \frac{\partial}{\partial f_p^{1+M_p}} \right], \]
for some integers \( M_i \) and strongly holomorphic function \( A \). A basic fact is that for a \((n-p, n-p)\) test form \( \xi \),
\[ d\eta \wedge \left[ \frac{1}{\eta_1^{1+M_1}} \wedge \cdots \wedge \frac{1}{\eta_p^{1+M_p}} \right] \xi = \int_{\eta=0} \partial_{\eta_1}^{M_1} \cdots \partial_{\eta_p}^{M_p} \xi, \]
where the derivative symbols refer to Lie derivatives. Now let \( d\eta \wedge \xi \) be an arbitrary \((n, n-p)\) test form supported outside \( W \) so that \((z, w)\) are coordinates on its support. Then, since \( d\eta \wedge \xi = \frac{1}{h} dw \wedge \xi \), the Leibniz rule gives that
\[ \mu.d\eta \wedge \xi = \int_{w=0} \sum_{\alpha_j \leq M_j} \frac{c_\alpha}{h} \partial_w^{M-\alpha}(A) \partial_w^\alpha \xi, \]
where \( M = (M_1, \ldots, M_p) \). We now want to express \( \phi \mu.d\eta \wedge \xi \) in terms of derivatives with respect to the variables \( \eta_i \) instead of \( w_i \). By the chain rule,
\[ \frac{\partial}{\partial w_j} = \frac{1}{h} \sum_k \gamma_{jk} \frac{\partial}{\partial \eta_k}. \]
Combining this with (8), we thus get operators \( Q_\alpha \) so that
\[ \phi \mu.d\eta \wedge \xi = \int_{w=0} \frac{1}{h} \sum_{\alpha_j \leq M_j} Q_\alpha(\phi) \partial_w^\alpha(\xi) = \]
\[ = \int_Z \sum_{\alpha_j \leq M_j} \frac{1}{h^{N_0}} L_\alpha(\phi) K_\alpha(\xi), \]
where \( N_0 \) is some (large) integer and \( L_\alpha = h^{N_0} Q_\alpha \) and \( K_\alpha = h^{N_0} \partial_w^\alpha \) are differential operators with respect to the variables \( \eta \) that are holomorphic across \( W \).

Clearly, the values of \( \partial_w^\alpha \xi \) can be prescribed on \( \{ w = 0 \} \). Therefore \( \phi \mu = 0 \) on \( Z \setminus W \) if and only if \( L_\alpha(\phi) = 0 \) on \( Z \setminus W \) for all \( \alpha \leq M \),
but by continuity and SEP, these relations hold if and only if they hold across $W$.

Let $\mathcal{J} \subset \mathcal{O}_{\mathbb{C}^n,0}$, assume that $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ is Cohen-Macaulay, and let $Z = Z(\mathcal{J})$. It is well-known, that there is a finite set $\mu_1, \ldots, \mu_n \in \mathcal{C}H_Z$ such that $\mathcal{J} = \bigcap \text{ann} \mu_j$. Hence we get a defining set for $\mathcal{J}$ by taking the union of the defining sets for $\text{ann} \mu_j$. In the next section, we will use the construction in [9] of a set $\{\mu_j\}$ associated to $\mathcal{J}$ as above. It is proved in [6] that this set actually generates the $\mathcal{O}$-module of all $\mu \in \mathcal{C}H_Z$ such that $\mathcal{J}\mu = 0$.

Finally, we extend, for later convenience, the formula (9) across $W$, i.e. so that the support of $\xi$ may intersect $W$. To simplify the argument we assume, without loss of generality, that $\phi = 1$. Then (9) agrees with

$$\lim_{\delta \to 0} \chi(|h|^2/\delta^2) \tau - \tau = 0.$$  

By expanding

$$\chi(|h|^2/\delta^2) \tau, \xi = \int_Z \frac{\mathcal{L}(\chi(|h|^2/\delta^2)\xi)}{h^{N_0}},$$

we get one term when all derivatives of $\mathcal{L}$ hit $\xi$, and clearly this term is precisely $\tau, \xi$ in the limit. We will now explain why all other contributions vanish; all these terms contain derivatives of $\chi(|h|^2/\delta^2)$ as a factor, and such a factor can be written as a sum of terms

$$\chi^{(k)} \left( \frac{|h|^2}{\delta^2} \right) \cdot \left( \frac{|h|^2}{\delta^2} \right)^k \frac{\sigma}{h^\kappa}.$$ 

for some integers $k$ and $\kappa$ and a smooth function $\sigma$. Let us define $\chi_0(x) := x^k \chi^{(k)}(x)$. This function is identically zero in some neighbourhoods of 0 and infinity. After applying the following lemma to $\chi_0$, we are done, since $\chi_0(\infty) = 0$.

**Lemma 2.2.** If $\tilde{\chi}$ is any bounded function on $[0, \infty)$ that vanishes identically near 0 and has a limit at $\infty$, then

$$\lim_{\delta \to 0} \tilde{\chi}(|h|^2/\delta^2) \frac{1}{f} \wedge [Z] = \tilde{\chi}(\infty) \frac{1}{f} \wedge [Z]$$

for any $h, f \in \mathcal{O}_{\mathbb{C}^n,0}$ that are generically non-vanishing on $Z$.

For a proof see [11], Lemma 2.
3. Residue currents associated to ideals

A summary of the machinery of residue currents needed to prove the Briançon-Skoda theorem on a singular variety, or in the present setting, appears in [7], Section 2. For the convenience of the reader, we give here an even shorter summary.

In [9] a method was presented for constructing currents $R$ and $U$ associated to any generically exact complex of hermitian vector bundles $(E_k)$ with maps $(f_k)$. These currents take values in the vector bundle $\text{End}_E := \text{End}(\bigoplus E_k)$. We define $f = \oplus f_j$ and $\nabla f = f - \partial$. The latter is an operator acting on $\text{End}_E$-valued forms and currents, and $U$ is related to $R$ by $\nabla f U = 1 - R$. Since $\nabla f_j = 0$, it therefore follows that that $\nabla f R = 0$. We restrict our attention to the case rank $E_0 = 1$ and define the ideal $J = \text{Im}(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$. Then $\text{ann} R \subset J$, but in general the inclusion is proper. However, a main result of [9] is that if $\mathcal{O}(E_k)$ with maps $(f_k)$ is a resolution of $\mathcal{O}_{\mathbb{C}^n}/J$, that is, an exact complex of sheaves, then $\text{ann} R = J$.

Now suppose that we start with any ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ with the assumptions mentioned before, that is, $J$ has pure codimension $p$ and $\mathcal{O}_{\mathbb{C}^n,0}/J$ is Cohen-Macaulay. We denote by $R^Z$ the current obtained from a resolution so that $\text{ann} R^Z = J$. We have a decomposition $R^Z = \sum_{k=0}^N R_k^Z$ so that $R_k^Z$ has bidegree $(0,k)$. The class of so called pseudomeromorphic currents $\mathcal{P}\mathcal{M}$ was introduced in [8]. Since $R_k^Z \in \mathcal{P}\mathcal{M}$ has support on $Z = Z(J)$, which has codimension $p$, $R_k^Z$ must vanish for $k < p$ by Corollary 2.4 in [8], so $R^Z = R^Z_p + R^Z_{p+1} + \ldots$. The same corollary implies that $R^Z$ has SEP. The Cohen-Macaulay condition means that there is a resolution of length $p$ of $\mathcal{O}_{\mathbb{C}^n}/J$, so $R_k^Z$ vanishes also for $k > p$, and thus $R^Z = R^Z_p$. Now $\nabla f R^Z = 0$ implies that $\overline{\partial R^Z} = 0$ by a consideration of bidegrees. By Proposition 2.3 in [8], $\overline{\partial R^Z} = \sqrt{J}$ annihilates $R^Z$. Collecting all information so far, we get that $R^Z$ is a tuple of Coleff-Herrera currents, since its components satisfy the three required conditions. Thus $J = \text{ann} R^Z = \bigcap \text{ann} \mu_j$, where $\mu_j$ are the components of $R^Z$, so by the proof of Björk’s theorem in Section 2 we have a defining set of operators for $J$, such that the current $R^Z$ can be represented as a principal value integral on $Z$ in terms of these defining operators, [9].

Next we construct currents $R^d_l$ and $U^d_l$ associated to a complex that is the Koszul complex for $l = 1$ and a slight modification otherwise. The current $R^d_l$ is of Bochner-Martinelli type and $\text{ann} R^d_l \subset a^d_l$, but in general we do not have equality. The restriction $u^{d} \mid U^{d}$ to the complement of $Z(a)$ is displayed in [8], and $R^d_l$ is obtained as the limit of $R_l^d = (1 - \chi^l_\delta) + \overline{\partial} \chi^l_\delta \wedge u^d$, where $\chi^l_\delta = \chi((a)^{2\delta_l}/\delta)$. We want to form the product of the currents $R$ and $R^d_l$, which corresponds to
restricting $R^{a_i}$ to $Z$ in a certain way. We thus define

$$R^{a_i} \wedge R^Z = \lim_{\delta \to 0} R^{a_i}_\delta \wedge R^Z.$$  

This product takes values in the product of the two complexes of the two factors, see [7].

It follows by Proposition 2.2 in [7] that $\phi R^{a_i} \wedge R^Z = 0$ for $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ implies that $\phi \in \mathcal{F} + a^i$, that is, the image of $\phi$ in $\mathcal{O}_{Z,x}$ belongs to $a^i$. Although $R^{a_i} \wedge R^Z$ is a current in $\mathbb{C}^n$, $\phi R^{a_i} \wedge R^Z$ depends only on the image of $\phi$. The proof of Theorem 1.2 is reduced to showing that indeed $\phi R^{a_i} \wedge R^Z = 0$ if $\phi$ satisfies (11), which we now begin to prove.

4. Proof of Theorem 1.2

Since $R^Z$ is, under our assumptions, a tuple of Coleff-Herrera currents, we can use (9) with $\xi = \omega \wedge R^{a_i}$ to calculate $\phi R^{a_i} \wedge \mu \omega$, where $\mu$ is a component of $R^Z$ and $\omega$ is a test form. We now choose a resolution $X \xrightarrow{r} Z$ such that $X$ is smooth and $\pi^*h$ is locally a monomial. Thus to show that $\phi R^{a_i} \wedge R^Z = 0$, it suffices by linearity to show that

$$I_\delta := \int_X \frac{\sigma}{s_1^{1+n_1} \cdots s_{n-p}^{1+n_{n-p}}} \wedge \pi^*(L_\gamma \phi) \pi^* \left( \partial_{\eta}^a R^{a_i}_\delta \right) \to 0,$$

where $\sigma$ is a smooth form with compact support. This integral is really a principal value integral with a smooth cut-off function as in (10). Recall that $|\alpha| \leq |M| = \sum M_i$, see (7). We want to integrate (13) by parts. The reciprocal of the monomial occurring in this integral is just a tensor product of one variable distributions and

$$\frac{\partial}{\partial z} \left[ \frac{1}{z^m} \right] = -m \left[ \frac{1}{z^{m+1}} \right], \quad m \geq 1.$$

This yields indeed that

$$I_\delta = \int_X \frac{ds_1 \wedge \cdots \wedge ds_{n-p}}{s_1 \cdots s_{n-p}} \wedge \partial_{\gamma}^{(m_1, \ldots, m_{n-p})} \left( \sigma \wedge \pi^*(L_\gamma \phi) \pi^* \left( \partial_{\eta}^a R^{a_i}_\delta \right) \right).$$

We now extend $\pi$ to a resolution $X' \xrightarrow{\gamma} X \xrightarrow{\pi} Z$ that principalizes $a$, and we call the generator in $\mathcal{O}_{X'}$ for $a_0$, i.e. $\gamma^* \pi^* a_j = a_0 a'_j$ and $|\gamma^* \pi^* a| \sim |a_0|$. Note that the form $\frac{ds_1 \wedge \cdots \wedge ds_{n-p}}{s_1 \cdots s_{n-p}}$ becomes a sum of similar forms when we pull it back to $X'$. If we show, for all integers $k_j \leq n_j$, $1 \leq j \leq n - p$, that the form

$$\gamma^* \left[ \partial_{\gamma}^{(k_1, \ldots, k_{n-p})} \left( \pi^*(L_\gamma \phi) \pi^* \left( \partial_{\eta}^a R^{a_i}_\delta \right) \right) \right]$$

is bounded on $X'$ by a constant independent of $\delta$, then an application of dominated convergence gives that $I_\delta \to 0$, and thereby concludes the proof. We assume that $k_j = n_j$ for $1 \leq j \leq n - p$. It will become apparent that this is the worst case.
The philosophy will be to write the expression in (15) as a product of factors of three types; factors whose modulii are equivalent to \(|a_0|\), or to \(|a_0|^{-1}\), and remaining factors, which we require to be bounded. We count the number of factors of the first type minus the number of factors of the second type, that is, the number of zeroes minus the number of singularities. Let us call this difference homogeneity. Since the factors of the third type may also be of the first type, the homogeneity depends on our factorization. However, we only want to find one factorization such that the homogeneity is non-negative, because then (15) is indeed bounded. Nothing will be lost if we instead get a finite number of terms which can be factorized in this way.

We examine first the factor \(L_i\phi\). Then we consider the second factor, \(\partial^\alpha\eta\), and add the two results.

Since \(X\) is a manifold, the usual Briançon-Skoda theorem and (11) gives that \(\pi^*(L_i\phi) \in (\pi^*a)^{N-\varrho+l}\), where \(\varrho = \min(m, \dim Z)\). The conclusion is that the term in (15) for which no derivatives with respect to \(s\) hit \(\pi^*(L_i\phi)\) gives a contribution of \(N - \varrho + l\) to the homogeneity. A term for which \(\pi^*(L_i\phi)\) is hit by \(k\) derivatives gives a contribution reduced by \(k\), since \(\partial^a a^{M+1} \subset a^M\) for any \(M\).

We proceed now with the second factor. For simplicity we begin with the case \((n_1, \ldots, n_{n-p}) = 0\), so that no derivatives appear with respect to the coordinates \(s\). We will then consider the general case and see that the derivatives \(\partial^s\), have the same effect on homogeneity as \(\partial^\alpha\eta\) – namely each derivative decreases the homogeneity by one.

From [3] we know that

\[
\nu^a = \sum_{\beta_1 + \cdots + \beta_p = k-1} \left( \frac{\bigwedge_{i=1}^l (\sum_{j=1}^m \bar{a}_j e_j^i) \wedge \bigwedge_{i=1}^l (\sum_{j=1}^m \partial \bar{a}_j \wedge e_j^i)^{\beta_i}}{|a|^{2(k+l-1)}} \right),
\]

where \(e_j^i\) are frames of trivial bundles, and \(a_j, 1 \leq j \leq m\), are generators for \(a\). For simplicity we consider only the most singular term of \(R^a_\delta = (1 - \chi^a_\delta) - \partial \chi^a_\delta \wedge u^a\), that is, the term of \(\partial \chi^a_\delta \wedge u^a\) for which \(k = \varrho\) above. All other terms can be treated similarly, but easier. We denote the most singular term by \(\nu_\delta\). Using the notation from Example [1,4] we get

\[
\nu_\delta = \chi' \left( \frac{|a|^2}{\delta^2} \right) \left( \frac{a \partial a}{\delta^2} \right) \wedge \sum_{|\beta| = \varrho-1} \frac{\theta_\beta(a)}{|a|^{2(\varrho+l-1)}},
\]

where

\[
\theta_\beta(a) = \bigwedge_{i=1}^l (\sum_{j=1}^m \bar{a}_j e_j^i) \wedge \bigwedge_{i=1}^l (\sum_{j=1}^m \partial \bar{a}_j \wedge e_j^i)^{\beta_i}.
\]

**Lemma 4.1.** The homogeneity of \(\gamma^* \pi^* \theta_\beta(a)\) is \(\varrho + l - 1\).
Proof. Recall that $\gamma^*\pi^*a_j = a_0a_j'$. All terms for which $\mathcal{J}$ hits $\overline{a}_0$ vanish since they contain the wedge product of $(\sum_{i=1}^{m_j}a_j')$ with itself, that is, the square of a 1-form. Thus all remaining terms are divisible by $a_0^{p+l-1}$.

By differentiating (17), one sees that $\partial_0^{\alpha}\nu_0$ is a sum of terms like

\[
(18) \quad \sigma \chi^{(1+|\alpha_1|)} \left( \frac{|a|^2}{2^2} \right) \left( \frac{a^i-|\epsilon|+|\alpha_1|}{\delta^{|\alpha_1|}+2} \right) \wedge \sum_{|\beta|=\phi-1} \frac{\bar{a}^{\alpha_2}\theta_0(a)}{|a|^{2(|\epsilon|+l-1)+2|\alpha_2|}},
\]

where $\sigma$ is a smooth function, and $\epsilon + \alpha_1 + \alpha_2$ are multi-indices such that $\epsilon + \alpha_1 + \alpha_2 \leq \alpha$. It is readily seen that the homogeneity of (18) only depends on $|\epsilon|+|\alpha_1|+|\alpha_2|$ (for $l$ and $\rho$ fixed), except if $1-|\epsilon|+|\alpha_1|$ is negative, but this only improves the estimate. For the worst term it is therefore $-l - \rho - |\alpha|$, due to Lemma 4.1. In other words, no matter which terms the derivatives hit, each one of them reduce the homogeneity by one (or zero). It remains to show that the homogeneity of $\gamma^*\partial_0^{\alpha}\pi^*\nu_0$ drops by at most $|\alpha_0|$, for any $\alpha_0$. This is however quite immediate; we apply $\pi^*$ to (18) and then differentiate with respect to $s$. The ideal $a$ is then replaced by $\pi^*a$ and since $\pi^*\theta_0(a)$ is anti-holomorphic, the differentiation follows exactly the same pattern as when we differentiated with respect to $\eta$ – the derivatives can hit $\chi^{(k)}$, or a power of $\pi^*a$ or $|\pi^*a|^2$, or a smooth function, so we already know that each derivative decreases the homogeneity by no more than one.

The total homogeneity of (18) is the sum of $N - \rho + l$ and $-l - \rho - |\alpha|$ less the number of derivatives with respect both to $s$ and to $\eta$, that is $N - 2\rho - |\alpha| - \sum_{j=1}^{n-p} n_j$. This is non-negative if $N \geq 2 \min(m, \dim Z) + |\alpha| + \sum_{j=1}^{n-p} n_j$, so any such integer $N$ has the desired property of Theorem 4.1, where $|\alpha|$ is maximized over the components of $R^2$ and $\sum_{j=1}^{n-p} n_j$ is maximized over an open covering of $\pi^{-1}(0)$ (recall that these numbers are constructed locally in the resolution).

5. AN IMPROVEMENT FOR SMOOTH ANALYTIC SPACES

We return now to the case where $Z_{\text{red}}$ is smooth. We then get a bound for $N$ and the proof also simplifies significantly. Recall from Section 3 that $R^2 = (\mu_1, \ldots, \mu_\nu)$ is tuple of Coleff-Herrera currents, and each of its components therefore gives rise to holomorphic differential operators $L_\alpha$, and all of these together form a defining set for $\mathcal{J}$ and satisfy (9). Note however that, since $Z_{\text{red}}$ is smooth, we can take $w = \eta$ in (6) so that $h = 1$ in (6). For each differential operator $L$ we denote by $\text{ord}(L)$ its order as a differential operator in the ambient space. This gives also a well-defined notion of order for the induced intrinsic mapping in $\mathcal{N}(Z)$.

**Theorem 5.1.** Let $Z$ be a germ of an analytic space such that $Z_{\text{red}}$ is pure-dimensional and smooth (which implies that $Z$ is Cohen-Macaulay).
Let $L_1 \ldots L_M$ be the Noetherian operators obtained from $R$ as before and let $d$ be the maximal order of these operators. Then if $l \geq 1$, and $a \subset \mathcal{O}_{Z,x}$ can be generated by $m$ elements,

(19) \[ |L_j \phi| \leq C|a|^\min(m, \dim Z) + d - \text{ord}(L_j) + l - 1, \quad 1 \leq j \leq M \]

implies that $\phi \in a^l$.

Note that we have already seen (19) in a slightly different appearance in (2).

**Proof.** We can assume that $Z_{\text{red}} = \mathbb{C}^{n-p} \subset \mathbb{C}^n$, and we call the last $p$ coordinates $w_1, \ldots, w_p$. Clearly these functions form a complete intersection, so if $\mu$ is a component of $R^Z$, we have

$$
\mu = A \left[ \prod_{i=1}^{p} \frac{1}{w_i^{1+M_i}} \right],
$$

for some holomorphic function $A$, cf. (7). We choose $A$ so that the numbers $M_i$ are minimal. Then the distribution order of $\mu$ is precisely $\sum M_i$, which, as we soon shall see, is also the highest order of the operators obtained from $\mu$. By Example 1 in [6], the components of $R^Z$ generate the $\mathcal{O}$-module of Coleff-Herrera currents on $Z$ which are annihilated by $J$. Hence, $d$ is an invariant of $Z$. We proceed as in the general case to construct a set of defining Noetherian operators for the ideal $J$ corresponding to $Z$. Since

$$
\mu. dw \wedge \xi = \int_{\omega} \sum_{\alpha \leq M} \partial^M - \alpha(A) \partial^\alpha w \xi,
$$

where $M = (M_1, \ldots, M_p)$, we get that

(20) \[ \phi \mu. dw \wedge \xi = \int_{\omega} \sum_{\alpha \leq M} L_\alpha(\phi) \partial^\alpha w \xi, \]

and

$$
L_\alpha = \sum_{\alpha \leq \gamma \leq M} C_{(\alpha, \gamma)} \partial_w^{M - \gamma} (A) \partial_{\alpha - \gamma} w,
$$

for some combinatorial constants $C_{(\alpha, \gamma)}$. Clearly $\text{ord}(L_\alpha) = |M| - |\alpha|$ and $L_0$ has the maximal order $\text{ord}(L_0) = |M|$. As before, we need to substitute $\xi = \omega \wedge R^a_{\delta}$ into (21). By principalizing $a$ as before, it follows that the limit of (20) is zero when $\delta$ goes to zero, if the integrand can be factored into two parts, one that is integrable, and one that is bounded. To see that the second factor is bounded we count its homogeneity as before. In the proof of the general case we used that

(21) \[ \partial \chi \left( \frac{|\alpha|^2}{\delta^2} \right) = \chi' \left( \frac{|\alpha|^2}{\delta^2} \right) \left( \frac{a \overline{\alpha}}{\delta^2} \right), \]
which was counted as a term of homogeneity $-1$. It is however well-known that $\partial a_0/|a_0|$ is integrable, so taking it as the first factor, it no longer counts into the homogeneity. Thus, the homogeneity contribution of (21) is now improved to 0. The most singular term of $\partial^a (R^a_0)$ has thus homogeneity $1 - \varrho - l - |\alpha|$ (one unit more than in the non-smooth case). Since $1 - \varrho - l - |\alpha| \geq 1 - \varrho - l - d + \text{ord}(L_\alpha)$, the homogeneity is non-negative if $|L_\alpha(\phi)| \leq C |a|^{\varrho + d - \text{ord}(L_\alpha) + l - 1}$, so we are done. □

Remark 5.2. In the non-smooth case the form in (21) has to be included into the bounded factor, since in that case we already had the form $ds/s$ occupying the integrable factor. Note also that we avoided using Hironaka desingularization and applying the classical Briançon-Skoda theorem in the resolution, which previously cost us $\varrho - 1$ units of homogeneity.

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