Conserved Quantities and Measurements

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Abstract

When a measurement is made on a system that is not in an eigenstate of the measured observable, it is often assumed that some conservation law has been violated. Discussions of the effect of measurements on conserved quantities often overlook the possibility of entanglement between the measured system and the preparation apparatus. The preparation of a system in any particular state necessarily involves interaction between the apparatus and the system. Since entanglement is a generic result of interaction, as shown by Gemmer and Mahler[1], and by Durt[2,3] one would expect some nonzero entanglement between apparatus and measured system, even though the amount of such entanglement is extremely small. Because the apparatus has an enormous number of degrees of freedom relative to the measured system, even a very tiny difference between the apparatus states that are correlated with the orthogonal states of the measured system can be sufficient to account for the perceived deviation from strict conservation of the quantity in question. Hence measurements need not violate conservation laws.

1 Introduction

Virtually all physical systems have a history of interacting with other systems. A generic result of such interaction is entanglement, as demonstrated by Gemmer and Mahler[1] and by Durt[2,3]. Hence, nearly all physical systems are entangled with other systems to some extent. The interactions that generate this entanglement result in the sharing of conserved quantities among the entangled systems. Therefore, any analysis of the effect of measurements on conserved quantities must take this entanglement into account.

To begin such an analysis it necessary to clearly define the quantities that are in question. The definition can be gleaned from the derivation of the conservation laws in quantum theory. To determine whether a quantity, \( q \), is conserved one must integrate the associated observable, \( Q \), over arbitrary states, \( |\psi\rangle \):

\[
\frac{dq}{dt} = \frac{d}{dt} \langle \psi | \hat{Q} | \psi \rangle.
\]  (1.1)
If the observable commutes with the (time-independent) Hamiltonian then the quantity, \( q \), is conserved.

\[
\frac{d}{dt} \langle \psi | \hat{Q} | \psi \rangle = \frac{i}{\hbar} \langle \psi | \hat{H} \hat{Q} - \hat{Q} \hat{H} | \psi \rangle = 0.
\] (1.2)

Thus, the proper definition of a conserved quantity is given by the expression, \( \langle \psi | \hat{Q} | \psi \rangle \). These quantities should not be seen merely as “expectation values”. They are the quantities that quantum theory implies are conserved, provided that they meet the commutation condition. To restrict the definition of ‘conserved quantity’ to situations in which a system that one can feasibly measure is in an exact eigenstate of the measured observable limits the application of conservation laws to a tiny (zero?) fraction of real cases. The idea that these laws hold only in a narrow statistical sense must be replaced by the recognition that quantum theory implies that they can only be tested in a probabilistic manner. The reason for this limitation is that the very small systems that are feasible to measure are almost invariably entangled (to some usually small extent) with much larger systems. To measure these larger systems would require even larger, more complex systems in a very long (possibly infinite) regress.

In addition to using the correct definition of conserved quantities we must also treat all physical systems as quantum systems. Classical boundaries typically act effectively as external potentials. So, to assume that a quantum system has a such a boundary typically involves the implicit assumption that a classical system can act on a quantum system without any reciprocal action by the quantum system on its surroundings. Such an assumption clearly violates any relevant conservation laws ab initio, so there would be no reason to expect these laws to hold.

The assumption that all systems are quantum systems does rule out a Copenhagen interpretation of quantum theory, but it is important to note that it does not commit one to either a decoherence or Everett-type viewpoint. Any adequate interpretation or extension must include some explanation of how the projection postulate and Born probability rule are related to the unitary evolution described by the Schrödinger equation. Everett-type approaches provide one sort of explanation, but both pilot wave theories and stochastic collapse equations offer alternative accounts that still treat all systems as quantum systems. In fact, for ease of illustration the argument presented here will frequently assume that wave function collapse is an actual physical occurrence. But, the way in which the argument would need to be altered to accommodate other interpretations should be readily apparent.

The effect of measurement on conservation laws has been examined previously by a number of authors. Although it reaches a different conclusion from the one argued for here, a recent paper by Carroll and Lodman contains a good discussion of many of the issues involved and includes references to much of the relevant literature. An account that seeks to insure strict conservation in a manner somewhat different from the current proposal has been described by Stoica. A method to restore
conservation of energy after measurement by including quantum clocks has been proposed by Gisin and Cruzeiro [13]. That work also contains an interesting illustration of the Wigner-Araki-Yanase (WAY) theorem [14, 15, 16], which deals with the effect of measurements of observables that do not commute with the Hamiltonian.

The next section will present a simple example illustrating the entanglement between an elementary system and a macroscopic apparatus that is almost always treated as classical. Section 3 deals with some technical issues involving entanglement. Section 4 ties the argument together, and presents some examples of how apparent deviations from strict conservation can be eliminated when all relevant interactions are taken into account. Section 5 is a brief summary.

2 A Simple Example

A Mach-Zehnder interferometer consists of a rectangular array with beam-splitters at two opposite corners and reflecting mirrors at the other two. A photon enters the apparatus through one of the beam-splitters. The two branches then proceed in orthogonal directions to the mirrors at opposite corners. After reflecting from the mirrors they are recombined at the second beam-splitter. By precisely adjusting the path lengths in the two arms of the apparatus it is possible to insure that the photon is either fully reflected or fully transmitted at the final beam splitter. This can be verified by placing detectors in the two possible exit paths. These devices have been used to demonstrate coherence in both classical and quantum contexts.

The (nearly) perfect coherence that can be achieved appears to indicate that the interactions with the beam-splitters and mirrors do not generate any significant entanglement between the apparatus and the photon since such entanglement would disrupt the interference pattern at the final beam-splitter.

Nevertheless, if we believe that interactions conserve momentum it must be acknowledged that the reflection of either of the photon branches by the mirrors or beam-splitters transfers some momentum to those devices. Therefore, the state of these macroscopic systems would be slightly altered. This implies that there must be some very small, but nonzero entanglement between the photon and these systems.

Let us now assume that a detector is placed somewhat downstream from the initial beam-splitter in the reflected branch, and that a detection is recorded. Prior to the detection the beam-splitter would be in a superposition of two very slightly different states, each of which is correlated with a different branch of the photon.

The uncertainty in the position of the reflecting surface of the beam-splitter must be extremely small - many times smaller than the dominant wavelength of the photon - in order to make a coherent recombination possible. This implies that the quantum state of the beam-splitter must span an extremely large spread in momentum. This explains why the exchange of momentum with the reflected branch of the photon makes such a tiny difference in the beam-splitter state.
If we assume that the inner product of the two resultant beam-splitter states has a magnitude of $|1 - \epsilon|$, where $\epsilon \ll 1$, but $\epsilon > 0$, then we can write an expression for the entanglement entropy. The biorthogonal decomposition can be determined by taking the sum and difference of the two beam-splitter states and then normalizing. These are correlated with the sum and difference of the photon branches. Since the photon branches have nearly equal amplitudes in order to achieve a coherent recombination, the entanglement calculation is fairly simple. The result can be approximated as $(\epsilon/2)[1 - \log(\epsilon/2)]$.

The essential point is that the collapse of the photon wave function brought about by the detection also collapses the beam splitter state to the state that is correlated with the reflected branch. Within the remaining branch momentum is completely conserved, and, depending on the details of the initial interaction of the beam-splitter and photon, it can be equal to the momentum in the combined system that existed prior to that initial interaction. The adaptation of this argument to other interpretations of quantum theory is pretty straightforward.

This example has been used in previous works by this author[17, 18], and it has also been used by Marletto and Vedral[19] in an article that makes an argument similar to the one presented here.

I believe that this example shows both how momentum can be conserved in a measurement process, and why it so easy to overlook the entanglement that makes conservation possible. In order to avoid some other misunderstandings surrounding these issues it is necessary to examine some technical aspects of entanglement.

3 Entangled or Factorizable?

Another factor that makes it very easy to overlook small amounts of entanglement between systems is that the question of whether two systems are entangled or not depends on the tensor product structure of the combined Hilbert space. This fact has been discussed by Dugić, Jekníc-Dugić, and Arsenijević,[20 21], and also by Thirring, Berthmann, Köhler, and Narnhofer[22 23]. It can be shown that (pure) entangled states can always be transformed to factorizable states by some unitary mapping[22].

These kinds of transformations are implied by the methods that are used to decouple the equations describing interacting systems. For example, in analyzing a situation with two interacting bodies one typically decouples the equations by redefining the system coordinates and related quantities. These redefinitions can be viewed as a rotation of the coordinate system in configuration space that alters the tensor product structure of the Hilbert space. The original division of the Hilbert space into component subspaces corresponding to each of the bodies is replaced by one in which the components are the center of mass and the reduced mass. If one of

\[1\]I am ignoring complications regarding the definition of photon wave functions.
the bodies has a mass that is much greater than the other, then after solution of the
equations there is a tendency to identify the center-of-mass coordinate with that of
the larger body, and the reduced mass with the smaller one.

We see this in both classical and quantum contexts. Informally, we say that
the earth revolves around the sun, but the more accurate characterization (ignoring
other celestial bodies) is that the earth and sun revolve around their common center
of mass. Something similar occurs in describing the hydrogen atom. In the simplest
calculation of the energy levels, the electron is simply placed in a central force field.

What is important to note about these approaches is that they eliminate interaction
from the description of the situation. The decoupling effectively places a single,
redefined system in an external potential. As already stated the use of external po-
tentials tends to obscure the issues surrounding the question of conservation. To
properly assess whether any conservation laws are violated it is necessary to describe
the evolution of the system strictly in terms of conservative interactions. As the
authors of [22] point out:

“Thus it’s only the interaction, which we consider to determine the density
matrix, or the measurement set-up, which fixes the factorization.”

The decomposition that is fixed in this way is the original one in terms of inter-
acting systems. In this decomposition the systems are entangled as a result of the
interactions, and that entanglement, and the sharing of conserved quantities between
entangled systems must be taken into account.

4 Entanglement Maintains Conservation laws

Interactions generate entanglement. Furthermore, they play the central role both in
defining the tensor product structure of the Hilbert space that determines whether
the wave function is entangled and in selecting the orthogonal basis in which entan-
glement relations are usually examined. This basis coincides with what is often called
the “decoherence basis”, but to avoid confusion it should be labeled simply as the
interaction basis.

Given these considerations the way in which conservation laws are maintained in
each (interacting) branch can be seen as follows. The conservative interactions that
generate a particular entangled branch are responsible for the exchange of conserved
quantities between the systems involved in the interaction. These exchanges redis-
tribute the relevant quantities among the various systems. When a measurement is

2The idea held by some that decoherence can only be defined at the stage at which a macroscopic
measurement apparatus interacts with “the environment” is needlessly restrictive, and at odds with
the way the term is used in the quantum computing community. If decoherence did not set in until
this late stage we would have had general purpose, large scale quantum computers over a decade
ago.
made on a particular target system it collapses the entire branch of the wave function to the branch that corresponds to the measurement result. The apparent nonconservation of a quantity in the target system is simply the selection of the value of that quantity associated with the system in that branch. This value is correlated with the values in the systems with which it has previously interacted. The distribution of the relevant quantities across the entire branch is consistent with strict conservation of the measured quantity. It is the impossibility of measuring the entire branch without the use of incredibly large and complex systems that creates the appearance of a violation of conservation laws.

I believe that the adaptation of this argument to Everett type interpretations or pilot wave theories is straightforward, although it requires that the wave function associated with each major branch must be renormalized following the measurement. If one were to assume that the universe began in a completely entangled state then the foregoing considerations would imply that conservation laws are strictly observed during measurements. Even without the assumption of a completely entangled initial state the long history of interaction behind almost all systems goes a long way toward guaranteeing almost perfect entanglement and conservation. The fact that conserved quantities have been thoroughly shared throughout the total system by those interactions implies that the system will approach a completely entangled state after a sufficient number of interactions take place. The process is similar in some respects to the approach to thermal equilibrium.

Application of these ideas to specific situations involving apparent violation is pretty straightforward. Nevertheless, the widespread belief that measurements necessarily violate conservation laws makes it worthwhile to demonstrate how these laws are maintained in a few situations in which these violations seem to occur. The following three scenarios have been described in an earlier work to make essentially the same point.

In many discussions of this issue an initial wave function for an elementary system is simply assumed. So, for the first scenario consider a freely propagating wave packet, initially localized in a small region. For definiteness let us stipulate that it is a Gaussian with standard deviation, \(a\), centered at the origin, \(x_0 = 0\), at time, \(t_0 = 0\), with net momentum, 0: 

\[
\psi_0(x) = N \cdot e^{-\frac{x^2}{4a^2}} ,
\]

where \(N\) is a normalization factor. By doing a Fourier expansion of the wave function one can see that at \(t_0\) the various frequency components interfere constructively at \(x_0\). The wave packet spreads as time increases, and only the lower frequency components continue to interfere constructively at the origin. Other components that are very close to each other in frequency move away from \(x_0\) with a velocity of about \(\vec{p}/m\) where \(\vec{p}\) is the average momentum associated with the components and \(m\) is the mass of the particle. At a later time, \(t\), this portion of the wave function will continue to interfere constructively.

\[\text{Wave function collapse is usually assumed to renormalize the wave function. However, linear collapse equations require an additional renormalization step.}\]
in a small region around a point, $\vec{x} = \vec{p} \ast t/m$. Other components, with momenta substantially different from $\vec{p}$ will interfere destructively at this location and time. If the particle is detected in this vicinity at this time then it appears that the initial (net) momentum, 0, has been replaced by $\vec{p}$ and that the initial energy of the particle that depended on the standard deviation, $a$, has been replaced by an energy dependent on the momentum, $\vec{p}$. Both the relevant conservation laws seem to be violated. (There can be an additional change in the momentum and energy of the target system brought about by the interaction between it and the measurement instrument. These kinds of changes have been frequently analyzed, and are accounted for by the measurement interaction.)

A little reflection shows that encountering an elementary system in a perfectly factorizable state with specific properties is extremely unlikely, and if one did so there is no way to know what those properties are without making a measurement. In virtually all cases an elementary system with (approximately) specific properties would have been prepared by a macroscopic apparatus. The wave function associated with the elementary system can be represented in any orthogonal basis that one chooses, such as an approximate position basis. Because the properties of the elementary system have been determined by interactions with the preparation apparatus there is entanglement between these two systems. The various orthogonal position states of the small system are correlated with very slightly different states of the apparatus. The detection of the particle at $\vec{x} = \vec{p} \ast t/m$ collapses both the state of the elementary system and that of the apparatus. The apparent violation of the conservation laws in the small system is offset by correlated adjustments in the relevant quantities in the large system.

The second type of example deals with both momentum and angular momentum. The primary conclusion of the original Stern-Gerlach experiments was that angular momentum is quantized, but slight variations of these experiments are often cited as examples of the nonconservation of this quantity. A spin-$\frac{1}{2}$ particle in an x-up state can be represented as being in a superposition of z-up and z-down states. For the purpose of illustration the existence and identification of such a particle will be simply assumed. Although the preparation of this system would be relevant in a full analysis of this situation it does not play a major role in the arguments that are usually offered. This x-up particle is sent through an inhomogeneous magnetic field that separates its z-up and z-down components, and detectors are placed downstream in the path of each of these. The particle is then found to be in just one of these two states. The vertical deflection of the particle appears to be a violation of momentum

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4This can be verified by calculating the wave function at time, $t$, expanding the wave function in the momentum basis, and determining which momentum components dominate the wave function at this point. Alternatively, one can renormalize the (small) detected segment of the wave function at $\vec{x}$ and $t$, and integrate the momentum operator over this renormalized segment.

5The approximate position basis corresponds to the interaction basis because the interaction potentials, including those of the measurement apparatus, depend on distance.
in the z-direction. Most notably, it appears that the conservation of both z and x angular momenta have been violated.

This appearance of a violation is created by allowing the magnetic fields to act on the particle, without acknowledging the action of the particle on the apparatus that generates the magnetic fields. As previously stated, to assess the effect of measurement on conserved quantities, we must assume that, prior to a measurement there is strict conservation of the quantity in question. Therefore, after the the interaction with the particle and prior to detection the apparatus is in a superposition of two very slightly different states. The differences in the momentum states are easy to understand. The upward deflection of the particle is correlated with a slight downward addition to the z-momentum state of the apparatus, and vice versa. A detailed analysis of the exchange of angular momentum is more involved. However, given the assumption of strict conservation at this stage we can see that the branch of the wave function with the z-up particle state must include an apparatus state that is altered from its pre-interaction state to reflect a transfer of x-up angular momentum and z-down angular momentum offsetting the changes in the particle state. The apparatus state in the other branch shows an increase in x-up angular momentum and z-down angular momentum. The detection of the particle renormalizes the branch in which it is detected. One can see that the initial x-up momentum of the particle has been transferred to the apparatus that generates the magnetic field. The changes of momentum and angular momentum in the z-direction in the particle state are offset by equal and opposite changes in the apparatus. The value of all of these quantities in the combined particle-apparatus system is the same after the detection as it was before the initial interaction.

The final example concerns an especially ingenious paradox constructed by Aharonov, Popescu, and Rohrlich[27]. The (apparent) paradox deals with a particle in a box, with the particle being assumed to be in a superposition of relatively low energy states. Initially, the box is completely closed, but there is a mechanism that allows the particle to escape from the box at some later time. The mechanism consists of an opener that moves along the top of the box and inserts a mirror for very brief periods. What makes this example particularly interesting is that when the particle escapes it can be detected (with very low probability) in an energy state that is much higher than any of the initial states that make up the particle’s wave function. The set-up is designed so that the opener cannot supply this extra energy to the particle. The high energy detection is made possible by the fact that the initial superposition is a rather special one that gives rise to superoscillations. These sorts of states can, at times, oscillate over limited regions at frequencies that are much greater than any of the original Fourier components.

In this example the particle and the opener are treated as quantum systems, but the box is regarded as classical. This feature points to the resolution of the paradox. A classical box is an external potential. Because the particle has been prepared in this special initial state there is entanglement between it and the box (and whatever else
was involved in the preparation). In addition, even though the entanglement between the particle and opener is quite small, it is not zero. Thus the opener, the particle, and the box are entangled. When the box and associated apparatus are treated as quantum systems one can see how the detection collapses the state of the total wave function that covers all of the systems involved. The extra energy is provided by the “external” potential.

It is interesting to note that the resolution to this paradox was actually presented by Gemmer and Mahler\[1\] several years before the paradox, itself, was described. They pointed out the interaction necessary to place a confined particle in an assumed particular state, and concluded that:

“Thus it is, strictly speaking, unjustified to describe a particle in a box, which is part of an interacting quantum system, by a wave-function” (italics added; “wave function” should be understood as a factorizable state).

There are many more examples of apparent violations of conservation laws, but it should now be clear that when all systems are treated as quantum systems that can exchange the relevant quantities through interactions, there is good reason to believe that these laws hold in all circumstances.

5 Discussion

The argument presented here is essentially a plea to apply the principles of quantum theory consistently on every scale. The fact that the measurement outcomes that we observe are probabilistic creates the impression that conservation laws are violated. But, entanglement is able to maintain these laws despite appearances.

We can see this in particularly simple cases. Two spin-$\frac{1}{2}$ particles in a singlet state have a total angular momentum of zero. If, after separation, the angular momentum of each particle is measured about a common axis the measurements give opposite results, consistent with the pre-existing value. However, if measurements are made along different axes it is less obvious how to interpret the result. The key to understanding is to broaden our view, and to recognize that the particular state (however created) has a history of interaction with other systems that still share some measure of entanglement.

It is, of course, difficult to track entanglement in detail when more than a few relevant interactions are involved. The difficulties in constructing quantum computers make this clear, but the progress that has been made indicates that the entanglement relations on which they depend extend beyond situations with just a few particles. This should give us confidence that the principles of quantum theory and the conservation laws that they entail can be applied to all physical systems.
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