The quotient Unimodular Vector group is nilpotent

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Abstract

Jose–Rao introduced and studied the Special Unimodular Vector group $SU_{\mathfrak{m}}(\mathbb{R})$ and $EU_{\mathfrak{m}}(\mathbb{R})$, its Elementary Unimodular Vector subgroup. They proved that for $r \geq 2$, $EU_{\mathfrak{m}}(\mathbb{R})$ is a normal subgroup of $SU_{\mathfrak{m}}(\mathbb{R})$. The Jose–Rao theorem says that the quotient Unimodular Vector group, $SU_{\mathfrak{m}}(\mathbb{R})/EU_{\mathfrak{m}}(\mathbb{R})$, for $r \geq 2$, is a subgroup of the orthogonal quotient group $SO_{2(r+1)}(\mathbb{R})/EO_{2(r+1)}(\mathbb{R})$. The latter group is known to be nilpotent by the work of Hazrat–Vavilov, following methods of A. Bak; and so is the former.

In this article we give a direct proof, following ideas of A. Bak, to show that the quotient Unimodular Vector group is nilpotent of class $\leq d = \dim(\mathbb{R})$. We also use the Quillen–Suslin theory, inspired by A. Bak’s method, to prove that if $\mathbb{R} = A[X]$, with $A$ a local ring, then the quotient Unimodular Vector group is abelian.$^1$

1 Introduction

$\mathbb{R}$ will be a commutative ring with 1, in which 2 is invertible. $Um_{r+1}(\mathbb{R})$ will denote the set of unimodular vectors $v \in \mathbb{R}^{r+1}$, i.e. those vectors $v$ for which there is a vector $w \in \mathbb{R}^{r+1}$, with $\langle v, w \rangle = v \cdot w^T = 1$.

Suslin introduced the Suslin matrix in ([18], §5), and indicated its properties as well as how he felt they will be useful.

In [9] we initiated the study of the special unimodular vector group $SU_{\mathfrak{m}}(\mathbb{R})$, which is a subgroup of $GL_{2r}(\mathbb{R})$ related to $Um_{r+1}(\mathbb{R})$. We

$^1$ §4 is part of the doctoral thesis of the first named author under the second named author; §5 is part of the doctoral thesis of the third named author under the fourth named author.
also introduced the elementary unimodular vector subgroup $EUm_r(R)$ of $SUm_r(R)$, which is related to the $(r+1)$-unimodular vectors which have a completion to an elementary matrix. We developed the calculus for $EUm_r(R)$ in [9], and got a nice set of generators for it. In [10] we showed that $EUm_r(R)$ is a normal subgroup of $SUm_r(R)$, for $r \geq 2$.

In [19] Suslin, inspired by Quillen’s methods in [13], applied them to the study of unstable $K_1$-theory of polynomial rings. He proved the $K_1$-analogue of the Local-Global Principle and the Monic Inversion Principle. The theory built up in [13, 19] is known as the Quillen–Suslin theory.

Using Quillen–Suslin Local Global principle, A. Bak established in [4], that the linear quotient $SL_n(R)/E_n(R)$, for $n \geq 3$, is nilpotent. This theme has been revisited several times for different classical groups, see [7], [17], and (6, §3.3) for instance.

Now we apply Bak’s approach to the pair $(SUm_r(R), EUm_r(R))$, for $r \geq 2$, when $R$ is a noetherian ring of Krull dimension $d$. We give a direct approach to reprove the result in [11] that the unimodular vector quotient $SUm_r(R)/EUm_r(R)$ is a nilpotent group of class $d$. (The latter had been established in [11] via the Jose–Rao theorem that the unimodular vector quotient group was a subgroup of the special orthogonal quotient group; which was nilpotent in view of [7].)

We also deduce a relative version of this result from the absolute case. This argument does not depend on the Excision ring argument of W. van der Kallen, which is normally used to deduce ‘relative’ results; and is much more flexible. (This approach evolved from the work [13] according to Anjan Gupta; who used it in his thesis (2, §2.2) to reprove a theorem of Chattopadhyay–Rao in [3]).

Finally, we consider $SUm_r(R)/EUm_r(R)$, the unimodular vector quotient group, when $R = A[X]$ is a polynomial extension of a local ring $A$. In this case we show, arguing as in [16] that the unimodular quotient group is an abelian group. A relative version for extended ideals is also deduced.

2 Recap about the Suslin matrix $S_r(v, w)$

Given two row vectors $v, w \in R^{r+1}$, A. Suslin constructed in [18, §5], a matrix $S_r(v, w)$, which is of determinant one if $\langle v, w \rangle = v \cdot w^T = 1$. He defined this inductively, as follows: Let $v = (a_0, a_1, \ldots, a_r) = (a_0, v_1)$, with $v_1 = (a_1, \ldots, a_r)$, $w = (b_0, b_1, \ldots, b_r) = (b_0, w_1)$, with $w_1 = (b_1, \ldots, b_r)$. Set
$S_0(v, w) = a_0$, and set

$$S_r(v, w) = \begin{pmatrix} a_0 I_{2^r-1} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^T & b_0 I_{2^r-1} \end{pmatrix}. $$

The reader will find more details about these matrices in this amazing §5; with several unresolved questions.

These matrices have been studied by Jose–Rao in [10] [11]. The survey article [15] gives a quick glimpse at the known results today.

We shall denote by $SUM_r(R)$ the subgroup of $GL_{2^r}(R)$ generated by the set \( \{ S_r(v, w) | v, w \in R^{r+1}, \langle v, w \rangle = 1 \} \), and $EU_m(R)$ its subgroup generated by the set \( \{ S_r(v, w) | v, w \in R^{r+1}, \langle v, w \rangle = 1, v = e_1 \varepsilon, \text{for some } \varepsilon \in E_{r+1}(R) \} \). It was shown in [10], that $EU_m(R)$ is a normal subgroup of $SU_m(R)$, for $r \geq 2$.

For a matrix $\alpha \in M_k(R)$, we define $\alpha^{\text{top}}$ as the matrix whose entries are the same as that of $\alpha$ above the diagonal, and on the diagonal, and is zero below the diagonal. Similarly, we define $\alpha^{\text{bot}}$. Moreover, we use $\alpha^{tb}$ for $\alpha^{\text{top}}$ or $\alpha^{\text{bot}}$.

In [9] a structure theorem for $EU_m(R)$ was proved. The following nice set of generators of $EU_m(R)$ was established:

For $2 \leq i \leq r + 1$, $\lambda \in R$, let

$$E(e_i)(\lambda) = S_r(e_i + \lambda e_i, e_i), \quad E(e_i^\ast)(\lambda) = S_r(e_i, e_i + \lambda e_i).$$

It was shown that the group $EU_m(R)$ can be generated by either

(a) $E(c)(x)$, $E(d)(x)S_r(e_i, e_i)^{-1}$, if $2$ is invertible in $R$, or by

(b) $E(c)(x)^{\text{top}}$, $E(c)(x)^{\text{bot}}$,

where $c = e_i$ or $e_i^\ast$, $d = e_i$ or $e_i^\ast$, $2 \leq i \leq r + 1$, $x \in R$.

In [9] [11] Jose–Rao noted a fundamental property which is satisfied by the Suslin matrices. Let $v, w, s, t \in M_{1,r+1}(R)$. Then

$$S_r(s, t)S_r(v, w)S_r(s, t) = S_r(v', w')$$

$$S_r(t, s)S_r(w, v)S_r(t, s) = S_r(w', v'),$$

for some $v', w' \in M_{1,r+1}(R)$, which depend linearly on $v, w$ and quadratically on $s, t$. Consequently, $v' \cdot w'^T = (s \cdot t^T)^2(v \cdot w^T)$.

This fundamental property enables one to define an involution $\ast$ on the group $SUM_r(R)$, details of which can be found in [11]. This involution is
then used to give an action of $SU_m(r)$ on the Suslin space, viz. the free $R$-module of rank $2(r+1)$

$$S = \{S_r(v,w)|v,w \in M_{1r+1}(R)\}.$$ (For a basis one can take $se_1, \ldots, se_{r+1}, se_1^*, \ldots, se_{r+1}^*$, where $se_i = S_r(e_i,0)$, $se_i^* = S_r(0,e_i)$, for $1 \leq i \leq r$.)

In [11] they associated a linear transformation $T_g$ of the Suslin space with a Suslin matrix $g$, via

$$T_g(x,y) = (x',y'),$$

where $gS_r(x,y)g^* = S_r(x',y')$. Moreover, if $g$ is a product of Suslin matrices $S_r(v_i,w_i)$, with $\langle v_i, w_i \rangle = 1$, for all $i$, then $T_g \in SO_{2(r+1)}(R)$, i.e.

$$\langle T_g(v,w), T_g(s,t) \rangle = \langle (v,w), (s,t) \rangle = v \cdot w^T + s \cdot t^T.$$  

3 Computation of the matrix of the linear transformation

In ([11], §4), via the fundamental property, Jose-Rao observed that the above action induces a canonical homomorphism

$$\varphi : SU_m(r) \rightarrow SO_{2(r+1)}(R),$$

$$\varphi(S_r(v,w)) = T_{S_r(v,w)} = \tau(v,w) \circ \tau(e_1,e_1),$$

where $\tau(v,w)$ is the standard reflection with respect to the vector $(v,w) \in R^{2(r+1)}$ (of length one) given by the formula

$$\tau(v,w)(s,t) = (v,w)(s,t) - (\langle v,t \rangle + \langle s,w \rangle)(v,w).$$

The following simple computation gives an alternate way to prove this:

**Lemma 3.1** Let $R$ be a commutative ring with 1. Let $v, w \in Um_{r+1}(R)$, then the matrix of the linear transformation $T_{S_r(v,w)}$ with respect to the (ordered) basis

$$\{S_r(e_1,0), S_r(e_2,0), \ldots, S_r(e_{r+1},0), S_r(0,e_1), S_r(0,e_2), \ldots, S_r(0,e_{r+1})\}$$

is

$$\begin{pmatrix}
I - \begin{pmatrix} v^T \\ w^T \end{pmatrix} & \begin{pmatrix} v \\ w \end{pmatrix} \\
\begin{pmatrix} e_1^T \\ e_1 \end{pmatrix} & \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}
\end{pmatrix}.$$
Proof: Let \( v = (a_0, a_1, \cdots, a_r) \), \( w = (b_0, b_1, \cdots, b_r) \). By the definition of \( T_{S_r(v,w)} \),

\[
T_{S_r(v,w)}(e_1,0) = \tau_{(v,w)} \circ \tau_{(e_1,e_1)}(e_1,0) = \tau_{(v,w)}(0,-e_1) = (0,-e_1) + a_0(v,w) = (a_0v,a_0w - e_1)
\]

\[
T_{S_r(v,w)}(e_j,0) = \tau_{(v,w)}(e_j,0) = (e_j,0) - b_{j-1}(v,w) = (e_j - b_{j-1}v, -b_{j-1}w)
\]

\[
T_{S_r(v,w)}(0,e_1) = \tau_{(v,w)}(0,e_1) = \tau_{(v,w)}(-e_1,0) = (-e_1,0) + b_0(v,w) = (b_0v - e_1, b_0w)
\]

\[
T_{S_r(v,w)}(0,e_j) = \tau_{(v,w)}(0,e_j) = \tau_{(v,w)}(0,e_1) = (0,e_j) - a_{j-1}(v,w) = (-a_{j-1}v, e_j - a_{j-1}w)
\]

Thus the matrix of \( T_{S_r(v,w)} \) is

\[
\begin{pmatrix}
  a_0v & e_2 - b_1v & \cdots & e_{r+1} - b_rv & b_0v - e_1 & -a_1v & \cdots & -a_rv \\
  a_0w - e_1 & -b_1w & \cdots & -b_rw & b_0w & e_2 - a_1w & \cdots & e_{r+1} - a_rw
\end{pmatrix}.
\]

Right multiply the above matrix by the matrix \( I - \left( \begin{pmatrix} e_1^T \\ e_1^T \end{pmatrix} \right) \) will interchange the 1-st and \((r + 2)\)-th columns with sign changed. Hence, the matrix of \( T_{S_r(v,w)} \) is

\[
\left( I - \begin{pmatrix} v^T \\ w^T \end{pmatrix} \right) \left( I - \begin{pmatrix} e_1^T \\ e_1^T \end{pmatrix} \right)
\]

as required. \( \square \)

**Notation:** We denote the matrix of the linear transformation \( T_{S_r(v,w)} \) by \([T_{S_r(v,w)}]\).

Let us recollect the matrix of the linear transformations corresponding to the generators of \( EU_{m_r}(R) \), \( r \geq 2 \), computed in [11].

For the sake of completeness we give a slightly simpler argument than the one given in [11] below. However, in this approach, unlike in [11], we need that 2 is invertible in \( R \).

**Lemma 3.2** For \( 2 \leq i, j \leq r+1 \), one has the following relations in \( EU_{m_r}(R) \):

\[
E(e_i^*)(-2\lambda)^{bot} = S_r(e_1 - e_j, e_1 - e_i)S_r((1+\lambda)e_1 + e_j, e_1 - \lambda e_j) \\
S_r(e_1 - e_j, e_1 + e_i)S_r((1-\lambda)e_1 + e_j, e_1 + \lambda e_j)
\]

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$$E(e_i)(-2\lambda)^{bot} = \begin{bmatrix} E(e_i^*)(\lambda), E(e_i^*)(1) \end{bmatrix}.$$ 

$$S_r(e_1 + \lambda e_j, (1 - \lambda)e_1 + e_j)S_r(e_1 + e_i, e_1 + e_j)$$

$$S_r(e_1 - \lambda e_j, (1 + \lambda)e_1 + e_j)S_r(e_1 - e_i, e_1 - e_j).$$

(Note that by reversing the elements in the product in the above relation we can obtain the formulae for $E(e_i^*)(-2\lambda)^{top}$ and $E(e_i)(-2\lambda)^{top}$.)

Proof: We prove the first relation; the others are verified similarly. Put $x = 1$, $y = \lambda$, and $z = 1$ in the proof of [9, Proposition 5.6], to get

$$E(e_i^*)(-2\lambda)^{bot}$$

$$= \{E(e_j)(1)^{-1}\}E(e_j)(1/2)E(e_i^*)(1/2)^{-1}E(e_j)(1/2)^{-1}E(e_j)(1/2)\}

\{E(e_j)(1)^{-1}\}\{S_r((1 + \lambda)e_1 + e_j, e_1 - \lambda e_j)\}\{E(e_j)(1)^{-1}\}

\{E(e_j)(1/2)E(e_i^*)(1/2)E(e_j)(1/2)\}\{E(e_j)(1)^{-1}\}

\{S_r((1 - \lambda)e_1 + e_j, e_1 + \lambda e_j)\} [E(e_j^*)(1), E(e_j^*)(\lambda)]^{-1}.$$

Now by [9, Lemma 5.2],

$$E(e_i^*)(-2\lambda)^{bot} = S_r(e_1 - e_j, e_1 - e_i)S_r((1 + \lambda)e_1 + e_j, e_1 - \lambda e_j)$$

$$S_r(e_1 - e_j, e_1 + e_i)S_r((1 - \lambda)e_1 + e_j, e_1 + \lambda e_j)$$

$$[E(e_j^*)(\lambda), E(e_j^*)(1)]$$

as required. \qed

**Corollary 3.3** ([11], Lemma 4.9, Proposition 4.10) Let $R$ be a commutative ring with 1 in which 2 is invertible. For $2 \leq i \leq r + 1$,

$$\text{the matrix of } T_X = \begin{cases} \text{oe}_{\pi(1)i}(\lambda) & \text{if } X = E(e_i^*)^{bot}(-\lambda) \\
\text{oe}_{\pi(1)}(-\lambda) & \text{if } X = E(e_i^*)^{top}(-\lambda) \\
\text{oe}_{1i}(\lambda) & \text{if } X = E(e_i^*)^{top}(-\lambda) \\
\text{oe}_{i1}(-\lambda) & \text{if } X = E(e_i^*)^{bot}(-\lambda) \end{cases}$$

Proof: By Lemma 3.1 the matrix $A$ of $T_{S_r(e_1 - e_j, e_1 - e_i)}$ is given by

$$A = \begin{pmatrix} I & (e_1 - e_j)^T \\
(e_1 - e_i)^T & (e_1 - e_i)^T \\
\end{pmatrix} \begin{pmatrix} I - (e_1^T e_1) \\
0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I + e_{1i} - e_{j1} - e_{ji} & e_{1j} - e_{j1} - e_{jj} \\
e_{1i} - e_{i1} - e_{ii} & I + e_{1j} - e_{i1} - e_{ij} \end{pmatrix}.$$
Similarly, the matrix $B$ of $T_{S_r(1+\lambda)e_1+e_j, e_1-\lambda e_j}$ is 

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$B_{11} = I + \lambda(\lambda + 2)e_{11} + \lambda(1 + \lambda)e_{1j} + (1 + \lambda)e_{j1} + \lambda e_{jj},$$

$$B_{12} = \lambda e_{11} - (1 + \lambda)e_{1j} + e_{j1} - e_{jj},$$

$$B_{21} = \lambda e_{11} + \lambda e_{1j} - (1 + \lambda)e_{j1} - \lambda^2 e_{jj},$$

$$B_{22} = I - e_{1j} - \lambda e_{j1} + \lambda e_{jj},$$

the matrix $C$ of $T_{S_r(e_1-\lambda e_j, e_1+e_i)}$ is

$$C = \begin{pmatrix} I - e_{1i} - e_{j1} + e_{ji} & e_{1j} - e_{j1} - e_{jj} \\ -e_{1i} + e_{i1} - e_{ii} & I + e_{1j} + e_{i1} + e_{ij} \end{pmatrix}$$

and the matrix $D$ of $T_{S_r((1-\lambda)e_1+e_j, e_1+\lambda e_j)}$ is

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where

$$D_{11} = I + \lambda(\lambda - 2)e_{11} + \lambda(\lambda - 1)e_{1j} + (1 - \lambda)e_{j1} - \lambda e_{jj},$$

$$D_{12} = -\lambda e_{11} + (\lambda - 1)e_{1j} + e_{j1} - e_{jj},$$

$$D_{21} = -\lambda e_{11} - \lambda e_{1j} + (1 - \lambda)e_{j1} - \lambda^2 e_{jj},$$

$$D_{22} = I - e_{1j} + \lambda e_{j1} - \lambda e_{jj}$$

Now $AB = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ where

$$\alpha_{11} = I + \lambda e_{11} + \lambda e_{1j} - \lambda e_{j1} - \lambda e_{jj} + e_{ii} - e_{ji},$$

$$\alpha_{12} = 0,$$

$$\alpha_{21} = e_{1i} - (1 + 2\lambda)e_{i1} - 2\lambda e_{ij} - e_{ii} - \lambda^2 e_{11} + \lambda(1 + \lambda)e_{1j} - \lambda(1 + \lambda)e_{j1} - \lambda^2 e_{jj},$$

$$\alpha_{22} = I - e_{i1} + e_{ij} - \lambda e_{j1} + \lambda e_{jj} - \lambda e_{11} + \lambda e_{1j}$$

Also $CD = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ where

$$\beta_{11} = I - \lambda e_{11} - \lambda e_{1j} + \lambda e_{j1} - e_{ii} + e_{ji},$$

$$\beta_{12} = 0,$$

$$\beta_{21} = -e_{1i} - \lambda^2 e_{11} - \lambda(1 + \lambda)e_{1j}(1 - 2\lambda)e_{i1} - 2\lambda e_{ij} - e_{ii} + \lambda(1 - \lambda)e_{j1} - \lambda^2 e_{jj},$$

$$\beta_{22} = I - e_{ij} - e_{1j} + \lambda e_{j1} - \lambda e_{jj} + \lambda e_{11} + e_{i1}.$$ 

Thus

$$ABCD = \begin{pmatrix} I & \\ 2\lambda e_{1i} - 2\lambda e_{i1} - 2\lambda e_{ij} + 2\lambda e_{ji} & I \end{pmatrix}.$$
Also by Lemma 3.1, the matrix $P$ of $T_{E(e^*_j)(\lambda)}$ is given by
\[
P = \begin{pmatrix}
I - \left( e^T_1 (e_1 + \lambda e_j) \right) (e_1 + \lambda e_j) & 0 \\
\lambda e_{1j} - \lambda e_{1j} - \lambda^2 e_{jj} & I + \lambda e_{j1}
\end{pmatrix}.
\]
Clearly $P^{-1} = \begin{pmatrix}
I + \lambda e_{1j} & 0 \\
-\lambda e_{j1} + \lambda e_{1j} - \lambda^2 e_{jj} & I - \lambda e_{j1}
\end{pmatrix}$, which is the matrix of $T_{E(e^*_j)(-\lambda)}$. Similarly, the matrix $Q$ of $T_{E(e^*_1)(1)}$ and its inverse $Q^{-1}$ of $T_{E(e^*_1)(-1)}$ are
\[
Q = \begin{pmatrix}
I - e_{1i} & 0 \\
-e_{1i} + e_{ii} - e_{ii} & I + e_{ii}
\end{pmatrix},
Q^{-1} = \begin{pmatrix}
I + e_{1i} & 0 \\
e_{1i} - e_{1i} - e_{ii} & I - e_{1i}
\end{pmatrix}.
\]
Thus the matrix
\[
[P, Q] = \begin{pmatrix}
I & 0 \\
2\lambda e_{ij} - 2\lambda e_{ji} & I
\end{pmatrix}.
\]
Hence the product of the matrices $ABCD$ and $[P, Q]$ is
\[
\begin{pmatrix}
I & 0 \\
2\lambda e_{1i} - 2\lambda e_{ii} & I
\end{pmatrix} = I + 2\lambda e_{\pi(1)i} - 2\lambda e_{\pi(i)1} = oe_{\pi(1)i}(2\lambda).
\]
Since $\varphi$ is a homomorphism, the matrix of $T_{E(e^*_1)(-2\lambda)bot}$ is $oe_{\pi(1)i}(2\lambda)$. This proves the first relation. The second relation is its transpose-inverse. Similarly, one can prove the third and fourth relations.

**Corollary 3.4** Let $R$ be a commutative ring with 1 in which 2 is invertible. For $2 \leq i \neq j \neq \pi(i) \leq r + 1$,

the matrix of $T_X = \begin{cases}
oe_{\pi(1)i}(\lambda)oe_{1i}(\lambda) & \text{if } X = E(e_i)(\lambda) \\
oe_{\pi(1)i}(\lambda)oe_{e_i}(\lambda) & \text{if } X = E(e^*_i)(\lambda) \\
oe_{\pi(i)}(\lambda)oe_{1\pi(i)}(\lambda) & \text{if } X = E(e_{1i})(\lambda) \\
\pi_{1i}(1)oe_{1i}(\lambda)oe_{1\pi(i)}(\lambda) & \text{if } X = E(e^*_1)(\lambda) \\
oe_{ij}(\lambda) & \text{if } X = [E(e_j)(\lambda)top, E(e_i)(1)bot] \\
oe_{\pi(j)}(\lambda) & \text{if } X = [E(e_j)(\lambda)top, E(e_i)(1)bot] \\
oe_{\pi(j)}(\lambda) & \text{if } X = [E(e_j)(\lambda)top, E(e_i)(1)bot].
\end{cases}
\]

(Here $\pi_{1i}(-1)$ denote the matrix of $T_{S_{e_i e_{1i}}}$.)

**Proof:** Follows immediately from Corollary 3.3. 

\[
\text{8}
\]
Proposition 3.5 Let \( R \) be a commutative ring with 1 in which 2 is invertible. Then the map \( \varphi : EUm_r(R) \rightarrow EO_{2(r+1)}(R) \) given by \( \varphi(S_r(v, w)) = TS_r(v, w) \) is surjective.

Proof: Follows from Corollary 3.3. \( \square \)

4 \( \text{SUM}_r(R)/EUM_r(R) \) is nilpotent

Notation: Let \( s \) be a non-zero divisor, \( \text{SUM}_r(R, s^nR) \) denote the subgroup of \( \text{SUM}_r(R) \) consisting of matrices which are identity modulo \( (s^n) \), and \( EUM_r(R, s^nR) \) denote the corresponding elementary subgroup.

Lemma 4.1 Let \( R \) be a commutative ring with 1. Let \( s \) be a non-zero divisor in Jacobson radical \( J(R) \) of \( R \) and \( \beta \in \text{SUM}_r(R, s^nR) \) for \( n \geq 0 \). Then the matrix of the linear transformation \( T_\beta \) is in \( SO_{2(r+1)}(R, s^nR) \).

Proof: Since \( \beta \in \text{SUM}_r(R, s^nR) \), \( \beta = S_r(v, w) \) where \( v \equiv e_1 \mod (s^n) \) and \( w \equiv e_1 \mod (s^n) \). Let \( v = (a_0, a_1, \ldots, a_r) \) and \( w = (b_0, b_1, \ldots, b_r) \), where \( a_0 \) and \( b_0 \) are \( \equiv 1 \mod (s^n) \), \( a_i \) and \( b_i \) are \( \equiv 0 \mod (s^n) \). By definition, the matrix of \( T_\beta \), \( [T_\beta] \in SO_{2(r+1)}(R) \) and by Lemma 3.1, \( [T_\beta] \)

\[
\begin{pmatrix}
I_{2(r+1)} - \begin{pmatrix}
v^T \\
w^T 
\end{pmatrix} & (w \ v)
\end{pmatrix}
\begin{pmatrix}
I_{2(r+1)} - \begin{pmatrix}
ev_1^T \\
(c_1 e_1)
\end{pmatrix} & (e_1 \ e_1)
\end{pmatrix}
\]

\[
= I_{2(r+1)} - \begin{pmatrix}
v^T \\
w^T 
\end{pmatrix} (w \ v) - \begin{pmatrix}
ev_1^T \\
(c_1 e_1)
\end{pmatrix} (e_1 \ e_1) + (a_0 + b_0) \begin{pmatrix}
v^T \\
w^T 
\end{pmatrix} (e_1 \ e_1)
\]

\[
= \begin{pmatrix}
I_{r+1} - v^T w - e_1^T e_1 + (a_0 + b_0)w^T e_1 & -v^T e_1 + (a_0 + b_0)w^T e_1 \\
-w^T e_1 + (a_0 + b_0)w^T e_1 & I_{r+1} - w^T e_1 + (a_0 + b_0)w^T e_1
\end{pmatrix}
\]

Therefore,

\[
[T_\beta] \mod (s^n) = \begin{pmatrix}
I_{r+1} - e_1^T e_1 - e_1^T e_1 + 2e_1^T e_1 & -e_1^T e_1 + e_1^T e_1 + 2e_1^T e_1 \\
-e_1^T e_1 - e_1^T e_1 + 2e_1^T e_1 & I_{r+1} - e_1^T e_1 - e_1^T e_1 + 2e_1^T e_1
\end{pmatrix}
\]

\[
= I_{2(r+1)}.
\]

Hence \( [T_\beta] \in SO_{2(r+1)}(R, s^nR) \). \( \square \)

Lemma 4.2 Let \( R \) be a commutative ring with 1. In \( EUm_r(R[X,Y,Z]) \), \( E(c)(Z)^{th} E(d)(X^3Y)^{th} E(c)(-Z)^{th} \), where \( c = e_i \) or \( e_i^* \) and \( d = e_j \) or \( e_j^* \), is a product of elementary generators in \( EUM_r(R[X,Y,Z]) \) each of which is \( \equiv I_{2r} \) modulo \( (X) \).

Proof: If necessary, the reader can consult [10], Lemma 3.1 for details. \( \square \)
Lemma 4.3 Let $R$ be a commutative ring with 1. Let $s$ be a non-zero divisor in Jacobson radical $J(R)$ of $R$. Then we can write $E(c)(1)^{bot} E(d)(s^3 x)^{top} E(c)(-1)^{bot}$, where $c = e_i$ or $e_i^*$, $d = e_j$ or $e_j^*$ and $x \in R$, as a product of elementary generators in $EUm_{r'}(R)$ which are $\equiv I_{2r}$ modulo $(s)$.

Proof: Put $Z = 1$, $X = s$ and $Y = x$ in Lemma 4.2 \hfill \Box

Lemma 4.4 Let $R$ be a commutative ring with 1. Let $s$ be a non-zero divisor in Jacobson radical $J(R)$ of $R$. If $u \equiv 1 \mod (s^9)$ where $u \in R$ with $u^2 = 1$, then $[u] \perp [u^{-1}]$ is a product of elementary generators in $EUm_{r'}(R)$ each of which is $\equiv I_{2r}$ modulo $(s)$.

Proof: Note that,

\[
[u] \perp [u^{-1}] = \{ E(e_2)(1 - u^{-1})^{bot} E(e_2^*)(1 - u^{-1})^{bot} \} \{ E(e_2)(1)^{bot} E(e_2)(-1)^{bot} \}
\]

\[
\{ E(e_2)(1 - u)^{top} E(e_2^*)(1 - u)^{top} \} \{ E(e_2)(1)^{bot} E(e_2^*)(1)^{bot} \}
\]

\[
\{ E(e_2)(1 - u^{-1})^{top} E(e_2^*)(1 - u^{-1})^{top} \}.
\]

Let $u^{-1} = u = 1 + s^9 x$ for some $x \in R$. Then

\[
[u] \perp [u^{-1}] = \{ E(e_2)(-s^9 x)^{bot} E(e_2^*)(-s^9 x)^{bot} \} \{ E(e_2)(1)^{bot} E(e_2)(-1)^{bot} \}
\]

\[
\{ E(e_2)(-s^9 x)^{top} E(e_2^*)(-s^9 x)^{top} \} \{ E(e_2)(1)^{bot} E(e_2^*)(1)^{bot} \}
\]

\[
\{ E(e_2)(-s^9 x)^{top} E(e_2^*)(-s^9 x)^{top} \} \alpha \{ E(e_2)(-s^9 x)^{bot} E(e_2^*)(-s^9 x)^{bot} \}
\]

where

\[
\alpha = \{ E(e_2)(-1)^{bot} E(e_2)(-1)^{bot} \} \{ E(e_2)(-s^9 x)^{top} E(e_2^*)(-s^9 x)^{top} \}
\]

\[
\{ E(e_2)(1)^{bot} E(e_2^*)(1)^{bot} \}
\]

\[
E(e_2^*)(-1)^{bot} \{ E(e_2)(-1)^{bot} E(e_2)(-s^9 x)^{top} E(e_2)(1)^{bot} \}
\]

\[
\{ E(e_2)(-1)^{bot} E(e_2^*)(-s^9 x)^{top} E(e_2)(1)^{bot} \} E(e_2^*)(1)^{bot}.
\]

By Lemma 4.3 each element in the bracket is a product of elementary generators in $EUm_{r'}(R)$ which are $\equiv I_{2r'}$ modulo $(s^3)$. Thus

\[
\alpha = E(e_2^*)(-1)^{bot} \left( \prod \alpha_i \prod \beta_i \right) E(e_2^*)(1)^{bot},
\]

where each $\alpha_i, \beta_i \in EUm_{r'}(R)$ with each one $\equiv I_{2r'} \mod (s^3)$. Also we can write,

\[
\alpha = \prod \left( E(e_2^*)(-1)^{bot} \alpha_i E(e_2^*)(1)^{bot} \right) \prod \left( E(e_2^*)(-1)^{bot} \beta_i E(e_2^*)(1)^{bot} \right).
\]
Thus \( \phi \) each element in the product of \( \alpha \) is a product of elementary generators in \( EUm_r(R) \) which are \( \equiv I_{2r} \mod (s) \). Thus \([u] \perp [u^{-1}] \) is a product of elementary generators in \( EUm_r(R) \) each of which are \( \equiv I_{2r} \mod (s) \).

**Lemma 4.5** Let \( R \) be a commutative ring with 1. Let \( s \) be a non-zero divisor in Jacobson radical \( J(R) \) of \( R \) and \( \beta \in SUm_r(R, s^nR) \) for \( n \gg 9 \). Then \( \beta \) can be written as a product of elementary generators in \( EUm_r(R) \) where each is \( \equiv I_{2r} \mod (s) \).

Proof: By Lemma 4.1, \( [T] \equiv \beta \) in \( SUm_r(R, s^nR) \). Thus by \([7] \), Lemma 2.2, \( \phi(\beta) = [T] = \varepsilon_1 \ldots \varepsilon_k \) where each \( \varepsilon_i \in EO_{2(r+1)}(R) \) which is \( \equiv I_{2r} \mod (s) \).

For sufficiently large \( n \), we may assume that each \( \varepsilon_i \equiv I_{2r} \mod (s^p) \) where \( n > p \geq 9 \). By Proposition 3.5, \( \varepsilon_i = \phi(\varepsilon_i') \) where each \( \varepsilon_i' \in EUm_r(R, s^pR) \).

Thus \( \phi(\beta) = \phi(\varepsilon_1 \ldots \varepsilon_k) \). Hence \( (\varepsilon_1' \ldots \varepsilon_k')^{-1} \in \ker \phi = Z(SUm_r(R)) \subseteq EUm_r(R) \). By \([1] \), Corollary 3.5), \( \beta(\varepsilon_1' \ldots \varepsilon_k')^{-1} = uI_{2r} \) where \( u \) is a unit with \( u^2 = 1 \). Since \( \beta \) and \( \varepsilon_i' \) are \( \equiv I_{2r} \mod (s^p) \) \((n > p \geq 9)\), \( u \equiv 1 \mod (s^p) \). Therefore, by Lemma 4.3, \( \beta = u\varepsilon_1' \ldots \varepsilon_k' \) is a product of elementary generators each of which is \( \equiv I_{2r} \mod (s) \). \( \square \)

**Lemma 4.6** Let \( R \) be a commutative ring with 1 in which 2 is invertible, \( s \in R \) a non-zero divisor and \( a \in R \). Then for \( n \gg 0 \) and \( c = e_i \), or \( e_i^* \),

\[
\left[ E(c) \left( \begin{array}{c} a \\ x \end{array} \right) , SUm_r(R, s^nR) \right] \subseteq EUm_r(R[X]).
\]

More generally, given \( p > 0 \), for \( n \gg 0 \),

\[
[EUm_r(R_s[X]), SUm_r(R, s^nR)] \subseteq EUm_r(R[X], s^pR[X])
\]

Proof: Let \( \alpha(X) = [E(c) \left( \begin{array}{c} a \\ x \end{array} \right) , \beta] \) where \( \beta \in SUm_r(R, s^nR) \). Then \( \phi(\beta) \in EO_{2(r+1)}(R, s^nR) \), where \( \phi : SUm_r(R, s^nR) \rightarrow EO_{2(r+1)}(R, s^nR) \) is the canonical homomorphism. By Corollary 3.3,

\[
\phi(E(c) \left( \begin{array}{c} a \\ x \end{array} \right) ) \in EO_{2(r+1)}(R_s[X]).
\]

Thus by \([7] \), Lemma 2.4,

\[
\phi(\alpha(X)) \in [EO_{2(r+1)}(R_s[X]), SO_{2(r+1)}(R, s^nR)] \subseteq EO_{2(r+1)}(R[X]),
\]

and hence by Proposition 3.5, there exists \( \varepsilon \in EUm_r(R[X]) \) such that \( \phi(\alpha(X)) = \phi(\varepsilon) \). This implies, \( \phi(X)\varepsilon^{-1} \in \ker \phi \subseteq Z(SUm_r(R[X])) \subseteq EUm_r(R[X]). \) Hence \( \alpha(X) \in EUm_r(R[X]) \). \( \square \)
Lemma 4.7 Let $R$ be a commutative ring with 1 in which 2 is invertible, $s \in R$ a non-zero divisor and $a \in R$. Then for $n \gg 0$ and $c = e_i$, or $e_i^*$, 
$$ 
\left[ E(c) \left( \frac{a}{s} \right), SU_m(r, s^n R) \right] \subseteq EU_m(r). 
$$

More generally, $[EU_m(r, R), SU_m(r, s^n R)] \subseteq EU_m(r)$ for $n \gg 0$.

Proof: Put $X = 1$ in Lemma 4.6.

In ([11], Corollary 4.15) the quotient group $SU_m(r)/EU_m(r)$, $r \geq 2$ was shown to be nilpotent. This was obtained as a consequence of the Jose–Rao Theorem in ([11], Theorem 4.14) which asserts that this quotient unimodular vector group is a subgroup of the orthogonal quotient group $SO_{2(r+1)}(R)/EO_{2(r+1)}(R)$; which has been shown to be nilpotent in [7]. (Also see [17] for another proof.) We give a direct proof of the result following Bak’s methods in [4].

Theorem 4.8 Let $R$ be a commutative noetherian ring with 1 in which 2 is invertible and let $\dim R = d$. Then the group $SU_m(r)/EU_m(r)$ is nilpotent of class $d$ for $r \geq 2$.

Proof: Let $G = SU_m(r)/EU_m(r)$. We prove that $Z^d = \{1\}$. We prove by induction on $d = \dim R$. When $d = 0$, the ring $R$ is Artinian, so is semilocal. Hence $Um_{r+1}(R) = e_1 E_{r+1}(R)$ and so any generator $S_r(v, w)$, $\langle v, w \rangle = 1$ is in $EU_m(r)$.

Suppose $d > 0$. Let $\alpha \in Z^d$, then $\alpha = [\beta, \gamma]$, where $\beta \in G$ and $\gamma \in Z^{d-1}$. Let $\beta'$ be the preimage of $\beta$ in $SU_m(r)$.

Choose a non-zero-divisor $s$ in $R$ such that $\beta'_s \in EU_m(r, R)$ (such $s$ exists as $d > 0$). Consider $G = \overline{SU_m(r, s^n R)}/\overline{EU_m(r, s^n R)}$ for some $n \gg 0$. By induction, $\overline{\gamma} = \{1\}$ in $\overline{G}$. Since $EU_m(r)$ is normal in $SU_m(r)$, by modifying $\gamma$ we may assume that $\gamma' \in SU_m(r, s^n R)$ where $\gamma'$ is the preimage of $\gamma$ in $SU_m(r, s^n R)$. Thus by Lemma 4.6 $[\beta', \gamma'] \in EU_m(r)$. Hence $\alpha = \{1\}$ in $G$.

The relative case

In this section we deduce the relative case of Theorem 4.8 from the absolute case. We use the ‘Excision ring’ $R \oplus I$ below instead of the usual non-noetherian Excision ring $\mathbb{Z} \oplus I$ as is usually done due to the work of van der Kallen in [8].
Notation 4.9 By ([9, Proposition 5.6]), the elementary generators,
\[ E(c)(x)^{\text{top}}, E(c)(x)^{\text{bot}}, \]
where \( c = e_i \) or \( e_j^* \), and for \( 2 \leq i \leq r + 1 \), and with \( x \in R \), generate the Elementary Unimodular vector group \( EUm_r(R) \). For simplicity, we shall denote these by \( ge_i(x) \) below.

Theorem 4.10 Let \( R \) be a commutative noetherian ring with 1 in which \( 2 \) is invertible and with \( \dim R = d \). Let \( I \) be an ideal of \( R \). Then the group \( SUm_r(R, I)/EUm_r(R, I) \) is nilpotent of class \( d \) for \( r \geq 2 \).

Proof: Let \( G = SUm_r(R, I)/EUm_r(R, I) \). We prove that \( Z^d = \{1\} \). We prove by induction on \( d = \dim R \). When \( d = 0 \), the ring \( R \) is Artinian, so is semilocal. Hence \( Um_{r+1}(R, I) = e_1E_{r+1}(R, I) \) and so any generator \( S_r(v, w), \langle v, w \rangle = 1 \) is in \( EUm_r(R, I) \).

Suppose \( d > 0 \), Let \( \alpha \in Z^d \), then \( \alpha = [\beta, \gamma] \), where \( \beta \in G \) and \( \gamma \in Z^{d-1} \). We can write \( \beta = Id + \beta', \gamma = Id + \gamma' \) for some \( \beta', \gamma' \in M_{2r}(I) \). Let \( \alpha = Id + \alpha' \) for some \( \alpha' \in M_{2r}(I) \). Let \( \tilde{\alpha} = (Id, \alpha') \in SUm_r(R \oplus I, 0 \oplus I) \). In view of ([I, Lemma 3.3]),
\[ \tilde{\alpha} \in EUm_r(R \oplus I) \cap SUm_r(R \oplus I, 0 \oplus I) = EUm_r(R \oplus I, 0 \oplus I) \]
as \( \frac{R \oplus I}{I} \simeq R \) is a retract of \( R \oplus I \). Thus,
\[ \tilde{\alpha} = \prod_{k=1}^{m} \varepsilon_k ge_i(0, a_k) \varepsilon_k^{-1}, \quad \varepsilon_k \in EUm_r(R \oplus I), \ a_k \in I. \]

Now, consider the homomorphism
\[ f: R \oplus I \longrightarrow R \]
\[ (r, i) \longmapsto r + i. \]

This \( f \) induces a map
\[ \tilde{f}: EUm_r(R \oplus I, 0 \oplus I) \longrightarrow EUm_r(R). \]

Clearly,
\[ \alpha = \tilde{f}(\tilde{\alpha}) \]
\[ = \prod_{k=1}^{m} \gamma_k ge_i(0 + a_k) \gamma_k^{-1} \]
\[ = \prod_{k=1}^{m} \gamma_k ge_i(a_k) \gamma_k^{-1} \in EUm_r(R, I); \text{ since } a_k \in I, \]
where, \( \gamma_k = \tilde{f}(\varepsilon_k) \). \( \square \)

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5 Abelian quotients over polynomial extensions of a local ring

In this section we use the Quillen–Suslin Local Global Principle, following the ideas of A. Bak in [4], to prove that if $R = A[X]$, with $A$ a local ring, then the quotient Unimodular Vector group is abelian. (The method is similar to the one in [16] where we had used it to analyse the quotients of the linear, symplectic, and orthogonal groups.

We begin with a few simple observations.

The following observation is well known, we record it here for future use:

\textbf{Lemma 5.1} Let $R$ be a commutative ring and $v, w \in Um_n(R)$ be such that $v \cdot w^t = 1$. If $v = e_1 \sigma$ for some $\sigma \in E_n(R)$ then there exists $\varepsilon \in E_n(R)$ such that $v = e_1 \varepsilon$ and $w = e_1 (\varepsilon^{-1})^t$.

Proof: In view of ([18, Corollary 2.8]), $w \zeta = e_1 (\sigma^{-1})^t$, where $\zeta = I_n + v^t (e_1 (\sigma^{-1})^t - w) \in E_n(R)$. We see that $v \zeta^t = v$. Thus $e_1 \sigma \zeta^t = e_1 \sigma = v$. Upon taking $\varepsilon = e_1 \sigma \zeta^t$, we have $v = e_1 \varepsilon$ and $w = e_1 (\varepsilon^{-1})^t$. \hfill \Box

\textbf{Corollary 5.2} Let $R$ be a local ring. For $r \geq 1$, $SUm_r(R) = EUm_r(R)$.

Proof: Let $\alpha = S_r(v, w) \in SUm_r(R)$. Since $R$ is a local ring, therefore $v = e_1 \sigma$ for some $\sigma \in E_{r+1}(R)$. Since $v \cdot w^t = 1$, by Lemma 5.1 there exists $\varepsilon \in E_{r+1}(R)$ such that $v = e_1 \varepsilon$ and $w = e_1 (\varepsilon^{-1})^t$. Thus $\alpha = S_r(v, w) = S_r(e_1 \varepsilon, e_1 (\varepsilon^{-1})^t) \in EUm_r(R)$. \hfill \Box

\textbf{Lemma 5.3} Let $R$ be a local ring and $\alpha(X), \beta(X) \in SUm_r(R[X])$. Then, for $r \geq 2$, the commutator,

$$[\alpha(X), \beta(X)] \in [\alpha(X)\alpha(0)^{-1}, \beta(X)\beta(0)^{-1}]EUm_r(R[X]).$$

Proof: Since $R$ is a local ring, in view of Corollary 5.2 $SUm_r(R) = EUm_r(R)$ for all $r \geq 1$. Thus $\alpha(0), \beta(0) \in EUm_r(R)$.

Let $\eta = \alpha(X)\alpha(0)^{-1}$, $\tau = \beta(X)\beta(0)^{-1}$. Then,

$$[\alpha(X), \beta(X)] = [\alpha(X)\alpha(0)^{-1}\alpha(0), \beta(X)\beta(0)^{-1}\beta(0)]$$

$$= \eta \alpha(0) \tau \beta(0) (\eta \alpha(0))^{-1} (\tau \beta(0))^{-1}$$

$$= \eta \tau \eta^{-1} \tau^{-1} (\tau \eta^{-1} \alpha(0) \tau^{-1} \eta^{-1}) (\tau \eta \beta(0) \alpha(0)^{-1} \eta^{-1} \tau^{-1}) (\tau \beta(0) \alpha(0)^{-1} \eta^{-1} \tau^{-1}).$$
By (11 Corollary 4.12), $EU_{m_r}(R[X])$ is a normal subgroup of $SU_m(R[X])$ for $r \geq 2$, hence

$$(\tau \eta \tau^{-1} \alpha(0) \tau^{-1} \tau^{-1}) \in EU_{m_r}(R[X]), (\tau \eta \beta(0) \alpha(0)^{-1} \eta^{-1} \tau^{-1}) \in EU_{m_r}(R[X]), (\beta(0)^{-1} \tau^{-1}) \in EU_{m_r}(R[X]).$$

Hence the result. \hfill \Box

**Theorem 5.4** Let $R$ be a local ring. Then the group $\frac{SU_r(R[X])}{EU_{m_r}(R[X])}$ is an abelian group for $r \geq 2$.

**Proof:** Let $\alpha(X), \beta(X) \in SU_m(R[X])$, we need to prove $[\alpha(X), \beta(X)] \in EU_{m_r}(R[X])$. In view of Lemma 5.3 we may assume that $\alpha(0) = \beta(0) = Id$. Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)].$$

Then for every maximal ideal $m$ of $R[X]$,

$$\gamma(X, T)_m = [\alpha(XT)_m, \beta(X)_m].$$

Since $\beta(X)_m \in SU_m(R[X]_m) = EU_{m_r}(R[X]_m)$, and in view of the normality of $EU_{m_r}(R[X]_m[T]) \trianglelefteq SU_m(R[X]_m[T])$, for $r \geq 2$, one has $\gamma(X, T)_m \in EU_{m_r}(R[X]_m[T])$ and $\gamma(X, 0) = Id$. Thus by the Local-Global Principle, (11 Corollary 4.11), $\gamma(X, T) \in EU_{m_r}(R[X, T])$, by putting $T = 1$, one gets, $\gamma(X, 1) = [\alpha(X), \beta(X)] \in EU_{m_r}(R[X])$. \hfill \Box

**Theorem 5.5** Let $R$ be a local ring and $I$ be an ideal of $R$. Then the group $\frac{SU_{m_r}(R[X], I[X])}{EU_{m_r}(R[X], I[X])}$ is an abelian group for $r \geq 2$.

**Proof:** Let $\alpha, \beta \in SU_{m_r}(R[X], I[X])$. We can write $\alpha = Id + \alpha', \beta = Id + \beta'$ for some $\alpha', \beta' \in M_{p_r}(I[X])$. Let $\sigma = [\alpha, \beta] = Id + \sigma'$ for some $\sigma' \in M_{p_r}(I[X])$. Let $\tilde{\sigma} = (Id, \sigma') \in SU_{m_r}(R[X] \oplus I[X], 0 \oplus I[X])$. In view of (11 Lemma 3.3) and Theorem 4.4 $\tilde{\sigma} \in EU_{m_r}(R[X] \oplus I[X]) \cap SU_{m_r}(R[X] \oplus I[X], 0 \oplus I[X]) = EU_{m_r}(R[X] \oplus I[X], 0 \oplus I[X])$ as $\frac{R[X] \oplus I[X]}{0 \oplus I[X]} \simeq R[X]$ is a retract of $R[X] \oplus I[X]$. Thus,

$$\tilde{\sigma} = \prod_{k=1}^{m} \varepsilon_k g_{\varepsilon_k} (0, a_k) \varepsilon_k^{-1}, \quad \varepsilon_k \in EU_{m_r}(R[X] \oplus I[X]), a_k \in I[X].$$

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Now, consider the homomorphism

\[ f : R[X] \oplus I[X] \longrightarrow R[X] \]

\[ (r, i) \longmapsto r + i. \]

This \( f \) induces a map

\[ \tilde{f} : EUm_r(R[X] \oplus I[X], 0 \oplus I[X]) \longrightarrow EUm_r(R[X]) \]

Clearly,

\[ \sigma = \tilde{f}(\tilde{\sigma}) = \prod_{k=1}^{m} \gamma_k g e_{i_k}(0 + a_k) \gamma_k^{-1} \]

\[ = \prod_{k=1}^{m} \gamma_k g e_{i_k}(a_k) \gamma_k^{-1} \in E(n, R, I); \quad \text{since } a_k \in I, \]

where, \( \gamma_k = \tilde{f}(\varepsilon_k). \)

\[ \square \]

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