Finding and classifying critical points of 2D vector fields: a cell-oriented approach using group theory

Felix Effenberger · Daniel Weiskopf

Received: 6 March 2009 / Accepted: 30 July 2010 / Published online: 25 May 2011
© Springer-Verlag 2011

Abstract We present a novel approach to finding critical points in cell-wise barycentrically or bilinearly interpolated vector fields on surfaces. The Poincaré index of the critical points is determined by investigating the qualitative behavior of 0-level sets of the interpolants of the vector field components in parameter space using precomputed combinatorial results, thus avoiding the computation of the Jacobian of the vector field at the critical points in order to determine its index. The locations of the critical points within a cell are determined analytically to achieve accurate results. This approach leads to a correct treatment of cases with two first-order critical points or one second-order critical point of bilinearly interpolated vector fields within one cell, which would be missed by examining the linearized field only. We show that for the considered interpolation schemes determining the index of a critical point can be seen as a coloring problem of cell edges. A complete classification of all possible colorings in terms of the types and number of critical points yielded by each coloring is given using computational group theory. We present an efficient algorithm that makes use of these precomputed classifications in order to find and classify critical points in a cell-by-cell fashion. Issues of numerical stability, construction of the topological skeleton, topological simplification, and the statistics of the different types of critical points are also discussed.

Keywords Vector field topology · Interpolation · Barycentric interpolation · Linear interpolation · Bilinear interpolation · Level sets · Higher-order singularities · Computational group theory · Colorings

1 Introduction

The visualization of vector field topology is a problem that arises naturally when studying the qualitative structure of flows that are tangential to some surface. As usual, we use the term surface for a real, smooth 2-manifold (equipped with an atlas consisting of charts), see for example [18] for an introduction to Riemannian Geometry. Having its roots in the theory of dynamical systems, the topological skeleton of a Hamiltonian flow on a surface with isolated critical points consists of these critical points and trajectories (streamlines) of the vector field that lie at the boundary of a hyperbolic sector and connect two of the critical points. Helman and Hesselink [9] introduced the concept of the topology of a planar vector field to the visualization community and proposed the following construction scheme: (1) critical points are located, (2) classified, and then (3) trajectories along hyperbolic sectors are traced and connected to their originating and terminating critical points or boundary points. Step (2)—the classification of a critical point—is usually based on the Jacobian of the vector field, see [19]. Trajectories of step (3) are typically constructed by solving an ordinary differential equation for particle tracing.

Although a vast body of previous work in the field of flow visualization focuses on the problem of how to extend
the method of Helman and Hesselink to vector fields on arbitrary surfaces as well as the second and third step of the above algorithm, not much attention has been paid to the first step. In this paper, we specifically address the identification and classification of critical points in parameter space.

As efficient computer-based visualization algorithms usually work with discrete parametrized versions of the surfaces involved—examples of popular discretization schemes are for example triangulated or quadrangulated versions of the surface—we will in this paper not focus on the well-researched field of how to parametrize a given surface (see [10] for a recent survey) but assume that a surface always comes equipped with a globally continuous discrete parametrization that allows a cell-wise (local) barycentric or bilinear interpolation scheme of a vector field tangential to the surface in parameter space.

While this task is rather easy for linear vector fields, the problem setting becomes more interesting for bilinearly interpolated fields. Bilinear interpolation is ubiquitous in scientific visualization because it is popular for widely used uniform or curvilinear grids representing planar or curved surfaces. Since bilinear interpolation is not linear, it can lead to higher-order critical points, which are neglected by often-used linearization approaches.

In this paper, we introduce a new method that locates and classifies all critical points within piecewise linearly (barycentrically) or bilinearly interpolated two-dimensional grids. Our method determines the index of a critical point without the need to evaluate the Jacobian of the vector field in the critical point in order to determine its Poincaré index. For the case of bilinearly interpolated vector fields, our method is able to detect higher-order critical points and the presence of two first-order critical points within one cell, which, to our knowledge, has not been achieved with the common methods [10] for the bilinear interpolation scheme before. Figure 1 shows a corresponding example and illustrates our classification method. Our approach is based on the idea that investigating the qualitative behavior of 0-level sets of the components of the interpolated vector field provides information needed to compute the Poincaré index of a critical point. All qualitatively different possibilities of this behavior and the types of critical points yielded by each possibility are completely classified using the computational group theory tool GAP (Groups, Algorithms, and Programming) [7]. See the enumeration of cases for marching cubes and generic substitope algorithms by Banks et al. [3] for a previous example of an application of computational group theory in the field of scientific visualization. Furthermore, we discuss a cell-based topology simplification method as well as the question of numerical stability. Our approach results in an efficient, accurate, and robust cell-based algorithm for detecting all critical points of barycentrically or bilinearly interpolated 2D vector fields.

The paper is organized as follows. First, we will give a short review of the visualization literature dealing with vector field topology. Then, the theoretical foundations of vector field topology, namely the theory of the qualitative behavior of second-order dynamical systems along with such fundamental notions as those of critical points, separatrices, and the Poincaré index of a critical point are reviewed. Following this, we will present our new approach—first the general framework will be discussed and then applied to two cases: barycentrically and bilinearly interpolated vector fields. Then, we will deal with open issues such as critical points on the boundary of cells and numeric stability followed by a more detailed description of the cell-based algorithm. We will conclude giving results and a short review of our method.

This paper has accompanying material in the form of online resources, namely the GAP programs used in this paper (Online Resource 3) and lists of equivalence classes of colored cells referred to in Theorem 2 (Online Resource 1) and Theorem 5 (Online Resource 2).

![Fig. 1](image-url) Classification of critical points for three intersection cases for the 0-level sets of the two vector field components \( f_1 \) (cyan) and \( f_2 \) (orange) of a bilinearly interpolated vector field \( \mathbf{f} = (f_1, f_2) \): a no intersection of the level sets, b touching of the level sets yielding one critical point, c double intersection of the level sets yielding two critical points (one saddle and one non-saddle). The colors of the vector arrows encode the types of characteristic areas as defined in Sect. 4.2 (green = ++, yellow = +−, blue = −+, red = −−), a set of streamlines is shown in gray, and critical points lying in the intersection set of the two level sets are shown as black dots. Each vertex of a square is marked with ++, +−, −+, −− according to the sign of \( f_1 \) and \( f_2 \) at that vertex.