Lattice Fermions and Chiral Symmetry

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We propose a formulation of lattice fermions with one-sided differences that is hermitian, chirally symmetric (barring a bare mass term) and completely free of doubling. To obtain the axial anomaly in perturbation theory it was necessary to break chiral symmetry on the lattice only through a bare mass term for the physical fermion. The chiral limit may be taken once the continuum limit is reached. We comment on the role of the mass term with examples elsewhere in field theory.

1. Introduction

The so-called ‘naive’ lattice fermions give rise to sixteen species and the absence of the axial anomaly in this case is well understood in terms of cancellation of contributions from different species \[1\]. The Wilson fermions avoid species ‘doubling’ by introducing generalized (momentum-dependent) mass terms through an irrelevant term that makes the doublers as heavy as the cutoff and the axial anomaly comes out correctly with the irrelevant Wilson mass term playing the role of a Pauli-Villars regulator mass \[2\]; while a naive analysis gives fifteen times the ‘conformal’ anomaly \[3\]. Apart from removing doublers, we will be concerned in the following with Ward identities in the continuum limit and the role of the bare mass term and irrelevant terms in deriving the chiral anomaly.

2. Hermitian Action with One-sided Differences

All earlier attempts with one-sided difference operators \[4\] to remove doublers lacked hermiticity and as a result may have problems with reality properties in the continuum limit and also with numerical simulations. Most importantly, the issue of chiral symmetry breaking and the chiral anomaly was not addressed.

It is important for anomaly-related issues \[5\] to keep intact, in the chiral limit, the chiral and hermiticity structure of the Dirac operator, \(i\mathcal{D}\). Accordingly, we propose to use, in Weyl basis, for free fermions:

\[
i\mathcal{D} = \begin{pmatrix} 0 & D \end{pmatrix} \equiv \begin{pmatrix} i\sigma_\mu \Delta^b_\mu & 0 \end{pmatrix}, \tag{1}\]

where, \(\Delta^b_\mu = (\delta_{y,x} - \delta_{y,x+\mu})/a\), \(\Delta^f_\mu = (\delta_{y,x+\mu} - \delta_{y,x})/a\), and \(\sigma_\mu = (i, \sigma_k)\) with \(\sigma_k\) the usual Pauli matrices, \(k = 1, 2, 3\). We shall see later the need to add, in the lattice-regularized theory, an explicit chiral symmetry breaking bare mass term.

For massive Dirac fermions, the Dirac operator follows from Eq. (1):

\[
i\mathcal{D} = i\gamma_\mu \Delta^a_\mu + i\gamma_5 \Delta^b_\mu + m, \tag{4}\]

where, \(\Delta^a_\mu(x,y) = 1/2a(\delta_{y,x+\mu} - \delta_{y,-x-\mu})\), \(\Delta^b_\mu = (\delta_{y,x} - \delta_{y,x+\mu})/a = -(\Delta^f_\mu)^\dagger\), \(\Delta^a_\mu = 1/2a(2\delta_{y,x} - \delta_{y,x+\mu} - \delta_{y,-x-\mu})\). \(\Delta^a_\mu\) produces an (irrelevant) term in the lattice action that formally goes to zero as \(a \to 0\).

In a vectorlike gauge theory, the fermionic part of the euclidean lattice action is then given by,

\[
S_F = \sum_{x,\mu} \frac{1}{2a} \bar{\psi}_x \gamma_\mu [U_{x+\mu} \psi_{x+\mu} - U^\dagger_{x-\mu,\mu} \psi_{x-\mu}] + \sum_{x,\mu} \frac{1}{2a} \bar{\psi}_x \gamma_5 [2\psi_x - U_{x\mu} \psi_{x+\mu} - U^\dagger_{x-\mu,\mu} \psi_{x-\mu}] - \frac{1}{a} \bar{\psi}_x \psi_x + m \sum_x \bar{\psi}_x \psi_x, \tag{7}\]
where $\psi_x$ and $\bar{\psi}_x$ are fermion fields at $x$ and $U_{x\mu}$ is the gauge field at the link $(x, \mu)$. The action has full chiral symmetry for $m = 0$.

The action (8) leads to the free fermion propagator $G_0$ given in momentum space by

$$G_0^{-1} = i\gamma_\mu s_\mu + \gamma_\mu \gamma_5 c_\mu + m$$

where $s_\mu = (\sin ak_\mu)/a$ and $c_\mu = (1 - \cos ak_\mu)/a$. To see that it is free from doublers, consider

$$(G_0 G_0^0)^{-1} = \sum_\mu (s_\mu^2 + c_\mu^2) + m^2 + 2\sigma_{\mu\nu}\gamma_\delta s_\mu c_\nu$$

Note that a zero of $G_0^{-1}$ for $m = 0$ is necessarily a zero of $Tr(G_0 G_0^0)^{-1}$. But the latter can vanish only when $k_\mu = 0$ for all $\mu$ in the first Brillouin zone. Our propagator is essentially different from the previous cases where the $\gamma_5$ in the irrelevant term is absent and the inverse propagator is expressible in the form $\gamma_\mu J_\mu$ or $\sigma_\mu J_\mu$, where $J_\mu$ is a trigonometric function of the momentum and there are at least some extra non-covariant excitations.

The irrelevant term in our action, however, breaks hypercubic and reflection symmetries. Reflection Symmetry is a necessary condition for reflection positivity, one of the assumptions of the Nielsen-Ninomiya theorem in the euclidean theory.

As a remedy we propose to use instead the Dirac operator $\gamma_\mu \Delta_\mu^s + \gamma_5 \gamma_5 \Delta_\mu^c$ with $\gamma_5^s = \gamma_\mu \epsilon_\mu$ ($\epsilon_\mu = \pm 1$) and correlation functions are to be evaluated only after averaging over $\epsilon_\mu$. In Weyl language, the prescription resolves the ambiguity, whether to take the forward or the backward difference operator for a particular Weyl component, by averaging correlations functions over such possibilities for each $\mu$-component of the difference operator while maintaining the (anti)-correlation of the finite differences for $L$- and $R$-handed fermions and hence preserving the chiral and hermiticity structure.

A similar averaging prescription has also been applied previously to recover covariance in case of point-split regularization. In actuality, in our case the $\epsilon_\mu$ averaging may not be needed after all in the continuum limit since only an irrelevant term is concerned.

3. The Chiral anomaly

The locally derived Ward identities for the theory (8) corresponding to the global U(1) vector and axial symmetries, found in the usual way, are

$$\langle \Delta_\mu^s J_\mu^\dagger(x) \rangle = \langle Y \rangle,$$

$$\langle \Delta_\mu^c J_\mu^{c\dagger}(x) \rangle = 2m \langle \bar{\psi}_x \gamma_5 \psi_x \rangle + \langle X \rangle,$$

with, $\langle Y \rangle = \langle \Delta_\mu^s J_\mu^{s\dagger}(x) \rangle$, $\langle X \rangle = \langle \Delta_\mu^c J_\mu^{c\dagger}(x) \rangle$.

$$J_{\mu 5}^\pm(x) = \frac{1}{2} \left[ \bar{\psi}_x \gamma_\mu \gamma_5 U_{x\mu} \psi_{x+\mu} \pm \bar{\psi}_{x+\mu} \gamma_\mu \gamma_5 U_{x\mu}^{\dagger} \psi_x \right].$$

$J_{\mu 5}^\pm$ and $J_{\mu 5}^{c\pm}$ are obtained respectively from $J_{\mu 5}^\pm$ and $J_{\mu 5}^{c\pm}$ by replacing $\gamma_\mu$ by $\gamma_\mu^s$ in the above expressions. Note that the naive continuum limit of the currents $J_{\mu 5}^\pm(x)$ and $J_{\mu 5}^{c\pm}(x)$ gives respectively the expected vector and axial vector currents in the continuum, while $J_{\mu 5}^{c\pm}$ go in the naive continuum limit to zero. We like to point out that the theory is fully defined by the action and the measure. The Ward identities simply follow from the expected vector and axial vector currents in the currents $J_{\mu 5}^\pm$ and $J_{\mu 5}^{c\pm}$.

In weak-coupling perturbation theory, we now calculate the $\langle Y \rangle$ and $\langle X \rangle$ in the limit $a \to 0$. We start from the operator representations:

$$\langle Y \rangle = Tr \gamma_5 \langle x \mid (G_F \not R^\epsilon + \not R^\epsilon G_F) \mid x \rangle,$$

$$\langle X \rangle = Tr \langle x \mid (G_F \not R^\epsilon - \not R^\epsilon G_F) \mid x \rangle,$$

where $\not R^\epsilon = \gamma_\lambda \epsilon_\lambda U_\lambda$ with

$$R_\lambda = (2 - U_\lambda e^{ip_\lambda a} - e^{-ip_\lambda a} U_\lambda^\dagger)/(2a).$$

The full fermion propagator $G_F$ is given by

$$G_F^{-1} = \not D + \not R^\epsilon \gamma_5 + m,$$

where, $D_\lambda = (U_\lambda e^{ip_\lambda a} - e^{-ip_\lambda a} U_\lambda^\dagger)/(2a)$. 

In \([3, 4]\) ‘Tr’ stands for trace over \(\gamma\)-matrices, and, in non-abelian gauge theory, also over the symmetry matrices.

Two factors play key role in the calculation of \(\langle X \rangle\) and \(\langle Y \rangle\): (i) terms odd in \(R_\lambda\) drop out because of \(\epsilon_\mu\)-averaging, and (ii) in the ‘physical’ sector of the loop momentum \(0 \leq |k_\lambda| \leq \pi/2a, R_\lambda\) is of \(0(a)\) whereas in the ‘doubler’ sector \(\pi/2a \leq |k_\lambda| \leq \pi/a\) it is of \(0(1/a)\) and behaves like the mass of a Pauli-Villars regulator field. One finally obtains \(\lim_{a \rightarrow 0} \langle Y \rangle = 0\) so that the U(1) vector current is conserved:

\[
\langle \partial_\mu J_\mu(x) \rangle = 0. \tag{20}
\]

However, \(\langle X \rangle = \)

\[
2i \text{tr}(F_{\lambda \rho} \tilde{F}_{\lambda \rho}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^4k}{(2\pi)^4} \frac{1}{\sum_\lambda (s_\lambda^2 + c_\lambda^2) + m^2} \left[ \sum_\lambda c_\lambda^2 H_\lambda \cos ak_\lambda \right] \]

\[
= (i/2 \pi^4) \text{tr}(F_{\lambda \rho} \tilde{F}_{\lambda \rho}) \sum_{\nu=1}^{4} (-1)^{\nu} 3 C_{\nu-1} I_\nu, \tag{21}
\]

where \(F_{\lambda \rho}\) and \(\tilde{F}_{\lambda \rho}\) are the field tensor and its dual, ‘tr’ stands for trace over symmetry matrices, and in the continuum limit \(I_\nu = \pi^2/2\nu\). Thus the model reproduces the anomalous Ward identity:

\[
\langle \partial_\mu J_\mu(x) \rangle = 2m \langle \bar{\psi}_x \gamma_5 \psi_x \rangle - \frac{i}{16\pi} \text{tr}(F_{\lambda \rho} \tilde{F}_{\lambda \rho}). \tag{22}
\]

Note that the subscript \(\nu\) of \(I_\nu\) has the meaning of the number of components of the loop momentum with support in the doubler sector \(\pi/2a \leq |k_\mu| \leq \pi/a\). For finite \(m\) the contribution from \(\nu = 0\) vanishes in the continuum limit. If, however, \(m\) is zero the latter would exactly cancel the anomaly. For a nonvanishing anomaly it is, therefore, essential that the continuum limit is taken first and the chiral limit \(m = 0\), if necessary, afterwards.

In Eq. \(\langle 22 \rangle\), the first or explicit symmetry breaking term is of course a direct consequence of the nonzero mass and the second or anomalous term results essentially from the square of the irrelevant term of the action. However, the irrelevant term alone cannot produce the anomaly unless \(m \neq 0\).

\section{4. Conclusions}

We have been able to achieve a doubler-free hermitian lattice fermion action producing the chiral anomaly in the continuum limit. Barring a bare mass term which can only receive multiplicative renormalization, the action is chirally symmetric and therefore may be adaptable to chiral gauge theories. The presence of the mass term on the regulator is crucial to get the chiral anomaly. The chiral limit may only be taken once the cutoff is removed. There is growing evidence in helicity-flip interactions in QED and QCD \(\[\[\) also suggesting that the role of mass terms is more than just soft symmetry breaking. In Wilson fermions too, the anomaly appears in a very similar way. Our analysis indicates that breaking chiral symmetry only in the physical sector may be enough unlike in the Wilson case where chiral symmetry is broken also by mass terms for the doublers.

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