VECTOR BUNDLES ON PROJECTIVE VARIETIES WHICH SPLIT ALONG \( q \)-AMPLE SUBVARIETIES

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ABSTRACT. Given a vector bundle on a complex, smooth projective variety, I prove that its splitting along a very general, \( q \)-ample subvariety (for appropriate \( q \)), which admits sufficiently many embedded deformations implies the global splitting. The results go far beyond previously known splitting criteria obtained by restricting vector bundles to subvarieties.

I discuss the particular cases of zero loci of sections in globally generated vector bundles, on one hand, and sources of multiplicative group actions (corresponding to Bialynicki-Birula decompositions), on the other hand. Finally, I elaborate on the symplectic and orthogonal Grassmannians; I prove that the splitting of any vector bundle on them can be read off from the restriction to a low dimensional 'sub'-Grassmannian.

INTRODUCTION

For a vector bundle \( \mathcal{V} \) on the irreducible projective variety \( X \), I consider its restriction \( \mathcal{V}_Y \) to an irreducible subvariety \( Y \subset X \), and investigate the following:

**Question.** Assuming that \( \mathcal{V}_Y \) splits, under what assumptions on \( Y \) does \( \mathcal{V} \) split too?

To my knowledge, there is not a single reference which addresses the question in this generality. Yet, the problem is interesting because it allows to probe the splitting of vector bundles on (high dimensional) varieties by restricting them to (possibly low dimensional) subvarieties. The goal of this article is to give a tentative answer to the question.

Horrocks' criterion \cite{15} is the most widely known result of this type. Splitting criteria, corresponding to restrictions to ample divisors, have been obtained in \cite{5}. Surprisingly, restrictions to higher codimensional subvarieties have not been considered. For this reason, I studied the case when \( Y \) is the zero locus of a regular section in a globally generated, ample vector bundle \( N \) on \( X \).

Unfortunately, this setting discards very basic situations, e.g. \( X \) is a product \( X' \times V \) and \( N = \mathcal{O}_{X'}(1) \). Second, in perfect analogy with Horrocks' criterion, I proved that the splitting of a vector bundle on any Grassmannian can be verified by restricting it to an arbitrary, standardly embedded Grass(2; 4); this is not an ample subvariety. These observations are the motivations to consider 'sufficiently positive' subvarieties of \( X \), which posses 'many' embedded deformations. Loosely speaking, the main result of this article is the following:

**Theorem.** Let \( \mathcal{V} \) be an arbitrary vector bundle on a projective variety \( X \), defined over an uncountable, algebraically closed field of characteristic zero. Let \( Y \) be a subvariety of \( X \) satisfying the following properties:

- \( Y \) is \( q \)-ample for appropriate \( q \) (e.g. \( q + 1 \leq \dim X - 3\text{codim}_X(Y) \), but not necessarily); (The definition of a \( q \)-ample subvariety is inspired from \cite{22, 25, 2}, also \cite{11}.);

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The embedded deformations of $Y$ cover an open subset of $X$, and their intersection pattern is sufficiently non-trivial. Then $\mathcal{V}$ splits if and only if it splits along a very general deformation of $Y$.

The precise statement can be found in theorem 2.8; applications to the cases of globally generated vector bundles and multiplicative group actions, with emphasis on homogeneous varieties, are stated in theorems 4.1 and 6.3 respectively.

The $q$-ampleness condition is asymptotic in nature, and involves thickenings of subvarieties. For this reason, the price to pay for the generality of the theorem is that of testing the splitting along subvarieties which are very general within their deformation space. Thus the $q$-ample case studied here can be characterized as probabilistic, in contrast with the case of ample vector bundles studied in [13], which is deterministic. Although the result is formulated algebraically, the proof uses complex analytic techniques; the final statement is deduced by base change.

The essential feature of the criterion is that of being intrinsic to the subvariety; it avoids additional data. This allows to treat in a unified way examples arising in totally different situations: zero loci of globally generated (not necessarily ample) vector bundles (cf. sections 3-4), on one hand; homogeneous subvarieties of homogeneous varieties (cf. sections 5-6), on the other hand. As a by-product, we obtain many examples of $q$-ample subvarieties which are not zero loci of regular sections in globally generated $q$-ample vector bundles.

The case of isotropic Grassmannians is detailed in section 7. As I already mentioned, a vector bundle on $\text{Gr}(u; w)$, $u \geq 2$, $w \geq 4$, splits if and only if its restriction to any embedded $\text{Gr}(2; 4)$ splits. Unfortunately, the same proof breaks down in the case of the symplectic and orthogonal Grassmannians. However, our ‘probabilistic’ approach works very well.

**Theorem.** Let $\mathcal{V}$ be a vector bundle either on the symplectic Grassmannian $\text{sp-Gr}(u; w)$, with $u \geq 2, w \geq 2u$, or the orthogonal Grassmannian $\text{o-Gr}(u; w)$, with $u \geq 3, w \geq 2u$. Then $\mathcal{V}$ splits if and only if it splits along a very generally embedded:

- $\text{sp-Gr}(2; 4)$, in the symplectic case;
- $\text{o-Gr}(3; 6)$, for $w = 2u$,
- $\text{o-Gr}(3; 7)$, for $w = 2u + 1$,
- $\text{o-Gr}(3; 8)$, for $w \geq 2u + 2$,

in the orthogonal case.

This dramatically simplifies the problem of deciding the splitting of vector bundles on Grassmannians. One should compare it with the cohomological criteria [21, 13, 14, 17], which involve a large number of vanishings.

Throughout this article, $X$ stands for a smooth, projective, irreducible variety over $\mathbb{C}$, except a few statements containing base change arguments. We consider a vector bundle $\mathcal{V}$ on $X$ of rank $r$, and denote by $\mathcal{E} := \mathcal{E}nd(\mathcal{V})$ the endomorphisms of $\mathcal{V}$. For a subvariety $Y \subset X$, we let $\mathcal{V}_Y := \mathcal{V} \otimes \mathcal{O}_Y$, etc.

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1. Short digression on $q$-ampleness

1.1. Why $q$-ample subvarieties? Before reviewing the relevant definitions, I briefly present the reasons which led to consider the notion of $q$-ampleness in this context. One says that $\mathcal{V}$ splits on $X$ if

$$\mathcal{V} = \bigoplus_{j \in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j}$$

with $\mathcal{L}_j \in \text{Pic}(X)$ pairwise non-isomorphic, and $\sum_{j \in J} m_j = r$.

It is easy to see that $\mathcal{V}$ splits if and only if it admits an endomorphism with $\text{rk}(\mathcal{V})$ distinct eigenvalues. The main tool to attack the splitting problem is the following observation.

Lemma 1.1 Let $S \subset X$ be a connected, projective subscheme, such that $\mathcal{V}_S$ splits. If the restriction homomorphism $\text{res}_S : \Gamma(X, \mathcal{E}) \rightarrow \Gamma(S, \mathcal{E}_S)$ is surjective, then $\mathcal{V}$ splits too.

The subschemes $S$ will usually be thickenings of smooth subvarieties of $X$.

Definition 1.2 For $m \geq 0$, the $m$-th order thickening $Y_m$ of a subvariety $Y \subset X$ is the closed subscheme defined by the sheaf of ideals $\mathcal{I}_m^{Y+1}|_Y$; with this convention, $Y_0 = Y$. The structure sheaves of two consecutive thickenings of $Y$ fit into the exact sequence

$$0 \rightarrow \mathcal{I}_m^Y/\mathcal{I}_m^{Y+1}|_Y \rightarrow \mathcal{O}_{Y_m} \rightarrow \mathcal{O}_{Y_{m-1}} \rightarrow 0. \quad (1.1)$$

The formal completion of $X$ along $Y$ is defined as $\lim_{\rightarrow} Y_m$, and is denoted by $\hat{X}_Y$.

To apply lemma 1.1 I adopt a two-step strategy:

(i) Prove that $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(\hat{X}_Y, \mathcal{E}_{\hat{X}_Y})$ is surjective; this step requires the $(\dim Y - 1)$-ampleness of $Y$. (Actually, we will consider also more general thickenings.)

(ii) Prove that $\Gamma(\hat{X}_Y, \mathcal{E}_{\hat{X}_Y}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$ is surjective. This step is more involved, requires less ampleness (more positivity) of $Y$, and also its genericity in the space of embedded deformations.

If $Y$ has the property that $H^1(X, \mathcal{F} \otimes \mathcal{I}_m^Y) = 0$ for all vector bundles $\mathcal{F}$ on $X$ ($m$ depending on $\mathcal{F}$), then the splitting of $\mathcal{V}$ along a sufficiently high order thickening of $Y$ implies the splitting on $X$. Indeed, simply take $\mathcal{F} = \mathcal{E}$. (For ample subvarieties, explicit lower bounds for $m$ are obtained in [13].)

Lemma 1.3 Assume that $Y$ is $(\dim Y - 1)$-ample (cf. [16] below). Then $\mathcal{V}$ splits on $X$ if and only if it does so along the formal completion of $X$ along $Y$.

The difficulty consists in proving that the splitting of $\mathcal{V}$ along a very general subvariety $Y \subset X$ implies the splitting of $\mathcal{V}$ along $\hat{X}_Y$. This step constitutes the body of the article.

1.2. Definition and first properties. The concept of $q$-ampleness for globally generated vector bundles was introduced in [23]; the $q$-ampleness in defined intrinsically in [2, 25] through cohomology vanishing properties. We recall the latter definition, the case of globally generated vector bundles being detailed in section 3.
Definition 1.4 (cf. [25, Theorem 7.1], [21, Lemma 2.1]) A line bundle \( \mathcal{L} \) on a (reduced) projective Gorenstein variety \( \tilde{X} \) is called \( \bar{q} \)-ample if for any coherent sheaf \( \mathcal{F} \) on \( \tilde{X} \) holds:

\[
H^t(\mathcal{F} \otimes \mathcal{O}_X(mE_Y)) = 0, \forall t > \bar{q}, \forall m \gg 0.
\]

We say that a vector bundle \( \mathcal{N} \) on \( X \) is \( q \)-ample if \( \mathcal{O}_{\mathbb{P}(\mathcal{N}/X)}(1) \) is \( q \)-ample.

Proposition 1.5 (cf. [25, Section 7], [22, Definition 3.1]). Let \( Y \subset X \) be an equidimensional subscheme of codimension \( \delta \), \( \tilde{X} := \text{Bl}_Y(X) \) be the blow-up of the ideal of \( Y \), and \( E_Y \subset \tilde{X} \) be the exceptional divisor. Assume that \( \tilde{X} \) is Gorenstein. The following statements are equivalent:

(i) For any locally free (hence also for any coherent) sheaf \( \mathcal{F} \) on \( \tilde{X} \) holds

\[
H^t(\tilde{X}, \mathcal{F} \otimes \mathcal{O}_X(mE_Y)) = 0, \forall t \geq q + \delta, \forall m \gg 0,
\]

(that is, \( E_Y \) is \( (q + \delta - 1) \)-ample);

(ii) For all locally free sheaves (that is vector bundles) \( \mathcal{F} \) on \( X \) holds

\[
H^t(X, \mathcal{F} \otimes \mathcal{O}_Y^m) = 0, \forall t \leq \dim Y - q, \forall m \gg 0.
\]

Proof. Is identical with [22, Lemma 2.2 and Proposition 6.2]. \( \square \)

If \( X \) is smooth and \( Y \subset X \) is a locally complete intersection (lci for short), then \( \tilde{X} \) is automatically Gorenstein. We are primarily interested in the cohomology vanishing property (1.3); for this reason, we introduce an ad hoc terminology.

Definition 1.6 We say that a lci subvariety \( Y \subset X \) is (has the property) \( p^{>0} \) if for any vector bundle \( \mathcal{F} \) on \( X \) holds:

\[
\exists m_{\mathcal{F}} \geq 1, \forall m \geq m_{\mathcal{F}}, \forall t \leq p, \quad H^t(X, \mathcal{F} \otimes \mathcal{O}_Y^m) = 0.
\]

Intuitively, \( Y \) is \( p^{>0} \) if its normal bundle at each point contains at least \( p \) ‘positive’ directions. Probably the appropriate name for this property of \( Y \) would be ‘\( q \)-ample subvariety’, for \( q = \dim Y - p \). (The case of ample subvarieties [22] corresponds to \( q = 0 \).) This choice is made to emphasize the amount of positivity of the various objects which which appear subsequently.

Remark 1.7 If an effective Cartier divisor \( D \subset \tilde{X} \) is \( 1^{>0} \) (that is, \( D \) is \( (\dim \tilde{X} - 2) \)-ample), then \( D \) is connected. (Indeed, take \( \mathcal{F} = K_\tilde{X} \) in (1.4) and apply Serre duality.) Hence, if \( Y \subset X \) is lci such that \( E_Y \) is \( 1^{>0} \), then \( Y \) is connected and also equidimensional. In particular, \( Y \) is smooth implies that it is irreducible.

Proposition 1.8 (i) Let \( Y \subset X \) be irreducible, lci of codimension \( \delta \). Then holds:

\[
Y \text{ is } p^{>0} \Leftrightarrow \begin{cases} \text{the normal bundle } \mathcal{N} = \mathcal{N}_{Y/X} \text{ is } (\dim Y - p)\text{-ample}, \\ \text{the cohomological dimension } \text{cd}(X \setminus Y) \leq \dim X - (p + 1). \end{cases}
\]

(ii) Let \( Z \subset Y \) is \( p^{>0} \), \( Y \subset X \) is \( r^{>0} \), and are both irreducible lci, then

\[
\begin{cases}
\mathcal{N}_{Z/X} \text{ is } (\dim Y + \dim Z - (r + p))\text{-ample}, \\
\text{cd}(X \setminus Z) \leq \dim X - (\min\{r, p\} + 1). 
\end{cases}
\]

In particular, \( Z \subset X \) is \((p - (\dim Y - r))^{>0}\).
Proof. (i) \(\Rightarrow\) Let \(\mathcal{F}\) be a vector bundle on \(X\). Since \(E_Y = \mathbb{P}(N)\), the exact sequence
\[
0 \rightarrow \mathcal{O}_X((m - 1)E_Y) \rightarrow \mathcal{O}_X(mE_Y) \rightarrow \mathcal{O}_{\mathbb{P}(N)}(-m) \rightarrow 0
\]
implies \(H^t(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(N)}(-m)) = 0\), for all \(t \geq \dim X - p\) and \(m \gg 0\). The equality
\[
H^{\delta + t}(\mathbb{P}(N), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(N)}(-\delta - m) \otimes (\mathcal{F} \otimes \det(N)^{-1})) = H^{t+1}(\mathbb{P}(N')', \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(N')}(m)),
\]
(\text{cf. \cite[(4.1)]{22}}) yields \(H^{t+1}(\mathbb{P}(N'), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(N')}(m)) = 0\), for all \(t \geq \dim Y - p\). The second implication is \cite[Proposition 5.1]{22}.

(\(\Leftarrow\)) The \((\dim Y - p)\)-ampleness of \(\mathcal{O}_{\mathbb{P}(N')}(1)\), combined with (1.7), implies
\[
H^t(\mathcal{F} \otimes \mathcal{O}_X((m - 1)E_Y)) \xrightarrow{\sim} H^t(\mathcal{F} \otimes \mathcal{O}_X(mE_Y)), \text{ for } t \geq \dim X - p \text{ and } m \gg 0,
\]
so \(H^t(\mathcal{F} \otimes \mathcal{O}_X(mE_Y)) = \lim_{k \to 0} H^t(\mathcal{F} \otimes \mathcal{O}_X(ke_Y))\); according to \cite[(5.1)]{22}, the right hand side is \(H^t(X \setminus Y, \mathcal{F})\), which vanishes.

(ii) In the sequence \(0 \rightarrow N_{Z/Y} \rightarrow N_{Z/X} \rightarrow N_{Y/X}|_Z \rightarrow 0\), the extremities are \((\dim Z - p)\), respectively \((\dim Y - r)\)-ample; the subadditivity of the ampleness \cite[Theorem 3.1]{22} yields the conclusion. The bound on the cohomological dimension follows by repeating \textit{ad litteram} \cite[Proposition 6.4]{22}. For the last statement, \(Z\) satisfies the conditions (i). \(\square\)

The lack of sufficient positivity of the normal bundle \(N_{Z/X}\) prevents to conclude that \(Z \subset X\) is \(\min\{r, p\} > 0\). However, the next proposition shows that it is close to be so.

**Definition 1.9**

(i) For any vector bundle \(\mathcal{F}\) on \(X\), define
\[
\tilde{H}^t(Z_t, \mathcal{F}_t) := \left\{ h \in H^t(\mathcal{F}_t) \mid \exists U \supseteq Z \text{ open subset of } X, \exists \tilde{\alpha} \in H^t(\mathcal{F}_U) \text{ such that } \alpha = \text{res}_{Z_t}(\tilde{\alpha}) \right\}.
\]
The set \(U\) is allowed to be either a Zariski or an analytic (tubular) open neighbourhood of \(Z\).

(ii) We say that a subscheme \(Y \subset X\) is \(p = 0\) (approximately \(p > 0\)) if there is a decreasing sequence of sheaves of ideals \((\mathcal{J}_n)\) on \(X\) such that the following hold:
- \(\forall m, n \geq 1 \exists m' > m, n' > n\) such that \(\mathcal{J}_m \subset \mathcal{J}_{m'} \subset \mathcal{J}_{n'} \subset \mathcal{J}_n\);
- for any vector bundle \(\mathcal{F}\) on \(X\), \(\exists m_{\mathcal{F}} \geq 1 \forall m \geq m_{\mathcal{F}}, \forall t \leq p, H^t(X, \mathcal{F}) \rightarrow H^t(X, \mathcal{F} \otimes \mathcal{O}_X(\mathcal{J}_m))\) is: - an isomorphism, for \(t \leq p - 1\), - injective, for \(t = p\). \hfill (1.9)

**Proposition 1.10**

Let \(Z \subset Y, Y \subset X\) be lci, and both \(p > 0\). Then \(Z \subset X\) is \(p > 0\), and for all vector bundles \(\mathcal{F}\) on \(X\) holds:
\[
\text{res}_{\mathcal{J}_m}^X : H^t(\mathcal{F}) \rightarrow H^t(\mathcal{F}_{\mathcal{J}_m}) \text{ is: - an isomorphism, for } t \leq p - 1, - injective, for } t = p.
\]

**Proof.** Since \(Z \subset Y \subset X\) are lci, for any \(l \geq a\) one has the exact sequence:
\[
0 \rightarrow \mathcal{J}_Y^a / \mathcal{J}_Y^{a+1} \rightarrow \mathcal{J}_Y^a / \mathcal{J}_Y^{a+1} / \mathcal{J}_Y^a \rightarrow \mathcal{J}_Z / \mathcal{J}_Y^a \rightarrow 0. \hfill (1.10)
\]
Indeed, just use \((\mathcal{J}_Z / \mathcal{J}_Y^a) / \mathcal{J}_Y^a \cong \mathcal{J}_Z / \mathcal{J}_Y^a \cdot \mathcal{J}_Z / \mathcal{J}_Y^a\). Note that the left hand side is an \(\mathcal{O}_Y\)-module; \(\mathcal{J}_Z / \mathcal{J}_Y = \mathcal{J}_{Z \cap Y}\) is the ideal of \(Z \subset Y\), and that \(\mathcal{J}_Y^a / \mathcal{J}_Y^{a+1} = \text{Sym}^a N_Y / X\) is the symmetric power.
of co-normal bundle. The $p^{>0}$ assumption implies that there are integers $k_\sigma, l_\sigma$, and a linear function $l(k) = \lambda k + \mu$ ($\lambda, \mu$ independent of $\mathcal{F}$) with the following properties:

$$
H^t(\mathcal{F} \otimes \mathcal{I}_Y^k) = 0, \quad \forall t \leq p, \quad \forall k \geq k_\sigma,
$$

$$
H^t(\mathcal{F}_Y \otimes \mathcal{I}_{Z \cap Y}^l) = 0, \quad \forall t \leq p, \quad \forall l \geq l_\sigma,
$$

$$
H^t(\mathcal{F}_Y \otimes \text{Sym}^a \mathcal{N}_{Y/X}^\nu \otimes \mathcal{I}_{Z \cap Y}^{l-a}) = 0, \quad \forall t \leq p, \quad \forall a \leq k, \quad \forall l \geq l(k).
$$

The last claim uses the uniform $q$-ampleness property [25, Theorem 6.4] and the subadditivity of the regularity [25, Theorem 3.4]: first, there is a linear function $l(r)$ such that for any vector bundle $\mathcal{F}$ with regularity $\text{reg}(\mathcal{F}_Y) \leq r$ holds

$$
H^t(\mathcal{F}_Y \otimes \mathcal{I}_{Z \cap Y}^l) = 0, \text{ for } t \leq p, \quad l \geq l(r);
$$

second, for $a \leq k$, $\text{reg}(\mathcal{F}_Y \otimes \text{Sym}^a \mathcal{N}_{Y/X}^\nu) \leq \text{linear}(k)$. Recursively, for $a = 1, \ldots, k$, the exact sequence (1.10) yields:

$$
H^t(\mathcal{F} \otimes \mathcal{I}_Y^k) = 0, \quad \forall t \geq l(k).
$$

Now plug in this into $0 \to \mathcal{I}_Y^k \to \mathcal{I}_Z^k + \mathcal{I}_Y^1 \to \mathcal{I}_Z^k \otimes \mathcal{I}_Y^1 \to 0$ (tensored by $\mathcal{F}$), and deduce that

$$
H^t(\mathcal{F} \otimes (\mathcal{I}_Y^k + \mathcal{I}_Z^l)) = 0, \quad \forall t \leq p, \quad \forall k \geq k_\sigma, \quad \forall l \geq l(k).
$$

We denote by $Z_{l,k}$ the subscheme defined by $\mathcal{I}_Y^k + \mathcal{I}_Z^l$; it is an ‘asymmetric’ thickening of $Z$ in $X$. For any $l$ as above, there is $m > l$ such that $\mathcal{I}_Y^l + \mathcal{I}_Z^m \subset \mathcal{I}_Y^l \subset \mathcal{I}_Y^l + \mathcal{I}_Z^l = Z_{l,k} \subset Z_l \subset Z_{m,l}$.

For $m > l > k$ as above, one has the commutative diagram

$$
\begin{array}{ccc}
\text{res}_{m,l}^X & \xrightarrow{\xi} & \text{res}_{m,l}^Y \\
\text{res}_{l,k}^X & \xrightarrow{\xi} & \text{res}_{l,k}^Y \\
H^t(\mathcal{F}) & \xrightarrow{} & H^t(\mathcal{F}_Z) \\
\end{array}
$$

Notice that $\text{res}_{l,k}^X \circ \xi = \text{res}_{l,k}^Y$, which is injective, so $\xi$ is injective. It remains to prove that $\xi$ maps onto $H^t(\mathcal{F}_Z)$, for $t \leq p - 1$: assume $\alpha = \text{res}_{l,k}^Y(\tilde{\alpha})$, where $U \supset Z$ is open and $\tilde{\alpha} \in H^t(\mathcal{F}_U)$; then $\alpha = \text{res}_{l,k}^X(\text{res}_{m,l}^Y(\tilde{\alpha}))$, and $\text{res}_{m,l}^Y(\tilde{\alpha})$ is in the image of $\text{res}_{m,l}^X$.

**Lemma 1.11** Let $\varphi : X \to X'$ be a flat, surjective morphism, with $X, X'$ smooth, whose fibres are $d$-dimensional. If $Y' \subset X'$ is lci and $p^{>0}$, then $Y = \varphi^{-1}(Y') \subset X$ is lci and $p^{>0}$.

**Proof.** Since $f$ is flat, $Y$ is still lci in $X$ and $\text{codim}_X(Y) = \text{codim}_{X'}(Y') = \delta$. Let us check the property (1.2). The universality property of the blow-up (cf. [14, Ch. II, Corollary 7.15]) yields the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} = \text{Bl}_Y(X) & \xrightarrow{\tilde{\varphi}} & \tilde{X}' = \text{Bl}_{Y'}(X') \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & X',
\end{array}
$$

and $\varphi^* \mathcal{O}_{\tilde{X}}(E_{Y'}) = \mathcal{O}_{\tilde{X}}(E_Y)$. Let $\tilde{\mathcal{F}}$ be a coherent sheaf on $\tilde{X}$: then $R^i \tilde{\varphi}_* (\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = R^i \varphi_* \mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_{Y'})$ and $R^{>d} \varphi_* \mathcal{F} = 0$; as $Y' \subset X'$ is $p^{>0}$, holds

$$
H^i(\tilde{X}', R^i \tilde{\varphi}_* (\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_{Y'}))) = 0, \text{ for } i = 0, \ldots, d, \quad t \geq (\dim X' - p), \quad \text{and } m \gg 0.
$$

Then the Leray spectral sequence implies that $E_Y$ is $((\dim X' - p) + d - 1)$-ample. \qed
1.3. Criterion for the positivity of a subvariety. Most examples of \(q\)-ample subvarieties are zero loci of regular sections in (Sommese) \(q\)-ample vector bundles. (This will be detailed in section 3.) However, is necessary to have a test for the \(q\)-ampleness of a subvariety in more general circumstances. Fortunately, an elementary criterion was already used in [13].

**Proposition 1.12** Let the situation be as in Proposition 1.12. Assume there is an irreducible variety \(V\) and a morphism \(b : \tilde{X} \to V\) such that \(\mathcal{O}_{\tilde{X}}(E_Y)\) is \(b\)-relatively ample. Then \(Y \subset X\) is \(p\)-ample, for \(p := \dim X - \dim b(\tilde{X}) - 1\).

**Proof.** Let \(\mathcal{F}\) be a coherent sheaf on \(\tilde{X}\), and \(j \geq \dim X - p > \dim b(\tilde{X})\). Then
\[
R^tb_*(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = 0, \text{ for } t > 0 \text{ and } m \gg 0,
\]
implies that
\[
H^j(\tilde{X}, \mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = H^j(V, b_*(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y))).
\]
But the right hand side vanishes, because \(\text{Supp } b_*(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y))\) is at most \(\dim b(\tilde{X})\)-dimensional.

**Remark 1.13** In section 2 we will be interested in families of \(p\)-ample subvarieties of \(X\); the proposition generalizes straightforwardly. Let \(\mathcal{Y} \subset S \times X\) be a smooth family of subvarieties of \(X\) indexed by some parameter space \(S\). Assume that there is an \(S\)-variety such that
\[
\mathcal{O}(E_Y)\text{ is relatively ample for a morphism } \text{Bl}_Y(S \times X) \to V.
\]
Then \(Y_s\) is \((\dim(S \times X) - \dim V - 1)^{>0}\), for all \(s \in S\) such that \(Y_s\) is lci.

The criterion covers the situations discussed in sections 3 and 5, that is the case of zero sections in globally generated vector bundles, and the case of sources of \(G_m\)-actions.

1.4. \(q\)-positive line bundles. Intuitively, the hypotheses in 1.12 allow \(\mathcal{O}_{\tilde{X}}(E_Y)\) to be ‘negative’ in at most \(\dim b(\tilde{X}) < \dim X - p\) directions. As we will see below, this stronger positivity property allows to control \(\text{Pic}(Y)\).

**Definition 1.14** (cf. [11]) A line bundle \(\mathcal{L}\) on a smooth projective variety \(\tilde{X}\) is \(q\)-positive if it admits a Hermitian metric whose curvature form has at most \(q\) negative (or zero) eigenvalues.

It is generally true (cf. [1, Proposition 28], [11, Proposition 2.1]) that a \(q\)-positive line bundle \(\mathcal{L}\) satisfies \(H^{q+1}(\tilde{\mathcal{F}} \otimes \mathcal{L}^m) = 0\), for any vector bundle \(\tilde{\mathcal{F}}\) and \(m \gg 0\), that is \(\mathcal{L}\) is \(q\)-positive \(\Rightarrow \) \(\mathcal{L}\) is \(q\)-ample.

**Proposition 1.15** Let the situation be as in Proposition 1.12, with \(X, Y\) smooth. Then \(\mathcal{O}_{\tilde{X}}(E_Y)\) is \(\dim b(\tilde{X})\)-positive.

**Proof.** One may assume that \(V\) is smooth; otherwise, take an embedding into a smooth variety. As \(\mathcal{O}_{\tilde{X}}(E_Y)\) is \(b\)-relatively ample, there is an embedding \(\tilde{X} \hookrightarrow \mathbb{P}^N \times V\) (over \(V\)) such that \(\mathcal{O}_{\tilde{X}}(m_0E_Y) = \iota^*(\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{M})\), for some \(m_0 > 0\) and \(\mathcal{M} \in \text{Pic}(V)\). Take a strictly positive Hermitian metric on \(\mathcal{O}_{\mathbb{P}^N}(1)\), an arbitrary on \(\mathcal{M}\), and the product metric on \(\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{M}\).

The maximal rank of \(\text{db}_{\tilde{x}}\) is \(\dim b(\tilde{X})\), attained on a dense open subset of \(\tilde{X}\), so \(\text{rk}(\text{db}_{\tilde{x}}) \leq \dim b(\tilde{X})\) for all \(\tilde{x} \in \tilde{X}\).

Since \(\iota\) is an embedding, the curvature of the pull-back metric on \(\mathcal{O}_{\tilde{X}}(m_0E_Y)\) is positive definite on \(\text{Ker}(\text{db}_{\tilde{x}})\), and \(\dim \text{Ker}(\text{db}_{\tilde{x}}) \geq \dim X - \dim b(\tilde{X})\). So, at each point of \(\tilde{X}\), there are at most \(\dim b(\tilde{X})\) negative eigenvectors.

**Lemma 1.16** Let \(\varphi : X \to X'\) be a smooth morphism of relative dimension \(d\). If \(\mathcal{L}'\) is a \(q\)-positive line bundle on \(X'\), then \(\varphi^*\mathcal{L}'\) is \((q + d)\)-positive on \(X\).
Lemma 1.18 Let \( H \) be an isomorphism if \( H \) is the pull-back res

\[ Y_{st} := Y_s \cap Y_t, \quad Y_{ost} := Y_o \cap Y_s \cap Y_t, \quad \forall o, s, t \in S. \]  

(1.11)

Lemma 1.17 Let \( Y_o, Y_s, Y_t \) be \( \delta \)-codimensional, \( p^{>0} \) lci subvarieties of \( X \) such that their double and triple intersections are equidimensional lci (if non-empty). If \( 2\delta + 1 \leq p \), then \( Y_{os} \) and \( Y_{ost} \) are indeed non-empty and connected.

Proof. The following diagram is commutative

\[ \begin{CD}
\text{Bl}_{Y_{ost}}(Y_{ot}) @>>> \text{Bl}_{Y_{os}}(Y_o) @>>> \text{Bl}_Y(X) \\
\downarrow \downarrow \downarrow \downarrow \\
Y_{ot} @>>> Y_o @>>> X,
\end{CD} \]  

(1.12)

and the exceptional divisors are obtained by restriction, thus \( Y_{os} \subset Y_o \) and \( Y_{ost} \subset Y_{ot} \) are \( q \)-ample (if non-empty). If \( Y_{os} = \emptyset \), then \( Y_o \subset X \setminus Y_o \), which yields

\[ \text{cd}(X \setminus Y_o) \geq \dim X - \delta \implies \dim X - p - 1 \geq \dim X - \delta. \]

The contradiction shows that the double intersections are non-empty.

If \( Y_{ost} = \emptyset \), then \( Y_{os} \subset Y_o \setminus Y_{ot} \), and we get another contradiction:

\[ \text{cd}(Y_o \setminus Y_{ot}) \geq \dim Y_o \implies \dim X - p - 1 \geq \dim X - 2\delta. \]

The connectedness follows from the remark 1.7; the assumption \( p \geq 2\delta + 1 \) implies that all the exceptional divisors in (1.12) are at least \( 1^{>0} \), so they are connected.

1.5.2. Picard group of \( q \)-ample subvarieties. Let \( Y \subset X \) be smooth. We are interested when is the pull-back \( \text{res}^Y_X : \text{Pic}(X) \to \text{Pic}(Y) \) an isomorphism.

Lemma 1.18 Let \( Y \subset X \) be a subvariety. Then \( \text{res}^Y_X : \text{Pic}(X) \to \text{Pic}(Y) \) is an isomorphism as soon as \( H^t(X; \mathbb{Z}) \to H^t(Y; \mathbb{Z}), \ t = 1, 2 \), are isomorphisms. In particular, \( \text{res}^X_Y \) is an isomorphism if \( H_{2\dim C \cdot X - t}(X \setminus Y; \mathbb{Z}) = 0 \) for \( 1 \leq t \leq 3 \).

Proof. The hypothesis implies \( H^t(X; \mathbb{C}) \to H^t(Y; \mathbb{C}) \), for \( t = 1, 2 \); the Hodge decomposition yields \( H^t(X; \mathcal{O}_X) \to H^t(Y; \mathcal{O}_Y) \). Now compare the exponential sequences for \( X \) and \( Y \):

\[ \begin{array}{ccc}
H^1(X; \mathbb{Z}) & \to & H^1(Y; \mathcal{O}_X) \\
\text{res}^X_Y & \downarrow & \text{res}^Y_X \\
H^1(Y; \mathbb{Z}) & \to & H^1(Y; \mathcal{O}_X)
\end{array} \]

(1.13)

Remark 1.19 The restriction \( H^t(X; \mathbb{Q}) \to H^t(Y; \mathbb{Q}) \) is an isomorphism for \( t \leq p - 1 \), for any \( p^{>0} \) subvariety \( Y \) (cf. [22, Corollary 5.2]). Hence, if \( Y \) is \( 3^{>0} \), \( \text{Pic}^0(X) \to \text{Pic}^0(Y) \) is a finite morphism, and \( \text{NS}(X) \to \text{NS}(Y) \) has finite index.
Theorem 1.20 If $\mathcal{L}$ is a $q$-positive line bundle on a smooth projective variety $X$ and the zero locus $D$ of $s \in \Gamma(\mathcal{L})$ is smooth, then
\[ D \text{ is } (\dim_{\mathbb{C}} X - q - 1)^{\geq 0} \text{ and } H^t(X; \mathbb{Z}) \xrightarrow{\cong} H^t(D; \mathbb{Z}), \text{ for } t \leq \dim_{\mathbb{C}} X - q - 2. \]
Let $Y \subset X$ be a smooth subvariety. Assume there is a smooth variety $V$ and a morphism $\tilde{X} = \text{Bl}_Y(X) \to b \to V$ such that $\mathcal{O}_{\tilde{X}}(E_Y)$ is $b$-relatively ample. Then holds
\[ H^t(X; \mathbb{Z}) \xrightarrow{\cong} H^t(Y; \mathbb{Z}), \text{ for } t \leq p - 1 \text{ and } p := \dim X - \dim \tilde{X} - 1. \]
Thus $\text{Pic}(X) \to \text{Pic}(Y)$ is an isomorphism, for $p \geq 3$. If $Y'$ is a small, smooth deformation of $Y$, then $\text{Pic}(X) \to \text{Pic}(Y')$ is still an isomorphism, and $Y'$ is $p > 0$ too.

Proof. The isomorphism is proved in [8, Theorem III], [22, Lemma 10.1]. For the second statement, $\mathcal{O}_{\tilde{X}}(E_Y)$ is $b(\tilde{X})$-positive, by proposition 1.15, apply the previous step for $D = E_Y$. 

2. The splitting criterion

Let $\mathcal{Y} = \bigoplus_{j \in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j}$ be a split vector bundle, with $\mathcal{L}_j \in \text{Pic}(X)$ pairwise non-isomorphic. In this case, $\mathcal{Y}_j' := \mathcal{L}_j \otimes \mathbb{C}^{m_j}$, $j \in J$, are called the isotypical components of the splitting. If $\bigoplus_{j \in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j}$ and $\bigoplus_{j' \in J'} \mathcal{L}_{j'} \otimes \mathbb{C}^{m_{j'}}$ are two splittings, then there is a bijective function $J \to J'$ such that $\mathcal{L}_{\epsilon(j)}' \cong \mathcal{L}_j$ and $m_{\epsilon(j)}' = m_j$ for all $j \in J$. However, the isotypical components are not uniquely defined, because the global automorphisms of $\mathcal{Y}$ send a splitting into a new one. We consider the partial order `$\prec$' on $\text{Pic}(X)$:
\[ \mathcal{L} \prec \mathcal{L}' \quad \text{if} \quad \mathcal{L} \neq \mathcal{L}' \text{ and } \Gamma(X, \mathcal{L}^{-1} \mathcal{L}') \neq 0. \quad (2.1) \]
The isotypical components corresponding to the maximal elements, are canonically defined.

Lemma 2.1 Let $M \subset J$ be the subset of maximal elements with respect to $\prec$. Then there is a natural, injective homomorphism of vector bundles
\[ \text{ev}_M : \bigoplus_{j \in M} \mathcal{L}_j \otimes \Gamma(X, \mathcal{L}_j^{-1} \otimes \mathcal{Y}) \to \mathcal{Y}. \quad (2.2) \]

Consequently, given a family of subvarieties $\{Y_s\}_{s \in S}$ of $X$ such that $\mathcal{Y}$ splits along each of them, we can glue together the maximal isotypical components of $\mathcal{Y}_s$ (cf. lemma 2.6 below).

2.1. Gluing of split vector bundles. Let $S$ be an irreducible quasi-affine variety, and $\mathcal{Y} \subset S \times X$ be subvariety (denote by $\pi, \rho$ the morphisms to $S$ and $X$, respectively) with the following properties:

(0) $\text{Pic}(S)$ is trivial. (This can be achieved by shrinking $S$.)
(i) The morphism $\mathcal{Y} \to S$ is proper, smooth, with connected fibres.
(ii) For $o \in S$, define $S(o) := \{s \in S \mid Y_s \cap Y_o \neq \emptyset\}$. (It is a subscheme of $S$.) Assume that the very general point $o \in S$ has the following properties:
(a) $Y_{os} = Y_o \cap Y_s$ is irreducible, for all $s \in S(o)$, and the intersection is transverse for $s \in S(o)$ generic;
(b) For $s \in S(o)$ generic, all the arrows below are isomorphisms:
Remark 2.2 If all the triple intersections are non-empty and connected (sufficient conditions are given in [17]), then \( S(o) = S \), and (1-arm), (no-\( \Delta \)) are automatically satisfied. This property holds in the case of zero loci of sections in ample vector bundles, which was studied in [13]. More general situations when (2.3) is satisfied appear in propositions 3.6, 5.6.

However, for the symplectic and orthogonal Grassmannian studied in section 7, the generic double and triple intersections are empty. The analysis of these cases led to the weaker conditions above.

Definition 2.3 (i) The geometric generic fibre of \( \pi \) is defined by the Cartesian diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & \mathcal{Y} \\
\Spec(\overline{k}) & \xrightarrow{\pi} & S, \quad \overline{k} := \mathbb{C}(S).
\end{array}
\]

A very general point of \( S \) refers to a point outside a countable union of subvarieties of \( S \).

(ii) The double (resp. triple) geometric generic self-intersections of \( \mathcal{Y} \), denoted \( \mathcal{Y}_2 \) and \( \mathcal{Y}_3 \) respectively, are defined by means of the diagrams:

\[
\begin{array}{ccc}
\mathcal{Y}_2 := (\mathcal{Y} \times_X \mathcal{Y}) \setminus \text{diag}(\mathcal{Y}) & \xrightarrow{\pi_2} & \mathcal{Y} \times_X \mathcal{Y} \xrightarrow{\text{diag}} X \\
S_2 := \text{Supp}(\pi \times \pi)_* \mathcal{O}_{\mathcal{Y}_2} & \hookrightarrow & S \times S
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{Y}_3 := \mathcal{Y} \times_X \mathcal{Y} \times_X \mathcal{Y} & \xrightarrow{\text{diag}} & X \\
S \times S \times S.
\end{array}
\]

(2.4)
Thus there are natural morphisms $\mathcal{Y}_2 \supseteq \mathcal{Y}$ (the projections onto the first and second components), and $\mathcal{Y}_3 \cong \mathcal{Y}_2$. (One may think off $\mathcal{Y}_2$ as $Y_{st} = Y_s \cap Y_t$, for very general $(s, t) \in S_2$, and the morphisms are the inclusions into $Y_s, Y_t$ respectively; similarly for $\mathcal{Y}_3$.)

For $o \in S$, $S(o)$ introduced at (2.3) is $((\text{pr}_{S_2}^S)^{-1}(o) \subset \{o\} \times S) \cap S_2$.

**Lemma 2.4** There is an open analytic subset (a ball) $B \subset S$, which can be chosen arbitrarily in some Zariski open subset of $S$, such that the following hold:

(i) If $(\rho^* \mathcal{V}) \otimes \mathcal{O}_B$ splits on $\mathcal{Y}$, then $(\rho^* \mathcal{V})_B$ splits.

(ii) If $\text{Pic}(X \otimes \mathcal{O}_B) \rightarrow \text{Pic}(\mathcal{Y})$ and $\text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(\mathcal{Y}_2)$ are isomorphisms, then (2.3)($\text{Pic}$) is satisfied for points in $B$.

This condition is satisfied in the particular case when $\mathcal{Y} \subset S \times X$ is a $\delta$-codimensional, $p^{\geq 0}$ family of subvarieties of $X$ as in [1.13], with $p \geq \delta + 3$.

**Proof.** Since $(\rho^* \mathcal{V})_{\mathcal{Y}}$ splits, there are $\ell_1', \ldots, \ell_r' \in \text{Pic}(\mathcal{Y})$ such that $(\rho^* \mathcal{V})_{\mathcal{Y}} = \ell_1' \oplus \ldots \oplus \ell_r'$; the right hand side is defined over a finitely generated (algebraic) extension of $\mathbb{C}(S)$. Thus there is an open affine $S^0 \subset S$, and a finite morphism $\pi : S' \rightarrow S^0$ such that $\ell_1', \ldots, \ell_r'$ are defined over $\mathbb{C}[S']$, and $(\rho^* \mathcal{V})_{S'}$ splits on $\mathcal{Y}_{S'}$. Then there are open balls $B' \subset S'$ and $B \subset S^0$ such that $B' \supseteq B$ is an analytic isomorphism, and the splitting of $(\rho^* \mathcal{V})_B$ on $\mathcal{Y}_B$ descends to $(\rho^* \mathcal{V})_B$ on $\mathcal{Y}_B$. The second statement is analogous.

In the situation [1.13] Pic$(X_B) \rightarrow \text{Pic}(\mathcal{Y})$ is an isomorphism for $p \geq 3$, cf. theorem [1.20] the isomorphism $\text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(\mathcal{Y}_2)$, requires $p - \delta \geq 3$. \hfill $\square$

Henceforth, for $o \in S$ very general, we replace $S$ by $S(o)$ (actually by an irreducible component containing $o$) and $\mathcal{Y}$ by $\mathcal{Y}_x S(o)$; the restrictions of $\pi, \rho$ are denoted the same.

**Lemma 2.5** Then $\text{Pic}(X) \xrightarrow{\rho^*} \text{Pic}(\mathcal{Y}_S)$ is an isomorphism, for $S$ small enough.

**Proof.** The composition $\text{Pic}(X) \xrightarrow{\rho^*} \text{Pic}(\mathcal{Y}) \xrightarrow{\text{res}_o} \text{Pic}(Y_o)$ is bijective, so $\rho^*$ is injective. For the surjectivity, take $\ell \in \text{Pic}(\mathcal{Y})$. If $\ell_{Y_o} \cong O_{Y_o}$, then $\{s \in S \mid \ell_s \cong O_Y\} = \{s \in S \mid h^0(\ell_{Y_s}) = 0\}$ is open, by semi-continuity, so $\{s \in S \mid \ell_s \cong O_Y\}$ is closed. On the other hand, after restricting to $Y_{os}$, the hypothesis (Pic) implies that this set is dense; thus it is the whole $S$. It follows that $\ell \cong \pi^* \bar{\ell}$, with $\bar{\ell} \in \text{Pic}(S)$. But Pic$(S)$ is trivial for $S$ sufficiently small, so $\ell \cong 0$. If $\ell \in \text{Pic}(\mathcal{Y})$ is arbitrary, take $L \in \text{Pic}(X)$ such that $\ell_{Y_o} \cong L_{Y_o}$, so $(\rho^* L)^{-1}\ell|_{Y_o}$ is trivial. \hfill $\square$

Let $V$ be a vector bundle on $X$, such that $\rho^* V$ splits; by the previous lemma,

$$\rho^* V \cong \rho^* (\bigoplus_{j \in J} L_j^{\oplus m_j}), \text{ with } L_j \in \text{Pic}(X) \text{ pairwise non-isomorphic.} \quad (2.5)$$

For any $s \in S$, let $M_s \subset J$ be the subset of maximal elements, for $V_{Y_s}$. By semi-continuity, there is a neighbourhood $S_s \subset S$ of $s$ such that $M_s \subset M_{s'}$ for all $s' \in S_s$; thus there is a largest subset $M \subset J$, and an open subset $S' \subset S$ such that $M = M_s$, for all $s \in S'$. Hence, possibly after shrinking $S$, the set of maximal isotypical components of $V_{Y_s}$ with respect to (2.1) is independent of $s \in S$.

**Lemma 2.6** Let the situation be as above, and $B \subset S$ be a standard (analytic) ball. We consider the (analytic) open subset $U := \rho(\mathcal{Y}_B) \subset X$. Then the following statements hold:
(i) There is a pointwise injective homomorphism \( \bigoplus_{\mu \in M} \mathcal{L}_\mu \oplus \mathcal{O}_\mathcal{U} \rightarrow \mathcal{V} \otimes \mathcal{O}_\mathcal{U} \) whose restriction to \( \mathcal{Y}_s \) is the natural evaluation \((\mathcal{V} \otimes \mathcal{O}_\mathcal{U})_s \), for all \( s \in \mathbb{B} \).

(ii) \( \mathcal{V} \otimes \mathcal{O}_\mathcal{U} \) is obtained as a successive extension of \( \{\mathcal{L}_j\}_{j \in J} \subset \text{Pic}(X) \).

Proof. (i) First we prove that \( \mathcal{U} \) is indeed open. Since \( \rho \) is an algebraic morphism, \( \rho(\mathcal{Y}_s) \subset X \) is a locally closed analytic subvariety (it is constructible). If it is not open, the (local) components of \( \rho|_{\mathcal{Y}_s} \) satisfy a non-trivial algebraic relation. This relation holds on whole \( \mathcal{Y} \), since \( \mathcal{Y}_s \subset \mathcal{Y} \) is open, which contradicts the hypothesis that \( \rho(\mathcal{Y}) \subset \mathcal{Y} \) is open.

Now we proceed with the proof of the lemma. For all \( s \in \mathbb{B} \), the restriction of \( \text{ev} : \bigoplus_{\mu \in M} \rho^* \mathcal{L}_\mu \otimes \pi^* \pi_* \rho^*(\mathcal{L}_\mu^{-1} \otimes \mathcal{V})_{\mathbb{B}} \rightarrow (\rho^* \mathcal{V})_{\mathbb{B}} \) to \( \mathcal{Y}_s \) is the homomorphism \((\mathcal{V} \otimes \mathcal{O}_\mathcal{U})_s \). The maximality of \( \mu \in M \) implies that \( \pi_* \rho^*(\mathcal{L}_\mu^{-1} \otimes \mathcal{V}) \cong \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu}, \forall \mu \in M \), and \( \text{ev} \) is pointwise injective. We claim that, after suitable choices of bases in \( \pi_* \rho^*(\mathcal{L}_\mu^{-1} \otimes \mathcal{V}) \), the homomorphism \( \text{ev} \) descends to \( \mathcal{U} \). We deal with each \( \mu \in M \) separately, the overall basis being the direct sum of the individual ones.

Consider \( \mu \in M \) and a base point \( o \in \mathbb{B} \). Then \( \mathcal{V}'' := \mathcal{L}_\mu^{-1} \otimes \mathcal{V} \) has the properties:

\[-(\rho^* \mathcal{V}'')_{\mathbb{B}} \cong \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu} \oplus \bigoplus_{j \in J \setminus \{\mu\}} \rho^*(\mathcal{L}_j^{-1} \mathcal{L}_j)_{\mathbb{B}}^{\oplus m_j}.\]

We choose an isomorphism \( \alpha_\mathcal{Y} \) between them.

\[-\pi_* (\rho^* \mathcal{V}'')_{\mathbb{B}} \cong \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu}. \]

\[-\pi_* (\rho^* \mathcal{V}'') \rightarrow (\rho^* \mathcal{V}'')_{\mathbb{B}} \text{ is pointwise injective; let } \mathcal{J} \subset (\rho^* \mathcal{V}'')_{\mathbb{B}} \text{ be its image.} \]

We choose a complement \( \mathcal{W} \cong \bigoplus_{j \in J \setminus \{\mu\}} \rho^*(\mathcal{L}_j^{-1} \mathcal{L}_j)_{\mathbb{B}}^{\oplus m_j} \) of \( \mathcal{J} \), so \( (\rho^* \mathcal{V}'')_{\mathcal{Y}_s} = \mathcal{J} \oplus \mathcal{W} \). (\( \mathcal{W} \) is defined up to \( \text{Hom}(\rho^* \mathcal{V}'/\mathcal{J}, \mathcal{J}). \) Then \( \alpha_\mathcal{Y} \) above determines the pointwise injective homomorphism \( \alpha : \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu} \rightarrow (\rho^* \mathcal{V}'')_{\mathcal{Y}_s} \) whose second component vanishes, since \( \Gamma(\mathcal{Y}_s, \mathcal{W}) = 0 \). The left inverse \( \beta : (\rho^* \mathcal{V}'')_{\mathcal{Y}_s} \rightarrow \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu} \) of \( \alpha \) with respect to the splitting, satisfies \( \alpha \circ \beta|_\mathcal{J} = \text{Id}_\mathcal{J} \).

Claim After a suitable change of coordinates in \( \mathcal{O}_{\mathcal{Y}_s}^{\oplus m_\mu} \), the homomorphisms \( \alpha \) descends to \( \rho(\mathcal{Y}_s) \subset X \). Indeed, for any \( s \in \mathbb{B} \), we consider the diagram (recall that \( \mathcal{Y}_{os} \neq \emptyset \)):

\[
\begin{array}{c}
\mathcal{O}_{\mathcal{Y}_{os}}^{\oplus m_\mu} \\
\downarrow \alpha_s
\end{array} \xrightarrow{\alpha} \begin{array}{c}
\mathcal{Y}'_{Y_{os}}
\end{array}
\]

with \( \alpha_s := \beta_s \circ \alpha_o \in \text{End}(\mathbb{C}^m) \).

Similarly, we let \( \alpha_s' := \beta_o \circ \alpha_s \). Then holds \( \alpha_s' \alpha_s = \beta_o \alpha_s \beta_s \alpha_o = \beta_o \alpha_o = \text{Id} \) (the second equality uses \( \text{Im}(\alpha_o) = \mathcal{Y}_{Y_{os}} \), \( \text{Im}(\alpha_s) = \mathcal{Y}_{Y_{os}} \), and \( \alpha_s \beta_s|_\mathcal{J} = \text{Id} \)) and similarly \( \alpha_s' \alpha_s' = \text{Id} \). Thus \( \alpha_s \in \text{Gl}(m; \mathbb{C}) \) for all \( s \in \mathbb{B} \), and the new trivialization \( \tilde{\alpha}_s := \alpha \circ a \) of \( \mathcal{J} \) satisfies

\[
\tilde{\alpha}_s = \tilde{\alpha}_o \text{ along } \mathcal{Y}_{os}, \quad \forall \ s \in \mathbb{B},
\]

because \( \tilde{\alpha}_s|_{\mathcal{Y}_{os}} = (\alpha_s \beta_s) \alpha_o|_{\mathcal{Y}_{os}} = \alpha_o|_{\mathcal{Y}_{os}} = \alpha_o|_{\mathcal{Y}_{os}} \).

Also, for all \( s, t \in \mathbb{B} \) such that \( \mathcal{Y}_{st} \neq \emptyset \), the trivializations of \( \mathcal{J}_{Y_{st}} \) induced by \( \tilde{\alpha} \) from \( Y_s \) and \( Y_t \), coincide; equivalently, the following diagram commutes:

\[
\begin{array}{c}
\mathcal{O}_{\mathcal{Y}_{st}}^{\oplus m_\mu} \\
\downarrow \tilde{\alpha}_t
\end{array} \xrightarrow{\tilde{\alpha}_s} \begin{array}{c}
\mathcal{J}_{Y_{st}}
\end{array} \subset \begin{array}{c}
\mathcal{Y}_{Y_{st}}'
\end{array}
\]

\[
\iff \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{\mathcal{Y}_{st}} = \text{Id} \in \text{Gl}(r; \mathbb{C}).
\]
Indeed, $Y_{or}$ is non-empty and connected by (no-$\triangle$), so is enough to prove that the restriction of (2.7) to $Y_{or}$ is the identity. After restricting (2.6) to $Y_{or}$, we deduce

$$\tilde{\alpha}_s|_{Y_{or}} = \tilde{\alpha}_o|_{Y_{or}} = \tilde{\alpha}_t|_{Y_{or}} = \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{or}} = 1.$$ 

Now we can conclude that the trivialization $\tilde{\alpha}$ of $\pi(s)\rho^*(\mathcal{L}_\mu \otimes \mathcal{V})$ descends to $\mathcal{U} := \rho(\mathcal{V}_B)$, as announced. Indeed, define $\tilde{\alpha} : \mathcal{O}^{\oplus m}_U \to \mathcal{V} \otimes \mathcal{O}_U$, $\tilde{\alpha}(x) := \tilde{\alpha}_s(x)$ for some $s \in B$ such that $x \in Y_s$. The diagram (2.7) implies that $\tilde{\alpha}(x)$ is independent of $s \in B$ with $s(x) = 0$.

(ii) Apply repeatedly the first part.

**Lemma 2.7** Let the situation be as in lemma 2.6, and assume $Y_0 \subset \mathcal{U}$ is $2 \geq 0$ (cf. 1.4). Let $A, B, C$ be vector bundles on $X$, whose restriction to $\mathcal{U}$ fit into $0 \to A_U \to B_U \to C_U \to 0$. Then $B$ is an extension of $C$ by $A$ on $X$.

**Proof.** As $Y_0 = 2 \geq 0$, $\text{Ext}^1(C, A) \to \text{Ext}^1(C_{\mathcal{Y}_m}, A_{\mathcal{Y}_m})$ is an isomorphism for an increasing sequence of thickenings $\{\mathcal{Y}_m\}_{m \geq m_0}$ of $Y_0$. The restriction of $B_U \in \text{Ext}^1(C_U, A_U)$ to $\mathcal{Y}_m$ yields the extension $0 \to A \to B' \to C \to 0$ over $X$, with the property that $B'_{\mathcal{Y}_m} \cong B_{\mathcal{Y}_m}$, for all $m \geq m_0$. As $Y_0 = 2 \geq 0$, it follows $B \cong B'$.

2.2. **The splitting criterion.** Let $\mathbb{F} \hookrightarrow \mathbb{C}$ be a finitely generated extension of $\mathbb{Q}$, such that $X, \mathcal{V}, \mathcal{V}$ are defined over $\mathbb{F}$; its algebraic closure $\overline{\mathbb{F}} \subset \mathbb{C}$ is countable.

**Theorem 2.8** Let $X$ be a a smooth, projective variety, and assume the following:

(i) the situation is as in (2.3);

(ii) $\mathcal{V}$ splits on $\mathbb{Y}$; alternatively, $\mathcal{V}_{\mathcal{Y}_s}$ splits, for a very general $s \in S$;

(iii) $\mathbb{Y} \subset X \otimes \mathbb{C} \overline{\mathbb{R}}$ is $2 \geq 0$ (in particular, it is $2 \geq 0$).

Then $\mathcal{V}$ is obtained by successive extensions of line bundles on $X$. If, moreover, $X$ has the property that $H^1(X, \mathcal{L}) = 0$ for all $\mathcal{L} \in \text{Pic}(X)$, then $\mathcal{V}$ splits.

The very same statements remain valid if $X$ is defined over an uncountable, algebraically closed field, rather than over $\mathbb{C}$.

**Proof.** First assume that $\mathcal{V}$ splits. By lemma 2.3, there is a ball $B \subset S$, such that $(\rho^*\mathcal{V})_{Y_s}$ splits; lemma 2.6 implies that $\mathcal{V}$ is a successive extension of line bundles on a tubular neighbourhood of $Y_o$, $o \in B$. It remains to apply lemma 2.7.

Let $\tau : S \to S_\mathbb{F}$ be the trace morphism; for $s \in S$, let $\mathbb{K}_0 := \overline{\mathbb{F}}(\tau(s))$ be the residue field of $\tau(s) \in S_\mathbb{F}$. For $s$ very general, $\tau(s)$ is the generic point of $S_\mathbb{F}$, so $\mathbb{K}_0 = \overline{\mathbb{F}}(S_\mathbb{F})$. Assume $\mathcal{V}_s$ splits; in the Cartesian diagram below $\mathcal{V}_s = \mathcal{V}_{\mathbb{Y}_0} \times \mathbb{K}_0 \mathbb{C}$:

$$\begin{array}{ccc}
Y_s & \longrightarrow & \mathbb{Y}_0 \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(\mathbb{K}_0) \longrightarrow S_\mathbb{F}.
\end{array}$$

The splitting of a vector bundle commutes with base change, for varieties defined over algebraically closed fields (cf. [13]). The previous discussion implies that $\mathcal{V}_{\mathbb{K}_0}$ splits on $\mathbb{Y}_{\mathbb{K}_0}$ (the geometric generic fibre of $\mathbb{Y}_\mathbb{F} \to S_\mathbb{F}$); hence the same holds for $\mathcal{V}_s$ on $\mathcal{Y} = \mathcal{Y}_{\mathbb{K}_0} \times \mathbb{C}$. This brings us back to the previous case. For the last statement, we use once more that the splitting property commutes with base change. 

$\square$
The proof of the theorem even precises the meaning of the term ‘very general’: if \( \mathcal{F} \) is the field of definition of \( X, Y, \mathcal{F} \), then, in local affine coordinates coming from \( S_{\mathcal{F}} \), the coordinates of \( s \in S \) should be algebraically independent over \( \mathcal{F} \).

**Definition 2.9** We say that the variety \( X \) is 1-splitting if \( H^1(X, \mathcal{L}) = 0 \), for all \( \mathcal{L} \in \text{Pic}(X) \).

The simplest examples of 1-splitting varieties are the Fano varieties of dimension at least two with cyclic Picard group, and products of such. In [6,4] are obtained examples of homogeneous 1-splitting varieties with larger Picard groups.

**Remark 2.10** The genericity assumption in 2.8 plays an essential role. If one is interested in the same result for arbitrary \( Y \), one needs stronger positivity hypotheses, in order to apply effective cohomology vanishing results; in [13] I obtained a similar result for ample, globally generated vector bundles. In this case, the situation [2.3] holds automatically.

Theorem 2.8 will be illustrated in the following sections with concrete examples.

### 3. Positivity Properties of Zero Loci of Sections in Vector Bundles

Throughout this section, \( N \) stands for a globally generated vector bundle on \( X \) of rank \( \nu \).

#### 3.1. Sommese’s \( q \)-ampleness for globally generated vector bundles

We briefly review the \( q \)-ampleness concept introduced in [23].

**Proposition 3.1** (cf. [23, Proposition 1.7]). The following statements are equivalent:

(i) For all coherent sheaves \( \mathcal{F} \) on \( X \) holds

\[
H^t(X, \mathcal{F} \otimes \text{Sym}^m(N)) = 0, \quad \forall t \geq q + 1, \quad \forall m \gg 0; \tag{3.1}
\]

(ii) The fibres of the morphism \( \mathbb{P}(N^\nu) \to |\mathcal{O}_{\mathbb{P}(N^\nu)}(1)| \) are at most \( q \)-dimensional.

A vector bundle with these properties is called \( q \)-ample.

**Proposition 3.2** Assume \( N \) is \( q \)-ample. Then \( \mathcal{O}_{\mathbb{P}(N^\nu)}(1) \) is \( q \)-positive (as in 1.14). If \( Y \) is the zero locus of a regular section in \( N \), then \( Y \subset X \) is \( (\dim X - \nu - q)^{>0} \).

**Proof.** The first statement is proved in [19, Theorem 1.4]. Since the section is regular, \( \text{codim}_X(Y) = \nu \). For any coherent sheaf \( \mathcal{F} \) on \( X \) holds

\[
H^{\nu+t}(\mathbb{P}(N), \mathcal{O}(-\nu - m) \otimes \pi^*(\mathcal{F} \otimes \text{det}(N)^{-1})) = H^{t+1}(X, \mathcal{F} \otimes \text{Sym}^m(N)),
\]

so the \( H^{\geq \nu+q} \)-cohomology on \( \mathbb{P}(N) \) vanishes; the same holds for \( \tilde{X} \subset \mathbb{P}(N) \). \( \square \)

For \( \nu = 1 \), the line bundle \( N \) is \( q \)-ample if an only if the morphism \( X \to |N| \) has at most \( q \)-dimensional fibres. This property is easy to check, and convenient for concrete applications. In contrast, for \( \nu \geq 2 \), the criterion is not effective; the \( q \)-ampleness test for \( N \) is too restrictive to check the positivity of the zero loci of its sections (cf. remarks 3.4, 7.4)

#### 3.2. The positivity criterion [1.12]

Suppose \( Y \subset X \) is lci of codimension \( \delta \), and the zero locus of a section \( s \) in \( N \), of rank \( \nu \geq 2 \); we do not assume that \( s \) is regular, so we allow \( \delta \leq \nu \).
In this context, the situation 1.12 arises as follows: since $Y$ is the zero locus of $s \in \Gamma(N)$, the blow-up $\tilde{X}$ fits into

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & \mathbb{P}(N) = \mathbb{P}\left(\bigwedge^{\nu} N \otimes \det(N)^{\vee}\right) \\
\downarrow & & \downarrow \\
X & \xrightarrow{b} & \mathbb{P} := \mathbb{P}\left(\bigwedge^{\nu-1} \Gamma(N)^{\vee}\right),
\end{array}
$$

(3.2)

and holds

$$
\mathcal{O}_{\tilde{X}}(E_Y) = \mathcal{O}_{\mathbb{P}(N)}(-1)|_{\tilde{X}} = (\det(N) \otimes \mathcal{O}_\mathbb{P}(-1))|_{\tilde{X}}.
$$

(3.3)

The assumptions of the proposition 1.12 are satisfied, and we deduce the following.

**Proposition 3.3** Suppose $\det(N)$ is ample. If the dimension of the generic fibre of $b$ is $p + 1$, then $\mathcal{O}_{\tilde{X}}(E_Y)$ is dim $b(\tilde{X})$-positive, and $Y$ is $p^0$.

Observe that the propositions 3.1 and 3.3 deal with complementary situations: $\mathcal{O}_{\mathbb{P}(N^{\nu})}(1)$ is the pull-back of an ample line bundle; $\mathcal{O}_{\tilde{X}}(E_Y)$ is relatively ample for some morphism.

Let $W \subseteq \Gamma(N)$ be a vector subspace which generates $N$; let $\dim W = \nu + u + 1$. A globally generated vector bundle $N$ on $X$ is equivalent to a morphism $f: X \to \text{Gr}(W; \nu)$ into the Grassmannian of $\nu$-dimensional quotients of $W$; $\det(N)$ is ample if and only if $\varphi$ is finite onto its image.

For $\text{Gr}(W; \nu)$ and $N$ the universal quotient bundle on it, we can write the morphism $b$ in (3.2) explicitly: $\mathbb{P}(N) \to \mathbb{P}$ is defined by

$$(x, \langle e_x \rangle) \mapsto \det(N_x/\langle e_x \rangle)^{\vee} \subset \bigwedge^{\nu-1} N_x^{\vee} \subset \bigwedge^{\nu-1} W^{\vee}.
$$

(3.4)

$$(\langle e_x \rangle) \text{ stands for the line generated by } e_x \in N_x, \ x \in \text{Gr}(W; \nu).$$

The restriction to the Grassmannian corresponds to the commutative diagram

$$
\begin{array}{cccccc}
0 & \xrightarrow{\beta s} & W/\langle s \rangle \otimes \mathcal{O}_{\text{Gr}(W; \nu)} & \xrightarrow{\beta} & \mathcal{N} & \xrightarrow{\beta s} & \mathcal{N}/\langle \beta s \rangle & \xrightarrow{\beta} & 0
\end{array}
$$

(3.5)

Thus $b$ is the desingularization of the rational map

$$
g_s: \text{Gr}(W; \nu) \dashrightarrow \text{Gr}(W/\langle s \rangle; \nu - 1), \quad [W \to N] \mapsto [W/\langle s \rangle \to N/\langle \beta s \rangle].
$$

(3.6)

followed by the Plücker embedding of $\text{Gr}(W/\langle s \rangle; \nu - 1)$; the indeterminacy locus of $b$ is $\text{Gr}(W/\langle s \rangle; \nu) \subseteq \text{Gr}(W; \nu)$.

**Remark 3.4** We mentioned that Sommese’s $q$-ampleness criterion is not effective for $\nu \geq 2$. For $X = \text{Gr}(\nu + u + 1; \nu)$, Sommese’s criterion implies that $N$ is $q$-ample, with $q = \dim \mathbb{P}(N^{\nu}) - \mathbb{P}^{\nu+u} = \dim X - (u + 1)$; hence $Y$, the zero locus of a generic section of $N$, is $(u + 1 - \nu)^{\geq 0}$. On the other hand, the criterion 1.12 implies that $Y$ is actually $u^{>0}$.

There is a ‘universal’ rational map $g_{\text{univ}}$ containing the maps $g_s$ above, as $s$ varies:

$$
g_{\text{univ}}: \mathbb{P}(W) \times \text{Gr}(W; \nu) \dashrightarrow \text{Flag}(W; \nu + u, \nu - 1),
$$

$$
((s), [W \to N]) \mapsto [W \to \frac{W}{s} \to \frac{N}{\langle \beta s \rangle}].
$$

(3.7)
(The right hand side denotes the flag variety of successive quotients of \(W\).) It is undefined on the incidence variety \(J := \{(s, N) \mid s \in \text{Ker}(W \to N)\} \).

Back to the general case of a globally generated vector bundle on a variety \(X\). By varying \(s \in W\), one obtains the family of subvarieties of \(X\) (over \(\mathbb{P}(W)\))
\[
\mathcal{Y} := (\mathbb{P}(W) \times X) \times_{\mathbb{P}(W) \times \text{Gr}(W; \nu)} J,
\]
and the situation mentioned at \([1.13]\)
\[
\begin{array}{c}
\text{Bl}_y(\mathbb{P}(W) \times X) \\
\downarrow \pi
\end{array}
\begin{array}{c}
\text{Bl}_J(\mathbb{P}(W) \times \text{Gr}(W; \nu))
\end{array}
\begin{array}{c}
\text{Flag}(W; \nu + u; \nu - 1).
\end{array}
\]
\[\text{(3.8)}\]
\[
\begin{array}{c}
\mathbb{P}(W) \times X
\end{array}
\begin{array}{c}
\varphi
\end{array}
\begin{array}{c}
\mathbb{P}(W) \times \text{Gr}(W; \nu)
\end{array}
\begin{array}{c}
\xrightarrow{\text{univ}}
\end{array}
\begin{array}{c}
\text{Flag}(W; \nu + u; \nu - 1).
\end{array}
\]
\[\text{(3.9)}\]

(The intersection \(X \cap \text{Gr}(W/(s); \nu)\) may not be transverse for certain \(s \in W\).) Proposition \([3.3]\) can be restated as follows.

**Corollary 3.5** Let \(\varphi : X \to \text{Gr}(W; \nu), \nu \geq 2\), be a morphism finite onto its image. If the general fibre of \(g_{\text{univ}} \circ \varphi : \mathbb{P}(W) \times X \to \text{Flag}(W; \nu + u; \nu - 1)\) is at least \((p + 1)\)-dimensional, then \(Y_s\) is \(p > 0\), for all \(s \in W\) such that \(Y_s\) is irreducible lci in \(X\).

### 3.3. Picard groups, and the diagram \([2.3]\)

The sheaf \(\mathcal{K}\) defined by
\[
0 \to \mathcal{K} := \text{Ker}(\text{ev}) \to W \otimes \mathcal{O}_X \xrightarrow{\rho} N \to 0
\]
is locally free, and the incidence variety
\[
\mathcal{Y} = \{(s, x) \mid s(x) = 0\} \subset X \times \mathbb{P}(W)
\]
is isomorphic to \(\mathbb{P}(\mathcal{K})\). We denote \(\pi\) and \(\rho\) the projections onto \(\mathbb{P}(W)\) and \(X\) respectively; actually we restrict ourselves to a sufficiently small open subset \(S \subset \mathbb{P}(W)\). The vector bundles \(N^\otimes 2\) and \(N^\otimes 3\) are generated by \(W^\otimes 2\) and \(W^\otimes 3\) respectively, and determine the double and triple self-intersection diagrams \([2.4]\).

**Proposition 3.6** Let \(\mathcal{Y} \subset S \times X\) be as in \([3.9]\), with \(S \subset \mathbb{P}(W)\) suitably small. Then the conditions of \([2.3]\) are fulfilled as soon as \(N\) satisfies any of the following conditions:

(i) \(N\) is Sommese-q-ample (cf. \([7.7]\)) and
\[
\text{dim } X - q \geq 3\nu + 1 \text{ (for } \nu \geq 2\text{), or dim } X - q \geq 5 \text{ (for } \nu = 1\text{).}
\]

or

(ii) \(\nu \geq 2\), and the generic fibre of \(\mathbb{P}(W) \times X \to \text{Flag}(W; \nu + u; \nu - 1)\) is at least \(2(\nu + 1)\)-dimensional (cf. \([3.8]\)).

The condition is satisfied by any \(X \subset \text{Gr}(W; \nu)\), with \(\text{codim}_{\text{Gr}(W; \nu)}(X) + 2\nu + 1 \leq u\).

**Proof.** (i) The non-emptiness of the triple intersections requires \(\text{dim } X - q \geq 3\nu + 1\), by lemma \([1.17]\) the isomorphism of Picard groups requires \(\text{dim } X - q \geq 2\nu + 3\), by theorem \([1.20]\).

(ii) In this case, \(Y_s \subset X\) is \((2\nu + 1)^{>0}\), so the triple intersections are non-empty (cf. \([1.17]\)); the isomorphism of Picard groups requires \((2\nu + 1) - \nu \geq 3\) (cf. \([2.4]\)). \(\square\)

### 4. Splitting along zero loci of globally generated vector bundles

Let \(\mathcal{F}\) be an arbitrary vector bundle on \(X\). The previous discussion immediately yields the following criterion.
Theorem 4.1 Let $X$ be 1-splitting variety (cf. [23]), $\mathcal{N}$ be a globally generated vector bundle of rank $\nu$ on $X$ such that $\det(\mathcal{N})$ is ample, and $W \subset \Gamma(X, \mathcal{N})$ be a generating vector subspace. Assume that $\mathcal{N}$ satisfies one of the conditions in proposition [3.6].

Then $\mathcal{V}$ splits on $X$, if its restriction to the zero locus of a very general $s \in W$ splits. (If $X$ is not 1-splitting, $\mathcal{V}$ is a successive extension of line bundles.)

Is natural to ask what happens if one drops the hypothesis that $\det(\mathcal{N})$ is ample. The Stein factorization of $\varphi : X \to \text{Gr}(W; \nu)$ decomposes it into a morphism with connected fibres followed by a finite map.

Theorem 4.2 Let $\varphi : X \to X'$ be a smooth morphism of relative dimension $d$, $\mathcal{N}'$ a globally generated vector bundle on $X'$ of rank $\nu$ with $\det(\mathcal{N}')$ ample, and $\mathcal{N} = \varphi^*\mathcal{N}'$.

Assume that $X$ is 1-splitting, and $\mathcal{N}'$ satisfies proposition 3.6. Then $\mathcal{V}$ splits on $X$ as soon as $\mathcal{V}$ splits along $Y_s$, for $s$ very general.

Proof. The conditions of theorem [23] are fulfilled. Indeed, consider the family $\mathcal{V}'$ of subvarieties of $X'$ as above and let $\mathcal{V} := \varphi^{-1}(\mathcal{V}')$. The positivity is preserved under pull-back (cf. lemma 1.11, 1.16). For the condition (Pic), the morphism $\tilde{\varphi} : X \to X'$ induced at the level of the blows-up is still smooth of relative dimension $d$, so $\mathcal{O}_{X'}(E_Y) = \tilde{\varphi}^*\mathcal{O}_{X'}(E_Y')$ is $(q + d)$-ample; it remains to apply 1.20.

It is interesting that this yields new results even in the simplest case $X = X' \times V$, with $X'$, $V$ smooth, and $\mathcal{N}' = \mathcal{O}_{X'}(1)$ is globally generated, ample. If $X$ is 1-splitting, then $X'$, $V$ are both 1-splitting; if either $X'$ or $V$ are simply connected, the converse is true.

Corollary 4.3 Assume that $X = X' \times V$ is 1-splitting, $\dim X' \geq 5$, and $\mathcal{O}_{X'}(1)$ is ample, globally generated. Then $\mathcal{V}$ splits on $X$, if it splits along a very general divisor in $|\varphi^*\mathcal{O}_{X'}(1)|$.

Example 4.4 (i) (Vector bundles on multi-projective spaces). A cohomological splitting criterion for vector bundles $\mathcal{V}$ on $X := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ is obtained in [10] Theorem 4.7. It generalizes Horrocks’ criterion, and involves the vanishing of $(n_1 + 1) \ldots (n_r + 1)$ cohomology groups. By repeatedly applying corollary 4.3 for the pull-back of $\mathcal{O}_{\mathbb{P}^{n_1}}(1)$, we deduce that $\mathcal{V}$ splits if and only if it does so along a very general $Y := \mathbb{P}^2 \times \ldots \times \mathbb{P}^2$. (As formulated, the corollary yields the restriction to a product of copies of $\mathbb{P}^4$. We can restrict to a product of copies of $\mathbb{P}^2$ because the condition (Pic) in the diagram [2.3] is fulfilled; see also [5, 13].) Thus the number of cohomological tests for the splitting of $\mathcal{V}$ is reduced to $3^r$.

(ii) (Vector bundles on products of projective spaces and quadrics). The results of [10] have been extended in [6, Theorem 2.14, 2.15] for vector bundles on the product $X_1 \times X_2$, where $X_1$ is as above and $X_2$ is a product of hyper-quadrics $Q_n \subset \mathbb{P}^{n+1}$. Again, the splitting criterion involves a very large number of cohomological conditions.

Our result implies that a vector bundle on $X_1 \times X_2$ splits if and only if it splits when restricted to a very general $X'_1 \times X'_2 \subset X_1 \times X_2$, where $X'_1$ is a product of projective planes $\mathbb{P}^2$ and $X'_2$ is a product of copies of $Q_3$. (The reduction from $Q_4$ to $Q_3$ is implied by [5, 13].) Hence, the number of necessary cohomological tests is dramatically reduced again.

5. Positivity properties of sources of $G_m$-actions

Another (totally different) framework which leads to the situation [23] arises in the context of actions of the multiplicative group on (almost) homogeneous varieties.
5.1. Basic properties of the BB-decomposition. We start with general considerations which should justify the appearance of the homogeneous varieties in the next section. Let $G$ be a connected reductive linear algebraic group, and $T \subset B$ be a maximal torus and a Borel subgroup of it. Also, consider a 1-parameter subgroup (1-PS for short) $\lambda : G_m \to G$, where $G_m = \mathbb{C}^*$ is the multiplicative group; we assume $\lambda(G_m) \subset T$. Finally, let $X$ be a smooth projective $G$-variety with an effective $G$-action $\mu : G \times X \to G \times X$.

The 1-PS induces the action $\lambda : G_m \times X \to X$, which determines the so-called Bialynicki-Birula (BB for short) decomposition of $X$ into locally closed subsets. Below are summarized its basic properties (cf. [13, 14]):

- For any $x \in X$, the rational map $\mathbb{P}^1 \dashrightarrow X, t \mapsto \lambda(t) \times x = \mu(\lambda(t), x)$, extends to $\mathbb{P}^1$.
- The specializations at $\{0, \infty\} = \mathbb{P}^1 \setminus G_m$ are denoted $\lim_{t \to 0} \lambda(t) \times x$ and $\lim_{t \to \infty} \lambda(t) \times x$; they are both fixed by $\lambda$.
- The fixed locus $X^\lambda$ of the action is a disjoint union $\bigsqcup_{s \in S_{\text{BB}}} Y_s$ of smooth subvarieties.

For $s \in S_{\text{BB}}, Y_s^+ := \{x \in X \mid \lim_{t \to 0} \lambda(t) \times x \in Y_s\}$ is locally closed in $X$ (a BB-cell).

- $X = \bigsqcup_{s \in S_{\text{BB}}} Y_s^+$, and the morphism $Y_s^+ \to Y_s, x \mapsto \lim_{t \to 0} \lambda(t) \times x$ is a locally trivial, affine space fibration. It is not necessarily a vector bundle, that is the transition functions may be non-linear.
- The *source* $Y := Y_{\text{source}}$ and the *sink* $Y_{\text{sink}}$ of the action are uniquely characterized by the fact that $Y^+ = Y_{\text{source}}^+ \subset X$ is open and $Y^+_{\text{sink}} = Y_{\text{sink}}$. 

We denote:

$$
G(\lambda) := \{ g \in G \mid g^{-1} \lambda(t) g = \lambda(t), \forall t \in G_m \} \text{ the centralizer of } \lambda \text{ in } G,
$$

$$
P(\pm \lambda) := \{ g \in G \mid \lim_{t \to 0} \lambda(t)^{\pm} g \lambda(t)^{\mp} \text{ exists in } G \},
$$

$$
U(\pm \lambda) := \{ g \in G \mid \lim_{t \to 0} \lambda(t)^{\pm} g \lambda(t)^{\mp} = e \in G \}.
$$

Then $G(\lambda)$ is a connected, reductive subgroup of $G$, $P(\pm \lambda) \subset G$ are parabolic subgroups, $G(\lambda)$ is their Levi-component, and $U(\pm \lambda)$ the unipotent radical (cf. [24 §13.4]). The adjoint action of $\lambda$ on $\text{Lie}(G)$ decomposes it into the direct sum of its weight spaces; we group them into the zero, strictly positive, and negative weight spaces:

$$
\text{Lie}(G) = \text{Lie}(G)_0^\lambda \oplus \text{Lie}(G)_1^\lambda \oplus \text{Lie}(G)_\lambda, \text{ and } \text{Lie}(P(\pm \lambda)) = \text{Lie}(G)_0^\lambda \oplus \text{Lie}(G)_\lambda^\pm.
$$

Lemma 5.1 (i) $Y$ is invariant under $P(-\lambda)$, thus for $G(\lambda)$ too.
(ii) $Y_s^+$ is $P(\lambda)$-invariant and $U(\lambda)$ preserves the fibration $Y_s^+ \to Y_s$, for all $s \in S_{\text{BB}}$.

Proof. (i) First we prove that $G(\lambda)$ leaves $Y$ invariant; for $y \in Y$, and $c \in G(\lambda)$ holds:

$$
y = (c^{-1} \lambda(t) c) y \Rightarrow cy = \lambda(t) \cdot (cy) \forall t \in G_m \Rightarrow cy \in X^\lambda; \text{ thus } G(\lambda)y \subset X^\lambda.
$$

But $X^\lambda$ is the disjoint union of its components, $G(\lambda)y$ is connected, and contains $y \in Y$; hence $G(\lambda)Y = Y$. We claim that $Y$ is also $P(-\lambda)$-invariant; for $g \in P(-\lambda)$ holds:

$$
c := \lim_{t \to 0} \lambda(t)^{-1} g \lambda(t) \in G(\lambda) \Rightarrow cy = \lim_{t \to 0} \lambda(t)^{-1} g \lambda(t) y \in Y \Rightarrow \lim_{t \to 0} \lambda(t)^{-1} (gy) \in Y.
$$

We claim that $gy \in Y$; otherwise $gy \in X \setminus Y$ is ‘repelled’ from $Y$, since $Y$ is the source of $\lambda$, and the limit belongs to some other component of $X^\lambda$.

(ii) The previous argument shows also that $Y_s$ is $G(\lambda)$-invariant, for any $s \in S_{\text{BB}}$. Now consider $x \in X$ with $\lim_{t \to 0} \lambda(t)x = y \in Y_s \subset X^\lambda, g \in P(\lambda)$:
If $E$ is relatively ample.

One may check that, $\lim_{t \to 0} \lambda(t)x = y$ implies that $\lim_{t \to 0} \lambda(t)(gx) = y$, for all $g \in U(\lambda)$. 

Theorem 5.2 requires that the embedded deformations of $Y$ sweep out an open subset of $X$; this is achieved if $GY$, defined set theoretically as $\{gy \mid g \in G, y \in Y\}$, is open in $X$.

**Lemma 5.2** $GY$ has a natural structure of a closed subscheme of $X$. Therefore $GY \subset X$ is open if and only if $GY = X$.

**Proof.** Indeed $GY$ is the image of $\mu : G \times Y \to X$. Since $Y$ is $P(-\lambda)$-invariant, it factorizes $(G \times Y)/P(-\lambda) \to X$, for $p \times (g, y) := (gp^{-1}, py)$. But $P(-\lambda)$ is parabolic, so $(G \times Y)/P(-\lambda)$ is projective, hence $\text{Im}(\mu)$ is a closed in $X$. 

**Remark 5.3** Since $P(\lambda)P(-\lambda)Y = P(\lambda)Y = U(\lambda)G(\lambda)Y = U(\lambda)Y$, and $P(\lambda)P(-\lambda)$ is open in $G$, it follows:

$$GY = X \iff U(\lambda)Y \subset X$$

This observation hints towards the fact that the $G$-varieties satisfying the lemma \[5.2\] should be homogeneous (or, at least, have an open $B$-orbit).

### 5.2. The positivity criterion \[1.12\]

Now we follow the same steps as in section 3.

**Proposition 5.4** Assume $G_m$ acts effectively on the smooth projective variety $X$, and $Y$ is the source of the action. Then $Y$ is $p^{\to 0}$, with $p := \dim X - \dim(X \setminus Y^+) - 1$.

Observe that:

$$\dim X - \dim(X \setminus Y^+) = \dim X - \max\{\dim Y_s^+ \mid s \neq \text{source}\}$$

$$= \min_{s \neq \text{source}} \left\{ \begin{array}{c}
\text{number of strictly negative } \\
\text{weights of } \lambda \text{ on } T_X|_Y
\end{array} \right\}.$$ 

(5.3)

**Proof.** Take a $G_m$-equivariant embedding of $X$ into some $\mathbb{P}^N$, such that $X$ is not contained in a hyperplane. (Linearize the $G_m$-action in a very ample line bundle on $X$.) In coordinates $z_{N_1} \in \mathbb{C}^{N_1}, \ldots , z_{N_r} \in \mathbb{C}^{N_r}$, the $G_m$-action on $\mathbb{P}^N$ is:

$$t \times [z_{N_1}, z_{N_2}, \ldots , z_{N_r}] = [z_{N_1}, t^{m_2}z_{N_2}, \ldots , t^{m_r}z_{N_r}], \quad \text{with } 0 < m_2 < \ldots < m_r.$$ 

(5.4)

The source and sink of $\mathbb{P}^N, X$ are respectively:

$$\mathbb{P}^N_{\text{source}} = \{[z_{N_1}, 0, \ldots , 0]\}, \quad \mathbb{P}^N_{\text{sink}} = \{[0, \ldots , 0, z_{N_r}]\},$$

$$Y = Y_{\text{source}} = X \cap \mathbb{P}^N_{\text{source}}, \quad Y_{\text{sink}} = X \cap \mathbb{P}^N_{\text{sink}}.$$ 

Note that $\mathbb{P}^N_{\text{source}}$ is the indeterminacy locus of the rational map

$$\mathbb{P}^N \dashrightarrow \mathbb{P}^{N'}, \quad [z_{N_1}, z_{N_2}, \ldots , z_{N_r}] \mapsto [z_{N_2}, \ldots , z_{N_r}],$$

which can be resolved by a simple blow-up. By restricting to $X$, we get the situation \[1.12\]:

$$\tilde{X} = \text{Bl}_Y(X) \xrightarrow{\iota} \mathbb{P}^N := \text{Bl}_{\mathbb{P}^N_{\text{source}}}(\mathbb{P}^N) \xrightarrow{\text{b}} \mathbb{P}^{N'}$$ 

(5.5)

If $E \subset \mathbb{P}^N$ is the exceptional divisor, $\mathcal{O}_{\tilde{X}}(E) = \mathcal{O}_{\mathbb{P}^N}(E)|_{\tilde{X}} = \mathcal{O}_b(1) \otimes b^*\mathcal{O}_{\mathbb{P}^{N'}}(-1)|_{\tilde{X}}$, and $\mathcal{O}_b(1)$ is relatively ample.

It remains to understand $(bh)(\tilde{X})$; we claim that it is a component of $X \setminus Y^+$. Indeed, $(\mathbb{P}^N_{0})^+ = \{z = [z_{N_1}, z_{N_2}, \ldots , z_{N_r}] \mid z_{N_1} \neq 0\}$ and $\mathbb{P}^N \setminus (\mathbb{P}^N_{0})^+$ is the set of limit points
\[
\lim_{t \to \infty} t \times z, \text{ where } z \in (\mathbb{P}_0^N)^+ \text{ and } t \text{ acts by scalar multiplication (not (5.4)) on the coordinates } z_{N_1}, \ldots, z_{N_r}. \text{ The situation on } X \text{ is similar: } Y^+ = X \cap (\mathbb{P}_0^N)^+ \text{ and } Y^+ \to Y \text{ is an affine space fibration (cf. [4, Theorem 2.5]); the scalar multiplication on fibres make sense (at least generically), and the corresponding limit points (for } t \to \infty \text{) are contained in } X \setminus Y^+. \]

5.3. The Picard group, and the diagram (2.3).

**Theorem 5.5** Let the situation be as in proposition 5.4 with \( \dim X - \dim(X \setminus Y^+) \geq 2. \) Then \( \text{Pic}(X) \to \text{Pic}(Y) \) is an isomorphism.

**Proof.** Proposition 5.4 implies that \( Y \) is \( 1 > 0, \) so the cohomological dimension of \( X \setminus Y \) is at most \( \dim X - 2, \) cf. 1.8(i); in particular, \( X \setminus Y \) contains no divisors. On the other hand, the inclusion \( Y^+ \subset X \) yields the exact sequence
\[
0 \to \text{Pic}(X \setminus Y^+) = \text{Pic} \left( \bigcup_{s \neq \text{source}} Y_s^+ \right) \to \text{Pic}(X) \to \text{Pic}(Y^+) \cong \text{Pic}(Y) \to 0.
\]
On the right hand side, the isomorphism holds because \( Y^+ \to Y \) is an affine space fibration. The left hand side is the free abelian group generated by the divisors \( Y_s^+, s \in S_{BB} \setminus \{\text{source}\}; \) but we saw that such divisors, contained in \( X \setminus Y^+, \) do not exist. \( \square \)

Note that the inequality in theorem 1.20 is ‘twice weaker’ than in 5.5 above. This is due to the fact that \( G_m \)-actions yield *complex* deformation retracts of \( X \setminus Y \) onto subvarieties of \( X, \) in contrast with the *real* retracts arising in the Morse theoretic proof of 1.20.

**Proposition 5.6** Consider the \( G \)-subvariety \( \mathcal{Y} := \mu(G \times Y) \subset G \times X. \) Assume that \( \mathcal{Y} \xrightarrow{\mu} X \) is smooth, and \( \dim X - \dim(X \setminus Y^+) \geq 2\text{codim}_X(Y) + 2 \geq 6. \) Then the conditions in (2.3) are satisfied.

**Proof.** By lemma 1.17, the double and triple self-intersections (2.4) of \( \mathcal{Y} \) are non-empty and connected; the generic intersections are also smooth. It remains to prove the isomorphisms of the Picard groups. For \( Y \) (so also \( Y_g = gY, g \in G \)) this follows from 5.5 for the double intersections \( Y \cap Y_g, g \in G \) generic, the isomorphism is implied by 1.20. \( \square \)

6. Splitting criteria for vector bundles on homogeneous varieties

In this section we apply the conclusions of the previous section to homogeneous varieties. Assume \( X = G/P, \) where \( G \) is connected, reductive, and \( P \) is a parabolic subgroup, and consider a 1-PS \( \lambda \) of \( G. \) For any parabolic subgroup \( Q \) of \( G, \) denote \( \text{Weyl}(Q) := \text{Weyl}(\text{Levi}(Q)). \)

**Lemma 6.1** The following statements hold:

(i) The components of \( X^\lambda \) are homogeneous for the action of \( G(\lambda). \)
(ii) The source \( Y \) contains \( \hat{e} \in G/P \) if and only if \( \lambda \subset P \) and \( \text{Lie}(G)^{\lambda} \subset \text{Lie}(P). \)

**Proof.** (i) The differential of the multiplication \( d\mu : \text{Lie}(G) \to T_yX \) is surjective at any \( y \in X^\lambda, \) and is \( \lambda \)-equivariant for the adjoint action on \( \text{Lie}(G). \) Both sides decompose into direct sums of weight spaces; in particular, \( d\mu_y(\text{Lie}(G)^{\lambda}) = (T_yX)^{\lambda}, \) so the differential of \( \mu : G(\lambda) \to Y \) is surjective at any \( y \in X^\lambda. \) Hence all the \( G(\lambda) \)-orbits are open in \( X^\lambda, \) therefore the components of \( X^\lambda \) are homogeneous under \( G(\lambda). \)

(ii) The point \( \hat{e} \) belongs to \( Y \) if and only if:

- \( \hat{e} \) is fixed by \( \lambda, \) that is \( \lambda \subset P; \)
Let the latter cells are pairwise disjoint. It follows that each Bruhat cell equals some BB-cell. Y belongs to a component and g
union of locally closed orbits under the action of Q

If Q, P \subset G are two parabolic subgroups, G/P decomposes into the following finite disjoint union of locally closed orbits under the action of Q:

\[ G/P = \bigcup_{w \in S_{\text{Bruhat}}} wP, \quad \text{with } S_{\text{Bruhat}} = \text{Weyl}(Q) \setminus \text{Weyl}(G)/\text{Weyl}(P). \]

Actually, S_{\text{Bruhat}} parameterizes the Weyl(Q)-orbits in (G/P)^T. Each double coset in S_{\text{Bruhat}} contains a unique representative of minimal length; for each w \in S_{\text{Bruhat}} of minimal length,
\[ \dim(wP) = \text{length}(w) + \dim\left(\text{Levi}(Q)/\text{Levi}(Q) \cap wPw^{-1}\right). \]

**Proposition 6.2** The Bialynicki-Birula decomposition of G/P for the action of \( \lambda \) coincides with the Bruhat decomposition for the action of P(\( \lambda \)).

**Proof.** First, each Bruhat cell is the P(\( \lambda \))-orbit of some \( x \in (G/P)^T \); second, any such \( x \) belongs to a component \( Y_s \subset (G/P)^\lambda \); finally, the BB-cells \( Y_s^+ \) are P(\( \lambda \))-invariant. Hence each Bruhat cell is contained in a unique BB-cell. But the union of the former is G/B, and the latter cells are pairwise disjoint. It follows that each Bruhat cell equals some BB-cell. □

For homogeneous varieties, the criterion 2.8 yields the following.

**Theorem 6.3** Let \( \mathcal{V} \) be an arbitrary vector bundle on G/P. Consider a 1-parameter subgroup of T, such that Lie(G)_\( \lambda \) \subset Lie(P) (so \( \hat{e} \in G/P \) belongs to the source Y). We assume that
\[ \dim(X) - \dim(X \setminus Y^+) \geq 2\text{codim}_X(Y) + 2 \geq 6. \] (6.1)
and \( g^*\mathcal{V} \) splits on Y, for a very general \( g \in G \). Then \( \mathcal{V} \) is a successive extension of line bundles. If the variety G/P is 1-splitting, then \( \mathcal{V} \) splits.

Since the BB- and the Bruhat-decompositions coincide, (6.1) can be expressed in terms of the root datum of G. A pleasant feature of the criterion is that the splitting of \( \mathcal{V} \) is reduced to the splitting along a homogeneous subvariety of G/P, so the procedure can be iterated (see 6.5 below). Explicit calculations are performed in section 7.

### 6.1. When is G/P a 1-splitting variety?

To settle this question, we need some notations.

\[ X^*(T) := \text{the group of characters of } T; \text{ similarly for } B, P, G, \text{ etc.}; \]
\[ (\cdot, \cdot) \text{ the Weyl}(G)\text{-invariant scalar product on } X^*(T)_Q; \]
\[ \Psi := \text{the roots of } G, \Delta \subset \Psi \text{ the simple roots}; \]
\[ \Lambda := \text{the weights } \{ \omega \in X^*(T) \mid (\beta, \omega) \in \mathbb{Z}, \forall \beta \in \Delta \}; \]
\[ \{ \omega_\alpha \}_{\alpha \in \Delta} \text{ the fundamental weights, that is } (\beta, \omega_\alpha) = \text{Kronecker}_\beta \alpha; \]
\[ \Lambda_+ := \text{the dominant weights } \{ \omega \in \Lambda \mid (\beta, \omega) \geq 0, \forall \beta \in \Delta \}; \]
\[ \Lambda_+(I) := \bigcap_{\alpha \in I} \alpha^\perp \text{ the } I\text{-face of } \Lambda_+, \forall I \subset \Delta; \]
\[ \langle \Lambda_+(I) \rangle := \text{the vector space generated by } \Lambda_+(I). \]

The parabolic subgroup P corresponds to a subset \( I \subset \Delta \) (cf. [24 Section 8.4]); we denote it by \( P_I \). Its Weyl group \( W_I \) is generated by the reflections \( \tau_\alpha, \alpha \in I \).
Proposition 6.4  The homogeneous space $X = G/P_I$ is 1-splitting (cf. [24]) if and only if there is no simple root perpendicular to $\{\alpha \mid \alpha \in I\}$. Equivalently, define

$$\tilde{I} := I \cup \{\beta \in \Delta \setminus I \mid \beta \text{ is adjacent to some } \alpha \in I \text{ in the Dynkin diagram of } G\}.$$  

Then $G/P_I$ is 1-splitting if and only if $\tilde{I} = \Delta$. (For Dynkin diagrams, see [24] §9.5.)

Proof. For $\chi \in \mathcal{X}^*(P_I)$, let $\mathcal{L}_\chi$ be the line bundle $(G \times \mathbb{C})/P_I$, where $(g, z) \sim (g p^{-1}, \chi(p) z)$. By the Borel-Weil-Bott theorem [9, 12], $H^1(G/P_I, \mathcal{L}_\chi) = H^1(G/B, \mathcal{L}_\chi) \neq 0$ if and only if there is $\beta \in \Delta$ and $\chi_+ \in \Lambda_+$ such that

$$\chi = \tau_\beta(\chi_+ + \rho) - \rho = \tau_\beta(\chi_+ + \beta) \iff \chi_+ = \tau_\beta(\chi + \beta).$$

Let $T_\beta$ be the transformation $\chi \mapsto \tau_\beta(\chi + \beta)$; it is the reflection in the plane orthogonal to $\beta$, passing through $-\frac{\beta}{2}$. The pull-back of $\mathcal{L}_\chi$ to $G/B$ corresponds to the image of $\chi$ by $w_I : \mathcal{X}^*(P_I) \to \mathcal{X}^*(B) = \mathcal{X}^*(T)$, so $G/P_I$ is 1-splitting if and only if

$$\Lambda_+ \cap T_\beta(\text{Im}(w_I)) = \emptyset, \forall \beta \in \Delta.$$  

But $\mathcal{X}^*(P_I) = \mathcal{X}^*(\text{Levi}(P_I))$, so $\text{Im}(w_I) \subset \mathcal{X}^*(T)$ consists of the $W_I$-invariant elements. Since $W_I$ is generated by the reflections $\tau_\alpha$ in the hyperplanes $\alpha^\perp$, $\alpha \in I$, it follows $\text{Im}(w_I) = \langle \Lambda_+(I) \rangle$, so we must have

$$\Lambda_+ \cap T_\beta(\langle \Lambda_+(I) \rangle) = \emptyset, \forall \beta \in \Delta.$$  

For $\beta \in I$, this condition is automatically satisfied: $\langle \Lambda_+(I) \rangle \subset \Lambda_+ \cap \beta^\perp$, so $T_\beta(\langle \Lambda_+(I) \rangle) \subset \{\langle \beta, \cdot \rangle < 0\}$ and $\Lambda_+ \subset \{\langle \beta, \cdot \rangle \geq 0\}$.

For $\beta \in \Delta \setminus I$, let $\beta_\perp$ be the component of $\beta$ on $\langle I \rangle$ with respect to the orthogonal decomposition $\langle \mathcal{X}^*(T) \rangle = \langle I \rangle \oplus \langle \Lambda_+(I) \rangle$. Then $\beta_\perp$ is also the orthogonal projection of $0$ to the affine space $\beta + \langle \Lambda_+(I) \rangle$; hence $\Lambda_+$ and $T_\beta(\langle \Lambda_+(I) \rangle) = \tau_\beta(\beta + \langle \Lambda_+(I) \rangle)$ are disjoint if and only if they are on different sides of the hyperplane $\langle \tau_\beta \beta_\perp \rangle$:

\begin{align*}
(i) \quad & (\beta_\perp, \beta) > 0, \quad (ii) \quad (\tau_\beta \beta_\perp, \omega_\alpha) = (\beta_\perp, \tau_\beta \omega_\alpha) \leq 0, \forall \beta \in \Delta \setminus I, \forall \alpha \in \Delta. \quad (6.2)
\end{align*}

Claim  The inequality (ii) is automatically satisfied.

Case $\alpha \neq \beta$: $\tau_\beta \omega_\alpha = \omega_\alpha$.

- Assume $\alpha \notin I$. Then holds $\omega_\alpha \in \Lambda_+(I) \Rightarrow (\beta_\perp, \omega_\alpha) = 0$.
- Assume $\alpha \in I$. Since any two vectors in $I$ make an angle of at least $90^\circ$ (they are simple roots), and $(-\beta_\perp, c) = -(\beta, c) \geq 0$, for all $c \in I$, we deduce that $-\beta_\perp$ is in the cone $\sum_{c \in I} \mathbb{R}_{\geq 0} c$. Thus holds:

$$-\beta_\perp = k_\alpha \alpha + \sum_{c \in I \setminus \{\alpha\}} k_c c, \quad k_\alpha \geq 0 \Rightarrow (\beta_\perp, \omega_\alpha) = -k_\alpha (\alpha, \omega_\alpha) \leq 0.$$  

Case $\alpha = \beta$: $\omega_\beta \in \Lambda_+(I) \Rightarrow \beta_\perp \perp \omega_\beta \Rightarrow (\beta_\perp, \tau_\beta \omega_\beta) = (\beta_\perp, \omega_\beta - \beta) = -(\beta_\perp, \beta) \leq 0$.

For the last step: the angle between a vector and its projection to any plane is at most $90^\circ$.

Hence the only relevant condition in (6.2) is the first one. However, $(\beta_\perp, \beta) \geq 0$ from the very construction, so we must eliminate the case $(\beta_\perp, \beta) = 0$. This happens precisely when $0 \in \beta + \langle \Lambda_+(I) \rangle \iff \beta \in \langle \Lambda_+(I) \rangle \iff \beta \perp \alpha, \forall \alpha \in I.$

Corollary 6.5  Assume $T \subset B \subset P_I \subset G$, the variety $X = G/P_I$ is 1-splitting, and $2 \cdot \# I \geq 1 + \# \Delta$ (here $\#$ stands for the cardinality).
Then there is \( \lambda : G_m \to T \) such that the source \( Y \) of the action has the properties:

(i) \( Y = G(\lambda)/G(\lambda) \cap P_I \);
(ii) \( Y \) is 1-splitting;
(iii) \( G(\lambda) \cap P_I \) corresponds to the simple roots \( I \setminus \{ \alpha_0 \} \), \( \alpha_0 \in I \).

Proof. If \( \Psi_I \) stands for the roots generated by \( I \), the (positive) roots of the unipotent radical \( U_I \subset P_I \) are \( \Psi^+ \setminus \Psi_I^+ \) (cf. [24, Theorem 8.4.3]). It holds:
\[ \hat{e} \in Y \iff \text{Lie}(G) \setminus \text{Lie}(P_I) \iff \Psi^+ \setminus \Psi_I^+ \subset \text{Lie}(G(\lambda)). \]

The simplest way to get this situation is if the simple roots of \( \text{Lie}(G(\lambda)) \) are \( \Delta \setminus \{ \alpha_0 \} \), with \( \alpha_0 \in I \). It remains to impose that \( Y \) is 1-splitting, that is:
\[ \exists \beta \in \Delta \setminus I \text{ such that } \beta \text{ is adjacent only to } \alpha_0. \]

Let us prove that there is such an \( \alpha_0 \in I \). Otherwise, for all \( \alpha \in I \) there is \( \beta \in \Delta \setminus I \) adjacent only to \( \alpha \); this yields an injective function \( I \to \Delta \setminus I \), so \( \#I \leq \#(\Delta \setminus I) \), which contradicts the hypothesis.

\[ \square \]

7. Splitting criteria for vector bundles on Grassmannians

The Grassmannian plays a central role because it is homogeneous, and is the ‘universal target’ for pairs \( (X, N) \) consisting of a variety and a globally generated vector bundle on it. Hence the results of both sections 4, 5 apply. In this section we obtain splitting criteria for vector bundles on isotropic (symplectic and orthogonal) Grassmannians. The degenerate case, when the bilinear form has kernel, is also included, to demonstrate that theorem 2.8 is not restricted only to the situations discussed in sections 4, 5.

Cohomological splitting criteria have been obtained in [21, 18, 3, 17]; however, they involve a large number of conditions. The results below are interesting for their simplicity: indeed, the problem of deciding the splitting of a vector bundle on a Grassmann variety, which is a high dimensional object, is reduced to the splitting along a (very) low dimensional subvariety. Throughout this section, \( W \) stands for a \( w + 1 = \nu + u + 1 \)-dimensional vector space.

7.1. The Grassmannian of linear subspaces. This case is discussed in [13], where is proved that things are as good as possible, without any genericity assumptions.

Theorem 7.1 The vector bundle \( \mathcal{V} \) on \( \text{Gr}(u; \mathbb{C}^w), u \geq 2, w \geq u + 2 \), splits if and only if its restriction to an arbitrary \( \text{Gr}(2; \mathbb{C}^1) \subset \text{Gr}(u; \mathbb{C}^w) \) does so.

This is in perfect analogy with Horrocks’ criterion. However, the proof uses a (fortunate) cohomology vanishing, and can not be extended directly.

7.2. The symplectic-isotropic Grassmannian. Let \( \omega \) be a skew-symmetric bilinear form on \( W \). If \( \dim W \) is even, we assume that \( \omega \) is non-degenerate (\( \omega \) is a usual symplectic form), while for \( \dim W \) odd, we assume that \( \dim \ker(\omega) = 1 \) (\( \omega \) is symplectic on \( W/\ker(\omega) \) of dimension \( w \)). Let \( X := \text{sp-Gr}(u + 1; W) \) be the variety of \( \omega \)-isotropic, \( (u + 1) \)-dimensional subspaces of \( W \). It is a Fano variety with
\[ \dim(X) = \frac{(u + 1)(2w - 3u)}{2}. \]

If \( \varphi : \text{sp-Gr}(u + 1; W) \to \text{Gr}(u + 1; W) \) stands for the natural embedding, then \( \mathcal{O}_X(1) \) is the pull-back of the corresponding line bundle on the Grassmannian.

Denote by \( G := \text{Sp}_{(w+1)/2} \) the symplectic group \( ([\cdot] \text{ stands for the integral part}) \). If \( \dim W \) is even, \( X \) is homogeneous for \( G \): \( X = G/P \), where \( P \) is the stabilizer of the flag.
\[ \{ (1, \ldots, 0, 1, \ldots, 0 | 0, \ldots, 0), (0, \ldots, 1, 0, \ldots, 0 | 0, \ldots, 0) \} \subset \mathbb{C}^{\frac{w+1}{2}} \oplus \mathbb{C}^{\frac{w+1}{2}}. \]

If \( \dim W \) is odd, \( X \) has two orbits under the \( G \)-action: the open orbit of subspaces which intersect \( \text{Ker}(\omega) \) trivially, and the closed orbit of subspaces containing \( \text{Ker}(\omega) \).

**Lemma 7.2** If \( w \geq 2u + 1 + \dim \text{Ker}(\omega) \), then \( \text{Pic}(X) = \mathbb{Z} \cdot O_X \).

**Proof.** For \( \omega \) non-degenerate, this is clear. For \( \text{Ker}(\omega) = \langle s_0 \rangle \),

\[ \text{sp-Gr}(u + 1; W) = \{ U | s_0 \in W \} \cup \{ U | s_0 \notin U \}. \]

The first term is a subvariety, isomorphic to \( \text{sp-Gr}(u; W/\langle s_0 \rangle) \), of codimension \( w - 2u \geq 2 \); thus \( \text{Pic}(X) \) is isomorphic to the Picard group of the open stratum. The morphism

\[ \{ U | s_0 \notin U \} \to \text{sp-Gr}(u + 1, W/\langle s_0 \rangle), \quad [U \subset W] \mapsto [\langle s_0 \rangle + U/\langle s_0 \rangle \subset W/\langle s_0 \rangle] \]

is an affine fibration, and the base has cyclic Picard group. \( \square \)

The quotient bundle \( N \) and the tautological bundle \( \mathcal{U} := \text{Ker}(W \otimes O_X \to N) \) on \( X \) are the pull-back by \( \varphi \) of their counterparts on the Grassmannian. An element \( s \in W \setminus \text{Ker}(\omega) \) defines a section in \( N \), whose zero set is the ‘smaller’ isotropic Grassmannian:

\[
\{ U \in \text{sp-Gr}(u; W) | s \in U \} = \{ U \in \text{sp-Gr}(u; W) | s \in U \subset U^\perp \subset \langle s \rangle^\perp \} \\
= \text{sp-Gr}(u; \langle s \rangle^\perp/\langle s \rangle) \cap \text{Gr}(u; W/\langle s \rangle),
\]

with \( \langle s \rangle^\perp := \{ t \in W | \omega(s, t) = 0 \} \).

An element \( \sigma \in W^\vee \setminus \{ 0 \} \) determines a section in \( W \), with zero locus

\[ \text{sp-Gr}(u + 1, \sigma^\perp) = \{ U \in \text{sp-Gr}(u + 1; W) | U \subset \sigma^\perp \}, \text{ where } \sigma^\perp := \text{Ker}(\sigma). \]  

(7.2)

In particular, \( s \in (W/\text{Ker}(\omega)) \setminus \{ 0 \} \) determines \( \sigma_s(\cdot) := \omega(s, \cdot) \in W^\vee \); in this case, \( \sigma_s^\perp = \langle s \rangle^\perp \).

For \( \omega \) non-degenerate, \( s \mapsto \sigma_s \) defines an isomorphism \( W \to W^\vee \); however, if \( \dim \text{Ker}(\omega) = 1 \), the image of this map is the hyperplane in \( W^\vee \). In this latter case, for generic \( \sigma \in W^\vee \), \( \omega|_{\sigma^\perp} \) is non-degenerate, so \( \sigma^\perp \) is a symplectic subspace of \( W \). In general, it holds

\[ \dim \text{Ker}(\omega|_{\sigma^\perp}) = 1 - \dim \text{Ker}(\omega), \text{ for generic } \sigma. \]  

(7.3)

We start by explicitly computing the positivity of some subvarieties of \( X \). Deliberately, we consider both zero loci of sections and sources of \( G_m \)-actions, to illustrate the general theory developed in the previous sections.

**Lemma 7.3** (i) For \( s \in W^\vee \setminus \{ 0 \} \) and \( u \geq 1 \),

\[ \text{sp-Gr}(u + 1; s^\perp) \subset \text{sp-Gr}(u + 1; W) \text{ is } (w - 2u - 1)^>0. \text{ (Recall } s^\perp := \text{Ker}(s).) \]

(ii) Assume that either \( \text{Ker}(\omega) = \langle s \rangle \subset W \), or \( \omega \) is non-degenerate and \( s \neq 0 \). Then

\[ \text{sp-Gr}(u; \langle s \rangle^\perp/\langle s \rangle) \subset \text{sp-Gr}(u + 1; W) \text{ is } w^>0, \text{ for } u \geq 1. \]

(iii) Decompose \( W = W' \oplus W'' \) into the sum of Lagrangian subspaces of dimension \( (w + 1)/2 \). Then \( \text{Gr}(u + 1; W') \subset \text{sp-Gr}(u + 1, W) \text{ is } \left( \frac{w - 2u - 1}{2} \right)^>0. \)

**Proof.** (i) In the diagram

\[ \text{sp-Gr}(u + 1; W) \to \text{Gr}(u + 1; W) \]

(7.4)

\[ \text{sp-Gr}(u + 1; W) \leftarrow \text{sp-Gr}(u; s^\perp) \]

(7.5)
g is undefined on \( \text{Gr}(u + 1; s^+) \), and \( b \) is undefined on \( \{ U \mid U \subset s^+ \} = \text{sp-Gr}(u + 1; s^+) \). The blow-up of \( \text{Gr}(u + 1; W) \) along \( \text{Gr}(u + 1; s^+) \) resolves \( g \), hence \( b \). It remains to apply proposition 3.3, the fibres of \( b \) are at least \( \frac{(u+1)(2u-3u)}{2} - \frac{u(2w-3u+1)}{2} = w - 2u \) dimensional.

(ii) Case \( \ker(\omega) = \langle s \rangle \). In the diagram

\[
\begin{array}{ccc}
\text{sp-Gr}(u + 1; W) - - - & - - - & \text{sp-Gr}(u + 1; W/\langle s \rangle) \\
\uparrow \phi & & \uparrow \phi \\
\text{Gr}(u + 1; W) - - - & - - - & \text{Gr}(u + 1; W/\langle s \rangle).
\end{array}
\]

\( b \) is not defined on \( \text{Gr}(u; W/\langle s \rangle) \cap \text{sp-Gr}(u + 1; W) = \text{sp-Gr}(u; W/\langle s \rangle) \). The blow-up of \( \text{Gr}(u; W/\langle s \rangle) \) resolves \( g \), hence \( b \). Now apply 3.3 again: the general fibre of \( b \) is \((u+1)\)-dimensional.

Case \( W \) is symplectic. Decompose \( W \) into a direct sum of Lagrangian subspaces

\[
W = \mathbb{C}^{(w+1)/2}_{\text{left}} \oplus \mathbb{C}^{(w+1)/2}_{\text{right}},
\]

and assume \( s = (1, \ldots, 0) \mid (0, \ldots, 0) \). Then \( Y := \text{sp-Gr}(u, \langle s \rangle) \) is the source of the \( G_m \)-action, corresponding to the 1-PS:

\[
\lambda : G_m \to \text{Sp}_{(w+1)/2}(\mathbb{C}), \quad \lambda(t) = \text{diag} \left[ t^{-1}, \mathbb{1}_{(w-1)/2}, t, \mathbb{1}_{(w-1)/2} \right]
\]

(for generic \( U, s \in \ker(\lambda(t)U) \).

The complement of the open BB-cell is \( X \setminus Y^+ = \{ U \in X \mid s \notin \lim_{t\to0} \lambda(t)U \} \). Consider a basis in \( U \) such that the corresponding column matrix is lower triangular; then one can see that \( X \setminus Y^+ = \{ U \mid U \subset \mathbb{C}^{(w-1)/2}_{\text{left}} \oplus \mathbb{C}^{(w+1)/2}_{\text{right}} \} \). The conclusion follows from the proposition 5.4 because \( \dim X - \dim(X \setminus Y^+) = u + 1 \).

(iii) With the same notations, \( Y := \text{Gr}(u + 1, \mathbb{C}^{(w+1)/2}_{\text{left}}) \) is the source of the \( G_m \)-action

\[
\lambda : G_m \to \text{Sp}_{(w+1)/2}(\mathbb{C}), \quad \lambda(t) = \text{diag} \left[ t^{-1}, \mathbb{1}_{(w-1)/2}, t, \mathbb{1}_{(w-1)/2} \right].
\]

It follows \( X \setminus Y^+ = \{ U \mid \ker(\text{pr} : U \to \mathbb{C}^{(w+1)/2}_{\text{left}}) \neq 0 \} \). The minimal degeneration is when \( \ker(\text{pr}) \) is one-dimensional; the corresponding stratum maps onto \( \mathbb{P}(\mathbb{C}^{(w+1)/2}_{\text{right}}) \), with fibres isomorphic to \( \text{sp-Gr}(u; \mathbb{C}^{w-1}) \), thus \( \dim X - \dim(X \setminus Y^+) = \frac{w-2u+1}{2} \).

Remark 7.4 (i) Sommese’s criterion 3.1 implies that the quotient bundle on \( \text{sp-Gr}(u + 1; W) \) is \( q \)-ample, for \( q = \dim \text{sp-Gr}(u + 1; W) - (u + 1) \). Hence \( \text{Gr}(u; \langle s \rangle) \) is \( p \)-ample, with \( p = (u + 1) - \dim \text{sp-Gr}(u; \langle s \rangle) = 2u - w + 1 < 0 \).

This shows that this test is week compared with proposition 3.3

(ii) The conclusion of 7.3(i), for \( \omega \) non-degenerate, can not be obtained by using a 1-PS of \( G \), because \( \text{sp-Gr}(u + 1; s^+) \) is not homogeneous.

(iii) At 7.3(ii), the section \( s \) is neither generic (since \( \langle s \rangle = \ker(\omega) \)), nor regular (transverse to the zero section): indeed \( \text{sp-Gr}(u; W/\langle s \rangle) \) is \((w - 2u)\)-codimensional in \( \text{sp-Gr}(u + 1; W) \), rather than \( \text{rk}(\mathbb{N}) = w - u \) (cf. subsection 3.2).

Proposition 7.5 Let \( \omega \) be a skew-symmetric bilinear form on \( W \), \( \dim W = w + 1 \), and \( \kappa := \dim \ker(\omega) \leq 1 \). Let \( X := \text{sp-Gr}(u + 1; W) \), and \( \mathcal{V} \) be an arbitrary vector bundle on \( X \).

In the following cases, \( \mathcal{V} \) splits if and only if \( \mathcal{V}_s \) splits.
(i) \( Y_s := \text{sp-Gr}(u + 1; s^\perp) \), with \( s \in W^\vee \) very general, and 
\[ w + 1 > 2(u + 1) + \kappa, \text{ so } w \geq 2u + 3 + \kappa, \text{ and } u \geq 1. \]
(ii) \( Y_s := \text{sp-Gr}(u; \langle s \rangle^\perp/\langle s \rangle) \), with \( s \in W \setminus \text{Ker}(\omega) \) very general, and \( w \geq 2u + 1 + \kappa, u \geq 2. \)

**Proof.** In both cases, we verify (2.3) and apply theorem 2.8 directly.

(i) Take \( S \subset W^\vee \setminus \{0\} \) open subset such that (7.3) holds, and \( \mathcal{Y} \subset S \times X \) be the zero locus of the universal section in \((pr_X^S \times X)^*U^\vee\). The morphisms \( \mathcal{Y} \overset{\pi}{\to} S \) and \( \mathcal{Y} \overset{\rho}{\to} X \) are both open, and \( \pi^{-1}(s) = Y_s \) is projective, connected; also, \( \dim \text{Ker}(\omega_{s^\perp}) = 1 - \kappa. \)

– We claim that, for all \( o, s, t \in S \),
\[ Y_{os} = \{ U \in X \mid U \subset \langle o, s \rangle^\perp \} \cong \text{sp-Gr}(u + 1; \langle o, s \rangle^\perp), \text{ and } \]
\[ Y_{ost} = \text{sp-Gr}(u + 1; \langle o, s, t \rangle^\perp) \cong \text{sp-Gr}(u + 1; \langle o, s, t \rangle^\perp) \]
are connected and non-empty. The connectedness is clear, since they are quasi-homogeneous, with finitely many orbits, for actions of appropriate subgroups of \( \text{Sp}(W) \). We verify that \( Y_{ost} \neq \emptyset \) (thus (1-arm) and (no-\( \Delta \)) are satisfied), that is
\[ \exists U \in X \text{ such that } U \subset \langle o, s, t \rangle^\perp. \]

Let \( \omega_{ost} \) be the restriction of \( \omega \) to \( \langle o, s, t \rangle^\perp \); then \( \dim \text{Ker}(\omega_{ost}) = 1 - \kappa. \) We verify:
\[ \dim \langle o, s, t \rangle^\perp / \text{Ker}(\omega_{ost}) \geq 2((u + 1) - \dim \text{Ker}(\omega_{ost})) \Leftrightarrow w + \dim \text{Ker}(\omega_{ost}) \geq 2u + 4. \]
The last inequality is fulfilled, by hypothesis.

– The diagram (2.3)(Pic) consists of isomorphisms (cf. lemma 7.2).
– Finally, \( Y_s \subset X \) is \((w - 2u - 1) > 0 \) (cf. lemma 7.3(i)), so is at least \( 2 > 0. \)

(ii) Let \( S := W \setminus \text{Ker}(\omega) \), and \( \mathcal{Y} \) be the zero locus of the universal section in \((pr_X^S \times X)^*N. \) The situation is analogous, with a few differences. Indeed, for \( o, s \in S \) holds:
\[ Y_{os} \neq \emptyset \Leftrightarrow o \perp s, \text{ so generically } Y_{os} = \emptyset. \]

For \( o \in S \), we check the properties (2.3) for
\[ S(o) = \{ s \in S \mid Y_s \neq \emptyset \} = S \cap \langle o \rangle^\perp. \]
– The diagram (Pic) consists of isomorphisms (cf. lemma 7.2).
– The condition (1-arm), that is \( \rho(Y_{S(o)}) = X, \) is the following:
\[ \forall U \in X \exists s \in S \cap \langle o \rangle^\perp \exists V \in X \text{ such that } s \in U, \langle o, s \rangle \subset V. \]

Indeed, \( \dim(U \cap \langle o \rangle^\perp) \geq u, \text{ so } U \cap \langle o \rangle^\perp \neq 0. \) Take \( s \in (U \cap \langle o \rangle^\perp) \setminus \text{Ker}(\omega), \) non-zero; then \( \langle o, s \rangle \subset W \) is an isotropic subspace. There exists \( V \in X \) containing it, as \( u \geq 2. \)

– The condition (no-\( \Delta \)) reads:
\[ [s \perp o, t \perp o, s \perp t] \Rightarrow \exists V \in X, \langle o, s, t \rangle \subset V. \]
The left hand side implies that \( \langle o, s, t \rangle \subset W \) is isotropic subspace; then \( V \) exists, since \( u \geq 2. \)
– Finally, by lemma 7.3(ii), \( Y_s \subset X \) is \( u > 0, \) so at least \( 2 > 0. \)

**Theorem 7.6** Let \( \omega \) be a skew-symmetric bilinear form on \( \mathbb{C}^w, \) with \( \kappa = \dim \text{Ker}(\omega) \leq 1. \)
We consider the isotropic Grassmannian \( X = \text{sp-Gr}(u; \mathbb{C}^w), \) with \( u \geq 2, \) and an arbitrary vector bundle \( \mathcal{Y} \) on it. Then \( \mathcal{Y} \) splits if and only if it does so along a very general subvariety \( Y \cong \text{sp-Gr}(2; 4 + \kappa) \) of \( X. \)

Note that \( \text{sp-Gr}(2; 4) \), the Lagrangian 2-planes in \( \mathbb{C}^4 \), is isomorphic through the Plücker embedding to the 3-dimensional quadric.

**Proof.** By applying repeatedly the first part of previous proposition (after replacing \( w + 1 \sim w \) and \( u + 1 \sim u \)), we deduce that \( \mathcal{Y} \) splits if and only if \( \mathcal{Y}_Z \) splits, for some very general subvariety \( Z \cong \text{sp-Gr}(u; 2u + \kappa) \) of \( X. \) Now apply the second part to deduce the
splitting problem from \( Z \) to \( Y \cong sp\text{-}Gr(2, 4 + \kappa) \). (For this latter, the process cannot be iterated anymore.)

7.3. **The orthogonal-isotropic Grassmannian.** The situation is similar to the previous case: let \( \beta \) be a symmetric, non-degenerate, bilinear symmetric form on \( W \), and consider \( X := o\text{-}Gr(u + 1; W) \) be the variety of isotropic \((u + 1)\)-dimensional subspaces of \( W \); assume \( u \geq 1 \), \( \dim W \geq 5 \). (If \( w + 1 = 2(u + 1) \), the full space of Lagrangian planes in \( \mathbb{C}^{2(u+1)} \) has two connected components, and we consider only one of them.) It is a homogeneous variety for \( G = SO(\beta) \), with

\[
\dim X = \frac{(u + 1)(2w - 3u - 2)}{2}, \quad \text{and Pic}(X) = Z \cdot O_X(1).
\]

Similar arguments as before yield the following.

**Lemma 7.7**

(i) If \( u \geq 1 \), \( w \geq 2u + 2 \), then

\[
o\text{-}Gr(u + 1; (s)\perp) \subset o\text{-}Gr(u + 1; W) \text{ is } (w - 2u - 2) > 0.
\]

The form \( \beta\mid_{(s)\perp} \) is non-degenerate if and only if \( s \) is a non-isotropic vector.

(ii) Let \( s \in W \) be isotropic, \( u \geq 2 \). Then \( o\text{-}Gr(u; (s)\perp/(s)) \subset o\text{-}Gr(u + 1; W) \) is

\[
\begin{align*}
(1 - u)^{> 0} & \quad \text{for } w = 2u + 1, \\
\text{for } w \geq 2u + 2.
\end{align*}
\]

**Proof.** (i) \( \dim o\text{-}Gr(u + 1; W) - \dim o\text{-}Gr(u; (s)\perp) = w - 2u - 1 \) (cf. lemma 7.3(i)).

(ii) Decompose \( W = \mathbb{C}^{(w+1)/2} \oplus \mathbb{C}^{(w+1)/2} \) into the sum of two Lagrangian subspaces and consider \( \lambda: G_m \rightarrow SO_{(w+1)/2} \) as in (7.7). The source of \( \lambda \) is \( Y = \{ U \mid (1, 0, \ldots, 0) \in U \} \), and \( X \setminus Y^+ = \{ U \mid U \subset W^* := \mathbb{C}^{(w-1)/2} \oplus \mathbb{C}^{(w+1)/2} \} \). Note that \( \beta\mid_{W^*} \) has a 1-dimensional kernel \( \langle s' \rangle \); if \( w \in \{2u + 1, 2u + 2\} \), then \( s' \in U \) for all \( U \in X \setminus Y^+ \). By using this remark one finds that \( \dim X - \dim(X \setminus Y^+) \) equals: \( u \) for \( w = 2u + 1 \), and \( u + 1 \) for \( w \geq 2u + 2 \).

**Proposition 7.8** Let \( \beta \) be a non-degenerate symmetric bilinear form on \( W \), with \( \dim W = w + 1 \). Let \( X := o\text{-}Gr(u + 1; W) \), and \( Y \) be an arbitrary vector bundle on \( X \). In the following cases, \( Y \) splits if and only if \( Y_{s_0} \) splits.

(i) \( Y_s := o\text{-}Gr(u + 1; (s)\perp) \), with \( s \in W \) very general, non-isotropic, and \( u \geq 2u + 4 \).

(ii) \( Y_s := o\text{-}Gr(u; (s)\perp/(s)) \), with \( s \) very general isotropic, and \( u \geq 2u + 1, \ u \geq 3 \).

**Proof.** Again we check (2.3) and apply (2.8) directly. Let \( Q_\beta := \{ s \in W \mid \beta(s, s) = 0 \} \) be the isotropic cone.

(i) Take \( S := W \setminus Q_\beta \) and let \( Y \subset S \times X \) be the universal family, with \( Y_s = o\text{-}Gr(u + 1, (s)\perp) \).

- For \( o, s, t \in S \),

\[
Y_o = \{ U \in X \mid U \subset \langle o, s \rangle\perp \}, \quad Y_{ot} = \{ U \in X \mid U \subset \langle o, s, t \rangle\perp \}.
\]

They are quasi-homogeneous for the action of appropriate subgroups of \( SO(\beta) \), thus connected. As \( w \geq 2u + 4 \), we deduce \( \dim(o, s, t)\perp \geq 2 \dim U \), so \( Y_{ot} \) is always non-empty; in particular, \((1\text{-arm})\) and \((\text{no-})\) are satisfied.

- Finally, the diagram \((\text{Pic})\) consists of isomorphisms.

(ii) Here we choose \( S := Q_\beta \setminus \{0\} \); the situation is similar to (7.3)ii). For \( o, s \in S \),

\[
Y_o \neq \emptyset \iff s \in \langle o \rangle\perp \cap S; \quad \text{let } S(o) := \langle o \rangle\perp \cap S.
\]

We check the conditions (2.3) for \( Y_{S(o)} \).
\(- W_{os} := \langle o, s \rangle / \langle o, s \rangle \) has an induced non-degenerate orthogonal form, so \( Y_{os} \) is connected.

- The diagram \((\text{Pic})\) consists of isomorphisms.

- The condition \((1\text{-arm}), \) that is \( \rho(Y_{S(o)}) = X \) is:
  \[ \forall U \in X \exists s \in S(o) \exists V \in X \text{ such that } s \in U, \langle o, s \rangle \subset V. \]

Indeed, take \( s \in U \cap \langle o \rangle \neq 0, \) so \( \langle o, s \rangle \subset W \) is isotropic; now take any \( V \) containing it.

- The condition \((\text{no-}\triangle)\) reads: \([ o \perp s, o \perp t, s \perp t ] \Rightarrow Y_{ost} \neq \emptyset. \)

Indeed, \( \langle o, s, t \rangle \subset W \) is isotropic, so there is \( U \in X \) containing it because \( u + 1 \geq 3. \)

- By lemma \( \text{[7.7(ii)]}, Y_{os} \subset X \) is at least \( 2^{+0}. \)

**Theorem 7.9** Let \( \omega \) be a non-degenerate bilinear form on \( \mathbb{C}^w. \) We consider the isotropic Grassmannian \( X = \text{spGr}(u; \mathbb{C}^w), \) with \( u \geq 3, \) and an arbitrary vector bundle \( \mathcal{V} \) on it. Then \( \mathcal{V} \) splits if and only if \( \mathcal{V}_Y \) splits, with \( Y \) very general, where:

- \( Y \cong \text{o-Gr}(3, \mathbb{C}^6), \) if \( w = 2u; \)
- \( Y \cong \text{o-Gr}(3, \mathbb{C}^7), \) if \( w = 2u + 1; \)
- \( Y \cong \text{o-Gr}(3, \mathbb{C}^8), \) if \( w \geq 2u + 2. \)

**Proof.** Assume that \( w \geq 2u + 2. \) Then the first part of previous proposition implies (after replacing \( w + 1 \leadsto w \) and \( u + 1 \leadsto u \)) that \( \mathcal{V} \) splits if and only if \( \mathcal{V}_Z \) splits, for some very general subvariety \( Z \cong \text{spGr}(u, 2u + 2) \) of \( X. \) There remain three possibilities: \( \text{o-Gr}(u, 2u), \text{o-Gr}(u, 2u + 1), \text{o-Gr}(u, 2u + 2). \) The theorem follows now from the second part of the proposition. \( \square \)

The somewhat non-uniform formulation of the theorem, compared to \( \text{[7.6]} \) is due to the lack of sufficient positivity of \( Y = \text{o-Gr}(u; 2u + 1) \subset \text{o-Gr}(u, 2u + 2) = X, \) which is only \( 1^{+0}. \)

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