Scalar field as a Bose-Einstein condensate?

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Abstract. We discuss the analogy between a classical scalar field with a self-interacting potential, in a curved spacetime described by a quasi-bounded state, and a trapped Bose-Einstein condensate. In this context, we compare the Klein-Gordon equation with the Gross-Pitaevskii equation. Moreover, the introduction of a curved background spacetime endows, in a natural way, an equivalence to the Gross-Pitaevskii equation with an explicit confinement potential. The curvature also induces a position dependent self-interaction parameter. We exploit this analogy by means of the Thomas-Fermi approximation, commonly used to describe the Bose-Einstein condensate, in order to analyze the quasi bound scalar field distribution surrounding a black hole.

Keywords: GR black holes, quantum gravity phenomenology, dark matter theory

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1 Introduction

Since half a century ago, when the Jordan-Brans-Dicke scalar-tensor theory of gravity was proposed [1, 2], scalar fields have experienced a long and controversial life. Nowadays, they appear in the formulation of many phenomena in gravitational theories. On the one hand, a scalar field is always present in the context of Dirac’s large number hypothesis, and also in all higher-dimensional unified field theories; they appear as dilatons in string theory and as inflatons or dark matter in cosmology [3]. Nevertheless, they have remained until now as exotic matter. It was only in the last year that the Higgs boson was detected [4–6], a very important fact in the development of the scalar field theory.

It has also been found that there exist fundamental relations between particle physics, cosmology and condensed matter [7–9]. Different condensed matter systems — such as acoustics in flowing fluids, light in moving dielectrics or quasi particles in moving superfluids — can be shown to reproduce some aspects of General Relativity and cosmology [10–13]. They can be conceived as laboratory toy models that make some features of quantum field theory on curved spacetime experimentally accessible [14].

In this context, one of the most peculiar phenomena in physics, discovered in the last century, is Bose-Einstein condensation [15, 16], which is a macroscopic quantum phenomenon that was first discovered theoretically by Bose [15] and Einstein [16] in the 1920’s, where it was applied through the new concept of Bose statistics to a non-interacting gas of identical atoms which were at thermal equilibrium and trapped in a box. It was predicted that, at sufficiently low temperatures, the particles would accumulate in the lowest quantum state in the box and would merge into a giant superatom. Locked together, moving as one, this condensate of atoms would become a new state of matter, different from solid, liquid or gas. Thus, large numbers of bosons can collapse into the same quantum state to form a condensate, while two fermions cannot be in the same quantum state — they obey the Pauli principle [17]. Bose-Einstein condensation is only possible for massive bosonic particles. The
particle density at the center of a Bose-Einstein condensate is typically of the order $10^{13} - 10^{15}$ cm$^{-3}$ [18], not that far from the density of molecules in air at room temperature and atmospheric pressure, which is of the order $10^{19}$ cm$^{-3}$, or the density of atoms in liquids and solids, which is about $10^{22}$ cm$^{-3}$ [19, 20]. It was a great achievement when this theoretical idea was finally realized in the laboratory, 75 years later [21–23].

The Gross-Pitaevskii equation [24, 25] is usually applied to investigate the physical properties of Bose-Einstein condensates in trapped ultra-cold atoms with a temperature, $T$, of about 100 nK. This equation was derived independently by Gross [24] and Pitaevskii [25] in 1961. Its validity is based on the conditions that, for a diluted gas, the s-wave scattering length must be much smaller than the average distance between atoms, and that the number of atoms in the condensate must be much larger than one. It can be used at very low temperature (including absolute zero) to explore the macroscopic behavior of the system, characterized by variations of the order parameter over distances larger than the mean distance between atoms [18, 26–28].

It is interesting to note that, after several approximations and assumptions, it can be shown that the Klein-Gordon equation governing the dynamics of a classical scalar field can be reduced to the Schrödinger equation, from which the Gross-Pitaevskii equation that governs the dynamics of a Bose-Einstein condensate follows. It is also remarkable that, as mentioned above, we face a classical relativistic scalar field (with second order time derivatives), and a quantum Newtonian BEC (with first order time derivatives), with different concepts of time [29, 30]. In this way, the Klein-Gordon equation contains the Gross-Pitaevskii one. However, the physical meaning being described changes quantitatively and qualitatively between these ideas.

Even though the Schrödinger equation can be derived from the Klein-Gordon equation, they describe very different physical phenomena. There are, however, macroscopic and quantum systems which, under certain circumstances, are described by the same formal equation, with the same mathematical structure. This is the case between a scalar field distribution surrounding a black hole, the so called quasi-bound states [31], and the Bose-Einstein condensate described by the Gross-Pitaevskii equation. We remark that in the former case, it is the space-time curvature, together with the scalar field features, that form an effective potential which contains the scalar field distribution.

At least until now, it has not been proper to call the quasi bound distributions ‘cosmological condensates’, as long as such distributions are described by a classical field without any reference to particles or quantum states. Even though the quasi bound distributions satisfy a very similar equation to the one satisfied by the stationary Bose-Einstein condensate, they have a very different origin, describe very different phenomena and, as mentioned above, are conceptually different. The similarity between the equations which describes each case is, however, worthy of deeper research, see for example [34, 35]. In fact, the scalar field description of dark matter and its possible relation with Bose-Einstein condensates is yet to be fully understood [36–38].

In this paper we will use this similarity to investigate the possible implications in determining the dynamics of a scalar field, using the techniques applied in solving the dynamics of a Bose-Einstein condensate. The mentioned similarity will work in static and stationary situations where the role played by the time coordinate is not relevant. We will see how it is possible to express, under certain circumstances, the Klein-Gordon equation as a Gross-Pitaevskii-like equation. As we mentioned above, the presence of a gravitational background provides, in a natural way, a trapping potential in the effective Gross-Pitaevskii equation.
Once the analogy is established, we will apply the usual techniques of atomic physics to the description of the quasi bound state.

We consider the pathological one-dimensional case, where the Thomas-Fermi approximation breaks down, but we are able to obtain analytical solutions for the different descriptions, so that we can, in principle, compare them.

Finally, we must mention that the scalar field collapse can form stable compact objects: an oscillaton in the case of a real scalar field, and a boson star if the collapsing scalar field is complex; see for instance [39, 40]. These configurations could also be compared with Bose-Einstein condensates, and we could try to find analogies. However, in this work we are stressing the mathematical similarity between the final equation describing quasi-stationary scalar distributions in a curved background, and the Gross-Pitaevskii one. An analogy between the Bose-Einstein condensate and the physical compact object, such as the boson star, could also be an interesting line worth pursuing, but it is beyond the scope of the present paper and will be discussed in future works.

The outline of the paper is as follows: in section 2, the flat space analogy between a classical scalar field and a Bose-Einstein condensate is described. In section 3, a curved space analogy is considered. In section 4 the Thomas-Fermi approximation is applied to the scalar field equation in a gravitational background. In section 5, the conclusions and outlook are presented.

2 Flat space analogy

Let us first consider the flat spacetime case, i.e., without a gravitational background. We will find an exact solution of the Gross-Pitaevskii equation, obtained from the Klein-Gordon equation, for the case of a scalar field trapped in a one-dimensional box. It is worthwhile to mention that the solution has the form of the one obtained for the case of a Bose-Einstein condensate trapped in a one-dimensional box. This solution is known as the static soliton.

2.1 One-dimensional scalar field trapped in a box: static soliton

First, we reduce the Klein-Gordon equation for a classical scalar field to a Gross-Pitaevskii-like equation for a Bose-Einstein condensate in one spatial dimension. Following [41], we analyze the problem of a scalar field trapped in a box.

It is worthwhile to note that in the one dimensional flat case, the Gross-Pitaevskii-like equation does not have a confinement potential other than the boundary walls of the box.

Consider a complex scalar field, $\Phi(t, x)$, satisfying the following Klein-Gordon equation

$$\frac{-1}{c^2} \Phi + \Phi'' - \frac{dV_{1d}}{d\Phi^*} = 0,$$

(2.1)

where the prime denotes a derivative with respect to $x$, and the dot a derivative with respect to $ct$. We restrict ourselves to the case of harmonic time dependence of a scalar field, which is coupled to a scalar self-interacting potential

$$\Phi(t, x) = e^{i\omega t} \chi(x),$$

(2.2)

$$V_{1d} = \frac{\sigma^2}{2} \Phi \Phi^* + \frac{\lambda}{4} (\Phi \Phi^*)^2,$$

(2.3)

$$= \frac{\sigma^2}{2} \chi^2 + \frac{\lambda}{4} \chi^4.$$

(2.4)
Additionally, the condition \( (d V_1/d d \Phi^*) = e^{i \omega t} (d V_1/d d \chi) \) reduces the Klein-Gordon equation to the following Gross-Pitaevskii-like equation; see eq. (3.12) below (the generalization to three dimensions is straightforward, see for instance refs. [42–45])

\[
\chi'' - \left[ \left( \sigma^2 - \frac{\omega^2}{c^2} \right) \chi + \lambda \chi^3 \right] = 0,
\]

which can be integrated directly leading to

\[
\frac{1}{2} \chi'^2 - \frac{\lambda}{4} \left( \chi^2 + \frac{\sigma^2 - \frac{\omega^2}{c^2}}{\lambda} \right)^2 = C_1,
\]

where we added a term \(-\sigma^2 - (\omega^2/c^2)^2/4\lambda\) to both sides of the equation and defined a new integration constant \(C_1\).

Eq. (2.6) can be solved directly, leading to a JacobiSN solution, which should be restricted to the case of vanishing integration constants. Thus, by imposing analogous conditions for the relation between the chemical potential \(\mu\) and the parameter that describes the interaction \(U_0 = 4\pi \hbar^2 a/m\), with \(a\) being the s-wave scattering length as for the usual Gross-Pitaevskii equation, i.e., \(\mu = U_0 |\psi(x)|^2 = U_0 n\), and \(n\) the corresponding particle density, we obtain the following condition for our system

\[
\mu_{\text{eff}} \equiv \frac{\omega^2}{c^2} - \sigma^2 = \lambda |\chi_0|^2 \equiv \lambda \rho_N,
\]

where we have defined an effective chemical potential \(\mu_{\text{eff}}\), together with a new constant \(|\chi_0|^2\), which corresponds, as in the usual case, to the wave function far away from the wall, where the kinetic energy term becomes negligible.

The balance between the kinetic term and the interaction energy characterized by the coupling constant \(\lambda\) in eq. (2.5) allows us to fix a typical distance over which the system can heal, as in the usual Gross-Pitaevskii equation. In our case this is

\[
\xi_{\text{flat}} = \frac{1}{\sqrt{\lambda \rho_N}}.
\]

It is interesting to notice that the associated healing length \(\xi_{\text{flat}}\) is independent of the parameter \(\sigma\) due to the functional form of eq. (2.5).

Finally, in this scenario, the solution to the Klein-Gordon equation for a classical scalar field with a self-interacting potential reduces to following expression

\[
\chi(x) = |\chi_0| \tanh(x/\sqrt{2\xi_{\text{flat}}}),
\]

which is precisely the kink solution obtained for a Bose-Einstein condensate trapped in a box [20].

We see that in this simple case, for a harmonic-like solution, and an infinite barrier potential, we can relate the Klein-Gordon equation, and the scalar field solution, to the Gross-Pitaevskii equation and the order parameter (that is, the solution to the Gross-Pitaevskii equation inside the potential). The chemical potential, \(\mu_{\text{eff}}\), is identified with the subtraction of the mass parameter and the oscillation frequency, as seen in eq. (2.7).
3 Curved space analogy

In order to continue exploring the analogy between the Klein-Gordon equation and the Gross-Pitaevskii equation, we now present the Klein-Gordon equation in a curved spacetime.

One starts from the Lagrangian density for a complex scalar field

$$\mathcal{L} = \frac{c^4}{16 \pi G} \left( \frac{1}{2} \nabla \Phi \nabla \Phi^* - V(\Phi \Phi^*) \right),$$

(3.1)

where $V(\Phi \Phi^*)$ is the scalar field potential. From this Lagrangian one obtains, by $\partial \mathcal{L} / \partial g^{\mu \nu}$, the stress energy tensor:

$$T_{\mu \nu} = \frac{c^4}{16 \pi G} \left[ \Phi_\mu \Phi^*_\nu + \Phi_\nu \Phi^*_\mu - g_{\mu \nu} \left( g^{\alpha \beta} \Phi_\alpha \Phi^*_\beta + 2V(\Phi \Phi^*) \right) \right],$$

(3.2)

which can be used in Einstein’s equations, and its conservation equation, $T^\mu_{\nu ; \mu} = 0$ lead us to the Klein-Gordon equation for the complex scalar field.

The classical scalar field can also be considered as a test particle/field, i.e., it only feels gravity, while its own gravity is neglected. Its dynamics is determined by a Klein-Gordon equation in a curved background spacetime. In this work we consider the scalar field as a test particle/field in a curved background.

In the previous section we displayed, for a box potential as container, an analogy between a classical scalar field satisfying a Klein-Gordon equation and the Gross-Pitaevskii equation for a Bose-Einstein condensate. This fact reinforces our guess about the existence of an analogy between classical scalar field configurations with the order parameter associated with the Bose-Einstein condensate theory. Now we show that this analogy is actually even more remarkable.

Consider a spherically-symmetric-static background spacetime

$$ds^2 = - F(r) c^2 dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2,$$

(3.3)

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and $c$ the speed of light in vacuum. By solving the vacuum Einstein field equations including cosmological constant

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = 0,$$

(3.4)

one can determine the explicit form of the geometric function, $F$.

We consider the dynamics of a scalar test field, $\Phi$, with a scalar self-interacting potential given by

$$V(\Phi \Phi^*) = \frac{\sigma^2}{2} \phi^* \phi + \frac{\lambda}{4} [\phi^* \phi]^2.$$

(3.5)

That is, the scalar field satisfies a Klein-Gordon equation in the curved spherically symmetric background spacetime given by eq. (3.3), which reads

$$\left[ g^{\mu \nu} \nabla_\mu \nabla_\nu - (\sigma^2 + \lambda \rho_n) \right] \Phi = 0,$$

(3.6)

where we used the following definition of the number density $\rho_n$

$$\rho_n = \Phi^* \Phi.$$

(3.7)
Let us restrict our attention to the monopolar component of the scalar field with harmonic time dependence
\[ \Phi = e^{i\omega t} \frac{u(r)}{r}. \]

The Klein-Gordon equation reduces to a non-linear Schrödinger-like equation, which is a kind of Gross-Pitaevskii-like equation
\[ \left(- \frac{d^2}{dr^*^2} + V_{\text{eff}} + \lambda F \rho_n \right) u = \frac{\omega^2}{c^2} u. \]

Here, we identified the particle/field density \( \rho_n = \frac{u^2}{r^2} \), and introduced the \( r^* \) coordinate
\[ r^* = \int \frac{dr}{F}. \]

Hence, the effective trapping potential reads
\[ V_{\text{eff}} = F \left( \sigma^2 + \frac{F'}{r^*} \right), \]
where now the prime stands for a derivative with respect to \( r^* \).

In order to obtain stationary (or quasi stationary) solutions for the scalar field, the curvature of the spacetime itself should confine the scalar field. Indeed, it is not necessary to introduce "by hand" an external potential to confine the scalar field in the Klein-Gordon equation; the gravitational background can do the work, with some background spacetimes able to confine the scalar field. In what follows we set \( r^* \rightarrow r \) to simplify the notation.

Eq. (3.9) is a Gross-Pitaevskii-like equation
\[ - \frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r) + U_0|\psi(r)|^2 \psi(r) = \mu \psi(r). \]

We identify the effective potential \( V_{\text{eff}} \) of eq. (3.9) with the trapping potential \( V(r) \) of the Gross-Pitaevskii equation, eq. (3.12). The Klein-Gordon self-interaction term is given by \( \lambda F \), which includes a geometric coefficient. In our Gross-Pitaevskii-like equation, this is identified with \( U_0 \). The particle density \( \rho_n \) is identified with \( |\psi(r)|^2 \); the frequency, \( (\omega/c)^2 \), together with the mass parameter, \( \sigma^2 \), as we saw in the 1D case, is identified with the chemical potential, \( \mu \); and the radial dependence of the scalar field \( u \) is related to the order parameter, \( \psi \).

Notice from eq. (3.9) that the curvature of the space-time induces also a spatially dependent interaction, through the parameter \( \lambda F \). This effective interaction parameter \( \lambda F \) (position dependent) could be interpreted as some kind of gravitational Feshbach resonance induced by the curvature of space-time, and could affect, for instance, the stability of the system, as in usual condensates [20]. Clearly, this particular issue deserves deeper analysis, to be presented elsewhere, due to the fact that it is intimately related with the structure of the system and could be tested, in principle, as dark matter in more realistic scenarios.

In the cases where the effective potential holds the scalar field quasi-stationary configuration, the analogy between the scalar field in a curved background, satisfying eq. (3.9), and the order parameter describing a Bose-Einstein condensate, satisfying the Gross-Pitaevskii stationary equation, eq. (3.12), is remarkable.

We present relevant examples of such spacetimes in the following subsections.
### 3.1 Schwarzschild-de Sitter spacetime

Let us consider the case of a Schwarzschild black hole within a de Sitter spacetime, for which the metric coefficient $F$ of eq. (3.3) has the form

$$F = 1 - 2 \frac{M G}{c^2 r} - \frac{\Lambda}{3} r^2,$$

where $M$ is the mass of the black hole and $\Lambda$ is the cosmological constant. Choosing a mass scale, $M_0$, and a distance scale, $R_0$, we construct the dimensionless quantity

$$q = \frac{G M_0}{c^2 R_0},$$

and the mass of the black hole under study is then a factor of the mass scale, $M = n M_0$, and the distance is a multiple of the distance scale $r = x R_0$, with $n, x$ as dimensionless constants.

Since $\Lambda$ has units of curvature — that is, inverse area — we construct the dimensionless quantity

$$\nu = \frac{\Lambda}{R_0^2},$$

so the metric coefficient $F$ given in eq. (3.13) reduces to the following dimensionless form

$$F = 1 - 2 q \left( \frac{n x}{\nu} \right) - \nu x^2.$$  

(3.14)

Assume that $\Lambda$ represents the dark energy in our model. Using Planck data [46], the definition for the critical density of the universe, $\rho_{\text{critical}} = 3 H_0 \text{Planck}^2/(8 \pi G)$ and the gravitational constant value $G = 4.29 \times 10^{-9} \text{ (km/s)}^2 \text{ Mpc}/\text{M}_\text{Sun}$ [47], we obtain the value $\rho_{\text{critical}} = 1.26 \times 10^{11} \text{M}_\text{Sun}/\text{Mpc}^3$. According to Planck priors, the ratio $\rho_\Lambda/\rho_{\text{critical}} = 68.3\%$, standing for the dark energy sector, is a quantity that we can use to compute the corresponding value for the cosmological constant via $\Lambda = 8\pi G c^{-2} \rho_\Lambda$. Finally, we obtain the value $\Lambda = 1.036 \times 10^{-7} \pm 4.302 \times 10^{-9} \text{ Mpc}^{-2}$.

Choosing the distance scale as $R_0 = 1\text{ Mpc}$, the dimensionless metric coefficient takes the value $\nu = 3.452 \times 10^{-6} \pm 1.434 \times 10^{-9}$.

In the Schwarzschild-de Sitter case, we deal with two kinds of horizons; one related to the black hole, and the other associated with $\Lambda$. This external horizon, as long as the cosmological constant $\Lambda$ has a definite value, is fixed, and amounts to $x_{\text{ext}} = 5386.37$ in the absence of black hole, which gives the size of a Universe dominated by the cosmological constant, $R_{\text{max}} = 5.38 \times 10^3 \text{ Mpc}$. When a black hole is present, and if we consider $q = 1$, this implies that our mass scale is $10^{19}$ solar masses. In this case there exist two horizons, and when one considers a more massive black hole, the internal horizon grows towards the external one in such a way that, for a value of $n = 1036.6$, the two horizons merge into one, and we have a critical Schwarzschild-de Sitter spacetime.

Now, the effective potential, eq. (3.11) for this spacetime reads

$$V_{\text{eff,sds}} = \frac{1}{R_0^2} \left( \alpha^2 - 2 \nu + \frac{2 q n}{x^3} \right) \left( 1 - \frac{2 q n}{x} - \nu x^2 \right),$$

(3.15)

where we define $\alpha = R_0 \sigma$. For the scalar potential we use the same distance scale as the one used for the spacetime parameters, so that the parameter $\alpha$ is dimensionless. For $\alpha < \sqrt{2} \nu$, the asymptotic behavior of the effective potential is positive, and for $x \equiv 2 q n$ we have a characteristic black hole barrier. Thus, we expect to have regions where bound states of the scalar distribution could exist.

We also present a less realistic case, characterized by a huge black hole mass and a very large scalar mass parameter. Despite its unrealistic features, it is interesting that this limit shows clearly how the effective potential induces the formation of bounded regions near the black hole, depending on the value of the scalar mass parameter. For larger radius the effective potential grows and then starts to decline, reaching zero value at the cosmological horizon. The results are presented in figure 1. In this way, the gravitational field generated
Figure 1. Effective potential eq. (3.15), for a large black hole mass and scalar mass parameter \( \alpha \). We see how the potential forms confinement regions depending on the values of \( \alpha \) and \( \Lambda \). Here \( q = n = 1 \) and \( \nu = 3.452 \times 10^{-8} \). Left: the five curves represent the potential behavior for \( \alpha = 0.27, 0.24, 0.19, 0.16, 0.1 \), in descending order. For the first value there are no bounded regions and the latter value disappears. Right: the corresponding behavior for large radii, \( 10^3 \leq x \), in which the potential decreases until it reaches the cosmological horizon at \( x_{\text{ext}} = 5386.37 \), where it is equal to zero for any value of \( \alpha \).

by the black hole mass, the cosmological constant, \( \Lambda \), and the scalar field mass parameter \( \alpha \), is endowed with trapped regions for the scalar field. In the case of a black hole spacetime, [32] showed that these trapped regions can host quasi-stationary distributions of the scalar field, lasting even for cosmological periods.

This fact also strengthens the analogy between the scalar field in curved backgrounds and the order parameter describing Bose-Einstein condensates, since there do exist quasi-stationary distributions of the scalar field in a curved background, which behaves in an analogous way to a Bose-Einstein condensate.

3.2 Schwarschild spacetime

This case has been already discussed in detail [31, 32] within the context of scalar field distributions which remain surrounding a black hole for cosmological times. The metric coefficient in the line element, eq. (3.3) reduces to the form (\( \Lambda = \nu = 0 \))

\[
F = 1 - \frac{2 M G}{c^2 r} = 1 - 2 q \left( \frac{n}{x} \right),
\]

(3.16)

where now the mass \( M_0 \) and distance scale \( R_0 \) are not constrained to be large. The effective potential eq. (3.11) reduces to the following form

\[
V_{\text{eff}} = \frac{1}{R_0^2} \left( \alpha^2 + \frac{2 q n}{x^3} \right) \left( 1 - \frac{2 q n}{x} \right),
\]

(3.17)

where the asymptotic value of the effective potential is the square of the scalar mass parameter \( \alpha^2 \). In this case, the potential presents confinement regions, as shown in figure 2. A detailed discussion has been given in refs. [31, 32], describing how one does indeed have quasi-stationary scalar field distributions which accrete towards the black hole, but at such a slow rate that they can last for cosmological times.

We remark that the solutions are quasi-stationary as long as the stationary solutions are forbidden by the black hole no-hair theorems [48]. Indeed, such scalar field distributions are
Figure 2. Effective potential eq. (3.17). We have have taken $R_0^2 = q = n = 1$, and the case $\alpha = 0.17$. We observe the confinement region for small values of $x$.

not completely stationary, and so are not excluded by those theorems; they are not “hair”, and can last for very long times, depending on the mass of the black black hole and the scalar mass parameter. Hence they have been dubbed “wigs” [32]

3.3 De Sitter spacetime

The usual cosmological solution to Einstein field equations, eq. (3.4), proposed originally by de Sitter, reads

$$ds^2 = -c^2dT^2 + e^{\Lambda T} \left( dR^2 + R^2 d\Omega^2 \right),$$

which can be rewritten as a static-like line element with the form given by eq. (3.3), with

$$F(r) = 1 - \frac{\Lambda}{3} r^2 = 1 - \nu x^2,$$

so that the effective potential and the coordinate transformation to $r^*$ reduces to the following form

$$V_{\text{eff,as}} = \frac{1}{R_0^2} \left( \alpha^2 - 2 \nu \right) \left( 1 - \nu x^2 \right),$$

The asymptotic behavior in this case reads $- (\alpha^2 - 2 \nu) \nu x^2 / R_0^2$, and as long as there is only one extreme, $x = 0$, the scalar mass parameter, $\alpha^2$, must be less than the cosmological one, $\nu$, in order to have a trapped region. This behavior is shown in figure 3.

We do not expect to have scalar field distributions in this case, as long as the potential is always negative.

4 Thomas-Fermi approximation for the scalar field equation in curved backgrounds

Following the analogy between the quasi-stationary scalar field distributions in curved background and Bose-Einstein condensates described by the Gross-Pitaevskii equation, we are able to explore how the Thomas-Fermi approximation can be used to study and, in some sense, to experimentally observe scalar field distributions in curved spacetimes. The Thomas-Fermi approximation is very useful for exploring some relevant thermodynamical properties of Bose-Einstein condensates in the presence of interactions. In usual condensates, an accurate description of the system may be obtained by neglecting the kinetic energy term in
the Gross-Pitaevskii equation from the very beginning. Such an approximation is valid for sufficiently large clouds, and when the scattering length $a$, which describes the interaction among the particles within the system, is much smaller that the mean inter-particle spacing; in other words, when the system is diluted enough and contains a large number of atoms. Finally, we must add that the Thomas-Fermi approximation fails for trapped condensates near the edge of the cloud, due to the divergent behavior of the kinetic energy (i.e. the total kinetic energy per unit area diverges on the boundary of the system).

If we assume that the Thomas-Fermi approximation is valid for our system, then we can obtain from eq. (3.9) the following expression

$$\left(V_{\text{eff}} + \lambda F \rho_n\right) u = \frac{\omega^2}{c^2} u,$$

where we have used the definition given in eq. (3.7) and, consistently with the analogy between the classical scalar field and the condensate, we call $\rho_n$ the particle density which, with the plane wave ansatz, eq. (3.8), is given by $\rho_n = u^2(r)/r^2$.

Within the Thomas-Fermi approximation, one transforms a differential equation into an algebraic one, with a solution

$$\rho_n = \frac{\omega^2}{\lambda F} - V_{\text{eff}}.$$

Finding the values where $\rho_n = 0$, we obtain the region where the scalar field is contained in this approximation. It is clear that there are differences between the solution obtained within this approximation and the actual solution to the Klein-Gordon equation in curved spacetimes. For example, eq. (4.2) diverges at the horizon with $F = 0$, which is an unacceptable behavior for a stable scalar field distribution. However, this sort of problem already occurs in the Thomas-Fermi approximation in usual condensates. The approximation is not valid at the borders. We can expect, at most, that the Thomas-Fermi approximation describes the scalar field distribution in the regions where the density has a maximum — roughly, where the potential has a minimum.

There are several definitions of density used in each context and, as we are using ideas from different fields, it is important to have clear definitions from each, and to understand how they are related to one another.

On the one hand, there is the particle density in nuclear physics, $\rho_n$, eq. (3.7), used in the Thomas-Fermi approximation, eq. (4.2), which is related to the probability density from

Figure 3. Effective potential eq. (3.20). We have have taken $R_0^2 = q = n = 1$ and $\nu = 3.452 \times 10^{-8}$. The case for $\alpha = 2.5 \times 10^{-4}$ shows a confinement region where the effective potential takes negative values.
quantum mechanics. On the other hand, there is the energy density defined in the relativistic context: \( c^2 \rho_E = -T_{00} \). From eq. (3.2), it is straightforward to show, for the space time given by eq. (3.3), that the energy density takes the form

\[
\rho_E = \frac{c^2}{16\pi G} \left( \Phi \Phi^* \frac{\Phi' \Phi^{*'}}{c^2 F} + \sigma^2 \Phi \Phi^* + \frac{\lambda}{2} (\Phi \Phi^*)^2 \right).
\]

(4.3)

This is the expression for the density in General Relativity for static and spherically symmetric spacetimes. One can see that it includes terms involving the particle density from nuclear physics, \( \Phi \Phi^* \), and also spatial and temporal derivatives which, according to the theory, also determine the energy. Indeed, using the plane wave ansatz and the normalization for the scalar function given in eq. (3.8), we obtain the relationship between the energy density and the particle density, eq. (3.7)

\[
\rho_E = \frac{c^2}{16\pi G} \left[ \left( \sigma^2 + \frac{\omega^2}{c^2 F} \right) \rho_n + \frac{\lambda}{2} \rho_n^2 + \frac{\rho_n \rho_n'}{4} \right].
\]

(4.4)

The above can define a mass density, \( \rho_{\text{mass}} \) (with units of density), related to the particle density,

\[
\rho_{\text{mass}} = \frac{c^2 \sigma^2}{16\pi G} \rho_n.
\]

(4.5)

To obtain the order of magnitude of this mass density, we can consider an ultra-light scalar field with a mass of around \( m_\phi c^2 \equiv 10^{-24} \text{eV} \). For \( \hbar \sigma/c = m_\phi \), we find that the corresponding parameter \( \sigma \) for this ultra-light scalar mass is \( 5.06 \times 10^{-18} \text{m}^{-1} \), and \( c^2 \sigma^2/16\pi G = 6.86 \times 10^{-13} \text{grs/cm}^3 \). Notice that as long as the scalar field is dimensionless, the particle density is defined as \( \rho_n = \Phi \Phi^* = u^2/v^2 \).

When studying the validity of the Thomas-Fermi approximation in the Klein-Gordon equation in curved spacetimes in this way, the first question is: how is the number density related to the actual energy density? The energy density reduces to the mass density under certain conditions. For instance, consider a weak gravitational field region, \( F \equiv 1 \), where the mass density has small gradients and the self-interaction term is negligible compared with the first order term. In this case,

\[
\rho_E \sim \left( 1 + \frac{\omega^2}{\sigma^2 c^2} \right) \rho_{\text{mass}}.
\]

(4.6)

which is a relation that could, in principle, be probed, once one has the solution to the Klein-Gordon equation.

Following this line of thought, we also define an expected size and a possible number of particles for the scalar field in a curved background.

We consider the extremes of the density, defined for those values of the distance, \( x_i \) and \( x_f \), where the frequency, \( \omega \), intersects the effective potential. Then, we define the number of particles as the integral of the density between those extremes by using eq. (4.5)

\[
N = \frac{c^2 \sigma^2}{4G} \int_{x_i}^{x_f} x^2 \rho_{\text{mass}} dx.
\]

(4.7)

Let us now apply these concepts to the concrete spacetimes we already presented.
4.1 Schwarzschild-de Sitter spacetime

Within the Thomas-Fermi approximation, a direct application of the particle density definition leads to the following expression for the particle density of the scalar field

$$\rho_{n \text{Sch-deSitt}} = \frac{\omega^2 - \frac{1}{R_0^2} \left( \alpha^2 - 2 \nu + \frac{2 q n}{x^r} \right) \left( 1 - \frac{2 q n}{x} - \nu x^2 \right)}{\lambda \left( 1 - \frac{2 q n}{x} - \nu x^2 \right)}.$$  

(4.8)

From the effective potential plot figure 1, we can choose a value for the frequency which intersects the potential, thus determining the extrema of the density. Then, choosing the value of the parameter \( \lambda \), one obtains a sketch of the mass density distribution. This procedure results in the behavior shown in figure 4.

The cosmological constant modifies the shape of the effective potential in a manner that is noticeable in the external region. It is also worth noting that it can be interpreted as modifying the value of the mass parameter of the scalar field, making it “lighter”, as seen by comparing the effective potential in this case, eq. (3.15), and the one in Schwarzschild, eq. (3.17).

4.2 Schwarzschild spacetime

For this case the function \( F \) takes the form given by eq. (3.17). Then, the solution reads

$$\rho_{n \text{Sch}} = \frac{\omega^2 - \frac{1}{R_0^2} \left( \alpha^2 + \frac{2 q n}{x^r} \right) \left( 1 - \frac{2 q n}{x} \right)}{\lambda \left( 1 - \frac{2 q n}{x} \right)}.$$  

(4.9)

A sketch of the density eq. (4.9) is presented in figure 5.

It is remarkable how straightforward it is to derive the density distribution in this approximation. According to it, the scalar field forms a shell surrounding the black hole, with a maximum at a few Schwarzschild radii. One expects that the actual solution has this form, but that it goes to zero at the horizon and fades smoothly in the external region.

In this way, we expect that the Thomas-Fermi approximation describes the shape, and perhaps even the position, near the maximum of the density distribution of a quasi stationary
Figure 5. Density distribution eq. (4.9) (solid line) and the effective potential eq. (3.17) (dashed line). The values considered for this case are $q = n = 1$, $\alpha = 0.17$, $\omega = 0.158$, $\lambda = 0.01$.

scalar field surrounding a black hole. A work comparing the density derived from the actual numerical solution of the Gross-Pitaevskii equation with the Thomas-Fermi approximation shows that the approximation is indeed excellent in all regions except the extrema [49]. We do not expect such excellent accordance, but we think it certainly hints at the actual behaviour of the scalar field distribution in curved spacetimes.

Let us add that the trapping potential in usual condensates induces a density peak at the center of the trap, i.e. near to $r \sim 0$, and in this scenario, the effects caused by interactions, in a dilute system, are expected to be significant [18]. We must mention that in our case, there is a shift in the corresponding density peak which is caused by the curvature of the space-time, see figures 4 and 5. Notice also that the density peak observed in figures 4 and 5 seems to be located around the minimum of the induced confinement region. The shift in the density peak caused by the geometry of the space-time, together with the position-dependent interaction parameter $\lambda F$ obtained above, deserves deeper regarding the stability of the system; this will be presented elsewhere, as it could lead to observable manifestations.

Finally, concerning the de Sitter background spacetime, we see that there is no value where the particle density vanishes, since the effective potential is always negative, so that a boundary is not well defined and the analogy evidently does not hold in this case.

5 Conclusions and outlook

We presented the different analogies existing between the solution to the Klein-Gordon equation with the solution to the Gross-Pitaevskii equation.

For the case of a particle trapped in a one dimensional box potential with a large separation between the walls (a static soliton kink solution), we showed that both solutions are formally related.

Additionally, we have shown that the Klein-Gordon equation in some classes of curved background spacetimes is such that the gravitational background induces a kind of confinement effective potential which allows quasi-stationary states for the scalar field, in close analogy to what happens with Bose-Einstein condensates in atomic physics. It was shown that a Schwarzschild-de Sitter black hole background spacetime, together with the scalar field potential parameters, provides an effective trapping potential which allows the existence of such quasi-stationary scalar field distributions.
The curved background also induces an effective self-interaction parameter, which clearly modifies the strength of interactions within the system. This fact could in principle be interpreted as a kind of Feshbach resonance caused by the curvature of the space time. In usual Bose-Einstein condensates, the Feshbach resonances make it possible to tune scattering lengths and other quantities, adjusting an external field such as the magnetic field \cite{20}. This could affect the stability of the cloud. Thus, it is interesting to explore if the system is stable, taking into account the effects caused by the induced effective self-interaction parameter. This topic deserves deeper investigation. In particular, in order that our system can be compared with observations, it is mandatory to derive a numerical solution, along the lines of the one in \cite{49}, and compare with the corresponding Thomas-Fermi solution presented in this work. Indeed, the stability or quasi-stability of the system, show in the case without self interaction, is also a question which has to be solved, i.e. what is the influence of the $\lambda F$ term on the stability of the system?

Finally, this approach can be extended to more general scenarios, in which the spacetime has rotation, and an analogy could be made with phenomena associated with Bose-Einstein condensates such as vorticity and superfluidity, along the lines of \cite{50}. It is worth noting that these ideas could imply the possibility that a scalar field considered as a Bose-Einstein condensate could account for dark matter halos surrounding galaxies \cite{31–33}. Clearly these topics deserve deeper investigation, which will be presented elsewhere \cite{51}.

To summarize: in this work we considered a classical relativistic scalar field as a test particle/field. For some particular background spacetimes, a remarkable analogy between the Klein-Gordon equation for a test scalar particle/field and the Gross-Pitaevskii equation for the order parameter of a Bose-Einstein condensate trapped by an external potential, can be made. It is important to stress that the gravitational background provides, in a natural way, an effective confinement potential for the scalar field, together with an effective self-interaction parameter between the constituents of the system.

It would be desirable to have a deeper understanding of the analogy by means of a special relativistic formulation of the Bose-Einstein condensates, and to look for some phenomena which are seen in the Bose-Einstein condensates in the laboratories. This could point to an observable signature in the astrophysical realm.

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References

[1] T.F. Jordan and G. Sudarshan, Lie group dynamical formalism and the relation between quantum mechanics and classical mechanics, Rev. Mod. Phys. 33 (1961) 515 [inSPIRE].

[2] C.H. Brans and R.H. Dicke, Mach’s principle and a relativistic theory of gravitation, Phys. Rev. 124 (1961) 925 [inSPIRE].

[3] C.H. Brans, Gravity and the tenacious scalar field, gr-qc/9705069 [inSPIRE].
[4] R. Brout, F. Englert and C. Truffin, *Chiral symmetry and linear trajectories*, Phys. Lett. B 29 (1969) 590 [insPIRE].
[5] P.W. Higgs, *Broken symmetries, massless particles and gauge fields*, Phys. Lett. 12 (1964) 132 [insPIRE].
[6] H. Abreu on behalf of the ATLAS collaboration, *ATLAS Higgs searches*, PoS(QFTHEP 2013)001.
[7] W.G. Unruh, *Experimental black hole evaporation*, Phys. Rev. Lett. 46 (1981) 1351 [insPIRE].
[8] W.G. Unruh and R. Schützhold eds., *Quantum analogues analogues: from phase transitions to black holes and cosmology*, Lecture Notes in Physics 718 Springer, Berlin Germany (2007).
[9] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, *From Navier-Stokes To Einstein*, JHEP 07 (2012) 146 [arXiv:1101.2451] [insPIRE].
[10] J. Anandan, *Curvature Effects in Interferometry*, Phys. Rev. D 30 (1984) 1615 [insPIRE].
[11] G. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford U.K. (2003).
[12] C. Barceló, S. Liberati and M. Visser, *Analogue Gravity*, Living Rev. Relativ. 8 (2005) 12.
[13] G.T. Horowitz, *Surprising Connections Between General Relativity and Condensed Matter*, Class. Quant. Grav. 28 (2011) 114008 [arXiv:1010.2784] [insPIRE].
[14] R. Schützhold, *Emergent horizons in the laboratory*, Class. Quant. Grav. 25 (2008) 114011 [arXiv:1004.2586] [insPIRE].
[15] S.N. Bose, *Plancks Gesetz und Lichtquantenhypothese*, Z. Phys. 26 (1924) 178.
[16] A. Einstein, *Quantentheorie des einatomigen idealen Gases 2, Sitzberg. K. Preuss. Aka. (1925) 3.
[17] R.K. Pathria, and P.D. Beale, *Statistical Mechanics*, third edition, Academic Press (2011).
[18] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys. 71 (1999) 463 [insPIRE].
[19] L. Vandevenne, *Bose-Einstein Condensation under the influence of attractive boundary conditions*, Ph.D. Thesis (2005).
[20] C.J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge University Press (2002).
[21] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman and E.A. Cornell, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science 269 (1995) 198 [insPIRE].
[22] K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee et al., *Bose-Einstein condensation in a gas of sodium atoms*, Phys. Rev. Lett. 75 (1995) 3969 [insPIRE].
[23] C.C. Bradley, C.A. Sackett, J.J. Tollett and R.G. Hulet, *Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions*, Phys. Rev. Lett. 75 (1995) 1687 [insPIRE].
[24] E.P. Gross, *Structure of Quantized Vortex in Boson Systems*, Nuovo Cim. 20 (1961) 454.
[25] L.P. Pitaevskii, *Vortex Lines in an Imperfect Bose Gas*, Sov. Phys. JETP 40 (1961) 451.
[26] L.J. Garay, J.R. Anglin, J.I. Cirac and P. Zoller, *Black holes in Bose-Einstein condensates*, Phys. Rev. Lett. 85 (2000) 4643 [gr-qc/0002015] [insPIRE].
[27] L.J. Garay, J.R. Anglin, J.I. Cirac and P. Zoller, *Sonic black holes in dilute Bose-Einstein condensates*, Phys. Rev. A 63 (2001) 023611 [gr-qc/0005131] [insPIRE].
[28] Y. Kurita, M. Kobayashi, T. Morinari, M. Tsubota and H. Ishihara, *Spacetime analogue of Bose-Einstein condensates: Bogoliubov-de Gennes formulation*, Phys. Rev. A 79 (2009) 043616 [arXiv:0810.3088] [insPIRE].
(29) A. Macás and H. Quevedo, *Time paradox in quantum gravity*, in *Quantum Gravity — Mathematical models and experimental bounds*, B. Fauser, J. Tolksdorf, E. Zeidler eds., Birkhäuser, Basel, Switzerland (2006), pg. 41–60.

(30) A. Macás and A. Camacho, *On the incompatibility between quantum theory and general relativity*, *Phys. Lett. B* 663 (2008) 99 [arXiv:1108.0931] [inspire].

(31) J. Barranco, A. Bernal, J.C. Degollado, A. Diez-Tejedor, M. Megevand et al., *Are black holes a serious threat to scalar field dark matter models?*, *Phys. Rev. D* 84 (2011) 083008 [arXiv:1108.0931] [inspire].

(32) J. Barranco, A. Bernal, J.C. Degollado, A. Diez-Tejedor, M. Megevand et al., *Schwarzschild black holes can wear scalar wigs*, *Phys. Rev. Lett.* 109 (2012) 081102 [arXiv:1207.2153] [inspire].

(33) J. Barranco, A. Bernal, J.C. Degollado, A. Diez-Tejedor, M. Megevand et al., *Schwarzschild scalar wigs: spectral analysis and late time behavior*, *Phys. Rev. D* 89 (2014) 083006 [arXiv:1312.5808] [inspire].

(34) M. Khlopov, B.A. Malomed and I. Zeldovich, *Gravitational instability of scalar fields and formation of primordial black holes*, *Mon. Not. Roy. Astron. Soc.* 215 (1985) 575 [inspire].

(35) I. Dymnikova and M. Khlopov, *Decay of cosmological constant as Bose condensate evaporation*, *Mod. Phys. Lett. A* 15 (2000) 2305 [astro-ph/0102094] [inspire].

(36) V. Sahni and L.-M. Wang, *A New cosmological model of quintessence and dark matter*, *Phys. Rev. D* 62 (2000) 103517 [astro-ph/9910097] [inspire].

(37) L.A. Ureña-López, *Bose-Einstein condensation of relativistic Scalar Field Dark Matter*, *JCAP* 01 (2009) 014 [arXiv:0806.3093] [inspire].

(38) L. Arturo Ureña-López, *Nonrelativistic approach for cosmological scalar field dark matter*, *Phys. Rev. D* 90 (2014) 027306.

(39) E. Seidel and W.-M. Suen, *Dynamical Evolution of Boson Stars. 1. Perturbing the Ground State*, *Phys. Rev. D* 42 (1990) 384 [inspire].

(40) M. Alcubierre, R. Becerril, S.F. Guzman, T. Matos, D. Núñez et al., *Numerical studies of $\Phi^2$ oscillatons*, *Class. Quant. Grav.* 20 (2003) 2883 [gr-qc/0301105] [inspire].

(41) E. Kopylova, private communication.

(42) T. Matos and A. Suarez, *Finite temperature and dissipative corrections to the Gross-Pitaevskii equation from $\Phi^4$ one loop contributions*, *Europhys. Lett.* 96 (2011) 56005 [arXiv:1110.3114] [inspire].

(43) T. Matos and E. Castellanos, *Bose gas to Bose-Einstein Condensate by the Phase Transition of the Klein-Gordon equation*, arXiv:1207.4416 [inspire].

(44) E. Castellanos and T. Matos, *Klein-Gordon Fields and Bose-Einstein Condensates: Thermal Bath Contributions*, *Int. J. Mod. Phys.* B 27 (2013) 1350060.

(45) T. Matos and E. Castellanos, *Phase transition from the symmetry breaking of charged Klein-Gordon fields*, AIP Conf. Proc. 1577 (2014) 181 [inspire].

(46) PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XVI. Cosmological parameters*, *Astron. Astrophys.* (2014) [arXiv:1303.5076] [inspire].

(47) D. Nunez, and J.D. Degollado, *General Relativity*, (2013) available in spanish at http://www.nucleares.unam.mx/~nunez/cursos/relatividad/RG.pdf.

(48) A. García, E. Hackmann, C. Lämmerzahl and A. Macías, *No-hair conjecture for Einstein-Plebanski nonlinear electrodynamics static black holes*, *Phys. Rev. D* 86 (2012) 024037 [inspire].
[49] Z. Marojević, E. Gökliü and C. Lämmerzahl, *Energy eigenfunctions of the 1D Gross-Pitaevskii equation*, *Comput. Phys. Commun.* 184 (2013) 1920 [arXiv:1208.2123] [SPIRE].

[50] C. Xiong, M.R.R. Good, Y. Guo, X. Liu and K. Huang, *Relativistic superfluidity and vorticity from the nonlinear Klein-Gordon equation*, arXiv:1408.0779 [SPIRE].

[51] E. Castellanos, C. Escamilla-Rivera, E. Santos and D. Núñez, *Bose Einstein Condensate around Black Holes*, in preparation.