Lattice-QCD based Schwinger-Dyson approach for Chiral Phase Transition

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Dynamical chiral-symmetry breaking in QCD is studied with the Schwinger-Dyson (SD) formalism based on lattice QCD data, i.e., LQCD-based SD formalism. We extract the SD kernel function $K(p^2)$ in an Ansatz-independent manner from the lattice data of the quark propagator in the Landau gauge. As remarkable features, we find infrared vanishing and intermediate enhancement of the SD kernel function $K(p^2)$. We apply the LQCD-based SD equation to thermal QCD with the quark chemical potential $\mu_q$. We find chiral symmetry restoration at $T_c \simeq 100\text{MeV}$ for $\mu_q = 0$. The real part of the quark mass function decreases as $T$ and $\mu_q$. At finite density, there appears the imaginary part of the quark mass function, which would lead to the width broadening of hadrons.

1. Introduction

The Schwinger-Dyson (SD) approach has been used as a nonperturbative calculation method for dynamical chiral-symmetry breaking (DCSB) in QCD. The SD formalism consists of an infinite series of nonlinear integral equations which determine the $n$-point Green function of quarks and gluons. Hence, it includes the infinite-order effect of the gauge coupling constant $g$.

The SD equation for the quark propagator $S(p)$ is described with the nonperturbative gluon propagator $D_{\mu\nu}(p)$ and the nonperturbative quark-gluon vertex $g\Gamma_{\nu}(p, q)$ as

$$S^{-1}(p) = S_0^{-1}(p) + g^2 \int_q \gamma_\mu S(q) D_{\mu\nu}(p-q) \Gamma_{\nu}(p, q),$$

where $S_0(p)$ denotes the bare quark propagator and the simple notation $\int_q = \int \frac{d^4q}{(2\pi)^4}$ has been used in the Euclidean metric.

In the practical calculation for QCD, however, the SD formalism is drastically truncated: the perturbative gluon propagator and the one-loop running coupling are used instead of the nonperturbative quantities in the original formalism. By this simplification, some nonperturbative QCD effects will be neglected and therefore the results may not be trustable.

In this paper, we formulate the SD equation based on the recent lattice QCD (LQCD) results, i.e., the LQCD-based SD equation \[3\], and apply it to DCSB at finite temperatures and densities.

2. The Quark Propagator in Lattice QCD

In the Euclidean metric, the quark propagator $S(p)$ in the Landau gauge is generally given by

$$S(p) = Z(p^2)/\{\not{p} + M(p^2)\},$$

with the quark mass function $M(p^2)$ and the wave-function renormalization factor $Z(p^2)$.

Recent quenched lattice QCD \[7\] indicates

$$M(p^2) = M_0/\{1 + (p/\bar{p})^\gamma\}$$

with $M_0 \simeq 260\text{MeV}$, $\bar{p} \simeq 870\text{MeV}$ and $\gamma \simeq 3.04$ in the range of $0 \leq p \leq 4\text{GeV}$ in the chiral limit.

The infrared quark mass $M(p^2 = 0) = M_0 \simeq 260\text{MeV}$ seems consistent with the constituent quark mass in the quark model. Using this lattice result of $M(p^2)$, the pion decay constant is calculated as $f_\pi \simeq 87\text{MeV}$ with the Pagels-Stokar formula, and the quark condensate is obtained as $\langle \bar{q}q \rangle_{\Lambda=1\text{GeV}} \simeq -(220\text{MeV})^3$. These quantities on DCSB seem consistent with the standard values.

3. The Schwinger-Dyson Equation

In the Landau gauge, the Euclidean gluon propagator is generally expressed by

$$D_{\mu\nu}(p^2) = d(p^2)/p^2 \cdot (\delta_{\mu\nu} - p_\mu p_\nu/p^2),$$

where we refer to $d(p^2)$ as the gluon polarization factor. For the quark-gluon vertex, we assume the chiral-preserving vector-type vertex,

$$\Gamma_{\mu}(p, q) = \gamma_\mu \Gamma((p - q)^2).$$
which keeps the chiral symmetry properly. (Recent lattice QCD studies indicate dominance of the vector vertex with some nontrivial structure in the quark-gluon vertex.) Taking the trace of Eq. (1), one gets

\[
\frac{M(p^2)}{Z(p^2)} = 3C_F \int_q \frac{Z(q^2)M(q^2)K((p - q)^2)}{q^2 + M^2(q^2)} \frac{K((p - q)^2)}{(p - q)^2},
\]

where we define the kernel function

\[
K(p^2) \equiv g^2 \Gamma(p^2)d(p^2)
\]

as the product of the quark-gluon vertex function \(\Gamma(p^2)\) and the gluon polarization factor \(d(p^2)\).

The color factor of quarks is \(C_F = 4/3\).

4. Extraction of the SD Kernel Function

In usual, the SD equation is used to obtain the quark mass function \(M(p^2)\). We note however that the SD equation is the relation between the quark propagator and \(K(p^2)\). Then, we extract the kernel function \(K(p^2)\) in the SD equation using the quark propagator obtained in lattice QCD. By shifting the integral variable from \(q\) to \(\tilde{q} \equiv p - q\), we obtain

\[
\frac{M(p^2)}{Z(p^2)} = \frac{3C_F}{8\pi} \int_0^\infty d\tilde{q}^2 \Theta(p, \tilde{q}) K(\tilde{q}^2),
\]

where \(\Theta(p, q)\) is defined with \(M(p^2)\) and \(Z(p^2)\) as

\[
\Theta(p, q) \equiv \int_0^\pi d\theta \sin^2 \theta \frac{Z(p^2 + q^2 - 2pq \cos \theta)M(p^2 + q^2 - 2pq \cos \theta)}{p^2 + q^2 - 2pq \cos \theta + M^2(p^2 + q^2 - 2pq \cos \theta)}
\]

From the knowledge of \(M(p^2)\) and \(Z(p^2)\) in lattice QCD, we can calculate \(K(p^2)\) with Eq. (8).

Figure 1 shows the kernel function \(K(p^2)\) extracted from the quark propagator in lattice QCD. As remarkable features, we find “infrared vanishing” and “intermediate enhancement” in the SD kernel \(K(p^2)\). In fact, \(K(p^2)\) seems consistent with zero in the very infrared region as \(K(p^2 \sim 0) \simeq 0\), while \(K(p^2)\) exhibits a large enhancement in the intermediate-energy region around \(p \sim 0.6\,\text{GeV}\).

Note that, to extract the kernel function \(K(p^2) \equiv g^2 \Gamma(p^2)d(p^2)\), we use only the quark propagator and never use the gluon propagator. Nevertheless, we find infrared vanishing and intermediate enhancement, which are also observed in the polarization factor \(d(p^2)\) in the gluon propagator in the Landau gauge. In fact, these two features of \(K(p^2)\) are embedded in the information of the quark propagator in lattice QCD.

Figure 1 corresponds to the usage of the perturbative gluon propagator \(d(p^2) = 1\) and the one-loop running coupling \(g_{\text{run}}(p^2)\), i.e., \(K(p^2) = g_{\text{run}}^2(p^2)\), which largely differs from the present result based on lattice QCD both in the infrared and in the intermediate-energy regions. Hence, the simple version of the SD equation using the perturbative gluon propagator and the one-loop running coupling would be too crude for the quantitative study of QCD.

5. Chiral Symmetry at Finite Temperature

We demonstrate a simple application of the LQCD-based SD equation to chiral symmetry restoration in finite-temperature QCD.

At a finite temperature \(T\), the field variables obey the (anti-)periodic boundary condition in the imaginary-time direction, which leads to the
SD equation for the thermal quark mass \(M_n(p^2)\) of the Matsubara frequency \(\omega_n \equiv (2n + 1)\pi T\) and spatial three momentum \(p\) as
\[
\frac{M_n(p^2)}{Z_n(p^2)} = \int_{m,q} \frac{3C_F Z_m(q^2)M_m(q^2)}{\omega_{nm}^2 + \hat{q}^2 + M_m^2(q^2)} \frac{K(\omega_{nm}^2 + \hat{q}^2)}{\omega_{nm}^2 + \hat{q}^2} \quad (10)
\]
with \(\int_{m,q} = T \sum_{m=-\infty}^{\infty} \int \frac{d^3q}{(2\pi)^3}, \omega_{nm} \equiv \omega_n - \omega_m\) and \(\hat{q} \equiv p - q\).

Using the kernel function \(K(p^2)\) obtained in the previous section, we solve Eq. (10) for the thermal infrared quark mass \(M_0(p^2 = 0)\) plotted against the temperature \(T\). We thus find chiral symmetry restoration at a critical temperature \(T_c \approx 100\text{MeV}\).

Note that there appears the imaginary part in the SD equation due to the explicit breaking of charge conjugation of the system. Then, we write the quark mass function \(M_n(p^2) \equiv \mathbb{C}\) as
\[
\tilde{M}_n(p^2, \mu_q) \equiv M_n(p^2, \mu_q) + i\Gamma_n(p^2, \mu_q) \quad (13)
\]
with real functions, \(M_n(p^2, \mu_q)\) and \(\Gamma_n(p^2, \mu_q)\).

Here, we expand the SD equation by the chemical potential \(\mu_q\), and calculate the quark mass function order by order. As a merit of the expansion by \(\mu_q\), the formalism is extremely simplified. In principle, this expansion can be done to arbitrary order of \(\mu_q\). In general, the even-order contribution is real and the odd-order contribution is pure imaginary, because of the reflection symmetry of the SD equation in terms of \(\mu_q \rightarrow -\mu_q\) and the complex conjugation, i.e., \(\tilde{M}^*_n(p^2, -\mu_q) = \tilde{M}_n(p^2, \mu_q)\). Then, the quark mass function can be written as
\[
M_n(p^2, \mu_q) \equiv M_n^{(0)}(p^2) + \mu_q^2 M_n^{(2)}(p^2) + \cdots \quad (14)
\]
\[
\Gamma_n(p^2, \mu_q) \equiv \mu_q \Gamma_n^{(1)}(p^2) + \mu_q^3 \Gamma_n^{(3)}(p^2) + \cdots. \quad (15)
\]

We calculate the quark mass function up to the second order of the quark chemical potential \(\mu_q\). The 0th-order SD equation for \(M_n^{(0)}(p^2)\) is the same as the SD equation (10) at zero density.

The 1st-order SD equation for \(\Gamma_n^{(1)}(p^2)\) reads
\[
\Gamma_n^{(1)}(p^2) = \int_{m,q} \frac{K(\omega_{nm}^2, (p - q)^2)}{\omega_{nm}^2 + (p - q)^2} \frac{3CF}{D_m(q^2)} \omega_{nm} \Gamma_n^{(1)}(q^2) + 2\omega_m M_n^{(0)}(q^2) \}
\]
with \(D_m(q^2) \equiv \omega_{nm}^2 + q^2 + M_n^{(0)}(q^2)\).

The 2nd-order SD equation for \(M_n^{(2)}(p^2)\) reads
\[
M_n^{(2)}(p^2) = \int_{m,q} \frac{K(\omega_{nm}^2, (p - q)^2)}{\omega_{nm}^2 + (p - q)^2} \frac{3CF}{D_m(q^2)} \omega_{nm} M_n^{(2)}(q^2) + C_n^{(2)}(q^2) \}
\]
with the non-homogeneous term,
\[
C_n^{(2)}(q^2) \equiv -\frac{4M_n^{(0)}(q^2)(M_n^{(0)}(q^2)\Gamma_n^{(1)}(q^2) - \omega_m)^2}{D_m(q^2)} + M_n^{(0)}(q^2) - 2\omega_m \Gamma_n^{(1)}(q^2) + 3M_n^{(0)}(q^2) \Gamma_n^{(1)}(q^2).
\]

Figure 3(a) shows the real part of the quark mass function, \(M_n(p^2, \mu_q) = \text{Re} \tilde{M}_n(p^2, \mu_q)\), for...
heavy-ion collision experiment at the CERN SPS [10]. This change seems to be explained as the “width broadening” rather than the “mass shift” of the vector meson [11]. In this point, our result supports the width broadening together with a small mass shift at finite density.

7. Summary and Concluding Remarks

We have studied DCSB in QCD with the SD formalism based on lattice QCD data, i.e., LQCD-based SD formalism. We have extracted the SD kernel function $K(p^2)$ in an Ansatz-independent manner from the lattice data of the quark propagator in the Landau gauge. We have found that the SD kernel $K(p^2)$ exhibits infrared vanishing and a large enhancement at the intermediate-energy region around $p \sim 0.6\text{GeV}$.

We have applied the LQCD-based SD formalism to thermal and dense QCD. We have found that the real part of the quark mass function decreases as $T$ and $\mu_q$. At finite density, the quark mass function has an imaginary part, which would lead to the width broadening of hadrons.

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