THE TETRABLOCK AS A SPECTRAL SET

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Abstract. We study a commuting triple of bounded operators \((A,B,P)\) which has the tetrablock as a spectral set.

1. Introduction

We follow Arveson’s terminologies from \cite{10}.

A compact subset \(K\) of \(\mathbb{C}^n\) is called a spectral set for a commuting \(n\)-tuple of operators \(\mathbf{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)\) acting on a Hilbert space \(\mathcal{H}\) if the Taylor joint spectrum of \(\mathbf{\delta}\) is contained in \(K\) and

\[
\|r(\delta_1, \delta_2, \ldots, \delta_n)\| \leq \|r\|_{\infty,K} = \sup\{|r(z_1, z_2, \ldots, z_n)| : (z_1, z_2, \ldots, z_n) \in K\}
\]

(1.1)

for any rational function \(r\) in \(\text{rat}(K)\). For an \(m \times m\) matrix valued function \(r\), we can make sense of \(r(\delta_1, \delta_2, \ldots, \delta_n)\). Any such \(r\) is a matrix of functions of the form

\[
r = [r_{ij}]_{i,j=1}^m
\]

and hence

\[
r(\delta_1, \delta_2, \ldots, \delta_n) = [r_{ij}(\delta_1, \delta_2, \ldots, \delta_n)]_{i,j=1}^m.
\]

This is then a block operator matrix acting as an operator on the direct sum of \(n\) copies of \(\mathcal{H}\). The set \(K\) is called a complete spectral set if the von Neumann type inequality (1.1) above holds for all rational functions taking values in matrices of any order with the norm now being

\[
\|r\|_{\infty,K} = \sup\{\|r(z_1, z_2, \ldots, z_n)\| : (z_1, z_2, \ldots, z_n) \in K\}.
\]

Say that \(K\) has the property \(P\) if the following holds. "If \(K\) is a spectral set for a commuting tuple \(\mathbf{\delta}\), then it is a complete spectral set for \(\mathbf{\delta}.\)" It has historically intrigued operator theorists which subsets of the complex plane or of higher dimensional Euclidean space have this property.

(1) In dimension one, the unit disk (von Neumann \cite{26}) has this property. Berger, Foias and Lebow extended that result to any simply connected domain. Whether a domain in the plane with a single hole enjoyed the property \(P\) was open for a long time till Agler showed in \cite{3} that the annulus has this property. Dritschel and McCullough proved in \cite{15} that a multiply connected domain in general does not. They use Gel’fand-Naimark-Segal construction combined with a cone-separation argument. Agler, Harland and Raphael in \cite{4} have shown that there exists a planar domain with two holes and an operator \(\delta\) on a Hilbert space of dimension

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4 for which the property \( P \) fails. Thus, this branch of generalization from the disc to multiply connected domain is now complete.

(2) The other direction of generalization is to higher dimensions. In dimension two, the bidisc (Ando [9]) and the symmetrized bidisc have this property, see Agler and Young [5 - 7].

(3) No subset of \( C^3 \) with property \( P \) is known.

(4) If \( K \) is the closed unit ball of some norm on \( C^n \) where \( n > 2 \), then \( K \) cannot have this property. See Paulsen [21] and Pisier [23].

Theorem 1.2.2 and the remark following it in Arveson [10] gives a novel idea that has been taken up and pursued heavily by operator theorists. A simultaneous dilation of \( \delta \) consists of \( n \) commuting bounded operators \( \Delta_1, \Delta_2, \ldots, \Delta_n \) on a Hilbert space \( K \) containing \( H \) in such a way that

\[
P_H \Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_n^{k_n} = \delta_1^{k_1} \delta_2^{k_2} \cdots \delta_n^{k_n}
\]

for all non-negative integers \( k_1, k_2, \ldots, k_n \). The dilation is called minimal if

\[
\mathcal{K} = \overline{\text{span}} \{ \Delta_1^{k_1} \Delta_2^{k_2} \cdots \Delta_n^{k_n} h : k_1, k_2, \ldots, k_n \text{ are non-negative integers and } h \in H \}.
\]

Dilation was a game-changing new geometric concept introduced by Sz.-Nagy and has made effective yet unpredictable appearances in many places. For more details, see the classic [20]. Naturally, one wants the dilation operators to be nicer than the given ones. The dilation \( (\Delta_1, \Delta_2, \ldots, \Delta_n) \) is called a normal dilation if its Taylor joint spectrum is contained in the Shilov boundary of \( K \) (relative to \( A(K) \)) and the \( \Delta_i \) are normal operators. Arveson showed that \( K \) is a complete spectral set for \( \delta \) if and only if \( \delta \) has a normal dilation.

Similarities and dissimilarities of the symmetrized bidisc and the tetrablock have attracted recent attention from both complex analysts and operator theorists. The geometry of them has been studied by Edigarian, Kosinski and Zwonek in [16], [17], [18] and [27]. Young’s work on the symmetrized bidisc, the \( \Gamma \)-contractions and the tetrablock with his many co-authors (see [1], [2], [5], [6] and [7]) has been from operator theoretic point of view.

The symmetrized bidisc

\[
G = \{ (z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D} \}
\]

and the tetrablock

\[
E = \{ (a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is in the classical Cartan domain of type II} \}
\]

are both non-convex domains. The Lempert function and the Caratheodory distance coincide on them. But they cannot be exhausted by domains biholomorphic to convex ones.

Those are the similarities. The dissimilarities appear in operator theory on these domains. Normal dilation was constructed for a pair of commuting operators, having the symmetrized bidisc as a spectral set, in [11]. Now we show that given a triple of commuting operators, having the tetrablock as a spectral set, its fundamental operators need to satisfy certain commutativity conditions so that there is a normal dilation.
2. Preliminaries

Definition 2.1. Let \((A, B, P)\) be a triple of commuting bounded operators on a Hilbert space \(H\). We call it a tetrablock contraction if \(E\) is a spectral set for \((A, B, P)\).

A tetrablock contraction consists of commuting contractions. This can be seen easily if we take the polynomial \(f_i(x_1, x_2, x_3) = x_i\) for any \(i = 1, 2, 3\). From the definition of \(E\), it is evident that \(\|f_i\|_{\infty, E} \leq 1\). Thus \(\|A\| = \|f_1((A, B, P))\| \leq 1\). Similarly, \(B\) and \(P\) are contractions. Let \(D_P = (I - P^*P)^{1/2}\) be the defect operator.

An added source of pleasure consists of numerous relations of the tetrablock to the symmetrized bidisk and of tetrablock contractions to \(\Gamma\)-contractions. The closed symmetrized bidisc will be denoted by \(\Gamma\). This is a polynomially convex subset of \(C^2\) with the distinguished boundary (the Shilov boundary relative to \(A(\Gamma)\)) being

\[
b\Gamma = \{(z_1 + z_2, z_1z_2) : |z_1| = |z_2| = 1\}.
\]

A commuting pair of bounded operators \((S, P)\) is called

1. a \(\Gamma\)-contraction if the closed symmetrized bidisc \(\Gamma\) is a spectral set for \((S, P)\),
2. a \(\Gamma\)-unitary if \(S\) and \(P\) are normal operators with the Taylor joint spectrum of the pair \((S, P)\) contained in \(b\Gamma\),
3. a \(\Gamma\)-isometry if it is the restriction of a \(\Gamma\)-unitary to a joint invariant subspace.

Agler and Young studied the \(\Gamma\)-contractions in a series of papers, see [5] - [7]. They proved that if \(\Gamma\) is a spectral set for \((S, P)\), then it is a complete spectral set for \((S, P)\). Thus, by Arveson’s theorem, a \(\Gamma\)-contraction has a normal dilation. In other words, a \(\Gamma\) contraction has a \(\Gamma\) unitary dilation. The change in point of view introduced in [11] explicitly constructed a \(\Gamma\)-isometric dilation of a \(\Gamma\)-contraction by finding solution of an operator equation. We pick out the salient features from there as well as from earlier papers of Agler and Young ([7] and [5]) which we summarize below. For a contraction \(P\) and bounded operator \(S\) commuting with \(P\), define \(\rho(S, P) = 2(I - P^*P) - (S - S^*P) - (S^* - P^*S)\).

Theorem 2.2. Let \((S, P)\) be a pair of commuting bounded operators on \(H\) with \(P\) being a contraction. Then the following are equivalent.

1. \(\Gamma\) is a spectral set for the commuting pair \((S, P)\).
2. \(I - P^*P \geq \text{Re}\beta(S - S^*P)\) for all \(\beta\) on the unit circle. In other words, \(\rho(\beta S, \beta^2 P) \geq 0\) for all \(\beta\) on the unit circle.
3. The operator equation

\[
S - S^*P = (I - P^*P)^{1/2}\Phi(I - P^*P)^{1/2}
\]

called its fundamental equation has a unique solution \(\Phi \in B(\text{Ran}(I - P^*P)^{1/2})\) with numerical radius of \(\Phi\) being not greater than one.
4. There is a pair of commuting normal operators \(R\) and \(U\) on a bigger Hilbert space \(K\) containing \(H\) such that the Taylor joint spectrum \(\sigma(R, U)\) is contained in the distinguished boundary of \(\Gamma\) and

\[
P_{H}R^{m}U^{n}|_{H} = S^{m}P^{n}
\]

for all non-negative integers \(m\) and \(n\).

Explicit construction of the \(\Gamma\) unitary \((R, U)\) involves the fundamental operator \(\Phi\) obtained above. Certain routine facts will be used without further ado. Some of these are
(1) if \((S, P)\) is a \(\Gamma\)-contraction and \(P\) is a unitary, then \((S, P)\) is a \(\Gamma\)-unitary,
(2) if \((S, P)\) is a \(\Gamma\)-contraction and \(P\) is an isometry, then \((S, P)\) is a \(\Gamma\)-isometry.

3. The fundamental equations for a tetrablock contraction

The main content of this section is Theorem 3.5 which solves the fundamental equations for a tetrablock contraction. The tetrablock \(E\) has several characterizations as stated below from the papers [1] and [2]. It follows from the definition of \(E\) that \(1 - x_1 z \neq 0 \neq 1 - x_2 w\) for any \(x \in E\) and any \(z \in \mathbb{D}\). Thus the holomorphic functions

\[
\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1} \quad \text{and} \quad \Upsilon(z, x) = \frac{x_3 z - x_2}{x_1 z - 1}
\]

(3.1)
can be defined on \(\mathbb{D} \times E\). Let \(\mathbb{M}_2\) denote the algebra of all \(2 \times 2\) complex matrices equipped with the operator norm \(\| \cdot \|\). Define \(\pi : \mathbb{M}_2 \rightarrow \mathbb{C}^3\) by

\[
\pi \left( \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = (a_{11}, a_{22}, \det A)
\]

We quote from Abouhajar, White and Young a number of ways for deciding membership of an \(x \in E\).

**Theorem 3.1** (Abouhajar, White, Young). For \(x = (x_1, x_2, x_3) \in \mathbb{C}^3\), the following are equivalent. And they all guarantee membership in \(E\) (respectively \(\in \mathbb{E}\)).

1. \(1 - x_1 z - x_2 w + x_3 zw \neq 0\) whenever \(|z| < 1\) and \(|w| < 1\),
2. \(\|\Psi(z, x)\|_{H^\infty} < 1\) (respectively \(\leq 1\)),
3. \(|x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3| < 1 - |x_2|^2\) (respectively \(\leq 1 - |x_2|^2\)),
4. \(|x_2 - \overline{x}_1 x_3| + |x_1 x_2 - x_3| < 1 - |x_1|^2\) (respectively \(\leq 1 - |x_1|^2\)),
5. \(|x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_1 - \overline{x}_2 x_3| < 1\) (respectively \(\leq 1\)) and \(|x_1| < 1\) (respectively \(\leq 1\)),
6. \(|x_1|^2 - |x_3|^2 + 2|x_1 x_2 - x_3| < 1\) (respectively \(\leq 1\)) and \(|x_3| < 1\) (respectively \(\leq 1\)),
7. \(x = \pi(A)\) for an \(A \in \mathbb{M}_2\) with \(\|A\| < 1\) (respectively \(\leq 1\)),
8. \(x = \pi(A)\) for a symmetric \(A \in \mathbb{M}_2\) with \(\|A\| < 1\) (respectively \(\leq 1\)),
9. \(|x_3| < 1\) (respectively \(\leq 1\)) and there are complex numbers \(\beta_1\) and \(\beta_2\) with \(|\beta_1| + |\beta_2| < 1\) (respectively \(\leq 1\)) such that

\[
x_1 = \beta_1 + \overline{\beta}_2 x_3 \quad \text{and} \quad x_2 = \beta_2 + \overline{\beta}_1 x_3.
\]

The theorem above will be crucial for the purpose of this note. And we have a neat new criterion for membership of an \((x_1, x_2, x_3)\) in \(E\).

**Lemma 3.2.** The triple \((x_1, x_2, x_3)\) is in \(E\) if and only if the pair \((x_1 + z x_2, x_3)\) is in \(G\) for every \(z\) on the unit circle.

**Proof.** Theorem 1.1 in [7] gives several characterizations of when a pair \((s, p)\) will be in \(G\). Of those characterizations, the one most suitable for our present purpose is that

\[
|s - \overline{s} p| < 1 - |p|^2 \quad \text{and} \quad |s| < 2.
\]

(3.2)
Let \( s_z = x_1 + zx_2 \) and \( p_z = zx_3 \). Let \((x_1, x_2, x_3)\) be in \( E \). By part (9) of Theorem 3.1 we have \( |s_z| < 2 \). Now
\[
|s_z - \overline{s_z}p_z| = |x_1 + zx_2 - (\overline{x_1} + \overline{zx_2})zx_3| = |x_1 - \overline{x_2}x_3 + z(x_2 - \overline{x_1}x_3)| \leq |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3|.
\]
Now appeal to part (6) of Theorem 3.1 above to complete the proof that \((s_z, p_z)\) is in \( G \).

Conversely, let \((s_z, p_z)\) be in \( G \) (respectively \( \Gamma \)) for all \( z \) on the circle. By the characterization (3.2), we have a function \( \beta \) on the circle which satisfies
\[
s_z - \overline{s_z}p_z = \beta(z)(1 - |p|^2) \text{ and } |\beta(z)| < 1.
\]
Now
\[
\beta(z) = \frac{s_z - \overline{s_z}p_z}{1 - |p|^2} = \frac{x_1 + zx_2 - \overline{x_1} + \overline{zx_2}zx_3}{1 - |x_3|^2} = \frac{(x_1 - \overline{x_2}x_3) + z(x_2\overline{r_1}x_3)}{1 - |x_3|^2}.
\]
Since this holds for all \( z \) on the circle, we have \( \beta(z) = \beta_1 + z\beta_2 \), where
\[
\beta_1 = \frac{x_1 - \overline{x_2}x_3}{1 - |x_3|^2} \text{ and } \beta_2 = \frac{x_2 - \overline{x_1}x_3}{1 - |x_3|^2}.
\]
Clearly,
\[
x_1 = \beta_1 + \beta_2x_3 \text{ and } x_2 = \beta_2 + \beta_1x_3.
\]
Moreover, for any \( z_1 \) and \( z_2 \) on the unit circle, we have
\[
|z_1\beta_1 + z_2\beta_2| = |z_1| |\beta_1 + z_2\beta_2| = |\beta(z_2)\beta(z_1)| < 1.
\]
Now if we choose \( z_1 \) and \( z_2 \) so that \( z_1\beta_1 + z_2\beta_2 = |\beta_1| + |\beta_2| \) and apply part (9) of Theorem 3.1 that finishes the proof. \( \square \)

The first thing to observe about tetrablock contractions is that the defining criterion can be greatly simplified.

**Lemma 3.3.** A commuting triple of bounded operators \((A, B, P)\) is a tetrablock contraction if and only if
\[
\|f(A, B, P)\| \leq \|f\|_{\infty, E} = \sup\{|f(x_1, x_2, x_3)| : (x_1, x_2, x_3) \in \overline{E}\} \tag{3.3}
\]
for any holomorphic polynomial \( f \) in three variables.

**Proof.** If \((A, B, P)\) is a tetrablock contraction, then of course (3.3) just follows from definition.

The converse proof can be easily done by using polynomial convexity of \( \overline{E} \). Indeed, if the Taylor spectrum \( \sigma((A, B, P)) \) is not contained in \( \overline{E} \), then there is a point \((\lambda_1, \lambda_2, \lambda_3)\) in \( \sigma((A, B, P)) \) that is not in \( \overline{E} \). By polynomial convexity of \( \overline{E} \), there is a polynomial \( f \) such that \( |f(\lambda_1, \lambda_2, \lambda_3)| > \|p\|_{\infty, \overline{E}} \). By polynomial spectral mapping theorem,
\[
\sigma(f((A, B, P))) = \{f(x_1, x_2, x_3) : (x_1, x_2, x_3) \in \sigma((A, B, P))\}
\]
and hence the spectral radius of \( f((A, B, P)) \) is bigger than \( \|f\|_{\infty, \overline{E}} \). But then \( \|f((A, B, P))\| > \|f\|_{\infty, \overline{E}} \), contradicting the fact that \( \overline{E} \) is a spectral set for \((A, B, P)\).

By polynomial convexity of \( \overline{E} \), a triple satisfying (3.3) will also satisfy
\[
\|f(A, B, P)\| \leq \|f\|_{\infty, \overline{E}}
\]
for any function holomorphic in a neighbourhood of \( \overline{E} \). Indeed, Oka-Weil theorem (Theorem 5.1 of [19]) allows us to approximate \( f \) uniformly by polynomials. Then Theorem
9.9 of Chapter III of [25] about functional calculus in several commuting operators seals the rest of the deal.

As a matter of detail, we note that the restriction of a tetrablock contraction to a joint invariant subspace is a tetrablock contraction. Indeed, if \( M \) is a such a subspace, and \( f \) is any polynomial in three variables, then

\[
\| f(A|_M, B|_M, P|_M) \| = \| f((A, B, P))|_M \| \leq \| f((A, B, P)) \| \leq \| f \|_\infty.
\]

The other thing that follows immediately from the lemma above is that the adjoint triple \((A^*, B^*, P^*)\) is a tetrablock contraction too.

The numerical radius \( w(T) \) of a bounded operator \( T \) features naturally. It is defined as

\[
w(T) = \sup_{\|x\| \leq 1} | \langle Tx, x \rangle | .
\]

It is well known that \( w(T) \) and \( \| T \| \) define equivalent norms. Indeed,

\[
r(T) \leq w(T) \leq \| T \| \leq 2w(T)
\]

for all bounded operators \( T \) where \( r(T) \) denotes the spectral radius of \( T \).

The main result of this section is the following theorem which gives a chain of one way implications. The main content of the theorem is the existence and uniqueness of solutions of the fundamental equations for a given tetrablock contraction.

**Definition 3.4.** For a commuting triple of contractions \( \mathbf{T} = (T_1, T_2, T_3) \), the equations

\[
T_1 - T_2^*T_3 = DF_1D, \quad \text{and} \quad T_2 - T_1^*T_3 = DF_2D
\]

are called the first fundamental equation and the second fundamental equation respectively.

Given two bounded operators \( X \) and \( Y \), let \([X, Y]\) denote the commutator \( XY - YX\).

**Theorem 3.5.** Let \( A, B \) and \( P \) be three commuting contractions on a Hilbert space \( \mathcal{H} \). Then, in the following, (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4).

1. The triple \((A, B, P)\) is a tetrablock contraction.
2. The operator functions \( \rho_1 \) and \( \rho_2 \) defined by
   \[
   \rho_1((A, B, P)) = I - P^*P + (B^*B - A^*A) - 2 \operatorname{Re} (B - A^*P)
   \]
   and
   \[
   \rho_2((A, B, P)) = I - P^*P + (A^*A - B^*B) - 2 \operatorname{Re} (A - B^*P).
   \]
   satisfy
   \[
   \rho_1(A, zB, zP) \geq 0 \quad \text{and} \quad \rho_2(A, zB, zP) \geq 0 \quad \text{for all} \quad z \in \overline{\mathbb{D}}.
   \]
3. For any \( z \in \mathbb{C} \), if we define a pair of operators by
   \[
   S_z = A + zB \quad \text{and} \quad P_z = zP.
   \]
   then \((S_z, P_z)\) is a \( \Gamma \)-contraction for every \( z \) on the unit circle.
4. The fundamental equations (3.4) have unique solutions \( F_1 \) and \( F_2 \) in \( \mathcal{B}(\operatorname{Ran}D_P) \). Moreover, the \( \mathcal{B}(\operatorname{Ran}D_P) \) valued function \( F_1 + zF_2 \) has numerical radius not greater than 1 for all \( z \in \overline{\mathbb{D}} \).
Proof. (1) ⇒ (2): It is enough to prove that ρ₁(A, zB, zP) ≥ 0 for all z ∈ ℍ. The proof for ρ₂ is the same.

Consider the function Ψ defined in 3.1. If z ∈ ℍ, then Ψ(z, ·) is a holomorphic function on E with

\[ Ψ(z, (A, B, P)) = (zP - A)(I - zB)^{-1}. \]

Because ℍ is a spectral set for (A, B, P), we know that \( \|Ψ(z, (A, B, P))\| \leq 1 \) which in other words means that

\[ (I - \bar{z}B^*)^{-1}(\bar{z}P^* - A^*)(zP - A)(I - zB)^{-1} ≤ I \]

which translates to

\[ (\bar{z}P^* - A^*)(zP - A) ≤ (I - \bar{z}B^*)(I - zB) \]

and finally

\[ |z|^2P^*P - \bar{z}P^*A - zA^*P + A^*A ≤ I - zB - \bar{z}B^* + |z|^2B^*B \]

which is nothing but ρ₁(A, zB, zP) ≥ 0. Since this holds for all z in the disk and since the function ρ₁ is continuous, the inequality holds on ℍ. That finishes the proof of this step.

Lemma 3.2 showed that the tetrablock is intimately connected with the symmetrised bidisc. In the following, we shall see that the same is true for tetrablock contractions and Γ-contractions. Many facets of the theory of Γ-contractions will be used in this paper.

(2) ⇒ (3): One of the criteria for \((S_z, P_z)\) to be a Γ-contraction is that ρ₁₁₁(S, α₂P_z) ≥ 0 for all α on the unit circle, see Theorem 2.2. Since we have

\[ ρ₁₁₁(A, z_1B, z_1P) ≥ 0 \]

and

\[ ρ₂₂₂(A, z_2B, z_2P) ≥ 0 \]

for all z₁ and z₂ on the circle, adding these two, we get

\[ D_p² ≥ \text{Re } (z_1Σ₁ + z_2Σ₂) = \text{Re } (z_1(Σ₁ + \bar{z}_2Σ₂)). \]

So for all α and z on the circle, we have

\[ D_p² ≥ \text{Re } (α(Σ₁ + zΣ₂)) \]

\[ = \text{Re } (α(A - B^*P + z(B - A^*P))) \]

\[ = \text{Re } (α((A + zB) - (zA^* + B^*))P)) \]

\[ = \text{Re } (α((A + zB) - z(A^* + \bar{z}B^*)P)) \]

\[ = \text{Re } (α((A + zB) - (A + zB)^*zP)). \]

Thus we have

\[ D_p² ≥ \text{Re } α(S_z - S_z^*P_z) \]

for all α on the circle which is the same as saying that ρ₁₁₁(S_z, α₂P_z) ≥ 0 for all α on the unit circle. That finishes the proof.

(3) ⇒ (4): The uniqueness part is the simplest. Indeed, let \( F_1 \) and \( F'_1 \) be two bounded operators on \( \text{Ran}D_p \) both of which satisfy the first fundamental equation \( Σ₁ = D_pXD_p \). Then \( F = F_1 - F'_1 \) satisfies \( D_pFD_p = 0 \). Now, for x and y in H, we have \( \langle F_D_p x, D_p y \rangle = \langle D_pFD_px, y \rangle = 0 \) and hence \( F_1 - F'_1 = 0 \). The existence part of this proof starts from the fact that \((S_z, P_z)\) is a Γ-contraction for any z on the circle. Appeal to Theorem 2.2 again. The Γ-contraction \((S_z, P_z)\) has a fundamental operator, call it \( F(z) \). It satisfies

\[ S_z - S_z^*P_z = D_{P_z}F(z)D_{P_z} \] and \( w(F(z)) ≤ 1 \)
where \( w \) stands for the numerical radius. Recalling what \( S_z \) and \( P_z \) are, the last equation becomes
\[
\Sigma_1 + z\Sigma_2 = D_p F(z)D_p.
\]
This holds for all \( z \) on the unit circle. Integrating over the unit circle, we see that \( \Sigma_1 = D_p F_1 D_p \) where \( F_1 \) is the integration of the function \( F \) over the unit circle. Thus
\[
z\Sigma_2 = D_p (F(z) - F_1) D_p
\]
for all \( z \) on the unit circle and hence putting \( z = 1 \), we get that \( \Sigma_2 = D_p F_2 D_p \) where \( F_2 = F(1) - F_1 \). Thus we see that \( F \) is a linear function \( F(z) = F_1 + zF_2 \) for some bounded operators \( F_1 \) and \( F_2 \) on \( \text{Ran} D_p \) and
\[
\Sigma_1 = D_p F_1 D_p \text{ and } \Sigma_2 = D_p F_2 D_p.
\]
\[\square\]

**Remark 3.6.** By a computation similar to part (3), we have that \((zA + B, zP)\) is a \( \Gamma \)-contraction for \( z \) on the unit circle. This can also be seen as a consequence of part (4). Indeed,
\[
zA + B - (zA + B)^* zP = zA + B - (\Sigma A^* + B^*) zP = z(A - B^* P) + B - A^* P = D_p (zF_1 + F_2) D_p,
\]
thereby showing that the pair \((zA + B, zP)\) satisfies fundamental equation which is a necessary and sufficient condition for it to be a \( \Gamma \)-contraction.

In the next section, we shall obtain a characterization of the fundamental operators.

### 4. New results about \( \Gamma \)-contractions

This section first proves new results about \( \Gamma \) contractions. Then these results are applied to tetrablock contractions.

The fundamental operator \( \Phi \) of a \( \Gamma \)-contraction \((S, P)\) is the unique bounded operator on \( D_p \) that satisfies \( S - S^* P = D_p \Phi D_p \). The lemma below gives a different characterization of \( \Phi \).

**Lemma 4.1.** The fundamental operator \( \Phi \) of a \( \Gamma \)-contraction \((S, P)\) is the unique bounded linear operator on \( D_p \) that satisfies the operator equation
\[
D_p S = XD_p + X^* D_p P.
\]

**Proof.** To see that \( \Phi \) satisfies the equation \([4.1]\) we shall use a relation which originally appeared in the proof of Theorem 4.3 of \([11]\). The relation says that if \( \Phi \) is the fundamental operator of a \( \Gamma \)-contraction \((S, P)\), then
\[
D_p S = \Phi D_p + \Phi^* D_p P.
\]

The proof of this relation is simple. Let \( G = \Phi^* D_p P + \Phi D_p - D_p S \). Then \( G \) is defined from \( \mathcal{H} \to D_p \). Since \( \Phi \) is the fundamental operator of the \( \Gamma \)-contraction \((S, P)\), we have
\[
D_p G = D_p \Phi^* D_p P + D_p \Phi D_p - D_p^2 S = (S^* - P^* S) P + (S - S^* P) - (I - P^* P) S = 0.
\]

Now \( \langle Gh, D_p h' \rangle = \langle D_p Gh, h' \rangle = 0 \) for all \( h, h' \in \mathcal{H} \). This shows that \( G = 0 \) and hence \( \Phi^* D_p P + \Phi D_p = D_p S \).

Conversely, let \( X \) satisfy \( D_p S = XD_p + X^* D_p P \). We need to show that \( X = \Phi \).

Since we just proved that \( \Phi \) satisfies the equation, we have \( \Phi D_p + \Phi^* D_p P = D_p S =
\[XD_P + X^*D_PP. \] Consequently, \((X - \Phi)D_P + (X - \Phi)^*D_PP = 0.\] Let \(Y = X - \Phi.\) So 
\[YD_P + Y^*D_PP = 0.\] We need to show that \(Y = 0.\) We have 
\[YD_P + Y^*D_PP = 0\]

or 
\[YD_P = -Y^*D_PP\]

or 
\[D_PYD_P = -D_PY^*D_PP = P^*D_PYD_PP = P^{*2}D_PYD_PP^2 = \cdots\]

Thus we have 
\[DPYD_P = P^{*n}DPYP^P = 4.3\]

for all \(n = 1, 2, \ldots.\) Now consider the series 
\[
\sum_{n=0}^{\infty} \|DP^nDh\|^2 = \sum_{n=0}^{\infty} \langle DP^nDh, DP^nDh \rangle
\]

\[= \sum_{n=0}^{\infty} \langle P^{*n}DP^2h, h \rangle
\]

\[= \sum_{n=0}^{\infty} \langle P^{*n}(I - P^*)P^nDh, h \rangle
\]

\[= \sum_{n=0}^{\infty} \langle (P^{*nP} - P^{*nP+h} + P^{*nP+h})h, h \rangle
\]

\[= \sum_{n=0}^{\infty} (\|P^nDh\|^2 - \|P^{*nP+h}\|^2)
\]

\[= \|h\|^2 - \lim_{n \to \infty} \|P^nDh\|^2.
\]

Now \(\|h\| \geq \|Ph\| \geq \|P^2h\| \geq \cdots \geq \|P^nDh\| \geq \cdots \geq 0.\) So \(\lim_{n \to \infty} \|P^nDh\|^2 \) exists. So the series is convergent. So \(\lim_{n \to \infty} \|DP^nDh\|^2 = 0.\) So

\[\|DPYD_Ph\| = \|P^{*n}DPYP^P\| \text{ by } (4.3)
\]

\[\leq \|P^{*n}\| \|DPY\| \|DP^nDh\| \leq \|DPY\| \|DP^nDh\| \to 0.
\]

So \(DPYD_P = 0.\) So \(Y = 0.\) \(\square\)

**Corollary 4.2.** The fundamental operators \(F_1\) and \(F_2\) of a tetrablock contraction \((A, B, P)\) are the unique bounded linear operators on the range closure of \(DP\) that satisfy the pair of operator equations

\[DP = X_1DP + X_2^*DP^P \text{ and } DP = X_2DP + X_1^*DP^P.
\]

**Proof.** Consider the \(\Gamma\)-contraction \((S_z, P_z)\) defined in (3.5). Its fundamental operator is \(F_1 + zF_2.\) Thus by the first part of the proof the lemma above, we have

\[DP(A + zB) = (F_1 + zF_2)DP + (F_1 + zF_2)^*DP^P = F_1DP + F_2^*DP^P + z(F_2^*DP^P + F_1^*DP^P)
\]

because \(z\) comes from the unit circle. Hence \(F_1\) and \(F_2\) satisfy the given operator equations. Now take a pair \((X_1, X_2)\) that satisfies the equations. Then \(X_1 + zX_2\) satisfies (4.1) for the \(\Gamma\)-contraction \((S_z, P_z).\) Indeed, remembering that \(|z| = 1,\) we have

\[DP(S_z = DP(A + zB) = DP + zDP^P = X_1DP + X_2^*DP^P + z(X_2DP + X_1^*DP^P)
\]

\[= X_1DP + zX_2DP + X_2^*DP^P + zX_1^*DP^P
\]

\[= (X_1 + zX_2)DP + (X_1 + zX_2)^*DP^P.
\]
By uniqueness, \( X_1 + zX_2 = F_1 + zF_2 \). Since this holds for all \( z \) on the unit circle, we have \( X_1 = F_1 \) and \( X_2 = F_2 \). \( \square \)

**Lemma 4.3.** Let \( S_1, S_2 \) and \( P \) be three commuting bounded operators such that \((S_1, P)\) and \((S_2, P)\) are \( \Gamma \)-contractions with commuting fundamental operators \( \Phi_1 \) and \( \Phi_2 \). Then
\[
S_1^*S_2 - S_2^*S_1 = D_P(F_1^*F_2 - F_2^*F_1)D_P.
\]

**Proof.** Using commutativity of \( S_1 \) and \( S_2 \), we get \( S_2^*S_1P = S_1^*S_2P \). Now we use the fundamental equation for \( \Gamma \)-contractions to get that
\[
S_2^*(S_1 - D_P\Phi_1D_P) = S_1^*(S_2 - D_P\Phi_2D_P).
\]
Thus, using \( \text{(4.2)} \), we get that \( S_2^*S_1 - S_1^*S_2 \) is the same as \((D_P\Phi_2^* + P^*D_P\Phi_2)\Phi_1D_P - (D_P\Phi_1^* + P^*D_P\Phi_1)\Phi_2D_P\) which is equal to \( D_P(\Phi_2^*\Phi_1 - \Phi_1^*\Phi_2)D_P \) in view of commutativity of the fundamental operators. That proves the lemma. \( \square \)

**Corollary 4.4.** Let \((A, B, P)\) be a tetrablock contraction with commuting fundamental operators \( F_1 \) and \( F_2 \). Then
\[
A^*A - B^*B = D(F_1^*F_1 - F_2^*F_2)D.
\]

**Proof.** This follows from the Lemma \( \text{(4.3)} \). With \( S_1 = A + zB, S_2 = zA + B \) and \( P = zP \), we shall get
\[
S_1^*S_2 - S_2^*S_1 = (z - \bar{z})(A^*A - B^*B) \quad \text{and} \quad \Phi_1^*\Phi_2 - \Phi_2^*\Phi_1 = (z - \bar{z})(F_1^*F_1 - F_2^*F_2).
\]
That finishes the proof. \( \square \)

Armed with the fundamental operators \( F_1 \) and \( F_2 \) of a tetrablock contraction \((A, B, P)\), we are ready to investigate those tetrablock contractions which are special. A unitary operator is a special kind of contraction because it is normal and its spectrum is contained in the unit circle. Indeed, that is a characterization. The next section completely unravels the structure of a commuting triple which consists of normal operators and whose Taylor joint spectrum is contained in the distinguished boundary of \( E \).

5. Tetrablock unitaries and tetrablock isometries

The beginning of this section warrants a discussion on what the distinguished boundary of a domain \( \Omega \) is. Any study of a dilation involves those special tuples of operators whose joint spectrum is contained in the distinguished boundary \( b\Omega \) of \( \Omega \). Let \( A(\Omega) \) be the algebra of continuous scalar functions on \( \overline{\Omega} \) that are holomorphic on \( \Omega \). A boundary for \( \Omega \) is a subset \( C \) of \( \Omega \) such that every function in \( A(\Omega) \) attains its maximum modulus on \( C \). It is well-known that for a polynomially convex \( \Omega \), there is a smallest closed boundary of \( \Omega \), contained in all the closed boundaries of \( \Omega \). This is called the distinguished boundary of \( \Omega \). We need characterizations of the distinguished boundary of \( \overline{E} \) and this is given in \( \text{[1]} \). We quote parts of Theorem 7.1 from there.

**Theorem 5.1** (Abouhajar, White, Young). For \( \underline{x} = (x_1, x_2, x_3) \in \mathbb{C}^3 \), the following are equivalent.

1. \( \underline{x} \in bE \),
2. \( x_1 = \overline{x}_2x_3, |x_3| = 1 \) and \( |x_2| \leq 1 \);
3. there exists a \( 2 \times 2 \) unitary matrix \( U \) such that \( x = \pi(U) \);
4. there exists a symmetric \( 2 \times 2 \) unitary matrix \( U \) such that \( x = \pi(U) \);
Motivated by Lemma 3.2, it is natural to ask whether an analogous characterization holds for the distinguished boundary and the answer is yes.

**Lemma 5.2.** A triple \( \mathbf{x} = (x_1, x_2, x_3) \in bE \) if and only if \((x_1 + zx_2, zx_3) \in b\Gamma \) for all \( z \) from the unit circle.

**Proof.** Indeed, if \((x_1, x_2, x_3) \in bE\), then we know from Lemma 3.2 that \((s_z, p_z) \in \Gamma \) and we know from Theorem 5.1 that \(|p_z| = 1\). These two together form a characterization for \((s_z, p_z)\) to be in \( b\Gamma \) (see Theorem 1.3 of [7]). Conversely, if for every \( z \) on the unit circle, \((s_z, p_z) \in b\Gamma\), then by Lemma 3.2 we have \((x_1, x_2, x_3) \in E\). That along with the fact \(|x_3| = 1\) implies that \((x_1, x_2, x_3)\) has to be in \( bE \) by part (5) of Theorem 5.1. \(\square\)

We begin the study of those tetrablock contractions which are special in the same sense that unitaries are special among contractions. So these special tetrablock contractions are the candidates for dilation.

**Definition 5.3.** A tetrablock unitary is a commuting triple of normal operators \( \mathbf{N} = (N_1, N_2, N_3) \) such that its Taylor joint spectrum \( \sigma(\mathbf{N}) \) is contained in \( bE \).

**Theorem 5.4.** Let \( \mathbf{N} = (N_1, N_2, N_3) \) be a commuting triple of bounded operators. Then the following are equivalent:

1. \( \mathbf{N} \) is a tetrablock unitary,
2. \( N_3 \) is a unitary, \( N_2 \) is a contraction and \( N_1 = N_2^* N_3 \),
3. there is a \( 2 \times 2 \) unitary block operator matrix \( [U_{ij}] \) where \( U_{ij} \) are commuting normal operators and \( \mathbf{N} = (U_{11}, U_{22}, U_{11}U_{22} - U_{21}U_{12}) \),
4. \( N_3 \) is a unitary and \( \mathbf{N} \) is a tetrablock contraction,
5. the family \( \{ (R_z, U_z) : |z| = 1 \} \) where \( R_z = N_1 + zN_2 \) and \( U_z = zN_3 \) is a commuting family of \( \Gamma \)-unitaries.

**Proof.** (1) \(\Rightarrow\) (2): By definition of a tetrablock unitary, \( N_1, N_2 \) and \( N_3 \) are commuting normal operators and their Taylor joint spectrum is contained in \( bE \). By spectral mapping theorem, \( \sigma(N_3) = P_3 \sigma(\mathbf{N}) \) where \( P_3 \) is the projection onto the third co-ordinate. Since \( \sigma(\mathbf{N}) \) is contained in \( bE \), we have \(|\lambda| = 1\) for all \( \lambda \in \sigma(N_3) \). So \( N_3 \) is a normal operator with its spectrum contained in the unit circle. So it is a unitary.

Consider the \( C^* \)-algebra \( C \) generated by the commuting normal operators \( N_1, N_2 \) and \( N_3 \). This commutative \( C^* \)-algebra is isometrically isomorphic, by the Gelfand map, to \( C(\sigma(\mathbf{N})) \). The Gelfand map takes \( N_i \) to the co-ordinate function \( x_i \) for \( i = 1, 2, 3 \). The co-ordinate functions satisfy \( x_1 = \pi_{2x_3} \) on the whole of \( bE \) and hence on \( \sigma(\mathbf{N}) \) which is contained in \( bE \). Thus \( N_1 = N_2^* N_3 \).

(2) \(\Rightarrow\) (3): We first note that \( N_2 \) is normal. Indeed, using the fact that \( N_1 = N_2^* N_3 \), we get that \( N_1 N_2 = N_2^* N_3 N_2 = N_2^* N_2 N_3 \). On the other hand, \( N_2 N_1 = N_2 N_2^* N_3 \). Since \( N_1 \) and \( N_2 \) commute, we have \( N_2^* N_2 N_3 = N_2 N_2^* N_3 \). Multiplying both sides on the right by \( N_2^* \), we get \( N_2 \) to be normal. Now just take

\[
U = \begin{pmatrix} N_2^* N_3 & -D_{N_2} \\ N_3 D_{N_2} & N_2 \end{pmatrix}.
\]

(5) \( \mathbf{x} \in \overline{E} \) and \(|x_3| = 1\).
(3) ⇒ (4): We shall verify that \( \mathcal{N} \) is a tetrablock contraction by using Lemma 3.3. First note that, because \( U \) is a unitary, we have
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = U^*U = \begin{pmatrix}
U_1^* & U_2^* \\
U_{12}^* & U_{22}^*
\end{pmatrix}
\begin{pmatrix}
U_1 & U_2 \\
U_{12} & U_{22}
\end{pmatrix} = \begin{pmatrix}
U_1^*U_1 + U_2^*U_2 & U_1^*U_2 + U_2^*U_2 \\
U_{12}^*U_1 + U_{22}^*U_2 & U_{12}^*U_2 + U_{22}^*U_2
\end{pmatrix}.
\]
So
\[
U_1^*U_1 + U_2^*U_2 = I = U_{12}^*U_{12} + U_{22}^*U_{22}
\] (5.1)
and
\[
U_1^*U_{12} + U_2^*U_{22} = 0.
\] (5.2)
Since \( U_{11}, U_{12}, U_{21} \) and \( U_{22} \) are commuting normal operators, given any \((z_{11}, z_{12}, z_{21}, z_{22}) \in \sigma(U_{11}, U_{12}, U_{21}, U_{22})\), we have
\[
(z_{11}, z_{12}, z_{21}, z_{22}, \overline{z}_{11}, \overline{z}_{12}, \overline{z}_{21}, \overline{z}_{22}) \in \sigma(U_{11}, U_{12}, U_{21}, U_{22}, U_1^*, U_2^*, U_{12}^*, U_{22}^*).
\]
Thus by the relations (5.1) and (5.2), we have
\[
|z_{11}|^2 + |z_{21}|^2 = 1 = |z_{12}|^2 + |z_{22}|^2 \text{ and } \overline{z}_{12}z_{11} + \overline{z}_{22}z_{21} = 0.
\]
Thus the scalar matrix \( Z = [\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}] \) is a unitary. Let \( p \) be a polynomial in three variables. Then
\[
\| p(N_1, N_2, N_3) \|
= r(p(N_1, N_2, N_3)) \text{[by normality]}
\leq \sup \{|p(\pi(Z))| : Z = [\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}] \text{ with } (z_{11}, z_{12}, z_{21}, z_{22}) \in \sigma(U_{11}, U_{12}, U_{21}, U_{22})\}
\leq \sup \{|p(\lambda_1, \lambda_2, \lambda_3)| : (\lambda_1, \lambda_2, \lambda_3) \in bE\} \text{[by the discussion above]}
\leq \sup \{|p(\lambda_1, \lambda_2, \lambda_3)| : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}\} = \|p\|_{\infty, E}
\]
proving that \( \mathcal{N} \) is a tetrablock contraction.

(4) ⇒ (5): From Theorem 3.5 we know that \((R_z, U_z)\) is a \( \Gamma \)-contraction for every \( z \) on the unit circle. Moreover, \( U_z \) is a unitary. A \( \Gamma \)-contraction whose second component is a unitary has to be a \( \Gamma \)-unitary, see part (4) of Theorem 2.5 of [11]. The commutativity is clear.

(5) ⇒ (1): First note that \( N_3 \) is a unitary by putting \( z = 1 \). Since \( R_1 \) and \( R_{-1} \) are commuting normal operators, \( N_1 = (R_1 + R_{-1})/2 \) and \( N_2 = (R_1 - R_{-1})/2 \) are commuting normal operators. It remains to see that the joint spectrum \( \sigma(N) \) is contained in \( bE \).

The proof of that will depend on the observation that \((x_1, x_2, x_3) \in bE \) if and only if for every \( z \) on the unit circle, \((s_z, p_z) \in b\Gamma \) where \( s_z = x_1 + z x_2 \) and \( p_z = z x_3 \). Let \((x_1, x_2, x_3) \) be a point in the Taylor joint spectrum \( \sigma(N) \) of \( \mathcal{N} \). Let \( z \) be from the unit circle. Then the Taylor joint spectrum of \((R_z, U_z)\) is the set \{\((s_z, p_z) : s_z = x_1 + z x_2 \) and \( p_z = z x_3 \)\} which is contained in \( b\Gamma \) because \((R_z, U_z)\) is a \( \Gamma \)-unitary. Thus any point \((x_1, x_2, x_3) \) in \( \sigma(N) \) has the property that \((x_1 + z x_2, z x_3) \) is in \( b\Gamma \) for every \( z \) on the unit circle. By Lemma 5.2 above, \((x_1, x_2, x_3) \) then has to be in \( bE \) and that completes the proof. \( \square \)

The class of tetrablock contractions that are natural candidates for dilation are the tetrablock unitaries for reasons that have been amply described. However, we can simplify our lives by enlarging the class to include the following.
Definition 5.5. A tetrablock isometry is the restriction of a tetrablock unitary to a joint invariant subspace.

This is also expressed by saying that a tetrablock isometry is a triple of commuting bounded operators which has a simultaneous extension to a tetrablock unitary. Thus a tetrablock isometry $\underline{V} = (V_1, V_2, V_3)$ always consists of commuting subnormal operators. Moreover, $V_3$ has to be an isometry. Call $\underline{V}$ a pure tetrablock isometry if $V_3$ is a pure isometry, i.e., a shift of some multiplicity. When we dilate a tetrablock contraction, it will be enough to dilate only to a tetrablock isometry because extension of a dilation is a dilation again as we shall see when we define dilation in the next section. It is for this reason that we need to understand the structure of tetrablock isometries completely. The rest of this section does that,

Theorem 5.6 (Wold decomposition for a tetrablock isometry). Let $\underline{V} = (V_1, V_2, V_3)$ be a tetrablock isometry on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$ into a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ satisfying the following two conditions.

1. The subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are reducing subspaces for each of the $V_i$.
2. If $N_i = V_i|_{\mathcal{H}_1}$ and $W_i = V_i|_{\mathcal{H}_2}$, then the triple $\underline{N}$ is a tetrablock unitary and the triple $\underline{W}$ is a pure tetrablock isometry.

Proof. Let $V_3 = N_3 \oplus W_3$ be the Wold decomposition of the isometry $V_3$ into its unitary part $N_3$ and the shift part $W_3$. Suppose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the corresponding decomposition of the whole space $\mathcal{H}$. Thus $V_3$ has the block matrix decomposition

$$V_3 = \begin{pmatrix} N_3 & 0 \\ 0 & W_3 \end{pmatrix}.$$ 

If we now write $V_2$ according to this decomposition of the space, then let its block matrix form be

$$V_2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$ 

By commutativity of $V_2$ with $V_3$, the off diagonal entries $A_{12}$ and $A_{21}$ end up commuting with the unitary $N_3$ and the shift $W_3$. It is well known that no non-zero operator can do that because $(W_3^*)^n$ converges to 0 strongly as $n$ tends to $\infty$. Thus $V_2$ is block diagonal too, say $V_2 = N_2 \oplus W_2$ where $N_2$ and $W_2$ are contractions. Note that $V_1 = V_2^*V_3$ because it inherits this property from its normal extension. Indeed, by definition of a tetrablock isometry, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a tetrablock unitary $(M_1, M_2, M_3)$ on $\mathcal{K}$ such that $M_i|_{\mathcal{H}} = V_i$ for $i = 1, 2, 3$. Thus for $h_1$ and $h_2$ in $\mathcal{H}$, we have

$$\langle V_1 h_1, h_2 \rangle = \langle M_1 h_1, h_2 \rangle = \langle M_2^* M_3 h_1, h_2 \rangle = \langle M_3 h_1, M_2 h_2 \rangle = \langle V_3 h_1, V_2 h_2 \rangle = \langle V_2^* V_3 h_1, h_2 \rangle.$$ 

Consequently, $V_1 = N_1 \oplus W_1$ where $N_1 = N_2^* N_3$ and $W_1 = W_2^* W_3$. By part (2) of Theorem 5.3, we have that $(N_1, N_2, N_3)$ is a tetrablock unitary. Thus our given tuple $\underline{V}$ has now been written as the direct sum of a tetrablock unitary $\underline{N}$ and a tuple $\underline{W}$ whose third component $W_3$ is a shift. That completes the proof. \hfill $\square$

However, given a triple, how does one decide whether it is a tetrablock isometry or not. The following result gives necessary and sufficient criteria. The fourth part will be very handy when we construct the dilation.

Theorem 5.7. Let $\underline{V} = (V_1, V_2, V_3)$ be a commuting triple of bounded operators. Then the following are equivalent.
(1) $V$ is a tetrablock isometry.
(2) $V$ is a tetrablock contraction and $V_3$ is an isometry.
(3) $V_3$ is an isometry, $V_2$ is a contraction and $V_1$ is the same as $V_2^*V_3$.
(4) $V_3$ is an isometry, $r(V_1) \leq 1$, $r(V_2) \leq 1$ and $V_1$ is the same as $V_2^*V_3$ where $r$ stands for spectral radius.

Remark 5.8. Note that post facto, (3) and (4) above imply that $V_2 = V_1^*V_3$ as well.

Proof. We shall prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and then (3) $\Rightarrow$ (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2): Given a tetrablock isometry $V = (V_1, V_2, V_3)$ on a Hilbert space $\mathcal{H}$, by its definition, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a tetrablock unitary $N = (N_1, N_2, N_3)$ acting on $\mathcal{K}$ for which $\mathcal{H}$ is an invariant subspace and $V_i = N_i|_\mathcal{H}$ for $i = 1, 2, 3$. It is then clear that $V_3$ is an isometry because it is the restriction of the unitary $N_3$ to the invariant subspace $\mathcal{H}$. It is also clear that $V_2$ is a tetrablock contraction because it is the restriction of a tetrablock contraction $N$ to an invariant subspace.

(2) $\Rightarrow$ (3): This part will require the solutions of the fundamental equations. This is a major departure from the theory of $\Gamma$-contractions because properties of $\Gamma$-isometries were deduced before the fundamental equation for a $\Gamma$-contraction was introduced. In the case of a tetrablock isometry, it is simplest to use the fundamental equations since existence and uniqueness of fundamental operators have already been deduced in Section 2. Since $V$ is a tetrablock contraction, the first fundamental equation has a solution. Since $V_3$ is an isometry, the right hand side of that equation vanishes and hence we have $V_1 = V_2^*V_3$ (of course, we also have $V_2 = V_1^*V_3$ from the second fundamental equation, but this is redundant). Contractivity of $V_2$ (or equivalently $V_1$) holds because all components of a tetrablock contraction are contractions.

(3) $\Rightarrow$ (1): Given a commuting triple $V = (V_1, V_2, V_3)$ on a Hilbert space $\mathcal{H}$ consisting of an isometry $V_3$, a contraction $V_2$ commuting with $V_3$ and $V_1 = V_2^*V_3$, we invoke the Wold decomposition of the isometry $V_3$. This gives a decomposition of the space $\mathcal{H}$ into a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ according to which $V_3$ decomposes as the direct sum of a normal operator $N_3$ and a shift $W_3$. We find that $V_2$, because it commutes with $V_3$, has $\mathcal{H}_1$ and $\mathcal{H}_2$ as reducing subspaces. Similarly, for $V_1$. This part of the argument has been detailed above in the proof of the Wold decomposition of a tetrablock isometry. Let $N_1, N_2, W_1$ and $W_2$ be as in that proof. Moreover, the relations $N_1 = N_2^*N_3$ and $W_1 = W_2^*W_3$ follow because of the given relation $V_1 = V_2^*V_3$. Then the commuting triple $N$ consists of a unitary $N_3$, a contraction $N_2$ and $N_1 = N_2^*N_3$. Thus it is a tetrablock unitary. Thus to show that $V$ is a tetrablock isometry, we need to show that the tuple $V$ can be extended to a tetrablock unitary, since the other part $N$ is already a tetrablock unitary.

Realize $W_3$ as multiplication by the co-ordinate function $z$ on a vector valued Hardy space $H^2(E)$ where the dimension of $E$ is the multiplicity of the shift $W_3$. Since $W_1$ and $W_2$ commute with this shift and with each other, there are commuting $H^\infty(E)$ functions $\varphi_1$ and $\varphi_2$ such that $W_i = T_{\varphi_i}$, the multiplication on $H^2(E)$ by $\varphi_i$ for $i = 1, 2$. Moreover, the $H^\infty$ norms of the operator valued functions $\varphi_1$ and $\varphi_2$ are not greater than one since $W_1$ and $W_2$ are contractions. Because $W_1 = W_2^*W_3$, or equivalently $M_{\varphi_1}^E = M_{\varphi_2}^E M_z^E$, we have

$$\varphi_1(z) = \varphi_2^*(z)z \text{ for all } z \in \mathbb{T}.$$  \hspace{1cm} (5.3)

Consider on $L^2(E)$, the multiplication operators $U_{\varphi_1}^E, U_{\varphi_2}^E$ and $U_z^E$, multiplications by $\varphi_1(z), \varphi_2(z)$ and $z$ respectively. Obviously $U_z^E$ is a unitary operator on $L^2(E)$. Because of the relation $\text{[5.3]}$ that the functions $\varphi_1$ and $\varphi_2$ satisfy, we have $U_{\varphi_1}^E = (U_{\varphi_2}^E)^* U_z^E$. 
Altogether the triple \((U^E_{\varphi_1}, U^E_{\varphi_2}, U^E_z)\) makes a tetrablock unitary and it extends the triple \(W\). So we are done.

Thus we have proved \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)\). For the remaining part, first note that \((3)\) implies \((4)\) trivially. For the converse, since we have \(V_1 = V_2^* V_3\), multiplying both sides from the left by \(V_3^*\), we get \(V_3^* V_1 = V_3^* V_2^* V_3\) which by commutativity is the same as \(V_2^* V_3^* V_3\) which again is just \(V_2^*\) because \(V_3^*\) is an isometry. Thus \(V_2 = V_1^* V_3\) as well. Consider the two operators

\[
B_1 = \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} V_3 & 0 \\ 0 & V_3 \end{pmatrix}.
\]

By the relations \(V_1 = V_2^* V_3\) and \(V_2 = V_1^* V_3\), this pair satisfies \(B_1 = B_1^* B_2\). This immediately implies that \(B_1\) is hyponormal. Indeed, \(B_1 B_1^* = B_2 B_3^* B_3 B_1 \leq B_1^* B_1\) and this is the defining property of hyponormality. In fact, the pair \((B_1, B_2)\) is a \(\Gamma\)-contraction, but that is besides the point.

Now we use a remarkable theorem due to Stampfli.

**Theorem 5.9 (Stampfli).** If \(X\) is a hyponormal operator, then \(\|X^n\| = \|X\|^n\) and so \(\|X\| = r(X)\).

For a proof of this theorem, see Proposition 4.6 of [12]. We shall use it with \(X = B_1\) to get that \(\|B_1\| = r(B_1)\). The operator norm of \(B_1\) is as follows.

\[
\|B_1\|^2 = \| \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & V_2^* \\ V_1^* & 0 \end{pmatrix} \| = \| \begin{pmatrix} V_1 V_1^* & 0 \\ 0 & V_2 V_2^* \end{pmatrix} \| = \max\{\|V_1\|^2, \|V_2\|^2\}.
\]

Thus \(\|B_1\| = \max\{\|V_1\|, \|V_2\|\}\). Now, for the spectral radius, we apply the spectral radius formula. A straightforward computation using commutativity of \(V_1\) and \(V_2\) shows that

\[
B_1^{2n} = \begin{pmatrix} (V_1 V_2)^n & 0 \\ 0 & (V_1 V_2)^n \end{pmatrix}.
\]

Consequently, \(r(B_1) = \lim \|B_1^{2n}\|^{1/2n} = \lim \|V_1 V_2\|^n\|^{1/2n} = r(V_1 V_2)^{1/2}\). Because \(V_1\) and \(V_2\) commute, the spectrum of \(V_1 V_2\) is contained in the set \(\{\lambda \mu : \lambda \in \sigma(V_1)\text{ and }\mu \in \sigma(V_2)\}\). Thus, given that both \(V_1\) and \(V_2\) have spectral radii not greater than one, the same is true for \(V_1 V_2\). Consequently, \(r(B_1) \leq 1\). thus by Stampfli’s result, both \(V_1\) and \(V_2\) are contractions.

We have a structure theorem for pure tetrablock isometries to go with the result above.

**Theorem 5.10.** Let \(V = (V_1, V_2, V_3)\) be a triple of bounded operators on a separable Hilbert space \(H\). Then \(V\) is a pure tetrablock isometry if and only if there is a separable Hilbert space \(E\), a unitary \(U : H \to H^2(E)\) and two bounded operators \(\tau_1\) and \(\tau_2\) on \(E\) such that

1. the \(H^\infty\) norm of the operator valued function \(\tau_1 + \tau_2 z\) is at most 1,
2. \(V_3 = U^* M_{\varphi_1}^E U, V_2 = U^* M_{\varphi_2}^E U\) and \(V_1 = U^* M_{\varphi_1}^E U\) where \(\varphi_1(z) = \tau_1 + \tau_2 z\) and \(\varphi_2(z) = \tau_2 + \tau_1^* z\),
3. \(\tau_1 \tau_2 = \tau_2 \tau_1\) and \([\tau_1, \tau_2^*] = [\tau_2, \tau_1^*] = [\tau_2, \tau_2^*]\).

**Proof.** First suppose we are given a separable Hilbert space \(E\), a unitary \(U : H \to H^2(E)\) and two bounded operators \(\tau_1\) and \(\tau_2\) on \(E\) such that

1. the \(H^\infty\) norm of the operator valued function \(\tau_1 + \tau_2 z\) is at most 1,
(2) $V_3 = U^* M^E z U, V_2 = U^* M^E w U$ and $V_1 = U^* M^E z U$ where $\varphi_1(z) = \tau_1 + \tau_2 z$ and $\varphi_2(z) = \tau_2 + \tau_1 z$.

(3) $\tau_1 \tau_2 = \tau_2 \tau_1$ and $[\tau_1, \tau_2] = [\tau_2, \tau_1]$.

In that case, $V_3$ is an isometry, $V_2$ is a contraction and $V_1$ is the same $V_2 V_3$. Moreover, the condition (3) above implies that $V$ is a commuting triple. Consequently, we can invoke part (3) of Theorem 5.7 to conclude that $V$ is a tetrablock isometry. Moreover, the pureness of $V_3$ now implies that $V$ is a pure tetrablock isometry.

Conversely, let $V$ be a pure tetrablock isometry. The existence of $E$ is due to the fact that $V_3$ is a pure isometry and hence is necessarily isomorphic to multiplication by $z$ on $H^2(E)$ for some $E$.

By the commutativity of $V_1$ and $V_2$ with $V_3$, we have that

$$V_1 = M^E_{\varphi_1} \text{ and } V_2 = M^E_{\varphi_2}$$

for some $\varphi_1$ and $\varphi_2$ in $H^\infty(E)$. Because $V$ is a tetrablock isometry, it satisfies $V_1 = V_2 V_3$. This, when translated in terms of the functions, by using the power series expansion of the holomorphic functions $\varphi_1$ and $\varphi_2$, gives us that the functions necessarily have to be of the form

$$\varphi_1(z) = \tau_1 + \tau_2 z \text{ and } \varphi_2(z) = \tau_2^* + \tau_1^* z.$$

Their $H^\infty$ norms do not exceed 1 because $V_1$ and $V_2$ are contractions. Moreover, commutativity of $V_1$ and $V_2$ now gives the condition (3) above.

After deciphering the structure of two special kinds of tetrablock contractions, we are ready to construct the dilation.

6. Dilation

As we have noted already, given a tetrablock contraction, the dilation triple $(\Delta_1, \Delta_2, \Delta_3)$ needs to be a tetrablock unitary. Note that a tetrablock contraction has a tetrablock unitary dilation if and only if it has a tetrablock isometric dilation. This is true because a tetrablock isometry is nothing but the restriction of a tetrablock unitary to a joint invariant subspace. In other words, a tetrablock isometry can, by definition, be extended to a tetrablock unitary. It is elementary that extension of a dilation is a dilation. Thus if a tetrablock contraction has a tetrablock isometric dilation, then it also has a tetrablock unitary dilation. We shall first construct a dilation assuming that $(A, B, P)$ is a tetrablock contraction whose fundamental operators $F_1$ and $F_2$ satisfy the conditions $[F_1, F_2] = 0$ and $[F_1, F_1^*] = [F_2, F_2^*]$. This is akin to Schäffer’s construction of minimal isometric dilation of a contraction.

**Theorem 6.1.** Let $(A, B, P)$ be a tetrablock contraction on $\mathcal{H}$ with fundamental operators $F_1$ and $F_2$. Let $D_P$ be the closure of the range of $D_P$. Let $K = \mathcal{H} \oplus D_P \oplus D_P \oplus \cdots = \mathcal{H} \oplus l^2(D_P)$. Consider the operators $V_1, V_2$ and $V_3$ defined on $K$ by

$$V_1(h_0, h_1, h_2, \ldots) = (Ah_0, F_1 D_P h_0 + F_1 h_1, F_2 h_1 + F_1 h_2, F_2 h_2 + F_1 h_3, \ldots)$$

$$V_2(h_0, h_1, h_2, \ldots) = (Bh_0, F_1^* D_P h_0 + F_2 h_1, F_1^* h_1 + F_2 h_2, F_1^* h_2 + F_2 h_3, \ldots)$$

$$V_3(h_0, h_1, h_2, \ldots) = (Ph_0, D_P h_0, h_1, h_2, \ldots).$$

Then

(1) $V = (V_1, V_2, V_3)$ is a minimal tetrablock isometric dilation of $(A, B, P)$ if $[F_1, F_2] = 0$ and $[F_1, F_1^*] = [F_2, F_2^*]$.
(2) If there is a tetrablock isometric dilation $W = (W_1, W_2, W_3)$ of $(A, B, P)$ such that $W_3$ is the minimal isometric dilation of $P$, then $W$ is unitarily equivalent to $V$. Moreover, $[F_1, F_2] = 0$ and $[F_1, F_3^*] = [F_2, F_3^*]$.

Proof of (1): It is evident from the definition that $V_3$ on $\mathcal{K}$ is the minimal isometric dilation of $P$ in the Schäffer form. Schäffer wrote down the unitary dilation in [24] and we only have the isometry part of it here.

Obviously the adjoints of the three operators on $\mathcal{K}$ are

\[
V_1^*(h_0, h_1, h_2, \ldots) = (A^*h_0 + D_PF_2h_1, F_1^*h_1 + F_3h_2, F_2h_3, \ldots)
\]
\[
V_2^*(h_0, h_1, h_2, \ldots) = (B^*h_0 + D_PF_1h_1, F_2^*h_1 + F_1h_2, F_2h_2 + F_1h_3, \ldots)
\]
\[
V_3^*(h_0, h_1, h_2, \ldots) = (P^*h_0 + D_Ph_1, h_2, h_3, \ldots).
\]

The space $\mathcal{H}$ can be embedded inside $\mathcal{K}$ by the map $h \mapsto (h, 0, 0, \ldots)$. It is clear that $\mathcal{H}$, considered as a subspace of $\mathcal{K}$ is co-invariant under $V_1, V_2$ and $V_3$. Moreover, $V_1^*|_\mathcal{H} = A^*, V_2^*|_\mathcal{H} = B^*$ and $V_3^*|_\mathcal{H} = P^*$. This of course immediately implies (1.2). The job now is to show that $V$ is a tetrablock isometry.

Since $V_3$ is an isometry, in order to show that $V$ is a tetrablock isometry, one has to justify the following:

1. $V$ is a commuting triple,
2. $V_1 = V_2 V_3$,
3. $r(V_1) \leq 1$ and $r(V_2) \leq 1$.

If we can show these, then by part (4) of Theorem 5.7, $V$ will be a tetrablock isometry.

\[
V_1 V_3(h_0, h_1, h_2, \ldots) = V_1(P h_0, D_P h_0, h_1, h_2, \ldots) = (A P h_0, F_2^* D_P P h_0 + F_1 D_P h_0, F_2^* D h_0 + F_1 h_1, F_2^* h_1 + F_1 h_2, F_2 h_2 + F_1 h_3, \ldots).
\]

\[
V_3 V_1(h_0, h_1, h_2, \ldots) = V_3(A h_0, F_2^* D h_0 + F_1 h_1, F_2^* h_1 + F_1 h_2, F_2 h_2 + F_1 h_3, \ldots) = (P A h_0, D_P A h_0, F_2^* D_P h_0 + F_1 h_1, F_2^* h_1 + F_1 h_2, F_2 h_2 + F_1 h_3, \ldots).
\]

Thus, to show that $V_1$ and $V_3$ commute, we only need to show that $D_PA = F_2^* D_P P + F_1 D_P$. Similarly, for $V_2$ and $V_3$ to commute, the criterion is that $D_PB = F_1^* D_P P + F_2 D_P$.

The proofs of these identities will use the formula (4.2) for the $\Gamma$-contraction $(A + zB, zP)$ and its fundamental operator $F_1 + zF_2$ where $z$ is on the unit circle. We get

\[
D_P (A + zB) = (F_1 + z F_2)^* D_P z P + (F_1 + z F_2) D_P = F_2^* D_P P + F_1 D_P + z (F_1^* D_P P + F_2 D_P).
\]

This holds for every $z$ on the unit circle. Therefore, the required criteria for commutativity of $V_1$ and $V_3$ are fulfilled. The commutativity of $V_1$ and $V_2$ is more difficult.

\[
V_1 V_2(h_0, h_1, h_2, \ldots) = V_1(B h_0, F_2^* D h_0 F_2 h_1, F_1^* h_1 + F_2 h_2, F_2^* h_2 + F_3 h_3, \ldots)
\]
\[
= (A B h_0, (F_2^* D P B + F_1 F_1^* D_P) h_0 + F_1 F_2 h_1, F_2^* F_1^* D_P h_0
\]
\[
+ (F_2^* F_2 + F_1 F_1^*) h_1, F_2^* F_1^* h_1 + (F_2^* F_2 + F_1 F_1^*) h_2, \ldots)
\]
we show that by virtue of fundamental equations. By commutativity of \( V \) which we know to be true from part (2) above. Having gotten commutativity of the

Or, in other words,

The first two are part of assumption. For the third one, we have to prove that

The first fundamental equation, we have

Thus, to show that \( V_1 \) and \( V_2 \) commute, we need

(1) \( F_1 \) and \( F_2 \) commute,
(2) \( F_2F_2 + F_1F_1^* = F_1F_1^* + F_2F_2^* \) and
(3) \( F_1^*D_A + F_2F_2^*D_P = F_2^*D_PB + F_1F_1^*D_P \).

The first two are part of assumption. For the third one, we have to prove that

Or, in other words,

since the ranges of \( F_1 \) and \( F_2 \) are contained in \( \mathcal{D}_P \). This last thing is the same as

by virtue of fundamental equations. By commutativity of \( A \) and \( B \), this is the same as

In view of \( F_1F_2 = F_2F_1 \) and Corollary 4.1, this is equivalent to

which we know to be true from part (2) above. Having gotten commutativity of the \( V_i \), we show that \( V_2V_3 \) is \( V_1 \). This is a straightforward computation.

By the first fundamental equation, we have \( B^*P + D_PF_1D_P = A \). Therefore we have \( V_2V_3 = V_1 \).

We now show that \( r(V_1) \leq 1 \) and \( r(V_2) \leq 1 \). It is clear from the definition that \( V_1 \) has the matrix form

with respect to the decomposition \( \mathcal{H} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \ldots \) of \( \mathcal{K}_0 \). Thus \( V_1 \) can be written as

we have by Lemma 1 of [14] that \( \sigma(T_A) \subseteq \sigma(A) \cup \sigma(E_1) \). We shall be done if we show that \( r(A) \) and \( r(E_1) \) are not greater than 1. We shall show that the numerical radius of
$E_1$ is not greater than 1. Since spectral radius is not greater than the numerical radius, we shall be done. Let us define

$$\varphi : \mathbb{D} \to \mathcal{B}(\mathcal{D})$$

$$z \to F_1 + F_2^* z.$$  

Clearly $\varphi$ is holomorphic, bounded and continuous on the boundary $\partial \mathbb{D} = \mathbb{T}$ of the disc. Under the Hilbert space isomorphism which sends $\mathcal{D} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \ldots$ to $H^2(\mathbb{D}) \otimes \mathcal{D}$, the operator $E_1$ goes to multiplication by the function $\varphi$. Now $w(M_{\varphi}) \leq \sup \{ w(\varphi(z)) : z \in \mathbb{T} \}$. Let us see what $w(\varphi(z))$ is. Recall that the numerical radius of an operator $X$ is not greater than one if and only if the real part of the operator $zX$ is not bigger than identity for every $z$ on the unit circle see [S]. Since we know that $w(F_1 + zF_2) \leq 1$, we have that $w(z_1F_1 + z_2F_2) \leq 1$ for every $z_1$ and $z_2$ on the unit circle. Thus

$$(z_1F_1 + z_2F_2) + (z_1F_1 + z_2F_2)^* \leq 2.$$  

In other words,

$$(z_1F_1 + \overline{z}F_2^*) + (z_1F_1 + \overline{z}F_2^*)^* \leq 2$$

which is the same as

$$z_1(F_1 + zF_2^*) + \overline{z}_1(F_1 + zF_2^*)^* \leq 2$$

for every $z$ and $z_1$ on the unit circle. And that by Ando’s result again ([S]), implies that $w(F_1 + zF_2^*) \leq 1$.

Proof of (2): Suppose a tetrablock contraction $(A, B, P)$ is given on a Hilbert space $\mathcal{H}$. Suppose that it has a tetrablock isometric dilation $W = (W_1, W_2, W_3)$ on a Hilbert space $\mathcal{K}$. Suppose $W_3$ is actually the minimal isometric dilation of $P$. Then we can obviously take $W_3 = \begin{pmatrix} P & 0 \\ C_3 & E_3 \end{pmatrix}$ with respect to the decomposition $\mathcal{H} \oplus l^2(\mathcal{D}_P)$ of $\mathcal{K}$, where

$$C_3 = \begin{pmatrix} D_P \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

from $\mathcal{H} \to \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \ldots$ and $E_3 = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ I & 0 & 0 & \ldots \\ 0 & I & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$

on $\mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \ldots$. Using this special form of $W_3$ and using the fact that $W_1$ and $W_2$ commute with $W_3$, it takes a straightforward computation to see that $W_1$ and $W_2$ have the operator matrix forms

$$W_1 = \begin{pmatrix} A & 0 \\ C_1 & E_1 \end{pmatrix} \quad \text{and} \quad W_2 = \begin{pmatrix} B & 0 \\ C_2 & E_2 \end{pmatrix}$$

for some $C_i$ and $E_i$ for $i = 1, 2$ with respect to the decomposition of $\mathcal{K}$ as $\mathcal{H} \oplus l^2(\mathcal{D}_P)$.

Here, we shall many times use the natural identification between Hardy space $H^2(\mathcal{D}_P)$ of $\mathcal{D}_P$ valued functions on the unit disk and $l^2(\mathcal{D}_P) = \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \ldots$. This Hilbert space isomorphism will be used without further mention. Under this Hilbert space isomorphism the operator $E_3$ is the same as the multiplication operator $M_{z}^{\mathcal{D}_P}$ on $H^2(\mathcal{D}_P)$. Because $W$ is a tetrablock isometry, we use the characterization obtained in Theorem 3.7 to get $W_1 = W_2^* W_3$ and hence

$$\begin{pmatrix} A & 0 \\ C_1 & M_{\varphi_1}^{\mathcal{D}_P} \end{pmatrix} = \begin{pmatrix} B^* & C_2^* \\ 0 & (M_{\varphi_2}^{\mathcal{D}_P})^* \end{pmatrix} \begin{pmatrix} P & 0 \\ C_3 & M_{z}^{\mathcal{D}_P} \end{pmatrix} = \begin{pmatrix} B^*P + C_2^*C_3 & C_2^*M_{z}^{\mathcal{D}_P} \\ (M_{\varphi_2}^{\mathcal{D}_P})^*C_3 & (M_{\varphi_2}^{\mathcal{D}_P})^*M_{z}^{\mathcal{D}_P} \end{pmatrix},$$
which gives

\[
\begin{array}{l}
(i) A - B^*P = C_2^*C_3 \\
(ii) C_1 = (M_{\psi_1}^{D_P})^*C_3 \\
(iii) M_{\psi_1}^{D_P} = (M_{\psi_2}^{D_P})^*M_z.
\end{array}
\] (6.1)

From (6.1)-(iii), it is clear by considering the power series expansions of $\psi_1$ and $\psi_2$ that $\psi_1(z) = F_1 + F_2^*z$ and $\psi_2(z) = F_2 + F_1^*z$ for some $F_1$ and $F_2$ in $\mathcal{B}(D_P)$. Thus

\[
E_1 = \begin{pmatrix}
F_1 & 0 & 0 & \ldots \\
F_2^* & F_1 & 0 & \ldots \\
0 & F_2^* & F_1 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\quad \text{and} \quad
E_2 = \begin{pmatrix}
F_2 & 0 & 0 & \ldots \\
F_1^* & F_2 & 0 & \ldots \\
0 & F_1^* & F_2 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

on $D_P \oplus D_P \oplus D_P \oplus \ldots$. Combining this with (6.1)-(ii), we get that

\[
W_1 = \begin{pmatrix}
A & 0 & 0 & \ldots \\
E_1^*C_3 & E_3
\end{pmatrix}
\quad \text{on} \quad \mathcal{H} \oplus l^2(D_P).
\]

Considering the stated matrix forms of $E_1$ and $C_3$ above, we get $E_1^*C_3 = \begin{pmatrix} F_2^*D_P \\ 0 \\ \vdots \end{pmatrix}$.

Hence with respect to the decomposition $\mathcal{H} \oplus D_P \oplus D_P \oplus \ldots$ of $K_0$, we have

\[
W_1 = \begin{pmatrix}
A & 0 & 0 & 0 & \ldots \\
F_2^*D_P & F_1 & 0 & 0 & \ldots \\
0 & F_2^* & F_1 & 0 & \ldots \\
0 & 0 & F_2^* & F_1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

A similar computation gives that

\[
W_2 = \begin{pmatrix}
B & 0 & 0 & 0 & \ldots \\
F_1^*D_P & F_2 & 0 & 0 & \ldots \\
0 & F_1^* & F_2 & 0 & \ldots \\
0 & 0 & F_1^* & F_2 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

Since $W_1$ commutes with $W_3$, we have

\[
\begin{pmatrix}
A & 0 & 0 & 0 & \ldots \\
F_2^*D_P & F_1 & 0 & 0 & \ldots \\
0 & F_2^* & F_1 & 0 & \ldots \\
0 & 0 & F_2^* & F_1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix}
P & 0 & 0 & 0 & \ldots \\
P & 0 & 0 & 0 & \ldots \\
D_P & I & 0 & 0 & \ldots \\
0 & I & 0 & 0 & \ldots \\
0 & 0 & I & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
= \begin{pmatrix}
P & 0 & 0 & 0 & \ldots \\
D_P & I & 0 & 0 & \ldots \\
0 & I & 0 & 0 & \ldots \\
0 & 0 & I & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 & 0 & \ldots \\
F_2^*D_P & F_1 & 0 & 0 & \ldots \\
0 & F_2^* & F_1 & 0 & \ldots \\
0 & 0 & F_2^* & F_1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]
Equating the entries in the second row and first column on both sides, we have \( F^*_2 D_P P + F_1 D_P = D_P A \). Similarly, from the fact that \( W_2 \) commutes with \( W_3 \), we have \( F^*_1 D_P P + F_2 D_P = D_P B \). Now by Corollary 4.2, we know that \( F_1 \) and \( F_2 \) have to be the fundamental operators of the tetrablock contraction \((A, B, P)\). This immediately tells us that \( W \) and \( V \) are same modulo the unitaries hidden in the arguments above. Since \( W_1 \) and \( W_2 \) commute, equating the diagonal entries of \( W_1 W_2 \) and \( W_2 W_1 \), we get that \( [F_1, F_1^*] = [F_2, F_2^*] \). Equating the subdiagonal entries of \( W_1 W_2 \) and \( W_2 W_1 \), we get that \( [F_1, F_1^*] = [F_2, F_2^*] \). That completes the proof.

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