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AUBRY SETS VS MATHER SETS
IN TWO DEGREES OF FREEDOM

DANIEL MASSART

ABSTRACT. Let $L$ be an autonomous Tonelli Lagrangian on a closed manifold of dimension two. Let $C$ be the set of cohomology classes whose Mather set consists of periodic orbits, none of which is a fixed point. Then for almost all $c$ in $C$, the Aubry set of $c$ equals the Mather set of $c$.

1. Introduction

1.1. Motivation. We study Tonelli Lagrangian systems on closed manifolds, along the lines of [Mr91]. The Aubry set is a specific invariant set of the Euler-Lagrange flow, defined in [F]. Roughly speaking, it is the obstruction to push a Lagrangian submanifold inside a convex hypersurface of the cotangent bundle of a closed manifold without changing its cohomology class (see [PPS03]). Various nice results hold when the Aubry set is a finite union of hyperbolic, periodic orbits:

- asymptotic estimates for near-optimal periodic geodesics ([A03]), if the Lagrangian is a metric of negative curvature on a surface
- existence of ‘physical’ solutions of the Hamilton-Jacobi equation ([AIPS03])
- existence of $C^\infty$ subsolutions of the Hamilton-Jacobi equation ([Be07]).

By [CI99] when there is a minimizing periodic orbit, a small perturbation makes it hyperbolic while still minimizing. The trouble is to find minimizing periodic orbits.

While this seems out of reach for the time being, there is a particular case when this difficulty is easily overcome: that is when the dimension of the configuration space is two, for then Proposition 2.1 of [CMP04] says that any minimizing measure with a rational homology class is supported on periodic orbits.

Even then, yet another problem arises: the Aubry set always contains the union of the supports of all minimizing measures (Mather set), but the inclusion may be proper. The purpose of this paper is to clarify the relationship between the Aubry set and the Mather set, when the latter consists of periodic orbits. In loose terms our main result says that in that case (and in two degrees of freedom) they almost always coincide. See the next paragraph.

1.2. Definitions and precise statements. After Fathi and Bernard we call autonomous Tonelli Lagrangian on a closed manifold $M$ a $C^2$ function
L from $TM$ to $\mathbb{R}$ which is fiberwise superlinear and such that $\partial^2 L/\partial v^2$ is positive definite everywhere. Let

- $L$ be an autonomous Tonelli Lagrangian on a closed manifold $M$
- $\phi_t$ be the Euler-Lagrange flow of $L$
- $p$ be the canonical projection $TM \to M$.

The first object one encounters when using variational methods is Mañé’s action potential: for each nonnegative $t$, and $x, y$ in $M$, define

$$h_t(x, y) := \inf \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds$$

over all absolutely continuous curves $\gamma : [0, t] \to M$ such that $\gamma(0) = x$, $\gamma(t) = y$. The infimum is in fact a minimum due to the fiberwise strict convexity and superlinearity of $L$, and the curves achieving the minimum are pieces of orbits of $\phi_t$.

Looking for orbits that realize the action potential between any two of their points, one is led to consider the Peierls barrier ([F])

$$h(x, y) := \liminf_{t \to \infty} h_t(x, y).$$

The projected Aubry set is then defined ([F]) as

$$A(L) := \{ x \in M : h(x, x) = 0 \}.$$

Mather’s Graph Theorem ([F], Theorem 5.2.8) then says that for any $x \in A(L)$, there exists a unique $v \in T_x M$ such that $p \circ \phi_t(x, v) \in A(L)$ for all $t \in \mathbb{R}$. The set

$$\tilde{A}(L) := \{(x, v) \in M : p \circ \phi_t(x, v) \in A(L) \ \forall \ t \in \mathbb{R}\}$$

is called the Aubry set of $L$, it is compact and $\phi_t$-invariant.

As noticed by Mather, it is often convenient to deal with invariant measures rather than individual orbits. Define $\mathcal{M}_{\text{inv}}$ to be the set of $\Phi_t$-invariant, compactly supported, Borel probability measures on $TM$. Mather showed that the function (called action of the Lagrangian on measures)

$$\mathcal{M}_{\text{inv}} \to \mathbb{R}$$

$$\mu \mapsto \int_{TM} L \, d\mu$$

is well defined and has a minimum. A measure achieving the minimum is called $L$-minimizing. The value of the minimum, times minus one, is called the critical value of $L$, and denoted $\alpha(L)$. The Mather set $\mathcal{M}(L)$ of $L$ is then defined as the closure of the union of the supports of all minimizing measures. It is compact, $\phi_t$-invariant, and contained in $\tilde{A}(L)$.

The minimization procedure may be refined as follows. Mather observed that if $\omega$ is a closed one-form on $M$ and $\mu \in \mathcal{M}_{\text{inv}}$ then the integral $\int_{TM} \omega \, d\mu$ is well defined, and only depends on the cohomology class of $\omega$. By duality this endows $\mu$ with a homology class: $[\mu]$ is the unique $h \in H_1(M, \mathbb{R})$ such that

$$\langle h, [\omega] \rangle = \int_{TM} \omega \, d\mu$$

for any closed one-form $\omega$ on $M$. Besides, for any $h \in H_1(M, \mathbb{R})$, the set

$$\mathcal{M}_{h, \text{inv}} := \{ \mu \in \mathcal{M}_{\text{inv}} : [\mu] = h \}$$
is not empty. Again the action of the Lagrangian on this smaller set of measures has a minimum, which is a function of $h$, called the $\beta$-function of the system. A measure achieving the minimum is called $(L,h)$-minimizing, or $h$-minimizing for short.

When the dimension of $M$ is two, we get a bit of help from the topology. Let $\Gamma$ be the quotient of $H_1(M,\mathbb{Z})$ by its torsion (we do not assume $M$ to be orientable), $\Gamma$ embeds as a lattice into $H_1(M,\mathbb{R})$. A homology class $h$ is said to be 1-irrational if there exist $h_0 \in \Gamma$ and $r \in \mathbb{R}$ such that $h = rh_0$.

Proposition 1. Let $M$ be a closed surface, possibly non-orientable, and let $L$ be a Tonelli Lagrangian on $M$. If $h$ is a 1-irrational homology class and $\mu$ is an $h$-minimizing measure, then the support of $\mu$ consists of periodic orbits, or fixed points.

There is a dual construction: if $\omega$ is a closed one-form on $M$, then $L - \omega$ is a Tonelli Lagrangian, and furthermore $L - \omega$ has the same Euler-Lagrange flow as $L$. The Aubry set, Mather set, and critical value of $L$ are denoted $\tilde{A}(L, c, h)$, $\tilde{M}(L, c, h)$, respectively, or just $\tilde{A}(c)$, $\tilde{M}(c)$, $\alpha(c)$ when no ambiguity is possible. An $(L - \omega)$-minimizing measure is also called $(L, \omega)$-minimizing, $(L, c)$-minimizing, or just $c$-minimizing for short if $c$ is the cohomology of $\omega$. In formal terms we have defined

\begin{align*}
\beta_L : H_1(M, \mathbb{R}) &\rightarrow \mathbb{R} \\
h &\mapsto \min \{ \int_{TM} Ld\mu : [\mu] = h \}
\end{align*}

\begin{align*}
\alpha_L : H^1(M, \mathbb{R}) &\rightarrow \mathbb{R} \\
c &\mapsto -\min \{ \int_{TM}(L - \omega)d\mu : [\omega] = c \}.
\end{align*}

Mather proved that $\alpha_L$ and $\beta_L$ are convex, superlinear, and Fenchel dual of one another. In particular $\min \alpha = -\beta(0)$, and we have the Fenchel inequality:

$$\alpha_L(c) + \beta_L(h) \geq \langle c, h \rangle \quad \forall c \in H^1(M, \mathbb{R}), \ h \in H_1(M, \mathbb{R}).$$

Given $c \in H^1(M, \mathbb{R})$ (resp. $h \in H_1(M, \mathbb{R})$), the set of $h \in H_1(M, \mathbb{R})$ (resp. $c \in H^1(M, \mathbb{R})$) achieving equality in the Fenchel equality is called the Legendre transform of $c$ (resp. $h$), and denoted $\mathcal{L}(c)$ (resp. $\mathcal{L}(h)$).

The main geometric features of a convex function are its smoothness and strict convexity, or lack thereof. In general, the maps $\alpha_L$ and $\beta_L$ are neither strictly convex, nor smooth ([Mt97]). The regions where either map is not strictly convex are called flats (see Section 2 for precise definitions). A flat is a convex subset of a linear space, hence it makes sense to speak of its relative interior, or interior for short. The sets $\mathcal{L}(c)$ for $c \in H^1(M, \mathbb{R})$ (resp. $\mathcal{L}(h)$ for $h \in H_1(M, \mathbb{R})$), if they contain more than one point, are non-trivial instances of flats; conversely, by the Hahn-Banach Theorem, any flat is contained in the Legendre transform of some point.

Note that if two cohomology classes lie in the relative interior of a flat $F$ of $\alpha_L$, by [Mr91] their Mather sets coincide. We denote by $\tilde{M}(F)$ the common Mather set to all the cohomologies in the relative interior of $F$. For any $c$ in $F$, the Mather set of $c$ contains the Mather set of $F$. We say a flat is rational if its Mather set consists of periodic orbits or fixed points. It is easy to see that any rational flat of $\alpha_L$ is contained in $\mathcal{L}(h)$ for some
1-irrational $h$. A partial converse is true when the dimension of $M$ is two (see Lemma 22).

As to Aubry sets, Proposition 6 of \cite{Mt03} reads:

**Proposition 2.** If a cohomology class $c_1$ belongs to a flat $F_c$ of $\alpha_L$ containing $c$ in its interior, then $\mathcal{A}(c) \subset \mathcal{A}_{c_1}$. In particular, if $c_1$ lies in the interior of $F_c$, then $\mathcal{A}(c) = \mathcal{A}(c_1)$. Conversely, if two cohomology classes $c$ and $c_1$ are such that $\tilde{\mathcal{A}}(c) \cap \tilde{\mathcal{A}}(c_1) \neq \emptyset$, then $\alpha_L$ has a flat containing $c$ and $c_1$.

So for any flat $F$ of $\alpha$ and any $c_1, c_2$ in the interior of $F$, the Aubry sets $\tilde{\mathcal{A}}(c_1)$ and $\tilde{\mathcal{A}}(c_2)$ coincide. We denote by $\tilde{\mathcal{A}}(F)$ the common Aubry set to all the cohomologies in the interior of $F$.

A flat of $\alpha_L$ is called singular if its Mather set contains a fixed point of the Euler-Lagrange flow. A homology class $h$ is called singular if its Legendre transform is a singular flat. So the set of singular classes is either empty, or it contains zero and is compact. When there are fixed points, we lose some of the perks of the low dimension, which explains why we have to exclude singular classes from our main result. The purpose of this paper is to prove that the Aubry set of a nonsingular rational flat equals its Mather set.

**Theorem 3.** Assume
- $M$ is a closed surface
- $L$ is an autonomous Tonelli Lagrangian on $M$
- $h$ is a 1-irrational, nonsingular homology class.

Then $\tilde{\mathcal{A}}(\mathcal{L}(h)) = \tilde{\mathcal{M}}(\mathcal{L}(h))$, and $\mathcal{A}(\mathcal{L}(h))$ is a union of periodic orbits.

So in the interior of a nonsingular rational flat, the Aubry set is as small as possible since it must contain the Mather set. Note that the boundary of a convex set $C$ is negligible in $C$, in any reasonable sense of negligible, which accounts for the phrase ‘almost always coincide’ we used in the ‘Motivation’ subsection.

Here is an outline of this paper. In Section 2 we have gathered some technical results about Fenchel dual pairs of convex functions. In Section 3 we prove the lemmas we need so as to include the case of non-orientable surfaces. In Section 4 we take a close look at the structure of the Aubry set in a neighborhood of a periodic orbit which is not a fixed point. In Section 5 we consider the flats of $\beta_L$. This is needed to prove that $\mathcal{L}(h)$ is a rational flat for any non-singular 1-irrational $h$. In the final section we put it all together and prove our main theorem.

### 2. Convex and superlinear functions

Let
- $E$ be a finite dimensional Banach space
- $A : E \rightarrow \mathbb{R}$ be a convex and superlinear map.

Then the Fenchel transform of $A$, defined by the formula

$$B : E^* \rightarrow \mathbb{R}$$

$$y \mapsto \sup_{x \in E} \langle y, x \rangle - A(x)$$

is well-defined, convex and superlinear. The Legendre transform (with respect to $A$) of a point $x$ in $E$ is the set $\mathcal{L}(x)$ of $y \in E^*$ which achieve the
Given a flat $F$ of $x$. Let us show the converse inclusion holds true. Take \( t \in F \) such that \( A(x) + B(y) = \langle y, x \rangle \). We want to show that \( y \in \mathcal{F}(F) \), that is, \( A(x) + B(y) = \langle y, x \rangle \) for all \( x \in F \). Take \( x \in F \). Since \( x_0 \) lies in the interior of \( F \), there exists \( x' \) in \( F \) and \( 0 < t < 1 \) such that \( x_0 = tx + (1-t)x' \). Since any flat is contained in a face, there exists \( y_0 \) such that \( F \subset \{ x \in E : A(x) + B(y_0) = \langle y_0, x \rangle \} \), so

\[
A(x) + B(y_0) = \langle y_0, x \rangle
\]

Summing \( t \) times the first equation with \((1-t)\) times the second equation yields \( tA(x) + (1-t)A(x') + B(y_0) = \langle y_0, x_0 \rangle \), but since \( x_0 \in F \), we have \( A(x_0) + B(y_0) = \langle y_0, x_0 \rangle \) whence \( A(x_0) = tA(x) + (1-t)A(x') \).
On the other hand by definition of $B$ we have
\[
A(x) + B(y) \geq \langle y, x \rangle \\
A(x') + B(y) \geq \langle y, x' \rangle.
\]
Summing $t$ times the first inequality with $(1 - t)$ times the second inequality yields the equality $A(x_0) + B(y) = \langle y, x_0 \rangle$, thus both inequalities are equalities, which proves the lemma. \hfill \square

Our next lemmas are Lemma 4.1 and 4.2 of \[BM\].

**Lemma 5.** Let $F_1$ and $F_2$ be two flats of $A$, both containing a point $x_0$, $x_0$ being an interior point of $F_1$. Then there exists a flat $F$ containing $F_1 \cup F_2$.

**Proof.** Let $y_i \in E^*$ be such that for all $x$ in $F_2$, we have $B(y) + A(x) = \langle y, x \rangle$. In particular we have $B(y) + A(x_0) = \langle y, x_0 \rangle$ so $y$ lies in the Legendre transform of $x_0$. Thus by Lemma 4 $y \in F(F_1)$. This means that for all $x$ in $F_1$, we have $B(y) + A(x) = \langle y, x \rangle$. Thus the face $L(y)$ contains $F_1 \cup F_2$. \hfill \square

**Lemma 6.** Let $F_1$ and $F_2$ be two flats of $A$, both containing a point in their interiors. Then there exists a flat $F$ containing $F_1 \cup F_2$ such that $x$ is an interior point of $F$.

3. **What we need to know about non-orientable surfaces**

Assume $M$ is non-orientable. Let $\pi : M \to M$ be the orientation cover of $M$. Then $M_0$ is an orientable surface endowed with a fixed-point free, orientation-reversing involution $I$. Let $I_*$ be the involution of $H_1(M_0, \mathbb{R})$ induced by $I$, and let $E_1$ (resp. $E_{-1}$) be the eigenspace of $I_*$ for the eigenvalue 1 (resp. $-1$). Likewise, let $I^*$ be the involution of $H^1(M_0, \mathbb{R})$ induced by $I$, and let $E_1$ (resp. $E_{-1}$) be the eigenspace of $I^*$ for the eigenvalue 1 (resp. $-1$). We have (\[BM\], 2.2)

\[
\ker \pi_* = E_{-1} \subset H_1(M_0, \mathbb{R}) \quad \text{and} \quad \pi^* \left( H^1(M, \mathbb{R}) \right) = E_1 \subset H^1(M_0, \mathbb{R})
\]

Let
- $T\pi$ denote the tangent map of $\pi$
- $L'$ denote the Lagrangian $L \circ T\pi$ on $TM_0$.

Likewise we denote with primes the Aubry and Mather sets of $L'$. Proposition 4 of \[F98\] says that

\[
A_0' = \pi^{-1}(A_0), \tilde{A}_0' = T\pi^{-1}\left(\tilde{A}_0\right).
\]

**Lemma 7.** We have
\[
\forall c \in H^1(M_0, \mathbb{R}), \quad \alpha_o(I^*c) = \alpha_o(c) \\
\forall h \in H_1(M_0, \mathbb{R}), \quad \beta_o(I_*h) = \beta_o(h)
\]

**Proof.** Take
- $c \in H^1(M_0, \mathbb{R})$
- a smooth one-form $\omega$ on $M_0$ such that $[\omega] = c$
- a $I^*c$-minimizing measure $\mu$. 

We have
\[ -\alpha_o(I^*c) = \int_{TM_o} (L' - I^*\omega) \, d\mu = \int_{TM_o} (L' - \omega) \, dI_*\mu \geq -\alpha_o(c) \]
where the second equality owes to the \( I \)-invariance of \( L' \). So \( \alpha_o(I^*c) \leq \alpha_o(c) \), whence \( \alpha_o(c) = \alpha_o(I^*I^*c) \leq \alpha_o(I^*c) \), which proves the first statement of the lemma.

Now take \( h \in H_1(M_o, \mathbb{R}) \) and an \( h \)-minimizing measure \( \mu \). We have \([I_*\mu] = I_\ast h\) thus
\[ \beta_o(I_\ast h) \leq \int_{TM_o} L'dI_*\mu = \int_{TM_o} L'd\mu = \beta_o(h) \]
whence \( \beta_o(h) = \beta_o(I_\ast I_\ast h) \leq \beta_o(I_*h) \), which ends the proof of the lemma. \( \square \)

**Lemma 8.** For all \( c \in H^1(M, \mathbb{R}) \), \( \alpha(c) = \alpha_o(\pi^*c) \).

**Proof.** Take \( c \in H^1(M, \mathbb{R}) \) and a smooth one-form \( \omega \) on \( M \) such that \( [\omega] = c \). Then the lifted Lagrangian corresponding to \( L - \omega \) is \( L' - \pi^*\omega \). By [F98], Proposition 4, \( L - \omega \) and \( L' - \pi^*\omega \) share the same critical value, that is, \( \alpha(c) = \alpha_o(\pi^*c) \).

\( \square \)

**Lemma 9.** For all \( h \in E_1 \subset H_1(M_o, \mathbb{R}) \), \( \beta_o(h) = \beta(\pi hồ) \), and if \( \mu \) is an \( h \)-minimizing measure, then \( \pi_*\mu \) is \( \pi_*h \)-minimizing.

**Proof.** Take
- \( h \in E_1 \subset H_1(M_o, \mathbb{R}) \)
- an \( h \)-minimizing measure \( \mu \)
- \( c \in H^1(M_o, \mathbb{R}) \) such that \( \alpha_o(c) + \beta_o(h) = \langle c, h \rangle \).

Then by Lemma 8, \( \alpha_o(I^*c) + \beta_o(I_*h) = \langle c, h \rangle \) and \( \langle I^*c, I_*h \rangle = \langle c, h \rangle \) since \( I^* \) and \( I_* \) are dual of one another. Besides, \( I_*h = h \) because \( h \in E_1 \). Setting \( c_1 := 2^{-1}(c + I^*c) \), we have
\[ \alpha_o(c_1) \leq \frac{1}{2} (\alpha_o(c) + \alpha_o(I^*c)) = \alpha_o(c) \]
by convexity of \( \alpha \), but on the other hand
\[ \frac{1}{2} (\alpha_o(c) + \alpha_o(I^*c)) + \beta_o(h) = \langle c_1, h \rangle \leq \alpha_o(c_1) + \beta_o(h) \]
whence
\[ \alpha_o(c_1) = \frac{1}{2} (\alpha_o(c) + \alpha_o(I^*c)) = \alpha_o(c) \]
and
\[ \langle c_1, h \rangle = \alpha_o(c_1) + \beta_o(h). \]
Since \( c_1 \in E_1 \subset H^1(M_o, \mathbb{R}) \), there exists \( c_2 \) in \( H_1(M, \mathbb{R}) \) such that \( \pi^*c_2 = c_1 \). By lemma 8, \( \alpha_o(c_1) = \alpha(c_2) \) so
\[ \alpha(c_2) + \int_{TM_o} L'd\mu = \langle \pi^*c_2, h \rangle \]
that is,
\[ \alpha(c_2) + \int_{TM} Ld\pi_*\mu = \langle c_2, \pi_*h \rangle \]
which proves that \( I_*\mu \) is \( \pi_*h \)-minimizing and \( \beta_o(h) = \beta(\pi_*h) \). \( \square \)
Lemma 10. Let $h$ be an element of $H_1(M_0, \mathbb{R})$. We have
$$\pi^*(L(\pi_*h)) = \mathcal{L}(h) \cap E_1.$$ 

Proof. Take $c$ in $\mathcal{L}(\pi_*h)$. We have
$$\alpha(c) + \beta(\pi_*h) = \langle c, \pi_*h \rangle$$
whence by Lemmas 8, 9
$$\alpha_o(\pi^*c) + \beta_o(h) = \langle \pi^*c, h \rangle$$
that is, $\pi^*c \in \mathcal{L}(h)$. Therefore
$$\pi^*(\mathcal{L}(\pi_*h)) \subset \mathcal{L}(h) \cap E_1.$$ 

Now take $c \in \mathcal{L}(h) \cap E_1$. Since $c \in E_1$, there exists $c_1 \in H^1(M, \mathbb{R})$ such that $\pi^*c_1 = c$. We have
$$\alpha_o(c) + \beta_o(h) = \langle c, h \rangle$$
whence $\alpha(c_1) + \beta(\pi_*h) = \langle c_1, \pi_*h \rangle$ so $c \in \pi^*(\mathcal{L}(\pi_*h))$, which concludes the proof of the lemma.

Lemma 11. Let $h$ be an element of $H_1(M, \mathbb{R})$, and $h'$ be an element of $E_1 \subset H_1(M_0, \mathbb{R})$ be such that $\pi_*h' = h$. We have
$$\mathcal{F}(\mathcal{L}(h)) = \pi_*\left(\mathcal{F}(\mathcal{L}(h')) \cap E_1\right)$$

Proof. Let
- $h_1$ be an element of $\pi_*(\mathcal{F}(\mathcal{L}(h')) \cap E_1)$
- $h_2$ be an element of $\mathcal{F}(\mathcal{L}(h')) \cap E_1$ such that $\pi_*(h_2) = h_1$
- $c$ be an element of $\mathcal{L}(h)$.

By Lemma 10, $\pi^*c \in \mathcal{L}(h')$ so
$$\alpha_o(\pi^*c) + \beta_o(h_2) = \langle \pi^*c, h_2 \rangle$$
whence by Lemmas 8, 9
$$\alpha(c) + \beta(h_1) = \langle c, h_1 \rangle$$
thus $h_1 \in \mathcal{F}(\mathcal{L}(h))$, hence
$$\mathcal{F}(\mathcal{L}(h)) \supset \pi_*\left(\mathcal{F}(\mathcal{L}(h')) \cap E_1\right).$$

Conversely, let
- $h_1$ be an element of $\mathcal{F}(\mathcal{L}(h))$
- $h_2$ be an element of $E_1 \subset H_1(M_0, \mathbb{R})$ such that $\pi_*(h_2) = h_1$
- $c'$ be an element of $\mathcal{L}(h')$.

Setting $c_2 := 2^{-1}(c + I^*c)$, we see, as in the proof of Lemma 8, that $c_2 \in \mathcal{L}(h') \cap E_1$. Since $c_2 \in E_1 \subset H^1(M_0, \mathbb{R})$, there exists $c_1$ in $H_1(M, \mathbb{R})$ such that $\pi^*c_1 = c_2$. Then by Lemmas 8, 9 $c_1 \in \mathcal{L}(h)$. Since $h_1 \in \mathcal{F}(\mathcal{L}(h))$ we have
$$\alpha(c_1) + \beta(h_1) = \langle c, h_1 \rangle$$
thus
$$\alpha_o(c_2) + \beta_o(h_2) = \langle c_2, h_2 \rangle$$
whence the two inequalities
$$\alpha_o(c') + \beta_o(h_2) \geq \langle c_2, h_2 \rangle$$
$$\alpha_o(I^*c') + \beta_o(h_2) \geq \langle c_2, h_2 \rangle$$
sum to an equality, so both inequalities are equalities. Therefore $h_2 \in \mathcal{F}(\mathcal{L}(h'))$, so

$$\mathcal{F}(\mathcal{L}(h)) \subset \pi_\ast (\mathcal{F}(\mathcal{L}(h')) \cap E_1).$$

4. Local structure of the Aubry set at periodic orbits

We shall use Lemma 10 of [BM], and quote it below for the commodity of the reader. In [BM] only geodesic flows are considered but the proof extends without modification to the case of Lagrangian flows.

Let $\gamma_0$ be a closed, minimizing extremal on an oriented surface $M$, such that $\gamma_0$ is not a fixed point. Let $U_0$ be a neighborhood of $\gamma_0$ in $M$ homeomorphic to an annulus.

Lemma 12. There exists a neighborhood $V_0$ of $(\gamma_0, \gamma_0)$ in $TM$ such that $p(V_0) = U_0$ and, for any simple extremal $\gamma$, if $(\gamma, \dot{\gamma})$ enters (resp. leaves) $V_0$ then either $\gamma$ intersects transversally $\gamma_0$ or $\gamma$ is forever trapped in $p(V_0)$ in the future (resp. past), that is

$$\exists t_0 \in \mathbb{R}, \forall t \geq (\text{resp.} \leq)t_0, \gamma(t) \in p(V_0).$$

Besides, all intersections with $\gamma_0$, if any, have the same sign with respect to the chosen orientation.

4.1. Let $\gamma_0$ be a $C^1$ simple closed curve (not a fixed point) in an oriented surface $M$. We say a $C^1$ curve $\alpha: \mathbb{R} \rightarrow M \setminus \gamma_0$ is positively asymptotic to $\gamma_0$ on the right if the $\omega$-limit set of $\alpha$ is $\gamma_0$ and there exists some $t_0 \in \mathbb{R}$ such that, for any $t \geq t_0$, $\alpha(t)$ lies in the right-hand side (with respect to the chosen orientation of $M$) of a tubular neighborhood of $\gamma_0$. Similar definitions can be made replacing positively with negatively, and right with left. The lemma below will be used in the proof of Lemma 14.

Lemma 13. Let $\gamma_0$ be a $C^1$ simple closed curve in an oriented surface $M$. Any extremal curve $\alpha: \mathbb{R} \rightarrow M \setminus \gamma_0$ positively asymptotic to $\gamma_0$ on the right intersects transversally any extremal curve $\alpha: \mathbb{R} \rightarrow M \setminus \gamma_0$ negatively asymptotic to $\gamma_0$ on the right.

Proof. Let

- $\alpha_0: \mathbb{R} \rightarrow M \setminus \gamma_0$ be a $C^1$ curve positively asymptotic to $\gamma_0$ on the right
- $\alpha_1: \mathbb{R} \rightarrow M \setminus \gamma_0$ be a $C^1$ curve negatively asymptotic to $\gamma_0$ on the right
- $\delta$ be a $C^1$ transverse segment to $\gamma_0$, oriented so that its transverse intersection with $\gamma_0$ is positive.

Intersecting transversally with a given sign is an open property, so there exists a neighborhood $U$ of $(\gamma_0, \gamma_0)$ in $TM$ such that for any $C^1$ arc $\alpha$ in $M$, if $(\alpha(t), \dot{\alpha}(t))$ is contained in $U$ for a sufficiently long time, then $\alpha$ intersects $\delta$ transversally with positive sign.

Since $\alpha_1$ is negatively asymptotic to $\gamma_0$ on the right, there exists a tubular neighborhood $V$ of $\gamma_0$ in $M$ such that $\alpha_1$ eventually leaves the right-hand side of $V$. Restricting $U$ if necessary, we assume $p(U) \subset V$. Take $t_1, t_2$ two consecutive intersection points of $\alpha_0$ with $\delta$. 

□
Consider the topological annulus $A$ bounded by $\gamma_0$ on the left, and on the right, by $\alpha_0([t_1, t_2])$ glued to the segment of $\delta$ comprised between $\alpha_0(t_2)$ and $\alpha_0(t_1)$. Since $\alpha_1$ eventually leaves the right-hand side of $V$, it must leave $A$. In so doing it cannot intersect $\delta$ for then the intersection of $\delta$ with $\alpha_1$ would be negative. Therefore it must intersect $\alpha_0$, which proves the lemma. \qed

4.2. Periodic orbits which are not fixed points. Besides the Aubry set, another set of note is the Mañé set $\tilde{\mathcal{N}}(L)$; all we need to know about it is that

- it is compact and $\phi_L$-invariant
- it contains $\tilde{\mathcal{A}}(L)$
- no orbit contained in $\tilde{\mathcal{A}}(L)$ intersects tranversally an orbit contained in $\tilde{\mathcal{A}}(L)$ (Theorem 5.2.4)
- it is lower semi-continuous with respect to the Lagrangian, that is, for any neighborhood $U$ of $\tilde{\mathcal{N}}(L)$, for any sequence $L_n$ of Tonelli Lagrangians converging to $L$ in the $C^2$ topology, for $n$ large enough $\tilde{\mathcal{N}}(L_n) \subset U$.

**Lemma 14.** Let $c$ be a cohomology class and let $\gamma_i, i \in I$ be a collection of $c$-minimizing periodic orbits which are not fixed points. Then there exists a face $F$ of $\alpha_0$ containing $c$, and a neighborhood $V$ of $\cup_{i \in I} \gamma_i$ in $TM$, such that $\tilde{\mathcal{A}}(F) \cap V$ consists of closed orbits whose projections to $M$ are homologous to $\gamma_i$ for some $i$.

**Proof.** We may assume that $c = 0$. Note that by Mather’s Graph Theorem, the $\gamma_i$ have no self-intersections and are pairwise disjoint. Hence they can be partitioned into a finite number of homology classes $h_0, \ldots, h_k$. Let $h_m$ be any barycenter with positive coefficients of $h_0, \ldots, h_k$. The face $F := \mathcal{L}(h_m)$ of $\alpha$ contains zero (not necessarily in its interior) because the curves $\gamma_i$ are zero-minimizing. Choose one of the $\gamma_i$; assume, without loss of generality, that it is $\gamma_0$ and its homology class is $h_0$.

**Case 1 : $M$ is orientable.**

Denote by $\text{Int}$ the symplectic intersection form induced on $H_1(M, \mathbb{R})$ by the algebraic intersection number of closed curves.

**Case 1.1 : $\gamma_0$ does not separate $M$.**

Pick $h_1 \in \Gamma$ such that $\text{Int}(h_0, h_1) = 1$. Denote $h_n^\pm := nh_m \pm h_1 \in \Gamma, n \in \mathbb{N}$. Take, for each $n \in \mathbb{N}$,

- $n^{-1} h_n^\pm$-minimizing measures $\mu_n^\pm$
- $c_n^\pm \in H^1(M, \mathbb{R})$ such that $\mu_n^+$ (resp. $\mu_n^-$) is $c_n^+$-minimizing (resp. $c_n^-$-minimizing).

The homology classes $n^{-1} h_n^\pm$ remain in a compact subset of $H_1(M, \mathbb{R})$ so the cohomology classes $c_n^\pm$ remain in a compact subset of $H^1(M, \mathbb{R})$. Therefore the supports of the measures $\mu_n^\pm$, which lie in the energy levels $\alpha(c_n^\pm)$ by [Ca80], remain in a compact subset of $TM$. Hence the sequences $\mu_n^\pm, n \in \mathbb{N}$, have weak* limit points $\mu^\pm$. Likewise, we may assume the sequences $c_n^\pm$ converge to $c^\pm \in H^1(M, \mathbb{R})$. Since the homology class is a continuous
function of the measure, we have $[\mu^\pm] = h_m$. Besides, since $\mu_n^\pm$ is $c_n^\pm$ minimizing,

$$\langle c_n^+, h_n^+ \rangle = \alpha(c_n^+) + \beta(h_n^+)$$

$$\langle c_n^-, h_n^- \rangle = \alpha(c_n^-) + \beta(h_n^-)$$

whence, taking limits,

$$\langle c^\pm, h_m \rangle = \alpha(c^\pm) + \beta(h_m)$$

that is, $c^\pm \in L(h_m)$.

By the semi-continuity of the Mañé set with respect to the Hausdorff distance on compact sets,

$$\lim_{n \to +\infty} \text{supp}\mu_n^+ \subset \mathcal{N}(c^+)$$

$$\lim_{n \to +\infty} \text{supp}\mu_n^- \subset \mathcal{N}(c^-).$$

On the other hand, by Proposition 4.

$$\mathcal{A}(F) \subset \mathcal{A}(c^+) \subset \mathcal{N}(c^+)$$

$$\mathcal{A}(F) \subset \mathcal{A}(c^-) \subset \mathcal{N}(c^-)$$

and furthermore, no orbit contained in $\mathcal{N}(c^\pm)$ intersects transversally an orbit in $\mathcal{A}(F)$. Now by Lemma 12 there exists a neighborhood $V$ of $\gamma_0$ in $TM$, such that $\mathcal{M}(F) \cap V$ consists of closed orbits homotopic to $\gamma_0$. Therefore $\mathcal{A}(F) \cap V$ consists of closed orbits homotopic to $\gamma_0$ and orbits asymptotic to one of the latter.

Now since $\text{Int}(h_0, [\mu_n^+]) > 0$, the Hausdorff limit of $\text{supp}\mu_n^+$ must contain an orbit asymptotic to $\gamma_0$ positively on the right, and an orbit asymptotic to $\gamma_0$ negatively on the left. Likewise, since $\text{Int}(h_0, [\mu_n^-]) < 0$, the Hausdorff limit of $\text{supp}\mu_n^-$ must contain an orbit asymptotic to $\gamma_0$ positively on the left, and an orbit asymptotic to $\gamma_0$ negatively on the right. Thus by Lemma 13 any orbit asymptotic to a closed curve homotopic to $\gamma_0$ must intersect transversally either $\mathcal{N}(c^+) \cup \mathcal{N}(c^-)$, and thus cannot be in $\mathcal{A}(F)$. So $\mathcal{A}(F) \cap V$ consists of closed orbits homotopic to $\gamma_0$.

**Case 1.2**: $\gamma_0$ separates $M$.

**Remark 15.** In that case the result is stronger: there exists a neighborhood of $(\gamma_0, \hat{\gamma}_0)$ in $TM$ such that $\mathcal{A}_0 \cap V$ consists of closed orbits whose projection to $M$ are homotopic to $\gamma_0$.

Let $U$ be a tubular neighborhood of $\gamma_0$ in $M$. Let $V$ be the neighborhood of $(\gamma_0, \hat{\gamma}_0)$ in $TM$ given by Lemma 12, such that $p(V) = U$. Let $(x, v)$ be a point of $\mathcal{A}_0 \cap V$. Then by Lemma 12, $\Phi_t(x, v)$ is either a closed orbit homotopic to $\gamma_0$, or is asymptotic to a closed orbit homotopic to $\gamma_0$. In the former case, we are done, so assume the latter occurs, say, $\Phi_t(x, v)$ is positively asymptotic to a closed orbit $\gamma_1$ homotopic to $\gamma_0$. Let $U_1$ a tubular neighborhood of $\gamma_1$ in $M$ such that $U_1$ is properly contained in $U$ and $x \notin U_1$. Let $V_1$ be the neighborhood of $(\gamma_1, \hat{\gamma}_1)$ in $TM$ given by Lemma 13, such that $p(V_1) = U_1$. In particular there exists $t_1$ such that for all $t \geq t_1$, $\Phi_t(x, v) \in V_1$. 
Assume, by replacing \( L \) with \( L - \alpha(0) \) if needed, that the critical value \( \alpha(0) \) of \( L \) is zero. Then, by definition of \( A_0 \) there exists a sequence of extremal curves

\[
\alpha_n: [0, t_n] \to M \quad \text{such that}
\]
\[
x = \alpha_n(0) = \alpha_n(t_n) \text{ and } \lim_{n \to +\infty} \int_0^{t_n} L(\alpha_n(t), \dot{\alpha}_n(t))dt = 0.
\]

Note that \( \lim_{n \to +\infty} \dot{\alpha}_n(0) = v \). So for \( n \) large enough, there exists some \( t \) such that \( (\alpha_n(t), \dot{\alpha}_n(t)) \in V_1 \). Then, by Lemma [12],

- either \( (\alpha_n(t), \dot{\alpha}_n(t)) \) is trapped in \( V_1 \)
- or \( \alpha_n \) intersects \( \gamma_0 \) with constant sign.

The latter case never occurs, for \( \gamma_0 \) separates \( M \), so the algebraic intersection of \( \gamma_0 \) with any closed curve is zero. Neither does the former case, since \( \alpha_n \) must return to its initial point, which lies outside of \( U_1 \). This proves that \( \Phi_t(x, v) \) is a closed orbit homotopic to \( \gamma_0 \). The case when \( \Phi_t(x, v) \) is negatively asymptotic to a closed orbit \( \gamma_1 \) homotopic to \( \gamma_0 \) is treated similarly.

Now let us finish the proof of the orientable case of the lemma. For every \( i \in I \), we have found a neighborhood \( V_i \) of \( (\gamma_i, \dot{\gamma}_i) \) in \( TM \) such that \( \tilde{A}(L(h_m)) \cap V_i \) consists of periodic orbits homologous to \( \gamma_i \). Define \( V \) to be the union over \( i \in I \) of the \( V_i \), then \( V \) is a neighborhood of \( \bigcup_{i \in I} (\gamma_i, \dot{\gamma}_i) \) in \( TM \), and \( \tilde{A}(L(h_m)) \cap V \) consists of periodic orbits homologous to \( \gamma_i \) for some \( i \).

**Case 2 : \( M \) is not orientable.** Let

- \( \delta_j, j \in J \) be the collection of all lifts to \( M_o \) of the \( \gamma_i \)
- \( \{h'_1, \ldots, h'_l\} \) be the set of all homology classes of the \( \delta_j, j \in J \)
- \( h'_m \) be \( l^{-1}(h'_1 + \ldots + h'_l) \)
- \( h_m \) be \( \pi_*(h'm) \).

The set \( \{\delta_j : j \in J\} \) is invariant under \( I \), thus the set \( \{h'_1, \ldots, h'_l\} \) is invariant under \( I_* \). Therefore \( h'_m \in E_1 \subset H_1(M_o, \mathbb{R}) \). The \( \delta_j, j \in J \) are minimizing ([F98]) so by the orientable case of the lemma, there exists a neighborhood \( V \) of \( \bigcup_{j \in J} (\delta_j, \dot{\delta}_j) \) in \( TM_o \) such that \( \tilde{A}(L(h'_m)) \cap V \) consists of periodic orbits homologous to \( \delta_j \) for some \( j \). Taking a smaller \( V \) if we have to, we assume that \( V \) is invariant under \( I \). Lemma [13] says

\[
\pi^*(L(h_m)) = \mathcal{L}(h'm) \cap E_1.
\]

Now take \( c \) in the relative interior of \( \mathcal{L}(h_m) \). Then, \( \pi^* \) being a linear isomorphism onto \( E_1 \subset H^1(M, \mathbb{R}) \), \( \pi^* c \) lies in the relative interior of \( \mathcal{L}(h'm) \cap E_1 \). Since \( h'_m \in E_1 \subset H_1(M_o, \mathbb{R}) \), it is easy to see that \( \mathcal{L}(h'm) \) is \( I^* \)-invariant. Therefore \( \pi^* c \) lies in the relative interior of \( \mathcal{L}(h'm) \), so

\[
\tilde{A}(L(h'_m)) = \tilde{A}(\pi^*c).
\]

By [F98],

\[
T\pi(\tilde{A}(\pi^*c)) = \tilde{A}(c)
\]

that is

\[
T\pi(\tilde{A}(L(h'_m))) = \tilde{A}(L(h_m)).
\]
Since $V$ is invariant under $I$, we have
\[ T\pi \left( \hat{A}(\mathcal{L}(h_m')) \cap V \right) = T\pi \left( \hat{A}(\mathcal{L}(h_m')) \right) \cap T\pi(V). \]

The projection $\pi$ is a local diffeomorphism so $T\pi(V)$ is a neighborhood of $\cup_{i \in I} (\gamma_i, \gamma_i)$ in $TM$. This finishes the proof of Lemma 14. □

The following corollary of Lemma 14 reduces the proof of our main result to proving that $\mathcal{L}(h)$ is a rational flat when $h$ is 1-irrational and nonsingular.

**Corollary 16.** Assume that for some $c$ in $H^1(M, \mathbb{R})$, $\hat{\mathcal{M}}(c)$ consists of periodic orbits which are not fixed points $\gamma_i, i \in I$. Let $h$ be any barycenter with positive coefficients of the homology classes of $\gamma_i, i \in I$. Then
\[ \hat{A}(\mathcal{L}(h)) = \hat{\mathcal{M}}(\mathcal{L}(h)) = \hat{\mathcal{M}}(c). \]

**Proof.** By Lemma 14 there exists a neighborhood $V$ of $\hat{\mathcal{M}}(c)$, such that (1)
\[ \hat{A}(\mathcal{L}(h)) \cap V = \hat{\mathcal{M}}(c). \]

Hence
\[ \hat{\mathcal{M}}(c) \subset \hat{A}(\mathcal{L}(h)), \text{ so } \hat{\mathcal{M}}(c) \subset \hat{\mathcal{M}}(\mathcal{L}(h)). \]

On the other hand $c \in \mathcal{L}(h)$, thus
\[ \hat{\mathcal{M}}(c) \supset \hat{\mathcal{M}}(\mathcal{L}(h)), \text{ so } \hat{\mathcal{M}}(c) = \hat{\mathcal{M}}(\mathcal{L}(h)). \]

Now $\hat{A}(\mathcal{L}(h))$ consists of $\hat{\mathcal{M}}(\mathcal{L}(h))$, and orbits homoclinic to $\hat{\mathcal{M}}(\mathcal{L}(h))$. Such orbits enter any neighborhood of $\hat{\mathcal{M}}(\mathcal{L}(h))$, so (1) implies
\[ \hat{A}(\mathcal{L}(h)) = \hat{\mathcal{M}}(\mathcal{L}(h)) = \hat{\mathcal{M}}(c). \]

□

5. **Faces and flats of $\beta$**

5.1. **Notations.** Let $M$ be a closed oriented surface and let $L$ be a Tonelli Lagrangian on $M$. For $h \in H_1(M, \mathbb{R}) \setminus \{0\}$, we define the maximal radial face $R_h$ of $\beta$ containing $h$ as the largest subset of the half-line $\{th: t \in [0, +\infty]\}$ containing $h$ (not necessarily in its relative interior) in restriction to which $\beta$ is affine. Beware that $R_h$ is a flat of $\beta$, but may not be a face of $\beta$. The possibility of radial flats is the most obvious difference between the $\beta$ functions of Riemannian metrics ([M97], [BM]) and those of general Lagrangians. An instance of radial flat is found in [CL99]. We define the Mather set $\hat{\mathcal{M}}(R_h)$ as the closure in $TM$ of the union of the supports of all $th$-minimizing measures, for $th \in R_h$. The next lemma implies that if $h$ is non-singular, then so is any element of $R_h$.

**Lemma 17.** For any $h \in H_1(M, \mathbb{R})$, for any $t$ such that $th \in R_h$, we have $\mathcal{L}(h) = \mathcal{L}(th)$. Consequently,
\[ R_h \subset \mathcal{F}(\mathcal{L}(h)). \]

**Proof.** Take $t \in \mathbb{R}$ such that $th \in R_h$. By definition of $R_h$ there exists $c \in H^1(M, \mathbb{R})$ such that
\[ \alpha(c) + \beta(h) = \langle c, h \rangle \]
\[ \alpha(c) + \beta(th) = \langle c, th \rangle. \]
The first equality says that \( c \in \mathcal{L}(h) \). Take \( c' \in \mathcal{L}(h) \). Let us show that \( c' \in \mathcal{L}(th) \), which proves that \( \mathcal{L}(h) \subset \mathcal{L}(th) \), whence \( th \in \mathcal{F}(\mathcal{L}(h)) \).

Since \( L \) is autonomous, by Ca95 \( \alpha(c') = \alpha(c) \) and \( \langle c', h \rangle = \langle c, h \rangle \). So

\[
\alpha(c') + \beta(th) = \alpha(c) + \beta(th) = \langle c, th \rangle = \langle c', th \rangle
\]

that is, \( c' \in \mathcal{L}th \).

Conversely, let us prove that \( \mathcal{L}(h) \supset \mathcal{L}(th) \). Take \( c' \in \mathcal{L}(th) \). Since \( L \) is autonomous, by \( \text{Ca95} \) \( \alpha(c') = \alpha(c) \) and \( \langle c', th \rangle = \langle c, th \rangle \). Therefore

\[
\alpha(c') + \beta(h) = \langle c, h \rangle,
\]

so \( c' \in \mathcal{L}(h) \). \( \square \)

Now we look at some consequences of Proposition 4. Assume \( h \) is 1-irrational. Then for all \( t \) such that \( th \in R_h \), \( th \) is also 1-irrational. Furthermore \( R_h \) is contained in a face of \( \beta \), so Mather’s Graph Theorem and Proposition 4 combine to say that \( \mathcal{M}(R_h) \) is a union of pairwise disjoint periodic orbits \( \gamma_i, i \in I \) where \( I \) is some set, not necessarily finite. We denote by \( V(R_h) \) the linear subspace of \( H_1(M, \mathbb{R}) \) generated by \( [\gamma_i], i \in I \). Since the \( \gamma_i \) are pairwise disjoint, there exist homology classes \( h_1, \ldots, h_k \) with \( k \leq 3/2(b_1(M) - 2) \), such that \( \forall i \in I, \exists j = 1, \ldots, k, [\gamma_i] = h_j \).

Let \( T_i \) be the least period of \( \gamma_i \). Then the invariant measure \( \mu_i \) supported by \( \gamma_i \) has homology \( T_i^{-1} [\gamma_i] \). By Lemma 3 the convex hull \( C \) of \( T_i^{-1} [\gamma_i], i \in I \), is a flat of \( \beta \) containing \( th \) in its interior whenever \( th \) is contained in the interior of \( R_h \).

5.2. Faces of \( \beta \). The following lemma is a rewriting of Lemma 12 of Mt97.

**Lemma 18.** Let \( F \) be a flat of \( \beta \). Assume \( F \) contains a 1-irrational homology class \( h_0 \) in its interior. Then for all \( h \in F \), for all \( h \)-minimizing measure \( \mu \), the support of \( \mu \) consists of periodic orbits, or fixed points.

**Proof.** Let \( c \) be a cohomology class such that for all \( h \in F \), \( \alpha(c) + \beta(h) = \langle c, h \rangle \). Take \( h \) in \( F \). Since \( h_0 \) lies in the interior of \( F \), there exist \( h' \) in \( F \) and \( 0 < t < 1 \) such that \( h_0 = th + (1 - t)h' \). Take an \( h \)-minimizing (resp. \( h' \)-minimizing) measure \( \mu \) (resp. \( \mu' \)), so we have \( \beta(h) = \int Ld\mu, \beta(h') = \int Ld\mu' \). Thus

\[
\alpha(c) + \int Ld\mu = \langle c, h \rangle
\]

\[
\alpha(c) + \int Ld\mu' = \langle c, h' \rangle
\]

so

\[
\alpha(c) + \int Ld(t\mu + (1 - t)\mu') = \langle c, th + (1 - t)h' \rangle = \langle c, h_0 \rangle
\]

that is, the probability measure \( t\mu + (1 - t)\mu' \) is \( h_0 \)-minimizing. Since \( h_0 \) is 1-irrational, Proposition 4 implies that the support of \( t\mu + (1 - t)\mu' \), hence that of \( \mu \), consists of periodic orbits, or fixed points. \( \square \)

Here is a version of Theorem 6.1 of BM for general Lagrangians.

**Theorem 19.** Let

- \( M \) be a closed oriented surface
- \( L \) be a Tonelli Lagrangian on \( M \)
- \( h_0 \) be a 1-irrational, nonsingular homology class of \( M \)
- \( V_0 \) be \( V(R_{h_0}) \)
On the other hand, since $\lambda = h_{\text{interior of } R}$, the segment $[h_0, t(h_0, h)h_0 + s(h_0, h)h]$ is contained in a face of $\beta$.

Proof. We use the notation of Paragraph 5.1.

First case : $h \in V_0$. Take $t_0$ such that $t_0h_0$ lies in the relative interior of $R_{h_0}$. Then $t_0h_0$ lies in the relative interior of the convex hull $C$ of $T_i^{-1}[\gamma_i]$, $i \in I$, so there exists a finite subset of $I$, say $\{1, \ldots, n\}$, and $\lambda_1, \ldots, \lambda_n$ in $]0, 1[$ such that

- $\lambda_1 + \ldots + \lambda_n = 1$
- $t_0h_0 = \lambda_1T_1^{-1}[\gamma_1] + \ldots + \lambda_nT_n^{-1}[\gamma_n]$
- $[\gamma_1], \ldots, [\gamma_n]$ generate $V_0$.

On the other hand, since $h \in V_0$, there exist real numbers $x_1, \ldots, x_n$ such that $h = x_1T_1^{-1}[\gamma_1] + \ldots + x_nT_n^{-1}[\gamma_n]$. Take $\epsilon > 0$ such that $\forall i = 1, \ldots, n$, $\epsilon x_i + \lambda_i > 0$. Then $(\epsilon x_1 + \lambda_1 + \ldots + \epsilon x_k + \lambda_k)^{-1}(t_0h_0 + \epsilon h)$ lies in the relative interior of $C$. Thus there exists a face of $\beta$ containing $h_0$ and $th_0 + \epsilon h$, where

$$t := \frac{1}{\sum \lambda_i} \epsilon, \quad s := \frac{\epsilon}{t}.$$

Second case : $h \notin V_0$, that is, $h \in V_0^\perp \setminus V_0$.

Remark 20. In that case the dimension of $V_0$ must be less than $b_1(M)/2$.

For any $n \in \mathbb{N}^*$ let us denote $h_n := h_0 + n^{-1}h$. Let $\mu_n$ be an $h_n$-minimizing measure. For each $i \in I$ let $V_i$ be the neighborhood of $(\gamma_i, \hat{\gamma}_i)$ given by Lemma 12. Let $V$ be the union over $i \in I$ of the $V_i$. Be sure to take the $V_i$ small enough so $V$ is a disjoint union of annuli.

First let us prove that $V \cap \text{supp}(\mu_n)$ is $\phi_\gamma$-invariant and consists of periodic orbits homotopic to some or all of the $\gamma_i$. Indeed by Lemma 12 a minimizing orbit $\gamma$ that enters $V$ is either

- asymptotic to one of the $\gamma_i$
- or homotopic to one of the $\gamma_i$
- or cuts one of the $\gamma_i$ with constant sign.

The first case is impossible since $\gamma$ is contained in the support of an invariant measure (see Lemma 5.5 of [BM]). The third case is impossible since it would imply $\text{Int}([\mu_n], [\gamma_i]) \neq 0$, which contradicts $h \in V_0^\perp$. So $V \cap \text{supp}(\mu_n)$ is $\phi_\gamma$-invariant and consists of periodic orbits homotopic to some $\gamma_i$.

Now let us show that for $n$ large enough, $0 < \mu_n(V) < 1$. Note that any limit point, in the weak* topology, of the sequence $\mu_n$ is an $h_0$-minimizing measure, hence supported in $V$, so $\mu_n(V)$ tends to 1. On the other hand, if $\mu_n(V)$ were 1, then $\mu_n$ would be supported in $V$. By the Graph Theorem any minimizing measure supported inside $V$ may be viewed as an invariant measure of a vector field defined in $p(V)$. But $p(V)$ is a union of annuli, hence by the Poincaré-Bendixon Theorem any minimizing measure supported inside $V$ is supported on fixed points, or periodic orbits homotopic to some $\gamma_i$. Note that fixed points are ruled out by the nonsingularity of $h$, which implies that $h_n$ is non-singular for $n$ large enough. In particular if $\mu_n(V) = 1$, $[\mu_n] \in V_0$, which is a contradiction. So $0 < \mu_n(V) < 1$ and we may set, for
any Borelian subset $A$ of $TM$,

$$\mu_{n,1}(A) := \frac{\mu_n(A \cap V)}{\mu_n(V)}$$

$$\mu_{n,2}(A) := \frac{\mu_n(A \setminus V)}{\mu_n(TM \setminus V)}$$

$$\lambda_n := \mu_n(V).$$

Then $\mu_{n,1}$ and $\mu_{n,2}$ are two probability measures on $TM$. They are invariant by the Lagrangian flow because $V \cap \text{supp}(\mu_n)$, as well as its complement in $\text{supp}(\mu_n)$, is $\phi_t$-invariant. There exists a face of $\beta$ containing $[\mu_{n,1}]$ and $[\mu_{n,2}]$ because

$$\mu_1 = \lambda_n \mu_{n,1} + (1 - \lambda_n) \mu_{n,2}. $$

Let $\mu_{0,1}$ and $\mu_{0,2}$ be limit points, in the weak* topology, of the sequences $\mu_{n,1}$ and $\mu_{n,2}$. Then $\mu_{0,1}$ is an $h_0$-minimizing measure, and there exists a face of $\beta$ containing $h_0 = [\mu_{0,1}]$ and $[\mu_{0,2}]$.

Now we prove that $[\mu_{0,2}] \notin V_0$. Assume to the contrary. Then, as in the first case, the face $F_0$ containing $h_0$ and $[\mu_{0,2}]$ contains $th_0$ in its interior for some $t$ such that $th_0$ lies in $R_0$. Take $\lambda \in ]0,1[$ and $h'$ in $F_0$ such that $th_0 = \lambda [\mu_{0,2}] + (1 - \lambda) h'$.

Take an $h'$-minimizing measure $\mu'$. Then $\lambda \mu_{0,2} + (1 - \lambda) \mu'$ is a $th_0$-minimizing measure, hence is supported inside $V$, which is impossible since $\mu_{n,2}$ is supported outside $V$ for all $n$.

Thus there exist $v \in V_0$ and $x \neq 0$ such that $[\mu_{0,2}] = v + xh$. Take $t_0$ such that $t_0h_0$ lies in the relative interior of the convex hull $C$ of $T_i^{-1} [\gamma_i]$, $i \in I$, so there exists a positive $\epsilon$ such that $t_0h_0 - \epsilon v$ lies in the relative interior of $C$. Lemma 8 now says that there is a face of $\beta$ containing $h_0$, $t_0h_0 - \epsilon v$ and $[\mu_{0,2}] = v + xh$. Such a face must contain $h_0$ and

$$\frac{1}{1 + \epsilon} (t_0h_0 - \epsilon v) + \left(1 - \frac{1}{1 + \epsilon}\right) (v + xh) = \frac{t_0}{1 + \epsilon} h_0 + \frac{x\epsilon}{1 + \epsilon} h,$$

Now set

$$t(h_0, h) := \frac{t_0}{1 + \epsilon}, \quad s(h_0, h) := \frac{x\epsilon}{1 + \epsilon}$$

and the theorem is proved.

6. PROOF OF THE MAIN THEOREM

Let $h$ be a homology class. Recall that $\mathcal{L}(h)$ is the Legendre transform of $h$ with respect to $\beta$, and that $\mathcal{F}(\mathcal{L}(h))$ is the set of homology classes $h'$ that lie in $\mathcal{L}(c)$ for all $c \in \mathcal{L}(h)$. By Lemma 8, $\mathcal{F}(\mathcal{L}(h))$ is a face of $\beta$. It is clear that $h$ lies in $\mathcal{F}(\mathcal{L}(h))$, although possibly on the boundary.

**Lemma 21.** Let $L$ be a Tonelli Lagrangian on a closed surface $M$. Let $h$ be a $1$-irrational, nonsingular homology class. Then the relative interior of $R_h$ is contained in the relative interior of the face $\mathcal{F}(\mathcal{L}(h))$.

**Proof.** Orientable case

Assume $M$ is oriented. Take $h_i, i = 1 \ldots k$ such that $h + h_i$ lies in $\mathcal{F}(\mathcal{L}(h))$ for all $i = 1, \ldots k$, and the convex hull of $h + h_i$, $i = 1, \ldots k$, contains an
open subset of $\mathcal{F}(\mathcal{L}(h))$. Pick one of the $h_i$. Then the segment $[h, h + h_i]$ is contained in a face of $\beta$ so by Mather’s graph Theorem $h_i \in V_0^\perp$. Then $-h_i$ lies in $V_0^\perp$ so by Theorem 13 there exist $t_i \in \mathbb{R}, s_i := s(h, h_i) > 0$ such that the segment $[h, t_i h - s_i h_i]$ is contained in a face of $\beta$. So there exists $c \in H^1(M, \mathbb{R})$ such that

$$
\alpha(c) + \beta(h) = \langle c, h \rangle \quad \text{and} \quad \alpha(c) + \beta(t_i h - s_i h_i) = \langle c, t_i h - s_i h_i \rangle.
$$

The first equality above says that $c \in \mathcal{L}(h)$. Since $h + h_i$ lies in $\mathcal{F}(\mathcal{L}(h))$, we also have

$$
\alpha(c) + \beta(h + h_i) = \langle c, h + h_i \rangle.
$$

Therefore $\mathcal{F}(\mathcal{L}(h))$ contains the convex hull $C_i$ of $h, h + h_i, t_i h - s_i h_i$ for $i = 1, \ldots, k$.

We claim that for each $i = 1, \ldots, k$, $C_i$ contains some $th$ in its interior. Indeed, the following three cases may occur:

- $h_i \notin \mathbb{R}h, t_i \neq 1$ : then $C_i$ is a 2-simplex containing $th$ in its relative interior, for some $t \in [1, t_i[$.
- $h_i \notin \mathbb{R}h, t_i = 1$ : then $C_i$ is the segment $[h + h_i, h - s_i h_i]$, which contains $h$ in its relative interior.
- $h_i \in \mathbb{R}h$ : then $C_i$ is a segment of the straight line $\mathbb{R}h$, hence it contains some $th$ in its relative interior.

Now the convex hull $C$ of $\cup_{i=1}^k C_i$ is contained in $\mathcal{F}(\mathcal{L}(h))$ by convexity of the latter. The relative interior of $C$ is open in $\mathcal{F}(\mathcal{L}(h))$ by our hypothesis on $h_i, i = 1 \ldots k$. On the other hand $C$ contains an interior point of $R_h$ in its relative interior. So the intersection of the relative interiors of $R_h$ and $\mathcal{F}(\mathcal{L}(h))$ is non-empty, convex, and contains $h$ in its closure. Hence its closure is the whole of $R_h$.

**Non-orientable case**

Take

- $h$ a 1-irrational, nonsingular homology class of $M$
- $h' \in E_1 \subset H_1(M_o, \mathbb{R})$ such that $\pi_* h' = h$
- $c \in H^1(M, \mathbb{R})$ such that $\alpha(c) + \beta(h) = \langle c, h \rangle$.

Since $\pi_*$ sends an integer class to an integer class, and is one-to-one on $E_1$, $h'$ is 1-irrational.

Any support of an $h'$-minimizing measure is the lift to $M_o$ of the support of an $h$-minimizing measure by $[F98]$, hence $h'$ is nonsingular.

The orientable case of the lemma then says that for any $t$ such that $th$ is in the relative interior of $R_{h'}$, $th'$ lies in the relative interior of $\mathcal{F}(\mathcal{L}(h'))$. Furthermore $h' \in E_1$ so $th'$ lies in the relative interior of $\mathcal{F}(\mathcal{L}(h')) \cap E_1$. Thus, $\pi_*$ being linear, $\pi_*(th') = th$ lies in the relative interior of $\pi_*(\mathcal{F}(\mathcal{L}(h')) \cap E_1)$. This combines with Lemma 13 to end the proof.

A consequence of our last lemma is that for any 1-irrational, nonsingular homology class $h$, $\mathcal{L}(h)$ is a nonsingular rational flat of $\alpha_L$:

**Lemma 22.** Let $h$ be a 1-irrational, nonsingular homology class. We have

$$
\mathcal{M}(\mathcal{L}(h)) = \mathcal{M}(R_h).
$$
Proof. Let \( \mu \) be a minimizing measure supported in \( \tilde{\mathcal{M}}(\mathcal{L}(h)) \). Then for all \( c \in \mathcal{L}(h) \) we have
\[
\alpha(c) + \beta(\mu) = \langle c, [\mu] \rangle
\]
whence \( [\mu] \in \mathcal{F}(\mathcal{L}(h)) \). Take \( t \) such that \( th \) lies in the interior of \( \mathcal{F}(\mathcal{L}(h)) \) such that
\[
th = \lambda [\mu] + (1 - \lambda)h'.
\]
Take an \( h' \)-minimizing measure \( \mu' \). Then \( \lambda\mu + (1 - \lambda)\mu' \) is a \( th \)-minimizing measure, hence its support is contained in \( \tilde{\mathcal{M}}(R_h) \).
\[ \square \]

6.1. Proof of Theorem 3. Take a nonsingular, 1-irrational homology class \( h \). Note that \( th \) is 1-irrational for any \( t \). For any measure \( \mu \) supported in \( \mathcal{M}(\mathcal{L}(h)) \), the homology class of \( \mu \) lies in \( \mathcal{F}(\mathcal{L}(h)) \). Thus by Lemma 21 and Lemma 18, the support of \( \mu \) consists of periodic orbits and fixed points. The latter are ruled out by the nonsingularity of \( th \). Thus \( \mathcal{M}(\mathcal{L}(h)) \) is a union of non-trivial minimizing periodic orbits. Note that by Lemmas 21 and 22, if \( th \) lies in the relative interior of \( R_h \), then \( th \) is a barycenter with positive coefficients of the homology classes of the periodic orbits contained in \( \mathcal{M}(\mathcal{L}(h)) \). So by Corollary 18 and Lemma 17,
\[
\tilde{\mathcal{M}}(\mathcal{L}(h)) = \tilde{\mathcal{M}}(\mathcal{L}(th)) = \tilde{\mathcal{A}}(\mathcal{L}(th)) = \tilde{\mathcal{A}}(\mathcal{L}(h))
\]
which finishes the proof of Theorem 3.
\[ \square \]

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