Positive Solutions for a Hadamard Fractional \( p \)-Laplacian Three-Point Boundary Value Problem

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Abstract: This article is to study a three-point boundary value problem of Hadamard fractional \( p \)-Laplacian differential equation. When our nonlinearity grows \((p - 1)\)-superlinearly and \((p - 1)\)-sublinearly, the existence of positive solutions is obtained via fixed point index. Moreover, using an increasing operator fixed-point theorem, the uniqueness of positive solutions and uniform convergence sequences are also established.

Keywords: hadamard fractional differential equations; \( p \)-Laplacian boundary value problems; positive solutions; fixed point index

1. Introduction

In this paper, we study the existence and uniqueness of positive solutions for the Hadamard fractional \( p \)-Laplacian three-point boundary value problem

\[
\begin{aligned}
D^\alpha (\phi_p(D^\beta u(t))) &= f(t, u(t)), \quad t \in (1, e), \\
D^\beta u(1) &= D^\beta u(e) = 0, \quad u(1) = u'(1) = 0, \quad u(e) = au(\xi),
\end{aligned}
\]  

where \( \alpha \in (1, 2], \beta \in (2, 3], \) and \( D^\alpha, D^\beta \) are respectively the Hadamard fractional derivatives of orders \( \alpha, \beta; \xi \in (1, e), \) and \( a \geq 0 \) with \( a(\log \xi)^{\beta - 1} \in [0, 1); \) note \( \phi_p(s) = |s|^{p-2} s \) is the \( p \)-Laplacian for \( p > 1, s \in \mathbb{R}. \)

Arafa et al. \cite{1} introduced a fractional-order HIV-1 infection of CD4+T cells dynamics model and then used the generalised Euler method to find a numerical solution of the HIV-1 infection fractional order model: the model is

\[
\begin{aligned}
D^\alpha_1(T) &= s - KVT - dT + bI, \\
D^\alpha_2(I) &= KVT - (b + \delta)I, \\
D^\alpha_3(V) &= N\delta I - cV,
\end{aligned}
\]

where \( D^\alpha_1(i = 1, 2, 3) \) are fractional-order derivatives. Nonlinear analysis methods (such as fixed-point theorems, Leray–Schauder alternative, subsolution and supersolution methods and iterative techniques) are used to study various kinds of fractional-order equations (most of these results involve the Riemann–Liouville and Caputo-type fractional derivatives); see \cite{2–52} and the
where $D^\alpha_0, D^\beta_0$ are the standard Riemann–Liouville derivatives. For the unique solution, they constructed uniform convergent sequences, and provided estimates on the error and the convergence rate. In [3], the authors adopted some fixed-point theorems on cones to study the unique solution for the fractional $p$-Laplacian boundary value problem

$$
\begin{align}
\begin{cases}
D^\alpha_0 u(t) + f(t, u(t), u'(t)) = 0, & t \in (0, 1), \\
u(0) = u'(0) = 0,
\end{cases}
\end{align}
$$

where $D^\alpha_0$ denotes the Riemann–Liouville fractional derivative. Positive solutions [16–35] and nontrivial solutions [36–52] were also studied for fractional-order equations. For example, the authors in [16] used the Guo–Krasnosel’skii’s fixed-point theorem and the Leggett–Williams fixed-point theorem to study the existence and multiplicity of positive solutions for the fractional boundary-value problem

$$
\begin{align}
\begin{cases}
D^\alpha(D^\beta u(t)) = f(t, u(t)), & t \in [0, 1], \\
u(0) = u(1) = 0,
\end{cases}
\end{align}
$$

and obtained existence and nonexistence of positive solutions, and considered the impact of parameters on solutions. In [36], the authors used the Kuratowski noncompactness measure and the Sadovskii fixed-point theorem to study the impulsive fractional differential equations with the $p$-Laplacian operator

$$
\begin{align}
\begin{cases}
D^\alpha_0 u(t) + f(t, x(t), x'(t)) = 0, & t \in (0, 1), \\
\Delta x(t)|_{t=k} = I_k(x(t_k)), & x(a) = x(0) = I^1_0 a_1(x(s))ds, x(1) = x'(1) = \int_0^1 a_2(x(s))ds.
\end{cases}
\end{align}
$$
Hadamard fractional-order problems were briefly discussed in the literature; see [53–72] and the references therein. Yang in [53] used the comparison principle and the monotone iterative technique combined with the subsolution and supersolution method to study the existence of extremal solutions for Hadamard fractional differential equations with Cauchy initial value conditions

\[
\begin{cases}
(D^{\alpha}_{a+} x)(t) = f(t, x(t), y(t)), (J^{1-\alpha}_{a+} x)(a+) = x_0^*, \alpha \in (0, 1], t \in (a, b], \\
(D^{\alpha}_{a+} y)(t) = g(t, x(t), y(t)), (J^{1-\alpha}_{a+} y)(a+) = y_0^*, \alpha \in (0, 1], t \in (a, b],
\end{cases}
\]

where \(D^{\alpha}_{a+}, J^{1-\alpha}_{a+}\) are the left-sided Hadamard fractional derivative and integral of order \(\alpha\), respectively. In [54], the authors used fixed point methods to study the existence of positive solutions for Hadamard fractional integral boundary value problems

\[
\begin{cases}
D^\beta (\varphi_p(D^\alpha u(t))) = f(t, u(t)), t \in (1, e), \\
u(1) = D^\alpha u(1) = u'(1) = u'(e) = 0, \varphi_p(D^\alpha u(e)) = \mu \int_1^e \varphi_p(D^\alpha u(t)) \frac{dt}{t}.
\end{cases}
\]

In this paper, we study the existence of positive solutions for the Hadamard fractional \(p\)-Laplacian three-point boundary value problem (1). Note: (i) we establish some relations from the corresponding fractional integral boundary value problems

\[
\begin{cases}
D^\alpha g(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s) \frac{ds}{s}, n-1 < q < n,
\end{cases}
\]

where \(n = [q] + 1, [q]\) denotes the integer part of the real number \(q\) and \(\log(\cdot) = \log_\varepsilon(\cdot)\).

In what follows, we calculate the Green’s functions associated with (1). We let \(\varphi_p(D^\beta u(t)) = -v(t)\) for \(t \in [1, e]\). Then, from (1) we obtain

\[
\begin{cases}
-D^\alpha v(t) = f(t, u(t)), t \in (1, e), \\
v(1) = v(e) = 0.
\end{cases}
\]

**Lemma 1.** The boundary value problem (10) takes the form

\[
v(t) = \int_1^e G_n(t, s) f(s, u(s)) \frac{ds}{s},
\]

where

\[
G_n(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases}
(\log t)^{n-1} (1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, & 1 \leq s \leq t \leq e, \\
(\log t)^{n-1} (1 - \log s)^{\alpha-1}, & 1 \leq t \leq s \leq e.
\end{cases}
\]

**Proof.** We use ideas in Lemma 2 of [59]. For some \(c_i \in \mathbb{R}(i = 1, 2)\), we have

\[
v(t) = c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} f(s, u(s)) \frac{ds}{s}.
\]
Lemma 2. The boundary value problem

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where

\[ c_1 - 1 \int_1^t (1 - \log s)^{a-1} f(s, u(s)) \frac{ds}{s} = 0. \]

Then, substituting \( \epsilon \) into the above equation, and using \( u(\epsilon) = 0 \), we obtain

\[ v(\epsilon) = c_1 - 1 \int_1^\epsilon (1 - \log s)^{a-1} f(s, u(s)) \frac{ds}{s} = 0. \]

Then, we have

\[ c_1 = \frac{1}{\Gamma(\alpha)} \int_1^\epsilon (1 - \log s)^{a-1} f(s, u(s)) \frac{ds}{s} = 0. \]

Consequently, we have

\[
\begin{align*}
v(t) &= \frac{1}{\Gamma(\alpha)} \int_1^\epsilon (1 - \log s)^{a-1} f(s, u(s)) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^t (1 - \log s)^{a-1} f(s, u(s)) \frac{ds}{s} \\
&= \int_1^\epsilon G_a(t, s) f(s, u(s)) \frac{ds}{s}.
\end{align*}
\]

This completes the proof. \( \Box \)

Note that \( \varphi_p(D^\beta u(t)) = -v(t) \). Then, \( \varphi_p(-D^\beta u(t)) = v(t) \) and \( -D^\beta u(t) = \varphi_q(v(t)) \), where \( q \) is a constant with \( q^{-1} + p^{-1} = 1 \). Then, from (1), we have

\[
\begin{align*}
&D^\beta u(t) = \varphi_q(v(t)), \quad t \in (1, \epsilon), \\
&u(1) = u'(1) = 0, \quad u(\epsilon) = au(\xi).
\end{align*}
\]

(12)

**Lemma 2.** The boundary value problem (12) is equivalent to the integral equation

\[
u(t) = \int_1^\epsilon G_0(t, s) \varphi_q(v(s)) \frac{ds}{s}, \tag{13}\]

where

\[
\begin{align*}
G_0(t, s) &= \Gamma(\beta) \\
G_0(t, s) &= \Gamma(\beta) (1 - \log s)^{-1} - (1 - \log s)^{-1}, \\
G_0(t, s) &= \Gamma(\beta) (1 - \log s)^{-1}, \\
G_0(t, s) &= \Gamma(\beta) (1 - \log s)^{-1} - (1 - \log s)^{-1} \varphi_q(v(s)).
\end{align*}
\]

**Proof.** We follow the ideas in Lemma 1. For some \( c_i \in \mathbb{R} (i = 1, 2, 3) \), we have

\[
u(t) = c_1 (\log t)^{\beta-1} + c_2 (\log t)^{\beta-2} + c_3 (\log t)^{\beta-3} - \frac{1}{\Gamma(\beta)} \int_1^\epsilon (1 - \log s)^{\beta-1} \varphi_q(v(s)) \frac{ds}{s},
\]

Then, \( u(1) = u'(1) = 0 \) implies \( c_2 = c_3 = 0 \). Consequently, we have

\[
u(t) = c_1 (\log t)^{\beta-1} - \frac{1}{\Gamma(\beta)} \int_1^\epsilon (1 - \log s)^{\beta-1} \varphi_q(v(s)) \frac{ds}{s}.
\]

Substituting \( \epsilon, \xi \) into the above equation, and using \( u(\epsilon) = au(\xi) \), we obtain

\[
c_1 = \frac{1}{\Gamma(\beta)} \int_1^\epsilon (1 - \log s)^{\beta-1} \varphi_q(v(s)) \frac{ds}{s} = ac_1 (\log \xi)^{\beta-1} - \frac{a}{\Gamma(\beta)} \int_1^\xi (1 - \log s)^{\beta-1} \varphi_q(v(s)) \frac{ds}{s}.
\]

Solving this equation, we have
A is a solution for (1). Therefore, in what follows, we turn to study the existence of fixed points of the $u$ operator. Moreover, if there is a $P$ existence of fixed points of the $u$ operator. Moreover, if there is a $P$

Let $E$ be a Banach space and let $f : E 	imes E 	o E$ be a nonlinear map. We consider the following integral equation:

$$u(t) = \int_1^t G(t,s) f(s,u(s)) \, ds, \quad t \in [1,e].$$

From (15), we define an operator $A : E \to E$ as follows:

$$(Au)(t) = \int_1^t G(t,s) f(s,u(s)) \, ds, \quad u \in E.$$ 

Note that our functions $G$ are continuous, so the operator $A$ is a completely continuous operator. Moreover, if there is a $u \in E$ is a fixed point of $A$, then from Lemmas 1–2, we have that $u$ is a solution for (1). Therefore, in what follows, we turn to study the existence of fixed points of the operator $A$.

**Lemma 3** (see [21] (Lemma 3.2)). Let $\beta \in (n-1, n]$, and $n \geq 3$. Then, the function $G$ has the properties:

(R1) $G(t,s) = G(1-s, 1-t), \quad t, s \in [0, 1],$

(R2) $t^{\beta-1}(1-t)s(1-s)^{\beta-1} \leq \Gamma(\beta)G(t,s) \leq (\beta-1)s(1-s)^{\beta-1}, \quad t, s \in [0, 1],$

(R3) $t^{\beta-1}(1-t)s(1-s)^{\beta-1} \leq \Gamma(\beta)G(t,s) \leq (\beta-1)t^{\beta-1}(1-t), \quad t, s \in [0, 1],$

where

$$G(t,s) = \begin{cases} 
\frac{1}{\Gamma(\beta)} \int_1^t (1-s)^{\beta-1} \phi_t(v(s)) \, ds, & 0 \leq s \leq t \leq 1, \\
\frac{1}{\Gamma(\beta)} \int_1^t (1-s)^{\beta-1} \phi_t(v(s)) \, ds, & 0 \leq t \leq s \leq 1.
\end{cases}$$
Lemma 4. Let \( \phi(t) = \frac{\log(1 - t)}{t(1 - \log t)} \), \( \eta(s) = \frac{\log(1 - s)}{s} \), for \( t, s \in [1, e] \). Then, the functions \( G_\alpha, G_\beta \) have the properties:

(I) \( G_\alpha \in C([1, e] \times [1, e], \mathbb{R}^+) \) and \( \Gamma(a)G_\alpha(t, s) \leq 1 \), for \( t, s \in [1, e] \),

(II) \( \phi(t)\eta(s) = \frac{(1 - a\log t)^{\beta - 1} + a\phi(t)}{1 - a\log t} \leq G_\beta(t, s) \leq \frac{(a + (1 - a)(\log \xi)^{\beta - 1})\Gamma(\beta)}{(1 - a\log \xi)^{\beta - 1} \Gamma(\beta)} \eta(s) \), for \( t, s \in [1, e] \),

(III) \( G_\beta(t, s) \leq \frac{(1 - a\log t)^{\beta - 1} + a\phi(t)}{1 - a\log \xi} \Gamma(\beta) \), for \( t, s \in [1, e] \).

Proof. From the definition of \( G_\alpha \), we easily have (I). From Lemma 3, in \( G(t, s) \), using \( \log t, \log s \) to replace \( t, s \), we have

\[
\Gamma(\beta)\phi(t)\eta(s) \leq G_{1\beta}(t, s) \leq (\beta - 1)\eta(s), \quad \text{for} \ t, s \in [1, e],
\]

(17)

and

\[
G_{1\beta}(t, s) \leq (\beta - 1)\phi(t), \quad \text{for} \ t, s \in [1, e].
\]

(18)

Consequently, from (17), we have

\[
G_\beta(t, s) = G_{1\beta}(t, s) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log t)^{\beta - 1} \Gamma(\beta)} G_1(t, s)
\]

\[
\leq (\beta - 1)\eta(s) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log t)^{\beta - 1} \Gamma(\beta)}(\beta - 1)\eta(s)
\]

\[
= \frac{a + (1 - a)(\log \xi)^{\beta - 1})\Gamma(\beta)(\beta - 1)}{(1 - a\log \xi)^{\beta - 1} \Gamma(\beta)} \eta(s), \quad \text{for} \ t, s \in [1, e],
\]

and

\[
G_\beta(t, s) = G_{1\beta}(t, s) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log \xi)^{\beta - 1} \Gamma(\beta)} G_1(t, s)
\]

\[
\geq \Gamma(\beta)\phi(t)\eta(s) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log t)^{\beta - 1} \Gamma(\beta)} \Gamma(\beta)\phi(t)\eta(s)
\]

\[
= \Gamma(\beta)\phi(t)\eta(s) + \frac{a}{1 - a\log t^\beta - 1} \Gamma(\beta)\phi(t)\eta(s)
\]

\[
= \Gamma(\beta)\phi(t)\eta(s) \frac{1 - a(\log \xi)^{\beta - 1} + a\phi(t)}{1 - a(\log \xi)^{\beta - 1}}, \quad \text{for} \ t, s \in [1, e].
\]

This implies that (II) holds. Finally, from (18), we obtain

\[
G_\beta(t, s) = G_{1\beta}(t, s) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log \xi)^{\beta - 1} \Gamma(\beta)} G_1(t, s)
\]

\[
\leq (\beta - 1)(\log t)^{\beta - 1}(1 - \log t) + \frac{a(\log t)^{\beta - 1}}{(1 - a\log \xi)^{\beta - 1} \Gamma(\beta)}(\beta - 1)\phi(t)
\]

\[
= \frac{a}{1 - a\log t^\beta - 1} \Gamma(\beta)\phi(t)\eta(s)
\]

Thus, (III) holds. This completes the proof. \( \square \)
For convenience, we define three positive constants
\[
\kappa_1 = \frac{(1 - a(\log \xi)\beta^{-1} + a\phi(\xi))\Gamma(\beta)}{1 - a(\log \xi)\beta^{-1}}, \quad \kappa_2 = \frac{(a + (1 - a(\log \xi)\beta^{-1})\Gamma(\beta))(\beta - 1)}{(1 - a(\log \xi)\beta^{-1})\Gamma(\beta)},
\]
\[
\kappa_3 = \frac{(1 - a(\log \xi)\beta^{-1} + a\phi(\xi))(\beta - 1)}{(1 - a(\log \xi)\beta^{-1})\Gamma(\beta)}.
\]

Lemma 5. Let \( z \in P \) and \( \mu(t) = \int_1^e \eta(s)G_\alpha(s,t)\frac{ds}{s} \), for \( t \in [1,e] \). Then, we have the following two integral inequalities
\[
\int_1^\epsilon G_\beta(t,s) \int_1^\epsilon G_\alpha(s,\tau)z(\tau) \frac{d\tau ds}{s} \geq \kappa_1 \phi(t) \int_1^\epsilon z(\tau) \mu(\tau) \frac{d\tau}{\tau}, \text{ for } t \in [1,e],
\]
and
\[
\int_1^\epsilon G_\beta(t,s) \int_1^\epsilon G_\alpha(s,\tau)z(\tau) \frac{d\tau ds}{s} \leq \kappa_2 \int_1^\epsilon z(\tau) \mu(\tau) \frac{d\tau}{\tau}, \text{ for } t \in [1,e].
\]
This is a direct result from Lemma 4[I2], so we omit the details.

Lemma 6 (see [74] (Lemma 2.6)). Let \( \theta > 0 \) and \( \varphi \in P \). Then,
\[
\left( \int_0^1 \varphi(t)dt \right)^\theta \leq \int_0^1 \varphi(t)dt, \text{ if } \theta \geq 1, \text{ and } \left( \int_0^1 \varphi(t)dt \right)^\theta \geq \int_0^1 \varphi(t)dt, \text{ if } 0 < \theta \leq 1.
\]

Lemma 7 (see [75]). Let \( E \) be a real Banach space and \( P \) a cone on \( E \). Suppose that \( \Omega \subset E \) is a bounded open set and that \( A : \Omega \cap P \rightarrow P \) is a continuous compact operator. If there exists a \( \omega_0 \in P \setminus \{0\} \) such that
\[
\omega - A\omega \neq \lambda \omega_0, \forall \lambda \geq 0, \omega \in \partial \Omega \cap P,
\]
then \( i(A, \Omega \cap P, P) = 0 \), where \( i \) denotes the fixed point index on \( P \).

Lemma 8 (see [75]). Let \( E \) be a real Banach space and \( P \) a cone on \( E \). Suppose that \( \Omega \subset E \) is a bounded open set with \( 0 \in \Omega \) and that \( A : \Omega \cap P \rightarrow P \) is a continuous compact operator. If
\[
\omega - \lambda A\omega \neq 0, \forall \lambda \in [0,1], \omega \in \partial \Omega \cap P,
\]
then \( i(A, \Omega \cap P, P) = 1 \).

Lemma 9 (see [75]). Let \( E \) be a partially ordered Banach space, and \( x_0, y_0 \in E \) with \( x_0 \leq y_0 \), \( D = [x_0, y_0] \).
Suppose that \( A : D \rightarrow E \) satisfies the following conditions:
(i) \( A \) is an increasing operator;
(ii) \( x_0 \leq Ax_0, y_0 \geq Ay_0 \), i.e., \( x_0 \) and \( y_0 \) is a subsolution and a supersolution of \( A \);
(iii) \( A \) is a completely continuous operator.
Then, \( A \) has the smallest fixed point \( x^* \) and the largest fixed point \( y^* \) in \([x_0, y_0]\), respectively. Moreover, \( x^* = \lim_{n \to \infty} A^n x_0 \) and \( y^* = \lim_{n \to \infty} A^n y_0 \).

3. Positive Solutions for (1)
For convenience, let
\[
\kappa_3 = \kappa_1 \int_1^\epsilon \frac{\mu(t)\phi(t)}{\tau} dt, \quad \kappa_4 = \kappa_2 \int_1^\epsilon \frac{\mu(t)\phi(t)}{t} dt.
\]
First, we list assumptions for our nonlinearity \( f \):
(H1) \( f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+) \),
Lemma 10. Suppose that (H1) holds. Then, \( A \mathbb{P} \subset P_0 \).

Proof. From Lemma 4(12), for \( u \in P \), we have

\[
(Au)(t) = \int_1^c G_\beta(t,s) \varphi_q \left( \int_1^c G_a(s,\tau) f(\tau,u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \leq \int_1^c \frac{(a + (1-a) (\log \xi)^{\delta-1} \Gamma(\beta)) (\beta - 1)}{(1 - a (\log \xi)^{\delta-1} \Gamma(\beta))} \eta(s) \varphi_q \left( \int_1^c G_a(s,\tau) f(\tau,u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s},
\]

and

\[
(Au)(t) \geq \int_1^c \Gamma(\beta) \phi(t) \eta(s) \frac{1 - a (\log \xi)^{\delta-1} + a \phi(\xi)}{1 - a (\log \xi)^{\delta-1} \Gamma(\beta)} \varphi_q \left( \int_1^c G_a(s,\tau) f(\tau,u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \geq \left( 1 - a (\log \xi)^{\delta-1} + a \phi(\xi) \right) \Gamma(\beta) \cdot \varphi_q \left( \int_1^c G_a(s,\tau) f(\tau,u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\]

For \( t \in [1,c] \), we have

\[
(Au)(t) \geq \left( 1 - a (\log \xi)^{\delta-1} + a \phi(\xi) \right) \Gamma(\beta) \cdot \varphi_q \left( \int_1^c G_a(s,\tau) f(\tau,u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\]
Therefore, \( (Au)(t) \geq \frac{(1-a)(\log z)^{p-1} + a \phi(t)) \Gamma(z)}{(a+(1-a)(\log z)^{p-1}) \Gamma(z)} \phi(t) \|Au\|, \) for \( t \in [1, e]. \) This completes the proof.  \qed

Let \( B_\rho = \{ u \in P : \|u\| < \rho \}, \) for \( \rho > 0. \)

**Theorem 1.** Suppose that (H1)–(H3) hold. Then, (1) has at least one positive solution.

**Proof.** Let \( S_1 = \{ u \in P : u = Au + \lambda \psi, \ \forall \lambda \geq 0 \}, \) where \( \psi \in P_0 \) is a fixed element. We prove that \( S_1 \) is bounded in \( P. \) If \( u \in S_1, \) then, from Lemma 10, we have \( u \in P_0, \) and \( u(t) \geq (Au)(t) \) for \( t \in [1, e]. \) Now, we consider two cases.  \qed

**Case 1.** Let \( p \geq 2. \) Then, we have \( \frac{1}{\rho^{p-1}} \in (0, 1]. \) From (H2), we have

\[
f \frac{1}{\rho^{p-1}} (t, z) + c \frac{1}{\rho^{p-1}} \geq (f(t, z) + c_1) \frac{1}{\rho^{p-1}} \geq (a_1 z^{p-1}) \frac{1}{\rho^{p-1}} = a_1 \frac{1}{\rho^{p-1}} z, \quad \text{for } (t, z) \in [1, e] \times \mathbb{R}^+.
\]

Consequently, from (19) and Lemma 6, we obtain

\[
u(t) \geq \int_1^e G_\beta(t, s) \left( \int_1^e G_\alpha(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
= \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t, s) \left( \int_1^e \Gamma(\alpha) G_\alpha(s, \tau) f(\tau, u(\tau)) d\tau \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
= \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t, s) \left( \int_1^e \Gamma(\alpha) G_\alpha(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
\geq \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t, s) \int_0^1 (\Gamma(\alpha) G_\alpha(s, e^x)) \frac{1}{p-1} f(\frac{1}{p-1}, e^x, u(e^x)) dx \frac{ds}{s}
\]

\[
\geq \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t, s) \int_0^1 (\Gamma(\alpha) G_\alpha(s, e^x)) \frac{1}{p-1} f(\frac{1}{p-1}, e^x, u(e^x)) dx \frac{ds}{s}
\]

\[
= \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e G_\beta(t, s) \int_1^e G_\alpha(s, \tau) f(\frac{1}{p-1}, \tau, u(\tau)) \frac{d\tau}{\tau} \frac{ds}{s}
\]

\[
\geq \kappa_1 \phi(t) \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e \mu(t) f(\frac{1}{p-1}, \tau, u(\tau)) \frac{d\tau}{\tau}
\]

\[
\geq \kappa_1 \phi(t) \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e \mu(t) \left( a \frac{1}{p-1} u(t) - c_1 \frac{1}{p-1} \right) \frac{d\tau}{\tau}
\]

Multiplying by \( \mu(t) \) on both sides of (21) and integrating over \([1, e],\) we obtain

\[
\int_1^e u(t) \mu(t) \frac{dt}{T} \geq \kappa_1 \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e \mu(t) \phi(t) \frac{dt}{T} \int_1^e \mu(t) \left( a \frac{1}{p-1} u(t) - c_1 \frac{1}{p-1} \right) \frac{dt}{T}
\]

\[
= \kappa_3 \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e \mu(t) \left( a \frac{1}{p-1} u(t) - c_1 \frac{1}{p-1} \right) \frac{dt}{T}.
\]

Solving this inequality, we have

\[
\int_1^e u(t) \mu(t) \frac{dt}{T} \leq \frac{\kappa_3 \Gamma^{-\frac{1}{p-1}}(\alpha) \int_1^e \mu(t) \frac{dt}{T}}{\kappa_3 a \frac{1}{p-1} \Gamma^{-\frac{1}{p-1}}(\alpha) - 1}.
\]
Note that, for $u \in P_0$, we get

$$\int_1^e \frac{k_1}{k_2} \phi(t) \|u\| \mu(t) \frac{dt}{t} \leq \frac{k_3 c_1^{\frac{1}{p}}}{k_3 a_1^{\frac{1}{p}}} \frac{p-2}{p-1} (a) \int_1^e \mu(t) \frac{dt}{t}, \quad \text{and} \quad \|u\| \leq \frac{k_3 c_1^{\frac{1}{p}}}{k_3 a_1^{\frac{1}{p}}} \frac{p-2}{p-1} (a) - 1.$$  

**Case 2.** Let $p \in (1, 2]$. Then, we have $p - 1 \in (0, 1]$. Note that $\frac{G_{\alpha}(t, s)}{s} \leq 1$, for $t, s \in [1, e]$, by (H2), (19) and Lemma 6 we have

$$u^{p-1}(t) \geq \left( \int_1^e G_{\beta}(t, s) \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{1}{s} d\tau \right) \frac{1}{s} - 1,$$

$$= \kappa_3^{p-1} \left( \int_1^e \frac{G_{\beta}(t, s)}{k_3} \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{1}{s} d\tau \right) \frac{1}{s} - 1,$$

$$= \kappa_3^{p-1} \left( \int_1^e \frac{G_{\beta}(t, e^\tau)}{k_3} \left( \int_1^e G_{\alpha}(e^\tau, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{1}{s} d\tau \right) \frac{1}{s} - 1,$$

$$\geq \kappa_3^{p-2} \phi(t) \int_1^e f(\tau, u(\tau)) \frac{d\tau}{\tau} \frac{1}{s} - 1,$$

$$\geq \kappa_1 \kappa_3^{p-2} \phi(t) \int_1^e f(\tau, u(\tau)) \frac{d\tau}{\tau} \frac{1}{s} - 1.$$

Multiplying by $\mu(t)$ on both sides of (22) and integrating over $[1, e]$, we conclude that

$$\int_1^e \mu(t) u^{p-1}(t) \frac{dt}{t} \geq \kappa_1 \kappa_3^{p-2} \int_1^e \mu(t) \phi(t) \frac{dt}{t} \int_1^e \mu(t) (a_1 u^{p-1}(t) - c_1) \frac{dt}{t}.$$  

Solving this inequality, we have

$$\int_1^e \mu(t) u^{p-1}(t) \frac{dt}{t} \leq \frac{\kappa_3^{p-1} c_1 \int_1^e \mu(t) \phi(t) \frac{dt}{t}}{\kappa_3^{p-1} a_1 - 1}.$$  

Noting that $u \in P_0$, we have

$$\|u\|^{p-1} \leq \frac{\kappa_3^{p-1} k_2^{p-1} \kappa_1^{p-1} c_1 k_4}{\kappa_3^{p-1} a_1 - 1} \left( \int_1^e \mu(t) \phi(t) \frac{dt}{t} \right)^{-1}.$$  

The above two cases imply that $S_1$ is bounded in $P$. Then, we can choose

$$R_1 = \begin{cases} \left( \frac{\kappa_3^{p-1} k_2^{p-1} \kappa_1^{p-1} c_1 k_4}{\kappa_3^{p-1} a_1 - 1} \left( \int_1^e \mu(t) \phi(t) \frac{dt}{t} \right)^{-1} \right)^{-1}, & 1 < p \leq 2, \\ \left( \frac{\kappa_3^{p-1} k_2^{p-1} \kappa_1^{p-1} c_1 k_4}{\kappa_3^{p-1} a_1 - 1} \left( \int_1^e \mu(t) \phi(t) \frac{dt}{t} \right)^{-1} \right)^{-1}, & p \geq 2, \end{cases}$$

such that

$$u \neq Au + \lambda \psi, \text{ for } u \in \partial B_{R_1} \cap P, \forall \lambda \geq 0.$$
As a result, Lemma 7 implies that
\[ i(A, B_{R_1} \cap P, P) = 0. \] (23)

For \( r_1 \) in (H3), we now prove that
\[ u \neq \lambda Au \text{ for } u \in \partial B_{r_1} \cap P, \forall \lambda \in [0, 1]. \] (24)

If this claim isn’t true, then there exist \( u \in \partial B_{r_1} \cap P \) and \( \lambda \in [0, 1] \) such that \( u = \lambda Au \), and \( u(t) \leq (Au)(t) \), for \( t \in [1, e] \). Now, we consider two cases.

**Case 1.** Let \( p \geq 2 \). Then, we have \( p - 1 \geq 1 \). From (20), (H3) and Lemma 6, we get
\[
\int_1^e \mu(t) u^{p-1}(t) \, dt \leq \kappa_3^{-p-1} \left( \int_1^e G_\beta(t, s) \left( \int_s^e G_a(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{1}{\tau} \, ds \right)^{p-1}.
\]

Multiplying by \( \mu(t) \) on both sides of (25) and integrating over \([1, e]\), we find
\[
\int_1^e \mu(t) u^{p-1}(t) \, dt \leq \kappa_3^{-p-2} a_2 \kappa_4 \int_1^e \mu(t) u^{p-1}(t) \, dt.
\]

This implies that
\[
\int_1^e \mu(t) u^{p-1}(t) \, dt = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e],
\]
since \( \mu(t) \neq 0 \), for \( t \in [1, e] \). This contradicts \( u \in \partial B_{r_1} \cap P, r_1 > 0 \).

**Case 2.** Let \( p \in (1, 2] \). Then, we have \( \frac{1}{p-1} \geq 1 \). From (20), (H3) and Lemma 6, we obtain
\[
u(t) \leq \int_1^e G_\beta(t, s) \left( \int_s^e G_a(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{1}{\tau} \, ds.
\]

Multiplying by \( \mu(t) \) on both sides of the preceding inequalities and integrating over \([1, e]\), we find
\[
\int_1^e \mu(t) u(t) \, dt \leq \Gamma^{p-2} (\alpha) \kappa_4 a_2^{-1} \int_1^e \mu(t) u(t) \, dt.
\]
Note that \( \mu(t) \not\equiv 0 \), for \( t \in [1, e] \), and this implies that
\[
\int_1^e \mu(t)u^p(t) \frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e].
\]
This contradicts \( u \in \partial B_{r_1} \cap P, r_1 > 0 \).
Combining the above two cases, we have that (24) holds. Then, from Lemma 8, we obtain
\[
i(A, B_{r_1} \cap P, P) = 1.
\]  \( \text{(27)} \)

Note that we can also take \( R_1 > r_1 \) such that (23) is still true. Thus, from (23) and (27), we have
\[
i(A, (B_{R_1} \setminus B_{r_1}) \cap P, P) = i(A, B_{r_1} \cap P, P) - i(A, B_{r_1} \cap P, P) = -1,
\]
and hence \( A \) has at least one fixed point in \((B_{R_1} \setminus B_{r_1}) \cap P\), i.e., (1) has at least one positive solution. This completes the proof.

**Theorem 2.** Suppose that (H1), and (H4)–(H5) hold. Then, (1) has at least one positive solution.

**Proof.** We can use similar methods as in Theorem 1 to provide the proof. We first prove that
\[
u \neq Au + \lambda \varphi, \text{ for } u \in \partial B_{r_2} \cap P, \forall \lambda \geq 0,
\]  \( \text{(28)} \)
where \( \varphi \in P \) is a given element, and \( r_2 \) is defined in (H4). Otherwise, there exist \( u \in \partial B_{r_2} \cap P \) and \( \lambda \geq 0 \) such that \( u = Au + \lambda \varphi \), and thus \( u(t) \geq (Au)(t) \), for \( t \in [1, e] \). Now, we consider two cases. \( \square \)

**Case 1.** Let \( p \geq 2 \). Then, we have \( \frac{1}{p-1} < (0, 1] \). Using (21) and (H4), we conclude
\[
u(t) \geq \kappa_1 \phi(t) \Gamma^{\frac{p-2}{p-1}}(a) \int_1^e \mu(\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \geq \kappa_1 \phi(t) \Gamma^{\frac{p-2}{p-1}}(a) \int_1^e \mu(\tau) a_3^\frac{1}{p-1} u(\tau) \frac{d\tau}{\tau}.
\]
Multiplying by \( \mu(t) \) on both sides of the preceding inequalities and integrating over \([1, e]\), we find
\[
\int_1^e \mu(t)u(t) \frac{dt}{t} \geq a_3^\frac{1}{p-1} \kappa_1 \Gamma^{\frac{p-2}{p-1}}(a) \int_1^e \phi(t) \mu(t) \frac{dt}{t} \int_1^e \mu(t)u(t) \frac{dt}{t}.
\]
This implies that
\[
\int_1^e \mu(t)u(t) \frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e],
\]

since \( \mu(t) \not\equiv 0 \), for \( t \in [1, e] \). This contradicts \( u \in \partial B_{r_2} \cap P, r_2 > 0 \).

**Case 2.** Let \( p \in (1, 2] \). Then, we have \( p - 1 \in (0, 1] \). Using (22) and (H4), we obtain
\[
u^p(t) \geq \kappa_1 \kappa_3^{p-2} \phi(t) \int_1^e \mu(\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \geq \kappa_1 \kappa_3^{p-2} \phi(t) \int_1^e \mu(\tau) a_3 u^{p-1}(\tau) \frac{d\tau}{\tau}.
\]
Multiplying by \( \mu(t) \) on both sides of the preceding inequalities and integrating over \([1, e]\), we find
\[
\int_1^e \mu(t)u^{p-1}(t) \frac{dt}{t} \geq a_3 \kappa_1 \kappa_3^{p-2} \int_1^e \phi(t) \mu(t) \frac{dt}{t} \int_1^e \mu(t)u^{p-1}(t) \frac{dt}{t}.
\]
This implies that
\[
\int_1^e \mu(t)u^{p-1}(t) \frac{dt}{t} = 0, \text{ and } u(t) \equiv 0, \text{ for } t \in [1, e],
\]
since \( \mu(t) \not\equiv 0 \), for \( t \in [1, e] \). This contradicts \( u \in \partial B_{r_2} \cap P, r_2 > 0 \).
As a result, we have that (28) holds, and Lemma 7 implies that

\[ i(A, B_{2} \cap P, P) = 0. \] (29)

Let \( S_{2} = \{ u \in P : u = \lambda Au, \forall \lambda \in [0, 1] \} \). Then, we claim that \( S_{2} \) is bounded in \( P \). Indeed, if \( u \in S_{2} \), then from Lemma 10 we have \( u \in P_{0} \) and \( u(t) \leq (Au)(t) \), for \( t \in [1, e] \). Now, we consider two cases.

**Case 1.** Let \( p \geq 2 \). Then, we have \( p - 1 \geq 1 \). Using (25) and (H5), we have

\[ u^{p-1}(t) \leq \kappa_{3}^{p-2} k_{2} \int_{1}^{t} \mu(\tau)(a_{4}u^{p-1}(\tau) + c_{2}) \frac{d\tau}{\tau}. \]

Multiplying by \( \mu(t) \) on both sides of the preceding inequalities and integrating over \([1, e]\), we find

\[
\int_{1}^{e} \mu(t)u^{p-1}(t) \frac{dt}{t} \leq \kappa_{3}^{p-2} k_{4} \int_{1}^{e} \mu(t)(a_{4}u^{p-1}(t) + c_{2}) \frac{dt}{t}.
\]

Solving this inequality, we have

\[
\int_{1}^{e} \mu(t)u^{p-1}(t) \frac{dt}{t} \leq \frac{\kappa_{3}^{p-2} c_{2} k_{4}}{1 - \kappa_{3}^{p-2} a_{4} k_{4}} \left( \int_{1}^{e} \mu(t) \phi^{p-1}(t) \frac{dt}{t} \right)^{-1}.
\]

Note that \( u \in P_{0} \), and we have

\[
\|u\|^{p-1} \leq \frac{\kappa_{3}^{p-2} c_{2} k_{4}}{1 - \kappa_{3}^{p-2} a_{4} k_{4}} \left( \int_{1}^{e} \mu(t) \phi^{p-1}(t) \frac{dt}{t} \right)^{-1}.
\]

**Case 2.** Let \( p \in (1, 2] \). Then, we have \( \frac{1}{p-1} \geq 1 \). Using (26) and (H5), we obtain

\[
u(t) \leq \Gamma^{p-2}(a) k_{2} \int_{1}^{e} \mu(\tau)(a_{4}u^{p-1}(\tau) + c_{2}) \frac{1}{\tau^{1+1/p}} d\tau \leq \Gamma^{p-2}(a) 2^{p-2} k_{2} \int_{1}^{e} \mu(t)(a_{4}^{1/p} u(t) + c_{2}^{1/p}) \frac{1}{\tau^{1+1/p}} dt.
\]

Multiplying by \( \mu(t) \) on both sides of the preceding inequalities and integrating over \([1, e]\), we find

\[
\int_{1}^{e} \mu(t)u(t) \frac{dt}{t} \leq \Gamma^{p-2}(a) 2^{p-2} k_{4} \int_{1}^{e} \mu(t)(a_{4}^{1/p} u(t) + c_{2}^{1/p}) \frac{1}{\tau^{1+1/p}} dt.
\]

Solving this inequality, we have

\[
\int_{1}^{e} \mu(t)u(t) \frac{dt}{t} \leq \frac{\Gamma^{p-2}(a) 2^{p-2} c_{2}^{1/p} k_{4}}{k_{2}(1 - \Gamma^{p-2}(a) 2^{p-2} a_{4}^{1/p} k_{4})}.
\]

Noting that \( u \in P_{0} \), we have

\[
\|u\| \leq \frac{\Gamma^{p-2}(a) 2^{p-2} c_{2}^{1/p} k_{4}}{k_{3}(1 - \Gamma^{p-2}(a) 2^{p-2} a_{4}^{1/p} k_{4})}.
\]
Combining the above two cases, we have proved that $S_2$ is bounded in $P$. Then, we can choose $R_2 > r_2$ and

$$R_2 > \left\{ \begin{array}{ll}
\frac{(p-2)\frac{p-1}{p} \frac{1}{2} \prod_{k=1}^{4} g_k^p}{k_1^{1-p} \prod_{k=1}^{4} g_k^{1-p}} & , \quad 1 < p \leq 2,
\frac{(p-2)\frac{p-1}{p} \frac{1}{2} \prod_{k=1}^{4} g_k^p}{k_1^{1-p} \prod_{k=1}^{4} g_k^{1-p}} & , \quad p \geq 2,
\end{array} \right. $$

such that

$$u \neq \lambda Au, \quad u \in \partial R_2 \cap P, \forall \lambda \in [0, 1].$$

(30)

Then, from Lemma 8, we have

$$i(A, B_{R_2} \cap P, P) = 1.$$  

(31)

Thus, from (29) and (31), we have

$$i(A, (B_{R_2} \setminus \overline{B}_r) \cap P, P) = i(A, B_{R_2} \cap P, P) - i(A, B_r \cap P, P) = 1,$$

and hence $A$ has at least one fixed point in $(B_{R_2} \setminus \overline{B}_r) \cap P$, i.e., (1) has at least one positive solution. This completes the proof.

In what follows, we consider the uniqueness of positive solutions for (1) with the boundary conditions $D^\beta u(1) = D^\beta u(e) = 0, u(1) = u'(1) = u(e) = 0$. This problem is equivalent to the Hammerstein type integral equation

$$u(t) = \int_1^e G_\beta(t,s) \frac{q(s) \left( \int_1^e G_\alpha(s, \tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right)}{s} ds,$$

(32)

where $G_\beta(t,s) = G_{1,\beta}(t,s)$ for $t, s \in [1, e]$. Note that here we still use the operator $A$ as in (16).

**Lemma 11.** Let $w_0(t) = \int_1^t G_\beta(t,s) \frac{ds}{s}$ for $t \in [1, e]$. Then, for all nonnegative functions $w \in C[1, e] \neq 0$, there exist two positive $a_w, b_w$ such that $a_w w_0(t) \leq \int_1^e G_\beta(t,s) w(s) \frac{ds}{s} \leq b_w w_0(t)$, for $t \in [1, e]$.

**Proof.** We first calculate $w_0$. From (14), we have

$$\int_1^e G_\beta(t,s) \frac{ds}{s} = \frac{1}{\Gamma(\beta)} \int_1^t \left[ (\log t)^{\beta-1} (1 - \log s)^{\beta-1} - (\log t - \log s)^{\beta-1} \right] \frac{ds}{s}$$

$$+ \frac{1}{\Gamma(\beta)} \int_t^e (\log t)^{\beta-1} (1 - \log s)^{\beta-1} \frac{ds}{s}$$

$$= \frac{1}{\Gamma(\beta)} \int_1^t (\log t)^{\beta-1} (1 - \log s)^{\beta-1} \frac{ds}{s} - \frac{1}{\Gamma(\beta)} \int_1^e (\log t - \log s)^{\beta-1} \frac{ds}{s}$$

$$= \frac{(\log t)^{\beta-1} (1 - \log t)}{\beta \Gamma(\beta)}.$$

Using (17) and (18), we have

$$\int_1^e \Gamma(\beta) \phi(t) \eta(s) w(s) \frac{ds}{s} \leq \int_1^e G_\beta(t,s) w(s) \frac{ds}{s} \leq \int_1^e (\beta - 1) \phi(t) w(s) \frac{ds}{s}.$$

Therefore, let $a_w = \beta \Gamma(\beta) \int_1^e \eta(s) w(s) \frac{ds}{s}$, and $b_w = \beta (\beta - 1) \int_1^e w(s) \frac{ds}{s}$, then, we have that (33) holds. This completes the proof. 

**Theorem 3.** Suppose that (H1), (H6)–(H7) hold. Then, (1) has a unique positive solution.
Proof. Note that (H7) implies that $A$ is an increasing operator, and 0 isn’t a fixed point for $A$. Next, we shall prove that $A$ has a subsolution and a supersolution. Let

$$
\xi(t) = \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \rho(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s},
$$

where

$$
\rho(t) = \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \text{ for } t \in [1,e].
$$

From Lemma 11, there exist $a_\rho > 0, b_\rho > 0$ such that

$$
a_\rho \rho(t) \leq \xi(t) \leq b_\rho \rho(t), \text{ for } t \in [1,e].
$$

Take $\xi_1(t) = \delta_1 \xi(t), \xi_2(t) = \delta_2 \xi(t)$, where $0 < \delta_1 < \min \left\{ \frac{1}{a_\rho}, \frac{1}{b_\rho} \right\}, \delta_2 > \max \left\{ \frac{1}{a_\rho}, \frac{1}{b_\rho} \right\}$. Then, we have

$$
\begin{align*}
(A\xi_1)(t) &= \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \xi_1(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&= \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \delta_1 \xi(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&= \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f\left( \tau, \frac{\delta_1 \xi(\tau)}{\rho(\tau)} \rho(\tau) \right) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\geq \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) \left( \frac{\delta_1 \xi(\tau)}{\rho(\tau)} \right)^{k(p-1)} f(\tau, \rho(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\geq (\delta_1 a_\rho)^k \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \rho(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\geq \delta_1 \xi(t),
\end{align*}
$$

and

$$
A\xi_1 \geq \xi_1, \text{ i.e., } \xi_1 \text{ is a subsolution of } A.
$$

In addition, we have

$$
\begin{align*}
(A\xi_2)(t) &= \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \xi_2(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&= \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \delta_2 \xi_2(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\leq \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) \delta_2^{p-1} \delta_2^{1-p} f(\tau, \delta_2 \xi_2(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\leq \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) \delta_2^{p-1} f(\tau, \delta_2 \xi_2(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\leq \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \rho(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\leq \delta_2 \int_1^t G_\beta(t,s) \varphi_q \left( \int_1^t G_a(s,\tau) f(\tau, \rho(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s},
\end{align*}
$$

and

$$
A\xi_2 \leq \xi_2, \text{ i.e., } \xi_2 \text{ is a supersolution of } A.
$$
As a result, from Lemma 9, A has the smallest fixed point \( u^* \) and the largest fixed point \( u^* \) in \([\xi_1, \xi_2]\), respectively. Moreover, \( u_1 = \lim_{n \to \infty} A^n \xi_1 \) and \( u^* = \lim_{n \to \infty} A^n \xi_2 \).

Next, we claim that \( u_1(t) = u^*(t) \), for \( t \in [1, e] \). We only prove that \( u_1(t) \geq u^*(t) \). Note that they are fixed points for \( A \), so

\[
\begin{align*}
  u_1(t) &= \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u_1(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, \\
  u^*(t) &= \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u^*(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{align*}
\]

Then, from Lemma 11, there exists \( b_1 \geq a_1 (i = 1, 2) \) such that

\[
a_1w_0 \leq u_1 \leq b_1w_0, \quad a_2w_0 \leq u^* \leq b_2w_0.
\]

Hence, \( u_1 \geq \frac{a_1}{b_1} u^* \). Let \( \mu_0 := \sup \{ \mu > 0 : u_1 \geq \mu u^* \} \). Then, \( \mu_0 > 0 \), and \( u_1 \geq \mu_0 u^* \). Next, we claim that \( \mu_0 \geq 1 \). If it is not true, then \( \mu_0 \in (0, 1) \). Using (H6), (H7), we have

\[
\begin{align*}
  u_1(t) &= \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u_1(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
  &\geq \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, \mu_0 u^*(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
  &\geq \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) \mu_0^{k(p-1)} f(\tau, u^*(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
  &= \mu_0^k \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u^*(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}.
\end{align*}
\]

Let

\[
g(t) = \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, \mu_0 u^*(\tau)) \frac{d\tau}{\tau} \right) - \mu_0^k \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u^*(\tau)) \frac{d\tau}{\tau} \right).
\]

Then, from (H6) and Lemma 11, we have

\[
a_2w_0(t) \geq \int_1^e G_{\beta}(t, s) g(s) \frac{ds}{s} \leq b_2w_0(t).
\]

Consequently,

\[
\begin{align*}
  u_1(t) &\geq \int_1^e G_{\beta}(t, s) g(s) \frac{ds}{s} + \mu_0^k \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u^*(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
  &\geq \frac{a_3}{b_2} u^*(t) + \mu_0^k u^*(t) \\
  &\geq \left( \frac{a_3}{b_2} + \mu_0 \right) u^*(t).
\end{align*}
\]

This contradicts the definition of \( \mu_0 \), and \( u_1(t) \geq \mu_0 u^*(t) \geq u^*(t) \). Therefore, \( A \) has a unique positive fixed point in \([\xi_1, \xi_2] \), and (1) has also a unique positive solution in \([\xi_1, \xi_2] \). This completes the proof. \( \square \)

**Theorem 4.** Suppose all the assumptions in Theorem 3 hold. Let \( \tilde{u} \) is a unique positive solution in \([\xi_1, \xi_2] \). Then, for any \( u_0 \in [\xi_1, \xi_2] \) with \( f(t, u_0(t)) \neq 0 \), the sequence

\[
u_n(t) = \int_1^e G_{\beta}(t, s) \varphi_q \left( \int_1^e G_{\alpha}(s, \tau) f(\tau, u_{n-1}(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s}, n = 1, 2, ...
\]
uniformly converges to $\tilde{u}(t)$, for $t \in [1, e]$. 

**Proof.** From Theorem 3, we have $\tilde{u} = \lim_{n \to \infty} A^n \xi_1 = \lim_{n \to \infty} A^n \xi_2$. Note that $A$ is increasing, so, if $u_0 \in [\xi_1, \xi_2]$, we have 

$$A^n \xi_1 \leq A^n u_0 \leq A^n \xi_2, \quad \forall n \in \mathbb{N}.$$ 

This implies that $A^n u_0 \to \tilde{u}$ as $n \to \infty$. From the definition of $A$, we have $u_n(t) = (Au_{n-1})(t) = A(Au_{n-2})(t) = \cdots = (A^n u_0)(t)$, and thus $u_n(t) \to \tilde{u}(t)$ uniformly on $t \in [1, e]$. This completes the proof. \qed

### 4. Conclusions

In this paper we investigate the existence and uniqueness of positive solutions for the Hadamard fractional $p$-Laplacian three-point boundary value problem (1). We first establish some relations from the corresponding problem without the $p$-Laplacian operator, and use some $(p-1)$—superlinearly and $(p-1)$—sublinearly conditions for the nonlinearity to obtain positive solutions to problem (1). After, using an increasing operator fixed-point theorem, we obtain the unique solution to problem (1), and establish uniform converged sequences for this solution.

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