Bass-Ihara Zeta functions for non-uniform tree lattices

Antonius Deitmar

Abstract: It is shown that the Euler product giving the Bass-Ihara zeta function of a finite graph also converges in the case of a graph with cusps. It turns out to be a rational function, generally with zeros and poles, in contrast to the compact case. The determinant formulas of Bass and Ihara hold true if one defines the determinant as limit of all finite principal minors.

Contents

1 Cuspidal tree lattices 5
2 The Bass-Ihara zeta function 7
3 Determinant class operators 8
4 The zeta function as a determinant 12
5 The Ihara formula 16
6 L-functions 18
7 An arithmetic example 19
Introduction

The Ihara zeta function, introduced by Yasutaka Ihara in the nineteen-sixties \cite{iha66a,iha66b} is a zeta function counting prime elements in discrete subgroups of rank one $p$-adic groups. It can be interpreted as a geometric zeta function for the corresponding finite graph, which is a quotient of the Bruhat-Tits building attached to the $p$-adic group \cite{ser03}. Over time it has been generalized in stages by Hashimoto, Bass and Sunada \cite{hh89,has89,has90,has92,has93,bas92,ks00}. Comparisons with number theory can be found in the papers of Stark and Terras \cite{st96,st00,ts07}. This zeta function is defined as the product

$$Z(u) = \prod_p (1 - u^{l(p)})^{-1},$$

where $p$ runs through the set of prime cycles in a finite graph $X$. The product, being infinite in general, converges to a rational function, actually the inverse of a polynomial, and satisfies the famous \textit{Ihara determinant formula}

$$Z(u)^{-1} = \det(1 - uA + u^2Q)(1 - u^2)^{-\chi},$$

where $A$ is the adjacency operator of the graph, $Q+1$ is the valency operator and $\chi$ is the Euler number of the graph. One of the most remarkable features of the Ihara formula is, that in the case of $X = \Gamma \backslash Y$, where $Y$ is the Bruhat-Tits building of a $p$-adic group $G$ and $\Gamma$ is a cocompact arithmetic subgroup of $G$, then the right hand side of the Ihara formula equals the non-trivial part of the Hasse-Weil zeta function of the Shimura curve attached to $\Gamma$, thus establishing the only known link between geometric and arithmetic zeta- or L-functions.

In recent years, several authors have asked for a generalization of these zeta functions to infinite graphs. The paper \cite{sch99} considers the arithmetic situation, where the graph is the union of a compact part and finitely many cusps. The zeta function is defined by plainly ignoring the cusps, so indeed, it is a zeta function of a finite graph. In \cite{cms01} and \cite{cla09}, the zeta function of a finite graph is generalized to an $L^2$-zeta function where a finite trace on a group von-Neumann algebra is used to define a determinant. In \cite{g04}, an infinite graph is approximated by finite ones and the zeta function is defined as a suitable limit. In \cite{gil08a,gil08b}, a relative version of the zeta function is considered on an infinite graph which is acted upon by a group with finite quotient. In \cite{CJK}, finally, the idea of the Ihara zeta
function is extended to infinite graphs by counting not all cycles, but only those which pass through a given point.

In the present paper it is shown that the original definition by Bass actually yields a convergent Euler product in the case of graphs of “Lie type”, i.e., quotients of Bruhat-Tits buildings by lattices in rank one $p$-adic groups. These graphs are “cuspidal” in the sense that they consist of a compact part and a finite number of cusps. It is this combinatorial description that is used in the text. It turns out that these zeta functions actually are rational functions, which in the infinite-dimensional context is a quite surprising result. In order to derive determinant expressions of the zeta function we introduce the notion of operators of “determinant class”, which means that the net of all finite principal minors converges, the limit being called the determinant of the operator. It turns out that with this notion, the classical determinant formulas of Bass and Ihara actually hold without change.

Let us now introduce the geometric idea behind this approach. The classical predecessor of Ihara’s zeta function is the zeta function of Selberg, which counts closed geodesics in compact Riemann surfaces. The latter generalises to non-compact surfaces as long as they have finite hyperbolic volume. In this case such a surface is a union of a compact set and finitely many cusp sectors and the typical behavior of a closed geodesic is that it winds around a cusp, going out for a while and then winds back to the compact core. Below we have drawn this in the case of the quotient of the upper half plane by $\text{SL}_2(\mathbb{Z})$. The first picture shows how a geodesic in the universal covering, runs through translates of the standard fundamental domain.

\begin{center}
\includegraphics[width=0.5\textwidth]{FundamentalDomain.png}
\end{center}

Instead, one can also leave the fundamental domain fixed and replace the geodesic by the union of its translates, as in the second picture.
Finally, to understand the behavior of the geodesic in the quotient, one only looks at what happens in the fundamental domain, where one now nicely sees, how, after identifying the left and right boundary of the domain, the geodesic winds up and down again.

The analogue of the upper half plane in the $p$-adic setting is the Bruhat-Tits tree $Y$ of a rank one group, together with a lattice $\Gamma$ acting on the tree. In this setting, a cusp sector in the quotient $X = \Gamma \backslash Y$ is an infinite ray emanating from a compact core. The following picture shows an example of a graph with 4 cusps, the compact core is not drawn.
Now winding up and down along a cusp means that one considers cycles which move out on the ray, then return once and go back to the compact core. This corresponds nicely to what happens to a “geodesic” in the tree when projected down to the quotient graph. Note that non-compact quotients with cusps can only occur when the group $\Gamma$ has torsion, so $\Gamma$ is not the fundamental group of the quotient $X$ which is why one cannot consider the quotient alone but has to take the action of $\Gamma$ on $Y$ into the picture. This is the idea behind Hyman Bass’s approach to the zeta function [Bas92].

In the following paper we decided to present the case of the untwisted zeta function first, as it is easier to understand. The twisted version, or $L$-function case is then presented in a separate section, where only the necessary changes to the original concepts are given. We find it easier to read this way and hope the reader will think so, too.

1 Cuspidal tree lattices

A tree lattice is a group $\Gamma$ together with an action on a tree $Y$ such that all stabilizer groups $\Gamma_e$ of edges $e$ are finite and such that

$$\sum_e \frac{1}{|\Gamma_e|} < \infty.$$ 

As an additional condition, we always assume that the tree $Y$ be uniform, i.e., the quotient graph $G\setminus Y$ is finite, where $G = \text{Aut}(Y)$ is the automorphism group of the tree $Y$. The compact-open topology makes $G$ a totally disconnected locally compact group. The action of $\Gamma$ on $Y$ defines a group homomorphism $\alpha : \Gamma \to G$ and $\Gamma$ is a tree lattice if and only if $\alpha$ has finite kernel and the image $\alpha(\Gamma)$ is a lattice in the group $G$, i.e., $\alpha(\Gamma)$ is a discrete subgroup such that on $G/\Gamma$ there exists a $G$-invariant Radon-measure of finite positive volume.

By replacing $\Gamma$ with a subgroup of index two if necessary, we can assume that $\Gamma$ acts orientation preservingly on $Y$. We will always assume this, as it simplifies the presentation. In this way oriented edges of the quotient graph $X = \Gamma\setminus Y$ will be $\Gamma$-orbits of oriented edges on $Y$.

A tree lattice of Lie type is a lattice in a semisimple $p$-adic group $H$ of rank one, acting on the Bruhat-Tits building $Y$ of $H$, see [BL01].

Let $Y$ be a uniform tree. A path in $Y$ is a sequence $p = (e_1, e_2, \ldots, e_n)$
of oriented edges such that the end point $t(e_j)$ of $e_j$ is the starting point $o(e_{j+1})$ of $e_{j+1}$ for each $j \in \{1, \ldots, n-1\}$. We say that the path is reduced, if $e_{j+1} \neq e_j^{-1}$ for every $1 \leq j \leq n-1$, where $e^{-1}$ denotes the reverse of the oriented edge $e$. A ray in $Y$ is an infinite reduced path $r = (r_1, r_2, \ldots)$. Two rays $r, s$ are equivalent, if they join at some point, i.e., if there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $r_{j+k} = s_j$ holds for all $j \geq N$. An equivalence class of rays is called an end of $Y$. The set $\partial Y$ of all ends is called the boundary, or visibility boundary of $Y$. For a given vertex $x_0$ and any point $c \in \partial Y$ there exists a unique ray in $c$ which starts at the point $x_0$. So the visibility boundary is what you see, when you look around from any point in the tree.

Let $c \in \partial Y$ be a boundary point. Two vertices $x, y$ of $Y$ lie in the same horocycle with respect to $c$ if there are rays $r, s \in c$ such that $r_0 = x$, $s_0 = y$ and an $n \in \mathbb{N}_0$ such that $r_n = s_n$. So a horocycle is the set of all vertices which have the “same distance” to the boundary point $c$. Let $P_c$ be the stabilizer of the boundary point $c$ in the automorphism group $G = \text{Aut}(Y)$. Let $N_c$ be the subgroup of $P_c$ of all elements that stabilize a horocycle (and hence all horocycles) with respect to $c$.

The point $c$ is called a cusp of the tree lattice $\Gamma$, if $\Gamma \cap P_c = \Gamma \cap N_c$ and there exists a horocycle $H$ on which $\Gamma \cap N_c$ acts transitively. It will then act transitively on every horocycle which is nearer to $c$ than $H$. In this case, any ray in $c$, or rather its image in $X = \Gamma \setminus Y$ is called a cusp section of $\Gamma$. Let $c$ be a ray in $Y$ giving a cusp section in $\Gamma$. Recall the valency of a vertex $x$ is the number of edges ending at $x$. The valency of the vertices in the ray $c$ can generally behave irregularly. We say that $c$, and so the cusp is periodic, if the sequence of valencies is eventually periodic, i.e., if the vertices of $c$ are $(x_1, x_2, \ldots)$ then $c$ is periodic if there exists $N \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\text{val}(x_n) = \text{val}(x_{n+k})$ holds for all $n \geq N$, where $\text{val}(x)$ is the valency of the vertex $x$. The smallest possible such $k$ is called the period of the cusp. If $Y$ is the Bruhat-Tits tree of a semisimple $p$-adic group, then every cusp is periodic of period one or two [Lub91].

We say that the tree lattice $\Gamma$ is cuspidal, if $X = \Gamma \setminus Y$ is the union of a finite graph and finitely many periodic cusp sections. Any tree lattice of Lie-type is cuspidal.
2 The Bass-Ihara zeta function

Let $\Gamma$ be a cuspidal tree lattice of the uniform tree $Y$ and let $X = \Gamma \backslash Y$ be the quotient graph. A path $p = (e_1, \ldots, e_n)$ in $X$ is called closed, if $t(e_n) = o(e_1)$. Then the shifted path $\tau p = (e_2, \ldots, e_n, e_1)$ is closed again and the shift induces an equivalence relation on the set of closed paths, whereby two closed paths are equivalent if one is obtained from the other by finitely many shifts. A cycle is an equivalence class of closed paths. If $c$ is a cycle, any power $c^m, n \in \mathbb{N}$, which you obtain by running the same path for $n$ times, is again a cycle and a given cycle $c_0$ is called a prime cycle, if it is not a power of a shorter one. For every given cycle, there exists a unique prime cycle $c_0$ such that $c = c_0^m$ for some $m \in \mathbb{N}$. The number $m = m(c)$ is called the multiplicity of $c$. A path $p = (e_1, \ldots, e_n)$ is called reduced, if $e_{j+1} \neq e_j^{-1}$ holds for every $1 \leq j \leq n - 1$. A cycle is called reduced if it consists of reduced paths only.

For an edge $e$ of $Y$ we set
\[ w(e) = |\Gamma_{o(e)} e|, \]
this is the cardinality of the $\Gamma_{o(e)}$-orbit of $e$. For an edge $e$ of $X = \Gamma \backslash Y$ we write $w(e) = w(\tilde{e})$, where $\tilde{e}$ is any preimage of $e$ in $Y$. For two consecutive edges $e, e'$, i.e., if $t(e) = o(e')$, we write
\[ w(e, e') = \begin{cases} w(e') & e' \neq e^{-1}, \\ w(e') - 1 & e' = e^{-1}. \end{cases} \]

For a closed path $p = (e_1, \ldots, e_n)$ let
\[ w(p) = \prod_{j \mod n} w(e_j, e_{j+1}). \]

**Definition 2.1.** The Bass-Ihara zeta function for the tree lattice $\Gamma$ is defined to be
\[ Z(u) = \prod_c \left(1 - w(c)u^{l(c)}\right)^{-1}, \]
where the product runs over all prime cycles and $l(c)$ is the length of the cycle $c$. In [Bas92] it is shown that this in general infinite product converges for $|u|$ small to a rational function if $X$ is compact. We will now show the same in the cuspidal case.
Note the special case of all stabilizer groups being trivial. In this case one gets
\[ Z(u) = \prod_{c} \left(1 - u^{l(c)}\right)^{-1}, \]
where now the product runs over reduced prime cycles only.

**Theorem 2.2.** Suppose that the tree lattice \( \Gamma \) is cuspidal. Then the product \( Z(u) \) converges for small \( |u| \) to a rational function.

The proof requires a new form of determinant for operators on infinite dimensional spaces, which is given in the next section.

### 3 Determinant class operators

For a given set \( I \) consider the formal complex vector spaces
\[ P = P(I) = \prod_{i \in I} \mathbb{C}i, \quad S = S(I) = \bigoplus_{i \in I} \mathbb{C}i. \]

We shall denote elements of these spaces as formal sums \( \sum_{i \in I} a_i i \), where in the second case the sums are finite. For a linear operator \( A : S \to P \) and any finite subset \( F \subset I \) we define the operator \( A_F \) as the composition
\[ S(F) \hookrightarrow S(I) \xrightarrow{A} P(I) \to P(F). \]

**Definition 3.1.** We say that \( A \) is of **determinant class**, if the limit of all principal minors,
\[ \det(A) \overset{\text{def}}{=} \lim_{F} \det(A_F) \]
exists.

Define a pairing \( \langle , \rangle : P \times S \to \mathbb{C} \) by
\[ \left\langle \sum_{i \in I} a_i i, \sum_{i \in I} b_i i \right\rangle = \sum_{i \in I} a_i b_i. \]

A permutation \( \sigma : I \to I \) is called **finite**, if \( \sigma(i) = i \) outside a finite set \( F \). In this case we write \( \text{sgn}(\sigma) \) for the sign of the permutation \( \sigma|_F \). It does not depend on the choice of \( F \).
Lemma 3.2 (Computation of the determinant). Let \( A : S \to P \) be a linear operator such that

\[
\sum_{i \in I} |\log \langle A_i, i \rangle| < \infty \quad \text{and} \quad \sum_{\sigma \text{ finite}} \left| \prod_{i \in I} \langle A_i, \sigma_i \rangle \right| < \infty.
\]

Then \( A \) is of determinant class and we have

\[
\det(A) = \sum_{\sigma \text{ finite}} \text{sgn}(\sigma) \prod_{i \in I} \langle A_i, \sigma_i \rangle.
\]

Proof. Suppose that \( A \) satisfies the conditions of the lemma. Then

\[
\sum_{\sigma} \text{sgn}(\sigma) \prod_{i \in I} \langle A_i, \sigma_i \rangle = \lim_F \left( \sum_{\sigma \in \text{Per}(F)} \text{sgn}(\sigma) \prod_{i \in F} \langle A_i, \sigma_i \rangle \prod_{i \in I \setminus F} \langle A_i, i \rangle \right) = \det(A_F) \to 1.
\]

The second factor tends to one, so we get the claim. \( \square \)

Examples 3.3.

- If \( I \) is finite, every operator is of determinant class and this formula gives the usual determinant.

- If the set \( I \) is an orthonormal basis of a Hilbert space \( H \) on which \( T : H \to H \) is a linear operator, then \( T \) gives rise to an operator in the algebraic sense above, again written \( T \) and if \( T \) is of trace class, then \( 1 - T \) is of determinant class and the determinant coincides with the Fredholm determinant

\[
\det(1 - T) = \sum_{k=0}^{\infty} (-1)^k \text{tr}^k T.
\]

- The operators we consider here can not generally be composed. There are, however, two classes of operators which allow composition. Firstly, an operator \( A \) as above is called a finite column operator, if it maps
\( S(I) \) to \( S(I) \subset P(I) \). The name derives from the fact that the corresponding matrix indeed has finite columns, i.e., for every \( i \in I \) the set \( \{ j \in I : \langle A_i, j \rangle \neq 0 \} \) is finite.

If, on the other hand, for each \( j \in I \) the set \( \{ i \in I : \langle A_i, j \rangle \neq 0 \} \) is finite, we call \( A \) a finite row operator. A finite row operator possesses a canonical extension to a linear operator \( P \to P \).

- Let \( I = \mathbb{N} \), then any operator is given by an infinite matrix. If this matrix is upper triangular with ones on the diagonal, the operator will be of determinant class. The same holds true for lower triangular matrices. The product of a lower triangular times an upper triangular will, however, not always be of determinant class. This means that the determinant class is not closed under multiplication.

For an operator \( T : S \to P \) we say that \( T \) is traceable, if \( \sum_{i \in I} | \langle T_i, i \rangle | < \infty \).

In the case that \( T \) is traceable, we define its trace by

\[
\text{Tr}(T) = \sum_{i \in I} \langle T_i, i \rangle.
\]

**Lemma 3.4.** Suppose that \( A, B \) are operators such that \( A \) has finite rows or \( B \) has finite columns, so that the product \( AB \) exists. Assume further that \( A, B \) and \( AB \) are of determinant class. Then

\[
\det(AB) = \det(A) \det(B).
\]

**Proof.** Viewed as infinite matrices, we have

\[
\langle AB_j, i \rangle = (AB)_{i,j} = \sum_k A_{i,k}B_{k,j}.
\]

Under either condition this sum is finite. In the rest of the proof we assume that \( A \) is a finite row operator, the other case being similar. For a finite set \( E \subset I \) we write \( A^E \) for the operator given by \( A_{i,k}^E = \delta_{i,k} \) unless \( i, k \) are both in \( E \), in which latter case we have \( A_{i,j}^E = A_{i,j} \). Then \( A^E \) has finite rows and columns and \( \det(A^E) \) tends to \( \det(A) \) as \( E \to I \). Now if \( F \supset E \), then one gets \( (A^E B)_F = A^E_B F \) and so

\[
\det(A^E B) = \lim_{F} \det(A^E_B F) = \det(A^E) \det(B).
\]

The right hand side tends to \( \det(A) \det(B) \) as \( E \to I \). The fact that \( A \) is of finite rows, implies that for every finite set \( F \subset I \) there exists a finite set \( E \) with \( F \subset E \subset I \) such that \( (A^E B)_F = (A)_F \). This implies that \( \det(A^E B) \to \det(AB) \) as \( E \to I \) and the lemma is proven. \( \square \)
Connectedness

Let $A : S(I) \rightarrow P(I)$ be an operator. A subset $F \subset I$ is called $A$-connected, if for each disjoint decomposition $F = E \cup H$ such that the resulting decomposition $S(F) = S(E) \oplus S(H)$ is stable under $A_F$, one has $E = \emptyset$ or $H = \emptyset$.

**Definition 3.5.** The operator $A$ is of connected determinant class, if $I$ is $A$-connected and $\lim_F \det(A_F)$ exists where the limit is taken over all $A$-connected finite subsets $F \subset I$. If this is the case, we still write

$$\det(A) = \lim_F \det(A_F),$$

where the limit is taken over connected sets $F$ only.

**Example 3.6.** Let $I$ be the vertex set of a graph $X$ and let $A$ be the adjacency operator, i.e.,

$$Ax = \sum_{x'} x',$$

where the sum runs over all neighbors $x'$ of $x$. Then for every subset $F \subset I$ we write $X_F$ for the full subgraph of $X$ with vertex set $F$. One has

$$X_F \text{ is connected } \iff F \text{ is } A\text{-connected}.$$

**Proposition 3.7.** If $I$ is $A$-connected and $A$ is of determinant class, then $A$ is of connected determinant class. There are operators, which are of connected determinant class, but not of determinant class.

**Proof.** In order to prove the first statement, we need to show that $A$-connected finite sets are cofinal in the set of all finite subsets of $I$. So we need to show that each finite subset $F \subset I$ there exists an $A$-connected finite subset $C \subset I$ with $F \subset C$. For this let $X$ be the graph with vertex set $I$, where $i, j \in I$ are connected if $| \langle Ai, j \rangle | + | \langle Aj, i \rangle |$ is non-zero. For $F \subset I$ let $X_F$ be the full subgraph with vertex set $F$. Then $F$ is $A$-connected if and only if the graph $X_F$ is connected. So the assertion boils down to the fact that in a connected graph each finite set of vertices is contained in a connected finite subgraph.

An example of an operator which is of connected determinant class, but not of determinant class, will be given in Section 5. \hfill $\square$
4 The zeta function as a determinant

We now apply the theory of determinant class operators to give a proof of Theorem 2.2. So we assume that \( \Gamma \) is a cuspidal tree lattice acting on the tree \( Y \). We write \( X = \Gamma \backslash Y \) for the quotient graph. Let \( I = \text{OE}(X) \) be the set of all oriented edges of \( X \) and define the operator \( T : S(I) \to S(I) \) by

\[
Te = \sum_{e'} w(e, e')e',
\]

where the sum runs over all edges \( e' \) with \( o(e') = t(e) \). Note that \( T \) is exactly the pushdown of the operator \( \tilde{T} \) on \( S(J) \), where \( J = \text{OE}(Y) \) given by

\[
\tilde{T}e = \sum_{e'} e',
\]

where here the sum runs over all edges \( e' \neq e^{-1} \) with \( o(e') = t(e) \). We will make this a bit more precise. As \( T \) is an operator of finite rows and columns, we can as well consider it as \( T : P(I) \to P(I) \). Now \( P(I) = P(\text{OE}(X)) \) can be identified with

\[
P(\text{OE}(Y))^\Gamma,
\]

i.e., the \( \Gamma \)-invariants in \( P(J) \), where \( J = \text{OE}(Y) \). Hence any operator on \( P(J) \) which commutes with the \( \Gamma \)-action, defines an operator on \( P(I) \). In this way the operator \( T \) corresponds to the operator \( \tilde{T} : P(J) \to P(J) \) given above.

**Lemma 4.1.** For any given \( n \in \mathbb{N} \) the operator \( T^n \) is traceable and the trace is

\[
\text{Tr}(T^n) = \sum_{c(l(c)=n)} l(c_0)w(c),
\]

where the sum runs over all cycles \( c \) of length \( n \) and \( c_0 \) is the underlying prime cycle of a given cycle \( c \).

**Proof.** Recall that \( X \) is a union of a finite graph \( X_{\text{fin}} \) and a finite number of cusp sections. The best way of thinking of \( T \) is that it sends potentials from an edge to the following edges and therefore an edge \( e \in I \) only gives a non-zero contribution to the trace, i.e., \( \langle T^n e, e \rangle \neq 0 \), if \( e \) lies on some cycle of length \( n \). Now if \( e \) lies on a cusp section and its distance to \( X_{\text{fin}} \) is bigger than \( n \), then it cannot lie on such a cycle, as potentials on a cusp section, which move inward, i.e., towards \( X_{\text{fin}} \), cannot reverse on the cusp section.
but have to move all the way to $X_{\text{fin}}$ before returning. Therefore the sum $\sum_e \langle T^n e, e \rangle$ is actually finite and so $T^n$ is traceable. The claimed formula is clear.

With this lemma we compute, formally at first,

$$Z(u)^{-1} = \prod_{c_0} \left( 1 - w(c_0) u^{l(c_0)} \right) = \exp \left( \sum_{c_0} \log \left( 1 - w(c_0) u^{l(c_0)} \right) \right)$$

$$= \exp \left( - \sum_{c_0} \sum_{n=1}^{\infty} \frac{w(c_0) u^{nl(c_0)}}{n} \right) = \exp \left( - \sum_{c} \frac{w(c) u^{l(c)}}{l(c)} \right)$$

$$= \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{c|d(c)=n} l(c) \right) = \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \text{Tr}(T^n) \right)$$

We say that a sequence $A_n$ of operators converges weakly to an operator $A : S \to P$, if for all $i, j \in I$ the sequence of complex numbers $\langle A_n i, j \rangle$ converges to $\langle A i, j \rangle$.

**Lemma 4.2.** There is $\alpha > 0$ such that for $u \in \mathbb{C}$ with $|u| < \alpha$ the series $-\sum_{n=1}^{\infty} \frac{u^n}{n} T^n$ converges weakly to an operator we call $\log(1 - uT)$. This operator is traceable and we have

$$Z(u)^{-1} = \exp \left( \text{Tr} \left( \log(1 - uT) \right) \right)$$

for every $u \in \mathbb{C}$ with $|u| < \alpha$.

**Proof.** As $Y$ is uniform, there is an upper bound $M$ to the valency of vertices. It follows that $\sum_{e'} w(e, e') \leq M$ for every edge $e$, where the sum runs over all edges $e'$ with $o(e') = t(e)$. It follows that for any two $i, j \in I$ one has

$| \langle T^n i, j \rangle | \leq M^n C_n(i, j)$,

where $C_n(i, j)$ is the number of paths of length $n$ connecting $i$ and $j$. Let $r$ denote the number of cusp sections in $X$ and let $s$ be the number of oriented edges in $X_{\text{fin}}$. For counting the number of paths connecting any two given edges, it suffices to replace each cusp section with a vertex and one oriented edge going out and one in. Going out a long stretch on a cusp section then is replaced by iterating the loop. In that way one sees that

$| \langle T^n i, j \rangle | \leq M^n (s + 2r)^n$, 


from which the convergence assertion follows. The trace assertion really is an assertion of changing the order of summation because the trace is itself a sum over $I$. For $u > 0$ all summands are positive, so there is no problem with this interchange of order, for general $u$ one uses absolute convergence, i.e., a Fubini-argument, to reach the same conclusion. 

**Theorem 4.3.** For $|u|$ small enough, the operator $1 - uT$ is of determinant class and one has

$$Z(u)^{-1} = \det(1 - uT).$$

This is a rational function of $u$.

**Proof.** We consider one cusp at a time and for simplicity assume that the period is one. Then all vertices along the cups can be assumed to have the same valency, say $q + 1$. The modifications for the general case are easy. Let $(e_0, e_1, e_2, \ldots)$ be a ray which represents the cusp. Write $f_j = e^{-1}$ then we get

$$Te_j = e_{j+1} + (q - 1)f_j, \quad j \geq 0,$$
$$Tf_j = qf_{j-1}, \quad j \geq 1.$$

We find that the operator $1 - uT$ is represented by the matrix

| $A$ | $e_0$ | $f_0$ | $e_1$ | $f_1$ | $e_2$ | $f_2$ |
|-----|-------|-------|-------|-------|-------|-------|
| $e_0$ | $\beta$ | 1     |       |       |       |       |
| $f_0$ | $a$   | 1     | 0     | $a + b$|       |       |
| $e_1$ | $b$   | 1     |       |       |       |       |
| $f_1$ | $a$   | 1     |       | $a + b$|       |       |
| $e_2$ | $b$   | 1     |       |       |       |       |
| $f_2$ |       |       |       |       |       |       |

where $a = -u$ and $b = -u(q - 1)$. Further the operator $A$ represents what is going on outside the current cusp, $\alpha$ is a finite column vector and $\beta$ a finite row vector. It follows that a finite permutation $\sigma$ which gives a non-zero contribution to the determinant must satisfy $\sigma(e_0) = e_0$ or $\sigma(e_0) = e_1$. In the first case it follows that $\sigma$ is the identity on the whole cusp section. So, if $\sigma$ is not the identity on the cusp section, the factor $\langle Te_0, \sigma e_0 \rangle$ gives a factor $b = u(q - 1)$. We find that there are not very many choices for such a $\sigma$ and that they come with growing powers of $u$, which for $|u|$ small, force in
IHARA ZETA FOR NON-UNIFORM TREE-LATTICES

convergence of the determinant series. Therefore $1 - uT$ is of determinant class. We compute

$$\det(1 - uT) = \lim_F \det(1 - uT_F)$$

$$= \lim_F \exp \left( - \sum_n \frac{u^n}{n} \text{tr} T^n_F \right)$$

$$= \exp \left( - \lim_F \sum_n \frac{u^n}{n} \text{tr} T^n_F \right),$$

where we have used the continuity of the exponential function. Next the limit can be interchanged with the sum for small $|u|$ by using dominated convergence by means of a crude estimate of $\text{tr} T^n_F$ similar to the proof of Lemma 4.2. Finally we have $\lim_F \text{tr} T^n_F = \text{Tr} T^n$ by the same estimate, so that we end up with the claim. The other cusps are dealt with in the same fashion.

It remains to show that $\det(1 - uT)$ is a rational function in $u$. For this we again look at one cusp only and again we assume it to be of period one. As in the proof of Theorem 4.3 we see that $\det(1 - uT)$ is the determinant of the matrix

$$\begin{array}{c|cccccc}
A & e_0 & f_0 & e_1 & f_1 & e_2 & f_2 \\
\hline
e_0 & \beta & 1 & & & & \\
f_0 & a & 0 & a + b & & & \\
e_1 & b & 1 & & & & \\
f_1 & a & 1 & a + b & & & \\
e_2 & b & 1 & & & & \\
f_2 & & & & & & \\
\end{array}$$

But starting the cusp section one step later, i.e., moving out on the cusp, we can assume that the vectors $\alpha$ and $\beta$ each have at most one non-zero entry. Then there exists a submatrix $A'$ of $A$ and a number $c \in \mathbb{C}$ such that the determinant above is equal to $\det(A)$ plus $cu^2 \det(A')$ times the determinant
of the infinite matrix

\[
\begin{pmatrix}
  a & 0 & a + b \\
  b & 1 & \\
  a & 1 & a + b \\
  b & 1 & \\
  a & 1 & a + b \\
  b & 1 & \\
  & & \\
\end{pmatrix}
\]

Let’s call this latter determinant \( D \). Using Laplace expansion along the first row we see that \( D \) equals \( a \) plus \((a + b)\) times the determinant of

\[
\begin{pmatrix}
  b & 1 & \\
  a & a + b & \\
  b & 1 & \\
  a & 1 & a + b \\
  b & 1 & \\
  a & 1 & \\
  b & & \\
\end{pmatrix}
\]

But this is \( b \) times \( D \), so we get \( D = a + (a + b)bD \), or

\[
D = \frac{a}{1 - (a + b)b} = \frac{u}{u^2q(q - 1) - 1}
\]

The proof of Theorem 4.3 and therefore of Theorem 2.2 is finished.

5 The Ihara formula

In [Bas92] it is shown that in the case of a uniform tree lattice \( \Gamma \) acting on a tree \( Y \), the zeta function satisfies

\[
Z(u)^{-1} = \frac{\det(1 - uA + u^2Q)}{(1 - u^2)^\chi},
\]

where \( A : S(VY) \to S(VY) \) is the adjacency operator of \( Y \), i.e.,

\[
Ay = \sum_{y'} y',
\]
where the sum runs over all vertices \( y' \) adjacent to \( y \in VY \). Further, \( Q \) is the valency operator minus one, i.e.,

\[
Q(y) = (\text{val}(y) - 1)y,
\]

where \( \text{val}(y) \) is the valency of the vertex \( y \). Finally, \( \chi \) is the Euler number of the finite graph \( X \). Here we use the identification of \( S(VX) \) with the set of \( \Gamma \)-invariants in \( S(VY) \).

**Theorem 5.1.** Let \( \Gamma \) be a cuspidal tree lattice. For \( |u| \) small enough, the operator \( 1 - uA + u^2Q \) is of connected determinant class and one has

\[
Z(u)^{-1} = \frac{\det(1 - uA + u^2Q)}{(1 - u^2)\chi(X_{\text{fin}})},
\]

where \( X_{\text{fin}} \) is the finite part of \( X \), i.e., it is \( X \) minus the cusp sections. If \( X \) has at least one cusp, then \( 1 - uA + u^2Q \) is not of determinant class, providing the example promised in Proposition 3.7.

**Proof.** For any finite subset \( F \) of \( VX \) let \( X_F \) be the full finite subgraph with vertex set \( F \). Let \( Y_F \) be the preimage of \( X_F \) in \( Y \). then each of the finitely many connected components of \( Y \) is a tree, acted upon by \( \Gamma \). So Bass’s theorem applies to each connected component, giving

\[
Z_F(u)^{-1} = \frac{\det(1 - uA_F + u^2Q_F)}{(1 - u^2)\chi(X_F)},
\]

If \( X_F \) is connected and contains \( X_{\text{fin}} \), then \( \chi(X_F) = \chi(X_{\text{fin}}) \) as cusps do not contribute to the Euler number. Therefore we conclude that, as \( \lim_F \det(1 - uT_F) \) exists, the connected limit over \( \det(1 - uA_F + u^2Q_F) \) also exists, proving all but the last assertion of the theorem. It remains to show that \( 1 - uA_F + u^2Q_F \) is not of determinant class. For this let \( F \) be large enough that \( X_F \) contains \( X_{\text{fin}} \). Then each connected component of \( X_F \), which does not contain \( X_{\text{fin}} \) contributes a factor \( 1 - u^2 \) to the rational function \( (1 - u^2)\chi(X_F) \).

So one sees that

\[
\frac{\det(1 - uA_F + u^2Q_F)}{(1 - u^2)\pi_0(X_F)}
\]

converges as \( F \to I \), where \( \pi_0(X_F) \) is the set of connected components of \( X_F \). As the denominator alone doesn’t converge, the numerator won’t either. \( \Box \)
6 L-functions

The Bass-Ihara zeta function can be twisted with a finite dimensional unitary representation $\omega: \Gamma \to \text{GL}(V)$ of the tree lattice $\Gamma$. For better distinction, we will in this section denote oriented edges of $Y$ by $e, e', e_1, e_2, \ldots$ and oriented edges of $X$ by $f, f', f_1, f_2, \ldots$. For each $e \in \text{OE}(Y)$ we denote by $V_e$ a copy of the space $V$, so between any $V_e$ and any $V_{e'}$ there is a natural identification $V_e \cong V_{e'}$. For each $f \in \text{OE}(X)$ we let

$$V_f = \left( \prod_{e \in f} V_e \right)^\Gamma,$$

the space of $\Gamma$-invariants in the product of which $\Gamma$ acts by $(\gamma v)_e = \omega(\gamma)v_{\gamma^{-1}e}$. Recall that $f$ is an edge of $\Gamma \setminus Y$, so $f$ is a $\Gamma$-orbit of edges in $Y$. The space $V_f$ is finite-dimensional, isomorphic with $V_{f_e}^\Gamma$ for any $e \in f$. If $f, f'$ are consecutive edges in $X$, so $t(f) = o(f')$, then we define a map $W(f, f'): V_f \to V_{f'}$ by

$$W(f, f')v_f = \sum_{e \in f} \sum_{e' : e \rightarrow e'} v_{e'}.$$

For a closed path $p = (f_1, \ldots, f_n)$ in $X$ we define $W(p): V_{f_1} \to V_{f_1}$ by

$$W(p) = W(f_n, f_1) \circ \cdots \circ W(f_2, f_3) \circ W(f_1, f_2).$$

Then $W(p)$ does depend on the path $p$, but $\det(1 - u(\ell(p))W(p))$ only depends on the cycle of $p$. Therefore the product

$$L(\omega, u) = \prod_c \det(1 - u(\ell(c))W(c))^{-1}$$

is well defined as a product over all prime cycles in $X$. On the space $\bigoplus_{f \in \text{OE}(X)} V_f$ we consider the operator

$$T(\omega, v_f) = \sum_{f'} W(f, f')v_f,$$

where the sum runs over all $f' \in \text{OE}(X)$ with $o(f') = t(f)$.

Similarly, for a vertex $y$ if $Y$ we let $V_y$ denote a copy of $V$ and for $x$ of $X$ we set

$$V_x = \left( \prod_{y \in x} V_y \right)^\Gamma.$$
On the space $\bigoplus_{x \in V} V_x$ we consider the adjacency operator

$$A_\omega(v_x) = \sum_{y \in x} \sum_{y'} v_{y'},$$

where the first sum runs over all $y$ in the $\Gamma$-orbit $x$ and the second is extended over all neighbors $y'$ of $y$ in $Y$. Finally, $v_{y'}$ is the image of $v_x$ under the canonical identification $V_x \cong V \cong V_{y'}$. Further, let

$$Q(v_x) = (\text{val}(y) - 1)v_x,$$

where $y$ is any element of $x$ and $\text{val}(y)$ is the valency of the vertex $y$ in the tree $Y$.

**Theorem 6.1.** Let $\Gamma$ be a cuspidal lattice. For $|u|$ small enough, the operator $1 - uT_\omega$ is of determinant class and one has

$$L(\omega, u)^{-1} = \det(1 - uT_\omega).$$

This is a rational function of $u$. For $|u|$ small enough, the operator $1 - uA + u^2Q$ is of connected determinant class and one has the Ihara formula,

$$L(\omega, u)^{-1} = \frac{\det(1 - uA_\omega + u^2Q)}{(1 - u^2)^{\chi(X_{\text{fin}})}},$$

where $X_{\text{fin}}$ is the finite part of $X$, i.e., it is $X$ minus the cusp sections.

**Proof.** Analogous to Lemma 4.1 one sees that

$$\text{Tr}(T^n) = \sum_{c, l(c) = n} l(c_0) \text{tr}(W(c)),$$

where the sum runs over all cycles $c$ of length $n$ and $l(c_0)$ is the prime cycle underlying $c$. The first identity follows as in Theorem 4.3. The argument for rationality is analogous to the proof of Theorem 4.3, and the Ihara formula follows as in Theorem 5.1. \qed

### 7 An arithmetic example

For background material on this section see [Ser03]. Let $q = p^k$ be a prime power and let $\mathbb{F} = \mathbb{F}_q$ be the finite field of $q$ elements. On the function field $\mathbb{F}_q(t)$ we put the discrete valuation corresponding to the “point at infinity”:

$$v\left(\frac{a}{b}\right) = \text{deg}(b) - \text{deg}(a),$$

.polynomial degree
Let $K = \hat{\mathbb{F}}(t)$ denote the local field one gets by completing $\mathbb{F}(t)$ and let $\mathcal{O} \subset K$ be the corresponding complete discrete valuation ring

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}.$$ 

Then $\pi = 1/t$ is a uniformizer in $\mathcal{O}$. We consider the locally compact group $G = \text{GL}_2(K)$. Its Bruhat-Tits tree $Y$ can be described as follows. The vertices are homothety classes of $\mathcal{O}$-lattices in $K^2$. Two such lattice classes $[L], [L']$ are connected by an edge if and only if the representatives may be chosen in a way that

$$\pi L \subset L' \subset L$$

holds. The graph described in this way is a tree $Y$ which has constant valency $q + 1$. The natural action of $G$ on $\mathcal{O}$-lattices induces an action of $G$ on the tree $Y$. The group $\Gamma = \text{GL}_2(\mathbb{F}[t])$ is a discrete subgroup of $G$. In [Ser03], the quotient $\Gamma \backslash Y$ is described as follows. For each $n \in \mathbb{N}_0$ let

$$L_n = \mathcal{O}e_1 \oplus \pi^n \mathcal{O}e_2,$$

where $e_1, e_2$ is the standard basis of $K^2$. Write $x_n$ for the vertex given by the class $[L_n]$. Then $L_0, L_1, \ldots$ is a complete set of representatives for $\Gamma \backslash Y$, the only edges being $(L_n, L_{n+1})$ for $n \geq 0$. Put $\Gamma_0 = \text{GL}_2(\mathbb{F})$ and for $n \geq 1$,

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}^\times, b \in \mathbb{F}[t], \deg(b) \leq n \right\}.$$ 

Then for each $n \geq 0$, the group $\Gamma_n$ is the stabilizer group of $x_n$. The group $\Gamma_0$ acts transitively on the set of edges with origin $x_0$ and for $n \geq 1$ the edge $(L_n, L_{n+1})$ is fixed by $\Gamma_n$. Finally, the group $\Gamma_n$ acts transitively on the set of edges with origin $x_n$ distinct from $(x_n, x_{n+1})$. For this see [Ser03] Proposition I.1.3.

So the quotient $X = \Gamma \backslash Y$ is a single ray. As in the proof of Theorem 4.3 we let $a = -u$ and $b = -(q-1)u$ and we see that $\text{det}(1 - uT)$ equals the determinant of

$$\begin{bmatrix} 1 & a + 2b \\ a & 1 & 0 & a + b \\ b & 1 & a & a + b \\ & b & 1 & a + b \\ & & b & 1 \\ & & & \ddots \end{bmatrix}$$
Again as in the theorem, we see that

\[ Z(u)^{-1} = \det(1 - uT) = \frac{u^2(q^2 + 1) - 1}{u^2q(q - 1) - 1} \]

References

[Bas92] Hyman Bass, The Ihara-Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992), no. 6, 717–797.

[BL01] Hyman Bass and Alexander Lubotzky, Tree lattices, Progress in Mathematics, vol. 176, Birkhäuser Boston Inc., Boston, MA, 2001. With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits.

[CJK] G. Chinta, J. Jorgenson, and A. Karlsson, Heat kernels on regular graphs and generalized Ihara zeta functions, available at http://arxiv.org/abs/1302.4644.

[CMS01] Bryan Clair and Shahriar Mokhtari-Sharghi, Zeta functions of discrete groups acting on trees, J. Algebra 237 (2001), no. 2, 591–620.

[Cla09] Bryan Clair, Zeta functions of graphs with \( \mathbb{Z} \) actions, J. Combin. Theory Ser. B 99 (2009), no. 1, 48–61.

[GŽ04] Rostislav I. Grigorchuk and Andrzej Źuk, The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps, Random walks and geometry, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 141–180.

[GIL08a] Daniele Guido, Tommaso Isola, and Michel L. Lapidus, Ihara zeta functions for periodic simple graphs, C*-algebras and elliptic theory II, Trends Math., Birkhäuser, Basel, 2008, pp. 103–121.

[GIL08b] , Ihara’s zeta function for periodic graphs and its approximation in the amenable case, J. Funct. Anal. 255 (2008), no. 6, 1339–1361.

[HH89] Ki-ichiro Hashimoto and Akira Hori, Selberg-Ihara’s zeta function for p-adic discrete groups, Automorphic forms and geometry of arithmetic varieties, Adv. Stud. Pure Math., vol. 15, Academic Press, Boston, MA, 1989, pp. 171–210.

[Has89] Ki-ichiro Hashimoto, Zeta functions of finite graphs and representations of p-adic groups, Automorphic forms and geometry of arithmetic varieties, Adv. Stud. Pure Math., vol. 15, Academic Press, Boston, MA, 1989, pp. 211–280.

[Has90] , On zeta and \( L \)-functions of finite graphs, Internat. J. Math. 1 (1990), no. 4, 381–396.

[Has92] , Artin type \( L \)-functions and the density theorem for prime cycles on finite graphs, Internat. J. Math. 3 (1992), no. 6, 809–826.

[Has93] , Artin \( L \)-functions of finite graphs and their applications, Sûrikaisekikenkyûsho Kôkyûroku 840 (1993), 70–81 (Japanese). Algebraic combinatorics (Japanese) (Kyoto, 1992).

[Iha66a] Yasutaka Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966), 219–235.
IHARA ZETA FOR NON-UNIFORM TREE-LATTICES

[1] IHARA ZETA FOR NON-UNIFORM TREE-LATTICES

[1] Discrete subgroups of PL(2, \(k_\ell\)), Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 272–278.

[2] Motoko Kotani and Toshikazu Sunada, Zeta functions of finite graphs, J. Math. Sci. Univ. Tokyo 7 (2000), no. 1, 7–25.

[3] Alexander Lubotzky, Lattices in rank one Lie groups over local fields, Geom. Funct. Anal. 1 (1991), no. 4, 406–431.

[4] Ortwin Scheja, On zeta functions of arithmetically defined graphs, Finite Fields Appl. 5 (1999), no. 3, 314–343.

[5] Jean-Pierre Serre, Trees, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell; Corrected 2nd printing of the 1980 English translation.

[6] H. M. Stark and A. A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996), no. 1, 124–165, DOI 10.1006/aima.1996.0050.

[7] H. M. Stark and A. A. Terras, Zeta functions of finite graphs and coverings. II, Adv. Math. 154 (2000), no. 1, 132–195, DOI 10.1006/aima.2000.1917.

[8] A. A. Terras and H. M. Stark, Zeta functions of finite graphs and coverings. III, Adv. Math. 208 (2007), no. 1, 467–489, DOI 10.1016/j.aim.2006.03.002.

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de

February 20, 2014