Bounds on the Distinguishing Number of Orthogonality Graphs

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Abstract

A graph $G$ is said to be $d$-distinguishable if there is a labeling of the vertices with $d$ labels so that only the trivial automorphism preserves the labels. The smallest such $d$ is the distinguishing number, Dist($G$). A set of vertices $S \subseteq V(G)$ is a determining set for $G$ if every automorphism of $G$ is uniquely determined by its action on $S$. The size of a smallest determining set for $G$ is called the determining number, Det($G$). The orthogonality graph $\Omega_{2k}$ has vertices which are bitstrings of length $2k$ with an edge between two vertices if they differ in precisely $k$ bits. This paper shows that Det($\Omega_{2k}$) = $2^{2^k-1}$ and that if $\binom{m}{2} \geq 2^k$ if $k$ is odd or $\binom{m}{2} \geq 2k + 1$ if $k$ is even then $2 < \text{Dist}(\Omega_{2k}) \leq m$.

1 Introduction

A labeling of the vertices of a graph $G$ with the integers 1, $d$ is called a $d$-distinguishing labeling if no non-trivial automorphism of $G$ preserves the labels. A graph is called $d$-distinguishable if it has a $d$-distinguishing labeling. The distinguishing number of $G$, Dist($G$), is the fewest number of labels necessary for distinguishing labeling. Albertson and Collins introduced graph distinguishing in [3]. Over the last few decades, this topic has generated significant interest and abundant results.

Most of the work in the last few decades has been in studying large families of graphs and showing that all but a finite number in each family have distinguishing number 2. Examples of this for finite graphs include: hypercubes $Q_n$ with $n \geq 4$ [4], Cartesian powers $G^n$ for a connected graph $G \neq K_2, K_3$ and $n \geq 2$ [2, 9, 11], Kneser graphs $K_{n,k}$ with $n \geq 6, k \geq 2$ [1], and (with seven small exceptions) 3-connected planar graphs [7]. Examples for infinite graphs include: the denumerable random graph [10], the infinite hypercube [10], locally finite trees with no vertex of degree 1 [14], and denumerable vertex-transitive graphs of connectivity 1 [13].

Exhaustion shows that the cycles $C_3, C_4, C_5$ and the hypercubes $Q_2, Q_3$ each have distinguishing number 3. Some infinite graph families that are not 2-distinguishable are $K_n$ (Dist($K_n$) = $n$) and the complete bipartite graph $K_{m,n}$. 

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(Dist(\(K_{m,n}\)) = \max\{m,n\} for \(m \neq n\), and Dist(\(K_{n,n}\)) = n + 1). We will see in Section 3 that orthogonality graphs \(\Omega_{2k}\) are also not 2-distinguishable.

A useful tool used in finding distinguishing classes is the determining set \([5]\), a set of vertices whose pointwise stabilizer is trivial. The determining number of a graph \(G\), \(\text{Det}(G)\), is the size of a smallest determining set. For some families we only have bounds on the determining number. For instance, for the Kneser graph, \(\log_2(n+1) \leq \text{Det}(K_{n:k}) \leq n - k\) with both upper and lower bounds sharp \([5]\). However, there are families for which we know the determining number exactly. For instance, for the Kneser graph, \(\log_2(2n+1)\) with both upper and lower bounds sharp \([5]\). However, there are families for which we know the determining number exactly. For instance, in Cartesian products, \(\text{Det}(Q_n) = \lceil \log_2 n \rceil + 1\), and \(\text{Det}(K_3^n) = \lceil \log_3(2n+1) \rceil + 1\) \([6]\).

Determining sets and distinguishing number were introduced at different times, by different authors, and for distinct purposes. However, Albertson and Boutin connected them in \([1]\) by noting that if \(G\) has a determining set of size \(d\), then there is a \((d+1)\)-distinguishing labeling for \(G\). Thus \(\text{Dist}(G) \leq \text{Det}(G) + 1\). We will find this relationship useful in pursuing the distinguishing number of orthogonality graphs.

The orthogonality graph on the \(n\)-dimensional hypercube, \(\Omega_n\), has the same vertex set as an \(n\)-dimensional hypercube but with vertices adjacent when they are orthogonal. That is, two vertices of \(\Omega_n\) are adjacent if their Hamming distance is \(\frac{n}{2}\). The graph \(\Omega_n\) with \(n = 2^r\) is used in quantum information theory to study the cost of simulating a specific quantum entanglement on \(r\) qubits. With quantum computing as inspiration, the independence number and chromatic numbers of \(\Omega_n\) were studied in \([8]\). In this paper, we study the determining and distinguishing numbers.

The paper is organized as follows. Definitions and facts about determining sets, distinguishing labelings, and orthogonality graphs are given in Section 2. Section 3 examines pairs of twin vertices in \(\Omega_{2k}\), proves \(\text{Det}(\Omega_{2k}) = 2^{2k-1}\), and shows that \(\Omega_{2k}\) is not 2-distinguishable. Section 4 discusses odd and even vertices in \(\Omega_n\) and introduces a quotient graph \(\tilde{\Omega}_{2k}\). Section 5 shows that \(\text{Det}(\tilde{\Omega}_{2k}) = 2^{2k-2}\). Finally, Section 6 provides the upper bound for \(\text{Dist}(\Omega_{2k})\). Section 7 provides some open problems for future work.

## 2 Background

### 2.1 Determining Sets and Distinguishing Labelings

Let \(G\) be a graph. A subset \(S \subseteq V(G)\) is said to be a determining set for \(G\) if whenever \(\varphi, \psi \in \text{Aut}(G)\) so that \(\varphi(x) = \psi(x)\) for all \(x \in S\), then \(\varphi = \psi\). Thus every automorphism of \(G\) is uniquely determined by its action on the vertices of a determining set. The determining set is an example of a base of a permutation group action. Every graph has a determining set since a set containing all but one vertex of the graph is determining. The determining number of \(G\), \(\text{Det}(G)\), is the minimum size of a determining set for \(G\).

Recall that the set stabilizer of \(S \subseteq V(G)\) is the set of all \(\varphi \in \text{Aut}(G)\) for which \(\varphi(x) \in S\) for all \(x \in S\). In this case we say that \(S\) is invariant under \(\varphi\) and
we write \( \varphi(S) = S \). The pointwise stabilizer of \( S \) is the set of all \( \varphi \in \text{Aut}(G) \) for which \( \varphi(x) = x \) for all \( x \in S \). It is easy to see that \( S \subseteq V(G) \) is a determining set for \( G \) if and only if the pointwise stabilizer of \( S \) is trivial.

A labeling \( f : V(G) \to \{1, \ldots, d\} \) is said to be \( d \)-distinguishing if only the trivial automorphism preserves the label classes. Every graph has a distinguishing labeling since each vertex can be assigned a distinct label. A graph is called \( d \)-distinguishable if it has a \( d \)-distinguishing labeling. The distinguishing number of \( G \), \( \text{Dist}(G) \), is the fewest number of labels necessary for a distinguishing labeling.

**Lemma 1.** Let \( G \) be a graph, \( \alpha \in \text{Aut}(G) \), and \( f : V(G) \to \{1, \ldots, d\} \) a vertex labeling. Then \( f \) is distinguishing if and only if \( f \circ \alpha \) is distinguishing.

**Proof.** It is straightforward to verify that \( \varphi \in \text{Aut}(G) \) preserves the label classes of \( f \circ \alpha \) if and only if \( \alpha \circ \varphi \circ \alpha^{-1} \) preserves the label classes of \( f \).

The following ties together determining sets and distinguishing labelings and facilitates the work in this paper.

**Theorem 1.** \([1]\) \( G \) is \( d \)-distinguishable if and only if it has a determining set \( S \) that can be labeled in such a way that any automorphism of \( \text{Aut}(G) \) that preserves the labeling classes of \( S \) fixes \( S \) pointwise.

**Corollary 1.** \( \text{Dist}(G) \leq \text{Det}(G) + 1 \).

**Proof.** Suppose \( S \) is a smallest determining set for \( G \). Label each of the vertices of \( S \) with a different label. Label each of the remaining vertices of \( G \) by the label \( d + 1 \). If \( \varphi \in \text{Aut}(G) \) preserves the label classes of \( S \), then \( \varphi \) is the identity. Since \( S \) is a determining set, this means \( \varphi \) is the identity. Thus our labeling is a \( (d + 1) \)-distinguishing labeling for \( G \).

As we’ll see in Lemma 2 below, twin vertices play a significant role in the study of graph symmetry.

**Definition 1.** Two vertices \( u, v \in V(G) \) are called twins if they have identical sets of neighbors. That is, \( u \) and \( v \) are twins if \( N(u) = N(v) \).

**Lemma 2.** Let \( u, v \in V(G) \) be twins. Then the function on \( V(G) \) that interchanges \( u \) and \( v \) and acts as the identity on all other vertices is a graph automorphism. Thus in any distinguishing labeling members of a twin pair must have different labels, and further, any determining set must contain one and only one member of each twin pair.

The proof is elementary.

### 2.2 Orthogonality Graphs

**Definition 2.** The orthogonality graph \( \Omega_{2k} \) has as its vertex set all bitstrings of length \( 2k \),

\[
V(\Omega_{2k}) = \{ u = u_1u_2 \ldots u_{2k} \mid u_i \in \{0, 1\} \} = \mathbb{Z}_2^{2k},
\]
with two vertices adjacent if the corresponding bitstrings differ in exactly \( k \) bits. Note that \( \Omega_{2k} \) has order \( 2^{2k} \) and is \((\binom{2k}{k})\)-regular.

**Example 1.** The smallest orthogonality graph \( \Omega_2 \) occurs when \( k = 1 \) and is isomorphic to \( C_4 \).

**Example 2.** The orthogonality graph \( \Omega_4 \) is a 6-regular graph of order 16 and consists of two isomorphic components, each of which is a copy of the circulant graph \( C_{8\cdot[1,2,3]} \).

**Definition 3.** The (Hamming) weight of \( u \in V(\Omega_{2k}) \), denoted \( wt(u) \), is the number of 1s in its bitstring. Let 0 and 1 be the bitstrings of length \( 2k \) of weight 0 and \( 2k \) respectively. The support of \( u \) is the set of indices of the bits where its 1’s occur. That is, \( supp(u) = \{ i \mid u_i = 1 \} \subseteq \{1,2,\ldots,2k\} \).

Note that \( wt(u) = |supp(u)| \). Also, note that for any \( u,v \in V(\Omega_{2k}) \), \( supp(u+v) = supp(u) \triangle supp(v) \), where \( \triangle \) denotes the symmetric difference. In particular, \( supp(u+1) \) is the complement of \( supp(u) \). Further, \( u,v \in V(\Omega_{2k}) \) are adjacent if and only if \( wt(u+v) = |supp(u) \triangle supp(v)| = k \).

It is easy to verify that the vertex maps described below are automorphisms of \( \Omega_{2k} \).

• **Permutation automorphisms.** For any permutation \( \sigma \in S_{2k} \), let \( \sigma \) act on vertices of \( \Omega_{2k} \) by permuting the order of the bits; that is, \( \sigma(u) = \sigma(u_1 u_2 \ldots u_{2k}) = u_{\sigma(1)} u_{\sigma(2)} \ldots u_{\sigma(2k)} \).

• **Translation automorphisms.** For any \( u \in V(\Omega_{2k}) \), define \( \tau_u : V(\Omega_{2k}) \to V(\Omega_{2k}) \) by

\[
\tau_u(w) = u + w = (u_1 + w_1)(u_2 + w_2)\ldots(u_{2k} + w_{2k}),
\]

where all bit-sums are taken modulo 2.

For \( k \geq 2 \), these two families of automorphisms do not exhaust \( \text{Aut}(\Omega_{2k}) \). For example, we will see in Section 3, that there is an automorphism \( \pi_0 \) that transposes 0 and 1 and leaves all other vertices fixed. The following argument shows that \( \pi_0 \) is not in the subgroup generated by permutation automorphisms and translation automorphisms.

Clearly any composition of translation automorphisms is itself a translation automorphism; the same goes for permutation automorphisms. Note also that for all \( u,w \),

\[
(\sigma \circ \tau_u)(w) = \sigma(u + w) = \sigma(u) + \sigma(w) = (\tau_{\sigma(u)} \circ \sigma)(w).
\]

Thus any automorphism in the subgroup generated by permutations and translations can be written in the form \( \tau_u \circ \sigma \). If \( \tau_0 \) is in this subgroup, then there exists \( u \in V(\Omega_{2k}) \) and \( \sigma \in S_{2k} \) so that \( \pi_0 = \tau_u \circ \sigma \). Note that \( \pi_0(0) = 1 \) while \( \tau_u \circ \sigma(0) = \tau_u(0) = u \). Thus \( u = 1 \).
However, $\pi_0$ fixes all vertices other than 0 and 1 while $\tau_1 \circ \sigma$ takes vertices of weight 1 to vertices of weight $2k - 1$. Since $k > 1$, this shows $\pi_0$ is not in this subgroup.

Orthogonality graphs are highly symmetric in the sense that they are arc-, edge-, and vertex-transitive. Suppose $(x, y)$ and $(u, w)$ are arcs (directed edges) of $\Omega_{2k}$. Since $(x, y)$ and $(u, w)$ are edges, each of $x + y$ and $u + w$ has weight $k$. Since their supports have the same size, there is a permutation $\sigma \in S_{2k}$ taking the support of $x + y$ to the support of $u + w$. Also denote by $\sigma$ the corresponding permutation automorphism of $\Omega_{2k}$. Then $\sigma(x + y) = u + w$. Now consider the automorphism $\tau_u \circ \sigma \circ \tau_x$:

$$(\tau_u \circ \sigma \circ \tau_x)(x) = \tau_u(\sigma(0)) = \tau_u(0) = u;$$

$$(\tau_u \circ \sigma \circ \tau_x)(y) = \tau_u(\sigma(x + y)) = \tau_u(u + w) = w.$$ 

Thus the automorphism $\tau_u \circ \sigma \circ \tau_x$ maps the arc $(x, y)$ to $(u, w)$ proving that $\Omega_{2k}$ is arc-transitive. The edge- and vertex-transitivity of $\Omega_{2k}$ follows from its arc-transitivity.

### 3 Det($\Omega_{2k}$) and a Lower Bound on Dist($\Omega_{2k}$)

To approach the determining number and distinguishing number of $\Omega_{2k}$ we will first want to study the twin vertices in the graph.

**Lemma 3.** The vertices of $\Omega_{2k}$ can be partitioned uniquely into twin pairs of the form $\{u, u + 1\}$ for $u \in \Omega_{2k}$. In particular, $u$ and $w$ are twins if and only if $w = u + 1$, and there is no set of vertices of size three or more which are pairwise twins.

**Proof.** Note that the Hamming distance between $u$ and $u + 1$ is $2k$, so $u$ and $u + 1$ are nonadjacent. Suppose $v \in N(u)$. Then $\text{wt}(v + u) = |\text{supp}(v + u)| = k$. Then $\text{wt}((v + u) + 1) = |\text{supp}((v + u) + 1)| = 2k - k = k$. Thus $v \in N(u + 1)$. Similarly, $v \in N(u + 1)$ means that $v \in N(u)$. Thus $N(u) = N(u + 1)$ so $u$ and $u + 1$ are twins.

Suppose that $w \neq u + 1$ and that $w$ is not adjacent to $u$. We will show that there is some $y \in N(w)$ so that $y \not\in N(u)$.

Let $\text{wt}(w + u) = \ell$. Since $w$ is not adjacent to $u$, $\ell \neq k$. Let $r$ be the smaller of $\ell$ and $k$. Choose $x \in V(\Omega_{2k})$ of weight $k$ so that its support overlaps with $r$ positions in the support of $w + u$. Let $y = w + x$. Since $\text{wt}(x) = k$, $y \in N(w)$. By our choice of support for $x$, $\text{wt}(y + u) = \text{wt}((w + x) + u) = \text{wt}((w + u) + x) = |k - \ell|$.

Further, since $\ell \not\in \{0, 2k\}$, we get that $\text{wt}((w + u) + x) \neq k$. Thus $y \not\in N(u)$. Thus $w$ and $u$ are not twins. In particular, each vertex $u$ in $\Omega_{2k}$ has a unique twin, $u + 1$.

Thus the vertices of $\Omega_{2k}$ can be partitioned uniquely into twin pairs of the form $\{u, u + 1\}$. 

\qed
Together Lemma 2 and Lemma 3 prove the following.

**Theorem 2.** A subset of $V(\Omega_{2k})$ is a determining set for $\Omega_{2k}$ if and only if it contains precisely one vertex from each twin pair. Thus $\text{Det}(\Omega_{2k}) = 2^{2k-1}$.

The following lemma helps us understand how automorphisms of $\Omega_{2k}$ interact with twin pairs.

**Lemma 4.** Any automorphism $\alpha \in \text{Aut}(\Omega_{2k})$ respects twin pairs; that is, for all $u \in V(\Omega_{2k})$, $\alpha(u + 1) = \alpha(u) + 1$.

**Proof.** Since automorphisms preserve adjacency and nonadjacency, $\alpha(u)$ and $\alpha(u + 1)$ must be nonadjacent vertices with exactly the same neighbors. By Lemma 3, the only other vertex with exactly the same neighbors as $\alpha(u)$ is its twin $\alpha(u) + 1$.

Our knowledge of twin pairs will also help us prove below that $\Omega_{2k}$ is not 2-distinguishable.

**Theorem 3.** $\text{Dist}(\Omega_{2k}) > 2$.

**Proof.** Suppose there exists a distinguishing 2-labeling $f$ of $\Omega_{2k}$; we will call the labels red and green. Since twin vertices must get different labels, exactly half the vertices are red and exactly half the vertices are green. For any vertex $u$, let $\pi_u$ be the automorphism that interchanges $u$ and its twin $u + 1$, and leaves all other vertices fixed. By Lemma 2, $f \circ \pi_u$ is also a distinguishing 2-labeling. For each $u$ in our red label class that does not have a 1 at its first bit, apply $\tau_u$. This process leads us to a distinguishing 2-labeling, $f'$ in which the red label class is precisely the set of vertices with a 1 in their first bit.

Let $\sigma$ be the cyclic permutation $(2 \ 3 \ \cdots \ (2k)) \in S_{2k}$. The corresponding permutation automorphism is nontrivial and fixes the first bit of each vertex. Thus $\sigma$ preserves the label classes. Hence the labeling $f'$ is not distinguishing, and thus by Lemma 1, $f$ is also not a distinguishing labeling.

4 Structure of $\Omega_{2k}$

To achieve an upper bound on the distinguishing number of $\Omega_{2k}$, we study more carefully the structure of $\Omega_{2k}$ as well as the structure of its quotient graph $\tilde{\Omega}_{2k}$ achieved by identifying twin pairs. In particular we will look at the vertices in $\Omega_{2k}$ in terms of the parity of their weights and we will extend this to the vertices of the quotient graph.

4.1 Odd and Even Vertices

**Lemma 5.** Let $E(\Omega_{2k})$ and $O(\Omega_{2k})$ denote the subset of vertices of $\Omega_{2k}$ having even and odd weight respectively. We call vertices in $E(\Omega_{2k})$ even vertices and vertices in $O(\Omega_{2k})$ odd vertices.
1. If \( k \) is even, then \( \Omega_{2k} \) consists of two isomorphic connected components, namely the subgraphs induced by \( E(\Omega_{2k}) \) and \( O(\Omega_{2k}) \), which we refer to as the even and odd component respectively.

2. If \( k \) is odd, then \( \Omega_{2k} \) is connected and bipartite, with bipartition \( V(\Omega_{2k}) = E(\Omega_{2k}) \sqcup O(\Omega_{2k}) \), which we refer to as the even and odd partite respectively.

Proof. Recall that the neighbors of a vertex \( u \) consist of all vertices of the form \( u + x \) where \( \text{wt}(x) = k \). For such a vertex \( x \),

\[
\text{Parity}(\text{wt}(u + x)) = \text{Parity}(\text{wt}(u) + k)
\] (1)

First, assume \( k \) is even. By (1), no even vertex can be adjacent to an odd vertex. However, any two vertices \( u \) and \( w \) differing in an even number of bits are connected by a path. To prove this, we must show that there exist vertices \( x_1, x_2, \ldots, x_t \), all of weight \( k \), such that

\[
u + (x_1 + x_2 + \cdots + x_t) = w;
\]

the corresponding path will then be the vertex sequence

\[(u, u + x_1, (u + x_1) + x_2, \ldots, (u + x_1 + x_2 + \cdots + x_{t-1}) + x_t = w).\]

Equivalently, we must show that \( u + w \), which has even weight, can be expressed as a sum of vertices of weight \( k \). By concatenating paths, it suffices to prove this when \( u + w \) has weight 2. Since permutations of indices is a graph automorphism, it suffices to show that \( 1100 \ldots 0 \) can be represented as a sum of vertices of weight \( k \). For \( k = 2 \), this is obvious; for \( k = 4 \),

\[
11000000 = 10001110 + 01001110.
\]

We can generalize this pattern for all \( k \geq 4 \) as follows:

\[
1100 \ldots 0 = 100 \ldots 0 1 \ldots 1 0 + 010 \ldots 0 1 \ldots 1 0.
\]

The translation automorphism \( \tau_{100 \ldots 0} \) shows that the even component and odd component are isomorphic.

Next assume \( k \) is odd. By (1), even vertices can only be adjacent to odd vertices and vice versa. To show connectedness, by an argument similar to the one above, it suffices to show that two vertices differing in exactly one bit are connected by a path. For this, it suffices to show that \( 10 \ldots 0 \) can be expressed as a sum of vertices of weight \( k \). If \( k = 3 \), then

\[
100000 = 111000 + 110100 + 101100.
\]
To generalize this, let $\ell = \frac{k-1}{2}$, or equivalently $k = 2\ell + 1$. Then
\[
1 \, 0 \ldots 0 = 1 \, 0 \ldots 0 \, 0 \ldots 0 \, 0 \\
\begin{array}{c}
\ell \\
2k-1
\end{array}
= 1 \, 1 \ldots 1 \, 0 \ldots 0 \ldots 0 +
1 \, 1 \ldots 1 \, 0 \ldots 0 \ldots 0 \\
\begin{array}{c}
\ell \\
\ell
\end{array}
+ 1 \, 0 \ldots 1 \, 1 \ldots 1 \, 0 \ldots 0 \\
\begin{array}{c}
\ell \\
\ell
\end{array}.
\]

4.2 The Quotient Graph $\widetilde{\Omega}_{2k}$

Given any equivalence relation $\sim$ on the vertex set $V$ of a graph $G = (V, E)$, we can define a corresponding quotient graph $G/\sim$ whose vertices are the equivalence classes of $V$, with classes $[u]$ and $[w]$ being adjacent if there exist $u', w' \in V$ with $u \sim u'$, $w \sim w'$ and $uw \in E$. The quotient graph is smaller and simpler, yet preserves some structure of the original graph.

In $V(\Omega_{2k})$ we identify each vertex with its twin. That is, we define $u \sim u + 1$. It is easy to verify that this is an equivalence relation. We denote the resulting quotient graph by $\widetilde{\Omega}_{2k}$. Note that $\widetilde{\Omega}_{2k}$ has order $2^{2k-1}$ and is $\frac{1}{2}(\binom{k}{2})$-regular.

Example 3. The quotient graph $\widetilde{\Omega}_2$ is $K_2$.

Example 4. The quotient graph $\widetilde{\Omega}_4$ is a 3-regular graph of order 8, consisting of two isomorphic components. By degree considerations alone, $\widetilde{\Omega}_4$ is the disjoint union of two copies of $K_4$.

Below we preview methods used in Theorem 4 in Section 6 to find an upper bound on $\text{Dist}(\Omega_{2k})$ using $\text{Det}(\widetilde{\Omega}_{2k})$.

Proposition 1. $\text{Det}(\widetilde{\Omega}_4) = 5$ and $\text{Dist}(\Omega_4) = 4$.

Proof. Each $K_4$ component has distinguishing number 4; to distinguish between the two isomorphic components, we need 5 labels in total. Note that using any 4 labels we can create $\binom{4}{2} = 6$ distinct label pairs and then use 5 of these to label the vertices of $\widetilde{\Omega}_4$. This 5-distinguishing labeling of $\widetilde{\Omega}_4$ with label pairs extends naturally to a 4-distinguishing labeling of $\Omega_4$ with twin pairs in $\widetilde{\Omega}_4$ assigned the labels from the label pairs assigned to vertices of $\widetilde{\Omega}_4$: see Figure 1.

Note that if $\Omega_4$ is 3-distinguishable, then so are each of its components. Let $C$ be the component of even vertices. Recall from Example 2, that $C = C_6(1, 2, 3)$, so $V(C)$ consists of 4 twin pairs with an edge between every pair of vertices that are not twins. Suppose we label $C$ with 3 labels. Since there are precisely $\binom{3}{2} = 3$ distinct label pairs for the 4 distinct twin pairs, two twin pairs, say $\{u, u + 1\}$ and $\{w, w + 1\}$, are assigned the same pair of labels. Without loss of generality we can assume that the labels on $u$ and $w$ are red and the labels on $u + 1$ and $w + 1$ are green (or replace $w$ with $w + 1$). Let $\alpha$ be the vertex map of $C$ that
transposes $u$ and $w$, transposes $u + 1$ and $w + 1$, and fixes all other vertices. Since the complement of $C$ is a set of 4 disjoint edges between twin pairs, and since $\alpha$ transposes two of these edges, $\alpha$ is an automorphism of $\overline{C}$ and thus of $C$ itself. Further $\alpha$ preserves label classes. Thus this is not a 3-distinguishing labeling of $C$. We conclude that there is no distinguishing 3-labeling for $C$ and therefore none for $\Omega_4$. Thus we have proved that $\text{Dist}(\Omega_4) = 4$.

Figure 1: $\overline{\Omega}_4$ with a 5-distinguishing labeling and $\Omega_4$ with a 4-distinguishing labeling

Since $\text{wt}(u + 1) = 2k - \text{wt}(u)$, we see that $u \in E(\Omega_{2k})$ if and only if $u + 1 \in E(\Omega_{2k})$. Hence the vertices of the quotient graph can still be partitioned into even and odd vertices. Moreover, if $k$ is even, $\overline{\Omega}_{2k}$ still consists of an even and an odd component, and if $k$ is odd, $\overline{\Omega}_{2k}$ is still bipartite with an even and an odd partite.

By Lemma 3,

$$u \text{ and } x \text{ are adjacent } \iff u \text{ and } x + 1 \text{ are adjacent}$$

$$\iff u + 1 \text{ and } x \text{ are adjacent}$$

$$\iff u + 1 \text{ and } x + 1 \text{ are adjacent}.$$ (2)

This gives a stronger interpretation of the adjacency of $[u]$ and $[x]$ than is prescribed in the definition of a quotient graph. One implication of this is given below.

**Lemma 6.** $\overline{\Omega}_{2k}$ is twin-free.
Proof. Suppose \( N([v]) = N([u]) \) in \( \tilde{\Omega}_{2k} \). Then for all \( w \in V(\Omega_{2k}) \),

\[
\text{w} \in N(v) \iff [w] \in N([v]) \iff [w] \in N([u]) \iff w \in N(u).
\]

Hence \( N(v) = N(u) \). By Lemma 3, \( v = u + 1 \), which means \([v] = [u]\) in \( \tilde{\Omega}_{2k} \).

Another implication of the equivalences in (2) is that \( \Omega_{2k} \) is the wreath product of \( \tilde{\Omega}_{2k} \) and \( N_2 \), the null graph on two vertices; that is, \( \Omega_{2k} = \tilde{\Omega}_{2k} \wr N_2 \). From [12], this implies that the wreath product \( \text{Aut}(\Omega_{2k}) \wr \text{Aut}(N_2) = \text{Aut}(\tilde{\Omega}_{2k} \wr Z_2) \) is a subgroup of \( \text{Aut}(\Omega_{2k}) \). In fact, the argument below shows that in this case, we have equality. By Lemma 4, any \( \alpha \in \text{Aut}(\Omega_{2k}) \) induces an automorphism \( \tilde{\alpha} \in \text{Aut}(\tilde{\Omega}_{2k}) \) given by

\[
\tilde{\alpha}([u]) = \tilde{\alpha}([u, u + 1]) = \{\alpha(u), \alpha(u + 1)\} = \{\alpha(u), \alpha(u) + 1\} = [\alpha(u)].
\]

Let \( u \in V(\Omega_{2k}) \). Recall that \( \pi_u \) is the automorphism of \( \Omega_{2k} \) that interchanges \( u \) and \( u + 1 \) while fixing all other vertices. Since for any \( \alpha \in \text{Aut}(\Omega_{2k}) \), we can see that \( \alpha \) and \( \alpha \circ \pi_u \) each induce \( \tilde{\alpha} \), there are \( |V(\Omega_{2k})| \) distinct automorphisms in \( \text{Aut}(\Omega_{2k}) \) that induce the same automorphism in \( \text{Aut}(\tilde{\Omega}_{2k}) \). Thus by a counting argument \( \text{Aut}(\Omega_{2k}) = \text{Aut}(\tilde{\Omega}_{2k} \wr Z_2) \).

5 Determining \( \tilde{\Omega}_{2k} \)

Definition 4. For any \([x] \in V(\tilde{\Omega}_{2k})\), we define

\[
\text{wt}([x]) = [\text{wt}(x), \text{wt}(x + 1)] = [\text{wt}(x), 2k - \text{wt}(x)].
\]

To eliminate ambiguity, we assume \( \text{wt}(x) \leq k \) and thus that \( \text{wt}(x) \leq 2k - \text{wt}(x) \).

For example, \( \{00101100, 11010011\} \in V(\tilde{\Omega}_8) \) has weight \([3, 5]\).

In what follows, we will be concentrating on the odd vertices in \( \tilde{\Omega}_{2k} \). Our goal is to show every odd vertex in \( \tilde{\Omega}_{2k} \) has a unique set of neighbors among the set of vertices of weight \([k - 1, k + 1]\).

In the original orthogonality graph, let \( u \) be a vertex of weight \( m \), where \( 1 \leq m \leq k \), \( m \) odd, and let \( v \) be a vertex of weight \( k \), so that \( u + v \) is a neighbor of \( u \). If \( |\text{supp}(u) \cap \text{supp}(v)| = t \), then the neighbor \( u + v \) of \( u \) has weight exactly \( m + k - 2t \).

The number of neighbors of \( u \) of weight \( m + k - 2t \) is the number of \( v \) such that \( |\text{supp}(u) \cap \text{supp}(v)| = t \), which is

\[
\binom{m}{t} \binom{2k - m}{k - t}.
\]

Now, \( m + k - 2t = k - 1 \iff t = \frac{m + 1}{2} \) and \( m + k - 2t = k + 1 \iff t = \frac{m - 1}{2} \).

By Pascal’s Identity,

\[
\binom{m}{\frac{m + 1}{2}} \binom{2k - m}{k - \frac{m + 1}{2}} = \binom{m}{\frac{m - 1}{2}} \binom{2k - m}{k - \frac{m - 1}{2}},
\]

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Figure 2: The weight of a neighbor of \( u \) is \( m + k - 2t \).

which makes sense because the neighbors of \( u \) of weight \( k + 1 \) and of weight \( k - 1 \) can be matched up in twin pairs by Lemma 3. Thus, in \( \Omega_{2k} \), the number of neighbors of \( [u] \) of weight \([k-1,k+1]\) is the common value of the expression in the equation above.

Note that all theorems and propositions are written for \( k \geq 1 \). However, because their statements involve an odd integer \( m \) with either \( 1 < m \leq k \) or \( 1 < m < k \), for Lemma 7 and Corollary 2 technically \( k \geq 3 \), while for Lemma 8 and Corollary 3 technically \( k \geq 4 \). This does not change the fact that all theorems are true for all \( k \geq 1 \).

**Lemma 7.** For distinct odd \( m,n \), both less than or equal to \( k \),

\[
\binom{m}{\frac{m+1}{2}} \binom{2k-m}{k-\frac{m+1}{2}} \neq \binom{n}{\frac{n+1}{2}} \binom{2k-n}{k-\frac{n+1}{2}}.
\]

The proof is by binomial computation and is contained in Appendix A.

**Corollary 2.** For distinct odd \( m,n \), both less than or equal to \( k \), vertices in \( \tilde{\Omega}_{2k} \) of weight \([m,2k-m]\) have a different number of neighbors of weight \([k-1,k+1]\) than vertices in \( \Omega_{2k} \) of weight \([n,2k-n]\).

We next consider distinct odd vertices in \( \tilde{\Omega}_{2k} \) of the same weight. We start with a technical lemma about vertices in the original orthogonality graph \( \Omega_{2k} \).

**Lemma 8.** Let \( 1 < m < k \), with \( m \) odd. Let \( u \) and \( w \) be distinct vertices in \( \Omega_{2k} \) with \( \text{wt}(u) = \text{wt}(w) = m \). Then there exists \( y \in V(\Omega_{2k}) \) with \( \text{wt}(y) = k - 1 \) that is adjacent to \( u \) but not to \( w \).

**Proof.** We divide into two cases.

**Case 1.** Assume \( \text{supp}(u) \cap \text{supp}(w) = \emptyset \). To find a neighbor \( y \) of \( u \) with \( \text{wt}(y) = k - 1 \), we must find \( v \in V(\Omega_{2k}) \) of weight \( k \) such that

\[
\text{wt}(y) = \text{wt}(u + v) = |\text{supp}(u + v)| = |\text{supp}(u) \triangle \text{supp}(v)| = k - 1.
\]

Using the cardinality variables shown in Figure 3, we have the equations \( a + b = d + e = m \) (because \( \text{wt}(u) - \text{wt}(w) = m \)), \( b + c + d = k \) (because \( \text{wt}(v) = k \)) and \( a + c + d = k - 1 \) (because \( \text{wt}(u + v) = k - 1 \)). Together these
Figure 3: Case 1: $\text{supp}(u) \cap \text{supp}(w) = \emptyset$

imply that $b - a = 1$ or $b = a + 1$, which in turn implies that $m = 2a + 1$, or equivalently,

$$a = \frac{m - 1}{2} \text{ and } b = \frac{m + 1}{2}.$$  

Arguing indirectly, the neighbor $y = u + v$ of $u$ will also be a neighbor of $w$ if and only if $\text{wt}(y + w) = \text{wt}((u + v) + w) = k$. Since

$$[\text{supp}(u) \triangle \text{supp}(v)] \triangle \text{supp}(w) = \text{supp}(u) \triangle \text{supp}(v) \triangle \text{supp}(w),$$  

this condition can be expressed as $a + c + e = k$. Given that $a + c + d = k - 1$,

$$a + c + e = k \iff e - d = 1 \iff e = d + 1 \iff d = \frac{m - 1}{2} \text{ and } e = \frac{m + 1}{2}.$$  

Thus, to ensure that $y = u + v$ is not a neighbor of $w$, all we have to do is choose $v$ in such a way that this is not true. That is, we choose $v$ so that it has 1’s in $\frac{m+1}{2}$ bits in which $u$ also has 1’s, and does not have $\frac{m-1}{2}$ 1’s in bits where $w$ has 1’s.

For example, if $k = 9$, $m = 7$,

$$u = 11111100000000000$$
$$w = 00000000001111111,$$

we can let $v = 11110000000011111$. In this example, $d = 5 \neq 3 = \frac{m-1}{2}$.

**Case 2.** Assume $\text{supp}(u) \cap \text{supp}(w) \neq \emptyset$. Again, the neighbor $y = u + v$ of $u$ will also be a neighbor of $w$ if and only if $\text{wt}((u + v) + w) = k$. Using the (reassigned) cardinality variables in Figure 4, this can be expressed in this case as $a + c + g = k$.

Using the fact that $a + b + g + f = k - 1$ (because $\text{wt}(u + v) = k - 1$),

$$a + c + g = k \iff c - (b + f) = 1 \iff c = b + f + 1.$$  

Substituting this into the equation $b + c + e + f = m$ gives $2(b + f) + 1 + e = m$. Since we assumed $m$ odd, this implies that $e$ is even. So to ensure that $y$ is not
Figure 4: Case 2: \( \text{supp}(u) \cap \text{supp}(w) \neq \emptyset \)

A neighbor of \( w \), it suffices to choose \( v \) so that \( e \) is odd. By our assumption that \( \text{supp}(u) \cap \text{supp}(w) \neq \emptyset \), we know \( b + e > 0 \), so this is possible.

For example, if \( k = 9, m = 7 \),

\[
\begin{align*}
  u &= 11111100000000000 \\
  w &= 0011111110000000000,
\end{align*}
\]

we can let \( v = 00111110000011111 \). In this example, \( e = 5 \).

By passing to the quotient graph, we have the following.

**Corollary 3.** Let \( 1 < m < k \), with \( m \) odd. Let \( [u] \) and \( [w] \) be distinct vertices in \( \tilde{\Omega}_{2k} \) of the same weight \( [m, 2k - m] \). Then there exists a vertex \( [y] \) of weight \( [k - 1, k + 1] \) that is adjacent to \( [u] \) but not to \( [w] \).

Combining Corollaries 2 and 3 achieves our goal.

**Proposition 2.** Each odd vertex in \( \tilde{\Omega}_{2k} \) has a unique set of neighbors among the set of vertices of weight \( [k - 1, k + 1] \).

For \( i \in \{1, \ldots, 2k\} \), let \( x_i \) denote the vertex of \( \Omega_{2k} \) represented as a bitstring with a 1 in position \( i \) and 0’s elsewhere, with \( [x_i] \) being the corresponding vertex of \( \tilde{\Omega}_{2k} \).

**Proposition 3.** Let \( D = \{ [x_1], [x_2], \ldots, [x_{2k - 1}] \} \), a subset of the odd vertices of \( \Omega_{2k} \). If \( k \) is even, then \( D \) is a determining set for the odd component of \( \tilde{\Omega}_{2k} \). If \( k \) is odd, then \( D \) is a determining set for \( \tilde{\Omega}_{2k} \).

**Proof.** Assume \( \alpha \in \text{Aut}(\Omega_{2k}) \) fixes pointwise the vertices in \( D \). Any graph automorphism of \( \tilde{\Omega}_{2k} \) must respect its separation into two components if \( k \) is even, or its bipartition if \( k \) is odd. Thus, since \( \alpha \) fixes \( D \), \( \alpha \) must map odd vertices to odd vertices and even vertices to even vertices.

Any neighbor of a vertex in \( D \) has weight \( [k - 1, k + 1] \). Conversely, let \( y \in V(\Omega_{2k}) \) be a vertex of weight \( k + 1 \). Then \( y \) is adjacent to \( x_i \) if and only if \( i \in \text{supp}(y) \); equivalently \( y \) can be uniquely identified either by which \( k + 1 \) of
the $x_i$ it is adjacent to, or by which $k - 1$ of the $x_i$ it is not adjacent to. In the quotient graph,

$$\{[y]\} = \bigcap\{N([x_i]) \mid i \in \text{supp}(y)\}.$$  

If $2k \notin \text{supp}(y)$, then $[y]$ is the unique common neighbor of $k + 1$ elements of $D$. If $2k \in \text{supp}(y)$, then $[y]$ can still be identified by which $k$ elements of $D$ it is adjacent to and which $k - 1$ elements it is not adjacent to. Thus fixing $D$ fixes all vertices in $\tilde{\Omega}_{2k}$ of weight $k - 1, k + 1$. Then by Proposition 2, $\alpha$ must fix every odd vertex of $\Omega_{2k}$. If $k$ is even, then we are done.

If $k$ is odd, then $\tilde{\Omega}_{2k}$ is bipartite with any even vertex having only odd neighbors. By Lemma 6, since $\Omega_{2k}$ is twin-free, no two nonadjacent (i.e., even) vertices of $\Omega_{2k}$ have the same neighborhood. Hence $\alpha$ also fixes all even vertices and we are done.

Although the preceding proposition does not assert that $D$ is a minimum determining set, it is a minimal determining set. Without loss of generality let $D' = \{[x_1], [x_2], \ldots, [x_{2k-2}]\}$. Let $\sigma \in S_{2k}$ be the transposition permutation that interchanges $2k - 1$ and $2k$. Then the corresponding nontrivial permutation automorphism on $\Omega_{2k}$ fixes $x_1, \ldots, x_{2k-2}$ and so the induced nontrivial automorphism on $\tilde{\Omega}_{2k}$ fixes the elements of $D'$.

Corollary 4. $\text{Det}(\tilde{\Omega}_{2k}) \leq 2k - 1$.

6 Distinguishing $\Omega_{2k}$

Theorem 4. $2 < \text{Dist}(\Omega_{2k}) \leq m$, where $m$ satisfies

$$\binom{m}{2} \geq \begin{cases} 2k, & k \text{ odd}, \\ 2k + 1, & k \text{ even}. \end{cases}$$

Proof. First assume $k$ is odd. By Proposition 3, $D$ is a determining set of $\tilde{\Omega}_{2k}$. The subgraph of $\tilde{\Omega}_{2k}$ induced by $D$ is a null graph and so has distinguishing number $|D| = 2k - 1$. Thus by Theorem 1, $\tilde{\Omega}_{2k}$ can be $2k$-distinguished.

Next assume $k$ is even. By Proposition 3, $D$ is a determining set of the odd component of $\tilde{\Omega}_{2k}$. If $k = 2$, then the subgraph of $\tilde{\Omega}_{2k}$ induced by $D$ is a complete graph, and otherwise it is a null graph. In all cases, it has distinguishing number $2k - 1$. Thus by Theorem 1, the odd component of $\tilde{\Omega}_{2k}$ can be $2k$-distinguished.

Since the even component is an isomorphic copy of the odd component, $\tilde{\Omega}_{2k}$ can be $(2k + 1)$-distinguished.

Suppose there exists an $\ell$-distinguishing labeling $\tilde{f}$ of $\tilde{\Omega}_{2k}$. To extend it to a distinguishing labeling on $\Omega_{2k}$, recall that by Lemma 2, twin vertices in $\Omega_{2k}$ must be assigned different labels in any distinguishing labeling.

If $m$ satisfies

$$\binom{m}{2} \geq \ell,$$
then we can create $\ell$ different label-pairs from $m$ different labels. We assign these label-pair to vertices in $\tilde{\Omega}_{2k}$ according to $\tilde{f}$, then randomly assign one label from each label-pair to the two vertices of $\Omega_{2k}$.

The following argument shows that this creates an $m$-distinguishing labeling of $\Omega_{2k}$. Suppose $\alpha \in \text{Aut}(\Omega_{2k})$ satisfies $f(u) = f(\alpha(u))$ for all $u \in V(\Omega_{2k})$. Then by Lemma 4,

$$f(u + 1) = f(\alpha(u + 1)) = f(\alpha(u) + 1),$$

and so

$$\tilde{f}([u]) = \{f(u), f(u + 1)\} = \{f(\alpha(u)), f(\alpha(u) + 1)\} = \tilde{f}([\alpha(u)]) = \tilde{f}(\tilde{\alpha}([u])).$$

By the assumption that $\tilde{f}$ is distinguishing, $\tilde{\alpha}$ is the identity on $\tilde{\Omega}_{2k}$, which means that either $\alpha(u) = u$ or $\alpha(u) = u + 1$. Since twin vertices have different labels under $f$ and $\alpha$ respects $f$, $\alpha$ must be the identity on $\Omega_{2k}$. \qed

The table below shows minimum values of the upper bound $m$ for $2 \leq k \leq 18$.

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-----|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| $m$ | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 | 9 | 9 |

So when $k = 18$, $\Omega_{36}$ has $2^{36} \approx 69$ billion vertices, each of degree $\binom{36}{18} \approx 9$ billion, and yet it has distinguishing number no bigger than 9.

### 7 Open Questions

**Question 1.** Is $\text{Det}(\Omega_{2k}) = 2k - 1$ or can it be smaller?

**Question 2.** Let $k \geq 2$; let $m$ be the smallest integer so that $\binom{m}{2} \geq \begin{cases} 2k, & \text{k odd,} \\ 2k + 1, & \text{k even.} \end{cases}$

For which $k \geq 2$ does $\text{Dist}(\Omega_{2k}) = m$?

### A Proof of Lemma 7

**Lemma 7** For distinct odd $m, n$, both less than or equal to $k$,

$$\binom{m}{\frac{m+1}{2}} \binom{2k-m}{\frac{k-m+1}{2}} \neq \binom{n}{\frac{n+1}{2}} \binom{2k-n}{\frac{k-n+1}{2}}.$$
Proof. It suffices to show that the sequence
\[
\binom{n-2}{k-n+2} \binom{2k-n+2}{k-n+1} > \binom{n}{k+1} \binom{2k-n}{k-n+1}
\]
is monotone decreasing (where \(m\) is largest odd number satisfying \(m \leq k\)), and for this it suffices to show that for \(n\) odd, \(1 < n \leq k\),
\[
\binom{n-2}{k-n+2} \binom{2k-n+2}{k-n+1} > \binom{n}{k+1} \binom{2k-n}{k-n+1}.
\]
We use some combinatorial algebra to rewrite the binomial coefficients:
\[
\binom{n+1}{k+1} = \frac{n!}{(n+1)!} = \frac{n(n-1)(n-2)!}{(n+1)!} = \frac{(n-1)(n-2)!}{(n+1)!} = \frac{n(n-1)}{(n+1)!} \binom{n+1}{k+1} = \frac{n(n-1)}{n+1} \binom{n+1}{k+1}.
\]
Similar algebraic manipulations yield
\[
\binom{2k-n+2}{k-n+1} = \frac{4(2k-n+2)}{2k-n+3} \binom{2k-n}{k-n+1}.
\]
Substituting in, we are trying to show that
\[
\binom{n-2}{k-n+2} \binom{2k-n+2}{k-n+1} > \binom{n}{k+1} \binom{2k-n}{k-n+1}.
\]
Cancelling equal terms and cross-multiplying, this holds if and only if
\[
(2k-n+2)(n+1) > n(2k-n+3),
\]
which simplifies to \(k+1 > n\). Since we assumed \(n \leq k\), we are done. \(\square\)

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