Tunneling of Bell Particles, Page Curve and Black Hole Information

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Abstract

We propose that the quantum states of black hole responsible for the Bekenstein-Hawking entropy are given by a thin shell of Bell particles located at the region just underneath the horizon. We argue that the configuration can be stabilized by a new kind of degeneracy pressure which is suggested by a noncommutative geometry in the interior of the black hole. Black hole singularity is avoided. We utilize the work of Parikh and Wilczek [1] to include the effect of tunneling on the Bell particles. We show that partially tunneled Bell particles give the Page curve of Hawking radiation, and the entirety of information initially stored in the black hole is returned to the outside via the Hawking radiation. In view of entropic force, the location of these Bell states is naturally related to the island and the quantum extremal surface.

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1 Introduction

There are mounting theoretical evidences that a black hole obeys the first law of thermodynamics with an entropy

\[ S_{\text{BH}} = \frac{A}{4G} \]

and a temperature \( T = \frac{1}{8\pi M} \) due to a thermal Hawking radiation [2]. Despite remarkable progress that has been made in microstate counting [3], it is still not known what is the nature of the gravitational degrees of freedom that are being counted by (1.1).

Another outstanding problem of the black hole is the information problem [4]. Consider a black hole form from a pure state. Assuming that the Hawking radiation is thermal, then the entanglement entropy outside the black hole increases monotonically for the entire course of life of the black hole. This \textit{Hawking curve} violates the fundamental unitarity principle of quantum mechanics: the fine-grained entanglement entropy should not exceed the coarse-grained black hole entropy. On the other hand, Page argued that [5, 6] if the black hole evolution process is unitary, then the total system of black hole and radiation must go back to a pure state at the end of the evaporation. As a result, unitarity of quantum black hole requires the following properties for the Hawking radiation: 1. Page curve: the entanglement entropy of Hawking radiation should initially rise until the so called Page time \( t_P \) when it starts to drop down to zero as the black hole completely vanishes. 2. Recovery of information: at the end of the black hole life, one must be able to recover from the Hawking radiation all the information of the initial pure state. It
was Page’s remarkable insight that the Page curve behavior of the Hawking radiation, due to its accessibility to external observers, can be especially useful as a decisive criteria for unitary theory of quantum gravity.

Recently, by using a fully quantum form [7] of the Ryu-Takayanagi entropy formula [8,9], the entanglement entropy of the Hawking radiation was computed and shown to obey the Page curve [10–12,12]. Central to this analysis is the emergence of island, regions of spacetime that are completely disconnected and spacelike separated from the region of the Hawking radiation. From the semi-classical point of view, island originated from the wormhole saddle in the replica path integral for the entanglement entropy [13]. It is interesting to note that the island is located just beneath the horizon for evaporating Schwarzschild black hole [10–12].

That the Page curve is obtained gives confidence that the AdS/CFT correspondence and the generalized entropy formula constitutes a credible quantum gravity framework to study black holes. Yet, it still leaves many questions unanswered. For example, it is believed that the Bekenstein-Hawking entropy (1.1) is coarse-grained in nature [14,15]. From this perspective, what role does island play in coarse graining the initial pure state? How does it help in returning the information to the environment? The main motivations of this work have been to understand better the origin of the Page curve and the return of information in terms of explicit spacetime quantum mechanical processes.

2 Quantum pressure and interior distribution of degrees of freedom

General relativity predicts that any object that has collapsed beyond a certain point would form a black hole, inside which there is a singularity. For a spherical symmetric black hole, the spacetime is given by the Schwarzschild metric, a vacuum solution ($T_{\mu\nu} = 0$) of the Einstein equation, and the singularity at the origin represents a place where matter is compressed infinitely and the classical description of spacetime breaks down. In the absence of a new source of pressure to counterbalance the gravitational pull, the collapse of black hole to a singularity is unavoidable. However it should be noted that the singularity theorem [16] is based on the canonical assumptions of general relativity and certain energy conditions on the matter. It has been widely speculated that quantized gravity would resolve the singularity and provide a complete description of the interior of black hole. The singularity may also be resolved in the form of wormhole due to the existence of quantum energy. Given little is known inside the black hole, we consider in this paper the possibility of a novel kind of degeneracy pressure for the degrees of freedom in the interior of a black hole that can counter-balance the gravitational collapse.
2.1 Degeneracy Pressure of Neutron Star

To get motivated, let us briefly review the collapse process of a compact star. Let \( M \) be the mass and \( \rho \) be the radius of the star. The gravitational energy is of the order of \( E_g \sim -GM^2/\rho \). This gives rise to a gravitational pressure \( P_g = -\frac{\partial E_g}{\partial V} \)

\[
P_g \sim -\frac{GM^2}{\rho^4}.
\]

(2.1)

The collapse would be halted if the gravitational pressure is balanced out by some pressure produced by the star matter. The strongest known matter pressure is the neutron degeneracy pressure. It is useful to review its origin. Consider a gas of free particles with a dispersion relation of the form \( E = d_0 p^\beta \), where \( E \) is the kinetic energy, and \( d_0, \beta \) are constants. This covers the non-relativistic case \( E = p^2/2m \), fully relativistic case \( E = p \), as well as possible quantum deformed dispersion relations (QDR) with \( \beta \neq 1, 2 \). The density of states is \( g(E)dE = g_sV4\pi p^2dp/(2\pi\hbar)^3 \), \( g_s = 2s + 1 \) is a spin factor, then takes the form

\[
g(E) = c_0 V E^\alpha,
\]

(2.2)

where \( \alpha = 3/\beta - 1 \) and \( c_0 \) is a constant. For a fully degenerate system of fermions, the occupation number \( n(E) \) is given by

\[
n(E) = \begin{cases} 
1, & E \leq \mu, \\
0, & E > \mu,
\end{cases}
\]

(2.3)

where \( \mu \) is the Fermi level at zero temperature. This gives the total number of particles \( N = \int dE g(E)n(E) \) and the total kinetic energy \( U = \int dE g(E)n(E)E \)

\[
N = \frac{c_0 V}{\alpha + 1} \mu^{\alpha + 1},
\]

(2.4)

\[
U = \frac{\alpha + 1}{\alpha + 2} N \mu.
\]

(2.5)

As a result, the matter pressure \( P_m = -(\partial U/\partial V)_N \) reads

\[
P_m \approx -N \frac{\partial \mu}{\partial V},
\]

(2.6)

up to an order 1 proportional constant \( \frac{\alpha + 1}{\alpha + 2} \). Therefore, the degeneracy pressure is determined by the volume dependence of the Fermi level \( \mu \). For non-relativistic neutron star, \( \alpha = 1/2 \). (2.4) gives \( \mu = \frac{1}{2m}(6\pi^2 N/V)^{2/3} \) and

\[
P_m \sim \frac{N^{5/3} \hbar^2}{m \rho^{5/3}}
\]

(2.7)

where \( m \) is the mass of neutron. The collapse is halted if \( P_g + P_m = 0 \) and this is possible if the mass limit \( M \lesssim 2M_\odot \) is satisfied. Physically, the degeneracy pressure (2.7) which stabilizes the neutron star arises from the fact that all the available energy levels of the system are filled such that no further addition of states is possible.
2.2 Black Hole Interior

Back to the case of black hole. The fact that black hole has an entropy and a temperature suggests that black hole is not a classical vacuum as described in general relativity, but a nontrivial quantum system of microstates. Obviously these cannot be elementary particles of the standard model since ordinary energy-momentum cannot provide a pressure strong enough to withstand the collapse of gravity. Instead, these quanta should have an universal nature that is independent of the matter that has been collapsed to form the black hole. It then seems natural that these are the elementary quanta of black hole spacetime itself. In the following, we will ignore their mutual interaction and model the quantum black hole as a gas of free particles. As these microstates arises from the quantization within the compact interior of black hole, it is natural that they carry an average energy of the order \( \mu \sim 1/\rho \). We will now see that this simple assumption tells us something interesting about the nature of these microstates.

The area dependence of the Bekenstein-Hawking entropy has led to the formulation of the holographic principle [17,18] and has been the chief guiding route to the understanding of quantum gravity. For a microcanonical ensemble, the entropy takes the form \( S = N \ln \Omega \) where \( N \) is the number of degrees of freedom and \( \Omega \) is the phase space volume available to each individual degrees of freedom. Normally the entropy of a many bodies continuum quantum system is divergent since there is an infinite volume of phase space available to each degree of freedom. That the black hole entropy (1.1) is finite means that not just the degrees of freedom making up the black hole is finite, but also the phase space volume available to each degree of freedom is finite. In fact, in the holographic picture of [18], there is a two states spin system (hence \( \Omega = 2 \)) associated with each site of a two dimensional lattice of quanta (called “partons”) living on the horizon area. Now in our model, using the fact that the energy of the system must be equal to the mass \( M \) of the black hole, we obtain immediately from (2.5)

\[
N \sim M \rho \sim A/G
\]  

and the entropy (1.1) is reproduced if the number of available states to each particle is a finite constant. This simple interpretation of the formula (1.1) suggests that the entirety of the black hole microstates are distributed over a thin shell near the horizon. In this case, the total energy of the system is \( E_m \sim N\hbar c/\rho \) and (2.6) gives the degeneracy pressure,

\[
P_m \sim N\hbar/\rho^4,
\]

which cancels that (2.1) of gravity up to numerical factor of order 1. This simple analysis suggests that the collapsed matter can indeed be stabilized just underneath the horizon by the degeneracy pressure.

So what is the reason for this exclusion of states? One possible origin of this is noncommutative geometry: that the interior of black hole is in fact described by a quantized space with an uncertainty relation

\[
\Delta V \gtrsim \ell_P^3.
\]
In this case, the number of states that is available in the region of space underneath the horizon and with a thickness \( w \sim l_P \) is given by \( N \sim \rho^2/l_P^2 \). It explains (1.1). We note that in this picture, the horizon is at the junction between the exterior commutative geometry and the interior quantum geometry, and the degeneracy pressure may be thought of as some kind of interface pressure.

It is amazing that the quantum extremal surface for Schwarzschild black hole is also located just underneath the horizon. One may speculate that this is how the quantum space and the collapsed matter are described as island and quantum extreme surface in holography. Recall the location of island is determined by minimizing the generalized entropy with contributions from both gravity and matters, i.e. \( \delta S_{\text{gen}} = 0 \) \[15\]. In the viewpoint of entropic force \[19\], this means the quantum extreme surface is the force balance surface \( F = T dS_{\text{gen}}/d\rho = 0 \). It is consistent with our discussions of quantum pressure \( P_g + P_m = F/A = 0 \), where \( P_g \) and \( P_m \) originated from the entropy of gravity and matters, respectively.

Summarizing, we propose that, instead of a singularity, the interior of black hole is described by a quantum geometry with the uncertainty relation (2.10). The resulting maximal occupancy of states provides a novel degeneracy pressure which stabilizes the collapsed matter at a thin region (thickness \( w \sim l_P \)) right beneath the horizon.

### 3 Black hole interior as thin shell of Bell particles

Consider a black hole form in flat space from a pure state and decays under Hawking radiation. If unitarity of quantum mechanics is not violated in the black hole formation process, then the black hole interior and exterior together is in a pure state. Due to our ignorance of the interior, this give rises to a number \( S_{\text{BH}} \) of coarse-grained states with entropy (1.1). By definition, the knowledge of these states together with the knowledge of the exterior constitutes a pure state. Eventually, the black hole is exhausted by the Hawking radiation. If we assume that unitarity is preserved throughout, then the final state of the Hawking radiation cannot be purely thermal, but it must encompass the information contained in the initial set of coarse-grained states so that, together with the outside information, a pure state can be reconstructed.

It has been suggested that some form of nonlocality is needed to resolve the black hole information problem (see \[20\] for a review). As quantum nonlocality is best captured by entanglement, it seems natural to consider the coarse-grained degrees of freedom to be rich in entanglement content. We propose in this paper that the black hole interior degrees of freedom are given by maximally entangled Bell pairs of particles localized on the internal side of the horizon. Is this observable to the outside? Our main idea is that tunneling can reveal the interior information of black hole to the outside world.

Let us briefly comment on the Hawking radiation. In addition to being an effect of QFT in curved spacetime, a particularly transparent understanding of the origin of Hawking radiation
is the quantum mechanical picture of Parikh and Wilczek [1], where the interior of black hole is considered to be in a vacuum state and the effect of tunneling on the virtually created particles resulted in the Hawking radiation. For an outgoing particle with energy \( \omega \) created just inside the horizon, they found that it can travels cross the horizon and results in a nonvanishing imaginary part for the particle action:

\[
\text{Im} S = 4\pi\omega(M - \frac{\omega^2}{2}),
\]

This corresponds to a tunneling process with the tunneling rate \( \Gamma = \Gamma_0 e^{-8\pi\omega(M - \frac{\omega^2}{2})} \), where \( \Gamma_0 \) is a prefactor which can be computed from a more detailed knowledge of the dynamics. The leading exponential \( \omega \) dependence in \( \Gamma \) registers a Boltzmann thermal distribution with the Hawking temperature \( T \). The \( \omega^2 \) term is a back reaction term, suggesting that the spectrum is slightly deviated from the thermal one.

In addition, pair creation outside the horizon was also considered. In this so called anti-particle channel, the anti-particle follows a time reversed ingoing geodesic crosses the horizon and also makes contribution to the Hawking radiation. Note that while the original analysis of [1] is for massless particles, it applies to massive particles as well and the tunneling rate is the same.

Although this picture of tunneling explains very physically the existence of Hawking radiation and its temperature, it also leads to the Hawking curve. To see this, consider a virtual pair created in the interior side of the horizon (particle channel). Since the pair was created entangled, an entanglement between the black hole and the Hawking radiation is created when the particle tunnels through and the anti-particle got absorbed by the black hole. The rate of increase of the entanglement entropy of the Hawking radiation is given by \( \alpha(\omega)A \), where \( \alpha(\omega) \) is the creation rate per unit area of virtual pairs on the interior side of horizon times the tunneling probability. Similarly, there is a contribution \( \beta(\omega)A \) from the anti-particle channel, where \( \beta(\omega) \) is the creation rate of virtual pairs on the exterior side of horizon times the tunneling probability. As \( \alpha, \beta \) are positive, the entanglement entropy of Hawking radiation increases monotonically. This Hawking curve violates the unitarity of quantum mechanics.

We note that the main difference between the consideration of [1] and ours is in the assumption concerning the black hole interior. Instead of a vacuum as considered in [1], we propose that there is a thin shell of Bell pairs sitting right beneath the horizon. We will next show how tunneling in our model affects the Bell pairs and leads to the Page curve and the recovery of the full entanglement content of the coarse-grained states in late time of the Hawking radiation.

4 Tunneling and entanglement swapping of Bell pairs

Due to their close proximity to the horizon, both particles of the Bell pairs can tunnel and leave the black hole as Hawking radiation. In addition, the Bell particles also have an interesting effect of entanglement swapping. Consider the particle channel, the anti-particle \( \bar{p} \) that is left behind can get annihilated by a particle \( b_2 \) of one of the Bell pairs. As a result, the particle \( p \) becomes entangled with the other partner \( b_1 \) of the Bell pair, and entanglement is swapped from the Bell pair and the virtual pair to one between the Hawking particle and the remaining Bell particle inside the black hole. Similarly, entanglement swapping occurs in the anti-particle channel. See
Both of these processes will not only alter the entropy content of the black hole, but also introduce an entanglement entropy for the Hawking radiation due to the partially escaped entangled pairs. To study the dynamics of the entangled pairs of particles, let us denote the energies of the particles of each entangled pair by $0 \leq \omega_1, \omega_2 \leq M_0$, where $M_0$ is the original mass of the black hole. Let $n_1(\omega_1, \omega_2, t)d\omega_1 d\omega_2$ be the number of entangled pairs that are located entirely inside the horizon and have energies within the intervals $(\omega_1, \omega_1 + d\omega_1)$ and $(\omega_2, \omega_2 + d\omega_2)$. Similarly, let us denote by $n_2(\omega_1, \omega_2, t)d\omega_1 d\omega_2$ the number of entangled pairs that has the $\omega_1$-particle inside the horizon and $\omega_2$-particle outside, and $n_3(\omega_1, \omega_2, t)d\omega_1 d\omega_2$ the number of entangled pairs that are located entirely outside the horizon. In order to avoid over counting, the domain of energies for $n_1(\omega_1, \omega_2), n_3(\omega_1, \omega_3)$ is given by $D := \{(\omega_1, \omega_2) | 0 \leq \omega_1 \leq \omega_2 \leq M_0\}$. As for $n_2(\omega_1, \omega_2, t)$, the first (resp. second) argument $\omega_1$ (resp. $\omega_2$) refers to the energy of the particle that is inside (resp. outside) the horizon. There is no constraint on the size of $\omega_1, \omega_2$ and the domain for $n_2$ is given by $D' := \{(\omega_1, \omega_2) | 0 \leq \omega_1, \omega_2 \leq M_0\}$. See Figure 2.

Taking into account of tunneling of the Bell particles and entanglement swapping, it is easy
to obtain
\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= -(\Gamma_1 + \Gamma_2) n_1 - (\delta_1 + \delta_2) A, \\
\frac{\partial n_2}{\partial t} &= \Gamma_2 n_1 - \Gamma_1 n_2 + \delta_1 A, \quad \text{for } \omega_1 \leq \omega_2, \\
\frac{\partial n_2}{\partial t} &= \Gamma_2 \tilde{n}_1 - \Gamma_1 n_2 + \delta_1 A, \quad \text{for } \omega_1 > \omega_2, \\
\frac{\partial n_3}{\partial t} &= \Gamma_1 n_2 + \Gamma_2 \tilde{n}_2,
\end{align*}
\]
where \( \Gamma_a := \Gamma(\omega_a, M(t)) \ (a = 1, 2) \) is the tunneling rate for a particle of energy \( \omega_a \) from a black hole of mass \( M(t) \). \( \alpha_a := \alpha(\omega_a, M(t)) \) (resp. \( \beta_a := \beta(\omega_a, M(t)) \)) is the production rate per unit area of the conventional vacuum created Hawking radiation in the anti-particle channel (resp. particle channel), and \( \delta_a := \alpha_a + \beta_a \) is the total production rate from both channels. Here \( n_i = n_i(\omega_1, \omega_2, t) \), while \( \tilde{n}_{1,2} = n_{1,2}(\omega_2, \omega_1, t) \) has its arguments reversed. Physically, the \( n_i \) terms and the \( A \) terms on the RHS of (4.1) (resp. (4.2) or (4.3)) represent the effect of direct tunneling and quantum swapping on the entangled pairs that are entirely inside the black hole (resp. partially inside or entirely outside the black hole).

The total number of each types of entangled pairs is
\[
N_i(t) = \int_{D_i} d\omega_1 d\omega_2 n_i(\omega_1, \omega_2, t), \quad i = 1, 2, 3, \quad (4.4)
\]
where, as explained above, \( D_1 = D_3 = D \) and \( D_2 = D' \). Physically, \( (2N_1 + N_2) \) represent the amount of coarse-grained entropy of the black hole at time \( t \), the \( N_2 \)-part of which is entangled with the outside observer and gives the entanglement entropy of Hawking radiation. As for \( N_3 \), it represents the amount of entanglement information contained in the Hawking radiation. In our model, the mass of the black hole is given by
\[
M(t) = \int d\omega_1 d\omega_2 [(\omega_1 + \omega_2) n_1 + \omega_1 n_2] \quad (4.5)
\]
where the appropriate domains of integration is understood. In a consistent analysis, (4.5) will have to be considered together with (4.1) - (4.3). This is quite a complicated system to solve. It turns out that without assuming the form of \( \Gamma(\omega, M) \) and \( \delta(\omega, M) \), and without solving the system, one can immediately show that the Hawking radiation obeys the Page curve and that the complete information of the black hole is returned.

Let us consider a black hole formed at \( t = 0 \), implying the conditions \( n_2 = n_3 = 0 \) initially. Consider first (4.1). As the RHS of (4.1) is non-positive, \( n_1 \) will continue to decrease until it reaches zero, where also \( \partial n_1 / \partial t = 0 \). Denote this time as \( t_1(\omega_a) \) and define \( t_1 := \text{Supp}(\omega_1, \omega_2) \in D(t_1, \omega_a) \). We have
\[
N_1 = 0 \quad \text{for } t \geq t_1. \quad (4.6)
\]
Next consider (4.2). Starting with the initial conditions \( n_2 = 0 \) and \( n_1 > 0 \), \( n_2 \) is set to increase initially. As time progresses, there will be a crossover over time where the RHS of (4.2) vanishes and becomes negative subsequently. Then \( n_2 \) will continue to decrease until it reaches zero at some time \( t_{2,\omega_a} \). Define \( t_E := \text{Supp}_{(\omega_1, \omega_2) \in D'}(t_{2,\omega_a}) \), we have

\[ N_2 = 0 \quad \text{for} \quad t \geq t_E. \quad (4.7) \]

Finally consider (4.3). It is clear that \( n_3 \) increases monotonically until \( n_2 \) reaches zero at \( t = t_{2,\omega_a} \), then it remains a constant. Therefore we have

\[ N_3 = N_{3f} \quad \text{a constant} \quad \text{for} \quad t \geq t_E. \quad (4.8) \]

It is clear that \( t_1 \) is when the black hole’s degrees of freedom become completely entangled with the environment and \( t_E \) is the end time that the black hole dies.

Note that as \( N_2 \) starts zero at \( t = 0 \) and ends at zero again at \( t = t_E \), it must follow an inverted V-shape curve and reaches a maximum at some intermediate time \( t_P \). This is generic and we obtain the Page curve for the Hawking radiation, with \( t_P \) being the Page time. By expressing \( N_2 \) as integrals over the domain \( D \), it is easy to establish the conservation equation:

\[ \frac{d}{dt}(N_1 + N_2 + N_3) = 0. \quad (4.9) \]

It is remarkable that this leads immediately to

\[ N_{3f} = N_{10}, \quad (4.10) \]

meaning that all the entanglement information originally stored in the Bell pairs are returned to the exterior observer via the Hawking radiation. We note that it is crucial in our model to include the thin shell of Bell particles to start with and allow the tunneling of them. Without these, our equations (4.1)-(4.3) will reduce to the equation \( \partial n_2 / \partial t = \delta_1 A \) as in the conventional tunneling model of Hawking radiation. There would then be the Hawking curve and it would not possible to return information via \( n_3 \). In the supplementary material, we justify that \( t_E \) is finite in our model. We also show the explicit time dependence of the \( n_i \)'s for some typical values of the energies, which confirm the generic behavior discussed here.

Note that in our model, the Hawking radiation is non-thermal and contains correlations that are necessarily highly nonlocal as they could arise from tunneling process that occurs at very different times over the life of the black hole. We remark that an infalling observer will encounter the thin shell of entangled particles located underneath the horizon and get assimilated there as new Bell particles. In this sense the shell of Bell particles acts like a firewall [22]. We have considered and proposed that the wave function of the thin shell matter can be written in terms of 2-qubit Bell states. It is interesting to understand if this is true and how the firewall makes it.

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A Semiclassical analysis

In a semiclassical estimation, Hawking radiation reduces the mass of the black hole as [21]

\[
\frac{dM}{dt} = -\frac{Q}{3M^2},
\]

(A.1)

where \( Q = 3\alpha/G^2 \) and \( \alpha \) is some numerical constant. This gives

\[ M(t) = M_0 \left[ 1 - \frac{Qt}{M_0^3} \right]^{1/3} \]

(A.2)

and a finite lifetime of the black hole \( t_E = M_0^3/Q \). In our model, this should come out from the consistent set of equations (4.1)-(4.3) and (4.5). Although the complete analysis is quite complicated, we can see that our model does roughly give (A.1) near the final stage of evaporation and so \( t_E \) is finite. In fact, using (4.1) and (4.2), we have

\[
\frac{dM}{dt} = \int_D -\Gamma_1\omega_1(n_1 + n_2) - \Gamma_2\omega_2(n_1 + \tilde{n}_2),
\]

(A.3)

where we have ignored the much smaller terms \(-(\delta_2\omega_1 + \delta_1\omega_2)A\) on the RHS since \( \delta \ll \Gamma \) as \( \delta \) involves an additional vacuum creation rate. Near the final stage of evaporation, it is \( \int n_1 \sim \int n_2 \sim O(1) \) and we can use the mean value theorem of calculus to estimate that \( \int \Gamma_1\omega_1n_1 = \frac{\Gamma_1\omega_1}{\Gamma_1\omega_1} \int n_1 \sim 1/M^2 \), where we have used \( \omega_1 \sim M \) and that \( \Gamma \sim \Gamma_0 \) for small \( M \). Here the prefactor of the tunneling rate has a dependence \( \Gamma_0 \sim 1/M^3 \) coming from the phase space volume. As a result, (A.3) is consistent with the semiclassical result (A.1) at time close to the end point and so \( t_E \) is finite.

To get a better feeling of the time evolution of the Bell pairs, let us consider the approximate mass function \( M(t) \) (A.2) with \( M_0 = 1, Q = 0.05 \) so that \( t_E = 20 \). In Figure 3, we show the plots of \( n_1, n_2, \tilde{n}_2, n_3 \) for two typical set of energies \( (\omega_1, \omega_2) \) given by (a): \( (0.001, 0.1) \) and (b): \( (0.05, 0.05) \). Since \( \delta_a \ll \Gamma_a \), we have taken \( \delta_a = 0 \) here for simplicity. Including \( \delta_a \neq 0 \) won’t affect these plots much. We see that the generic behaviors of the \( n_i \)’s discussed above do get captured very well here, showing that (A.2) is a decent approximation. However, we can see that for the set (a) of energies, the time \( t_2 \) for \( n_2 \) and \( \tilde{n}_2 \) to decrease to zero is actually slightly larger than \( t_E = 20 \), showing that (A.2) is not entirely consistent with (4.1)-(4.3). Nevertheless (A.2) is a pretty good approximation and a perturbative scheme can in principle be devised to solve the system (4.1)-(4.3), (4.5).

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Figure 3: Evolution of Bell states for the energies \((\omega_1, \omega_2) = (a) (0.001, 0.1), (b) (0.05, 0.05)\). In case (b), the curves for \(n_2\) and \(\tilde{n}_2\) overlap and is represented by the red line. The mass \((A.2)\) with \(M_0 = 1, Q = 0.05\) is adopted for these plots.

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