Improved generating technique for \( D = 5 \) supergravities and squashed Kaluza-Klein Black Holes

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Abstract

Recently we suggested a solution generating technique for five-dimensional supergravity with three Abelian vector fields based on the hidden \( SO(4,4) \) symmetry of the three-dimensionally reduced theory. This technique generalizes the \( G_{2(2)} \) generating technique developed earlier for minimal 5D supergravity (A. Bouchareb, G. Clément, C-M. Chen, D. V. Gal’tsov, N. G. Scherbluk, and Th. Wolf, Phys. Rev. D 76, 104032 (2007)) and provides a new matrix representation for cosets forming the corresponding sigma-models in both cases. Here we further improve these methods introducing a matrix-valued dualisation procedure which helps to avoid difficulties associated with solving the dualisation equations in the component form. This new approach is used to generate a five-parametric rotating charged Kaluza-Klein black hole with the squashed horizon adding one parameter more to the recent solution by Tomizawa, Yasui and Morisawa which was constructed using the previous version of the \( G_{2(2)} \) generating technique.

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I. INTRODUCTION

Although all supersymmetric solutions to minimal five-dimensional supergravity were classified \cite{1}, to find non-BPS configurations is a more difficult problem which attracts attention in view of the discovery of black rings and their generalizations \cite{2}. An efficient tool to construct exact solutions to multidimensional Einstein equations consists in dimensional reduction based on the assumption of existence of a sufficient number of commuting Killing symmetries (toroidal reduction). Starting with D-dimensional Einstein equations coupled to scalar and vector fields and assuming the existence of $D - 3$ Killing vectors one is able to derive three-dimensional gravity coupled sigma-models in which the target scalar fields incorporate the initial scalars, vectors and moduli of the toroidal reduction. With some luck, the sigma-model target space happens to be a coset space $G/H$ where $G$ is a semi-simple group known as the hidden symmetry group (for a recent review see \cite{3}). The hidden symmetry can be used directly to generate new solution form known ones with the same three-dimensional metric. Another way of using the dimensional reduction consists in solving the sigma-model equations on the some submanifold of the target space (e.g. the null geodesic method \cite{3} which is an alternative way to find BPS solutions). Finally, one can use the integrability methods assuming one more Killing symmetry which allows to construct two-dimensional systems (for a concise formulation see, e.g., \cite{4}).

Sigma-model technique for minimal five-dimensional supergravity was developed in \cite{5},\cite{6}, for an earlier discussion of hidden symmetries in this theory see \cite{7, 8, 9, 10, 11, 12}. The hidden symmetry is this case is the non-compact version $G_{2(2)}$ of the lowest exceptional group $G_2$. To formulate the solution generating technique one has to use some matrix representation of the coset. Representing the seed solution in the matrix terms and acting by hidden symmetry transformations one can extract the sigma-model variables for new solutions. In \cite{5},\cite{6} an explicit $7 \times 7$ representation of the $G_{2(2)}$ was used found earlier by Gunyadin and Gursey \cite{13}.

The generalization to the case of five-dimensional supergravity with three $U(1)$ vector and two scalar fields was given in \cite{14}. Apart from being more general, this theory is interesting by the fact that the corresponding hidden symmetry is given by such a familiar group as $SO(4, 4)$ whose structure is simpler than that of $G_{2(2)}$. Actually, one of the ways to construct matrix representation of $G_{2(2)}$ consists in using $SO(4, 4)$ as a starting point \cite{13}
and imposing some constraints. It turns out that dealing with the unconstrained \(SO(4, 4)\) is in some sense easier. The matrix representation of the relevant three-dimensional coset \(SO(4, 4)/SO(4) \times SO(4)\) was given in terms of the \(8 \times 8\) matrices split into the \(4 \times 4\) blocks. Freezing the scalar moduli and identifying the vector fields reduces this theory back to minimal 5D supergravity thus providing an alternative to the technique of [3] in terms of \(8 \times 8\) matrices.

Although construction of the new target space variables via hidden symmetry transformations is a relatively simple problem, certain complications may arise in solving the dualisation equations for the one-form fields to scalars [3, 14]. These have to be solved twice: for the seed solution to obtain its description in terms of the coset matrix, and for the transformed solution in order to extract the metric and the matter fields from the transformed coset matrix. Though these equations are just linear partial differential equations, solving them may present difficulties in the case of complicated transformations. To remedy this problem, here we propose to pass to dualized variables in the matrix form. Such a possibility is suggested by the fact that the three dimensional dual to the sigma-model matrix-valued current one-form is closed by virtue of the equations of motion. Then locally it is an exact form, and this provides the matrix-valued one-form whose exterior derivative is dual to the the initial matrix current. This new matrix transforms under the global action of the hidden symmetry by some related transformation, this allows us to find it independently. From this matrix one can read out the metric and matter fields of the transformed solution algebraically, thus avoiding the inverse dualisation problem.

We apply this approach to generate new Kaluza-Klein squashed black holes. Such black holes look as five-dimensional near the event horizon possessing the \(S^3\) angular section, but asymptotically \(S^3\) collapses to a twisted bundle of \(S^1\) over the base space \(S^2\) with a constant radius of \(S^1\) and growing radius of \(S^2\). Thus they become four-dimensional objects at infinity with a compactified fifth dimension. Such a solution to five-dimensional Einstein-Maxwell system was proposed by Ishihara and Matsuno [15] in the non-rotating case. Its physical parameters and thermodynamical properties were investigated in [16]. Black objects with such a topology is another alternative to usual five-dimensional black holes and black rings. A certain class (but not all) of squashed black holes can be obtained be the so-called squashing transformation. This procedure was applied to asymptotically flat [15, 17, 18] and non-asymptotically flat solutions such as Kerr-Gödel black holes [19, 20, 21]. In an attempt
to enlarge the class of solutions, more recently Tomizawa, Yasui and Morisawa [22] applied $G_{2(2)}$ transformations of [3] to construct a generalization of the charged Rasheed black hole [23] obtaining a new solution with four independent parameters: mass, angular momentum, Kaluza-Klein parameter $\beta$ (in the notation of [23]) and an electric charge. In this paper we will derive a more general five-parametric solution adding as an independent parameter the quantity $\alpha$ of [23], which corresponds to an electric charge in the four-dimensional interpretation of the Rasheed solution.

The paper is organized as follows. In the following section we give the brief review of the derivation of the $SO(4, 4)$ sigma-model. Then (Sec. III) we introduce the dualisation equations in the matrix form and present the transformation law for the new matrix-valued one-form. In Sec. IV we explore which isometries preserve the 5D Kaluza-Klein asymptotic behavior and describe the strategy of the solution generation. Finally in Sec. V we demonstrate how the Rasheed solution can be obtained from the Kerr metric using $SO(4, 4)$ transformations and construct a new five-parametric squashed black hole.

II. 3D SIGMA-MODEL

A. Dimensional reduction

In this section we briefly review the derivation of the 3D sigma-model for the $U(1)^3$ theory [14]. This theory may be regarded as a truncated toroidal compactification of the 11D supergravity:

$$I_{11} = \frac{1}{16\pi G_{11}} \int \left( R_{11} \star_{11} 1 - \frac{1}{2} F_{[4]} \wedge \star_{11} F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right),$$

where $F_{[4]} = d A_{[3]}$. Assuming an ansatz for the metric

$$ds_{11}^2 = ds_5^2 + X^1 (dz_1^2 + dz_2^2) + X^2 (dz_3^2 + dz_4^2) + X^3 (dz_5^2 + dz_6^2),$$

and the form field

$$A_{[3]} = A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 + A^3 \wedge dz_5 \wedge dz_6,$$

where all functions are independent of $z$, we obtain the the bosonic sector of 5D supergravity coupled to three scalar moduli $X^I$ ($I = 1, 2, 3$), satisfying the constraint $X^1 X^2 X^3 = 1$, and
to three vector fields $A^I$:

$$I_5 = \frac{1}{16\pi G_5} \int \left( R_5 \ast_5 1 - \frac{1}{2} G_{IJ} dX^I \wedge \ast_5 dX^J - \frac{1}{2} G_{IJ} F^I \wedge \ast_5 F^J - \frac{1}{6} \delta_{IJK} F^I \wedge F^J \wedge A^K \right), \quad (3)$$

$G_{IJ} = \text{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right), \quad F^I = dA^I, \quad I, J, K = 1, 2, 3.$

Here the Chern-Simons coefficients $\delta_{IJK} = 1$ for the indices $I, J, K$ being a permutation of 1, 2, 3, and zero otherwise. Contraction of the above theory to minimal 5D supergravity is effected via an identification of the vector fields:

$$A^1 = A^2 = A^3 = \frac{1}{\sqrt{3}} A,$$

and freezing out the moduli: $X^1 = X^2 = X^3 = 1$. This leads to the Lagrangian

$$L_5 = R_5 \ast_5 1 - \frac{1}{2} F \wedge \ast_5 F - \frac{1}{3\sqrt{3}} F \wedge F \wedge A.$$

It is worth noting that the 5D Einstein-Maxwell theory without the Chern-Simons term does not lead to the three-dimensional sigma model with a semi-simple hidden symmetry group, so in this case the solution generating technique can be formulated only for the static truncation of the theory. This explains why the charged rotating black hole solutions are not known analytically.

Consider now further reduction to three dimensions assuming the existence of two more commuting Killing symmetries. An overall assumption for the 11D manifold will be $\mathcal{M}_{11} = T^6 \times \Sigma \times \mathcal{M}_3$ where $\Sigma$ is $T^2$ if both these Killing vectors are asymptotically space-like, or $T^1 \times \mathbb{R}$ if one of them is asymptotically time-like. The full set of 11D coordinates $x^\mu$, $\mu = 1, \ldots, 11$ is thus split into $z^a \in T^6$, $a = 1, \ldots, 6$, $x^i \in \mathcal{M}_3$, $i = 1, \ldots, 3$ and $z^p \in \Sigma$, $p = 7, 8$. The decomposition of the 5D metric is given by

$$ds^2_5 = \lambda_{pq} (dz^p + a^p) (dz^q + a^q) - \kappa \tau^{-1} h_{ij} dx^i dx^j, \quad (4)$$

where all metric functions are independent on $z^a$ and $z^p$. The 5D metric components are parameterized by the KK one-forms $a^p = a^p_i dx^i$, the three-dimensional metric $h_{ij}$ of $\mathcal{M}_3$, three moduli fields $X^I$, $I = 1, 2, 3$ and the scalars $\varphi_1, \varphi_2, \chi$, which are arranged in the following $2 \times 2$ matrix

$$\lambda = e^{-\frac{2}{\sqrt{3}} \varphi_1} \begin{pmatrix} 1 & \chi \\ \chi^2 + \kappa e^{\sqrt{3} \varphi_1 - \varphi_2} & \chi \end{pmatrix}, \quad \det \lambda \equiv -\tau = \kappa e^{-\frac{2}{\sqrt{3}} \varphi_1 - \varphi_2},$$
where \(\kappa = \pm\) is responsible for the signature: \(\kappa = 1\) for space-like \(z^8\), and \(\kappa = -1\) for time-like \(z^8\). The ansatz (2) leads to the five-dimensional action (3). The 5D U(1) gauge fields \(A^I\) reduce to the 3D one-forms \(A^I(x^i)\) and the six axions collectively denoted as the 2D-covariant doublet \(\psi^I_p = (u^I, v^I)\) with the index \(p\) relative to the metric \(\lambda_{pq}\)

\[
A^I(x^i, z^7, z^8) = A^I(x^i) + \psi^I_p dz^p = A^I(x^i) + u^I dz^7 + v^I dz^8.
\]

### B. Dualisation equations

To obtain the three-dimensional sigma-model one has to dualize the electro-magnetic (EM) one-forms \(A^I\) and the KK one-forms \(a^p\) to scalars, which will be denoted as \(\mu_I\) and \(\omega_p\). The dualisation equations read:

\[
\tau \lambda_{pq} da^q = * V_p,
\]

\[
dA^I = d\psi^I_q \wedge a^q + \tau^{-1} G^{IJ} \star G_J,
\]

where the one-forms \(G_I\) and \(V_p\) are given by

\[
G_I = d\mu_I + \frac{1}{2} \delta_{IJK} d\psi^J_p \psi^K_q \varepsilon^{pq},
\]

\[
V_p = d\omega_p - \psi^I_p \left( d\mu_I + \frac{1}{6} \delta_{IJK} d\psi^J_q \psi^K_r \varepsilon^{qr} \right).
\]

In the component form the Eqs. (5) read:

\[
\lambda_{pq} \partial^{[i} a^{j]q} = \frac{1}{2 \tau \sqrt{h}} \varepsilon^{ijk} \left[ \partial_k \omega_p - \psi^I_p \left( \partial_k \mu_I + \frac{1}{6} \delta_{IJK} \partial_k \psi^J_q \psi^K_r \varepsilon^{qr} \right) \right],
\]

\[
\partial^{[i} A^{j]I} = a^{[ij]q} \partial_q \psi^I_p + \frac{1}{2 \tau \sqrt{h}} \varepsilon^{ijk} G^{IJ} \left( \partial_k \mu_J + \frac{1}{2} \delta_{JKL} \partial_r \psi^K_p \psi^L_q \varepsilon^{pq} \right).
\]

Substituting the metric \(ds_5^2\) in the form (4) into the 5D action (3) and performing dualisation via Eqs. (5) one derive the 3D gravity coupled sigma-model:

\[
I_3 = \frac{1}{16 \pi G_3} \int \sqrt{|h|} \left( R_3 - G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right) d^3 x,
\]

1 the antisymmetrization is assumed with 1/2.
where the Ricci scalar \( R_3 \) is build using the 3-dimensional metric \( h_{ij} \). The set of potentials \( \Phi^A = (\vec{\phi}, \psi^I, \mu_I, \chi, \omega_p) \), \( A, B = 1, \ldots, 16 \) realizes the harmonic map \( \Phi^A : x^i \in M_3 \rightarrow \Phi^A(x^i) \in M_{scal} \) between the 3D space-time \( M_3 \) and the target space (TS) \( M_{scal} \) with the metric \( G_{AB}(\Phi^C) \). The target space line element \( ds^2 = G_{AB}d\Phi^A d\Phi^B \) has the form

\[
dl^2 = \frac{1}{2} G_{IJ} (dX^I dX^J + d\psi^T \lambda^{-1} d\psi^I) - \frac{1}{2} \tau^{-1} G^{IJ}G_I G_J + \frac{1}{4} \text{Tr} (\lambda^{-1} d\lambda \lambda^{-1} d\lambda)
+ \frac{1}{4} \tau^{-2} d\tau^2 - \frac{1}{2} \tau^{-1} V^T \lambda^{-1} V. \tag{8}
\]

It is invariant under the action of the 28-parametric isometry group \( SO(4,4) \). The target space manifold \( M_{scal} \) is isomorphic to the coset \( M = SO(4,4)/H \), where the isotropy group \( H = SO(4) \times SO(4) \) for \( \kappa = 1 \) and \( SO(2,2) \times SO(2,2) \) for \( \kappa = -1 \). That is there is an isomorphic map \( \pi : \Phi^A \rightarrow \pi(\Phi^A) \in M \). Moreover if \( g \in SO(4,4) \) is some constant element of the isometry group then the following transformations

\[\pi \rightarrow \pi' = g \circ \pi, \quad ds_3^2 \rightarrow ds_3^2\]

leave invariant the action (7).

C. Matrix representation

As a convenient representative of the coset \( \pi(\Phi^A) \in M \) one can choose the matrix representation \( \gamma : \pi \rightarrow \gamma(\pi) \equiv \mathcal{V} \), where \( \mathcal{V} \) is the upper triangular matrix. We assume that \( \mathcal{V} \) transforms under the global action of the symmetry group \( SO(4,4) \) by the right multiplication and under the local action of the isotropy group \( H \) by the left multiplication: \( \mathcal{V} \rightarrow \mathcal{V}' = h(\Phi) \mathcal{V} g \), where \( g \) and \( h \) belong to the matrix representation \( \gamma \) of \( SO(4,4) \) and \( H \) respectively. Given this representative, one can construct the \( H \) invariant matrix (which we denote the same symbol \( M \) as the coset space)

\[M = \mathcal{V}^T K \mathcal{V},\]

\footnote{The set \( \vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4) \) comprises four scalars related to previously introduced \( \varphi_1, \varphi_2, \chi \) and \( X^I \) via

\[
\phi_1 = \frac{1}{\sqrt{2}} \left( -\ln(X^3) + \frac{1}{\sqrt{3}} \varphi_1 + \varphi_2 \right), \quad \phi_2 = \frac{1}{\sqrt{2}} \left( \ln(X^3) - \frac{1}{\sqrt{3}} \varphi_1 + \varphi_2 \right),
\phi_3 = \frac{1}{\sqrt{2}} \left( \ln(X^3) + \frac{2}{\sqrt{3}} \varphi_1 \right), \quad \phi_4 = \frac{1}{\sqrt{2}} \ln \frac{X^1}{X^2}.
\]
where $K$ is an involution matrix invariant under $H$: $h(\Phi)^T K h(\Phi) = K$, and dependent on the coset signature parameter $\kappa$. Then the transformation of the matrix $\mathcal{M}$ under $SO(4, 4)$ will be

$$\mathcal{M} \rightarrow \mathcal{M}' = g^T \mathcal{M} g. \quad (9)$$

The target space metric in terms of the matrix $\mathcal{M}$ will read

$$d l^2 = -\frac{1}{8} \text{Tr}(d \mathcal{M} d \mathcal{M}^{-1}). \quad (10)$$

Choosing suitable $8 \times 8$ matrix representation $\gamma$ of the isometry group $SO(4, 4)$ we construct (see [14] for details) the matrix representation of the coset $\mathcal{M}$ in terms of the $4 \times 4$ block matrices $\mathcal{P} = \mathcal{P}^T$ and $\mathcal{Q} = -\mathcal{T}^T \mathcal{Q}$ as follows

$$\mathcal{M} = \begin{pmatrix} \mathcal{P} & \mathcal{P} \mathcal{Q} \\ \mathcal{Q}^T \mathcal{P} & \mathcal{T} + \mathcal{Q}^T \mathcal{P} \mathcal{Q} \end{pmatrix},$$

where the block matrices are given explicitly in the Appendix.

**III. DUALISATION IN THE MATRIX FORM**

As we have discussed, the dualisation equations may present difficulties in applications of the solution generating technique. We can improve the situation performing dualisation in the matrix form. Introducing the matrix-valued current one-form $\mathcal{J}$

$$\mathcal{J} = J_i dx^i = M d M^{-1}$$

we can rewrite the 3-dimensional sigma-model action (7) in the following form

$$I_3 = \frac{1}{16 \pi G_3} \int \left( R_3 * 1 - \frac{1}{8} \text{Tr}(\mathcal{J} \wedge * \mathcal{J}) \right).$$

In this expression the Hodge dual $*$ is assumed with respect to the 3-dimensional metric $h_{ij}$. Variation of this action with respect to $\mathcal{J}$ shows that the two-form $* \mathcal{J}$ is closed:

$$d * \mathcal{J} = 0. \quad (11)$$

Variation with respect to the metric leads to three-dimensional Einstein equations:

$$(R_3)_{ij} = \frac{1}{8} \text{Tr}(\mathcal{J}_i \mathcal{J}_j). \quad (12)$$

$^3 \mathcal{P}$ denotes transposition with respect to the minor diagonal
The first equation (11) means that the matrix-valued two-forms \( \star J \) is locally exact, i.e., it can be presented as the exterior derivative of some matrix-valued one-form \( \mathcal{N} \), that is
\[
\star J = \mathcal{M} \star d \mathcal{M}^{-1} = d \mathcal{N}.
\] (13)

The matrix \( \mathcal{N} \) is defined up to adding an arbitrary matrix-valued closed one-form, which can be determined by choosing suitable asymptotic conditions. Now comparing the matrix dualisation equation (13) with the initial dualisation equations (6) we find the following purely algebraic relations between certain components of the matrix \((\mathcal{N})_{ab}\), \(a, b = 1, \ldots, 8\) are and the previous variables \(a^p\) and \(A^I\), namely
\[
a^7 = (\mathcal{N})_{16}, \quad a^8 = (\mathcal{N})_{17},
A^1 = \psi_1^a a^p + (\mathcal{N})_{15}, \quad A^2 = \psi_2^a a^p + (\mathcal{N})_{14}, \quad A^3 = \psi_3^a a^p - (\mathcal{N})_{26}.
\] (14)

Thus, if one manages to find the matrix \( \mathcal{N} \), the metric and matter fields can be extracted algebraically.

For the following it is important that the definition (13) and the transformation law for the matrix \( \mathcal{M} \) (9) under the global transformations \( g \in SO(4, 4) \) imply the following global transformation of the matrix \( \mathcal{N} \):
\[
\mathcal{N} \rightarrow \mathcal{N}' = g^T \mathcal{N} (g^T)^{-1}.
\]

IV. SOLUTION GENERATING TECHNIQUE

The sigma-model presented in the previous sections gives rise to generating technique which allows to construct new solutions from the known ones. Let the metric \( h_{ij} \) and the set of potentials \( \Phi^A \) combined in the coset matrix \( \mathcal{M} \) correspond to the metric and the three-form of some 11D seed solution. One has to extract part of the TS potentials from the seed solution algebraically and solve the differential dualisation equations (5) to find the remaining potentials. Using the action of the TS isometries one can then construct a new solution of the sigma-model with the same three-metric \( h'_{ij} = h_{ij} \) and the coset matrix
\[
\mathcal{M}' = g^T \mathcal{M} g \quad (\text{or } \mathcal{M}' = g \mathcal{M} g^T), \quad g \in SO(4, 4).
\]
Note that five TS variables $\phi_1, \phi_2, \phi_3, \phi_4, \chi$ enter the eleven-dimensional metric algebraically, via the moduli $X^I, \lambda_{pq}$:

$$ds^2_{11} = \sum_{I,a,a'} X^I \left( (dz^a)^2 + (dz^{a'})^2 + \lambda_{pq}(dz^p + a^p)(dz^q + a^q) + \tau^{-1} h_{ij}dx^i dx^j \right), \quad aa' = (12, 34, 56),$$

while the KK vectors $a^p$ in the $T^2$ sector are related to the TS potentials $\omega_p$ via dualisation. Similarly, in the form-field sector,

$$A_3 = (A^1 + \psi_1^p dz^p) \wedge dz^1 \wedge dz^2 + (A^2 + \psi_2^p dz^p) \wedge dz^3 \wedge dz^4 + (A^3 + \psi_3^p dz^p) \wedge dz^5 \wedge dz^6$$

the six quantities $\psi_1^p$ are the TS potentials, while the remaining one forms $A^I$ are related to the potentials $\mu_I$ via dualisation. So the set of transformed potentials $\lambda'_{pq}, (X'^I)$ and $(\psi'^I)$ can be explicitly extracted from the coset matrix $\mathcal{M}'$. The remaining components of the transformed metric $(ds^2_{11})'$ and the 3-form $(A_3')$ which are parametrized as the KK one-forms $(a^p)'$ and the EM fields $(A^I)'$ are determined by the dualisation equations (5). The inverse dualisation via the Eqs. (6) may be very difficult technically. Fortunately, this problem can be reduced to a purely algebraic one using the dualisation in the matrix form (13) as described in the previous section. Taking into account that the matrix $\mathcal{N}$ transforms as

$$\mathcal{N}' = g^T \mathcal{N} (g^T)^{-1} \quad \text{(or } \mathcal{N}' = g \mathcal{N} g^{-1}) \), \quad g \in SO(4, 4)$$

and using the relations (14) one can easily obtain the desired quantities $(a^p)'$ and $(A^I)'$.

We will denote the 28 generators of the $so(4, 4)$ algebra as

$$T = (H_1, H_2, H_3, H_4, P^\pm I, W_{\pm I}, Z_{\pm I}, \Omega_{\pm p}, X^\pm),$$

with $I = 1, 2, 3, p = 7, 8$. Their matrix representation can be found in the Appendix. The corresponding one-parametric transformations $g = e^{\alpha T}$, where $\alpha$ is a transformation parameter, give the set of the target space isometries.

### A. Asymptotic conditions

An important question is how to identify the isometries we need to use in order to construct solutions with the desired properties. These are usually associated with asymptotic conditions. In this paper we consider asymptotic conditions corresponding to 5D Kaluza-Klein black holes with squashed horizons embedded into eleven dimensions which correspond
to the following asymptotic manifold: $T^6 \times \mathbb{R}^1 \times S_{sq}$, where $S_{sq}$ is a squashed $S^3$. We will assume that TS potentials have the following general asymptotic behavior

$$\lambda \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta \lambda}{r}, \quad \omega_7 \sim \frac{\delta \omega_7}{r}, \quad \omega_8 \sim \frac{\delta \omega_8}{r^2}, \quad A_{[3]} = 0,$$

(15)

where $\delta \lambda$, $\delta \omega_7$ and $\delta \omega_8$ are constant. The asymptotic behavior with $\delta \lambda = \delta \omega_7 = \delta \omega_8 = 0$ correspond to the trivial $S^1$ bundle over a 4D Minkowski space-time. The asymptotic coset matrix for this case is $M_{as} = K$ which is preserved under the isometries belonging to the isotropy group $H$ of the $SO(4,4)$:

$$P^I + P^{-I}, \quad Z_+ + Z_-, \quad W_I - W_{-I}, \quad X^+ + X^- \quad \Omega^7 + \Omega^{-7} \quad \Omega^8 - \Omega^{-8}.$$

For more general asymptotic behavior such as (15) one have use the above transformations with some constraints on the parameters.

To apply these isometries in the case of minimal 5D supergravity one needs to find the relevant embedding of the $G_{2(2)}$ subgroup into $SO(4,4)$. As was shown in [14], the following combinations of the $SO(4,4)$ generators realize the positive and negative root generators of $G_{2(2)}$:

$$P^\pm \sim \sum P^\pm I, \quad Z_\pm \sim \sum Z_{\pm I}, \quad W_\pm \sim \sum W_{\pm I}, \quad \Omega^{\pm p}, \quad X^\pm.$$

Thus the isometries

$$P^+ + P^-, \quad Z_+ + Z_-, \quad W_+ - W_-, \quad X^+ + X^- \quad \Omega^7 + \Omega^{-7} \quad \Omega^8 - \Omega^{-8}$$

can be used to generate new KK solutions in the minimal 5D supergravity.

V. CONSTRUCTING FIVE-PARAMETRIC SQUASHED BLACK HOLE

A. From Kerr to Rasheed solution

First we would like to demonstrate how to construct using our technique the rotating dyonic black hole of [23] from the Kerr metric. We define the coordinates $z^7 = x^5$, $z^8 = t$ and $x^i = (r, \theta, \phi)$. In this basis the Kerr solution of the mass $M_K$ and the angular momentum $J_K = aM_K$ smeared into the fifth dimension reads

$$ds_5^2 = (dx^5)^2 - (1 - Z) \left( dt + \frac{aZ \sin^2 \theta}{1 - Z} d\phi \right)^2 + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \frac{\Delta}{1 - Z} \sin^2 \theta d\phi^2,$$
where
\[ \rho = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2M_K r + a^2, \quad Z = \frac{2M_K r}{\rho}. \]

The corresponding TS variables are:
\[ \lambda_{pq} = \begin{pmatrix} 1 & 0 \\ 0 & Z - 1 \end{pmatrix}, \quad \tau = 1 - Z, \]
\[ \omega_\gamma = 0, \quad \omega_8 = \frac{2M_K a \cos \theta}{\rho}, \quad (a_\phi^7 = 0, \quad a_\phi^8 = aZ \sin^2 \theta). \]

The above definitions of the TS potentials lead to the following blocks of the coset matrix \( \mathcal{M} \)
\[ Q = \begin{pmatrix} 0 & 0 & \frac{2M_K a \cos \theta}{\rho} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{1}{Z-1} & 0 & 0 & 0 \\ 0 & \frac{1}{Z-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

One can easily obtain the dual matrix \( \mathcal{N} \) solving the Eq. (13):
\[ \mathcal{N} \equiv \begin{pmatrix} -\frac{2M_K \Delta \cos \theta}{\rho(1-Z)} & 0 & 0 & 0 & 0 & Za \sin^2 \theta & 0 \\ 0 & -\frac{2M_K \Delta \cos \theta}{\rho(1-Z)} & 0 & 0 & 0 & 0 & -\frac{Za \sin^2 \theta}{1-Z} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} d\phi. \]

The dyon solution \([23]\) was generated by applying the constrained \( SO(1, 2) \) transformations to the smeared Kerr solution. The constraint ensures asymptotic flatness and absence of the NUT parameter in the four-dimensional solution. The latter requirement means that the asymptotic of the \( \phi \)-component of the KK one-form \( a^8 \) decays as \( O(\frac{1}{r}) \) after applying the transformation. This leads to one relation between three parameters of the general \( SO(1, 2) \) transformation so only two parameters remain independent.

The \( \sigma \)-model with the \( SO(4, 4) \) isometry group reduces to that of the five-dimensional KK gravity (embedded into 11D) with the \( SL(3, \mathbb{R}) \) isometry group under the conditions \( X^I = 1, \quad A_{[3]} = 0 \). So there exists the \( SO(1, 2) \) subgroup of the \( SO(4, 4) \) group which
produces a dyonic rotating KK black hole. One finds that the this subgroup is generated by the elements $X^+ + X^-$, $\Omega^7 + \Omega^{-7}$, $\Omega^8 - \Omega^{-8}$.

We perform the transformation $g = g_1 g_2 g_3$, $g \in SO(1, 2)$, where

$$
g_1 = e^{\alpha (X^+ + X^-)}, \quad g_2 = e^{\beta (\Omega^7 + \Omega^{-7})}, \quad g_3 = e^{\gamma (\Omega^8 - \Omega^{-8})},
$$

assuming that the matrices $\mathcal{M}$ and $\mathcal{N}$ are transformed under $g$ as $\mathcal{M}_1 = g \mathcal{M} g^T$ and $\mathcal{N}_1 = g \mathcal{N} g^{-1}$ respectively. Then we demand that $g$ preserve the $O(\frac{1}{r})$ asymptotic behavior of $a_8$ or, equivalently, the $O(\frac{1}{r^2})$ asymptotic behavior of $\omega_8$. This give the same relation between three parameters $\alpha, \beta, \gamma$ as in [23]:

$$
\tan 2\gamma = \tanh \alpha \sinh \beta.
$$

The metric of [23] now can be extracted from the transformed matrices $\mathcal{M}_1$ and $\mathcal{N}_1$ as usual. But for further application of other isometries of the TS we need to know only the matrices $\mathcal{M}_1$ and $\mathcal{N}_1$.

**B. Charging the Rasheed solution**

Our improved generating technique allows us to construct the charged Rasheed solution without the restriction $\alpha = 0$ (corresponding to absence of the electric charge $Q$ in the four-dimensional interpretation of the solution [23]) which was imposed in [22] apparently because of the difficulties with inverse dualisation. Using our Eq. (5) we can extract the final solution from the target space potentials purely algebraically.

To obtain the charged solution from the vacuum Rasheed solution we apply the following global transformation with the parameter $\delta$ to the matrices $\mathcal{M}_1$ and $\mathcal{N}_1$:

$$
\mathcal{M}_2 = \Pi \mathcal{M}_1 \Pi^T, \quad \mathcal{N}_2 = \Pi \mathcal{N}_1 \Pi^{-1}, \quad \Pi = e^{\delta \sum_i (Z_i + Z_{-i})}.
$$

As was shown in [14], the transformation $\Pi$ is equivalent to the action of the one-parametric $G_{2(2)}$ subgroup, which generate an electric charge. Then extracting the TS variables from
$\mathcal{M}_2$ one can find the transformed potentials $\psi_p'$ and $\lambda_{pq}'$:

\[
v' = v_1' = v_2' = v_3' = csD^{-1}\left((\mathcal{M}_1)_{11} + (\mathcal{M}_1)_{33}\right),
\]

\[
u' = u_1' = u_2' = u_3' = -csD^{-1}\left(s(\mathcal{M}_1)_{17} + c(\mathcal{M}_1)_{23}\right),
\]

\[
\lambda_{88}' = D^{-2}(\mathcal{M}_1)_{11}(\mathcal{M}_1)_{33},
\]

\[
\lambda_{78}' = -\lambda_{88}' \left(\frac{c^3(\mathcal{M}_1)_{23}}{(\mathcal{M}_1)_{33}} - s^3(\mathcal{M}_1)_{17}\right)\frac{(\mathcal{M}_1)_{11}(\mathcal{M}_1)_{33}}{D^2} \left(\frac{c^3(\mathcal{M}_1)_{23}}{(\mathcal{M}_1)_{33}} - s^3(\mathcal{M}_1)_{17}\right),
\]

\[
\tau' = -D^{-1}, \quad (X^I)' = 1.
\]

where $c = \cosh \delta$, $s = \sinh \delta$ and

\[
D = c^2(\mathcal{M}_1)_{11} + s^2(\mathcal{M}_1)_{33}.
\]

The explicit expressions for the coset components $(\mathcal{M}_1)_{ab}$, $a, b = 1, \ldots, 8$ are

\[
(\mathcal{M}_1)_{11} = \frac{1}{p(Z-1)} \left\{ p + \frac{2M_K}{\rho} \left( a \cos \theta s_\alpha c_\beta^2 s_\beta + r (c_\beta^2 c_\alpha - p) - M_K c_\beta^2 (c_\alpha - p) \right) \right\},
\]

\[
(\mathcal{M}_1)_{33} = -\frac{1}{p(Z-1)} \left\{ p + \frac{2M_K}{\rho} \left( a \cos \theta s_\alpha s_\beta (1 + c_\alpha^2 c_\beta^2) \right. \right.
\]

\[
\left. \quad + \left( r - M_K \right) \left[ c_\alpha(c_\beta^2 - p^2) + p(c_\alpha^2 c_\beta^2 - 1) \right] - r pc_\alpha^2 c_\beta^2 \left. \right\},
\]

\[
(\mathcal{M}_1)_{23} = \frac{2M_K}{\rho(Z-1)} \left\{ c_\alpha s_\beta p c \theta - r p s_\alpha + M_K s_\alpha(p - c_\alpha^2 c_\beta^2) \right\},
\]

\[
(\mathcal{M}_1)_{17} = \frac{2M_K c_\beta}{\rho(Z-1)} \left\{ a p c \theta + s_\alpha s_\beta M_K \right\},
\]

where

\[ p = \sqrt{c_\alpha^2 + s_\alpha^2 s_\beta^2}, \quad c_* = \cosh *, \quad s_* = \sinh *.
\]

Similarly, the relations (14) give the transformed KK one-forms $(a^p)'$ and the five-dimensional one-form $A'$:

\[
(a^7)' = (\mathcal{N}_1)_{16}, \quad (a^8)' = c^3(\mathcal{N}_1)_{17} + s^3(\mathcal{N}_1)_{32},
\]

\[
A' = u'dx^5 + v'dt + d\phi[u'(a_\phi^7)'+v'(a_\phi^8)'] - cs \left( c(\mathcal{N}_1^\phi)_{17} + s(\mathcal{N}_1^\phi)_{32} \right),
\]
where

\[(N_1)_{16} = \frac{2M_K}{\rho(Z - 1)p} \left\{ ac_\beta s_\alpha \sin^2 \theta \left[ p(r - M_K) + M_K c_\alpha c_\beta^2 \right] - \Delta s_\beta c_\beta \cos \theta \right\} d\phi,\]

\[(N_1)_{17} = -\frac{2M_K}{\rho(Z - 1)} \left\{ ac_\beta \sin^2 \theta \left[ M_K p + c_\alpha (r - M_K) \right] \right\} d\phi,\]

\[(N_1)_{32} = -\frac{2M_K}{\rho(Z - 1)} \left\{ as_\beta \sin^2 \theta \left[ c_\alpha M_K p + M_K - r \right] + \Delta s_\alpha p \cos \theta \right\} d\phi.\]

To write the resulting metric we introduce the functions $A, B, C, E, W, X, Y$ via

\[A = (M_1)_{11}, \quad B = (M_1)_{33}, \quad C = (M_1)_{17}, \quad E = (M_1)_{23},\]

\[W = (N_1)^\phi_{16}, \quad X = (N_1)^\phi_{17}, \quad Y = (N_1)^\phi_{32}.\]

In terms of them the eleven-dimensional metric will read

\[ds_{11}^2 = \sum_{a,a'} \left( (dz^a)^2 + (dz'^a)^2 \right) \]

\[+ f(dt + \Omega')^2 + \frac{1}{fD} (dx^5 + W d\phi)^2 - D \left( \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \frac{\Delta}{1 - Z} \sin^2 \theta d\phi^2 \right),\]

with

\[f = \frac{AB}{D^2}, \quad \Omega' = \Omega_5 dx^5 + \Omega_\phi d\phi, \quad D = A c^2 + B s^2,\]

\[\Omega_5 = \frac{C}{A} s^3 - \frac{E}{B} c^3, \quad \Omega_\phi = \frac{WC + YA}{A} s^3 + \frac{XB - WE}{B} c^3.\]

The corresponding 3-form $A_{[3]}$ can be written as

\[A_{[3]} = \sum_{a,a'} A' \wedge dz^a \wedge dz'^a,\]

where the five-dimensional one-form $A'$ reads

\[A' = \frac{cs}{D} \left\{ (A + B) dt - (sC + cE) dx^5 + \left[ c(XB - WE) - s(WC + YA) \right] d\phi \right\}.\]

Our new solution contains five free parameters $M_K, a, \alpha, \beta, \delta$ and reduces to that of [22] if $\alpha = 0$. Physical properties of the new solution will be discussed in a separate publication.

VI. CONCLUSIONS

In this paper we have suggested an improved solution generating technique for the 5D $U(1)^3$ supergravity based on the 3D sigma-model with the $SO(4,4)$ isometry group.
The main new ingredient is the matrix dualisation equation which opens a way to avoid the inverse dualisation problem in constructing new solutions from old. As an application we have obtained the five-parametric Kaluza-Klein black hole of the minimal 5D supergravity generalizing the solution by Tomizawa, Yasui and Morisawa [22].

Note that we did not use the most general transformations preserving the desired asymptotic structure \( T^6 \times \mathbb{R}^1 \times S_{sq} \). Namely, the combination of the transformations \( e^{\delta_1 \sum_i (P^I + P^{-I})} \), \( e^{\delta_2 \sum_i (Z_I + Z^{-I})} \) and \( e^{\delta_3 \sum_i (W_I - W^{-I})} \) will give more general solutions with the same asymptotic behavior provided some constraint on the parameters \( \delta_1, \delta_2, \delta_3 \) holds.

Also, we have restricted attention here by the case of minimal 5D supergravity. The corresponding generalization to the \( U(1)^3 \) theory is straightforward.

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APPENDIX A: 8 \times 8 MATRIX REPRESENTATION

We choose the following 8 \times 8 matrix representation of the so(4,4) algebra

\[
E = \begin{pmatrix} A & B \\ C & -A^\hat{T} \end{pmatrix},
\]

where \( A, B, C \) are the 4 \times 4 matrices, \( A, B \) being antisymmetric, \( B = -B^T, C = -C^T \), and the symbol \( \hat{T} \) in \( A^\hat{T} \) means transposition with respect to the minor diagonal. The diagonal matrices \( \vec{H} \) are given by the following \( A \)–type matrices (with \( B = 0 = C \)):

\[
A_{H_1} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{H_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{H_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{H_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Twelve generators corresponding to the positive roots are given by the upper-triangular matrices \( E_k, k = 1, \ldots, 12 \). From these the generators labeled by \( k = 2, 4, 6, 7, 9, 12 \) are of
pure $A$-type (with $B = 0 = C$):

$$
A_{E_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_6} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
A_{E_7} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_9} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{E_{12}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

while the other six are of pure $B$ type (with $A = 0 = C$):

$$
B_{E_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{E_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{E_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
$$

$$
B_{E_8} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_{E_{10}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{E_{11}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The correspondence with the previously introduced generators is as follows ($I = 1, 2, 3, \ p = 7, 8$):

$$
P^I \leftrightarrow E_I, \quad W_I \leftrightarrow E_{I+3}, \quad Z_I \leftrightarrow E_{I+6}, \quad \Omega^p \leftrightarrow E_{p+3}, \quad X^+ \leftrightarrow \ E_{12}.
$$

In this representation, the matrices corresponding to the negative roots,

$$
P^{-I} \leftrightarrow E_{-I}, \quad W_{-I} \leftrightarrow E_{-(I+3)}, \quad Z_{-I} \leftrightarrow E_{-(I+6)}, \quad \Omega^{-p} \leftrightarrow E_{-(p+3)}, \quad X^- \leftrightarrow E_{-12},
$$

are transposed with respect to the positive roots matrices:

$$
E_{-k} = (E_k)^T.
$$

The following normalization conditions are assumed:

$$
\text{Tr}(H_i, H_j) = 4\delta_{ij}, \quad i, j = 1 \ldots 4, \quad \text{Tr}(E_k, E_{-k}) = 2,
$$
and the involution matrix $K$ is chosen as

$$K = \text{diag}(\kappa, \kappa, 1, 1, 1, 1, \kappa).$$

The generators of the isotropy subgroup are selected by the equation $h(\Phi)^T K h(\Phi) = K$. They are given by the following linear combinations of the generators:

$$P_1^I - \kappa P^{-1}, \quad Z_1 - \kappa Z_{-1}, \quad W_1 - W_{-1}, \quad X^+ - \kappa X^-, \quad \Omega^7 - \kappa \Omega^{-7}, \quad \Omega^8 - \Omega^{-8}.$$  

**APPENDIX B: MATRIX REPRESENTATION OF COSET $\mathcal{M}$**

$$\mathcal{M} = \left( \begin{array}{cc} \mathcal{P} & \mathcal{P} \mathcal{Q} \\ \mathcal{Q}^T \mathcal{P} & \mathcal{Q}^T \mathcal{P} \mathcal{Q} \end{array} \right),$$

where the $4 \times 4$ blocks $\mathcal{P}$ and $\mathcal{Q}$ are

$$\mathcal{Q} = \begin{pmatrix} \mu_1 + \frac{u^3 v^2 - v^3 u^2}{2} & \omega_7 - \frac{u^3 v^2 - 2 u^3 v^2 u^1 + v^3 u^1 u^2}{6} - u^2 \mu_2, & \omega_8 + \frac{v^3 u^1 v^2 - 2 v^3 u^1 v^2 u^1 + u^3 v^1 u^2}{6} - v^2 \mu_2 \\ -v^2 & -\mu_3 + \frac{v^4 u^2 - u^1 v^2}{2} & 0 \\ -u^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{P} = \begin{pmatrix} \Psi^T \Lambda \Psi, & \Psi^T \Lambda \Phi \\ \Phi^T \Lambda \Psi, & \Phi^T \Lambda \Phi + e^{\sqrt{2} \phi_4} \end{pmatrix}, \quad \mathcal{P}^T = (\mathcal{P}^{-1})^T.$$

Here $\Psi$ and $\Lambda$ are the $3 \times 3$ matrices

$$\Psi = \begin{pmatrix} 1 & u^3 & -v^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \kappa \begin{pmatrix} e^{\sqrt{2} \phi_1} & 0 & 0 \\ 0 & e^{\sqrt{2} \phi_2} & -\chi e^{\sqrt{2} \phi_2} \\ 0 & -\chi e^{\sqrt{2} \phi_2} & e^{\sqrt{2} \phi_2} \chi^2 + \kappa e^{\sqrt{2} \phi_3} \end{pmatrix},$$

and $\Phi$ is the 3-column

$$\Phi = \begin{pmatrix} \mu_2 + \frac{1}{2}(u^1 v^3 - u^3 v^1) \\ -v^1 \\ -u^1 \end{pmatrix}.$$

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