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On the stability of Baer subplanes

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Abstract

A blocking set in a projective plane is a point set intersecting each line. The smallest blocking sets are lines. The second smallest minimal blocking sets are Baer subplanes (subplanes of order $\sqrt{q}$). Our aim is to study the stability of Baer subplanes in $\text{PG}(2,q)$. If we delete $\sqrt{q} + 1 - k$ points from a Baer subplane, then the resulting set has $(\sqrt{q} + 1 - k)(q - \sqrt{q})$ 0-secants. If we have somewhat more 0-secants, then our main theorem says that this point set can be obtained from a Baer subplane or from a line by deleting somewhat more than $k$ points and adding some points. The motivation for this theorem comes from planes of square orders, but our main result is valid also for non-square orders. Hence in this case the point set contains a relatively large collinear subset.

1 Introduction

A blocking set is a point set intersecting each line. It is easy to see that the smallest blocking sets of projective planes are lines. A blocking set is non-trivial if it contains no line. A blocking set is minimal, when no proper subset of it is a blocking set. Using combinatorial arguments Bruen proved that the smallest non-trivial blocking sets of $\text{PG}(2,q)$ have at least $q + \sqrt{q} + 1$

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points. When \( q \) is a square, minimal blocking sets of this size exist; they are the points of a Baer subplane, that is a subplane of order \( \sqrt{q} \).

There are lots of interesting results on blocking sets, for a survey see [1], [10], [2] and [3]. For a set \( S \), a line meeting \( S \) in \( i \) points is called an \( i \)-secant. Instead of 0-secants, we sometimes use the term skew lines or external lines.

The stability question for blocking sets would mean that sets having few 0-secants can be obtained from blocking sets by deleting a relatively small number of points. Some results of this type can be found in [11] and [12].

The next theorem of Erdős and Lovász shows the stability of lines.

**Theorem 1.1** (Erdős–Lovász [5]) If \( S \) is a set of \( q + k \) points, \( 0 \leq k \leq \sqrt{q} + 1 \), and the number of 0-secants is less than \((\sqrt{q} + 1 - k)(q - \lfloor \sqrt{q} \rfloor)\), where \( k \leq \sqrt{q} + 1 \), then the set contains at least \( q + k - \lfloor \sqrt{q} \rfloor + 1 \) collinear points.

The result is sharp for \( q \) square: deleting \( \sqrt{q} + 1 - k \) points from a Baer subplane gives this number of 0-secants. For the proof the reader is referred to [2].

The aim of this paper is to study the stability of Baer subplanes. So we have a set that has a little more 0-secants than what is guaranteed by the Erdős-Lovász bound. Then we wish to prove that it can be obtained from a Baer subplane (or a line) by deleting and adding some points.

The exact formulation of our main result is the following.

**Theorem 1.2** Let \( B \) be a point set in \( \text{PG}(2, q) \), with cardinality \( q + k \), \( 0 \leq k \leq 0.6\sqrt{q} \) and \( 1600 \leq q \). Assume that the number of skew lines of \( B \) is less than \((q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)\), where \( 0 \leq c \leq 0.05\sqrt{q} - 2 \). Then \( B \) contains more than \( q + 1 - (\sqrt{q} - k + c + 1) \) points from a line or \( q + \sqrt{q} + 1 - (\sqrt{q} - k + c + 1) \) points from a Baer subplane.

## 2 Preliminaries

Here we collect some results from [11], which will be used later.

**Lemma 2.1** Let \( S \) be a point set of size less than \( 2q \) in \( \text{PG}(2, q) \), \( q \geq 81 \), and assume that the number of external lines \( \delta \) of \( S \) is less than \((q^2 - q)/2\). Denote by \( s \) the number of external lines of \( B \) passing through a point \( P \). Then \((2q + 1 - |S| - s)s \leq \delta\).
When $\delta$ is relatively small, for example of $O(q^{1/2})$, then after solving the second order inequality in the lemma above, we get that $s$ is either relatively small ($O(\sqrt{q})$) or it is relatively large ($q - O(\sqrt{q})$). Note that if we delete few points from a blocking set, then this is exactly the case; the number of skew lines through a deleted point is of $O(q)$ and small otherwise.

Theorem 2.2 ([11]) Let $B$ be a point set in $\text{PG}(2, q)$, $q \geq 16$, of size less than $\frac{3}{2}(q + 1)$. Denote the number of 0-secants of $B$ by $\delta$, and assume that

$$\delta < \min \left( (q - 1) \frac{2q + 1 - |B|}{2(|B| - q)}, \frac{1}{2} \left( q - \sqrt{q} \right)^{3/2} \right).$$

Then $B$ can be obtained from a blocking set by deleting at most $\frac{\delta}{2q + 1 - |B|} + \frac{1}{2}$ points of it.

Observe that if the size of $B$ is less than $q + \sqrt{q}/2$, then Theorem 1.1 is stronger than Theorem 2.2. Also note that when the size of $B$ is around $q + \sqrt{q}$, then Theorem 1.1 gives almost nothing, while the theorem above still gives some reasonable bound on the number of 0-secants. The situation is similar with our main theorem for $|B| > q + 0.6\sqrt{q}$, that is our main theorem is weaker in this case than Theorem 2.2.

3 The stability of Baer subplanes

The aim of this section is to prove a stability version of the Erdős–Lovász bound. That is, we show that a set of $q + k$ points having at most $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$ skew lines either contains a relatively large collinear set or it contains $q + k - c$ points from a Baer subplane. Note that if we delete $\sqrt{q} - k + c + 1$ points from a Baer subplane and add $c$ points, then our point set $B'$ will have at most $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$ skew lines. Hence it is natural to assume that $c \geq 0$, which we assume throughout this paper and so from $c \leq 0.05\sqrt{q} - 2$ in Theorem 3.10, it follows that $q \geq 1600$.

Note that for the point set $B'$, the points through which there pass at least $(q - \sqrt{q} - c)$ 0-secants are exactly the points that were deleted from the Baer subplane. In the first theorem we assume that there are no such points.

Theorem 3.1 Let $B$ be a point set in $\text{PG}(2, q)$, $1600 \leq q$, with cardinality $q + k$, $0 \leq k \leq 0.6\sqrt{q}$. Assume that the number $\delta$, of skew lines of $B$ is at
most \((q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)\), where \(0 \leq c \leq 0.05\sqrt{q} - 2\). Furthermore, suppose that there is no point in \(\text{PG}(2, q)\), through which the number of skew lines is at least \(q - \sqrt{q} - c\). Then \(B\) contains a Baer subplane or a line.

**Proof.** We will prove the theorem through a sequence of lemmas. The upper bound on \(\delta\) and solving the quadratic inequality in Lemma 2.1 give that the number of skew lines through a point cannot be in a certain interval.

**Corollary 3.2** The number of skew lines to \(B\) through a point is either at most \(\sqrt{q} - k + c + 1\) or at least \(q - \sqrt{q} - c\). If the assumptions of Theorem 3.1 hold, then the latter case cannot occur. ■

Using a similar argument we can say something about the number of lines through a point \(P\) in \(B\), which intersect \(B\) in at least two points. This will be called the degree of \(P\).

**Lemma 3.3** The degree of a point in \(B\) is either at most \(\sqrt{q} + c + 2\) or at least \(q - \sqrt{q} - c + k - 2\).

**Proof.** Let \(P\) be a point in \(B\). The number of skew lines, \(\delta'\) to the point set \(B \setminus \{P\}\) is at most \(\delta + q\) and hence by Lemma 2.1, the number of skew lines through \(P\) to \(B \setminus \{P\}\) is less than \(\sqrt{q} - k + c + 3\) or larger than \(q - \sqrt{q} - c - 1\) and so the proof follows. ■

**Lemma 3.4** There are at most \(2(\sqrt{q} + c + 2)\) points in \(B\) with degree at least \(q - \sqrt{q} - c + k - 2\).

**Proof.** Let \(L_1, L_2, \ldots, L_{q^2+q+1}\) be the lines of \(\text{PG}(2, q)\) and denote by \(n_i\) the number of points of \(B\) on the line \(L_i\). Then

\[
\sum_{n_i>1}(n_i-1) = |B|(q+1) - (q^2 + q + 1) + \delta = k(q+1) + \delta - 1. \tag{2}
\]

Note that \(\sum_{n_i>1} n_i\) is the sum of the degrees of the points in \(B\), and it is \(\sum_{n_i>1}(n_i-1) + \sum_{n_i>1} 1\) and so \(2 \sum_{n_i>1}(n_i-1)\) is an upper bound on the sum of the degrees. Hence there can be at most \(2(k(q+1)+\delta-1)/(q-\sqrt{q}+c+k-2)\) points in \(B\) with degree at least \(q-\sqrt{q}-c+k-2\), which is at most \(2(\sqrt{q}+c+2)\). ■

The next lemma summarizes some important properties of \(B\).
Lemma 3.5  
(i) Every line intersects $B$ in at most $\sqrt{q} + c + 2$ points or there is a line contained in $B$.
(ii) The intersection of any two lines, each intersecting $B$ in more than $\frac{\sqrt{q}+c+2}{2}$ points, lies in $B$.

Proof. By Lemma 3.4, there are at least $|B| - 2(\sqrt{q} + c + 2)$ points with degree at most $\sqrt{q} + c + 2$. Let $\ell$ be a line and $P$ be a point of $B \setminus \ell$. If $P$ has degree at most $\sqrt{q} + c + 2$, then clearly $|B \cap \ell| \leq \sqrt{q} + c + 2$. If each $P \notin \ell$ has large degree, then $|B \cap \ell| \geq |B| - 2(\sqrt{q} + c + 2)$. Now assume that $\ell$ is a line and $|\ell \cap B| \geq |B| - 2(\sqrt{q} + c + 2)$. We show that $\ell \subseteq B$. Suppose to the contrary that $P \in \ell \setminus B$. Then the number of 0-secants through $P$ is at least $q - 2(\sqrt{q} + c + 2)$. By Corollary 3.2, through such a point there are at least $q - \sqrt{q} - c$ 0-secants, but this contradicts the assumption of Theorem 3.1; hence we proved (i).

To prove (ii), assume to the contrary that through a point $P \notin B$ there are two lines both intersecting $B$ in more than $\frac{\sqrt{q}+c+2}{2}$ points. Then the number of skew lines through $P$ is more than $q + 1 - 2 - (|B| - (\sqrt{q} + c + 2))$, that contradicts Corollary 3.2 and the assumption of Theorem 3.1. ■

From now on we assume that there is no line contained in $B$.

Lemma 3.6  Let $P$ be a point in $B$ with degree at most $\sqrt{q} + c + 2$. Then there are more than $0.8(\sqrt{q} + c + 2)$ lines through $P$ intersecting $B$ in more than $\frac{\sqrt{q}+c+2}{2}$ points.

Proof. Assume to the contrary that there is a point $P$ in $B$ not satisfying the lemma. Using Lemma 3.5 (i) and counting the points of $B$ on the lines through $P$, we get that $B$ has at most $0.8(\sqrt{q} + c + 2)(\sqrt{q} + c + 2) + (1 - 0.8)(\sqrt{q} + c + 2)^2$ points (here $P$ was counted degree of $P$ times), which is a contradiction since $(0.8 + \frac{1-0.8}{2})(\sqrt{q} + c + 2)^2 < q \leq q + k$. ■

Two lines meeting $B$ in more than $\frac{\sqrt{q}+c+2}{2}$ points intersect in a point of $B$, hence if we take these lines through two points of $B$ (and disregard the line joining the two points) then we get a relatively large “grid” $R'$ inside $B$. Lemma 3.9 will show that such grids can be embedded in a somewhat larger subgroup grid $R$, which has transitive automorphism group. Finally, we will show that there can be only few points that are not in the intersection of $B$
and the subgroup grid and so it will follow that the subgroup grid is relatively large and it is contained in $B$. For the construction of the subgroup grid, Kneser’s theorem is needed.

**Result 3.7** (Kneser [8]) Let $(G, +)$ be an Abelian group, $\emptyset \neq A, B$ be finite subsets of $G$. Then there is a subgroup $H$ of $G$ such that $A + B = A + B + H$ and $|A + B| \geq |A + H| + |B + H| - |H|$. ■

**Corollary 3.8** Let $M$ and $N$ be subsets of the Abelian group $(G, +)$. Assume that $|M| = |N|$ and that $|M + N| < \frac{3}{2}|M|$. Then there exists a subgroup $H$, so that $M + N = M + N + H$ and both $M$ and $N$ lie in one-one coset of $H$.

A similar result can be found in [13].

**Proof.** Kneser’s theorem assures that there is a subgroup $H$ of $G$, so that $M + N = M + N + H$ and $|M + H| + |N + H| - |H| \leq |M + N|$. As $|M| \leq |M + H|$ and $|N| \leq |N + H|$, the above inequality and the assumption that $\frac{3}{2}|M + N| < 2|M|$ imply that $\frac{1}{2}|M + N| < |H|$. Since $M + N$ is the union of some cosets of $H$, the above inequality implies that $M + N$ is either one coset of $H$ or the union of two cosets. The first case immediately yields the corollary. Now assume to the contrary that $|M + N| = 2|H|$. The condition $|M + N| < \frac{3}{2}|M|$ (and $|M + N| = 2|H|$) implies, $|M| = |N| > |H|$. Hence $|M + H| \geq 2|H|$ and $|N + H| \geq 2|H|$, so Kneser’s theorem gives that $|M + N| \geq 3|H|$; a contradiction. ■

Let $P_0, P_1, P_2$ be three collinear points having small degree. Such points exist, since by Lemma 3.6 through a point with small degree, there are lots of relatively long lines and by Lemma 3.4 there are only few points in total with large degree. Hence we can easily find a line intersecting $B$ in more than $\sqrt{q} + c + 2$ points and containing almost only small degree points. Let the points $P_0, P_1, P_2$ be $(0, 1, 0), (0, 0, 1)$ and $(0, 1, -1)$ (after a suitable coordinate transformation). We will disregard the line $\ell$ containing the points $P_0, P_1, P_2$. The lines through $P_0$ intersecting $B$ in more than $(\sqrt{q} + c + 2)/2$ points have homogeneous coordinates $[c, 0, 1], c \in C$. Similarly, the lines through $P_1$ intersecting $B$ in more than $(\sqrt{q} + c + 2)/2$ points have homogeneous coordinates $[d, 1, 0], d \in D$. The lines through $P_2$ intersecting $B$ in at least 2 points have homogeneous coordinates $[e, 1, 1], e \in E$. Note that $|C|, |D| \geq 0.8(\sqrt{q} + c + 2)$ and $|E| \leq \sqrt{q} + c + 2$. Observe also, that the lines $[c, 0, 1]$,
$[d, 1, 0]$ and $[e, 1, 1]$ are concurrent if and only if $c + d = e$. Now let $G$ be the additive group of GF$(q)$, and we may assume that $|C| = |D|$ by disregarding some lines, hence we can apply the corollary of Kneser’s theorem (Corollary 3.8) to deduce that there exists a subgroup $H$ of the additive group of GF$(q)$, so that $C$ and $D$ are contained in one-one coset of $H$. So $|E| = |H|$ and hence $|H| \leq \sqrt{q} + c + 2$.

This proves the following lemma. To this we will denote by $R'$ the grid obtained by the intersection points of the lines in $C$ and the lines in $D$.

**Lemma 3.9** The grid $R' \subseteq B$ is contained in a subgroup grid $R$. $R'$ contains at least $\left\lceil 0.8(\sqrt{q} + c + 2) \right\rceil - 1$ points of $R \cap B$. The subgroup grid $R$ has a transitive automorphism group.

**Proof.** From the discussion preceding the lemma the existence of $R$ is clear, we have to take the lines belonging to a coset of an additive subgroup $H'$ through $P_1$ and $P_2$. So the subgroup grid $R$ consists of the points $\{(x, y) : x \in a + H', y \in b + H'\}$. The fact that $R'$ contains at least $\left\lceil 0.8(\sqrt{q} + c + 2) \right\rceil - 1$ points of $R \cap B$ is clear from the construction. The automorphism group that acts regularly on $R$ is just the group $\{\alpha_h, k : (x, y) \mapsto (h + x, k + y) : h, k \in H'\}$. $\blacksquare$

**Proof of Theorem 3.1** Finally, we will show that the subgroup grid $R$ of Lemma 3.9 is a Baer subplane minus one line, which is contained in $B$ and consequently the entire Baer subplane must be contained in $B$.

Note that the size of the subgroup $H$ defining $R$ should divide $q$ and $|H| \leq \sqrt{q} + c + 2$ implying $|H| \leq \sqrt{q}$. On the other hand, by Lemma 3.9, $|R'|$ is at least $\left\lceil 0.8(\sqrt{q} + c + 2) \right\rceil - 1$, which shows that $|H| = \sqrt{q}$.

Let $P_0, P_1, P_2$ be three collinear points having small degree (see the argument after Corollary 3.8). Construct the grid $R'$ and the subgroup grid $R$ containing it as in Lemma 3.9. First we will show that there are few points in $B \setminus R$ and not on the line $\ell$ containing $P_0, P_1, P_2$. Let $P$ be a point in $B \setminus (R \cup \ell)$. Applying the automorphisms $\{\alpha_{h,k} : h, k \in H\}$ of $R$ (see the proof of Lemma 3.9), shows that the orbit of $P$ under the automorphism group of $R$ has size at least $|R|$. Hence there must be a point $Q$ in the orbit of $P$ that is not in $B$, otherwise $B$ would have at least $|R| + |R \cap B| \geq 2|R \cap B| > 2\left\lceil 0.8(\sqrt{q} + c + 2) \right\rceil - 1 \geq 2(\left\lceil 0.8(\sqrt{q} + 1) \right\rceil - 1)^2$ points, that is larger than $q + k$. It follows from Corollary 3.2, that through $Q$ there are at least $q - \sqrt{q} + k - c$ lines intersecting $B$, hence the total number
of points on the lines passing through $Q$ and having more than one point of $B$ is at most $|B| - (q - \sqrt{q} + k - c)$. This shows that there are at least $|B \cap R| - ((|B| - (q - \sqrt{q} + k - c)) = |B \cap R| - (\sqrt{q} + \delta) \text{ lines through } Q$ that intersect $B \cap R$, which means that also through $P$ there must be at least this many lines intersecting $R$. Hence at least $(|B \cap R|-(\sqrt{q}+\delta))-(|R|-|B \cap R|) = 2|B \cap R| - q - \sqrt{q} - c \text{ lines through } P \text{ contain a point of } B \cap R$. So the degree of $P$ is definitely larger than $\sqrt{q} + c + 2$ and so by Lemma 3.3 it must be at least $q - \sqrt{q} - c + k$. By Lemma 3.4, there are at most $2(\sqrt{q} + c + 2)$ such points.

For simplicity, let $\ell$ be the line at infinity and consider the directions determined by the grid $R$. Due to the transitivity of the automorphism group of $R$, it is clear that the lines of one parallel class intersects $R$ in the same number of points. So through a determined direction there can be at most $q/2$ lines intersecting $R$. If such a determined infinite point was not in $B$, then the number of $0$-secants through it would be at least $q/2 - 2(\sqrt{q} + c + 2)$.

By Corollary 3.2, through such a point there would pass at least $q - \sqrt{q} - c$ $0$-secants through $B$, but as there are no such points by the assumption of Theorem 3.1, determined directions should be in $B$. Hence $R$ determines at most $\sqrt{q} + c + 2$ directions and by the results of Rédei (see [6]) $R$ is either an affine Baer subplane or it determines at least $2\sqrt{q}$ directions. If there is a point in $R \setminus B$, then through it there pass at least $q - \sqrt{q} - 2(\sqrt{q} + c + 2)$ $0$-secants of $B$, but such a point does not exist by Corollary 3.2. 

**Theorem 3.10** Let $B$ be a point set in $\PG(2,q)$, $1600 \leq q$, with cardinality $q + k$, $0 \leq k \leq 0.6\sqrt{q}$. Assume that the number, $\delta$ of skew lines of $B$ is less than $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$, where $0 \leq c \leq 0.05\sqrt{q} - 2$. Then $B$ contains more than $q + 1 - (\sqrt{q} - k + c + 1)$ points from a line or $q + \sqrt{q} + 1 - (\sqrt{q} - k + c + 1)$ points from a Baer subplane.

**Proof.** Corollary 3.2 implies that the number of skew lines through a point not in $B$ is either at most $\sqrt{q} - k + c + 1$ or at least $q - \sqrt{q} - c$. Let $R$ be the set of points with at least $q - \sqrt{q} - c$ skew lines through it. Let $\delta(k) := (q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$. Let us add a point $P$ of $R$ to $B$. Then $B \cup \{P\}$ has $q + k'$ points, where $k' = k + 1$ and the number of skew lines is at most $\delta(k')$. So again Corollary 3.2 implies that the number of skew lines through a point not in $B$ is either at most $\sqrt{q} - k' + c + 1$ or at least $q - \sqrt{q} - c$. Thus we can continue adding the points of $R$ one by one to $B$. 


If $k'$ reaches $0.6\sqrt{q}$, then by Theorem 2.2 this point set can be obtained from a blocking set by deleting at most $\frac{\delta(k')}{2q+1-|B|} + \frac{1}{2} < 0.5\sqrt{q}$ points. Note that the size of this blocking set is less than $q+1 + q^{1/3}\left[\frac{q^{2/3}+1}{q^{1/3}+1}\right]$, when $q > 1600$; hence by Sziklai ([9]), Corollary 5.1 each line intersects it in $1 \mod \sqrt{q}$ points and so by Bruen [4] this blocking set is either a Baer subplane or a line. If $k'$ does not reach $0.6\sqrt{q}$ at all, then applying Theorem 3.1 for the set $B \cup R$, we get that it contains a line or a Baer subplane. Hence in both cases there is a set $R'$, such that $B \cup R'$ is either a Baer subplane or a line. Note that if we delete $\sqrt{q} - k + x + 1$ points from a Baer subplane and add $x$ points outside then the number of skew lines are at least $(q - \sqrt{q} - x)(\sqrt{q} - k + x + 1)$. Hence $x < c$. Similarly, if we delete $x$ points from a line and add $k - x + 1$ points, then the number of skew lines will be at least $(q - k + 1 - x)x$. Hence $x < \sqrt{q} - k + c + 1$. \[\Box\]

**Remark 3.11** We may extend our main result to negative $k$, that is for sets with size less than $q$. To prove Lemma 3.6 we need that

\[(0.8 + 1 - 0.8 \frac{1}{2})(\sqrt{q} + c + 2)^2 < q + k \tag{3}\]

So for example if $c < 0.02\sqrt{q} - 2$ and $-0.06q < k$, then the previous inequality remains true. As $c$ has to be at least 0, this means that $q > 10000$.

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