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Particle with spin 1 in a magnetic field
on the hyperbolic plane $H_2$
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Abstract

There are constructed exact solutions of the quantum-mechanical equation for a spin $S = 1$ particle in 2-dimensional Riemannian space of constant negative curvature, hyperbolic plane, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels describing quantization of the motion of the vector particle in magnetic field on the 2-dimensional space $H_2$ has been found, nonrelativistic and relativistic equations have been solved.

1. Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3, 4]. In 1985 – 2010, a more general problem in a curved Riemannian background, hyperbolic and spherical planes, was extensively studied [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], providing us with a new system having intriguing dynamics and symmetry, both on classical and quantum levels.

Extension to 3-dimensional hyperbolic and spherical spaces was performed recently. In [25, 26, 27], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky $H_3$ and Riemann $S_3$ spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [28, 29, 30]. In the present paper, we consider a quantum-mechanical problem a particle with spin 1/2 described by the Dirac equation in 3-dimensional Lobachevsky and Riemann space models in presence of the external magnetic field.

In the present paper, we will construct exact solutions for a vector particle described by 10-dimensional Duffin–Kemmer equation in external magnetic field on the background of 2-dimensional spherical space $H_2$.

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10-dimensional Duffin–Kemmer equation for a vector particle in a curved space-time has the form \[ \tag{31} \]
\[
\left\{ \beta^c \left[ i \left( e^\beta_{(c)} \partial_\beta + \frac{1}{2} J^{ab}_{\gamma_{abc}} \right) + \frac{e}{\hbar c} A_{(c)} \right] - \frac{mc}{\hbar} \right\} \Psi = 0 ,
\]
where \( \gamma_{abc} \) stands for Ricci rotation coefficients, \( A_a = e^\beta_{(a)} A_{\beta} \) represent tetrad components of electromagnetic 4-vector \( A_{\beta} \); \( J^{ab} = \beta^a \beta^b - \beta^b \beta^a \) are generators of 10-dimensional representation of the Lorentz group. For shortness, we use notation \( \epsilon / \hbar c \rightarrow e, \ mc / \hbar \rightarrow M \).

In the space \( H_3 \) we will use the system of cylindric coordinates \[ \tag{32} \]
\[
dS^2 = c^2 dt^2 - \cosh^2 z (dr^2 + \sinh^2 r \ d\phi^2) - dz^2 ,
\]
\[
u_1 = \cosh z \sinh r \cos \phi , \quad \nu_2 = \cosh z \sinh r \sin \phi ,
\]
\[
u_3 = \sinh z , \quad \nu_0 = \cosh z \cosh r ; \tag{1.2}
\]
\[
G = \{ r \in [0, +\infty) , \ \phi \in [0, 2\pi] , \ z \in ( -\infty , +\infty ) \} .
\]

Generalized expression for electromagnetic potential for an homogeneous magnetic field in the curved model \( H_3 \) is given as follows
\[
A_{\phi} = -2B \sinh^2 \frac{r}{2} = -B (\cosh r - 1) . \tag{1.3}
\]

We will consider the above equation in presence of the field in the model \( H_3 \). Corresponding to cylindric coordinates \( x^\alpha = (t, r, \phi, z) \) a tetrad can be chosen as
\[
e^\beta_{(a)} (x) = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & \cosh^{-1} z & 0 & 0 \\
0 & 0 & \cosh^{-1} z \sinh^{-1} r & 0 \\
0 & 0 & 0 & 1 \\
\end{vmatrix} . \tag{1.4}
\]

Taking into account relations
\[
\Gamma_{jk}^r = \begin{vmatrix}
0 & 0 & \tanh z & \\
-\sinh r \cosh r & 0 & \\
0 & 0 & \\
\tanh z & 0 & \\
\end{vmatrix} , \quad \Gamma_{jk}^\phi = \begin{vmatrix}
0 & \coth r & 0 \\
\coth r & 0 & \tanh z \\
0 & \tanh z & 0 \\
\end{vmatrix} , \\
\Gamma_{jk}^z = \begin{vmatrix}
-\sinh z \cosh z & 0 & 0 \\
0 & 0 & \sinh z \cosh z \sinh^2 r & 0 \\
0 & 0 & 0 \\
\end{vmatrix} ,
\]
\[
\gamma_{122} = \frac{1}{\cosh z \tanh r} , \quad \gamma_{311} = \tanh z , \quad \gamma_{322} = \tanh z , \tag{1.5}
\]
eq. (1.1) reduces to the form
\[
\begin{align*}
\left\{ i \beta^0 \frac{\partial}{\partial t} + \frac{1}{\cosh z} \left( i \beta^1 \frac{\partial}{\partial r} + \beta^2 i \partial_\phi - eB(\cosh r - 1) + i J^{12} \cosh r r \right) \right\} +
\end{align*}
\]
$$+i\beta^3 \frac{\partial}{\partial z} - i \frac{\sinh z}{\cosh z} (\beta^1 J^{13} + \beta^2 J^{23}) - M \right \} \Psi = 0.$$ \hfill (1.6)

To separate the variables in eq. (1.5), we are to employ an explicit form of the Duffin–Kemmer matrices $\beta^a$; it will be most convenient to use so called cyclic representation \[34\], where the generator $J^{12}$ is of diagonal form (we specify matrices by blocks in accordance with $(1 - 3 - 3 - 3)$-splitting)

$$\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^1 = \begin{pmatrix} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{pmatrix} \quad \beta^2 = \begin{pmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_i \\ e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{pmatrix} \quad \beta^3 = \begin{pmatrix} 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & \tau_i \\ e_3^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{pmatrix}, \quad (1.7)$$

where $e_i$, $e_i^t$, $\tau_i$ denote

$$e_1 = \frac{1}{\sqrt{2}} (-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}} (1, 0, 1), \quad e_3 = (0, i, 0),$$

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = s_3.$$ \hfill (1.8)

The generator $J^{12}$ explicitly reads

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 =$$

$$= \begin{pmatrix} (-e_1 e_2^+ + e_2 e_1^+) & 0 & 0 & 0 \\ 0 & (-\tau_1 \tau_2 + \tau_2 \tau_1) & 0 & 0 \\ 0 & 0 & (-e_2^+ \cdot e_2 + e_2^+ \cdot e_1) & 0 \\ 0 & 0 & 0 & (-\tau_1 \tau_2 + \tau_2 \tau_1) \end{pmatrix} =$$

$$= -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{pmatrix} = -i S_3.$$ \hfill (1.9)

2. Restriction to 2-dimensional model

Let us restrict ourselves to 2-dimensional case, spherical space $H_2$ (formally it is sufficient in eq. (1.5) to remove dependence on the variable $z$ fixing its value by $z = 0$)

$$\left[ i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial \phi}{\sinh r} - eB (\cosh r - 1) + iJ^{12} \cosh r \right] \Psi = 0.$$ \hfill (2.1)
With the use of substitution

\[ \Psi = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \Phi_0(r) \\ \Phi(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{vmatrix}, \quad (2.2) \]

eq. (2.1) assumes the form (introducing notation \( m + B(\cosh r - 1) = \nu(r) \))

\[ \begin{vmatrix} \epsilon \beta^0 + i\beta^1 \frac{\partial}{\partial r} - \beta^2 \frac{\nu(r) - \cosh r S_3}{\sinh r} - M \end{vmatrix} \begin{vmatrix} \Phi_0(r) \\ \Phi(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{vmatrix} = 0 . \quad (2.3) \]

Eq. (2.3) reads

\[ \begin{vmatrix} \epsilon & 0 & 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} + i \begin{vmatrix} 0 & 0 & 0 & 0 & \tau_1 \\ 0 & 0 & 0 & 0 & 0 \\ -e_1 & 0 & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 & 0 \end{vmatrix} \frac{\partial}{\partial r} - \frac{1}{\sinh r} \begin{vmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2 & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{vmatrix} (\nu - \cosh r S_3) - M \begin{vmatrix} \Phi_0 \\ \Phi \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0 , \quad (2.4) \]

or in a block form

\[ \begin{align*}
ie_1 \partial_r \vec{E} - \frac{1}{\sinh r} e_2 (\nu - \cosh r s_3) \vec{E} &= M \Phi_0 , \\
\epsilon \cosh z \vec{E} + i\tau_1 \partial_r \vec{H} - \frac{\tau_2}{\sinh r} (\nu - \cosh r s_3) \vec{H} &= M \vec{\Phi} , \\
-\epsilon \cosh z \vec{\Phi} - ie_1^+ \partial_r \Phi_0 + \frac{\nu}{\sinh r} e_2^+ \Phi_0 &= M \vec{E} , \\
-i\tau_1 \partial_r \vec{\Phi} + \frac{(\nu - \cosh r s_3)}{\sinh r} \tau_2 \vec{\Phi} &= M \vec{H} .
\end{align*} \quad (2.5) \]

After separation of the variables we get

\[ \gamma \left( \frac{\partial E_1}{\partial r} - \frac{\partial E_2}{\partial r} \right) - \frac{\gamma}{\sinh r} [(\nu - \cosh r) E_1 + (\nu + \cosh r) E_3] = M \Phi_0 , \]

\[ + i\epsilon \cosh z E_1 + i\gamma \frac{\partial H_2}{\partial r} + i\gamma \frac{\nu}{\sinh r} H_2 = M \Phi_1 , \]

\[ + i\gamma E_2 + i\gamma \frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r} - \frac{i\gamma}{\sinh r} [(\nu - \cosh r) H_1 - (\nu + \cosh r) H_3] = M \Phi_2 , \]

\[ 4 \]
\[ +i\varepsilon E_3 + i\gamma \frac{\partial H_2}{\partial r} - i\gamma \frac{\nu}{\sinh r} H_2 = M\Phi_3 \]  

(2.6)

\[ -i\varepsilon\Phi_1 + \gamma \frac{\partial \Phi_0}{dr} + \gamma \frac{\nu}{\sinh r} \Phi_0 = ME_1, \]
\[ -i\varepsilon\Phi_2 = ME_2, \]
\[ -i\varepsilon\Phi_3 - \gamma \frac{\partial \Phi_0}{dr} + \gamma \frac{\nu}{\sinh r} \Phi_0 = ME_3, \]  

(2.7)

\[ -i\gamma \frac{\partial \Phi_2}{dr} - i\gamma \frac{\nu}{\sinh r} \Phi_2 = M\cosh z H_1, \]
\[ -i\gamma \left( \frac{\partial \Phi_1}{dr} + \frac{\partial \Phi_3}{dr} \right) + \frac{i\gamma}{\sinh r} \left[ (\nu - \cosh r)\Phi_1 - (\nu + \cosh r)\Phi_3 \right] = MH_2, \]
\[ -i\gamma \frac{\partial \Phi_2}{dr} + i\gamma \frac{\nu}{\sinh r} \Phi_2 = MH_3. \]  

(2.8)

With the notation

\[ \frac{1}{\sqrt{2}} \left( \frac{\partial}{dr} + \frac{\nu - \cosh r}{\sinh r} \right) = \hat{a}_-, \quad \frac{1}{\sqrt{2}} \left( \frac{\partial}{dr} + \frac{\nu + \cosh r}{\sinh r} \right) = \hat{a}_+, \quad \frac{1}{\sqrt{2}} \left( \frac{\partial}{dr} + \frac{\nu}{\sinh r} \right) = \hat{a}, \]
\[ \frac{1}{\sqrt{2}} \left( -\frac{\partial}{dr} + \frac{\nu - \cosh r}{\sinh r} \right) = \hat{b}_-, \quad \frac{1}{\sqrt{2}} \left( -\frac{\partial}{dr} + \frac{\nu + \cosh r}{\sinh r} \right) = \hat{b}_+, \quad \frac{1}{\sqrt{2}} \left( -\frac{\partial}{dr} + \frac{\nu}{\sinh r} \right) = \hat{b}, \]

the above system reads

\[ -\hat{b}_- E_1 - \hat{a}_+ E_3 = M\Phi_0, \]
\[ -i\hat{b}_- H_1 + i\hat{a}_+ H_3 + i\varepsilon E_2 = M\Phi_2, \]
\[ i\hat{a}_- H_2 + i\varepsilon E_1 = M\Phi_1, \]
\[ -i\hat{b}_- H_2 + i\varepsilon E_3 = M\Phi_3, \]
\[ \hat{a}\Phi_0 - i\varepsilon \Phi_1 = ME_1, \]
\[ -i\hat{a}\Phi_2 = MH_1, \]
\[ \hat{b}\Phi_0 - i\varepsilon \Phi_3 = ME_3, \]
\[ i\hat{b}\Phi_2 = MH_3, \]
\[ -i\varepsilon\Phi_2 = ME_2, \]
\[ i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3 = MH_2. \]  

(2.9)

(2.10)
3. Nonrelativistic approximation

Excluding non-dynamical variables $\Phi_0, H_1, H_2, H_3$ with the help of equations

\[-\hat{b}_- E_1 - \hat{a}_+ E_3 = M \Phi_0 ,
\]
\[-i \hat{a} \Phi_2 = M H_1 ,
\]
\[i \hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3 = M H_2 ,
\]
\[i \hat{b} \Phi_2 = M H_3 ,
\]

we get 6 equations (grouping them in pairs)

\[i \hat{a} (i \hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3) + i \epsilon M E_1 = M^2 \Phi_1 ,
\]
\[\hat{a} (-\hat{b}_- E_1 - \hat{a}_+ E_3 - i \epsilon M \Phi_1) = M^2 e_1 ,
\]

\[\tag{3.2a}
\]
\[-i \hat{b}_- (-i \hat{a} \Phi_2) + i \hat{a}_+ (i \hat{b} \Phi_2) + i \epsilon M E_2 = M^2 \Phi_2 ,
\]
\[-i \epsilon M \Phi_2 = M^2 E_2 ,
\]

\[\tag{3.2b}
\]
\[-i \hat{b} (i \hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3) + i \epsilon M E_3 = M^2 \Phi_3 ,
\]
\[\hat{b} (-\hat{b}_- E_1 - \hat{a}_+ E_3) - i \epsilon M \Phi_3 = M^2 E_3 ,
\]

\[\tag{3.2c}
\]

Now we introduce big and small constituents

\[\Phi_1 = \Psi_1 + \psi_1 , \quad iE_1 = \Psi_1 - \psi_1 ,
\]
\[\Phi_2 = \Psi_2 + \psi_2 , \quad iE_2 = \Psi_2 - \psi_2 ,
\]
\[\Phi_3 = \Psi_3 + \psi_3 , \quad iE_3 = \Psi_3 - \psi_3 ;
\]

besides we should separate the rest energy by formal change $\epsilon \rightarrow \epsilon + M$; summing and subtracting equation within each pair (3.2) and ignoring small constituents $\psi_i$ we arrive at three equations for big components

\[\left(-2 \hat{a} \hat{b}_- + 2 \epsilon M \right) \Psi_1 = 0 ,
\]
\[\left(-\hat{b}_- \hat{a} + \hat{a}_+ \hat{b} + 2 \epsilon M \right) \Psi_2 = 0 ,
\]
\[\left(-2 \hat{a} \hat{b}_+ + 2 \epsilon M \right) \Psi_3 = 0 .
\]

\[\tag{3.4}
\]

It is a needed Pauli-like system for the spin 1 particle.

Explicitly they read

\[\left[\frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - \frac{1}{\sinh r} \frac{d\nu}{dr} - \frac{1 - 2 \nu \cosh r}{\sinh^2 r} - \frac{\nu^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_1 = 0 ,
\]
\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - \frac{\nu^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_2 = 0,
\]
\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} + \frac{1}{\sinh r} \frac{d\nu}{dr} - \frac{1 + 2 \nu \cosh r}{\sinh^2 r} - \frac{\nu^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_3 = 0.
\]

(3.5)

Allowing for \( \nu(r) = m + B \left( \cosh r - 1 \right) \) we arrive at
\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - B - \frac{1 - 2 \left[ m + B \left( \cosh r - 1 \right) \right] \cosh r - \left[ m + B \left( \cosh r - 1 \right) \right]^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_1 = 0,
\]
\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - \left[ m + B \left( \cosh r - 1 \right) \right]^2 \frac{1 + 2 \nu \cosh r - \nu^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_2 = 0,
\]
\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} + B - \frac{1 + 2 \left[ m + B \left( \cosh r - 1 \right) \right] \cosh r - \left[ m + B \left( \cosh r - 1 \right) \right]^2}{\sinh^2 r} + 2 \epsilon M \right] \Psi_3 = 0.
\]

(3.6)

The first and the third equations are symmetric with respect to formal change \( m \rightarrow -m, \ B \rightarrow -B \).

In the new variable \( 1 - \cosh r = 2y \), they look
\[
y(1 - y) \frac{d^2\Psi_1}{dy^2} + (1 - 2y) \frac{d\Psi_1}{dy} +
\]
\[
+ \left[ B^2 - B - 2 \epsilon M - \frac{1}{4} \frac{(2B - m - 1)^2}{1 - y} - \frac{1}{4} \frac{(m - 1)^2}{y} \right] \Psi_1 = 0,
\]

(3.7a)

\[
y(1 - y) \frac{d^2\Psi_2}{dy^2} + (1 - 2y) \frac{d\Psi_2}{dy} +
\]
\[
+ \left[ B^2 - 2 \epsilon M - \frac{1}{4} \frac{(2B - m)^2}{1 - y} - \frac{1}{4} \frac{m^2}{y} \right] \Psi_2 = 0,
\]

(3.7b)

\[
y(1 - y) \frac{d^2\Psi_3}{dy^2} + (1 - 2y) \frac{d\Psi_3}{dy} +
\]
\[
+ \left[ B^2 + B - 2 \epsilon M - \frac{1}{4} \frac{(2B - m + 1)^2}{1 - y} - \frac{1}{4} \frac{(m + 1)^2}{y} \right] \Psi_3 = 0.
\] (3.7c)

Eq. (3.7a) with the substitution

\[
\Psi_1 = y^{C_1} (1 - y)^{A_1} f_1
\]

leads to

\[
y (1 - y) \frac{d^2 \Psi_1}{dy^2} + \left[ 2C_1 + 1 - (2A_1 + 2C_1 + 2) y \right] \frac{dB_1}{dy} + \\
+ \left[ B^2 - B - 2 \epsilon M - (A_1 + C_1) (A_1 + C_1 + 1) + \\
\frac{1}{4} \frac{4A_1^2 - (2B - m - 1)^2}{1 - y} + \frac{1}{4} \frac{4C_1^2 - (m - 1)^2}{y} \right] \Psi_1 = 0.
\] (3.8)

At \( A_1, C_1 \) obeying

\[
A_1 = \pm \frac{1}{2} (2B - m - 1), \quad C_1 = \pm \frac{1}{2} (m - 1),
\]

eq. (3.8) becomes simpler

\[
y (1 - y) \frac{d^2 \Psi_1}{dy^2} + \left[ 2C_1 + 1 - (2A_1 + 2C_1 + 2) y \right] \frac{dB_1}{dy} + \\
+ \left[ B^2 - B - 2 \epsilon M - (A_1 + C_1) (A_1 + C_1 + 1) \right] \Psi_1 = 0,
\] (3.9a)

what is hypergeometric equation with parameters

\[
\alpha_1 = A_1 + C_1 + \frac{1}{2} + \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}},
\]

\[
\beta_1 = A_1 + C_1 + \frac{1}{2} - \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}},
\]

\[
\gamma_1 = 2C_1 + 1.
\] (3.9b)

To have finite and single-valued solutions one must impose restrictions \( A_1 < 0, C_1 > 0 \). Besides, one must get \( n \)-order polynomials and satisfy the inequality \( A_1 + C_1 + n < 0 \).

Four different possibilities for \( A_1, C_1 \) are (for definiteness let it be \( B > 0 \)):

1. \( A_1 = -\frac{1}{2} (2B - m - 1), \quad C_1 = -\frac{1}{2} (m - 1), \)

2. \( A_1 = +\frac{1}{2} (2B - m - 1), \quad C_1 = -\frac{1}{2} (m - 1), \)

3. \( A_1 = +\frac{1}{2} (2B - m - 1), \quad C_1 = +\frac{1}{2} (m - 1), \)

4. \( A_1 = -\frac{1}{2} (2B - m - 1), \quad C_1 = +\frac{1}{2} (m - 1), \)
4. \[ A_1 = -\frac{1}{2} (2B - m - 1), \quad C_1 = +\frac{1}{2} (m - 1). \]

To describe bound state, only variants 1 and 4 are appropriate:

\[
\begin{align*}
\alpha_1 &= -B + \frac{3}{2} + \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}}, \\
\beta_1 &= -B + \frac{3}{2} - \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}}, \\
\gamma_1 &= -m + 2,
\end{align*}
\]

spectrum \[ \alpha_1 = -n, \quad \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}} = B - \frac{3}{2} - n, \quad (3.10a) \]

\[ \epsilon M = B - 1 + n \left( B - \frac{3}{2} - \frac{n}{2} \right); \]

\[
\begin{align*}
\alpha_1 &= -B + m + \frac{1}{2} + \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}}, \\
\beta_1 &= -B + m + \frac{1}{2} - \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}}, \\
\gamma_1 &= m,
\end{align*}
\]

spectrum \[ \alpha_1 = -n, \quad \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}} = B - \frac{1}{2} - (n + m), \quad (3.10b) \]

\[ \epsilon M = (m + n) \left( B - \frac{1}{2} - \frac{1}{2} (m + n) \right). \]

Formulas (3.10a,b) can be jointed into single one

\[ \sqrt{B^2 - B - 2 \epsilon M + \frac{1}{4}} = -n - \frac{1}{2} - \left| \frac{2B - m - 1}{2} \right| + \left| m - 1 \right|. \quad (3.10c) \]

From eq. (3.7b) with the substitution

\[ \Psi_2 = y^{C_2} (1 - y)^{A_2} f_2 \]

we get

\[ y \left( 1 - y \right) \frac{d^2 f_2}{dy^2} + \left[ 2C_2 + 1 - (2A_2 + 2C_2 + 2) y \right] \frac{df_2}{dy} + \]

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\[ + \left[ B^2 - 2 \epsilon M - (A_2 + C_2) (A_2 + C_2 + 1) + \frac{1}{4} \frac{4 A_2^2 - (2 B - m)^2}{1 - y} + \frac{1}{4} \frac{4 C_2^2 - m^2}{y} \right] f_2 = 0. \]  

(3.11)

At

\[ A_2 = \pm \frac{1}{2} (2 B - m), \quad C_2 = \pm \frac{m}{2}, \]

eq. (3.11) becomes simpler

\[ y (1 - y) \frac{d^2 f_2}{dy^2} + [2 C_2 + 1 - (2 A_2 + 2 C_2 + 2) y] \frac{d f_2}{d y} + \]

\[ + \left[ B^2 - 2 \epsilon M - (A_2 + C_2) (A_2 + C_2 + 1) \right] f_2 = 0 \]  

(3.12a)

which is recognized as of hypergeometric type

\[ \alpha_2 = A_2 + C_2 + \frac{1}{2} + \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]

\[ \beta_2 = A_2 + C_2 + \frac{1}{2} - \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]

\[ \gamma_2 = 2 C_2 + 1. \]  

(3.12b)

From four variants

1. \[ A_2 = -\frac{1}{2} (2 B - m), \quad C_2 = -\frac{m}{2}, \]

2. \[ A_2 = +\frac{1}{2} (2 B - m), \quad C_2 = -\frac{m}{2}, \]

3. \[ A_2 = +\frac{1}{2} (2 B - m), \quad C_2 = +\frac{m}{2}, \]

4. \[ A_2 = -\frac{1}{2} (2 B - m), \quad C_2 = +\frac{m}{2} \]

only 1 and 4 seem to be approptiate to describe bound states:

\[ 1, \quad m < 0, \]

\[ \alpha_2 = -B + \frac{1}{2} + \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]

\[ \beta_2 = -B + \frac{1}{2} - \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]

\[ \gamma_2 = -m + 1, \]

spectrum \[ \alpha_2 = -n, \quad \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}} = B - \frac{1}{2} - n, \]  

(3.13a)
\[ \epsilon M = \frac{B}{2} + n \left( B - \frac{1}{2} - \frac{n}{2} \right); \]

\[ 4, \quad 0 < m < B, \]
\[ \alpha_2 = -B + m + \frac{1}{2} + \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]
\[ \beta_2 = -B + m + \frac{1}{2} - \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}}, \]
\[ \gamma_2 = m + 1, \]

spectrum \[ \alpha_2 = -n, \quad \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}} = B - \frac{1}{2} - (n + m), \quad (3.13b) \]

\[ \epsilon M = \frac{B}{2} + (m + n) \left( B - \frac{1}{2} - \frac{1}{2} (m + n) \right). \]

Formulas (3.13a,b) can be joint into a single one

\[ \sqrt{B^2 - 2 \epsilon M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{|2B - m| + |m|}{2}. \quad (3.13c) \]

The region for allowed values of \( m \) for bound states can be illustrated by Fig. 1.

Fig. 1. Bound states at \( B > 0 : m < B \)
At $B < 0$, we should have different Fig. 2.

\begin{align*}
\left| m \right| - \left| 2B - m \right| < 0
\end{align*}

Fig. 2. Bound states at $B < 0$: $B < m$

Similar Figures can be given in connection with the functions $\Psi_1(y)$ and $\Psi_3$ as well. In case of (3.7c), with substitution

$$
\Psi_3 = y^{C_3}(1 - y)^{A_3}f_3,
$$

we will obtain

$$
y (1 - y) \frac{d^2 f_3}{dy^2} + [2 C_3 + 1 - (2 A_3 + 2 C_3 + 2) y] \frac{df_3}{dy} + \\
+ \left[ B^2 + B - 2 \epsilon M - (A_3 + C_3) (A_3 + C_3 + 1) + \\
+ \frac{1}{4} \frac{4 \ A_3^2 - (2 B - m + 1)^2}{1 - y} + \frac{1}{4} \frac{4 \ C_3^2 - (m + 1)^2}{y} \right] f_3 = 0. \quad (3.14)
$$

At $A_3, C_3$

$$
A_3 = \pm \frac{1}{2} (2 B - m + 1), \quad C_3 = \pm \frac{1}{2} (m + 1),
$$

eq (3.14) will read

$$
y (1 - y) \frac{d^2 f_3}{dy^2} + [2 C_3 + 1 - (2 A_3 + 2 C_3 + 2) y] \frac{df_3}{dy} + \
\frac{1}{4} \frac{4 \ A_3^2 - (2 B - m + 1)^2}{1 - y} + \frac{1}{4} \frac{4 \ C_3^2 - (m + 1)^2}{y} f_3 = 0.
$$
\[ + \left[ B^2 + B - 2 \epsilon M - (A_3 + C_3) (A_3 + C_3 + 1) \right] f_3 = 0 \quad (3.15a) \]

that is a hypergeometric equation

\[
\begin{align*}
\alpha_3 &= A_3 + C_3 + \frac{1}{2} + \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\beta_3 &= A_3 + C_3 + \frac{1}{2} - \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\gamma_3 &= 2C_3 + 1.
\end{align*}
\]

(3.15b)

From four possibilities

1. \( A_3 = -\frac{1}{2} (2B - m + 1), \quad C_3 = -\frac{1}{2} (m + 1), \)

2. \( A_3 = +\frac{1}{2} (2B - m + 1), \quad C_3 = -\frac{1}{2} (m + 1), \)

3. \( A_3 = +\frac{1}{2} (2B - m + 1), \quad C_3 = +\frac{1}{2} (m + 1), \)

4. \( A_3 = -\frac{1}{2} (2B - m + 1), \quad C_3 = +\frac{1}{2} (m + 1). \)

only 1 and 4 are appropriate to describe bound states:

\[
\begin{align*}
\alpha_3 &= -B - \frac{1}{2} + \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\beta_3 &= -B - \frac{1}{2} - \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\gamma_3 &= -m,
\end{align*}
\]

spectrum \( \alpha_3 = -n, \quad \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}} = B + \frac{1}{2} - n, \) \quad (3.16a)

\[
\epsilon M = n \left( B + \frac{1}{2} - \frac{n}{2} \right);
\]

\[
\begin{align*}
\alpha_3 &= -B + m + \frac{1}{2} + \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\beta_3 &= -B + m + \frac{1}{2} - \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}}, \\
\gamma_3 &= m + 2,
\end{align*}
\]

\[4, \quad 0 < m < B,\]
\begin{align}
\text{spectrum} \quad \alpha_3 = -n, \quad \sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}} = B - \frac{1}{2} - (n + m), \quad (3.16b)
\end{align}

\begin{align}
\epsilon M = B + (m + n) \left( B - \frac{1}{2} - \frac{1}{2} (m + n) \right).
\end{align}

Again, formulas (3.16a,b) can be joint into a single one

\begin{align}
\sqrt{B^2 + B - 2 \epsilon M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{2B - m + 1}{2} + \frac{m + 1}{2}.
\end{align} (3.16c)

4. Solution of radial equations in relativistic case

Let start with eqs. (2.4)-(2.5)

\begin{align}
\hat{b}_- E_1 - \hat{a}_+ E_3 = M \Phi_0,
-\hat{b}_- H_1 + i\hat{a}_+ H_3 + i\epsilon E_2 = M \Phi_2,
\hat{a} H_2 + i\epsilon E_1 = M \Phi_1,
-\hat{b} H_2 + i\epsilon E_3 = M \Phi_3,
\hat{a} \Phi_0 - i\epsilon \Phi_1 = M E_1,
-\hat{a} \Phi_2 = M H_1,
\hat{b} \Phi_0 - i\epsilon \Phi_3 = M E_3,
\hat{b} \Phi_2 = M H_3.
\end{align} (4.1)

Excluding six components \(E_i, H_i\), we derive four second order equations for \(\Phi_a\):

\begin{align}
\begin{aligned}
(-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2) \Phi_2 &= 0, \\
(-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} - M^2) \Phi_0 + i\epsilon(\hat{b}_- \Phi_1 + \hat{a}_+ \Phi_3) &= 0, \\
(-\hat{a} \hat{b}_- + \epsilon^2 - M^2) \Phi_1 + \hat{a} \hat{a}_+ \Phi_3 + i\epsilon \hat{a} \Phi_0 &= 0, \\
(-\hat{b} \hat{a}_+ + \epsilon^2 - M^2) \Phi_3 + \hat{b} \hat{b}_- \Phi_1 + i\epsilon \hat{b} \Phi_0 &= 0.
\end{aligned}
\end{align} (4.3)

Once, it should be noted existence of a simple solution of the system

\begin{align}
\Phi_0 = 0, \quad \Phi_1 = 0, \quad \Phi_3 = 0, \\
(-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2) \Phi_2 = 0.
\end{align} (4.4a)

and simple expressions for tensors components

\begin{align}
E_1 = 0, \quad H_1 = -iM^{-1} \hat{a} \Phi_2,
\end{align}
\[ E_3 = 0, \quad H_3 = iM^{-1}\hat{b} \Phi_2, \]
\[ E_2 = -i\epsilon M^{-1}\Phi_2, \quad H_2 = 0. \]  

Let us turn to (4.3) and act on the third equation from the left by operator \( \hat{b}_- \), and on the forth equation by operator \( \hat{a}_+ \). Thus, introducing the notation
\[ \hat{b}_- \Phi_1 = Z_1, \quad \hat{a}_+ \Phi_3 = Z_3, \]
instead of (4.3) we obtain
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0,
\]
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} - M^2)\Phi_0 + i\epsilon(Z_1 + Z_3) = 0,
\]
\[
(-\hat{b}_-\hat{a}_+ + \epsilon^2 - M^2)Z_1 + \hat{b}_-\hat{a}_+ Z_3 + i\hat{b}_-\hat{a}_-\Phi_0 = 0,
\]
\[
(-\hat{a}_+ \hat{b} + \epsilon^2 - M^2)Z_3 + \hat{a}_+ \hat{b} Z_1 + i\epsilon\hat{a}_+ \hat{b}\Phi_0 = 0. \]  

Instead of \( Z_1, Z_3 \), let us introduce new functions
\[ Z_1 = \frac{f + g}{2}, \quad Z_3 = \frac{f - g}{2}, \]
the the above system reads
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0,
\]
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} - M^2)\Phi_0 + i\epsilon f = 0,
\]
\[
-\hat{b}_-\hat{a}_+ \frac{f + g}{2} + (\epsilon^2 - M^2) \frac{f + g}{2} + \hat{b}_-\hat{a}_- \frac{f - g}{2} + i\epsilon\hat{b}_-\hat{a}_-\Phi_0 = 0,
\]
\[
-\hat{a}_+ \hat{b} \frac{f - g}{2} + (\epsilon^2 - M^2) \frac{f - g}{2} + \hat{a}_+ \hat{b} \frac{f + g}{2} + i\epsilon\hat{a}_+ \hat{b}\Phi_0 = 0. \]  

After elementary manipulations with equation 3 and 4 we get
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0,
\]
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} - M^2)\Phi_0 + i\epsilon f = 0,
\]
\[
-\hat{b}_-\hat{a}_+ g + (\epsilon^2 - M^2) \frac{f + g}{2} + i\epsilon\hat{b}_-\hat{a}_-\Phi_0 = 0,
\]
\[
\hat{a}_+ \hat{b} g + (\epsilon^2 - M^2) \frac{f - g}{2} + i\epsilon\hat{a}_+ \hat{b}\Phi_0 = 0. \]

Now, summing and subtracting equations 3 and 4, we obtain
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0,
\]
\[
(-\hat{b}_-\hat{a}_- - \hat{a}_+ \hat{b} - M^2)\Phi_0 + i\epsilon f = 0,
\]

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\begin{align*}
(-\hat{b}_-\hat{a} + \hat{a}_+\hat{b})g + (\epsilon^2 - M^2) f + i\epsilon(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) \Phi_0 &= 0, \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b})g + (\epsilon^2 - M^2) g + i\epsilon(\hat{b}_-\hat{a} - \hat{a}_+\hat{b})\Phi_0 &= 0, \\
\text{(4.7)}
\end{align*}

Taking into account identities

\begin{align*}
-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} &= \Delta_2 = ... \\
-\hat{b}_-\hat{a} + \hat{a}_+\hat{b} &= 2B
\end{align*}

we arrive at the system

\begin{align*}
\left(\Delta_2 + \epsilon^2 - M^2\right) \Phi_2 &= 0, \\
\left(\Delta_2 - M^2\right) \Phi_0 + i\epsilon f &= 0, \\
2B g + (\epsilon^2 - M^2) f - i\epsilon\Delta_2 \Phi_0 &= 0, \\
\Delta_2 g + (\epsilon^2 - M^2) g - 2i\epsilon B \Phi_0 &= 0, \\
\text{(4.9)}
\end{align*}

From the second equation, with the use of expression for \(\Delta_2\Phi_0\) according to the first equation, we derive linear relation between three functions

\begin{align*}
2B g - M^2 f - i\epsilon M^2 \Phi_0 &= 0. \\
\text{(4.10)}
\end{align*}

Let us exclude \(f\)

\begin{align*}
f &= \frac{2B}{M^2} g - i\epsilon \Phi_0
\end{align*}

so we get

\begin{align*}
\left(\Delta_2 + \epsilon^2 - M^2\right) g &= 2i\epsilon B \Phi_0, \\
\left(\Delta_2 - \epsilon^2 - M^2\right) \Phi_0 &= -\frac{2i\epsilon B}{M^2} g.
\end{align*}

With notation \(\gamma = \epsilon^2 / M^2\), the system can be presented in a matrix form as follows

\begin{align*}
\left(\Delta_2 + \epsilon^2 - M^2\right) \begin{vmatrix}
g \\ \epsilon \Phi_0
\end{vmatrix} &= \begin{vmatrix}
0 & 2iB \\
-2iB\gamma & 0
\end{vmatrix} \begin{vmatrix}
g \\ \epsilon \Phi_0
\end{vmatrix}. \\
\text{(4.13)}
\end{align*}

or symbolically

\begin{align*}
\Delta f &= Af \\
\Delta f' &= SAS^{-1} f' \\
f' &= S f.
\end{align*}

It remains to find a transformation reducing the matrix \(A\) to a diagonal form

\begin{align*}
SAS^{-1} &= \begin{vmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{vmatrix}, \\
S &= \begin{vmatrix}
a & d \\
c & b
\end{vmatrix};
\end{align*}

the problem is equivalent to the linear system

\begin{align*}
-\lambda_1 a - 2i\gamma B d &= 0, \\
2iB a - \lambda_1 d &= 0;
\end{align*}

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\[-\lambda_2 \ c - 2i\gamma B \ b = 0, \]
\[2iB \ c - \lambda_2 \ b = 0. \]

Its solutions can be chosen in the form
\[
\lambda_1 = + \frac{2\epsilon B}{M}, \quad \lambda_2 = - \frac{2\epsilon B}{M},
\]
\[
S = \begin{vmatrix} \epsilon & +iM \\ -iM & \epsilon \end{vmatrix}, \quad S^{-1} = \frac{1}{-2i\epsilon M} \begin{vmatrix} -iM & -iM \\ -\epsilon & \epsilon \end{vmatrix}.
\] (4.14)

New (primed) function satisfy the following equations
\[
1) \quad \left( \Delta_2 + \epsilon^2 - M^2 - \frac{2\epsilon B}{M} \right) g' = 0, \quad (4.15a)
\]
\[
2) \quad \left( \Delta_2 + \epsilon^2 - M^2 + \frac{2\epsilon B}{M} \right) \Phi'_0 = 0. \quad (4.15b)
\]

they are independent from each other, therefore there exist two solutions
\[
1) \quad g' \neq 0, \quad \Phi'_0 = 0, \quad (4.16a)
\]
\[
2) \quad g' = 0, \quad \Phi'_0 \neq 0. \quad (4.16b)
\]

The initial functions for these two cases assume respectively the form
\[
g = \frac{1}{2\epsilon} g' + \frac{1}{2i\epsilon} \epsilon \Phi'_0, \quad \epsilon \Phi_0 = \frac{1}{2iM} g' - \frac{1}{2iM} \epsilon \Phi'_0. \quad (4.17)
\]

In cases 1) and 2) they assume respectively the form
\[
1) \quad g = \frac{1}{2\epsilon} g', \quad \epsilon \Phi_0 = \frac{1}{2iM} g'. \quad (4.18a)
\]
\[
2) \quad g = \frac{1}{2i\epsilon} \epsilon \Phi'_0, \quad \epsilon \Phi_0 = -\frac{1}{2iM} \epsilon \Phi'_0. \quad (4.18b)
\]

To obtain explicit solutions for these differential equation, we need not any additional calculations, instead it suffices to perform simple formal changes as pointed below

\[
\left[ \frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} - \frac{[m + B (\cosh r - 1)]^2}{\sinh^2 r} + 2\epsilon M \right] f(r) = 0,
\]
\[
\sqrt{B^2 - 2\epsilon M + \frac{1}{4}} = -n - \frac{1}{2} - \frac{2B - m}{2} + \frac{1}{2} \quad (4.19)
\]
\[
2\epsilon M \quad \Rightarrow \quad \begin{cases} 
\left( \epsilon^2 - M^2 - \frac{2\epsilon B}{M} \right) & \quad (4.19) \\
\left( \epsilon^2 - M^2 \right) & \quad (4.15a) \\
\left( \epsilon^2 - M^2 + \frac{2\epsilon B}{M} \right) & \quad (4.15b)
\end{cases} \quad (4.20)
\]
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