1 Abstract

This paper finds a symmetry relation (between quantiles of a random variable and its negative) that is intuitively appealing. We show this symmetry is quite useful in finding new relations for quantiles, in particular an equivariance property for quantiles under continuous decreasing transformations.

Keywords: Quantile function, distribution function, symmetry, equivariance

2 Introduction

The traditional definition of quantiles for a random variable $X$ with distribution function $F$,

$$lq_X(p) = \inf\{x|F(x) \geq p\},$$

appears in classic works as [4]. We call this the “left quantile function”. In some books (e.g. [5]) the quantile is defined as

$$rq_X(p) = \inf\{x|F(x) > p\} = \sup\{x|F(x) \leq p\},$$

this is what we call the “right quantile function”. Also in robustness literature people talk about the upper and lower medians which are a very specific case of these definitions. Chapter 5 of [2] considers both definitions, explore their relation and shows that considering both has several advantages. In particular it provides a proof of the following lemma regarding the properties of the quantiles.

Lemma 2.1: (Quantile Properties Lemma) Suppose $X$ is a random variable on the probability space $(\Omega, \Sigma, P)$ with distribution function $F$: 
Section 3 presents the desirable “Quantile Symmetry Theorem”, a result that could be obtained only by considering both left and right quantiles. This relation can help us prove several other useful results regarding quantiles. Also using the quantile symmetry theorem, we find a relation for the equivariance property of quantiles under non-increasing continuous transformations.

In order to motivate why this relation is intuitively appealing we give the following example.

A scientist asked two of his assistants to summarize the following data regarding the acidity of rain:
Tab. 1: Rain acidity data

$pH$ is defined as the cologarithm of the activity of dissolved hydrogen ions ($H^+$).

\[ pH = -\log_{10} aH. \]

In the data file handed to the assistants (Table 1), the data is sorted with respect to $pH$ in increasing order from top to bottom. Hence the data is arranged decreasingly with respect to $aH$ from top to bottom.

The scientist asked the two assistants to compute the 20th and 80th percentile of the data to get an idea of the variability of the acidity. The first assistant used the $pH$ scale and the traditional definition of the quantile function:

\[ q_F(p) = \inf\{x \mid F(x) \geq p\}, \]

where $F$ is the empirical distribution of the data. He obtained the following numbers

\[ q_F(0.2) = 4.8327 \text{ and } q_F(0.8) = 5.2901, \] (1)

which are positioned in row 2 and 8 respectively.

The second assistant also used the traditional definition of the quantile function and the $aH$ scale to get

\[ q_F(0.2) = 2.6724 \times 10^{-6} \text{ and } q_F(0.8) = 14.1514 \times 10^{-6}, \] (2)
which correspond to row 9 and 3.

The scientist noticed that the assistants had used different scales. Then he thought since one of the scales is in the opposite order of the other and 0.2 and 0.8 have the same distance from 0 and 1 respectively, he must get the other first assistant’s result by transforming the second’s. So he transformed the second assistant’s results given in Equation 2 (or by simply looking at the corresponding rows, 9 and 3 under pH), to get

\[ 5.5731 \text{ and } 4.8492, \]

which is not the same as the first assistant’s result in Equation 1. He noticed that the position of these values are off by only one position from the previous values (being in row 9 and 3 instead of 8 and 2).

Then he tried the same himself for 25th and 75th percentile using both scales

\[ pH : q_F(0.25) = 4.8492 \text{ and } q_F(0.75) = 5.2901, \]

which are positioned at 3rd and 8th row.

\[ aH : q_F(0.25) = 5.1274 \times 10^{-6} \text{ and } q_F(0.25) = 14.1514 \times 10^{-6}, \]

which are also positioned at 8th and 3rd row. This time he was surprised to observe the symmetry he expected. He wondered when such symmetry exist and what can be said in general. He conjectured that the asymmetric definition of the traditional quantile is the reason of this asymmetry. He also conjectured that the symmetry property is off at most by one position in the dataset.

\section{3 Quantile symmetries}

This section studies the symmetry properties of distribution functions and quantile functions. Symmetry is in the sense that if \(X\) is a random variable, some sort of symmetry should hold between the quantile functions of \(X\) and \(-X\). We only treat the quantile functions for distributions here but the results can readily be applied to data vectors by considering their empirical distribution function.

Here we consider different forms of distribution functions. The usual one is defined to be \(F_X(x) = P(X \leq x)\). But clearly one can also consider \(F_X(x) = P(X <
In a random variable, $X$, $G_X^c(x) = P(X \geq x)$ or $G_X^o(x) = P(X > x)$ to characterize the distribution of a random variable. We call $F^c$ the left-closed distribution function, $F^o$ the left-open distribution function, $G^c$ the right-closed and $G^o$ the right-open distribution function. Like the usual distribution function these functions can be characterized by their limits in infinity, monotonicity and one-sided continuity.

First note that

$$F^c_X(x) = P(-X \leq x) = P(X \geq -x) = G_X^c(-x).$$

Since the left hand side is right continuous, $G_X^c$ is left continuous. Also note that

$$F^c_X(x) + G^o_X(x) = 1 \implies G^o_X(x) = 1 - F^c_X(x),$$
$$F^c_X(x) + G^c_X(x) = 1 \implies F^c_X(x) = 1 - G^c_X(x).$$

The above equations imply the following:

a) $G^o$ and $F^c$ are right continuous.
b) $F^o$ and $F^c$ are left continuous.
c) $G^o$ and $G^c$ are non-decreasing.
d) $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$ for $F = F^o, F^c$.
e) $\lim_{x \to \infty} G(x) = 0$ and $\lim_{x \to -\infty} G(x) = 1$ for $G = G^o, G^c$.

It is easy to see that the above given properties for $F^o, G^o, G^c$ characterize all such functions. The proof can be given directly using the properties of the probability measure (such as continuity) or by using arguments similar to the above.

Another lemma about the relation of $F^c, F^o, G^o, G^c$ is given below.

**Lemma 3.1:** Suppose $F^o, F^c, G^o, G^c$ are defined as above. Then

a) if any of $F^c, F^o, G^o, G^c$ are continuous, all the other ones are continuous too.
b) $F^c$ being strictly increasing is equivalent to $F^o$ being strictly increasing.
c) $F^c$ being strictly increasing is equivalent to $G^o$ being strictly decreasing.
d) $G^c$ being strictly decreasing is equivalent to $G^o$ being strictly increasing.

**Proof**  a) Note that

$$\lim_{y \to x^-} F^c(x) = \lim_{y \to x^-} F^o(x),$$

and

$$\lim_{y \to x^+} F^o(x) = \lim_{y \to x^+} F^o(x).$$
If these two limits are equal for either $F^c$ or $F^o$ they are equal for the others as well. 

b) If either $F^c$ or $F^o$ are not strictly increasing then they are constant on $[x_1, x_2]$, $x_1 < x_2$. Take $x_1 < y_1 < y_2 < x_2$. Then 

\[ F^o(x_1) = F^o(x_2) \Rightarrow P(y_1 \leq X \leq y_2) = 0 \Rightarrow F^o(y_1) = F^o(y_2). \]

Also we have 

\[ F^c(x_1) = F^c(x_2) \Rightarrow P(y_1 \leq X \leq y_2) = 0 \Rightarrow F^o(y_1) = F^o(y_2). \]

c) This is trivial since $G^o = 1 - F^c$. 

d) If $G^c$ is strictly decreasing then $F^o$ is strictly increasing since $G^c = 1 - F^o$. By Part b), $F^c$ strictly is increasing. Hence $G^o = 1 - F^c$ is strictly decreasing. 

The relationship between these distribution functions and the quantile functions are interesting and have interesting implications. It turns out that we can replace $F^c$ by $F^o$ in some definitions. 

**Lemma 3.2**: Suppose $X$ is a random variable with open and closed left distributions $F^o, F^c$ as well as open and closed right distribution functions $G^o, G^c$. Then 

a) $lq_X(p) = \inf \{x | F_X^o(x) \geq p\}$. In other words, we can replace $F^c$ by $F^o$ in the left quantile definition. 

b) $rq_X(p) = \inf \{x | F_X^o(x) > p\}$. In other words, we can replace $F^c$ by $F^o$ in the right quantile definition. 

**Proof** a) Let $A = \{x | F_X^o(x) \geq p\}$ and $B = \{x | F_X^c(x) \geq p\}$. We want to show that $\inf A = \inf B$. Now 

\[ A \subset B \Rightarrow \inf A \geq \inf B. \]

But 

\[ \inf B < \inf A \Rightarrow \exists x_0, y_0, \text{ inf } B < x_0 < y_0 < \inf A. \]

Then 

\[ \inf B < x_0 \Rightarrow \exists b \in B, \ b < x_0 \Rightarrow \exists b \in \mathbb{R}, \ p \leq P(X \leq b) \leq P(X \leq x_0) \Rightarrow P(X \leq x_0) \geq p \Rightarrow P(X < y_0) \geq p. \]
On the other hand

\[ y_0 < \inf A \Rightarrow y_0 \notin A \Rightarrow P(X < y_0) < p, \]

which is a contradiction, thus proving a).

b) Let \( A = \{ x | F_X^a(x) > p \} \) and \( B = \{ x | F_X^c(x) > p \} \). We want to show \( \inf A = \inf B \).

Again,

\[ A \subset B \Rightarrow \inf A \geq \inf B. \]

But

\[ \inf B < \inf A \Rightarrow \exists x_0, y_0, \inf B < x_0 < y_0 < \inf A. \]

Then

\[ \inf B < x_0 \Rightarrow \exists b \in B, b < x_0 \Rightarrow \exists b \in \mathbb{R}, p < P(X \leq b) \leq P(X \leq x_0) \Rightarrow P(X \leq x_0) > p \Rightarrow P(X < y_0) > p. \]

On the other hand,

\[ y_0 < \inf A \Rightarrow y_0 \notin A \Rightarrow P(X < y_0) \leq p, \]

which is a contradiction.$\blacksquare$

Using the above results, we establish the main theorem of this section which states the symmetry property of the left and right quantiles.

**Theorem 3.1**: (Quantile Symmetry Theorem) Suppose \( X \) is a random variable and \( p \in [0, 1] \). Then

\[ lq_X(p) = -rq_{-X}(1 - p). \]

**Remark.** We immediately conclude

\[ rq_X(p) = -lq_{-X}(1 - p), \]

by replacing \( X \) by \(-X\) and \( p \) by \( 1 - p \).

**Proof**
4 Equivariance of quantiles under decreasing transformations

It is widely claimed that (e.g. in [3] or [1]) the traditional quantile function is equivariant under monotonic transformations. [2] shows that this does not hold even for strictly increasing functions. However he proves that the traditional quantile function is equivariant under non-decreasing left continuous transformations. He also shows that the right quantile function is equivariant under non-decreasing right continuous transformations. In other words

Now we show how these symmetries can become useful to derive other relationships for quantiles.

Lemma 3.3: Suppose $X$ is a random variable with distribution function $F$. Then

$$lq_X(p) = \sup\{x | F^c(x) < p\}.$$  

Proof

$$lq_X(p) = -rq_{-X}(1-p) = -\inf\{x | F^c_{-X}(x) > 1-p\} =$$

$$-\inf\{x | 1 - G_{-X}(x) > 1-p\} = \sup\{-x | G_{-X}(x) < p\} =$$

$$\sup\{-x | P(-X \geq x) < p\} = \sup\{x | P(X \leq x) < p\} =$$

$$\sup\{x | F^c(x) < p\}.$$
\[ lq_{\phi(X)}(p) = \phi(lq_X(p)), \]

where \( \phi \) is non-decreasing left continuous. Also

\[ rq_{\phi(X)}(p) = \phi(rq_X(p)), \]

for \( \phi : \mathbb{R} \to \mathbb{R} \) non-decreasing right continuous.

Using the quantile symmetry, a similar neat result is found for continuous decreasing transformations using the Quantile Symmetry Theorem.

**Theorem 4.1:** (Decreasing transformation equivariance)

a) Suppose \( \phi \) is non-increasing and right continuous on \( \mathbb{R} \). Then

\[ lq_{\phi(X)}(p) = \phi(rq_X(1-p)). \]

b) Suppose \( \phi \) is non-increasing and left continuous on \( \mathbb{R} \). Then

\[ rq_{\phi(X)}(p) = \phi(lq_X(1-p)). \]

**Proof**  
a) By the Quantile Symmetry Theorem, we have

\[ lq_{\phi(X)}(p) = -rq_{-\phi(X)}(1-p). \]

But \(-\phi\) is non-decreasing right continuous, hence the above is equivalent to

\[ -(-\phi(rq_X(1-p))) = \phi(rq_X(1-p)). \]

b) By the Quantile symmetry Theorem

\[ rq_{\phi(X)}(p) = -lq_{-\phi(X)(1-p)} = -(-\phi(lq_X(1-p))) = \phi(lq_X(p)), \]

since \(-\phi\) is non-decreasing and left continuous.
References

[1] L. Hao and D. Q. Naiman. *Quantile Regression*. Quantitative Applications in the Social Sciences Series. SAGE publications, 2007.

[2] R. Hosseini. *Statistical Models for Agroclimate Risk Analysis*. PhD thesis, Department of Statistics, UBC, 2009.

[3] R. Koenker. *Quantile Regression*. Cambridge university press, 2005.

[4] E. Parzen. Nonparametric statistical data modeling. *Journal of the American Statistical Association*, 74:105–121, 1979.

[5] T. Rychlik. *Projecting statistical functionals*. Springer, 2001.