A Study of Vehicle Vibration with Multiple Degrees of Freedom Using OCTAVE and ANSYS

Alexander Wray¹, Jacob Reed¹, Dr. Shengyong Zhang¹

¹Department of Mechanical Engineering, Purdue University Northwest, 1401 US–421, Westville, IN 46391, USA

Abstract. Vibration vehicle models provide a great opportunity to integrate vehicle-based vibrations into a mechanical engineering vibrations course. This paper documents and discusses an approach to a multiple-degree-of-freedom (MDOF) in regards to pitch and bounce of the vehicle body with dampening and stiffness observed. The approach utilizes hand calculations, OCTAVE, and ANSYS to solve speed-bump induced vibration and to calculate vibrations modes. The problem involves a bicycle-model approach to a vehicle’s response when subject to a speed bump, or a sudden impulse load upon the forward-face of the model. These responses are captured and analyzed on pitch and bounce of the model at the center-of-gravity and the front/rear sides of the model. Graphical models capture responses of the loading conditions and the modal equations of the model providing a method of studying and understanding vibrational responses under provided stiffness. Using ANSYS as an approach of the model to capture damping response, further graphical models can be captured and used to describe motions of the vehicle model’s center-of-gravity, frontward and rearward wheel assembly, and the pitch (rotation) of the center-of-gravity to the introduced responses. A successful study with analysis of each graphical element and further dissemination of the results is concluded.

1. Introduction
Vibrations are prevalent in the world and attribute to the performance of machinery. In vehicles, vibrations can impact the bounce and pitch of the vehicle when subject to excitation. When approaching vehicle bodies for analysis, an MDOF approach is taken by attributing vibrations to each mass-point. The approach treats the problem in matrices/arrays and calculates it as a set of equations for each degree-of-freedom observed. In approaching vibrations regarding MDOF equations, the use of eigenvectors and eigenvalues can calculate natural frequencies and modes. As well, through the use of domain-shifting, vibration equations can be calculated based upon shifting the real domain variables to a modal domain, done through what is called modal analysis [1]. Figure 1 shows the Bicycle-Model approach, in which it is assumed that the vibrations are in symmetry along the car’s long axis (relative right side vibrations are equal to the relative left side vibrations). The model will be used in calculating the responses of the automobile when subject to an initial displacement for OCTAVE, and when subject to force excitations for FEA Analysis. To relate the model to a real-world automobile the DeLorean DMC-12 will be used. The properties are detailed below for the DeLorean DMC-12. Note that the tires are different types front-to-back of the DeLorean DMC-12 to account for its 65%/35% weight distribution [2]. The moment of inertia is also detailed for the FEA analysis further on. The values for the spring stiffness and spring damping are acquired through shock absorption catalogs for standard sedans [3] and SAE Baja forums [4]. To get the true values would require testing in real life of a DeLorean DMC-12, but for the purposes of the analysis the values are treated as theoretical and unverified. As well, when conducting FEA of the DeLorean DMC-12, the vehicle structure above the suspensions is simulated by two beam elements joined at the center-of-gravity in FEA analysis; the moment of inertia of the Beam element is based on the mass of the vehicle (curbed weight of DeLorean DMC-12 at 1,233 kilograms) times the beam area of a square, solid beam at a dimension of roughly 0.70 meters. This beam area is chosen as theoretical and assumed and provided the best representation of vibration response for the purposes of analysis.
Below, a force-body diagram with degrees of freedom are drawn below and modelled. Additionally, the properties to be captured for modelling the vibrations are below for the DeLorean DMC-12 [5]. For the load conditions, a road excitation taken from bicycle shock vibrations recorded [6] is scaled and modelled into a Finite-Element Analysis (FEA) software called ANSYS. Additionally, a speed bump excitation will be observed that is theoretical.

![Vehicle Bicycle Model with 4 Degrees-of-Freedom](image1)

**Figure 1. Vehicle Bicycle Model with 4 Degrees-of-Freedom**

![DeLorean DMC-12 Properties](image2)

**Figure 2. DeLorean DMC-12 Properties**

2. **Hand calculations for vibration modal analysis**

The analysis will begin by looking into the Free-Body Diagrams (FBD) of the unsprung and sprung masses. The model assumes for the purposes of the analysis that the car body can be modelled as a beam element, that the tire does not have damping as an element of the rubber, and that the model is treated as homogenous. In calculating the system equation of vibrations, balance of forces is observed. When observing each free-body diagram, it is the summation of all forces acting upon the body, followed by systems equations based on Newton’s Second Law. The direction of rotation and acceleration is marked for each free-body diagram. Additionally, the final equations for each FBD will be written rather than step-by-step arithmetically solved.
The following are equations as a result of the FBD Diagrams, where Equation 1 represents Figure 3, Equation 2 represents Figure 4, Equation 3 represents Figure 5, and Equation 4 represents Figure 6.

\[ m_1 \ddot{x}_1 + c_1 \dot{x}_1 + (k_{t1} + k_1)x_1 + a_1c_1\dot{\theta} + a_1k_1\theta - c_1x_{\text{car}} - k_1x_{\text{car}} = 0 \]  

\[ m_2 \ddot{x}_2 + c_2 \dot{x}_2 + (k_{t2} + k_2)x_2 - a_2c_2\dot{\theta} - a_2k_2\theta - c_2x_{\text{car}} - k_2x_{\text{car}} = 0 \]  

\[ m_{\text{car}}\ddot{x}_{\text{car}} + (c_1 + c_2)x_{\text{car}} + (k_1 + k_2)x_{\text{car}} + (a_2c_2 - a_1c_1)\dot{\theta} + (a_2k_2 - a_1k_1)\theta - c_1\dot{x}_1 - c_2\dot{x}_2 - k_1\dot{x}_1 - k_2\dot{x}_2 = 0 \]  

\[ I\ddot{\theta} + (a_1^2c_1 + a_2^2c_2)\dot{\theta} + (a_1^2k_1 + a_2^2k_2)\theta + (a_2c_2 - a_1c_1)x_{\text{car}} - (a_2k_2 - a_1k_1)x_{\text{car}} + a_1c_1\dot{x}_1 - a_2c_2\dot{x}_2 + a_1k_1\dot{x}_1 - a_2k_2\dot{x}_2 = 0 \]
With the formulated equations, matrices can be formed for analysis and use in modal analysis. The mass, damping coefficient and stiffness matrices are defined below after summation of the above equations for the following modal analysis.

\[
M = \begin{bmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & m_{eq} & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\] (5)

\[
C = \begin{bmatrix}
c_1 & 0 & -c_1 & a_1c_1 \\
0 & c_2 & -c_2 & a_2c_2 \\
-c_1 & c_2 & c_1 + c_2 & a_2c_2 - a_1c_1 \\
a_1c_1 & -a_2c_2 & a_2c_2 - a_1c_1 & a_2^2c_2 + a_1^2c_1
\end{bmatrix}
\] (6)

\[
K = \begin{bmatrix}
k_{11} + k_{c1} & 0 & -k_1 & a_2k_1 \\
0 & k_{c2} & -k_2 & a_2k_2 \\
-k_1 & -k_2 & k_{c1} + k_{c2} & a_2k_2 - a_1k_1 \\
a_1k_1 & -a_2k_2 & a_2k_2 - a_1k_1 & a_2^2k_2 + a_1^2k_1
\end{bmatrix}
\] (7)

These matrices utilize variable data from Figure 2 but are left symbolic for representing the modal analysis approach below. The variables will be initialized with their data from Figure 2 for OCTAVE.

Continuing, upon acquiring the force balance of matrices, modal analysis can begin. To simplify the modal analysis, only the mass and stiffness matrices will be observed as the damping coefficient matrix is ignored. The damping will be considered in FEA to best capture damping effects upon the Bicycle-Model approach.

Modal analysis can allow the system of equations to be ‘decoupled’, where each vibrational equation can be acquired separately. This is done through acquiring the eigenvector matrix P, which allows the vibration equations to be converted into the modal domain, and by using Cholesky decomposition, the system of equations for vibration response can be calculated. [1]

\[
\tilde{K} = M^{-1/2} \left( KM^{-1/2} \right) \quad (8)
\]

\[
P^T \tilde{K} P = \Lambda \quad (9)
\]

\[
\Lambda = \text{diag}(\lambda_i) = \text{diag}(\omega_i^2) \quad (10)
\]

Where \( \tilde{K} \) is the mass normalized stiffness matrix, \( \Lambda \) the spectral matrix of \( \tilde{K} \), and \( \omega_i \) the \( i \)th natural frequency.

Upon calculating the eigenvalues, the square root of the diagonal of eigenvalues yields the natural frequencies of the system. These natural frequencies are then used to calculate the resulting modal equations of the system and conduct modal analysis. The mode shape matrix is calculated first.

\[
S = M^{-1/2} P \quad (11)
\]

With initial conditions considered, the initial conditions for the mode shape are calculated by converting to the modal domain. Then, the modal vibration equations are calculated, for each DOF equation defined. These modal equations can then be converted to the real domain by the use of the mode shape matrix [1].

Where \( x_i(t) \) are the real domain vibration equations of the MDOF system of equations. These equations are then processed directly into OCTAVE and used to calculate the vibration responses under initial conditions.
3. MATLAB graphical analysis

The vibration responses are plotted to separate figures for each equation of motion. The figures are in the order of Front Tire, Rear Tire, Car Bounce, and Car Pitch. First, the Front Tire Vibration Motion is plotted and captured, with a time period of analysis of 1.0 second. By using OCTAVE, an open-source array and graphical mathematics program. When the problem is modeled, damping is not factored in the OCTAVE analysis. Damping will be modeled through FEA software discussed later in the paper. The graphical responses below represent the system when subject to the initial conditions of a displacement of 50 millimeters on the front tire unsprung mass.

\[
A = \sqrt{\omega_i^2 r_i(0)^2 + \dot{r}_i(0)^2} \tag{12}
\]

\[
\varphi = \frac{\omega_i r_i(0)}{\dot{r}_i(0)} \tag{13}
\]

\[
r_i(t) = A \cdot \sin \left(\omega_i t + \tan^{-1}(\varphi)\right) \tag{14}
\]

\[
x_i(t) = S \cdot r_i(t) \tag{15}
\]

**Figure 7.** Front Unsprung Mass Bounce Response, in meters.

**Figure 8.** Rear Unsprung Mass Bounce Response, in meters.

**Figure 9.** Car Mass Bounce Response, in meters.

**Figure 10.** Car Mass Pitch Response, in radians.
The front tire bounce oscillates +/- 50 mm; it oscillates at a bounce of 50 mm, as this is the introduced displacement. The rear tire, while under the effect of displacement from the front tire, is far lower in bounce, oscillating at +/- 0.02 mm. The car vibration response is less erratic, and the bounce is less pronounce than the front tire, only peaking at 0.065 mm. As before, due to no damping the response will infinite oscillate at +/- 0.0065 mm. The rotational response is that of +/- 0.0002 radians, or roughly +/- 0.011 degrees.

The plots represent the system when observed from the perspective of stiffness only. As such, there is no apparent damping upon the system, so it will infinitely oscillate. In real-world conditions, and as to be modelled with FEA to account for damping, the initial response would match close to the initial condition, but eventually dampen to an equilibrium of a reference height (in this case, to an equilibrium of zero displacement, or 0.0 meters). As well, the natural frequencies are focused upon the stiffness only. Natural frequencies change if damping is factored into the balance of forces. The OCTAVE code is published in the appendix of this report. The code can be ran in MATLAB, but care must be taken on figure generation and for/end statements, as MATLAB translates this syntax differently for processing graphical elements and equations.

4. FEA Analysis
To help factor for damping in the analysis, an FEA model will be used to approach the introduction of damping by the suspension system. The initial conditions, plotting, and structure of the FEA will be detailed in this section, and the load steps modelled for the analysis will also be observed. The FEA software used is ANSYS Mechanical Academic 19.1; the FEA in question will have the initial conditions, the model structure, and constraints listed.

In ANSYS, nodes are created to symbolize the bicycle model as described in Figure 1. Note that the center-of-gravity of the bicycle model is modelled at a node ⅔ from the left (Front) of the vehicle as to match its weight distribution. The node locations are detailed in Table 1.

| Table 1. X, Y, and Z Coordinates of Bicycle-Car Model. Locations are in meters. |
|---|---|---|---|
| Node | X  | Y  | Z  |
| 1    | 0.00 | 0.00 | 0.00 |
| 2    | 0.00 | 1.00 | 0.00 |
| 3    | 0.00 | 2.00 | 0.00 |
| 4    | 2.43 | 2.00 | 0.00 |
| 5    | 2.43 | 1.00 | 0.00 |
| 6    | 2.43 | 0.00 | 0.00 |
| 20   | 1.60 | 2.00 | 0.00 |

As well, the Bicycle Model shape in ANSYS is below with the nodes shown. It has LINE elements to connect between nodes. LINE elements between nodes 3-20 and 20-4 are modelled as a BEAM element with a strength to treat it as Rigid. LINE elements between nodes 1-2, 2-3, 4-5, and 5-6 are modelled as COMBINATION elements containing the coefficients to model spring stiffness and damping ($K$ and $C$ values). Lines 1-2 and 5-6 represent the unsprung tire stiffness values ($k_{tire}$ values, $c = 0$), and Lines 2-3 and 4-5 represent the suspension which contains the spring and shock values ($k$ and $c$ values). The nodes 2 and 5 represent the mass of the unsprung tires, and node 20 represents the mass and moment of inertia of the vehicle body. The values for the COMBINATION (Spring/Damper) elements and the masses and moment of inertia are acquired from Figure 2.
The first analysis will be conducted by modelling a ‘push-down’ impulse force upon the front of the vehicle. The motion is matching that of a weight pushed onto the front of the vehicle and immediately removed. The Dirac-Impulse occurs in a time frame of 0.01 seconds, and the force is introduced in the -UY direction. The load is applied as a Stepped load of -500N, occurs between 0 to 0.01 seconds, then force is removed and goes to zero from 0.01 to 5.00 seconds. This force is applied onto the front of the vehicle body at rear end.

**Figure 11.** ANSYS FEA Model; this is the model made in ANSYS, taking Table 1’s node locations and drawing it in a 3D workspace.

The analysis will have constraints in such a way that the only perceived or recorded motion is in the UY direction. The tracked motions are the displacement of the center-of-gravity, and the pitch relative to the center of the gravity. When the analysis is ran to completion, the following are the resulting Bounce and Pitch values of the Vehicle’s Center-of-gravity. Note that displacement is modelled in meters, and the Pitch is modelled in radians. As well, the capture of Displacement and Pitch ends at 2 seconds due to equilibrium being closely achieved on displacement at that time.
Figure 13. Bounce Graph of Dirac-Impulse Analysis. As seen, the largest displacement is, in millimeters, -0.23 mm. The vehicle then rebounds in an overshoot to roughly 0.02 mm, before going to equilibrium.

Figure 14. Pitch Graph of Dirac-Impulse Analysis. Here, the vehicle oscillates and reaches peak at 0.001 radians, before slowly damping over time. If given enough time the response should reach equilibrium.

The impulse force demonstrates how damping the shock system is to reduce the vibrations upon the vehicle body. However, because no damping exists upon the tires, the oscillatory behavior that the pitch undergoes is more pronounced. As well, the displacement of the vehicle’s center-of-gravity is small, which indicates a highly stiff response to the impulse loading. The next analysis covers another set of Impulses but mimics a theoretical speed bump for a vehicle and applies the force directly to the tire nodes. A force of 1000 Newtons is used upon both tires at separate intervals of time. The first impulse applies to the front tire at 0.1 seconds, is applied for 0.03 seconds, then goes to 0N. Then, another impulse is applied separately to the rear tire at 1000N, applied at 0.3
seconds for 0.03 seconds, then goes to zero. The first impulse applies to node 2, the second impulse applies to node 5.

As described in the first analysis, the same constraints are applied to the model for this analysis. The center-of-gravity node is observed for its displacement and pitch, recorded in meters and radians respectively.

**Figure 15.** Front Tire Impulse Load, where the impulse of 1000 Newtons is applied to the Front Tire at time of 0.1 second and is applied for 0.03 seconds before returning to 0 Newtons.

**Figure 16.** Rear Tire Impulse Load, where the impulse of 1000 Newtons is applied to the Rear Tire at time of 0.3 seconds and is applied for 0.03 seconds before returning to 0 Newtons.

After running the analysis, Figures 17 and 18 are the vibration responses to the impulses. The window capture is 2.0 seconds.

In observing the response, it is seen how oscillatory the responses are: due to the lack of damping with the tire, the vehicle body rapidly oscillates as seen in the pitch and displacement. As well, when introduced to rapid impulses much like a theoretical speed bump, the displacement is more pronounced. What can also be drawn from the response is the near identical overshoot periods when the vehicle recedes to equilibrium, even under different impulse loadings.
Figure 17. Bounce of Speed-bump Analysis: the vehicle exhibits an initial displacement (in millimeter) of 0.26mm, starts to rebound after the impulse, then rises in response to the 2nd impulse to a peak of 0.5 mm. After the impulse, the vehicle recedes to equilibrium with overshoot.

Figure 18. Pitch of Speed-bump Analysis: here, similar responses of impulse loading is factored into the pitch. The vehicle, upon receiving the first impulse, pitches to a peak -0.0025 radians (0.143 degrees), before receding and oscillating. Then, upon receiving the 2nd impulse, the pitch rises to a peak 0.0016 radian (0.091 degrees), before receding and oscillating to equilibrium.

5. Conclusion & thoughts
The FEA proved very useful in modelling vibration models of a vehicle with the symmetry model, and if necessary, can also be used to model the entirety of a vehicle’s suspension, tire, and shocks/struts design. A good case-study to continue off this project is increasing the number of masses and couples that would mimic a real vehicle’s conditions, such as shock struts, roll-bars or additional axle/mechanisms, or modelling different embankment or turning scenarios, or even modelling impure road conditions like gravel, or differing pavements. The Bicycle-Model approach helps to visualize and introduce vibrational models of vehicle bodies. It additionally helps to demonstrate the impact of altering stiffness and damping coefficients on a vehicular body and is a good start into teaching students how springs, stiffness and damping components affect vibration. The OCTAVE analysis helps to prove the theory and give ideal scenarios of analysis, and the FEA Analysis model allows for more complex scenarios that cannot be easily calculated. Key areas to be addressed in this project would be correcting for, or even researching into proper stiffness and damping values of the suspension system of the vehicle,
as well as research into damping coefficients to best represent the tire in this model. Overall, the project captured the vibration model of a Bicycle-Model approach to a vehicle and was a success in doing so.

6. References

[1] Inman, D J. *Engineering Vibration 4th Edition*. Englewood Cliffs, N.J: Prentice Hall, 2016. Print.

[2] Bayrakdar, Özgür. *Random Vibration of a Road Vehicle*. Master Thesis. İzmir Institute of Technology, Turkey. 2010.

[3] Clark, Samuel K. *Mechanics of Pneumatic Tires*. National Bureau of Standards and US Department of Commerce; Washington D.C., 1971. Print.

[4] Dixon, John C. *Shock Absorber Handbook 2nd Edition*. Society of Automotive Engineering (SAE) International. West Sussex, England. 2007. Print.

[5] https://en.wikipedia.org/wiki/DMC_DeLorean#Suspension. Wikimedia Organization. *Wikipedia*. Accessed Jan 2021.

[6] Gogola M. Analysing the Vibration of Bicycles on Various Road Surfaces in the City of Zilina. The Archives of Automotive Engineering – Archiwum Motoryzacji. 2020;88(2):77-97. doi:10.14669/AM.VOL88.ART6.

APPENDIX 1: OCTAVE CODE

```octave
## OCTAVE Code for Vibration Analysis on a Bicycle-Model Approach
## with given attributes and variables.

%M % clear all; close all; clc; warning off;
m1 = 7.96; m2 = 11.34; mcar = 1231.4; I = 606;
k1 = 38500; k2 = 78750;
kt1 = 343000; kt2 = 392000;
a1 = 0.33 * 2.43; a2 = 0.66 * 2.43;

%M % Initialization of Matrices for the Code %
M = [m1 0 0 0; 0 m2 0 0; 0 0 mcar 0; 0 0 0 I];
K = [(k1 + kt1) 0 -k1 a1*k1; 0 (k2 + kt2) -k2 -a2*k2; ...
   -k1 -k2 (k1 + k2) (a2*k2 - a1*k1); ...
   a1*k1 -a2*k2 (a2*k2 - a1*k1) ((a1^2)*k1 + (a2^2)*k2)];

%M % Modification of M and K Matrices to Mmod and Kweb Matrix %
Mmod = M^(-1/2); Kweb = Mmod * K * Mmod;

%M % Eigenvector and Eigenvalue Matrices of the Kweb Matrix. Note that V = P from Modal Analysis arrays. %
[V,D] = eig(Kweb);
w = real(sqrt(D));

%M % Modal Q Domain to R Domain %
Ii = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1];
S = Mmod * V; Smod = inv(S);

%M % Initial Conditions for Analysis %
initx = [1;0;0;0]; initv = [0;0;0;0];
rxinit = Smod * initx;
rvinit = Smod * initv;

%M % Acquiring the A Matrix Coefficients to be used with the R Domain %
for(j=1:4)
  A(j)= sqrt((w(j,j)^2) * (rxinit(j)^2) + (rvinit(j)^2))/w(j,j);
endfor
A = transpose(A);
%M % Conversion of R Domain to X (Real) Domain %
```


\[ X = S \ast A; \]

%% Graphical Preparations (may not work with MATLAB). Note that the equations are phased to cosine, as the phase in sine equations would be equal to 90 degrees due to no initial velocities. %%

\[ t = \text{linspace}(0,1,300); \]
\[ \text{Eq1} = X(1) \ast \cos(w(1,1) \ast t); \]
\[ \text{Eq2} = X(2) \ast \cos(w(2,2) \ast t); \]
\[ \text{Eq3} = X(3) \ast \cos(w(3,3) \ast t); \]
\[ \text{Eq4} = X(4) \ast \cos(w(4,4) \ast t); \]

%% Plot Resulting Real Equations %%

\text{figure}(1)
\text{plot}(t, \text{Eq1}, 'b');
\text{xlabel}('Time (sec) ');
\text{ylabel}('Displacement (m)');
\text{title}('Front Tire Vibration Motion');
\text{grid on}

\text{figure}(2)
\text{plot}(t, \text{Eq2}, 'b')
\text{xlabel}('Time (sec) ');
\text{ylabel}('Displacement (m)');
\text{title}('Rear Tire Vibration Motion');
\text{grid on}

\text{figure}(3)
\text{plot}(t, \text{Eq3}, 'b')
\text{xlabel}('Time (sec) ');
\text{ylabel}('Displacement (m)');
\text{title}('Car Vibration Motion');
\text{grid on}

\text{figure}(4)
\text{plot}(t, \text{Eq4}, 'b')
\text{xlabel}('Time (sec) ');
\text{ylabel}('Pitch (rad)');
\text{title}('Car Rotation Motion');
\text{grid on}