MOTIVIC RANDOM VARIABLES AND REPRESENTATION STABILITY II: HYPERSURFACE SECTIONS

SEAN HOWE

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Abstract. We prove geometric and cohomological stabilization results for the universal smooth degree \( d \) hypersurface section of a fixed smooth projective variety as \( d \) goes to infinity. We show that relative configuration spaces of the universal smooth hypersurface section stabilize in the completed Grothendieck ring of varieties, and deduce from this the stabilization of the Hodge Euler characteristic of natural families of local systems constructed from the vanishing cohomology. We conjecture explicit formulas using a probabilistic interpretation, and prove them in some cases. These conjectures have natural analogs in point counting over finite fields, which we prove. We explain how these results provide new geometric examples of a weak version of representation stability for symmetric, symplectic, and orthogonal groups. This interpretation of representation stability was studied in the prequel [14] for configuration spaces.
1. Introduction

1.1. Cohomological stabilization. Let $Y$ be a polarized smooth projective variety — i.e. a smooth projective variety equipped with a very ample line bundle $L$ giving a closed embedding $Y \to \mathbb{P}(\Gamma(Y, L)^*)$. Let $U_d$ be the space of smooth hypersurface sections of degree $d$ of $Y$ (with respect to this embedding). The non-constant part of the cohomology of the universal smooth hypersurface section $Z_d/U_d$ gives rise to a local system, the vanishing cohomology $\mathcal{V}_{\text{van}, Q}$ on $U_d(\mathbb{C})$. The corresponding monodromy representation is to a symmetric, symplectic, or orthogonal group, depending on whether $\dim Y$ is zero, even and positive, or odd — we will refer to this group as the algebraic monodromy group.

If $\pi$ is a representation of the algebraic monodromy group, then by composing the monodromy representation with $\pi$ we obtain a new local system $\mathcal{V}_{\pi, \text{van}, Q}$ on $U_d(\mathbb{C})$. In any of these situations, a partition $\sigma$ gives rise to a family of irreducible representations $\pi_{\sigma, d}$ of the monodromy groups for $U_d$ (by the theory of Young tableaux for symmetric groups, as in the theory of representation stability [8], and by highest weight theory for symplectic and orthogonal groups – cf. Section 3).

Each of the local systems $\mathcal{V}_{\pi_{\sigma, d}}$ is equipped with a natural variation of Hodge structure, and thus its cohomology is equipped with a mixed Hodge structure. In particular, denoting the weight filtration by $W$, 

$$\text{Gr}_W H^i_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, Q})$$

is a direct sum of polarizable Hodge structures.

We define $K_0(\text{HS})$ to be the Grothendieck ring of polarizable Hodge structures, which is the quotient of the free $\mathbb{Z}$–module with basis given by isomorphism classes $[V]$ of polarizable $\mathbb{Q}$–Hodge structures $V$ by the relations $[V_1 \oplus V_2] = [V_1] + [V_2]$. It is a ring with $[V_1] : [V_2] = [V_1 \otimes V_2]$.

We denote by $\mathbb{Q}(-1) = H^2(\mathbb{A}^1)$, the Tate Hodge structure of weight 2, and $\mathbb{Q}(n) = \mathbb{Q}(-1)^{\otimes -n}$.

By the above considerations, we obtain for each $d$ a compactly supported Euler characteristic

$$\chi_{\text{HS}}(H^*_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, Q}^{\pi_{\sigma, d}})) := \sum_i (-1)^i [\text{Gr}_W H^i_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, Q}^{\pi_{\sigma, d}})] \in K_0(\text{HS})$$

Our first main theorem states that this class stabilizes as $d \to \infty$ in the completion $K_0(\text{HS})$ of $K_0(\text{HS})$ for the weight filtration.

**Theorem A.** If $Y/\mathbb{C}$ is a polarized smooth projective variety of dimension $n \geq 1$, then for any partition $\sigma$,

$$\lim_{d \to \infty} \frac{\chi_{\text{HS}}(H^*_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, Q}^{\pi_{\sigma, d}}))}{[\mathbb{Q}(- \dim U_d)]}$$

exists in $K_0(\text{HS})$.

In Subsection 1.3 we conjecture an explicit universal formula for the value of this limit, which we can verify in some cases.

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1. To simplify some arguments we use the theory of Arapura [2] rather than that of Saito [21,22] to produce this mixed Hodge structure – cf. Subsection 2.2.
Example 1.1.1. If $Y = \mathbb{P}^n$, then $\mathcal{V}_{\text{van}, \mathbb{Q}}$ is the local system of primitive cohomology coming from the universal smooth hypersurface. In this case, we deduce (cf. Example A.0.5 in Appendix A) that

$$\lim_{d \to \infty} \frac{\chi_{\text{HS}}(H^i_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, \mathbb{Q}}))}{[\mathbb{Q}(- \dim U_d)]} = 0.$$ 

We also obtain the point counting analog. We view this as strong evidence that the individual cohomology groups $H^i_c(U_d(\mathbb{C}), \mathcal{V}_{\text{van}, \mathbb{Q}})$ are themselves stably trivial. We note that in this case $H^1$ is known to be stably trivial by Nori’s connectivity theorem [19, Corollary 4.4].

Remark 1.1.2.
1. Theorem A is new except for the trivial local system (i.e. the cohomology of $U_d(\mathbb{C})$ itself), corresponding to $\sigma = \emptyset$, where it is due to Vakil and Wood [24]. The methods of [24] will play an important role in our proof.
2. Because the $U_d$ are smooth but not proper, the Hodge Euler characteristic can potentially contain less information than the cohomology groups (with mixed Hodge structures) themselves. Nevertheless, in practice it seems a large amount of information is retained: for example, for $Y = \mathbb{P}^n$ and the trivial local system, Tommasi [23] has shown the individual cohomology groups also stabilize and from her computation one finds that there is no cancellation between degrees. We refer the reader to [24, 1.2] for more on this Occam’s razor principle for Hodge structures.

The local systems $\mathcal{V}_{\text{van}, \mathbb{Q}}^{\pi, d}$ also have $l$-adic incarnations $\mathcal{V}_{\text{van}, \mathbb{Q}}^{\pi, d, l}$ in étale cohomology, and we study these over finite fields. We obtain the point-counting analog

**Theorem B.** If $Y/\mathbb{F}_q$ is a polarized smooth projective variety of dimension $n \geq 1$ and $\sigma$ is a partition, then

$$\lim_{d \to \infty} q^{-\dim U_d} \sum \chi TrFrob_q \subset H^i_c(U_d(\mathbb{F}_q), \mathcal{V}^{\pi, d, \mathbb{Q}}_{\text{van}, \mathbb{Q}})$$

exists in $\mathbb{Q}$ (here the limit is of elements of $\mathbb{Q}$ in the archimedean topology). Furthermore, for a fixed $\sigma$ and dimension $n$, the limit is given by an explicit, computable universal formula. The universal formula is a rational function of $q$ and symmetric functions of the eigenvalues of Frobenius acting on the cohomology of $Y$.

Remark 1.1.3. For the trivial local system, Theorem B is due to Poonen [20]. Furthermore, one of the main technical inputs in our proof of Theorem B is Poonen’s sieving Bertini theorem with Taylor conditions [20, Theorem 1.2] (cf. Remark 1.3.1 below).

Remark 1.1.4. We also conjecture a universal formula for the limits in the Hodge case (Theorem A) matching the universal formula in Theorem B, however at the present we are unable to prove it except for tensor powers of the standard representation (which are not typically irreducible, but which we can make sense of as a family in a similar way).

Both Theorem A and Theorem B are deduced from corresponding geometric stabilization results (Theorems C and D below). The universal formulas are more
natural in the geometric setting and we state them below in Theorem D and Conjecture E. In Appendix A, we explain an algorithm to obtain the cohomological universal formulas of Theorem B (and the corresponding conjectural universal formulas for the limits in Theorem A) for a given $\sigma$ and $n$.

1.2. Motivic stabilization. Returning to the case of $Y/\mathbb{C}$, our goal now is to systematically understand the stabilization of natural varieties over $U_d$ produced from the universal family $Z_d$ (e.g., relative configuration spaces).

To study this, we will use the notion of a relative Grothendieck ring of varieties. For $S/\mathbb{C}$ a variety, we denote by $K_0(\text{Var}/S)$ the ring spanned by isomorphism classes $[X/S]$ of varieties $X/S$ (i.e. maps of varieties $f : X \to S$) modulo the relations

$$[X/S] = [X \setminus Z/S] + [Z/S]$$

for any closed subvariety $Z \subset X$. It is a ring with

$$[X_1/S] \cdot [X_2/S] = [X_1 \times_S X_2/S].$$

We will write $K_0(\text{Var})$ for $K_0(\text{Var}/\mathbb{C})$.

For any $S/\mathbb{C}$ there is a natural pre-$\lambda$ structure on the Grothendieck ring $K_0(\text{Var}/S)$ which gives an efficient way to work with constructions such as relative symmetric powers and relative configuration spaces (cf. [14, Subsection 3.1]). This pre-$\lambda$ structure can be interpreted as a set-theoretic pairing

$$(, ) : \Lambda \times K_0(\text{Var}/S) \to K_0(\text{Var}/S)$$

where

$$\Lambda = \mathbb{Z}[h_1, h_2, h_3, ...]$$

is the ring of symmetric functions [17] (the $h_i$ are the complete symmetric functions). If we fix the second variable, the pairing gives a ring homomorphism

$$\Lambda \to K_0(\text{Var}/S)$$

characterized by

$$(h_k, [T/S]) = [\text{Sym}^k_S T/S]$$

where the subscript on the symmetric power denotes that it is taken relative to $S$.

Our geometric stabilization will take place in

$$\widehat{\mathcal{M}_L} := K_0(\text{Var})[L^{-1}]$$

where $L = [\mathbb{A}^1]$ and the completion is with respect to the dimension filtration (cf. [24]). For $x \in K_0(\text{Var}/S)$, we will denote by $x_{\widehat{\mathcal{M}_L}}$ the element of $\widehat{\mathcal{M}_L}$ obtained by forgetting the structure morphism to $S$ (i.e. by applying the map $[T/S] \mapsto [T]$).

Our main motivic stability theorem is then

**Theorem C.** If $Y/\mathbb{C}$ is a polarized smooth projective variety of dimension $n \geq 1$, then, for any symmetric function $f \in \Lambda$,

$$\lim_{d \to \infty} \frac{\langle f, [Z_d/\mathcal{U}_d] \rangle_{\widehat{\mathcal{M}_L}}}{L^{\dim \mathcal{U}_d}}$$

exists in $\widehat{\mathcal{M}_L}$. 

Remark 1.2.1. For $f = 1$, Theorem C is due to Vakil-Wood [24], who compute
\[
\lim_{d \to \infty} \frac{[U_d]}{L^{\dim U_d}} = \zeta_Y(n+1)^{-1}.
\]
Here $\zeta_Y(n+1)$ is obtained by substituting $t = L^{-(n+1)}$ in the Kapranov zeta function
\[
Z_Y(t) = 1 + [Y] \cdot t + [\text{Sym}^2 Y] \cdot t^2 + \ldots \in 1 + tK_0(\text{Var})[[t]].
\]
Example 1.2.2 (Relative generalized configuration spaces stabilize). For any generalized partition $\tau = \lambda_1 \cdot \ldots \cdot \lambda_m$ we denote by $\text{Conf}_\tau S \otimes T$ the relative generalized configuration space which parameterizes collections of $\sum l_i$ distinct points lying in a fiber of $T \to S$, of which $l_i$ are labeled by $\lambda_i$ for each $1 \leq i \leq m$.

As described in [14, Subsection 3.1], there is a $c_\tau \in \Lambda$ such that for all $T \otimes S$,
\[
(c_\tau, [T/S]) = [\text{Conf}_\tau S \otimes T/S].
\]
If we apply Theorem C with the element $c_\tau$, we find
\[
\lim_{d \to \infty} \frac{[\text{Conf}_\tau U_d Z_d]}{L^{\dim U_d}}
\]
exists in $\overline{M}_L$. (In fact, we will prove Theorem C by showing this statement and using that the $c_\tau$ form a basis for $\Lambda$).

As we will see in Subsection 1.3 below, in some cases we can describe the limit in Theorem C explicitly.

Example 1.2.3. For $Y = \mathbb{P}^n$ and $f = h_1$, so that we are considering $Z_d$ the universal family of smooth hypersurfaces in $\mathbb{P}^n$, we obtain (using Remark 1.2.1 and Example 1.3.7 below)
\[
\lim_{d \to \infty} \frac{[Z_d]}{L^{\dim U_d}} = [\mathbb{P}^{n-1}] \cdot \zeta_{\mathbb{P}^n}(n+1)
\]
where $\zeta_{\mathbb{P}^n}(n+1)$ is the Kapranov zeta function of $\mathbb{P}^n$ evaluated at $L^{-(n+1)}$.

Remark 1.2.4. The cohomological Theorem A is deduced from the geometric Theorem C by taking $f$ to be a (symmetric, symplectic, or orthogonal) Schur polynomial $s_\sigma$ then carefully modifying the result to remove the contribution of the constant part of the cohomology of $Z_d$ (which can be described explicitly in terms of the cohomology of $Y$). The key tool that makes this extraction possible is the fact that the Adams operations on a pre-$\lambda$ ring, given by pairing with power sum symmetric functions, are additive.

1.3. Probabilistic interpretation. In Theorem D below we give the point-counting analog of Theorem C, but with explicit formulas for the limit given using probabilistic language. In Conjecture E below we make analogous conjectures for the values of the limits in Theorem C, which we can prove in some cases.

We now explain our geometric stabilization result over finite fields using probabilistic language to give explicit formulas. For $Y/\mathbb{F}_q$, a polarized smooth projective variety, we consider the subring $K_0(\text{Loc}_{\mathbb{F}_q} U_d)'$ of the Grothendieck ring of lisse $l$-adic sheaves consisting of virtual sheaves with integral characteristic series of Frobenius at every closed point. It is a pre-$\lambda$ ring, and it admits an algebraic probability measure $\mu_d$ with values in $\mathbb{Q}$ where the expectation $\mathbb{E}_{\mu_d}$ is given by averaging the
traces of Frobenius over $\mathbb{F}_q$-rational points (or, equivalently by the Grothendieck-Lefschetz formula, by computing the alternating sum of traces on the compactly supported cohomology then dividing by $\#U_d(\mathbb{F}_q)$).

In $K_0(\text{Loc}_{\mathbb{Q}}, U_d)'$, there is a class $[Z_d/U_d]$ given by the alternating sum of the cohomology local systems of $Z_d$. We can view $K_0(\text{Loc}_{\mathbb{Q}}, U_d)'$ as a ring of motivic random variables lifting the ring of classical random variables on the discrete probability space $U_d(\mathbb{F}_q)$ with uniform distribution, and from this perspective the random variable $[Z_d/U_d]$ lifts the classical random variable assigning to a smooth hypersurface section $u \in U_d(\mathbb{F}_q)$ the number of $\mathbb{F}_q$ points on $Z_{d,u}$.

Let

$$p_k' := \frac{1}{k} \sum_{d|k} \mu(d/k)p_k \in \Lambda$$

be the Mobius inverted power sum polynomials. We can view $(p_k'[Z_d/U_d])$ as a motivic lift of the classical random variable assigning to a smooth hypersurface section $u \in U_d(\mathbb{F}_q)$ the number of degree $k$ closed points on $Z_{d,u}$.

**Theorem D.** Let $Y/\mathbb{F}_q$ be a polarized smooth projective variety of dimension $n \geq 1$. Then, for a formal variable $t$,

$$\lim_{d \to \infty} \mathbb{E}_{\mu,d} \left[ (1 + t)^{p_k'[Z_d/U_d]} \right] = \left( 1 + \frac{q^{nk} - 1}{q^{(n+1)k} - 1} \right)^{\# \text{ closed points of degree } k \text{ on } Y}$$

and the $(p_k'[Z_d/U_d])$ are asymptotically independent for distinct $k$, i.e. for formal variables $t_1, t_2, \ldots, t_m$,

$$\lim_{d \to \infty} \mathbb{E}_{\mu,d} \left[ \prod_k (1 + t_k)^{p_k'[Z_d/U_d]} \right] = \prod_k \lim_{d \to \infty} \mathbb{E}_{\mu,d} \left[ (1 + t_k)^{p_k'[Z_d/U_d]} \right].$$

Here exponentiation is interpreted in terms of the standard power series

$$(1 + t)^a = \exp(\log(1 + t) \cdot a) = \sum_i \binom{a}{i} t^i,$$

and $\mathbb{E}_{\mu,d}$ and limits are applied individually to each coefficient of a power series.

In particular, for any symmetric function $f \in \Lambda$,

$$\lim_{d \to \infty} \mathbb{E}_{\mu,d}[f[Z_d/U_d]]$$

exists in $\mathbb{Q}$, and can be computed explicitly by expressing $f$ as a polynomial in the $p_k'$ and applying asymptotic independence plus the explicit distributions above.

**Remark 1.3.1.** The probabilities in Theorem D have a simple geometric origin: Each closed point $y$ of degree $k$ in $Y$ defines an indicator Bernoulli random variable on $U_d(\mathbb{F}_q)$ that is 1 at $u \in U_d(\mathbb{F}_q)$ if $y$ is contained in the fiber $Z_{d,u}$ and 0 otherwise. These random variables are asymptotically independent each with asymptotic probability of being 1 determined by the proportion of the number of smooth first order Taylor expansions vanishing at $t$, $q^{nk} - 1$, to the total number of smooth first order Taylor expansions at $t$, $q^{(n+1)k} - 1$. The asymptotic independence of these Bernoulli random variables is a simple consequence of a result of Poonen [20, Theorem 1.2], and using this it is straightforward to prove Theorem D.
Example 1.3.2 (The average smooth hypersurface in \( \mathbb{P}^n \), point counting version). Taking \( Y = \mathbb{P}^n \) and \( f = p_1 \) in Theorem D, we find
\[
\lim_{d \to \infty} \frac{\sum_{u \in U_d(F_q)} \#Z_{d,u}(F_q)}{\#U_d(F_q)} = \lim_{d \to \infty} \mathbb{E}_{\mu_d}[[Z_d]] = \#\mathbb{P}^n(F_q) \cdot \frac{q^n - 1}{q^{n+1} - 1} = \#\mathbb{P}^{n-1}(F_q)
\]
Thus, in this sense the average smooth hypersurface in \( \mathbb{P}^n \) is \( \mathbb{P}^{n-1} \).

Remark 1.3.3. Explicit formulas in Theorem B can be deduced from Theorem D as in Remark 1.2.4. We carefully describe an algorithm for this in Appendix A.

We now return to \( Y \subset \mathbb{P}^n_\mathbb{C} \). Because \( U_d \) is an open in affine space, \( [U_d] \) is invertible in \( \overline{M}_\mathbb{C} \). Thus we obtain an algebraic probability measure (in the sense of [14, 4]) on \( K_0(\text{Var}/U_d) \) with values in \( \overline{M}_\mathbb{C} \), characterized by the expectation function
\[
\mathbb{E}_{\mu_d}[x] = \frac{x_{\overline{M}_\mathbb{C}}}{[U_d]}
\]
We make the following conjecture describing the stabilization in Theorem C after tensoring with \( \mathbb{Q} \):

Conjecture E. Let \( Y \subset \mathbb{P}^n_\mathbb{C} \) be a smooth projective variety of dimension \( n \geq 1 \). For \( t \) a formal variable,
\[
\lim_{d \to \infty} \mathbb{E}_{\mu_d} \left[ (1 + t)^{(p'_k, [Z_d/U_d])} \right] = \left( 1 + \frac{\prod_{k} (n k + 1) - 1}{\prod_{k} (n + k - 1) - 1} \right)^{(p'_k, [Y])}
\]
and the \( (p'_k, [Z_d/U_d]) \) are asymptotically independent for distinct \( k \), i.e. for formal variables \( t_1, t_2, \ldots, t_m \),
\[
\lim_{d \to \infty} \mathbb{E}_{\mu_d} \left[ \prod_{k} (1 + t_k)^{(p'_k, [Z_d/U_d])} \right] = \prod_{k} \lim_{d \to \infty} \mathbb{E}_{\mu_d} \left[ (1 + t_k)^{(p'_k, [Z_d/U_d])} \right],
\]
Here exponentiation is interpreted in terms of the standard power series
\[
(1 + t)^a = \exp(\log(1 + t) \cdot a) = \sum_i \binom{a}{i} t^i,
\]
and \( \mathbb{E}_{\mu_d} \) and limits are applied individually to each coefficient of a power series.

In particular, for any symmetric function \( f \in \Lambda \),
\[
\lim_{d \to \infty} \mathbb{E}_{\mu_d}[(f, [Z_d/U_d])]
\]
can be computed explicitly in \( \overline{M}_\mathbb{C} \otimes \mathbb{Q} \) by expressing \( f \) as a polynomial in the \( p'_k \) and applying asymptotic independence plus the explicit distributions above.

Remark 1.3.4. As in [14], the significance of the terms \( (p'_k, \bullet) \) appearing in Conjecture E are that they give the exponents in the naive Euler product for the Kapranov zeta function. It is in this sense that they are natural motivic avatars for the number of closed points of a fixed degree on a variety over a finite field.

In Theorem 6.3.3 we prove the asymptotic binomial distribution of Conjecture E for \( p'_1 = p_1 \). This leads to

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We thank Jesse Wolfson for pointing out this simple fact to us.
Example 1.3.5. We denote by $\text{PConf}^k_{U_d} \mathbb{Z}_d$ the relative configuration space of $k$ distinct ordered points in $\mathbb{Z}_d$. In the notation of Example 1.2.2,

$$\text{PConf}^k_{U_d} = \text{Conf}_{U_d}^{a_1, \ldots, a_k} \mathbb{Z}_d.$$

We have the identity

$$[\text{PConf}^k_{U_d} \mathbb{Z}_d/U_d] = ([\mathbb{Z}_d/U_d]) \cdot ([\mathbb{Z}_d/U_d] - 1) \cdot \ldots ([\mathbb{Z}_d/U_d] - k + 1)$$

$$= ((p_1)(p_1 - 1)(p_1 - k + 1), [\mathbb{Z}_d/U_d]).$$

Thus, from the part of Conjecture E proven in Theorem 6.3.3, we obtain

$$\lim_{d \to \infty} \frac{[\text{PConf}^k_{U_d} \mathbb{Z}_d]}{[U_d]} = \text{PConf}^k Y \cdot \left(\frac{L^n - 1}{L^{n+1} - 1}\right)^k.$$

Using Theorem D, we can also obtain the corresponding point-counting result.

Remark 1.3.6. Theorem 6.3.3 also leads to an explicit computation of the limit in Theorem A for tensor powers of the standard representations, as alluded to in Remark 1.1.4.

Example 1.3.7 (The average smooth hypersurface in $\mathbb{P}^n$, motivic version). Taking $Y = \mathbb{P}^n$ and $k = 1$ in Example 1.3.5, we obtain

$$\lim_{d \to \infty} \frac{[\mathbb{Z}_d]}{[U_d]} = \left[\mathbb{P}^n\right] \cdot \left(\frac{L^n - 1}{L^{n+1} - 1}\right)$$

$$= \left[\mathbb{P}^{n-1}\right].$$

Thus, in this sense the average smooth hypersurface in $\mathbb{P}^n$ is $\mathbb{P}^{n-1}$ (cf. Example 1.3.2 for the point counting version).

Remark 1.3.8. Imitating the strategy used in [14] for configuration spaces, in Proposition 6.3.1 we reduce Conjecture E to a combinatorial identity using the power structure of the Grothendieck ring of varieties.

1.4. Connections with representation stability and the prequel. As noted in the introduction to the prequel [14], one consequence of representation stability for configuration spaces (as studied in [7,8]) is cohomological stabilization for certain families of local systems factoring through the monodromy representation. Our results give an analog of this for the spaces $U_d$, at least at the level of the Hodge Euler characteristic: Generically the image of the monodromy representation is a lattice in the algebraic monodromy group, and thus, by superrigidity, when we consider algebraic representations we are really considering all natural local systems which factor through the monodromy representation. We view Theorems A and B together as strong evidence that the individual cohomology groups of the local systems $\mathcal{Y}_{\mathbb{P}^n, \mathbb{Q}}$ themselves stabilize, and that the unstable cohomology has sub-exponential growth (this kind of relation between point-counting and cohomological stability is studied in [13]).

As with the analogous results of the prequel [14] for configurations spaces, the form of Theorem D and Conjecture E suggests that the stabilization is of modules over the (conjectural) stable cohomology of $U_d$, and that it should be possible to find explicit generators. Hypersurface sections are more difficult to analyze using representation stability because there are no obvious stabilizing maps, however, it is natural to hope that with a more sophisticated approach one might find a richer structure explaining and strengthening our results.
1.5. **Related work.** Besides the works of Poonen [20] and Vakil and Wood [24] discussed above, we highlight some other related threads in the literature:

1.5.1. **Plane curves.** Bucur, David, Feigon, and Lalin [3] use Poonen’s sieve with refined control over the error term to show that the distribution of the number of degree 1 points on a random smooth plane curve over $\mathbb{F}_q$ is asymptotically Gaussian (letting both $q$ and $d$ go to infinity in a controlled way). They use the same probabilistic interpretation of Poonen’s results that we adopt here, and in particular Theorem 1.1 of [3] is a refined version of the statement that the number of degree 1 points on a smooth curve of degree $d$ over $\mathbb{F}_q$ is asymptotically distributed as a sum of $\#\mathbb{P}^2(\mathbb{F}_q)$ independent Bernoulli random variables as $d \to \infty$.

1.5.2. **Moduli of curves.** Achter, Erman, Kedlaya, Wood, and Zureick-Brown [1] use results on the stable cohomology of moduli of genus $g$ curves with $n$ marked points $M_{g,n}$ to make conjectures about the asymptotic distribution of degree 1 points on curves over $\mathbb{F}_q$ as the genus $g \to \infty$. They give a heuristic explanation of why the unstable cohomology should not contribute and deduce an asymptotic distribution from the stable cohomology under this assumption.

The stable cohomology of moduli of curves is well understood with explicit generators, even with coefficients [16]. It would be interesting to use the stable cohomology to compute the asymptotic distributions, and hopefully asymptotic independence, of the natural motivic random variables coming from the universal curve giving an analog of Conjecture E in this setting.

We highlight here that the state of knowledge for moduli of curves is the opposite of that for smooth hypersurface sections: For moduli of curves one knows all of the desired results on cohomological stability, but because there is no ambient space containing all curves, sieve methods are not available to prove the corresponding stable point counts. Conversely, for smooth hypersurfaces sections we have shown that sieve methods can be used to produce all stable point counts, and the methods of [24] to give stabilization in the Grothendieck ring of Hodge structures, but the only stability result known for the individual cohomology groups is Tommasi’s [23] result for the trivial local system on the space of all smooth hypersurfaces. In both cases one of the main obstacles of moving directly between the cohomological stability results and the stable point counting results is a lack of control over the unstable cohomology.

1.5.3. **Complete intersections.** Bucur and Kedlaya [4] have generalized Poonen’s sieve results to complete intersections and used them to study the number of degree 1 points on a random complete intersection over a finite field. Using their results, one finds point-counting stabilization results analogous to Theorems B and D for the vanishing cohomology and relative configuration spaces of the universal smooth complete intersection. We expect that the techniques of Vakil and Wood, as extended in this paper to prove Theorems A and C, can also be applied to complete intersections, but we do not carry this out in the present work.

1.5.4. **Branched covers.** There has been some interesting work on cyclic branched covers of $\mathbb{P}^n$. The monodromy representations coming from these covers are studied by Carlson and Toledo [5] in their work on fundamental groups of discriminant complements. In the case $n = 1$, Chen [6] has recently proved a stability result for
the cohomology with coefficients in the local system corresponding to the first cohomology of the universal family of cyclic branched covers. From this he deduces the corresponding point counting result, which he phrases using probability. It would be interesting to extend this computation to the natural local systems constructed out of the first cohomology via the $\lambda$-ring structure, and to extend the probabilistic interpretation to a motivic setting.

For $n = 1$, it would also be interesting to study these problems for Hurwitz spaces, where cohomological stabilization with trivial coefficients is shown in \[12\].

1.6. Outline. In Section 2 we introduce the algebraic probability measures (in the sense of \[14\]) used in our proofs. In Section 3 we describe some aspects of the representation theory of symmetric, symplectic, and orthogonal groups. In Section 4 we recall the construction of vanishing cohomology and explain in more detail the construction of the local systems corresponding to a representation $\pi$. In Section 5 we prove our point counting Theorems B and D. Finally, in Section 6 we prove our stabilization results over $\mathbb{C}$, Theorems A and C, and discuss Conjecture E and some partial results towards it.

1.7. Notation. For partitions and configuration spaces we follows the conventions of Vakil and Wood \[24\], except that where they would write $w_\tau$, we write $\text{Conf}_\tau$. Also, we tend to avoid the use of $\lambda$ to signify a partition to avoid conflicts with the theory of pre-$\lambda$ rings.

A variety over a field $K$ is a finite-type scheme over $K$. It is quasi-projective if it can be embedded as a locally closed subvariety of $\mathbb{P}^n_K$.

Our notation for pre-$\lambda$ rings and power structures is described in \[14, \text{Section 2}\]. We highlight the following point here: if $f \in 1 + (t_1, t_2, ...) R[[t_1, t_2, ...]]$, then $f^r$ will always denote the naive exponential power series

$$\exp (r \cdot \log f) \in 1 + (t_1, t_2, ...) R[[t_1, t_2, ...]].$$

If $R$ is a pre-$\lambda$ ring and we want to denote an exponential taken in the associated power structure, then we write it as $f^{\text{Pow}^r}$.

We note, however, that the power structure will play a less prominent role in the present work than in \[14\].

Our notation for Grothendieck rings of varieties and motivic measures is explained in \[14, \text{Section 3}\].

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This paper and its prequel \[14\] grew out of an effort to systematically understand the explicit predictions for the cohomology of local systems on $U_d(\mathbb{C})$ that can be obtained from Poonen’s sieving results \[20\] (as in Theorem B). In particular, the results in the complex setting began as conjectures motivated by the finite field
case, and it was only after reading Vakil and Wood’s paper [24] that we realized some predictions could be proved by working in the Grothendieck ring of varieties. We would like to acknowledge the intellectual debt this article owes to both [20] and [24], which will be obvious to the reader familiar with these works.

2. SOME PROBABILITY SPACES

In this section we define the algebraic probability spaces we use in the rest of the paper. For basic results on algebraic probability theory, we refer to [14, Section 4]. For convenience, here we recall that for a ring $R$, an $R$-probability measure $\mu$ on an $R$-algebra $A$ with values in an $R$-algebra $A'$ is given by an expectation map

$$E_\mu : A \to A'$$

which is a map of $R$-modules sending $1_A$ to $1_{A'}$. Here we think of $A$ as being an algebra of random variables.

In Subsection 2.1, we discuss the spaces used in Section 5 to prove our finite field Theorems B and D. The main point is to work in a setting rich enough to handle cohomology groups of algebraic varieties, but small enough so that we can discuss convergence in the archimedean topology on $\mathbb{Q}$ without too many gymnastics. We reach this middle ground by using the subring of the Grothendieck ring of lisse $l$-adic sheaves consisting of virtual sheaves with everywhere integral characteristic power series of Frobenius.

In Subsection 2.1, we discuss the spaces used to prove Theorem A on the stabilization of the Hodge structures in the cohomology of the variations of Hodge structure $V_{\pi, d, \text{van}}$. We note that the standard approach for producing the mixed Hodge structure on these cohomology groups is via Saito’s [21, 22] theory of mixed Hodge modules, however, for our purposes it is simpler to use the geometric theory of Arapura [2]. Besides introducing less technical overhead, the main advantage of Arapura’s theory for us is that compatibility with Deligne’s [9] mixed Hodge structures and the Leray spectral sequence is built-in from the start. We use this compatibility when we deduce the cohomological stabilization in Theorem A from the geometric stabilization in Theorem C (cf. Lemma 2.2.1 for the precise statement used). As indicated in the introduction of [2], the mixed Hodge structures obtained via Saito’s theory should agree with those used here.

In Subsection 2.3 we discuss the motivic random variables used in our motivic stabilization result, Theorem C (cf. also [14, Section 3]).

2.1. Lisse $l$-adic sheaves. For a variety $S/\mathbb{F}_q$, denote by $\text{Loc}_{\mathbb{Q}_l} S$ the category of lisse $\mathbb{Q}_l$-sheaves on $S$. The Grothendieck ring

$$K_0(\text{Loc}_{\mathbb{Q}_l})$$

is a $\lambda$-ring with

$$\sigma_k([V]) = [\text{Sym}^k V].$$

Trace of Frobenius at a closed point $u \in S$ gives a map of rings

$$K_0(\text{Loc}_{\mathbb{Q}_l}) \to \mathbb{Q}_l$$

$$K \mapsto \text{TrFrob}_u \subset K_\pi$$

which sends $\sigma_1(K)$ to the characteristic power series of Frobenius acting on $K_\pi$. We denote by

$$K_0(\text{Loc}_{\mathbb{Q}_l} S)$$


the sub-\(\lambda\)-ring consisting of elements \(K\) such that this character power series is in \(\mathbb{Z}[[t]]\) for all closed points. By [10, Theorem 1.6] and smooth proper base change, if \(f : T \to S\) is smooth and proper, then for any \(i\),

\[
[R^if_*\mathbb{Q}_l] \in K_0(\text{Loc}_{\mathbb{Q}_l}S)'.
\]

For any smooth proper \(f : T \to S\), we will denote

\[
[T] := [Rf_*\mathbb{Q}_l] = \sum_i (-1)^i [R^if_*\mathbb{Q}_l] \in K_0(\text{Loc}_{\mathbb{Q}_l}S)'.
\]

Pullback via \(S \to \text{Spec}\mathbb{F}_q\) induces a map of \(\lambda\)-rings

\[
K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)' \to K_0(\text{Loc}_{\mathbb{Q}_l}S)'
\]

with image the constant sheaves.

In the other direction, there is a map of \(K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)\)'-modules

\[
K_0(\text{Loc}_{\mathbb{Q}_l}S)' \xrightarrow{\chi_S} K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'
\]

\[
[V] \mapsto \sum_i (-1)^i [H^i_c(S_{\mathbb{F}_q}, V)].
\]

By the Grothendieck-Lefschetz fixed point formula, the composition of \(\chi_S\) with \(\text{TrFrob}_q\) to \(\mathbb{Z}\) is given by

\[
K \to \sum_{w \in S(\mathbb{F}_q)} \text{TrFrob}_q \subset K_{\pi}
\]

(in particular, this verifies that the image of \(\chi_S\) is actually in \(K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)\)'). The image of \(\mathbb{Q}_l\) under this composed map is \(#S(\mathbb{F}_q)\).

In particular, if we consider \(\mathbb{Q}\) as an algebra over \(K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'\) via \(\text{TrFrob}_q\), we obtain a \(\mathbb{Q}\)-valued probability measure \(\mu\) on \(K_0(\text{Loc}_{\mathbb{Q}_l}/S)\)' via

\[
\mathbb{E}_\mu : K \to \frac{\text{TrFrob}_q \subset \chi_S(K)}{#S(\mathbb{F}_q)}.
\]

If we fix a \(K \in K_0(\text{Loc}_{\mathbb{Q}_l}S)'\), then we obtain a map of \(K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'\) algebras

\[
\Lambda_{K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'} \to K_0(\text{Loc}_{\mathbb{Q}_l}S)'
\]

\[
g \mapsto (g, K)
\]

and, by composition with \(\mathbb{E}_\mu\), a \(\mathbb{Q}\)-valued \(K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'\) measure on \(\Lambda_{K_0(\text{Loc}_{\mathbb{Q}_l}\mathbb{F}_q)'}\).

2.2. Variations of Hodge structure. Let \(S/\mathbb{C}\) be a smooth variety. Following Arapura [2], we denote by \(\text{GVSH}(S)\) the category of geometric polarizable variations of \(\mathbb{Q}\)-Hodge structure on \(S(\mathbb{C})\) – it is the full subcategory of variations of Hodge structures consisting of those isomorphic to a direct summand of \(R^i f_* \mathbb{Q}\) for a smooth projective \(f : T \to S\).

By [2], Theorem 5.1, for \(V \in \text{GVSH}(S)\), \(H^i(S, V)\) is equipped with a natural mixed Hodge structure. For \(V = R^q f_* \mathbb{Q}\), it is compatible with Leray spectral sequences for smooth proper maps and the Deligne [9] mixed Hodge structures on the cohomology of smooth varieties.

For \(V \in \text{GVSH}(S)\) of weight \(w\), we define the Hodge structure on compactly supported cohomology using Poincaré duality

\[
H^i_c(S(\mathbb{C}), V) := H^{2n-i}(S, V)^*(-w - \dim S).
\]
We denote by $K_0(GVSH(S))$ the corresponding Grothendieck ring, which is a $\lambda$-ring with 
\[ \sigma_k([V]) = [\text{Sym}^k V]. \]
That $\text{Sym}^k V$ is also geometric for $V$ geometric follows by finding it as a direct summand of the relative cohomology of a product, as in the proof of Corollary 5.5 of [2].

For $S = \mathbb{C}$, this is the Grothendieck ring of geometric polarizable $\mathbb{Q}$-Hodge structures $K_0(\text{GHS})$.

For any $T/S$ smooth and projective, $R^if_*T$ is naturally a variation of Hodge structure, and thus gives a class 
\[ [R^if_*T] \in K_0(GVSH(S)). \]
In this setting, we denote 
\[ [T/S]_{GVSH} := \sum_i (-1)^i [R^if_*T] \in K_0(GVSH(S)) \]
(here the subscript is to distinguish from the class $[T/S]$ in $K_0(\text{Var} S)$).

Pullback via $S \to \text{Spec} \mathbb{C}$ gives a map of $\lambda$-rings 
\[ K_0(\text{GHS}) \to K_0(GVSH(S)) \]
with image the constant geometric variations of Hodge structure on $S$.

In the other direction, there is a map of $K_0(\text{GHS})$-modules 
\[ K_0(\text{GVSH}_Q S) \xrightarrow{\times S} K_0(\text{HS}) \]
\[ [V] \mapsto \sum_i (-1)^i [H^i_c(S(\mathbb{C}), V)] \]

Given a smooth projective $T/S$, we will later want to deduce stabilization results for the variations of Hodge structures constructed from $T$ from motivic stabilization results for $T$. To do so, we need a lemma that says certain pre-$\lambda$ ring structures are compatible:

**Lemma 2.2.1.** For $T/S$ smooth and projective, the following diagram commutes.

\[ \begin{array}{ccc}
K_0(\text{Var}/S) & \xrightarrow{\text{forget}} & K_0(\text{Var}) \\
\Lambda & \xrightarrow{f \to (f, [T/S])} & K_0(GVSH(S)) \\
\downarrow & & \downarrow \\
K_0(\text{GVSH}_Q S) & \xrightarrow{\times S} & K_0(\text{HS})
\end{array} \]

**Proof.** It suffices to verify that for any $l_1, \ldots, l_m$, the two maps from $\Lambda$ to $K_0(\text{HS})$ agree on $h_{l_1} \ldots h_{l_m}$. We denote $n = \sum l_i$, and $S_\Gamma = S_{l_1} \times \ldots \times S_{l_m}$.

Via the top arrows, we obtain the cohomology of 
\[ T^{\times n}/S_\Gamma. \]
Here the superscript on $T$ denotes the $n$-fold fiber product over $S$.

This is equivalent to the $S_\Gamma$-invariants of the cohomology of $T^{\times n}$, which is computed by the $S_\Gamma$-invariants of the Leray spectral sequence for 
\[ T^{\times n} \to S. \]
On the $E_2$ page of the spectral sequence, the $S_r$-invariants are the cohomology of the $S_r$-invariants of the local systems coming from push-forward of $T^{\times s_2}$. Using this and the relative Kunneth formula for these local systems, one can verify we obtain the same value by following the bottom arrows. \[\square\]

Finally, since \[\frac{[S]}{[S]_{HS}} = \chi_S([S]_{GVSȩ}) = \chi_S(\mathcal{Q})\]
is invertible in $\widetilde{K_0}(\text{HS})$, we obtain a $K_0(\text{GHS})$-probability measure on $K_0(\text{GVSȩ}(S))$ with values in $K_0(\text{HS})$ by

\[
\mathbb{E}_\mu: K_0(\text{GVSȩ}(S)) \rightarrow K_0(\text{HS}) \\
K \mapsto \frac{\chi_S(K)}{[S]_{HS}}.
\]

2.3. **Motivic random variables.** If $S/\mathbb{K}$ is a variety, we consider the relative Grothendieck ring of varieties $K_0(\text{Var}/S)$, which we view as the ring of *motivic random variables* on $S$. It is an algebra over $K_0(\text{Var})$, the Grothendieck ring of varieties over $\mathbb{K}$. If 

\[\phi: K_0(\text{Var}) \rightarrow R\]
is a motivic measure such that $[S]_\phi$ is invertible in $R$, we obtain an $R$-valued $K_0(\text{Var})$-probability measure on $K_0(\text{Var}/S)$ by

\[
\mathbb{E}_\mu[[Y/S]] = \frac{[Y]_\phi}{[S]_\phi}.
\]

In this article, the most important example of $\phi$ will be the natural map 

\[K_0(\text{Var}) \rightarrow \widetilde{\mathcal{M}}_L.\]

When $\mathbb{K}$ is of characteristic zero, the relative Kapranov zeta function gives $K_0(\text{Var}/S)$ the structure of a pre-$\lambda$ ring, and given $Y/S$, we can pull back $\mathbb{E}_\mu$ via 

\[f \mapsto (f, [Y/S])\]
to obtain an $R$-valued $\mathbb{Z}$-probability measure on $\Lambda$.

For more details on the relative Grothendieck ring $K_0(\text{Var}/S)$ and the pre-$\lambda$ structure, cf. [14, Section 3].

3. **Some representation theory**

In this section we explain how to parameterize irreducible representations of symmetric, orthogonal, and symplectic groups by partitions $\sigma$ in order to produce the representations $\pi_{\sigma,d}$ described in Subsection 1.1. The key point in common between these groups is that each such group $G$ admits a standard representation $V_{\text{std}}$ inducing via the pre-$\lambda$ structure a surjection 

\[\Lambda \twoheadrightarrow K_0(\text{Rep}G)\]

\[f \mapsto (f, [V_{\text{std}}]).\]

In other words, the representation ring is spanned by products of symmetric powers of the standard representation. Through these surjections, families of irreducible representations are naturally parameterized by certain Schur polynomials. For orthogonal and symplectic groups, the results in this section follow from work of Koike and Terada [15], and for symmetric groups from work of Marin [18].
Let $G$ be a linear algebraic group over a field $F$ of characteristic zero, and consider the category $\text{Rep} G$ of algebraic representations of $G$ on finite dimensional vector spaces over $F$. For us, the most important examples are $F = \mathbb{Q}$ or $\mathbb{Q}_l$ and

$$G = \begin{cases} 
S_N, \text{ a symmetric group} \\
\text{Sp}(V, \langle , \rangle), \text{ automorphisms of a non-degenerate symplectic form} \\
O(V, \langle , \rangle), \text{ automorphisms of a non-degenerate symmetric form}
\end{cases}$$

In the latter two cases we will also consider the corresponding group of homotheties $G'$.

In each of the cases $\bullet = \{S, \text{Sp}, O\}$ of Equation (3.0.1), we have a way to assign to a partition $\sigma = (\sigma_1, ..., \sigma_k)$ a family of irreducible representations $V_{G, \sigma}$ of the groups in $\bullet$ of sufficient size depending on $\sigma$ (here size means $N$ for symmetric groups and the dimension of $V$ for symplectic and orthogonal groups):

1. For symmetric groups, we use the method of [8, Section 2.1]. The irreducible representation of $S_N$ attached to $\sigma$ is obtained via the theory of Young tableaux from the partition $(N - |\sigma|, \sigma_1, ..., \sigma_k)$. In particular, it is defined only for $|N| \geq |\sigma| + \sigma_1$.

2. For symplectic groups (all of which are split over $F$), we use highest weight theory as in [15, Section 1] (see also [8, Section 2.2]). The partition $\sigma$ defines a dominant integral weight of $\text{Sp}_k$ by padding with $m$ zeros at the end, and from this we obtain an irreducible representation.

3. For orthogonal groups, there are two complications: first, the group is not connected (the components correspond to $\det = \pm 1$), and second, we will typically be discussing orthogonal groups which are not split over $F$. Nevertheless, after padding the right by zeroes, the partition $\sigma$ will give rise as in [15, Section 1] to a representation of $\text{SO}(V_F)$ obtained as the restriction of an irreducible representation of $O(V_F)$, and we will see below in Remark 3.0.2 that there is a natural way to choose a representation of $O(V)$ (and even the group of homotheties) restricting to this representation and defined over $F$.

For $G$ as in Equation (3.0.1), we denote by $V_{\text{std}}$ the standard representation – when $G = S_N$ it is the space of functions on $\{1, ..., N\}$ summing to zero\footnote{It is sometimes more natural to take the full permutation representation on $\{1, ..., N\}$ as the standard representation in this case.} and when $G$ is the automorphism group of a pairing on $V$ it is given by $V$ itself.

The Grothendieck ring $K_0(\text{Rep} G)$ is equipped with a pre-\lambda ring structure by

$$\sigma_k([V]) = [\text{Sym}^k V].$$

It has the following useful interpretation: if we identify a virtual representation $K$ with its trace in $F[G]^G$ (action by conjugation), then for $f \in \Lambda$, $(f, [V])$ is identified with the function which sends an element $g \in G(F)$ to the symmetric function $f$ evaluated on the eigenvalues of $g$ acting on $V$. 

Proposition 3.0.1. For $G$ as in Equation (3.0.1), the map

$$
\Lambda \rightarrow K_0(\text{Rep}G) \\
f \mapsto (f, [V_{\text{std}}])
$$

is surjective. Furthermore, if we fix one of the three cases $\bullet \in \{S, \text{Sp}, O\}$ of Equation (3.0.1), then for any partition $\sigma$, there is a Schur polynomial $s_{\bullet, \sigma}$ such that

$$(s_{\bullet, \sigma}, [V_{\text{std}}]) = V_{\sigma}$$

for any $G$ (of sufficient size) of the form $\bullet$.

Remark 3.0.2. In the case of orthogonal groups, we had only defined $V_{\sigma}$ up to restriction to $\text{SO}(V_{\tau})$. The orthogonal Schur polynomials of Proposition 3.0.1 pick out a lift to $\text{O}(V_{\tau})$, and show that it is defined over $F$.

Furthermore, for both symplectic and orthogonal groups, $V_{\text{std}}$ is naturally a representation of the homothety group $G'$, and thus Proposition 3.0.1 picks out a natural lift of $V_{\sigma}$ to a representation of $G'$ by

$$[V_{\sigma}] = (s_{\bullet, \sigma}, [V_{\text{std}}]) \in K_0(\text{Rep}G').$$

Proof. We first handle the symplectic and orthogonal cases. To prove surjectivity we may base change $G$ to $\overline{F}$. The surjectivity in the symplectic case then follows from [15, Proposition 1.2.6], noting that the generator $p_i$ there corresponds to $r_{\text{Sym}^i[V_{\text{std}}]} = (h_i, [V_{\text{std}}])$ in our notation. In the orthogonal case, we caution the reader that the ring $R(\text{O}(m))$ of [15] is the image of $K_0(\text{Rep}(\text{O}(m)))$ in $K_0(\text{RepSO}(m))$. Surjectivity onto this ring follows from the same proposition, noting that the generator $e_i$ there corresponds to $[\Lambda^i V_{\text{std}}] = (e_i, [V_{\text{std}}])$

in our notation. To obtain surjectivity onto $K_0(\text{Rep}(\text{O}(m)))$, we observe that two irreducible representations of $\text{O}(m)$ with equal restriction to $\text{SO}(m)$ differ by tensoring with $\det$, and thus if one is in the image of $\Lambda$, we obtain the other via multiplication by $e_m$. We define the Schur polynomials using the formulas of [15, Theorems 1.3.2 and 1.3.3], together with the observation of [15, Remark 1.3.4] that for the dimension large relative to $\sigma$, the formulas are independent of the group.

For symmetric groups, surjectivity and the existence of Schur polynomials follows from the proof of [18, Proposition 4.1], which proves surjectivity essentially by inductively constructing such Schur polynomials from the elementary symmetric polynomials $e_k$. □

Remark 3.0.3. For symmetric groups, we observe that if $X_i$ is the function counting cycles of length $i$, then the character of $\Lambda^k(V_{\text{std}} \oplus 1)$ is given by

$$\sum_{(\tau_1, \ldots, \tau_i) \text{ s.t. } \sum i \tau_i = k} \binom{X}{\tau}$$

where

$$\binom{X}{\tau} = \prod_i \binom{X_i}{\tau_i}.$$ 

Combining this with [17, Section 1.7, Example 14], one can obtain an explicit formula for the Schur polynomials as polynomials in the $X_i$ (which correspond, as in [14], to Mobius inverted power sum polynomials $p'_i$ in $\Lambda_Q$).
It would be interesting to compare further the two integral structures on character polynomials occurring naturally here – the first coming from $\Lambda$, the second coming from the span of the

$$\begin{pmatrix} X \\ \tau \end{pmatrix},$$

which give rise to integer valued class functions.

4. VANISHING COHOMOLOGY

In this section we recall the construction of the local systems $\mathcal{V}_{\text{van},\mathbb{Q}}$ and $\mathcal{V}_{\text{van},\mathbb{Q}_l}$ of vanishing cohomology and establish some of the related notation used in the rest of this paper.

Let $Y$ be a polarized smooth projective variety of dimension $n$ over a field $K$, with polarization denoted by $\mathcal{L}$. For any $d \in \mathbb{Z}_{\geq 1}$, we define

$$U_d \subset \mathbb{A}(\Gamma(Y, \mathcal{L}^\otimes d)).$$

to be the locus of smooth sections. It a Zariski open subvariety.

There is a universal smooth projective hypersurface section $f : Z_d \to U_d$ fitting into a commutative diagram

$$
\begin{array}{ccc}
Z_d & \xrightarrow{f} & Y \times U_d \\
\downarrow & & \downarrow \\
U_d & \xrightarrow[\text{rel. Gysin map}]{} & Y
\end{array}
$$

such that for $u \in U_d(\overline{K})$, the fiber

$$Z_{d,u} \to Y$$

is the smooth closed subvariety of $Y_{\overline{K}}$ corresponding to $u$.

**Definition 4.0.4.** Let $l$ be coprime to the characteristic of $K$. The *étale vanishing cohomology* $\mathcal{V}_{\text{van},\mathbb{Q}_l}$ is the lisse $\mathbb{Q}_l$-sheaf on $U_d$

$$\mathcal{V}_{\text{van},\mathbb{Q}_l} := \ker R^{n-1} f_* \mathbb{Q}_l \to H^{n+1}(Y_{\overline{K}}, \mathbb{Q}_l)(1)$$

where the underline denotes the constant sheaf and the map is the relative Gysin map. If $K = \mathbb{C}$, the *Betti vanishing cohomology* $\mathcal{V}_{\text{van},\mathbb{Q}}$ is the $\mathbb{Q}$-local system on $U_d(\mathbb{C})$

$$\mathcal{V}_{\text{van},\mathbb{Q}} := \ker R^{n-1} f_{\text{an}*} \mathbb{Q} \to H^{n+1}(Y(\mathbb{C}), \mathbb{Q})(1)$$

where $f_{\text{an}}$ denotes the analytification of $f$.

**Remark 4.0.5.** Geometrically, the Gysin map is given fiberwise by the natural map of homology

$$H_{n-1}(Z_u) \to H_{n-1}(Y)$$

after writing homology as dual to cohomology and then using Poincaré duality. The kernel is generated by the classes of vanishing spheres in a Lefschetz pencil through $Z_u$, thus the name vanishing cohomology. For more on this geometric interpretation, cf. e.g. [25, Section 2.3.3].
Example 4.0.6. If $Y = \mathbb{P}^n$, then $U_d$ parameterizes smooth hypersurfaces of degree $d$, and $\mathcal{V}_{\text{van},\mathbb{Q}}$ (resp. $\mathcal{V}_{\text{van},\mathbb{Q}}$) is the primitive part; in particular, if $n = \dim Y$ is odd then the vanishing cohomology is equal to $R^{n-1}f_\ast\mathbb{Q}_l$ (resp. $R^{n-1}f_{\text{an},\ast}\mathbb{Q}$). The latter holds also for any $Y$ of odd dimension that is a complete intersection in $\mathbb{P}^m$.

The local system $R^{n-1}f_\ast\mathbb{Q}_l$ (resp $R^{n-1}f_{\text{an},\ast}\mathbb{Q}$) is equipped with a non-degenerate pairing to $\mathbb{Q}_l(-(n-1))$ (resp $\mathbb{Q}(-(n-1))$) whose restriction to $\mathcal{V}_{\text{van},\mathbb{Q}}$ (resp $\mathcal{V}_{\text{van},\mathbb{Q}}$) is also non-degenerate; we denote this restricted pairing by $\langle \cdot, \cdot \rangle$. It is symmetric if $n - 1$ is even and anti-symmetric if $n - 1$ is odd.

By [11, Corollaire 4.3.9] (resp. by the hard Lefschetz theorem over $\mathbb{C}$), we have
\[
\mathcal{V}_{\text{van},\mathbb{Q}}^\perp \cong H^{n-1}(\mathbb{V}_{\mathbb{K},\mathbb{Q}_l})
\]
(resp.
\[
\mathcal{V}_{\text{van},\mathbb{Q}}^\perp \cong H^{n-1}(\mathbb{V}_{\mathbb{C},\mathbb{Q}}),
\]
and there is a direct sum decomposition
\[
R^{n-1}f_\ast\mathbb{Q}_l \cong \mathcal{V}_{\text{van},\mathbb{Q}_l} \oplus H^{n-1}(\mathbb{V}_{\mathbb{K},\mathbb{Q}_l})
\]
(resp.
\[
R^{n-1}f_{\text{an},\ast}\mathbb{Q} \cong \mathcal{V}_{\text{van},\mathbb{Q}} \oplus H^{n-1}(\mathbb{V}_{\mathbb{C},\mathbb{Q}}).
\]

If we fix a base point $u_0 \in U(\mathbb{K})$ and a trivialization of $\mathbb{Q}_l(1)|_{u_0}$, the local system $\mathcal{V}_{\text{van},\mathbb{Q}_l}$ (resp. $\mathcal{V}_{\text{van},\mathbb{Q}}$) is determined by the monodromy representation
\[
\rho_1 : \pi_1,\text{ét}(U_d, u_0) \to \text{Aut}'(\mathcal{V}_{\text{van},\mathbb{Q}_l,u_0}, \langle \cdot, \cdot \rangle)
\]
(resp.
\[
\rho : \pi_1(U_d(\mathbb{C}), u_0) \to \text{Aut}(\mathcal{V}_{\text{van},\mathbb{Q},u_0}, \langle \cdot, \cdot \rangle),
\]
where by $\text{Aut}'$ we denote the group of homotheties of the pairing.

The local system $\mathcal{V}_{\text{van},\mathbb{Q}_l}$ (resp. $\mathcal{V}_{\text{van},\mathbb{Q}}$) or, equivalently, the corresponding monodromy representation $\rho_1$ (resp. $\rho$) is absolutely irreducible. In fact, we can say much more: by Deligne [11, Théorèmes 4.4.1 and 4.4.9], the image of $\pi_1,\text{ét}(U_d, \mathbb{K})$ under $\rho_1$ is either open or finite and equal to the Weyl reflection group of a root system of type $A, D, E$ embedded in $\mathcal{V}_{\text{van},\mathbb{Q}_l,u_0}, \langle \cdot, \cdot \rangle$ (the vanishing cycles). The latter can occur only when $n - 1$ is even, so that the pairing is symmetric. Open image comes as a consequence of Zariski density, and the argument that the image is either Zariski dense or a reflection group as above given in [11] is valid also in the topological setting for $\rho$ (c.f., e.g., [25, Section 3.2], for the vanishing cycles input). For the symmetric group case, [25, Theorem 3.3.4] shows the monodromy is surjective.

Let $G_{\mathbb{Q}_l}$ denote the algebraic group over $\mathbb{Q}_l$ (resp. over $\mathbb{Q}$)
\[
G_{\mathbb{Q}_l} := \text{Aut}(\mathcal{V}_{\text{van},\mathbb{Q}_l,u_0}, \langle \cdot, \cdot \rangle)
\]
(resp.
\[
G := \text{Aut}(\mathcal{V}_{\text{van},\mathbb{Q},u_0}, \langle \cdot, \cdot \rangle).
\]
Denote by $G'_{\mathbb{Q}_l}$ (resp. $G'$) the corresponding group of homotheties.

Let $\text{Rep}G_{\mathbb{Q}_l}$ (resp. $\text{Rep}G$) denote the category of algebraic representations of $G_{\mathbb{Q}_l}$ on $\mathbb{Q}_l$-vector spaces (resp. $\mathbb{Q}$-vector spaces) and by $\text{Loc}_{\mathbb{Q}_l}U_d$ (resp. $\text{Loc}_{\mathbb{Q}}U_d(\mathbb{C})$) the
category of lisse $\mathbb{Q}_l$-local systems on $U_d$ (resp. $\mathbb{Q}$-local systems on $U_d(\mathbb{C})$). We obtain a functor
\[
\text{Rep}G_{\mathbb{Q}_l} \to \text{Loc}_{\mathbb{Q}_l}U_d, \pi \to (\mathcal{V}_{\text{van},\mathbb{Q}_l}|_{U_d,\pi})^\pi
\]
where $(\mathcal{V}_{\text{van},\mathbb{Q}_l}|_{U_d,\pi})^\pi$ is the local system corresponding to the representation
\[
\pi \circ \rho|_{\pi_1,\pi(U_d,\overline{\mathbb{Q}_l})}.
\]

If $\mathbb{K}$ does not contain the $l$-power roots of unity, then in general the image of the monodromy representation is contained in $G'_{\mathbb{Q}_l}$ but not in $G_{\mathbb{Q}_l}$ (unless $n = 0$). Thus we obtain a functor
\[
\text{Rep}G'_{\mathbb{Q}_l} \to \text{Loc}_{\mathbb{Q}_l}U_d, \pi' \to \mathcal{V}_{\text{van},\mathbb{Q}_l}
\]
fitting into a commutative diagram
\[
\begin{array}{ccc}
\text{Rep}G'_{\mathbb{Q}_l} & \xrightarrow{\pi' \mapsto \mathcal{V}_{\text{van},\mathbb{Q}_l}} & \text{Loc}_{\mathbb{Q}_l}U_d \\
\pi' \mapsto \pi'|_{\mathcal{V}_{\text{van},\mathbb{Q}_l}} & & \pi' \mapsto (\mathcal{V}_{\text{van},\mathbb{Q}_l}|_{U_d,\pi})^\pi \\
\end{array}
\]
\[
\begin{array}{ccc}
\text{Rep}G_{\mathbb{Q}_l} & \xrightarrow{\pi \mapsto \mathcal{V}_{\text{van},\mathbb{Q}_l}} & \text{Loc}_{\mathbb{Q}_l}U_d, \pi
\end{array}
\]

Similarly, if $\mathbb{K} = \mathbb{C}$, then $\mathcal{V}_{\text{van},\mathbb{Q}}$ is equipped with a natural polarized variation of $\mathbb{Q}$-Hodge structure. The map from $\text{Rep}G$ to local systems is enriched to a map
\[
\text{Rep}G' \to \text{VHS}_{U_d(\mathbb{C})}
\]
where the right-hand side denotes the category of polarizable variations of Hodge structure on $U_d(\mathbb{C})$. It fits into a commutative diagram
\[
\begin{array}{ccc}
\text{Rep}G' & \xrightarrow{\pi' \mapsto \mathcal{V}_{\text{van},\mathbb{Q}_l}} & \text{VHS}_{\mathbb{Q}_l}U_d(\mathbb{C}) \\
\pi' \mapsto \pi'|_{\mathcal{V}_{\text{van},\mathbb{Q}_l}} & & \pi \mapsto \mathcal{V}_{\text{van},\mathbb{Q}_l} \\
\end{array}
\]
\[
\begin{array}{ccc}
\text{Rep}G & \xrightarrow{\pi \mapsto \mathcal{V}_{\text{van},\mathbb{Q}} \text{忘记 Hodge filtration}} & \text{Loc}_{\mathbb{Q}_l}U_d(\mathbb{C})
\end{array}
\]

5. Point counting results

In this section we prove Theorems B and D. As noted in Remark 1.3.1, after setting up the necessary language Theorem D is a simple consequence of results of Poonen [20]. Our main contributions here are the reinterpretation of Poonen’s results in the language of asymptotic independence and the pre-$\lambda$ ring structure, and from this the deduction of Theorem B.

We use the setup of Section 4 with $\mathbb{K} = \mathbb{F}_q$.

5.1. Geometric stabilization. In this subsection we prove Theorem D.

We consider the classical discrete uniform probability measures
\[
\begin{array}{ccc}
\text{Map}(U_d(\mathbb{F}_q), \mathbb{Q}) & \xrightarrow{\mathbb{E}_{\mu_d}} & \mathbb{Q} \\
f & \mapsto & \frac{1}{\#U_d(\mathbb{F}_q)} \sum_{u \in U_d(\mathbb{F}_q)} f(u)
\end{array}
\]
Denote by $A(Y)$ the free polynomial ring over $\mathbb{Q}$ with generators $X_y$ for $y$ running over the closed points of $Y$.

For each $d$, we obtain a map

$$A(Y) \to \text{Map}(U_d(\mathbb{F}_q), \mathbb{Q})$$

given by sending a closed point $X_y$ to the indicator random variable $X_y$ on $U_d(\mathbb{F}_q)$ defined by

$$X_y(u) = \begin{cases} 1 & \text{if } y \in Z_{d,u} \\ 0 & \text{if } y \notin Z_{d,u}. \end{cases}$$

By composition, we can think of $\mu_d$ as a sequence of $\mathbb{Q}$-probability measures on $A(Y)$.

**Theorem 5.1.1** (Poonen). The random variables in $A(Y)$

$$\{X_y\}_{y \in Y} \text{ a closed point}$$

are asymptotically independent with asymptotic Bernoulli distributions

$$E_{\mu_x}[(1 + t)^{X_y}] = 1 + \frac{q^{\deg y} - 1}{q^{(n+1)\deg y} - 1} \cdot t.$$ 

In particular, because the $X_y$ generate $A(Y)$ as a $\mathbb{Q}$-algebra, the asymptotic measure $\mu_\infty$ is defined on $A(Y)$ and, for any given $a \in A(Y)$, $E_{\mu_\infty}[a]$ can be computed explicitly by writing it as a sum of monomials in the $X_y$.

There are also natural point counting random variables in $\text{Map}(U_d(\mathbb{F}_q), \mathbb{Q})$ coming from the universal family:

$$X_k(u) = \# \text{ closed points of degree } i \text{ on } Z_{d,u}.$$ 

This is the sum of the indicator variables over all of the closed points of a fixed degree, and thus we can consider $X_k$ as an element of $A(Y)$:

$$X_k := \sum_{y \in Y \text{ closed of degree } i} X_y.$$ 

The random variables in $\mathbb{Q}[X_1, X_2, ...]$ are, in a precise sense, the random variables coming from the universal family $Z_d$: there is a map

$$K_0(\text{Loc}_{\mathbb{Q}}, U_d) \to \text{Map}(U_d(\mathbb{F}_q), \mathbb{Q})$$

$$K \mapsto u \mapsto \text{trFrob}_q \subset K_\pi$$

and the random variable $X_1$ is the image of $[Rf_*\mathbb{Q}_l]$ under this map. From the $\lambda$-ring structure on $K_0(\text{Loc}_{\mathbb{Q}}, U_d)$ we obtain

$$\Lambda_\mathbb{Q} \to \text{Map}(U_d(\mathbb{F}_q), \mathbb{Q})$$

sending $h_i$ to

$$u \mapsto \#\text{Sym}^i Z_{d,u}(\mathbb{F}_q).$$

It is a standard computation that $p_k'$ maps to the random variable attached to $X_k$. Because the $p_k'$ form a polynomial basis for $\Lambda_\mathbb{Q}$, this lifts to a map

$$\Lambda_\mathbb{Q} \to A(Y)$$

$$p_k' \mapsto X_k.$$ 

which induces an isomorphism between $\Lambda_\mathbb{Q}$ and $\mathbb{Q}[X_1, X_2, ...]$. 
From Theorem 5.1.1, we deduce that the the $X_k = (p'_k, \{Z_d/U_d\})$ are asymptotically independent with asymptotic binomial distributions, the sum over the closed points of degree $k$ of Bernoulli random variables that are $1$ with probability

$$\frac{q^{(n+1)i} - 1}{q^{(n+1)i} - 1}.$$ 

Thus, we have proven Theorem D.

5.2. Cohomological stabilization. In this subsection we prove Theorem B.

By Poonen [20],

$$\lim_{d \to \infty} \#U_d(F_q)/q^{d\dim U_d} = \zeta_Y(n+1)^{-1},$$

and thus we may normalize by $\#U_d(F_q)$ instead of $q^{d\dim U_d}$. Applying the Grothendieck-Lefschetz fixed point theorem to the numerator, we are then studying

$$\lim_{d \to \infty} \frac{1}{\#U_d(F_q)} \sum_{u \in U_d(F_q)} \TrFrob_q \subset V^{\pi, d}_{\text{van}, Q_l, \pi}$$

where $\pi$ is the $F_q$ point obtained by composition of $u$ with $F_q \hookrightarrow F_q$.

In particular, if we consider the random variable $X_\tau$ in

$$\text{Map}(U_d(F_q), \mathbb{Q})$$

given by

$$X_\tau(u) = \TrFrob_q \subset V^{\pi, d}_{\text{van}, Q_l, \pi}$$

then we are studying the asymptotic behavior of

$$\mathbb{E}_{\mu_d}(X_\tau).$$

Our goal now is to deduce the stabilization of this quantity from Theorem D.

The $\mathbb{Q}$-probability measure on $\Lambda_{\mathbb{Q}}$ given by

$$g \mapsto \mathbb{E}_{\mu_d}[(g, [Z_d/U_d])]$$

is the same as the measure of the previous section. It naturally extends to a $K_0(\text{Loc}_{\mathbb{Q}} F_q)$-probability measure on $\Lambda_{K_0(\text{Loc}_{\mathbb{Q}} F_q)}$ with values in $\mathbb{Q}$ and by Theorem D the measures $\mu_d$ on $\Lambda_{K_0(\text{Loc}_{\mathbb{Q}} F_q)}$ converge to a measure $\mu_\infty$.

Our strategy now is clear: we will construct an element $s_{\tau, Y}$ of $\Lambda_K$ such that for all $d$,

$$(s_{\tau, Y}, [Z_d/U_d]) = [V^{\pi, d}_{\text{van}, Q_l}],$$

which implies

$$\mathbb{E}_{\mu_d}[s_{\tau, Y}] = \mathbb{E}[X_\tau].$$

We denote

$$[Y_{\text{old}}] := \sum_{i<n-1} (-1)^i \cdot \left( [H^i(Y_{\text{reg}}, Q_l)] + [H^{2n-i}(Y_{\text{reg}}, Q_l)](1) \right) + (-1)^{n-1}[H^{n-1}(Y_{\text{reg}}, Q_l)] \in K_0(\text{Loc}_{\mathbb{Q}} F_q).$$

By Equation 4.0.2 and the weak Lefschetz theorem, it is the constant part of the cohomology of $Z_d$: in $K_0(\text{Loc}_{\mathbb{Q}} U_d)$, we have the identity

$$[Z_d] = [Y_{\text{old}}] + [V_{\text{van}, Q_l}].$$

Recall that $[V^{\pi, d}_{\text{van}, Q_l}] = (s_{\tau, [V_{\text{van}, Q_l}]}$ for an appropriate Schur polynomial $s_{\tau} \in \Lambda$ as in Proposition 3.0.1.
We observe that \((p'_k, \_\_\_)\) is additive, so that
\[
(p'_k, [\mathcal{V}_{\text{van}, \mathbb{Q}_L}]) = (p'_k, [Z_d]) - (p'_k, [Y^{\text{old}}])
\]
and thus
\[
(p'_k, [\mathcal{V}_{\text{van}, \mathbb{Q}_L}]) = (p'_k) - (p'_k, [Y^{\text{old}}]), [Z_d]).
\]
In particular, if we let \(s_{\tau,Y}\) be the element in \(\Lambda_{K_0(\text{Loc}, \mathbb{F}_q)}\) given by expressing \(s_{\tau}\) as a polynomial (with \(\mathbb{Q}\)-coefficients) in \(p'_k\) and then substituting \(p'_k, Y\):
\[
E_{\mu_d}[s_{\tau,Y}] = \frac{1}{\#U_d(\mathbb{F}_q)} \sum_{u \in U_d(\mathbb{F}_q)} \text{TrFrob} \mathcal{V}_{\text{van}, \mathbb{Q}_L}^{-\mu_d, \pi}
\]
This stabilizes to \(E_{\mu_d}[s_{\tau,Y}]\) as \(d \to \infty\). Moreover, since the \((p'_k, [Y^{\text{old}}])\) are constants and the \(p'_k\) are independent for \(\mu, \pi\), so are the \(p'_k, Y\). The asymptotic falling moment generating function for \(p'_k, Y\) is given by multiplying the asymptotic function for \(p'_k\) in Theorem D by the moment generating function for the constant random variable \((p'_k, [Y^{\text{old}}])\) which is
\[
(1 + t)^{-\text{TrFrob} \mathcal{V}_{(p'_k, [Y^{\text{old}}])}^{-\mu_d, \pi}} = (1 + t)^{-\sum_{d|k} \mu(k/d) \text{TrFrob}^k \mathcal{V}_{(p'_k, [Y^{\text{old}}])}^{-\mu_d, \pi}}.
\]
Thus we obtain Theorem B.

6. HODGE THEORETIC RESULTS

In this section we prove Theorems A and Theorems C. In Subsection 6.3 we also prove some partial results towards Conjecture E.

We use the setup of Section 4 with \(K = \mathbb{C}\).

6.1. Geometric stabilization.

Proof of Theorem C. The elements \(c_{\tau} \in \Lambda\) described in Example 1.2.2 form a basis for \(\Lambda\) (cf. [14, Subsection 3.1]). Thus it suffices to verify that for any \(\tau,\)
\[
\lim_{d \to \infty} \frac{(c_{\tau}, [Z_d/U_d])}{\text{dim } U_d}
\]
exists in \(\mathcal{M}_L\). Because
\[
(c_{\tau}, [Z_d/U_d]) = [\text{Conf}_{U_d}^\tau Z_d/U_d],
\]
we must verify
\[
\lim_{d \to \infty} \frac{[\text{Conf}_{U_d}^\tau Z_d]}{\text{dim } U_d}
\]
exists in \(\mathcal{M}_L\). To do so, we adopt the proof strategy of [24, Theorem 1.13]. Recall, in particular, that we use the notation for partitions from [24, Section 2].

Let
\[
V_d := \Lambda(\Gamma(Y, \mathcal{O}(d))).
\]
Given partitions \(\alpha\) and \(\beta\) we denote by \(W^\beta_\alpha\) the subvariety of
\[
V_d \times \text{Conf}^\alpha Y \times \text{Conf}^\beta Y
\]
consisting of \((s, a, b)\) such that \(s\) vanishes at the points in \(a\) and such that \(s\) is singular at the points in \(b\). We denote by \(W^\alpha_\beta\) the locally closed subset such that \(s\)
vanishes at the points in $a$ and is singular at exactly the points in $\beta$. In particular, we have

$$W^\alpha_\emptyset = \text{Conf}^\alpha Z_d$$

and for any $k \in \mathbb{Z}_{>1}$

$$(6.1.2) \quad [W^\alpha_{\geq \beta}] = [W^\alpha_\beta] + [W^\alpha_{\beta \cdot a_1}] + [W^\alpha_{\beta \cdot a_2}] + \ldots + [W^\alpha_{\beta \cdot a_k}] .$$

Given two partitions $\alpha = (k_1, k_2, \ldots)$ and $\beta = (l_1, l_2, \ldots)$, we consider formalizations

$$\alpha = a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots$$

and

$$\beta = b_1^{l_1} \cdot b_2^{l_2} \cdot \ldots$$

in the free commutative monoid $M_{\alpha, \beta}$ on $a_1, \ldots, b_1, \ldots$. Denote by $S_{\alpha, \beta}$ the set of generalized partitions $\gamma$ in $M_{\alpha, \beta}$ with

$$\sum_{\gamma} = \sum \alpha \cdot \beta \in M_{\alpha, \beta}$$

and such that only terms of the form $a_i$, $b_j$, and $a_ib_j$ appear in $\gamma$. The partitions $\gamma \in S_{\alpha, \beta}$ describe the possible overlap of a configuration of type $\alpha$ and a configuration of type $\beta$, and thus index a decomposition

$$(6.1.3) \quad \text{Conf}^\alpha Y \times \text{Conf}^\beta Y = \bigsqcup_{\gamma \in S_{\alpha, \beta}} w_\gamma$$

where $w_\gamma$ is the image of $\text{Conf}^\gamma Y$ in $\text{Conf}^\alpha Y \times \text{Conf}^\beta Y$ under the map which on the first coordinate is induced by the map $M_{\alpha, \beta} \rightarrow M_\alpha$ sending $b_i$ to 0 (forgetting the $b$ labels) and the second by the map $M_{\alpha, \beta} \rightarrow M_\beta$ sending $a_i$ to 0 (forgetting the $a$ labels).

**Lemma 6.1.1.** If $d >> |\beta| + |\alpha|$ then

$$(6.1.4) \quad [W^\alpha_{\geq \beta}] \equiv \sum_{\gamma \in S_{\alpha, \beta}} L^{-c(\gamma)} \cdot [\text{Conf}^\gamma Y] \mod \text{Fil}^{-(n-1)-|\alpha|+|\beta|}$$

in $\mathcal{M}$, where $c(\gamma)$ is sum over $\gamma$ of the set functional on $M_{\alpha, \beta}$ which sends $a_i$ to 1, $b_i$ to $n+1$, $a_ib_j$ to $n+1$, and all other elements of $M_{\alpha, \beta}$ to 0.

In particular,

$$[W^\alpha_{\geq \beta}] / L^{\dim U_d} \equiv 0 \mod \text{Fil}^{-(n-1)-|\alpha|+|\beta|}$$

**Proof.** We can write

$$W^\alpha_{\geq \beta} = \bigsqcup_{\gamma \in S_{\alpha, \beta}} W^\gamma_\prime$$

where $W^\gamma_\prime$ is the locally closed set in $W^\alpha_{\geq \beta}$ lying over $w_\gamma$ in the decomposition (6.1.3). (We are breaking up the space based on the overlap of the configuration of vanishing points $a \in \text{Conf}^\gamma Y$ and the configuration of singular points $b \in \text{Conf}^\beta Y$).

Arguing as in [24, Lemma 3.2], we find that for $d$ sufficiently large, $W^\gamma_\prime$ is a vector bundle of rank $\dim V_d - c(\gamma)$ over $\text{Conf}^\gamma Y$, so that

$$[W^\gamma_\prime] = L^{\dim V_d - c(\gamma)} \cdot [\text{Conf}^\gamma Y].$$

Here the formula for $c(\gamma)$ comes from the fact that marking a point as being on a hypersurface imposes one linear condition on $V_d$, whereas marking a point as singular imposes $n+1$ linear conditions on $V_d$, and *includes* the condition of vanishing at that point.
Since \( \dim U_d = \dim V_d \), we obtain the equality (6.1.4) by summing these expressions for \( [W^r_\beta] \) and dividing by \( \mathbb{L}^\dim U_d \).

To prove the final statement of the lemma, we note that for any \( \gamma \in S_{\alpha, \beta} \),
\[
-c(\gamma) + \dim \text{Conf}^\gamma Y \leq (n - 1) \cdot |\alpha| - |\beta|.
\]

To see this, first observe that equality holds when \( \gamma \) represents the partition where there is no overlap between \( \alpha \) and \( \beta \) so that
\[
c(\gamma) = |\beta|(n + 1) + |\alpha|
\]
and
\[
\dim \text{Conf}^\gamma Y = (|\alpha| + |\beta|)(n).
\]

Then, for any point added to the overlap, we take away \( n \) from \( \dim \text{Conf}^\gamma Y \), and 1 from \( c(\gamma) \), so that
\[
-c(\gamma) + \text{Conf}^\gamma Y
\]
drops by \( n - 1 \).

We can now prove the limit in Equation (6.1.1) exists. Fix a \( m \), and let \( N \) be large enough such that Lemma 6.1.1 holds for \( \alpha = \tau \) and all \( |\beta| \leq m + 1 \) and \( d \geq N \).

Let \( d \geq N \). Using Equation 6.1.2 iteratively to replace terms of the form \( r_{W^\tau_\beta} \beta \) with \( r_{W^\tau_\beta} \beta \),
\[
\begin{aligned}
&\lim_{d \to \infty} \frac{[\text{Conf}^\gamma U_d Z_d/U_d]}{\mathbb{L}^\dim U_d} = L^{-\dim U_d} \left( [W^r_{\geq 0}] - [W^r_{\geq 1}] - [W^r_{\geq 2}] - \ldots - [W^r_{\geq m}] \right) \\
&= L^{-\dim U_d} \left( [W^r_{\geq 0}] - [(W^r_{\geq 1}) - [W^r_{\geq 2}] - \ldots - [W^r_{\geq m-1}] - ([W^r_{\geq 2}] - [W^r_{\geq m-1}]) \right) \\
&= \ldots = L^{-\dim U_d} \sum_{\mu \in Q, |\mu| \leq m} -1^{||\mu||} [W^r_{\geq \mu}] = \sum_{\mu \in Q, |\mu| \leq m} -1^{||\mu||} \sum_{\gamma \in S_{r, \mu}} L^{-c(\gamma)} [\text{Conf}^\gamma Y].
\end{aligned}
\]

Thus, we conclude
\[
\lim_{d \to \infty} \frac{[\text{Conf}^\gamma U_d Z_d/U_d]}{\mathbb{L}^\dim U_d} = \sum_{\mu \in Q, |\mu| \leq m} -1^{||\mu||} \sum_{\gamma \in S_{r, \mu}} L^{-c(\gamma)} [\text{Conf}^\gamma Y].
\]

### 6.2. Cohomological stabilization

In this section we prove Theorem A. The proof mirrors the deduction of Theorem B from Theorem D given in Subsection 5.2. As in the point counting case, our method also gives an algorithm to compute an explicit formula for the limit, but in this case the formula is conditional on Conjecture E (cf. Appendix A for a description of this algorithm).

**Proof of Theorem A.** We use the notation developed in Subsection 2.2 and Section 4. We denote
\[
[Y^\text{old}]_{\text{GHS}} := \sum_{i<n-1} (-1)^i \cdot (H^i(Y(\mathbb{C}), \mathbb{Q}) + [H^{2n-i}(Y(\mathbb{C}), \mathbb{Q})(1)])
\]
\[
+ (-1)^{n-1} [H^i(Y(\mathbb{C}), \mathbb{Q})] \in K_0(\text{GHS}).
\]
We can view $[Y_{\text{old}}]^\text{GHS}$ as a constant variation of Hodge structure on $U_d$. By Equation (4.0.3) and the weak Lefschetz theorem,

$$[Y_{\text{old}}]^\text{GHS} + [\nu_{\text{van},Q}] = [Z_d/U_d]^\text{GVSH},$$

and thus, using the additivity of $(p_k')$,,

$$(p_k' - (p_k'[Y_{\text{old}}]), [Z_d/U_d]^\text{GVSH}) = (p_k'[Z_d/U_d]^\text{GVSH}) - (p_k'[Y_{\text{old}}]) = (p_k'[\nu_{\text{van},Q}]).$$

Denote by $s_{\tau,Y}$ the polynomial in $\Lambda_0(\text{GHS})$ obtained by expressing $s_{\tau} \in \Lambda$ as a polynomial in $p_k'$, then substituting $(p_k'[Y_{\text{old}}])$ for $p_k'$. By the above, we have

$$(s_{\tau,Y}, [Z_d/U_d]^\text{GVSH}) = (s_{\tau}, [\nu_{\text{van},Q}]) = [\nu_{\text{van},Q}].$$

Thus,

$$\lim_{d \to \infty} \frac{\chi^\text{HS}(H^\bullet_c(U_d(\mathbb{C}), \nu_{\text{van},Q})]}{\mathbb{Q}(\dim U_d]} = \lim_{d \to \infty} \frac{\chi^\text{HS}([s_{\tau,Y}, [Z_d/U_d]^\text{GVSH}])}{\mathbb{Q}(\dim U_d]}.$$ 

It suffices to verify that the limit on the right exists for any $f \in \Lambda$, since by linearity we can then extend the scalars on $\Lambda$ to $K_0(\text{GHS})$ to obtain the result for $s_{\tau,Y}$. For $f \in \Lambda$, by Lemma 2.2.1,

$$\lim_{d \to \infty} \frac{\chi^\text{HS}([f, [Z_d/U_d]^\text{GVSH}])}{\mathbb{Q}(\dim U_d]} = \lim_{d \to \infty} \frac{\chi^\text{HS}([f, [Z_d/U_d]^\text{GVSH}])}{\mathbb{Q}(\dim U_d]}.$$ 

Applying the Hodge realization to Theorem C, we see the limit on the right exists.

\[ \square \]

6.3. Towards Conjecture E. In this subsection we explain an approach to Conjecture E, and reduce it to a simple conjectural combinatorial statement (Proposition 6.3.1). We also prove in Theorem 6.3.3 that $p_1'$ has the asymptotic distribution predicted by Conjecture E.

Following the construction of Subsection 2.3, we denote by $E_{\mu_d}$ the measure on $\Lambda_0$ obtained by pulling back the uniform measure

$$[X/U_d] \mapsto [X]/[U_d]$$

on $K_0(\text{Var}/U_d)$ with values in $\hat{\mathcal{M}}_k \otimes \mathbb{Q}$ via

$$f \mapsto ([f, [Z_d/U_d]]).$$

Conjecture E is then concerned with the behavior of the random variables $p_k'$ under

$$f \mapsto \lim_{d \to \infty} E_{\mu_d}[f].$$

As in [14], to show Conjecture E, it suffices to verify that it correctly predicts these limits on the elements $c_{\tau} \in \Lambda$ since these form a basis for $\Lambda$ over $\mathbb{Z}$.

By [14, Lemma 3.5],

$$\prod_k (1 + t^k + t^{2k} + \ldots)^p_k = \sum_f c_f t^f.$$ 

In particular, for $\tau = a_1^{\tau_1} \cdot \ldots \cdot a_m^{\tau_m}$,

$$\frac{\partial^\tau}{\tau!} |_{t^\tau = 0} \prod_k (1 + t^k + t^{2k} + \ldots)^p_k = c_\tau,$$
and thus Conjecture E predicts that
\begin{equation}
(6.3.1) \lim_{d \to \infty} E_{\mu_d}[c_r] = \frac{\partial t^r}{\tau!}\bigg|_{t^* = 0} \prod_k \left(1 + \frac{L_k(n) - 1}{L_k(n+1) - 1} (t_1^k + t_2^k + \ldots)^{(p_k^r)[Y]} \right).
\end{equation}

We have
\[(c_r, [Z_d/U_d]) = [\text{Conf}_d^r Z_d/U_d],\]
and, adopting the strategy of [14], one hopes to establish Equation 6.3.1 by using the power structure on $K_0(\text{Var})$ to write down a generating function for the limits
\[\lim_{d \to \infty} \frac{[\text{Conf}_d^r Z_d/U_d]_{\bar{M}_L}}{L_{\dim U_d}}\]
using the expression for these limits obtained in Equation 6.1.5. In Proposition 6.3.1 below, we produce a candidate for this generating function which would allows us to carry through this argument.

We recall (cf. Remark 1.2.1) that, by [24],
\begin{equation}
(6.3.2) \lim_{d \to \infty} \frac{[U_d]}{L_{\dim U_d}} = Z_Y \left( L^{-1} \right) \in \bar{M}_L.
\end{equation}

**Proposition 6.3.1.** If
\[\left(1 - s \cdot (1 + t_1 + t_2 + \ldots) + z \cdot (t_1 + t_2 + \ldots)\right)^{p_{\infty}[Y]} = \sum_I \sum_{\mu} -1^{||\mu||} \sum_{\gamma \in S_{r, I, \mu}} [\text{Conf}_I^r Y] z^{b(\gamma)} s^{||-b(\gamma)||} t^I\]
then Conjecture E holds.

**Proof.** Dividing both sides by $(1 - s)^{p_{\infty}[Y]}$ we obtain
\[\left(1 \frac{z - s}{1 - s} (t_1 + t_2 + \ldots)\right)^{p_{\infty}[Y]} = (1 - s)^{p_{\infty}[Y]} \sum_I \sum_{\mu} -1^{||\mu||} \sum_{\gamma \in S_{r, I, \mu}} [\text{Conf}_I^r Y] z^{b(\gamma)} s^{||-b(\gamma)||} t^I\]
We have $(1 - s)^{-[Y]} = Z_Y(s)$, so plugging in
\[s = L^{-1}, \quad z = L^{-1}\]
and comparing with Equations (6.1.5) and (6.3.2), we find the coefficient of $t^I$ on the righthand side becomes
\[\lim_{d \to \infty} E_{\mu_d}[c_r].\]
Using [14, Lemma 2.7], the left hand side is equal to
\[\prod_k \left(1 - \frac{z^k - s^k}{1 - s^k} (t_1 + t_2 + \ldots)^{(p_k^r)[Y]} \right)\]
and, plugging in
\[s = L^{-1}, \quad z = L^{-1}\]
we obtain the expression of Equation (6.3.1) for the coefficient of $t^I$. \hfill \Box

**Remark 6.3.2.** To verify the identity in the hypothesis of Proposition 6.3.1, it would suffice, by [14], Lemma 3.4 to verify it under point counting for varieties over finite fields.
For a partition $\tau = b_1 \cdot b_2 \cdots b_k$ consisting of $k$ distinct elements,

$$c_\tau = p_1 (p_1 - 1) \cdots (p_1 - k + 1)$$

and thus studying its limits corresponds to studying the falling moments of $p_1 = p'_1$. By a direct computation with these elements, we are able to verify the asymptotic distribution predicted by Conjecture E for $p'_1 = p_1$:

**Theorem 6.3.3.** $p'_1 = p_1$ is asymptotically a binomial random variable, the “sum of” $[Y]$ independent Bernoulli random variables that are 1 with probability

$$\frac{L^n - 1}{L^{n+1} - 1},$$

i.e.,

$$E_{\mu, \tau} [(1 + t)^p_1] = \left( 1 + \frac{L^n - 1}{L^{n+1} - 1} \right)^{[Y]}.$$  

*Proof.* In fact, we will prove a slightly stronger statement without tensoring with $Q$, namely that, for each $k$,

(6.3.3) $E_{\mu, \tau} [(p_1)(p_1 - 1) \cdots (p_1 - k + 1)] =

\left( [Y] \right) \left( [Y] - 1 \right) \cdots \left( [Y] - k + 1 \right) \left( \frac{L^n - 1}{L^{n+1} - 1} \right)^k.$

The variable on the inside is $c_\tau$ for $\tau = b_1 \cdot b_2 \cdots b_k$, i.e. for $\tau$ corresponding to ordered configurations of $k$ points. For such a $\tau$, we consider the quantity

$$\sum_{\mu \in Q} -1^{||\mu||} \sum_{\gamma \in S_{\tau, \mu}} [\text{Conf}^\gamma Y] z^{b(\gamma)} s^{\gamma - b(\gamma)}.$$  

We split it up by the power of $s$ as

$$\sum_{j=0}^{k} \sum_{\mu \in Q} -1^{||\mu||} \sum_{\gamma \in S_{\tau, \mu, j}} [\text{Conf}^\gamma Y] z^j s^{\gamma - j}.$$  

Where $S_{\tau, \mu, j}$ consists of those elements of $S_{\tau, \mu}$ containing exactly $j$ elements of the form $b_i$ (i.e. where exactly $j$ of the marked points are non-singular). The remaining $k - j$ labels $b_i$ then each appear together with a label from $\mu$. If we think of $\mu$ as piles of blocks, then for each of these labels we’re picking one of the piles to take it from, and if we denote by $\mu' \in Q$ the piles obtained by removing these labeled blocks and squeezing the piles together, we have that $m(\gamma) = m(\mu' \cdot \tau)$. In particular, instead of summing over $\mu$ and the ways to remove $k - j$ labeled blocks, we can sum over $\mu'$ and ways to insert $k - j$ labeled blocks. Each time we add a labeled block to a partition of length $m$ in $Q$, there are $m$ ways to add it without increasing the length (put it on top of a pile), and $m + 1$ ways to add it while increasing the length by 1 (put it between any two piles, to the left of the first pile, or to the right of the last pile). Iterating, we find that if we add an even number of blocks, then there is 1 more way to do it resulting in a partition with the same length (number of block piles) parity than ways to do it resulting in a partition with the opposite length parity, and if we add an odd number of blocks, then there is 1 more way to do it resulting in a partition with the opposite length parity. Thus, we find
\[
\sum_{j=0}^{k} \sum_{\mu \in \mathcal{Q}} -1^{\|\mu\|} \sum_{\gamma \in S_{\tau, \mu, j}} \left[ \text{Conf}^\gamma Y \right] z^j s^{\gamma|j| - j} = \sum_{\mu' \in \mathcal{Q}} -1^{\|\mu'\|} \text{Conf}^\mu \tau \sum_{j=0}^{k} \binom{k}{j} \cdot (-1)^{k-j} z^j s^{\mu|k-j}} = (z-s)^k \sum_{\mu' \in \mathcal{Q}} -1^{\|\mu'\|} \text{Conf}^\mu \tau s^{\mu'}
\]

In the notation of [24, 3.9],
\[
Z_{Y, \tau}^{-1}(s) := \sum_{\mu' \in \mathcal{Q}} -1^{\|\mu'\|} \text{Conf}^\mu \tau s^{\mu'},
\]
and by [24, Remark 3.24],
\[
Z_{Y, \tau}^{-1}(s) = \frac{s^k}{(1-s)^k} \cdot Z_{Y}^{-1}(s) \cdot ([Y]([Y]-1)\ldots([Y]-k+1).
\]

Thus, we have shown
\[
\sum_{\mu \in \mathcal{Q}} -1^{\|\mu\|} \sum_{\gamma \in S_{\tau, \mu}} \left[ \text{Conf}^\gamma Y \right] z^b(\gamma) s^{\gamma|b(\gamma)} = (z-s)^k \frac{1}{s^k} Z_{Y, \tau}^{-1}(s)
\]
\[
= \frac{(z-s)^k}{(1-s)^k} \cdot Z_{Y}^{-1}(s) \cdot ([Y]([Y]-1)\ldots([Y]-k+1).
\]

If we divide by $Z_{Y}^{-1}(s)$ and substitute in
\[
s = \mathbb{L}^{-(n+1)}, \quad z = \mathbb{L}^{-1},
\]
then, comparing with Equations (6.1.5) and (6.3.2), we obtain Equation (6.3.3) as desired.

**Appendix A. Algorithms and Explicit Computations.**

In this appendix we give an algorithm for computing the explicit universal formulas in Theorem B, extracted from the proof in Subsection 5.2. With minor modifications, this also computes the analogous limits in Theorem C implied by Conjecture E.

**Algorithm.**

**Input:** A partition $\sigma$ and dimension $n$.

**Output:** A universal formula computing
\[
\lim_{d \to \infty} q^{-\dim U_d} \sum_{i} (-1)^i \text{TrFrob}_q \left( H^*_* (U_d, \mathbb{F}_q) \right) \nabla_{\sigma}^{\text{van,}d, \text{Q}_l}
\]
for polarized smooth projective varieties $Y/\mathbb{F}_q$ of dimension $n$ in terms of symmetric functions of the eigenvalues of Frobenius acting on the cohomology of $Y$.

**Step 1:** As $n-1 = 0$, is even positive, or odd, compute the appropriate Schur polynomial $s_\sigma$ of Proposition 3.0.1 as a binomial polynomial in the $p_k'$. 
Step 2: For each “binomial monomial”

\[
\left( \frac{p}{l} \right) = \prod \left( \frac{p_k}{l_k} \right)
\]

appearing in this expression, substitute the coefficient of \( t_k^{l_k} \) appearing in

\[
\prod_k \left( 1 + \frac{q^{nk} - 1}{q^{(n+1)k} - 1} t_k^{(r'_e,Y[Y])} \right)^{(p'_e,Y[Y])} \cdot (1 + t_k)^{-(p'_e,Y[Y^\text{old}]})
\]

Step 3: Multiply the resulting expression by \( \zeta_Y(n+1) \). (Here we use that this is the alternating product of the characteristic series of Frobenius acting on the cohomology of \( Y \), evaluated at \( q^{-(n+1)} \)).

Remark A.0.4. Usually it is more convenient to omit Step 3, which corresponds to normalizing by \( \#U_d(F_q) \) instead of \( q^{-\dim U_d} \).

Example A.0.5. For the standard representation, corresponding in all cases to the partition \((1)\), we obtain

\[
E_{\mu_x} [s_r,Y] = E_{\mu_x} [p'_1 - (p'_1,Y[Y^\text{old}])]
\]

\[
= \frac{q^n - 1}{q^{n+1} - 1} \#Y(F_q) - \text{TrFrob}_q \subset [Y^\text{old}].
\]

For example, if \( Y = \mathbb{P}^n \) this gives,

\[
E_{\mu_x} [s_r,\mathbb{P}^n] = \frac{q^n - 1}{q^{n+1} - 1} \#F_q^n - \#F_q^{n-1} = 0.
\]

By Theorem 6.3.3, we obtain a similar result for Hodge structures (cf. Example 1.1.1).

We note that for the standard representation, \( H^1 \) is known to stabilize by Nori’s connectivity theorem [19, Corollary 4.4], and our results are compatible with the stable values that appear.

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Department of Mathematics, University of Chicago, Chicago IL 60637
E-mail address: seanpkh@gmail.com