Unitarity at small Bjorken $x$

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This paper presents a solution to the nonlinear small $x$ “projectile side” evolution equations as derived by Balitskii in 1996. The solution is based on functional Fokker-Planck methods. The fixed point at small $x$ is explicitly calculated and all correlation functions in this limit are determined. They show clear saturation and unitarization properties. Scaling laws that hold during the saturation phase and throughout the whole course of the evolution are established. The corresponding Langevin equations are given as a basis for numerical simulations opening the field for future studies of dynamical issues of the evolution not analytically accessible. The methods used may be extended to the “target side” equations of Jalilian-Marian, Kovner, Leonidov and Weigert.

1 Introduction

Recent years have seen a growing activity centered around systems and experiments involving large numbers of partons. Experimental activities contributing to this newly focussed interest have been conducted and are planned with widely varying focus such as the deep inelastic scattering (DIS) experiments at HERA, the heavy ion experiments at CERN with LHC and RHIC at BNL. One common denominator is that at all these experiments, be it because of high energies in $ep$ collisions or because of intrinsically higher numbers of participants in $eA$ and $AA$ collisions, QCD effects are dominated by the large numbers of gluons involved. This fact will clearly gain importance as the energy or the size of the participants grows with newer experiments at RHIC and LHC.

As a consequence, one has to face up to the presence of these many degrees of freedom if one wants to understand the qualitative changes in the behavior of these systems as compared to lower energy and smaller targets. The main new feature expected is a “saturation of gluon densities” with implications for experiment ranging from global features such as unitarization effects in cross sections to their more subtle and harder to measure aspects in exclusive measurements.

There have been many attempts to calculate, or at least to formulate the tools to do so, the energy dependence—in the guise of Bjorken $x$ dependence—of cross sections and other observables as well as their $A$ dependence. Characteristic for all these attempts is the basic fact that they, by necessity, use nonlinear features of QCD to incorporate high density effects. This activity has reached a point at which $eA$ options for the HERA and RHIC programs are being actively considered.

Still, besides many qualitative steps forward that all seem to confirm the generic ideas of “gluon saturation,” the theoretical understanding remains limited. Partly this is due to the limited information presently available from experiment. The bulk of data used today is concerned with the total DIS cross sections and was measured at HERA. Although these are perfectly compatible with very tightly constrained saturation fits [1, 2], they are not fully conclusive, since conventional interpretations of the same data using Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution appear to be equally successful in reproducing the general trends in the data. This latter evolution is intrinsically linear and thus has no knowledge of the density effects important for unitarization and saturation. However, in comparing with experiment, the DGLAP machinery has been applied down to very small momentum transfers $Q^2$ and serious conceptual criticism on the use of these equations so far beyond their region of validity is in order. Nevertheless, the fact remains that it has not
been conclusively shown that DGLAP evolution is violated in the present data. What one needs then is further experiments to provide the means to see DGLAP evolution fail and to conclusively demonstrate the presence of nonlinear effects, rather than just their compatibility with available data. The main effort on the theoretical side, however, has to focus on more quantitative treatments of nonlinear effects at small $x$ and/or large $A$.

Theoretical approaches which may form the basis for such endeavors have been formulated using widely different techniques. This often obscures their basic similarities and hinders comparison. To list but a few, nonlinear small $x$ evolution equations have been presented by Mueller et al. [3,4], Balitskii [5,6], Kovchegov [7], Gribov et al. [8], Levin et al. [9–11], Bartels and Ewerz [12], as well as Jalilian-Marian, Kovner, Leonidov and Weigert (JKLW) [13–17]. All these equations generalize the linear Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the gluon distribution to nonlinear—often infinite systems of coupled—equations for more general objects. They all show saturation and unitarization behavior at small $x$, although this often has only been demonstrated in simple limits such as the double logarithmic approximation.

The relationships between many of these approaches has recently been investigated in Ref. [18] which has finally put them on the same map. It had been known that the dipole approach of Mueller’s as applied by Kovchegov to DIS [7] was contained in Balitskii’s approach to small $x$ evolution. Ref. [18] has allowed to identify [5,7] as a limiting case of the JKLW approach. This is not to say that the JKLW approach or Balitskii’s equations have superseded the “simpler” evolution equations, quite to the contrary, there are clearly regimes in which the latter do capture the important physics. Being simpler, they will play their rôle in analyzing what is happening in new experiments.

Moreover, although all these nonlinear equations have been received with interest, they have all mostly remained in the virgin state of a newly derived equation, without much knowledge about their solutions and concrete applications to physical problems. The more general and hence complicated the set of equations, the more this is true. The use of the more general approaches has up to now been—to some extent—more as a tool to organize simpler variants into genealogical trees than to provide new physics insight. It is simply the sheer complexity of the full nonlinear equations of Refs. [5] and [13–17] that has hindered progress. To fully appreciate the scale of the problem, suffice it to say that they all result in infinite hierarchies of coupled equations for QCD correlators. Solving those is tantamount to solving a nonlinear field theory, albeit a bit simpler than QCD itself. However, the very way this “field theory” is presented has also precluded any numerical work that does not take recourse to truncations and other simplifications. Such approximations, however, would inherently limit the value of any results obtained, as it would at least partly eliminate the feature introduced with great care to capture the density effects, the nonlinearities themselves.

This paper tries to tear down the brick wall which made progress come so slow. As it turns out, there exist methods to completely solve at least one of these infinite sets of evolution equations, the equations given in Ref. [5]. As will become clear while doing so, the same methods should provide a good starting point to also tackle the somewhat more complex JKLW equations.

In the early stages of rederiving the evolution equations of Ref. [5] in preparation for Ref. [18] it became clear that there is a much more compact way of presenting the results, quite analogous to the small $x$ Wilson RG approach of JKLW, but with far less involved evolution kernels. It is this relative simplicity that makes these evolution equations the ideal candidate to search for further insight into their fixed point structure and to lay the groundwork for future numerical studies. The lessons learned in this example will undoubtedly prove useful in the further study of the JKLW equation, but will simultaneously give results for all the limiting cases already contained here. It will, in particular, provide a way to judge their respective regions of validity. The most prominent example in this case being dipole evolution as formulated by Ref. [3,4] and applied to DIS in Ref. [7].

In analogy with the JKLW equation, Ref. [5]’s set of evolution equations can be written in form of a functional Fokker-Planck (FP) equation for a statistical weight that characterizes the expectation values of operators in the wave function of the target. This formulation provides the cornerstone for the progress reported here. It will lead to an understanding of the limiting behavior at small $x$ and at the same time provide a formulation amenable to numerical treatment.

To set the scene for these developments, Sec. 2 sketches a simple rederivation of Balitskii’s infinite set of coupled RG equations for correlation functions at small $x$. In doing so a very brief discussion of the
relationships of the main nonlinear approaches so far in existence is provided.

The rederivation itself has the purpose of emphasizing the basic physics ideas and concepts involved and provides a formulation in terms of a generating functional much more compact than the original version. While this in itself does not add anything to the physics results already known, it provides a convenient framework in which to extract all the known limiting or special cases and restricted versions of such evolution equations in a very transparent way. A short discussion on how to do this for a few select cases and of the physics implications of the approximations involved is given in Sec. 3.

Sec. 4 steps beyond the known results by converting the RG equation for the generating functional into an equation for a statistical weight, a formulation very similar to that used in the approach of JKLW \[13–17\].\footnote{For a more thorough comparison of these two approaches see [18].} Here emphasis is put on the fact that the resulting equation is of the form of a functional Fokker-Planck (FP) equation. This allows to interpret the statistical weight as a probability function(al) (in a very concrete sense) for the relevant operators at small \(x\) and to formulate the question of the existence and nature of a fixed point of the RG equation in terms of what one would normally call the equilibration properties of the FP operator involved. This indeed turns out to be a very powerful analogy that allows to identify the small \(x\) asymptotics of this evolution equation in full generality and at the same time paves the way for a new numerical approach. All subsequent results are based on this fact. It is obtained in two separate steps, the first, given in Sec. 5, is to find that there is no “restoring” force and hence one is dealing with a case of generalized Brownian motion. In a second step it is shown in Sec. 6 that this indeed corresponds to a case of ordinary Brownian motion for physical initial conditions allowing to eliminate some redundancy of the original presentation and to provide a yet simpler formulation. It is now a trivial consequence that there is a fixed point in this evolution and easy to show that it is attractive: the Brownian process asymptotically fills all available configuration space and one ends up with a very simple uniform statistical weight. In turn this means that in the approximation underlying this evolution, Ref. [18], QCD exhibits universality at small \(x\). Cross sections and other measurable quantities become –in certain features– independent of the target at small enough \(x\). This is a useful result even if it turns out that the approximations break down before the asymptotic limit is reached as it provides a clear overall tendency for the flow of observables within the region of applicability. More than that, the nature of the evolution kernel itself will allow to extract scaling laws that apply also far away from the fixed point and gives rise to predictions wherever there is a window of applicability of the evolution equations as such.

Sec. 7 is devoted to demonstrating one of the main new tools now available: The new approach allows to write down the Langevin equation corresponding to this FP/RG evolution. This on the one hand gives a very intuitive picture for the whole evolution –allowing to deduce scaling laws for the growth of the target and the cross section in the unitarization region. On the other hand one is then in a position to apply numerical simulations in order to study “dynamical” issues such as the rate of approach to the fixed point or, in other words, the practical aspects of how soon the system loses its memory of target specific scales and properties.

Sec. 8 discusses the asymptotic form of the correlators in the unitarization region, given some mild assumptions about the negligibility of edge effects that should be good for sufficiently large targets. This exemplifies the scaling laws found in Sec. 7 and sheds some more light on the unitarization mechanism discussed in the previous sections. This also highlights some of the limitations in extracting quantitative information analytically that can be overcome by numerical simulations, such as the precise way evolution erases details of the initial conditions and reaches the asymptotic region.

Sec. 9 will summarize the results and try to put them into context. Several appendices contain technical and notational odds and ends.

2 A compact summary of Balitskii’s equations

The goal of this section is to set the stage for the fundamentally new results developed starting with Sec. 4 by summarizing the physics ideas behind the small \(x\) evolution mechanism underlying the results of [5]. The presentation in itself, however, is new and has not been given in this form before. It provides for a very
A concise way of laying out the calculations involved and leads to the most compact formulation of the results within a single equation. I therefore hope this formulation will provide an easier access point to this field than other already published variants. It certainly makes it much easier to extract limiting cases and provides the simplest possible starting point for the studies laid out below.

To begin with a physics motivation, I will take recourse to the paradigm example for saturation questions in small $x$ physics, deep inelastic scattering (DIS) of virtual photons on hadronic targets of any size. The problem is set up in the frame in which the photon fluctuates into an energetic quark-antiquark pair long before it reaches the target, but where most of the energy resides in the target hadron which moves very fast. The scattering of the quark-antiquark pair is dominated by its interaction with the gluons in the target. Since the target hadron moves fast, the time evolution of the gluon fields is slowed by Lorentz time dilation. Also, due to Lorentz contraction, the gluon fields are well localized in the plane perpendicular to the direction of motion, which is taken to be the positive $x$ axis. The target can, therefore, be modeled by a distribution of static gluon fields localized at $x^- = 0$. As the scattering energy increases ($x$ decreases) the gluon fields of the target change due to contributions of quantum fluctuations. It is this evolution in $x$ of the hadronic ensemble that one intends to describe in terms of the evolution equation.

It proves convenient to use the light cone gauge $A^- = 0$. In this gauge, loosely following Ref. [5], one takes the vector potentials representing the relevant gluon field configurations to be of the form

$$A = b + \delta A \quad \text{with} \quad b^i = 0, \quad b^+ = \beta(x)\delta(x^-).$$

(1)

The question if this form is the only relevant one is in fact nontrivial and determines the region of validity of the present approach. This is discussed in detail in [18].

The DIS structure function (which determines the cross section) can be written in the following general form

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int \frac{dz dx dy}{4\pi} \Phi(x - y, z) N(x, y, y).$$

(2)

Here, $x$ and $y$ are the transverse coordinates of the quark and the antiquark in the pair, $z$ is the fraction of the pairs longitudinal momentum carried by the quark and $y$ is the rapidity of the slowest particle in the pair. Also, $\Phi(x - y, z)$ is the square of the “wave function” of the photon — the probability that the virtual photon fluctuates into the pair with given coordinates and momenta — and $\int d^2(x + y)N(x, y, y)$ is the cross section for the scattering of the pair. This formula is best illustrated diagrammatically through

$$F_2 \sim.$$  

(3)

The layout is chosen such as to reflect the predominant space time geometry of the collision in the kinematical situation at small $x$, with the $x^+$ axis going from lower left to upper right and $x^-$ from the lower right to upper left.

The wave function $\Phi$ is well known. It is given for example in Ref. [7], but its explicit form will not be of interest here. The focus of the discussion here will be on the “dipole” scattering cross section $\int N$ and its generalizations. If the quark-antiquark pair is energetic enough the scattering cross section is eikonal and determined via the scattering amplitude

$$N(x, y) = \text{tr} \langle 1 - U(x)U(y) \rangle_A,$$

(4)

\footnote{Note that the average over $A$, denoted by $\langle \ldots \rangle_A$, implies the inclusion of higher order gluonic corrections that are expected to be important at small $x$. It is their influence one tries to trace in the small $x$ evolution equation to be derived below.}

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perturbative corrections to expectation values like the above. Being interested in small carry out such averages explicitly without solving QCD, so the best to hope for is to being able to calculate of the target as indicated by the average over relevant configurations split into more than just a single q pair. This will also be the case for all the fields in the rest of this section. For simplicity, the \( x^+ \) coordinate will be suppressed in the following.

In fact this “eikonalization” of operators related to the probe at small \( x \) is completely generic, and so, in order to describe correlators appearing in any expectation value with the above small \( x \) characteristic features, it is sufficient to know the corresponding correlations of \( U^{(1)} \) operators.

So, to abstract from the concrete example of DIS, one in general is trying to calculate expectation values of the form

\[
\langle U_{\omega_1} \otimes \ldots \otimes U_{\omega_n} \rangle_A
\]

with \( U^{(1)} \) in the fundamental representation. To picture this, just imagine a probe like in Eq. (3) that has split into more than just a single \( q\bar{q} \) pair. This again is to be averaged over the field in the wave function of the target as indicated by the average over relevant configurations \( \langle \ldots \rangle_A \). Clearly one can not expect to carry out such averages explicitly without solving QCD, so the best to hope for is to being able to calculate perturbative corrections to expectation values like the above. Being interested in small \( x \) then means one would like to find \( \alpha_s \ln(1/x) \) corrections to scattering events.

This can be done in two complementary ways (or of course any suitable combination thereof). One may either try to directly calculate corrections to the target’s characteristic field \( A \) (for each member of the ensemble separately) and this way find the change to Eq. (6) at small \( x \), or one may try to find how the expectation values change by trying to modify the wave function of the projectile, which in this setting is the \( U^{(1)} \) content thereof. The former approach leads to the JKLW equation, the latter to [5] and is also the route that I will follow here.

To be a bit more precise, there is a difference in the two approaches concerning the technical setup which, although it seems innocent enough at first sight, will lead to differences in the type of nonlinearities one is able to include in the end. That difference, although not inherent in writing down Eq. (6) nevertheless appears quite naturally and has in fact been used in Refs. [5–7, 18]. Let me first describe the situation and then clarify why it is fairly hard –on purely technical grounds– to step beyond these limitations.

Any attempt to write down evolution purely in terms of \( U^{(1)} \) correlators appears most natural if combined with an asymmetric picture of the evolution: One allows that the target’s field, which is responsible for the nontrivial exponent of the link factors may be large. At the same time the field generated by the projectile is assumed to be small, it plays no rôle in finding the \( U^{(1)} \) correlators. Clearly such an assumption would break down if evolution were to generate large fields on the projectile side. As it will turn out gluons are produced well spread out over rapidity and hence the projectile’s field may be considered to be small throughout. There is another problem, however, that will become crucial as projectile rapidities approach target rapidities and the two objects –components in the tails of their wavefunctions really– cease to be distinguishable. Then this picture will clearly break down. Evolution in the JKLW sense, on the other hand, evolves the large field of the target into an even larger field, taking all the nonlinearities into account already in the emission process. Technically speaking the fluctuations propagator around the dominant background field configurations contains additional nonlinearities that go beyond the “independent emission” scenario of the weak field perspective imposed on the projectile side.

\[ U_{ab} = 2t \epsilon [\epsilon U^{(1)} U^\dagger]. \]

\( ^3 \)This is completely general as one may write any adjoint links as a combination of two fundamental ones by virtue of

\[
\langle U(x^+ = 0, x) = \mathcal{P} \exp \left[ -ig \int_{-\infty}^{+\infty} dx^- A^+(x^+ = 0, x, x^-) \right] \]

\((y\text{ dependence suppressed})\) where \( U (U^\dagger) \) is the eikonal phase for the scattering of the energetic quark (antiquark)
To overcome this limitation in the projectile evolution picture, one would have to allow both fields—the target’s and the projectile’s—to be large. Due to the nonlinear nature of QCD this would require to determine the relevant dominant field configurations due to two strong sources and find the fluctuations propagator around such configurations, a feat of which not even the first step has been accomplished analytically.

From this perspective, there are three degrees of complications with correspondingly increased amounts of nonlinearities included: Evolution from a projectile with a weak field in the presence of a strong target field as done in [5–7, 18]; evolution from the target with a strong field as in the JKLW case and evolution with both fields strong, a case which has not yet been looked at in earnest.

Returning from this general discussion of the scope and content of different approaches to matters at hand, let me continue to set up the calculation in the projectile evolution perspective. Here the idea is to calculate small $x$ corrections to correlators of the type Eq. (6) and infer their change from the resulting equations, without explicitly tracing how the target fields change as one proceeds.

As one is interested in the behavior of arbitrary correlators of the form Eq. (6) it would be most economical to find the generic behavior of the generating functional of such objects

$$\mathcal{Z}[J^1, J] := \langle e^{\mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J]} \rangle_b. \tag{7}$$

Here

$$\mathcal{S}^{q\bar{q}}_{\text{ext}}[A, J^1, J] = \int d^2 x \left\{ \text{tr} \left( (J^1_x)^t U_x [A^+] \right) + \text{tr} \left( J^1_x U_x^t [A^+] \right) \right\} \tag{8}$$

is an external source term, correlators (Eq. (6)) are extracted via $J^{(1)}$ derivatives and the $(\ldots)_A$ average is taken with the QCD action.

So far the physics of small $x$ is encoded in the type of correlation functions considered—exclusively correlators of link operators $U^{(1)}$—all the rest is mathematical convenience. The sole purpose of this machinery is to get a clearer, more streamlined picture of what one can do to calculate these correlators and how to do it in practice. Simple, diagrammatic interpretations are close at hand.

As it is obviously impossible to perform the field average in one go and the form of $\mathcal{S}^{q\bar{q}}_{\text{ext}}[A, J^1, J]$ has the small $x$ limit already built in, one will look at Eq. (7) expanding the gluon field $A$ around $b$ (whose correlators are assumed to be known) with fluctuations only taken into account to order $\alpha_s \ln(1/x)$. That is to say that one will expand around $b$ to one loop accuracy and select the terms carrying a $\ln(1/x)$ factor. This way one will be able to infer the change of correlation functions as one lowers $x$. This is the second statement about physics or rather what one can learn about it through such an approach.

Now turn back to the actual calculation. At one loop one needs at most second order in fluctuations:

$$\langle e^{\mathcal{S}^{q\bar{q}}_{\text{ext}}[b + \delta A, J^1, J]} \rangle_{b, \delta A} = \langle 1 + \delta A_x \frac{\delta}{\delta b_x} + \frac{1}{2} \delta A_x \frac{\delta}{\delta b_x} \delta A_y \frac{\delta}{\delta b_y} + O(\delta A^3) \rangle e^{\mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J]}_{b, \delta A} = \langle e^{\mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J]} \rangle_b + \frac{1}{2} \langle \delta A_x \frac{\delta}{\delta b_x} A_y \rangle_{\delta A} [b] \times \left( 2 \frac{\delta}{\delta b_x} \mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J] \frac{\delta}{\delta b_y} \mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J] + \frac{\delta}{\delta b_x} \frac{\delta}{\delta b_y} \mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J] \right) \times e^{\mathcal{S}^{q\bar{q}}_{\text{ext}}[b, J^1, J]}_b + \ldots \tag{9}$$

$^4$Note that I purposely write $\text{tr} A^t B$ to unclutter notation in what follows, not $\text{tr} A B$, having in mind that $\frac{\delta}{\delta A_{ij}} \text{tr} A^t B = \frac{\delta}{\delta A_{ij}} A_{kl} B_{kl} = B_{ij}$ while the other variant would give $B_{ji}$. Also, $x^+$ has been suppressed, assuming all $U$’s to be located at $x^+ = 0$ as discussed above.

$^5$In writing Eq. (9) one has anticipated that $\langle a^+_i \rangle_{\delta A} [b] = 0$ as in the free case. This is a consequence of the structure of the propagator in this background field and has been used repeatedly [5, 18].
The second term is the calculable perturbative correction to $\bar{Z}[J^1, J]$ the generating functional for all generalized distribution functions of the target, an object which can not be calculated as such with present tools. Arbitrary correlators of course are extracted according to

$$\begin{align*}
\delta \frac{\delta S_\text{ext}^{\bar{q}q}[U, U^\dagger, J^1, J]}{\delta J} & = U^\dagger e^{S_\text{ext}^{\bar{q}q}[U, U^\dagger, J^1, J]}, \\
\delta \frac{\delta S_\text{ext}^{\bar{q}q}[U, U^\dagger, J^1, J]}{\delta J^I} & = U e^{S_\text{ext}^{\bar{q}q}[U, U^\dagger, J^1, J]}
\end{align*}$$

(10a, 10b)

or

$$\frac{\delta}{\delta J_1 \ldots \delta J_n \delta J_1^I \ldots \delta J_m^I} \bar{Z}[J^1, J] = \langle U_1^\dagger \otimes \ldots \otimes U_n^\dagger \otimes U_1 \otimes \ldots \otimes U_m \rangle_b .$$

(10c)

To understand the individual terms in Eq. (9) in detail requires the calculation of the one loop corrections to path ordered exponentials $U[b]^{(1)}$ in the presence of a background field $b$ of the form Eq. (1). Before diving into this it may be helpful to specialize once more to the DIS example from above to illustrate the physics content of the terms in Eq. (9). Here one needs to look at the averaging procedure $\langle \ldots \rangle_b$ is $x$ dependent as the average over $\delta A$ brings in $1/x$ dependent terms denoted by “quantum corrections”, a special case of the second term in Eq. (9). This sets the task: If one is able to calculate these quantum corrections before taking the $b$ average, that is, for all relevant $b$, one can deduce how the $U^{(1)}$ correlation functions or, as will become clear in Sec. 4, the weight $Z[U, U^\dagger]$ that defines $\langle \ldots \rangle_b$ evolves with $x$.

Clearly there are two generic types of corrections corresponding to the two terms in

$$\begin{align*}
\frac{1}{2} \langle \delta A_x \delta A_y \rangle_{\delta A} \bigg| & \frac{\delta}{\delta b_x} S_\text{ext}^{\bar{q}q}[b, J^1, J] \frac{\delta}{\delta b_y} S_\text{ext}^{\bar{q}q}[b, J^1, J] + \frac{\delta}{\delta b_x} \frac{\delta}{\delta b_y} S_\text{ext}^{\bar{q}q}[b, J^1, J] \bigg| \langle 1 - U_x[b] U_y[b] \rangle_b \\
\times e^{S_\text{ext}^{\bar{q}q}[b, J^1, J]} .
\end{align*}$$

(12)

The first term represents contributions where the gluon propagator connects two quarks ($U$s) or antiquarks ($U^\dagger$s) as well as a quark to an antiquark. In addition there are pure self energy corrections dressing one quark or antiquark line instead of connecting two of them, represented by the second term.

Particular correlators are again selected taking any number of $J$ derivatives and setting $J$ to zero. One point functions get contributions from the second term only, while both terms contribute to anything with more than two $J^{(1)}$ derivatives. Clearly it is sufficient to calculate

$$\begin{align*}
\frac{1}{2} \frac{\delta}{\delta J_2^{(1)}} \frac{\delta}{\delta J_y} \bigg|_{J^{(1)}=0} & \frac{1}{2} \langle \delta A_u \delta A_v \rangle_{\delta A} \bigg| \frac{\delta}{\delta b_u} S_\text{ext}^{\bar{q}q}[b, J^1, J] \frac{\delta}{\delta b_v} S_\text{ext}^{\bar{q}q}[b, J^1, J] \\
\times \left( 2 \frac{\delta}{\delta b_u} S_\text{ext}^{\bar{q}q}[b, J^1, J] + \frac{\delta}{\delta b_u} \frac{\delta}{\delta b_v} S_\text{ext}^{\bar{q}q}[b, J^1, J] \right) \bigg|
\end{align*}$$

(13)

and

$$\frac{\delta}{\delta J_x} \bigg|_{J^{(1)}=0} \frac{1}{2} \langle \delta A_u \delta A_v \rangle_{\delta A} \bigg| \frac{\delta}{\delta b_u} S_\text{ext}^{\bar{q}q}[b, J^1, J] \frac{\delta}{\delta b_v} S_\text{ext}^{\bar{q}q}[b, J^1, J]$$

(14)

to completely reconstruct Eq. (12) as these terms are precisely second respectively first order in $J^{(1)}$. Eq. (12) then defines the change in all other correlators and one has reached the goal of finding the fluctuation induced corrections.

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\[\text{After integration over impact parameter (simply } x + y, \text{ if one puts the target at the origin in transverse space) this is the dipole cross section that features prominently in calculations of } \gamma^* p \text{ or } \gamma^*A \text{ cross sections at small } x \text{ [7, 19, 20].}\]
The task is clear and the terms appearing in the actual calculation are best visualized diagrammatically. For the gluon exchange diagrams corresponding to Eq. (13) one defines

\[ \alpha_s \ln(1/x) \cdot \bar{\chi}^{qq}_{xy} := \quad \text{b} \quad ; \quad \alpha_s \ln(1/x) \cdot \bar{\chi}^{\bar{q}q}_{xy} := \quad \text{b} \] (15)

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As already seen in [5,18], these split naturally into \( x^- \) ordered contributions when one combines the structure of the vertices (the \( b \) derivatives of \( S_{\text{ext}} \) in Eqs. (13) and (14)) and the gluon propagator in the background field. Take \( \bar{\chi}^{qq} \) as an example:

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\[ \mathbf{b} \quad = \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad . \] (16)

The self energy corrections corresponding to Eq. (14) and their \( x^- \) ordered sub-contributions are given by

\[ \alpha_s \ln(1/x) \quad \bar{\sigma}_x^q = \quad \mathbf{b} \quad = \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad . \] (17a)

\[ \alpha_s \ln(1/x) \quad \bar{\sigma}_x^{\bar{q}} = \quad \mathbf{b} \quad = \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad + \quad \mathbf{b} \quad . \] (17b)

Again, the various components of \( \bar{\chi} \) correspond to gluon exchange diagrams while the \( \bar{\sigma} \)s are self energy corrections. As for the time ordered ones, there are those with and without interaction with the target (with the gluon line crossing the \( x^+ \) axis or not respectively as shown in Eqs. (16) and (17)). These latter versions determine the \( U^{(1)} \) dependence of \( \bar{\chi} \) and \( \bar{\sigma} \) in detail. They have been calculated several times using different methods. The common feature of all of them is that diagrams without target interaction leave the number of \( U^{(1)} \)s invariant while those with target interaction insert an additional adjoint \( U^{(1)} \) at the point of interaction. This implies a nonlinearity in the evolution: Any correlator with a fixed number of fundamental representation \( U^{(1)} \)s will couple to other correlators with two additional factors of fundamental representation \( U^{(1)} \)s in each step in \( x \).

It is probably most efficient to read off the results for \( \bar{\chi} \) and \( \bar{\sigma} \) from Ref. [5] or [18]. For convenience, define the integral kernel

\[ K_{xyz} := \frac{(x-z)(z-y)}{(x-z)(y-z)^2} = K_{yzx} . \] (18)

\[ \text{Representations for } \bar{\chi}^{qq}_{xy}, \bar{\chi}^{\bar{q}q}_{xy} \text{ and } \bar{\chi}^{\bar{q}q}_{xy} \text{ result from reversing the quark lines accordingly.} \]
This allows to write explicit expressions for the components of \( \tilde{\chi} \)

\[
\alpha_s [\tilde{\chi}_{xy}]_{ij kl} := \frac{\alpha_s}{2\pi^2} \int d^2 z \ K_{xzy}
\]

\[
\times \left( [U_x U_y]_{ij} [U_x U_y]_{k} - [U_x U_y]_{ij} [U_x U_y]_{k} - [U_x U_y]_{ij} [U_x U_y]_{k} \delta_{ij} - \delta_{ij} [U_x U_y]_{ij} \right)
\]

\[
\times \left( [U_x U_y]_{ij} [U_x U_y]_{k} + [U_x U_y]_{ij} [U_x U_y]_{k} - [U_x U_y]_{ij} [U_x U_y]_{ij} \right)
\]

\[
\times \left( [U_x U_y]_{ij} [U_x U_y]_{k} - [U_x U_y]_{ij} [U_x U_y]_{ij} \right)
\]

\[
\times \left( [U_x U_y]_{ij} [U_x U_y]_{k} - [U_x U_y]_{ij} [U_x U_y]_{ij} \right)
\]

as well as those of \( \tilde{\sigma} \):

\[
\alpha_s [\tilde{\sigma}]_{ij} := - \frac{\alpha_s}{2\pi^2} \int d^2 z \ K_{xzy} \left( [U_x]_{ij} \text{tr}(U_x U_y) - N_c [U_x]_{ij} \right)
\]

\[
\alpha_s [\tilde{\sigma}]_{ij} := - \frac{\alpha_s}{2\pi^2} \int d^2 z \ K_{xzy} \left( [U_x]_{ij} \text{tr}(U_x U_y) - N_c [U_x]_{ij} \right)
\]

At this point all the calculational effort pays off: Using the above definitions for \( \tilde{\chi} \) and \( \tilde{\sigma} \), and remembering that

\[
\langle e^{S_{\text{ext}}_{[b,J^1,J]} } \rangle_{b,J^1,J}
\]

\[
+ \frac{1}{2} \langle \delta A_x \delta A_y \rangle_{b,J^1,J} \left( - \frac{1}{2} \delta_{x} S_{\text{ext}}_{[b,J^1,J]} + \delta_{y} S_{\text{ext}}_{[b,J^1,J]} \right)
\]

\[
\times e^{S_{\text{ext}}_{[b,J^1,J]} } + \ldots
\]

\[
= \alpha_s \ln \left( \frac{x}{x} \right) \left\{ \left( \frac{1}{2} \right) \left[ \tilde{\chi}_{x} \tilde{\chi}_{y} \right] \right\} \left( U^J \right) \left( U^{J^1} \right)
\]

\[
(21)
\]

The logarithm comes from an integration over \( \delta A \) with momenta in a finite interval of \( x \) values. Accordingly the equation has the form of a finite difference equation for \( \tilde{Z}[J^1,J] \) with respect to \( \ln 1/x \).

This gives rise to the final RG equation in differential form, which I write using condensed notation with \( J = (J^1,J) \) and \( \frac{d}{dx} = (\frac{d}{\delta x}, \frac{d}{\delta y}) \) as

\[
\frac{\partial}{\partial \ln 1/x} \tilde{Z}[J] = \alpha_s \left\{ \left( \frac{1}{2} J_u J_v \tilde{\chi}_{uw} \frac{\delta}{\delta J_v} + J_u \tilde{\sigma}_{u} \frac{\delta}{\delta J} \right) \right\} \tilde{Z}[J]
\]

(22)

where \( u \) and \( v \) now also stand for \( q \) and \( \tilde{q} \) in addition to color and transverse coordinates. Eq. (22) represents a full set of evolution equations for all correlators of \( U^{J^1} \) that can be extracted by \( J^{(1)} \) derivatives evaluated at \( J^{(1)} = 0 \) as demonstrated in Eq. (11) for the dipole cross section.\(^9\) It is probably worthwhile to recapitulate the surprisingly few steps involved in arriving at the quite complex result displayed in Eq. (22).

- The first step consists of writing down the generating functional of all the correlators important at small \( x \). It is then a small step to recognize that quantum corrections at small \( x \) can be calculated directly as corrections to this object rather than for all the correlators separately.

\(^8\)Note the minus signs in \( K_{xzy} = -1/(x-z)^2 \).

\(^9\)The convention for index contraction again is defined in order to simplify notation. Products are defined as in \( J^1 \tilde{\sigma}^q := [J^1]_{ij} [\tilde{\sigma}^q]_{ij} \) with integration over transverse coordinates implied.
• Simply expanding in fluctuations allows to identify the main players in the evolution $\bar{\chi}$ and $\bar{\sigma}$. Their calculation requires some effort, but also that can be streamlined considerably [18].

• Even before calculating their values, with just their basic presence known, one then may write the RG equation in terms of $\bar{\chi}$ and $\bar{\sigma}$.

The physical content of the RG is equally transparent. Eq. (22) describes an infinite set of coupled RG equations for correlators of $U$ fields averaged over the dominant color fields in the target $G^{+\dagger}[b](w) \leftrightarrow b^+(w) = \beta(w)\delta(w^--)$. As is clear from Eq. (21) one can describe this in two complementary ways: First as an evolution in which the target is left unchanged – the $\langle \ldots \rangle_x$ average stays the same in an evolution step – while the corrections induce correlations between different operators characterizing the projectile wave function. This description has been used many times in the literature. Second, and this interpretation is closer to the spirit of Eq. (22), as an evolution in which the correlators of $U$ operators are changed through an infinite set of coupled RG equations. As one is talking about correlators of $U$ averaged according to $\langle \ldots \rangle_b$ this implies a change in this averaging procedure as $x$ runs. In turn this is nothing but a change of the dominant configurations contributing to this average and hence a change in the target wave function.

Any way one looks at it, the only operators involved are correlators of $U = U[b]$ averaged over $b$. Accordingly, during the entire evolution, one will encounter nontrivial correlators only within the targets $-x$ dependent–radius.

During the course of this work it will become possible to make much farther reaching statements about this evolution. For now I will return from generalities to concrete examples that may help compare individual expressions to Ref. [5] and [7] and should give a better idea of the structures involved when trying to represent Eq. (22) as a system of coupled equations.

- The one point functions evolve according to

\[
\frac{\partial}{\partial \ln 1/x} \delta J_x \frac{\delta}{\delta J_x} \bar{Z}[J] \bigg|_{J=0} = \alpha_s \bar{\sigma}_x \frac{\delta}{\delta J_x} \bar{Z}[J] \bigg|_{J=0}
\]

(23a)

or equivalently, in the language of correlators

\[
\frac{\partial}{\partial \ln 1/x} (U_x)_b = \alpha_s \langle \bar{\sigma}_x [U] \rangle_b
\]

(23c)

or

\[
\frac{\partial}{\partial \ln 1/x} (U^+_x)_b = \alpha_s \langle \bar{\sigma}_x [U] \rangle_b.
\]

(23d)

To emphasize this point once more: since the $\bar{\sigma}$s are nonlinear in $U^{(1)}$ these equations do not close among themselves but couple (by induction) to all other evolution equations for higher n-point correlators.

- An example for the evolution of a two point correlator (there are 3 more generic cases involving the other components of $\bar{\chi}$) is given by

\[
\bar{Z}[J] \bigg|_{J=0} = \alpha_s \left\{ \bar{\chi}_{xy} \frac{\delta}{\delta J_x} + \bar{\sigma}_x \frac{\delta}{\delta J_x} \right\} \bar{Z}[J] \bigg|_{J=0} + \alpha_s \left\{ \bar{\chi}_{xy} \frac{\delta}{\delta J_y} + \bar{\sigma}_y \frac{\delta}{\delta J_y} \right\} \bar{Z}[J] \bigg|_{J=0}
\]

(24a)

or, again equivalently,

\[
\frac{\partial}{\partial \ln 1/x} (U_x \otimes U^+_y)_b = \alpha_s \langle \bar{\chi}_{xy}^\dagger [U] \otimes \bar{\sigma}_y^\dagger [U] \rangle_b.
\]

(24b)

This last equation contains the evolution of the dipole cross section alluded to in Eq. (11), a connection that will be explored in somewhat closer detail in Sec. 3.

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10This can not be true down to arbitrarily small $x$ where projectile and target strongly overlap and limits the region of applicability of this particular RG equation, see [18].
3 Limiting cases

The full RG equation is quite general and contains various limiting cases. Two distinct types are of particular interest,

- the first one which still captures saturation and unitarization effects, the dipole evolution for $\gamma^*A$ collisions based on a large $N_c$ argument and
- a second regime in which expansions at small “densities” simplify the equations. This may be called the BFKL limit. For obvious reasons this regime will only be useful to gain insight at moderately small $x$ before saturation effects set in.

To turn to the nonlinear $N_c$ dipole evolution for $\gamma^*A$ collisions first, recall Eq. (24b). It allows to extract the equation for the dipole cross section without much effort. One reads off that

$$\left[\hat{\chi}_{x y}^{q q}(U)\right]_{i j} = \frac{1}{2\pi^2} \int d^2 z \left[ \left(\frac{1}{x-z} \right)^2 \left(\frac{1}{y-z} \right)^2 \right]_{i j} \times 2 \left[ \text{tr}(U_{x}^{1} U_{y}^{1}) \text{tr}(U_{y}^{1} U_{z}^{1}) - N_c \text{tr}(U_{x}^{1} U_{z}^{1}) \right]$$

while the sigma contributions are given by

$$\left[\hat{\sigma}_{x}^{q}(U)\right]_{i j} \left[\hat{\bar{\sigma}}_{y}^{q}(U)\right]_{i j} = \frac{1}{2\pi^2} \int d^2 z \left[ \left(\frac{1}{x-z} \right)^2 \left(\frac{1}{y-z} \right)^2 \right]_{i j} \times 2 \left[ \text{tr}(U_{x}^{1} U_{y}^{1}) \text{tr}(U_{y}^{1} U_{z}^{1}) - N_c \text{tr}(U_{x}^{1} U_{z}^{1}) \right]$$

Adding these according to Eq. (24b) leads to

$$\frac{\partial}{\partial \ln 1/x} \left(\text{tr}(U_{x}^{1} U_{y}^{1})\right)_{b} = \alpha_s \left(\hat{\chi}_{x y}^{q q}(U) + \hat{\sigma}_{x}^{q}(U) U_{y}^{1} + U_{x}^{1} \hat{\bar{\sigma}}_{y}^{q}(U) \right)_{b}$$

This equation exhibits the well known dipole kernel [3,4] and turns into known closed evolution equations, if one

- factorizes at large $N_c$ according to $\left(\text{tr}(U_{x}^{1} U_{y}^{1})\right)_{b} \xrightarrow{N_c \to \infty} \langle \text{tr}(U_{x}^{1} U_{y}^{1}) \rangle_{b} \langle \text{tr}(U_{z}^{1} U_{z}^{1}) \rangle_{b}$ to decouple the equation for the two point correlator from higher order ones to close the equation.
- expresses the resulting equation in terms of the dipole scattering amplitude $N_{x y} := \langle \text{tr}(1-U_{x}^{1} U_{y}^{1}) \rangle_{b}/N_c$.

As a result the RG equation of Ref. [6,7] for DIS emerges

$$\frac{\partial}{\partial \ln 1/x} N_{x y} = \alpha_s N_c \int d^2 z \left[ \left(\frac{1}{x-z} \right)^2 \left(\frac{1}{y-z} \right)^2 \right] \left( \text{tr}(U_{x}^{1} U_{y}^{1}) - N_c \text{tr}(U_{x}^{1} U_{z}^{1}) \right)$$

This equation has hallmarks of an equation that incorporates saturation effects. Schematically this is quite simple: Starting from small $N$, the linear terms are the only relevant ones and grow according to the BFKL kernel they are subject to. At the same time the $N^2$ term grows as well and will catch up with the linear one, at which point one has reached saturation. Although transverse dynamics will be important for the details of when this happens, it will not alter this basic mechanism.

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11Note that the cancellations leading to Eq. (25d) provide a cross check on the relative sign of the $\hat{\chi}^{qq}$ and both the $\hat{\sigma}$ contributions via [7]. Unfortunately the relative signs of the other $\hat{\chi}$ components are not fixed by this, but there will be a surprise waiting when I come to look at the force term in the FP formulation.
The second limit of interest is the small “density” limit in which one may start expanding all $U^{(1)}$ factors around $1$. Alternatively one may attempt to decouple the RG equation hierarchy by expanding in powers of $U^{(1)}$. Both of these perspectives will have only a very limited region of applicability as will be seen below and will most likely yield similar results there. I will limit myself to the latter.

To see how these expansions affect the RG, define the generating functional for connected correlators via

$$Z[J] = e^{-\mathcal{G}[J]}$$

and view $\mathcal{G}[J]$ as a power series in $J = (J^\dagger, J)$. Similarly think of $\bar{\chi}[U, U^\dagger]$ and $\bar{\sigma}[U]$ as a series in $U = (U, U^\dagger)$. For example for $\bar{\chi}$ one writes the expansion

$$\bar{\chi}_{uv}[U] = \sum_n \frac{1}{n!} \bar{\chi}_{uv;\alpha_1\ldots\alpha_n} U_{\alpha_1} \otimes \ldots \otimes U_{\alpha_n}$$

with $\bar{\chi}_{uv;\alpha_1\ldots\alpha_n}$ the $n$-th derivative of $\bar{\chi}_{uv}[U]$ with respect to its argument and similarly for $\bar{\sigma}$ and $\mathcal{G}$.

Truncating $\bar{\chi}$ and $\bar{\sigma}$ at lowest order (quadratic and linear in $U$ respectively) they reduce to real and virtual parts of the BFKL kernel in the “dipole” formulation. One then finds

$$-\frac{\partial}{\partial \ln 1/x} \mathcal{G}[J] = \alpha_s \left\{ \frac{1}{2} J^\dagger \chi^{(2)} U \chi^{(2)} \left(-\mathcal{G}_{\alpha\beta}[J] + \mathcal{G}_{\alpha\beta}[J] \mathcal{G}_{\alpha\beta}[J]\right) - J^\dagger \bar{\sigma}^{(1)} \mathcal{G}_{\alpha\alpha}[J] \right\} .$$

This still represents a whole hierarchy of equations; one equation for each power in $J$. Assuming a vanishing one point function, the hierarchy starts with the BFKL equation for $\mathcal{G}^{(2)}$. The equation for the three and four point functions then get feedback from the BFKL equation via $\mathcal{G}^{(2)}$, a pattern of lower order n-point functions coupling into the equations of higher order ones that continues on to arbitrary $\mathcal{G}^{(n)}$. This is the very same pattern that shows up in [12], although here one is looking at “projectile” evolution. Had one started this type of truncation scheme with the JKLW equation, one would have directly reproduced the “target” setup.

This identification together with what has been established in Ref. [18] should now cover all presently known equations for small $x$ evolution to leading order in $\alpha_s \ln 1/x$ in which target and projectile are treated asymmetrically.

Expanding the $U^{(1)}$ factors around $1$ will lead to very similar equations for the lower order correlators, although clearly higher orders will show differences for the two approaches. However, it is probably pointless to try and contrast the two versions based on these higher orders. Clearly as it becomes important to include such contributions there will no longer be any justification to truncate the full evolution Eq. (22) in any form and both versions lose their validity, although one might outlast the other by some margin. This—as well as potential saturation properties of these truncations—is not of much interest in the present context, as the new results to be developed will be for Eq. (22) directly, without the need for any truncation. For this same reason I will not comment on other cases, such as the doubly logarithmic limit, but rather refer to Ref. [17] for the JKLW case and Ref. [18] for a comparison between the target and projectile evolution pictures also in that limit. The result there nicely mirrors the the hierarchy of increasing complexity described at the outset of this derivation of evolution equations.

## 4 A Fokker-Planck form for the evolution equations: Probability distributions & fixed point equations

If one recalls the definition of $\mathcal{Z}[J] := \langle e^{S^{(\text{tgt})}[b] J] \rangle_b$ and the initial idea of interpreting $\langle \ldots \rangle_b$ as a statistical average with associated statistical weight $\mathcal{Z}$ according to

$$\langle \ldots \rangle_b = \langle \ldots \rangle_U = \int D[U] \mathcal{Z}[U] \langle \ldots \rangle_U$$

12
one may try to rewrite Eq. (22) as an equation for $\bar{Z}$. Indeed –remembering that $S_{\text{ext}}^{\bar{q}}[b, J] = S_{\text{ext}}^{\bar{q}}[U, J]$ is actually given as a functional of $U^{(1)}$—

\[
\frac{\partial}{\partial \ln 1/x} \int D[U] \bar{Z}[U] e^{S_{\text{ext}}^{\bar{q}}[U, J]} = \alpha_s \int D[U] \bar{Z}[U] \times \left\{ \frac{1}{2} \chi_{uv}[U] J_u J_v + \sigma_u[U] J_u \right\} e^{S_{\text{ext}}^{\bar{q}}[U, J]} .
\]

Using the fact that if one treats $U$ and $U^\dagger$ as independent variables one finds, conjugate to Eqs. (10), that

\[
\frac{\delta}{\delta U^\dagger} e^{S_{\text{ext}}^{\bar{q}}[U, J]} = J^\dagger e^{S_{\text{ext}}^{\bar{q}}[U, J]} ,
\]

\[
\frac{\delta}{\delta U} e^{S_{\text{ext}}^{\bar{q}}[U, J]} = J e^{S_{\text{ext}}^{\bar{q}}[U, J]} .
\]

Hence, Eq. (31) can be rewritten as an equation for $\bar{Z}$. Note that now both $U^{(1)} \in GL(N_c, \mathbb{C})$, not $SU(N_c)$, otherwise the condition $UU^\dagger = 1$ would invalidate Eqs. (32). The restriction to $SU(N_c)$, which of course has to be present for consistency of the whole approach, will have to come from the properties of $\bar{\chi}$ and $\bar{\sigma}$ and hence the nature of evolution for $\bar{Z}$. Leaving this for later, the evolution equation for $\bar{Z}$ is given by

\[
\frac{\partial}{\partial \ln 1/x} \bar{Z}[U] = \alpha_s \left\{ \frac{1}{2} \delta \frac{\delta}{\delta U_u} \chi_{uv}[U] - \delta \frac{\delta}{\delta U_u} \bar{\sigma}_u[U] \right\} \bar{Z}[U] ,
\]

where the $U^{(1)}$ derivatives act on both the kernels $\bar{\chi}, \bar{\sigma}$ and $\bar{Z}$. This equation exhibits exactly the same structures as the JKLW equation, although the variables and details of the kernels are, of course, different. Note that this has the form of a Fokker-Planck (FP) equation, simply due to the fact that the original equation as represented in Eq. (31) had only terms quadratic and linear in $J$.

As with any other FP equation, the fact that the right hand side is a total derivative implies that the normalization of $\bar{Z}$—its $U, U^\dagger$ integral— is conserved under $x$ evolution and, if one chooses this norm to be 1, allows to interpret $\bar{Z}$ as a probability distribution at any stage of the evolution.

Clearly this formulation is completely equivalent to the original version involving the generating functional $\bar{Z}$. Instead of obtaining equations for correlators by functional differentiation, one now extracts them, again as in any FP formulation, by multiplying with the desired monomial in $U^{(1)}$ followed by integrating the result over $U, U^\dagger$.

Let me conclude this section by rewriting the FP equation Eq. (33) in what I would like to call its canonical form by pulling one of the derivatives in the $\bar{\chi}$ term to the right and factor the other derivative out to the left. The FP equation then reads

\[
\frac{\partial}{\partial \ln 1/x} \bar{Z}[U] = \alpha_s \left\{ \frac{1}{2} \chi_{uv}[U] \frac{\delta}{\delta U_v} + \left[ \frac{1}{2} \left( \frac{\delta}{\delta U_v} \bar{\chi}_{uv}[U] \right) - \bar{\sigma}_u[U] \right] \right\} \bar{Z}[U] .
\]

Technically, this form is particularly useful when searching for fixed points, but the reason for this is of course rooted in physics: Written in this form the terms have the explicit interpretation of a stochastic and an external force term. Only the latter would be capable of balancing the diffusion caused by the first term and could lead to fixed point structure that does not correspond to pure diffusion. However, upon translation of Eq. (33) into a Langevin equation, it is still $\bar{\sigma}$ that shows up as a (possibly “apparent”) force, while $\bar{\chi}$ determines the correlator of the purely stochastic fluctuations. However, I will delay writing down this Langevin equation—the prerequisite for any numerical simulation for such evolution—until a better understanding of the properties of the RG equation has been reached.

Any reader familiar with typical examples of FP equations, be it in classical physics, stochastic quantization and its applications to numerical simulation of field theories or any others, will by now have begun to wonder about the the “equilibration” behavior of the FP equation (33). In the present RG context, this is nothing but the question for the existence and nature of the small $x$ fixed point of the evolution.
A sufficient condition for the existence of a fixed point of the RG is simply given by the vanishing of the driving term of the RG, i.e. the r.h.s. of Eq. (34). In other words, a fixed point exists if

$$\frac{1}{2} \bar{\chi}_{uv}[U] \frac{\delta}{\delta U_v} Z[U] = - \left[ \frac{1}{2} \left( \frac{\delta}{\delta U_v} \bar{\chi}_{uv}[U] \right) - \bar{\sigma}_u[U] \right] Z[U]$$

has a solution. Implicitly this determines the fixed point value of ln $\bar{Z}$ via a functional differential equation in which one has to integrate back from the force term to get the “potential” that determines the “equilibrium” or fixed point. It depends on the nature of this solution whether the fixed point in question is attractive and hence reached by evolving any arbitrary initial condition or not. Besides the mere existence of a fixed point it is this fact which determines whether Eq. (34) predicts universality in the sense of target independence of QCD in scattering events at small $x$ or not. Instead of pronounced regularities, a repulsive fixed point would imply runaway solutions and indicate an increasing target dependence of measurable quantities with smaller and smaller $x$. That way the force term on the r.h.s. of these equations will decide the general nature of QCD cross section at small $x$ in the leading ln $1/x$ approximation employed.

To answer any of these open question one will therefore have to carefully study the properties of both the force and stochastic terms.

5  A case of Brownian motion?

Indeed, there remains quite a bit to be learned about the character of the evolution described by the above FP equation before one can satisfactorily answer this question, but a first observation is very easily extracted, just by looking at the force term.

To this end one has to evaluate $U^{(1)}$ derivatives acting on $\bar{\chi}$ and, quite surprisingly, one finds

$$\frac{1}{2} \left( \frac{\delta}{\delta U_v} \bar{\chi}_{uv}[U] \right) - \bar{\sigma}_u[U] = 0.$$  \hspace{1cm} (36)

(For both the $q$ and $\bar{q}$ components.) During this calculation, which is exemplified in App. B, one observes very intricate cancellations of infinities between the derivative terms before the finite remainder cancels against $\bar{\sigma}$.

As a result, the force terms vanish completely and one is dealing with a Fokker-Planck equation with a nontrivial kernel, but without a force term. The cancellation of the latter is not at all an accident. In fact it relies on a very deep relation between the $\bar{\chi}$ and $\bar{\sigma}$ terms. Tracing back what is left of this cancellation in the BFKL limit, one finds that there it is responsible for the infrared finiteness of the BFKL kernel.\(^{12}\) There gauge invariance guarantees that one finds a cancellation of “real” against “virtual” diagrams in the infrared. The situation and the cancellation just observed here is even more general, as it goes beyond the BFKL limit, that is the limit of the RG equations where $U$ is in the vicinity of 1 to include the full set of nonlinearities of this evolution equation.

With the force term vanishing, the Fokker-Planck equation now deceptively looks like a case of Brownian motion:

$$\frac{\partial}{\partial \ln {1/x}} \bar{Z}[U] = \alpha_s \frac{1}{2} \frac{\delta}{\delta U_v} \bar{\chi}_{uv}[U] \frac{\delta}{\delta U_v} \bar{Z}[U].$$  \hspace{1cm} (37)

One might be tempted to jump to the conclusion that $\bar{Z} = \text{const}$ is the “equilibrium” limit in standard Fokker-Planck parlance – the fixed point at small $x$ in this case. In particular as one is forced to take $U$ and $U^\dagger$ derivatives independently, thus violating for instance the condition that $UU^\dagger = 1$, one seems to be in danger of losing all memory of the fact that QCD has a $SU(N_c)$ gauge symmetry, a possibility strikingly at odds with what has just been identified as the source of the cancellations leading into this very dilemma.

Barring any hope for an inconsistency an unfriendly reader might harbor, there must be further structure hidden in Eq. (37) that prevents this from happening and the only place left for that to hide is within the nontrivial kernel $\bar{\chi}$.

\(^{12}\)This statement is strictly true only in the “target” formulation. For the present “projectile” or “dipole” formulation the same statement refers to a cancellation of unphysical ultraviolet divergences.
6 Uncovering the gauge group

6.1 Independent derivatives? RG flow and Lie derivatives

The fact that in writing the Fokker-Planck equation one had to take independent derivatives with respect to \( U \) and \( U^\dagger \), raises the question of how the evolution knows about the phase factors being \( SU(N_c) \) and not some arbitrary complex \( N_c \times N_c \) matrix. Generically there are two options:

- To outline the first possibility let me turn to simple mechanics analogue, a Fokker-Planck/Langevin problem in a 2d harmonic oscillator with a stochastic term acting only in a 1d subspace. This in turn implies that the system is completely deterministic in the other direction and the restoring force there will asymptotically move the particle into the line with stochastic perturbations, the rate of approach being proportional to the force. The probability distribution (for all times) will factorize into a deterministic and a truly probabilistic part. The deterministic part has the form of a time dependent delta function following the particles trajectory from its initial value into the minimum. If the force becomes infinite (say by making the potential infinitely steep in this direction) this factor will become time independent and fix the position at the minimum in the non-stochastic direction. This is to say that an infinite force signals a constraint in the system.

Although this looked like an attractive option when discovering the infinite contributions to the individual terms in Eq. (36) it is emphatically ruled out as a consequence of the remarkable cancellations occurring when adding all of them together.

- The second option is simpler and more elegant: Imagine the forces (both deterministic and stochastic alike) to act only “along” \( SU(N_c) \), not out of it in the sense that if you start with an initial condition “within,” evolution will not push the system “outside.” Clearly this would have to result from intrinsic properties of both \( \bar{\chi} \) and \( \bar{\sigma} \), although in the present case the burden lies solely with \( \bar{\chi} \) as the only independent ingredient left in the equation.

In this scenario evolution would separate the whole (apparent) configuration space into invariant orbits. Given an initial condition in one of them, evolution would keep the system within this same orbit. To achieve what is needed for consistency, one of these orbits would have to be \( SU(N_c) \) as embedded into the complex \( N_c \times N_c \) matrices or \( GL(N_c, \mathbb{C}) \).

Given this property, one would be able to reduce the above equations to the orbit of the Fokker-Planck operator on an invariant subspace. In other words it should be possible to write the FP equation in a more compact form within what is of interest here: the physical subspace of \( SU(N_c) \) matrices. This will be done below.

The first clue that a separation into invariant orbits, with \( SU(N_c) \) being one of them, might indeed be realized is the observation that

\[
\delta_{uu}[U]\frac{\delta}{\delta U_{uv}}[U_w U_{vw}^\dagger]mn \bigg|_{UU^\dagger=1} = 0
\]  

(38)

for both \( q \) and \( \bar{q} \) components.

What one needs in fact is, that for physical distributions\(^{13}\) of the form

\[
\tilde{Z}_{\text{phys}}[U, U^\dagger] = \delta(UU^\dagger - 1) \delta(\det U - 1) \tilde{Z}[U],
\]

(39)

the constraint factor \( \delta(UU^\dagger - 1) \delta(\det U - 1) \) is left invariant by the evolution operator, i.e. that in some sense one may commute it though the delta functions.

As it turns out the evolution operator can be pulled through the constraint and does simplify tremendously during this procedure. Only then will one be able to see if the Brownian motion concept is actually realized. Pulling the evolution operator through the constraint is in fact the main technical problem left.

\(^{13}\) all others would be illegal and spurious in any case
In order to get a feeling for what is involved it is useful to collect some ideas of what to expect if this were actually possible. The main step in rewriting the evolution within the group consists of turning $\frac{\delta}{\delta U}$ and $\frac{\delta}{\delta U^\dagger}$ from independent to dependent derivatives via $UU^\dagger = 1$. What is to replace them?

Pondering this question a little (c.f. App. C for a pedestrian approach), one inevitably ends up considering the natural derivatives along $SU(N_c)$, its Lie derivatives. They would be given by\(^\text{14}\)

$$i\nabla_{\bar{U}} := [Ut^a]_{ij} \frac{\delta}{\delta U} = [-t^aU^{-1}]_{ij} \frac{\delta}{\delta U} = \frac{\delta}{\delta U} ,$$

so that as a first step in this direction one would like to reexpress things via $[Ut^A]_{ij} \frac{\delta}{\delta U}$ and $[-t^A U^\dagger]_{ij} \frac{\delta}{\delta U}$ instead of $\frac{\delta}{\delta U}$ and $\frac{\delta}{\delta U^\dagger}$. Indeed, if one now just tries out what this transformation will do, one finds remarkable results for the transformed components of $\bar{\chi}$.

To achieve this technically, one needs to be able to represent all of $\delta/\delta U_{ij}^{(t)}$’s components. Accordingly one has to replace the index $a \in \{1, \ldots, N_c^2 - 1\}$ by $A \in \{0, \ldots, N_c^2 - 1\}$ and supply $t^0 = 1/\sqrt{2N_c}$, such that $\text{tr}(t^A B) = \delta^{AB}/2$ for all $A, B$. Then the transformation is easily accomplished (see App. C for details).

As already advertised, the results are quite striking:

- in $SU(N_c)$ all four of the transformed components are equal.
- in $SU(N_c)$ all of them vanish if either $A$ or $B$ is 0. Only octet components survive. All matrices involved now are explicitly in the adjoint representation.

- The unique common form even factorizes naturally into the “square” of a much simpler factor. The result will be denoted by $\hat{\chi}_{xy}^{ab}[U]$ and reads

$$\alpha_s \hat{\chi}_{xy}^{ab}[U] := \frac{\alpha_s}{\pi^2} \int d^2 z \left( \frac{(x - z)_i}{(x - z)^2} \left[ 1 - \tilde{U}_{xy}^{-1} \tilde{U}_z \right]^{ac} \right) \left( \frac{(z - y)_i}{(z - y)^2} \left[ 1 - \tilde{U}_{xy}^{-1} \tilde{U}_y \right]^{cb} \right) .$$

The diagonalizing factors of $\bar{\chi}$ deserve a name for further reference

$$[\hat{\chi}_{xy}^{ab}] := \sqrt{\frac{\alpha_s}{\pi^2}} \left( \frac{(x - y)_i}{(x - y)^2} \left[ 1 - \tilde{U}_{xy}^{-1} \tilde{U}_y \right]^{ab} \right) .$$

If the FP equation can be written entirely within $SU(N_c)$ – and this will be demonstrated below – one may also expect that a description in terms of a Langevin process with a completely decorrelated Gaussian random noise becomes possible due to the factorized form of (41) via (42).

### 6.2 A Fokker Planck equation within the gauge group: The small $x$ fixed point

The results of the above transformation of $\bar{\chi}$ are so suggestive that it seems obvious what the form of the FP operator on physical configurations – modulo some chiarvoyance as regards the prefactor – has to be:

$$\alpha_s \frac{1}{2} i \nabla_{\bar{U}}^{a} \hat{\chi}_{xy}^{ab} \nabla_{U_y}^{b} \ .$$

The chain of argument to prove this is the following. One sets out to show that

$$\frac{\delta}{\delta U_u} \bar{\chi}_{uv}^{ab} \frac{\delta}{\delta U_v} \delta(UUU^\dagger - 1) \delta(\det U - 1) = \delta(UUU^\dagger - 1) \delta(\det U - 1) \nabla_{\bar{U}}^{a} \hat{\chi}_{xy}^{ab} \nabla_{U_y}^{b} \ .$$

This amounts to verifying that

$$\frac{\delta}{\delta U_u} \bar{\chi}_{uv}^{ab} \frac{\delta}{\delta U_v} \left. U^{(1)} \right|_{UUU^\dagger = 1} \cdots \left. U^{(1)} \right|_{UUU^\dagger = 1} = \nabla_{\bar{U}}^{a} \hat{\chi}_{xy}^{ab} \nabla_{U_y}^{b} \left. U^{(1)} \right|_{UUU^\dagger = 1} \cdots \left. U^{(1)} \right|_{UUU^\dagger = 1}$$

\(^{14}\text{It is easy to see that this is the correct form as both variants amongst themselves and with each other satisfy the }SU(N_c)\text{ commutation relations at each point.}\)
for arbitrary monomials \( U^{(t)}_{w_1} \otimes \ldots \otimes U^{(t)}_{w_n} \), a task that once more is best performed via a generating functional. This strategy has the added benefit that it makes it readily apparent how the two operators lead to the same hierarchy of equations for the correlators which provided the starting point for all these deliberations. The details of this exercise are given in App. D.

This leads to the central result of this paper, the RG/FP equation on the physical configuration space, where

\[
\hat{Z}_{\text{phys}}[U,U^\dagger] = \delta(UU^\dagger - 1)\delta(\det U - 1)\hat{Z}[U].
\]  
(46)

The constraint factor may now be absorbed into the measure to form a functional Haar measure according to

\[
\hat{D}[U] := D[U]\delta(UU^\dagger - 1)\delta(\det U - 1)
\]  
(47)

This constitutes the natural measure to use for averaging correlators of the form

\[
\langle U^{(-1)}_{w_1} \otimes \ldots \otimes U^{(-1)}_{w_n} \rangle_U
\]  

with the probability distribution \( \hat{Z}[U] \). The RG/FP equation simplifies to

\[
\frac{\partial}{\partial \ln 1/x} \hat{Z}[U](x) = -H_{\text{FP}} \hat{Z}
\]  
(49)

where

\[
H_{\text{FP}} := -\frac{\alpha_s}{2} i\nabla_{U_x}^a \chi^{ab}_{xy} \nabla_{U_y}^b \hat{Z}[U](x)
\]  
(50)

Now define

\[
\left[ P^a_x \right]_i := \left[ 1 - U_x^{-1} U_y \right]^{ab}_{xy} \left[ i\partial_i \frac{2\pi}{\rho^2} \right]_{xy} i\nabla_{U_y}^b
\]  
(51)

to recast \( H_{\text{FP}} \) as

\[
H_{\text{FP}} = -\frac{\alpha_s}{2} i\nabla_{U_x}^a \chi^{ab}_{xy} \nabla_{U_y}^b = \frac{\alpha_s}{4\pi^2} P^a \cdot P.
\]  
(52)

Hence, the spectrum of of \( H_{\text{FP}} \) is bounded from below.\(^{16}\) This is what one was hoping for: Now the system will “equilibrate”, the small \( x \) fixed point will be attractive: A finite evolution step is described by

\[
\hat{Z}(y) = e^{-H_{\text{FP}}(y-y_0)} \hat{Z}(y_0),
\]  
(53)

leading to a damping of all but the zero eigenmode, which will become the fixed point. This fixed point however is trivial: With \( H_{\text{FP}} \) acting as a derivative, the asymptotic solution to this evolution equation is simply

\[
\hat{Z}[U] \xrightarrow{x \to 0} 1,
\]  
(54)

\(^{15}\) \( \partial \frac{1}{\rho^2} \left( x-y \right) = \partial \frac{1}{\rho^2} \ln((x-y)^2/\rho^2) = \frac{1}{2\pi} \frac{(x-y)}{(x-y)^2} \)

\(^{16}\) Note how crucial the sign here becomes: If it were negative, the Langevin formulation would break entirely. Instead of getting random kicks in the Lie algebra, one would see an additional factor of \( i \) there and such a formulation becomes meaningless. It is instructive to note that the wrong sign here would simultaneously imply a decreasing BFKL pomeron as one easily discovers when tracing this sign back to what it corresponds to in Eq. (26). A growing BFKL pomeron and positivity of \( H_{\text{FP}} \) are the same thing.
where it has been assumed that the Haar measure has been normalized to yield unit probability within $SU(N_c)$. While Eq. (54) obviously is a zero mode of $H_{FP}$, it is much less obvious that it is a unique one. The argument here rests on the fact that the Lie derivatives in $H_{FP}$ induce local gauge transformations with respect to the transverse coordinates, while any change in the $\pm$ plane is immediately related to the coarse graining step of the RG procedure. For the details, see App. E.

It is definitely worth taking a step back to contemplate this result: The fixed point probability distribution has no $U$ dependence. Its physics content lies in the fact that the distribution of $U$’s completely fills configuration space and the “phases” randomize entirely, a standard Brownian motion scenario. This may sound trivial, but in fact it will have intriguing consequences. On the one hand this is the root of any universality argument i.e. the hope for target independent features at small $x$. On the other hand, as will be shown in Sec. 7, this limit of infinite energy is not the most interesting piece of information to be learned about the asymptotics of this evolution. Already the fact that this limiting solution exists and is attractive leads to regular features in a larger region of energy, well before the fixed point itself is reached. To expose this additional structure, however, one needs to establish a few further insights.

One is, however, already able to get a glimpse on how the damping exhibited by Eq. (53) can be reconciled with the known strong growth of the BFKL limit. This in fact will expose the main unitarization mechanism. Note that it is the dipole kernel $\tilde{K}$ which governs the BFKL limit of Eqs. (25d), (26) not $K$, the integral kernel in $\chi$. They are related according to (c.f. Eq. (25a))

$$K_{xzy} = \frac{1}{2} [\tilde{K}_{xzy} + K_{yzy} + K_{xzx}] .$$

(55)

In contrast to $K$, $\tilde{K}$ is not a square and in fact not bounded at zero. This is in direct correspondence to the same lack of boundedness at zero of the BFKL kernel. This observation sheds some light on the unitarization mechanism and the lack thereof in a strict BFKL truncation. To see how this comes about take Eq. (26), but before factorization. Define the dipole operator $N_{xy} := \text{tr}(1 - U_x U_y^\dagger)/N_c$ (not the amplitude). Its evolution equation is given by

$$\frac{\partial}{\partial \ln 1/x} \langle N_{xy} \rangle = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \left\{ \langle N_{xz} + N_{zy} - N_{xy} \rangle - \langle N_{xz} N_{zy} \rangle \right\}$$

(56)

and clearly couples to all higher order ones through the nonlinear term. Now think of the full evolution equations as an infinite set again, more precisely, think of it as a matrix representation on the space of correlators. Eq. (56) only reduces to the BFKL equation if one ignores the nonlinearity and hence the way this particular piece of the matrix equation couples to the other components of this matrix representation. That is what breaks positivity and leads to the runaway solutions.

This, however, also explains –along the same lines as in the large $N_c$ truncation of [6,7]– how a unitarization mechanism might work here: As long as the correlators are close to the free ones, the nonlinearities are unimportant. The growing mode of $\tilde{K}$ can work its magic and violently push the system towards a situation in which the nonlinearities become important. Then “positivity” of $H_{FP}$ takes over and slows down all further evolution drastically.

Complementary to Eq. (54) there is one thing one knows about the initial conditions, and that is that all $U$ factors outside the transverse tail of the wave function are unity. There

$$\hat{Z}[U] \rightarrow \delta(U - 1) .$$

(57)

This and the notion of what is outside the target will change during the course of the evolution, as $\chi$ has non vanishing contributions everywhere, due to its nonlocal nature, c.f. Eqs. (41), (42). Together with the initial BFKL like growth, this will be an integral part of the way unitarization is reached. I will come back to this after having spelled out the Langevin equations.

Eq. (54) together with Eq. (57) implies a full knowledge of all correlators as one enters the saturation region with $x \rightarrow 0$, provided some information about the transition region and hence the transverse tail of the wave function is put in via the initial condition.
I will come back to use these correlators below when discussing the consequences for unitarization in DIS. As seen above, impact parameter space and relative coordinates play a very different role in the evolution. The latter only enter in a scale invariant manner into the kernel. The interplay of these two facts will lead to scaling laws with respect to target “size.” As this applies to the evolution of all correlators, it will also have consequences for the DIS cross sections.

7 Langevin equations: the basis for numerical simulations, scaling laws

Before discussing the generic form of correlators in the saturation region, there is one more useful perspective to be explored that can be harvested without much effort, directly from the the wealth of methods in the stochastic physics tool box. Besides providing a generic method for numerical simulation, it will also provide a very intuitive physics picture. This will directly lead to scaling laws governing the whole of the evolution and provide a very nice, novel perspective to view the unitarization mechanism.

There is an extensive literature on numerical simulations of field theories with stochastic methods, that can be adapted to simulate the generic small $x$ evolution equation Eq. (48). This is an enormous step beyond what was feasible up to now [21]. The key tool to do so lies in the possibility to describe the same stochastic process encoded in Eq. (48) via a Langevin equation with Gaussian noise.

To prepare translation into Langevin form let me write the RG again in standard form

$$\frac{\partial}{\partial \ln(1/x)} \hat{Z}[U] = -\alpha_s \nabla^a \left[ \frac{1}{2} \nabla^b \chi_{xy}^{ab} - \left( \frac{1}{2} \nabla^b \chi_{xy}^{ab} \right) \hat{Z}[U] \right]$$

(58)

and define

$$\hat{\sigma}_x^a := \frac{1}{2} \nabla^b \chi_{xy}^{ab} = -i \left( \frac{1}{2} \frac{1}{\pi^2} \int d^2 z \frac{1}{(x-z)^2} \text{tr}(\hat{\tau}^i \hat{U}_x^1 \hat{U}_z) \right)$$

(59)

(the “∼” indicating adjoint matrices and traces). The Langevin equation then reads

$$\frac{\partial}{\partial \ln 1/x} [U_x]_{ij} = [U_x i t^a]_{ij} \left[ \omega^a_x + \alpha_s \hat{\sigma}_x^a \right] = [U_x i t^a]_{ij} \left[ \int d^2 y \left( \mathcal{E}_{xy}^{ab} \right)_{ij} \left| \chi_{xy}^b \right| k + \alpha_s \hat{\sigma}_x^a \right]$$

(60)

where both $\omega$ and $\xi$ are Gaussian random variables with correlators determined according to

$$\langle \ldots \rangle_\omega = \det \chi^{1/2} \int D[\omega] \ldots e^{-\frac{1}{2} \omega^i \chi^{-1} \omega} \quad \text{and} \quad \langle \ldots \rangle_\xi = \int D[\xi] \ldots e^{-\frac{1}{2} \xi \xi} \,. \quad \text{(61)}$$

Also note the factor of $i$ which is essential to render this an equation for an infinitesimal change of an element of $SU(N_c)$. In fact the components of $\omega^a_x$ can be directly interpreted as the “angles” parametrizing a local gauge transformation in transverse space.

It is of particular importance to note that the possibility to formulate the stochastic term via a completely decorrelated Gaussian noise $\xi$, that is to say with $\langle \xi_x \xi_y \rangle = \delta^{ij} \delta_x^{(2)}$, reduces the numerical cost for a simulation like this considerably.

Note how the Langevin equation induces transverse spreading: Even at points initially outside the target, where $U_x$ starts out as unity, the noise term generates nontrivial phases due to the nonlocality in $[\mathcal{E}_{xy}^{ab}]$. This growth is of the order of $\sqrt{\ln 1/x}$ according to the typical scaling behavior of all Brownian processes, by a textbook argument. How does this knowledge of $\omega \sim \xi \sim O(\sqrt{\ln 1/x})$, which is the change of the exponent of $U_x$, affect spreading, which, naively, one also would expect to follow this trend?

The key lies in Eq. (60), used outside the target. Take the noise term as an example: There the $U_x$ in $\mathcal{E}_{xy}$ and elsewhere in Eq. (60) is 1. However, when $y$ is inside the target while integrating, one gets a contribution
that changes $U_x$ away from 1 via (60) since $U_y$ is nontrivial. Imagine the target is characterized by some radius parameter, then this integral over $y$ would be cut off there as $1 - U_y$ is different from zero only inside:

$$\int d^2 y \frac{(x - y)}{(x - y)^2} (1 - U_y^{-1} U_y) \xi_y \rightarrow 0 \text{ if } |y| > R$$

Clearly, here the $1 - U_y$ factor determines the region of non vanishing random noise contributing to the evolution. It is a simple exercise to see that the force term $\hat{\sigma}$ follows the same pattern.

All these facts combine into a scaling argument: using the natural scaling properties of the ingredients in (60) allows to scale the size of $\xi \sim O(\sqrt{\ln 1/x})$ out of the equation almost everywhere using $x \rightarrow \sqrt{\ln 1/x}$, $y \rightarrow \sqrt{\ln 1/x} y$.$^{17}$ Everywhere, except in the radius parameter which is now the only ingredient affected by the typical growth rate according to $R \rightarrow \sqrt{\ln 1/x} R$. This argument does not at all depend on the way such a radius parameter is introduced, it only relies on the fact that the only scale invariance breaking feature involved is the size of the target. Note, however, that it refers to the area of activity only –the area where $U_y$ is nontrivial– not the measured cross section. These two notions become equal only when the blackness limit has been reached.

The picture of evolution then is the following: During the whole evolution, the systems area of activity is growing like $\ln 1/x$, the measurable cross section, however, will only start to follow this trend once the blackness limit has been reached. From then on $\ln 1/x$ type unitarization is visible in the cross sections, with little to distinguish a proton from uranium targets up to a global rescaling with the cross sections at the “unitarization limit.” One thus expects scaling laws that relate the evolution of different targets in this region. Unitarization is thus viewed as the limit in which the area over which the system appears to be black catches up with the total area, which scales with $\ln 1/x$ no matter which $x$ one looks at. This is a remarkably concrete and hands on scenario to emerge from the rather mathematical elegance of Eqs. (48) and (60).

The focus for further study then will become the question for saturation “efficiency,” the speed at which an initially grey system becomes black on the one hand and the relative size of not yet saturated tails on the other. The negligibility of such tails is the main approximation underlying any scaling laws referred to above.

The present equations reduce to a BFKL scenario at small densities and therefore imply power like saturation “efficiencies” when starting from moderate $x$. Note that the actual location of the unitarization limit will depend on target properties. As one is not talking about the asymptotic region of the BFKL equation –quite to the contrary– different components of the wave function will grow with different speed and it will depend on the amount of overlap with fast growing eigenfunctions if the unitarization limit is reached sooner or later.

To turn back from general features to prospective numerical simulations, there is one item left to be supplied: an initial condition for the evolution, which will have to be adapted to the physical process to be explored. What one has to do is to generate an initial ensemble of configurations that, when averaged over, yields some physical cross section. At this point there are a few possibilities readily available. The first is to just follow [21] as closely as possible. This would limit the discussion to DIS per se. The second somewhat more sophisticated approach would start with the McLerran-Venugopalan (MV) model for large nuclei [22,23]. This also has something to tell about about the dipole cross section for fixed, small $x$ as rudimentarily laid out in [20], but is more generic. This approach, however, should be updated to incorporate overall color neutrality as suggested in [24]. Although both approaches –the present RG and the MV model– are formulated with the help of statistical weights, the variables are not the same. The MV model is formulated in terms of what essentially corresponds to the exponent of $U$, the variables of this RG. This needs to be adapted.

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$^{17}$ The scaling dimension of $\xi$ has to be length$^{-1}$, c.f. Eq. (61). This is also consistent with having a scaling dimension for $U$ of zero through Eq. (60).
8 Implications: correlators at vanishing x, universality and scaling laws exemplified

The first and most obvious consequence of of the known form of $\hat{Z}$ at vanishing $x$, Eq. (54) inside the target, the area of activity, and Eq. (57) outside, allows to make generic statements about the correlators in the “unitarization region,” provided that the influence of tails that have not reached the saturation limit is small. This seems to be borne out by simulations in the large $N_c$ limit [21]. One would also expect this to be of less importance with increasing target size $A$, as the system there starts from a situation with more gluons involved even initially and the tails proportionately smaller.

One finds then that generic correlators with all distinct coordinates vanish inside the target and become equally trivial outside. Knowing that the area of activity will grow logarithmically as parametrized by a radius parameter $R_A(x)$ one may write in the saturation region

$$\langle U^{(i)}_{x_1} \otimes \ldots \otimes U^{(i)}_{x_n} \rangle_b \rightarrow \theta(\text{any of } \{|x_1|, \ldots, |x_n|\} < R_A(x))$$

$$+ 1 \otimes \ldots \otimes 1 \theta(\text{all of } \{|x_1|, \ldots, |x_n|\} > R_A(x))$$

(63)

iff all the $x_i$ are distinct. Here the “$\theta$ functions” are one if the condition indicated is met and zero otherwise. Exceptions to the above occur if one allows to have factors of $U$ and $U^\dagger$ at the same point as for example in

$$\langle [U_{x,i}^\dagger [U^i_{y,j}]_{kl}] \rangle_b \rightarrow \delta^{(2)}_{xy} \delta_{jk} \delta_{ij} \theta(|x| < R_A(x))$$

$$+ \theta(|x|, |y| > R_A(x)) \delta_{ij} \delta_{kl},$$

(64)

due to the $U_x U^\dagger_x = 1$ constraint in the measure. Similarly, the other constraint in the Haar measure, $\det U = 1$, induces nontrivial expectation values with exact multiples of $N_c$ factors of $U$s or $U^\dagger$s inside. For example

$$\langle [U_{x,i}^\dagger [U^i_{x,j} \ldots [U_{x,N_c}^i [U_{x,N_c,j}^i]_{ij,jk} \ldots \rangle_b \rightarrow \delta^{(2)}_{x_1,x_2} \ldots \delta^{(2)}_{x_{N_c,i},x_{N_c,j},i_1,i_2 \ldots j_{N_c}, \ldots , j_{N_c}, \ldots , j_{N_c}} \theta(x_1 < R_A(x))$$

$$+ \delta_{i_1,i_j} \ldots \delta_{i_{N_c},j_{N_c}} \theta(\text{all of } \{x_1, \ldots, x_{N_c}\} > R_A(x)).$$

(65)

Any improvement beyond the naive step function scenario would naturally describe delayed saturation in the transverse tails of the wave function and would lead to replacing the step functions by some other effective “profile function” of the target.

With this proviso one may calculate the asymptotics of specific correlators such as the dipole cross section in DIS, $\langle \text{tr} (1 - U_x U_y) \rangle_b$. The impact parameter integral will be cut off by the “outside” term in Eq. (64) and one finds

$$\int d^2(x+y) \langle \text{tr} (1 - U_x U_y) \rangle_b \rightarrow f_A(\pi R_A(x)) N_c (1 - \delta^{(2)}_{xy}).$$

(66)

This is obviously universality of the dipole cross section, but not “saturation”, a feature also observed in the large $N_c$ limit as reported in Ref. [21]. The lack of saturation, of course, is a direct consequence of the global $\ln 1/x$ growth of the active area observed above.

Here $f_A(\pi R_A^2)$ has the interpretation of the integral over a profile function for the target, the knowledge of which depends on whatever one is able to correctly implement when writing the initial condition for the evolution. To find the total cross section or the $F_2$ structure function, this is to be convoluted with the correct photon wave function factors, but even without doing so, one more striking consequence is obvious: The above expression is only sensitive to the value of this wave function at its large effective momenta via the $\delta^{(2)}_{xy}$ and vanishing effective momenta via the 1 term. This leads to a complete loss of information about the momentum transfer of DIS, $Q^2$, in the limit of vanishing $x$.

To come back to the scaling argument, one should expect that cross sections such as the above scale with their respective integrated profile functions $f_A(\pi R_A^2)$ (or equivalently with their ratios at the initial value of $x$). $x$ evolution should be universal with this scaling law. Interestingly, this feature is already present in the large $N_c$ limit, as demonstrated numerically in [21] for the dipole cross section in DIS. It it worth noting
that the simple scaling behavior found there according to the ratio of areas $R_A^2 / R_B^2$ may be a consequence of the simple scaling behavior put into the initial conditions. More sophisticated approaches based on the McLerran-Venugopalan model [22, 23], as used in [20] may lead to more complicated scaling factors with $A$ dependence varying from $A$ to $A^{2/3}$. It is intriguing to note that in this numerical study scaling seems to hold everywhere, implying that saturation is very efficient and tails do not play an important rôle. It deserves further investigation to find out if this is due to the initial conditions chosen there or a rather more generic feature of these evolution equations.

9 Conclusions

The bridge spanned by this paper, from the presentation of an infinite hierarchy of evolution equations given originally in Ref. [5], summarized here in a single equation for a generating functional, to the functional Fokker-Planck or, more precisely, Brownian motion and Langevin representations has led one to a point from which the whole content of this “small $x$ effective field theory” is much easier to understand.

This has allowed to prove the existence and attractiveness of a fixed point at small $x$ using simple arguments like positivity of the associated Fokker Planck Hamiltonian. This on its own makes it possible that there are universality features in experimental cross sections as one goes to smaller $x$. The nature of the evolution mechanism, including the driving force behind the unitarization mechanism have become apparent at this stage.

In particular the Langevin equation has proven to be a formulation that encodes much of the physics intuition about what should be going on in this dynamical regime in formulae in a one to one correspondence.

It has allowed to describe the unitarization mechanism at small $x$ in very physical terms as the limit in which the area over which the system appears to be black catches up with the total area, which itself scales with $\ln 1/x$ throughout evolution. The mechanism driving the race towards this limit may depend on the observable at hand, for the DIS cross section it has been demonstrated that it is the BFKL evolution that drives the system towards unitarization, quite in agreement with standard expectations.

With such a clear distinction between the onset of unitarization and the strict mathematical fixed point of the Brownian process itself, with the dynamical variables completely filling the configuration space, the latter has been found to be of only limited interest, as it refers to the limit at infinite energy with equally infinite total area and cross section. It is the much larger and experimentally accessible saturation region with its $\ln 1/x$ growth of the cross section that is the physics signal to be looked for.

The results obtained for correlators in the saturation region clearly exemplify how the system loses information about the initial conditions already at this stage and give a clear idea about the kind of universality to be expected at small $x$.

It is the $\sqrt{\ln 1/x}$ erasing behavior of the Brownian process that drives and characterizes these aspects of the evolution. But there is also conceptual clarity encoded in this formulation: Any such stochastic process is accompanied by a loss of information, a fact nicely in correspondence with the Wilson RG interpretation underlying the integrating out of degrees of freedom used to (re)derive the initial version of the equations in this paper.

At the same time the effective field theory nature encoded in the evolution equations has been complemented with a method that opens the doors to numerical simulations. The Langevin equations derived are suitable for this task and should help answer many if not all of the remaining “dynamical” questions. First on the list there is the question of saturation efficiencies and target dependence. This should give quantitative backing to expectations based on the BFKL limit and clarify open issues such as saturation in the transverse tails of the targets wave function as well as the amount of target dependence.

There are of course farther ranging options. One of them is the possibility to use the results of such evolution as an input to describe the rapidity dependence of nuclear collisions in an extension of the work laid out in [25] and made feasible numerically in [26, 27]. This would constitute one step deeper into the hierarchy of complexity detailed in Sec. 2.
Another intriguing question is that for the application of the methods developed here to the JKLW equation. With the basic correspondence of the approaches known, an at least numerical Langevin type analysis should be possible. To do so, an analysis of the force terms, which need not be vanishing there, and possibly a reformulation in terms of eikonal operators are to be performed. This, however, needs more effort and space than available here and will have to be left for the future.

Acknowledgements: This work was funded by the EC TMR Program, contract ERB FMRX-CT96-0008. I want to thank D.T. Son for this optimistic first time look at these functional RG equations which made me try again. As usual the "before" looked messier than the well ordered "after" snapshot. Thanks also to Y. Kovchegov and A. Kovner for discussions on various topics. A lot of help came from Olav Syljuasen, who proved time and again how much high energy/nuclear and condensed matter types have to teach each other. Kari Rummukainen encouraged me with his open reception of rather clumsy initial ideas for numerical simulations. Many thanks also to Larry McLerran for his questions when discussing a draft version of this paper.

A Feynman rules

Schematic Feynman rules used for these diagrams are given in the table below. All algebraic expressions needed in calculations are given in the main text.

| Expression                                                                 | Diagram |
|---------------------------------------------------------------------------|---------|
| \( \langle \delta A_x \delta A_y \rangle s \phi \) \( b \), the full gluon propagator in the presence of a "background" field \( b \) | ![Diagram](image) |
| Free gluon propagator                                                    | ![Diagram](image) |
| a quark line or \( U \) factor along \( x^- \)                           | ![Diagram](image) |
| an antiquark line or \( U^\dagger \) factor along \( x^- \)             | ![Diagram](image) |
| The location of the color field \( (G^{+\dagger}) \) of the target in \( x^- \) ordered diagrams | ![Diagram](image) |
B The vanishing of force terms

In this appendix, the vanishing of the force terms in the FP equation is demonstrated by looking at the \( q \) component in somewhat greater detail. There one finds

\[
\frac{1}{2} \int d^2 y \left( \frac{\delta}{\delta U^y_{ij}} \left[ \chi_{xy}^{gq} \right]_{ij kl} + \frac{\delta}{\delta U^y_{ij}} \left[ \chi_{xy}^{gq} \right]_{ij kl} \right) - \left[ \sigma_2^g \right]_{ij} = \\
= \frac{1}{2\pi^2} \int d^2 z \left\{ \frac{1}{2} \kappa_{\bar{x}z z} \left( -2 N_c [U_{\bar{x}z}]_{ij} + 2 [U_{\bar{x}z}^T tr(U_{\bar{x}z}^T U_{\bar{x}z})] \right) + \kappa_{\bar{x}z z} N_c \left( [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) \right. \\
+ \int d^2 y \kappa_{xy y} N_c \left( -2 [U_{\bar{x}z}]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) \delta_{yy}^{(2)} \\
+ \frac{1}{2} \left\{ - \kappa_{\bar{x}z z} N_c \left( [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) - \int d^2 y \kappa_{xy y} N_c \left( -2 [U_{\bar{x}z}]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) \delta_{yy}^{(2)} \\
+ \kappa_{\bar{x}z z} \left( [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) \right. \\
\left. \left. \left( [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} + [U_{\bar{x}z} U_{\bar{x}z}^T]_{ij} \right) \right) \right. \\
- \kappa_{\bar{x}z z} \left( [U_{\bar{x}z}]_{ij} tr(U_{\bar{x}z}^T U_{\bar{x}z}) - N_c [U_{\bar{x}z}]_{ij} \right) \right. \\
\right. \\
= 0 .
\]

A similar cancellation occurs in the \( \tilde{q} \) component. Note how the divergent terms cancel among the derivatives of \( \bar{\chi} \) before the finite remainder cancels against \( \bar{\sigma} \).

C Transforming the evolution kernel

C.1 Lie derivatives are the natural objects to consider

Suppose you never heard of Lie derivatives before. Do they come up naturally? They do: The first step is to express derivatives “along” \( SU(N_c) \) via the independent \( \frac{\delta}{\delta U} \) and \( \frac{\delta}{\delta U^T} \). This is done observing that, if one wants to respect the constraint \( U U^T = 1 \), one needs to replace for instance \( \frac{\delta}{\delta U^T} \) by

\[
\frac{\delta}{\delta U} \rightarrow \left( \frac{\delta}{\delta U^T} - U^T \delta \frac{\delta}{\delta U^T} \right)
\]

which clearly does the trick

\[
\left( M_{ij} \frac{\delta}{\delta U^T} - [U^T MU^T]_{ij} \right) \left\{ (U^T)_{\alpha\beta} \right\} = \left\{ (M U^T - U U^T M U^T)_{\alpha\beta} \right\} \rightarrow 0
\]

on \( SU(N_c) \). The only thing missing to turn the above into Lie derivatives is that they do not obey the correct commutation rules. This, however, can be achieved for the price of a variable change. Proper Lie derivatives on \( SU(N_c) \) may be written just like above, but with \( U^{(1)} \) dependent \( M \). For this example letting \( M \rightarrow U^{(A)} \), one finds

\[
\left( M_{ij} \frac{\delta}{\delta U^T} - [U^T MU^T]_{ij} \right) \left\{ \left[U^{(A)}\right]_{\alpha\beta} \right\} \frac{\delta}{\delta U^T} \rightarrow \left\{ \left[U^{(A)}\right]_{\alpha\beta} \right\} \frac{\delta}{\delta U^T} \left\{ U^{(A)} \right\} \frac{\delta}{\delta U^T} \rightarrow 0
\]

With \( A = a \in \left\{ 1, \ldots, N_c^2 - 1 \right\} \) one easily extracts the \( SU(N_c) \) commutation relations for \( [U^a]^T \delta U^T_{ij} - [U^T MU^T]_{ij} \delta U^T_{ij} \) for \( U \in SU(N_c) \) while still taking the derivatives with respect to \( U \) and \( U^T \) independently.
C.2 A little algebra

There is a little catch that one has to deal with when trying to reexpress things in these terms which has already been anticipated in the above notation. In order to be able to represent all of \( \delta/\delta U_{ij} \) components one has to allow \( A \) to run over \( A = \{0, \ldots, N_c^2 - 1\} \) and supply \( t^0 = 1/\sqrt{2N_c} \), such that \( \text{tr}(t^A t^B) = \delta^{AB}/2 \) for all \( A, B \in \{0, N_c^2 - 1\} \). To perform the transformation, one writes

\[
M_{ij} = M^A_{ij} =: N^A [U t^A]_{ij} \xrightarrow{2\text{tr}(t^B) \to M^B} N^A 2\text{tr}(t^B U t^A)
\]

\[
M_{ij} = M^A_{ij} =: \tilde{N}^A [-t^A U^\dagger]_{ij} \xrightarrow{2\text{tr}(t^B) \to M^B} -\tilde{N}^A 2\text{tr}(t^B t^A U^\dagger)
\]

and one is left with inverting \( \text{tr}(t^B U t^A) \) and \( \text{tr}(t^B t^A U^\dagger) \). Using completeness

\[
t^A_{\alpha\beta\gamma\delta} t^0_{\alpha\beta\gamma\delta} + t^a_{\alpha\beta\gamma\delta} = \frac{1}{2} \delta_{\alpha\delta} \delta_{\beta\gamma} \Rightarrow 2\text{tr}(K t^B) 2\text{tr}(t^B L) = 2\text{tr}(KL)
\]

one gets

\[
2\text{tr}(t^A U t^B) 2\text{tr}(t^B U^{-1} t^C) = 2\text{tr}(t^A U U^{-1} t^C) = \delta^{AC}
\]

\[
2\text{tr}(t^A t^B U^\dagger) 2\text{tr}(t^B t^C (U^\dagger)^{-1}) = \delta^{AC}.
\]

The transformation is then done according to either variant in

\[
M_{ij} = 2\text{tr}(t^A U^{-1} M) [U t^A]_{ij} = -2\text{tr}((U^\dagger)^{-1} t^A M) [-t^A U^\dagger]_{ij}, \tag{71}
\]

taking proper account of any issues of chain rule when used for the Lie derivatives. Note that here “\(-1\)” is really “\(-1\)” as in “inverse” not “\(\dagger\)”.

To see how (71) affects the FP equation, explore how \( \bar{\chi} \) transforms. When doing so, chain rule terms appear in combinations like

\[
\frac{\delta}{\delta U_{ij}} M_{ij} = \frac{\delta}{\delta U_{ij}} (2\text{tr}(t^A U^{-1} M) [U t^A]_{ij})
\]

\[= [U t^A]_{ij} \frac{\delta}{\delta U_{ij}} (2\text{tr}(t^A U^{-1} M)) + N_c \text{tr}(U^{-1} M) \tag{72a}
\]

or

\[
\frac{\delta}{\delta U^\dagger_{ij}} M_{ij} = \frac{\delta}{\delta U^\dagger_{ij}} (2\text{tr}((U^\dagger)^{-1} t^A M) [-t^A U^\dagger]_{ij})
\]

\[= [-t^A U^\dagger]_{ij} \frac{\delta}{\delta U^\dagger_{ij}} (-2\text{tr}((U^\dagger)^{-1} t^A M)) + N_c \text{tr}((U^\dagger)^{-1} M) \tag{72b}
\]

where I have used \( \frac{\delta}{\delta U_{ij}} [U t^A]_{ij} = \delta_{ik} \delta_{jl} \frac{\delta}{\delta U_{ij}} [U t^A]_{kl} = N_c \text{tr}(t^A) = \delta^A 0 N_c \sqrt{N_c/2} \) to get the last term.
C.3 Results for the evolution kernel

Chain rule aside for a moment, the components of the transformed $\bar{\chi}$ are then of the form

$$\alpha_s^2 \left[ t^A U^{-1}_x \right]_{ji} \left[ \bar{\chi}^{\text{rev}}_{xy} \right]_{kl} (2 [U^+_y] - 1 [t^B]_{kl}) = -4 \frac{\alpha_s}{2 \pi^2} \int d^2 z \ K_{xzy} \tag{73a}$$

$$\times \left( \text{tr}(t^A U^{-1}_x U^+_z t^B U^{-1}_z U_x) + \text{tr}(t^A U^+_z U^{-1}_y t^B U^{-1}_y U_x) - \text{tr}(t^A U^{-1}_x) - \text{tr}(t^B U^{-1}_y U_x) \right)$$

$$\alpha_s (-2 [t^A U^{-1}_x]_{ji} \left[ \bar{\chi}^{\text{rev}}_{xy} \right]_{kl} 2 [t^B U^{-1}_y]_{kl} = -4 \frac{\alpha_s}{2 \pi^2} \int d^2 z \ K_{yzy} \tag{73b}$$

$$\times \left( \text{tr}(t^A U^{-1}_x U^+_y t^B U^{-1}_y U_z) + \text{tr}(U^+_z U^{-1}_y U^+_y U_z) - \text{tr}(t^A U^{-1}_x U^+_y t^B U^{-1}_y U_z) \right)$$

$$\alpha_s^2 [t^A U^{-1}_x]_{ji} \left[ \bar{\chi}^{\text{rev}}_{xy} \right]_{kl} 2 [t^B U^{-1}_y]_{kl} = -4 \frac{\alpha_s}{2 \pi^2} \int d^2 z \ K_{xyz} \tag{73c}$$

$$\times \left( \text{tr}(t^A U^{-1}_x U^+_y U^+_z U_x) + \text{tr}(U^+_z U^{-1}_y U^+_y U_x) - \text{tr}(t^A U^{-1}_x U^+_y U^+_y U_x) \right)$$

The key features of these expressions are as follows:

- in $SU(N_c)$ all of them vanish if either $A$ or $B$ is 0. Only octet components survive. All matrices involved now are explicitly in the adjoint representation. This does not hold outside $SU(N_c)$ for the terms involving $U$'s at $z$.
- Combining Eqs. (72) with Eqs. (73) one concludes that all chain rule terms also vanish in $SU(N_c)$.
- in $SU(N_c)$ all four of the above are equal.
- This unique common form even factorizes naturally into the square of a much simpler factor. It will be denoted by $\bar{\chi}^{ab}_{xy}[U]$ and reads

$$\alpha_s \bar{\chi}^{ab}_{xy}[U] := - \frac{\alpha_s}{\pi} \int d^2 z \ K_{xzy} \left( \bar{U}^{-1}_x U_y + \bar{U}^{-1}_x U_z - \bar{U}^{-1}_x U_y - 1 \right)^{ab}. \tag{74}$$

The factorized version is given in Eq. (41).

D Reduction to the gauge group: details of proof

It is actually instructive to see how Eq. (45) comes about. The cancellations involved will make it clear how evolution equations for correlators are recovered from the new form of the evolution equation. The easiest way to prove Eq. (45) is again by doing it for a generating functional for all those correlators in one go. In other words, one simply compares the action of the two operators on arbitrary correlators in the guise of $e^{S^{\text{ext}}[U,J]}$. The identity to prove then is simply

$$\frac{1}{2} \frac{\delta}{\delta U_u} \bar{\chi}^{ab}_{uv} \frac{\delta}{\delta U_v} e^{S^{\text{ext}}[U,J]} \bigg|_{U=U^+} = \frac{1}{2} \nabla_u \nabla_a \bar{\chi}^{ab}_{xy} \nabla_y \nabla_b e^{S^{\text{ext}}[U,J]} \bigg|_{U=U^+} \tag{75}$$

With the vanishing of the force term known, Eq. (36), the l.h.s. is straightforwardly given by

$$\frac{1}{2} \frac{\delta}{\delta U_u} \bar{\chi}^{ab}_{uv} \frac{\delta}{\delta U_v} e^{S^{\text{ext}}[U,J]} \bigg|_{U=U^+} = \left\{ J_a \bar{\sigma}_u [U] + J_a J_b \left[ \frac{1}{2} \bar{\chi}^{ab}_{uv} [U] \right] e^{S^{\text{ext}}[U,J]} \bigg|_{U=U^+} \right\} \tag{76}$$

\[^{18}\text{Again, the arbitrariness of the correlators results from that of the sources } J \text{ and the equations for correlators follow by differentiation w.r.t. } J \text{ at } J = 0.\]
In summary, all the evolution does in a finite interval is to evolve any $U_x(y)$ into

$$U_x(y) = U_x(y_0) V_x(y)$$  \hspace{1cm} \text{(82)}$$

while the r.h.s is

$$\frac{1}{2} i \nabla^a_{U_x} \hat{\chi}^{ab}_{xy} i \nabla^b_{U_y} S_{ext}[U,J] |_{UU=1} = \frac{1}{2} \left\{ \left( i \nabla^a_{U_x} \hat{\chi}^{ab}_{xy} \right) i \nabla^b_{U_y} + \hat{\chi}^{ab}_{xy} i \nabla^a_{U_x} i \nabla^b_{U_y} \right\} S_{ext}[U,J] |_{UU=1}$$

$$= \frac{1}{2} \left\{ \left( i \nabla^a_{U_x} \hat{\chi}^{ab}_{xy} \right) + \hat{\chi}^{ab}_{xy} \right\} \text{tr} \left( (J^1)_y U_y^{b \dagger} - J^1_y U_y^{b \dagger} U_y^{-1} \right) S_{ext}[U,J] |_{UU=1}$$

$$= \frac{1}{2} \left\{ \left( i \nabla^a_{U_x} \hat{\chi}^{ab}_{xy} \right) \text{tr} \left( (J^1)_y U_y^{b \dagger} U_y^{-1} \right) + \hat{\chi}^{ab}_{xy} \text{tr} \left( (J^1)_y U_y^{b \dagger} U_y^{-1} \right) \delta_{xy}^{(2)} \right\}$$

$$+ \left\{ (J^1)_x U_y^{a \dagger} U_y^{a \dagger} - J^1_x U_y^{a \dagger} U_y^{a \dagger} \right\} \hat{\chi}^{ab}_{xy} \text{tr} \left( (J^1)_y U_y^{b \dagger} U_y^{-1} \right) S_{ext}[U,J] |_{UU=1}$$

$$= \left\{ J_x \hat{\sigma}_a [U] + J_x \hat{J}_x \frac{1}{2} \hat{\chi}^{ab}_{xy} \right\} S_{ext}[U,J] |_{UU=1}$$

\textit{q.e.d.}

Note how the $\hat{\sigma}$ terms emerge. Quite a surprising combination.

### E From Lie derivatives to zero modes of the FP Hamiltonian

#### E.1 Lie derivatives and transverse local gauge transformations

There is more to be said besides\(^{19}\)

\begin{align}
    i \nabla^a_{U_x} R(U_y) &= R(U_x) R(t^a) \delta_{xy}^{(2)} \\
    i \nabla^a_{U_x} R(U_y^{-1}) &= - R(t^a) R(U_x^{-1}) \delta_{xy}^{(2)}
\end{align}

(78)

to characterize the Lie derivatives as an infinitesimal gauge transformation. They truly do implement \textit{transverse} local gauge transformations, a feature that can be summarized by

\begin{align}
    \omega_x^a i \nabla^a_{U_x} \left( \partial_i R(U) \right)_y &= R(U_x) \omega_x^a R(t^a) \left( \partial_i \delta_{xy}^{(2)} \right) \\
    \omega_x^a i \nabla^a_{U_x} \left( \partial_i R(U^{-1}) \right)_y &= - \omega_x^a R(t^a) R(U_x^{-1}) \left( \partial_i \delta_{xy}^{(2)} \right)
\end{align}

(79)

This implements an infinitesimal change to $U$ in such a way that (restricting to the fundamental representation for brevity)

$$\omega_x^a i \nabla^a_{U_x} U_y^{-1} \left( \partial_i U \right)_y = \left[ \partial_i^a + U_y^{-1} \left( \partial_i U \right)_x \omega_x^a t^a \right] \delta_{xy}^{(2)}$$

(80)

Clearly this is an infinitesimal local gauge transformation (with respect to the transverse coordinate) of a pure gauge potential $b_i = U_x^{-1} \left( \partial_i U \right)_x$.

This is in one to one correspondence with the Langevin equation which also implements an infinitesimal transverse local transformation according to

$$U_x \rightarrow U_x \left( 1 + it^a \omega_x^a \right)$$

(81)

in a (randomly) $\ln 1/x$ dependent manner in each step of the evolution.

In summary, all the evolution does in a finite $x$ interval is to evolve any $U_x(y_0)$ into

$$U_x(y) = U_x(y_0) V_x(y)$$

\(^{19}\) $R(U_y)$ indicates $U_y$ in a given representation $R$. Just leave it off for the fundamental representation.
where $y = \ln 1/x$ is the rapidity.

It is important to note, however, that this notion of a local gauge transformation only applies to the transverse coordinate dependence. Any $y$ dependence is related to coarse graining in the $\pm$ plane and will not behave like a gauge transformation with respect to these coordinates.

### E.2 A unique zero mode

With the nature of the evolution clearly established, the zero modes of the FP Hamiltonian are also quite easily listed: Any function of only scalar, (transversally) locally gauge invariant operators will do as a candidate for $\hat{Z}_\infty[U]$. The list of such arguments is in practice much shorter that it might initially appear. Indeed, purely (transversally) 2-d operators are generically trivial: There the only ones one could write would have to contain at least one factor of $F_{ij}[U^{-1}\partial U]$ and that is, of course, zero.

All other apparent candidates would necessarily involve a $\pm$ derivative, say via $F^{+i}$, and will necessarily not be invariant.\footnote{In particular there is no argument that the evolution would leave the support of $F^{+i}$ invariant as would be the case had the transverse and $\pm$ coordinates been treated on equal footing. Accordingly the way in which the area where $F^{+i}$ has nonvanishing contributions will grow. It will grow with probably with the same rate as the transverse region where $U_x(y)$ and hence $U_x^{-1}(y)(\partial U)_x(y)$ or, for example, the dipole amplitude $1 - U_x(y)U_y(y)$ and hence the corresponding cross section have their support. The $\ln 1/x$ scaling law which follows from the scaling argument introduced in Sec. 7 also applies here.} In particular the weight of the MV model is not an invariant of this RG, neither in its initial form nor its “color neural” setup according to [24]. The same argument excludes any “natural” topological term of the Wess Zumino type $\int \epsilon^{ABC} \text{tr} (U^{-1}(\partial_A U)U^{-1}(\partial_B U)U^{-1}(\partial_C U))$ in which the third coordinate would have to be taken again from within the $\pm$ plane. This exhausts the list of all possible candidates of any dimension and leaves a “trivial” but unique solution: $\hat{Z}_\infty[U] = 1$.

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