\textbf{X: AN EQUIVARIANT K-THEORY FUNCTORS FROM SPACES TO \(\lambda\)-RINGS}

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\textbf{Abstract.} For a compact Hausdorff space \(X\), let
\[
\mathcal{X}(X) = \bigoplus_{n \geq 0} K_{\Sigma_n}(X^n).
\]
We show that \(\mathcal{X}\) takes values in \(\lambda\)-rings and satisfies a Thom isomorphism. In the case that \(X\) is constructed entirely of even-dimensional cells, \(\mathcal{X}(X)\) is the free \(\lambda\)-ring on generators in one-to-one correspondence with those cells.

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1. Introduction

Grothendieck first defined \(\lambda\)-rings in some unpublished work in 1957 and published in [4] under the name “special \(\lambda\)-rings”. In short a \(\lambda\)-ring is a ring \(A\), with certain set functions \(\lambda^n : A \rightarrow A\) that satisfy the relations that are satisfied by the exterior powers of vector spaces. Examples of \(\lambda\)-rings include complex representation rings of groups and K-theory groups of spaces. In both cases the \(\lambda\)-ring functions come from exterior power of vector spaces.

In this paper, we introduce a homotopy functor from suitable pairs of topological spaces to \(\lambda\)-rings. For a compact, Hausdorff \(G\)-space \(X\) and a closed subspace \(A\), let \(K_G(X, A)\) be the \(G\)-equivariant K-theory of the pair \((X, A)\). The notation \((X, A)^n\) denotes the pair
\[
(X^n, (X^{n-1} \times A) \cup (X^{n-2} \times A \times X) \cup \ldots \cup (A \times X^{n-1})).
\]
These are \(\Sigma_n\)-spaces. We then let
\[
\mathcal{K}(X, A) = \bigoplus_{n \geq 0} K_{\Sigma_n}((X, A)^n)
\]
and write \(\mathcal{K}(X, \emptyset)\) as \(\mathcal{K}(X)\).

1.1. Main Results.

Our first result is

\textbf{Theorem 1.} \(\mathcal{K}(X, A)\) is a graded \(\lambda\)-ring. Furthermore, if each \(K_{\Sigma_n}((X, A)^n)\) is a finitely generated and free and \(K_{\Sigma_n}^1((X, A)^n) = 0\) for all \(n\), then \(\mathcal{K}(X, A)\) is also equipped with a coproduct that makes it into a Hopf algebra.
The structure on $\mathcal{K}(X, A)$ arises as follows. For each $i, j \geq 0$ we have an inclusion $\Sigma_i \times \Sigma_j \leq \Sigma_{i+j}$. This induces maps

$$K_{\Sigma_i}((X, A)^i) \otimes K_{\Sigma_j}((X, A)^j) \to K_{\Sigma_i \times \Sigma_j}((X, A)^{i+j}) \xrightarrow{\text{Ind}_{\Sigma_i \times \Sigma_j}} K_{\Sigma_{i+j}}((X, A)^{i+j})$$

and

$$K_{\Sigma_{i+j}}((X, A)^{i+j}) \xrightarrow{\text{Res}_{\Sigma_i \times \Sigma_j}} K_{\Sigma_i \times \Sigma_j}((X, A)^{i+j}) \to K_{\Sigma_i}((X, A)^i) \otimes K_{\Sigma_j}((X, A)^j).$$

These give the product, and in the cases listed above, the coproduct on $\mathcal{K}(X, A)$.

It is also good to note that the functor $\mathcal{K}$ satisfies excision. A functor of pairs, $F$, satisfies excision if when $X = A \cup B$ is the union of two closed subspaces, then

$$F(X, A) \cong F(B, B \cap A).$$

So, suppose that $X = A \cup B$. Then

$$X^n = B^n \cup [(X^{n-1} \times A) \cup \ldots \cup (A \times X^{n-1})]$$

and

$$B^n \cap [(X^{n-1} \times A) \cup \ldots \cup (A \times X^{n-1})] = [B^{n-1} \times (A \cap B)] \cup \ldots \cup [(A \cap B) \times B^{n-1}].$$

Since each $K_{\Sigma_n}$ is a cohomology functor, it satisfies excision. Using the above, we have

$$K_{\Sigma_n}((X, A)^n) \cong K_{\Sigma_n}((B, B \cap A)^n).$$

Adding these up we get

**Theorem 2.** (Excision) If $X = A \cup B$ is a union of two closed subspaces, then

$$\mathcal{K}(X, A) \cong \mathcal{K}(B, B \cap A).$$

We obtain a $\lambda$-ring structure as follows. If $E \in K_{\Sigma_i}((X, A)^i)$, the $n$-fold exterior product $E^{\otimes_n} \in K_{\Sigma_i \Sigma_n}((X, A)^{in})$. Then $\lambda^n(E) \in K_{\Sigma_n}((X, A)^{in})$ is defined by

$$\lambda^n(E) = \text{Ind}_{\Sigma_i \Sigma_n}^{\Sigma_n} (\text{sgn}_n \otimes E^{\otimes_n})$$

where $\text{sgn}_n$ is the sign representation of $\Sigma_n$.

Our next main result is that the functor $\mathcal{K}$ satisfies a Thom isomorphism.

**Theorem 3.** If $E \to X$ is a vector bundle with associated disk and sphere bundles $D(E)$ and $S(E)$, there is an isomorphism of $\lambda$-rings:

$$\mathcal{K}(X, A) \to \mathcal{K}(D(E), S(E) \cup D(E)|_A).$$

Our final result concerns the restriction of $\mathcal{K}$ to a certain subclass of spaces. Let $\mathfrak{F}$ be the free $\lambda$-ring functor from the category of abelian groups to $\lambda$-rings.

**Theorem 4.** If $X$ is obtained from $A$ by attaching finitely many even-dimensional cells, then the natural map of $\lambda$-rings

$$\mathfrak{F}(K(X, A)) \to \mathcal{K}(X, A)$$

is an isomorphism.

### 1.2. Relation to the Literature.

This functor $\mathcal{K}$ is not new. We remark that

$$\mathcal{K}(\text{pt}) = \bigoplus_{n \geq 0} R(\Sigma_n)$$

has been known to be the free $\lambda$-ring on one generator at least since [6] and our work presents another proof of this fact.

In response to a comment by Grojnowski [3], Graeme Segal, studied the object

$$\bigoplus_{n \geq 0} K_{\Sigma_n} (X^n) \otimes \mathbb{C}$$
in an unpublished note [8]. Weiqiang Wang replaced $\Sigma_n$ with $G \wr \Sigma_n$, the wreath product of $\Sigma_n$ with a finite group $G$ [11]. He proves that

$$\bigoplus_{n \geq 0} K_{G; \Sigma_n}(X^n) \otimes \mathbb{C}$$

is the free lambda ring on $K_G(X)$. Moreover, it is shown that $\mathcal{K}(X) \otimes \mathbb{C}$ is a Fock space associated to a certain Heisenberg algebra.

Another connection to the literature is as follows: There are maps

$$K_{\Sigma_n}((X, A)^n) \rightarrow K(E\Sigma_n \times_{\Sigma_n} (X, A)^n)$$

which are algebraic completions by the Atiyah-Segal Completion Theorem [10] Proposition 4.2]. Let

$$\mathbb{P}(X, A) = \prod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} (X, A)^n.$$ 

Combined, the above maps define a natural map

$$\mathcal{K}(X, A) \rightarrow K(\mathbb{P}(X, A)).$$

Furthermore, if $X$ is path connected, $K(\mathbb{P}(X, pt))$ identifies with $K(\Omega^\infty \Sigma^\infty X)$. This isomorphism will not necessarily be an isomorphism of $\lambda$-rings. In general the former will be an "associated graded" version of the latter, so that $K(\Omega^\infty \Sigma^\infty X)$ can be calculated as an appropriate algebraic completion of $\mathcal{K}(X, pt)$.

In 2011, $\mathcal{K}(X)$ has shown up in the work of Nora Ganter and Mikhail Kapranov as the Grothendieck group of the symmetric powers in the category of coherent sheaves on a smooth projective variety [2].

### 1.3. Methods of Proof.

We prove that $\mathcal{K}(X, A)$ is a $\lambda$-ring by showing that it is a $\tau$-ring in the sense of [5] and use his result that every $\tau$-ring has the structure of a $\lambda$-ring. The categories are, in fact, equivalent. We do this in Section 3 after a brief introduction to $\lambda$-rings and $\tau$-rings in Section 2. We prove Theorem 3 in Section 4 and Theorem 4 in Section 5. We prove Theorem 3 by showing that the usual Thom isomorphism can be extended to the structure on $\mathcal{K}(X, A)$. We prove Theorem 4 by induction on cells.

### 2. $\lambda$-Rings and $\tau$-Rings

We begin with the definition of a $\lambda$-ring.

**Definition 2.1.** A $\lambda$-ring is a commutative ring with unit, $A$, and a set of functions for each $n = 0, 1, 2, \ldots$

$$\lambda^n : A \rightarrow A$$

satisfying:

(i) $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for $x \in A$

(ii) $\lambda^k(x + y) = \sum_{i=0}^k \lambda^i(x)\lambda^{k-i}(y)$ for $x, y \in A$

(iii) $\lambda^k(xy) = \prod_{i,j} (1 + a_i b_j t) = \sum_k p_k(\sigma_1(\bar{a}), \ldots, \sigma_k(\bar{a}), \sigma_1(\bar{b}), \ldots, \sigma_k(\bar{b})) t^k$

The $\sigma_i$ are the elementary symmetric polynomials.

(iv) $\lambda^k(\lambda^n(x)) = q_{k,l}(\lambda^1(x), \ldots, \lambda^l(x))$ for $x \in A$. Here $q_{k,l}$ is the unique polynomial such that

$$\prod_{i_1 < \cdots < i_t} (1 + x_{i_1} \cdots x_{i_t} t) = \sum_k q_{k,l}(\sigma_1(\bar{x}), \ldots, \sigma_k(\bar{x})) t^k$$

in $\mathbb{Z}[x_1, \ldots, x_n, t]$. 
A basic example of a $\lambda$-ring is the $G$-equivariant $K$-theory of a compact Hausdorff space, $X$, where the $\lambda$-operations are the exterior powers of vector bundles.

Now for the definition of a $\tau$-ring. Suppose that $A$ is a commutative ring with unit. Define

$$AP = \prod_{n \geq 0} A \otimes R(\Sigma_n)$$

where $R(\Sigma_n)$ is the representation ring of the $n$th symmetric group. Furthermore let

$$AP^{(2)} = \prod_{n, m \geq 0} A \otimes R(\Sigma_n) \otimes R(\Sigma_m), \quad AQ = \prod_{n \geq 0} A \otimes R(\Sigma_n \lhd \Sigma_n)$$

We have a variety of maps that we must define before we can define a $\tau$-ring. First of all, we have a multiplication map

$$\times : AP \otimes AP \to AP.$$

The $n$th coordinate of this map is given by the composite

$$AP \to \prod_{i + j = n} A \otimes R(\Sigma_i) \otimes A \otimes R(\Sigma_j) \to A \otimes R(\Sigma_n).$$

Where the first map is projection and the second map is the induction multiplication in $R_n$.

Our next multiplication comes from the levelwise internal tensor product on each $R(\Sigma_n)$. This will induce a map

$$\cdot : AP \otimes AP \to AP$$

with $n$th coordinate

$$AP \otimes AP \to A \otimes R(\Sigma_n) \otimes A \otimes R(\Sigma_n) \to A \otimes R(\Sigma_n)$$

where the first map is projection and the second sends $a_1 \otimes V \otimes a_2 \otimes W$ to $a_1 a_2 \otimes (V \otimes W)$.

We have a map

$$\square : AP \to AQ.$$

This is induced by the restriction maps $R(\Sigma_{mn}) \to R(\Sigma_m \lhd \Sigma_n)$ and has $(n, m)$ coordinate function:

$$AP \to A \otimes R(\Sigma_{mn}) \to A \otimes R(\Sigma_m \lhd \Sigma_n).$$

Where the first map is projection and the second is the aforementioned restriction.

From the coproduct structure on $\oplus R(\Sigma_n)$ we obtain a map

$$\Delta : AP \to AP^{(2)},$$

with $(i, j)$ coordinate function

$$AP \to A \otimes R(\Sigma_n) \to A \otimes R(\Sigma_i) \otimes R(\Sigma_j).$$

Where $n = i + j$; the first map is projection; and the second map is the coproduct.

Next, we have a map

$$\mu : AP \otimes AP \to AP^{(2)}.$$

The $(i, j)$ coordinate is the composite

$$AP \otimes AP \to A \otimes R(\Sigma_i) \otimes A \otimes R(\Sigma_j) \to A \otimes R(\Sigma_i) \otimes R(\Sigma_j).$$

Where the first map is projection and the second is multiplication in $A$.

Finally, let $e_n$ denote the trivial representation of $R(\Sigma_n)$, with the convention that $e_0 = 1 \in \mathbb{Z}$. With this notation we may now introduce the notion of a $\tau$-ring.

**Definition 2.2.** A $\tau$-ring is a commutative ring with unit, $A$, and a function

$$\tau : A \to AP$$

satisfying:

1. $\tau(x) \in 1 \otimes e_0 + x \otimes e_1 + \prod_{n \geq 2} A \otimes R(\Sigma_n)$ for all $x \in A$.
2. $\tau(x + y) = \tau(x) \times \tau(y)$, for $x, y \in A$.
(3) The following commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\tau^{(2)}} & AP \otimes AP \\
\downarrow \tau & & \downarrow \mu \\
AP & \xrightarrow{\Delta} & AP^{(2)}
\end{array}
\]

where \(\tau^{(2)}\) is the map \(a \mapsto \tau(a) \otimes \tau(a)\).

(4) \(\tau(xy) = \tau(x) \cdot \tau(y)\) for \(x, y \in A\).

(5) The following commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\tau} & AP \\
\downarrow \tau & & \downarrow \hat{\tau} \\
AP & \xrightarrow{\Box} & AQ
\end{array}
\]

The definition of \(\hat{\tau}\) is a little involved: Assume \(\tau : A \to AP\) is a map that satisfies the first three properties for a \(\tau\)-ring. Let \(\tau^k\) be the \(k\)th component of \(\tau\). We have the map

\[A \times R(\Sigma_l) \to A \otimes R(\Sigma_k) \otimes R(\Sigma_l \wr \Sigma_k)\]

which takes \((x, V)\) to \(\tau^k(x) \otimes V^{\otimes k}\). Next we have a projection map

\[p : \Sigma_l \wr \Sigma_k \to \Sigma_k\]

that induces a pullback map

\[p^* : R(\Sigma_k) \to R(\Sigma_l \wr \Sigma_k)\]

This leads to another map

\[\text{Id}_A \otimes p^* \otimes \text{Id}_{R(\Sigma_l \wr \Sigma_k)} : A \otimes R(\Sigma_k) \otimes R(\Sigma_l \wr \Sigma_k) \to A \otimes R(\Sigma_l \wr \Sigma_k) \otimes R(\Sigma_l \wr \Sigma_k)\]

Finally, we note that there is an internal tensor product on \(R(\Sigma_l \wr \Sigma_k)\), call it \(\gamma\). Our last map is:

\[\text{Id}_A \otimes \gamma : A \otimes R(\Sigma_l \wr \Sigma_k) \otimes R(\Sigma_l \wr \Sigma_k) \to A \otimes R(\Sigma_l \wr \Sigma_k)\]

The composition of all the above maps is denoted

\[\hat{\tau}^k : A \times R(\Sigma_l) \to A \otimes R(\Sigma_l \wr \Sigma_k)\]

We define \(\hat{\tau}\) to be the product over all \(k\). Since \(\tau\) satisfies the first three properties of a \(\tau\)-ring, it is bi-additive from addition to cross product. Thus it extends uniquely to a function

\[\hat{\tau} : A \otimes R(\Sigma_l) \to A \otimes \prod_{k \geq 0} R(\Sigma_l \wr \Sigma_k)\]

Put

\[\hat{\tau} = \prod \hat{\tau}_i\]

With our definition of a \(\tau\)-ring in place, let us now show how to obtain a \(\lambda\)-ring from a \(\tau\)-ring. For \(n = 1, 2, 3, \ldots\) let

\[g_n : AP \to A\]

be the composite

\[AP \to A \otimes R(\Sigma_n) \xrightarrow{g'_n} A\]

Here the first map is projection and the second is

\[g'_n(a \otimes \nu) = \begin{cases} a & \text{if } \nu = \text{sgn}_n \\ 0 & \text{otherwise} \end{cases}\]

Here the \(\nu_i\) are the irreducible representations of \(\Sigma_n\) and \(\text{sgn}_n\) is the sign representation of \(\Sigma_n\). We may define \(\lambda\) operations by \(\lambda^n(a) = g_n(\tau(a))\).
3. \( \mathcal{K}(X, A) \) is a \( \tau \)-ring

The proof presented is only for \( \mathcal{K}(X) \), but is easily seen to be valid for the functor of pairs \( \mathcal{K}(X, A) \). It is presented in this way to provide ease of exposition.

Let us introduce some notation. Let

\[
\mathcal{K}P = \prod_{i \geq 0} \mathcal{K}(X) \otimes R(\Sigma_i),
\]

\[
\mathcal{K}P^{(2)} = \prod_{i,j \geq 0} \mathcal{K}(X) \otimes R(\Sigma_i) \otimes R(\Sigma_j),
\]

and

\[
\mathcal{K}Q = \prod_{k,l \geq 0} \mathcal{K}(X) \otimes R(\Sigma_l \wr \Sigma_k).
\]

Maps \( \tau_n^m \) will be defined for \( n, m \geq 1 \). Then, \( \tau \) will simply be the product over all \( n \) and \( m \).

Let \( E \) be a \( \Sigma_n \)-bundle over \( X^n \). First we have the \( m \)-th power map for \( m \geq 1 \):

\[
P_n^m : K_{\Sigma_n}(X^n) \to K_{\Sigma_n \wr \Sigma_m}(X^{nm})
\]

that sends \( E \) to \( E^{\otimes m} \), the external tensor product. Here \( \Sigma_m \) acts by \( g \cdot (v_i) = (v_{g^{-1}i}) \) and \( \Sigma_n \) acts diagonally. The power map and some of its elementary properties are discussed in [7, Section 1]. A second family of maps is constructed as follows. Embed \( \Sigma_n \wr \Sigma_m \) as a subgroup of \( \Sigma_m \times \Sigma_m \), via the product of the canonical quotient map \( \Sigma_n \wr \Sigma_m \to \Sigma_m \) with the usual inclusion map \( \Sigma_n \wr \Sigma_m \to \Sigma_m \) given by concatenation. If we regard \( X^{nm} \) as a \( \Sigma_m \times \Sigma_m \) space where the first factor acts trivially and the second acts in the standard manner, the restriction of this action to our embedded subgroup is the standard action of \( \Sigma_n \wr \Sigma_m \) on \( X^{nm} \). We can therefore define the map for \( m \geq 1 \):

\[
\delta_n^m = \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_m \times \Sigma_m} : \mathcal{K}_{\Sigma_n \wr \Sigma_m}(X^{nm}) \to \mathcal{K}_{\Sigma_m \times \Sigma_m}(X^{nm}) \cong \mathcal{K}_{\Sigma_m}(X^{nm}) \otimes R(\Sigma_m).
\]

The induction map is also discussed in [7, Section 1] and is a generalization of the induction in the groups setting. We then define \( \tau_n^m \) to be the composition of \( P_n^m \) followed by \( \delta_n^m \). The \( \tau \) map is then the product over all \( m \) and \( n \) of \( \tau_n^m \).

Before we begin the proof, it should be noted that the resulting \( \lambda \)-ring structure is simply

\[
\lambda^m(E) = \text{Ind}_{\Sigma_m \times \Sigma_m}^{\Sigma_m \times \Sigma_m}(\text{sgn}_m \otimes E^{\otimes m}).
\]

In the case that \( X = \text{pt} \), this is the plethysm operation.

In order to show that this is indeed a \( \tau \)-ring structure a few things must be shown. Each will be introduced and proven in turn. First we show that it is a pre-\( \tau \)-ring, i.e. it satisfies these first three properties:

1. \( \tau(x) \in 1 \otimes e_0 + x \otimes e_1 + \prod_{i \geq 2} \mathcal{K}(X) \otimes R(\Sigma_i) \), where \( e_0 \) and \( e_1 \) are the trivial representations of \( \Sigma_0 \) and \( \Sigma_1 \).

   \textbf{Proof.} First, we have assumed that \( \tau_0^0 \) is the constant map \( 1 \otimes e_0 \). Now, the map \( P_n^1 \) is just the identity map on \( K_{\Sigma_n}(X^n) \). And for \( m = 1 \) the embedding of \( \Sigma_n = \Sigma_n \wr \Sigma_1 \) in \( \Sigma_m = \Sigma_1 \times \Sigma_n \) is again the identity and so \( x \in K_{\Sigma_n}(X^n) \) certainly maps to \( x \otimes e_1 \).

2. \( \tau(x + y) = \tau(x) \times \tau(y) \)

   \textbf{Proof.} Notice that we already have a cross product structure on \( \prod_{i \geq 0} K_{\Sigma_i \wr \Sigma_i}(X^{ni}) \) given by the composition of:

\[
K_{\Sigma_n \wr \Sigma_m}(X^{nm}) \otimes K_{\Sigma_k \wr \Sigma_k}(X^{nk}) \xrightarrow{\otimes} K_{\Sigma_n \wr \Sigma_m \times \Sigma_k \wr \Sigma_k}(X^{nm} \times X^{nk})
\]

with

\[
K_{\Sigma_n \wr \Sigma_m \times \Sigma_k \wr \Sigma_k}(X^{nm} \times X^{nk}) \xrightarrow{\text{Ind}} K_{\Sigma_m \wr \Sigma_k}(X^{n(k+m)}).
\]

Since \( \delta_n^m \) is a ring homomorphism with the appropriate multiplications we need only show the claim for \( P_n^m \). For \( m = 0 \), the equation

\[
P^0(x + y) = P^0(x) \times P^0(y)
\]
is trivial. For \( m > 0 \), it follows from the isomorphism
\[
(E_0 \oplus E_1)^\Sigma_m \cong \bigoplus_{i=0}^m W_i
\]
of \( \Sigma_n \rtimes \Sigma_m \)-bundles over \( X^{nm} \), where \( W_i = \bigoplus J_i E_i \boxtimes \cdots \boxtimes E_{im} \), and the index set, \( J_i \), is
\[
\{(i_1, \ldots, i_m) \mid i = 0 \text{ or } 1, \sum_i i = i \}.
\]
Each \( W_i \) is a direct sum of \( \binom{m}{i} \) sub-bundles which are permuted transitively by \( \Sigma_n \rtimes \Sigma_m \).

Also one of the summands is \( E_0^\Sigma_m \rtimes E_1^{\Sigma_1} \) which is invariant under the inclusion of \( \Sigma_n \rtimes \Sigma_{m-1} \rtimes \Sigma_i \) in \( \Sigma_n \rtimes \Sigma_m \). Therefore \( W_i \cong P^{m-1}(E_0) \times P^i(E_1) \). The result then follows. \( \square \)

3. The following commutes:

\[
\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{(2)} & \mathcal{K}P \otimes \mathcal{K}P \\
\tau & \downarrow & \downarrow \mu \\
\mathcal{K}P & \xrightarrow{\Delta} & \mathcal{K}P(2)
\end{array}
\]

where \( \mu \) is multiplication in the left hand factor, \( \mathcal{K}(X) \), of \( \mathcal{K}P \) and \( \Delta \) is induced by the restrictions maps from \( \Sigma_m \) to \( \Sigma_{m-i} \times \Sigma_i \).

**Proof.** To prove this we shall split the diagram into two easier pieces:

\[
\begin{array}{ccc}
K_{\Sigma_n}(X^n) & \xrightarrow{p(2)} & \prod_{i,j} K_{\Sigma_i(\Sigma_j \times \Sigma_n)}(X^{ni} \times X^{nj}) \\
\downarrow \Delta & & \downarrow \mu \\
\prod_i K_{\Sigma_i(\Sigma_j)}(X^{ni}) & \xrightarrow{\delta} & \mathcal{K}P \otimes \mathcal{K}P
\end{array}
\]

where the \( \Delta \) maps are the “coproducts” induced by the restrictions to the subgroups \( \Sigma_i \times \Sigma_j \leq \Sigma_{i+j} \) and \( \Sigma_n \rtimes \Sigma_i \times \Sigma_j \leq \Sigma_n \rtimes \Sigma_{i+j} \). The left triangle commutes due to the fact that the canonical isomorphism \( E^{\otimes i+j} \cong E^{\otimes i} \boxtimes E^{\otimes j} \) is an isomorphism of \( \Sigma_n \rtimes \Sigma_i \times \Sigma_n \rtimes \Sigma_j \)-bundles.

Now suppose \( E \) is a \( \Sigma_n \rtimes \Sigma_i \)-bundle over \( X^{nl} \). Then
\[
\Delta(E) = \bigoplus_{i+j=l} \text{Res}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_n}(E).
\]

Then \( \delta(2) \) followed by \( \mu \) is just
\[
\bigoplus_{i+j=l} \text{Ind}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_n}(\text{Res}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_n}(E)).
\]

On the other hand
\[
\delta(E) = \text{Ind}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}(E)
\]
and
\[
\Delta(\delta(E)) = \bigoplus_{i+j=l} \text{Res}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_j} \text{Ind}^{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}_{\Sigma_n \rtimes \Sigma_i \times \Sigma_j}(E).
\]

Conveniently
\[
(\Sigma_n \rtimes \Sigma_i) \cap (\Sigma_n \rtimes \Sigma_i \rtimes \Sigma_j) = \Sigma_n \rtimes \Sigma_i \times \Sigma_n \rtimes \Sigma_j
\]
in \( \Sigma_i \times \Sigma_n \). So, an application of Mackey’s Lemma will tell us that these two bundles are isomorphic.

\( \square \)

There are two more things to prove in order to show that \( \mathcal{K}(X) \) is a \( \tau \)-ring. First let’s record certain lemmas about wreath products that will be useful. Note that if there is a group homomorphism \( H \to G \) we get an induced map \( H \rtimes \Sigma_n \to G \rtimes \Sigma_n \).
Lemma 5. Let $H$ be a subgroup of a finite group $G$, and $X$ be a $G$-space. The following commutes:

$$K_H(X) \xrightarrow{p^k} K_{H \wr \Sigma_k}(X^k)$$

Lemma 6. Proof. If $E \in \text{Vect}_H(X) \cong \text{Vect}_G(G \times H \times X)$, then $\text{Ind}_{\mu}^G(E)$ is a product of $E$'s. Let $s_1, \ldots, s_r$ be a list of coset representatives. Then

$$\text{Ind}_{\mu}^G(E) = \bigoplus_{i=1}^r E_{s_i}.$$ 

An element $g \in G$ acts on a vector $(e_{s_1}, \ldots, e_{s_r})$ diagonally

$$g \cdot (e_{s_1}, \ldots, e_{s_r}) = (ge_{g^{-1}s_1}, \ldots, ge_{g^{-1}s_r}).$$

We then apply $P^k$. Then $(g_1, \ldots, g_k, \sigma) \in G \wr \Sigma_k$ acts by sending a vector

$$(e_{s_1}, \ldots, e_{s_r}) \boxtimes \cdots \boxtimes (e_{s_1}, \ldots, e_{s_r})$$

to

$$(g_1e_{g_1^{-1}s_1-1(i)}, \ldots, g_k e_{g_k^{-1}s_k-1(i)} \boxtimes \cdots \boxtimes (g_1e_{g_1^{-1}s_1-1(k)}, \ldots, g_k e_{g_k^{-1}s_k-1(k)})).$$

In order for a vector

$$\boxtimes_{i=1}^k (e_{s_1}, \ldots, e_{s_r})$$

to be invariant under the action of $H \wr \Sigma_k$, we must have $e_{s_i} = e_{s_k}$ for all $j$ and $k$. But, this is isomorphic to $P^k(E)$. \hfill \Box

Lemma 6. The following diagram commutes:

$$\begin{array}{ccc} K_{\Sigma_n}(X^n) & \xrightarrow{p^l} & K_{\Sigma_{n+l}}(X^{nl}) \\
\downarrow \text{Ind} & & \downarrow \text{Ind} \\
K_{\Sigma_{n+l}}(X^{nkl}) & \xrightarrow{\text{Res}} & K_{\Sigma_{n+l} \wr \Sigma_k}(X^{kl}) \\
\end{array}$$

Proof. The isomorphism $(E^{\boxtimes l})^{\boxtimes k} \cong E^{\boxtimes kl}$ is a map of $(\Sigma_n \wr \Sigma_l) \wr \Sigma_k$-bundles. \hfill \Box

Now that we have these lemmas in place, we are equipped to show that $X(X)$ satisfies the final two properties.

4. $\tau(xy) = \tau(x) \cdot \tau(y)$

Proof. We must show that the following diagram commutes:

$$\begin{array}{ccc} K_{\Sigma_i}(X^i) \otimes K_{\Sigma_j}(X^j) & \xrightarrow{\times} & K_{\Sigma_{i+j}}(X^{i+j}) \\
\downarrow \delta^i \otimes \delta^j & & \downarrow \delta^{i+j} \\
K_{\Sigma_{i+j} \wr \Sigma_k}(X^{i+j}) & \xrightarrow{\text{Res}} & K_{\Sigma_{i+j} \wr \Sigma_k}(X^{i+j}) \otimes R(\Sigma_k) \\
\end{array}$$

$$\begin{array}{ccc} K_{\Sigma_{i+j} \wr \Sigma_k}(X^{i+j}) \otimes R(\Sigma_k) & \xrightarrow{\text{Ind}_{\Sigma_{i+j} \wr \Sigma_k}^\Sigma_{i+j} \otimes \text{Res}_{\Sigma_k}^\Sigma_{i+j}} & K_{\Sigma_{i+j} \wr \Sigma_k}(X^{i+j}) \otimes R(\Sigma_k) \\
\end{array}$$
This new multiplication, *, is given by the composition of the isomorphism
\[ K_{\Sigma_i \Sigma_k}(X^i) \times K_{\Sigma_j \Sigma_k}(X^j) \xrightarrow{\cong} K_{\Sigma_i \Sigma_k \times \Sigma_j \Sigma_k}(X^{i+j}) \]
with
\[ K_{\Sigma_i \Sigma_k \times \Sigma_j \Sigma_k}(X^{i+j}) \xrightarrow{\text{Res}} K_{(\Sigma_i \times \Sigma_j) \Sigma_k}(X^{i+j}) \xrightarrow{\text{Ind}} K_{\Sigma_{i+j} \Sigma_k}(X^{i+j}). \]
For the bottom triangle and middle square we again apply Mackey’s Lemma to
\((\Sigma_{k(i+j)} \times \Sigma_k) \cap ((\Sigma_i \times \Sigma_k) \times (\Sigma_j \times \Sigma_k)) = (\Sigma_i \times \Sigma_j) \cap \Sigma_k.\]

The top square is a little more involved. The map \(P^k(x \times y)\) is bi-additive in the variables \(x\) and \(y\) into the cross product invertibles, \(1 + \prod_{k \geq 1} K_{\Sigma_{i+j} \Sigma_k}.\) Also,
\[ P^k(x + y) * P^k(z) = \sum_{i=0}^{k} (P^i(x) * P^{k-i}(y)) \ast P^i(z) \]
= \[\sum_{i=0}^{k} (P^i(x) * P^i(z)) \ast (P^{k-i}(y) \ast P^{k-i}(z))\]
So \(P^k(x) \ast P^k(y)\) is bi-additive. This bi-additivity property allows us to divide up the square into:
\[
\begin{array}{c}
\text{Vect}_{\Sigma_i} \otimes \text{Vect}_{\Sigma_j} \xrightarrow{P^k \otimes P^k} K_{\Sigma_i \Sigma_k}(X^i) \otimes K_{\Sigma_j \Sigma_k}(X^j) \\
\downarrow \\
\text{Vect}_{\Sigma_i \times \Sigma_j}(X^{i+j}) \xrightarrow{P^k} K_{(\Sigma_i \times \Sigma_j) \Sigma_k}(X^{i+j}) \\
\downarrow \\
\text{Vect}_{\Sigma_{i+j}} \xrightarrow{P^k} K_{\Sigma_{i+j} \Sigma_k}(X^{i+j})
\end{array}
\]
The top square is the fact that \((E \boxtimes F)^{\Sigma_k} \to E^{\Sigma_k} \boxtimes F^{\Sigma_k}\) is a map of \((\Sigma_i \times \Sigma_j) \cap \Sigma_k\) bundles. The bottom square is Lemma 5. 

5. The following commutes
\[
\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{\tau} & \mathcal{K}P \\
\downarrow \tau & & \downarrow \hat{\tau} \\
\mathcal{K}P & \xrightarrow{\square} & \mathcal{K}Q
\end{array}
\]
A definition for \(\hat{\tau}\) is given in Section 2. We repeat the details here. We treat \(R(\Sigma_i)\) as the \(K\)-theory ring \(K_{\Sigma_i}(pt)\). Then we get a map
\[ \tau^k : \mathcal{K}(X) \times K_{\Sigma_i}(pt) \to \mathcal{K}(X) \otimes R(\Sigma_k) \otimes K_{\Sigma_i \Sigma_k}(pt). \]
We follow this map by the map
\[ \text{Id} \otimes \text{Ind}_{\Sigma_k} : \mathcal{K}(X) \otimes R(\Sigma_k) \otimes K_{\Sigma_i \Sigma_k}(pt) \to \mathcal{K}(X) \otimes K_{\Sigma_i \Sigma_k} \otimes K_{\Sigma_i \Sigma_k}. \]
Then take the map \(\text{Id}_{\mathcal{K}(X)} \otimes \cdot\) where \(\cdot\) is the internal tensor product in the ring \(K_{\Sigma_i \Sigma_k}(pt)\).
We define the map \(\hat{\tau}_i^k\) to be the composite of all these maps
\[ \hat{\tau}_i^k : \mathcal{K}(X) \times K_{\Sigma_i}(pt) \to \mathcal{K}(X) \otimes K_{\Sigma_i \Sigma_k}(pt). \]
We define \(\hat{\tau}I\) to be the product over all \(k\). Then \(\hat{\tau}_I\) is bi-additive from the additions to the cross product. These maps extend to \(\mathcal{K}(X) \otimes K_{\Sigma_i}(pt)\) and we define \(\hat{\tau}\) to be the product of these maps over all \(I\).
The map \(\square\) is induced by the maps \(R(\Sigma_{kl}) \to R(\Sigma_l \cap \Sigma_k)\).

Proof. We divide the diagram up and show that the pieces commute:
Now that most of our structure is in place, we can provide some results. The first of these will be a Thom Isomorphism Theorem.

Square (1) commutes by Lemma [X] (2) commutes Lemma [Y] and (4) is trivial. Square (3) is another application of Mackey’s Lemma using

\[(\Sigma_m \wr \Sigma_{kl}) \cap \Sigma_{klm} \times (\Sigma_k \wr \Sigma_k) = (\Sigma_m \wr \Sigma_k) \wr \Sigma_k\]

embedded in \(\Sigma_{klm} \times \Sigma_{kl}\).

Square (5) commutes as follows. Note first that both maps are homomorphic from + to the double cross product, when we product over all \(k\). For a proof of the right side the reader is referred to [Z] page 137. The left side follows from the fact that \(\Pi_k P^k(x + y) = \Pi_k P^k(x) \times \Pi_k P^k(y)\) and the diagram:

\[
\begin{align*}
K_{\Sigma_{klm} \times \Sigma_{kl}}(X^{lmj}) &\xrightarrow{p_{kl}} K_{\Sigma_m \times \Sigma_{kl}}(X^{lm}) \xrightarrow{p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \\
K_{\Sigma_{klm} \times \Sigma_{kl}}(X^{lm}) &\xrightarrow{p_{kl}} K_{\Sigma_m \times \Sigma_{kl}}(X^{lm}) \xrightarrow{p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \\
K_{\Sigma_{klm} \times \Sigma_{kl}}(X^{lm}) &\xrightarrow{p_{kl}} K_{\Sigma_m \times \Sigma_{kl}}(X^{lm}) \xrightarrow{p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm})
\end{align*}
\]

Therefore it suffices to check commutativity for \(K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times R(\Sigma_l) \subseteq K_{\Sigma_{klm} \times \Sigma_l}(X^{lm})\). The appropriate diagram is now:

\[
\begin{align*}
K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times R(\Sigma_l) &\xrightarrow{p_{kl} \times p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times K_{\Sigma_{klm} \times \Sigma_k}(X^{lk}) \\
K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times R(\Sigma_l) &\xrightarrow{p_{kl} \times p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times K_{\Sigma_{klm} \times \Sigma_k}(X^{lk}) \\
K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times R(\Sigma_l) &\xrightarrow{p_{kl} \times p_k} K_{\Sigma_{klm} \times \Sigma_l}(X^{lm}) \times K_{\Sigma_{klm} \times \Sigma_k}(X^{lk})
\end{align*}
\]

The right hand commutes by another application of Mackey’s Lemma, this time with

\[((\Sigma_{lm} \wr \Sigma_k) \times (\Sigma_l \wr \Sigma_k)) \cap (\Sigma_{klm} \times (\Sigma_l \wr \Sigma_k)) = (\Sigma_{lm} \times \Sigma_l) \wr \Sigma_k.\]

The left hand commutes due to the fact that \((E \boxtimes F)^{\oplus 2k} \cong E^{\oplus 2k} \boxtimes F^{\oplus 2k}\) is an isomorphism of \((\Sigma_{lm} \times \Sigma_l) \wr \Sigma_k\)-bundles.

\[\square\]

4. A Thom Isomorphism

Now that most of our structure is in place, we can provide some results. The first of these will be a Thom Isomorphism Theorem.
In what follows it will be more convenient to use a different definition of $K_G(X, A)$, in keeping with definitions given in [1], [9]. Here, we let $X$ be any compact, Hausdorff space, with a closed subspace $A$. By a complex of $G$-vector bundles over the pair $(X, A)$, we shall mean a complex of vector bundles, $E_i$, over $X$:

$$
\cdots 0 \rightarrow E_i \xrightarrow{d} E_{i+1} \xrightarrow{d} \cdots \xrightarrow{d} E_{i+k} \rightarrow 0 \rightarrow \cdots
$$

such that $d^2 = 0$. We also require the complex to be acyclic when restricted to $A$. A morphism of two complexes, $E_\ast$ and $F_\ast$, is a sequence of functions, $f_i : E_i \rightarrow F_i$, that commute with the differential $d$. As with the other definition of the $K$ groups, the isomorphism classes of complexes form an abelian semigroup. Two isomorphism classes, $E_\ast$ and $F_\ast$, are homotopic if there is complex of $G$-vector bundles over $(X \times [0, 1], A \times [0, 1])$ such that the restriction to $X \times \{0\}$ is $E_\ast$ and the restriction to $X \times \{1\}$ is $F_\ast$. Furthermore, two homotopy classes of complexes, $E_\ast$ and $F_\ast$, are said to be equivalent if there are two complexes, $E'_\ast$ and $F'_\ast$, acyclic over $X$, such that

$$E_\ast \oplus E'_\ast \simeq F_\ast \oplus F'_\ast.$$  

The set of equivalence classes can be shown to be isomorphic to $K_G(X, A)$, [9 page 148]. The isomorphism is simply $E_\ast \mapsto \sum_k (-1)^k E_k$.

In what follows we will assume that $A = \emptyset$. At the end of the section, we will discuss how to prove the result for arbitrary $A$.

Given a $\Sigma_n$ complex, $E_\ast$, over $X^n$ and a $\Sigma_m$ complex, $F_\ast$, over $X^m$, define the product of $E_\ast$ and $F_\ast$ over $X^{n+m}$ to have $k$th term

$$(E_\ast F_\ast)_k = \bigoplus_{p+q=k} \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (E_p \boxtimes F_q).$$

The $\tau$-ring structure, $\lambda$-ring structure, and comultiplication (when appropriate) are defined similarly.

Recall that if $E \rightarrow X$ is a $G$-vector bundle then we have an additive isomorphism

$$T_E : K_G(X) \rightarrow K_G(D(E), S(E))$$

called the Thom isomorphism. Here $D(E)$ and $S(E)$ are the associated disk and sphere bundles. We need to recall precisely how this is defined.

Let the bundle map be $p : E \rightarrow X$. Then the pullback bundle $p^* E \rightarrow E$ has a natural section along the diagonal $\delta : E \rightarrow E \times_X E$. Now we form the Koszul complex

$$\Lambda^E_x = \cdots \rightarrow 0 \rightarrow \mathbb{C} \xrightarrow{d} \Lambda^1 p^* E \xrightarrow{d} \Lambda^2 p^* E \xrightarrow{d} \cdots,$$

where $d(v) = v \wedge \delta(x)$ for $v \in \Lambda^1 p^* E_x$. For our purposes, we must restrict each $\Lambda^i p^* E$ to $D(E)$, but we will keep the same notation. Since $\delta$ is non-vanishing on $S(E)$ this complex will be acyclic when restricted to the sphere bundle, [1 page 99]. If $F_\ast$ is a complex on $X$ then $p^* F_\ast$ is a complex on $E$ (again restrict down to $D(E)$) and the Thom homomorphism is then

$$T_E(F_\ast) = \Lambda^E_x \otimes p^* F_\ast.$$  

Note that, if $E$ is a bundle over $X$, then $E^n$ is a $\Sigma_n$-bundle over $X^n$. Therefore $T_{E^n}$ is an isomorphism

$$K_{\Sigma_n}(X^n) \rightarrow K_{\Sigma_n}(D(E^n), S(E^n)).$$

But, for each $n$, $D(E^n) \cong D(E)^n$ and

$$[D(E)^{n-1} \times S(E)] \cup \cdots \cup [S(E) \times D(E)^{n-1}] \cong S(E^n).$$

This gives a homeomorphism of pairs $(D(E^n), S(E^n)) \cong (D(E), S(E))^n$. With a slight abuse of notation, we have isomorphisms for all $n$:

$$T_{E^n} : K_{\Sigma_n}(X^n) \xrightarrow{\cong} K_{\Sigma_n}((D(E), S(E))^n).$$

Adding all these together, we get an isomorphism of abelian groups

$$T : \mathcal{K}(X) \rightarrow \mathcal{K}(D(E), S(E)).$$

We now wish to show that this isomorphism agrees with both the ring and the $\lambda$-ring structure.
For the ring structure we begin with two elements $V_* \in K_{\Sigma_n}(X^n)$ and $W_* \in K_{\Sigma_m}(X^m)$. If we do the Thom homomorphism to each separately and then multiply we get:

$$ \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( \bigoplus_{p+q=k} \left( \bigoplus_{i+j=p, s+t=q} ((\Lambda^i p^* E^n \otimes V_j) \boxtimes (\Lambda^s p^* E^m \otimes W_t)) \right) \right) $$

$$ \cong \bigoplus_{i+j+s+t=k} \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( ((\Lambda^i p^* E^n \otimes V_j) \boxtimes (\Lambda^s p^* E^m \otimes W_t)) \right) $$

$$ \cong \bigoplus_{i+j+s+t=k} \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( ((\Lambda^i p^* E^n \boxtimes \Lambda^s p^* E^m) \otimes (V_j \boxtimes W_t)) \right) $$

$$ \cong \bigoplus_{j+t+p=k} \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( \left( \bigoplus_{i+s=p} (\Lambda^i p^* E^n \boxtimes \Lambda^s p^* E^m) \right) \otimes (V_j \boxtimes W_t) \right) $$

$$ \cong \bigoplus_{j+t+p=k} \Lambda^p p^* E^{n+m} \otimes \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (V_j \boxtimes W_t), $$

as $T(V_*W_*)_k$. The second isomorphism is due to the fact that

$$(E_1 \otimes E_2) \boxtimes (F_1 \otimes F_2) \cong (E_1 \boxtimes F_1) \otimes (E_2 \boxtimes F_2)$$

as $\Sigma_n \times \Sigma_m$ bundles. The fourth isomorphism is due jointly to the facts that

$$p^* E^{n+m} \cong (p^* E)^{n+m}$$

and

$$\Lambda^p F^{n+m} \cong \bigoplus_{i+j=p} \Lambda^i F^n \boxtimes \Lambda^j F^m$$

as $\Sigma_n \times \Sigma_m$ bundles. The final isomorphism is an instance of Frobenius reciprocity:

$$\text{Ind}_{\mu}^G \left( \text{Res}_{\mu}^G (E_1) \otimes E_2 \right) \cong E_1 \otimes \text{Ind}_{\mu}^G (E_2)$$

for a $G$ bundle $E_1$ and an $H$ bundle $E_2$, $H$ a subgroup of a group $G$.

Since

$$ \bigoplus_{j+t+p=k} \Lambda^p p^* E^{n+m} \otimes \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (V_j \boxtimes W_t) $$

is what we get by multiplying and then applying the Thom homomorphism, we are done.

We now move onto the $\lambda$-ring isomorphism. If we start by taking $V_* \in K_{\Sigma_n}(X^n)$, apply $\lambda^m$, and then do the Thom homomorphism we get:

$$ \bigoplus_{p+q=k} \Lambda^p p^* E^{n+m} \otimes \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( \text{sgn}_m \otimes \bigoplus_{i_1+i_2+\cdots+i_m=q} V_{i_1} \boxtimes V_{i_2} \boxtimes \cdots \boxtimes V_{i_m} \right) $$

$$ \cong \text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \left( \text{sgn}_m \otimes \bigoplus_{i_1+i_2+\cdots+i_m+j_1+\cdots+j_m=k} (\Lambda^{i_1} p^* E^n \otimes V_{j_1} \boxtimes \cdots \boxtimes (\Lambda^{i_m} p^* E^n \otimes V_{j_m}) \right) $$

as $(T(\lambda^m(V_*)))_k$. Since the second line is precisely what we get when we do $T$ then $\lambda^m$, we are done. Thus $T$ is an isomorphism as $\lambda$-rings.

We observe that the discussion generalizes to the pair $(X, A)$. So that our work will fit on the page, let us denote

$$A(n) = (X^{n-1} \times A) \cup (X^{n-2} \times A \times X) \cup \ldots \cup (A \times X^{n-1}).$$
At level \( n \) we have a Thom isomorphism

\[
T_{E^n} : K_{\Sigma_n}(X^n, A(n)) \to K_{\Sigma_n}(D(E^n), S(E^n) \cup D(E^n)|_{A(n)}).
\]

When \( n = 1 \), this is

\[
K(D(E), S(E) \cup D(E)|_A).
\]

As before we need that \( D(E^n) \cong D(E)^n \) and \( S(E^n) \cong S(E)^n \). What we also need is that

\[
[D(E)^{n-1} \times (S(E) \cup D(E)|_A)] \cup \ldots \cup [(S(E) \cup D(E)|_A) \times D(E)^{n-1}] \cong S(E)^n \cup D(E)^n|_{A(n)}.
\]

But, this is also true: the collection

\[
(D(E)^{n-1} \times S(E)) \cup \ldots \cup (S(E) \times D(E)^{n-1})
\]

will give us \( S(E)^n \). Now all we would need is for

\[
(D(E)^{n-1} \times D(E)|_A) \cup \ldots \cup (D(E)|_A \times D(E)^{n-1}) \cong D(E)^n|_{A(n)}.
\]

The proof of the Thom isomorphism above with \( A = \emptyset \) generalizes to give

\[
\mathcal{K}(X, A) \cong \mathcal{K}(D(E), S(E) \cup D(E)|_A).
\]

5. Spaces with Even Cells

By induction on cells it is not hard to see that if \( X \) is a CW-complex made only from finitely many, even dimensional cells, then \( K(X) \) is the free abelian group on generators in correspondence with those cells and \( K^1(X) = 0 \). What will be shown here is that, if \( X \) has only even dimensional cells, then \( \mathcal{K}(X) \) is the free \( \lambda \)-ring on \( K(X) \) as defined in Section 4.

If \( X \) is a point, then we already know that \( \mathcal{K}(X) \cong \oplus R\Sigma_n \), which is the free \( \lambda \)-ring on one generator.

Let \( X \) be a space constructed out of even cells. Assume \( \mathcal{K}(X) \) is isomorphic to \( \mathcal{K}(K(X)) \). We assume that \( Y \) is obtained from \( X \) by adding one even dimensional cell, denoted by \( D \), via attaching along a sphere \( S \), and proceed by induction. The strategy of the proof will be to first show that \( \mathcal{K}(Y) \) is a polynomial ring. We can then more easily show that the induced map \( \mathcal{K}(K(Y)) \to \mathcal{K}(Y) \) is an isomorphism. We shall also prove along the way that \( \mathcal{K}^1(Y) = 0 \), where we define

\[
\mathcal{K}^1(Y, X) = \bigoplus_{n \geq 0} K^1_{\Sigma_n}((Y, X)^n).
\]

We then have the additional assumption that \( \mathcal{K}^1(X) = 0 \). Again we know this to be true for \( X = \text{pt} \):

In this case

\[
K^1_{\Sigma_n}(\text{pt}) \cong K^1(\text{pt}) \otimes R(\Sigma_n) = 0
\]

To avoid confusion we shall begin writing \( \mathcal{K}^0(X) \) for \( \mathcal{K}(X) \).

Now, \( Y^n \) has a filtration:

\[
\emptyset = F^n_0 \subseteq F^n_1 \subseteq \ldots \subseteq F^n_{n-1} \subseteq F^n_n = Y^n
\]

where \( F^n_k \) is the set of points in \( Y^n \) such that at least \( n-k \) of the coordinates are in \( X \), or at most \( k \) of them are in \( Y \) but not \( X \). This leads to a nested set of pairs:

\[
(Y^n, F^n_0) \subseteq (Y^n, F^n_1) \subseteq \ldots \subseteq (Y^n, F^n_{n-1}) \subseteq (Y^n, F^n_n)
\]

which gives rise to a filtration of \( K^*_{\Sigma_n}(Y^n) \):

\[
K^*_{\Sigma_n}(Y^n, F^n_0) \supseteq K^*_{\Sigma_n}(Y^n, F^n_1) \supseteq \ldots \supseteq K^*_{\Sigma_n}(Y^n, F^n_{n-1}) \supseteq K^*_{\Sigma_n}(Y^n, F^n_n)
\]

It might also be good to note that this filtration is

\[
K^*_{\Sigma_n}(Y^n) \supseteq K^*_{\Sigma_n}(Y^n, X^n) \supseteq \ldots \supseteq K^*_{\Sigma_n}((Y, X)^n) \supseteq K^*_{\Sigma_n}(Y^n, Y^n).
\]
The resulting abelian groups of the spectral sequence are:

\[ E_1^{n,m,*} = K_{\Sigma_{n+m}}^* \left( Y^{n+m}, F_n^{n+m} \right) \]

The following lemma shows that this is actually a sequence of commutative algebras.

**Lemma 7.** The induction product gives a map

\[ K_{\Sigma_n}^* (Y^n, F_k^n) \otimes K_{\Sigma_m}^* (Y^m, F_l^m) \rightarrow K_{\Sigma_{n+m}}^* (Y^{n+m}, F_{k+l}^{n+m}) \]

**Proof.** Take \( K_{\Sigma_n}^* (Y^n, F_k^n) \) and \( K_{\Sigma_m}^* (Y^m, F_l^m) \). Then the external tensor product goes

\[ K_{\Sigma_n}^* (Y^n, F_k^n) \otimes K_{\Sigma_m}^* (Y^m, F_l^m) \rightarrow K_{\Sigma_{n+m}}^* (Y^{n+m}, (Y^n \times F_l^m) \cup (F_k^n \times Y^m)) \]

But,

\[ (Y^n \times F_l^m) \cup (F_k^n \times Y^m) \supseteq F_{k+l}^{n+m} \]

Thus we get a map

\[ K_{\Sigma_{n+m}}^* (Y^{n+m}, (Y^n \times F_l^m) \cup (F_k^n \times Y^m)) \rightarrow K_{\Sigma_{n+m}}^* (Y^{n+m}, F_{k+l}^{n+m}) \]

The induction map then lands in

\[ K_{\Sigma_{n+m}}^* (Y^{n+m}, F_{k+l}^{n+m}) \].

\[ \square \]

So, we now have a spectral sequence of commutative algebras. Since

\[ K_{\Sigma_m}^1 (Y^m, F_n^m) = K_{\Sigma_m}^1 (F_n^m) = 0, \]

the long exact sequence of the pair gives a map of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_{\Sigma_m}^* (Y^m, F_n^{m+1}) & \longrightarrow & K_{\Sigma_m}^* (Y^m) & \longrightarrow & K_{\Sigma_m}^* (F_n^{m+1}) & \longrightarrow & 0 \\
& & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & K_{\Sigma_m}^* (Y^m, F_n^m) & \longrightarrow & K_{\Sigma_m}^* (Y^m) & \longrightarrow & K_{\Sigma_m}^* (F_n^m) & \longrightarrow & 0
\end{array}
\]

The Snake Lemma then gives us a sequence:

\[ 0 \longrightarrow K_{\Sigma_m}^* (F_n^{m+1}, F_n^m) \xrightarrow{\partial} E_1^{m,m-n,*} \longrightarrow 0. \]

Therefore

\[ E_1^{n,m,*} \cong K_{\Sigma_{n+m}}^* (F_n^{m+n}, F_n^{n+m}). \]

Our goal is to show that \( E^{*,*,0} \) is polynomial and \( E^{*,*,1} = 0 \). Then we will know that \( K^0(Y) \) is polynomial and \( K^1(Y) = 0 \). We do this by proving \( K^* (Y, X) \otimes K^* (X) \cong E_1 \) and then show that \( K^* (Y, X) \) is polynomial with \( K^1(Y, X) = 0 \). The ring \( K^0(X) \) is polynomial with \( K^1(X) = 0 \) by hypothesis. First we show:

**Lemma 8.** The external tensor product

\[ K_{\Sigma_n}^* (Y^n, X^n) \otimes K_{\Sigma_m}^* (X^m) \rightarrow K_{\Sigma_{n+m}}^* (F_n^{n+m}, F_n^{n-1}) \]

is an isomorphism.

**Proof.** We have maps of \( \Sigma_{n+m} \)-spaces

\[ \Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (F_n^{n-1} \times X^m) \rightarrow F_n^{n+m} \]

and

\[ \Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (Y^n \times X^m) \rightarrow F_n^{n+m} \]
that fit into a pushout square

\[
\begin{array}{ccc}
\Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (F_{n-1}^n \times X^m) & \longrightarrow & F_{n-1}^n \\
\downarrow & & \downarrow \\
\Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (Y^n \times X^m) & \longrightarrow & F_{n+m}^n
\end{array}
\]

Therefore,

\[
K^*_{\Sigma_{n+m}} (F_{n}^{m+n}, F_{n-1}^{n+m}) \cong K^*_{\Sigma_{n+m}} \left( \Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (Y^n \times X^m), \Sigma_{n+m} \times_{\Sigma_n \times \Sigma_m} (F_{n-1}^n \times X^m) \right)
\]

\[
\cong K^*_{\Sigma_{n+m}} \left( (Y^n \times X^m), (F_{n-1}^n \times X^m) \right)
\]

\[
\cong K^*_{\Sigma_{n+m}} ((Y, X)^n) \otimes K^*_{\Sigma_{m}} (X^m)
\]

The third isomorphism is given by the external product:

\[
K_{\Sigma_{n+m}} ((Y, X)^n) \otimes K_{\Sigma_{m}} (X^m) \longrightarrow K_{\Sigma_{n+m}} (Y^n \times X^m, F_{n-1}^n \times X^m).
\]

The Thom isomorphism tells us that \( K_{\Sigma_{n+m}} ((Y, X)^n) \) is finitely generated and free as an abelian group. So, the external product is an isomorphism by the Kunneth formula.

\[\square\]

We have now shown that \( E^{n,m,1} = 0 \), hence \( \mathcal{A}^1(Y) = 0 \). Due to this we will stop using \( \mathcal{A}^0 \) and go back to just \( \mathcal{A} \). Since \( K_{\Sigma_{n+m}} (F_{n}^{m+n}, F_{n-1}^{n+m}) \cong E^{n-1,m+1}_1 \), we have shown that the map

\[\mathcal{A}(Y, X) \otimes \mathcal{A}(X) \longrightarrow E_1\]

is an isomorphism of bigraded abelian groups. It remains to show that it is an isomorphism of bigraded commutative algebras. The next lemma demonstrates that.

**Lemma 9.** The following diagram commutes:

\[
\begin{array}{ccc}
K_{\Sigma_{n+m}} ((Y, X)^n) \otimes K_{\Sigma_{m}} (X^m) & \cong & K_{\Sigma_{n+m}} (F_{n}^{m+n}, F_{n-1}^{n+m}) \\
\downarrow & & \downarrow \\
K_{\Sigma_{n+m}} ((Y, X)^n) \otimes K_{\Sigma_{m}} (Y^m) & \longrightarrow & K_{\Sigma_{n+m}} (Y_{n+m}^n, F_{n-1}^{n+m}) \\
\downarrow & & \downarrow \\
K_{\Sigma_{n}} (Y^n) \otimes K_{\Sigma_{m}} (Y^m) & \longrightarrow & K_{\Sigma_{n+m}} (Y_{n+m}^n)
\end{array}
\]

where the vertical maps are induced by inclusion, the top map is the isomorphism from Lemma 8, the bottom map is the ring multiplication, and the middle map is external tensor

\[
K_{\Sigma_{n}} ((Y, X)^n) \otimes K_{\Sigma_{m}} (Y^m) \underbrace{\cong}_{\text{Eq}} K_{\Sigma_{n+m}} (Y_{n+m}^n, (F_{n-1}^n \times Y^m))
\]

followed by

\[
K_{\Sigma_{n+m}} (Y_{n+m}^n, (F_{n-1}^n \times Y^m)) \subseteq K_{\Sigma_{n+m}} (Y_{n+m}^n, F_{n-1}^{n+m}) \underbrace{\text{Ind}}_{\text{Ind}} K_{\Sigma_{n+m}} (Y_{n+m}^n, F_{n-1}^{n+m}).
\]
Proof. We need the fact that \((Y, X)^n = (Y^n, F^n_{n-1})\). To show that the top square commutes, break it up into the following:

\[
\begin{array}{ccc}
K_{\Sigma_n} (Y^n, F^n_{n-1}) \otimes K_{\Sigma_m} (Y^m) & \longrightarrow & K_{\Sigma_n} (Y^n, F^n_{n-1}) \otimes K_{\Sigma_m} (X^m) \\
\oplus & & \oplus \\
K_{\Sigma_n \times \Sigma_m} (Y^{n+m}, F^{n+m}_{n-1} \times Y^m) & \longrightarrow & K_{\Sigma_n \times \Sigma_m} (Y^n \times X^m, F^n_{n-1} \times X^m) \\
\downarrow \text{Ind} & & \downarrow \text{Ind} \\
K_{\Sigma_n + m} (Y^{n+m}, F^{n+m}_{n-1}) & \longrightarrow & K_{\Sigma_n + m} (\Sigma_{n+m} \times \Sigma_n \times \Sigma_m) (Y^n \times X^m, F^n_{n-1} \times X^m)) \\
f & & f \\
& & \\
K_{\Sigma_n + m} (F^{n+m}_n, F^{n+m}_{n-1}) & \longrightarrow & K_{\Sigma_n + m} (F^{n+m}_n, F^{n+m}_{n-1})
\end{array}
\]

The top two squares commute trivially. It takes more care to show that the bottom square commutes. To show this we will show that the diagram

\[
\begin{array}{ccc}
K_{\Sigma_n \times \Sigma_m} (Y^{n+m}, F^{n+m}_{n-1}) & \longrightarrow & K_{\Sigma_n \times \Sigma_m} (\Sigma_{n+m} \times \Sigma_n \times \Sigma_m) (Y^n \times X^m, F^n_{n-1} \times X^m) \\
\downarrow \text{Ind} & & \downarrow \text{Ind} \\
K_{\Sigma_n + m} (Y^{n+m}, F^{n+m}_{n-1}) & \longrightarrow & K_{\Sigma_n + m} (\Sigma_{n+m} \times \Sigma_n \times \Sigma_m) (Y^n \times X^m, F^n_{n-1} \times X^m)) \\
f & & f \\
& & \\
K_{\Sigma_n + m} (F^{n+m}_n, F^{n+m}_{n-1}) & \longrightarrow & K_{\Sigma_n + m} (F^{n+m}_n, F^{n+m}_{n-1})
\end{array}
\]

commutes where the map

\[
K_{\Sigma_n + m} (F^{n+m}_n, F^{n+m}_{n-1}) \longrightarrow K_{\Sigma_n + m} (\Sigma_{n+m} \times \Sigma_n \times \Sigma_m) (Y^n \times X^m, F^n_{n-1} \times X^m))
\]

is the isomorphism from the pushout diagram in Lemma 8 which is inverse to the isomorphism in the diagram we now consider. We start with a bundle in \(K_{\Sigma_n \times \Sigma_m} (Y^{n+m}, F^{n+m}_{n-1})\) which is of the form \(E \boxtimes F\) where \(E\) restricted to \(F^n_{n-1}\) is trivial. We take this bundle and induce up \(\text{Ind}_{\Sigma_n \times \Sigma_m} (E \boxtimes F)\). We then restrict this bundle down to \(Y^n \times X^m\) and only consider the action of \(\Sigma_n \times \Sigma_m\). The induced bundle is a direct sum of copies of \(E \boxtimes F\), one for each coset of \(\Sigma_{n+m} / (\Sigma_n \times \Sigma_m)\). Write this as

\[
\xi = (E \boxtimes F)_{\sigma_1} \oplus \cdots \oplus (E \boxtimes F)_{\sigma_r},
\]

where \(\sigma_i\) are coset representatives. If we have a point

\[
\bar{a} = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \in Y^n \times X^m
\]

we may write the preimage of \(\bar{a}\) under the projection

\[
\Sigma_{n+m} \times \Sigma_n \times \Sigma_m Y^{n+m} \longrightarrow Y^n \times X^m
\]

as the collection of points

\[
(\sigma_1, a_{\sigma_1^{-1}(1)}, \ldots, a_{\sigma_1^{-1}(n+m)}
\]

for \(i = 1, \ldots, r\). If \(\sigma_i \notin \Sigma_n \times \Sigma_m\) then for some \(k\), with \(1 \leq k \leq n\), we have \(\sigma_i(k) > n\). But this forces \(E\) to be restricted to a subspace of \(F^n_{n-1}\) over which it is trivial. Therefore \(\xi\) is isomorphic to \(E \boxtimes F\) restricted to the subspace \(Y^n \times X^m\) which is what we get when we go along the top of the square.
To complete the lemma we need only show that the following commutes:

\[
\begin{align*}
K_{\Sigma_n} (Y^n, F_n^{n-1}) \otimes K_{\Sigma_n} (Y^n) & \longrightarrow K_{\Sigma_n} (Y^n) \otimes K_{\Sigma_n} (Y^n) \\
K_{\Sigma_n \times \Sigma_m} (Y^{n+m}, F_n^{n-1} \times Y^m) & \longrightarrow K_{\Sigma_n \times \Sigma_m} (Y^{n+m}) \\
K_{\Sigma_n \times \Sigma_m} (Y^{n+m}, F_n^{n-1}) & \longrightarrow K_{\Sigma_n + \Sigma_m} (Y^{n+m}) \\
K_{\Sigma_{n+m}} (Y^{n+m}, F_n^{n-1}) & \longrightarrow K_{\Sigma_{n+m}} (Y^{n+m})
\end{align*}
\]

But, this is just the fact that induction is natural. \(\Box\)

We are nearing the end. Recall that \(\mathcal{F}(K(Y))\) is the free \(\lambda\)-ring on \(K(Y)\). We have the following commutative diagram:

\[
\begin{align*}
\mathcal{F}(K(Y, X)) & \longrightarrow \mathcal{F}(K(Y)) \longrightarrow \mathcal{F}(K(X)) \\
\cong & \quad \cong \\
\mathcal{K}(Y, X) & \longrightarrow \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)
\end{align*}
\]

where the left map is an isomorphism induced by the map of pairs \((D, S) \longrightarrow (Y, X)\) and the right map is an isomorphism by assumption. By definition \(\mathcal{F}(K(Y))\) is free as an algebra over \(\mathcal{F}(K(Y, X))\). And \(\mathcal{K}(Y)\) is free as an algebra over \(\mathcal{K}(Y, X)\) by the work above. The three lemmas of the section have demonstrated that we have an isomorphism

\[\mathcal{K}(Y) \otimes_{\mathcal{K}(Y, X)} \mathbb{Z} \cong \mathcal{K}(X).\]

This fits neatly into a diagram

\[
\begin{align*}
\mathcal{F}(K(Y)) \otimes_{\mathcal{F}(K(Y, X))} \mathbb{Z} & \longrightarrow \mathcal{F}(K(X)) \\
\cong & \quad \cong \\
\mathcal{K}(Y) \otimes_{\mathcal{K}(Y, X)} \mathbb{Z} & \longrightarrow \mathcal{K}(X)
\end{align*}
\]

Therefore we have that

\[\mathcal{F}(K(Y)) \otimes_{\mathcal{F}(K(Y, X))} \mathbb{Z} \longrightarrow \mathcal{K}(Y) \otimes_{\mathcal{K}(Y, X)} \mathbb{Z}\]

is an isomorphism. Since \(\mathcal{F}(K(Y, X)) \cong \mathcal{K}(Y, X)\) naturally, the isomorphism becomes

\[\mathcal{F}(K(Y)) \otimes_{\mathcal{F}(K(Y, X))} \mathbb{Z} \cong \mathcal{K}(Y) \otimes_{\mathcal{F}(K(Y, X))} \mathbb{Z}.\]

Because everything is a polynomial algebra, we may conclude that \(\mathcal{F}(K(Y))\) is isomorphic to \(\mathcal{K}(Y)\).

**Theorem 10.** If \(X\) is a CW-complex, constructed from a finite number of even dimensional cells, then the natural map of \(\lambda\)-rings

\[\mathcal{F}(K(X)) \longrightarrow \mathcal{K}(X)\]

induces an isomorphism.

Just as in the case of the Thom isomorphism, we can extend these results to pairs, \((X, A)\). In this case, Theorem 10 becomes:

**Theorem 11.** If \(X\) is a CW-complex, constructed by attaching a finite number of even dimensional cells to a finite CW-complex \(A\), then the natural map of \(\lambda\)-rings

\[\mathcal{F}(K(X, A)) \longrightarrow \mathcal{K}(X, A)\]

induces an isomorphism.
The way to see this is that the base case $\mathcal{K}(A, A)$ is clearly true. One then checks that the lemmas from this section can be appropriately modified. So, if we know it to be true for the pair $(X, A)$, and $Y$ is built from $X$ by attaching an even dimensional cell, our new diagram becomes:

\[
\begin{align*}
\mathcal{K}(Y, X) & \longrightarrow \mathcal{K}(Y, A) \longrightarrow \mathcal{K}(X, A) \\
\mathcal{K}(Y, Y) & \longrightarrow \mathcal{K}(Y, A) \longrightarrow \mathcal{K}(X, A)
\end{align*}
\]

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