Abstract

We begin from the SL(3) Chern-Simons higher spin theory, which contains the first-order formulation of Einstein gravity theory in three dimensions. Then the dimensional reduction in this Chern-Simons theory gives the SL(3) reparametrization invariant Schwarzian theory, which is the boundary theory of an interacting theory between the spin-2 and spin-3 fields at the infrared or massless limit. We show that the Lorentzian SL(3) Schwarzian theory is dual to the integrable model, SL(3) open Toda chain theory. Finally, we demonstrate the application of open Toda chain theory from the SL(2) case. The numerical result confirms that the SYK model in the holographic limit, \( N \gg \beta J \gg 1 \), where \( N \) is the number of Majorana fermion fields, \( \beta \) is the inverse temperature, and \( J \) is the coupling constant, and the JT gravity theory with a constant boundary dilaton should be integrable and not have the random matrix spectrum by computing the spectral form factor.
1 Introduction

A theory is topological if it only has global degrees freedom and does not depend on the metric of spacetime. The path integral of the topological theory [1] has the semiclassically exact description [3]. Hence this feature localizes the integration [2] to a finite dimensional manifold. By now, the topological systems provide us with an exact approach to study quantum gravity theory and applications to strongly correlated condensed matter systems [4]. Therefore, the topological theory is still interesting in many hot topics due to the solvable property.

An interesting direction is the first-order formulation of the three-dimensional Einstein gravity theory, which describes the gravitation. This theory could be formulated by the three-dimensional Chern-Simons theory with the SL(2) group [5]. Although the three-dimensional Einstein gravity theory is not renormalizable, the Chern-Simons gravity theory avoids this problem, which enables us to study quantum gravity theory from this perspective. Moreover, the Chern-Simons gravity theory also gets attention from the study of three-dimensional Anti-de Sitter/ two-dimensional conformal field theory (AdS$_3$/CFT$_2$) correspondence [6]. This correspondence states that a weakly coupled theory in the AdS$_3$ background, which is a solution of Einstein gravity theory with a negative cosmological constant, can be dual to a strongly coupled CFT$_2$.

The study in the exact boundary dual theory [7] of AdS$_3$ Einstein gravity theory [8] begins with the two-dimensional Liouville theory [9]. The Liouville theory is CFT$_2$ and has a continuous spectrum without a normalizable vacuum. On the contrary, the AdS$_3$ Einstein gravity theory has a discrete spectrum with a normalizable vacuum. Therefore, the bulk theory contradicts the fact of the Liouville theory. Hence this exact study does not provide a reliable result. The other possibility of the boundary dual theory, CFT$_2$, of AdS$_3$ Einstein gravity theory is the monster theory [10] for the most negative possible value of the cosmological constant [11]. Moreover, this approach also did not provide a clear answer.

The compactification of the Euclidean AdS$_3$ Einstein gravity theory provides the nearly AdS$_2$/CFT$_1$ correspondence describing the relationship between the Euclidean Jackiw-Teitelboim (JT) gravity theory [12] and the Sachdev-Ye-Kitaev (SYK) model [13]. The JT gravity theory describes the linear interaction between a dilaton field and two-dimensional Einstein gravity theory with the Dirichlet boundary condition. From the
JT gravity theory, one derived the Euclidean Schwarzian theory on the boundary by integrating out the dilaton field [13]. It is particularly interesting to point the emergence picture from that the solution of the boundary dilaton could be obtained from the JT gravity theory and the Euclidean Schwarzian theory both [13]. The SYK model has all-to-all interaction from a four-Majorana fermion coupled term with a Gaussian random coupling constant. The SYK (or SYK4) model also gives the Euclidean Schwarzian theory in the holographic limit, which is $N \gg \beta J \gg 1$, where $N$ is the number of Majorana fermion field, $\beta$ is the inverse temperature, and $J$ is the coupling constant [16, 17]. The Euclidean Schwarzian theory is not conformal field theory and was found to be dual to the one-dimensional Liouville theory [17, 18], which can be obtained from the two-dimensional Liouville theory through the dimensional reduction. The technology of the path integration in the Euclidean Schwarzian theory [19] was developed by using the Duistermaat-Heckman formula [20], which is the inverse Fourier transformation of a symplectic measure [21] that can be exactly given by the stationary phase approximation [20]. This was also generalized to the equivalent cohomology to enable doing integration which is equivalent to summing over all fixed points of the Hamiltonian [22]. The perturbation with respect to the low-energy parameter ($\beta J \gg 1$) also gives a deformation of the JT gravity theory with a square dilaton term [23]. Therefore, these results provide an interesting exact study going beyond the AdS/CFT correspondence and combine the qubit physics [24] with the gravity theory. Moreover, the first-order formulation of JT gravity theory could be constructed from the BF gravity theory [25], which was also obtained from the dimensional reduction of the single Chern-Simons term [26].

More recently, the boundary theory of the AdS3 Einstein gravity theory has been derived [27]. The boundary theory appears as a double copy of the two-dimensional Schwarzian-like theory, similar to the Schwarzian theory [27]. Hence the holographic study is the nearly AdS/CFT correspondence. The problem of the exact study in the two-dimensional Liouville theory is that the gauge fields have the SL(2) redundancy [27]. Ones thus need to identify these field configurations to obtain additional constraints in the path integration [27].

The generalization from the SL(2) to SL(M) in the Chern-Simons formulation provides massless interacting fields of the spin-2, spin-3, \ldots, and spin-M [28]. This provides the equations of motion which are similar to the equations of motion of four-dimensional higher spin theory in the flat background limit. Ones expect that the three-dimension
higher spin theory has the interacting information without meeting the no-go theorem in four dimensions because the theory was built in the AdS background. The similar higher spin extension was also built in the JT gravity theory [29]. Since the higher spin theory has infinite constraints from the equations of motion and ultraviolet complete quantum gravity theory should be unique, ones expect that the higher spin theory should be contained in string theory at infinity stringy correction ($\alpha' \to \infty$). The string theory describes a one-dimensional string moving on two-dimensional worldsheet and the fluctuation of target space giving spectrum containing the Einstein gravity theory. The spectrum of three-dimensional higher spin theory was already reproduced by string theory. Therefore, a study of the higher spin theory should enhance our understanding of ultraviolet quantum gravity theory.

The central interesting direction that we would like to address in this article is: Could we use the Schrödinger equation in the integrable quantum mechanics to study the different limits of string theory, higher spin theory, as in the JT gravity theory? In this paper, we show that the action of Chern-Simons higher spin theory after the dimensional reduction is the action of SL(3) Schwarzian theory, which is not CFT. We also explicitly show that the Lorentzian SL(3) Schwarzian theory is dual to the integrable quantum mechanics, SL(3) open Toda chain theory. Moreover, we discuss the application of open Toda chain theory from the SYK perspective and quantum chaos.

The dynamics of SYK model provides the interesting random matrix spectrum in the spectral form factors [30, 31] numerically. One of the required conditions of the quantum chaos, which excludes the integrability, is the random eigenvectors [32]. This required condition can be realized from the dip-ramp-plateau behavior of the spectral form factor at the high-temperature limit, which first decreases, then increases, and then reaches to an approximately constant in the time evolution [31]. This realization was justified from the SYK$_2$ model [33]. Recently, one got an exact solution of the spectral form factor in the Gaussian unitary ensemble matrix model to confirm the dip-ramp-plateau behavior [34]. Since the SL(2) open Toda chain theory has infinite energy levels, it is similar to quantum field theory. It is still unclear whether our experience of quantum chaos can be directly applied to quantum field theory. Hence we want to justify the behavior in quantum chaos and the holographic limit.

In this paper, we do the simulation in the SL(2) open Toda chain theory to calculate the spectral form factor. The result does not contradict with the integrability and also
shows that the large $N$ SYK model in the holographic limit and the JT gravity theory with a constant boundary dilaton do not have the random matrix spectrum. Because the simulation of the random matrix spectrum in the SYK model was still confined by a finite $N$, which is still far from the holographic limit [31], we should provide the first numerical study to the large $N$ SYK model.

2 Chern-Simons Higher Spin Theory and SL(3) Schwarzian Theory

The action of the SL($M$) Chern-Simons formulation in the Lorentzian spacetime is given by [5]

$$
S_G = \frac{k}{2\pi} \int d^3x \epsilon^{tr\theta} \text{Tr} \left( A_t F_{r\theta} - \frac{1}{2} (A_r \partial_t A_\theta - A_\theta \partial_t A_r) \right) \\
- \frac{k}{2\pi} \int d^3x \epsilon^{tr\theta} \text{Tr} \left( \bar{A}_t \bar{F}_{r\theta} - \frac{1}{2} (\bar{A}_r \partial_t \bar{A}_\theta - \bar{A}_\theta \partial_t \bar{A}_r) \right) \\
- \frac{k}{4\pi} \int dt d\theta \text{Tr}(A_\theta^2) \\
- \frac{k}{4\pi} \int dt d\theta \text{Tr}(\bar{A}_\theta^2),
$$

(1)
in which the boundary conditions of the gauge fields $A$ and $\bar{A}$ are given by $A_\mu = A_t - A_\theta = 0$ and $\bar{A}_\mu = \bar{A}_t + \bar{A}_\theta = 0$. The time direction is $t$, and two spatial directions are $r$ and $\theta$. Each bulk term, which lives in three dimensions, is equivalent to the action of the Chern-Simons theory up to a total derivative term. We assume that the gauge fields are given by $A_\mu \equiv A_\mu^a J_a$ and $\bar{A}_\mu = \bar{A}_\mu^a \bar{J}_a$, in which the spacetime indices are labeled by $\mu = t, r, \theta$, and the Lie algebra indices are labeled by $a = 1, 2, \cdots, M^2 - 1$. The algebra indices are raised or lowered by $\eta^{ab} \equiv \text{diag}(-1, 1, \cdots, 1)$. The $J^a$ and $\bar{J}^a$ are the generators of the SL($M$) Lie algebras. The constant $k$ is $l/(4G_3)$ with $1/l^2 \equiv -\Lambda$, where the cosmological constant is denoted by $\Lambda$, and three-dimensional gravitational constant is denoted by $G_3$. The gauge fields are defined by the vielbeins $e_\mu^a$ and spin connections $\omega_\mu^a$ [3]: $A_\mu = J_a \left( e_\mu^a / l + \omega_\mu^a \right)$ and $\bar{A}_\mu = \bar{J}_a \left( e_\mu^a / l - \omega_\mu^a \right)$. The path integral measure of the SL($M$) Chern-Simons formulation is $\int \mathcal{D}A \mathcal{D}\bar{A}$.

When $M = 2$, the theory is the spin-2 theory, which is the first-order formulation of Einstein gravity theory [5]. The boundary action of spin-2 theory is just an SL(2)
invariant double copy of the two-dimensional Schwarzian-like theory [27]. One could compactify the Euclidean time direction to obtain the Schwarzian theory [26, 27].

In this paper, we focus on $M = 3$. We are interested in the gauge fields in the Lorentzian theory, which provides the Lorentzian AdS$_3$ metric on the boundary ($r \to \infty$), namely the asymptotically AdS solution [28]. The Lorentzian AdS$_3$ geometry is:

$$ds_{\text{AdS}_3}^2 = -(r^2 + 1)dt^2 + \frac{dr^2}{r^2 + 1} + r^2d\theta^2 = -\left( e^\rho + \frac{1}{4}e^{-\rho} \right)^2 dt^2 + d\rho^2 + \left( e^\rho - \frac{1}{4}e^{-\rho} \right)^2 d\theta^2, \quad (2)$$

where $r \equiv e^\rho - e^{-\rho}/4$. The Lorentzian AdS$_3$ geometry corresponds to the gauge fields:

$$A_{\text{AdS}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} d\rho + \begin{pmatrix} 0 & -\frac{1}{2}e^{-\rho} & 0 \\ e^\rho & 0 & -\frac{1}{2}e^{-\rho} \\ 0 & e^\rho & 0 \end{pmatrix} dx^+,$$

$$\bar{A}_{\text{AdS}_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} d\rho + \begin{pmatrix} 0 & 2e^\rho & 0 \\ -\frac{1}{4}e^{-\rho} & 0 & 2e^\rho \\ 0 & -\frac{1}{4}e^{-\rho} & 0 \end{pmatrix} dx^-, \quad (3)$$

where $x^+ \equiv t + \theta$ and $x^- \equiv t - \theta$. We thus impose the following conditions on the gauge fields:

$$A_{\rho \to \infty} = A_{\text{AdS}_3}, \quad \bar{A}_{\rho \to \infty} = \bar{A}_{\text{AdS}_3}, \quad (4)$$

which is known as the fall-off condition of the asymptotically AdS solution.

The equations of motion of Chern-Simons theory provide $F^a_{\mu \nu} = \bar{F}^a_{\mu \nu} = 0$, where $F^a_{\mu \nu}$ and $\bar{F}^a_{\mu \nu}$ are the field strengths associated to the one-form gauge fields, $A$ and $\bar{A}$. Hence the action can be rewritten as the following

$$S_G = -\frac{k}{4\pi} \int d^3x \ e^{trA} \text{Tr} \left( -g^{-1}(\partial_r g)g^{-1}(\partial_t g)g^{-1}(\partial_\theta g) + g^{-1}(\partial_r g)g^{-1}(\partial_\theta g)g^{-1}(\partial_t g) \right) + \frac{k}{4\pi} \int d^3x \ e^{tr\bar{A}} \text{Tr} \left( -\bar{g}^{-1}(\partial_r \bar{g})\bar{g}^{-1}(\partial_t \bar{g})\bar{g}^{-1}(\partial_\theta \bar{g}) + \bar{g}^{-1}(\partial_r \bar{g})\bar{g}^{-1}(\partial_\theta \bar{g})\bar{g}^{-1}(\partial_t \bar{g}) \right) + \frac{k}{2\pi} \int dt d\theta \ \text{Tr} \left( g^{-1}(\partial_\theta g)g^{-1}(\partial_+ g) \right) - \frac{k}{2\pi} \int dt d\theta \ \text{Tr} \left( \bar{g}^{-1}(\partial_\theta \bar{g})\bar{g}^{-1}(\partial_+ \bar{g}) \right), \quad (5)$$

5
in which we integrate out $A_0$ and $\bar{A}_0$, which is equivalent to using the solutions of equations of motion, $A_I = g^{-1}\partial_I g$ and $\bar{A}_I = \bar{g}^{-1}\partial_I \bar{g}$. The spatial index is labeled by $I$. For $M = 3$, the bulk terms of the higher spin action cannot be just described by a lower dimension term, which is not as in $M = 2$. We have 16 independent variables to describe the $M = 3$ theory, which is too complicated to analyze. For simplicity, we do the dimensional reduction [26] in the Euclidean time direction [27] to study the $M = 3$ theory. Because the Lorentzian and Euclidean Einstein gravity theories have the opposite overall signs in the actions, the action of the $M = 3$ theory in the Euclidean time, which becomes $\theta$ in the boundary theory, is $S_{G1} = (k/2) \int d\theta \text{Tr}(g^{-1}(\partial_\theta g)g^{-1}(\partial_\theta \bar{g}))$ in the infrared limit.

We use the Gauss parameterization of the SL(3) group elements

$$g_{\text{SL(3)}}(F, \lambda, \Psi) = \begin{pmatrix} 1 & 0 & 0 \\ F_1 & 1 & 0 \\ F_2 & F_3 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1 \lambda_2} \end{pmatrix} \begin{pmatrix} 1 & \Psi_1 & \Psi_2 \\ 0 & 1 & \Psi_3 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

The boundary condition [4] thus leads to:

$$g_{\text{SL(3)}}^{-1}(F, \lambda) \partial_\theta g_{\text{SL(3)}} = A_\theta|_{\rho \to \infty} = \begin{pmatrix} 0 & 0 & 0 \\ r & 0 & 0 \\ 0 & r & 0 \end{pmatrix}. \quad (7)$$

This gives the following boundary conditions on the fields ($F$, $\lambda$, and $\Psi$):

$$F_3 = \frac{\partial_\theta F_2}{\partial_\theta F_1}, \quad \lambda_2^3 = \frac{\partial_\theta F_1}{\partial_\theta F_3}, \quad \lambda_1^3 = r^3 \frac{1}{(\partial_\theta F_1)^2(\partial_\theta F_3)}, \quad \lambda_1 = r \left( \frac{\partial_\theta \lambda_1}{\lambda_1} + \frac{\partial_\theta \lambda_2}{\lambda_2} \right). \quad (8)$$

We first use the conditions, $\lambda_1 \lambda_2^2 \partial_\theta F_3 = (\lambda_1/\lambda_2) \partial_\theta F_1$ and $F_3 \partial_\theta F_1 = \partial_\theta F_2$, in the Lorentzian action and then do the Wick rotation ($\theta \to -i\theta$) to obtain the action of the higher spin theory in the Euclidean time

$$\frac{k}{2} \int d\theta \text{Tr}\left(g^{-1}(\partial_\theta g)g^{-1}(\partial_\theta \bar{g})\right) = \frac{k}{2} \int d\theta \left( \frac{\lambda_1^2}{\lambda_2} (\partial_\theta \lambda_1)^2 + \frac{2}{\lambda_1 \lambda_2} (\partial_\theta \lambda_1)(\partial_\theta \lambda_2) + \frac{2}{\lambda_2^2} (\partial_\theta \lambda_2)^2 \right. \left. + 2 \frac{\lambda_1}{\lambda_2} (\partial_\theta F_1)(\partial_\theta \Psi_1) + 2 \frac{\lambda_1}{\lambda_2} (\partial_\theta F_1)(\partial_\theta \Psi_3) \right). \quad (9)$$
Then we use the conditions, \( \lambda_1 = r (\partial_\theta F_1)^{-\frac{2}{3}} (\partial_\theta F_3)^{-\frac{1}{3}} \) and \( \lambda_2 = (\partial_\theta F_1)^{\frac{1}{3}} (\partial_\theta F_3)^{-\frac{1}{3}} \), to obtain the action of the SL(3) Schwarzian theory in the Euclidean time

\[
S_{G1} = -k \int d\theta \left[ \frac{\partial^3 F_1}{\partial \theta F_1} + \frac{\partial^3 F_3}{\partial \theta F_3} - \frac{4}{3} \left( \frac{\partial^2 F_1}{\partial \theta F_1} \right)^2 - \frac{4}{3} \left( \frac{\partial^2 F_3}{\partial \theta F_3} \right)^2 - \frac{1}{3} \left( \frac{\partial F_1}{\partial \theta F_1} \right) \left( \frac{\partial F_3}{\partial \theta F_3} \right) \right]. \tag{10}
\]

The measure of the SL(3) Schwarzian theory is given by the following

\[
\int d\lambda_1 \wedge d\lambda_2 \wedge dF_1 \wedge dF_2 \wedge dF_3 \wedge dF_1 \wedge dF_2 \wedge dF_3 \wedge \lambda_1 \lambda_2 \lambda_3
\times \delta \left( F_3 - \frac{\partial F_2}{\partial F_1} \right) \delta \left( \lambda_2^3 \left( \frac{\partial F_3}{\partial F_1} \right)^2 - \frac{r^3}{\partial F_3} \right) \delta \left( \frac{\lambda_2^2 \partial F_3}{\partial F_1} - 1 \right)
\times \delta \left( r \Psi_1 - \frac{\partial \lambda_1}{\lambda_1} \right) \delta \left( \Psi_2 - \frac{\partial \lambda_1}{r^2 \lambda_1} \right) \delta \left( \Psi_3 - \frac{1}{r} \left( \frac{\partial \lambda_1}{\lambda_1} + \frac{\partial \lambda_2}{\lambda_2} \right) \right)
\sim \int dF_1 \wedge dF_2 \frac{1}{\left( \partial F_1 \right)^2 \left[ \frac{1}{\partial F_2} \left( \frac{\partial F_2}{\partial F_1} \right) \right]}, \tag{11}
\]

which is invariant under the SL(3) transformation. The fields \( (F, \lambda, \Psi) \) are transformed by that \( h_{SL(3)} \cdot g_{SL(3)}(F, \lambda, \Psi) = g_{SL(3)}(\tilde{F}, \tilde{\lambda}, \tilde{\Psi}) \), where \( h_{SL(3)} \) is an SL(3) transformation and does not dependent on the fields. The transformed fields are denoted by \( \tilde{F}, \tilde{\lambda}, \) and \( \tilde{\Psi} \). Moreover, the transformations of \( F_1 \) and \( F_2 \) reproduce the standard transformations in the SL(3) Schwarzian theory.

### 3 SL(3) Open Toda Chain Theory

We begin from the action in the Euclidean time

\[
S_{OTC1} = 4k \int d\theta \left( (\partial_\theta \phi_1)^2 + (\partial_\theta \phi_2)^2 + (\partial_\theta \phi_1)(\partial_\theta \phi_2) + \Pi_{F_1} (\partial_\theta F_1 - e^{2\phi_1 - 2\phi_2}) + \Pi_{F_2} (\partial_\theta F_2 - e^{4\phi_2 + 2\phi_1}) \right). \tag{12}
\]

The measure is \( \int (dF_1 \wedge dF_2)(d\phi_1 \wedge d\phi_2)(d\Pi_{F_1} \wedge d\Pi_{F_2}) \left( 1/\partial_\theta \phi_2 \right) \). If we integrate out the momenta, \( \Pi_{F_1} \) and \( \Pi_{F_2} \), the action goes back to the action \( S_{G1} \) up to a total derivative term and its measure. Now we integrate out the fields, \( F_1 \) and \( F_2 \), and choose \( \Pi_{F_1} = \Pi_{F_2} = -1 \). After doing the Wick rotation \( (\theta \rightarrow i\theta) \), we obtain the action in the Lorentzian time

\[
S_{OTC2} = 4k \int d\theta \left( (\partial_\theta \phi_1)^2 + (\partial_\theta \phi_2)^2 + (\partial_\theta \phi_1)(\partial_\theta \phi_2) - e^{2\phi_1 - 2\phi_2} - e^{4\phi_2 + 2\phi_1} \right). \tag{13}
\]
The measure becomes \( \int d\phi \wedge d\phi_2 (1/\partial_\theta \phi_2) \). Finally, imposing the constraint \( \phi_1 + \phi_2 + \phi_3 = 0 \) gives the action of the SL(3) open Toda chain theory

\[
S_{OTC} = 4k \int d\theta \left( \frac{1}{2} \sum_{i_1=1}^{3} (\partial_\theta \phi_{i_1})^2 - \sum_{i_2=1}^{2} e^{2(\phi_{i_2} - \phi_{i_2+1})} \right). \tag{14}
\]

The measure of the SL(3) open Toda Chain theory is \( \int d\phi \wedge d\phi_2 \wedge d\phi_3 \delta(\phi_1 + \phi_2 + \phi_3) / \partial_\theta \phi_2 \).

4 Spectral Form Factor in the SL(2) Open Toda Chain Theory

The spectral form factor \([30, 31] \) is defined by \( g(t) \equiv |Z(\beta, t)|^2 / |Z(\beta, 0)|^2 \), where

\[
Z(\beta, t) \equiv \text{Tr} \left( \exp \left( - (\beta - it)H \right) \right)
\]

is the un-normalized thermal average of operator \( \exp(itH) \) and \( H \) is the Hamiltonian of a system.

The exact solution of the spectral form factor in the SL(2) open Toda chain theory

\[
g(t) = \left( \frac{\beta^3}{(\beta^2 + t^2)^{3/2}} \right) \cdot \exp \left( - \pi^2 t^2 / (\beta (\beta^2 + t^2)) \right) \tag{31}
\]

can be obtained by the partition function of the SL(2) Schwarzian theory \( Z_S(\beta, 0) \sim \exp(\pi^2/(2\beta))/\beta^{3/2} \) through the localization \([19] \), in which we choose \( k = 1/4 \), and its analytical continuation on time. Therefore, we can find that the spectral form factor only has the dip except for \( \beta = 0 \). Since the partition function was obtained by the initial time with a finite temperature, the exact solution cannot extract the information at \( \beta = 0 \). This should not be problematic by using the lattice model. Hence we discretize the SL(2) open Toda chain theory to study the numerical solution of spectral form factor at \( \beta = 0 \). Since the spectrum is continuum, we should use the density of states with a continuous integral \( \int_0^\infty dE \rho(E) \) to define the trace operation \([19] \). Therefore, we do the rough approximation

\[
\int_0^\infty dE \rho(E) \approx \sum_{E=0}^{\infty} \delta N(E) \equiv \sum_{E=0}^{\infty} (N(E + \delta E) - N(E)), \quad N(E) \equiv \int_0^E dE \rho(E) \tag{15}
\]

to get the qualitative result.

The Hamiltonian of the SL(2) open Toda chain theory is \( H_{OTC} = p^2 + e^{4\phi} \), where \( p \equiv -i\partial/\partial \phi \). In our numerical study, we put the theory on a lattice defined as the
following: 

\[-L \leq \phi_j < L, \quad \phi_1 = -L, \quad \phi_{j+1} \equiv \phi_j + a, \quad \phi_0 \equiv \phi_n, \quad \phi_{n+1} \equiv \phi_1, \quad \text{and} \quad 2L = n \cdot a.\]

The lattice index is labeled by \( j = 1, 2, \cdots, n \). The lattice spacing is denoted by \( a \), and the lattice size is denoted by \( L \). The kinetic term on a lattice is written as

\[ p_l^2 \equiv -\left( \psi_{j+1} - 2\psi_j + \psi_{j-1} \right)/a^2, \]

where \( p_l \) is the lattice canonical momentum, and \( \psi_j \) is the lattice eigenfunction.

Since the theory is put in a box in our numerical study, the only realizable energy modes are the ones which are larger than the potential. To provide a physical result, we first fix a number of the low-lying modes and check the lattice spacing effects with a given lattice size in the spectral form factors. We fix the inverse temperature \( \beta = 0 \) and choose the lattice size \( L = 4 \), the number of lattice points \( n = 1024 \). In Fig. 1, we plot the spectral form factor to show the dip-ramp-plateau behavior for 32, 64, and 128 low-lying modes. Because the ramp time changes obviously by choosing different numbers of low-lying modes, the ramp time should go to zero under the continuum and the infinite size limits. Therefore, dip-ramp-plateau behavior in Fig. 1 should be due to the lattice artifact. The relation between the JT gravity theory with a constant boundary dilaton [12] and the SL(2) Toda chain theory is exact [14, 15, 17]. Hence spectrum of the JT gravity theory should be integrable. This also suggests that the spectrum in SYK model [13] in the holographic limit should not have the random matrix structure.

Hence these results do not contradict with the fact of integrability. Our numerical study possibly not be accurate enough to reach the quantitative level, but the physical interpretation and qualitative behaviors should not be modified from the lattice size effect.
5 Outlook

Because the SYK model [13] in the holographic limit and the JT gravity with a boundary dilaton is dual to the SL(2) Toda chain theory [17, 18], we justified the random matrix spectrum and duality. A suitable quantity for diagnosing chaos should be valid for any duality. Because the dip-ramp-plateau behavior on the spectral form factor is related to the randomness [33], this should be a condition of the irregular motion under the classical limit [32]. The irregular motion in the classical chaos in the interval is a purely topological condition without affecting by any mapping. Hence our result should demonstrate a useful application in the open Toda chain theory. It should be interesting to apply the SL(3) Toda chain theory to the different limit of string theory, higher spin theory.

Although the open Toda chain theory is a quantum mechanical system, the theory has infinite energy levels. The integrability and quantum chaos in quantum field theory still have ambiguity in the definition. Hence the open Toda chain theory is also a good toy model to explore the definition of quantum chaos and integrability in quantum field theory.

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