Abstract

In their recent paper (GandALF 2018), Goubault, Ledent, and Rajsbaum provided a formal epistemic model for distributed computing. Their logical model, as an alternative to the well-studied topological model, provides an attractive framework for refuting the solvability of a given distributed task by means of logical obstruction: One just needs to devise a formula, in the formal language of epistemic logic, that describes a discrepancy between the model of computation and that of the task. However, few instances of logical obstruction were presented in their paper and specifically logical obstruction to the wait-free 2-set agreement task was left as an open problem. Soon later, Nishida affirmatively answered to the problem by providing inductively defined logical obstruction formulas to the wait-free $k$-set agreement tasks.

The present paper refines Nishida’s work and devises logical obstruction formulas to $k$-set agreement tasks for superset-closed adversaries, which supersede the wait-free model. These instances of logical obstruction formulas exemplify that the logical framework can provide yet another feasible method for showing impossibility of distributed tasks, though it is currently being confined to one-round distributed protocols. The logical method has an advantage over the topological method that it enjoys a self-contained, elementary induction proof. This is in contrast to topological methods, in which sophisticated topological tools, such as Nerve lemma, are often assumed as granted.

1 Introduction

In the last few decades, the topological theory of distributed computing has been successful in giving a range of fundamental insights and results, most notably the simplicial complex model of distributed tasks, protocols as simplicial subdivisions, and the impossibility results on a significant family of distributed tasks [18, 16]. Though it has been recognized since the earliest work by Saks and Zaharoglou [25] that the topological model is an interpretation of epistemic knowledge held by distributed processes, the rigorous connection was established only very recently by Goubault, Ledent, and Rajsbaum in [13]. They defined the task solvability in terms of a multi-agent dynamic epistemic logic (DEL) [26], establishing the isomorphism between the topological models and an appropriate class of logical models of epistemic knowledge.

This new logical model yields a formal means to refute task solvability: One is obliged to find an epistemic logic formula $\psi$, called logical obstruction, such that $\psi$ holds for the model of the task but does not for the model of the protocol that is relevant to the system of concern. However, in [13], concrete instances of logical obstruction are given only for the wait-free binary consensus task but none for others. Particularly the logical obstruction to the general wait-free $k$-set agreement task was left as an open problem. Soon later, Nishida settled it positively by devising concrete logical obstruction formulas to the wait-free $k$-set agreement tasks [24], where he carried out the proof by elementary inductive argument on inductively defined formulas.

This paper presents an extension of Nishida’s result to the computation model of superset-closed adversaries [10], which generalizes the basic wait-free model. More specifically, we provide logical
obstruction formulas to show that $k$-set agreement tasks for a sufficiently small $k$ are not solvable under superset-closed adversaries, by a single round execution of the round operator [17].

The logical obstruction we are to present for the adversarial model is a generalization of the one devised by Nishida for the wait-free model. It is somewhat surprising that essentially the same logical obstruction, up to appropriate generalization, works well for the models of different levels of complexity. (See Section 4.2 for a discussion of how the underlying simplicial structures are different in the two models.) This is made possible by exploiting the notion of permutation subset, a combinatorial feature intrinsic to set agreement tasks [24].

The impossibility proofs by Nishida’s and ours demonstrate that the logical method proposed by by Goubault, Ledent, and Rajsbaum [13] serves as yet another feasible method of refuting task solvability, where the logical obstruction is expressed in a formal language of epistemic logic. In the logical framework, one just needs to devise a logical obstruction formula to the task of concern and is obliged to prove that the formula is indeed an obstruction. In this paper, we show that a simple elementary inductive argument suffices, at least the unsolvability of $k$-set agreement tasks for the adversarial model is concerned.

We have to remark that this paper discusses task unsolvability is confined to single round protocols solely and does not concern multiple round protocols, i.e., iterated execution of a single round protocol. This is in contrast to the preceding work by Herlihy and Rajsbaum [17] that provides a topological proof for the unsolvability of set agreement tasks for multiple round protocols. We would need a further technical development for effectively dealing with multiple round protocols in the logical framework and leave this topic for future investigation.

Related work. Topological methods have been successfully applied to show the impossibility of set agreement tasks for the wait-free model [18] and for the adversary model [17]. The general proof strategy is to find a ‘topological’ obstruction that detects the topological inconsistency between the models of the task and the protocol. In the case of set agreement tasks, the unsolvability comes from the fact that the image of the carrier map [16] of the task, i.e., a functional specification of the task that maps an input simplex to an output complex, is less connected than that of the protocol. In contrast, in the logical framework proposed by Goubault et al. [13], one is asked to devise a logical obstruction formula in the product update model, which encodes the input/output relation of a carrier map by a complex obtained by a relevant product construction. (See Section 2.3.2 for the formal definition of product update model.) It is often difficult to translate a topological inconsistency into a logical obstruction formula, since the complex of a product update model tends to have a fairly different, more complicated simplicial structure than that of the original one.

The impossibility results presented in [24] and this paper are demonstrated by a self-contained, elementary proof. Once the semantics of the formal language of epistemic logic is learnt, one is immediately accessible to every detail of the proof. In contrast, impossibility proofs in topological methods tend to resort to sophisticated theorems from topology. For instance, the impossibility proof of set agreement tasks given in [17] critically depends on the Nerve lemma, which is by no means an elementary topological result. (See [4] for a combinatorial proof of Nerve lemma and [20] for a proof from the perspective of modern algebraic topology.)

It should be noticed that the logic-based solution provided in this paper does not fully substitute for the topological method. First, as we mentioned earlier, our logical obstruction concerns single round protocols only and is not immediately applicable to multiple round protocols. This is because a different logical obstruction formula is required for each specific epistemic structure that varies at every incremental round step. In contrast, topological method is more robust to such incremental evolution in the underlying structure, resorting to a certain topological invariant that is kept intact throughout the steps of rounds. Second, the product update model of the logical framework is only able to relate each input facet (i.e., a simplex of maximal dimension) to an output complex (of the same dimension). This means the logical framework assumes certain crash-free systems in which no process execution fails, as opposed to the usual crash-prone assumption on distributed systems. The logical framework, however, still allows us to derive impossibility results for set agreement tasks: Under the asynchronous setting, faulty processes can be regarded just as processes which execute so “slowly” that correct processes are ignorant of them. [22, 11]

Naturally, the development of the present paper much owes to Nishida’s original work [24]. Our contribution is to generalize his insight in the construction of logical obstruction formulas so that it can encompass adversarial models. This also contributes to refine Nishida’s original proof with
appropriate level of generalization. We hope the present paper would help the significant idea in Nishida’s paper, which is of limited accessibility, to reach a wider audience of interest.

The rest of the paper is organized as follows. Section 2 reviews the simplicial semantic model for epistemic logic and the theorem for task solvability in the model of dynamic update of knowledge, as introduced in [13]. In Section 3, we reproduce a classical impossibility result in the DEL framework by means of logical obstruction, showing that a simple combinatorial argument suffices for demonstrating the obstruction proof, without recourse to topological arguments. In Section 4, generalizing Nishida’s, we provide logical obstruction formulas to $k$-set agreement tasks for superset-closed adversaries and provides an inductive proof on the cardinality of adversaries. Section 5 concludes the paper with a summary and a discussion on directions for future research.

2 The DEL Model for Distributed Computing

Throughout the paper, we assume that a distributed system consists of $n+1$ asynchronous processes ($n \geq 0$), which are given unique ids.

2.1 Simplicial topology for distributed computing

A simplicial complex (complex for short) $C$ is a family of finite sets of vertexes, called simplexes, closed under set inclusion, that is, $Y \subseteq X$ and $X \in C$ implies $Y \in C$ for any pair of simplexes $X$ and $Y$. We say that a complex $D$ is a subcomplex of $C$, if $D \subseteq C$. A simplex $X \in C$ is of dimension $n$ if $|X| = n + 1$. We call a simplex $X \in C$ a facet, if $X$ is a maximal simplex in $C$, i.e., $X \subseteq Y$ implies $X = Y$ for any $Y \in C$. A complex $C$ is called pure (of dimension $n$), if every facet of $C$ has the same dimension $n$. We write $V(C)$ for the set of all vertexes in $C$ and $F(C)$ for the set of all facets in $C$.

In topological methods for distributed computing [16], a simplex is used for modeling a system state of the collection of asynchronous processes, where each vertex stands for a state of an individual process. A simplicial complex stands for the set of possible states of a distributed system. In this paper, we are solely concerned with the so-called colored distributed tasks. Colored tasks are modeled by chromatic simplicial complex, where the coloring function $\chi$ assigns a color $\chi(v)$ for each vertex $v$ so that different vertexes contained in the same simplex are distinctively colored, that is, whenever $u, v \in X$ for a simplex $X$, $\chi(u) = \chi(v)$ implies $u = v$. The coloring function models the assignment of unique ids to individual processes.

Throughout the paper, we are solely concerned with pure $n$-dimensional chromatic complex, which we simply call complexes in the rest of the paper. A chromatic complex is formally denoted by a pair $(C, \chi)$. Since a (pure chromatic) complex $C$ is equally defined by the set of its facets $F(C)$, so that $C = \{ Y \mid Y \subseteq X \text{ for some } X \in F(C) \}$, we may also write a complex as $(F(C), \chi)$, or even in the abridged notation $C$, leaving the coloring function $\chi$ implicit.

A simplicial map $\mu : V(C) \rightarrow V(D)$, where $C$ and $D$ are complexes, is a vertex map such that $\mu(X)$ is a simplex of $D$ for any simplex $X$ of $C$. In addition, we postulate that $\mu$ is color-preserving, that is, $\chi(v) = \chi(\mu(v))$ for every vertex $v \in V(C)$. Hence $\mu$ preserves the dimension of simplexes, in particular, it maps a facet of $C$ to a facet of $D$. Furthermore, simplicial map commutes with intersection on simplexes, namely, $\mu(X \cap Y) = \mu(X) \cap \mu(Y)$.

2.2 Epistemic logic and its semantics

The (multi-agent) epistemic logic [26] is a logic for formal reasoning of knowledge of an individual agent or a group of agents. We assume a set $AP$ of atomic propositions: An atomic proposition is a propositional symbol that states whether a certain property is qualified. The following grammar defines the formal language of epistemic logic formulas:

$$\varphi ::= p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \neg \varphi \mid K_a \varphi \mid C_A \varphi \mid D_A \varphi,$$

where $p$ ranges over $AP$, $a$ ranges over a finite set $Ag$ of agents, and $A$ ranges over the powerset $2^{AP}$. As a usual convention, we write `false` for abbreviation of $p \land \neg p$, where $p$ is an arbitrary atomic proposition.

In addition to propositional connectives, epistemic logic formulas are equipped with a few modal operators regarding agents’ knowledge: $K_a \varphi$ is the ‘knowledge’ operator for a single agent, reading
“the agent a knows ϕ”; \(C_A ϕ\) and \(D_A ϕ\) are operators defined for a group \(A\) of agents, called ‘common knowledge’ and ‘distributed knowledge’ operators, respectively.

The semantics for epistemic logic formulas is defined over an appropriate Kripke frame. A Kripke frame is formally a pair \((S, ∼)\), where \(S\) is a collection of possible worlds and \(∼\) is a family of equivalence relations \(\{∼_a\}_{a ∈ A}\) over \(S\). Each \(∼_a\) is called an indistinguishability relation, meaning that the worlds \(X\) and \(Y\) are not distinguishable by the agent \(a\) iff \(X ∼_a Y\). A Kripke model \(M = (S, ∼, L)\) is a Kripke frame along with a function \(L : S → 2^{AP}\), which assigns a set \(L(X)\) of atomic propositions that are true in each world \(X ∈ S\).

In the following, let \(∼_{C_A}\) denote the reflexive transitive closure of \(\bigcup_{a ∈ A} ∼_a\), that is, \(X ∼_{C_A} Y\) iff \(X = X_0 ∼_{a_0} X_1 ∼_{a_1} \cdots ∼_{a_n} X_n = Y\) for some \(X_0, X_1, \ldots, X_n ∈ S\) and \(a_0, a_1, \ldots, a_n ∈ A\) \((n ≥ 0)\). Further, we define the relation \(∼_{D_A}\) by: \(X ∼_{D_A} Y\) iff \(X ∼_a Y\) for every \(a ∈ A\). Then for each particular world \(X\) of Kripke model \(M\), the semantics, or truth valuation, of an epistemic logic formula \(ϕ\) is given by the assertion \(M, X \models ϕ\), which is defined below by induction on the structure of \(ϕ\).

**Definition 2.1.** Given a Kripke model \(M = (S, ∼, L)\), the satisfaction relation \(M, X \models ϕ\) is defined as follows, by induction on \(ϕ\).

- \(M, X \models ϕ \iff p \in L(X)\);
- \(M, X \models ϕ \lor ψ \iff M, X \models ϕ \text{ or } M, X \models ψ\);
- \(M, X \models ϕ \land ψ \iff M, X \models ϕ \text{ and } M, X \models ψ\);
- \(M, X \models ¬ϕ \iff M, X \not\models ϕ\);
- \(M, X \models K_a ϕ \iff M, Y \models ϕ \text{ for every } Y \text{ such that } X ∼_a Y\);
- \(M, X \models C_A ϕ \iff M, Y \models ϕ \text{ for every } Y \text{ such that } X ∼_{C_A} Y\);
- \(M, X \models D_A ϕ \iff M, Y \models ϕ \text{ for every } Y \text{ such that } X ∼_{D_A} Y\).

We write \(M, X \not\models ϕ\) to mean \(M, X \not\models ϕ\) does not hold. A formula \(ϕ\) is called valid (in \(M\)), if \(M, X \not\models ϕ\) for every \(X ∈ S\); \(ϕ\) is called invalid, if \(M, X \not\models ϕ\) for some \(X ∈ S\).

We say an epistemic formula positive, if its every subformula of negated form \(¬ψ\) has no occurrences of modal operators \(K_a\), \(C_A\), and \(D_A\). For example, assuming \(p, q ∈ AP\), \(¬q \land C_A ¬p\) is a positive formula, while \(p \lor (q ∨ C_A p)\) is not, because the latter contains a negated subformula \(¬(q ∨ C_A p)\), whose scope of negation embraces a knowledge operator \(C_A\).

### 2.3 Dynamic epistemic logic and its simplicial model

In [13], Goubault et al. have shown that there is a tight correspondence between Kripke frame and simplicial complex. Restricted to an appropriate class of Kripke models, they are indeed the dual of each other in a suitable category theoretic setting [14]. In short, a simplicial complex and a Kripke model are just different representations of the same structure: colors correspond to agents, facets to possible worlds, and adjacency of facets to relation over possible worlds.

In the rest of the paper, \(Π = \{0, 1, \ldots, n\}\) is assumed to be the set of unique ids given to \(n + 1\) individual processes. We also write \(\text{Value}\) to denote the set of possible local values held by the processes. For convenience, we may often denote a colored vertex in a simplicial complex by a pair \((a, v) ∈ Π × \text{Value}\), which is intended to mean a process of id \(a\) that holds a local value \(v\). The coloring function is defined by \(χ((a, v)) = a\).

#### 2.3.1 Simplicial model induced by complex

In this paper, we assume the set of atomic propositions is given by \(AP = \{\text{input}_a^v \mid a ∈ Π, v ∈ \text{Value}\}\), where each atomic proposition \(\text{input}_a^v\) is intended to assert that the value \(v\) is held by the agent \(a\) (that is, the process whose id is \(a\)).

The Kripke model that is dual to a given complex \(C = (C, χ)\) is induced as below.

**Definition 2.2 (Simplicial model).** For any complex \(C\), we can induce a Kripke model \((F(C), ∼, L)\) such that:

\(^1\)There is another operator worth mentioning, called ‘group knowledge’, which is of no use in the present paper and is omitted.
the set of agents is taken as the set of colors, i.e., \( \text{Ag} = \Pi \),

- the set of possible worlds are the set of facets \( F(C) \),

- the equivalence relation is defined by:

\[
X \sim_a Y \text{ iff } a \in \chi(X \cap Y), \text{ where } X, Y \in F(C) \text{ and } a \in \text{Ag, and}
\]

\[
X \sim_b Y \text{ iff } \{ \text{input}_i, \text{input}_j, \text{input}_k \} \text{ if and only if } X \text{ is a facet comprising of vertexes } w_i, w_j, w_k.
\]

We call this induced Kripke model a \textit{simplicial model}. In abuse of notation, we also write \( C \) for the simplicial model \((F(C), \sim, L)\).

Although the entire results of this paper can be fully worked out without any help of topological intuition, it is instructive to see here the topological implication of epistemic formulas. Figure 1 illustrates a complex \( C \) of dimension 2, from which a simplicial model is induced. The complex consists of five facets \( X_1, X_2, X_3, X_4, \) and \( X_5 \), which are the possible worlds of the induced Kripke model. The three different agents (processes) are distinguished by colors 0, 1, and 2, and in the figure each vertex colored 0 (resp., 1 and 2) is depicted in white (resp., blue and red). A vertex denoted by \( w_i \) (resp., \( b \) and \( r_j \)) indicates that the value \( i \) is held by the vertex whose color is white (resp., blue and red). We assume \( L(X) = \{ \text{input}_0, \text{input}_1, \text{input}_2 \} \).

In the induced model, it holds that \( C, X_1 \models K_0 (\forall a \in \Pi \text{ input}_1), \) because there are only two facets related with \( X_1 \) by \( \sim_0 \) via the vertex \( w_2 \), namely \( X_1 \) itself and \( X_2 \), and \( \forall a \in \Pi \text{ input}_1 \) holds in both facets. On the other hand, \( C, X_3 \models K_2 (\forall a \in \Pi \text{ input}_1) \) does not hold, because \( X_5 \sim_2 X_3 \) but the value 1 is not held by none of vertexes of \( X_3 \). As for distributed knowledge, since \( \sim_{D(0,1)} \) relates \( X_3 \) with solely itself and \( \sim_{D(0,2)} \) further relates \( X_3 \) with \( X_4 \), we have \( C, X_3 \models D_{(0,1)} \text{ input}_1 \) but \( C, X_3 \nmodels D_{(0,2)} \text{ input}_1 \) because \( C, X_4 \nmodels \text{ input}_1 \). An assertion on common knowledge \( C, X_5 \models C_\Pi (\forall a \in \Pi \text{ input}_1) \) holds, because \( \sim_{C_0} \) relates \( X_5 \) with those facets that belong to the connected component of \( X_5 \), namely all the facets in \( C \), and \( \forall a \in \Pi \text{ input}_1 \) holds for every facet.

As for positive epistemic formulas, it can be shown that the epistemic knowledge never increases along with an appropriate simplicial map.

**Theorem 2.1** (Knowledge gain[13]). Suppose we are given a pair of simplicial models \((F(C), \sim, L_C)\) and \((F(D), \sim, L_D)\) that are induced from complexes \( C \) and \( D \), respectively. We call a function \( \delta : C \to D \) a \textit{morphism}, if it is a color-preserving simplicial map from \( V(C) \) to \( V(D) \) such that \( L_D(\delta(X)) = L_C(X) \) for every facet \( X \in F(C) \).

For any morphism \( \delta : C \to D \) and any positive formula \( \varphi, \ D, \delta(X) \models \varphi \implies C, X \models \varphi \), for every facet \( X \in F(C) \).

**Proof.** Proof is by induction on \( \varphi \). We only examine a few cases that have not been demonstrated in [13].

To prove the case of disjunction \( \varphi \lor \psi \), suppose \( D, \delta(X) \models \varphi \lor \psi \). Without loss of generality, we may assume \( D, \delta(X) \models \varphi \). By induction hypothesis, \( C, X \models \varphi \) and thus we have \( C, X \models \varphi \lor \psi \).

For the case of distributed knowledge operator, suppose \( D, \delta(X) \models D_A \varphi \). In order to show \( C, X \models D_A \varphi \), we assume \( Y \subseteq F(C) \) is an arbitrary facet such that \( X \sim_{D_A} Y \), i.e., \( A \subseteq \chi(X \cap Y) \), and show \( C, Y \models \varphi \). Since \( \delta \) is a color-preserving simplicial map, \( A \subseteq \chi(X \cap Y) \Rightarrow A \subseteq \chi(\delta(X \cap Y)) = \chi(\delta(X) \cap \delta(Y)) \), which means \( \delta(X) \sim_{D_A} \delta(Y) \). By the assumption \( D, \delta(X) \models D_A \varphi \), we have \( D, \delta(Y) \models \varphi \); hence \( C, Y \models \varphi \) by induction hypothesis.

This completes the proof.
From a topological perspective, it can be understood that the knowledge gain theorem stems from the fact that, for any morphism $\delta$, a positive epistemic formula is less likely to hold in $\delta(\mathcal{D})$, which is a more densely connected model than $\mathcal{C}$. Thus we can make use of positive epistemic formulas for detecting discrepancy in connectivity of simplicial models, as we will demonstrate in later sections.

2.3.2 The action model and product update

The dynamic epistemic logic (DEL) [26] is an epistemic logic with possible updates in the knowledge model. Following [13], we make use of product update model generated by epistemic actions [2, 26] for modeling the change in the knowledge structure incurred by an action of information exchange among communicating processes.

To define product update model, let us first define the cartesian product of simplicial complexes. Suppose $(\mathcal{C}, \chi_{\mathcal{C}})$ and $(\mathcal{D}, \chi_{\mathcal{D}})$ are complexes. For each pair of simplexes $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that $\chi_{\mathcal{C}}(X) = \chi_{\mathcal{D}}(Y)$, we define cartesian product $X \times Y$ as a simplex whose each vertex is a pair of vertexes of matching color from original simplexes, namely,

$$X \times Y = \{(u, v) \mid u \in X, v \in Y, \chi_{\mathcal{C}}(u) = \chi_{\mathcal{D}}(v)\}.$$  

The coloring on vertexes in the cartesian product inherits that on the original ones, namely, $\chi((u, v)) = \chi_{\mathcal{C}}(u)$. The cartesian product $\mathcal{C} \times \mathcal{D}$ of complexes of $\mathcal{C}$ and $\mathcal{D}$ is then defined by $F(\mathcal{C} \times \mathcal{D}) = F(\mathcal{C}) \times F(\mathcal{D}) = \{X \times Y \mid X \in F(\mathcal{C}), Y \in F(\mathcal{D})\}$.

We can define the pair of projection maps $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ by $\pi_{\mathcal{C}}((u, v)) = u$ and $\pi_{\mathcal{D}}((u, v)) = v$ for every $(u, v) \in V(\mathcal{C} \times \mathcal{D})$, respectively. They are both color-preserving simplicial maps.

**Definition 2.3** (Simplicial action model and product update). A simplicial action model $\mathcal{D} = \langle F(\mathcal{D}), \sim^{\mathcal{D}}, \text{pre} \rangle$ is an action model, where the possible worlds are facets of some complex $\mathcal{D}$ and $\sim^{\mathcal{D}}$ is the family of relations $\{\sim_{a} \mid a \in A_{g}\}$ induced from $\mathcal{D}$, i.e., $X \sim_{a} Y$ iff $a \in \chi(X \cap Y)$, where $a \in A_{g}$ and $X, Y \in F(\mathcal{D})$, and $\text{pre}$ is a function that assigns an epistemic logic formula, called a precondition, to each facet $X$ of $\mathcal{D}$.

The product update model of an initial simplicial model $\mathcal{C} = \langle \mathcal{C}, \sim^{\mathcal{C}}, L \rangle$ by an action model $\mathcal{D} = \langle F(\mathcal{D}), \sim^{\mathcal{D}}, \text{pre} \rangle$ is a simplicial model $\mathcal{C}[\mathcal{D}] = \langle \mathcal{C}[\mathcal{D}], \sim, L' \rangle$, where $\langle \mathcal{C}[\mathcal{D}], \sim \rangle$ is the Kripke frame induced from the complex $\mathcal{C}[\mathcal{D}] = \{z \in \mathcal{C} \times \mathcal{D} \mid \chi_{\mathcal{C}}(Z) \models \text{pre}(\pi_{\mathcal{D}}(Z))\}$ and $L'(Z) = L(L_{\mathcal{C}}(Z))$ for every $Z \in \mathcal{C} \times \mathcal{D}$.

The product update model provides an alternative way for specifying distributed tasks. In the topological setting, a task is modeled by a carrier map $\mathcal{C} \to 2^{\mathcal{D}}$, which maps each input simplex in $\mathcal{C}$ to a subcomplex $\mathcal{D}$ consisting of possible output simplexes [16]. The product update model $\mathcal{C}[\mathcal{D}]$ encodes the carrier map as a subset of the cartesian product $\mathcal{C} \times \mathcal{D}$, where $\mathcal{C}, X \models \text{pre}(Y)$ determines how an input simplex $X \in \mathcal{C}$ maps to an output simplex $Y \in \mathcal{D}$.

2.4 Task solvability in DEL semantics

Let us consider the task solvability for distributed system of $n + 1$ asynchronous processes, whose colors (process ids) are given by the set $\Pi = \{0, 1, \ldots, n\}$. We assume that, at the initial configuration, every process is given an arbitrary input taken from a finite set $\text{inp}$ of values and that the system solves a task by means of a particular protocol, a certain distributed procedure for exchanging values among processes, so that each process in the system decides the final output that conforms to the requirement by the task.

The initial simplicial model $\langle \mathcal{I}^{\text{inp}}, \sim^{\mathcal{I}^{\text{inp}}}, L \rangle$ is the simplicial model induced from the complex $\langle \mathcal{I}^{\text{inp}}, \chi_{\mathcal{I}^{\text{inp}}} \rangle$ such that

- The facets of $\langle \mathcal{I}^{\text{inp}}, \chi_{\mathcal{I}^{\text{inp}}} \rangle$ comprises of simplexes $\{(0, v_{0}), (1, v_{1}), \ldots, (n, v_{n})\}$ of dimension $n$, where $v_{0}, v_{1}, \ldots, v_{n} \in \text{inp}$ and the color of each vertex $(a, v)$ is determined by $\chi_{\mathcal{I}^{\text{inp}}}(a, v) = a$ and
- $L(X) = \{\text{input}^{a}_{v} \mid (a, v) \in X\}$ for every facet $X$ of $\mathcal{I}^{\text{inp}}$.  

6
A protocol is modeled by a simplicial action model \( (C, \sim^C, \text{pre}_C) \), called a communicative action model, and a task is modeled by a simplicial action model \( (T, \sim^T, \text{pre}_T) \), where \( C \) (resp., \( T \)) models the possible results of the execution of the protocol (resp., the possible outputs of the task) with \( \text{pre}_C \) (resp., \( \text{pre}_T \)) relating facets of \( I^{\text{inp}} \) with those of \( C \) (resp., \( T \)) appropriately.

**Definition 2.4** (task solvability). Let \( I^{\text{inp}} \) be an initial simplicial model and \( C \) and \( T \) be action models for a protocol and a task, respectively, as defined above. Then a task \( I^{\text{inp}}[T] \) is solvable by \( I^{\text{inp}}[C] \), if there exists a morphism \( \delta : I^{\text{inp}}[C] \rightarrow I^{\text{inp}}[T] \) such that \( \pi_{I^{\text{inp}}} = \pi_{I^{\text{inp}}} \circ \delta \).

This definition of task solvability by product update conforms to the topological definition based on carrier maps [13]. (The cartesian product of complexes is indeed a categorical product, whose universality is equivalent to the topological definition. [14])

In the rest of this paper, we are solely concerned with uniform communicative action models, as for product update models of protocols. A communicative action model \( (C, \sim^C, \text{pre}_C) \) is called uniform, if each of its action point (i.e., a facet of \( C \)) is uniquely denoted by \( X^r \), which is a pair of facet \( X \in F(I^{\text{inp}}) \) and an index \( r \in J \), where \( J \) is a fixed finite index set. The precondition is defined by \( \text{pre}_C(X^r) = \bigwedge_{u \in \Pi} \text{input}^u_s \) for each \( X = \{0, v_1, \ldots, u, v_n\} \).

The product update model \( I^{\text{inp}}[C] \) with a uniform communicative action \( C \) is intended to model a protocol that produces a uniform output for each input, independent to the initial input values: A pair \( X, Y \) and \( Y^r \) output generate the subcomplexes \( \{X^r \mid r \in J\} \) and \( \{Y^r \mid r \in J\} \), respectively, of different output values but of isomorphic combinatorial structure. In this way, every facet of the product update model \( I^{\text{inp}}[C] \) is given as a product \( X \times X' \). For brevity, we write \( X^r \) to denote a facet of \( I^{\text{inp}}[C] \), suppressing the duplicate occurrence of \( X \).

### 3 Logical Obstruction to Wait-free Binary Consensus Task

By the task solvability definition in DEL (Definition 2.4) and the knowledge gain theorem (Theorem 2.1), we obtain a logical method for refuting the solvability of distributed task.

**Theorem 3.1** ([13]). Let \( I \) be an input simplicial model, \( C \) be a communicative action model for a protocol, and \( T \) be a task action model. Then, the task \( I[T] \) is not solvable by the protocol \( I[C] \) if there exists a positive epistemic logic formula \( \varphi \) such that \( I[T] \models \varphi \) but \( I[C] \not\models \varphi \); That is, \( \varphi \) holds in every world of \( I[T] \) but \( \varphi \) is falsified for some world in \( I[C] \).

The positive epistemic formula \( \varphi \) is called logical obstruction to the solvability of the task \( I[T] \) for the protocol \( I[C] \).

As a demonstration of impossibility proof by means of logical obstruction, we examine the wait-free distributed binary consensus task (i.e., 1-set agreement task with exactly 2 possible initial input values). The well-known fact that the binary consensus task is not wait-free solvable in the read-write shared-memory model with two or more processes has been shown by varying methods such as valency argument [8, 15] and combinatorial topology [18, 1, 16].

Here we demonstrate that this well-known result is also reproduced in the logical framework of DEL. As Goubault et al. [13] have already shown, the logical obstruction to the binary consensus task can be argued based on a topological intuition: The input to the task has a connected combinatorial structure, but the output does not. Here we give a concrete logical obstruction formula in the formal language of epistemic logic and thereby demonstrate how a purely combinatorial argument works, without recourse to topological infrastructure.

We assume \( \Pi = \{0, 1, \ldots, n\} \) (\( n \geq 1 \)) and the set of binary inputs is \{0, 1\}. To show the unsolvability of binary consensus task in the wait-free read-write shared memory model, we consider the full-information protocol of the execution model, namely the immediate snapshot protocol [6, 5]. In the immediate snapshot protocol, the \( n + 1 \) processes are arbitrarily divided into an ordered set partition \( S_1 \mid S_2 \mid \cdots \mid S_m \), which is an ordered sequence of nonempty, pairwise disjoint subsets \( S_i \)'s such that \( \Pi = \bigcup_{i=1}^{m} S_i \). Each partition \( S_i \) represents a concurrency class and the snapshots are taken per each partition one after another in the order, where every process in the same concurrency class \( S_i \) simultaneously writes its local value to the shared memory and then collects the view, i.e., the set of the values that have been written so far. As a sequel, every process that belongs to a concurrency class \( S_k \) obtains the view \( \{(a, v_a) \mid a \in \bigcup_{i=1}^{k} S_i\} \), where \( v_a \) is the initial input value to process \( a \).
It is well-known that a single round execution of the immediate snapshot protocol corresponds to a subdivision of the input complex, a.k.a. the standard chromatic subdivision \[18, 21\]. (See Figure 2 for a standard chromatic subdivision and its correspondence to ordered set partitions.) We write \(\text{Ch} \langle C, \chi \rangle\) for the standard chromatic subdivision of a complex \(\langle C, \chi \rangle\). We designate a vertex of \(\text{Ch} \langle C, \chi \rangle\) by a pair \((a, \text{view}_a)\), where \(a\) is the color of the vertex and \(\text{view}_a\) is the snapshot view of the process \(a\).

A facet of \(\text{Ch} \langle C, \chi \rangle\) is a \(\{ (a, \text{view}_a) | a \in \Pi \} \), where the snapshot views \(\text{view}_a\) are derived from a single common ordered set partition.

The immediate snapshot protocol is modeled by the product update \(I\{0, 1\}[IS]\), where \(\langle F(IS), \sim, \text{pre}_{IS} \rangle\) is the uniform communicative action model such that:

- The action points are facets of the standard chromatic subdivision \(IS = \text{Ch} I\{0, 1\}\), where we denote each facet by \(X S_1 | S_2 | \cdots | S_m\) with \(X \in F(I\{0, 1\})\) and \(S_1 | S_2 | \cdots | S_m\) being an ordered set partition of \(\Pi\);
- The preconditions are given as in the definition of uniform action model in Section 2.4.

In what follows, for each facet \(X S_1 | S_2 | \cdots | S_m \in F(I\{0, 1\})\), we write \(\text{View}_{X S_1 | S_2 | \cdots | S_m}(a)\) for \(\{ v \in X | a \in S_k, \chi(v) \in \bigcup_{i=1}^k S_i \}\), which denotes the corresponding view of process \(a\).

The binary consensus task is modeled by the product update \(I\{0, 1\}[BC]\), where \(\langle F(BC), \sim, \text{pre}_{BC} \rangle\) is the action model such that:

- The underlying complex \(BC\) consists of solely two facets \(\hat{0} = \{(a, 0) | a \in \Pi \}\) and \(\hat{1} = \{(a, 1) | a \in \Pi \}\);
- The preconditions are defined by \(\text{pre}_{BC}(\hat{0}) = \bigvee_{a \in \Pi} \text{input}_a^0\) and \(\text{pre}_{BC}(\hat{1}) = \bigvee_{a \in \Pi} \text{input}_a^1\).

The action model \(BC\) defines the binary consensus task, whose output must be a unanimous agreement to one of input values, either 0 or 1.

**Theorem 3.2.** There is a logical obstruction formula to the binary consensus task \(I\{0, 1\}[BC]\) for the protocol \(I\{0, 1\}[IS]\). This refutes the solvability of the binary consensus task by the single round immediate snapshot protocol.

**Proof.** We will show the following formula works as logical obstruction.

\[
\Psi = \neg \left( \bigwedge_{a \in \Pi} \text{input}_a^0 \right) \lor C_{\Pi} \left( \bigvee_{a \in \Pi} \text{input}_a^0 \right)
\]

We are obliged to show that \(I\{0, 1\}[BC] \models \Psi\) but \(I\{0, 1\}[IS] \not\models \Psi\).

We first show \(I\{0, 1\}[IS] \not\models \Psi\). Let us write \(Y = \{(0, 0), (1, 0), \ldots, (n, 0)\}\), \(W = \{(0, 0), (1, 1), \ldots, (n, 1)\}\), and \(Z = \{(0, 1), (1, 1), \ldots, (n, 1)\}\), which are the facets of \(I\{0, 1\}\). It suffices to disprove
Without loss of generality, we may assume that the input values are taken from $\Pi = \{0, \ldots, n\}$. Obviously, $X_1 \not\models \bigwedge_{a \in \Pi} \text{input}_a^0$. Let us consider facets $X_2 = W^{0, \ldots, n}$, $X_3 = W^{0, \ldots, n}$, $X_4 = W^{1, \ldots, n}$, and $X_5 = Z^{1, \ldots, n}$, which are related in $\mathcal{I}^{(0,1)}[\mathcal{I}S]$ as follows. (See Figure 3 for the corresponding adjacency of facets, for $n = 3$.)

- $X_1 \sim_{\mathcal{I}^{(0,1)}[\mathcal{I}S]} X_2$, because $\text{View}_{X_1}(0) = \text{View}_{X_2}(0) = \{(0,0)\}$;
- $X_2 \sim_{\mathcal{I}^{(0,1)}[\mathcal{I}S]} X_3$, because $\text{View}_{X_2}(n) = \text{View}_{X_3}(n) = \{(0,0), (1,1), \ldots, (n,1)\}$;
- $X_3 \sim_{\mathcal{I}^{(0,1)}[\mathcal{I}S]} X_4$, because $\text{View}_{X_3}(0) = \text{View}_{X_4}(0) = \{(0,0), (1,1), \ldots, (n,1)\}$;
- $X_4 \sim_{\mathcal{I}^{(0,1)}[\mathcal{I}S]} X_5$, because $\text{View}_{X_4}(0) = \text{View}_{X_5}(0) = \{(1,1), \ldots, (n,1)\}$.

These entail that $X_1 \sim_{\mathcal{I}^{(0,1)}[\mathcal{I}S]} X_5$. Since $\mathcal{I}^{(0,1)}[\mathcal{I}S]$, $X_5 \not\models \bigvee_{a \in \Pi} \text{input}_a^0$, we conclude that $\mathcal{I}^{(0,1)}[\mathcal{I}S]$, $X_1 \not\models \Psi$.

We then prove $\mathcal{I}^{(0,1)}[\mathcal{B}C] \models \Psi$. Suppose, in contradiction, $\mathcal{I}^{(0,1)}[\mathcal{B}C]$, $X_0 \not\models \Psi$ for some facet $X_0$ in $\mathcal{I}^{(0,1)}[\mathcal{B}C]$. $X_0$ must be a product $Y_0 \times \hat{1}$ for some $Y_0 \in F(\mathcal{I}^{(0,1)})$, because pre$(0) = \bigvee_{a \in \Pi} \text{input}_a^0$ means $\mathcal{I}^{(0,1)}[\mathcal{B}C]$, $X_0 \models \text{input}_a^0$ for some $a \in \Pi$. Furthermore, it must be $Y_0 = \{(0,0), (1,0), \ldots, (n,0)\}$. Since $\mathcal{I}^{(0,1)}[\mathcal{B}C]$, $X_0 \not\models C_{\Pi}(\bigvee_{a \in \Pi} \text{input}_a^0)$, there must exist $X_1, \ldots, X_m \in \mathcal{I}^{(0,1)}[\mathcal{B}C]$ and $a_1, \ldots, a_m \in \Pi$ ($m > 0$) such that $X_0 \sim_{\mathcal{I}^{(0,1)}[\mathcal{B}C]} X_1 \cdots \sim_{\mathcal{I}^{(0,1)}[\mathcal{B}C]} X_m$ and $\mathcal{I}^{(0,1)}[\mathcal{B}C]$, $X_m \not\models \bigvee_{a \in \Pi} \text{input}_a^0$. Hence it must be $X_m = Y_m \times \hat{1}$ and $Y_m = \{(0,1), (1,1), \ldots, (n,1)\}$ and therefore $X_k = Y_k \times \hat{1}$ and $X_{k+1} = Y_{k+1} \times \hat{1}$ for some $k$. However, $X_k \sim_{\mathcal{I}^{(0,1)}[\mathcal{B}C]} X_{k+1}$ implies that the $X_k$ and $X_{k+1}$ share a common vertex $(v, u) \in X_k \cap X_{k+1}$, which leads to a contradiction $0 = u = 1$. \hfill \square

4 Impossibility of General Set Agreement Tasks

This section applies the logical method introduced in previous sections to set agreement tasks. A $k$-set agreement is a distributed computation such that, when each process is initially given an arbitrary input value, processes decide at most $k$ different values taken from the initial inputs. As before, we assume $n + 1$ processes, whose unique process ids are given by the set $\Pi = \{0, 1, \ldots, n\}$ of colors. Without loss of generality, we may assume that the input values are taken from $\Pi = \{0, 1, \ldots, n\}$, renaming the input values appropriately.

The initial simplicial model is then given by $\mathcal{I}^\Pi$, as defined in Section 2.4. Throughout this section, for brevity, we write $\mathcal{I}$ for $\mathcal{I}^\Pi$. The simplicial action model for $k$-set agreement is given by $(F(S\mathcal{A}_k), \sim^S\mathcal{A}_k, \text{pre})$ such that:

- the action points are the facets of a complex $S\mathcal{A}_k$ that are given by
  
  \[ F(S\mathcal{A}_k) = \{ (d_0, \ldots, d_n) \mid d_0, \ldots, d_n \in \Pi, |\{d_0, \ldots, d_n\}| \leq k \}, \]

  where we write $\langle d_0, \ldots, d_n \rangle$ to denote a facet $\{(0, d_0), \ldots, (n, d_n)\}$, whose vertexes are colored by $\chi(a, d) = a$;

- the precondition is given by $\text{pre}(\langle d_0, \ldots, d_n \rangle) = \bigwedge_{a \in \Pi} \left( \bigvee_{a' \in \Pi} \text{input}_a^{d_n} \right)$.
We study the unsolvability of set agreement tasks under adversary schedulers, generalizing the basic with the superset-closed adversaries and a superset-closed adversary is simply called an adversary.

4.1 Logical obstruction to wait-free set agreement

In [24], Nishida presented inductively defined epistemic formulas as logical obstruction to the wait-free k-set agreement tasks. The primary insight in his impossibility proof is that each particular execution of a k-set agreement task is associated with a permutation subset, whose formal definition is given below.

Definition 4.1 (Permutation subset). We say a function \( g \) a permutation of a set \( S \), if \( g \) is a bijection over \( S \). We call a nonempty subset \( A \) of a finite set \( U \) is a permutation subset for a function \( f: U \rightarrow U \), if \( f(A) = A \), i.e., \( A \) is a fixed point of \( f \).

Lemma 4.1. Let \( f: U \rightarrow U \) be a function over a nonempty finite set \( U \). Then there exists a permutation subset \( A \) for \( f \).

Proof. Trivially, \( f \) is a monotonic function over \( 2^U \), that is, \( A \subseteq A' \) implies \( f(A) \subseteq f(A') \). By the monotonicity, it follows that \( f^i(U) \supseteq f^{i+1}(U) \neq \emptyset \) from \( U \supseteq f(U) \) by induction. Since \( U \) is finite, there exists \( m \) such that \( f^i(U) = f^{i+1}(U) \neq \emptyset \) for every \( i \geq m \). This implies that the set \( A = \bigcap_{i=0}^{\infty} f^i(U) \) is a nonempty fixed point, i.e. \( f(A) = A \). \( A \) is indeed the greatest fixed point of \( f \).

As an immediate consequence of this lemma, \( \text{Output}_X: \Pi \rightarrow \Pi \) has a permutation subset for any facet \( X \) of \( \mathcal{T}[\mathcal{S}_A_k] \).

Below we give Nishida’s logical obstruction \( \Phi \) to the wait-free k-set agreement task:

\[
\Phi \triangleq \bigvee_{a=0}^{n} \neg \text{Input}^a_0 \lor \bigvee_{m=1}^{k} \bigvee_{A \subseteq \Pi} \bigvee_{|A|=m} \Psi^{(m)}_A,
\]

where \( \Psi^{(m)}_A \)'s are a class of formulas such that \( A \subseteq \Pi \) and \( |A| = m \) that is defined inductively as follows:

\[
\Psi^{(m)}_A \triangleq \mathcal{D}_A \left( \bigvee_{a \in \Pi \setminus A} \neg \text{Input}^a_0 \lor \bigvee_{a \in \Pi \setminus A} K_a \left( \bigvee_{j \in A} \bigvee_{a' \in \Pi} \text{Input}^j_{a'} \right) \lor \bigvee_{i=m+1}^{n} \bigvee_{B \subseteq \Pi \setminus A} \left| B \right| = i-m \right).
\]

We refrain from repeating his original proof herein, because in the rest of this section we provide a generalized form of this logical obstruction for superset-closed adversaries. By the appropriate level of generalization, our presentation is much clearer and more succinct than the original one.

4.2 Obstruction to adversarial set agreement task

We study the unsolvability of set agreement tasks under adversary schedulers, generalizing the basic wait-free scheduler to the ones that allow nonuniform failures.

We model such an adversary scheduler by a set \( \mathcal{A} \), called an adversary [10], of possible subsets of correct processes, that is, \( \mathcal{A} \) is a nonempty subset of \( 2^n \) such that \( \emptyset \not\in \mathcal{A} \). An adversary \( \mathcal{A} \) is called superset-closed, if \( P \in \mathcal{A} \) and \( P \subseteq P' \subseteq \Pi \) implies \( P' \in \mathcal{A} \). In this paper, we are solely concerned with the superset-closed adversaries and a superset-closed adversary is simply called an adversary.

Nishida called such a fixed point a family of cycles [24], because of the well-known fact that every permutation is decomposed into one or more cycles.
An adversary \( \mathcal{A} \) has two different characterizations by survivor sets and core sets [19]. An adversary \( \mathcal{A} \) can be specified by survivor sets, where a survivor set \( S \) is a minimal set contained in \( \mathcal{A} \), that is, \( S \supseteq P \) for none of \( P \in \mathcal{A} \); Dually, an adversary \( \mathcal{A} \) can be specified by core sets, where a core set \( C \) is a minimal subset of \( \Pi \) satisfying \( C \cap P \neq \emptyset \) for every \( P \in \mathcal{A} \). Remark that the wait-free scheduler is just an instance of (uniform) adversary that is characterized by survivor sets \( \{ \{ a \} \mid a \in \Pi \} \).

We write \( \text{csize}(\mathcal{A}) \) to denote the minimum core size of an adversary \( \mathcal{A} \), i.e., the minimum cardinality of the core sets defined by \( \text{csize}(\mathcal{A}) = \min \{ |C| \mid C \text{ is a core set of } \mathcal{A} \} \).

Following [17], we model a single round execution of distributed processes against a given adversary \( \mathcal{A} \) by a protocol specified by the round operator \( \mathcal{R}_\mathcal{A} \). For each facet \( X \in \mathcal{I} \), which corresponds to an initial configuration of inputs to processes, the round operator \( \mathcal{R}_\mathcal{A} \) associates an output facet \( \{ (a, \text{view}_a) \mid a \in \Pi \} \), where each \( \text{view}_a \) is a subset of \( X \) assigned to the \( a \)-colored process and the set of views satisfy the following properties:

**(survival)** For each \( a \in \Pi \), \( \text{view}_a \) is a subset of \( X \) that subsumes a survivor set, i.e., \( \chi(\text{view}_a) \in \mathcal{A} \);

**(self-inclusion)** For each \( a \in \Pi \), \( a \in \chi(\text{view}_a) \);

**(containment)** Either \( \text{view}_a \subseteq \text{view}_{a'} \) or \( \text{view}_a \supseteq \text{view}_{a'} \) holds, for every \( a, a' \in \Pi \).

Note that the wait-free immediate snapshot is further required to satisfy the immediacy condition: For every \( a, a' \in \Pi \), \( \text{view}_{a'} \subseteq \text{view}_a \) holds whenever \( a' \in \chi(\text{view}_a) \). Missing the immediacy condition, the underlying simplicial structure of the round operator is more involved in general. Even under the wait-free adversary, the round operator results in a non-manifold, as studied in [3]. To cope with this extra complexity, our impossibility proof refines Nishida’s original one. Specifically we need to show an additional property, as stated in lemma 4.2.

We define the round operator as the product update model \( \mathcal{I}[\mathcal{R}_\mathcal{A}] \), where, in abuse of notation, \( \mathcal{R}_\mathcal{A} \) denotes the uniform communicative action model \( \langle \mathcal{F}(\mathcal{R}_\mathcal{A}), \sim^{\mathcal{R}_\mathcal{A}}, \text{pre} \rangle \) such that:

- Each facet in \( \mathcal{F}(\mathcal{R}_\mathcal{A}) \) is denoted by \( X^{\langle S_0, \ldots, S_n \rangle} \), where \( X \in \mathcal{F}(\mathcal{I}) \) and the vector \( \langle S_0, \ldots, S_n \rangle \) of subsets of \( \Pi \) satisfies:
  - for each \( a \in \Pi \), \( S_a \) subsumes a survivor set, i.e., \( S_a \in \mathcal{A} \),
  - for each \( a \in \Pi \), \( a \in S_a \), and
  - either \( S_a \subseteq S_{a'} \) or \( S_a \supseteq S_{a'} \) holds, for every \( a, a' \in \Pi \);
- The preconditions are given as in the definition of uniform action models in Section 2.4.

In the following, we may occasionally write \( \vec{S} \) to denote the vector \( \langle S_0, \ldots, S_n \rangle \) for brevity.

For a facet \( X^{\langle S_0, \ldots, S_n \rangle} \), let us define a pair of functions \( \text{Input}_{X^{\langle S_0, \ldots, S_n \rangle}} \) and \( \text{View}_{X^{\langle S_0, \ldots, S_n \rangle}} \) by \( \text{Input}_{X^{\langle S_0, \ldots, S_n \rangle}}(a) = v \) if and only if \( (a, v) \in X \) and also by \( \text{View}_{X^{\langle S_0, \ldots, S_n \rangle}}(a) = S_a \), respectively. Then, \( X^{\vec{S}} \sim^{\mathcal{I}[\mathcal{R}_\mathcal{A}]} Y^{\vec{S}} \) holds iff \( \text{View}_{X^{\vec{S}}}(a) = \text{View}_{Y^{\vec{S}}}(a) \) and \( \text{Input}_{X^{\vec{S}}}(a') = \text{Input}_{Y^{\vec{S}}}(a') \) for every \( a' \in \text{View}_{X^{\vec{S}}}(a) \). In particular, \( X^{\vec{S}} \sim^{\mathcal{I}[\mathcal{R}_\mathcal{A}]} X^{\vec{S}} \) iff \( \text{View}_{X^{\vec{S}}}(a) = \text{View}_{X^{\vec{S}}}(a) \).

Assuming an adversary \( \mathcal{A} \) is fixed, we define a class of positive epistemic formulas indexed by \( A \in 2^\Pi \):

\[
\Psi_A \triangleq \begin{cases} 
\text{false} & \text{if no survivor set is subsumed by } \Pi \setminus A; \\
D_A \psi_A & \text{otherwise,}
\end{cases}
\]

where \( \psi_A \)'s are formulas given below

\[
\psi_A \triangleq \bigvee_{a \in \Pi \setminus A} \neg \text{input}_a^A \lor \bigvee_{a \in \Pi \setminus A} K_a \left( \bigvee_{j \in A} \bigvee_{a' \in \Pi} \text{input}_{a'}^j \right) \lor \bigvee_{B \supseteq A} \Psi_B,
\]

defined by induction on \( A \), ordered by inverse set inclusion (with \( \psi_{\Pi} \) being the base case of induction).

In what follows, we claim that the following formula is the logical obstruction against the adversary \( \mathcal{A} \):

\[
\Phi \triangleq \bigvee_{a \in \Pi} \neg \text{input}_a^A \lor \bigvee_{A \in [\Pi]^{<c}} \Psi_A.
\]

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where \( c = \text{cs}(A) \) and \([\Pi]^{<c} = \{ A \subseteq \Pi \mid 0 < |A| < c \} \).

Let us write \( \minView(X^S) \) to denote the minimum of the views of a facet \( X^S \in F(\mathcal{I}[\mathcal{R}_A]) \), i.e.,

\[
\minView(X^S) = \bigcap_{a \in \Pi} \text{View}_{X^S}(a).
\]

**Lemma 4.2.** Suppose a facet \( X^S \in F(\mathcal{I}[\mathcal{R}_A]) \) satisfies

(i) \( I[\mathcal{R}_A], X^S \models \bigwedge_{a \in \Pi} \text{input}_a^a \), and

(ii) for any \( a \in \minView(X^S) \), \( \text{View}_{X^S}(a) = \minView(X^S) \).

Then we have \( I[\mathcal{R}_A], X^S \not\models \psi_{\Pi \setminus \minView(X^S)} \).

**Proof.** Let us write \( A \) for the minimum view \( \minView(X^S) \). Note that \( A \) subsumes a survivor set.

We proceed the proof by induction on \( A \). Let us consider the case that \( A \) is exactly a survivor set. For any \( B \) that is a proper superset of \( \Pi \setminus A \), \( \Pi \setminus B \) is a proper subset of \( A \) and hence contains no survivor set. Thus by the assumption (i), \( \psi_{\Pi \setminus A} \) is logically equivalent to \( X \equiv \bigvee_{a \in A} K_a \left( \bigvee_{j \in \Pi \setminus A} \text{input}_a^j \right) \).

We may suppose \( A \neq \Pi \), because \( X \) does not hold otherwise. To see \( I[\mathcal{R}_A], X^S \not\models \psi_{\Pi} \), it suffices to show that there is a facet \( Y^S \) satisfying \( Y^S \sim_a X^S \) for every \( a \in A \) and also \( \text{Input}_{Y^S}(a') \in A \) for every \( a' \in \Pi \). Take \( k \in A \) arbitrarily. Then such a facet is instantiated by \( Y^{(S_1, \ldots, S_k)} \) where \( Y = \{(a, a) \mid a \in A\} \cup \{(a, k) \mid a \in \Pi \setminus A\} \), \( S_0 = S = A \) if \( a \in A \), and \( S_0 = \Pi \) if \( a \in \Pi \setminus A \).

Let us then consider the case \( A \) is a proper superset of a survivor set. Similarly as above, we can prove \( I[\mathcal{R}_A], X^S \not\models \psi_{\Pi \setminus B} \) for every proper subset \( B \) of \( A \). For such a proper subset \( B \), we define \( S^S \) by \( S^S_a = B \) if \( a \in B \) and otherwise \( S^S_a = S_0 \). Then, the properties (i) and (ii) are held by the facet \( X^S \), and hence by induction hypothesis we have \( I[\mathcal{R}_A], X^S \not\models \psi_{\Pi \setminus B} \). Since \( X^S \sim_{\Pi \setminus B} X^S \), we conclude that \( I[\mathcal{R}_A], X^S \not\models \psi_{\Pi \setminus B} \).

**Proposition 4.3.** \( I[\mathcal{R}_A] \not\models \Phi \).

**Proof.** Let \( X^S \) be the facet of \( I[\mathcal{R}_A] \) such that \( X = \{(a, a) \mid a \in \Pi\} \) and \( S_a = \Pi \) for every \( a \in \Pi \). We show \( I[\mathcal{R}_A], X^S \not\models \Phi \). Clearly \( I[\mathcal{R}_A], X^S \not\models \bigvee_{a \in \Pi} \neg \text{input}_a^a \) and thus it suffices to show \( I[\mathcal{R}_A], X^S \not\models \bigwedge_{A \in [\Pi]^{<c}} \Phi \). Take any \( A \in [\Pi]^{<c} \) arbitrarily. Note that \( \Pi \setminus A \) subsumes a survivor set. Let us define \( S^S \) by \( S^S_a = \Pi \setminus A \) if \( a \in \Pi \setminus A \) and \( S^S_a = S_0 \) otherwise. Then \( I[\mathcal{R}_A], X^S \not\models \psi_A \) follows from lemma 4.2. Since \( X^S \sim_{\Pi} X^S \), we have \( I[\mathcal{R}_A], X^S \not\models \psi_A \). As we have taken \( A \in [\Pi]^{<c} \) arbitrarily, this concludes \( I[\mathcal{R}_A], X^S \not\models \bigwedge_{A \in [\Pi]^{<c}} \Phi \).

**Lemma 4.4.** Suppose \( X \times \vec{d} \) is a facet of \( I[\mathcal{S}_A] \). Then, for every \( a \in A \) and \( A \subseteq A \), \( \text{Output}_{X \times \vec{d}}(a) \in A \) implies \( I[\mathcal{S}_A], X \times \vec{d} \models K_a \left( \bigvee_{j \in A} \text{input}_a^j \right) \).

**Proof.** Let \( Y \times \vec{d} \) be any facet such that \( X \times \vec{d} \sim_a I[\mathcal{S}_A] Y \times \vec{d} \). Then, \( \text{Output}_{Y \times \vec{d}}(a) = \text{Output}_{X \times \vec{d}}(a) \in A \). Since \( \text{Input}_{Y \times \vec{d}}(\Pi) \supseteq \text{Output}_{Y \times \vec{d}}(\Pi) \), it follows that \( \text{Input}_{Y \times \vec{d}}(a') \in A \) for some \( a' \in \Pi \). This implies \( I[\mathcal{S}_A], Y \times \vec{d} \models \bigvee_{j \in A} \text{input}_a^j \). Hence we are done.

**Proposition 4.5.** Let \( A \) be any adversary. Then, \( I[\mathcal{S}_A] \models \Phi \) for every \( k < \text{cs}(A) \).

**Proof.** Let \( k < \text{cs}(A) \) and \( X \times \vec{d} \) be any facet of \( I[\mathcal{S}_A] \). We will show \( I[\mathcal{S}_A], X \times \vec{d} \models \Phi \). We may suppose \( I[\mathcal{S}_A], X \times \vec{d} \not\models \bigwedge_{a \in A} \neg \text{input}_a^a \), because \( I[\mathcal{S}_A], X \times \vec{d} \models \Phi \) immediately holds otherwise. By lemma 4.1, the function \( \text{Output}_{X \times \vec{d}} \) has a permutation subset \( A \). Since \(|A| \leq \max \{|\text{Output}_{X \times \vec{d}}(Z)| \mid Z \subseteq A \} < k < \text{cs}(A) \), it suffices to show that \( I[\mathcal{S}_A], X \times \vec{d} \models \psi_A \) for this permutation subset \( A \).

We proceed by induction on the size of \( A \) \( \setminus A \). First, consider the base case \(|A| = k \). Let us show that \( I[\mathcal{S}_A], Y \times \vec{d} \models \psi_A \) holds for any facet \( Y \times \vec{d} \) such that \( Y \times \vec{d} \sim_{\Pi \setminus A} X \times \vec{d} \).

Since \( A = \text{Output}_{X \times \vec{d}}(A) \subseteq \text{Output}_{Y \times \vec{d}}(A) \subseteq \text{Output}_{Y \times \vec{d}}(\Pi) \) and \(|\text{Output}_{Y \times \vec{d}}(\Pi)| \leq k \), we have \( \text{Output}_{Y \times \vec{d}}(A) \subseteq A \). Hence by lemma 4.4, \( I[\mathcal{S}_A], Y \times \vec{d} \models K_a \left( \bigvee_{j \in A} \text{input}_a^j \right) \) holds for every \( a \in A \). This entails \( I[\mathcal{S}_A], Y \times \vec{d} \models \psi_A \).
Next, consider the case $|A| < k$. Again, let us show that $I[S_{A_k}], Y \times \bar{d} \models \psi_A$ holds for any facet $Y \times \bar{d}$ such that $Y \times \bar{d} \sim_{D_{A_k}} X \times \bar{d}$. Since $Output_{X \times \bar{d}}$ has $A$ as a permutation subset, so does $Output_{Y \times \bar{d}}$. We will show that $I[S_{A_k}], Y \times \bar{d} \models \bigvee_{B \supseteq A} \Psi_B$ holds, assuming $I[S_{A_k}], Y \times \bar{d} \models \bigvee_{a \in \Pi \setminus A} \text{input}^a_a \lor \bigvee_{a \in \Pi \setminus A} K_a \left( \bigvee_{j \in \Pi} \bigvee_{a' \in \Pi} \text{input}^a_{a'} \right)$. It follows from $I[S_{A_k}], Y \times \bar{d} \models \bigvee_{a \in \Pi \setminus A} K_a \left( \bigvee_{j \in \Pi} \bigvee_{a' \in \Pi} \text{input}^a_{a'} \right)$ that $Output_{Y \times \bar{d}}(\Pi \setminus A) \subseteq \Pi \setminus A$ by lemma 4.4. Thus by lemma 4.1, $Output_{Y \times \bar{d}}$ has a permutation subset $A'$, that is, $Output_{Y \times \bar{d}}(A') = A' \subseteq \Pi \setminus A$. Therefore $A \cup A'$ is a proper superset of $A$ and is also a permutation subset for $Output_{Y \times \bar{d}}$. By induction hypothesis we have $I[S_{A_k}], Y \times \bar{d} \models \Psi_{A \cup A'}$, which entails $I[S_{A_k}], Y \times \bar{d} \models \bigvee_{B \supseteq A} \Psi_B$. \hfill $\Box$

By proposition 4.3 and 4.5, $\Phi$ is a logical obstruction and hence the following impossibility result follows.

**Theorem 4.6.** Let $A$ be an adversary. If $k < csize(A)$, the $k$-set agreement task $I[S_{A_k}]$ is not solvable the protocol $I[R_{A_k}]$, a single round of the round operator.

5 Conclusion and Future Work

We have applied the logical method developed by Goubault, Ledent, and Rajsbaum [13] to show the impossibility of the set agreement tasks for superset-closed adversaries, by giving concrete logical obstruction formulas, which are generalization of Nishida’s logical obstruction for the wait-free model [24]. The method based on logical obstruction allows an elementary inductive proof, without recourse to sophisticated topological tools. The instances of logical obstruction exemplify that logical method would serve as a feasible alternative to topological method.

There are several topics to pursue that merit further investigation. First, it is quite interesting what varieties of impossibility can be proven by devising concrete logical obstruction formulas. To date, the topological method has been extensively applied to show impossibility results and still keep expanding its application area, e.g., network computing [7, 9]. Finding concrete logical obstruction formulas to these particular instances would be an interesting topic of its own right.

We would also expect that logical obstruction could give a deeper logical understanding on the nature of unsolvability of distributed tasks. As observed in [12], the original logical framework of [13], whose atomic propositions are allowed to mention input values only, cannot refute the solvability of the equality negation task [23], while an extended framework that allows atomic propositions to mention output decision values has logical obstruction. This seems to suggest that solvability of distributed tasks could be classified in terms of expressibility of epistemic logic.

As we have mentioned in Introduction, this paper solely concerns single round protocols, leaving multiple round protocols for future investigation. In principle we are able to discuss multiple round protocols in the epistemic logic setting, but we would need to work on a different epistemic model for each incremental round step. It would be a challenging topic to give a remedy for this by providing, say, a series of obstruction formulas indexed by the number of round steps.

Acknowledgment

The second author would like to thank Yutaro Nishida, who left academia just after writing up his Master’s thesis, for fruitful discussion on the topic. The second author is supported by JSPS KAKENHI Grant Number 20K11678.

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