The variational structure of the space of holonomic measures

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Abstract

We examine a setting appropriate for the analysis of many variational problems. Roughly speaking, we work on the closure of the space of measures induced by immersions of submanifolds. We characterize the space of variations for these objects. We use this characterization to deduce some results for the critical points of the action of very general Lagrangians.

1 Introduction

In this paper we consider the space of holonomic measures, which are roughly all measures that can be approximated by closed, immersed submanifolds. These are defined carefully in Section 2.

We study the ways in which these measures can be deformed, thus characterizing the velocity vectors of all curves in the space of holonomic measures that are differentiable in a certain sense. We are thus able to give a good description of what would be the tangent space to the space of holonomic measures. We do this in Section 3.

This study is fruitful, as is shown by an initial set of applications presented in Section 4. Among other things, we are able to show that the conditions we obtain for criticality are effectively more general than the classical Euler-Lagrange equations.

Related literature. Geometric measure theory and variational analysis are vast subjects, so a discussion about how this research fits in those contexts is in place. However, since it seems impossible to give an exhaustive discussion, we choose to instead give just a brief one and hence minimize the number of mistakes we make in the process due to our lack of expertise in all these fields. Also, we will not define all the objects involved, but rather we will just mention them in the hope that readers familiar with these concepts
will find the information they are looking for, while readers not familiar with them will be happy to ignore the discussion.

Throughout this paper, $d \geq 1$ will denote the dimension of the ambient manifold $M$, while $1 \leq n \leq d$ will denote the dimension of the holonomic measures. This roughly means that we are considering submanifolds of dimension $n$.

Holonomic measures appeared in the $n = 1$ case in Mather’s [14] version of Mather-Aubry theory for minimizers of the action of Lagrangians on the torus. The theory of holonomic measures was extended by others; for example by Mañé [5,13], Bangert [3], Bernard [4]. A certain case of codimension one of Mather-Aubry theory was considered by Moser [16–18].

In the more general context we treat here, in which $n$ can be arbitrary, a similar theory should exist for a large class of Lagrangians. Under rather mild conditions in the Lagrangian (such as convexity, superlinearity, tightness) minimizers exist in all holonomy classes with coefficients in the real numbers $\mathbb{R}$. However, analogues of Mather’s $\alpha$ and $\beta$ functions are probably only defined for a very restricted set of Lagrangians.

Holonomic measures induce superpositions of currents (cf. [10,15]) on a manifold $M$ in an obvious way. However, they carry more information than currents because they take into account the parameterization of the minimizers, and hence allow for the study of anisotropic Lagrangians.

Holonomic measures also induce varifolds (cf. [1,2,21]). Again, they carry more information because they record not only the tangent planes, but also the velocity vectors of a ‘parameterization.’

With holonomic measures the issue of rectifiability is not a concern since rectifiability is built into them. Whether or not one can find their volume (or the action of a Lagrangian) depends on the question of whether this function is integrable with respect to them. They have empty perimeter, so finiteness of perimeter is also not a concern.

Holonomic measures are suitable for the treatment of many problems that could be approached parametrically using functions for example in Sobolev or Lipschitz spaces (cf. [7,8,11,12]).

Superpositions of Young measures (cf. [1,23]) are a special case of holonomic measures.

In Section 4.2 we deduce a sort of general Hamilton-Jacobi equation, a case of which has been studied to great depth (see for example [6,9]).

The definition of differentiability of families of measures (i.e., of variations) that we use is only one possibility of many; see for example [22] for an exploration of other possibilities.
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2 Preliminaries

2.1 Setting

Phase space. Let $M$ be a compact, oriented $C^\infty$ manifold of dimension $d \geq 1$, possibly with boundary $\partial M$. Denote by $TM$ its tangent bundle and, for $n \geq 1$, denote by $T^n M$ the direct sum bundle

$$T^n M = \underbrace{TM \oplus \cdots \oplus TM}_n$$

of $n$ copies of $TM$. The dimension of $T^n M$ is $d(n + 1)$. An element in $T^n M$ can be denoted $(x, v_1, v_2, \ldots, v_n)$, where $x$ is a point in $M$ and $v_1, v_2, \ldots, v_n \in T_x M$ are vectors tangent to $x$. When taking local coordinates, we will write

$$x = (x_1, x_2, \ldots, x_d) \quad \text{and} \quad v_i = (v_{i1}, v_{i2}, \ldots, v_{id}).$$

Sometimes for brevity we will write $(x, v)$ instead of $(x, v_1, v_2, \ldots, v_n)$.

The projection $\pi : T^n M \to M$ is given by $\pi(x, v_1, \ldots, v_n) = x$. We denote by $\Omega^n(M)$ the space of smooth differential $n$-forms on $M$. We will often consider these forms as smooth functions on $T^n M$.

Throughout, when referring to functions on these objects, we will use the term smooth to mean $C^\infty$. We will denote by $C^\infty(X, Y)$ the space of all smooth functions $X \to Y$. If $Y$ is the real line $\mathbb{R}$, we will sometimes omit it in our notation. We will denote by $C^\infty_c(X)$ the set of all real-valued, compactly-supported, smooth functions on the set $X$.

Riemannian structure. We fix, once and for all, a Riemannian metric $g \in C^\infty(T^2 M)$ on $M$ and its corresponding Levi-Civita connection $\nabla$. We
denote the operation of covariant differentiation in the direction of a vector field $F$ by $\nabla_F$.

We will denote $|v| = \sqrt{g(v, v)}$ for $v \in T_x M$, and we extend this norm to $T^n_x M$ by letting

$$|(v_1, v_2, \ldots, v_n)| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}.$$  

**Forms.** We will denote by $\Omega^k(M)$ the space of smooth differential $k$-forms on $M$. On this space we define a norm $\| \cdot \|$ by letting, for $\omega \in \Omega^k(M)$,

$$\|\omega\| = \sup \{ \omega_x(v_1, \ldots, v_k) : (x, v_1, \ldots, v_k) \in T^n_M, |v_i| \leq 1 \}.$$  

2.2 Definition of holonomic measures and their topology

**Subpower functions.** We let $\mathcal{P}_n$ be the space of *subpower functions*, that is, the space of real-valued continuous functions $f \in C^0(T^n M)$ such that

$$\sup_{(x,v) \in T^n M} \frac{|f(x,v)|}{1 + |v|^n} < +\infty.$$  

Note that all differential $n$-forms on $M$ belong to $\mathcal{P}_n$ when regarded as functions on $T^n M$. We endow $\mathcal{P}_n$ with the supremum norm and its induced topology.

**Mild measures.** A signed measure $\mu$ on $T^n M$ is *mild* if

$$\int_{T^n M} 1 + |(v_1, \ldots, v_n)|^n d|\mu| < +\infty,$$

where $|\mu| = \mu^+ + \mu^-$ is the absolute value of the measure with Hahn decomposition $\mu = \mu^+ - \mu^-$. For positive measures $\mu^+$ and $\mu^-$. We denote the space of mild measures by $\mathcal{M}_n$. We define the *mass* $M(\mu)$ of $\mu \in \mathcal{M}_n$ to be

$$M(\mu) = \int_{T^n M} \left[ \sup_{\omega \in \Omega^n(M), \|\omega\| \leq 1} \omega_x(v_1, \ldots, v_n) \right] d|\mu|(x,v_1,\ldots,v_n)$$

$$= \int_{T^n M} \text{vol}_n(v_1, v_2, \ldots, v_n) d|\mu|(x,v_1,\ldots,v_n).$$  

This is always a nonnegative number.

The space $\mathcal{M}_n$ is naturally embedded in the dual space $\mathcal{P}_n^*$ and we endow it with the topology induced by the weak* topology on $\mathcal{P}_n^*$. Although the topology on $\mathcal{P}_n^*$ is not metrizable, the topology on $\mathcal{M}_n$ is. We can give a
metric in $\mathcal{M}_n$ by picking a sequence of functions $\{f_i\}_{i \in \mathbb{N}} \subset C_c^\infty(T^n M)$ that are dense in $\mathcal{P}_n$, and then letting

$$\text{dist}_{\mathcal{M}_n}(\mu_1, \mu_2) = M(\mu_1 - \mu_2) + \sum_{k=1}^\infty \frac{1}{2^k \sup |f_k|} \left| \int f_k d\mu_1 - \int f_k d\mu_2 \right|. \quad (1)$$

**Holonomic measures.** A mild measure $\mu \in \mathcal{M}_n$ is holonomic if it is a probability (that is, a positive measure such that $\mu(T^n M) = 1$), and if for every differential $(n-1)$-form $\omega \in \Omega^{n-1}(M)$,

$$\int_{T^n M} d\omega d\mu = 0. \quad (2)$$

The space $\mathcal{H}$ of holonomic measures is convex. By the Banach-Alaoglu theorem, it is also compact, since it is a closed subset of the unit ball of $\mathcal{P}_n^*$ (it is cut out by the closed condition (2)).

The motivation for this definition is given by Proposition 1 below.

**Cellular complexes.** An $n$-dimensional cell (or $n$-cell) $\gamma$ is a smooth map

$$\gamma : D \subseteq \mathbb{R}^n \to M,$$

where $D$ is a subset of $\mathbb{R}^n$ homeomorphic to a closed ball, together with a choice of coordinates $t = (t_1, t_2, \ldots, t_n)$ on $D$. A chain of $n$-cells is a formal linear combination of the form

$$a_1 \gamma_1 + a_2 \gamma_2 + \cdots + a_k \gamma_k$$

for real numbers $a_1, a_2, \ldots, a_k$ and $n$-cells $\gamma_1, \gamma_2, \ldots, \gamma_k$. We will say that a chain is positive if $a_i > 0$.

Let $\gamma : D \subseteq \mathbb{R}^n \to M$ be an $n$-cell. Denote by $d\gamma$ the differential map associating, to each element in $D$, an element in $T^n M$. Explicitly, if we have coordinates $t = (t_1, t_2, \ldots, t_n)$ on $D$, then

$$d\gamma(t) = \left( \gamma(t), \frac{\partial \gamma}{\partial t_1}(t), \frac{\partial \gamma}{\partial t_2}(t), \cdots, \frac{\partial \gamma}{\partial t_n}(t) \right).$$

This map does depend on our choice of coordinates $t$.

To an $n$-cell $\gamma$, we associate a measure $\mu_\gamma$ on $T^n M$ defined by

$$\int_{T^n M} f d\mu_\gamma = \int_D f(d\gamma(t)) dt,$$
where \( dt = dt_1 \wedge \cdots \wedge dt_n \). Similarly, to a chain of \( n \)-cells \( \alpha = \sum_{i=1}^{k} a_i \gamma_i \), we associate the measure \( \mu_\alpha \) given by

\[
\mu_\alpha = \sum_{i=1}^{k} a_i \mu_{\gamma_i}.
\]

The measure \( \mu_\alpha \) is an element of \( \mathcal{M}_n \). We will say that the chain \( \alpha \) is a cycle if for all forms \( \omega \in \Omega^{n-1}(M) \),

\[
\int_{T^nM} d\omega d\mu_\alpha = 0.
\]

In other words, the chain \( \alpha \) is a cycle if \( \mu_\alpha \) is holonomic.

**Proposition 1** ([20]). Assume that \( 1 \leq n \leq d \). Let \( \mu \in \mathcal{M}_n \) be a probability measure on \( T^nM \). Then the following conditions are equivalent:

1. The measure \( \mu \) is holonomic.
2. There exists a sequence \( \{ \alpha_k \}_{k \in \mathbb{N}} \) of cycles such that \( \mu_{\alpha_k} \to \mu \) as \( k \to \infty \) in the topology induced by the distance \( \| \cdot \| \).

In other words, the space of holonomic measures is precisely the closure of the space of measures \( \mu_\alpha \) induced by cycles \( \alpha \).

## 3 The tangent space

### 3.1 Distributions

A *distribution* on the open set \( U \subseteq \mathbb{R}^m \) is a functional \( \eta: C_0^\infty(U) \to \mathbb{R} \) such that for each compact set \( K \subset U \) there are some constants \( N > 0 \) and \( C > 0 \) (depending only on \( K \) and \( \eta \)) such that

\[
|\langle \eta, f \rangle| \leq C \sum_{|I| \leq N} \sup_{p \in U} |\partial^I f(p)|
\]

for all \( f \in C_0^\infty(U) \). Here, the sum is taken over all multiindices \( I \) with \( n \) nonnegative entries adding up to at most \( N \), and \( \partial^I \) denotes the iterated partial derivatives in the corresponding directions in \( \mathbb{R}^n \).

Let \( P \) be a \( C^\infty \) manifold. For a chart \( \varepsilon: U \to W \) from the open set \( U \subset P \) to the open set \( W \subset \mathbb{R}^n \), the *pushforward* \( \varepsilon_* \eta \) is defined by

\[
\langle \varepsilon_* \eta, f \rangle = \langle \eta, f \circ \varepsilon^{-1} \rangle
\]
for \( f \in C^\infty_c(W) \).

A distribution on the manifold \( P \) is a functional \( \eta: C^\infty_c(P) \to \mathbb{R} \) such that for each chart \( \varepsilon \) as above, \( \varepsilon_* \eta \) is a distribution on \( W \). We will denote by \( \mathcal{D}'(P) \) the space of distributions on \( T^nM \). The topology on \( \mathcal{D}'(P) \) is induced by the seminorms

\[
\eta \mapsto |\langle \eta, f \rangle|
\]

for \( f \in C^\infty_c(T^nM) \).

Now take the case in which \( P = T^nM \). A partition of unity in \( T^nM \) is a set of nonnegative functions \( \{ \psi_i \} \subset C^\infty(T^nM) \) such that for all \( x \in T^nM \)

\[
\sum_i \psi_i(x) = 1.
\]

Given a distribution \( \eta \in \mathcal{D}'(T^nM) \), we want to make sense of its value at a form \( \omega \in \Omega^n(M) \). We let

\[
\langle \eta, \omega \rangle = \sum_i \langle \eta, \psi_i \omega \rangle,
\]

We denote by \( \mathcal{D}'_n \subset \mathcal{D}'(T^nM) \) the set of distributions for which the series in the right-hand-side converges absolutely for all \( \omega \in \Omega^n(M) \). This is independent of our choice of partition of unity \( \{ \psi_i \} \). Also, the spaces of mild measures \( \mathcal{M}_n \) and of holonomic measures \( \mathcal{H} \) are subsets of \( \mathcal{D}'_n \).

A family of measures \( \mu_t \in \mathcal{M}_n \) is differentiable at 0 if there is a distribution \( \eta \in \mathcal{D}'_n \) such that for all \( f \in C^\infty_c(T^nM) \)

\[
\left. \frac{d}{dt} \right|_{t=0} \int f \, d\mu_t = \langle \eta, f \rangle.
\]

3.2 Variations of holonomic measures

**Theorem 2.** Let \( \mu \) be a holonomic measure in \( T^nM \) and let \( \eta \in \mathcal{D}'_n \) be a distribution on \( T^nM \). Then there exists a family of holonomic measures \( \mu_t \in \mathcal{M}_n, t \in \mathbb{R} \), such that \( \mu_0 = \mu \) and

\[
\left. \frac{d}{dt} \right|_{t=0} \int f \, d\mu_t = \langle \eta, f \rangle \tag{3}
\]

for all \( f \in C^\infty_c(T^nM) \) if, and only if, the following conditions are satisfied:

- (Pos) For all nonnegative \( f \in C^\infty_c(T^nM) \) that vanish on \( \text{supp} \mu \), \( \langle \eta, f \rangle = 0 \).
- (Hol) For all differential forms \( \omega \in \Omega^{n-1}(M) \), \( \langle \eta, d\omega \rangle = 0 \).
(Prob) $\langle \eta, 1 \rangle = 0$.

**Remark 3.** In other words, the tangent space to the space of holonomic measures at the point $\mu$ is characterized by Conditions (Pos), (Hol), and (Prob).

**Proof.** By [19, Theorem 7], Condition (Pos) is necessary. If $\mu_t$ exists, then we have

$$0 = \frac{d}{dt} \bigg|_{t=0} \int d\omega d\mu_t = \langle \eta, d\omega \rangle$$

for all $\omega \in \Omega^{n-1}(M)$. Hence, Condition (Hol) is also necessary. Condition (Prob) is necessary because we want $\mu_t(T^nM) = 1$ for all $t$.

To prove that Conditions (Pos), (Hol), and (Prob) are sufficient, assume that they are satisfied. Then by [19, Theorem 7] we have a family of probability measures $\theta_t$ for $t$ in some interval that contains $0$, with $\theta_0 = \mu$ and with (3). Moreover, the proofs of Theorem 7 and Lemma 9 in [19] show that $\theta_t$ can be assumed to be in $\mathcal{M}_n$ for all $t$. Now we need to modify $\theta_t$ so that it is also a family of holonomic measures.

There exists a family of measures $\nu_t$ such that for all $\omega \in \Omega^{n-1}(M)$ and all $t$

$$\int d\omega d\theta_t + \int d\omega d\nu_t = 0.$$  

The measure $\nu_t$ can for example be obtained from $\theta_t$ as follows. For each $x \in M$, let $r_x : T^n_x M \to T^n_x M$ be some reflection such that the multivector $r_x(v)$ has the opposite orientation as the multivector $v \in T^n_x M$. These reflections can be chosen in a piecewise-continuous (and hence measurable) way with respect to the variable $x$. Then one can take the family of measures determined by $\nu_t|_{T^n_x M} = r_x^*(\theta_t|_{T^n_x M})$.

For $a > 0$, let $\lambda_a : T^n_x M \to T^n_x M$ be the map given by

$$\lambda_a(x, v_1, v_2, \ldots, v_n) = (x, av_1, v_2, \ldots, v_n).$$

The measure $\nu_t^a = \lambda_a^* \nu_t/a$ satisfies

$$\int d\omega d\nu_t^a = \frac{1}{a} \int d\omega(x, av_1, \ldots, v_n) d\nu_t = \int d\omega d\nu_t$$

for all $\omega \in \Omega^{n-1}(M)$. However, as $a \to \infty$, the mass $\int d\nu_t^a$ of $\nu_t^a$ tends to $0$. It is hence possible to find a function $b : \mathbb{R} - \{0\} \to \mathbb{R}_+$ such that $\nu_t^b(t)$ is a family of measures with

$$\frac{d\nu_t^b(t)}{dt} \bigg|_{t=0} = 0 \quad \text{and} \quad \lim_{t \to 0} \frac{1}{t^2} \int d\nu_t^b(t) = 0.$$
We let
\[ \mu_t = \frac{\theta_t + \nu_t \mu_t}{1 + \int d\nu_t \mu_t} \]
for \( t \neq 0 \) and \( \mu_0 = \mu \). This is a family of measures as in the statement of the theorem.

4 Examples

Results in this section are valid for measures that are critical with respect to the action of a general smooth Lagrangian \( L \in C^\infty(T^nM) \). Unless explicitly stated, we do not require, for example, that \( L \) be convex.

A variation of a holonomic measure \( \mu \in \mathcal{H} \) is a family \( \mu_t \) of holonomic measures that is defined for \( t \) in an interval \( I \subseteq \mathbb{R} \) containing 0 and is differentiable at 0.

We denote by \( A_L \) the action of the Lagrangian \( L \),
\[ A_L(\mu) = \int_{T^nM} L \, d\mu. \]
We say that \( \mu \in \mathcal{H} \) is critical for \( A_L \) if for every variation \( \mu_t \) with \( \mu_0 = \mu \)
\[ \frac{d}{dt} \bigg|_{t=0} A_L(\mu_t) = 0. \] (4)

By Theorem 2, \( \mu \) is critical if, and only if, for all distributions \( \eta \in \mathcal{D}' \) that satisfy Conditions (Pos), (Hol), and (Prob), we have
(Crit) \( \langle \eta, L \rangle = 0 \).

Homology. A holonomic measure \( \mu \in \mathcal{H} \) is assigned its homology class \( \rho(\mu) \in H^n(M; \mathbb{R}) \) by requiring
\[ \langle \rho(\mu), \omega \rangle = \int \omega \, d\mu \]
for all closed forms \( \omega \in \Omega^n(M) \), \( d\omega = 0 \). If for each \( t \) the measure \( \mu_t \) has the same associated homology class as \( \mu_0 \), \( \rho(\mu_t) = \rho(\mu_0) \), then we say that the variation \( \mu_t \) is homology preserving. Clearly, for this to happen the following condition is necessary on the derivative \( \eta = d\mu_t/dt \big|_{t=0} \):
(Hom) \( \langle \eta, \omega \rangle = 0 \) for all \( \omega \in \Omega^n(M) \) with \( d\omega = 0 \).
Conjecture 4. Conditions (Pos), (Hol), (Prob), and (Hom) are sufficient for the existence of a homology preserving variation $\mu_t$.

We will say that $\mu \in \mathcal{H}$ is critical for $A_L$ within its homology class if equation (4) holds for every homology variation $\mu_t$ of $\mu$. In particular, if $\mu$ is critical for $A_L$, then it is also critical within its homology class.

4.1 Horizontal variations

Let $X : M \to TM$ be a smooth vector field on $M$. For $f \in C^\infty_c(T^n M)$, denote by $Xf$ the Lie derivative in the (horizontal) direction $X$. This is given by $Xf = df(X)$, and is independent of the Riemannian metric on $M$. For a differential form $\omega \in \Omega^n(M)$, the action of $X$ on $\omega$ is also defined, and it is equal to the Lie derivative $L_X \omega = i_X d\omega + di_X \omega$. Here, $i_X$ denotes the contraction.

Let $\mu$ be a holonomic measure on $T^n M$. The distribution $\eta$ given by

$$\langle \eta, f \rangle = \int_{T^n M} Xf \, d\mu$$

for $f \in C^\infty_c(T^n M)$ clearly satisfies Conditions (Pos) and (Prob). It also satisfies Condition (Hol) because for all $\omega \in \Omega^{n-1}(M)$,

$$\langle \eta, d\omega \rangle = \int L_X d\omega \, d\mu = \int i_X d^2 \omega + di_X d\omega \, d\mu = 0.$$

Therefore, $\eta$ is in the tangent space to $\mu$.

It also satisfies Condition (Hom) because, if $\omega$ is a closed $n$-form,

$$\frac{d}{ds} \int \omega \, d\mu_s = \int L_X \omega \, d\mu_s = \int i_X d\omega + di_X \omega \, d\mu_s = 0.$$

The last equality is true since $d\omega = 0$ because $\omega$ is closed, and $\int di_X \omega \, d\mu_s = 0$ because $\mu_s$ is holonomic.

In fact, it is easy to explicitly construct a family $\mu_t$ with derivative $\eta$ and $\mu_0 = \mu$. To do this, take the flow $\phi_t : \mathbb{R} \times M \to M$ of $X$ on $M$, determined by $\phi_0(x) = x$, $\frac{d}{dt} \phi_t(x) = X(x)$, for $x \in M, t \in \mathbb{R}$.

Extend this to an isotopy $r : \mathbb{R} \times T^n M \to T^n M$ by

$$r_t(x, v_1, \ldots, v_n) = (\phi_t(x), d\phi_t(v_1), \ldots, d\phi_t(v_n)),$$

where $d\phi_t : T_x M \to T_{\phi_t(x)} M$ denotes the derivative of $\phi_t$ at $x$. Then we can simply let $\mu_t = r^*_t \mu$. From this construction and Proposition 1 it is clear that $\mu_t$ is homology preserving. We thus have
**Proposition 5.** If \( \mu \) is critical for \( A_L \) within its homology class, then Condition (Crit) must hold for all distributions \( \eta \) of the form given in equation (5).

**Euler-Lagrange equations.** Assume that the holonomic measure \( \mu \) is induced by a cycle \( \alpha \), that is, \( \mu = \mu_\alpha \).

We will now recover the traditional Euler-Lagrange equations in this special case.

For \( t \in \mathbb{R}, \ r_t \circ \alpha \) denote the cycle that results from the operation of composing each of the \( n \)-cells \( \gamma_i \) that appear in \( \alpha \) with the isotopy \( r_t \):  

\[
\text{if } \alpha = \sum_i c_i \gamma_i, \ c_i \in \mathbb{R}, \text{ then } r_t \circ \alpha = \sum_i c_i r_t \circ \gamma_i.
\]

The variation \( \mu_t = r_t^* \mu_\alpha \) constructed above is precisely the same as \( \mu_{r_t \circ \alpha} \).

We want to examine what happens when the measure \( \mu \) is critical for \( A_L \) with respect to all such variations \( \mu_t \) for all vector fields \( X \). For clarity, we use the time variable \( s \) instead of \( t \), and we use the variables \( t_j \) on the domain of \( \gamma_i \). Also, we write \( dt = dt_1 \cdots dt_n \). We denote the partial derivatives of \( L \) by \( L_x \) and \( L_{v_i} \). For each such variation have:

\[
0 = \left. \frac{d}{ds} \right|_{s=0} \int L \, d\mu_s = \left. \frac{d}{ds} \right|_{s=0} \sum_i c_i \int L(d(r_s \circ \gamma_i)) \, dt
\]

\[
= \sum_i c_i \int \left[ L_x(d\gamma_i) \left. \frac{\partial r_s \circ \gamma_i}{\partial s} \right|_{s=0} + \sum_j L_{v_j}(d\gamma_i) \left. \frac{\partial^2 (r_s \circ \gamma_i)}{\partial s \partial t_j} \right|_{s=0} \right] \, dt
\]

\[
= \sum_i c_i \int \left[ L_x(d\gamma_i) - \sum_j \frac{\partial L_{v_j}(d\gamma_i)}{\partial t_j} \right] \left. \frac{\partial (r_s \circ \gamma_i)}{\partial s} \right|_{s=0} \, dt
\]

\[
= \sum_i c_i \int (E-L) \left. \frac{\partial r_s}{\partial s} \right|_{s=0} \, dt
\]

where

\[
(E-L) := \frac{\partial L}{\partial x} - \sum_{i=1}^n \left( \frac{\partial^2 L}{\partial x \partial v_i} v_i + \sum_{j=1}^n \frac{\partial^2 L}{\partial v_i \partial v_j} X_{ij} \right),
\]

and a point in the vector space \( T_{(v_1,\ldots,v_n)}(T^n_x M) \) has coordinates \( X_{ij}, 1 \leq i, j \leq n \). Since the above is true for all smooth vectorfields \( X = \partial r_s / \partial s \big|_{s=0} \),
we conclude that (E-L) must vanish identically throughout the support of \( \mu = \mu_\alpha \).

In other words, Condition (Crit) for measures \( \mu_\alpha \) and for distributions of the form (5) is equivalent to the Euler-Lagrange equations (E-L).

Remark 6. In the case of an arbitrary holonomic measure (not necessarily induced by a cycle) we have no information about the second derivatives, so we find no clear way to give this deduction in that general case. While it can be ascertained that these equations must be respected in a ‘weak’ way (if \( \mu = \lim \mu_\alpha \), the measures \( \mu_\alpha \) will asymptotically satisfy Euler-Lagrange in the sense of distributions, so (E-L) must vanish \( \mu \)-almost everywhere), it is not clear to us how this can be useful.

4.2 Vertical variations

Let \( \mu \) be a holonomic measure in \( T^n M \).

We introduce the Hilbert space \( \mathcal{H} \) of all functions \( u : \text{supp} \mu \subseteq T^n M \to T^n M \) such that \( u(x,v) \in T^n_x M \) for all \( (x,v) \in T^n M \), and \( \int g(u,u) \, d\mu < +\infty \), where \( g \) is the Riemannian metric on \( M \). The inner product in \( \mathcal{H} \) is defined by

\[
(u_1, u_2) = \int g(u_1, u_2) \, d\mu.
\]

The set of gradients \( \nabla_v \omega \) of exact differential forms (viewed as functions on \( T^n M \)) is a subspace \( F \) of \( \mathcal{H} \).

Each function \( u \) in \( \mathcal{H} \) induces a distribution \( \eta^u \) of the form

\[
\langle \eta^u, f \rangle = \int_{T^n M} g(u, \nabla_v f) \, d\mu.
\]

for \( f \in C_0^\infty (T^n M) \). This distribution clearly satisfies Conditions (Pos) and (Prob). The set of all functions \( u \) in \( \mathcal{H} \) such that \( \eta^u \) satisfies Condition (Hol) as well are exactly the orthogonal complement \( F^\perp \) to \( F \) in \( \mathcal{H} \) because Condition (Hol) is

\[
0 = \langle \eta^u, d\omega \rangle = \int g(u, \nabla_v d\omega) \, d\mu = (u, d\omega).
\]

for \( \omega \in \Omega^{n-1} (M) \).

It follows that, if Condition (Crit) is satisfied for all \( \eta^u \) satisfying Conditions (Pos), (Hol), and (Prob), then \( \nabla_v L \) must be contained in the space \( F^\perp \perp \), which coincides with the topological closure \( \overline{F} \). We have proved
Proposition 7 ("$L_v = d\omega$"). If $\mu$ is a holonomic measure that is critical for $A_L$, then there exist a sequence $\{\omega^j\}_i \subset \Omega^{n-1}(M)$ such that

$$\nabla_v L|_{\text{supp } \mu} = \lim_{i \to \infty} \nabla_v d\omega^j.$$ 

The limit is taken in $\mathcal{H}$.

It is possible to produce an explicit variation $\mu^u_t$ of $\mu$ with derivative $\eta^u$ by letting

$$\int_{T^n M} f \, d\mu^u_t = \int_{T^n M} f(x, v + su(x, v)) \, d\mu(x, v)$$

for all $f \in C^\infty_c(T^n M)$ and $s \in \mathbb{R}$. It follows from the construction that this variation preserves homology whenever Condition (Hom) holds. That is, whenever $u$ is such that

$$0 = \langle \eta^u, \omega \rangle = (u, \omega)$$

for all closed forms $\omega \in \Omega^n(M)$. Hence, the same argument as before yields

Proposition 8. If $\mu$ is a holonomic measure that is critical for $A_L$ within its homology class, then there exists a sequence of closed $n$-forms $\{\omega^j\}_i \subset \Omega^n(M)$, $d\omega^j = 0$, such that

$$\nabla_v L|_{\text{supp } \mu} = \lim_{i \to \infty} \nabla_v \omega^j.$$ 

The limit is taken in $\mathcal{H}$.

4.3 Transpositional variations

Let $\mu \in \mathcal{H}$ again be a holonomic measure, and let $L$ be a Lagrangian.

Let $\sigma \in C^\infty_c(T^n M)$ and fix some $1 \leq i \leq n$. We consider the distribution on $T^n M$ given by

$$\langle \eta, f \rangle = \int \sigma f \, d\mu - \int \sigma \frac{\partial f}{\partial v_i} \cdot v_i \, d\mu - \int f \, d\mu \int \sigma \, d\mu$$

for $f \in C^\infty_c(T^n M)$. The distribution $\eta$ clearly satisfies Conditions (Pos) and (Prob). To see that it also satisfies Condition (Hol), we compute, for $\omega \in \Omega^{n-1}(M)$,

$$\langle \eta, d\omega \rangle = \int \sigma d\omega \, d\mu - \int \sigma d\omega \, d\mu - \int d\omega \, d\mu \int \sigma \, d\mu = 0.$$
Here, we used that $\partial d\omega / \partial v_i = d\omega$ by linearity, and we also used the fact that $\mu$ is holonomic.

If $\mu$ is critical for $A_L$, it must satisfy Condition (Crit) for all variations arising in this way from any $\sigma \in C^\infty_c(T^n M)$. This translates to

$$0 = \langle \eta, L \rangle = \int \sigma L d\mu - \int \sigma L_{v_i} \cdot v_i d\mu - \int L d\mu \int \sigma d\mu.$$ 

If the domain of $\sigma$ is very small around a point $(x, v) \in T^n M$, this can be very well approximated by

$$0 \approx \int \sigma d\mu \left( L(x, v) - v_i L_{v_i}(x, v) - \int L d\mu \right).$$

This is how we deduce

**Proposition 9** (Energy conservation). If a holonomic measure is critical with respect to all transpositional variations, then its support is a subset of the set where

$$v_i \cdot L_{v_i} - L = -A_L(\mu).$$

**Remark 10.** In the cases in which we can define the change of variables $p_i = L_{v_i}$ (for example, in the case of convex, superlinear Lagrangians), we can also define the Hamiltonians

$$H_i(x, p_i) = H_i(x, p_i; v_1, \ldots, \tilde{v}_i, \ldots, v_n) = \max_{v_i \in T^n_M} p_i v_i - L(x, v_1, \ldots, v_n),$$

and what we have here is just a higher-dimensional version of the usual energy conservation principle.

**Remark 11** (Hamilton-Jacobi equation). We can form the full Hamiltonian $H = \sum_i H_i$, which in the case of a convex, superlinear Lagrangian $L$ is the convex dual to the Lagrangian $nL$. Then it follows from Proposition 7 that there is a sequence of $(n-1)$-forms $\omega^k$ such that, abusing the notation a little,

$$\lim_{k \to \infty} H(x, d\omega^k) = -nA_L(\mu)$$

on $\text{supp} \mu$. This is a generalized form of the Hamilton-Jacobi equation.

The distribution $\eta$ is in fact the derivative of the variation $\mu_t^\sigma$ given by

$$\int f d\mu_t^\sigma = \int \frac{(1-t\sigma)f(x, v_1, \ldots, v_{i-1}, (1-t\sigma)^{-1}v_i, v_{i+1}, \ldots, v_n) d\mu}{\int (1-t\sigma) d\mu}$$

for $f \in C^\infty_c(T^n M)$ and for $t$ in an open interval that contains 0.
If we require the variation $\mu^\sigma_t$ to preserve homology, then we find that we must require $\int \sigma \, d\mu = 0$ because

$$\int \omega \, d\mu^\sigma_t = \frac{\langle \rho(\mu), \omega \rangle}{\int (1 - t\sigma) \, d\mu}$$

must be constant for each closed form $\omega \in \Omega^*(M)$, $d\omega = 0$. It follows that if $\mu$ is critical for $A_L$ within its homology class then it must satisfy

$$\int \sigma (L - L_{vi} \cdot v_i) \, d\mu = 0$$

for all $\sigma$ with $\int \sigma \, d\mu = 0$. We can use $\sigma d\mu$ to approximate the derivative at any point in $\text{supp} \mu$ arbitrarily well. Hence, we get

**Proposition 12** (Energy conservation for homological minimizers). *If a holonomic measure is critical for $A_L$ within its homology class, then there are some $c_1, \ldots, c_n \in \mathbb{R}$ such that the support of $\mu$ contained in the set where

$$L - v_i \cdot L_{vi} = c_i, \quad i = 1, 2, \ldots, n.$$*

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