Flavor neutrinos as unstable particles

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Abstract. In this paper we review flavor-energy uncertainty relations for neutrino oscillations in quantum field theory, putting in evidence the analogy with the case of unstable particles. Our study reveals that flavor neutrinos are intrinsically characterized by an energy distribution with a non-vanishing width. In the ultrarelativistic limit, the energy width is bounded from below by the inverse of the oscillation length, which plays the same role as the half-life for unstable particles.

1. Introduction

The idea of neutrino oscillations was originally introduced by B. Pontecorvo [1] and currently represents one of the few established examples of physics beyond Standard Model [2, 3]. Some fundamental issues regarding the nature of flavor mixed particles and their formal description in quantum field theory (QFT), were firstly investigated in Ref. [4] and then in many subsequent works (see e.g. Ref. [5]). In that context, mixing transformation is recognized as a canonical (Bogoliubov) transformations and flavor neutrinos are described as excitations of flavor vacuum (zero-flavor state), which presents a non trivial condensate structure. This last feature suggests that field mixing could have a dynamical origin [6], and it has been recently proved that such a structure naturally emerges in chirally symmetric models [7]. It is worthy noting that a similar condensate structure was also derived in Ref. [8] in the context of grand unified theories.

In Ref. [9] flavor-energy uncertainty relations (FEUR) for neutrino oscillations were derived, by generalizing the quantum mechanical (QM) reasoning of Ref. [10] on Mandelstam–Tamm time energy uncertainty relations (TEUR). An important upshot of this is that flavor neutrinos has to be described by an energy distribution with a finite width, whose lower bound is dictated by FEUR. In the ultrarelativistic limit, such a lower bound is proportional to the inverse of the oscillation length. This suggests an analogy with the case of unstable particles, whose energy is also not sharply defined and the width of the energy distribution is proportional to the inverse of particle life-time.

In this paper we review key points of Ref. [9], emphasizing the parallelism between flavor neutrinos and unstable particles. In order to illustrate our point, we explicitly construct in
Section 2 the vacuum state for unstable particles. This, so called De Filippo-Vitiello vacuum is obtained in the framework of Thermo-field dynamics (TFD) [12] like formalism which makes use of the doubling of degrees of freedom. TEUR is derived in this case, improving some results of Ref. [13]. In Section 3 flavor mixing in QFT is reviewed and FEUR is explicitly derived and computed in particular cases.

2. Mandelstam–Tamm TEUR for unstable particles

From the Heisenberg equation of motion

$$\frac{dO(t)}{dt} = i[H, O(t)] ,$$

and the Cauchy-Schwarz inequality, it follows a particular form of the TEUR [14]:

$$\Delta E \Delta t \geq \frac{1}{2} ,$$

where

$$\Delta E \equiv \sigma_H \quad \Delta t \equiv \sigma_O / \left| \frac{d\langle O(t) \rangle}{dt} \right| .$$

Here $O(t)$ represents the “clock observable” whose dynamics quantifies temporal changes in a system and $\Delta t$ is the characteristic time interval over which the mean value of $O$ changes by a standard deviation [15].

Although TEUR is invoked very often in the study of unstable particles, see, e.g. Ref. [16], the use of Mandelstam–Tamm TEUR to derive such inequality in a rigorous way was proposed only in Ref. [13]. There the author derived the Heisenberg-like relation:

$$\Delta E T_h \geq \frac{\pi}{4} ,$$

where $T_h$ is the time when the survival probability of the decaying particle equals 1/2. Here we will repeat similar passages by means of the approach of Ref. [11].

The basic idea is the following: a system of unstable particles has to be regarded as an open quantum system. Therefore, it is not sufficient to study the system alone, in terms of excitations of some fields on a vacuum state, but such a system is always coupled with its double, as it is usually done in Thermo field dynamics (TFD) [12]. It is important to note that a similar construction can be repeated in different areas of physics as dissipative quantum systems [17], QFT in curved spacetime [18], and brain dynamics [19]. We remark that the following analysis does not rely on a specific Lagrangian and therefore it is quite general.

Let us introduce the fermion operators$^1$ $a_k$, $a_k^{\dagger}$, which permit to describe the annihilation and creation of a decaying particle at $t = 0$ and which satisfy the canonical anticommutation relations:

$$\left\{ a_k, a_p^{\dagger} \right\} = \delta (k - p) ,$$

with all other anticommutators equal to zero. Their double will be denoted as $\tilde{a}_k$, $\tilde{a}_k^{\dagger}$. These follow analogous anticommutation relations. The vacuum state is defined so that

$$a_k |0\rangle = \tilde{a}_k |0\rangle = 0 ,$$

$|0\rangle$ has a tensor product structure as $|0\rangle = |0 \rangle \otimes |\tilde{0}\rangle$, where

$$a_k |0\rangle = \tilde{a}_k |\tilde{0}\rangle = 0 .$$

$^1$ In this section, for notational simplicity, we drop out helicity indexes.
The idea is to consider a state $|0(\varphi)\rangle$ such that:

$$\mathcal{N}_k(t) = \langle 0(\varphi) | N_k | 0(\varphi) \rangle = \exp(-\Gamma_k t) \, ,$$

(8)

where

$$N_k = a_k^\dagger a_k \, ,$$

(9)

is the number operator for particles with a fixed momentum and $\Gamma_k$ is the inverse of particle lifetime\(^2\). In order to get this result, we introduce a new set of annihilation and creation operators via the Bogoliubov transformation:

$$a_k(\varphi) = \cos \varphi_k a_k - \sin \varphi_k \tilde{a}_k^\dagger \, ,$$

(10)

$$\tilde{a}_k(\varphi) = \cos \varphi_k \tilde{a}_k + \sin \varphi_k a_k^\dagger \, ,$$

(11)

which reminds the one encountered in TFD [12]. The vacuum in the new representation is defined by the condition:

$$a_k(\varphi) | 0(\varphi) \rangle = \tilde{a}_k(\varphi) | 0(\varphi) \rangle \, ,$$

(12)

and it is explicitly given by the expression:

$$|0(\varphi)\rangle = \prod_k \left( \cos \varphi_k + \sin \varphi_k a_k^\dagger \tilde{a}_k^\dagger \right) |0\rangle \, .$$

(13)

Note that $\varphi_k \equiv \varphi_k(t)$, because of Eq. (8). From that equation it also follows that:

$$\mathcal{N}_k(t) = \sin^2 \varphi_k(t) = \exp(-\Gamma_k t) \, ,$$

(14)

which relates $\varphi_k$ with the physical quantities. The meaning of $|0(\varphi)\rangle$ can be understood looking at limiting cases. For example:

$$|0(\varphi)\rangle|_{t=0} = \prod_k |0(\varphi_k)\rangle \, ,$$

(15)

where

$$|0(\varphi_k)\rangle = a_k^\dagger \tilde{a}_k^\dagger |0\rangle \, ,$$

(16)

is the state describing an unstable particle. Moreover,

$$|0(\varphi)\rangle|_{t=\infty} = |0\rangle \, ,$$

(17)

which is the zero particle state.

Note that in the Heisenberg representation, the expression (14) should remain untouched, while $|0(\varphi)\rangle$ should be fixed as in Eq. (15). Therefore, the number operator (9) must be time dependent. We can thus take $O(t) = \mathcal{N}_k(t)$ in Eq. (2). We get:

$$\sigma_N^2 = \langle 0(\varphi) | \mathcal{N}_k^2(t) | 0(\varphi) \rangle - \langle 0(\varphi) | \mathcal{N}_k(t) | 0(\varphi) \rangle^2 = \mathcal{N}_k(t) (1 - \mathcal{N}_k(t)) \, ,$$

(18)

and then

$$\left| \frac{d\mathcal{N}_k(t)}{dt} \right| \leq 2\Delta E \sqrt{\mathcal{N}_k(t) (1 - \mathcal{N}_k(t))} \, .$$

(19)

\(^2\) For a derivation of an explicit form of $\Gamma_k$, e.g. $\Gamma_k = M \Gamma_0 / \sqrt{|k|^2 + M^2}$ (with $\Gamma_0 \equiv \Gamma_{k=0}$), see Refs. [20, 21], where non-exponential corrections to the decay law are also investigated.
This result is the same found in Ref. [13]. There it is also noticed that the r.h.s. is maximized when \( N_k(T_h) = 1/2 \). Therefore, the weaker inequality

\[
\Delta E \geq \left| \frac{dN_k(t)}{dt} \right|, \tag{20}
\]

holds. Using the triangular inequality and integrating both sides from 0 to \( T \), we get

\[
\Delta ET \geq \int_0^T dt \left| \frac{dN_k(t)}{dt} \right| \geq \left| \int_0^T dt \frac{dN_k(t)}{dt} \right|, \tag{21}
\]

Therefore, one finds

\[
\Delta ET \geq 1 - N_k(T). \tag{22}
\]

For \( T = T_h \), we finally have

\[
\Delta ET_h \geq \frac{1}{2}, \tag{23}
\]

which is even stronger than (4) and, in addition, it has Heisenberg-like lower bound. From Eq.(14), one has

\[
T_h = \frac{1}{\Gamma_k} \log 2 = \tau_k \log 2, \tag{24}
\]

where \( \tau_k \) is the particle life-time. TEUR (23) thus reads

\[
\Delta E \geq \frac{1}{\tau_k \log 4}. \tag{25}
\]

This relation poses a lower-bound on energy distribution width for unstable particles, which is fundamental and cannot be removed via improved measurement techniques or better instrumentation. We will see that similar consideration can be repeated for neutrinos.

3. FEUR for neutrino oscillations in QFT

One can use a similar line of reasonings to arrive at the TEUR for the neutrino oscillations [9]. These are derived from the fundamental incompatibility of flavor charges and Hamiltonian operators. We will find that, as in the case of unstable particles, seen in the previous section, such inequality establishes a lower bound on neutrino energy precision measurement, which cannot be removed. This is in accord with the view of neutrinos as internal lines of Feynman diagrams [22] and then as off-shell particles.

3.1. Neutrino mixing in QFT

In this subsection we briefly review basics of neutrino mixing transformation in QFT [4, 5]. The two-flavor mixing transformation for neutrinos is

\[
\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \tag{26}
\]

\[
\nu_\mu(x) = \nu_2(x) \cos \theta - \nu_1(x) \sin \theta, \tag{27}
\]

where \( \nu_1, \nu_2 \) indicate neutrino fields with definite masses and \( \nu_e, \nu_\mu \), the ones with a definite flavor. The fields with definite masses have the usual mode expansion

\[
\nu_j(x) = \sum_r \int \frac{d^3k}{(2\pi)^{3/2}} \left[ u_{k,j}(t) \alpha_{k,j}(t) + v^r_{-k,j}(t) \beta^r_{-k,j}(t) \right] e^{ik \cdot x}, \quad j = 1, 2. \tag{28}
\]
The **mass vacuum** is defined by the condition:

\[ \alpha_{k,j}^r |0\rangle_{1,2} = \beta_{k,j}^r |0\rangle_{1,2} = 0 \quad j = 1, 2. \quad (29) \]

From (28) and (26), (27) it follows that flavor fields can be expanded as:

\[ \nu_{\sigma}(x) = \sum_r \int \frac{d^3k}{(2\pi)^3} \left[ u_{k,j}^r(t) \alpha_{k,\sigma}^r(t) + v_{k,j}^r(t) \beta_{-k,\sigma}^r(t) \right] e^{ikx}, \quad (30) \]

with \((\sigma, j) = (e, 1), (\mu, 2),\) and flavor ladder operators given by \(^3\)

\[ \begin{align*}
\alpha_{k,e}^r(t) &= \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k(t) \alpha_{k,2}^r + \epsilon^r V_k(t) \beta_{-k,2}^r \right), \\
\alpha_{k,\mu}^r(t) &= \cos \theta \alpha_{k,2}^r - \sin \theta \left( U_k(t) \alpha_{k,1}^r - \epsilon^r V_k(t) \beta_{-k,1}^r \right), \\
\beta_{k,1}^r(t) &= \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k(t) \beta_{-k,2}^r - \epsilon^r V_k(t) \alpha_{k,2}^r \right), \\
\beta_{k,2}^r(t) &= \cos \theta \beta_{-k,2}^r - \sin \theta \left( U_k(t) \beta_{-k,1}^r + \epsilon^r V_k(t) \alpha_{k,1}^r \right),
\end{align*} \quad (31)-(34) \]

where \(\epsilon^r = (-1)^r\) and we defined

\[ U_k(t) = |U_k| e^{i(\omega_{k,2} - \omega_{k,1})t}, \quad V_k(t) = |V_k| e^{i(\omega_{k,1} + \omega_{k,2})t}, \quad (35) \]

\[ |U_k| = \mathcal{A}_k \left(1 + \frac{|k|^2}{\omega_{k,1} + m_1 \omega_{k,2} + m_2} \right), \quad (36) \]

\[ |V_k| = \mathcal{A}_k \left(1 + \frac{|k|^2}{\omega_{k,1} + m_1 \omega_{k,2} + m_2} \right), \quad (37) \]

\[ \mathcal{A}_k = \sqrt{\frac{\omega_{k,1}^2 + m_1^2}{2\omega_{k,1}}} \frac{\omega_{k,2}^2 + m_2^2}{2\omega_{k,2}}. \quad (38) \]

Here \(m_j\) and \(\omega_{k,j} = \sqrt{|k|^2 + m_j^2}\) are (mass) neutrino masses and energy, respectively.

Eqs.(31)-(34) form a four modes Bogoliubov transformation, which generalizes the simpler ones (10), (11). The **flavor vacuum** is defined as:

\[ \alpha_{k,\sigma}^r(t)|0(t)\rangle_{e,\mu} = \beta_{-k,\sigma}^r(t)|0(t)\rangle_{e,\mu} = 0 \quad \sigma = e, \mu, \quad (39) \]

and its explicit expression, at the reference time \(t = 0\), is given by:

\[ |0\rangle_{e,\mu} = \prod_k \prod_r \left( 1 - \sin^2 \theta |V_k|^2 \right) - \epsilon^r \sin \theta \cos \theta |V_k| \left( \alpha_{k,1}^{\dagger} \beta_{-k,2}^{\dagger} + \alpha_{k,2}^{\dagger} \beta_{-k,1}^{\dagger} \right) + \epsilon^r \sin^2 \theta |V_k||U_k| \left( \alpha_{k,1}^{\dagger} \beta_{-k,1}^{\dagger} - \alpha_{k,2}^{\dagger} \beta_{-k,2}^{\dagger} \right) + \sin^2 \theta |V_k|^2 \left( \alpha_{k,1}^{\dagger} \beta_{-k,2}^{\dagger} + \alpha_{k,2}^{\dagger} \beta_{-k,1}^{\dagger} \right) |0\rangle_{1,2}. \quad (40) \]

This has to be compared with Eq. (13). Both the expressions represents condensates of particles and are coherent states of Perelomov type [23], related to the group \(SU(2)\).

\(^3\) Here we choose a Lorentz frame so that \(k = (0, 0, |k|)\).
Flavor eigenstates can be explicitly constructed as

$$|\nu^r_{k,\sigma}\rangle = \alpha^r_{k,\sigma}|0\rangle_{e,\mu}.$$  (41)

where flavor operators and vacuum state are taken at reference time $t = 0$. This definition is enforced by introducing the flavor charges [24]:

$$Q_{\nu_\sigma}(t) = \int d^3x : \nu^\dagger_\sigma(x) \nu_\sigma(x) :.$$  (42)

In fact, one can easily verify that

$$Q_{\nu_\sigma}(0)|\nu^r_{k,\sigma}\rangle = |\nu^r_{k,\sigma}\rangle,$$  (43)

i.e. flavor states are eigenstates of the flavor charges at the reference time.

It follows that oscillation formula can be found by taking the expectation value of the flavor charges [24, 25]

$$Q_{\sigma \rightarrow \rho}(t) = \langle Q_{\nu_\sigma}(t)\rangle_{\rho},$$  (44)

where $\langle \cdots \rangle_\sigma = \langle \nu^r_{k,\sigma}|\cdots|\nu^r_{k,\sigma}\rangle$, which gives

$$Q_{\sigma \rightarrow \rho}(t) = \sin^2(2\theta)\left[|U_k|^2 \sin^2(\omega_k^{-} t) + |V_k|^2 \sin^2(\omega_k^{+} t)\right],$$

$$Q_{\sigma \rightarrow \sigma}(t) = 1 - Q_{\sigma \rightarrow \rho}(t), \quad \sigma \neq \rho,$$  (45)

where $\omega_k^\pm = (\omega_{k,2} \pm \omega_{k,1})/2$.

The oscillation formula (45) quantifies the changing of flavor with respect the one at $t = 0$ and it corresponds to Eq.(8) for unstable particles. This formula presents oscillations on two different time-scales: $T_− = 2\pi/\omega_k^-$, which is the main one, observed also in the standard treatment, and $T_+ = 2\pi/\omega_k^+$, due to the interaction with the flavor vacuum condensate [4]. It is worthy to remark that the same formula can also be derived in a QM context, by taking into account that negative frequency modes cannot be disregarded when computing oscillation probability [26, 27].

### 3.2. TEUR for neutrino oscillations in QFT

In this subsection we review FEUR, closely following Ref. [9] and putting in evidence the analogies with the case of unstable particles, as presented in Section 2.

From the non-conservation of the flavor-charges

$$[Q_{\nu_\sigma}(t), H] = i \frac{dQ_{\nu_\sigma}(t)}{dt} \neq 0,$$  (46)

we derive the flavor–energy uncertainty relation

$$\sigma_H \sigma_Q \geq \frac{1}{2} \left| \frac{dQ_{\sigma \rightarrow \sigma}(t)}{dt} \right|.$$  (47)

One can explicitly compute

$$\sigma_Q^2 = \langle Q_{\nu_\sigma}^2(t)\rangle_\sigma - \langle Q_{\nu_\sigma}(t)\rangle_\sigma^2$$

$$= Q_{\sigma \rightarrow \sigma}(t) \left(1 - Q_{\sigma \rightarrow \sigma}(t)\right).$$  (48)
Eq. (48) quantifies dynamical (flavor) entanglement for neutrino states in QFT, (cf. Ref. [29]). This should be compared to the analogue result (19) for unstable particles. Following the same passages which led to Eq.(20), one gets

$$\Delta E \geq \left| \frac{dQ_{\sigma \rightarrow \rho}(t)}{dt} \right|. \quad (49)$$

By integrating as in Section 2, we arrive at the Mandelstam–Tamm TEUR in the form [9]

$$\Delta E T \geq Q_{\sigma \rightarrow \rho}(T), \quad \sigma \neq \rho. \quad (50)$$

When $m_i/|k| \rightarrow 0$, i.e. in the relativistic case, we get

$$|U_k|^2 \approx 1 - \varepsilon(k), \quad |V_k|^2 \approx \varepsilon(k), \quad (51)$$

with $\varepsilon(k) \equiv (m_1 - m_2)^2/4|k|^2$. In the same limit

$$\omega_k^+ \approx \frac{\delta m^2}{4|k|}, \quad \omega_k^- \approx |k|, \quad (52)$$

where $\delta m^2 \equiv m_2^2 - m_1^2$ and $L_{osc} \equiv 4\pi|k|/\delta m^2$. At the leading order (ultra-relativistic limit) $|U_k|^2 \rightarrow 1$, $|V_k|^2 \rightarrow 0$. In this case, one recovers the standard oscillation formula [3]

$$Q_{\sigma \rightarrow \rho}(t) \approx \sin^2(2\theta) \sin^2 \left( \frac{\pi L}{L_{osc}} \right), \quad \sigma \neq \rho, \quad (53)$$

where we put $t \approx L$. The RHS of (53) reaches its maximum at $L = L_{osc}/2$ and the inequality (50) reads

$$\Delta E \geq \frac{2\sin^2(2\theta)}{L_{osc}}. \quad (54)$$

Note that $\Delta E$ is time independent and then the inequality (54) is valid at every time. This relation has to be compared with the analogous one (25) for unstable particles. From this comparison we figure out that the ratio $\frac{2\sin^2(2\theta)}{L_{osc}}$ plays a role similar to $\Gamma_k$, in the present context.

Inequalities of the form (54) are usually interpreted as conditions of neutrino oscillations [10, 28]. Here we, however, propose a new interpretation: From the inequality (54) we infer that flavor neutrinos have an intrinsic energy uncertainty, which bounds every future experiment. In order to clarify this statement, note that [4]

$$\lim_{V \rightarrow \infty} \langle \nu_{k,i}^{\sigma} | \nu_{k,i}^{\sigma} \rangle \neq 0, \quad i = 1, 2, \quad (56)$$

where neutrino mass states are defined as $|\nu_{k,i}^{\sigma}\rangle \equiv \alpha_{k,i}^{\dagger} |0\rangle$. Eq.(56) tells us that flavor neutrino states cannot be written as a linear superposition of single-particle mass eigenstates. The orthogonality condition (56) does not hold for the standard Pontecorvo flavor states [1]

$$|\nu_{k,e}^{\sigma}\rangle_{P} = \cos \theta |\nu_{k,1}^{\sigma}\rangle + \sin \theta |\nu_{k,2}^{\sigma}\rangle, \quad (57)$$

$$|\nu_{k,\mu}^{\sigma}\rangle_{P} = -\sin \theta |\nu_{k,1}^{\sigma}\rangle + \cos \theta |\nu_{k,2}^{\sigma}\rangle. \quad (58)$$
for which
\[
\lim_{V \to \infty} \langle \nu^r_{k,1}|\nu^r_{k,e}\rangle_p = \cos \theta .
\] (59)

Eqs. (57), (58) are obtained from the definition (41), in the ultrarelativistic limit. It is then clear that, in our case
\[
\lim_{m_i/|k| \to 0} \lim_{V \to \infty} \lim_{m_j/|k| \to 0} = \lim_{V \to \infty} \lim_{m_j/|k| \to 0} ,
\] (60)
which means that the ultra-relativistic limit cannot be taken once the “thermodynamical” QFT limit is performed, but has to be considered just as QM approximation, which does not hold for systems with an infinite number of degrees of freedom. Eq. (56) should be thus understood as
\[
\langle \nu^r_{k,i}|\nu^r_{k,e}\rangle = 1, i.e., the bound on energy decreased by 10%.
\] (54)

Furthermore, field expansions, such as (28) and (30), are, in fact, integrated over all momenta. Said above, this corresponds to a selection of a single particle sub-space from the Hilbert space. Moreover, field expansions, such as (28) and (30), are, in fact, integrated over all momenta. Beyond the QM, single particle view, such an approximation and standard definition of neutrino state cannot work.

Let us now consider corrections beyond the ultra-relativistic limit. The oscillation formula (45) in the next-to-leading relativistic order reads [31]
\[
Q_{\sigma \to \rho}(t) \approx \sin^2(2\theta) \left[ \sin^2 \left( \frac{\pi t}{L_{osc}} \right) (1 - \varepsilon(k)) + \varepsilon(k) \sin^2(\|k|t) \right], \quad \sigma \neq \rho .
\] (62)

By setting \( T = L_{osc}/2 \), the relation (50) takes the form
\[
\Delta E \geq \frac{2 \sin^2 2\theta}{L_{osc}} \left[ 1 - \varepsilon(k) \cos^2 \left( \frac{\|k|L_{osc}}{2} \right) \right] ,
\] (63)
i.e. the bound on the energy is lowered with respect to (54). For neutrino masses \( m_1 = 0.0497 \text{eV} \), \( m_2 = 0.0504 \text{eV} \), and \( |k| = 1 \text{MeV} \), then \( \varepsilon(k) = 2 \times 10^{-19} \).

Relevant effects can be obtained looking at the non-relativistic regime where interaction with the vacuum condensate cannot be disregarded. Let us consider, e.g. \( |k| = \sqrt{m_1 m_2} \). In this case,
\[
|U_k|^2 = \frac{1}{2} + \frac{\xi}{2} = 1 - |V_k|^2 , \quad \xi = \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} ,
\] (64)
and we can rewrite (50) as
\[
\Delta E T \geq \frac{\sin^2 2\theta}{2} \left[ 1 - \cos(\tilde{\omega}_1 T) \cos(\tilde{\omega}_2 T) - \xi \sin(\tilde{\omega}_1 T) \sin(\tilde{\omega}_2 T) \right] ,
\] (65)
with \( \tilde{\omega}_j = \sqrt{m_j(m_1 + m_2)} \). To compare it with the ultra-relativistic case, we take \( T = \tilde{L}_{osc}/4 \), with \( \tilde{L}_{osc} = 4\pi \sqrt{m_1 m_2}/\delta m^2 \), obtaining
\[
\Delta E \geq \frac{2 \sin^2 2\theta}{\tilde{L}_{osc}} (1 - \chi) .
\] (66)

Here
\[
\chi = \xi \sin(\tilde{\omega}_1 \tilde{L}_{osc}/4) \sin(\tilde{\omega}_2 \tilde{L}_{osc}/4) + \cos(\tilde{\omega}_1 \tilde{L}_{osc}/4) \cos(\tilde{\omega}_2 \tilde{L}_{osc}/4) .
\] (67)

Substituting the same numerical values of neutrino masses which we previously used, we obtain \( \chi = 0.1 \), i.e. the original bound on energy decreased by 10%.

\( ^4 \) This values for neutrino masses are taken from [32], in the case of inverted hierarchy.
4. Conclusions
We have pointed out the parallelism between QFT description of unstable particles and flavor mixed neutrinos. In both cases the vacuum state is a condensate, with an explicit time dependance, and the application of Mandelstam–Tamm TEUR fixes a lower bound on the energy variance: flavor neutrinos, as unstable particles do not have a sharp energy (and hence the rest mass), and the width of energy distribution can be estimated by TEUR. In the case of flavor mixed neutrinos, this inequality quantifies the uncertainty of the incompatible observables energy and flavor and is indicated as FEUR.

It is worthy to remark that such considerations can be also extended to curved spacetimes [33]. In that case the lower bound of energy variance for flavor neutrinos is modified with respect to the Minkowski case, by the introduction of an effective oscillation length, which depends on the details of the spacetime metric. In other words, neutrinos “life-time” is not fundamental but depends on the particular spacetime geometry. An analogue issue was also investigated in Ref. [34], where it was shown that decaying properties of unstable particles depend on the chosen reference frame.

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