Graphs with at Most Three Distance Eigenvalues Different from $-1$ and $-2$

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Abstract Let $G$ be a connected graph on $n$ vertices, and let $D(G)$ be the distance matrix of $G$. Let $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$ denote the eigenvalues of $D(G)$. In this paper, we characterize all connected graphs with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$. In the course of this investigation, we determine all connected graphs with at most three distance eigenvalues different from $-1$ and $-2$.

Keywords Distance matrix · The third largest distance eigenvalue · The second least distance eigenvalue

Mathematics Subject Classification 05C50

1 Introduction

Let $G$ be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Denote by $d_G(v_i, v_j)$ the length of the shortest path connecting $v_i$ and $v_j$ in $G$. The distance between $v \in V(G)$ and $H$, a connected subgraph of $G$, is defined to be $d(v, H) = \min\{d_G(v, w) \mid w \in V(H)\}$. Furthermore, we define the diameter and distance matrix of $G$ as $d(G) = \max\{d_G(v_i, v_j) \mid v_i, v_j \in V(G)\}$ and $D(G) = [d_G(v_i, v_j)]_{n \times n}$, respectively. Then the characteristic polynomial $\Phi_G(x) = \det(xI - D(G))$ of $D(G)$ is also called the distance polynomial ($D$-polynomial for short) of $G$.

Since $D(G)$ is a real and symmetric, its eigenvalues can be conveniently denoted and arranged as $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$. These eigenvalues are also called
the distance eigenvalues (D-eigenvalues for short) of $G$. The distance spectrum (D-spectrum for short) of $G$, denoted by Spec$_D(G)$, is the multiset of D-eigenvalues of $G$. If $\alpha_1 > \alpha_2 > \ldots > \alpha_s$ denote all the distinct $D$-eigenvalues (with multiplicities $m_1, m_2, \ldots, m_s$, respectively) of $G$, then the $D$-spectrum of $G$ can be written as \[\text{Spec}_D(G) = \{[\alpha_1]^{m_1}, \ldots, [\alpha_s]^{m_s}\}.\] Two connected graphs are said to be distance cospectral (D-cospectral for short) if they share the same $D$-spectrum, and the graph $G$ is called determined by its $D$-spectrum (DDS for short) if any connected graph distance cospectral with $G$ must be isomorphic to it.

Throughout this paper, we denote by $G^c$ the complement of $G$, $tG$ the disjoint union of $t$ copies of $G$, $N_G(v)$ the neighborhood of $v \in V(G)$, $G[X]$ the induced subgraph of $G$ on $X \subseteq V(G)$, and $D_G(X)$ the principal submatrix of $D(G)$ corresponding to $G[X]$. Also, we denote by $P_n$ the path of order $n$, $K_n$ the complete graph on $n$ vertices, and $K_{n_1, \ldots, n_k}$ the complete $k$-partite graph with parts of order $n_1, \ldots, n_k$, respectively.

For a connected graph $G$ whose vertices are labeled as $v_1, v_2, \ldots, v_n$, and a sequence of graphs $H_1, H_2, \ldots, H_n$, the corresponding generalized lexicographic product $G[H_1, \ldots, H_n]$ is defined as the graph obtained from $G$ by replacing $v_i$ with the graph $H_i$ for $1 \leq i \leq n$, and connecting all edges between $H_i$ and $H_j$ if $v_i$ is adjacent to $v_j$ for $1 \leq i \neq j \leq n$. For example, Fig. 1 illustrates the graph $P_4[K_2, K_2^c, K_2^c, K_2]$.

Connected graphs whose $D$-eigenvalues possess special properties have received a lot of attention in the recent years. Lin et al. [13] (see also Yu [22]) proved that $\partial_n(G) = -2$ if and only if $G$ is a complete multipartite graph, and conjectured that complete multipartite graphs are DDS. Recently, Jin and Zhang [9] confirmed the conjecture. Lin et al. [12,14] characterized all connected graphs with $\partial_n(G) \geq -2 - \sqrt{2}$ and $\partial_{n-1}(G) = -1$, respectively, and showed that these graphs are DDS. Li and Meng [11] extended the result to connected graphs with $\partial_n(G) \geq -\frac{1+\sqrt{17}}{2}$. Xing and Zhou [21] determined all connected graphs with $\partial_2(G) < -2 + \sqrt{2}$, and Liu et al. [15] generalized the result to $\partial_2(G) \leq \frac{17 - \sqrt{329}}{2}$ and proved that these graphs are DDS. Very recently, Lu et al. [16] characterized all connected graphs with $\partial_3(G) \leq -1$ and $\partial_n(G) \geq -3$. It is worth noticing that most of the graphs mentioned above are of diameter 2.

On the other hand, in the past two decades, connected graphs with few distinct eigenvalues have been investigated for several graph matrices since such graphs always have pretty combinatorial properties [20]. For some recent works on this topic, we refer the reader to [2–4,8,17,18]. With regard to distance matrix, Koolen et al. [10] determined all connected graphs with three distinct $D$-eigenvalues of which two are simple; Lu et al. [16] determined all connected graphs with exactly two $D$-eigenvalues different from $-1$ and $-3$ (which are also DDS); Alazemi et al. [1] characterized distance-regular graphs with diameter three having exactly three distinct $D$-eigenvalues, and also bipartite distance-regular graphs with diameter four having three distinct $D$-eigenvalues.
In this paper, we completely characterize the connected graphs with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$ (the diameter of these graphs could be 2 or 3). As a by-product, we also determine all connected graphs with at most three $D$-eigenvalues different from $-1$ and $-2$, which gives new classes of graphs with few distinct $D$-eigenvalues.

2 Main Tools

First of all, we present some results about the bounds of $\partial_n(G)$ and $\partial_{n-1}(G)$, which are useful in the subsequent sections.

**Lemma 2.1** (Lin [12]) Let $G$ be a connected graph on $n$ vertices. Then $\partial_n(G) \leq -d(G)$ where $d(G)$ is the diameter of $G$ and the equality holds if and only if $G$ is a complete multipartite graph.

In particular, for graphs of diameter 2, we have

**Lemma 2.2** (Lin et al. [13]) Let $G$ be a connected graph on $n$ vertices. Then $\partial_n(G) = -2$ with multiplicity $n - k$ if and only if $G$ is a complete $k$-partite graph for $2 \leq k \leq n - 1$.

The following lemma determines all connected graphs with $\partial_{n-1}(G) \leq -1$.

**Lemma 2.3** (Lin et al. [14]) Let $G$ be a connected graph on $n$ vertices. If $n \geq 4$, then $\partial_{n-1}(G) \leq -1$ and the equality holds if and only if $G = K_r \vee (K_s \cup K_t)$ with $r \geq 1$.

The following result is well known under the name *Cauchy Interlace Theorem* (see Hamburger and Grimshaw [7]).

**Theorem 2.1** Let $A$ be a Hermitian matrix of order $n$, and $B$ a principal submatrix of $A$ of order $m$. If $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of $A$ and $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$ the eigenvalues of $B$, then $\lambda_i(A) \geq \mu_i(B) \geq \lambda_{n-m+i}(A)$ for $i = 1, \ldots, m$.

From Theorem 2.1 one can easily deduce the following result.

**Lemma 2.4** If $H$ is a connected induced subgraph of $G$ with diameter 1 or 2, then the $D$-eigenvalues of $H$ interlace those of $G$.

Note that $\partial_2(K_{1,2}) = 1 - \sqrt{3}$. By Lemma 2.4, we have

**Lemma 2.5** Let $G$ be a connected graph with $n \geq 2$ vertices. Then $\partial_2(G) < 1 - \sqrt{3}$ if and only if $G$ is the complete graph $K_n$.

**Lemma 2.6** Let $G$ be a connected graph. If $S_1, \ldots, S_r$ ($|S_i| \geq 2$) are disjoint independent sets (resp. cliques) of $V(G)$ such that, for each $1 \leq i \leq r$, $N_G(u) \setminus S_i = N_G(v) \setminus S_i$ for any $u, v \in S_i$, then $-2$ (resp. $-1$) is a $D$-eigenvalue of $G$ with multiplicity at least $\sum_{i=1}^{r} |S_i| - r$. 

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Proof. We only prove the lemma for the case that $S_1, \ldots, S_r$ are disjoint independent sets since the case of cliques can be dealt with similarly. For $1 \leq i \leq r$, set $S_i = \{u_1^i, \ldots, u_{|S_i|}^i\}$, $N_i = N_G(u_1^i) \backslash S_i = \{v_1^i, \ldots, v_{|N_i|}^i\}$ and $T_i = V(G) \backslash (S_i \cup N_i) = \{w_1^i, \ldots, w_{|T_i|}^i\}$. According to the assumption, $S_i \cup N_i \cup T_i$ is a vertex partition of $G$. Furthermore, we have $d_G(u_j^i, v_k^i) = 2$ for $1 \leq j \neq k \leq |S_i|$, $d_G(u_j^i, v_k^i) = 1$ for $1 \leq j \leq |S_i|$ and $1 \leq k \leq |N_i|$, and $d_G(u_j^i, w_k^i) = d_G(u_j^i, v_k^i) = a_j^i$ for $1 \leq j, j' \leq |S_i|$ and $1 \leq k \leq |T_i|$. Thus the distance matrix of $G$ can be written as

$$D(G) = \begin{bmatrix} 2(J_{|S_i|} - I_{|S_i|}) & J_{|S_i| \times |N_i|} & D(S_i, T_i) \\ J_{|N_i| \times |S_i|} & D(N_i, N_i) & D(N_i, T_i) \\ D(S_i, T_i)^T & D(N_i, T_i)^T & D(T_i, T_i) \end{bmatrix} S_i, N_i, T_i,$$

where $J$ and $I$ denote the all ones matrix and identity matrix, respectively, and $D(S_i, T_i)$ is given by

$$D(S_i, T_i) = \begin{bmatrix} a_1^i & a_2^i & \cdots & a_{|T_i|}^i \\ a_1^i & a_2^i & \cdots & a_{|T_i|}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_1^i & a_2^i & \cdots & a_{|T_i|}^i \end{bmatrix}.$$

For $1 \leq i \leq r$ and $2 \leq l \leq |S_i|$, let $x_i^j$ be the vector defined on $V(G)$ with $x_i^j(u_1^i) = 1$, $x_i^j(u_l^i) = -1$ and $x_i^j(u) = 0$ for $u \in V(G) \backslash \{u_1^i, u_l^i\}$. Then we have $D(S_i, T_i)^T(x_i^j(u_1^i), \ldots, x_i^j(u_{|S_i|}^i))^T = 0$, and so $D(G) \cdot x_i^j = (-2) \cdot x_i^j$. Clearly, $\{x_i^j \mid 1 \leq i \leq r, 2 \leq l \leq |S_i|\}$ is a set of linearly independent vectors because $S_1, \ldots, S_r$ are disjoint. Hence we may conclude that $-2$ is a $D$-eigenvalue of $G$ with multiplicity at least $\sum_{i=1}^r |S_i| - r$.

For a connected graph $G$ of order $n$, the vertex partition $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is called a distance equitable partition if, for any $v \in V_i$, $\sum_{u \in V_j} d_G(v, u) = b_{ij}$ is a constant only dependent on $i, j$ ($1 \leq i, j \leq k$). The matrix $B_\Pi = (b_{ij})_{k \times k}$ is called the distance divisor matrix of $G$ with respect to $\Pi$. The characteristic matrix $\chi_\Pi$ of $\Pi$ is the $n \times k$ matrix whose columns are the character vectors of $V_1, \ldots, V_k$.

The following lemma is an analogue of the result for adjacency matrix (see Godsil and Royle [5], pp. 195–198, or Haemers [6]), which states that the eigenvalues of $B_\Pi$ are also that of $D(G)$.

**Lemma 2.7** Let $G$ be a connected graph with distance matrix $D(G)$, and let $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ be a distance equitable partition of $G$ with distance divisor matrix $B_\Pi$. Then $\Phi_\Pi(x) = \det(xI - B_\Pi)|\Phi_G(x) = \det(xI - D(G))$, and the largest eigenvalue of $B_\Pi$ equals to $\partial_1(G)$. In particular, the matrix $D(G)$ has the following two kinds of eigenvectors:

(i) the eigenvectors in the column space of $\chi_\Pi$, and the corresponding eigenvalues coincide with the eigenvalues of $B_\Pi$;

(ii) the eigenvectors orthogonal to the columns of $\chi_\Pi$. 

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Let $G$ be a graph with vertex set $V(G)$. For any $X \subseteq V(G)$, we say that $X$ is $G$-connected if the induced subgraph $G[X]$ is connected.

**Lemma 2.8** (Seinsche [19]) Let $G$ be a graph. The following statements are equivalent.

(i) $G$ has no induced subgraph isomorphic to $P_4$.
(ii) Every subset of $V(G)$ with more than one element is not $G$-connected or not $G^c$-connected.

Let $G_1$ and $G_2$ be two vertex disjoint graphs. The join of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by connecting all edges between $G_1$ and $G_2$. Let $G$ be a connected graph containing no induced $P_4$. Then $V(G)$ is a subset of itself and so is $G$-connected, by Lemma 2.8, we know that $G^c$ is disconnected. Thus we obtain the following result.

**Lemma 2.9** If $G$ is a connected graph containing no induced $P_4$, then $G$ must be the join of two graphs.

### 3 Graphs with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$

In this section, we focus on characterizing those graphs with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$. To achieve this goal, we need the following two crucial lemmas.

**Lemma 3.1** If $G$ is a connected graph on $n$ vertices with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$, then the graphs $F_1$–$F_7$ shown in Fig. 2 cannot be induced subgraphs of $G$.

**Proof** By simple computation, it is seen that each $F_i$ ($1 \leq i \leq 7$) has third largest $D$-eigenvalue greater than $-1$ or second least $D$-eigenvalue less than $-2$ (see Fig. 2). Then the result follows by Lemma 2.4 due to $d(F_i) = 2$ for each $i$.

**Lemma 3.2** If $G$ is a connected graph on $n$ vertices with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$, then each matrix listed below cannot be the principal submatrix of $D(G)$.

$$
A_1 = \begin{bmatrix}
0 & 1 & 2 & 3 & 1 \\
2 & 0 & 0 & 1 & 2 \\
3 & 2 & 1 & 0 & 2 \\
1 & 2 & 2 & 2 & 0
\end{bmatrix},
A_2 = \begin{bmatrix}
0 & 1 & 2 & 3 & 1 \\
2 & 1 & 0 & 1 & 2 \\
3 & 2 & 1 & 0 & 3 \\
1 & 2 & 2 & 3 & 0
\end{bmatrix},
A_3 = \begin{bmatrix}
0 & 1 & 2 & 3 & 1 \\
2 & 1 & 0 & 1 & 2 \\
3 & 2 & 1 & 0 & 2 \\
1 & 2 & 3 & 2 & 0
\end{bmatrix}
$$
$$\begin{array}{cccc}
A_4 & A_5 & A_6 \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_7 & A_8 & A_9 \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{10} & A_{11} & A_{12} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{13} & A_{14} & A_{15} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{16} & A_{17} & A_{18} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{19} & A_{20} & A_{21} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{22} & A_{23} & A_{24} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{25} & A_{26} & A_{27} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{28} & A_{29} & A_{30} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{31} & A_{32} & A_{33} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{34} & A_{35} & A_{36} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{37} & A_{38} & A_{39} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$

$$\begin{array}{cccc}
A_{40} & A_{41} & A_{42} \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}$$
Table 1  The third largest or second least eigenvalues of $A_1$–$A_{51}$

| $A_i$ | $\partial_3$ or $\partial_5$ | $A_i$ | $\partial_3$ or $\partial_5$ | $A_i$ | $\partial_3$ or $\partial_5$ | $A_i$ | $\partial_3$ or $\partial_5$ |
|-------|-----------------|-------|-----------------|-------|-----------------|-------|-----------------|
| $A_1$ | $\partial_3 = -0.6557$ | $A_2$ | $\partial_3 = -0.9321$ | $A_3$ | $\partial_3 = -0.6286$ | $A_4$ | $\partial_3 = -0.6012$ |
| $A_5$ | $\partial_5 = -0.8365$ | $A_6$ | $\partial_5 = -2.5294$ | $A_7$ | $\partial_5 = -2.4413$ | $A_8$ | $\partial_5 = -2.4413$ |
| $A_9$ | $\partial_5 = -2.3224$ | $A_{10}$ | $\partial_3 = -0.7666$ | $A_{11}$ | $\partial_3 = -0.7520$ | $A_{12}$ | $\partial_3 = -2.0671$ |
| $A_{13}$ | $\partial_3 = -0.6851$ | $A_{14}$ | $\partial_5 = -2.1099$ | $A_{15}$ | $\partial_5 = -2.1725$ | $A_{16}$ | $\partial_5 = -2.1099$ |
| $A_{17}$ | $\partial_5 = -2.0898$ | $A_{18}$ | $\partial_3 = -0.5714$ | $A_{19}$ | $\partial_5 = -2.4413$ | $A_{20}$ | $\partial_3 = -0.6851$ |
| $A_{21}$ | $\partial_5 = -2.3224$ | $A_{22}$ | $\partial_5 = -2.6712$ | $A_{23}$ | $\partial_5 = -3.4142$ | $A_{24}$ | $\partial_5 = -2.5829$ |
| $A_{25}$ | $\partial_5 = -3.1708$ | $A_{26}$ | $\partial_5 = -2.4216$ | $A_{27}$ | $\partial_5 = -2.3862$ | $A_{28}$ | $\partial_3 = -0.4353$ |
| $A_{29}$ | $\partial_3 = -0.8401$ | $A_{30}$ | $\partial_3 = -0.4523$ | $A_{31}$ | $\partial_3 = -0.6010$ | $A_{32}$ | $\partial_3 = -0.8303$ |
| $A_{33}$ | $\partial_3 = -0.6712$ | $A_{34}$ | $\partial_3 = -0.6535$ | $A_{35}$ | $\partial_3 = -0.4679$ | $A_{36}$ | $\partial_5 = -2.3391$ |
| $A_{37}$ | $\partial_5 = -2.5829$ | $A_{38}$ | $\partial_5 = -2.5829$ | $A_{39}$ | $\partial_5 = -3.1708$ | $A_{40}$ | $\partial_3 = -0.8636$ |
| $A_{41}$ | $\partial_3 = -0.8401$ | $A_{42}$ | $\partial_3 = -0.7720$ | $A_{43}$ | $\partial_5 = -2.3770$ | $A_{44}$ | $\partial_3 = -0.7465$ |
| $A_{45}$ | $\partial_3 = -0.7466$ | $A_{46}$ | $\partial_5 = -2.3770$ | $A_{47}$ | $\partial_3 = -0.4607$ | $A_{48}$ | $\partial_3 = 0$ |
| $A_{49}$ | $\partial_3 = -0.6535$ | $A_{50}$ | $\partial_3 = -0.4679$ | $A_{51}$ | $\partial_3 = -0.7720$ | — | — |

Proof  According to Table 1, each $A_i$ $(1 \leq i \leq 51)$ has third largest eigenvalue greater than $-1$ or second least eigenvalue less than $-2$. Thus the result follows by Theorem 2.1.

Now we begin to prove the main result of this section.

Proposition 3.1  Let $G$ be a connected graph on $n \geq 3$ vertices with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$. Then one of the following occurs:

1. $d(G) \leq 2$ and $G \in \{I_i \mid 1 \leq i \leq 7\}$, where $I_1 = K_n$ $(n \geq 3)$, $I_2 = K_a \lor K_b^c$ $(a, b \geq 2)$, $I_3 = K_a \lor (K_b \lor K_c)$ $(a, b, c \geq 2)$, $I_4 = K_a \lor (K_b \lor K_c^c)$ $(a, b \geq 2, c \geq 1)$, $I_5 = K_a^c \lor K_b^c$ $(a + b \geq 3)$, $I_6 = K_a^c \lor (K_b \lor K_c)$ $(a \geq 1, b, c \geq 2)$ and $I_7 = K_a^c \lor (K_b \lor K_c^c)$ $(a, c \geq 1, b \geq 2)$;

2. $d(G) = 3$ and $G \in \{J_i \mid 1 \leq i \leq 8\}$, where $J_1 = P_4[K_a^c, K_b, K_c, K_d^c]$, $J_2 = P_4[K_a^c, K_b, K_c, K_d^c]$, $J_3 = P_4[K_c^c, K_b, K_c, K_d]$, $J_4 = P_4[K_a^c, K_b, K_c, K_d]$, $J_5 = P_4[K_a^c, K_b, K_c, K_d]$, $J_6 = P_4[K_a, K_b, K_c, K_d]$ and $J_7 = P_4[K_a, K_b, K_c, K_d]$, where $a, b, c, d \geq 1$. 

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Proof If $G = K_n = I_1$, then $G$ obviously satisfies the above condition due to $\partial_3(K_n) = \partial_{n-1}(K_n) = -1$. Now suppose $G \neq K_n$. Let $d(G)$ be the diameter of $G$. If $d(G) \geq 4$, then $D(P_5)$ is a principal submatrix of $D(G)$, and so $-0.7639 = \partial_3(P_5) \leq \partial_3(G) = -1$ by Theorem 2.1, which is impossible. Thus it remains to consider the following two cases.

Case 1. $d(G) = 2$.

First of all, we prove that $G$ cannot contain $P_4$ as its induced subgraph. Suppose to the contrary that $P_4 = v_1 v_2 v_3 v_4$ is an induced subgraph of $G$. Then there exists some vertex $v \in V(G)$ which is adjacent to both $v_1$ and $v_4$ because $d_G(v_1, v_4) = 2$ due to $d(G) = 2$. Thus at least one of $\{F_1, F_2, F_3\}$ is the induced subgraph of $G$, which is impossible by Lemma 3.1. Therefore, from Lemma 2.9, there exist two graphs $G_1$ and $G_2$ such that $G = G_1 \lor G_2$. We only need to discuss the following two situations.

Subcase 1.1 Both $G_1$ and $G_2$ contain no induced $P_3$.

Since $P_3$ is not an induced subgraph of $G_1$ and $G_2$, we claim that both $G_1$ and $G_2$ are the disjoint unions of some complete graphs. Further, if $G_1$ or $G_2$ contains $2K_2 \cup K_1$ as its induced subgraph, then $G = G_1 \lor G_2$ contains induced $F_4$, which is a contradiction by Lemma 3.1. Thus, for $i = 1,2$, we conclude that $G_i$ is one of the following graphs: $K_a$ ($a \geq 2$), $K_a$ ($a \geq 1$), $K_a \cup K_b$ ($a, b \geq 2$) and $K_a \cup K_b$ ($a \geq 2, b \geq 1$). As $G \neq K_n$ and $n \geq 3$, all the possible forms of $G$ are $I_2 = K_a \lor K_b$ ($a, b \geq 2$), $I_3 = K_a \lor (K_b \cup K_c)$ ($a, b, c \geq 2$), $I_4 = K_a \lor (K_b \lor K_c)$ ($a, b \geq 2, c \geq 1$), $I_5 = K_a \lor K_d$ ($a + b \geq 3$), $I_6 = K_a \lor (K_b \cup K_c)$ ($a \geq 1, b, c \geq 2$), $I_7 = K_a \lor (K_b \cup K_c)$ ($a, c \geq 1, b \geq 2$), $I_8 = (K_a \cup K_b) \lor (K_c \cup K_d)$ ($a, b, c, d \geq 2$), $I_9 = (K_a \cup K_b) \lor (K_c \lor K_d)$ ($a, b, c \geq 2, d \geq 1$) and $I_{10} = (K_a \cup K_c) \lor (K_b \lor K_d)$ ($a, c \geq 2, b, d \geq 1$). By Lemma 3.1, we know that $F_3$ cannot be an induced subgraph of $G$, which implies that $G \notin \{I_8, I_9, I_{10}\}$. Hence, we may conclude that $G \in \{I_i \mid 2 \leq i \leq 7\}$ in this situation.

Subcase 1.2 At least one of $G_1$ and $G_2$ contains induced $P_3$.

Without loss of generality, we assume that $G_1$ contains induced $H = P_3 = u_1 u_2 u_3$. Then $G_2$ contains no induced $2K_1$ because $F_6 = P_3 \lor (2K_1)$ cannot be the induced subgraph of $G$ by Lemma 3.1. This implies that $G_2$ is a complete graph $K_a$ ($a \geq 1$).

Now consider the structure of $G_1$. For any vertex $v \in V(G_1) \setminus V(H)$, we claim that $v$ is adjacent to at least one vertex of $H$, since otherwise $F_7$ will be an induced subgraph of $G$, which is impossible by Lemma 3.1. Thus, for any $v \in V(G_1) \setminus V(H)$, we have $G_1[[v_1, v_2, v_3, v]] \notin \{H_1, H_2, H_3, H_4, H_5\}$. Obviously, $G_1[[v_1, v_2, v_3, v]] \neq H_1$ because $G$ contains no induced $P_4$. Furthermore, we see that $G_1[[v_1, v_2, v_3, v]] \neq H_4$ because $G$ cannot contain $F_6$ as its induced subgraph. Thus $G_1[[v_1, v_2, v_3, v]] \in \{H_2, H_3, H_5\}$ and we have the following claim.

Claim 1.1 For any $v \in V(G_1) \setminus V(H)$, $N_{G_1}(v) \cap V(H) = \{v_2\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$ or $\{v_1, v_2, v_3\}$.

Denote by $V_1$, $V_2$, $V_3$ and $V_4$ the sets of $v \in V(G_1) \setminus V(H)$ such that $N_{G_1}(v) \cap V(H) = \{v_2\}$, $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$ and $\{v_2, v_3\}$, respectively. Then $V(G_1) \setminus V(H) = V_1 \cup V_2 \cup V_3 \cup V_4$. Now we begin to analyse the structure of $G_1$.

If $G_1[V_1]$ contains an induced $P_3 = u_1 u_2 u_3$, then $G_1[[v_1, v_2, u_1, u_2, u_3]] = F_7$, which is impossible by Lemma 3.1. This implies that $G_1[V_1]$ is the disjoint union of some complete graphs. Moreover, we see that $G_1[V_1]$ contains no induced $2K_2$.
because $F_4$ is not an induced subgraph of $G$. Therefore, if $V_1 \neq \emptyset$ then $G_1[V_1]$ is one of the following graphs: $K_a$ ($a \geq 2$), $K_a^c$ ($a \geq 1$) and $K_a \cup K_b^c$ ($a \geq 2$, $b \geq 1$).

For any $u, v \in V_2$, if $u$ and $v$ are not adjacent, then $G_1[v_1, v_2, v_3, u, v] = F_7$, a contradiction. Thus $G_1[V_2]$ (if $V_2 \neq \emptyset$) is a complete graph, and so is $G_1[V_4]$ by the symmetry. Similarly, we see that $G_1[V_3]$ (if $V_3 \neq \emptyset$) is also a complete graph because $F_6$ cannot be the induced subgraph of $G$.

For any $u \in V_1$ and $v \in V_2$ (resp. $v \in V_4$), if $u$ and $v$ are adjacent, then $G_1[(v_1, v_2, v_3, u, v)] = F_7$, a contradiction. Thus there are no edges connecting $V_1$ and $V_2 \cup V_4$. Moreover, every vertex of $V_1$ is adjacent to every vertex of $V_3$ again because $F_7$ cannot be the induced subgraph of $G$.

For any $u \in V_2$ (resp. $u \in V_4$) and $v \in V_3$, if $u, v$ are not adjacent, then $G_1[(v_1, v_2, v_3, u, v)] = F_3$, which is a contradiction. Thus every vertex of $V_2 \cup V_4$ is adjacent to every vertex of $V_3$. Moreover, we claim that there are no edges connecting $V_2$ and $V_4$ again because $G$ contains no induced $F_3$.

By the definition of $V_i$ ($1 \leq i \leq 4$), we see that $v_1$ is adjacent to every vertex of $V_2 \cup V_3$ but none of $V_1 \cup V_4$, $v_2$ is adjacent to every vertex of $V_1 \cup V_2 \cup V_3 \cup V_4$, and $v_3$ is adjacent to every vertex of $V_3 \cup V_4$ but none of $V_1 \cup V_2$. Put $V_2' = V_2 \cup \{v_1\}$, $V_3' = V_3 \cup \{v_2\}$ and $V_4' = V_4 \cup \{v_3\}$. Then $V(G_1) = V_1 \cup V_2' \cup V_3' \cup V_4'$.

Summarizing above results, we see that $G_1[V_1]$ (if $V_1 \neq \emptyset$) is of the from $K_a$ ($a \geq 2$), $K_a^c$ ($a \geq 1$) or $K_a \cup K_b^c$ ($a \geq 2$, $b \geq 1$), and $G_1[V_i']$ ($|V_i'| \geq 1$) is a complete graph for $i = 2, 3, 4$; every vertex of $V_3'$ is adjacent to every vertex of $V_1 \cup V_2' \cup V_4'$ and there are no edges connecting $V_1$, $V_2'$ and $V_4'$. Therefore, $G_1 = G_1[V_2'] \cup G_1[V_1 \cup V_2' \cup V_4']$ is of one form listed below: $K_a \cup (K_b \cup K_c)$ with $a, b, c \geq 1$, $K_a \cup (K_b \cup K_c \cup K_d)$ with $a, c, d \geq 1$ and $b \geq 2$, $K_a \cup (K_b \cup K_c \cup K_d \cup K_e)$ with $a, c, d, e \geq 1$ and $b \geq 2$. Considering that $G$ (and so $G_1$) cannot contain $F_4$ as its induced subgraph, we have $G_1 = K_a \cup (K_b \cup K_c)$ with $a, b, c \geq 1$, $K_a \cup (K_b \cup K_c^c)$ with $a \geq 1, b, c \geq 2$ or $K_a \cup K_b^c$ with $a \geq 1, b \geq 3$. Recalling that $G_2$ is a complete graph and $G = G_1 \cup G_2$, we obtain $G = K_a \cup (K_b \cup K_c)$ with $a \geq 2, b, c \geq 1$, $K_a \cup (K_b \cup K_c^c)$ with $a \geq 2, b, c \geq 2$ or $K_a \cup K_b^c$ with $a \geq 2, b \geq 3$. Thus we also have $G \in \{I_i \mid 2 \leq i \leq 7\}$.

Case 2. $d(G) = 3$.

Let $H = P_4 = v_1v_2v_3v_4$ be a diameter path of $G$. Then $H$ is an induced subgraph of $G$ and $D(H) = D_G([v_1, v_2, v_3, v_4])$ is a principal submatrix of $D(G)$. Firstly, we have the following claim.

Claim 2.1 $d(v, H) = 1$ for any $v \in V(G) \setminus V(H)$.

If not, we have $2 \leq d(v, H) \leq 3$ since $d(G) = 3$. Let $d_i = d_G(v, v_i)$ for $i = 1, 2, 3, 4$. Then $d_i \in \{2, 3\}$ for each $i$, and the principal submatrix of $G$ corresponding to $G[[v_1, v_2, v_3, v_4]]$ is of the form

$$D_G([v_1, v_2, v_3, v_4]) = \begin{bmatrix} 0 & 1 & 2 & 3 & d_1 \\ 1 & 0 & 1 & 2 & d_2 \\ 2 & 1 & 0 & 1 & d_3 \\ 3 & 2 & 1 & 0 & d_4 \\ d_1 & d_2 & d_3 & d_4 & 0 \end{bmatrix}.$$
In Table 2, we list the possible values of the second least eigenvalue of $D_G([v_1, v_2, v_3, v_4, v])$, which are all less than $-2$. From Theorem 2.1 we get $\partial_{n-1}(G) \leq \partial_4(D_G([v_1, v_2, v_3, v_4, v])) < -2$, contrary to $\partial_{n-1}(G) \geq -2$. Hence, each vertex in $V(G) \setminus V(H)$ must be adjacent to at least one vertex of $H$.

Note that $d_G(v_1, v_4) = 3$. From Claim 2.1 and the symmetry of $v_1$ and $v_4$ (resp. $v_2$ and $v_3$), for any $v \in V(G) \setminus V(H)$, we can suppose $G[[v_1, v_2, v_3, v_4, v]] \in \{H_6, H_7, H_8, H_9, H_{10}, H_{11}\}$ (see Fig. 3). If $G[[v_1, v_2, v_3, v_4, v]] = H_6$, then $d_G(v, v_1) = 1, d_G(v, v_2) = 2$, and $d_G(v, v_3), d_G(v, v_4) \in \{2, 3\}$. Thus $D(G)$ has $D_G([v_1, v_2, v_3, v_4, v]) \in \{A_1, A_2, A_3, A_4\}$ as its principal submatrix, which is impossible by Lemma 3.2. Similarly, if $G[[v_1, v_2, v_3, v_4, v]] = H_9$, then the corresponding principal submatrix is given by $D_G([v_1, v_2, v_3, v_4, v]) = A_5$, a contradiction. Hence, $G[[v_1, v_2, v_3, v_4, v]] \in \{H_7, H_8, H_{10}, H_{11}\}$ for any $v \in V(G) \setminus V(H)$. Again by considering the symmetry of $v_1$ and $v_4$ (resp. $v_2$ and $v_3$), we have the following claim.
Claim 2.2 For any \( v \in V(G) \setminus V(H), N_G(v) \cap V(H) = \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\} \) or \( \{v_2, v_3, v_4\} \).

Denote by \( V_{11}, V_{12}, V_{21}, V_{22}, V_{31}, V_{32}, V_{41} \) and \( V_{42} \) the sets of \( v \in V(G) \setminus V(H) \) such that \( N_G(v) \cap V(H) = \{v_2\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\} \) and \( \{v_3, v_4\} \), respectively. Let \( V_i = V_{i1} \cup V_{i2} \) for \( i = 1, 2, 3, 4 \). Then \( V(G) \setminus V(H) = V_1 \cup V_2 \cup V_3 \cup V_4 \).

For any \( u, v \in V_{11} \), if \( u \) and \( v \) are adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{12} \) (see Fig. 3), and the corresponding principal submatrix \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \) belongs to \( \{A_6, A_7, A_8, A_9\} \) because \( d_G(u, v_1) = d_G(v, v_1) = d_G(u, v_3) = d_G(v, v_3) = 2, d_G(u, v_2) = d_G(v, v_2) = 1, d_G(u, v_4), d_G(v, v_4) \in \{2, 3\} \) and \( d_G(u, v) = 1 \), which contradicts Lemma 2.7. Thus \( V_{11} \) is an independent set, and so is \( V_4 \) by the symmetry. Similarly, if \( u, v \in V_{12} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{13} \) and the corresponding principal submatrix \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \) belongs to \( \{A_{10}, A_{11}, A_{12}, A_{13}\} \), implying that \( V_{12} \) is a clique and so is \( V_{42} \). Furthermore, if neither \( V_{11} \) nor \( V_{12} \) is empty, then \( H_{14} \) or \( H_{15} \) is an induced subgraph of \( G \), and the corresponding principal submatrix is one of \( \{A_i \mid 14 \leq i \leq 21\} \), a contradiction. Thus at least one of \( V_{11} \) and \( V_{12} \) (resp. \( V_{41} \) and \( V_{42} \)) by the symmetry is empty.

For any \( u \in V_{11} \) and \( v \in V_{21} \), if \( u \) and \( v \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{16} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_{22}, A_{23}, A_{24}, A_{25}\} \), which is impossible and so each vertex of \( V_{11} \) is adjacent to every vertex of \( V_{12} \). Similarly, if \( u \in V_{11} \) and \( v \in V_{22} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{17} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) = \{A_{26}, A_{27}\} \); if \( u \in V_{12} \) and \( v \in V_{21} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{18} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_{28}, A_{29}\} \); if \( u \in V_{12} \) and \( v \in V_{22} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{19} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_{30}, A_{31}\} \). Thus all these cases are impossible, and we conclude that every vertex of \( V_1 = V_{11} \cup V_{12} \) is adjacent to every vertex of \( V_2 = V_{21} \cup V_{22} \), and by the symmetry, every vertex of \( V_4 = V_{41} \cup V_{42} \) is adjacent to every vertex of \( V_3 = V_{31} \cup V_{32} \).

As above, if \( u \in V_1 = V_{11} \cup V_{12} \) and \( v \in V_3 = V_{31} \cup V_{32} \) are adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] \in \{H_{20}, H_{21}, H_{22}, H_{23}\} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_{32}, A_{33}, A_{34}, A_{35}\} \); if \( u \in V_1 = V_{11} \cup V_{12} \) and \( v \in V_4 = V_{41} \cup V_{42} \) are adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] \in \{H_{24}, H_{25}, H_{26}\} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_i \mid 36 \leq i \leq 41\} \). Therefore, there are no edges in \( G \) connecting \( V_1 \) and \( V_3 \cup V_4 \), and symmetrically, there are no edges connecting \( V_2 \) and \( V_1 \cup V_3 \).

For any \( u, v \in V_{21} \), if \( u \) and \( v \) are adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{27} \), and the corresponding principal submatrix is \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) = A_{43}, \) a contradiction. Thus \( V_2 \) is an independent set and so is \( V_3 \) by the symmetry. Similarly, if \( u, v \in V_{22} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] = H_{28} \) and the corresponding principal submatrix \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \) is equal to \( A_{44} \), which implies that \( V_{22} \) is a clique and so is \( V_{32} \). Furthermore, if neither \( V_{21} \) nor \( V_{22} \) is empty, then \( H_{29} \) or \( H_{30} \) is an induced subgraph of \( G \), and the corresponding principal submatrix is \( A_{45} \) or \( A_{46} \). Thus at least one of \( V_{21} \) and \( V_{22} \) (resp. \( V_{31} \) and \( V_{32} \)) by the symmetry is empty.

Also, if \( u \in V_2 = V_{21} \cup V_{22} \) and \( v \in V_3 = V_{31} \cup V_{32} \) are not adjacent, then \( G[\{v_1, v_2, v_3, v_4, u, v\}] \in \{H_{31}, H_{32}, H_{33}\} \) and \( D_G(\{v_1, v_2, v_3, v_4, u, v\}) \in \{A_{47}, A_{48}, A_{49}, A_{50}\} \), implying that every vertex of \( V_2 \) is adjacent to every vertex of \( V_3 \). Moreover, we claim that either \( V_{21} \) or \( V_{31} \) is empty, since otherwise \( H_{34} \) will
be an induced subgraph of \( G \) and the corresponding principal submatrix is \( A_{51} \), a contradiction.

Summarizing above results, we have the following claim.

**Claim 2.3** The graph \( G \) has the properties P1–P4:

(P1) every vertex of \( V_2 \) is adjacent to every vertex of \( V_1 \) and \( V_3 \), and every vertex of \( V_3 \) is adjacent to every vertex of \( V_2 \) and \( V_4 \);

(P2) there are no edges connecting \( V_1 \) and \( V_3 \cup V_4 \), and \( V_2 \) and \( V_4 \);  

(P3) for each \( 1 \leq i \leq 4 \), \( V_i \cup V_{i+1} = \emptyset \) or \( V_i = \emptyset \), and if \( V_i \neq \emptyset \) (resp. \( V_i = \emptyset \)) then \( V_1 \) (resp. \( V_2 \)) is an independent set (resp. clique);

(P4) \( V_1 = \emptyset \) or \( V_3 = \emptyset \).

Not put \( V'_i = V_i \cup \{v_i\} \) for \( i = 1, 2, 3, 4 \). Then \( V(G) = V'_1 \cup V'_2 \cup V'_3 \cup V'_4 \). From the definition of \( V_i = V_1 \cup V_2 \) and \( V'_i \) (\( 1 \leq i \leq 4 \)), we see that \( v_i \) is adjacent to every vertex of \( V_2 \) but none of \( V_i \) (resp. \( v_4 \)) is adjacent to every vertex of \( V'_2 \) (resp. \( V'_3 \)), \( v_2 \) (resp. \( v_3 \)) is adjacent to every vertex of \( V'_1 \cup V_3 \) (resp. \( V'_2 \cup V'_4 \)). Combining this with Claim 2.3, we may conclude that \( G = P_4(G_1, G_2, G_3, G_4) \), where \( G_i = G[V_i] \) is a complete graph or a union of some isolated vertices for \( 1 \leq i \leq 4 \), and \( G_2, G_3 \) cannot be the union of some isolated vertices at the same time if \( |V'_2|, |V'_3| \geq 2 \). By the symmetry of \( V'_1 \) and \( V'_4 \) (resp. \( V'_2 \) and \( V'_3 \)), without loss of generality, we can suppose that \( G \) is one of the following graphs: \( J_1 = P_4[K_a^c, K_b, K_c^c, K_d^e], J_2 = P_4[K_a^c, K_b, K_c, K_d^e], J_3 = P_4[K_a^c, K_b, K_c^c, K_d], J_4 = P_4[K_a^c, K_b^c, K_c, K_d], J_5 = P_4[K_a^c, K_b, K_c, K_d], J_6 = P_4[K_a^c, K_b, K_c^c, K_d] \) and \( J_7 = P_4[K_a, K_b, K_c, K_d] \), where \( a, b, c, d \geq 1 \).

We complete the proof.

**Proposition 3.2** The \( D \)-polynomials of \( I_1–I_7 \) and \( J_1–J_7 \) (see Proposition 3.1) are listed in Table 3.

**Proof** We only show how to obtain the \( D \)-polynomial of \( J_1 \). For the remaining graphs, the methods are similar and so we omit the process of computation.

It is easily seen that \( J_1 = P_4[K_a^c, K_b, K_c^c, K_d^e] \) has the distance divisor matrix

\[
 B_{\Pi} = \begin{bmatrix}
 2(a - 1) & b & 2c & 3d \\
 a & b - 1 & c & 2d \\
 2a & b & 2(c - 1) & d \\
 3a & 2b & c & 2(d - 1)
\end{bmatrix}.
\]

By Lemma 2.7, we have \( \Psi_{J_1}(x) = \det(xI - B_{\Pi})\Phi_{J_1}(x) = \det(xI - D(J_1)) \), where \( \Psi_{J_1}(x) = x^4 + (7 - b - 2c - 2d - 2a)x^3 + (ab - 6b - 10c - 10d - 10a - 5ad + bc - 2bd + 3cd + 18)x^2 + (4ab - 12b - 16c - 16d - 16a - 15ad + 4bc - 8bd + 9cd + 3abd + 8acd + 3bcd + 20)x - 8a - 8b - 8c - 8d + 4ab - 10ad + 4bc - 8bd + 6cd + 6abd + 8acd + 6bcd - 4abc + 3 \), furthermore, from Lemma 2.6 we know that \( -1 \) and \( -2 \) are \( D \)-eigenvalues of \( J_1 \) with multiplicities at least \( b - 1 \) and \( a + c + d - 3 \), respectively. Thus the \( D \)-polynomial of \( J_1 \) is equal to \( \Phi_{J_1}(x) = (x + 1)^{b-1}(x + 2)^{a+c+d-3}\Psi_{J_1}(x) \) since the constructed eigenvectors we use to prove Lemma 2.6 are of the second kind according to Lemma 2.7.

Combining Propositions 3.1 and 3.2, we now give the main result of this section.
Table 3  The $D$-polynomials of $I_1$–$I_7$ and $J_1$–$J_7$

| $G$ | $\Phi_G(x)$ |
|-----|------------|
| $I_1 = K_n$ ($n \geq 3$) | $(x - n + 1)(x + 1)^{n-1}$; |
| $I_2 = K_a \vee K_b$ ($a, b \geq 2$) | $(x + 1)^a - 1(x + 2)^{b-1}[x^2 + (3 - 2b - a)x - 2a - 2b + ab + 2]$; |
| $I_3 = K_a \vee (K_b \cup K_c)$ ($a, b, c \geq 2$) | $(x + 1)^{a+b+c-3}x^3 + (3 - b - c - a)x^2 + (3 - 2b - 2c - 3bc - 2a)x - a - b - c - 3bc + abc + 1$; |
| $I_4 = K_a \vee (K_b \cup K_c)$ ($a, b, c \geq 1$) | $(x + 1)^{a+b-2}(x + 2)^{c-1}[x^3 + (4 - b - 2c - a)x^2 + (ac - 3b - 4c - 3a - 2bc + 5)x - 2a - 2b - 2c + ac - 2bc + abc + 2]$; |
| $I_5 = K_a^c \vee K_b^c$ ($a + b \geq 3$) | $(x + 2)^{a+b-2}[x^2 + (4 - 2b - 2a)x - 4a - 4b + 3ab + 4]$; |
| $I_6 = K_a^c \vee (K_b \cup K_c)$ ($a \geq 1, b, c \geq 2$) | $(x + 1)^{b+c-2}(x + 2)^{a-1}[x^3 + (4 - b - c - 2a)x^2 + (ab - 3b - 3c - 4a + ac - 3bc + 5)x - 2a - 2b - 2c + ab + ac - 6bc + 4abc + 2]$; |
| $I_7 = K_a^c \vee (K_b \cup K_c)$ ($a, c \geq 1, b \geq 2$) | $(x + 1)^{b-1}(x + 2)^{a+c-2}[x^3 + (5 - b - 2c - 2a)x^2 + (ab - 4b - 6c - 6a + 3ac - 2bc + 8)x - 4a - 4b - 4c + 2ab + 3ac - 4bc + 3abc + 4]$; |
| $J_1 = P_4[K_a^c, K_b, K_c, K_d^c]$ | $(x + 1)^{b-1}(x + 2)^{a+c+d-3}[x^4 + (7 - b - 2c - 2d - 2a)x^3 + (ab - 6b - 10c - 10d - 10a - 5ad + bc - 2bd + 3cd + 18)x^2 + (4ab - 12b - 16c - 16d - 16a - 15ad + 4bc - 8bd + 9cd + 3abd + 8acd + 3bcd + 20)x - 8a - 8b - 8c - 8d + 4ab - 10ad + 4bc - 8bd + 6cd + 6abd + 8acd + 6bcd - 4abcd + 8]$; |
| $J_2 = P_4[K_a^c, K_b, K_c, K_d^c]$ | $(x + 1)^{b+c-2}(x + 2)^{a+d-2}[x^4 + (6 - b - c - 2d - 2a)x^3 + (ab - 5b - 5c - 8d - 8a - 2ac - 5ad - 2bd + cd + 13)x^2 + (3ab - 8b - 8c - 10d - 10a - 6ac - 10ad - 6bd + 3cd + abc + 3abd + 3acd + bcd + 12)x - 4a - 4b - 4c - 4d + 2ab - 4ac - 5ad - 4bd + 2cd + 2abc + 3abd + 3acd + 2bcd - abcd + 4]$; |
Table 3 continued

| \(G\) | \(\Phi_G(x)\) |
|---|---|
| \(J_3 = P_4[K_a^c, K_b, K_c^c, K_d]\) | \((x + 1)^{a+b+d-2}(x + 2)^{a+c-2}[x^4 + (6 - b - 2c - d - 2a)x^3 + (ab - 5b - 8c - 5d - 8a - 7ad + bc - 3bd + cd + 13)x^2 + (3ab - 8b - 10c - 8d - 10a - 21ad + 3bc - 12bd + 3cd + 4abd + 8acd + 4bcd + 12)x - 4a - 4b - 4c - 4d + 2ab - 14ad + 2bc - 12bd + 2cd + 8abd + 8acd + 8bcd - 4abcd + 4]\); |
| \(J_4 = P_4[K_a^c, K_b^c, K_c, K_d]\) | \((x + 1)^{a+b+d-2}(x + 2)^{a+b-2}[x^4 + (6 - 2b - c - d - 2a)x^3 + (3ab - 8b - 5c - 5d - 8a - 2ac - 7ad + bc - 2bd + 13)x^2 + (6ab - 10b - 8c - 8d - 10a - 6ac - 21ad + 3bc - 6bd + 3abc + 11abd + acd + bcd + 12)x - 4a - 4b - 4c - 4d + 3ab - 4ac - 14ad + 2bc - 4bd + 3abc + 11abd + 2acd + 2bcd - abcd + 4]; |
| \(J_5 = P_4[K_a^c, K_b, K_c, K_d]\) | \((x + 1)^{b+c+d-3}(x + 2)^{a-1}[x^4 + (5 - b - c - d - 2a)x^3 + (ab - 4b - 4c - 4d - 6a - 2ac - 7ad - 3bd + 9)x^2 + (2ab - 5b - 5c - 5d - 6a - 4ac - 14ad - 9bd + abc + 4abd + acd + bcd + 7)x - 2a - 2b - 2c - 2d + ab - 2ac - 7ad - 6bd + abc + 4abd + acd + 2bcd + 2]; |
| \(J_6 = P_4[K_a, K_b, K_c^c, K_d]\) | \((x + 1)^{a+b+d-3}(x + 2)^{a-1}[x^4 + (5 - b - c - d - a)x^3 + (bc - 4b - 6c - 4d - 2ac - 8ad - 4a - 3bd + cd + 9)x^2 + (2bc - 5b - 6c - 5d - 4ac - 24ad - 5a - 9bd + 2cd + abc + abd + 9acd + 4bcd + 7)x - 2a - 2b - 2c - 2d - 2ac - 16ad + bc - 6bd + cd + abc + 2abd + 9acd + 4bcd + 2]; |
| \(J_7 = P_4[K_a, K_b, K_c, K_d]\) | \((x + 1)^{a+b+c+d-4}[x^4 + (4 - b - c - d - a)x^3 + (6 - 3b - 3c - 3d - 3ac - 8ad - 3bd - 3a)x^2 + (abc - 3b - 3c - 3d - 6ac - 16ad - 6bd - 3a - abd + acd + bcd + 4)x - a - b - c - d - 3ac - 8ad - 3bd + abc + abd + acd + bcd + abcd + 1]. |
Theorem 3.1 Let $G$ be a connected graph on $n \geq 3$ vertices. Then $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$ if and only if

(1) $d(G) \leq 2$ and $G \in \{I_i \mid 1 \leq i \leq 7\}$, where $I_1 = K_n$ ($n \geq 3$), $I_2 = K_a \cup K_b^c (a, b \geq 2)$, $I_3 = K_a \cup (K_b \cup K_c) (a, b, c \geq 2)$, $I_4 = K_a \cup (K_b \cup K_c^c) (a, b \geq 2, c \geq 1)$, $I_5 = K_a^c \cup K_b^c (a + b \geq 3)$, $I_6 = K_a^c \cup (K_b \cup K_c) (a \geq 1, b, c \geq 2)$ and $I_7 = K_a^c \cup (K_b \cup K_c^c) (a, c \geq 1, b \geq 2)$; or

(2) $d(G) = 3$ and

(2.1) $G = J_1 = P_4[K_a^c, K_b, K_c^c, K_d^c]$, where $b \geq 1$, and $a = c = 1$, $d \geq 1$ or $a = 1$, $c = 2$, $d \leq 2$ or $a = 1$, $c \geq 3$, $d = 1$ or $a = 2$, $c = 1$, $d \leq 2$ or $a \geq 3$, $c = 1$, $d = 1$; or

(2.2) $G = J_2 = P_4[K_a^c, K_b, K_c, K_d^c]$, where $b, c \geq 1$, and $a = 1$, $d \geq 1$ or $a = 2$, $d \leq 2$ or $a \geq 3$, $d = 1$; or

(2.3) $G = J_3 = P_4[K_a^c, K_b, K_c^c, K_d]$, where $b, d \geq 1$, and $a = 1$, $c \geq 1$ or $a \geq 2$, $c = 1$; or

(2.4) $G = J_4 = P_4[K_a^c, K_b^c, K_c, K_d^c]$, where $c, d \geq 1$, and $a = 1$, $b \geq 1$ or $a = 2$, $b \leq 2$ or $a \geq 3$, $b = 1$; or

(2.5) $G = J_5 = P_4[K_a^c, K_b, K_c, K_d^c]$, where $a, b, c, d \geq 1$; or

(2.6) $G = J_6 = P_4[K_a, K_b, K_c^c, K_d^c]$, where $a, b, c, d \geq 1$; or

(2.7) $G = J_7 = P_4[K_a, K_b, K_c, K_d]$, where $a + b + c + d - 3ac - 8ad - 3bd - abc - abd - acd - bcd + abcd + 1 \leq 0$.

Proof According to Proposition 3.1, to determine the graphs with $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$, it suffices to identify such graphs from $\{I_i, J_i \mid 1 \leq i \leq 7\}$ by using Proposition 3.2. Here we only check the graphs $I_7, J_1$ and $J_7$, and the remaining graphs could be checked in a similar way and so the detail is omitted.

First suppose $G = I_7 = K_a^c \cup (K_b \cup K_c^c) (a, c \geq 1, b \geq 2)$. Then the D-polynomial of $G$ is equal to $\Phi_G(x) = (x + 1)^{b-1}(x + 2)^{a+c-2}\Psi_G(x)$ (see Table 3), where $\Psi_G(x) = x^3 + (5 - b - 2c - 2a)x^2 + (ab - 4b - 6c - 6a + 3ac - 2bc + 8)x - 4a - 4b - 4c + 2ab + 3ac - 4bc + 3abc + 4$. Let $\alpha_1 > \alpha_2 > \alpha_3$ be the three zeros of $\Psi_G(x)$. Note that $\alpha_1 = \partial_1(G) > 0$ by Lemma 2.7. Since $\Psi_G(-1) = ab - b^2 + 2bc > 0$ and $\Psi_G(-2) = 3abc - 3ac > 0$, we have $\alpha_1 > \alpha_2 > -1$ and $\alpha_3 < -2$, which implies that $\partial_3(G) \leq -1$ and $\partial_{n-1}(G) \geq -2$.

Next suppose $G = J_1 = P_4[K_a^c, K_b, K_c^c, K_d^c]$, where $a, b, c, d \geq 1$. The D-polynomial of $G$ is $\Phi_G(x) = (x + 1)^{b-1}(x + 2)^{a+c+d-3}\Psi_G(x)$ (see Table 3), where $\Psi_G(x) = x^4 + (7 - b - 2c - 2d - 2a)x^3 + (ab - 6b - 10c - 10d - 10a - 5ad + bc - 2bd + 3cd + 18)x^2 + (4ab - 12b - 16c - 16d - 16a - 15ad + 4bc - 8bd + 9cd + 3abd + 8acd + 3bcd + 20)x - 8a - 8b - 8c - 8d + 4ab - 10ad + 4bc - 8bd + 6cd + 6abd + 8acd + 6bcd - 4abcd + 8$. Let $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$ be the four zeros of $\Psi_G(x)$. Note that $\alpha_1 = \partial_1(G) > 0$, and $\alpha_4 = \partial_{n-1}(G) \leq -3$ by Lemma 2.1. Also note that $\alpha_2 = \partial_2(G) \geq 1 - \sqrt{3} > -1$ by Lemma 2.5 since $G$ is not compete. By simple computation, we get $\Psi_G(-2) = -8acd - 4abcd < 0$, which implies that $\alpha_2 > -2$ since we have obtained $\alpha_1 > \alpha_2 > -1$ and $\alpha_4 \leq -3$, and so $\partial_{n-1}(G) \geq -2$.

Furthermore, we see that $\partial_3(G) \leq -1$ if and only if $\alpha_3 \leq -1$, which is the case if and only if $\Psi_G(-1) = ab - b + bc - 2bd + 3abd + 3bcd - 4abcd > 0$ by above arguments. Now it suffices to determine those $a, b, c, d$ such that $\partial_3(G) \leq -1$. If $a, c \geq 2$, then $D(P_4[2K_1, K_2, 2K_1, K_1])$ is a principal submatrix of $D(G)$, which implies that
−0.8990 = \partial_3(\mathcal{P}_4[2K_1, K_1, 2K_1, K_1]) \leq \partial_3(G) \text{ by Theorem 2.1, a contradiction.}

Then we can suppose that \( a = 1 \) or \( c = 1 \). If \( a = 1 \), then \( \Psi_G(-1) = bc + bd – bcd \), and so \( \Psi_G(-1) \geq 0 \) if and only if \( b \geq 1 \), and \( c = 1, d \geq 1 \) or \( c = 2, d \leq 2 \) or \( c \geq 3, d = 1 \) by simple computation. Similarly, if \( c = 1 \), then \( \Psi_G(-1) = ab + bd – abd \geq 0 \) if and only if \( b \geq 1 \), and \( a = 1, d \geq 1 \) or \( a = 2, d \leq 2 \) or \( a \geq 3, d = 1 \). Combining above results, if \( G = J_1 = \mathcal{P}_4[K\alpha, K\beta, K\gamma, K\delta] \), then \( \partial_3(G) \leq -1 \) and \( \partial_{n-1}(G) \geq -2 \) if and only if \( b \geq 1 \), and \( a = c = 1, d \geq 1 \) or \( a = 1, c = 2, d \leq 2 \) or \( a = 1, c \geq 3, d = 1 \); \( a = 1, c = 2, d \leq 2 \) or \( a \geq 3, c = 1, d = 1 \).

Finally, we suppose \( G = J_7 = \mathcal{P}_4[K\alpha, K\beta, K\gamma, K\delta] \), where \( a, b, c, d \geq 1 \). Then \( \Phi_G(x) = (x + 1)^a b + c + d - 4 \psi_G(x) \), where \( \psi_G(x) = x^4 + (4 - b - c - d - a)x^3 + (6 - 3b - 3c - 3d - 3ac - 8ad - 3bd - 3a)x^2 + (abc - 3b - 3c - 3d - 6ac - 16ad - 6bd - 3a + abd + acd + bcd + 4a)x - a - b - c - d - 3ac - 8ad - 3bd + abc + abd + acd + bcd + abcd + 1 \). Let \( \alpha_1 > \alpha_2 \geq \alpha_3 \geq \alpha_4 \) be the four zeros of \( \psi_G(x) \). As above, we have \( \alpha_1 = \partial_1(G) > 0 \), \( \alpha_2 = \partial_2(G) > -1 \) and \( \alpha_4 = \partial_4(G) \leq -3 \). By simple computation, we have \( \Psi_G(-1) = abcd + 0 \), which implies that \( \alpha_3 < -1 \), and so \( \partial_3(G) \leq -1 \). Moreover, we see that \( \partial_{n-1}(G) \geq -2 \) if and only if \( \alpha_3 \geq -2 \), which is the case if and only if \( \psi_G(-2) = a + b + c + d - 3ac - 8ad - 3bd + abc + abd - acd - bcd + abcd + 1 \) \( \leq 0 \) by above arguments. Therefore, we have \( \partial_3(G) \leq -1 \) and \( \partial_{n-1}(G) \geq -2 \) if and only if \( a + b + c + d - 3ac - 8ad - 3bd + abc + abd - acd - bcd + abcd + 1 \) \( \leq 0 \).

**Remark 3.1** To investigate whether the graphs with \( \partial_3(G) \leq -1 \) and \( \partial_{n-1}(G) \geq -2 \) are DDS, it remains to compare the \( D \)-polynomials of \( I_1-I_7 \) and \( J_1-J_7 \) according to Theorem 3.1. The process of computation is complicated and tedious, so we do not discuss the DDS-property of these graphs in this paper. Indeed, there exist some non-isomorphic \( D \)-cospectral graphs belonging to this class.

### 4 Graphs with at Most Three \( D \)-Eigenvalues Different from \(-1\) and \(-2\)

For a connected graph \( G \) on \( n \) vertices, we denote by \( m_G(\partial) \) the multiplicity of \( \partial \) as a \( D \)-eigenvalue of \( G \). In this section, we focus on characterizing the graphs with at most three \( D \)-eigenvalues different from \(-1\) and \(-2\), that is, the graphs with \( m_G(-1) + m_G(-2) \geq n - 3 \), which gives new families of graphs with few distinct \( D \)-eigenvalues. Clearly, we have \( m_G(-1) + m_G(-2) \leq n - 1 \). If \( m_G(-1) + m_G(-2) = n - 1 \), then \( \partial_2(G) \leq -1 < 1 - \sqrt{3} \), implying that \( G \) is the complete graph \( K_n \) by Lemma 2.5. Thus it suffices to determine those graphs with \( m_G(-1) + m_G(-2) \in \{n - 2, n - 3\} \).

**Theorem 4.1** Let \( G \) be a connected graph on \( n \geq 4 \) vertices. Then \( m_G(-1) + m_G(-2) = n - 2 \) if and only if \( G = K_{s,n-s} \) \((1 \leq s \leq n-1)\) or \( G = K_{s}^{\gamma} \lor K_{n-s}^{\gamma} \) \((2 \leq s \leq n-2)\).

**Proof** Clearly, \( G \) is not a complete graph due to \( m_G(-1) \leq n - 1 \). We consider the following three cases.

**Case 1.** \( m_G(-1) = n - 2 \) and \( m_G(-2) = 0 \).
By Lemma 2.1, we have $\partial_n(G) \leq -2$ because $d(G) \geq 2$. This implies that $\partial_2(G) = -1$ because $\partial_1(G) > 0$ and $m_G(-1) = n - 2 > 0$, and thus $G$ is a complete graph by Lemma 2.5, which is a contradiction.

Case 2. $m_G(-1) = 0$ and $m_G(-2) = n - 2$.

In this situation, we can suppose that $\text{Spec}_D(G) = \{\alpha, \beta, [-2]^{n-2}\}$ with $\alpha > \beta > -2$ or $\text{Spec}_D(G) = \{\alpha, [-2]^{n-2}, \beta\}$ with $\alpha > -2 > \beta$. For the former, we have $\partial_n(G) = -2$ and so $G$ is a complete bipartite graph $K_{s,n-s}$ ($1 \leq s \leq n-1$) according to Lemma 2.2. Conversely, it is easy to verify that $-1$ is not a $D$-eigenvalue of $K_{s,n-s}$ due to $n \geq 4$. For the later, we have $\partial_2(G) = -2 < 1 - \sqrt{3}$, and so $G$ is a complete graph, which is impossible.

Case 3. $m_G(-1) \geq 1$, $m_G(-2) \geq 1$ and $m_G(-1) + m_G(-2) = n - 2$.

In this situation, the $D$-spectrum of $G$ has three possible forms, i.e., $\text{Spec}_D(G) = \{\alpha, \beta, [-1]^{m_1}, [-2]^{m_2}\}$ with $\alpha > \beta > -1$, $\text{Spec}_D(G) = \{\alpha, [-1]^{m_1}, \beta, [-2]^{m_2}\}$ with $\alpha > -1 > \beta > -2$ or $\text{Spec}_D(G) = \{\alpha, [-1]^{m_1}, [-2]^{m_2}, \beta\}$ with $\alpha > -1 > -2 > \beta$, where $m_1 = m_G(-1) \geq 1$, $m_2 = m_G(-2) \geq 1$ and $m_1 + m_2 = n - 2$.

We claim that the last two forms cannot occur since otherwise we have $\partial_2(G) = -1$, which is impossible because $G$ cannot be a complete graph. For the first form, we have $\partial_n(G) = -2$, and so $G$ is a complete $(n - m_2)$-partite $(n - m_2 \geq 3)$ graph according to Lemma 2.2. Moreover, we claim that $G$ cannot contain $K_{2,2,1} = F_6$ (see Fig. 2) as its induced subgraph by Lemma 3.1 since $\partial_3(G) = -1$. Thus we may conclude that $G = K_{g_1}, \ldots, K_{g_s}$, where $s = m_2 + 1 \in [2, n - 2]$ because we have known that $G$ is a complete $(n - m_2)$-partite graph. Conversely, as in Proposition 3.2, one can easily check that $\text{Spec}_D(K_{g_1}^c \cup K_{n-s}) = \{\alpha,\beta,[-1]^{m_1-1},[-2]^{m_2-1}\}$, where $\alpha, \beta$ are the two zeros of $x^2 - (n + s - 3)x - s^2 + sn - 2(n - 1)$ satisfying $\alpha > \beta > -1$ due to $2 \leq s \leq n - 2$.

We complete the proof.

**Theorem 4.2** Let $G$ be a connected graph with $n \geq 5$ vertices. Then $m_G(-1) + m_G(-2) = n - 3$ if and only if $G$ is one of the following graphs: $K_a \cup (K_b \cup K_c)$ where $a + b + c \geq 5$ and $b + c \geq 3$; $K_{a,b,c}$ where $a + b + c \geq 5$; $K_a^c \cup K_b^c \cup K_c^c$ where $a, b, c \geq 2$; $I_4 = K_a^c \cup (K_b \cup K_c)$ where $a, b, c \geq 2$; $I_7 = K_a^c \cup (K_b \cup K_c)$ where $a + c \geq 3$ and $b \geq 2$; $J_1 = P_4[K_a^c, K_b, K_c, K_d]$ where $b \geq 1$ and $a = 2; d = 2$ or $a = 2, c = 1, d = 2$; $J_2 = P_4[K_a^c, K_b, K_c, K_d]$ where $b, c \geq 1$ and $a = d = 2$; $J_4 = P_4[K_a^c, K_b^c, K_c, K_d]$ where $a + b + c + d \geq 5$ and $a + b + c + d - 3ac - 8ad - 3bd - abc - abd - acd - bcd + cbc + 1 = 0$.

**Proof** Clearly, $G$ is not a complete graph due to $m_G(-1) < n - 1$. We consider the following three cases.

Case 1. $m_G(-1) = n - 3$ and $m_G(-2) = 0$.

Since $\partial_n(G) \leq -2$ and $\partial_1(G) > 0$, we can suppose that $\text{Spec}_D(G) = \{\alpha, [-1]^{n-3}, \beta, \gamma\}$ with $\alpha > -1 > \beta \geq \gamma$ or $\text{Spec}_D(G) = \{\alpha, \beta, [-1]^{n-3}, \gamma\}$ with $\alpha > \beta > -1 > \gamma$. Note that $G$ is not a complete graph. The former case cannot occur, and the later case implies that $\partial_{n-1}(G) = -1$ and so $G = K_a \cup (K_b \cup K_c)$ $(a, b, c \geq 1$ and $a + b + c = n \geq 5)$ by Lemma 2.3. Conversely, it is easy to check that $-1$ is a $D$-eigenvalue of $K_a \cup (K_b \cup K_c)$ with multiplicity $n - 3$, and $-2$ is a $D$-eigenvalue.
of \( K_a \lor (K_b \cup K_c) \) if and only if \( b = c = 1 \). Therefore, in this situation, we obtain that \( G = K_a \lor (K_b \cup K_c) \), where \( a + b + c = n \geq 5 \) and \( b + c \geq 3 \).

**Case 2.** \( m_G(-1) = 0 \) and \( m_G(-2) = n - 3 \).

By Lemma 2.5, we see that \(-2\) cannot be the second largest \( D\)-eigenvalue of \( G \).

Thus it suffices to consider the following two situations.

**Subcase 2.1** \( \text{Spec}_D(G) = \{ \alpha, \beta, \gamma, [-2]^{n-3} \} \), where \( \alpha > \beta \geq \gamma > -2 \).

Since \( \partial_n(G) = -2 \) with multiplicity \( n - 3 \), from Lemma 2.2 we have \( G = K_{a,b,c} \), where \( a + b + c = n \geq 5 \). Also, it is easy to check that \(-1\) cannot be a \( D\)-eigenvalue of \( K_{a,b,c} \) because \( a + b + c > 3 \), and so our result follows.

**Subcase 2.2** \( \text{Spec}_D(G) = \{ \alpha, \beta, [-2]^{n-3}, \gamma \} \), where \( \alpha > \beta > -2 > \gamma \).

In this situation, we have \( \partial_3(G) = -2 \). First we claim that \( G \) contains no induced \( P_4 \). If not, let \( P_4 = v_1v_2v_3v_4 \) be an induced subgraph of \( G \). Then \( 2 \leq d_G(v_1, v_4) \leq 3 \).

If \( d_G(v_1, v_4) = 3 \), then \( D(P_4) \) is a principal submatrix of \( D(G) \), and so \(-1.1623 = \partial_3(P_4) \leq \partial_3(G) \leq -2 \) by Theorem 2.1, a contradiction. If \( d_G(v_1, v_4) = 2 \), then one of \( [F_1, F_2, F_3] \) is the induced subgraph of \( G \) (see Fig. 2), which is impossible because \( \partial_3(F_i) > -2 \) for \( i = 1, 2, 3 \). Thus \( G \) contains no induced \( P_4 \), and we can suppose \( G = G_1 \lor G_2 \) by Lemma 2.9. Moreover, we conclude that both \( G_1 \) and \( G_2 \) contain no edges since \( G \) contains no induced \( K_3 \) due to \( \partial_3(K_3) = -1 > -2 = \partial_3(G) \).

Then \( G \) is a complete bipartite graph, and so \( \partial_n(G) = -2 \) by Lemma 2.2, which contradicts \( \partial_n(G) = \gamma < -2 \). Therefore, there are no graphs satisfying \( \text{Spec}_D(G) = \{ \alpha, \beta, [-2]^{n-3}, \gamma \} \), where \( \alpha > \beta > -2 > \gamma \).

**Case 3.** \( m_G(-1) \geq 1, m_G(-2) \geq 1 \) and \( m_G(-1) + m_G(-2) = n - 3 \).

By Lemma 2.5 we know that \( \partial_2(G) \neq -1 \) because \( G \) is not complete. Thus we only need to consider the following three cases.

**Subcase 3.1** \( \text{Spec}_D(G) = \{ \alpha, \beta, \gamma, [-1]^{m_1}, [-2]^{m_2} \} \), where \( \alpha > \beta \geq \gamma > -1 \) and \( m_1, m_2 \geq 1 \).

Since \( \partial_n(G) = -2 \), from Lemma 2.2 we obtain that \( G \) is a complete \((n - m_2)\)-partite \((n - m_2) \geq 4\) graph. Furthermore, we claim that \( G \) cannot contain \( K_{2,2,2,1} \) as its induced subgraph since otherwise we have \(-0.8730 = \partial_4(K_{2,2,2,1}) \leq \partial_4(G) = -1 \) by Lemma 2.4, which is a contradiction. Thus we may conclude that \( G = K_{a,b,\ldots,\ldots} = (K_a^c \lor K_b^c) \lor K_c \), where \( a + b = m_2 + 2 \in [3, n - 2] \) and \( c = n - a - b \in [2, n - 3] \) because we have known that \( G \) is a complete \((n - m_2)\)-partite graph. By simple computation, we obtain \( \Phi_G(x) = (x + 2)^{a+b-2}(x+1)^{c-1}\Psi_G(x) \), where \( \Psi_G(x) = x^3 + (5 - 2b - c - 2a)x^2 + (3ab - 6b - 4c - 6a + ac + bc + 8)x - 4a - 4b - 4c + 3ab + 2ac + 2bc - abc + 4 \).

Let \( \alpha_1, \alpha_2, \alpha_3 \) be the three zeros of \( \Psi_G(x) \). Then \( \alpha_1 > 0 \) and \( \alpha_2 > -1 \) because \( G \) is not complete. Also note that \( \Psi_G(-2) = -3ab - abc < 0 \) and \( \Psi_G(-1) = -(a - 1)(b - 1)c \leq 0 \). Then we have \( \alpha_3 > -1 \) if and only if \( a, b \geq 2 \).

Therefore, in this situation, we obtain that \( G = (K_a^c \lor K_b^c) \lor K_c \), where \( a, b, c \geq 2 \). As a result, we have that \( G = K_{s,1,\ldots,\ldots} = K_s^c \lor K_{n-s} \), where \( s = m_2 + 1 \in [2, n - 3] \) because we have known that \( \text{Spec}_D(K_s^c \lor K_{n-s}) = \{ \alpha, \beta, [-1]^{n-s-1}, [-2]^{s-1} \} \), contrary to \( m_1 + m_2 = n - 3 \). Thus there are no graphs in this situation.
Subcase 3.3 \( \text{Spec}_D(G) = \{\alpha, \beta, [-1]^{m_1}, [-2]^{m_2}, \gamma\} \), where \( \alpha > \beta > -1 > -2 > \gamma \) and \( m_1, m_2 \geq 1 \).

In this situation, we have \( \partial_3(G) = -1 \) and \( \partial_{n-1}(G) = -2 \). Then \( G \) is one of the graphs listed in Theorem 3.1. Therefore, it suffices to select from Theorem 3.1 those graphs whose \( \gamma \) with the help of Proposition 3.2, one can easily check that all the required graphs are: \( I_4 = K^c_{a} \cup (K^c_b \cup K^c_c) \) with \( a, b, c \geq 2 \); \( I_6 = K^c_{a} \cup (K^c_b \cup K^c_c) \) with \( a, b, c \geq 2 \); \( I_7 = K^c_{a} \cup (K^c_b \cup K^c_c) \) with \( a+c \geq 3 \) and \( b \geq 2 \); \( J_1 = P_{a}[K^c_{a}, K^c_{b}, K^c_{c}, K^c_{d}] \) with \( b \geq 1 \) and \( a = 1, c = 2, d = 2 \) or \( a = 2, c = 1, d = 2 \); \( J_2 = P_{a}[K^c_{a}, K^c_{b}, K^c_{c}, K^c_{d}] \) with \( c, d \geq 1 \) and \( a = b = 2 \); \( J_4 = P_{a}[K^c_{a}, K^c_{b}, K^c_{c}, K^c_{d}] \) with \( a + b + c + d \geq 5 \) and \( a + b + c + d - 3ac - 8ad - 3bd - abc - abd - acd - bcd + abcd + 1 = 0 \).

We complete the proof.

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