A recurring pattern in natural numbers of a certain property

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Abstract

Natural numbers satisfying an unusual property are mentioned in [1], in which their infinity is also proved. In this paper, we start with an arbitrary natural number which is not a multiple of 10 and non-palindromic, form numbers by repeating its decimal digits, and investigate which of them have the unusual property. In particular, the pattern of which of them have the unusual property recurs.

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1 Introduction

An unusual property which some natural numbers, e.g. 198, satisfy are defined in [1]. We see that

\[ 198 = 2 \cdot 3^2 \cdot 11, \]
\[ 891 = 3^4 \cdot 11, \]

and

\[ 2 + (3 + 2) + 11 = (3 + 4) + 11. \]

That is, the sum of the numbers appearing in the prime factorizations of the two numbers are equal. Notice that the exponents 1 “does not appear”. In general, the definition is, that a natural number \( n \) has this property if \( 10 \nmid n \), \( n \) is non-palindromic, and that the sum of the numbers appearing in the prime factorization of \( n \) is equal to that of the number formed by reversing its decimal digits. In [1], the infinitude of such numbers are proved, in particular

\[ 18, 1818, 181818, \ldots \]
all have this property. In this paper, we start with an arbitrary non-palindromic natural number \(10 \nmid n\), form, like in (1.1), numbers by repeating its decimal digits, and show that there is a recurring pattern in which of them have this property. More precisely, whether one of them have this property depends only on the number of times the digits of \(n\) are repeated to form it modulo some natural number.

2 Definition of the unusual property

In this section we will recall the definition in [1] of the unusual property. In the following,

\[
\mathbb{N}_{\neq 10} = \{n \in \mathbb{N} : 10 \nmid n\},
\]

\[
\mathbb{Z}_{\geq 0} = \{z \in \mathbb{Z} : z \geq 0\}.
\]

**Definition 2.1.** For \(n \in \mathbb{N}_{\neq 10}\) with decimal representation \(n = d_{k-1} \ldots d_1 d_0\), we put

\[ r(n) = d_0 d_1 \ldots d_k. \]

That is, \(r(n)\) is the number formed by writing the decimal digits of \(n\) in reverse order. Hence we have \(r : \mathbb{N}_{\neq 10} \to \mathbb{N}_{\neq 10}\). We define \(n\) to be palindromic if \(n = r(n)\).

**Definition 2.2.** We put

- \(v(p) = p\) for \(p\) a prime,
- \(v(p^e) = p + e\) for \(p\) a prime and \(e \geq 2\),

and insist that \(v : \mathbb{N} \to \mathbb{Z}_{\geq 0}\) be an additive arithmetic function. If we put

\[ \iota(e) = \begin{cases} 0 & (e = 1) \\ e & (e \geq 2), \end{cases} \]

then we may combine the above two points and just put

\[ v(p^e) = p + \iota(e). \]

Let \(n \geq 2\) be a natural number with prime factorization

\[ n = p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m}. \] (2.1)

Then

\[ v(n) = \sum_{i=1}^{m} v(p_i^{e_i}) = \sum_{i=1}^{m} (p_i + \iota(e_i)). \]

Hence \(v(n)\) is the sum of the numbers appearing in the prime factorization of \(n\), not counting exponents which are 1.

We may now define the unusual property, which we call \(v\)-palindromic:

**Definition 2.3.** A natural number \(n\) is \(v\)-palindromic if \(n \in \mathbb{N}_{\neq 10}\), \(n \neq r(n)\), and \(v(n) = v(r(n))\).

It is clear that if \(n\) is \(v\)-palindromic then so is \(r(n)\). As noted in the Introduction, 198 and the numbers (1.1) are \(v\)-palindromic numbers. In the next section we shall state our main theorem.
3 Statement of the main theorem

In this section we shall define some notations to state our main theorem.

Definition 3.1. For $c, k \geq 1$, put

$$\rho_{c,k} = \overbrace{10\ldots01\ldots01\ldots01}^{c \times (k-1) \times (k-1) \times (k-1)},$$

meaning that 1 appears $c$ times and that between each consecutive pair of them 0 appears $k - 1$ times.

It is clear that if $n$ is a $k$-digit number then the number formed by repeating its digits $c$ times is just $n\rho_{c,k}$. We may now state our main theorem:

Theorem 3.1. Let $n$ be a natural number with $k$ digits and with $n \in \mathbb{N}_{\geq 10}$ and $n \neq r(n)$. Then there exists a natural number $\omega > 0$ such that for every $c \geq 1$, $n\rho_{c,k}$ is $v$-palindromic if and only if $n\rho_{c+\omega,k}$ is. In other words, whether $n\rho_{c,k}$ is $v$-palindromic depends only $c$ modulo $\omega$.

We make the following definition based on the truth of the above theorem:

Definition 3.2. A natural number $\omega > 0$ satisfying the condition of the above theorem is called a period of $n$ and denoted $\omega(n)$. If there exists a $c \geq 1$ such that $n\rho_{c,k}$ is $v$-palindromic, the smallest one is called the order of $n$ and denoted $c(n)$.

We have the following:

Theorem 3.2. The set of all periods of $n$ is $\{q\omega(n) : q \in \mathbb{N}\}$.

We prove our main theorem in Section 7. Before that, we need some preparation. In Section 4 we investigate some divisibility properties of the numbers $\rho_{c,k}$. In Section 6.1 we first consider the case $n = 819$ of the main theorem; the proof of the main theorem is essentially a generalization of this.

4 Divisibility properties of $\rho_{c,k}$

We consider the divisibility of the numbers $\rho_{c,k}$ by prime powers $p^a$. Recall that $\text{ord}_p(n)$ is the largest integer $\beta$ with $p^\beta | n$. We have:

Lemma 4.1. Let $p^a$ be a prime power, with $p \neq 2, 5$. Let $k \geq 1$, let $\beta = \text{ord}_p(10^k - 1)$, and let $h$ be the order of $10^k$ regarded as an element of $(\mathbb{Z}/p^{a+\beta}\mathbb{Z})^\times$. Then $h > 1$ and for $c \geq 1$, $p^a | \rho_{c,k}$ if and only if $h \mid c$. 

Proof. We first show that $h > 1$. That $h = 1$ means that

$$10^k \equiv 1 \pmod{p^{\alpha+\beta}} \iff p^{\alpha+\beta} \mid 10^k - 1 \iff p^{\alpha+\text{ord}_p(10^k-1)} \mid 10^k - 1,$$

which cannot be. Whence $h > 1$. We have

$$(10^k - 1)\rho_{c,k} = (10^k - 1) \sum_{i=0}^{c-1} 10^{ki} = 10^{kc} - 1.$$  

As $\beta = \text{ord}_p(10^k - 1)$,

$$p^{\alpha} \mid \rho_{c,k} \iff 10^{kc} - 1 \equiv 0 \pmod{p^{\alpha+\beta}} \iff 10^{kc} \equiv 1 \pmod{p^{\alpha+\beta}} \iff h \mid c,$$

where the last $\iff$ is due to the structure of cyclic subgroups. \[ \square \]

Remark 4.2. In Lemma 4.1, if $p = 2, 5$, then $10^k$ cannot be regarded as an element of $(\mathbb{Z}/p^{\alpha+\beta}\mathbb{Z})^\times$. But obviously for every $c \geq 1$, $p^{\alpha} \nmid \rho_{c,k}$. Also, let us denote the $h$ in the lemma by $h_{p^{\alpha},k}$.

Using Mathematica we have:

| $p^{\alpha}$ | 7 | 7² | 13 | 13² | 17 | 17² |
|-------------|---|----|----|-----|----|-----|
| $h_{p^{\alpha},3}$ | 2 | 14 | 2 | 26 | 16 | 272 |

Regarding divisibility in general, not just for $\rho_{c,k}$, we recall that:

Lemma 4.3. Let $n$ be a natural number, let $p$ be a prime, and let $g = \text{ord}_p(n)$.

1. $g = 0 \iff p \nmid n$,
2. $g = 1 \iff p \mid n$ and $p^2 \nmid n$,
3. $g \leq 1 \iff p^2 \nmid n$,
4. $g \geq 1 \iff p \mid n$,
5. $g \geq 2 \iff p^2 \mid n$.

We will need this lemma later.
5 The functions $\varphi_{p,\delta}$

Fix a prime $p$, the sequence of powers of $p$ is

$$1, p, p^2, \ldots, p^\alpha, \ldots$$

Applying $v$ to them yields

$$0, p, p + 2, \ldots, p + \alpha, \ldots$$

Now we take differences of consecutive terms to get

$$p, 2, 1, \ldots, 1, \ldots$$  \hspace{1cm} (5.1)

with all 1’s from the third term onwards. We give notation for terms of this sequence

**Definition 5.1.** For a prime $p$ and integer $\alpha \geq 0$, put

$$\varphi_{p,1}(\alpha) = v(p^\alpha+1) - v(p^\alpha).$$

In this notation then, the sequence (5.1) is $(\varphi_{p,1}(\alpha))_{\alpha=0}^{\infty}$. More generally we define:

**Definition 5.2.** For a prime $p$, an integer $\alpha \geq 0$, and a $\delta \geq 1$, put

$$\varphi_{p,\delta}(\alpha) = v(p^{\alpha+\delta}) - v(p^\alpha).$$

In this notation, for instance, the sequence $(\varphi_{p,3}(\alpha))_{\alpha=0}^{\infty}$ is

$$p + 3, 4, 3, \ldots, 3, \ldots$$  \hspace{1cm} (5.2)

with all 3’s from the third term onwards. More generally, for $\delta \geq 2$, the sequence $(\varphi_{p,\delta}(\alpha))_{\alpha=0}^{\infty}$ is just

$$p + \delta, \delta + 1, \delta, \ldots, \delta, \ldots$$

We may view, for a prime $p$ and $\delta \geq 1$, $\varphi_{p,\delta} : \mathbb{Z}_{\geq 0} \to \mathbb{N}$ as a function of $\alpha \in \mathbb{Z}_{\geq 0}$.

Rephrasing (5.1) and (5.2), the values of $\varphi_{p,\delta}$ may be summarized as follows.

$$\varphi_{2,1}(\alpha) = \begin{cases} 2 & (\alpha = 0, 1) \\ 1 & (\alpha \geq 2) \end{cases}$$  \hspace{1cm} (5.3)

$$\varphi_{p,1}(\alpha) = \begin{cases} p & (\alpha = 0) \\ 2 & (\alpha = 1) \\ 1 & (\alpha \geq 2) \end{cases}$$  \hspace{1cm} (5.4)

and

$$\varphi_{p,\delta}(\alpha) = \begin{cases} p + \delta & (\alpha = 0) \\ \delta + 1 & (\alpha = 1) \\ \delta & (\alpha \geq 2) \end{cases}$$  \hspace{1cm} (5.5)

Where we have deliberately distinguished between the cases where the values are distinct.

We give a notation for the ranges of $\varphi_{p,\delta}$:
Definition 5.3. For a prime \( p \) and \( \delta \geq 1 \) put \( R_{p,\delta} = \varphi_{p,\delta}(\mathbb{Z}_{\geq 0}) \).

Remark 5.1. In view of (5.3), (5.4), and (5.5), it is clear that \( |R_{2,1}| = 2 \) and \( |R_{p,\delta}| = 3 \) otherwise. Also, any nonempty fiber of \( \varphi_{p,\delta} \) is one of

\[
\{0\}, \{1\}, \{0, 1\}, \mathbb{Z}_{\geq 2} = \{z \in \mathbb{Z} : z \geq 2\}.
\]

Following directly from (5.3), (5.4), and (5.5), we have the following:

Lemma 5.2. Let \( p \) be a prime, \( \delta \geq 1 \), \( u \in R_{p,\delta} \), and \( \mu \geq 0 \). Then we have:

1. In case \( \varphi_{p,\delta}^{-1}(u) = \{0\} \), for \( g \geq 0 \),

\[
\varphi_{p,\delta}(\mu + g) = u \iff \mu + g = 0 \iff \begin{cases} g = 0 \quad (\mu = 0) \\ \text{impossible} \quad (\mu \geq 1), \end{cases}
\]

(5.6)

2. In case \( \varphi_{p,\delta}^{-1}(u) = \{1\} \), for \( g \geq 0 \),

\[
\varphi_{p,\delta}(\mu + g) = u \iff \mu + g = 1 \iff \begin{cases} g = 1 - \mu \quad (\mu = 0, 1) \\ \text{impossible} \quad (\mu \geq 1), \end{cases}
\]

(5.7)

3. In case \( \varphi_{p,\delta}^{-1}(u) = \{0, 1\} \), for \( g \geq 0 \),

\[
\varphi_{p,\delta}(\mu + g) = u \iff \mu + g \in \{0, 1\} \iff \begin{cases} g \leq 1 \quad (\mu = 0) \\ g = 0 \quad (\mu = 1), \\ \text{impossible} \quad (\mu \geq 2), \end{cases}
\]

(5.8)

4. In case \( \varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2} \), for \( g \geq 0 \),

\[
\varphi_{p,\delta}(\mu + g) = u \iff \mu + g \geq 2 \iff \begin{cases} g \geq 2 - \mu \quad (\mu = 0, 1) \\ \text{always true} \quad (\mu \geq 2). \end{cases}
\]

(5.9)

Here impossible means that no \( g \geq 0 \) can be found to fulfill \( \varphi_{p,\delta}(\mu + g) = u \), and that always true means that all \( g \geq 0 \) fulfills \( \varphi_{p,\delta}(\mu + g) = u \).

6. The case of \( n = 819 \)

We consider the case \( n = 819 \) of the main theorem Theorem 3.1. We have the prime factorizations

\[
819 = 3^2 \cdot 7 \cdot 13,
\]

\[
918 = 2 \cdot 3^3 \cdot 17.
\]

Let the prime factorization of \( \rho_{c,3} \) be

\[
\rho_{c,3} = 3^{\delta_1} \cdot 7^{\delta_2} \cdot 13^{\delta_3} \cdot 17^{\delta_4} \cdot b,
\]
where \((b, 3 \cdot 7 \cdot 13 \cdot 17) = 1\). The \(g_1, g_2, g_3, g_4, b\) obviously depends on \(c\), but we have suppressed the notation for simplicity. Now

\[
819 \rho_{c,3} = 3^{2+g_1} \cdot 7^{1+g_2} \cdot 13^{1+g_3} \cdot 17^{g_4} \cdot b,
\]

\[
r(819 \rho_{c,3}) = 918 \rho_{c,3} = 2 \cdot 3^{3+g_1} \cdot 7^{g_2} \cdot 13^{g_3} \cdot 17^{1+g_4} \cdot b.
\]

Applying the additive function \(v\) to these equations

\[
v(819 \rho_{c,3}) = v(3^{2+g_1}) + v(7^{1+g_2}) + v(13^{1+g_3}) + v(17^{g_4}) + v(b),
\]

\[
v(r(819 \rho_{c,3})) = v(918 \rho_{c,3}) = v(2) + v(3^{3+g_1}) + v(7^{g_2}) + v(13^{g_3}) + v(17^{1+g_4}) + v(b).
\]

Hence \(819 \rho_{c,3}\) is a \(v\)-palindromic number if and only if the above two quantities are equal, that is, after rearranging

\[
(v(7^{1+g_2}) - v(7^{g_2})) + (v(13^{1+g_3}) - v(13^{g_3})) = 2 + (v(3^{3+g_1}) - v(3^{2+g_1})) + (v(17^{1+g_4}) - v(17^{g_4})).
\]

In terms of the functions \(\varphi_{p,0}\) of Section 5, this becomes

\[
\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 2 + \varphi_{3,1}(2 + g_1) + \varphi_{17,1}(g_4).
\]

Since \(2 + g_1 \geq 2\), by (5.4), \(\varphi_{3,1}(2 + g_1) = 1\), therefore (6.1) becomes

\[
\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 3 + \varphi_{17,1}(g_4).
\]

Now consider the equation

\[
u_2 + u_3 = 3 + u_4.
\]

We want to solve it for \(u_2 \in R_{7,1}, u_3 \in R_{13,1}\), and \(u_4 \in R_{17,1}\). In view of (5.4),

\[R_{7,1} = \{7, 2, 1\}, R_{13,1} = \{13, 2, 1\}, R_{17,1} = \{17, 2, 1\}.
\]

By trying all possibilities we see that the only solutions are \((u_2, u_3, u_4) = (7, 13, 17), (2, 2, 1)\). Whence (6.2) is satisfied if and only if

\[
(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17) \text{ or } (\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1).
\]

We first consider when \((\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)\). By Lemmas 5.2 (or more easily just by looking at (5.4), 4.3, 4.1, and Table,

\[
\varphi_{7,1}(g_2) = 7 \iff g_2 = 0 \iff 7 \nmid \rho_{c,3} \iff h_{7,3} \nmid c \iff 2 \nmid c
\]

\[
\varphi_{13,1}(g_3) = 13 \iff g_3 = 0 \iff 13 \nmid \rho_{c,3} \iff h_{13,3} \nmid c \iff 2 \nmid c
\]

\[
\varphi_{17,1}(g_4) = 17 \iff g_4 = 0 \iff 17 \nmid \rho_{c,3} \iff h_{17,3} \nmid c \iff 16 \nmid c.
\]

Hence \((\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)\) simply when \(c\) is odd. We next consider when \((\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1)\). Similarly we have

\[
\varphi_{7,1}(g_2) = 2 \iff g_2 = 1 \iff 7 \mid \rho_{c,3} \text{ and } 7^2 \nmid \rho_{c,3} \iff 2 \mid c \text{ and } 14 \nmid c
\]

\[
\varphi_{13,1}(g_3) = 2 \iff g_3 = 1 \iff 13 \mid \rho_{c,3} \text{ and } 13^2 \nmid \rho_{c,3} \iff 2 \mid c \text{ and } 26 \nmid c
\]

\[
\varphi_{17,1}(g_4) = 1 \iff g_4 \geq 2 \iff 17^2 \mid \rho_{c,3} \iff 272 \mid c.
\]

Hence \((\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1)\) precisely when \(272 \mid c\) and \((c, 7 \cdot 13) = 1\). Hence we have established that
Theorem 6.1. $819 \varrho_{c,3}$ is $v$-palindromic if and only if $c$ is odd or if $272 \mid c$ and $(c, 7 \cdot 13) = 1$.

From the above theorem, we immediately see that $c(819) = 1$ (refer to definitions in Definition 3.2).

We see that $819 \varrho_{c,3}$ is $v$-palindromic if and only if all 3 conditions in (7.8) hold, or if all 3 conditions in (7.9) hold. Now these conditions have the same truth values when $c$ increases by lcm$(16, 14, 26, 272) = 24752$. Hence $\omega = 24752$ is a period of 819. With some work, it can be shown that actually it is the smallest period, that is, $\omega(819) = 24752$.

7 Proof of the main theorem

We now enter the proof of the main theorem and this is essentially writing the discussion about 819 in the previous section in the general setting.

Let the prime factorizations of $n$ and $r(n)$ be

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m},$$
$$r(n) = p_1^{f_1} p_2^{f_2} \cdots p_m^{f_m},$$

where we have done the factorization over the set of primes which divide one of $n$ or $r(n)$, setting $e_i = 0$ or $f_i = 0$ if necessary. Since $n \neq r(n)$, $e_i \neq f_i$ for some $i$. Let the set of $i$ such that $e_i = f_i$

$$i_1 < i_2 < \ldots < i_t. \quad (7.1)$$

Let the prime factorization of $\rho_{c, k}$ be

$$\rho_{c, k} = p_1^{g_1} p_2^{g_2} \cdots p_m^{g_m} b, \quad (7.2)$$

where $(b, p_1 p_2 \cdots p_m) = 1$. The $g_1, g_2, \ldots, g_m, b$ obviously depends on $c$, but we suppress it from our notation for simplicity. Then

$$n \rho_{c, k} = p_1^{e_1+g_1} p_2^{e_2+g_2} \cdots p_m^{e_m+g_m} b,$$
$$r(n \rho_{c, k}) = r(n) \rho_{c, k} = p_1^{f_1+g_1} p_2^{f_2+g_2} \cdots p_m^{f_m+g_m} b.$$

Taking their $v$, we have

$$v(n \rho_{c, k}) = \sum_{i=1}^{m} v(p_i^{e_i+g_i}) + v(b),$$
$$v(r(n \rho_{c, k})) = \sum_{i=1}^{m} v(p_i^{f_i+g_i}) + v(b).$$

Hence $n \rho_{c, k}$ is $v$-palindromic, that is, $v(n \rho_{c, k}) = v(r(n \rho_{c, k}))$, if and only if

$$\sum_{i=1}^{m} (v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i})) = 0. \quad (7.3)$$
Whence \( e_i = f_i \), of course the term \( v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i}) = 0 \), so by (7.1), (7.3) is equivalent to
\[
\sum_{j=1}^{t} (v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j})) = 0. \tag{7.4}
\]

But this is a cumbersome notation, so we just write \( p_i j \) as \( p_j \), \( e_i j \) as \( e_j \), \( f_i j \) as \( f_j \), and \( g_i j \) as \( g_j \), which will not cause confusion from here on because we will not be referring to the other prime factors or exponents hereafter. Consequently (7.4) becomes
\[
\sum_{j=1}^{t} (v(p_j^{e_j}) - v(p_j^{f_j})) = 0. \tag{7.5}
\]
We also let
\[
\begin{align*}
\delta_j &= e_j - f_j, \\
\mu_j &= \min(e_j, f_j), \\
\alpha_j &= \mu_j + g_j,
\end{align*}
\]
for \( 1 \leq j \leq t \). Then it is clear that the left-hand-side of (7.5) can be rewritten, using the functions \( \varphi_{p,\delta} \) of Section 5 as
\[
\sum_{j=1}^{t} (v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j})) = \sum_{j=1}^{t} \operatorname{sgn}(\delta_j)(v(p_j^{a_j}) - v(p_j^{a_j})) = \sum_{j=1}^{t} \operatorname{sgn}(\delta_j)\varphi_{p_j,\delta_j}(\alpha_j), \tag{7.6}
\]
where \( \operatorname{sgn} \) is the sign function with \( \operatorname{sgn}(\delta_j) = 1 \) if \( \delta_j > 0 \) and \( \operatorname{sgn}(\delta_j) = -1 \) if \( \delta_j < 0 \). Now consider the equation
\[
\sum_{j=1}^{t} \operatorname{sgn}(\delta_j)u_j = 0. \tag{7.7}
\]
Supposedly we can solve it for
\[
(u_1, u_2, \ldots, u_t) \in R_{p_1,|\delta_1|} \times R_{p_2,|\delta_2|} \times \cdots R_{p_t,|\delta_t|}.
\]
Let the set of all solutions be
\[
U = \{ u = (u_1, \ldots, u_t) \}.
\]
Then we see that
\[
\sum_{j=1}^{t} \operatorname{sgn}(\delta_j)\varphi_{p_j,\delta_j}(\alpha_j) = 0
\]
holds if and only if for some \( u \in U \),
\[
\varphi_{p_j,\delta_j}(\alpha_j) = u_j \quad \forall 1 \leq j \leq t.
\]
Summarizing up to now, we have shown that
Lemma 7.1. \(n\rho_{c,k}\) is \(v\)-palindromic if and only if for some \(u \in U\), \(\varphi_{p_i,b_j}(\alpha_j) = u_j\) for all \(1 \leq j \leq t\).

Now let us consider just the "atomic" condition \(\varphi_{p_i,b_j}(\alpha_j) = \varphi_{p_i,b_j}(\mu_j + g_j) = u_j\). By Lemmas 5.2, 4.3, and 4.1

\[
\varphi_{p_i,b_j}(\mu_j + g_j) = u_j \iff \begin{cases} 
  g_j = 0, & \text{(if (5.6) and } \mu_j = 0, \text{ or (5.7) and } \mu_j = 1, \text{ or (5.8) and } \mu_j = 1) \\
  g_j = 1, & \text{(if (5.7) and } \mu_j = 0) \\
  g_j \leq 1, & \text{(if (5.8) and } \mu_j = 0) \\
  g_j \geq 1, & \text{(if (5.9) and } \mu_j = 1) \\
  g_j \geq 2, & \text{(if (5.9) and } \mu_j = 0) \\
  \text{impossible,} & \text{(otherwise)} \\
  \text{always true.} & \text{(if (5.9) and } \mu_j \geq 2)
\end{cases}
\]

(7.8)

As the last two cases, "impossible" and "always true", never change (as \(c\) varies), we exclude them from our consideration. By Lemma 5.2 we can continue the equivalences in (7.8) respectively (here we do not write out the cases as in (7.8)), recalling that \(g_j = \text{ord}_{p_i}(\rho_{c,k})\)

\[
\varphi_{p_i,b_j}(\mu_j + g_j) = u_j \iff \begin{cases} 
  p_j \nmid \rho_{c,k}, \\
  p_j \mid \rho_{c,k} \text{ and } p_j^2 \nmid \rho_{c,k}, \\
  p_j^2 \mid \rho_{c,k}, \\
  p_j \mid \rho_{c,k}, \\
  p_j^2 \mid \rho_{c,k}.
\end{cases}
\]

(7.9)

In case \(p_j \neq 2, 5\), we can use Lemma 4.3 to (7.9) to obtain, respectively

\[
\varphi_{p_i,b_j}(\mu_j + g_j) = u_j \iff \begin{cases} 
  h_{p_i,k} \nmid c, \\
  h_{p_i,k} \mid c \text{ and } h_{p_i^2,k} \nmid c, \\
  h_{p_i^2,k} \nmid c, \\
  h_{p_i,k} \mid c, \\
  h_{p_i^2,k} \mid c.
\end{cases}
\]

(7.10)

However, in case \(p_j = 2, 5\), by the Remark 4.2 (7.9) becomes

\[
\varphi_{p_i,b_j}(\mu_j + g_j) = u_j \iff \begin{cases} 
  \text{always true,} \\
  \text{impossible,} \\
  \text{always true,} \\
  \text{impossible,} \\
  \text{impossible.}
\end{cases}
\]

(7.11)

Since in general \(a \mid b \iff a \mid (b + b')\) if \(a \mid b'\) \((a, b, b' \geq 1\) arbitrary integers, the \(b\) not the one introduced in (7.2)), we see that the truth of \(\varphi_{p_i,b_j}(\mu_j + g_j) = u_j\) does not change if we increase \(c\) by

\[\omega = \text{lcm}[h_{p_i,k}, h_{p_i^2,k} : p_j \neq 2, 5].\]

In view of Lemma 7.1 whether \(n\rho_{c,k}\) is \(v\)-palindromic depends only on the truths of the individual \(\varphi_{p_i,b_j}(\mu_j + g_j) = u_j\). Hence this \(\omega\) serves as a possible \(\omega\) as required by the main theorem.
Remark 7.2. We found constructively a possible $\omega$ as required by the main theorem. However whether or not this $\omega$ is the smallest period is still unclear.

References

[1] D. Tsai, *Natural numbers satisfying an unusual property*, Sugaku seminar 57, no.11 (2018), 35–36 (written in Japanese).

[2] Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).