Compact weak $G_2$-manifolds with conical singularities

Adel Bilal$^1$ and Steffen Metzger$^{1,2}$

$^1$ CNRS - Laboratoire de Physique Théorique, École Normale Supérieure
24 rue Lhomond, 75231 Paris Cedex 05, France

$^2$ Sektion Physik, Ludwig-Maximilians-Universität
Munich, Germany

e-mail: adel.bilal@lpt.ens.fr, metzger@physique.ens.fr

Abstract

We construct 7-dimensional compact Einstein spaces with conical singularities that preserve 1/8 of the supersymmetries of M-theory. Mathematically they have weak $G_2$-holonomy. We show that for every non-compact $G_2$-holonomy manifold which is asymptotic to a cone on a 6-manifold $Y$, there is a corresponding weak $G_2$-manifold with two conical singularities which, close to the singularities, looks like a cone on $Y$. Our construction provides explicit metrics on these weak $G_2$-manifolds. We completely determine the cohomology of these manifolds in terms of the cohomology of $Y$. 
1 Introduction

When discussing compactifications of M-theory on a 7-manifold $X$ down to 4 dimensions one is mainly interested in preserving exactly $\mathcal{N} = 1$ supersymmetry. If the 4-form field $G$ of 11-dimensional supergravity has a vanishing expectation value, it is well-known that the compactification manifold $X$ must be Ricci-flat with holonomy group $G_2$. This can however be generalized, similarly to the Freund-Rubin solution, to a non-vanishing background value with $G_{\mu\nu\rho\sigma}$ being proportional to the 4-dimensional $\epsilon$-tensor. In this latter case the compactification manifold must have so-called weak $G_2$-holonomy if we want to preserve exactly $\mathcal{N} = 1$ supersymmetry. Weak $G_2$-holonomy manifolds are Einstein spaces with positive curvature, in agreement with the fact that the non-vanishing 4-form field induces a non-vanishing energy-momentum tensor.

M-theory on $G_2$-holonomy manifolds has been much discussed recently and many such manifolds are known with more or less explicit metrics. Complete non-compact metrics were first given in [1], and compact spaces, though no explicit metrics, were first constructed in [2], see also [3].

M-theory/supergravity compactified on smooth $G_2$-holonomy manifolds only has abelian gauge groups and no charged chiral fermions, and hence is rather uninteresting. Introducing $ADE$-orbifold singularities, however, leads to non-abelian gauge groups, the symmetry enhancement being provided by M2-branes that wrap the vanishing two-cycles [4]. The presence of conical singularities was shown to lead to charged chiral fermions [5, 6, 7], and hence the theory could be potentially anomalous. The issue of anomaly cancellation is discussed in [8, 9]. The drawback of these discussions is that no explicit examples of compact $G_2$-holonomy manifolds with conical singularities are known. Instead one considers [6] the non-compact manifolds of [1] which in a limit become cones on some 6-manifold $Y$. One then assumes that compact $G_2$-holonomy manifolds also can develop conical singularities and, close enough to the singularities, look like one of these cones. While these discussions are very elegant, it still would be nice to have some explicit examples of compact manifolds with conical singularities at our disposal.

Since the basic examples of conical singularities are limits of the non-compact manifolds of [1] one could try to start with these manifolds and somehow make them compact. This can indeed be done, as we will show in this paper, at the price of
introducing positive curvature, deforming the $G_2$-holonomy to weak $G_2$-holonomy.

Our strategy to construct the compact weak $G_2$-holonomy manifolds is the following: we begin with any non-compact $G_2$-holonomy manifold $X$ that asymptotically, for “large $r$” becomes a cone on some 6-manifold $Y$. The $G_2$-holonomy of $X$ implies certain properties of the 6-manifold $Y$ which we deduce. In fact, $Y$ can be any Einstein space of positive curvature with weak $SU(3)$-holonomy. Then we use this $Y$ to construct a compact weak $G_2$-holonomy manifold $X_\lambda$ with two conical singularities that, close to the singularities, looks like a cone on $Y$.

We go on to study in detail the cohomology of these manifolds. Since, up to scale, $X_\lambda$ is completely determined in terms of $Y$ it is not surprising that the cohomology of $X_\lambda$ is determined by that of $Y$. Due to the singularities, however, one has to specify which class of forms one is going to allow on $X_\lambda$. Physically it is clear that one is interested in square-integrable forms. We prove that all $L^2$-harmonic $p$-forms on $X_\lambda$ for $p \leq 3$ are given by the trivial extensions of the harmonic $p$-forms on $Y$. In particular, $b^2(X_\lambda) = b^2(Y)$ and $b^3(X_\lambda) = b^3(Y)$, while $b^4(X_\lambda) = 0$. For $p \geq 4$ the $L^2$-harmonic $p$-forms on $X_\lambda$ are just the Hodge duals of the previous ones. We also give a simple generalisation of these cohomological results to analogous constructions in arbitrary dimensions of spaces with two conical singularities.

Our construction provides examples, with explicitly known metrics and cohomology, of compact manifolds with conical singularities that are Einstein spaces and preserve $1/8$ of the supersymmetries of M-theory. Actually, our results can easily be extended and applied more widely. One could go on and further quotient $X_\lambda$ by some $\Gamma_{ADE}$ to obtain $ADE$-orbifold singularities. Almost the whole discussion of [6, 7, 8, 9] about non-abelian gauge groups, chiral fermions and anomalies could then be repeated in this setting, but now with the advantage of having well-defined explicit examples at hand. Other situations for which our results yield compact examples are those discussed in refs. [10, 11].

This paper is organized as follows. Section 2 is a brief review of $G_2$- and weak $G_2$-holonomy. In section 3 we explicitly carry out the construction of the weak $G_2$-holonomy manifolds, and in section 4 we discuss the cohomology of these manifolds in detail and prove the above-mentioned results. In section 5, we draw some conclusions and discuss further developments. The appendix contains some technical material which is needed in the main text.
2 A brief review of weak $G_2$-holonomy

Compactifications of M-theory on 7-manifolds $X$ of $G_2$-holonomy preserve $\frac{1}{8}$ of the 32 supersymmetries if the expectation value of the four-form $G$ vanishes. It is known, however, that one can have non-vanishing $G$-flux on the four-dimensional space-time $M_4$ and still preserve the same amount of supersymmetry if the 7-manifold $X_\lambda$ has weak $G_2$-holonomy and $M_4$ is $AdS_4$. There are several equivalent ways to characterise these manifolds. For $G_2$-holonomy we have (exactly) one covariantly constant spinor $\eta$

$$\left( \partial_j + \frac{1}{4} \omega_j^{ab} \gamma^{ab} \right) \eta = 0 ,$$

from which one can construct (see appendix 6.2) a closed and co-closed three-form $\Phi$:

$$d\Phi = 0 , \quad d^* \Phi = 0 .$$

Here $\omega^{ab}$ is the spin-connection 1-form on the 7-manifold, and $a, b = 1, \ldots 7$ are flat indices, while $i, j = 1, \ldots 7$ are curved ones. This implies that $X$ is Ricci flat.

For weak $G_2$ we have instead

$$\left( \partial_j + \frac{1}{4} \omega_j^{ab} \gamma^{ab} \right) \eta = i \frac{\lambda}{2} \gamma_j \eta , \quad \lambda \in \mathbb{R} ,$$

which implies the existence of a three-form $\Phi_\lambda$ obeying

$$d\Phi_\lambda = 4 \lambda^* \Phi_\lambda ,$$

as well as $X_\lambda$ being Einstein:

$$R_{ij} = 6 \lambda^2 g_{ij} .$$

It can be shown that the converse statements are also true, namely that eq. (2.2) implies (2.1), and eq. (2.4) implies (2.3). Note that for $\lambda \to 0$, at least formally, weak $G_2$ goes over to $G_2$-holonomy.

Physically, a non-vanishing Ricci tensor is due to a non-vanishing energy-momentum tensor $T_{MN}$. Indeed, eq. (2.3) precisely is the condition (see e.g. [12]1) for preserving $N = 1$ supersymmetry in four dimensions if the four-form $G$ has a background value

$$\langle G \rangle = -6 \lambda \text{vol}_4$$

1In [12] $\lambda$ was normalised differently: $\lambda_{\text{BDS}} = 4 \lambda$. 

3
where \( \text{vol}_4 \) is the volume form on \( M_4 \), i.e. \( \langle G_{\mu \nu \rho \sigma} \rangle = -6\lambda \epsilon_{\mu \nu \rho \sigma} \) and all other components vanish. This induces a non-vanishing energy-momentum tensor, \( T_{\mu \nu} \neq 0 \) and \( T_{ij} \neq 0 \). Einstein’s equations are

\[
R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{96} \left( 8 G_{MPQR} G_N^{PQR} - g_{MN} G_{PQRS} G^{PQRS} \right)
\]

and, for the background (2.6), they imply

\[
\langle R_{\mu \nu} \rangle = -12\lambda^2 \langle g_{\mu \nu} \rangle, \quad \langle R_{ij} \rangle = 6\lambda^2 \langle g_{ij} \rangle.
\]

This is consistent with eq. (2.5) and the fact that \( M_4 \) is \( \text{AdS}_4 \).

3 Construction of weak \( G_2 \)-holonomy manifolds with singularities

One method to construct \( G_2 \)-holonomy or weak \( G_2 \)-holonomy metrics is based on a certain self-duality condition for the spin connection, as explained in [12]. This condition can be written as

\[
\psi_{abc} \omega^{bc} = -2\lambda \epsilon^a
\]

where \( \epsilon^a \) is the 7-bein on \( X \) and the \( \psi_{abc} \) are the structure constants of the imaginary octonions. The latter are completely antisymmetric and equal \( \pm 1 \) or 0. An explicit choice is \( \psi_{123} = \psi_{516} = \psi_{624} = \psi_{435} = \psi_{471} = \psi_{673} = \psi_{572} = 1 \). Self-dual and anti self-dual projections are explained in appendix 6.1. It was shown in [12] that \( G_2 \)-holonomy/weak \( G_2 \)-holonomy is equivalent to the existence of a local \( SO(7) \) frame where the corresponding \( \omega^{bc} \) satisfy eq. (3.1). We will call such a frame a self-dual frame. Most examples of \( G_2 \) metrics (\( \lambda = 0 \)) known in the literature actually are written in a self-dual frame. Examples of weak \( G_2 \)-metrics naturally written with self-dual frames are those with principal orbits being the Aloff-Walach spaces \( SU(3)/U(1)_{k,l} \) as given in [13], as well as the one described in [12]. These examples are smooth manifolds. They can be obtained by starting with an 8-dimensional manifold of \( Spin(7) \)-holonomy that is a cone over a 7-manifold \( X \). An analogous self-duality condition for the 8-manifold then reduces to the self-duality condition (3.1) of \( X \) with \( \lambda \neq 0 \) which has weak \( G_2 \)-holonomy. In this construction the original \( Spin(7) \) metric is a cohomogeneity-one metric so that the resulting metric on \( X \) is cohomogeneity-zero.

\(^2\)Note that \( a, b, c \) are “flat” indices with euclidean signature, and upper and lower indices are equivalent.
A basic role in establishing $G_2$- or weak $G_2$-holonomy is played by the above-
mentioned 3-form $\Phi$ or $\Phi_\lambda$. Hence, it is worthwhile to note the following result. The
3-form $\Phi$ of eq. (2.2) or $\Phi_\lambda$ of (2.4) is given by

$$\Phi_\lambda = \frac{1}{6} \psi_{abc} \hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c \quad (3.2)$$

if and only if the 7-beins $e^a$ are a self-dual frame. The proof is simple and is given in
appendix 6.2. Later, for weak $G_2$, we will consider a frame which is not self-dual and
thus the 3-form $\Phi_\lambda$ will be slightly more complicated than (3.2).

Our goal is to construct compact weak $G_2$-holonomy manifolds with conical singu-
larities. Some specific examples of such manifolds were constructed in [14]. Here
we want to extend this construction. We will show that, for every non-compact $G_2$-
manifold that is asymptotic to a cone on $Y$, one can construct a corresponding compact
weak $G_2$-manifold with conical singularities, that close to each of the singularities be-
comes a cone on $Y$.

We start with any $G_2$-manifold $X$ which asymptotically is a cone on a compact
6-manifold $Y$:

$$ds_X^2 \sim dr^2 + r^2 ds_Y^2. \quad (3.3)$$

Since $X$ is Ricci flat, $Y$ must be an Einstein manifold with $R_{\alpha \beta} = 5 \delta_{\alpha \beta}$. In practice [1],
$Y = \mathbb{CP}^3$, $S^3 \times S^3$ or $SU(3)/U(1)^2$, with explicitly known metrics. On $Y$ we introduce
6-beins

$$ds_Y^2 = \sum_{a=1}^{6} \tilde{e}^a \otimes \tilde{e}^\alpha, \quad (3.4)$$

and similarly on $X$

$$ds_X^2 = \sum_{a=1}^{7} \hat{e}^a \otimes \hat{e}^a. \quad (3.5)$$

Our conventions are that $a, b, \ldots$ run from 1 to 7 and $\alpha, \beta, \ldots$ from 1 to 6. The various
manifolds and corresponding vielbeins are summarized in the table below. Since $X$
has $G_2$-holonomy we may assume that the 7-beins $\hat{e}^a$ are chosen such that the $\omega^{ab}$ are
self-dual, and hence we know from the above remark that the closed and co-closed
3-form $\Phi$ is simply given by eq. (3.2), i.e. $\Phi = \frac{1}{6} \psi_{abc} \hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c$. Although there is
such a self-dual choice, in general, we are not guaranteed that this choice is compatible
with the natural choice of 7-beins on $X$ consistent with a cohomogeneity-one metric

\footnote{Here we want to treat both cases in parallel and we simply write $\Phi_\lambda$ with the understanding that $\Phi_\lambda|_{\lambda=0} = \Phi$.}
as (3.3). (For any of the three examples cited above, the self-dual choice actually is compatible with a cohomogeneity-one metric.)

| manifolds | $X$ | $Y$ | $X_c$ | $X_\lambda$ |
|-----------|-----|-----|-------|-------------|
| vielbeins | $\hat{e}^a$ | $\hat{e}^\alpha$ | $\hat{e}^a$ | $e^a$ |
| 3-forms   | $\Phi$ | $\phi$ | $\Phi_\lambda$ |

**Table 1**: The various manifolds, corresponding vielbeins and 3-forms that enter our construction.

Now we take the limit $X \to X_c$ in which the $G_2$-manifold becomes exactly a cone on $Y$ so that $\hat{e}^a \to \overline{e}^a$ with

$$\overline{e}^a = r\hat{e}^a, \quad \overline{e}^7 = dr. \quad (3.6)$$

In this limit the cohomogeneity-one metric can be shown to be compatible with the self-dual choice of frame (see appendix 6.3) so that we may assume that (3.6) is such a self-dual frame. More precisely, we may assume that the original frame $\hat{e}^a$ on $X$ was chosen in such a way that after taking the conical limit the $\overline{e}^a$ are a self-dual frame. Then we know that the 3-form $\Phi$ of $X$ becomes a 3-form $\phi$ of $X_c$ given by the limit of (3.2), namely

$$\phi = r^2 dr \wedge \xi + r^3 \zeta \quad (3.7)$$

with the 2- and 3-forms on $Y$ defined by

$$\xi = \frac{1}{2} \psi_{\tau\alpha\beta} \hat{e}^\alpha \wedge \hat{e}^\beta$$

$$\zeta = \frac{1}{6} \psi_{\alpha\beta\gamma} \hat{e}^\alpha \wedge \hat{e}^\beta \wedge \hat{e}^\gamma \quad (3.8)$$

The dual 4-form is given by

$$^{*}\phi = r^4 \, ^*\!_{\gamma}^\tau \zeta - r^3 dr \wedge \, ^*\!_{\gamma}^\tau \xi \quad (3.9)$$

where $^{*}\!_{\gamma}^\tau \xi$ is the dual of $\xi$ in $Y$ (see appendix 6.4 for an explanation of the powers of $r$ and the sign). As for the original $\Phi$, after taking the conical limit, we still have $d\phi = 0$ and $d^{*}\phi = 0$. This is equivalent to

$$d\xi = 3\zeta$$

$$d^{*}\!_{\gamma}^\tau \zeta = -4 \, ^*\!_{\gamma}^\tau \xi \quad (3.10)$$
These are properties of appropriate forms on $Y$, and they can be checked to be true for any of the three standard $Y$’s. Actually, these relations show that $Y$ has weak $SU(3)$-holonomy. Conversely, if $Y$ is a 6-dimensional manifold with weak $SU(3)$-holonomy, then we know that these forms exist. This is analogous to the existence of the 3-form $\Phi_\lambda$ with $d\Phi_\lambda = 4\lambda^*\Phi_\lambda$ for weak $G_2$-holonomy. These issues were discussed e.g. in [15]. Combining the two relations (3.10), we see that on $Y$ there exists a 2-form $\xi$ obeying

$$d^{*\gamma}d\xi + 12^{*\gamma}\xi = 0$$

$$d^{*\gamma}\xi = 0 \quad (3.11)$$

This equation can equivalently be written as $\Delta_Y \xi = 12\xi$ where $\Delta_Y = -^{*\gamma}d^{*\gamma}d - d^{*\gamma}d^{*\gamma}$ is the Laplace operator on forms on $Y$. Note that with $\zeta = \frac{1}{3}d\xi$ we actually have $\phi = d\left(\zeta^3\right)$ and $\phi$ is cohomologically trivial. This was not the case for the original $\Phi$.

We now construct a manifold $X_\lambda$ with a 3-form $\Phi_\lambda$ that is a deformation of this 3-form $\phi$ and that will satisfy the condition (2.4) for weak $G_2$-holonomy. This general construction is inspired by the examples of [14]. Since weak $G_2$-manifolds are Einstein manifolds we need to introduce some scale $r_0$ and make the following ansatz for the metric on $X_\lambda$

$$ds^2_{X_\lambda} = dr^2 + r_0^2 \sin^2 \hat{r} \, ds^2_Y$$

with

$$\hat{r} = \frac{r}{r_0} \quad , \quad 0 \leq r \leq \pi r_0 \quad (3.13)$$

Clearly, this metric has two conical singularities, one at $r = 0$ and the other at $r = \pi r_0$. Obviously also, $X_\lambda$ is a compact manifold since $Y$ is compact and $r$ ranges over a closed finite interval with the metric remaining finite as $r \to 0$ or $r \to \pi r_0$. More specifically, $X_\lambda$ has finite volume which is easily computed to be

$$\text{vol}(X_\lambda) = \int_{X_\lambda} \sqrt{g} = \text{vol}(Y) \int_0^{\pi r_0} dr \, (r_0 \sin \hat{r})^6 = \frac{5}{16} r_0^7 \text{vol}(Y) \quad (3.14)$$

As shown in appendix 6.5, the metric 3.12 is “unique” in the following sense: let $Y$ be an Einstein space with $\mathcal{R}_{\alpha\beta} = 5\delta_{\alpha\beta}$ and let

$$ds^2_{X_\lambda} = dr^2 + h^2(r) \, ds^2_Y$$

Then $X_\lambda$ is an Einstein space with $\mathcal{R}_{ab} = 6\lambda^2 \delta_{ab}$ iff

$$h^2(r) = r_0^2 \sin^2 \left(\frac{r}{r_0}\right) \quad \text{with} \quad \lambda^2 = r_0^{-2} \quad (3.16)$$
(up to a possible shift of $r$). This remains true in the limit $r_0 \to \infty$ with $\lambda = 0$ and $h(r) = r$ where one gets back the cone.

We see from (3.12) that we can choose 7-beins $e^a$ on $X_\lambda$ that are expressed in terms of the 6-beins $\tilde{e}^\alpha$ of $Y$ as

$$e^\alpha = r_0 \sin \hat{r} \tilde{e}^\alpha, \quad e^7 = dr.$$  \hfill (3.17)

Although this is the natural choice, it should be noted that it is not the one that leads to a self-dual spin connection $\omega^{ab}$ that satisfies eq. (3.1). We know from \cite{12} that such a self-dual choice of 7-beins must exist if the metric (3.12) has weak $G_2$-holonomy but, as noted earlier, there is no reason why this choice should be compatible with cohomogeneity-one, i.e. choosing $e^7 = dr$. Actually, it is easy to see that for weak $G_2$-holonomy, $\lambda \neq 0$, a cohomogeneity-one choice of frame and self-duality are incompatible: a cohomogeneity-one choice of frame means $e^7 = dr$ and $e^\alpha = h_{(\alpha)}(r) \tilde{e}^\alpha$ so that $\omega^{\alpha\beta} = \frac{h_{(\alpha)}(r)}{h_{(\beta)}(r)} \tilde{\omega}^{\alpha\beta}$ and $\omega^{7\alpha} = h'_{(\alpha)}(r) \tilde{e}^\alpha$. But then the self-duality condition for $a = 7$ reads $\psi_{7\alpha\beta} h_{(\alpha)}(r) \tilde{\omega}^{\alpha\beta} = -2 \lambda dr$. Since $\tilde{\omega}^{\alpha\beta}$ is the spin connection on $Y$, associated with $\tilde{e}^\alpha$, it contains no $dr$-piece, and the self-duality condition cannot hold unless $\lambda = 0$. Indeed, the examples of self-dual $\omega^{ab}$ for weak $G_2$ mentioned above were all for cohomogeneity-zero.

Having defined the 7-beins on $X_\lambda$ in terms of the 6-beins on $Y$, the Hodge duals on $X_\lambda$ and on $Y$ are related accordingly. As shown in appendix 6.4, if $\omega_p$ is a $p$-form on $Y$, we have

$$* (dr \wedge \omega_p) = (r_0 \sin \hat{r})^{6-2p} * \gamma \omega_p$$

$$* \omega_p = (-)^p (r_0 \sin \hat{r})^{6-2p} dr \wedge * \gamma \omega_p,$$ \hfill (3.18)

where we denote both the form on $Y$ and its trivial ($r$-independent) extension onto $X_\lambda$ by the same symbol $\omega_p$.

Finally, we are ready to determine the 3-form $\Phi_\lambda$ satisfying $d\Phi_\lambda = \lambda^* \Phi_\lambda$. Inspired by the examples considered in \cite{14}, we make the ansatz

$$\Phi_\lambda = (r_0 \sin \hat{r})^2 dr \wedge \xi + (r_0 \sin \hat{r})^3 (\cos \hat{r} \zeta + \sin \hat{r} \theta).$$ \hfill (3.19)

Here, the 2-form $\xi$ and the 3-forms $\zeta$ and $\theta$ are forms on $Y$ which are trivially extended to forms on $X_\lambda$ (no $r$-dependence). Note that this $\Phi_\lambda$ is not of the form (3.2) as
the last term is not just \( \zeta \) but \( \cos \hat{r} \zeta + \sin \hat{r} \theta \). This was to be expected since the cohomogeneity-one frame cannot be self-dual. The Hodge dual of \( \Phi_\lambda \) then is given by

\[
^\ast \Phi_\lambda = (r_0 \sin \hat{r})^4 \ast_\gamma \xi - (r_0 \sin \hat{r})^3 \mathrm{d}r \wedge (\cos \hat{r} \ast_\gamma \zeta + \sin \hat{r} \ast_\gamma \theta)
\]  

(3.20)

while

\[
d\Phi_\lambda = (r_0 \sin \hat{r})^2 \mathrm{d}r \wedge (-\mathrm{d}\xi + 3\zeta) + (r_0 \sin \hat{r})^3 (\cos \hat{r} \mathrm{d}\zeta + \sin \hat{r} \mathrm{d}\theta)
\]

\[ + \frac{4}{r_0} (r_0 \sin \hat{r})^3 \mathrm{d}r \wedge (\cos \hat{r} \theta - \sin \hat{r} \zeta) \]  

(3.21)

In the last term, the derivative \( \partial_r \) has exchanged \( \cos \hat{r} \) and \( \sin \hat{r} \) and this is the reason why both of them had to be present in the first place.

Requiring \( d\Phi_\lambda = 4\lambda^\ast \Phi_\lambda \) leads to the following conditions

\[
d\xi = 3\zeta
\]  

(3.22)

\[
d\theta = 4\lambda r_0 \ast_\gamma \xi
\]  

(3.23)

\[
\theta = -\lambda r_0 \ast_\gamma \zeta
\]  

(3.24)

\[
\zeta = \lambda r_0 \ast_\gamma \theta.
\]  

(3.25)

Equations (3.24) and (3.25) require

\[ r_0 = \frac{1}{\lambda} \]  

(3.26)

and \( \zeta = ^\ast_\gamma \theta \iff \theta = -^\ast_\gamma \zeta \) (since for a 3-form \( ^\ast_\gamma (^\ast_\gamma \omega_3) = -\omega_3 \)). Then (3.23) is \( \mathrm{d}\theta = 4^\ast_\gamma \xi \), and inserting \( \theta = -^\ast_\gamma \zeta \) and eq. (3.22) we get

\[
d^\ast_\gamma \mathrm{d}\xi + 12^\ast_\gamma \xi = 0.
\]  

(3.27)

But we know from (3.11) that there is such a two-form \( \xi \) on \( Y \). Then pick such a \( \xi \) and let \( \zeta = \frac{1}{3} \mathrm{d}\xi \) and \( \theta = -^\ast_\gamma \zeta = -\frac{1}{3} ^\ast_\gamma \mathrm{d}\xi \). We conclude that

\[
\Phi_\lambda = \left( \frac{\sin \lambda r}{\lambda} \right)^2 \mathrm{d}r \wedge \xi + \frac{1}{3} \left( \frac{\sin \lambda r}{\lambda} \right)^3 (\cos \lambda r \mathrm{d}\xi - \sin \lambda r ^\ast_\gamma \mathrm{d}\xi)
\]  

(3.28)

satisfies \( d\Phi_\lambda = 4\lambda^\ast \Phi_\lambda \) and that the manifold with metric (3.12) has weak \( G_2 \)-holonomy. Thus we have succeeded to construct, for every non-compact \( G_2 \)-manifold that is asymptotically (for large \( r \)) a cone on \( Y \), a corresponding compact weak \( G_2 \)-manifold \( X_\lambda \) with two conical singularities that look, for small \( r \), like cones on the same \( Y \). Of course, one could start directly with any 6-manifold \( Y \) of weak \( SU(3) \)-holonomy.

The quantity \( \lambda \) sets the scale of the weak \( G_2 \)-manifold \( X_\lambda \) which has a size of order \( \frac{1}{\lambda} \). As \( \lambda \to 0 \), \( X_\lambda \) blows up and, within any fixed finite distance from \( r = 0 \), it looks like the cone on \( Y \) we started with.
4 Cohomology of the weak $G_2$-manifolds

We now want to investigate the cohomology of the weak $G_2$-manifolds $X_\lambda$ we have constructed. We will show that it is entirely determined by the cohomology of $Y$. As before, we always assume that $Y$ is a non-singular compact Einstein space with positive curvature. This implies (see e.g. p. 63 of [3]) that there are no harmonic 1-forms on $Y$ and $b^1(Y) = 0$.

Essentially, we will show that, on $X_\lambda$, harmonic $p$-forms with $p \leq 3$ are given by those on $Y$, while harmonic $p$-forms with $p \geq 4$ are given by their Hodge duals on $X_\lambda$. In particular, this means that there are no harmonic 1- or 6-forms on $X_\lambda$.

There are various ways to define harmonic forms which are all equivalent on a compact manifold without singularities where one can freely integrate by parts. Since $X_\lambda$ has singularities we must be more precise about the definition we adopt and about the required behaviour of the forms as the singularities are approached.

Physically, when one does a Kaluza-Klein reduction of an eleven-dimensional $k$-form $C_k$ one first writes a double expansion $C_k = \sum_{p=0}^{k} \sum_i A_{k-p}^i \wedge \phi_p^i$ where $A_{k-p}^i$ are $(k-p)$-form fields in four dimensions and the $\phi_p^i$ constitute, for each $p$, a basis of $p$-form fields on $X_\lambda$. It is convenient to expand with respect to a basis of eigenforms of the Laplace-operator on $X_\lambda$. Indeed, the standard kinetic term for $C_k$ becomes

$$\int_{M_4 \times X_\lambda} dC_k \wedge *dC_k = \sum_{p,i} \left( \int_{M_4} dA_{k-p}^i \wedge *dA_{k-p}^i \right) \int_{X_\lambda} \phi_p^i \wedge *\phi_p^i \right) + \int_{M_4} A_{k-p}^i \wedge *A_{k-p}^i \int_{X_\lambda} d\phi_p^i \wedge *d\phi_p^i \right).$$

Then a massless field $A_{k-p}$ in four dimensions arises for every closed $p$-form $\phi_p^i$ on $X_\lambda$ for which $\int_{X_\lambda} \phi_p^i \wedge *\phi_p^i$ is finite. Moreover, the usual gauge condition $d^*C_k = 0$ leads to the analogous four-dimensional condition $d^*A_{k-p} = 0$ provided we also have $d^*\phi_p = 0$.

We are led to the following definition:

**Definition**: An $L^2$-harmonic $p$-form $\phi_p$ on $X_\lambda$ is a $p$-form such that

$$(i) \quad ||\phi_p||^2 \equiv \int_{X_\lambda} \phi_p \wedge *\phi_p < \infty, \quad \text{and}$$

$$(ii) \quad d\phi_p = 0 \quad \text{and} \quad d^*\phi_p = 0.$$ 

Note that this definition is manifestly invariant under Hodge duality. We will prove the following proposition.
Proposition: Let $X_\lambda$ be a 7-dimensional manifold with metric given by (3.12), (3.13), i.e.

$$ds^2_{X_\lambda} = dr^2 + r_0^2 \sin^2 \hat{r} \, ds_Y^2,$$

where $\hat{r} = \frac{r}{r_0}, \quad 0 \leq r \leq \pi r_0.$

Then all $L^2$-harmonic $p$-forms $\phi_p$ on $X_\lambda$ for $p \leq 3$ are given by the trivial ($r$-independent) extensions to $X_\lambda$ of the $L^2$-harmonic $p$-forms $\omega_p$ on $Y$. For $p \geq 4$ all $L^2$-harmonic $p$-forms on $X_\lambda$ are given by $*\phi_{7-p}$.

Since there are no harmonic 1-forms on $Y$ we immediately have the

Corollary: The Betti numbers on $X_\lambda$ are given by those of $Y$ as

$$b^0(X_\lambda) = b^7(X_\lambda) = 1, \quad b^1(X_\lambda) = b^6(X_\lambda) = 0,$$

$$b^2(X_\lambda) = b^5(X_\lambda) = b^2(Y), \quad b^3(X_\lambda) = b^4(X_\lambda) = b^3(Y).$$

Before embarking on the rather lengthy proof, let us make some remarks. If we do not require square-integrability on $X_\lambda$, it is obvious that harmonic $p$-forms on $Y$ ($p = 0, \ldots, 6$) carry over to harmonic $p$-forms on $X_\lambda$ as $\phi_p = \omega_p$. This yields $b^p(Y)$ harmonic $p$-forms on $X_\lambda$. Their Hodge duals on $X_\lambda$ then give another set of harmonic $(7 - p)$-forms on $X_\lambda$ as $\phi_{7-p} = (-)^p h(r) 6^{-2p} dr \wedge *\omega_p$. Since $*\omega_p$ is a harmonic $(6 - p)$-form on $Y$, there are $b^{6-p}(Y)$ such forms. Setting $q = 7 - p$ ($q = 1, \ldots, 7$), we see that this yields $b^{q-1}(Y)$ harmonic $q$-forms on $X_\lambda$. Thus the first set gives $b^p(Y)$ and the second set gives $b^{p-1}(Y)$ harmonic $p$-forms on $X_\lambda$. The Betti numbers then would be related as $b^p(X_\lambda) = b^p(Y) + b^{p-1}(Y)$. This agrees with the Künneth formula for a space of topology $[0, \pi r_0] \times Y$. What changes this result is the requirement of square-integrability which eliminates the harmonic $p$-forms on $X_\lambda$ for $p = 4, 5, 6$ in the first set, and those with $q = 1, 2, 3$ in the second set.

Another subtle point is the following. We also have to show that on $X_\lambda$ there are no other $L^2$-harmonic forms than those that come from the harmonic forms on $Y$. This would follow easily from standard arguments if we could freely integrate by parts with respect to $r$. However, due to the singularities at $r = 0$ and $r = \pi r_0$ this is not allowed. Instead we have to carefully solve the “radial” differential equation and show that there are no “square-integrable” solutions. Fortunately, the differential equation can be reduced to the hypergeometric equation where we have explicit formulae for the asymptotic behaviour of the solutions at our disposal.

Finally, we remark that it is straightforward to generalise the proposition to $(n+1)$- and $n$-dimensional manifolds $X$ and $Y$ and/or to more general metrics where $r_0 \sin \hat{r}$
is replaced by a different function $h(r)$ with the same linear asymptotics, so that one still has conical singularities. We will give the precise statement as a corollary, after having completed the proof.

**Preliminaries:** We begin by considering general forms on $Y$, not just harmonic ones. For some fixed $p$, let $\omega^k_p$ (resp. $\omega^{i}_{p-1}$) be a basis of $p$-forms (resp. $(p-1)$-forms) on $Y$. Since we will only be interested in square-integrable forms on $X_\lambda$, without loss of generality, we will only consider forms on $Y$ that are square-integrable on $Y$ with respect to the standard inner product for $q$-forms ($q = p$ or $p-1$):

$$
(\omega^k_q, \omega^l_q) = \int_Y \omega^k_q \wedge \ast_Y \omega^l_q.
$$

We will assume that the basis is chosen to be orthonormal with respect to this inner product, which will simplify the discussion below. Furthermore, we will encounter $(d\omega^{i}_{p-1}, d\omega^{j}_{p-1})$. Since we can freely integrate by parts on the smooth compact manifold $Y$ and as $\ast_Y= (-)^{p-1} \omega_{p-1}$, we have

$$
(d\omega^{i}_{p-1}, d\omega^{j}_{p-1}) = (\omega^{i}_{p-1}, -\ast_Y d \ast_Y d\omega^{j}_{p-1}).
$$

Thus $-\ast_Y d \ast_Y d$ is a symmetric differential operator on forms on $Y$ and we may as well assume that the basis of $\omega^{i}_{p-1}$ has been chosen as a basis of orthonormal eigenforms of this operator:

$$
-\ast_Y d \ast_Y d \omega^{i}_{p-1} = \mu_i \omega^{i}_{p-1}, \quad \mu_i \geq 0,
$$

so that by (4.7) we have $(d\omega^{i}_{p-1}, d\omega^{j}_{p-1}) = \mu_i \delta^{ij}$.

**Proof:** To prove the proposition, we consider the most general $p$-form $\phi_p$ on $X_\lambda$:

$$
\phi_p = dr \wedge G_i(r) \omega^i_{p-1} + g_k(r) \omega^k_p
$$

where sums over repeated indices $i$ or $k$ are understood. We want to see which restrictions are imposed on the functions $G_i(r)$ and $g_k(r)$ by equations (4.2) and (4.3) defining an $L^2$-harmonic $p$-form on $X_\lambda$.

First, $d\phi_p = 0$ implies

$$
G_i(r) d\omega^i_{p-1} = g_k(r) \omega^k_p \quad \text{and} \quad g_k(r) d\omega^k_p = 0.
$$

We are only interested in forms with real coefficients, but it is trivial to extend the whole discussion to forms with complex coefficients. Then the inner product has to be appropriately modified by including complex conjugation of the first factor.
We define $\tilde{F}_i(r) = \int_r^* G_i(\tilde{r}) d\tilde{r}$ for some $r^* \in [0, \pi r_0]$. Integrating the first relation (4.10) gives

$$g_k(r)\omega^k_p = \tilde{F}_i(r) d\omega^i_{p-1} + \omega_p,$$

where $\omega_p = g_k(r^*)\omega^k_p$, and by the second equation (4.10) we have $d\omega_p = g_k(r^*)d\omega^k_p = 0$. Hence, $\omega_p$ is closed on $Y$ and we can write $\omega_p = \omega_p^{(0)} + d(a_i\omega^i_{p-1})$, where $\omega_p^{(0)}$ is a harmonic $p$-form on $Y$. Finally, if we let $F_i(r) = \tilde{F}_i(r) + a_i$, we see from equations (4.9) and (4.11) that

$$\phi_p = \omega_p^{(0)} + d \left( F_i(r)\omega^i_{p-1} \right)$$

$$= \omega_p^{(0)} + F_i(r)d\omega^i_{p-1} + F'_i(r)dr \wedge \omega^i_{p-1},$$

with $\omega_p^{(0)}$ harmonic on $Y$. Note that, in general, $F_i(r)\omega^i_{p-1}$ is not square-integrable so that $\phi_p - \omega_p^{(0)}$ is not $L^2$-exact.

The Hodge dual of such a $\phi_p$ on $X_\lambda$ is (recall eq. (3.18))

$$\ast \phi_p = (-)^p (r_0 \sin \hat{r})^6\ast r d\omega_p^{(0)} + (\ast \omega_p^{(0)} + F_i(r) \ast r d\omega^i_{p-1}) + (r_0 \sin \hat{r})^8 - 2p F'_i(r) \ast \omega^i_{p-1}.$$

Then $d\ast \phi_p = 0$ implies

$$(r_0 \sin \hat{r})^8 - 2p F'_i(r) \ast \omega^i_{p-1} = (-)^p (r_0 \sin \hat{r})^6\ast r d\omega^i_{p-1}$$

$$(r_0 \sin \hat{r})^8 - 2p F'_i(r) d\ast \omega^i_{p-1} = 0.$$  

At this point it is useful that we have chosen the $\omega^i_{p-1}$ as a basis of eigenforms of the differential operator $-\ast r d\ast d r$ with eigenvalue $\mu_i$ so that eq. (4.14) becomes

$$(r_0 \sin \hat{r})^8 - 2p F'_i(r) \ast \omega^i_{p-1} = \mu_i (r_0 \sin \hat{r})^6 - 2p F_i(r) \quad \text{(no sum over $i$).}$$

Then every solution to eqs (4.15) and (4.16) which leads to a square-integrable $\phi_p$ yields an $L^2$-harmonic $p$-form on $X_\lambda$. From eqs (4.12) and (4.13) we get

$$||\phi_p||^2 = \int_0^{\pi r_0} (r_0 \sin \hat{r})^6 - 2p dr \int_Y \omega_p^{(0)} \wedge \ast \omega_p^{(0)}$$

$$+ \sum_i \int_0^{\pi r_0} \left[ (r_0 \sin \hat{r})^6 - 2p (F'_i(r))^2 \mu_i + (r_0 \sin \hat{r})^8 - 2p (F'_i(r))^2 \right] dr,$$  

where we used eqs (4.7), (4.8) and the orthonormality of the $\omega^i_{p-1}$. The cross-terms, linear in $\omega_p^{(0)}$ dropped out, upon integrating by parts on $Y$. Since (4.17) is a sum of non-negative terms, finiteness of $||\phi_p||^2$ implies finiteness of each of them separately.
In the sum over \( i \) we will distinguish the indices \( i = 1, \ldots, i_0 \) that we will take to correspond to \( \mu_i = 0 \), from those \( i > i_0 \) with \( \mu_i > 0 \). Since \( \mu_i = 0 \) means \( d\omega_{p-1}^i = 0 \), eq. (4.12) can be rewritten as

\[
\phi_p = \omega_p^{(0)} + \sum_{i=1}^{i_0} \int_{0}^{r_0} F_i(r) \, dr \land \omega_p^i + \sum_{i > i_0} \left( \left( F_i(r) \, dr \land \omega_p^i + F_i(r) \, d\omega_p^i \right) \right) \\
\equiv \phi_p^{(1)} + \phi_p^{(2)} + \phi_p^{(3)} .
\] (4.18)

We will show below that if \( \phi_p^{(3)} \neq 0 \) (with \( F_i \) solutions of (4.16) with \( \mu_i > 0 \)) then \( ||\phi_p||^2 \) diverges. Anticipating this result, we must take \( \phi_p^{(3)} = 0 \). Then only \( F_i(r) \) with \( i = 1, \ldots, i_0 \) appear in eq. (4.18). They are solutions of eq. (4.16) with \( \mu_i = 0 \) so that

\[
(r_0 \sin \hat{r})^{8-2p} F_i(r) = c_i , \quad i = 1, \ldots, i_0 , \tag{4.19}
\]

(with real \( c_i \)) and from eq. (4.15)

\[
d^{*y} \tilde{\omega}_{p-1}^{(0)} = 0 \quad \text{with} \quad \tilde{\omega}_{p-1}^{(0)} = \sum_{i=1}^{i_0} c_i \omega_{p-1}^i . \tag{4.20}
\]

Note that this implies that \( \tilde{\omega}_{p-1}^{(0)} \) is harmonic on \( Y \) and \( *\phi_p^{(2)} = *^{y} \tilde{\omega}_{p-1}^{(0)} \). Then

\[
||\phi_p||^2 = \int_0^{\pi r_0} dr \left[ (r_0 \sin \hat{r})^{6-2p} \int_Y \omega_p^{(0)} \land *y \omega_p^{(0)} + (r_0 \sin \hat{r})^{2p-8} \sum_{i=1}^{i_0} c_i^2 \right] . \tag{4.21}
\]

This converges for \( p \leq 3 \) if and only if \( c_i = 0 \), i.e. \( \tilde{\omega}_{p-1}^{(0)} = 0 \), and for \( p \geq 4 \) if and only if \( \omega^{(0)}_p = 0 \). Hence, \( \phi_p \) is \( L^2 \)-harmonic if and only if

\[
\phi_p = \omega_p^{(0)} \quad \text{for} \quad p \leq 3 , \\
\phi_p = *(y \tilde{\omega}_{p-1}^{(0)}) \quad \text{for} \quad p \geq 4 . \tag{4.22}
\]

Since \( *y \tilde{\omega}_{p-1}^{(0)} \) is a harmonic \((7-p)\)-form on \( Y \), the \( L^2 \)-harmonic \( \phi_p \) for \( p \geq 4 \) are exactly the Hodge duals of the \( L^2 \)-harmonic \( \phi_q \) with \( q \leq 3 \).

Finally, we show that in the decomposition (4.18) of an \( L^2 \)-harmonic \( p \)-form \( \phi_p \) the third part \( \phi_p^{(3)} \) must vanish. For this it is enough to show that \( \phi_p^{(3)} \) by itself cannot be \( L^2 \). In order to do so, we assume \( F_i(r) \neq 0 \) for one or more \( i > i_0 \) (with \( \mu_i > 0 \)). For these \( F_i \), equations (4.15) and (4.16) must hold. While eq. (4.15) involves a sum

\[\sum_{i=1}^{i_0} F_i(r) \land \omega_p^i = \sum_{i=1}^{i_0} \left( \sum_{i=1}^{i_0} F_i(r) \land \omega_p^i \right) \text{ is square-integrable,}
\]

but \( \sum_{i=1}^{i_0} F_i(r) \land \omega_p^i \) is not, confirming the remark below eq. (4.12).
over all \( i \), eq. (4.16) holds for every \( i \) separately. We want to show that there are no (non-vanishing) solutions of (4.16) that lead to a finite norm \(^6\) of \( \phi_p^{(3)} \).

To proceed, we now reduce eq. (4.16) to the well-known hypergeometric equation. The required change of variables is two-to-one, so first we have to reformulate the problem on half the interval, i.e. on \([0, \pi r / 2]\). Since eq. (4.16) is a second-order differential equation there are two linearly independent solutions. Given the symmetry \( \mathcal{P} \) that maps \( \hat{r} \to \pi - \hat{r} \), one solution can be chosen \( \mathcal{P} \)-even and the other \( \mathcal{P} \)-odd. The general solution \( F_i \) then is \( F_i = \alpha_i F_i^{(\text{even})} + \beta_i F_i^{(\text{odd})} \). When computing \( \| \phi_p^{(3)} \|^2 \) no cross-terms \( F_i^{(\text{even})} F_i^{(\text{odd})} \) or \((F_i^{(\text{even})})'(F_i^{(\text{odd})})'\) survive and it is enough to show that neither \( F_i^{(\text{even})} \) nor \( F_i^{(\text{odd})} \) leads to a finite norm. With either choice, \( F_i = F_i^{(\text{even})} \) or \( F_i = F_i^{(\text{odd})} \), the integrand in (4.17) is \( \mathcal{P} \)-even and can be rewritten as twice the integral over half the interval:

\[
\| \phi_p \|^2_{F_i \text{- contribution}} = 2 \int_{0}^{\pi r / 2} dr \left( (r_0 \sin \hat{r})^{6-2p} (F_i(r))^{2} \mu_i + (r_0 \sin \hat{r})^{8-2p} (F_i'(r))^{2} \right),
\]

and it is enough to know the solution \( F_i \) for \( 0 \leq r \leq \pi r / 2 \). Note that eq. (4.23) only holds if \( F_i \) is the even or the odd solution, but not for an arbitrary linear combination thereof. If \( F_i \) is \( \mathcal{P} \)-even then \( F_i'(\pi r_0 / 2) = 0 \) and if \( F_i \) is \( \mathcal{P} \)-odd then \( F_i(\pi r_0 / 2) = 0 \). Hence we can solve the differential equation (4.16) on the smaller interval \([0, \pi r_0 / 2]\) and select the even or odd \( F_i \) through the appropriate boundary condition at \( \pi r_0 / 2 \).

Now we make the change of variables

\[
x = (\sin \hat{r})^{-2}, \quad F_i(r) = f(x).
\]

This is one-to-one on the smaller interval \([0, \pi r_0 / 2]\): as \( r \) goes from 0 to \( \pi r_0 / 2 \), our new variable \( x \) goes from \( \infty \) to 1. One has \( r_0 \partial_r = -2 \sqrt{x-1} x \partial_x \) and eq. (4.16) becomes

\[
4x(x-1)f''(x) + 2((5-2p)(1-x) + 1) f'(x) - \mu_i f(x) = 0,
\]

which is the standard hypergeometric equation. The vanishing of \( F_i(\pi r_0 / 2) \) translates into \( f(1) = 0 \), and the vanishing of \( F_i'(\pi r_0 / 2) \) into the vanishing of \( \sqrt{x-1} \partial_x f(x) \) at

---

\(^6\)The usual argument for this type of problem goes like this: Assume \( F_i \) satisfies eq. (4.16) with \( \mu_i > 0 \). From the asymptotics of the differential equation one immediately deduces that the general solution behaves as \( F_i \sim \alpha_i r^{\nu^+} + \beta_i r^{\nu^-} \) as \( r \to 0 \) with \( \nu_{\pm} = \tilde{p} \pm \sqrt{\tilde{p}^2 + \mu_i} \) where \( \tilde{p} = p - \frac{7}{2} \), and similarly \( F_i \sim \gamma_i (\pi r_0 - r)^{\nu^+} + \delta_i (\pi r_0 - r)^{\nu^-} \) as \( r \to \pi r_0 \). We may choose the coefficients \( \alpha_i \) and \( \beta_i \) but then \( \gamma_i \) and \( \delta_i \) are determined through the actual solution of the full differential equation. Square-integrability (in the sense of eq. (4.17)) requires \( \beta_i = 0 \) and \( \delta_i = 0 \). We may freely choose \( \beta_i = 0 \), and then determine \( \delta_i \). If \( \delta_i \neq 0 \) then there is no square-integrable solution. The argument we present in the text is somewhat different, although perfectly equivalent.
Thus we have to find the solutions of the hypergeometric equation satisfying one or the other of the following boundary conditions:

\[
\text{as } x \to 1 : \quad f(x) \to 0 \quad \text{or} \quad \sqrt{x-1} \partial_x f(x) \to 0 ,
\]

(4.26)

and show that both solutions lead to an infinite norm of \( \phi_p^{(3)} \). The contribution of a given solution \( f(x) \) to \( \| \phi_p^{(3)} \|^2 \) is

\[
\left| \left| \phi_p^{(3)} \right| \right|_{F_i- \text{ contribution}}^2 = r_0^{-2p} \int_1^\infty \left[ f^2(x) \mu_i + 4x(x-1)(f'(x))^2 \right] \frac{x^{p-4}}{\sqrt{x-1}} \, dx .
\]

(4.27)

We can integrate the second term by parts and use the differential equation. It is precisely such that the whole integrand vanishes leaving us only with the boundary terms

\[
\left. \left| \left| \phi_p^{(3)} \right| \right|_{F_i- \text{ contribution}}^2 \right|_{x=\infty}^\infty = 2r_0^{-2p}x^{p-3} \sqrt{x-1} \partial_x f^2(x) \bigg|_{x=\infty}^\infty
\]

\[
- 2r_0^{-2p}x^{p-3} \sqrt{x-1} \partial_x f^2(x) \bigg|_{x=1} .
\]

(4.28)

But due to the boundary condition (4.26), only the term at \( \infty \) remains. So all we have to do is to determine the two solutions of the hypergeometric differential equation (4.25) that satisfy one or the other boundary condition (4.26) and check whether \( x^{p-3} \sqrt{x-1} \partial_x f^2(x) \) has a finite limit as \( x \) goes to \( \infty \). This is done in appendix 6.6 with the result that, for \( \mu_i > 0 \), \( x^{p-3} \sqrt{x-1} \partial_x f^2(x) \big|_{x=\infty} \) always diverges. We conclude that all \( F_i \) with \( i > i_0 \) must vanish, i.e. \( \phi_p^{(3)} = 0 \). This completes the proof of our proposition.

As already noted, it is straightforward to generalise the proposition. First, in the metric one can replace \( r_0 \sin \hat{r} \) by a more general function \( h(r) \) provided it also vanishes linearly in \( r \) as the singularities are approached. Indeed, this is all that was used to decide the convergence or divergence of the integrals. Furthermore, if \( Y \) is \( n \)-dimensional, the exponents \( 6 - 2p \) and \( 8 - 2p \) are replaced by \( n - 2p \) and \( n + 2 - 2p \). The only novelty occurs for odd \( n \). Then, if \( p \) is the middle value \( \frac{n+1}{2} \), both terms in the equation replacing (4.21) have a factor \( (h(r))^{-1} \) and diverge. Also, in this case, changing variables from \( r \) to \( \xi \) with \( \frac{dr}{d\xi} = (h(r))^{-1} \) and \( -\infty < \xi < \infty \), eq. (4.16) gets replaced by \( \partial_\xi^2 F_i(\xi) = \mu_i F_i(\xi) \) and the r.h.s. of (4.23) by \( \int_\infty^- d\xi [F_i^2 \mu_i + (\partial_\xi F_i)^2] \). Evidently, no solution of finite norm exists. We conclude that the following generalisation holds:
**Corollary:** Let $Y$ be any smooth compact $n$-dimensional manifold and $X$ an $(n+1)$-dimensional manifold with metric given by

$$ds_X^2 = dr^2 + h(r)^2 ds_Y^2, \quad r_1 \leq r \leq r_2,$$

(4.29)

with a smooth function $h(r)$ that is non-vanishing for $r_1 < r < r_2$ and behaves as $h(r) \sim r - r_1$ as $r \to r_1$ and $h(r) \sim r_2 - r$ as $r \to r_2$. We require that $h(r)$ is such that for $\mu > 0$ the differential equation

$$\left((h(r))^{n+2-2p} f'(r)\right)' = \mu (h(r))^{n-2p} f(r)$$

(4.30)

has no solution $f$ for which

$$\int_{r_1}^{r_2} dr \left[(h(r))^{n-2p}(f(r))^2 + (h(r))^{n+2-2p}(f'(r))^2\right] = (h(r))^{n+2-2p} f(r) f'(r) \bigg|_{r_1}^{r_2}$$

(4.31)

is finite. Then the $L^2$-harmonic $p$-forms on $X$ are given, for $p \leq \frac{n}{2}$, by the trivial extensions of the harmonic $p$-forms on $Y$, while for $p \geq \frac{n+2}{2}$ they are given by their Hodge duals on $X$. If $n$ is odd, there are no $L^2$-harmonic $\frac{n+1}{2}$-forms on $X$.

5 Conclusions and discussion

The compactifications of M-theory on manifolds of $G_2$- or weak $G_2$-holonomy lead to $\mathcal{N} = 1$ supersymmetry in four dimensions. If the compact $G_2$-holonomy manifolds have conical singularities, interesting four-dimensional physics emerges, involving charged chiral fermions and anomalies. As a matter of fact, no explicit metric on such singular spaces exists. In this paper, we constructed metrics on compact seven-manifolds with two conical singularities which carry weak $G_2$-holonomy. To do so, we started from any (non-compact) $G_2$-holonomy manifold $X$ which is asymptotic to a cone on some $Y$ and derived various properties of $Y$. The corresponding weak $G_2$-holonomy manifolds then can be taken to be the direct product of an interval and the six-manifold $Y$, where the metric involves a warp factor $\sin^2 \frac{r}{r_0}$.

Although the compactification on these manifolds results in a four-dimensional anti-de Sitter space, we can read off useful information from models of that kind. In particular, we again expect charged chiral fermions living at the singularities which will in general lead to anomalies. A better understanding of the mechanism of anomaly cancellation might be obtained by studying these explicitly known weak $G_2$-manifolds.
One question that arose while studying anomalies on singular spaces [8] was the
determination of the gauge group $H^2(X;U(1))$ and more generally of the cohomol-
ogy of $X$. For the manifolds $X_\lambda$ we constructed, these questions were answered
unambiguously. The reason is that the structure of the weak $G_2$-manifold is al-
most entirely determined by the 6-manifold $Y$. In particular, the cohomology of $X_\lambda$
can be inferred from the cohomology of $Y$, with the Betti numbers being related as
$b^p(X_\lambda) = b^{7-p}(X_\lambda) = b^p(Y)$ for $p \leq 3$. Thus, Poincaré duality and vanishing first Betti
number are maintained although $X_\lambda$ is singular. An important ingredient was the
physically motivated restriction to square-integrable forms only. We also generalised
this result to arbitrary $\text{dim}(Y)$ and more general warp factors $h^2(r)$. As an interesting
consequence of these facts we find that the gauge group $H^2(X_\lambda;U(1))$ coincides with
$H^2(Y;U(1))$. For the standard examples $Y = \mathbb{CP}^3$, $SU(3)/U(1)^2$ and $S^3 \times S^3$ we thus
get gauge groups $U(1)$, $U(1)^2$ and no gauge group, respectively.

To generate non-abelian gauge groups one could divide the weak $G_2$-manifolds by
$\Gamma_{ADE}$, with $\Gamma_{ADE}$ acting non-trivially on $Y$. It is rather straightforward to discuss
the resulting orbifolds and what happens at the two conical singularities, but this is
beyond the scope of the present paper.

Acknowledgements
Steffen Metzger gratefully acknowledges support by the Studienstiftung des deutschen
Volkes and by the Gottlieb Daimler- and Karl Benz-Stiftung.

6 Appendix
In this appendix we collect some useful formulas and prove some results needed in the
main text.

6.1 Projectors on the 14 and 7 of $G_2$
Every antisymmetric tensor $A^{ab}$ transforming as the 21 of $\text{SO}(7)$ can always be decom-
posed [12] into a piece $A^{ab}_+$ transforming as the 14 of $G_2$ (called self-dual) and a piece
$A^{ab}_-$ transforming as the 7 of $G_2$ (called anti-self-dual):

\begin{align}
A^{ab} &= A^{ab}_+ + A^{ab}_- \\
A^{ab}_+ &= \frac{2}{3} \left( A^{ab} + \frac{1}{4} \dot{\psi}^{abcd} A^{cd} \right), \quad \psi^{abc} A^{bc}_+ = 0,
\end{align}

(A.1) (A.2)
\[ A^{ab} = \frac{1}{3} \left( A^{ab} - \frac{1}{2} \hat{\psi}^{abcd} A^{cd} \right) = \frac{1}{6} \hat{\psi}_{abcd} A^{de} , \]  
\hspace{1cm} (A.3)\]

where
\[ \hat{\psi}_{abcd} = \frac{1}{3!} \epsilon^{abcdefg} \psi_{efg} . \]  
\hspace{1cm} (A.4)\]

In particular, one has
\[ \omega^{ab} \gamma^{ab} = \omega^{ab} \gamma^{ab} + \omega^{ab} \gamma^{ab} . \]  
\hspace{1cm} (A.5)\]

### 6.2 Self-duality and the 3-form

It will be useful to have an explicit representation for the \( \gamma \)-matrices in 7 dimensions. A convenient representation is in terms of the \( \psi_{abc} \) as [12]
\[ (\gamma_a)_{AB} = i(\psi_{aAB} + \delta_a A \delta_B - \delta_a B \delta_A) . \]  
\hspace{1cm} (A.6)\]

Here \( a = 1, \ldots, 7 \) while \( A, B = 1, \ldots, 8 \) and it is understood that \( \psi_{aAB} = 0 \) if \( A \) or \( B \) equals 8. One then has [12]
\[ (\gamma_{ab})_{AB} = \hat{\psi}_{abAB} + \psi_{abA} \delta_B - \psi_{abB} \delta_A + \delta_a A \delta_B - \delta_a B \delta_A \]  
\hspace{1cm} (A.7)\]
\[ (\gamma_{abc})_{AB} = i \psi_{abc}(\delta_{AB} - 2 \delta_A \delta_{B}) - 3i \psi_{[abc]B} - 3i \psi_{[ab]c} \delta_A + \hat{\psi}_{abcA} \delta_B - \hat{\psi}_{abcB} \delta_A . \]  
\hspace{1cm} (A.8)\]

Now we can prove the following:

**Lemma** : Let \( \Phi \) be the 3-form that satisfies \( d\Phi = 0 \), \( d^* \Phi = 0 \), for \( G_2 \)-holonomy, or \( d\Phi = 4\lambda^* \Phi \), for weak \( G_2 \)-holonomy.\(^7\) Then \( \Phi \) is given by
\[ \Phi = \frac{1}{6} \psi_{abc} e^a \wedge e^b \wedge e^c \]  
\hspace{1cm} (A.9)\]

if and only if the spin connection is self-dual, i.e. satisfies \( \psi_{abc} \omega^{ab} = -2\lambda e^c \).

As explained in [12], from the covariantly constant spinor (2.1) one can always construct a covariantly constant 3-form

\[ \Phi = \frac{i}{6} \eta^T \gamma_{abc} \eta e^a \wedge e^b \wedge e^c \]  
\hspace{1cm} (A.10)\]

This implies that \( \Phi \) is closed (\( d\Phi = 0 \)) and co-closed (\( d^* \Phi = 0 \)). For weak \( G_2 \)-holonomy, one defines the 3-form \( \Phi \) in exactly the same way by (A.10), but now \( \eta \)
\hspace{1cm} \(^7\)Here we do not distinguish between \( \Phi \) and \( \Phi_\lambda \) but use the same symbol \( \Phi \) since both cases are treated in parallel.
obeys the Killing spinor property (2.3). It then follows that $\Phi$ obeys $d\Phi = 4\lambda^*\Phi$. Thus (A.10) is the correct 3-form $\Phi$. To see under which condition it reduces to (A.9) we use the explicit representation for the $\gamma$-matrices (A.6) given above. It is then easy to see that $\gamma_T^{\alpha\beta\gamma} \eta_{\alpha\beta} \sim \psi_{\alpha\beta\gamma}$, if and only if $\eta_A \sim \delta_{8A}$. This means that our 3-form $\Phi$ is given by (A.10) if and only if the covariantly constant, resp. Killing spinor $\eta$ only has an eighth component, which then must be a constant which we can take to be 1. With this normalisation we have

$$\eta_T^{\alpha\beta\gamma} \eta = -i\psi_{\alpha\beta\gamma}, \quad (A.11)$$

so that $\Phi$ is correctly given by (A.10). From the above explicit expression for $\gamma_{ab}$ one then deduces that $(\gamma_{ab})_{AB} \eta_B = \psi_{abA}$ and $\omega_{ab}(\gamma_{ab})_{AB} \eta_B = \omega_{ab} \psi_{abA}$. Also, $i(\gamma_c)_{AB} \eta_B = -\delta_{cA}$, so that eq. (2.1), resp. (2.3) reduces to $\omega_{ab} \psi_{abc} = -2\lambda \epsilon^c$.

### 6.3 Compatibility of self-duality and cohomogeneity-one

Next we want to investigate the compatibility of cohomogeneity-one and self-dual choices of frame. A cohomogeneity-one choice of frame is one where

$$\bar{e}^\gamma = dr, \quad e^\alpha = h_{(\alpha)}(r) \bar{e}^\alpha, \quad (A.12)$$

so that

$$\omega^{\alpha\beta} = \frac{h_{(\alpha)}(r)}{h_{(\beta)}(r)} \bar{\omega}^{\alpha\beta}, \quad \omega^{\alpha\gamma} = h'_{(\alpha)}(r) \bar{e}^\alpha \quad (A.13)$$

We have already seen in the main text that for weak $G_2$-holonomy with $\lambda \neq 0$ this is incompatible with the self-duality condition $\psi_{\gamma\alpha\beta} \omega^{\alpha\beta} = -2\lambda \epsilon^\gamma$. So let now $\lambda = 0$. Then the self-duality conditions become

$$\psi_{\gamma\alpha\beta} \frac{h_{(\alpha)}(r)}{h_{(\beta)}(r)} \bar{\omega}^{\alpha\beta} = 0$$

$$2\psi_{\alpha\delta\gamma} h'_{(\delta)}(r) \bar{e}^\delta + \psi_{\alpha\beta\gamma} \frac{h_{(\beta)}(r)}{h_{(\gamma)}(r)} \bar{\omega}^{\beta\gamma} = 0 \quad (A.14)$$

In general, this provides constraints on the $h_{(\alpha)}(r)$ since all $r$ dependence is carried by them. One simple solution to solve the $r$-dependence in these equations is to take all $h_{(\alpha)}(r)$ equal and $h'_{(\alpha)}(r)$ to be a constant which can be chosen to equal 1. This solution of course corresponds to the case where $h_{(\alpha)}(r) = r$ and $X$ is a cone on $Y$. Then equations (A.14) simply translate into conditions on the choice of frame on $Y$:

$$\psi_{\gamma\alpha\beta} \bar{\omega}^{\alpha\beta} = 0, \quad \psi_{\alpha\beta\gamma} \bar{\omega}^{\beta\gamma} = -2\psi_{\gamma\alpha\beta} \bar{e}^{\gamma} \quad (A.15)$$
These conditions look similar to the self-duality conditions for weak $G_2$-holonomy, but they are conditions in 6 dimensions on $Y$. Actually, as pointed out in the main text, if $X$ is a cone on $Y$, the $G_2$-holonomy of $X$ implies that $Y$ has weak $SU(3)$-holonomy [15]. Similarly as in [12] one can show that for a weak $SU(3)$-holonomy manifold one can always chose a frame such that (A.15) holds. This then shows that for a $G_2$-holonomy manifold that is a cone on $Y$, the cohomogeneity-one frame can also be chosen to be self-dual.

6.4 Relating Hodge duals

We need to relate Hodge duals on the 7-manifolds $X$, $X_c$ or $X_\lambda$ to the Hodge duals on the 6-manifold $Y$. For the present purpose, we do not need to specify the 7-manifold and just call it $X_7$. We assume that the 7-beins of $X_7$, called $e^a$, and the 6-beins of $Y$, called $\tilde{e}^{\alpha}$ can be related by

$$e^7 = dr, \quad e^\alpha = h(r)\tilde{e}^{\alpha}. \tag{A.16}$$

We denote the Hodge dual of a form $\pi$ on $X_7$ simply by $^*\pi$ while the 6-dimensional Hodge dual of a form $\sigma$ on $Y$ is denoted $^{*Y}\sigma$.

The duals of $p$-forms on $X_7$ and on $Y$ are defined in terms of their respective vielbein basis, namely

$$^* (e^{a_1} \wedge \ldots \wedge e^{a_p}) = \frac{1}{(7-p)!} \epsilon^{a_1 \ldots a_p}_{b_1 \ldots b_{7-p}} e^{b_1} \wedge \ldots \wedge e^{b_{7-p}} \tag{A.17}$$

and

$${}^{*Y}(\tilde{e}^{\alpha_1} \wedge \ldots \wedge \tilde{e}^{\alpha_p}) = \frac{1}{(6-p)!} \epsilon^{a_1 \ldots a_p}_{\beta_1 \ldots \beta_{6-p}} \tilde{e}^{\beta_1} \wedge \ldots \wedge \tilde{e}^{\beta_{6-p}}. \tag{A.18}$$

Here the $\epsilon$-tensors are the “flat” ones that equal $\pm 1$. Expressing the $e^a$ in terms of the $\tilde{e}^\alpha$ provides the desired relation. In particular, for a $p$-form $\omega_p$ on $Y$ we have

$$^* (dr \wedge \omega_p) = h(r)^{6-2p} {}^*{}^Y \omega_p$$

$$^* \omega_p = (-)^p h(r)^{6-2p} dr \wedge {}^*{}^Y \omega_p, \tag{A.19}$$

where we denote both the form on $Y$ and its trivial extension onto $X_\lambda$ by the same symbol $\omega_p$. 

21
6.5 Curvature in \((n + 1)\) dimensions from curvature in \(n\) dimensions

Suppose that the metric \(ds_X^2\) on an \((n + 1)\)-dimensional manifold \(X\) is given in terms of the metric \(ds_Y^2\) on an \(n\)-dimensional manifold \(Y\) by

\[
ds_X^2 = dr^2 + h(r)^2 ds_Y^2.
\]

Similarly as before, denote vielbeins on \(Y\) by \(\tilde{e}^\alpha, \alpha = 1, \ldots, n\) and those on \(X\) by \(e^a, a = 0, 1, \ldots, n\) so that \(e^0 = dr\) and \(e^\alpha = h(r) \tilde{e}^\alpha\). Denote the spin-connections by \(\tilde{\omega}^{\alpha\beta}\) and \(\omega^{ab}\) respectively. Then

\[
\omega^{\alpha\beta} = \tilde{\omega}^{\alpha\beta}, \quad \omega^{00} = \frac{h'}{h} e^\alpha.
\]

It follows that the curvature 2-forms \(R^{ab}\) of \(X\) and \(\tilde{R}^{\alpha\beta}\) of \(Y\) are related as

\[
R^{\alpha\beta} = \tilde{R}^{\alpha\beta} - \left(\frac{h'}{h}\right)^2 e^{\alpha} \wedge e^{\beta}, \quad R^{00} = -\frac{h''}{h} e^0 \wedge e^0.
\]

Note that this implies that the components of the curvature tensors are related as

\[
R^{\alpha\beta\gamma\delta} = \frac{1}{h^2} \tilde{R}^{\alpha\beta\gamma\delta} - \left(\frac{h'}{h}\right)^2 \left(\delta^{\alpha}_\gamma \delta^{\beta}_\delta - \delta^{\alpha}_\delta \delta^{\beta}_\gamma\right), \quad R^{00\gamma0} = -\frac{h''}{h} \delta^{\alpha}_\gamma.
\]

This yields for the Ricci tensor

\[
\mathcal{R}_\gamma^\alpha = \frac{1}{h^2} \left(\tilde{\mathcal{R}}_\gamma^\alpha - (n - 1) h'^2 \delta^{\alpha}_\gamma - hh'' \delta^{\alpha}_\gamma\right), \quad \mathcal{R}_0^0 = \frac{1}{h^2} (-nhh''),
\]

and \(\mathcal{R}_\alpha^0 = \mathcal{R}_0^\alpha = 0\). Now suppose \(Y\) is an Einstein space with \(\tilde{\mathcal{R}}_\gamma^\alpha = \mu^2 \delta^{\alpha}_\gamma\) (\(\mu\) is real or imaginary, and necessarily vanishes if \(n = 1\)) so that

\[
\mathcal{R}_\gamma^\alpha = \frac{1}{h^2} \left(\mu^2 - (n - 1) h'^2 - hh''\right) \delta^{\alpha}_\gamma.
\]

Then \(X\) is an Einstein space with \(\mathcal{R}_c^a = n \lambda^2 \delta^{a}_c\) (\(\lambda\) real or imaginary) if and only if

\[
-\frac{h''}{h} = \lambda^2 \quad \text{and} \quad (n - 1) h'^2 = \mu^2 - (n - 1) \lambda^2 h^2.
\]

The first equation is solved by

\[
h(r) = r_0 \sin \lambda (r - r_1)
\]

and then, for \(n \geq 2\), the second equation implies

\[
r_0^2 \lambda^2 = \frac{\mu^2}{n - 1}.
\]

For \(n = 6\) and \(\mu^2 = 5\) we get eq. (3.16).
6.6 Properties of the hypergeometric equation

Here, we want to show that all solutions \( f(x) \) of the hypergeometric equation (4.25) with \( \mu_i > 0 \) that satisfy either of the two boundary conditions (4.26) are such that \( x^{b-3} \sqrt{x-1} \partial_x f_i^2(x) \) diverges as \( x \to \infty \).

The standard form of the hypergeometric equation is [16]

\[
x(1-x)f''(x) + [c-(a+b+1)x]f'(x) - abf(x) = 0 ,
\]

(A.29)

The coefficients of our hypergeometric equation (4.25) correspond to

\[
a = \frac{1}{2} \left( \tilde{p} + \sqrt{\tilde{p}^2 + \mu_i} \right) , \quad b = \frac{1}{2} \left( \tilde{p} - \sqrt{\tilde{p}^2 + \mu_i} \right) , \quad c = p - 3 ,
\]

(A.30)

with \( \tilde{p} \equiv p - \frac{7}{2} \). As is well-known [16], there is a systematic way to obtain pairs of linearly independent solutions of the hypergeometric differential equation with cuts on various combinations of the intervals \((-\infty, 0], [0, 1] \) and \([1, \infty) \). They are given by the Kummer solutions \( u_1, \ldots u_6 \) each of which can be expressed in four different ways. Since we implicitly assumed that the \( F_i(r) \) are real, we have to be careful to choose solutions \( f(x) \) that are real on the positive real axis for \( x \geq 1 \).

For our present purpose, it is convenient to consider the following pair of linearly independent solutions

\[
\begin{align*}
  u_2(x) &= F(a, b; a + b + 1 - c; 1 - x) , \\
  u_6(x) &= (1-x)^{c-a-b}F(c-a, c-b; c + 1 - a - b; 1 - x) ,
\end{align*}
\]

(A.31) (A.32)

where \( a, b, c \) are given above and \( F \) is the usual hypergeometric function. The function \( F \) has a cut when its argument \( 1-x \) is in the interval \([1, \infty) \). But this corresponds to \( x < 0 \) and is irrelevant to us. However, for \( u_6 \), the power of \( 1-x \) in front of \( F \) has a simple square-root branch cut from 1 to \( +\infty \) since \( c - a - b = \frac{1}{2} \). Hence we define

\[
\begin{align*}
  f_1(x) &= u_2(x) \\
  f_2(x) &= e^{i\pi(c-a-b)} u_6(x + i\epsilon) = (x - 1)^{c-a-b} F(c-a, c-b; c + 1 - a - b; 1 - x)
\end{align*}
\]

(A.33)

which are real and defined without any ambiguity for real \( x \geq 1 \) and satisfy the boundary condition (4.26) since

\[
\begin{align*}
  f_1(1) &= 1 , \quad 2\sqrt{x-1} \partial_x f_1(x) \big|_{x=1} = 0 , \\
  f_2(1) &= 0 , \quad 2\sqrt{x-1} \partial_x f_2(x) \big|_{x=1} = 1 .
\end{align*}
\]

(A.34)
To investigate the behaviour of these solutions as \( x \to \infty \), one uses the formula [16] for the analytic continuation expressing \( u_2 \) or \( u_6 \) as linear combinations of \( u_3 \) and \( u_4 \), both of which behave as powers of \( x \) as \( x \to \infty \):

\[
\begin{align*}
    f_1(x) & \sim \frac{\Gamma(a + b + 1 - c)\Gamma(b - a)}{\Gamma(b + 1 - c)\Gamma(b)} (x^{-a} + \ldots) + \frac{\Gamma(a + b + 1 - c)\Gamma(a - b)}{\Gamma(a + 1 - c)\Gamma(a)} (x^{-b} + \ldots), \\
f_2(x) & \sim \frac{\Gamma(c + 1 - a - b)\Gamma(b - a)}{\Gamma(1 - a)\Gamma(c - a)} (x^{-a} + \ldots) + \frac{\Gamma(c + 1 - a - b)\Gamma(a - b)}{\Gamma(1 - b)\Gamma(c - b)} (x^{-b} + \ldots),
\end{align*}
\]

(A.35)

where the dots indicate terms that are subleading by integer powers of \( x \). In our case, \(-b > -a\) and the leading terms in \( f_1 \) and \( f_2 \) as \( x \to \infty \) are the \( x^{-b}\)-terms, unless their coefficients, which we will call \( d_1 \) and \( d_2 \), happen to vanish. This can only happen if \( a + 1 - c, \ a, \ 1 - b \) or \( c - b \) is a non-positive integer, which is impossible since \( \sqrt{\tilde{\rho}^2 + \mu_i} > \tilde{\rho} \). We conclude that, as \( x \to \infty \), for both solutions we have

\[
x^{p-3} \sqrt{x - 1} \partial_x f_i^2(x) \sim d_i^2 x^{\tilde{\rho} - 2b} = d_i^2 x^{\sqrt{\tilde{\rho}^2 + \mu_i}} \quad \text{as} \ x \to \infty \ , \ i = 1, 2 .
\]

(A.36)

This diverges for all \( p \) and \( \mu_i > 0 \) and, by eq. (4.28), so does the corresponding contribution to \( \| \phi^{(3)}_p \|^2 \).

**References**

[1] R. Bryant and S. Salomon, *On the construction of complete metrics with exceptional holonomy*, Duke Math. Journal 58 (1989) 829; G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on \( S^3 \), \( R^3 \) and \( R^4 \) bundles*, Comm. Math. Phys. 127 (1990) 529.

[2] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy \( G_2 \)* I, J. Diff. Geom. 43 (1996) 291, and *idem* II, J. Diff. Geom. 43 (1996) 329.

[3] D.D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford 2000.

[4] B.S. Acharya, *M-theory, Joyce orbifolds and super Yang-Mills*, Adv. Theor. Math. Phys. 3, 227, hep-th/9812205.
[5] M. Cvetic, G. Shiu and A.M. Uranga, *Chiral Four-Dimensional N=1 Supersymmetric Type IIA Orientifolds from Intersecting D6-Branes*, Nucl. Phys. **B615** (2001) 3, hep-th/0107166.

[6] M. Atiyah and E. Witten, *M-theory dynamics on a manifold with G_2-holonomy*, Adv. Theor. Math. Phys. **6** (2003) 1, hep-th/0107177.

[7] B.S. Acharya and E. Witten, *Chiral Fermions from Manifolds of G_2 Holonomy*, hep-th/0109152.

[8] E. Witten, *Anomaly cancellation on G_2-manifolds*, hep-th/0108165.

[9] A. Bilal and S. Metzger, *Anomalies in M-theory on singular G_2-manifolds*, hep-th/0303243.

[10] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory flop as a large N duality*, J. Math. Phys. **42** (2001) 3209, hep-th/0011256.

[11] T. Friedmann, *On the quantum moduli space of M-theory compactifications*, Nucl. Phys. **B635** (2002) 384, hep-th/0203256.

[12] A. Bilal, J.-P. Derendinger and K. Sfetsos, *(Weak) G_2-holonomy from self-duality, flux and supersymmetry*, Nucl. Phys. **B628** (2002) 11, hep-th/0111274.

[13] H. Kanno and Y. Yasui, *On Spin(7) holonomy based on SU(3)/U(1)*, J. Geom. Phys. **43** (2002) 293, hep-th/0108226.

[14] R. Cleyton and A. Swann, *Cohomogeneity-one G_2 structures*, J. Geom. Phys. **44** (2002) 202, math.DG/0111056.

[15] N. Hitchin, *Stable forms and special metrics*, in: Proc. Congress in memory of Alfred Gray, (eds M. Fernandez and J. Wolf), AMS Contemporary Mathematics Series, math.DG/0107101.

[16] A. Erdelyi *et al*, *Higher Transcendental Functions*, vol. I, McGraw Hill, New York, 1953-1955.