SEMI-INVERTIBLE EXTENSIONS AND ASYMPOTIC HOMOMORPHISMS

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Abstract. We consider the semigroup Ext($A, B$) of extensions of a separable C*-algebra $A$ by a stable C*-algebra $B$ modulo unitary equivalence and modulo asymptotically split extensions. This semigroup contains the group Ext$^{-1/2}(A, B)$ of invertible elements (i.e. of semi-invertible extensions). We show that the functor Ext$^{1/2}(A, B)$ is homotopy invariant and that it coincides with the functor of homotopy classes of asymptotic homomorphisms from $C(T) \otimes A$ to $M(B)$ that map $SA \subseteq C(T) \otimes A$ into $B$.

1. Introduction

This is a study of a general structure in the extensions of a separable C*-algebra by another separable and stable C*-algebra. The significance of such extensions comes from many applications, but is perhaps best illustrated by the fact that all the common homotopy invariant and stable functors on the category of separable C*-algebras admit descriptions in terms of such C*-extensions. To explain our viewpoint on these extensions, which originates from our work in [15] and the problems which it naturally leads us to consider, we must put the results and methods from [15] into perspective.

The main discovery in [15] was that the E-theory of Connes and Higson is the quotient of the unitary equivalence classes of extensions by the asymptotically split extensions, provided the C*-algebras that play the roles of quotient and ideal in the extensions are, respectively, suspended and stable. This reveals that if the role of the split extensions, which has served as the natural trivial extensions since the work of Brown, Douglas and Fillmore, [2], [3], are replaced by the asymptotically split extensions, then the question about invertibility of the extensions disappear, at least when the quotient is a suspended C*-algebra. The significance of this is stressed by the (albeit slowly) growing number of examples of extensions which are not invertible in the BDF sense, [4], [19], [17], [10], [8], [6]. Among these, Kirchbergs examples are the most striking in our optic because they show that the BDF semi-group of extensions fail to be a group for a large class of naturally occuring C*-algebras, in cases where the homotopy classes of extensions do form a group.

An important point concerning the methods used in [15] is that they provide proofs of homotopy invariance in the class of unitary equivalence classes of extensions modulo the asymptotically split extensions by using the relation to asymptotic homomorphisms given by the Connes-Higson construction, [4]. This is a completely new approach to homotopy invariance in the theory of C*-algebra extensions which is independent of the methods which were developed for this in [4] and [9].

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However, the methods in [15] require in an essential way that the \( C^* \)-algebra which plays the role of the quotient in the extension is a suspended \( C^* \)-algebra. This is annoying because it means that general \( C^* \)-algebra extensions must be suspended in order to become amenable to the methods and results of [15], and this is particularly frustrating because the key tool from [15], the Connes-Higson construction, is available for any \( C^* \)-extension. The most crucial reason for the success of the methods developed in [15] is that every extension is semi-invertible, in the sense that it can be made asymptotically split by adding another extension to it, when the quotient is a suspended \( C^* \)-algebra. One of the main questions left open by [15] is therefore

**Question:** Does the Connes-Higson construction, in the general case, provide us with an isomorphism, from unitary equivalence classes of semi-invertible extensions modulo asymptotically split extensions to homotopy classes of asymptotic homomorphisms?

At present we do not know, in the general case, if every extension of a separable \( C^* \)-algebra by a separable stable \( C^* \)-algebra is semi-invertible. All the examples mentioned above of extensions that fail to be invertible in the BDF sense may very well turn out to be semi-invertible. In fact, it follows from [15] that the examples of Kirchberg, [10], are semi-invertible. Thus we must also ask:

**Question:** Are all extensions of a separable \( C^* \)-algebra by a separable stable \( C^* \)-algebra semi-invertible?

The main purpose here is to answer the first question by a qualified 'Yes'. More precisely we show that a variant of the Connes-Higson construction, which takes the semi-invertibility of the extensions into account, does give rise to an isomorphism. Unfortunately this does not, in itself, answer the question for the genuine Connes-Higson map.

### 2. The group of semi-invertible extensions

Let \( A \) and \( B \) be separable \( C^* \)-algebras, \( B \) stable. Let \( M(B) \) be the multiplier algebra of \( B \) and \( Q(B) = M(B)/B \) the generalized Calkin-algebra of \( B \). Let \( q_B : M(B) \to Q(B) \) be the quotient map. The extensions of \( A \) by \( B \) will be identified with \( \text{Hom}(A,Q(B)) \); the \(*\)-homomorphisms from \( A \) to \( Q(B) \). Two extensions \( \varphi, \psi \in \text{Hom}(A,Q(B)) \) are unitary equivalent when there is a unitary \( u \in M(B) \) such that \( \text{Ad} q_B(u) \circ \varphi = \psi \). An extension \( \psi \in \text{Hom}(A,Q(B)) \) is asymptotically split when there is an asymptotic homomorphism \( \pi = (\pi_t)_{t \in [1,\infty]} : A \to M(B) \) such that \( q_B \circ \pi_t = \psi \) for all \( t \). Thanks to the stability of \( B \), the unitary equivalence classes in \( \text{Hom}(A,Q(B)) \) form an abelian semi-group: Choose isometries \( V_1, V_2 \in M(B) \) such that \( V_1 V_1^* + V_2 V_2^* = 1 \), and define \( \varphi \oplus \psi \) to be the extension \( a \mapsto \text{Ad} q_B(V_1) \circ \varphi(a) + \text{Ad} q_B(V_2) \circ \psi(a) \). Then the addition in the unitary equivalence classes in \( \text{Hom}(A,Q(B)) \) is given by \( [\varphi] + [\psi] = [\varphi \oplus \psi] \). This addition is independent of the choice of isometries \( V_1, V_2 \), subject to the condition that \( V_1 V_1^* + V_2 V_2^* = 1 \). We say that the extension \( \varphi \in \text{Hom}(A,Q(B)) \) is semi-invertible when there is another extension \( \psi \) such that \( \varphi \oplus \psi \) is asymptotically split. Both the semi-invertible and the asymptotically split extensions represent a semi-group in the unitary equivalence classes of extensions; the latter contained in the first, and we denote the 'quotient' by \( \text{Ext}^{-1/2}(A,B) \). Thus two semi-invertible extensions, \( \varphi \) and \( \psi \), define the same element of \( \text{Ext}^{-1/2}(A,B) \) if and only if there are asymptotically split extensions, \( \lambda_1, \lambda_2 \), such that \( \varphi \oplus \lambda_1 \) is unitarily equivalent to \( \psi \oplus \lambda_2 \). The main goal of the paper is to obtain a description of \( \text{Ext}^{-1/2}(A,B) \) in terms of asymptotic homomorphisms. For this purpose we set \( \text{Ext}^{-1/2}(A,D) = \text{Ext}^{-1/2}(A,D \otimes \mathbb{K}) \), where \( \mathbb{K} \) is the \( C^* \)-algebra of compact operators on a separable infinite dimensional Hilbert
space, when $D$ is a separable $C^*$-algebra which is not stable. Note that $\text{Ext}^{-1/2}(A, D)$ is functorial (contravariantly) in an obvious way in the first variable $A$. In the second variable, $D$, there is a priori only functoriality with respect to quasi-unital $*$-homomorphisms, cf. [14], in a way we now describe. Given a quasi-unital $*$-homomorphism $\varphi : D \to D_1$, the tensor product, $\varphi \otimes \text{id}_K : D \otimes K \to D_1 \otimes K$, of $\varphi$ with the identity on $K$ is again quasi-unital and admits therefore an extension $\varphi \otimes \text{id}_K : M(D \otimes K) \to M(D_1 \otimes K)$ which, in turn, defines a $*$-homomorphism $\hat{\varphi} \otimes \text{id}_K : Q(D \otimes K) \to Q(D_1 \otimes K)$. We set $\varphi_*[\psi] = [\varphi \otimes \text{id}_K \circ \psi]$. When $e \in K$ is a minimal non-zero projection, we define $s_D : D \to D \otimes K$ by $s_D(d) = d \otimes e$.

**Lemma 2.1.** $s_{D*} : \text{Ext}^{-1/2}(A, D) \to \text{Ext}^{-1/2}(A, D \otimes K)$ is an isomorphism.

**Proof.** This is all very standard: As is well-known, there is an isometry $V \in M(D \otimes K \otimes K)$ and an isomorphism $\gamma : D \otimes K \to D \otimes K \otimes K$ such that $\text{Ad} V \circ \gamma = s_D \otimes \text{id}_K$. It suffices therefore to show that conjugation by the isometry $V$ induces the identity map of $\text{Ext}^{-1/2}(A, D \otimes K)$, and this is clear because conjugation by $V$ is just addition by the trivial extension 0.

In other words, the functor $\text{Ext}^{-1/2}(A, -)$ is stable, and there is no reason to distinguish between $\text{Ext}^{-1/2}(A, B)$ and $\text{Ext}^{-1/2}(A, B \otimes K)$ when $B$ is a stable separable $C^*$-algebra.

### 3. Pairing $\text{Ext}^{-1/2}$ with $KK$-Theory

In this section we prove homotopy invariance of $\text{Ext}^{-1/2}$ in the second variable. Homotopy invariance in the first variable is an immediate consequence. Unlike the approach taken in [15], the proof hinges on Kasparov’s homotopy invariance result from [9], in the more abstract guise it was given by Higson in [7].

Recall that an asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [1, \infty)} : A \to B$ between $C^*$-algebras is **equi-continuous** when the family of maps, $\varphi_t : A \to B, t \in [1, \infty)$, is an equi-continuous family of maps. As is well-known any asymptotic homomorphism is asymptotically equal to one which is equi-continuous. We shall make use of the following generalisation of this fact. The proof exploits the so-called asymptotic algebra of a given $C^*$-algebra $E$, via the Bartle-Graves selection theorem. Let $C_b([1, \infty), E)$ be the $C^*$-algebra of continuous and norm-bounded $E$-valued function on $[1, \infty)$ and $C_0([1, \infty), E)$ the ideal in $C_b([1, \infty), E)$ consisting of elements $f$ for which $\lim_{t \to \infty} \|f(t)\| = 0$. The **asymptotic algebra** $\text{as}(E)$ of $E$ is the quotient

$$\text{as}(E) = C_b([1, \infty), E) / C_0([1, \infty), E).$$

**Lemma 3.1.** Let $A, B, D$ be $C^*$-algebras, $A_0 \subset A$, $B_0 \subset B$ $C^*$-subalgebras and $\chi : B \to D$ a $*$-homomorphism. Let $\pi = (\pi_t)_{t \in [1, \infty)} : A \to B$ be an asymptotic homomorphism and $\mu : A \to D$ a $*$-homomorphism such that $\chi \circ \pi_t = \mu$ for all $t \in [1, \infty)$. Assume that $\pi_t(A_0) \subset B_0$ for all $t \in [1, \infty)$.

It follows that there is an equi-continuous asymptotic homomorphism $\tilde{\pi} : A \to B$ such that

1) $\lim_{t \to \infty} \pi_t(a) - \tilde{\pi}_t(a) = 0$ for all $a \in A$,
2) $\chi \circ \tilde{\pi}_t = \mu$ for all $t \in [1, \infty)$,
3) $\tilde{\pi}_t(A_0) \subset B_0$ for all $t \in [1, \infty)$. 

Proof. Set $P = \{(a, b) \in A \oplus B : \mu(a) = \chi(b)\}$, and $\tilde{\pi}_t(a) = (a, \pi_t(a)) \in P$. Then $\tilde{\pi}$ is an asymptotic homomorphism, and defines in a natural way a $\ast$-homomorphism $\tilde{\pi} : A \to \text{as}(P)$ into the asymptotic algebra of $P$. Since $\pi_t(A_0) \subseteq B_0$ by assumption,

$$\pi_t(A_0) \subseteq \text{as}(P_0),$$  

where $P_0 = \{(a, b) \in P : a \in A_0, \ b \in B_0\}$. It follows from the Bartle-Graves selection theorem that there is a continuous lift $\Phi : A \to C_b([1, \infty), P)$ of $\pi$. For a detailed account of the Bartle-Graves selection theorem we refer to [13], where there is also an important remark, Remark 2 on p. 114, that we shall use: Because of (3.1) we can choose $\Phi$ such that $\Phi(A_0) \subseteq P_0$. Set $\tilde{\pi}_t(a) = p(\Phi(a)(t))$, where $p : P \to B$ is the projection to the second coordinate.

Let $A, B, D$ be separable $C^*$-algebras, $B$ and $D$ stable. Let $\psi' \in \text{Hom}(A, Q(B))$ be a semi-invertible extension, and $x$ an element of $KK(B, D)$. $x$ is then represented, in the picture of $KK$-theory obtained in [18], by a pair of $\ast$-homomorphisms $\pi_{\pm} : M(B) \to M(D)$ such that

$$\pi_\pm(b) - \pi_\mp(b) \in D$$  

for all $b \in B$. Since $\psi'$ is semi-invertible, there is an asymptotic homomorphism $\psi = (\psi_t)_{t \in [1, \infty)} : A \to M_2(M(B))$, given in matrix notation as

$$\psi_t = \begin{pmatrix} \psi_{11}^t & \psi_{12}^t \\ \psi_{21}^t & \psi_{22}^t \end{pmatrix},$$

such that $q_{M_2(B)} \circ \psi_t$ is $t$-independent and $q_B \circ \psi_{11}^t = \psi'$ for all $t \in [1, \infty)$. In particular,

$$\psi_{12}^t(a), \psi_{21}^t(a) \in B$$  

for all $t, a$. Note that by Lemma 3.1 we can assume that $\psi$ is equi-continuous. We will refer $\psi$ as a trivialization of $\psi'$. Set

$$(\pi_{\pm} \times \psi)_t(a) = q_{M_2(D)} \begin{pmatrix} \pi_+ (\psi_{11}^t(a)) & \pi_+ (\psi_{12}^t(a)) \\ \pi_+ (\psi_{21}^t(a)) & \pi_+ (\psi_{22}^t(a)) \end{pmatrix},$$

Lemma 3.2. $((\pi_{\pm} \times \psi)_t)_{t \in [1, \infty)} : A \to M_2(Q(D))$ is an asymptotic homomorphism.

Proof. Calculating modulo $M_2(D)$ we find that

$$\begin{pmatrix} \pi_+ (\psi_{11}^t(a)) & \pi_+ (\psi_{12}^t(a)) \\ \pi_+ (\psi_{21}^t(a)) & \pi_+ (\psi_{22}^t(a)) \end{pmatrix} = \begin{pmatrix} \pi_+ (\psi_{11}^t(a)\psi_{11}^t(b) + \psi_{12}^t(a)\psi_{21}^t(b)) & \pi_+ (\psi_{11}^t(a)\psi_{12}^t(b) + \psi_{12}^t(a)\psi_{22}^t(b)) \\ \pi_+ (\psi_{21}^t(a)\psi_{11}^t(b)) + \pi_+ (\psi_{22}^t(a)\psi_{21}^t(b)) & \pi_+ (\psi_{21}^t(a)\psi_{12}^t(b) + \psi_{22}^t(a)\psi_{22}^t(b)) \end{pmatrix} = \begin{pmatrix} \pi_+ (\psi_{11}^t(a)\psi_{11}^t(b) + \psi_{12}^t(a)\psi_{21}^t(b)) & \pi_+ (\psi_{11}^t(a)\psi_{12}^t(b) + \psi_{12}^t(a)\psi_{22}^t(b)) \\ \pi_+ (\psi_{21}^t(a)\psi_{11}^t(b) + \psi_{22}^t(a)\psi_{21}^t(b)) & \pi_+ (\psi_{21}^t(a)\psi_{12}^t(b) + \psi_{22}^t(a)\psi_{22}^t(b)) \end{pmatrix},$$

thanks to (3.2) and (3.3). Since $\psi$ is an asymptotic homomorphism the last expression is asymptotically equal to

$$\begin{pmatrix} \pi_+ (\psi_{11}^t(ab)) & \pi_+ (\psi_{12}^t(ab)) \\ \pi_+ (\psi_{21}^t(ab)) & \pi_+ (\psi_{22}^t(ab)) \end{pmatrix}.$$
Let $\hat{\Theta} : M_2(Q(D)) \to Q(D)$ be a $*$-isomorphism induced by two isometries $V_1, V_2 \in M(D)$ such that $V_1 V_1^* + V_2 V_2^* = 1$.

**Lemma 3.3.** Assume that $\psi' \in \text{Hom}(A, Q(B))$ is asymptotically split. It follows that there are asymptotic homomorphisms $\mu, \nu : A \to M(D)$ such that

$$
(\hat{\Theta} \circ (\pi_\pm \times \psi)_t) \oplus (q_D \circ \mu_t) = (q_D \circ \nu_t)
$$

for all $t$.

**Proof.** Since $\psi'$ is asymptotically split, there is an asymptotic homomorphism $\theta : A \to M(B)$ such that $\psi'_t(a) - \theta_t(a) \in B$ for all $t, a$. By Lemma 3.1, we can assume that $\theta$ is equi-continuous. Let

$$
E_t = \pi_-(B),
$$

$$
\mathcal{F} = \{\pi_- \circ \theta_t(a) + C\pi_-(1) : t \in [1, \infty), a \in A\},
$$

and

$$
X = \{\theta_t(a) + \psi'_t(a') + C1 : t \in [1, \infty), a, a' \in A\}.
$$

Both $X$ and $\mathcal{F}$ are separable sets since the involved asymptotic homomorphisms are equi-continuous. Let $E_2$ be the $C^*$-algebra generated by $\{\pi_-(x) - \pi_+(x) : x \in X\}$. Then $E_1 E_2 \subseteq D$ and $\mathcal{F} E_1 \subseteq E_1$. Thus Kasparov’s technical theorem, [9], provides us with $M, N \in M(D)$ such that $M, N \geq 0, M + N = 1, ME_1 \subseteq D, NE_2 \subseteq D$ and $[N, \mathcal{F}] \subseteq D$. Set

$$
U = q_{M_2(D)} \begin{pmatrix} \pi_-(1) \sqrt{M} & -\pi_-(1) \sqrt{N} \\ \pi_-(1) \sqrt{N} & \pi_+(1) \sqrt{M} \end{pmatrix}.
$$

Since $\pi_-(1) \in \mathcal{F}$, we see that both $N$ and $M$ commute with $\pi_-(1)$ modulo $D$. Since $1 \in X$ and $NE_2 \subseteq D$, we see that $N\pi_-(1) = N\pi_+(1)$ modulo $D$. In particular, $N$ and $M$ both commute with $\pi_-(1)$ and $\pi_+(1)$ modulo $D$. It follows that $U + U^* \geq 0$. Using this, and that $UU^* = U^* U = q_{M_2(D)} \begin{pmatrix} \pi_-(1) & 0 \\ \pi_+(1) & 0 \end{pmatrix}$, we can lift $U$ to an element $V \in M_2(M(D))$ such that $V + V^* \geq 0$ and $VV^* = V^* V = \begin{pmatrix} \pi_-(1) & 0 \\ \pi_+(1) & 0 \end{pmatrix}$. Set $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V$, and note that

$$
S = \begin{pmatrix} \pi_-(1) \sqrt{N} & \pi_+(1) \sqrt{M} \\ \pi_-(1) \sqrt{M} & -\pi_-(1) \sqrt{N} \end{pmatrix} \text{ modulo } M_2(D).
$$

It follows that

$$
W = (S_1) + \begin{pmatrix} 0 & 1-\pi_-(1) \\ 1-\pi_-(1) & 0 \end{pmatrix}
$$

is a unitary in $M_3(M(D))$ such that

$$
W = \begin{pmatrix} \pi_-(\theta_t(a)) & 0 \\ 0 & \pi_+(\psi_1^t(a)) \pi_+(\psi_2^t(a)) \pi_+(\psi_2^t(a)) \end{pmatrix} = \begin{pmatrix} \pi_+(\theta_t(a)) & 0 \\ 0 & \pi_-(\psi_1^t(a)) \pi_-(\psi_1^t(a)) \pi_-(\psi_1^t(a)) \end{pmatrix} \text{ modulo } M_3(D).
$$

It follows from the properties of $N$ and $M$ that

$$
W \begin{pmatrix} \pi_-(\theta_t(a)) & 0 \\ 0 & \pi_+(\psi_1^t(a)) \pi_+(\psi_2^t(a)) \end{pmatrix} = \begin{pmatrix} \pi_+(\theta_t(a)) & 0 \\ 0 & \pi_-(\psi_1^t(a)) \pi_-(\psi_1^t(a)) \pi_-(\psi_1^t(a)) \end{pmatrix} W
$$

□
modulo $M_3(D)$ for all $t, a$. Set $\mu_t = q_D \circ \pi_+ \circ \theta_t$ and

$$\nu_t^0 = \begin{pmatrix} \pi_+ (\theta_t(a)) & 0 \\ 0 & \pi_- (\psi_1^t(a)) \end{pmatrix}.$$

Finally, we choose an appropriate isomorphism $\Theta_0 : M_3(M(D)) \to M(D)$ and set $\nu_t = \Theta_0 \circ \nu_t^0$.

**Lemma 3.4.** Let $\varphi', \psi' \in \text{Hom}(A, Q(D))$ be semi-invertible extensions with trivializations $\varphi$ and $\psi$, respectively. There is then a trivialization $\lambda$ of $\varphi' \oplus \psi'$ such that $\pi_\lambda \times \lambda$ is unitarily equivalent to $(\pi_\varphi \times \psi) \oplus (\pi_\psi \times \varphi)$.

**Proof.** Let $V_1, V_2 \in M(B)$ be the isometries used to define the addition in $\text{Ext}^{-1/2}(A, B)$. Then $\psi' \oplus \varphi' = q_D \circ (\text{Ad} V_1 \circ \psi_1^t + \text{Ad} V_2 \circ \varphi_1^t)$ for all $t$, and

$$\lambda_t = \begin{pmatrix} \text{Ad} V_1 \circ \psi_1^t + \text{Ad} V_2 \circ \varphi_1^t & \text{Ad} V_1 \circ \psi_2^t + \text{Ad} V_2 \circ \varphi_2^t \\ \text{Ad} V_1 \circ \psi_2^t + \text{Ad} V_2 \circ \varphi_1^t & \text{Ad} V_1 \circ \psi_1^t + \text{Ad} V_2 \circ \varphi_2^t \end{pmatrix}$$

is a trivialization of $\psi' \oplus \varphi'$. Note that

$$(\pi_\varphi \times \lambda)_t = (\text{Ad} \pi_+(V_1) \circ \pi_+ \circ \psi_1^t + \text{Ad} \pi_+(V_2) \circ \pi_+ \circ \varphi_1^t, \text{Ad} \pi_+(V_1) \circ \pi_+ \circ \psi_1^t + \text{Ad} \pi_+(V_2) \circ \pi_+ \circ \varphi_1^t),$$

Modulo $D$ we have that

$$\pi_+(V_1) (\pi_+ \circ \psi_1^t(a)) \pi_-(V_1^*) = \pi_+(V_1 \psi_1^t(a)) \pi_-(V_1^*)$$

(by (3.2) and (3.3))

$$= \pi_-(V_1 \psi_1^t(a)) \pi_-(V_1^*)$$

(by (3.2) and (3.3))

$$= \text{Ad} \pi_+(V_1) \circ \pi_+ \circ \psi_1^t(a).$$

Via similar considerations regarding $\text{Ad} \pi_+(V_2) \circ \pi_+ \circ \varphi_1^t$, $\text{Ad} \pi_+(V_1) \circ \pi_+ \circ \psi_2^t$, and $\text{Ad} \pi_+(V_2) \circ \pi_+ \circ \varphi_2^t$, we see that

$$(\pi_\varphi \times \lambda)_t = \text{Ad} \left(\begin{pmatrix} \pi_+(V_1) & 0 \\ 0 & \pi_-(V_1) \end{pmatrix} \right) \circ \left(\begin{pmatrix} \pi_+ \circ \psi_1^t & \pi_+ \circ \psi_2^t \\ \pi_+ \circ \psi_2^t & \pi_+ \circ \psi_1^t \end{pmatrix} \right) + \text{Ad} \left(\begin{pmatrix} \pi_+(V_2) & 0 \\ 0 & \pi_-(V_2) \end{pmatrix} \right) \circ \left(\begin{pmatrix} \pi_+ \circ \varphi_1^t & \pi_+ \circ \varphi_2^t \\ \pi_+ \circ \varphi_2^t & \pi_+ \circ \varphi_1^t \end{pmatrix} \right),$$

modulo $M_2(D)$. We conclude that

$$(\pi_\varphi \times \lambda)_t = \text{Ad} S_1 \circ (\pi_\varphi \times \psi)_t + \text{Ad} S_2 \circ (\pi_\varphi \times \varphi)_t$$

modulo $M_2(D)$, where

$$S_i = \begin{pmatrix} \pi_+(V_i) & - \pi_-(V_i) \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Thus, up to unitary equivalence, we have that $\pi_\varphi \times \lambda = (\pi_\varphi \times \psi) \oplus (\pi_\varphi \times \varphi)$. 


Lemma 3.2 of [16]. Given such a pair, there is a sequence for all $i$ when $\lim_{i} D$ of norm cf. Lemma 3.3 of [16]. We say that the pair for all $a,b$ $\sqrt{\phi}$ the basic construction is called $\phi$ morphism $\phi$ $\phi$. We now introduce the basic construction of [16]. Given an equi-continuous asymptotic homomorphism $\varphi = (\varphi_{t})_{t \in [1,\infty)} : A \to Q(D)$, the basic construction gives us a genuine extension $\varphi^{\prime} \in \text{Hom}(A,Q(D))$. The construction goes as follows: Let $b$ be a strictly positive element in $D$ of norm $\leq 1$. A unit sequence (cf. [15]) in $D$ is a sequence $\{u_{n}\}_{n=0}^{\infty} \subseteq E$ such that

u1) there is a continuous function $f_{n} : [0,1] \to [0,1]$ which is zero in a neighbourhood of 0 and $u_{n} = f_{n}(b)$,
u2) $u_{n+1}u_{n} = u_{n}$ for all $n$,
u3) $\lim_{n \to \infty} u_{n}x = x$ for any $x \in D$.

Unit sequences exist by elementary spectral theory. Given a unit sequence $\{u_{n}\}$ we set $\Delta_{0} = \sqrt{u_{0}}$ and $\Delta_{j} = \sqrt{u_{j} - u_{j-1}}, j \geq 1$. Note that u2) implies that

$$\Delta_{i}\Delta_{j} = 0, |i - j| \geq 2. \quad (3.4)$$

Let $\hat{\varphi}_{t} : A \to M(D)$ be an equi-continuous lift of $\varphi$, cf. Lemma 2.1 of [16]. There exists a sequence $t_{1} < t_{2} < t_{3} < \ldots$ in $[1,\infty)$ such that $\{\varphi_{t_{n}}\}_{n=1}^{\infty}$ is a discretization of $\varphi$ and
t1) $\lim_{n \to \infty} \sup_{t \in [t_{n},t_{n+1}]} \|\hat{\varphi}_{t}(a) - \hat{\varphi}_{t_{n}}(a)\| = 0$ for all $a \in A$, and
t2) $t_{n} \leq n$ for all $n \in \mathbb{N}$,

cf. Lemma 3.3 of [16]. We say that the pair $(\{\hat{\varphi}_{t}\}_{t \in [1,\infty)}, \{u_{n}\}_{n=0}^{\infty})$ is a compatible pair for $\varphi$ when $\lim_{n \to \infty} \sup_{t \in [1,n+1]} \|u_{n}\hat{\varphi}_{t}(a) - \hat{\varphi}_{t}(a)u_{n}\| = 0$ for all $a \in D$. Compatible pairs exist by Lemma 3.2 of [16]. Given such a pair, there is a sequence $n_{0} < n_{1} < n_{2} < \ldots$ in $\mathbb{N}$ such that

$$n_{i} - n_{i-1} > i + 1$$

for all $i \geq 1$,

$$\lim_{i \to \infty} \sup_{j \geq n_{i}} \sup_{t \in [1,i+3]} \| (1 - u_{j}) (\hat{\varphi}_{t}(a)\hat{\varphi}_{t}(b) - \hat{\varphi}_{t}(ab)) - \| \varphi_{t}(a)\varphi_{t}(b) - \varphi_{t}(ab)\| = 0,$$

$$\lim_{i \to \infty} \sup_{j \geq n_{i}} \sup_{t \in [1,i+3]} \| (1 - u_{j}) (\hat{\varphi}_{t}(a) + \lambda \hat{\varphi}_{t}(b) - \hat{\varphi}_{t}(a + \lambda b)) - \| \varphi_{t}(a) + \lambda \varphi_{t}(b) - \varphi_{t}(a + \lambda b)\| = 0,$$

$$\lim_{i \to \infty} \sup_{j \geq n_{i}} \sup_{t \in [1,i+3]} \| (1 - u_{j}) (\hat{\varphi}_{t}(a^{*}) - \hat{\varphi}_{t}(a^{*})) - \| \varphi_{t}(a^{*}) - \varphi_{t}(a^{*})\| = 0$$

for all $a,b \in A$ and all $\lambda \in \mathbb{C}$, cf. Lemma 3.4 of [16]. The quadruple

$$\left(\{\hat{\varphi}_{t}\}_{t \in [1,\infty)}, \{u_{n}\}_{n=0}^{\infty}, \{n_{i}\}_{i=0}^{\infty}, \{t_{i}\}_{i=1}^{\infty}\right)$$

is called the folding data. Given the folding data there is then an extension $\varphi^{\prime} : A \to Q(D)$ such that

$$\varphi^{\prime}(a) = q_{D} \left( \sum_{j=0}^{\infty} \Delta_{j}\hat{\varphi}_{t_{j+1}}(a)\Delta_{j} \right)$$

for all $a \in A$, cf. Lemma 3.5 of [16]. We will refer to $\varphi^{\prime}$ as a folding of $\varphi$.

We claim that we can define a map

$$\text{Ext}^{-1/2}(A,B) \ni [\psi^{\prime}] \mapsto \pi_{\pm} \cdot [\psi^{\prime}] \in \text{Ext}^{-1/2}(A,D)$$
by setting \( \pi_\pm \bullet [\psi'] = \left[ (\hat{\Theta} \circ (\pi_\pm \times \psi))' \right] \), where \( \psi \) is an arbitrary trivialization of \( \psi' \). To see that this recipe is well-defined we must show that \( \left[ (\hat{\Theta} \circ (\pi_\pm \times \psi))' \right] \) is independent of all the choices involved in its construction, and in fact only depends on the class \([\psi']\) of \( \psi' \in \text{Ext}^{-1/2}(A, B) \). For the first purpose, let \( \varphi' \in \text{Hom}(A, Q(B)) \) be an extension such that \( \varphi' \oplus \psi' \) is asymptotically split, and let \( \varphi \) and \( \psi \) be trivializations of \( \varphi' \) and \( \psi' \), respectively. It follows then from Lemma 3.3 and Lemma 3.4 that there are asymptotic homomorphisms \( \nu^1, \nu^2 : A \to M(D) \) such that

\[
\left( \hat{\Theta} \circ (\pi_\pm \times \varphi) \right) + \left( \hat{\Theta} \circ (\pi_\pm \times \psi) \right) \oplus q_D \circ \nu^1 = q_D \circ \nu^2
\]

for all \( t \in [1, \infty) \). Then Lemma 4.4 of [16] implies that

\[
\left[ (\hat{\Theta} \circ (\pi_\pm \times \psi))' \right] = - \left[ \left( \left( \hat{\Theta} \circ (\pi_\pm \times \varphi) \right) \oplus (q_D \circ \nu^1) \right)' \right] = - \left[ \left( \hat{\Theta} \circ (\pi_\pm \times \varphi) \right)' \right]
\]

in \( \text{Ext}^{-1/2}(A, D) \). Thus \( \left[ (\hat{\Theta} \circ (\pi_\pm \times \psi))' \right] \) only depends on \( \psi' \in \text{Hom}(A, Q(D)) \). To show that \( \left[ (\hat{\Theta} \circ (\pi_\pm \times \psi'))' \right] \) only depends on the class of \( \psi' \) in \( \text{Ext}^{-1/2}(A, B) \), it suffices now, thanks to Lemma 3.4 and Lemma 4.5 of [16], only to show that the class is not changed when \( \psi' \) is replaced by an extension unitarily equivalent to it. We leave this to the reader.

We want to show that the map \( \pi_\pm \bullet - : \text{Ext}^{-1/2}(A, B) \to \text{Ext}^{-1/2}(A, D) \) only depends on the class of \( (\pi_+, \pi_-) \) in \( KK(B, D) \). For this purpose, we shall use the homotopy-invariance theorem of Higson, cf. Section III of [17].

Thanks to Lemma 2.1, our construction above gives rise to a pairing

\[
\text{Ext}^{-1/2}(A, C \otimes B) \to \text{Ext}^{-1/2}(A, C)
\]

with quasi-unital Fredholm modules for \( B \) in the sense of [17] for all separable \( C^* \)-algebras \( A, C \) and \( B \). This goes as follows: Let \( \varphi_\pm : B \to M(\mathbb{K}) \) be a quasi-unital Fredholm pair in the sense of [17], i.e. \( \varphi_\pm \) are quasi-unital \(*\)-homomorphisms such that \( \varphi_+(b) - \varphi_-(b) \in \mathbb{K} \) for all \( b \in B \). Then \( \text{id}_C \otimes \varphi_\pm : C \otimes B \to C \otimes M(\mathbb{K}) \subseteq C \otimes (C \otimes \mathbb{K}) \) are quasi-unital \(*\)-homomorphisms, and admit canonical extensions \( \hat{\text{id}}_C \otimes \varphi_\pm : M(C \otimes B) \to M(C \otimes \mathbb{K}) \). Note that \( \hat{\text{id}}_C \otimes \varphi_+(y) - \hat{\text{id}}_C \otimes \varphi_-(y) \in C \otimes \mathbb{K} \) for all \( y \in C \otimes B \). Hence \( x \mapsto s_{C \otimes \mathbb{K}}^{-1}(\hat{\text{id}}_C \otimes \varphi_\pm \bullet x) \) is a homomorphism, defining the desired pairing with quasi-unital Fredholm modules. By the quasi-\( K \)-version of Higson’s result, Theorem 3.1.4 and the remarks in the first paragraph of Section 3.3 in [17], homotopy invariance of \( \text{Ext}^{-1/2}(A, -) \) will now follow if we can show that the pairing constructed above has the following properties (cf. 3.1.3a – 3.1.3f of [17]):

a) \( h^*(\pi_\pm \bullet [\psi']) = (\hat{h} \circ \pi_\pm) \bullet [\psi'] \), when \( h : D \to D' \) is a quasi-unital \(*\)-homomorphism with canonical extension \( \hat{h} : M(D) \to M(D') \);

b) \( \pi_\pm \bullet [\psi'] + \varphi_\pm \bullet [\psi'] = (\pi_+, \varphi_-) \bullet [\psi'] \), when \( \pi_- = \varphi_+ \);

c) \( (\pi_+ \circ \pi) \bullet [\psi'] = \pi_\pm \bullet [\psi'] \) for every \(*\)-homomorphism \( \pi : M(B) \to M(D) \);

d) \( \pi_\pm \bullet [\psi'] = [\psi'] \) when \( \pi_+ = \text{id} : M(\mathbb{K}) \to M(\mathbb{K}) \) and \( \pi_- = 0 \);

e) \( (\text{Ad} U \circ \pi_\pm) \bullet [\psi'] = \pi_\pm \bullet [\psi'] \) when \( U \in M(D) \) is a unitary;

f) \( (\pi, \text{Ad} V \circ \pi) \times [\psi'] = 0 \), when \( \pi : M(B) \to M(D) \) is a \(*\)-homomorphism and \( V \in M(D) \) is a unitary such that \( V = 1 \) modulo \( D \).
Of these, d) and f) are trivial and c) and e) follow from Lemma 4.4 and Lemma 4.5 of [16], respectively. To prove b), we use Kasparov’s technical theorem in the following way: Let $E_2 \subseteq M_2(M(D))$ be the $C^*$-algebra generated by elements of the form

$$
\begin{pmatrix}
\pi_+ (\psi^t_{11}(a)) & \pi_+ (\psi^t_{12}(a)) \\
\pi_+ (\psi^t_{21}(a)) & 0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\pi_- (\psi^t_{11}(a)) & \pi_- (\psi^t_{12}(a)) \\
\pi_- (\psi^t_{21}(a)) & 0
\end{pmatrix},
$$

$t \in [1, \infty)$, $a \in A$, and $\mathcal{F} \subseteq M_2(M(D))$ the subspace spanned by elements of the form

$$
\begin{pmatrix}
\pi_+ (\psi^t_{11}(a)) & \pi_+ (\psi^t_{12}(a)) \\
\pi_+ (\psi^t_{21}(a)) & \pi_+ (\psi^t_{22}(a))
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\varphi_+ (\psi^t_{11}(a)) & \varphi_+ (\psi^t_{12}(a)) \\
\varphi_+ (\psi^t_{21}(a)) & \varphi_+ (\psi^t_{22}(a))
\end{pmatrix},
$$

$t \in [1, \infty)$, $a \in A$. Let $E \subseteq M(D)$ be the $C^*$-subalgebra consisting of the elements $m \in M(D)$ with the property that $\pi_+(b)m, m\pi_+(b) \in D$ for all $b \in B$. Since $\pi_-(b) = \pi_+(b) = \varphi_+(b) = \varphi_-(b)$ modulo $D$ when $b \in B$, we might as well have used $\pi_-, \varphi_-$ or $\varphi_+$ instead of $\pi_+$ to define $E$. Note that $\pi_-(\psi^t_{22}(a)) - \varphi_-(\psi^t_{22}(a)) \in E$ for all $a \in A$ and all $t$, and that

$$
\mathcal{F} \left( \frac{D}{D} \right) \cup \left( \frac{D}{D} \right) \mathcal{F} \subseteq \left( \frac{D}{D} \right).
$$

We can therefore choose a separable $C^*$-subalgebra $E_1$ of $\left( \frac{D}{D} \right)$ containing

$$
\begin{pmatrix}
0 & 0 \\
0 & \pi_-(\psi^t_{22}(a)) - \varphi_-(\psi^t_{22}(a))
\end{pmatrix}
$$

for all $a \in A$ and all $t$ such that $[\mathcal{F}, E_1] \subseteq E_1$. Note that $E_1 E_2 \subseteq M_2(D)$. Kasparov’s technical theorem gives us elements $0 \leq N, M \in M_2(M(D))$ such that $N + M = 1$, $[M, \mathcal{F}] \subseteq M_2(D)$, $M E_1 \subseteq M_2(D)$ and $N E_2 \subseteq M_2(D)$. Then

$$
U = \begin{pmatrix}
-\sqrt{M} & \sqrt{N} \\
\sqrt{N} & \sqrt{M}
\end{pmatrix}
$$

is a unitary in $M_4(M(D))$ with the property that

$$
U \begin{pmatrix}
\pi_+ (\psi^t_{11}(a)) & \pi_+ (\psi^t_{12}(a)) \\
\pi_+ (\psi^t_{21}(a)) & \pi_+ (\psi^t_{22}(a))
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & \varphi_+ (\psi^t_{11}(a)) - \varphi_+ (\psi^t_{12}(a))
\end{pmatrix} =
\begin{pmatrix}
\pi_+ (\psi^t_{11}(a)) & \pi_+ (\psi^t_{12}(a)) \\
\pi_+ (\psi^t_{21}(a)) & \pi_+ (\psi^t_{22}(a))
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & \varphi_+ (\psi^t_{11}(a)) - \varphi_+ (\psi^t_{12}(a))
\end{pmatrix}
$$

modulo $M_4(D)$, for all $t$ and $a$. This shows that $\pi_+ \times \psi) \oplus (\varphi_+ \times \psi)$ is unitarily equivalent to $((\pi_+, \varphi_-) \times \psi) \oplus (q_D \circ \nu)$, where $\nu : A \to M(D)$ is an asymptotic homomorphism. It follows then from Lemma 4.4 and Lemma 4.5 of [16] that $\left[ \left( \hat{\Theta} \circ (\pi_+ \times \psi) \right)^t \right] + \left[ \left( \hat{\Theta} \circ (\varphi_+ \times \psi) \right)^t \right] = \left[ \left( \hat{\Theta} \circ ((\pi_+, \varphi_-) \times \psi) \right)^t \right]$ in $\text{Ext}^{-1/2}(A, D)$, proving b). To prove a), note that $\hat{\Theta}' \circ \left( \hat{h} \circ (\pi_+ \times \psi) \right)_t = \text{Ad} q_{D'}(U) \circ \hat{h} \circ \hat{\Theta} \circ (\pi_+ \times \psi)_t$ for all $t$, where $U \in M(D')$ is the unitary $U = W_1 \left[ \hat{h}(V_1^*) + 1 - \hat{h}(1) \right] + W_2 \hat{h}(V_2)$, when $V_1, V_2$ and $W_1, W_2$ are the isometries used to define $\Theta$ and $\Theta'$, respectively, and $\hat{h} : Q(D) \to Q(D)$ is induced by $\hat{h}$. Thanks to Lemma 4.5 of
it remains therefore only to prove that 

\[\left[\hat{h} \circ \left(\hat{\Theta} \circ (\pi_\pm \times \psi)\right)\right]^f = \left[\hat{h} \circ \left(\hat{\Theta} \circ (\pi_\pm \times \psi)\right)\right]^f,\]

or if we set \(\varphi = \hat{\Theta}(\pi_\pm \times \psi)\), that 

\[\left[\hat{h} \circ \varphi^f\right] = \left[\left(\hat{h} \circ \varphi\right)^f\right]\]

in \(\text{Ext}^{-1/2}(A, D')\). Let \(\chi : A \to Q(D)\) be an equi-continuous asymptotic homomorphism with the property that \(\varphi \oplus \chi\) asymptotically splits. Since \(\hat{h}\) is strictly continuous on norm-bounded sets, we see that

\[\hat{h} \circ \varphi^f(a) = qD'\left(\sum_{j=0}^{\infty} \hat{h}(\Delta_j) \hat{h}(\hat{\varphi}_{t_j+1}(a)) \hat{h}(\Delta_j)\right),\]

for a given tuple of folding data \((\hat{\varphi}_t)_{t \in [1, \infty)}, \{u_n\}_{n=0}^\infty, \{n_i\}_{i=0}^\infty, \{t_i\}_{i=1}^\infty\). Then the proof of Lemma 4.4 in [16] shows that

\[\left[\hat{h} \circ \varphi^f\right] = -\left[\hat{h} \circ \chi^f\right]\]

in \(\text{Ext}^{-1/2}(A, D')\). Since \(\left[\hat{h} \circ \chi^f\right] = -\left[\left(\hat{h} \circ \varphi\right)^f\right]\) by Lemma 4.4 of [16], we obtain the desired conclusion.

For any \(C^*-\)algebra \(E\) we denote in the following the \(C^*-\)algebra \(C[0, 1] \otimes E\) by \(IE\). It follows that the functor \(\text{Ext}^{-1/2}(A, -)\) is homotopy invariant, in the sense that the point evaluations \(\pi_t : IB \to B, t \in [0, 1]\), induce the same maps \(\pi_{t*} : \text{Ext}^{-1/2}(A, IB) \to \text{Ext}^{-1/2}(A, B)\) for any separable \(C^*-\)algebra \(B\). When this is established it is easy to make \(\text{Ext}^{-1/2}(A, -)\) functorial with respect to arbitrary \(*\)-homomorphisms; if \(h : B \to B_1\) is a \(*\)-homomorphism it follows from [18] that \(h \otimes \text{id}_K : B \otimes K \to B_1 \otimes K\) is homotopic to a quasi-unital \(*\)-homomorphism \(g : B \otimes K \to B_1 \otimes K\), unique up to homopy, and we set \(h_*(\psi) = [g \circ \psi]\), when \(\psi \in \text{Hom}(A, Q(B \otimes K))\). Thus we have obtained the following.

**Theorem 3.5.** For every separable \(C^*-\)algebra \(A\), \(\text{Ext}^{-1/2}(A, -)\) is a homotopy invariant functor, from the category of separable \(C^*-\)algebras to the category of abelian groups.

It follows also from the homotopy invariance that the pairing \(\pi_{\pm} \cdot - : \text{Ext}^{-1/2}(A, B) \to \text{Ext}^{-1/2}(A, D)\) constructed above only depends on the class of \(\pi_{\pm}\) in \(\text{KK}(B, D)\). Thus we have in fact a pairing

\[\text{KK}(B, C) \times \text{Ext}^{-1/2}(A, B) \to \text{Ext}^{-1/2}(A, C)\]

for all separable \(C^*-\)algebras \(A, B\) and \(C\).

It follows from Theorem 3.5 that two semi-invertible extensions of \(A\) by \(B \otimes K\) define the same element of \(\text{Ext}^{-1/2}(A, B)\) if and only if they are homotopic via a semi-invertible homotopy. Specifically, two semi-invertible extensions \(\varphi, \psi : A \to Q(B \otimes K)\), define the same element of \(\text{Ext}^{-1/2}(A, B)\) if and only if there is a semi-invertible extension \(\Phi : A \to Q(IB \otimes K)\) such that \(\tilde{\pi}_i \circ \Phi = \psi\) and \(\tilde{\pi}_1 \circ \Phi = \varphi\), where \(\tilde{\pi}_i : Q(IB \otimes K) \to Q(B \otimes K), i = 0, 1\), are the \(*\)-homomorphisms induced by the point evaluations \(\pi_0, \pi_1 : IB \otimes K \to B \otimes K\). It is this consequence of Theorem 3.5 that we shall make intensive use of in the following. But let us point out that the homotopy invariance of \(\text{Ext}^{-1/2}(A, B)\) in the second variable, \(B\), implies the homotopy invariance in the first variable.

**Theorem 3.6.** For every separable \(C^*-\)algebra \(B\), \(\text{Ext}^{-1/2}(-, B)\) is a homotopy invariant functor, from the category of separable \(C^*-\)algebras to the category of abelian groups.
Proof. Let $\varphi, \psi : A \to D$ be homotopic $*$-homomorphisms between separable $C^*$-algebras. Thus there is a $*$-homomorphism $\Phi : A \to ID$ such that $\pi_0 \circ \Phi = \varphi, \pi_1 \circ \Phi = \psi$. Let $\chi : D \to Q(B \otimes K)$ be a semi-invertible extension. Let $\tau : IQ(B \otimes K) \to Q(IB \otimes K)$ be the canonical inclusion. Then

$$\pi_0 \circ \tau \circ (id_I \otimes \chi) \circ \Phi = \varphi$$

and

$$\pi_1 \circ \tau \circ (id_I \otimes \chi) \circ \Phi = \psi,$$

so $[\psi] = \pi_{1*}[\tau \circ (id_I \otimes \chi) \circ \Phi] = \pi_{0*}[\tau \circ (id_I \otimes \chi) \circ \Phi] = [\varphi]$ in $\text{Ext}^{-1/2}(A, B)$ by Theorem 3.6.

\[ \square \]

4. Extended asymptotic homomorphisms

In this section $A$ and $B$ are separable $C^*$-algebras. Let $J \subseteq A$ be a $C^*$-subalgebra of $A$. An asymptotic homomorphism $\varphi = (\varphi_t)_{t \in [1, \infty)} : A \to M(B)$ is extended from $J$ when $\varphi_t(J) \subseteq B$ for all $t \in [1, \infty)$. If the context identifies the subalgebra $J$, we say simply that $\varphi$ is extended. If $\varphi$ is extended from $J$ and $q_B \circ \varphi_t = q_B \circ \varphi_1$ for all $t$, or equivalently, $\varphi_t(x) - \varphi_1(x) \in B$ for all $x \in A$ and all $t$, we say that $\varphi$ is constantly extended from $J$ or just constantly extended. Two (constantly) extended asymptotic homomorphisms $\varphi, \psi : A \to M(B)$ are homotopic when there is an (constantly) extended asymptotic homomorphism $\Phi : A \to M(IB)$ such that $\pi_0 \circ \Phi_t = \varphi_t$ and $\pi_1 \circ \Phi_t = \psi_t$ for all $t \in [1, \infty)$, where $\pi_s : M(IB) \to M(B)$ is the $*$-homomorphism induced by the point evaluation $\pi_s : IB \to B$. Homotopy is an equivalence relation in both cases, and we denote by $[[A, J; B]]$ the homotopy classes of extended asymptotic homomorphisms, and by $[[\{A, J; B]\}]$ the homotopy classes of constantly extended asymptotic homomorphisms. The set $[[A, J; B]]$ has been introduced and studied in [5] in relation to relative E-theory.

Theorem 4.1. The canonical (forgetful) map $[[\{A, J; B]\}] \to [[A, J; B]]$ is a bijection.

Proof. Surjectivity: Let $\varphi : A \to M(B)$ be an extended asymptotic homomorphism. To show that $\varphi$ is homotopic to a constantly extended asymptotic homomorphism we may assume that $\varphi$ is equi-continuous since it is asymptotically identical, and hence homotopic, to such an extended asymptotic homomorphism by Lemma 4.1. By Lemma 4.1 of [10] there is a continuous increasing function $r : [1, \infty) \to [1, \infty)$ such that $(\varphi_{r(t)})_{t \in [1, \infty)}$ is uniformly continuous, in the sense that the function $t \mapsto \varphi_{r(t)}(a)$ is uniformly continuous for all $a \in A$. Since $\psi = (\varphi_{r(t)})_{t \in [1, \infty)}$ is homotopic to $\varphi$, it suffices to show that $\psi$ is homotopic to a constantly extended asymptotic homomorphism. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ be a sequence of finite sets with dense union in $A$, such that $\bigcup_n F_n \cap J$ is dense in $J$. Let $\epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \ldots$ be a sequence in $[0, 1]$ chosen so small that

$$\|[a, \psi_t(x)]\| \leq \epsilon_i, \|[b, \psi_t(x)]\| \leq \epsilon_i \Rightarrow \|\sqrt{b - a}, \psi_t(x)\| \leq 2^{-i-1},$$

for all $t \in [1, i + 2], x \in F_i$, and

$$\|a \psi_t(x) - \psi_t(x)\| \leq \epsilon_i, \|b \psi_t(x) - \psi_t(x)\| \leq \epsilon_i \Rightarrow \|\sqrt{b - a} \psi_t(x)\| \leq 2^{-i-1},$$

for all $t \in [1, i + 2], x \in F_i \cap J$, when $0 \leq a \leq b \leq 1$. Let $v_0, v_1, v_2, \ldots$, be a unit sequence in $B$ such that

$$\|[v_t, \psi_t(x)]\| \leq \epsilon_i, t \in [1, i + 2], x \in F_i,$$
\[ \|v_i \psi_t(x) - \psi_t(x)\| \leq \epsilon_i, \; t \in [1, i+2], \; x \in F_i \cap J. \]  
(4.4)

Let \( n_0 < n_1 < n_2 < \ldots \) be a sequence in \( \mathbb{N} \) such that \( n_i - n_{i-1} > i + 1 \) for all \( i \geq 1 \).

We claim that there are continuous paths \( u_i(t), t \in [1, \infty), i = 0, 1, 2, \ldots \), in \( B \), such that 
\[ u_0(t) \leq u_1(t) \leq u_2(t) \leq \ldots \] is a unit sequence in \( B \) for all \( t \),
\[ u_i(1) = v_{n_i}, \]  
(4.5)

and for \( t \in [n, n+1] \) one has
\[ u_i(t) \in \text{co}\{v_j : j \geq n\} \]  
(4.6)

for all \( i = 0, 1, 2, \ldots \), and
\[ u_i(t) = u_i(1), \; i \geq n + 1. \]  
(4.7)

(In particular, at integer points we have the following equations:
\begin{align*}
  u_0(1) &= v_{n_0}, \quad u_0(2) = v_{n_1}, \quad u_0(3) = v_{n_2}, \quad u_0(4) = v_{n_3}, \quad u_0(5) = v_{n_4}, \quad \ldots; \\
  u_1(1) &= v_{n_1}, \quad u_1(2) = v_{n_1+1}, \quad u_1(3) = v_{n_2+1}, \quad u_1(4) = v_{n_3+1}, \quad u_1(5) = v_{n_4+1}, \quad \ldots; \\
  u_2(1) &= v_{n_2}, \quad u_2(2) = v_{n_2+1}, \quad u_2(3) = v_{n_2+2}, \quad u_2(4) = v_{n_3+2}, \quad u_2(5) = v_{n_4+2}, \quad \ldots; \\
  u_3(1) &= v_{n_3}, \quad u_3(2) = v_{n_3+1}, \quad u_3(3) = v_{n_4}, \quad u_3(4) = v_{n_4+1}, \quad u_3(5) = v_{n_4+3}, \quad \ldots;
\end{align*}

etc.) The construction is the same as the construction of \( \{w_i(t)\}_{i=0}^\infty \) in the proof of Lemma 4.4 in \cite{16}: Assume that \( \{u_i(t)\}_{i=0}^\infty, \; t \in [1, k], \) have been constructed, and that \( v_{n_{k-1}} \leq u_0(k) \leq u_k(k) = v_{n_k} \). We construct then \( \{u_i(t)\}_{i=0}^\infty, \; t \in [k, k+1], \) as follows. Since \( n_{k+1} - n_k > k + 1 \), we have that
\[ v_{n_k} = u_k(k) \leq v_{n_k+1} \leq v_{n_k+2} \leq \cdots \leq v_{n_k+k+1} = u_{k+1}(k+1) = v_{n_{k+1}}. \]

Set \( u_i(t) = u_i(1) = v_{n_i}, t \in [k, k+1], \) when \( i \geq k + 1 \). Set \( I_j = [k + \frac{j}{k+1}, k + \frac{j+1}{k+1}], \; j = 0, 1, 2, \ldots, k \). On the interval \( I_j \), \( u_k-j(t), t \in I_j \), is the straight line from \( u_{k-j}(k) \) to \( v_{n_k+k-j}, \) i.e.
\[ u_{k-j}(t) = (j + 1 - (k + 1)(t - k))u_{k-j}(k) + ((k + 1)(t - k) - j)v_{n_k+k-j}, \]
\( t \in I_j \) and other \( u_m(t), m \neq k-j, \) are constants. The construction of \( \{u_i(t)\}_{i=0}^\infty, \; t \in [1, \infty), \) can then proceed by induction.

Set \( \Delta_0(t) = \sqrt{u_0(t)}, \; \Delta_i(t) = \sqrt{u_i(t)} - u_{i-1}(t), \; i \geq 1 \). Let \( \{\psi_t\}_{t=0}^\infty \) be a discretization of \( \psi \) such that \( t_i \leq i \) for all \( i \geq 1 \). Set
\[ \Psi_t(a) = \sum_{i=0}^\infty \Delta_i(t)\psi_{\max\{t,t_i\}}(a)\Delta_i(t). \]

The sequence converges in the strict topology of \( M(B) \) by Lemma 3.1 of \cite{16}. Note that it follows from \cite{17} that for each \( n \in \mathbb{N} \) there is an \( N_n \in \mathbb{N} \) such that
\[ \Psi_t(x) - \Psi_s(x) = \sum_{i=0}^{N_n} \Delta_i(t)\psi_{\max\{t,t_i\}}(x)\Delta_i(t) - \Delta_i(s)\psi_{\max\{s,t_i\}}(x)\Delta_i(s) \]
for all \( s, t \in [1, n] \). This shows that \( \Psi_t(x) - \Psi_s(x) \in B \) for all \( s, t \in [1, \infty) \) and that \( t \mapsto \Psi_t(x) \) is continuous. We claim that \( \Psi_t(J) \subseteq B \) for all \( t \). By Lemma 3.1 of \cite{16}, \( \psi_t \in [1, \infty), \) is an equi-continuous family since \( \psi_t, t \in [1, \infty), \) is, so it suffices to show that \( \Psi_t(x) \in B \) when \( x \in F_t \cap J \). As we know that \( \Psi_t(x) - \Psi_1(x) \in B \), we must show that \( \Psi_1(x) \in B \). It follows from \cite{15}, \cite{17} and \cite{18} that
\[ \|\Delta_1(t)\psi_t(x)\| \leq 2^{-i-1} \]
when \( i \geq k \). This shows that \( \sum_{i=0}^\infty \Delta_i(1)\psi_t(x)\Delta_1(1) \) converges in norm, proving that \( \Psi_1(x) \in B \).
To show that $\Psi$ is asymptotically multiplicative, it suffices, by equi-continuity of $\Psi_t$, $t \in [1, \infty)$, to check for $x, y \in F_k$. In the following we write $a \sim_\delta b$ when $a$ and $b$ are elements of the same $C^*$-algebra and $\|a - b\| \leq \delta$. Let $t \in [m, m + 1]$, $m \geq k$. When $i > m$, we have that $\max\{t, t_i\} \leq i$, and hence
\[
\Delta_i(t) \psi^{\max\{t, t_i\}}_{x}(x) \Delta_i(t) \sim_{2^{-i}} \psi^{\max\{t, t_i\}}_{x}(x) \Delta_i(t)^2,
\]
thanks to (4.4), (4.3), (4.2) and (4.1). Similarly,
\[
\Delta_i(t) \psi^{\max\{t, t_{i-1}\}}_{x}(x) \Delta_i(t) \sim_{2^{-i}} \psi^{\max\{t, t_{i-1}\}}_{x}(x) \Delta_i(t)^2,
\]
and both estimates also hold with $x$ replaced by $y$. When $i \leq m$, $\max\{t, t_i\} \leq m + 1$, while $u_i(t), u_{i-1}(t) \in \co \{v_j : j \geq m\}$ by (4.6). It follows therefore from (4.3) and (4.1) that
\[
\Delta_i(t) \psi^{\max\{t, t_i\}}_{x}(x) \Delta_i(t) \sim_{2^{-m}} \psi^{\max\{t, t_i\}}_{x}(x) \Delta_i(t)^2.
\]
Similarly,
\[
\Delta_i(t) \psi^{\max\{t, t_{i-1}\}}_{x}(x) \Delta_i(t) \sim_{2^{-m}} \psi^{\max\{t, t_{i-1}\}}_{x}(x) \Delta_i(t)^2,
\]
and both estimates also hold with $x$ replaced by $y$. Set
\[
\delta_1(t) = \sup_j \|\psi^{\max\{t, t_j\}}_{x}(y) - \psi^{\max\{t, t_{j+1}\}}_{x}(y)\|,
\]
\[
\delta_2(t) = \sup_j \|\psi^{\max\{t, t_j\}}_{x}(x) \psi^{\max\{t, t_j\}}_{x}(y) - \psi^{\max\{t, t_j\}}_{x}(xy)\|,
\]
and
\[
k_x = \sup_t \|\psi_t(x)\|.
\]
Using Lemma 3.1 of [16] and the above estimates we find that
\[
\Psi_t(x)\Psi_t(y) = \left(\sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(x) \Delta_j(t)\right) \left(\sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(y) \Delta_j(t)\right)
\]
\[
= \sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(x) \Delta_j(t)^2 \psi^{\max\{t, t_j\}}_{x}(y) \Delta_j(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(x) \Delta_j(t) \Delta_{j+1}(t) \psi^{\max\{t, t_{j+1}\}}_{x}(y) \Delta_{j+1}(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_{j+1}(t) \psi^{\max\{t, t_{j+1}\}}_{x}(x) \Delta_{j+1}(t) \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(y) \Delta_j(t)
\]
\[
\sim \delta_{k_x} m^{-2} \sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(x) \psi^{\max\{t, t_j\}}_{x}(y) \Delta_j(t)^2 \Delta_j(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_j(t) \psi^{\max\{t, t_j\}}_{x}(x) \psi^{\max\{t, t_{j+1}\}}_{x}(y) \Delta_j(t) \Delta_{j+1}(t) \Delta_{j+1}(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_{j+1}(t) \psi^{\max\{t, t_{j+1}\}}_{x}(x) \psi^{\max\{t, t_j\}}_{x}(y) \Delta_{j+1}(t) \Delta_j(t) \Delta_j(t)
\]
\[
\sim 2k_2 \delta_1(t) \sum_{j=0}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(x) \psi_{\max\{t, t_j\}}(y) \Delta_j(t) \Delta_j(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(x) \psi_{\max\{t, t_j\}}(y) \Delta_j(t) \Delta_{j+1}(t) \Delta_{j+1}(t)
\]
\[
+ \sum_{j=1}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(x) \psi_{\max\{t, t_j\}}(y) \Delta_j(t) \Delta_{j-1}(t) \Delta_{j-1}(t)
\]
\[
\sim 2k_2 \delta_2(t) \sum_{j=0}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(xy) \Delta_j(t) \Delta_j(t)
\]
\[
+ \sum_{j=0}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(xy) \Delta_j(t) \Delta_{j+1}(t) \Delta_{j+1}(t)
\]
\[
+ \sum_{j=1}^{\infty} \Delta_j(t) \psi_{\max\{t, t_j\}}(xy) \Delta_j(t) \Delta_{j-1}(t) \Delta_{j-1}(t)
\]
\[
= \Psi_t(xy).
\]
Since \(6k_2 m 2^{-m} + 2k_2 \delta_1(t) + 3\delta_2(t)\) goes to zero as \(m\) tends to infinity, we conclude that \(\lim_{t \to \infty} \Psi_t(x)\Psi_t(y) - \Psi_t(xy) = 0\). Asymptotic linearity and self-adjointness follow in the same way. Thus \(\Psi\) is a constantly extended asymptotic homomorphism. For \(a \in A, s \in [0, 1]\), define \(\Lambda_t(a)(s) \in M(B)\) by the strictly convergent sequence
\[
\Lambda_t(a)(s) = \sum_{i=0}^{\infty} \Delta_i(t) \psi_{s \max\{t, t_i\} + (1-s)t}(a) \Delta_i(t).
\]
Since \(s \mapsto \Lambda_t(a)(s)\) is a strictly continuous and normbounded function, we have defined a family of maps \(\Lambda_t : A \to M(IB), t \in [1, \infty)\). It follows from (4.7) that for fixed \(n\) there is an \(N_n\) so large that
\[
\Lambda_t(a)(s) - \Lambda_{t'}(a)(s) = \sum_{i=0}^{N_n} (\Delta_i(t) \psi_{s \max\{t, t_i\} + (1-s)t}(a) \Delta_i(t) - \Delta_i(t') \psi_{s \max\{t', t_i\} + (1-s)t'}(a) \Delta_i(t'))
\]
\[
+ \sum_{i=N_n+1}^{\infty} \Delta_i(t) (\psi_{s \max\{t, t_i\} + (1-s)t}(a) - \psi_{s \max\{t', t_i\} + (1-s)t'}(a)) \Delta_i(t)
\]
for all \(a\) and \(s\), provided \(t, t' \in [1, n]\). When \(t\) tends to \(t'\), the first term converges to 0 in norm, uniformly in \(s\), for obvious reasons, and the second term does the same thanks to Lemma 3.1 of [16] and the continuity of \(t \mapsto \psi_t(a)\). Thus \(t \mapsto \Lambda_t(a)\) is normcontinuous. Lemma 3.1 of [16] also shows that the family \(\Lambda_t\), \(t \in [1, \infty]\), is equi-continuous since \((\psi_t)\) is. To show that \(\Lambda_t(J) \subseteq IB\) for all \(t\), we must give an argument different from the one used above since \(\Lambda_t - \Lambda_1\) does not map \(J\) into \(IB\). Note that \(s \max\{t, t_i\} + (1-s)t \leq \max\{t, i\} \leq i\) when \(i \geq t\). According to (4.7) and (4.5), \(u_i(t) = v_{n_i} \geq v_i\) when \(i \geq t + 1\), so we conclude from (4.4) and (4.2) that \(\sup_{s} \|\Delta_i(t) \psi_{s \max\{t, t_i\} + (1-s)t}(x)\| \leq 2^{-i}\) for all large enough \(i\), when \(x \in \bigcup_k F_k \cap J\). Hence the sum defining \(\Lambda_t(x)\) converges in norm to an element of \(IB\). By continuity of \(\Lambda_t\), we conclude that \(\Lambda_t(J) \subseteq IB\). The arguments that proved that \(\Psi\) is an asymptotic homomorphism show
the same about $\Lambda$, thanks to the uniform continuity of $\psi$. (The uniform continuity is used to show that the analog of $\delta_1(t)$ tends to 0 when $t$ goes to infinity.) $\Lambda$ is consequently an extended asymptotic homomorphism given us a homotopy connecting $\Psi$ to $(\sum_{i=0}^{\infty} \Delta_i(t)\psi_i(\cdot)\Delta_{i}(t))_{t \in [1,\infty)}$.

For each $a, s \in [0, 1]$, define $\mu_t(a)(s) \in M(B)$ by

$$
\mu_t(a)(s) = \begin{cases} 
\sum_{i=0}^{\infty} \Delta_i(t - \log s)\psi_i(a)\Delta_i(t - \log s), & s \neq 0, \\
\psi_i(a), & s = 0.
\end{cases}
$$

(4.8)

Since $\Delta_0(t)$ strictly tends to 1 and $\Delta_i(t)$, $i > 0$, strictly tend to 0, as $t \to \infty$, the formula (4.8) defines an extended asymptotic homomorphism $\mu = (\mu_t)_{t \in [1,\infty)} : A \to M(IB)$ providing a homotopy between $(\sum_{i=0}^{\infty} \Delta_i(t)\psi_i(\cdot)\Delta_{i}(t))_{t \in [1,\infty)}$ and $(\psi_t)_{t \in [1,\infty)}$.

Injectivity: Let $\varphi, \psi : A \to M(B)$ be constantly extended asymptotic homomorphisms that are homotopic as extended asymptotic homomorphisms, and let $\Phi : A \to M(IB)$ be an extended asymptotic homomorphism realizing a homotopy between the two. Disregarding a few considerations concerning equi-continuity and uniform continuity, the construction from the proof of surjectivity gives us a homotopy of constantly extended asymptotic homomorphisms between

$$
(\sum_{j=0}^{\infty} \Delta_j(t)\varphi_{i}(\cdot)\Delta_{j}(t))_{t \in [1,\infty)}
$$

and

$$
(\sum_{j=0}^{\infty} \Delta_j(t)\psi_{i}(\cdot)\Delta_{j}(t))_{t \in [1,\infty)},
$$

where the $\Delta_i$’s arise from appropriately chosen continuous paths of unit sequences in $B$. To complete the proof it suffices therefore to check that the asymptotic homomorphism $\mu$ of (4.8) is constantly extended when $\psi$ is. So assume this is the case and let $\epsilon > 0$ be given. Let $t, t' \in [1, \infty)$. By equi-continuity it suffices to show that $\mu_t(x) - \mu_{t'}(x) \in IB$ when $x \in F_k$. Take $n \in \mathbb{N}$ such that $n \geq \max\{t, t'\}$. It follows then from (4.3) that $u_i(t - \log s) \in \mathbb{C}\{v_j : j \geq n\}$ for all $i \in \mathbb{N}$ and all $s \in [0, 1]$. It follows therefore from (4.3) and (4.1) that $\|[(\Delta_i(t - \log s), \psi_i(x)])\| \leq 2^{-i}$ when $i \geq \max\{n, k\}$. As a consequence there is an $N \in \mathbb{N}$ so large that

$$
\psi_i(x) (1 - u_N(t - \log s)) + \sum_{i=0}^{N} \Delta_i(t - \log s)\psi_i(x)\Delta_i(t - \log s) \sim \epsilon \sum_{i=0}^{\infty} \Delta_i(t - \log s)\psi_i(a)\Delta_i(t - \log s)
$$

for all $s \in [0, 1]$. By increasing $N$ we may assume that the same estimate holds with $t$ replaced by $t'$. Thus $\mu_t(x) - \mu_{t'}(x)$ has distance less than $2\epsilon$ to the element of $M(IB)$ given by the strictly continuous map $f : [0, 1] \to M(B)$, where

$$
f(s) = \sum_{i=0}^{N} (\Delta_i(t - \log s)\psi_i(x)\Delta_i(t - \log s) - \Delta_{i}(t' - \log s)\psi_{i'}(x)\Delta_i(t' - \log s))
$$

$$
+ \psi_i(x) (1 - u_N(t - \log s)) - \psi_{i'}(x) (1 - u_N(t' - \log s)),
$$

(4.9)

when $s \in [0, 1]$, and

$$
f(s) = \psi_i(x) - \psi_{i'}(x),
$$

(4.10)

when $s = 0$. Note that for each $s$, $f(s)$ is in $B$ since $\psi_i(x) - \psi_{i'}(x)$ is, and that $f$ is obviously norm-continuous on $[0, 1]$. It suffices now to show that (4.9) converges in norm to (4.10) when $s$ tends to zero. To see that this is the case note that (4.6), (4.3) and (4.1) imply that

$$
\lim_{s \to 0} \left( \sum_{i=0}^{N} \Delta_i(t - \log s)\psi_i(x)\Delta_i(t - \log s) - \sum_{i=0}^{N} \psi_i(x) (\Delta_i(t - \log s)\Delta_i(t - \log s))^2 \right) = 0.
$$
The same conclusion holds with \( t \) replaced by \( t' \) so (4.9) approaches (4.10) as \( s \to 0 \) because
\[
\sum_{i=0}^{N} (\Delta_i(t - \log s))^2 + (1 - u_N(t - \log s)) = \sum_{i=0}^{N} (\Delta_i(t' - \log s))^2 + (1 - u_N(t' - \log s)) = 1
\]
for all \( s \in ]0, 1[ \). The proof is complete.

\[ \square \]

Theorem 4.1 serves as our excuse for not distinguishing very strictly between \([A, J; B]\) and \([\{A, J; B\}]\) in the following.

When \( B \) is stable both \([A, J; B]\) and \([\{A, J; B\}]\) are equipped with a semi-group structure in the familiar way: When \( \varphi, \psi : A \to M(B) \) are (constantly) extended asymptotic homomorphisms and \( V_1, V_2 \in M(B) \) are isometries such that \( V_1V_1^* + V_2V_2^* = 1 \), we can define a (constantly) extended asymptotic homomorphism \( \varphi \oplus \psi : A \to M(B) \) by \( (\varphi \oplus \psi)_t(a) = V_1\varphi_t(a)V_1^* + V_2\psi_t(a)V_2^* \). The compositions defined in this way in \([A, J; B]\) and \([\{A, J; B\}]\) are commutative and associative, and are independent of the choice of isometries \( V_1, V_2 \). In this case the bijection of Theorem 4.1 is an isomorphism of abelian semi-groups. In the following we assume that \( B \) is stable.

**Lemma 4.2.** \([A, 0; B]\) = 0.

**Proof.** Consider an asymptotic homomorphism \( \pi : A \to M(B) \) such that \( \pi_t(0) \in B \) for all \( t \). Then \( \pi'_t(a) = \pi_t(a) - \pi_t(0) \) defines an asymptotic homomorphism \( \pi' : A \to M(B) \) with the property that \( \pi'_t(0) = 0 \) for all \( t \), and \([\pi] = [\pi']\) in \([A, 0; B]\). By Lemma 1.3.6 of [11] there is a strictly continuous family \( V_s, s \in ]0, 1[ \), of isometries in \( M(B) \) such that \( V_1 = 1 \) and
\[
\lim_{s \to 0} V_sV_s^* = 0 \text{ in the strict topology.}
\]
Set
\[
\Phi_t(a)(s) = \begin{cases} V_s\pi'_t(a)V_s^*, & s \in ]0, 1[, \\ 0, & s = 0. \end{cases}
\]
Note that \( s \mapsto \Phi_t(a)(s) \) is strictly continuous and norm-bounded. Thus \( \Phi = (\Phi_t)_{t \in ]1, \infty[} : A \to M(IB) \) is an asymptotic homomorphism such that \( \Phi_t(0) = 0 \) for all \( t \), giving us a homotopy connecting \( \pi' \) to 0.

\[ \square \]

We denote the \( C^*-\)algebras \( C(\mathbb{T}) \otimes A \) and \( C_0(0, 1) \otimes A \) by \( TA \) and \( SA \), respectively. Note that there is an extension
\[
0 \longrightarrow SA \longrightarrow TA \xrightarrow{ev} A \longrightarrow 0,
\]
where \( ev : TA \to A \) is evaluation at \( 1 \in \mathbb{T} \). We shall often identify \( TA \) with \( \{ f \in IA : f(0) = f(1) \} \) in the obvious way.

**Lemma 4.3.** \([TA, SA; B]\) and \([\{TA, SA; B\}]\) are groups.

**Proof.** Since the bijection of Theorem 4.1 is an isomorphism of semi-groups with zero, it suffices to show that \([TA, SA; B]\) is a group. Let \( \varphi : TA \to M(B) \) be an extended asymptotic homomorphism. Let \( \alpha \in \text{Aut} TA \) be the automorphism which changes orientation on the circle, i.e. \( \alpha(f)(s) = f(1 - s), f \in T, s \in ]0, 1[ \). Then the homomorphism \( \gamma : TA \to M_2(TA) \) given by
\[
\gamma(f) = \begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix}
\]
is homotopic to the $*$-homomorphism
\[ f \mapsto \begin{pmatrix} f(0) \\ f(0) \end{pmatrix}, \]
via a path of $*$-homomorphisms which all send $SA$ into $M_2(SA)$. Since $\varphi \oplus (\varphi \circ \alpha) = \bar{\Theta} \circ (\text{id}_{M_2} \otimes \varphi) \circ \gamma$, where the $*$-isomorphism $\bar{\Theta} : M_2(M(B)) \to M(B)$ is given by
\[
\bar{\Theta}(a_{11}^{11} a_{12}^{12}) = V_1 a_{11} V_1^* + V_1 a_{12} V_2^* + V_2 a_{21} V_1^* + V_2 a_{22} V_2^*,
\]
we conclude that $\varphi \oplus (\varphi \circ \alpha)$ is homotopic as an extended asymptotic homomorphism to $(\varphi \circ c \circ \text{ev}) \oplus (\varphi \circ c \circ \text{ev})$, where the $*$-homomorphisms $c : A \to TA$ and $\text{ev} : TA \to A$ are given by $c(a)(t) = a, t \in T$, and $\text{ev}(f) = f(0)$, respectively. Since $[\varphi \circ c] = 0$ in $[[A,0;B]]$ by Lemma 4.2 it follows that $[(\varphi \circ c \circ \text{ev}) \oplus (\varphi \circ c \circ \text{ev})] = 0$ in $[[TA,SA;B]]$.

\[\square\]

5. Various maps

Let $A$ and $B$ be separable $C^*$-algebras, $B$ stable. In this section we obtain our main result which is that an appropriate modification of the Connes-Higson construction gives rise to an isomorphism between $\text{Ext}^{-1/2}(A, B)$ and $[[TA,SA;B]]$.

5.1. The Connes–Higson map. Let $\phi : A \to Q(B)$ be a semi-invertible extension. Then there exists an extension $\psi : A \to Q(B)$ and an equi-continuous asymptotic homomorphism $\pi = (\pi_t)_{t \in [1,\infty)} : A \to M_2(M(B))$ such that $q_{M_2(B)} \circ \pi = \phi \oplus \psi$. Denote the matrix elements of $\pi_t$ by $\pi_t^{ij}, i, j = 1, 2$. Note that $\pi_t^{12}(A) \cup \pi_t^{21}(A) \subseteq B$ for all $t$. It follows from the equi-continuity of $\pi$ and the separability of $A$ and $B$ that there exists an approximate unit $(u_t)_{t \in [1,\infty)} \subseteq B$ such that
\[
\lim_{t \to \infty} \left[ f(u_t), \pi_t^{11}(a) \right] = 0, \tag{5.1}
\]
\[
\lim_{t \to \infty} \pi_t^{12}(a) (f(u_t) - f(1)) = 0, \tag{5.2}
\]
and
\[
\lim_{t \to \infty} \pi_t^{21}(a) (f(u_t) - f(1)) = 0, \tag{5.3}
\]
for all $f \in C[0,1]$ and all $a \in A$. Then $\begin{pmatrix} f(u_t) \\ f(0) \end{pmatrix}$ and $\pi_t(a)$ asymptotically commute for all $f \in T$ and $a \in A$. We use here and in the following $T$ to denote the $C^*$-algebra $C(T)$. Note that
\[
\pi_t(a) - \pi_1(a) \in M_2(B)
\]
and
\[
\begin{pmatrix} f(u_t) \\ f(0) \end{pmatrix} - \begin{pmatrix} f(0) \\ f(0) \end{pmatrix} \in M_2(B)
\]
for all $a \in A, f \in T$. Set
\[
X = \{ g \in C_b([1,\infty), M_2(M(B))) : g(t) - g(1) \in M_2(B) \ \forall t \}.
\]
It follows that there is a $*$-homomorphism $\Phi : TA \to X/C_0([1,\infty), M_2(B))$ such that $\Phi(f \otimes a)$ is the image of the element in $X$ given by the function
\[
t \mapsto \begin{pmatrix} f(u_t) \\ f(0) \end{pmatrix} \pi_t(a).
\]
It follows from the Bartle-Graves selection theorem that there is a continuous map \( \chi : X/C_0([1, \infty), M_2(B)) \rightarrow X \) which is a right-inverse for the quotient map \( X \rightarrow X/C_0([1, \infty), M_2(B)) \). By Remark 2 on page 114 of [13] we can assume that \( \chi \) maps the asymptotic algebra of \( M_2(B) \), which is a C*-subalgebra of \( X \), into \( C_b([1, \infty), M_2(B)) \). Set

\[
CH(\varphi)(a) = \tilde{\Theta}((\chi \circ \Phi(a)(t))),
\]

where \( \tilde{\Theta} : M_2(M(B)) \rightarrow M(B) \) is the *-isomorphism [12]. Note that \( CH(\varphi) \) is an equi-continuous asymptotic homomorphism \( CH(\varphi) : TA \rightarrow M(B) \) such that \( CH(\varphi)(SA) \subseteq B \) and \( CH(\varphi)(x) - CH(\varphi)(x) \in B \) for all \( t \) and all \( x \in TA \). In short, \( CH(\varphi) \) is an asymptotic homomorphism which is constantly extended from \( SA \), and defines an element of \( \{TA, SA; B\} \). It is easy to see that the construction gives us a well-defined group homomorphism

\[
CH : \text{Ext}^{-1/2}(A, B) \rightarrow \{TA, SA; B\}.
\]

When composed with the obvious forgetful map \( \{TA, SA; B\} \rightarrow \{SA, B\} \) obtained by restricting asymptotic homomorphisms to \( SA \), we get the usual Connes-Higson map.

5.2. The \( E \)-map. Let \( \varphi = (\varphi_t)_{t \in [1, \infty)} : TA \rightarrow M(B) \) be an asymptotic homomorphism which is constantly extended from \( SA \), i.e. \( \varphi \) is an asymptotic homomorphism such that \( \varphi_t(SA) \subseteq B \) and \( \varphi_t(x) - \varphi_1(x) \in B \) for all \( t \in [1, \infty) \) and all \( x \in TA \). By Lemma 3.1 we may assume that \( \varphi \) is equi-continuous. We will use \( \varphi \) to define a semi-invertible extension of \( T^2A \) by \( B \), where \( T^2A = T(TA) = C(T^2) \otimes A \). To do this we choose first a discretization \( \varphi_{t_0}, \varphi_{t_1}, \varphi_{t_2}, \ldots \) such that \( \lim_{i \rightarrow \infty} t_i = \infty \) and \( \lim_{i \rightarrow \infty} \sup_{t \in [t_i, t_{i+1}]} \|\varphi(t) - \varphi_{t_i}(a)\| = 0 \) for all \( a \in TA \). To define from such a discretization a map \( \Phi : TA \rightarrow M(B \otimes \mathbb{K}) \) we identify \( \mathbb{K} \) with the compact operators on the Hilbert space \( l^2(\mathbb{Z}) \), and introduce the corresponding matrix units \( e_{i,j} \in \mathbb{K}, i, j \in \mathbb{Z} \). Then both sums in

\[
\Phi(f) = \sum_{i \geq 1} \varphi_{t_i}(f) \otimes e_{i,i} + \sum_{i \leq 0} \varphi_{t_{-i}}(f(0)) \otimes e_{i,i}
\]

converge in the strict topology and \( [5.3] \) defines a map \( \Phi : TA \rightarrow M(B \otimes \mathbb{K}) \). Observe that \( \Phi \) is a *-homomorphism modulo \( B \otimes \mathbb{K} \). Furthermore, \( \Phi(a) \) commutes modulo \( B \otimes \mathbb{K} \) with the two-sided shift \( \mathcal{T} = \sum_{j \in \mathbb{Z}} e_{j,j-1} \). So we get in this way an extension

\[
E(\varphi) : T^2A \rightarrow Q(B \otimes \mathbb{K})
\]

such that

\[
E(\varphi)(g \otimes f) = q_{B \otimes \mathbb{K}}(g(\mathcal{T}) \Phi(f))
\]

for all \( g \in T, f \in TA \).

**Lemma 5.1.** \( E(\varphi) \) is semi-invertible, and its class in \( \text{Ext}^{-1/2}(T^2A, B) \) does not depend on the chosen discretization of \( \varphi \).

**Proof.** The inverse \(-E(\varphi)\) is given by the formula

\[
-E(\varphi)(g \otimes f) = q_{B \otimes \mathbb{K}}(g(\mathcal{T}) \Psi(f))
\]

where

\[
\Psi(f) = \sum_{i \leq 0} \varphi_{t_{-i}}(f) \otimes e_{i,i} + \sum_{i \geq 1} \varphi_{t_i}(f(0)) \otimes e_{i,i}.
\]

To see that \( \tilde{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi))) \) is asymptotically split it is appropriate to view \( M_2(B \otimes \mathbb{K}) \) as \( M_2(B) \otimes \mathbb{K} \). Let \( z \) denote the identity function on the circle \( \mathbb{T} \), so that \( z \) generates \( T \).
as a $C^*$-algebra. Then $\hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi)))$ is determined by the condition that $\hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi))) = q_{M_2(B \otimes K)} \circ \Psi$, where

$$
\Psi(z^k \otimes f) = \left( \sum_{i \in A} \left( \frac{1}{0} \otimes e_{n,n-1} \right) \right)^k \left( \sum_{i \geq 1} \left( \frac{\varphi_t(f)}{0} \varphi_t(f(0)) \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \frac{\varphi_t(f(0))}{0} \varphi_t(f) \right) \otimes e_{i,i} \right),
$$

modulo $B \otimes K$ for all $k \in \mathbb{Z}$ and all $f \in TA$. Set

$$
\varphi_t^i = \begin{cases} 
\varphi_{\max\{t, i\}}, & i \geq 0, \\
\varphi_{\max\{t, i\}}, & i \leq 0.
\end{cases}
$$

Without loss of generality we may assume that the discretization $\{t_i\}_{i=0}^\infty$ satisfies $t_0 = t_1 = 1$, so that $\varphi_t^0 = \varphi_t^1 = \varphi_t$. Define a continuous family of unitaries, $S_t$, by

$$
S_t = \sum_{n \neq 1} \left( \frac{1}{0} \otimes e_{n,n-1} \right) + \left( \frac{1-u_t}{\sqrt{2u_t-u_t^2}} \frac{2u_t-u_t^2}{1-u_t} \right) \otimes e_{1,0},
$$

where $u_t, t \in [1, \infty)$, is a continuous approximate unit in $B$. Then

$$
\left[ S_t, \left( \sum_{i \geq 1} \left( \frac{\varphi_t^i(f)}{0} \varphi_t^i(f(0)) \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \frac{\varphi_t^i(f(0))}{0} \varphi_t^i(f) \right) \otimes e_{i,i} \right) \right] = \sum_{i \leq 0} \left( \frac{\varphi_t^{i-1}(f) - \varphi_t^i(f)}{0} \varphi_t^{i-1}(f(0)) - \varphi_t^i(f(0)) \right) \otimes e_{i,i-1}
$$

$$
+ \sum_{i > 1} \left( \frac{\varphi_t^{i-1}(f(0)) - \varphi_t^i(f(0))}{0} \varphi_t^{i-1}(f) - \varphi_t^i(f) \right) \otimes e_{i,i-1}
$$

$$
+ \left( (1-u_t)\varphi_t(f) - \varphi_t(f(0))(1-u_t) \right) \left[ \sqrt{2u_t-u_t^2}, \varphi_t(f(0)) \right] \left( 1-u_t \right) \varphi_t(f(0)) - \varphi_t(f)(1-u_t)) \otimes e_{1,0}. \tag{5.8}
$$

The first two terms in the right-hand side of (5.8) vanishes as $t \to \infty$ due to the choice of the discretization. Since $A$ is separable and $\varphi_t(f) - \varphi_t(f(0)) \in B$, we can choose $(u_t)_{t\in[1,\infty)}$ such that

$$
\lim_{t \to \infty} [u_t, \varphi_t(f)] = \lim_{t \to \infty} [u_t, \varphi_t(f(0))] = 0
$$

and

$$
\lim_{t \to \infty} (1-u_t) (\varphi_t(f) - \varphi_t(f(0))) = 0
$$

for all $f \in TA$. Such a choice ensures that the last term in (5.8) also vanishes as $t \to \infty$, so we see that $S_t$ asymptotically commutes with $\sum_{i \geq 1} \left( \frac{\varphi_t(f)}{0} \varphi_t(f(0)) \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \frac{\varphi_t(f(0))}{0} \varphi_t(f) \right) \otimes e_{i,i}$. The last expression, as well as $S_t$, is constant in $t$ when taken modulo $B \otimes K$, so we can define an asymptotic splitting $A$ for $\hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi)))$ such that $\Lambda_t(z^k \otimes f)$ asymptotically agrees with

$$
S_t^k \left( \sum_{i \geq 1} \left( \frac{\varphi_t^i(f)}{0} \varphi_t^i(f(0)) \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \frac{\varphi_t^i(f(0))}{0} \varphi_t^i(f) \right) \otimes e_{i,i} \right)
$$
for all $k \in \mathbb{Z}$ and all $f \in TA$. The method used to reach this conclusion will be used several times in the following, so we give a detailed account here: Set

$$X = \left\{ f \in C_0([1, \infty), M_2(M(B \otimes \mathbb{K}))) \mid f(1) - f(t) \in M_2(B \otimes \mathbb{K}) \quad \forall t\right\},$$

which is a $C^*$-algebra containing $C_0([1, \infty), M_2(B \otimes \mathbb{K}))$ as an ideal. Since $S_t$ asymptotically commutes with $\sum_{i \geq 1} \left( \begin{array}{cc} \varphi_i(f) & 0 \\ 0 & \varphi_i(f(0)) \end{array} \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \begin{array}{cc} \varphi_i(f(0)) & 0 \\ 0 & \varphi_i(f) \end{array} \right) \otimes e_{i,i},$ which is an asymptotic homomorphism, we get straightforwardly a $\ast$-homomorphism $\Phi$ from $T^2A$ into the asymptotic algebra of $M_2(M(B \otimes \mathbb{K}))$ such that

$$\Phi(z^k \otimes f) = S_t^k \left( \sum_{i \geq 1} \left( \begin{array}{cc} \varphi_i(f) & 0 \\ 0 & \varphi_i(f(0)) \end{array} \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \begin{array}{cc} \varphi_i(f(0)) & 0 \\ 0 & \varphi_i(f) \end{array} \right) \otimes e_{i,i} \right), \quad (5.9)$$

modulo $C_0([1, \infty), M_2(M(B \otimes \mathbb{K})))$, for all $k \in \mathbb{Z}, f \in TA$. Since the right-hand side of (5.9) is constant in $t$, modulo $M_2(B \otimes \mathbb{K})$, it follows that $\Phi$ takes values in $X/C_0([1, \infty), M_2(B \otimes \mathbb{K}))$, which is - or should be considered as - a $C^*$-subalgebra of the asymptotic algebra of $M_2(M(B \otimes \mathbb{K}))$. By the Bartle-Graves selection theorem there is a continuous section $\chi : X/C_0([1, \infty), M_2(B \otimes \mathbb{K})) \to X$ for the quotient map $X \to X/C_0([1, \infty), M_2(B \otimes \mathbb{K}))$. Set $\Pi_t(x) = \chi \circ \Phi(x)(t)$. Then $\Pi$ is an asymptotic homomorphism such that $q_{M_2(B \otimes \mathbb{K})} \circ \Pi_t = \hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi)))$ for all $t$.

It follows that $E(\varphi)$ is semi-invertible. That its class in $\text{Ext}^{-1/2}(T^2A, B)$ is independent of the choice of discretization follows from the homotopy invariance of $\text{Ext}^{-1/2}$, Theorem 3.5, by using, for example, the construction of homotopy from Lemma 5.3 in \[14\].

It follows from Lemma 5.4 that there is a well-defined group homomorphism

$$E : \{[TA, SA; B]\} \to \text{Ext}^{-1/2}(T^2A, B)$$

such that $E[\varphi] = [E(\varphi)]$.

**Remark 5.2.** For use in arguments below we give another proof of the semi-invertibility of $E(\varphi)$, i.e. of the fact that that $\hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi)))$ is asymptotically split. For each $m \in \mathbb{N}$, define a sequence $\alpha_i(m), i \in \mathbb{Z}$, of real numbers as follows:

$$\alpha_i(m) = \begin{cases} 1, & i \geq 0 \\ 1 + \frac{i}{m}, & 0 > i > -m \\ 0, & i \leq -m \end{cases}$$

For $t \in [m, m + 1]$, set

$$\alpha_i(t) = (m + 1 - t)\alpha_i(m) + (t - m)\alpha_i(m + 1).$$

For each $t \in [1, \infty)$, set

$$U_t = \sum_{i \in \mathbb{Z}} \left( \sqrt{\alpha_i(t)} 1 \right) \otimes e_{i,i}.$$ 

Each $U_t$ is a unitary in $M_2(M(B \otimes \mathbb{K}))$ and

$$\lim_{t \to \infty} \left[ U_t \sum_{i \in \mathbb{Z}} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes e_{i,i-1} \right] = 0. \quad (5.10)$$
Set
\[ \Gamma_t(f) = \left( \sum_{i \geq 1} \left( \varphi_i(f) \ 0 \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \varphi_i(f) \ 0 \right) \otimes e_{i,i} \right) \]
and note that
\[ U_t \Gamma_t(f) U_t^* - \left( \sum_{i \geq 1} \left( \varphi_i(f) \ 0 \right) \otimes e_{i,i} + \sum_{i \leq 0} \left( \varphi_i(f) \ 0 \right) \otimes e_{i,i} \right) \in M_2(B \otimes \mathbb{K}). \]
Then
\[ \hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi))) \left( z^k \otimes f \right) = q_{M_2(B \otimes \mathbb{K})} \left( \left( \sum_{i \in \mathbb{Z}} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes e_{n,n-1} \right)^k U_t \Gamma_t(f) U_t^* \right), \quad (5.11) \]
for all \( k \in \mathbb{Z}, f \in TA \) and all \( t \). By the method used at the end of the proof of Lemma 5.1 we see from this that \( \hat{\Theta}^{-1} \circ (E(\varphi) \oplus (-E(\varphi))) \) is asymptotically split.

5.3. The Bott maps. We need a version of the Bott isomorphism in one of its many guises. The one which best serves our purpose is based on a particular projection in \( M_2(T^2) \) which we now describe.

Given two commuting unitaries \( S, T \) in a \( C^* \)-algebra, we define a projection \( P(S, T) \) in the \( 2 \times 2 \) matrices over the \( C^* \)-algebra generated by \( S \) and \( T \) in the following way. Let \( s, c_0, c_1 : [0, 1] \to \mathbb{R} \) be the functions
\[ c_0(t) = |\cos(\pi t)|1_{[0, \frac{1}{2}]}(t), \quad c_1(t) = |\cos(\pi t)|1_{(\frac{1}{2}, 1]}(t), \quad s(t) = \sin(\pi t). \]
Set \( \tilde{g} = sc_0, \tilde{h} = sc_1 \) and \( \tilde{f} = s^2 \). Since \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \) are continuous and 1-periodic they give rise to continuous functions, \( f, g, h, \) on \( \mathbb{T} \) (we identify \( \tilde{f} \) on \( [0, 1] \) with \( f \) on \( \mathbb{T} \) in such a way that if \( S = e^{2\pi ix} \) then \( f(S) = \tilde{f}(x) \)). Set
\[ P(S, T) = \left( \begin{array}{cc} f(S) & g(S) + h(S)T \\ T^* h(S) + g(S) & 1 - f(S) \end{array} \right), \]
cf. [13]. When we apply the recipe to the canonical generating unitaries of \( C(\mathbb{T}^2) \), we get the desired projection \( P \in C(\mathbb{T}^2) \otimes M_2 \). Then \( P \) takes the form
\[ P(t, z) = \left( \begin{array}{cc} f(t) & g(t) + h(t)z \\ h(t)z + g(t) & 1 - f(t) \end{array} \right), \quad (5.12) \]
t \in [0, 1], \( z \in \mathbb{T} \). Let \( P_0 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \in M_2(T^2) \). Given a semi-invertible extension \( \phi : T^2 A \to Q(B) \) we set
\[ \phi_1(a) = \hat{\Theta} \left( (\text{id}_{M_2} \otimes \phi)(P \otimes a) \right), \quad \phi_0(a) = \hat{\Theta} \left( (\text{id}_{M_2} \otimes \phi)(P_0 \otimes a) \right), \quad (5.13) \]
a \in A, where \( \hat{\Theta} : M_2(Q(B)) \to Q(B) \) is the isomorphism induced by the isomorphism \( \hat{\Theta} \) [1, 12]. It is easy to see that \( \varphi_1 \) and \( \varphi_0 \) are both semi-invertible since \( \phi \) is. We define the Bott map \( \text{Bott} : \text{Ext}^{-1/2}(T^2 A, B) \to \text{Ext}^{-1/2}(A, B) \) such that
\[ \text{Bott}(\phi) = [\phi_1] - [\phi_0], \]
where \([\phi_i], i = 0, 1, \) are the classes of \( \phi_i \) in \( \text{Ext}^{-1/2}(A, B) \).

We can also use the projections \( P \) and \( P_0 \) to define a Bott map
\[ \text{Bott} : [[T^3 A, ST^2 A; B]] \to [[TA, SA; B]] \]
such that $\text{Bott}([\varphi]) = [\varphi^1] - [\varphi^0]$, where

$$\varphi^1_t(a) = \tilde{\Theta} \left( (\text{id}_{M_2} \otimes \varphi_t) (P \otimes a) \right),$$

and

$$\varphi^0_t(a) = \tilde{\Theta} \left( (\text{id}_{M_2} \otimes \varphi_t) (P_0 \otimes a) \right)$$

for all $t$ and all $a \in TA$. It is easy to see that the diagram

$$\begin{array}{c}
\text{Ext}^{-1/2}(T^2A, B) \xrightarrow{\text{Bott}} \text{Ext}^{-1/2}(A, B) \\
\downarrow \text{CH} \quad \quad \downarrow \text{CH}
\end{array}$$

(5.14)

commutes.

6. From semi-invertible extensions to asymptotic homomorphisms and back

In this section we prove our main result which is the following theorem.

**Theorem 6.1.** Let $A$ and $D$ be separable $C^*$-algebras. Then

$$\text{CH} : \text{Ext}^{-1/2}(A, D) \to [[TA, SA; D \otimes \mathbb{K}]]$$

is an isomorphism.

We will prove Theorem 6.1 by establishing the commutativity of the following two diagrams:

$$\begin{array}{c}
\text{Ext}^{-1/2}(A, B) \xrightarrow{\text{CH}} [[TA, SA; B]] \xrightarrow{E} \text{Ext}^{-1/2}(T^2A, B \otimes \mathbb{K}) \xrightarrow{\text{Bott}} \text{Ext}^{-1/2}(A, B \otimes \mathbb{K}),
\end{array}$$

(6.1)

and

$$\begin{array}{c}
[[TA, SA; B]] \xrightarrow{E} \text{Ext}^{-1/2}(T^2A, B \otimes \mathbb{K}) \xrightarrow{\text{Bott}} \text{Ext}^{-1/2}(A, B \otimes \mathbb{K}) \xrightarrow{\text{CH}} [[TA, SA; B \otimes \mathbb{K}]],
\end{array}$$

(6.2)

where $e$ in both cases is an isomorphism induced by the stabilizing map $b \mapsto b \otimes e_{11}$ for some minimal non-zero projection $e_{11} \in \mathbb{K}$. From the commutativity of the first diagram we conclude that $\text{CH}$ is injectivity, and from the commutativity of the latter that $\text{CH}$ is surjective.

6.1. $\text{CH}$ is injective. For simplicity of notation we shall ignore the $*$-isomorphism $\Theta$ in the definition of $\text{CH}$, and consider instead $\text{CH}$ as a map $\text{CH} : \text{Ext}^{-1/2}(A, B) \to [[TA, SA; M_2(B)]]$. Similarly, we will consider Bott as a map $\text{Bott} : \text{Ext}^{-1/2}(T^2A, B) \to \text{Ext}^{-1/2}(A, M_2(B))$. Let $\varphi \in \text{Hom}(A, Q(B))$ be a semi-invertible extension. There is then an equi-continuous and essentially constant asymptotic homomorphism

$$\left( \begin{array}{cc} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{array} \right)_{t \in [1, \infty)} : A \to M_2(M(B))$$
such that \( \varphi = q_B \circ \alpha_t \) for all \( t \). By ‘essentially constant’ we refer to the fact that \( (\alpha_t / \beta_t) \) is \( t \)-independent modulo \( M_2(B) \). \( E \circ \text{CH}(\varphi) \in \text{Ext}^{-1/2}(T^2 A, M_2(B) \otimes \mathbb{K}) \) is given by a continuous approximate unit \( \{u_t\}_{t \in [1, \infty)} \) in \( B \) and a sequence \( t_0 < t_1 < t_2 < t_3 < \ldots \) in \( [1, \infty) \) such that

\[
E \circ \text{CH}(\varphi)(f \otimes z^k \otimes a) = q_{M_2(B \otimes \mathbb{K})}(\left( \begin{array}{cc} T^k & 0 \\ 0 & T^k \end{array} \right) \sum_{n \in \mathbb{Z}} \left( \begin{array}{cc} f(u_n)\alpha_t(a) & f(0)\beta_t(a) \\ f(0)^*\gamma_t(a) & f(0)^*\delta_t(a) \end{array} \right) \otimes e_{n,n}),
\]

when \( f \in T = \{ g \in C[0,1] : g(0) = g(1) \} \), \( k \in \mathbb{Z} \), \( a \in A \). In this expression \( t_n = |t_n| \) when \( n \leq 0 \), \( u_n = u_{tn} \) when \( n \geq 1 \), and \( u_n = 0 \) when \( n < 1 \).

To describe \( \text{Bott} \circ E \circ \text{CH}(\varphi) \in \text{Ext}^{-1/2}(A, M_2(B) \otimes \mathbb{K}) \) we will ignore the *-isomorphism \( \hat{\Theta} \) appearing in the definition of \( \text{Bott} \). Then

\[
\text{Bott} \circ E \circ \text{CH}(\varphi) \in \text{Ext}^{-1/2}(A, M_4(B) \otimes \mathbb{K})
\]

is the difference \( x - x_0 \) of two elements, \( x, x_0 \), corresponding to the projections \( P \) and \( P_0 \), respectively. Using the explicit description of \( P \), we see that \( x = [q_{M_4(B) \otimes \mathbb{K}} \circ \psi] \), where \( \psi : A \to M(M_4(B) \otimes \mathbb{K}) \) is given, modulo \( M_4(B) \otimes \mathbb{K} \), by the formula

\[
\psi(a) = \sum_{n \in \mathbb{Z}} \left( \begin{array}{ccc} s^2(u_n)\alpha_t(a) & 0 & \text{sc}_0(u_n)\alpha_t(a) \\ 0 & 0 & 0 \\ \text{sc}_0(u_n)\alpha_t(a) & 0 & 0 \end{array} \right) \otimes e_{n,n}.
\]

Similarly, \( x_0 = [q_{M_4(B) \otimes \mathbb{K}} \circ \psi_0] \), where \( \psi_0 : A \to M(M_4(B) \otimes \mathbb{K}) \) is given, modulo \( M_4(B) \otimes \mathbb{K} \), by the formula

\[
\psi_0(a) = \sum_{n \in \mathbb{Z}} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_t(a) \end{array} \right) \otimes e_{n,n}.
\]

Define \( \alpha, \beta, \gamma, \delta : A \to M(B \otimes \mathbb{K}) \) by

\[
\alpha(a) = \sum_{n \in \mathbb{Z}} \alpha_t(a) \otimes e_{n,n}, \quad \beta(a) = \sum_{n \in \mathbb{Z}} \beta_t(a) \otimes e_{n,n}, \quad \gamma(a) = \sum_{n \in \mathbb{Z}} \gamma_t(a) \otimes e_{n,n}, \quad \delta(a) = \sum_{n \in \mathbb{Z}} \delta_t(a) \otimes e_{n,n}.
\]
and \( S, C_0, C_1 \in M(B \otimes \mathbb{K}) \) by
\[
S = \sum_{n \in \mathbb{Z}} s(u_n) \otimes e_{n,n}, \quad C_0 = \sum_{n \in \mathbb{Z}} c_0(u_n) \otimes e_{n,n}, \quad C_1 = \sum_{n \in \mathbb{Z}} c_1(u_n) \otimes e_{n,n}.
\]

Since \( \alpha_{t_n}(a) - \alpha_{t_{n+1}}(a) \in B \) and \( \lim_{n \to \pm \infty} \alpha_{t_n}(a) - \alpha_{t_{n+1}}(a) = 0 \) for all \( a \in A \), we see that \( [\alpha(a), T] \in B \otimes \mathbb{K} \) for all \( a \in A \), i.e. \( T \) essentially commutes with \( \alpha \). The same is true, for the same reason, for \( \beta, \gamma \) and \( \delta \). Similarly, we can assume that \( \lim_{n \to \infty} u_n - u_{n+1} = 0 \), which implies that also \( S, C_0 \) and \( C_1 \) essentially commute with \( T \). Note that
\[
\psi'(a) = \begin{pmatrix}
S^2 \alpha(a) & SC_1 T \alpha(a) + SC_0 \alpha(a) & 0 & 0 \\
SC_1 T^* \alpha(a) + SC_0 \alpha(a) & (C_0^2 + C_1^2) \alpha(a) & 0 & \beta(a) \\
0 & 0 & \gamma(a) & 0 \\
0 & \delta(a)
\end{pmatrix},
\]
while
\[
\psi_0(a) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha(a) & \beta(a) & 0 \\
0 & \gamma(a) & \delta(a)
\end{pmatrix}.
\]

Set \( T_0 = \sum_{n \geq 1} e_{n,n-1} + \sum_{n \leq -1} e_{n,n} \), and note that \( T_0 \in M(B \otimes \mathbb{K}) \) is an isometry such that \( T_0 T_0^* = 1 - e_{0,0} \). Like \( T \), also \( T_0 \) commutes with \( S, C_0, C_1, \alpha(a), \beta(a), \gamma(a) \) and \( \delta(a) \), modulo \( B \otimes \mathbb{K} \). Set
\[
W_+ = \begin{pmatrix}
S \\
C_0 + C_1 T^*
\end{pmatrix} - C_1 T, \quad W_- = \begin{pmatrix}
e_{1,1} \\
T_0^* \\
0
\end{pmatrix} \in M_2(M(B \otimes \mathbb{K})).
\]

Then \( W_- \) is a unitary while \( W_+ \) is unitary modulo \( M_2(B \otimes \mathbb{K}) \).

Furthermore, a calculation shows that
\[
\text{Ad} \left( \begin{pmatrix} W_+^* & 0 \\ 0 & W_+ \\ W_-^* & 0 \end{pmatrix} \right) \circ \psi'(a) = \begin{pmatrix}
\alpha(a) & 0 & [C_0 T_0^* + C_1 T T_0^*]_2 \beta(a) & 0 \\
0 & \gamma(a) & S T_0^* \beta(a) & 0 \\
0 & 0 & T_0 T_0^* \delta(a) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
modulo \( M_4(B) \otimes \mathbb{K} \). Since \( \lim_{n \to \infty} \beta_{t_n}(a)s(u_n) = \lim_{n \to \infty} \gamma_{t_n}(a)s(u_n) = 0 \), we see that \( S \beta(a), S \gamma(a) \in B \otimes \mathbb{K} \). It follows that \( S T_0^* \beta(a), S T_0^* \gamma(a) \in B \otimes \mathbb{K} \). Similarly, since \( \lim_{n \to \infty} c_0(u_n) \beta_{t_n}(a) = \lim_{n \to \infty} c_0(u_n) \gamma_{t_n}(a) = 0 \), we find that \( C_0 \beta(a) = P_- \gamma(a) \) and \( C_0 \beta(a) = P_- \beta(a), \) modulo \( B \otimes \mathbb{K} \), where \( P_- = \sum_{n \leq 0} e_{n,n} \). For a similar reason, we find that \( C_1 \beta(a) = P_+ \beta(a) \) and \( C_1 \gamma(a) = P_+ \gamma(a) \), modulo \( B \otimes \mathbb{K} \), where \( P_+ = \sum_{n \geq 1} e_{n,n} \). It follows that \( [C_0 T_0^* + C_1 T T_0^*]_2 \beta(a) = P_- \beta(a) + P_+ \beta(a) = \beta(a) \) and \( [T_0 C_0 + T_0 T^* C_1] \gamma(a) = P_- \gamma(a) + P_+ \gamma(a) = \gamma(a) \), modulo \( B \otimes \mathbb{K} \). Consequently,
\[
\text{Ad} \left( \begin{pmatrix} W_+^* & 0 \\ 0 & W_- \end{pmatrix} \right) \circ \psi'(a) = \begin{pmatrix}
\alpha(a) & 0 & \beta(a) & 0 \\
0 & 0 & 0 & 0 \\
\gamma(a) & 0 & T_0 T_0^* \delta(a) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
modulo $M_4(B) \otimes \mathbb{K}$. Since conjugation by the unitary $q_{M_4(B)}^{W_+ W_-}$ does not change the class in $	ext{Ext}^{-1/2}(A, M_4(B))$, we conclude that $x = [q_{M_4(B) \otimes \mathbb{K}} \circ \psi']$, where

$$\psi''(a) = \begin{pmatrix} \alpha(a) & \beta(a) \\ \gamma(a) & (1 - e_{0,0})\delta(a) \end{pmatrix}.$$  

Clearly, $x_0 = [q_{M_4(B) \otimes \mathbb{K}} \circ \psi'_0]$, where

$$\psi'_0(a) = \begin{pmatrix} \alpha(a) & \beta(a) \\ \gamma(a) & \delta(a) \end{pmatrix}.$$  

It follows that $x_0 - x$ is represented by $a \mapsto q_{B \otimes \mathbb{K}}(e_{0,0}\delta(a))$, which represents $e(-[\varphi])$ since $q_B \circ \delta$ represents $-[\varphi]$. Hence $x - x_0 = e([\varphi])$. We have now shown that (6.1) commutes, and it follows that $CH$ is injective.

### 6.2. CH is surjective.  

Let $P, P_0 \in M_2(T^2)$ be the projections used to define Bott, cf. Section 5.3. We can then define a unitary $U \in M_2(T^3) = \{f \in C([0,1], M_2(C(T^2))) : f(0) = f(1)\}$ by

$$U(s) = e^{2\pi isP}e^{-2\pi isP_0} = (1 + (e^{2\pi is} - 1)P)(1 + (e^{-2\pi is} - 1)P_0).$$  

Note that $U(0) = U(1) = 1$. Since $P(0, z) = P(1, z) = P_0$ for all $z \in \mathbb{T}$, it follows that $U - 1 \in M_2(STS)$. Consequently the $*$-homomorphism $\phi_U : T \to M_2(T^3)$ given by $\phi_U(f) = f(U)$ has the property that $\phi_U(S) \subseteq M_2(STS)$. We can therefore define a map $B : [[T^3A, STSA; B]] \to [[TA, SA; B]]$ such that $B[\varphi] = [\varphi']$, where

$$\varphi'_t(a) = \tilde{\Theta} \circ (\text{id}_{M_2} \otimes \varphi_t)((\phi_U \otimes \text{id}_A)(a)),$$

for $a \in TA$. To compare $B$ with the other maps we have in play, let $j : [[T^3A, ST^2A; B \otimes \mathbb{K}]] \to [[T^3A, STSA; B \otimes \mathbb{K}]]$ be the forgetful homomorphism obtained from the fact that $STS A \subseteq ST^2 A$.

We claim that the diagram

$$
\begin{array}{ccc}
[[T^3A, ST^2A; B]] & \xrightarrow{\text{Bott}} & [[TA, SA; B]] \\
\downarrow{j} & & \downarrow{B} \\
[[T^3A, STSA; B]] & & 
\end{array}
$$

commutes. To see this define $\pi_1, \lambda_1 : T \to M_2(T^3)$ by $\pi_1(f) = fP$ and $\lambda_1(f) = fP + f(0)(1 - P)$. Similarly, we set $\pi_0(f) = fP_0$ and $\lambda_0(f) = fP_0 + f(0)(1 - P_0)$. Note that all these $*$-homomorphisms take $S$ into $M_2(ST^2)$. Let $z \in T$ denote the canonical unitary generator; the identity function on $T$. For $\theta \in [0, \frac{\pi}{2}]$, set

$$V_\theta = \begin{pmatrix} zP + 1 - P & 1 + \frac{\cos \theta \sin \theta}{\sin \theta \cos \theta} & zP_0 + 1 - P_0 \\ -\frac{\cos \theta \sin \theta}{\sin \theta \cos \theta} & 1 & \frac{\cos \theta \sin \theta}{\sin \theta \cos \theta} \\
1 + \frac{\cos \theta \sin \theta}{\sin \theta \cos \theta} & -\frac{\cos \theta \sin \theta}{\sin \theta \cos \theta} & zP_0 + 1 - P_0 \end{pmatrix},$$

which gives a homotopy of unitaries in $M_4(T^3)$ connecting $(U_{zP_0 + 1 - P_0})$ to $(zP + 1 - P)$. Note that when we substitute 1 for $z$ in the formula for $V_\theta$, we get 1 for each $\theta$. It follows that $\phi_U \oplus \lambda_0$ is homotopic to $\lambda_1 \oplus \text{ev}$, where $\text{ev}(f) = f(0)$, via a path of $*$-homomorphisms taking $S$ into $M_2(ST^2)$. Since $[[\text{id}_{M_2} \otimes \varphi] \circ \pi_1] = [[\text{id}_{M_2} \otimes \varphi] \circ \pi_i], i = 0, 1, \text{ in } [[TA, SA; B]]$ for all $\varphi$ by Lemma 4.2, we conclude that

$$\tilde{\Theta} \circ (\text{id}_{M_2} \otimes \varphi_t)((\phi_U \otimes \text{id}_A)(a)) \oplus \tilde{\Theta} \circ (\text{id}_{M_2} \otimes \varphi_t)(P_0 \otimes a)$$
defines the same element of \([TA, SA; B]\) as \(\tilde{\Theta} ((\text{id}_{M_2} \otimes \varphi_t) (P \otimes a))\). This establishes the commutativity of (6.3).

Let \(\varphi = (\varphi_t)_{t \in [1, \infty)} : TA \rightarrow M(B)\) be an asymptotic homomorphism, constantly extended from \(SA\). Let \((\varphi_n)_{n \in \mathbb{N}}\) be a discretization of \(\varphi\). For each \(a \in TA, t \in [1, \infty), \) set
\[
\overline{\varphi}_t(a) = \sum_{n \geq 1} \varphi_{\max\{t_n, t\}}(a) \otimes e_{n,n} + \sum_{n \leq 0} \varphi_{\max\{t_{n}, t\}}(a) \otimes e_{n,n},
\]
which is an element of \(M(B \otimes \mathbb{K})\). Then \(\overline{\varphi} = (\overline{\varphi}_t)_{t \in [1, \infty)}\) is an asymptotic homomorphism which is essentially constant, i.e. \(\overline{\varphi}_t(a) - \overline{\varphi}_s(a) \in B \otimes \mathbb{K}\) for all \(t, s \in [1, \infty)\). Furthermore, \(\overline{\varphi}_t(a)\) commutes with the two-sided shift \(\mathcal{T}\) modulo \(B \otimes \mathbb{K}\). For each \(n \geq 1, \) set
\[
v_n = \sum_{i \in \mathbb{Z}} v_n^{(i)} \otimes e_{i,i},
\]
where
\[
v_n^{(i)} = \begin{cases} 1, & i \leq 0 \\ \frac{n-i}{n}, & 1 \leq i \leq n, \\ 0, & i \geq n. \end{cases}
\]
Then set
\[
v_t = (t - n)v_{n+1} + (n + 1 - t)v_n, \quad (6.4)
\]
\(t \in [n, n+1].\) It follows that \([v_t, \overline{\varphi}_s(a)] = 0\) for all \(a, s, t,\) and that \(\lim_{t \rightarrow \infty}[\mathcal{T}, f(v_t)] = 0\) for all \(f \in C[0,1]\) for which \(f(0) = f(1).\) We can therefore define an asymptotic homomorphism \(\beta(\varphi) : T^3A \rightarrow M(B \otimes \mathbb{K})\) determined, up to asymptotic equality, by the condition that
\[
\lim_{t \rightarrow \infty} \beta(\varphi)_t(f \otimes z^k \otimes a) - \overline{\varphi}_t(a)f(v_t)\mathcal{T}^k = 0
\]
when \(f \in C(\mathbb{T}), k \in \mathbb{Z}\) and \(a \in TA.\) Since \(\overline{\varphi}_t(a)f(v_t)\mathcal{T}^k - \overline{\varphi}_s(a)f(v_s)\mathcal{T}^k \in B \otimes \mathbb{K}\) for all \(s, t, a, f, k,\) and \(\overline{\varphi}_t(a)f(v_t)\mathcal{T}^k \in B \otimes \mathbb{K},\) when \(a \in SA\) and \(f \in S,\) we can arrange that \(\beta(\varphi)\) is essentially constant and that
\[
\beta(\varphi)_t(\text{STSA}) \subseteq B \otimes \mathbb{K},
\]
for all \(t \in [1, \infty),\) cf. the construction in Remark 5.2. We get in this way a map \(\beta : [[TA, SA; B]] \rightarrow [[T^3A, STSA; B \otimes \mathbb{K}]]\) such that \(\beta[\varphi] = [\beta(\varphi)].\) We claim that the diagram
\[
\begin{array}{ccc}
[[TA, SA; B]] & \xrightarrow{E} & \text{Ext}^{-1/2}(T^2A, B \otimes \mathbb{K}) \\
\text{CH} & \downarrow & \text{Bott} \\
[[T^3A, ST^2A; B \otimes \mathbb{K}]] & \xrightarrow{\text{Bott}} & [[TA, SA; B \otimes \mathbb{K}]] \\
\end{array}
\]
commutes. Since the square commutes by the naturality of the extended Connes-Higson construction, cf. (5.14), and the right triangle commutes by (6.3), it suffices to show that the left triangle commutes, i.e. we must show that \(\beta = j \circ \text{CH} \circ E.\) Let therefore \(\varphi : TA \rightarrow M(B)\) be a constantly extended asymptotic homomorphism. \(E(\varphi)\) is given by (5.4) and (5.5) for an appropriate discretization \((\varphi_n)_{n \in \mathbb{N}}\) of \(\varphi,\) and the inverse \(-E(\varphi)\) is given by (5.6) and (5.7). We shall use the constructions of Remark 5.2 in order to get a workable description of \(\text{CH} \circ E(\varphi).\)
In particular, we refer to Remark 4.2 for the notation used in the following. Let \( \{u_t\}_{t \in [1, \infty)} \) be a continuous approximate unit in \( B \otimes K \) satisfying the requirements needed to define \( CH \circ E(\varphi) \), cf. (5.1)-(5.3). Then \( CH \circ E[\varphi] \) is represented by an asymptotic homomorphism \( \tilde{\Theta} \circ \psi \), where the asymptotic homomorphism \( \psi : T^3A \to M_2(M(B \otimes K)) \) satisfies that \( \psi_t(h \otimes z^k \otimes f) \) asymptotically agrees with
\[
\left( \left( \frac{h(u_t)}{h(1)} \right) \left( T^k \right) \right) \sum_{i \in Z} \left( \frac{\alpha_i(t)\varphi^i(f) + (1 - \alpha_i(t))\varphi^i(0)}{\sqrt{\alpha_i(t) - \alpha_i(t)^2}} \left( \varphi^i(f) - \varphi^i(0) \right) \right) \otimes e_{i,i},
\]
for all \( h \in T \), \( k \in \mathbb{Z} \) and \( f \in TA \). Note that
\[
U_t \Gamma_t(f) U_t^* = \sum_{i \in \mathbb{Z}} \left( \frac{\alpha_i(t)\varphi^i(f) + (1 - \alpha_i(t))\varphi^i(0)}{\sqrt{\alpha_i(t) - \alpha_i(t)^2}} \left( \varphi^i(f) - \varphi^i(0) \right) \right) \otimes e_{i,i}.
\]
Let \( F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots \) be a sequence of finite sets with dense union in \( TA \). Let \( v_n, n \in \mathbb{N}, \) be an approximate unit in \( B \) such that
\[
\|v_n, \varphi_t(f)\| \leq \frac{1}{n}
\]
and
\[
\|(v_n - 1) (\varphi_t(f) - \varphi_t(0))\| \leq \frac{1}{n}
\]
for all \( t \in [1, 3n], f \in F_n \). Set
\[
\beta_i = \begin{cases} 
1, & i \in \{-n, i - n + 1, \ldots, n - 1, n\} \\
\frac{2n - i}{n}, & i \in \{n + 1, n + 2, \ldots, 2n\} \\
\frac{2n + i}{n}, & i \in \{-2n - 1, -2n + 1, \ldots, -n - 1\} \\
0, & |i| > 2n,
\end{cases}
\]
\[
\tilde{v}_n = \sum_{i \in \mathbb{Z}} \beta_i v_n \otimes e_{i,i}, \text{ and } \quad w_t = (n + 1 - t)\tilde{v}_n + (n - t)\tilde{v}_{n+1},
\]
when \( t \in [n, n + 1] \). Then \( \{w_t\}_{t \in [0, \infty)} \) is a continuous approximate unit in \( B \otimes K \) such that the requirements needed to define \( CH \circ E(\varphi) \), cf. (5.1)-(5.3) hold for \( w_t \) in place of \( u_t \). We can therefore work with this path instead of \( \{u_t\}_{t \in [1, \infty)} \) in (6.6). Thanks to (5.10), (6.6) then becomes asymptotically the same as
\[
U_t \left[ \left( h(u_t) \right) (T^k \right) \sum_{i \in \mathbb{Z}} \left( \varphi^i(f) \varphi^i(0) \right) \otimes e_{i,i} \right] U_t^*, \tag{6.7}
\]
Note that conjugation by \( U_t \) induces the identity map in \([[T^3A, ST^2A; M_2(B \otimes K)]]\). We see therefore from (6.7) that \( CH \circ E[\varphi] \) is represented by an asymptotic homomorphism \( \tilde{\Theta} \circ \psi' \), such that \( \psi'_t(h \otimes z^k \otimes f) \) asymptotically agrees with
\[
h(Y_t) \left( \sum_{i \in \mathbb{Z}} \left( 1 \right) \otimes e_{i,i-1} \right) k \left( \sum_{i \in \mathbb{Z}} \left( \varphi^i(f) \varphi^i(0) \right) \otimes e_{i,i} \right),
\]
where
\[
Y_t = U_t^* (w_t) U_t.
\]
We want to substitute \( Y_t \) with something else. To this end write
\[
\varphi_t(f) = \sum_{i \in \mathbb{Z}} \varphi_i^t(f) \otimes e_{i,i},
\]
and let
\[
Y_t = \begin{pmatrix} \varphi_{11}^t & \varphi_{12}^t \\ \varphi_{21}^t & \varphi_{22}^t \end{pmatrix}
\]
be the 2 \( \times \) 2-matrix decomposition of \( Y_t \). Set \( Q = \sum_{i \leq 0} e_{i,i} \). The significant properties of \( Y_t \) are the following:

0) \( 0 \leq Y_t \leq 1 \),
1) \( \lim_{t \to \infty} [Y_{11}^t, \varphi_t(f)] = \lim_{t \to \infty} [Y_{22}^t, \varphi_t(f(0))] = 0 \) for all \( f \in TA \),
2) \( \lim_{t \to \infty} Y_{12}^t (\varphi_t(f) - \varphi_t(f(0))) = \lim_{t \to \infty} Y_{21}^t (\varphi_t(f) - \varphi_t(f(0))) = 0 \) for all \( f \in TA \),
3) \( \lim_{t \to \infty} [Y_t, (\varphi_t - \varphi_t(0))] = 0 \),
4) \( Y_t (\varphi_t(f)_0) = (\varphi_{t,i}(f)_0) \) modulo \( M_2(B \otimes K) \) for all \( t \) and all \( f \in SA \),

which are all easy to check. Note that 4) implies

5) \( [Y_t, (\varphi_t - \varphi_t(0))] (\varphi_t(f)_0) \in M_2(B \otimes K) \) for all \( t \) and all \( f \in SA \),

since
\[
[(\varphi_t - \varphi_t(0)), (\varphi_t(f)_0)], [(\varphi_t - \varphi_t(0)), (Q\varphi_t(f)_0)] \in M_2(B \otimes K)
\]
for all \( t \) and all \( f \in SA \).

Put
\[
Y_t^\lambda = (1 - \lambda)Y_t + \lambda (v_0 \otimes 0),
\]
where \( v_t \) is defined by (6.4). Then \( Y_t^\lambda \) satisfies 0)-4) for all \( \lambda \in [0, 1] \). It follows from 0)-3) that we can define an asymptotic homomorphism \( \Phi : T^3 A \to M_2(M(IB \otimes K)) \) such that
\[
\Phi_t (h \otimes z^k \otimes f)(\lambda), \lambda \in [0, 1],
\]
asymptotically agrees with
\[
h(Y_t^\lambda) (\varphi_t(f)_0) \otimes (\varphi_t(f(0)))
\]
for all \( h \in T, k \in \mathbb{Z} \) and \( f \in TA \). We claim that we can arrange that
\[
\Phi_t (STSA) \subseteq M_2(IB \otimes K)
\]
for all \( t \). Since \( S = \{ \mu \in C[0, 1] : \mu(1) = \mu(0) = 0 \} \) is generated by the function \( s \mapsto e^{2\pi is} - 1 \), it suffices for this purpose to check that
\[
(e^{2\pi is}Y_t^\lambda - 1) (\varphi_t(f)_0) \otimes (\varphi_t(f(0)))
\]
is in \( M_2(B \otimes K) \) for \( \lambda \in [0, 1], k \in \mathbb{Z} \) and \( f \in SA \). This follows from 5) and 4) because we see that
\[
(e^{2\pi is}Y_t^\lambda - 1) (\varphi_t(f)_0) \otimes (\varphi_t(f(0)))
\]
= \(
(\varphi_t(f)_0) \otimes (e^{2\pi is}Y_t^\lambda - 1) (\varphi_t(f(0)))
\]
= \(
(\varphi_t(f)_0) \otimes (e^{2\pi is}Y_t^\lambda - 1) (\varphi_t(f(0)))
\]
modulo \( M_2(B \otimes K) \). Thus \( \Phi \) gives us a homotopy of, not necessarily constantly, extended asymptotic homomorphisms. At both ends the asymptotic homomorphisms are constantly extended so we can conclude from Theorem 4.1 that \( j \circ CH \circ E[\varphi] \in [T^3 A, STSA; B \otimes K] \)
is represented by an asymptotic homomorphism \( \psi \) such that \( \psi_t(h \otimes z^k \otimes f) \) essentially (i.e. modulo \( B \otimes \mathbb{K} \)) and asymptotically agrees with

\[
h(v_t)T^k \psi_t(f)
\]

when \( h \in T, k \in \mathbb{Z} \) and \( f \in TA \). But this is \( \beta(\varphi) \), so we have shown that the diagram (5.3) commutes.

It suffices now to show that \( B \circ \beta = \epsilon \). To this end define for each \( f \in TA \) an element \( H(f) \in M(IB \otimes \mathbb{K}) \) such that

\[
H(f)(\lambda) = \sum_{i \in \mathbb{Z}} \varphi_{\lambda \max\{t_i, t\} + (1-\lambda)_t}(f) \otimes e_{i,i}.
\]

We can then define an asymptotic homomorphism \( \Psi : T^3A \rightarrow M(IB \otimes \mathbb{K}) \) such that \( \Psi_t(h \otimes z^k \otimes f) \) asymptotically agrees with

\[
h(v_t)T^k H(f).
\]

Since \( h(v_t)T^k H(f) \in IB \otimes \mathbb{K} \), when \( h \in S, f \in SA \), we get a homotopy of (typically not constantly) extended asymptotic homomorphisms showing that \( \beta(\varphi) \) defines the same element in \([T^3A, STSA; B\otimes \mathbb{K}]\) as an asymptotic homomorphism \( \psi \) with the property that \( \psi_t(h \otimes z^k \otimes f) \) asymptotically agrees with

\[
h(v_t)T^k \sum_{i \in \mathbb{Z}} \varphi_t(f) \otimes e_{i,i}
\]

for all \( h \in T, k \in \mathbb{Z}, f \in TA \). To compare this with \( \varphi \), define an asymptotic homomorphism \( \varphi \otimes \text{id}_{\mathbb{K}^+} : TA \otimes \mathbb{K}^+ \rightarrow M(B \otimes \mathbb{K}) \) such that \( (\varphi \otimes \text{id}_{\mathbb{K}^+})_t(f \otimes x) \) asymptotically agrees with

\[
\iota(\varphi_t(f) \otimes x),
\]

where \( \iota : M(B) \otimes \mathbb{K}^+ \rightarrow M(B \otimes \mathbb{K}) \) is the canonical embedding. Since \( \varphi_t(SA) \subseteq B \) we can arrange that \( (\varphi \otimes \text{id}_{\mathbb{K}^+})_t(SA \otimes \mathbb{K}) \subseteq B \otimes \mathbb{K} \) for all \( t \). Define also a continuous path \( A_t, t \in [1, \infty) \), of contractions in \( M_2(T\mathbb{K}^+) \) by

\[
A_t = [zQ_t + 1 - Q_t] [z^*P_0 - 1 - P_0],
\]

where \( z \in T \) is the identity function, \( P_0 = (0, 1) \) and

\[
Q_t = \begin{pmatrix}
    s^2(v_t) & sc_0(v_t) + sc_1(v_t)T \\
    T^*sc_1(v_t) + sc_0(v_t) & 1 - s^2(v_t)
\end{pmatrix}.
\]

Then, by definition, \( B(\psi) = \tilde{\Theta} \circ \psi' \), where \( \psi' : TA \rightarrow M_2(M(B \otimes \mathbb{K})) \) is an asymptotic homomorphism such that \( \psi'_t(z^k \otimes a) \) asymptotically agrees with

\[(\text{id}_{M_2} \otimes (\varphi \otimes \text{id}_{\mathbb{K}^+}))(A_t \otimes a)
\]

for all \( k \in \mathbb{Z}, a \in A \). Since \( v_t \) asymptotically commutes with \( T \), we see that \( \lim_{t \to \infty} \|Q_t^2 - Q_t\| = 0 \). Hence a standard application of spectral theory gives us a continuous path \( \{P_t\}_{t \in [1, \infty)} \) of projections in \( M_2(\mathbb{K}^+) \) such that

\[
\lim_{t \to \infty} \|P_t - Q_t\| = 0.
\]

Since \( Q_t - P_0 \in M_2(\mathbb{K}) \) we can arrange (or rather, the standard procedure will automatically ensure) that

\[
P_t - P_0 \in M_2(\mathbb{K}).
\]

It follows that

\[
U_t = [zP_t + 1 - P_t] [z^*P_0 - 1 - P_0]
\]
Our main result, Theorem 6.1, shows that the map $\text{Ext}^{-1}$ represents zero in $[TA, SA]$. If $U_t - 1 \in M_2(S\mathbb{K})$, then $U_t$ is homotopic, within the indicated subgroup of the unitary group of $M_2(T\mathbb{K}^+)$, to the unitary

$$R = \begin{pmatrix} ze_{00} + \sum_{i \in \mathbb{Z} \setminus \{0\}} e_{ii} & 0 \\ 0 & \sum_{i \in \mathbb{Z}} e_{ii} \end{pmatrix}.$$ 

Consequently $[\psi'] = e[\varphi] + [\varphi_0]$ in $[[TA, SA; B \otimes \mathbb{K}]]$, where $\varphi_0 : TA \to M(B \otimes \mathbb{K})$ is an asymptotic homomorphism which factors through the evaluation map $ev : TA \to A$. $\varphi_0$ represents zero in $[[TA, SA; B]]$ by Lemma 4.2 and we conclude that $[\psi'] = e[\varphi]$.

7. Conclusion

Our main result, Theorem 6.1, shows that the map $\text{Ext}^{-1/2}(A, B) \to [[SA, B]]$ arising from the Connes-Higson construction as defined in [4] factors as

$$\text{Ext}^{-1/2}(A, B) \xrightarrow{\text{restriction map}} [[TA, SA; B]] \xrightarrow{\text{homotopy classes of asymptotic homomorphisms}} [[SA, B]]$$

such that the horizontal map is an isomorphism. Thus the question whether or not the vertical $CH$-map is an isomorphism has been transformed to a question which solely involves homotopy classes of asymptotic homomorphisms. Specifically the question is now whether or not the restriction map $[[TA, SA; B]] \to [[SA, B]]$ is an isomorphism. It is with some regret that we must report that we haven’t been able to decide the latter.

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