ON THE SATURATION CONJUNCTURE FOR Spin(2n)

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Abstract. In this paper we examine the saturation conjecture on decompositions of tensor products of irreducible representations for complex semisimple algebraic groups of type D (the even spin groups: Spin(2n) for n \geq 4 an integer), extending work done by Kumar-Kapovich-Millson on Spin(8). Our main theorem is that the saturation conjecture holds for Spin(10): for all triples of dominants weights \(\lambda, \mu, \nu\) such that \(\lambda + \mu + \nu\) is in the root lattice, and for any \(N > 0\),

\[(V(\lambda) \otimes V(\mu) \otimes V(\nu))^\text{Spin(10)} \neq 0\]

if and only if

\[(V(N\lambda) \otimes V(N\mu) \otimes V(N\nu))^\text{Spin(10)} \neq 0.\]

Some preliminary results are also obtained for Spin(12), and a question is formulated regarding the Hilbert basis of the saturated tensor cone for all Spin(2n). Some related results for groups of other types are listed as well.

1. Introduction

In this paper we examine the saturation conjecture on decompositions of tensor products of irreducible representations for complex semisimple algebraic groups of type D (the spin groups: Spin(2n) for n \geq 4 an integer), extending work done by Kapovich-Kumar-Millson in [KKM09] on Spin(8). Our main theorem is that the saturation conjecture holds for Spin(10). Some preliminary results are also obtained for Spin(12), and Question 6.3 is formulated regarding the Hilbert basis of the saturated tensor cone for all Spin(2n). Some related results for groups of other types are listed as well.

The saturation conjecture can be approached by studying a certain polyhedral cone, the saturated tensor cone, whose defining inequalities are known to be minimally parametrized by products in the cohomology ring of relevant spaces \(G/P\) (see [BK06] and [Res10], as well as the survey [Kum14]). We introduce a computationally feasible method (based on the polynomial realization of [BGG73]) for calculating cup products in the singular (or deformed) cohomology of any \(G/P\), and we indicate some pseudocode for implementing this method on a computer in order to find the desired inequalities. The method lends itself to (partial) parallelization, and it was with the crucial aid of the parallel-capable supercomputer Longleaf, maintained at the University of North Carolina, that we obtained the aforementioned results.

1.1. The Saturation Conjecture. Let \(G\) be a semisimple algebraic group over \(\mathbb{C}\). Fix a Borel subgroup \(B \subset G\) and maximal torus \(H \subset B\). Let \(W = N_G(H)/H\) be the Weyl group of \(H\) in \(G\). The choice of \(H \subset B\) determines a root system \(\Phi \subset \mathfrak{h}^*\) with base (simple roots) \(\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi\), where \(r = \dim H\) is the rank of \(G\). The \(\mathbb{Z}\)-span of the \(\alpha_i\) is called the root lattice.

Let \(\omega_1, \ldots, \omega_r \in \mathfrak{h}^*\) be the associated dominant fundamental weights. The weights of the \(\mathbb{Z}_{\geq 0}\)-span of \(\{\omega_1, \ldots, \omega_r\}\) are the dominant weights, and to each such \(\lambda\) is associated a unique, irreducible representation of \(G\), denoted by \(V(\lambda)\).

It is a standard problem to determine when an irreducible component \(V(\nu)\) appears in a tensor product of irreducible representations \(V(\lambda) \otimes V(\mu)\). This question is, in general, hard to answer. However, the question whether \(V(N\nu)\) appears in \(V(N\lambda) \otimes V(N\mu)\) for some \(N \geq 1\) has a well-known answer. One then may ask if and how \(N\) may be controlled. Replacing \(\nu\) by its dual weight, the questions may be posed symmetrically, which we now make precise.

Definition 1.1. Define the saturated tensor cone \(\mathcal{C}(G)\) to be the set of triples \(\lambda_1, \lambda_2, \lambda_3\) of dominant weights whose sum is in the root lattice and for which

\[(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G \neq 0\]

for some integer \(N > 0\). Define the tensor cone \(\mathcal{R}(G)\) to be the set of triples \(\lambda_1, \lambda_2, \lambda_3\) of dominant weights satisfying (1) with \(N = 1\) (in which case \(\lambda_1 + \lambda_2 + \lambda_3\) must lie in the root lattice).

There exists a system of inequalities (listed below) determining this cone as a subset of \((\mathfrak{h}^*)^3\); an overview can be found in the survey paper of Kumar [Kum14, Section 6].

The cone \(\mathcal{C}(G)\) has a natural additive structure via \((\lambda_1, \lambda_2, \lambda_3) + (\lambda'_1, \lambda'_2, \lambda'_3) = (\lambda_1 + \lambda'_1, \lambda_2 + \lambda'_2, \lambda_3 + \lambda'_3)\), making it a monoid with identity \((0, 0, 0)\). This follows from the Borel-Weil theorem, and the same argument shows that \(\mathcal{R}(G)\) is a monoid as well. Note that, by definition, \(\mathcal{R}(G) \subseteq \mathcal{C}(G)\). The saturation conjecture asks about the converse:
Conjecture 1.2. For $G$ simple, simply-connected, and simply-laced,
\begin{equation}
\mathcal{R}(G) = \mathcal{C}(G).
\end{equation}

For $G$ of type $A$, Conjecture 1.2 is true, as demonstrated by Knutson and Tao in [KT99]. Furthermore, Kapovich, Kumar, and Millson proved this conjecture for $G = \text{Spin}(8)$ (type $D_4$) [KKM09]. It is known that if $G$ is not of simply-laced type, 1.2 fails: see [El92], [KM06], and the discussion in [Kum14]. The question is still open for types $D$ and $E$ in general. The main theorem of this paper is

Theorem 1.3. The saturation conjecture for $G = \text{Spin}(10)$ (type $D_5$) holds.

The proof of this theorem will be given in Section 5, following the approach of [KKM09]. The proof reduces to finding a finite set of generators for $\mathcal{C}(G)$ and verifying that these generators each belong to $\mathcal{R}(G)$. One can ask a generalized saturation question for embedded subgroups, and Pasquier and Ressayre have employed a similar method for answering specific instances of this question in [PR13].

Additionally, we list a summary of computational results - number of (irredundant) inequalities, number of Hilbert basis elements, number of extremal rays - pertaining to the saturated tensor cones of types $A$, $C$, and $D$ and of small rank. For several of these examples, such computations have already been presented in the literature, and we verify that our results agree. In principal, similar computational results could be obtained for type $B$ (dual to type $C$) and the exceptional types $E$, $F$, $G$.

Finally, we include a discussion of certain Hilbert basis elements for $\mathcal{C}(\text{Spin}(10))$ which fail to have the “Fulton scaling property.” It was conjectured and proven that all elements of type $A$ cones have this property, but a strictly weaker statement holds for general type.

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2. Notation and Preliminaries

We fix the following notation:

- $G$ is a simply-connected complex semisimple algebraic group, with Lie algebra $\mathfrak{g}$;
- $B$ is a fixed Borel subgroup of $G$, with Lie algebra $\mathfrak{b}$;
- $H \subset B$ is a fixed maximal torus of $G$, with Lie algebra $\mathfrak{h}$;
- $\mathfrak{h}^*$ is the vector space dual to $\mathfrak{h}$;
- $\Phi \subset \mathfrak{h}^*$ is the root system of $\mathfrak{h}$ in $\mathfrak{g}$;
- $\Phi^+$ is the set of positive roots w.r.t. $\mathfrak{b}$, and $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ is the set of simple roots;
- $\{\omega_1, \ldots, \omega_r\} \subset \mathfrak{h}^*$ is the set of dominant fundamental weights;
- $\{x_1, \ldots, x_r\}$ is the dual basis for $\mathfrak{h}$ relative to $\Delta$;
- $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ is the dual basis for $\mathfrak{h}$ relative to the fundamental weights;
- $W = N_G(H)/H$ is the Weyl group of $G$, with longest element $w_0$;
- $\sigma_\gamma \in W$ is the reflection across the hyperplane $\gamma = 0 \subset \mathfrak{h}$ for $\gamma \in \mathfrak{h}^*$;
- $\ell(w)$ denotes the length of an element $w \in W$;
- if $P \supset B$ is a standard parabolic subgroup with Lie algebra $\mathfrak{p}$, then $\Delta(P)$ denotes the subset of simple roots whose negatives appear in $\mathfrak{p} \subset \mathfrak{g}$;
- $L$ denotes the Levi subgroup of $P$, and $L^{ss}$ the semisimple part of $L$;
- $W_P \subset W$ denotes the Weyl group of $P$ (that is, $N_{L(P)}(H)/H$), with longest element $w_0^P$;
- $W_P$ is the set of minimal-length left coset representatives of $W_P$ in $W$;
- if $w \in W$ (resp., $w \in W_P$), then define $X_w = \overline{BwB}$ (resp., $X_w^P = \overline{BwB}$), a subvariety of $G/B$ (resp., $G/B$) of dimension $\ell(w)$;
- $\mu(X_w)$ (resp., $\mu(X_w^P)$) is the associated fundamental class in $H_2(\mathfrak{g}B/G)$ (resp., $H_2(\mathfrak{g}B)$);
- $[X_w]$ (resp., $[X_w^P]$) is the Poincaré dual to $\mu(X_w)$ (resp., $\mu(X_w^P)$) and is an element in $H^2(\dim G/B - \ell(w))\mathbb{Z}(G/B)$ (resp., $H^2(\dim G/B - \ell(w))\mathbb{Z}(G/B)$);
- the cup product in $H^*(G/B)$ or $H^*(G/P)$ will be denoted by $\cup$, and the deformed cup product by $\odot$;
- $\rho$ is the half-sum of positive roots, and $\rho^L$ is the half-sum of positive roots for the root system of $L$;
- for $w \in W_P$, $\chi_w := \rho - 2\rho^L + w^{-1}\rho$.
2.1. Inequalities for the Tensor Cone. Let $\lambda_1, \lambda_2, \lambda_3$ be dominant weights whose sum is in the root lattice. Then by [BK06],

$$(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G \neq 0$$

for some integer $N > 0$ (i.e., $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{C}(G)$) if and only if for every maximal standard parabolic $P = P_i \subset G$ and every triple $(w_1, w_2, w_3) \in (W^P)^3$ satisfying

$$(3) \quad [X_{w_1}^P] \cap_0 [X_{w_2}^P] \cap_0 [X_{w_3}^P] = [X_{\hat{\epsilon}}^F],$$

the inequality

$$(4) \quad \left(\sum_{j=1}^{3} w_j^{-1}\lambda_j\right)(x_i) \leq 0$$

holds. Note that this elucidates the monoidal structure of $\mathcal{C}(G)$, since the inequalities (4) are linear.

By the definition of the deformed product $\cap_0$, a triple $(w_1, w_2, w_3) \in (W^P)^3$ satisfies (3) if and only if it satisfies

$$(5) \quad [X_{w_1}^P] \cdot [X_{w_2}^P] \cdot [X_{w_3}^P] = [X_{\hat{\epsilon}}^F]$$

and $(\chi_{w_1} + \chi_{w_2} + \chi_{w_3} - \chi_1)(x_i) = 0$;

where $\chi_w$ is as defined above.

Fixing a basis for $h^*$ will allow the inequalities to be understood by a computer; see Section 4 for discussion of the inequalities. The computer software Normaliz [BIR⁺], among others, is capable of reporting various characteristics of the cone $\mathcal{C}(G)$ given these defining inequalities.

2.2. Facets of the Tensor Cone. It was demonstrated by Ressayre in [Res10] that the inequalities (4) are irredundant. Therefore, the subcones

$$\mathcal{F}(\vec{w}, P) = \left\{ \vec{x} \in \mathcal{C}(G) \mid \left(\sum_{j=1}^{3} w_j^{-1}\lambda_j\right)(x_i) = 0 \right\}$$

given by $(w_1, w_2, w_3), P$ satisfying (3) form regular facets of $\mathcal{C}(G)$; i.e., they are codimension 1 faces (hence facets) and not contained in any dominant chamber wall $\{\lambda_i(\alpha_j^\vee) = 0\}$ (hence regular). The only other facets of $\mathcal{C}(G)$ are those coming from the dominant criterion: each $\lambda_i$ must be a dominant weight; i.e., $\lambda_i(\alpha_j^\vee) \in \mathbb{Z}_{\geq 0}$ for each $j$.

3. Reduction to smaller groups

The inequalities of Section 2.1 determine an integral polyhedral cone. Therefore once those inequalities have been obtained, standard techniques and the aid of a computer yield the Hilbert basis of the cone $\mathcal{C}(G)$. The Hilbert basis is the unique minimal set of monoid generators (over $\mathbb{Z}$) of the cone $\mathcal{C}(G)$.

Once the Hilbert basis is obtained, the question remains whether each basis element is in fact a member of $\mathcal{R}(G)$. Greatly reducing that burden is the following result of Roth (see [Rot11]).

**Theorem 3.1.** Suppose $(w_1, w_2, w_3), P$ satisfy (3). Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{F}(\vec{w}, P)$. Define $L^{ss}$ to be the semisimple part of $P$, and set $\sum_j w_j^{-1}\lambda_j, j = 1, 2, 3$. Then there exists an isomorphism

$$(6) \quad (V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3))^G \cong (V(\sum_j w_j^{-1}\lambda_j))^L^{ss}.$$

Roth’s original theorem is more general; there are lower-dimensional regular faces $\mathcal{F}(\vec{w}, P)$ of $\mathcal{C}(G)$ coming from non-maximal parabolics $P$, and the same theorem holds there, too. The following application of Roth’s theorem was brought to the author’s attention by S. Kumar; details were discussed by the author and P. Belkale:

**Corollary 3.2.** Suppose $\vec{x} \in \mathcal{C}(G)$ lies on a regular face $\mathcal{F}(\vec{w}, P)$, and suppose the saturation conjecture holds for $L^{ss}$. Then $\vec{x} \in \mathcal{R}(G)$.

**Proof.** Because $\lambda \in \mathcal{C}(G)$, there exists $N > 0$ so that

$$(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G \neq 0.$$ 

Since $(N\lambda_1, N\lambda_2, N\lambda_3)$ is also in $\mathcal{F}(\vec{w}, P)$, and since $\sum_j w_j^{-1}\lambda_j = \sum_j w_j^{-1}\lambda_j$, Theorem 3.1 gives

$$(V(N\sum_j w_j^{-1}\lambda_j))^L^{ss} \neq 0.$$ 

Recall that, for any $w \in W$ and dominant weight $\lambda$, $\lambda - w\lambda$ is in the root lattice (see [Hum72]). Therefore

$$\sum_j \lambda_j - \sum_j w_j^{-1}\lambda_j = \sum_j (\lambda_j - w_j^{-1}\lambda_j)$$
is in the root lattice. Since $\lambda_1 + \lambda_2 + \lambda_3$ is in the root lattice, $\overline{\lambda_1 + \lambda_2 + \lambda_3}$ is in the root lattice for $G$. Furthermore, the equation defining $F(w, P)$ implies that $\overline{\lambda_1 + \lambda_2 + \lambda_3}$ is indeed in the root lattice for $L^{ss}$. So $(\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}) \in C(L^{ss})$ and therefore also lies in $\mathcal{R}(L^{ss})$. Thus

$$(V(\overline{\lambda_1}) \otimes V(\overline{\lambda_2}) \otimes V(\overline{\lambda_3}))^{L^{ss}} \neq 0,$$

and the result follows from another application of Theorem 3.1.

4. Calculation of Inequalities

We discuss here some pseudocode used to generate the inequalities (4). A ring isomorphism $H^*(G/B) \cong R/J$ is described thanks to [BGG73], and we explain a method for our specific computations in $R/J$ using only arithmetic. We then indicate how a computer might use these calculations to explicitly parametrize the desired inequalities.

4.1. Polynomial realization of $H^*(G/P)$. The ring $H^*(G/P)$ may be described by polynomials, cf. [BGG73]. There is a ring homomorphism $\pi^* : H^*(G/P) \to H^*(G/B)$ induced by the standard projection $\pi : G/B \to G/P$. Because of the Bruhat decomposition, $\pi^*$ is an injection, and it satisfies

$$\pi^* \left( [X_{w_0 w}^{P}] \right) = [X_{w_0 w}]$$

for any $w \in W^P$. Furthermore, there is a ring isomorphism

$$R/J \cong H^*(G/B; \mathbb{Q}),$$

where $R = \text{Sym}^*(\mathfrak{h}^*) = \mathbb{Q}[\alpha_i]$ and $J$ is the ideal generated by all $W$-invariant polynomials with no constant term. Since $H^*(G/B)$ is a free $\mathbb{Z}$-module, $H^*(G/B; \mathbb{Q}) = H^*(G/B) \otimes \mathbb{Q}$ and no products in $H^*(G/B)$ are trivialized in $H^*(G/B; \mathbb{Q})$; that is, we are free to calculate coefficients of products in $H^*(G/B; \mathbb{Q}) = R/J$ and interpret them as coefficients of the corresponding products in $H^*(G/B)$.

4.2. Polynomials and integration. For any $\gamma \in \Phi$, define $A_\gamma : R \to R$ by

$$A_\gamma(f) = \frac{f - \sigma_\gamma f}{\gamma},$$

where $\sigma_\gamma \in W$ is the reflection across the hyperplane perpendicular to $\gamma$. As shown in [BGG73], $A_\gamma$ is well-defined and, if $w = \sigma_{\gamma_1} \cdots \sigma_{\gamma_t}$ is a minimal length decomposition of $w$,

$$A_w := A_{\gamma_1} \circ \cdots \circ A_{\gamma_t}$$

does not depend on the choice of minimal decomposition. The operators $A_w$ descend to well-defined operators on $R/J$, and one easily checks that $A_{\gamma}^2 = 0$. These operators generate a good basis of $R/J$:

**Definition 4.1.** Define $\tilde{P}_{w_0} = \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha \in R$, and define

$$\tilde{P}_w := A_{w^{-1} w_0} P_{w_0}$$

for all other $w \in W$. Let $P_w$ denote the image of $\tilde{P}_w$ in $R/J$.

We record various properties of the $P_w$:

**Proposition 4.2.** (a) The collection $\{P_w\}$ forms a $\mathbb{Q}$-basis for $R/J$.

(b) Under the isomorphism (7), $P_w \mapsto [X_{w_0 w}]$.

(c) Each $\tilde{P}_w$ is homogeneous of degree $\ell(w)$.

(d) Any $f \in R$ may be written as

$$f = \sum \tilde{P}_w f_w,$$

where each $f_w$ is $W$-invariant.

(e) For any $w \in W$, $P_w P_{w w_0} = P_{w_0}$.

Now define a linear functional $\Psi : R \to \mathbb{Q}$ as follows:

$$\Psi(f) = \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \sigma(f)|_0,$$

where $|_0$ means evaluation of a polynomial in $\text{Sym}^*(\mathfrak{h}^*)$ at $0 \in \mathfrak{h}$. It is known that the linear operators on $R$

$$A_{w_0} \text{ and } \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \sigma(\cdot)$$

coincide (see, for example, [Las03]). The following properties of $\Psi$ follow readily.
Proposition 4.3. The map $\Psi$ is well-defined, and $\Psi(f)$ is the $P_{w_0}$-coefficient of $f \in R/J$. If $f \in R$ has degree $\leq \deg \bar{P}_{w_0}$, evaluation at 0 may be replaced by evaluation at any element of $\mathfrak{h}_Q$.

Proof. $\Psi$ is well-defined since $A_{w_0}$ is. For any $f \in R$, write $f = \sum \bar{P}_w f_w$ as in Proposition 4.2(d). Since $A_{w_0} \bar{P}_w = 0$ for any $w \neq w_0$, $A_{w_0} f_w^{[0]} = f_w(0)$, which is the $P_{w_0}$-coefficient of $\bar{f}$ in $R/J$.

If $f$ has degree $\leq \deg(\bar{P}_{w_0})$, then $A_{w_0} f$ is a constant by degree considerations and evaluation at 0 may be replaced with evaluation at any element of $\mathfrak{h}_Q$.

Remark 4.4. See the appendix for another, more direct, derivation of the above proposition.

Since $\Psi$ vanishes on $J$, we write $\Psi$ again for the induced operator $R/J \to \mathbb{Q}$. The following corollaries explain that $\Psi$ may be viewed as integration of forms on $G/B$ and how this is useful for products in $H^*(G/P)$.

Corollary 4.5. Viewed as a linear functional $H^*(G/B; \mathbb{Q}) \to \mathbb{Q}$, $\Psi$ is the same as capping with the fundamental class $\mu(X_e) \in H_*(G/B; \mathbb{Q})$.

Corollary 4.6. Given $w_1, w_2, w_3 \in W$ such that $\ell(w_1) + \ell(w_2) + \ell(w_3) = \ell(w_0)$, the number $c$ in

$$[X_{w_0 w_1}] \cdot [X_{w_0 w_2}] \cdot [X_{w_0 w_3}] = c[X_c]$$

may be computed as

$$c = \mu(X_e) \cap c[X_c] = \Psi(\bar{P}_{w_1} \bar{P}_{w_2} \bar{P}_{w_3})$$

furthermore, since $\bar{P}_{w_1} \bar{P}_{w_2} \bar{P}_{w_3}$ has degree no greater than $\ell(w_0)$,

$$c = \frac{1}{|W|P_{w_0}(h)} \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \bar{P}_{w_1}(\sigma^{-1}h) \bar{P}_{w_2}(\sigma^{-1}h) \bar{P}_{w_3}(\sigma^{-1}h)$$

for any $h \in \mathfrak{h}_Q$.

Corollary 4.7. Given $w_1, w_2, w_3 \in W^P$ such that $\ell(w_1) + \ell(w_2) + \ell(w_3) = \ell(w_0^P)$, the number $c$ in

$$[X_{w_0 w_1}^P] \cdot [X_{w_0 w_2}^P] \cdot [X_{w_0 w_3}^P] = c[X_c^P]$$

is the same as $c$ in (under $\pi^*$)

$$[X_{w_0 w_1}] \cdot [X_{w_0 w_2}] \cdot [X_{w_0 w_3}] = c[X_c]$$

which is the number $c$ in

$$[X_{w_0 w_1}] \cdot [X_{w_0 w_2}] \cdot [X_{w_0 w_3}] \cdot [X_{w_0 w_0 w_3}^P] = c[X_c^P] \cdot [X_{w_0 w_0 w_3}^P] = c[X_c]$$

therefore

$$c = \frac{1}{|W|P_{w_0}(h)} \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \bar{P}_{w_1}(\sigma^{-1}h) \bar{P}_{w_2}(\sigma^{-1}h) \bar{P}_{w_3}(\sigma^{-1}h) \bar{P}_{w_0^P}(\sigma^{-1}h)$$

for any $h \in \mathfrak{h}_Q$ such that $\bar{P}_{w_0}(h) \neq 0$.

This last corollary is what we use to calculate the coefficient $c$ in cohomology products, via the method explained below.

4.3. Pseudocode for obtaining inequalities. Given a computer package with sufficient knowledge of root systems and their associated Weyl groups (as available through Sage [TSD17], for example), one can deduce the defining inequalities (4) for $\mathcal{C}(G)$ once one knows the set of all triples $(w_1, w_2, w_3) \in (W^P)^3$ satisfying (3) (or, equivalently, (5)), for all maximal standard $P$. The question of computing the cup product in (5) is reduced to computing a sum of polynomials evaluated on a fixed vector $h \in \mathfrak{h}_Q$ as in Corollary 4.7.

Some math software (such as Sage) is capable of polynomial manipulation and simplification. However, the following pseudocode illustrates that the need for polynomial handling can be replaced with rudimentary data storage and arithmetic.

```plaintext
dict = {}; # this dictionary will hold values of \( \bar{P}_w \) for each \( w \in W \).
weylgroup.sort(); # list the elements of \( W \) in order of decreasing length.
h = rho; # the half-sum of positive roots (or set h to anything not in the root hyperplanes)
val = 1;
for a in positivroots:
  val = val*a(h);
val = val/len(weylgroup);
dict[weylgroup[0]] = [val*(\-1)^length(s) for s in weylgroup]; # list of values for \( \bar{P}_{w_0} \)
```

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for \( w \) in weylgroup[1:]: # all except the longest element
    # find simple reflection \( s_\gamma \) so that \( \ell(ws_\gamma) > \ell(w) \).
    # then use \( A_\gamma \tilde{P}_w = \tilde{P}_w \) to compute the values \( \tilde{P}_w(th), t \in W \).
    for \( i \) in \([1,\ldots,\text{rank}]\):
        \( s = \text{simplereflections}[i] \);
        if length\((w*s)\) > length\((w)\):
            exit for
        listofvals = [];
        for \( t \) in weylgroup:
            # here \( j(t) \) returns the index so that weylgroup[\( j(t) \)] = \( t \).
            listofvals += [(dict[w*s][j(t)] - dict[w*s][j(s*t)])/simpleroots[i](h)];
    dict[w] = listofvals;

The dictionary \( \text{dict} \) now contains a list for each \( w \in W \); that list is the set of values \( \tilde{P}_w \) where \( t \) ranges over all elements of \( W \). The above algorithm can be quasi-parallelized: subsequent dictionary entries need only a single previous entry to be populated before going forward. Integration is now straightforward:

```python
def integrate(w1,w2,w3,i):
    # here \( i \) is such that \( P_i \).
    # Below \( w_0(i) \) is the longest element of \( W_p \).
    sum = 0;
    for \( j \) in \([0,\ldots,\text{len(weylgroup)}-1]\):
        sum += (-1)^length(weylgroup[\( j \)])*dict[w1][\( j \])*dict[w2][\( j \])*dict[w3][\( j \])*dict[w0(i)][\( j \)];
    return sum/(\text{len(weylgroup)}*dict[w0][0]);
```

The algorithm for generating the inequalities coming from (4) is also straightforward:

```python
ineqs = [];
for \( i \) in \([1,\ldots,\text{rank}]\):
    for \( w1,w2,w3 \) in weylgroup:
        # such that \( \ell(w_1) + \ell(w_2) + \ell(w_3) + \ell(w_0) = \ell(w) \)
        \( c = \text{integrate}(w1,w2,w3,i) \);
        if \( c == 1 \):
            if (chi(w1)+chi(w2)+chi(w3)-chi(1))(x(i)) == 0:
                \( v1,v2,v3 = (w0*w1*w0(i),w0*w2*w0(i),w0*w3*w0(i)) \);
                ineqs += [v1*x(i)+v2*x(i)+v3*x(i)]; # express the \( v_j x_i \) in the \( \omega_k \)s, then concatenate.
```

5. Proof of Theorem 1.3

Via computer (code written in Sage 8.0 [TSD17]), we obtained the following conditions governing the cone \( C(\text{Spin}(10)) \):

- 1967 inequalities coming from (4), using the algorithm described above
- 15 chamber inequalities
- 2 equalities (for ensuring the sum is in the root lattice)

Submitting these inequalities to the freely available software Normaliz [BIR+], and with the aid of supercomputer Longleaf, we found that the Hilbert basis for \( C(\text{Spin}(10)) \) consists of 505 elements, all of which lie on some regular facet. The computation failed to complete on a regular computer but successfully finished in 4 hours on the supercomputer.

The possible \( L^* \) subgroups arising from regular facets are of the following types: \( D_4, A_1 \times A_3, A_2 \times A_1 \times A_1, \) and \( A_4 \). It is known that the saturation conjecture holds for each of these (see [KKM09], [KT99]), so by Corollary 3.2, each Hilbert basis element \( \tilde{\lambda} \) is in \( R(G) \). This shows \( C(G) \subseteq R(G) \), and the result follows.

**Remark 5.1.** We also checked explicitly (using the freely available software LiE [vLCL]) that each Hilbert basis element \( \tilde{\lambda} \in R(G) \).

6. Related Results

6.1. The saturated tensor cones for type A of small rank. Using a computer (code written in Sage 8.0 [TSD17]) and the procedures described above, the following results were obtained for \( G = \text{SL}(n+1) \) (type \( A_n \)), for \( n = 1,2,3,4,5 \). The total number of inequalities is expressed as \( a + b \), where \( a \) is the number of inequalities coming
from (4) and $b$ is the number of chamber inequalities (always $3 \times \text{rank}$). In each case below, there is also one equality that ensures the sum in the root lattice condition.

| rank | total ineqs. | H.b. elements | extremal rays | H.b. elements not on a regular facet |
|------|--------------|---------------|---------------|--------------------------------------|
| 1    | $3 + 3$      | 3             | 3             | 0                                    |
| 2    | $12 + 6$     | 8             | 8             | 0                                    |
| 3    | $41 + 9$     | 18            | 18            | 0                                    |
| 4    | $142 + 12$   | 42            | 42            | 0                                    |
| 5    | $521 + 12$   | 112           | 112           | 0                                    |

The counts of inequalities for ranks 2 and 3 agree the results listed in [Kum14] and [KLM03]. The number of extremal rays for rank 2 agrees with [KLM09].

6.2. The saturated tensor cones for type $C$ of small rank. In the same fashion, the following results were obtained for $G = \text{Sp}(2n)$ (type $C_n$), for $n = 2, 3, 4, 5$. In each case below, there is also one equality that ensures the sum in the root lattice condition.

| rank | total ineqs. | H.b. elements | extremal rays | H.b. elements not on a regular facet |
|------|--------------|---------------|---------------|--------------------------------------|
| 2    | $18 + 6$     | 13            | 12            | 1                                    |
| 3    | $93 + 9$     | 58            | 51            | 1                                    |
| 4    | $474 + 12$   | 302           | 237           | 2                                    |
| 5    | $2421 + 15$  | 1598          | 1122          | 16                                   |

The results for ranks 2 and 3 above agree with those found in [KLM09], [Kum14], and [KLM03].

**Remark 6.1.** The saturated tensor cones for type $C$ and type $B$ are isomorphic due to the duality at the level of root systems, so the above data may be interpreted as results for type $B$ as well.

**Remark 6.2.** It is known that the saturation conjecture fails for the aforementioned cones. For each of $n = 2, 3, 4, 5$, we verified this fact by finding Hilbert basis elements which fail to lie in $\mathcal{R}(\text{Sp}(2n))$.

6.3. Preliminary results for type $D_6$. In similar fashion, we used a computer to obtain the following conditions governing the cone $C(\text{Spin}(12))$ (type $D_6$):

- 12144 inequalities from (4)
- 18 chamber inequalities
- 2 equalities (for ensuring the sum is in the root lattice)

Probably due to the large number of inequalities and high dimensionality (the cone lives in $\mathbb{R}^{18}$) of this problem, it has not yet been computationally feasible to determine the Hilbert basis and extremal rays of this cone. The following table summarizes known features of the cones for type $D$ of small rank (starting at rank 4):

| rank | total ineqs. | H.b. elements | extremal rays | H.b. elements not on a regular facet |
|------|--------------|---------------|---------------|--------------------------------------|
| 4    | $294 + 12$   | 82            | 81            | 1                                    |
| 5    | $1967 + 15$  | 505           | 492           | 0                                    |
| 6    | $12144 + 18$ | ??            | ??            | $\geq 1$                             |

The results for rank 4 agree with the 306 inequalities, 82 H.b. elements, and 81 extremal rays given in [KKM09].

The only Hilbert basis element for $C(\text{Spin}(8))$ which does not lie on a regular facet is $(\omega_2, \omega_2, \omega_2)$. Because $\omega_n$ is self-dual for type $D_n$, $n$ even, the element $(\omega_{n-2}, \omega_{n-2}, \omega_{n-2})$ will always be a Hilbert basis element. We checked directly that $(\omega_4, \omega_4, \omega_4)$ does not lie on any regular facet for type $D_6$ as well. Naïvely, we ask the following
Question 6.3. Let $G$ be simple, simply-connected of type $D_n$.

For $n$ even: is $(\omega_{n-2, \omega_{n-2}, \omega_{n-2}})$ the only Hilbert basis element of $C(G)$ not lying on a regular facet?

For $n$ odd: are there never Hilbert basis elements of $C(G)$ not lying on a regular facet?

6.4. “Non-Fultonian” Hilbert basis elements in $C(\text{Spin}(10))$.

Definition 6.4. Say a triple $(\lambda_1, \lambda_2, \lambda_3) \in C(G)$ has the Fulton scaling property if

$$\dim(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G = 1,$$

for every $N \geq 1$. Call such a triple “Fultonian” for short.

In type $A$, it is known that

$$\dim(V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3))^G = 1 \implies (\lambda_1, \lambda_2, \lambda_3) \text{ is Fultonian.}$$

This was conjectured by Fulton - hence the name - and first proved by Knutson-Tao-Woodward [KTW04]. The direct generalization of this conjecture for arbitrary $G$ does not hold; this implies that some cones $C(G)$ contain non-Fultonian elements. We list here certain elements of $C(\text{Spin}(10))$ which are non-Fultonian, including some which cause the implication (8) to fail. All claims were verified using LiE [vLCL].

6.4.1. The following 13 Hilbert basis elements $(\lambda_1, \lambda_2, \lambda_3)$ satisfy

$$\dim(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G = \left\lfloor \frac{N}{2} \right\rfloor + 1,$$

for $N = 1, 2, \ldots, 5$. They are listed only up to permutation:

$$\begin{align*}
(\omega_2, \omega_3, \omega_5) & \quad (\omega_1 + \omega_3, \omega_4, \omega_5) \\
(\omega_2, \omega_3, \omega_2) & \quad (2\omega_2, 2\omega_3, \omega_2 + \omega_4 + \omega_5).
\end{align*}$$

Therefore each of these Hilbert basis elements is non-Fultonian and, furthermore, fails implication (8). Interestingly, these 13 Hilbert basis elements are the same 13 ($= 505 - 492$) which are not extremal rays. It is not known whether formula (9) holds for all $N \geq 1$ for these elements.

6.4.2. The following 3 Hilbert basis elements $(\lambda_1, \lambda_2, \lambda_3)$ satisfy

$$\dim(V(N\lambda_1) \otimes V(N\lambda_2) \otimes V(N\lambda_3))^G = N + 1,$$

for $N = 1, 2, \ldots, 5$. They are the 3 permutations of the single element

$$(\omega_3, \omega_3, \omega_4 + \omega_5).$$

Therefore each of these is non-Fultonian. It is not known whether formula (10) holds in general for these three.

These Hilbert basis elements also give extremal rays of the cone. As discussed in [BK18], all extremal rays of $C(G)$ lie on a facet $F$. The extremal rays on a facet may be classified as either “Type I” or “Type II”; however, it is possible for rays to be Type I on one facet and Type II on another. Because every Type I ray is Fultonian, the three aforementioned Hilbert basis elements give examples of extremal rays which are not Type I on any facet.

**Appendix A. Another proof of Proposition 4.3**

Directly from the definition of $\Psi$, one may deduce the properties in the proposition as follows. For a fixed $w \neq w_0$, write $w^{-1}w_0 = \sigma_1 \sigma_{\gamma_1} \cdots \sigma_{\gamma_t}$ as a reduced word. Then $P_w = A_1Q$, where $Q = A_{\sigma_{\gamma_t}} \cdots A_{\sigma_{\gamma_1}}$. In particular, $A_1P_w = 0$.

Now let $W'$ be a set of representatives for the cosets $W/\langle \sigma_{\gamma} \rangle$. We can therefore write

$$\frac{1}{\prod_{\alpha \in \Phi^+} \alpha} \sum_{\sigma \in W'} (-1)^{\ell(\sigma)} \sigma(P_w) = \frac{1}{\prod_{\alpha \in \Phi^+} \alpha} \sum_{\sigma \in W'} \left( (-1)^{\ell(\sigma)} \sigma(P_w) - (-1)^{\ell(\sigma)} \sigma(\gamma) \sigma(P_w) \right)$$

$$= \frac{1}{\prod_{\alpha \in \Phi^+} \alpha} \sum_{\sigma \in W'} (-1)^{\ell(\sigma)} \sigma \left( \frac{P_w - \gamma P_w}{\gamma} \right) \sigma(\gamma)$$

$$= \frac{1}{\prod_{\alpha \in \Phi^+} \alpha} \sum_{\sigma \in W'} (-1)^{\ell(\sigma)} \sigma(\gamma) \sigma(A_1 P_w)$$

$$= 0.$$ 

Furthermore, one easily checks that

$$\sum_{\sigma \in W'} \sigma(\gamma) = 1.$$
Since the expression \( \sum_{\sigma \in W} \frac{c(\sigma)}{P_{\sigma \circ \phi} + c^+} \) is linear in \( f = \sum \tilde{P}_w f_w \), \( \Psi(f) \) is well-defined and equals \( f_{w_0}(0) \) (here we use that the \( f_w \) are \( W \)-invariant). If \( f \) is of degree \( \leq \deg \tilde{P}_{w_0} \), then \( f_{w_0} \) can be assumed constant. In such a case,

\[
\frac{1}{\prod_{\alpha \in \Phi^+} \alpha} \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \sigma(f) = f_{w_0},
\]

and evaluating at any point of \( \mathfrak{h}_Q \) gives \( \Psi(f) = f_{w_0} \), which is also the \( P_{w_0} \)-coefficient of \( \bar{f} \in R/J \).

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