ON THE EXISTENCE OF ORTHOGONAL POLYNOMIALS FOR OSCILLATORY WEIGHTS ON A BOUNDED INTERVAL

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Abstract. It is shown that the orthogonal polynomials, corresponding to the oscillatory weight $e^{i\omega x}$, exists if $\omega$ is a transcendental number and $\tan \omega/\omega \in \mathbb{Q}$. Also, it is proved that such orthogonal polynomials exist for almost every $\omega > 0$, and the roots are all simple if $\omega > 0$ is either small enough or large enough.

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1. Introduction

We consider the problem of existence of orthogonal polynomials and Gaussian quadrature rules (in the standard form) for the following inner product:

$$(f, g)_{\omega} = \int_{-1}^{1} f(x)g(x)e^{i\omega x} dx,$$  \hspace{1cm} (1)

with $\omega > 0$. More precisely, we seek a monic polynomial $p_n^\omega$ of a given degree $n$ such that

$$\int_{-1}^{1} p_n^\omega(x)x^j e^{i\omega x} dx = 0, \quad j = 0, 1, \ldots, n - 1.$$  \hspace{1cm} (2)

The following results on the existence of $p_n^\omega$ are due to [1]:

Proposition 1: $p_1^\omega$ exists for any $\omega$ except when $\omega$ is a multiple of $\pi$;
Proposition 2: $p_2^\omega$ exists for all $\omega$;
Conjecture 1: $p_n^\omega$ with $n$ even exists for all $\omega$;
Conjecture 2: $p_n^\omega$ with $n$ odd does not exists for some $\omega$. 


In this paper, we give a sufficient condition on $\omega$ for which $p_n^\omega$ exists for all $n$. According to Conjecture 1, this condition is not necessary. We show that $p_n^\omega$ exists for almost every $\omega > 0$. If the existence of $p_n^\omega$ is assumed, it is shown that all of its roots are simple when $\omega > 0$ is either small enough or large enough.

Throughout the paper, we frequently suppress the dependence of objects on $\omega$ for simplification in notations.

## 2. Orthogonal polynomials

A necessary and sufficient condition for existence of the orthogonal polynomial $p_n^\omega$ is that the Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}$$

(3)

does not vanish. The moment $\mu_k := \int_{-1}^{1} x^k e^{i\omega x} \, dx$ is defined recursively (see [1]):

$$\mu_0 = \frac{2 \sin \omega}{\omega},$$

(4a)

$$\mu_k = \frac{1}{k \omega} (e^{i\omega} - (-1)^k e^{-i\omega}) - \frac{k}{i \omega} \mu_{k-1}, \quad k \geq 1.$$  

(4b)

It is easy to show that

$$\mu_k = \frac{(-1)^k k!}{(i \omega)^k} \sum_{\nu=0}^{k} \frac{(-i \omega)^\nu s_\nu}{\nu!},$$

(5)

where

$$s_\nu := \frac{1}{i \omega} (e^{i\omega} - (-1)^\nu e^{-i\omega}) = \begin{cases} \frac{2 \sin \omega}{\omega}, & \text{for } \nu \text{ even}, \\ \frac{2 \cos \omega}{i \omega}, & \text{for } \nu \text{ odd}. \end{cases}$$
Then we can expand (5) into

\[ \mu_k = \frac{(1 \pm 1)k!}{((-1)^{k+1}k)!} \left( \cos \omega \sum_{\nu = 1}^{k} \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\sin \omega}{\omega} \left( 1 + \sum_{\nu = 2}^{k} \frac{(-i\omega)^{\nu}}{\nu!} \right) \right). \]

(6)

Now consider the matrix corresponding to the Hankel determinant \( \Delta_n \). If we take from the \( r \)th row the factor \( \frac{(-1)^{r-1}}{i\omega} \), and from the \( s \)th column the factor \( \frac{(-1)^{s-1}}{i\omega} \), then we arrive at a new Hankel determinant \( \tilde{\Delta}_n \) with the moments

\[ \tilde{\mu}_k := -2k! \left( \cos \omega \sum_{\nu = 1}^{k} \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\sin \omega}{\omega} \left( 1 + \sum_{\nu = 2}^{k} \frac{(-i\omega)^{\nu}}{\nu!} \right) \right). \]

(7)

The relation between \( \Delta_n \) and \( \tilde{\Delta}_n \) is then

\[ \Delta_n = \left( \frac{1}{i\omega} \right)^{n(n-1)} \tilde{\Delta}_n. \]

Thus, \( \tilde{\Delta}_n \neq 0 \) if and only if \( \Delta_n \neq 0 \). If \( \omega \) is such that each \( \tilde{\mu}_k \) is a polynomial in \( i\omega \) with rational coefficients, then \( \tilde{\Delta}_n \) is a polynomial in \( i\omega \) with rational coefficients. As the proof of Theorem 2.3 in [2], we employ the fact that transcendental numbers can not be zeros of a polynomial with rational coefficients. Then we seek a set \( S \) of transcendental \( \omega \), for which \( \tilde{\mu}_k \) is a polynomial in \( i\omega \) with rational coefficients. Clearly, any multiplier of \( \pi \) falls in \( S \).

If \( \omega \in S \), then \( \cos \omega \neq 0 \). Then the moments can be rewritten as

\[ \mu_k = \frac{2(-1)^{k+1}k! \cos \omega}{(i\omega)^k} \left( \sum_{\nu = 1}^{k} \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\tan \omega}{\omega} \left( 1 + \sum_{\nu = 2}^{k} \frac{(-i\omega)^{\nu}}{\nu!} \right) \right). \]

(8)

Again using the above idea, it is enough to determine \( \omega > 0 \) not belonging to \( Q \) (the field of rational numbers) for which

\[ \tilde{\mu}_k := -2k! \left( \sum_{\nu = 1}^{k} \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\tan \omega}{\omega} \left( 1 + \sum_{\nu = 2}^{k} \frac{(-i\omega)^{\nu}}{\nu!} \right) \right). \]

(9)
is a polynomial in $i\omega$. Thus, the problem is to find transcendental numbers $\omega > 0$ not belonging to $\{m\pi : m = 1, 2, \ldots\}$, such that $\tan \omega/\omega \in \mathbb{Q}$.

Transcendental numbers can be zeros of a polynomial with rational coefficients if and only if the polynomial is identically zero. Thus it is enough to show that $\Delta_n, \Delta_n$, as functions of $i\omega$, are not identically zero for $n > 1$. This can be shown by a discussion similar to that carried out in the proof of Theorem 2.3 in \cite{2}. Thus we have the following result.

**Proposition 2.1.** For any transcendental $\omega > 0$ with $\tan \omega/\omega \in \mathbb{Q}$, the orthogonal polynomial $p_n^\omega$ exists.

**Remark 2.1.** The converse is not necessarily true. There are examples of $\omega$ with $\tan \omega/\omega \notin \mathbb{Q}$ while $\Delta_n \neq 0$, i.e., the orthogonal polynomial $p_n^\omega$ exists. For example, $p_2^\omega$ exists for any $\omega > 0$ \cite{1}.

The set $S$ determined in Proposition 2.1 is at most countable due to countability of $\mathbb{Q}$. However, our numerical experiences show that $p_n^\omega$ exists for almost every $\omega > 0$. In the following, we establish this result.

**Theorem 2.2.** $p_n^\omega$ exists for almost every $\omega > 0$.

**Proof.** By induction on the index $k$, we can show from (4) that the moments $\mu_k$, as functions of $\omega$, are analytic in $D$, an arbitrary connected neighborhood of the semi-axis $\omega > 0$. The same result holds then for the Hankel determinant $\Delta_n = \Delta_n(\omega)$. Since zeros of any analytic function (if it is not identically zero) are isolated, it is enough to show that $\Delta_n(\omega)$ is not identically zero in $D$. Since $\Delta_n$ is analytic and then continuous, it is enough to show that $\Delta_n(0) \neq 0$; and this can be done similar to the proof of Theorem 2.3 in \cite{2}. \hfill $\square$
3. Gaussian quadrature rules

Since the weight function in (1) is not positive, we can not readily claim that the roots of $p_n^\omega$ (if exists) are all simple. If $p_n^\omega$ have some multiple zeros, then the $n$-point Gaussian quadrature rule can be written in the following form:

$$G_n(g) = \sum_{\nu=1}^{n} \sum_{k=0}^{m_{\nu}-1} w_{\nu,k} f^{(k)}(x_{\nu}),$$

where $m_{\nu}$ is the multiplicity of the node $x_{\nu}$, and the weights $w_{\nu,k}$ are such that the rule is exact if $f$ is replaced by a polynomial of degree at most $2n-1$. Here in the notations, we suppressed the dependence of the nodes and the weights on $n$. This rule, however, is rarely of practical interest since determining the multiplicities of the nodes is not an easy task. Our numerical experiences show that the roots of $p_n^\omega$ (if exists) are all simple.

This result can be established if we assume the existence of $p_n^\omega$ for all $\omega > 0$. According to Conjecture 2, this result most probably holds for $n$ even. From our numerical experiences, the same result can be drawn too. We have computed the absolute values of the Hankel determinant for $n = 2, 4, 6$; for each $n$, the graph has been drawn for some increasing $\omega$ (see Figure 1). As it is seen, the graphs never cut the horizontal axis, i.e., the Hankel determinants never vanish.

**Lemma 3.1.** For a given integer $n > 0$, assume that the orthogonal polynomial $p_n^\omega(x)$ exists for all $\omega > 0$. Then all coefficients of $p_n^\omega(x)$ as functions of $\omega$ are continuous.

**Proof.** If $p_n^\omega(x) = x^n + \sum_{k=0}^{n-1} a_k(\omega)x^k$, then the coefficients $a_0(\omega), \ldots, a_{n-1}(\omega)$ satisfy the linear system

$$[v_0(\omega), \ldots, v_{n-1}(\omega)] u_n(\omega) + v_n(\omega) = 0, \quad (10)$$
where

\[ v_k(\omega) = [\mu_k, \mu_{k+1}, \ldots, \mu_{k+n-1}]^T, \quad u_n(\omega) = [a_0(\omega), \ldots, a_{n-1}(\omega)]^T. \]

Then

\[ u_n(\omega) = -\frac{1}{\Delta_n} [V_n(\omega)]^T v_n(\omega), \quad (11) \]

where \( V_n(\omega) \) is the cofactor matrix of \([v_0(\omega), \ldots, v_{n-1}(\omega)]\). All entries of the matrix \( V_n(\omega) \) are continuous with respect to \( \omega \) due to the continuity of the moments \( \mu_k \), the entries of \([v_0(\omega), \ldots, v_{n-1}(\omega)]\). Since the denominator \( \Delta_n \) does not vanishes for any \( \omega > 0 \), the result follows from (11).

\[ \square \]

Theorem 3.2. For a given integer \( n > 0 \), assume that the orthogonal polynomial \( p_n^\omega(x) \) exists for all \( \omega > 0 \). If \( \omega > 0 \) is small enough or large enough, then all of the roots of the orthogonal polynomial \( p_n^\omega(x) \) are simple.

Proof. It is well-known that the roots of a polynomial vary continuously as the coefficients of the polynomial change continuously. Thus, Lemma 3.1 implies that the trajectories of the roots of \( p_n^\omega(x) \), as \( \omega > 0 \) increases, are all continuous. Since the roots corresponding to \( \omega = 0 \) as well as \( \omega \to \infty \) are all distinct \([1]\), then the result follows. \[ \square \]
We have shown that the orthogonal polynomial $p^\omega_n$, corresponding to the oscillatory weight $e^{i\omega x}$, exists if $\omega$ is a transcendental number and $\tan \omega/\omega \in \mathbb{Q}$. The set of such $\omega$ is nonempty since it contains the multipliers of $\pi$. Determining other members is not an easy task, so the main problem is still unsolved: For which values of $\omega$ does $p^\omega_n$ exist?

We have also shown that $p^\omega_n$ exist for almost every $\omega$.

In order to arrive at an $n$-point Gaussian quadrature rule of standard form, it is necessary that all the roots of $p^\omega_n$ (if exists) to be simple. The simplicity of the roots of $p^\omega_n$ is established only when $\omega > 0$ is small enough or when it is large enough. The problem is unsolved for an arbitrary $\omega > 0$. We believe that the more properties of $p^\omega_n$ one knows, the higher chance he has to solve the problem. For instance, the symmetricity of $p^\omega_n$ (cf. [1]) implies that the coefficients of $p^\omega_n(z)$ (starting from 1, the coefficient of $z^n$) are real and pure imaginary, alternatively. Also form the three-term recurrence relation,

$$p^\omega_k(z) = (z - \alpha_k-1)p^\omega_{k-1}(z) - \beta_k-1p^\omega_{k-2}(z),$$

(12)

and Theorem 3.3 of [1], it is easy to show that $\alpha_k$ and $\alpha'_k$ are pure imaginary numbers; $\beta_k$ and $\beta'_k$ are real. Here the prime sign indicates the derivative with respect to $\omega$.

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