Scalar Curvature of Manifolds with Boundaries: Natural Questions and Artificial Constructions

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I describe here in writing what came out of a conversation between Chao Li, Pengzi Miao, André Neves, Christina Sormani, and myself which took place during the workshop Emerging Topics: Scalar Curvature and Convergence in October 15-19, 2018 at IAS in Princeton.

That conversation was about a proper formulation of and a possible approach to the solution of the following problems.

**Problem A.** Let $Y = (Y,h)$ be a closed $n$-dimensional Riemannian manifold, which, besides a Riemannian metric $h$, carries a continuous function $M = M(y)$ on it.

Find condition(s) on $(Y,h,M)$ that would allow/wouldn’t allow a filling of $Y$ by a compact Riemannian $(n + 1)$-dimensional manifold $X = (X,g)$, where "filling" means that

$$\partial X = Y,$$

where

- the restriction of the Riemannian metric $g$ to $Y$ is equal to $h$;
- the mean curvature of $Y$ in $X$ is equal to $M$,
- (by our sign convention convex boundaries have $M \geq 0$) and where the essential property required of $X$ is
- non-negativity of the scalar curvature,

$$Sc(X) = Sc(g) \geq 0,$$

or, more generally, a lower bound $Sc(X) \geq \sigma$ for a given $\sigma \in (-\infty, \infty)$.

**Problem B.** Granted that a filling $X$ with $Sc(X) \geq 0$ (or with $Sc(X) \geq \sigma$) exists, what are the constraints on the geometry of such an $X$ imposed by $(Y,h,M)$?

For example:

**A$_1$.** Does sufficient, depending on $(Y,g)$, mean convexity, i.e. "large positivity" of the mean curvature of $Y$, rule out such fillings?

**B$_1$.** Is there a lower bound on the volume of filling manifolds $X$, in terms of $(Y,h,M)$ and the lower bound on the scalar curvature of $X$ (with a particular attention to the the case $Sc(X) \geq 0$)?

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1 This question was raised by Pengzi Miao, as I recall.
C. Is there a relation(s) between \( A_1 \) and lower bounds on the dihedral angles of Riemannin manifolds \( X \) with corners, where

\[
Sc(X) \geq 0 \text{ and } Mncurv(\partial X) \geq 0^2
\]

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1 Bound on the Size of \( \partial X \) by the Scalar Curvature and the Mean Curvature

As it turns out, the solution to \( A_1 \) for spin manifolds \( X \) with \( Sc \geq 0 \) can be obtained by confronting several known results as follows.

Let \( X \) be a compact orientable \((n + 1)\)-dimensional Riemannian manifold with boundary \( Y = \partial X \) and let us introduce the following three invariants of the manifold \( Y \).

1. The lower bound on the scalar curvature of \( Y \) with the Riemannin metric induced from \( X \), denoted

\[
\underline{S} = \underline{S}(Y) = \inf_{y \in Y} Sc(Y) = \inf_{y \in Y} Sc(Y, y).
\]

2. The lower bound on the mean curvature of \( Y \subset X \),

\[
\underline{M} = \underline{M}(Y) = \inf_{y \in Y} Mncurv(Y) = \inf_{y \in Y} Mncurv(Y).
\]

3. The spherical radius of \( Y \), denoted \( Rad_{S^n}(Y) \), that is the maximal radius \( R \) of a sphere, such that \( Y \) admits a distance decreasing map \( Y \to S^n(R) \) of non-zero degree.

\[2 \text{This question by Christina Sormani started our conversation.}\]
Recall that if $Y$ is spin, then, by Llarull’s theorem,

$$\mathbf{S}(Y) \cdot \text{Rad}_{S^n}(Y) \leq n(n-1),$$

where the equality implies that $Y$ is isometric to the sphere $S^n(R)$ of radius $R = \text{Rad}_{S^n}(Y)$.

But this by itself has nothing to do with $X \supset Y$ – what we want to know is what happens when $\underline{M} = \inf M\text{curve}(Y=\partial X)$ and $\text{Sc}(X)$ enter the picture.

**Rough Bound on Mean Curvature.** There exists a continuous function $M_n(\sigma, S, R)$, which is monotone decreasing in $\sigma$, $S$ and $R$, where $-\infty < \sigma, S < \infty$, and $R > 0$, such that the mean curvatures of the boundaries of all compact Riemannian $(n+1)$-dimensional spin manifolds $X$ with boundaries, where $\text{Sc}(X) \geq \sigma$ and $\text{Sc}(\partial X) \geq \underline{S}$, satisfy

$$\bigcirc_{\underline{M}} \underline{M} = \inf_{y \in \partial X} M\text{curve}(\partial X) \leq M_n(\sigma, \underline{S}, \text{Rad}_{S^n}(\partial X)).$$

Let us now reformulate $[\bigcirc_{\underline{M}}]$ for $\text{Sc}(X) \geq 0$ and $\text{Sc}(\partial X) \geq 0$ with a more precise description of the function $M_n$.

(Since the function $M_n$ is *continuous* at $(0,0,R)$, this will give us some information about $M_n(\sigma, \underline{S}, R)$ for all values of $\sigma$ ($\leq \text{Sc}(X)$) and $\underline{S}$ ($\leq \text{Sc}(Y)$), since these can be made arbitrarily small by scaling (the metric of) $X$ by a large constant.)

Let $X$ be a compact Riemannian $(n+1)$-dimensional spin manifolds $X$ with boundary $Y = \partial X$, such that $\text{Sc}(X) \geq 0$ and $\text{Sc}(Y) \geq 0$. Then the mean curvature of the boundary $Y$ of $X$ is bounded in terms of the spherical radius of $Y$ as follows.

$$\bigcirc_{++} \underline{M}(Y) = \inf_{y \in Y} M\text{curve}(Y, y) \leq \frac{10n^2}{\text{Rad}_{S^n}(Y)}.$$

*About the Proofs.* We state and prove in section 5 a more precise version of $[\bigcirc_{++}]$ by applying Llarull’s theorem to an auxiliary closed manifold, denoted $X_\delta$, construction of which is described in section 3. Also in section 3, we explain how $[\bigcirc_{\underline{M}}]$ reduces to $[\bigcirc_{++}]$.

**Generalisation.** The (large) size of the mean curvature only matters on the part of $Y$ that contributes to the spherical radius of $Y$.

Namely, if $Y$ admits a distance decreasing map $Y \rightarrow S^n(R)$ of non-zero degree (as in the definition of $\text{Rad}_{S^n}(Y)$), such that the complement of a domain $U \subset Y$ is sent to a point, then, assuming that $M\text{curve}(Y) > 0$ on all of $Y$ and that $\text{Sc}(Y, y) \geq 0$ for $y \in U$, one can replace

$$\underline{M}(Y) = \inf_{y \in Y} M\text{curve}(Y, y)$$

in the above inequalities by

$$\underline{M}(U) = \inf_{y \in U} M\text{curve}(Y, y).$$
But with the available argument, one has to pay the price of requiring a bound on the geometry of $U$ (similar to that in section 8) in a neighbourhood of $\partial U \subset Y$: a bound on the injectivity radius, by 1 from below, and on the absolute values of the sectional curvatures of $Y$, also by one, now from above, with both bounds required in the $1$-neighbourhood of $\partial U$.

The arguments in the proofs of $[\bigcirc M]$ and $[\bigcirc \ast \ast]$ in sections 3-5 allow other refinements and generalisations, but the following remains problematic.

(a) Is "spin" necessary? (See $[\Box - 1]$ in section 8 for what can be expected without spin.)

(b) Is the lower bound on $Sc(Y = \partial X)$ essential? Is, for instance,

$$M(Y) \leq \frac{\text{const}_n}{\text{Rad}_{S^n}(Y)}$$

whenever $Sc(X) \geq 0$?

Possibly (b) and (c) are related, since if there were stable minimal hypersurfaces with free boundaries in $Y$, we could proceed by Schoen-Yau-style inductive dimension descent argument. (See remark (B) in section 5.)

(c) Can one substitute $\inf_Y M_{\text{ncurv}}(Y,y)$ by some integral (or spectral) characteristic of the function $M_{\text{ncurv}}(Y,y)$? (Compare with $\star \star$ below.)

(d) What are simple (non-simple?) examples of extremal $(n+1)$-manifolds $X$ with boundaries, such that $Sc(X) = \sigma > 0$ and $M_{\text{ncurv}}(\partial X) = M > 0$, where "extremal" means the following.

$X$ is extremal, if whenever a Riemannian $(n+1)$-dimensional (spin?) manifold $X'$ with boundary satisfies:

(i) $Sc(X') \geq \sigma$;

(ii) $M_{\text{ncurv}}(\partial X') \geq M$;

(iii) $\text{Rad}_{S^n}(\partial X') \geq \text{Rad}_{S^n}(\partial X)$,

then

$$Sc(X') = \sigma, \ M_{\text{ncurv}}(\partial X') = M, \ \text{Rad}_{S^n}(\partial X') = \text{Rad}_{S^n}(\partial X).$$

And $X$ is regarded rigid, if (i)-(iii) imply that $X'$ is isometric to $X$.

It may be admissible to strengthen (iii) by the requirement that $\partial X'$ is isometric to $\partial X$. But even then the extremality/non-extremality problem remains difficult, even for convex balls in $S^{n+1}$.

In fact, the subtlety of the extremality/rigidity problem for balls in $S^n$ was revealed by the results by Brendle, Marques and Neves [BMN 2010] and by Brendle and Marques [BM 2011]:

the existence/nonexistence of a deformation of the spherical metric in the interior of such a ball with increase of the scalar curvature depends on the radius of the ball.

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3The argument we have in mind is an obvious modification of the proof of $[\bigcirc \ast \ast]$ given in sections 3-5

4As usual, "$X$ is spin" can be relaxed to "the universal covering $\tilde{X}$ of $X$ is spin".
2 Rigidity and Extendability of Metrics with Lower Bounds on the Scalar Curvature.

Prior to proving the inequalities $[\textcircled{A}]$ and $[\textcircled{B}]$, let us remind several earlier results by Miao, Shi-Tam, and Mantoulidis-Miao on manifolds with boundaries, which motivated the above A and B.

Let $Y_0$ be a smooth closed hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, let $X$ be a compact manifold with the boundary $Y$ isometric to $Y_0$ and let $M_0 = M_0(y)$, $y \in Y$, be the mean curvature of $Y_0$ transported to $Y$ by our isometry $Y \leftrightarrow Y_0$.

⋆ Mean Curvature Rigidity Theorem. If $Sc(X) \geq 0$, then the mean curvature $M$ of $Y \subset X$ can't be greater than that of $Y_0 \subset \mathbb{R}^{n+1}$, $M(y) \neq M_0(y)$, unless $M(Y) = M(Y_0)$ and $X$ is isometric to the domain $X_0 \subset \mathbb{R}^{n+1}$ bounded by $Y_0 \subset \mathbb{R}^{n+1}$.

This inequality, which in the case $Y_0 = S^n(R)$ represents the sharp version of $[\textcircled{B}]$, is proved in [M 2003] and in [ST 2003] in three steps, roughly, as follows.

⋆ Attach $X$ to the complement $Z_0 = \mathbb{R}^{n+1} \setminus X_0$ by the "↔"-isometry $Y = \partial X \leftrightarrow \partial Z = Y_0$.

⋆⋆ Smooth the $Y$-corner in the resulting manifold $W = X \cup Y Z$

while keeping $Sc(W)$ everywhere $\geq 0$.

*** Use the Euclidean Rigidity theorem, formerly Geroch conjecture:

⋆ If a complete $C^3$-smooth Riemannian manifold $W$ with $Sc(W) \geq 0$ is isometric to $\mathbb{R}^{n+1}$ at infinity, then $W$ is isometric to $\mathbb{R}^{n+1}$.

This is a special case of the Schoen-Yau positive mass theorem which also (trivially) follows from non-existence of non-flat metrics with $Sc \geq 0$ on the torus and on overtorical manifolds.

The following more satisfactory version of ⋆ is proven in [M 2003] and [ST 2003], for convex hypersurfaces.

⋆ ⋆ Integral Mean Curvature Rigidity Theorem. If $Y_0$ is convex, then

$$\int_Y M(y)dy \leq \int_Y M_0(y)dy,$$

where the equality $\int M(y) = \int M_0(y)$ implies that $X$ is isometric to $X_0$.

Hyperbolic Remarks. (H1) It is shown in [MM 2016], Proposition 3.2, that,

Min-Oo’s rigidity and positive mass theorems for hyperbolic spaces

\footnote{A simple reduction of the full positive mass theorem to the rigidity theorem was found in 1999 by J. Lohkamp: Scalar curvature and hammocks, https://link.springer.com/article/10.1007%2Fs002080050266}
allow

\( \star_{-\kappa} \) extensions of theorems \( \star \) and \( \star \star \) to hypersurfaces \( Y \) in the hyperbolic spaces \( \mathbb{H}^{n+1}_{-\kappa} \) with the sectional curvature \( -\kappa \) and manifolds \( X \) with \( \text{Sc}(X) \geq -n(n+1)\kappa \).

This is can be reformulated in an especially pleasant manner for \( n = 2 \) if \( Y = \partial X \) is homeomorphic to \( S^2 \) and has \( \text{sect}\text{.curve}(Y) \geq -\kappa \) due to

Pogorelov's existence (and essential uniqueness) theorem for isometric embeddings of spherical surfaces with metrics having sectional curvatures \( > -\kappa \) to \( \mathbb{H}^{3}_{-\kappa} \).

\((\mathbf{H}_2)\) The above mentioned results from [M 2003], [ST 2003] and [MM 2016] allow small perturbations of the metrics on \( Y \) which can be deformed back to their original values with a minimal loss of positivity of the scalar and the mean curvatures.

Thus, one easily shows, for instance, that

given a smooth closed hypersurface \( Y_0 \) in \( \mathbb{H}^{n+1}_{-\kappa} \), there is an effectively computable \( \Delta = \Delta_{\kappa,n}(Y) > 0 \), such that if a metrics \( h \) on \( Y_0 \) is \( C^2 \)-close to the original metric \( h_0 \) on \( Y_0 \subset \mathbb{H}^{n+1}_{-\kappa} \) induced from \( \mathbb{H}^{n+1}_{-\kappa} \)

\[ \|h - h_0\|_{C^2} \leq \delta < \Delta, \]

then \( Y \) admits no filling by manifolds \( X \) with \( \text{Sc}(X) \geq -n(n+1)\kappa \) and with the mean curvature of \( Y = \partial X \) in \( X \) bounded from below by \( M_\delta \) for function \( M_\delta \) on \( Y_0 \) which converges to the mean curvature of \( Y_0 \subset \mathbb{H}^{n+1}_{-\kappa} \) for \( \delta \to 0 \).

Whenever applies, this inequality is sharper than \([O_\mathcal{M}]\) and \([O_{++}]\). But, unfortunately, even for the unit sphere \( S^n \subset \mathbb{H}^{n+1}_{-1} \), this \( \Delta \) is pretty small, something like \( 1/n^2 \), while available more conclusive results need strong topological conditions (see sections 6 and 7).

\((\mathbf{H}_3)\) Min-Oo’s proof of his rigidity and the positive mass theorems uses a version of Witten’s Dirac operator method, thus, it needs the spin condition. Accordingly, the original proofs in [M 2003], [ST 2003] and [MM 2016] needed \( X \) to be spin, but, as we know now, the spin condition is redundant according to [SY 2017].

Moreover, Min-Oo’s rigidity theorem remains valid for manifolds \( Z = \mathbb{H}^{n+1}_{-1}/\Gamma \) for all discrete parabolic isometry groups\(^6\) where it reads as follows.

\((*)\) Generalised Min-Oo Rigidity Theorem. Parabolic quotient manifolds \( Z = \mathbb{H}^{n+1}_{-1}/\Gamma \) admit no non-trivial ”deformations” with compact supports and with \( \text{Sc} \geq -n(n+1) \),

where these ”deformations" may arbitrarily change the topology of (a compact region in) \( Z \) with no condition on spin.

Notice, that the original Min-Oo rigidity theorem corresponds to the case where \( \Gamma = \{id\} \) and that the rigidity for all \( \Gamma \), including Min-Oo rigidity itself, trivially follows from the rigidity of ”hyperbolic cusps", i.e. where \( \Gamma \) is isomorphic to \( \mathbb{Z}^n \).

\(^6\) An isometry group \( \Gamma \) of \( \mathbb{H}^{n+1}_{-1} \) is parabolic if there is a horosphere in \( \mathbb{H}^{n+1}_{-1} \) invariant under the action of \( \Gamma \), or, equivalently, if all isometries \( \gamma \in \Gamma \) except \( \text{id} \) keep fixed a unique common fixed point in the ideal boundary of \( \mathbb{H}^{n+1}_{-1} \).

Besides, in the present context, we also require that the actions of \( \Gamma \) on \( \mathbb{H}^{n+1}_{-1} \) are free. But this, in fact, becomes unnecessary, if instead of deformations of \( \mathbb{H}^{n+1}_{-1}/\Gamma \), we speak of \( \Gamma \)-equivariant deformations of \( \mathbb{H}^{n+1}_{-1} \), such that the \( \Gamma \)-quotients of the supports of these deformations are compact.
Thus all of (*) reduces to the case of cusps, that was proven in §5.5 of [G 1996] with a use of minimal "soap bubbles" under the condition $n+1 \leq 7$, and that it is now guarantied for all $n$ by the Schoen-Yau regularity theorem [SY 2017]; possibly, Lohkamp’s theorem [L 2018] or his argument can be used here as well.

3 Collars, Doubles and Area Contracting Maps.

Given a compact $(n+1)$-manifold with boundary $Y = \partial X$, let $X_\delta$ be obtained by "gluing" "$\delta$-negative collar" to its boundary, where this "collar" is the cylinder $Y \times [-\delta,0]$, which is attached to $X$ along $Y = \partial X$ by identifying $Y = \partial X$ with $Y = Y_\delta = Y \times \{-\delta\} \subset Y \times [-\delta,0]$.

Observe that $\partial X_\delta = Y_0$ for $Y_0 = Y \times \{0\}$, while

$$ Y = Y_\delta = Y \times \{-\delta\} \subset X_\delta $$

separates $X \subset X_\delta$ from $Y \times [-\delta,0] \subset X_\delta$.

Let $h_t$, $t \in [-\delta, 0]$, be a smooth family of Riemannin metrics on $Y$ and let $g = g_t(y,t) = dt^2 + h_t(y)$ be the corresponding metric on $Y \times [-\delta,0]$.

Observe that the two boundaries, $Y_\delta$ and $Y_0$ of $Y \times [-\delta,0]$, are $\delta$-equidistant with respect to $g_+$.

Let

$$ X_{\delta\delta} = X_\delta \cup_{Y_0} X_\delta $$

be the double of $X_\delta$.

Let

$$ F : X_{\delta\delta} \to S^{n+1}(R) $$

be a smooth map from $X_{\delta\delta}$ to the sphere $S^{n+1}(R)$ of a certain radius $R$ with the following three properties.

(i) The map $F$ is "supported" on $Y \times [-\delta, \delta] \subset X_{\delta\delta}$, which means it is constant on the original manifold $X \subset X_\delta \subset X_{\delta\delta}$ as well as on the second copy of $X$ in $X_{\delta\delta}$.

In fact, the map $F$, as we shall construct it, will send $X$ to the south pole of the sphere, and the second copy of $X$ will go to the north pole.

(ii) The map $F$ radially sends the "$\delta$-negative collar" $Y \times [-\delta,0]$ to the lower hemisphere $S^n_{n+1} \subset S^{n+1}$, where "radially" means the following.

The segments $y \times [-\delta,0] \subset Y \times [-\delta,0]$, $y \in Y$, are sent by $F$ to geodesic segments in $S^{n+1}$ between the south pole and the north pole while the $t$-copies of $Y$, that are

$$ Y_t = Y \times \{t\} \subset Y_{\delta\delta}, \ t \in [-\delta,0], $$

go to the $n$-spheres concentric to the equator.

(iii) The map $F$ is symmetric with respect to the obvious involutions in $X_{\delta\delta}$ and in $S^n = S^n_{n+1} \cup S^n_{n+1}$.

Thus, the map $F$ is determined by the following two maps.

1. The first one is a map from $Y$ to the $n$-sphere, call it $f : Y \to S^n$.

To be sign consistent, we denote the south pole, that is the bottom of the sphere, by $S^n_{n+1}$ and the north pole – the top – by $S^n_{n+1}$.
2. The second map goes from $[-\delta, 0]$ to the segment $[-\frac{\delta R}{2}, 0]$, call it $\psi : [-\delta, 0] \to \left[-\frac{\pi R}{2}, 0\right]$. Let $f_t : (Y_t, h_t) \to S^n_{\psi(t)} \subset S^{n+1}(R)$, denote the restrictions of $F$ to submanifolds $Y_t \subset Y \times [-\delta, 0] \subset X_{\delta\delta}$, $t \in [-\delta, 0]$, where $S^n_{\psi(t)}$ are spheres concentric to the equator $S^n(R) \subset S^{n+1}(R)$.

Observe that the area expansion/dilation by this $F = F_{\psi}$ at a point $(y, t) \in Y \times [-\delta, 0]$, call it $\|\wedge^2 dF(y, t)\|$ is bounded by the product of the Lipschitz constants of the maps $f_t$ and $\psi(t)$,

$$\|\wedge^2 dF(y, t)\| \leq \text{Lip}(f_t; y) \cdot \text{Lip}(\psi; t),$$

where for smooth maps,

$$\text{Lip}(f_t; y) = \|df_t(y)\|, \text{ Llip}(\psi; t) = \left|\frac{d\psi(t)}{dt}\right|$$

and

$$\|\wedge^2 dF(y, t)\| = \|df_t(y)\| \cdot \left|\frac{d\psi(t)}{dt}\right|.$$

We shall see in section 5 that the assumption $\text{Mncurv}(Y) > \frac{10n}{\text{Rad}_{S^n}(Y)}$ allows a choice of $h_t$ and $\psi(t)$ that would contradict the Llarull theorem applied to $F$; thus, we shall prove the inequality $[\O_{++}]$.

**About Llarull Theorem.** Let us formulate this theorem in the form adapted to the present application.

Let $W$ be a complete orientable Riemannian $(n+1)$-manifold without boundary with $\text{Sc}(W) \geq 0$ and let $F : W \to S^{n+1}$ be a smooth map which is locally constant at infinity, i.e. outside a compact subset in $W$.

If $\text{deg}(F) \neq 0$, and

$$\text{Sc}(W, w) \geq n(n + 1) \cdot \|\wedge^2 dF(w)\|, w \in W,$$

then, provided the universal covering of $X$ is spin, the equality holds:

$$\text{Sc}(W, w) = n(n + 1) \cdot \|\wedge^2 dF(w)\|;$$

moreover, (this we don't need) the map $F$ is an isometry.\(^8\)

**Reduction of $[\O_M]$ to $[\O_{++}]$.** Let the scalar curvatures of $X$ and of $Y = \partial X$ be bounded from below by negative constants $\sigma$ and $\underline{\text{Sc}}$ and let us multiply $X$ by the sphere $S^2(R)$ for $R = -\frac{1}{\max(\sigma, \underline{\text{Sc}})}$.

Observe that the resulting product manifold $X' = X \times S^2(R)$ has $\text{Sc}(X') \geq 0$, $\text{Sc}(\partial X') \geq 0$ and $\text{Mncurv}(\partial X') = \text{Mncurv}(\partial X)$, while

$$\text{Rad}_{S^n}(\partial X') \geq \min(R, \text{Rad}_{S^n} \partial X).$$

\(^8\)This is the second exterior power of the differential $dF$ for smooth $F$.

\(^9\)The main point in the proof of Llarull's theorem is a sharp and rather painful algebraic calculation of the curvature term in the Lichnerowicz formula for the Dirac operator twisted with a particular vector bundle. If we don’t care for sharp constants in $[\O_{++}]$, then the general hypersphericity inequality from [GL 1983], which needs no specific calculation, will do.
Therefore, the inequality $[O_+]$ for $X'$ implies $[O_M]$ for $X$.

**Warning Exercise.** Find a counterexample to the following "proposition" and then find the mistake in its "proof".

Let $X$ be a closed orientable Riemannian spin $(n+1)$-manifold with $Sc(X) \geq -\epsilon$ and let $U \subset X$ be an open connected subset, such that the scalar curvature (function) of $X$ is large on $U$, say

$$Sc(X, u) \geq 1000n(n+1) \text{ for all } u \in U.$$ 

"Proposition". If $\epsilon > 0$, is sufficiently small, say $\epsilon \leq 0.001$, then $X$ admits no area decreasing map $f : X \to S^{n+1} = S^{n+1}(1)$ of non-zero degree that sends the complement of $U$ to a point $s_0 \in S^{n+1}$.

"Proof". Assume otherwise, let $X' = X \times S^2$ and compose the map $f' = (f, id) : X' \to S^{n+1} \times S^2$ with a smooth distance decreasing map $S^{n+1} \times S^2 \to S^{n+3} \left(\frac{1}{10}\right)$ of degree one, which collapses $(s_0) \times S^2$ to a point. Since the resulting map $F' : X' \to S^{n+3}$ is area decreasing and since the scalar curvature of $X'$ is everywhere positive, say for $\epsilon < 1$, and since

$$Sc(X') > 100(n+2)(n+3) = Sc\left(S^{n+3} \left(\frac{1}{10}\right)\right)$$

at all points where the differential of $F'$ doesn't vanish, namely in $U \times S^2$, the "proof" follows by the contradiction with Llarull's theorem.

**Hint.** The above argument becomes valid if "no area decreasing map $f$" is replaced by "no distance decreasing map $f$". 

## 4 Curvatures of Riemannian Bands and Tubes.

Let us recall *Hermann Weyl’s tube formula* (see below) which allows an effective expression of the scalar curvature of $dt^2 + h_t$ in terms of $Sc(h_t)$ and the first and second $t$-derivatives of $h_t$, which will be used in section 5 for the choice of $\delta$, $h_t$ and $\psi(t)$ in the construction of $F : X_{M} \to S^{n+1}(R)$ needed for the proof of $[O_{+,+}]$ from section 1.

**Codimension One Tube Formulae.** Let $h_t$, $t \in [0, d]$, be a family of Riemannian metric on an $n$-dimensional manifold $Y$ and let us incorporate $h_t$ to the metric $g = dt^2 + h_t$ on $Y \times [0, d]$.

Notice that an arbitrary Riemannian metric on an $(n + 1)$-manifold $X$ admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of a compact hypersurface $Y \subset X$.

The following two formulae, in conjunction with the Gauss *Theorema Egregium* for the hypersurfaces $Y_t \subset Y \times [0, d]$, allow a simple computation of the scalar curvature of $g$.

(I) *First Variation Formula.* The $t$-derivative of $h_t$ is equal to twice the second fundamental form of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, d]$, denoted

$$A_t^* = A^*(Y_t)$$

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10 The method of minimal hypersurfaces (which needs no spin) applied to $U$ allows an alternative proof of this in most (but not all!) cases, see [GL 1983] and [G 2918].
and regarded as a quadratic differential form on $Y = Y_t$. In writing,

$$\frac{dh_t}{dt} = 2A^*_t.$$  

(II) Weyl Formula. The $t$-derivative of the shape operator $A_t$ on $Y = Y_t$, which corresponds to $A^*_t$, satisfies:

$$\text{trace} \left( \frac{dA_t}{dt} \right) = -\text{trace} A^2_t - \text{Ricci}_g \left( \frac{d}{dt} \frac{d}{dt} \right).$$

Remark. Weyl’s Formula is obtained by tracing what can be justly called

The Second Main Formula of Riemannian Geometry.

$$\frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where $B_t$ is the quadratic differential form on $Y = Y_t$, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y = Y_t)$ are equal to the values of the sectional curvature of $g$ at (the 2-planes spanned by) the bivectors $(\tau, \frac{d}{dt})$.

(C) Umbilic Boundaries. The Weyl’s Formula becomes most transparent where the family $h_t$ remains almost constant in $t$, yet, having large derivatives $\frac{dh_t}{dt}$ and $\frac{d^2h_t}{dt^2}$.

This allows, for example, a $C^0$-slight perturbation of a given Riemannian metric $g$ on $X$ near the boundary $Y = \partial X$, such that

- the restriction $h'$ of the perturbed metric $g'$ to $Y$ remains $C^\infty$-close to the original metric $h = g|_Y$ on $Y$;
- the second fundamental form $A^*_g$ becomes proportional to the Riemannian metric on $Y$, that is $A^*_g = a \cdot h'$ for a (positive or negative) constant $a$.
- The scalar curvature of the metric $g'$ is almost as large as that of $g$,

$$\text{Sc}(g') - \text{Sc}(g) \geq -\varepsilon,$$

where this $\varepsilon > 0$ can be chosen as small as one wishes.

Then, as far as our purpose is concerned, one needs to deal only with families of the form $h_t = \varphi^2(t)h$ on $X = Y \times \mathbb{R}$ with functions $\varphi(t) > 0$, where the Weyl and Gauss formulas yield the following (standard) expression for the scalar curvature of the (warped product) metric $g = dt^2 + \varphi^2(t)h$ on $X$ in terms of the scalar curvature of the metric $h$ on $Y$ and of the first and second derivatives of the function $\varphi(t)$.

$$\text{Sc}(g) = \frac{1}{\varphi^2} \text{Sc}(h) - 2n \left( \frac{\varphi'}{\varphi} \right) - n(n+1) \left( \frac{\varphi'}{\varphi} \right)^2,$$

11 The first main formula is Gauss’ Theorema Egregium.

12 This formula can be taken for the definition of the sectional curvature and it also allows a fast proof of the basic Riemannian comparison theorems along with their standard corollaries. However, for some inexplicable reason, this formula is absent from most (all?) textbooks on Riemannian geometry where, instead, the old-fashioned and cumbersome language of Jacobi fields is used.

13 See 11.5 in [G 2018], but beware: there is a sign confusion in my GAFA paper which is corrected in the later version.
where, recall, \( n = \dim(Y) \) and the mean curvature of \( Y_t \) is

\[
Mncurv(Y_t \subset X) = \frac{\varphi'(t)}{\varphi(t)}.
\]

This gives you a full control over the scalar curvature of \( X \) and the mean curvatures of \( Y_t \) and the existence of the family \( h_t \) needed for the proof of \([\circ,+]\) easily follows. (We explain this in detail in the next section.)

**Questions.** Let \( h_0 \) and \( h_1 \) be two Riemannin metrics on a closed \( n \)-manifold \( Y \) and \( M_0, M_1 \) be smooth functions on it.

When does there exist a smooth homotopy \( h_t \) of metrics on \( Y \) that joins \( h_0 \) with \( h_1 \), which has

\[
\text{trace}_{h_t} \frac{dh_t}{dt}|_{t=0} = 2M_0, \quad \text{trace}_{h_t} \frac{dh_t}{dt}|_{t=1} = 2M_1
\]

and such that the metric \( g = dt^2 + h_t \) on \( Y \times [0,1] \) has \( Sc(g) \geq 0 \)?

When does, in addition to \( h_t \), there exist a smooth function \( \psi(y,t) > 0 \), such that the above holds with the metric \( G = \psi(y,t) dt^2 + h_t \) instead of \( g = dt^2 + h_t \):

\[
- Sc(G) \geq 0;
- \text{the restrictions of } G \text{ to } Y \times \{0\} \text{ and } Y \times \{1\} \text{ are equal to } h_0 \text{ and } h_1 \text{ correspondingly};
- \text{the } G\text{-mean curvatures of } Y \times \{0\} \text{ and } Y \times \{1\} \text{ with respect to the vector field } \frac{\partial}{\partial t}\text{ are equal to } M_0 \text{ and } M_1 \text{ respectively?}

**Categorical Motivation.** The true problem is understanding (quantisation?) the category, where the objects are manifolds \( Y \) with Riemannian metrics \( h \) and functions \( M \) on them and where morphisms are cobordisms (h-cobordisms?) \((X,g), \partial X = Y_0 \cup Y_1\), where \( g \) is a Riemannian metric on \( X \) with \( Sc \geq 0 \), which restricts to \( h_0 \) to \( h_1 \) on \( Y_0 \) and \( Y_1 \) and where the the mean curvature of \( Y_0 \) with inward coorientation is equal to \( -M_0 \) while the mean curvature of \( Y_1 \) with the outward coorientation is equal to \( M_1 \).

More generally, there is a kind of \( A_\infty \)-category \( P^\circ \) (I am not certain about terminology), where objects are Riemannian manifolds with corners and where, besides prescribing/controlling the mean curvatures of the codimension one faces, one keeps track of the dihedral angles between these faces (see [G 2014]).

Conceivably, the [SY 2918]-variational techniques for "flags" of hypersurfaces or its generalisation(s), may have a meaningful interpretation in \( P^\circ \), while a suitably adapted Dirac operator method may serve for quantisation of \( P^\circ \).

### 5 Sharpening Inequality \([\bigcirc,++]\) and Conclusion of its Proof.

Let us describe the best possible bound on \( Mncurv(\partial Y) \), \( Y = \partial X \), in terms of

\[
S = \inf Sc(\partial Y) \quad \text{and} \quad \text{Rad}_{S^2}(Y), \text{ that is achievable by the argument from section 3 under assumption } Sc(X) \geq 0.
\]

We shall express this in terms of the geometry of \( n \)-spherically radial metrics

\[
g = dt^2 + \varphi(t)^2 dS_2^2
\]
on the cylinder $S^n \times [-\delta, 0]$, where the derivative $\varphi'(t)$ vanishes at $t = 0$ and $\varphi$ is defined for $\delta < \frac{\pi}{2}$ and it is equal to the spherical metric in the band between two equidistant hyperspheres in $S^{n+1}$.

(b) Another useful family of metrics (compare §12 in [GL 1983]) is where
\[
(\log \phi)' = \frac{\varphi'}{\varphi} = \tan \frac{n+1}{2} \delta,
\]

which is defined for $\delta < \frac{\pi}{n+1}$ and where the scalar curvature of this metric satisfies:
\[
Sc\left(dt^2 + \varphi(t)^2 dS^2_n\right) = n(n+1) + \frac{n(n-1)}{\varphi^2}.
\]

Let $\psi : [-\delta, 0] : [-\frac{\pi}{2}, 0]$ be a smooth map, which sends $-\delta \mapsto -\frac{\pi}{2}$ and $0 \mapsto 0$ and let
\[
\Psi : S^n \times [-\delta, 0] \rightarrow S^{n+1}_-\]
be the corresponding radial map $\Psi = (id, \psi)$.

Denote by $\|\wedge^2 d\Psi(x)\|$ the norm of the exterior square of the differential of the map $\Psi$, that is the area dilation by $\Psi$ at $x = (t,s)$ and observe that this dilation is easily expressible in terms of $a = \phi'(t)$ and the ratio $b = -\frac{\cos \phi(t)}{\phi'(t)}$ as follows.
\[
\|\wedge^2 d\Psi(t,s)\| = \max(b\phi', b^2).
\]

**Definitions.** Call a normal spherically radial metric $g$ area-wise large if there exists a smooth map $\psi : [-\delta, 0] : [-\frac{\pi}{2}, 0]$ as above, such that
the corresponding radial map $\Psi : S^n \times [-\delta, 0] \rightarrow S^{n+1}_-$ contracts the area at least proportionally to the ratio of the scalar curvatures of $S^n \times [-\delta, 0]$ and $S^{n+1}_-$ at the corresponding points;
In writing,
\[
\|\wedge^2 d\Psi(x)\| \leq \frac{Sc\left(x\right)}{n(n+1)}, \ x \in S^n \times [-\delta, 0].
\]

Denote by $M_{max}(S, R)$ the supremum of $\frac{\varphi'(\delta)}{\varphi(\delta)}$ over all and $\delta > 0$ and all smooth positive functions $\varphi$ on $[-\delta, 0]$ with the following properties.

- $\bullet_R \varphi(\delta) \leq R$;
- $\bullet_S \frac{n(n-1)}{\varphi'(\delta)} \leq S$;
- $\bullet_\varphi$ the metric $g = dt^2 + \varphi^2 dS_n^2$ is area-wise large.

**Reformulation and Proof of Inequality** $[\Omega_{\ast}\ast]$. Let $\varphi_*$ and $\psi_*$ be extremal (almost extremal will do) for $M_{max}$.

Reduce the case of general Riemannian $X$ with boundary $Y$ to that where $R = \text{Rad}_{S^n}(Y) = 1$ by scaling the Riemannin metric in $X$ by $\text{Rad}_{S^n}(Y)^{-2}$.

Let $G$ be a smooth metric on $X_{\delta\delta}$ from section 3, which restricts to the original $g$ on $X \subset X_{\delta\delta}$, which is equal to $dt^2 + \varphi_*(t)h$ on $Y \times [-\delta, 0]$ and which is
symmetric under the involution on $X_{\delta \delta}$, where the existence of such a (smooth!) metric (modulo an arbitrary small perturbation of $g$) is possible in view of (\bigcirc) from the previous section.

Then the application of the Llarull theorem to the map $F = F_{f, \psi} : (X_{\delta \delta}, G) \to S^{n+1}(R = 1)$, where $f : Y \to S^{n}(1)$ is a distance decreasing map of non-zero degree, implies the following.

**Sharpened Inequality** [$\bigcirc_{++}$]. Let $X$ be a compact Riemannian $(n + 1)$-dimensional spin manifolds with boundary $Y = \partial X$, such that $Sc(X) \geq 0$ and $Sc(Y) \geq 0$. Then the mean curvature of $Y$ is bounded in terms of the spherical radius of $\partial X$ and the lower bound $\underline{S} = \inf Sc(\partial X)$ as follows.

$$\underline{M}(Y) = \inf_{y \in Y} M_{\text{ncurv}}(Y, y) \leq M_{\text{max}}(\underline{S}, \text{Rad}_{S^{n}}(\partial X)).$$

**Derivation of [$\bigcirc_{++}$] from [$\bigcirc_{max}$]** is performed with $h_t$ taken from the above example (b), which gives the lower bound on $M_{\text{max}}$ needed for [$\bigcirc_{++}$], that, recall, under assumptions $Sc(X) \geq 0$ and $Sc(Y) \geq 0$, reads:

$$\underline{M}(Y) = \inf_{y \in Y} M_{\text{ncurv}}(Y, y) \leq \frac{10n^2}{\text{Rad}_{S^{n}}(Y)}.$$

**Remarks** (A) Despite its name, the inequality [$\bigcirc_{max}$], is almost never sharp. If it has a chance for optimality, then only if

$$Sc(Y) = \frac{n(n-1)}{\text{Rad}_{S^{n}}(Y)^2},$$

which makes $Y$ isometric to a $S^{n}(R)$, and where the manifold $X$ is Riemannian flat.

But in truth, I am afraid, the chance for this optimality is close to zero for $n + 1 \geq 3$ due to the term $\frac{n(n-1)}{\text{Rad}_{S^{n}}(Y)^2}$ in (\bigstar). Apparently, there is a wide gap of empty space between the Miao-Shi-Tam theorem and the inequality [$\bigcirc_{max}$].

(Yet a Witten style argument could, conceivably, deliver a derivation of the positive mass theorem expressed in polar coordinates from Llarull’s theorem, as it is suggested by a formal similarity between the algebra involved in the arguments by Witten and by Llarull, as well as the analysis behind their proofs – both arguments rely on god-given equivariant spinor fields and on stability of the index under deformations, rather then on the Atiyah-Singer index formula.)

(B) Besides being unsharp and using spin, the above proof of $\bigcirc_{max}$ seems artificial as it depends on two preliminary modifications $X \leadsto X_{\delta \delta} \leadsto X_{\delta \delta}$. Hopefully, this can be done better with minimal surfaces but, apparently, these work only in the ambience of "complicated topology", e.g. where $X$ is homeomorphic to the product of the disk by the $(n-1)$-torus, or, more generally, if $X$ admits a map with non-zero degree to this product, to

$$(X, \partial X) \sim (B^2 \times T^{n-1}, \partial B^2 \times T^{n-1}).$$

In this case, the symmetrisation version of the Schoen-Yau descent argument from [G 2018] reduces the general case to the one where the metric on $X$ is $T^{n-1}$-invariant, for which, probably, the proof must be not difficult.
This is indeed simple for \( n + 1 = 3 \), but where, instead of symmetrization, one uses the Gauss-Bonnet theorem, which, when applied to area minimizing minimal surfaces \( \Sigma \subset X \) with boundaries in \( Y = \partial X \), shows the following.

If \( \text{Sc}(X) \geq 0 \) and \( \text{Mncurv}(Y) \geq M > 0 \), then the total length \( L \) of the boundary curves of \( \Sigma \) is bounded by \( L \leq \frac{2\pi}{M} \).

This, say for \( X \) homeomorphic to \( B^2 \times S^1 \), shows that the boundary torus \( Y \) of \( X \) has \( \text{Rad}_{S^2}(Y) \leq \frac{1}{M} \).

(C) If \( X \) is homeomorphic to the 3-ball, one can use, instead of Llarull’s, the Marques-Neves area bound for spherical minimal surfaces in \( X_{\delta \delta} \) with Morse index one (theorem 4.9 in [MN 2011])\(^{14}\) and, possibly, there is a version of this bound applicable to non-spherical 3-manifolds.

(D) It is not hard to generalise \( M_{\text{max}} \) and the above (B) to non-compact complete manifolds \( X \) with boundaries, e.g. to \( X \) diffeomorphic to \( Y \times [0, \infty] \), where \( Y \) may be a closed manifold (this removes the necessity of \( Y \) being bordant to zero) or, if you wish, non-compact complete one\(^{15}\).

6 Manifolds with Negative Scalar Curvatures Bounded From Below.

Let \( X \) be a compact orientable Riemannin \((n + 1)\)-manifold with, possibly disconnected, boundary, such that

- the scalar curvature of \( X \) is bounded from below by that of the hyperbolic \((n + 1)\)-space \( H^{n+1}_{\gamma} \),

\[
\text{Sc}(X) \geq -n(n + 1),
\]

- the mean curvature of the boundary is bounded from below by that of the complement of a horoball in \( H^{n+1}_{\gamma} \),

\[
\text{Mncurv}(\partial X) \geq -n;
\]

- there is a connected component of the boundary \( \partial X \), call it \( Y_* \), the mean curvature of which is bounded from below by a constant greater than \(-n\), say

\[
\text{Mncurv}(Y) \geq \sigma_*.
\]

- If \( Y_* \) admits a map with non-zero degree to the \( n \)-torus and if the inclusion homomorphism between the fundamental groups

\[
\pi_1(Y_*) \to \pi_1(X)
\]

is injective, then

\[
\sigma_* \leq n,
\]

where the equality holds if and only if the universal covering of \( X \) is isometric to a "band" between too equidistant horospheres in \( H^{n+1}_{\gamma} \).

---

\(^{14}\)The Marques-Neves inequality, which implies a sharp(!) bound on the \( \delta \)-waist of \( X_{\delta \delta} \), requires, a priori, less of \( Y = \partial X \), than Llarull’s theorem.

\(^{15}\)If \( Y \) is non-compact, the present proof of \( M_{\text{max}} \) needs the sectional curvature of \( Y \) to be bounded from above and the injectivity radius of \( Y \) bounded from below: \( \text{sectcurv}(Y) \leq \text{const} \leq \infty \) and \( \text{injrad}(Y) \geq \rho > 0 \).
This is proved in §5 of [G 1996] for \( n + 1 \leq 7 \) by a "soap bubble" argument which extends to all \( n \) in view of [SY 2017].

**Example and Generalisation.** Let \( X_0 \) be a compact hyperbolic (i.e. \( \text{sect}, \text{curv}(X_0) = -1 \)) manifold with totally geodesic boundary \( Y_0 \).

Observe that the equidistant deformations \( Y_d \) for \( d \geq 0 \) of \( Y_0 \) in the ambient complete hyperbolic manifold \( X \supset X_0 \) have \( \text{Mncurv}(Y_d) \to n \) for \( d \to \infty \).

If \( Y_0 = \partial X_0 \), or a finite covering of it admits a non-zero degree map to the \( n \)-torus, then, by the above, no Riemannin manifold \( X \) homeomorphic to \( X_0 \), or just admitting a map with non-zero degree to \( X_0 \), can have \( \text{Sc}(X) \geq -n(n+1) \) and \( \text{Mncurv}(\partial X) \geq n \).

This doesn’t quite work if no map \( Y_0 \to T^n \) with \( \text{deg} \neq 0 \) exists. However, since \( X_0 \) is isoenlargeable, a combination of the arguments from [G 1996] and [G 2018] (with a use of [SY 2017] for \( n+1 \geq 9 \)) shows that the above conclusion still holds for these \( X \), namely,

\[
\text{Sc}(X) \geq -n(n+1) \Rightarrow \text{Mncurv}(Y) \not> n.
\]

**Application of Symmetrization.** In many (all?) cases ●, can be proved by the torical symmetrization argument from [G 2018] with no use of "bubbles" at all. A characteristic example is as follows.

Let \( X \) be an \((n+1)\)-dimensional Riemannin manifold homeomorphic to \( Y \times [0,1] \) for \( Y = T^n \).

If \( \text{Sc}(X) \geq -n(n+1) \) and the mean curvature of \( Y_1 = Y \times \{1\} \subset \partial X \) is bounded from below by \( M > n \), then

\[
dist(Y_0 = Y \times \{0\}, Y_1) \leq l = \frac{2}{n+1} \text{coth}^{-1} \frac{M}{n},
\]

where \( \text{coth}^{-1} \) denotes the inverse function of \( \text{coth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \).

In fact (see [G 2018]), the extremal Riemannin metric \( g_{\text{ext}} \) with the maximal \( l \) is defined on the manifold \( T^n \times (0,l] \), where it is invariant under the obvious action of the torus \( T^n \). Hence, \( g_{\text{ext}} = dt^2 + \varphi^2 h_{\text{flat}} \), where one may assume that \( \phi(t) \to 0 \) for \( t \to 0 \).

Since,

\[
\text{Mncurv}(g_{\text{ext}}) = \frac{n \varphi}{\varphi'},
\]

by the first variation formula ((I) in section 4) and \( \text{Sc}(g_{\text{ext}}) = -n(n+1) \), the Weyl formula ((II)) shows that

\[
-n(n+1) = \text{Sc}(g_{\text{ext}}) = -2n \frac{\varphi''}{\varphi} - n(n-1) \frac{\varphi'^2}{\varphi^2}.
\]

It follows that the function \( f = \frac{\varphi}{\varphi'} \) satisfies the equation

\[
\frac{f''}{1-f^2} = (\text{coth}^{-1} f)' = \frac{n+1}{2},
\]

which implies that

\[
f(t) = \text{coth} \frac{t(n+1)}{2},
\]
where the domain of definition of this \( f \) is the segment \((0, l]\) for \( l = \frac{2n}{n+1} \coth^{-1} \frac{M}{n} \).

Cuspidal Boundary Rigidity Theorem. Let \( X \) be an \((n+1)\)-dimensional complete orientable Riemannian manifold with compact boundary \( Y \), such that \( Sc(X) \geq -n(n+1) \) and \( M\text{curv}(Y) \geq n \).

\(*\) If some connected component \( Y_0 \) of \( Y \) admits a map to the \( n \)-torus \( T^n \) with non-zero degree, which continuously extends to a map \( X \to T^n \), then the above argument shows that the universal covering of \( X \) is isometric to a horoball in the hyperbolic space \( H^{n+1}_2 \).

(If \( X \) is compact and \( Y = \partial X \) is connected, then maps \( Y \to T^n \) with non-zero degrees do not extend to \( X \), but this can be sometimes remedied by passing to an infinite coverings \( p : \tilde{X} \to X \), such that the pullbacks \( p^{-1}(Y) \subset \tilde{X} \) are disjoint union of compact manifolds.)

Remark. The advantage of \(*\) over the above \( \bullet \) is admission of complete non-compact manifolds \( X \). In fact, one may also allow here additional boundary components with \( M\text{curv} \geq -n \), and then \(*\) will imply \( \bullet \).

Question. Is there a Dirac operator proof of \(*\) à la Witten and Min-Oo?

7 Problems with \( B^2 \times T^{n-1} \)-Manifolds.

Let us discuss a possibility of extension of the above to a class of compact orientable \((n+1)\)-dimensional Riemannian manifolds \( X \) with boundaries \( Y = \partial X \) as in (B) from section 5.

Namely,

\( ^{(n+1)} \) Let an \( X \) admit a map with non-zero degree to \( B^2 \times T^{n-1} \), where \( B^2 \) denotes the disc.

If \( Sc(X) \geq n(n+1) \) and \( M\text{curv}(Y) \geq M > n \), then, conjecturally, the lengths \( L \) of the closed curves in \( Y \), which are non-homologous to zero in \( Y \) but homologous to zero in \( X \supset Y \), are bounded by the length of the circle \( Y_0 \) in the hyperbolic plane \( H^2_{-1} \) with \( M\text{curv}(Y_0) = M_0 = \frac{M}{n} \).

If \( n + 1 = 2 \), this is a baby version of the Miao-Shi-Tam rigidity theorem which reads as follows.

\( ^{(2)} \) Let a compact connected surface \( X \) with boundary \( Y \) has \( Sc(X) \geq -2 \) and \( M\text{curv} \geq M > 1 \). Then \( X \) is homeomorphic to the disc and the length of \( Y \) is bounded by that of the circle \( Y_0 \) in the hyperbolic plane \( H^2_{-1} \) with \( M\text{curv}(Y_0) = M \).

In fact, this is seen by attaching \( X \) to the complement of the ball in \( H^2_{-1} \) with the boundary length equal the length of \( Y \) and using the (obvious) rigidity of the hyperbolic metric under curvature increasing deformations, which have compact supports in \( H^2_{-1} \).

Exercise. State and prove the "dual" inequality for metrics with sectional curvatures bounded from above.

If \( n + 1 \geq 3 \), then some (very limited) results can be derived by Miao-Shi-Tam gluing argument combined with the rigidity of the manifold \( Z = H^2_{n+1} / \mathbb{Z}^{n-1} \) that
was stated in (H₃) in section 2.

Example. Think of \(\mathbb{H}^{n+1}/\mathbb{Z}^{n-1}\) as \(\mathbb{H}^2 \times \mathbb{T}^{n-1}\) with a warped (in particular, \(\mathbb{T}^{n-1}\)-invariant) metric and let \(X_0 = B^2 \times \mathbb{T}^{n-1}\) for some disc \(B^2 \subset \mathbb{H}^2\).

Notice that the boundary \(Y_0 = \partial X_0\) is the \(n\)-torus \((\partial B^2 \times S^1) \times \mathbb{T}^{n-1}\) with a warped product metric where the certain warping function on the closed curve \(S^1 = \partial B^2\) depends on the position and shape of this curve in the hyperbolic plane \(\mathbb{H}^2\).

Then, by the rigidity of \(Z = \mathbb{H}^{n+1}/\mathbb{Z}^{n-1}\),

no Riemannian manifold with the boundary isometric to \(Y_0\) can have the scalar curvature \(\geq -n(n+1)\) and the mean curvature of the boundary greater than that of the original \(M \text{ncurv}(Y_0 \subset Z)\). 

Also notice that here, similarly to what was indicated in (H₃) in section 2, one may allow a \(C^2\)-small perturbation \(h_{\text{new}}\) of the original metric \(h_0\) in \(Y_0\) induced from \(Z\). Namely,

no Riemannian manifold with the boundary isometric to \((Y_0, h_{\text{new}})\) can have the scalar curvature \(\geq -n(n+1)\) and the mean curvature of the boundary greater than that of the original \(M \text{ncurv}(Y_0 \subset Z)\) plus one.

Our formulation of (*₂) as well as distinguishing the case of \(\mathbb{T}^{n-1}\)-invariant torical domains is motivated by the possibility of symmetrisation of an arbitrary \(X\) with no decrease of its scalar curvature and of the mean curvature of the boundary \(\partial X\).

Recall (see [G 2018]), that "the ultimately" symmetric manifold \(Z_0\) carries the (warped product ) metric \(dt^2 + (\sinh^2 t) \cdot h_{\text{flat}}, 0 < t < \infty\), which has constant sectional curvature only for \(n+1 = 2\), where, if \(n+1 = 2\), this metric remains non-singular at \(t = 0\) if \(h_{\text{flat}}\) is the metric of the circle \(S^1_{2\pi}\) of length (exactly) \(2\pi\).

In general, we assign \(h_{\text{flat}}\) to the torus \(\mathbb{T}^{n-1} \times S^1_{2\pi}\) and think of \(Z_0\) as a torical warped product over the hyperbolic plane minus a point.

But we don’t know

whether such metrics are rigid with respect to deformations with compact (bounded?) supports and \(\text{Sc} \geq -n(n+1)\).

Another, even more annoying, problem is that the symmetrization in the present case terminates at a warped product metric \(g_0\) on \(X_0 = \Sigma^2 \times \mathbb{T}^{n-1}\), where \(\Sigma^2\) is a disc with a Riemannian metric \(g_0\), where we don’t control either \(g_0\) or the warping factor that is a function on \(\Sigma^2\).

Ideally, we would like the warping factor \(w(\sigma), \sigma \in \Sigma^2\), of this \(g_0\) on the (circular) boundary of \(\Sigma^2\) to match the warping function on \(\partial B^2 \subset \mathbb{H}^2\) in the above example.

But, apparently, \(w(\sigma)\) for \(\sigma \in \partial \Sigma^2\) depends on the geometry of all of \(X\), not only on \(\partial X\). This prevents us from a proof of an even non-sharp version of (*₂) by the gluing argument used for \(n+1 = 2\).

On the other hand one may think of an alternative geometric approach, which, in particular, would deliver a proof for the case of \(n+1 = 2\) by analysing the intrinsic geometry of the surface \(X\) from (*₂) rather than by attaching something to it.\(^{16}\)

\(^{16}\)A non-sharp inequality can be derived from "strong generalised concavity" of the distance function \(\text{dist}_X(s_1, s_2)\) for \(s_1, s_2 \in \partial X\), as in the "proof" of unproven corollary in section 9.
On Surfaces in 3-Manifolds. If \( n + 1 = 3 \), then additional possibilities are opened by the existence of isometric imbeddings of "many" non rotationally symmetric Riemannian metric on 2-tori to \( \mathbb{H}^3/\mathbb{Z} \). (Probably, the space of the embeddable metrics has codimension one in the space of all metrics.)

The situation is more satisfactory for imbedding into \( \mathbb{H}^3/\mathbb{Z}^2 \), where instead of an ambiguous "many" one can definitely say "all".

In fact, let \( Y = (\mathbb{T}^2, h) \) be a 2-torus with a Riemannian metric where \( \text{sectcurv}(h) > -1 \). Then, by the torical version of Pogorelov’s isometric embedding theorem,

there exists a hyperbolic cusp with the sectional curvature \(-1\) and an (essentially unique) isometric embedding \( Y \to Z \).

(This can be exploited similarly to how that was done in [MM 2016] and mentioned in \( \star \kappa \) in section 2.)

8 Manifolds with Corners

Let us briefly discuss here what can be done about Sormani’s question C from the introduction.

Basic examples of Riemannian \((n+1)\)-manifolds with corners are (convex) polyhedra in \( \mathbb{R}^{n+1} \) with smooth Riemannian metrics on them.

In general, a corner structure \( P \) on a smooth manifold \( X \) with boundary \( Y = \partial X \) is given by the shadow \( P \) of \( P \), that is a partition of \( Y \) into locally closed submanifolds corresponding to the actual faces of \( P \).

For example, the shadow of the corner structure of the cube \([-1,1]^{n+1}\) on the unit \( n \)-sphere \( S^n \subset \mathbb{R}^{n+1} \) can be seen by radially projecting the boundary of the cube to this sphere.

More generally, let \( X \) be a smooth \((n+1)\)-manifold with boundary \( Y \) and let \( f : Y \to S^n \) be a smooth map which is transversal to all faces of such a shadow \( P \) in \( S^n \). Then the pullbacks of these faces define a shadow of a corner structure on \( X \), where the corresponding corner structure on \( X \) is called cubical if \( X \) is orientable and the map \( f \) has non-zero degree.

\((\square_6)\) The simplest instance of this, that we shall use below, is where the map \( f \) is a diffeomorphism from the interior of a ball \( B \subset Y \) onto \( S^n \) minus the south pole \( s_* \subset S^n \) and where all of the complement of \( B \) goes to \( s_* \).

Besides the scalar curvature of \( X \) and the mean curvatures of its \( n \)-faces an essential geometry of a Riemannian manifold with corners is carried by the dihedral angles \( \alpha \) at the "edges" that are the \((n-1)\)-faces of \( X \), where the difference \( \pi - \alpha \) plays the role of the mean curvature.

Here is a property of cubical manifolds that motivates what follows in this section.

Hyperbolic Subrectangular Theorem. Let \( X \) be a compact cubical Riemannian manifold, where

(i) the dihedral angles are \( \leq \frac{\pi}{2} \);
(ii) all \( n \)-faces but one have positive mean curvatures;

The set of the isometry classes of these "cusps" that are the quotient manifolds \( \mathbb{H}^3/\Gamma \), \( \Gamma = \mathbb{Z}^2 \), is naturally parametrised by the modular curve of conformal structures on \( \mathbb{T}^2 \), where the modular parameter of \( Z \), which receives an isometric embedding of \((\mathbb{T}^2, h)\), depends on \( h \).
(iii) the exceptional face have $M_{\text{curv}} \geq -n$;
(iv) the opposite face, which is also called "exceptional", has $M_{\text{curv}} > n$.

Then the scalar curvature of $X$ satisfies:

$$\inf_{x \in X} Sc(X, x) < -n - n.$$  

Idea of the proof. Reflect $X$ around $2n$ non-exceptional faces, smooth the resulting $C^0$-metric corners and apply $\bullet$ from section 6 to the resulting "sub-hyperbolic band".

In truth, however, such a smoothing is a mess; a technically less demanding approach is explained in [G 2014] and in section 11.10 of [G 2018].

(This proof of the theorem is immediate for $n + 1 = 2$, where it is seen by looking at minimal geodesic segments between two exceptional faces).

Rigidity Question. If (iv) is relaxed to $M_{\text{curv}} \geq n$, and if $\text{Sc}(X, x) \geq -n(n + 1)$, then, most likely, $X$ is isometric to a parabolic rectangular solid in $H^{n+1}$, that is defined in the horospherical coordinates $x_0, x_1, \ldots, x_n$ by the inequalities

$$0 \leq x_i \leq a_i, \quad i = 0, 1, \ldots, n,$$

where, recall, the hyperbolic metric $g$ in these coordinates is

$$g = dx_0^2 + e^{2x_0} \sum_{i=0}^{n} dx_i^2.$$  

We fail short of directly proving this because of, a priori possible, presence of singularities of the boundaries of extremal $X$, which is due to a use of the Schoen-Yau style variational argument in an essential step of our proof.

On the other hand, a suitable adapted Kazdan-Warner perturbation argument would, probably, reduce the rigidity problem for $\text{Sc}(X) \geq -n(n + 1)$ to that, where all faces but one are convex and $\text{Ricci}(X) \geq -ng$; then the proof would follow by Weyl’s tube formula.

Unproven Corollary. Let $X$ be a compact Riemannian $(n + 1)$-manifold with boundary $Y = \partial X$, such that

$$\text{Sc}(X) \geq -1 \text{ and } M_{\text{curv}}(Y) \geq -1.$$  

Let $B = B_{B_0}(r) \subset Y, \; y \in Y$, be a ball of radius $r$, such that

-the sectional curvatures of $Y$ in this ball are bounded in absolute values by a constant $\kappa > 0$;
-the exponential map $T_y(Y) \to Y$ is one-to-one on the ball $B_0(r) \in T_y(Y)$ to $B$.

Then

"the infimum of the mean curvature of $Y$ in the ball $B$ satisfies"

$$\left[\Box_{-1}\right] \inf_{y \in B} M_{\text{curv}}(Y, y) \leq \mathcal{M}_n(\kappa + r^{-1})$$

for some (possibly very large) universal continuous function $\mathcal{M}_n$.

How we Want to Prove it. Start by observing that our "corollary" for $n + 1 = 2$, generalises a non-sharp version of (*2) from section 7 and that the actual proof for $n + 1 = 2$ is easy.
In fact, let $X$ be a surface with circular boundary $Y = \partial X$, and let $Y_+ \subset Y$ be a segment, such that the following conditions are satisfied.

1. $\text{sect.curv}(X) \geq -1$;
2. the curvature of the segment $Y_+$ is $\geq M_+ = 1 + \varepsilon_+, \varepsilon_+ > 0$;
3. the curvature of $Y$ in the complement of $Y_0$ is $\geq -1$.

Then the length of $Y_+$ is bounded by $1000$ times the length of the circle $Y_M \subset \mathbb{H}^2_1$ with the curvature $\text{curv}(Y_M) = M_+$.

**Proof.** Assume that the curvature of $Y$ is constant on $Y_+$ (this is justified as in section 4) and let us endow the cylinder $Y \times \mathbb{R}$, to $X$ with a (canonical) metric with sectional curvature $-1$, and such that the boundary $Y \times \{0\}$ of this cylinder admits an isometry, i.e. a length preserving map

$$Y \times \{0\} \leftrightarrow Y,$$

where this isometry matches the curvature of these curves, which makes the resulting metric on the extended manifold

$$X_+ = X \cup Y \times \mathbb{R}_+$$

$C^1$-smooth.

If the length of the segment $Y_+$, now positioned in $Y \times \mathbb{R}_+$, were sufficiently long then, by an elementary argument, there would exist a deformation of $Y$ in $Y \times \mathbb{R}_+$ to a curve

$$Y' \subset Y \times \mathbb{R}_+ \subset X_+,$$

such that

- there are four $90^\circ$ corners on this curve $Y'$.
- the curvatures of one pair of disjoint segments bounded by the corner points are $\geq 0$, while the curvatures of the second pair of segments, call them $Y'_+ \text{ and } Y'_-$ are bounded by

$$\text{curv}(Y'_+) \geq 1 + 0.01 \varepsilon_+ \text{ and } \text{curv}(Y'_-) \geq -1.$$

This contradicts the (obvious in this case) the above (sub)rectangular theorem, applied to the domain $X'$ bounded by the curve $Y'$ in $X_+$, and thus, the proof is concluded.

**Questions.** (i) Which functions can be realised as curvatures of boundaries $Y = \partial X$ of surfaces $X$, where $\text{sect.curv}(X) \geq \kappa$?

To get an idea of the expected patterns of segments of $Y$ with large and small curvatures, let $Y \subset \mathbb{R}^2$ be a simple closed curve, which contains a segment $Y_+$ of length $L_+$, where the curvature of $Y$ is $\geq 1$.

It is not hard to show that $Y$ must contain an open subset $Y_+$ (possibly, with arbitrarily many connected components) of length $L_+$, such that $L_+ \geq (1 - \varepsilon)L_+$ and

$$\inf \text{curv}(Y_-) = \inf \limits_{y \in Y_-} \text{curv}(Y, y) \leq -1 + \delta,$$

where $\varepsilon, \delta \to 0$ for $L_+ \to \infty$.

Probably, all surfaces $X$ with $\text{sect.curv} \geq \kappa$ display similar patterns of distributions of curvatures of their boundaries $Y$ for $\text{length}(Y) >> -\kappa$.

(ii) What would be analogues of the above for (distribution of) the mean curvatures of boundaries $Y$ of $(n+1)$-manifolds $X$ with lower bounds on the sectional, Ricci or scalar curvatures of $X$?
(The case of bounded domains $X \subset \mathbb{R}^{n+1}$, $n+1 \geq 3$, already seems interesting.)

**Idea of a Possible Proof of $[\square_{-1}]$.** We want to extend the above "cornering" argument to $n+1 \geq 3$ and create a (sub)rectangular corner structure on $X$, the shadow of which will be induced by a map $B \to S^n$ as in the above ($\square_\varnothing$).

It seems manageable to bend $Y$ along individual edges, that are the $(n-1)$-faces of $X$ (compare with the constructions in section 11.3, 11.4 in [G 2018]), but it less clear how to do it (I think it is unpleasant but possible) at the corners, where the edges meet.

However, similarly how this was bypassed in [G 2014] and in [G 2018] one can, probably, avoid this problem by performing the constructions of corners, reflections and smoothings interchangeably.

For instance, let $X$, topologically, be the 3-ball and let us create two circular corners on its (spherical) boundary corresponding to the (circular) boundaries of two non-exceptional faces.

Then double the resulting $X'$ over this pair, smooth it and arrive at $X_1$, which is now homeomorphic to $B^2 \times S^1$, where the shadow of the corner structure is induced from that on the disc $B^2$.

Proceed as before, now with the remaining pair of non-exceptional faces and, thus, arrive at $X_2$, which is homeomorphic to $S^1 \times S^1 \times [0,1]$ and to which $\square_\varnothing$ from section 6 applies.

Conceivably, the proof of $[\square_{-1}]$ can be achieved along these lines. But artificiality of the argument and the issuing non-sharpness of the result leave the problem of the correct (re)formulation and of the proof of $[\square_{-1}]$ open.

On the other hand, the geometry behind this may be interesting in its own right as suggested by following.

**Question.** Let $X = (X,g)$ be a Riemannian manifold diffeomorphic to the $(n+1)$-ball and let the scalar curvature of $X$ in a neighbourhood $U \subset X$ of $Y = \partial X$ be positive. Let $V$ be a closed smooth $(n+1)$-manifold.

A packing with $Sc > 0$ of $V$ by (copies of) $X$ is a Riemannian metric $g_+$ on $V$, along with isometric embeddings of several copies of $(X,g)$ to $(V,g_+)$, such the metric $g_+$ has positive scalar curvature in the complement of the images of these embeddings.

Under what conditions on the intrinsic geometry of $Y = \partial X$, on the mean curvature of $Y$ and on the topology of $V$, can $V$ be packed with $Sc > 0$ by $X$?

More specifically, let $\text{lgeo}(Y)$ denote the maximum of two invariants of $Y$

\[
\text{lgeo}(Y) = \max \left( \min \left( \text{sect.curve}(V), \frac{1}{\text{injrad}(V)} \right), \right.
\]

let $\text{top}(V)$ means "topology of $V"$ and let $\mathcal{M}_{\text{top}(V)}(*)$ be a (large) continuous function.

For which $V$, is the inequality $M_{\text{ncurv}}(Y) \geq \mathcal{M}_{\text{top}(V)}(\text{lgeo}(Y))$ sufficient for the existence of such packing?

Notice in this regard, that our cornering argument, if it works, implies the existence of such packings of the $(n+1)$-torus, provided $\mathcal{M} = \mathcal{M}_{\mathbb{T}^{n+1}}$ is sufficiently large.$^{18}$

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$^{18}$The packings associated with such "cornering", may have, if you wish, the complements to the balls contained in the $\varepsilon$-neighbourhood of these balls for arbitrarily small $\varepsilon > 0$. 

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Also, such packings are likely to exist in other "reflection manifolds". Contrariwise, the largeness of the curvature of the boundary curves serves as an obstruction for packing with $Sc > 0$ of 2-spheres by discs.

More generally, aspherical$^{19}$ manifolds $V$ of all dimensions have a chance to be packable, (more realistically, approximately packable in some sense), under similar conditions but it is rather improbable for the spheres, even if "locgeo" is replaced by a more comprehensive global invariant of $X$. But finding relevant examples of non-extendability of metrics with $Sc > 0$ seems difficult.

(All what matters for the existence of an extension with $Sc > 0$ is the geometry of $X$ near $Y = \partial X$. But what happens in $X$ far from $Y$ may harbour obstructions to such extensions.)

Bound on Hight of Fat Cylinders. Let us indicate another application of the cornering construction, where there is an instance where no higher order corners difficulty appears.

Let $X$ be a compact connected orientable $(n + 1)$-manifold with boundary $Y$ and let $Y_{\text{bot}}$ and $Y_{\text{top}}$ be two disjoint smooth connected domains in $Y = \partial X$ called the bottom and the top.

The basic example is where $X$ is the cylinder, $X_0 = B^n \times [0, 1]$; in general, $X = (X, Y_{\text{bot}}, X_{\text{top}})$ is called "cylinder" if it admits a continuous map $f : X \to B^n \times [0, 1]$, such that $\partial X$ is sent to $\partial X_0$,

$$f^{-1}(B^n \times \{0\}) = Y_{\text{bot}}, \quad f^{-1}(B^n \times \{1\}) = Y_{\text{top}}$$

and $\deg(f) \neq 0$.

Denote by $Y_0 \subset Y$ the "side" of the "cylinder" $X$, that is the pullback $f^{-1}(S^{n-1} \times [0, 1])$ for the boundary sphere $S^{n-1} = \partial B^n$ and let the following conditions be satisfied.

- $Sc(X) \geq n(n + 1)$,
- $locgeo(Y_0) \leq 1$,
- $Mncurv(Y_0) \geq M$.

**Second Unproven Corollary.** The "hight" of $X$ is bounded by a universal function of $M$:

$$\text{dist}_X(Y_{\text{bot}}, Y_{\text{top}}) \leq \mathcal{H}_n(M),$$

such that

$$\mathcal{H}_n(M) \to \frac{2\pi}{n + 1} \text{ for } M \to \infty.$$

Idea of a Possible Proof. If $n + 1 = 3$, bend $Y_0$ along four segments between the top and the bottom of $X$ and, thus, produce four edges with dihedral angles $\leq \frac{\pi}{2}$. This seems non-difficult.

More problematic is to create similar $n$-cubical structure for $n + 1 \geq 4$ along the lines indicated in the above "possible proof" of $[\square_{-1}]$.

But if this is accepted, then the sub-rectangular $\frac{2\pi}{n}$-inequality from section 11.10 in [G 2018] could be applied and the proof would follow.

Remarks/Questions. (a) The apparent examples suggest that

$$\text{dist}_X(Y_{\text{bot}}, Y_{\text{top}}) \to 0 \text{ for } M \to \infty,$$

$^{19}$A topological space is called aspherical if its universal covering is contractible.
where it is, indeed, so for $n + 1 = 2$ by an elementary argument.

But it is unclear whether this is true or false for $n + 1 \geq 3$.

(b) Can one replace the bound on $\text{locgeo}(Y_{\Theta})$ by a lower bound on some kind of size of $Y_{\Theta}$?

Such a "size" can be conveniently defined with the above map $f$ in the definition of the "cylinder" structure on $X$ as follows.

Restrict this $f$ to $Y_{\Theta}$, observe that it sends $Y_{\Theta} \to S^{n-1} \times [0, 1]$, compose this with the projection $S^{n-1} \times [0, 1] \to S^{n-1}$ and denote the resulting map by $\tilde{f} : Y_{\Theta} \to S^{n-1}$.

If $n + 1 \geq 3$, denote by $\text{Rad}_{S^{n-1}}(Y_{\Theta})$ the supremum of the numbers $R$ such that $\tilde{f}$ is homotopic to a smooth map $\tilde{f}' : Y_{\Theta} \to S^{n-1}$, such that the norm of the differential of $\tilde{f}'$ is bounded by

$$\sup_{y \in Y_{\Theta}} \|d\tilde{f}'(y)\| \leq \frac{1}{R}.$$ 

And if $n + 1 = 2$, let $\text{Rad}_{S^{n-1}}(Y_{\Theta}) = 1$ for all $X$.

**Conjecture.** Let $S\text{c}(X) \geq 0$ and $\text{Mncurv}(Y) \geq M$. If the product

$$\Theta(X) = M \cdot \text{Rad}_{S^{n-1}}(Y_{\Theta})$$

(this $\Theta$ speaks for "fatness" of $X$) is sufficiently large, say

$$\Theta(X) > \Theta_{\text{crit}},$$

then

$$\text{dist}_X(Y_{\text{bot}}, Y_{\text{top}}) \leq \lambda_n(\Theta(X))$$

for some continuous monotone decreasing function $\lambda_n(\Theta)$ defined for $\Theta > \Theta_{\text{crit}}$.

The best one may expect here is that

$$\Theta_{\text{crit}} = n - 1$$

and that

$$\lambda_n(\Theta) \to 0 \text{ for } \Theta \to \infty.$$ 

But any bound $\Theta_{\text{crit}} \leq \text{const}_n$ will be welcome and the inequality

$$\lim_{\Theta \to \infty} \lambda_n(\Theta) \leq \frac{2\pi}{n + 1}$$

will be quite satisfactory.

And it is conceivable that all of the above holds for $S\text{c}(X) \geq \sigma$, for all $-\infty < \sigma < \infty$, with $\Theta_{\text{crit}} = \Theta_{\text{crit}}(\sigma)$ and $\lambda_n = \lambda_n(\Theta, \sigma)$.

### 9 Further Conjectures and Problems.

Probably, all we know (and don’t know) about manifolds with boundaries extends to manifolds with corners, albeit by no means automatically. In fact, it will be more productive to study manifolds with corners for their own sake rather than as intermediates in the arguments concerning smooth manifolds.
For instance, problem A from the introduction becomes even more interesting for manifolds with corners where it reads as follows.

Let $X$ be a compact $(n + 1)$-dimensional Riemannian manifold with corners and $Y_0$ be an $n$-dimensional face of $X$.

Find an upper bound on the mean curvature of this face, in terms of lower bounds on the scalar curvature of $X$ and the mean curvatures of the remaining $n$-faces of $X$ and the dihedral angles along $(n - 1)$-faces, as well as some geometric invariants of the face $Y_0$, e.g. its "size" and its own scalar curvature.

The topologically simplest instances of this are as follows.

- The half-ball $B_0^n + 1$, that is the intersection of the ball with the halfspace $x_1 \geq 0$ where the $n$-ball $B_0^n \cap \mathbb{R}^n$ is taken for $Y_0$;
- the cylinder $X = B_0^n \times [0, 1]$, where $B_0^n \times \{0\}$ is taken for $Y_0$;
- products of smooth manifolds with boundaries, e.g. cubes $[0, 1]^{n+1}$ and close relatives of cubes – $(n + 1)$-diamonds and $(n + 1)$-simplices.

But it is more challenging to understand the geometry of $X$ when the combinatorics of the corner structure becomes more complicated.

For instance, define $\text{CombRad}_n(Y)$, for a closed orientable $n$-manifold $Y$ partitioned into faces, as the supremum of the numbers $R$, such that $Y$ admits a continuous map of non-zero degree to the sphere $S^n(R)$, such that the images of all faces have diameters at most one.

○ Conjecture. Let $X$ be a compact orientable Riemannian $(n + 1)$-manifold with corners, such that $\text{Sc}(X) \geq 0$, and all $n$-faces have positive mean curvatures.

Let $\overline{\alpha}(X)$ denote the supremum of the dihedral angles of $X$ taken over all points of all edges ($(n - 1)$-faces) of $X$.

Then

$$\pi - \overline{\alpha}(X) \leq \frac{\text{const}_n}{\text{CombRad}_n(\partial X)}$$

for some universal (possibly large) constant $\text{const}_n$.

Admission. I haven’t check this even in the case of convex polyhedra $X \subset \mathbb{R}^{n+1}$.

The following is a version of ○ for manifolds without corners, which may be also overoptimistic.

○ Conjecture. There exists a (possibly very large) function $\mathcal{M}_n(\sigma, R) > 0$, $-\infty < \sigma < \infty$, $R > 0$, such that

all compact Riemannian $(n + 1)$-manifolds $X$ with boundaries, where

$\text{Sc}(X) \geq \sigma$, $\text{Rad}_{S^n}(\partial X) \geq R$ and $\text{Mcurv}(\partial X) \geq M > 0$, satisfy

$$M \leq \mathcal{M}_n(\sigma, R).$$

This doesn’t seem clear even for smooth bounded (and unbounded?) domains $X \subset \mathbb{R}^{n+1}$, where one naturally (naively?) expects that

round $n$ spheres $S^n(R)$ have maximal spherical radii among all closed (embedded!) hypersurfaces with $\text{Mcurv} \geq \frac{\pi}{R}$.

Below is another kind of conjecture, where the difficulty resides in the gap between a use of Dirac theoretic methods and those of minimal hypersurfaces.
Conjecture. There exists a (possibly very large) constant $\sigma_n > 0$ with the following property.

If a compact orientable Riemannian $(n+1)$-manifold with boundary $Y$ has $\text{Sc}(X) \geq \sigma_n$, then every smooth area decreasing map $f$ from $X$ to the unit $(n+1)$-sphere, which sends the 1-neighbourhood (unit collar) of $Y \subset X$ to a single point, has degree $\deg(f) = 0$.

We conclude by formulating an instance of a general stability problem for $\text{Sc} \geq \sigma$, in the spirit of [S 2016], which may have a satisfactory solution, at least in dimension 3.

Spherical Stability Problem. Describe the geometry of closed Riemannian orientable $n$-dimensional manifolds $Y_i$, such that

$$\inf \text{Sc}(Y_i) \to n(n-1) \quad \text{and} \quad \text{Rad}_{S^n}(Y_i) \to 1.$$  

One expects here, in view of Llarull’s theorem, that manifold $Y_i$ with $\text{Sc}(Y_i) \geq n(n-1) - \varepsilon_i$ and $\text{Rad}_{S^n}(Y_i) \geq 1 - \varepsilon_i$ must look, approximately, as the unit sphere $S^n$ with an extra staff attached by "$\varepsilon_i$-narrow bridges" to it.

In other words, certain equidimensional submanifolds $U_i \subset Y_i$ with (small?) boundaries must somehow converge (sub-converge?), e.g. in the intrinsic flat topology (see [S 2016]), to $S^n$.

Besides $S^n$, the stability problem arises for all, not only spherical, length extremal and area extremal closed manifolds with positive scalar curvatures as well as for extremal manifolds with boundaries (as in (d) of section 1), where the situation is even less clear.

It seems helpful for developing an idea of what happens to scalar curvature in this regard, up to a point of formulating conjectures, to look at the corresponding problem(s) for hypersurfaces with positive mean curvatures.

For instance,

what is the geometry of (limits of sequences of) smooth bounded domains $X_i \subset \mathbb{R}^{n+1}$, which contain the unit ball $B_1^{n+1} \subset \mathbb{R}^{n+1}$ and such that the mean curvatures of the boundaries of these domains are bounded from below by $M_i \to n$ for $i \to \infty$.

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