MONOTONICITY AND CRITICAL POINTS OF THE PERIOD FUNCTION FOR POTENTIAL SYSTEM

JIHUA WANG*

Abstract: This paper is concerned with the analytic behaviors (monotonicity, isochronicity and the number of critical points) of period function for potential system $\ddot{x} + g(x) = 0$. We give some sufficient criteria to determine the monotonicity and upper bound to the number of critical periods. The conclusion is based on the semi-group properties of (Riemann-Liouville) fractional integral operator of order $\frac{1}{2}$ and Rolle’s Theorem. In polynomial potential settings, bounding the number of critical periods of potential center can be reduced to counting the real zeros of a semi-algebraic system. From which we prove that if nonlinear potential $g$ is odd, the potential center has at most $\frac{\text{deg}(g) - 3}{2}$ critical periods. To illustrate its applicability some known results are proved in more efficient way, and the critical periods of some hyper-elliptic Hamiltonian systems of degree five with complex critical points are discussed, it is proved the system can have exactly two critical periods.

Keywords Period function · Potential system · Monotonicity · Critical period · Semi-algebraic system · Fractional calculus.

Mathematics Subject Classification 34C10 · 34C15 · 37G15.

1. Introduction and statement of the results

This paper is concerned with the period of periodic solutions of conservative second order equation $\ddot{x} + g(x) = 0$ or equivalently the period function of centers of planar potential system with the form

\begin{equation}
\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = g(x).
\end{equation}

The system has one degree of freedom “kinetic+potential” Hamiltonian

\begin{equation}
H(x, y) = \frac{y^2}{2} + G(x) = h,
\end{equation}

where $G(x) = \int_0^x g(s)ds$ is a primitive function of $g(x)$ which denotes the potential energy of this classical mechanical system.

*Corresponding author: wangjh2000@163.com (J. Wang).
Recall that a singular point of planar differential system is a center if it has a punctured neighborhood foliated with a continuous band of periodic orbits $\Gamma_h$ which is determined by equation (1.2). The largest punctured neighborhood with this property is called period annulus, denoted by $\mathcal{P}$. It is obvious that the vector field generated by system (1.1) is symmetric with respect to $x-$axis, we denote the projection of $\mathcal{P}$ onto $x-$axis by $(x_m, x_M)$.

Parameterizing the periodic orbits by energy level $h$ and assigning the minimal positive period of the motion along each of the periodic orbit $\Gamma_h$, then we have a energy-period function $T : (0, h_s) \mapsto (0, +\infty)$. As $g$ is a polynomial, the period of $\Gamma_h$ is given by the following Abelian integral

$$T(h) = -\oint_{\Gamma_h} \frac{dx}{y} = \sqrt{2} \int_{x_-(h)}^{x_+(h)} \frac{dx}{\sqrt{h - G(x)}},$$

where $x_-(h) < 0 < x_+(h)$ are the abscissas of the intersection points between $\Gamma_h$ and $x-$axis.

We say $T$ is monotone increasing (or decreasing) if for any couple of periodic orbits $\Gamma_{h_1}$ and $\Gamma_{h_2}$ in $\mathcal{P}$, there holds $T(h_1) < T(h_2)$ for $h_1 < h_2$. (resp. $T(h_1) > T(h_2)$). The local maximum or minimum of the period function is called critical period, the periodic orbit with this property is called critical periodic orbit [27]. If the period of oval $\Gamma_h$ is independent with the energy level $h$, namely $T(h)$ is constant, then the center is called isochronous, see the survey paper [10].

Aside from their intrinsic interest, the analytical properties (monotonicity, isochronicity or critical points) of period function arise from the study of oscillation features such as the soft or hard springs, also occur in the study of subharmonic bifurcations of periodic oscillations and the bifurcation of steady-state solutions of a reaction-diffusion equation, in the problem on the existence and uniqueness of solutions in boundary value problem [1, 2]. In some sense the study of critical periodic orbits is analogous to the study of limit cycles [4, 5], which is the main concern of the celebrated Hilbert’s 16th problem and its various weakened versions.

In what follows, we suppose that the annulus $\mathcal{P}$ only surrounds the center and no other equilibria, it follows the equality $G(x_m) = G(x_M)$ holds. Without loss of generality, we assume the center of system (1.1) locates at the origin, it is known that the multiplicity of $x = 0$ as the zero of $g(x)$ must be odd. Therefore we assume

**Assumption 1.1.** The potential $g(x)$ is smooth enough in $(x_m, x_M)$ and $k$ is non-negative integer.
Under the Assumption 1.1 if $k = 0$ the origin is an elementary (or nondegenerate) center for the linear part of the singular point has a pair of conjugate pure imaginary eigenvalues (or the Jacobian matrix at which is nondegenerate, i.e., its determinant does not vanish).

If $k \geq 1$ system (1.1) at the origin $O$ has nilpotent linear part $-\frac{2}{g^2}x^2$, accordingly it is called a nilpotent center. If the period annulus $\mathcal{P}$ is unbounded, the center is said to be global. Hamiltonian (1.2) has a local minimum $H(0,0) = 0$ at the origin, we assert that $G(x) = \frac{A}{2}x^{2k+2} + h.o.t$ and $A > 0$, thus there exists an analytic involution $\sigma$ on $(x_m, x_M)$ such that

\begin{equation}
G(x) = G(z), \quad z = \sigma(x) = -x + o(x), \quad x \in (x_m, x_M) \setminus \{0\}.
\end{equation}

Recall that an involution $\sigma$ is a diffeomorphism with a unique fixed point satisfying $\sigma \circ \sigma = \text{Id}$ and $\sigma \neq \text{Id}$. Following the terminology proposed in [24], we define the balance of function $f$ with respect to the involution $\sigma$ as

\begin{equation}
B_\sigma(f)(x) = f(x) - f(z).
\end{equation}

we can state the main results of the present paper.

**Theorem 1.1.** Consider the potential system (1.1), suppose $g$ is smooth in $(x_m, x_M)$ and satisfies Assumption 1.1 for all $x \in (0, x_M)$,

1. if $B_\sigma(\delta)(x) > 0 (< 0$, resp.) then the period function $T(h)$ is monotone increasing (or decreasing, resp.) in $(0, h_s)$;
2. if $B_\sigma(\delta)(x)$ has $l$ zeros, then system (1.1) has at most $l$ critical periods; moreover if $B_\sigma(\frac{G}{g^2})(x)$ has $l$ zeros as well, then the system (1.1) has exactly $l$ critical periods;
3. if $B_\sigma(\delta)(x) \equiv 0$, then the origin of system (1.1) is an isochronous center.

where

\begin{equation}
\delta(x) = \left(\frac{G}{g^2}\right)'(x) = \frac{g^2 - 2Gg'}{g^3}(x).
\end{equation}

Note that the function $\frac{G}{g^2}(x)$ and its derivative $(\frac{G}{g^2})'(x)$ play an important role in the study of the period function, which appear in many works in this field, see Loud [1], Coppel and Gavrilov [8], Chicone [9], Zeng and Jing [11], A. Cima et al. [14], Li and Lu [17], Yang...
and Zeng [18], Villadelprat and Zhang [27] and the references therein. If $k = 0$ from the Assumption 1.1, it can be derived that $\delta(x)$ is continuous in $(x_m, x_M)$ and continuously differentiable in $(x_m, 0)$ and $(0, x_M)$. In fact, by applying L’Hospital’s rule and Assumption 1.1 we get

$$
\delta(0) = -\frac{1}{3} \frac{g''(0)}{(g'(0))^2}, \quad \delta'(0) = \frac{5(g'')^2 - 3g'g''}{12(g')^3}(0).
$$

If $k \geq 1$, it is easy to deduce from Assumption 1.1 that

$$
\delta(0^+) = -\infty, \quad \delta(0^-) = \infty.
$$

Note that the outer boundary of period annulus must contain singular points. Thus from Assumption 1.1 if $g(x_m)g(x_M) = 0$ therefore

$$
B_\sigma(\delta)(x_M) = \delta(x_M) - \delta(x_m) = \infty.
$$

Consider potential system $\ddot{x} + g(x) = 0$, the authors [2] proved for generic $g$, the period function is morse function, which means all the critical points of $T$ are nondegenerate. If the potential $g$ is a polynomial in $x$, it is proved in [3] that the unique polynomial isochronous center of potential system is the linear and global one. Recall that only nondegenerate center can be isochronous. C. Chicone, F. Dumortier proved in Theorem 1.2 of [13] that the Hamiltonian system (1.1) with polynomial potential $g$ of degree two or more has finite number of critical periods. Chow and Sanders [3] ask whether there exists a bound on the number of critical periods and assume that the bound if it exists, depends on the degree of the polynomial $g(x)$. In what follows, we propose a positive answer to the problem. By applying Theorem 1.1 together with Rolle’s Theorem and its generalization, Theorem 1.3 we state the second main results as follows

**Theorem 1.2.** Consider the potential system (1.1) with polynomial potential $g$ of degree $\geq 2$, under the assumption (1.1), we have

1. if the potential $g$ is odd function, thus the system (1.1) has at most $\frac{\text{deg}(g)-3}{2}$ critical periods;

2. if all the zeros of $g$ are real, then system (1.1) has at most one critical period.

It is worth mentioning that the authors [3] give even potential conjecture that if the potential energy $G(x)$ is even, the center of the system at the origin has at most $\frac{\text{deg}(g)-3}{2}$ local critical periods. Here we confirm the result holds globally.
To ensure the monotonicity of period function, i.e., the absence of critical periodic orbits, from Theorem 1.1(1), it suffices to show

\[ B_\sigma(\delta)(x) = \delta(x) - \delta(\sigma(x)) > 0, \quad (< 0, \text{resp.}) \]

where \( x_m < \sigma(x) < 0 < x < x_M \). Hence it is easy to get Chicone’s criterion \[9\]

**Corollary 1.1.** If \( \delta'(x) > 0 \) (or \( \delta'(x) < 0 \), respectively) in \( (x_m, x_M) \), then \( T(h) \) is monotone increasing (or decreasing, respectively) in \( (0, h_s) \).

This criterion provides a sufficient condition for monotonicity which is easy to verify, while its disadvantages is that the condition is far away from being necessary, as a generalization see Theorem 1 and corollary 1 of \[11\] for instance.

One necessary condition on the monotonicity of period for the potential center \[1.1\] was stated in Theorem 2.6 of \[25\] as follows

**Lemma 1.1.** If the period function \( T(h) \) of system \[1.1\] is monotone in \( (0, h_s) \), then the function \( x \mapsto G(x) \), where \( G(x) = h_s \) corresponds to the energy level of outer contour of period annulus.

In literatures it is noted in \[13\] that any given analytic vector field of center type has a finite number of critical periods on a period annulus contained in a compact region. To our knowledge, consider a family of centers with critical periods, the method often applied therein bases on proposition where the period function satisfies some kind of Picard-Fuchs differential equation for algebraic curves \[15, 16\]. This approach seems hard to be used in the cases of polynomial potential \( g \) with high degree or non-polynomial settings. Cima et al.,\[14\] obtain some lower bounds for the number of critical periods of the families of potential, reversible and Liénard centers by perturbing linear one. Concerning the Hamiltonian systems with separable variables \[7, 27\], the results there exists at most one critical period are proved by means of studying the period function’s convexity. Mañosas and Villadelprat \[25\] propose a criterion to bound the number of critical periods, while the criterion function is defined recursively which increases computational complexity rapidly.

Theorem \[1.1(2)\] proposes an efficient criterion for (sharp) upper bound to the number of critical periods, which can be reduced to counting the number of zeros of balance function. If potential \( g \) is polynomial, it reduces to solving the following semi-algebraic system (SAS
for short).

\begin{equation}
G(x) = G(z), \quad \delta(x) = \delta(z)
\end{equation}

in the domain

\[0 < x < X_M, \quad x_m < z < 0.\]

By SAS, we mean the system consists of polynomial equations, polynomial inequations and inequalities \[28\]. There are various problems in both practice and theory which need to solving a semi-algebraic system. A SAS formally can be represented as \[[F, N, P, H](U, X)\] where \(U = (u_1, u_2, \cdots, u_d)\) is vector of parameters, and \(X = (x_1, x_2, \cdots, x_n)\) is vector of variables. \(F, N, P\) and \(H\) stands for the equations \([f_1, f_2, \cdots, f_s]\), non-negative inequalities \([g_1, g_2, \cdots, g_r]\), positive inequalities \([g_{r+1}, \cdots, g_t]\) and inequations \([h_1, h_2, \cdots, h_m]\), respectively, where \(f_i, g_j, h_k \in Z(U)[X]\) are all polynomials with integer coefficients with subordinates \(d, r, t, m \geq 0\) and \(n, s \geq 1\). If \(d = 0\), the SAS is called constant. If \(d > 0\), the SAS is called parametric, see Appendix A of \[32\] for more details.

In the viewpoint of symbolic computation, how to isolate and count the distinct roots of a SAS are two crucial issues. By using command \texttt{RealRootClassification} of the package \texttt{SemiAlgebraicSetTools} in computer algebra system Maple 13 or its higher versions, one can solve the SAS with parameters, Parametric SAS for short. \texttt{RealRootClassification} computes conditions on the parameters for the system to have a given number of real solutions. The algorithm implemented by Maple is described in \[29\]. The algorithm is complete in theory, however, the command \texttt{RealRootClassification} may quit in some case due to too heavy computation. If the parameters are assigned, the constant SAS can be easily solved by the command \texttt{RealRootCounting} of the package \texttt{SemiAlgebraicSetTools} which returns the number of distinct real solutions. The corresponding algorithm is described in the paper coauthored by Xia and Hou \[30\]. In order to know exactly the distinct real roots, the \texttt{RealRootIsolate} command returns a list of boxes. Each box isolates exactly one real root of semi-algebraic system. All the calling sequence, descriptions and examples of command can be viewed by consulting “?Command name” in programme interface of Maple.

Alternatively one can seek some (explicit) conditions on \(g\) so that \(B_\sigma(\delta)(x)\) is non-vanishing or has zeros. To this end Rolle’s Theorem or Intermediate-value Theorem are often applied to count the number of zeros of univariate functions. To solve the SAS \[1.8\] in a geometric way,
it need to count the number of intersections of two planar smooth curves, thus the following
generalized Rolle’s Theorem (See Theorem 6.11 in [31] for instance) is useful.

**Theorem 1.3. (Generalized Rolle’s Theorem.)** Let $D = (x_l, x_r) \times (y_b, y_t)$. Let $F(x, y), G(x, y)$
be two functions continuous on closure $\bar{D}$ and smooth in $D$. Suppose that $F'_x(x, y) \neq 0$, $F'_y(x, y) \neq 0$ in $D$. Consider the number of solution of system

$$F(x, y) = 0, \ G(x, y) = 0, \ (x, y) \in D.$$ 

Denote the number by \(\#\{(F, G)|(x, y) \in D\}\) and let the Jacobian

$$J[F, G](x, y) = F'_x(x, y)G'_y(x, y) - F'_y(x, y)G'_x(x, y),$$

then

$$\#\{(F, G)|(x, y) \in D\} \leq 1 + \#\{(F, J[F, G])|(x, y) \in D\}.$$ 

The rest part of paper is organized as follows. In section 2 we consider the properties of
(Riemann-Liouville) fractional derivative with which to prove Theorem 1.1, where properties of Variation Diminishing of Abel’s integral operator or fractional calculus with the order $\frac{1}{2}$ is given. This is the highlight of present paper. In Section 3 some asymptotical results of period function near the center and the outer boundary are proposed, together with Theorem 1.1, the global analytical behaviors of period function can be determined. The proof of main results of present paper, Theorem 1.1 is given in Section 4, the main idea is to deduce the formula (4.5) about $T'(h)$ which is formulated as the integral of Balance function with order $\frac{1}{2}$. As the applications of Theorem 1.1 in Section 5 some known results are proved in more efficient way. To the best of our knowledges, there are few works on existence of multiple critical periods in literatures. The period function of some hyperelliptic Hamiltonians of degree five with complex critical points are considered, it is proved the system can have two critical periods. Some comments are stated in Section 6.

2. Fractional calculus and properties of Variation Diminishing of Abel’s integral operator

On account of the upper bound to the number of critical periods of potential center (1.1),
Theorem 3.7 in [18] takes advantage of Zhang’s differential operator to deduce the higher order derivative of period function. Theorem A of [25] bases on the Chebyshev properties of
the tuple of criterion functions defined recursively. In this section we intend to consider an integral operator with fractional order with which to prove Theorem 1.1(2).

Recall that in studying problem of Isochrone N. H. Abel at 1823 derived Abel’s integral equation

\begin{equation}
A(k)(h) := \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{h} \frac{k(x)}{\sqrt{h-x}} \, dx, \quad \forall k \in C^{0}[0,x_{M}].
\end{equation}

where \(A(k)(h)\) denotes the Riemann-Liouville integral of function \(k(x)\) with order \(\frac{1}{2}\). In the viewpoint of inverse problem, it is a singular Volterra integral equation of convolution type with weakly singular integral kernel \(\frac{1}{\sqrt{h-x}}\). If free term \(A(k)(h)\) is continuously differentiable and vanishes at \(h = 0\) then equation (2.1) has exactly one continuous solution formed as

\begin{equation}
k(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \cdot \frac{d}{dx} \int_{0}^{x} \frac{A(k)(t)}{\sqrt{x-t}} \, dt = \frac{1}{\sqrt{\pi}} \left[ \frac{A(k)(0)}{\sqrt{x}} + \int_{0}^{x} \frac{A(k)'(t)}{\sqrt{x-t}} \, dt \right], \quad x \in [0,x_{M}).
\end{equation}

Note that \(A(k)(h)\) vanishes at \(h = 0\) is the necessary condition to the existence of solution (2.2) at \(x = 0\).

Fractional calculus [19] used to denote calculus of non-integer order which is in contrast to the classical calculus as created independently by Newton and Leibniz. We recall the definitions of fractional integral and derivative as follows [19, 20].

**Definition 2.1.** Suppose \(\alpha > 0\) and \(f(x) \in L^{1}(0, +\infty)\). Then for all \(x > 0\) we call

\begin{equation}
I^{\alpha}_{0}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(s)}{(x-s)^{1-\alpha}} \, ds
\end{equation}

the (below) Riemann-Liouville integral of \(f\) of order \(\alpha\).

**Definition 2.2.** Suppose \(\alpha > 0\) and suffices \(n - 1 \leq \alpha < n\) where \(n\) is a positive integer. if \(f(x) \in C^{n}(R)\) we call

\begin{equation}
D^{\alpha}_{0}(f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dx^{n}} \int_{0}^{x} \frac{f(s)}{(x-s)^{\alpha+n-1}} \, ds
\end{equation}

the (below) Riemann-Liouville derivative of \(f\) of order \(\alpha\).

Remark that the order of fractional differ-integration can be rational, irrational or complex number. When order \(\alpha\) is a positive integer, the differ-integration (2.3) and (2.4) are in accordance with the classical case. Noting that fractional derivatives are non-local in contrast to the classical derivative. For example, the fractional derivative of constant function is not zero. Riemann–Liouville derivative depends on a free parameter which relies on global
information of the function. Nevertheless the (below) Riemann-Liouville fractional differential operator still satisfies the following classical proposition [20].

**Lemma 2.1.** The (below) Riemann-Liouville fractional integral operator is commutable, suppose that function \(f\) is continuous in \((a, b)\) for any \(\mu, \nu > 0\) there holds

\[
I_\mu^a I_\nu^a f(x) = I_{\mu+\nu}^a f(x) = I_\nu^a I_\mu^a f(x).
\]

**Proof.** By Definition 2.3 and change the order of double integral we get

\[
I_\mu^a I_\nu^a (f(t)) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_a^t (t-\tau)^{\mu-1} \int_a^\tau f(s)(\tau-s)^{\nu-1}dsd\tau
\]

\[
= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_a^t f(s) \int_s^t (t-\tau)^{\mu-1}(\tau-s)^{\nu-1}d\tau ds
\]

By applying change of coordinates \(\xi = (\tau - s)/(t - s)\) and proposition of Beta function, furthermore we obtain

\[
I_\mu^a I_\nu^a (f(t)) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_a^t (t-s)^{\mu+\nu-1}f(s) \int_0^1 (1-\xi)^{\mu-1}\xi^{\nu-1}d\xi ds
\]

\[
= \frac{B(\mu, \nu)}{\Gamma(\mu)\Gamma(\nu)} \int_a^t (t-s)^{\mu+\nu-1}f(s) ds
\]

\[
= I_{\mu+\nu}^a f(t).
\]

where \(\Gamma(\cdot), B(\cdot, \cdot)\) represent gamma function and Beta function, respectively. \(\square\)

As in classical calculus, we have the equality \(D_\alpha^a[I_\alpha^a f(x)] = f(x)\) for any \(\alpha > 0\) as well, namely the derivative operator \(D_\alpha^a\) is the left inverse of \(I_\alpha^a\). It is obvious \(I_\alpha^a\) is a positive linear operator and the set \(\{I_\alpha^a, \alpha > 0\}\) is formed as strongly continuous semi-group with respect to \(\alpha\).

From the Lemma 2.1 we assert the following results.

**Theorem 2.1.** Abel’s integral operator (2.1) is Variation Diminishing. Suppose that function \(k \in L^1(0, h_s)\), then the number of zeros of function \(k\) in \((0, h_s)\) is an upper bound to that of image function \(A(k)(h)\) (i.e., the Riemann-Liouville integral of \(k(x)\) with order \(\frac{1}{2}\)); moreover the bound is sharp when the primitive function of \(k\) has the same number of zeros as \(k\) in \((0, h_s)\).

**Proof.** We prove the conclusion by contradiction. Denote the number of zeros of function \(f\) in the interval \((a, b)\) by \(\sharp f|_{(a,b)}\). Assume that

\[
\sharp A(k)|_{(0,h_s)} > \sharp k|_{(0,h_s)}.
\]
Without of generality say $\sharp A(k)_{|_{(0,h_s)}} = \sharp k_{|_{(0,h_s)}} + 1$. Note that $I_0^1(k) = A \circ A(k)$ is the primitive function of $k$. By assumption we have

$$\sharp I_0^1(k)_{|_{(0,h_s)}} \geq \sharp k_{|_{(0,h_s)}} + 2,$$

which contradicts the classical Rolle’s Theorem. It leads to the first conclusion.

From the above discussions, we have

$$\sharp I_0^1(k)_{|_{(0,h_s)}} \leq \sharp A(k)_{|_{(0,h_s)}} \leq \sharp k_{|_{(0,h_s)}},$$

The second result follows.

3. The asymptotical results of period function near the boundary

Note that Theorem 1.1(2) proposes an practical approach to bound the number of critical periods. If taking into account the monotonicity of period function near the two endpoints of interval $(0, h_s)$, one can determine analytical behaviors of $T(h)$ globally in $(0, h_s)$. On the Poincaré-Lyapunov disc, the period annulus of center has two connect components, the inner one is center itself and the outer boundary is a polycycle.

The first terms of the Taylor expansions of period function near the nondegenerate center is well known, see Proposition 3.5 in [18] and formulas (25a) and (25b) of [6], Theorem 4.2 [13] for instance. For completeness we state as follows.

**Lemma 3.1.** Assume function $g(x)$ of potential system (1.1) is analytic and satisfies the Assumption 1.1.

If $k = 0$, then period function (1.3) is analytic in $(0, h_s)$ and can be analytically extended to $h = 0$. Moreover function $T(h)$ and its derivation $T'(h)$ as $h \to 0^+$ satisfy the following limits

$$T(0^+) = \frac{2\pi}{\sqrt{g'(0)}}, \quad T'(0^+) = \pi \frac{5(g''(x))^2 - 3 g'(x)g'''(x)}{12(g'(x))^2} \big|_{x=0}.$$

If $k \geq 1$, then

$$T(0^+) = \lim_{h \to 0^+} h^{\frac{k}{2k+2}} \lambda_k = +\infty, \quad T'(0^+) = \lim_{h \to 0^+} -\frac{k\lambda_k h^{\frac{2k+2}{2k+2}}}{2k+2} = -\infty.$$

where $\lambda_k$ is a positive constant.

Concerning the limit of the period function $T(h)$ as $h$ tends to energy level $h_s$ corresponding to the outer boundary of period annulus $\Gamma_s$, we discuss in two cases.
THE PERIOD FUNCTION FOR POTENTIAL SYSTEM

If \( h_s < +\infty \) which means the outer boundary \( \Gamma_s \) is a polycycle connecting to some singularities. Let us mention a well known result in the following lemma, the proof is referred to Lemma 5.1 of [18].

**Lemma 3.2.** If potential \( g \) of system (1.1) is polynomial and satisfies Assumption 1.1, in addition it exists some other isolated zeros, then the period annulus is bounded. The period of center \( O(0,0) \) is divergent to infinity near the outer boundary.

If \( h_s = +\infty \), we have the following conclusions [22].

**Lemma 3.3.** Consider conservative second order autonomous duffing equation \( \ddot{x} + g(x) = 0 \), suppose that \( g \) is locally Lipschitz and \( xg(x) > 0 \) in \((-\infty, +\infty)\). In addition if it satisfies

(i). the superlinear condition \( \lim_{|x| \to +\infty} \frac{g(x)}{x} = +\infty \), the system (1.1) is said to be superlinear, then \( \lim_{h \to +\infty} T(h) = 0 \);

(ii). the sublinear condition \( \lim_{|x| \to +\infty} \frac{g(x)}{x} = 0 \), the system (1.1) is said to be sublinear, then \( \lim_{h \to +\infty} T(h) = +\infty \);

(iii). the semilinear condition \( 0 < \liminf_{|x| \to +\infty} \frac{g(x)}{x} \leq \limsup_{|x| \to +\infty} \frac{g(x)}{x} < +\infty \), the system (1.1) is said to be semilinear, then \( 0 < \liminf_{h \to +\infty} T(h) \leq \limsup_{h \to +\infty} T(h) < +\infty \).

In this respect we refer the reader to see Theorem 5.1 of [27] or Theorem 1.2 of [18] for more details.

4. THE PROOF OF THEOREM 1.1

From the expression (1.3) of period function \( T(x) \), since \( x \to \frac{1}{\sqrt{h-G(x)}} \) has singularities at \( x_-(h) \) and \( x_+(h) \), it is necessary to make some transformations. Following the change in coordinates applied in [18], let

\[
G(x) = r \sin^{2k+2}\theta, \quad \frac{y}{\sqrt{2}} = \sqrt{r} \cos \theta,
\]

\[
x \sin \theta \geq 0, \quad \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}], \quad r \in (0, h_s).
\]

Under which the periodic orbit

\[
\Gamma_h : \quad \frac{y^2}{2} + G(x) = h
\]

is mapped to a cycle \( r = h \). Along the cycle we have

\[
g(x)dx = 2(k+1)h \sin^{2k+1}\theta \cos \theta d\theta,
\]
then the period function $T(h)$ yields to

\begin{equation}
T(h) = 2(k + 1)\sqrt{2h} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^{2k+1}\theta}{g(x)\sqrt{1 + \sin^2 \theta + \cdots + \sin^{2k} \theta}} d\theta,
\end{equation}

where $x = x(\theta, h)$ is implicitly determined by the mapping (4.1), recall that formula (4.2) is proved as the formula (3.11) of Proposition 3.2 in [18].

On the other hand, from the mapping (4.1), we get

\[
\frac{\partial x}{\partial h} = \sin^{2k+2} \theta g(x) = G(x) \frac{g(x)}{g(x)h}
\]

Hence we get $T'(h)$ can be represented as follows, see also formula (3.14) of Proposition 3.3 in [18].

\begin{equation}
T'(h) = (k + 1)\sqrt{2h} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\delta(x) \sin^{2k+1}\theta}{\sqrt{1 + \sin^2 \theta + \cdots + \sin^{2k} \theta}} d\theta,
\end{equation}

where

\[
\delta(x) = \left(\frac{G}{g^2}\right)'(x) = \frac{g^2 - 2Gg'}{g^3}(x), \quad x \in (x_m, x_M).
\]

Note that

\[
G(x) = G(\sigma(x)) = h \sin^{2k+2} \theta, \quad x_-(h) < \sigma(x) < 0 < x < x_+(h),
\]

and $x \sin \theta > 0$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ which implies $x(-\theta, h) = \sigma(x(\theta, h))$. It yields to

\begin{align*}
T'(h) &= (k + 1)\sqrt{2h} \left( \int_{-\frac{\pi}{2}}^{0} + \int_{0}^{\frac{\pi}{2}} \right) \frac{\delta(x) \sin^{2k+1}\theta}{\sqrt{1 + \sin^2 \theta + \cdots + \sin^{2k} \theta}} d\theta, \\
&= (k + 1)\sqrt{2h} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{B_\sigma(\delta)(x) \sin^{2k+1}\theta}{\sqrt{1 + \sin^2 \theta + \cdots + \sin^{2k} \theta}} d\theta,
\end{align*}

where $x \in (0, x_M)$ and $B_\sigma(\delta)(x)$ is the balance of function $\delta$ with respect to the involution $\sigma$.

From the mapping $G(x) = h \sin^{2k+2} \theta$, differentiating both sides we get

\[
g(x)dx = 2(k + 1)h \sin^{2k+1} \theta \cos \theta d\theta.
\]

combining with (4.4), it brings to

\begin{equation}
T'(h) = \frac{1}{\sqrt{2h}} \int_{0}^{x_+(h)} B_\sigma(\delta)(x) \frac{g}{\sqrt{h - G}}(x) dx, \quad x_+(h) \in (0, x_M).
\end{equation}

The expression (4.5) seems not to be new, while it is an improvement of the expression (7) of [11] based on Loud's result [1] where an additional condition is required that the center is elementary.
Therefore from formula (4.5), the conclusions (1),(3) of Theorem 1.1 are obvious. It is crucial to derive the second conclusion about the (sharp) upper bound to the number of critical periods.

If the center is elementary, $T'(h)$ is continuous in $(0, h_s)$ and can be continuously extended to $h = 0$. Moreover $B_\sigma(\delta)(0) = 0$.

If the center is nilpotent, from formula (1.6) then we have

$$B_\sigma(\delta)(0) = G(0) = 0.$$  

Concerning the expression (4.5) of $T'(h)$, we have

$$\frac{\partial x}{\partial h} = \frac{1}{g(x)} > 0 \text{ in } (0, x_M),$$

it implies $G := (0, x_M) \rightarrow (0, h_s)$ is a diffeomorphism preserving orientation which does not change the number of zeros. From expression (4.5), we get

\begin{equation}
\sqrt{2h} T'(h) = \int_0^h B_\sigma \delta(x^{-1}(u)) \frac{du}{\sqrt{h-u}},
\end{equation}

where $x^{-1}(u) \in (0, x_M)$ is the inverse function determined by $G(x) = u, u \in (0, h_s)$. Thus by Theorem 2.1 we obtain the number of zeros of function $B_\sigma \delta(x^{-1}(u))$ in $(0, h_s)$ (therefore $B_\sigma(\delta)(x)$ in $(0, x_M)$) is an upper bound to that of $T'(h)$ in $(0, h_s)$. Moreover the bound is achievable if function

$$\int_0^h B_\sigma \delta(x^{-1}(u))du = \int_{x^{-}(h)}^{x^{-}(h)} \delta(x)dx = \frac{G}{g^2(x)} - \frac{G}{g^2}(z)$$

also has $n$ zeros in $(0, x_M)$ where $G(x) = G(z), x_m \leq z < 0 < x \leq x_M$. It completes the proof of conclusion (2) of Theorem 1.1.

To assert that the period function of center is monotone indeed (absence of critical periods), one sufficient condition on monotonicity of period through verifying the convexity of $T(h)$ is helpful in many literatures \cite{7, 27}. Similarly as above deductions, we get the derivative of $T(h)$ of order two as follows

\begin{equation}
T''(h) = \frac{1}{\sqrt{2h^2}} \int_0^{x_+(h)} B_\sigma(\phi)(x) \frac{g}{\sqrt{h-G}}(x) dx, \quad x_+(h) \in (0, x_M),
\end{equation}

where

$$\phi(x) = \delta'(x) \frac{G}{g}(x) - \frac{\delta(x)}{2} = \frac{12(Gg')^2 - 4gG(g''G + g'g) - g^4}{2g^5}(x).$$

**Proof of Theorem 1.2** We will prove in the sequel by applying Rolle's Theorem and its generalization.

Firstly we generalize Lemma 4.5 of \cite{18} to the following
Lemma 4.1. Assume that \( g(x) \) is a polynomial of degree no less than 2, all its zeros are real and satisfies the condition (1.1), then
\[
\delta'(x) > 0, \quad x \in (x_m, 0) \cup (0, x_M).
\]

Proof. (1) Suppose that the potential \( g(x) \) is an odd polynomial with degree no less than three, recall that \( g(0) = 0 \), that follows \( g(x) \) has at most \( \frac{\deg(g) - 1}{2} \) positive zeros. From Theorem 1.1(2) we get the number of critical periods for system (1.1) is bounded by the number of positive zeros of function
\[
\delta(x) = \left(\frac{G}{g^2}\right)'(x) = \frac{g^2 - 2Gg'}{g^3}(x).
\]
at \((0, x_M)\). Let \( N(x) = (g^2 - 2Gg')(x) \) be the numerator of function \( \delta(x) \), note that \( g(x) > 0 \) at \((0, x_M)\). Then \( N'(x) = -2Gg''(x) \), recall that \( G(x) \) is positive at \((0, x_M)\) and \( N(0) = 0 \) holds, thus the number of zeros of function \( N'(x) \) is no more than \( \frac{\deg(g) - 3}{2} \) at \((0, x_M)\), that follows function \( \delta(x) \) has at most \( \frac{\deg(g) - 3}{2} \) zeros at \((0, x_M)\).

(2) By using Generalized Rolle’s Theorem 1.3 the number of solutions of SAS
\[
\#\{(x, z) \in (0, x_M) \times (x_m, 0) \mid G(x) = G(z), \quad \delta(x) = \delta(z)\}
\]
is no more than one plus the number of the solutions of the SAS
\[
\#\{(x, z) \in (0, x_M) \times (x_m, 0) \mid G(x) = G(z), \quad \delta'(x)g(z) = \delta'(z)g(x)\}.
\]

Since for all \( x \in (x_m, 0) \) and \((0, X_M)\) there holds \( xg(x) > 0 \) and \( \delta'(x) > 0 \) deduced by Lemma 4.1. Then from Theorem 1.3 we assert the former SAS (1.8) has at most one zero, therefore Theorem 1.1(2) concludes that period function \( T(h) \) has at most one critical points. \( \square \)

5. Applications

In this section we give several applications of Theorem 1.1. By means of analytical and computational techniques, the analytical behaviors of period function are proved.

Firstly to illustrate our approaches, we propose more simpler proofs to two known results.

Example 5.1 We consider the family of potential system with even potential energy
\[
\begin{align*}
\frac{dx}{dt} &= -y, \\
\frac{dy}{dt} &= x + kx^2 + x^5.
\end{align*}
\]

For \( k \in (-2, +\infty) \) the system (5.1) is superlinear and the origin \( O(0,0) \) is a global center. In Theorem 1.1(b) of [16] and Theorem 1.5 of [21], it is proved that the system (5.1) associated
with the global center has at most one critical period. By using the Harmonic Balance Method
Theorem 1.5 of [21] is stated as follows

(i) The function is monotonous decreasing for $k \geq 0$.
(ii) System (5.1) has exactly one critical period for $k \in (-2, 0)$, the period function starts
increasing, until a maximum (a critical period) and then decreases towards zero.

From Lemma 3.1 and 3.3 we have

\begin{align*}
(5.2) \quad \lim_{h \to 0^+} T(h) &= 2\pi, \quad \lim_{h \to 0^+} T'(h) = -\frac{3k\pi}{2}, \quad \lim_{h \to +\infty} T(h) = 0.
\end{align*}

An easy computation shows for all $x \in (0, +\infty)$,

$$B_\sigma(\delta)(x) = 2\delta(x) = \frac{x(4x^6 + 9kx^4 + 3k^2x^2 + 20x^2 + 9k)}{3(x^4 + kx^2 + 1)^3}.$$ 

It is obvious for $k \geq 0$ Theorem 1.1(1) implies that period function of system (5.1) is mono-
tone decreasing globally.

As $-2 < k < 0$ we get from the expressions (5.2) that system (5.1) has at least one critical
period. In what follows we intend to prove the assertion that the function $B_\sigma(\delta)(x)$ has exactly
one positive zero. By change of variable $x^2 \to u$ the sextic polynomial $4x^6 + 9kx^4 + 3k^2x^2 +
20x^2 + 9k$ is changed to cubic polynomial with parameter $k \in (-2, 0)$

$$\omega(u, k) = 4u^3 + 9ku^2 + (3k^2 + 20)u + 9k.$$ 

By Intermediate-Value Theorem it has at least one positive zero. Furthermore by Descartes’
law of sign it might even have three positive zeros (multiplicities taken into account). Differenti-
ating function $\omega(u, k)$ with respect to $u$ we get

$$\frac{\partial \omega(u, k)}{\partial u} = 12u^2 + 18ku + (3k^2 + 20),$$ 

which does not vanish at $(0, \infty)$ for all $k \in (-2, 0)$. The contradiction confirms our assertion.
Hence from Theorem 1.1(2) system (5.1) has exactly one critical period for $k \in (-2, 0)$ which
is a maximum.

**Example 5.2** We consider a potential system with isochronous center where the potential
is not polynomial.

\begin{align*}
(5.3) \quad \frac{dx}{dt} &= -y, \quad \frac{dy}{dt} = (1 + \frac{x}{4}) - (1 + \frac{x}{4})^{-3}.
\end{align*}

By direct computation we get the potential energy

$$G(x) = (x + \frac{x^2}{8}) + 2(1 + \frac{x}{4})^{-2} - 2.$$
The origin $O(0,0)$ is nondegenerate center surrounded by an unbounded period annulus whose projection onto $x$-axis is interval $(-4, +\infty)$. By direct computation, we get

$$\delta(x) = \frac{G}{g^2}'(x) = \frac{128(4 + x^3)}{(x^2 + 8x + 32)^3}.$$  

Since $xg(x) > 0$ holds for $x \in (-4, +\infty) \setminus \{0\}$, there exists an analytic involution $\sigma$ satisfying $z = \sigma(x) \in (-4,0)$ and $G(x) = G(z)$. It is easy to verify that Semi-algebraic system (1.8) always have solutions for all $(x, z) \in (0, \infty) \times (-4, 0)$. As a matter of fact the equations $G(x) = G(z)$ and $\delta(x) = \delta(z)$ have common factor $xz + 4(x + z)$ excluding $x - z$. It implies $B_\sigma(\delta)(x)$ identically equals to zero for all $x \in (0, +\infty)$, thus from Theorem 1.1(3) we assert the origin is an isochronous center.

Note that system (5.3) can be changed to the so-called dehomo genized Loud’s system with parameters $(D, F) = (0, \frac{1}{4})$ by some coordinate transformations, as proposed in Lemma 2.3 of [26]. It is known that the origin is one of four Loud’s isochronous centers.

It is remarked in [12] (pp 382–383) that perhaps the easiest families of potential Hamiltonian other than the families of polynomial function that one could study next are those potential energy $G$ is an entire function or a rational function without real poles. An interesting question for further research is to construct the isochronous center in these families differ from the linear one.

**Example 5.3.** As the applications of Theorem 1.2 and Lemma 4.1 we consider system (1.1) in case of all the zeros of the potential $g$ are real.

(1) When $g(x) = x(1-x^2)$, the origin of system (1.1) is an elementary center, test function $\delta'(x) > 0$ for all $x \in (-1,1)$, thus the system has a monotone increasing period function, i.e. absence of critical periods.

(2) when $g(x) = x^3(1-x^2)$ or $x^3(x+1)$, the origin of system (1.1) is nilpotent center, test function $\delta'(x) > 0$ for all $x \in (x_m, 0)$ or $(0, x_M)$, the system has exactly one critical period which is a minimum.

In what follows, we consider the hyperelliptic Hamiltonian system of degree five

$$\begin{align*}
\frac{dx}{dt} &= -y, \\
\frac{dy}{dt} &= x(x+1)(x^2 + \beta x + \alpha)
\end{align*}$$  

with real parameters $\alpha, \beta$. The normal form of hyperelliptic Hamiltonian of degree five was proposed in [23]. As $\beta^2 - 4\alpha \geq 0$, it was proved in [17, 18] the period function of any period annulus of the system has at most one critical point, and it has exactly one if and only if the period annulus surrounds three equilibria, counted with multiplicities. The proof in
latter work is purely analytical, different from that computer algebra is utilized in the former work. Moreover it is conjectured in [17] the system (5.4) has at most two critical periods. As $\beta^2 - 4\alpha < 0$, system (5.4) has a pair of conjugated complex critical points, it has only one phase portrait topologically: the period annulus surrounds a non-degenerate center $O(0,0)$ and terminates at a homoclinic loop connecting to a hyperbolic saddle $S(-1,0)$. In this case the period function may be monotone, or have one or two critical points, depending on the values of parameters $\alpha$, $\beta$.

The phrase portrait of system (5.4) with $\beta^2 - 4\alpha < 0$ is sketched as Fig.1 of [32].

Noticing that when polynomial potential is asymmetric, the involution $\sigma$ in general can not be explicitly stated, the problem on critical period can be reduced to solving (constant or parameteric) Semi-algebraic system.

Recall that the conclusion is known in [17, 18] where the period function of potential system (5.4) with $\beta^2 - 4\alpha \geq 0$ was tackled by different approaches. The proofs therein are long and highly nontrivial.

Example 5.4. In literature there are few results concerning the existence of exact two critical period orbits. we have the following conclusion.

**Theorem 5.1.** when $\beta = -1$, $\alpha = \frac{1}{2}$ the system (5.4) has exactly two critical periods.

**Proof.** When $\beta = -1$, $\alpha = \frac{1}{2}$, the system (5.4) has a pair of complex critical points. By Lemma 3.1 we get near the elementary center $O$ it holds

$$T'(0^+) = \frac{\pi}{6} \left(10\alpha^2 + 11\alpha\beta + 10\beta^2 - 9\alpha\right) \alpha^{7/2}.$$ 

In this case we have $T'(0^+) > 0$. Note that $T(h^-) = +\infty$. Hence $T(h)$ can be monotone increasing or have even number of critical periods in $(0, h_a)$, counted with multiplicities.

Recall that the author of present paper in [32] considered how to bound the number of limit cycles bifurcating from the period annulus under small polynomial perturbations, whether the system (5.4) with $\beta = -1$, $\alpha = \frac{1}{2}$ has two critical periods is left open.

By an easy computation, we get $x_M \approx 0.9239964237$. By using Maple 18, we get

$$\frac{\delta(x) - \delta(z)}{x - z} = \Psi(x, z),$$

where $\Psi(x, z)$ is a symmetric polynomial of $x, z$ with degree 13. By command RealRootIsolate in Maple 18 we assert the equations $U(x, z)$ and $\Psi(x, z)$ in the domain $(0, x_M) \times (-1, 0)$
have two isolated zeros contained in the following domains

\[(0.3560526240, -0.2935057702) \in \left[\frac{25}{128}, \frac{51}{256}\right] \times \left[-\frac{91}{128}, -\frac{181}{256}\right]\]

and

\[(0.7682670211, -0.6425079942) \in \left[\frac{51}{256}, \frac{13}{64}\right] \times \left[-\frac{229}{256}, -\frac{57}{64}\right],\]

respectively. Note that \(x_1^N \approx 0.3560526240\) and \(x_2^N \approx 0.7682670211\) are all nodal zeros of function \(B_\sigma(\delta)(x)\) in \((0, x_M)\). Hence from Theorem 1.1(2), the system (5.4) can have at most two critical periods.

We can refine the conclusion by excluding the period function is monotonic. On account of Lemma 1.1 if the center of system (5.4) at the origin has a monotone period function, then the test function \(P(x, z) = \frac{G(x)}{(x-z)^2}, \, z = \sigma(x)\) should be monotone in \((0, x_M)\) as well. With the aid of Maple we obtain a negative result through verifying that its total derivative with respect to \(x\), namely \(P_x(x, z) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} \frac{dz}{dx}\) has two common zeros with \(G(x) = G(z)\) at domain \((x, z) \in (0, X_M) \times (-1, 0)\). Therefore we assert that when \(\beta = -1, \, \alpha = \frac{1}{2}\) the center of system (5.4) at the origin has exactly two critical periods. \(\Box\)

6. Remarks and Discussions

We remark here that Theorem 1.1(2) proposes an efficient criterion for bounding the number of critical periods of potential center, while how to obtain the sharp bound is much more difficult.

Consider the critical periods of the center of Hyperelliptic Hamiltonian system (5.4) of degree 5, we leave Some unsolved cases. For instance when \(\beta = \frac{7}{5}\) or \(-\frac{7}{5}\) and \(\alpha = \frac{1}{2}\) an easy computation lead to \(T'(0^+) > 0\) and \(T(h^-) = +\infty\) as well. Whether the period function of center at the origin is monotone increasing or has two critical points is left open. The trials and try fails we can assert by Theorem 1.1(2) it has at most two critical periods. While we can not dismiss the claims that the period function is monotone increasing as the mentioned above case \(\beta = -1, \, \alpha = \frac{1}{2}\) by Lemma 1.1 for the test function \(P(x, z)\) is monotone in \((0, X_M)\).

The effort to confirm monotonicity of period has also defeated by verifying the convexity of period function through expression (4.7), unfortunately in these cases test function \(B_\sigma(\phi)(x)\) has two zeros in \((0, X_M)\).

How to derive more efficient and convenient approaches to determine the sharp bound to the number of critical periods, it is an important and challenging topic, we leave it for further research.
Acknowledgements

The author would like to thank the anonymous referees for their helpful corrections and valuable suggestions which improve the presentation of this paper.

References

[1] Loud, W. S.: Periodic solution of \( x'' + cx' + g(x) = \varepsilon f(t) \), Mem. Ams. Math. Soc. 31, 1–57 (1959)
[2] Brunovsky, P., Chow, S. N.: Generic properties of stationary state solutions of reaction-diffusion equation, J. Differ. Equ. 53, 1–23 (1984)
[3] S. N. Chow, J. A. Sanders, On the number of critical points of the period, J. Differential equations 64 (1986) 51–66.
[4] Chicone, C., Jacobs, M.: Bifurcation of critical periods for plane vector fields, Trans. Amer. Math. Soc. 312 (2), 433–486 (1989)
[5] Grau, M., Villadelprat, J.: Bifurcation of critical periods from Pleshkan’s isochrones, J. London Math. Soc. 81 (2), 142–160 (2010)
[6] Rothe, F.: Remarks on periods of planar Hamiltonian systems, SIAM J. Math. Anal. 24, 129–154 (1993)
[7] M. Sabatini, Period function’ convexity for Hamiltonian centers with separable variables, Ann. Polon. Math. 85, 153–163 (2005)
[8] Coppel, W. A., Gavrilov, L.: The period function of a Hamiltonian quadratic system, Differential Integral Equations 6, 1357–1365 (1993)
[9] Chicone, C.: The monotonicity of the period function for planar Hamiltonian vector fields, J. Differ. Equ. 69, 310–321 (1987)
[10] Chavarriga, J., Sabatini, M.: A survey of isochronous centers, Qual. Theory Dyn. Syst. 1 (1), 1–70 (1999)
[11] Zeng, X., Jing, Z.: Monotonicity and critical points of period, Progr. Natur. Sci. 6, 401–407 (1996)
[12] Cima, A., Mañosas, F., Villadelprat, J.: Isochronicity for several classes of Hamiltonian systems, J. Differ. Equ. 157, 373–413 (1999)
[13] Chicone, C., Dumortier, F.: Finiteness for critical periods of planar analytic vector fields, Nonlinear Anal. 20, 315–335 (1993)
[14] Cima, A., Gasull, A., Silva, P. R. da.: On the number of critical periods for planar polynomial systems, Nonlinear Anal. 69, 1889–1903 (2008)
[15] Gavrilov, L.: Remark on the number of critical points of the period, J. Differ. Equ. 101, 58–65 (1993)
[16] Mañosas, F., Villadelprat, J.: A note on the critical periods of potential systems, Internat. J. Bifur. Chaos, Appl. Sci. Engrg. 16 (3), 765–774 (2006)
[17] Li, C., Lu, K.: The period function of hyperelliptic Hamiltonians of degree 5 with real critical points, Nonlinearity 21, 465–483 (2008)
[18] Yang, L., Zeng, X.: The period function of potential systems of polynomials with real zeros, Bull. Sci. Math. 133, 555–577 (2009)
[19] Oldham K. B., Spanier J.: The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press, New York, 1974.

[20] Igor Podlubny, Fractional Differential Equations [M]. Academic Press, New York, NY, USA, 1999.

[21] Johanna D. García-Saldaña, Armengol Gasull, The period function and the Harmonic Balance Method, Bull. Sci. math. 139 33–60 (2015)

[22] Ding, T., Huang, H., Zanolin, F.: A priori bounds and periodic solutions for a class of planar systems with applications to Lotka-Volterra Equations, Disc. Cont. Dyn. Syst. 1, 103–117 (1995)

[23] Gavrilov, L., Iliev, I. D.: Complete hyperelliptic integrals of the first kind and their non-oscillation, Trans. Amer. Math. Soc. 356, 1185–1207 (2004)

[24] Grau, M., Mañosas, F., Villadelprat, J.: A Chebyshev criterion for Abelian integrals, Trans. Amer. Math. Soc. 363, 109–129 (2011)

[25] Mañosas, F., Villadelprat, J.: Criteria to bound the number of critical periods, J. Differ. Equ. 246, 2415–2433 (2009)

[26] Mañosas, F., Villadelprat, J.: The bifurcation set of the period function of the dehomogenized Loud’s centers is bounded, Proc. Amer. Math. Soc. 136, 1631–1642 (2008)

[27] Villadelprat, J., Zhang, X.: The period function of Hamiltonian systems with separable variables, J. Dyn. Diff. Eqns, 32, 741–767 (2020)

[28] Yang, L.: Recent advances on determining the number of real roots of parametric polynomials, J. Symb. Comput. 28, 225–242 (1999)

[29] Yang, L., Hou, X., Xia, B.: A complete algorithm for automated discovering of a class of inequality-type theorems, Sci. China Ser. F, 44, 33–49 (2001)

[30] Xia, B., Hou, X.: A complete algorithm for counting real solutions of polynomial systems of equations and inequalities, Computers and Mathematics with applications, 44, 633–642 (2002)

[31] Zhu, H., Rousseau, C.: Finite cyclicity of graphics with a nilpotent singularity of saddle or elliptic type. J. Differ. Eqn. 178, 325–436 (2002)

[32] Wang, J.: Bound the number of limit cycles bifurcating from center of polynomial Hamiltonian system via interval analysis. Chaos, Solitons & Fractals. 87, 30–38 (2016)