Differentially Private Top-k Selection via Canonical Lipschitz Mechanism

Michael Shekelyan 1  Grigoris Loukides 1

Abstract
Selecting the top-k highest scoring items under differential privacy (DP) is a fundamental task with many applications. This work presents three new results. First, the exponential mechanism, permute-and-flip and report-noisy-max, as well as their oneshot variants, are unified into the Lipschitz mechanism, an additive noise mechanism with a single DP-proof via a mandated Lipschitz property for the noise distribution. Second, this new generalized mechanism is paired with a canonical loss function to obtain the canonical Lipschitz mechanism, which can directly select k-subsets out of d items in O(dk + d log d) time. The canonical loss function assesses subsets by how many users must change for the subset to become top-k. Third, this composition-free approach to subset selection improves utility guarantees by an $\Omega(\log k)$ factor compared to one-by-one selection via sequential composition, and our experiments on synthetic and real-world data indicate substantial utility improvements.

1. Introduction
Let $\{1, \ldots, d\}$ be a set of items and $\vec{x} \in \mathbb{R}^d$ be a data vector comprised of numerical scores for these items. Depending on the application domain the items can be thought of as features, policies, models, or in some cases physical objects as the term suggests. The top-k selection problem seeks to select $k$ highest scoring items, i.e., $\arg\max_{i \in \{1, \ldots, d\}} [k](\vec{x}_i)$. It is a fundamental problem with myriads of applications (see Ilyas et al. 2008 for a survey) and also a building block in analytic tasks including classification, summarization, and content extraction (Wu et al. 2007; Fujiwara et al. 2013). The applications of top-k selection are typically fueled by user data and thus have raised privacy concerns (Narayanan & Shmatikov 2009). To address such concerns, it is crucial that top-k selection preserves privacy. This is possible by enforcing differential privacy (DP) (Dwork et al. 2014) which, informally speaking, ensures that the selected items do not depend heavily on any single user’s private information.

A data vector $\vec{x}$ is derived from some object $\hat{x} \in X$ and is influenced through the private information of a set of users $\text{USERS}(\hat{x})$. This influence is a central concept in DP, which is essentially the sensitivity of the underlying object $\hat{x}$ to the selection of $\vec{x}$. While the underlying object $\hat{x}$ can often be thought of as a database of user records, it may also lack a natural division into user-specific parts. For example, $\vec{x}$ can contain numerical features of some video $\hat{x}$ in which the individuals $\text{USERS}(\hat{x})$ appear in. In this case, there are no “user records”, but one can still model each individual as a “user” who influences the numerical features $\vec{x}$.

1 King’s College London, Department of Informatics, London, United Kingdom. Correspondence to: Michael Shekelyan <michael.shekelyan@kcl.ac.uk>. 

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This ensures that if two objects $\hat{a}$ and $\hat{b}$ differ in the participation of at most one user, then observing $Y = y$ provides little insight into the participation status of a user due to the bounded log-likelihood ratio of participation scenarios $\hat{a}$ vs. $\hat{b}$, i.e., $|\log \frac{Pr[Y = y|\hat{a}]}{Pr[Y = y|\hat{b}]}| \leq \varepsilon$. Consider the well-known example of the Netflix prize dataset (Bennett & Lanning, 2007) and queries asking for top-$k$ movies/patterns, based on movie ratings and other user information. This $\varepsilon$-DP modifies query answers to limit inferences on user data.

Our work considers the top-$k$ selection problem under $\varepsilon$-DP, proposing novel methods that reduce utility loss both in theory (cf. Section 5) and in practice (cf. Figure 1 for results on the Netflix prize dataset) and are amenable to simple, efficient implementations. Our main contributions are:

I. Lipschitz Mechanism. Inspired by the report-noisy-max mechanism (Dwork & Roth, 2013), equivalences of the exponential mechanism (McSherry & Talwar, 2007) and permute-and-flip mechanism (McKenna & Sheldon, 2020) to their additive noise formulations (Durfee & Rogers, 2019; Ding et al., 2020), we sought a property that unifies them and guarantees $\varepsilon$-DP. We discovered that adding noise $X$ to each score and reporting the index of the maximal noisy score is $\varepsilon$-DP if $\log(1 - Pr[X \leq x])$ is 1-Lipschitz (cf. Theorem 4.1). Hence, we call this additive noise method the Lipschitz mechanism. This mechanism instantiates many popular mechanisms and novel variants via different choices for noise distributions and parameter $\kappa$:

| Noise Distr. | top-1 with $\kappa = 1$ | top-$k$ with $\kappa = k$ |
|--------------|-------------------------|---------------------------|
| Gumbel       | EXP. MECHANISM           | PEELING                   |
| Laplace      | REPORT-NOISY-MAX (new)   | ONESHOTLaplace (new)      |
| (Half-)Logistic | PERMUTE-AND-FLIP (new)  | ONESHOTExp (new)          |

Reporting the indices of the $\kappa$ largest utility values is also $\varepsilon$-DP ($\kappa$ reduces the $\varepsilon$ internally) and instantiates the faster oneshot mechanisms, such as the ONESHOTLaplace mechanism (Qiao et al., 2021). The Lipschitz mechanism yields the first $\varepsilon$-DP proof of a oneshot variant of permute-and-flip (ONESHOTExp). Due to the versatility of the Lipschitz mechanism, we use it both to instantiate existing work, as well as new top-$k$ methods that apply the mechanism directly on the set of $k$-subsets as the selection domain (with $\kappa = 1$).

II. Canonical Lipschitz Mechanism for top-$k$. Applying general selection mechanisms, such as the Lipschitz mechanism, over an exponentially large selection domain is prohibitive in cost. Thus, specialized mechanisms for suitable loss functions are needed. A natural loss function is inspired by the well-known trick that a single user can change the number of user changes needed (for a solution to become optimal) at most by one. We investigated how to deploy such a canonical loss function over the domain of $k$-subsets, as recent works indicate strong utility guarantees for canonical functions in general (Asi & Duchi, 2020) and (Medina & Gillenwater, 2020). Prior work on related loss functions for top-$k$ over $d$ items (Joseph et al., 2021) specialized the exponential mechanism to improve the sampling time from $O(d^k)$ to $O(dk^3)$. As this is still too costly for large $k$, we develop faster methods. By plugging the canonical loss function into the Lipschitz mechanism with $\kappa = 1$, we obtain the Canonical Lipschitz mechanism (CANONICAL) and its faster variant (CANONICAL-$\gamma=1$):

| Runtime | Prior Mechanisms | Contribution |
|---------|------------------|--------------|
| $O(d^k)$ | naively applied mechanisms | - |
| $O(dk^3)$ | JOINT (Joseph et al., 2021) | - |
| $O(dk)$ | PEELING (Durfee & Rogers, 2019) | CANONICAL |
| $O(d)$ | ONESHOTLaplace (Qiao et al., 2021) | CANON.-$\gamma=1$ |

CANONICAL can be sampled in $O(dk)$ and CANONICAL-$\gamma=1$ in $O(d)$ time, both storing only a handful auxiliary values (cf., Appendix A.1). Their subset probabilities (when instantiating the exponential mechanism) can be obtained in time and space matching their sampling time complexities.

III. Composition-Free vs Sequential Composition. Traditionally, top-$k$ has been approached by $k$ repeated ($\varepsilon/k$)-DP selections without-replacement (PEELING), which is $\varepsilon$-DP via sequential composition (McSherry, 2009). We show that composition-free methods like CANONICAL improve utility guarantees by a factor $\Omega(\log k)$ compared to PEELING and our experiments on synthetic and real-world data show similarly conclusive results.

\footnote{We consider two different forms of analysis. One is based on investigating noise terms exploiting the additive noise formulation of the Lipschitz mechanism (cf. Theorem 5.5) and the other is based on classical utility loss bounds (cf. Theorem 5.6). In both analyses, we obtain the result of an improvement by a factor $\Omega(\log k)$ for $\frac{\varepsilon}{k} = O(1)$ .}
2. Lipschitz Mechanism for Discrete Selection

**Definition 2.1 (Lipschitz Mechanism).** Let $F, F^{-1}$ be a pair of a cumulative and an inverse distribution function for which $\log(1 - F(x))$ is 1-Lipschitz, i.e., for any $x, c \in \mathbb{R}$:

$$|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|$$

Let $\varepsilon \in \mathbb{R}_{>0}, \kappa \in \mathbb{N}, \hat{x} \in \mathbb{X}$ be a data object, $\mathbb{Y}$ be the selection domain of the mechanism, and $U_y \sim Unif(0, 1)$ be independent for each possible output $y \in \mathbb{Y}$. Let $\text{LOSS}(y | \hat{x})$ for $y \in \mathbb{Y}$ have sensitivity $\Delta_{\text{LOSS}}$. Then the output of the Lipschitz mechanism is:

$$\{Y_1, \ldots, Y_\kappa\} = \arg \max_{y \in \mathbb{Y}} \left\{ \frac{\text{LOSS}(y | \hat{x})}{-2\kappa \Delta_{\text{LOSS}} / \varepsilon} + F^{-1}(u_y) \right\}$$

Each LOSS value is negated/rescaled and then distorted by adding noise terms generated via inverse transform sampling. Then the items with the $\kappa$ largest noisy values are reported. The $\varepsilon$-DP proof (cf. Theorem 4.1) follows from the Lipschitz condition. This condition is for instance satisfied by the standard Laplace, Gumbel, Exponential Distribution and (Half-)Logistic distribution functions (cf. Appendix A.3).

The standard exponential distribution and $F^{-1}(p) = -\log(1 - p)$ satisfies the Lipschitz condition tightly ($\log(1 - F(x)) = -x$), has the smallest variance amongst the considered distributions, and consistently performed best in our experiments (cf. Figure 2 in Appendix A.2). The Laplace distribution matches the exponential distribution except that it selects a random sign i.e., absolute values are still exponentially distributed. Laplace noise tends to distort the loss values more than exponential noise due to opposing signs. The Gumbel distribution with $F^{-1}(p) = -\log(-\log(p))$ is often used as the default to simplify the analysis and because it allows to easily derive probabilities as it instantiates the exponential mechanism and probabilities are therefore proportional to simple exponential terms. These benefits come at the price of slightly higher loss value distortion.

2.1. Lipschitz mechanism for Discrete Selection over exponentially large selection domains

Naively sampling the Lipschitz mechanism over a selection domain $\mathbb{Y}$ takes $O(\kappa |\mathbb{Y}|)$ time. This is problematic if $\mathbb{Y}$ is exponentially large as is the domain of top-$k$ selection. Yet, it can be evaded if $\text{LOSS}$-values can be grouped into few classes with distinct $\text{LOSS}$-values. For large groups with the same loss value, only the group members with the $\kappa$ largest noise values can ever be selected and each group member has the same probability to receive those noise values. Hence, it suffices to generate the $\kappa$ largest noise values per group. This can be done efficiently using order statistics for uniform random variables, because $F^{-1}$ is strictly increasing and each needed order statistic can be generated sequentially in $O(1)$ arithmetic operations (Lurie & Hartley, 1972).

In the special case of $\kappa = 1$, only one noise term per group needs to be generated. Let $U_0, \ldots, U_m$ be i.i.d. standard uniform random variables used to generate the noise terms of a group of size $m$ via inverse transform sampling. From order statistics of standard uniform variables, it follows that $U_0^{1/m}$ is distributed equally to $\max\{U_1, \ldots, U_m\}$. Thus, $F^{-1}(U_0^{1/m})$ directly generates the maximal noise term of the group and its elements have a uniform probability $\frac{1}{m}$ of receiving this maximal noise term.

3. Canonical Lipschitz Mechanism for Top-$k$

3.1. Canonical Loss Function

Let $f$ be some scoring function over a set of items $\{1, \ldots, d\}$ with sensitivity $\Delta_f$. Let $\mathbb{Y}$ be all $k$-subsets of $\{1, \ldots, d\}$. Let for any $y \in \mathbb{Y}$ the utility loss $\text{LOSS}(y | \hat{x})$ be a shorthand for $\text{LOSS}(y | \hat{x})$, where each component $\hat{x}_j = f(j | \hat{x})/\Delta_f$ for $j \in \{1, \ldots, d\}$.

A common loss function for many problems is to quantify how much the data object $\hat{x}$ would need to change in terms of participants’ indices $y_1, \ldots, y_k$. The canonical loss function is then the distance to the nearest data vector in $OPT^{-1}(y)$:

$$\text{LOSS}(y | \hat{x}) = \min_{\bar{\hat{x}} \in OPT^{-1}(y)} ||\hat{x} - \bar{\hat{x}}||_\infty$$

which has sensitivity 1 (see Lemma A.14 in Appendix A.8).

This definition of canonical loss is real-valued unlike in (Asi & Duchi 2020[a], Medina & Gillenwater 2020). However, after rounding up the value can be interpreted as the number of users needed to transform $\hat{x}$ into $\bar{\hat{x}}$ s.t. $y \in OPT(\bar{\hat{x}})$.
Lemma 3.1. The function $\text{LOSS}^*(y \mid \vec{x}) = \ceil{\text{LOSS}(y \mid \vec{x})}$ has sensitivity $\Delta_{\text{LOSS}^*} = 1$.

Proof. The canonical loss function $\text{LOSS}(y \mid \vec{x}) \geq 0$ and has sensitivity 1. Thus, if $a \in \mathbb{R}_{\geq 0}$ is replaced with a value in $[\max(0, a-1), a+1]$, then $b = \lceil a \rceil$ is replaced by an integer between $b-1 = \lceil a-1 \rceil$ and $b+1 = \lceil a+1 \rceil$. □

3.2. Top-$k$

Definition 3.2 (score vector). Let $f$ be a scoring function with sensitivity $\Delta_y$. Items $\{1, \ldots, d\}$ are assigned scores $\vec{x} \in \mathbb{R}^d$ with $\vec{x}_j = f(j \mid \vec{x})/\Delta_y$ for $j \in \{1, \ldots, d\}$. The (descending) order statistics of the components of $\vec{x}$ are:

$$\vec{x}_{[1]} \geq \vec{x}_{[2]} \geq \ldots \geq \vec{x}_{[d]}$$

Let $j_1, \ldots, j_d \in \{1, \ldots, d\}$ be indices such that:

$$\vec{x}_{j_1} = \vec{x}_{[1]}, \vec{x}_{j_2} = \vec{x}_{[2]}, \ldots, \vec{x}_{j_d} \approx \vec{x}_{[d]}$$

The top-$k$ of a score vector are its $k$-largest components:

Definition 3.3 (top-$k$). Let $\vec{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$. Then:

$$\text{OPT}(\vec{x}) = \{j_1, \ldots, j_k\} = \arg \max \{k \mid \{\vec{x}_i\} \}$$

As releasing the top-$k$ under DP is not always possible, it is useful to identify groups of subsets that are good approximations. For this purpose, all $k$-subsets are partitioned into disjoint utility classes based on an integer $h \in \{1, \ldots, k-1\}$ and an integer $t \in \{k, \ldots, d\}$, where $k$ is only allowed if $h = k-1$. The integer $h$ relates to the highest missing rank from the subset and $t$ is the lowest present rank in the subset:

Definition 3.4 (utility class). The utility class $C_{h,t}$ is comprised of all subsets of the form $\{j_1, \ldots, j_h\} \cup B \cup \{j_t\}$ with $B \subseteq \{j_{h+1}, \ldots, j_{t-1}\}$ and $h + \lceil B \rceil + 1 = k$.

Thus, each subset $y \in C_{h,t}$ fully contains the top-$h$, while the remaining $k-h$ items are contained in the top-$t$. The subset $\{j_1, \ldots, j_h\}$ can be thought of as the "head", $B$ as the "body" and $j_t$ as the "tail" of a subset.

Let $y \in C_{h,t}$ be a $k$-subset of $\{1, \ldots, d\}$, then (cf. Lemma A.23 in Appendix A.8):

$$\text{LOSS}(y \mid \vec{x}) = \min_{\vec{v} \in \text{OPT}^{-1}(y)} \|\vec{x} - \vec{v}\|_\infty = \frac{\vec{x}_{[h+1]} - \vec{x}_{[t]} + 1}{2}.$$  

Intuitively, $\vec{x}_{[h+1]}$ is the best item missing from $y$ and even the worst item $\vec{x}_{[t]}$ in $y$ needs to overtake $\vec{x}_{[h+1]}$. The factor $\frac{1}{2}$ is due to the fact that users can both increase $\vec{x}_{[t]}$ by 1 and decrease $\vec{x}_{[h+1]}$ by 1 such that overtaking takes half as much effort. This can be generalized by a parameter $\gamma \in [0, 1]$ (matching the previous definition for $\gamma = \frac{1}{2}$):

**Theorem 3.5** (top-$k$ canonical loss function). Let $y \in C_{h,t}$.

$$\text{LOSS}(y \mid \vec{x}) = (1 - \gamma) \vec{x}_{[h+1]} - \gamma \vec{x}_{[t]}$$

has sensitivity 1 for any $\gamma \in [0, 1]$.

Proof. If all values of a set change at most by $C \in \mathbb{N}$, then their extrema can also change at most by $C$ (see Lemma A.22 in Appendix A.8). As the values of $\vec{x}_j$ for $j \in \{1, \ldots, d\}$ and the values for $\vec{x}_j$ for $j \in y$ change at most by 1, the values of $\vec{x}_{[h+1]}$ and $\vec{x}_{[t]}$ also change at most by 1, since $\vec{x}_{[h+1]} = \max_j \in \{1, \ldots, d\} \cup y \vec{x}_j$ and $\vec{x}_{[t]} = \min_j \in y \vec{x}_j$.

The term $(1 - \gamma) \vec{x}_{[h+1]}$ can therefore change at most by $(1 - \gamma)$ and the term $\gamma \vec{x}_{[t]}$ can change by at most $\gamma$. Thus, the difference can change by at most $(1 - \gamma) + \gamma = 1$. □

By plugging the function in Theorem 3.5 into the Lipschitz mechanism with $\kappa = 1$ and using the techniques in Section 2.4 to deal with exponentially large selection domains, we obtain the Canonical Lipschitz mechanism, which can be sampled in $O(dk)$ (cf. Theorem A.1 in Appendix A.1).

Numerical precision can be either achieved by taking computations into the log-space\(^3\) or via special libraries.

While Theorem A.1 applies to any $\gamma \in [0, 1]$, the mechanism can be sampled in $O(d)$ time for $\gamma = 1$, as only each $t \in \{k, \ldots, d\}$ has a distinct loss value. In this case there are $\sum_{h=0}^{k-1} |C_{h,t}| = \binom{d-1}{k-1}$ subsets for each loss value $\vec{x}_{[t]}$.

4. Differential Privacy Guarantees

All approaches considered in this work are instantiated through the Lipschitz mechanism, which only requires a distribution choice s.t. $\log(1 - F(x))$ is 1-Lipschitz for any $x \in \mathbb{R}$ and the LOSS function to have some known sensitivity $\Delta_{\text{LOSS}}$. Thus, it suffices to prove:

**Theorem 4.1.** The Lipschitz mechanism from Definition 2.1 is $\varepsilon$-DP for any LOSS function with finite sensitivity $\Delta_{\text{LOSS}}$.

Proof. We use here the same variables as in Definition 2.1. Let $\Delta$ be a shorthand for $\Delta_{\text{LOSS}}$. Let $a = a_1, \ldots, a_k$ be an arbitrary $k$-subset of $\mathbb{Y}$ and $b = \{b_1, \ldots, b_{d-k}\}$ be the $d-k$ items missing from $a$. Let the loss $\vec{a} = \text{LOSS}(a_1 \mid \hat{x}), \ldots, \text{LOSS}(a_k \mid \hat{x})$ and $\vec{b} = \text{LOSS}(b_1 \mid \hat{x}), \ldots, \text{LOSS}(b_{d-k} \mid \hat{x})$. Let the noisy no-

\(^3\)One can use $F^{-1}(p) = -\log(-\log(p))$ from the Gumbel distribution where it simplifies to $F^{-1}(U_{b_1}^{1/(C_{b,1})}) = \log(C_{b,1}) + F^{-1}(U_{b_1})$. As $|C_{b,1}|$ are binomial coefficients that can be computed via multiplications, it is trivial to take the computations into the log-space.
The integration variable \( x \) is the \((\kappa + 1)\)-largest noisy value overall. Thus, it covers all possible events where \( \tilde{A} \) are the \( \kappa \)-largest noisy values and all events are disjoint, differing at least in the \((\kappa + 1)\)-largest noisy value.

Let \( \omega \in [-1, +1] \). Then replacing \( \bar{a} \) with \( \bar{a} + 2\omega \Delta \) maximizes a user’s impact on \( Pr[Y = a] \), due to the following properties that follow from the additive noise framework:

1. Monotonicity: Replacing \( \bar{a} \) with \( \bar{a} + \omega \Delta \) and \( \bar{b} + \omega \Delta \) has no effect on \( Pr[Y = a] \).

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(i.) Monotonicity: Replacing \( \bar{a} \) with \( \bar{a} + \omega \Delta \) and \( \bar{b} \) with \( \bar{b} - \omega \Delta \) maximizes a user’s impact on \( Pr[Y = a] \).

(ii.) Shift-Invariance: Replacing both \( \bar{a} \) and \( \bar{b} \) with \( \bar{a} + \omega \Delta \) and \( \bar{b} + \omega \Delta \) has no effect on \( Pr[Y = a] \).

It is easy to see that (i.i.) can be used to cancel out changes to \( \bar{b} \) in (i.) by doubling down on changes to \( \bar{a} \). For \( \kappa = 1 \), this corresponds to the claims about regular mechanisms in (McKenna \& Sheldon 2020).

As \( Pr[\max(\tilde{B}) = x] \) is independent of \( \bar{a} \), it is only left to show that \( Pr[\min(\tilde{A}) > x] \) changes by at most a multiplicative factor \( \exp(\epsilon) \) due to a single user.

From inverse transform sampling with \( U_i \sim Unif(0, 1) \) follows \( Pr[A_{i} > x] = Pr[(2\Delta/\kappa)\bar{a}_{i} + F^{-1}(U_i) > x] \) is for any \( i \in \{1, \ldots, \kappa\} \) equal to:

\[
Pr[F^{-1}(U_i) > x - \frac{\epsilon}{2\Delta\kappa} \bar{a}_{i}] = 1 - F(x - \frac{\epsilon}{2\Delta\kappa} \bar{a}_{i})
\]

Replacing the normalized loss \( \tilde{a} \) with \( \bar{a} + 2\omega \Delta \) replaces \( \frac{\epsilon}{2\Delta\kappa} \tilde{a} \) with \( \frac{\epsilon}{2\Delta\kappa} (\bar{a} + 2\omega \Delta) = \frac{\epsilon}{2\Delta\kappa} \bar{a} + \omega \frac{\epsilon}{\kappa} \). This means that the noisy normalized loss \( \tilde{A} \) is replaced with \( \tilde{A} + \omega \frac{\epsilon}{\kappa} \) and the probability \( Pr[\min(\tilde{A} + \omega \frac{\epsilon}{\kappa}) > x] \) is equal to:

\[
\prod_{i=1}^{\kappa} Pr[A_{i} + \omega \frac{\epsilon}{\kappa} > x] = \prod_{i=1}^{\kappa} \left[ 1 - F(x - \frac{\epsilon}{2\Delta\kappa} \bar{a}_{i} - \omega \frac{\epsilon}{\kappa} ) \right]
\]

The 1-Lipschitz condition on \( \log(1 - F(x)) \) implies \( \frac{1 - F(x)}{\exp(c)} \leq 1 - F(x + c) \leq \exp(c)(1 - F(x)) \) for any \( c \in \mathbb{R}_{\geq 0} \). Thus, with \( c = \omega \frac{\epsilon}{\kappa} \) the value of \( 1 - F(x - \frac{\epsilon}{2\Delta\kappa} \bar{a}_{i} - c ) \) fluctuates at most by a multiplicative factor \( \exp(c) \leq \exp(\epsilon/\kappa) \) and \( Pr[\min(\tilde{A} > x)] \) fluctuates at most by a multiplicative factor \( \prod_{i=1}^{\kappa} \exp(\epsilon/\kappa) = \exp(\epsilon) \) due to \( \omega \in [-1, +1] \).

5. Theoretical Comparison

5.1. Comparison of Canonical for different \( \gamma \)

The subset classes \( C_{h,t} \) have better utility for larger \( h \) and smaller \( t \). The parameter \( \gamma \) governs how much CANONICAL prioritizes \( t \) against \( h \). Inversely, for small values of \( \gamma \) the mechanism improves \( h \), but neglets \( t \). By trading privacy loss for utility CANONICAL with smaller \( \gamma \) can become as good as CANONICAL with larger \( \gamma \) at minimizing \( t \):

**Lemma 5.1** (Internal superiority). For \( \Gamma_2 > \Gamma_1 \) and \( \epsilon_1 = (\Gamma_2/\Gamma_1)\epsilon_2 < \epsilon_2 \), \( \epsilon_1 \)-DP CANONICAL,\( \gamma_1 = \Gamma_1 \), has superior utility to \( \epsilon_2 \)-DP CANONICAL,\( \gamma_1 = \Gamma_2 \), i.e., for any \( y, y' \in \mathcal{Y} \)

\[
\frac{|Pr[y | \hat{x}, \gamma_1 = \Gamma_1, \epsilon = \epsilon_1] - Pr[y' | \hat{x}, \gamma_1 = \Gamma_1, \epsilon = \epsilon_1]|}{|Pr[y | \hat{x}, \gamma_2 = \Gamma_2, \epsilon = \epsilon_2] - Pr[y' | \hat{x}, \gamma_2 = \Gamma_2, \epsilon = \epsilon_2]|} > 1
\]

**Proof.** Let \( y \in C_{h,t} \) and \( y' \in C_{h',t'} \) (cf. Definition 3.4). Then the following holds:

\[
\frac{|\epsilon_1 \text{ LOSS}(y | \hat{x}, \gamma = \Gamma_1) - \epsilon_1 \text{ LOSS}(y' | \hat{x}, \gamma = \Gamma_1)|}{\epsilon_2 \text{ LOSS}(y | \hat{x}, \gamma = \Gamma_2) - \epsilon_2 \text{ LOSS}(y' | \hat{x}, \gamma = \Gamma_2)} > 1
\]

Since \( \Gamma_2 > \Gamma_1 \), then \((1 - \Gamma_1) > (1 - \Gamma_2)\) and \(\frac{\epsilon}{\Gamma_2} \) just needs to be large enough to scale \( \Gamma_1 \) to \( \Gamma_2 \). This is achieved due to \( \frac{\epsilon}{\Gamma_2} = \frac{\Gamma_2}{\Gamma_1} \), which is implied by \( \epsilon_1 = (\Gamma_2/\Gamma_1)\epsilon_2 \).

**Corollary 5.2.** \( \epsilon_1 \)-DP CANONICAL,\( \gamma_1 = 1 \), has superior utility to \( \epsilon_2 \)-DP CANONICAL,\( \gamma_1 = 1 \) as shown in Lemma 5.1.

5.2. Noise Analysis of Canonical vs Peeling

Let \( \epsilon = 2\Delta = 1 \) without loss of generality. For the Lipschitz mechanism with Gumbel noise, the maximum \( G_{i(n)} \) of \( n \) i.i.d. standard Gumbel random variables (RVs) \( G_1, \ldots, G_n \) is distributed \( G_{i(n)} \sim \log(n) + \text{Gumbel}(0, 1) \) and the difference of two standard Gumbel RVs follows a standard Logistic RV. We seek a specific difference that instructs how far non-top-\( k \) options leap forward compared to top-\( k \) ones, hence called Logistic Leap.

To obtain simpler formulas we add for CANONICAL an additional subset with the same loss value as the previously worst one to the selection domain \( \mathcal{Y} \), which can only disadvantage CANONICAL.
Lemma 5.3 (Canonical Logistic Leap). Let $\mathcal{Y}$ be the selection domain with an additional (dummy) subset s.t. $|\mathcal{Y}| = (d)_{k} + 1$. Let $y \in \mathcal{Y}$ be the subset selected by CANONICAL with parameter $\gamma \in (0, 1]$. Let:

$$X \sim \frac{1}{\gamma} \left( k \log(d/k) + \log(c_{d,k}) + \text{Logistic}(0, 1) \right)$$

with $0 < \log(c_{d,k}) = \log \left( \frac{d}{k} \right) \leq k$. Then:

$$Pr \left[ |\text{OPT} \cap y| > 0 \right] \geq Pr[X > \bar{x}_1 - \bar{x}_d]$$

$$Pr \left[ |\text{OPT} \cap y| < k \right] \leq Pr[X < \bar{x}_k - \bar{x}_{k+1}]$$

Proof. Let $N$ be a $1 \times |\mathcal{Y}|$ matrix whose entries are independent noise terms: $N = (N_{1,1}, N_{1,2}, \ldots, N_{1,|\mathcal{Y}|})$.

Without loss of generality we can fix $\gamma = 1$ and then replace $\varepsilon$ by $\frac{\varepsilon}{\gamma}$ as according to Corollary this incurs no utility loss. Replacing $\varepsilon$ with $\frac{\varepsilon}{\gamma}$ is equivalent to multiplying all noise terms by $\frac{1}{\gamma}$.

$N_{1,1}$ is the noise-term received by the top-$k$, while the submatrix $S$ of $N$ with $N_{1,j}$ where $2 \leq j \leq |\mathcal{Y}|$ are the noise terms received by the non-top-$k$ subsets.

Let $N_{1,m} = \max(S)$. For CANONICAL-$\gamma=1$, we can define $X = N_{1,m} - N_{1,1}$. The non-top-$k$ subset that receives the noise term $N_{1,m}$ stops the top-$k$ subset if $X > \bar{x}_1 - \bar{x}_d$, but fails to overtake the top-$k$ subset if $X < \bar{x}_k - \bar{x}_{k+1}$. Due to these implications, the probability inequalities in the statement hold.

With $n = |\mathcal{Y}| - 1 = (d)_{k}$ non-top-$k$ subset noise terms we get $X = \log((d)_{k}) + \text{Logistic}(0, 1)$ and we can then write $\log((d)_{k}) = k \log(d/k) + \log(c_{d,k})$ with $0 < \log(c_{d,k}) \leq k$.

As PEELING removes in each round an item and we want to simplify it, we strictly remove the item with utility $\bar{x}_d$, which results in the elimination of more noise terms assigned to non-top-$k$ items (helping PEELING) as would occur due to the item selection:

Lemma 5.4 (Peeling Logistic Leap). Let $y$ be subset selected by PEELING from items $\{1, \ldots, d\} \setminus \{j_d\}$. Let:

$$X' \sim k \log((d - 1 - k)k) + k \cdot \text{Logistic}(0, 1)$$

Then:

$$Pr \left[ |\text{OPT} \cap y| > 0 \right] \geq Pr[X' > \bar{x}_1 - \bar{x}_d]$$

$$Pr \left[ |\text{OPT} \cap y| < k \right] \leq Pr[X' < \bar{x}_k - \bar{x}_{k+1}]$$

Proof. Let $N'$ be a $k \times d$ matrix whose entries are independent noise terms:

$$N' = \begin{pmatrix} N_{1,1}' & N_{1,2}' & \cdots & N_{1,k}' \\ N_{2,1}' & N_{2,2}' & \cdots & N_{2,k}' \\ \vdots & \vdots & \ddots & \vdots \\ N_{k,1}' & N_{k,2}' & \cdots & N_{k,k}' \end{pmatrix}$$

For PEELING, $N'_{i,j}$ is the noise term received by the item with score $\bar{x}_i$ in round $j$, while $S'$ is the submatrix of $N'$ (depicted as right half) with $N'_{i,j}$ where $1 \leq i \leq k$ and $k + 1 \leq j \leq d$ with the bottom noise terms, i.e., noise terms of items in the bottom partition “after” the top-$k$. Let $N'_{i,m} = \max(S')$. For PEELING, we can define $X' = N'_{1,m} - N'_{1,1}$. The bottom item $m$ that receives the noise term $N'_{1,m}$ leaps ahead of top item $j$ in round $r$ if $X' > \bar{x}_j - \bar{x}_m$, but fails to overtake $j$ if $X' < \bar{x}_j - \bar{x}_m$. Similarly, if $X' > \bar{x}_1 - \bar{x}_{d} \geq \bar{x}_j - \bar{x}_m$, then $j$ is displaced and if $X' < \bar{x}_k - \bar{x}_{k+1} \leq \bar{x}_j - \bar{x}_m$ the top-item $j$ is not displaced by any bottom item. The probability inequalities in the statement follow from these implications.

With $n' = (d - 1 - k)k$ bottom noise terms we get $X' = k \log((d - 1 - k)k) + \text{Logistic}(0, 1)$. □

Theorem 5.5. Let $X, X'$ be logistic leap RVs from Lemma 5.3 and Lemma 5.4. Then for $\frac{d}{k} = O(1)$ it holds:

$$\frac{E[X']}{E[X]} = \Omega(\log k)$$

Proof. Since $\gamma \in (0, 1]$, $\frac{1}{\gamma} = O(1)$. Then, for $\frac{d}{k} = O(1)$, we rewrite $X$ via $L \sim \text{Logistic}(0, 1)$ as $X = \log(k + L)$ and $X' = O(k \log(k) + kL)$. As $E[L] = 0$, the claim then follows due to $\log k = \Omega(\log k)$ and $k = \Omega(\log k)$. □

5.3. Utility Loss Bounds for Canonical vs Peeling

Based on standard utility guarantees for the exponential mechanism one can derive:

Theorem 5.6. Let $Y_1, \ldots, Y_k$ be the selected set by canonical with $\gamma \in (0, 1]$ supposing $\frac{1}{\gamma} = O(1)$. Let $Y'_1, \ldots, Y'_k$ be the outputted set by PEELING. Let $T = \arg\min_{i \in \{1, \ldots, k\}} \bar{x}'_i$ and $T' = \arg\min_{i \in \{1, \ldots, k\}} \bar{x}_i$.

Let $\alpha \in (0, 0.1]$ be the failure rate. Then with at least probability $1 - \alpha$ it holds that $\bar{x}_T < \bar{x}'_k + \frac{2\Delta}{\gamma \epsilon}$ and $\bar{x}[T'] < \bar{x}_k + \frac{2\Delta}{\gamma \epsilon}$ with utility loss terms:

$$\mathcal{E} = \left( k \log(d/k) + \log \frac{1 - \alpha}{\alpha} + k \right)$$

$$\mathcal{E}' = \left( k \log(dk) + k \log \frac{1}{\alpha} - \frac{6}{100}k \right)$$
Also, for $\frac{d}{k} = O(1)$, it holds that $\mathcal{E}'/\mathcal{E} = \Omega(\log k)$.

Proof. The proof follows from several Theorems and Lemmas in Appendix A.8. For CANONICAL:

- Lemma A.19 adopts standard theorems for the exponential mechanism to obtain general utility guarantees (instantiated by Lipschitz mechanism with $F^{-1}$ from the Gumbel distribution).

- Lemma A.20 plugs the loss function with $\gamma = 1$ over the $\mathcal{F}_k$ subsets from Theorem 3.5 into Lemma A.19 to obtain the inequality for $\bar{x}_{[T]}$ as in the claim. It is then generalized to $\gamma \in (0, 1]$ via Corollary 5.2.

For PEELING, Lemma A.21 plugs the score function over the $d$ items into Lemma A.19 but uses a reduced failure rate $\alpha'$ s.t. $k$ selections have a joint success rate $(1 - \alpha')^k \geq 1 - \alpha$.

Due to independence one can here use the Šidák correction $\alpha' = 1 - (1 - \alpha)^{1/k}$ for family wise error rates of hypothesis tests. As this leads to terms that complicate comparisons, $\alpha'$ is rewritten via the Bonferroni correction $\alpha/k$ and a ratio between both corrections $r_{\alpha,k}$ is used to restore the Šidák correction. We derive in Lemma A.18 that the Bonferroni correction is as expected a very good approximation and that $r_{\alpha,k} \leq \log(\frac{1}{1 - \alpha})/\alpha$ which even for $\alpha < 0.1$ is smaller than 1.06 such that $\log(r_{\alpha,k}) < \log(1.06) < \frac{6}{\alpha^2}$. From then follows the inequality for $T'$ in the claim. For $\frac{d}{k} = O(1)$, $\mathcal{E}' = O\left(k + \log \frac{1}{\alpha'}\right)$ and $\mathcal{E}' = O\left(k \log(k) + k \log \frac{1}{\alpha'}\right)$. Thus, $\frac{\mathcal{E}}{\mathcal{E}'} = \Omega(\log k)$.

6. Related Work

The report-noisy-max mechanism (Dwork & Roth, 2013) adds Laplace noise to utility values and then selects the item with the maximal noisy value. Other popular mechanisms can be formulated in a similar way, i.e., by adding instead Gumbel noise (Dong et al., 2021) one gets the exponential mechanism (EM) and by adding Exponential noise (Ding et al., 2021) one gets the permute-and-flip (P&F) mechanism. In this work, we extend these results and unification efforts via the proposed Lipschitz mechanism. We model it as a single mechanism rather than a framework or family of mechanisms as the DP proof is independent of instantiations. We show that in the context of the Lipschitz mechanism asymmetric sensitivities (Dong et al., 2020) can be reduced to ordinary sensitivities, generalizing results on monotonic functions (Dwork & Roth, 2013; McKenna & Sheldon, 2020) that are treated to have sensitivity $\frac{1}{\alpha}$. Due to its generality, the Lipschitz mechanism also instantiates oneshot variants that select the $k$ largest noisy values (Qiao et al., 2021), e.g., the oneshot variant of P&F (McKenna & Sheldon, 2020) did not have a DP proof although it promises the best utility amongst oneshot mechanisms.

The joint EM (Joseph et al., 2021) is a mechanism that directly selects $k$-subsets based on a loss function akin to canonical loss. Aside from the problem definition, the joint EM paper employs a loss function that counts how many users are needed to change the utility values of the subset to match the utility values of the top-$k$. Catching up with the top-$k$ may not displace all items of the top-$k$ and instead yields a mix of top-$k$ and subset items. In contrast, the canonical loss function requires all missing top-$k$ items to be displaced by subset items, which is desirable as it requires more users (cf. Lemma A.24 in Appendix A.8). The joint EM can be sampled in $O(\min(\alpha \cdot k^2, d^2))$ time, whereas all proposed methods with the canonical loss function require only $O(\min(\alpha \cdot dk, d^2))$ time. The authors of (Joseph et al., 2021) mention as a caveat of joint EM that it may be difficult to avoid exponentially large values in the matrix multiplications that are needed to compute loss value multiplicities. In contrast, the canonical loss value multiplicities (defined from $C_{d,h}$) are binomial coefficients which can be computed with simple methods in the log-space to avoid large values.

Previous works did not theoretically compare direct subset selection with composition methods, but have shown that canonical loss functions offer general utility guarantees (Asi & Duchi, 2020a, 2020b; Medina & Gillenwater, 2020) if they are plugged into the EM (Asi & Duchi, 2020a). Our motivation to generalize beyond the EM are results for P&F which show it consistently improves upon the EM (McKenna & Sheldon, 2020). Our experiments also indicate for the Lipschitz mechanism that using $F^{-1}$ from the Exponential distribution leads to best utility (cf. Figure 4 in Appendix A.2).

Last, there are several works (Chaudhuri et al., 2014; Carvalho et al., 2020; Cesar & Rogers, 2021) that focus on approximate DP (Dwork et al., 2014; Beimel et al., 2016), i.e., $(\varepsilon, \delta)$-DP with $\delta > 0$. This work focuses on pure $\varepsilon$-DP, leaving the $\delta > 0$ related question open.

7. Empirical Comparison

We compared four top-$k$ mechanisms:

- **PEELING** (Durfee & Rogers, 2019) is one-by-one $\frac{d}{k}$-DP selection without replacement via Exp. Mechanism.

- **ONESHOTExp** is a oneshot variant (Qiao et al., 2021) of permute-and-flip (McKenna & Sheldon, 2020).

- **CANONICAL** from Section 3 (cf. Appendix A.1) for details with $\gamma = \frac{1}{2}$ and $F^{-1}$ from Gumbel distribution.

- **CANONICAL$\gamma=1$** as CANONICAL except with $\gamma = 1$.

We did not compare to the Joint Exponential Mechanism (Joseph et al., 2021), due to its prohibitive cost for large $k$. Apart from its runtime, it is in any case very similar to
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Figure 2. For a fixed privacy budget $\varepsilon = 1$, the data distribution of $\vec{x} \in \mathbb{R}^d$ with $d = 10^4$ is varied from uniform values ($s = 0$) to linear values ($s = 1$) for $k \in \{10, 100, 1000\}$.

Figure 3. Follow-up to Figure 2 with $s = 0.04$ and varying $\varepsilon$ for $\varepsilon = 1$ and $s \geq 0.04$ all compared mechanisms have $Pr[\text{good-10}] \geq Pr[\text{top-10}] \geq 99.9\%$.

7.1. Top-$k$: Real-World data

In this experiment, $\vec{x} \in \mathbb{R}^d$ is based on half a million users who rated $d = 17700$ movies in the Netflix prize dataset (see Introduction) between 1 and 5. Each component $x_i$ is equal to the number of users that gave the movie $i$ a 5/5 rating such that the sensitivity is $\frac{1}{2}$ (cf. Theorem A.15 in Appendix). As can be seen in Figure 1 the new methods CANONICAL / CANONICAL$_{\gamma=1}$ almost certainly return the correct top-1000 with a privacy budget of $\varepsilon \leq 1$, whereas classical methods require an up to $81 \times$ larger privacy budget to achieve the same feat. For $k = 100$, the privacy budget of our methods is $34 \times$ smaller and for $k = 10$ it is $6 \times$ smaller. In the Appendix, we report similar results for five additional real-world datasets (cf. Table 1 and Figures 6, 7, 8, 9, 10). In conclusion, the new methods show vast improvements.

7.2. Top-$k$: Synthetic Data

We generated $\vec{x} \in \mathbb{R}^d$ with $d = 10^4$ directly by Zipf’s law, i.e., $f(i) \propto i^{-\lambda}$ and $\vec{x}_i = (1.5 \cdot 10^6) f(i)$. Figure 2 shows the results for varying parameter $s$ of Zipfian distribution, which controls how challenging the distribution is; scores are uniform when $s = 0$. CANONICAL and CANONICAL$_{\gamma=1}$ appear clearly superior to the existing mechanisms, being able to sample the top-100 or top-100 with a probability many times larger. For top-10, the difference is not obvious for $\varepsilon = 1$, but it becomes much larger for smaller $\varepsilon$ (see Figure 3). Besides top-$k$, we considered a “good”-$k$ subset as one that replaces at most half of the top-$k$ elements with elements from the top-$\left\lceil \frac{3}{2}k \right\rceil$ without touching the top-$\left\lceil \frac{1}{100}k \right\rceil$, i.e., $y \in C_{h,t}$ with $h \geq \frac{1}{100}k$ and $t \leq \frac{3}{2}k$ (cf. Definition 3.4). The new mechanisms again outperform classical ones, particularly for large $k$ and small $\varepsilon$ which is more demanding, but with a smaller margin than for top-$k$.

7.3. Runtimes

In our experiments with 2776 runtime measurements $^4$ CANONICAL was as fast as PEELING (average < 1s) and CANONICAL$_{\gamma=1}$ was as fast as ONESHOT (average < 3ms). As all methods are simple, the recorded runtimes can be predicted with < 80% relative error by $(d \cdot k) \mu s$ for $\tilde{O}(dk)$ methods PEELING / CANONICAL and by $(d) \mu s$ for $\tilde{O}(d)$ methods ONESHOT / CANONICAL$_{\gamma=1}$.

8. Discussion

We investigated three questions in this work. If it is possible to unify existing discrete selection mechanisms with an additive noise framework, if it is possible to operate selection mechanisms efficiently over subsets as an immediate selection domain, and if there are theoretical differences between approaches that select a $k$-subset directly or independently in $k$ steps (PEELING). The Lipschitz mechanism conclusively answers the first question, the Canonical Lipschitz mechanism (CANONICAL) is itself an affirmative answer to the second question, and our analysis shows an $\Omega(\log k)$ factor improvement of CANONICAL (our direct approach) over PEELING when $\varepsilon = O(1)$, i.e., when $d$ does not grow faster than $k$. Experimental results also indicate clear practical benefits, i.e., being able to quickly and reliably obtain high utility subsets with a far smaller privacy budget.

Open questions include if the Lipschitz mechanism could be generalized to non-finite selection domains, if other loss functions over subsets could yield similar efficiency and utility results, and if the theoretical analysis could be extended beyond Gumbel noise, which is used to instantiate the exponential mechanism. Specifically, it appears all noise distributions perform fairly similarly. Also, it would be interesting to see if unification, efficiency, and theoretical results in the same spirit could be achieved for other differential privacy (DP) notions such as approximate DP.

$^4$Running on a laptop with 3.2Ghz Apple M1 processor and Python 3.9.9 interpreter using one single thread/core. The reported runtimes do not include sorting the data vector $\vec{x}$, which in our experiments took less than $(d \log_2 d) \mu s$ (1 $\mu s = 10^{-6}$ secs).
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A. Appendix

A.1. Implementation

Theorem A.1. The Canonical Lipschitz Mechanism for top-k can be sampled in \(O(dk)\) time for \(d\) pre-sorted scores.

Proof. The mechanism releases a subset from class \(C_{h,t}\) (cf. Definition 3.4) with \((H, T)\) equal to:

\[
\arg\max_{(h,t)} \left\{ \frac{(1-\gamma)x_{[h+1]} - (\gamma)x_{[h]}}{-2/\varepsilon} + F^{-1}(U_{h,t} 1/|C_{h,t}|) \right\}
\]

with \(|C_{k-1,k}| = 1\) and \(|C_{h,t}| = \binom{t-h-2}{k-1-h} \) for \(h \in \{0, \ldots, k-1\}, t \in \{k+1, \ldots, d\}\).

The binomial coefficients for each \(t\) can be computed in \(O(k)\) by starting with \(h = k-1\) and \((\binom{t}{h}) = 1\) and decreasing \(h\) until \(h = 0\). In each step, \(h\) decreases by 1 and \((\binom{t}{h}) = \frac{t!}{(t-h)!h!}\) can be used to update the binomial coefficient. The number of subsets in each class \(C_{h,t}\) is equal to the distinct number of possibilities for the body \(B\), which is equal to the number of \((k-1-h)\)-subsets out of \(|B| = \left| \{j_{h+2}, \ldots, j_{t-1}\} \right| = (t-h-2)\) items (counted as 1 as \(|B| = 0\)). All subsets in the class \(C_{h,t}\) from Definition 3.4 have the same loss value. As the time complexity of implementations hinges upon the distinct number of loss values, one can see that there is the class \(C_{k-1,k}\) with the top-k and otherwise \(h \in \{0, \ldots, k-1\}\) and \(t \in \{k+1, \ldots, d\}\). Thus, the total number of classes is

\[
1 + k(d-k) = 1 + dk - k^2 = O(dk).
\]

The time can be reduced to \(O(d)\) for \(\gamma = 1\) as only each \(t \in \{k, \ldots, d\}\) has a distinct loss value. In this case \(\sum_{k=0}^{k-1} |C_{h,t}| = \binom{t-1}{k-1}\) which can be also computed via \(O(1)\) time updates by considering that for \(n, k \in \mathbb{N}\):

\[
\begin{align*}
\binom{n}{k} &= \binom{k}{k} = 1 \\
\binom{n}{k} &= \binom{n-1}{k} = \frac{n}{n-k} \binom{n-1}{k-1} \\
\binom{n}{k-1} &= \binom{n}{k} = \frac{n-k}{n+1-k} \binom{n}{k+1} \\
\binom{n-1}{k} &= \frac{n-k}{n} \binom{n-1}{k} = \frac{n+1-k}{n+1} \binom{n}{k} \\
\binom{n-1}{k-1} &= \frac{n-k}{n} \binom{n}{k} = \frac{n+1-k}{n+1} \binom{n}{k+1} \\
\end{align*}
\]

When instantiating the exponential mechanism, subset probabilities \(Pr(y \in C_{h,t}) \propto \exp\left(\frac{(1-\gamma)x_{[h+1]} - (\gamma)x_{[h]}}{-2/\varepsilon}\right)\).

For sampling large numbers can be avoided by taking \(F^{-1}(U^{\exp(-m)})\) into the logspace:

Lemma A.2. Let \(F^{-1}(p) = -\log(-\log(p))\), then \(F^{-1}(U^{\exp(-m)}) = m + F^{-1}(U)\)

Algorithm 1: Canonical Lipschitz Mechanism for top-k (with \(F^{-1}\) for additive noise generation and \(\gamma \in [0, 1]\))

Input: scoring function \(f: \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}\) over \(d\) items, subset size \(k \in \{1, \ldots, d-1\}\) and \(\varepsilon \in \mathbb{R}_{\geq 0}\)

Let \(\Delta_f\) be the sensitivity of \(f\).

Let \(\bar{x}_i = f(i | \hat{x})/\Delta_f\) for any \(i \in \{1, \ldots, d\}\).

Let \(\bar{x}_{[1]} \geq \ldots \geq \bar{x}_{[d]}\) sort \(\bar{x}\) in \(O(d\log d)\).

if \(\gamma < 1\) then

\[
\begin{align*}
&\text{/* CANONICAL sampling */} \\
&\text{/* in O(dk) time and O(1) space */} \\
&\text{Let } U_{h,t} \sim Unif(0, 1) \text{ be i.i.d. for any} \\
&\text{ } h \in \{0, \ldots, k-1\}, t \in \{k, \ldots, d\} \\
&\text{Initiate } H = k-1 \text{ and } T = k. \\
&\text{Let } \varepsilon_1 = (1-\gamma)\varepsilon \text{ and } \varepsilon_2 = \gamma \cdot \varepsilon \text{ s.t. } \varepsilon_1 + \varepsilon_2 = \varepsilon. \\
&\text{Initiate } v = \frac{\varepsilon_1 - 1}{\varepsilon_2} \bar{x}_{[k]} + F^{-1}(U_{H,T}) \\
&\text{foreach } t \in \{k+1, \ldots, d\} \text{ do} \\
&\text{Initiate } m = 0 / / m = \log \binom{t-h-1}{k-1} \\
&\text{Initiate } h = k-1 \text{ while } h \geq 0 \text{ do} \\
&\text{if } k-(h+1) > 0 \text{ then} \\
&\text{Update } m = m + \log \frac{t-h-1}{k-(h+1)} \\
&\text{Let } X = F^{-1}(U_{h,t} \exp(-m)) \\
&\text{Let } v' = \frac{\varepsilon_2}{\varepsilon_1} \bar{x}_{[k]} + X \\
&\text{if } v' > v \text{ then} \\
&\text{Update } v = v' \text{ and } T = t \\
&\text{Update } h = h-1 \\
&\text{Report random subset in } C_{H,T}. \\
\end{align*}
\]

else

\[
\begin{align*}
&\text{/* CANONICAL\gamma=1 sampling */} \\
&\text{/* in O(d) time and O(1) space */} \\
&\text{Let } U_t \sim Unif(0, 1) \text{ be i.i.d. for } t \in \{k, \ldots, d\}. \\
&\text{Initiate } T = k \text{ and } v = \bar{x}_{[k]} + F^{-1}(U_T) \\
&\text{Initiate } m = 0 / / m = \log \binom{t-1}{k-1} \\
&\text{Initiate } t = k+1 \text{ while } t \leq d \text{ do} \\
&\text{Initiate } m = m + \log \frac{t-1}{(t-1)-(k-1)} \\
&\text{if } (t-1) \geq (k-1) \text{ then} \\
&\text{Let } X = F^{-1}(U_t \exp(-m)) \\
&\text{Let } v' = \frac{\varepsilon_2}{\varepsilon_1} \bar{x}_{[k]} + X \\
&\text{if } v' > v \text{ then} \\
&\text{Update } v = v' \text{ and } T = t \\
&\text{Update } t = t + 1 \\
&\text{if } T = k \text{ then} \\
&\text{Report } \{1, \ldots, k\} \\
&\text{else} \\
&\text{Report random subset in } (C_{0,T} \cup C_{1,T} \cup \ldots \cup C_{k-1,T}).
\end{align*}
\]
We aggregate $(\text{General})$ Lipschitz Mechanism (with $F^{-1}$ for additive noise generation)

**Algorithm 2:** (General) Lipschitz Mechanism (with $F^{-1}$ for additive noise generation)

**Input:** scoring function $f : \mathbb{N} \times X \rightarrow \mathbb{R}$ over $d$ items, subset size $\kappa \in \{1, \ldots, d-1\}$ and $\epsilon \in \mathbb{R}_{\geq 0}$

Let $\bar{x}_i = f(i | x)/\Delta$, for any $i \in \{1, \ldots, d\}$.

Let $U_i \sim \text{Unif}(0,1)$ be i.i.d. for any $t \in \{k, \ldots, d\}$.

Maintain heap structure for top-$\kappa$ based on scores

**foreach** $i \in \{1, \ldots, d\}$

Let $Z_i = \frac{1}{\epsilon \Delta} \bar{x}_i + F^{-1}(U_i)$

Consider $i$ for top-$\kappa$ based on noisy score $Z_i$

Report top-$\kappa$.

**Proof.**

$$F^{-1}(U^{\exp(-m)}) = -\log(-\log(U^{\exp(-m)}))$$

$$= -\log(-\log(U)) \exp(-m))$$

$$= -\log(-\log(U)) - \log(\exp(-m))$$

$$= -\log(-\log(U)) - (-m)$$

$$= m - \log(-\log(U))$$

$$= m + F^{-1}(U)$$

\[
\Box
\]

**A.2. Replicability and Additional Experimental Results**

Datasets are described in Table 1. The sensitivities of all featured (count-based) datasets are presumed to be $\Delta = \frac{1}{2}$ via the shifting trick (cf. Theorem A.15).

We aggregate $C_{h,t}$ classes from Definition 3 into high utility predicates:

- **TOP**($y$) $\Leftrightarrow y \in C_{k-1,k}$ $\Leftrightarrow y \in \text{OPT}(\bar{x})$
- **GREAT**($y$) $\Leftrightarrow y \in C_{h,t}$ with $h \geq \frac{1}{10} k$ and $t \leq k + \frac{k}{10}$
- **GOOD**($y$) $\Leftrightarrow y \in C_{h,t}$ with $h \geq \frac{1}{100} k$ and $t \leq k + \frac{k}{2}$

The predicate TOP($y$) mandates $y$ to be the exact top-$k$. The predicate GREAT($y$) mandates for $y$ the inclusion of the top-$\lfloor \frac{1}{10} k \rfloor$ and exclusion of of items outside of top-$\lfloor \frac{1}{10} k \rfloor$. For $k = 100$ this means inclusion of all top-10 items and the remaining 90 items must come from top-110. The predicate GOOD($y$) mandates for $y$ the inclusion of the top-$\lfloor \frac{1}{100} k \rfloor$ and exclusion of items outside of top-$\lfloor \frac{1}{100} k \rfloor$.

Workflow of how each plot (with $\epsilon$ in x-axis) is generated:

- The vector $\bar{x}$ is fixed for one of the datasets and given as input to all mechanisms.
- The subset size $k \in \{10, 100, 1000\}$ is fixed
- The privacy loss $\epsilon \in \mathbb{R}_{\geq 0}$ is then varied with sufficient precision for plotting purposes
- For each data point $\bar{x}, k, \epsilon$ either the probability distribution over $C_{h,t}$ is computed (CANONICAL, CANONICAL-$\gamma$=1) or probabilities are estimated via Monte Carlo methods (using ideas from Section 2.1) with 10000 generated subset classes (ONESHOT, PEELING). Additionally, a subset is sampled for runtime measurements and validation purposes.

- Each plot (for fixed $\bar{x}, k$) then shows along the x-axis the $\epsilon$ (except Figure 2 and Figure 3 where the synthetic data distribution is varied) and along the y-axis the probability of either TOP-$k$, GREAT-$k$ or GOOD-$k$.

**Implementation details:**

- ONESHOTExp is implemented as the $\epsilon$-DP mechanism in Algorithm 2 with $\kappa = k$.
- PEELING samples $k$ times without replacement from the $\epsilon$-DP mechanism in Algorithm 2 with $\kappa = 1$.
- CANONICAL as the $\epsilon$-DP mechanism in Algorithm 1 with $\gamma = \frac{1}{2}$.
- CANONICAL-$\gamma$=1 as the $\epsilon$-DP mechanism in Algorithm 1 with $\gamma = 1$.

The inverse distribution function $F^{-1}$ used in the approaches (always standard distribution parameters):

| Approach     | $F^{-1}(p)$ | Distribution |
|--------------|-------------|--------------|
| ONESHOTExp   | $-\log(1-p)$ | Exponential  |
| PEELING      | $-\log(-\log(p))$ | Gumbel      |
| CANONICAL    | $-\log(-\log(p))$ | Gumbel      |
| CANONICAL-$\gamma$=1 | $-\log(-\log(p))$ | Gumbel      |

**Additional plots:**

- Figure 4 compares the Lipschitz mechanism with $\kappa = k \in \{1, 10, 100, 1000\}$ for different choices of $F^{-1}$.
- Figure 5 supplements Figure 4 with GREAT-$k$ results.
- Figures 6, 7, 8, 9, 10 replicate results from the paper for five additional datasets. Some datasets lack a sufficient number of users to reach good utility with any $\epsilon$-DP methods with $\epsilon < 1$. In some rare instances the top-$k$ can be one out of many arbitrary subsets (we break ties arbitrarily for top-$k$), because we do not modify $\bar{x}$ to break ties between uniform values.
Table 1. Featured real-world data sets. The first dataset is the Netflix Prize dataset (Bennett & Lanning, 2007). We used as input to all approaches a data vector \( \vec{x} \in \mathbb{R}^d \) with \( d \) scores. The other datasets and their corresponding data vectors were obtained from McKenna & Sheldon (2020).

**Figure 4.** Lipschitz mechanism with different noise distributions (\( \vec{x} = \text{NETFLIX} \in \mathbb{R}^{17770}, k = \kappa \in \{1, 10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0} \)). Different noise distribution instantiate many selection mechanisms from the literature (cf. Section A.3).
Figure 5. Comparison ($\bar{x} = \text{NETFLIX} \in \mathbb{R}^{17700}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$).

Figure 6. Comparison ($\bar{x} = \text{PATENT} \in \mathbb{R}^{4096}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$)
Differentially Private Top-k Selection via Canonical Lipschitz Mechanism

Figure 7. Comparison ($\vec{x} = \text{SEARCHLOGS} \in \mathbb{R}^{4096}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$)

Figure 8. Comparison ($\vec{x} = \text{MEDCOST} \in \mathbb{R}^{4096}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$)
Figure 9. Comparison ($\vec{x} = \text{INCOME} \in \mathbb{R}^{4096}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$)

Figure 10. Comparison ($\vec{x} = \text{HEPTH} \in \mathbb{R}^{4096}, k \in \{10, 100, 1000\}, \varepsilon \in \mathbb{R}_{>0}$)
A.3. The Lipschitz Mechanism: Overview

The standard Exponential, Gumbel, Laplace, Logistic and Half-Logistic distribution are examples of distributions that satisfy the Lipschitz property mandated in the Lipschitz mechanism:

- **Theorem A.3** The standard Exponential distribution with \( F(x) = 1 - \exp(-x) \) for \( x \in \mathbb{R}_{\geq 0} \) and \( F(x) = 0 \) for \( x \in \mathbb{R}_{< 0} \) and \( F^{-1} = -\log(1 - p) \) for \( p \in [0, 1) \) satisfies the Lipschitz condition from the Lipschitz mechanism.

- **Theorem A.6** The standard Gumbel distribution with \( F(x) = \frac{1}{2} \exp(-x) \) for \( x \in \mathbb{R}_{\geq 0} \) and \( F(x) = 1 - \frac{1}{2} \exp(-|x|) \) for \( x \in \mathbb{R}_{< 0} \) and \( F^{-1}(p) = \text{sgn}(p - \frac{1}{2}) \log(1 - |2p - 1|) \) for \( p \in [0, 1) \) satisfies the Lipschitz condition from the Lipschitz mechanism.

- **Theorem A.11** The standard Laplace distribution with \( F(x) = \frac{1}{\pi} \exp(-x) \) for \( x \in \mathbb{R}_{\geq 0} \) and \( F(x) = 1 - \frac{1}{\pi} \exp(-|x|) \) for \( x \in \mathbb{R}_{< 0} \) and \( F^{-1}(p) = \text{sgn}(p) \sqrt{2|1 - p|} \) for \( p \in [0, 1) \) satisfies the Lipschitz condition from the Lipschitz mechanism.

- **Theorem A.12** The standard Half-Logistic distribution with \( F(x) = \frac{1}{2} \exp(-x) \) for \( x \in \mathbb{R}_{\geq 0} \) and \( F(x) = 0 \) for \( x \in \mathbb{R}_{< 0} \) and \( F^{-1} = \log(1 + p) - \log(1 - p) \) for \( p \in [0, 1) \) satisfies the Lipschitz condition from the Lipschitz mechanism.

- **Theorem A.13** The standard Logistic distribution with \( F(x) = \frac{1}{1 + \exp(-x)} \) for \( x \in \mathbb{R} \) and \( F^{-1} = -\log(p/(1 - p)) \) for \( p \in [0, 1) \) satisfies the Lipschitz condition from the Lipschitz mechanism.

For some of these distributions the Lipschitz instantiates popular mechanisms from the literature:

- **Theorem A.7** The Lipschitz mechanism with \( F^{-1}(p) = -\log(-\log(p)) \) from the standard Gumbel distribution instantiates the exponential mechanism (for \( \kappa = 1 \)) and peeling technique (Durfee & Rogers, 2019) (for \( \kappa = k \)).

- **Theorem A.9** The Lipschitz mechanism with \( F^{-1}(p) = \text{sgn}(p - \frac{1}{2}) \log(1 - |2p - 1|) \) from the standard Laplace distribution instantiates/matches report-noisy max mechanism (Dwork et al., 2014) (for \( \kappa = 1 \)) and one-shot Laplace mechanism (Qiao et al., 2021) (for \( \kappa = k \)).

- **Theorem A.5** The Lipschitz mechanism with \( F^{-1}(p) = -\log(1 - p) \) from the standard Exponential distribution instantiates permute-and-flip (McKenna & Sheldon, 2020).

A.4. Exponential Lipschitz Mechanism: Permute-And-Flip Mechanism

The Lipschitz mechanism is \( \varepsilon \)-DP when adding exponentially distributed noise:

**Theorem A.3** (Lipschitz condition: Exponential distribution). Let \( F(x) = 1 - \exp(-x) \) and \( x, c \in \mathbb{R} \).

Then \( |\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c| \).

**Proof.**

\[
\begin{align*}
\exp(c) &= \exp(-x - (-x - c)) \\
\exp(c) &= \frac{\exp(-x)}{\exp(-x - c)} \\
\exp(c) &= \frac{1 - (1 - \exp(-x))}{1 - (1 - \exp(-x - c))} \\
\exp(c) &= \frac{1 - F(x)}{1 - F(x + c)}
\end{align*}
\]

From the last equality we obtain:

\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.
\]

The Lipschitz condition follows for \( c \in \mathbb{R} \) if it is met for \( c \geq 0 \):

**Lemma A.4.** Let \( F(x) \) be a strictly increasing function.

If \( \frac{1 - F(x)}{1 - F(x + c)} \leq \exp(c) \) for \( c \geq 0 \), then for \( x, c \in \mathbb{R} \) follows:

\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.
\]

**Proof.** case \( c \geq 0 \): As \( F(x) \) is strictly increasing it follows that \( \exp(-c) \leq 1 - \frac{1 - F(x)}{1 - F(x + c)} \). Also, from the statement we know that \( \frac{1 - F(x)}{1 - F(x + c)} \leq \exp(c) \) for \( c \geq 0 \). Thus, \( \exp(-c) \leq \frac{1 - F(x)}{1 - F(x + c)} \leq \exp(c) \), which implies

\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.
\]

case \( c < 0 \): Let \( c' = -c \geq 0 \) and \( x' = x + c \).

\[
1 \leq \frac{1 - F(x')}{1 - F(x' + c')} \leq \exp(c')
\]

\[
1 \leq \frac{1 - F(x + c)}{1 - F(x + c + c)} \leq \exp(-c)
\]

\[
1 \leq \frac{1 - F(x + c)}{1 - F(x)} \leq \exp(-c)
\]

\[
\exp(c) \leq \frac{1 - F(x)}{1 - F(x + c)} \leq 1 \leq \exp(-c)
\]
from which we obtain
\[ c \leq \log(1 - F(x)) - \log(1 - F(x + c)) \leq -c \]
and \(|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.

The Lipschitz mechanism with \( k = 1 \) instantiates the Permute-And-Flip mechanism \cite{McKenna2020} when adding exponentially distributed noise (confirming the results of \cite{Ding2021}):

**Theorem A.5** (Permute-and-Flip via Exponential Noise). If \( F^{-1}(p) = -\log(1 - p) \) then \( \Pr[Y = y] \) matches the Permute-and-Flip mechanism.

**Proof.** Let \( q_y = -\frac{1}{\Delta} \text{LOSS}(y) \) for a \text{LOSS} function with sensitivity \( \Delta \).

\[
Y = \arg\max_{y \in Y} \{ q_y - \log(1 - U_y) \} \\
= \arg\max_{y \in Y} \{ \exp(q_y) - \log(1 - U_y) \} \\
= \arg\max_{y \in Y} \{ \exp(q_y) \} / (1 - U_y) \\
= \arg\min_{y \in Y} \{ (1 - U_y) \exp(-q_y) \}
\]

This means for each \( y \) a (uniform) random number \( R_y \) between 0 and \( \exp(-q_y) \) is drawn and the smallest is selected. Let \( q_* = \max_{y \in Y} q_y \). Then any \( R_y > \exp(-q_*) \) will be certainly rejected and all non-rejected ones have the same probability of being the smallest. Let \( N \) be the number of accepted elements, such that \( N = \sum_{y \in Y \setminus \{y\}} B_z \) is a Poisson Binomial random variable where each summation term \( B_z \) is a Bernoulli random variable with success probability \( 1 - \Pr[Y \neq z] \). Then \( \Pr[Y = y|N] = \frac{1 - \Pr[Y \neq y]}{N+1} \). The rejection probability \( \Pr[Y \neq y] \) is then:

\[
\Pr[Y \neq y] = \Pr[R_y > \exp(-q_*)] \\
= \Pr[1 - U_y > \exp(-q_y) > \exp(-q_*)] \\
= \Pr[(1 - U_y) > \exp(q_y - q_*)] \\
= \Pr[U_y < 1 - \exp(q_y - q_*)] \\
= 1 - \exp(q_y - q_*)
\]

Selecting a random non-rejected item is equivalent to selecting the first non-rejected item if items are in a random order. This matches Permute-and-Flip, which goes through items in random order, rejects each item \( y \) with probability \( 1 - \exp(q_y - q_*) \) and then selects the first item that does not get rejected.

A.5. Gumbel Lipschitz Mechanism: Exponential Mechanism

The Lipschitz mechanism is \( \varepsilon \)-DP when adding Gumbel distributed noise terms \( F^{-1}(p) = -\log(1 - p) \):

**Theorem A.6** (Lipschitz condition: Gumbel distribution). Let \( F(x) = \exp(-\exp(-x)) \) and \( x, c \in \mathbb{R} \). Then:

\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|
\]

**Proof.** Due to Lemma A.4 one can presume without loss of generality that \( c \geq 0 \).

Let \( \alpha = 1 - F(x) \in [0, 1] \) and \( k = \exp(c) \geq 1 \), then from Lemma A.18 follows that:

\[
\frac{1 - (1 - \alpha)^{1/k}}{\alpha/k} \geq 1 \\
\frac{1 - (1 - \alpha)^{1/k}}{\alpha} \leq k \\
\frac{1 - F(x)}{1 - F(x) \exp(-c)} \leq \exp(c) \\
\frac{1 - F(x)}{1 - F(x) \exp(-c)} \leq \exp(c)
\]

The claim follows due to Lemma A.4.

The Lipschitz mechanism with \( \kappa = k = 1 \) instantiates the Exponential Mechanism \cite{McSherry2007} when adding the Gumbel distributed noise terms \( F^{-1}(p) = -\log(-\log(p)) \) for \( \kappa = k > 1 \) it matches the Peeling technique using the Exponential Mechanism \cite{Durfee2019}.
Theorem A.7 (Exponential Mechanism via Gumbel trick). If $F^{-1}(p) = -\log(-\log(p))$ for the Lipschitz mechanism with $k = 1$, then $Pr[Y = y] \propto \exp\left(\frac{-\text{LOSS}(y)}{\varepsilon/2\Delta}\right)$.

Proof. Let $\mathbb{Y}$ be the selection domain and $\lambda_y = \exp\left(-\frac{\text{LOSS}(y)}{\varepsilon/2\Delta}\right)$ for any $y \in \mathbb{Y}$. Then:

$$Y = \arg\max_{y \in \mathbb{Y}} \left\{ \frac{-\text{LOSS}(y)}{\varepsilon/2\Delta} - \log(-\log(U_y)) \right\}$$

$$= \arg\max_{y \in \mathbb{Y}} \left\{ \log(\lambda_y) - \log(-\log(U_y)) \right\}$$

$$= \arg\max_{y \in \mathbb{Y}} \left\{ \exp(\log(\lambda_y)) - \log(-\log(U_y)) \right\}$$

$$= \arg\max_{y \in \mathbb{Y}} \left\{ -\frac{\text{LOSS}(y)}{\varepsilon/2\Delta} \right\}$$

$$= \arg\min_{y \in \mathbb{Y}} \left\{ -\log(U_y)/\lambda_y \right\}$$

Each $\arg\min$ term $-\log(U_y)/\lambda_y$ is then an exponential random variable with rate $\lambda_y = \exp\left(-\frac{\text{LOSS}(y)}{\varepsilon/2\Delta}\right)$ and $Pr[Y = y] \propto \lambda_y$. A proof for this property can be found in the following Lemma A.8.

For independent events with exponentially distributed time delays, each event’s probability of preceding the others is proportional to their rate. This well-known property has for instance been used to prove the Gumbel Trick (Balog et al., 2017):

Lemma A.8 (Exponential clocks). Let $U_y$ be i.i.d. $U_y \sim Unif(0, 1)$ and $\mathbb{Y}$ be a finite set. Let $Y = \arg\min_{y \in \mathbb{Y}} \left\{ -\log(U_y)/\lambda_y \right\}$ supported over $\mathbb{Y}$. Then $Pr[Y = y] \propto \lambda_y$.

Proof. Let $E_y = -\log(U_y)/\lambda_y$.

As this is how one would generate an exponential random variable with rate $\lambda_y$, it follows that the density $Pr[E_y = x] = \lambda_y \exp(-x\lambda_y)$, cumulative $Pr[E_y \leq x] = 1 - \exp(-\lambda_y x)$ and complementary cumulative $Pr[E_y > x] = \exp(-\lambda_y x)$. The probability from the claim can then be written as:

$$Pr[Y = y] = Pr[\min_{z \in \mathbb{Y}} E_z]$$

$$= \int_{0}^{\infty} \prod_{z \in \mathbb{Y} \setminus \{y\}} \Pr[\max E_z > x] dx$$

$$= \int_{0}^{\infty} \lambda_y \exp(-x\lambda_y) \prod_{z \in \mathbb{Y} \setminus \{y\}} \exp(-x\lambda_z) dx$$

$$= \lambda_y \int_{0}^{\infty} \exp(-x \sum_{z \in \mathbb{Y}} \lambda_z) dx$$

$$= \lambda_y / \sum_{z \in \mathbb{Y}} \lambda_z$$

The last equality follows from $\int a \exp(-bx) dx = -\frac{a}{b} \exp(-bx) + c$ for any $a, b, c \in \mathbb{R}$ with $b > 0$. Let $a = \lambda_y$ and $b = \sum_{z \in \mathbb{Y}} \lambda_z$. Then $\int_{0}^{\infty} a \exp(-bx) dx = (\lim_{x \to \infty} -\frac{a}{b} \exp(-bx) + c) - (\frac{a}{b} \exp(-b \cdot 0) + c) = c - (\frac{a}{b} + c) = \frac{a}{b} = \lambda_y / \sum_{z \in \mathbb{Y}} \lambda_z$. Hence, $Pr[Y = y] \propto \lambda_y$.

A.6. Laplace Lipschitz Mechanism: Report Noisy Max Mechanism

Theorem A.9. Trivially, the Lipschitz mechanism with $k = 1$ instantiates the Report Noisy Max mechanism (Dwork et al., 2014) and the Oneshot Laplace Mechanism (Qiao et al., 2021) when adding Laplace distributed noise.

As the Lipschitz condition limits how fast a distribution function can change, satisfying the Lipschitz condition is inherited by doubled/mirrored distribution:

Lemma A.10. Let $F(x)$ be a strictly increasing function that satisfies the Lipschitz condition $|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|$.

Let $F_{\text{double}}(x) = \begin{cases} F(x) & x < 0 \\ \frac{1}{2} + \frac{F(x)}{2} & x \geq 0 \end{cases}$.

Then $F_{\text{double}}(x)$ also satisfies the Lipschitz condition $|\log(1 - F_{\text{double}}(x)) - \log(1 - F_{\text{double}}(x + c))| \leq |c|$.

Proof. Due to Lemma A.4, one can presume without loss of generality that $c \geq 0$.

Case $x \geq 0$ (which implies $x + c \geq 0$) where $F_{\text{double}}(x) = \frac{1}{2} + \frac{F(x)}{2}$.
The Lipschitz mechanism is \( \varepsilon \)-DP when adding Halflogistic distributed noise:

**Theorem A.12** (Lipschitz condition: Halflogistic distribution). Let \( F(x) = \frac{1-\exp(-x)}{1+\exp(-x)} \) and \( x, c \in \mathbb{R} \). Then 
\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.
\]

**Proof.** The exponential distribution satisfies the Lipschitz condition (see Theorem A.3) and the Laplace distribution satisfies the Lipschitz condition by being the double exponential distribution (see Lemma A.10). \( \square \)

### A.7. Logistic Lipschitz Mechanism

The Lipschitz mechanism is \( \varepsilon \)-DP when adding Halflogistic distributed noise:

**Theorem A.13** (Lipschitz condition: Logistic distribution). Let \( F(x) = \frac{1-\exp(-x)}{1+\exp(-x)} \) and \( x, c \in \mathbb{R} \). Then 
\[
|\log(1 - F(x)) - \log(1 - F(x + c))| \leq |c|.
\]

**Proof.** Follows from Lemma A.10 and Theorem A.12 as the Logistic distribution is defined as the “double” Halflogistic distribution. \( \square \)

### Proof of A.11

In the second step it is exploited that adding the same positive value to numerator and denominator can only move the ratio closer to 1 (see Lemma A.16).

Case \( x < 0 \) and \( x + c < 0 \) where \( F_{\text{double}}(x) = \frac{F(-x)}{2} \).

\[
\begin{align*}
\exp(-c) &\leq \frac{1 - F(-x)}{1 - F(-x - c)} \leq \exp(c) \\
\exp(-c) &\leq \frac{2 - F(-x)}{2 - F(-x - c)} \leq \exp(c) \\
\exp(-c) &\leq \frac{1 - F(-x)/2}{1 - F(-x - c)/2} \leq \exp(c) \\
\exp(-c) &\leq \frac{1 - F_{\text{double}}(x)}{1 - F_{\text{double}}(x + c)} \leq \exp(c)
\end{align*}
\]

The Lipschitz mechanism is \( \varepsilon \)-DP when adding Laplace distributed noise:

**Theorem A.11** (Lipschitz condition: Laplace distribution). Let \( F(x) = 1 - \exp(-x) \) and \( x, c \in \mathbb{R} \).

Let \( F_{\text{double}}(x) = \begin{cases} 
\frac{F(-x)}{2} & x < 0 \\
\frac{1}{2} + \frac{F(x)}{2} & x \geq 0
\end{cases} \).

Then 
\[
|\log(1 - F_{\text{double}}(x)) - \log(1 - F_{\text{double}}(x + c))| \leq |c|.
\]
A.8. Additional Theorems and Proofs

Lemma A.14 (canonical loss function). Let \( \mathcal{Y} \) be some discrete-valued output domain, \( y \in \mathcal{Y} \) and \( \text{OPT}^{-1}(y) \) comprise any score vectors \( \hat{x} \) s.t. \( y \in \text{OPT}(\hat{x}) \), where \( \text{OPT}(\hat{x}) \) is the optimal k-subset. Let \( \hat{x} \) have sensitivity \( \Delta_x \).

Then the function \( \text{LOSS} \) defined in the following has sensitivity \( \Delta_{\text{LOSS}} = 1 \):

\[
\text{LOSS}(y \mid \hat{x}) = \min_{\hat{v} \in \text{OPT}^{-1}(y)} \|\hat{x} - \hat{v}\|_\infty
\]

Proof. Let \( C \in \mathbb{R} \). In the following, \( \hat{u} \pm C \) is used as a shorthand for the subspace \( \hat{u} + \mathbb{R} \cap [-C, +C]^d \).

As \( \hat{x}_j = f(j \mid \hat{x})/\Delta_f \) for \( j \in \{1, \ldots, d\} \), a single user can change each component of \( \hat{x} \) by at most \( 1 \). If \( \hat{x} \) is replaced by \( \hat{x} \pm 1 \) then each term \( \|\hat{x} - \hat{v}\|_\infty \) is replaced by \( \|\hat{x} \pm 1 - \hat{v}\|_\infty \).

Based on the definition of the \( L_\infty \) norm, it then follows that \( \|\hat{x} - \hat{v}\|_\infty \leq \|\hat{x} \pm 1 - \hat{v}\|_\infty \leq 1 \).

Then the values \( \|\hat{x} - \hat{v}\|_\infty \) for different \( \hat{v} \in \text{OPT}^{-1}(y) \) form a set over which a minimum is taken. If all values of a set change by at most \( C \), then their extremum also change at most by \( C \) (see Lemma A.22 in supplementary material). Hence the sensitivity of \( \text{LOSS}(y \mid \hat{x}) \) is equal to \( \Delta_{\text{LOSS}} = 1 \). \( \square \)

In the context of shift-invariant selection mechanisms, one can apply the following shifting trick to obtain a reduced sensitivity analysis for Canonical Lipschitz functions and alike:

Theorem A.15 (asymmetric sensitivity). Let \( \hat{a}, \hat{b} \in \mathbb{X} \) with \( \text{users}(\hat{a}) \subset \text{users}(\hat{b}) \) and \( |\text{users}(\hat{b})| = |\text{users}(\hat{a})| + 1 \).

Let \( f : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R} \), \( \Delta_1, \Delta_2 \in \mathbb{R}_{\geq 0} \) and \( f(y \mid \hat{b}) - f(y \mid \hat{a}) \leq \Delta_1 \) and \( f(y \mid \hat{b}) - f(y \mid \hat{a}) \leq \Delta_2 \) for any \( y \in \mathcal{Y} \). Then the function \( g(y \mid \hat{x}) = f(y \mid \hat{x}) + |\text{users}(\hat{x})| \Delta_1 - \Delta_2 \) has sensitivity \( \Delta_1 + \Delta_2 \).

Proof. By definition \( |\text{users}(\hat{b})| - |\text{users}(\hat{a})| = 1 \). Thus, for any \( y \in \mathcal{Y} \) it holds that \( g(y \mid \hat{b}) - g(y \mid \hat{a}) = f(y \mid \hat{b}) - f(y \mid \hat{a}) + \Delta_1 - \Delta_2 \leq \Delta_1 + \Delta_2 \) and \( g(y \mid \hat{b}) - g(y \mid \hat{a}) = f(y \mid \hat{b}) - f(y \mid \hat{a}) - \Delta_1 + \Delta_2 \leq \Delta_1 - \Delta_2 \leq \Delta_1 + \Delta_2 \). \( \square \)

For positive reals \( a, b, c \) the ratio of \( \frac{b+c}{a+c} \) is smaller than \( \frac{b}{a} \), because \( b+c \) and \( a+c \) are closer to each other than \( a \) and \( b \).

Lemma A.16. Let \( a, b, c \) be positive reals with \( b > a \). Then:

\[
1 \leq \frac{b+c}{a+c} \leq \frac{b}{a}
\]

Proof. Assume that \( \frac{b+c}{a+c} > \frac{b}{a} \). Thus, we get \( (b+c)a > (a+c)b \), \( ab + ca > ab + cb \), and \( a > b \) which contradicts \( b > a \) in the statement. Thus, \( \frac{b+c}{a+c} \leq \frac{b}{a} \). From \( b > a \), we get \( \frac{b+c}{a+c} > 1 \) and thus \( \frac{b+c}{a+c} \geq 1 \). Therefore, we obtain

\[
1 \leq \frac{b+c}{a+c} \leq \frac{b}{a}
\]

Lemma A.17. Let \( n, x \in \mathbb{R} \) with \( n \geq 1 \) and \( x > -n \). Then:

\[
1 + x \leq \left( 1 + \frac{x}{n} \right)^n \leq \exp(x)
\]

Proof. A variant of Bernoulli’s inequality is \( 1 + nx' \leq (1 + x')^n \) for any reals \( n \geq 1 \) and \( x' \geq 1 \). Due to \( x > -n \), one can pick \( x' = \frac{x}{n} \geq -1 \) to obtain \( 1 + x \leq (1 + \frac{x}{n})^n \).

Then \( 1 + x \leq \exp(x) \) for any \( x \in \mathbb{R} \) is a well-known inequality due to the following. The derivative of \( f(n) = (1 + \frac{x}{n})^n \) with respect to \( n \) is \( f'(n) = (1 + \frac{x}{n})^{n-1} \) and \( f'(n) \geq 0 \) as \( 1 + \frac{x}{n} \geq 0 \) for \( x > -n \). Thus, for \( x > -n \) the function \( f(n) \) is increasing and base case \( f(1) = 1 + x \). The rest follows from \( \lim_{n \to \infty} (1 + \frac{x}{n})^n = \exp(x) \) which is a well-known identity that offers one way to define the exponential function. It follows a simple proof using L’Hôpital’s rule:

\[
\lim_{n \to \infty} (1 + \frac{x}{n})^n = \exp(\lim_{n \to \infty} \log(1 + \frac{x}{n}))
\]

\[
= \exp(\lim_{n \to \infty} \frac{\log(1 + \frac{x}{n})}{1/n})
\]

\[
= \exp(\lim_{n \to \infty} \frac{\frac{dx}{dn} \log(1 + \frac{x}{n})}{\frac{1}{1/n}})
\]

\[
= \exp(\lim_{n \to \infty} \frac{x(-1/n^2)}{-1/n^2})
\]

\[
= \exp(x)
\]

\( \square \)

Lemma A.18. Let \( \alpha \in [0, 1], k \in \mathbb{N} \). Then:

\[
1 < \frac{1 - (1 - \alpha)^{1/k}}{\alpha/k} \leq \log\left(\frac{1}{1 - \alpha}\right)/\alpha
\]

which follows that \( r_{\alpha,k} = \frac{1 - (1 - \alpha)^{1/k}}{\alpha/k} < \log(4) \) for \( \alpha \leq 0.5 \) and \( r_{\alpha,k} < 1.06 \) for \( \alpha \leq 0.1 \).
Proof. This relates to Bonferroni and Šidák corrections for family-wise error rates (FWER) in hypothesis testing. The Šidák correction $\alpha_s = 1 - (1 - \alpha)^k$ is exact in case of independence, i.e., $(1 - \alpha_s)^k = 1 - \alpha$, whereas the Bonferroni correction $\alpha_b = \alpha/k$ is in that case conservative, i.e., $(1 - \alpha_b) > 1 - \alpha$ (the success rate is unnecessarily large). This also means $\alpha_s > \alpha_b$ and their ratio $R = r_{\alpha,k} = \frac{\alpha_s}{\alpha_b} > 1$. According to Lemma A.17 for $x \in \mathbb{R}$, $k \in \mathbb{N}$ with $x \geq -k$ it holds that $(1 - \frac{x}{k})^k \leq \exp(-x)$. Thus, we can set $x = -\alpha \cdot R = -\alpha \frac{\alpha_s}{\alpha_b} = -k \alpha_s \geq -k$ to obtain:

$$(1 - \frac{\alpha}{k} R)^k \leq \exp(-\alpha \cdot R)$$

Clearly, $\frac{\alpha}{k} R = \alpha_b \frac{\alpha_s}{\alpha_b} = \alpha_s$. And we know $1 - \alpha = (1 - \alpha_s)^k$. Therefore by inserting $\alpha_s = \frac{\alpha}{k} R$ we get $1 - \alpha = (1 - \frac{\alpha}{k} R)^k$. Thus, we can continue with:

$$
(1 - \alpha) = (1 - \frac{\alpha}{k} R)^k \leq \exp(-\alpha \cdot R)
$$

$$
\log(1 - \alpha) \leq -R
$$

$$
-\frac{1}{1 - \alpha} \leq -\frac{1}{R}
$$

$$
R \leq \frac{1}{(1 - \alpha) / \alpha}
$$

The rest follows from $\log \frac{1}{(1 - \alpha) / \alpha}$ being strictly decreasing. □

**Lemma A.19 (EM utility guarantees).** Let $\mathcal{Y}$ be the selection domain, $\varepsilon \in \mathbb{R}_{\geq 0}$ and $\Delta$ be the sensitivity of the loss function $\text{LOSS}(y \mid \hat{x})$.

If $Y$ is a random variable supported over $\mathcal{Y}$ with $P_r[Y = y] \propto \exp(\varepsilon \text{LOSS}(y \mid \hat{x}))$, then with probability $1 - \alpha$ it holds that $\text{LOSS}(Y) \leq \text{LOSS}(\text{OPT}) + \mathcal{E}$ with:

$$
\mathcal{E} = \frac{2\Delta}{\varepsilon}(\log\left(\frac{|\mathcal{Y}|}{|\text{OPT}|}\right) + \log\left(\frac{1}{\alpha}\right))
$$

where OPT are all selection options with minimal loss.

Proof. Let $u(y) = -\text{LOSS}(y \mid \hat{x})$. Theorem 3.11 in [Dwork & Roth 2013):

$$
P_r[u(Y) \leq u(\text{OPT}) - \frac{2\Delta}{\varepsilon}(\log\left(\frac{|\mathcal{Y}|}{|\text{OPT}|}\right) + c) \leq \exp(-c)
$$

$$
P_r[u(Y) > u(\text{OPT}) - \frac{2\Delta}{\varepsilon}(\log\left(\frac{|\mathcal{Y}|}{|\text{OPT}|}\right) + c) \geq 1 - \exp(-c)
$$

With probability $1 - \exp(-c)$:

$$
u(Y) > u(\text{OPT}) - \frac{2\Delta}{\varepsilon}(\log\left(\frac{|\mathcal{Y}|}{|\text{OPT}|}\right) + c)
$$

**Lemma A.20 (CANONICAL utility loss bounds).** Let $Y_1, \ldots, Y_k$ be the selected set by CANONICAL with $\gamma \in (0, 1]$ and $F^{-1}$ from the Gumbel distribution with tail item $T = \arg \min_{i \in \{Y_1, \ldots, Y_k\}} \hat{x}[i]$. With at least probability $1 - \alpha$:

$$
\hat{x}[T] < \hat{x}[i] + \frac{2\Delta}{\varepsilon}(\log(d/k) + \log\left(\frac{1}{\alpha}\right) + \log(c_{d,k}))
$$

Proof. For $\gamma = 1$ it follows directly that the loss value of each subset is $\hat{x}[i]$, that the optimal loss is $-\text{LOSS}(\text{OPT} \mid \hat{x}) = \hat{x}[k]$ and the logarithm of the domain size is $\log k \gamma \leq \log\left(\frac{d}{\gamma}\right) = k \log(d/k) + \log(c_{d,k})$ with $1 < c_{d,k} = \frac{(d/\gamma)^k}{F(\gamma)} \leq \exp(k)$. Due to Corollary 5.2 for $\gamma < 1$ the privacy loss $\gamma$ must simply be replaced with $\gamma/e$.

**Lemma A.21 (PEELING utility loss bounds).** Let $Y_1, \ldots, Y_k$ be the selected set by PEELING with $T = \arg \min_{j \in \{Y_1, \ldots, Y_k\}} \hat{x}[j]$. Let $r_{\alpha,k} = \frac{1 - (1 - \alpha)^{1/k}}{\alpha/k}$. Then with probability $(1 - \alpha)$:

$$
\hat{x}[T] < \hat{x}[k] + \frac{2\Delta}{\varepsilon}(\log(d/k) + \log\left(\frac{1}{\alpha}\right) - k \log(r_{\alpha,k}))
$$

Proof. Each selection has $|\mathcal{Y}| = d$ and $-\text{LOSS}(\text{OPT}) \geq \hat{x}[k]$. Let $\alpha' = 1 - (1 - \alpha)^{1/k}$. If each of the $k$ selections $Y_i$ satisfies $\text{LOSS}(Y_i) < \hat{x}[k] + \frac{2\Delta \log(d/\alpha')}{\varepsilon}$ with probability $\alpha'$, then all items $Y_1, \ldots, Y_k$ satisfy $\text{LOSS}(Y_i) < \hat{x}[k] + \frac{2\Delta \log(d/\alpha)}{\varepsilon}$ with probability $\alpha$, which includes the tail item $T \in \{Y_1, \ldots, Y_k\}$. By replacing $\log(\alpha')$ in $\text{LOSS}(Y_i \mid \hat{x}) < \hat{x}[k] + \frac{2\Delta \log(d/\alpha')}{\varepsilon}$ with $\log(1 - \sqrt{1 - \alpha}) = \log(\alpha/k) + \log(r_{\alpha,k})$ one then obtains the claim. □

**Lemma A.22.** Let $A, B, C \in \mathbb{R}$ and $(A \pm B)$ be a shorthand for the interval $[A - B, A + B]$. Let $a, b, \ldots, z \in \mathbb{R}$. Then if $a' \in (a \pm C), b' \in (b \pm C), \ldots, z' \in (z \pm C)$, it holds that:

$$
\min\{a', b', \ldots, z'\} \in (\min\{a, b, \ldots, z\} \pm C)
$$

$$
\max\{a', b', \ldots, z'\} \in (\max\{a, b, \ldots, z\} \pm C)
$$
Proof. Follows from $\min\{a' - 1, \ldots, z' - 1\} \leq \min\{a', \ldots, z'\} \leq \min\{a' + 1, \ldots, z' + 1\}$ and analogously $\max\{a' - 1, \ldots, z' - 1\} \leq \max\{a', \ldots, z'\} \leq \max\{a' + 1, \ldots, z' + 1\}$.

Lemma A.23 (top-$k$ canonical loss function). Let $y \in C_{h,t}$ be a $k$-subset of $\{1, \ldots, d\}$.

\[
\text{LOSS}(y \mid \bar{x}) = \min_{\bar{v} \in \text{OPT}^{-1}(y)} \|\bar{x} - \bar{v}\|_\infty = \frac{\bar{x}_{[h+1]} - \bar{x}_{[t]}}{2}
\]

Proof. If $t = k$, then $y \in \text{OPT}(\bar{x})$ and $\text{LOSS}(y \mid \bar{x}) = 0$.

If $t > k$, then from $y \in C_{h,t}$ follows that $\{j_1, \ldots, j_h\} \subset y$, but the top-$k$ item $j_{h+1} \notin y$. In order for $y$ to become optimal all of its items need to catch up with the missing top-$k$ item $j_{h+1}$. The tail item $\bar{x}[t]$ has the largest gap to $\bar{x}_{[h+1]}$. Let $g = \bar{x}_{[h+1]} - \bar{x}[t]$ be the gap between $\bar{x}_{[h+1]}$ and $y$’s tail $\bar{x}[t]$ that must become 0 for $y$ to become an optimal solution.

Let $\bar{a} \in \mathbb{R}^d$ with $\bar{a}_i = \begin{cases} 1 & i \in y \\ 0 & \text{otherwise} \end{cases}$.

Let $\bar{v} = \bar{x} + \frac{g}{2} \bar{a} - \frac{g}{2}(1 - \bar{a})$, which increases any $\bar{x}_j$ with $j \in y$ by $\frac{g}{2}$ and decreases all others by $\frac{g}{2}$. Through algebraic reformulations one gets $\bar{v} = \bar{x} + g\bar{a} - \frac{g}{2} \bar{a}$.

Then $y \in \text{OPT}(\bar{v})$ for $\bar{v} = \bar{x} + \frac{g}{2} \bar{a} - \frac{g}{2}(1 - \bar{a}) = \bar{x} + g\bar{a} - \frac{g}{2} \bar{a}$.

\[
\text{LOSS}(y \mid \bar{x}) = \|\bar{x} - \bar{v} + g(\bar{a} - \bar{a})\|_\infty = \frac{\|\bar{x}_{[h+1]} - \bar{x}_{[t]}\|_\infty}{2}
\]

Due to the definition of the $L_\infty$ norm $\|\bar{x} - \bar{a}\|_\infty = \frac{g}{2}$, because $\max(\frac{g}{2} - g\bar{a}) = \frac{g}{2}$ and $\min(\frac{g}{2} - g\bar{a}) = \frac{g}{2} - g$. Hence one gets:

\[
\text{LOSS}(y \mid \bar{x}) = \frac{g}{2} = \frac{\bar{x}_{[h+1]} - \bar{x}_{[t]}}{2}
\]

Lemma A.24. Let $k, d \in \mathbb{N}$ with $k < d$. Let $\bar{x} \in \mathbb{R}^d$ and $y = \{y_1, \ldots, y_k\} \subset \{1, \ldots, d\}$. Let $\bar{y} \in \mathbb{R}^k$ and $\bar{y}_i = \bar{x}_{y_i}$ for $\ell \in \{1, \ldots, k\}$. Let $j_1, \ldots, j_d$ be indices $\{1, \ldots, d\}$ sorted by $\bar{x}$ and $i_1, \ldots, i_k$ be indices $\{1, \ldots, k\}$ sorted by $\bar{y}$ such that:

$\bar{x}_{j_1} \geq \bar{x}_{j_2} \geq \ldots \geq \bar{x}_{j_d}$ and $\bar{y}_{i_1} \geq \bar{y}_{i_2} \geq \ldots \geq \bar{y}_{i_k}$

Let $\bar{x}[\ell] = \bar{x}_{j_\ell}$ for any $\ell \in \{1, \ldots, d\}$ and $\bar{y}[\ell] = \bar{y}_{i_\ell}$ for any $\ell \in \{1, \ldots, k\}$. Let $C_{h,t}$ for any $h \in \{0, \ldots, k - 1\}$ and $t \in \{k, \ldots, d\}$ be defined as in Definition 3.4 which implies $\{j_1, \ldots, j_h\} \subset y$ and $y[y_h] = t$. Let:

\[
\text{CANONICAL}(y \mid \bar{x}) = \frac{\bar{x}_{[h+1]} - \bar{y}[t]}{2} \text{ for } y \in C_{h,t}
\]

\[
\text{JOINT}(y \mid \bar{x}) = \max_{\ell \in \{1, \ldots, k\}} \frac{\bar{x}_{[\ell]} - \bar{y}[\ell]}{2}
\]

Let $\text{OPT}^{-1}(y)$ be a $d$-dimensional vector space where the indices $y$ have the $k$ largest values of each vector (allowing for ties), then:

(i) $\forall \bar{x} \in \mathbb{R}^d, y = \{y_1, \ldots, y_k\} \subset \{1, \ldots, d\}$:

\[
\text{JOINT}(y \mid \bar{x}) < \text{CANONICAL}(y \mid \bar{x})
\]

Proof. Claim (i) follows directly from Lemma A.23 where CANONICAL is matches the subset loss function $\text{LOSS}(y \mid \bar{x})$.

Claim (ii) follows from the following example.

Let $\bar{x} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]^T$ and $y = \{1, 5, 10\}$. As the index 9 is missing from $y$, each vector $\bar{z} \in \text{OPT}^{-1}(y)$ cannot have a larger value for the component with index 9 than for the component with index 1, i.e., $z_9 \leq z_1$. Then:

\[
\min_{\bar{z} \in \text{OPT}^{-1}(y)} \|\bar{x} - \bar{z}\|_\infty = \frac{9 - 1}{2} = \frac{8}{2} = 4
\]

\[
\text{CANONICAL}(y \mid \bar{x}) = \frac{9 - 1}{2} = \frac{8}{2} = 4
\]

\[
\text{JOINT}(y \mid \bar{x}) = \max\{10 - 10, 9 - 5, 8 - 1\} = \frac{7}{2}
\]

Intuitively, $\frac{7}{2}$ is the maximal change to the scores $\{1, 5, 10\}$ to make them as good as $\{8, 9, 10\}$, i.e., $\{1 + \frac{7}{2}, 5 + \frac{7}{2}, 10\} = \{4.5, 8.5, 10\}$ is not worse than $\{8 - \frac{7}{2}, 9 - \frac{7}{2}, 10\} = \{4.5, 5.5, 10\}$. It does not match $\min_{\bar{z} \in \text{OPT}^{-1}(y)} \|\bar{x} - \bar{z}\|_\infty$, because raising $\bar{x}$ in all components of $y$ by $\frac{7}{2}$ and decreasing all others by $\frac{7}{2}$ will not produce a vector in $\text{OPT}^{-1}(y)$, because $9 - \frac{7}{2} = 5.5$ is still larger than $1 + \frac{7}{2} = 4.5$ and that index is not featured in $y$. In contrast, $9 - 4 = 5$ is not larger than $1 + 4 = 5$.

The function JOINT matches the right-hand-side of the equation in Lemma 5 of (Joseph et al., 2021). The factors $\frac{1}{2}$ in the proof are due to the $L_\infty$ norm, i.e., $\arg \min_{\bar{z} \in \text{OPT}^{-1}(y)} \|\bar{x} - \bar{z}\|_\infty$ cannot only have larger values than $\bar{x}$ for indices contained in $y$, but also smaller values for indices missing from $y$. This corresponds to users being able to both raise and lower all scores by the sensitivity value (cf. Definition 1.1), which for count-based functions can be halved in the context of selection mechanisms (cf. Theorem A.13).