Recognition of collapsible complexes is NP-complete

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Abstract

We prove that it is NP-complete to decide whether a given (3-dimensional) simplicial complex is collapsible. This work extends a result of Malgouyres and Francés showing that it is NP-complete to decide whether a given simplicial complex collapses to a 1-complex.

1 Introduction

A classical question often considered in algebraic topology is whether some topological space is contractible. When we consider this question as an algorithmic question, that is, we consider the topological space as an input for an algorithm (given as a finite simplicial complex), then it turns out that this question is algorithmically undecidable. (Since the author is not aware of a reference, we offer a proof of such result in Appendix A.)

Although the contractibility question is undecidable, it is still important to recognize contractible simplicial complexes whenever this is possible. An important tool, introduced by Whithead, is collapsibility. Roughly speaking, a simplicial complex is collapsible if it can be shrunk to a point by a sequence of face collapses, preserving the homotopy type. (Precise definition is given in the following section.) In particular collapsible complexes are contractible.

We consider the collapsibility problem as an algorithmic question and we show that this question is NP-complete. More precisely, we obtain NP-completeness even if we restrict ourselves to 3-dimensional complexes.

Theorem 1. It is NP-complete to decide whether a given 3-dimensional simplicial complex is collapsible.

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1The author is aware of two research articles claiming that this result appears in [Hak73]. However, in the author’s opinion, these references are faulty (most probably because of confusion of contractible and simply-connected complexes).
It is easy to see that this problem belongs to NP (it is just sufficient to guess a right sequence of elementary collapses), thus the core of our paper relies on showing NP-hardness.

**Previous work.** Eğecioğlu and Gonzalez [EG96] have shown that it is (strongly) NP-complete to decide whether a given 2-dimensional complex can be collapsed to a point by removing at most \( k \) 2-faces where \( k \) is a part of the input. This problem, however, becomes polynomial-time solvable when \( k \) is fixed as pointed out by Joswig and Pfetsch [JP06] or Malgouyres and Francés [MF08]. In particular, deciding whether a 2-dimensional complex collapses to a point is polynomial-time solvable. The same approach yields to the fact that deciding whether a \( d \)-complex collapses to a \((d-1)\)-complex is polynomial time solvable. Since the author is not aware of a reference, we include a simple proof here; see Proposition 5.

Malgouyres and Francés [MF08] have shown that it is NP-complete to decide whether a given 3-dimensional complex collapses to some 1-complex. Naturally, they asked what is the complexity of the problem deciding whether a given 3-complex is collapsible. Theorem 1 answers this question in terms of NP-completeness.

Our approach relies, in a significant part, on the work of Malgouyres and Francés. We sketch their proof as well as point out the differences in Section 3.

**Links to Morse theory.** Using the result of Eğecioğlu and Gonzalez [EG96], Joswig and Pfetsch [JP06] proved that it is (strongly) NP-complete to decide whether there exists a Morse matching with at most \( c \) critical cells where \( c \) is a part of the input. If we use the fact that a simplicial complex is collapsible if and only if it admits a perfect Morse matching (see Forman [For98]), we obtain a reformulation of the main result.

**Theorem 2.** It is NP-complete to decide whether a given 3-dimensional simplicial complex admits a perfect Morse matching.

For further details on Morse matchings in the computational complexity context we refer to Joswig and Pfetsch [JP06].

**Links to shape-reconstruction.** The task of shape reconstruction is to reconstruct a shape from a set of points that sample the shape. An important subtask is to reconstruct the homotopy type or the homeomorphism type of the shape. In a recent work of Attali and Lieutier [AL12], the task is to collapse the Rips complex or the Čech complex of the sampling set to a complex homeomorphic with the shape. In this context, Theorem 1 mean certain limitations for results one can expect. In particular, specific treatment using properties of Rips and Čech complexes seem important.

**Another notion of collapsibility.** In the context of Helly-type theorems in discrete geometry Wegner [Weg75] introduced a notion of \( d \)-collapsibility. This notion shares some properties with collapsibility, but for example, it does not preserve the homotopy type. The author has shown in [Tan10] that the recognition of \( d \)-collapsible complexes...
is NP-hard for $d \geq 4$. We remark that the approach in that case is different and the result from [Tan10] should not be confused with the result presented here on classical (Whitehead’s) collapsibility.

2 Preliminaries

**Simplicial complexes.** We work with finite (abstract) simplicial complexes, that is, with set systems $K \subseteq 2^V$ such that $V$ is a finite set, and if $\alpha \in K$ and $\beta \subset \alpha$, then $\beta \in K$. We recall few basic definitions; however, we also assume that the reader is familiar some basic properties of simplicial complexes. Otherwise we refer to any of the books [Hat01, Mat03, Mun84]. In particular, we assume that the reader is familiar with correspondence of abstract simplicial complexes and geometric simplicial complexes since it will be very convenient in the further text to define some simplicial complexes by pictures.

Elements of a simplicial complex $K$ are faces (or simplices). A $k$-face is a face of dimension $k$, that is, a face in $K$ of size $k + 1$. 0-dimensional, 1-dimensional, and 2-dimensional faces are vertices, edges, and triangles respectively.

When we consider a simplicial complex as an input for an algorithm, it is given by a list of all faces.

**Collapsibility.** Let $\sigma$ be a nonempty non-maximal face of $K$. We say that $\sigma$ is free if it is contained in only one maximal face $\tau$ of $K$. Simplicial complex $K'$ obtained by removing $\sigma$ and all faces above $\sigma$,

$$K' := K \setminus \{\varnothing \in K : \sigma \subseteq \varnothing\},$$

is an elementary collapse of $K$.

We say that a complex $L$ is a collapse of $K$ (or that $K$ collapses to $L$) if there exist a sequence of complexes $(K_1 = K, K_2, \ldots, K_{m-1}, K_m = L)$ such that $K_{i+1}$ is an elementary collapse of $K_i$ for any $i \in \{1, \ldots, m-1\}$. A simplicial complex $K$ is collapsible if it collapses to a point.

Let $(K_1 = K, K_2, \ldots, K_{m-1}, K_m = L)$ be a sequence as above. Then for every $\eta \in K \setminus L$ there is a unique complex $K_i$ such that $\eta \in K_i$ and $\eta \notin K_{i+1}$. Then we say that $\eta$ collapses in this step. In particular, we will often use phrases such as ‘$\eta_1$ collapses before $\eta_2$’.

**Collapsibility with constrains.** In our constructions, we will often encountered the following situation: We will be given a complex $L$ glued to some other complexes forming a complex $M$. We will know some collapsing sequence of $L$ and we will want to use this collapsing sequence for $M$. This might or might not be possible. We will set up a condition when it is possible.
Definition 3. Let $M$ be a simplicial complex and $L$ be a subcomplex of $M$. We define the constrain complex of pair $(M, L)$ as

$$\Gamma = \Gamma(M, L) := \{ \vartheta \in L : \vartheta \subseteq \eta \text{ for some } \eta \in M \setminus L \}.$$ 

The constrain complex is obviously a subcomplex of $L$. Now we can present condition when collapsing of $L$ induces collapsing of $M$.

Lemma 4. Let $M$ be a complex, $L$ subcomplex of $M$ and $\Gamma$ be the constrain complex of $(M, L)$. We also assume that $L$ collapses to $L'$ containing $\Gamma$. Then $M$ collapses to $M' := L' \cup (M \setminus L)$.

Proof. Let $(L_1 = L, L_2, \ldots, L_{m-1}, L_m = L')$ be a sequence of elementary collapses. Let $\sigma_i$ be the face of $L_i$ which is collapsed in order to obtain $L_{i+1}$ and let $\tau_i$ be the unique maximal face in $L_i$ containing $\sigma_i$. We also set $M_i = L_i \cup (M \setminus L)$. Condition from the statement ensure us that all superfaces of $\sigma_i$ in $M_i$ belong to $L_i$. Therefore $(M_1 = M, M_2, \ldots, M_{m-1}, M_m = M')$ is a sequence of elementary collapses still induced by $\sigma_i$ and $\tau_i$. \hfill \Box

Collapsibility in codimension 1. Here we show that collapsibility in codimension 1 is polynomial-time solvable. For this we need the following proposition. The proposition implies that we can collapse an input $d$-complex $K$ greedily, and with this greedy algorithm, we obtain a $(d-1)$-complex if and only if $K$ collapses to a $(d-1)$-complex $L$.

Proposition 5. Let $K$ be a $d$-complex which collapses to a $(d-1)$-complex $L$ and to some $d$-complex $M$. Then $M$ collapses to a $(d-1)$-complex.

Proof. Let $(K_1 = K, K_2, \ldots, K_{m-1}, K_m = L)$ be a sequence of elementary where the collapse from $K_i$ to $K_{i+1}$ is induced by faces $\sigma_i$ and $\tau_i$. Note that every $d$-dimensional face of $K$ is $\tau_i$ for some $i$.

Let $j$ be smallest index such that $\tau_j$ belongs to $M$ and let $\eta_j$ be a $(d-1)$-face with $\sigma_j \subseteq \eta_j \subset \tau_j$. Note also that no $\tau_i$ with $i < j$ belongs to $M$ due to our choice of $j$. However, since $\sigma_j$ is free in $K_j$ and therefore $\eta_j$ is free as well, the only $d$-faces of $K$ containing $\eta_j$ might be the faces $\tau_i$ with $i \leq j$. Altogether, $\tau_j$ is the unique $d$-face of $M$ containing $\eta_j$ and we can collapse $\eta_j$ (removing $\tau_j$). If $M$ is still $d$-dimensional, we repeat our procedure. After finitely many steps we obtain a $(d-1)$-complex. \hfill \Box

3 Approach of Malgouyres and Francés

In this section we describe the approach of Malgouyres and Francés [MF08]. In some steps we follow their approach almost exactly; however, there are also steps that have
to be significantly modified in order to obtain the result. We will emphasize the steps where our approach differs.

Given a 3-CNF formula $\Phi$, Malgouyres and Francés construct a 3-dimensional complex $C(\Phi)$ such that $C(\Phi)$ collapses to a 1-complex if and only if $\Phi$ is satisfiable. They compose $C(\Phi)$ of several smaller complexes that we will call gadgets. For every literal $\ell$ in the formula they introduce literal gadget $C(\ell)$ (as well as $C(\overline{\ell})$ is introduced for $\overline{\ell}$) so that $C(\ell)$ and $C(\overline{\ell})$ are glued along an edge so that a major part of only $C(\ell)$ or only $C(\overline{\ell})$ can be collapsed in the first phase of collapsing. Another gadget is a conjunction gadget $C_{\text{and}}$ glued to literal gadgets via clause gadgets so that $C_{\text{and}}$ can be collapsed at some step if and only if every clause contains a literal $\ell$ such that the major part of $C(\ell)$ was already collapsed, that is, if and only if $\Phi$ is satisfiable. As soon as $C_{\text{and}}$ is collapsed, it makes few other faces of literal gadgets free which enables the whole complex to collapse to a 1-dimensional complex. As it follows from the construction of Malgouyres and Francés, the resulting 1-complex contains many cycles and therefore it cannot be further collapsed to a point.

Our idea relies on filling the cycles of the resulting 1-complex so that we can further proceed with collapsings. However, we cannot fill the cycles completely naively, since we do not know in advance which 1-complex we obtain. In addition filling these cycles naively could possibly introduce new collapsing sequences starting with edges on the boundaries of the filled cycles which could possibly yield to collapsing the complex even if the formula were not satisfiable. Therefore, we have to be very careful with our construction (which unfortunately means introducing few more technical steps).

We are going to construct a simplicial complex $K(\Phi)$ such that $K(\Phi)$ collapses to a point if and only if $\Phi$ is satisfiable. In fact, our complex $K(\Phi)$ will always be contractible, independently of $\Phi$ (although we do not need this fact in our reduction; and therefore we do not prove it). We reuse literal and conjunction gadgets of Malgouyres and Francés (only with minor modifications regarding distinguished subgraphs). Unfortunately, we need to replace very simple clause gadget of Malgouyres and Francés (it consists of two triangles sharing an edge or is even simpler). For this we need to introduce Bing’s house with three rooms and three thick walls. We also need disk gadgets which fill the cycles in the resulting 1-complex. We remark that the disk gadgets will not be topological disks but only some contractible complexes. However, we keep the name disk gadgets because of the idea of filling the cycles.

4 Bing’s rooms and Bing’s house with three rooms

In our reduction we will need several auxiliary constructions that we suitably glue together. We present them in this section.

**Bing’s rooms.** We will consider Bing’s house as a simplicial complex obtained by gluing two smaller simplicial complexes called Bing’s rooms. Later we will use these
rooms for building more complicated Bing’s house with three rooms. Bing’s room with a thin wall is a complex depicted in Figure 1 on the left and Bing’s room with a thick wall is in the middle. The room with a thin wall contains only 2-dimensional faces whereas the room with a thick wall contains one 3-dimensional block obtained by thickening one of the walls. Both rooms contain two holes in the ground floor and on hole in the roof. If we suitably collapse the thick wall of Bing’s room with a thick wall we obtain Bing’s room with collapsed thick wall on the right. (Note that the left bottom edge of the collapsed thick wall is still present although it is not contained in any 2-dimensional face.)

**Bing’s house with one thin and one thick wall.** If we rotate the ground floor of one of the rooms and we glue the two rooms together along the ground floor, we obtain Bing’s house with one thin and one thick wall as introduced by Malgouyres and Francés [MF08]. See Figure 2. Similarly, we can obtain Bing’s house with two thin walls or Bing’s house with two thick walls.

**Bing’s house with three rooms and three thick walls.** As an auxiliary construction, we also need to introduce Bing’s house with three rooms. First we consider base floor depicted in Figure 3. It consists of three quadrilaterals with holes, glued together.
For simplicity of explanations, we will assume that all these three quadrilaterals are squares as well as holes are squares. Now we consider three Bing’s rooms with thick walls labeled 1, 2 and 3. The Bing room with label $i$ is glued to the two squares with label $i$ so that the grey part of one of the squares with label $i$ is the place where the thick wall of the room is glued to the base floor. Here, it is important that we do not have to distinguish whether the rooms are glued to the base floor from below or from above, since we could not place them in such a way (in 3D) simultaneously. The resulting complex we call Bing’s house with three rooms (and three thick walls). We remark that Bing’s house with three rooms is contractible which can be shown in a similar way as contractibility of classical Bing’s house. Later on, we will need specific collapsing sequence of Bing’s house with three rooms. Existence of such a sequence implies contractibility as well.

**Bing’s house with three collapsed walls.** For further purposes it will be convenient to work with Bing’s house with three rooms where the thick walls are collapsed. We let collapse each of the thick walls to the edge on the base floor. This way we obtain Bing’s house with three collapsed walls. We provide the reader with a drawing with two rooms only (this can still be done in three dimensions); see Figure 4. We also distinguished edges $x_1$, $x_2$ and $x_3$ such that $x_i$ is the only remaining edge of the thick wall in room $i$ after collapsing the wall. Note that $x_1$, $x_2$ and $x_3$ are the only free faces of Bing’s house with three collapsed walls. See also Figure 9 for the base floor.

**Triangulations.** In order to obtain simplicial complexes we need to triangulate our gadgets that we obtain from Bing’s rooms, Bing’s houses etc. It will not be important for us, how we precisely triangulate pieces in the construction of dimensions 2 or less. For example, the middle level of Bing’s house with one thick and one thin wall can be triangulated as suggested in Figure 5 keeping in mind that the triangulations of particular 2-cells have to be compatible on intersections. In some cases, we will require gadgets with many required edges where the number of edges depends on the size of ...
the 3-CNF formula we will work with (see Figure 5 or 10). In such cases we require that the size of the triangulation is polynomial in the number of required edges.

The only 3-cells appearing in our construction are thick walls of Bing’s rooms (houses). For these thick walls we use particular triangulations of Malgouyres and Frances [MF08]. The thick wall is subdivided into four prisms 012389, 014589, 236789 and 456789. See Figure 6, left. Each prism is further subdivided into two simplices (which are not shown on the picture). This triangulation allows collapsing the thick wall into two smaller complexes from Figure 6, middle and right (in the middle picture, the edge 01 is contained in no 2-cell; in the right one, the edge 89 is contained in no 2-cell). In particular, the collapsing from the middle picture is used when obtaining Bing’s room with a collapsed thick wall from Bing’s room with a thick wall.

5 Construction

Here we start filling in details of the construction sketched in Section 3.

Given a 3-CNF formula $\Phi$ we construct a three-dimensional simplicial complex $K(\Phi)$ such that $K(\Phi)$ is collapsible if and only if $\Phi$ is satisfiable. We assume that every
clause of $\Phi$ contains exactly three literals and also that no clause contains a literal and its negation.

The complex $K(\Phi)$ will consist of several gadgets described below. For each of the gadgets we also need to find some suitable collapsing sequence. We usually postpone the proofs that such sequences exist to Section 7 so that the main idea can be explained while the technical details are left to the end.

**Literal gadget.** First we establish the literal gadget $K(\ell, \bar{\ell})$ for every pair of literals $\ell$ and $\bar{\ell}$. This gadget is by Malgouyres and Francés [MF08], we only glue it to other gadgets in a different way. It consists of two smaller gadgets $X(\ell)$ and $X(\bar{\ell})$ suitably glued together.

We set $X(\ell)$ to be Bing’s house with two thick walls as in Figure 7. It contains two distinguished edges $e(\ell)$ and $f(\ell)$. Furthermore, a path $p(\ell)$ joining the common vertex of $e(\ell)$ and $f(\ell)$ with the upper thick wall is distinguished (this path contains neither $e(\ell)$ nor $f(\ell)$). Let us emphasize that in this case, we use particular triangulation by Malgouyres and Francés [MF08] that subdivides the upper thick wall into four prisms which are further triangulated. The path $p(\ell)$ enters the upper wall in vertex 0 of this triangulation and it continues to vertex 8. For $\bar{\ell}$ we construct $X(\bar{\ell})$ analogously.

The complex $K(\ell, \bar{\ell})$ is composed of $X(\ell)$ and $X(\bar{\ell})$ glued together along edge 89. The common vertex 8 will be important for further constructions; and therefore we rename it to $u_{\ell, \bar{\ell}}$ emphasizing dependency on $\ell$ and $\bar{\ell}$. The following lemma describes a particular sequence of collapsing the literal gadget that we will use later. It also says that at least one of the edges $f(\ell)$, $f(\bar{\ell})$ has to be collapsed before collapsing the literal gadget to a 2-complex.

**Lemma 6.**

1. $K(\ell, \bar{\ell})$ collapses to a complex that contains only path $p(\ell)$ and edges $e(\ell)$ and
Figure 7: Complex $X(\ell)$ from the literal gadget with distinguished edges and paths. Detailed triangulation of the upper thick wall is on the right.

$f(\ell)$ from $X(\ell)$ while it contains almost all $X(\bar{\ell})$ with exception that the upper thick wall of $X(\ell)$ was collapsed to a thin wall keeping only rectangles 0462, 0451, 4576, 2673, and 0132\footnote{For simplicity of notation, we keep the same numbers of vertices either for upper thick wall of $X(\ell)$ or of $X(\bar{\ell})$. However, we once more emphasize that that these two walls share the edge 89 only.} (Consult Figure 7 right, if you remove the edge 89.) The role of $\ell$ and $\bar{\ell}$ can be interchanged.

2. Let $L(\ell, \bar{\ell})$ be the complex resulting in item 1 without the edge $e(\ell)$. This complex further collapses to the union of the paths $p(\ell)$, $p(\bar{\ell})$ and the edge $e(\bar{\ell})$.

3. Let $T(\ell)$ be any of the two triangles containing $e(\ell)$ and $T(\bar{\ell})$ be any triangle containing $e(\bar{\ell})$. Before collapsing both $T(\ell)$ and $T(\bar{\ell})$ at least one of the edges $f(\ell)$, $f(\bar{\ell})$ must be collapsed.

Proof. We postpone the proof of items 1 and 2 on precise collapsing sequences to Section\[7\] Item 3 is already proved by Malgouyres and Francés; see Remark 1, Example 3 and the proof of Theorem 4 in [MF08]. We sketch here that if neither $f(\ell)$ nor $f(\bar{\ell})$ is collapsed, then the only one of the two upper thick walls, one of $X(\ell)$ and one of $X(\bar{\ell})$, can be collapsed so that its edge 01 becomes free. Hence, only one of the triangles $T(\ell)$ and $T(\bar{\ell})$ might become free before collapsing $f(\ell)$ or $f(\bar{\ell})$.

Conjunction gadget. Next we define the conjunction gadget $K_{\text{and}}$. It is Bing’s house with one collapsed thick wall and one thin wall. See Figure 8 on the left. We also distinguish several edges and vertices of the gadget.

As an auxiliary construction, for every pair $\ell, \bar{\ell}$ of literals, we create anchor-shaped tree $A(\ell, \bar{\ell})$ formed of $u_{\ell, \bar{\ell}}$, $p(\ell)$, $p(\bar{\ell})$, $f(\ell)$ and $f(\bar{\ell})$ from $K(\ell, \bar{\ell})$ and furthermore of newly introduced edge $a(\ell, \bar{\ell})$ and vertex $v_{\text{and}}$. See Figure 9 on the right. We glue all trees $A(\ell, \bar{\ell})$ in vertex $v_{\text{and}}$ obtaining a tree $A$. 

\[\text{Figure 7: Complex } X(\ell) \text{ from the literal gadget with distinguished edges and paths. Detailed triangulation of the upper thick wall is on the right.}\]
Finally, we let $e_{\text{and}}$ to denote the only free edge of $K_{\text{and}}$ and we glue $A$ to the lower left wall of $K_{\text{and}}$ as on Figure 8 on the left. Note that, in particular, every literal gadget is glued to the conjunction gadget.

After we introduce remaining gadgets, we will see that $K_{\text{and}}$ is glued to other gadgets only along $A$ and $e_{\text{end}}$. The following lemma states that if we want to collapse $K_{\text{and}}$ at some phase of collapsing, we have to make $e_{\text{end}}$ free first in whole $K(\Phi)$ and only then we can continue with collapsing $K_{\text{and}}$. On the other hand, as soon as we make $e_{\text{and}}$ free, we can collapse the complex to $A$.

**Lemma 7.**

1. $K_{\text{and}}$ collapses to $A$.

2. Before collapsing any triangle containing one of the edges $f(\ell, \bar{\ell})$, the edge $e_{\text{and}}$ has to be collapsed.

**Proof.** We prove item 1 in Section 7. Item 2 of the lemma is explained in [MF08, Remark 1].

**Clause gadget.** We proceed with introducing the clause gadget. For a clause $c = (\ell_1 \lor \ell_2 \lor \ell_3)$ we set $K(c)$ to be Bing’s house with three collapsed walls as described in Section 4 and with several distinguished edges and paths; see Figure 9. Namely, the only three free edges of $K(c)$ are labeled $(\ell_i, c)$. We also distinguish three paths $p(\ell_i, c)$ connecting the center of the base floor with $(\ell_i, c)$ (we assume that $(\ell_i, c)$ is not contained in the path). We also distinguish one other edge emanating from the center inside the base floor and we label it by $e_{\text{and}}$. This last edge $e_{\text{and}}$ is glued together with the edge of the conjunction gadget labelled $e_{\text{and}}$ so that the central vertex of the base floor becomes the vertex $v_{\text{and}}$ of the conjunction gadget.

**Lemma 8.**
1. $K(c)$ collapses to a complex composed of $e_{\text{and}}$, three paths $p(\ell_i, c)$ and two of the three edges $(\ell_i, c)$.

2. Any collapsing of $K(c)$ starts with one of the edges $(\ell_i, c)$.

Proof. We again postpone the proof of item 1 to Section 7. Item 2 is obvious as soon as we realize that the only free faces of $K(c)$ are the three edges $(\ell_i, c)$.

**Disk gadgets.** Finally, for every pair of literals $\ell, \bar{\ell}$ we construct the disk gadget $D(\ell, \bar{\ell})$ filling empty cycles in the construction of Malgouyres and Francés [MF08]. As we mentioned before, these gadgets will not be topological disks. However, they are contractible and play a similar role as disks.

Disk gadgets are composed of several smaller pieces that we will describe now.

We start with Bing’s house with one collapsed thick wall and one thin wall; see Figure 10. We label the only free face of this complex with $e(\ell)$ and glue it to the edge $e(\ell)$ of $K(\ell, \bar{\ell})$. We pick a vertex on the edge connecting the left and the bottom wall and label it $v_{\text{and}}$. We also glue this vertex to $v_{\text{and}}$ vertex of the conjunction gadget. The edge connecting $v_{\text{and}}$ and one of the vertices of $e(\ell)$ is labelled $b(\ell)$. Next, for every clause $c_j$ containing the literal $\ell$ we make a copy of path $p(\ell, c_j)$ and edge $(\ell, c_j)$ (where the template comes from $K(c_j)$) starting in vertex $v_{\text{and}}$. In particular, $B(\ell)$ is glued to the complexes $K(c_j)$ along these paths and edges. The resulting complex is denoted by $B(\ell)$. We perform an analogous construction for $B(\bar{\ell})$.

This complex can be collapsed (inside whole $K(\Phi)$) as soon as the edge $e(\ell)$ is free. Then it collapses to a complex composed of the distinguished edges and paths as the following lemma summarizes.

**Lemma 9.**

1. $B(\ell)$ collapses to the 1-complex composed of $b(\ell)$, paths $p(\ell, c_j)$ and edges $(\ell, c_j)$. 
2. Any collapsing of $B(\ell)$ starts with the edge labelled $e(\ell)$.

Proof. As usual, item 1 is proved in Section 7. Item 2 is true since $e(\ell)$ is the only free edge of $B(\ell)$. \qed

Now we can finally construct $D(\ell, \bar{\ell})$; see Figure 11. We fill two cycles with a disk. The first cycle is formed by $b(\ell)$, $p(\ell)$ and $a(\ell, \bar{\ell})$, the second cycle by $b(\bar{\ell})$, $p(\bar{\ell})$ and $a(\ell, \bar{\ell})$. This finishes the construction of $D(\ell, \bar{\ell})$ and since we have already described all gluings, it also finishes the construction of $K(\Phi)$.  

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6 Collapsibility of $K(\Phi)$

In this section we prove that $K(\Phi)$ is collapsible if and only if $\Phi$ is satisfiable. Thereby we prove Theorem 1.

Satisfiable formulas. Let us first assume that $\Phi$ is satisfiable and fix one satisfying assignment of $\Phi$. We construct a collapsing sequence for $K(\Phi)$. We proceed in several steps (each step will still consist of many elementary collapses). By $K^{(i)}(\Phi)$ we denote the complex obtained after performing $i$th step of collapsing. We use similar notation for gadgets, for example, $K^{(i)}_{and}$ is the remaining part of $K_{and}$ after $i$th step, that is, $K^{(i)}(\Phi) \cap K_{and}$.

Step 1. For every literal $\ell$ we start with “partial” collapsing of $K(\ell, \bar{\ell})$ such as in Lemma 6 (1). Note that that the constrain complex of the pair $(K(\Phi), K(\ell, \bar{\ell}))$ consist of $p(\ell), p(\bar{\ell}), e(\ell), e(\bar{\ell}), f(\ell)$ and $f(\bar{\ell})$; therefore Lemma 4 induces collapsing on whole $K(\Phi)$. If $\ell$ has positive occurrence in the assignment, we let $X(\ell)$ collapse to $p(\ell), e(\ell)$ and $f(\ell)$ while in $X(\bar{\ell})$ only the upper thick wall of $X(\ell)$ collapses to a thin wall. This collapsing makes the edge $e(\ell)$ free.

We gradually perform this collapsing for all literals with positive occurrence. Note that by considering positive occurrences only, we do not “miss” negative ones since for every variable $u$ exactly one literal among $u$ and $\neg u$ has positive occurrence.

Step 2. We continue with collapsing $B(\ell)$ as stated in Lemma 9 (1). Observe that at this stage, the constrain complex for the pair $(K^{(1)}(\Phi), B^{(1)}(\ell)) = (K^{(1)}(\Phi), B(\ell))$ contains only $b(\ell), paths p(\ell, c_j)$ and edges $\ell, c_j$ (in particular the edge $e(\ell)$ is not in it as well as the vertex of $e(\ell)$ which is not adjacent to $b(\ell)$), therefore collapsing from Lemma 9 induces collapsing of $K^{(1)}(\Phi)$ by Lemma 4.

In further text we will use Lemma 4 many times in a similar fashion without explicit mentioning it. (We will describe the constrain complex only.)

Step 3. Now, since the assignment is satisfying for every clause $c$ at least one of the edges $(\ell_i, c)$ became free. Therefore, every clause gadget collapses to a 1-complex described in Lemma 8 (1). The constrain complex for the pair $(K^{(2)}(\Phi), K^{(2)}(c))$ is a subcomplex of the complex formed by paths $p(\ell_j, c)$ and edges $\ell_j, c$ with $j \neq i$.

Step 4. Now we focus on the edge $e_{and}$. At the beginning, it was contained in triangles in clause gadgets and in a single triangle of the conjunction gadget. All triangles of clause gadgets were collapsed, therefore $e_{and}$ is free now. According to Lemma 7 (1), we can collapse the conjunction gadget $K_{and}$ to $A$ now (checking that the constrain complex for $(K^{(3)}(\Phi), K^{(3)}_{and})$ is $A$).

Step 5. In this step, we will collapse literal and disk gadgets. The important fact is that the edges $f(\ell)$ and $f(\bar{\ell})$ are already free. Therefore, we can proceed with collapsing $K^{(4)}(\ell, \bar{\ell})$ according to Lemma 3 (2). This leaves $e(\bar{\ell})$ free as well as all remaining edges.
of paths $p(\ell)$ and $p(\bar{\ell})$. Now we can easily collapse $B(\bar{\ell})$ according to Lemma 9 (1) and consequently also the $D(\ell, \bar{\ell})$ (having all boundary edges free).

**Step 6.** Now we have a collection of paths emanating from $v_{\text{and}}$ (which are remainders of clause gadgets). This collection can be easily collapsed to a point, say $v_{\text{and}}$.

**Non-satisfiable formulas.** Now we show that $K(\Phi)$ is not collapsible for non-satisfiable formulas. More precisely, we assume that $K(\Phi)$ is collapsible and we deduce that $\Phi$ is satisfiable.

If $K(\Phi)$ is collapsible, then in particular some triangle of $K_{\text{and}}$ has to be collapsed. We investigate what had to be collapsed before collapsing a first triangle of $K_{\text{and}}$. According to Lemma 7 (2) the edge $e_{\text{and}}$ has to be made free before this step. Lemma 8 (2) implies that for every clause $c$ there is a literal $\ell(c)$ in this clause such that the edge $(\ell(c), c)$ was made free prior this step. This means by Lemma 9 (2) that the edge $(\ell(c))$ had to be made free previously. Now we recall that no triangle of $K_{\text{and}}$ was collapsed yet (including triangles of $K_{\text{and}}$ attached to $f(\ell)$ and $f(\bar{\ell})$). Therefore, Lemma 9 (3) implies that only one of the edges $e(\ell)$ and $e(\bar{\ell})$ can be collapsed at this stage. This gives a satisfying assignment to $\Phi$ by setting a variable $u$ to be TRUE if $e(u)$ was collapsed (before collapsing a triangle from $K_{\text{and}}$) and FALSE otherwise. Existence of $\ell(c)$ implies that every clause $c$ is indeed satisfied.

### 7 Collapsing sequences

Here we prove technical lemmas used previously in the text. It is convenient to change the order of the proofs.

**Proof of Lemma 7 (1).** We recall that our task is to collapse the conjunction gadget from Figure 8 to the tree $A$. We start with collapsing the wall below the edge $e_{\text{and}}$ and then the lowest floor (except edges belonging to $A$). We continue with collapsing all walls that used to be attached to the lowest floor. At this step we have a complex depicted in Figure 12. This complex is already a 2-sphere with a hole and with $A$ attached to it. It is easy to collapse it to $A$ in the directions of arrows.

**Proof of Lemma 8 (1).** We recall that our task is to collapse the clause gadget from Figure 9. We will provide a collapsing to the union of paths $p(\ell_i, c)$ and edges $(\ell_2, c)$ and $(\ell_3, c)$. Other cases are analogous. For sake of picture, we will assume that Bing’s room number 1 is above the base floor and Bing’s room number 2 is below the base floor as in Figure 13 on the left.

Since $(\ell_1, c)$ is allowed to be collapsed, we can collapse the left wall of room 2 and then the bottom wall. In next step, we can collapse all walls of room 2 perpendicular to the base floor. We obtain complex as in Figure 13 on the right. Next we collapse the interior of the 23 square so that the edges left of $(\ell_2, c)$ become free. This means...
that the room 3 can be collapsed in a similar fashion as we collapsed room 2 (note that after this step only \((\ell_2, c), p(\ell_2, c)\) and part of \(p(\ell_1, c)\) remain of the 23 square). Finally we can collapse room 1 in a similar fashion taking care that the edge \(e_{\text{and}}\) remains uncollapsed.

**Proof of Lemma 6 (1).** First we collapse the thick wall of \(X(\bar{\ell})\) in the way on Figure 6 on the right. This makes the common edge 89 of \(X(\ell)\) and \(X(\bar{\ell})\) free. Then the thick wall of \(X(\ell)\) can be collapsed so that the upper Bing’s room of \(X(\ell)\) becomes Bing’s room with collapsed thick wall. Then the rest of \(X(\ell)\) can be collapsed in very same way as in the proof of Lemma 7 (1) while keeping \(p(\ell)\) and \(f(\ell)\).

**Proof of Lemma 6 (2).** As soon as we are allowed to collapse \(f(\bar{\ell})\), the lower thick wall of \(X(\bar{\ell})\) can be collapsed obtaining Bing’s room with collapsed thick wall from the lower Bing’s room. Now \(X(\bar{\ell})\) can be collapsed in analogous way as was presented in the proof of Lemma 7 (1) while keeping the required subcomplex (the role of the lower and upper room are interchanged).

**Proof of Lemma 9 (1).** We use almost the same collapsing procedure as in the proof of Lemma 7 (1). We just remark, that the wall below \(e(\ell)\), split by \(b(\ell)\) is collapsed in two stages. First the half containing \(e(\ell)\) is collapsed; then the lowest floor of \(B(\ell)\) is collapsed and finally, the second half of this wall is collapsed.

8 Conclusion

We have shown that it is NP-hard to decide whether a 3-dimensional complex collapses to a point. Here we mention few (simple) corollaries of our construction as well as several related questions.

**Collapsing** \(d\)-**complexes to** \(k\)-**complexes.** Motivated by a question of Malgouyres and Francés [MF08] about higher dimensions, we set up question \((d, k)\)-COLLAPSIBILITY
asking whether a given \(d\)-dimensional complex collapses to some \(k\)-dimensional complex where \(d > k \geq 0\) are fixed parameters.

Our result shows that \((3,0)\)-COLLAPSIBILITY is NP-complete; however, it is not difficult to observe that our result can be extended to showing that \((d,k)\)-COLLAPSIBILITY is NP-complete for any \(d \geq 3\) and \(k \in \{0,1\}\). For this it is sufficient to attach a \(d\)-simplex to \(v_{\text{and}}\) and remark that if \(\Phi\) is not satisfiable, then any collapsing of \(K(\Phi)\) yields a complex of dimension 2 or more. (We also remark that the case \(d \geq 3, k = 1\) can be already obtained from the construction of Malgouyres and Francés.)

As we mentioned in the introduction, it is not hard to see that \((d,k)\)-COLLAPSIBILITY is polynomial time solvable whenever \(d \leq 2\), and also in codimension 1 case (see Proposition 5).

In the remaining cases, \(d \geq k+2 \geq 2\), it is reasonable to believe that NP-hardness reduction can be obtained with higher dimensional analogues of the gadgets in our construction. However, it does not yield from our construction immediately, therefore we pose this case as a question.

**Question 10.** What is the complexity status of \((d,k)\)-collapsibility for \(d \geq k+2 \geq 2\)?

**Collapsing to a fixed 1-complex.** In fact, our construction also shows that it is NP-complete to decide whether a 3-dimensional complex collapses to a fixed 1-complex. For this, it is sufficient to attach the fixed 1-complex to \(v_{\text{and}}\) (and eventually a \(d\)-simplex for \(d \geq 3\) again if we want to reach higher dimension).

**Collapsing of complexes from a specific class.** In general we can consider two collections of simplicial complexes, the initial collection \(I\) and the target collection \(T\). The \((I,T)\)-COLLAPSIBILITY question asks whether the given input complex from
\( \mathcal{T} \) collapses to some complex from \( \mathcal{I} \). It would be interesting to know whether this question is polynomial time solvable for some natural choices of \( \mathcal{T} \) and \( \mathcal{I} \). One natural choice, in the author’s opinion, is when \( \mathcal{T} \) is a collection of triangulated \( d \)-balls for \( d \geq 3 \) and \( \mathcal{I} \) is simply a point. Note that even in this setting the question is non-trivial since there exist non-collapsible \( d \)-balls; see, e.g., [Ben12, Corollary 4.25]. However, even in this case we suspect NP-hardness.

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A Unrecognizability of contractible complexes

Here we show the following theorem on unrecognizability contractible complexes.

Theorem 11. It is algorithmically undecidable whether a given simplicial complex of dimension 5 is contractible.

Theorem 11 relatively easily follows from known results. However, the author is not aware any reference to it, therefore we prove it here.

First, we need few notions from PL topology.

PL structures. For purposes of this section, it is important to distinguish a simplicial complex $K$ and its geometric realization $|K|$. Let $K, L$ be two simplicial complexes. A continuous map $|K| \to |L|$ is called PL (piecewise-linear) if it is linear on the simplices of a subdivision $K'$ of $K$. A PL map which is a homeomorphism is called a PL homeomorphism.
A PL $d$-ball is a simplicial complex PL homeomorphic to the $d$-simplex $\Delta^d$. A PL $d$-sphere is a simplicial complex PL homeomorphic to the boundary of the $d$-simplex $\partial \Delta^d$. We remark that for $d$ large enough there are known examples of simplicial complexes homeomorphic to the $d$-ball (resp. $d$-sphere) but which are not a PL $d$-ball (resp. PL $d$-sphere). When we remove a simplex of maximum dimension from a PL $d$-sphere we obtain a PL $d$-ball [RS72, Corollary 3.13].

Homology spheres. A homology $d$-sphere is a (topological) $d$-manifold with the same singular homology as the $d$-sphere.

Our main tool is the following form of Novikov’s theorem [VKF74], taken from [MTW11].

**Theorem 12 (Novikov).** Let $d \geq 5$ be a fixed integer. There is an effectively constructible sequence of simplicial complexes $\Sigma_i$, $i \in \mathbb{N}$, with the following properties:

1. Each $|\Sigma_i|$ is a homology $d$-sphere (in particular a manifold).
2. For each $i$, either $\Sigma_i$ is a PL $d$-sphere, or the fundamental group of $\Sigma_i$ is nontrivial (in particular, $\Sigma_i$ is not homeomorphic to the $d$-sphere).
3. There is no algorithm that decides for every given $\Sigma_i$ which of the two cases holds.

A proof of Theorem 12 follows from the exposition by Nabutovsky; see the appendix of [Nab95]. Indeed Nabutovsky constructs a sequence of polynomials such that it is algorithmically undecidable whether their zero set is homeomorphic to a $d$-sphere. These zero sets are always smooth manifolds, and if they are homeomorphic to a $d$-sphere, they are in addition diffeomorphic to the standard $d$-sphere. Such smooth manifolds have a natural PL-structure [Whi40] and their triangulations can be found algorithmically [BPR06, Remark 11.19] (see Remark 12.35 if you consult the first edition). We conclude by remarking that in case of triangulating standard (smooth) $d$-sphere we obtain a PL-sphere.

**Proof of Theorem 11.** Let $(\Sigma_i)$ be the sequence from Theorem 12. Let $\Theta_i$ be a complex obtained from $\Sigma_i$ by removing a single simplex of dimension $d$.

It is easy to see that $\Theta_i$ is contractible if and only if $\Sigma_i$ is a $d$-sphere. If $\Sigma_i$ is a PL sphere, then $\Theta_i$ is a PL $d$-ball as we mentioned in the brief overview of PL structures. If $\Sigma_i$ is not a PL sphere, then it has a nontrivial fundamental group and, in particular $\Theta_i$ is not contractible.

Therefore, if there was an algorithm for recognition contractible 5-complexes, we could use it on $\Theta_i$ recognizing whether $\Sigma_i$ is a sphere. This contradicts Theorem [12] (3).
Remark 13. The dimension 5 in Theorem 11 can be dropped to 4, if we greedily collapse 5-dimensional simplices of $\Theta_i$ via some of their 4-dimensional faces. Note that we cannot get stuck on a 5-dimensional complex, since $\Theta_i$ is connected. (A significant influence on this idea is by Bruno Benedetti.)