Klein–Gordon equation with mean field interaction. Orbital and asymptotic stability of solitary waves

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Received 19 April 2021, revised 15 February 2022
Accepted for publication 31 March 2022
Published 16 June 2022

Abstract
We investigate orbital and asymptotic stability of solitary wave solutions to the U(1)-invariant nonlinear Klein–Gordon equation with mean field interaction.

Keywords: nonlinear Klein–Gordon equation, mean-field interaction, solitary waves, orbital stability, asymptotic stability
Mathematics Subject Classification numbers: 35L70, 47F05.

(Some figures may appear in colour only in the online journal)

1. Introduction
We study U(1)-invariant nonlinear Klein–Gordon equation on a line with mean field self-interaction:

\[ \ddot{\psi}(x, t) = \partial_x^2 \psi(x, t) - m^2 \psi(x, t) + \rho(x) F(\langle \psi(\cdot, t) \rangle), \quad \psi(x, t) \in \mathbb{C}, \quad \rho(x) \in \mathbb{R}, \quad x, t \in \mathbb{R}, \]

(1.1)

where \( m > 0 \), and \( \langle \psi, \rho \rangle = \int \psi(x) \rho(x) dx \). We write the equation as the dynamical system:

\[ \dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ -m^2 & 0 \end{bmatrix} \Psi(t) + \rho(x) \begin{bmatrix} 0 \\ F(\langle \psi(\cdot, t) \rangle) \end{bmatrix}, \quad \Psi(x, t) = \begin{bmatrix} \psi(x, t) \\ \pi(x, t) \end{bmatrix} \in \mathbb{C}^2. \]

(1.2)

∗Research supported by the Austrian Science Fund (FWF) under Grant No. P 34177.
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Recommended by Dr Jean-Claude Saut.

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Equation (1.2) admits finite energy solutions of the form
\[ \Psi_m(x, t) = e^{-i\omega t}\varphi_m(x) = e^{-i\omega t}\begin{bmatrix} \varphi(x, \omega) \\ -i\omega\varphi(x, \omega) \end{bmatrix}, \quad \omega \in (-m, m), \]
called solitary waves. The solitary waves form a two-dimensional solitary manifold in the Hilbert space of finite energy states of the system (the set \( S \) in (2.11) below).

In the first part of the article we study the orbital stability of the solitary waves. We recall that, orbital stability of solitary wave solution \( e^{-i\omega t}\phi_\omega \) means the stability of the \( \phi_\omega \)-orbit \( \{e^{-i\omega t}\phi_\omega; t \in \mathbb{R} \} \) (see [7]). Denote \( E := H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \).

**Definition 1.1.** The \( \phi_\omega \)-orbit is stable if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \|\Psi(0) - \phi_\omega\|_E < \delta \) then
\[ \sup_{0 < t < \infty} \inf_{\varepsilon \in \mathbb{R}} \|\Psi(t) - e^{i\varepsilon}\phi_\omega\|_E < \varepsilon. \]
Otherwise the \( \phi_\omega \)-orbit is called unstable.

We show that for equation (1.2), the standard criterion for orbital stability (see [10] and references therein) holds:
\[ \partial_\omega Q(\phi_\omega) = \partial_\omega(\omega \|\varphi_\omega\|_{L^2(\mathbb{R})}^2) < 0, \tag{1.3} \]
where the charge \( Q(\Psi) = \text{Im} \int \psi(x)\bar{\pi}(x) dx \) is conserved for solutions to (1.2). We express the condition in term of nonlinearity (condition (4.13) below). In particular, in the case when \( F(z) = |z|^{2\kappa} \) and \( \rho \) is close to \( \delta \)-function, the condition holds for any \( \kappa < 0 \) and \( \omega \in (-m, m) \), if \( 0 < \kappa < 1 \) it holds only for \( m\sqrt{\kappa} < |\omega| < m \).

In the second part of the paper, we prove the scattering asymptotics of type
\[ \Psi(t) \sim \Psi_{\infty}(t) + W(t)\Xi, \quad t \to \pm \infty, \tag{1.4} \]
where \( W(t) \) is the dynamical group of the free Klein–Gordon equation, \( \Xi \in E \) are the corresponding asymptotic scattering states, and the remainder decays to zero as \( O(|t|^{-1/2}) \) in global norm of \( E \). The asymptotics holds for the solutions with initial states close to the stable part of the solitary manifold, extending the results of [2, 3, 5, 15, 17] to equation (1.2).

For the proof we develop the approach [2, 3] to equation (1.2). This approach is essentially based on properties of linearized dynamics at a solitary wave \( e^{-i\omega t}\phi_\omega \). Let us note that the model (1.2) allows us to explicitly check the most of them. Namely, we prove weighted energy decay for the solution of the linearized equation. For linearization operator (operator \( A(\omega) \) in (3.6)) we prove the absence of virtual levels at the embedding threshold \( \pm i(m + |\omega|) \) and give the criterion for the absence of virtual levels at the endpoints \( \pm i(m - |\omega|) \) of the essential spectrum (condition (A.25)). Moreover, we prove the limiting absorption principle for the corresponding resolvent. We show also that under condition (1.3), the linearized operator \( A(\omega) \) has no real nonzero eigenvalues, and zero eigenvalue is of multiplicity 2 in the case when \( \partial_\omega Q(\phi_\omega) \neq 0 \).

The only assumption we postulate is the absence of pure imaginary eigenvalues of the linearized operator. In appendix A we provide examples when this assumption holds. Namely, in the case when \( F(z) = |z|^{2\kappa} \), and \( \rho(x) \) is close to \( \delta(x) \), the condition holds for any \( \kappa \leq -1/2 \) and \( \omega \in (-m, m) \); if \( \kappa > -1/2, \kappa \neq 0 \), it holds only for \( m^{1+2\kappa} < |\omega| < m \).

The well-posedness of the model (1.2), as well as the global attraction to the set of all solitary waves in local seminorms of \( H^{1-\varepsilon}(\mathbb{R}) \oplus H^{-\varepsilon}(\mathbb{R}) \) were proved in [8]. Let us emphasize that such global attraction is completely different from asymptotics (1.4) with fixed solitary wave and asymptotic scattering state which happens in the global energy norm.
2. The model

We suppose that \( \rho(\cdot) \in C(\mathbb{R}) \) is real-valued even coupling function, satisfying

\[
|\rho(x)| \leq C(x)^{-3-\epsilon}, \quad \text{where } \langle x \rangle := (1 + |x|^2)^{1/2}, \ x \in \mathbb{R}
\]

with some \( \epsilon > 0 \). Moreover, we suppose that

\[
\hat{\rho}(\xi) = \int e^{i\xi x} \rho(x) dx \neq 0 \quad \text{for any } \xi \in \mathbb{R}.
\]

(2.2)

We consider the nonlinearity \( F \) of a special form

\[
F(z) := a(|z|^2)z, \quad z \in \mathbb{C},
\]

(2.3)

where \( a(\tau) \in C(\mathbb{R}) \) is real-valued. Then the nonlinearity admits a real-valued potential:

\[
F(z) = -\nabla_{\text{Re}, \text{Im}} U(z), \quad U(z) = u(|z|^2), \quad u(\tau) = -\frac{1}{2} \int_0^\tau a(s)ds.
\]

(2.4)

In this case the system (1.2) is formally Hamiltonian with the Hamiltonian functional

\[
\mathcal{H}(\Psi) = \frac{1}{2} \int (|\pi|^2 + |
abla \psi|^2 + m^2 |\psi|^2) dx + U(\langle \psi, \rho \rangle), \quad \Psi = (\psi, \pi) \in E.
\]

(2.5)

We assume that

\[
U(z) \geq A - B|z|^2 \quad \text{for } z \in \mathbb{C}, \quad \text{where } A \in \mathbb{R}, \ 0 \leq B < \frac{m}{2 \|ho\|_{L^2(\mathbb{R})}}.
\]

(2.6)

The global existence result for the Cauchy problem for equation (1.2) was proved in [8, theorem 1.3]:

**Theorem 2.1.** Let \( F(z) \in C^1(\mathbb{C}) \) satisfy conditions (2.3), (2.4) and (2.6), and \( \Psi_0 \in E \). Then

(a) Equation (1.2) has a unique solution \( \Psi \in C_b(\mathbb{R}, E) \) such that \( \Psi(0) = \Psi_0 \).

(b) The map \( W(t) : \Psi(0) \mapsto \Psi(t) \) in continuous in \( E \).

(c) The energy is conserved,

\[
\mathcal{H}(\Psi(t)) = \text{const}, \quad t \in \mathbb{R}.
\]

(2.7)

(d) There exists \( \Lambda(\Psi_0) > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \|\Psi(t)\|_E \leq \Lambda(\Psi_0) < \infty.
\]

(2.8)

Bound states of equation (1.2), known as solitary waves [7], are finite energy solutions of the form

\[
\Psi(x, t) = e^{-i\omega t} \phi_\omega(x), \quad \phi_\omega(x) = \begin{bmatrix} \varphi_\omega(x) \\ -i\omega \varphi_\omega(x) \end{bmatrix}, \quad \varphi_\omega \in H^1(\mathbb{R}), \ \omega \in \mathbb{R}.
\]

(2.9)
By (1.1), the frequency $\omega \in \mathbb{R}$ and the amplitude $\varphi_\omega(x)$ solve the following nonlinear eigenvalue problem:

$$\omega^2 \varphi_\omega(x) = -\partial^2_x \varphi_\omega(x) + m^2 \varphi_\omega(x) - \rho(x) F(\varphi_\omega(x), \rho(x)), \quad x \in \mathbb{R}. \quad (2.10)$$

**Definition 2.2.** $S$ denotes the set of all amplitudes $\phi_\omega(x) = \left[ \varphi_\omega(x) - i\omega \varphi_\omega(x) \right].$

**Lemma 2.3 (see [8, proposition 1.5]).** The set $S$ is given by

$$S = \left\{ \phi_\omega(x) e^{i\theta} = \left[ \varphi_\omega(x) - i\omega \varphi_\omega(x) \right] e^{i\theta}, \quad |\omega| < m, \quad \theta \in [0, 2\pi) \right\}, \quad (2.11)$$

where

$$\hat{\varphi}_\omega(\xi) = \frac{c_\omega \hat{\rho}(\xi)}{\xi^2 + m^2 - \omega^2}, \quad \xi \in \mathbb{R}, \quad (2.12)$$

and $c_\omega > 0$ is a root of the equation

$$\sigma(\omega) a(c_\omega^2 \sigma^2(\omega)) = 1, \quad (2.13)$$

with

$$\sigma(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\rho}^2(\xi)}{\xi^2 + m^2 - \omega^2} d\xi > 0. \quad (2.14)$$

We consider a real version of (1.2). Namely, we define $\Psi = \left[ \begin{array}{cccc} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{array} \right] \in \mathbb{R}^4$, and write the Cauchy problem for (1.2) in the form:

$$\dot{\Psi}(t) = \Sigma E'(\Psi(t)), \quad \Sigma = \left[ \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right], \quad \Psi|_{t=0} = \Psi_0 = \left[ \begin{array}{c} \Re \psi_0 \\ \Im \psi_0 \\ \Re \pi_0 \\ \Im \pi_0 \end{array} \right]. \quad (2.15)$$

Here

$$F_j(\Psi) = a((\Psi_1, \rho)^2 + (\Psi_2, \rho_1^2) \langle \Psi_j, \rho \rangle), \quad j = 1, 2. \quad (2.16)$$

The equation (2.15) can be written formally as Hamiltonian system

$$\dot{\Psi}(t) = \Sigma E'(\Psi(t)), \quad \Sigma = \left[ \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right], \quad (2.17)$$

with Hamiltonian functional

$$E(\Psi) = \frac{1}{2} \int_\mathbb{R} \left( (\partial_x \Psi_1(x))^2 + m^2 \Psi_1^2(x) + \Psi_3^2(x) + \Psi_4^2(x) \right) dx + u(\langle \Psi_1, \rho \rangle^2 + (\Psi_2, \rho_1^2)), \quad (2.18)$$
which is conserved for finite energy solutions \( \Psi \in E := H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) by theorem 2.1. Equation (2.15) is \( U(1) \)-invariant: if \( \Psi(x,t) \) is a solution, then so is \( e^{i\theta}\Psi(x,t) \) for any \( \theta \in \mathbb{R} \), where

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\] (2.19)

The Noether theorem implies the charge conservation: the value of the functional

\[
Q(\Psi) := \int_{\mathbb{R}} (\Psi_2(x)\Psi_3(x) - \Psi_1(x)\Psi_4(x)) \, dx
\] (2.20)

is conserved for solutions to (2.15) (cf [8, theorem 2.1]).

The bound states of equation (2.15) corresponding to (2.9) read as follow

\[
\Psi(x,t) = e^{i\omega t + \theta} \Phi_{\omega}(x), \quad \Phi_{\omega}(x) = \begin{bmatrix} \varphi_{\omega}(x) \\ 0 \\ 0 \\ -\omega \varphi_{\omega}(x) \end{bmatrix}, \quad \theta \in [0, 2\pi).
\] (2.21)

Remark 2.4. One can readily check that \( \Phi_{\omega} \) is solution to the equation

\[
E'(\Phi_{\omega}) = \omega Q'(\Phi_{\omega}).
\] (2.22)

In the terminology of [7] (see definition on p 166), the solution \( T(\omega t)\Phi_{\omega}, \) with \( T(\omega t) = e^{i\omega t} \), is the bound state solution to (1.2).

We will write \( \| \cdot \| = \| \cdot \|_{L^2(\mathbb{R})} \).

Definition 2.5.

(a) \( E_\sigma = H^1_\sigma \oplus L^2_\sigma \), \( \sigma \in \mathbb{R} \), is the Hilbert space of the states \( \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \) with the finite norm

\[
\| \Psi \|_{E_\sigma} := \| \langle x \rangle^\sigma \partial_x \psi \| + \| \langle x \rangle^\sigma \pi \| + \| \langle x \rangle^\sigma \pi \|.
\]

(b) \( E_\sigma = H^1_\sigma \oplus H^1_\sigma \oplus L^2_\sigma \oplus L^2_\sigma \), is the Hilbert space of the states \( \Psi = \begin{bmatrix} \psi_1(x) \\ \pi_1(x) \\ \psi_2(x) \\ \pi_2(x) \end{bmatrix} \) with the finite norm

\[
\| \Psi \|_{E_\sigma} := \sum_{j=1}^2 \left( \| \langle x \rangle^\sigma \partial_x \psi_j \| + \| \langle x \rangle^\sigma \pi_j \| + \| \langle x \rangle^\sigma \pi_j \| \right).
\]

(c) \( W^{1,1} \) is the Sobolev space with the finite norm

\[
\| \psi \|_{W^{1,1}} = \| \psi \|_{L^1(\mathbb{R})} + \| \partial_x \psi \|_{L^1(\mathbb{R})} < \infty
\] (2.23)

Remark 2.6. Assumption (2.1) and formula (2.12) implies that \( \Phi_{\omega} \in E_\sigma \).
3. Linearization at a solitary wave

Substituting

$$\Psi(x,t) = e^{i\omega t + i \theta} (\Phi_0(x) + Z(x,t)),$$

into (2.15), we obtain:

$$J_\omega(\Phi_0 + Z) + \dot{Z} = \begin{bmatrix} 0 & \mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\Phi_0 + Z) + \rho(x) \begin{bmatrix} 0 \\ 0 \\ F_1(\Phi_0 + Z) - F_1(\Phi_0) \\ F_2(\Phi_0 + Z) \end{bmatrix}.$$  (3.1)

Further, equation (2.10) leads to

$$\dot{Z} = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & -\omega & 0 \end{bmatrix} Z + \rho(x) \begin{bmatrix} 0 \\ 0 \\ F_1(\Phi_0 + Z) - F_1(\Phi_0) \\ F_2(\Phi_0 + Z) \end{bmatrix}.  \quad (3.2)$$

Denote

$$C_\omega = c_\omega \sigma(\omega) = \langle \varphi_\omega, \rho \rangle, \quad \alpha = a(C_2^\omega) > 0, \quad \kappa = \kappa(\omega) := \frac{C_2^\omega a'(C_2^\omega)}{a(C_2^\omega)}.  \quad (3.3)$$

Then the first order part of (3.2) is given by

$$\dot{X}(x,t) = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & -\omega & 0 \end{bmatrix} X(x,t) + \rho(x) \begin{bmatrix} 0 \\ 0 \\ (1 + 2\kappa)\alpha(X(1),\rho) \\ \alpha(X(1),\rho) \end{bmatrix}.  \quad (3.4)$$

**Remark 3.1.** We note that the above definition of $\kappa$ is compatible with the pure power case $a(\tau) = \tau^\alpha$, $\alpha \neq 0$, $\tau > 0$, when $a'(z^2)z^2 = \kappa a(z^2)$.

Denote

$$L_\omega(\omega) = -\partial_x^2 + m^2 - \rho \alpha(1 + 2\kappa)(X(1),\rho).  \quad (3.5)$$

In terms of operators (3.5), the system (3.4) reads as follows

$$\dot{X}(x,t) = A(\omega) X(x,t), \quad A(\omega) := \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ -L_\omega(\omega) & 0 & 0 & -\omega \\ 0 & -L_\omega(\omega) & \omega & 0 \end{bmatrix}.  \quad (3.6)$$

Theorem 2.1 generalizes to equation (3.6): for every initial function $X(x,0) = X_0(x) \in E$, the equation admits a unique solution $X(x,t) \in C_b(\mathbb{R}, E)$. Note that $A(\omega)$ is factored into

$$A(\omega) = \Sigma H(\omega),  \quad (3.7)$$
where $\Sigma$ is defined in (2.17) and

$$
H(\omega) = \begin{bmatrix}
L_\kappa(\omega) & 0 & 0 & \omega \\
0 & L_0(\omega) & -\omega & 0 \\
0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1
\end{bmatrix}.
$$

(3.8)

We note that there is a standard relation $H(\omega) = E''(\Phi_\omega) - \omega Q''(\Phi_\omega)$. (cf [7, equation(2.17)].)

4. Orbital stability

Here we will study the orbital stability of solitary waves following the approach of Grillakis–Shatah–Strauss [7]. First, we investigate the spectral properties of the operator $H(\omega)$ from (3.8).

**Theorem 4.1.**

(a) The essential spectrum of $H(\omega)$ is given by

$$
\sigma_{\text{ess}}(H(\omega)) = [c^- (\omega), 1] \cup [c^+ (\omega), \infty), \quad \omega \in (-m, m),
$$

where

$$
c^\pm (\omega) = m^2 + 1 \pm \sqrt{(m^2 - 1)^2 + 4\omega^2} > 0.
$$

(b) For $\kappa \in \mathbb{R} \setminus \{0\}$ the kernel of $H(\omega)$ is spanned by $T'(0)\Phi_\omega = J\Phi_\omega$.

(c) For $\kappa = 0$ the kernel of $H(\omega)$ is spanned by $J\Phi_\omega$ and $\Phi_\omega$.

(d) The operator $H(\omega)$ has no negative eigenvalues for $\kappa \leq 0$, and it has exactly one simple negative eigenvalue for $\kappa > 0$.

We start with the spectrum of the operators $L_\kappa(\omega)$.

**Lemma 4.2.** Let $\omega \in (-m, m)$. The operators $L_\kappa(\omega)$ is self-adjoint and satisfies

$$
\sigma_{\text{ess}}(L_\kappa(\omega)) = [m^2, +\infty), \quad \sigma_{\text{p}}(L_\kappa(\omega)) = \begin{cases}
\emptyset, & \kappa \leq -1/2, \\
\Lambda_\kappa(\omega), & \kappa > -1/2,
\end{cases}
$$

where the eigenvalue $\Lambda_\kappa(\omega)$ is simple. Besides, $\Lambda_0(\omega) = \omega^2$, and $\Lambda_\kappa(\omega) < \omega^2$ for $\kappa > 0$.

**Proof.** Solving the equation $L_\kappa \psi = \Lambda \psi$, we find that, up to a nonzero coefficient, the eigenfunction is given by

$$
\hat{\psi}_\Lambda(\xi) = \frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - \Lambda}
$$

(4.2)

with $\Lambda \in \mathbb{R}$ obtained from the condition

$$
1 = \alpha(1 + 2\kappa)\bar{\sigma}(\Lambda), \quad \bar{\sigma}(\Lambda) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\rho}^2(\xi)}{\xi^2 + m^2 - \Lambda} d\xi > 0.
$$

(4.3)

Due to (2.2), the $L^2$-norm of $\hat{\psi}_\Lambda$ is infinite for $\Lambda \geq m^2$; therefore, $\Lambda < m^2$. The function $\bar{\sigma}(\Lambda)$ is monotonically growing with $\Lambda$, satisfying

$$
\lim_{\Lambda \to -\infty} \bar{\sigma}(\Lambda) = 0, \quad \lim_{\Lambda \to m^2} \bar{\sigma}(\Lambda) = +\infty.
$$
Hence, for $\kappa > -1/2$, equation (4.3) has exactly one root $\Lambda = \Lambda_\kappa(\omega) \in (-\infty, m^2)$. According to (2.13), the value $\kappa = 0$ corresponds to $\Lambda_0(\omega) = \omega^2$. For $\kappa > 0$, comparing (4.3) and $1 = \alpha \sigma(\omega^2)$, (see (2.13) and (2.14)), we conclude that $\sigma(\Lambda_\kappa(\omega)) < \sigma(\omega^2)$, and then $\Lambda_\kappa(\omega) < \omega^2$. □

Further, we denote

$$G_1 = G_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad G_2 = G_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix},$$

$$G = G_2 G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \end{bmatrix},$$

and consider the operator which is similar to $H(\omega)$:

$$\tilde{H}(\omega) = G H(\omega) G^{-1} = \begin{bmatrix} L_0(\omega) & \omega & 0 & 0 \\ \omega & L_0(\omega) & 0 & 0 \\ 0 & 0 & L_0 & \omega \\ 0 & 0 & \omega & 1 \end{bmatrix}.$$

Thus, the spectral problem for $H(\omega)$ is reduced to studying the spectrum of

$$H_\kappa(\omega) = \begin{bmatrix} L_\kappa(\omega) & \omega \\ \omega & 1 \end{bmatrix}.$$  (4.6)

**Lemma 4.3 (cf [4, lemma 3.4]).**

(a) For any $\kappa \in \mathbb{R}$, the essential spectrum of $H_\kappa(\omega)$ is given by

$$\sigma_{es}(H_\kappa(\omega)) = \begin{cases} [m^2, +\infty), & \omega = 0, \\ [c^-(\omega), 1] \cup [c^+(\omega), +\infty), & \omega \in (-m, m) \setminus \{0\}, \end{cases}$$

where $c^\pm(\omega)$ is defined in (4.1). We note that

$$0 < c^-(\omega) \leq \min\{1, m^2\}, \quad c^+(\omega) \geq \max\{1, m^2\}.$$

(b) The point spectrum of $H_\kappa(\omega)$ is given by

$$\sigma_p(H_\kappa(\omega)) = \begin{cases} \{0\}, & \kappa \leq -1/2, \ \omega \in (-m, m) \setminus \{0\}, \\ \{1\}, & \kappa \leq -1/2, \ \omega = 0, \\ \{\lambda_\kappa(\omega)\}, & \kappa > -1/2, \ \omega \in (-m, m), \end{cases}$$

with the eigenvalues $\lambda_\kappa(\omega)$ given by

$$\lambda_\kappa(\omega) = \frac{1}{2} \left( \Lambda_\kappa + 1 \pm \sqrt{\Lambda_\kappa + 1)^2 - 4\Lambda_\omega} \right), \quad \kappa > 1/2$$

with $\Lambda_\kappa(\omega)$ from lemma 4.2.

(c) The multiplicity of eigenvalue $\lambda = 1$ is infinite (it is present in the spectrum if and only if $\omega = 0$). The corresponding eigenspace contains $\begin{bmatrix} 0 \\ u_2 \end{bmatrix}$ with arbitrary $u_2 \in L^2(\mathbb{R})$. 


(d) All eigenvalues of $H_n(\omega)$ which are different from $\lambda = 1$ are simple. The corresponding eigenfunctions are given by (cf (4.2))

$$u^\dagger(x) = \left[ \begin{array}{c} 1 - \frac{\lambda e}{\omega} \\ -\omega \end{array} \right] \psi_{\lambda e}(x).$$

(\epsilon) $0 \in \sigma_p(H_n(\omega))$ if and only if $\kappa = 0$, and $\sigma_p(H_0(\omega)) = \{0\} \cup \{\omega^2 + 1\}$, with the corresponding eigenvectors

$$u_0(x) = \left[ \begin{array}{c} \varphi_\omega(x) \\ -\omega \varphi_\omega(x) \end{array} \right], \quad u_{0,1}(x) = \left[ \begin{array}{c} \omega \varphi_\omega(x) \\ \varphi_\omega(x) \end{array} \right].$$

**Remark 4.4.** The lemma was proved in [4] in the case $\rho(x) = \delta(x)$. However, the proof it still valid in our case.

Denote

$$\Xi_1 = \left[ \begin{array}{c} 0 \\ -\omega \varphi_\omega(x) \end{array} \right], \quad \Xi_2 = \left[ \begin{array}{c} \varphi_\omega(x) \\ -\omega \varphi_\omega(x) \\ 0 \\ 0 \end{array} \right].$$

Due to (4.5) and (4.9),

$$\ker \tilde{H}(\omega) = \begin{cases} \text{Span}\{\Xi_1\}, & \kappa \neq 0, \\ \text{Span}\{\Xi_1, \Xi_2\}, & \kappa = 0. \end{cases}$$

To complete the proof of theorem 4.1, it remains to note that

$$G^{-1} \Xi_1 = -iJ\Phi_\omega, \quad G^{-1} \Xi_2 = \Phi_\omega.$$  \hspace{1cm} (4.12)

**Corollary 4.5.** The geometric multiplicity of $\lambda = 0 \in \sigma_p(H(\omega))$ equals 1 if $\kappa \neq 0$; it equals 2 if $\kappa = 0$.

The following theorem gives the orbital stability result.

**Theorem 4.6.** Assume that there is an interval $I \subset (-\infty, \infty)$ such that for $\omega \in I$ there are bound states $e^{-i\omega t}\phi_{\omega}(x)$ of equation (1.2). If $\kappa \leq 0$ then the bound state is orbitally stable. In the case $\kappa > 0$ the bound state is orbitally stable if $\partial_\omega(\omega \|\phi_\omega\|^2) < 0$; it is orbital unstable if $\partial_\omega(\omega \|\phi_\omega\|^2) > 0$.

**Proof.** We will follow the Grillakis–Shatah–Strauss theory [7].

Let us first consider the case $\kappa < 0$. In this case, by theorem 4.1, the operator $H(\omega)$ has simple eigenvalue $\lambda = 0$ with the corresponding eigenvector $J\Phi_\omega$ and the rest of its spectrum is positive and bounded away from zero. Then all assumption of [7, theorem 1] are satisfied, and that theorem shows that in this case the bound state $e^{Jt}\Phi_\omega$ is orbitally stable. Obviously, the stability of $\Phi_\omega$-orbit of (2.15) implies the stability of $\phi_\omega$-orbit of (1.2).

In the case $\kappa > 0$, theorem 4.1 implies that $H(\omega)$ has exactly one simple negative eigenvalue $\lambda_{\omega}(\omega)$, and has the kernel spanned by $T(0)\Phi_\omega$ while the rest of its spectrum is positive and separated away from zero. Then all assumption of [7, theorems 2 and 3] are satisfied. According to these theorems, the $\Phi_\omega$-orbit is stable (and consequently so is $\phi_\omega$-orbit) if the function $d(\omega) = E(\phi_\omega) - \omega Q(\phi_\omega)$ satisfies the condition $d''(\omega) > 0$ and is unstable if $d''(\omega) < 0$. It remains to note, that $d''(\omega) = -Q(\Phi_\omega) = -\omega \|\phi_\omega\|^2$ by (2.20) and (2.22).
To treat the case $\kappa = 0$, we consider the linearization of equation (1.2) at a solitary wave (2.9) directly, substituting $\Psi(x, t) = e^{-i\omega t} \left( [\phi(x)] - i\omega \phi(x) \right)$ with $\zeta(x, t) \in \mathbb{C}^2$ into (1.2).

The linearized equation on $\zeta$,

$$
\dot{\zeta}(x, t) = A(\omega) \zeta(x, t), \quad \zeta(x, t) \in \mathbb{C}^2,
$$

is $\mathbb{C}$-linear, with the linearization operator $A(\omega)$ given by

$$
A(\omega) = J\tilde{H}_0(\omega), \quad \text{where} \quad \tilde{H}_0(\omega) = \begin{bmatrix} L_0(\omega) & -i\omega \\ i\omega & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

Due to (4.6), the operator $\tilde{H}_0(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ is similar to $\Sigma H_0(\omega)$ and hence has the same spectrum. Namely, by lemma 4.3,

$$
\sigma_{\text{ess}}(\tilde{H}_0(\omega)) = [c^-(\omega), 1] \cup [c^+(\omega), +\infty), \quad \sigma_p(\tilde{H}_0(\omega)) = \{0\} \cup \{\omega^2 + 1\},
$$

with eigenvalue $\lambda = 0$ being simple. Hence, [7, theorem 1] applies, showing that the solitary wave solution $e^{-i\omega t} \phi_0(x)$ to equation (1.2) is orbitally stable.

**Remark 4.7.** Using (2.12) and (2.14), the condition $\partial_{\varphi}(\omega\|\varphi\|_2^2) < 0$ can be written as

$$
\partial_{\varphi}(e^2 \sigma'(\omega)) < 0. \quad (4.13)
$$

### 5. Spectral stability

Recall that the solitary wave $e^{-i\omega t} \phi_0$ is called **spectrally stable** if the linearization operator $A$ has purely imaginary spectrum.

**Theorem 5.1.** The bound state $e^{-i\omega t} \phi_0(x)$ is spectrally stable for $\kappa \leq 0$. In the case $\kappa > 0$, the bound state is spectrally stable if and only if $\partial_{\varphi}(\omega\|\varphi\|_2^2) < 0$.

**Proof.** We follow the arguments of Kolokolov [11] and Grillakis–Shatah–Strauss [7].

**Step 1.** We note that if $\lambda$ belongs to $\sigma_p(A(\omega))$, then so do $\lambda, -\lambda, \text{ and } -\lambda$, since the spectrum of $A(\omega)$ is symmetric with respect to both $\mathbb{R}$ and $i\mathbb{R}$: indeed, $A(\omega)$ has real coefficients, while, by (3.7),

$$
A^*(\omega) = (\Sigma H(\omega))^* = -H(\omega) \Sigma = -\Sigma^{-1}(\Sigma H(\omega)) \Sigma
$$

is similar to $-\Sigma H(\omega)$. Let us prove that

$$
\sigma_p(A(\omega)) \subset \mathbb{R} \cup i\mathbb{R}. \quad (5.1)
$$

To achieve this, we consider the eigenvalue problem for

$$
\tilde{A}(\omega) = \tilde{\Sigma} \tilde{H}(\omega) = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_0(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix}, \quad (5.2)
$$

where $\tilde{H}(\omega)$ is defined in (4.5), $\tilde{\Sigma} = G\Sigma G^{-1} = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}$, and $\sigma_1$ is the first Pauli matrix.
If $\kappa \leq 0$, then $\tilde{H}(\omega)$ is nonnegative and selfadjoint, hence one can extract the square root which is also nonnegative and selfadjoint; therefore, since $\tilde{H}^{1/2}(\omega)\Sigma\tilde{H}^{1/2}(\omega)$ is antiselfadjoint, 
\[
\sigma_p(\tilde{A}(\omega)) \backslash \{0\} = \sigma_p\left(\tilde{H}^{1/2}(\omega)\Sigma\tilde{H}^{1/2}(\omega)\right) \backslash \{0\} \subset \mathbb{R}.
\]
Now we concentrate on the case $\kappa > 0$. The equation $\tilde{A}(\omega)\Psi = \lambda\Psi$ reads
\[
-1\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}, \quad \Psi = \begin{bmatrix} u \\ v \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^2). (5.3)
\]
Eliminating $v$, we get
\[
-\sigma_1 H_0(\omega)\sigma_1 H_\kappa(\omega)u = \lambda^2 u, \quad u \in L^2(\mathbb{R}, \mathbb{C}^2). (5.4)
\]
For $\lambda \neq 0$, one can see that $u$ is orthogonal to 
\[
\ker(\sigma_1 H_0(\omega)\sigma_1) = \ker\left(\begin{bmatrix} 1 & \omega \\ \omega & L_0 \end{bmatrix}\right) = \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} (5.5)
\]
(cf (2.10)), hence we can write
\[
H_\kappa(\omega)u = -\lambda^2 (\sigma_1 H_0(\omega)\sigma_1)^{-1}u = -\lambda^2 \sigma_1 H_0^{-1}(\omega)\sigma_1 u.
\]
Coupling this relation with $u$ and taking into account that $\langle u, (\sigma_1 H_0(\omega)\sigma_1)^{-1}u \rangle > 0$, we see that $\lambda^2 \in \mathbb{R}$.

Step 2. To find whether $-\lambda^2$ can be negative in the case $\kappa > 0$ and $\omega \neq 0$ (thus corresponding to linear instability), one considers the minimization problem
\[
\mu = \inf \left\{ \langle u, H_\kappa u \rangle : \langle u, u \rangle = 1, \ u \in (\ker(\sigma_1 H_0(\omega)\sigma_1))^\perp = \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} \right\}, (5.6)
\]
which implies that $u$ satisfies
\[
H_\kappa u = \mu u + \nu \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, (5.7)
\]
with $\mu, \nu \in \mathbb{R}$ the Lagrange multipliers. We note that if $\mu \leq 0$, then $\nu \neq 0$, or else one would have $\mu = \lambda^-_\kappa$ (the only negative eigenvalue of $H_\kappa(\omega)$), which is not possible since $u^-(x)$ from (4.8) corresponding to eigenvalue $\lambda^-_\kappa$ is not orthogonal to $\ker(\sigma_1 H_0(\omega)\sigma_1)$: one has
\[
(u^-)^* \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} = \begin{bmatrix} 1 - \lambda^-_\kappa \\ -\omega \\ -\omega \end{bmatrix} \psi_{\lambda^-_\kappa}(x)\varphi_{\lambda^-_\kappa}(x) = -(2 - \lambda^-_\kappa)\omega \psi_{\lambda^-_\kappa}(x)\varphi_{\lambda^-_\kappa}(x)
\]
where $2 - \lambda^-_\kappa$ is strictly positive, and $\psi_{\lambda^-_\kappa}(0)\varphi_{\lambda^-_\kappa}(0) \neq 0$ by (2.2) and (4.2).
So, if $\mu \leq 0$, then $\nu \neq 0$. Then one concludes from (5.6) that $\mu \geq \lambda^-_\kappa$. We rewrite (5.7) as
\[
(H_\kappa - \mu)u = \nu \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, \quad u = \nu(H_\kappa(\omega) - \mu)^{-1} \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}.
\]
The sign of $\mu$ could be found from the condition that $u$ is orthogonal to $\ker(\sigma_1 H_0(\omega)\sigma_1)$:
\[
\left\langle \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, (H_\kappa - z)^{-1} \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} \right\rangle = 0.
\]
We consider
\[ h(z) = \left\langle \begin{bmatrix} -\omega \varphi_{\omega} \\ \varphi_{\omega} \end{bmatrix}, (H_{\kappa} - z)^{-1} \begin{bmatrix} -\omega \varphi_{\omega} \\ \varphi_{\omega} \end{bmatrix} \right\rangle, \quad z \in (\lambda_{-}, c^{-}) \subset \rho(H_{\kappa}). \]

Since \( h(z) \) is monotonically increasing on \( \rho(H_{\kappa}) \), the sign of \( \mu \) is opposite to the sign of \( h(0) \), which is given by
\[ h(0) = \left\langle \begin{bmatrix} -\omega \varphi_{\omega} \\ \varphi_{\omega} \end{bmatrix}, (H_{\kappa} - 1) \begin{bmatrix} -\omega \varphi_{\omega} \\ \varphi_{\omega} \end{bmatrix} \right\rangle = \partial_{\omega} (\omega \| \varphi_{\omega} \|^2). \]

Hence, there is \( \lambda^2 > 0 \) if and only if \( h(0) = \partial_{\omega} (\omega \| \varphi_{\omega} \|^2) > 0 \). Above, we used the relation
\[ H_{\kappa} \begin{bmatrix} -\partial_{\omega} \varphi_{\omega} \\ \omega \partial_{\omega} \varphi_{\omega} + \varphi_{\omega} \end{bmatrix} = \begin{bmatrix} L_{\kappa} - \omega \\ -1 \end{bmatrix} \begin{bmatrix} -\partial_{\omega} \varphi_{\omega} \\ -\omega \partial_{\omega} \varphi_{\omega} + \varphi_{\omega} \end{bmatrix} = \begin{bmatrix} -\omega \varphi_{\omega} \\ \varphi_{\omega} \end{bmatrix} \]

which in turn follows from (2.10).

**Step 3.** It remains to consider the case \( \kappa > 0 \) and \( \omega = 0 \). In this case (5.4) reads
\[ -\begin{bmatrix} 0 & 1 \\ L_{0}(0) & 0 \end{bmatrix} u = -\begin{bmatrix} L_{\kappa}(0) & 0 \\ 0 & L_{0}(0) \end{bmatrix} u = \lambda^2 u. \]

By lemma 4.2, one has:
\[ -\lambda^2 \in \sigma_p(L_{\omega}(0)) \cup \sigma_p(L_{0}(0)) = \{ 0, \Lambda_{\kappa} \}, \quad \kappa > 0, \]
where \( \Lambda_{\kappa} < 0 \). This leads to the existence of eigenvalues \( \lambda = \pm \sqrt{-\Lambda_{\kappa}} \in \mathbb{R} \), showing that there is linear instability □

Below, we will need the following spectral property.

**Lemma 5.2.** Assume that \( \partial_{\omega} Q(\varphi) \neq 0 \). Then the zero eigenvalue of \( A(\omega) \) is of algebraic multiplicity two. In the case \( \kappa \neq 0 \), the generalized null space spanned by the vectors \( J \Phi_{\omega} \) and \( \partial_{\omega} \Phi_{\omega} \) which satisfy
\[ A(\omega) J \Phi_{\omega} = 0, \quad A(\omega) \partial_{\omega} \Phi_{\omega} = J \Phi_{\omega}. \]

In the case \( \kappa = 0 \), \( \ker(A(\omega)) = \text{Span}\{J \Phi_{\omega}, \Phi_{\omega}\} \).

**Proof.** First consider the case \( \kappa \neq 0 \). By (4.11), \( \ker(A(\omega)) = \text{Span}\{\Xi_{1}\} = \text{Span}\{\Xi_{1}\} \).

Hence, \( \ker(A(\omega)) = \text{Span}\{G^{-1} \Xi_{1}\} = \text{Span}\{J \Phi_{\omega}\}. \)

**Proof.** First consider the case \( \kappa \neq 0 \). By (4.11), \( \ker(A(\omega)) = \text{Span}\{\Xi_{1}\} = \text{Span}\{\Xi_{1}\} \).

Denote \( \Xi := -G \partial_{\omega} \Phi_{\omega} = \begin{bmatrix} -\partial_{\omega} \varphi_{\omega} \\ \omega \partial_{\omega} \varphi_{\omega} + \varphi_{\omega} \\ 0 \end{bmatrix} \). Applying (4.10) and (5.8), we obtain
\[ \tilde{A}(\omega) \Xi = -i \begin{bmatrix} 0 & \sigma_{1} \\ \sigma_{1} & 0 \end{bmatrix} \begin{bmatrix} H_{\omega}(\omega) & 0 \\ 0 & H_{0}(\omega) \end{bmatrix} \Xi = -i \Xi_{1}. \]

Further, the relation \( \tilde{A}(\omega) = GA(\omega)G^{-1} \) together with (4.12) and (5.12) imply that
\[ A(\omega) \partial_{\omega} \Phi_{\omega} = -A(\omega) G^{-1} \Xi = -G^{-1} \tilde{A}(\omega) \Xi = iG^{-1} \Xi_{1} = J \Phi_{\omega}. \]
Hence, \((5.10)\) follows by \((5.11)\) and \((5.13)\). Thus \(\lambda = 0\) is an eigenvalue of \(A(\omega)\) of multiplicity at least two. To be able to extend this Jordan chain, we need to make sure that \(\Xi\) is orthogonal to the kernel of \(\tilde{A}^\ast(\omega)\), which is given by (see \((4.11)\))

\[
\ker \left( \tilde{A}^\ast(\omega) \right) = \ker \left( \tilde{H}(\omega)\tilde{\Sigma} \right) = \text{Span}\{\tilde{\Sigma}\Xi_1\} = \text{Span}\left\{ \begin{bmatrix} -\omega\varphi_{\omega} \\ \varphi_{\omega} \\ 0 \\ 0 \end{bmatrix} \right\};
\]

thus, the condition to have a Jordan block of a larger size is

\[
0 = \begin{bmatrix} -\omega\varphi \\ \varphi \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\partial_{\omega}\varphi \\ \omega\partial_{\omega}\varphi + \varphi \\ 0 \\ 0 \end{bmatrix} = \partial_{\omega} \left( \omega\|\varphi_{\omega}\|^2 \right).
\]

\((5.14)\)

In the case \(\kappa = 0\), \(\ker \left( \tilde{A}^\ast(\omega) \right) = \text{Span}\{\tilde{\Sigma}\Xi_1, \tilde{\Sigma}\Xi_2\}\). Therefore, the condition to have a Jordan block corresponding to \(\Xi_j\) for some \(j = 1, 2\) reads as follows:

\[
(\Xi, \tilde{\Sigma}\Xi_k) = 0, \quad 1 \leq k \leq 2.
\]

Evidently, \((\Xi, \tilde{\Sigma}\Xi_k) = 0\) for all \(j \neq k\) and for any \(\omega \in (-m, m)\), while \((\Xi, \tilde{\Sigma}\Xi_j) = 2i\omega\|\phi_{\omega}\|^2 = 0\) if and only if \(\omega = 0\). Thus, if \(\kappa = 0\), the eigenvalue \(\lambda = 0\) is of algebraic multiplicity larger than two if and only if \(\omega = 0\).

\(\Box\)

6. Asymptotic stability

In the remaining part of the paper, we will prove asymptotic stability of solitary waves and scattering asymptotics \((1.4)\). In the appendix A.1 we show that the continuous spectrum of \(A(\omega)\) coincides with

\[
C := i \left( \mathbb{R} \setminus (-m + |\omega|, m - |\omega|) \right).
\]

We assume that the following spectral conditions hold

**Assumption 6.1 (spectral conditions).** There is an open interval \(\mathcal{I} \subset (-m, m)\) such that for \(\omega \in \mathcal{I}\) there are bound states \(e^{-i\omega t}\phi_{\omega}(x)\) of equation \((1.2)\), and moreover \(\forall \omega \in \mathcal{I}\)

(a) \(\partial_{\omega}(\omega\|\varphi_{\omega}\|^2) \neq 0\);

(b) \(\sigma_p(A(\omega)) = \{0\}\);

(c) There are no virtual levels at \(\lambda = \pm i(m - |\omega|)\).

**Remark 6.2.** Recall that there are the virtual level at a threshold point \(\mu\) of the essential spectrum of an operator \(K\) (i.e. the endpoint of the essential spectrum or the point where the continuous spectrum changes its multiplicity) if the equation \(K\psi = \mu\psi\) has a nonzero solution \(\psi \in L^\infty(\mathbb{R})\setminus L^2(\mathbb{R})\).
Lemma 6.3. Suppose that for $\omega = 0$ there is stationary state $\phi_0(x)$ of equation (1.2). Then conditions (a)–(c) hold if and only if $\kappa < -1/2$.

Proof. First, note that $\partial_\omega(\|\varphi_\omega\|^2)_{|\omega=0} = \|\phi_0\|^2 \neq 0$ by (2.12). Further, (5.9) and lemma 4.2 imply that $\sigma_\nu(A(0)) = \{0\}$ if and only in $\kappa < -1/2$. It remains to consider the endpoints $\pm im$. Due to (3.5) and (3.6), the equation $A(0)X = \pm imX$ reads

$$\begin{align*}
-X''^i &= \rho_0(1 + \ell \kappa) \langle X_1, \rho \rangle = 0, \\
-X''^2 &= \rho_0(X_2, \rho) = 0, \\
X_1 &= \pm imX_1, \\
X_2 &= \pm imX_2.
\end{align*}$$

The equation has nonzero solution $X \in (L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})) \oplus \mathbb{C}^4$ if and only if $\kappa = -1/2$. In this case $X_1 = 1, X_3 \pm im, X_4 = X_5 = 0$. □

Remark 6.4. For any $\omega \in (-m, m)$, the absence of virtual levels at the embedded thresholds $\pm i(m + |\omega|)$ follows from lemma A.3. Besides, in lemma A.3 we prove a criterion for absence of virtual levels at endpoints $\pm i(m - |\omega|)$.

Remark 6.5. In appendix A.5 we prove that assumption 6.1 holds in the case when $\rho(x)$ is close to $\delta(x), \kappa \in (-\infty, 0) \cup (0, 2^{-1/2})$ and $a(\tau) = \tau^\kappa$ (for $0 \leq \tau \leq 1$ if $\kappa \in (0, 2^{-1/2})$ and for $\tau \geq c_2 > 0$ if $\kappa < 0$).

Theorem 6.6 (asymptotic stability). Assume that the spectral conditions (assumption 6.1) hold in an interval $I \subset (-m, m)$. Let (2.6) hold, and let $\Psi(x, t) \in C(\mathbb{R}, E)$ be the solution to equation (1.2) with the initial value $\Psi_0(x) = \Psi(x, 0) \in E^{2+}$, which is close to a solitary wave $e^{-i\omega_0}e^{i\theta_0}e^{-i\phi_0_\omega_0}(x) = e^{-i\omega_0} \begin{bmatrix} \varphi_\omega_0 \\ -i\omega_0\varphi_\omega_0 \end{bmatrix}$ with $\omega_0 \in I$ and $\theta_0 \in \mathbb{R} \mod 2\pi$:

$$\nu := \|\Psi_0 - e^{-i\omega_0} \phi_\omega_0\|_{E^{2+}} \ll 1. \quad (6.2)$$

Then for $\nu$ sufficiently small the solution admits the following asymptotics:

$$\Psi(x, t) = e^{-i(\omega_0t + \theta_0)}\phi_\omega_0(x) + W(t)\Xi_\pm + r_\pm(t), \quad t \to \pm \infty, \quad (6.3)$$

where $\Xi_\pm \in E$ are the corresponding asymptotic scattering states and

$$\|r_\pm(t)\|_E = O(|t|^{-1/2}), \quad t \to \pm \infty. \quad (6.4)$$

Above, $W(t)$ the dynamical group of the free Klein–Gordon equation

$$\begin{bmatrix} \dot{\psi} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \partial_x - m^2 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \pi \end{bmatrix}. \quad (6.5)$$

Here and below we denote by $a \pm$ any number $a \pm \epsilon$ with an arbitrary small, but fixed $\epsilon > 0$.

7. Invariant subspace of the discrete spectrum

The real version of the set $S$ reads

$$S = \left\{ e^{i\theta} \Phi_\omega(x), \ |\omega| < m, \ \theta \in [0, 2\pi) \right\}.$$
The tangent space to $S$ at the point $e^{i\theta} \Phi_\omega$ with parameters $\omega, \theta$ is the linear span of the derivatives with respect to $\theta$ and $\omega$:

$$T_{e^{i\theta} \Phi_\omega} S := \text{Span} \left\{ J e^{i\theta} \Phi_\omega(x), e^{i\theta} \partial_\omega \Phi_\omega(x) \right\}.$$ 

Notice that the operator $A(\omega)$ corresponds to $\theta = 0$ since we have extracted the phase factors $e^{i\theta}$ from the solution in the process of linearization (3.1). The tangent space to $S$ at the point $\Phi_\omega$ with parameters $(\omega, 0)$ is spanned by the vectors

$$e_0(\omega) := J \Phi_\omega, \quad e_1(\omega) := \partial_\omega \Phi_\omega.$$ (7.1)

We introduce the symplectic form $\Omega$ for the real vectors $X, Y \in L^2(\mathbb{R}) \otimes \mathbb{R}^4$ by the integral

$$\Omega(X, Y) = \langle \Sigma X, Y \rangle = \int \Sigma X(x) \cdot Y(x) \, dx,$$ (7.2)

where '$\cdot$' stands for the scalar product on $\mathbb{R}^4$. By assumption 6.1(a),

$$\Omega(e_0, e_1) = \Omega(J \Phi_\omega, \partial_\omega \Phi_\omega) = \langle \Sigma J \Phi_\omega, \partial_\omega \Phi_\omega \rangle = \begin{bmatrix} -\omega \varphi_\omega & 0 \\ 0 & 0 \\ 0 & -\omega \partial_\omega \varphi_\omega \end{bmatrix},$$

$$= -\partial_\omega (\omega \| \varphi_\omega \|^2) \neq 0. \quad (7.3)$$

It follows that $\Omega$ is a nondegenerate symplectic form on the tangent space $T_{e^{i\theta} \Phi_\omega} S$. There is a symplectic projection $P^0(\omega) : L^2(\mathbb{R}) \otimes \mathbb{R}^4 \to T_{e^{i\theta} \Phi_\omega} S$, represented by the formula

$$P^0(\omega) \Psi = \frac{1}{\langle \Sigma J \Phi_\omega, \partial_\omega \Phi_\omega \rangle} \left( -\langle \Sigma \Psi, \partial_\omega \Phi_\omega \rangle J \Phi_\omega + \langle \Sigma \Psi, J \Phi_\omega \rangle \partial_\omega \Phi_\omega \right). \quad (7.4)$$

Due to remark 2.6 the symplectic projection $P^0(\omega)$ is well defined for $\Psi \in E_{-5/2}$.  

**Corollary 7.1.**

$$P^c (\omega) = I - P^0(\omega) \quad (7.5)$$

is also a symplectic projector.

**Remark 7.2.** By lemma 5.2, on the generalized null space of $A(\omega)$, one has $A^2(\omega) = 0$, and so the semigroup $e^{A(\omega)t}$ reduces to $I + A(\omega)t$.

### 8. Time decay in the continuous spectrum

Here and below we will write $A$ instead of $A(\omega)$, $P^0$ instead of $P^0(\omega)$, etc. Due to remark 7.2, the solutions $X(t) = e^{At}X_0$, where $A = A(\omega)$, of the linearized equation (3.6) do not decay as $t \to \infty$ if $P^0X_0 \neq 0$. On the other hand, we do expect time decay of $P^c X(t)$, as a consequence of the Laplace representation for $P^c e^{At}$:

$$P^c e^{At} = -\frac{1}{2\pi i} \int_C e^{\lambda t} (R(\lambda + 0) - R(\lambda - 0)) \, d\lambda. \quad (8.1)$$
Here the integral is taken over the continuous spectrum $C$ of the operator $A$ defined in (6.1), and $R(\lambda \pm 0), \lambda \in C,$ are the right and the left limits as $\varepsilon \to 0$ of the resolvent $R(\lambda \pm \varepsilon) = (A - \lambda \mp \varepsilon)^{-1}$. We prove the existence these limits in appendix A.2.

**Proposition 8.1.** Assume that the conditions (b) and (c) of assumption 6.1 hold. Then

$$\|P^t e^{At}\|_{E_{1/2} \to E_{-1/2}} \leq C(1 + t)^{-3/2}, \quad t \in \mathbb{R}. \quad (8.2)$$

We prove the proposition in appendix A.4.

## 9. Modulation equations

In this section we present the modulation equations which allow us to construct solutions $\Psi(x, t)$ of equation (2.15) such that, at each time $t$, it remains close to a soliton, with time varying (‘modulating’) parameters $(\omega, \theta) = (\omega(t), \theta(t))$. It will be assumed that $\Psi(x, t)$ is a given weak solution of (2.15) as provided by theorem 2.1, so that the map $t \mapsto \Psi(\cdot, t), \mathbb{R}_+ \to E$, is continuous.

We look for a solution to (2.15) in the form

$$\Psi(x, t) = e^{i\theta(t)} (\Phi_{\omega(t)}(x) + Z(x, t)) = e^{i\theta(t)} Y(x, t), \quad Y(x, t) = \Phi_{\omega(t)}(x) + Z(x, t), \quad (9.1)$$

where $Z$ is small and

$$\theta(t) = \int_0^t \omega(s) ds + \gamma(t), \quad (9.2)$$

with $\gamma$ treated perturbatively. The choice of parameters $(\omega(t), \theta(t))$ is determined by restriction $Z(t)$ to lie in the image of the projection operator onto the continuous spectrum $P^t = P(\omega(t))$

or equivalently that

$$P^t|Z(t) = 0, \quad P^0 = P^0(\omega(t)) = I - P(\omega(t)) \quad (9.3)$$

with the projection operators defined in (7.4). Now we give a system of modulation equations for $\omega(t)$ and $\gamma(t)$ which ensure that the conditions (9.3) are preserved by the time evolution.

**Lemma 9.1.**

(a) Assume given a solution of (2.15) with regularity as described in theorem 2.1, which can be written in the form (9.1)–(9.3) with continuously differentiable $\omega(t), \theta(t)$. Then

$$\dot{Z} = AZ - \dot{\omega} \partial_x \Phi_\omega - \dot{\gamma} J(\Phi_\omega + Z) + Q, \quad A = A(\omega(t)), \quad (9.4)$$

where

$$Q = Q(Z, \omega) = \rho(x) \left( \begin{array}{c} 0 \\ 0 \\ F_1(\Phi_\omega + Z) - F_1(\Phi_\omega) \\ F_2(\Phi_\omega + Z) \\ \alpha(1 + 2\kappa)(Z_1, \rho) \\ \alpha(Z_2, \rho) \end{array} \right), \quad (9.5)$$

$$\dot{\omega} = \frac{\langle P^t Q, \Sigma Y \rangle}{\langle \partial_x \Phi_\omega - \partial_x P^t Z, \Sigma Y \rangle}, \quad (9.6)$$

$$\dot{\gamma} = \frac{\langle \Sigma P^t (\partial_x \Phi_\omega - \partial_x P^t Z), P^t Q \rangle}{\langle \partial_x \Phi_\omega - \partial_x P^t Z, \Sigma Y \rangle}. \quad (9.7)$$

3608
(b) Conversely, given $\Psi$ a solution of (2.15) as in theorem 2.1 and continuously differentiable functions $\omega(t), \theta(t)$ which satisfy (9.6) and (9.7), then $Z$ defined by (9.1) satisfies (9.4) and the condition (9.3) holds at all times if it holds initially.

**Proof.** Substituting (9.1) into (2.15), we obtain

$$\dot{\gamma}(JY + \dot{Y} = AZ + Q, \tag{9.8}$$

which implies (9.4). Further, the scalar product of (9.8) with $(P_i^0)^* \Sigma JY$ gives

$$\dot{\gamma}(JY, (P_i^0)^* \Sigma JY) + \langle \dot{Y}, (P_i^0)^* \Sigma JY \rangle = \langle Q, (P_i^0)^* \Sigma JY \rangle. \tag{9.9}$$

Taking into account $\Sigma P_i^0 = (P_i^0)^* \Sigma$, and $(P_i^0)^2 = P_i^0$, we get

$$\langle JY, (P_i^0)^* \Sigma JY \rangle = \langle P_i^0 JY, (P_i^0)^* \Sigma JY \rangle = \langle P_i^0 JY, \Sigma P_i^0 JY \rangle = 0.$$

Then (9.9) becomes

$$\langle P_i^0 \dot{Y}, \Sigma JY \rangle = \langle P_i^0 Q, \Sigma JY \rangle.$$}

Notice also that $\partial_i (P_i^0 Z) = P_i^0 Z + \dot{\omega} \partial_i P_i^0 Z = 0$. Moreover, $P_i^0 \dot{\Phi}_\omega = \dot{\omega} \partial_i \Phi_\omega$. Hence,

$$P_i^0 \dot{Y} = P_i^0 \dot{\Phi}_\omega + P_i^0 \dot{Z} = (\partial_i \Phi_\omega - \partial_i P_i^0 Z) \omega. \tag{9.10}$$

This immediately implies (9.6). Finally, taking the scalar product of (9.8) with $\Sigma P_i^0 \dot{Y}$, we obtain

$$\dot{\gamma}(JY, \Sigma P_i^0 \dot{Y}) = \langle Q, \Sigma P_i^0 \dot{Y} \rangle \tag{9.11}$$

since

$$\langle \dot{Y}, \Sigma P_i^0 \dot{Y} \rangle = -(\langle \dot{Y}, P_i^0 \dot{Y} \rangle = -(\langle P_i^0 \Sigma \dot{Y}, P_i^0 \dot{Y} \rangle = -(\langle \Sigma P_i^0 \dot{Y}, P_i^0 \dot{Y} \rangle = 0,$$

and

$$\langle AZ, \Sigma P_i^0 \dot{Y} \rangle = -(\langle \Sigma P_i^0 AZ, P_i^0 \dot{Y} \rangle = -(\langle \Sigma AP_i^0 Z, P_i^0 \dot{Y} \rangle = 0.$$}

Substituting (9.10) into (9.11) leads to (9.7).

It remains to show that for the initial data sufficiently close to a soliton there exist solutions to (9.6) and (9.7), at least locally. To achieve this, we observe that if the spectral conditions from assumption 6.1(a) hold, then the denominator appearing on the right-hand side of (9.6) and (9.7) does not vanish for small $\|Z\|_{E_{5/2-}}$ by (7.3). This has the consequence that the orthogonality conditions can be satisfied for small $Z$ because they are equivalent to a locally well-posed set of ordinary differential equations for $t \to (\theta(t), \omega(t))$. This implies the following corollary:

**Corollary 9.2.**

(a) In the situation of lemma 9.1(a) assume that assumption 6.1 hold. If $\|Z(t)\|_{E_{5/2-}}$ is sufficiently small, the right hand sides of (9.6) and (9.7) are smooth in $\theta, \omega$ and there exists $C(\omega, Z) > 0$ which depends continuously on $\omega$ and $Z$ such that

$$|\dot{\gamma}(t)| \leq C(\omega, Z)\|Z(t)\|_{E_{5/2-}}, \quad |\dot{\omega}(t)| \leq C(\omega, Z)\|Z(t)\|_{E_{5/2-}}^2.$$

(b) Assume given $\Psi$, a solution of (2.15) as in theorem 2.1. If $\omega_0$ satisfies (7.3) and $Z(x, 0) = e^{-J\Phi}(x, 0) - \Phi_\omega(x)$ is small in $E_{5/2+}$ norm and satisfies (9.3) there is a time interval on which there exist $C^1$ functions $t \mapsto (\omega(t), \gamma(t))$ which satisfy (9.6) and (9.7).
10. Time decay for the transversal dynamics

We deduce theorem 6.6 from the following time decay of the transversal component \( Z(t) \) in the nonlinear setting,

**Theorem 10.1.** Let all the assumptions of theorem 6.6 hold. There are \( \nu_0 > 0 \) and \( c > 0 \) such that for \( \nu \in (0, \nu_0) \) there exist \( C^1 \)-functions \( t \mapsto (\omega(t), \gamma(t)) \) defined for \( t \geq 0 \) such that the solution \( \Psi(x, t) \) of (2.15) can be written as in (9.1)–(9.3) with (9.6)–(9.7) satisfied, and there exists \( M > 0 \), depending only on the initial data, such that

\[
m(T) = \sup_{0 \leq t \leq T} \left[ (1 + t)^{3/2} ||Z(t)||_{E_{5/2}} + (1 + t)^{3} \left( |\dot{\gamma}| + |\dot{\omega}| \right) \right] \leq M, \quad \forall T \geq 0, \tag{10.1} \]

and moreover \( M \leq cv \).

Similarly to [2, lemma 10.1], we can assume that (9.3) holds initially without loss of generality. Then the local existence asserted in corollary 9.2 implies the existence of an interval \( [0, t_1] \) on which are defined \( C^1 \) functions \( t \mapsto (\omega(t), \gamma(t)) \) satisfying (9.6) and (9.7) and such that \( M(t_1) = \delta \) for some \( t_1 > 0 \) and \( \delta > 0 \). By continuity we can make \( \delta \) as small as we like by making \( \nu \) and \( t_1 \) small. In section 11 below we will prove the following proposition

**Proposition 10.2.** In the situation of theorem 10.1 let \( M(t_1) \leq \delta \) for some \( t_1 > 0 \) and \( \delta > 0 \). Then there exist numbers \( \nu_1 > 0 \) and \( \delta_1 > 0 \), independent of \( t_1 \), such that

\[
m(t_1) \leq \delta/2 \tag{10.2} \]

if \( \nu = ||Z(0)||_{E_{5/2}} < \nu_1 \) and \( \delta \in (0, \delta_1) \).

Assuming the truth of proposition 10.2 for now theorem 10.1 will follow from the next argument. Consider the set \( T \) of \( t_1 \geq 0 \) such that \( (\omega(t), \gamma(t)) \) are defined on \( [0, t_1] \) and \( M(t_1) \leq \delta \). This set is relatively closed by continuity. On the other hand, (10.2) and corollary 9.2 with sufficiently small \( \delta \) and \( \nu \) imply that this set is also relatively open, and hence \( \sup T = +\infty \), completing the proof of theorem 10.1.

11. Proof of proposition 10.2

11.1. Frozen linearized equation

Note that linear part of (9.4) is non-autonomous. Therefore (following [3]) we introduce a small modification of (9.1), which leads to an autonomous linearized equation. Namely, we represent the solution in the form

\[
\Psi(x, t) = e^{J(t)}(\Phi_{\omega_1}(x) + e^{-J(t-\theta_1(t))}X(x, t)), \tag{11.1}
\]

where \( \theta(t) \) is defined in (9.2), \( \theta_1(t) = \omega_1 t + \gamma(0) \) with \( \omega_1 = \omega(t_1) \). Thus,

\[
X = e^{J(\theta-\theta_1)}Z, \quad \text{and} \quad Z = e^{-J(\theta-\theta_1)}X. \tag{11.2}
\]

Since \( \tilde{Z} = e^{-J(\theta-\theta_1)}(X - (\omega + \dot{\gamma} - \omega_1)JX) \), equation (9.4) implies

\[
\tilde{X} = (\omega_1 - \omega)JX, \end{align*}
\]

\[
\begin{align*}
\tilde{X} &= (\omega_1 - \omega)JX + e^{J(\theta-\theta_1)}(\dot{\gamma}JX + \dot{\omega_1}J\Phi_{\omega_1} - Q(e^{-J(\theta-\theta_1)}X, \omega)) \tag{11.3}
\end{align*}
\]
The matrices $A$ and $e^{J(\theta-\theta_1)}$ do not commute:
\[
A e^{J(\theta-\theta_1)} - e^{J(\theta-\theta_1)} A = 2\kappa \rho \sin(\theta - \theta_1)P, \quad \text{where} \quad P = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \langle \cdot, \rho \rangle & 0 & 0 \\
\langle \cdot, \rho \rangle & 0 & 0 & 0
\end{bmatrix}, \quad (11.4)
\]
and $\kappa(t) = \kappa(\omega(t))$ is defined in (3.3). Hence, (11.3) implies
\[
\dot{X} = (\omega - \omega_1) JX + AX - e^{J(\theta-\theta_1)} (2\kappa \rho \sin(\theta - \theta_1)P X + \dot{\gamma} J \Phi_\omega \\
+ \dot{\omega} \partial_\omega \Phi_\omega - Q(e^{-J(\theta-\theta_1)}X, \omega)) . \quad (11.5)
\]
To obtain an autonomous equation we rewrite the first two terms on the right-hand side by freezing the coefficients at $t = t_1$. Note that $(\omega - \omega_1) J + A = A_1 + \rho (V - V_1)$.

Here $\omega_1 = \omega(t_1)$, $A_1 = A(\omega_1)$, and
\[
V = \alpha \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(1 + 2\kappa) \langle \cdot, \rho \rangle & 0 & 0 & 0 \\
0 & \langle \cdot, \rho \rangle & 0 & 0
\end{bmatrix}, \quad V_1 = \alpha_1 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(1 + 2\kappa_1) \langle \cdot, \rho \rangle & 0 & 0 & 0 \\
0 & \langle \cdot, \rho \rangle & 0 & 0
\end{bmatrix}, \quad (11.6)
\]
where $\alpha(t) = \alpha(\omega(t))$ is defined in (3.3), $\alpha_1 = \alpha(t_1)$, $\kappa_1 = \kappa(t_1)$. Now equation (11.5) can be written in the following frozen form
\[
\dot{X} = A_1 X + f_1 \quad (11.7)
\]
where
\[
f_1 = \rho (V - V_1) X - e^{J(\theta-\theta_1)} (2\kappa \rho \sin(\theta - \theta_1)P X + \dot{\gamma} J \Phi_\omega \\
+ \dot{\omega} \partial_\omega \Phi_\omega - Q(e^{-J(\theta-\theta_1)}X, \omega)) . \quad (11.8)
\]
The following lemma show that the additional terms in $f_1$ can be estimated as small uniformly in $t_1$.

**Lemma 11.1.** *In the situation of proposition 10.2 there exists $c > 0$, independent of $t_1$, such that for $0 \leq t \leq t_1$\n\[
|\alpha(t) - \alpha_1| + |\kappa(t) - \kappa_1| + |\theta(t) - \theta_1(t)| \leq c\delta.
\]

**Proof.** By (10.1)
\[
\sup_{0 \leq t \leq t_1} (1 + t^3) \max (|\gamma_1(t)| + |\dot{\omega}(t)|) \leq M(t_1) = \delta. \quad (11.9)
\]
Therefore
\[
|\alpha(t) - \alpha(t_1)| = \left| \int_t^{t_1} \dot{\alpha}(\tau)d\tau \right| \leq c \left( \sup_{0 \leq \tau \leq t_1} (1 + \tau^2)|\dot{\omega}(\tau)| \right) \int_t^{t_1} \frac{d\tau}{1 + \tau^2} \leq c\delta,
\]
since \( |\dot{\alpha}(\tau)| \leq c|\dot{\omega}(\tau)| \). The difference \( |\alpha(t) - \kappa(t_1)| \) can be estimated similarly. Finally,
\[
\theta(t) - \theta_1(t) = \int_0^t \omega(\tau)d\tau + \gamma(t) - \omega(t_1)t - \gamma(0)
= \int_0^t (\omega(\tau) - \omega(t_1))d\tau + \int_0^t \dot{\gamma}(\tau)d\tau
= \int_0^t \dot{\omega}(s)dsd\tau + \int_0^t \dot{\gamma}(\tau)d\tau.
\]
Then \( |\theta(t) - \theta_1(t)| \leq c\delta \) by (11.9).

**Lemma 11.2.** In the situation of proposition 10.2 there exists \( c > 0 \), independent of \( t_1 \), such that for \( 0 \leq t \leq t_1 \)
\[
\|Q(e^{-3(\theta - \theta_1)}X(t), \omega(t))\|_{E_{5/2}^+} \leq C\|X(t)\|_{E_{5/2}^+}. \tag{11.10}
\]

**Proof.** Note that
\[
e^{-3(\theta - \theta_1)}X(t) = Z(t) = e^{-3(\theta(t) - \theta_1)}\psi(t - \Phi_{\omega(t)}(t_{\omega(t)})) \in C_b([0, t_1], X)
\]
by (2.8) and (11.2). Hence, \( Z(t) = (Z_1(t), Z_2(t)) \in C_b([0, t_1], \mathbb{L}^\infty(\mathbb{R}) \oplus \mathbb{L}^\infty(\mathbb{R})) \), and (9.5) implies
\[
|Q(Z(x, \omega(t)))| \leq C |\rho(x)|(|Z_1(x, t)|^2 + |Z_2(x, t)|^2)
\leq C_1 |\rho(x)|(|Z_1|_{L^2_{5/2}^+}^2 + |Z_2|_{L^2_{5/2}^+}^2), \quad 0 \leq t \leq t_1.
\]
Hence, (11.10) follows.

11.2. Projection onto discrete and continuous spectral spaces

Let \( P_1^d = P_d^1 \) and \( P_1^c = P_c^1 \) be the symplectic projections onto the discrete and continuous spectral subspaces defined by the operator \( A_1 \) and write, at each time \( t \in [0, t_1] \):
\[
X(t) = X^d_1(t) + X^c_1(t) \tag{11.11}
\]
with \( X^d_1(t) = P_1^dX(t) \) and \( X^c_1(t) = P_1^cX(t) \). The following lemma shows that it is only necessary to estimate \( X^c_1(t) \).

**Lemma 11.3.** In the situation of proposition 10.2, assume that
\[
\Delta := \sup_{0 \leq t \leq t_1} (|\omega(t) - \omega_1| + |\theta(t) - \theta_1(t)|) \tag{11.12}
\]
is sufficiently small. Then for \( 0 \leq t \leq t_1 \) there exists \( c(\Delta, \omega_1) \) such that
\[
c(\Delta, \omega_1)^{-1} \|X_1^c(t)\|_{E_{\sigma}} \leq \|X(t)\|_{E_{\sigma}} \leq c(\Delta, \omega_1) \|X_1^c(t)\|_{E_{\sigma}, \sigma \in \mathbb{R}}. \tag{11.13}
\]
Proof. We write
\[ X(t) = X_0^c(t) + X_1^c(t), \quad X_0^c(t) = b_0(t)e_0(\omega_1) + b_1(t)e_1(\omega_1), \]
where \( b_0(t) \), \( b_1(t) \) are chosen at each time \( t \) to ensure that \( \Omega(X_1^c(t), e_0(\omega_1)) = \Omega(X_1^c(t), e_1(\omega_1)) = 0 \).

Using the fact that
\[ \Omega(e^{-J(0-\theta)}X(t), e_j(\omega(t))) = 0, \quad j = 0, 1 \]
(since \( P^j_0 Z(t) = 0 \)) this means that \( b_j \) are determined by
\[ -\Omega(e_0(\omega_1), e_1(\omega_1)) b_0(t) = \Omega(X(t), e_1(\omega_1)) = \Omega(X(t), e_1(\omega_1) - e^{J(0-\theta)} e_1(\omega(t))), \]
\[ \Omega(e_0(\omega_1), e_1(\omega_1)) b_1(t) = \Omega(X(t), e_0(\omega_1)) = \Omega(X(t), e_0(\omega_1) - e^{J(0-\theta)} e_1(\omega(t))). \]

From these it follows that there exists \( c > 0 \) such that \( \|X_0^c(t)\|_{E^c} \leq c\Delta\|X(t)\|_{E_0} \) and hence (11.13) follows as claimed.

11.3. Estimation of \( M \)

Here we show that both terms in \( M \) (see (10.1)) are bounded by \( \delta/4 \), uniformly in \( t_1 \). As in corollary 9.2, we have
\[ |\dot{\gamma}(t)| + |\dot{\omega}(t)| \leq c_0Z(t)\|X_{E_0}^c\|_{E_{-5/2}^c} \leq c_0\frac{M^2(t)}{(1 + |t|)^3}, \quad t \leq t_1. \]

Taking \( \delta_1 < 1/(4c_0) \), we bound the second term in \( M \) by \( \delta/4 \). By lemma 11.3, to estimate the first term in \( M \) it is enough to estimate \( X_1^c(t) \). Let us apply the projection \( P^c_1 \) to both sides of (11.7). Then the equation for \( X_1^c(t) \) reads
\[ \dot{X}_1^c = \mathbf{A}_1^c X_1^c + P^c_1 f_1. \]

Now to estimate \( X_1^c \), we use the Duhamel representation:
\[ X_1^c(t) = e^{\mathbf{A}_1^c t}X_1^c(0) + \int_0^t e^{\mathbf{A}_1^c(t-s)} P^c_1 f_1(s) \, ds, \quad t \leq t_1. \]

Recall that \( P^c_1 X_1^c(t) = 0 \) for \( t \in [0, t_1] \). Therefore,
\[ \|e^{\mathbf{A}_1^c t}X_1^c(0)\|_{E_{-5/2}^-} \leq c(1 + t)^{-3/2}\|X_1^c(0)\|_{E_{5/2}^+} \leq c(1 + t)^{-3/2}\|X(0)\|_{E_{5/2}^+}. \]

(11.17)

by proposition 8.1 and (11.13). Further, representation (11.8) for \( f_1 \), theorem 8.1, corollary 9.2 and lemmas 11.1 and 11.2 imply
\[ \|e^{\mathbf{A}_1^c(t-s)} P^c_1 f_1\|_{E_{-5/2}^-} \leq c(1 + t - s)^{-3/2}\|f_1(t)\|_{E_{5/2}^+} \leq c(1 + t - s)^{-3/2}\|Z(t)\|_{E_{-5/2}^-} \leq c(1 + t - s)^{-3/2}(\|Z(t)\|_{E_{-5/2}^-}^2 + \delta\|Z(t)\|_{E_{-5/2}^-}), \quad t \leq t_1. \]

(11.18)
Now (11.16)-(11.18) and (11.13) yield
\[
\|X(t)\|_{E_{\infty}/2} \leq c(1 + t)^{-3/2}\|X(0)\|_{E_{\infty}/2} + c_1 \int_0^t \frac{ds}{(1 + t - s)^{3/2}} \left( \|X(s)\|^2_{E_{\infty}/2} + \delta \|X(s)\|_{E_{\infty}/2} \right).
\]
We multiply the above by \((1 + t)^{3/2}\) to deduce
\[
(1 + t)^{3/2}\|X(t)\|_{E_{\infty}/2} \leq c\nu + c_1 \int_0^t \frac{(1 + t)^{3/2}(1 + s)^{-3}}{(1 + t - s)^{3/2}} (1 + s)^3 \|X(s)\|^2_{E_{\infty}/2} \, ds
+ c_1 \delta \int_0^t \frac{(1 + t)^{3/2}(1 + s)^{-3/2}}{(1 + t - s)^{3/2}} (1 + s)^3 \|X(s)\|_{E_{\infty}/2} \, ds
\]
since \(\|X(0)\|_{E_{\infty}/2} \leq \nu\). We introduce the majorant
\[
m(t) := \sup_{[0,t]} (1 + s)^{3/2}\|X(s)\|_{E_{\infty}/2}, \quad t \leq t_1.
\]
Then
\[
m(t) \leq c\nu + c_1 m^2(t) \int_0^t \frac{(1 + t)^{3/2}(1 + s)^{-3}}{(1 + t - s)^{3/2}} \, ds + c_1 \delta m(t) \int_0^t \frac{(1 + t)^{3/2}(1 + s)^{-3/2}}{(1 + t - s)^{3/2}} \, ds,
\]
where both these integrals are bounded uniformly in \(t\). Thus (11.19) implies that there exist \(c_2, c_3\), independent of \(t_1\), such that
\[
m(t) \leq c\nu + c_2 m(t) + c_3 m^2(t), \quad t \leq t_1.
\]
Recall that \(m(t_1) \leq \delta \leq \delta_0\) by assumption. Therefore this inequality implies that \(m(t) \leq c\nu\) for \(t \leq t_1\) when \(\nu\) and \(\delta\) are sufficiently small. The constant \(c_4\) does not depend on \(t_1\). We choose \(c_4\) such that \(\nu < \delta/(4c_4)\). Therefore,
\[
\sup_{[0,t_1]} (1 + t)^{3/2}\|X(t)\|_{E_{\infty}/2} < \delta/4
\]
if \(\nu\) and \(\delta\) are sufficiently small.

Since \(Z = e^{-\overline{\theta}(t_1)/4}X\), the last inequality bounds the first term in \(M\) as \(\delta/4\) and hence \(M(t_1) < \delta/2\), completing the proof of proposition 10.2.

12. Soliton asymptotics

Here we prove our main theorem 6.6 using the bounds (10.1) from theorem 10.1. For a solution \(\Psi(x,t)\) to (1.2), we define the accompanying soliton as \(S(x,t) = e^{-\overline{\theta}(t)\phi_{\infty}(x)}\), where \(\overline{\theta}(t) = \int_0^t \overline{\omega}(\tau) \, d\tau + \gamma(t)\). Then for the difference \(D(x,t) = \Psi(x,t) - S(x,t)\) we obtain from equations (1.2) and (2.10)
\[
\dot{D}(x,t) = \left[ \begin{array}{cc} 0 & 1 \\ \partial_t^2 - m^2 & 0 \end{array} \right] D(x,t) + R(x,t), \quad (12.1)
\]
where
\[ R(x,t) = i\dot{\gamma}(t)S(x,t) + \dot{\omega}(t)\partial_x S(x,t) + \rho(x) \left[ F(\Psi_1(x,t)) - F(S_1(x,t)) \right]. \]

Then
\[ D(t) = W(t)D(0) + \int_0^t W(t-\tau)R(\tau)\,d\tau \]
\[ = W(t) \left( D(0) + \int_0^\infty W(-\tau)R(\tau)\,d\tau \right) \]
\[ - \int_t^\infty W(t-\tau)R(\tau)\,d\tau = W(t)\Xi_+ + r_+(t), \]

where \( W(t) \) is the dynamical group of the free Klein–Gordon equation (6.5). Since \( \gamma(t) - \gamma_+ = O(t^{-2}) \), \( \omega(t) - \omega_+ = O(t^{-2}) \), and therefore \( \theta(t) - \omega_+ t - \gamma_+ = O(t^{-1}) \) as \( t \to \infty \), to establish the asymptotic behaviour (6.3) it suffices to prove that
\[ E_{\Xi_+} + \| r_+(t) \|_E = O(t^{-1/2}), \quad t \to \infty. \] (12.2)

Due to (10.1),
\[ \| \dot{\gamma}(t)S(t) \|_E + \| \dot{\omega}(t)\partial_x S(t) \|_E \leq C(1 + t)^{-3}. \]

\[ \| \rho(x)(F(\langle \psi(t), \rho \rangle) - F(\langle S_1(t), \rho \rangle)) \|_{L^2(\mathbb{R})} \]
\[ \leq C |a(\langle \psi(t), \rho \rangle)^2 \langle \psi(t), \rho \rangle - a(\langle S_1(t), \rho \rangle)^2 \langle S_1(t), \rho \rangle | \]
\[ \leq C_1 |\langle D_1(t), \rho \rangle| \leq C_2 \| D_1(t) \|_{L^2(\mathbb{R})} \leq C_3(1 + t)^{-3/2}, \quad t > 0. \]

Hence, the ‘unitarity’ in \( E \) of the group \( W(t) \) implies (12.2).

**Appendix A. Spectrum of the linearization**

**A.1. Free resolvent**

For \( \omega \in (-m, m) \), denote
\[ A_0 = A_0(\omega) = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ \partial_x^2 - m^2 & 0 & 0 & -\omega \\ 0 & \partial_x^2 - m^2 & \omega & 0 \end{bmatrix}, \]
\[ \hat{A}_0 = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ -\xi^2 - m^2 & 0 & 0 & -\omega \\ 0 & -\xi^2 - m^2 & \omega & 0 \end{bmatrix}. \]
We have
\[
\det(\hat{A}_0 - \lambda) = (\xi^2 + m^2 + \lambda^2 - \omega^2 - 2i\lambda\omega) (\xi^2 + m^2 + \lambda^2 - \omega^2 + 2i\lambda\omega).
\]
Hence,
\[
(\hat{A}_0 - \lambda)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{P^+(\lambda)}{2(\xi^2 + m^2 + \lambda^2 - \omega^2 - 2i\lambda\omega)} + \frac{P^-(\lambda)}{2(\xi^2 + m^2 + \lambda^2 - \omega^2 + 2i\lambda\omega)),
\]
where
\[
P^\pm(\lambda) = \frac{1}{2} \begin{bmatrix} \pm i(\omega \pm i\lambda) & - (\omega \pm i\lambda) & -1 & \mp i \\ (\omega \pm i\lambda) & \pm i(\omega \pm i\lambda) & \pm i & -1 \\ \mp i(\omega \pm i\lambda)^2 & \mp i(\omega \pm i\lambda)^2 & \pm i(\omega \pm i\lambda) & - (\omega \pm i\lambda) \\ \pm i(\omega \pm i\lambda)^2 & \pm i(\omega \pm i\lambda)^2 & \omega \pm i\lambda & \pm i(\omega \pm i\lambda) \end{bmatrix}.
\]

Let \( R_0(\mu) = (-\partial_x^2 - \mu)^{-1}, \mu \in \mathbb{C} \setminus [0, \infty) \) be the Schrödinger resolvent:
\[
R_0(\mu) = \frac{e^{\sqrt{\mu}|x-y|}}{2i\sqrt{\mu}}, \quad \text{Im} \sqrt{\mu} > 0.
\]
Then in the case \( 0 < |\omega| < m \), (A.1) implies
\[
R_0(\lambda) = R_0(\omega, \lambda) := (A_0 - \lambda I)^{-1}
\]
\[
= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R_0^+(\lambda) + R_0^-(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathcal{C},
\]
where
\[
R_0^\pm(\lambda) = \frac{1}{2} R_0((\omega \pm i\lambda)^2 - m^2) P^\pm(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathcal{C}_\pm,
\]
and
\[
\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-.
\]

with
\[
\mathcal{C}_+ := (-i\infty, -i(m - \omega)] \cup [i(m + \omega), \infty),
\]
\[
\mathcal{C}_- := (-i\infty, -i(m + \omega)] \cup [i(m - \omega), \infty).
\]

Remark A.1. In the case \( \omega = 0 \), \( \mathcal{C} = \mathcal{C}_- = \mathcal{C}_+ = (-i\infty, -im] \cup [im, \infty) \), and
\[
R_0^\pm(\lambda) = \frac{1}{2} R_0(-\lambda^2 - m^2) P^\pm(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathcal{C}.
\]
The well-known properties of the Schrödinger resolvent \( R_0 \) (see [1, 14]) imply:
Similarly to the case of free Klein–Gordon (6.5), the dynamical group\( e^{iH_\varepsilon t} \) of equation\( \dot{X} = A_\varepsilon X \) admits the following representation (cf [16]):

\[
e^{A_\varepsilon t} = \sum_\pm e^{A_\varepsilon^\pm t}, \quad e^{A_\varepsilon^\pm t} = \frac{e^{\pm i\varepsilon t}}{2} \begin{bmatrix} \dot{U}(t) & \mp i\dot{U}(t) & U(t) & \pm iU(t) \\ \mp i\dot{U}(t) & \dot{U}(t) & \mp iU(t) & U(t) \\ \pm i\dot{U}(t) & U(t) & \dot{U}(t) & \pm iU(t) \\ \mp i\dot{U}(t) & \dot{U}(t) & \mp iU(t) & U(t) \end{bmatrix}.
\]

Here \( U(t) \) is an integral operator with the integral kernel

\[
U(x, y, t) = \frac{1}{2} \theta(t - |x - y|)J_0(m\sqrt{t^2 - |x - y|^2}), \quad x, y \in \mathbb{R}, \quad (A.11)
\]

\( J_0 \) is the Bessel function of order 0, and \( \theta \) is the Heaviside function. Moreover, for \( X \in E_{1/2^+} \), the following integral representation holds

\[
e^{A_\varepsilon^\pm t} X = -\frac{1}{2\pi i} \int_{\mathbb{C}^+} e^{\lambda t} \left[ R_0^\pm(\lambda + 0) - R_0^\pm(\lambda - 0) \right] X d\lambda, \quad t \in \mathbb{R}, \quad (A.12)
\]

where the integrals converge in the sense of distributions of \( t \in \mathbb{R} \) with the values in \( E_{-1/2^-} \).

Let \( \zeta \in C_0^\infty \) be an even function, such that

\[
\zeta(\lambda) = \begin{cases} 1, & \lambda \in [-i(2m + 1), i(2m + 1)] \\ 0, & \lambda \in [-i(2m + 2), i(2m + 2)] \end{cases} \quad (A.13)
\]

Then (A.12) implies

\[
e^{A_\varepsilon^\pm t}(1 - \zeta(A_0))X = -\frac{1}{2\pi i} \sum_\pm \int_{\mathbb{C}^+} (1 - \zeta(\lambda)) e^{\lambda t} \left[ R_0^\pm(\lambda + 0) - R_0^\pm(\lambda - 0) \right] X d\lambda.
\]
× [R_0^+(λ + 0) - R_0^+(λ - 0)]Xdλ, \quad X \in E_{1/2^+}.

Applying [13, lemma 2.3], we obtain the following high energy decay,

\[ \|e^{A(t)}(1 - ζ(A_0))\|_{L∞ \rightarrow L∞} \leq C(1 + |t|)^{-σ}, \quad σ > 1/2, \quad t \in \mathbb{R}. \]

(A.14)

**A.2. Limiting absorption principle**

Here we prove that for λ ∈ C, the resolvent R(λ ± ε) = (A - λ ± ε)^{-1} has right and left limits R(λ ± 0) as ε → 0:

**Proposition A.2.** For κ ∈ R and 0 ≤ |ω| < m, the convergence holds:

\[ R(λ ± ε) → R(λ ± 0), \quad ε → 0+, \quad λ \in C \setminus \{±i(m + ω), ±i(m - ω)\} \]

in E_{1/2^+} → E_{-1/2}.

**Proof.** Due to (3.6), A = A_0 + ρV, where V is defined in (11.6). Then the Born decomposition implies

\[ R(λ) = [1 + R_0(λ)ρV]^{-1}R_0(λ). \]

(A.16)

Because of (A.8) it suffices to prove that

\[ [1 + R_0(λ ± 0)ρV]^{-1} \leftrightharpoons [1 + R_0(λ ± 0)ρV]^{-1}, \]

\[ ε → +0, \quad λ \in C \setminus \{±i(m + ω), ±i(m - ω)\} \]

in E_{-1/2} → E_{-1/2}. The convergence holds if and only if both limiting operators 1 + R_0(λ ± 0)ρV : E_{-1/2} → E_{-1/2} are invertible for λ ∈ C \{±(m + ω), ±(m - ω)\}. According to the Fredholm theorem and compactness of the operators R_0(λ ± 0)ρV : E_{-1/2} → E_{-1/2}, it remains to prove that the equations

\[ (1 + R_0(λ ± 0)ρV)X = 0, \quad λ \in C \setminus \{±i(m + ω), ±i(m - ω)\} \]

(A.17)

admit only zero solution in E_{-1/2}.

Due to the structure of matrices V and P^± (see (A.1) and (A.2)), it suffices to prove that the restriction of (A.17) on \( \tilde{X} = (X_1, X_2) \),

\[ (1 + \tilde{R}_0(λ ± 0)ρ\tilde{V})\tilde{X} = 0, \quad λ \in C \setminus \{±i(m + ω), ±i(m - ω)\}, \]

(A.18)

admit only zero solution in H^1_{-1/2} ⊕ H^1_{1/2}. Here

\[ \tilde{V} = \begin{bmatrix} α(1 + 2κ)\langle., ρ \rangle & 0 \\ 0 & α\langle., ρ \rangle \end{bmatrix}, \]

\[ \tilde{R}_0(λ) = \frac{1}{2} \sum_{±}^\circ R_0(\ω ± iλ)^2 - m^2)P^±, \quad P^± = \begin{bmatrix} -1 & ±i \\ ±i & -1 \end{bmatrix}. \]

(A.19)

(A.20)

Taking the scalar product of (A.18) with ρ\tilde{V}\tilde{X}, we obtain

\[ \langle ρ\tilde{V}\tilde{X}, \tilde{X} \rangle + \langle ρ\tilde{V}\tilde{X}, \tilde{R}_0(λ ± 0)ρ\tilde{V}\tilde{X} \rangle = 0, \]

(A.21)
We denote 
\[
C_1 = \alpha(1 + 2\kappa)(X_1, \rho), \quad C_2 = \alpha(X_2, \rho)
\]
and consider the case ‘+’ only. In this case, (A.4) and (A.5) imply
\[
\langle \rho \tilde{V} \tilde{X}, \tilde{R}_0(\lambda + 0) \rangle = 0.
\] (A.22)

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\]
and consider the case ‘+’ only. In this case, (A.4) and (A.5) imply
\[
\langle \rho \tilde{V} \tilde{X}, \tilde{R}_0(\lambda + 0) \rangle = 0.
\] (A.22)
A.3. Virtual levels

Here we study virtual levels (also known as a threshold resonance) of the operator \( \mathbf{A}(\omega) \). Recall that in the case \( \omega = 0 \), there are virtual levels at \( \pm im \) if an only if \( \kappa = -1/2 \) (see lemma 6.3).

Now we consider the case \( \omega \neq 0 \). We prove that there are no virtual levels of the operator \( \mathbf{A} \) at the embedded thresholds \( \pm i(m + |\omega|) \) and give the condition on absence virtual levels at the edge points points \( \pm i(m - |\omega|) \). For concreteness, we consider \( 0 < \omega < m \) and the points \( \lambda_{\pm} = i(m \pm \omega) \). Formulæ (A.4) and (A.5) imply

\[
R_0(\lambda) = \frac{B^0_0}{\sqrt{\lambda - \lambda_{\pm}}} + B^0_1 + B^0_2 \sqrt{\lambda - \lambda_{\pm}} + \cdots, \quad \lambda \to \lambda_{\pm}, \; \lambda \in \mathbb{C}\setminus\mathbb{C}_{\pm}
\]

with appropriate operators \( B^\pm_j \). Following [14, formula (3.1)], we introduce generalized eigenspaces

\[
\mathbf{M}^\pm := \{ X \in H_{1/2-0}^1 \otimes \mathbb{C}^2 : (1 + B^\pm_1 \rho \mathbf{V})X \in \text{Ran}(B^\pm_0), B^\pm_0 \rho \mathbf{V}X = 0 \},
\]

where \( \text{Ran}(B^\pm_0) \) is the range of \( B^\pm_0 \). By [14, theorem 7.2], the condition \( \mathbf{M}^\pm = 0 \) is equivalent to the absence of virtual levels of \( \mathbf{A} \) at the points \( \lambda_{\pm} \).

**Lemma A.3.** \( \mathbf{M}^+ = 0 \) for any \( 0 < \omega < m \) and for any \( \kappa \in \mathbb{R} \), while \( \mathbf{M}^- \neq 0 \) only for \( \omega = \omega_\kappa \), where \( \omega_\kappa \) is the solution to

\[
\int \frac{d\xi}{\xi^2 + \omega(m - \omega)} = \frac{\kappa + 1}{\kappa + \frac{1}{2}} \int \frac{d\xi}{\xi^2 + m^2 - \omega^2}.
\]

(A.25)

In particular, \( \mathbf{M}^- = 0 \) for \( \kappa \leq -1/2 \).

**Proof.** Due to the structure of \( \mathbf{P}^\pm \) and \( \mathbf{V} \) it suffices to prove the lemma for \( \tilde{\mathbf{M}}^\pm \) instead of \( \mathbf{M}^\pm \), where

\[
\tilde{\mathbf{M}}^\pm &= \{ \tilde{X} \in H_{1/2-0}^1 \otimes \mathbb{C}^2 : (1 + \tilde{B}^\pm_1 \rho \tilde{\mathbf{V}})\tilde{X} \in \text{Ran}(\tilde{B}^\pm_0), \tilde{B}^\pm_0 \rho \tilde{\mathbf{V}}\tilde{X} = 0 \},
\]

\( \tilde{\mathbf{V}} \) is defined in (A.19), and \( \tilde{B}^\pm_j, j = 0, 1 \) are the upper right corners of matrix operators \( B^\pm_j \).

One has

\[
e^{\sqrt{(\omega \pm \lambda)^2 - m^2} (x - y)}
\]

\[
\frac{4i \sqrt{(\omega \pm m)^2 - m^2}}{4i \sqrt{(\omega \pm i \lambda)^2 - m^2}} + \frac{|x - y|}{4} + O(\sqrt{(\omega \pm i \lambda)^2 - m^2})
\]

\[
e \frac{1}{4 \sqrt{2m} \sqrt{\lambda - i(m \pm \omega)}} + \frac{|x - y|}{4}
\]

\[+ O(\sqrt{\lambda - i(m \pm \omega)}), \quad \lambda \to i(m \pm \omega).
\]

Hence, (A.3) and (A.5) imply

\[
\tilde{B}^\pm_0(x, y) = \frac{1}{4 \sqrt{2m}} \tilde{P}^\pm, \quad \tilde{B}^\pm_1(x, y) = \frac{|x - y|}{4} \tilde{P}^\pm + \frac{e^{2i \sqrt{\omega \pm m}(x - y)}}{8i \omega \sqrt{\omega \pm m}} \tilde{P}^\pm.
\]

(3620)
where $\tilde{P}^\pm$ are defined in (A.20). For arbitrary $\tilde{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \tilde{M}^\pm$,

$$\tilde{B}_i^\pm \rho \tilde{V} \tilde{X} = \frac{\rho(x) \ dx}{4\sqrt{2m}} \left[ -\alpha(1 + 2\kappa)(X_1, \rho) \mp i\alpha(X_2, \rho) \right] = 0$$

if and only if

$$\langle X_2, \rho \rangle = \pm i(1 + 2\kappa)(X_1, \rho). \quad (A.26)$$

In this case

$$\tilde{V} \tilde{X} = \alpha(1 + 2\kappa)(X_1, \rho) \left[ \frac{1}{\pm i} \right], \quad \tilde{B}_i^\pm \rho \tilde{V} \tilde{X} = \alpha(1 + 2\kappa)\langle X_1, \rho \rangle \gamma^\pm \left[ \frac{1}{\pm i} \right],$$

where

$$\gamma^\pm(\alpha) = \int e^{2i\sqrt{\omega(\pm m)|\xi - \alpha|^2}} \rho(y) \ dy = \frac{2}{2\sqrt{\omega(\pm m)}} \delta(\xi - \pm \sqrt{\omega(\pm m)}) + \frac{2}{2\sqrt{\omega(\pm m)}} \delta(\xi + \pm \sqrt{\omega(\pm m)}) \quad (A.27)$$

Hence, the condition $(1 + \tilde{B}_i^\pm \rho \tilde{V}) \tilde{X} \in \text{Ran}(\tilde{B}_i^\pm)$ reads

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \alpha(1 + 2\kappa)(X_1, \rho) \gamma^\pm \left[ \frac{1}{\pm i} \right] = \begin{bmatrix} C \\ \mp iC \end{bmatrix}, \quad C \in \mathbb{C}, \quad (A.28)$$

which implies that

$$X_2 = \pm iX_1 \mp 2iC. \quad (A.29)$$

Hence, $\langle X_2, \rho \rangle = \pm i(X_1, \rho) \mp 2i\hat{C}\hat{\rho}(0)$. Comparing this with (A.26), we get

$$(1 + 2\kappa)(X_1, \rho) = (X_1, \rho) - 2C\hat{\rho}(0).$$

Hence,

$$C\hat{\rho}(0) = -\kappa(X_1, \rho).$$

Substituting this into the first line of (A.28) and getting scalar product with $\rho$, we obtain

$$\langle X_1, \rho \rangle \left( 1 + \kappa + \alpha(1 + 2\kappa)(\gamma^\pm, \rho) \right) = 0. \quad (A.30)$$

Denote

$$\eta_+(\xi) = \int e^{i\xi x} e^{2i\sqrt{\omega(\pm m)|\xi - \alpha|^2}} \ dx = \frac{1}{2} \delta \left( \xi - 2\sqrt{\omega(\omega + m)} \right) + \frac{1}{2} \delta \left( \xi + 2\sqrt{\omega(\omega + m)} \right),$$

$$\eta_-(\xi) = \int e^{i\xi x} e^{-2i\sqrt{\omega(\omega - m)|\xi - \alpha|^2}} \ dx = \frac{1}{2} \delta \left( \xi - 2\sqrt{\omega(\omega - m)} \right) + \frac{1}{2} \delta \left( \xi + 2\sqrt{\omega(\omega - m)} \right).$$
Then
\[
\langle \gamma_+, \rho \rangle = \frac{1}{2\pi} \langle \gamma_+, \hat{\rho} \rangle = \frac{1}{8\pi i \sqrt{\omega (\omega + m)}} \int \eta_+ (\xi) |\hat{\rho}(\xi)|^2 d\xi
\]
\[
= \sum_{\pm} |\hat{\rho}(\pm 2\sqrt{\omega (\omega + m)})|^2.
\]
\[
\langle \gamma_-, \rho \rangle = \frac{1}{2\pi} \langle \gamma_-, \hat{\rho} \rangle = -\frac{1}{8\pi \sqrt{\omega (m - \omega)}} \int \eta_- (\xi) |\hat{\rho}(\xi)|^2 d\xi
\]
\[
= -\frac{1}{2\pi} \int \frac{|\hat{\rho}(\xi)|^2 d\xi}{4\omega (m - \omega) + \xi^2}.
\]

In mind that \( \langle \gamma_+, \rho \rangle \) is purely imaginary, \( 1 + \kappa + \alpha (1 + 2\kappa) \langle \gamma_+, \rho \rangle \neq 0 \) for any \( \kappa \in \mathbb{R} \), and (A.30) implies that \( \langle X_1, \rho \rangle = 0 \) in the ‘+’ case. Hence, \( C = 0 \) by (A.38), and \( X_1 = X_2 = 0 \) by (A.28) and (A.29). Therefore, \( M^+ = 0 \) for \( 0 < \omega < m \) and \( \kappa \in \mathbb{R} \).

Further, substituting (A.31) into the equation \( 1 + \kappa + \alpha (1 + 2\kappa) \langle \gamma_-, \rho \rangle = 0 \), we obtain
\[
\frac{1}{4\pi} \int \frac{|\hat{\rho}(\xi)|^2 d\xi}{\xi^2 + \omega (m - \omega)} = \frac{1 + \kappa + \alpha (1 + 2\kappa)}{\alpha (1 + 2\kappa)} \int \frac{|\hat{\rho}(\xi)|^2 d\xi}{\xi^2 + m^2 - \omega^2}
\]
by (2.13) and (2.14). Hence, (A.25) follows. \( \square \)

A.4. Proof of proposition 8.1

We split \( P^e e^{\lambda t} \) as
\[
P^e e^{\lambda t} = P^e e^{\lambda t} \zeta (A) + P^e e^{\mu^j (1 - \zeta (A))}
\]
\[
= -\frac{1}{2\pi i} \int \zeta (\lambda) e^{\lambda t} (R(\lambda + 0) - R(\lambda - 0)) d\lambda
\]
\[
- \frac{1}{2\pi i} \int (1 - \zeta (\lambda)) e^{\lambda t} (R(\lambda + 0) - R(\lambda - 0)) d\lambda,
\]
where \( \zeta \in C^C \) satisfies (A.13). Then proposition 8.1 follows form two lemmas below.

Lemma A.4. Let condition (c) of assumption 6.1 hold. Then
\[
\| P^e e^{\lambda t} \zeta (A) \|_{E_{4/2} \rightarrow E_{-5/2}} = O(t^{-3/2}), \quad t \rightarrow \infty.
\]

Proof. The absence of virtual levels at the edge points \( \mu_{\pm}^{(1)} = \pm i(m - |\omega|) \) and at the embedded thresholds \( \mu_{\pm}^{(2)} = \pm i(m + |\omega|) \) implies the boundedness of the resolvent \( R(\lambda) \) at the vicinity of these points. More precisely, the following asymptotics hold
\[
R(\lambda) \rightarrow R(\mu_{\pm}^{(j)}), \quad j = 1, 2, \, \lambda \in C \setminus C_{\pm}
\]
\[
R'(\lambda) = O(|\lambda - \mu_{\pm}^{(j)}|^{-1/2}), \quad \lambda \rightarrow \mu_{\pm}^{(j)}
\]
\[
R''(\lambda) = O(|\lambda - \mu_{\pm}^{(j)}|^{-3/2})
\]
in \( E_{4/2} \rightarrow E_{-5/2} \). The asymptotics can be proved by standard way (see, for example, [9, 12]). Finally, (A.33) implies (A.32) by [6, lemma 10.2] (see also [10, lemma 22.5]). \( \square \)
Lemma A.5. Let condition (b) of assumption 6.1 holds. Then
\[
\|P^* \mathcal{F}^*(1 - \zeta(\lambda))\|_{E_{1+1/2+} \to E_{-1/2-}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{A.34}
\]

Step (i). First we obtain the decay of type (A.10) for \( R(\lambda) \). Namely, we will prove that
\[
\|(R^{(k)}(\lambda \pm 0)X)\|_{L^2_{1/2-k-}} \leq C|\lambda|^{-1}\|X\|_{E_{1/2+k+}}, \quad \lambda \in i\mathbb{R}, \ |\lambda| \geq 2m+1, \ j = 1, 2. \tag{A.35}
\]

First, for \( k = 0, 1, 2 \),
\[
\|\rho VR_0^{(k)}(\lambda \pm 0)X\|_{E_{1/2+k+}} \leq C\|\rho\|_{L^2_{1/2+k+}} \left( \|(R_0^{(k)}(\lambda \pm 0)X)\| + \|(R_0^{(k)}(\lambda \pm 0)X)\| \right)

\leq C_1 \left( \|(R_0^{(k)}(\lambda \pm 0)X)\|_{L^2_{1/2-k-}} + \|(R_0^{(k)}(\lambda \pm 0)X)\|_{L^2_{1/2-k-}} \right)

\leq C_1 |\lambda|^{-1}\|X\|_{E_{1/2+k+}}, \quad \lambda \in i\mathbb{R}, \ |\lambda| \geq 2m+1 \tag{A.36}
\]

by (11.6) and (A.10). Therefore, the norm of the inverse operator
\[
(1 + \rho VR_0(\lambda \pm 0))^{-1} : E_{1/2} \to E_{1/2}
\]
is bounded for large \( \lambda \in i\mathbb{R} \). Using the Born decomposition: \( R = R_0(1 + \rho VR_0)^{-1} \) and (A.10) with \( k = 0 \), we obtain (A.35) with \( k = 0 \). To estimate the first derivative, we apply the formula (cf [10, formula (17.9)])
\[
R' = R_0' - R_0\rho VR_0 - R_0'\rho VR + R_0'\rho VR_0'\rho VR. \tag{A.37}
\]

Due to (A.10) with \( k = 1 \) and (A.35) with \( k = 0 \),
\[
\|(R(\lambda \pm 0)\rho VR_0^{(k)}(\lambda \pm 0)X)\|_{L^2_{1/2-k-}} \leq C|\lambda|^{-1}\|\rho\|_{L^2_{1/2-k-}} \sum_{i=1,2} \|(R_0^{(k)}(\lambda \pm 0)X)\|_{L^2_{3/2-i-}}

\leq C|\lambda|^{-2}\|X\|_{E_{1/2+k+}}, \quad \lambda \geq 2m+1, \ j = 1, 2. \tag{A.38}
\]

Similarly,
\[
\|(R_0^{(k)}(\lambda \pm 0)\rho VR(\lambda \pm 0)X)\|_{L^2_{3/2-i-}} + \|(R(\lambda \pm 0)\rho VR_0^{(k)}(\lambda \pm 0)\rho VR(\lambda \pm 0)X)\|_{L^2_{1/2-k-}}

\leq C|\lambda|^{-2}\|X\|_{E_{1/2+k+}}, \quad \lambda \geq 2m+1, \ j = 1, 2. \tag{A.39}
\]

Formula (A.37), the bound (A.10) with \( k = 1 \) and (A.38) and (A.39) imply (A.35) with \( k = 1 \). To obtain (A.35) with \( k = 2 \), we apply the formula
\[
R'' = R_0'' - R_0\rho VR_0'' - R_0'\rho VR + R_0'\rho VR_0'\rho VR - 2R_0'\rho VR_0 + 2R_0'\rho VR_0'\rho VR
\]
(cf [10, formula (17.11)]) together with bounds (A.10) with \( k = 1, 2 \) and (A.35) with \( k = 0, 1 \).
we obtain
\[
P^c e^{i\lambda(1-\zeta(A))} = \frac{1}{2\pi i} \int_C (1 - \zeta(\lambda))e^{i\lambda} [R_0(\lambda + 0) - R_0(\lambda - 0)] d\lambda
\]
\[
- \frac{1}{2\pi i} \int_C (1 - \zeta(\lambda))e^{i\lambda} [R_0(\lambda + 0)\rho VR_0(\lambda + 0) - R_0(\lambda - 0)\rho VR_0(\lambda - 0)] d\lambda
\]
\[
- \frac{1}{2\pi i} \int_C (1 - \zeta(\lambda))e^{i\lambda} [R_0(\lambda + 0)\rho VR_0(\lambda + 0) - R_0(\lambda - 0)\rho VR_0(\lambda - 0)] d\lambda
\]
\[
= \mathcal{W}_1(t) + \mathcal{W}_2(t) + \mathcal{W}_3(t), \quad t \in \mathbb{R}.
\]

where
\[
\|\mathcal{W}_1(t)\|_{E_{3/2-} \to E_{-3/2-}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}
\]
by (A.14). The decay of \(\mathcal{W}_2(t)\) follows from convolution representation
\[
\mathcal{W}_2(t)X = i \int_0^t \mathcal{W}_1(t-\tau)\rho \mathcal{W}_1(\tau)X d\tau, \quad t \in \mathbb{R}
\]
(cf representation (3.19) in [9]). Applying (A.40) to the integrand in (A.41), we obtain
\[
\|\mathcal{W}_1(t-\tau)\rho \mathcal{W}_1(\tau)X\|_{E_{-3/2-}} \leq \frac{C \left( \|\mathcal{W}_1(\tau)X\|_{1, \rho} + \|\mathcal{W}_1(\tau)X\|_{2, \rho} \right) \|X\|_{E_{3/2+}}}{(1 + |t-\tau|)^{3/2}}
\]
\[
\leq \frac{C_1(1 + |\tau|)^{-3/2}}{(1 + |t-\tau|)^{3/2}}.
\]
Hence, (A.41) implies
\[
\|\mathcal{W}_2(t)X\|_{E_{-3/2-}} \leq C(1 + |t|)^{-3/2}\|X\|_{E_{3/2+}}, \quad t \in \mathbb{R}.
\]

It remains to estimate \(\mathcal{W}_3(t)\). Applying (A.9) and (A.35), we get
\[
\|R_0^{(\rho)}(\lambda \pm 0)\rho VR_0^{(\rho)}(\lambda \pm 0)\rho VR^{(\rho)}(\lambda \pm 0)X\|_{E_{-3/2-}}
\]
\[
\leq C\|\rho\|_{l_{1/2}^{2,+}} \sum_{j=1,2} \|\langle R_0^{(\rho)}(\lambda \pm 0)\rho VR^{(\rho)}(\lambda \pm 0)X_j, \rho \rangle \|_{l_{1/2}^{2,+}}
\]
\[
\leq C_1 \sum_{j=1,2} \|\langle R_0^{(\rho)}(\lambda \pm 0)\rho VR^{(\rho)}(\lambda \pm 0)X_j, \rho \rangle \|_{l_{1/2}^{2,+}}
\]
\[
\leq C_2 |\lambda|^{-1} \|\rho\|_{l_{1/2}^{2,+}} \sum_{j=1,2} \|\langle R^{(\rho)}(\lambda \pm 0)X_j, \rho \rangle \|_{l_{1/2}^{2,+}}
\]
\[
\leq C_3 |\lambda|^{-1} \|\langle R^{(\rho)}(\lambda \pm 0)X_j, \rho \rangle \|_{E_{-1/2-}}
\]
\[
\leq C_4 |\lambda|^{-1} \|X\|_{E_{1/2,+}}, \quad 0 \leq k+n+\ell \leq 2, \quad \lambda \in i\mathbb{R}, \quad |\lambda| \geq 2m+1.
\]
Hence, two times partial integration implies that
\[ \|V_\delta(t)x\|_{E_{3/2}} \leq C(1 + |t|)^{-3/2}\|x\|_{E_{3/2}}, \quad t \in \mathbb{R}. \]

This completes the proof of proposition 8.1.

### A.5. Examples

In [4] the operator \( A_\delta(\omega) = A_0(\omega) + \delta(x)V_\delta(\omega) : H^1 \otimes \mathbb{C}^d \to (H^1 \otimes \mathbb{C}^2) \oplus (H^{-1} \otimes \mathbb{C}^2), \)

\[ \text{corresponding to } \rho(x) = \delta(x), \text{ was considered. Here} \]

\[ V_\delta(\omega) = 2\sqrt{m^2 - \omega^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1 + 2\kappa)(\cdot, \delta) & 0 & 0 & 0 \\ 0 & (\cdot, \delta) & 0 & 0 \end{bmatrix}. \]

For \( A_\delta(\omega) \) the following spectral properties hold (figure 1):

(a) The eigenvalue \( \lambda = 0 \) is of algebraic multiplicity 4 if \( \kappa \geq 0 \) and \( |\omega| = m\sqrt{\kappa} \), and of algebraic multiplicity 2 otherwise.

(b) For \( \kappa \in \mathbb{R} \) and \( 0 < |\omega| < m \) there are no virtual levels at the embedded thresholds \( \lambda = \pm i(m + |\omega|) \). There are virtual levels at \( \lambda = \pm i(m - |\omega|) \) if and only if \( \kappa \in \left[-\frac{1}{2}, \frac{1}{\sqrt{2}}\right] \) and \( |\omega| = T_\kappa := m\left(\frac{1+2\kappa}{3+4\kappa}\right)^2 \).

(c) There are embedded eigenvalues \( \lambda = \pm 2\omega \) if and only if \( \kappa = 0 \) and \( |\omega| \geq m/3 \).

(d) \( \sigma_1(A_\delta(\omega)) = \{0\} \) in \( \Xi := \{\kappa < -1/2; \omega \in (-m,m)\} \cup \{\kappa \in [-1/2,0) \cup (0,1/\sqrt{2}) \}; \omega \in \left[T_\kappa, m\right) \).

**Remark A.6.** For \( \rho(x) = \delta(x), \) \( (A.25) \) reads
\[ \frac{1}{2\sqrt{|\omega|\sqrt{m^2 - |\omega|^2}}} = \frac{1 + \kappa}{\sqrt{m^2 - \omega^2(1 + 2\kappa)}}, \quad \kappa \in (-1/2, \infty), \quad 0 < |\omega| < m, \]

since \( \hat{\rho}(\xi) = 1. \) This is precisely the condition \( |\omega| = T_\kappa. \) We note that \( 0 < T_\kappa < m \) as long as \(-1/2 < \kappa < 1/\sqrt{2}. \)

Now we take \( m < 1/2 \) so that
\[ \frac{1}{2\pi} \int \frac{d\xi}{\xi^2 + m^2} = \frac{1}{2m} > 1, \]

and suppose that \( \rho \) is close to \( \delta(x) \) in \( H^{-1}(\mathbb{R}) \) (or, equivalently, \( \hat{\rho} \) is close to 1 in \( L^2_{-1} \)), such that
\[ \sigma_0 = \sigma(0) = \frac{1}{2\pi} \int \frac{\hat{\rho}^2(\xi)d\xi}{\xi^2 + m^2} > 1. \]

Note that we cannot take as an example a purely power function \( a(t) = t^\kappa, \kappa \neq 0, \) since this function does not satisfy the condition of theorem 2.1. So we will change it a little.

(a) Let \( \kappa > 0, \) and let \( a(\tau) \in C^1([0, \infty)) \) be a monotonically increasing function such that
\[ a(\tau) = \begin{cases} \tau^\kappa, & 0 \leq \tau \leq 1, \\ \text{const}, & \tau \geq 2. \end{cases} \]
Now the conditions of the theorem 2.1 are satisfied. Let us show that $c_\omega = \sigma_\omega^{\frac{2\kappa+1}{\kappa}}$, $\sigma_\omega = \sigma(\omega)$, is the unique positive solution to (2.13). Indeed,

$$c_\omega^2 \sigma_\omega^2 = \sigma_\omega^{-\frac{1}{\kappa}} \leq \sigma_0^{-\frac{1}{\kappa}} < 1,$$

and (2.13) holds:

$$\sigma_\omega a(c_\omega^2 \sigma_\omega^2) = \sigma_\omega \left( \sigma_\omega^{-\frac{1}{\kappa}} \right)^{\kappa} = 1.$$

For the uniqueness, it remains to note that equation (2.13) has no solution for $c_\omega^2 \sigma_\omega^2 > 1$, since in this case $a(c_\omega^2 \sigma_\omega^2) > 1$, and $\sigma_\omega \geq \sigma_0 > 1$.

(b) In the case $\kappa < 0$, we take monotonically decreasing function $a(\tau) \in C^1([0, \infty))$ such that

$$a(\tau) = \begin{cases} \tau^\kappa, & \tau \geq 1, \\ \text{const}, & 0 \leq \tau \leq 1/2, \end{cases}$$

Then the conditions of the theorem 2.1 theorem are also satisfied, and $c_\omega = \sigma_\omega^{\frac{2\kappa+1}{\kappa}}$ is a root of equation (2.13) since

$$c_\omega^2 \sigma_\omega^2 = \sigma_\omega^{-\frac{1}{\kappa}} \geq \sigma_0^{-\frac{1}{\kappa}} \geq 1.$$

The solution is unique because $\sigma_\omega a(c_\omega^2 \sigma_\omega^2) > 1$ for $c_\omega^2 \sigma_\omega^2 < 1$. 

3626
Thus, \( c_\omega = \sigma_\omega \frac{\kappa^2 + 1}{\kappa} \) is well defined for \( \kappa \neq 0 \) for any \( \omega \in (-m, m) \) as well as solitary wave
\[
\varphi_\omega(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i \xi x} \frac{c_\omega \rho(\xi)}{\xi^2 + m^2 - \omega^2} \, d\xi.
\] (A.42)

Denote \( \Delta = \|\rho - \delta\|_{H^{-1}} = \|\rho - 1\|_{L^2} \).

**Lemma A.7.** For any small \( \varepsilon > 0 \) there exists \( \nu(\varepsilon) \) such that if \( \Delta < \nu \) then \( \partial_\omega(\omega \|\varphi_\omega\|)^2 \neq 0 \) outside the domain \( \{ |\omega| < \varepsilon, |\kappa - \frac{\omega^2}{m^2}| \leq \varepsilon \} \). In particular, \( \partial_\omega(\omega \|\varphi_\omega\|)^2 \neq 0 \) in \( \mathbb{R} \) for sufficiently small \( \Delta \).

**Proof.** Note that
\[
\frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{-m} \, d\xi = \frac{1}{\pi}, \quad \text{where} \quad \varkappa := 2 \sqrt{m^2 - \omega^2}.
\]
Hence, (2.14) implies
\[
|\partial_\omega(\sigma - \varkappa^{-1})| = \frac{1}{2\pi} \left| \int (\hat{\rho}^2(\xi) - 1) \partial_\omega^j \left( \frac{1}{\xi^2 + m^2 - \omega^2} \right) \, d\xi \right| \\
\leq C \Delta \left| \partial_\omega^j \left( \frac{1}{\xi^2 + m^2 - \omega^2} \right) \right|_{L^1} \\
\leq C(\varepsilon) \Delta, \quad |\omega| \leq m - \varepsilon, \quad j = 0, 1, 2.
\]
Moreover,
\[
\alpha = a(c_\omega^2 \sigma_\omega^2) = c_\omega^2 \sigma_\omega^{-2} = \sigma_\omega^{-1} \leq \sigma_0^{-1}.
\] (A.43)

Therefore,
\[
|\partial_\omega^j (\alpha - \varkappa)| = \left| \partial_\omega^j \left( \frac{\varkappa}{\sigma_\omega} (\sigma_\omega - \varkappa^{-1}) \right) \right| \leq C(\varepsilon) \Delta, \quad |\omega| \leq m - \varepsilon, \quad j = 0, 1, 2.
\] (A.44)

Further, comparing (2.12) and (2.14), we obtain by (A.43) that
\[
\omega \|\varphi(x, \omega)\|^2 = \frac{1}{2\pi} \omega \|\hat{\varphi}(\xi, \omega)\|^2 = \frac{c_\omega^2}{2\pi} \int \frac{\omega \hat{\rho}(\xi)^2 \, d\xi}{(\xi^2 + m^2 - \omega^2)^2} = \frac{c_\omega^2}{2} \sigma_\omega
\]
\[
= -\frac{c_\omega^2}{2} \alpha^2 \frac{\alpha'}{\alpha^2} \frac{\alpha''}{\alpha^2} = -\frac{1}{2} \alpha^2 \frac{\alpha'}{\alpha^2} \frac{\alpha''}{\alpha^2}.
\]
Hence, \( \partial_\omega(\omega \|\varphi(x, \omega)\|^2) = -\frac{1}{2 \Delta} \kappa \alpha^2 + \left( \alpha' \right)^2 \) if and only if \( \kappa = \kappa_\omega = -\frac{\omega^2}{m^2} \alpha' \). Finally, (A.44) implies
\[
\left| \kappa_\omega - \frac{\omega^2}{m^2} \right| \leq \left| \frac{(\alpha')^2}{\alpha \alpha''} - \left( \frac{\alpha'}{\alpha''} \right)^2 \right| \leq C(\varepsilon) \Delta, \quad |\omega| \leq m - \varepsilon.
\]

**Lemma A.8.**

(a) Let \( \kappa \leq -1/2 < \varepsilon < m \), and \( \Delta \leq \nu(\kappa, \varepsilon) \) is sufficiently small. Then for any \( \omega \in (-m + \varepsilon, m - \varepsilon) \),

1. \( \sigma_\rho(\Lambda(\omega)) = \{0\} \);
2. There are no virtual levels at \( \lambda = \pm \imath (m - |\omega|) \).

(b) Let \( \kappa \in (-1/2, 0) \cup (0, 1/\sqrt{2}) \), \( 0 < \varepsilon < \frac{3}{4} (m - \mathcal{T}_0) \), and \( \Delta \leq \nu(\kappa, \varepsilon) \) is sufficiently small. Then for any \( |\omega| \in (\mathcal{T}_\kappa + \varepsilon, m - \varepsilon) \), properties (1) and (2) hold.

**Proof.** For any \( \psi \in H^1(\mathbb{R}) \),
\[
|\langle \psi, \rho \rangle - \langle \psi, \delta \rangle| \leq C \| \psi \|_{H^1(\mathbb{R})} \| \rho - \delta \|_{H^{-1}(\mathbb{R})} = C \Delta \| \psi \|_{H^1(\mathbb{R})},
\]
Therefore,
\[
\| (\rho V - \delta V \delta) X \|_{H^{-1}(\mathbb{R})} \leq C(\omega) \Delta (\| X_1 \|_{H^1(\mathbb{R})} + \| X_2 \|_{H^1(\mathbb{R})}), \quad j = 3, 4,
\]
(A.45)
where \( V \) is defined in (11.6). Recall that \( [(\rho V - \delta V \delta) X] = 0, j = 1, 2 \).

Suppose that for some fixed values \( \kappa \) and \( \omega \), the point \( \lambda \) belongs to the resolvent set of \( A_\delta(\omega) \). Then the resolvent \( R_\delta(\lambda) = R_\delta(\omega, \lambda) := (A_\delta(\omega) - \lambda)^{-1} \) is bounded operator \( (H^1 \otimes C^2) \oplus (H^{-1} \otimes C^2) \rightarrow H^1 \otimes C^2 \):
\[
\| R_\delta(\lambda) X \|_{H^1 \otimes C^2} \leq C \| X \|_{(H^1 \otimes C^2) \oplus (H^{-1} \otimes C^2)}, \quad \text{(A.46)}
\]
Hence, (A.45) and (A.46) imply that
\[
\| (\rho V - \delta V \delta) R_\delta(\lambda) X \|_{H^{-1}} \leq C \Delta \| X \|_{(H^1 \otimes C^2) \oplus (H^{-1} \otimes C^2)}, \quad j = 3, 4,
\]
\[
[(\rho V - \delta V \delta) R_\delta(\lambda) X] = 0, \quad j = 1, 2.
\]
Therefore, the operator
\[
1 + (\rho V - \delta V \delta) R_\delta(\lambda) : (H^1 \otimes C^2) \oplus (H^{-1} \otimes C^2) \rightarrow (H^1 \otimes C^2) \oplus (H^{-1} \otimes C^2)
\]
is invertible and bounded for sufficiently small \( \Delta \). Finally, the relation
\[
R(\lambda) = R_\delta(\lambda) (1 + (\rho V - \delta V \delta) R_\delta(\lambda))^{-1}
\]
implies that \( \lambda \) belongs to the resolvent set of \( A(\omega) \).

\[\square\]

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3628
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