FACIAL STRUCTURES OF A GENERALIZED WEDGE OF POLYTOPES

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Abstract. In this paper, we study face and flag information of a generalized wedge of two polytopes over a face. In particular, we provide the explicit formula for the \(cd\)-index of the wedge of a polytope and a one-dimensional simplex over an arbitrary face.

1. Introduction

When \(P\) is a \(d\)-polytope in \(\mathbb{R}^d\) and \(F\) is a facet of \(P\), a wedge over \(P\) with foot \(F\) is the polytope in \(\mathbb{R}^{d+1}\) that is obtained by intersecting the cylinder \(P \times [0, \infty)\) with a halfspace \(J\) in \(\mathbb{R}^{d+1}\) such that \(J\) contains \(P \times \{0\}\), \(J\)’s bounding hyperplane \(H\) contains \(F \times \{0\}\), and \(H\) intersects the open rays \(\{x\} \times [0, \infty)\) for each \(x \in P \setminus F\). Combinatorially, the wedge may be obtained from the product \(P \times [0, 1]\) by (in effect) collapsing \(F \times [0, 1]\) to \(F \times \{0\}\). The wedge construction was used to study the Hirsch conjecture [4]. Paffenholz generalized this construction for any face of the polytope to study faces of Birkhoff polytopes [6].

In this paper, we consider a generalized wedge which is obtained from the product of two polytopes \(P\) and \(Q\) by collapsing \(\sigma \times Q\) to \(\sigma\) when \(\sigma\) is a face of \(P\). This construction was used to define the wedge product [9]. We study the faces and flags of a generalized wedge of polytopes over a face.

In Section 2, we define a generalized wedge and provide the results about its faces. In particular, we give the \(f\)-vector of a generalized wedge. In Section 3, we define the \(cd\)-index which compactly encodes information about the flag vector of a polytope and provide a formula for the \(cd\)-index of a generalized wedge when \(Q\) is a one-dimensional simplex.

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Figure 1. Schlegel diagram for the generalized wedge of a pentagon and a triangle at a vertex

2. A generalized wedge

In this section, we give a definition of the generalized wedge and investigate its faces.

**Definition 2.1.** Let $P$ be a $d$-polytope in $\mathbb{R}^d$ with $m$ facets given by the inequality system $Ax \leq 1$, and let $Q$ be a $d'$-polytope in $\mathbb{R}^{d'}$ with $m'$ facets given by $By \leq 1$. Let $\sigma$ be the face of $P$ defined by the hyperplane $cx = 1$. The generalized wedge $P \triangleleft_\sigma Q$ of $P$ and $Q$ at $\sigma$ is the $(d + d')$-dimensional polytope in $\mathbb{R}^{d + d'}$ defined by

$$P \triangleleft_\sigma Q := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d + d'} : \begin{pmatrix} A' & 0 \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

where $C = 1c$ is the $m' \times d$ matrix all of whose rows are equal to $c$, and $A'$ is the matrix $A$ if $\sigma$ is not a facet and $A'$ is the matrix $A$ without the row $a_0$ if $\sigma$ is a facet defined by $a_0x = 1$.

Rörig showed the following combinatorial interpretation of the generalized wedge in his thesis [7].

**Proposition 2.2.** If $P$ and $Q$ are polytopes of dimension $d$ (resp. $d'$), then the generalized wedge $P \triangleleft_\sigma Q$ comes with a projection to $P$ (to the first $d$ coordinates) such that the fiber over every point of $P$ is an affine copy of $Q$, except that it is a single point $\{\ast\}$ above every point of $\sigma$.

From this proposition, one can think the generalized wedge $P \triangleleft_\sigma Q$ as a polytope obtained from the product $P \times Q$ by collapsing the face $\sigma \times Q$ to $\sigma$. Figure 1 shows the Schlegel diagram for the generalized
wedge $P \triangleleft_\sigma Q$ when $P$ is a pentagon, $Q$ is a triangle and $\sigma$ is a vertex of $P$.

The next proposition describes the vertices and facets of $P \triangleleft_\sigma Q$.

**Proposition 2.3.** [7] Let $P \triangleleft_\sigma Q$ be the generalized wedge of $P$ and $Q$ at $\sigma$.

1. If $P, Q, \sigma$ have $n, n', \bar{n}$ vertices, respectively, then $P \triangleleft_\sigma Q$ has $(n - \bar{n})n' + \bar{n}$ vertices. These belong to two families

$$u_{k\ell} = \begin{cases} \binom{v_k}{0} & \text{for } v_k \in \sigma, \quad 0 \leq k < n \\ \binom{v_k}{1-cv_kw_\ell} & \text{for } v_k \notin \sigma, \quad 0 \leq k < n, 0 \leq \ell < n' \end{cases},$$

where $v_k$ is a vertex of $P$ and $w_\ell$ is a vertex of $Q$.

2. The inequalities defining $P \triangleleft_\sigma Q$ are of two different kinds:

   (a) If $a_i x = 1$ define the facet $F_i \neq \sigma$ of $P$, then $a_i x = 1$ defines a facet of the generalized wedge combinatorially equivalent to

      (i) the product $F_i \times Q$ if $F_i \cap \sigma = \emptyset$, and

      (ii) the generalized wedge $F_i \triangleleft_{F_i \cap \sigma} Q$ if $F_i \cap \sigma \neq \emptyset$.

   (b) If $b_j y = 1$ define the facet $F_j$ of $Q$, then $cx + b_j y = 1$ defines a facet combinatorially equivalent to the generalized wedge $P \triangleleft_{\sigma} F_j$.

We generalize the above result to describe all the faces of $P \triangleleft_\sigma Q$.

**Proposition 2.4.** The set of $i$-faces of the generalized wedge $P \triangleleft_\sigma Q$ is in one-to-one correspondence with the set of following polytopes:

1. the $i$-faces of $\sigma$,
2. the product $\mu \times \tau$ for a face $\mu$ of $P$ and a face $\tau$ of $Q$ satisfying $\mu \cap \sigma = \emptyset$ and $\dim \mu + \dim \tau = i$,
3. the generalized wedge $\mu \triangleleft_{\mu \cap \sigma} \tau$ for a face $\mu$ of $P$ and a face $\tau$ of $Q$ satisfying $\mu \cap \sigma \neq \emptyset$ and $\dim \mu + \dim \tau = i$.

**Proof.** First, we consider the case when $\sigma$ is not a facet. Then $P \triangleleft_\sigma Q$ is determined by

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \preceq \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

A face $\omega$ of $P \triangleleft_\sigma Q$ is obtained by intersecting some facets of $P \triangleleft_\sigma Q$. After relabeling, we may assume that $\omega$ is obtained by intersecting facets corresponding to the rows $1, \ldots, k, m + 1, \ldots, m + \ell$ where $k \leq m$ and $\ell \leq m'$. Let $F_1, \ldots, F_m$ be the facets of $P$ corresponding to the rows of $A$, and $F'_1, \ldots, F'_m$ be the facets of $Q$ corresponding to the rows of $B$. Let $\mu$ be the intersection of $F_1, \ldots, F_k$ and $\tau$ be the intersection of $F'_1, \ldots, F'_\ell$. There are three cases:
Case 1. When $\tau = \emptyset$:
In this case, the intersection of facets corresponding to the rows $m + 1, \ldots, m + \ell$ is $\sigma$. Thus $\omega$ is a face of $\sigma$.

Case 2. When $\tau \neq \emptyset$ and $\mu \cap \sigma = \emptyset$:
In this case, one can see that the image of the vertices of $\omega$ under the projection to $P$ is the vertices of $\mu$ and the image of them under the projection to $Q$ is the vertices of $\tau$. Thus the vertices of $\omega$ are the vertices in the fiber over the vertices of $\mu$ which correspond to the vertices of $\tau$. Thus $\omega$ is combinatorially equivalent to the product $\mu \times \tau$.

Case 3. When $\tau \neq \emptyset$ and $\mu \cap \sigma \neq \emptyset$:
In this case, the vertices of $\omega$ are either
(a) vertices of $\mu \cap \sigma$, or
(b) vertices in the fiber over the vertices of $\mu \setminus \sigma$ which correspond to the vertices of $\tau$.
Thus $\omega$ is combinatorially equivalent to the the generalized wedge $\mu \prec_{\mu \cap \sigma} \tau$.

The proof for the case when $\sigma$ is a facet is similar and is omitted. □

Let $f_i(P)$ be the number of $i$-dimensional faces of a $d$-polytope $P$. By convention, $f_{-1}(P) = 1$. The tuple $(f_{-1}(P), f_0(P), \ldots, f_d(P))$ is called the $f$-vector of $P$. One can easily derive the following corollary about the generalized wedge.

**Corollary 2.5.** The $f$-vector of the generalized wedge $P \prec_{\sigma} Q$ is given by

$$f_i(P \prec_{\sigma} Q) = f_i(\sigma) + \sum_{j+k=i} [f_j(P) - f_j(\sigma)] \cdot f_k(Q).$$

**Example 2.6.** Let $P$ be a 5-gon with vertices 1, 2, 3, 4, and 5 and $Q$ be a triangle with vertices $a, b, c$. The generalized wedge $P \prec_{1} Q$ is shown in Figure 1. The vertices except the vertex 1 are labeled by $ij$, where $2 \leq i \leq 4$ and $j = a, b,$ or $c$. Then the face 1 and the empty face are the only faces of type $(1)$. The face with vertices $2a, 2b, 3a, 3b$ is a face of type $(2)$, while the face with vertices $1, 2a, 2b$ is a face of type $(3)$.

Corollary 2.5 implies

$$f_2(P \prec_{1} Q) = f_2(1) + \sum_{j+k=2} [f_j(P) - f_j(1)] \cdot f_k(Q) = 0 + 1 \cdot 3 + 5 \cdot 3 + 4 \cdot 1 = 22$$

and one can easily verify this result.
3. The cd-index of generalized wedges

In this section, we define the $cd$-index for Eulerian posets and give the $cd$-index of the generalized wedge when $Q$ is a one-dimensional simplex.

Let $P$ be a graded poset of rank $n + 1$ with the rank function $\rho$. For a subset $S$ of $[n]$, define $f_p(S)$ to be the number of chains of $P$ whose ranks are exactly given by the set $S$. The function $f_p : 2^{[n]} \to \mathbb{N}$ is called the flag $f$-vector of $P$. The flag $h$-vector is defined by the identity

$$h_p(S) = \sum_{T \subseteq S} (-1)^{|S\setminus T|} \cdot f_p(T).$$

Since this identity is equivalent to the relation

$$f_p(S) = \sum_{T \subseteq S} h_p(T),$$

the flag $f$-vector and the flag $h$-vector contain the same information.

For a subset $S$ of $[n]$, define the noncommutative $ab$-monomial $u_S = u_1 \cdots u_n$, where

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

The $ab$-index of the poset $P$ is defined to be the sum

$$\Psi(P) = \sum_{S \subseteq [n]} h_p(S) \cdot u_S.$$

An alternative way of defining the $ab$-index is as follows. For a chain $c := \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$, we give a weight $w_p(c) = w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} b & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ a - b & \text{otherwise}. \end{cases}$$

Define the $ab$-index of the poset $P$ to be the sum

$$\Psi(P) = \sum_c w(c),$$

where the sum is over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in $P$. Recall that a poset $P$ is Eulerian if its Möbius function $\mu$ is given by $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ (see [10] for more details). One important class of Eulerian posets is face lattices of convex polytopes (see [5, 8]). It is known that the ab-index of an Eulerian poset $P$ can be written uniquely as a polynomial of $c = a + b$ and $d = ab + ba$ (see [1]). When the $ab$-index can be written as a polynomial in $c$ and $d$, we call $\Psi(P)$...
the cd-index of \( P \). We will use the notation \( \Psi(P) \) for the cd-index of the face poset of a convex polytope \( P \).

The cd-index of a polytope is an invariant which compactly encodes all information about the flag \( f \)-vector of the given polytope. There are explicit formulas for how the cd-index changes after applying a geometric operations to the polytope. These operations include prism, pyramid, cutting off a face and contracting a face into a vertex [2, 3]. Note that contracting a face into a vertex is different from the generalized wedge construction we consider since it gives a CW-complex which is not a polytope in general.

**Example 3.1.** Let \( P \) be a square pyramid in \( \mathbb{R}^3 \). Then the cd-index of \( P \) is

\[
\Psi(P) = a^3 + 4a^2b + 7aba + 4ab^2 + 4ba^2 + 7bab + 4b^2a + b^3 \\
= (a + b)^3 + 3(a + b)(ab + ba) + 3(ab + ba)(a + b) \\
= c^3 + 3cd + 3dc.
\]

Let \( v \) be a vertex of a polytope \( Q \) and let \( l(x) = c \) be a supporting hyperplane of \( Q \) defining \( v \). The vertex figure \( Q/v \) of \( v \) is defined by

\[
Q/v = Q \cap \{ l(x) = c + \delta \}
\]

where \( \delta \) is an arbitrary small positive number. For a face \( \sigma \) of \( Q \), the face figure \( Q/\sigma \) of \( \sigma \) is defined by

\[
Q/\sigma = (\cdots ((Q/\sigma_0)/\sigma_1) \cdots )/\sigma_k
\]

where \( \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k = \sigma \) is a maximal chain with \( \dim \sigma_i = i \). For faces \( \sigma \) and \( \tau \) of \( Q \) with \( \sigma \subseteq \tau \), the face lattice of the face figure \( \tau/\sigma \) is the interval \([\sigma, \tau]\).

**Proposition 3.2.** Let \( P \) be a polytope in \( \mathbb{R}^n \) and \( \sigma \) be a face of \( P \). Also let \( I \) be the interval \([-1, 1]\) in \( \mathbb{R} \). Then the cd-index of \( P \downarrow_\sigma I \) is

\[
\Psi(P \downarrow_\sigma I) = \Psi(P) \cdot c + \sum_{\tau} \Psi(\tau) \cdot d \cdot \Psi(P/\tau),
\]

where the sum is over all faces \( \tau \) of \( P \) which are not in \( \sigma \).

**Proof.** Since \( I \) is determined by the equation

\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix} y \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

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\[
\Psi(P \downarrow_\sigma I) = \Psi(P) \cdot c + \sum_{\tau} \Psi(\tau) \cdot d \cdot \Psi(P/\tau),
\]

where the sum is over all faces \( \tau \) of \( P \) which are not in \( \sigma \).

**Proof.** Since \( I \) is determined by the equation

\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix} y \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
the generalized wedge $P <_{\sigma} I$ is determined by
\[
\begin{pmatrix}
A' & 0 \\
f & -1 \\
f & 1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
\leq
\begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix},
\]
where $fx = 1$ is the hyperplane that defines the face $\sigma$. Let $F^-$ be the facet determined by the equation $\begin{pmatrix} f & -1 \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}^T = 1$ and $F^+$ be the facet determined by the equation $\begin{pmatrix} f & 1 \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}^T = 1$. Let
\[
c = \{\emptyset < \tau_1 < \cdots < \tau_{k-1} < \tau_k = P <_{\sigma} I\}
\]
be a chain of faces of $P <_{\sigma} I$.

From Propositions 2.2 and 2.4, we may define $\hat{\tau}$ for a face $\tau$ of $P <_{\sigma} I$ as the image of $\tau$ under the projection to $P$. Let $\hat{c}$ be the chain in the face poset of $P$ obtained from the set
\[
\{\emptyset, \hat{\tau}_1, \ldots, \hat{\tau}_k = P\}.
\]
Note that this set may have repeated elements.

Let $s$ be the smallest index such that $\tau_i$ meets both $F^-$ and $F^+$. There are two cases, each having two subcases.

1. When $s = k$ and $\tau_{k-1}$ is not contained in $F^- \setminus F^+$:
   (a) The first subcase is when $\tau_{k-1} = F^-$:
       In this case, $\tau_1, \ldots, \tau_{k-1}$ are faces of $F^-$ and the weight of the chain is
       \[
w(c) = w_P(\hat{c}) \cdot b.
\]
   (b) The second subcase is when $\tau_{k-1} \neq F^-$:
       In this case, let $c'$ be the chain $c - \{F^+\}$ and let $c''$ be the chain $c \cup \{F^+\}$. Then the sum of the weights of the chains $c'$ and $c''$ is
       \[
w(c') + w(c'') = w_P(\hat{c}) \cdot a.
\]

2. When $s < k$, or $s = k$ but $\tau_{k-1}$ is contained in $F^- \setminus F^+$:
   (a) The first subcase is when $s = k$, or $s < k$ and $\tau_{s-1}$ is a nonempty face contained in $F^- \setminus F^+$:
       In this case, let $\tau = \hat{\tau}_{s-1}$. Then $\tau$ is a face of $P$ which is not contained in $\sigma$. If $\tau \cap \sigma = \emptyset$, let $\tau^I$ be the face of $P <_{\sigma} I$ corresponding to the product $\tau \times I$. Otherwise, let $\tau^I$ be the face of $P <_{\sigma} I$ corresponding to the generalized wedge $\tau <_{\tau \cap \sigma} I$. Then the element $\tau^I$ may or may not be in the chain $c$. Let $c'$ be the chain $c - \{\tau^I\}$ and $c''$ be the chain $c \cup \{\tau^I\}$. Moreover, let $c_1$ be the chain $\{\emptyset < \hat{\tau}_1 < \cdots < \hat{\tau}_{s-2} < \hat{\tau}\}$ and let $c_2$ be the chain
\{\hat{\tau} < \hat{\tau}_s < \cdots < \hat{\tau}_{k-1} < \hat{\tau}_k = P\}. Then the sum of the weights of the chains \(c'\) and \(c''\) is given by
\[
w(c') + w(c'') = w(\emptyset, \hat{\tau}](c_1) \cdot b \cdot a \cdot w[\hat{\tau}, P](c_2).
\]

(b) The second subcase is when \(s = 1\), or \(1 < s < k\) and \(\tau_{s-1}\) is not contained in \(F^- \setminus F^+\):

In this case, let \(\tau = \hat{\tau}_s\). Then \(\tau\) is a face of \(P\) which is not contained in \(\sigma\). Let \(\tau^+_s\) be the face \(\tau_s \cap F^+\). Let \(c'\) be the chain \(c - \{\tau^+_s\}\) and let \(c''\) be the chain \(c \cup \{\tau^+_s\}\). Moreover, let \(c_1\) be the chain \(\{\emptyset < \hat{\tau}_1 < \cdots < \hat{\tau}_{s-1} < \hat{\tau}\}\) and let \(c_2\) be the chain \(\{\hat{\tau} < \hat{\tau}_{s+1} < \cdots < \hat{\tau}_{k-1} < \hat{\tau}_k = P\}\). Then the sum of the weights of the chains \(c'\) and \(c''\) is given by
\[
w(c') + w(c'') = w(\emptyset, \hat{\tau}](c_1) \cdot b \cdot a \cdot w[\hat{\tau}, P](c_2).
\]

Now summing over all chains of \(P \triangleleft_\sigma I\), we obtain the desired formula. \(\square\)

4. Future works

We would like to generalize Proposition 3.2 to the general case. As the general case could be too complicated, we will first try to understand the case when \(Q\) is a \(k\)-simplex for \(k \geq 2\). This case would be useful to understand the cd-index of certain matroid base polytopes.

It is well-known that a (loopless) rank 2 matroid on \([n]\) is determined up to isomorphism by a partition \(\alpha\) of \(n\) that gives the sizes \(\alpha_i\) of its parallelism classes. The matroid base polytope \(\mathcal{P}(M)\) of a matroid \(M\) on \([n]\) is the polytope in \(\mathbb{R}^n\) whose vertices are the incidence vectors of the bases of \(M\).

Let \(M\) be a rank 2 matroid on \([n]\) with a parallelism class of size \(k\). Let \(M'\) be a matroid obtained from \(M\) by removing all but one element \(e\) from the given parallelism class. Then one can see that \(\mathcal{P}(M)\) is the generalized wedge of \(\mathcal{P}(M')\) and the \((k-1)\)-dimensional simplex at certain face of \(\mathcal{P}(M')\) which is the matroid base polytope for the deletion of \(e\). Thus we are interested in generalization of Proposition 3.2 to the case when \(Q\) is a simplex.

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