Distributed Kalman Filters with State Equality Constraints: Time-based and Event-triggered Communications

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In this paper, we investigate a distributed estimation problem for multi-agent systems with state equality constraints (SECs). We propose a distributed Kalman filter design based on a covariance intersection approach by combining a filtering structure and a fusion-projection scheme for time-varying dynamics, in order to overcome the strong correlation between the estimates of agents and timely provide an upper bound of the error covariance matrix of each agent. It is shown that all SECs will be satisfied as the fusion-projection number goes to infinity. Further, given a finite fusion-projection number, the SECs will be satisfied as time goes to infinity. Based on the extended collective observability, we prove the Gaussianity, consistency, upper boundedness of the covariance matrix and convergence of the proposed distributed time-based filter, and show how the SEC improves the estimation performance and relaxes the observability condition. Moreover, to reduce the communication cost, we propose a distributed event-triggered filter with SECs for time-invariant dynamics, and provide the Gaussianity, consistency and upper boundedness of the covariance matrix for the proposed filter under the extended collective observability. We also show that a smaller triggering threshold leads to a smaller upper bound of the covariance matrix. Finally, we study the distributed tracking on a land-based vehicle for illustration. The simulation results demonstrate the effectiveness of the proposed algorithms.

Index Terms—multi-agent systems, distributed Kalman filter, time-varying dynamics, state equality constraint, event-triggered communication, extended collective observability, covariance intersection

I. INTRODUCTION

State estimation problems are very important, related to parameter identification, signal reconstruction, target monitoring and control design, which have been studied for several decades. In recent years, distributed state estimation of multi-agent systems has received more and more attention because of its broad range of applications in engineering systems such as communication networks, sensor networks and smart grids. Among the distributed estimation methods, Kalman-filter-based estimation plays a key role due to its ability of real-time estimation and non-stationary process tracking.

Existing distributed Kalman filters can be roughly classified into two categories: the Kalman-consensus filter (KCF) and the Kalman-diffusion filter (KDF). KCF is a design to include consensus terms to the Kalman filter structure (see [2][10]). For example, [3] constructed a KCF, with estimate of each agent obtained by consensus on measurement information to approximate centralized estimate, while [2] proposed an optimal KCF, which yet was not scalable (or of scalability) as it is based on all-to-all communications. Furthermore, [4][5] dealt with communication uncertainties like intermittent observation and communication faults, respectively. Nevertheless, [2][4][5] assumed the local observability (i.e., the dynamical system is observable for each agent), which may be unsuitable in large-scale networks. Even though [3] took a collective observability condition, the agents have to communicate infinitely times between any two sampling instants. Some works extended the local observability assumption to a collective observability assumption for the filter design. For example, [9] constructed a consensus+innovations distributed estimator based on some knowledge of a global observation model, under a collective observability condition. [6] considered the KCF with consensus on the inverse covariance matrix and the information vector, while [7] extended the results to a class of nonlinear systems. Under collective observability assumptions, [6][7] proved the upper boundedness of covariance matrix of the KCFs, with requirement of the non-singularity of the system matrix. Additionally, [12][13] studied the design for switching communication topologies. In [14], the authors studied gossip interactive Kalman filter for networked systems based on a random communication scheme, and provided some theoretical results on the asymptotic properties of error process. On the other hand, the KDF is to fuse the neighbor’s estimates updated by standard Kalman filter (referring to [15]-[18]). Compared with many results on KCF [2][4][5][10][19], the designs of KDF usually do not require the computation of correlation matrices between agents and do not need that all agents should obtain the same estimate in steady-state. For example, [15] discussed the design of the fusion weights and the convergence of the KDF algorithm, while [16] proposed a KDF based on a covariance intersection (CI) scheme. However, the observability assumptions of [15][17] are not global conditions. An effective distributed filter based on optimized weights for time-varying topology was proposed in [19]. Although many effective distributed Kalman filters have been developed, more scalable and practical distributed filters relying collective observability conditions still need more attention.

In practice, we may obtain some information or knowledge on the state constraints in advance from physical laws, geometric relationship, and environment constraints of the considered system, which can improve the estimation performance in 1Scalability is the capability of a system, network, or process to handle a growing amount of work, or its potential to be enlarged to accommodate that growth.
the filter design. Many practical examples can be formulated with SECs, such as quaternion-based attitude estimation [20], magnetohydrodynamic data assimilation [21], tracking and navigation [22–23] and aeronautics [24–25]. A typical example of linear equality constraints arises in target tracking problem. A car may be traveling off-road, or on an unknown road, where the problem is unconstrained. However, most of the time it may be traveling along a given road, where the estimation problem is constrained. Another example is tracking a train, where the railway can be treated as constraints. There are various methods incorporating the information of state constraints into the Kalman filter structure, such as the pseudo-observation [26], projection [27–28], and moving horizon [29]. A survey on the conventional design of the Kalman filter with state constraints provided in [30] showed the usefulness of the constraints as additional information to improve the estimation performance or accuracy. It is known that the moving horizon method performs the best, but at the cost of high computational complexity [29–30]. In some cases, one may formulate state constraints as equalities and project the unconstrained estimate onto the constrained surface under SECs [27–30]. Linear state inequality constraints can be transformed to SECs in some conditions by using active set methods [30]. Although the Kalman filter with state constraints has drawn much research attention, the corresponding distributed version, i.e., distributed Kalman filter with state constraints, has not yet been adequately investigated, to the best of our knowledge.

Communication may spend much cost or energy in distributed design, and therefore, a well-known scheme, called event-triggered scheme, is developed to reduce the communication costs, where the communication is carried out only when some predefined event conditions are satisfied. A variance-based triggering scheme of state estimation was studied in [31], which analyzed the convergence of the switching Riccati equation of estimation variance for a scalar system. With the Gaussian properties of the a priori conditional distribution, a deterministic event-triggered schedule was proposed in [32], and moreover, to overcome the limitation of the Gaussian assumption, a stochastic event-triggered scheme was developed and the corresponding exact MMSE estimator was obtained in [33]. Additionally, a set-valued measurement Kalman filter for remote state estimation problem was studied in [34], which built some relationship between the estimation performance and triggering thresholds. Also, Send-on-Delta (SoD), a typical event-based regulation proposed in [35], has been widely utilized in the triggering condition design of estimation algorithms [36–38]. A recursive distributed filter with SoD triggering scheme on innovation was studied in [36], where the relationship between triggering thresholds and an upper bound of error covariance was analyzed. Additionally, a stochastic SoD triggering rule was proposed in [37], where an upper bound and a lower bound of the mean square error were derived, but the performance was not studied. Although quite a few effective event-triggered filters were developed, the relationship between the estimation performance and the triggering thresholds still need further investigations in a distributed framework under collective observable condition.

The objective of this paper is to give a distributed estimation design with SECs for time-varying systems under the extended collective observability, and then consider a distributed event-triggered design for time-invariant systems. We focus on developing CI-based distributed Kalman filters with SECs, analyzing the estimation performance, and proving the upper boundedness of the covariance matrix of the algorithms. The contributions of this paper are summarized as follows:

1) We solve the distributed estimation problem with SECs for a class of time-varying systems. Different from the design in [9–39], we propose a fully distributed Kalman filter (i.e., the proposed algorithm does not rely on the global information), which overcomes the strong correlation between the estimates of agents and provides an upper bound of the estimation covariance in real time. We prove that for the proposed algorithm, all SECs will be satisfied if the fusion-projection number goes to infinity. Further, given a finite fusion-projection number, the SECs will be satisfied as time goes to infinity. In fact, our work can be viewed as a fully distributed extension of the conventional Kalman filter with SECs [24–27–30].

2) We prove the Gaussianity, consistency, upper boundedness of the covariance matrix, and convergence (Theorems 1–4) of the proposed filter under the extended collective observability assumption, and moreover, show that the SECs can improve the dynamic performance of the estimation and also can relax either the collective observability assumptions given in [35, 39] or the local observability conditions given in [32–35]. Simulation results of tracking on a land-based vehicle show that the estimation performance of our proposed distributed Kalman filter with SECs is even better than the centralized Kalman filter which is the optimal filter in some sense.

3) To reduce the communication cost, for linear time-invariant systems, we propose an event-triggered scheme based on the idea that each agent transmits its own data only when the information increment from the latest transmitting moment is accumulated to a certain degree. We then analyze the Gaussianity, consistency and upper boundedness of the covariance matrix (Theorems 5–7) on the distributed event-triggered state estimation under the extended collective observability, which is an open problem [36–40] due to difficulties in connecting the triggering thresholds with the estimation performance. We also show that a smaller triggering threshold leads to a smaller upper bound of the covariance matrix.

The remainder of the paper is organized as follows. The problem formulation is given in section II. Then a fully distributed Kalman filter for a time-varying system with SECs is proposed and analyzed in section III, while a distributed Kalman filter for a time-invariant system with a novel event-triggered communication scheme is given and discussed in section IV. Following that, numerical simulations on a constrained moving vehicle are shown in section V. Finally, the conclusions of this paper are provided in section VI.

Notations: The superscript “T” represents the transpose. If \( A \)
and $B$ are both symmetric matrices, $A \geq B$ (or $A > B$) means that $A - B$ is a positive semidefinite (or positive definite) matrix. $I_n$ stands for the identity matrix with $n$ rows and $n$ columns, and $\otimes$ stands for the Kronecker product. $E\{x\}$ denotes the mathematical expectation of the stochastic variable $x$. $\text{blockdiag}\{\cdot\}$ and $\text{diag}\{\cdot\}$ represent the diagonalizations of block elements and scalar elements, respectively. $tr(P)$ is the trace of the matrix $P$ and $\text{var}(x)$ is the variance of $x$. The integer set from $a$ to $b$ is denoted as $[a:b]$. $\mathbb{G}$ is the pseudo inverse of $G$, and if $G$ is a nonsingular matrix, $G^- = G^{-1}$. $\text{dis}(x,y)$ stands for the Euclidean distance between $x$ and $y$. $P_D[x]$ means the projected value of $x$ onto the set $D$. $\mathbb{R}$ and $\mathbb{Z}^+$ stand for the set of real scalars and positive integers, respectively. $N(\mu, \sigma^2)$ represents the probability distribution with the mean $\mu$ and variance $\sigma^2$. $1.0e4$ stands for $1.0 \times 10^4$.

II. PROBLEM FORMULATION AND FILTER DESIGN

In this section, we provide some preliminary knowledge and formulate a distributed Kalman filter problem with SECs. Then we provide a filtering structure for the design.

A. Problem Statement

Consider the following time-varying stochastic dynamics

$$x_{k+1} = A_k x_k + \omega_k,$$

where $x_k \in \mathbb{R}^n$ is the state vector, $A_k \in \mathbb{R}^{n \times n}$ is the known system matrix, $\omega_k \in \mathbb{R}^n$ is the zero-mean process noise, and $E\{\omega_k|\omega_k^T\} \leq Q_k \in \mathbb{R}^{n \times n}$, where $0 < Q_1 \leq Q_k \leq Q_2 < +\infty$. The state $x_k$ is observed by a multi-agent network of $N$ agents, whose measurement model is given as

$$y_{k,i} = H_{k,i} x_k + v_{k,i}, i = 1, 2, \ldots, N,$$

where $y_{k,i} \in \mathbb{R}^{m_i}$ is the measurement vector obtained by agent $i$, $H_{k,i} \in \mathbb{R}^{m_i \times n}$ is the observation matrix of agent $i$ and $v_{k,i} \in \mathbb{R}^{m_i}$ is the zero-mean observation noise. Suppose that $x_0$, $\{\omega_k\}_{k=0}^\infty$ and $\{v_{k,i}\}_{k=0}^\infty$ are Gaussian and uncorrelated. Also, $E\{(x_0 - E\{x_0\})(x_0 - E\{x_0\})^T\} \leq P_0 \in \mathbb{R}^{n \times n}$ and $E\{v_{k,i}v_{k,i}^T\} \leq R_{k,i} \in \mathbb{R}^{m_i \times m_i}$, where $R_{k,i} > 0$. It is noted that $v_{k,i}$ and $v_{k,j}$ may be correlated at each time $k$.

The communication between agents in the multi-agent network is modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, which consists of the set of agents or nodes $\mathcal{V} = \{1, 2, \ldots, N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacent matrix $A = [a_{i,j}]$. In the weighted adjacent matrix $A$, all the elements are nonnegative, row stochastic and the diagonal elements are all positive, i.e., $a_{i,j} > 0$, $a_{i,i} \geq 0$, $\sum_{j \in \mathcal{V}} a_{i,j} = 1$. If $a_{i,j} > 0, j \neq i$, there is an edge $(i, j) \in \mathcal{E}$, which means node $i$ can directly receive the information of node $j$, and node $j$ is called the neighbor of node $i$. All the neighbors of node $i$ can be represented by the set $\{j \in \mathcal{V} | (i,j) \in \mathcal{E}\} \equiv N^0_i$, whose size is denoted as $|N^0_i|$. Also, $N^0_i \cup \{i\} \equiv N_i$. $\mathcal{G}$ is called strongly connected if for any pair nodes $(i_1, i_2)$, there exists a directed path from $i_1$ to $i_2$ consisting of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{l-1}, i_l)$. An undirected graph $\mathcal{G}$ is simply called connected if it is strongly connected.

In practical applications, many examples can be formulated with SECs [20, 25]. Given the dynamics [1], there may exist equality constraints on the overall state for each agent. The SECs can be expressed as

$$D_{k,i} x_k = d_{k,i}, i = 1, 2, \ldots, N,$$

where $D_{k,i} \in \mathbb{R}^{s_i \times n}$, $d_{k,i} \in \mathbb{R}^{s_i}$ are known constraint matrix and constraint vector of agent $i$, respectively, and $s_i$ is the number of constraints of agent $i$. Clearly, the number of constraints is no larger than the number of the state elements, i.e., $s_i \leq n$. Also, $D_k = \bigcap_{i \in \mathcal{V}} D_{k,i} \neq \emptyset$, where $D_{k,i} = \{x_k | D_{k,i} x_k = d_{k,i}\}$. The constraint equality $D_k x_k = d_k$ contains all the local constraints $D_k x_k = d_k, i \in \mathcal{V}$. Without loss of generality, $D_k$ is supposed to be either zero or of full row rank, and $D_k$ is assumed to be of full row rank.

For convenience, we denote

$$\Phi_{k,i} = I_n, \Phi_{k+1,i} = A_k, \Phi_{j,j-k} = \Phi_{j,j-k-1} \cdots \Phi_{k+1,k}(j > k),$$

$H_k = [H_{k,1}^T, H_{k,2}^T, \ldots, H_{k,n}^T]^T$, $D_k = [D_{k,1}^T, D_{k,2}^T, \ldots, D_{k,n}^T]^T$, $R_k = \text{blockdiag}\{R_{k,1}, R_{k,2}, \ldots, R_{k,n}\}$.

In the distributed framework, each agent aims to timely estimate the dynamics (1) based on the knowledge of local measurements (2) and constraints (3) as well as the information communication with neighbors. To solve the problem, the following assumptions are adopted throughout the paper.

**Assumption 1.** The directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ of multi-agent network is strongly connected.

**Assumption 2.** (Extended collective observability) There exist a positive integer $\bar{N}$ and a positive constant $\alpha > 0$ such that for any $k \geq 0$,

$$\sum_{j=k}^{k+\bar{N}} \Phi_{j,k}^T (H_j^T R_j^{-1} H_j + D_j^T D_j) \Phi_{j,k} \geq \alpha I_n > 0.$$ (4)

Even if the observability conditions in [6, 7, 10, 16, 19, 41] are not satisfied, the extended collective observability condition in Assumption 2 can still be fulfilled. In other words, Assumption 2 is more general than the existing observability conditions because the mildest observability conditions given in the existing papers such as [6, 7] still depend on $(A_k, H_k)$, which is a special case of Assumption 2 satisfied if $D_{k,i} = 0, i \in \mathcal{V} = k, 1, 2, \ldots$.

**Assumption 3.** There exist a sequence set $\mathcal{K} = \{k_l, l \geq 1\}$, an integer $\bar{L} \geq N + N$ and two scalars $\beta_1 \geq \beta_2 > 0$, such that

- $\sup_{l \geq 1} (k_{l+1} - k_l) < \infty$
- $\inf_{l \geq 1} (k_{l+1} - k_l) > 0$
- $\lambda_{\max}(A_{k_l} A_{k_l}^T) \leq \beta_1, \forall k \geq 0$ and $\lambda_{\min}(A_{k_l+s} A_{k_l+s}^T) \geq \beta_2, \forall k_l \in \mathcal{K}, s = 0, \ldots, \bar{L} - 1$. 


Different from the assumption in [7] that the system matrices \( \{ A_k \}_{k=1}^{\infty} \) belong to a non-singular compact set, Assumption [3] relaxes the non-singularity restriction of \( A_k \). In other words, if \( \{ A_k \}_{k=1}^{\infty} \) are satisfied with the assumption in [7], Assumption [3] in this paper holds naturally. Extended Kalman filter is considered for estimation of dynamics \( x_{k+1} = f(x_k) + \omega_k \), which after linearization leads to the case with \( A_k \) being the Jacobian of the partial derivatives of \( f(x_k) \) w.r.t \( x \). Thus Assumption [3] extends the application range of the estimation algorithms noting that in the linearization process \( A_k \) can probably be singular at some moments.

### B. Filter Design

Given the dynamics (1) with observations of \( N \) agents (2) and constraints (3), we propose the following general distributed filtering structure in Table I.

**TABLE I**

**DISTRIBUTED FILTERING STRUCTURE**

| Prediction | Measurement update |
|------------|-------------------|
| \( \hat{x}_{k,i} = A_{k-1,i} \hat{x}_{k-1,i} \) | \( \bar{x}_{k,i} = \hat{x}_{k,i} + K_{k,i}(y_{k,i} - H_{k,i}(\bar{x}_{k,i})) \) |
| **Local fusion:** | Perform \( L \) steps of fusion-projection: |
| \( \bar{x}_{k, i} = \sum_{j \in N_i} W_{k,i,j} \bar{x}_{k,j} \) | \( l = 0, 1, \cdots, L - 1 \) |
| **Projection:** | \( \bar{x}_{k,i} = \arg \min_{x} (x - \bar{x}_{k,i})^T (M_{k,i})^{-1} (x - \bar{x}_{k,i}) \), s.t. \( D_{k,i} x = d_{k,i} \) |
| **Output:** | \( \bar{x}_{k,i} = \bar{x}_{k,i} \) |

In Table I, \( \bar{x}_{k,i}, \hat{x}_{k,i}, \bar{x}_{k,j} \) and \( \bar{x}_{k,i} \) are the state estimates in the state prediction, state update, state fusion and state projection of agent \( i \) at the \( k \)th moment, respectively. Additionally, \( \bar{x}_{k,i} \) is the estimate to be fused by agent \( i \) under time-based or event-triggered schemes. The matrices \( K_{k,i}, W_{k,i,j} \) and \( M_{k,i} \) are to be designed.

**Remark 1.** The projection method in Table I has two advantages. First, the state estimates of one agent can be strictly satisfied with the state equality constraints of its own \([3][27]\). Second, the projection method can better handle the time-varying and dimensional changing constraints \([32]\).

In the Kalman filter, the error covariance can be represented by the parameter \( P_k \). In \([14]\), \( P_k \) stands for the error covariance due to the particular communication scheme (i.e., exchanging information with and only with one neighbor randomly). However, in the general distributed Kalman filter \([3][41][42]\), the relationship between \( P_{k,i} \) and the error covariance matrix is uncertain.

Unlike \([9][41][43]\) without analyzing the estimation error covariance in real time, we seek to evaluate the error covariance by consistency, which is defined as follows.

**Definition 1.** (Consistency) Suppose \( x_k \) is a random vector. Let \( \hat{x}_k \) and \( P_k \) be the estimate of \( x_k \) and the estimate of the corresponding error covariance matrix. Then the pair \((\hat{x}_k, P_k)\) is said to be consistent at the \( k \)th moment if

\[
E\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T\} \leq P_k. \tag{5}
\]

Existing results did not provide many scalable and fully distributed filters for the time-varying dynamics (1) with the capability in error evaluation, because of the strong correlation between the estimations of agents. Furthermore, since the difficulty in satisfying with the SECs \([3]\), existing distributed methods \([2][9][15][18]\) cannot be directly applied to the distributed state estimation under constraints. Additionally, due to the difficulty in connecting the triggering thresholds and the estimation performance, the analysis on the upper boundedness of the covariance matrix of distributed state estimation with event-triggered scheme under collective observability is an open problem \([36][40]\).

In fact, intuitively, the prior information of the constraints \([3]\) can contribute to the improvement of the estimation performance. It is quite necessary for us to develop fully distributed filters by taking advantage of the SECs over multi-agent networks. In the sequel, we try to answer the following important questions:

1. How to design \( K_{k,i}, W_{k,i,j} \) and \( M_{k,i} \) of Table I such that the proposed distributed Kalman filter is a fully distributed design and subject to consistency at each moment?
2. What benefits can SECs provide? And how do the equality constraints relax the existing collective observability conditions? For each agent, how to satisfy the global SECs simultaneously?
3. How to design an event-triggered scheme? And what is the relationship between event-triggered thresholds and the estimation performance under collective observability?

### III. DISTRIBUTED FILTER WITH TIME-BASED COMMUNICATIONS

For the time-varying system (1)–(2) with constraints (3), we provide a time-based communication strategy for each agent to exchange information with its neighbors at each sampling moment in this section. To be specific, regarding the time-based strategy on the distributed filtering structure in Table I, we propose a scalable distributed algorithm called “time-based projected distributed Kalman filter” (TPDKF) in Table II with the communication topology in Fig. 1 which illustrates the message transmission between neighbors. It can be seen that agent \( i \) obtains the pair \((\bar{x}_{k,j}, P_{k,j})\) from its neighbor agent \( j \), \( j \in N_i \). Different from \([19]\), where the measurement information pair \((H_{k,i} R_{k,i}^{-1} H_{k,j}, R_{k,i}^{-1} y_{k,i})\) need to be transmitted, our transmission messages do not contain any measurement information, thus this communication protocol can protect data privacy \([45][46]\). Through the information communication over networks, each agent can achieve effective tracking on the stochastic dynamics.

A few remarks on the considered algorithm of Table II are given in the following.

**Remark 2.** On the proposed TPDKF in Table II, we find the following facts.

- The positive parameter \( \varepsilon_i \) in TPDKF is adjustable, and it can be set sufficiently small like the setting in [27]:

\[
\varepsilon_i = \alpha_i N_i^{-1}, \tag{6}
\]
TABLE II
TIME-BASED PROJECTED DISTRIBUTED KALMAN FILTER (TPDKF)

| Initialization: |
|-----------------|
| $(\hat{x}_{0,k}, P_{0,k})$ is satisfied with $P_{0,k} \geq (1 + \theta_k)\tilde{P}_0 + \frac{\theta_k}{\theta_k} P_0^\ast, \theta_k > 0$, where $P_0^\ast \geq (\hat{x}_{0,k} - E(x_{0,k})) (\hat{x}_{0,k} - E(x_{0,k}))^T$. |

| Input: |
|-----------------|
| $(\hat{x}_{k-1,i}, P_{k-1,i}, e_i)$. |

| Prediction: |
|-----------------|
| $\bar{x}_{k,i} = \hat{x}_{k-1,i} + Q_{k-1,i}^{-1} \bar{y}_{k,i}$, $P_{k,i} = \hat{x}_{k-1,i} P_{k-1,i} hatscript{}_{k-1} + Q_{k-1,i}^{-1}$. |

| Measurement update: |
|-----------------|
| $x_{k,i} = \hat{x}_{k,i} + K_{k,i} (y_{k,i} - H_{k,i} \hat{x}_{k,i})$ $K_{k,i} = P_{k,i} H_{k,i}^T (H_{k,i} (P_{k,i} H_{k,i}^T + R_{k,i})^{-1})$, $P_{k,i} = (I - K_{k,i} H_{k,i}) P_{k,i}$. |

| Perform $L$ steps of fusion-projection with $x_{k,i}^0 = \hat{x}_{k,i}, P_{k,i}^0 = \tilde{P}_{k,i}$: |
|-----------------|
| **Local fusion:** Receive $(\hat{x}_{k,i}^l, P_{k,i}^l)$ from neighbors $j \in N_i$, $\hat{x}_{k,i}^l = \tilde{P}_{k,i}^l \sum_{j \in N_i} a_{i,j} (P_{k,j}^l)^{-1} \hat{x}_{k,j}^l$, $l = 0, 1, \cdots, L - 1$, $P_{k,i}^l = (\sum_{j \in N_i} a_{i,j} (P_{k,j}^l)^{-1})^{-1}$, $l = 0, 1, \cdots, L - 1$. |

| **Projection:** |
|-----------------|
| $\hat{x}_{k,i}^{l+1} = \hat{x}_{k,i}^l + P_{k,i}^l - P_{k,i}^l A_{k,i}^T (P_{k,i}^l + D_{k,i}^2)_{k,i}^{-1} (P_{k,i}^l - A_{k,i}^T D_{k,i}^2 \hat{x}_{k,i}^l - d_{k,i})$, $P_{k,i}^{l+1} = P_{k,i}^l - P_{k,i}^l A_{k,i}^T D_{k,i}^2 (P_{k,i}^l + D_{k,i}^2)_{k,i}^{-1} (P_{k,i}^l - A_{k,i}^T \hat{x}_{k,i}^l)$, $P_{k,i}^{l+1}$. |

| Output: |
|-----------------|
| $\hat{x}_{k,i} = \hat{x}_{k,i}^L, P_{k,i} = \tilde{P}_{k,i}^L$. |

**Fig. 1.** An illustration of the communication topology for TPDKF.

**Table II:** 
**TIME-BASED PROJECTED DISTRIBUTED KALMAN FILTER (TPDKF)**

- The matrix $P_{k,i}$ in TPDKF does not stand for the error covariance of agent $i$. The relationship between $P_{k,i}$ and the error covariance matrix will be shown in Theorem 2.
- Unlike [9] which depends on the agent number of the system and [39] which requires the network structure, TPDKF is a fully distributed algorithm, since the design of the filtering gain $K_{k,i}$, the parameter matrices $M_{k,i}^l$ and $W_{k,i}^l$ simply depend on local information and does not rely on any global knowledge of the system and network topology.

**Remark 3.** The proposed algorithm in Table II does not require the assumption of [19] that the measurement noises of agents are uncorrelated.

**Remark 4.** The fusion step in [19] has the form of $P_{k,i}^H = \left( \sum_{j \in N_i} a_{i,j} P_{k,j}^H + \sum_{j \in N_i} H_{k,j}^T R_{k,j}^{-1} H_{k,j} \right)^{-1}$. We can see that $P_{k,i} \geq P_{k,i}^H$, i.e., $P_{k,i}^H$ in [19] is a less conservative upper bound of estimation covariance compared with our algorithm. However, the fusion step in our algorithm has meaningful information-theoretic interpretation [6], which is achieved as a consensus of Gaussian distributions under measure of Kullback-Leibler divergence. For example, two normal distributions $\mathbb{N}(0, 10000)$ and $\mathbb{N}(10, 10000)$ are almost indistinguishable with the Euclidean distance 10. In contrast, the distributions $\mathbb{N}(0, 0.01)$ and $\mathbb{N}(0.1, 0.01)$ barely overlap, but this is not reflected in the Euclidean distance, which is only 0.1. Namely, fusion of probability distributions is more suitable under the measure of Kullback-Leibler divergence than that in Euclidean space.

**Proposition 1.** Under Assumption [7] for the proposed TPDKF in Table II

$$\lim_{L \to +\infty} \text{dis}(\hat{x}_{k,i}^L, P_{D_k,\hat{x}_{k,i}^L}) = 0, \forall k \geq 1$$

where $D_k = \bigcap_{i \in N} D_{k,i}$ and $D_{k,i} = \{ x_k | D_{k,i} x_k = d_{k,i} \}$.

**Proof.** See the proof in Appendix A.

**Remark 5.** Proposition [7] shows that the estimate of each agent $\hat{x}_{k,i} = \bar{x}_{k,i}$ will satisfy the global equality constraints, if $L \to \infty$. In what follows, it is also shown that $\hat{x}_{k,i}$ will satisfy the global equality constraints as $k \to \infty$ even if $L < \infty$ (see Corollary [7]).

The following result shows the Gaussianity of the estimation error of TPDKF.

**Theorem 1.** *(Gaussianity)* For TPDKF of the system $\{1\} - [2]$ with constraints $\{3\}$, the estimation error $e_{k,i} = \hat{x}_{k,i} - x_k$ is Gaussian, for any $i \in V, k = 1, 2, \ldots$.

**Proof.** See the proof in Appendix B.

Due to the strong correlation between the estimates of agents, it is quite difficult to obtain the error covariance matrix $E\{e_{k,i} e_{k,i}^T\}$. Although the optimal Kalman-consensus filters proposed in $\{4\}, [4]$ tried to obtain the covariance matrix, the computational and communication costs are not scalable in $N$ (the number of agents). In this paper, the next result shows that the proposed TPDKF possesses consistency defined in Definition 1 and the parameter matrix $P_{k,i}$ in TPDKF.

**Theorem 2.** *(Consistency)* For TPDKF of the system $\{1\} - [2]$ with constraints $\{3\}$, the pair $(\hat{x}_{k,i}, P_{k,i})$ is consistent. Moreover, if $L \geq 1$, $\hat{P}_{k,i}^{l+1} \leq \hat{P}_{k,i}^l, l = 0, 1, \cdots, L - 1, \forall i \in V, k = 1, 2, \ldots$.

**Proof.** See the proof in Appendix C.
The analysis of upper boundedness of the covariance matrix is one of the most difficult issues in the distributed estimation due to the time-varying dynamics and the strong correlation between the errors of the agents. Because of the consistency of TPDKF, we can turn to prove that $P_{k,i}$ is upper bounded so as to ensure upper boundedness of the covariance matrix of TPDKF.

**Theorem 3. (Boundedness)** For TPDKF, under Assumptions $[$4$]$, $[$5$]$ and the fusion-projection number $L$ is greater than zero, then there exists a positive definite matrix $\hat{P}$ such that

$$ P_{k,i} \leq \hat{P} < \infty, \quad \forall i \in \mathcal{V}, \forall k = 0, 1, \ldots.$$  

Proof. For brevity, the proof is given for the case $L = 1$. It is straightforward to extend the results for $L \geq 1$.

Under Assumption $[$3$]$ there exists a subsequence set $\{k_{l,m}, m \geq 1\}$ from the sequence set $\mathcal{K} = \{k_l, l \geq 1\}$, where $L \leq k_{l,m+1} - k_{l,m}$. As $\sup_{l \geq 1} (k_{l+1} - k_l) < \infty$, there exists a sufficiently large integer $L$, such that $L \leq k_{l,m+1} - k_{l,m} \leq L$. Without loss of generality, we suppose that the set $\{k_l, l \geq 1\}$ has this property in the following, i.e., $L \leq k_{l+1} - k_L \leq L$, $\forall l \geq 1$. To prove the boundedness of $P_{k,i}$, we divide the sequence set $\{k_l, l \geq 1\}$ into two bounded and non-overlapping set: $\{k_l + L, l \geq 1\}$ and $\bigcup_{i \geq 1} [k_l + L + 1 : k_{l+1} + L - 1]$

1) First, we consider the case of $k = k_l + L, l \geq 1$. For convenience, denote $k_l + L \equiv \tilde{k}$. At the $k$th moment, by exploiting the matrix inverse formula on $P_{k,i}$ and $P_{\tilde{k},i}$, respectively, we obtain

$$ P_{\tilde{k},i}^{-1} \geq \sum_{j \in \mathcal{N}_i} a_{i,j}(P_{k,j}^{-1} + S_{k,j}) + \frac{D_{k,i}^T D_{k,i}}{\varepsilon_i}, $$

where $S_{k,j} = H_{k,j}^T R_{k,j}^{-1} H_{k,j}$, and $0 < \eta < 1$, which is obtained similarly to Lemma 1 in $[$6$]$ by noting the upper boundedness of $Q_k$ and lower boundedness of $A_{k} A_{k}^T$. By recursively applying $[$4$]$ for $\tilde{k}$ times, one has

$$ P_{\tilde{k},i}^{-1} \geq \eta^{\tilde{k}} \Phi_{\tilde{k},k_l}^{-1} \left[ \sum_{j \in \mathcal{V}} a_{i,j} \tilde{P}_{\tilde{k},j}^{-1} \right] \Phi_{\tilde{k},k_l}^{-1} + \tilde{P}_{\tilde{k},i}^{-1},  
$$

where

$$ \tilde{P}_{\tilde{k},i}^{-1} = \sum_{s = 1}^{\tilde{k}} \eta^{s-1} \Phi_{s, \tilde{k}}^{-1} \left[ \sum_{j \in \mathcal{V}} (a_{i,j} s S_{k,s+1,j} + \eta^{s-1} D_{k,s+1,j}^T D_{k,s+1,j}) \right] \Phi_{s, \tilde{k}}^{-1}, $$

and $\Phi_{s, \tilde{k}}$ is the state transition matrix, $a_{i,j}, s$ is the $(i, j)$th element of $\Phi^s$. According to Assumption $[$4$]$ and $[$47$]$, $a_{i,j} > 0$ for $s \geq N - 1$. Since the first term on the right hand side of $[$9$]$ is positive definite, we consider the second term $\tilde{P}_{\tilde{k},i}^{-1}$. From $[$10$]$, one can obtain

$$ \tilde{P}_{\tilde{k},i}^{-1} \geq a_{\min} \eta^{\tilde{k}-1} \sum_{s = N}^{\tilde{k}} \Phi_{s, \tilde{k}}^{-1} \left[ \sum_{j \in \mathcal{V}} (a_{i,j} s S_{k,s+1,j} + \eta^{s-1} D_{k,s+1,j}^T D_{k,s+1,j}) \right] \Phi_{s, \tilde{k}}^{-1}, $$

where $a_{\min} = \arg \min_{j \in \mathcal{V}_i} a_{i,j} > 0, s \in [N - 1 : \tilde{k}]$ and $\mathcal{Y} = \text{blockdiag} \{ \varepsilon_i, i \in \mathcal{V} \}$. Under Assumption $[$2$]$ for the given $\{\varepsilon_i, i \in \mathcal{V}\}$, there exists an $\tilde{\alpha} > 0$, such that

$$ \sum_{j = k_l+1}^{k_l+N} \Phi_{s, k_l+1}^{-1} + \tilde{P}_{\tilde{k},i}^{-1} + \tilde{P}_{\tilde{k},i}^{-1} \geq \sum_{j = k_l+1}^{k_l+N} \Phi_{s, k_l+1}^{-1} + \tilde{P}_{\tilde{k},i}^{-1} \geq \alpha I, \alpha > 0, $$

where $\tilde{G}_{k_l+1} = \{H_{k_l+1}^T, D_{k_l+1}^T, D_{k_l+1}^T, \tilde{H}_{k_l+1}^T, D_{k_l+1}^T, \tilde{Y}, D_{k_l+1}^T, \tilde{D}_j \}$, $\tilde{H}_{k_l+1} = \text{blockdiag} \{ R_{k_l+1, \mathcal{Y}}, R_{k_l+2, \mathcal{Y}}, \ldots, R_{k_l+1+N, \mathcal{Y}} \}$.

Since $A_k$ is nonsingular if $k \in [k_l : k_l + L - 1]$, due to $N + N \leq L$, the matrix $F_{k_l+1}$ can be well defined as $F_{k_l+1} = \Phi_{k_l+1}^{-1}$. Under Assumption $[$3$]$ for $k \in [k_l : k_l + L - 1]$, $A_k^{-1}$ lies in a compact set. Then there exists a positive real such, that $F_{k_l+1} F_{k_l+1} I_{k_l+1} = \alpha I, \alpha > 0$. One can obtain that

$$ \sum_{j = k_l+1}^{k_l+N} \Phi_{s, k_l+1}^{-1} + \tilde{P}_{\tilde{k},i}^{-1} \geq \alpha I, \alpha > 0, $$

Since $L \geq N + N$, $\tilde{G}_{k_l+1}$ in $[$11$]$ is lower bounded by a constant positive definite matrix. Thus, $P_{k_l+1, \mathcal{V}_i}$ is upper bounded by a constant positive definite matrix $\hat{P}$, i.e., $P_{k_l+1, \mathcal{V}_i} \leq \hat{P}$.  

2) Second, $k \in [k_l + L + 1 : k_l+1 + L - 1]$, $l \geq 1$. Since the processes in measurement update and projection do not enlarge the parameter $P$ of the algorithm, and the local fusion process has no effect when fusing the common upper bound of $P$ of each agent, we focus on analyzing the prediction process by enlarging its upper bound. Under Assumption $[$3$]$ $A_k A_k^T \leq \beta_1 I$. Without lose of generality, let $\beta_1 \geq 1$. Due to $Q_k \leq Q_2 < \infty$, for $k \in [k_l + L + 1 : k_l+1 + L - 1],$

$$ P_{k,l} \leq \sum_{s = N}^{k_l} \Phi_{s, \tilde{k}}^{-1} \left[ \sum_{j \in \mathcal{V}} (a_{i,j} s S_{k,s+1,j} + \eta^{s-1} D_{k,s+1,j}^T D_{k,s+1,j}) \right] \Phi_{s, \tilde{k}}^{-1}, $$

and $\Phi_{s, \tilde{k}}$ is the state transition matrix, $a_{i,j}, s$ is the $(i, j)$th element of $\Phi^s$. According to Assumption $[$4$]$ and $[$47$]$, $a_{i,j} > 0$ for $s \geq N - 1$. Since the first term on the right hand side of $[$9$]$ is positive definite, we consider the second term $\tilde{P}_{\tilde{k},i}^{-1}$. From $[$10$]$, one can obtain

$$ P_{k,i} \leq \alpha I, \alpha > 0, $$

where $\alpha = \sum_{j = k_l+1}^{k_l+N} \Phi_{s, k_l+1}^{-1} + \tilde{P}_{\tilde{k},i}^{-1} \geq \alpha I, \alpha > 0, $
compact set and the fusion-projection number $L$ is larger than zero, then there is

$$
\lim_{k \to +\infty} (\tilde{x}_{k,i} - x_k) = 0, \quad \forall i \in V.
$$

(12)

**Proof.** Under the conditions of this theorem, the conclusion of Theorem 3 holds. Thus, $P_{k,i}$ is uniformly upper bounded. Then $\tilde{P}_{k,i}$ is uniformly upper bounded and lower bounded due to the compactness of $\{A_k\}_{k=0}^\infty$ and $0 < Q_1 \leq Q_k \leq Q < \infty$. Consider the Lyapunov function $v_{k,i}(\tilde{e}_{k,i}) = \tilde{e}_{k,i}^T \tilde{P}_{k,i}^{-1} \tilde{e}_{k,i}$. In the noise-free case, according to the fact (iii) of Lemma 1 in [6], there exists a $\bar{\beta} \in (0, 1)$, such that

$$
V_{k+1,i}(\tilde{e}_{k+1,i}) = \tilde{P}_{k+1,i}^{-1} \tilde{e}_{k+1,i} \leq \bar{\beta} \tilde{e}_{k,i}^T \tilde{P}_{k,i}^{-1} \tilde{e}_{k,i}.
$$

(13)

Then according to (12) and (13),

$$
\tilde{e}_{k,i}^{l+1} = P_{k,i}^l \tilde{P}_{k,i}^{-1} \sum_{j \in V} a_{i,j} (\tilde{P}_{k,i})^{-1} \tilde{e}_{k,j}.
$$

(14)

where $P_{k,i}^l = I_n - \sum_{j=0}^{l-1} T_k D_{k,l} (P_{k,i}^l P_{k,i}^T D_{k,l} T_k).$

According to Lemma 1 and Lemma 2 in [6] to the right hand of (13), one can obtain that

$$
(\tilde{e}_{k,i}^{l+1})^T (\tilde{P}_{k,i}^{-1})^{-1} \tilde{e}_{k,i}^{l+1} \leq \sum_{j \in V} a_{i,j} (\tilde{e}_{k,j}^T (\tilde{P}_{k,i})^{-1} \tilde{e}_{k,j}).
$$

(15)

By recursively applying (15) for $l = 0, 1, \ldots, L - 1$, we have

$$
e_{k,i}^T e_{k,i} P_{k,i}^{-1} e_{k,i} = \left( \tilde{e}_{k,i}^T (\tilde{P}_{k,i})^{-1} \tilde{e}_{k,i} \right) = \sum_{j \in V} a_{i,j} \tilde{e}_{k,i}^T (\tilde{P}_{k,i})^{-1} \tilde{e}_{k,j} = \sum_{j \in V} a_{i,j} \tilde{e}_{k,i}^T (\tilde{P}_{k,i})^{-1} \tilde{e}_{k,j}.
$$

(16)

Since $\tilde{P}_{k,i} = (I - K_{k,i} H_{k,i}) P_{k,i}$, one can get $\tilde{P}_{k,i}^{-1} (I - K_{k,i} H_{k,i}) = \tilde{P}_{k,i}^{-1}$, From TDKF, we have $\tilde{P}_{k,i} = \sum_{l \in N} a_{i,j} (\tilde{P}_{k,j})^{-1} \leq \sum_{l \in N} a_{i,j} (\tilde{P}_{k,j})^{-1}$. Considering $\tilde{e}_{k,i} = (I - K_{k,i} H_{k,i}) e_{k,i}$, [13] and (16), it can be obtained that

$$
V_{k+1,i}(\tilde{e}_{k+1,i}) = \beta \sum_{j \in V} a_{i,j} V_{k,j}(\tilde{e}_{k,j}).
$$

(17)

Summing up (17) for $i = 1, 2, \ldots, N$, we have

$$
V_{k+1}(\tilde{e}_{k+1}) \leq \tilde{\beta} A^L V_k(\tilde{e}_k), \quad 0 < \tilde{\beta} < 1,
$$

(18)

where $V_k(\tilde{e}_k) = \text{col}(V_{k,1}(\tilde{e}_{k,1}), \ldots, V_{k,N}(\tilde{e}_{k,N}))$.

Since $A$ is a row stochastic matrix at each moment, thus the spectral radius of $\mathcal{A}$ is always 1. Due to $0 < \tilde{\beta} < 1$, $\lim_{k \to +\infty} \tilde{e}_{k+1,i} = 0$. Under the equation $\tilde{e}_{k+1,i} = A_k \tilde{e}_{k,i}$ and the assumption that $\{A_k\}_{k=0}^\infty$ belongs to a nonsingular compact set, the conclusion of this theorem holds. □

**Corollary 1.** Under the same conditions as Theorem 2

$$
\lim_{k \to +\infty} \text{dis}(\tilde{x}_{k,i}, P_{D_k}(\tilde{x}_{k,i})) = 0, \quad \text{where} \quad D_k = \bigcap_{i \in V} D_{k,i} \quad \text{and} \quad D_{k,i} = \{x_k | D_{k,i} x_k = d_{k,i}\}.
$$

□

**Theorem 4.** (Convergence) Under Assumptions 1 and 2 and the noise-free situation, if $\{A_k\}_{k=0}^\infty$ belongs to a nonsingular compact set and the fusion-projection number $L$ is larger than zero, then there is

$$
\lim_{k \to +\infty} (\tilde{x}_{k,i} - x_k) = 0, \quad \forall i \in V.
$$

(12)
Therefore, the conclusion holds due to (19).

Remark 8. It can be seen from Corollary 4 that the state estimate of each agent will eventually lie in the intersection space of all local equality constraints.

IV. DISTRIBUTED FILTER WITH EVENT-TRIGGERED COMMUNICATIONS

In the commonly-used time-based distributed communication mechanism, the local messages of each agent are broadcasted to its neighbors at each sampling moment. Such a communication strategy may lead to some unnecessary communication data flow in the network, which unavoidably increases the communication burden and energy consumption. In fact, it is quite necessary to employ efficient strategies to reduce communication rate and save energy, because of the practical bandwidth and energy constraints (e.g., in wireless sensor network). To study event-triggered schemes for this purpose, we focus on time-invariant systems, though the method can be extended to time-varying cases. However, if the system is time-varying, the triggered scheme will be very complicated and time-varying, which make the implementation very complicated. In other words, we give the following assumption for simplicity.

Assumption 4. The matrices in the system (1)–(2) and the constraints (3) satisfy $A_k = A$, $Q_k = Q$, $H_k,i = H_i$, $R_k,i = R_i$, $D_{k,i} = D_i$, $\forall i \in V, k = 1, 2, \ldots$.

Then the following two lemmas are quite straightforward.

Lemma 2. Under Assumptions 2 and 4, $[A, \hat{H}]$ is observable, where $\hat{H} = [H^T, D^T]^T$, $H = [H_1, H_2, \ldots, H_N]^T$, $D = [D_1^T, D_2^T, \ldots, D_N^T]^T$.

Lemma 3. Under Assumptions 3 and 4, the system matrix $A$ is non-singular.

Different from many existing results with event-triggered schemes [36, 50], whose assumptions on the upper boundedness of the covariance matrix are usually related to the existence of the solutions of Riccati equations or Hurwitz stability of $A$, the collective observability of $[A, \hat{H}]$ of the system can be verified before the implementation of algorithms. Clearly, the collective observability of $[A, \hat{H}]$ is a time-invariant version of the extended collective observability in Assumption 2, which is weaker than the assumption that $(A, \hat{H})$ is observable given in [36, 38, 40]. The non-singularity of $A$ can be guaranteed through discretization from general continuous systems.

For TPKF in Table II with $L = 1$, each agent can decide whether to broadcast the message $(\tilde{x}_{k,j}, \tilde{P}_{k,j})$ to its neighbors or not, thus we need to give a criterion to determine what message is worthy to broadcast. Here, we utilize the SoD regulation triggering mechanism, which is an event-triggered principle and related to the information increment. Different from the SoD methods in [36, 38] depending on some stochastic variables, a novel SoD method is given in this paper. The event triggers of proposed SoD method is indeed deterministic and based on $\tilde{P}_{k,j}$, where one can judge the information increment according to its variation degree.

Let the pair $(\tilde{x}_{t,j}, \tilde{P}_{t,j})$ be the latest message broadcasted by agent $j$ to its neighbors. Define the following triggering mechanism function for agent $j$ as $g_{k,j}(\cdot) : \mathbb{R}^{n \times n} \times \mathbb{R} \to \mathbb{R}$,

$$g_{k,j}(\tilde{P}_{k,j}, \tilde{P}_{k,j}, \delta_j) = \lambda_{max}(\tilde{P}_{k,j} - \tilde{P}_{k,j}) - \delta_j, j \in V,$$

where $\delta_j \geq 0$ is the triggering threshold of agent $j$ which is usually predefined, and

$$\tilde{P}_{k,j} \triangleq A^{k-t} \tilde{P}_{t,j}(A^{k-t})^T + \sum_{l=1}^{k-1} A^{k-1-l}Q(A^{k-1-l})^T$$

is the multi-step prediction matrix.

The event for agent $j$ at the $k$th moment is triggered if $g_{k,j} > 0$. In other words, if $g_{k,j} > 0$, agent $j$ broadcasts its information message to its neighbors at the $k$th moment. Suppose that the event is triggered at the initial moment. Define the pair $(\tilde{x}_{k,j}, \tilde{P}_{k,j})$ as

$$\begin{align*}
(\tilde{x}_{k,j}, \tilde{P}_{k,j}) &= \begin{cases} (A^{k-t} \tilde{x}_{t,j}, \tilde{P}_{t,j}), & \text{if } g_{k,j} \leq 0, \\
(\tilde{x}_{k,j}, \tilde{P}_{k,j}), & \text{if } g_{k,j} > 0.
\end{cases}
\end{align*}$$

If $g_{k,j} > 0$, the event for agent $j$ is triggered at this moment and $(\tilde{x}_{k,j}, \tilde{P}_{k,j})$ is broadcasted to the neighbors of agent $j$, and then each neighbor receives $(\tilde{x}_{k,j}, \tilde{P}_{k,j}) = (\tilde{x}_{k,j}, \tilde{P}_{k,j})$. Otherwise, these neighbors obtain no information of agent $j$ at this moment and they make a multi-step prediction using the latest received pair $(\tilde{x}_{k,j}, \tilde{P}_{k,j})$ to obtain $(\tilde{x}_{k,j}, \tilde{P}_{k,j}) = (A^{k-t} \tilde{x}_{t,j}, \tilde{P}_{t,j})$. Based on the above discussion, we propose an event-triggered projected distributed Kalman filter (EPDKF) in Table III with an illustration of communication topology in Fig. 2.

![Fig. 2. An illustration of the communication topology for EPDKF](image-url)

In this section, for the sake of energy saving, each agent can at most communicate with its neighbors once at one time instant.
TABLE III
EVENT-TRIGGERED PROJECTED DISTRIBUTED KALMAN FILTER (EPDKF)

| Initialization: | \( (\tilde{x}_{k,i}, P_{k,i}) \) is satisfied with \( P_{0,i} \geq (1 + \theta_i)P_0 + \frac{\theta_i}{\theta_i - 1}P_{0,i}^* \), \( \theta_i > 0 \), where \( P_{0,i}^* \geq (\tilde{x}_{0,i} - E\{x(0)\})(\tilde{x}_{0,i} - E\{x(0)\})^T \). |
|----------------|----------------------------------------------------------------------------------|
| Input: | \( (\tilde{x}_{k-1,i}, P_{k-1,i}, \epsilon_i, \delta_i) \). |
| Prediction: | \( \hat{x}_{k,i} = A\hat{x}_{k-1,i} \), \( P_{k,i} = AP_{k-1,i}A^T + Q \). |
| Measurement Update: | \( \tilde{x}_{k,i} = \tilde{x}_{k,i} + K_{k,i}(y_{k,i} - H_i\tilde{x}_{k,i}) \), \( K_{k,i} = P_{k-1,i}H_i^T(1 + P_{k-1,i}H_i^T) \), \( P_{k,i} = (I - K_{k,i}H_i)P_{k-1,i} \). |
| Local Fusion: | Obtain the pair \((\tilde{x}_{k,j}, \tilde{P}_{k,j})\) through (22) |
| Output: | \( (\tilde{x}_{k,i}, P_{k,i}) \). |

Remark 9. To calculate \( \tilde{P}_{k,i} \) in multi-step prediction, the neighbors of agent \( j \) only utilize the latest received \( \tilde{P}_{i,j} \) and the system dynamical structure through (27).

Clearly, given threshold \( \delta_i \), the design of the filtering gain, the parameter matrices and the event-triggered scheme, depends simply on the local available information without the global knowledge of the system or the network topology. Hence, the proposed algorithm EPDKF is a distributed filtering algorithm.

The following result shows the Gaussian properties of estimation error for EPDKF in Table III.

Theorem 5. (Gaussianity) For EPDKF of the system (20)–(22) with constraints (3), the estimation error \( e_{k,i} = \tilde{x}_{k,i} - x_{k,i} \) is Gaussian, \( \forall i \in \mathcal{V}, k = 1, 2, \ldots \)

Proof. See the proof in Appendix [9].

Theorem 6. (Consistency) For EPDKF of the system (20)–(22) with constraints (3), the pair \((\tilde{x}_{k,i}, P_{k,i})\) is consistent. Moreover, \( P_{k,i} \leq \tilde{P}_{k,i}, \forall i \in \mathcal{V}, k = 1, 2, \ldots \)

Proof. See the proof in Appendix [9].

Like TPDKF, the estimation error covariances of EPDKF in the steps of prediction, measurement update and local fusion are also upper bounded by the matrices \( \tilde{P}_{k,i} \), \( P_{k-1,i} \) and \( \tilde{P}_{i,k} \). Thus, these parameter matrices can be employed to evaluate the estimation error in real time under the error distribution illustrated in Theorem 5. Additionally, for EPDKF, the inverse of \( \tilde{P}_{k,i} \) can be treated as a lower bound of information matrix, which contributes much in the design of event-triggered mechanism (20). Similar to Theorem 2 Theorem 6 also shows that the SECs in (3) can reduce the upper bound of the estimation error covariance in the event-triggered scheme.

The following result shows upper boundedness of the covariance matrix of the proposed EPDKF algorithm.

Theorem 7. (Boundedness) For EPDKF in Table III under Assumptions 1–4 there exists a positive definite matrix \( \tilde{P} \), such that

\[
0 < \tilde{P}_{k,i} \leq \tilde{P} < \infty, \quad \forall k \geq 0, i \in \mathcal{V},
\]

if the triggering thresholds \( \delta_i, j \in \mathcal{V} \) satisfy

\[
\sum_{j \in \mathcal{V}} \delta_j M_{i,j} < \tilde{M}_i, \forall i \in \mathcal{V},
\]

where

\[
M_{i,j} = \sum_{\tau=1}^{k^*} \beta^{-1} \alpha_{ij,\tau}(A^{-1})^T A^{-1},
\]

\[
\tilde{M}_i = \sum_{\tau=1}^{k^*} \beta^{-1}
\]

\[
\sum_{j \in \mathcal{V}} (A^{-1})^T \alpha_{ij,\tau} H_j^T R_j^{-1} H_j + \frac{\alpha_{ij,\tau-1}}{\epsilon_j} D_j^T D_j A^{-1}.
\]

for all \( k^* \geq N + \beta \) given in (29) and \( \alpha_{ij,\tau} \) is the \((i,j)\)th element of \( A^\tau \).

Proof. With EPDKF and the triggering condition (20), it follows by exploiting the matrix inverse formula on \( \tilde{P}_{k,i} \) and \( \tilde{P}_{i,k} \) that

\[
P_{k,i}^{-1} = a_{i,i} \tilde{P}_{k,i}^{-1} + \sum_{j \in \mathcal{N}_i} a_{i,j} (\tilde{P}_{k,j}^{-1})^{-1} + \frac{1}{\epsilon_i} D_i^T D_i
\]

\[
\geq a_{i,i} \tilde{P}_{k,i}^{-1} + \sum_{j \in \mathcal{N}_i} a_{i,j} (\tilde{P}_{k,j}^{-1} - \delta_j) + \frac{1}{\epsilon_i} D_i^T D_i
\]

\[
\geq \sum_{j \in \mathcal{N}_i} a_{i,j} (\tilde{P}_{k,j}^{-1} + H_j^T R_j^{-1} H_j - \delta_j) + \frac{1}{\epsilon_i} D_i^T D_i.
\]

By Lemma 1 in [6], there exists a real scalar \( \beta \in (0, 1) \) such that

\[
(\alpha P_{k,i} A^T + Q)^{-1} \geq \beta A^{-T} P_{k,i}^{-1} A^{-1}, i \in \mathcal{V}, \forall k \geq 0.
\]

From (27),

\[
P_{k,i}^{-1} \geq \beta \sum_{j \in \mathcal{N}_i} a_{i,j} A^T P_{k-1,j}^{-1} A^{-1}
\]

\[
+ \sum_{j \in \mathcal{N}_i} a_{i,j} (H_j^T R_j^{-1} H_j - \delta_j) + \frac{1}{\epsilon_i} D_i^T D_i.
\]

By recursively applying the above inequality for \( k^* \) times, we obtain

\[
P_{k,i}^{-1} \geq \beta a_{i,j} A^T \left[ (A^{-k^*})^T \tilde{P}_{k-k^*,j}^{-1} A^{-k^*} \right] + \tilde{P}_{i}^{-1} - \beta^{k^*} \sum_{j \in \mathcal{V}} a_{i,j,k}^* \left[ (A^{-k^*})^T \tilde{P}_{k-k^*,j}^{-1} A^{-k^*} \right] + \tilde{P}_{i}^{-1}.
\]

where

\[
\tilde{P}_{i}^{-1} = \sum_{\tau=1}^{k^*} \beta^{-1} \sum_{j \in \mathcal{V}} (A^{-1})^T \alpha_{ij,\tau} H_j^T R_j^{-1} H_j + \frac{\alpha_{ij,\tau-1}}{\epsilon_j} D_j^T D_j A^{-1}.
\]
If $\hat{P}^{-1}$ is positive definite, we have $P_{k,i} \leq \hat{P}$. Thus, there exists a matrix $P_k^*$ such that $P_{k,i} \leq P_k^*, \forall i \in \mathcal{V}$. To guarantee the positiveness of $\hat{P}_i^{-1}$, $\delta_j, j \in \mathcal{V}$ can be designed such that

$$
\sum_{\tau=1}^{k'} \beta^{\tau-1} \sum_{j \in \mathcal{V}} a_{ij,\tau}(A^{1-\tau})^T \delta_j A^{1-\tau} < \hat{P}_i^{-1},
$$

(30)

where $\hat{P}_i^{-1} = \sum_{\tau=1}^{k'} \beta^{\tau-1} \sum_{j \in \mathcal{V}} (A^{1-\tau})^T (a_{ij,\tau} H_j^T R_j^{-1} H_j + \frac{a_{ij,\tau}}{\beta^{j-1}} D_j^T D_j) A^{1-\tau}$. Then the conclusion in (24) is reached. For the existence of a positive upper bound of $\delta_j, j \in \mathcal{V}$, one needs to show the positiveness of $\hat{P}_i^{-1}$. According to Assumption 1 and (47), (48), we have $a_{ij,s} > 0$, $s \geq N - 1$. Thus,

$$
\hat{P}_i^{-1} \geq a_{\min} \beta^{k'-1} \sum_{\tau=N+1}^{k'} (A^{1-\tau})^T \left( \sum_{j \in \mathcal{V}} (H_j^T R_j^{-1} H_j + \frac{a_{ij,\tau-1}}{\beta^{j-1}} D_j^T D_j) A^{1-\tau} \right) = a_{\min} \beta^{k'-1} (A^{-N})^T D_H A^{-N},
$$

where

$$
\begin{align*}
\{ & \begin{array}{l}
R = \text{blockdiag}\{R_1, \ldots, R_N\} \\
\{ & \begin{array}{l}
\{ & \begin{array}{l}
a_{\min} = \arg \min \ a_{ij,\tau} > 0, \tau \in [N, k^*] \\
D_H = \sum_{i=0}^{k^*-N-1} (A^{-i})^T (H^T R^{-1} H + D^T Y^{-1} D) A^{-i}.
\end{array}
\end{array}
\end{array}
\end{align*}
$$

Consider the observability matrix

$$
O_n = \sum_{i=0}^{n-1} (A^i)^T (H^T R^{-1} H + D^T Y^{-1} D) A^i = G_n \bar{R}_n^{-1} G_n,
$$

where $\bar{R}_n = I_n \otimes \text{blockdiag}(R, \mathcal{Y})$, and $G_n = (H^T, D^T, \ldots, (H A^{-1})^T (D A^{-1})^T)^T$. Under Assumption 1 it easily follows that $O_n > 0$ and $G_n$ is of column full rank. Define the matrix $F_n = A^{-n}$, and then the matrix $G_n F_n$ is still of column full rank, where $G_n F_n = (H A^{-1})^T (D A^{-1})^T + \ldots + (D A^{-1})^T$. Thus, $(F_n^T O_n F_n + D_{Hi})$, let $k^* \geq N + n$, then the positiveness of $\hat{P}_i^{-1}$ is verified. For $0 \leq k \leq k^*$, there exists a sufficiently large $P_k$ such that $0 < P_{k,i} \leq P_k$, $\forall i \in \mathcal{V}, \forall k = 0, 1, 2, \ldots, k^*$. Recall that $P_{k,i} \leq P^{*}, k > k^*, \forall i \in \mathcal{V}$. Thus, under the triggering condition (24), it is straightforward to guarantee (23).

**Corollary 2.** With the same conditions given in Theorem 7 and let $\delta_j = \delta_j, k_j \in \mathcal{V}$, if

$$
\delta < \lambda_{\min}(M_i^{-\frac{1}{2}} M_j M_i^{-\frac{1}{2}}), \forall i \in \mathcal{V},
$$

(31)

1) there exists a positive definite matrix $\hat{P}_\delta$ such that

$$
0 < P_{k,i} \leq \hat{P}_\delta < \infty, \quad \forall k \geq 0, \forall i \in \mathcal{V},
$$

(32)

2) $\hat{P}_\delta$ is non-decreasing with respect to $\delta$, i.e.,

$$
P_{k_1} \leq P_{k_2}, \delta_{k_1} \leq \delta_{k_2},
$$

(33)

with $M_i = \sum_{j \in \mathcal{V}} M_{i,j}$, where $M_{i,j}$ and $M_i$ are defined in (25) and (26), respectively.

**Proof.** The first conclusion is directly obtained from Theorem 7. Considering (29), the second conclusion can be easily proved. \( \square \)

Theorem 7 and Corollary 2 point out that the parameter matrix $P_{k,i}$ of EPDKF can be eventually upper bounded by a constant matrix under the proper design of event triggering thresholds, which guarantees the upper boundedness of the parameter matrix $P_{k,i}$. Considering the consistency of the proposed algorithm EPDKF through Theorem 6, i.e., the estimation error covariance matrix is upper bounded by $P_{k,i}$, the upper boundedness of the covariance matrix of the proposed EPDKF in Table III is proved.

**Remark 10.** Different from many existing results, the offline conditions of triggering threshold (24) and (31) are given to guarantee an upper bound of the covariance matrix of each agent. Although the upper bounds (24) and (31), which the triggering conditions depend on, are related to the network topology and overall system, through centralized design, they can be checked before the implementations of the filters by each agent.

**Remark 11.** The conditions (24) and (31) essentially provide reasonable upper bounds of the event-triggering thresholds $\delta_i, i \in \mathcal{V}$. Under collective observability conditions, $\delta_i, i \in \mathcal{V}$ cannot be set too large. Otherwise, suppose $\delta_i, i \in \mathcal{V}$ are sufficiently large. Then, agents will not communicate with each other, which may lead to instability of estimates under collective observability conditions.

**Remark 12.** We provided the covariance boundedness conclusions on the proposed EPDKF under the extended collective observability condition, which is an open problem [36, 40] due to the difficulty in connecting the triggering thresholds and the estimation performance. Additionally, by presenting the novel deterministic event-triggered scheme [20], EPDKF can keep a trade-off between the communication rate and estimation performance.

**V. NUMERICAL SIMULATION**

In this section, we consider a state constraint navigation problem of a land-based vehicle, which was widely studied in [24, 30], in order to illustrate the effectiveness of the proposed TPDKF and EPDKF. The vehicle dynamics and measurements of multiple agents (sensors) can be approximated through the following equations:

$$
\begin{align*}
x_{k+1} &= \begin{bmatrix}
1 & 0 & 0
\end{bmatrix} x_k + \omega_k,

y_{k,i} &= H_{k,i} x_k + v_{k,i}, i = 1, \ldots, N,
\end{align*}
$$

(34)

where the first two state elements of $x_k$ are the north and east positions, and the last two are the north and east velocities, $\omega_k$ is the process noise whose covariance matrix is upper bounded by $Q = \text{diag}\{4, 4, 1, 1\}$; and $v_{k,i}$ is the measurement noise of $i$th agent with the covariance matrix $R_i = 90$. The time
interval of measurements is [0, 25s] with the sampling period \( T = 0.1s \), which means \( k \in [0 : 250] \), and the initial state is \( x_0 = [0, 0, 10 \tan \theta, 10]^T \). The situation illustration is shown in Fig. 3 where the heading \( \theta \) is set as constant 60 degrees.

![Fig. 3. Situation illustration](image)

If one agent has the knowledge of the vehicle running on a road with the heading of \( \theta \) like Fig. 3 then \( \tan \theta = x_k(1)/x_k(2) = x_k(3)/x_k(4) \). The constraint can be rewritten in the form of \( D_k x_k = 0 \) with

\[
D_k = \begin{pmatrix}
1 - \tan \theta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - \tan \theta
\end{pmatrix}, \quad k = 1, 2, \ldots , \quad (35)
\]

In the following, the elements of the weighted adjacent matrix \( A \) are set as \( a_{i,j} = \frac{1}{\lvert \mathcal{N}_i \rvert}, \quad j \in \mathcal{N}_i \). The initial settings of the algorithms are

\[
\begin{align*}
\hat{x}_{0,i} &= [0, 0, 10 \tan \theta, 10]^T, \\
P_{0,i} &= \text{diag}\{90, 90, 4, 4\}, \\
\xi_i &= 0.01, \quad i = 1, \ldots , N.
\end{align*}
\quad (36)
\]

To better show the estimation performance of the proposed TPDKF, in the following, we compare the proposed TPDKF with the centralized Kalman filter (CKF) and Distributed State Estimation with Consensus on the Posteriors (DSEA-CP) [6]. CKF is the minimum-variance centralized filter for linear dynamic systems and the algorithm DSEA-CP considers the distributed filter based on consensus. To compare the three algorithms under the same communication rate, we consider both the consensus number of DSEA-CP and fusion-projection number of TPDKF to be once every other sampling moment. We conduct the numerical simulation through Monte Carlo experiment, in which 500 Monte Carlo trials for TPDKF, CKF and DSEA-CP are performed, respectively. The mean square error, averaged over all the agents, is defined as

\[
\text{MSE}_k = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{500} \sum_{j=1}^{500} (\hat{x}_{k,i}^j - x_k^j)^T (\hat{x}_{k,i}^j - x_k^j) \right], \quad (37)
\]

where \( \hat{x}_{k,i}^j \) is the state estimate of the \( j \)th trail of agent \( i \) at the \( k \)th moment. In addition, considering the proposed TPDKF, the upper bound of MSE defined above is given as \( \text{tr}(P_k) = \frac{1}{N} \sum_{i=1}^{N} \text{tr}(P_{k,i}) \).

### A. Performance Evaluation: Case 1

In this subsection, we focus on the extended collective observability condition [4] for the performance of the proposed algorithms by only considering three agents in the network shown in Fig. 4. The state constraints of the agents are assumed to be

\[
\begin{align*}
D_{k,1} &= \begin{pmatrix} 1 - \tan \theta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - \tan \theta \end{pmatrix} \\
D_{k,2} &= \begin{pmatrix} 0 & 0 & 1 - \tan \theta \\
0 & 1 & 0 \\
1 - \tan \theta & 0 & 0 \end{pmatrix} \\
D_{k,3} &= D_{k,1}, \quad dk,i = 0, \quad i = 1, 2, 3.
\end{align*}
\quad (38)
\]

The observation matrices are supposed to have the following forms

\[
\begin{align*}
H_{k,1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
\end{pmatrix} \\
H_{k,2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
H_{k,3} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\
\end{pmatrix}.
\end{align*}
\quad (39)
\]

![Fig. 4. The communication illustration of a agent network with 3 agents](image)

The MSE comparison of TPDKF, CKF and DSEA-CP with the observation matrices (39) is given in Fig. 5. It can be seen that the algorithms CKF and DSEA-CP are both divergent, but the MSE of TPDKF and the corresponding upper bound (i.e., \( \sum_{i=1}^{N} \text{tr}(P_{k,i}) \)) still remain stable. Since the collective observability condition based on (34) and (39) is not satisfied, the CKF is divergent. Yet the extended collective observability in Assumption 2 holds due to the contribution of state constraints (38). The results demonstrate that the state constraints can relax the observability condition of the algorithms and TPDKF in this paper can efficiently employ the information of state constraints.

![Fig. 5. Performance comparison among TPDKF, DSEA-CP and CKF](image)
The event triggering thresholds are set to $\delta_1 = 0.3, \delta_2 = 0.4, \delta_3 = 0.8$. By simulation, the communication rate $\lambda$ is $\lambda = 0.4077$, which effectively reduces the communication burden of the network. Fig. 6 shows the triggering instants of each agent. Fig. 7 reveals that the estimation error of EPDKF remains stable and the consistency holds. Fig. 6 illustrates that there is a periodic transmission behavior of each agent. As a result, Fig. 7 shows some periodicity of the estimation performance. Table IV shows the relationships between $\delta_i, i = 1, 2, 3$ and $\text{trace}(P_e)$, $\text{trace}(\text{MSE}_e)$.

$$\text{trace}(P_e) = 3.76e4, 3.77e3, 179.19, 141.99, 101.53,$$

$$\text{trace}(\text{MSE}_e) = 2.01e4, 2.77e3, 124.40, 78.35, 49.71.$$ 

Fig. 8 reveals the dynamic changing of estimation performance along with the communication rate $\lambda$. It can be seen that the event triggering thresholds decrease, and meanwhile, the estimation error decreases, as the communication rate increases.

$$\lambda \in [0, 1],$$ 

$$\text{trace}(P_e) = 3.76e4, 3.77e3, 179.19, 141.99, 101.53,$$

$$\text{trace}(\text{MSE}_e) = 2.01e4, 2.77e3, 124.40, 78.35, 49.71.$$ 

The average communication rate $\lambda \in [0, 1]$, defined as total number of communications averaged by moments and agent number, is utilized to evaluate the influence of event triggering thresholds to the communication cost.

Table IV shows the relationships between $\delta_i, i = 1, 2, 3$ and $\text{trace}(P_e)$, $\text{trace}(\text{MSE}_e)$.

Table IV

| $\delta_i$ | $\text{trace}(P_e)$ | $\text{trace}(\text{MSE}_e)$ |
|-----------|-------------------|-------------------|
| $\delta_1$ | 3.76e4            | 2.01e4            |
| $\delta_2$ | 3.77e3            | 2.77e3            |
| $\delta_3$ | 179.19            | 124.40            |
| $\delta_4$ | 141.99            | 78.35             |
| $\delta_5$ | 101.53            | 49.71             |

The network communication topology is illustrated in Fig. 9. It is noted that for the system (34), regarding all the observation matrices listed below, the traditional collective observability condition is satisfied.

$$\begin{align*}
H_{k,1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \\
H_{k,2} &= \begin{pmatrix} 0 & 0.3 & 0 & 0 \end{pmatrix} \\
H_{k,3} &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix},
\end{align*}$$

(40)

$$\begin{align*}
D_{k,1} &= \begin{pmatrix} 1 - \tan \theta & 0 & 0 \\
0 & 0 & 1 - \tan \theta \end{pmatrix} \\
D_{k,2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix},
\end{align*}$$

(41)

$\delta_i = \delta, i = 1, 2, 3$,

$\lambda = 0.4077$.
illustrated. Additionally, although the three algorithms are all stable, the proposed TPDKF has better estimation performance than DSEA-CP and even than CKF, which effectively shows that the state constraints in our algorithm can improve the estimation precision. From Fig. [11] it can be seen that the state estimates of 20 agents using the proposed TPDKF with different initial values can well track the stochastic dynamics of the system.

VI. CONCLUSIONS

In this paper, we investigated the problem of distributed estimation with SECs under time-based and event-triggered communication schemes, respectively. We proposed a fully distributed Kalman filter for a time-varying dynamics based on CI scheme, called TPDKF, and then proved its Gaussianity, consistency, upper boundedness of the covariance matrix and convergence, under the extended collective observability condition of the system. Also, we further showed that the traditional collective observability assumption can be relaxed with additional SECs. Additionally, we proved that the global SECs can be satisfied if the fusion-projection number goes to infinity or time goes to infinity. Moreover, we proposed a new deterministic event-triggered scheme and a distributed event-triggered filter with SECs, called EPDKF. Different from existing works, our trigger mechanism was based on a lower bound of information matrix, with building a direct connection between estimation performance and the event triggering thresholds. Furthermore, we provided an offline design principle for the triggering thresholds to keep the upper boundedness of the covariance matrix of the distributed estimation under the extended collective observability condition. In the numerical simulation, we studied the distributed tracking on a land-based vehicle, which verified the developed method and showed the effectiveness of the proposed algorithms.

APPENDIX A

PROOF OF PROPOSITION[1]

Using the matrix inverse formula on $\hat{P}_k^{l+1}$ in TPDKF for $l = L - 2$, one can obtain that $\hat{P}_k^{L-1}^{-1} = \sum_{j \in N_i} \alpha_{i,j}(\hat{P}_k^{L-2}^{-1} + \frac{1}{\epsilon_j} D_{k,j}^T D_{k,j})$. Recursively utilizing the equation for $L-1$ steps, it then follows $\hat{P}_k^{L-1}^{-1} = \sum_{j \in V} \alpha_{i,j} L^{-1} (\hat{P}_k^{L-2}^{-1} + S_L)$, where $S_L = \sum_{j=0}^{L-2} \sum_{i,j \in V} \alpha_{i,j} \epsilon_j D_{k,j}^T D_{k,j}$, $\alpha_{i,j}$ is the $(i,j)$th element of $A^s, s \in Z^+$, and $\alpha_{i,j,0} = 0, j \in N_i^0$. According to Assumption 1 and [47, 48], $a_{i,j,s} > 0$ for $s \geq N - 1$. Denote $d$ as the row rank of $D_k$, where the constraint equality $D_k x_k = d_k$ contains all the local constraints $D_{k,i} x_k = d_{k,i}, i \in V$. Then if $L \rightarrow +\infty$, the rank of $S_L$ will be $d$ and $d$ eigenvalues of $S_L$ will go to infinity. There exists an orthogonal matrix $T_1 \in R^{n \times n}$, such that $\hat{P}_k^{L-1}^{-1} = T_1^T \begin{pmatrix} I_{n-d} & 0 \\ 0 & F_L \end{pmatrix} T_1$, where $F_L \in R^{d \times d}$ and $F_L \rightarrow +\infty$ as $L \rightarrow +\infty$. Let $T_1 x = \tilde{x} = [x_1^T x_2^T]^T$, where $x_2$ corresponds to the part of $F_L$. Since $M_k^{L-1} = \hat{P}_k^{L-1}$, from Table I the projection operator is given as

$$\min_{x, z \in \hat{P}_k} \left[ (c_k^1)^T, (c_k^2)^T \right] \begin{pmatrix} I_{n-d} & 0 \\ 0 & F_L \end{pmatrix} \left[ (c_k^1)^T, (c_k^2)^T \right]^T$$

where $c_k^1 = x_1 - \tilde{x}_{k,1,1}^L, c_k^2 = x_2 - \tilde{x}_{k,2,1}^L$. Since $F_L \rightarrow +\infty$ as $L \rightarrow +\infty$ and $D_k = \bigcap_{i \in V} D_{k,i}$, the minimization of the projection operator onto $D_{k,i}$ leads to $c_k^{L2} \rightarrow 0$ as $L \rightarrow +\infty$. Therefore, the conclusion in [4] holds.

APPENDIX B

PROOF OF THEOREM[1]

According to TPDKF in Table I let $l \in \{0 : L-1\}$, then

$$\hat{e}_{k,i}^{l+1} = \tilde{x}_{k,i}^{l+1} - x_k$$

$$= \tilde{x}_{k,i} - \hat{P}_{k,i}^l D_{k,i}^T (D_{k,i} \hat{P}_{k,i}^l D_{k,i}^T)^{-1} (D_{k,i} \tilde{x}_{k,i} - d_{k,i}) - x_k$$

$$= \left[ I_n - \hat{P}_{k,i}^l D_{k,i}^T (D_{k,i} \hat{P}_{k,i}^l D_{k,i}^T)^{-1} D_{k,i} \right] \hat{e}_{k,i}^l,$$ (42) where the estimation error of the local fusion $\hat{e}_{k,i}$ satisfies

$$\hat{e}_{k,i}^l = \tilde{x}_{k,i} - x_k = \hat{P}_{k,i}^l \sum_{j \in N_i} \alpha_{i,j} (\hat{P}_{k,j}^l)^{-1} e_{k,j}^l,$$ (43)

The error of the measurement update $\hat{e}_{k,i} = \tilde{x}_{k,i} - x_k$ can be derived through

$$\hat{e}_{k,i} = \phi_{k,i} - x_k = (I - K_{k,i} H_{k,i}) \hat{e}_{k,i} + K_{k,i} u_{k,i}.$$ (44)

Additionally, the prediction error follows from

$$\hat{e}_{k,i} = \tilde{x}_{k,i} - x_k = A_{k-1} \tilde{e}_{k-1,i}^l + \omega_{k-1}.$$ (45)
Thus, the conclusion follows.

It is straightforward to see $E(\hat{x}_{0,i} - x_0) = \hat{x}_{0,i} - x_0$ are Gaussian by employing the inductive method and the property of Gaussian distribution.

**APPENDIX C**

**PROOF OF THEOREM [2]**

We use the inductive method to show the consistency of the pair $(\hat{x}_{k,i}, P_{k,i})$.

At the initial moment, given $\theta_i > 0$,

$$E\{(\hat{x}_{0,i} - x_0)(\hat{x}_{0,i} - x_0)^T\}$$

$$\leq \frac{1}{\theta_i}(\hat{x}_{0,i} - E\{x_0\})^T(\hat{x}_{0,i} - E\{x_0\}) + (1 + \theta_i)E\{(x_0 - E\{x_0\})(x_0 - E\{x_0\})^T\}$$

$$\leq \frac{1}{\theta_i}P_{0,i} + (1 + \theta_i)P_0,$$

where $E\{(\hat{x}_{0,i} - x_0)(\hat{x}_{0,i} - x_0)^T\} \leq P_{0,i}$ and $E\{(x_0 - E\{x_0\})(x_0 - E\{x_0\})^T\} \leq P_0$.

Let us assume that, at the $k$th moment, the pair $(\hat{x}_{k-1,i}, P_{k-1,i})$.

$$E\{(\hat{x}_{k-1,i} - x_{k-1})(\hat{x}_{k-1,i} - x_{k-1})^T\} \leq P_{k-1,i},$$

For the prediction estimation error at the $k$th moment, from (45), $e_{k,i} = A_{k-1}e_{k-1,i} + \omega_{k-1}$. As $E(\omega_{k-1}\omega_{k-1}^T) \leq Q_{k-1}$, one can obtain $E(e_{k,i}^2) = \frac{1}{\theta_i}(A_{k-1} - I)A_{k-1}^T + Q_{k-1} \leq P_{k,i}$. According to (44), $e_{k,i} = (I - K_{k,i}H_{k,i})e_{k,i} + K_{k,i}v_k$, since $E\{v_k, v_k^T\} \leq R_{k,i}$, it then follows that

$$E\{e_{k,i}e_{k,i}^T\} \leq (I - K_{k,i}H_{k,i})E(e_{k,i}e_{k,i}^T)(I - K_{k,i}H_{k,i})^T + K_{k,i}R_{k,i}K_{k,i}^T$$

$$\leq (I - K_{k,i}H_{k,i})P_{k,i}(I - K_{k,i}H_{k,i})^T + K_{k,i}R_{k,i}K_{k,i}^T$$

$$\leq P_{k,i}.$$ (47)

From (43), (47) and the consistent estimation of CI strategy (51), one can obtain that

$$E\{e_{k,i}^T e_{k,i}\} \leq P_{k,i}^l, l = 0, 1, \ldots, L - 1.$$ (48)

From (42), the estimation error satisfies $e_{k,i}^{l+1} = P_{k,i}^D e_{k,i}^l$, where

$$P_{k,i}^D = I_n - \hat{P}_{k,i}^D \hat{D}_{k,i}^T (D_{k,i} \hat{P}_{k,i}^D \hat{D}_{k,i}^T)^{-1} D_{k,i}.$$ Consider

$$E\{e_{k,i}^{l+1} e_{k,i}^{l+1\dagger}\} = P_{k,i}^D E\{e_{k,i}^l e_{k,i}^l\} (P_{k,i}^D)^T$$

$$\leq P_{k,i}^D E\{e_{k,i}^l e_{k,i}^l\} (P_{k,i}^D)^T$$

$$= P_{k,i}^D e_{k,i}^l e_{k,i}^T = P_{k,i}^l.$$ (49)

Then $E\{e_{k,i}^l e_{k,i}^T\} \leq P_{k,i}^l$, which is straightforward to see $P_{k,i}^l \leq P_{k,i}^{l+1}$. Also, it is straightforward to see $P_{k,i}^{l+1} \leq P_{k,i}^{l+2}$. This, the conclusion follows.

**APPENDIX D**

**PROOF OF THEOREM [5]**

According to EPDKF in Table III

$$e_{k,i} = \hat{x}_{k,i} - x_k$$

$$\tilde{e}_{k,i} = \hat{x}_{k,i} - x_k$$

$$= \tilde{P}_{k,i} (a_{i,i} \tilde{P}_{k,i}^{-1} e_{k,i} + \sum_{j \in N_0} a_{i,j} (\tilde{P}_{k,j}^{-1} e_{k,j})^T)$$

(50)

where

$$\tilde{e}_{k,i} = \hat{x}_{k,i} - x_k = A^{k-t} \hat{x}_{t,i} - A^{k-t} \hat{x}_t - \sum_{l=t}^{k-1} A^{k-1-l} w_l$$

$$A^{k-t} \hat{x}_{t,i} - A^{k-t} \hat{x}_t - \sum_{l=t}^{k-1} A^{k-1-l} w_l.$$ (51)

The error of the measurement update $\tilde{e}_{k,i}$ can be derived through

$$\tilde{e}_{k,i} = \hat{x}_{k,i} - x_k = (I - K_{k,i}H_i)e_{k,i} + K_{k,i}v_k.$$ (52)

Additionally, the prediction error follows from

$$\tilde{e}_{k,i} = \hat{x}_{k,i} - x_k = A e_{k-1,i} + \omega_{k-1}.$$ (53)

Based on (49) – (53), we obtain the iteration equation of estimation error as follows

$$e_{k,i} = P_{k,i} \tilde{P}_{k,i} (a_{i,i} \tilde{P}_{k,i}^{-1} e_{k,i} + \sum_{j \in N_0} a_{i,j} (\tilde{P}_{k,j}^{-1} e_{k,j})^T)$$

(54)

where

$$P_{k,i}^D = [I - \tilde{P}_{k,i} D_{k,i}^T (D_{k,i} \tilde{P}_{k,i} D_{k,i}^T)^{-1}]$$

$$\tilde{e}_{k,i} = (I - K_{k,i}H_i)(A e_{k-1,i} + \omega_{k-1}) + K_{k,i}v_k$$

$$= A^{k-t} (I - K_{t,j}H_j)(A e_{t,j} + \omega_{t-1}) + K_{t,j}v_{t,j}$$

$$- \sum_{l=t}^{k-1} A^{k-1-l} w_l.$$ (51)

Since $e_{k-1,i}$, $v_k$, and $e_0,i = \hat{x}_{0,i} - x_0$ are Gaussian, $e_{k,i}$ is also Gaussian by employing the inductive method and the property of Gaussian distribution acting on (54).

**APPENDIX E**

**PROOF OF THEOREM [6]**

The proof of Theorem 6 has similar steps as the proof of Theorem 2 hence we only consider the fusion part for convenience. Considering the event triggered scheme, agent $i$ can obtain the prediction error of agent $j$, $j \in N_i$, based on the received latest $\tilde{e}_{t,j}$, then $e_{k,j}^{l+1} = A^{k-t} \tilde{e}_{t,j} - \sum_{l=t}^{k-1} A^{k-1-l} w_l, t \leq k$. Then considering $E\{\tilde{e}_{t,j} w_l^T\} = 0, l \geq t$, from (22) and (47),

$$E\{e_{k,j}^{l+1} e_{k,j}^{l+1\dagger}\}$$

$$\leq A^{k-t} E\{\tilde{e}_{t,j} \tilde{e}_{t,j}^T\} A^{k-t} + \sum_{l=t}^{k-1} A^{k-1-l} Q(\bar{A}^{k-1-l})^T$$

$$\leq A^{k-t} E\{\tilde{e}_{t,j} \tilde{e}_{t,j}^T\} A^{k-t} + \sum_{l=t}^{k-1} A^{k-1-l} Q(\bar{A}^{k-1-l})^T$$
\[ \leq A^{k-t} \tilde{P}_{t,j}(A^{k-t})^T + \sum_{l=t}^{k-1} A^{k-l-1}Q(A^{k-l-1})^T \]
\[ = \tilde{P}_{t,j}. \]  
(55)

From (50) and the proposed EPDKF in Table III,
\[ \tilde{e}_{k,i} = \tilde{P}_{k,i} \left( a_{i,j} \tilde{P}_{k,j}^{-1} \tilde{e}_{k,j} + \sum_{j \in \mathcal{N}_{i}} a_{i,j} ( ˜P_{k,j}^{-1})^{-1} \varepsilon_{k,j} \right). \]  
(56)

where \( \tilde{P}_{k,i} = \left( a_{i,i} \tilde{P}_{k,i}^{-1} + \sum_{j \in \mathcal{N}_{i}} a_{i,j} ( ˜P_{k,j}^{-1})^{-1} \right)^{-1} \). According to the consistent estimation of CI strategy (51), we have
\[ E\{\tilde{e}_{k,i} \tilde{e}_{k,j}^T\} \leq P_{k,i}. \]

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