A RIEMANN-HURWITZ-PLÜCKER FORMULA

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Abstract. We prove a simultaneous generalization of the classical Riemann-Hurwitz and Plücker formulas, addressing the total ramification of a morphism from a (smooth, projective) curve to an arbitrary (smooth, projective) higher-dimensional variety. Our definition of ramification is relative to an algebraic family of divisors on the target variety, and our formula is obtained using the theory of refined top Chern classes. In assigning multiplicities to ramification points, we frequently have to consider excess degeneracy loci, but we are able to show nonetheless that the multiplicities are always nonnegative, and are positive under very mild hypotheses.

1. Introduction

We work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. The classical Riemann-Hurwitz and Plücker formulas describe the total amount of ramification of a morphism between (smooth, projective) curves, and of a morphism from a curve to projective space, respectively (see for instance [EH83, Proposition 1.1] for the latter). These are both of fundamental importance to the study of algebraic curves, and they overlap in the case of a morphism from a curve to the projective line, where they give the same formula. The aim of this paper is to give a simultaneous generalization of both formulas, to treat morphisms from a curve $C$ to an arbitrary variety $X$ (still assumed smooth and projective). We hope that this will constitute a useful new tool in the study of curves on higher-dimensional varieties.

There does not appear to be any intrinsic notion of ramification of such a morphism, so our approach is to define ramification relative to a family of divisors on $X$. Accordingly, we work throughout in the following situation:

Situation 1.1. Let $X$ be a smooth, projective variety and $S$ a projective variety of dimension $n$, and $\mathcal{D} \subseteq X \times S$ a flat family of divisors on $X$.

Definition 1.2. If we are given a smooth, projective curve $C$, and a morphism $f : C \to X$, we say that a $\mathbb{K}$-point $p \in C$ is a ramification point relative to $\mathcal{D}$ if there exists a $z \in S$ with corresponding divisor $D = D_z \subseteq X$ such that $f^*D$ has multiplicity at least $n + 1$ at $p$.

Here we allow the possibility that $f(C) \subseteq D$ for some $D$ in $\mathcal{D}$, in which case our convention is that every point of $C$ is considered a ramification point. Thus, this situation will not be interesting for us.

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Although this setup depends on the extra data of the divisor family $D$, in practice it can often be made canonical, or dependent only on standard data. By taking $S$ to be a connected component of the Hilbert scheme of $X$ (provided it is irreducible), we can reduce the extra data to an element of the Neron-Severi group. If the Neron-Severi group of $X$ is cyclic and $X$ is projective, this element can be chosen canonically as the ample generator. This is precisely what happens in the case of the Riemann-Hurwitz and Plücker formulas. Alternatively, it is also natural to consider principally polarized abelian varieties, where our divisor family is determined by the polarization. We examine this case in §5.

Our formula for the total ramification of such a morphism will be expressed in terms of the following quantities.

**Definition 1.3.** For a natural number $m$, let

$$D^m_S = D \times_S D \times_S \cdots \times_S D \subseteq X^m \times S.$$ 

Let $\text{proj}_m : X^m \times S \to X^m$ be the projection morphism, and $\delta_m : X \to X^m$ the small diagonal. Then let

$$h_0(X, D) = \delta^*_m \text{proj}_{n+1} [D^m_S] \in A^0(X),$$

and

$$h_1(X, D) = \delta^*_m \text{proj}_{n+1} [D^{m+1}_S] \in A^1(X).$$

Here $[D^m_S]$ denotes the Chow class associated to the scheme $D^m_S$, which is pure of dimension $m(\dim X - 1) + n$. Above and throughout, we use $A^k(X)$ as shorthand for $A_{\dim X-k}(X)$. In addition, $\delta^*_m$ is the Gysin construction for regular imbeddings; see Example 5.2.1 of [Fu98].

Note that $h_0(X, D)$ is an effective codimension-0 class in $X$, which we can then canonically identify with a nonnegative integer, and interpret as the number of divisors in $D$ containing $n$ general points in $X$. On the other hand, $h_1(X, D)$ is a codimension-1 class, hence an element of $\text{Pic}(X)$, which is essentially the pullback to $X$ under the small diagonal immersion of the divisor consisting of $(n + 1)$-tuples of points on $X$ contained in some divisor in $D$.

Our main theorem is then the following.

**Theorem 1.4 (Main Theorem).** In Situation 1.1, if we are given any smooth proper curve $C$, and any morphism $f : C \to X$ such that the set $R \subseteq C$ of ramification points relative to $D$ is finite, then it is possible to assign nonnegative integer multiplicities $m_p$ to all $p \in R$ such that

$$\sum_{p \in R} m_p \sim f^*h_1(X, D) + \left(\frac{n+1}{2}\right)h_0(X, D)K_C,$$

where $\sim$ denotes linear/rational equivalence.

Moreover, if $D$ has the property that every divisor occurs at most finitely many times in the family, then all the $m_p$ are strictly positive.

The quantities $m_p$ will be constructed naturally using refined top Chern classes of a section of a vector bundle, and we will show in Corollary 4.9 below a more specific version of positivity. We will also show:
**Theorem 1.5.** The righthand side of (1) specializes to the classical Riemann-Hurwitz and Plücker formulas in the special cases that $X$ is a curve, or that $X$ is a projective space, respectively. Moreover, our general construction of $m_p$ recovers the usual multiplicities occurring in these formulas.

Finally, our main theorem only applies when a morphism has finitely many ramification points, so it is a natural to analyze when one can guarantee that this will be the case. In fact, it turns out to be possible for a particular $D$ to have that every morphism $f : C \to X$ is everywhere ramified with respect to $D$. In this case, our formula still provides a 'virtual' number of ramification points, but this number is never realized! In §6, we discuss these issues, and in particular we apply a theorem of Ein, Mustaţă and Yasuda to show that provided the divisors $D_z$ aren’t too singular for any $z \in S$, there will always be morphisms $f$ which are not everywhere ramified with respect to $D$.

Our fundamental technique throughout is to study the ramification in terms of a section of a suitable rank-$(n + 1)$ jet bundle on $C \times S$. We obtain the righthand side of (1) by computing the top Chern class of this bundle, and we use the notion of refined Chern classes relative to a section in order to define the multiplicities. What makes this rather delicate is that the zero set of the section in question is frequently positive-dimensional, so that for instance positivity of the multiplicities is not automatic. Indeed, this happens already in the situation of the classical Plücker formula. We are therefore forced to make direct use of the construction of refined Chern classes given in Lemma 3.2 of [Fu98], taking advantage of the usual filtration of jet bundles to obtain some control over the geometry.

**Remark 1.6.** Our construction enjoys various functoriality properties, and in particular if $D$ happens to be a base-point free linear series on $X$, then considering the composed morphism $C \to X \to \mathbb{P}^n$, our formula will simply recover the usual Plücker formula. From this point of view, our formula is most interesting in cases that $D$ varies non-trivially in an algebraic equivalence class. However, considering nonlinear subfamilies of linear series can also lead to instructive examples; see for instance Example 6.8 below.

**Conventions.** All varieties (and in particular curves) are assumed irreducible, and in particular connected.

## 2. Preliminaries

Before getting started on the actual proof of Theorem 1.4, we record for future use a version of the fact that pushforwards and pullbacks commute in intersection theory, granted that certain conditions hold. We haven’t strived for generality – on the contrary, this is an exercise in specializing results from Fulton’s book [Fu98].

**Lemma 2.1.** Let $X_1, X_2, X_3, X_4$ be varieties over $\mathbb{K}$ and consider a diagram

\[
\begin{array}{ccc}
X_4 & \xrightarrow{\xi_{43}} & X_3 \\
\downarrow{\xi_{42}} & & \downarrow{\xi_{31}} \\
X_2 & \xrightarrow{\xi_{21}} & X_1
\end{array}
\]

Assume that

1. the diagram is cartesian;
(2) \(\xi_{31}\) is proper and hence so is \(\xi_{42}\);
(3) \(\xi_{21}\) and \(\xi_{43}\) are lci morphisms;
(4) the alternating sum of dimensions vanishes, i.e.
\[
\dim X_1 + \dim X_4 = \dim X_2 + \dim X_3.
\]

Then \(\xi_{21} \circ \xi_{31,*} = \xi_{42,*} \circ \xi_{43,*}\) as group homomorphisms \(A_*(X_3) \to A_*(X_2)\).

**Proof.** We ought to clarify that \(\xi_{21,*}\) and \(\xi_{43,*}\) are unrefined Gysin maps, cf. condition (3). The key observation, which we leave to the reader, is that the excess normal bundle [Fu98, page 113] of our cartesian square has rank
\[
\dim X_1 - \dim X_2 - \dim X_3 + \dim X_4 = 0
\]
thanks to condition (4) above. Then the general form of the excess intersection formula [Fu98, Proposition 6.6.(c), so see also Theorem 6.3] shows that
\[
\xi_{43,*} : A_*(X_3) \to A_*(X_4)
\]
coincides with \(\xi_{21,*} : A_*(X_3) \to A_*(X_4)\).

When invoking the excess intersection formula, the reader should imagine a second (trivial) cartesian square on top of ours. Thus we’ve boiled down the claim to
\[
\xi_{21} \circ \xi_{31,*} = \xi_{42,*} \circ \xi_{21,*},
\]
which is [Fu98, Proposition 6.6.(c), so see also Theorem 6.2.(a)] if we draw another trivial cartesian square, this time under ours. \(\Box\)

**Remark 2.2.** Conditions (1), (2) and (4) will be trivial to check in practice. Regarding condition (3), the following easy criterion for lci-ness will suffice for our purposes: if \(X, Y\) are varieties over \(K\) which are also smooth \(S\)-schemes, then any \(S\)-morphism \(X \to Y\) is lci. We briefly justify this assertion as follows. If the map \(X \to Y\) happens to be a closed immersion, then it is necessarily a regular closed immersion. In general, factor the map as \(X \to X \times_S Y \to Y\) via the “graph map” and note that the former map is a regular closed immersion by the previous sentence, while the latter map is smooth, so \(X \to Y\) is an lci morphism, as desired.

We will also make extensive use of relative jet bundles, so we take the opportunity to recall a couple of standard facts about them. Note that formation of jet bundles does not commute with pullback in general; however, relative jet bundles do behave well with respect to base change.

**Proposition 2.3.** Let \(X\) be an \(S\)-scheme, and \(\mathcal{F}\) a coherent sheaf on \(X\), and \(n \geq 0\). Then

1. for all morphisms \(S' \to S\), if \(X' = X \times_S S'\), and \(f : X' \to X\) is the induced morphism, then we have
   \[
   J^n_{X'/S}(f^*\mathcal{F}) = f^*J^n_{X/S}(\mathcal{F});
   \]
2. for all line bundles \(\mathcal{M}\) on \(S\), if \(\pi : X \to S\) is the structure morphism, we have
   \[
   J^n_{X/S}(\mathcal{F} \otimes \pi^*\mathcal{M}) = \left(J^n_{X/S}(\mathcal{F})\right) \otimes \pi^*\mathcal{M}.
   \]

**Proof.** For (i), see Proposition 16.4.5 of [GD67]. (ii) is an immediate consequence of the definition of jet bundles and the projection formula for tensoring with a locally free sheaf. \(\Box\)
3. A Chern class computation

In this section, we will compute the top Chern class of a jet bundle on \( C \times S \), recovering the righthand side of (1). We also motivate this computation by observing that the set of ramification points can be studied via the zero set of a section of this jet bundle.

We begin with an analysis of ramification in a family. For this, we work with a straightforward extension of the setup given in the introduction. This is not done so much for the sake of generality, but rather for clarity.

**Situation 3.1.** Let \( S \) be a finite type scheme over \( K \) of pure dimension \( n \). Let \( \mathcal{X} \) be a smooth projective family of varieties over \( S \) and \( D \subseteq \mathcal{X} \) a family of divisors, flat over \( S \). Finally, let \( C \) be a smooth projective family of curves over \( S \) and \( f : C \rightarrow \mathcal{X} \) an \( S \)-morphism.

**Definition 3.2.** In Situation 3.1, a **ramification point** is a \( K \)-point \( r \in C \), with image \( z \in S \), such that the \( n \)th order thickening of \( r \) inside its fiber \( C_z \subseteq C \) is contained inside \( f^{-1}(D) \subseteq C \).

Our immediate task is to clarify the scheme structure on the set of ramification points, and show that it comes equipped with a natural \( 0 \)-class. In fact, we will exhibit the set of ramification points as the vanishing locus of a section of a rank- \((n+1)\) vector bundle, which settles both issues at once thanks to the theory of refined Chern classes. This will also give a formula for the constructed virtual class. The key observation is the following.

**Proposition 3.3.** Let \((\mathcal{L}, s)\) be a pair of a line bundle on \( \mathcal{X} \) together with a global section whose vanishing locus is precisely \( D \). Let \( J^n_{C/S} f^* \mathcal{L} \) be the relative jet bundle of \( f^* \mathcal{L} \) over \( S \), and let \( \tilde{s} \in \Gamma(\mathcal{X}, J^n_{C/S} f^* \mathcal{L}) \) be the section induced by \( f^*s \).

Then the set of ramification points is precisely the (set of \( K \)-points of the) zero locus of \( \tilde{s} \), and

\[
(2) \quad c_{n+1}(J^n_{C/S} f^* \mathcal{L}) = \prod_{k=0}^{n} (f^*[D] + kc_1(\omega_{C/S})).
\]

**Proof.** The first statement is clear from the definitions: since we are taking a relative jet bundle, the construction commutes with restriction to a fiber over any \( z \in S \), and then we have that \( f^*s \) vanishes precisely on \( f^{-1}D_z \), so that \( \tilde{s} \) vanishes precisely on those \( p \in C \) for which \( f^{-1}D_z \) contains an \((n+1)\)st order thickening of \( p \), as desired.

For the second, we use the well-known short exact sequence

\[
0 \longrightarrow f^* \mathcal{L} \otimes \omega_{C/S}^k \longrightarrow J^k_{C/S} f^* \mathcal{L} \longrightarrow J^{k-1}_{C/S} f^* \mathcal{L} \longrightarrow 0
\]

which holds for each \( k \). Applying this inductively, we obtain

\[
(4) \quad J^n_{C/S} f^* \mathcal{L} \sim f^* \mathcal{L} \otimes \bigoplus_{k=0}^{n} \omega_{C/S}^k
\]

in \( K \)-theory, hence the two sides have equal Chern classes.

Clearly, \( c_1(\omega_{C/S}^k) = kc_1(\omega_{C/S}) \), so the Whitney formula implies that

\[
(5) \quad c_j \left( \bigoplus_{k=0}^{n} \omega_{C/S}^k \right) = e_j(0, 1, ..., n)c_1(\omega_{C/S})^j,
\]
where the $e_j$ denote the elementary symmetric polynomials. Then (2) follows from (4), (5), and the well-known Lemma 3.4 recalled below, whose proof is a trivial application of the splitting principle.

Lemma 3.4. If $\mathcal{L}$ is a line bundle on a (possibly singular) variety $X$ and $\mathcal{E}$ is a vector bundle of rank $r$ on $X$, then

$$c_r(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^{r} c_1(\mathcal{L})^j c_{r-j}(\mathcal{E}).$$

Returning to the more specific setup of Situation 1.1, we have a flat family of divisors $D \subseteq X \times S$. We will now analyze the classes $h_0(X, D)$ and $h_1(X, D)$ more closely.

Proposition 3.5. In Situation 1.1, for any $m \geq n$, we have the identity

$$\delta_m^* \text{proj}_{m,*}[D_S^m] = \text{proj}_{1,*}([D]^m).$$

in $A^{m-n}(X)$.

Note that the $m$th power on the right-hand side is intersection product; this makes sense even when $S$ is singular because we are intersecting Cartier divisor classes.

Proof. Let $\pi_j : X^m \to X$ be the projection to the $j$th factor, $j \leq m$. Then

$$[D_S^m] = \prod_{j=1}^{m} (\pi_j \times 1_S)^* [D] \in A^m(X^m \times S),$$

using that $D_S^m$ is flat over $S$ and $X$ is smooth to conclude that the intersection of schemes commutes with passing to Chow classes. Thus, if $\delta_m : X \to X^m$ is the small diagonal immersion, we have

$$(\delta_m \times 1_S)^* [D_S^m] = [D]^m.$$

Lemma 2.1 and Remark 2.2 yield the commutativity of pushforward and pullback in the cartesian diagram

$$\begin{array}{ccc}
X \times S & \xrightarrow{\delta_m \times 1_S} & X^m \times S \\
| & & | \\
X & \xrightarrow{\delta_m} & X^m,
\end{array}$$

and the desired identity then follows from (6). □

We are now ready to complete our Chern class calculation. Of course, Proposition 3.3 and the subsequent definitions will be applied to $\mathcal{X} = X \times S$, $\mathcal{C} = C \times S$ etc. For simplicity, we write $f_S = f \times 1_S$, the map we were calling “$f$” in Situation 3.1.

Corollary 3.6. In Situation 1.1, with notation as in Proposition 3.3, we have

$$\text{proj}_{C \times S/C,*} e_{n+1}(J_{C \times S/S} f^* \mathcal{L}) = f^* h_1(X, \mathcal{D}) + \binom{n+1}{2} h_0(X, \mathcal{D}) K_C$$

in $A_0(C)$. 

Here, technically we should consider $c_{n+1}(J^n_{C \times S/S} f^* \mathcal{L})$ as an operator on cycles, and apply it to the fundamental class of $C \times S$ in order to obtain a 0-cycle.

**Proof.** Since $C = C \times S$, we have

$$c_1(\omega_{C/S}) = \text{proj}_{C \times S/C}^* K_C \text{ and } c_1(\omega_{C/S})^2 = 0,$$

so formula (2) becomes

$$c_{n+1}(J^n_{C \times S/S} f^* \mathcal{L}) = (f^n_S[\mathcal{D}])^{n+1} + \binom{n+1}{2} (f^n_S[\mathcal{D}])^n \cdot \left( \text{proj}_{C \times S/C}^* K_C \right).$$

Hence, by the projection formula, if we push forward to $C$ we obtain

$$\text{proj}_{C \times S/C}^* c_{n+1}(J^n_{C \times S/S} f^* \mathcal{L}) = \text{proj}_C^* (f^n_S[\mathcal{D}])^{n+1} + \binom{n+1}{2} (f^n_S[\mathcal{D}])^n \cdot \left( \text{proj}_{C \times S/C}^* K_C \right).$$

On the other hand, Lemma 2.1 and Remark 2.2 give us commutativity of pushforward and pullback in the cartesian diagram

$$\begin{array}{ccc}
C \times S & \xrightarrow{f_S} & X \times S \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & X,
\end{array}$$

so Proposition 3.5 gives us that

$$\text{proj}_{C \times S/C, *}(f^n_S[\mathcal{D}])^{n+1} = \text{proj}_{X \times S/X, *}(f^n_S[\mathcal{D}])^n = f^* h_{m-n}(X, \mathcal{D}/S)$$

for $m = n, n+1$. Hence the previous formula becomes

$$\text{(7) } \text{proj}_{C \times S/C, *} c_{n+1}(J^n_{C \times S/S} f^* \mathcal{L}) = f^* h_1(X, \mathcal{D}/S) + \binom{n+1}{2} f^* h_0(X, \mathcal{D}/S) K_C.$$

Since $h_0(X, \mathcal{D}/S)$ is essentially an integer, we can drop the $f^*$ preceding it, and we recover the desired formula. \hfill \Box

### 4. Multiplicities

All that remains to do is to assign the integer multiplicities $m_p$ to the ramification points, and prove the desired nonnegativity and positivity statements.

In light of Proposition 3.3, we make the following definitions.

**Definition 4.1.** In the situation of Proposition 3.3, let $\mathcal{R}$ be the zero locus of $\mathfrak{s}$, and define $[\mathcal{R}]^\text{vir} \in A_0(\mathcal{R})$ to be the refined top Chern class of $\mathfrak{s} \in \Gamma(\mathcal{X}, J^n_{C \times S/S} f^* \mathcal{L})$ (applied to the fundamental class of $\mathcal{C}$).

Note that it follows from the theory of refined Chern classes that if $\iota$ denotes the immersion of $\mathcal{R}$ into $\mathcal{C}$, then

$$\iota_* [\mathcal{R}]^\text{vir} = c_{n+1}(J^n_{C \times S/S} f^* \mathcal{L})$$

in $A_0(\mathcal{C})$. 

A pleasant feature of the situation at hand is that, just under the assumption that the number of ramification points on $C$ is finite, the potentially virtual class $[\mathcal{R}]^{\text{vir}}$ gives an actual (non-virtual) class on $C$. The key observation is that, since the projection $C \times S \to C$ sends $\mathcal{R}$ to a finite set of points $R$, the subscheme $\mathcal{R}$ must be a disjoint union

$$\mathcal{R} = \bigsqcup_{p \in R} \mathcal{R}_p$$

such that $\mathcal{R}_p$ is contained set-theoretically in $\{p\} \times S$. Hence there exist uniquely determined $[\mathcal{R}_p]^{\text{vir}} \in A_0(\mathcal{R}_p)$ such that $[\mathcal{R}]^{\text{vir}} = \sum_{p \in R} [\mathcal{R}_p]^{\text{vir}} \in A_0(\mathcal{R})$, where the new classes are implicitly pushed forward to $\mathcal{R}$. We can therefore make the following definition.

**Definition 4.2.** In the situation of Theorem 1.4, and with notation as in Definition 4.1 and as above, for each $p \in R$ set

$$m_p = \deg [\mathcal{R}_p]^{\text{vir}} \in \mathbb{Z}.$$  

Then

$$\sum_{p \in R} m_pp = \sum_{p \in R} \deg [\mathcal{R}_p]^{\text{vir}} p = \text{proj}_{\mathcal{R}/C, *} [\mathcal{R}]^{\text{vir}}$$

in $A_0(C)$ and (1) follows from Corollary 3.6. It thus remains to show the desired nonnegativity/positivity statements on the $m_p$.

We first recall the construction of a refined top Chern class from Lemma 3.2 of [Fu98]. The general situation will be the following:

**Situation 4.3.** Let $X$ be a scheme, $\mathcal{E}$ a vector bundle of rank $r$ on $X$, $s \in \Gamma(X, \mathcal{E})$ a section, and $Y \subseteq X$ the zero set of $X$. Assume further that $\mathcal{E}$ has a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_r = \mathcal{E},$$

so that $\mathcal{L}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is a line bundle for $1 \leq i \leq r$.

Then for any $\alpha \in A_k(X)$, Fulton constructs a class $\alpha' \in A_{k-r}(Y)$ such that the pushforward of $\alpha'$ to $X$ is equal to $c_r(\mathcal{E}) \cdot \alpha$ in $A_{k-r}(X)$ (see Lemma 3.2 and (*) in the proof of Theorem 3.2 of [Fu98]). The construction proceeds inductively as follows: first, let $\sum a_i Z_i$ be a representative of $\alpha$, and observe that $s$ induces a section $s_r$ of $\mathcal{L}_r$. For each $i$, if $Z_i \subsetneq Z(s_r)$, then $s_r|Z_i \in \Gamma(Z_i, \mathcal{L}_r|Z_i)$ induces a Cartier divisor, whose associated Weil divisor is a $(k-1)$-cycle supported by construction on $Z(s_r)$. On the other hand, if $Z_i \subsetneq Z(s_r)$, then choosing any (not necessarily effective) Cartier divisor representing $\mathcal{L}_r|Z_i$, and taking its associated Weil divisor, we again obtain a $(k-1)$-cycle supported on $Z(s_r)$. Summing over all $i$, we obtain an element of $A_{k-1}(Z(s_r))$. Now, on $Z(s_r)$, the section $s$ induces a section $s'_r \in \Gamma(Z(s_r), \mathcal{E}_{r-1}|Z(s_r))$, and $Y = Z(s) = Z(s'_r) \subseteq Z(s_r)$, so we replace $X$ by $Z(s_r)$, $\mathcal{E}$ by $\mathcal{E}_{r-1}$, and $s$ by $s'_r$, and iterate the procedure until we obtain the desired $\alpha' \in A_{k-r}(Y)$.

In our case, we will assume that the ambient scheme is a variety of dimension equal to the rank of $\mathcal{E}$, and we will set $\alpha$ to be the fundamental class, so that we will canonically obtain a class in $A_0(Y)$. We will be interested in studying the degree of this class on each connected component $Y'$ of $Y$, and an important observation is that for this purpose, we can work up to algebraic equivalence rather than rational equivalence. Another important observation is that we can obtain effectivity and positivity results from the properties of $\mathcal{L}_i|_{Y'}$. Our basic lemma is the following.
Lemma 4.4. In Situation 4.3, with $X$ a variety of dimension $d$, and $\alpha = [X]$, the associated refined top Chern class $\beta \in A_{d-r}(Y)$ can be represented as follows: there exist irreducible closed subsets $Y_1, \ldots, Y_m$ of $Y$, positive integers $a_1, \ldots, a_m$, and subsets $S_1, \ldots, S_m \subseteq \{1, \ldots, r\}$, such that $r = |S_j| + \dim X - \dim Y_j$ for each $j$, and

$$\beta = \sum_{j=1}^{m} i_j \left( a_j \prod_{i \in S_j} [\mathcal{L}_i|_{Y_j}] \right),$$

where $i_j : Y_j \to Y$ is the inclusion.

More formally, $\prod_{i \in S_j} [\mathcal{L}_i|_{Y_j}]$ denotes the class on $Y_j$ obtained by iteratively applying $c_1(\mathcal{L}_i|_{Y_j})$ to the fundamental class of $Y_j$, or equivalently, starting with (the Weil divisor associated to) a Cartier divisor representing $\mathcal{L}_i|_{Y_j}$ for the maximal $i \in S_j$, and then iteratively taking the product with the pseudodivisors $(\mathcal{L}_i|_{Y_j}, Y_j, 0)$ for each of the smaller $i \in S_j$ (see §5.2.2–2.5 of [Fu98]).

Proof. The proof is by induction on $r$, noting that after the $k$th step of the construction described above, what we have done is exactly the construction for the refined top Chern class of $S_r$ supported on the zero set of the section induced by $s$. The base case $r = 0$ is trivial. Now suppose we have the lemma for $r - 1$, and let $s' = S_r$, $s'$ the section of $s'$ induced by $s$, $Y' = Z(s')$, and $\beta' \in A_{d-(r-1)}(Y')$ the associated refined class. Let $s_1 \in \Gamma(Y', \mathcal{E}_1|_{Y'})$ be the section induced by $s$ (note that $\mathcal{L}_1 = \mathcal{E}_1$). By the inductive hypothesis, we have some $Y_1', \ldots, Y_{m'}', a_1', \ldots, a_{m'}'$, and $S_1', \ldots, S_{m'}' \subseteq \{2, \ldots, r\}$ such that $r - 1 = |S_j| + \dim X - \dim Y_j'$ for each $j$, and

$$\beta' = \sum_{j=1}^{m'} i'_j \left( a'_j \prod_{i \in S'_j} [\mathcal{L}_i|_{Y'_j}] \right).$$

By definition, we have

$$\beta = (\mathcal{L}_1|_{Y'}, Y', s_1) : \beta',$$

so we have to show that the operation of product with the pseudodivisor $(\mathcal{L}_1|_{Y'}, Y', s_1)$ leads to an expression of the desired form, which we carry out one $j$ at a time.

For those $j$ with $Y_j' \subseteq Z(s_1) = Y$, the product operation simply takes a divisor representative of $\mathcal{L}_1|_{Y_j'}$, and in this case we get the desired expression simply by setting $Y_j = Y_j'$, $a_j = a_j'$, and $S_j = \{1\} \cup S_j'$.

On the other hand, for those $j$ with $Y_j' \not\subseteq Z(s_1)$, then $Z(s_1)$ induces an effective Cartier divisor on $Y_j'$, say $b_1 Z_1 + \cdots + b_\ell Z_\ell$, with each $b_i > 0$. Then commutativity of product with pseudodivisors (Corollary 2.4.2 of [Fu98]) says that the product of $(\mathcal{L}_1|_{Y'}, Y', s_1)$ with $\prod_{i \in S'_j} [\mathcal{L}_i|_{Y'_j}]$ is the same as iteratively taking the product of each $[\mathcal{L}_i|_{Y_j'}]$ with $b_1 Z_1 + \cdots + b_\ell Z_\ell$. We thus see that if we split the $j$ term of our sum into $\ell$ parts, with each $Z_j'$ in place of $Y_j'$, and $b_j a_j'$ in place of $a_j'$ (and leaving $S_j'$ unchanged), we obtain an expression of the desired form.

To apply the lemma, we will need the following.

Definition 4.5. We say a line bundle $\mathcal{L}$ on a variety $X$ is algebraically base-point free if for all $p \in X$, there exists an effective divisor $D$ on $X$, with $p \notin D$, and such that $D$ is algebraically equivalent to (a representative of) $\mathcal{L}$. 

Our main criterion is then as follows.
Corollary 4.6. In Situation 4.3, with $X$ a variety, $r = \dim X$, and $\alpha = [X]$, let $\beta \in A_0(Y)$ be the associated refined top Chern class. Suppose that $Y'$ is a connected component of $Y$ such that $\mathcal{L}|_{Y'}$ is algebraically base-point free for all $i$. Then $\deg \beta|_{Y'} \geq 0$. If in addition for every $i$ we have that $\mathcal{L}|_{Y'}$ has positive degree on every curve in $Y'$, then $\deg \beta|_{Y'} > 0$.

Proof. According to Lemma 4.4, the degree of $\beta|_{Y'}$ is computed by (taking a positive linear combination of classes obtained by) intersecting representatives of restrictions of the $\mathcal{L}_s$ to various irreducible closed subsets $Y_j$ of $Y'$. However, since degree is invariant under algebraic equivalence, we may replace these representatives by any algebraically equivalent classes. By hypothesis, for each $i$ there are effective classes on $Y'$ algebraically equivalent to representatives for $\mathcal{L}_i|_{Y'}$, and not containing any given point, so choosing these iteratively we can realize the degree as a proper intersection, which is therefore nonnegative.

Next, if $\mathcal{L}|_{Y'}$ has positive degree on every curve, then we see that for each $Y_j$, every stage of the intersection on $Y_j$ must yield a nonzero class, so we obtain the desired positivity. \qed

Remark 4.7. The algebraically base-point free condition is actually slightly stronger than necessary for Corollary 4.6: indeed, it is enough to assume that the ‘algebraic base locus’ of $\mathcal{L}$ does not contain any curves.

Note also that the condition of having positive degree on every curve does not imply the algebraically base-point free condition. Indeed, Mumford constructed an example of a (smooth, projective) surface $X$ and line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L}$ has positive degree on every curve on $S$, but $\mathcal{L} \cdot \mathcal{L} = 0$; in particular, $\mathcal{L}$ cannot have a representative which is algebraically equivalent to an effective divisor on $X$. See Example 1.5.2 of [La04].

We now apply Corollary 4.6 to our situation.

Proposition 4.8. In Situation 1.1, let $\mathcal{L}$ be as in Proposition 3.3. Then for all $p \in X$, we have that $\mathcal{L}|_{\{p\} \times S}$ is algebraically base-point free.

Moreover, if every divisor in $\mathcal{D}$ occurs over only finitely many values of $z \in S$, then for all $p \in X$, we have that $\mathcal{L}|_{\{p\} \times S}$ has positive degree on every curve in $S$.

Proof. Let $s \in \Gamma(X \times S, \mathcal{L})$ also be as in Proposition 3.3. Then by definition, for any $p' \in X$, we have that $s|_{\{p'\} \times S}$ vanishes precisely on the set of $z \in S$ such that $p' \in D_z \subseteq X$. Given $z \in S$, choose any $p' \in X \setminus D_z$, so that $z$ is not in $Z(s|_{\{p'\} \times S})$, which is an effective divisor representing $\mathcal{L}|_{\{p'\} \times S}$. Now, since $X$ is connected, we have that $\mathcal{L}|_{\{p\} \times S}$ and $\mathcal{L}|_{\{p'\} \times S}$ are in the same connected component of $\text{Pic}(S)$, so it follows that the former is algebraically base-point free, as desired.

Next, let $Z \subseteq S$ be a curve. Then the restriction of $\mathcal{L}|_{\{p\} \times S}$ to $Z$ has the same degree as the restriction $\mathcal{L}|_{\{p'\} \times S}$ to $Z$ for any $p' \in X$. Under our finiteness hypothesis, we have that the divisors $D_z$ must move nontrivially as $z$ varies in $Z$, so if we choose any $z \in Z$ and $p' \in X \setminus D_z$, there are necessarily a positive, finite number of $z' \in Z$ such that $p' \in D_{z'}$, so we have that $Z(s|_{\{p'\} \times S})$ meets $Z$ in a positive, finite number of points, and hence that the degree of $\mathcal{L}|_{\{p'\} \times S}$ on $Z$ is positive, as desired. \qed

Putting everything together, we conclude the following:
Corollary 4.9. In the situation of Theorem 1.4, all the $m_p$ are nonnegative. Under the further hypothesis that every divisor appears at most finitely many times in $D$, then for any ramification point $p \in C$, we have that $m_p$ is at least equal to the number of connected components of $R|_{\{p\} \times S}$, and in particular $m_p$ is positive.

Proof. Iterating (3) induces a complete flag of quotient bundles for our jet bundles, and taking kernels gives us a filtration. Note that the quotient line bundles from this filtration are precisely the same as the kernels of (3), and also that the $\omega_{C \times S/S}$ term drops out upon restriction to a fiber $\{p\} \times S$. Then applying Proposition 4.8 and Corollary 4.6, we conclude the desired statements on the $m_p$. □

This completes the proof of Theorem 1.4, together with the promised refinement of the positivity statement for the $m_p$. Morally, the refinement says that if more than one divisor in $D$ realizes the ramification at $p$, they each contribute to $m_p$. However, this is not what contributes to higher multiplicities in either the Riemann-Hurwitz or Plücker settings: indeed, in both these cases $R|_{\{p\} \times S}$ is always connected.

Remark 4.10. Note that it is important that Corollary 4.6 only requires the weaker notion of being algebraically base-point free: in our situation, if $X$ is not rational, the line bundles $\mathcal{L}|_{\{p\} \times S}$ themselves are frequently not base-point free, and indeed in interesting cases (as in the Riemann-Hurwitz situation, or theta divisors on principally polarized abelian varieties) often have only one effective representative.

Similarly, it is important that we do not assume that $\mathcal{L}$ is algebraically base-point free on all of $C \times S$, since in our situation we may get negative contributions from $\omega_{C \times S/S}$ when $C$ is rational.

5. Special cases

In this section, we prove Theorem 1.5, showing that our formula reduces to the classical Riemann-Hurwitz and Plücker formulas in the appropriate special cases. The most substantive issue is verifying that our multiplicities agree with the classical definitions of the multiplicities. As a new class of examples, we also consider the case of a principally polarized abelian variety, with the family of divisors induced by the polarization.

Recovering the Riemann-Hurwitz formula. Let $X$ be a smooth curve, $S = X$, and $D$ the diagonal in $X \times S = X \times X$; i.e. the family of divisors consists of all single points on $X$. Clearly, $h_0(X, D) = 1$. Moreover, $h_1(X, D) = -K_X$ because the divisor in $X \times X$ of pairs of points contained in a divisor of $D$ (formally, the pushforward to $X \times X$ of the small diagonal in $X \times X \times X$) is precisely the diagonal $\Delta$, and the pullback of $\mathcal{O}_{X \times X}(\Delta)$ under the diagonal immersion $X \to X \times X$ is the tangent bundle $\mathcal{T}_X$ of $X$. Then Theorem 1.4 reads

\[ \sum_{p \in R} m_p p \sim -f^* K_X + K_C, \]

which is the (algebraic form of the) Riemann-Hurwitz formula, provided that the multiplicities have the same meaning as in the Riemann-Hurwitz formula. In particular, $\sum_{p \in R} m_p = 2g - 2 - d(2h - 2)$. 


To prove that that the multiplicities agree, we first review how they appear in Riemann-Hurwitz. The differential
\[ df : \mathcal{T}_C \to f^* \mathcal{T}_X \]
is a global section of \( Hom(\mathcal{T}_C, f^* \mathcal{T}_X) = f^* \omega_X^* \otimes \omega_C \) and its vanishing locus is the usual ramification divisor (one then further relates this divisor to the ramification indices of \( f \), but this is not relevant to us at the moment). Now, in our setup we have
\[ f^*_X \mathcal{O}_{X \times X}(\Delta) = \mathcal{O}_{C \times X}(\Gamma_f), \]
where \( \Gamma_f \subseteq C \times X \) is the graph of \( f \). Thus our construction takes the vanishing locus of the induced section \( s \) of \( J^1_{C \times X/X} \mathcal{O}_{C \times X}(\Gamma_f) \), specifically the section induced by the unique (up to scalars) global section of \( \mathcal{O}_{C \times X}(\Gamma_f) \) vanishing precisely on \( \Gamma_f \).

To compare the two notions of multiplicity, we simply express both sections in local coordinates. Let \( p \in C \) be a ramification point, and \( q = f(p) \in X \). Let \( u \) and \( v \) be local coordinates on \( C \) at \( p \) and on \( X \) at \( q \), respectively. Then \( \omega_C \) and \( \omega_X \) are trivialized locally by \( du \) and \( dv \), and if we write \( f^*v = g(u) \in \mathbb{K}[u] \), the map \( f^* \omega_X \to \omega_C \) from the usual Riemann-Hurwitz formula is expressed locally by \( g'(u) \), so that the corresponding multiplicity is simply \( \text{ord}_p g'(u) \). On the other hand, on \( C \times X \) the section \( s \) of the sheaf \( \mathcal{O}_{C \times X}(\Gamma_f) \) which vanishes precisely on \( \Gamma_f \) has a local expression \( v - g(u) \) in a suitable local trivialization, and the induced section \( s \) of \( J^1_{C \times X/X} \mathcal{O}_{C \times X}(\Gamma_f) \) is given in terms of the usual trivialization simply by \( (v - g(u), -g'(u)) \). Taking the zero set of the first term restricts to \( \Gamma_f \), while the second term then vanishes to order \( \text{ord}_p g'(u) \) at \( p \), showing the desired agreement.

**Recovering the Plücker formula.** Let \( X = \mathbb{P}^r, S = (\mathbb{P}^r)^*, \) and \( \mathcal{D} \) the family of all hyperplanes on \( \mathbb{P}^r \), so \( n = r \). Then \( h_0(\mathbb{P}^r, \mathcal{D}) = 1 \) and \( h_1(\mathbb{P}^r, \mathcal{D}) = (r + 1)h \), where \( h \) denotes the hyperplane class. The former is trivial – there is one hyperplane through \( r \) general points in \( \mathbb{P}^r \). For the latter, the locus of co(hyper)planar \( r + 1 \)-tuples of points in \( \mathbb{P}^r \) is precisely the vanishing locus of the determinant, i.e. \( \{ p_1 \wedge p_2 \wedge \cdots \wedge p_{r+1} = 0 \} \subseteq (\mathbb{P}^r)^{r+1} \), whose associated line bundle \( \mathcal{O}_{\mathbb{P}^r}(1)^{\otimes r+1} \) pulls back to \( \mathcal{O}_{\mathbb{P}^r}(r + 1) \) under the diagonal immersion. Then (1) reads
\[ \sum_{p \in R} m_p p \sim (r + 1)f^*h + \left( \frac{r + 1}{2} \right) K_C. \]
Taking degrees, we obtain \( \sum_{p \in R} m_p = (r+1)d+(r+1)(2g-2) \), which is precisely the classical Plücker formula \([\text{EH}83, \text{Proposition } 1.1] \), provided that the multiplicities have the same meaning.

Now, note that if we set \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^r \times S}(\mathcal{D}) \) as usual, and \( \mathcal{L}' := f^* \mathcal{O}_{\mathbb{P}^r}(1) \), then we have
\[ f^*_S \mathcal{L} = \mathcal{L}' \otimes \mathcal{O}_S(1), \]
and hence
\[ J^r_{C \times S/S} f^*_S \mathcal{L} = J^r_{C \times S/S} (\mathcal{L}' \otimes \mathcal{O}_S(1)) = (J^r_C \mathcal{L}') \otimes \mathcal{O}_S(1) \]
by Proposition 2.3.

Setting \( \mathcal{E} = J^r_C \mathcal{L}' \) and \( \mathbb{P}V = S = (\mathbb{P}^r)^* \), the desired agreement of the definition of multiplicities is then immediate from the below lemma, together with the usual argument for the Plücker formula \([\text{EH}83, \text{Proposition } 1.1] \), which expresses the
ramification divisor as the vanishing divisor of a section of a line bundle (specifically, the determinant line bundle of $J_C$).

**Lemma 5.1.** Let $\mathcal{E}$ be a locally free sheaf on $C$ of rank $r + 1$, and $V$ an $(r + 1)$-dimensional $K$-vector space. Then pushing forward to $C$ induces an isomorphism

$$\Gamma(C \times \mathbb{P}V, \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}V}(1)) \cong \text{Hom}_{\mathbb{C}}(V \otimes \mathcal{O}_C, \mathcal{E}).$$

Moreover, suppose we have a section $s \in \Gamma(C \times \mathbb{P}V, \mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}V}(1))$ such that the zero locus of $s$ is supported over finitely many points of $C$, and let $\tilde{s} \in \text{Hom}_{\mathbb{C}}(V \otimes \mathcal{O}_C, \mathcal{E})$ be the image of $s$ under the above isomorphism, and $p \in C$ any point. Then $\text{ord}_p \det \tilde{s}$ (as a section of the line bundle $\det \mathcal{E}$) is equal to the multiplicity of the part of the refined top Chern class of $s$ which is supported over $p$.

**Proof.** We write $\text{proj}_C$ and $\text{proj}_V$ respectively for the projections to the two factors. Using the projection formula, we have

$$\text{proj}_{C,*}(\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}V}(1)) = \text{proj}_{C,*}(\text{proj}_C^* \mathcal{E} \otimes (\mathcal{O}_C \boxtimes \mathcal{O}_{\mathbb{P}V}(1)))$$

$$= \mathcal{E} \otimes \text{proj}_{C,*}(\mathcal{O}_C \boxtimes \mathcal{O}_{\mathbb{P}V}(1))$$

$$= \mathcal{E} \otimes (V^* \otimes \mathcal{O}_C) = \text{Hom}_{\mathbb{C}}(V \otimes \mathcal{O}_C, \mathcal{E}).$$

Since $\text{Hom}$ requires no sheafification, by taking global sections we obtain the isomorphism asserted in the first statement of the lemma.

For the second statement, it suffices to work locally on $C$, so we may restrict to a neighborhood $U$ of $C$ on which we have a trivialization of $\mathcal{E}$, and the only zero(s) of $s$ lie over $p$. Choosing coordinates $x_i$ on $\mathbb{P}V$ we may represent $s$ as

$$(\sum_{i=0}^r f_i^0 x_1, \ldots, \sum_{i=0}^r f_i^r x_1),$$

where the $f_i^j$ are regular functions on $U$. Thus, the zero set of $s$ is the intersection of the zero sets of the $r + 1$ forms $\sum_{i=0}^r f_i^j x_i$ for $j = 0, \ldots, r$. Consequently, we observe that the zero set of $s$ is supported over a point $p' \in U$ if and only if $\det(f_i^j(p')) = 0$. At the same time, the section $\tilde{s}$ is represented by the matrix $(f_i^j)$, so we obtain a set-theoretic identity which we want to extend to multiplicities.

We prove this by induction on $r$. The base case $r = 0$ is trivial. For $r > 0$, the refined zero locus of $s$ is defined by successive refined intersection of the $f_i^j x_i$, which we order from $j = r$ to $j = 0$. Note that the hypothesis that the zero set of $s$ is empty over $U \setminus \{p\}$ implies that we cannot have $\sum_{i=0}^r f_i^j x_i$ vanishing identically, so it has a naive associated effective divisor on $U \times \mathbb{P}V$, which consists of a multiple of $\{p\} \times \mathbb{P}V$ together with a hyperplane bundle $Z$ inside $U \times \mathbb{P}V$ (more formally, a section of $U \times \mathbb{P}(V^*)$). Explicitly, if $t$ is a local coordinate on $C$ at $p$, and $e_r = \min \text{ord}_p f_i^j$, then factoring out $t^{e_r}$ from all the $f_i^j$ induces the stated decomposition, with the multiplicity of $\{p\} \times \mathbb{P}V$ being given by $e_r$. The definition of the refined zero locus of $s$ entails computing the contributions from each component separately, and taking the sum.

For the component $e_r(\{p\} \times \mathbb{P}V)$, we claim we always get a contribution of precisely $e_r$. Indeed, each subsequent form will be restricted to this fiber to compute the iterated intersection, so is simply a linear form over the base field. We then see that each iterated intersection simply yields (the class of) a linear subspace of one smaller dimension: the zero set is either a hyperplane in the previously constructed cycle, in which case the new cycle is equal to this hyperplane (with multiplicity 1), or it contains the previously constructed cycle, in which case the new cycle is
defined by restricting the line bundle to the old cycle and choosing a representative divisor, which again yields a hyperplane with multiplicity 1.

We then claim that the contribution from $Z$ consists of $(\text{ord}_p \det(f_r^j)) - e_r$ points, which yields the desired statement. First note that if we set

$$\tilde{f}_i^j = \begin{cases} t^{-e_r} f_i^j & \text{if } j = r \\ f_i^j & \text{if } j < r, \end{cases}$$

we see that $(\text{ord}_p \det(f_i^j)) - e_r = \text{ord}_p \det(\tilde{f}_i^j)$, and by definition $Z = Z(\sum_i \tilde{f}_i^r x_i)$. Also note that because the zero set of $s$ is empty over $U \setminus \{p\}$, the zero sets of $\sum_i f_i^j x_i$ cannot contain $Z$ for any $j < r$. It follows that the refined intersection of these zero sets with each other and with $Z$ is equal to the refined intersection of the zero sets of $(\sum_i f_i^r x_i)|_Z$ for $j = 0, \ldots, r - 1$. By hypothesis, we have $f_r^j(p) \neq 0$ for some $i$; without loss of generality, suppose that this holds for $i = r$. Restricting $U$ if necessary, we then have that $f_r^r$ is invertible, and $x_0, \ldots, x_{r-1}$ induce coordinates on $Z$. In terms of these coordinates, for each $j < r$ we have that

$$\left(\sum_i f_i^j x_i\right)|_Z = \sum_{i=0}^{r-1} \left(f_i^j - f_i^j \frac{\tilde{f}_i^r}{f_r^r}\right) x_i.$$

By the induction hypothesis, the desired multiplicity is equal to

$$\text{ord}_p \det \left(f_i^j - f_i^j \frac{\tilde{f}_i^r}{f_r^r}\right)_{i,j<r},$$

but we see that if we add a row consisting of $f_r^j$ on the bottom and a column consisting of $0$'s in all but the last entry (and $f_r^r$ in the last entry) on the right, the resulting matrix can be obtained by row operations from $f_i^j$, so the above determinant differs from $\det(f_i^j)$ by a multiple of $f_r^r$, which doesn’t affect the order of vanishing at $p$. We thus conclude the desired statement. \qed

**Abelian varieties.** For our final class of examples, let $X = (A, \theta)$ be a principally polarized abelian variety and let $D$ be the family of translates of $\theta$. Then $S = \text{Pic}^0(A)$ is a homogeneous space for $A^\vee$, which is isomorphic to $A$ via the principal polarization. Let

$$\omega : A \times \text{Pic}^0(A) \to \text{Pic}^0(A)$$

be the action map via translation. This time we compute the $b$-classes from Proposition 3.5. Note that $D = \omega^{-1}(D)$, where $D$ is the divisor which parametrizes line bundles $\mathcal{L} \in \text{Pic}^0(A)$ whose (unique up to scalars) section vanishes at 0, and that $D$ is the image of $\theta$ under the isomorphism $A \to \text{Pic}^0(A)$ given by $x \mapsto T_x^* \mathcal{O}(\theta)$. Then the isomorphism $\mathcal{L} \mapsto \mathcal{L}(-\theta)$ from $\text{Pic}^0(A)$ to $A^\vee$ maps $D$ to a principal polarization of $A^\vee$, so Riemann-Roch for abelian varieties (see §16 of [Mu08]) says that we have $D^n = n!$. Then $[D]^m = \omega^*[D]^m$, so $b_1(A, 1) = 0$ trivially and $b_0(A, 1) = n!$, the latter since the projection of $A \times \text{Pic}^0(A)$ to the first factor restricts to isomorphisms on the fibers of $\omega$. In conclusion, if $C$ is any curve of genus $g$, and $f : C \to A$ is a morphism which is not everywhere ramified, then

$$\sum_{p \in R} m_p p \sim n! \binom{n+1}{2} K_C,$$

so the number of ramification points counted appropriately is $(g-1)n(n+1)!$. 

Note that the case \( n = 1 \) overlaps with the Riemann-Hurwitz formula, when the target is an elliptic curve. In this case, \( K_X \) is trivial, so we see that the two formulas agree, as they must.

Moreover, in characteristic 0, we see that our main theorem is not vacuous for this family of examples. Indeed, provided \( A \) is not a product of lower-dimensional abelian varieties, Ein and Lazarsfeld have shown that \( \theta \) is normal \([\text{EL97}, \text{Theorem } 1]\) (using also the decomposition theorem \([\text{BL04}, \text{Theorem } 4.3.1]\)), and Kollár had previously shown that \((A, \theta)\), and hence \( \theta \), is log canonical \([\text{Ko95}, \text{Theorem } 17.13]\). Thus, by Corollary 6.7 below we conclude that there exists curves \( C \) and morphisms \( f : C \to A \) which have only finitely many ramification points.

6. Discussion of degeneracy

One issue which we have not addressed is the notion of degeneracy, or how to characterize when it will be the case that every point of \( C \) is a ramification point relative to \( D \). Obviously, this will occur in the case that \( f(C) \) is wholly contained in some divisor in \( D \), so we make the following definition.

**Definition 6.1.** We say that the morphism \( f : C \to X \) is degenerate with respect to \( D \) if there exists \( z \in S \) such that \( f(C) \subseteq D_z \).

This is the classical notion of degeneracy which is implicitly disallowed by the Riemann-Hurwitz and Plücker formulas, where we consider non-constant morphisms of curves, or morphisms to projective space with image not contained in any hyperplane, respectively. In both of these situations, it is part of the classical statement of the respective formulas that (in characteristic 0) the set of ramification points is always finite. This leads us to the following question:

**Question 6.2.** Under what hypotheses on \((X, D)\) is it the case that whenever \( f \) is nondegenerate, we have only finitely many ramification points relative to \( D \)?

We see that some additional hypotheses are certainly necessary. For instance, it is in general possible to have \((X, D)\) such that every \( f : C \to X \) is everywhere ramified.

**Example 6.3.** Suppose that \( X \) is a curve, and \( D = \{2q : q \in X\} \), with \( S = X \), so that \( n = 1 \). In this case, every morphism \( f : C \to X \) will be everywhere ramified.

This leads us to the following definition.

**Definition 6.4.** We say that the family \( D \) is totally degenerate if for all \( C \) and \( f : C \to X \), we have that \( f \) is everywhere ramified with respect to \( D \).

The above behavior can be understood in terms of jet schemes as follows.

**Notation 6.5.** Let \( X_n \) denote the scheme of \( n \)-jets on \( X \). Let \( D_n \subseteq X_n \) denote the set of \( n \)-jets which occur as \( n \)-jets for some \( D \in D_n \).

Then \( X_n \) is smooth of dimension \((\dim X)(n + 1)\), and \( D_n \) can be understood equivalently as the image in \( X_n \) of the relative \( n \)-jet scheme of \( D \), which is a closed subset of \( X_n \times S \). In particular, since \( S \) is assumed proper we have that \( D_n \) is closed in \( X_n \). We then have:

**Proposition 6.6.** The family \( D \) is totally degenerate if and only if \( D_n = X_n \).
Proof. Any \( f \) induces a morphism \( C_n \to X_n \), and a point \( p \in C \) is a ramification point with respect to \( D \) if and only if the image of \( C_n|_p \) is contained inside \( D_n \).

Thus, if \( D_n = X_n \), it is clear that \( D_n \) is totally degenerate. Conversely, if \( D_n \subsetneq X_n \), if we choose an \( n \)-jet in \( X_n \setminus D_n \) based at some \( p \in X \) which induces a nonzero tangent vector at \( p \), then cutting inductively by hypersurfaces we can produce a curve in \( X \) which is smooth at \( p \) and induces the given jet; we can then let \( C \) be the normalization of this curve. \( \square \)

If all the divisors in \( D \) are smooth, then \( D_n \) has dimension at most \( n + (\dim X - 1)(n + 1) = (\dim X)(n + 1) - 1 \), so has codimension at least 1 in \( X_n \). In this case, we see from Proposition 6.6 that \( D \) cannot be totally degenerate. As it turns out, this generalizes considerably thanks to a criterion of Ein, Mustaţă and Yasuda, who have shown that (in characteristic 0) a divisor in \( X \) which is normal and has log canonical singularities has a jet scheme which is pure of the expected dimension \( (\dim X - 1)(n + 1) \) \[EMY03, \text{Remark 3.4}\]. We can therefore conclude the following:

**Corollary 6.7.** If \( \mathbb{K} \) has characteristic 0, and \( D_z \) is normal with at worst log canonical singularities for all \( z \in S \), then \( D \) is not totally degenerate.

Proof. Applying the Ein-Mustaţă-Yasuda theorem, the relative jet scheme will be pure of the expected dimension \( n + (\dim X - 1)(n + 1) \), so as in the smooth case above, we necessarily have \( D_n \subsetneq X_n \) and conclude the corollary from Proposition 6.6. \( \square \)

However, even when \( D \) is not totally degenerate, there may be particular morphisms which are nondegenerate but nonetheless everywhere ramified.

**Example 6.8.** Suppose that \( X = \mathbb{P}^2 \), and \( f : C \to \mathbb{P}^2 \) is a closed immersion. Let \( D \) be the dual curve to \( C \); i.e., the set of lines in \( \mathbb{P}^2 \) occurring as tangent lines to \( C \). Then \( n = 1 \), and although \( f(C) \) is not contained in any line, we do have that \( f \) is everywhere ramified relative to \( D \).

Thus, we conclude that the hypotheses required to address Question 6.2 must include some nontrivial conditions on the divisors in \( D \) moving with sufficient flexibility.

**Remark 6.9.** The jet scheme approach discussed above constitutes an alternate, more geometric approach to everything we do. However, we have found that our current approach using jet bundles seems to be more tractable overall.

We conclude with an additional question. In both the Riemann-Hurwitz and Plücker formulas, the multiplicities \( m_p \) can be expressed explicitly in terms of suitably defined ramification indices (at least, in characteristic 0). We are therefore led to ask:

**Question 6.10.** Can one give explicit formulas for the \( m_p \), for instance in terms of the multiplicities at \( p \) that occur among the \( f^*D \) for \( D \in \mathcal{D} \)?

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