A family of mass-critical Keller–Segel systems

Michael Winkler

Abstract
The no-flux initial-boundary value problem for the quasilinear Keller–Segel system

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), \\
    v_t &= \Delta v - v + u,
\end{align*}
\]

is considered in smoothly bounded domains \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \), where \( D \in C^2([0,\infty)) \) and \( S \in C^2([0,\infty)) \) are such that \( D > 0 \) on \( [0,\infty) \) and that \( S(0) = 0 < S(s) \) for all \( s > 0 \). A particular focus is on cases in which there exist \( \nu > 0, C_{SD} > 0 \) and \( f \in L^1((1,\infty)) \) such that

\[
    -f(s) \leq D(s) \frac{\nu}{S(s)} - \frac{C_{SD}}{s^{2/n}} \quad \text{for all } s \geq 1.
\]

It is first shown that then there exists \( m_0 > 0 \) such that whenever \( u_0 \) and \( v_0 \) are reasonably regular and nonnegative with \( \int_\Omega u_0 < m_0 \), within a suitably generalized concept of solvability one can always find a global solution for which \( u \) remains bounded with respect to the norm in \( L^{2(n-1)/n}(\Omega) \). Second, in radially symmetric settings this is complemented by a converse result on nonexistence of such solutions for some appropriately large initial data, hence leading to the conclusion that any pair \( (D, S) \) from the family of nonlinearities fulfilling (**) gives rise to a critical mass phenomenon in (*). Remarkably, this does not only include cases of arbitrarily strong diffusion degeneracies due to fast decay of \( D(s) \) as \( s \to \infty \),...
but according to the considerably wide funnel described by (*) this moreover indicates that mass criticality in Keller–Segel systems is far more than a nongeneric phenomenon limited to precise functional forms of model ingredients hardly to be found in nature. Applications to concrete scenarios include the detection of mass-critical probability distribution functions of the form \(0 \leq s \mapsto \exp(-s^{(n-2)/n})\) in the version of (**) accounting for so-called volume-filling effects.

MSC 2020
35B33 (primary), 35B44, 35K65, 35Q92, 92C17 (secondary)

1 | INTRODUCTION

Critical mass thresholds in the context of blow-up phenomena belong to the apparently most striking subtleties going along with the interplay of diffusion and self-enhanced cross-diffusion in frameworks of Keller–Segel type chemotaxis systems [29, 42, 45]. On the one hand, such a dependence of the possibility to enforce singularity formation on total population masses seems to reflect fairly well the presence of spatially global quorum-sensing type mechanisms of apparent relevance for taxis-driven aggregation in various experimental settings [21, 29, 43]. Beyond this, however, a considerable motivational aspect of essentially mathematical nature rests on the circumstance that effects of this flavor apparently presuppose quite a precise balance of those system ingredients that incorporate the strength of diffusion and of cross-diffusive interaction, respectively.

Constituting the undoubtedly most prominent example in this regard, the spatially two-dimensional version of the classical Keller–Segel system [32]

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \nabla v), \\
\tau \frac{\partial v}{\partial t} &= \Delta v - v + u,
\end{align*}
\]

is known to exhibit critical mass phenomena with respect to the occurrence of unbounded solutions both in its fully parabolic variant with \(\tau = 1\), and in its parabolic-elliptic simplification with \(\tau = 0\). In fact, when posed along with homogeneous Neumann boundary conditions in bounded planar domains \(\Omega\), the initial value problem for (1.1) with \(\tau = 1\) possesses global bounded solutions for all reasonably regular initial data whenever \(\int_{\Omega} u(\cdot, 0) < 4\pi\) [40], whereas for all values of \(m \in (4\pi, \infty) \setminus \{4k\pi \mid k \in \mathbb{N}\}\), one can find unbounded solutions fulfilling \(\int_{\Omega} u(\cdot, 0) = m\), provided that \(\Omega\) is simply connected [30]; in the simpler case when \(\tau = 0\), the knowledge in this regard is even much more complete, inter alia asserting the existence of solutions genuinely blowing up within finite-time, without any restriction on the prescribed size \(m > 4\pi\) of their initial mass, and on the shape of the domain ([39]; cf. also [23] and [10] for results on mass-dependent bubbling in the associated fully stationary problem). Some further simplifications of the second equation allow for yet more comprehensive results on corresponding Cauchy problems posed in the whole plane (see, for example, [5, 6, 8, 9, 12, 42] and the references therein).
Beyond these particular scenarios in close connection to (1.1), however, only few rigorous detections of mass criticality in specific chemotaxis systems significantly different from (1.1) can be found in the literature. This may be viewed as partially reflecting the substantial mathematical challenges linked to proving occurrence of blow-up in systems less simple than (1.1), especially in fully parabolic cases in which, besides the equation for the population density $u$, also that describing the evolution of the signal concentration $v$ is parabolic. Accordingly, the essentially only further result in this direction that addresses such a parabolic–parabolic Keller–Segel system asserts finite-time blow-up of some large-mass solutions in three- and four-dimensional variants of (1.1) involving certain critical nonlinear cell diffusion mechanisms for which the resulting diffusion operator is exactly of porous medium type [37]. Apart from that, rigorous proofs for the occurrence of critical mass phenomena seem available only for some relatives of (1.1) with certain particular structures, inter alia presupposing that the signal evolution is governed by elliptic equations (cf., for instance, [7, 14, 2, 13, 36, 47] and [3]).

Even in the quasilinear generalization of (1.1), in the framework of a no-flux initial-boundary value problem in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ given by

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), \quad x \in \Omega, \quad t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \quad t > 0, \\
    D(u) \frac{\partial u}{\partial \nu} - S(u) \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

and arising for different choices of the parameter functions $D$ and $S$ in refined models for chemotactic migration in various contexts [28], despite a certain structural proximity to (1.1), to be described in more detail below, available results seem to essentially concentrate on identifying conditions on $D$ and $S$ that ensure either global existence and boundedness of widely arbitrary solutions on the one hand, or the occurrence of unboundedness phenomena at arbitrarily small mass levels on the other. Quite well understood in this regard seems the subclass of (1.2) determined by the prototypical choices

\[
D(s) = (s + 1)^{m-1} \quad \text{and} \quad S(s) = s(s + 1)^{q-1}, \quad s \geq 0,
\]

for which, namely, it is known that whenever $m \in \mathbb{R}$ and $q \in \mathbb{R}$ are such that $m > q + 1 - \frac{2}{n}$, given any suitably regular nonnegative initial data one can find a global classical solution for which $u$ is bounded throughout $\Omega \times (0, \infty)$ [31, 33, 44, 46], while if conversely $m < q + 1 - \frac{2}{n}$, and if moreover $n \geq 2$ and $\Omega$ is a ball, then for arbitrary $m > 0$, it is possible to find classical solutions for which $u$ cannot remain uniformly bounded [50]. Further studies even have provided further information on subcases of the latter in which the respective blow-up phenomenon either must occur within finite time [16–18], or only arises in the sense of an infinite-time grow-up [16, 18, 54]; cf. also [35, 15, 22, 24, 25] for related results concerned with parabolic-elliptic analogues, one-dimensional versions, and small-data solutions in some supercritical parameter settings).

Especially in the context of modeling procedures based on the inclusion of volume-filling effects in the style of the seminal work [41] in this regard, however, diffusion and cross-diffusion rates with quite rapid decay at large population densities arise in a natural manner when saturation effects in cell motility at large population densities are accounted for; in fact, the particular
approach in [41] (cf. also [55]) suggests to link $D$ and $S$ through the relationships

$$D(s) = Q(s) - sQ'(s) \quad \text{and} \quad S(s) = sQ(s), \quad s \geq 0,$$

(1.4)

where $Q(s)$ represents the probability for a cell, when positioned at a place with current population density $s$, to find space in some neighboring site. Accordingly, functions $D$ and $S$ exhibiting exponential or even faster decay appear whenever $Q$ is chosen so as to reflect strong saturation in decreasing faster than algebraically.

For such more general classes of diffusion rates $D$ and chemotactic sensitivity functions $S$ in (1.2), the knowledge seems much sparser; while the outcome of [50] actually extends so as to provide unboundedness in the sense described above, also of small-mass solutions, under quite mild conditions on $D$ and $S$ which essentially reduce to the assumption that

$$\frac{S(s)}{D(s)} \geq Ks^{\frac{2}{n} + \eta} \quad \text{for all } s \geq 1 \text{ and some } K > 0 \text{ and } \eta > 0,$$

(1.5)

a comparably exhaustive result on boundedness in the presence of correspondingly subcritical asymptotics of $\frac{S}{D}$ is yet lacking: Only under the crucial overall assumption that $D(s)$ be bounded from below by a function decreasing at most algebraically fast as $s \to \infty$, the hypothesis that

$$\frac{S(s)}{D(s)} \leq ks^{\frac{2}{n} - \eta} \quad \text{for all } s \geq 1 \text{ and some } k > 0 \text{ and } \eta > 0$$

(1.6)

has been shown to imply comprehensive statements on global existence of bounded classical solutions so far [46]. The lack of farther-reaching knowledge in this regard appears to be inherently linked to missing information on suitably regularizing features of the associated diffusion operators in (1.2) especially in cases when $D(s)$ decays at rates faster than algebraic as $s \to \infty$, in which accessibility to standard arguments, for example, based on Moser-type iteration procedures, seems limited; accordingly, even for simple choices of $D$ and $S$ exhibiting essentially exponential decay only partial results concerned with issues of boundedness and unboundedness in frameworks of classical solutions are available [19, 20, 53].

Main results. The purpose of the present work is to reveal that under quite mild assumptions on $D$ and $S$ not substantially exceeding the requirement that

$$\begin{cases} D \in C^2([0, \infty)) \text{ is such that } D > 0 \text{ on } [0, \infty) \quad \text{and} \\ S \in C^2([0, \infty)) \text{ satisfies } S(0) = 0 \text{ and } S > 0 \text{ on } (0, \infty), \end{cases}$$

(1.7)

any three- or higher dimensional constellation in which

$$\left| \frac{S(s)}{D(s)} - ks^{\frac{2}{n}} \right| \leq \varepsilon(s) \quad \text{for all } s > 1$$

holds with some $k > 0$ and some $\varepsilon$ decaying suitably fast as $s \to \infty$ enforces a certain mass criticality, and that hence a separation between the two mass-insensitive scenarios discussed above around (1.5) and (1.6) is achieved within a considerably large family of model ingredients, rather than merely nongeneric lines. In fact, such settings will be shown to go along with some critical mass phenomenon with respect to an $L^p$ boundedness property in a suitable context of
generalized solvability (Theorem 1.3, Theorem 1.2 and Corollary 1.4), and for essentially power-type nonlinearities such as those in (1.3) this phenomenon will be seen to actually take place in contexts of classical solutions and their pointwise boundedness (Theorem 1.3 and Corollary 1.4).

To make this more precise, and to simultaneously sketch some ideas fundamental to our approach, we note that if (1.7) holds, and if, for some $T \in (0, \infty]$, $(u, v) \in (C^{2,1}(\Omega \times (0, T)))^2$ is a classical solution of the boundary value problem in (1.2) on $\Omega \times (0, T)$ with $u > 0$ in $\Omega \times (0, T)$, then according to a straightforward computation, we have

$$
\frac{d}{dt} F(u(\cdot, t), v(\cdot, t)) = - \int_{\Omega} v^2 - \int_{\Omega} S(u) \left| \frac{\nabla u}{h(u)} - \nabla v \right|^2 \text{ for all } t \in (0, T),
$$

where we have set

$$
F(\phi, \psi) := \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{2} \int_{\Omega} \psi^2 - \int_{\Omega} \phi \psi + \int_{\Omega} G(\phi)
$$

for $\phi \in L^1(\Omega)$ and $\psi \in W^{1,2}(\Omega)$ such that $\phi > 0$ a.e. in $\Omega$, (1.9)

with

$$
G(s) := \int_1^s \int_1^\sigma \frac{d\tau d\sigma}{h(\tau)}, \quad s > 0,
$$

and

$$
h(s) := \frac{S(s)}{D(s)}, \quad s \geq 0.
$$

(1.10)

Now it can be observed (Lemma 2.1) that if

$$
\frac{S(s)}{D(s)} \leq K_{SD} s^{\frac{2}{n}} \text{ for all } s > 0
$$

(1.12)

with some $K_{SD} > 0$, then at suitably small mass levels the negative contribution to $F$ in (1.9) is essentially dominated by the respective positive summands, which through (1.8) will lead to time-independent a priori bounds for $u$ and $v$ with respect to the norms in $L^{(2n-2)/n}(\Omega)$ and $W^{1,2}(\Omega)$, respectively.

In the case when additionally $D$ is assumed to be algebraically bounded from above and below, this basic information can be seen to be sufficient as a starting point for an iterative argument ultimately leading to $L^\infty$ estimates for $u$, in Section 3 hence implying the following first of our main results.

**Theorem 1.1.** Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $K_{SD} > 0$. Then there exists $m_0 = m_0(K_{SD}) > 0$ such that if $D$ and $S$ satisfy (1.7) and (1.12) as well as

$$
\liminf_{s \to \infty} s^{\beta} D(s) > 0 \quad \text{and} \quad \limsup_{s \to \infty} s^{-\gamma} D(s) < \infty
$$

(1.13)

with some $\beta \geq 0$ and $\gamma \geq 0$, then whenever

$$
\begin{cases}
  u_0 \in W^{1,\infty}(\Omega) \text{ is such that } u > 0 \text{ in } \overline{\Omega} \text{ and } \\
  v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative in } \Omega
\end{cases}
$$

(1.14)
with
\[ \int \Omega u_0 < m_0, \quad (1.15) \]
the problem (1.2) possesses a global classical solution \((u, v)\) such that
\[
\begin{cases}
    u \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)) & \text{and} \\
    v \in \bigcap_{r>1} C^0([0, \infty); W^{1,r}(\Omega)) \cap C^{2,1}(\Omega \times (0, \infty)),
\end{cases}
\]
that \(u > 0\) and \(v \geq 0\) in \(\Omega \times [0, \infty)\), and that
\[ \sup_{t>0} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty. \quad (1.16) \]
If, furthermore, there exists \(R > 0\) such that \(\Omega = B_R(0)\), and if
\[ u_0 \text{ and } v_0 \text{ are radially symmetric with respect to } x = 0, \quad (1.17) \]
then \((u(\cdot, t), v(\cdot, t))\) are radially symmetric with respect to \(x = 0\) for all \(t > 0\).

If the additional requirement (1.13) is dropped, however, then we shall see in Section 4 that the above basic regularity properties, together with (1.9), after all allow for the construction of a certain global generalized solution, the first component of which remains bounded in some reflexive \(L^p\) space:

**Theorem 1.2.** Suppose that \(n \geq 3\) and that \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary, and let \(D\) and \(S\) be such that (1.7) and (1.12) hold with some \(K_{SD} > 0\). Then there exists \(m_0 > 0\) with the property that whenever \((u_0, v_0)\) satisfies (1.14) with (1.15), the problem (1.2) admits, in the sense of Definition 4.2, at least one global very weak energy solution having the additional property that
\[ \text{ess sup}_{t>0} \int \Omega u^{2n-2}_\eta (\cdot, t) < \infty. \quad (1.18) \]
If, moreover, \(\Omega = B_R(0)\) with some \(R > 0\) and (1.17) holds, then this solution even is a radial global very weak energy solution in the flavor of Definition 4.2.

We shall next complement the above by making sure that the requirements on \(S/D\) and on smallness of \(\int \Omega u_0\) both in Theorem 1.1 and in Theorem 1.2 actually cannot be substantially relaxed. To this end, in Section 5 we shall first see by means of an explicit construction that if an inequality opposite to that in (1.12) holds, then the energy functional \(\mathcal{F}\) has a certain property of unboundedness from below, even when restricted to radial functions (Lemma 5.1). Intending to use such as initial data, in Lemma 5.10 we shall perform a Pohozaev-type testing procedure to derive, at each fixed level of the total population mass, an \(a \text{ priori}\) bound from below for energies of suitably regular radially symmetric steady state solutions to (1.2) in balls, where in order to be able to include the critical relationship with \(\eta = 0\) in (1.5), in contrast to previous related approaches [50], we will rely on some decisive refinement which at its core rests on a simple consequence
of standard elliptic regularity to be stated in Lemma 5.9. Combined with knowledge on natural connections between \( \omega \)-limit sets of global solutions and steady states, to be carefully derived in the considered context of poor regularity information in Section 5.2, this will lead to our following main result on nonexistence of solutions enjoying boundedness features in the flavor of (1.18).

**Theorem 1.3.** Suppose that \( \Omega = B_R(0) \subset \mathbb{R}^n \) with some \( n \geq 3 \) and \( R > 0 \), and that \( D \) and \( S \) satisfy (1.7) and, with some \( k_{SD} > 0 \) and \( K_{\ell} > 0 \),

\[
\frac{S(s)}{D(s)} \geq k_{SD}s^\frac{2}{n} \quad \text{for all } s \geq 1
\]

as well as

\[
\ell'(s) \leq \frac{n-2}{n}G(s) + K_{\ell} \cdot (s+1) \quad \text{for all } s \geq 1,
\]

where \( G \) is as defined in (1.10), and where

\[
\ell'(s) := \int_1^s \frac{\sigma d\sigma}{h(\sigma)}, \quad s > 0.
\]

Then for any choice of \((u_0, v_0)\) fulfilling (1.14) and (1.17), one can find functions \(u_{0j} \) and \(v_{0j}, j \in \mathbb{N}\), such that \((u_{0j}, v_{0j})\) complies with (1.14) and (1.17) for all \( j \in \mathbb{N}\), that

\[
u_{0j} \rightarrow u_0 \text{ in } L^p(\Omega) \quad \text{and} \quad v_{0j} \rightarrow v_0 \text{ in } W^{1,q}(\Omega) \quad \text{as } j \rightarrow \infty \quad \text{for all } p \in (0, 1) \text{ and } q \in [1, \frac{n}{n-1}),
\]

and such that for each \( j \in \mathbb{N}\), the problem (1.2) does not admit a radial global very weak energy solution \((u, v)\) which has the additional boundedness property that

\[
es\sup_{t > 0} \int_{\Omega} u^p(\cdot, t) < \infty \quad \text{for some } p > \frac{2n}{n+2}.
\]

As a fairly immediate consequence, in cases where the respective hypotheses of the above existence and nonexistence results are satisfied, we infer the presence of genuine critical mass phenomena, depending on whether (1.13) holds either in classical or generalized solution frameworks:

**Corollary 1.4.** Let \( n \geq 3 \), \( R > 0 \) and \( \Omega = B_R(0) \subset \mathbb{R}^n \), and suppose that (1.7) and

\[
k_{SD}s^\frac{2}{n} \leq \frac{S(s)}{D(s)} \leq K_{SD}s^\frac{2}{n} \quad \text{for all } s \geq 1
\]

as well as (1.20) hold with some \( k_{SD} > 0 \), \( K_{SD} \geq k_{SD} \) and \( K_{\ell} > 0 \). Then

\[
m_{c}^{(\text{weak})} := \sup \{ m > 0 \mid \text{for all } (u_0, v_0) \text{ fulfilling (1.14) and (1.17) with } \int_{\Omega} u_0 = m, (1.2) \text{ admits a radial global very weak energy solution satisfying (1.23)}\}.
\]
is finite and positive. If, moreover, there exist $\beta \geq 0$ and $\gamma \geq 0$ such that (1.13) is satisfied, then even

$$m_c^{(\text{classical})} := \sup \left\{ m > 0 \mid \text{for all } (u_0, v_0) \text{ fulfilling (1.14) and (1.17) with } \int_\Omega u_0 = m, \right.$$ 

$$(1.2) \text{ possesses a global classical solution fulfilling (1.16)} \}$$

is finite and positive.

Here a considerably large fund of concrete examples can be generated by observing that the structural condition (1.20) indeed is fulfilled in quite a large number of situations in which $S/D$ approaches some multiple of the critical function $0 < s \mapsto s^{2/n}$ suitably fast:

**Corollary 1.5.** Suppose that $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and that $D$ and $S$ satisfy (1.7) as well as

$$-f(s) \leq \frac{D(s)}{S(s)} - \frac{\kappa}{s^{2/n}} \leq \frac{C_{SD}}{s} \quad \text{for all } s \geq 1 \tag{1.25}$$

with some $\kappa > 0$ and $C_{SD} > 0$ and some $f \in L^1((1, \infty))$. Then the number $m_c^{(\text{weak})}$ from Corollary 1.4 is finite and positive, and if furthermore (1.13) holds with some $\beta \geq 0$ and $\gamma \geq 0$, then also $m_c^{(\text{classical})}$ from Corollary 1.4 is finite and positive.

**Remark.** Let us emphasize that according to the remarkable sizes of the funnel-type regions determined by (1.25), the latter can be interpreted as confirming that the occurrence of critical mass phenomena in chemotaxis systems is not limited to constellations involving ingredients of precise functional nature, such as those addressed in precedent literature. Indeed, by admitting not only fairly arbitrary behavior of $D$ and $S$ as long as merely $D/S$ remains asymptotically near $\kappa/s^{2/n}$ for some $\kappa > 0$, but by furthermore even including cases of noticeably strong oscillations in $S/D$, Corollary 1.5 indicates a considerable stability of mass criticality in quasilinear Keller–Segel systems with respect to the model components.

When applied to the prototypical situation determined by the power-type laws in (1.3), Corollary 1.5 asserts mass criticality in the classical sense along the entire line $m = q + 1 - \frac{2}{n}$ in the $(m, q)$ plane whenever $n \geq 3$, thus manifesting a remarkable contrast to the corresponding one-dimensional analogue in which at least the particular point $(m, q) = (0, 1)$ is known to belong to the regime of unconditional boundedness of solutions [4]:

**Corollary 1.6.** Let $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and let

$$D(s) := (s + 1)^{q - \frac{2}{n}} \quad \text{and} \quad S(s) := s(s + 1)^{q - 1}, \quad s \geq 0, \tag{1.26}$$

with some $q \in \mathbb{R}$. Then $m_c^{(\text{classical})}$ as introduced in Corollary 1.4 is finite and positive.

With regard to the particularly application-relevant version of (1.2) that accounts for volume-filling effects through the relations in (1.4), our results finally reveal mass criticality whenever the probability distribution function $Q$ therein exhibits a certain type of exponentially fast decay:
Corollary 1.7. Let \( n \geq 3, R > 0 \) and \( \Omega = B_R(0) \subset \mathbb{R}^n \), and let \( D \) and \( S \) be defined through (1.4) with
\[
Q(s) := a \exp \left( -bs^{\frac{n-2}{n}} \right), \quad s \geq 0,
\]
where \( a > 0 \) and \( b > 0 \). Then (1.2) exhibits a critical mass phenomenon in the sense that the number \( m_{c(\text{weak})} \) from Corollary 1.4 is finite and positive.

2 ENERGY-INDUCED BOUNDS FOR SMALL-MASS SOLUTIONS

The core of our existence analysis both in classical and in generalized frameworks can be found in the following lower estimate for the energy functional from (1.8) and (1.9) at suitably small mass levels of the first among its arguments.

Lemma 2.1. Let \( K_{SD} > 0 \). Then there exist \( m_0 = m_0(K_{SD}) > 0 \) and \( C = C(K_{SD}) > 0 \) such that if (1.7) and (1.12) hold, then with \( \mathcal{F} \) taken from (1.9), we have
\[
\mathcal{F}(\phi, \psi) \geq \frac{1}{C} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{C} \int_{\Omega} \psi^2 + \frac{1}{C} \int_{\Omega} \phi^{\frac{2n-2}{n}} - C \tag{2.1}
\]
for all \( \psi \in W^{1,2}(\Omega) \) and each \( \phi \in C^0(\overline{\Omega}) \) fulfilling \( \phi > 0 \) in \( \overline{\Omega} \) as well as
\[
\int_{\Omega} \phi \leq m_0. \tag{2.2}
\]

Proof. We first estimate
\[
\int_1^S \int_1^\sigma \frac{d\tau d\sigma}{\tau^{\frac{2}{n}}} = \frac{n^2}{2(n-1)(n-2)} s^{\frac{2n-2}{n}} - \frac{n}{n-2} s + \frac{n}{2(n-1)} \geq c_1 s^{\frac{2n-2}{n}} - c_2 s \quad \text{for all } s > 0 \tag{2.3}
\]
with \( c_1 := n^2/[2(n-1)(n-2)] \) and \( c_2 := n/(n-2) \), and then rely on the continuity of the embedding \( W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega) \) to fix \( c_3 > 0 \) such that
\[
\|\psi\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq c_3 \cdot \left\{ \int_{\Omega} |\nabla \psi|^2 + \int_{\Omega} \psi^2 \right\}^{\frac{1}{2}} \quad \text{for all } \psi \in W^{1,2}(\Omega). \tag{2.4}
\]
Then given \( K_{SD} > 0 \), we choose \( m_0 = m_0(K_{SD}) > 0 \) small enough such that
\[
m_0^{\frac{2}{n}} \leq \frac{c_1}{2c_3^2 K_{SD}}, \tag{2.5}
\]
and assume that (1.7) and (1.12) hold, and that \( \psi \in W^{1,2}(\Omega) \) and \( \phi \in C^0(\overline{\Omega}) \) are such that \( \phi > 0 \) in \( \Omega \) and that (2.2) is valid. Then according to (1.12), the function \( G \) in (1.10) satisfies

\[
G(s) \geq \frac{1}{K_{SD}} \int_1^s \int_1^\sigma \frac{d\tau d\sigma}{\tau^{\frac{2}{n}}} \geq \frac{c_1}{K_{SD}} s^{\frac{2n-2}{n}} - \frac{c_2}{K_{SD}} s \quad \text{for all } s > 0,
\]

so that by (1.9),

\[
\mathcal{F}(\phi, \psi) \geq \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{2} \int_{\Omega} \psi^2 - \int_{\Omega} \phi \psi + \frac{c_1}{K_{SD}} \int_{\Omega} \phi^{\frac{2n-2}{n}} - \frac{c_2}{K_{SD}} \int_{\Omega} \phi.
\]

Now combining the Hölder inequality with (2.4) and Young’s inequality, we see that here

\[
\int_{\Omega} \phi \psi \leq \|\phi\|_{L^{\frac{2n}{n+2}}(\Omega)} \|\psi\|_{L^{\frac{2n}{n-2}}(\Omega)}
\]

\[
\leq c_3 \|\phi\|_{L^{\frac{2n}{n+2}}(\Omega)} \left\{ \int_{\Omega} |\nabla \psi|^2 + \int_{\Omega} \psi^2 \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{4} \int_{\Omega} \psi^2 + c_2^2 \|\phi\|^2_{L^{\frac{2n}{n+2}}(\Omega)},
\]

where again due to the Hölder inequality,

\[
c_2^2 \|\phi\|^2_{L^{\frac{2n}{n+2}}(\Omega)} \leq c_3^2 \|\phi\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \|\phi\|^2_{L^1(\Omega)}
\]

\[
\leq \frac{c_1}{2K_{SD}} \|\phi\|^2_{L^{\frac{2n}{n-2}}(\Omega)}
\]

because of (2.2) and (2.5). Therefore, (2.6) entails that

\[
\mathcal{F}(\phi, \psi) \geq \frac{1}{4} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{4} \int_{\Omega} \psi^2 + \frac{c_1}{2K_{SD}} \int_{\Omega} \phi^{\frac{2n-2}{n}} - \frac{c_2}{K_{SD}} \int_{\Omega} \phi
\]

and hence entails (2.1) with \( C \equiv C(K_{SD}) := \max\{4, \frac{2K_{SD}}{c_1}, \frac{c_2 m_0}{K_{SD}}\}. \)

Indeed, due to the energy identity (1.8), the latter implies some basic information on integrability properties of \( u \) and \( \nabla v \).

**Lemma 2.2.** Let \( K_{SD} > 0 \), and suppose that (1.14) holds with \( \int_{\Omega} u_0 < m_0(K_{SD}) \), where \( m_0(K_{SD}) > 0 \) is as in Lemma 2.1. Then there exists \( C > 0 \) such that whenever (1.7) and (1.12) hold and \( (u, v) \) is a classical solution of (1.2) in \( \Omega \times (0, T) \) for some \( T \in (0, \infty] \), we have

\[
\int_{\Omega} u^{\frac{2n-2}{n}}(\cdot, t) \leq C \quad \text{for all } t \in (0, T)
\]
\[
\int_\Omega |\nabla v(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T).
\] (2.8)

**Proof.** By means of Lemma 2.1, we can choose \( c_1 = c_1(K_{SD}) > 0 \) and \( c_2 = c_2(K_{SD}) > 0 \) such that the functional from (1.9) has the property that

\[
F(\phi, \psi) \geq c_1 \int_\Omega |\nabla \psi|^2 + c_1 \int_\Omega \phi^{\frac{2n-2}{n}} - c_2
\]

for all \( \psi \in W^{1,2}(\Omega) \) and any positive \( \phi \in C^0(\overline{\Omega}) \) fulfilling (2.2). According to (1.8), for the solution under consideration, we thus have

\[
c_1 \int_\Omega |\nabla v(\cdot, t)|^2 + c_1 \int_\Omega u^{\frac{2n-2}{n}}(\cdot, t) \leq F(u(\cdot, t), v(\cdot, t)) + c_2 \leq F(u_0, v_0) + c_2
\]

for all \( t \in (0, T) \). □

Especially, through the bound in (2.8), the above lemma can be relied on to derive further regularity properties by means of the quasi-energy inequality in the following statement. This will be used in Lemma 3.2 to address classical solutions, and later on again in Lemma 4.5 in the course of our construction of very weak energy solutions.

**Lemma 2.3.** Let \( p \geq 2, K_{SD} > 0 \) and \( M > 0 \). Then there exists \( C(p, K_{SD}, M) > 0 \) with the property that if (1.7) and (1.12) hold, and if for some \( T \in (0, \infty] \) and some \((u_0, v_0)\) fulfilling (1.14), we are given a classical solution \((u, v)\) of (1.2) in \( \Omega \times (0, T) \) which is such that

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq M \quad \text{and} \quad \|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leq M \quad \text{for all } t \in (0, T),
\] (2.9)

then for all \( t \in (0, T) \),

\[
\frac{d}{dt} \left\{ \int_\Omega \Phi_p(u) + \int_\Omega |\nabla v|^2 q \right\} + \frac{1}{C(p, K_{SD}, M)} \cdot \left\{ \int_\Omega u^{p-2} |\nabla u|^2 + \int_\Omega |\nabla v|^2 q \right\} \leq C(p, K_{SD}, M),
\] (2.10)

where \( q := \frac{np}{2(n-1)} \) and

\[
\Phi_p(s) := \int_0^s \int_0^\sigma \frac{\tau^{p-2}}{D(\tau)} d\tau d\sigma, \quad s \geq 0.
\] (2.11)

**Proof.** By (1.2) and the definition of \( \Phi_p \), due to (1.12) and Young’s inequality we have

\[
\frac{d}{dt} \int_\Omega \Phi_p(u) = \int_\Omega \Phi'_p(u) \nabla \cdot \left\{ D(u) \nabla u - S(u) \nabla v \right\}
= -\int_\Omega \Phi''_p(u) D(u) |\nabla u|^2 + \int_\Omega \Phi''_p(u) S(u) \nabla u \cdot \nabla v
= -\int_\Omega u^{p-2} |\nabla u|^2 + \int_\Omega u^{p-2} \frac{S(u)}{D(u)} \nabla u \cdot \nabla v.
\]
\[
\leq -\frac{1}{2} \int_{\Omega} u^{p-2}|\nabla u|^2 + \frac{1}{2} \int_{\Omega} u^{p-2} \frac{S^2(u)}{D^2(u)} |\nabla u|^2 \\
\leq -\frac{1}{2} \int_{\Omega} u^{p-2}|\nabla u|^2 + \frac{K_{SD}^2}{2} \int_{\Omega} u^{p-2+\frac{4}{n}} |\nabla u|^2 \quad \text{for all } t \in (0, T),
\] (2.12)

while a routine computation on the basis of the second equation in (1.2) shows that

\[
\frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2q-1} \nabla v \cdot (\nabla \Delta v - \nabla v + \nabla u) \\
= \frac{1}{2} \int_{\Omega} |\nabla v|^{2q-2} \Delta |\nabla v|^2 - \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 - \int_{\Omega} |\nabla v|^{2q} \\
- \int_{\Omega} u |\nabla v|^{2q-4} \cdot \left\{ 2(q-1) \nabla v \cdot (D^2 \cdot \nabla v) + |\nabla v|^2 \Delta v \right\} \\
= -\frac{2(q-1)}{q^2} \int_{\Omega} |\nabla \nabla v|^q \cdot \left[ \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 - \int_{\Omega} |\nabla v|^{2q} \\
- \int_{\Omega} u |\nabla v|^{2q-4} \cdot \left\{ 2(q-1) \nabla v \cdot (D^2 \cdot \nabla v) + |\nabla v|^2 \Delta v \right\} \\
+ \frac{1}{2} \int_{\partial \Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial v} \quad \text{for all } t \in (0, T). \] (2.13)

Here relying on the compactness of the embedding \(W^{\frac{1}{2}, 2}(\Omega) \hookrightarrow L^2(\partial \Omega)\), and on the existence of \(c_1 > 0\) such that in accordance with a well-known result we have \(\frac{\partial |\phi|^q}{\partial v} \leq c_1 |\nabla \phi|^2\) on \(\partial \Omega\) for all \(\phi \in C^2(\overline{\Omega})\) such that \(\frac{\partial \phi}{\partial v} = 0\) on \(\partial \Omega\) [38], we can find \(c_2 = c_2(p) > 0\) such that

\[
-\frac{1}{2} \int_{\Omega} |\nabla \phi|^{2q-2} \frac{\partial |\nabla \phi|^2}{\partial v} \leq \frac{q-1}{q^2} \int_{\Omega} |\nabla \nabla \phi|^q \cdot \left\{ \int_{\Omega} |\nabla \phi|^2 \right\}^q \quad \text{for all } \phi \in C^2(\overline{\Omega})
\]

fulfilling \(\frac{\partial \phi}{\partial v} = 0\) on \(\partial \Omega\).

As furthermore, writing \(c_3 = c_3(p) := 2(q-1) + \sqrt{n}\), we have

\[
- \int_{\Omega} u |\nabla v|^{2q-4} \cdot \left\{ 2(q-1) \nabla v \cdot (D^2 \cdot \nabla v) + |\nabla v|^2 \Delta v \right\} \\
\leq c_3 \int_{\Omega} u |\nabla v|^{2q-2} |D^2 v| \leq \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + \frac{c_3^2}{4} \int_{\Omega} u^2 |\nabla v|^{2q-2} 
\]

for all \(t \in (0, T)\) by Young’s inequality, from (2.12), (2.13) and (2.9), we thus obtain that there exists \(c_4 = c_4(p, K_{SD}, M) > 0\) such that

\[
\frac{d}{dt} \left\{ \int_{\Omega} \Phi_p(u) + \int_{\Omega} |\nabla v|^{2q} \right\} + \frac{1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \int_{\Omega} |\nabla v|^{2q} + \frac{q-1}{q^2} \int_{\Omega} |\nabla |\nabla v||^q \right\} \\
\leq c_4 \int_{\Omega} u^{p-2+\frac{4}{n}} |\nabla v|^2 + c_4 \int_{\Omega} u^2 |\nabla v|^{2q-2} + c_4 \quad \text{for all } t \in (0, T). \] (2.14)
Now once more due to Young’s inequality,

\[
c_4 \int_{\Omega} u^{p-2} |\nabla v|^2 + c_4 \int_{\Omega} u^2 |\nabla v|^{2q-2} \leq c_4 \int_{\Omega} u^{p+\frac{2}{n}} + c_4 \int_{\Omega} |\nabla v|^\frac{np+2}{n-1} + c_4 \int_{\Omega} u^{p+\frac{2}{n}} + c_4 \int_{\Omega} |\nabla v|^\frac{np+2}{np-2n+4} (2q-2) \]

\[
= 2c_4 \int_{\Omega} u^{p+\frac{2}{n}} + 2c_4 \int_{\Omega} |\nabla v|^\frac{np+2}{n-1} \quad \text{for all } t \in (0, T)
\]

(2.15)

according to our choice of \( q \). Here we may combine the Gagliardo–Nirenberg inequality with Young’s inequality and (2.9) to find \( c_i = c_i(p, K_{SD}, M) > 0, \ i \in \{5, \ldots, 10\} \), such that since \( p \geq 2 \) implies that \( a := \frac{2np}{(n+2)(2-n+np)} \) and \( b := \frac{np(np-2n+4)}{(n+2)(2-n+np)} \) both belong to \((0,1)\) and satisfy

\[
\frac{2(np+2)a}{np} = \frac{4}{2-n+np} \leq \frac{4}{2+n} < 2
\]

as well as

\[
\frac{(np+2)b}{(n-1)q} - 2 = \frac{8-4n}{n^2(p-2)+6n-4} < 0,
\]

we have

\[
2c_4 \int_{\Omega} u^{p+\frac{2}{n}} = 2c_4 \| u^\frac{p}{2} \|_{L^{\frac{2(np+2)}{np}}(\Omega)}^{\frac{2(np+2)}{np}} \leq c_5 \| \nabla u^\frac{p}{2} \|_{L^{\frac{2}{p}}(\Omega)} \| u^\frac{p}{2} \|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{n}} + c_5 \| u^\frac{p}{2} \|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{n}} \leq \frac{1}{4} \int_{\Omega} u^{p-2} |\nabla u|^2 + c_7 \quad \text{for all } t \in (0, T)
\]

and

\[
2c_4 \int_{\Omega} |\nabla v|^\frac{np+2}{n-1} = 2c_4 \| |\nabla v|^q \|_{L^{\frac{(np+2)b}{(n-1)q}}(\Omega)}^{\frac{np+2}{(n-1)q}} \leq c_8 \| \nabla |\nabla v|^q \|_{L^{\frac{(np+2)b}{(n-1)q}}(\Omega)} \| |\nabla v|^q \|_{L^{\frac{2}{q}}(\Omega)}^{\frac{(np+2)b}{(n-1)q}} + c_8 \| |\nabla v|^q \|_{L^{\frac{2}{q}}(\Omega)} \leq c_9 \| \nabla |\nabla v|^q \|_{L^{\frac{(np+2)b}{(n-1)q}}(\Omega)} + c_9 \leq \frac{q-1}{q^2} \int_{\Omega} |\nabla |\nabla v|^q |^2 + c_{10} \quad \text{for all } t \in (0, T).
\]
In view of (2.15), from (2.14) we thus infer that
\[
\frac{d}{dt} \left\{ \int_\Omega \Phi_p(u) + \int_\Omega |\nabla v|^2 q \right\} + \frac{1}{4} \int_\Omega u^{p-2} |\nabla u|^2 + \int_\Omega |\nabla v|^2 q \leq c_4 + c_7 + c_{10} \quad \text{for all } t \in (0, T)
\]
and conclude as intended. □

3 GLOBAL BOUNDED CLASSICAL SOLUTIONS. PROOF OF THEOREM 1.1

Now in order to suitably utilize the above in classical solution frameworks, let us recall from standard theory [1, 34] that smooth solutions exist at least locally in time.

Lemma 3.1. Let \( n \geq 3 \) and \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary, and suppose that \( D, S, u_0 \) and \( v_0 \) satisfy (1.7) and (1.14). Then there exist \( T_{\max} \in (0, \infty] \) and functions
\[
\begin{align*}
\{ u &\in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \quad \text{and} \\
v &\in \bigcap_{r>1} C^0([0, T_{\max}); W^{1,r}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))
\end{align*}
\]
such that \( u > 0 \) and \( v \geq 0 \) in \( \overline{\Omega} \times [0, T_{\max}) \), that \( (u, v) \) solves (1.2) classically in \( \Omega \times (0, T_{\max}) \), and that
\[
\text{if } T_{\max} < \infty, \quad \text{then } \limsup_{t \to T_{\max}} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,r}(\Omega)} \right\} = \infty \text{ for all } r > 1. \tag{3.1}
\]
Moreover,
\[
\int_\Omega u(\cdot, t) = \int_\Omega u_0 \quad \text{for all } t \in (0, T_{\max}), \tag{3.2}
\]
and if additionally \( \Omega = B_R(0) \) with some \( R > 0 \) and (1.17) holds, then \( (u(\cdot, t), v(\cdot, t)) \) is radially symmetric with respect to \( x = 0 \) for all \( t \in (0, T_{\max}) \).

Then under the additional hypothesis in (1.13), appropriate selection of the parameter \( p \) in Lemma 2.3 enables us to derive bounds for the first component of these solutions, as well as for the gradient of the second, in Lebesgue spaces involving arbitrary finite integrability exponents, provided that the assumption on smallness of mass from Lemma 2.3 is satisfied.

Lemma 3.2. Assume (1.7), (1.12) and (1.13) with some \( K_{SD} > 0, \beta \geq 0 \) and \( \gamma \geq 0 \), and suppose that (1.14) holds with \( \int_\Omega u_0 < m_0 \), where \( m_0 = m_0(K_{SD}) > 0 \) is as in Lemma 2.1. Then for all \( r > 1 \) there exists \( C(r) > 0 \) such that
\[
\int_\Omega u^r(\cdot, t) \leq C(r) \quad \text{for all } t \in (0, T_{\max}) \tag{3.3}
\]
and
\[
\int_\Omega |\nabla v(\cdot, t)|^r \leq C(r) \quad \text{for all } t \in (0, T_{\max}). \tag{3.4}
\]
Proof. Given \( r > 1 \), we fix \( p = p(r) \geq 2 \) large such that
\[
p > 1 - \beta, \quad p > \frac{(n-2)\beta}{2}, \quad p > r + \gamma \quad \text{and} \quad p \geq \frac{(n-1)r}{n},
\] (3.5)
and invoke Lemma 2.3 along with (3.2) and Lemma 2.2 to see that writing \( q = \frac{np}{2(n-1)} \) we can find \( c_1 = c_1(r) > 0 \) and \( c_2 = c_2(r) > 0 \) such that the function \( \Phi_p \) introduced in (2.11) satisfies
\[
\frac{d}{dt} \left\{ \int_\Omega \Phi_p(u) + \int_\Omega |\nabla v|^{2q} \right\} + c_1 \int_\Omega |\nabla u|^p + c_1 \int_\Omega |\nabla v|^{2q} \leq c_2 \quad \text{for all } t \in (0, T_{max}).
\] (3.6)
We moreover use (1.13) to see that with some positive constants \( c_i = c_i(r), i \in \{3, 4, 5, 6\} \), we have
\[
c_3 s^{p+\beta-2} + c_4 \geq \frac{s^{p-2}}{D(s)} \geq c_5 s^{p-\gamma-2} - c_6 \quad \text{for all } s > 0
\]
and hence
\[
c_3 \int_\Omega u^{p+\beta} + c_4 |\Omega| \geq \int_\Omega \Phi_p(u) \geq c_5 \int_\Omega u^{p-\gamma} - c_6 |\Omega| \quad \text{for all } t \in (0, T_{max}).
\] (3.7)
We furthermore introduce
\[
a := \frac{np(p + \beta - 1)}{(2 - n + np)(p + \beta)}
\]
and note that the first two restrictions in (3.5) ensure that \( a \in (0, 1) \), and that thus the Gagliardo–Nirenberg inequality applies so as to show that once more due to (3.2), with some \( c_7 = c_7(r) > 0 \) and \( c_8 = c_8(r) > 0 \), we have
\[
\int_\Omega u^{p+\beta} = \|u^\frac{p}{2(p+\beta)}\|_{L^\frac{2(p+\beta)}{p}(\Omega)}^{2(p+\beta)} \leq c_7 \|\nabla u^\frac{p}{2(\beta+1-a)}\|_{L^\frac{2(\beta+1-a)}{p}(\Omega)} \|u^\frac{p}{2(\beta+1-a)}\|_{L^\frac{2(p+\beta)}{p}(\Omega)} + c_7 \|u^\frac{p}{2(\beta+1-a)}\|_{L^\frac{2(p+\beta)}{p}(\Omega)} \]
\[
\leq c_8 \|\nabla u^\frac{p}{2(\beta+1-a)}\|_{L^\frac{2(p+\beta)}{p}(\Omega)} + c_8 \quad \text{for all } t \in (0, T_{max}).
\]
Writing \( \lambda := \min\{\frac{p}{p+\beta a}, 1\} \), by using Young’s inequality we thus infer from (3.7) that
\[
y(t) := \int_\Omega \Phi_p(u(\cdot, t)) + \int_\Omega |\nabla v(\cdot, t)|^{2q}, \quad t \in [0, T_{max}),
\]
satisfies
\[
y^\lambda(t) \leq 2^\lambda \cdot \left\{ \int_\Omega \Phi_p(u) \right\}^\lambda + 2^\lambda \cdot \left\{ \int_\Omega |\nabla v|^{2q} \right\}^\lambda
\leq 2^\lambda \cdot \left\{ \int_\Omega \Phi_p(u) \right\}^\frac{p}{p+\beta a} + 2^\lambda \int_\Omega |\nabla v|^{2q} + 2^{\lambda+1}
\]
\[ \leq 2^2 \cdot \left\{ c_3 c_8 \| \nabla u_p^2 \|_{L^2(\Omega)}^{\frac{p}{p+\beta a}} + c_3 c_8 + c_4 |\Omega| \right\}^{\frac{p}{p+\beta a}} + 2^2 \int_\Omega |\nabla v|^{2q} + 2^{\lambda+1} \]

\[ \leq c_9 \int_\Omega |\nabla u^p|^2 + c_9 \int_\Omega |\nabla v|^{2q} \quad \text{for all } t \in (0, T_{\text{max}}) \]

with \( c_9 = c_9(r) := \max\{2^2 \cdot (2c_3 c_8)^{\frac{p}{p+\beta a}} + 2^2 \cdot (2c_3 c_8 + 2c_4 |\Omega|)^{\frac{p}{p+\beta a}} + 2^{\lambda+1}\} \). Therefore, (3.6) entails that

\[ y'(t) + \frac{c_1}{c_9} y^\lambda(t) \leq c_1 + c_2 \quad \text{for all } t \in (0, T_{\text{max}}) \]

and that hence, by comparison,

\[ y(t) \leq c_{10} = c_{10}(r) := \max \left\{ y(0), \left\{ \frac{(c_1 + c_2)c_9}{c_1} \right\}^{\frac{1}{\lambda}} \right\} \quad \text{for all } t \in (0, T_{\text{max}}). \]

In view of (3.7), this shows that

\[ c_5 \int_\Omega u^{p-\gamma} + \int_\Omega |\nabla v|^{2q} \leq c_6 |\Omega| + c_{10} \quad \text{for all } t \in (0, T_{\text{max}}) \]

and thereby establishes (3.3) and (3.4), because

\[ \int_\Omega u^r \leq \int_\Omega u^{p-\gamma} + |\Omega| \quad \text{for all } t \in (0, T_{\text{max}}) \]

and

\[ \int_\Omega |\nabla v|^r \leq \int_\Omega |\nabla v|^{2q} + |\Omega| \quad \text{for all } t \in (0, T_{\text{max}}) \]

according to Young’s inequality and the two rightmost conditions in (3.5). \□

Once more due to (1.13), a standard iterative argument thereupon becomes applicable so as to assert even an \( L^\infty \) bound for the first component of any such small-mass solution.

**Lemma 3.3.** Suppose that (1.7), (1.12) and (1.13) hold with some \( K_{SD} > 0, \beta \geq 0 \) and \( \gamma \geq 0 \), and that (1.14) is satisfied with \( \int_\Omega u_0 < m_0 \), where \( m_0 = m_0(K_{SD}) > 0 \) is as in Lemma 2.1. Then there exists \( C > 0 \) such that

\[ \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \]

**Proof.** Once more relying on the algebraic lower decay estimate for \( D \) contained in (1.13), this can be derived by employing a well-known result of a Moser-type iteration procedure ([46]) using Lemma 3.2, applied to suitably large \( r > 1 \), as a starting point. \□
Proof of Theorem 1.1. In light of Lemma 3.1, the statement is a direct consequence of Lemma 3.3 and Lemma 3.2.

4 | GLOBAL VERY WEAK ENERGY SOLUTIONS. PROOF OF THEOREM 1.2

4.1 | The notion of global very weak energy solutions

In order to facilitate applicability of Lemmas 2.2 and 2.3 also in the presence of asymptotic behavior of $D$ that is too degenerate to be compatible with (1.13), we shall resort to a generalized solution concept which with regard to the fulfillment of the differential equations in (1.2) is very weak, especially through the requirement that the first sub-problem in (1.2) be rather satisfied in the sense of some weak supersolution property only, after all accompanied with an additional information on mass conservation. Thus orienting our approach along precedents of solution constructions in potentially quite irregular chemotaxis systems [49, 52], we thereby particularly rely on a notion of solution that is even yet weaker than that of natural weak solvability, but, as will become important in our nonexistence arguments in the context of Theorem 1.3, despite this substantial weakening we will retain some crucial information on the energy structure expressed in (1.8) in a sense that will turn out to be suitable for our subsequent approximation procedure.

Our formulation thereof will be prepared by the following lemma which introduces an auxiliary function $\Sigma$, and which for later reference in Section 5.2 summarizes some evident basic properties thereof.

**Lemma 4.1.** Assume (1.7), and with $h$ taken from (1.11), let

$$\Sigma(s) := \begin{cases} K\Sigma \sqrt{\frac{S(s)}{S(s)+1}}, & s \in [0, 1], \\ \sqrt{\frac{S(s)}{S(s)+1}} \cdot \frac{h(s)}{h(s)+1}, & s > 1, \end{cases}$$

(4.1)

where $K\Sigma := \frac{h(1)}{h(1)+1} \in (0, 1)$. Then $\Sigma \in C^0([0, \infty))$ with $\Sigma(s) > 0$ for all $s > 0$, and we have $\Sigma^2(s) \leq S(s)$ and $\Sigma(s) \leq 1$ for all $s \geq 0$ as well as $\frac{\Sigma(s)}{h(s)} \leq 1$ for all $s > 1$ and $\sup_{s \in (0, 1)} \frac{h(s)}{\Sigma(s)} < \infty$.

We can now specify the solution concept to be subsequently pursued.

**Definition 4.2.** Assume (1.7) and (1.14), and let

$$\begin{cases} u \in L^\infty_{loc}([0, \infty); L^1(\Omega)) \\ v \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)) \end{cases}$$

(4.2)

be such that $u > 0$ and $v \geq 0$ almost everywhere (a.e.) in $\Omega \times (0, \infty)$, that with $\Sigma$ and $h$ as in (4.1) and (1.11), we have

$$\frac{\Sigma(u)}{h(u)} \nabla u \in L^2_{loc}(\Omega \times [0, \infty); \mathbb{R}^n) \quad \text{and} \quad v_t \in L^2_{loc}(\Omega \times [0, \infty)),$$

(4.3)
and that there exists a null set \( N_* \subset (0, \infty) \) such that

\[
v(\cdot, t) \in W^{1,2}(\Omega), \quad u(\cdot, t)v(\cdot, t) \in L^1(\Omega) \quad \text{and} \quad G(u(\cdot, t)) \in L^1(\Omega) \quad \text{for all} \quad t \in (0, \infty) \setminus N_*,
\]

where \( G \) is as defined in (1.10). Then we will call \((u, v)\) a global very weak energy solution of (1.2) if with \( F \) taken from (1.9) and with

\[
D(t) := \int_{\Omega} v_t^2(\cdot, t) + \int_{\Omega} \left| \Sigma(u(\cdot, t)) \left( \frac{\nabla u(\cdot, t)}{h(u(\cdot, t))} - \frac{\nabla u(\cdot, t)}{h(u(\cdot, t))} \right) \right|^2, \quad t > 0,
\]

we have

\[
F(u(\cdot, t), v(\cdot, t)) + \int_0^t D(s) \, ds \leq F(u_0, v_0) \quad \text{for all} \quad t \in (0, \infty) \setminus N_*,
\]

if \( \text{ess lim}_{t \downarrow 0} \| v(\cdot, t) - v_0 \|_{L^2(\Omega)} = 0 \) and

\[
\int_0^\infty \int_{\Omega} v \, \varphi = - \int_0^\infty \int_{\Omega} v \, \nabla \varphi - \int_0^\infty \int_{\Omega} \varphi \, \nabla v + \int_0^\infty \int_{\Omega} u \varphi
\]

for all \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \) and

\[
\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0 \quad \text{for all} \quad t \in (0, \infty) \setminus N_*,
\]

and if for each \( \chi \in C^\infty([0, \infty)) \) such that \( \chi' \) is nonnegative and has compact support with \( \chi'' \leq 0 \), the inequality

\[
- \int_0^\infty \int_{\Omega} \chi(u) \varphi_t - \int_{\Omega} \chi(u_0) \varphi(\cdot, 0)
\geq - \int_0^\infty \int_{\Omega} \chi''(u) D(u) |\nabla u|^2 \varphi + \int_0^\infty \int_{\Omega} \chi''(u) S(u)(\nabla u \cdot \nabla v) \varphi
- \int_0^\infty \int_{\Omega} \chi'(u) D(u) \nabla u \cdot \nabla \varphi + \int_0^\infty \int_{\Omega} \chi'(u) S(u) \nabla u \cdot \nabla \varphi
\]

is valid for any nonnegative \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \). If furthermore \( u(\cdot, t) \) and \( v(\cdot, t) \) are radially symmetric with respect to \( x = 0 \) for all \( t \in (0, \infty) \setminus N_* \), then we say that \((u, v)\) is a radial global weak energy solution of (1.2).

**Remark.**

(i) Since \( \Sigma \) is bounded, the regularity assumptions in (4.2) and (4.3) indeed ensure that each of the integrals making up (4.6)–(4.8) is well defined. Apart from that, (4.3) moreover warrants that \( \rho(u) \nabla u \in L^2_{\text{loc}}(\Omega \times [0, \infty); \mathbb{R}^n) \) for any \( \rho \in C^\infty([0, \infty)) \) and that thus, again by (4.2), also all expressions in (4.9) are meaningful.
(ii) If \((u, v)\) is a global very weak energy solution which additionally belongs to \((C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)))^2\), then a standard reasoning shows that due to (4.7) we have \(v_t = \Delta v - v + u\) in \(\Omega \times (0, \infty)\) with \(\frac{\partial u}{\partial \nu}|_{\partial \Omega} = 0\) and \(v|_{t=0} = v_0\), and that (4.9) implies that \(u_t \geq \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v)\) in \(\Omega \times (0, \infty)\) with \((D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu})|_{\partial \Omega} \geq 0\) and \(u|_{t=0} \geq u_0\). The mass conservation (3.2) thereupon guarantees that in fact the latter three inequalities must actually be identities (cf. [52, Lemma 2.1] for a detailed argument addressing a closely related setting).

4.2 Construction of global very weak energy solutions. Proof of Theorem 1.2

To construct such generalized solutions via approximation by smooth solutions to suitably regularized problems, given \(D\) and \(S\) complying with (1.7) we let

\[
D_\varepsilon(s) := \frac{D(s) + \varepsilon}{1 + \varepsilon D(s)}, \quad s \geq 0, \quad \varepsilon \in (0, 1),
\]

(4.10)

and

\[
S_\varepsilon(s) := \frac{S(s)}{1 + \varepsilon D(s)}, \quad s \geq 0, \quad \varepsilon \in (0, 1),
\]

(4.11)

as well as

\[
h_\varepsilon(s) := \frac{S(s)}{D(s) + \varepsilon}, \quad s > 0, \quad \varepsilon \in (0, 1),
\]

(4.12)

and then begin with the following collection of elementary observations.

**Lemma 4.3.** Suppose that (1.7) and (1.12) hold with some \(K_{SD} > 0\). Then with \(D_\varepsilon\) and \(S_\varepsilon\) as in (4.10) and (4.11), we have

\[
\frac{S_\varepsilon(s)}{D_\varepsilon(s)} \leq \frac{S(s)}{D(s)} \leq K_{SD} s^{\frac{2}{n}} \quad \text{for all } s \geq 0 \text{ and } \varepsilon \in (0, 1)
\]

(4.13)

and

\[
\varepsilon \leq D_\varepsilon(s) \leq \frac{1}{\varepsilon} \quad \text{for all } s \geq 0 \text{ and } \varepsilon \in (0, 1)
\]

(4.14)

as well as

\[
D_\varepsilon(s) \geq \frac{D(s)}{1 + D(s)} \quad \text{for all } s \geq 0 \text{ and } \varepsilon \in (0, 1).
\]

(4.15)

Moreover, for \(\varepsilon \in (0, 1)\), writing

\[
\Sigma_\varepsilon(s) := \begin{cases} 
K_{\Sigma\varepsilon} \sqrt{\frac{S_\varepsilon(s)}{S_\varepsilon(s) + 1}}, & s \in [0, 1], \\
\sqrt{\frac{S_\varepsilon(s)}{S_\varepsilon(s) + 1}}, & \varepsilon \in (0, 1),
\end{cases}
\]

(4.16)
with \( K_{\varepsilon} := \frac{\varepsilon}{h_{\varepsilon}(1) + \varepsilon} \), we have \( \Sigma_\varepsilon \in C^0([0, \infty)) \) with

\[
0 < \Sigma_\varepsilon \leq \sqrt{S_\varepsilon} \quad \text{on} \ (0, \infty)
\]

for all \( \varepsilon \in (0, 1) \), and \( \Sigma_\varepsilon \to \Sigma \) in \( L^\infty_{\text{loc}}([0, \infty)) \) as \( \varepsilon \searrow 0 \), where \( \Sigma \) is as in (4.1).

Proof. While (4.13) and (4.15) are obvious, (4.14) follows from the observation that \( \varepsilon \leq \frac{\xi + \varepsilon}{1 + \xi \varepsilon} \leq \frac{1}{\varepsilon} \) for all \( \xi \geq 0 \) and \( \varepsilon \in (0, 1) \). The statements concerning \( \Sigma_\varepsilon \) are evident.

When applied to the regularized variants of (1.2) given by

\[
\begin{cases}
  u_{\varepsilon t} = \nabla \cdot (D_\varepsilon(\varepsilon) \nabla u_{\varepsilon}) - \nabla \cdot (S_\varepsilon(\varepsilon) \nabla v_{\varepsilon}), & x \in \Omega, \ t > 0, \\
  v_{\varepsilon t} = \Delta v_{\varepsilon} - v_{\varepsilon} + u_{\varepsilon}, & x \in \Omega, \ t > 0, \\
  \frac{\partial u_{\varepsilon}}{\partial \nu} = \frac{\partial v_{\varepsilon}}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
  u_{\varepsilon}(x, 0) = u_0(x), \ v_{\varepsilon}(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\]

for \( \varepsilon \in (0, 1) \), in conjunction with Theorem 1.1, the latter particularly ensures global classical solvability therein:

**Lemma 4.4.** Assume (1.7) and (1.12) with some \( K_{SD} > 0 \), and let \( m_0 = m_0(K_{SD}) > 0 \) be as in Lemma 2.1. Then given any \( (u_0, v_0) \) fulfilling (1.14), for each \( \varepsilon \in (0, 1) \), one can find a pair \( (u_{\varepsilon}, v_{\varepsilon}) \) of functions

\[
\begin{cases}
  u_{\varepsilon} \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \\
  v_{\varepsilon} \in \bigcap_{q \geq 1} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\end{cases}
\]

such that \( u_{\varepsilon} > 0 \) and \( v_{\varepsilon} \geq 0 \) in \( \overline{\Omega} \times [0, \infty) \), and that \( (u_{\varepsilon}, v_{\varepsilon}) \) solves (4.18) in the classical sense in \( \Omega \times (0, \infty) \). Moreover,

\[
\int_{\Omega} u_{\varepsilon}(-, t) = \int_{\Omega} u_0 \quad \text{for all} \ t > 0,
\]

and

\[
F_{\varepsilon}(u_{\varepsilon}(-, t), v_{\varepsilon}(-, t)) + \int_0^t \int_{\Omega} v_{\varepsilon}^2 dt + \int_0^t \int_{\Omega} \sqrt{S_\varepsilon(\varepsilon) \frac{\nabla u_{\varepsilon}}{h_{\varepsilon}(\varepsilon)} - \sqrt{S_\varepsilon(\varepsilon) \nabla v_{\varepsilon}}}^2 dt
= F_{\varepsilon}(u_0, v_0) \quad \text{for all} \ t > 0,
\]

where

\[
F_{\varepsilon}(\phi, \psi) := \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{2} \int_{\Omega} \psi^2 - \int_{\Omega} \phi \psi + \int_{\Omega} G_\varepsilon(\phi),
\]

for \( \phi \in L^1(\Omega) \) and \( \psi \in W^{1,2}(\Omega) \) such that \( \phi > 0 \) a.e. in \( \Omega \).
and where with \((h_\varepsilon)_{\varepsilon \in (0,1)}\) as in (4.12)

\[
G_\varepsilon(s) := \int_1^s \int_1^\sigma \frac{d\tau d\sigma}{h_\varepsilon(\tau)}, \quad s > 0, \ \varepsilon \in (0,1).
\]

Finally, if in addition \(\Omega = B_R(0)\) with some \(R > 0\), and if (1.17) holds, then \((u_\varepsilon(\cdot, t), v_\varepsilon(\cdot, t))\) is radially symmetric with respect to \(x = 0\) for all \(t > 0\).

**Proof.** In view of Lemma 4.3 and (1.8), this readily follows on applying Theorem 1.1 to \(\beta := 0\) and \(\gamma := 0\).

In the context of rapidly decreasing diffusion rates \(D\), due to an apparently lacking appropriate dissipative action in (2.10) that might act as a growth restriction for the functional \(\int_\Omega \Phi_p(u) + \int_\Omega |\nabla v|^{2q}\) therein, for example, in the style of the argument from Lemma 3.2, corresponding temporally uniform and \(\varepsilon\)-independent spatial \(L^r\) estimates for the above approximate solutions seem not available, at least not on the basis of Lemma 2.3. By making adequate use of the dissipated quantity in (2.10), however, after all the following spatio-temporal *a priori* information can be gained.

**Lemma 4.5.** Assume (1.7) and (1.12) with some \(K_{SD} > 0\), let \(m_0 = m_0(K_{SD}) > 0\) be as in Lemma 2.1, and suppose that \((u_0, v_0)\) satisfies (1.14) with \(\int_\Omega u_0 < m_0\). Then for all \(p \geq 2\), there exists \(C(p) > 0\) such that

\[
\int_0^T \int_\Omega u_\varepsilon^p \leq C(p) \cdot (T + 1) \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1)
\]

and

\[
\int_0^T \int_\Omega |\nabla u_\varepsilon^p|^2 \leq C(p) \cdot (T + 1) \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1).
\]

**Proof.** Due to (4.13), we may invoke Lemma 2.3 to find \(c_1 = c_1(p) > 0\) and \(c_2 = c_2(p) > 0\) such that writing \(q := \frac{n p}{2(n-1)}\) and

\[
\Phi_{p,\varepsilon}(s) := \int_0^s \int_0^\sigma \frac{\tau^{p-2}}{D_\varepsilon(\tau)} d\tau d\sigma, \quad s \geq 0, \ \varepsilon \in (0,1),
\]

we have

\[
\frac{d}{dt} \left\{ \int_\Omega \Phi_{p,\varepsilon}(u_\varepsilon) + \int_\Omega |\nabla v_\varepsilon|^{2q} \right\} + c_1 \int_\Omega |\nabla u_\varepsilon^p| \leq c_2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\]

which on integration, by nonnegativity of \(\Phi_{p,\varepsilon}\), entails that

\[
c_1 \int_0^T \int_\Omega |\nabla u_\varepsilon^p| \leq \int_\Omega \Phi_{p,\varepsilon}(u_0) + \int_\Omega |\nabla v_0|^{2q} + c_2 T \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1).
\]
Here we note that due to (4.15),
\[ \Phi_{p, \epsilon}(s) \leq \Phi_p(s) := \int_0^s \int_0^\sigma \frac{\tau^{p-2}(1 + D(\tau))}{D(\tau)} d\tau d\sigma \quad \text{for all } s \geq 0 \text{ and } \epsilon \in (0, 1), \]
and that due to a Poincaré inequality and (4.19), there exists \( c_3 = c_3(p) > 0 \) such that
\[ \int_\Omega u_\epsilon^p \leq c_3 \int_\Omega |\nabla u_\epsilon|^2 + c_3 \quad \text{for all } t > 0 \text{ and each } \epsilon \in (0, 1). \]
Therefore, (4.25) yields the inequality
\[ \frac{c_1}{2} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + \int_0^T \int_\Omega u_\epsilon^p - \frac{c_1 T}{2} \leq \int_\Omega \Phi_p(u_0) + \int_\Omega |\nabla v_0|^2 + c_2 T \quad \text{for all } T > 0 \text{ and } \epsilon \in (0, 1) \]
and thus implies both (4.23) and (4.24).

Some further implications of the latter are now quite straightforward. First, through standard parabolic theory, the estimate (4.23) entails a second-order regularity information on \( v \), at least in regions away from the temporal origin:

**Lemma 4.6.** Suppose that (1.7) and (1.12) hold with some \( K_{SD} > 0 \), and let \((u_0, v_0)\) satisfy (1.14) with \( \int_\Omega u_0 < m_0 \), where \( m_0 = m_0(K_{SD}) > 0 \) is as in Lemma 2.1. Then for all \( p \geq 2 \), each \( \tau > 0 \) and any \( T > \tau \), one can find \( C(p, \tau, T) > 0 \) such that
\[ \int_\tau^T \|v_\epsilon(\cdot, t)\|_{W^{2, p}(\Omega)}^p dt \leq C(p, \tau, T) \quad \text{for all } \epsilon \in (0, 1). \]

**Proof.** This is an immediate consequence of Lemma 4.4 when combined with well-known maximal Sobolev regularity estimates [26] and a standard localization argument.

Due to (1.14), corresponding first-order regularity features, even up to \( L^\infty \) spaces, are available even down to \( t = 0 \):

**Lemma 4.7.** Let (1.7) and (1.12) be fulfilled with some \( K_{SD} > 0 \), let \( m_0 = m_0(K_{SD}) > 0 \) be as in Lemma 2.1, and let \((u_0, v_0)\) satisfy (1.14) with \( \int_\Omega u_0 < m_0 \). Then for all \( T > 0 \), there exists \( C(T) > 0 \) with the property that
\[ |\nabla v_\epsilon(x, t)| \leq C(T) \quad \text{for all } x \in \Omega, t \in (0, T) \text{ and } \epsilon \in (0, 1). \]

**Proof.** This follows on applying standard smoothing estimates for the Neumann heat semigroup to the second equation in (4.18), and employing Lemma 4.4 with suitably large \( p \geq 2 \).

Together with Lemma 4.5, these estimates in turn imply some favorable properties related to regularity of \( u_\epsilon \) in time.
Lemma 4.8. Assume (1.7) and (1.12) with some $K_{SD} > 0$, let $m_0 = m_0(K_{SD}) > 0$ be as in Lemma 2.1, and suppose that $(u_0, v_0)$ satisfies (1.14) with $\int_{\Omega} u_0 < m_0$. Moreover, let $l \in \mathbb{N}$ be such that $l > \frac{n}{2}$, and let $\chi \in C^\infty([0, \infty))$ be such that $\text{supp} \chi'$ is compact. Then for each $T > 0$, there exists $C(l, \chi, T) > 0$ such that

$$
\int_0^T \left\| \partial_t \chi(u_\varepsilon(\cdot, t)) \right\|_{(W^{1,2}(\Omega))^*} \, dt \leq C(l, \chi, T) \quad \text{for all } \varepsilon \in (0, 1).
$$

Proof. For $\psi \in C^\infty(\overline{\Omega})$, according to (4.18), Young's inequality and the Cauchy–Schwarz inequality, we have

$$
\left| \int_{\Omega} \partial_t \chi(u_\varepsilon) \psi \right| = \left| - \int_{\Omega} \chi''(u_\varepsilon) D_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 \psi + \int_{\Omega} \chi''(u_\varepsilon) S_\varepsilon(u_\varepsilon)(\nabla u_\varepsilon \cdot \nabla v_\varepsilon) \psi \\
- \int_{\Omega} \chi'(u_\varepsilon) D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \psi + \int_{\Omega} \chi'(u_\varepsilon) S_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \psi \right|
$$

$$
\leq \left\{ \int_{\Omega} |\chi''(u_\varepsilon)| D_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 \right\} \cdot \|\psi\|_{L^\infty(\Omega)}
$$

$$
+ \left\{ \int_{\Omega} \chi''^2(u_\varepsilon) S^2_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{1}{4} \int_{\Omega} |\nabla v_\varepsilon|^2 \right\} \cdot \|\psi\|_{L^\infty(\Omega)}
$$

$$
+ \left\{ \int_{\Omega} \chi'^2(u_\varepsilon) D^2_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \frac{1}{4} \right\} \cdot \|\nabla \psi\|_{L^2(\Omega)}
$$

$$
+ \left\{ \int_{\Omega} \chi'^2(u_\varepsilon) S^2_\varepsilon(u_\varepsilon) |\nabla v_\varepsilon|^2 + \frac{1}{4} \right\} \cdot \|\nabla \psi\|_{L^2(\Omega)}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Since $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, this implies the existence of $c_1 > 0$ such that for all $t > 0$ and $\varepsilon \in (0, 1)$,

$$
\left\| \partial_t \chi(u_\varepsilon) \right\|_{(W^{1,2}(\Omega))^*} \leq c_1 \cdot \left\{ \int_{\Omega} |\chi''(u_\varepsilon)| D_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \int_{\Omega} \chi''^2(u_\varepsilon) S^2_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \int_{\Omega} |\nabla v_\varepsilon|^2 \\
+ \int_{\Omega} \chi'^2(u_\varepsilon) D^2_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2 + \int_{\Omega} \chi'^2(u_\varepsilon) S^2_\varepsilon(u_\varepsilon) |\nabla v_\varepsilon|^2 \right\}.
$$

(4.27)

where we observe that due to (4.10) and (4.11) we have $D_\varepsilon \leq D + 1$ and $S_\varepsilon \leq S$ on $[0, \infty)$, so that taking $c_2 > 0$ such that $\text{supp} \chi' \subset [0, c_2]$ we can estimate

$$
|\chi''| D_\varepsilon \leq c_3 := \|\chi''\|_{L^\infty((0, \infty))} \|D + 1\|_{L^\infty((0, \varepsilon^2))} \quad \text{for all } \varepsilon \in (0, 1)
$$

and

$$
\chi''^2 S_\varepsilon^2 \leq c_4 := \|\chi''^2\|_{L^\infty((0, \infty))} \|S\|_{L^\infty((0, \varepsilon^2))} \quad \text{for all } \varepsilon \in (0, 1)
$$

as well as

$$
\chi'^2 D_\varepsilon^2 \leq c_5 := \|\chi'^2\|_{L^\infty((0, \infty))} \|D + 1\|_{L^\infty((0, \varepsilon^2))}^2 \quad \text{for all } \varepsilon \in (0, 1)
$$
\[ \chi' S_\epsilon^2 \leq c_6 := \| \chi' \|^2_{L^\infty((0,\infty))} \||S|\|^2_{L^\infty((0,\epsilon^2))} \quad \text{for all } \epsilon \in (0,1). \]

Therefore, (4.27) entails that for all \( t > 0 \) and \( \epsilon \in (0,1) \),

\[ \| \partial_t \chi(u_\epsilon) \|_{(W^{1,2}(\Omega))^*} \leq c_1 (c_3 + c_4 + c_5) \int_\Omega |\nabla u_\epsilon|^2 + c_1 (1 + c_6) \int_\Omega |\nabla v_\epsilon|^2 + c_1, \]

from which (4.26) follows on an integration using Lemmas 4.5 and 4.7.

Apart from that, as the function \( G \) might be singular at \( s = 0 \), for our construction it will be convenient to get hold of the \( \epsilon \)-independent positivity information contained in the following estimate.

**Lemma 4.9.** Assume (1.7) and (1.12) with some \( K_{SD} > 0 \), let \( m_0 = m_0(K_{SD}) > 0 \) be as in Lemma 2.1, and suppose that (1.14) holds with \( \int_\Omega u_0 < m_0 \). Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that

\[ \int_\Omega L(u_\epsilon(\cdot, t)) \leq C(T) \quad \text{for all } t \in (0,T) \text{ and } \epsilon \in (0,1), \]

where \( L \in C^2((0,\infty)) \) is the nonnegative function defined by letting

\[ L(s) := \begin{cases} \ln \frac{1}{s} - \frac{s^2}{2} + 2s - \frac{3}{2}, & s \in (0,1), \\ 0, & s \geq 1. \end{cases} \]

**Proof.** It can readily be verified that indeed \( L \) is nonnegative and belongs to \( C^2((0,\infty)) \) with

\[ L'(s) = -\frac{1}{s} - s + 2 \quad \text{and} \quad L''(s) = \frac{1}{s^2} - 1 > 0 \quad \text{for all } s \in (0,1), \]

so that integrating by parts in (4.18) and using Young’s inequality, we can estimate

\[ \frac{d}{dt} \int_\Omega L(u_\epsilon) = -\int_\Omega L''(u_\epsilon) D_\epsilon(u_\epsilon) |\nabla u_\epsilon|^2 + \int_\Omega L''(u_\epsilon) S_\epsilon(u_\epsilon) \nabla u_\epsilon \cdot \nabla v_\epsilon \]

\[ \leq \frac{1}{4} \int_\Omega L''(u_\epsilon) S_\epsilon^2(u_\epsilon) \|D_\epsilon(u_\epsilon) \|^2 \quad \text{for all } t > 0 \text{ and } \epsilon \in (0,1). \]

Here we note that due to (4.15), (4.11) and our overall requirement from (1.7) that \( S(0) = 0 \),

\[ \frac{S_\epsilon^2(s)}{s^2 D_\epsilon(s)} \leq \frac{1 + D(s)}{D(s)} \cdot \frac{S_\epsilon^2(s)}{s^2} \]

\[ \leq c_1 := \left\{ \frac{1}{\min_{\sigma \in [0,1]} D(\sigma)} + 1 \right\} \cdot \|S'|_{L^\infty((0,1))}^2 \quad \text{for all } s \in (0,1) \text{ and } \epsilon \in (0,1), \]
so that $L''(s) \leq \frac{1}{s^2}$ for all $s \in (0, 1)$ by (4.30), from (4.31) we infer that

\[
\frac{d}{dt} \int_{\Omega} L(u_\varepsilon) \leq \frac{c_1}{4} \int_{\Omega} |\nabla v_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
\]

and that thus (4.28) is a consequence of Lemma 4.7. \qed

As a final preparation for our limit procedure, in the following lemma we will inter alia observe that indeed the approximate counterpart of (4.6) holds, and that the $u_\varepsilon$ furthermore enjoy a temporally uniform integrability property consistent with (1.18).

**Lemma 4.10.** Let (1.7) and (1.12) hold with some $K_{SD} > 0$, and let (1.14) be satisfied with $\int_{\Omega} u_0 < m_0$, and with $m_0 = m_0(K_{SD}) > 0$ taken from Lemma 2.1. Then

\[
\mathcal{F}_\varepsilon(u_\varepsilon(\cdot, t), v_\varepsilon(\cdot, t)) + \int_0^t D_\varepsilon(s) ds \leq \mathcal{F}_\varepsilon(u_0, v_0) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
\]

(4.32)

and there exists $C > 0$ such that

\[
\int_{\Omega} u_\varepsilon^{\frac{2n-2}{n}}(\cdot, t) \leq C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\]

(4.33)

and

\[
\int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^2 \leq C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\]

(4.34)

and

\[
\int_0^t \int_{\Omega} v_{\varepsilon, t}^2 \leq C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\]

(4.35)

as well as

\[
\int_0^t \int_{\Omega} \left| \frac{\Sigma(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(\Sigma(u_\varepsilon) \nabla v_\varepsilon) \right|^2 \leq C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
\]

(4.36)

where $\mathcal{F}_\varepsilon$ and $h_\varepsilon$ are as in (4.21) and (4.12), and where

\[
D_\varepsilon(t) := \int_{\Omega} v_{\varepsilon, t}^2(\cdot, t) + \int_{\Omega} \left| \frac{\Sigma(u_\varepsilon(\cdot, t))}{h_\varepsilon(u_\varepsilon(\cdot, t))} \frac{\nabla u_\varepsilon(\cdot, t)}{h_\varepsilon(\cdot, t)} - \Sigma(u_\varepsilon(\cdot, t)) \nabla v_\varepsilon(\cdot, t) \right|^2, \quad t > 0, \quad \varepsilon \in (0, 1).
\]

(4.37)

**Proof.** In the identity (4.20), we use (4.17) to verify the pointwise inequality

\[
\left| \sqrt{\frac{S_\varepsilon(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)}} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \right|^2 \geq \frac{1}{2} \left| \sqrt{\frac{S_\varepsilon(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)}} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} \right|^2
\]

\[+ \frac{1}{2} \left| \Sigma(u_\varepsilon) \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(\Sigma(u_\varepsilon) \nabla v_\varepsilon) \right|^2 \quad \text{in } \Omega \times (0, \infty), \varepsilon \in (0, 1),
\]
and observe that furthermore, by (4.10) and (4.11),

$$R_\varepsilon(u_0, v_0) \leq \frac{1}{2} \int_\Omega |\nabla v_0|^2 + \frac{1}{2} \int_\Omega v_0^2 + \int_\Omega \overline{G}(u_0) \quad \text{for all } \varepsilon \in (0, 1)$$

with

$$\overline{G}(s) := \begin{cases} \int_s^1 \int_\sigma^1 \frac{D(\tau)}{S(\tau)} d\tau d\sigma, & s \in (0, 1), \\ \int_1^s \int_1^\sigma \frac{D(\tau) + 1}{S(\tau)} d\tau d\sigma, & s \geq 1. \end{cases}$$

In view of Lemma 2.1 and our assumption on $\int_\Omega u_0$, we therefore obtain (4.32) as well as (4.33)–(4.36) from (4.20).

We are thus prepared for the extraction of a suitably convergent subsequence that indeed approaches a global solution in the sense of Definition 4.2.

**Lemma 4.11.** Assume (1.7) and (1.12) with some $K_{SD} > 0$, let $m_0 = m_0(K_{SD}) > 0$ be as in Lemma 2.1, and suppose that $(u_0, v_0)$ satisfies (1.14) with $\int_\Omega u_0 < m_0$. Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ and functions

$$\begin{cases} u \in L^\infty((0, \infty); L^{\frac{2n-2}{n}}(\Omega)) \cap \bigcap_{p>1} L^p_{loc}(\bar{\Omega} \times [0, \infty)) \cap L^2_{loc}([0, \infty); W^{1,2}(\Omega)) \quad \text{and} \\
 v \in L^\infty((0, \infty); W^{1,2}(\Omega)) \cap L^\infty_{loc}([0, \infty); W^{1,\infty}(\Omega)) \cap L^2_{loc}([0, \infty); W^{2,2}(\Omega) \cap \bigcap_{p>1} L^p_{loc}((0, \infty); W^{2,p}(\Omega)) \quad \text{such that } u > 0 \text{ and } v \geq 0 \text{ a.e. in } \Omega \times (0, \infty) \text{ and that } \varepsilon_j \searrow 0 \text{ as } j \to \infty, \text{ that} \end{cases}$$

$$\begin{align*} &u_\varepsilon \to u \quad \text{in } \bigcap_{p>1} L^p_{loc}(\bar{\Omega} \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty), \quad (4.39) \\
 &u_\varepsilon(\cdot, t) \to u(\cdot, t) \quad \text{in } \bigcap_{p>1} L^p(\Omega) \text{ and a.e. in } \Omega \text{ for a.e. } t > 0, \quad (4.40) \\
 &\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)), \quad (4.41) \\
 &v_\varepsilon \to v \quad \text{in } L^\infty_{loc}([0, \infty); L^2(\Omega)) \cap \bigcap_{p>1} L^p_{loc}([0, \infty); W^{1,\infty}(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty), \quad (4.42) \\
 &v_\varepsilon(\cdot, t) \to v(\cdot, t) \quad W^{1,\infty}(\Omega) \text{ for a.e. } t > 0, \quad (4.43) \end{align*}$$
\[ v_\varepsilon \rightharpoonup v \quad \text{in} \quad \bigcap_{p>1} L^p_{\text{loc}}((0, \infty); W^{2,p}(\Omega)) \quad \text{and} \]

\[ v_{\varepsilon t} \rightharpoonup v_t \quad \text{in} \quad L^2(\Omega \times (0, \infty)) \quad (4.45) \]

as \( \varepsilon = \varepsilon_j \downarrow 0 \), and that \((u, v)\) is a global very weak energy solution of (1.2) in the sense of Definition 4.2. If moreover (1.17) holds with \( \Omega = B_R(0) \) and some \( R > 0 \), then \((u, v)\) even is a radial global very weak energy solution of (1.2).

**Proof.** Let \((\chi_k)_{k \in \mathbb{N}} \subset C^\infty(0, \infty)\) be such that \(\chi_k(s) = s\) for all \(s \in [0, k]\) and that \(\text{supp} \chi'_k\) is bounded for all \(k \in \mathbb{N}\). Then from Lemma 4.5, it follows that for each \(k \in \mathbb{N}\),

\[ (\chi_k(u_\varepsilon))_{\varepsilon \in (0, 1)} \text{ is bounded in } L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)), \]

while Lemma 4.8 asserts that if we fix \(l \in \mathbb{N}\) such that \(l > \frac{n}{2}\), then for any such \(k\),

\[ (\partial_t \chi_k(u_\varepsilon))_{\varepsilon \in (0, 1)} \text{ is bounded in } L^1_{\text{loc}}([0, \infty); (W^{1,2}(\Omega))^*). \]

In view of our choice of \((\chi_k)_{k \in \mathbb{N}}\), an application of an Aubin–Lions type lemma [48] along with a straightforward diagonal sequence extraction procedure therefore provides \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) and a nonnegative measurable function \(u\) on \(\Omega \times (0, \infty)\) such that \(\varepsilon_j \downarrow 0\) as \(j \to \infty\) and that

\[ u_\varepsilon \rightharpoonup u \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_j \downarrow 0, \quad (4.46) \]

which in conjunction with Lemma 4.9 and Fatou’s lemma readily shows that with \(L\) as defined in (4.29) we have \(L(u) < \infty\) a.e. in \(\Omega \times (0, \infty)\) and hence \(u > 0\) a.e. in \(\Omega \times (0, \infty)\). Since relying on Lemma 4.10 and again on Lemma 4.5 we even know that \((u_\varepsilon)_{\varepsilon \in (0, 1)}\) is bounded in \(L^\infty((0, T); L^{\frac{2n-2}{n}}(\Omega)) \cap L^p(\Omega \times (0, T))\) for all \(T > 0\) and \(p > 1\), by using the Vitali convergence theorem together with (4.46) we see on passing to a subsequence if necessary that we can also achieve that \(u\) enjoys the regularity properties claimed in (3.38), and that (4.39)–(4.41) hold.

Likewise, combining Lemma 4.10 with Lemma 4.5, Lemma 4.6 and (4.18), we find that \((v_\varepsilon)_{\varepsilon \in (0, 1)}\) is bounded in \(L^\infty((0, \infty); W^{1,2}(\Omega))\), \(L^\infty_{\text{loc}}((0, \infty); W^{1,\infty}(\Omega))\), \(L^2_{\text{loc}}((0, \infty); W^{2,2}(\Omega))\) and \(L^p_{\text{loc}}((0, \infty); W^{2,p}(\Omega))\),

and that \((v_{\varepsilon t})_{\varepsilon \in (0, 1)}\) is bounded in \(L^2(\Omega \times (0, \infty))\),

whence again due to an Aubin–Lions Lemma, and due to the Arzelà–Ascoli Theorem, possibly after a further extraction we may also assume that (4.42)–(4.45) are valid as \(\varepsilon = \varepsilon_j \downarrow 0\) with some nonnegative \(v\) complying with (4.38).

Clearly, \((u, v)\) inherits the claimed radial symmetry of \((u_\varepsilon, v_\varepsilon)\) whenever (1.17) holds and \(\Omega = B_R(0)\), \(R > 0\), so that it remains to show that \((u, v)\) is a global very weak energy solution of (1.2). To this end, we note that the inclusions in (4.2) are implied by (4.28), whereas those in (4.4) and
(4.3) as well as the inequality in (4.6) readily result from Lemma 4.10, (4.39)–(4.43), (4.45), Fatou’s lemma and lower semicontinuity of $L^2$ norms with respect to weak convergence. The identities in (3.2) and (4.7) directly follow on taking $\varepsilon = \varepsilon_j \searrow 0$ and using (4.39), (4.42) and (4.45) in (4.19) and the respective weak formulation of the second sub-problem from (4.18), and from the first convergence property contained in (4.42) we readily obtain that $\text{ess sup}_{t \in (0,t_0)} \| v_{\varepsilon_j} (\cdot, t) - v_0 \|_{L^2(\Omega)} \leq \liminf_{j \to \infty} \sup_{t \in (0,t_0)} \| v_{\varepsilon_j} (\cdot, t) - v_0 \|_{L^2(\Omega)} \to 0$ as $t_0 \searrow 0$. Moreover, given an arbitrary nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$ and any $\chi \in C^\infty([0, \infty))$ such that $\text{supp} \chi' \subset \subset [0, \infty)$ and $\chi' \geq 0$ as well as $\chi'' \leq 0$, from the first equation in (4.18), we obtain that

$$- \int_0^\infty \int_\Omega \chi''(u_{\varepsilon}) D(u_{\varepsilon}) \nabla u_{\varepsilon} \varphi = - \int_0^\infty \int_\Omega \chi(u_{\varepsilon}) \varphi_t - \int_\Omega \chi(u_0) \varphi(\cdot, 0)$$

$$- \int_0^\infty \int_\Omega \chi''(u_{\varepsilon}) S(u_{\varepsilon}) (\nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}) \varphi$$

$$+ \int_0^\infty \int_\Omega \chi'(u_{\varepsilon}) D(u_{\varepsilon}) \nabla u_{\varepsilon} \cdot \nabla \varphi$$

$$- \int_0^\infty \int_\Omega \chi'(u_{\varepsilon}) S(u_{\varepsilon}) \nabla v_{\varepsilon} \cdot \nabla \varphi \quad \text{for all} \; \varepsilon \in (0, 1).$$

(4.47)

Here the compactness of $\text{supp} \chi'$ ensures that $\limsup_{s \to \infty} \frac{|\chi(s)|}{s} < \infty$, which together with (4.39) and the Vitali convergence theorem ensures that

$$- \int_0^\infty \int_\Omega \chi(u_{\varepsilon}) \varphi_t \to - \int_0^\infty \int_\Omega \chi(u) \varphi_t \quad \text{as} \; \varepsilon = \varepsilon_j \searrow 0. \quad (4.48)$$

Apart from that, this compactness property guarantees, again by (4.39) and the Vitali convergence theorem, that

$$\chi''(u_{\varepsilon}) S(u_{\varepsilon}) \to \chi''(u) S(u) \quad \text{in} \; L^4_{\text{loc}}(\overline{\Omega} \times [0, \infty))$$

as well as

$$\chi'(u_{\varepsilon}) D(u_{\varepsilon}) \to \chi'(u) D(u) \quad \text{in} \; L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty))$$

and

$$\chi'(u_{\varepsilon}) S(u_{\varepsilon}) \to \chi'(u) S(u) \quad \text{in} \; L^4_{\text{loc}}(\overline{\Omega} \times [0, \infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$, so that since $\nabla u_{\varepsilon} \to \nabla u$ in $L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty))$ and $\nabla v_{\varepsilon} \to \nabla v$ in $L^4_{\text{loc}}(\overline{\Omega} \times [0, \infty))$ as $\varepsilon = \varepsilon_j \searrow 0$ due to (4.41) and (4.42), we see that

$$- \int_0^\infty \int_\Omega \chi''(u_{\varepsilon}) S(u_{\varepsilon}) (\nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}) \varphi \to - \int_0^\infty \int_\Omega \chi''(u) S(u) (\nabla u \cdot \nabla v) \varphi \quad (4.49)$$
as well as
\[
\int_0^\infty \int_\Omega \chi'(u_\varepsilon)D_\varepsilon u_\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi \to \int_0^\infty \int_\Omega \chi'(u)D(u) \nabla u \cdot \nabla \varphi \tag{4.50}
\]
and
\[
-\int_0^\infty \int_\Omega \chi'(u_\varepsilon)S_\varepsilon u_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi \to -\int_0^\infty \int_\Omega \chi'(u)S(u) \nabla v \cdot \nabla \varphi \tag{4.51}
\]
as \(\varepsilon = \varepsilon_j \searrow 0\). Since \(\chi'' \leq 0\) and \(\varphi \geq 0\), and since (4.41) also warrants that \(\sqrt{-\chi''(u_\varepsilon)D_\varepsilon u_\varepsilon} \to \sqrt{-\chi''(u)D(u)\nabla u}\) in \(L^2_{\text{loc}}(\Omega \times [0, \infty))\) as \(\varepsilon = \varepsilon_j \searrow 0\), once more based on a lower semicontinuity property we infer that
\[
-\int_0^\infty \int_\Omega \chi''(u)D(u)|\nabla u|^2 \varphi \leq \liminf_{\varepsilon = \varepsilon_j \searrow 0} \left\{ -\int_0^\infty \int_\Omega \chi''(u_\varepsilon)D_\varepsilon u_\varepsilon|\nabla u_\varepsilon|^2 \varphi \right\}.
\]
Along with (4.48)–(4.51), this shows that for any such \(\varphi\) and \(\chi\), (4.9) is a consequence of (4.47). \(\square\)

We have thereby already proved our main result on global existence of generalized solutions emanating from small-mass initial data:

**Proof of Theorem 1.2.** All statements have been covered by Lemma 4.11 already. \(\square\)

## 5 | NONEXISTENCE. PROOF OF THEOREM 1.3

### 5.1 | Constructing initial data at large negative energy levels

Our approach toward the statement on nonexistence in Theorem 1.3 will be based on the observation that contrary to the small-mass scenario addressed in Lemma 2.1, throughout favorably large sets of radial initial data, thus necessarily involving large-mass functions at least in cases of critical behavior of \(\frac{S}{D}\), the energy functional from (1.9) is unbounded from below. This will be seen by means of an essentially explicit construction which, especially in order to cover widely general and also critical nonlinearities, relies on an approximation process quite different from related precedents in the literature [18, 30, 35, 51, 54].

**Lemma 5.1.** Let \(\Omega = B_R(0) \subset \mathbb{R}^n\) with \(n \geq 3\) and \(R > 0\), and assume (1.19) with some \(k_{SD} > 0\). Then for any choice of \((u_0, v_0)\) fulfilling (1.14) and (1.17), one can find \((u_{0j})_{j \in \mathbb{N}} \subset W^{1,\infty}(\Omega)\) and \((v_{0j})_{j \in \mathbb{N}} \subset W^{1,\infty}(\Omega)\) such that \((u_{0j}, v_{0j})\) satisfies (1.14) and (1.17) for all \(j \in \mathbb{N}\), that

\[
\begin{align*}
&u_{0j} \to u_0 \text{ in } L^p(\Omega) \quad \text{and} \quad v_{0j} \to v_0 \text{ in } W^{1,q}(\Omega) \text{ as } j \to \infty \\
&\text{for all } p \in (0, 1) \text{ and } q \in \left[1, \frac{n}{n-1}\right],
\end{align*}
\]

that

\[
\sup_{j \in \mathbb{N}} \int_\Omega u_{0j} < \infty, \tag{5.2}
\]
and that
\[ F(u_{0j}, v_{0j}) \to -\infty \quad \text{as } j \to \infty, \tag{5.3} \]
where again \( F \) is taken from (1.9).

Proof. Since \( S \) is positive on \((0, \infty)\) and hence the function \( G \) in (1.10) \( G \) is bounded in \([s_0, 1]\) with \( s_0 := \min_{x \in \Omega} u_0(x) \) being positive by (1.14), according to (1.19) we can readily find \( c_1 > 0 \) such that
\[ G(s) \leq c_1 s^{\frac{2n-2}{n}} \quad \text{for all } s \geq s_0. \tag{5.4} \]
We moreover fix an arbitrary nontrivial radially symmetric function \( \phi \in C^\infty(\mathbb{R}^n) \) such that \( \phi \geq 0 \) and \( \text{supp} \phi \subset \Omega \), and abbreviate
\[ c_2 := \int_{\mathbb{R}^n} \phi^{\frac{2n-2}{n-2}}, \quad c_3 := \int_{\mathbb{R}^n} |\nabla \phi|^2 \quad \text{and} \quad c_4 := \int_{\mathbb{R}^n} \phi^2. \]
We then let
\[ a := 2^{\frac{2n-2}{n}} c_1 + \frac{2(c_3 + c_4)}{c_2} \tag{5.5} \]
and define
\[ u_{0j}(x) := u_0(x) + a^n j^n \phi^{\frac{n-2}{n-2}}(jx), \quad x \in \overline{\Omega}, \ j \in \mathbb{N}, \tag{5.6} \]
as well as
\[ v_{0j}(x) := v_0(x) + a^{n-1} j^{n-2} \phi(jx), \quad x \in \overline{\Omega}, \ j \in \mathbb{N}. \tag{5.7} \]
Then \((u_{0j}, v_{0j})\) clearly satisfies (1.14) and (1.17) for all \( j \in \mathbb{N} \), and since \( u_{0j} \geq u_0 \geq s_0 \) in \( \Omega \) for all \( j \in \mathbb{N} \), we may employ (5.4) to estimate
\[
\int_{\Omega} G(u_{0j}) \leq c_1 \int_{\Omega} u_{0j}^{\frac{2n-2}{n}} \\
\leq 2^{\frac{n-2}{n}} c_1 \int_{\Omega} u_0^{\frac{2n-2}{n}} + 2^{\frac{n-2}{n}} c_1 \int_{\Omega} \left\{ a^n j^n \phi^{\frac{n-2}{n-2}}(jx) \right\}^{\frac{2n-2}{n}} dx \\
= 2^{\frac{n-2}{n}} c_1 \int_{\Omega} u_0^{\frac{2n-2}{n}} + 2^{\frac{n-2}{n}} c_1 c_2 a^{2n-2} j^{n-2} \quad \text{for all } j \in \mathbb{N}. \tag{5.8} \]
Moreover,
\[
\frac{1}{2} \int_{\Omega} |\nabla v_{0j}|^2 \leq \int_{\Omega} |\nabla v_0|^2 + \int_{\Omega} \left| a^{n-1} j^{n-2} \nabla \phi(jx) \right|^2 dx \\
= \int_{\Omega} |\nabla v_0|^2 + c_3 a^{2n-2} j^{n-2} \quad \text{for all } j \in \mathbb{N}.
and
\[
\frac{1}{2} \int_{\Omega} v_{0j}^2 \leq \int_{\Omega} v_0^2 + \int_{\Omega} \left| a^{n-1} j^{n-2} \phi(jx) \right|^2 dx
\]
\[= \int_{\Omega} v_0^2 + c_4 a^{2n-2} j^{n-4} \]
\[\leq \int_{\Omega} v_0^2 + c_4 a^{2n-2} j^{n-2} \quad \text{for all } j \in \mathbb{N},\]
whereas, again by nonnegativity of \(\phi\),
\[
\int_{\Omega} u_0 j v_0 j \geq \int_{\Omega} \left( a^n j^n \phi^{n \frac{n}{n-2}}(jx) \right) \cdot \left( a^{n-1} j^{n-2} \phi(jx) \right) dx
\]
\[= c_2 a^{2n-1} j^{n-2} \quad \text{for all } j \in \mathbb{N}.\]

Therefore, our definition (5.5) ensures that abbreviating \(c_5 := \int_{\Omega} |\nabla v_0|^2 + \int_{\Omega} v_0^2 + 2 \frac{n-2}{n} c_1 \int_{\Omega} \frac{u_0^{2n-2}}{u_0^n}\), we have
\[
F(u_0, v_0) \leq c_5 + (c_3 + c_4) a^{2n-2} j^{n-2} - c_2 a^{2n-1} j^{n-2} + 2 \frac{n-2}{n} c_1 c_2 a^{2n-2} j^{n-2}
\]
\[= c_5 + \left\{ \frac{c_3 + c_4 + 2 \frac{n-2}{n} c_1 c_2}{a} - c_2 \right\} \cdot a^{2n-1} j^{n-2}
\]
\[= c_5 - \frac{c_2}{2} a^{2n-1} j^{n-2} \quad \text{for all } j \in \mathbb{N},\]
which due to our restriction \(n \geq 3\) entails (5.3).

To derive (5.1), we fix \(p \in (0, 1)\) and \(q \in [1, \frac{n}{n-1})\) and once more go back to (5.6) and (5.7) to find that
\[
\int_{\Omega} |u_0j - u_0|^p = \int_{\Omega} \left| a^n j^n \phi^{n \frac{n}{n-2}}(jx) \right|^p dx = a^{np} j^{n(p-1)} \int_{\mathbb{R}^n} \phi^{np} \quad \text{for all } j \in \mathbb{N}
\]
and
\[
\int_{\Omega} |\nabla v_0j - \nabla v_0|^q = \int_{\Omega} \left| a^{n-1} j^{n-2} \nabla \phi(jx) \right|^q dx = a^{(n-1)q} j^{(n-1)q-n} \int_{\mathbb{R}^n} |\nabla \phi|^q \quad \text{for all } j \in \mathbb{N}
\]
as well as
\[
\int_{\Omega} |v_0j - v_0| = \int_{\Omega} \left| a^{n-1} j^{n-2} \phi(jx) \right| dx = a^{n-1} j^{-2} \int_{\mathbb{R}^n} \phi \quad \text{for all } j \in \mathbb{N}.
\]
Since \(n(p-1) < 0\) and \((n-1)q - n < 0\), these identities immediately lead to (5.1).

Finally, (5.2) results on observing that, again by (5.6),
\[
\int_{\Omega} u_0j - \int_{\Omega} u_0 = \int_{\Omega} a^n j^n \phi^{n \frac{n}{n-2}}(jx) dx = a^n \int_{\mathbb{R}^n} \phi^{n \frac{n}{n-2}}
\]
for all \(j \in \mathbb{N}\. \square
5.2 Lower bounds for energy levels of ω-limits

The core of our analysis can now be found in this section, the purpose of which is to establish \textit{a priori} bounds from below for all conceivable energy levels of initial data which evolve into global very weak solutions enjoying the additional boundedness property in (1.23). According to the non-increase of \( F \), this essentially reduces to the derivation of corresponding bounds for respective \( ω \)-limits, and the main part of our analysis in this direction will be focused on the challenge to appropriately cope with the circumstance that all conclusions need to be based on the considerably poor regularity information provided by (1.23) and Definition 4.2) for such solutions. Indeed, significant efforts appear necessary already at the first stage of our considerations in this regard, aiming at a verification of the intuitive guess that the energy inequality (4.6) should enforce any such solution to possess a nonempty \( ω \)-limit set consisting of solutions \((u_∞, v_∞)\) to a corresponding stationary problem, located at some energy level below that of the initial data.

Only after this has been accomplished in Lemmas 5.3 and 5.6, it becomes at all meaningful to substantiate the overall strategy to control \( F(u_∞, v_∞) \) by rewriting \( F(u_∞, v_∞) = -\frac{1}{2} \int_Ω |\nabla v_∞|^2 - \frac{1}{2} \int_Ω v_∞^2 + \int_Ω G(u_∞) \) (Lemma 5.10), and relating the crucial nonpositive contribution herein to the nonnegative summand \( \int_Ω G(u_∞) \) by means of the identity

\[
\frac{n - 2}{2} \int_Ω |\nabla v_∞|^2 = \int_Ω \Delta v_∞(x \cdot \nabla v_∞),
\]

and the inequality

\[
\int_{[u_∞ > 1]} \mathcal{E}(u_∞) + 2R \int_Ω |\nabla v_∞| \tag{5.10}
\]

through the hypothesis (1.20) (Lemmas 5.8 and 5.7).

In a first preliminary step toward this, we observe that for an arbitrary global very weak solution, in the second sub-problem from (1.2) an evaluation at almost every point on the time semi-axis is possible in the following sense.

\textbf{Lemma 5.2.} Suppose that (1.7) holds, and let \((u, v)\) be a global weak energy solution of (1.2) with some \((u_0, v_0)\) fulfilling (1.14). Then there exist \((ψ_l)_{l∈\mathbb{N}} \subset C^1(\overline{Ω})\) and a null set \(N \subset (0, ∞)\) such that \(\{ψ_l | l ∈ \mathbb{N}\}\) is dense in \(W^{1,2}(Ω)\), and such that

\[
\int_Ω ψ_l(\cdot, t)ψ_l = -\int_Ω \nabla v(\cdot, t) \cdot \nabla ψ_l - \int_Ω v(\cdot, t)ψ_l + \int_Ω u(\cdot, t)ψ_l
\]

for all \(t ∈ (0, ∞) \setminus N\) and any \(l ∈ \mathbb{N}\). \( \tag{5.11} \)

\textbf{Proof.} Relying on the separability of \(W^{1,2}(Ω)\), we may take any \((ψ_l)_{l∈\mathbb{N}} \subset C^1(\overline{Ω})\) with the claimed density property, and use that \(\{ψ_l | l ∈ \mathbb{N}\}\) is countable in choosing a null set \(N \subset (0, ∞)\) in such a way that each \(t_0 ∈ (0, ∞) \setminus N\) is a common Lebesgue point of \(0 < t \mapsto \int_Ω v_l(\cdot, t)ψ_l\) and of \(0 < t \mapsto \int_Ω (-\nabla v(\cdot, t) \cdot \nabla ψ_l - v(\cdot, t)ψ_l + u(\cdot, t)ψ_l)\) for all \(l ∈ \mathbb{N}\). Then fixing any \(t_0 ∈ (0, ∞) \setminus N\), we use that by a completion argument the identity (4.7) extends so as to remain valid for \(φ = φ_δ, δ > 0,\)
where \( \varphi_\delta(x, t) := 0 \) for \( (x, t) \in \overline{\Omega} \times ([0, t_0] \cup [t_0 + \delta, \infty)) \) and \( \varphi_\delta(x, t) := \frac{1}{\delta} \psi_\delta(x) \) for \( (x, t) \in \overline{\Omega} \times (t_0, t_0 + \delta), \delta > 0 \). In consequence, we obtain that

\[
\frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \int_\Omega v_t \psi_\delta = \frac{1}{\delta} \int_0^{t_0 + \delta} \int_\Omega (-\nabla v \cdot \nabla \psi_\delta - v \psi_\delta + u \psi_\delta) \quad \text{for all } \delta > 0,
\]

from which, due to our choice of \( N \), (5.11) results on taking \( \delta \searrow 0 \).

From now on concentrating on radial solutions fulfilling (1.23), we can utilize the latter lemma to indeed make sure that along suitably chosen divergent sequences of times, a limit \((u_\infty, v_\infty)\) is approached in a suitable sense, and that this limit at least satisfies a standard weak formulation of the second equation in (1.2).

**Lemma 5.3.** Let \( \Omega = B_R(0) \subset \mathbb{R}^n \) with some \( n \geq 3 \) and \( R > 0 \), and suppose that (1.7), (1.14) and (1.17) hold, and that \((u, v)\) is a radial global weak energy solution of (1.2) such that

\[
\text{ess sup}_{t > 0} \int_\Omega u^p(\cdot, t) < \infty \quad (5.12)
\]

for some \( p > \frac{2n}{n+2} \). Then with \( N_* \subset (0, \infty) \) taken from Definition 4.2, we can find \((t_k)_{k \in \mathbb{N}} \subset (0, \infty) \setminus N_* \) and nonnegative radially symmetric functions \( u_\infty \in L^p(\Omega) \) and \( v_\infty \in W^{2,p}(\Omega) \) such that \( t_k \to \infty \) as \( k \to \infty \), that

\[
u(\cdot, t_k) \rightharpoonup u_\infty \quad \text{in } L^p(\Omega) \quad \text{as } k \to \infty \quad (5.13)
\]

and

\[
\sup_{k \in \mathbb{N}} \int_\Omega G(u(\cdot, t_k)) < \infty, \quad (5.14)
\]

that

\[
v(\cdot, t_k) \to v_\infty \quad \text{in } W^{2,p}(\Omega) \quad \text{as } k \to \infty \quad (5.15)
\]

and

\[
v(\cdot, t_k) \to v_\infty \quad \text{in } W^{1,2}(\Omega) \quad \text{as } k \to \infty, \quad (5.16)
\]

and that

\[
\Sigma(u(\cdot, t_k)) \frac{\nabla u(\cdot, t_k)}{h(u(\cdot, t_k))} - \Sigma(u(\cdot, t_k)) \nabla v(\cdot, t_k) \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } k \to \infty. \quad (5.17)
\]

Moreover,\n
\[
\int_\Omega u_\infty = \int_\Omega u_0 \quad (5.18)
\]

and

\[
\int_\Omega \nabla v_\infty \cdot \nabla \psi + \int_\Omega v_\infty \psi = \int_\Omega u_\infty \psi \quad \text{for all } \psi \in W^{1,2}(\Omega).
\]

(5.19)
Proof. Since $\frac{2n}{n+2} < 2$, it is sufficient to consider the case $p < 2$ only. Then according to (5.12), we can fix a null set $N_1 \subset (0, \infty)$ and $c_1 > 0$ such that $N_1 \supset N_\ast$ and that \{\( u(\cdot, t) \mid t \in (0, \infty) \setminus N_1 \)\} $\subset L^p(\Omega)$ with
\[
\int_\Omega u^p(\cdot, t) \leq c_1 \quad \text{for all } t \in (0, \infty) \setminus N_1,
\]
noting that in view of Definition 4.2, the inclusion $N_\ast \subset N_1$ ensures that
\[
\int_\Omega u(\cdot, t) = \int_\Omega u_0 \quad \text{for all } t \in (0, \infty) \setminus N_1,
\]
and that for all $t \in (0, \infty) \setminus N_1$ we have $v(\cdot, t) \in W^{1,2}(\Omega)$, $u(\cdot, t)v(\cdot, t) \in L^1(\Omega)$ and $G(u(\cdot, t)) \in L^1(\Omega)$, with
\[
P(u(\cdot, t), v(\cdot, t)) + \int_0^t \int_\Omega v^2 \, dt + \int_0^t \int_\Omega \left| \frac{\nabla u}{h(u)} - \Sigma(u)\nabla v \right|^2 \leq P(u_0, v_0) \quad \text{for all } t \in (0, \infty) \setminus N_1;
\]
in view of the additional regularity requirements in Definition 4.2, and due to Lemma 5.2, on enlarging $N_1$ if necessary we may assume that furthermore
\[
L^2(\Omega) \ni v_1(\cdot, t) = \Delta v(\cdot, t) - v(\cdot, t) + u(\cdot, t) \quad \text{a.e. in } \Omega \quad \text{for all } t \in (0, \infty) \setminus N_1
\]
and
\[
\Sigma(u(\cdot, t))\frac{\nabla u(\cdot, t)}{h(u(\cdot, t))} - \Sigma(u(\cdot, t))\nabla v(\cdot, t) \in L^2(\Omega; \mathbb{R}^n) \quad \text{for all } t \in (0, \infty) \setminus N_1,
\]
and that with $(\psi_l)_{l \in \mathbb{N}}$ as provided by Lemma 5.2, we have
\[
\int_\Omega v_l(\cdot, t)\psi_l = -\int_\Omega \nabla v(\cdot, t) \cdot \nabla \psi_l - \int_\Omega v(\cdot, t)\psi_l + \int_\Omega u(\cdot, t)\psi_l \quad \text{for all } t \in (0, \infty) \setminus N_1 \text{ and each } l \in \mathbb{N}.
\]
Now since $W^{1,2} \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ due to our assumption that $p > \frac{2n}{n+2}$, the Hölder inequality together with (5.20) and Young’s inequality shows that with some $c_2 > 0$, we have
\[
\int_\Omega uv \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{\frac{p}{p-1}}(\Omega)} \leq c_1^\frac{1}{p} \|v\|_{L^{\frac{p}{p-1}}(\Omega)}.
\]
\[
\leq c_2 \cdot \left\{ \int_{\Omega} |\nabla v|^2 + \int_{\Omega} v^2 \right\}^{1/2} \\
\leq \frac{1}{2} \cdot \left\{ \int_{\Omega} |\nabla v|^2 + \int_{\Omega} v^2 \right\} + \frac{c_2^2}{2} \quad \text{for all } t \in (0, \infty) \setminus N_1.
\]

From the definition of \( F \), we thus obtain that
\[
F(u(\cdot, t), v(\cdot, t)) \geq \int_{\Omega} G(u(\cdot, t)) - \frac{c_2^2}{2} \quad \text{for all } t \in (0, \infty) \setminus N_1,
\]
so that by nonnegativity of \( G \), (5.22) entails that since \((0, \infty) \setminus N_1\) is unbounded,
\[
\int_0^\infty \int_{\Omega} v_t^2 + \int_0^\infty \int_{\Omega} \left| \frac{\nabla u}{h(u)} - \Sigma(u) \nabla v \right|^2 \leq c_3 := F(u_0, v_0) + \frac{c_2^2}{2}
\]
and that thus
\[
\int_t^{t+1} \int_{\Omega} v_t^2 + \int_t^{t+1} \int_{\Omega} \left| \frac{\nabla u}{h(u)} - \Sigma(u) \nabla v \right|^2 \to 0 \quad \text{as } t \to \infty.
\]

We moreover observe that again due to Young’s inequality, due to the restriction \( p < 2 \), we can use (5.20), (5.23) and (5.27) to estimate
\[
\int_t^{t+1} \int_{\Omega} |\Delta v - v|^p \leq 2^{p-1} \int_t^{t+1} \int_{\Omega} |v_t|^p + 2^{p-1} \int_t^{t+1} \int_{\Omega} u^p
\]
\[
\leq 2^{p-1} \int_t^{t+1} \int_{\Omega} |v_t|^2 + 2^{p-1} |\Omega| + 2^{p-1} \int_t^{t+1} \int_{\Omega} u^p
\]
\[
\leq 2^{p-1} c_3 + 2^{p-1} |\Omega| + 2^{p-1} c_1 \quad \text{for all } t > 0,
\]
whence elliptic regularity theory [27] ensures the existence of \( c_4 > 0 \) such that
\[
\int_t^{t+1} \|u(\cdot, s)\|_{W^{2,p}(\Omega)}^p ds \leq c_4 \quad \text{for all } t > 0.
\]

Now combining this with (5.20) and (5.28), by means of a straightforward extraction procedure, we obtain \((t_k)_{k \in \mathbb{N}} \subset (0, \infty) \setminus N_1 \subset (0, \infty) \setminus N_*\) such that with some nonnegative and radially symmetric \( u_\infty \in L^p(\Omega) \) and \( v_\infty \in W^{2,p}(\Omega) \), we have \( t_k \to \infty \) as \( k \to \infty \) as well as (5.13) and (5.15), and such that moreover (5.17) holds and that
\[
v_t(\cdot, t_k) \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } k \to \infty.
\]

Since \( W^{2,p}(\Omega) \) is compactly embedded into \( W^{1,2}(\Omega) \) due to the fact that \( p > \frac{2n}{n+2} \), (5.15) also entails (5.16), whereas (5.18) is a consequence of (5.13) and (5.21), and (5.14) results from (5.26) and (5.22).
For the verification of (5.19), we first fix $l \in \mathbb{N}$ and use (5.25) along with (5.29) to see that

$$- \int_{\Omega} \nabla u(\cdot, t_k) \cdot \nabla \psi_l - \int_{\Omega} v(\cdot, t_k) \psi_l + \int_{\Omega} u(\cdot, t_k) \psi_l = \int_{\Omega} v_l(\cdot, t_k) \psi_l \to 0 \quad \text{as } k \to \infty,$$

which together with (5.16) and (5.13) shows that

$$- \int_{\Omega} \nabla v_{\infty} \cdot \nabla \psi_l - \int_{\Omega} v_{\infty} \psi_l + \int_{\Omega} u_{\infty} \psi_l = 0$$

for any such $l$. Recalling that $(\psi_l)_{l \in \mathbb{N}}$ is dense in $W^{1,2}(\Omega)$, from this we readily infer by another completion argument that the identity in (5.19) indeed holds for each $\psi \in W^{1,2}(\Omega)$.

In contrast to the above argument confirming (5.19), our derivation of a corresponding equilibration property with respect to the first equation in (1.2) will turn out to be substantially more involved; in fact, the weak convergence property described in (5.13) seems insufficient for any expedient limit passage verifying the identity $\nabla u_{\infty} = h(u_{\infty}) \nabla v_{\infty}$ that is to be expected as a result of (5.17).

Fortunately, in the presently considered radial framework, some additional regularity features are available outside the spatial origin: According to the second-order information contained in (5.15) and one-dimensional Sobolev embeddings, the component $v_{\infty}$ can easily be seen to actually possess the first-order properties listed in the following lemma which, besides being used in Lemma 5.6 addressing the above identity, will moreover be relied on in Lemmas 5.7 and 5.8.

**Lemma 5.4.** Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and let $p > 1$ and $q \geq 1$ be such that $(n - 2p)q < np$. Then there exist $C(p) > 0$ and $C(p, q) > 0$ such that if $V \in W^{2,p}(\Omega)$ is radially symmetric, then $V \in C^1(\overline{\Omega} \setminus \{0\})$ with

$$|\nabla V(x)| \leq C(p) \cdot \|V\|_{W^{2,p}(\Omega)} \cdot |x|^{\frac{n-p}{p}} \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}$$

and

$$\int_{\Omega} |x \cdot \nabla V|^q \leq C(p, q) \cdot \|V\|_{W^{2,p}(\Omega)}^q.$$  \hspace{1cm} (5.31)

**Proof.** Let $\phi = \phi(r) \in C^2(\overline{\Omega})$ be radial. Then by the Hölder inequality, writing $\omega_n := n |B_1(0)|$, we see that

$$|r^{n-1} \phi_r(r)| = \left| \int_0^r (\rho^{n-1} \phi_r)(\rho) d\rho \right|$$

$$\leq \left\{ \int_0^r \rho^{-(n-1)(p-1)} |(\rho^{n-1} \phi_r)(\rho)|^p d\rho \right\} \cdot \left\{ \int_0^r \rho^{n-1} d\rho \right\}^{\frac{p-1}{p}}$$

$$= \left\{ \int_0^r \rho^{n-1} \left| \rho^{1-n} (\rho^{n-1} \phi_r)(\rho) \right|^p d\rho \right\} \cdot \left( \frac{r^n}{n} \right)^{\frac{p-1}{p}} \quad \text{for all } r \in (0, R]$$
and hence

\[ |\phi_r(r)| \leq c_1 \|\Delta \phi\|_{L^p(\Omega)} r^{-\frac{n-p}{p}} \quad \text{for all } r \in (0, R], \tag{5.32} \]

where \( c_1 := n \frac{-p-1}{p} \omega_n^{-\frac{1}{p}} \). Upon integration, this entails that

\[ \int_0^R r^{n-1} |r \phi_r(r)|^q dr \leq c_1^q c_2 \|\Delta \phi\|_{L^p(\Omega)}^q, \tag{5.33} \]

with \( c_2 := \int_0^R r^{n-1-(n-2p)q} \frac{1}{p} dr \) being finite due to our restriction on \( q \).

To derive the claim of the lemma, given a radial \( V \in W^{2,p}(\Omega) \) let us choose radial functions \( V_l \in C^2(\Omega), l \in \mathbb{N} \), such that \( V_l \to V \) in \( W^{2,p}(\Omega) \) as \( l \to \infty \). Then an application of (5.32) to \( \phi := V_l - V_{l'} \) for \( l, l' \in \mathbb{N} \) shows that \( (V_l)_{l \in \mathbb{N}} \) is a Cauchy sequence in \( C^1(\overline{\Omega} \setminus B_{2}(0)) \) for all \( \delta \in (0, R) \), thus implying that indeed \( V \in C^1(\overline{\Omega} \setminus \{0\}) \). Thereupon, taking \( \phi := V_l, l \in \mathbb{N} \), and letting \( l \to \infty \) in (5.32) and (5.33) yields (5.30) and (5.31). □

In Lemma 5.6, we shall furthermore draw on the following elementary observation.

**Lemma 5.5.** Let \( I \subset \mathbb{R} \) be an open interval, and suppose that \( (\psi_k)_{k \in \mathbb{N}} \subset W^{1,1}(I) \) is such that as \( k \to \infty \) we have \( \psi_k(r) \to \psi(r) \) for all \( r \in I \) and \( \psi'_k \to \phi \) in \( L^1(I) \) with some \( \psi : I \to \mathbb{R} \) and some \( \phi \in C^0(I) \). Then \( \psi \in C^1(I) \) with \( \psi' \equiv \phi \) in \( I \).

By means of an appropriately careful analysis of the limit process documented in (5.17), again explicitly relying on radial symmetry and one-dimensional Sobolev inequalities, we can now pass to the announced verification of asymptotic stationarity in the first equation from (1.2), and simultaneously confirm the ordering of final and initial energy levels that has been suggested by (4.6).

**Lemma 5.6.** Assume that \( \Omega = B_R(0) \subset \mathbb{R}^n \) for some \( n \geq 3 \) and \( R > 0 \), that (1.7), (1.14) and (1.17) hold, and that \( (u, v) \) is a radial global weak energy solution of (1.2) such that (5.12) holds for some \( p > \frac{2n}{n+2} \), and let \( (t_k)_{k \in \mathbb{N}} \) as well as \( u_\infty \) and \( v_\infty \) be as in Lemma 5.3. Then \( u_\infty \in C^0(\overline{\Omega} \setminus \{0\}; [0, \infty]) \) and \( u_\infty \in C^1((\overline{\Omega} \setminus \{0\}) \cap \{u_\infty < \infty\}) \) with

\[ \nabla u_\infty = h(u_\infty) \nabla v_\infty \quad \text{in } (\overline{\Omega} \setminus \{0\}) \cap \{u_\infty < \infty\}. \tag{5.34} \]

Moreover, \( G(u_\infty) \in L^1(\Omega) \) and

\[ F(u_\infty, v_\infty) \leq F(u_0, v_0). \tag{5.35} \]

**Proof.** We let \( c_1 := \frac{\Sigma(1)}{h(1)} \) and define

\[ H(s) := \begin{cases} c_1 \cdot (s - 1), & s \in [0, 1], \\ \int_1^s \frac{\Sigma(\sigma)}{h(\sigma)} d\sigma, & s > 1. \end{cases} \]
Then $H \in C^1([0, \infty))$ with $H' > 0$ on $[0, \infty)$, $H(0) = -c_1$ and $H(s) / c_2 := \int_{\xi}^{\infty} \frac{\Sigma(\sigma)}{h(\sigma)} d\sigma \in (0, \infty)$ as $s \to \infty$, so that there exists a continuous function $H^{-1} : [-c_1, c_2] \to [0, \infty)$ such that $H^{-1}(H(s)) = s$ for all $s \in [0, \infty)$ and $H(H^{-1}(\xi)) = \xi$ for all $\xi \in [-c_1, c_2)$.

Now abbreviating $u_k := u(\cdot, t_k)$, $v_k := \psi(\cdot, t_k)$ and $g_k := \Sigma(u_k)[\nabla u_k / h(u_k)] - \Sigma(u_k)\nabla v_k$ for $k \in \mathbb{N}$, and taking $c_3 > 0$ such that in accordance with Lemma 4.1 we have $\frac{h^2(s)}{\Sigma^2(s)} \leq c_3$ for all $s \in [0, 1]$, we can estimate

$$
\int_{\Omega} |\nabla H(u_k)|^2 = c_1^2 \int_{\{u_k \leq 1\}} |\nabla u_k|^2 + \int_{\{u_k > 1\}} \frac{\Sigma^2(u_k)}{h^2(u_k)} |\nabla u_k|^2
$$

$$
= c_1^2 \int_{\{u_k \leq 1\}} \frac{h^2(u_k)}{\Sigma^2(u_k)} \cdot \frac{\Sigma^2(u_k)}{h^2(u_k)} |\nabla u_k|^2 + \int_{\{u_k > 1\}} \frac{\Sigma^2(u_k)}{h^2(u_k)} |\nabla u_k|^2
$$

$$
\leq (c_1^2 c_3 + 1) \int_{\Omega} \frac{\Sigma^2(u_k)}{h^2(u_k)} |\nabla u_k|^2
$$

$$
= (c_1^2 c_3 + 1) \int_{\Omega} |g_k + \Sigma(u_k)\nabla v_k|^2
$$

$$
\leq 2(c_1^2 c_3 + 1) \int_{\Omega} |g_k|^2 + 2(c_1^2 c_3 + 1) \int_{\Omega} \Sigma^2(u_k) |\nabla v_k|^2
$$

$$
\leq 2(c_1^2 c_3 + 1) \int_{\Omega} |g_k|^2 + 2(c_1^2 c_3 + 1) \int_{\Omega} |\nabla v_k|^2
$$

for all $k \in \mathbb{N}$, because $\Sigma^2 \leq 1$ on $[0, \infty)$ by Lemma 4.1. Since Lemma 5.3 guarantees that $(\nabla v_k)_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega)$ and that

$$
g_k \to 0 \quad \text{in} \ L^2(\Omega) \quad \text{as} \ k \to \infty, \quad (5.36)
$$

this entails the existence of $c_4 > 0$ such that

$$
\int_{\Omega} |\nabla H(u_k)|^2 \leq c_4 \quad \text{for all} \ k \in \mathbb{N}. \quad (5.37)
$$

As Lemma 4.1 moreover warrants that $H(s) \leq \int_{\xi}^{\infty} d\sigma \leq s$ for all $s > 1$ and that hence

$$
\int_{\Omega} |H(u_k)| = c_1 \int_{\{u_k \leq 1\}} (1 - u_k) + \int_{\{u_k > 1\}} H(u_k)
$$

$$
\leq c_1 |\Omega| + \int_{\Omega} u_k \quad \text{for all} \ k \in \mathbb{N},
$$

for example, relying on (5.12), we moreover find $c_5 > 0$ such that

$$
\int_{\Omega} |H(u_k)| \leq c_5 \quad \text{for all} \ k \in \mathbb{N},
$$
so that (5.37) in particular implies boundedness of \((H(u_k))_{k \in \mathbb{N}}\), when viewed as a sequence of functions defined on the interval \([0, R]\), in \(W^{1,2}((\delta, R))\) for all \(\delta \in (0, R)\). According to the compactness of the embedding \(W^{1,2}((\delta, R)) \hookrightarrow C^0([\delta, R])\) for any such \(\delta\), we thus obtain that \((H(u_k))_{k \in \mathbb{N}}\) is relatively compact in \(C^0_{\text{loc}}((0, R])\), and that whenever \((H(u_{k_l}))_{l \in \mathbb{N}}\) is a subsequence thereof fulfilling \(H(u_{k_l}) \to z_\infty\) in \(C^0_{\text{loc}}((0, R])\) as \(l \to \infty\) with some \(z_\infty \in C^0((0, R]; [-c_1, c_2])\), by continuity of \(H^{-1}\), we must have

\[ u_{k_l} = H^{-1}(H(u_{k_l})) \to H^{-1}(z_\infty) \quad \text{in } C^0_{\text{loc}}((0, R] \cap \{z_\infty < c_2\}) \quad \text{as } l \to \infty. \]

Since from (5.12) and (5.13) we already know that \(u_{k_l} \to u_\infty\) in \(L^1_{\text{loc}}((0, R])\) as \(l \to \infty\), by Egorov’s theorem this ensures that in fact \(u_\infty = H^{-1}(z_\infty)\) a.e. in \((0, R) \cap \{z_\infty < c_2\}\). As thus all such accumulation points of \((H(u_k))_{k \in \mathbb{N}}\) must coincide, it thus follows that if we suitably redefine \(u_\infty\) on a null set if necessary, then in fact we must have \(H(u_\infty) \in C^0((0, R]; [-c_1, c_2])\) and \(H(u_k) \to H(u_\infty)\) in \(C^0_{\text{loc}}((0, R])\) along the entire sequence \(k \to \infty\), so that \(\{u_\infty < \infty\}\) is relatively open in \((0, R]\), that \(u_\infty \in C^0((0, R]; [0, \infty])\), and that

\[ u_k \to u_\infty \quad \text{in } C^0_{\text{loc}}((0, R] \cap \{u_\infty < \infty\}) \quad \text{as } k \to \infty. \quad (5.38) \]

Since clearly \(|\{u_\infty = \infty\}| = 0\) by (5.18), this in particular entails that \(u_k \to u_\infty\) a.e. in \(\Omega\) as \(k \to \infty\), so that using Fatou’s lemma along with (5.14) we infer that \(G(u_\infty) \in L^1(\Omega)\) and that

\[ \int_\Omega G(u_\infty) \leq \liminf_{k \to \infty} \int_\Omega G(u_k). \]

Since furthermore \(v_k \to v_\infty\) in \(W^{1,2}(\Omega)\) by (5.16) and thus

\[ \frac{1}{2} \int_\Omega |\nabla v_k|^2 \to \frac{1}{2} \int_\Omega |\nabla v_\infty|^2 \quad \text{and} \quad \frac{1}{2} \int_\Omega v_k^2 \to \frac{1}{2} \int_\Omega v_\infty^2 \]

as \(k \to \infty\), and since this also ensures that

\[ \int_\Omega u_k v_k \to \int_\Omega u_\infty v_\infty \quad \text{as } k \to \infty, \]

due to the continuity of the embedding \(W^{1,2}(\Omega) \hookrightarrow L^{2n/(n+2)}(\Omega)\) and the fact that \(u_k \to u\) in \(L^{2n/(n+2)}(\Omega)\) as \(k \to \infty\) by (5.13), the inequality (5.35) thereby becomes a consequence of (4.6).

To finally derive the claimed differentiability property and (5.34), we fix any open interval \(I \subset (0, R] \cap \{u_\infty < \infty\}\) such that \(\bar{I} \subset (0, R) \cap \{u_\infty < \infty\}\), and use (5.38) to see that by continuity of \(h\) on \([0, \infty)\),

\[ h(u_k) \to h(u_\infty) \quad \text{and} \quad \frac{h(u_k)}{\Sigma(u_k)} \to h(u_\infty) \frac{1}{\Sigma(u_\infty)} \quad \text{in } C^0(\bar{I}) \quad \text{as } k \to \infty, \]

while from Lemma 5.3, we know that

\[ v_{kr} \to v_{\infty r} \quad \text{in } L^2(I) \quad \text{as } k \to \infty, \]
and that \(g_k \equiv \Sigma(u_k)\frac{u_k}{h(u_k)} - \Sigma(u_k)v_{kr}\) satisfies
\[
g_k \to 0 \quad \text{in} \quad L^2(I) \quad \text{as} \quad k \to \infty.
\]
Therefore,
\[
u_{kr} = h(u_k)v_{kr} + \frac{h(u_k)}{\Sigma(u_k)}g_k \to h(u_\infty)v_{\infty,r} \quad \text{in} \quad L^2(I) \quad \text{as} \quad k \to \infty,
\]
where we note that in view of the inclusion \(v_\infty \in W^{2,p}(\Omega)\), as asserted by Lemma 5.3, applying Lemma 5.4 we find that after rearranging \(v_\infty\) on a null set we can achieve that the function \(h(u_\infty)v_{\infty,r}\) becomes continuous throughout \((0, R] \cap \{u_\infty < \infty\}\). Together with again (5.38), this enables us to invoke Lemma 5.5 to infer that indeed \(u_\infty \in C^1(I)\) with \(u_{\infty,r} = h(u_\infty)v_{\infty,r}\), in \(I\). Since \(I\) was an arbitrary interval with the indicated properties, this completes the proof. \(\square\)

Building on the latter and assuming (1.20) now, we can proceed to verify the inequality in (5.10).

**Lemma 5.7.** Let \(\Omega = B_R(0) \subset \mathbb{R}^n\) for some \(n \geq 3\) and \(R > 0\), and suppose that (1.7), (1.14) and (1.17) are satisfied, and that (1.20) holds with some \(\mu > 0\) and \(K_{\varepsilon G} > 0\). Then whenever \((u, v)\) is a radial global weak energy solution of (1.2) fulfilling (5.12) for some \(p > \frac{2n}{n+2}\), the functions \(u_\infty\) and \(v_\infty\) from Lemma 5.3 have the property that \(u_\infty(x \cdot \nabla v_\infty)\) belongs to \(L^1(\Omega)\) with
\[
-\int_\Omega u_\infty(x \cdot \nabla v_\infty) \leq n \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + 2R \int_\Omega |\nabla v_\infty|.
\]

**Proof.** Again using that \(q := \frac{p}{p-1}\) satisfies \(q < \frac{np}{(n-2)p}\) due to the fact that \(p > \frac{2n}{n+2}\), an application of Lemma 5.4 shows that indeed \(u_\infty(x \cdot \nabla v_\infty) \in L^1(\Omega)\), because \(u_\infty \in L^p(\Omega)\) by Lemma 5.3. To verify (5.39), we choose \((\zeta_\eta)_{\eta \in (0,R)} \subset C^0_{\text{c}}(\overline{\Omega} \setminus \{0\})\) such that \(0 \leq \zeta_\eta \leq 1\) in \(\overline{\Omega}\), \(\zeta_\eta \equiv 1\) in \(\overline{\Omega} \setminus B_R(0)\) and \(|\nabla \zeta_\eta| \leq \frac{2}{\eta} \) in \(\overline{\Omega}\) for all \(\eta \in (0, R)\), and we moreover fix \((\chi_\delta)_{\delta \in (0, \frac{1}{2})} \subset C^\infty([0, \infty))\) such that for all \(\delta \in (0, \frac{1}{2})\) we have \(0 \leq \chi_\delta \leq 1\) in \([0, \infty)\), \(\chi_\delta \equiv 0\) in \([0, 1] \cup [2, \infty)\) and \(\chi_\delta \equiv 1\) in \([\frac{1}{2}, \infty)\), and such that \(\chi_\delta \not\equiv \chi\) in \([0, \infty)\) and for some \(\chi \in C^\infty([0, \infty))\) fulfilling \(\chi \equiv 0\) in \([0,1]\) and \(\chi \equiv 1\) in \([2, \infty)\). Then introducing
\[
\ell_\delta(s) := \int_1^s \frac{\sigma \chi_\delta(\sigma)}{h(\sigma)}d\sigma, \quad s \geq 0, \quad \delta \in \left(0, \frac{1}{2}\right),
\]
we know from Lemma 5.6 that \(\nabla \ell_\delta(u_\infty) \in C^0(\overline{\Omega} \setminus \{0\})\) with
\[
\nabla \ell_\delta(u_\infty) = \chi_\delta(u_\infty)u_\infty \nabla v_\infty \quad \text{in} \quad \overline{\Omega} \setminus \{0\},
\]
so that we may integrate by parts to see that for all \(\eta \in (0, R)\) and \(\delta \in (0, \frac{1}{2})\),
\[
-\int_\Omega \zeta_\eta(x)\chi_\delta(u_\infty)u_\infty(x \cdot \nabla v_\infty) = -\int_\Omega \zeta_\eta(x) (x \cdot \nabla \ell_\delta(u_\infty))
\]
\[
= n \int_\Omega \zeta_\eta(x) \ell_\delta(u_\infty) + \int_\Omega \ell_\delta(u_\infty)(x \cdot \nabla \zeta_\eta(x)) - \int_{\partial \Omega} \ell_\delta(u_\infty)(x \cdot \nu) \\
\leq n \int_{\{u_\infty \geq 1\}} \ell_\delta(u_\infty) + \int_\Omega \ell_\delta(u_\infty)(x \cdot \nabla \zeta_\eta(x)),
\]
(5.41)
because \(\nabla \cdot x \equiv n\) in \(\Omega\), \(0 \leq \zeta_\eta \leq 1\) in \(\Omega\), \(\ell_\delta \equiv 0\) in \((0,1)\), \(0 \leq \ell_\delta \leq \ell\) in \([1, \infty)\) and \(x \cdot \nu = R > 0\) on \(\partial \Omega\).

Now since (1.20) along with (5.18) and the inclusion \(G(u_\infty) \in L^1(\Omega)\), as guaranteed by Lemma 5.6, asserts that \(\int_{\{u_\infty \geq 1\}} \ell_\delta(u_\infty) < \infty\), and since \(\ell_\delta \equiv 0\) on \([0,1]\) as well as
\[
|\ell_\delta(u_\infty(x))(x \cdot \nabla \zeta_\eta(x))| \leq 2\ell(u_\infty(x)) \quad \text{for all } x \in \{u_\infty \geq 1\}, \eta \in (0,R) \text{ and } \delta \in \left(0, \frac{1}{2}\right),
\]
by means of the dominated convergence theorem, we see that herein
\[
\int_\Omega \ell_\delta(u_\infty)(x \cdot \nabla \zeta_\eta(x)) \to 0 \quad \text{as } \eta \searrow 0 \quad \text{for all } \delta \in \left(0, \frac{1}{2}\right).
\]
(5.42)

Apart from that, by the same token,
\[
\int_\Omega (1 - \zeta_\eta(x)) u_\infty(x \cdot \nabla v_\infty) \to 0 \quad \text{as } \eta \searrow 0 \quad \text{for all } \delta \in \left(0, \frac{1}{2}\right)
\]
due to the majorization \(|(1 - \zeta_\eta(x)) u_\infty(x \cdot \nabla v_\infty)| \leq u_\infty|x \cdot \nabla v_\infty| \in L^1(\Omega)\), and moreover
\[
\left| - \int_\Omega \zeta_\eta(x)(1 - \zeta_\delta(u_\infty)) u_\infty(x \cdot \nabla v_\infty) \right| \\
\leq \int_{\{u_\infty < 2\} \cup \{u_\infty > 1\}} u_\infty|x \cdot \nabla v_\infty| \\
\leq 2R \int_\Omega |\nabla v_\infty| + \int_{\{u_\infty > \frac{1}{2}\}} u_\infty|x \cdot \nabla v_\infty| \quad \text{for all } \eta \in (0,R) \text{ and } \delta \in \left(0, \frac{1}{2}\right),
\]
with
\[
\int_{\{u_\infty > \frac{1}{2}\}} u_\infty|x \cdot \nabla v_\infty| \to 0 \quad \text{as } \delta \searrow 0
\]
again by the dominated convergence theorem and the fact that \(u_\infty\) and \(u_\infty|x \cdot \nabla v_\infty|\) belong to \(L^1(\Omega)\). Decomposing
\[
- \int_\Omega u_\infty(x \cdot \nabla v_\infty) = - \int_\Omega \zeta_\eta(x) \chi_\delta(u_\infty) u_\infty(x \cdot \nabla v_\infty) - \int_\Omega (1 - \zeta_\eta(x)) u_\infty(x \cdot \nabla v_\infty) \\
- \int_\Omega \zeta_\eta(x)(1 - \zeta_\delta(u_\infty)) u_\infty(x \cdot \nabla v_\infty), \quad \eta \in (0,R), \delta \in \left(0, \frac{1}{2}\right),
\]
we thus infer (5.39) from (5.41) and (5.42) on taking \(\eta \searrow 0\) and then \(\delta \searrow 0\).
In order to facilitate an application of this in the intended line of arguments, it remains to verify the identity \((5.9)\), which in fact can be achieved for widely arbitrary radial functions from \(W^{2,p}(\Omega)\) with vanishing normal derivative, provided that \(p\) satisfies the inequality \(p > \frac{2n}{n+2}\) which is of crucial importance here.

**Lemma 5.8.** Let \(\Omega = B_R(0) \subset \mathbb{R}^n\) for some \(n \geq 3\) and \(R > 0\), and suppose that \(p > \frac{2n}{n+2}\) and that \(V \in W^{2,p}(\Omega)\) is radially symmetric with \(\frac{\partial V}{\partial n} = 0\) on \(\partial \Omega\). Then \(\Delta V(x \cdot \nabla V) \in L^1(\Omega)\) with

\[
\int_{\Omega} \Delta V(x \cdot \nabla V) = \frac{n-2}{2} \int_{\Omega} |\nabla V|^2. \tag{5.43}
\]

**Proof.** Since \(p > 1\) and \(V\) is radial, from Lemma 5.4 and our hypothesis we know that \(V \in C^1(\overline{\Omega} \setminus \{0\})\) with \(\nabla V = 0\) on \(\partial \Omega\), and that if we choose a sequence \((V_l)_{l \in \mathbb{N}} \subset C^2(\overline{\Omega})\) of radially symmetric functions \(V_l\) fulfilling

\[
V_l \to V \quad \text{in} \ W^{2,p}(\Omega) \quad \text{as } l \to \infty, \tag{5.44}
\]

then also

\[
\nabla V_l \to 0 \quad \text{in} \ C^0(\partial \Omega) \quad \text{as } l \to \infty. \tag{5.45}
\]

Apart from that, making use of the assumption \(p > \frac{2n}{n+2}\) we infer from \((5.44)\) that writing \(q := \frac{p}{p-1}\) we have

\[
\int_{\Omega} |x \cdot \nabla V_l - x \cdot \nabla V|^q \to 0 \quad \text{as } l \to \infty, \tag{5.46}
\]

because \(\frac{(n-2)p}{n+2} - 1 = \frac{n-2p}{n(p-1)} - 1 = \frac{2n-(n+2)p}{n(p-1)} < 0\).

Now for each \(l \in \mathbb{N}\), two integrations by parts show that

\[
\int_{\Omega} \Delta V_l(x \cdot \nabla V_l) = -\int_{\Omega} |\nabla V_l|^2 - \frac{1}{2} \int_{\Omega} x \cdot \nabla |\nabla V_l|^2 + \int_{\partial \Omega} (x \cdot \nabla V_l)(\nabla V_l \cdot \nu) \\
= \frac{n-2}{2} \int_{\Omega} |\nabla V_l|^2 - \frac{1}{2} \int_{\partial \Omega} |\nabla V_l|^2(x \cdot \nu) + \int_{\partial \Omega} (x \cdot \nabla V_l)(\nabla V_l \cdot \nu), \tag{5.47}
\]

where by \((5.45)\),

\[
-\frac{1}{2} \int_{\partial \Omega} |\nabla V_l|^2(x \cdot \nu) + \int_{\partial \Omega} (x \cdot \nabla V_l)(\nabla V_l \cdot \nu) \to 0 \quad \text{as } l \to \infty,
\]

and where due to our definition of \(q\), \((5.44)\) together with \((5.46)\) guarantees that

\[
\int_{\Omega} \Delta V_l(x \cdot \nabla V_l) \to \int_{\Omega} \Delta V(x \cdot \nabla V) \quad \text{as } l \to \infty.
\]
Since the assumption \( p > \frac{2n}{n+2} \) ensures continuity of the embedding \( W^{2,p}(\Omega) \hookrightarrow W^{1,2}(\Omega) \), and since thus from (5.44) we furthermore obtain that also

\[
\frac{n-2}{2} \int_{\Omega} |\nabla V_l|^2 \rightarrow \frac{n-2}{2} \int_{\Omega} |\nabla V|^2 \quad \text{as} \ l \rightarrow \infty,
\]
on taking \( l \rightarrow \infty \) in (5.47), we obtain (5.43).

Now a key toward our exploitation of (5.9) and (5.10) especially in the presence of critical relationships between \( S \) and \( D \), and hence marking a crucial difference to the approach in [50], consists in the following quite straightforward application of standard elliptic regularity theory.

**Lemma 5.9.** There exists \( B > 0 \) such that whenever \( V \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega} \setminus \{0\}) \) and \( f \in L^1(\Omega) \) are radially symmetric functions satisfying

\[
\int_{\Omega} \nabla V \cdot \nabla \psi + \int_{\Omega} V \psi = \int_{\Omega} f \psi \quad \text{for all} \ \psi \in W^{1,2}(\Omega),
\]
we have

\[
\int_{\Omega} |\nabla V| \leq B \int_{\Omega} |f|
\]
and

\[
|V| \leq B \int_{\Omega} |f| \quad \text{throughout} \ \partial \Omega.
\]

**Proof.** According to a well-known result on \( W^{1,p} \) regularity in elliptic boundary value problems with right-hand sides in \( L^1 \) [11], there exists \( c_1 > 0 \) such that whenever \( V \in W^{1,2}(\Omega) \) and \( f \in L^1(\Omega) \) satisfy (5.48), the inequality

\[
\int_{\Omega} |\nabla V| + \int_{\Omega} |V| \leq c_1 \int_{\Omega} |f|
\]
holds. Besides directly implying (5.49), this ensures that if in addition \( V = V(r) \) is radial and continuous on \( (0, R) \), then there exists \( r_0 = r_0(V) \in (R_2, R) \) such that

\[
|V(r_0)| = \frac{1}{|\Omega \setminus B_{R/2}(0)|} \int_{\Omega \setminus B_{R/2}(0)} |V| \leq c_2 \int_{\Omega} |f|
\]
with \( c_2 := \frac{c_1}{|\Omega \setminus B_{R/2}(0)|} \). As (5.51) moreover entails that writing \( \omega_n := n |B_1(0)| \) and \( c_3 := \frac{2^{n-1} c_1}{R^{n-1} \omega_n} \) we have

\[
\int_{r_0}^r |V(r)| dr \leq r_0^{1-n} \int_{r_0}^R r^{n-1} |V(r)| dr \leq \frac{2^{n-1} c_1}{R^{n-1} \omega_n} \int_{\Omega \setminus B_{R/2}(0)} |\nabla V| \leq c_3 \int_{\Omega} |f|,
\]
this guarantees that

\[ |V(R)| = \left| V(r_0) + \int_{r_0}^{R} V_r(\rho) d\rho \right| \leq (c_2 + c_3) \int_{\Omega} |f| \]

and hence establishes (5.50).

As a consequence of Lemmas 5.7 and 5.8, this will now provide a lower bound, depending on \((u_0, v_0)\) only through the number \(\int_{\Omega} u_0\), for the initial energy level of any conceivable radial global very weak energy solution, quite in the style of the outcomes in [50] or also [30], but now in frameworks of possibly critical and much more general nonlinearities.

**Lemma 5.10.** Suppose that (1.7) and additionally (1.20) hold with some \(K_\varphi > 0\), and assume that for some \((u_0, v_0)\) fulfilling (1.14) and (1.17), \((u, v)\) is a radial global very weak energy solution of (1.2) satisfying (1.23). Then

\[
P(u_0, v_0) \geq -\frac{\omega_n R^n B^2}{2(n-2)} \cdot \left\{ \int_{\Omega} u_0 \right\}^2 - \frac{2(RB + nK_\varphi)}{n-2} \cdot \int_{\Omega} u_0,
\]

(5.52)

where \(B > 0\) is as in Lemma 5.9 and \(\omega_n := n|B_1(0)|\).

**Proof.** Taking \(p > \frac{2n}{n+2}\) such that the inequality in (1.23) holds, we let \((u_\infty, v_\infty) \in L^p(\Omega) \times W^{2,p}(\Omega)\) be as correspondingly provided by Lemma 5.3, and note that then \(\Delta v_\infty = v_\infty - u_\infty\) a.e. in \(\Omega\) according to (5.19) and standard elliptic regularity theory ([27]). Since \(x \cdot \nabla v_\infty \in L^{(p/p-1)}(\Omega)\) due to Lemma 5.4 and the fact that \((n-2p) \cdot \frac{p}{p-1} < np\) according to our restriction \(p > \frac{2n}{n+2}\), we may multiply this by \(x \cdot \nabla v_\infty\) to see that due to Lemmas 5.8 and 5.7,

\[
\frac{n-2}{2} \int_{\Omega} |\nabla v_\infty|^2 = \int_{\Omega} v_\infty(x \cdot \nabla v_\infty) - \int_{\Omega} u_\infty(x \cdot \nabla v_\infty)
\leq \int_{\Omega} v_\infty(x \cdot \nabla v_\infty) + n \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + 2R \int_{\Omega} |\nabla v_\infty|.
\]

(5.53)

Here an integration by parts shows that since \(\nabla \cdot x = n\) in \(\Omega\) and \(x \cdot \nu = R\) on \(\partial \Omega\),

\[
\int_{\Omega} v_\infty(x \cdot \nabla v_\infty) = \frac{1}{2} \int_{\Omega} x \cdot \nabla v_\infty^2
\]

\[
= -\frac{n}{2} \int_{\Omega} v_\infty^2 + \frac{1}{2} \int_{\partial \Omega} v_\infty^2 (x \cdot \nu)
\]

\[
= -\frac{n}{2} \int_{\Omega} v_\infty^2 + \frac{R}{2} \int_{\partial \Omega} v_\infty^2.
\]

Using Lemma 5.9 together with (5.19) and (5.18), from (5.53) we thus infer that with \(m := \int_{\Omega} u_0\) we have

\[
\frac{n-2}{2} \int_{\Omega} |\nabla v_\infty|^2 \leq -\frac{n}{2} \int_{\Omega} v_\infty^2 + \frac{R}{2} \cdot \omega_n R^{n-1} \cdot B^2 m^2 + n \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + 2RBM
\]

\[\square\]
and hence
\[
\frac{1}{2} \int_{\Omega} |\nabla v_\infty|^2 + \frac{1}{2} \int_{\Omega} v_\infty^2 \leq \left\{ \frac{1}{2} - \frac{n}{2(n-2)} \right\} \cdot \int_{\Omega} v_\infty^2 \\
+ \frac{n}{n-2} \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + \frac{\omega_n R^n B^2 m^2}{2(n-2)} + \frac{2RBm}{n-2} \\
\leq \frac{n}{n-2} \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + \frac{\omega_n R^n B^2 m^2}{2(n-2)} + \frac{2RBm}{n-2},
\]

because \( \frac{n}{n-2} \geq 1 \). Now since
\[
\frac{n}{n-2} \int_{\{u_\infty \geq 1\}} \ell(u_\infty) \leq \int_{\{u_\infty \geq 1\}} G(u_\infty) + \frac{nK_\ell}{n-2} \int_{\{u_\infty \geq 1\}} (u_\infty + 1) \leq \int_{\Omega} G(u_\infty) + \frac{2nK_\ell m}{n-2}
\]
according to (1.20) and the nonnegativity of \( G \), and since testing (5.19) by \( v_\infty \) shows that
\[
\int_{\Omega} |\nabla v_\infty|^2 + \int_{\Omega} v_\infty^2 = \int_{\Omega} u_\infty v_\infty,
\]
along with (5.35), this implies that indeed
\[
\mathcal{F}(u_0, v_0) \geq \mathcal{F}(u_\infty, v_\infty) = -\frac{1}{2} \int_{\Omega} |\nabla v_\infty|^2 - \frac{1}{2} \int_{\Omega} v_\infty^2 + \int_{\Omega} G(u_\infty) \\
\geq -\frac{\omega_n R^n B^2 m^2}{2(n-2)} - \frac{2(RB + nK_\ell)m}{n-2},
\]
as claimed.

When viewed together with Lemma 5.1, this finally entails our main result on nonexistence of global very weak energy solutions throughout large sets of initial data:

**Proof of Theorem 1.3.** For fixed \((u_0, v_0)\) fulfilling (1.14) and (1.17), taking \(B > 0\) from Lemma 5.9 we may employ Lemma 5.1 to find \((u_{0j})_{j \in \mathbb{N}}\) and \((v_{0j})_{j \in \mathbb{N}}\) such that \((u_{0j}, v_{0j})\) satisfies (1.14) and (1.17) for all \(j \in \mathbb{N}\), that (1.22) holds, and that
\[
\mathcal{F}(u_{0j}, v_{0j}) < -\frac{\omega_n R^n B^2}{2(n-2)} \cdot \left\{ \int_{\Omega} u_{0j} \right\}^2 - \frac{2(RB + nK_\ell)}{n-2} \cdot \int_{\Omega} u_{0j} \quad \text{for all} \quad j \in \mathbb{N}.
\]
Then Lemma 5.10 asserts that a radial global very weak energy solution fulfilling (1.23) indeed cannot exist.

\[ \square \]

6 \ | \ **CRITICAL MASS PHENOMENA: CONCRETE EXAMPLES**

For nonlinearities fulfilling both (1.12) and (1.19) together with (1.20), a straightforward combination of Theorem 1.3 with Theorem 1.1 and Theorem 1.2 yields our main result on critical mass phenomena in its general form stated in Corollary 1.4:
**Proof of Corollary 1.4.** Finiteness and positivity of $m_c^{\text{weak}}$ is an immediate consequence of Theorems 1.3 and 1.2. Since global classical solutions are clearly also global very weak energy solutions according to Definition 4.2, and since (1.16) implies (1.23) by boundedness of $\Omega$, however, Theorem 1.3 can moreover be combined with Theorem 1.1 so as to assert finiteness and positivity also of $m_c^{\text{classical}}$ whenever (1.13) holds with some $\beta \geq 0$ and $\gamma \geq 0$.

The verification of our basis for concrete examples, as thereby implied, now reduces to quite an elementary estimation of the integrals making up $\ell'$ and $G$:

**Proof of Corollary 1.5.** In view of Corollary 1.4, we only need to make sure that (1.20) holds with some appropriately large $K_{\varphi} > 0$. To this end, we first use the definition (1.21) of $\ell'$ to see that the second inequality in (1.25) implies that

$$
\ell'(s) \leq \kappa \int_1^s \sigma^{-\frac{2}{n}} d\sigma + C_{SD} \int_1^s d\sigma
$$

$$
= \frac{n\kappa}{2(n-1)} \cdot \left( s^{\frac{2(n-1)}{n}} - 1 \right) + C_{SD} \cdot (s - 1)
$$

$$
\leq \frac{n\kappa}{2(n-1)} s^{\frac{2(n-1)}{n}} + C_{SD} \cdot s \quad \text{for all } s \geq 1,
$$

and then go back to (1.10) to infer that due to the first inequality in (1.25),

$$
\frac{n-2}{n} G(s) \geq \frac{(n-2)\kappa}{n} \int_1^s \int_1^\sigma \tau^{-\frac{2}{n}} d\tau d\sigma - \frac{n-2}{n} \int_1^s \int_1^\sigma f(\tau) d\tau d\sigma
$$

$$
= \frac{n\kappa}{2(n-1)} s^{\frac{2(n-1)}{n}} - \frac{n\kappa}{2(n-1)} + \kappa - \frac{n-2}{n} \int_1^s \int_1^\sigma f(\tau) d\tau d\sigma
$$

$$
\geq \frac{n\kappa}{2(n-1)} s^{\frac{2(n-1)}{n}} - \frac{n\kappa}{2(n-1)} - \kappa - \frac{(n-2)c_1}{n} \cdot s \quad \text{for all } s \geq 1
$$

with $c_1 := \int_1^\infty |f(\tau)| d\tau$. Combining this with (6.1) readily entails (1.20) on an evident choice of $K_{\varphi}$. 

We can thereby verify the occurrence of a critical mass phenomenon in (1.2)–(1.26) in a rather handy manner:

**Proof of Corollary 1.6.** Since, by concavity of $0 < \tau \mapsto \tau^{-\frac{2}{n}}$,

$$
0 \leq (1 + s)^{\frac{n-2}{n}} - s^{\frac{n-2}{n}} \leq \frac{n-2}{n} \cdot s^{\frac{n-2}{n}} \leq \frac{n-2}{n} \quad \text{for all } s \geq 1
$$

we can estimate

$$
0 \leq \frac{D(s)}{S(s)} - \frac{1}{s^{2/n}}
$$
\[
= \frac{1}{s} \cdot \left\{ (s + 1) \frac{n+2}{n} - s \frac{n-2}{n} \right\} \\
\leq \frac{n-2}{n} \cdot \frac{1}{s} \quad \text{for all } s \geq 1.
\]

As thus (1.25) holds with \( f \equiv 0, \kappa := 1 \) and \( C_{SD} := \frac{n-2}{n} \), observing that in addition clearly (1.13) is satisfied with some \( \beta \geq 0 \) and \( \gamma \geq 0 \), we may directly derive the claim from Corollary 1.5. \( \square \)

Likewise, confirming mass criticality of the nonlinearities specified in (1.4) and (1.27) now also reduces to a simple computation:

**Proof of Corollary 1.7.** By means of (1.4) and (1.27), we compute

\[
D(s) = a \cdot \left( 1 + \left( \frac{n-2}{n} \cdot \frac{1}{s^{2/n}} \right) \cdot \exp \left( -bs^{\frac{n-2}{n}} \right) \right) \quad \text{and} \quad S(s) = as \exp \left( -bs^{\frac{n-2}{n}} \right), \quad s \geq 0,
\]

to see that

\[
\frac{D(s)}{S(s)} = \left( \frac{n-2}{n} \right) b \cdot \frac{1}{s^{2/n}} + \frac{1}{s} \quad \text{for all } s > 0.
\]

Therefore, (1.25) holds with \( f \equiv 0, \kappa := \frac{(n-2)b}{n} \) and \( C_{SD} := 1 \), so that an application of Corollary 1.5 immediately leads to the claimed conclusion. \( \square \)

**ACKNOWLEDGEMENTS**

The author is grateful to the anonymous reviewer for a very thorough evaluation of this manuscript. He furthermore acknowledges support of the Deutsche Forschungsgemeinschaft in the context of the project Emergence of structures and advantages in cross-diffusion systems (Project No. 411007140, GZ: WI 3707/5-1).

**JOURNAL INFORMATION**

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. H. Amann, *Dynamic theory of quasilinear parabolic systems III. Global existence*, Math. Z. **202** (1989), 219–250.
2. N. Bellomo and M. Winkler, *Finite-time blow-up in a degenerate chemotaxis system with flux limitation*, Trans. Amer. Math. Soc. **B 4** (2017), 31–67.
3. N. Bellomo and M. Winkler, *A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up*, Commun. Partial Differential Equations **42** (2017), 436–473.
4. B. Bieganowski, T. Cieślak, K. Fujie, and T. Senba, *Boundedness of solutions to the critical fully parabolic quasilinear one-dimensional Keller-Segel system*, Math. Nachr. **292** (2019), 724–732.
5. P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), 715–743.
6. P. Biler, G. Karch, Ph. Laurençot, and T. Nadzieja, *The $8\pi$ -problem for radially symmetric solutions of a chemotaxis model in the plane*, Math. Meth. Appl. Sci. 29 (2006), 1563–1583.

7. A. Blanchet, J. A. Carrillo, and Ph. Laurençot, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differential Equations 35 (2009), 133–168.

8. A. Blanchet, J. A. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$*, Comm. Pure Appl. Math. 61 (2008), 1449–1481.

9. A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equ. 11 (2011), 1–32.

10. D. Bresch and Ph. Laurençot, *Boundedness of the critical chemotaxis system in the plane*, J. Differential Equations 258 (2015), 277–306.

11. H. Brézis and W. A. Strauss, *Semi-linear second-order elliptic equations in $L^1$*, J. Math. Soc. Japan 25 (1973), 565–590.

12. V. Calvez and L. Corrias, *The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$*, Comm. Math. Sci. 6 (2008), 417–446.

13. V. Calvez, V. Perthame, and S. Masmoudi, *Critical mass for the parabolic-elliptic Keller-Segel model in dimension 2*, J. Funct. Anal. 252 (2012), 3112–3130.

14. T. Cieślak and Ph. Laurençot, *Finite time blow-up for radially symmetric solutions to a critical quasilinear Smoluchowski-Poisson system*, C. R. Math. Acad. Sci. Paris 347 (2009), 237–242.

15. T. Cieślak and Ph. Laurençot, *Finite time blow-up for a one-dimensional parabolic-parabolic chemotaxis system*, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 1, 437–446.

16. T. Cieślak and C. Stinner, *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions*, J. Differential Equations 252 (2012), no. 10, 5832–5851.

17. T. Cieślak and C. Stinner, *Finite-time blowup in a supercritical quasilinear parabolic-parabolic Keller-Segel system in dimension 2*, Acta Appl. Math. 129 (2014), 135–146.

18. T. Cieślak and C. Stinner, *New critical exponents in a fully parabolic quasilinear Keller-Segel system and applications to volume filling models*, J. Differential Equations 258 (2015), no. 6, 2080–2113.

19. T. Cieślak and M. Winkler, *Global bounded solutions in a two-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity*, Nonlinear Anal. Real World Appl. 35 (2017), 1–19.

20. T. Cieślak and M. Winkler, *Stabilization in a higher-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity*, Nonlinear Anal. Theory Meth. Appl. 159 (2017), 129–144.

21. M. H. Cohen and A. Robertson, *Wave propagation in the early stages of aggregation of cellular slime molds*, J. Theoret. Biol. 31 (1971), 101–118.

22. L. Corrias and B. Perthame, *Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces*, Math. Comput. Mod. 47 (2008), 755–764.

23. M. Del Pino, A. Pistoia, and G. Vaira, *Large mass boundary condensation patterns in the stationary Keller-Segel system*, J. Differential Equations 261 (2016), 3414–3462.

24. M. Ding and X. Zhao, *$L^p$-measure criteria for boundedness in a quasilinear parabolic-parabolic Keller-Segel system with supercritical sensitivity*, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 5297–5315.

25. M. Ding and S. Zheng, *$L^p$-measure criteria for boundedness in a quasilinear parabolic-elliptic Keller-Segel system with supercritical sensitivity*, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 2971–2988.

26. Y. Giga and H. Sohr, *Abstract $L^p$ estimates for the cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal. 102 (1991), 72–94.

27. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin/Heidelberg, 2001.

28. T. Hillen and K. J. Painter, *A user’s guide to PDE models for chemotaxis*, J. Math. Biol. 58 (2009), 183–217.

29. D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I*, Jahresberichte DMV 105 (2003), no. 3, 103–165.

30. D. Horstmann and G. Wang, *Blow-up in a chemotaxis model without symmetry assumptions*, Eur. J. Appl. Math. 12 (2001), 159–177.

31. S. Ishida, K. Seki, and T. Yokota, *Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains*, J. Differential Equations 256 (2014), 2993–3010.

32. E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. 26 (1970), 399–415.
33. R. Kowalczyk and Z. Szymańska, On the global existence of solutions to an aggregation model, J. Math. Anal. Appl. 343 (2008), 379–398.
34. J. Lankeit, Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion, J. Differential Equations 262 (2017), 4052–4084.
35. J. Lankeit, Infinite time blow-up of many solutions to a general quasilinear parabolic-elliptic Keller-Segel system, Discrete Contin. Dyn. Syst. Ser. S 13 (2020), 233–255.
36. Ph. Laurençot, Global bounded and unbounded solutions to a chemotaxis system with indirect signal production, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 6419–6444.
37. Ph. Laurençot and N. Mizoguchi, Finite time blowup for the parabolic-parabolic Keller-Segel system with critical diffusion, Ann. Inst. H. Poincaré, Anal. Non Linéaire 34 (2017), 197–220.
38. N. Mizoguchi and Ph. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaré, Anal. Non Linéaire 31 (2014), 851–875.
39. T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, J. Inequal. Appl. 6 (2001), 37–55.
40. T. Nagai, T. Senba, and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkc. Ekvacij Ser. Int. 40 (1997), 411–433.
41. K. J. Painter and T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, Can. Appl. Math. Q. 10 (2002), 501–543.
42. B. Perthame, Transport equations in biology, Birkhäuser, Basel, 2007.
43. L. M. Prescott, J. P. Harley, and D. A. Klein, Microbiology, Wm. C. Brown Publishers, Chicago/London, 1996.
44. T. Senba and T. Suzuki, A quasi-linear system of chemotaxis, Abstr. Appl. Anal. 2006 (2006), 1–21.
45. T. Suzuki, Free energy and self-interacting particles, Birkhäuser Boston Inc., Boston, 2005.
46. Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, J. Differential Equations 252 (2012), no. 1, 692–715.
47. Y. Tao and M. Winkler, Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production, J. Eur. Math. Soc. 19 (2017), 3641–3678.
48. R. Temam, Navier-Stokes equations. Theory and numerical analysis, vol. 2, North-Holland, Amsterdam, 1977.
49. Y. Wang, Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system with subcritical sensitivity, Math. Models Methods Appl. Sci. 27 (2017), 2745–2780.
50. M. Winkler, Does a ‘volume-filling effect’ always prevent chemotactic collapse? Math. Meth. Appl. Sci. 33 (2010), 12–24.
51. M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl. 100 (2013), 748–767.
52. M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, SIAM J. Math. Anal. 47 (2015), 3092–3115.
53. M. Winkler, Global existence and slow grow-up in a quasilinear Keller-Segel system with exponentially decaying diffusivity, Nonlinearity 30 (2017), 735–764.
54. M. Winkler, Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities, J. Differential Equations 266 (2019), 8034–8066.
55. D. Wrzosek, Volume filling effect in modelling chemotaxis, Math. Mod. Nat. Phenom. 5 (2010), 123–147.