Effective action for strongly correlated electron systems

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The $su(2|1)$ coherent-state path-integral representation of the partition function of the $t - J$ model of strongly correlated electrons is derived at finite doping. The emergent effective action is compared to the one proposed earlier on phenomenological grounds by Shankar to describe holes in an antiferromagnet (Nucl.Phys. B330 (1990) 433). The $t - J$ model effective action is found to have an important "extra" factor with no analogue in Shankar’s action. It represents the local constraint of no double electron occupancy and reflects the rearrangement of the underlying phase-space manifold due to the presence of strong electron correlation. This important ingredient is shown to be essential to describe the physics of strongly correlated electron systems.

Keywords: $t - J$ model of strongly correlated electrons; $su(2|1)$ coherent-state path integral

I. INTRODUCTION

In this work, we discuss a path-integral representation of the partition function for strongly correlated electron systems. In particular, we are interested in the effective low-energy action to describe a lightly doped antiferromagnet (AF). Strong electron correlations essentially determine the low-energy physics of high-$T_c$ superconductors [2], heavy-fermion systems [3], itinerant ferromagnets [4], as well as some optical lattices [5]. Accordingly, many various approaches have been proposed to study those corresponding low-energy effective actions. We will comment on some of them at the end of the paper.

Our work has actually been motivated by Shankar’s contribution [1]. Namely, two decades ago Shankar put forward a conjecture assuming that the low-energy action to describe a doped quantum AF involves spinless fermions locally coupled to a compact $U(1)$ gauge field. This gauge field is driven by the AF fluctuations controlled by a nonlinear sigma-model. This approach was further discussed in [6]. A natural question then arises as to whether that action can be derived directly from the $t - J$ model for strongly correlated electrons at finite doping. After all, this microscopical model is widely believed to capture the essential physics of a lightly doped AF.

In the present paper we show that Shankar’s effective action can indeed be derived from the $su(2|1)$ path-integral representation of the partition function of the $t - J$ model. However, the resulting path-integral measures differ by an important factor. This distinction is a manifestation of the strong coupling nature of the problem due to the no double occupancy (NDO) constraint. It may seem that this constraint is already fully accounted in Shankar’s theory by the requirement that the fermions are spinless, since no double fermion occupation is possible in this case. However, the NDO constraint modifies the original on-site Hilbert space as well as the on-site path-integral phase space, which is not explicitly taken into consideration in [1]. Shankar proposed instead an effective action right in terms of conventional fermion and spin fields to describe a doped AF. This of course implies a standard measure in the path integral. Although, his action may presumably describe some unconstrained spin-fermion models, it is not appropriate for strongly correlated electrons.

Our aim is to demonstrate in what way the NDO constraint modifies the theory discussed in [1]. To start with, we illustrate our point by considering a path-integral representation of the partition function for a simple single-site Hamiltonian. After that, we address the $t - J$ model close to half filling which is precisely the physically most relevant situation for strongly correlated electrons. This model is believed to capture the low-energy physics of lightly doped quantum AF.

Quite plainly, an electron system is said to be strongly correlated if the leading energy scale in the problem is the on-site Coulomb repulsion energy $U$. In this case the low-energy sector of the underlying on-site Hilbert space should be modified to exclude doubly occupied states. Such a modification results in an entirely new physics to account for the relevant low-energy excitations. Formally, strong correlations are encoded into the projected electron (Hubbard) operators. They act directly in the restricted Hilbert space as opposed to the conventional electron operators which describe the unconstrained system. In contrast with the conventional fermion operators which generate the standard fermionic algebra, the new operators obey more complicated commutation/anticommutation relations and are closed into a superalgebra $su(2|1)$. It is therefore natural to seek a path-integral formalism that takes the structure of that superalgebra fully into consideration. In analogy with the conventional $su(2)$ spin path integral [7], this can be carried out by employing an appropriate coherent-state basis associated with the $su(2|1)$ superalgebra representations. To make our exposition self-contained we employ some notation and results in connection with a continuum $su(2|1)$
path integral already reported elsewhere. However, the conclusion we reach below requires a more sophisticated approach based on a carefully defined time-lattice representation of the corresponding path integral.

Just to get an idea about what strong correlations really are, consider the $U = \infty$ Hubbard model which is known to capture the extreme limit for strongly correlated electrons. The Hamiltonian reads

$$H_i = -\sum_{ij,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_i (1 - n_i), \quad n_i = \sum_\sigma n_{i\sigma} \leq 1. \quad (1)$$

Here $c_{i\sigma}^\dagger (c_{i\sigma})$ is an on-site creation (annihilation) operator of an electron excitation with the spin projection $\sigma = \uparrow, \downarrow$. The hopping amplitudes $t_{ij}$ represent jumps between nearest-neighbor (nn) and next-to-nearest-neighbor (nnn) sites and are zero otherwise. Throughout this paper we will be considering a $D$ dimensional bipartite lattice, $L = A \oplus B$. The chemical potential $\mu$ controls the total number of vacancies (empty sites).

The infinitely large on-site Coulomb repulsion is accounted for by the local NDO constraint, $n_i \leq 1$, that restricts the on-site Hilbert space to states with at most one electron per site. This is the essence of strong electron correlations. In the absence of this constraint, this model simply describes a system of noninteracting electrons and it can be trivially diagonalized in the momentum space. The NDO constraint makes this problem nontrivial and its exact solution is still unknown for spatial dimensions $> 1D$. The physics behind the model (1) is certainly far from trivial. Indeed, one of the few exact results was proved by Nagaoka who showed that for one hole the ground state of the Hubbard model is a fully saturated ferromagnet. This provides an interesting example of a quantum system with ferromagnet ordering due to a purely kinetic-energy effect driven by hole hopping (itinerant ferromagnetism). Unfortunately, despite very extensive work over the years, both this model and itinerant ferromagnetism are still poorly understood. One of the open important questions related to this concerns the thermodynamic stability of the Nagaoka phase. That is, whether or not the Nagaoka state is stable when the density of holes is finite in the thermodynamic limit.

The local NDO constraint can be explicitly incorporated into the theory by projecting the Hamiltonian (1) onto the restricted Hilbert space $\prod_i \mathcal{H}_{i\text{phys}}^0$, where the 3D on-site Hilbert space $\mathcal{H}_{i\text{phys}}^0$ is spanned by the vectors $|0\rangle_i$ (empty site), $|\uparrow\rangle_i$ (spin-up electron), and $|\downarrow\rangle_i$ (spin-down electron):

$$H \rightarrow H = \mathcal{P} H \mathcal{P}, \quad \mathcal{P} = \prod_i \mathcal{P}_i,$$

where the Gutzwiller projection operator $\mathcal{P}_i$ is given by

$$\mathcal{P}_i = 1 - n_{i\sigma} n_{i-\sigma},$$

so that

$$\mathcal{P}_i c_{i\sigma} \mathcal{P}_i = : \hat{c}_{i\sigma} = c_{i\sigma}(1 - n_{i-\sigma}).$$

In this 3D subspace the constrained electron operators $\hat{c}_{i\sigma}$ can be identified with the Hubbard operators,

$$X_{i\sigma}^0 = (X_{i\sigma}^0)^\dagger = |0\rangle \langle \sigma|, \quad \sigma = \uparrow, \downarrow.$$

It follows that $\hat{c}_{i\sigma} = X_{i\sigma}^\dagger$, provided $n_{i\sigma} n_{i-\sigma} = 0$. The last requirement eliminates the doubly occupied fermionic states. As a result the Hamiltonian (1) in the constrained physical space takes the form

$$H_i = -t \sum_{ij,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} - \mu \sum_i (1 - \sum_\sigma \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma})$$

$$= -t \sum_{ij,\sigma} X_{i\sigma}^0 X_{j\sigma}^0 - \mu \sum_i X_{i0}^0. \quad (2)$$

Here $X_{i0}^0 = X_{i0}^0 X_{i0}^0 = |0\rangle \langle 0| = 1 - \sum_\sigma \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}$ stands for the on-site vacancy number operator, so that the concentration of vacancies becomes $\delta = \frac{1}{N_\sigma} \sum_i \langle X_{i0}^0 \rangle$. The on-site operator $X_{i0}^0$ removes an electron with the spin projection $\sigma$ and creates a vacancy (empty state) which is a spin singlet.

The important point is that the fermionic Hubbard operators $X_{i0}^\sigma = (X_{i0}^\sigma)^\dagger$ along with the bosonic ones, $X_{i\sigma}\sigma'$, $X_{i0}^0$, are closed under commutation/anticommutation relations into the superalgebra $su(2|1)$ [3]. The $su(2|1)$ superalgebra can be thought of as the simplest possible extension of the conventional spin $su(2)$ algebra to incorporate fermionic degrees of freedom. Namely, the bosonic sector of the $su(2|1)$ consists of three bosonic superspin operators,

$$Q^+ = X_{\uparrow\downarrow}, \quad Q^- = X_{\downarrow\uparrow}, \quad Q^z = \frac{1}{2} (X_{\uparrow\uparrow} - X_{\downarrow\downarrow}) \quad (3)$$
closed into $su(2)$, and a bosonic operator $X^{00}$ that generates a $u(1)$ factor of the maximal even subalgebra $su(2) \times u(1)$ of $su(2|1)$. The fermionic sector is constructed out of four operators $X^{\sigma\sigma}$, $X^{0\sigma}$ that transform in a spinor representation of $su(2)$. We do not intend here to discuss a general theory of the $su(2|1)$ irreducible representations that can be found elsewhere \[10\]. We will focus instead on a specific $su(2|1)$ factor of the maximal even subalgebra $su(2)$ of $su(2|1)$. The special important property of this representation is that $X^{\lambda\lambda'}X^{\lambda''\lambda'''} = \delta_{\lambda\lambda'}X^{\lambda\lambda'''}$.

II. $su(2|1)$ COHERENT-STATE MANIFOLD

The normalizable coherent states (CS's) associated with the lowest irreducible representation of $su(2|1)$ superalgebra spanned by Hubbard operators take the form

$$|z, \xi\rangle = (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2} \exp(z X^{\dagger} + \xi X^0) |\uparrow\rangle$$

$$= (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2} (|\uparrow\rangle + |\downarrow\rangle + \xi |0\rangle),$$

(5)

where a complex even Grassmann parameter $z$ and an odd complex Grassmann parameter $\xi$ are the inhomogeneous (proper) coordinated of a point on a supersphere, $(z, \xi) \in S^{2|2} \simeq CP^{1|1} = SU(2|1)/U(1|1)$. Here $CP^{1|1}$ stands for a complex projective superspace with a complex dimension $(1, 1)$. It can be thought of as a minimal superextension of an ordinary projective space $CP^1$ homeomorphic to a two-sphere, $CP^1 \simeq S^2$. The odd Grassmann parameter appears in (4) due to the fact that $X^{10}$ is a fermionic operator in contrast with the operator $X^{1\dagger}$. The product $\xi X^{0\dagger}$ represents therefore a bosonic quantity as required.

One can use as well the homogeneous coordinates $(z^1, z^2, \theta)$ on the supersphere, so that $z = z^1/z^2, \xi = \theta/z^2$, with $z^2 \neq 0$. Then the coset $CP^{1|1}$ manifold is defined by the equation

$$|z^1|^2 + |z^2|^2 + \theta \bar{\theta} = 1.$$  

The supergroup $SU(2|1)$ acts on $CP^{1|1}$ according to $Z \rightarrow Z' = gZ$, where $Z = (z^1, z^2, \theta) \in CP^{1|1}$ and $g \in SU(2|1)$. This generates a corresponding transformation of the inhomogeneous coordinates, $(z, \xi)$. In particular, if one chooses $g$ to represent a pure spin rotation,

$$g = \left( \begin{array}{ccc} u & v & 0 \\ \bar{v} & \bar{u} & 0 \\ 0 & 0 & 1 \end{array} \right), \quad \left( \begin{array}{ccc} u & v \\ -\bar{v} & \bar{u} \end{array} \right) \in SU(2),$$

one gets

$$z \rightarrow \frac{uz + v}{-\bar{v}z + \bar{u}}, \quad \xi \rightarrow \frac{\xi}{-\bar{v}z + \bar{u}}.$$  

(6)

Note that both the bosonic and fermionic fields transform themselves under $SU(2)$ spin rotations. At $\xi = 0$, the $su(2|1)$ CS reduces to the ordinary spin $su(2)$ CS,

$$|z, \xi = 0\rangle = |z\rangle_{s=1/2} \equiv |z\rangle = \frac{1}{\sqrt{1 + |z|^2}} \exp(zS^-)|\uparrow\rangle = \frac{1}{\sqrt{1 + |z|^2}} (|\uparrow\rangle + z |\downarrow\rangle),$$

(7)

where the complex number $z$ is a stereographic coordinate of a point on an ordinary sphere, $z \in S^2 \simeq CP^1 = SU(2)/U(1)$. The spin operators $\vec{S}$ obey the standard commutation relations

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z, \quad S^2 = 3/4.$$  

(8)

These operators coincide with bosonic generators $\tilde{Q}_i$ of $su(2|1)$ at half filling, in which case the on-site Hilbert space is reduced and spanned only by the vectors $|\uparrow\rangle, |\downarrow\rangle$. In contrast, at $z = 0$, the state $|\xi\rangle \equiv |z = 0, \xi\rangle$ represents a pure fermionic CS.
At this stage it is helpful to introduce the important notion of the covariant (Berezin) symbol for a Hubbard operator $X$. It can also be referred to as a coherent-state symbol and is defined as follows

$$X_{\text{cov}} := \langle z, \xi \mid X \mid z, \xi \rangle. \quad (9)$$

Explicitly, we find

$$X_{\text{cov}}^{0\downarrow} = -\frac{z \xi}{1 + |z|^2}, \quad X_{\text{cov}}^{0\uparrow} = -\frac{\bar{z} \xi}{1 + |z|^2}, \quad X_{\text{cov}}^{1\downarrow} = -\frac{\xi}{1 + |z|^2}, \quad X_{\text{cov}}^{1\uparrow} = -\frac{\bar{\xi}}{1 + |z|^2} \quad (10)$$

$$Q_{\text{cov}}^+ = S_{\text{cov}}^+ (1 - X_{\text{cov}}^{00}), \quad Q_{\text{cov}}^- = S_{\text{cov}}^- (1 - X_{\text{cov}}^{00}), \quad Q_{\text{cov}}^z = S_{\text{cov}}^z (1 - X_{\text{cov}}^{00}), \quad (11)$$

where the covariant symbol of the hole number operator reads

$$X_{\text{cov}}^{00} = \frac{\bar{z} \xi}{1 + |z|^2}. \quad (12)$$

The corresponding CS symbols of the $su(2)$ generators are evaluated to be $(S_{\text{cov}} := \langle z | S | z \rangle)$:

$$S_{\text{cov}}^+ = \frac{z}{1 + |z|^2}, \quad S_{\text{cov}}^- = \frac{\bar{z}}{1 + |z|^2}, \quad S_{\text{cov}}^z = \frac{1}{2} \left( \frac{1 - |z|^2}{1 + |z|^2} \right). \quad (13)$$

The important point is that the covariant symbol of the Hamiltonian associated with the algebra generators enters the path-integral action for the partition function. For a compact simple algebra those symbols are in one-to-one correspondence with the algebra generators. This is the case for both the $su(2|1)$ and $su(2)$ algebras. Note also that at half filling, $\delta = 0$, we get $X_{\text{cov}}^{00} = 0$ so that the symbol of the $\bar{Q}$ operator reduces to that of the conventional spin operator, $\bar{Q}_{\text{cov}} = \bar{S}_{\text{cov}}$. In general this is not the case. Although the generalized spin operators $\bar{Q}$ fulfill the same commutation relations as operators $\bar{S}$ do, the operator $\bar{Q}^2$ is not a c-number. Namely, $\bar{Q}^2 = S^2 (1 + X^{00})$. This means that operators $\bar{Q}$ are closed into an algebra larger than $su(2)$.

The $su(2|1)$ CS is parameterized by the coordinates of a point on the coherent-state supermanifold $CP^{1|1}$. The latter appears as a classical phase space for the constrained electrons. The symplectic structure is given by a canonical symplectic two-form,

$$\Omega = d (i(z, \xi | d(z, \xi)), = -i \left[ dz(\alpha) d\bar{z} + dz(\beta) d\bar{\xi} + d\bar{z}(\gamma) d\xi + d\bar{\xi}(\delta) d\xi \right], \quad (14)$$

where the external derivative

$$d = dz \frac{\partial}{\partial z} + d\bar{z} \frac{\partial}{\partial \bar{z}} + d\xi \frac{\partial}{\partial \xi} + d\bar{\xi} \frac{\partial}{\partial \bar{\xi}}$$

and

$$a = \frac{1 + \xi \bar{\xi}}{(1 + |z|^2 + \xi \bar{\xi})^2}, \quad \alpha = \frac{\xi}{(1 + |z|^2 + \xi \bar{\xi})^2}, \quad \gamma = \frac{\bar{\xi}}{(1 + |z|^2 + \xi \bar{\xi})^2}, \quad y = -\frac{1}{1 + |z|^2}.$$ 

We use the convention that a superform on $CP^{1|1}$ is $Z \times Z$ graded, where the $Z$-gradation is the usual gradation of the de Rham complexes, while the $Z_2$-gradation is a natural gradation of Grassmann algebra $[11]$. We thus have

$$dz d\bar{z} = -d\bar{z} dz, \quad d\bar{z} d\bar{\xi} = -d\xi d\bar{\xi}$$

Classical dynamics on $CP^{1|1}$ is governed by the Poisson brackets generated by the symplectic structure. Namely, for any two functions $g$ and $h$ that represent classical observables on $CP^{1|1}$, we get

$$\{g, h\}_P = +i X_g | dh,$$  \quad (15)

where $X_g$ denotes a vector field associated to $g$ and the symbol $\mid$ stands for the interior product (or contraction), so that $[12]$

$$X_g | \Omega = dg.$$
In particular, the covariant symbols (10,11) are closed into the $su(2|1)$ superalgebra under the action of brackets (15). Up to a multiplicative constant the $SU(2|1)$ invariant measure is given by

$$d\mu \propto |\Omega_{ab}|d\bar{z}dxd\bar{\xi}d\xi \propto \frac{d\bar{z}dxd\bar{\xi}d\xi}{1+\bar{z}z+\xi\bar{\xi}},$$

where "sdet" stands for the superdeterminant (or Berezian) [13], while $||\Omega_{ab}||$ stands for the supermatrix form of $\Omega$, namely $\Omega = dx^a\Omega_{a\bar{b}}dx^{\bar{b}}, x^a = z,\xi$.

**III. $su(2|1)$ PATH INTEGRAL: PRELIMINARIES**

We now seek a path-integral representation of the partition function $Z = tr e^{-\beta H}$ where $H$ is expressed in terms of Hubbard operators. This implies that the NDO constraint is explicitly resolved from the outset, so that no redundant gauge dependent variables emerge within that approach. To work out the path-integral formalism to deal directly with the Hubbard operators, consider the on-site 3D physical Hilbert space $H_{phys}$ spanned by the vectors $|\lambda\rangle, \lambda = \uparrow, \downarrow, 0$. The Hubbard operators act in this space and are naturally $Z_2$ graded. As was already mentioned, there are bosonic or even-graded and fermionic or odd-graded operators. Accordingly, the basis in $H_{phys}$ becomes graded as well. We choose the spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ states to be even-graded, whereas the empty state (vacancy) $|0\rangle$ is considered to be odd-graded. This means that odd Grassmann parameters commute with the spin states and anti-commute with the vacancy vector. From now on we consider the grading of the physical 3D subspace to be fixed in accordance with this. For any operator $H$ acting in $H_{phys}$ one then gets

$$tr e^{-\beta H} = \sum_\lambda \langle \lambda | e^{-\beta H} | \lambda \rangle,$$

and

$$str e^{-\beta H} = \sum_\lambda (-)^{grad(|\lambda\rangle)} \langle \lambda | e^{-\beta H} | \lambda \rangle,$$

By using the $su(2|1)$ CS [5], Eq.(17) can be rewritten in the form

$$tr e^{-\beta H} = \int d\mu (z,\xi) e^{-\beta H}(z,\xi).$$

Here the $SU(2|1)$ invariant measure

$$d\mu_{su(2|1)} \equiv d\mu = \frac{d\bar{z}dxd\bar{\xi}d\xi}{2\pi i(1+\bar{z}z+\xi\bar{\xi})}$$

agrees with Eq. (19). Integration over $\bar{z},z$ is performed on a complex plane. The integration over Grassmann parameters is carried out according to the standard rules [13],

$$\int d\xi = \int d\bar{\xi} = 0, \quad \int d\xi = \int d\bar{\xi} = 1.$$

Notice that in contrast with Eq.(19)

$$str e^{-\beta H} = \int d\mu (z,\xi) e^{-\beta H}(z,\xi).$$

The distinction between Eqs. (19) and (21) results in different boundary conditions (anti-periodic versus periodic) for Grassmann amplitudes in the corresponding path integrals for the partition/superpartition functions.

In the course of the derivation of the partition function we essentially rely on a certain change of the path-integral variables. To justify it, we start with an example of a simple single-site Hamiltonian that also admits a treatment in terms of ordinary multiple integrals. After that, we reconsider the problem in terms of the time-continuous path integral.

Consider the single-site Hamiltonian

$$H = \mu X^{00}. $$
Operator $X^{00} = |0\rangle\langle 0|$ is represented in $\mathcal{H}_{\text{phys}}$ by a $3 \times 3$ matrix with eigenvalues 0, 0, 1. As a result, the partition function reduces to

$$Z_0 = 2 + e^{-\beta \mu},$$

(23)

and the vacancy occupation number becomes

$$\delta = \langle X^{00} \rangle = -\frac{1}{\beta} \partial_\mu \log Z_0 = \frac{1}{2e^{\beta \mu} + 1}.$$

(24)

This is to be compared with the conventional spinless fermion occupation number,

$$n_F = \frac{1}{e^{\beta \mu} + 1}.$$

Since $(X^{00})^2 = X^{00}$ we get

$$e^{-\beta \mu X^{00}} = 1 + X^{00}(e^{-\beta \mu} - 1).$$

Equation (19) then gives

$$\text{tre}^{-\beta \mu X^{00}} = \int d\mu(z,\xi) (1 + X^{00}(e^{-\beta \mu} - 1)) |z, -\xi\rangle$$

$$= \int \frac{d\bar{z} d\bar{\xi} d\xi}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})} \left[ \frac{1 + \bar{z} z - \xi \bar{\xi}}{1 + \bar{z} z + \xi \bar{\xi}} - \frac{\xi \bar{\xi}}{1 + \bar{z} z + \xi \bar{\xi}}(e^{-\beta \mu} - 1) \right].$$

(25)

Here we have used that

$$\langle z', \xi' | z, \xi \rangle = \frac{1 + \bar{z}' z + \bar{\xi}' \xi}{\sqrt{1 + \bar{z}' z' + \bar{\xi}' \xi'}} \frac{1 + \bar{z} z + \xi \bar{\xi}}{\sqrt{1 + \bar{z} z + \xi \bar{\xi}}}.$$

(26)

Although integral (25) can be calculated immediately, it is instructive at this stage to make a change of variables

$$z \to z, \quad \xi \to \xi \sqrt{1 + |z|^2}$$

(27)

to bring the measure into a more tractable form,

$$d\mu = \frac{d\bar{z} d\bar{\xi} d\xi}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})} \to \frac{d\bar{z} d\bar{\xi} d\xi}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})^2}(1 - \xi \bar{\xi}).$$

(28)

Equation (26) then becomes

$$\text{tre}^{-\beta \mu X^{00}} = \int \frac{d\bar{z} d\bar{\xi} d\xi}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})^2} (1 - \xi \bar{\xi}) \left[ \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} - \xi \bar{\xi} (e^{-\beta \mu} - 1) \right]$$

$$= \int \frac{d\bar{z} d\bar{\xi} d\xi}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})^2} \left[ 1 - 3 \xi \bar{\xi} - \xi \bar{\xi} (e^{-\beta \mu} - 1) \right] = 3 + e^{-\beta \mu} - 1 = 2 + e^{-\beta \mu},$$

(29)

which agrees with (23).

Under the change (27) we get $X^{00}_{\text{cov}} \to \xi \bar{\xi}$, so that it may seem we have reduced the problem to that of spinless fermions. However, this is not the case. The measure (25) involves the extra factor $(1 - \xi \bar{\xi})$ comparing with the standard product of the $SU(2)$ invariant spin and spinless fermion measures,

$$\frac{d\bar{z} d\bar{\xi}}{2\pi i(1 + \bar{z} z + \xi \bar{\xi})^2}.$$

This extra factor reflects a nontrivial symplectic structure of the underlying path-integral phase space. Namely, the transformation (27) brings the symplectic two-form in

$${\Omega \to \bar{\Omega} = -i \left[ dz(\bar{\alpha}) d\bar{\xi} + dz(\bar{\gamma}) d\bar{\xi} + d\xi(\bar{\gamma}) d\bar{\xi} + d\bar{\xi}(\bar{\beta}) d\bar{\xi} \right]},$$

(30)
where
\[ \tilde{a} = \frac{1 - \tilde{\xi}^2 + \tilde{\xi}^2 |z|^2}{(1 + |z|^2)^2}, \quad \tilde{\alpha} = \frac{z\tilde{\xi}}{1 + |z|^2}, \quad \tilde{\gamma} = \frac{z\tilde{\xi}}{1 + |z|^2}, \quad \tilde{y} = -1. \]

It then follows immediately that
\[ d\mu \to d\tilde{\mu} \propto s\text{det}|\bar{\Omega}_{ab}|dzd\tilde{z}d\tilde{\xi} \propto \frac{1 - \tilde{\xi}^2}{(1 + |z|^2)^2} dzd\tilde{z}d\tilde{\xi}, \]
which agrees with Eq. (28). Poisson brackets with respect to (30) then give us
\[ \{\xi, \tilde{\xi}\}_{PB} = 1 - \tilde{\xi}^2 |z|^2, \quad \{\xi, \bar{S}_{\text{cov}}\}_{PB} \neq 0, \]
which clearly implies that the amplitudes \( \xi, \tilde{\xi} \) do not represent spinless fermions independent of lattice spins: vacancies and lattice spins are correlated due to the NDO constraint. For example, the destruction of a vacancy necessarily results in the creation of a lattice spin. As we see below, a calculation of a purely fermionic correlator involves both the \( \xi \) and \( z \) variables in a nontrivial manner.

If we ignored the factor \((1 - \tilde{\xi}^2)\) in the measure we would end up with a partition function of a spinless fermion, \( Z = 1 + e^{-\beta \mu} \). The conventional fermionic amplitudes \( f, \tilde{f} \) obey the conventional rules, \( \{f, \tilde{f}\}_{PB} = 1, \quad \{f, \bar{S}_{\text{cov}}\}_{PB} = 0 \), with the symplectic structure taking on the standard form,
\[ \Omega \propto \frac{dzd\tilde{z}}{(1 + |z|^2)^2} - dfd\tilde{f}. \]

The composite field \((z, \xi)\) parameterizes a point on a supersphere, \( S^{2|2} \simeq CP^{1|1} \). For the conventional spinless fermions coupled to \( su(2) \) spins the underlying phase space is instead given by a direct product of the ordinary sphere, \( S^2 \simeq CP^1 \), and a complex Grassmann plane.

### IV. \( su(2|1) \) PATH INTEGRAL: PARTITION FUNCTION

We turn now to a derivation of the path-integral representation of the partition function. A key ingredient in constructing the CS path integral is the resolution of unity in \( H_{\text{phys}} \),
\[ I = \int d\mu |z, \xi\rangle \langle z, \xi|, \quad (31) \]
where the measure is given by (29). This equation can be used repeatedly to compute \( \langle z, \xi|e^{-\beta H}|z, -\xi\rangle \), considering \( H \) to be a local on-site Hamiltonian expressible in terms of the \( su(2|1) \) generators. Following standard procedure, let us break up the interval \([0, \beta]\) into \( N \) small pieces of length \( \epsilon = \beta/N \), \( N \to \infty \). Then Eq. (19) can be rewritten in the form
\[ \text{tr} e^{-\beta H} = \int d\mu \int \prod_{k=0}^{N} d\mu_k \langle z, \xi|N\rangle \langle N|N - 1 \rangle \cdots \langle 0|z, -\xi\rangle e^{-\epsilon \sum_k H(k, k-1)} + O(\epsilon^2). \quad (32) \]

Here
\[ H(k, k - 1) = \frac{\langle k|H|k - 1 \rangle}{\langle k|k - 1 \rangle}, \quad |k\rangle := |z_k, \xi_k\rangle, \quad z_k = z(\epsilon k), \quad \xi_k = \xi(\epsilon k), \quad k = 1, \ldots, N, \quad (33) \]
and equation (19) tells us that
\[ \int d\mu |z, \xi|N\rangle \langle 0|z, -\xi\rangle = \text{tr} |N\rangle \langle 0| = \langle z_0, \xi_0|z_N, -\xi_N\rangle. \]

Finally, integrating over \( d\mu \) in (32) yields
\[ \text{tr} e^{-\beta H} = \int d\mu_0 \prod_{k=1}^{N} d\mu_k \langle z_0, \xi_0|z_k, \xi_k|z_N, -\xi_N\rangle e^{-\epsilon \sum_k H(k, k-1)}. \quad (34) \]
From now on we drop the $O(\epsilon^2)$ contribution to the partition function having in mind that the continuum limit will be taken eventually. We further notice that the kernel $\langle z, \xi | z', \xi' \rangle$ acts as a delta function with respect to measure $d\mu$. To see this consider any vector $|\psi\rangle$ that belongs to $H_{phys}$. Then resolution of unity implies that

$$\langle \psi | = \int d\mu(\psi|z,\xi) \langle z,\xi|,$$

which in components reads simply

$$\psi(z',\xi') = \int d\mu \psi(z,\xi) \langle z,\xi|z',\xi', \rangle, \quad \psi(z,\xi) := \langle \psi|z,\xi\rangle.$$

Having this in mind, Eq. (33) becomes

$$\text{tr} e^{-\beta H} = \int \prod_{k=1}^{N} d\mu_k \langle k|k-1 \rangle e^{-\epsilon \sum_k H(k,k-1)} |_{z_0=z_N, \xi_0=-\xi_N} = \int \prod_{k=1}^{N} d\mu_k \exp \sum_{k=1}^{N} [\log \langle k|k-1 \rangle - \epsilon H(k,k-1) ] |_{z_0=z_N, \xi_0=-\xi_N} \tag{35}$$

Performing here a formal time-continuum limit $\epsilon \to 0$ yields

$$\text{tr} e^{-\beta H} = \int D\mu e^{\int_0^\beta L d\tau},$$

where

$$L = -\langle z,\xi| \frac{\partial}{\partial \tau} \rangle + H|z,\xi\rangle \tag{37}$$

and

$$D\mu_{su(2|1)}(z,\xi) \equiv D\mu(z,\xi) = \prod_\tau \frac{d\bar{z}(\tau) dz(\tau) d\bar{\xi}(\tau) d\xi(\tau)}{2\pi i (1 + \bar{z}z + \bar{\xi}\xi)} \tag{38}$$

The $SU(2|1)$ symplectic potential explicitly reads

$$\langle z,\xi| - \frac{\partial}{\partial \tau} |z,\xi\rangle = \frac{1}{2} \left( \frac{\bar{z}z - \bar{\xi}\xi - \bar{z}\xi - \bar{\xi}z}{1 + |z|^2 + \xi \bar{\xi}} \right), \tag{39}$$

and $\langle z,\xi|H|z,\xi\rangle = H_{cov}$.

We now specify Hamiltonian to be that of the $U = \infty$ Hubbard model given by Eq. (2) and make the change of variables (27). We then get

$$\text{tr} e^{-\beta H} = \int \prod_{i} \prod_{k=1}^{N} \frac{d\bar{z}_i(ek) dz_i(ek) d\bar{\xi}_i(ek) d\xi_i(ek)}{2\pi i (1 + \bar{z}_i(ek) z_i(ek))^2} (1 - \xi_i(ek) \bar{\xi}_i(ek))$$

$$\times \exp \sum_{k=1}^{N} \left[ \sum_i \log A_i(k) - \epsilon H_{cov}(k, k-1) \right], \tag{40}$$

accompanied with the boundary conditions $z_0 = z_N, \xi_0 = -\xi_N$. Here

$$H_{cov}(k, k-1) = \sum_{ij} \bar{\xi}_j(ek) \xi_i(ek-1) \langle z_i(ek)|z_j(ek-1) \rangle - \mu \sum_i \bar{\xi}_i(ek) \xi_i(ek-1), \tag{41}$$

and

$$A_i(k) = \frac{1 + \bar{z}_i(ek) z_i(ek-1)}{\sqrt{(1 + |z_i(ek)|^2)(1 + |\bar{z}_i(ek-1)|^2)}} \left( 1 - \frac{1}{2} \xi_i(ek) \bar{\xi}_i(ek) \right) \left( 1 - \frac{1}{2} \xi_i(ek-1) \bar{\xi}_i(ek-1) \right) + \bar{\xi}_i(ek) \xi_i(ek-1).$$
Let us recall that the indices $i, j$ denote the lattice sites, whereas the index $k$ numerates time slices only.

In the continuum $\epsilon \to 0$ limit, Eq. (40) becomes

$$tre^{-\beta H} = \int D\mu(z, \xi) e^{\int_0^\beta L(z, \xi) d\tau},$$

(42)

where

$$D\mu(z, \xi) = \prod_{i, \tau} \frac{d\bar{\xi}_i(\tau) d\xi_i(\tau) d\bar{\xi}_i(\tau) d\xi_i(\tau)}{2\pi i (1 + |\xi|^2)^2} (1 - \bar{\xi}_i \xi_i)$$

(43)

stands for the measure with the boundary conditions, $z_i(0) = z_i(\beta)$, $\xi_i(0) = -\xi_i(\beta)$. The Lagrangian now reads

$$L = \sum_i i a_i^{(0)}(\tau) - \sum_i \bar{\xi}_i \left( \partial_\tau + \mu + i a_i^{(0)} \right) \xi_i - H_{cov}.$$  

(44)

The first piece of the action involves the time component of the Berry connection to be discussed later,

$$ia_i^{(0)} = -\langle z| \partial_\tau |z \rangle = \frac{1}{2} \frac{\bar{z} \dot{z} - \dot{\bar{z}} z}{1 + |z|^2}.$$  

In the time-discretized representation, it reads

$$ia_i^{(0)}(\epsilon k) = ia_k = \frac{1}{2} \frac{\bar{\xi}_k \dot{z}_k - \delta_k \bar{z}_k}{1 + |z_k|^2} = \log \langle z_k | z_{k-1} \rangle + \mathcal{O}(\delta_k^2), \quad \delta_k = z_k - z_{k-1}.$$  

(45)

The dynamical part of the action is given by

$$H_{cov} = t \sum_{i \neq j} \bar{\xi}_j \xi_i \langle z_i | z_j \rangle - \mu \sum_i \bar{\xi}_i \xi_i.$$  

(46)

Here $\langle z_i | z_j \rangle$ stands for the product of the spin coherent states,

$$\langle z_i | z_j \rangle = \frac{1 + \bar{z}_i \bar{z}_j}{\sqrt{(1 + |z_i|^2)(1 + |z_j|^2)}}.$$  

(47)

The covariant symbol of the on-site electron spin operator reduces to

$$\hat{G}^\text{cov}_i = \hat{G}^\text{cov}_i (1 - \bar{\xi}_i \xi_i).$$  

(48)

Now we move on to a practical calculation of path integral (42). However, to be more accurate we do that right on the time lattice. As we see below, that will enables us to clarify some subtle points concerning the structure of the path-integral action which is not seen in the naive continuum limit.

The major problem is of course the factor $\prod_k (1 - \bar{\xi}_k \xi_k)$ presented in the on-site measure. Since one can rewrite this factor as

$$\exp \left( - \sum_k \bar{\xi}_k \xi_k \right) = \exp \left( - \frac{1}{\epsilon} \sum_k \bar{\xi}_k \xi_k \epsilon \right) \to \exp \left( - \frac{1}{\epsilon} \int_0^\beta \bar{\xi} \xi d\tau \right), \quad \epsilon = \beta/N \to 0, \quad N \to \infty$$

it might seem that in the continuum limit it simply amounts to an additive renormalization of the chemical potential and as such can be discarded. However, this is not the case. To start with, the renormalization is infinite which makes the whole procedure rather formal. Besides, in the chemical potential term from representation (41) the arguments of the $\bar{\xi}$ and $\xi$ fields are shifted by one time step, whereas the factor in the measure involves the $\bar{\xi}$, $\xi$ fields at coinciding time moments. This slight difference happens to affect the final results drastically.

To see this, consider path integral (42) with the action

$$S = \int_0^\beta ia_i^{(0)}(\tau) d\tau - \int_0^\beta \bar{\xi}_i \left( \partial_\tau + \mu + ia_i^{(0)} \right) \xi_i d\tau.$$  

(49)

With this action the path-integral evaluation of the partition function should reproduce the earlier result

$$Z = tre^{-\beta \mu X^{(0)}} = 2 + e^{-\beta \mu}.$$
which we already calculated in Eq. (29) using ordinary integrals. By a $U(1)$ phase transformation of the fermionic fields the potential $a^{(0)}(\tau)$ can be brought into a time independent form,

$$a^{(0)} \to a^{(0)} - \dot{\phi} = \frac{1}{\beta} \int_0^\beta a d\tau,$$

where

$$\phi(\tau) = -\frac{\tau}{\beta} \int_0^\beta a^{(0)} d\tau + \int_0^\tau a^{(0)} d\tau. \quad (50)$$

Note that $\phi(0) = \phi(\beta)$. The effective action then becomes

$$S = \int_0^\beta i a^{(0)}(\tau) d\tau - \int_0^\beta \bar{\xi} (\partial_\tau + \bar{\mu}) \xi d\tau, \quad (51)$$

where $\bar{\mu} = \mu + \frac{1}{\beta} \int_0^\beta i a^{(0)} d\tau$. Here we cannot simply integrate out the $\xi$ fields in (51) by standard means, since now the measure contains the extra factor, $\exp \left( -\sum_k \tilde{\xi}_k \xi_k \right)$. To figure out how the path integral works in this case we must look back at the defining time-discretized representation (40). It becomes

$$\begin{aligned}
&\text{tre}^{-\beta \mu N^0} = \int \prod_{k=1}^N d\tilde{\xi}_k d\xi_k e^{i a_k} e^{-S}, \\
&\text{where} \\
&S = \epsilon \sum_{k=2}^N \tilde{\xi}_k \left( \frac{\xi_k - \xi_{k-1}}{\epsilon} + \beta \xi_{k-1} \right) + \epsilon \tilde{\xi}_1 \left( \frac{\xi_1 + \xi_N}{\epsilon} - \beta \xi_N \right) + \sum_{k=1}^N \xi_k \xi_k, \quad \xi_k = \xi(\epsilon k). \quad (53)
\end{aligned}$$

Here we have taken into account that $\xi_0 = -\xi_N$. The last term in the action represents the contribution that originates from the extra measure factor, so that the integral over the fermionic fields reduces simply to

$$\lim_{N \to \infty} \int \prod_{k=1}^N d\tilde{\xi}_k d\xi_k e^{-\sum_{k,l=1}^N \tilde{\xi}_k S_{kl} \xi_l} = \lim_{N \to \infty} \det S.$$  

In our case the $N \times N$ matrix $S$ reads

$$S = \begin{pmatrix} 2 & 0 & \cdots & b \\ -b & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -b & 2 \end{pmatrix}, \quad (54)$$

where $b = 1 - \frac{4}{N^2 \beta}$. The determinant is evaluated to yield

$$\lim_{N \to \infty} \det S = \lim_{N \to \infty} \left[ 2^N + (1 - \frac{\beta}{N \beta})^N \right]. \quad (55)$$

If, in contrast, we absorb the extra factor produced by the measure in the chemical potential term, we, instead, end up with

$$\lim_{N \to \infty} \left[ 1 + (1 - \frac{\beta}{N \beta})^N \right] = 1 + e^{-\beta \beta},$$

which yields the familiar result for non-interacting spinless fermions with the chemical potential

$$\mu \to \bar{\mu} = \mu + \frac{1}{\beta} \int_0^\beta i a^{(0)} d\tau.$$  

The $2^N$ factor in Eq. (55) is produced by the modification of the measure. As discussed earlier in our analysis of the time discretized representation, it cannot be absorbed in the chemical potential term. Besides, integrating out the
Fermi fields formally results in a divergent expression in the limit $N \to \infty$. However, as we show below the remaining integral in (52) over the bosonic fields $\bar{z}_k, z_k$ precisely cancels such a divergence.

Explicitly, we get
\[
\text{tre}^{-\beta \mu X^{00}} = \lim_{N \to \infty} \int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k e^{ia_k}}{2\pi i (1 + \bar{z}_k z_k)^2} \det S = \lim_{N \to \infty} \int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k e^{ia_k}}{2\pi i (1 + \bar{z}_k z_k)^2} \left( 2N + e^{-\beta \mu} e^{-i \sum_k a_k} \right)
\]
\[
= \lim_{N \to \infty} 2^N \int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k}{2\pi i (1 + \bar{z}_k z_k)^2} e^{ia_k} + e^{-\beta \mu},
\]

since
\[
\int \frac{d\bar{z} d\bar{z}}{2\pi i (1 + \bar{z}_k z_k)^2} = 1.
\]

Using the result
\[
\int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k}{2\pi i (1 + \bar{z}_k z_k)^2} e^{ia_k} = \int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k}{2\pi i (1 + \bar{z}_k z_k)^2} \langle z_k | z_{k-1} \rangle |_{z_0 = z_N}
\]
\[
= \int \prod_{k=1}^{N} \frac{d\bar{z}_k dz_k}{2\pi i (1 + \bar{z}_k z_k)^2} \langle z_N | z_{N-1} \cdots z_1 | z_N \rangle = 2^{-N+1},
\]

where
\[
\int \frac{(2s + 1)d\bar{z} d\bar{z}}{2\pi i (1 + \bar{z}_k z_k)^2} | z \rangle_s \langle z | = \int \frac{2d\bar{z} d\bar{z}}{2\pi i (1 + \bar{z}_k z_k)^2} | z \rangle_{s=1/2} \langle z |_{s=1/2} = I,
\]

finally gives
\[
\text{tre}^{-\beta \mu X^{00}} = 2 + e^{-\beta \mu},
\]
as desired. On the other hand, if we suppress the extra measure factor we find instead the incorrect result,
\[
\text{tre}^{-\beta \mu X^{00}} = e^{-\beta \mu}.
\]

At this point it is instructive to compare the action (49) with its formal analog for conventional spinless fermions coupled to $SU(2)$ spins through the Berry’s potential:
\[
S_{s/f} = \int_0^\beta i a^{(0)}(\tau) d\tau - \int_0^\beta \bar{f} \left( \partial_\tau + \mu + ia^{(0)}(\tau) \right) f d\tau.
\]

The standard path-integral representation of the pertinent partition function reads
\[
Z_{s/f} = \int D\mu_{\text{fermion}}^{\text{su}(2)} e^{S_{s/f}},
\]
where
\[
D\mu_{\text{su}(2)}^{\text{fermion}} = \prod_{\tau} \frac{2d\bar{z}(\tau) dz(\tau)}{2\pi i (1 + \bar{z}_k z_k)^2} d\bar{f}(\tau) df(\tau)
\]
is a standard measure in the CS space generated by the basis $| z \rangle_s = 1/2 \times | f \rangle$. Here $| f \rangle$ stands for the normalized CS associated with the fermionic algebra,
\[
| f \rangle = \frac{1}{\sqrt{1 + ff}} e^{ff} | 0 \rangle_F = \frac{1}{\sqrt{1 + ff}} (| 0 \rangle_F + f | 1 \rangle_F),
\]
where \( \{ \hat{f}, \hat{f}^\dagger \} = 1 \). Within Shankar’s approach, \( \hat{f}^\dagger \) represents a hole creation operator. Accordingly, the one-particle state \( |1\rangle_F \) corresponds to a hole excitation, whereas \( |0\rangle_F \) stands for a hole-vacuum state. However, this vacuum state is unphysical, since it is not present in the 3D on-site reduced Hilbert space for strongly correlated electrons which is spanned by the spin-up, spin-down and hole (vacancy) state vectors.

The integral (60) can be calculated on the time lattice along the lines depicted earlier on and yields the divergent result,

\[
Z_{s/f} = \lim_{N \to \infty} 2(1 + 2^{N-1}e^{-\beta \mu}).
\]

This clearly indicates that action (59) cannot represent a physical system. The Berry’s potential in the last term of Eq. (59) takes into account the fact that the \( SU(2) \) spins and spinless fermions are correlated within Shankar’s theory due to the NDO constraint, even in the absence of a direct interaction between them. However, that correlation is not taken into a full account there. The projection of the spin-fermion Hamiltonian onto the physically constrained Hilbert space manifests itself both in the appearance of the Berry’s phase in the fermionic action as well as in the modification of the measure in the path integral (32) for the partition function.

It is also important to realize that the spin part of the measure in Eq. (61) contains the extra factor of \( 2 = (2s+1)_{s=1/2} \) in the numerator compared to the spin part of the on-site version of Eq. (63). It comes from the normalized \( SU(2) \) spin measure as given by the resolution of the identity in the spin space (58). The fermionic part of the measure (61) is standard and comes from the resolution of the identity in the fermionic Hilbert space,

\[
\int df \, |f\rangle \langle f| = \int df \, |f\rangle \langle f| = 1.
\]

This factorization of the path-integral measure (61) into pure spin and fermionic parts reflects the fact that the Hilbert space of the whole system is represented by a direct product of the spin and fermion subspaces. In contrast, the measure (43) comes from the resolution of the unity in the whole superspace \( CP^{11} \) as given by Eq. (31). It cannot be factorized into the spin-fermion parts and the resulting effective action represents a unique composed object. For example, the integration over just the fermionic amplitudes in (52) diverges.

Finally, let us evaluate the Green’s function

\[
G^{(0)}(\tau_q - \tau_r) = Z_0^{-1} \int D\mu(z, \xi) \xi(\tau_q) \tilde{\xi}(\tau_r) e^{S_0(z, \xi)},
\]

where

\[
D\mu(z, \xi) = \prod_\tau \frac{dz(\tau) dz(\tau) d\xi(\tau) d\bar{\xi}(\tau)}{2\pi i(1 + |z|^2)^2} (1 - \xi \bar{\xi}), \quad z(0) = z(\beta), \; \xi(0) = -\xi(\beta)
\]

and

\[
S_0 = \int_0^\beta i a^{(0)}(\tau) d\tau - \int_0^\beta \tilde{\xi} \left( \partial_\tau + \mu + ia^{(0)} \right) \xi d\tau.
\]

Making the change of variables, \( \xi(\tau) \rightarrow \xi(\tau)e^{-i\phi(\tau)} \), where \( \phi(\tau) \) is given by (53), this can be brought into the form

\[
G^{(0)}(\tau_q - \tau_r) = Z_0^{-1} \int D\mu(z, \xi) \xi(\tau_q) \tilde{\xi}(\tau_r) e^{-i\phi(\tau_q) + i\phi(\tau_r)} e^{S_0(z, \xi)},
\]

where \( \tau_\alpha = \frac{\alpha}{N} \tau, \alpha \) is integer, and \( S_0 \) is given by (51). This expression is a continuum limit of the time-discretized representation

\[
G^{(0)}(\tau_q - \tau_r) = Z_0^{-1} \lim_{N \to \infty} \int \prod_{k=1}^N \frac{d\tilde{z}_k dz_k e^{ia_k \tilde{z}_k - i\phi(\tau_q) + i\phi(\tau_r)}}{2\pi i(1 + \tilde{z}_k z_k)^2} (\det S) S_{qr}^{-1},
\]

where the inverse of \( S \) is

\[
S_{qr}^{-1} = (\det S)^{-1} \begin{pmatrix}
2^{N-1} & -b^{N-1} & \cdots & -b^{2N-2} \\
2^{N-2}b & 2^{N-1} & \cdots & -b^{2N-1} \\
\vdots & \vdots & \ddots & \vdots \\
b^{N-1} & 2b^{N-2} & \cdots & 2^{N-1}
\end{pmatrix}, \quad \det S = 2^N + b^N.
\]
Hence, for \( q > r \)
\[
S^{-1}_{qr} = \frac{2^{N-1-(q-r)}b^{q-r}}{2^N + b^N},
\]
and this gives
\[
G^{(0)}(\tau_q - \tau_r) = (2 + e^{-\beta\mu})^{-1} \lim_{N \to \infty} \int \prod_{k=1}^{N} \frac{d\xi_k dz_k}{2\pi i(1 + \xi_k^2)^2} 2^{N-1-(q-r)}e^{-\mu(\tau_q - \tau_r)}
\times \prod_{k=1}^{N} \langle z_k|z_{k-1} \rangle \bigg( \prod_{k=r}^{q} \langle z_k|z_{k-1} \rangle \bigg)^{-1}.
\]

Since
\[
\int \prod_{k=1}^{N} \frac{d\xi_k dz_k}{2\pi i(1 + \xi_k^2)^2} 2^{N-1-(q-r)}e^{-\mu(\tau_q - \tau_r)} \prod_{k=1}^{N} \langle z_k|z_{k-1} \rangle \bigg( \prod_{k=r}^{q} \langle z_k|z_{k-1} \rangle \bigg)^{-1} = 2^{-N+(q-r)+2},
\]
we finally get
\[
G^{(0)}(\tau_q - \tau_r) \mid_{q>r} = e^{-\mu(\tau_q - \tau_r)}(1 - \delta),
\]
where the vacancy occupation probability \( \delta \) is determined by
\[
\delta = \langle X^{00} \rangle = \frac{1}{2e^{\beta\mu} + 1}.
\]
Similarly, for \( q < r \), we find
\[
G^{(0)}(\tau_q - \tau_r) \mid_{q<r} = -e^{-\mu(\tau_q - \tau_r)}\delta.
\]
Combining these results, we finally get
\[
G^{(0)}(\tau - \tau') = e^{-\mu(\tau - \tau')} \left[ \theta(\tau - \tau' - \eta)(1 - \delta) - \theta(\tau' - \tau + \eta)\delta \right],
\]
where the infinitesimal parameter \( \eta = 0^+ \) is a reminder that the variables \( \xi(\tau') \) and \( \xi(\tau) \) in the path integral \[12\] are associated with variables displaced by one time step, \( \xi(ek) \) and \( \xi(\epsilon(k-1)) \), respectively. At equal times we find
\[
-G^{(0)}(0^-) = \langle \xi(\tau)\xi(\tau) \rangle = \langle X^{00} \rangle = \delta.
\]
as it should. In contrast, for the conventional spinless fermions governed by the Hamiltonian \( H = \mu \hat{\mathbf{f}}\hat{\mathbf{f}} \) one obtains Eq.\[73\] with \( \delta \) replaced by \( n_F \).

Because of the extra factor \( \prod_{\tau}(1 - \xi(\tau)\xi(\tau)) \) in the measure \[19\], the fields \( \hat{\xi}, \hat{\xi} \) separately do not have a direct physical meaning. They don’t represent spinless fermions, for instance. However, the bilinear combination \( \xi(\tau)\xi(\tau') \) represents, at equal times, a physical observable, the covariant symbol of the vacancy number operator \( X^{00} \). We see that the spin degrees of freedom are nontrivially involved in the evaluation of a purely fermionic vacancy propagator \[74\]. Notice, however, that this spin-charge correlation due to the NDO constraint does not prevent the spin-charge separation observed in the 1D Hubbard model. The physical elementary excitations that represent separately the spin and charge degrees of freedom in the 1D Hubbard model are not simply the \( z \) and \( \xi \) field excitations, but are, instead, nonlocal string objects composed, simultaneously, of both types of bare elementary excitations \[13\].

Finally, let us say a few words concerning the meaning of the continuum path-integral representation \[12\]. Strictly speaking, Eq.\[12\] is only symbolic: the measure in that path integral cannot be defined in a mathematically rigorous way. This observation has a deep physical meaning. If a continuum path integral could be defined as a bona-fide integral with respect to a genuine measure insensitive to discrete approximations, it would immediately provide a one-to-one correspondence between classical and quantum physics thereby making the latter an unnecessary ingredient. However, classical dynamics is known to give rise to different quantum theories if one follows different quantization schemes. A specific quantization scheme is encoded into a discrete representation of the continuum path integral that should be taken as its true definition. In our case the basic discrete representation is given by our Eq.\[10\]. Since
\[
\xi(\tau + \eta) = \xi(\tau) + \frac{d\xi(\tau)}{d\tau} \eta + O(\eta^2), \quad \eta \to 0,
\]
it might seem that one could safely ignore that shift in the low-energy limit, $\frac{d\omega}{dt} \propto \omega \to 0$, where $\omega$ stands for a characteristic frequency of the fermionic degrees of freedom [16]. However, this is not the case: the path-integral variables are not in general smooth functions. There may exist different discrete approximations to one and the same continuum action that result in completely different low-energy dynamics. Some explicit examples can be found in [17]. Even in cases where the continuum limit of a path integral does make sense, i.e. in a semiclassical or perturbation theory, the relative shift of the arguments of the path-integral variables still cannot be ignored. The discontinuity of the correlators such as $<\xi(\tau)\xi(\tau')>$, at equal time arguments, should be dealt with according to rules following from the defining discrete approximation. As soon as the discrete approximation is fixed, no “operator ordering problem” shows up. In our case the order is fixed from the very beginning and manifests itself in the Hamiltonian function defined by Eq. (53).

V. 1D EXAMPLE

The $U = \infty$ Hubbard model [2] is known to be exactly solvable in 1D. We show below that the exact ground-state energy can be recovered within the path-integral representation [42-44]. To start with, the ground state of the 1D $U = \infty$ Hubbard model is known to be degenerate with respect to spin. To calculate the path integral we can therefore choose any spin configuration, e.g., the ferromagnetic (FM) one. We thus put $z_i = z_j$ in Eq. (45), which yields

$$Z_{U=\infty} = \int D\mu(z, \xi) e^{S(z, \xi)}$$

where the measure is given by Eq. (43) and the action reads

$$S = \int_0^{\beta} tr \, i a^{(0)} d\tau - \int_0^{\beta} \tilde{\chi} \left( \partial_\tau + a^{(0)} + T \right) \chi d\tau.$$  (78)

Here $T$ and $a^{(0)}$ are the $N_s \times N_s$ matrices,

$$(a^{(0)})_{ij} = a_i^{(0)} \delta_{ij}, \quad (T)_{ij} = t_{ij} - \mu \delta_{ij},$$

with $i, j = 1, 2, \ldots, N_s$ numbering the lattice sites. The vector $\chi = (\xi_1, \xi_2, \ldots, \xi_N)^T$, and the trace is taken over the lattice site indices. On the time lattice, the integral over the $\chi$ fields can be carried out to yield

$$Z_{U=\infty} = \lim_{N \to \infty} \int \prod_{k=1}^{N_s} \prod_{j} d\xi_k(j) d\zeta_k(j) \det e^{ia_{ij}} \det(2^N + e^{-\beta T - i \sum_k a_k})$$

$$= \lim_{N \to \infty} \prod_{k=1}^{N_s} \prod_{j} \frac{d\xi_k(j) d\zeta_k(j)}{2\pi N} \det(2^N e^{i \sum a_k} + e^{-\beta T}), \quad \zeta_k(j) \equiv z_j(ck).$$  (80)

Since the matrix $\langle 2^N e^{i \sum a_k} + e^{\beta \mu} \rangle$ at $t_{ij} = 0$ is diagonal, Eq. (80) reduces to

$$Z_{U=\infty}(t = 0) = (2 + e^{\beta \mu})^{N_s} = e^{\beta \mu N_s (1 + o(1))}, \quad \beta \to \infty, \mu > 0,$$

where $N_s$ is fixed. This result is a direct consequence of Eq. (57). Accordingly, at $t_{ij} \neq 0$ the negative eigenvalues of the matrix $T$ determine the asymptotic behaviour of the partition function. Consequently,

$$Z_{U=\infty} = \prod_{p, T_p < 0} e^{-\beta T_p (1 + o(1))}, \quad \beta \to \infty,$$

where $T_p = -t_p - \mu$, $t_p = 2t \cos p$, $p \in BZ$. The ground-state energy then becomes

$$E_{gr}/N_s = -\frac{2t}{\pi} \sin(\pi \delta), \quad \delta = \frac{1}{\pi} \arccos\left(\frac{\mu}{2t}\right), \quad t \geq 0,$$

which coincides with the exact 1D result for the $U = \infty$ Hubbard model [18].
VI. DOPED ANTIFERROMAGNET

Let us now turn our attention to a derivation of the low-energy effective action of a doped AF, starting right from the microscopic \( t - J \) model,

\[
H_{t-J} = H_t + H_J = -t \sum_{i,j} \xi_i^j \xi_{ij} + J \sum_{\langle i,j \rangle} \bar{Q}_i \cdot \bar{Q}_j, \tag{82}
\]

where \( \bar{Q}_i \) stands for the local electron spin operators given by Eq. (3) and \( J \geq 0 \) describes the nn exchange interaction. The parameter \( J \sim \mathcal{O}(1/U) \) and the bare constants are chosen such that \( t \gg J \).

Because of the NDO constraint there are no charge fluctuations at half filling (\( \delta = 0 \)), and precisely in this limit, the \( t - J \) model reduces to a Heisenberg AF model

\[
H_{t-J} \to H_J^{t=0} = J \sum_{\langle i,j \rangle} \bar{S}_i \cdot \bar{S}_j, \tag{83}
\]

with no restriction on \( J \), the sole energy scale, apart from its positive sign. In the low-energy long-wavelength limit \( H_J \) gives rise to the action of the nonlinear sigma-model.

Proceeding as discussed in the preceding section we arrive at the representation of the \( t - J \) partition function,

\[
Z_{t-J} = \int \mathcal{D}\mu(z, \xi) e^{\int_0^\beta \phi_{t-J}(z, \xi) d\tau}, \tag{84}
\]

where the measure factor \( D\mu(z, \xi) \) is given by Eq. (43). The Lagrangian now reads

\[
L_{t-J} = \sum_i \bar{a}_i^{(0)}(\tau) - \sum_i \xi_i \left( \partial_\tau + \mu + ia_i^{(0)} \right) \xi_i - H^{\text{cov}}, \tag{85}
\]

where

\[
H^{\text{cov}} = \frac{t}{2} \sum_{i \neq j} \xi_i \xi_j |z_i|z_j| + \frac{J}{2} \sum_{i \neq j} \bar{S}_i^{\text{cov}} \bar{S}_j^{\text{cov}} (1 - \bar{\xi}_i \xi_j)(1 - \bar{\xi}_i \xi_j). \tag{86}
\]

The summation in (86) is extended over nn and nnn sites. As a result we find

\[
\langle z_i | z_j \rangle = \langle z(\vec{r}_i) | z(\vec{r}_i + \delta \vec{r}) \rangle = 1 + \langle z(\vec{r}_i) \rangle \frac{d}{d\vec{r}_i} |z(\vec{r}_i)| \delta \vec{r} + \mathcal{O}(\delta \vec{r}^2) = 1 + \langle z(\vec{r}_i) \rangle \left( \frac{\partial z_i}{\partial \vec{r}_i} \frac{\partial}{\partial z_i} + \frac{\partial z_i}{\partial \vec{r}_i} \frac{\partial}{\partial \vec{r}_i} \right) |z(\vec{r}_i)| \delta \vec{r} + \mathcal{O}(\delta \vec{r}^2)
\]

\[
= 1 - i\vec{a}_i \delta \vec{r} + \mathcal{O}(\delta \vec{r}^2) = \exp(-i\vec{a}_i \delta \vec{r}) + \mathcal{O}(\delta \vec{r}^2),
\]

where \( \vec{r}_j = \vec{r}_i + \delta \vec{r} \). Vector \( \vec{a} \) represents the spatial components of the pull-back of the Berry’s connection one-form, \( \varphi^* A = a \), with \( \varphi : (\tau, \vec{r}) \to z(\tau, \vec{r}) \) and

\[
A = i\langle z|d|z \rangle.
\]

Here \( d \) stands for the exterior derivative,

\[
d = dz \frac{\partial}{\partial z} + d\bar{z} \frac{\partial}{\partial \bar{z}}.
\]

The effective hole action becomes

\[
S_t = \sum_{\vec{r}_i} \int_0^\beta \bar{a}_i^{(0)}(\tau) d\tau - \sum_{\vec{r}_i} \int_0^\beta \bar{\xi}_i \left( \partial_\tau + \mu + ia_i^{(0)} \right) \xi_i d\tau
\]

\[
- \frac{t}{2} \sum_{\vec{r}_i, \delta \vec{r}} \bar{z}_{\vec{r}_i + \delta \vec{r}} \xi_{\vec{r}_i} e^{-i\vec{a}_i \delta \vec{r}} d\tau + \mathcal{O}(\delta \vec{r}^2). \tag{87}
\]

Here \( a_i^{(0)}(\tau) := a^{(0)}(\vec{r}_i, \tau), \nu = 0, x, y, \) with \( i\bar{a}_i = -\langle z_i | \frac{\partial}{\partial z_i} | z_i \rangle \). Note also that \( z_i := z(\vec{r}_i) \) and \( \vec{r}_j = \vec{r}_i + \delta \vec{r} \) where \( \delta \vec{r} \propto a \) with \( a \) being the lattice spacing.
The whole action should be complemented by the $J$-term. Right at half filling the only surviving term gives rise to the nonlinear sigma-model action to describe the quantum AF. The perfect Neel ordering prevents interlattice hole hopping. This follows directly from Eq. (86). Namely, the AF long-range order, $\vec{S}_i = -\vec{S}_j = \vec{S}$, implies

$$z_i = z, \quad z_j = -1/z, \quad i \in A, j \in B.$$  

(88)

In view of (47), this results in $\langle z_i | z_j \rangle = \langle z |-1/z \rangle = 0$, and there is no hopping between the $A$ and $B$ sublattices. Only intralattice nn hopping of vacancies is possible in this case. One can also check that

$$\langle z|d|z \rangle = -(-1/z|d| -1/z),$$  

(89)

which means that $a_i^\dagger = -a_i^\dagger$.

Suppose now that we lightly dope the AF with holes, with the dominant energy scale in the problem continuing to be the exchange coupling $J$. We assume that the AF ordered lattice spins are slightly perturbed by a small amount of vacancies. In this case for a small enough hole concentration $\delta$, one gets

$$z_i = -1/z_j + O(\delta),$$

where $i, j$ are the nn sites. This in turn implies

$$\langle z_i | z_j \rangle = O(\delta), \quad \delta \rightarrow 0.$$  

Therefore, very close to half filling, the interlattice hopping effectively results in the renormalization of the hopping amplitude $t \rightarrow \delta t$ and, as a result, this term can be discarded.

Summing this all up, the partition function of the $t-J$ model close to half filling reads

$$Z_{t-J} = \int D\mu(z, \xi) e^{\int_0^\beta L_{t-J}(z, \xi) d\tau}. $$  

(90)

Here

$$L_{t-J} = \sum_{\vec{r}, \in A} i a_i^{(0)}(\tau) - \sum_{\vec{r}, \in A} \xi_{\vec{r}} \left( \partial_\tau + \mu + ia_i^{(0)} \right) \xi_{\vec{r}}$$  

$$- t \sum_{\vec{r}, \in A} \xi_{\vec{r}_i + \delta \vec{r}} \xi_{\vec{r}_j} e^{-i a_i^\dagger \delta \vec{r}} + (A \rightarrow B, a_i \rightarrow -a_i)$$  

$$- J \sum_{\vec{r}, \in A} \xi_{\vec{r}_i} \xi_{\vec{r}_j} e^{-i a_i^\dagger \delta \vec{r}}, \quad J = J(1 - \delta)^2, $$

(91)

and the measure is given by Eq. (83).

$$D\mu(z, \xi) = \prod_{i, \tau} \frac{d\bar{z}_i(\tau)dz_i(\tau)}{2\pi i(1 + |z_i|^2)^2} d\xi_i(\tau) d\xi_i(1 - \bar{\xi}_i \xi_i). $$

Within Shankar’s approach, the underdoped AF is described by the partition function

$$Z_{Shankar} = \int D\mu_{\text{fermion}}(z, f) e^{\int_0^\beta L_{Shankar}(z, f) d\tau}. $$

(92)

Here

$$L_{Shankar} = \sum_{\vec{r}, \in A} i a_i^{(0)}(\tau) - \sum_{\vec{r}, \in A} f_{\vec{r}_i} \left( \partial_\tau + \mu + ia_i^{(0)} \right) f_{\vec{r}_i}$$  

$$- t \sum_{\vec{r}, \in A} f_{\vec{r}_i + \delta \vec{r}} f_{\vec{r}_j} e^{-i a_i^\dagger \delta \vec{r}} + (A \rightarrow B, a_i \rightarrow -a_i)$$  

$$- J \sum_{\vec{r}, \in A} f_{\vec{r}_i} f_{\vec{r}_j} e^{-i a_i^\dagger \delta \vec{r}}, $$

(93)

and the unconstrained spin-fermion measure reduces to

$$D\mu_{\text{fermion}} = \prod_{i, \tau} \frac{2d\bar{z}_i(\tau)dz_i(\tau)}{2\pi i(1 + \bar{z}_i z_i)^2} df_i(\tau) df_i(\tau). $$

(94)
Here the amplitudes denoted by \( f_i \) describe the conventional spinless fermions, whereas the \( z_i \) fields correspond to the \( su(2) \) spins. This is Shankar’s result for the underdoped AF (with \( s = 1/2 \)).

Formally, the Lagrangians (11) and (13) appear to be identical. However, due to the different measures, the corresponding partition functions are completely different from each other. The fermionic extra term in our measure cannot be ignored. Physically, it reflects a rearrangement of the underlying Hilbert space induced by the NDO constraint. This clearly cannot be treated perturbatively. In a true spin-fermion Hamiltonian, the on-site Hilbert space is represented by a 4D direct product of the 2D spin and the 2D spinless fermion subspaces. However, the NDO constraint reduces it to a 3D on-site Hilbert space composed of three state vectors: the spin-up, spin-down, and the vacancy states. As shown above, it is precisely the extra measure term that takes explicit care of that distinction.

As an example, consider partition functions (90) and (92) in the limiting case \( t = J = 0 \):

\[
L_{t-J}(t = J = 0) = \sum_{\vec{r}_i} i a_i^{(0)}(\tau) - \sum_{\vec{r}_i} \xi_{\vec{r}_i} \left( \partial_{\vec{r}} + \mu + ia_i^{(0)} \right) \xi_{\vec{r}_i},
\]

and

\[
L^{Shankar}(t = J = 0) = \sum_{\vec{r}_i} i a_i^{(0)}(\tau) - \sum_{\vec{r}_i} \bar{f}_{\vec{r}_i} \left( \partial_{\vec{r}} + \mu + ia_i^{(0)} \right) f_{\vec{r}_i}.
\]

In view of Eqs. (52)

\[
Z_{t-J}(t = J = 0) = (2 + e^{-\beta \mu})^{N_s},
\]

as it should. On the other hand, Eq. (63) tells us that \( Z^{Shankar}(t = J = 0) \) diverges, which is inappropriate for a system with a finite number of degrees of freedom. If one, however, drops the \( a_i^{(0)} \) potential in the fermionic part of Eq. (96), one arrives at the conventional partition function

\[
2^{N_s}(1 + e^{-\beta \mu})^{N_s}
\]

which describes entirely independent uncorrelated spin and fermionic degrees of freedom.

Only right at half filling, \( \delta = 0 \), our representation (50) and Shankar’s theory become identical. This can be seen as follows. Since there are no holes in this limit, the fermionic amplitudes \( f, \bar{f} \) describing their propagation throughout the lattice vanish identically. The partition function then becomes

\[
Z^{Shankar}_{\delta = 0} = \int D\mu_{su(2)} \sum_{t=0} \bar{L}^{Shankar}(\delta = 0) d\tau,
\]

where

\[
L^{Shankar}(\delta = 0) = \sum_i ia_i^{(0)}(\tau) - H^{cov}, \quad H^{cov} = J \sum_{i \neq j} \bar{S}_i^{cov} S_j^{cov},
\]

and the normalized spin measure is given by

\[
D\mu_{su(2)} = \prod_{i, \tau} \frac{2dz_i(\tau)dz_i(\tau)}{2\pi i (1 + |z_i|^2)^2}.
\]

On the other hand, at \( t = 0 \), the projected \( t - J \) Hamiltonian reads

\[
H_J = J \sum_{ij} \vec{Q}_i \cdot \vec{Q}_j + \mu \sum_i X_i^{00}.
\]

The half filling limit in this representation can be enforced by sending the chemical potential \( \mu \) to \( +\infty \). This has an immediate effect in allowing only the zero eigenvalues of the local vacancy number operator \( X_i^{00} \) to survive. In this way the partition function becomes

\[
Z_J = \int D\mu(z, \xi) e^{\int_0^\beta L_J d\tau},
\]

with

\[
L_J = \sum_i ia_i^{(0)}(\tau) - \sum_i \xi_i \left( \partial_{\vec{r}} + \mu + ia_i^{(0)} \right) \xi_i
- J \sum_{i \neq j} \bar{S}_i^{cov} S_j^{cov}, \quad \mu \rightarrow +\infty.
\]
The measure $D\mu(z, \xi)$ is given by Eq. (13). Note that it is normalized by the condition (31), so that the factor of $2 = (2s+1)_{s=1/2}$ is missing in the numerator of its spin part.

Notice that we cannot simply put $\xi = \bar{\xi} = 0$ in Eq. (102). This is because those fields have no physical meaning and do not represent holes directly as opposed to the amplitudes $f, \bar{f}$. We must instead integrate them out in the limit $\mu \to +\infty$. We do that on the time lattice, which gives

$$Z_J = \lim_{N \to \infty} 2^{2N_N_s} \int \prod_{k=1}^{N} \prod_{j} d\bar{z}_k(j) dz_k(j) \det e^{i\theta_k} \exp \left[-\epsilon \sum_k H_{\text{cov}}(k, k-1)\right].$$

(103)

Here

$$H_{\text{cov}}(k, k-1) = J \sum_{i \neq j} \bar{\mathcal{S}}^i_{\text{cov}}(\bar{z}_k, \bar{z}_{k-1}) \mathcal{S}^i_{\text{cov}}(\bar{z}_k, \bar{z}_{k-1}).$$

Note that the factor of $2^{2N_N_s}$ comes precisely from the fermionic extra factor in the measure (13). This factor can be absorbed back into the spin measure to turn it into a conventional normalized spin measure (31). As a result, the integral over the spin fields $z, \bar{z}$ in (103) becomes identical to that given by Eq. (97). The distinction between the representations (90) and (92) disappears at half filling since the corresponding on-site spin Hilbert spaces become identical.

Finally, the Berry’s gauge potential $a^\sigma = (\varphi^* A)^\sigma$ appears in the path-integral action (91) as an external gauge field and has no dynamical role. The gauge-theory approaches to treat the $t-J$ model basically fall into two categories. The first one emerges from the slave-particle representations of the constrained electron operators. The idea of that approach is to re-express these constrained operators in terms of the standard boson/fermion bilinears. This is equivalent to the so-called oscillatory representations of $su(2|1)$ superalgebra. For example, the physical electron operator can be represented by a product of a spinful boson and slave spinless fermion

$$\hat{c}_i^\sigma = f_i b_i^\dagger,\tag{104}$$

with standard commutation/anticommutation rules. However, this representation clearly increases the on-site number of degrees of freedom by a factor of 2. The emergent $U(1)$ local gauge field, $f_i, b_i^\sigma \to e^{i\theta_i} f_i, e^{i\theta_i} b_i^\sigma$, takes care of one redundant degrees of freedom (by fixing a gauge), while the NDO constraint

$$f_i^\dagger f_i + \sum_\sigma b_i^\dagger b_i^\sigma = 1$$

takes care of the other. However, the elementary excitations of the slave-particle fields do not represent physical excitations since they are gauge dependent. This compact dynamical $U(1)$ gauge field is generated by the NDO constraint. Within our approach the constraint is resolved explicitly and there is no need for an imposing the $U(1)$ gauge field theory. In fact, an explicit resolution of the NDO constraint within the slave-fermion theory results exactly in the $su(2|1)$ path-integral representation (8).

The gauge potential $a^\sigma = (\varphi^* A)^\sigma$ which appears in Eq. (91) has a different nature altogether. It is driven by the fluctuations of the spin background present in any $su(2)$ path-integral action regardless of the chosen coordinates and it takes care of the $U(1)$ redundancy of the spin quantum state $|\varphi\rangle$. From the geometric viewpoint, this can be stated as follows (19). The $su(2)$ CS’s can be considered as sections of the principle $U(1)$ bundle $P(CP^1, U(1))$ frequently referred to as a magnetic monopole bundle. The base space of this bundle, $M = CP^1$, appears as a classical phase space of the spin, whereas its covariantly constant sections $|z\rangle$ form a quantum Hilbert space for the spin, with

$$\nabla |z\rangle = 0, \quad \nabla := d + iA.$$ 

Since the manifold $CP^1 \cong S^2$ is topologically nontrivial, the monopole bundle is nontrivial as well. Notice that the one-form $A$ does not exist globally, and any two locally defined gauge potentials are related by a $U(1)$ gauge transformation. Fixing the spin CS by Eq. (2) amounts to fixing a local section of the bundle.

Under the global canonical $SU(2)$ transformations acting in the base space we get

$$z_i \to z_i^\eta = \frac{u z_i + v}{-v z_i + u}, \quad g = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \in SU(2). \tag{104}$$

When lifted to the bundle (quantum) space, this gives

$$g \to U_g : \quad U_g |z_i\rangle = e^{-i\zeta_i} |z_i^\eta\rangle, \quad \zeta_i = -i \log \sqrt{-v z_i + u \over -v z_i + u}. \tag{105}$$
The Berry’s connection transforms according to

$$A_i \rightarrow A_i + d\zeta_i. \quad (106)$$

and the Fermi field becomes

$$\xi_i \rightarrow e^{i\zeta_i}\xi_i, \quad (107)$$

leaving the whole action in Eq.(90) globally $SU(2)$ invariant.

**VII. DISCUSSION**

In this section we discuss some physical consequences that follow from representations (90) and (92). Let us start with Shankar’s theory (92). This theory naturally allows for the hopping of the conventional fermions throughout the lattice in the absence of the local spin degrees of freedom. The corresponding partition function follows from the representation (92) if one discards the spin degrees of freedom. In the momentum representation, it reads

$$Z_{S_{i=0}}^{Shankar} = \int \prod D\tilde{\psi}_f(\tau) D\bar{\psi}_f(\tau) \exp \left( - \sum_{\rho} \int_0^\beta \tilde{f}_\rho(\partial_\tau - \epsilon_\rho)\bar{\psi}_f d\tau \right). \quad (108)$$

Here $\epsilon_\rho = t_\rho - \mu$, and $t_\rho = 2t \sum a \cos(\vec{p} \cdot \vec{a})$, where $\vec{a}$ is a lattice vector and $\vec{p} \in BZ$. This path integral can be easily computed to yield a partition function for the conventional spinless fermions,

$$Z_{S_{i=0}}^{Shankar} = \prod_{\rho} (1 + e^{\beta t_\rho}).$$

Having this in mind, the low-energy long-wavelength limit can be taken explicitly to reduce Shankar’s action (93) in 1D to that of Dirac fermions coupled to the nonlinear sigma model via a compact $U(1)$ gauge field [1]. Following the usual procedure for 1D systems to take into account the low-energy fermionic excitations, we linearize the theory near the Fermi points $\pm k_F$. The spinless Fermi amplitudes $\psi = f/\sqrt{a}$ can be written as follows:

$$\psi(n) = e^{ik_F n}\psi_1(n) + e^{-ik_F n}\psi_2(n). \quad (109)$$

Here index $n$ stands for the lattice sites. The resulting action in the continuum limit reads

$$Z_{Shankar}^D = \int D\tilde{\psi}D\psi D\mu_{su(2)}(\hat{\tau}, \hat{z}, \bar{z}) e^{S_F + S_\theta},$$

$$S_F = \int [\tilde{\psi}_A(-\hat{\partial} - i\hat{a})\psi_A + \tilde{\psi}_B(-\hat{\partial} + i\hat{a})\psi_B] dx d\tau,$$

$$S_\theta = -\frac{1}{2} \int dx d\tau (c\partial_x \bar{z}\partial_x z + c^{-1}\bar{z}\partial_x \bar{z}) + i\theta W, \quad c = Ja, \quad (110)$$

where $S_\theta$ is the sigma model action including the topological $\theta$-term and $\hat{a} = a^\nu\gamma_\nu$. The chemical potential $\mu$ is incorporated in the theory through the relation $k_F = \arccos(-\mu/2t)$. The Euclidean $2 \times 2$ gamma matrices can be taken in the form $\gamma_0 = \sigma_y, \gamma_1 = \sigma_x$ so that $\gamma_3 = i\gamma_0\gamma_1 = \sigma_z$. Notice also that $\psi = (\psi_1, \psi_2)^t, \quad \bar{\psi} = \psi^\dagger \gamma_0$.

An interesting observation concerning the representation (110) is that the dependence on the parameter $\theta$ actually drops out from the theory (1). This can be seen as follows. The action to describe the massless fermions coupled with the spin background fields $\hat{z}, \bar{z}$,

$$S_F = \int [\tilde{\psi}(-\hat{\partial} \pm i\hat{a})\psi] dx d\tau,$$

is invariant under a chiral $U(1)$ transformation,

$$\psi \rightarrow e^{i\gamma_5 \phi}\psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\gamma_5 \phi}, \quad (111)$$

with $\phi$ being a parameter. However, this is no longer the case for the partition function

$$Z_F = \int D\bar{\psi}D\psi e^{S_F}. \quad (112)$$
The fields \( z(x,\tau), \bar{z}(x,\tau) \) map a compactified 2D plane \((x,\tau)\) homeomorphic to a two sphere \( S^2 \) onto a spin phase space which is also a two sphere, \( S^2 \to S^2 \). It is known that such maps can be classified by the integers \( W \) which define the number of times the the “space-time” sphere covers the “spin” sphere. Explicitly,

\[
W = \frac{1}{2\pi} \int_{S^2} da.
\]

The remarkable result in quantum field theory tells us that, at nonzero values of \( W \), the classical chiral symmetry cannot be promoted to quantum level. In fact,

\[
Z_F \to e^{i\delta W} Z_F,
\]

under the transformation \((111)\). This manifests the presence of a quantum anomaly. In view of the well known index theorem, the winding number \( W \) is given by the difference of positive and negative chirality zero modes of the Dirac equation,

\[
W = n_+ - n_-.
\]

This implies that the Dirac operator \( \hat{D} = -\hat{\sigma} \pm i\hat{a} \) has zero eigenvalues at \( W \neq 0 \), which kills the fermionic path integral in \((110)\). The theory survives only if \( W = 0 \). In this case \( \theta \) then multiplies zero and nothing can depend on it.

The direct consequence of that observation is that the difference between integer and half integer spins goes away and the low-energy excitations in the spin sector of the \((110)\) model become massive. This is in agreement with the mean-field theory of the 1D \( t-J \) model obtained within the slave-fermion framework \([20]\). However, in the exact excitation spectrum of the \( t-J \) model, both spin and charge excitations are gapless in 1D, at any hole concentration \([18, 21]\). Specifically, in the limit \( J \ll t \) the 1D Hubbard model reduces to a squeezed Heisenberg chain with an enlarged lattice constant \( \bar{a} = a/(1-\delta) \) and a renormalized superexchange coupling

\[
\tilde{J} = J(1-\delta) \left(1 - \frac{\sin 2\pi(1-\delta)}{2\pi(1-\delta)}\right).
\]

For small momentum the energy varies linearly with momentum, \( \epsilon(k) \propto \tilde{J} k \), \( k \to 0 \), leading to a linear-temperature contribution to the low-temperature specific heat. The predicted linear term in the specific heat has been experimentally observed in a 1D organic molecular solid \([22]\). This system can be described in terms of the 1D Hubbard model with a transfer integral of \( 2.1 \times 10^{-2} \) eV and an effective Coulomb interaction of \( 0.17 \) eV. The action \((110)\) predicts instead an exponential fall off of the specific heat and hence does not capture the low-energy physics of 1D strongly correlated electrons.

On the other hand, within the representation \((91)\) Eq.\((108)\) is replaced with

\[
Z_{F_{t-J}}^{\mu=0} = \int \prod_{\bar{\rho},\tau} D\xi_{\bar{\rho}}(\tau) D\bar{\xi}_{\bar{\rho}}(\tau) \exp \left( -\sum_{\bar{\rho}} \xi_{\bar{\rho}}(\tau) \bar{\xi}_{\bar{\rho}}(\tau) - \sum_{\bar{\rho}} \int_0^\beta \xi_{\bar{\rho}}(\tau) (\partial_\tau - \epsilon_\bar{\rho}) \bar{\xi}_{\bar{\rho}}(\tau) d\tau \right), \tag{115}
\]

where the above discussed shift of the arguments of the path-integral variables is implicitly understood. The time-lattice computation of this integral reduces to Eq.\((55)\) with the parameter \( \overline{\overline{\mu}} \) replaced by \( \mu \). As a result, the partition function \((115)\) is found to diverge. This divergency is of course readily compensated when the spin dynamics is turned on. However, the fermionic degrees of freedom taken alone cannot be considered as well-defined physical entities independently from spin variables. They act as auxiliary degrees of freedom to describe a composite object - a constrained electron. This also means that we cannot use the standard theory to describe the low-lying fermionic excitation as given by Eqs.\((108)\), \((109)\). The arguments that lead to Eq.\((113)\) are no longer applicable. Accordingly, the gapless low-lying spin excitations cannot be ruled out in the present case.

We have not yet discovered how to perform the path integral \((90)\) in the continuum limit in an analytically trustworthy way. Integration over the \( \xi \) or \( z \) variables separately from each other does not in general result in a physically relevant effective action. For example, the action \((19)\) corresponds to the constrained on-site Hamiltonian \([22]\). Integrating out the fermionic degrees of freedom results in the divergent expression given by Eq.\((55)\). A strongly correlated electron system at finite doping can hardly be represented solely in terms of the \( su(2) \) spin operators. This agrees with the observation that the spin-spin correlators in the 1D \( t-J \) model involve the conventional \( su(2) \) spin operators necessarily modified by fermionic ”string” operators. Weng et al. \([13]\) have shown that the effects due to the squeezing and rearrangement of spin configurations in 1D \( t-J \) model can be included by introducing a nonlocal fermionic ”string” field. These processes of first squeezing the \( t-J \) chain and then rearranging the spin configuration
cannot be described perturbatively and must be taken into account by introducing string-like fields. Formally, this can be thought of as a manifestation of the fact that the constrained electron (Hubbard) operator cannot be split into a convolution of conventional fermion and su(2) spin operators. The bare charge and spin degrees of freedom are not independent and interact with each other very strongly due to the NDO constraint. Within the path-integral approach, this statement reasserts itself in the appearance of the extra factor in the measure of the path integral.

VIII. CONCLUSION

Let us now summarize the main results of our work. If one assumes on some phenomenological grounds that the low-energy physics of a strongly correlated electron system allows for a description in terms of the conventional fermions coupled to the conventional lattice spins, one quite naturally ends up with the action first suggested by Shankar. Strong correlations are encoded there as follows. The theory comprises the spinless fermions which, even in the absence of the direct spin-fermion interaction, are coupled to the lattice spins through the Berry’s phase potential. Consequently, the doped t – J model does not admit a representation in terms of spinless fermions coupled to the local AF fluctuations via a compact U(1) gauge field even in the low-energy limit. The explicit resolution of the NDO constraint results in a rather involved path-integral representation for the t – J partition function. It is not yet evident how one can proceed with its direct calculation, except in some trivial cases. This appears as an evident consequence of the strong coupling nature of the physical low-energy excitations of a system of the constrained electrons.

Within the slave-particle approach, this problem is reflected in the lack of a controlled treatment for the emergent gauge field that strongly couples holons to spinons. The NDO constraint which actually gives rise to that gauge field is of a crucial importance right in the underdoped (δ ≪ 1) region which, by no accident, is the most interesting region of the phase diagram. It therefore brings in intractable strong interactions in this region, which in turn makes analytical calculations completely uncontrolled. Essentially, the gauge theory based on the slave-particle representation examines fluctuations around a mean-field solution that corresponds to a mean-field treatment of the NDO constraint. In general, such a theory is not stable against quantum fluctuations and the self-energy corrections are infinite due to the low-frequency gauge field fluctuations. The slave particles are not truly present in the physical spectrum, and cannot be treated as quasiparticles weakly coupled to the gauge field.

On the other hand, the Hartree-Fock approximations in the fermionic path integral for the Hubbard model,

\[ H_{\text{HF}} = -t \sum_{ij,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}, \]  

are known to recover successfully some well-known mean-field approximations such as, e.g., Stoner magnetism. Those approaches are essentially based on the Hubbard-Stratonovich decomposition of the U-quartic term which preserves the global spin rotation symmetry. The Hubbard-Stratonovich fields are then considered within the mean-field theory, which basically amounts to a saddle-point approximation accompanied by an integration over the corresponding Gaussian fluctuations around its saddle point solution. This is equivalent to a random-phase approximation (RPA) and it is normally controlled by an appropriately chosen 1/Nf expansion, where Nf stands for the number of the relevant field components. However, it is very unlikely that the hole dynamics in the strongly correlated regime, close to half filling, admits such a mean-field description. In fact, the physics of Nagaoka’s phase can hardly be recovered within such a mean-field theory.

In our view, the most appropriate way to proceed is to address the problem of the low-energy dynamics of the t – J model directly in terms of the superfield (ξ, z) ∈ CP^1|1] as dictated by Eqs. After all, it is this composite field that represents the true physical degrees of freedom - the constraint electron excitations. In this way, one is supposed to end up with a sigma model with the CP^1|1] target space. In fact, the 2D nonlinear sigma models with the CP^m+1|1] target superspaces have been discussed to describe percolations, polymers as well as some other
problems in statistical mechanics. This program can be explicitly carried out for a supersymmetric $t-J$ model that exhibits a global $SU(2|1)$ invariance at $J = 2t$. Given a ground state of the lattice $t-J$ model, the low-energy dynamics of the fluctuations around that vacuum state is supposed to be controlled by the nonlinear $CP^{1|1}$ sigma model. However, in a physically relevant case, $t \gg J$, the vacuum state is still unknown (in fact, there exist different competing vacuum states depending on the doping regime they are associated with). Therefore, it is still not clear what kind of low-energy action actually emerges in these cases.

To summarize, there is still no complete analytical effective theory to describe the low-energy dynamics for the underdoped Mott insulator in general. Such a theory should account for a simultaneous existence of a few competing nontrivial features: short-range AF order, superconductivity, uniform and modulated spin-liquid regime and a pseudogap phase. The important ingredient behind that picture is the local NDO constraint that essentially affects the low-energy physics close to half filling, and no trustworthy mean-field treatment is yet available to tackle this problem.

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