On the onset of the Jeans instability in a two-component fluid

J. P. M. de Carvalho and P. G. Macedo
Grupo de Cosmologia do C.A.U.P. ⋆
R. do Campo Alegre, 823
4100 Porto - Portugal

Abstract. Conditions for the establishment of small density perturbations in a self-gravitating two component fluid mixture are studied using a dynamical system approach.

It is shown that besides the existence of exponentially growing and decaying modes, which are present for values of the perturbation wave-number $k$ smaller than a critical value $k_M$, two other, pure oscillatory, modes exist at all scales. For $k < k_M$, the growing mode always affects both components of the fluid and not only one of them.

Due to the existence of a resonance between the baryonic and the dark perturbations, it is shown that the onset of structure formation in the post recombination epoch is substantially enhanced in a narrow scale band around another critical value $k_c$. For dark matter particles having a mass $\sim 30$ eV, the corresponding critical mass scale for the establishment of density perturbations at the time of recombination is of the same order of magnitude as the galactic one.

Key words: gravitation – hydrodynamics – instabilities – galaxies: formation – dark matter

1. Introduction

The study of the mechanisms responsible for the formation of structures in the universe was started by the pioneer work of Jeans (1902, 1928). Studying the conditions under which a cloud of gas (in a static background) becomes gravitationally unstable under its own gravity Jeans concluded that perturbations with masses smaller than a "critical" value $M_J$ (the Jeans mass) do not grow and behave like acoustic waves whereas perturbations with a mass $M$ bigger than $M_J$ grow under the effect of their self-gravity, leading to gravitationally bound structures.

Since then, the mechanism proposed by Jeans has been extensively applied as a criterion of stability in several models of galaxy formation. However, by the recombination epoch (which is believed to be the one after which baryonic perturbations could begin to grow), the critical scale predicted by the classical Jeans theory is not at all related to the galactic scale. Instead, it is well known that just before recombination the critical Jeans mass is $\sim 10^{16-17} M_\odot$ (i.e. the mass of rich clusters or even superclusters of galaxies), and immediately after recombination the Jeans mass drops abruptly, more than 10 orders of magnitude, to a value $\sim 10^{5-6} M_\odot$, which is related to the mass of globular clusters (Weinberg 1972; Zel’dovich & Novikov 1983).

The origin of this discrepancy between the Jeans mass, before and after recombination, and the mass of a typical galaxy, a few $10^{11} M_\odot$, has not been, up to now, well understood.

One should emphasize, however, that these values for the Jeans mass just before and just after recombination are obtained assuming a baryon-dominated universe.

On the other hand, the observation of flat galactic rotation curves at large galactic radii as well as high galaxy velocity dispersions in clusters has led to the missing mass problem and to the dark matter conjecture to solve it. This fact has, in particular, led to the hypothesis that dark matter halos exist around galaxies.

The existence of dark matter therefore also implies that when studying the dynamics and the formation of structures in the universe one cannot use the simple Jeans stability criterion for a one component gas, but one has to study how does the gravitational instability arise in a mixture of at least two components. This formulation of the problem leads to a different Jeans mass, as well as to a better understanding of the origin of galactic mass spectrum. This will be the main goal of this paper.

We must note that, after we finished this work, it was pointed out to us by Dr. Varun Sahni that this prob-
lem had already been studied, using a different formalism, by some Russian authors like Grishchuk & Zel’dovich (1981) or Polyachenko & Fridman (1981). Although having reached some of our conclusions, such studies lack most of the features present in this work. In particular they do not refer to the existence of the resonance we shall point out in this paper.

Since it is generally accepted that dark matter is constituted by WIMPs, (Weakly Interacting Massive Particles) in this study we shall consider the case for which the two components of the cosmic fluid interact only gravitationally.

The two fluid components that we shall consider will be a baryonic one and a hot dark matter (HDM) one, hereafter component $B$ and $D$ respectively. The latter being assumed to be made of neutrino-like particles.

Although the numerical results obtained in this paper refer to the particular case of HDM, one should point out that the formalism here developed is quite general and applies to any type of a two-component mixture.

In Sect. 2, we shall establish the linear coupled equations which describe the dynamical system for the density perturbations in the two-component fluid we wish to study. We shall then develop, in Sect. 3, a qualitative analysis of the dynamical system.

In Sect. 4, we shall study the behaviour, in phase-space, of the trajectories representing the general solution of the dynamical system. This analytical study will be complemented with some numerical plots in order to clarify some of the qualitative results obtained.

For numerical purposes we shall consider that the onset of gravitational instability occurs in a flat ($\Omega = 1$) universe at recombination epoch, after the decoupling between radiation and baryonic matter, when the radiation temperature is $\sim 3000$ K (Kolb & Turner 1990). We shall assume that the density parameter $\Omega_B$ has a value in agreement with the limits imposed by standard nucleosynthesis (Walker et al. 1991). Assuming $0.5 \leq h \leq 1$, where $h$ is the Hubble constant in units of 100 Km/s/Mpc, i.e. $h = H_0/(100$ Km/s/Mpc), such a value is $\sim 0.05$.

The neutrino-like particles which constitute the dark matter component are assumed to have a mass compatible with the assumption of being already non-relativistic by the recombination epoch. It will be shown later in this paper that this value for the mass of the dark matter particles seems to provide a good fit for the galactic mass spectrum.

In Sect. 5, we shall describe the occurrence of a resonance which, in our opinion, is responsible for the formation of structure at the typical galactic scales.

The relation between the existence of the resonance, the galactic scale and the mass of dark matter particles is discussed in Sect. 6.

Finally in Sects. 7 and 8, we shall point out the main results presented in this paper and outline some open questions as well as the research paths we intend to follow in the near future.

2. Dynamical equations for a two component fluid

The dynamics of a non-relativistic self-gravitating fluid can be described by the usual set of hydrodynamical equations: continuity equation, Euler equation and Poisson’s equation.

In this paper we shall neglect the expansion of the universe, since we shall be interested only in the qualitative behaviour, at the onset of the process of growth of density perturbations in the post recombination epoch, and not in the exact dynamics of the process.

In this case, for a fluid made of two components, $D$ and $B$, the above set of equations becomes:

\[ \frac{\partial \rho_D}{\partial t} + \nabla \cdot (\rho_D \mathbf{v}_D) = 0 \]  \hspace{1cm} (1)

\[ \frac{\partial \rho_B}{\partial t} + \nabla \cdot (\rho_B \mathbf{v}_B) = 0 \]  \hspace{1cm} (2)

\[ \frac{\partial \mathbf{v}_D}{\partial t} + (\mathbf{v}_D, \nabla) \mathbf{v}_D = -\frac{1}{\rho_D} \nabla p_D - \nabla \Phi \]  \hspace{1cm} (3)

\[ \frac{\partial \mathbf{v}_B}{\partial t} + (\mathbf{v}_B, \nabla) \mathbf{v}_B = -\frac{1}{\rho_B} \nabla p_B - \nabla \Phi \]  \hspace{1cm} (4)

\[ \nabla^2 \Phi = 4\pi G (\rho_D + \rho_B) \]  \hspace{1cm} (5)

where $\Phi$ is the total self-gravitational potential, $\rho_j, \mathbf{v}_j$ and $p_j$ are respectively the density, velocity and pressure fields for fluid elements of component $j$, (the index $j$ taking the values $D$ or $B$).

Using a perturbative analysis, to first order in the perturbation, one can write the dynamical variables as follows:

\[ \rho_j = \rho_{j0} + \delta \rho_j \]
\[ \mathbf{v}_j = \mathbf{v}_{j0} + \delta \mathbf{v}_j \]
\[ p_j = p_{j0} + \delta p_j \]
\[ \Phi_r = \Phi_{r0} + \delta \Phi_r \]  \hspace{1cm} (6)

where the index "0" indicates the unperturbed value, and $\delta$ the first order perturbation.

It’s common to consider the zero order solution to be an infinite fluid at rest, ($\mathbf{v}_0 = 0$), for which the density and pressure are the same everywhere (unperturbed fluid). We must point out, as it is well known, that this static solution for the unperturbed state of the above equations is not mathematically correct, except for $\rho_0 = 0$. However, since this problem is removed when considering the more realistic case of a fluid embedded in the expanding universe, (which we shall discuss in a forthcoming paper),
one can start perturbing this static configuration in order to obtain some qualitative information on the fluid behaviour.

We shall assume that the solutions of the above coupled system of Eqs. \( \{1\} - \{5\} \) are square integrable functions of the spatial variables and therefore can be Fourier analysed with respect to these variables.

To first order in the perturbation, after the Fourier analysis in the spatial variables and some manipulation, the set of Eqs. \( \{1\} - \{5\} \) leads to the following equations:

\[
\begin{aligned}
\ddot{\Delta}_D + \left(c_D^2 k^2 - W_D\right) \Delta_D - W_B \dot{\Delta}_B &= 0 \\
\ddot{\Delta}_B + \left(c_B^2 k^2 - W_B\right) \Delta_B - W_D \dot{\Delta}_D &= 0
\end{aligned}
\tag{7}
\]

where the double dot stands for second time derivative, \( k \) is the wave number of the perturbation. For each of the components \( j \), the parameter \( W_j \) is given by \( W_j = 4\pi G \rho_{j0} \) and \( \Delta_j \) is the density contrast \( \delta \rho_j / \rho_{j0} \) and \( c_j \) the adiabatic sound speed given by \( c_j = (\partial p_j / \partial \rho_j)^{1/2} \).

The value of the parameter \( W_j \), at the cosmic time of recombination, \( t_{\text{rec}} \), can be related to the density parameter, \( \Omega_j \), of component \( j \), by

\[
W_j = \frac{2}{3} \frac{\Omega_j}{t_{\text{rec}}^2}
\tag{8}
\]

The adiabatic sound speed in component \( j \) at recombination, \( c_{j,\text{rec}} \), is the sound speed of a monoatomic ideal gas, which is given by:

\[
c_{j,\text{rec}} = \left(\frac{5}{3} \frac{K_{\text{Bol}} T_{\text{rec}}}{m_j}\right)^{1/2}
\tag{9}
\]

where \( K_{\text{Bol}} \) is the Boltzmann constant and \( T_{\text{rec}} \), and \( m_j \) are respectively the temperature at recombination and mass of the particles of component \( j \).

One should emphasize, that from the decoupling between the neutrino-like particles and radiation until the epoch at which these particles become non-relativistic, (which occurs for a red-shift \( z_{NR} \sim 6 \times 10^4 \), Doroshkevich et al. 1988), their temperature \( T_D \) is given by \( T_D = (4/11)^{1/3} T_r \), where \( T_r \) is the radiation temperature. Since then, it decays with \( a^{-2} \), where \( a \) is the scale factor of the universe (Gao & Ruffini 1981; Padmanabhan 1993).

One can therefore estimate that, at the recombination epoch, such a dark component \( D \) will have a temperature \( T_{D,\text{rec}} \sim 40 \) K

The above system \( \{1\} \) of two linear differential equations of second order can be transformed into the following autonomous dynamical system of four first order differential equations which one can write in a matrix form as:

\[
\dot{x} = Ax
\tag{10}
\]

where \( x \equiv (x_1, x_2, x_3, x_4)^T \equiv (\Delta_D, \Delta_D, \dot{\Delta}_B, \Delta_B)^T \), the superscript \( T \) standing for transpose. \( A \) is a matrix whose components are only functions of the parameter \( k \) and has the form:

\[
A \equiv A(k) = \begin{pmatrix}
0 & W_D - c_D^2 k^2 & 0 & W_B \\
1 & 0 & 0 & 0 \\
0 & W_D & 0 & W_B - c_B^2 k^2 \\
0 & 0 & 1 & 0
\end{pmatrix}
\tag{11}
\]

We shall be particularly interested in the solutions which represent the density perturbations in components \( D \) and \( B \). These solutions are given by the second and fourth components, \( x_2(t) \) and \( x_4(t) \), of the general solution \( x(t) \), of the dynamical system \( \{10\} \).

Using methods of qualitative theory of differential equations one can analyse the dynamical system and obtain some qualitative information on the behaviour of the perturbations. This will help understanding the stability, against density perturbations, of the two-component system. Even in a case like this one where an explicit solution exists, this is worth doing.

Alternatively the autonomous dynamical system can be numerically studied provided we specify the initial condition \( x_0 \equiv x(t_0) \), where \( t_0 \) is the cosmic time at which the equilibrium state of the fluid is perturbed.

We shall analyse the system qualitatively and, whenever it proves convenient, to clarify any result, we shall illustrate this study with some numerical plots.

3. Qualitative study of the dynamical system

The above matrix \( A(k) \) has 4 different eigenvalues for all values of \( k \) except for \( k = 0 \) and for a critical value \( k = k_M \), which shall be defined below. Therefore for \( k > 0 \) and \( k \neq k_M \) there are 4 linearly independent eigenvectors. In this case the general solution \( x(t) \) of the system \( \{10\} \) is well known (Arnold 1973).

If all the eigenvalues are real ones, that solution can be written in the form of a linear expansion of particular solutions \( e^{\lambda_j t} \xi_j \), i.e.:

\[
x(t) = \sum_{j=1}^{4} \alpha_j e^{\lambda_j t} \xi_j
\tag{12}
\]

where \( \alpha_j \) are real-valued functions of the parameter \( k \), which are determined by the initial conditions, and \( \lambda_j \) and \( \xi_j \) are respectively the eigenvalues of the matrix \( A(k) \) and its associated eigenvectors.

If some of the eigenvalues are complex conjugates, one can easily see that the real and imaginary parts of the complex function \( e^{\lambda_j t} \xi_j \) are two real-valued particular solutions of the dynamical system associated with the complex conjugate pair of eigenvalues \( \lambda_j \) and \( \bar{\lambda}_j \).

In order to obtain some qualitative insight about the behaviour of the perturbations, one must find the critical or equilibrium points of the system \( \{10\} \) and calculate the eigenvalues of matrix \( A(k) \) as well as their associated eigenvectors.
3.1. Critical points, eigenvalues and eigenvectors

The critical points of the dynamical system (11) are obtained solving the equation

$$\dot{x} = 0.$$  \hspace{1cm} (13)

The only critical point which does not depend on the parameter $k$ is the origin, $x \equiv (0, 0, 0, 0)^T$.

The parameter-dependent critical points are given by the solutions of the equation $\det(A) = 0$. In the case when the parameter $k$ has a value given by:

$$k_m^2 = k_B^2 + k_D^2 = \frac{W_B}{c_B^2} + \frac{W_D}{c_D^2},$$  \hspace{1cm} (14)

any point $x \equiv (0, x_2, 0, x_4)^T$ of the $(x_2, x_4)$ plane, is a critical point of the above system.

In Eq. (14), $k_B$ and $k_D$ are respectively the Jeans wave numbers of components $B$ and $D$ when taken separately, and we shall call $k_m$ the Jeans wave number for the mixture of the two components.

The eigenvalues of the matrix $A(k)$ are the roots of the characteristic polynomial, obtained solving the equation

$$\det(A - \lambda I) = 0$$  \hspace{1cm} (15)

where $I$ is the identity matrix.

The 4 eigenvalues of $A(k)$, which we shall represent by $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$, are therefore given by:

$$\begin{cases}
\lambda_1 = -\lambda_2 = \frac{1}{2}\sqrt{f + \sqrt{f^2 + 4k^2g}} \\
\lambda_3 = -\lambda_4 = \frac{1}{2}\sqrt{f - \sqrt{f^2 + 4k^2g}}
\end{cases}$$  \hspace{1cm} (16)

where $f$ and $g$ are the following functions of $k$:

$$f(k) = W_B + W_D - k^2 (c_B^2 + c_D^2)$$  \hspace{1cm} (17)

$$g(k) = W_B c_B^2 + W_D c_D^2 - k^2 c_B^2 c_D^2$$  \hspace{1cm} (18)

The eigenvectors $\xi_j$ of the matrix $A(k)$, associated to the eigenvalues $\lambda_j$, must satisfy the condition

$$A \xi_j = \lambda_j \xi_j,$$  \hspace{1cm} (19)

and can be written in the following form:

$$\begin{align*}
\xi_1 &\equiv (\beta_1, \lambda_1, \beta_1, 1)^T \\
\xi_2 &\equiv (\beta_2, \lambda_2, \beta_2, 1)^T \\
\xi_3 &\equiv (\beta_3, \lambda_3, \beta_3, 1)^T \\
\xi_4 &\equiv (\beta_4, \lambda_4, \beta_4, 1)^T
\end{align*}$$  \hspace{1cm} (20)

where $\beta_j$ are also functions of $k$ and are given by:

$$\begin{align*}
\beta_1 &= \beta_2 = \frac{1}{2W_D} \left( h + \sqrt{h^2 + 4W_B W_D} \right) \\
\beta_3 &= \beta_4 = \frac{1}{2W_D} \left( h - \sqrt{h^2 + 4W_B W_D} \right)
\end{align*}$$  \hspace{1cm} (21)

and $h = h(k)$ is given by:

$$h(k) = W_D - W_B + k^2 (c_B^2 - c_D^2)$$  \hspace{1cm} (22)

Table 1. The signs of real and imaginary parts, $(\Re$ and $\Im$ respectively), of the four eigenvalues of matrix $A(k)$ as a function of parameter $k$. Note that $\Im(\lambda_3) = -\Im(\lambda_4) \neq 0$ for all $k \neq 0$. (See text for discussion.)

| $k$ | 0 | $0 < k < k_M$ | $k = k_M$ | $k > k_M$ |
|----|---|---------------|-----------|-----------|
| $\lambda_1$ | $\Re$ | + | + | 0 | 0 |
| $\Im$ | 0 | 0 | + | + |
| $\lambda_2$ | $\Re$ | – | + | + | 0 |
| $\Im$ | 0 | 0 | 0 | 0 |
| $\lambda_3$ | $\Re$ | 0 | 0 | 0 | 0 |
| $\Im$ | 0 | 0 | – | – |
| $\lambda_4$ | $\Re$ | 0 | 0 | 0 | 0 |
| $\Im$ | 0 | 0 | – | – |

3.2. Qualitative results

Qualitative information about the behaviour of the dynamical system can be obtained studying the signs of the real and imaginary parts of the eigenvalues $\lambda_j$.

The only zero of the eigenvalues $\lambda_1$ and $\lambda_2$ is obtained when $k$ is the Jeans wave-number of the mixture, $(k = k_M)$.

Looking at Eqs. (16) - (18) one can see that for $k < k_M$ the function $g(k)$ is positive and therefore $\lambda_1$ and $\lambda_2$ are real nonzero eigenvalues, $(\lambda_1 > 0, \lambda_2 < 0)$. For $k > k_M$, $\lambda_1$ and $\lambda_2$ are two pure imaginary conjugate eigenvalues.

The eigenvalues $\lambda_3$ and $\lambda_4$ do not have zeros, except for the asymptotic value $k = 0$. Since $f - \sqrt{f^2 + 4k^2g}$ is always negative, for all other values of $k$ $\lambda_3$ and $\lambda_4$ are pure imaginary conjugate eigenvalues.

Physically, this means that one has two paired wave modes (which correspond to acoustic oscillations), for all values of the wave number $k$ and two other modes which are acoustic for $k > k_M$ and become one growing and one decaying mode for $k < k_M$.

These results are summarized in Table 1.

One can therefore distinguish two, very distinct, situations concerning the four-dimensional phase-space according to the values of the parameter $k$, i.e.:

1. For $k > k_M$, the four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$, are pure imaginary and therefore the phase-space is a center space, which is spawned by the eigenvectors associated to those eigenvalues.

Solutions in a center subspace oscillate at constant amplitude, and therefore density perturbations with wave number $k > k_M$ can’t grow. This case corresponds to the existence of two pairs of acoustic wave modes propagating in the fluid with different phase velocities. In the limit of small wavelengths, it is easy to see
that these modes have the same dispersion relations and compositions as two completely uncoupled acoustic waves propagating in pure D and B fluids. This is due to the fact that the role played by their self gravity, which is the only mechanism coupling the two components, is not important at small scales.

2. For \( k < k_M \), (see Table 1), the eigenvalue \( \lambda_1 \) is real and positive, \( \lambda_2 \) is real and negative and \( \lambda_3 \) and \( \lambda_4 \) are pure imaginary (conjugates) eigenvalues. In this case the phase-space is, therefore, the direct product of three distinct subspaces:

(a) An unstable subspace of dimension one spawned by the eigenvector associated to the eigenvalue with positive real part, \( \lambda_1 \).

Solutions in an unstable subspace always grow. Since \( \lambda_1 \) is a real time-independent eigenvalue the growth of these particular solutions is exponential without oscillations.

(b) A stable subspace also of dimension one spawned by the eigenvector associated to the eigenvalue with negative real part, \( \lambda_2 \).

Solutions in a stable subspace always decay. In our case the eigenvalue associated to this subspace, \( \lambda_2 \), is also a real valued one and therefore the decay of these particular solutions is exponential without oscillations.

(c) A center subspace which for this range of \( k \) is of dimension two and is spawned by the eigenvectors associated to the complex eigenvalues with null real part, \( \lambda_3 \) and \( \lambda_4 \). Solutions in a center subspace oscillate.

This case corresponds to the existence of a pair of acoustic modes plus one growing and one decaying mode.

The analysis of the composition of these modes shows that all of them involve both components D and B, and are completely coupled through gravity, which, at these large scales plays an important role.

One can then conclude that growth of density perturbations is possible only for \( k < k_M \). Note that for all \( k < k_M \) this growth affects the two components (i.e. the mixture) and not only one of them. This is the reason why we called \( k_M \) the Jeans wave number of the mixture. The corresponding physical critical scale length below which no structure could develop is \( l_M = 2\pi/k_M \).

It is interesting to note that although this scale length is related to the Jeans lengths, \( l_D \) and \( l_B \), of the two components when taken separately it does not coincide, in the general case, with none of them. In fact from relation (14) one can see that \( l_M = (l_D^{-2} + l_B^{-2})^{-1/2} \).

Since we are mainly interested in structure formation we shall consider, in what follows, only the case when \( k < k_M \), (i.e. \( l > l_M \)).

4. Behaviour of the general solution

In order to study the behaviour with scale of the dynamical system at the onset of the structure formation process, one needs to analyse the \( k \)-dependence of the eigenvalues \( \lambda_j \), the eigenvectors \( \xi_j \) and the coefficients \( \alpha_j \), which appear in Eq. (12).

Since \( \lambda_2 = -\lambda_1 \) and \( \lambda_4 = -\lambda_3 \) we shall only study the dependence with scale of the real eigenvalue \( \lambda_1 \) and of the imaginary part of \( \lambda_3 \).

The eigenvalue \( \lambda_1 \), being real and positive, is the one related with the growth rate of the perturbations, is a growing function of scale. However \( \lambda_1 \) does not grow at the same rate for all scales. In fact there are two particular scales at which a significant change in its growth rate occurs. The first one is the Jeans scale-length of the mixture, \( l_M \), where, after changing from an imaginary to a real value, \( \lambda_1 \) increases very fast and then slows down, remaining approximately constant until the second peculiar scale is reached, which occurs near the Jeans scale-length of the dark matter component, \( l_D \), where \( \lambda_1 \) grows faster again. Beyond this region the growth of \( \lambda_1 \) slows down again and approaches the asymptotic value \( W_B + W_D \) as \( l \to \infty \); see Eqs. (18) and (17).

The eigenvalue \( \lambda_3 \) is a decaying function of scale and the only change in its behaviour occurs at a scale which coincides with the second peculiar scale for \( \lambda_1 \), (i.e. for a scale near \( l_D \)), where a faster decrease than for any other region of the scale range, is observed. For greater scales it approaches zero as \( l \to \infty \). Since \( \lambda_3 \) is the frequency associated to the acoustic modes one can see, from its dependence with scale, that although oscillations are present in all scales, for large scales acoustic waves have their frequency approaching zero, i.e. their period tends to infinity. See Fig. 1.

Additional information about the system behaviour can be obtained studying the trajectories in phase-space corresponding to the general solution \( \mathbf{x}(t) \); unfortunately, since the phase-space is of dimension four, they are hard to visualise. However, since, as referred above, we are particularly interested in the qualitative behaviour of the second and fourth components of the general solution \( \mathbf{x}(t) \), i.e. the behaviour of the density perturbations \( \Delta_B(t) \) and \( \Delta_D(t) \), we shall infer some properties of the dynamical evolution of the density perturbations analysing the behaviour of the projection, on the plane \( (x_4, x_2) \equiv (\Delta_B, \Delta_D) \), of the phase-space four-dimensional trajectories.

Looking at the components of the eigenvectors, given by the relations (14) - (22), one can see that the projection, on the plane \( (x_4, x_2) \), of the stable subspace coincides with that of the unstable subspace (since \( \beta_1 = \beta_2 \)).

The common projection of these subspaces is the straight line given by the equation:

\[ \Delta_D = \beta_1 \Delta_B. \]
The center subspace has a projection on the same plane \((\Delta_B, \Delta_D)\) which is also a straight line, defined by
\[\Delta_D = \beta_3 \Delta_B.\] (24)

These equations, (23) and (24), for the projection of the stable/unstable and center subspaces can give us the relative composition (relative percentage of components \(B\) and \(D\)), in the growing and decaying modes, Eq. (23), as well as in the acoustic modes, Eq. (24).

From relations (21), one can see that since \(h^2 + 4W_B W_D\) is always positive, \(\beta_2\) are, therefore, real-valued functions for all \(k > 0\), with \(\beta_1 > 0\) and \(\beta_3 < 0\). The condition \(\beta_1 > 0\) means that both components participate positively in the collapse of the mixture as well as in the decaying mode, which being transient will not interest us. On the other hand \(\beta_3 < 0\) means that, in the acoustic modes, the two components oscillate with opposite phases; when the density in one of them is growing, the density in the other is decreasing.

The direct product of the stable and unstable subspaces forms a two dimensional subspace which is usually called a saddle subspace and solutions in such a subspace are simultaneously pulled towards the equilibrium point along the stable axis and pushed away from it along the unstable axis. The result is that when the time \(t\) varies from \(-\infty\) to \(+\infty\) the orbits representing these particular solutions are initially attracted by the critical point reaching a closest approach to the origin at a time \(t_c\) after which they are repelled away.

Choosing initial conditions such that \(x_1 = x_3 = 0\), i.e. \(x_0\) lies on the plane \((x_1, x_2)\), and given the symmetry, in first and third components, between the eigenvectors \(\xi_1\) and \(\xi_2\), one concludes that the closest approach from the origin of these orbits, is attained at initial time \(t_0\), (i.e. \(t_c = t_0\)). Therefore, in this case, for \(t > t_0\) the system is monotonically driven away from the origin with time. It is worth noting that this condition (\(x_0\) lying on the \((x_1, x_2)\) plane), corresponds to the reasonable assumption that the mixture starts to collapse from rest.

Moreover, since the effect of the center subspace is to force the orbit to oscillate, one shall have a projected path which oscillates around the projection of the center subspace, i.e. around the straight line given by Eq. (23), see Fig. 3.

According to the initial conditions, \(\Delta_D\) and \(\Delta_D\), one can see that, if \(\Delta_D > \beta_3 \Delta_D\), the system will collapse to form a massive structure. On the other hand, if \(\Delta_D < \beta_3 \Delta_D\), it will evolve to form a void, as represented in Fig. 3.

Since \(\beta_1\) and \(\beta_3\) depend on the perturbation’s wave-number \(k\), it is clear that the direction of the projection of the above mentioned subspaces also changes with scale, and therefore the relative growth of the perturbations in the two components, as well as the composition of the collapsing and oscillating modes, is also scale dependent.

The change in the relative growth of perturbations with scale can be seen analysing the behaviour of \(\beta_1\) with wave number \(k\), or equivalently with scale length \(l\).

The straight lines defined by Eqs. (23) and (24) make angles with the direction of the \(x_4\)-axis, which are given respectively by \(\theta_1 = \arctan(\beta_1)\) and \(\theta_3 = \arctan(\beta_3)\).

If one assumes that \(c_B < c_D\), (case of hot dark matter), the derivative
\[
\frac{d\beta_1}{dl} = 4\pi^2 \frac{c_B^2}{c_D^2} \left(1 + \frac{h}{\sqrt{h^2 + 4W_B W_D}}\right) \frac{1}{l^3}
\] (25)
is always positive and since \(\beta_1(l_{\text{spin}}) = c_B^2/c_D^2\) and \(\lim_{l \to \infty} \beta_1 = 1\) one concludes that \(\theta_1\) is a monotonic function of \(l\), growing from \(\arctan(c_B^2/c_D^2)\) until the asymptotic value \(\lim_{l \to \infty} \theta_1 = 45^\circ\).

By a similar study of \(\beta_3\) one concludes that
\[
\frac{d\beta_3}{dl} = 4\pi^2 \frac{c_B^2}{c_D^2} \left(1 - \frac{h}{\sqrt{h^2 + 4W_B W_D}}\right) \frac{1}{l^3}
\] (26)
is also always positive and therefore \(\theta_3\) grows with scale from \(\theta_3(l_{\text{spin}}) = -\frac{W_B c_B^2}{W_D c_D^2}\) until the asymptotic value \(\lim_{l \to \infty} \theta_3 = \arctan(- W_B c_B^2 / W_D c_D^2)\). This is illustrated in Fig. 3.

It is important to note, however, that the rates of change with scale of both \(\beta_1\) and \(\beta_3\) are not constant. The same is obviously true for \(\theta_1\) and \(\theta_3\) as shown in the plot of Fig. 3. In fact from this plot, or alternatively analysing expressions (23) and (24), one can notice that for \(l_{\text{spin}} < l \ll l_D\) (i.e. \(k_{\text{spin}} > k \gg k_D\)), \(\beta_1\) and \(\beta_3\) are slowly increasing functions, while when \(l\) reaches the order of \(l_D\) they become much steeper functions of \(l\). For values of \(l\) above this scale the rate of change of \(\beta_1\) and \(\beta_3\) slows down again as \(\beta_1\) and \(\beta_3\) approach the asymptotic values 1 and \(- W_B / W_D\), respectively.

This means that the composition of the mixture in the collapsing mode changes substantially when the perturbation scale is of the order of \(l_{\text{spin}}\), starting to involve, at this scale, a much greater part of the dark material which, for smaller scales, was mainly oscillating in the acoustic mode.

Further on, it will be shown that there is a critical scale \(l_c\) where a resonance between the two components occurs\(^1\). It is for this particular scale, and larger ones, that the acoustic modes tend to be devoided in favour of the collapse mode.

\(^1\) This scale \(l_c\) is in the particular case we are considering of the order of \(l_D\) as shall be seen later. Moreover it will be shown that, if the mass of hot dark matter particles is \(m_D \sim 30\text{ eV}\), the scale \(l_c\) has a corresponding mass of the same order as that of typical galaxies.
As a matter of fact, in the limit of very large scales ($k \to 0$) one has:

$$\lim_{k \to 0} \frac{\Delta_{D_c}}{\Delta_{B_c}} = 1$$

(27)

where $\Delta_{D_c}$ and $\Delta_{B_c}$ are the density contrast of the dark and baryonic components participating in the collapse mode.

Denoting by $\delta_{D_c}$ and $\delta_{B_c}$ the density perturbations of the dark and baryonic components in the collapse mode, one has:

$$\lim_{k \to 0} \frac{\delta \rho_{D_a}}{\delta \rho_{B_a}} = \frac{\rho_{D_0}}{\rho_{B_0}}$$

(28)

and for the acoustic mode, by a similar reasoning, one obtains

$$\lim_{k \to 0} \frac{\delta \rho_{D_a}}{\delta \rho_{B_a}} = -1$$

(29)

where $\delta_{D_a}$ and $\delta_{B_a}$ are the density perturbations in the dark and baryonic components in the acoustic modes.

These results mean that, for very large scales, the composition of the perturbed region, in the collapsing mode, is identical to the one in the unperturbed state, as shown by Eq. (25). On the other hand the perturbations in both components, oscillating in the acoustic modes, cancel each other, since they oscillate with opposite phase and the same amplitude.

5. The structure formation resonance

The coefficients $\alpha_j$ in Eq. (22) can be obtained by specifying the initial conditions $x_j(t_0) = x_{i0}$. Substituting in the general solution given by Eq. (22), one obtains:

$$x_{i0} = \sum_j \alpha_j v_{(j)i}$$

(30)

where $v_{(j)i}$ is the $i^{th}$ component of eigenvector $j$.

Solving the above linear system of equations along with the already mentioned assumption $x_{10} = x_{30} = 0$ one obtains the coefficients $\alpha_j$, which are given by:

$$\alpha_1 = \frac{x_{20}}{2} \frac{1-Q_0 \beta_3}{\beta_1 - \beta_3} e^{-\lambda_1 t_0}$$

$$\alpha_2 = \frac{x_{20}}{2} \frac{1-Q_0 \beta_3}{\beta_1 - \beta_3} e^{\lambda_1 t_0}$$

$$\alpha_3 = x_{20} \frac{Q_0 \beta_3 - 1}{\beta_1 - \beta_3} \cos(i \lambda_3 t_0)$$

$$\alpha_4 = x_{20} \frac{1-Q_0 \beta_3}{\beta_1 - \beta_3} \sin(i \lambda_3 t_0)$$

(31)

where $Q_0$ is the ratio between the density contrast in the two components at $t = t_0$, (i.e. $Q_0 = x_{40}/x_{20} = \Delta_{B_0}/\Delta_{D_0}$), and $i = \sqrt{-1}$.

Substituting these expressions in the second and fourth components of the general solution (22), one obtains after some straightforward manipulation:

$$\Delta_D(\tau) = x_2(\tau) = x_{20} [\zeta_1 (e^{\lambda_1 \tau} + e^{-\lambda_1 \tau}) + \zeta_2 \cos(i \lambda_3 \tau)]$$

(32)

and

$$\Delta_B(\tau) = x_4(\tau) = x_{40} [\zeta_3 (e^{\lambda_1 \tau} + e^{-\lambda_1 \tau}) + \zeta_4 \cos(i \lambda_3 \tau)]$$

(33)

where we have used a new time variable, $\tau = t - t_0$, by translating the time origin to the beginning of the process. The new coefficients $\zeta_j$, which are the mode amplitudes for the perturbations in components $D$ and $B$, are given by the following expressions

$$\zeta_1 = \frac{\beta_1}{2} \frac{1-Q_0 \beta_3}{\beta_1 - \beta_3}$$

$$\zeta_2 = \frac{\beta_3}{2} \frac{Q_0 \beta_3 - 1}{\beta_1 - \beta_3}$$

$$\zeta_3 = \frac{1}{2} \frac{Q_0 \beta_3 - 1}{\beta_1 - \beta_3}$$

$$\zeta_4 = \frac{\beta_1 - Q_0 \beta_3}{\beta_1 - \beta_3}$$

(34)

For simplicity it shall be assumed, throughout this section, that $Q_0$ is positive, although massive structures can also originate even in the cases when it is negative, (see the previous section, Fig. 3).

One must also point out that if one allows $x_{10} \neq 0$ and/or $x_{30} \neq 0$ the expressions (21) for $\alpha_j$, and consequently the expressions (34) for $\zeta_j$, become more complicated, and therefore difficult to be studied analytically.

The general analytical study of the mode amplitudes, in the case when all the $x_{i0} \neq 0$, is under development and we intend to publish it in a forthcoming paper. However preliminary numerical results seem to indicate that no significant change occurs in the behaviour of the mode amplitudes.

Looking at the behaviour of the amplitudes $\zeta_j$, one should first of all point out that, from Eqs. (18), one can see that the pairs $(\zeta_1, \zeta_2)$ and $(\zeta_3, \zeta_4)$ are related by:

$$2 \zeta_1 + \zeta_2 = 2 \zeta_3 + \zeta_4 = 1$$

(35)

This means that a maximum of $\zeta_1$ corresponds to a minimum of $\zeta_2$ and a maximum of $\zeta_3$ corresponds to a minimum of $\zeta_4$ or inversely.

This can be seen in Fig. 3, where the numerical plot illustrates the dependence of $\zeta_3$ on scale, and shows its resonant behaviour for $l \sim l_c$, and in Fig. 4, where the sharp minimum occurring for the coefficient $\zeta_4$ is observed at the same value of the scale length. In the numerical plot of Fig. 4, we also illustrate the behaviour of the coefficients $\zeta_1$ and $\zeta_2$. 
Physically, this means that the occurrence of a resonance (presence of maxima in \( \zeta_1 \) or \( \zeta_3 \)), is related with the fact that at the resonant scale the ratio between the number of particles which populate the collapse and decaying modes and the number of particles of the same component which populate the acoustic mode grows substantially when compared with the same ratio in nearby scales. Therefore one can say that most of the resonant material is collapsing and little is left oscillating, with the oscillating mode helping to devoid the background to make a greater percentage of the particles participate in the collapse.

The occurrence of such a resonance can be interpreted as a consequence of the following:

As pointed out by Grishchuk & Zel’dovich (1981), the dynamical system (10) represents two gravitationally coupled oscillators. If one takes the two components \( B \) and \( D \) separately, the dispersion relations, for perturbations in those separate components, give us the natural frequencies of oscillation, i.e. the ones the components would have if uncoupled, which are given by

\[
\omega_B = \left( c_B^2 k^2 - W_B \right)^{1/2} \tag{36}
\]

and

\[
\omega_D = \left( c_D^2 k^2 - W_D \right)^{1/2} \tag{37}
\]

where \( \omega_B \) and \( \omega_D \) are the natural frequencies for components \( B \) and \( D \) respectively.

From Eqs. (36) and (37) one can see that the two natural frequencies coincide for a wave number \( k_c \) given by

\[
k_c = \left( \frac{W_B - W_D}{c_B^2 - c_D^2} \right)^{1/2}. \tag{38}
\]

This means that \( k_c \) is the characteristic wave number for which perturbations in the two components, taken separately, would have the same collapse times scales. Thus it is reasonable to expect a resonance to occur, at least in some of the mode amplitudes of the perturbations, for a value of \( k \) near \( k_c \). In fact it will be shown below that such a resonance does indeed occur.

The scale length corresponding to \( k_c \) is given by

\[
l_c = 2\pi \left( \frac{c_B^2 - c_D^2}{W_D - W_B} \right)^{1/2} = \left[ \frac{l_B^2 - (W_B/W_D) l_D^2}{1 - (W_B/W_D)} \right]^{1/2}. \tag{39}
\]

In the case when \( c_B \ll c_D \) and \( W_B \ll W_D \) one can see that \( l_c \) is of the order of \( l_D \), as noted in the last section.

This resonance has physical meaning only if the value of \( k_c \) is real and positive. Looking at Eq. (38) one can see that this can only happen in the case when the gravitationally dominant component, i.e. the one with a greater value of \( W \), has also a greater sound speed, otherwise the scale for which the two components have equal natural frequencies will be a physical meaningless imaginary one. Therefore, in what follows, it will be assumed that \( W_D > W_B \) and \( c_D > c_B \). This is a necessary condition for the existence of a resonance in the system. When applied to our universe, for which the observational evidences seem to indicate to be dominated by dark matter, (i.e. \( W_D > W_B \)), this means that such a resonance cannot occur in a cold dark matter (CDM) dominated universe.

As can be seen in Eqs. (24), the peculiar behaviour of \( \beta_1 \) and \( \beta_3 \) for \( l \sim l_c \) mentioned in the preceding section, is responsible for the fact that, from all the perturbations which can start to grow at the recombination epoch, those with masses of the order of \( M_c \) are the ones which are going to present, at the onset of the gravitational instability, the biggest amplitude, in the collapse mode. In this sense the scale \( l_c \) is a privileged one.

However one must be careful in interpreting this result since it is valid only for the onset of the instability and cannot be extrapolated in time. To have a correct idea about the dynamical evolution of the perturbations one has to solve a more complicated problem, i.e. we must include, in the equations, the terms due to the expansion of the universe, which allows us to obtain the correct growth rate of the perturbations. On the other hand another difficulty arises because we solve the equations only for one Fourier component of the general solution which separates a perturbation into distinct wavelengths that, in principle, have distinct growth rates. Therefore at any epoch the density contrast in a perturbed region of scale \( \sim 1/k \) must be integrated in all the region, i.e. for all \( k \) in the range \( 0 < k < 1/k \), (see Padmanabhan 1993).

In particular, before a careful analysis of these two points is made, one cannot know which perturbations are the first to reach the non linear regime. This is a very important question that we shall address in a forthcoming paper (in preparation).

Since the growing mode is the one that leads to structure formation we shall be particularly interested in the behaviour of the coefficients \( \zeta_1 \) and \( \zeta_3 \) which are the ones associated to that mode.

We shall therefore proceed by analysing the functions \( \zeta_1 \) and \( \zeta_3 \). Looking at their derivatives with respect to \( k \), one can see that their maxima occur for values of \( k \) given by:

\[
k (s_1^{\text{max}}) = \left[ k_c^2 - \frac{2 W_D}{Q_0 \left( c_D^2 - c_B^2 \right)} \right]^{1/2} \tag{40}
\]

for \( \zeta_1 \), and

\[
k (s_3^{\text{max}}) = \left[ k_c^2 + \frac{2 W_D}{q \left( c_D^2 - c_B^2 \right)} \right]^{1/2} \tag{41}
\]

for \( \zeta_3 \), where \( q = 1/Q_0 = \Delta_{D_o}/\Delta_{B_o} \).
Substituting Eq. (38) into Eq. (40) one can see that the existence of a maximum for $\zeta_1$ in the range $0 < k < k_M$, can only occur if $q$ satisfies the following condition:

$$q < \frac{1}{2} \left( 1 - \frac{W_B}{W_D} \right)$$

(42)

On the other hand, substituting Eq. (38) into Eq. (41) one can see that the existence of a maximum for $\zeta_3$ in the same range, can only occur if:

$$q > 2 \left( \frac{c_D^2}{c_B^2} - \frac{W_B}{W_D} \frac{c_B^2}{c_D^2} \right)^{-1}$$

(43)

It is reasonable to assume that the composition of the initial perturbation is the same as the composition of the unperturbed state (i.e., $q = W_D/W_B$). In this case, it is easy to see that $\zeta_1$ will never have a maximum in the desired range, while $\zeta_3$ can have one, only if the following condition is satisfied:

$$\frac{c_D^2}{c_B^2} > \frac{W_B}{W_D} + \sqrt{\frac{W_B^2}{W_D^2} + \frac{W_D}{W_B}}$$

(44)

From this analysis one concludes that, in a hot dark matter dominated-universe, the resonance will only affect the baryonic component. We shall, therefore, from now on study only the behaviour of the mode amplitude $\zeta_3$ which is the one associated to the growing mode in the baryonic component.

Substituting the value of $\zeta_3^{\text{max}}$ given by Eq. (13) into Eq. (44), one obtains for the function $\zeta_3$ its maximum value $\zeta_3^{\text{max}}$, given by:

$$\zeta_3^{\text{max}} = \frac{1}{4} \left( 1 + \sqrt{1 + \frac{W_B}{W_D} q^2} \right)$$

(45)

which depends only on the unperturbed densities of the components and on the initial conditions.

In the case of a universe dominated by dark matter ($W_D/W_B \gg 1$) and for the initial conditions specified above, one can say that $\zeta_3^{\text{max}} \approx \frac{1}{4} \left( \frac{W_D}{W_B} \right)^{3/2}$.

From this expression it is clear that the height of the resonant peak increases as the parameter $q$, (in this case of the order of $W_D/W_B$), increases. This means that the more the Universe is dominated by dark matter, the more the effect of the structure formation resonance in the baryonic material will be important.

In Fig. 3 the amplitude $\zeta_3$, of the growing mode in the baryonic component, is plotted as a function of the mass scale of the perturbation, for $q = 19$, which corresponds to assume that $q = W_D/W_B = \Omega_D/\Omega_B$, using for $\Omega_B$ and $\Omega_D$ the values 0.05 and 0.95 respectively. This is to be compared with the plots in Fig. 8 where other values of the parameter $q$ were used.

The importance of the resonance can be also measured by the width of its band: a larger resonant band corresponds to a weaker resonance (Feynman 1966). We shall define the resonant band width $\gamma$ as the width of the curve of $\zeta_3$ at half the maximum of its height.

The two values of $k$ ($k_{1,2}$), which define the resonant band width $\gamma$ as $|k_1 - k_2|$, must therefore satisfy the following condition:

$$\zeta_3^2 (k_{1,2}) = \frac{1}{2} (\zeta_3^{\text{max}})^2$$

(46)

and are given by:

$$k_{1,2} = \left[ k_3^2 + \frac{2 W_D q}{d} \pm \left( \sqrt{2} - 4 \zeta_3^{\text{max}} \right) \sqrt{\frac{d}{2}} \right]^{1/2}$$

(47)

where

$$d = 4 \zeta_3^{\text{max}} \left( 2 \zeta_3^{\text{max}} - \sqrt{2} \right) (c_D^2 - c_B^2)$$

(48)

and

$$r = \left[ \frac{W_B^2 q^2}{d} + 2 \left( \sqrt{2} - 1 \right) W_B W_D + 2 \left( \sqrt{2} - 1 \right) \sqrt{W_B W_D (W_D^2 q^2 + W_B W_D)} \right]^{1/2}$$

(49)

For given sound speeds of the two components one can see that this width is very sensitive to the product $W_B W_D$. On the other hand, the further apart the sound speeds $c_D$ and $c_B$ in the two components are, the smaller the resonant band width will be, indicating a stronger and sharper resonance.

Another feature one should point out is the behaviour of $\zeta_3$ to the left and to the right of the resonant scale. Since there is no growing mode for $k > k_M$, we shall be interested in studying the behaviour of $\zeta_3$ for the limits $k \to 0$ and $k \to k_M$.

The value of $\zeta_3$ in the limit $k \to 0$, depends on the values of $W_B$ and $W_D$ and is given by:

$$\lim_{k \to 0} \zeta_3 = \frac{1}{2} \frac{q W_D + W_B}{W_D + W_B}$$

(50)

whereas the value of $\zeta_3$ for $k = k_M$ depends on both the sound speeds and on the densities of the two components, and is given by:

$$\zeta_3 (k_M) = \frac{1}{2} \frac{q + \frac{W_B}{W_D} \frac{c_D^2}{c_B^2}}{\frac{W_B}{W_D} \frac{c_D^2}{c_B^2}}$$

(51)

Assuming, as above, that $q \sim W_D/W_B$, the asymptotic value $\lim_{k \to 0} \zeta_3 \gg 1/2$, as can be seen from Eq. (50), while, if one allows $q$ to be much smaller than 1, then $\lim_{k \to 0} \zeta_3 \ll 1/2$. 
On the other hand, in the case \( W_D/W_B \ll c_D/c_B \), and for reasonable values of \( q \), the value of \( \zeta_3(k_M) \) is always of the order of \( 1/2 \).

One can therefore conclude that the \( \zeta_3 \) curve has a different behaviour on the far left and on the far right of the resonant scale. For small scales the value of \( \zeta_3 \) does not vary very much with initial conditions, while for great scales that value is very sensitive to initial conditions. The smaller the initial perturbation in the dark component when compared with the initial perturbation in the baryonic component, the smaller will be the value of the mode amplitude in the growing mode of the baryonic component. For scales larger than the resonant one, favouring, in consequence, the onset of perturbations at the resonant scale or smaller (relative to greater ones). The inverse occurs if the perturbation in the dark component increases relative to the perturbation in the baryonic component, i.e. in this case the favoured scales for the onset of perturbations are the resonant one or greater (see Fig. 3).

6. The resonance, the galactic scale and the mass of dark matter particles

We believe that the resonance described in Sect. 5 is responsible for setting up the growth, at the recombination epoch, of massive structures at the galactic scale. The resonance therefore provides the mechanism to select out this particular scale, among all possible scales which could undergo the gravitational collapse. In other words this resonance provides an explanation why do galaxies, with the typical masses one observes, form around the recombination epoch, even if the Jeans mass for baryons, and even the Jeans mass for the mixture, either before or just after this epoch, is either much larger or smaller than typical galactic masses.

The typical galactic masses should, according to this work, correspond to the mass of the perturbations, which, at the recombination epoch, would have a scale near the above mentioned resonant scale. One would therefore have no need to assume that the mass spectrum of perturbations at that time was already populated in a way similar to the one we observe now. Instead, one can say that the typical galactic mass scale would simply result from the scale selected by this resonance.

Of course we need a better and deeper understanding of the physics of the resonant mechanism and how it can affect the power spectrum of the density perturbations, not only at recombination epoch but also in the period following it. Therefore one has to go from separate Fourier amplitudes to quantities integrated over the wavelength scale. Since one is interested in knowing how the spectrum of inhomogeneities will transform after recombination, one must follow the dynamics of several Fourier components of the density perturbations from recombination until the epoch when the perturbations become non linear.

Note that in the matter dominated period, before recombination, no resonance can exist. This is because, due to the coupling between baryonic matter and radiation in that period, the component with greater sound speed is the baryonic one. On the other hand, in a flat universe like the one we have assumed, the baryons are not gravitationally dominant. Therefore one can see from Eq. (38) that the resonance cannot occur in that period.

Thus, in order to have a detailed understanding of galaxy formation one only needs to study the dynamics of the Fourier components after recombination epoch. Before recombination one can assume that the power spectrum was any reasonable one predicted by another model, like, for instance, some type of inflationary model.

Since this is a very important point which must be carefully analysed. Since it lies out of the main goal of this paper, we intend to treat it thoroughly in a separate paper.

One can relate the scale of the resonance with the mass of the dark matter particle. The scale at which the resonance peak occurs depends on \( \zeta_3 \), see Eq. (11), which in turn is a function of the value of the mass of dark matter particles \( m_D \). From Eq. (38) we have already concluded that in order for the resonance to occur, the dark matter must necessarily be hot.

The total mass \( M \) inside a spherical region of radius \( \ell/2 = \pi/k \) is given by

\[
M = \frac{4}{3} \pi^4 \frac{\rho}{k^3},
\]

where \( k \) is the wave number and \( \rho = \rho_B + \rho_D \) is the total mean density in that region.

In order to obtain bounds for the mass of dark matter particles we shall assume that galaxies form by means of the above described resonance and therefore have mass scales which correspond to the resonant scale for \( \zeta_3 \). The range of galactic masses which one observes in the universe will therefore imply a specific range for the mass of hot dark matter particles.

Substituting in Eq. (12) the value of the wave number \( k \) by its resonant value \( k_{\text{res}}(\zeta_3^{\text{max}}) \), given by Eq. (11), one can establish the following relation between \( M \) (in solar masses) and \( m_D \) (in eV):

\[
m_D \simeq 210 \pi^{5/6} \frac{\rho^{1/3}}{\rho_B^{1/2} M^{1/3}},
\]

In deriving this approximate equation \( \text{2} \), we have assumed that \( W_D/W_B \gg W_B \) and that \( c_D \gg c_B \). The sound speeds \( c_j \) at recombination, appearing in Eq. (11), are given by Eq. (10).

---

2 Although an exact, but more complicated, expression can be derived for the mass of dark matter particles, the use of these assumptions, which are valid for the case of HDM, allows us to derive a simple but accurate expression for \( m_D \).
One should emphasize that from the decoupling, between the neutrino-like particles and radiation, until the epoch at which these particles become non-relativistic, which for \( m_\nu \sim 30\,\text{eV} \) occurs at a red-shift \( z_{\nu R} \sim 6 \times 10^4 \) (Doroshkevich et al. 1988), their temperature \( T \) is given by: \( T_D = \left(4/11\right)^{1/3} T_7 \), where \( T_7 \) is the radiation temperature. Since then, it decays with \( a^{-2} \), where \( a \) is the scale factor of the universe (Gao & Ruffini 1981; Padmanabhan 1993). One can therefore estimate that, at recombination, such a dark component \( D \) will have a temperature \( T_{D, rec} \sim 40\,\text{K} \).

In Fig. 8 is plotted the scale \( M \) at which the resonant peak occurs (which we have assumed to be the galactic total mass) versus the mass of hot dark matter particles \( m_D \).

Assuming that a typical galaxy formed at recombination epoch has a baryonic mass of a few \( 10^{11} M_\odot \), one can estimate from the plot in Fig. 8 that the mass of the dark matter particles is of the order of 30 eV. It is remarkable that this value agrees quite well with the cosmological upper bounds for neutrino’s masses (Peebles 1993) as well as the values predicted by Sciama’s Decaying Dark Matter Hypothesis (Sciama 1990a; Sciama 1990b).

7. Conclusions

It is clear from the results obtained in this work that it is incorrect to use the Jeans criterion for a 1-component fluid to a 2-component mixture, for two main reasons:

1. The Jeans scale for the mixture does not correspond to any of the Jeans scales for the components taken separately.
2. Moreover, in a 2-component mixture there are always present, at all scales, oscillations in the two components, which correspond to acoustic waves. The same is not true for a one component system.

However the main result of this work is, as we have shown, that, in a two-component fluid, a resonance between the two components of the fluid must always occur provided that the component with a greater density is the one with a greater sound speed, see Eq. (58). One can therefore put some constraints on the nature of the dark matter if galaxies are to be formed by such a resonant effect.

In fact, assuming that the density parameter of the universe is \( \Omega = 1 \) and adopting a value for the density of the baryonic component in agreement with standard light-element nucleosynthesis, one is led to the conclusion that the universe is dominated by dark matter, i.e. \( W_D > W_B \). In such a case, if \( c_D > c_B \) (case of CDM), one concludes from Eq. (58) that no resonance between the two components is possible and therefore, the galactic scale would not be a preferred one, as it seems to be. One is therefore led to believe that most, if not all, of the dark matter present in the universe is hot, i.e. it is constituted by light neutrino-like particles as implied by the condition for the existence of a strong resonance, i.e. \( c_D \gg c_B \), (see Eq. (77) and comments following it).

This resonance is linked to the fact that at the critical scale \( l_c \), there is an enhancement of the number of particles in both components, (but particularly in the baryonic one), which participate in the collapse mode, instead of participating in the acoustic modes as it happens for other scales. This effect is only significant in a narrow resonant band in the length, or mass, scale of the perturbations. This, we believe, is the reason for the formation of galactic structures at the time of recombination with their characteristic range of mass scales.

This effect occurs at the galactic scales provided that the dark matter component is made of neutrino-like particles with masses of the order of 30 eV. This value for the mass of the neutrino-like particles is below the upper bounds imposed by cosmological constraints on the neutrino’s mass (Peebles 1993), assuming a flat universe, and remarkably close to the value predicted by Sciama’s Decaying Dark Matter Hypothesis (Sciama 1990a; Sciama 1990b). We must also note that, from the numerical results obtained in the preceding sections, this mass scale is of the order of \( 10^{13} M_\odot \) which gives a baryonic mass, in the same scalelength, of the order of \( \frac{\rho_B}{\rho_D} 10^{13} M_\odot \), which lies in the range of the masses of typical galaxies (a few \( 10^{11} M_\odot \)).

One should point out, however, that the resonant scale \( l_c \) is particularly sensitive to the mass of the particles constituting the dark component, see Fig. 8.

8. Future work

This study has left a lot of open questions, some of which are already under investigation. Below we briefly comment on some of them.

First: One should point out that this study assumes galaxies to have started their formation during recombination. However, this can well be not the case and they could have started their formation over a wider time interval starting from recombination. In that case, the unperturbed densities \( \rho_j \) and the sound speeds \( c_j \) in the two components would vary during such a period, allowing the scale of the resonance to be shifted. This might in turn provide an explanation for the observed wide mass range for galactic structures. This is a problem which we intend to address in a future paper (in preparation), where the effects due to the expansion of the universe are taken into account.

Second: Although the analysis of possible effects of this resonant mechanism on the power spectrum is out of the main purpose of this paper, the existence of the above mentioned resonance, or at least the change in behaviour of the mode amplitudes near the resonant scale is, on its own, a very important result. The occurrence of this resonance can have, in principle, important implications in the
process of galaxy formation, and, in particular, transform the power spectrum of density perturbations at recombination epochs. Therefore, though a better understanding of the physics underlying the resonant mechanism is needed, this point shall be thoroughly treated in a forthcoming paper of ours.

Third: One must note that it will be interesting to study what happens for the case of a 3-component system like the dark matter mixed models. We suspect that in such a case more than one structure formation resonance occurs. This is a question which we intend to address in the future.

Finally, one should also mention that although resonances between periodic coupled oscillators are common in all branches of physics, this type of resonance occurring in (non periodic) collapsing modes are, to our knowledge, yet unheard of, and we suspect that in other branches of physics such as in plasmas (where one has a 2-component fluid whose components are coupled by the electromagnetic interaction) similar resonances may perhaps occur, giving rise to instabilities. We think that such an open question is also worth investigating.

Acknowledgements. We would like to thank the referee (Dr. M. Tagger) for his helpful criticism which strongly helped us to clarify some important results presented in this work.

We also thank JNICT for financial support through its STRIDE projects program, and Centro de Astrofísica da Universidade do Porto (CAUP) for the facilities provided.

References

Arnold V. I., 1973, Ordinary Differential Equations, MIT Press, Cambridge MA
Doroshkevich A.G., Klypin A.A., Khlopov M.Yu 1988, SVA, 32(2), 127
Feynman R.P., Leighton R.B., Sands M., 1966, The Feynman Lectures on Physics, Vol.I, Addison-Wesley Pub. Comp., Massachusetts
Gao J. G., Ruffini R. 1981, Phys. Lett., 100B, 47
Grishchuk L. P., Zel'dovich Ya. B. 1981, Sov. Astron., 25(3)
Jeans J. 1902, Phil. Trans. R. Soc., 199A, 49
Jeans J. 1928, Astronomy and Cosmogony, Cambridge University Press, Cambridge
Kolb E., Turner M. 1990, Early Universe, Addison-Wesley, Massachusetts
Padmanabhan T. 1993, Structure Formation in the Universe, Cambridge University Press, Cambridge
Peebles P. J. E., 1993, Principles of Physical Cosmology, Princeton Series in Physics, Princeton University Press, Princeton
Polyachenko V. L., Fridman A. M. 1981, Sov. Phys. JETP, 54(1)
Sciana D. W. 1990a, MNRAS 246, 191
Sciana D. W. 1990b, Nat 348, 617
Walker T. P., Steigman G., Schramm D. N., Olive K. A., Kang H., 1991, Ap.J., 376, 51
Weinberg S. 1972, Gravitation and Cosmology, J. Wiley, New York
Zel'dovich Ya. B., Novikov I. D. 1983, Relativistic Astrophysics (Vol. 2), The University of Chicago Press, Chicago
Fig. 1. Dependence on scale (in Solar Masses), of the eigenvalue $\lambda_1$ and imaginary part of $\lambda_3$. Note the two relevant mass scales for $\lambda_1$ and only one for $\lambda_3$ which coincides with the second one for $\lambda_1$. The vertical arrow indicates the total mass $M_D$ inside a spherical region of diameter $l_D$.

Fig. 2. Typical projected phase-space trajectories, representing the general solution of the dynamical system, on the plane $(x_4, x_2)$. Also plotted are the projections of center and saddle subspaces. The left hand orbit corresponds to the situation $\Delta_p < \beta_3 \Delta_{p_0}$ and the right hand one to the inverse initial condition. (This particular plot corresponds to a mass scale of the perturbation of $\sim 10^{13} M_\odot$)

Fig. 3. Projection on the $(x_4, x_2)$ plane of the three subspaces described in text for $l > l_M$. The long-dashed lines represent the projection of the saddle subspace, in the limits $l = l_M$ and $l \to \infty$. The short-dashed ones are the projections of the center subspace for the same limits.

Fig. 4. Scale dependence of the angle between the projection of the saddle and center subspaces with the $x_4$-axis, respectively $\theta_1$ and $\theta_3$. The vertical arrow indicates the total mass $M_D$ inside a spherical region of diameter $l_D$. Note the very peculiar behaviour around $M_D$.

Fig. 5. Coefficient $\zeta_3$ and its dependence on scale. The vertical arrow indicates the total mass $M_c$ inside a spherical region of diameter $l_c$. Note the clear change in behaviour of this coefficient around this scale.

Fig. 6. Coefficient $\zeta_4$ and its dependence on scale. See Eq. (39) and the discussion following it. The vertical arrow indicates the total mass $M_c$ inside a spherical region of diameter $l_c$.

Fig. 7. Coefficients $\zeta_1$ and $\zeta_2$ and their dependence on scale. Note that the behaviour of these mode amplitudes, although not similar to the one of $\zeta_3$ and $\zeta_4$, is also peculiar for a scale around $l_c$. The vertical arrow indicates the total mass $M_c$ inside a spherical region of diameter $l_c$.

Fig. 8. Dependence on scale (in Solar Masses), of the growing mode amplitude $\zeta_3$, for distinct initial conditions described by different values of the parameter $q$. One can notice that for $q > 1$ the values of $\zeta_3$ to the right of the resonant scale are higher than their values to the left of it. The opposite occurs for $q < 1$.

Fig. 9. Dependence on scale of $M_c$ (in Solar Masses), at which the resonant peak in the amplitude $\zeta_3$ occurs, plotted as a function of the mass $m_D$ of the particles which constitute hot dark matter.

This article was processed by the author using Springer-Verlag \LaTeX{} A&A style file L-\AA A version 3.
Center Subspace
Formation of Voids

Saddle Subspace
Formation of Massive Structures

$\Delta D$

$\Delta B$

$10^{-3}$
Δ → l = l → ∞

Saddle Subspace

Center Subspace

l → ∞
