Any smooth knot $S^n \hookrightarrow \mathbb{R}^{n+2}$ is isotopic to a cubic knot contained in the canonical scaffolding of $\mathbb{R}^{n+2}$.

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Abstract

The $n$-skeleton of the canonical cubulation $\mathcal{C}$ of $\mathbb{R}^{n+2}$ into unit cubes is called the canonical scaffolding $\mathcal{S}$. In this paper, we prove that any smooth, compact, closed, $n$-dimensional submanifold of $\mathbb{R}^{n+2}$ with trivial normal bundle can be continuously isotoped by an ambient isotopy to a cubic submanifold contained in $\mathcal{S}$. In particular, any smooth knot $S^n \hookrightarrow \mathbb{R}^{n+2}$ can be continuously isotoped to a knot contained in $\mathcal{S}$.

1 Introduction

In this paper we consider smooth higher dimensional knots, that is, spheres $S^n$ smoothly embedded in $\mathbb{R}^{n+2}$. In $\mathbb{R}^{n+2}$ we have the canonical cubulation $\mathcal{C}$ by translates of the unit $(n+2)$-dimensional cube. We will call the $n$-skeleton $\mathcal{S}$ of this cubulation the canonical scaffolding of $\mathbb{R}^{n+2}$ (see section 2 for precise definitions). We consider the question of whether it is possible to continuously deform the smooth knot by an ambient isotopy so that the deformed knot is contained in the scaffolding. In particular, a positive answer to this question implies that knots can be embedded as cubic sub-complexes of $\mathbb{R}^{n+2}$, which in turn implies the well-known fact that smooth knots can...
be triangulated by a PL triangulation ([2]). The problem of embedding an abstract cubic complex into some skeleton of the canonical cubulation can be traced back to S.P. Novikov. A considerable amount of work has been done regarding this problem (see, for example [1]). The question is non-trivial; for instance, among cubic manifolds there are non-combinatorial ones and therefore non-smoothable ones. Also there is a series of very interesting papers by Louis Funar regarding cubulations of manifolds ([7], [8]). The possibility of considering a knot as a cubic submanifold contained in the $n$-skeleton of the canonical cubulation of $\mathbb{R}^{n+2}$ has many advantages. For instance, in the important case of classical knots $n = 1$, Matveev and Polyak [11] begin the exposition of finite type invariants from the “cubic” point of view and show how one can clearly describe invariants such as polynomial invariants, Vassiliev-Goussarov invariants and finite type invariants of three-dimensional integer homology spheres (in this regard see also the unpublished important paper by Fenn, Rourke and Sanderson [6]). Cubic complexes may play a role in extending these invariants to higher dimensional knots.

In this paper we prove that any smooth, compact, closed, $n$-dimensional submanifold of $\mathbb{R}^{n+2}$ with trivial normal bundle can be continuously isotoped by an ambient isotopy of $\mathbb{R}^{n+2}$ onto a cubic submanifold contained in $S$. In particular, any knot can be isotoped onto a cubic knot contained in $S$.

2 Cubulations for $\mathbb{R}^{n+2}$

A cubulation of $\mathbb{R}^{n+2}$ is a decomposition of $\mathbb{R}^{n+2}$ into a collection $\mathcal{C}$ of $(n+2)$-dimensional cubes such that any two of its hypercubes are either disjoint or meet in one common face of some dimension. This provides $\mathbb{R}^{n+2}$ with the structure of a cubic complex.

In general, the category of cubic complexes and cubic maps is similar to the simplicial category. The only difference consists in considering cubes of different dimensions instead of simplexes. In this context, a cubulation of a manifold is specified by a cubical complex PL homeomorphic to the manifold (see [1], [7], [11]).

The canonical cubulation $\mathcal{C}$ of $\mathbb{R}^{n+2}$ is the decomposition into hypercubes which are the images of the unit cube $I^{n+2} = \{(x_1, \ldots, x_{n+2}) \mid 0 \leq x_i \leq 1\}$ by translations by vectors with integer coefficients.
Consider the homothetic transformation $h_m : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}$ given by $h_m(x) = \frac{1}{m}x$, where $m > 1$ is an integer. The set $h_m(C)$ is called a subcubulation or cubical subdivision of $C$.

**Definition 2.1** The $n$-skeleton of $C$, denoted by $S$, consists of the union of the $n$-skeleta of the cubes in $C$, i.e., the union of all cubes of dimension $n$ contained in the faces of the $(n+2)$-cubes in $C$. We will call $S$ the canonical scaffolding of $\mathbb{R}^{n+2}$.

Any cubulation of $\mathbb{R}^{n+2}$ is obtained by applying a conformal transformation $x \mapsto \lambda A(x) + a$, $\lambda \neq 0$, $a \in \mathbb{R}^{n+2}$, $A \in SO(n+2)$ to the canonical cubulation.

In this section, we will prove that the union of the cubes of the canonical cubulation which intersect a fixed hyperplane is a closed tubular neighborhood (i.e., a bicollar). Hence the boundary of the union of these cubes has two connected components and therefore the hyperplane can be isotoped to any of them. This isotopy can be realized using normal segments to the hyperplane. The boundaries of this bicollar are contained in the $(n+1)$-skeleton of the canonical cubulation. If $P$ is a hyperplane of $\mathbb{R}^{n+2}$ and $\Pi \subset P$ is an $n$-dimensional affine subspace we use the same sort of ideas to show that the couple $(P, \Pi)$ can be isotopically deformed to the boundary of the bicollar above in such a way that $\Pi$ is deformed into the $n$-skeleton of the canonical cubulation.

These linear cases already contain the main ingredients of our proofs which in particular include convexity properties.

Later we will generalize these ideas to the case of any smooth, compact, closed codimension one submanifold $M$ of $\mathbb{R}^{n+2}$: the union of the cubes of a small cubulation (i.e., with cubes of sufficiently small diameter) which intersect $M$ is a bicollar neighborhood of $M$. We can deform $M$ to any of the boundary components using an adapted flow which is nonsingular in the tubular neighborhood and is transverse to $M$. Finally using the same type of ideas we prove at the end of the section that any smooth codimension two submanifold $N$ can be deformed into the $n$-skeleton of a sufficiently small cubulation.

**Proposition 2.2** Let $P$ be a hyperplane in $\mathbb{R}^{n+2}$. Let $C$ be the canonical cubulation of $\mathbb{R}^{n+2}$ and $Q_P = \bigcup\{Q \in C : Q \cap P \neq \emptyset\}$. Then:
1. \( P \) is contained in the interior of \( Q \).

2. Let \( Q \in C \) such that the distance \( d(Q, P) = m > 0 \). Then the set \( M := \{ q \in Q \mid d(q, P) = m \} \) is a cubic simplex, of some dimension, of the boundary of \( Q \).

3. \( \partial Q \) is the union of \((n+1)\)-dimensional cubes which are faces of cubes in \( Q \).

Proof.

1. Let \( p \in \mathbb{R}^{n+2} \), \( n \geq 1 \) and \( C \) be the canonical cubulation of \( \mathbb{R}^{n+2} \). Let us consider the set \( C_p = \bigcup \{ Q \in C : p \in Q \} \). Let us first show by induction that \( p \in Int(C_p) \). If \( n = 1 \) the result is obvious. Let \( n > 1 \). If \( p \in Int(Q) \) for some cube \( Q \) then the result follows immediately. If \( p \in \partial Q \) for some cube \( Q \) then \( p \) belongs to at least one \((n+1)\)-face \( F \) of \( Q \). The cubulation \( C \) induces a cubulation \( C_{n+1} \) in the hyperplane which contains the face \( F \). By induction hypothesis \( p \) is in the interior (relative to the hyperplane) of the union of the \((n+1)\)-cubes of \( C_{n+1} \) which contain \( p \). Then \( p \) is in the interior of the union of these cubes. Therefore, by induction, \( p \in Int(C_p) \). To prove 1 it is sufficient to observe that, by the preceding argument, \( p \in Int(C_p) \), since \( C_p \subset Q \).

2. Let \( P_m \) be the hyperplane parallel to \( P \) at distance \( m \) of \( P \) which intersects \( Q \). Then \( P_m \) is a support plane of \( Q \) and therefore there exists a linear functional \( \alpha : \mathbb{R}^{n+2} \rightarrow \mathbb{R} \) such that \( P_m = \alpha^{-1}(\{m\}) \) and \( \alpha(p) \geq m \) for all \( p \in Q \). Since \( Q \) is a convex polytope, it follows from standard facts of the geometry of convex sets and linear programming that \( Q \cap P_m \) is a face of \( Q \) (cubic simplex) of some dimension since it is the set where the linear function \( \alpha \), restricted to \( Q \), achieves its minimum.

3. We have that \( \partial Q \subset \bigcup_{Q \subset Q_P} \partial Q \). Therefore \( \partial Q \) is contained in a union of \((n+1)\)-faces. Each face \( F \) of a cube in \( C \) is a face of exactly two cubes of \( C \). Furthermore \( \partial Q_P \) consists of faces \( F \) of \((n+2)\)-cubes in \( Q_P \) with the property that \( F \) is also the face of a cube not belonging to \( Q_P \). This is true since such type of faces belong to \( \partial Q_P \) and every point in \( \partial Q_P \) is contained in one of those faces by the proof of 1 in proposition 2.2. ■

**Lemma 2.3** Let \( P \) be a hyperplane in \( \mathbb{R}^{n+2} \). Let \( C \) be the canonical cubulation of \( \mathbb{R}^{n+2} \) and \( Q_P = \bigcup \{ Q \in C : Q \cap P \neq \emptyset \} \). Let \( k \) be a point in \( P \).
and $L_k$ the normal line to $P$ at $k$. Then $J_k = L_k \cap Q_P$ is connected. By proposition 2.2, $P$ is contained in the interior of $Q_P$ and therefore $J_k$ is a non-trivial compact interval.

Proof. $P$ divides $\mathbb{R}^{n+2}$ in two open connected components $H^+$ and $H^-$. Let us consider one of these components, for instance $H^+$. Let $Q \in C$ and suppose that $Q \subset H^+$. Then $Q \cap P = \emptyset$ and therefore $d(Q, P) = m > 0$. By proposition 2.2 the set $M := \{q \in Q \mid d(q, P) = m\}$ is a cubic simplex, of some dimension contained in the boundary of $Q$. For each $(n+1)$-hyperface $F$ of $Q$ which intersects $M$, there is a hyperplane $P_F$ which supports $F$. Only one of the closed halfspaces determined by $P_F$ contains $Q$. We define $S(Q)$ as the intersection of all such closed halfspaces (see Figure 1).

![Figure 1: The convex set $S(Q)$.](image)

The set $S(Q)$ is a convex unbounded set which contains $Q$ and it is a union of cubes of $C$. We will show that $P_m$ is a support hyperplane of $S(Q)$ and $P_m \cap S(Q) = M$.

First we will prove:

**Claim 1.** $S(Q) \cap P = \emptyset$.

**Proof of Claim 1.** If $M$ is a $(n + 1)$-dimensional face of $Q$, the result follows easily. If $\dim(M) < n + 1$, then $M$ is contained in the intersection of two hyperplanes $P_{F_1}$ and $P_{F_2}$. One has that $P_{F_1} \cap P_{F_2} = P_{F_1} \cap P_m$ is of dimension $n$, and therefore divides $P_{F_1}$ into two components, only one of which contains $Q \cap P_{F_1}$. Therefore $S(Q)$ lies in the halfspace determined by $P_m$ which contains $Q$ and this halfspace does not contain $P$. This proves claim 1.

To continue with the proof let us describe $P$, $P_m$ and $S(Q)$ in terms of
linear equalities and linear inequalities.

Let us consider the set of hyperplanes $P_{F_i} (i = 1, \ldots, k)$ defining $S(Q)$ which are not perpendicular to $P$. Then the intersection of the halfspaces corresponding to these hyperplanes and containing $Q$, is defined by the set of points $x \in \mathbb{R}^{n+2}$ which satisfy the following set of inequalities:

\[
\langle x, n_1 \rangle \geq a_1 \\
\langle x, n_2 \rangle \geq a_2 \\
\vdots \\
\langle x, n_k \rangle \geq a_k,
\]

where $\langle \cdot, \cdot \rangle$ is the standard inner product and $\langle n_i, n_j \rangle = \delta_{ij}$ and $a_i \in \mathbb{R}$. If none of the hyperplanes defining $S(Q)$ is orthogonal to $P$ then $k = n + 2$.

**Definition:** The vector $n_i$ is called the *exterior normal vector* to the face $F_i \subset Q$ contained in the hyperplane $P_{F_i}$.

If a hyperplane $P_{F_j}$ is perpendicular to $P$ the hyperplane corresponding to the opposite face to $F_j$ is also perpendicular to $P$. Let $H_{F_j}$ be the closed halfspace determined by $P_{F_j}$ which contains $Q$.

**Obvious Remark.** If $L$ is a line which is perpendicular to $P$ then either $L \cap H_{F_j} = \emptyset$ or $L \cap H_{F_j} = L$.

Using a translation, if necessary, we can assume without loss of generality that the hyperplane $P$ is given by the linear equation:

\[ P = \{ x \in \mathbb{R}^{n+2} \mid \langle x, n \rangle = 0 \} \text{ where } |n| = 1, \]

where $n$ is chosen in such a way that $H^+ = \{ x \in \mathbb{R}^{n+2} \mid \langle x, n \rangle > 0 \}$ (see Figure 2).

The hyperplane $P_m$ divides $\mathbb{R}^{n+2}$ into two closed halfspaces, one of which contains $Q$ and the other contains $P$.

Then the hyperplane $P_m$ is given by the equation:

\[ P_m = \{ x \in \mathbb{R}^{n+2} \mid \langle x, n \rangle = m \}, \]  

and by hypothesis we have that $\langle n, n_i \rangle > 0$, $i = 1, \ldots, k$. 

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For each \( p \in \mathbb{R}^{n+2} \), let \( L_p \) be the normal line to \( P \) which contains \( p \). Then the line \( L_p \) can be parametrized by the function \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^{n+2} \) defined as follows:

\[
t \mapsto p + tn
\]  

(4)

**Claim 2.** \( L_p \cap S(Q) \) is either empty (it happens only if \( Q \) has a hyperface, of dimension \( n + 1 \), which is contained in a hyperplane perpendicular to \( P \)) or \( L_p \cap S(Q) \) is a ray i.e., a noncompact closed interval contained in \( L_p \).

**Proof of Claim 2.** By the obvious remark above we need to show that if \( L_p \cap S(Q) \neq \emptyset \) then \( L_p \cap S(Q) = \{ \gamma(t) \mid t \geq c \} \) where \( c \geq m > 0 \). But this fact is immediate since, by (1) and (4), it follows that if \( \gamma(t) \in L_p \cap S(Q) \) then since \( \langle n, n_i \rangle > 0 \), \( i = 1, \ldots, k \) we have that \( t \geq \frac{a_{ij} - \langle p, n_i \rangle}{\langle n, n_j \rangle} = c_j \).

We can choose \( c = \max\{c_j\} \). Since \( S(Q) \) is contained in the halfspace determined by \( P_m \) that contains \( Q \) we must have \( c \geq m \). This finishes the proof of claim 2.

Claim 2 implies that for each \( q \in Q \), \( L_q \cap S(Q) \) is a ray. Let \( T(Q) \) be the union of all such rays. By the above we have the following

**Corollary of Claim 2.** If \( Q' \in \mathcal{C} \) is a cube which intersects \( \text{Int}(T(Q)) \) then \( Q' \cap P = \emptyset \).

Now we are able to finish the proof of lemma 2.3. For \( k \in P \), let \( J_k^+ = L_k \cap H^+ \) with the order induced by the distance to \( P \). Let \( x \) be the first point on \( J_k^+ \) such that \( x \in J_k^+ \cap \partial Q_P \). By proposition 2.2 the boundary \( \partial Q_P \) is the union of \( (n + 1) \)-dimensional cubes, each of which is a hyperface of exactly two hypercubes in \( \mathcal{C} \): one intersecting \( P \) and one completely contained in \( H^+ \). Let \( C \) be the cube containing \( x \) which does not intersect \( P \). Then, by the above corollary, for all \( z > x \), all cubes containing
z do not intersect $P$. Hence, $x$ is the only point on $J_k^+$ which lies on $\partial Q_P$. Therefore $J_k^+ \cap Q_P$ is connected. ■

Lemma 2.3 implies the following corollary:

**Corollary 2.4** The lengths of $J_k \subset L_k$ and $J_k^+ \subset L_k$ vary continuously with $k$. These intervals have a natural order which provides them with a continuous orientation. We denote $J_k$ and $J_k^+$ by $[a(k), b(k)]$ and $[k, b(k)]$, respectively. Furthermore, the functions $\psi^+(k) := b(k)$ and $\psi^-(k) := a(k)$ are homeomorphisms from $P$ to $\partial Q_P \cap H^+$ and $\partial Q_P \cap H^-$, respectively.

As a consequence $Q_P$ is a closed tubular neighborhood (or bi-collar neighborhood) of $P$.

**Proof.** By lemma 2.3, $J_k^+ \cap \partial Q_P$ is connected. We will prove that $J_k^+ \cap \partial Q_P$ is in fact one point. Since the cubes in the cubulation $C$ are right-angled, if $F$ is a face of a cube orthogonal to $P$ then $P$ intersects $F$. This means that $F$ can not be contained in the boundary of $Q_P$. Therefore $J_k^+$, which is orthogonal to $P$, must intersect each $(n+1)$-cube in $\partial Q_P$ transversally.

We have proved that $J_k$ is connected and in particular, $J_k \cap \partial Q_P$ consists of two points $a(k)$ and $b(k)$. Let $b(k)$ be the corresponding point in $H^+$. We define the map $\psi^+ : P \to \partial Q_P \cap H^+$ such that $x \in P$ is sent to $b(x)$. Without loss of generality we can assume that $P = \{x = (x_1, \ldots, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2} \mid x_{n+2} = 0\}$. Thus can identify $P$ with $\mathbb{R}^{n+1}$. In terms of this identification we have $\psi^+(k) = (k, h^+(k))$, $k \in \mathbb{R}^{n+1}$, and $h^+(k) = d(k, \psi^+(k)) = d(k, b(k))$. Consequently $H^+ \cap \partial Q_P$ can be thought of as the graph of the function $h^+$ and since this graph is closed, it follows that $h^+$ is continuous. This implies that the length of $J_k^+$ varies continuously. Hence $\psi^+$ is a homeomorphism. Obviously all the preceding arguments remain valid if we had used $H^-$, $\psi^-$ and $J_-$. ■

**Corollary 2.5** Since $Q_P$ is a bi-collared neighborhood of $P$ we obtain that $P$ can be continuously isotoped onto any of the two connected components of the boundary $\partial Q_P$. This can be done by an ambient isotopy of $\mathbb{R}^{n+2}$ [3], [13].

We will use the following lemma to prove our main theorem.
LEMMA 2.6 Let $P_1$ and $P_2$ be two orthogonal codimension one hyperplanes in $\mathbb{R}^{n+2}$. Then $P_1 \cap P_2$ can be cubulated; more precisely, there is an ambient isotopy which takes $P_1 \cap P_2$ onto a cubic complex, contained in the scaffolding $S$ of the canonical cubulation $C$ of $\mathbb{R}^{n+2}$.

Proof. Let $Q_1$ be the cubic complex formed by the union of elements of $C$ intersecting $P_1$. By the above corollary, $Q_1$ is a tubular neighborhood (or bi-collar in this case) of $P_1$. Let $E$ be a connected component of the boundary of $Q_1$. Recall the isotopy $\psi = \psi^+$ constructed in corollary 2.4 from $P_1$ to $E$, which assigns to each point $x$ in $P_1$ the unique point on $E$ lying on the line normal to $P_1$ at $x$. The inverse of $\psi$ in this case is in fact the canonical projection $\pi_1$ of $\mathbb{R}^{n+2}$ onto $P_1$.

Since $P_1$ and $P_2$ are orthogonal, if $x$ is a point in $P_1 \cap P_2$ then the normal line at $x$ to $P_1$ will lie in $P_2$. Therefore, $\psi$ restricted to $P_1 \cap P_2$ is a homeomorphism which can be extended to an ambient isotopy between $P_1 \cap P_2$ and $E \cap P_2$. Now by proposition 2.2 $E$ is a cubic complex of dimension $n+1$. We will repeat the idea of lemma 2.3 for $P_2 \cap E$. Take the union $B$ of all $(n+1)$-dimensional cubes in $E$ that intersect $P_2 \cap E$. We will prove that $B$ is a bi-collar of $P_2 \cap E$ in $E$ and therefore one of its boundary components is ambient isotopic to $P_1 \cap P_2$. This boundary component is obviously contained in the $n$-skeleton. This would prove the lemma.

The intersection $P_1 \cap P_2$ is an $n$-dimensional hyperplane, and hence of codimension one in $P_1$. Consider the foliation of $P_1$ given by lines orthogonal to $P_1 \cap P_2$. Notice that the image of each of these lines under $\psi$ is a polygonal curve $\gamma$ in $E$, intersecting $E \cap P_2$ in one unique point.

As mentioned above, $E$ is cubulated as a union of $(n+1)$-faces of cubes of $C$. The projection $\pi_1$ of this $(n+1)$-dimensional cubulation gives a decomposition $\mathcal{P}$ of $P_1$ into parallelepipeds. Observe that, in general, this decomposition is not a cubulation. We will prove that a result analogous to the lemma 2.3 holds for $\mathcal{P}$. By this we mean the following: Let $Q$ be a parallelepiped in $\mathcal{P}$ which does not intersect $P_1 \cap P_2$, $x$ be a point in $Q$ and $l_x$ be the set of points $y$ on the line in $P_1$ normal to $P_1 \cap P_2$ which passes through $x$ such that $x < y$, with the order induced by the distance to $P_2$. We will prove that any $Q' \in \mathcal{P}$ which intersects $l_x$ does not intersect $P_1 \cap P_2$.

The parallelepiped $Q$ is the image under $\pi_1$ of an $(n+1)$-cube in $E$. This $(n+1)$-cube is a face of two $(n+2)$-cubes in $\mathbb{R}^{n+2}$, at least one of
which does not intersect $P_2$. Let $C$ be the $(n+2)$-cube not intersecting $P_2$. Then, the set $S(C)$ defined in the proof of the lemma 2.3 does not intersect $P_2$. Thus $\pi_1(S(C))$ does not intersect $P_1 \cap P_2$. Moreover, $l_x$ is contained in $\pi_1(S(C))$ and, any parallelepiped in $\mathcal{P}$ which intersects the interior of $\pi_1(S(C))$ is contained in $\pi_1(S(C))$, and therefore does not intersect $P_1 \cap P_2$. Hence, no cube in $\psi(\pi_1(S(C)))$ intersects $E \cap P_2$. That is, no $(n+1)$-cube in $E$ which intersects the polygonal ray $\psi(l_x)$ intersects $E \cap P_2$.

Proceeding as in corollaries 2.4 and 2.5, choosing one of these points for each $x \in E \cap P_2$ yields a continuous function from $E \cap P_2$ to the boundary of $B$. Hence, $B$ is a bi-collar of $E \cap P_2$ in $E$ and $E \cap P_2$ can be deformed by an ambient isotopy into one boundary component of $B$. Observe that this isotopic copy of $E \cap P_2$ is contained in the $n$-dimensional skeleton of the cubulation $C$. ■

**Remark 2.7** In lemmas 2.3 and 2.6 we only need to consider the subset of those cubes of $C$ whose distance to the corresponding hyperplanes $P$, $P_1$ and $P_2$ is sufficiently small, for instance, less or equal than $4\sqrt{n+2}$, i.e., the cubes between two parallel hyperplanes at distance $8\sqrt{n+2}$. The number $\sqrt{n+2}$ appears because it is the diameter of the unit cubes in $R^{n+2}$.

A modification of the methods of the preceding results for hyperplanes can extend lemma 2.3 to the more general case given by following theorem:

**Theorem 2.8** Let $M^{n+1} \subset R^{n+2}$ be a smooth, compact and closed manifold. Let $V(M)$ be a closed tubular neighborhood of $M$. We can assume that $V$ is the union of linear segments of equal length $c > 0$, centred at points of $M$, and orthogonal to $M$. Let $C$ be the canonical cubulation of $R^{n+2}$. For $m \in N$, let $C_m$ denote the corresponding subcubulation.

Let $Q_M = \cup \{Q \in C_m : Q \cap M \neq \emptyset \}$. Then, we can choose $m$ big enough such that $Q_M$ is a closed tubular neighborhood of $M$ and $M$ can be deformed by an ambient isotopy onto any of the two boundary components of this tubular neighborhood.

*Proof.* The idea of the proof is to “blow up” the manifold and its tubular neighborhood by a homothetic transformation so that inside balls of large (but fixed) radius the manifold is almost flat and the normal segments are almost parallel. Then, locally, the manifold is approximately a hyperplane and we can apply a modification of the methods of the previous lemmas and then we rescale back to the original size by the inverse homothetic transformation. The modification consists in replacing the normal segments to a
hyperplane by segments of the flowlines of a nonsingular vector field which is very close to these normal segments in the tubular neighborhood of $M$.

More precisely, let $\psi_m : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ be the homothetic transformation $\psi_m(x) = mx$ where $m$ is a positive integer. Let $M_m = \psi_m(M)$ and $V_m = \psi_m(V)$. The homothetic transformation $\psi_m$ is isotopic to the identity, hence $M_m$ is isotopic to $M$ and $V_m$ is isotopic to $V$. Given $\epsilon > 0$ we can choose $m$ large enough such that at each point of $M_m$ its sectional curvature is less than $\epsilon$. This means we can choose $m$ large enough such that for every point $p \in M_m$ the pair $(B(p), B(p) \cap M_m)$ is $C^\infty$-close to the pair $(B(p), B(p) \cap T_pM_m)$, where $B(p)$ denotes the closed ball centred at $p$ of radius sufficiently large, for instance of radius $10m\sqrt{n+2}$, and $T_pM$ is the tangent space of $M_m$ at $p$.

Let $x \mapsto n_x$, $x \in M_m$ be the unit normal vector field of $M_m$ with respect to an orientation of $M_m$. It follows from standard facts of differential geometry (see [12]) that the normal map $x \mapsto x + tn_x$, for $x \in B(p) \cap M_m$ has no focal points in $V_m$. This implies that the map $\mu_p : (B(p) \cap M_m) \times [-1, 1] \to V_m$ given by $(x, t) \mapsto x + tn_x$ is a diffeomorphism onto its image. Furthermore, there exists a constant $\delta_p > 0$ such that if $v_p$ is a unit vector satisfying $||v_p - n_p|| < \delta_p$ then the map $x + tv_p$ is still a diffeomorphism from $(B(p) \cap M_m) \times [-1, 1]$ onto its image and $(v_p, n) > 0$ for any normal vector to $M_m$ at a point $x \in B(p) \cap M_m$.

The map $\varphi : M_m \times [-mc/2, mc/2] \to V_m \subset \mathbb{R}^{n+2}$ given by $\varphi(x, t) = x + tn_x$ is a parametrization of $V_m$. Let $V_{1/2} = \varphi(M_m \times [-mc/4, mc/4])$ be the smaller neighborhood of width $mc/2$. The boundary of $V_{1/2}$ has two connected components

$$\partial V_{1/2}^- = \varphi(M \times \{-mc/4\})$$

$$\partial V_{1/2}^+ = \varphi(M \times \{mc/4\}).$$

Let $Q_{M_m} = \cup\{Q \in \mathcal{C} : Q \cap M_m \neq \emptyset\}$. Its boundary has also two connected components $\partial Q_{M_m}^+ := \partial Q_{M_m} \cap V_{1/2}^+$ and $\partial Q_{M_m}^- := \partial Q_{M_m} \cap V_{1/2}^-$. 

**Claim.** There exists a finite family of diffeomorphisms

$$\{\psi_i : \mathbb{R}^{n+1} \times [-1, 1] \to V_m\}_{i=1}^k$$

such that if we set $U_i := \psi_i(\mathbb{B}^{n+1} \times [-1, 1])$ then
1. $V_{1/2} \subset \bigcup_{i=1}^{k} \text{Int}(U_i)$

2. $\psi_i(\mathbb{R}^{n+1} \times \{0\}) \subset M_m$

3. For each $i$ the curves $t \mapsto \psi_i(y, t)$ are parallel to the unit vector $v_i$

4. For every cube $Q$ of $\mathcal{C}$ such that $Q \cap M_m = \emptyset$, $Q \cap \partial Q_{M_m}^+ \neq \emptyset$ and $Q \subset \text{Int}(U_i)$ we have that $\langle v_i, w \rangle > 0$ for every $w$ exterior normal to a face of $Q$ which is contained in $\partial Q_{M_m}^+$ (see definition in proof of lemma 2.3).

5. We define the set $S(Q)$ as in proof of lemma 2.3, where the set $\mathcal{M}$ is replaced by $Q \cap \partial Q_{M_m}^+$ (see also equation 1). We consider the convex set $S_{V_m}(Q) = S(Q) \cap V_m$. Then there exists $t_0 \in (-1, 1)$ such that $\psi_i(y, t) \in S_{V_m}(Q)$, $t \geq t_0$.

Proof of Claim. Since $M_m$ is compact, there exist a finite open subcovering $\{A_1, A_2, \ldots, A_k\}$ of the open covering $\{\text{Int}(B(p)) \cap M_m\}_{p \in M_m}$, where $A_i := \text{Int}(B(p_i)) \cap M_m$ and vectors $v_{p_1}, \ldots, v_{p_k}$ such that $||v_{p_i} - n_{p_i}|| < \delta_{p_i}$ and the maps $\tilde{\mu}_{p_i} : A_i \times [-1, 1] \to V_m$ given by $(x, t) \mapsto x + tv_i$ have the property that $V_{1/2} \subset \bigcup_{i=1}^{k} \tilde{\mu}(A_i \times [-1, 1])$. Furthermore, $v_{p_1}, \ldots, v_{p_k}$ can be chosen such that $\langle v_i, w \rangle > 0$ for every $w$ exterior normal to a face of $Q$ which is contained in $\partial Q_{M_m}^+$ and intersects $\mu_{p_i}(A_i \times [-1, 1])$ (see Figure 3). Notice that this can be done because $Q$ is a cube, so its faces meet at right angles. Using the fact that $A_i$ is homeomorphic to $\text{Int}(\mathbb{R}^{n+1})$, the claim follows.

![Figure 3: Brown lines $t \mapsto \psi_i(y, t)$ parallel to the same unit vector $v_i$.](image)

Remark: The above claim can be modified to prove a similar result for $\partial Q_{M_m}^+$.

For each $i = 1, \ldots, k$, let $f_i : \mathbb{R}^{n+2} \to [0, 1]$ be a smooth function such that $f_i(x) > 0$ if $x \in \text{Int}(U_i)$ and $f_i(x) = 0$ if $x \notin \text{Int}(U_i)$. Let $v_i(z) = v_i$ be the global constant vector field equal to $v_i$ at every point $z \in \mathbb{R}^{n+2}$. Let $\mathcal{V}(z) = \sum_{i=1}^{k} f_i(z)v_i$. 

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Then $\mathfrak{U}(z)$ is nonzero in $V_{1/2}$ and it has compact support inside $V$, hence it defines a global flow $\eta_t : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$.

**Claim.** Let $\partial V_{1/2}^-$ and $\partial V_{1/2}^+$ be the two boundary components of $V_{1/2}$. Then

1. If $z \in \partial V_{1/2}^-$ there exists $t_z > 0$ such that $\eta_{t_z}(z) \in \partial V_{1/2}^+$.

2. The flowlines of $\{\eta_t\}_{t \in \mathbb{R}}$ are transversal to $\varphi(M \times \{t\})$, $t \in [-c/4, c/4]$.

3. The flowlines of $\{\eta_t\}_{t \in \mathbb{R}}$ meet both $\partial V_{1/2}^-$ and $\partial V_{1/2}^+$ in one point and furthermore the flowlines meet both $\partial Q_{M_m}^+ := \partial Q_{M_m} \cap V_{1/2}^+$ and $\partial Q_{M_m}^- := \partial Q_{M_m} \cap V_{1/2}^-$ in one point.

**Proof of Claim.** First we remark that the flow is defined for all $t \in \mathbb{R}$ because the vector field $\mathfrak{U}$ has compact support. Every point of $V_m$ can be written uniquely in the form $x + tn_x$, $x \in M_m$, $t \in [-cm/2, cm/2]$. Let $p : V_m \to [-cm/2, cm/2]$ be the map $x + tn_x \mapsto t$. By the properties of $\mathfrak{U}$, we have that for fixed $y$ the function $t \mapsto p \circ \eta_t(y)$ is strictly increasing for $t \in [-cm/4, cm/4]$ since the derivative of the function is positive. This implies that the flowlines meet both $\partial V_{1/2}^+$ and $\partial V_{1/2}^-$ in one point. Furthermore, if $Q$ is a cube which meets $\partial Q_{M_m}^+$ but does not meet $M_m$ we have that $S(Q)$, defined above, is positively invariant under the flow, in fact $\eta_t(S(Q)) \subset \text{Int}(S(Q))$. Therefore the flowlines meet both $\partial Q_{M_m}^+$ and $\partial Q_{M_m}^-$ in exactly one point. This proves the claim.

We can rescale the flow by multiplying the vector field $\mathfrak{U}$ by a positive smooth function to obtain a new flow $\{\hat{\eta}_t\}_{t \in \mathbb{R}}$ such that for every $z \in M_m$ we have $\hat{\eta}_{-1}(z) \in \partial Q_{M_m}^-$ and $\hat{\eta}_1(z) \in \partial Q_{M_m}^+$.

Then the map $\Phi : M \times [-1, 1] \to \mathbb{R}^{n+2}$ defined by $\Phi(x, t) = \hat{\eta}_t(x)$ has the properties:

1. $\Phi(M \times [-1, 1]) = Q_{M_m}$ and therefore $Q_{M_m}$ is a closed tubular neighborhood of $M$.

2. $\Phi(M \times \{-1\}) = \partial Q_{M_m}^-$

3. $\Phi(M \times \{1\}) = \partial Q_{M_m}^+$

We have concluded that the set of cubes of the *canonical* cubulation of $\mathbb{R}^{n+2}$ that touch $M_m$ is a closed tubular neighborhood of $M_m$ and $M_m$ can
be deformed by an ambient isotopy onto any of the two boundary components of this tubular neighborhood. To finish the proof we now rescale our construction back to its original size using the inverse homothetic transformation $\hat{h}_1/m$. This transformation transforms $C$ onto the subcubulation $C_m$. ■

Let $M, N \subset \mathbb{R}^{n+2}, N \subset M$, be compact, closed and smooth submanifolds of $\mathbb{R}^{n+2}$ such that dimension $(M) = n + 1$ and dimension $(N) = n$. Since $M$ is codimension one in $\mathbb{R}^{n+2}$ it is oriented and we will assume that $N$ has a trivial normal bundle in $M$ (i.e., $N$ is a two-sided hypersurface of $M$). Then under these hypotheses we have the following theorem for pairs $(M, N)$ of smoothly embedded compact submanifolds of $\mathbb{R}^{n+2}$:

**THEOREM 2.9** There exists an ambient isotopy of $\mathbb{R}^{n+2}$ which takes $M$ into the $(n + 1)$-skeleton of the canonical cubulation $C$ of $\mathbb{R}^{n+2}$ and $N$ into the $n$-skeleton of $C$. In particular, $N$ can be deformed by an ambient isotopy into a cubical manifold contained in the canonical scaffolding of $\mathbb{R}^{n+2}$.

**Proof.** The proof is very similar to that of the previous theorem and lemma 2.6. As before, given $\epsilon > 0$ there exists $m \in \mathbb{N}$, large enough, such that if we consider the homothetic transformation $h_m(x) = mx$ then the sectional curvatures of both $h_m(M) := M_m$ and $h_m(N) := N_m$ are less than $\epsilon$. More precisely, for every $p \in N_m$ the triple $(B(p), B(p) \cap M_m, B(p) \cap N_m)$ is $C^\infty$-close to the triple $(B(p), B(p) \cap T_p M_m, B(p) \cap T_p N_m)$, where $B(p)$ denotes the closed ball centred at $p$ of radius sufficiently large, for instance of radius $10m\sqrt{n + 2}$, and $T_p M_m$ is the tangent space of $M_m$ at $p$ and $T_p N_m \subset T_p M_m$ is the tangent space to $N_m$ at $p$.

Let $Q_{M_m}, V_m, V_1/2, \{\hat{h}_t\}_{t \in \mathbb{R}}$, and $\Phi$ be as in the previous theorem. Let $f^+ : M_m \rightarrow \partial Q_{M_m}^+$ be given by $f^+(z) = \Phi(z, 1) = \hat{h}_1(z)$. Let $\hat{N}_m = f^+(N_m)$. From above we know that $\partial Q_{M_m}^+$ is isotopic to $M_m$ and it is cubulated since it is a union of faces of cubes of $C$. We have that $N_m$ can be deformed to $\hat{N}_m$ by a global isotopy of $\mathbb{R}^{n+2}$. Since $\hat{N}_m$ is contained in $\partial Q_{M_m}^+$ to prove the theorem we only need to find an isotopy $\{h_t : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}\}_{t \in [0, 1]}$ such that $h_t(\partial Q_{M_m}^+) \subset \partial Q_{M_m}^+, h_0 = \text{Identity}$ and $h_1(\hat{N}_m)$ is contained in the $n$-skeleton of $C$.

Let $Q_{\hat{N}_m}$ be the union of all faces of $\partial Q_{M_m}$ which intersect $\hat{N}_m$. Let $\partial Q_{\hat{N}_m}$ be the boundary of $Q_{\hat{N}_m}$ as a subset of $\partial Q_{M_m}$. Then $\partial Q_{\hat{N}_m}$ is a union of $n$-cubes.
Claim. $Q_{\hat{N}_m}$ is a closed tubular neighborhood of $\hat{N}_m$ in $\partial Q_{M_m}$.

Proof of Claim. The proof is similar to the proof of lemma 2.6. Let $A : N_m \times [-c,c] \to M_m$ be the parametrization of a tubular neighborhood of $N_m$ in $M_m$ given by $A(x,s) = \gamma_x(s)$, where $\gamma_x : \mathbb{R} \to M_m$ is the geodesic in $M_m$ (with respect to the induced metric of $\mathbb{R}^{n+2}$ in $M_m$) such that $\gamma_x(0) = x$ and $\gamma_x'(0) = w_x$ where $w_x$ is a unit vector at $x$ tangent to $N_m$ and transversal to $T_xN_m$. In other words we use a sort of Fermi coordinates for the tubular neighborhood. Given $d > 0$ we can choose $m$ large enough so that the curvature of both $M_m$ and $N_m$ is very small and the geodesics $\gamma_x(s)$ are very close (in the $C^2$-topology) to linear segments for $s \in [-d,d]$. We can take $c = d$. Let $y = f^+(x) \in \hat{N}_m$, $x \in N_m$ and let $\beta : [-c,c] \to \partial Q_{M_m}^+$ the function $\beta(s) = f^+(\gamma_x(s))$ and $J_y = \{ \beta(s) \mid s \in [0,c]\}$. Then just as in the proofs of lemma 2.6 and theorem 2.8 we can prove that we can choose the vector field $x \mapsto w_x$ in such a way that $J_y$ is homeomorphic to a non-trivial segment and intersects $\partial Q_{\hat{N}_m}$ in exactly one point (see Figure 4). The curves $J_y$ are rectifiable and the function $y \mapsto \text{length}(J_y)$ is continuous. Hence, $Q_{\hat{N}_m}$ is a bi-collar of $\hat{N}_m$ in $\partial Q_{M_m}$. This proves the claim.

By the above, $\hat{N}_m$ can be deformed to one connected component of $\partial Q_{\hat{N}_m}$ by an isotopy in $\partial Q_{M_m}^+$. By standard theorems (see [3], [15]), this isotopy can be extended to a global isotopy $\{h_t : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}\}_{t \in [0,1]}$. Observe that this isotopic copy of $\hat{N}_m$ is contained in the $n$-dimensional skeleton of the cubulation $\mathcal{C}$. To finish the proof we now rescale our construction back to its original size using the inverse homothetic transformation $h_{m^{-1}}$. This transformation transforms $\mathcal{C}$ onto the subcubulation $\mathcal{C}_m$. ■
3 Cubic knots

In classical knot theory, a subset $K$ of a space $X$ is a knot if $K$ is homeomorphic to a sphere $S^p$. Two knots $K, K'$ are equivalent if there is a homeomorphism $h : X \to X$ such that $h(K) = K'$; in other words $(X, K) \cong (X, K')$. However, a knot $K$ is sometimes defined to be an embedding $K : S^p \to S^n$ or $K : S^p \to \mathbb{R}^n$ (see [10], [14]). We are mostly interested in the codimension two smooth case $K : S^n \to \mathbb{R}^{n+2}$.

In this section we will prove the main theorem of this paper.

**Theorem 3.1** Let $C$ be the canonical cubulation of $\mathbb{R}^{n+2}$. Let $K \subset \mathbb{R}^{n+2}$ be a smooth knot of dimension $n$. Given any closed tubular neighborhood $V(K)$ of $K$ there exists an ambient isotopy $f_t : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ with support in $V(K)$ and $t \in [0, 1]$ such that $f_0 = \text{Id}$ and $f_1(K) := \bar{K}$ is contained in the $n$-skeleton of the subcubulation $C_m$ for some integer $m$. In fact $\bar{K}$ is contained in the boundary of the cubes of $C_m$ which are contained in $V(K)$ and intersect $K$. In particular, using the homothetic transformation $h_m(x) = mx$ we see that there exists a knot $\hat{K}$ isotopic to $K$, which is contained in the scaffolding $(n$-skeleton) of the canonical cubulation $C$ of $\mathbb{R}^{n+2}$.

**Proof.** Let $\Psi : S^n \to \mathbb{R}^{n+2}$ be a smoothly embedded knotted $n$-sphere in $\mathbb{R}^{n+2}$. We denote $K = \Psi(S^n)$ and endow it with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^{n+2}$.

Let $V(K)$ be a closed tubular neighborhood of $K$. By a theorem of Whitney ([10]) any embedded $S^2$ in $S^4$ has trivial normal bundle. Since $H^2(S^n, \mathbb{Z}) = 0$ for $n > 2$, any embedding of $S^n$ in $\mathbb{R}^{n+2}$ has trivial normal bundle. Hence $V(K)$ is diffeomorphic to $K \times D^2$. Let $\phi : K \times D^2 \to V(K)$ be a diffeomorphism and $p : V(K) \to K$ be the projection. We can assume that the fibers of $p$ are Euclidean disks of radius $\delta > 0$. If $0 < r < \delta$, the restriction of $\phi$ to $K \times D^2_r$ (where $D^2_r$ is the closed disk of radius $r$) is a closed tubular neighborhood $W(K)$ of $K$. Then $K \subset W(K) \subset \text{Int}(V(K))$. Take a section in $\partial W(K)$, i.e. for a fixed $\theta_0 \in S^1_r$ consider $\bar{K} = \phi(K \times \theta_0)$. Observe that $\bar{K}$ is isotopic to the knot $K$. Now consider the pair $(\partial W(K), \bar{K})$. By the above $\bar{K}$ has normal bundle in $\partial W(K)$ and applying theorem 2.9 yields the result. $\blacksquare$
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