e-mail: rtheo@dat.demokritos.gr
Abstract. A new method is reported by which it is possible to induce certain flux configurations of desired characteristics via electromagnetic means into the overall quantum probability current of a many-body system in the Madelung hydrodynamic picture. Some indicative applications are also considered with emphasis in HTC and gravitational wave research.

PACS. XX.XX.XX No PACS code given

1 Introduction

We know by the work of Madelung [1,2] and Schonberg [3] that the Schroedinger equation can be reduced to a pair of two "hydrodynamic" like PDEs. The pressure terms appearing there have an intimate connection with a non-local quantum potential that appears in a similar treatment in De Broglie - Bohm Mechanics [4]. Madelung ideal fluid equations are directly associated with Euler hydrodynamics and vorticity theory while direct formal associations between ideal fluid vorticity have been less explored.

In this short report we make explicit a particular scheme that can be used in order to construct an efficient flux controller for the velocity field in the hydrodynamic description of quantum many-body problems via electromagnetic means. To this aim we consider the induced phase shift by an arbitrary vector potential and we describe a methodology for finding appropriate closed forms for the associated magnetic field. We also prescribe some possible applications including the case of a gravito-magnetic dipole induced by a modulated superfluid flow.
2 Madelung Formalism

One starts with the "Madelung Transform" where the complete wavefunction is decomposed in a real amplitude and a complex phase through $\Psi(r, t) = R(r, t) \exp i S(r, t)$, $R = |\Psi|^2$. Then, from the Schrödinger picture we find the system

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \partial \mathbf{u} = -\frac{1}{\rho} \nabla \cdot \mathbf{P} - \frac{1}{m} \nabla U \quad (2)$$

In the above, $\rho = m R$ is the equivalent mass-charge density, $\mathbf{u}$ is the velocity field in the quantum probability space and $\mathbf{P} = -(\hbar^2/2m) \rho \nabla \nabla T \ln \rho$ is the corresponding non-local Pressure Tensor dyadic. Associated with the above picture is the quantum probability current given as

$$\mathbf{J} = \rho \mathbf{u} = R \nabla S \quad (3)$$

The above is also expressed in the Schrödinger picture through the antisymmetric 1-tensor

$$\mathbf{J} = -\frac{i\hbar}{2m}[\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] \quad (4)$$

It is a rarely mentioned fact that both of the above expressions are identical with the so called "Monge representation" of arbitrary vector flows first used in hydrodynamics. Then, the vorticity vector potential can be obtained in the so called "Clebsch variables" representation as

$$\nabla \times \mathbf{J} = \nabla \times \nabla S = -i \frac{\hbar}{m} \nabla \Psi \times \nabla \Psi^* \quad (5)$$

In the above, the two alternative representations, the scalars $R, S$ and $\Psi, \Psi^*$ stand for the Clebsh "Stream" and "Flux" potentials. It should be emphasized that this representation is local and non-unique. Using (3) one can rewrite the first of (5) in the equivalent form

$$\nabla \times \mathbf{J} = \frac{1}{\rho} \nabla \rho \times \mathbf{J} = \nabla \ln \rho \times \mathbf{J} \quad (6)$$

In the case of an irrotational flow, the above shows that the current will follow the gradient of the charge-mass density.

Assume next that we would like to shift the total flux in a new desirable form $\mathbf{J}'$ of which the vorticity will obey a certain prescribed relation. Therefore, one or more external driving fields must be added. At the moment we ignore the contribution of Spin magnetic moment. This is possible with the addition of an external non-stationary electromagnetic field having a vector potential $\mathbf{A}$ such that eq. (3) becomes

$$\mathbf{J} = R(\nabla S - \frac{q}{c} \mathbf{A}) \quad (7)$$

It is then expected that the current flow will be driven towards the desired flow pattern. In order to find electric and magnetic fields compatible with the Schrödinger operator one has to solve the above conditions together with the full Schrödinger equation and then solve the inverse problem for the Maxwell equations to define the external currents necessary for exciting these fields. Hence, a self-consistent formulation of the problem can be given by any solution of the non-linear eigenvalue problem.
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\[
\left( \nabla - \frac{q}{c} \mathbf{A} \left( \partial_t R, \partial_t S \right) \right)^2 \Psi = \frac{2m}{\hbar^2} \left( U(\mathbf{r}) - E \right) \Psi \tag{8}
\]

Adapting the generic form of eq. (7) in the superflow case gives then

\[
\mathbf{B} = \nabla \times \mathbf{A} = \lambda \nabla S \tag{10}
\]

The above immediately gives the magnetic field as a function of the phase gradient and the vorticity eigenvalue. We also see that the result demands the magnetic field to be parallel to the flow. As superfluidity implies global coherence one may take the \( S \) variable to be of the form \( \mathbf{k} \cdot \mathbf{r} \) from which one gets \( \mathbf{B} \approx \lambda(r) \mathbf{k} \).

In the above, we did not specifically account for the contribution of the Spin degrees of freedom. These can be accounted for through an additional term in (7) given by \( s = \langle \psi^* \hat{S} \psi \rangle \). At the moment we restrict attention in cases where it would have no contribution \( (\nabla \times s = 0) \). In the case of a direct coupling with the externally applied magnetic field, it will bring about an additional orientational coherence of the microscopic dipole terms thus resulting in a shift of the magnetic field. This can then be accounted for by an additional factor in (9) as

\[
\mathbf{B} \approx \lambda(r) \mathbf{k} - s \tag{11}
\]

Some caution is required with respect to the specific choice of \( \lambda \) which has to follow the below constraints

1. The eigenvalue \( \lambda \) is not allowed to be a constant as this would immediately render the magnetic field stationary as implied by (10). In the case of superfluidity we would not then be able to properly define a vector potential.

3 Electromagnetic Current Vortification

In the case of low temperature superfluidity [], Bose condensation obliges a global phase coherence so that the flux becomes quantized. As a result, we get \( R \hbar/m \) and the flow in (3) becomes irrotational. Using now (7) as a generic form of flux control, we want to move this flux into a new state satisfying a condition of the form

\[
\nabla \times \mathbf{J} = \lambda(r) \mathbf{J} \tag{9}
\]

In the above \( \lambda(r) \) is an arbitrary scalar. Such a state prescribes the vorticity eigenvalue of a curl eigen-field and it is a characteristic of ideal helical (toroidal-poloidal) flows of classical hydrodynamics. The choice of this particular class of flows has been made deliberately in order to satisfy an additional condition that causes the field to be contained inside the bulk of the flux region as explained further in section 4.
In order to guarantee the solenoidal character of the magnetic field \((\nabla \cdot \mathbf{B} = 0)\) as well as the vanishing of the divergence of relation (8) we should consider the following additional constraints over \(\lambda\)

\[
\nabla (\lambda \mathbf{J}) = 0 \quad (12)
\]

\[
\nabla \lambda \cdot \mathbf{k} = 0 \quad (13)
\]

The first condition if combined with the continuity equation results in

\[
\nabla \ln \lambda \cdot \nabla S = \partial_t (\ln R) \quad (14)
\]

Eqs (13) and (14) are compatible only in a confined solenoidal flow that satisfies \(\nabla \cdot \mathbf{J} = 0\) with a conserved total density \(R\).

In the simplest possible case of a toroidal flux tube, condition (13) is guaranteed by the choice of a functional dependence of the form \(\lambda(r, z)\) as long as the wavevector \(\mathbf{k}\) will be tangential in a coherent superfluid flow. A possible application of the above is discussed in section 5.

We first discuss the possibility of self-consistent solutions in the more general case of some hot fermion gas with electromagnetic control which can be discussed in the same spirit of the hydrodynamic Madelung picture with a non-quantized amplitude \(R\). Direct application of the curl into (7) and subsequent introduction in (8) results in the condition

\[
\nabla R \times (\nabla S - \frac{q}{c} \mathbf{A}) - \frac{q}{c} R \mathbf{B} = \lambda R (\nabla S - \frac{q}{c} \mathbf{A}) \quad (15)
\]

As the problem is quite more complicated we will hereafter examine exclusively the case of flows with constant \(\lambda\) and in particular we will choose to set \(\lambda = \lambda_0\) to simplify further examination. Relation (15) can be rewritten in the matrix form

\[
\mathbf{B} = \tilde{\Gamma} \cdot \mathbf{J}_0 \quad (16)
\]

\[
\mathbf{J}_0 = \frac{c}{q} \nabla S - \mathbf{A} \quad (17)
\]

\[
\tilde{\Gamma} = \frac{1}{R} [\partial R]_X - \lambda_0 \mathbf{I} \quad (18)
\]

In the above, the symbol \([\partial R]_X\) stands for the matrix representation of the cross product in terms of the derivatives of \(R\) given below as

\[
[\partial R]_X = \begin{pmatrix}
0 & -\partial_x R & \partial_y R \\
\partial_x R & 0 & -\partial_y R \\
-\partial_y R & \partial_x R & 0
\end{pmatrix} \quad (19)
\]

Separating terms relating to the external fields in (15) and taking the curl via the identity \(\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A}\) results in an inhomogeneous wave equation for the vector potential in the form

\[
\nabla^2 \mathbf{A} - \lambda_0 \mathbf{A} = \nabla R \times \mathbf{A} + \frac{c}{q} \tilde{\Gamma} \cdot \nabla S \quad (20)
\]

From the above we deduce that at least in the case of a single frequency mode \(\omega = c\sqrt{\lambda_0}\) the additional external current sources should have the form

\[
\mathbf{J}_e = \nabla R \times \mathbf{A} + \frac{c}{q} \tilde{\Gamma} \cdot \nabla S \quad (21)
\]

It is now possible to insert any solution of (20) into (8) for the scheme to be self-consistent. Unfortunately, no
known general solution of (20) exists in closed form so the problem must be tackled numerically.

Using again the matrix representation of the exterior product we may write (24) in the equivalent form

\[ \hat{G} \cdot J = 0 \quad (26) \]

\[ \hat{G} = \left[ \nabla \lambda \right]_x + \beta I \quad (27) \]

The above then can only satisfied if \( J \) is an eigenvector of the matrix form given in (19) but with the derivatives of \( \lambda \) as matrix elements. Such matrices have three distinct eigenvalues of which one is 0 and the other two have the imaginary values \( \pm i \sqrt{1 + |\nabla \lambda|^2} \).

For the current to correspond to such an eigenvector it would have to satisfy the additional constraint \( R \nabla S = e_\pm (\partial_i \lambda) \) where \( e_\pm \) the corresponding eigenvectors. In such a case, the constraint (13) and (14) imply that one should also have \( \nabla \lambda \cdot \nabla S = \nabla \lambda \cdot e_\pm / R \) or \( e_\pm (\partial_i \lambda) = R k \).

As this is a highly nonlinear equation that restricts the choice of \( \lambda \), it is expected that for most simple choices of \( \lambda \) it will not be satisfied. Alternatively we may add (24) in the constraints for the choice of \( \lambda \) thus guaranteeing the containment of the field components in the interior of the flux tube.

The significance of the above argument is that in this situation, the total energy of the system and the induced magnetic stresses cannot be minimised by radiative emissions thus forcing the total flux to relax into a state consistent with the externally applied field.
5 Applications

The method of control introduced in the previous sections could find applications in microelectronics and spintronics. It is important to observe that in the absence of experimental data, such vortification is not known to be either beneficial or disruptive for long range coherence. It is equally important in the case of high temperature superconductivity to know if the total net vorticity of the kind explained here would be beneficial in which case there are numerous potential applications, especially if it could significantly increase the transition temperatures. Moreover, the issue of any critical spatial scales at which such vorticity could be found to contribute to long range coherence should be treated with statistical and other methods although it is beyond the scope of the present report.

In this section, we also examine the applicability of the method in an idealised conceptual experiment utilising superfluidity. Such an experiment can be described in terms of a very fast rotating confined toroidal-poloidal flux field. Specifically, an ideal substance to be used in such experiments with external magnetic fields is the case of a ferrofluid. We may then assume a dense set of appropriate cylindrical electromagnet pairs surrounding a toroidal tube. The currents in each pair should then be driven by a modulator circuit to approximate the field configuration found in (10) or (11) of section 3. It should be noted that in actual experiments it has been found that ferrofluid condensates exhibit strong dipolar interactions. Yet we will not deal here with the problem of the exact contribution which requires a detailed examination of the exact interaction type that goes beyond the scope of the present report. We only notice that this might prove beneficial in reducing the requirements for the external driving field.

If this flux is reorganized accordingly to the helical flows characterizing certain solutions for an ideal Euler fluid as implied by eq. (9) then it might be possible to induce a certain gravitomagnetic component. In the example below we attempt a revival of an old proposal made by Robert Forward in his 1963 paper, ”Guidelines to Anti-gravity” [9]. There, the calculated gravitational dipole component was given as

\[ G_D = -\mu_G \frac{d}{4\pi} \left[ \frac{u}{R} \right]^2 \]  \hspace{0.5cm} (28)

In the above \( \mu_G = 4\pi G/c^2 \) is the gravitomagnetic equivalent of the permeability, and \( r, R \) the internal and external radius respectively of a confined toroidal flow. Taking the ratio \( \Lambda = r/R \) constant leads to

\[ G_D = -\frac{G\Lambda^2}{4\pi c^2} \frac{du}{dt} \]  \hspace{0.5cm} (29)

The main difficulty in actually measuring such an effect stems from the value of the coefficient being of the order of \( 10^{-28} \). We will discuss possible ways that this could be addressed below.

Eq (29) can be rewritten in terms of the internal variables of the ferrofluid dynamics through the direct use of (2) of which the lhs represents the total time derivative, giving
In the above we recognise the two main contributions coming from inhomogeneities in the mass flow and from the exact type of the interaction potential. In fact the first part is practically negligible such that only the interaction potential is of importance here. We can then approximate (30) with the simpler expression

$$G_D \approx \frac{GA^2}{4\pi mc^2} \left[ \frac{-h^2}{2} \nabla \nabla^T \ln \rho + \nabla U \right]$$  \hspace{1cm} (31)

In the above $U^* = U/mc^2$ represents the undimensioned ratio of the interaction energy to the total rest mass-energy of the circulating ferrofluid.

Eq (31) allows an important observation. However small this ratio, it is only the spatial variations of it that contribute to the overall effect. Hence, the desired result could be enhanced by any abrupt variations of the internal distribution of the kinetic and interaction terms including the contained field quantities. Such an enhancement could also be attributed to a large number of spatial discontinuities or flux defects that could be induced by additional means. One way to provide such a variation would then be to make the total flux inhomogeneous in the angular direction by causing a large number of "bead" like formations.

Despite the extreme technical difficulties of the above, it is interesting to notice that in some more modern versions of the so called Brans-Dicke or scalar-tensor theories of gravity the gravitomagnetic effect may be enhanced in the presence of EM fields by an additional mechanism.

It was recently reported by Raptis and Minotti \cite{14} that in such theories there is a direct coupling between the gravitational part and the electromagnetic part such that the total energy of an intense magnetic field enters as a source for variations of the metric components via a wave equation

$$\nabla \Theta - \frac{1}{c^2} \frac{\partial^2 \Theta}{\partial t^2} = \kappa \left[ B^2 - (E/c)^2 \right]$$  \hspace{1cm} (32)

Expected fluctuations of the local potential are then of the order of $c^2 \nabla \Theta$. In \cite{14} the above parameter $\kappa$ is estimated to be of the order of $10^{-4}$ so that the possibility of a combined action of both the rotating hyperfluid and the applied electromagnetic field contained inside the main circulating mass and resulting to high frequency gravitational waves, seems to be a promising alternative for further research should these theories be verified by other experiments like the one proposed in \cite{14}.

6 Conclusions

In this short report we examine a novel method of electromagnetically inducing a vortical structure in the velocity field of certain quantum fluids. As this does not appear to have ever been tested in the past, we believe that it will open new avenues for theoretical and experimental research. An advantage of the method is that it implies certain characteristics that are unique in a class of solutions for ideal Euler fluids. In particular, it allows
control over the geometrical and topological characteristics of such flows. Such a type of control can have important implications in several areas including the study of high temperature superconductivity and superfluidity. An extreme example of possible application in gravito-magnetism is also examined. Further numerical work is required for locating exact solutions of the proposed equations which will be given in a following report.

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