THE DENSITY OF SUPERCONDUCTIVITY IN THE BULK REGIME

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Abstract. In the asymptotic limit of a large Ginzburg-Landau parameter, we give a new asymptotic formula for the $L^2$-norm of the Ginzburg-Landau order parameter. The formula is valid in the bulk regime where the intensity of the applied magnetic field is of the same order as the Ginzburg-Landau parameter and strictly below the second critical field. Our formula complements the celebrated one of Sandier-Serfaty for the $L^4$-norm.

1. Introduction and main results

The Ginzburg-Landau model. The Ginzburg-Landau functional is defined as the sum of two functionals, the energy of the order parameter and the magnetic energy. It reads as follows,

$$E_{GL}(\psi, A) = E_{op}(\psi, A) + E_{mag}(A),$$

(1.1)

where

$$E_{op}(\psi, A) = \int_{\Omega} \left( |(\nabla - i\kappa H A)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx,$$

$$E_{mag}(A) = \kappa^2 H^2 \int_{\Omega} |\text{curl} A - 1|^2 \, dx.$$  

(1.2)

Here:

- $\Omega \subset \mathbb{R}^2$ is an open, bounded and simply connected set with a $C^\infty$ boundary; $\Omega$ is the cross section of a cylindrical superconducting sample placed vertically.
- $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ describes the state of superconductivity as follows: $|\psi|^2$ measures the local density of the superconducting Cooper pairs and $\text{curl} A$ measures the induced magnetic field in the sample.
- $\kappa > 0$ is the Ginzburg-Landau parameter, a material characteristic of the sample.
- $H > 0$ measures the intensity of the applied magnetic field.
- The applied magnetic field is $\kappa H \vec{e}$, where $\vec{e} = (0,0,1)$.

We introduce the ground state energy of the functional in (1.1):

$$E_{gs}(\kappa, h_{ex}) = \inf \{ E_{GL}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \}. $$

(1.3)

For a given $(\kappa, H)$, a configuration $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ satisfying $E_{GL}(\psi, A) = E_{gs}(\kappa, h_{ex})$ is called a minimizer of the functional $E_{GL}$ and we will denote it by $(\psi, A)_{\kappa, H}$ to emphasize its dependence on $\kappa$ and $H$. Such a minimizer is a solution of the following Ginzburg-Landau equations (we use the notation $\nabla^\perp = (\partial_x, -\partial_y)$)

$$
\begin{cases}
-(\nabla - i\kappa H A)^2 \psi = \kappa^2 (1 - |\psi|^2) \psi & \text{in } \Omega, \\
-\nabla^\perp \text{curl} A = (\kappa H)^{-1} \text{Im}(\overline{\psi} (\nabla - i\kappa H A) \psi) & \text{in } \Omega, \\
\nu \cdot (\nabla - i\kappa H A) \psi = 0 & \text{on } \partial \Omega, \\
\text{curl} A = B_0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.4)
Gauge invariant quantities. The physically relevant quantities, density, induced magnetic field, energy and supercurrent are invariant under the Gauge transformations. More precisely, the following quantities

\[ |\psi|^2, \quad \text{curl } A, \quad |(\nabla - i\kappa H)\psi|^2, \quad (1.5) \]

\[ j(\psi, A) = \text{Re}(-i\psi(\nabla - i\kappa H)\psi), \quad (1.6) \]

are invariant under the transformation \((\psi, A) \mapsto (e^{i\chi}, A - \nabla \chi)\) for every given \(\chi \in H^1(\Omega; \mathbb{R})\). This gauge invariance insures that all the quantities in (1.5) and (1.6) are smooth functions (cf. [23, Ch. 2]) when \((\psi, A)\) is a minimizer. The solution \((\psi, A)\) of (1.4) in the class such that \(\text{div } A = 0 \in \Omega\) and \(A \cdot \nu = 0\) on \(\partial \Omega\) is indeed \(C^\infty\).

Earlier results on the density. In this paper, we will study the asymptotics for the density in the following regime

\[ H = b\kappa, \quad (1.7) \]

where \(b \in (0, 1)\) is a fixed constant.

This corresponds to the situation of an external magnetic field with intensity strictly below the second critical field \(H_{c2}(\kappa) := \kappa\). The case where \(b > 1\) in (1.7) is related to the phenomenon of surface superconductivity which is extensively studied by many authors [4, 8, 10, 22].

When (1.7) holds, Sandier-Serfaty [25] proved the following formula for the ground state energy in (1.3):

\[ E_{gs}(\kappa, h_{ex}) = g(b)|\Omega|\kappa^2 + o(\kappa^2) \quad \text{as } \kappa \to +\infty, \quad (1.8) \]

where \(g(b)\) is an implicitly defined quantity that depends only on \(b\). Its precise definition will be given in (2.1). In particular, it satisfies:

\[ g(0) = -\frac{1}{2}, \quad g(1) = 0 \quad \text{and} \quad g(b) < 0 \quad \text{for } b \in (0, 1). \]

The convergence in (1.8) is uniform with respect to \(b\) on every interval \([\epsilon, 1]\), \(\epsilon > 0\). The uniform convergence fails on the interval \((0, 1)\). More details regarding the uniformity with respect to \(b\) are given by K. Attar in [5, 6].

Now suppose that (1.7) holds and that \((\psi, A)_{\kappa, H}\) is a minimizer of the functional in (1.1). The magnetic energy satisfies [5]:

\[ \kappa^2 H^2 \int_\Omega |\text{curl } A - 1|^2 dx \leq C \kappa^{7/4}, \quad (1.9) \]

for \(\kappa \geq \kappa_0\), where \(\kappa_0\) and \(C\) are two constants that depend only on the domain \(\Omega\) and the constant \(b\) in (1.7). Hence its contribution in the ground state energy is relatively small as \(\kappa \to +\infty\).

Again, if \(b \in [\epsilon, 1]\) for some \(\epsilon > 0\), the constants \(\kappa_0\) and \(C\) can be selected independently from \(b\), but they will depend on \(\epsilon\). More details can be found in [5, 6], where it is allowed for \(\epsilon\) to depend on \(\kappa\), \(\epsilon = \epsilon(\kappa)\), and approach 0 as \(\kappa \to +\infty\).

Using the Ginzburg-Landau equation for \(\psi\) (see (1.4)), we get the following simple relation between the energy and the \(L^2\)-norm of the density:

\[ \mathcal{E}_{\text{op}}(\psi, A) = -\frac{\kappa^2}{2} \int_\Omega |\psi(x)|^4 dx, \quad (1.10) \]

where \(\mathcal{E}_{\text{op}}\) is the energy of the order parameter introduced in (1.2). Consequently, combining the estimates in (1.8) and (1.9), we deduce the following formula regarding the \(L^2\)-norm of the density [25]:

\[ \int_\Omega |\psi(x)|^4 dx = -2g(b)|\Omega| + o(1) \quad \text{as } \kappa \to +\infty, \quad (1.11) \]

where the function \(o(1)\) is dominated by a function \(s(\kappa)\) such that \(s(\kappa)\) is independent of the choice of the minimizer \((\psi, A)_{\kappa, H}\) and \(s(\kappa) \to 0\) as \(\kappa \to +\infty\). When \(b \in [\epsilon, 1]\) for some \(\epsilon > 0\), the function \(s(\kappa)\) can be selected independently from \(b\). More details can be found in [5, 6], where
the case $\epsilon = \epsilon(\kappa)$ tending to 0 is considered. In particular the comparison of $\epsilon(\kappa)$ with the first critical field $H_{c_1}(\kappa) \approx \frac{\kappa}{\kappa}$ could play a role.

Furthermore, Sandier-Serfaty obtained the following weak-convergence of $|\psi|^4$ as $\kappa \rightarrow +\infty$ in the sense of distributions $[25]$:

$$|\psi|^4 \rightarrow -2g(b) \quad \text{in } D'(\Omega).$$

(1.12)

Open questions. Note that for $b = 0$ in (1.7), i.e. $H = 0$, every minimizer $(\psi, A)_{\kappa, H}$ satisfies $|\psi| = 1$ and $\text{curl} A = 1$. This is consistent with (1.11) and (1.9). Indeed, as $b \rightarrow 0_+$, we know that $g(b) \rightarrow -\frac{1}{2}$.

The regime $b \rightarrow 0_+$ (which corresponds to $H \ll \kappa$, see (1.7)) is thoroughly analyzed by Sandier-Serfaty in [26, 24]. In particular, it is proved that, for any minimizer $(\psi, A)_{\kappa, H}$, the density $|\psi|^2$ satisfies $|\psi|^2 \rightarrow 1$ in $L^2(\Omega)$ and it is close to 1 everywhere except in narrow regions of area $O(\kappa^{-1})$. The region where $|\psi|^2$ is not close to 1 consists of small defects accommodating isolated zeros of $\psi$, called vortices. These vortices are evenly distributed in the domain $\Omega$ along a lattice, and the distance between two vortices is $\approx H^{-1}$, much larger than $\kappa^{-1}$, the core size of the vortex.

The detailed analysis of the distribution of vortices is missing when (1.7) holds for a fixed constant $b \in (0, 1)$, even for small values of $b$. This is a challenging problem mainly for the following reason. For a minimizer $(\psi, A)_{\kappa, H}$, it is expected that $\psi$ will have isolated zeros/vortices filling up all the domain $\Omega$, but these zeros are separated by a distance $O(H^{-1}) = O(\kappa^{-1})$. At the same time, the core-size of every vortex is equal to $O(\kappa^{-1})$. Consequently, detecting the vortices in this regime becomes harder than when $H \ll \kappa$ (i.e. $b \rightarrow 0_+$ in (1.7)).

This problem is related to the one of the Abrikosov state near the critical field $H_{c_2} := \kappa$, where the transition to the normal state in the bulk occurs. This is visualized in the regime $b \rightarrow 1_-$ in (1.7) and is analyzed in many papers, [3, 12, 18, 19]. The same difficulty is encountered when trying to detect the vortices by the methods of Sandier-Serfaty, so that the analysis is shifted to the distribution of the density $|\psi|^2$ instead.

In this paper, we complement the results of Sandier-Serfaty by obtaining analogues of the formulas in (1.11) and (1.12) for the density $|\psi|^2$ (instead of the square of the density $|\psi|^4$), in the regime where (1.7) holds for a fixed constant $b \in (0, 1)$. Besides that such results are new and do not follow from the analysis by Sandier-Serfaty [25], they might be helpful in the analysis of the vortices. Related to these results is the asymptotics of the supercurrent $j(\psi, A)$ when (1.7) holds. Even in the particular regime $H \ll \kappa$ (i.e. $b \ll 1$ in (1.7)), the analysis of the distribution of the super-current is missing. Actually, Sandier-Serfaty [23, Ch. 8, Corol. 8.1] prove only that, in the regime $\frac{|\ln \kappa|}{\kappa} \ll H \ll \kappa$, $\text{curl} j \rightarrow 0$ in $D'(\Omega)$, as $\kappa \rightarrow +\infty$.

Main results. To state our main results, we recall some properties of $g$. The function $g$ is increasing and concave (cf. [13 Thm. 2.1]). Consequently, $g$ has at each point left- and right-sided derivatives $g'(b_-)$ and $g'(b_+)$ with

$$g'(b_+) \leq g'(b_-).$$

Therefore, we can introduce the set

$$\mathcal{R} = \{ b \in (0, 1) : g'(b_-) = g'(b_+) \}$$

(1.13)

whose complement in the interval $(0, 1)$ is countable. Assuming that $b \in \mathcal{R}$ and (1.7) holds, we will prove that every minimizer $(\psi, A)_{\kappa, H}$ of the G-L functional in (1.11) satisfies (compare with (1.11))

$$\int_{\Omega} |\psi(x)|^2 \, dx = \left( g'(b) - 2g(b) \right)|\Omega| + o(1) \quad \text{as } \kappa \rightarrow +\infty.$$  

(1.14)

The formula in (1.14) is consistent with the one given in [18, Eq. (1.6)] which is valid as $b \rightarrow 1_-$. We have indeed (see below (2.4)),

$$g(b) \sim E_{\text{Ab}}(b - 1)^2,$$
where $E_{\lambda b} \in [-\frac{1}{2}, 0]$ is a universal constant.

More precisely, our main result is:

**Theorem 1.1.** Let $b \in (0, 1)$. There exist $\kappa_0 > 0$ and a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\kappa \to \infty} \lambda(\kappa) = 0$ and the following is true.

If $(\psi, A)_{\kappa, H}$ is a minimizer of the functional in (1.1) for $H = b\kappa$ and $\kappa \geq \kappa_0$, then

1. 
   
   \[ g'(b) - \lambda(\kappa) \leq \frac{1}{\kappa^2|\Omega|} \int_{\Omega} |(\nabla - i\kappa H)A(\Omega)|^2 \leq g'(b) + \lambda(\kappa). \]

2. 
   
   \[ g'(b) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 \leq g'(b) - 2g(b) + \lambda(\kappa) \quad \text{as} \ \kappa \to +\infty. \]

3. 
   
   If $b \in R$, then as $\kappa \to \infty$, the following convergence holds in the sense of distributions
   
   $|\psi|^2 \to g'(b) - 2g(b)$ in $\mathcal{D}'(\Omega)$.

4. 
   
   The supercurrent satisfies
   
   \[ \frac{1}{\kappa^2|\Omega|} \int_{\Omega} |j(\psi, A)|^2 \leq g'(b) + \lambda(\kappa), \]

   and
   
   \[ \frac{1}{\kappa|\Omega|} \int_{\Omega} |j(\psi, A)| dx \leq \sqrt{g'(b)}(g'(b) - 2g(b)) + \lambda(\kappa). \]

**Remark 1.2.** [On the leading order term]

The coefficient of the leading term in (1.14) does not vanish. Actually, $g'(b) \geq 0$ since $g$ is increasing, and $g(b) < 0$ for $b \in (0, 1)$.

**Remark 1.3.** [On the $L^2$-norm of $1 - |\psi|^2$]

Using (1.11) and Hölder’s inequality, we get, for fixed $b$ and as $\kappa \to +\infty$,

\[ \frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 \leq |\Omega|^{-\frac{1}{2}} \left( \int_{\Omega} |\psi(x)|^4 \right)^{\frac{1}{2}} \leq (-2g(b))^{\frac{1}{2}} + o(1). \]

Combined with the lower bound in (1.14), we get (we use that $g'(b) \geq 0$)

\[ -2g(b) - o(1) \leq \frac{1}{|\Omega|} \int_{\Omega} |\psi(x)|^2 \leq (-2g(b))^{\frac{1}{2}} + o(1). \]

Now we find the following estimate for the $L^2$-norm of $1 - |\psi|^2$,

\[ \frac{1}{|\Omega|} \int_{\Omega} (1 - |\psi(x)|^2)^2 \leq 1 + 2g(b) + o(1), \]

with the principal term on the right hand side approaching 0 as $b \to 0_+$, since

\[ \lim_{b \to 0_+} g(b) = -\frac{1}{2}. \]

This is consistent with the behavior $|\psi|^2 \to 1$ in $L^2(\Omega)$ obtained in [26].

**Remark 1.4.** [On the potential energy]

When $b \in R$ (see (1.13)), we get from Theorem (1.1) that the potential energy satisfies

\[ \kappa^2 \int_{\Omega} \left( -|\psi(x)|^2 + \frac{1}{2} |\psi(x)|^4 \right) dx = \kappa^2 \left( g(b) - g'(b) \right)|\Omega|(1 + o(1)). \]
2. Preliminaries

2.1. The bulk energy. Here we give the definition of the reference bulk energy \( g(\cdot) \). This energy first appeared in [25] and was then extensively studied in [2, 13, 6, 7, 17].

Consider \( b \in (0, +\infty) \), \( r > 0 \) and \( Q_r = (-r/2, r/2) \times (-r/2, r/2) \). Define the functional,

\[
F_{b,Q_r}(u) = \int_{Q_r} \left( b |(\nabla - iA_0)u|^2 - |u|^2 + \frac{1}{2} |u|^4 \right) \, dx, \quad \text{for } u \in H^1(Q_r). \tag{2.1}
\]

Here, \( A_0 \) is the magnetic potential,

\[
A_0(x) = \frac{1}{2}(-x_2, x_1), \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.2}
\]

Define the two ground state energies,

\[
e_D(b, r) = \inf \{ F_{b,Q_r}(u) : u \in H^1_0(Q_r) \}, \quad e_N(b, r) = \inf \{ F_{b,Q_r}(u) : u \in H^1(Q_r) \}. \tag{2.3}
\]

The function \( g(\cdot) \) may be defined as follows (cf. [13, 25, 6]),

\[
\forall b > 0, \quad g(b) = \lim_{r \to +\infty} \frac{e_D(b, r)}{|Q_r|} = \lim_{r \to +\infty} \frac{e_N(b, r)}{|Q_r|}, \tag{2.4}
\]

where \( |Q_r| \) denotes the area of \( Q_r \) (\( |Q_r| = r^2 \)). Furthermore, there exists a constant \( C \) such that, for all \( r \geq 1 \) and \( b \in (0,1) \),

\[
g(b) \leq \frac{e_D(b, r)}{|Q_r|} \leq g(b) + C\sqrt{b} \quad \text{and} \quad e_D(b, R) - Cr\sqrt{b} \leq e_N(b, r) \leq e_D(b, r). \tag{2.5}
\]

Various properties satisfied by the function \( g(\cdot) \) are established in [7, 13, 20, 25]. In particular, the function \( g(\cdot) \) is a non decreasing continuous and locally Lipschitz function such that

\[
g(0) = -\frac{1}{2} \quad \text{and} \quad g(b) = 0 \quad \text{when } b \geq 1, \tag{2.6}
\]

and

\[
\lim_{b \to 0} \frac{g(b)}{(b-1)^2} = E_{Ab} \in [-\frac{1}{2}, 0]. \tag{2.7}
\]

2.2. A priori estimates and Gauge transformations. Here we collect useful estimates regarding the critical points of the Ginzburg-Landau functional (cf. [10] Prop. 10.3.1 and 11.4.4]).

Proposition 2.1. Let \( b \in (0, 1) \). There exist two constants \( C > 0 \) and \( \kappa_0 > 0 \) such that, if \( \kappa \geq \kappa_0 \), \( H = bk^2 \) and \((\psi, A)_{\kappa,H} \) is a critical point of (2.3), then:

\[
\| \psi \|_\infty \leq 1, \tag{2.8}
\]

\[
\| (\nabla - ikH)\psi \|_{C(\Omega)} \leq C_\kappa, \tag{2.9}
\]

\[
\| \text{curl} A - 1 \|_{C^1(\Omega)} \leq \frac{C}{\kappa}. \tag{2.10}
\]

As a consequence of Proposition 2.1, we may pick a useful gauge transformation in every ball with small radius:

Proposition 2.2. Let \( b \in (0, 1) \). There exist two constants \( C > 0 \) and \( \kappa_0 > 0 \) such that, for any \( x_0 \in \Omega \), there exists a function \( \varphi_0 \in C^1(\Omega) \) such that

\[
\forall \, x \in \Omega, \quad \left| A(x) - (A_0(x - x_0) - \nabla \varphi_0(x)) \right| \leq \frac{C}{\kappa} \max \left( |x - x_0|, |x - x_0|^2 \right),
\]

where \( A_0 \) is the vector field introduced in (2.2).
Proof. Let $B = \text{curl} A$. Choose a convex and open set $U \subset \mathbb{R}^2$ such that $\overline{\Omega} \subset U$. We may extend the function $B$ to a function $B_{\text{ext}} : U \to \mathbb{R}$ such that

$$\text{supp}(B_{\text{ext}}) \subset U \quad \text{and} \quad \|\nabla B_{\text{ext}}\|_{L^\infty(U)} \leq C \|\nabla B\|_{L^\infty(\Omega)}, \quad (2.11)$$

where $C$ is a constant that depends solely on $\Omega$ and $U$ (i.e. it is independent of $B$).

Define the vector field in $\Omega$

$$G(x) = 2 \left( \int_0^1 sB_{\text{ext}}(s(x - x_0) + x_0) \, ds \right) A_0(x - x_0).$$

It is easy to check that

$$\text{curl} G = B_{\text{ext}} = B \quad \text{in} \ \Omega.$$

Consequently, since $\Omega$ is simply connected, there exists a smooth function $\varphi_0$ such that,

$$A(x) = G(x) - \nabla \varphi_0(x).$$

Using (2.10), (2.11) and the mean value theorem, we get further

$$|G(x) - B_0(x_0)A_0(x - x_0)| \leq \frac{C}{\kappa}|x - x_0|^2.$$

Again, using (2.10), we write $|B_0(x_0) - 1|A_0(x - x_0)| \leq C\kappa^{-1}|x - x_0|$. This yields the inequality

$$|G(x) - A_0(x - x_0)| \leq \frac{C}{\kappa} \max\left(|x - x_0|, |x - x_0|^2\right). \ \square$$

Remark 2.3. We will use the inequality in Proposition 2.2 for $|x - x_0| \leq \ell$ and $\ell \ll 1$, which in turn reads as follows

$$|A(x) - (A_0(x - x_0) - \nabla \varphi_0(x))| \leq \frac{C}{\kappa} \ell.$$

3. ON THE LOCAL ENERGY OF MINIMIZERS

For any open set $D \subset \Omega$, we define the following local energy

$$\mathcal{E}_0(f, a; D) = \int_D \left( |\nabla - i\kappa Ha|^2 f^2 - \kappa^2 |f|^2 + \frac{\kappa^2}{2} |f|^4 \right) \, dx. \quad (3.1)$$

For $x_0 \in \mathbb{R}^2$ and $\ell > 0$, $Q_{\ell}(x_0) = x_0 + (-\ell/2, \ell/2)^2$ denotes the square of center $x_0$ and side-length $\ell$.

We will need the following result, essentially proved in [5] modulo a few adjustments.

Proposition 3.1. If $b \in (0, 1)$, there exist positive constants $C, R_0, \kappa_0 > 0$, such that for $\kappa \leq \kappa_0$, $H = bk$, $R_0\kappa^{-1} \leq \ell \leq \kappa_0^{-1}$, $x_0 \in \Omega$, and if $Q_{\ell}(x_0) \subset \Omega$, then the following inequalities hold

$$\left| \frac{1}{|Q_{\ell}(x_0)|} \mathcal{E}_0 \left( e^{i\kappa H \varphi_0} \psi, A_0^0; Q_{\ell}(x_0) \right) - \kappa^2 g(b) \right| \leq C \left( \ell + (\kappa\ell)^{-1} \right)^2 \kappa^2,$$

and

$$\left| \frac{1}{|Q_{\ell}(x_0)|} \int_{Q_{\ell}(x_0)} |\psi(x)|^4 \, dx + 2g(b) \right| \leq C \left( \ell + (\kappa\ell)^{-1} \right),$$

where $A_0^0(x) = A_0(x - x_0)$, $A_0$ is the vector field in (2.2), and $\varphi_0$ is the function constructed in Proposition 2.2.
Proof. In [3] Prop. 4.2 and 6.2, it is proved that
\[ \left| \frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0\left( \psi, A; Q_\ell(x_0) \right) - \kappa^2 g(b) \right| \leq C \left( \ell + (\ell \kappa)^{-1} \right) \kappa^2. \] (3.2)
The estimate of the remainder term in [3] was worse because the magnetic field was assumed non-constant and a variant of the inequality in Proposition 2.2 was used (with a worse error as well).

However, in our case of a constant magnetic field, we insert the inequality in Proposition 2.2 into the proof given in [5] and get the better remainder as in (3.2).

We write
\[ \mathcal{E}_0\left( \psi, A; Q_\ell(x_0) \right) = \mathcal{E}_0\left( \psi, A_0^{x_0} - \nabla \varphi_0 + (A - A_0^{x_0} + \nabla \varphi_0); Q_\ell(x_0) \right) \]
\[ \geq (1 - \ell) \mathcal{E}_0\left( \psi, A_0^{x_0} - \nabla \varphi_0; Q_\ell(x_0) \right) \]
\[ - \ell^{-1} \kappa^2 H^2 \int_{Q_\ell(x_0)} |A - A_0^{x_0} + \nabla \varphi_0|^2 |\psi|^2 \, dx - \ell \kappa^2 \int_{Q_\ell(x_0)} |\psi|^2 \, dx. \]

Using the gauge invariance, the bound $|\psi| \leq 1$ and the inequality in Proposition 2.2, we get the following lower bound
\[ \mathcal{E}_0\left( \psi, A; Q_\ell(x_0) \right) \geq (1 - \ell) \mathcal{E}_0\left( e^{-\kappa H \varphi_0} \psi, A_0^{x_0}; Q_\ell(x_0) \right) - C \kappa^2 \ell^3. \]

In a similar fashion, we prove the upper bound
\[ \mathcal{E}_0\left( \psi, A; Q_\ell(x_0) \right) \leq (1 + \ell) \mathcal{E}_0\left( e^{-\kappa H \varphi_0} \psi, A_0^{x_0}; Q_\ell(x_0) \right) + C \kappa^2 \ell^3. \]

Inserting the foregoing lower and upper bounds into (3.2), we get the first inequality in Proposition 3.1.

Now we prove the second inequality in Proposition 3.1. We multiply the first G-L equation in (1.4) by $\psi$ and integrate by parts in the integral over $Q_\ell(x_0)$. We get
\[ \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} |\psi(x)|^4 \, dx = \mathcal{E}_0\left( \psi, A; Q_\ell(x_0) \right) + \int_{\partial Q_\ell(x_0)} \bar{\psi} \left( \nu \cdot (\nabla - i \kappa H A) \psi \right) \, d\sigma(x). \]

Using the bounds $|\psi| \leq 1$ and $|\nabla - i \kappa H A| \psi | \leq C \kappa$ in Proposition 2.1 we get that the boundary term is bounded by $C \kappa \ell$, where $C$ is a constant.

Now, using (3.2), we get
\[ - \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} |\psi(x)|^4 \, dx - g(b) \kappa^2 |Q_\ell(x_0)| \leq C \left( \ell + (\ell \kappa)^{-1} \right) \kappa^2 |Q_\ell(x_0)|. \]

□

4. PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 has some similarities with the analysis of diamagnetism 11 and the computation of the quantum supercurrent 9.

For the proof of Theorem 1.1, it is easier to work with rescaled variables.

Definition 4.1. Let $x_0 \in \Omega$, $\ell > 0$ and $f \in H^1(\Omega)$ and suppose that $Q_\ell(x_0) \subset \Omega$. We define the new function $\tilde{f}$ on $Q_{\ell \sqrt{\kappa H}} := Q_{\ell \sqrt{\kappa H}}(0)$ as follows:
\[ \tilde{f}(y) = f \left( x_0 + \frac{y}{\sqrt{\kappa H}} \right). \]

For $H = \beta c$ and $R = \ell \sqrt{\kappa H}$, we have the following relation:
\[ \frac{1}{\kappa^2 |Q_\ell(x_0)|} \mathcal{E}_0\left( \tilde{f}, A_0^{x_0}; Q_\ell(x_0) \right) = \frac{1}{|Q_R|} \int_{Q_R} \left( b |(\nabla - i A_0) \tilde{f}|^2 - |\tilde{f}|^2 + \frac{1}{2} |\tilde{f}|^4 \right) \, dy. \] (4.1)
Lemma 4.2. For \( b \in (0,1) \), there exist \( \kappa_0, R_0 > 0 \) and a positive-valued function \( r(\cdot, \cdot) \) such that \( \lim_{(t-1,s) \to 0} r(t,s) = 0 \) and the inequality

\[
g'(b_+) - r(R, \ell) \leq \frac{1}{|Q_R|} \int_{Q_R} |(\nabla - iA_0)\tilde{f}|^2 \, dy \leq g'(b_-) + r(R, \ell),
\]

holds for (cf. Prop. 3.1).

Lemma 4.3. There exists a function \( \bar{r}(\cdot, \cdot) \) such that \( \lim_{(t-1,s) \to \bar{0}} \bar{r}(t,s) = 0 \) and, under the assumptions in Lemma 4.2, the following inequality holds

\[
g'(b_+) - \bar{r}(R, \ell) \leq \frac{1}{|Q_R|} \int_{Q_R} |\tilde{f}(y)|^2 \, dy \leq g'(b_-) + \bar{r}(R, \ell).
\]
Proof. By (4.1) and Proposition 3.1
\[ |F_{b,Q_R}(\tilde{f}) - g(b)|Q_R| \leq CR^{3/2}. \]
By the formula for the \( L^4 \)-norm of \( \psi \) in Proposition 3.1 and a change of variables, we have
\[ \left| \int_{Q_R} |\tilde{f}(y)|^4 \, dy + 2g(b)|Q_R| \right| \leq C(\ell + R^{-1})|Q_R|. \]
Combining the aforementioned formulae and the one in (4.2), we get the formula for the integral of \( |\tilde{f}|^2 \). \( \square \)

By rescaling, we deduce from Lemma 4.3.

**Theorem 4.4.** Let \( b \in (0,1) \). There exist \( C, R_0, \kappa_0 > 0 \) and a positive-valued function \( \lambda(\cdot) \) such that \( \lim_{\kappa \to +\infty} \lambda(\kappa) = 0 \) and the following is true.

Suppose that

- \( \kappa \geq \kappa_0 \) and \( H = b\kappa \);
- \( R_0 \kappa^{-1} \leq \ell \leq \kappa_0^{-1} \);
- \( Q_\ell \) is the interior of a square of side length \( \ell \) satisfying \( \overline{Q_\ell} \subset \Omega \);
- \((\psi, A)_{\kappa,H}\) is a minimizer of the functional in (1.1).

Then the following inequalities hold
\[ g'(b_+) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi(x)|^2 \, dx \leq g'(b_-) - 2g(b) + \lambda(\kappa). \]

**Proof of Theorem 4.4** The proof of the statements (2) and (3) regarding the estimate of the \( L^2 \)-norm of \( \psi \) and the weak convergence of \( |\psi|^2 \) both follow from Theorem 4.4 in a standard manner, see e.g. [4] Proof of Thm. 4.1.

Now, the proof of statement (1) regarding the \( L^2 \)-norm of the magnetic gradient is a consequence of statement (1) and the formulas in (1.10) and (1.11).

The first inequality in statement (4) regarding the supercurrent results from statement (1) and the following inequality
\[ |j(\psi,A)| \leq |(\nabla - i\kappa HA)\psi|, \]
which is a consequence of the definition of the supercurrent in (1.6) and the inequality in (2.8).

The other inequality for the \( L^1 \)-norm of the supercurrent results from the inequality
\[ |j(\psi,A)| \leq |\psi||\nabla - i\kappa HA|\psi|, \]
the Cauchy-Schwarz inequality and the conclusions in Statements (1) and (2). \( \square \)

5. New Properties of the Function \( g \)

5.1. **Universal estimates of \( g(b) \).** As a by-product of the result in Theorem 4.1 we get new properties of the function \( g(\cdot) \) introduced in (2.4).

Using the classical bound \( |\psi| \leq 1 \) (see (2.8)), we deduce from (1.14) that
\[ \forall b \in (0,1), \quad g'(b_+) - 2g(b) \leq 1. \quad (5.1) \]
We can obtain an upper bound on the left-derivative of \( g \) as well by expanding the square in the inequality
\[ \int_{\Omega} (1 - |\psi(x)|^2)^2 \, dx \geq 0 \text{ then using (1.11) and (1.14):} \]
\[ \forall b \in (0,1), \quad g'(b_-) \leq \frac{1}{2} + g(b). \quad (5.2) \]
Note that (5.2) is better than (5.1) since $g'(b_+) \leq g'(b_-)$, $g(b) \geq -\frac{1}{2}$, hence $\frac{1}{2} + g(b) \leq 1 + 2g(b)$.

5.2. **On the behavior of $g(b)$ as $b \to 0_+$.** Taking the limit as $b \to 0_+$ in (5.1) and noticing that $g'(b_{\pm}) \geq 0$ and $g(0) = -\frac{1}{2}$, we get

$$\lim_{b \to 0_+} g'(b_{\pm}) = 0.$$ 

Consequently, there exists a sequence $(b_n)_{n \geq 1} \subset \mathcal{R}$ such that $b_n \to 0$ and $g'(b_n) \to 0$ ($\mathcal{R}$ is defined in (2.3)). On the other hand, it is proved in [20] that as $b \to 0_+$,

$$g(b) = \frac{1}{2} + \frac{b}{4} \ln \frac{1}{b} + o \left( b \ln \frac{1}{b} \right).$$

We deduce from this that:

- $g'(0_+) = +\infty$;
- the function $b \mapsto g'(b_{\pm})$ is not continuous at $0$;
- The asymptotics in (5.3) can not be differentiated, i.e. the formula

$$g'(b) \sim \frac{1}{4} \ln b - \frac{1}{4}$$

does not hold as $b \to 0_+$. Simply, the aforementioned sequence $(b_n)$ violates this formula.

5.3. **The radial symmetry.** Next we try to extract more information about the function $g$ by exploiting the radial symmetry. The function $g$ may be expressed as follows

$$\forall \ b \in (0, 1), \quad g(b) = \lim_{R \to \infty} \frac{\epsilon_{\text{disc}}(b, R)}{\pi R^2},$$

where

$$\epsilon_{\text{disc}}(b, R) = \inf \{ F_{b, D_R}(u) : u \in H^1_0(D_R) \},$$

$D_R = \{ x \in \mathbb{R}^2 : |x| < R \}$ and $F_{b, D_R}$ is the functional introduced in (2.1). The proof of (5.3) is standard (see [2, 13]). It follows by covering the disc $D(0, R)$ with squares $(Q_{r, j})_j$ with side-length $1 \ll R' \ll R$ and using the estimates in (5.4) (for $r = R'$). We omit the technical details.

We restrict the functional $F_{b, D_R}(u)$ on configurations of the form

$$u(r, \theta) = e^{im\theta} f(r),$$

where $f : (0, R) \to \mathbb{C}$, $m \in \mathbb{Z}$ and $(r, \theta)$ denote the polar coordinates.

Note that $u \in H^1_0((B(0, R))$ if and only if $f \in \mathcal{D}_m, R$, where

$$\mathcal{D}_m, R = \left\{ f : \sqrt{r} f', \sqrt{r} f, \frac{m}{\sqrt{r}} f \in L^2((0, R); \mathbb{R}), \ f(R) = 0 \right\}.$$ 

(5.7)

Furthermore,

$$F_{b, D_R}(u) = G_{m,b,R}(f),$$

where

$$G_{m,b,R}(f) = 2\pi \int_0^R \left( b|f'(r)|^2 + b \left( \frac{m}{r} - \frac{r}{2} \right)^2 |f(r)|^2 - |f(r)|^2 + \frac{1}{2} |f(r)|^4 \right) r dr.$$ 

(5.8)

Consequently, we define the following ground state energy

$$\epsilon_{\text{ID}}(m, b, R) = \inf \{ G_{b,m,R}(f) : f \in \mathcal{D}_m, R \}. $$

(5.9)

A minimizer $f_{m,b,R}$ exists, can be selected real-valued and non-negative (because $|f_{m,b,R}|$ is a minimizer too) and satisfies the following ODE

$$-f''_{m,b,R}(r) - \frac{1}{r} f'(r) + \left( \frac{m}{r} - \frac{r}{2} \right)^2 f_{m,b,R}(r) = \frac{1}{b} \left( 1 - |f_{m,b,R}(r)|^2 \right) f_{m,b,R}(r) \quad \text{in (0, R)}.$$ 

(5.10)

When the magnetic field is absent (i.e. the term $\frac{m}{r}$ is dropped from (5.10)) and $R = +\infty$, (5.10) has been studied in many papers, for example [16].
Now we define
\[ g_m(b) = \limsup_{R \to +\infty} \frac{\epsilon^{1D}(m, b, R)}{\pi R^2}. \quad (5.11) \]
We then have,
\[ \forall \ b \in (0, 1), \ \forall \ m \in \mathbb{Z}, \ \ g(b) \leq g_m(b). \quad (5.12) \]

Remark 5.1. A natural question is then to determine if for any \( b \in (0, 1) \) there exists \( m \in \mathbb{Z} \) such that \( g(b) = g_m(b) \) and if the discontinuity of \( g' \) corresponds to the case when two \( m's \) satisfy this property.

6. Extension to three dimensional domains

The result in Theorem 1.1 can be easily extended to the three dimensional Ginzburg-Landau model. In this section, \( \Omega \subset \mathbb{R}^3 \) denotes a bounded smooth open set with a smooth boundary. We introduce the Ginzburg-Landau functional in \( \Omega \) as follows [10, 21]
\[ E^{3D}(\psi, A) = E^{3D}_{\kappa, H}(\psi, A) = \int_{\Omega} \left[ (\nabla - i\kappa H A)\psi^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right] dx + \kappa^2 H^2 \int_{\mathbb{R}^3} |\nabla A - \beta|^2 dx, \quad (6.1) \]
where \( \beta = (0,0,1) \).

The configuration \( (\psi, A) \) belongs to the space \( H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}, F}(\mathbb{R}^3) \) with \( H^1_{\text{div}, F}(\mathbb{R}^3) \) defined as follows. Let \( H^1(\mathbb{R}^3) \) be the homogeneous Sobolev space, i.e. the closure of \( C^\infty(\mathbb{R}^3) \) under the norm \( u \mapsto ||u||_{H^1(\mathbb{R}^3)} := ||\nabla u||_{L^2(\mathbb{R}^3)} \). Let further \( F(x) = (-x_2/2,x_1/2,0) \). Clearly \( \text{div} F = 0 \).

We define the space,
\[ H^1_{\text{div}, F}(\mathbb{R}^3) = \{ A : \text{div} A = 0, \text{ and } A - F \in H^1(\mathbb{R}^3) \}. \quad (6.2) \]

Now we define the ground state energy,
\[ E_{gs}(\kappa, h_{ex}, \kappa, H) = \inf \left\{ E^{3D}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}, F}(\mathbb{R}^3) \right\}. \quad (6.3) \]
This energy is estimated in [13] when \( H = bk, \ b \in (0,1) \) is a fixed constant and \( \kappa \to \infty \). Using the methods in [13], we may easily adapt the proof of Theorems 1.1 and 4.4 to get the following result:

**Theorem 6.1.** For \( b \in (0,1) \), there exist \( C, R_0, \kappa_0 > 0 \) and a positive-valued function \( \lambda(\cdot) \) such that \( \lim_{\kappa \to +\infty} \lambda(\kappa) = 0 \) and the following is true.

Suppose that
- \( \kappa \geq \kappa_0 \) and \( H = bk \);
- \( R_0 \kappa^{-1} \leq \ell \leq R_0^{-1} \);
- \( Q_\ell \) is the interior of a cube of side length \( \ell \) satisfying \( \overline{Q_\ell} \subset \Omega \);
- \( (\psi, A)_{\kappa, H} \) is a minimizer of the functional in (5.1).

Then the following inequalities hold
\[ g'(b_+) - 2g(b) - \lambda(\kappa) \leq \frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi|^2 \ dx \leq g'(b_-) - 2g(b) + \lambda(\kappa). \]

As a consequence of Theorem 6.1 we can get that the minimizer \( (\psi, A)_{\kappa, H} \) satisfies the following weak convergence for \( H = bk, \ b \in \mathcal{R} \) and \( \kappa \to \infty \):
\[ |\psi|^2 \to g'(b) - 2g(b) \quad \text{in} \ D'(\Omega). \]

This result is complementary to the results in [14] and [19] devoted respectively to the regimes \( b > 1 \) (surface superconductivity) and \( b \to 1^- \) (bulk superconductivity near \( H_{C_2} \)) for three dimensional superconducting samples.
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