SMOOTH RATIONAL SURFACES OF $d=11$ AND $\pi=8$ IN $\mathbb{P}^5$.

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Abstract. We construct a linearly normal smooth rational surface $S$ of degree 11 and sectional genus 8 in the projective fivespace. Surfaces satisfying these numerical invariants are special, in the sense that $h^1(O_S(1)) > 0$. Our construction is done via linear systems and we describe the configuration of points blown up in the projective plane. We also present a short list, generated by the adjunction mapping, of linear systems whom are the only possibilities for other families of surfaces with the prescribed numerical invariants.

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1. Introduction

A result due to Arrondo, Sols and Pedreira [AS92] states that there are only finitely many families of non-general smooth surfaces in $G(1,3)$. Since the Plücker embedding embeds $G(1,3)$ as a smooth quadric into $\mathbb{P}(\wedge^2 V) \cong \mathbb{P}^5$, it follows that there are finitely many non-general smooth surfaces contained within a smooth quadric in $\mathbb{P}^5$. The latter idea led Papantonopoulou, Verra, Arrondo, Sols, and Gross to classify smooth non-general surfaces of degree $\leq 10$ contained within smooth quadrics in $\mathbb{P}^5$. For a complete classification, see [AS92] for degree $\leq 9$ and [Gro93] for degree 10.

A continuation of this classification is to study smooth non-general surfaces of degree 11 in $\mathbb{P}^5$. As such, this paper is a study of smooth rational surfaces of degree 11 in $\mathbb{P}^5$. In particular, we are only interested in special surfaces $S$ i.e., surfaces with $h^1(O_S(1)) > 0$, such that a simple application of Riemann-Roch yields that the sectional genus must be $\geq 8$. On the other hand, another application of Riemann-Roch tells us that if the sectional genus $> 8$, then the hyperplane intersection lies on a quadric. Therefore, we study smooth rational surfaces of degree 11 with sectional genus 8 in this paper.

Our point of view is the study of complete linear systems. The complete linear systems are generated by some results on the adjunction mapping, due to Sommese and Van de Ven [SV87]. The study is based on describing the geometry of the points blown up in $\mathbb{P}^2$ to obtain the surface is question, similar to the classification of smooth surfaces in $\mathbb{P}^4$ done by Catanese and Hulek [CH97], and Popescu and Ranestad [PR96].

Our main result is the following:
Theorem 1. There exists at least one family of linearly normal smooth rational surfaces in $\mathbb{P}^5$ with degree 11 and sectional genus 8. The family of surfaces is isomorphic to $\mathbb{P}^2$ blown up in 17 points and the embedding complete linear system is

$$[7L - 2E_1 - \ldots - 2E_7 - E_8 - \ldots - E_{17}]$$

where $L$ is the pullback of a line in $\mathbb{P}^2$ and $E_i$ are the exceptional curves of the blowup. Conversely, every linearly normal smooth rational surface in $\mathbb{P}^5$ with degree 11 and sectional genus 8 has

$$-10 \leq K^2_S \leq -7$$

and is one of the following embedding complete linear systems

(1). $[7L - 2E_1 - \ldots - 2E_7 - E_8 - \ldots - E_{17}]$

(2). $[9L - 3E_1 - \ldots - 3E_6 - 2E_7 - 2E_8 - E_9 - \ldots - E_{16}]$, where $e \leq 2$.

(3). $|4B + (4 - 2e)F - 2E_1 - E_2 - \ldots - E_{18}|$, where $e = 0$.

(4). $|4B + (5 - 2e)F - 2E_1 - \ldots - 2E_4 - E_5 - \ldots - E_{17}|$, where $e \leq 3$.

(5). $|4B + (6 - 2e)F - 2E_1 - \ldots - 2E_7 - E_8 - \ldots - E_{16}|$, where $e \leq 5$.

where $B$ is a section with self-intersection $B^2 = e$ on $\mathbb{P}_e$, $F$ is a ruling on $\mathbb{P}_e$, $L$ is the pullback of a line in $\mathbb{P}^2$ and $E_i$ are the exceptional curves of the blowup.

In Section 2 we prove the existence of a smooth rational surface in $\mathbb{P}^5$ with the prescribed numerical invariants. The reader may refer to the proof of Theorem 7 for the details of the special configuration of the points blown up in $\mathbb{P}^2$. In [AS92], appendix to section 4, Arrondo and Sols outline an incomplete example of this surface without details. Our construction verifies their conjecture that there exists a quartic and sextic passing through the points blown up. Beside the construction of (1) in Theorem 1 H.Abo and K.Ranestad (unpublished) have used Macaulay2 to show that case (2) of Theorem 1 is in the linkage class of a singular quadric surface and two smooth cubic surfaces in $\mathbb{P}^5$.

In Section 3 we give two examples of complete linear systems whom cannot be both very ample and have six global sections simultaneously, by lifting sections of curves on the surface to global sections on the surface. In Section 4 we generate a finite list of complete linear systems satisfying the numerical invariants of the surfaces in question. Then we shorten the list to the five complete linear systems depicted in Theorem 1 by making use of a result on curves of low degree, due to Catanese and Franciosi, and the lifting examples in Section 3.

2. A construction

Let $S$ be a smooth rational surface. Given a line bundle $\mathcal{L}$ on $S$, it is in general difficult to decide whether $\mathcal{L}$ is very ample or not. There is however a versatile result, due to Alexander and Bauer, which provides us with sufficient conditions for $\mathcal{L}$ to be very ample. The idea behind Alexander and Bauer’s result is that if $\mathcal{L}$ restricts to a very ample line bundle on a suitable family of curves on $S$ then $\mathcal{L}$ is itself very ample on $S$, given some minor assumptions. This allows us to answer the question of $\mathcal{L}$ being very ample on $S$ by answering the question of $\mathcal{L}$ being very ample on some curves on $S$. We state their precise result.

Lemma 2 (Alexander-Bauer). An effective line bundle $\mathcal{O}_S(H) \cong \mathcal{O}_S(A_1) \otimes \mathcal{O}_S(A_2)$ on a smooth surface $S$ is very ample, if each one of the following is true:

(1). $h^0(\mathcal{O}_S(A_i)) \geq 2$, for some $i$.

(2). $\mathcal{O}_A(H)$ is very ample, for all $A \in |A_1| \cup |A_2|$.

(3). $H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_A(H))$ is surjective, for all $A \in |A_1| \cup |A_2|$.

Proof. See Proposition 5.1 in [CF93].
The Alexander-Bauer lemma does not give any information on how to determine if $L$ restricts to a very ample line bundle on curves. Recall that it follows from Riemann-Roch that $L$ is a very ample line bundle on an irreducible curve $C$ if $\deg(L \otimes \mathcal{O}_C) \geq 2p_a(C) + 1$. Catanese, Franciosi, Hulek and Reid have generalized the latter into a result which is true for both irreducible and reducible curves. The part of their result which we will be using is the following.

**Theorem 3** (Curve embedding). Let $A$ and $B$ be effective divisors on a smooth surface $S$. Then $\mathcal{O}_A(H)$ is very ample whenever $H.A' \geq 2p_a(A') + 1$, for every subcurve $A' \subset A$.

*Proof.* See Theorem 1.1 in [CFHR99].

Let $H$ be a very ample divisor on a smooth surface $S$. Recall that the degree of every curve $C$ on $S$ is $H.C > 0$. Catanese and Franciosi have improved the lower bound of $H.C$ when $C$ is a curve of small arithmetic genus. We state their result.

**Proposition 4.** Suppose $H$ is a very ample divisor on a smooth surface $S$. Then every curve $C$ on $S$ with arithmetic genus $p_a(C) \leq 2$ has degree $H.C \geq 2p_a(C) + 1$. In particular, if the degree $H.C \leq 3$ then the arithmetic genus $p_a(C) \leq 1$.

*Proof.* See Proposition 5.2 in [CF93].

Due to the usefulness of the decomposition $\mathcal{O}_S(H) \simeq \mathcal{O}_S(A_1) \otimes \mathcal{O}_S(A_2)$ in the Alexander-Bauer lemma, we make the following definition.

**Definition 5.** Let $H$ be an effective divisor on a smooth surface $S$. We say that $H \equiv A_1 + A_2$ is a nice decomposition if $A_1$ and $A_2$ are both effective divisors on $S$ such that $A_1$ is non-special on $S$, i.e. $h^1(\mathcal{O}_S(A_1)) = 0$, and the intersection product $H.A_2 = 2p_a(A_2) - 2$.

We make the following observation about nice decompositions.

**Lemma 6.** Let $H \equiv A + B$ be a nice decomposition of $H$ on a smooth surface $S$. Suppose the intersection product $H.A = 2p_a(A) - 2$ and the divisor $(B - K_S)|_A$ is effective on $A$. Then $\mathcal{O}_A(H) \simeq \omega_A$.

*Proof.* The adjunction formula tells us that $\omega_A \simeq \mathcal{O}_A(A + K_S)$, so it suffices to show that $\mathcal{O}_A(B - K_S) \simeq \omega_A$. Combining $H.A = 2p_a(A) - 2$ with the adjunction formula we get $A.(B - K_S) = 0$. Since $\mathcal{O}_A(B - K_S)$ is effective on $A$, we obtain $\mathcal{O}_A \simeq \mathcal{O}_A(B - K_S)$, or equivalently $\mathcal{O}_A(H) \simeq \mathcal{O}_A(A + K_S)$. \hfill \Box

We are now ready to construct a smooth rational surface with the prescribed numerical invariants. Furthermore, this construction will prove the statement about existence in Theorem 1.

**Theorem 7.** It is possible to choose points $x_1, \ldots, x_5, y_1, y_2, z_1, \ldots, z_{10} \in \mathbb{P}^2$ such that the divisor class

$$H \equiv 7L - \sum_{i=1}^{5} 2E_i - \sum_{i=1}^{2} 2F_i - \sum_{i=1}^{10} G_i$$

is very ample on $S$ and $|H|$ embeds $S$ as a rational surface of degree 11 and sectional genus 8 in $\mathbb{P}^5$, such that

$$\pi: S \rightarrow \mathbb{P}^2$$

denotes the morphism obtained by blowing up the points $x_1, \ldots, x_5, y_1, y_2, z_1, \ldots, z_{10}$, $E_i = \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$, $G_i = \pi^{-1}(z_i)$ and $L = \pi^*l$ where $l \subset \mathbb{P}^2$ is a line.

*Proof.* We begin by choosing $x_1, \ldots, x_5 \in \mathbb{P}^2$ in general position, such that

1. No two points $x_i$ are infinitely near.
2. No three points $x_i$ are collinear.

The part of their result which we will be using is the following.

**Proposition 4.** Suppose $H$ is a very ample divisor on a smooth surface $S$. Then every curve $C$ on $S$ with arithmetic genus $p_a(C) \leq 2$ has degree $H.C \geq 2p_a(C) + 1$. In particular, if the degree $H.C \leq 3$ then the arithmetic genus $p_a(C) \leq 1$.

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We make the following observation about nice decompositions.

**Lemma 6.** Let $H \equiv A + B$ be a nice decomposition of $H$ on a smooth surface $S$. Suppose the intersection product $H.A = 2p_a(A) - 2$ and the divisor $(B - K_S)|_A$ is effective on $A$. Then $\mathcal{O}_A(H) \simeq \omega_A$.

*Proof.* The adjunction formula tells us that $\omega_A \simeq \mathcal{O}_A(A + K_S)$, so it suffices to show that $\mathcal{O}_A(B - K_S) \simeq \omega_A$. Combining $H.A = 2p_a(A) - 2$ with the adjunction formula we get $A.(B - K_S) = 0$. Since $\mathcal{O}_A(B - K_S)$ is effective on $A$, we obtain $\mathcal{O}_A \simeq \mathcal{O}_A(B - K_S)$, or equivalently $\mathcal{O}_A(H) \simeq \mathcal{O}_A(A + K_S)$. \hfill \Box

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denotes the morphism obtained by blowing up the points $x_1, \ldots, x_5, y_1, y_2, z_1, \ldots, z_{10}$, $E_i = \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$, $G_i = \pi^{-1}(z_i)$ and $L = \pi^*l$ where $l \subset \mathbb{P}^2$ is a line.

*Proof.* We begin by choosing $x_1, \ldots, x_5 \in \mathbb{P}^2$ in general position, such that

1. No two points $x_i$ are infinitely near.
2. No three points $x_i$ are collinear.
Note that the open conditions O1 and O2 are satisfied for a general choice of five points in \( \mathbb{P}^2 \). Let \( \pi_1 : S_1 \to \mathbb{P}^2 \) denote the morphism obtained by blowing up \( x_1, \ldots, x_5 \) and denote \( E_i = \pi_1^{-1}(x_i) \). On the rational surface \( S_1 \) we study the complete linear systems associated to the following two divisor classes:

\[
-2K_{S_1} = 6L - 2E_1 - \ldots - 2E_5 \\
-L - K_{S_1} = 4L - E_1 - \ldots - E_5.
\]

Since \( (S_1, -K_{S_1}) \) is a quartic Del Pezzo surface, the anti-canonical divisor \(-K_{S_1}\) is very ample on \( S_1 \). In particular, this means that \(-2K_{S_1}\) is very ample by using the Segre embedding and recalling that \( \mathcal{O}_{\mathbb{P}^N_1 \times \mathbb{P}^N_2}(1) \simeq p_1^* \mathcal{O}_{\mathbb{P}^N_1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^N_2}(1) \), where \( p_i : \mathbb{P}^N_1 \times \mathbb{P}^N_2 \to \mathbb{P}^{N_i} \) is the \( i \)-th projection map. Furthermore, since \(|-L|\) is base-point free it follows that \( L - K_{S_1} \) is very ample by another use of the Segre embedding. So \(-2K_{S_1}\) and \( L - K_{S_1} \) are very ample on \( S_1 \). By Bertini’s theorem, Theorem 20.2 in [BPV84], a general choice of curves in \(|-2K_{S_1}|\) and \(|L - K_{S_1}|\) are both irreducible and smooth.

Next we choose the points \( y_1, y_2, z_1, \ldots, z_{10} \in \mathbb{P}^2 \) such that the complete linear systems

\[
\Delta_1 = |6L - 2x_i - \sum y_i - \sum z_i| \\
\Delta_2 = |4L - x_i - \sum y_i - \sum z_i|
\]

on \( \mathbb{P}^2 \) satisfy the following closed condition

(C1). There exists a pair \((D_1, D_2) \in \Delta_1 \times \Delta_2\) such that \( D_1 \) and \( D_2 \) share common tangent directions \( y_i' \) and \( y_2' \) at the points \( y_1 \) and \( y_2 \), and that \( D_1 \cap D_2 = \sum x_i + \sum y_i + \sum y_i' + \sum z_i \).

and if \( \pi_2 : S \to S_1 \) denotes the morphism obtained by blowing up \( y_1, y_2, z_1, \ldots, z_{10} \), then the following open conditions are satisfied.

(O3). \(|L - \sum_{i \in I} E_i - F_j| = \emptyset\), for \(|I| \geq 2\) and for all \( j \).

(O4). \(|2L - E_1 - \ldots - E_5 - F_j| = \emptyset\), for all \( j \).

(O5). \(|6L - 2E_1 - \ldots - 2E_5 - 2F_1 - 2F_2 - G_1 - \ldots - G_10| = \emptyset\).

(O6). \(|6L - 2E_1 - \ldots - 2E_5 - 3F_j - F_{3-j} - G_1 - \ldots - G_10| = \emptyset\), for all \( j \).

Note that if the linear systems in O3 or O4 are non-empty, then there exists a curve \( C \) on the surface \( S \) such that \( HC.C \leq 0 \). If the linear system in O5 is non-empty, then there exists a curve on the surface \( S \) satisfying \( p_a(C) = 2 \) and \( HC.C = 2p_a(C) \). Then Proposition 4 would imply that \( H \) could not be very ample. Now, we show that the closed condition C1 is a non-empty condition.

Start off by choosing a smooth and irreducible curve \( A_1 \in |-2K_{S_1}| \), recall that this is possible due to Bertini’s theorem, and consider the incidence \( \Lambda \subset A_1 \times A_1 \times |L - K_{S_1}| \), given by

\[
\Lambda = \{(y_1, y_2, B_1) \mid y_1, y_2 \in A_1 \cap B_1, A_1 \text{ and } B_1 \text{ share common tangent directions at } y_1 \text{ and } y_2\}.
\]

Now, choose a triple \((y_1, y_2, B_1) \in \Lambda\) such that \( B_1 \) is smooth and irreducible on \( S_1 \). Recall that this is possible due to Bertini’s theorem. Since \( \{B_1\}_{\Lambda} \) is base-point free, it follows that the curve \( B_1 \) is smooth at each closed point in the zero-dimensional scheme \( A_1 \cap B_1 \). So we may set

\[
A_1 \cap B_1 = \sum x_i + \sum y_i + \sum y_i' + \sum z_i
\]

where \( \{z_i\} \) are remaining points on the intersection of \( A_1 \) and \( B_1 \). Furthermore, there are \( \# \{z_i\} = 6 \cdot 4 - 2 \cdot 5 - 4 = 10 \) number of distinct points in \( \{z_i\} \). Now we blow up the points \( y_1, y_2, z_1, \ldots, z_{12} \in S_1 \) to obtain a morphism \( \pi_2 : S \to S_1 \), such that we may define \( \pi = \pi_1 \circ \pi_2 : S \to \mathbb{P}^2 \). Denote

\[
A \equiv 6L - 2E_1 - \ldots - 2E_5 - F_1 - F_2 - G_1 - \ldots - G_{10} \\
B \equiv 4L - E_1 - \ldots - E_5 - F_1 - F_2 - G_1 - \ldots - G_{10}
\]
as the divisor classes of the strict transforms of $A_1$ and $B_1$ on $S$, respectively. Then the sublinear systems $|\pi(A)| \subset \Delta_1$ and $|\pi(B)| \subset \Delta_2$ are both non-empty in $\mathbb{P}^2$, which in turn yields that $\Delta_1, \Delta_2 \neq \emptyset$. Due to our construction, the curves $\pi(A)$ and $\pi(B)$ both share common tangent directions and two points and the points $x_1, \ldots, x_5, y_1, y_2, z_1, \ldots, z_{10}$ lie on the complete intersection between $\pi(A)$ and $\pi(B)$. Therefore, the closed condition $C_1$ is non-empty.

On the surface $S$, we study the curves $A$ and the following two divisor classes:

$$C \equiv L - F_1 - F_2$$

$$H := A + C \equiv 7L - 2E_1 - \ldots - 2E_5 - 2F_1 - 2F_2 - G_1 - \ldots - G_{10}.$$  

Lemma 8. $\mathcal{O}_A(H) \cong \omega_A$

Proof. Note that, by construction, $A \cap B = y'_1 + y'_2$ on the surface $S$. Since tangent directions $y'_1, y'_2$ at the points $y_1, y_2 \in S$ corresponds to points on the exceptional divisors $F_1, F_2 \subset S$, we have $B_i \equiv (F_1 + F_2)_i$. Moreover, since $A_1, (B_1 - F_1 - F_2) = 0$ and $B_i \equiv A_i$ is effective by construction, Lemma 8 yields that $\mathcal{O}_A(B_1 - F_1 - F_2) \cong \omega_A$. On the other hand, $(C - K_S)_i \equiv (B_1 - F_1 - F_2)_i$ such that $\mathcal{O}_A(C - K_S) \cong \mathcal{O}_A(B_1 - F_1 - F_2)$. By twisting the latter sheaves by $\mathcal{O}_A(A + K_S)$, we obtain $\mathcal{O}_A(H) \cong \mathcal{O}_A(A + K_S)$ such that the adjunction formula implies that $\mathcal{O}_A(H) \cong \omega_A$. □

We are ready to show that $\mathcal{O}_S(H)$ admits six global sections on the surface $S$.

Lemma 9. $h^0(\mathcal{O}_S(H)) = 6$.

Proof. Consider the following short exact sequence

$$0 \to \mathcal{O}_S(C) \to \mathcal{O}_S(H) \to \mathcal{O}_A(H) \to 0.$$  

Here, $H^1(\mathcal{O}_S(C))$ since $C$ is a $(-1)$-curve on the surface $S$. Then $h^1(\mathcal{O}_S(H)) = h^1(\omega_A) = 1$, due to Lemma 8 combined with $\chi(\mathcal{O}_S(H)) = 5$ implies that $h^0(\mathcal{O}_S(H)) = 6$. □

To finish the proof it remains to show that $H$ is very ample on $S$. We show this by applying the Alexander-Bauer lemma and the curve embedding theorem to the decomposition $H \equiv A + C$. This requires us to determine the dimension of the complete linear system $|A|$.

Lemma 10. $h^0(\mathcal{O}_S(A)) = 2$.

Proof. It is suffices to show that $h^1(\mathcal{O}_S(A)) = 1$ since $\chi(\mathcal{O}_S(A)) = 1$. We take cohomology of the short exact sequence

$$0 \to \mathcal{O}_S(K_S - A) \to \mathcal{O}_S(K_S) \to \mathcal{O}_A(K_S) \to 0$$  

and note that the cohomology groups $H^i(\mathcal{O}_S(K_S))$ vanish whenever $0 \leq i \leq 1$, due to the rationality of $S$. Therefore, $h^0(\mathcal{O}_A(K_S)) = h^1(\mathcal{O}_S(K_S - A))$. Combining Lemma 8 and the adjunction formula, we have that $K_S|_A \equiv C_1|_A$. On the other hand, Serre duality yields that $h^1(\mathcal{O}_S(A)) = h^1(\mathcal{O}_S(K_S - A))$. Thus $h^1(\mathcal{O}_S(A)) = h^0(\mathcal{O}_A(C))$. Now, consider the cohomology of

$$0 \to \mathcal{O}_S(C - A) \to \mathcal{O}_S(C) \to \mathcal{O}_A(C) \to 0.$$  

We claim that the cohomology groups $H^i(\mathcal{O}_S(C - A)) = 0$, for $0 \leq i \leq 1$. To see this, note that $K_S - C + A = 2L - E_1 - \ldots - E_5 + F_1 + F_2$ is effective and non-special on $S$ such that $H^i(\mathcal{O}_S(C - A)) = 0$, for $0 \leq i \leq 1$, by Serre duality. Therefore, $h^0(\mathcal{O}_S(C)) = h^0(\mathcal{O}_A(C))$. Since $C$ is a non-special curve on $S$ it follows that $\chi(\mathcal{O}_S(C)) = h^0(\mathcal{O}_A(C)) = 1$. Hence $h^1(\mathcal{O}_S(A)) = 1$. □

Next we show that every assumption in the Alexander-Bauer lemma, except the very ampleness of $H|_A$, is satisfied. Then we proceed to show that $H|_A$ is very ample by using the Curve embedding theorem.
Lemma 11. The complete linear system \([H]\) restricts to a very ample linear system on \(C\) and the restriction maps \(H^0(\mathcal{O}_S(H)) \rightarrow H^0(\mathcal{O}_C(H))\) and \(H^0(\mathcal{O}_S(H)) \rightarrow H^0(\mathcal{O}_A(H))\) are surjective, for all \(A \in |A|\).

Proof. The first assertion is true since \(H.C > 2p_a(C) + 1\) and \(\dim |C| = 0\). For the second assertion, consider the short exact sequences

\[
0 \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_A(H) \rightarrow 0
\]

\[
0 \rightarrow \mathcal{O}_S(A) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_C(H) \rightarrow 0
\]

Since \(C\) is a non-special curve on \(S\), it follows from the first short exact sequence that \(H^0(\mathcal{O}_S(H)) \rightarrow H^0(\mathcal{O}_S(H))\) is surjective, for all \(A \in |A|\). For the map \(\alpha : \mathcal{O}^0(\mathcal{O}_S(H)) \rightarrow \mathcal{O}^0(\mathcal{O}_C(H))\), the exactness of the second short exact sequence and Lemma 9, we get \(\dim \ker(\alpha) = h^0(\mathcal{O}_S(A)) = 2\). Recalling Lemma 9, this means that \(\text{rank}(\alpha) = 4\). On the other hand, \(h^0(\mathcal{O}_C(H)) = \chi(\mathcal{O}_C(H)) = 4\) since \(H.C > 2p_a(C) - 2\). Thus \(\alpha\) is surjective.

To show that \(\mathcal{O}_A(H)\) is very ample, for all \(A \in |A|\), we partition \(|A|\) into the following two families of curves on \(S\) and consider each family separately.

\(\mathcal{A}_{\text{Good}} = \{D \in |A| : \text{Every } A' \leq D \text{ satisfies } A'.F_j \leq 1, \text{for all } j\}\).

\(\mathcal{A}_{\text{Bad}} = \{D \in |A| : \text{Some } A' \leq D \text{ satisfies } A'.F_j > 1, \text{for some } j\}\).

Lemma 12. \(\mathcal{O}_{F_j}(H), \mathcal{O}_{A-F_j}(H)\) and \(\mathcal{O}_{(A-F_j) \cup A}(H)\) are very ample, for all \(1 \leq j \leq 2\).

Proof. Note that \(\mathcal{O}_{F_j}(H)\) is very ample since \(H.F_j > 2p_a(F_j) + 1\). Next, we claim that the \(\mathcal{O}_{A-F_j}(H) \simeq \omega_{A-F_j}\). Taking cohomology of the short exact sequence

\[
0 \rightarrow \mathcal{O}_S(C + F_j) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_{A-F_j}(H) \rightarrow 0
\]

we note that \(C + F_j\) is a non-special curve on \(S\), such that \(h^1(\mathcal{O}_{A-F_j}(H)) = h^1(\mathcal{O}_S(H)) = 1\) due to Lemma 9. Combining the latter with \(H(A-F_j) = 2p_a(A-F_j) - 2\) we conclude that \(\mathcal{O}_{A-F_j}(H) \simeq \omega_{A-F_j}\).

Then the adjunction formula yields that

\[
\mathcal{O}_{A-F_j}(H) \simeq \mathcal{O}_{A-F_j}(A - F_j + K_S) \simeq \mathcal{O}_{A-F_j}(3L - E_1 - \ldots - E_5 - F_j).
\]

Now, let \(D\) be any subcurve of the curve \(A - F_j + K_S\) such that \(H.D \leq 2p_a(D)\). Then we may write

\[
D \equiv aL - b_1E_1 - \ldots - b_5E_5 - cF_j,
\]

where \(0 \leq a \leq 3\). If \(a \leq 2\), then we have \(H.D \leq 0\). This occurs if and only if at least three of \(E_1, \ldots, E_5, F_j\) are collinear or there exists a conic passing through \(E_1, \ldots, E_5, F_j\), or equivalently if and only if the open conditions O3 or O4 are not satisfied, respectively. So \(a = 3\). Then there are two cases for the arithmetic genus, namely \(0 \leq p_a(D) \leq 1\). If \(p_a(D) = 1\), then \(H.D \leq 2\) implies that \(\sum b_i + c \geq 9\) which implies that \(p_a(D) < 1\). If \(p_a(D) = 0\), then \(D\) meets exactly one of the exceptional curves \(E_1, \ldots, E_5, F_j\) twice such that \(H.D = 4\) which is greater than \(p_a(D)\). Thus, there are no such subcurves \(D\) of \(A - F_j - K_S\) and so the Curve embedding theorem yields that \(\mathcal{O}_{A-F_j}(H)\) is very ample. Next, we show that \(|H|\) embeds the intersection of \((A - F_j)\) and \(F_j\). Note that \(\varphi_H(H)\) spans a \(\mathbb{P}^3\) and \(\varphi_H(F_j)\) spans a \(\mathbb{P}^2\). So \(\text{Span}(\varphi_H(A_0 - F_j) \cup \varphi_H(F_j)) = \mathbb{P}^4\) implies that \(\varphi_H(A_0 - F_j) \cap \varphi_H(F_j)\) spans a line. The line meets the conic \(\varphi_H(F_j)\) is exactly two points, which are exactly the images of points in \((A - F_j) \cap F_j\) due to \((A - F_j).F_j = 2\).

Lemma 13. \(\mathcal{O}_A(H)\) is very ample, for all \(A \in \mathcal{A}_{\text{Bad}}\).
Proof. Let $A \in \mathcal{A}_{\text{Bad}}$. It is clear that $A$ is reducible since there exists a $A'$ subcurve of $A$ such that $A'.F_j > 1$, for some $j$, while $A.F_j = 1$ for all $j$. Then we may decompose

$$A \equiv (A - nF_1 - mF_2) + nF_1 + mF_2,$$

for some $n, m \in \mathbb{Z}_{\geq 0}$ such that $n + m \geq 1$. In fact, the open condition O6 implies that $n, m \neq 2$ which combined with the open condition O5 implies that $n + m \neq 2$. Moreover, if $n + m \geq 3$ then the inclusion map $H^0(\mathcal{O}_S(A_0 - nF_1 - mF_2)) \hookrightarrow H^0(\mathcal{O}_S(A_0 - 2F_j))$, for some $j$, yields that $n + m = 1$ such that

$$A \equiv (A - F_j) + F_j.$$ 

Now, recall that Lemma 12 implies that $|H|$ indeed embeds $(A - F_j), F_j$ and $(A - F_j) \cap F_j$. □

Lemma 14. $\mathcal{O}_A(H)$ is very ample, for all $A \in \mathcal{A}_{\text{Good}}$.

Proof. Let $A'$ be a subcurve of some $A \in \mathcal{A}_{\text{Good}}$. We claim that the blow-up morphism $\pi_2 : S \longrightarrow S_1$ defines an isomorphism $A' \simeq \pi_2(A')$. This is clearly the case for every point in the set $S\backslash \{y_1, y_2, z_1, \ldots, z_{10}\}$. Now, without loss of generality we may assume that $A'$ is irreducible in which case $A'.F_i \leq 1$ and $A'.G_j \leq 1$, for all $i, j$. Note that if $A'.F_i < 1$ (resp. $A'.G_j < 1$), then $A'$ does not meet $F_i$ (resp. $G_j$). So it suffices to show the isomorphism when $A'$ is restricted to exceptional divisors $\{F_i\}$ (resp. $\{G_j\}$) whom satisfy $A'.F_i = 1$ (resp. $A'.G_j = 1$). In the latter case it is straightforward to see that $\pi_2$ and $\pi_2^{-1}$ are inverses of each other, such that we get $A \simeq \pi(A)$. On the other hand, since $\pi(A) + K_S$ does not meet the points $y_1, y_2, z_1, \ldots, z_{10}$, it follows that the strict transform $A + K_S$ of $\pi(A) + K_S$ under $\pi_2$ and the proper transform $\pi_2^*(\pi(A) + K_S)$ of $A + K_S$ are linearly equivalent. In other words,

$$\pi_2^*\mathcal{O}_{\pi_2(A)}(-K_{S_1}) \simeq \mathcal{O}_A(H)$$

since $\pi(A)_2 = -2K_{S_1}$ and since $\mathcal{O}_A(H) \simeq \mathcal{O}_A(A + K_S)$ due to Lemma 8. Therefore, $\mathcal{O}_A(H)$ is very ample whenever $\mathcal{O}_{-2K_{S_1}}(-K_{S_1})$ is very ample. Now, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(K_{S_1}) \longrightarrow \mathcal{O}_{S_1}(-K_{S_1}) \longrightarrow \mathcal{O}_{-2K_{S_1}}(-K_{S_1}) \longrightarrow 0.$$ 

Taking cohomology and noting that $H^i(\mathcal{O}_{S_1}(K_{S_1}))$ vanishes, for $0 \leq i \leq 1$, we get that $H^0(\mathcal{O}_{S_1}(-K_{S_1})) \simeq H^0(\mathcal{O}_{-2K_{S_1}}(-K_{S_1}))$ such that $\mathcal{O}_{-2K_{S_1}}(-K_{S_1})$ is very ample if and only if $\mathcal{O}_{S_1}(-K_{S_1})$ is very ample. Now, recall that $\mathcal{O}_{S_1}(-K_{S_1})$ is indeed very ample since $(S_1, -K_{S_1})$ is a quartic Del Pezzo surface. □

This concludes the proof of Theorem 7.

3. Lifting examples

We present two examples of linear systems $|H|$ who cannot be both special and very ample on a surface with the prescribed numerical invariants. These examples will be used to rule out possible adjunction maps in the classification done in Theorem 25.

The idea behind the examples is to search for nice decompositions, $H \equiv A_1 + A_2$, such that a divisor $C$ on $S$ restricts to the zero divisor on one of the components $A_i$, and the section of $\mathcal{O}_{A_i}(C)$ lifts to a section of $\mathcal{O}_S(C)$. This in turn yields open conditions on the points blown up to obtain the surface, contradicting the very ampleness of $H$.

Proposition 15. Let $S$ be a smooth rational surface with $K_S^2 + 6 = 0$. Let $\pi : S \longrightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, \ldots, x_{15} \in \mathbb{P}^2$. Furthermore, let $E_i = \pi^{-1}(x_i)$ and $L = \pi^*1$, where $1 \subset \mathbb{P}^2$ is a line. Suppose the divisor class

$$H \equiv 10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j$$

has $h^0(\mathcal{O}_S(H)) = 6$. Then $H$ is not very ample on $S$. □
Proof. To begin with, note that the following condition is a necessity for \( H \) being very ample.

\((O1) \quad |6L - 2E_1 - \ldots - 2E_8 - E_9 - \ldots - E_{15}| = \emptyset.\)

If there were a curve \( C \) in the complete linear system depicted in \( O1 \), then the curve \( C \) would have arithmetic genus \( p_a(C) = 2 \) and degree \( H.C = 2 \) which would imply that \( H \) is not very ample, by Proposition\[4\]. The idea now is to show that the condition \( O1 \) is false, whenever \( h^0(\mathcal{O}_S(H)) = 6 \). To do this, we study the decomposition \( H \equiv A + B \), where
\[
A \equiv 7L - 3E_1 - 2E_2 - \ldots - 2E_9 - E_{10} - \ldots - E_{15} \\
B \equiv 3L - E_1 - \ldots - E_8.
\]

**Lemma 16.** The decomposition \( H \equiv A + B \) is nice.

Proof. Clearly \( |B| \) is non-empty, but there is a priori no reason to believe that \( |A| \) is non-empty since \( \chi(\mathcal{O}_S(A)) = 0 \). However, consider the short exact sequence
\[
0 \to \mathcal{O}_S(A) \to \mathcal{O}_S(H) \to \mathcal{O}_B(H) \to 0.
\]

Since \( H.B > 2p_a(B) - 2 \), it follows that \( h^1(\mathcal{O}_B(H)) = 0 \). The assumption that \( h^0(\mathcal{O}_S(H)) = 6 \) implies that \( h^1(\mathcal{O}_S(H)) = 1 \). Then the surjectivity of \( H^1(\mathcal{O}_S(A)) \to H^1(\mathcal{O}_S(H)) \) yields that \( h^1(\mathcal{O}_S(A)) \geq h^1(\mathcal{O}_S(H)) = 1 \) and the rationality of \( S \) tells us that \( h^2(\mathcal{O}_S(A)) = 0 \). Therefore \( h^0(\mathcal{O}_S(A)) = h^1(\mathcal{O}_S(A)) \geq 1 \), i.e. \( |A| \) is a non-empty complete linear system on \( S \). Furthermore, note that \( H.A = 2p_a(A) - 2 \) and note that \( B^2 > 2p_a(B) - 2 \) gives us \( h^1(\mathcal{O}_S(B)) = 0 \). So, the decomposition \( H \equiv A + B \) is indeed a nice decomposition. \(\Box\)

**Lemma 17.** \( \mathcal{O}_A(B - K_S) \simeq \mathcal{O}_A \)

Proof. Taking cohomology of the short exact sequence
\[
0 \to \mathcal{O}_S(B) \to \mathcal{O}_S(H) \to \mathcal{O}_A(H) \to 0
\]
we get that \( h^1(\mathcal{O}_S(H)) = h^1(\mathcal{O}_A(H)) = 1 \) since \( h^0(\mathcal{O}_S(H)) = 6 \) and \( B \) is non-special on \( S \). Using Serre duality we have \( h^1(\mathcal{O}_A(H)) = h^0(\mathcal{O}_A(K_A - H)) = 1 \). Then the adjunction formula tells us that \( K_A \equiv (A + K_S)|_A \), such that \( h^0(\mathcal{O}_A(B - K_S)) = 1 \). Now, Lemma\[5\] yields that \( \mathcal{O}_A(H) \simeq \omega_A \). Another use of the adjunction formula yields that \( \mathcal{O}_A(H) \simeq \mathcal{O}_A(A + K_S) \). Twisting the former with \( \mathcal{O}_A(-A - K_S) \) we obtain \( \mathcal{O}_S(B - K_S) \simeq \mathcal{O}_S(6L - 2E_1 - \ldots - 2E_8 - E_9 - \ldots - E_{15}) \simeq \mathcal{O}_A \). \(\Box\)

We proceed by showing that it is possible to lift the non-zero global section of \( (B - K_S)|_A \) to the surface \( S \), and thus showing that \( |B - K_S| \) is a non-empty complete linear system on \( S \).

**Lemma 18.** The divisor \( B - K_S \) is an effective divisor on \( S \).

Proof. In light of Lemma\[17\] we have the short exact sequence
\[
0 \to \mathcal{O}_S(B - K_S - A) \to \mathcal{O}_S(B - K_S) \to \mathcal{O}_A \to 0.
\]

Note that \( h^0(\mathcal{O}_S(B - K_S - A)) = 0 \). We claim that \( B - K_S - A \equiv -L + E_1 + E_9 \) is non-special on \( S \), i.e. \( h^1(\mathcal{O}_S(-L + E_1 + E_9)) = 0 \). To see this, we look at the short exact sequence
\[
0 \to \mathcal{O}_S(-L + E_1 + E_9) \to \mathcal{O}_S \to \mathcal{O}_{L - E_1 - E_9} \to 0.
\]

Since \( h^0(\mathcal{O}_S) = 1 \) and \( h^1(\mathcal{O}_S) = 0 \), due to the rationality of \( S \), the possibility \( h^1(\mathcal{O}_S(-L + E_1 + E_9)) > 0 \) occurs if and only if \( h^0(\mathcal{O}_{L - E_1 - E_9}) > 1 \). But the latter is false, since \( h^0(\mathcal{O}_{L - E_1 - E_9}) = 1 \). So \( h^1(\mathcal{O}_S(-L + E_1 + E_9)) = 0 \). Then the second last short exact sequence implies that \( h^0(\mathcal{O}_S(B - K_S)) = h^0(\mathcal{O}_A) = 1 \). \(\Box\)
Lemma 18 implies that the open condition O1 is false. Hence $H$ is not a very ample divisor on the surface $S$. This proves Proposition 15.

**Proposition 19.** Let $S$ be a smooth rational surface with $K_S^2 + 10 = 0$. Let $\pi : S \rightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, x_2, y_1, \ldots, y_{17} \in \mathbb{P}^2$. Furthermore, let $E_i = \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$ and $L = \pi^*l$, where $l \subset \mathbb{P}^2$ is a line. Suppose the divisor class

$$H \equiv 6L - \sum_{i=1}^{2} 2E_i - \sum_{i=1}^{17} F_i$$

has $h^0(\mathcal{O}_S(H)) = 6$. Then $H$ is not very ample on $S$.

**Proof.** To begin with, note that the following conditions are necessities for $H$ being very ample.

1. $|L - \sum_{i \in I} E_i - \sum_{j \in J} F_j| = \emptyset$, where $2|I| + |J| \geq 6$.
2. $|2L - \sum_{i \in I} E_i - \sum_{j \in J} F_j| = \emptyset$, where $2|I| + |J| \geq 12$.
3. $|4L - E_1 - E_2 - F_1 - \ldots - F_{17}| = \emptyset$.

If the conditions O1 and O2 were false then a curve in either of the complete linear systems would intersect non-positive with $H$. Note that the decompositions

$$\mathcal{O}_S(\mathcal{O}_S(H)) = 6,$$

implies that $\mathcal{O}_S(H)$ is not very ample on $H$, implying that $H$ would not be very ample. If the complete linear system in O3 was non-empty, then that would imply the existence of a curve $C$ on the surface $S$ with arithmetic genus $p_a(C) = 3$ and degree $H.C = 3$. Then $H$ could not be very ample, due to Proposition 4. The idea now is to show that if $h^0(\mathcal{O}_S(H)) = 6$, then at least one of the open conditions above is false. To do so, we study the decompositions $H \equiv A_{ij} + B_{ij}$, where $1 \leq i \leq 2$, $1 \leq j \leq 17$ and

$$A_{ij} \equiv 5L - 2E_1 - 2E_2 - F_1 - \ldots - F_{17} + E_i + F_j,$$

$$B_{ij} \equiv L - E_i - F_j.$$

Note that the decompositions $H \equiv A_{ij} + B_{ij}$ are all nice decompositions due to Riemann-Roch, $B_{ij}^2 > 2p_a(A_{ij}) - 2$ and that $H.A_{ij} = 2p_a(A_{ij}) - 2$.

**Lemma 20.** $\mathcal{O}_{A_{ij}}(B_{ij} - K_S) \simeq \mathcal{O}_{A_{ij}}$ for all $i, j$.

**Proof.** Taking cohomology of the short exact sequence

$$0 \rightarrow \mathcal{O}_S(B_{ij}) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_{A_{ij}}(H) \rightarrow 0$$

we have that $h^1(\mathcal{O}_S(H)) = h^1(\mathcal{O}_{A_{ij}}(H)) = 1$, since $h^1(\mathcal{O}_S(B_{ij})) = 0$ and $h^0(\mathcal{O}_S(H)) = 6$. Then Serre duality yields $h^1(\mathcal{O}_{A_{ij}}(H)) = h^0(\mathcal{O}_{A_{ij}}(K_{A_{ij}} - A_{ij})) = 1$. Combining the latter with the adjunction formula, which states that $K_{A_{ij}} \equiv (A_{ij} + K_S)|_{A_{ij}}$, it follows that $h^0(\mathcal{O}_{A_{ij}}(A_{ij} - K_S)) = 1$. Then Lemma 14 applies and yields that $\mathcal{O}_{A_{ij}}(H) \simeq \omega_{A_{ij}}$. Another use of the adjunction formula gives us $\mathcal{O}_{A_{ij}}(H) \simeq \mathcal{O}_{A_{ij}}(A_{ij} + K_S)$. Twisting the latter with $\mathcal{O}_{A_{ij}}(-A_{ij} - K_S)$ we obtain $\mathcal{O}_{A_{ij}}(B_{ij} - K_S) \simeq \mathcal{O}_{A_{ij}}(4L - E_1 - E_2 - F_1 - \ldots - F_{17} - E_i - F_j) \simeq \mathcal{O}_{A_{ij}}$ for all $i, j$.

**Lemma 21.** The divisors $L - K_S + F_j$ are effective on $S$, for all $j$.

**Proof.** Without loss of generality, we may assume $i = 2$. Let $A_j := A_{2j}$ and $B_j := B_{2j}$. Due to Lemma 20 we have the following short exact sequence

$$0 \rightarrow \mathcal{O}_S(B_j - K_S - A_j) \rightarrow \mathcal{O}_S(B_j - K_S) \rightarrow \mathcal{O}_{A_j}(B_j - K_S) \rightarrow 0$$

Twisting the sequence above with $\mathcal{O}_S(E_2 + 2F_j)$, we get

$$0 \rightarrow \mathcal{O}_S(-L + E_1) \rightarrow \mathcal{O}_S(L - K_S + F_j) \rightarrow \mathcal{O}_{A_j}(E_2 + 2F_j) \rightarrow 0.$$
Note that $h^0(\mathcal{O}_S(-L + E_1)) = 0$ due to the rationality of $S$. Furthermore, we claim that $h^1(\mathcal{O}_S(-L + E_1)) = 0$. The latter can be seen by taking cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(-L + E_1) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{L - E_1} \longrightarrow 0.$$ 

Since $S$ is rational, we have $h^0(\mathcal{O}_S) = 1$ and $h^1(\mathcal{O}_S) = 0$ for $i > 0$. Combining $\chi(\mathcal{O}_{L - E_1}) = 1$ with $h^1(\mathcal{O}_{L - E_1}) = p_a(L - E_1) = 0$ we obtain $h^0(\mathcal{O}_{L - E_1}) = 1$. Then we have $h^1(\mathcal{O}_S(-L + E_1)) = h^0(\mathcal{O}_S) - h^0(\mathcal{O}_{L - E_1}) = 0$. In particular, this means that

$$h^0(\mathcal{O}_S(L - K_S + F_j)) = h^0(\mathcal{O}_{A_j}(E_2 + 2F_j))$$

by the second to the last sequence above. Next, we claim that $\mathcal{O}_{A_j}(F_j) \simeq \mathcal{O}_{A_j}$ for all $j$. This can be seen by taking cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{O}_{A_j} \longrightarrow \mathcal{O}_{A_j}(F_j) \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is a zero-dimensional scheme supported on $A_j \cap F_j$. Since $A_j, F_j = 0$, we get $h^0(\mathcal{F}) = 0$, by taking Euler characteristics of the sequence above, such that $\mathcal{O}_{A_j} \simeq \mathcal{O}_{A_j}(F_j)$ for all $j$. This implies that $h^0(\mathcal{O}_S(L - K_S + F_j)) = h^0(\mathcal{O}_{A_j}(E_2)).$ Now we take cohomology of

$$0 \longrightarrow \mathcal{O}_S(E_2 - A_j) \longrightarrow \mathcal{O}_S(E_2) \longrightarrow \mathcal{O}_{A_j}(E_2) \longrightarrow 0$$

and note that $h^0(\mathcal{O}_S(E_2 - A_j)) = 0$ such that $H^0(\mathcal{O}_S(E_2)) \longrightarrow H^0(\mathcal{O}_{A_j}(E_2))$ is injective. Clearly the exceptional divisor $E_2$ is effective on $S$ such that $h^0(\mathcal{O}_{A_j}(E_2)) > 0$.

Denote $C_j := L - K_S + F_j.$ Then

$$C_j \equiv 4L - E_1 - E_2 - F_1 - \ldots - F_{17} + F_j$$

are effective divisors on $S$, for all $j$, by Lemma 21. We are now ready to show how this implies that at least one of the open conditions O1-O3 is false and thus proving the Proposition. The images of the curves $C_j$ under the blowdown morphism $\pi : S \longrightarrow \mathbb{P}^2$ are plane quartic curves passing through the points $x_1, x_2$ and 16 of the points $y_1, \ldots, y_{17}$. Bezout’s theorem implies that every two curves $\pi(C_i)$ and $\pi(C_j)$ share a common component, since the points $x_1, x_2$ and 15 of the points $y_1, \ldots, y_{17}$ lie on the set-theoretic intersection $\pi(C_i) \cap \pi(C_j)$, whenever $i \neq j$. In particular, this means that there exists a plane quartic passing through the points $x_1, x_2, y_1, \ldots, y_{17}$ which implies that the divisor

$$C \equiv 4L - E_1 - E_2 - F_1 - \ldots - F_{17}$$

is effective on the surface $S$. If the curve $C$ is irreducible, then we are done due to the open condition O3. So suppose $C$ is a reducible curve on $S$. Now, suppose $H$ is very ample. Then the complete linear system $|H|$ embeds $C$ as a cubic curve due to $H.C = 3$. The irreducible components of the curve $C$ are one of the following two cases: (1). Three lines. (2). A conic and a line. In the first case, by pigeonholing the points 19 points among the 3 lines, clearly at least one of the lines passes through at least 7 of the exceptional divisors which in turn contradict $H$ being very ample, due to O1. In the second case, again by pigeonholing the points, it follows that either the conic passes through at least 12 of the points or the line passes through at least 8 of the points, such that O2 or O1 would contradict the very ampleness of $H$. This proves Proposition 19.

\[\square\]

4. Classification

Let $S$ be a smooth rational surface, such that $i : S \hookrightarrow \mathbb{P}^a$ is an embedding, and denote $H$ as the divisor class on $S$ associated to $i^*\mathcal{O}_{\mathbb{P}^a}(1)$. To achieve a classification we determine explicit expressions for $H$, whenever the degree $H^2 = 11$ of the surface and the sectional genus $\chi_S = 8$. To do so we study the adjoint linear system $|H + K_S|$ on $S$ instead and make use of a result, due to Sommese and Van de Ven, which states that $|H + K_S|$ almost always defines a birational morphism to some projective space.
Theorem 22. Let $S$ be a smooth rational surface in $\mathbb{P}^n$, let $H$ denote the class of a hyperplane section of $S$ and let $K_S$ denote the class of a canonical divisor of $S$. Then the adjoint linear system $|H + K_S|$ defines a birational morphism

$$\varphi_{|H + K_S|} : S \to \mathbb{P}^N$$

onto a smooth surface $S_1$ and $\varphi_{|H + K_S|}$ blows down $(-1)$-curves $E$ on $S$ such that

$$K_{S_1}E = -1$$

and $H.E = 1$,

unless one of the following three cases occurs:

1. $S$ is a plane, or $S$ is a Veronese surface of degree 4, or $S$ is ruled by lines.
2. $(H + K_S)^2 = 0$, which occurs if and only if $S$ is a Del Pezzo surface or $S$ is a conic bundle.
3. $(H + K_S)^2 > 0$ and $S$ belongs to one of the following four families:
   - $S = \mathbb{P}^2(x_1, ..., x_7)$ is embedded by $H \equiv 6L - 2E_1 - \ldots - E_7$.
   - $S = \mathbb{P}^2(x_1, ..., x_8)$ is embedded by $H \equiv 6L - 2E_1 - \ldots - 2E_7 - E_8$.
   - $S = \mathbb{P}^2(x_1, ..., x_8)$ is embedded by $H \equiv 9L - 3E_1 - \ldots - 3E_8$.

Proof. See [SV87] for a proof.

A crucial difference between surfaces in $\mathbb{P}^4$ and surfaces in $\mathbb{P}^5$ is that every smooth surface can be embedded into $\mathbb{P}^5$ through generic projection. In particular, this implies that we cannot expect a relation between invariants of a surface in $\mathbb{P}^5$ similar to the double-point formula of $\mathbb{P}^4$, see Example 4.1.3 in [Har77], which states that $\text{codim}(S, \mathbb{P}^4) - (\deg S)^2 = 0$, when $S \subset \mathbb{P}^4$. The double-point formula for surfaces in $\mathbb{P}^4$ plays an important role since it completely determines $K_S^2$ whenever $\deg S$ and $\pi_S$ are given. Since we are considering surfaces in $\mathbb{P}^5$, our strategy is to limit ourselves to finitely many possibilities for $K_S^2$, whenever we are given $\deg S$ and $\pi_S$. In the next result, we give an upper and lower bound for $K_S^2$ and show some other generalities on the adjunction mapping.

Lemma 23. Suppose $H$ is very ample divisor on $S$ and suppose $\varphi_{|H + K_S|}$ is a birational morphism onto $S_1$. Then:

1. $\varphi_{|H + K_S|}$ maps $S$ into $\mathbb{P}^{\pi_S - 1}$.
2. $\pi_S - 2 - H.(H + 2K_S) \leq K_S^2 \leq [(H.K)^2/H^2]$.
3. $(H + K_S).K_S = (H + K_S).K_S$
4. $\pi_S = \pi_S + (H + K_S).K_S$
5. If $H.K \geq 2$, then $K_S^2 < 0$.

Proof. (1). Combine Riemann-Roch and the adjunction formula to get $\chi(S, \mathcal{O}_S(H + K_S)) = \pi_S$. The very ampleness of $H$ and smoothness of $S$ implies that $H^1(S, \mathcal{O}_S(H + K_S)) = 0$, due to the Kodaira vanishing theorem. Furthermore, $H^2(S, \mathcal{O}_S(H + K_S)) = 0$ follows from the rationality of $S$.

(2). The upper bound for $K_S^2$ is a direct consequence of the Hodge index inequality. The lower bound for $K_S^2$ is obtained by noting that the non-degeneracy of $S$ yields that $\text{codim}(S, \mathbb{P}^{\pi_S - 1}) + 1 \leq (H + K_S)^2$.

(3). Note that $(H + K_S).E = 0$ for all $(-1)$-curves $E$ such that $K_{S_1}.E = -1$ and $H.E = 1$. The equality now follows by recalling that $S_1$ is obtained by blowing down every such $(-1)$-curves $E$.

(4). Apply the adjunction formula twice and then use Lemma [23].

(5). Riemann-Roch yields that $h^0(\mathcal{O}_S(-K)) = 1 + K_S^2 + h^1(\mathcal{O}_S(-K_S)) > 0$. A curve $C \in |-K_S|$ has $p_a(C) = 1$, such that the adjunction formula implies that $H.C = -H.K > 2$.

To be able to determine the configuration of the points blown up to obtain a smooth rational surface, we will be needing the following result.

Lemma 24. Let $H$ and $B$ be effective divisors on a smooth surface $S$ and denote $A = H - B$. Suppose $h^1(\mathcal{O}_S(H)) + \chi(\mathcal{O}_S(A)) > 0$, suppose $h^2(\mathcal{O}_S(A)) = 0$ and suppose $H.B > 2p_a(B) - 2$. Then $A$ is an effective divisor on $S$.
Proof: The result is clear if \( \chi(\mathcal{O}_S(A)) > 0 \), due to \( h^2(\mathcal{O}_S(A)) = 0 \). So suppose \( \chi(\mathcal{O}_S(A)) \leq 0 \). The assumption \( H.B > 2p_g(B) - 2 \) implies that \( h^1(\mathcal{O}_B(H)) = 0 \), by Riemann-Roch. Taking cohomology of the short exact sequence

\[
0 \to \mathcal{O}_S(A) \to \mathcal{O}_S(H) \to \mathcal{O}_B(H) \to 0
\]

we obtain \( h^1(\mathcal{O}_S(A)) \geq h^1(\mathcal{O}_S(H)) \). Then \( h^1(\mathcal{O}_S(H)) + \chi(\mathcal{O}_S(A)) > 0 \) implies that \( h^0(\mathcal{O}_S(A)) > 0 \). \( \square \)

We are now ready to prove the converse statement in Theorem 1.

**Theorem 25.** Suppose there exists a linearly normal smooth rational surface \( S \) of degree 11 and sectional genus \( 8 \) embedded in \( \mathbb{P}^5 \). If \( i : S \hookrightarrow \mathbb{P}^5 \) is an embedding and \( i^*\mathcal{O}_{\mathbb{P}^5}(1) \) is the very ample line bundle associated to \( i \), then the associated very ample divisor \( H \) of \( i^*\mathcal{O}_{\mathbb{P}^5}(1) \) belongs to the following divisor classes:

| \( K_S^2 \) | Type | \( H \) |
|---|---|---|
| -10 | \([4,4 - 2e]; 2^1, 1^{17}\) | \( 4B + (4 - 2e)F - 2E_1 - E_2 - \cdots - E_{18} \) |
| -9 | \([4,5 - 2e]; 2^1, 1^{13}\) | \( 4B + (5 - 2e)F - 2E_1 - \cdots - 2E_4 - E_5 - \cdots - E_{17} \) |
| -8 | \([4,6 - 2e]; 2^1, 1^9\) | \( 4B + (6 - 2e)F - 2E_1 - \cdots - 2E_7 - E_8 - \cdots - E_{16} \) |
| -8 | \([7; 2^2, 1^{10}]\) | \( 7L - 2E_1 - \cdots - 2E_7 - E_8 - \cdots - E_{17} \) |
| -7 | \([9; 3^6, 2^2, 1^8]\) | \( 9L - 3E_1 - \cdots - 3E_6 - 2E_7 - 2E_8 - E_9 - \cdots - E_{16} \) |

Proof. We proceed with the proof in two parts. First part, we use Theorem 22 and Lemma 23 to produce divisor classes \( H \) belongs to. Second part, we show that all but the divisor classes stated in the theorem cannot both be very ample and special.

**Part 1 of the proof.**

Let \( S_i \) be a smooth rational surface and let \( \varphi_i : S_i \hookrightarrow \mathbb{P}^N \) be an embedding such that \( \mathcal{O}_{S_i}(H_i) \simeq \varphi^*_i\mathcal{O}_{\mathbb{P}^N}(1) \). Denote \( H_{i+1} \) as the adjoint divisor of \( H_i \) on \( S_i \), that is \( H_{i+1} := H_i + K_i \) where \( K_i := K_{S_i} \). Furthermore, let \( \varphi_{i+1} := \varphi_{|H_i+K_i|} \) whenever \( \varphi_{i+1} \) is a birational morphism such that \( \varphi_{i+1}(S_i) := S_{i+1} \) and \( \pi_i := \pi_{H_i} \). The idea now is to use Theorem 22 repeatedly to obtain sequences of birational morphisms

\[
S \xrightarrow{\varphi} S_1 \xrightarrow{\varphi_1} S_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{N_0-1}} S_{N_0}
\]

where each surface \( S_{i+1} \) is embedded into \( \mathbb{P}^{\pi_i-1} \) due to Lemma 23.1 and \( N_0 := \min\{ i \mid \pi_i \leq 5 \} \). This will then allow us to produce the divisor class of \( H \) by determining the invariants \( (K^2_0, K^2_1, \ldots, K^2_{N_0}) \), \( \deg(S_{N_0}) \) and \( \pi_{N_0-1} \). Note that Lemma 23.2 and Lemma 23.5 yields that

\[-11 \leq K^2_0 \leq -1.\]

Using Lemma 23.4 we have \( \pi_1 = 11 + K^2_0 \) such that \( N_0 = 2 \) if and only if \(-11 \leq K^2_0 \leq -6 \). Furthermore, it follows from Lemma 23.3 that \( H^2_2 = 23 + 3K^2_0 + K^2_1 \). Suppose \( H^2_2 = 0 \). It is then straightforward, by Lemma 23.5, to check that \( (K^2_0, K^2_1) \) takes the following values:

\[-(10, 7), -(9, 4), -(8, 1), -(7, -2), (-6, -5)\]

By case 2 of Theorem 22 it suffices to check whether \( S_1 \) is a Del Pezzo surface or a conic bundle for the pairs \( (K^2_0, K^2_1) \) above. If \( S_1 \) is a Del Pezzo surface, then the divisor class of \( H_0 \) is of the following form

\[H_0 \equiv 6L - 2E_1 - \cdots - 2E_9 - K^2_1 - E_{10} - K^2_2 - \cdots - E_{9-K^2_2} \]

Then the adjunction formula implies that \( \pi_0 = 8 \) if and only if \((9) - (9 - K^2_2) = 8 \), that is \( K^2_2 = 7 \). So, if \( S_1 \) is a Del Pezzo surface then \( H_0 \) is of type \([6; 2^2, 1^{17}]\). If \( S_1 \) is a conic bundle, then

\[H_1 \equiv 2B + aF - E_1 - \cdots - E_{8-K^2_2} \]
For the six remaining pairs of $(K - \varphi)$ surfaces, Theorem 1.1 in [Ale88]. Then we get the following divisor classes of $P$ surfaces in $K$

This covers the cases $K \leq -6$. Clearly $H_1$ does not belong to any of the four families stated in case 3 of Theorem [22] since that would yield $\pi_0 < 8$ or $H_2^3 < 11$. Thus we may assume that $\varphi_1 : S_1 \to S_2$ is a birational morphism and that $S_2$ is a smooth rational surface. Then Lemma [23] yields the following possibilities for $(K^2_0, K^2_2)$.

$(-11, K^2_2) = (-8, 2) (-7, 1) (-7, 0) (-6, 0) (-6, -1) (-6, 2)$

Suppose $K_0 = -11$, then $S_1$ is a surface of minimal degree in $\mathbb{P}^7$ since $\deg(S_1) = 6$. This implies that $S_1$ is a Veronese surface or a rational normal scroll. If $S_1$ is a Veronese surface then $H_1 \equiv 2L$, which contradicts $H_0^3 = 11$. If $S_1$ is a rational normal scroll then $H_1 \equiv B + (\alpha - e)F$, where $0 \leq e < \alpha$. Recall that every minimal degree $d$ satisfies $d = 2a - e$ by Corollary IV.2.19 in [Har77]. An exhaustion of the pairs $(\alpha, e)$ satisfying the latter relation yields no divisor classes $H_0$ of degree 11. So we may rule out the case $(-11, K^2_2)$.

For the six remaining pairs of $(K^2_0, K^2_2)$ we reproduce $H_0$ by using classifications of smooth rational surfaces in $\mathbb{P}^n$ for $n \leq 4$. In particular, when $n = 4$ then we use Alexander's list of non-special rational surfaces, Theorem 1.1 in [Ale88]. Then we get the following divisor classes of $H_0$ when $S_2 \subset \mathbb{P}^{\pi_1 - 1}$.

$$(K^2_0, K^2_2) \quad \deg(S_2) \quad \pi_1 \quad H_2 \quad \text{Type of } H_0$$

$(-8, 2) \quad 1 \quad 3 \quad L \quad [7, 2^6, 1^{10}]$
$(-7, 0) \quad 2 \quad 4 \quad B + F \quad [(5, 5)_0; 2^6, 1^7]$
$(-7, 1) \quad 3 \quad 4 \quad 3L - E_1 - \ldots - E_6 \quad [9; 3^6, 2^2, 1^9]$
$(-6, -2) \quad 3 \quad 5 \quad 2L - E_1 \quad [8; 3^{10}, 2^{14}]$
$(-6, -1) \quad 4 \quad 5 \quad 4L - 2E_1 - E_2 - \ldots - E_8 \quad [9; 3^5, 2^5, 1^5]$
$(-6, 0) \quad 5 \quad 5 \quad 4L - E_1 - \ldots - E_{10} \quad [10; 4^1, 3^7, 2^1, 1^6]$

This covers the cases $-11 \leq K_0^2 \leq -6$. Note that if $K_0^2 \geq -5$ then Lemma [23]5 applies, due to $H_1 K_1 \geq -2$, such that we have $K_2^2 \leq -1$. Using Lemma [23]3 and Lemma [23]4, we see that $N_0 = 3$ occurs whenever $-5 \leq K_0^2 \leq -4$. Now we consider the latter two values of $K_0^2$. Suppose $H_2^4 = 0$. Then $K_0^2 \leq K_1^2 \leq K_2^2$ and Lemma [23]2 yields the following possibilities for $(K_0^2, K_1^2, K_2^2)$.

$(-5, -1, -1) (-5, -2, 2) (-5, -3, 5) (-5, -4, 8) (-4, -3, 0) (-4, -4, 3)$

If $S_2$ is a Del Pezzo surface, then the adjunction formula yields that $\pi_0 = 1 + K_1^2 + 2K_2^2$. So $\pi_0 = 8$ occurs only in the case $(-5, -3, 5)$ such that $H_0$ is of type $[9; 3^4, 2^5, 1^7]$. If $S_2$ is a conic bundle, then $H_2 \equiv 2B + aF - E_1 - \ldots - E_8 - K_2^2$ for some $a \in \mathbb{Z}_{\geq 0}$. Then $H_2 K_2 = 3 + K_0^2 + K_1^2$ gives us $2a = 1 - 2e - \sum_1^3 K_1^2$, so that we may reproduce $H_0$.  

Suppose $H_0^2 > 0$ and $-5 \leq K_0^2 \leq -4$. Clearly, $H_0$ does not belong to any of the four families of case 3 in Theorem 22. So we may assume that $\varphi_3 : S_2 \to S_3$ is a birational morphism. By ruling out the cases when $1 \leq \pi_2 \leq 3$, we obtain the following possibilities for $(K_0^2, K_1^2, K_2^2)$.

$$
(-5, -4, 2^2) \quad (-5, -1, 0) \quad (-4, -3, 2) \quad (-4, -2, -1) \quad (-4, -2, 0)
$$

Suppose $(K_0^2, K_1^2) = (-5, -4)$. Then $S_2$ is a surface of minimal degree in $\mathbb{P}^5$. If $S_2$ is a Veronese surface, then $H_2 \equiv 2L$ which yields that $H_0$ is of type $[8; 2^{13}, 1^1]$. If $S_2$ is a minimal rational scroll, then $H_2 \equiv B + (\alpha - e)F$ where $0 \leq e < \alpha$ and $4 = 2\alpha - c$. An investigation of the pairs $(\alpha, e)$ reveal that $H_0^2 = 11$ is not true. For the remaining triples $(K_0^2, K_1^2, K_2^2)$ we may now reconstruct $H_0$.

| $(K_0^2, K_1^2, K_2^2)$ | $\deg(S_3)$ | $\pi_2$ | $H_3$ | Type of $H_0$ |
|------------------------|--------------|--------|-------|----------------|
| $(-5, -1, 0)$          | $1$          | $3$    | $L$   | $[10; 3^1, 2^1, 1^4]$ |
| $(-4, -3, 1)$          | $1$          | $3$    | $L$   | $[10; 3^2, 5^2, 1^1]$ |
| $(-4, -2, -1)$         | $2$          | $4$    | $B + F$ | $[7, 7]; 3^{10}, 2^1, 1^2$ |
| $(-4, -2, 0)$          | $3$          | $4$    | $3L - E_1 - \ldots - E_6$ | $[12; 4^2, 3^2, 2^1, 1^2]$ |

Suppose $K_0^2 = -3$. Then we have $-3 \leq K_1^2 \leq -1$ and $K_2^2 \leq K_0^2$. Note that the case $(-3, -3, K_2^2)$ satisfies $\pi_2 = 5$. Furthermore, every other case satisfies $H_2, K_2 \geq -2$ such that $K_2^2 \leq -1$. In particular this means that $N_0 = 4$ whenever $K_0^2 = -3$. Suppose $H_0^2 = 0$ and $K_0^2 = -3$. Then $(K_0^2, K_1^2, K_2^2, K_3^2)$ takes the following values.

$$
(-3, -2, -2, 2) \quad (-3, -2, -1, -1)
$$

If $S_3$ is a Del Pezzo surface, then $\pi_0 = 5 + \sum_1^3 i \cdot K_i^2$ implies that none of the tuples above satisfy $\pi_0 = 8$. If $S_3$ is a conic bundle and $H_3, F = a$, then $2a = 1 - 2e - \sum_0^3 K_i^2$ yields the following possibilities for $H_0$.

| $(K_0^2, K_1^2, K_2^2, K_3^2)$ | $a$ | $\deg(S_{N_0})$ | $\pi_{N_0-1}$ | $H_{N_0}$ | Type of $H_0$ |
|------------------------|-----|------------------|---------------|--------|----------------|
| $(-3, -2, -2, 2)$      | $3$ | $5$              | $2L - E_1$    | $[11; 4^2, 3^{10}, 2^1]$ |
| $(-3, -2, -1, -1)$     | $4$ | $5$              | $3L - E_1 - \ldots - E_5$ | $[12; 4^3, 3^2, 2^2]$ |
| $(-3, -3, 0)$          | $5$ | $5$              | $4L - 2E_1 - E_2 - \ldots - E_8$ | $[13; 5^1, 4^7, 3^1, 2^3]$ |
| $(-3, -2, -1, 0)$      | $1$ | $3$              | $L$          | $[13; 4^9, 3^1, 2^1, 1^1]$ |
| $(-3, -2, -1, -1)$     | $5$ | $5$              | $4L - 2E_1 - E_2 - \ldots - E_8$ | $[16; 6^1, 5^7, 4^2, 3^1, 2^1]$ |

Suppose $H_0^2 > 0$ and $K_0^2 = -3$. Clearly, none of the four families in case 3 in Theorem 22 occur. So we may assume $\varphi_4 : S_3 \to S_4$ is a birational morphism. Then we may reproduce $H_0$.

| $N_0$ | $(K_0^2, \ldots, K_{N_0-1}^2)$ | $\deg(S_{N_0})$ | $\pi_{N_0-1}$ | $H_{N_0}$ | Type of $H_0$ |
|-------|---------------------|------------------|---------------|--------|----------------|
| $3$   | $(-3, -3, 2)$       | $3$              | $5$           | $2L - E_1$    | $[11; 4^2, 3^{10}, 2^1]$ |
| $3$   | $(-3, -3, -1)$      | $4$              | $5$           | $3L - E_1 - \ldots - E_5$ | $[12; 4^3, 3^2, 2^2]$ |
| $3$   | $(-3, -3, 0)$       | $5$              | $5$           | $4L - 2E_1 - E_2 - \ldots - E_8$ | $[13; 5^1, 4^7, 3^1, 2^3]$ |
| $4$   | $(-3, -2, -1, 0)$   | $1$              | $3$           | $L$          | $[13; 4^9, 3^1, 2^1, 1^1]$ |
| $4$   | $(-3, -2, -1, -1)$  | $5$              | $5$           | $4L - 2E_1 - E_2 - \ldots - E_8$ | $[16; 6^1, 5^7, 4^2, 3^1, 2^1]$ |

It remains to consider the cases $-2 \leq K_0^2 \leq -1$. First, suppose $K_0^2 = -2$. Then $-2 \leq K_1^2 \leq -1$ and $K_2^2 \leq K_3^2 \leq -1$. In the case $(-2, -2, -2)$ we obtain $\pi_2 = 5$, that is $N_0 = 4$. In the remaining cases $H_3, K_3 \geq -2$ implies that $K_2 \leq K_3 \leq -1$ such that $K_2^2 = K_3^2 = -1$. In the case $(-2, -2, -1, -1)$ we obtain $\pi_4 < 5$, that is $N_0 = 5$. The remaining case $(-2, -2, -1, -1)$ yields that $K_3^2 \leq K_4^2 \leq 0$ in which
case \( \pi_5 < 5 \), that is \( N_0 = 6 \). Next, suppose \( K_0^2 = -1 \). Then \( H_i.K_i \geq -2 \), for \( 1 \leq i \leq 5 \), which implies that \( K_0 = \ldots = K_5 = -1 \). It then follows that \( N_0 = 7 \).

Suppose \( H_{N_0}^2 = 0 \) and \(-2 \leq K_0^2 \leq -1 \), for the respective \( N_0 \) above. Then we obtain the following choices for \( (K_0^2, \ldots, K_{N_0}^2) \):

\[
(-2, -2, -1, -1, -1), (-2, -2, -1, -1, 1)
\]

It is clear that \( S_{N_0} \) can not be a Del Pezzo surface for neither of the tuples above. If \( S_{N_0} \) is a conic bundle, where \( H_{N_0-1}.F = a \), then by using \( 2a = 1 - 2e - \sum K_i^2 \) we may reconstruct \( H_0 \).

| \( N_0 \) | \( (K_0^2, \ldots, K_{N_0}^2) \) | \( a \) | Type of \( H_0 \) |
|---|---|---|---|
| 5 | (\(-2, -2, -1, -1, -1\)) | 4 - \( e \) | \([10, 12 - 5e]_3; 5^3, 2^1\) |
| 6 | (\(-2, -2, -1, -1, -1, 8\)) | 4 - \( e \) | \([12, 10 - 6e]_3; 5^3, 2^1\) |

Now suppose \( H_{N_0}^2 > 0 \) and \(-2 \leq K_0^2 \leq -1 \). Clearly, none of the four families in case 3 in Theorem 22 occur such that we may assume \( \varphi_{N_0} : S_{N_0-1} \rightarrow S_{N_0} \) is a birational morphism. Then \( (K_0^2, \ldots, K_{N_0}^2) \) takes the following values:

\[
(-2, -2, -2, 0), (-2, -2, -2, 1), (-2, -2, -2, 2), (-2, -2, -1, 0)
\]

Reconstructing \( H_0 \) is each of the cases above, we obtain the following.

| \( N_0 \) | \( (K_0^2, \ldots, K_{N_0-1}^2) \) | \( \text{deg}(S_{N_0}) \) | \( \tau_{N_0-1} \) | \( H_{N_0} \) | Type of \( H_0 \) |
|---|---|---|---|---|---|
| 4 | \((-2, -2, -2, 0)\) | 3 | 5 | \( 2L - E_1 \) | \([14; 5^4, 4^{10}]\) |
| 4 | \((-2, -2, -2, 1)\) | 4 | 5 | \( 3L - E_1 - \ldots - E_5 \) | \([15; 5^5, 4^6, 3^1]\) |
| 4 | \((-2, -2, -2, -2)\) | 5 | 5 | \( 4L - 2E_1 - E_2 - \ldots - E_8 \) | \([16; 6^1, 5^7, 4^1, 3^2]\) |
| 5 | \((-2, -2, -1, -1, 0)\) | 1 | 3 | \( L \) | \([16; 5^6, 4^1, 2^1]\) |
| 6 | \((-2, -1, -1, -1, -1, -1)\) | 1 | 3 | \( L \) | \([19; 6^1, 5^1, 1^1]\) |
| 7 | \((-1, -1, -1, -1, -1, 0)\) | 4 | 5 | \( 3L - E_1 - \ldots - E_5 \) | \([24; 8^5, 7^5]\) |
| 7 | \((-1, -1, -1, -1, -1, -1)\) | 5 | 5 | \( 4L - 2E_1 - E_2 - \ldots - E_8 \) | \([25; 9^1, 8^7, 7^1, 6^1]\) |

This exhausts all possibilities for \( K_0^2 \) and therefore concludes the first part of the proof.

**Part 2 of the proof.**

The idea now is to show that every divisor class obtained in Part 1 of the proof, except the divisor classes in the statement of the Theorem, cannot simultaneously be both very ample and have six global sections on \( S \). We show this by finding an explicit decomposition \( H = A + B \), where \( A \) will be effective by applying Lemma 23 and the numerical invariants of \( A \) will contradict Proposition 4.
| Type of $H$ | Type of $A$ | $\chi(\mathcal{O}_S(A))$ | $p_a(A)$ | $H.A$ | $\chi(\mathcal{O}_S(B))$ | $p_a(B)$ | $H.B$ |
|-------------|-------------|----------------|----------|-------|----------------|----------|-------|
| $[14; 5^4, 4^{10}]$ | $[7; 3^4, 2^{10}]$ | 0 | 2 | 3 | 3 | 4 | 8 |
| $[12, 10 - 6e]; 5^9, 2^3$ | $[2, 2 - e]; 1^9$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[10, 12 - 5e]; 5^9, 2^{11}$ | $[2, 2 - e]; 1^9$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[16; 6^5, 5^7, 4^2, 2^1]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[13; 4^3, 3^1, 2^1, 1^1]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[13; 5^1, 4^7, 3^1, 2^3]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 3 | 3 | 4 | 8 |
| $[12; 4^3, 3^5, 2^2]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[11; 4^1, 3^{10}, 2^1, 0^1, 2^5]$ | $[3; 1^{10}]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[8, 10 - 4e]; 4^9, 2^1, 1^3$ | $[2, 2 - e]; 1^9$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[8, 9 - 4e]; 4^6, 3^4, 1^1$ | $[2, 2 - e]; 2^1, 1^9$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[12; 4^6, 3^3, 2^2, 1^2]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[11; 4^2, 3^8, 2^1, 1^2]$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[10; 3^8, 2^5, 1^1]$ | $[4; 2^1, 1^{13}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[10; 3^8, 2^1, 1^1, 2^5]$ | $[1; 1^{12}]$ | 1 | 0 | 0 | 4 | 0 | 4 |
| $[8; 2^1, 1^2]$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[6, 7 - 3e]; 3^5, 2^7$ | $[3, 4 - 1^4]; 2^4, 1^8$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[6, 8 - 3e]; 3^8, 2^3, 1^1$ | $[2, 2 - e]; 1^9$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[6, 5 - 3e]; 2^1, 1^2, 1^1$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[6, 6 - 3e]; 2^3, 2^8, 1^1$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[6, 7 - 3e]; 2^6, 2^4, 1^3$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[6, 8 - 3e]; 2^9, 1^4$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[9; 3^4, 2^8, 1^1, 2^5]$ | $[2; 1^3]$ | 1 | 0 | 0 | 4 | 0 | 4 |
| $[9; 3^5, 3^1, 1^1]$ | $[3; 1^{10}]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[8; 3^1, 2^{10}, 1^1]$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[8; 3^2, 2^1, 1^7, 2^4, 5]$ | $[2; 1^3]$ | 1 | 0 | 0 | 4 | 0 | 4 |
| $[4, 8 - 2e]; 2^3, 1^1$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[4, 7 - 2e]; 2^{10}, 1^5$ | $[2, 2 - e]; 1^{12}$ | 0 | 2 | 4 | 2 | 3 | 7 |

Note that we have written subscripts on several of the divisor classes in the table above. Subscript 1 means that combining $\chi(\mathcal{O}_S(A))$ with the rationality of $S$ implies that $A$ is an effective divisor contradicting Proposition [4]. Subscript 2 means that the divisor $A$ has to be chosen relative to the ordering $i \geq j$ if and only if $A.E_i \geq A.E_j.$ Subscript 3 occurs in three cases and means that $A.E_0 = 0.$ Subscript 4 means that we have made a basechange $\text{Pic}F_0 \longrightarrow \text{Pic}F^2$ by embedding $F_0$ as the smooth quadric surface in $\mathbb{P}^3$ by a use of the Sege embedding. Subscript 5 means that Lemma [24] implies that $B$ is an effective divisor, since $\chi(\mathcal{O}_S(B)) = 0$ and $H.A > 2p_a(A) - 2.$ Then it follows from $H.B > 2p_a(B) - 2$ and $A^2 > 2p_a(A) - 2$ that $h^1(\mathcal{O}_S(H)) = 1$ is false.

Now we illustrate how one may use the table above to obtain a contradiction for every divisor class without subscript 1 or 5. For instance, if $H$ is of type $[24; 8^5, 7^5]$ then $A$ is of type $[8; 3^5, 2^5]$ and $B$ is of type $[16; 5^{10}].$ Since $\chi(\mathcal{O}_S(A)) = 0$ and $H.B > 2p_a(B) - 2,$ Lemma [24] yields that $A$ is effective whenever $H$ is special. But this contradicts Proposition [4] since $p_a(A) = 2$ and $H.A \leq 2p_a(A).$ This in turn contradicts the very ampleness of $H.$

Taking into account Theorem [7] and the lifting examples, namely Proposition [15] and Proposition [19] this concludes the proof.
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