Absolute clock synchronisation and special relativity paradoxes

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May 5, 2014

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Abstract: Solving special relativity paradoxes requires rigorous analysis of event timing, due to relative simultaneity in consequence of the Lorentz transformation. Since clock synchronisation is a convention in special theory of relativity, instead of the Einstein’s procedure one may choose such that offers absolute simultaneity. We present in short the corresponding formalism in one spatial dimension. We show that paradoxes do not arise with this choice of synchronisation and descriptions of these issues are exceptionally simple and consistent for both observers involved.

1 Introduction

The coordinate time is defined by the procedure of clock synchronisation in a given reference frame [1],[2],[3],[4]. In the special theory of relativity (STR) [5], the Einstein’s clock synchronisation procedure reflects the fundamental assumption of constancy and isotropy of the velocity of light, c, in any inertial reference frame. Correctness of this assumption cannot
be verified experimentally since it is not possible to determine a one-way (i.e. open path) velocity of light, other than \( c \), using light signals: the velocity \( c \) is the average velocity of light over a closed path and thus can be determined using a single clock without any conventions.

The Einstein’s clock synchronisation convention leads to the Lorentz transformation and, in consequence, to the result that simultaneity is not absolute, i.e. events simultaneous in one reference frame are not simultaneous in another reference frame in relative motion w.r.t. to the former. After postulating STR, numerous problems have been formulated that possessed apparently paradoxical character, including the ladder–barn paradox, the twin paradox as well as the much later proposed Bell’s spaceship paradox. These paradoxes arise because of relative simultaneity in STR – their explanations require precise analyses regarding timing of events and are non-intuitive.

Since the procedure of synchronising distant clocks is a convention, one has freedom to adopt synchronisation procedures different from that of Einstein’s, provided the velocity of light over closed paths equals \( c \). Obviously, transformation equations would differ in such cases from those of the standard Lorentz transformation but qualitative and quantitative results must remain unaltered. If so, could one define a suitable procedure that would provide a more simple and more intuitive explanation of STR paradoxes? Since the notion of simultaneity is the underlying message in this context, it is natural to consider a synchronisation procedure that offers absolute simultaneity. It has been shown by Rembieliński [6],[7] that indeed, by appropriate redefinition of coordinate time, one can derive transformation equations satisfying the above requirement. The related clock synchronisation procedure is referred to as the absolute synchronisation. Since the complete (covariant) formalism is somewhat complex, a derivation of the corresponding transformation equations is presented, according to Rembieliński and Włodarczyk [3], in one spatial dimension and so are the subsequent discussions that constitute the essence of the present article.

The aim of the paper is threefold: (i) present and discuss basic properties of transformation equations for space-time intervals in arbitrary synchronisation; (ii) present the absolute clock synchronisation procedure; (iii) demonstrate exceptional simplicity in explaining selected STR paradoxes on these grounds.

2 Transformation equations in arbitrary synchronisation

**Synchronisation of distant clocks** Assume two identical clocks (i.e. having the same duration of the unit time), placed at distant locations \( A \) and \( B \). The Einstein’s procedure of synchronising these clocks is to send a light signal from \( A \) to \( B \), reflect it and receive at \( A \). Clocks are said to be synchronised when their readings are related by:

\[
t_E(B) = t_E(A) + \frac{1}{2} \Delta t_{ABA},
\]

\[1\] There are more advantages of considering that formalism: firstly – it offers a wider framework from which it is straightforward to move to STR; secondly – its application to the problem of localisation in quantum mechanics allows to define consistently covariant position and spin operators and formulate a covariant relativistic quantum mechanics free of inconsistencies [9].
where $\Delta t_{ABA}$ denotes the time of flight over the closed path $ABA$, measured at $A$; subscript $E$ identifies the Einstein’s synchronisation. A generalisation of the clock synchronisation relation was proposed by Reichenbach [1]. It consists in replacing the factor $1/2$ in (1) with a parameter $\varepsilon_R$, called the Reichenbach coefficient, leading to the following relation, below referred to as arbitrary synchronisation:

$$t_{\varepsilon_R}(B) = t_{\varepsilon_R}(A) + \varepsilon_R \Delta t_{ABA},$$

where $0 < \varepsilon_R < 1$. Merging (1) and (2) leads to the following relation between space-time intervals in the Einstein’s and arbitrary synchronisations:

$$\Delta t_E = \Delta t_{\varepsilon_R} + (1 - 2\varepsilon_R)\Delta x/c,$$

where $\Delta t_E = t_E(B) - t_E(A)$, $\Delta t_{\varepsilon_R} = t_{\varepsilon_R}(B) - t_{\varepsilon_R}(A)$, $\Delta x$ is the distance between $A$ and $B$ and $c$ is the average velocity of light over a closed path. Introducing a new synchronisation coefficient for convenience, $\varepsilon = 1 - 2\varepsilon_R$, where $-1 < \varepsilon < 1$, one finally obtains:

$$\Delta t_E = \Delta t_\varepsilon + \varepsilon \frac{\Delta x}{c}.$$  

Since space intervals are measured relative to a unit length (e.g. using a ruler), they are synchronisation independent, i.e. $\Delta x_E = \Delta x_\varepsilon \equiv \Delta x$.

It follows from (4) that relations between velocities in the Einstein’s and arbitrary synchronisations, $v_E = \Delta x/\Delta t_E$ and $v_\varepsilon = \Delta x/\Delta t_\varepsilon$, respectively, read:

$$v_E = \frac{v_\varepsilon}{1 + \varepsilon v_E/c}, \quad v_\varepsilon = \frac{v_E}{1 - \varepsilon v_E/c}.$$  

Note that changing sign of $v_E$ does not imply the same for $v_\varepsilon$ – a property reflecting breaking of the reciprocity principle which is valid in the Einstein’s synchronisation only. Given two inertial reference frames, $O$ and $O'$, if the velocity of $O'$ w.r.t. $O$ equals $+V_E$ in the Einstein’s synchronisation and $V_\varepsilon^+$ in arbitrary synchronisation, then the velocity of $O$ w.r.t. $O'$ equals $-V_E$ in the Einstein’s synchronisation and, following from (5), $V_\varepsilon^- = -V_\varepsilon^+/(1 + 2\varepsilon V_\varepsilon^+/c)$ in arbitrary synchronisation.

One-way velocity of light It is straightforward to show, using (4), that the one-way velocity of light from $A$ to $B$ or from $B$ to $A$ in arbitrary synchronisation is given by, respectively:

$$c_{AB} = \frac{c}{1 - \varepsilon}, \quad c_{BA} = \frac{c}{1 + \varepsilon}.$$  

As can be readily verified, the harmonic average over a closed path equals $c$, i.e. is independent of synchronisation convention.
Transformation equations

In order to derive transformation equations for space-time intervals in arbitrary synchronisation, \( \Delta t_E \) and \( \Delta x \), between two inertial reference frames, \( O \) and \( O' \), one inserts (4) into the standard Lorentz transformation equations:

\[
\begin{pmatrix}
\frac{c\Delta t'}{\Delta x'}
\end{pmatrix} = \gamma_E \begin{pmatrix}
1 & -V_E/c \\
-V_E/c & 1
\end{pmatrix} \begin{pmatrix}
\frac{c\Delta t}{\Delta x}
\end{pmatrix},
\]

(7)

where \( V_E \) denotes the velocity of \( O' \) w.r.t. \( O \) and \( \gamma_E = 1/\sqrt{1 - V_E^2/c^2} \) is the standard Lorentz factor. Taking into account that, in general, the synchronisation coefficient \( \varepsilon \) transforms too, the relation (4) in the reference frame \( O' \) reads: \( \Delta t'_E = \Delta t'_E \varepsilon' + \varepsilon' \Delta x'/c \). One thus obtains the following relation of time and space intervals in arbitrary synchronisations:

\[
\begin{pmatrix}
\frac{c\Delta t'}{\Delta x'}
\end{pmatrix} = \gamma(\varepsilon) \begin{pmatrix}
1 & (\varepsilon + \varepsilon')V_E/c \\
-(\varepsilon + \varepsilon')V_E/c & 1
\end{pmatrix} \begin{pmatrix}
\frac{c\Delta t}{\Delta x}
\end{pmatrix}, \tag{8a}
\]

while the inverse transformation becomes:

\[
\begin{pmatrix}
\frac{c\Delta t}{\Delta x}
\end{pmatrix} = \gamma(\varepsilon) \begin{pmatrix}
1 & -(\varepsilon + \varepsilon' + (\varepsilon^2 - 1)V_E/c) \\
V_E/c & 1 + (\varepsilon + \varepsilon')V_E/c
\end{pmatrix} \begin{pmatrix}
\frac{c\Delta t'}{\Delta x'}
\end{pmatrix}, \tag{8b}
\]

where

\[
\gamma(\varepsilon) = \frac{1}{\sqrt{(1 + \varepsilon V_E/c)^2 - (V_E/c)^2}}. \tag{9}
\]

Strictly speaking, the above do not yet constitute a complete set of transformation equations since those should comprise a transformation law for synchronisation coefficients, too. The standard Lorentz transformation can be recovered from (8) by putting \( \varepsilon = \varepsilon' = 0 \), i.e. adopting the Einstein’s synchronisation in both reference frames. The above equations are, in general, not reciprocal, which reflects breaking of the relativity principle in arbitrary synchronisation. Reciprocity is restored if \( \varepsilon = -\varepsilon' \) which comprises the Einstein’s synchronisation, \( \varepsilon = \varepsilon' = 0 \), as well as in the Galilean limit, \( c \to \infty \).

Measurement of time intervals and length

Let two observers be stationary in \( O \) and \( O' \) (\( \Delta x = 0 \) and \( \Delta x' = 0 \), respectively). According to (8), the former will state that time flows at the following rate for the latter w.r.t. his rate, \( \Delta t'_E \):

\[
\Delta t'_E = \gamma(\varepsilon) \left(1 + (\varepsilon + \varepsilon')V_E/c\right) \Delta t_E
\]

(10a)

while the latter will state that time flows at the following rate for the former w.r.t. his rate, \( \Delta t'_E \):

\[
\Delta t_E = \gamma(\varepsilon) \Delta t'_E.
\]

(10b)

Let an object of length \( L \) be positioned in \( O \) along the \( x \) axis, the length being the distance between its ends measured simultaneously in a given reference frame. Thus the
3 Absolute synchronisation and a preferred reference frame

Among possible synchronisation schemes one can distinguish such that satisfies the requirement of absolute simultaneity, i.e. fulfilling the following proportionality: \( \Delta t_\epsilon' \sim \Delta t_\epsilon \). This scheme will be called the absolute synchronisation. The corresponding transformation equations are easily derived under the condition that the spatial component does not participate in the transformation of the time component. This is achieved by imposing the following requirement in (8):

\[
\varepsilon - \varepsilon' - (1 - \varepsilon \varepsilon') \frac{V_\varepsilon}{c} = 0,
\]

which thereby enables to write the transformation law for the synchronisation coefficients. In consequence, one obtains the following transformation equations between \( O \) and \( O' \) in the absolute synchronisation (in what follows, quantities in this synchronisation are not marked by subscripts):

\[
\begin{pmatrix}
\frac{c \Delta t'}{\Delta x'} \\
\frac{1}{\gamma(\varepsilon)} 0 \\
-\gamma(\varepsilon) V/c \\
\gamma(\varepsilon)
\end{pmatrix}
\begin{pmatrix}
\frac{c \Delta t}{\Delta x} \\
0 \\
-\gamma(\varepsilon) V/c \\
\gamma(\varepsilon)
\end{pmatrix},
\]

\[
\varepsilon' = \varepsilon - (1 - \varepsilon^2) V/c.
\]

The condition \( \varepsilon = 0 \) marks a class of reference frames in which the Einstein’s synchronisation is valid and, according to (3), the one-way velocity of light equals \( c \) in both directions of the \( x \) axis. One has freedom to select one reference frame belonging to this class and assign it the status of the preferred frame (\( O_{PF} \)). It is implicitly assumed there exist no physical phenomena in favour of a given particular choice. If one considers motion of bodies with velocities smaller than \( c \), as this is done in the present paper, it is a matter of indifference which reference frame will be named \( O_{PF} \). However, once a preferred frame has been chosen, the relativity principle in the standard formulation is broken in consequence. One
can formulate instead a principle stating that any inertial reference frame may be assumed the preferred frame (provided the clocks in this reference frame are synchronised according to the Einstein’s convention).

Since there is freedom of choosing the preferred frame, it is convenient to identify one of the reference frames, involved in the formulation of STR problems, with \( O_{PF} \) (e.g. \( O \equiv O_{PF} \)), subsequently using the tilde sign to mark the corresponding quantities. Let the reference frame \( O' \) move with velocity \( \vec{V} \) w.r.t. \( O_{PF} \). The following applies in \( O' \): (i) inserting \( \varepsilon = 0 \) in (13b) leads to the solution: \( \varepsilon' = -\vec{V}/c \) which defines the procedure of absolute clock synchronisation in the moving frame; (ii) it follows from (6) that the one-way velocity of light is direction dependent in \( O' \):

\[
c_+ = c/(1 + \vec{V}/c), \quad c_- = c/(1 - \vec{V}/c),
\]

where \( c_+ \) and \( c_- \) are velocities of light in the positive and negative direction of the \( x' \) axis, respectively. The condition that the average velocity over a closed path equals \( c \) is satisfied: \( \frac{1}{2}(1/c_+ + 1/c_-) = 1/c \).

Transformation equations (13a) between \( O_{PF} \) and a moving frame \( O' \) reduce to a simple form:

\[
\begin{pmatrix}
  c\Delta t' \\
  \Delta x'
\end{pmatrix}
= \begin{pmatrix}
  1/\gamma_0 & 0 \\
  -\gamma_0 \vec{V}/c & \gamma_0
\end{pmatrix}
\begin{pmatrix}
  c\Delta \tilde{t} \\
  \Delta \tilde{x}
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
  c\Delta \tilde{t} \\
  \Delta \tilde{x}
\end{pmatrix}
= \begin{pmatrix}
  \gamma_0 & 0 \\
  \gamma_0 \vec{V}/c & 1/\gamma_0
\end{pmatrix}
\begin{pmatrix}
  c\Delta t' \\
  \Delta x'
\end{pmatrix},
\]

where \( \gamma_0 = 1/\sqrt{1 - (\vec{V}/c)^2} \).

Equation (12) has also one specific solution, \( \varepsilon = 1 \) and \( \varepsilon' = 1 \), which corresponds to \( \varepsilon_R = 0 \) and \( \varepsilon'_R = 0 \) in the synchronisation relation (2). This case corresponds to the instantaneous synchronisation or, equivalently, absolute coordinate time, \( t(B) = t(A) \). Equations (13a), in the limit \( c \to \infty \), describe the Galilean transformation in that case.

4 Paradoxes

4.1 The ladder–barn paradox

Einstein’s synchronisation A ladder (\( O' \)) of length \( L \) is moving through a barn (\( O \)) of length \( l < L \), entering through the front door and leaving through the back door. An observer in the barn would state that, due to the Lorentz contraction, the ladder can be fit in the barn instantaneously if its velocity is such that \( \gamma_E > L/l \), where \( \gamma_E \) is the standard Lorentz factor. When the end of the ladder enters the front door of the barn, both doors of the barn are closed simultaneously to mark the fact that the barn entirely contains the ladder. An observer on the ladder would state however that the ladder cannot be fit in
the barn because the length of the barn, owing to the Lorentz contraction, is yet smaller: \( l/\gamma_E < l < L \). The apparent paradox consists in that simultaneous closing the front and the back door of the barn are not simultaneous events in the reference frame of the ladder: the back barn door closes earlier than the front of the ladder leaves the barn while the front door closes just as the end of the ladder enters the barn. Explanation of the problem in \( O \) focuses on the notion of the length while understanding it in \( O' \) does not explicitly involve lengths but lies in non-simultaneity.

**Absolute synchronisation**  Making use of the freedom of choosing a preferred frame, assume that the barn is identified with \( O_{PF} \). Justification of this choice follows from the above reasoning since the problem of fitting the ladder in the barn is well defined in the reference frame of the barn. According to (15), an observer in the barn would measure the length of the ladder (\( \Delta \tilde{\tau} = 0 \)) as \( \tilde{L} = L/\gamma_0 \), i.e. the ladder will be contracted and will fit in the barn if \( \gamma_0 \geq L/l \). The observer on the ladder would state that the length of the barn, \( l' \), is elongated (\( \Delta t' = 0 \)): \( l' = \gamma_0 l \). Owing to absolute simultaneity, both observers perform measurements of the length and both will obtain the same condition for the ladder to fit in the barn. **Ergo**, one avoids the paradox when this problem is described in the absolute synchronisation. On the other hand, if one associates the preferred frame with the ladder, the description of the entire phenomenon would be similar to that for an observer on the ladder using the Einstein’s synchronisation. In the absolute synchronisation, the paradox does not appear also in this case since both observers, on the ladder and in the barn, would agree that the barn becomes contracted and the ladder elongated so the ladder cannot be fit in the barn. In both reference frames, the front and the back doors would be opened non-simultaneously allowing the ladder to pass through the barn.

### 4.2 The twin paradox

**Einstein’s synchronisation**  Given twins in uniform relative motion, it follows from STR that each one would state that the proper time flows at a lower rate for the other twin. Verification of respective proper time intervals elapsed from the beginning of the journey may be possible only when the twins meet again and compare clock readings. If the formulation of the problem possesses a kinematical asymmetry, such that e.g. one twin turns around (travelling twin) to catch up with the other who keeps moving uniformly, then their clock readings would be different when eventually compared side-by-side. The travelling twin will be younger because the time interval elapsed in his reference frame during his journey will be smaller by the standard \( \gamma_E \) factor, assuming the simplest case that the velocities of his motion, w.r.t. the other twin, in both directions had equal values.

**Absolute synchronisation**  Assume one twin at rest in \( O_{PF} \) while \( O' \) assigned to the other (travelling) twin. Time intervals elapsed in the reference frames of the travelling twin and the twin at rest are related as follows, according to (15a): \( \Delta t' = \Delta \tilde{\tau}/\gamma_0 \) while according to (15b): \( \Delta \tilde{\tau} = \gamma_0 \Delta t' \) (this is valid for both legs of the journey). Since these relations are equivalent, both twins agree already during the travel that time flows at a lower rate for the
4.3 The Bell’s spaceship paradox

Formulation The paradox was originally formulated by E. Dewan and M. Beran in 1959 [10]. It is presented in short below, following J. S. Bell [11]. Two spaceships at rest in a given reference frame, \( O \), separated by a distance \( L \), are connected with a taut string. Both spaceships accelerate simultaneously in such a way that the distance between them, as viewed in \( O \), remains constant and equal \( L \). Will the string break at a certain moment during the acceleration? The problem received attention by several authors. Dewan and Beran, as well as Bell, judged that the string will break. There were however also contrary opinions [12] and replies [13] (see also [14]).

Einstein’s synchronisation Associate \( O' \) with one of the spaceships and assume that it has reached velocity \( V_E \) after certain time of acceleration. It can be shown that the distance between the spaceships in the co-moving frame, \( L' \), has increased during acceleration: \( L' = \gamma_E L \). Since the string is attached to both spaceships, it has to break once its elasticity limit is exceeded. On the other hand, as viewed in \( O \), the distance between the spaceships remains the same but the elasticity limit of the string undergoes Lorentz contraction, as argued by Bell, leading to the same conclusion.

Absolute synchronisation Assume that \( O_{PF} \) is the reference frame w.r.t. which the spaceships were accelerated. Substituting \( \Delta \bar{x} = L \) and requesting \( \Delta \bar{t} = 0 \) in (15a) leads directly to the result \( L' = \gamma_0 L \). Since the condition \( \Delta \bar{t} = 0 \) implies \( \Delta t' = 0 \) then \( L' \) is the measured distance between the two spaceships in the moving frame at any moment of the accelerated motion and the string has to break when its elasticity limit is exceeded. This limit corresponds to a certain characteristic length in the rest frame of the string (\( O' \)). According to (15), this length is contracted by the factor of \( \gamma_0 \) for an observer in \( O_{PF} \) which makes a consistent explanation for the string to break. Interpretation of the Bell’s spaceship paradox in the absolute synchronisation is thus simple and non-controversial.

5 Summary and conclusions

Procedure of clock synchronisation is a convention in STR. One is allowed to adopt a convention suitable for solving a given problem. In the Einstein’s synchronisation simultaneity is relative, being sometimes a source of certain difficulties in analysing STR issues. It has been shown that, by choosing the absolute synchronisation and including the preferred frame in solving STR paradoxes, it is straightforward to explain them without analysing simultaneity. The paradoxical features do not appear in consequence of these choices and, of course, lead to identical results as in the Einstein’s synchronisation.

\(^2\)Divergencies in opinions among physicists have been noted by Bell [15].
Acknowledgements

We are indebted to J. Rembieliński for numerous discussions of the preferred frame mechanics.

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