Breakdown of the Action Method in Gauge Theories

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It is shown that the definition of physical integration measures via “exponential of minus the action times kinematical integration measure” typically contradicts properties of physical models. In particular, theories with uncountably many non-vanishing Wilson-loop expectation values cannot be gained this way. The results are rigorous within the Ashtekar approach to gauge field theories.

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Introduction

The functional integral approach to quantum field theories consists of two basic steps: first the determination of a “physical” Euclidian integration measure \( d\mu \) on the configuration space and second the reconstruction of the quantum theory via an Osterwalder-Schrader procedure. The latter issue has been treated rigorously in several approaches – first by Osterwalder and Schrader for scalar fields, recently by Ashtekar et al. for diffeomorphism invariant theories. However, in contrast to this, the former issue kept a problem that has been solved completely only for some examples. Typically, one tried to define this integration measure \( d\mu \) heuristically using the action method, this means (up to a normalization factor) simply by

\[
d\mu := e^{-S} \, d\mu_0,
\]

where \( S \) is the classical action of the theory under consideration and \( d\mu_0 \) is an appropriate kinematical measure on the configuration space. In this letter we will discuss why just this ansatz can prevent the rigorous description of a huge class of physical theories. More precisely, we are going to show that in every model with uncountably many non-vanishing Wilson-loop expectation values there is no function \( f \) at all describing such a theory via

\[
d\mu := f \, d\mu_0.
\]

The criterion above is obviously fulfilled for most of the known physical theories, irrespectively of the dimension of the underlying space-time. Consequently, typically the naive action method fails.

Framework

This letter is based on the Ashtekar approach to gauge field theories because it is best-suited for solving measure-theoretical problems. Its basic idea goes as follows: The continuum gauge theory is known as soon as its restrictions to all finite floating lattices are known. This means, in particular, that the expectation values of all observables that are sensitive only to the degrees of freedom of any fixed lattice can be calculated by the corresponding integration over these finitely many degrees of freedom. Examples for those observables are the Wilson loop variables \( tr h_\beta \) where \( \beta \) is some loop in the space or space-time and \( h_\beta \) is the holonomy along that loop.

The above idea has been implemented rigorously for compact structure groups \( G \) as follows: First the original configuration space of all smooth gauge fields (modulo gauge transforms) has been enlarged by distributional ones. This way the configuration space became compact and could now be regarded as a so-called projective limit of the lattice configuration spaces. These, on the other hand, consist as in ordinary lattice gauge theories of all possible assignments of parallel transports to the edges of the considered floating lattice (again modulo gauge transforms). Since every parallel transport is an element of \( G \), the Haar measure on \( G \) yields a natural measure for the lattice theories. Now the so-called Ashtekar-Lewandowski measure \( d\mu_0 \) is just that continuum measure whose restrictions to the lattice theories coincide with these natural lattice Haar measures. It serves as a canonical kinematical measure. The corresponding square-integrable functions build the Hilbert space of wave-functions.

An important class among these functions is given by the so-called cylindrical functions. These are (continuous) functions depending only on the degrees of freedom of a finite floating lattice. A particular example for cylindrical functions are the so-called spin-network states: Given some lattice, one labels every edge with some representation of the structure group \( G \) and every vertex with some contracting intertwiner between the representations of the adjacent edges. The spin network state is then defined by the corresponding contraction of the representation matrices of the parallel transports along the edges of the lattice. The importance of these functions comes from the fact that they form a complete orthonormal system for the space of wave-functions. Note, furthermore, that a Wilson loop \( tr h_\beta \) is just a special case of
a spin-network state. Here, the underlying graph is simply one loop $\beta$ without self-intersections, the only edge $\beta$ is labelled with the fundamental representation, and the contraction at the only vertex corresponds to taking the trace.

However, in contrast to the beautiful results in the formulation of quantum geometry \cite{1} (coupled or not with Yang-Mills fields \cite{18}) within this framework, the progress to date in the treatment of general continuum gauge theories here has been quite limited. Only for the two-dimensional Yang-Mills theory the complete quantization program has been performed explicitly \cite{14,15,17}. However, even there the full measure has not been defined directly via the action method, but using a regularization and a certain limit. This was necessary because no extension of the classical action $S = \frac{1}{2} \int F^2$ to distributive gauge fields is known. Probably the same problem will arise for more complicated models as well. Therefore we are going to investigate a more fundamental problem: What kind of models at all can be studied via the action method or might it be typical that the action method fails?

**Result**

Let us be given a pure gauge theory (or the pure sector of a gauge theory), i.e. some integration measure $d\mu$ such that the expectation values can be computed by integrating the respective observables over the space $\mathcal{A}/\mathcal{G}$ of all (distributive) gauge fields modulo gauge transforms:

$$\langle O \rangle = \int_{\mathcal{A}/\mathcal{G}} O \, d\mu.$$ 

Moreover, we assume that within this model there are uncountably many non-zero spin-network expectation values. Then there is no function $f$ with $d\mu = f \, d\mu_0$.

Before establishing this result, we remark that obviously there is at least no $d\mu_0$-square-integrable function $f$. Namely, if there were such a function $f$, it could be expanded into a generalized Fourier series over spin-network states $T$ (recall that these states span a complete orthonormal basis of square-integrable functions):

$$f = \sum_T \langle T, f \rangle T \equiv \sum_T \left( \int_{\mathcal{A}/\mathcal{G}} T \, d\mu_0 \right) T$$

$$= \sum_T \left( \int_{\mathcal{A}/\mathcal{G}} T \, d\mu \right) T = \sum_T \langle T \rangle \, T. \quad (1)$$

Here, $(\phi_1, \phi_2) := \int_{\mathcal{A}/\mathcal{G}} \phi_1 \phi_2 \, d\mu_0$ denotes the scalar product on the Hilbert space of wave-functions. Now, the coefficients in this series are just the spin-network expectation values (up to complex conjugation). Hence, by assumption, there are uncountably many non-zero Fourier coefficients in this series, but this is known to be impossible for Fourier series in any Hilbert spaces.

Let us now come to the general case and let us assume there were some (integrable) function $f$ fulfilling $d\mu = f \, d\mu_0$. It is well-known \cite{14,15,16} that every integrable function can be approximated by cylindrical functions. Let now $f_n$ be such a sequence of cylindrical functions with $f_n \to f$. Since $f$ is real, we can assume that all $f_n$ are real. (Otherwise simply take the real part of each $f_n$.) Obviously we have

$$\langle f_n, \phi \rangle \equiv \int_{\mathcal{A}/\mathcal{G}} f_n \phi \, d\mu_0 = \int_{\mathcal{A}/\mathcal{G}} \phi \, f_n \, d\mu_0$$

$$\to \int_{\mathcal{A}/\mathcal{G}} \phi \, f \, d\mu_0 = \int_{\mathcal{A}/\mathcal{G}} \phi \, d\mu \equiv \langle \phi \rangle \quad (2)$$

for every continuous function $\phi$ on $\mathcal{A}/\mathcal{G}$. We know, by assumption, that there are uncountably many non-zero spin-network expectation values ($T$). But, note that we have only countably many approximating functions $f_n$. This means by (2), there must be some $n$ such that $(f_n, T)$ is non-zero for uncountably many spin-network states $T$. But this is a contradiction, since first a cylindrical function that depends, say, on the lattice $\Gamma$ is orthogonal to all spin-network states that belong to a lattice different from $\Gamma$ and since second for every fixed lattice there exist only countably many spin-network states. Consequently, there is no function $f$ with $d\mu = f \, d\mu_0$ as claimed above.

**Implications**

Since every Wilson loop can be interpreted as a special spin network, we get immediately the following criterion: If there are uncountably many non-zero Wilson-loop expectation values in the theory under consideration, then the action method fails for that theory, i.e. the definition of the integration measure via $d\mu := \frac{1}{2} e^{-S} \, d\mu_0$ cannot yield the correct expectation values. This is true even if we would substitute the classical action $S$ by some other function, maybe a regularized or renormalized action.

The significance of this result comes from its wide-range applicability. Almost all known physical gauge theories formulated in terms of loops do have uncountably many non-vanishing Wilson-loop expectation values (WLEVs). Already in the easiest example of a gauge theory in two dimensions, this criterion is fulfilled. There the WLEVs of non-selfoverlapping loops are explicitly given by $\langle \text{tr} \, h_\beta \rangle = d \, e^{-\frac{g^2\pi^2}{2} |G_\beta|}$ with $g$ being the coupling constant, $c$ the Casimir invariant of $G$, $d$ the dimension of $G$ and $|G_\beta|$ being the area enclosed by $\beta$. \cite{2,3,19,22}

Although for gauge theories in higher dimensions, such as for the pure gauge boson sectors of QED or QCD in four dimensions, the WLEVs are mostly not known explicitly, the knowledge of their approximative behaviour
suffices to see that uncountably many of them do not vanish. Hence, even there the action method fails.

But so it does typically for all theories describing confinement or deconfinement. In the first case the WLEVs are generally expected to obey an area law \[24\], i.e. being more or less proportional to \(e^{-\text{const} |G_\beta|}\) for loops \( \beta \) growing in the time-direction. This, however, implies immediately the existence of a “continuous”, hence uncountable family of loops with non-zero WLEVs. The same argumentation can be used in the case of a length law being an indicator for deconfinement.

Finally, just the existence of some continuous (quantum) symmetry in the given theory should typically suffice for the failure of the action method. Namely, if there is some non-vanishing WLEV, say for the loop \( \beta \), then all other expectation values for the loops \( \Phi(\beta) \) with \( \Phi \) running through the symmetry group will be non-vanishing as well. If now \( \beta \) itself is not accidentally invariant under most of the symmetries, we get again uncountably many non-vanishing WLEVs.

**Reason**

What could be the deeper reason behind that behaviour? A striking hint comes from the observation that the above criterion is obviously non-applicable for gauge theories on a fixed lattice. Since here the number of basic loops is finite, both the number of spin networks and that of Wilson loops is infinite, but countable. (We assume that the dimension of the structure group \( G \) is finite.) Therefore, the uncountability assumption above cannot be fulfilled on the level of a finite lattice. And, indeed, typically one can find certain lattice actions \( A_0 \) such that the corresponding lattice integration measures \( d\mu_0 \) can be written as \( \frac{1}{Z} e^{-A_0} d\mu_{\text{Haar}} \). However, this just means that the deeper reason for the breakdown of the action method must be the continuum limit, i.e. the transition from a discrete space-time to a continuous space-time – or, in other words, the transition from countability to uncountability.

At the same time, this observation shows possibly the best way out for defining physical integration measures for gauge theories avoiding this problem: Construct first the lattice measures using the action method with some action adapted to the lattice, calculate then the corresponding expectation values (if necessary, by means of certain limiting processes), transfer them to the continuum and reconstruct here the full measure. In fact, this procedure has been successfully implemented, completely, e.g., in the two-dimensional case \[8,17\].

We only mention finally that also other attempts has been made using, e.g., block-spin and renormalization group techniques (cf. \[9\]), but here again the full integration measure has not been gained in general.

**Remarks**

The main result of this letter can even be strengthened under some additional assumptions \[11\]. For that purpose, let us consider a theory having the following three properties: First there is a universal (bare) coupling constant, i.e. the interaction in the classical regime between arbitrarily charged, composite matter particles is determined completely by the interaction between the elementary particles and the charge of the particles. (For instance, in the electromagnetic case this simply encodes the fact that the interaction between \( n \)-times charged particles equals \( n^2 \) the interaction between single-charged particles.) Second there are some loops that are independent random variables. And third in certain cases the WLEVs go (not too slowly) to 1 when the considered loops shrink. (This assumption is met, e.g., in every model describing confinement.)

If the first two suppositions are fulfilled, then the lattice measures still can be received via the action method. However, if all three conditions are given, then not only the action method fails in the continuum, but the continuum integration measure \( d\mu \) is even contained in a zero subset of the kinematical integration measure \( d\mu_0 \). This means, absolutely none information about the physical model can be extracted from integration over \( f d\mu_0 \) where \( f \) is some function, maybe \( f = \frac{1}{2} e^{-S} \).

The same singularity result can be deduced if we assume as above the existence of uncountably many non-zero WLEVs and are provided additionally with some symmetry group that acts ergodically on the configuration space \( \mathcal{C}/\mathcal{G} \) w.r.t. both \( d\mu \) and \( d\mu_0 \). \[23\] This is the case \[23\] for the heavy states of the free Maxwell field constructed from the polymer-like excitations of the diffeomorphism invariant quantum theory in the canonical framework.

Finally, we note that the physical integration measure \( d\mu \) is often concentrated near non-generic strata \[13,14\], i.e. certain singular gauge fields. \[11\]

**Conclusions**

As we have shown, the breakdown of the action method can be regarded as a typical property of the continuum: Assuming the existence of uncountably many non-zero Wilson loop expectation values, the definition of the physical interaction measure via \( d\mu := \frac{1}{2} e^{-S} d\mu_0 \) is impossible. If one uses the action method, one can at most “approximate” it by lattice integration measures constructed this way. For all that it is mostly tried to get \( d\mu \) via the action method on the continuum level. Perhaps adherence to the action method is a deeper reason for the problems with the continuum limit occurring
permanently up to now. The desired absolute continuity between \(d\mu\) and \(d\mu_0\) seems to be a deceivingly simple tool, since it hides important physical phenomena.

But, the singularity of a measure *per se* is completely harmless. Namely, there is no singularity in the dual picture, i.e. for the expectation values. Moreover, strictly speaking, an integration measure is not a physically relevant quantity; only expectation values are detectable. From the theoretical point of view it is completely sufficient to know that such a measure does *exist*. Insofar, our result is just a striking hint that not the usage of functional integrals itself, but their definition is to be revised.

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