Extremal Black Holes, Attractor Equations, and Harmonic Functions

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ABSTRACT

We review the construction of multi-centered black hole solutions through dimensional reduction over time. This method does not rely on Killing spinor equations or gradient flow equations, but on solving the second order field equations in terms of harmonic functions. The black hole attractor equations are obtained directly from the field equations.
1 Extremal Black Holes

The problem of constructing stationary solutions of a theory in $d+1$ dimensions can be reduced to the problem of constructing solutions of a Euclidean theory in $d$ dimensions [1, 2]. If all relevant fields fit into a scalar sigma model, one is essentially left with solving the second order field equations for the scalars. This method is very powerful and allows the construction of, for example, multi-centered extremal black hole solutions in a very simple and systematic way, provided that the scalar manifold satisfies certain integrability conditions. In [3] this approach was applied to a class of five-dimensional Einstein-Maxwell-Scalar type theories, which can be thought of as a natural generalization of (the bosonic sector of) five-dimensional vector supermultiplets. In this article we review the construction of the solutions and some of their properties.

1.1 Dimensional Reduction

Our starting part is a five-dimensional action of the following form:

$$S = \int d^5x \sqrt{g_5} \left( \frac{1}{2} R_5 - \frac{3}{4} a_{IJ}(\sigma) \partial_{\mu} h^I \partial^\mu h^J - \frac{1}{4} a_{IJ}(h) F_{\mu\nu}^I F_{J\mu\nu} + \cdots \right) \hat{V}(h) = 1. $$

Besides gravity this theory contains $n$ gauge fields $A^I_{\mu}$, with field strength $F^I_{\mu\nu}$, where $I = 1, \ldots, n$, and $n-1$ scalars. The scalars are assumed to take values on a smooth hypersurface of an $n$-dimensional manifold, which is given by the equation $\hat{V}(h) = 1$, where $h^I$ are coordinates on the manifold, and where $\hat{V}(h)$ is a homogeneous function of degree $p$. The metric on the $n$-dimensional space is of the form

$$a_{IJ}(h) = -\frac{1}{p} \frac{\partial^2 \log \hat{V}(h)}{\partial h^I \partial h^J}. $$

In the above action it is understood that the $n$ fields $h^I$ are subject to the hypersurface constraint $\hat{V}(h) = 1$. We could solve this constraint in terms of $n-1$ independent fields, but this would turn out to be inconvenient. If $\hat{V}(h)$ is a homogeneous polynomial of degree $p = 3$, the above terms are part of the action of five-dimensional supergravity with $n$ vector multiplets [4]. The bosonic part of the supergravity action also contains a Chern-Simons term, but as we are only interested in stationary solutions carrying electric, but no magnetic charge, then the above truncation is consistent. We will allow that $p \neq 3$ and thus consider a class of non-supersymmetric theories which have a similar structure, and which in particular are completely determined by the choice of a prepotential $\hat{V}(h)$.

We reduce the five-dimensional theory with respect to time, using the following decomposition of the metric

$$ds_5^2 = -e^{2\sigma} (dt + A_m dx^m)^2 + e^{-\sigma} ds_4^2, $$

and obtain the following four-dimensional Euclidean action:

$$S = \int d^4x \sqrt{g_4} \left( \frac{1}{2} R_4 - \frac{1}{2} N_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J - \partial_m b^I \partial^m b^J + \cdots) \right). $$

We have omitted the gauge fields because they correspond to magnetic degrees of freedom of the five-dimensional theory. The $n$ scalars $b^I$ descend from the five-dimensional electro-static potentials $A^I_0$ and have axionic shift symmetries $b^I \rightarrow
$b^I + C^I$, where $C^I$ are constant. The Kaluza Klein scalar $\bar{\sigma}$ has been absorbed by rescaling the five-dimensional scalars. The resulting four-dimensional scalars $\sigma^I = e^{\bar{\sigma}} h^I$ are therefore $n$ independent fields. The scalar metric $N_{IJ}(\sigma)$ is given by

$$N_{IJ}(\sigma) = \frac{-3}{2p} \frac{\partial^2 \log \hat{V}(\sigma)}{\partial \sigma^I \partial \sigma^J}.$$ 

To solve the four-dimensional Euclidean equations of motion we will assume that the metric is flat, $g^{(4)}_{mn} = \delta_{mn}$. This implies that the five-dimensional line element has a form which is known to occur for five-dimensional extremal black holes. To solve the four-dimensional Einstein equations, we need to impose that the energy-momentum tensor of the scalars vanishes identically. This condition is easily seen to be equivalent to

$$N_{IJ}(\sigma) \left( \partial_m \sigma^I \partial_n \sigma^J - \partial_m b^I \partial_n b^J \right) = 0 \quad (1)$$

The remaining scalar equations of motion take the form

$$\partial^m (N_{IJ} \partial_m \sigma^J) - \frac{1}{2} \partial_I N_{JK}(\partial_m \sigma^J \partial^n \sigma^I - \partial_m b^J \partial^n b^I) = 0 \quad \partial^m (N_{IJ} \partial_m b^J) = 0.$$

The equations of motion of $b^I$ are the current conservation equations corresponding to the shift symmetries $b^I \rightarrow b^I + C^I$, which are remnants of the five-dimensional gauge symmetries. We observe that the ansatz $\partial_m \sigma^I = \pm \partial_m b^I$ solves the constraint (1) and reduces the equations of motion to

$$\partial^m (N_{IJ} \partial_m \sigma^J) = 0.$$

It follows that if there exists dual fields $\sigma_I$, such that $\partial_m \sigma_I = N_{IJ} \partial_m \sigma^J$, then the equations of motion take the form of harmonic equations $\Delta \sigma_I = 0$, and the solution can be expressed in terms of $n$ harmonic functions $H_I(x)$. The existence of dual fields $\sigma_I$ requires that the integrability condition $\partial_{[m} (N_{IJ} \partial_{n]} \sigma^J) = 0$ is satisfied. This has two obvious classes of solutions: i) the solution only depends on one coordinate. This leads, for spherical symmetry, to single-centered black holes solutions, or, for translational symmetry, to domain-wall type solutions. ii) The scalar metric satisfies $\partial_I N_{JK} = 0$, which is the integrability condition for the existence of a Hesse potential $\hat{V}$ for the metric $N_{IJ} = \hat{V}_{IJ} := \partial^2_{IJ} \hat{V}$. In this case no condition needs to be imposed on the space-time geometry. By construction, the scalar metric considered here have the Hesse potential:

$$\hat{V} = -\frac{3}{2p} \log \hat{V}.$$ 

The dual scalars are given by the first derivatives of the Hesse potential:

$$\sigma_I = \hat{V}_I \simeq \frac{\hat{V}_I}{\hat{V}}.$$ 

Thus, given the Hesse potential, the dual fields can always be found explicitly. However, it is not guaranteed that we can express the original scalars $\sigma^I$ in closed form in terms of the dual scalars $\sigma_I$, and, hence in terms of the harmonic functions. If we assume that the scalar fields approach constant values at infinity, the simplest type of solution is given by multi-centered harmonic functions

$$\sigma_I(x) = H_I(x) = h_I + \sum_{a=1}^{N} \frac{q_{aI}}{(x - x(a))^2}. $$
Whereas \( h_I \) encode the values of the scalars at infinity, the coefficients \( q_{aI} \) measure the charges (with respect to the shift symmetries) located at centers \( x_{(a)} \). The total charges \( Q_I \) are obtained by summing over the centers. Note that the charges associated with the shift symmetries can be written as surface integrals:

\[
Q_I := \int d^4x \partial^m (N_{IJ} \partial_m b^J) = \pm \oint d^3 \Sigma^m \partial_m \sigma^I = \pm 2\pi^2 \sum_{a=1}^N (-2)q_{aI}.
\]

1.2 Dimensional Lifting

We now lift the solution of the four-dimensional Euclidean theory to five dimensions. The resulting line element is

\[
ds_5^2 = -e^{2\beta(x)} dt^2 + e^{-\beta} \delta_{mn} dx^m dx^n,
\]

where \( \beta \) is given in terms of the four-dimensional scalars by \( e^\beta = \hat{V}(\sigma)^{1/p} \).

Rewriting \( \sigma_I(x) = \mathcal{V}_I = H_I(x) \) in terms of the five-dimensional scalars \( h^I = e^{-\beta} \sigma^I \), we obtain by

\[
e^{-\beta} \frac{\partial \hat{V}}{\partial h^I} = H_I,
\]

which are, in the supersymmetric case, precisely the so-called generalized stabilization equations \([5]\). If we approach the center at \( x_{(a)} \), the asymptotic behaviour of the harmonic functions is \( H_I \approx q_{aI} r^2 \), where \( r = x - x_{(a)} \). In the limit \( r \to 0 \) we obtain the five-dimensional stabilization or attractor equations \([5]\)

\[
Z_a \frac{\partial \hat{V}}{\partial h^I} \bigg|_{x = x_{(a)}} = q_{aI},
\]

where \( Z_a = \lim_{(x-x_{(a)}) \to 0} (r^2 e^{-\beta}) \). Thus the asymptotic solution is completely determined by the charges at the center, which is a manifestation of the black hole attractor mechanism \([6]\). Note that the near center limit is (generically) finite, because \( \sigma_I \sim r^{-2} \) implies, by homogeneity of \( \hat{V} \), that \( e^{-\beta} \sim r^{-2} \). Homogeneity can be used to solve for \( Z_a \),

\[
Z_a = \frac{1}{d} q_{aI} h^I_{x = x_{(a)}},
\]

The asymptotic value of \( \beta \) is given by \( e^{-\beta} \approx \frac{Z_a}{r^2} \), and the resulting asymptotic metric at the center is \( AdS^2 \times S^3 \):

\[
ds^2 = -\frac{r^4}{Z_a^2} dt^2 + \frac{Z_a}{r^2} dr^2 + Z_a d\Omega_3^2
\]

This shows that we obtain, if all \( Z_a \) are positive, a static configuration of extremal black holes with charges \( q_{aI} \) and entropies

\[
S_a = \frac{1}{4} A_a = \frac{\pi^2}{2} Z_a^{3/2}.
\]
Positivity of the $Z_a$ imposes inequalities on the parameters $q_{dI}$, which are, up to normalization and overall sign, the electric charges carried by the centers. This is clear because the five-dimensional electric charges are proportional to the charges associated to the four-dimensional shift symmetry. The charges determine the asymptotic values of the scalars, and positive $Z_a$ guarantee that the fixed point values approached by the scalars at the center $x_{(a)}$ correspond to a regular point of the scalar manifold. Vanishing $Z_a$ can be arranged by non-generic choices of the charges, for example by putting sufficiently many charges to zero. This corresponds to a degenerate black hole horizon with vanishing area, and to scalar fields running off to infinity on the scalar manifold. For negative $Z_a$ this running off already occurs before the center is reached, resulting in a singularity of the space-time metric.

The mass of the multi-centered black hole solution can be computed by the ADM formula, with the result
\[ M_{\text{ADM}} = |h^I_{\infty} Q_I|, \]
where $h^I_{\infty}$ are the values of the scalars at infinity, and where $Q_I$ are the total charges obtained by summing over the centers.

1.3 Discussion and concluding remarks

We have seen that using dimensional reduction over time, it becomes surprisingly simple to construct multi-centered extremal black hole solutions and to obtain the attractor equations which govern the near horizon fixed point behaviour. Our method does not require supersymmetry, or the reduction of the field equations to first order equations, but uses an integrability condition imposed on the scalar metric in order to manipulate the second order field equations until they have been reduced to harmonic equations. Note that the integrability condition, while satisfied in supersymmetric theories, does not require that the underlying theory is supersymmetric, but applies to a larger class. A deeper understanding why the method, which we presented in a simple, pedestrian fashion in this article, works so surprisingly well, can be obtained by a detailed analysis of the geometry of the scalar sigma models occurring in the construction. For this the reader is referred to [7, 3], which contains a more comprehensive list of literature, which puts this work into the context of other recent work on black holes, and which discusses the four-dimensional Euclidean solutions in their own right.

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References

[1] G. Neugebauer and D. Kramer, Ann der Physik (Leipzig) 24 (1969) 62. P. Breitenlohner, D. Maison and G. Gibbons, Comm. Math. Phys. 120 (1988) 253.

[2] K. Stelle, \textit{BPS Branes in Supergravity}, hep-th/9803116

[3] T. Mohaupt and K. Waite, JHEP 0910 (2009) 058, arXiv:0906.3451
[4] M. Gunaydin, G. Sierra and P.K. Townsend, Nucl. Phys. B 242 (1984) 244.

[5] W.A. Sabra, Mod. Phys. Lett. A 13 (1998) 239, hep-th/9708103.
A.H. Chamseddine and W.A. Sabra, Phys. Lett. B 426 (1998) 36, hep-th/9801161.

[6] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D 52 (1995) 5412, hep-th/9508072.
S. Ferrara, G.W. Gibbons and R. Kallosh, Nucl. Phys. B 500 (1997) 75, hep-th/9702103.

[7] V. Cortés and T. Mohaupt, JHEP 0907 (2009) 066, arXiv:0905.2844.