A Fractional Analogue of Brooks’ Theorem

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Abstract

Let $\Delta(G)$ be the maximum degree of a graph $G$. Brooks’ theorem states that the only connected graphs with chromatic number $\chi(G) = \Delta(G) + 1$ are complete graphs and odd cycles. We prove a fractional analogue of Brooks’ theorem in this paper. Namely, we classify all connected graphs $G$ such that the fractional chromatic number $\chi_f(G)$ is at least $\Delta(G)$. These graphs are complete graphs, odd cycles, $C_{28}$, $C_5 \boxtimes K_2$, and graphs whose clique number $\omega(G)$ equals the maximum degree $\Delta(G)$. Among the two sporadic graphs, the graph $C_{28}$ is the square graph of cycle $C_8$ while the other graph $C_5 \boxtimes K_2$ is the strong product of $C_5$ and $K_2$. In fact, we prove a stronger result; if a connected graph $G$ with $\Delta(G) \geq 4$ is not one of the graphs listed above, then we have $\chi_f(G) \leq \Delta(G) - \frac{2}{67}$.

1 Introduction

The chromatic number of graphs with bounded degrees has been studied for many years. Brooks’ theorem perhaps is one of the most fundamental results; it is included by many textbooks on graph theory. Given a simple connected graph $G$, let $\Delta(G)$ be the maximum degree, $\omega(G)$ be the clique number, and $\chi(G)$ be the chromatic number. Brooks’ theorem states that $\chi(G) \leq \Delta(G)$ unless $G$ is a complete graph or an odd cycle. Reed [10] proved that $\chi(G) \leq \Delta(G) - 1$ if $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq \Delta_0$ for some large constant $\Delta_0$. This excellent result was proved by probabilistic methods, and $\Delta_0$ is at least hundreds. Before this result, Borodin and Kostochka [1] made the following conjecture.

Conjecture [1]: Suppose that $G$ is a connected graph. If $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq 9$, then we have

$$\chi(G) \leq \Delta(G) - 1.$$ 

If the conjecture is true, then it is best possible since there is a $K_8$-free graph $G = C_5 \boxtimes K_3$ (actually $K_7$-free, see Figure 1) with $\Delta(G) = 8$ and $\chi(G) = 8$.

Here we use the following notation of the strong product. Given two graphs $G$ and $H$, the strong product $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$, and $(a, x)$ is connected to $(b, y)$ if one of the following holds

- $a = b$ and $xy \in E(H)$,
Reed’s result \cite{10} settled Borodin and Kostochka’s conjecture for sufficiently large $\Delta(G)$, but the cases with small $\Delta(G)$ are hard to cover using the probabilistic method.

In this paper we consider a fractional analogue of this problem. The fractional chromatic number $\chi_f(G)$ can be defined as follows. A $b$-fold coloring of $G$ assigns a set of $b$ colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. We say a graph $G$ is $a:b$-colorable if there is a $b$-fold coloring of $G$ in which each color is drawn from a palette of $a$ colors. We refer to such a coloring as an $a:b$-coloring. The $b$-fold coloring number, denoted by $\chi_b(G)$, is the smallest integer $a$ such that $G$ has an $a:b$-coloring. Note that $\chi_1(G) = \chi(G)$. It was shown that $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$. The fractional chromatic number $\chi_f(G)$ is $\lim_{b \to \infty} \frac{\chi_b(G)}{b}$.

By the definition, we have $\chi_f(G) \leq \chi(G)$. The fractional chromatic number can be viewed as a relaxation of the chromatic number. Many problems involving the chromatic number can be asked again using the fractional chromatic number. The fractional analogue often has a simpler solution than the original problem. For example, the famous $\omega - \Delta - \chi$ conjecture of Reed \cite{9} states that for any simple graph $G$, we have

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$  

The fractional analogue of $\omega - \Delta - \chi$ conjecture was proved by Molloy and Reed \cite{8}; they actually proved a stronger result with ceiling removed, i.e.,

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}. \quad (1)$$

In this paper, we classify all connected graphs $G$ with $\chi_f(G) \geq \Delta(G)$.

**Theorem 1** A connected graph $G$ satisfies $\chi_f(G) \geq \Delta(G)$ if and only if $G$ is one of the following

1. a complete graph,
2. an odd cycle,
3. A graph with $\omega(G) = \Delta(G)$,
4. $C_8^2$,
5. $C_5 \boxtimes K_2$.

For the complete graph $K_n$, we have $\chi_f(K_n) = n$ and $\Delta(K_n) = n - 1$. For the odd cycle $C_{2k+1}$, we have $\chi_f(C_{2k+1}) = 2 + \frac{1}{k}$ and $\Delta(C_{2k+1}) = 2$. If $G$ is neither a complete graph nor an odd cycle but contains a clique of size $\Delta(G)$, then we have

$$\Delta(G) \leq \omega(G) \leq \chi_f(G) \leq \chi(G) \leq \Delta(G).$$

(2)

The last inequality is from Brooks’ theorem. The sequence of inequalities above implies $\chi_f(G) = \Delta(G)$.

If $G$ is a vertex-transitive graph, then we have

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)},$$

where $\alpha(G)$ is the independence number of $G$. Note that both graphs $C_8^2$ and $C_5 \boxtimes K_2$ are vertex-transitive and have the independence number 2. Thus we have

$$\chi_f(C_8^2) = 4 = \Delta(C_8^2) \quad \text{and} \quad \chi_f(C_5 \boxtimes K_2) = 5 = \Delta(C_5 \boxtimes K_2).$$

Figure 2: The graph $C_8^2$ and $C_5 \boxtimes K_2$.

Actually, Theorem 1 is a corollary of the following stronger result.

**Theorem 2** Assume that a connected graph $G$ is neither $C_8^2$ nor $C_5 \boxtimes K_2$. If $\Delta(G) \geq 4$ and $\omega(G) \leq \Delta(G) - 1$, then we have

$$\chi_f(G) \leq \Delta(G) - \frac{2}{67}.$$

**Remark:** In the case $\Delta(G) = 3$, Heckman and Thomas [5] conjectured that $\chi_f(G) \leq 14/5$ if $G$ is triangle-free. Hatami and Zhu [3] proved $\chi_f(G) \leq 3 - \frac{3}{2g}$ for any triangle-free graph $G$ with $\Delta(G) \leq 3$. The second and third authors showed an improved result $\chi_f(G) \leq 3 - \frac{3}{2g}$ in the previous paper [7]. Thus we need only consider the cases $\Delta(G) \geq 4$. For any connected graph $G$ with sufficiently large $\Delta(G)$ and $\omega(G) \leq \Delta(G) - 1$, Reed’s result [10] $\chi(G) \leq \Delta(G) - 1$ implies $\chi_f(G) \leq \Delta(G) - 1$. The method introduced in [4] and strengthened in [7], has a strong influence on this paper. The readers are encouraged to read these two papers [4, 7].
Let $f(k) = \inf_G \{\Delta(G) - \chi_f(G)\}$, where the infimum is taken over all connected graphs $G$ with $\Delta(G) = k$ and not one of the graphs listed in Theorem 1. Since $\chi_f(G) \geq \omega(G)$, by taking a graph with $\omega(G) = \Delta(G) - 1$, we have $f(k) \leq 1$. Theorem 2 says $f(k) \geq \frac{\Delta(G)}{2}$ for any $k \geq 4$. Reed’s result [10] implies $f(k) = 1$ for sufficiently large $k$. Heckman and Thomas conjectured $f(3) = 1/5$. It is an interesting problem to determine the value of $f(k)$ for small $k$. Here we conjecture $f(4) = f(5) = \frac{1}{4}$. If Borodin and Kostochka’s conjecture is true, then $f(k) = 1$ for $k \geq 9$.

Theorem 2 is proved by induction on $k$. Because the proof is quite long, we split the proof into the following two lemmas.

**Lemma 1** We have $f(4) \geq \frac{2}{67}$.

**Lemma 2** For each $k \geq 6$, we have $f(k) \geq \min\{f(k-1), \frac{1}{2}\}$. We also have $f(5) \geq \min\{f(4), \frac{1}{4}\}$.

It is easy to see the combination of Lemma 1 and Lemma 2 implies Theorem 2. The idea of reduction comes from the first author, who pointed out $f(k) \geq \min\{f(k-1), \frac{1}{2}\}$ for $k \geq 7$ based on his recent results [6]. The second and third authors originally proved $f(k) \geq \frac{C}{k}$ (for some $C > 0$) using different method in the first version; they also prove the reductions at $k = 5, 6$, which are much harder than the case $k \geq 7$. We do not know whether a similar reduction exists for $k = 4$.

The rest of this paper is organized as follows. In section 2, we will introduce some notation and prove Lemma 2. In section 3 and section 4, we will prove $f(4) \geq \frac{2}{67}$.

## 2 Proof of Lemma 2

In this paper, we use the following notation. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood of a vertex $v$ in $G$, denoted by $\Gamma_G(v)$, is the set $\{u : uv \in E(G)\}$. The degree $d_G(v)$ of $v$ is the value of $|\Gamma_G(v)|$. The independent set (or stable set) is a set $S$ such that no edge with both ends in $S$. The independence number $\alpha(G)$ is the largest size of $S$ among all the independent sets $S$ in $G$. When $T \subset V(G)$, we use $\alpha_G(T)$ to denote the independence number of the induced subgraph of $G$ on $T$. Let $\Delta(G)$ be the maximum degree of $G$. For any two vertex-sets $S$ and $T$, we define $E_G(S, T)$ as $\{uv \in E(G) : u \in S$ and $v \in T\}$. Whenever $G$ is clear under context, we will drop the subscript $G$ for simplicity.

If $S$ is a subset of vertices in $G$, then contracting $S$ means replacing vertices in $S$ by a single fat vertex, denoted by $\hat{S}$, whose incident edges are all edges that were incident to at least one vertex in $S$, except edges with both ends in $S$. The new graph obtained by contracting $S$ is denoted by $G/S$. This operation is also known as identifying vertices of $S$ in the literature. For completeness, we allow $S$ to be a single vertex or even the empty set. If $S$ only consists of a single vertex, then $G/S = G$; if $S = \emptyset$, then $G/S$ is the union of $G$ and an isolated vertex. When $S$ consists of 2 or 3 vertices, for convenience, we write $G/uv$ for $G/\{u, v\}$ and $G/uvw$ for $G/\{u, v, w\}$; the fat vertex will be denoted by $uv$ and $uvw$, respectively. Given two disjoint subsets $S_1$ and $S_2$, we can contract $S_1$ and $S_2$ sequentially. The order of contractions does not matter; let $G/S_1/S_2$ be the resulted graph. We use $C - S$ to denote the subgraph of $G$ induced by $V(G) - S$.

In order to prove Lemma 2, we need use the following theorems due to King [6].

**Theorem 3 (King [6])** If a graph $G$ satisfies $\omega(G) > \frac{2}{7}(\Delta(G) + 1)$, then $G$ contains a stable set $S$ meeting every maximum clique.

Theorem 3 (King [6]) If a graph $G$ satisfies $\omega(G) > \frac{2}{7}(\Delta(G) + 1)$, then $G$ contains a stable set $S$ meeting every maximum clique.
Theorem 4 (King [6])  For a positive integer \(k\), let \(G\) be a graph with vertices partitioned into cliques \(V_1, \ldots, V_r\). If for every \(i\) and every \(v \in V_i\), \(v\) has at most \(\min\{k, |V_i| - k\}\) neighbors outside \(V_i\), then \(G\) contains a stable set of size \(r\).

Lemma 3 Suppose that \(G\) is a connected graph with \(\Delta(G) \leq 6\) and \(\omega(G) \leq 5\). Then there exists an independent set meeting all induced copies of \(K_5\) and \(C_5 \boxtimes K_2\).

Proof: We first show that there exists an independent set meeting all copies of \(K_5\). If \(G\) contains no \(K_5\), then this is trivial. Otherwise, we can apply Theorem 4 to get the desired independent set since \(\omega(G) > \frac{2}{5}(\Delta(G) + 1)\) is satisfied.

Now we prove the Lemma by contradiction. Suppose the Lemma is false. Let \(G\) be a minimum counterexample (with the smallest number of vertices). For any independent set \(I\), let \(C(I)\) be the number of induced copies of \(C_5 \boxtimes K_2\) in \(G - I\). Among all independent sets which meet all copies of \(K_5\), there exists one such independent set \(I\) such that \(C(I)\) is minimized.

Since \(C(I) > 0\), there is an induced copy of \(C_5 \boxtimes K_2\) in \(G - I\); we use \(H\) to denote it. In \(C_5 \boxtimes K_2\), there is a unique perfect matching such that identifying the two ends of each edge in this matching results a \(C_5\). An edge in this unique matching is called a canonical edge. We define a new graph \(G'\) as follows: First we contract all canonical edges in \(H\) to get a \(C_5\), where its vertices are called fat vertices. Second we add five edges turning the \(C_5\) into a \(K_5\). Observe that each vertex in this \(C_5\) can have at most two neighbors in \(G - H\) and \(\Delta(G') \leq 6\). We will consider the following four cases.

Case 1: There is a \(K_6\) in the new graph \(G'\). Since the original graph \(G\) is \(K_6\)-free, the \(K_6\) is formed by the following two possible ways.

Subcase 1a: This \(K_6\) contains 5 fat vertices. By the symmetry of \(H\), there is an induced \(C_5\) in \(H\) such that the vertices in \(C_5\) contain a common neighbor vertex \(v\) in \(G \setminus V(H)\), see Figure 3. Since \(H\) is \(K_5\)-free, we can find \(x, y\) in this \(C_5\) such that \(x, y\) is a non-edge. Let \(I' := (I \setminus \{v\}) \cup \{x, y\}\); \(I'\) is also an independent set. Observe that \(v\) is not in any \(K_5\) in \(G - I'\). Thus the set \(I'\) is also an independent set and meets every \(K_5\) in \(G\). Since \(C_5 \boxtimes K_2\) is a 5-regular graph, any copy of \(C_5 \boxtimes K_2\) containing \(v\) must contain at least one of \(x\) and \(y\). Thus, \(C(I') < C(I)\). Contradiction!

Subcase 1b: This \(K_6\) contains 4 fat vertices. Let \(u, v\) be the other two vertices. By the symmetry of \(H\), there is a unique way to connect \(u\) and \(v\) to \(H\) as shown by Figure 4. Since \(uv\) is an edge, one of \(u\) and \(v\) is not in \(I\). We assume \(u \notin I\). Let \(\{x, y\} \subset \Gamma_G(v) \cap V(H)\) as shown in Figure 4 and \(I' = I \setminus \{v\} \cup \{x, y\}\). Observe that \(I'\) is an independent set and \(v\) is not in a \(K_5\) in \(G - I'\). Thus \(I'\) is an independent set meeting each \(K_5\) in \(G\). Since each \(C_5 \boxtimes K_2\) containing \(v\) must contain one of \(x\) and \(y\). Thus \(C(I') < C(I)\). Contradiction!

Case 2: There is a \(K_5\) intersecting \(H\) with 4 vertices. Let \(v\) be the vertex of this \(K_5\) but not in \(H\), see Figure 5. We have two subcases.
Subcase 2a: The vertex $v$ has another neighbor $y$ in $H$ but not in this $K_5$. Since $H$ is $K_5$-free, we can select a vertex $x$ in this $K_5$ such that $xy$ is not an edge of $G$. Let $I' := I \setminus \{v\} \cup \{x, y\}$. Note that $v$ is not in a $K_5$ in $G - I'$, and $I'$ is an independent set. Thus $I'$ is an independent set meeting each $K_5$ in $G$. Since any $C_5 \boxtimes K_2$ containing $v$ must contain one of $x$ and $y$, we have $C(I') < C(I)$. Contradiction!

Subcase 2b: All neighbors of $v$ in $H$ are in this $K_5$. Let $x$ be any vertex in this $K_5$ other than $v$, and $I' := I \setminus \{v\} \cup \{x\}$. In this case, there is only one $K_5$ containing $v$. Thus, $I'$ is also an independent set meeting every copy of $K_5$ in $G$. Observe that $\Gamma_G(v) \setminus \{x\}$ is disconnected. If $v \in H' = C_5 \boxtimes K_2$, then $\Gamma_G(v) \cap H'$ is connected. Thus $v$ is not in a $C_5 \boxtimes K_2$ in $G - I'$ and $C(I') < C(I)$. Contradiction!

![Figure 5: Case 2.](image1)

![Figure 6: Case 3.](image2)

Case 3: There is an induced subgraph $H'$ isomorphic to $C_5 \boxtimes K_2$ such that $H'$ and $H$ are intersecting, see Figure 4. Since $V(H) \cap V(H') \neq \emptyset$ and $H \neq H'$, we can find a canonical edge $uv$ of $H$ and a canonical edge $uv'$ of $H'$ such that $v \notin V(H')$ and $v' \notin V(H)$. If $uv'$ is a non-edge, then let $I' := I \setminus \{v'\} \cup \{u\}$. It is easy to check $I'$ is still an independent set. We also observe that any possible $K_5$ containing $v'$ must also contain $u$. Thus, $I'$ meets every copy of $K_5$ in $G$. We have $v'$ in no $C_5 \boxtimes K_2$ in $G - I'$ since $uv'$ is not an edge. We therefore get $C(I') < C(I)$. Contradiction! If $uv'$ is an edge, then locally there are two $K_5$ intersecting at $u$, $v$, and $v'$; say the other four vertices are $x_1, x_2, y_1, y_2$, where two cliques are $\{x_1, x_2, u, v, v'\}$ and $\{y_1, y_2, u, v, v'\}$, see Figure 5. Let $I' = I \setminus \{x_1, y_1\} \cup \{v'\}$. Note that $I'$ is an independent set and $v'$ is not in a $K_5$ in $G - I'$. Thus $I'$ is an independent set meeting each $K_5$ in $G$. Observe that any copies of $C_5 \boxtimes K_2$ containing $v'$ must contain one of $x_1$ and $y_1$; we have $C(I') < C(I)$. Contradiction!.

Case 4: This is the remaining case, $G'$ is $K_6$-free. We have $\omega(G') \leq 5$ and $|V(G')| < |V(G)|$. By the minimality of $G$, there is an independent set $I'$ of $G'$ meeting every copy of $K_5$ and $C_5 \boxtimes K_2$. In $I'$, there is a unique vertex $x$ of the $K_5$ obtained from contracting canonical edges of $H$. Let $uv$ be the canonical edge corresponding to $x$. Let $I'' = I' \setminus \{x\} \cup \{u\}$, we get an independent set $I''$ of $G$. Note that any $v \in H \setminus \{u\}$ is not in any $K_5$ of $G - I''$ by Case 2 as well as not in any $C_5 \boxtimes K_2$ of $G - I''$ by Case 3. Thus $I''$ hits each $K_5$ in $G$ and $C(I'') = 0$. Contradiction!

The following lemma extends Theorem 3 when $\omega(G) = 4$; a similar result was proved independently in [2].

**Lemma 4** Let $G$ be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there is an independent set $I$ hitting all copies of $K_4$ in $G$.

**Proof:** We will prove it by contradiction. If the lemma is false, then let $G$ be a minimum counterexample. If $G$ is $K_4$-free, then there is nothing to prove. Otherwise, we consider the clique graph $C(G)$, whose edge set is the set of all edges appearing in some copy of $K_4$. Because of $\Delta(G) = 5$, here are all possible connected component of $C(G)$. 

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I Proof: which meet all copies of $K_t$ since $u$ is an antipode such that $v$ do not form a triangle, then we let $I' := I \setminus \{v\} \cup \{u_0, u_4\}$; note that $v$ is not in any $K_4$ of $G - I'$. Thus $I'$ is an independent set meeting every copy of $K_4$. Since every copy of $C_8^2$ containing $v$ must contain one of $u_0$ and

3. There are four other types listed in Figure 7

![Figure 7: All types of components in the clique graph $C(G)$.](image)

For each component $C_i$ in $C(G)$, let $V_i$ be the set of common vertices in all $K_4$’s of $C_i$; for the leftmost figure in Figure 7, $V_i$ is the set of all 4 vertices; for the middle two figures, $V_i$ is the set of bottom three vertices; for the rightmost figure, $V_i$ consists of the left-bottom vertex and the middle-bottom vertex. Note that all $V_i$’s are pairwise disjoint. Let $G'$ be the induce subgraph of $G$ on $\bigcup_i V_i$. Note that $G'$ does not contains any vertex in $C_i \setminus V_i$. By checking each type, we find out that for each $i$ and each $v \in V_i$, $v$ has at most $\min\{2, |V_i| - 2\}$ neighbors outside $V_i$ in $G'$ (not in $G'$). Applying Theorem 4 to $G'$, we conclude that there exists an independent set $I$ of $G'$ meeting every $V_i$; thus $I$ meets every $K_4$ in $G$. Contradiction!

\[ \square \]

**Lemma 5** Let $G$ be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there exists an independent set meeting all induced copies of $K_4$ and $C_8^2$.

**Proof:** We will use proof by contradiction. Suppose the Lemma is false. Let $G$ be a minimum counterexample (with the smallest number of vertices). For any independent set $I$, let $C(I)$ be the number of induced copies of $C_8^2$ in $G - I$. Among all independent sets which meet all copies of $K_4$, there exists an independent set $I$ such that $C(I)$ is minimized. Since $C(I) > 0$, let $H$ be a copy of $C_8^2$ in $G - I$. The vertices of $H$ are listed by $u_i$ for $i \in \mathbb{Z}_8$ anticlockwise such that $u_iu_j$ is an edge of $H$ if and only if $|i - j| \leq 2$. The vertex $v_{i+4}$ is the antipode of $v_i$ for any $i \in \mathbb{Z}_8$.

**Case 1:** There exists a vertex $v \notin V(H)$ such that $v$ has five neighbors in $H$. By the Pigeonhole Principle, $\Gamma(v)$ contains a pair of antipodes. Without loss of generality, say $u_0, u_4 \in \Gamma(v)$. If the other three neighbors of $v$ do not form a triangle, then we let $I' := I \setminus \{v\} \cup \{u_0, u_4\}$; note that $v$ is not in any $K_4$ of $G - I'$. Thus $I'$ is an independent set meeting every copy of $K_4$. Since every copy of $C_8^2$ containing $v$ must contain one of $u_0$ and
u_4, we have C(I') < C(I). Contradiction! Hence, the other three neighbors of v must form a triangle. Without loss of generality, we can assume that the three neighbors are u_1, u_2, and u_3. Now we let I' := I \{v\} \cup \{u_0, u_3\}; note that v \notin K_4 \subset G - I'. Thus I' is also an independent set meeting every copy of K_4 of G. Since every copy of \( C_8^2 \) containing v must contain one of u_0 and u_3, we have C(I') < C(I). Contradiction!

**Case 2:** There exists a vertex v \notin V(H) such that v has exactly four neighbors in H. Since H is K_4-free, we can find u_i, u_j \in \Gamma(v) \cap V(H) such that u_iu_j is a non-edge. Let I' := I \{v\} \cup \{u_i, u_j\}; I' is also an independent set. Note that \( \Gamma(v) \setminus \{u_i, u_j\} \) can not be a triangle; v is not in any K_4 \subset G - I'. Thus I' meets every copy of K_4. Since every copy of \( C_8^2 \) containing v must contain one of u_i and u_j, we have C(I') < C(I). Contradiction!

**Case 3:** There exists a vertex v \notin V(H) such that v has exactly three neighbors in H. If the 3 neighbors do not form a triangle, then choose u_i, u_j \in \Gamma(v) \cap V(H) such that u_iu_j is a non-edge. Note that \( \Gamma(v) \setminus \{u_i, u_j\} \) can not be a triangle; v is not in any K_4 \subset G - I'. Let I' := I \{v\} \cup \{u_i, u_j\}; I' is also an independent set meeting every copy of K_4. Since every copy of \( C_8^2 \) containing v must contain one of u_i and u_j, we have C(I') < C(I). Contradiction! Else, the three neighbors form a triangle; let u_i be one of them and I' := I \{v\} \cup \{u_i\}; v is not in any K_4 \subset G - I'. Thus I' is an independent set meeting every copy of K_4. Note that \( \Gamma(v) \setminus \{u_i\} \) has only two vertices in H. The induced graph \( \Gamma(v) \setminus \{u_i\} \) is disconnected. However, for any vertex v in H' = C_8^2, the subgraph induced by \( \Gamma_G(v) \cap V(H') \) is a P_4. There is no C_8^2 in G - I' containing v. Thus, C(I') < C(I). Contradiction!

**Case 4:** Every vertex outside H can have at most 2 neighbors in H. We identify each pair of antipodes of H to get a new graph G' from G. After identifying, H is turned into a K_4; where the vertices of this K_4 are referred as fat vertices.

**Subcase 4a:** G' \neq C_{2l+1} \boxtimes K_2. Observe \( \Delta(G') \leq 5 \). We claim G' is K_{l+1}-free. Suppose not. Since every vertex in H has at most one neighbor outside H, then each fat vertex can have at most two neighbors outside H. Recall that the original graph G is K_{l+1}-free. If G' has some K_5, then this K_5 contains either 3 or 4 fat vertices. Let w be one of the other vertices in this K_5. We get w has at least three neighbors in H. However, this is covered by Case 1, Case 2, or Case 3. Thus, G' is K_{l+1}-free. Since |G'| < |G|, by the minimality of G, G' has an independent set I' meeting every copy of K_4 and C_8^2 in G'. There is exactly one fat vertex in I'. Now replacing this fat vertex by its corresponding pair of antipodal vertices, we get an independent set I''; we assume the pair of antipodal vertices are u_2 and u_6. It is easy to check that I'' is an independent set of G. Next we claim any v \in V(H) \setminus \{u_2, u_6\} is neither in a K_4 \subset V(G) - I'' nor in a C_8^2 \subset V(G) - I''. Suppose there is some v such that v \notin K_4 \subset G - I''. Recall each v \in V(H) has at most one neighbor outside H and H is K_4-free; there is some w \in V(H) such that w has at least three neighbors in H. This is already considered by Case 1, Case 2, or Case 3. We are left to show that v \notin C_8^2 \subset G - I'' for each v \in V(H) \setminus \{u_2, u_6\}. If not, there exists a copy H' of C_8^2 in G - I'' containing v. Note H' is 4-regular, any vertex in H' can have at most one neighbor in I' in particular, v \neq u_0, u_3. Without loss of generality, we assume v = u_3. Then there is a vertex w \notin V(H) such that u_3w is an edge, see Figure. Observe that the neighborhood of each vertex of an induced C_8^2 is is a P_4. Since u_1u_4 and u_1u_5 are two non-edges, we have wu_1 being an edge. Observe \( \Gamma_G(u_1) = \{w, u_0, u_2, u_3, w\} \). Since u_2 \notin H', we have u_0 \in H'; u_0 has two neighbors (u_2 and u_6) outside H', contradiction! Therefore, I'' meets every copy of K_4 and C_8^2 in G. Contradiction!

**Subcase 4b:** G' = C_{2l+1} \boxtimes K_2. The graph G can be recovered from G'. It consists of an induced subgraph H = C_8^2 and an induced subgraph P_{2l-1} \boxtimes K_2. For each vertex u in H, there is exactly one edge connecting it to one of the four end vertices of P_{2l-1} \boxtimes K_2; for each end vertex v of P_{2l-1} \boxtimes K_2, there are exactly two edges connecting v to the vertices in H. First, we take any maximum independent set I' of P_{2l-1} \boxtimes K_2. Observe that I' has exactly two end points of P_{2l-1} \boxtimes K_2; so I' has exactly four neighbors in H. In the remaining four
vertices of $H$, there exists a non-edge $u_i u_j$ since $H$ is $K_4$-free. Let $I := I' \cup \{u_i, u_j\}$. Clearly $I$ is an independent set of $G$ meeting every copy of $K_4$ and $C_8^2$. Contradiction! □

We are ready to prove Lemma 2.

**Proof of Lemma 2.** We need prove for $k \geq 5$ and any connected graph $G$ with $\Delta(G) = k$ and $\omega(G) \leq k - 1$ satisfies

$$\chi_f(G) \leq k - \min \left\{ f(k - 1), \frac{1}{2} \right\}.$$  \hspace{1cm} (3)

If $\omega(G) \leq k - 2$, then by inequality (1), we have

$$\chi_f(G) \leq \Delta(G) + \omega(G) + 1 \leq k - 1.$$  \hspace{1cm} (2)

Thus, inequality (3) is satisfied. From now on, we assume $\omega(G) = \Delta(G) - 1$.

For $\Delta(G) = k \geq 6$ and $\omega(G) = k - 1$, the condition $\omega(G) > \frac{3}{2} (\Delta(G) + 1)$ is satisfied. By Theorem 3, $G$ contains an independent set meeting every maximum clique. Extend this independent set to a maximal independent set and denote it by $I$. Note that $\Delta(G - I) \leq k - 1$ and $\omega(G - I) \leq k - 2$.

**Case 1:** $k \geq 7$. From the definition of $f(k - 1)$, we have $\chi_f(G - I) \leq \Delta(G - I) - f(k - 1)$. Thus,

$$\chi_f(G) \leq \chi_f(G - I) + 1 \leq k - 1 - f(k - 1) + 1 = k - f(k - 1).$$

Thus, we have $f(k) \geq \min \{ f(k - 1), 1/2 \}$.

**Case 2:** $k = 6$. By Lemma 3 we can find an independent set meets every copy of $K_5$ and $C_5 \boxtimes K_2$. We extend this independent set as a maximal independent set $I$. Note that $G - I$ contains no induced subgraph isomorphic $C_5 \boxtimes K_2$. We have $\chi_f(G - I) \leq 5 - f(5)$; it implies $\chi_f(G) \leq 6 - f(5)$. Thus, $f(6) \geq \min \{ f(5), 1/2 \}$ and we are done.

**Case 3:** $k = 5$. If $G = C_{2l + 1} \boxtimes K_2$ for some $l \geq 3$; then $G$ is vertex-transitive and $\alpha(G) = l$. It implies that

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)} = 4 + \frac{2}{l} \leq 5 - \frac{1}{3}.$$  \hspace{1cm} (4)

If $G \neq C_{2l + 1} \boxtimes K_2$, then by Lemma 5 we can find an independent set meeting every copy of $K_4$ and $C_8^2$. We extend it as a maximal independent set $I$. Note that $G - I$ contains no induced subgraph isomorphic $C_8^2$. We have $\chi_f(G - I) \leq 4 - f(4)$; it implies $\chi_f(G) \leq 5 - f(4)$. Thus, $f(3) \geq \min \{ f(4), 1/3 \}$ and we are finished. □

Figure 8: Subcase 4a.
3 The case $\Delta(G) = 4$

To prove $f(4) \geq \frac{2}{67}$, we will use an approach which is similar to those in [4, 7]. We will construct 133 4-colorable auxiliary graphs, and from these colorings we will construct a 134-fold coloring of $G$ using 532 colors.

It suffices to prove that the minimum counterexample does not exist.

Let $G$ be a graph with the smallest number of vertices and satisfying

1. $\Delta(G) = 4$ and $\omega(G) \leq 3$;
2. $\chi_f(G) > 4 - \frac{2}{67}$;
3. $G \neq C_8^2$.

By the minimality of $G$, each vertex in $G$ has degree either 4 or 3. To prove Lemma 1, we will show $\chi_f(G) \leq 4 - \frac{2}{67}$, which gives us the desired contradiction.

For a given vertex $x$ in $V(G)$, it is easy to color its neighborhood $\Gamma_G(x)$ using 2 colors. If $d_G(x) = 3$, then we pick a non-edge $S$ from $\Gamma_G(x)$ and color the two vertices in $S$ using color 1. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \geq 3$, then we pick an independent set $S$ in $\Gamma_G(x)$ of size 3 and assign the color 1 to each vertex in $S$. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then we pick two disjoint non-edges $S_1$ and $S_2$ from $\Gamma_G(x)$; we assign color 1 to each vertex in $S_1$ and color 2 to each vertex in $S_2$.

The following Lemma shows that $G$ has a key property, which eventually implies that this local coloring scheme works simultaneously for $x$ in a large subset of $V(G)$.

**Lemma 6** For each $x \in V(G)$ with $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, there exist two vertex-disjoint non-edges $S_1(x), S_2(x) \subset \Gamma_G(x)$ satisfying the following property. If we contract $S_1(x)$ and $S_2(x)$, then the resulting graph $G/S_1(x)/S_2(x)$ contains neither $K_5^-$ nor $G_0$. Here $K_5^-$ is the graph obtained from $K_5$ by removing one edge and $G_0$ is the graph shown in Figure 9.

![Figure 9: The graph $G_0$.](image)

The proof of this lemma is quite long and we will present its proof in section 4.

For each vertex $x$ in $G$, we associate a small set of vertices $S(x)$ selected from $\Gamma_G(x)$ as follows. If $d_G(x) = 3$, then let $S(x)$ be the endpoints of a non-edge in $\Gamma_G(x)$ and label the vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \geq 3$, then let $S(x)$ be any independent set of size 3 in $\Gamma_G(x)$ and label all vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then...
let $S(x) = S_1(x) \cup S_2(x)$, where $S_1(x)$ and $S_2(x)$ are guaranteed by Lemma[3] we label the vertices in $S_1(x)$ as 1 and the vertices in $S_2(x)$ as 2. For any $x \in V(G)$, we have $|S(x)| = 2$, 3, or 4.

The following definitions depend on the choice of $S(*)$, which is assumed to be fixed through this section. For $v \in G$ and $j \in \{1, 2, 3\}$, we define

$$\mathcal{N}_G^j(v) = \{u \mid \text{there is a path } v v_0 \ldots v_{j-2} u \text{ in } G \text{ of length } j \text{ such that } v_0 \in S(v) \text{ and } v_{j-2} \in S(u)\}.$$

We now define $\mathcal{N}_G^j(u)$ for $j \in \{4, 5, 7\}$; each $\mathcal{N}_G^j(u)$ is a subset of the $j$th neighborhood of $u$. For $j = 4, v \in \mathcal{N}_G^4(u)$ if $d_G(u) = 4, \alpha(\Gamma_G(u)) = 2$, and $v$ and $u$ are connected as shown in Figure[10]; otherwise $\mathcal{N}_G^4(u) = \emptyset$. In Figure[10] $w$ is connected to one of the two vertices in $\mathcal{S}_2(u)$. Similarly, in Figure[11] and Figure[12] a vertex is connected to a group of vertices if it is connected to any vertex in the group.

For $j = 5, v \in \mathcal{N}_G^5(u)$ if $d_G(w) = 4, \alpha(\Gamma_G(w)) = 2$ for $w \in \{u, v\}$ and $u$ and $v$ are connected as shown in Figure[11]; otherwise $\mathcal{N}_G^5(u) = \emptyset$.

For $j = 7, v \in \mathcal{N}_G^7(u)$ if $d_G(w) = 4, \alpha(\Gamma_G(w)) = 2$ for $w \in \{u, v\}$ and $u$ and $v$ are connected as shown in Figure[12]; otherwise $\mathcal{N}_G^7(u) = \emptyset$.

![Figure 10: 4-th neighborhood. Figure 11: 5-th neighborhood. Figure 12: 7-th neighborhood.](image)

Note that for $j \in \{1, 2, 3, 5, 7\}$, $v \in \mathcal{N}_G^j(u)$ if and only if $u \in \mathcal{N}_G^j(v)$; but this does not hold for $j = 4$. We have the following lemma.

**Lemma 7** For $u \in V(G)$ such that $d_G(u) = 4$ and $\alpha(\Gamma_G(u)) = 2$, we have $|\mathcal{N}_G^4(u) \cup \mathcal{N}_G^5(u) \cup \mathcal{N}_G^7(u) \cup \mathcal{N}_G^8(u) \cup \mathcal{N}_G^9(u) \cup \mathcal{N}_G^{10}(u)| \leq 96$.

**Proof:** It is clear that $|\mathcal{N}_G^4(u) \cup \mathcal{N}_G^5(u) \cup \mathcal{N}_G^7(u)| \leq 4 + 8 + 3 = 15$. We next estimate $|\mathcal{N}_G^8(u)|$. In Figure[10] we observe that $w$ is connected to one vertex of $\mathcal{S}_2(u)$ and $w \notin \Gamma_G(u)$. For a fixed $u$, there are at most four choices for $w$, at most three choices for $v$, and at most three choices for $z$. Therefore, we have $|\mathcal{N}_G^8(u)| \leq 4 \times 3 \times 3 = 36$.

Let us estimate $|\mathcal{N}_G^9(u)|$. In Figure[11] for a fixed $u$, we have four choices for $w$ and two choices for $z$. Fix a $z$. Assume $\Gamma_G(z) \setminus \{w\} = \{a, b, c\}$. Let $T_1 = \{a, b\}, T_2 = \{b, c\},$ and $T_3 = \{a, c\}$. We have the following claim.

**Claim** There are at most three $v \in \mathcal{N}_G^9(u)$ such that for each $v$ we have $\Gamma_G(z) \cap \Gamma_G(v) = T_i$ for some $1 \leq i \leq 3$ as shown in Figure[11].

**Proof of the claim:** For each $1 \leq i \leq 3$, there are at most three $v \in \mathcal{N}_G^9(u)$ such that $\Gamma_G(z) \cap \Gamma_G(v) = T_i$ as shown in Figure[11] since each vertex in $T_i$ has at most three neighbors other than $z$. If the claim is false, then there is $1 \leq i \neq j \leq 3$ such that $\Gamma_G(z) \cap \Gamma_G(v_i) = T_i$ and $\Gamma_G(z) \cap \Gamma_G(v_j) = T_j$ for some $v_i, v_j \in \mathcal{N}_G^9(u)$, and $\Gamma_G(z) \cap \Gamma_G(v_k) = T_k$ for some $v_k \in \mathcal{N}_G^9(u)$, where $v_i, v_j, v_k$ are distinct. Without loss of generality, we assume $\Gamma_G(z) \cap \Gamma_G(v_i) = \Gamma_G(z) \cap \Gamma_G(v'_i) = T_i$ for $v_i, v'_i \in \mathcal{N}_G^9(u)$, and $\Gamma_G(z) \cap \Gamma_G(v_2) = T_2$ for some $v_2 \in \mathcal{N}_G^9(u)$, see Figure[13]. Observe that $\Gamma_G(v_2) = \{v_1, v'_i, v_2, z\}$ for some $v_1, v'_i \in \mathcal{N}_G^9(u)$, and $\Gamma_G(z) \cap \Gamma_G(v_2) = T_2$ for some $v_2 \in \mathcal{N}_G^9(u)$, see Figure[13]. Observe that $\Gamma_G(b) = \{v_1, v'_i, v_2, z\}$. Since $\Gamma_G(z) \cap \Gamma_G(v_1) = T_1$
as shown in Figure 11, a and one of b’s neighbors form $S_i(v_1)$ for some $i \in \{1, 2\}$; we assume it is $S_1(v_1)$. Note $\{z, v_1, v'_1\} \subset \Gamma_G(a)$. Thus $S_1(v_1) = \{a, v_2\}$ and $v_2 \in \Gamma_G(v_1)$. Similarly, we can show $S_1(v'_1) = \{a, v_2\}$ and $v_2 \in \Gamma_G(v'_1)$. Now, observe that $\Gamma_G(v_2) = \{v_1, v'_1, b, c\}$. Since $\Gamma_G(z) \cap \Gamma_G(v_2) = T_2$ as shown in Figure 11, b and one of neighbors of $v_2$ form $S_i(v_2)$ for some $i \in \{1, 2\}$; we assume $i = 1$. Because $\{v_1, v'_1\} \subset \Gamma_G(b)$, then $S_1(v_2) = \{b, c\}$. However, $b$ and $c$ are not in the same independent set in the definition of $N^3_G(u)$, see Figure 11. This is a contradiction and this case cannot happen. The claim follows.

![Figure 13: The picture for the claim.](image)

Therefore, $|N^2_G(u)| \leq 4 \times 2 \times 3 = 24$.

In Figure 12 for a fixed $u$, we have two choices for the edge $e$, one choice for $w$, two choices for $z$, and three choices for the edge $f$. Fix $a$. By considering the degrees of the endpoints of $f$, there is at most one $f$ and at most one $v \in N^3_G(u)$ such that $|\Gamma_G(f) \cap \Gamma_G(v)| = 4$ as shown in Figure 12. Therefore, we have $|N^3_G(u)| \leq 2 \times 2 \times 1 = 4$.

Last, we estimate $|N^2_G(u) \cup N^7_G(u)|$. If there is some $v \in N^7_G(u)$, then we observe that there are at most five $z$’s (see Figure 11). We get the number of $v \in N^5_G(u)$ is at most $5 \times 3 = 15$. In this case, we have

$$|N^2_G(u) \cup N^7_G(u)| \leq 4 + 15 < 24.$$  

If $N^2_G(u) = \emptyset$, then also we have

$$|N^2_G(u) \cup N^7_G(u)| \leq 24.$$  

Therefore

$$|N^1_G(u) \cup N^2_G(u) \cup N^3_G(u) \cup N^4_G(u) \cup N^5_G(u) \cup N^7_G(u)| \leq 36 + 36 + 24 = 96.$$  

Based on the graph $G$, we define an auxiliary graph $G^*$ on vertex set $V(G)$. The edge set is defined as follows: $uv \in E(G^*)$ precisely if either $u \in N^1_G(v) \cup N^2_G(v) \cup N^3_G(v) \cup N^4_G(v) \cup N^5_G(v) \cup N^7_G(v)$, or $v \in N^1_G(u)$. We have the following lemma.

**Lemma 8** The graph $G^*$ is 133-colorable.

**Proof:** Let $\sigma$ be an increasing order of $V(G^*)$ satisfying the following conditions.

1: For $u$ and $v$ such that $d_G(u) = 3$ and $d_G(v) = 4$, we have $\sigma(u) < \sigma(v)$.

2: For $u$ and $v$ such that $d_G(u) = d_G(v) = 4$, $\alpha(\Gamma_G(u)) \geq 3$, and $\alpha(\Gamma_G(v)) = 2$, we have $\sigma(u) < \sigma(v)$.

We will color $V(G^*)$ according to the order $\sigma$. For each $v$, we have the following estimate on the number of colors forbidden to use for $v$.  

\[12\]
For \( v \) such that \( d_G(v) = 3 \), the number of colors forbidden to use for \( v \) is at most 
\[ |N_G^{1}(v) \cup N_G^{2}(v) \cup N_G^{3}(v)| \leq 3 + 9 + 27 = 39. \]

2: For \( v \) such that \( d_G(v) = 4 \) and \( \alpha(\Gamma_G(v)) \geq 3 \), the number of colors forbidden to use for \( v \) is at most 
\[ |N_G^{1}(v) \cup N_G^{2}(v) \cup N_G^{3}(v)| \leq 3 + 9 + 27 = 39. \]

3: For \( v \) such that \( d_G(v) = 4 \) and \( \alpha(\Gamma_G(v)) = 2 \), the number of colors forbidden to use for \( v \) is at most 
\[ |N_G^{1}(v) \cup N_G^{2}(v) \cup N_G^{3}(v)| \leq 3 + 9 + 27 = 39 \]
by Lemma 7.

Therefore, the greedy algorithm shows \( G^* \) is 133-colorable. \( \square \)

Let \( X \) be a color class of \( G^* \). We define a new graph \( G(X) \) by the following process.

1. For each \( x \in X \), if \( |S(x)| = 2 \) or \( |S(x)| = 3 \), then we contract \( S(x) \) as a single vertex, delete the vertices in \( \Gamma_G(v) \setminus S(v) \), and keep label 1 on the new vertex; if \( |S(x)| = 4 \), i.e., \( S(x) = S_1(x) \cup S_2(x) \), then we contract \( S_1(x) \) and \( S_2(x) \) as single vertices and keep their labels. After that, we delete \( X \). Let \( H \) be the resulting graph.

2. Note that \( \Gamma_H(x) \cap \Gamma_H(y) = \emptyset \) and there is no edge from \( \Gamma_H(x) \) to \( \Gamma_H(y) \) for any \( x, y \in X \) as \( X \) is a color class.

3. We identify all vertices with label \( i \) as a single vertex \( w_i \) for \( i \in \{1, 2\} \). Let \( G(X) \) be the resulted graph.

We have the following lemma on the chromatic number of \( G(X) \).

**Lemma 9** The graph \( G(X) \) is 4-colorable for each color class.

We postpone the proof of this lemma until the end of this section and prove Lemma 11 first.

**Proof of Lemma 11** By Lemma 9 there is a proper 133-coloring of \( G^* \). We assume \( V(G^*) = V(G) = \bigcup_{i=1}^{133} X_i \), where \( X_i \) is the \( i \)-th color class.

For each \( i \in \{1, \ldots, 133\} \), Lemma 9 shows \( G(X_i) \) is 4-colorable; let \( c_i : V(G(X_i)) \rightarrow T_i \) be a proper 4-coloring of the graph \( G(X_i) \). Here \( T_1, T_2, \ldots, T_{133} \) are pairwise disjoint; each of them consists of 4 colors. For \( i \in \{1, \ldots, 133\} \), the 4-coloring \( c_i \) can be viewed as a 4-coloring of \( G \setminus X_i \) since each vertex with label \( i \) receives the color \( c_i(u) \) for \( j = 1, 2 \) and each removed vertex has at most three neighbors in \( G \setminus X_i \).

Now we reuse the notation \( c_i \) to denote this 4-coloring of \( G \setminus X_i \). For each \( v \in X_i \), we have \( |\{u \in \Gamma_G(v) : c_i(u) \neq \} \leq 2 \). We can assign two unused colors, denoted by the set \( Y(v) \), to \( v \).

We define \( f_i : V(G) \rightarrow P(T_i) \) (the power set of \( T_i \)) satisfying
\[
\begin{align*}
 f_i(v) = \begin{cases} 
 c_i(v) & \text{if } v \in V(G) \setminus X_i, \\
 Y(v) & \text{if } v \in X_i.
\end{cases}
\end{align*}
\]

Observe that each vertex in \( X_i \) receives two colors from \( f_i \) and every other vertex receives one color. Let \( \sigma : V(G) \rightarrow P(\bigcup_{i=1}^{133} T_i) \) be a mapping such that \( \sigma(v) = \bigcup_{i=1}^{133} f_i(v) \). It is easy to verify \( \sigma \) is a 133-fold coloring of \( G \) such that each color is drawn from a palette of 532 colors; namely we have
\[
\chi_f(G) \leq \frac{532}{134} = \frac{2}{67}.
\]

The proof of Lemma 11 is finished. \( \square \)

Before we prove Lemma 9 we need the following definitions.

A block of a graph is a maximal 2-connected induced subgraph. A Gallai tree is a connected graph in which all blocks are either complete graphs or odd cycles. A Gallai forest is a graph all of whose components are Gallai trees. A \( k \)-Gallai tree (forest) is a Gallai
tree (forest) such that the degree of all vertices are at most \( k - 1 \). A \( k \)-critical graph is a graph \( G \) whose chromatic number is \( k \) and deleting any vertex can decrease the chromatic number. Gallai showed the following Lemma.

**Lemma 10** [3] If \( G \) is a \( k \)-critical graph, then the subgraph of \( G \) induced on the vertices of degree \( k - 1 \) is a \( k \)-Gallai forest.

**Proof of Lemma 9** We use proof by contradiction. Suppose that \( G(X) \) is not 4-colorable. The only possible vertices in \( G(X) \) with degree greater than 4 are the vertices \( w_1 \) and \( w_2 \), which are obtained by contracting the vertices with label 1 and 2 in the intermediate graph \( H \). The simple greedy algorithm shows that \( G(X) \) is always 5-colorable. Let \( G'(X) \) be a 5-critical subgraph of \( G(X) \). Applying Lemma 10 to \( G'(X) \), the subgraph of \( G'(X) \) induced on the vertices of degree 4 is a 5-Gallai forest \( F \). The vertex set of \( F \) may contain \( w_1 \) or \( w_2 \). Delete \( w_1 \) and \( w_2 \) from \( F \) if \( F \) contains one of them. Let \( F' \) be the resulting Gallai forest. (Any induced subgraph of a Gallai forest is still a Gallai forest.) The Gallai forest \( F' \) is not empty. Let \( T \) be a connected component of \( F' \) and \( B \) be a leaf block of \( T \). The block \( B \) is either a clique or an odd cycle from the definition of a Gallai tree.

Let \( v \) be a vertex in \( B \). As \( v \) has at most two neighbors \( (w_1, w_2) \) outside \( F' \) in \( G(X) \), we have \( d_{F'}(v) \geq 2 \). If \( v \) is not in other blocks of \( F' \), then we have \( d_B(v) \geq 2 \). It follows that \( |B| \geq 3 \). Since \( B \) is a subgraph of \( G \) and \( G \) is \( K_4 \)-free, the block \( B \) is an odd cycle.

Let \( v_1, v_2 \) be an edge in \( B \) such that \( v_1 \) and \( v_2 \) are not in other blocks. The degree requirement implies \( v_1, v_2 \) are edges in \( G(X) \) for all \( i, j \in \{1, 2\} \). For \( i = 1, 2 \), there are vertices \( x_i, y_i \in X \) satisfying \( S(x_i) \cap \Gamma_G(v_i) \neq \emptyset \) and \( S(y_i) \cap \Gamma_G(v_i) \neq \emptyset \); moreover either \( |S(x_i)| = 4 \) or \( |S(y_i)| = 4 \) since one of its neighborhood has label 2. Without loss of generality, we assume \( |S(x_i)| = 4 \) for \( i \in \{1, 2\} \). If \( x_i \neq y_1 \), then \( y_1 \in N_G^5(x_i) \), i.e., \( y_1 \in \Gamma_G(x_i) \); this contradicts \( X \) being a color class. Thus we have \( x_i = y_1 \) and \( |S(x_i)| = 4 \) for \( i \in \{1, 2\} \). For \( \{i, j\} = \{1, 2\} \), if \( x_i \neq y_j \), then \( y_j \in N_G^5(x_i) \), i.e., \( y_j \in \Gamma_G(x_i) \); this is a contradiction of \( X \) being a color class. Thus we have \( x_1 = x_2 = y_1 = y_2 \).

Let \( x \) denote this common vertex above. Then \( d_G(x) = 4 \) and \( \alpha(\Gamma_G(x)) = 2 \).

Let \( v_0 \) be the only vertex in \( B \) shared by other blocks. Since \( B - v_0 \) is connected, the argument above shows there is a common \( x \) for all edges in \( B - v_0 \). If \( \Gamma_G(x) \cap \{w_1, w_2\} \neq \emptyset \), then there is some vertex \( x_0 \in X \) such that \( S(x_0) \cup \Gamma_G(v_0) \neq \emptyset \). By the similar argument, we also have \( x_0 = x \).

Therefore, \( x \) depends only on \( B \). In the sense that for any \( y \in X \) and any \( v \in B \), if \( S(y) \cap \Gamma_G(v) \neq \emptyset \), then \( y = x \).

The block \( B \) is an odd cycle as we mentioned above. Suppose \( |B| = 2r + 1 \). Let \( v_0, v_1, \ldots, v_{2r} \) be the vertices of \( B \) in cyclic order and \( v_0 \) be the only vertex which may be shared by other block.

Let \( x \in X \) be the vertex determined by \( B \). Recall \( d_G(x) = 4 \) and \( \alpha(\Gamma_G(x)) = 2 \). Each vertex in \( \Gamma(x) \) can have at most 2 edges to \( B \). We get

\[
4r \leq |E(B, \Gamma(x))| \leq 8. \tag{4}
\]

We have \( r \leq 2 \). The block \( B \) is either a \( C_5 \) or a \( K_3 \). We claim both \( v_0w_1 \) and \( v_0w_2 \) are non-edges of \( G(X) \).

If \( B = C_5 \), then inequality (4) implies that \( v_0 \) has no neighbor in \( \Gamma(x) \) and the claim holds. If \( B = K_3 \), then the claim also holds; otherwise \( B \cup \{ S_1(x), S_2(x) \} \) forms a \( K_5 \) in \( G/S_1(x)/S_2(x) \), which is a contradiction to Lemma 8.
Let $u_1$ and $u_2$ be the two neighbors of $v_0$ in other blocks of $F'$. If $u_1$ and $u_2$ are in the same block, then this block is an odd cycle; otherwise, $v_0u_1$ and $v_0u_2$ are in two different blocks.

The union of non-leaf blocks of $T$ is a Gallai-tree, denoted by $T'$. The argument above shows every leaf block of $T'$ must be an odd cycle. Let $C$ be such a leaf block of $T'$. Now $C$ is an odd cycle, and $C$ is connected to $|C| − 1$ leaf blocks of $T$. Let $B$ and $B'$ be two leaf blocks of $T$ such that $B ∩ C$ is adjacent to $B' ∩ C$. Without loss of generality, we may assume $B$ is the one we considered before. By the same argument, $B'$ is an odd cycle of size $2r' + 1$ with $r' ∈ \{1, 2\}$. Let $v'_0, v'_1, \ldots, v'_{2r'}$ be the vertices of $B'$ and $v'_0$ be the only vertex in $B' ∩ C$. For $i$ in $\{1, 2, \ldots, 2r'\}$ and $j$ in $\{1, 2\}$, $v'_i w_j$ are edges in $G'(X)$. Similarly, there exists a vertex $x' ∈ X$ with $d_G(x') = 4$ and $\alpha(Γ_G(x')) = 2$ such that $|E(v_i, S_1(x'))| ≥ 1$ and $|E(v_i, S_2(x'))| ≥ 1$. We must have $x = x'$; otherwise $x' ∈ N^2_G(x)$, i.e., $x' ∈ Γ_G(x)$, and this contradicts the fact that $X$ is a color class in $D$. Now we have $|E(Γ(x), B)| ≥ 4r$ and $E(Γ(x), B')| ≥ 4r'$. By counting the degrees of vertices in $Γ(x)$ in $G$, we have

$$4r + 4r' + 4 + 4 ≤ 16.$$ 

We get $r = r' = 1$. Both $B$ and $B'$ are $K_4$-free. In this case, $G/S_1(x)/S_2(x)$ contains the graph $G_0$, see figure 15. This contradicts Lemma 6.

We can find the desired contradiction, so the lemma follows.

4 Proof of Lemma 6

In this section, we will prove Lemma 6. We first review a Lemma from [4].

Lemma 11 Let $G$ be a graph. Suppose that $G_1$ and $G_2$ are two subgraphs such that $G_1 ∪ G_2 = G$ and $V(G_1) ∩ V(G_2) = \{u, v\}$.

1. If $uv$ is an edge of $G$, then we have

$$χ_f(G) = \max\{χ_f(G_1), χ_f(G_2)\}.$$ 

2. If $uv$ is not an edge of $G$, then we have

$$χ_f(G) ≤ \max\{χ_f(G_1), χ_f(G_2 + uv), χ_f(G_2/uv)\},$$

where $G_2 + uv$ is the graph obtained from $G_2$ by adding edge $uv$ and $G_2/uv$ is the graph obtained from $G_2$ by contracting $\{u, v\}$.

Proof of Lemma 6. Recall that $G$ is a connected $K_4$-free graph with minimum number of vertices such that $G ≠ C_5$ and $χ_f(G) > 4 − \frac{2}{67}$. Note that $G$ is 2-connected. We will prove it by contradiction.

Suppose Lemma 6 fails for some vertex $x$ in $G$. Observe $Γ_G(x)$ is one of the graphs in Figure 15. Here we assume $Γ_G(x) = \{a, b, c, d\}$. Through the proof of the lemma, let $S_1$ and $S_2$ be two vertex-disjoint independent sets in $Γ_G(x)$, and $H$ is a triangle in $V(G) \\setminus (\{x\} ∪ Γ_G(x))$, then say $S_1, S_2, H$ is a bad triple if $S_1, S_2, H$ contains a $K_5$ in $G/S_1/S_2$.

If $Γ_G(x) = P_4$, then $\{a, d\}$ and $\{b, c\}$ is the only pair of disjoint non-edges. There is a triangle $H$ with $V(H) = \{y, z, w\}$ such that $\{\{a, d\}, \{b, c\}, H\}$ is a bad triple. Note that $|E(\{a, b, c, d\}, \{y, z, w\})| = 5$ or $6$. By an exhaustive search, the induced subgraph of $G$ on $\{x, a, b, c, d, y, z, w\}$ is one of the following six graphs (see Figure 15).

If $Γ_G(x) = 2 e$, then $\{\{a, c\}, \{b, d\}\}$ and $\{\{a, d\}, \{b, c\}\}$ are two pairs of disjoint non-edges. By considering the degrees of vertices in $Γ_G(x)$, there is only one triangle $H$ with
Figure 14: Three possible cases of $\Gamma_G(x)$.

$\Gamma_G(x) = P_4$  

$\Gamma_G(x) = 2e$  

$\Gamma_G(x) = C_4$

Figure 15: If $\Gamma_G(x) = P_4$, then there are six possible induced subgraphs.

Figure 16: If $\Gamma_G(x) = 2e$, then there are three possible induced subgraphs.
Lemma 11, we have $G/bcz$ of $V$ (an exhaustive search, the induced subgraph of $G$ on $\{x, a, b, c, d, y, z, w\}$ is one of the following three graphs (see Figure 10).

It suffices to show that $G$ cannot contain $H_i$ for $1 \leq i \leq 9$. Since all vertices in $H_i$ (and $H_2$) have degree 4, $H_1$ (and $H_2$) is the entire graph $G$. Observe that $H_1$ is isomorphic to $C_8^2$ and $H_2$ is 11:3-colorable (see Figure 17). Contradiction!

Now we consider the case $H_3$. Note $H_3 + bz$ is the graph $H_2$. We have $\chi_f(H_3) \leq \chi_f(H_2) \leq 11/3$. The graph $H_3$ must be a proper induced subgraph of $G$, and the pair $\{b, z\}$ is a vertex cut of $G$. Let $G'$ be the induced subgraph of $G$ by deleting all vertices in $H_3$ but $b, z$. We apply Lemma 11 to $G + bz$ with $G_1 = H_3 + bz = H_2$ and $G_2 = G' + bz$. We have

$$\chi_f(G + bz) \leq \max\{\chi_f(H_2), \chi_f(G' + bz)\}.$$ 

Note $\chi_f(H_2) \leq 11/3$ and $11/3 < \chi_f(G) \leq \chi_f(G' + bz)$. We have $\chi_f(G) \leq \chi_f(G' + bz)$. Both $b$ and $z$ have at most 2 neighbors in $G' + bz$. Thus $G' + bz$ is $K_4$-free; $G' + bz \neq C_8^2$ and has fewer vertices than $G$. This contradicts to the minimality of $G$.

Note $H_3 + cy = H_2$. The case $H_5$ is similar to the case $H_3$.

Note that $H_4, H_6$, and $H_8$ are isomorphic to each other. It suffices to show $G$ does not contain $H_4$. Suppose that $H_4$ is a proper induced subgraph of $G$. Let $G_1$ be the induced subgraph of $G$ by deleting all vertices in $H_4$. Note $C_8^2$ is not a proper subgraph of any graph in $G_4$. We have $G_1 \neq C_8^2$. Note that $c$ and $z$ have degree 3 while other vertices in $H_4$ have degree 4. Since $G$ is 2-connected, $c$ has a unique neighbor, denoted by $u$, in $V(G_1)$. Similarly, $z$ has a unique neighbor, denoted by $v$, in $V(G_1)$. Observe that the pair $\{u, v\}$ forms a vertex cut of $G$. Let $G_2$ be the induced graph of $G$ on $V(H_4) \cup \{u, v\}$. Applying Lemma 11 to $G$ with $G_1$ and $G_2$, we have

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}.$$ 

Figure 18 shows $\chi_f(G_2 + uv)$ and $\chi_f(G_2/uv)$ are at most $11/3$.

Since $\chi_f(G) > 11/3$, we have $\chi_f(G) \leq \chi_f(G_1)$. Now $G_1$ is $K_4$-free and has maximum degree at most 4; $G_1$ has fewer vertices than $G$. This contradicts the minimality of $G$.

If $G = H_4$, then $\chi_f(H_4) \leq 11/3$, since $H_4$ is a subgraph of $G_2 + uv$ in Figure 18.

Now we consider the last case $H_6$. First, we contract $b, c, z$ into a fat vertex denoted by $bcz$. We write $G/bcz$ for the graph after this contraction. Observe that $\{bcz, d\}$ is a vertex-cut of $G/bcz$. Let $G_4$ and $G_4'$ be two connected subgraphs of $G/bcz$ such that $G_4 \cup G_4' = G/bcz$, $G_4 \cap G_4' = \{bcz, d\}$, and $\{u, v\} \subset G_4'$. Note that $G_4$ is 11:3 colorable, see Figure 19. Now by Lemma 11 we have

$$\chi_f(G/bcz) \leq \max\{\chi_f(G_4), \chi_f(G_4')\}.$$
As \{b, c, z\} is an independent set, each a:b-coloring of \(G/bcz\) gives an a:b-coloring of \(G\), that is \(\chi_f(G/bcz) \geq \chi_f(G) > 11/3\). The graph \(G_4\) is 11:3-colorable; see Figure 19. Thus we have \(\chi_f(G_4') \geq \chi_f(G/bcz) \geq \chi_f(G)\). It is easy to check that \(G_4'\) has maximum degree 4, \(K_4\)-free, and it is not \(C_5^2\). Hence \(G_4'\) must contain a \(K_4\). Otherwise, it contradicts the minimality of \(G\).

Second, we contract \{b, d, z\} into a fat vertex \(bdz\) and denote the graph by \(G/bcz\). Let \(G_5\) and \(G_5'\) be two connected subgraphs of \(G/bdz\) such that \(G_5 \cup G_5' = G/bdz\), \(G_5 \cap G_5' = \{bdz, c\}\), and \(\{u, v\} \subset G_5\). Note that \(G_5\) is 11:3-colorable; see Figure 19. By a similar argument, \(G_5'\) must contain a \(K_4\).

![Diagram](image1)

**Figure 18:** Case \(H_4\): both graph \(G_2 + uv\) and \(G_2/u\) are 11:3-colorable.

The remaining case is that both \(G_4'\) and \(G_5'\) have a \(K_4\) when we contract \(b\) and \(z\). Since the original graph \(G\) is \(K_4\)-free, the \(K_4\) in \(G_4'\) (and in \(G_5'\)) must contain the fat vertex \(bdz\) (or \(bdz\)), respectively. Note that each of the four vertices \(b, c, d, z\) has at most one edge leaving \(H_0\). There must be a triangle \(uvp\) in \(G\) and these four outward edges are connected to some element of \{u, v, p\}. The graph \(G/bz\) must contain the subgraph \(G_6\) as drawn in Figure 19.

Note that \{u, v\} is a vertex-cut in \(G/bz\). Let \(G_6\) and \(G_6'\) be two connected subgraphs of \(G/bz\), which satisfy \(G_6 \cup G_6' = G\), \(G_6 \cap G_6' = \{u, v\}\), and \(bz \in G_6\). By Lemma 11, we have

\[
\chi_f(G/bz) \leq \max\{\chi_f(G_6), \chi_f(G_6')\}.
\]

Note that \(G_6\) is 11:3-colorable; see Figure 19. We also have \(\chi_f(G/bz) \geq \chi_f(G) > 11/3\). We obtain \(\chi_f(G_6') \geq \chi_f(G/bz) \geq \chi_f(G)\). Observe that \(G_6'\) is a subgraph of \(G\). We arrive at a contradiction of the minimality of \(G\).

If \(\Gamma_G(x) = C_4\), then the only possible choice for the two independent sets are \{a, c\} and \{b, d\}. If there is some triangle \(H\) such that \(\{(a, c), \{b, d\}, H\}\) is a bad triple, then we have

\[
|E(\Gamma_G(x), H)| \geq 5.
\]

However, \(|E(\Gamma_G(x), H)| \leq 4\). This is a contradiction. Thus the lemma follows in this case.
We can select two vertex disjoint non-edges $S_1$ and $S_2$ such that the graph $G/S_1/S_2$ contains no $K_5^-$. For these particular $S_1$ and $S_2$, if $G/S_1/S_2$ contains no $G_0$, then Lemma 6 holds.

Without loss of generality, we assume that $G/S_1/S_2$ does contain $G_0$. Let $s_i = S_i$ for $i = 1, 2$. Observe that both $s_1$ and $s_2$ have four neighbors $u, v, p, q$ other than $x$ in $G_0$. It follows that

$$|E(S_1 \cup S_2, \{u, v, p, q\})| \geq 8.$$

On the one hand, we have

$$|E(G | S_1 \cup S_2)| = \frac{1}{2} \left( \sum_{v \in S_1 \cup S_2} d(v) - |E(S_1 \cup S_2, \{u, v, p, q\})| - 4 \right) \leq \frac{1}{2}(16 - 8 - 4) = 2.$$

On the other hand, $\alpha(\Gamma(x)) = 2$ implies $G | S_1 \cup S_2$ contains at least two edges. Thus, we have $\Gamma_G(x) = 2 e$. Label the vertices in $\Gamma_G(x)$ by $a, b, c, d$ as in Figure 14. We assume $ab$ and $cd$ are edges while $ac, bd, ad, bc$ are non-edges. Observe that each vertex in $\{u, v, p, q\}$ has exactly two neighbors in $\{a, b, c, d\}$.

If one vertex, say $u$, has two neighbors forming a non-edge, say $ac$, then we can choose $S'_1 = \{a, c\}$ and $S'_2 = \{b, d\}$. It is easy to check that $G/S'_1/S'_2$ contains neither $G_0$ nor $K_5^-$. We are done in this case.

In the remaining case, we can assume that for each vertex $y$ in $\{u, v, p, q\}$, the neighbors of $y$ in $\{a, b, c, d\}$ always form an edge. Up to relabeling vertices, there is only one arrangement for edges between $\{u, v, p, q\}$ and $\{a, b, c, d\}$; see the graph $H_{10}$ defined in Figure 20. The graph $H_{10}$ is 11:3-colorable as shown in Figure 20. Since $\chi_f(G) > 11/3$, $H_{10}$ is a proper subgraph of $G$. Note in $H_{10}$, every vertices except $w$ and $r$ has degree 4; both $w$ and $r$ have degree 3. Thus, $\{w, r\}$ is a vertex cut of $G$. Let $G_1 = H_{10}$ and $G_2$ be the subgraph of $G$ by deleting vertices in $\{x, a, b, c, d, p, q, u, v\}$. Applying Lemma 11 with $G_1$ and $G_2$ defined above, we have

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2)\}.$$ 

Since $\chi_f(G) > 11/3$ and $\chi_f(G_1) \leq 11/3$ (see Figure 20), we must have $\chi_f(G_2) \geq \chi_f(G)$. Note that $G_2$ has fewer number of vertices than $G$. This contradicts the minimality of $G$. Therefore, the lemma follows. □
References

[1] O. Borodin and A. Kostochka, On an upper bound on a graph’s chromatic number, depending on the graph’s degree and density, *J. Comb. Theory Ser. B*, 23 (1977), 247-250.

[2] D. Christofides, K. Edwards, and A.D. King, A note on hitting maximum and maximal cliques with a stable set, *Submitted. Arxiv preprint 1109.3092*, (2011).

[3] T Gallai, Kritische graphen I, *Magyar Tud. Akad. Mat. Kutat ó Int. Közl*, 8 (1963), 165-192.

[4] H. Hatami and X. Zhu, The fractional chromatic number of graphs of maximum degree at most three, *SIAM. J. Discrete Math.*, 24 (2009), 1762-1775.

[5] C. C. Heckman and R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, *Discrete Math.*, 233 (2001), 233-237.

[6] Andrew D. King, Hitting all maximum cliques with a stable set using lopsided independent transversals, *J. Graph Theory*, 67-4 (2011), 300-305.

[7] L. Lu and X. Peng, The fractional chromatic number of triangle-free graphs with $\Delta \leq 3$, *submitted*.

[8] M. Molloy and B. Reed, *Graph colouring and the probabilistic method*, volume 23 of Algorithms and Combinatorics, Springer-Verlag, Berlin, 2002.

[9] B. Reed, $\omega$, $\Delta$, and $\chi$, *J. Graph Theory*, 27-4 (1998), 177-227.

[10] B. Reed, A strengthening of Brooks’ theorem, *J. Comb. Theory Ser. B*, 76 (1999), 136-149.

[11] E. R. Scheinerman and D. H. Ullman, *Fractional graph theory*, A rational approach to the theory of graphs, Wiley-Intersci. Ser. Discrete Math. Optim, John Wiley & Sons, Inc, New York, 1997.
A Fractional Analogue of Brooks’ Theorem

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Abstract

Let $\Delta(G)$ be the maximum degree of a graph $G$. Brooks’ theorem states that the only connected graphs with chromatic number $\chi(G) = \Delta(G) + 1$ are complete graphs and odd cycles. We prove a fractional analogue of Brooks’ theorem in this paper. Namely, we classify all connected graphs $G$ such that the fractional chromatic number $\chi_f(G)$ is at least $\Delta(G)$. These graphs are complete graphs, odd cycles, $C^2_8$, $C_5 \boxtimes K_2$, and graphs whose clique number $\omega(G)$ equals the maximum degree $\Delta(G)$. Among the two sporadic graphs, the graph $C^2_8$ is the square graph of cycle $C_8$ while the other graph $C_5 \boxtimes K_2$ is the strong product of $C_5$ and $K_2$. In fact, we prove a stronger result; if a connected graph $G$ with $\Delta(G) \geq 4$ is not one of the graphs listed above, then we have $\chi_f(G) \leq \Delta(G) - \frac{2}{67}$.

1 Introduction

The chromatic number of graphs with bounded degrees has been studied for many years. Brooks’ theorem perhaps is one of the most fundamental results; it is included by many textbooks on graph theory. Given a simple connected graph $G$, let $\Delta(G)$ be the maximum degree, $\omega(G)$ be the clique number, and $\chi(G)$ be the chromatic number. Brooks’ theorem states that $\chi(G) \leq \Delta(G)$ unless $G$ is a complete graph or an odd cycle. Reed [9] proved that $\chi(G) \leq \Delta(G) - 1$ if $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq \Delta_0$ for some large constant $\Delta_0$. This excellent result was proved by probabilistic methods, and $\Delta_0$ is at least hundreds. Before this result, Borodin and Kostochka [1] made the following conjecture.

Conjecture [1]: Suppose that $G$ is a connected graph. If $\omega(G) \leq \Delta(G) - 1$ and $\Delta(G) \geq 9$, then we have $\chi(G) \leq \Delta(G) - 1$.

If the conjecture is true, then it is best possible since there is a $K_8$-free graph $G = C_5 \boxtimes K_3$ (actually $K_7$-free, see Figure 1) with $\Delta(G) = 8$ and $\chi(G) = 8$.

Here we use the following notation of the strong product. Given two graphs $G$ and $H$, the strong product $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$, and $(a, x)$ is connected to $(b, y)$ if one of the following holds

- $a = b$ and $xy \in E(H)$,
- $ab \in E(G)$ and $x = y$.

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Reed’s result \cite{9} settled Borodin and Kostochka’s conjecture for sufficiently large $\Delta(G)$, but the cases with small $\Delta(G)$ are hard to cover using the probabilistic method.

In this paper we consider a fractional analogue of this problem. The fractional chromatic number $\chi_f(G)$ can be defined as follows. A $b$-fold coloring of $G$ assigns a set of $b$ colors to each vertex such that any two adjacent vertices receive disjoint sets of colors. We say a graph $G$ is $a:b$-colorable if there is a $b$-fold coloring of $G$ in which each color is drawn from a palette of $a$ colors. We refer to such a coloring as an $a:b$-coloring. The $b$-fold coloring number, denoted by $\chi_b(G)$, is the smallest integer $a$ such that $G$ has an $a:b$-coloring. Note that $\chi_1(G) = \chi(G)$. It was shown that $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$. The fractional chromatic number $\chi_f(G)$ is $\lim_{b \to \infty} \chi_b(G)$.

By the definition, we have $\chi_f(G) \leq \chi(G)$. The fractional chromatic number can be viewed as a relaxation of the chromatic number. Many problems involving the chromatic number can be asked again using the fractional chromatic number. The fractional analogue often has a simpler solution than the original problem. For example, the famous $\omega - \Delta - \chi$ conjecture of Reed \cite{8} states that for any simple graph $G$, we have

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$  

The fractional analogue of $\omega - \Delta - \chi$ conjecture was proved by Molloy and Reed \cite{7}; they actually proved a stronger result with ceiling removed, i.e.,

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}. \quad (1)$$

In this paper, we classify all connected graphs $G$ with $\chi_f(G) \geq \Delta(G)$.

**Theorem 1** A connected graph $G$ satisfies $\chi_f(G) \geq \Delta(G)$ if and only if $G$ is one of the following

1. a complete graph,
2. an odd cycle,
3. a graph with $\omega(G) = \Delta(G)$,
4. \( C_8^2 \),
5. \( C_5 \boxtimes K_2 \).

For the complete graph \( K_n \), we have \( \chi_f(K_n) = n \) and \( \Delta(K_n) = n - 1 \). For the odd cycle \( C_{2k+1} \), we have \( \chi_f(C_{2k+1}) = 2 + \frac{1}{k} \) and \( \Delta(C_{2k+1}) = 2 \). If \( G \) is neither a complete graph nor an odd cycle but contains a clique of size \( \Delta(G) \), then we have

\[
\Delta(G) \leq \omega(G) \leq \chi_f(G) \leq \chi(G) \leq \Delta(G).
\]

The last inequality is from Brooks’ theorem. The sequence of inequalities above implies \( \chi_f(G) = \Delta(G) \).

If \( G \) is a vertex-transitive graph, then we have \[10\]

\[
\chi_f(G) = \frac{|V(G)|}{\alpha(G)},
\]

where \( \alpha(G) \) is the independence number of \( G \). Note that both graphs \( C_8^2 \) and \( C_5 \boxtimes K_2 \) are vertex-transitive and have the independence number 2. Thus we have

\[
\chi_f(C_8^2) = 4 = \Delta(C_8^2) \quad \text{and} \quad \chi_f(C_5 \boxtimes K_2) = 5 = \Delta(C_5 \boxtimes K_2).
\]

Figure 2: The graph \( C_8^2 \) and \( C_5 \boxtimes K_2 \).

Actually, Theorem 1 is a corollary of the following stronger result.

**Theorem 2** Assume that a connected graph \( G \) is neither \( C_8^2 \) nor \( C_5 \boxtimes K_2 \). If \( \Delta(G) \geq 4 \) and \( \omega(G) \leq \Delta(G) - 1 \), then we have

\[
\chi_f(G) \leq \Delta(G) - \frac{2}{67}.
\]

**Remark:** In the case \( \Delta(G) = 3 \), Heckman and Thomas [4] conjectured that \( \chi_f(G) \leq 14/5 \) if \( G \) is triangle-free. Hatami and Zhu [9] proved \( \chi_f(G) \leq 3 - \frac{1}{64} \) for any triangle-free graph \( G \) with \( \Delta(G) \leq 3 \). The authors showed an improved result \( \chi_f(G) \leq 3 - \frac{1}{64} \) in the previous paper [10]. Thus we need only consider the cases \( \Delta(G) \geq 4 \). For any connected graph \( G \) with sufficiently large \( \Delta(G) \) and \( \omega(G) \leq \Delta(G) - 1 \), Reed’s result [9] \( \chi(G) \leq \Delta(G) - 1 \) implies \( \chi_f(G) \leq \Delta(G) - 1 \). The method introduced in [9] and strengthened in [6], has a strong influence on this paper. The readers are encouraged to read these two papers [3, 6].

Let \( f(k) = \inf_G \{ \Delta(G) - \chi_f(G) \} \), where the infimum is taken over all connected graphs \( G \) with \( \Delta(G) = k \) and not one of the graphs listed in Theorem 1. Since \( \chi_f(G) \geq \omega(G) \), by taking a graph with \( \omega(G) = \Delta(G) - 1 \), we have \( f(k) \leq 1 \). Theorem 2 says \( f(k) \geq \frac{2}{67} \) for any \( k \geq 4 \). Reed’s result [9] implies \( f(k) = 1 \) for sufficiently large \( k \). Heckman and Thomas [4]
conjectured \( f(3) = 1/5 \). It is an interesting problem to determine the value of \( f(k) \) for small \( k \). Here we conjecture \( f(4) = f(5) = 1/4 \). If Borodin and Kostochka’s conjecture is true, then \( f(k) = 1 \) for \( k \geq 9 \).

Theorem 2 is proved by induction on \( k \). Because the proof is quite long, we split the proof into the following two lemmas.

**Lemma 1** We have \( f(4) \geq \frac{2}{67} \).

**Lemma 2** For each \( k \geq 6 \), we have \( f(k) \geq \min \{ f(k-1), \frac{1}{7} \} \). We also have \( f(5) \geq \min \{ f(4), \frac{1}{3} \} \).

It is easy to see the combination of Lemma 1 and Lemma 2 implies Theorem 2. The idea of reduction comes from the first author, who pointed out \( f(k) \geq \min \{ f(k-1), \frac{1}{2} \} \) for \( k \geq 7 \) based on his recent results [5]. The second and third authors originally proved \( f(k) \geq C/k \) (for some \( C > 0 \)) using different method in the first version; they also prove the reductions at \( k = 5, 6 \), which are much harder than the case \( k \geq 7 \). We do not know whether a similar reduction exists for \( k = 4 \).

The rest of this paper is organized as follows. In section 2, we will introduce some notation and prove Lemma 2. In section 3 and section 4, we will prove \( f(4) \geq \frac{2}{67} \).

## 2 Proof of Lemma 2

In this paper, we use the following notation. Let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). The neighborhood of a vertex \( v \) in \( G \), denoted by \( \Gamma_G(v) \), is the set \( \{ u : uv \in E(G) \} \). The degree \( d_G(v) \) of \( v \) is the value of \( |\Gamma_G(v)| \). The independent set (or stable set) is a set \( S \) such that no edge with both ends in \( S \). The independence number \( \alpha(G) \) is the largest size of \( S \) among all the independent sets \( S \) in \( G \). When \( T \subset V(G) \), we use \( \alpha(G(T)) \) to denote the independence number of the induced subgraph of \( G \) on \( T \). Let \( \Delta(G) \) be the maximum degree of \( G \). For any two vertex-sets \( S \) and \( T \), we define \( E_G(S, T) \) as \( \{ uv \in E(G) : u \in S \text{ and } v \in T \} \). Whenever \( G \) is clear under context, we will drop the subscript \( G \) for simplicity.

If \( S \) is a subset of vertices in \( G \), then contracting \( S \) means replacing vertices in \( S \) by a single fat vertex, denoted by \( S \), whose incident edges are all edges that were incident to at least one vertex in \( S \), except edges with both ends in \( S \). The new graph obtained by contracting \( S \) is denoted by \( G/S \). This operation is also known as identifying vertices of \( S \) in the literature. For completeness, we allow \( S \) to be a single vertex or even the empty set. If \( S \) only consists of a single vertex, then \( G/S = G \); if \( S = \emptyset \), then \( G/S \) is the union of \( G \) and an isolated vertex. When \( S \) consists of 2 or 3 vertices, for convenience, we write \( G/uv \) for \( G/\{u, v\} \) and \( G/uvw \) for \( G/\{u, v, w\} \); the fat vertex will be denoted by \( uv \) and \(uvw \), respectively. Given two disjoint subsets \( S_1 \) and \( S_2 \), we can contract \( S_1 \) and \( S_2 \) sequentially. The order of contractions does not matter; let \( G/S_1/S_2 \) be the resulted graph. We use \( G - S \) to denote the subgraph of \( G \) induced by \( V(G) - S \).

In order to prove Lemma 2, we need use the following theorems due to King [5].

**Theorem 3 (King [5])** If a graph \( G \) satisfies \( \omega(G) > \frac{2}{7}(\Delta(G) + 1) \), then \( G \) contains a stable set \( S \) meeting every maximum clique.

**Theorem 4 (King [5])** For a positive integer \( k \), let \( G \) be a graph with vertices partitioned into cliques \( V_1, \ldots, V_r \). If for every \( i \) and every \( v \in V_i \), \( v \) has at most \( \min \{ k, |V_i| - k \} \) neighbors outside \( V_i \), then \( G \) contains a stable set of size \( r \).
Theorem 3 Suppose that $G$ is a connected graph with $\Delta(G) \leq 6$ and $\omega(G) \leq 5$. Then there exists an independent set meeting all induced copies of $K_5$ and $C_5 \boxtimes K_2$.

Proof: We first show that there exists an independent set meeting all copies of $K_5$. If $G$ contains no $K_5$, then this is trivial. Otherwise, we can apply Theorem 2 to get the desired independent set since $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ is satisfied.

Now we prove the Lemma by contradiction. Suppose the Lemma is false. Let $G$ be a minimum counterexample (with the smallest number of vertices). For any independent set $I$, let $C(I)$ be the number of induced copies of $C_5 \boxtimes K_2$ in $G - I$. Among all independent sets which meet all copies of $K_5$, there exists one such independent set $I$ such that $C(I)$ is minimized.

Since $C(I) > 0$, there is an induced copy of $C_5 \boxtimes K_2$ in $G - I$; we use $H$ to denote it. In $C_5 \boxtimes K_2$, there is a unique perfect matching such that identifying the two ends of each edge in this matching results a $C_5$. An edge in this unique matching is called a canonical edge.

We define a new graph $G'$ as follows: First we contract all canonical edges in $H$ to get a $C_5$, where its vertices are called fat vertices. Second we add five edges turning the $C_5$ into a $K_5$. Observe that each vertex in this $G'$ can have at most two neighbors in $G - H$ and $\Delta(G') \leq 6$. We will consider the following four cases.

Case 1: There is a $K_6$ in the new graph $G'$. Since the original graph $G$ is $K_6$-free, the $K_6$ is formed by the following two possible ways.

Subcase 1a: This $K_6$ contains 5 fat vertices. By the symmetry of $H$, there is an induced $C_5$ in $H$ such that the vertices in $C_5$ contain a common neighbor vertex $v$ in $G \setminus V(H)$, see Figure 3. Since $H$ is $K_5$-free, we can find $x, y$ in this $C_5$ such that $x, y$ is a non-edge. Let $I' := (I \setminus \{v\}) \cup \{x, y\}$; $I'$ is also an independent set. Observe that $v$ is not in any $K_5 \subset G - I'$. Thus the set $I'$ is also an independent set and meets every $K_5$ in $G$. Since $C_5 \boxtimes K_2$ is a 5-regular graph, any copy of $C_5 \boxtimes K_2$ containing $v$ must contain at least one of $x$ and $y$. Thus, $C(I') < C(I)$. Contradiction!

Subcase 1b: This $K_6$ contains 4 fat vertices. Let $u, v$ be the other two vertices. By the symmetry of $H$, there is a unique way to connect $u$ and $v$ to $H$ as shown by Figure 4. Since $uv$ is an edge, one of $u$ and $v$ is not in $I$. We assume $u \notin I$. Let $\{x, y\} \subset \Gamma_G(v) \cap V(H)$ as shown in Figure 4 and $I' = I \setminus \{v\} \cup \{x, y\}$. Observe that $I'$ is an independent set and $v \notin K_5 \subset G - I'$. Thus $I'$ is an independent set meeting each $K_5$ in $G$. Since each $C_5 \boxtimes K_2$ containing $v$ must contain one of $x$ and $y$. Thus, $C(I') < C(I)$. Contradiction!

Case 2: There is a $K_5$ intersecting $H$ with 4 vertices. Let $v$ be the vertex of this $K_5$ but not in $I$, see Figure 5. We have two subcases.

Subcase 2a: The vertex $v$ has another neighbor $y$ in $H$ but not in this $K_5$. Since $H$ is $K_5$-free, we can select a vertex $x$ in this $K_5$ such that $xy$ is not an edge of $G$. Let $I' := I \setminus \{v\} \cup \{x, y\}$. Note that $v \notin K_5 \subset G - I'$ and $I'$ is an independent set. Thus $I'$ is an independent set meeting each $K_5$ in $G$. Since any $C_5 \boxtimes K_2$ containing $v$ must contain one of $x$ and $y$, we have $C(I') < C(I)$. Contradiction!
Subcase 2b: All neighbors of $v$ in $H$ are in this $K_5$. Let $x$ be any vertex in this $K_5$ other than $v$, and $I' := I \setminus \{v\} \cup \{x\}$. In this case, there is only one $K_5$ containing $v$. Thus, $I'$ is also an independent set meeting every copy of $K_5$ in $G$. Observe that $\Gamma_G(v) \setminus \{x\}$ is disconnected. If $v \in H' = C_5 \boxtimes K_2$, then $\Gamma_G(v) \cap H'$ is connected. Thus $v \notin C_5 \boxtimes K_2 \subset G - I'$ and $C(I') < C(I)$. Contradiction!

![Figure 5: Case 2.](image1)

![Figure 6: Case 3.](image2)

Case 3: There is an induced subgraph $H'$ isomorphic to $C_5 \boxtimes K_2$ such that $H'$ and $H$ are intersecting, see Figure 6. Since $V(H) \cap V(H') \neq \emptyset$ and $H \neq H'$, we can find a canonical edge $vw$ of $H$ and a canonical edge $w'v'$ of $H'$ such that $v \notin V(H')$ and $v' \notin V(H)$. If $vw'$ is a non-edge, then let $I' := I \setminus \{v\} \cup \{u\}$. It is easy to check $I'$ is still an independent set. We also observe that any possible $K_5$ containing $v'$ must also contain $w$. Thus, $I'$ meets every copy of $K_5$ in $G$. We have $v' \notin C_5 \boxtimes K_2 \subset G - I'$ since $vw'$ is not an edge. We get $C(I') < C(I)$. Contradiction! If $vw'$ is an edge, then locally there are two $K_5$ intersecting at $u$, $v$, and $v'$; say the other four vertices are $x_1, x_2, y_1, y_2$, where two cliques are $\{x_1, x_2, u, v, v'\}$ and $\{y_1, y_2, u, v, v'\}$, see Figure 6. Let $I' = I \cup \{x_1, y_1\} \setminus \{v'\}$. Note that $I'$ is an independent set and $v' \notin K_5 \subset G - I'$. Thus $I'$ is an independent set meeting each $K_5$ in $G$. Observe that any copies of $C_5 \boxtimes K_2$ containing $v'$ must contain one of $x_1$ and $y_1$; we have $C(I') < C(I)$. Contradiction!.

Case 4: This is the remaining case, $G'$ is $K_6$-free. We have $\omega(G') \leq 5$ and $|V(G')| < |V(G)|$. By the minimality of $G$, there is an independent set $I'$ of $G'$ meeting every copy of $K_5$ and $C_5 \boxtimes K_2$. In $I'$, there is a unique vertex $x$ of the $K_5$ obtained from contracting canonical edges of $H$. Let $uw$ be the canonical edge corresponding to $x$. Let $I'' = I' \setminus \{x\} \cup \{u\}$, we get an independent set $I''$ of $G$. Note that any $v \in H \setminus \{u\}$ is not in any $K_5$ of $G - I''$ by Case 2 as well as not in any $C_5 \boxtimes K_2$ of $G - I''$ by Case 3. Thus $I''$ hits each $K_5$ in $G$ and $C(I'') = 0$. Contradiction!

Lemma 4 Let $G$ be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there is an independent set $I$ hitting all copies of $K_4$ in $G$.

Proof: We will prove it by contradiction. If the lemma is false, then let $G$ be a minimum counterexample. If $G$ is $K_4$-free, then there is nothing to prove. Otherwise, we consider the clique graph $C(G)$, whose edge set is the set of all edges appearing in some copy of $K_4$. Because of $\Delta(G) = 5$, here are all possible connected component of $C(G)$.

1. $C_t \boxtimes K_2$ for $t \geq 4$. If this type occurs, then every vertex in $C_t \boxtimes K_2$ has degree 5; thus, this is the entire graph $G$. If $t$ is even, then we can find an independent set $I$ meeting every $K_4$. If $t$ is odd, then it is impossible to find such an independent set. However, this graph is excluded from the assumption of the Lemma.

2. $P_t \boxtimes K_2$ for $t \geq 3$. In this case, all internal vertices have degree 5 while the four end vertices have degree 4. Consider a new graph $G'$ which is obtained by deleting all
internal vertices and adding four edges to make the four end vertices as a $K_4$. It is easy to check $\Delta(G') \leq 5$ and $\omega(G') \leq 4$. Since $|G'| < |G|$, there is an independent set $I$ of $G'$ meeting every copy of $K_4$ in $G'$. Note that there is exactly one end vertex in $I$. Observe that any one end vertex can be extended into a maximal independent set meeting every copy of $K_4$ in $P_t \boxtimes K_2$. Thus, we can extend $I$ to an independent set $I'$ of $G$ such that $I'$ meets every copy of $K_4$ in $G$. Hence, this type of component does not occur in $\mathcal{C}(G)$.

3. There are four other types listed in Figure 7

![Figure 7: All types of components in the clique graph $C(G)$.
](image)

For each component $C_i$ in $\mathcal{C}(G)$, let $V_i$ be the set of common vertices in all $K_4$’s of $C_i$; for the leftmost figure in Figure 7, $V_i$ is the set of all 4 vertices; for the middle two figures, $V_i$ is the set of left-bottom vertex and the middle-bottom vertex. Note that all $V_i$’s are pairwise disjoint. Let $G'$ be the induce subgraph of $G$ on $\cup_i V_i$. Note that $G'$ does not contains any vertex in $C_i \setminus V_i$. By checking each type, we find out that for each $i$ and each $v \in V_i$, $v$ has at most $\min\{2, |V_i| - 2\}$ neighbors outside $V_i$ in $G'$ (not in $G$!). Applying Theorem 4 to $G'$, we conclude that there exists an independent set $I$ of $G'$ meeting every $V_i$; thus $I$ meets every $K_4$ in $G$. Contradiction!

\[\Box\]

**Lemma 5** Let $G$ be a connected graph with $\Delta(G) \leq 5$ and $\omega(G) \leq 4$. If $G \neq C_{2l+1} \boxtimes K_2$ for some $l \geq 2$, then there exists an independent set meeting all induced copies of $K_4$ and $C_8^2$.

**Proof:** We will use proof by contradiction. Suppose the Lemma is false. Let $G$ be a minimum counterexample (with the smallest number of vertices). For any independent set $I$, let $C(I)$ be the number of induced copies of $C_8^2$ in $G - I$. Among all independent sets which meet all copies of $K_4$, there exists an independent set $I$ such that $C(I)$ is minimized. Since $C(I) > 0$, let $H$ be a copy of $C_8^2$ in $G - I$. The vertices of $H$ are listed by $u_i$ for $i \in Z_8$ anticlockwise such that $u_i u_j$ is an edge of $H$ if and only if $|i - j| \leq 2$. The vertex $v_i + 4$ is the antipode of $v_i$ for any $i \in Z_8$.

**Case 1:** There exists a vertex $v \notin V(H)$ such that $v$ has five neighbors in $H$. By the Pigeonhole Principle, $\Gamma(v)$ contains a pair of antipodes. Without loss of generality, say $u_0, u_4 \in \Gamma(v)$. If the other three neighbors of $v$ do not form a triangle, then we let $I' := I \setminus \{v\} \cup \{u_0, u_4\}$; note that $v$ is not in any $K_4$ of $G - I'$. Thus $I'$ is an independent set meeting every copy of $K_4$. Since every copy of $C_8^2$ containing $v$ must contain one of $u_0$ and $u_4$, we have $C(I') < C(I)$. Contradiction! Hence, the other three neighbors of $v$ must form a triangle. Without loss of generality, we can assume that the three neighbors are $u_1, u_2,$ and $u_3$. Now we let $I'' := I \setminus \{v\} \cup \{u_0, u_3\}$; note that $v \notin K_4 \subset G - I''$. Thus $I''$ is also an independent set meeting every copy of $K_4$ of $G$. Since every copy of $C_8^2$ containing $v$ must contain one of $u_0$ and $u_3$, we have $C(I'') < C(I)$. Contradiction!

**Case 2:** There exists a vertex $v \notin V(H)$ such that $v$ has exactly four neighbors in $H$. Since $H$ is $K_4$-free, we can find $u_i, u_j \in \Gamma(v) \cap V(H)$ such that $u_i u_j$ is a non-edge. Let
I' := I \ \{v\} \cup \{u_i, u_j\}; I' is also an independent set. Note that Π(v) \ \{u_i, u_j\} can not be a triangle, v is not in any K_4 \subset G - I'. Thus I' meets every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_i and u_j, we have C(I') < C(I). Contradiction!

**Case 3:** There exists a vertex v \notin V(H) such that v has exactly three neighbors in H. If the 3 neighbors do not form a triangle, then choose u_i, u_j \in Π(v) \cap V(H) such that u_i u_j is a non-edge. Note that Π(v) \ \{u_i, u_j\} can not be a triangle; v is not in any K_4 \subset G - I'. Let I' := I \ \{v\} \cup \{u_i, u_j\}; I' is also an independent set meeting every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_i and u_j, we have C(I') < C(I). Contradiction!

Else, the three neighbors form a triangle; let C := C \notin H. Note that v \notin H has three neighbors in this K_4 \subset G - I'. Let I' := I \ \{v\} \cup \{u_i, u_j\}; I' is also an independent set meeting every copy of K_4. Since every copy of C_8^2 containing v must contain one of u_i and u_j, we have C(I') < C(I). Contradiction!

**Case 4:** Every vertex outside H can have at most 2 neighbors in H. We identify each pair of antipodes of H to get a new graph G' from G. After identifying, H is turned into a K_4; where the vertices of this K_4 are referred as fat vertices.

**Subcase 4a:** G' \neq C_{2k+1} \sqcup K_2. Observe Δ(G') \leq 5. We claim G' is K_5-free. Suppose not. Since every vertex in H has at most one neighbor outside H, then each fat vertex can have at most two neighbors outside H. Recall that the original graph G is K_5-free. If G' has some K_5, then this K_5 contains either 3 or 4 fat vertices. Let w be one of the other vertices in this K_5. We get w has at least three neighbors in H. However, this is covered by Case 1, Case 2, or Case 3. Thus, G' is K_5-free. Since |G'| < |G|, by the minimality of G, G' has an independent set I meeting every copy of K_4 and C_8 in G'. There is exactly one fat vertex in I. Now replacing this fat vertex by its corresponding pair of antipodal vertices, we get an independent set I'; we assume the pair of antipodal vertices are u_2 and u_6. It is easy to check that I' is an independent set of G. Next we claim any v \in V(H) \ \{u_2, u_6\} is neither in a K_4 \subset V(G) - I' nor in a C_8^2 \subset V(G) - I'. Suppose there is some v such that v \in K_4 \subset G - I'. Recall each v \in V(H) has at most one neighbor outside H and H is K_4-free; there is some w \notin V(H) such that w has at least three neighbors in H. This is already considered by Case 1, Case 2, or Case 3. We are left to show that v \notin C_8^2 \subset G - I' for each v \in V(H) \ \{u_2, u_6\}. If not, there exists a copy H' of C_8^2 in G - I' containing v. Note H' is 4-regular, any vertex in H' can have at most one neighbor in I'; in particular, v \neq u_0, u_4. Without loss of generality, we assume v = u_3. Then there is a vertex w \notin V(H) such that u_3 w is an edge, see Figure 8. Observe that the neighborhood of each vertex of an induced C_8^2 is a P_4. Since u_1 u_4 and u_1 u_5 are two non-edges, we have wu_1 being an edge. Observe Π_G(u_1) = \{u_7, u_0, u_2, u_3, w\}. Since u_2 \notin H', we have u_0 \notin H'; u_0 has two neighbors (u_2 and u_6) outside H', contradiction! Therefore, I' meets every copy of K_4 and C_8 in G. Contradiction!

Figure 8: Subcase 4a.
Subcase 4b: $G' = C_{2l+1} \boxtimes K_2$. The graph $G$ can be recovered from $G'$. It consists of an induced subgraph $H = C_8^l$ and an induced subgraph $P_{2l-1} \boxtimes K_2$. For each vertex $u$ in $H$, there is exactly one edge connecting it to one of the four end vertices of $P_{2l-1} \boxtimes K_2$; for each end vertex $v$ of $P_{2l-1} \boxtimes K_2$, there are exactly two edges connecting $v$ to the vertices in $H$.

First, we take any maximum independent set $I'$ of $P_{2l-1} \boxtimes K_2$. Observe that $I'$ has exactly two end points of $P_{2l-1} \boxtimes K_2$; so $I'$ has exactly four neighbors in $H$. In the remaining four vertices of $H$, there exists a non-edge $u_i u_j$ since $H$ is $K_4$-free. Let $I := I' \cup \{u_i, u_j\}$. Clearly $I$ is an independent set of $G$ meeting every copy of $K_4$ and $C_8^l$. Contradiction! \hfill $\square$

We are ready to prove Lemma 4.

**Proof of Lemma 2.** We need prove for $k \geq 5$ and any connected graph $G$ with $\Delta(G) = k$ and $\omega(G) \leq k - 1$ satisfies

$$\chi_f(G) \leq k - \min\left\{ f(k - 1), \frac{1}{2} \right\}. \tag{3}$$

If $\omega(G) \leq k - 2$, then by inequality (1), we have

$$\chi_f(G) \leq \frac{\Delta(G) + \omega(G) + 1}{2} \leq k - \frac{1}{2}.$$ 

Thus, inequality (3) is satisfied. From now on, we assume $\omega(G) = \Delta(G) - 1$.

For $\Delta(G) = k \geq 6$ and $\omega(G) = k - 1$, the condition $\omega(G) > \frac{k}{2}(\Delta(G) + 1)$ is satisfied. By Theorem 3 $G$ contains an independent set meeting every maximum clique. Extend this independent set to a maximal independent set and denote it by $I$. Note that $\Delta(G - I) \leq k - 1$ and $\omega(G - I) \leq k - 2$.

**Case 1:** $k \geq 7$. From the definition of $f(k - 1)$, we have $\chi_f(G - I) \leq \Delta(G - I) - f(k - 1)$.

Thus, $\chi_f(G) \leq \chi_f(G - I) + 1 \leq k - 1 - f(k - 1) + 1 = k - f(k - 1)$.

Thus, we have $f(k) \geq \min\{f(k - 1), 1/2\}$.

**Case 2:** $k = 6$. By Lemma 3 we can find an independent set meets every copy of $K_5$ and $C_5 \boxtimes K_2$: we extend this independent set as a maximal independent set $I$. Note that $G - I$ contains no induced subgraph isomorphic $C_5 \boxtimes K_2$. We have $\chi_f(G - I) \leq 5 - f(6)$; it implies $\chi_f(G) \leq 6 - f(5)$. Thus, $f(6) \geq \min\{f(5), 1/2\}$ and we are done.

**Case 3:** $k = 5$. If $G = C_{2l+1} \boxtimes K_2$ for some $l \geq 3$; then $G$ is vertex-transitive and $\alpha(G) = l$.

It implies that

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)} = 4 + \frac{2}{l} \leq 5 - \frac{1}{3}.$$ 

If $G \neq C_{2l+1} \boxtimes K_2$, then by Lemma 5 we can find an independent set meeting every copy of $K_4$ and $C_8^l$: we extend it as a maximal independent set $I$. Note that $G - I$ contains no induced subgraph isomorphic $C_8^l$. We have $\chi_f(G - I) \leq 4 - f(4)$; it implies $\chi_f(G) \leq 5 - f(4)$.

Thus, $f(3) \geq \min\{f(4), 1/3\}$ and we are finished. \hfill $\square$

### 3 The case $\Delta(G) = 4$

To prove $f(4) \geq \frac{2}{4}$, we will use an approach which is similar to those in [3, 6]. It suffices to prove that the minimum counterexample does not exist.

Let $G$ be a graph with the smallest number of vertices and satisfying

1. $\Delta(G) = 4$ and $\omega(G) \leq 3$;
2. $\chi_f(G) > 4 - \frac{2}{3}$.
3. $G \neq C_5^2$

By the minimality of $G$, each vertex in $G$ has degree either 4 or 3. To prove Lemma 5, we will show $\chi_f(G) \leq 4 - \frac{2}{m}$, which gives us the desired contradiction.

For a given vertex $x$ in $V(G)$, it is easy to color its neighborhood $\Gamma_G(x)$ using 2 colors. If $d_G(x) = 3$, then we pick a non-edge $S$ from $\Gamma_G(x)$ and color the two vertices in $S$ using color 1. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \geq 3$, then we pick an independent set $S$ in $\Gamma_G(x)$ of size 3 and assign the color 1 to each vertex in $S$. If $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then we pick two disjoint non-edges $S_1$ and $S_2$ from $\Gamma_G(x)$; we assign color 1 to each vertex in $S_1$ and color 2 to each vertex in $S_2$.

The following Lemma shows that $G$ has a key property, which eventually implies that this local coloring scheme works simultaneously for $x$ in a large subset of $V(G)$.

**Lemma 6** For each $x \in V(G)$ with $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, there exist two vertex-disjoint non-edges $S_1(x), S_2(x) \subset \Gamma_G(x)$ satisfying the following property. If we contract $S_1(x)$ and $S_2(x)$, then the resulted graph $G/S_1(x)/S_2(x)$ contains neither $K_5^−$ nor $G_0$. Here $K_5^−$ is the graph obtained from $K_5$ by removing one edge and $G_0$ is a graph defined in Figure 4.

![Figure 9: The graph $G_0$.](image)

The proof of this lemma is quite long and we will present its proof in section 4.

For each vertex $x$ in $G$, we associate a small set of vertices $S(x)$ selected from $\Gamma_G(x)$ as follows. If $d_G(x) = 3$, then let $S(x)$ be the endpoints of a non-edge in $\Gamma_G(x)$ and label the vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) \geq 3$, then let $S(x)$ be any independent set of size 3 in $\Gamma_G(x)$ and label all vertices in $S(x)$ as 1; if $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$, then let $S(x) = S_1(x) \cup S_2(x)$, where $S_1(x)$ and $S_2(x)$ are guaranteed by Lemma 6. We label the vertices in $S_1(x)$ as 1 and the vertices in $S_2(x)$ as 2. For any $x \in V(G)$, we have $|S(x)| = 2, 3, \text{or } 4$.

The following definitions depend on the choice of $S(\ast)$, which is assumed to be fixed through this section. For $v \in G$ and $j \in \{1, 2, 3\}$, we define

$$N^j_G(v) = \{u\mid \text{there is a path } vv_0 \ldots v_{j-2} u \text{ in } G \text{ of length } j \text{ such that } v_0 \in S(v) \text{ and } v_{j-2} \in S(u)\}.$$ 

For $j = 4$, $v \in N^4_G(u)$ if $d_G(u) = 4$, $\alpha(\Gamma_G(u)) = 2$, $u$ and $v$ are connected as shown in Figure 10; otherwise $N^4_G(u) = \emptyset$. In Figure 11, $w$ is connected to one of the two vertices in $S_2(u)$. Similarly, in Figure 11 and 11, a vertex is connected to a group of vertices means it is connected to any vertex in this group.
For \( j = 5 \), \( v \in N^2_G(u) \) if \( d_G(w) = 4 \), \( \alpha(\Gamma_G(w)) = 2 \) for \( w \in \{u, v\} \) and \( u \) and \( v \) are connected as shown in Figure 11 otherwise \( N^5_G(u) = \emptyset \).

For \( j = 7 \), \( v \in N^2_G(u) \) if \( d_G(w) = 4 \), \( \alpha(\Gamma_G(w)) = 2 \) for \( w \in \{u, v\} \) and \( u \) and \( v \) are connected as shown in Figure 12 otherwise \( N^7_G(u) = \emptyset \).

Figure 10: 4-th neighborhood. Figure 11: 5-th neighborhood. Figure 12: 7-th neighborhood.

Note that for \( j \in \{1, 2, 3, 5, 7\} \), \( v \in N^2_G(u) \) if and only if \( u \in N^2_G(v) \); but this does not hold for \( j = 4 \). We have the following lemma.

**Lemma 7** For \( u \in V(G) \) such that \( d_G(u) = 4 \) and \( \alpha(\Gamma_G(u)) = 2 \), we have \( |N^2_G(u)| \leq |N^3_G(u)\cup N^4_G(u)\cup N^5_G(u)\cup N^7_G(u)| \leq 96 \).

**Proof:** It is clear that \( |N^2_G(u)| \leq |N^3_G(u)\cup N^4_G(u)\cup N^5_G(u)\cup N^7_G(u)| \leq 4 + 8 + 8 \times 3 = 36 \). We next estimate \( |N^2_G(u)| \). In Figure 11 observe that \( w \) is connected to one vertices of \( S_2(u) \) and \( w \notin \Gamma_G(u) \). For a fixed \( u \), there are at most four choices for \( w \), at most three choices for \( z \), and at most three choices for \( v \). Therefore, we have \( |N^2_G(u)| \leq 4 \times 3 \times 3 = 36 \).

Let us estimate \( |N^2_G(u)| \). In Figure 11 for a fixed \( u \), we have four choices for \( w \) and two choices for \( z \). Fix a \( z \). Assume \( \Gamma_G(z) \setminus \{w\} = \{a, b, c\} \). Let \( T_1 = \{a, b\} \), \( T_2 = \{b, c\} \), and \( T_3 = \{a, c\} \). We have the following claim.

**Claim** There are at most three \( v \in N^2_G(u) \) such that for each \( v \) we have \( \Gamma_G(z) \cap \Gamma_G(v) = T_i \) for some \( 1 \leq i \leq 3 \) as shown in Figure 11.

**Proof of the claim:** For each \( 1 \leq i \leq 3 \), there are at most three \( v \in N^2_G(u) \) such that \( \Gamma_G(z) \cap \Gamma_G(v) = T_i \) as shown in Figure 11 since each vertices in \( T_i \) has at most three neighbors other than \( z \). If the claim is false, then there is \( 1 \leq i \neq j \leq 3 \) such that \( \Gamma_G(z) \cap \Gamma_G(v_i) = T_i \) and \( \Gamma_G(z) \cap \Gamma_G(v_j) = T_j \) for some \( v_i, v_j \in N^2_G(u) \), and \( \Gamma_G(z) \cap \Gamma_G(v_j) = T_j \) for some \( v_j \in N^2_G(u) \), where \( v_i, v_j \) are distinct. Without loss of generality, we assume \( \Gamma_G(z) \cap \Gamma_G(v_i) = \Gamma_G(z) \cap \Gamma_G(v_j) = T_i \) for \( v_i, v_j \in N^2_G(u) \), and \( \Gamma_G(z) \cap \Gamma_G(v_j) = T_2 \) for some \( v_j \in N^2_G(u) \), see Figure 13. Observe that \( \Gamma_G(b) = \{v_1, v_1', v_2, z\} \). Since \( \Gamma_G(z) \cap \Gamma_G(v_j) = T_1 \) as shown in Figure 11, \( a \) and \( b \)'s neighbors form \( S_i(v_i) \) for some \( i \in \{1, 2\} \); we assume it is \( S_1(v_1) \). Note \( \{z, v_1, v_1'\} \subset \Gamma_G(a) \). Thus \( S_1(v_1) = \{a, v_2\} \) and \( v_2 \in \Gamma_G(v_1) \). Similarly, we can show \( S_1(v_1') = \{a, v_2\} \) and \( v_2 \in \Gamma_G(v_1') \). Now, observe that \( \Gamma_G(v_2) = \{v_1, v_1', b, c\} \). Since \( \Gamma_G(z) \cap \Gamma_G(v_2) = T_2 \) as shown in Figure 11, \( b \) and one of neighbors of \( v_2 \) form \( S_i(v_2) \) for some \( i \in \{1, 2\} \); we assume \( i = 1 \). Because \( \{v_1, v_1'\} \subset \Gamma_G(b) \), then \( S_1(v_2) = \{b, c\} \). However, \( b \) and \( c \) are not in the same independent set in the definition of \( N^3_G(u) \), see Figure 11. This is a contradiction and this case can not happen. The claim follows.

Therefore, \( |N^2_G(u)| \leq 4 \times 2 \times 3 = 24 \).

In Figure 12 for a fixed \( u \), we have two choices for the edge \( e \), one choice for \( w \), two choices for \( z \), and three choices for the edge \( f \). Fix a \( z \). By considering the degrees of the endpoints of \( f \), there is at most one \( f \) and at most one \( v \in N^2_G(u) \) such that \( |\Gamma_G(f) \cap \Gamma_G(v)| = 4 \) as shown in Figure 12. Therefore, we have \( |N^2_G(u)| \leq 2 \times 2 \times 1 = 4 \).
For therefore, the greedy algorithm shows $G$.

If there is some $v \in N^7_G(u)$, then we observe that there are at most five $z$’s (see Figure 11). We get the number of $v \in N^7_G(u)$ is at most $5 \times 3 = 15$. In this case, we have

$$|N^5_G(u) \cup N^7_G(u)| \leq 4 + 15 < 24.$$  

If $N^7_G(u) = \emptyset$, then also we have

$$|N^5_G(u) \cup N^7_G(u)| \leq 24.$$  

Therefore

$$|N^1_G(u) \cup N^2_G(u) \cup N^3_G(u) \cup N^4_G(u) \cup N^5_G(u) \cup N^7_G(u)| \leq 36 + 36 + 24 = 96.$$  

Last, we estimate $|N^2_G(u) \cup N^7_G(u)|$. If there is some $v \in N^7_G(u)$, then we observe that $|N^2_G(u) \cup N^7_G(u)| \leq 4 + 15 < 24$.

Based on the graph $G$, we define an auxiliary graph $G^*$ on vertex set $V(G)$. The edge set is following: $uv \in E(G^*)$ if either $u \in N^1_G(v) \cup N^2_G(v) \cup N^3_G(v) \cup N^4_G(v) \cup N^5_G(v) \cup N^7_G(v)$, or $v \in N^4_G(u)$, or $u \in N^4_G(v)$. We have the following lemma.

**Lemma 8** The graph $G^*$ is 133-colorable.

**Proof:** Let $\sigma$ be an increasing order of $V(G^*)$ satisfying the following conditions.

1: For $u$ and $v$ such that $d_G(u) = 3$ and $d_G(v) = 4$, we have $\sigma(u) < \sigma(v)$.

2: For $u$ and $v$ such that $d_G(u) = d_G(v) = 4$, $\alpha(G(u)) \geq 3$, and $\alpha(G(v)) = 2$, we have $\sigma(u) < \sigma(v)$.

We will color $V(G^*)$ according to the order $\sigma$. For each $v$, we have the following estimate on the number of colors forbidden to use for $v$.

1: For $v$ such that $d_G(v) = 3$, the number of colors forbidden to use for $v$ is at most $|N^1_G(v) \cup N^2_G(v) \cup N^3_G(v)| \leq 4 + 8 + 24 = 36$.

2: For $v$ such that $d_G(v) = 4$ and $\alpha(G(v)) \geq 3$, the number of colors forbidden to use for $v$ is at most $|N^1_G(v) \cup N^2_G(v) \cup N^3_G(v)| \leq 4 + 8 + 24 = 36$.

3: For $v$ such that $d_G(v) = 4$ and $\alpha(G(v)) = 2$, the number of colors forbidden to use for $v$ is at most $|N^1_G(v) \cup N^2_G(v) \cup N^3_G(v) \cup N^4_G(v) \cup N^5_G(v) \cup N^7_G(v)| + |N^4_G(v)| \leq 96 + 36 = 132$ by Lemma 7.

Therefore, The greedy algorithm shows $G^*$ is 133-colorable.

Let $X$ be a coloring class of $G^*$. We define a new graph $G(X)$ by the following process.
1. For each \( x \in X \), if \( |S(x)| = 2 \) or \( |S(x)| = 3 \), then we contract \( S(x) \) as a single vertex, delete the vertices in \( \Gamma_G(v) \setminus S(v) \), and keep labeling the new vertex as \( 1 \); if \( |S(x)| = 4 \), i.e., \( S(x) = S_1(x) \cup S_2(x) \), then we contract \( S_1(x) \) and \( S_2(x) \) as single vertices and keep their labels. After that, we delete \( X \). Let \( H \) be the resulted graph.

2. Note that \( \Gamma_H(x) \cap \Gamma_H(y) = \emptyset \) and there is no edge from \( \Gamma_H(x) \) to \( \Gamma_H(y) \) for any \( x, y \in X \) as \( X \) is a coloring class.

3. We identify all vertices with label \( i \) as a single vertex \( w_i \) for \( i \in \{1, 2\} \). Let \( G(X) \) be the resulted graph.

We have the following lemma on the chromatic number of \( G(X) \).

**Lemma 9** The graph \( G(X) \) is 4-colorable for each coloring class.

We postpone the proof of this lemma until the end of this section and prove the Lemma first.

**Proof of Lemma** By Lemma there is a proper 133-coloring of \( G^* \). We assume \( V(G^*) = V(G) = \bigcup_{i=1}^{133} X_i \), where \( X_i \) is the \( i \)-th coloring class.

For each \( i \in \{1, \ldots, 133\} \), Lemma shows \( G(X_i) \) is 4-colorable; let \( c_i : V(G(X_i)) \rightarrow T_i \) be a proper 4-coloring of the graph \( G(X_i) \). Here \( T_1, T_2, \ldots, T_{133} \) are pairwise disjoint; each of them consists of 4 colors. For \( i \in \{1, \ldots, 133\} \), the 4-coloring \( c_i \) can be viewed as a 4-coloring of \( G \setminus X_i \) since each vertex with label \( j \) receives the color \( c_i(w_i) \) for \( j = 1, 2 \) and each removed vertex has at most three neighbors in \( G \setminus X_i \).

Now we reuse the notation \( c_i \) to denote this 4-coloring of \( G \setminus X_i \). For each \( v \in X_i \), we have \( |\bigcup_{u \in \Gamma_G(v)} c_i(u)\| \leq 2 \). We can assign two unused colors, denoted by the set \( Y(v) \), to \( v \). We define \( f_i : V(G) \rightarrow \mathcal{P}(T_i) \) (the power set of \( T_i \)) satisfying

\[
f_i(v) = \begin{cases} \{c_i(v)\} & \text{if } v \in V(G) \setminus X_i, \\ Y(v) & \text{if } v \in X_i. \end{cases}
\]

Observe that each vertex in \( X_i \) receives two colors from \( f_i \) and every other vertex receives one color. Let \( \sigma : V(G) \rightarrow \mathcal{P}^{m=133}(\bigcup_{i=1}^{133} T_i) \) be a mapping such that \( \sigma(v) = \bigcup_{i=1}^m f_i(v) \). It is easy to verify \( \sigma \) is a 134-fold coloring of \( G \) such that each color is drawn from a palette of 532 colors; namely we have

\[
\chi_f(G) \leq \frac{532}{134} = 4 - \frac{2}{67}.
\]

The proof of Lemma is finished. \( \square \)

Before we prove Lemma we need the following definitions.

A block of a graph is a maximal 2-connected induced subgraph. A Gallai tree is a connected graph in which all blocks are either complete graphs or odd cycles. A Gallai forest is a graph all of whose components are Gallai trees. A \( k \)-Gallai tree (forest) is a Gallai tree (forest) such that the degree of all vertices are at most \( k - 1 \). A \( k \)-critical graph is a graph \( G \) whose chromatic number is \( k \) and deleting any vertex can decrease the chromatic number. Gallai showed the following Lemma.

**Lemma 10** If \( G \) is a \( k \)-critical graph, then the subgraph of \( G \) induced on the vertices of degree \( k - 1 \) is a \( k \)-Gallai forest.

**Proof of Lemma** We use proof by contradiction. Suppose that \( G(X) \) is not 4-colorable. The only possible vertices in \( G(X) \) with degree greater than 4 are the vertices \( w_1 \) and \( w_2 \), which are obtained by contracting the vertices with label 1 and 2 in the intermediate graph \( H \). The simple greedy algorithm shows that \( G(X) \) is always 5-colorable. Let \( G'(X) \) be a
5-critical subgraph of $G(X)$. Applying Lemma 10 to $G'(X)$, the subgraph of $G'(X)$ induced on the vertices of degree 4 is a 5-Gallai forest $F$. The vertex set of $F$ may contain $w_1$ or $w_2$. Delete $w_1$ and $w_2$ from $F$ if $F$ contains one of them. Let $F'$ be the resulted Gallai forest. (Any induced subgraph of a Gallai forest is still a Gallai forest.) The Gallai forest $F'$ is not empty. Let $T$ be a connected component of $F'$ and $B$ be a leaf block of $T$. The block $B$ is either a clique or an odd cycle from the definition of a Gallai tree.

Let $v$ be a vertex in $B$. As $v$ has at most two neighbors ($w_1$ and $w_2$) outside $F'$ in $G(X)$, we have $d_{F'}(v) \geq 2$. If $v$ is not in other blocks of $F'$, then we have $d_B(v) \geq 2$. It follows that $|B| \geq 3$. Since $B$ is a subgraph of $G$ and $G$ is $K_4$-free, the block $B$ is an odd cycle.

Let $v_1v_2$ be an edge in $B$ such that $v_1$ and $v_2$ are not in other blocks. The degree requirement implies $v_iv_j$ are edges in $G(X)$ for all $i, j \in \{1, 2\}$. For $i = 1, 2$, there are vertices $x_i, y_i \in X$ satisfying $S(x_i) \cap \Gamma_G(v_i) \neq \emptyset$ and $S(y_i) \cap \Gamma_G(v_i) \neq \emptyset$; moreover either $|S(x_i)| = 4$ or $|S(y_i)| = 4$ since one of its neighborhood has label 2. Without loss of generality, we assume $|S(x_i)| = 4$ for $i \in \{1, 2\}$. Let $x_i \neq y_i$, then $y_i \in N_G^-(x_i)$, i.e., $y_i \in \Gamma_G^-(x_i)$; this contradicts $X$ being a coloring class. Thus we have $x_i = y_i$ and $|S(x_i)| = 4$ for $i \in \{1, 2\}$.

For $i, j \in \{1, 2\}$, if $x_i \neq y_j$, then $y_i \in N_G^-(x_j)$, i.e., $y_i \in \Gamma_G^-(x_j)$; this is a contradiction of $X$ being a coloring class. Thus we have

$$x_1 = x_2 = y_1 = y_2.$$  

Let $x$ denote this common vertex above. Then $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$.

Let $v_0$ be the only vertex in $B$ shared by other blocks. Since $B - v_0$ is connected, the argument above shows there is a common $x$ for all edges in $B - v_0$. If $\Gamma_G(X)(v_0) \cap \{w_1, w_2\} \neq \emptyset$, there is some vertex $x_0 \in X$ such that $S(x_0) \cap \Gamma_G(v_0) \neq \emptyset$. By the similar argument, we also have $x_0 = x$.

Therefore, $x$ depends only on $B$. In the sense that for any $y \in X$ and any $v \in B$, if $S(y) \cap \Gamma_G(v) \neq \emptyset$, then $y = x$.

The block $B$ is an odd cycle as we mentioned above. Suppose $|B| = 2r + 1$. Let $v_0, v_1, \ldots, v_{2r}$ be the vertices of $B$ in cyclic order and $v_0$ be the only vertex which may be shared by other block.

Let $x \in X$ be the vertex determined by $B$. Recall $d_G(x) = 4$ and $\alpha(\Gamma_G(x)) = 2$. Each vertex in $\Gamma(x)$ can have at most 2 edges to $B$. We get

$$4r \leq |E(B, \Gamma(x))| \leq 8. \tag{4}$$

We have $r \leq 2$. The block $B$ is either a $C_5$ or a $K_3$. We claim both $v_0w_1$ and $v_0w_2$ are non-edges of $G(X)$.

If $B = C_5$, then inequality (3) implies that $v_0$ has no neighbor in $\Gamma(x)$ and the claim holds. If $B = K_3$, then the claim also holds; otherwise $B \cup \{S_1(x), S_2(x)\}$ forms a $K_5^-$ in $G/S_1(x)/S_2(x)$, which is a contradiction to Lemma 6.

Let $u_1$ and $u_2$ be the two neighbors of $v_0$ in other blocks of $F'$. If $u_1$ and $u_2$ are in the same block, then this block is an odd cycle; otherwise, $v_0u_1$ and $v_0u_2$ are in two different blocks.

The union of non-leaf blocks of $T$ is a Gallai-tree, denoted by $T'$. The argument above shows every leaf block of $T'$ must be an odd cycle. Let $C$ be such a leaf block of $T'$. Now $C$ is an odd cycle, and $C$ is connected to $|C| - 1$ leaf blocks of $T$. Let $B$ and $B'$ be two leaf blocks of $T$ such that $B \cap C$ is adjacent to $B' \cap C$. Without loss of generality, we may assume $B$ is the one we considered before. By the same argument, $B'$ is an odd cycle of size $2r' + 1$ with $r' \in \{1, 2\}$. Let $v_0', v_1', \ldots, v_{2r'}'$ be the vertices of $B'$ and $v_0'$ be the only vertex in $B' \cap C$. For $i$ in $\{1, 2, \ldots, 2r'\}$ and $j$ in $\{1, 2\}$, $v_iw_j$ are edges in $G'(X)$. Similarly, there exists a vertex $x' \in X$ with $d_G(x') = 4$ and $\alpha(\Gamma_G(x')) = 2$ such that $|E(v_i, S_1(x'))| \geq 1$ and $|E(v_i, S_2(x'))| \geq 1$. We must have $x = x'$; otherwise $x' \in N_G^-(x)$, i.e., $x' \in \Gamma_G^-(x)$, and this
contradicts the fact that $X$ is a coloring class in $D$. Now we have $|E(\Gamma(x), B)| \geq 4r$ and $E(\Gamma(x), B')| \geq 4r'$. By counting the degrees of vertices in $\Gamma(x)$ in $G$, we have

$$4r + 4r' + 4 + 4 \leq 16.$$ 

We get $r = r' = 1$. Both $B$ and $B'$ are $K_3$’s. In this case, $G/S_1(x)/S_2(x)$ contains the graph $G_0$, see figure 3. This contradicts Lemma 6.

We can find the desired contradiction; so the lemma follows. □

4 Proof of Lemma 6

In this section, we will prove Lemma 6. We first review a Lemma in [6].

Lemma 11 Let $G$ be a graph. Suppose that $G_1$ and $G_2$ are two subgraphs such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{u, v\}$.

1. If $uv$ is an edge of $G$, then we have

$$\chi_f(G) = \max\{\chi_f(G_1), \chi_f(G_2)\}.$$ 

2. If $uv$ is not an edge of $G$, then we have

$$\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\},$$ 

where $G_2 + uv$ is the graph obtained from $G_2$ by adding edge $uv$ and $G_2/uv$ is the graph obtained from $G_2$ by contracting $\{u, v\}$.

Proof of Lemma 6 Recall that $G$ is a connected $K_4$-free graph with minimum number of vertices such that $G \neq C_8$ and $\chi_f(G) > 4 - \frac{2}{7}$. Note that $G$ is 2-connected. We will prove it by contradiction.

Suppose Lemma 6 fails for some vertex $x$ in $G$. Observe $\Gamma_G(x)$ is one of the graphs in Figure 14. Here we assume $\Gamma_G(x) = \{a, b, c, d\}$. Through the proof of the lemma, let $S_1$ and $S_2$ be two vertex-disjoint independent sets in $\Gamma_G(x)$, $H$ be a triangle in $V(G) \setminus (\{x\} \cup \Gamma_G(x))$, then say $(S_1, S_2, H)$ is a bad triple if $\{S_1, S_2, H\}$ contains a $K_5$ in $G/S_1/S_2$.

If $\Gamma_G(x) = P_4$, then $\{a, d\}$ and $\{b, c\}$ is the only pair of disjoint non-edges. There is a triangle $H$ with $V(H) = \{y, z, w\}$ such that $\{a, d\}, \{b, c\}, H$ is a bad triple. Note that $|E(\{a, b, c, d\}, \{y, z, w\})| = 5$ or 6. By an exhaustive search, the induced subgraph of $G$ on $\{x, a, b, c, d, y, z, w\}$ is one of the following six graphs (see Figure 15).
If $\Gamma_G(x) = P_4$, then there are six possible induced subgraphs.

If $\Gamma_G(x) = 2e$, then $\{(a, c), \{b, d\}\}$ and $\{(a, d), \{b, c\}\}$ are two pairs of disjoint non-edges. By considering the degrees of vertices in $\Gamma_G(x)$, there is only one triangle $H$ with $V(H) = \{y, z, w\}$ such that $\{(a, c), \{b, d\}, H\}$ and $\{(a, d), \{b, c\}, H\}$ are two bad triples. By an exhaustive search, the induced subgraph of $G$ on $\{x, a, b, c, d, y, z, w\}$ is one of the following three graphs (see Figure 16).

It suffices to show that $G$ cannot contain $H_i$ for $1 \leq i \leq 9$. Since all vertices in $H_1$ (and $H_2$) have degree 4, $H_1$ (and $H_2$) is the entire graph $G$. Observe that $H_1$ is isomorphic to $C_8^2$ and $H_2$ is 11:3-colorable (see Figure 17). Contradiction!

In $H_7$, the vertex $d$ is the only vertex with degree less than 4. If $H_7$ is not the entire graph $G$, then $d$ is a cut vertex of $G$. This contradicts the fact that $G$ is 2-connected. Thus $G = H_7$. The graph $H_7$ is 11:3-colorable as shown by Figure 17. Contradiction!

Now we consider the case $H_3$. Note $H_3 + bz$ is the graph $H_2$. We have $\chi_f(H_3) \leq \chi_f(H_2) \leq 11/3$. The graph $H_3$ must be a proper induced subgraph of $G$, and the pair $\{b, z\}$ is a vertex cut of $G$. Let $G'$ be the induced subgraph of $G$ by deleting all vertices in $H_3$ but $b, z$. We
Lemma 11, we have

\[\chi_f(G + bz) \leq \max\{\chi_f(H_2), \chi_f(G' + bz)\}.\]

Note \(\chi_f(H_2) \leq 11/3\) and \(11/3 < \chi_f(G) \leq \chi_f(G' + bz)\). We have \(\chi_f(G) \leq \chi_f(G' + bz)\). Both \(b\) and \(z\) have at most 2 neighbors in \(G' + bz\). Thus \(G' + bz\) is \(K_4\)-free; \(G' + bz \neq C_8^2\) and has fewer vertices than \(G\). This contradicts to the minimality of \(G\).

Note \(H_5 \cup cy = H_2\). The case \(H_5\) is similar to the case \(H_3\).

Note that \(H_4, H_6,\) and \(H_8\) are isomorphic to each other. It suffices to show \(G\) does not contain \(H_4\). Suppose that \(H_4\) is a proper induced subgraph of \(G\). Let \(G_1\) be the induced subgraph of \(G\) by deleting all vertices in \(H_4\). Note \(C_8^2\) is not a proper subgraph of any graph in \(G_4\). We have \(G_1 \neq C_8^2\). Note that \(c\) and \(z\) have degree 3 while other vertices in \(H_4\) have degree 4. Since \(G\) is 2-connected, \(c\) has a unique neighbor, denoted by \(u\), in \(V(G_1)\). Similarly, \(z\) has a unique neighbor, denoted by \(v\), in \(V(G_1)\). Observe that the pair \(\{u, v\}\) forms a vertex cut of \(G\). Let \(G_2\) be the induced graph of \(G\) on \(V(H_4) \cup \{u, v\}\). Applying Lemma 11 to \(G\) with \(G_1\) and \(G_2\), we have

\[\chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2 + uv), \chi_f(G_2/uv)\}.\]

Figure 18 shows \(\chi_f(G_2 + uv)\) and \(\chi_f(G_2/uv)\) are at most 11/3.

Since \(\chi_f(G) > 11/3\), we have \(\chi_f(G) \leq \chi_f(G_1)\). Now \(G_1\) is \(K_4\)-free and has maximum degree at most 4; \(G_1\) has fewer vertices than \(G\). This contradicts the minimality of \(G\).

If \(G = H_4\), then \(\chi_f(H_4) \leq 11/3\), since \(H_4\) is a subgraph of \(G_2 + uv\) in Figure 18.

Now we consider the last case \(H_8\). First, we contract \(b, c, z\) into a fat vertex denoted by \(bcz\). We write \(G/bcz\) for the graph after this contraction. Observe that \(\{bcz, d\}\) is a vertex-cut of \(G/bcz\). Let \(G_4\) and \(G'_4\) be two connected subgraphs of \(G/bcz\) such that \(G_4 \cup G'_4 = G/bcz\), \(G_4 \cap G'_4 = \{bcz, d\}\), and \(\{u, v\} \subset G'_4\). Note that \(G_4\) is 11:3 colorable, see Figure 19. Now by Lemma 11, we have

\[\chi_f(G/bcz) \leq \max\{\chi_f(G_4), \chi_f(G'_4)\}.\]
As \{b, c, z\} is an independent set, each a:b-coloring of \(G/bcz\) gives an a:b-coloring of \(G\), that is \(\chi_f(G/bcz) \geq \chi_f(G) > 11/3\). The graph \(G_4\) is 11:3-colorable; see Figure 19. Thus we have \(\chi_f(G'_4) \geq \chi_f(G/bcz) \geq \chi_f(G)\). It is easy to check that \(G'_4\) has maximum degree 4, \(K_4\)-free, and it is not \(G'_2\). Hence \(G'_4\) must contain a \(K_4\). Otherwise, it contradicts the minimality of \(G\).

Second, we contract \{b, d, z\} into a fat vertex \(bdz\) and denote the graph by \(G/bcz\). Let \(G_5\) and \(G'_5\) be two connected subgraphs of \(G/bdz\) such that \(G_5 \cup G'_5 = G/bzd\), \(G_5 \cap G'_5 = \{bdz, c\}\), and \(\{u, v\} \subseteq G'_5\). Note that \(G_5\) is 11:3-colorable; see Figure 19. By a similar argument, \(G'_5\) must contain a \(K_4\).

The remaining case is that both \(G'_4\) and \(G'_5\) have a \(K_4\) when we contract \(b\) and \(z\). Since the origin graph \(G\) is \(K_4\)-free, the \(K_4\) in \(G'_4\) (and in \(G'_5\)) must contain the fat vertex \(bdz\) (or \(bdz\)), respectively. Note that each of the four vertices \(b, c, d, z\) has at most one edge leaving \(Q_5\). There must be a triangle \(uvp\) in \(G\) and these four outward edges are connected to some element of \{\(u, v, p\}\}. The graph \(G/bz\) must contain the subgraph \(G_6\) as drawn in Figure 19.

Note that \{\(u, v\}\} is a vertex-cut in \(G/bz\). Let \(G_6\) and \(G'_6\) be two connected subgraphs of \(G/bz\), which satisfy \(G_6 \cup G'_6 = G\), \(G_6 \cap G'_6 = \{u, v\}\), and \(bdz \subseteq G_6\). By Lemma 11, we have

\[\chi_f(G/bz) \leq \max\{\chi_f(G_6), \chi_f(G'_6)\}\]

Note that \(G_6\) is 11:3-colorable; see Figure 19. We also have \(\chi_f(G/bz) \geq \chi_f(G) > 11/3\). We obtain \(\chi_f(G_6) \geq \chi_f(G/bz) \geq \chi_f(G)\). Observe that \(G_6\) is a subgraph of \(G\). We arrive at a contradiction of the minimality of \(G\).

If \(\Gamma_G(x) = C_4\), then the only possible choice for the two independent sets are \{\(a, c\}\} and \{\(b, d\)\}. If there is some triangle \(H\) such that \((\{a, c\}, \{b, d\}, H)\) is a bad triple, then we have

\[|E(\Gamma_G(x), H)| \geq 5\]

However, \(|E(\Gamma_G(x), H)| \leq 4\). This is a contradiction. Thus the lemma follows in this case.

We can select two vertex disjoint non-edges \(S_1\) and \(S_2\) such that the graph \(G/S_1/S_2\) contains no \(K_5^-\). For these particular \(S_1\) and \(S_2\), if \(G/S_1/S_2\) contains no \(G_0\), then Lemma 5 holds.

Without loss of generality, we assume that \(G/S_1/S_2\) does contain \(G_0\). Let \(s_i = S_i\) for \(i = 1, 2\). Observe that both \(s_1\) and \(s_2\) have four neighbors \(u, v, p, q\) other than \(x\) in \(G_0\). It follows that

\[|E(S_1 \cup S_2, \{u, v, p, q\})| \geq 8\]

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Figure 19: Case \(H_9\): the graphs \(G_4\), \(G_5\), and \(G_6\) are 11:3-colorable.
On the one hand, we have

\[ |E(G \mid S_1 \cup S_2)| = \frac{1}{2} \left( \sum_{v \in S_1 \cup S_2} d(v) - |E(S_1 \cup S_2, \{u, v, p, q\})| - 4 \right) \leq \frac{1}{2} (16 - 8 - 4) = 2. \]

On the other hand, \( \alpha(\Gamma(x)) = 2 \) implies \( G \mid S_1 \cup S_2 \) contains at least two edges. Thus, we have \( \Gamma G(x) = 2 \). Label the vertices in \( \Gamma G(x) \) by \( a, b, c, d \). We assume \( ab \) and \( cd \) are edges while \( ac, bd, ad, bc \) are non-edges. Observe that each vertex in \( \{u, v, p, q\} \) has exactly two neighbors in \( \{a, b, c, d\} \).

In the remaining case, we can assume that for each vertex \( y \) in \( \{u, v, p, q\} \), the neighbors of \( y \) in \( \{a, b, c, d\} \) always form an edge. Up to relabeling vertices, there is only one arrangement for edges between \( \{u, v, p, q\} \) and \( \{a, b, c, d\} \); see the graph \( H_{10} \) defined in Figure 20. The graph \( H_{10} \) is 11:3-colorable as shown in Figure 20. Since \( \chi_f(G) > 11/3 \), \( H_{10} \) is a proper subgraph of \( G \). Note in \( H_{10} \), every vertices except \( w \) and \( r \) has degree 4; both \( w \) and \( r \) have degree 3. Thus, \( \{w, r\} \) is a vertex cut of \( G \). Let \( G_1 = H_{10} \) and \( G_2 \) be the subgraph of \( G \) by deleting vertices in \( \{x, a, b, c, d, p, q, u, v\} \). Applying Lemma 11 with \( G_1 \) and \( G_2 \) defined above, we have

\[ \chi_f(G) \leq \max\{\chi_f(G_1), \chi_f(G_2)\}. \]

Since \( \chi_f(G) > 11/3 \) and \( \chi_f(G_1) \leq 11/3 \) (see Figure 20), we must have \( \chi_f(G_2) \geq \chi_f(G) \). Note that \( G_2 \) has fewer number of vertices than \( G \). This contradicts the minimality of \( G \). Therefore, the lemma follows.

References

[1] O. Borodin and A. Kostochka, On an upper bound on a graph’s chromatic number, depending on the graph’s degree and density, J. Comb. Th. B., 23 (1977), 247-250.

[2] T Gallai, Kritische graphen I, Magyar Tud. Akad. Mat. Kutat ó Int. Közl, 8 (1963), 165-192.
[3] H. Hatami and X. Zhu, The fractional chromatic number of graphs of maximum degree at most three, SIAM. J. Discrete Math., 24 (2009), 1762-1775.

[4] C. C. Heckman and R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, Discrete Math., 233 (2001), 233-237.

[5] Andrew D. King, Hitting all maximum cliques with a stable set using lopsided independent transversals, J. of Graph Theory, 67-4 (2011), 300-305.

[6] L. Lu and X. Peng, The fractional chromatic number of triangle-free graphs with $\Delta \leq 3$, submitted.

[7] M. Molloy and B. Reed, Graph colouring and the probabilistic method, volume 23 of Algorithms and Combinatorics, Springer-Verlag, Berlin, 2002.

[8] B. Reed, $\omega$, $\Delta$, and $\chi$, J. of Graph Theory, 27-4 (1998), 177-227.

[9] B. Reed, A strengthening of Brooks’ theorem, J.Comb. Th. B., 76 (1999), 136-149.

[10] E. R. Scheinerman and D. H. Ullman, Fractional graph theory. A rational approach to the theory of graphs, Wiley-Intersci. Ser. Discrete Math. Optim, John Wiley & Sons, Inc, New York, 1997.