Existence of relativistic dynamics for two directly interacting Dirac particles in 1+3 dimensions

Matthias Lienert* and Markus Nöth†

March 14, 2019

Abstract

Here we prove the existence and uniqueness of solutions of a class of integral equations describing two Dirac particles in 1+3 dimensions with direct interactions. This class of integral equations arises naturally as a relativistic generalization of the integral version of the two-particle Schrödinger equation. Crucial use of a multi-time wave function \( \psi(x_{1}, x_{2}) \) with \( x_{1}, x_{2} \in \mathbb{R}^{4} \) is made. A central feature is the time delay of the interaction. Our main result is an existence and uniqueness theorem for a Minkowski half space, meaning that Minkowski spacetime is cut off before \( t = 0 \). We furthermore show that the solutions are determined by Cauchy data at the initial time; however, no Cauchy problem is admissible at other times. A second result is to extend the first one to particular FLRW spacetimes with a Big Bang singularity, using the conformal invariance of the Dirac equation in the massless case. This shows that the cutoff at \( t = 0 \) can arise naturally and be fully compatible with relativity. We thus obtain a class of interacting, manifestly covariant and rigorous models in 1+3 dimensions.

Keywords: relativistic quantum theory, interaction with time delay, multi-time wave functions, Dirac equation, Volterra-type integral equation, non-Markovian dynamics.

1 Introduction

The Dirac equation is perhaps the most important equation in relativistic quantum theory, thus it may seem surprising that no completely satisfactory mathematical mechanism of interaction has been found for it. Usually, interactions between many particles are implemented in one of the following ways: (a) adding a potential to the free Hamiltonian, (b) using a second quantized electromagnetic field which mediates the interaction. Both approaches face difficulties. Approach (a) corresponds to postulating the equation

\[
 i\partial_{t} \phi(t, x_{1}, x_{2}) = \left( H_{1}^{\text{Dirac}} + H_{2}^{\text{Dirac}} + V(t, x_{1}, x_{2}) \right) \phi(t, x_{1}, x_{2}),
\]

where \( V \) is a potential and \( H_{k}^{\text{Dirac}} \) the Dirac Hamiltonian acting on the variables of the \( k \)-th particle. Under appropriate circumstances, it is clear that (1) defines an interacting dynamics (see e.g. [1] and references therein). However, (1) is not Lorentz invariant.

Approach (b), on the other hand, easily leads to a Lorentz invariant dynamics. However, one encounters difficulties with ultraviolet divergences. These difficulties have led to the situation that, great efforts notwithstanding, it has so far only been possible to rigorously

---

*Fachbereich Mathematik, Eberhard-Karls-Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. E-mail: matthias.lienert@uni-tuebingen.de
†Mathematisches Institut, Ludwig-Maximilians-Universität, Theresienstr. 39, 80333 München, Germany. E-mail: noeth@math.lmu.de
Dirac particles in Minkowski spacetime have undergone considerable developments \cite{14–24}; an overview can be found in \cite{25}. For two Schwinger \cite{7}, has been studied by different authors over the years \cite{8–13} and has recently undergone considerable developments \cite{14–24}; an overview can be found in \cite{25}. For two Dirac particles in Minkowski spacetime $\mathbb{M}$, a multi-time wave function is a map

$$\psi : \mathbb{M} \times \mathbb{M} \to \mathbb{C}^4 \otimes \mathbb{C}^4 \cong \mathbb{C}^{16}, \quad (x_1, x_2) \mapsto \psi(x_1, x_2).$$  \hspace{1cm} (2)

$\psi$ can be considered a generalization of the single-time wave function $\varphi$ in the Schrödinger picture, as in Eq. (1). The relation of $\psi$ to $\varphi$ is straightforwardly given by

$$\varphi(t, x_1, x_2) = \psi((t, x_1), (t, x_2)).$$  \hspace{1cm} (3)

Contrary to the single-time wave function $\varphi$ (which refers to a frame), $\psi$ is a manifestly covariant object. Under a Poincaré transformation $(a, \Lambda)$, $\psi$ transforms as

$$\psi'(x_1, x_2) = S[\Lambda] \otimes S[\Lambda] \psi(\Lambda^{-1}(x_1 - a), \Lambda^{-1}(x_2 - a)),  \hspace{1cm} (4)$$

where $S[\Lambda]$ are the matrices appearing in the spinor representation of the Lorentz group.

For the present purposes, it is crucial that $\psi$ is defined on general space-time configurations $(x_1, x_2) \in \mathbb{M} \times \mathbb{M}$, not only on equal-time configurations as $\varphi$. By relating configurations $(x_1, x_2)$ with different time coordinates $x_1^0 \neq x_2^0$, one can express interactions with time delay. It has been pointed out in \cite{26} that in this way, direct relativistic interactions (unmediated by fields) can be expressed at the quantum level. In particular, it becomes possible to formulate a quantum analog of direct interactions along light cones, such as in the Wheeler-Feynman formulation of classical electrodynamics \cite{27, 28}, using values of $\psi(x_1, x_2)$ with $(x_1 - x_2)_p(x_1 - x_2)^p = 0$. This is not directly feasible using just $\varphi$. We thus note that new kinds of interacting quantum dynamics can be defined using a multi-time wave function.

An interesting class of such dynamics has recently been suggested in \cite{26} and has been subsequently analyzed rigorously in \cite{29, 30}: multi-time integral equations. But why study integral equations instead of PDEs? To answer this question, note that the initial value problem $\varphi(0, x_1, x_2) = \psi_0(x_1, x_2)$ of the single-time Schrödinger equation (1) can equivalently be formulated as the following integral equation:

$$\varphi(t, x_1, x_2) = \varphi^{\text{free}}(t, x_1, x_2) + \int_0^\infty dt' \int d^3x_1' d^3x_2' \gamma_1^0 s_1^0 \gamma_2^0 s_2^0 \Gamma(t - t', x_1 - x_1') \Gamma_2(t - t', x_2 - x_2') \times V(t', x_1', x_2') \varphi(t', x_1', x_2'),$$  \hspace{1cm} (5)

where $\varphi^{\text{free}}$ is the solution of the same initial value problem of the free equation (1) with $V = 0$ and $S_k^{\text{ret}}$ is the retarded Green’s function of the $k$-th Dirac operator.

Now, (5) possesses a straightforward manifestly covariant generalization in terms of a multi-time wave function, namely:

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int d^4x_1' d^4x_2' S_1(x_1 - x_1') S_2(x_2 - x_2') K(x_1', x_2') \psi(x_1', x_2'),  \hspace{1cm} (6)$$

define a Lorentz invariant dynamics for toy models in $1+1$ and $1+2$ spacetime dimensions (see e.g. \cite{2–4}). In $1+3$ dimensions, it has been an open problem to prove the existence of the dynamics for any interacting and completely relativistic model.

In this paper, we pursue a new approach to defining interacting dynamics, neither via potentials nor via second quantized fields, but rather through direct interactions with time delay, and prove the existence of dynamics for the simple case of two Dirac particles in $1+3$ dimensions. The key innovation is to make use of multi-time wave functions. This concept goes back to Dirac \cite{5}, played an important role in the works of Tomonaga \cite{6} and Schwinger \cite{7}, but has recently been suggested in \cite{26} and has been subsequently analyzed rigorously in \cite{29, 30}: an overview can be found in \cite{25}. For two Dirac particles in Minkowski spacetime $\mathbb{M}$, a multi-time wave function is a map

$$\psi : \mathbb{M} \times \mathbb{M} \to \mathbb{C}^4 \otimes \mathbb{C}^4 \cong \mathbb{C}^{16}, \quad (x_1, x_2) \mapsto \psi(x_1, x_2).$$  \hspace{1cm} (2)

$\psi$ can be considered a generalization of the single-time wave function $\varphi$ in the Schrödinger picture, as in Eq. (1). The relation of $\psi$ to $\varphi$ is straightforwardly given by

$$\varphi(t, x_1, x_2) = \psi((t, x_1), (t, x_2)).$$  \hspace{1cm} (3)

Contrary to the single-time wave function $\varphi$ (which refers to a frame), $\psi$ is a manifestly covariant object. Under a Poincaré transformation $(a, \Lambda)$, $\psi$ transforms as

$$\psi'(x_1, x_2) = S[\Lambda] \otimes S[\Lambda] \psi(\Lambda^{-1}(x_1 - a), \Lambda^{-1}(x_2 - a)),  \hspace{1cm} (4)$$

where $S[\Lambda]$ are the matrices appearing in the spinor representation of the Lorentz group.

For the present purposes, it is crucial that $\psi$ is defined on general space-time configurations $(x_1, x_2) \in \mathbb{M} \times \mathbb{M}$, not only on equal-time configurations as $\varphi$. By relating configurations $(x_1, x_2)$ with different time coordinates $x_1^0 \neq x_2^0$, one can express interactions with a time delay. It has been pointed out in \cite{26} that in this way, direct relativistic interactions (unmediated by fields) can be expressed at the quantum level. In particular, it becomes possible to formulate a quantum analog of direct interactions along light cones, such as in the Wheeler-Feynman formulation of classical electrodynamics \cite{27, 28}, using values of $\psi(x_1, x_2)$ with $(x_1 - x_2)_p(x_1 - x_2)^p = 0$. This is not directly feasible using just $\varphi$. We thus note that new kinds of interacting quantum dynamics can be defined using a multi-time wave function.

An interesting class of such dynamics has recently been suggested in \cite{26} and has been subsequently analyzed rigorously in \cite{29, 30}: multi-time integral equations. But why study integral equations instead of PDEs? To answer this question, note that the initial value problem $\varphi(0, x_1, x_2) = \psi_0(x_1, x_2)$ of the single-time Schrödinger equation (1) can equivalently be formulated as the following integral equation:

$$\varphi(t, x_1, x_2) = \varphi^{\text{free}}(t, x_1, x_2) + \int_0^\infty dt' \int d^3x_1' d^3x_2' \gamma_1^0 s_1^0 \gamma_2^0 s_2^0 \Gamma(t - t', x_1 - x_1') \Gamma_2(t - t', x_2 - x_2') \times V(t', x_1', x_2') \varphi(t', x_1', x_2'),$$  \hspace{1cm} (5)

where $\varphi^{\text{free}}$ is the solution of the same initial value problem of the free equation (1) with $V = 0$ and $S_k^{\text{ret}}$ is the retarded Green’s function of the $k$-th Dirac operator.

Now, (5) possesses a straightforward manifestly covariant generalization in terms of a multi-time wave function, namely:

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int d^4x_1' d^4x_2' S_1(x_1 - x_1') S_2(x_2 - x_2') K(x_1', x_2') \psi(x_1', x_2'),  \hspace{1cm} (6)$$
where $\psi^{\text{free}}$ is a solution of the equations $D_1 \psi^{\text{free}} = 0$, $D_2 \psi^{\text{free}} = 0$, $D_k = (i\gamma^\mu \partial_\mu - m_k)$ and $S_1, S_2$ are (retarded or other) Green’s functions of $D_1, D_2$, respectively. $K(x_1, x_2)$ denotes the so-called interaction kernel, a Poincaré invariant function (or distribution) which generalizes the potential in Eq. (5). It has been demonstrated in [26] that for $K(x_1, x_2) \propto \delta((x_1 - x_2)_\mu(x_1 - x_2)^\mu)$, the Dirac delta distribution along the light cone, one re-obtains (1) with $V(t, x_1, x_2) \propto \frac{1}{|x_1 - x_2|}$ if one neglects the time delay of the interaction. Thus, (6) constitutes a natural generalization of (5).

Further support for considering the integral equation (6) comes from the fact that the Bethe-Salpeter (BS) equation of QFT [31], which is usually considered an effective equation for a bound state, has a similar form as (6). That being said, there are also significant physical and mathematical differences between the two equations (see [26, sec. 3.3]).

**Previous results.** To the best of our knowledge, the first results about the existence and uniqueness of dynamics for Eq. (6) have been obtained in [29], for the case of a Minkowski half-space and Klein-Gordon (KG) particles. A "Minkowski half-space" means to use $\frac{1}{2}M \times \frac{1}{2}M$ with $\frac{1}{2}M = [0, \infty) \times \mathbb{R}^3$, i.e. Minkowski spacetime cut off before $t = 0$, as the domain of integration in (6). The KG case refers to replacing $S_1, S_2$ with (retarded) Green’s functions of the KG equation and $\psi^{\text{free}}$ with a solution of $\left(\Box_k + m_k^2\right)\psi^{\text{free}} = 0$, $k = 1, 2$. The main result in [29] was to show that for every $\psi^{\text{free}}$ which is $L^2$ in the spatial directions and $L^\infty$ in the time directions there is a unique solution $\psi$ with the same properties. In addition, at $t_1 = t_2 = 0$, $\psi^{\text{free}}$ and $\psi$ agree so that one actually has a Cauchy problem at the initial time. In order to obtain that result, the interaction kernel was assumed to be either bounded or to just have a $1/|x_1 - x_2|$ singularity. In 1+3 dimensions, only the massless case was treated. The proof was based on exploiting a Volterra property which appears for retarded Green’s functions and $\frac{1}{2}M$, i.e. the time integrations in (6) reach only from 0 to $x_1^0$ or $x_2^0$ (given by the time arguments of $\psi$ on the left hand side). This allowed an effective iteration scheme for Eq. (6), leading to a global existence and uniqueness result for a formidable-looking non-Markovian (history dependent) type of dynamics.

The cutoff of spacetime at $t = 0$ was introduced in [29] to obtain the Volterra property. While such a cutoff destroys Lorentz invariance, there could be physical justification for a beginning in time which is compatible with relativity. Such a justification has been provided in [30]. There, the integral equation was extended to curved spacetimes and analyzed in more detail for certain spacetimes which feature a Big Bang singularity, Friedman-Robertson-Lemaître-Walker (FLRW) spacetimes. The Big Bang then provides a natural cutoff in the cosmological time. In this way, the existence of certain classes of fully covariant dynamics for massless KG particles was demonstrated.

**Goal of the paper.** Here we would like to extend the previous results to the case of Dirac instead of KG particles. This is desirable as the Dirac equation describes actual elementary particles (fermions) while the KG equation is usually considered only a toy equation as its currents do not have the right properties to play the role of a probability current. Mathematically, the Dirac case is more challenging than the KG case as contrary to the latter, the Dirac Green’s functions contain distributional derivatives. A Green’s function of the Dirac equation is given by acting with the adjoint Dirac operator $\overline{D} = (-i\gamma^\mu \partial_\mu - m)$ on a Green’s function $G(x)$ of the KG equation, i.e.

$$S(x) = \overline{D}G(x). \quad (7)$$

Consequently, one has to define the integral operator in (6) on a function space where one can take certain weak derivatives. In contrast to most of non-relativistic physics, this also
concerns the time derivatives here. The choice of function space can be a tricky issue, as the convergence of an iteration scheme (our strategy of proof) requires the integral operator to preserve the regularity, so that the regularity needs to harmonize with the structure of the integral equation (see Sec. 2.2).

Further motivation.

1. It is quite challenging to set up an interacting dynamics for multi-time wave functions. The issue here is not only Lorentz invariance but rather the mere compatibility of the time evolutions in the various time coordinates. A no-go theorem [14, 20] for example rules out interaction potentials (which could be Poincaré invariant functions in the multi-time approach). Thus, interaction is more difficult to achieve for multi-time than for single-time wave functions. So far, the only rigorous, interacting and Lorentz invariant multi-time models for Dirac particles have been constructed in 1+1 spacetime dimensions [17, 18] (see, however, [11–13] for non-rigorous Lorentz invariant models in 1+3 dimensions and [24, chap. 3] for a not fully Lorentz invariant but rigorous model in 1+3 dimensions). Considering these difficulties, the multi-time aspect of our model is interesting in its own right.

2. Eq. (6) defines, in the case of retarded Green’s functions, a new class of Volterra-type equations which may be interesting also for researchers specializing in integral equations. It provides a reason why a multi-dimensional Volterra-type equation would be relevant for physics, and shows which properties to expect for applications.

Overview. The paper is structured as follows. In Sec. 2, we specify the integral equation (6) in detail. The difficulties with understanding the distributional derivatives are discussed and a suitable function space is identified. Sec. 3 contains our main results. In Sec. 3.1, we formulate an existence and uniqueness theorem (Thm. 3.4) for Eq. (6) on \( \mathbb{R}^+ \times \mathbb{R}^3 \). It is shown that the relevant initial data are equivalent to Cauchy data at \( t = 0 \). In Sec. 3.2, we provide a physical justification for the cutoff at \( t = 0 \) by extending the results to a FLRW spacetime. In the massless case, we show that an existence and uniqueness theorem can be obtained from the one for \( \mathbb{R}^+ \times \mathbb{R}^3 \) via conformal invariance. The result, Thm. 3.5, covers a fully relativistic interacting dynamics in 1+3 spacetime dimensions. The proofs are carried out in Sec. 4. Sec. 5 contains a discussion and an outlook on future research.

2 Setting of the problem

2.1 Definition of the integral operator on test functions

In this section, we show how the integral operator in (6) can be defined rigorously on test functions. We consider the integral equation (6) on the Minkowski half space \( \mathbb{R}^+ \times \mathbb{R}^3 \) equipped with the metric \( g = \text{diag}(1, -1, -1, -1) \). We focus on retarded Green’s functions of the Dirac equation, \( S_{\text{ret}}(x) = DG_{\text{ret}}(x) \) where \( G_{\text{ret}}(x) \) is the retarded Green’s function of the KG equation. Explicitly,

\[
G_{\text{ret}}^x(x) = \frac{1}{4\pi} \frac{\delta(x^0 - |x|)}{|x|} - \frac{m}{4\pi} H(x^0 - |x|) \frac{J_1(m \sqrt{x^2})}{\sqrt{x^2}}
\]  

(8)

where \( H \) denotes the Heaviside function, \( J_1 \) a Bessel function and \( x^2 = (x^0)^2 - |x|^2 \).

In order to define the meaning of the Green’s functions as distributions, we introduce a suitable space of test functions. Let

\[
\mathcal{S}_c = \mathcal{S}([0, \infty), \mathbb{C}) \otimes C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)
\]  

(9)
denote the space of smooth functions which are compactly supported in the spatial variables and rapidly decreasing in time.\(^1\) We shall use the following space of test functions:

\[
\mathcal{D}_x = (\mathcal{S}_x)^\otimes 2.
\]  

(10)

For a smooth interaction kernel \(K\) and a test function \(\psi \in \mathcal{D}_x\), we then understand (6) by formally integrating by parts so that all partial derivatives act on \(K\psi\):

\[
\psi(x_1, x_2) = \psi^\text{free}(x_1, x_2) + \int_{\mathbb{R}^3} d^3x_1' \int_{\mathbb{R}^3} d^3x_2' G^\text{ret}_1(x_1 - x_1')G^\text{ret}_2(x_2 - x_2')[D_1D_2(K\psi)](x_1', x_2') + \text{boundary terms},
\]  

(11)

where \(D_k = (i\gamma_k \partial_{x_k} - m_k)\), \(k = 1, 2\). The boundary terms result from the fact that \(\psi(x_1, x_2) \neq 0\) for \(x_1' = 0\) or \(x_2' = 0\) and are given by:

\[
\int_{\mathbb{R}^3} d^3x_1' \int_{\mathbb{R}^3} d^3x_2' i\gamma_1 G^\text{ret}_1(x_1 - x_1')i\gamma_2 G^\text{ret}_2(x_2 - x_2')(K\psi)(x_1', x_2')|_{x_1' = 0, x_2' = 0}
\]

\[
+ \int_{\mathbb{R}^3} d^3x_1' \int_{\mathbb{R}^3} d^3x_2' i\gamma_1 G^\text{ret}_1(x_1 - x_1')G^\text{ret}_2(x_2 - x_2')D_2(K\psi)(x_1', x_2')|_{x_1' = 0}
\]

\[
+ \int_{\mathbb{R}^3} d^3x_1' \int_{\mathbb{R}^3} d^3x_2' G^\text{ret}_1(x_1 - x_1')i\gamma_2 G^\text{ret}_2(x_2 - x_2')D_1(K\psi)(x_1', x_2')|_{x_2' = 0}.
\]  

(12)

Now, \(G^\text{ret}_k\) still contains the \(\delta\)-distribution. We use the latter to cancel the integrals over \(x_k', k = 1, 2\) in (11) in the following manner. Let \(f \in \mathcal{S}_x\). Then:

\[
\frac{1}{4\pi} \int_{\mathbb{R}^3} d^3x' \frac{\delta(x_0' - x_0 - |x - x'|)}{|x - x'|} f(x') = \frac{1}{4\pi} \int_{B_{x_0}(x)} d^3x' \frac{1}{|x - x'|} f(x')|_{x_0' = x_0 - |x - x'|}
\]

\[
= \frac{1}{4\pi} \int_{B_{x_0}(0)} d^3y \frac{1}{|y|} f(x + y)|_{y_0 = -|y|}.
\]  

(13)

Moreover,

\[
\frac{m}{4\pi} \int_{\mathbb{R}^3} d^4x' H(x_0' - x_0' - |x - x'|) \frac{J_1(m\sqrt{(x - x')^2})}{\sqrt{(x - x')^2}} f(x')
\]

\[
= \frac{m}{4\pi} \int_{[-x_0, x] \times \mathbb{R}^3} d^4y H(-y_0 - |y|) \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x + y)
\]

\[
= \frac{m}{4\pi} \int_{-x_0}^{x_0} dy \int_{B_{y_0}(0)} d^3y_k \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x + y).
\]  

(14)

For the boundary terms, we similarly use

\[
\frac{i\gamma_0}{4\pi} \int_{\mathbb{R}^3} d^3x' \frac{\delta(x_0' - x_0 - |x - x'|)}{|x - x'|} f(0, x') = \frac{i\gamma_0}{4\pi} \int_{\partial B_{x_0}(0)} d\sigma(y) \frac{f(0, x + y)}{x_0}
\]  

(15)

as well as

\[
\frac{i\gamma_0 m}{4\pi} \int_{\mathbb{R}^3} d^3x' H(x_0' - x_0' - |x - x'|) \frac{J_1(m\sqrt{(x - x')^2})}{\sqrt{(x - x')^2}} f(x')|_{x_0' = 0}
\]

\[
= \frac{i\gamma_0 m}{4\pi} \int_{B_{x_0}(0)} d^3y \frac{J_1(m\sqrt{(x_0')^2 - y^2})}{\sqrt{(x_0')^2 - y^2}} f(0, x + y).
\]  

(16)

\(^1\)\(\mathcal{S}\) denotes the space of Schwartz functions.
This yields the form of the integral equation which shall be the basis of our investigation:

$$\psi(x_1, x_2) = \psi_{\text{free}}(x_1, x_2) + (A\psi)(x_1, x_2).$$  \hfill (17)

The operator $A$ is first defined on test functions $\psi \in \mathcal{D}_\infty$ as

$$A\psi = \left( \bigotimes_{i=1,2} (A_i D_i + A_2(m_i) D_i + B_1 + B_2(m_i)) \right) K\psi$$  \hfill (18)

with $A_1, A_2(m), B_1, B_2(m): \mathcal{D}_\infty \to C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ and

$$\begin{align*}
(A_1 f)(x) &= \frac{1}{4\pi} \int_{B_{x_0}(0)} d^3 y \frac{1}{|y|} f(x+y) |y^0=|y^1|, \\
(A_2(m) f)(x) &= -\frac{m}{4\pi} \int_{-x^0}^0 dy^0 \int_{B_{|y^0|}(0)} d^3 y \sqrt{\frac{J_1(m\sqrt{y^2})}{y^2}} f(x+y), \\
(B_1 f)(x) &= i\gamma^0 \frac{1}{4\pi} \int_{\partial B_{x_0}(0)} d\sigma(y) \frac{f(0, x+y)}{x^0}, \\
(B_2(m) f)(x) &= -i\gamma^0 \frac{m}{4\pi} \int_{B_{x_0}(0)} d^3 y \sqrt{\frac{J_1(m\sqrt{(x^0)^2-\gamma^2})}{(x^0)^2-\gamma^2}} f(0, x+y). \hfill (22)
\end{align*}$$

We now turn to the question of a suitable Banach space for Eq. (17).

### 2.2 Choice of Banach space

In order to prove the existence and uniqueness of solutions, we would like to demonstrate the convergence of an iteration scheme. This requires to extend the integral operator $A$ to an operator on a suitable Banach space $\mathcal{B}$. The behavior of solutions $\psi_{\text{free}}(x_1, x_2)$ of the free Dirac equation in each spacetime variable $x_1, x_2$ suggests to choose the Bochner space (for some arbitrary final time $T > 0$)

$$\mathcal{B}_0 = L^\infty\left( [0, T]^2(x_1, x_2), L^2(\mathbb{R}^6, \mathbb{C}^{16})(x_1, x_2) \right)$$  \hfill (23)

with norm

$$\|\psi\|_{\mathcal{B}_0} = \text{ess sup}_{x_1^0, x_2^0 \in [0, T]} \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}. \hfill (24)$$

The reason for choosing $\mathcal{B}_0$ is that the spatial norm $\|\psi_{\text{free}}(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}$ of a solution of the free Dirac equations is constant in the two time variables $x_1^0, x_2^0$. A very similar space as $\mathcal{B}_0$ has been used for analyzing (6) in the KG case [29].

However, as (18) involves the Dirac operators $D_1, D_2$, $\mathcal{B}_0$ is not sufficient for our problem. An appropriate Banach space $\mathcal{B}$ must allow us to take at least weak derivatives of $\psi$. The choice of $\mathcal{B}$ is a delicate matter. One can easily go wrong with demanding too much regularity, as we shall illustrate now.

**Possible difficulties with the choice of function space.** The problem can best be illustrated with an example which is structurally related to (6) but otherwise simpler. Consider the equation

$$f(t, x) = f_{\text{free}}(t, z) + \int_0^t dz' K(z, z') \partial_t f(t, z'), \hfill (25)$$
where \( f^{\text{free}}, f, K : \mathbb{R}^2 \to \mathbb{C} \) and \( f^{\text{free}} \) is given. (25) is inspired by the term \( A_1D_1 \) in (18).

We would like to set up an iteration scheme for (25). As we cannot integrate by parts to shift the \( t \)-derivative to \( K \), we must demand at least weak differentiability of \( f \) with respect to \( t \). This suggests using a Banach space such as \( \mathcal{B} = H^1(\mathbb{R}^2) \). To prove that the integral operator in (25) maps \( \mathcal{B} \) to \( \mathcal{B} \) (the first step in every iteration scheme), we then have to estimate the \( L^2 \)-norm of

\[
\partial_1 \int_0^t dz' K(z, z') \partial_1 f(t, z') = K(t, t) (\partial_1 f)(t, t) + \int_0^t dz' K(z, z') \partial_1^2 f(t, z').
\]

(26)

This expression, however, contains \( \partial_1^2 f \). For this to make sense, we must be allowed to take the second weak time derivative of \( f \). This, in turn, requires to choose a different Sobolev space, such as \( H^2(\mathbb{R}^2) \), and to estimate the \( L^2 \)-norm of the second time derivative of the integral operator acting on \( f \) which involves \( \partial_1^2 f \), and so on. One is thus led to a Sobolev space where all weak \( n \)-th time derivatives have to exist. Such infinite-order Sobolev spaces have, in fact, been investigated in [32]. However, it does not seem realistic to get an iteration to converge on these spaces. We therefore take a different approach.

**A Banach space adapted to our integral equation.** Considering the form of the integral operator \( A \) (18), one can see that it is sufficient that the derivatives \( D_1\psi, D_2\psi \) and \( D_1D_2\psi \) exist in a weak sense. As we want to prove later that \( A \) maps the Banach space to itself, among other things, a suitable norm of \( D_1(A\psi) \). If \( \psi_0 \in \mathcal{B}_0 \) is a test function and \( K \) is smooth, we have

\[
D_1(A\psi)(x_1, x_2) = D_1 \int d^4x' \, d^4x'' S_1(x_1 - x_1', x_2 - x_2') S_2(x_1', x_2') K(x_1, x_2') \psi(x_1', x_2')
\]

\[
= \int d^4x' \, S_2(x_1 - x_1', x_2') K(x_1, x_2') \psi(x_1, x_2')
\]

(27)

where we have used \( D_1S_1(x_1 - x_1') = \delta^{(4)}(x_1 - x_1') \). The crucial point now is that (27) does not contain higher-order derivatives such as \( D_1^2\psi \). The same holds true also for \( D_2(A\psi) \) and \( D_1D_2(A\psi) \). Thus, the problem of the toy example (25) is avoided.

Let

\[
\mathcal{B}_T = C^\infty([0, T], \mathbb{C}) \otimes C^\infty_c(\mathbb{R}^3, \mathbb{C}^4).
\]

(28)

Together with the previous considerations about \( \mathcal{B}_0 \) (23), we are led to define the Banach space \( \mathcal{B}_T \) as the completion of

\[
\mathcal{B}_T = \mathcal{B}_0 \otimes \mathcal{B}_T^\otimes 2,
\]

(29)

with respect to the following Sobolev-type norm:

\[
\|\psi\|^2 = \text{ess sup} \left[ \psi^2(x_1^0, x_2^0) \right].
\]

(30)

Here we use the notation

\[
\left[\psi^2(x_1^0, x_2^0) = \sum_{k=0}^3 \| (D_k\psi)(x_1^0, \ldots, x_2^0) \|^2_{L^2(\mathbb{R}^6, \mathbb{C}^16)}
\]

(31)

with

\[
D_k = \begin{cases} 
1, & k = 0 \\
D_1, & k = 1 \\
D_2, & k = 2 \\
D_1D_2, & k = 3
\end{cases}
\]

(32)

The norm \( \| \cdot \| \) then extends to a norm on \( \mathcal{B}_T \).
Remark. One can see the purpose of integral equation (6) in determining an interacting correction to a solution \( \psi_{\text{free}} \) of the free multi-time Dirac equations \( D_i \psi_{\text{free}} = 0, \ i = 1, 2 \). Therefore, it is important to check that sufficiently many solutions of these free equations lie in \( \mathcal{B}_T \). This is the case, as smooth and compactly supported initial data stay smooth and compactly supported under the free time evolution of the Dirac equation.

Given the definition of \( A \) on \( \mathcal{D}_T \) as in Sec. 2.1, we shall now proceed with showing that \( A \) is bounded on this space. By the linear extension theorem we can then extend it to a bounded linear operator on \( \mathcal{B}_T \). Subsequently, we shall derive suitable estimates to show that \( \psi = \sum_{k=0}^{\infty} A^k \psi_{\text{free}} \) yields the unique solution of \( \psi = \psi_{\text{free}} + A \psi \) in \( \mathcal{B}_T \).

3 Results

3.1 Results for a Minkowski half space

Theorem 3.1 Let \( \psi \in \mathcal{D}_T, \ \hat{\psi}_k = \gamma_k \hat{\nu}_{k, \mu}, \ k = 1, 2 \) and let \( K: \mathbb{R}^8 \rightarrow \mathbb{C} \) be a smooth function with
\[
\| K \| := \sup_{x_1, x_2 \in [0, T] \times \mathbb{R}^3} \max \left\{ |K(x_1, x_2)|, |\hat{\psi}_1 K(x_1, x_2)|, |\hat{\psi}_2 K(x_1, x_2)|, |\hat{\psi}_1 \hat{\psi}_2 K(x_1, x_2)| \right\} < \infty.
\]
Then we have \( A \psi \in \mathcal{D}_T \). Moreover, the following estimate holds.
\[
[A \psi]^2(x_1^0, x_2^0) \leq \| K \|^2 \left( \bigotimes_{i=1,2} (1 + 8A(m_i) + 8B(m_i)) \right) \| \psi \|^2(x_1^0, x_2^0),
\]
where \( A(m) = A_1 + A_2(m) \) and \( B(m) = B_1 + B_2(m) \) with \( A_i, B_i \) as defined in (57).
Let \( \mu = \max \{m_1, m_2\} \). Then, furthermore:
\[
\| A \psi \| \leq \| K \| \left[ 1 + 8 \left( T^2 + \frac{1}{3} T^4 + \frac{\mu^4 T^6}{2^2 3^2} + \frac{\mu^4 T^8}{2^6 3^2} \right) \right] \| \psi \|.
\]

The proof can be found in Sec. 4.1.
We have thus shown that \( A \) is bounded on \( \mathcal{D}_T \). As \( \mathcal{D}_T \) lies, by definition, dense in \( \mathcal{B}_T \), we obtain the following result.

Corollary 3.2 The operator \( A \) can be extended to a linear operator \( A: \mathcal{B}_T \rightarrow \mathcal{B}_T \) which also satisfies (35).

Banach’s fixed point theorem thus readily yields:

Corollary 3.3 (Short-time existence and uniqueness of solutions.) Let \( \mu = \max \{m_1, m_2\} \) and assume
\[
\| K \| \left[ 1 + 8 \left( T^2 + \frac{1}{3} T^4 + \frac{\mu^4 T^6}{2^2 3^2} + \frac{\mu^4 T^8}{2^6 3^2} \right) \right] < 1.
\]
Then for every \( \psi_{\text{free}} \in \mathcal{B}_T \), the equation \( \psi = \psi_{\text{free}} + A \psi \) has a unique solution \( \psi \in \mathcal{B}_T \).
**Remark.** Contrary to Markovian dynamics where one can deduce the existence of solutions for arbitrary times from short-time existence provided the estimates are uniform in time, this is not the case here, as (6) is history dependent. We need to improve on corollary 3.3. Our main result is:

**Theorem 3.4 (Existence and uniqueness of dynamics on a Minkowski half space.)**

Let $T > 0$ be arbitrary and let $\|K\| < 1$. Then for every $\psi^{\text{free}} \in \mathcal{B}_T$, the equation $\psi = \psi^{\text{free}} + A\psi$ possesses a unique solution $\psi \in \mathcal{B}_T$.

Let furthermore $\mu = \max\{m_1, m_2\}$. Then there is a $T$-independent constant $C > 0$, given explicitly in Eq. (104), such that

$$\|\psi\| \leq C \|\psi^{\text{free}}\| \exp\left((1 + \mu^4)T^2 + 64T^4 + \frac{\mu^4}{3}T^8\right).$$

The proof is given in Sec. 4.2. It is based on iterating the estimate (34) to prove that the series $\sum_{k=0}^{\infty} A^k\psi^{\text{free}}$ converges in $\mathcal{B}_T$.

**Remarks.**

1. The main condition in Thm. 3.4 is $\|K\| < 1$. This means that the interaction must not be too strong (in a suitable sense). A condition of that kind is to be expected solely because of the contribution $\|(D_1D_2(A\psi))(x_0', x_1', x_0', x_2')\|_{L^2} = \|K\psi(x_0', x_1', x_0', x_2')\|_{L^2}$ to $[A\psi](x_1', x_2')$. Taking our iteration scheme for granted, we therefore think that one cannot avoid a condition on the interaction strength. Note that conditions on the interaction strength also occur at other places in quantum theory (albeit in a different sense). For example, the Dirac Hamiltonian plus a Coulomb potential is only self-adjoint if the prefactor of the latter is smaller than a certain value.

2. As $T > 0$ is arbitrary, the solution exists and is unique for all times. Moreover, the estimate (99) shows that $\left(\prod_{i=1,2} \exp\left(-\left(1 + m_i^4\right)T^2 - 64T^4 - \frac{m_i^4}{3}T^8\right)\right) \psi$ lies in $\mathcal{B}_\infty$, the completion of $\mathcal{B}_\infty$ with respect to the norm (30).

3. **Cauchy problem.** Thm. 3.4 shows that $\psi^{\text{free}}$ uniquely determines the solution $\psi$. However, specifying a whole function in $\mathcal{B}_T$ amounts to a lot of data. In case $\psi^{\text{free}}$ is a solution of the free multi-time Dirac equations $D_1\psi^{\text{free}} = 0 = D_2\psi^{\text{free}}$ much less data are needed. $\psi^{\text{free}}$ is then determined uniquely by Cauchy data, and hence $\psi$ is as well. Furthermore, if $\psi^{\text{free}}$ is differentiable, (6) yields

$$\psi(0, x_1, 0, x_2) = \psi^{\text{free}}(0, x_1, 0, x_2).$$

Thus, Cauchy data for $\psi^{\text{free}}$ at $x_0' = x_2' = 0$ are also Cauchy data for $\psi$. The procedure works for arbitrary Cauchy data which are appropriate for the free multi-time Dirac equations. Note, however, that a Cauchy problem for $\psi$ for times $x_0' = t_0 = x_2'$ with $t_0 > 0$ is not possible. The reason is that $\psi(t_0, x_1, t_0, x_2) \neq \psi^{\text{free}}(t_0, x_1, t_0, x_2)$ in general (and contrary to (38) the point-wise evaluation may not make sense for $\psi$).

**3.2 Results for a FLRW universe with a Big Bang singularity**

In this section we show that a Big Bang singularity provides a natural and covariant justification for the cutoff at $t = 0$. As this justification is our main goal, we make the point at the example of a particular class of Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes and do not strive to treat more general spacetimes here. The reason for studying
these FLRW spacetimes is that they are conformally equivalent to \( \frac{1}{2} \mathcal{M} \) \([33]\). Together with the conformal invariance of the massless Dirac operator this allows for an efficient method of calculating the Green’s functions which occur in the curved spacetime analog of the integral equation (6). By doing this, we show that the existence and uniqueness result on these spacetimes can be reduced to Thm. 3.4.

As shown in [30], Eq. (6) possesses a natural generalization to curved spacetimes \( \mathcal{M} \),

\[
\psi(x_1, x_2) = \psi^\text{free}(x_1, x_2) + \int dV(x'_1) \int dV(x'_2) \, G_1(x_1, x'_1) G_2(x_2, x'_2) K(x'_1, x'_2) \psi(x'_1, x'_2).
\]  

(39)

Here, \( dV(x) \) is the spacetime volume element, \( S_i \) are (retarded) Green’s functions of the respective free wave equation, i.e.

\[
DG(x, x') = [-g(x)]^{-1/2} \delta^{(4)}(x, x'),
\]  

(40)

where \( g(x) \) is the metric determinant, \( D \) the covariant Dirac operator on \( \mathcal{M} \), and \( \psi \) a section of the tensor spinor bundle over \( \mathcal{M} \times \mathcal{M} \).

In order to explicitly formulate (39), we need to know the detailed form of \( S^\text{ret} \). Note that results for general classes of spacetimes showing that \( S^\text{ret} \) is a bounded operator on a suitable function space are not sufficient to obtain a strong (global in time) existence and uniqueness result. We therefore focus on the case of a flat FLRW universe where it is easy to determine the Green’s functions explicitly. In that case, the metric is given by

\[
ds^2 = a^2(\eta) \left[ d\eta^2 - d\mathbf{x}^2 \right]
\]  

(41)

where \( \eta \) is cosmological time and \( a(\eta) \) denotes the so-called scale factor. The coordinate ranges are given by \( \eta \in [0, \infty) \) and \( \mathbf{x} \in \mathbb{R}^3 \). For a FLRW universe with a Big Bang singularity, \( a(\eta) \) is a continuous, monotonically increasing function of \( \eta \) with \( a(\eta) = 0 \), corresponding to the Big Bang singularity. The spacetime volume element reads

\[
dV(x) = a^4(\eta) \, d\eta \, d^3\mathbf{x}.
\]  

(42)

The crucial point now is that according to (41) the spacetime is globally conformally equivalent to \( \frac{1}{2} \mathcal{M} \), with conformal factor

\[
\Omega(x) = a(\eta).
\]  

(43)

In addition, for \( m = 0 \), the Dirac equation is known to be conformally invariant (see e.g. [34]). More accurately, consider two spacetimes \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) with metrics

\[
\tilde{g}_{ab} = \Omega^2 g_{ab}.
\]  

(44)

Then the massless Dirac operator \( D \) on \( \mathcal{M} \) is related to the massless Dirac operator \( \tilde{D} \) on \( \tilde{\mathcal{M}} \) by (see [35]):

\[
\tilde{D} = \Omega^{-5/2} D \Omega^{3/2}.
\]  

(45)

This implies the following transformation behavior of the Green’s functions:

\[
\tilde{G}(x, x') = \Omega^{-3/2}(x) \Omega^{-3/2}(x') G(x, x').
\]  

(46)

One can verify this easily using (45) and the definition of Green’s functions on curved spacetimes (40).
Denoting the Green’s functions of the Dirac operator on Minkowski spacetime by $G(x, x') = S(x - x')$ and using coordinates $\eta, x$ we thus obtain the Green’s functions $\tilde{G}$ on flat FLRW spacetimes as:

$$\tilde{G}(\eta, x; \eta', x') = a^{-3/2}(\eta)a^{-3/2}(\eta') S(\eta - \eta', x - x').$$

(47)

With this result, we can write out in detail the multi-time integral equation (39) for massless Dirac particles on flat FLRW spacetimes (using retarded Green’s functions):

$$\psi(\eta_1, x_1, \eta_2, x_2) = \psi^{\text{free}}(\eta_1, x_1, \eta_2, x_2) + a^{-3/2}(\eta_1)a^{-3/2}(\eta_2) \int_0^\infty d\eta_1' \int_0^\infty d\eta_2' \int_0^\infty d^3 x_1' \int_0^\infty d^3 x_2'$$

$$\times a_5^{5/2}(\eta_1')a_5^{5/2}(\eta_2') S_{1}^{\text{ret}}(\eta_1 - \eta_1', x_1 - x_1') S_{2}^{\text{ret}}(\eta_2 - \eta_2', x_2 - x_2')(K\psi)(\eta_1', x_1', \eta_2', x_2').$$

(48)

Note that we can regard $\psi$ as a map $\psi : (\frac{1}{2}M)^2 \to \mathbb{C}^1$ as the coordinates $\eta, x$ cover the flat FLRW spacetime manifold globally.

It seems reasonable to allow for a singularity of the interaction kernel, i.e.

$$K(\eta_1, x_1, \eta_2, x_2) = a^{-\alpha}(\eta_1)a^{-\alpha}(\eta_1) \tilde{K}(\eta_1, x_1, \eta_2, x_2).$$

(49)

Here, $\alpha \geq 0$. The singular behavior is motivated by that of the Green’s functions of the conformal wave equation$^2$. Recall from the introduction that the most natural interaction kernel on $\frac{1}{2}M$ would be $K(x_1, x_2)\delta((x_1 - x_2)_\mu(x_1 - x_2)^\mu)$ which is a Green’s function of the wave equation – a concept that can be generalized to curved spacetimes using the conformal wave equation. Now, under conformal transformations, Green’s functions of that equation transform as [36]

$$\tilde{G}(x, x') = \Omega^{-1}(x) \Omega^{-1}(x') G(x, x'),$$

(50)

which corresponds to $\alpha = 1$ in (49).

Considering (49), our integral equation becomes:

$$\psi(\eta_1, x_1, \eta_2, x_2) = \psi^{\text{free}}(\eta_1, x_1, \eta_2, x_2) + a^{-3/2}(\eta_1)a^{-3/2}(\eta_2) \int_0^\infty d\eta_1' \int_0^\infty d^3 x_1' \int_0^\infty d\eta_2' \int_0^\infty d^3 x_2'$$

$$\times a_5^{5/2}(\eta_1')a_5^{5/2}(\eta_2') S_{1}^{\text{ret}}(\eta_1 - \eta_1', x_1 - x_1') S_{2}^{\text{ret}}(\eta_2 - \eta_2', x_2 - x_2')(\tilde{K}\psi)(\eta_1', x_1', \eta_2', x_2').$$

(51)

Apart from the scale factors which produce a certain singularity of $\psi$ for $\eta_1, \eta_2 \to 0$, this integral equation has the form of (6) on $\frac{1}{2}M$. Indeed, we can use the transformation

$$\chi(\eta_1, x_1, \eta_2, x_2) = a^{3/2}(\eta_1)a^{3/2}(\eta_2) \psi(\eta_1, x_1, \eta_2, x_2)$$

(52)

to transform the two equations into each other. We arrive at the following result.

**Theorem 3.5 (Existence and uniqueness of dynamics on a flat FLRW universe)**

Let $T > 0$, $0 \leq \alpha \leq 1$ and let $a : [0, \infty) \to [0, \infty)$ be a differentiable function with $a(0) = 0$ and $a(\eta) > 0$ for $\eta > 0$. Moreover, assume that $\tilde{K} \in C^1 \left([0, \infty) \times \mathbb{R}^3\right)^2, \mathbb{C}$ with

$$\|a^{1-\alpha}(\eta_1)a^{1-\alpha}(\eta_2) \tilde{K}\| < 1.$$ (53)

Then for every $\psi^{\text{free}}$ with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}} \in \mathcal{B}_T$, (51) has a unique solution $\psi$ with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi \in \mathcal{B}_T$.

---

$^2$The conformal wave equation reads $(\Box - R/6)\phi = 0$ where $\Box$ is the d’Alembertian and $R$ the Ricci scalar of the respective spacetime.
Proof: Multiplying (51) with $a^{3/2}(\eta_1)a^{3/2}(\eta_2)$ and using the relation yields
\[
\chi(\eta_1, x_1, \eta_2, x_2) = \chi_{\text{free}}(\eta_1, x_1, \eta_2, x_2) + \int_0^\infty d\eta_1' \int d^3x_1 \int_0^\infty d\eta_2' \ a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2') \times \delta_{\text{ret}}^a(\eta_1 - \eta_1', x_1 - x_1')\delta_{\text{ret}}^a(\eta_2 - \eta_2', x_2 - x_2')(\tilde{K}_\alpha)(\eta_1', x_1', \eta_2', x_2').
\] (54)
This equation has the form of (6) on $}\frac{1}{2}\mathbb{M}$ with $K$ replaced by $a^{1-\alpha}(\eta_1')a^{1-\alpha}(\eta_2')\tilde{K}$. Thus, using the same distributional understanding of the Green’s functions as before, Thm. 3.4 yields the claim.

Remarks.
1. Both $\psi^{\text{free}}$ and $\psi$ have a singularity proportional to $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ for $\eta_1, \eta_2 \to 0$.
2. For $\alpha < 1$, $\tilde{K}$ has to compensate the singularities caused by $a^{-3/2}(\eta_1)a^{-3/2}(\eta_2)$ in order for (53) to hold. In the most natural case $\alpha = 1$, however, $\tilde{K}$ only needs to satisfy $\|\tilde{K}\| < 1$, i.e. the same condition as for $K$ in Thm. 3.4.
3. Like Thm. 3.4, also Thm. 3.5 implies the existence of a unique solution $\psi$ for all $\eta_1, \eta_2 \geq 0$. Here, $(\prod_{i=1,2}a^{3/2}(\eta_i) \exp\left(-(1 + m^4_i)\eta_i^2 - 64\eta_i^4 - \frac{m^4_i}{3}\eta_i^8\right))$ $\psi$ lies in $\mathcal{B}_\infty$.
4. Let $\chi^{\text{free}} = a^{3/2}(\eta_1)a^{3/2}(\eta_2)\psi^{\text{free}}$ be differentiable and let $\chi$ be the unique solution of (54). Then, by (54), we have:
\[
\chi^{\text{free}}(0, x_1, 0, x_2) = \chi(0, x_1, 0, x_2),
\] (55)
i.e. $\chi$ satisfies a Cauchy problem "at the Big Bang".
5. Remarkably, Thm. 3.5 covers a class of manifestly covariant, interacting integral equations in 1+3 dimensions. Then the interaction kernel $\tilde{K}$ has to be covariant as well. A class of examples (see also [30]) is given by $\alpha = 1$ and
\[
\tilde{K}(x_1, x_2) = \begin{cases} 
  f(d(x_1, x_2)) & \text{if } x_1, x_2 \text{ are time-like related} \\
  0 & \text{else},
\end{cases}
\] (56)
where $d(x_1, x_2) = ((\eta_1 - \eta_2) - |x_1 - x_2|)\int_0^1 d\tau (\tau \eta_1 + (1 - \tau)\eta_2)$ denotes the time-like distance of the spacetime points $x_1 = (\eta_1, x_1)$ and $x_2 = (\eta_2, x_2)$, and $f$ is an arbitrary smooth function which leads to $\|\tilde{K}\| < 1$.

4 Proofs

Throughout this section, let $\psi \in \mathcal{D}_T$ and $K : \mathbb{R}^8 \to \mathbb{C}$ be a smooth function.

4.1 Proof of Theorem 3.1

We begin with some lemmas which are useful for estimating $\|A\psi\|^2(x_1^0, x_2^0)$.

Lemma 4.1 Let the following operators be defined on $C(\mathbb{R})$:
\[
(A_1 f)(t) := t \int_0^t \rho (t - \rho)^2 f(\rho),
\]
\[
(A_2(m) f)(t) := \frac{m^4 t^4}{2^4 3^2} \int_0^t \rho (t - \rho)^3 f(\rho),
\]
\[
(B_1 f)(t) := t^2 f(0),
\]
\[
(B_2(m) f)(t) := \frac{m^4 t^6}{2^2 3^2} f(0).
\] (57)
Then we have for all $f \in C(\mathbb{R})$:

$$ \|A_1 f(t, \cdot)\|_{L^2}^2 \leq A_1 \|f(t, \cdot)\|_{L^2}^2, \quad (58) $$

$$ \|A_2(m) f(t, \cdot)\|_{L^2}^2 \leq A_2(m) \|f(t, \cdot)\|_{L^2}^2, \quad (59) $$

$$ \|B_1 f(t, \cdot)\|_{L^2}^2 \leq B_1 \|f(t, \cdot)\|_{L^2}^2, \quad (60) $$

$$ \|B_2(m) f(t, \cdot)\|_{L^2}^2 \leq B_2(m) \|f(t, \cdot)\|_{L^2}^2. \quad (61) $$

Here, it is understood that the operators $A_1, A_2$ are applied to the functions defined by the norms which follow them, e.g. $B_2(m) \|f(t, \cdot)\|_{L^2}^2 = \frac{n^4 L^3}{24 \pi^2} \|f(0, \cdot)\|_{L^2}^2$.

**Proof:** We begin with (58):

$$ \|A_1 f(x^0, \cdot)\|_{L^2}^2 = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left| \int_{B_{x^0}(0)} d^3y \frac{1}{|y|} f(x+y)|_{y^0=-|y|} \right|^2 $$

$$ \leq \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left( \int_{B_{x^0}(0)} d^3y \frac{1}{|y|^2} \right) \left( \int_{B_{x^0}(0)} d^3y |f|^2(x+|y|)|_{y^0=-|y|} \right) $$

$$ = \frac{1}{4\pi} \int_{\mathbb{R}^3} d^3x \int_{B_{x^0}(0)} d^3y |f|^2(x^0 - |y|, x+y) $$

$$ = \frac{1}{4\pi} \int_{\mathbb{R}^3} d^3y \int_{\mathbb{R}^3} d^3z |f|^2(x^0 - |y|, z) $$

$$ = \int_0^x \int_0^{x^0} dr \ r^2 |f(x^0 - r, \cdot)|_{L^2}^2 $$

$$ = \int_0^{x^0} d\rho \ (x^0 - \rho)^2 \|f(\rho, \cdot)\|_{L^2}^2 $$

$$ = A_1 \|f(x^0, \cdot)\|_{L^2}^2. \quad (62) $$

Next, we turn to (59). Using $|J_1(x)/x| \leq \frac{1}{x}$, we find:

$$ \|A_2(m) f(x^0, \cdot)\|_{L^2}^2 = \frac{m^4}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left| \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} f(x+y) \right|^2 $$

$$ \leq \frac{m^4}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left( \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y \left| \frac{J_1(m\sqrt{y^2})}{\sqrt{y^2}} \right|^2 \right) \left( \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y |f|^2(x+y) \right) $$

$$ = \frac{m^4}{(4\pi)^2} \frac{\pi(x^0)^4}{12} \int_{\mathbb{R}^3} d^3x \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y |f(x^0 + y^0, \cdot)|_{L^2}^2 $$

$$ = \frac{m^4(x^0)^4}{2^6 \sqrt{3} \pi} \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y f(x^0 + y^0, \cdot)_{L^2}^2 $$

$$ = \frac{m^4(x^0)^4}{2^4 \sqrt{3} \pi} \int_{-x^0}^{0} dy \int_{B_{x^0}(0)} d^3y f(x^0 + y^0, \cdot)_{L^2}^2 $$

$$ = \frac{m^4(x^0)^4}{2^4 \sqrt{3} \pi} \int_{-x^0}^{0} dy \ f(x^0, \cdot)_{L^2}^2 $$

$$ = A_2(m) \|f(x^0, \cdot)\|_{L^2}^2. \quad (63) $$
We come to (60). Using that the modulus of the largest eigenvalue of $\gamma^0$ is 1, we obtain:

$$\|B_1 f(x^0, \cdot)\|^2_{L^2} \leq \frac{1}{(4\pi)^2 (x^0)^2} \int_{\mathbb{R}^3} d^3x \left| \int_{\partial B_o(0)} d\sigma(y) |f(0, x + y)|^2 \right.$$  

$$\leq \frac{1}{(4\pi)^2 (x^0)^2} \int_{\mathbb{R}^3} d^3x \left( \int_{\partial B_o(0)} d\sigma(y) \right) \left( \int_{\partial B_o(0)} d\sigma(y) |f|^2(0, x + y) \right)$$  

$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} d^3x \int_{\partial B_o(0)} d\sigma(y) |f(0, \cdot)|^2_{L^2}$$  

$$= (x^0)^2 \|f(0, \cdot)\|^2_{L^2}$$  

$$= B_1 \|f(x^0, \cdot)\|^2_{L^2}. \quad (64)$$

Finally, for (61), we find:

$$\|B_2(m) f(x^0, \cdot)\|^2_{L^2} \leq \frac{m^2}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left| \int_{B_o(0)} d^3y \left| J_1(m \sqrt{(x^0)^2 - |y|^2}) \right| f(0, x + y) \right|^2$$  

$$\leq \frac{m^2}{(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left( \int_{B_o(0)} d^3y \left| J_1(m \sqrt{(x^0)^2 - |y|^2}) \right|^2 \right) \left( \int_{B_o(0)} d^3y |f|^2(0, x + y) \right)$$  

$$= \frac{m^2}{(4\pi)^2} \frac{\pi m^2(x^0)^3}{3} \int_{B_o(0)} d^3y \|f(0, \cdot)\|^2_{L^2}$$  

$$= \frac{m^4(x^0)^3}{24 \pi^3} \int_{B_o(0)} d^3y \|f(0, \cdot)\|^2_{L^2}$$  

$$= \frac{m^4(x^0)^6}{2^2 \pi^2} \|f(0, \cdot)\|^2_{L^2}$$  

$$= B_2(m) \|f(x^0, \cdot)\|^2_{L^2}. \quad (65)$$

\[ \square \]

**Lemma 4.2** Let $A(m) = A_1 + A_2(m)$ and $B(m) = B_1 + B_2(m)$. Then the following estimates hold:

$$\|(A\psi)(x_1^0, x_2^0, \cdot)\|^2_{L^2} \leq 64 \|K\|^2 (A(m_1) + B(m_1)) \otimes (A(m_2) + B(m_2)) \|\psi\|^2(x_1^0, x_2^0), \quad (66)$$

$$\|(D_1 A\psi)(x_1^0, x_2^0, \cdot)\|^2_{L^2} \leq 8 \|K\|^2 \mathbb{1} \otimes (A(m_2) + B(m_2)) \|\psi\|^2(x_1^0, x_2^0), \quad (67)$$

$$\|(D_2 A\psi)(x_1^0, x_2^0, \cdot)\|^2_{L^2} \leq 8 \|K\|^2 (A(m_1) + B(m_1)) \otimes \mathbb{1} \|\psi\|^2(x_1^0, x_2^0), \quad (68)$$

$$\|(D_1 D_2 A\psi)(x_1^0, x_2^0, \cdot)\|^2_{L^2} \leq \|K\|^2 \|\psi\|^2(x_1^0, x_2^0), \quad (69)$$

where $[\psi]^2(x_1^0, x_2^0)$ is regarded as a function to which the operators in front of it are applied.

**Proof:** We start with (66). Recalling (18), the expression $A\psi$ contains terms such as $D_1 D_2 (K\psi)$ and $D_i (K\psi)$, $i = 1, 2$. Recalling also the definition of $D_k$ (Eq. (32)), we have:

$$D_1 D_2 (K\psi) = \sum_{k=0}^3 (\nabla_{3-k} K)(D_k \psi) \quad (70)$$
with

$$\nabla_k := \begin{cases} 1, & k = 0 \\ i\delta_1, & k = 1 \\ i\delta_2, & k = 2 \\ -\delta_1\delta_2, & k = 3. \end{cases}$$  \hfill (71)

Hence, noting (33):

$$|D_1D_2\psi| \leq \|K\| \sum_{k=0}^{3} |D_k\psi|.$$  \hfill (72)

Similarly, we find:

$$D_i(K\psi) \leq \|K\| \sum_{k=0}^{3} |D_k\psi|, \quad i = 1, 2.$$  \hfill (73)

Considering the definition of $A_i, B_j, i, j = 1, 2$ it follows that

$$|\psi| \leq \|K\| \sum_{k=0}^{3} (A_1 + A_2(m_i) + B_1 + B_2(m_i)) |D_k\psi|.$$  \hfill (74)

In slight abuse of notation, we here use the same symbols for the operators $A_i, B_i$ acting on functions with and without spin components.

We have thus achieved an estimate where the remaining operators are direct tensor products. We proceed with considering the case that $\psi = \psi_1 \otimes \psi_2$ with $\psi_i \in \mathcal{D}_T, \ i = 1, 2$, is a product wave function. This allows us to make use of lemma 4.1. The resulting estimate can be extended to the whole tensor space $\mathcal{D}_T = \mathcal{T}^2_T$ by linearity. Note that for this to work, it is important that the operators $D_k$ factorize into operators acting on the tensor factors of $\mathcal{D}_T$.

In order to proceed with this strategy, we first note that by Young’s inequality for $a_1, ..., a_N \in \mathbb{R}$, we have $\left(\sum_{i=1}^{N} a_i\right)^2 \leq N \sum_{i=1}^{N} a_i^2$ and thus (writing, for the sake of notation $A_1 = A_1(m)$ and $B_1 = B_1(m)$):

$$|\psi(x_1, x_2)|^2 \leq 64 \|K\|^2 \sum_{i,j=1,2} \sum_{k=0}^{3} \left(|A_i(m_1) \otimes A_j(m_2)|D_k\psi|^2 + |A_i(m_1) \otimes B_j(m_2)|D_k\psi|^2 + |B_i(m_1) \otimes A_j(m_2)|D_k\psi|^2 + |B_i(m_1) \otimes B_j(m_2)|D_k\psi|^2\right).$$  \hfill (75)

Integrating over this expression and using lemma 4.1 we obtain, writing also $A_1 = A_1(m)$ and $B_1 = B_1(m)$:

$$\|(A\psi(x_1^0, x_2^0, \cdot, x_2^0, \cdot))\|_{L^2}^2 \leq 64 \|K\|^2 \sum_{i,j=1,2} \sum_{k=0}^{3} (|A_i(m_1) \otimes A_j(m_2)| + |A_i(m_1) \otimes B_j(m_2)| + |B_i(m_1) \otimes A_j(m_2)| + |B_i(m_1) \otimes B_j(m_2)|) \|(D_k\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2$$  \hfill (76)

Recalling the definition of $|\psi|^2(x_1^0, x_2^0)$, Eq. (31), and summarizing the operators in a single tensor product yields (66).

Next, we turn to (67). We start from the initial form of the integral equation (6) and use that as a distributional identity on test functions $\psi \in \mathcal{D}_T$, we have $D_1 S_{ret}(x_1 - x_1') = \delta^{(4)}(x_1 - x_1')$. Thus, we obtain:

$$(D_1 A\psi)(x_1, x_2) = \int_{\mathbb{M}} d^4 x_2' S_{ret}^{(2)}(x_2 - x_2')(K\psi)(x_1, x_2').$$  \hfill (77)
Proceeding similarly as for (18) we rewrite this as:

\[ D_1(A\psi) = 1 \otimes (A_1D_2 + A_2(m_2)D_2 + B_1 + B_2(M_2))(K\psi). \] (78)

Considering the form of \(A_i, B_i\) this implies:

\[ |D_1(A\psi)| \leq \|K\| \sum_{i=1,2} \sum_{k=0}^2 (1 \otimes A_i(m_2) + 1 \otimes B_i(m_2)) |D_k\psi|. \] (79)

We now square and use Young’s inequality, finding:

\[ |D_1(A\psi)|^2 \leq 8 \|K\|^2 \sum_{i=1,2} \sum_{k=0}^2 \left( (1 \otimes A_i(m_2)) |D_k\psi|^2 + (1 \otimes B_i(m_2)) |D_k\psi|^2 \right). \] (80)

Integrating and using lemma 4.1 (first for a product wave function \(\psi = \psi_1 \otimes \psi_2\) and then on the whole space \(\mathcal{D}_T\)) yields:

\[ \|D_1(A\psi)(x_1^0, x_2^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \leq 8 \|K\|^2 \sum_{i=1,2} \sum_{k=0}^2 (1 \otimes A_i(m_2) + 1 \otimes B_i(m_2)) \|D_k\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2. \] (81)

Adding the terms with \(k = 1, 3\) and using the definition of \([\psi]^2(x_1^0, x_2^0)\) now gives us (67).

The estimate (68) follows in an analogous way.

Finally, for (69) we also start from the initial integral equation (6) and use \(D_1S_k^{\text{ret}}(x_i - x_i') = \delta^{(4)}(x_i - x_i')\). This results in:

\[ D_1D_2(A\psi) = K\psi. \] (82)

Squaring and integrating gives us:

\[ \|D_1D_2(A\psi)(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \leq \|K\|^2 \|\psi(x_1^0, \cdot, x_2^0, \cdot)\|_{L^2}^2 \leq \|K\|^2 [\psi]^2(x_1^0, x_2^0), \] (83)

which yields (69).

We are now prepared for the proof Thm. 3.1. We begin with a lemma about the preservation of regularity.

**Lemma 4.3** Let \(\psi \in \mathcal{D}_T\). Then also \(A\psi \in \mathcal{D}_T\).

**Proof:** Let \(\psi \in \mathcal{D}_T\). In order to prove that \(A\psi \in \mathcal{D}_T\), we need to show that (i) \(A\psi\) is smooth, and (ii) \(A\psi\) has compact spatial support. Considering the definition of \(A\) (Eq. (18)) we note that, as \(K\) is smooth, (i) and (ii) follow if for all \(f \in \mathcal{F}_T\) the operators \(A_i, B_i\), \(i = 1, 2\) have the property that \(A_if, B_if\) are smooth and have compact spatial support.

**Smoothness.** To show the smoothness of \(A_if\) and \(B_if\), we consider the definitions (19)–(22). Writing out the integrals in these equations in spherical coordinates yields integral operators over finite and regular domains and with smooth integration kernels. (Note that \(J_1(x)/x\) defines a smooth function on \(\mathbb{R}\).) Hence \(A_if\) and \(B_if\) are smooth if \(f\) is.

**Compact spatial support.** One can read off from Eqs. (19)–(22) that \((A_if)(x)\) and \((B_if)(x)\), \(i = 1, 2\) depend only on the values of \(f\) in the past light cone of \(x\). Hence, if \(f(0, \cdot)\) has compact spatial support in a set \(S \subset \mathbb{R}^3\), then for \(x^0 \geq 0\), \((A_if)(x^0, \cdot)\) and \((B_if)(x^0, \cdot)\), \(i = 1, 2\) have compact spatial support in \(\bar{S} = \{x \in \mathbb{R}^3 : \exists y \in S : |y - x| \leq x^0\}\).
Proof of Thm. 3.1: The fact that \( \psi \in \mathcal{D}_T \) implies \( A\psi \in \mathcal{D}_T \) has just been shown.

Concerning (34), we use lemma 4.2 together with the definition of \( [\psi]^2(x_1^0, x_2^0) \) to obtain:

\[
[A\psi]^2(x_1^0, x_2^0) \leq (66) + (67) + (68) + (69).
\]  

(84)

Summarizing the operators in a single tensor product yields (34).

We now turn to the proof of (35). We start from (34) and take the supremum with respect to the time coordinates \( x_1^0, x_2^0 \in [0,T] \) in order to obtain \( \|A\psi\|^2 \) (see Eq. (30)). In all the expressions, we use \( [\psi]^2(x_1^0, x_2^0) \leq \|\psi\|^2 \). Considering (57), this leads to:

\[
\|A\psi\|^2 \leq \|K\|^2 \|\psi\|^2 \prod_{i=1,2} \left[ 1 + 8 \left( \frac{1}{3} T^4 + \frac{m_i^4 T^8}{2^6 3^2} + T^2 + \frac{m_i^4 T^6}{2^2 3^2} \right) \right].
\]  

(85)

Using \( \mu = \max\{m_1, m_2\} \) now immediately yields (35).

\[ \square \]

4.2 Proof of Theorem 3.4

Our strategy will be to iterate the estimate (34) (not (35)!) in order to obtain a bound on \( \sum_{k=0}^\infty [A^k \psi_{\text{free}}](t_1, t_2) \) and therefore, by taking the supremum with respect to \( t_1, t_2 \in [0,T] \), on \( \sum_{k=0}^\infty \|A^k \psi_{\text{free}}\| \).

The main problem is to estimate powers of the operator \( (1 + 8A(m) + 8B(m)) \). Now,

\[
(1 + 8A(m) + 8B(m))^n = \sum_{k=0}^n \binom{n}{k} 8^k (A(m) + B(m))^k.
\]  

(86)

For our estimates, the operator \( (A(m) + B(m))^k \) needs to be applied to a constant function, \( \|\psi\| \). The first application of \( (A(m) + B(m)) \) to \( f(t) = 1 \) yields:

\[
((A(m) + B(m)) 1)(t) = t^2 + 1 + \frac{t^4}{3} + \frac{m_1^4 t^8}{2^2 3^2} + \frac{m_1^4 t^6}{2^2 3^2} =: P(t).
\]  

(87)

Now we observe that \( B(m)Q(t) = 0 \) for every polynomial \( Q(t) \) with \( Q(0) = 0 \) (recalling (57)). Considering that the operator \( A(m) + B(m) \) maps a polynomial \( Q_1(t) \) with \( Q_1(0) = 0 \) to a polynomial \( Q_2(t) \) with \( Q_2(0) = 0 \) leads to the following simplification \( (k \geq 1): \)

\[
((A(m) + B(m))^k 1)(t) = (A(m) + B(m))^{k-1} P(t) = A(m)^{k-1} P(t).
\]  

(88)

This means, it suffices to estimate powers of \( A(m) = A_1 + A_2(m) \) applied to \( P(t) \). We continue doing this in a series of preparatory lemmas.

Lemma 4.4 Let \( n \in \mathbb{N}_0 \) and let \( M_n : \mathbb{R} \to \mathbb{R} \) denote the monomial \( x \mapsto x^n \). Then the operators \( A_1, A_2(m) \) satisfy the following identities:

\[
(A_1 M_n)(t) = \frac{2t^{n+4}}{(n+1)(n+2)(n+3)},
\]  

(89)

\[
(A_2(m) M_n)(t) = \frac{m^4 t^{n+8}}{24 (n+1)(n+2)(n+3)(n+4)}.
\]  

(90)

Furthermore, for all \( t > 0 \), the following estimate holds:

\[
(A_1 A_2(m) M_n)(t) < (A_2(m) A_1 M_n)(t).
\]  

(91)
Proof: Eqs. (89) and (90) can be verified easily using (57) and the definition of the β-function as well as its properties. For \( a, b \in \mathbb{N} \):

\[
\beta(a, b) = \int_0^1 dt \ t^{a-1}(1-t)^{b-1},
\]

\[
\beta(a, b) = \frac{(a - 1)!(b - 1)!}{(a + b - 1)!}
\] (92)

We then obtain (91) through a direct calculation. For all \( \beta \), we find for (89):

\[
\text{Lemma 4.5} \quad \text{For all } p \in \mathbb{N}, \text{ the properties. For } p \in \mathbb{N}, \text{ the properties.}
\]

\[
\text{which yields (94). We now turn to the proof of (95). Iterating (90) for } \beta \text{, we have:}
\]

\[
A_1 A_2(m) t^n < A_2(m) A_1 t^n
\Rightarrow
2m^4 t^{n+12} < 24 \frac{(n + 11)(n + 10)(n + 9)(n + 1)!(n + 2)(n + 3)(n + 4)}{(n + 8)(n + 7)(n + 6)(n + 5)(n + 1)! (n + 2)(n + 3) t^{n+12}}
\Rightarrow
(n + 8)(n + 7)(n + 6)(n + 5) < (n + 11)(n + 10)(n + 9)(n + 4),
\] (93)

which can easily be seen to hold true for every \( n \in \mathbb{N}_0 \).

\[
\text{Lemma 4.5} \quad \text{For all } t > 0, n, r \in \mathbb{N}_0 \text{ and } M_r : x \mapsto x^r, \text{ we have:}
\]

\[
\left( A_1^n M_r \right)(t) < \frac{(r + n)! 2^{3n} t^{4n+r}}{(r + 4n)!},
\]

\[
\left( A_2(m)^n M_r \right)(t) < \frac{r! (m^4 t^8)^n t^r}{24^n (4n + r)!}.
\] (95)

Proof: Let \( n \geq 1 \) (for \( n = 0 \), the claim is easy to see). We begin with (94). By iterating (89), we find for \( r \in \mathbb{N}_0 \), \( n \in \mathbb{N} \) and all \( t > 0 \):

\[
\left( A_1^n M_r \right)(t) = (2t^{4n})^n t^r \prod_{k=0}^{n-1} \frac{1}{(4k + 1 + r)(4k + 2 + r)(4k + 3 + r)}
\]

\[
= 2^n t^{4n+r} \prod_{l=0}^{r+n} \left( r + 4 + 4l \right) \prod_{k=r+1}^{n-1} \frac{1}{k}
\]

\[
= 2^n t^{4n+r} \prod_{l=0}^{r+n} \left( r + 1 + l \right) \frac{r!}{(r + 4n)!}
\]

\[
= 2^{3n} t^{r+4n} \frac{(r + n)!}{(r + 4n)!},
\]

which yields (94). We now turn to the proof of (95). Iterating (90) for \( n, r \in \mathbb{N}_0 \) and \( t > 0 \) gives us:

\[
\left( A_2(m)^n M_r \right)(t) = \frac{(m^4 t^8)^n t^r}{24^n} \prod_{k=0}^{n-1} \frac{1}{(8k + 1 + r)(8k + 2 + r)(8k + 3 + r)(8k + 4 + r)}
\]

\[
\leq \frac{(m^4 t^8)^n t^r}{24^n} \prod_{k=0}^{n-1} \frac{1}{(4k + 1 + r)(4k + 2 + r)(4k + 3 + r)(4k + 4 + r)} = \frac{(m^4 t^8)^n t^r}{24^n (4n + r)!}.
\] (96)
Lemma 4.6 For all $n, k, r \in \mathbb{N}_0$ with $n \geq k$ and all $t > 0$, we have:

\[
(A_2^k(m)A_1^{n-k}M_r)(t) \leq 2^{3(n-k)}\frac{(r+(n-k))!}{(4(n-k)+r)!} \frac{m^{4k}}{(4k+4(n-k)+r)!} t^{4(n+k)+r} \frac{1}{(n!)^3} (97)
\]

Proof: Pick $n, k, r$ according to the statement of the lemma. Moreover, let $n \geq 1$ (for $n = 0$ the claim is easy to verify). Using (94) and (95) (in that order), we obtain for all $t > 0$:

\[
(A_2^k(m)A_1^{n-k}M_r)(t) \leq 2^{3(n-k)}\frac{(r+(n-k))!}{(4(n-k)+r)!} (A_2^k(m)M_{4(n-k)+r})(t) \\
\leq 2^{3(n-k)}\frac{(r+(n-k))!}{(4(n-k)+r)!} \frac{m^{4k}}{(4k+4(n-k)+r)!} t^{4(n+k)+r} \\
= 2^{3n-6k}3^{-k}\frac{(r+n-k)!}{(4n+r)!} m^{4k} t^{4(n+k)+r} \\
\leq 2^{3n-6k}3^{-k}\frac{(r+n)!}{(4n+r)!} m^{4k} t^{4(n+k)+r} \\
\leq 2^{3n-6k}3^{-k} \frac{1}{(n!)^3} m^{4k} t^{4(n+k)+r}.
\]

Lemma 4.7 Let $\|K\| < 1$, $T > 0$ and $\mu = \max\{m_1, m_2\}$. Then, we have for all $\psi \in \mathcal{D}_T$:

\[
\|A^k \psi\| \leq 16 L_k(-1) \|K\|^{k}\|\psi\| \exp \left( (1 + \mu^4)T^2 + 64T^4 + \frac{\mu^4}{3}T^8 \right), \quad (98)
\]

where $L_k$ is the $k$-th Laguerre polynomial.

Furthermore, there is a constant $C > 0$, given explicitly in (104), such that for all $t_1, t_2 > 0$:

\[
\sum_{k=0}^{\infty} [A^k \psi](t_1, t_2) \leq C \|\psi\| \prod_{i=1,2} \exp \left( \frac{1 + m_i^4}{2} t_i^2 + 32 t_i^4 + \frac{m_i^4}{6} t_i^6 \right). \quad (99)
\]

as well as

\[
\sum_{k=0}^{\infty} \|A^k \psi\| \leq C \|\psi\| \exp \left( (1 + \mu^4)T^2 + 64T^4 + \frac{\mu^4}{3}T^8 \right). \quad (100)
\]

Proof: Let $\psi \in \mathcal{D}_T$. We begin with estimating $[A^k \psi](t_1, t_2)$. As $A$ maps $\mathcal{D}_T$ to $\mathcal{D}_T$ (see lemma 4.3), we can iterate (34). This results in (for all $t_1, t_2 > 0$):

\[
[A^k \psi]^2(t_1, t_2) \leq \|K\|^{2k} \left( \bigotimes_{i=1,2} (1 + 8A(m_i) + 8B(m_i))^k \right) [\psi]^2(t_1, t_2) \\
\leq \|\psi\|^2 \|K\|^{2k} \left( \bigotimes_{i=1,2} \sum_{l=0}^{k} \binom{k}{l} (A(m_i) + B(m_i))^l \right) (t_1, t_2) \\
= \|\psi\|^2 \|K\|^{2k} \left( \bigotimes_{i=1,2} \left( 1 + \sum_{l=1}^{k} \binom{k}{l} A(m_i)^{l-1} P \right) \right) (t_1, t_2). \quad (101)
\]

where we have used (87) and (88). Next, we employ (91) and (97) to estimate the sum in (101). Considering the definition of $P$ (see (87)) and using the results of the previous
polynomials have the following asymptotic behavior for large $C$:

\[ \sum_{l=1}^{\infty} \binom{k}{l} \left( A(m)^{l-1} P \right)(t) \leq \sum_{l=1}^{\infty} \binom{k}{l} \left( A_2(m)^{l-1} A_1^{l-1-c} P \right)(t) \]

Taking the supremum over $\mu$, this yields:

\[ \sum_{l=1}^{\infty} \binom{k}{l} \frac{1}{c} \left( \sum_{c=0}^{l-1} \binom{l-1}{c} (l-1)^c (c^{l-1}) \right) \frac{1}{(l-1)!^3} \left( t^2 + t^4/3 + \frac{m^4}{2232^6} + \frac{m^4}{2632^8} \right) \]

\[ \leq 8 \sum_{l=1}^{\infty} \binom{k}{l} \left( 64t^4 + \frac{m^4}{3} t^8 \right)^{l-1} \frac{1}{(l-1)!^2} \left( t^2 + t^4/3 + \frac{m^4}{2232^6} + \frac{m^4}{2632^8} \right) \]

\[ \leq 16 \sum_{l=1}^{\infty} \binom{k}{l} \frac{1}{l!} \exp \left( 1 + m^4 \right) t^2 + 64t^4 + \frac{m^4}{3} t^8 \]

\[ \leq -1 + 16 L_k(-1) \exp \left( 1 + m^4 \right) t^2 + 64t^4 + \frac{m^4}{3} t^8, \]

where $L_k(-1)$ is the $k$-th Laguerre polynomial $L_k(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l x^l$ at the position $x = -1$. Using this result in (101), we obtain:

\[ [A^k \psi](t_1, t_2) \leq 16 L_k(-1) \left| K \right|^k \| \psi \| \prod_{i=1,2} \exp \left( 1 + m^4 \right) t_i^2 + 64t_i^4 + \frac{m^4}{3} t_i^8 \right)^{1/2}. \]

Taking the supremum over $t_1, t_2 \in [0, T]$ and considering $m_1, m_2 \leq \mu$ already yields (98).

Next, we take the sum of (102) over $k \in \mathbb{N}_0$ and find:

\[ \sum_{k=0}^{\infty} [A^k \psi](t_1, t_2) \leq C \| \psi \| \prod_{i=1,2} \exp \left( \frac{1 + m_i^4}{2} t_i^2 + 32t_i^4 + \frac{m_i^4}{6} t_i^8 \right) \]

with

\[ C = 16 \sum_{k=0}^{\infty} \| K \|^k L_k(-1). \]

This yields (99), provided that $C < \infty$. To demonstrate this, note that the Laguerre polynomials have the following asymptotic behavior for large $n$ [37, eqs. (3) and (4)]:

\[ L_n(-1) = \frac{1}{2\sqrt{\pi} \sqrt{n+1}} e^{2\sqrt{n+1}} \left( 1 + O \left( \frac{1}{\sqrt{n+1}} \right) \right). \]

This means that there is an $M \in \mathbb{N}$ such that

\[ \forall n > M : L_n(-1) < e^{2\sqrt{n+1}}. \]

We now pick $Z \in \mathbb{N}$, $Z > M$ such that $2\sqrt{k+1} < k \left| \ln(\| K \|) \right| / 2$ for all $k \in \mathbb{N}$, $k > Z$. Then, considering $\| K \| < 1$, we find:

\[ C \leq \sum_{k=0}^{Z} \| K \|^k L_k(-1) + \sum_{k=Z+1}^{\infty} e^{k/2 \ln |K|} \leq \sum_{k=0}^{Z} \| K \|^k L_k(-1) + \frac{1}{1 - \sqrt{\| K \|}} < \infty. \]

Finally, (100) follows from (99) by taking the supremum over $t_1, t_2 \in [0, T]$ and considering $\mu = \max\{m_1, m_2\}$.
To complete the proof of the main theorem, we only need to assemble the arguments.

**Proof of Theorem 3.4:** We show that the series

$$\psi = \sum_{k=0}^{\infty} A^k \psi_{\text{free}}$$

(108)

converges in $\mathcal{B}_T$ for every $\psi_{\text{free}} \in \mathcal{B}_T$ and yields the unique solution of the equation

$$\psi = \psi_{\text{free}} + A \psi.$$

(109)

We begin with the convergence. To this end, let $\phi \in \mathcal{B}_T$. By lemma 4.3, we know that $A^k \phi \in \mathcal{B}_T$ and according to lemma 4.7, $A^k \phi$ satisfies the bound (98). Thus for every $k \in \mathbb{N}_0$, the operator $A^k$ can be extended to a bounded operator on $\mathcal{B}_T$ which satisfies the same estimate. Thus, in the same way as we proved (100) in lemma 4.7, we now obtain (108) for every $\psi \in \mathcal{B}_T$. This implies the convergence of (108) and furthermore yields (37).

Next, we show that (108) solves (109). As $A$ is bounded and thus continuous, we have:

$$A \psi = A \sum_{k=0}^{\infty} A^k \psi_{\text{free}} = \sum_{k=0}^{\infty} A^{k+1} \psi_{\text{free}} = \sum_{k=0}^{\infty} A^k \psi_{\text{free}} - \psi_{\text{free}} = \psi - \psi_{\text{free}},$$

(110)

which is equivalent to (109).

Finally, we come to the uniqueness. Let $\psi_1, \psi_2 \in \mathcal{B}_T$ be two solutions of (109) and let $\chi = \psi_1 - \psi_2$. Then $\chi$ satisfies $A \chi$. As (108) converges for every $\psi \in \mathcal{B}_T$, we find:

$$\sum_{k=0}^{\infty} \|A^k \chi\| = \sum_{k=0}^{\infty} \|\chi\| < \infty,$$

(111)

which implies $\chi = 0$, hence $\psi_1 = \psi_2$. \qed

## 5 Conclusion and outlook

Extending previous work for Klein-Gordon particles [29, 30] to the Dirac case, we have shown the existence of dynamics for a class of integral equations which express direct interactions with time delay at the quantum level. To obtain this result, we have assumed a cutoff of the spacetime before $t = 0$. It has been demonstrated that the Big Bang singularity can naturally provide such a cutoff. Remarkably, this yields a class of rigorous interacting models in 1+3 spacetime dimensions.

The main challenge in our work has been the non-Markovian nature of the dynamics. This has made it necessary to directly prove global existence in time instead of concatenating short-time solutions on small time intervals (which are much easier to obtain). Apart from this, the distributional derivatives in the Green’s functions of the Dirac equation have made the analysis substantially more difficult than in the Klein-Gordon case. Compared to the latter, we have also treated the massive case (which was not considered in [29] for 1+3 dimensions).

Our results are furthermore characterized as follows. We have shown that the wave function is determined by Cauchy data at the initial time (corresponding to the Big Bang singularity); however, no Cauchy problem is available at different times. The main requirement of our theorems is a smallness condition on the interaction kernel $K$, demanding that both $K$ and certain first and second order derivatives of $K$ must be bounded and not too large. This still admits a wide class of interaction kernels, and we emphasize that in no way
the interaction needs to be small compared to the size of the domain of the wave function. The latter is a common requirement for Fredholm integral equations but it would would make the result worthless for infinite spatio-temporal domains (in particular for $T \to \infty$).

Besides, we have assumed that $K$ is complex-valued while it could be matrix-valued in the most general case. The reason for this assumption is that our proof requires the integral operator $A$ to be a map from a certain Sobolev space onto itself in which weak derivatives with respect to the Dirac operators of the two particles can be taken. If $K$ were matrix-valued, it would not commute with these Dirac operators in general. Then $A\psi$ would contain new types of weak derivatives which cannot be taken in the initial Sobolev space. As illustrated in Sec. 2.2, this creates a situation where more and more derivatives have to be controlled, possibly up to infinite order where the success of an iteration scheme seems unlikely. At present, we do not know how to deal with this issue. Improving on this point, however, defines an important task for future research, as e.g. electromagnetic interactions involve interaction kernels proportional to $\gamma_1^\mu \gamma_2^\mu$ (see [26]).

In addition, it would be desirable to generalize our work in the following regards.

- **$N$ particle integral equations.** Our hope is that our work could contribute to the formulation of a rigorous relativistic many-body theory that can be applied for finite times, not only for scattering processes. An important step in this direction is to treat an arbitrary fixed number $N \in \mathbb{N}$ of particles (setting aside particle creation and annihilation). A class of possible $N$-particle integral equations has been suggested in [26]. It has the schematic form

$$
\psi(x_1, ..., x_N) = \psi^{\text{free}} + \sum_{i<j} \int d^4x'_i d^4x'_j S_i(x_i-x'_i) S_j(x_j-x'_j) K_{ij}(x'_i, x'_j) \psi(...x'_i, ..., x'_j,...). 
$$

(112)

It might well be possible to prove the existence and uniqueness of solutions for that equation using the methods developed in the present paper.

- **Singular interaction kernels.** The physically most natural interaction kernel is given by a delta function along the light cone, $K(x_1, x_2) = \delta((x_1 - x_2)_\mu(x_1 - x_2)^\mu)$. Getting closer to this case is one of our central goals. Apart from approaching the problem head-on by suitably interpreting the distributional expressions and trying to prove the existence of solutions of the resulting singular integral equation, which seems difficult, one could also try to make smaller steps first. For example, one could decompose $\delta((x_1 - x_2)_\mu(x_1 - x_2)^\mu)$ into

$$
\frac{1}{2|x_1 - x_2|} [\delta(x_1^0 - x_2^0 - |x_1 - x_2|) + \delta(x_1^0 - x_2^0 + |x_1 - x_2|)]
$$

and only then replace the delta functions with a peaked but smooth function, keeping the singular factor $1/|x_1 - x_2|$. This has been done in [29] for the Klein-Gordon case. In the Dirac case, the distributional derivatives make a generalization of that result difficult, and we have not attempted it here. However, it is conceivable that a suitable modification of our techniques could make it possible to treat this case.

Another interesting question is whether the smallness condition on $K$ can be alleviated such that arbitrarily peaked functions are admitted. This could allow taking a limit where $K$ approaches the delta function along the light cone.

**Acknowledgments.**

We would like to thank Stefan Teufel and Roderich Tumulka for helpful discussions. M. N. acknowledges funding from Cusanuswerk and from the Elite Network of Bavaria, through the Junior Research Group ‘Interaction Between Light and Matter’.

\[ 
\text{This project has received funding from the European Union’s Framework for Research and Innovation Horizon 2020 (2014–2020) under the Marie Skłodowska-Curie Grant Agreement No. 705295.} 
\]
References

[1] D.-A. Deckert and M. Oelker. Distinguished self-adjoint extension of the two-body Dirac operator with Coulomb interaction. Preprint: https://arxiv.org/abs/1805.09634 (2018).

[2] W. Thirring. A Soluble Relativistic Field Theory? Ann. Phys., 3:91–112, 1958.

[3] J. Glimm and A. Jaffe. Quantum Physics – A Functional Integral Point of View. Springer, 1987.

[4] A. Jaffe. Constructive Quantum Field Theory. https://www.arthurjaffe.com/Assets/pdf/CQFT.pdf.

[5] P. A. M. Dirac. Relativistic Quantum Mechanics. Proc. R. Soc. Lond. A, 136:453–464, 1932.

[6] S. Tomonaga. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. Prog. Theor. Phys., 1:27–42, 1946.

[7] J. Schwinger. Quantum Electrodynamics. I. A Covariant Formulation. Phys. Rev., 74(2162):1439–1461, 1948.

[8] M. Günther. Many-Times Formalism and Coulomb Interaction. Phys. Rev., 88(6):1411–1421, 1952.

[9] E. Marx. Many-Times Formalism and Coulomb Interaction. Int. J. of Theor. Phys., 9(3):195–217, 1974.

[10] S. Schweber. An Introduction to Relativistic Quantum Field Theory. Dover, 2005. Originally published in 1961.

[11] Ph. Droz-Vincent. Relativistic Wave Equations for a System of Two Particles with Spin 1/2. Lettere al Nuovo Cimento, 30:375–378, 1981.

[12] H. Sazdjian. Relativistic wave equations for the dynamics of two interacting particles. Phys. Rev. D, 33:3401–3424, 1986.

[13] H. W. Crater and P. Van Alstine. A tale of three equations: Breit, Eddington-Gaunt, and Two-Body Dirac. Found. Phys., 27:67–79, 1997.

[14] S. Petrat and R. Tumulka. Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials. J. Math. Phys., 55(032302), 2014. https://arxiv.org/abs/1308.1065.

[15] S. Petrat and R. Tumulka. Multi-Time Wave Functions for Quantum Field Theory. Ann. Phys., 345:17–54, 2014. https://arxiv.org/abs/1309.0802v3.

[16] S. Petrat and R. Tumulka. Multi-time formulation of pair creation. J. Phys. A: Math. Theor., 47(11):112001, 2014.

[17] M. Lienert. A relativistically interacting exactly solvable multi-time model for two massless Dirac particles in 1+1 dimensions. J. Math. Phys., 56(4):042301, 2015. https://arxiv.org/abs/1411.2833.
[18] M. Lienert and L. Nickel. A simple explicitly solvable interacting relativistic N-particle model. *J. Phys. A: Math. Theor.*, 48(32):325301, 2015. [https://arxiv.org/abs/1502.00917](https://arxiv.org/abs/1502.00917).

[19] M. Lienert. On the question of current conservation for the Two-Body Dirac equations of constraint theory. *J. Phys. A: Math. Theor.*, 48(32):325302, 2015. [https://arxiv.org/abs/1501.07027](https://arxiv.org/abs/1501.07027).

[20] D.-A. Deckert and L. Nickel. Consistency of multi-time Dirac equations with general interaction potentials. *J. Math. Phys.*, 57(7):072301, 2016. [https://arxiv.org/abs/1603.02538](https://arxiv.org/abs/1603.02538).

[21] M. Lienert, S. Petrat, and R. Tumulka. Multi-Time Wave Functions Versus Multiple Timelike Dimensions. *Found. Phys.*, 47:1582–1590, Oct 2017. [https://arxiv.org/abs/1708.03376](https://arxiv.org/abs/1708.03376).

[22] M. Lienert and R. Tumulka. Born’s Rule for Arbitrary Cauchy Surfaces, 2017. [https://arxiv.org/abs/1706.07074](https://arxiv.org/abs/1706.07074).

[23] M. Lienert and L. Nickel. Multi-time formulation of creation and annihilation of particles via interior-boundary conditions. Preprint: [https://arxiv.org/abs/1808.04192](https://arxiv.org/abs/1808.04192) (2018).

[24] L. Nickel. PhD thesis. On the Dynamics of Multi-Time Systems, 2019. Mathematical Institute, Ludwig-Maximilians-Universität, Munich, Germany.

[25] M. Lienert, S. Petrat, and R. Tumulka. Multi-time wave functions. *J. Phys. Conf. Ser.*, 880(1):012006, 2017. [https://arxiv.org/abs/1702.05282](https://arxiv.org/abs/1702.05282).

[26] M. Lienert. Direct interaction along light cones at the quantum level. *J. Phys. A: Math. Theor.*, 51(43):435302, 2018. [https://arxiv.org/abs/1801.00060](https://arxiv.org/abs/1801.00060).

[27] J. A. Wheeler and R. P. Feynman. Interaction with the Absorber as the Mechanism of Radiation. *Rev. Mod. Phys.*, 17:157–181, 1945.

[28] J. A. Wheeler and R. P. Feynman. Classical Electrodynamics in Terms of Direct Interparticle Action. *Rev. Mod. Phys.*, 21:425–433, 1949.

[29] M. Lienert and R. Tumulka. A new class of Volterra-type integral equations from relativistic quantum physics. To appear in *J. Integral Equations Applications*. [https://projecteuclid.org/euclid.jiea/1536804036](https://projecteuclid.org/euclid.jiea/1536804036) (2019).

[30] M. Lienert and R. Tumulka. Interacting relativistic quantum dynamics of two particles on spacetimes with a Big Bang singularity. To appear in *J. Math. Phys.* (2019). Preprint [https://arxiv.org/abs/1805.06348](https://arxiv.org/abs/1805.06348).

[31] E. E. Salpeter and H. A. Bethe. A Relativistic Equation for Bound-State Problems. *Phys. Rev.*, 84:1232–1242, 1951.

[32] Yu. A. Dubinskii. Sobolev spaces of infinite order. *Russ. Math. Surv.*, 46(6):107–147, 1991.

[33] M. Ibison. On the conformal forms of the Robertson-Walker metric. *J. Math. Phys.*, 48:122501, 2007. [https://arxiv.org/abs/0704.2788](https://arxiv.org/abs/0704.2788).
[34] R. Penrose and W. Rindler. *Spinors and Space-time, Volume 1*. Cambridge University Press, 1984.

[35] J. Haantjes. The conformal Dirac equation. *Proc. Kon. Nederl. Acad. Wetensch.*, 44:324–332, 1941.

[36] R. W. John. The Hadamard Construction of Green’s Functions on a Curved Space-time with Symmetries. *Ann. Phys.*, 48(7):531–544, 1987.

[37] D. Borwein, J. M. Borwein, and R. E. Crandall. Effective Laguerre Asymptotics. *SIAM J. Numer. Anal.*, 46(6):3285–3312, 2008. https://epubs.siam.org/doi/10.1137/07068031X.