THE TENSOR PRODUCT OF FUNCTION ALGEBRAS

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ABSTRACT. In this paper we study the tensor product of two $f$-algebras. We show that the Riesz Subspace generated by a subalgebra in an $f$-algebra is an algebra in order to prove that the Riesz tensor product of two $f$-algebras has a structure of an $f$-algebra.

1. Introduction.

The algebraic tensor product of two vector spaces $E, F$ is a vector space $E \otimes F$ with a bilinear map $\otimes : E \times F \rightarrow E \otimes F$ satisfying the following universal property:

for every vector space $G$ and a bilinear map $T : E \times F \rightarrow G$ there exists a unique linear map $T^\otimes : E \otimes F \rightarrow G$ such that $T = T^\otimes \circ \otimes$.

The Riesz tensor product of two Riesz spaces $E, F$ is a Riesz space $E \overline{\otimes} F$ with a Riesz bimorphism map $\otimes : E \times F \rightarrow E \overline{\otimes} F$ satisfying a similar universal property (where maps are assumed to respect the Riesz structure). The construction of this Tensor product is due to Fremlin in his fundamental paper [1]. This subject was studied several times later namely by Schaffer in [12], Grobbler and Labaushagne [7, 8].

It appears natural to ask about the tensor product of another class of ordered structure: $f$-algebras! That is, if $E$ and $F$ are $f$-algebras, does an $f$-algebra $E \overline{\otimes} f F$ with a algebra and Riesz bimorphism map $\otimes : E \times F \rightarrow E \overline{\otimes} f F$ satisfying a similar universal property (where maps are assumed to respect the Riesz and the algebra structures) exist?

The latter question motivated our study, therfore it is natural to focus on the most important case of $f$-algebras: the $C(X)$’s case. The Riesz tensor product $C(X) \overline{\otimes} C(Y)$ of $C(X)$ and $C(Y)$, where $X$ and $Y$ are topological spaces, is the Riesz sub-space of $C(X \times Y)$ generated by the algebraic tensor product $C(X) \otimes C(Y)$. The legitimate candidate to consider as an $f$-algebra tensor product of $C(X)$ and $C(Y)$ is the $f$-algebra generated by $C(X) \overline{\otimes} C(Y)$ in the $f$-algebra $C(X \times Y)$. Another question araise: is $C(X) \overline{\otimes} C(Y)$ itself an $f$-algebra? This leads to a more general question : let $B$ be a subalgebra of the $f$-algebra $A$. Is the Riesz subspace generated by $B$ in $A$ a subalgebra?

Section 2 gives, among other results, an affirmative question to the last question, where we use which seems to be a new construction a Riesz space generated by a subset.

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In the next section, we present some results on bilinear maps that will be useful in the last section.

Section 4 is devoted to the study of the $f$-algebra tensor product. We will prove that the Riesz tensor product of two semiprime $f$-algebras $A$ and $B$ has automatically a structure of $f$-algebra, and satisfies the appropriate universal property that is:

$$A \overline{\otimes} B = A \overline{\otimes} f B.$$ 

2. Preliminaries

We assume the reader to be familiar with the terminology of Riesz spaces and $f$-algebras and we refer to [3] and [12]. Recall that a linear mapping $T : E \rightarrow F$ between two Riesz spaces $E$ and $F$ is said to be lattice homomorphism if

$$T(|x|) = |T(x)| \quad \text{for all } x \in E.$$ 

A bilinear map $\varphi : E \times F \rightarrow G$ is called positive if $\varphi(x, y) \geq 0$ for all $(x, y) \in E^+ \times F^+$ and it is called a Riesz bimorphism whenever $\varphi(., y)$ and $\varphi(x, .)$ are Riesz homomorphism for all $(x, y) \in E^+ \times F^+$.

The (relatively) uniform topology on Riesz spaces plays a key role in the context of this work. Let us therefore recall the definition. Let $E$ be a Riesz space. A sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $E$ is said to converge relatively uniformly to $f \in E$ if there exists $v \in E$ such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n - f| \leq \varepsilon v$ for all $n \geq N$. Uniform limits are unique if and only if $E$ is Archimedean. For this reason all vector lattices and lattice-ordered algebras under consideration are assumed to be Archimedean.

The following lemma turns out to be useful for later purposes.

**Lemma 2.1.** Let $E$, $F$ and $G$ be Archimedean Riesz spaces and $\varphi : A \times B \rightarrow C$ be a positive bilinear map. Let $(f_n)_{n \in \mathbb{N}} \in A^\mathbb{N}$ and $(g_n)_{n \in \mathbb{N}} \in B^\mathbb{N}$ be two sequences which converges relatively uniformly to $f$ and $g$ respectively. Then $(\varphi(f_n, g_n))_{n \in \mathbb{N}}$ converges relatively uniformly to $\varphi(f, g)$.

**Proof.** There exists $u \in A$ and $v \in B$ such that for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $|f_n - f| \leq \varepsilon u$ and $|g_n - g| \leq \varepsilon v$. Observe that the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded, that is there exist $w \in B$ such that $|g_n| \leq w$ for all $n \in \mathbb{N}$. It follows from the bilinearity and the positivity of $\varphi$ that for all $n \geq N$,

$$|\varphi(f_n, g_n) - \varphi(f, g)| \leq \varphi(|f_n - f|, |g_n|) + \varphi(|f|, |g_n - g|)$$

$$\leq \varepsilon (\varphi(u, w) + \varphi(|f|, v)).$$

This implies the desired result. \[\square\]

The next paragraph deals with the notion of function-algebras (or $f$-algebras). A Riesz space $A$ is called a lattice ordered algebra if there exists an associative multiplication in $A$ with the usual algebraic properties such that $fg \geq 0$ for all $f, g \in A^+$. The lattice-ordered algebra $A$ is said to be an $f$-algebra if $f \land g = 0$ and $0 \leq h \in A$ imply $fh \land g = hfg = 0$. The most classical example of an $f$-algebra is the algebra $C(X)$ of all real-valued continuous functions on a topological space $X$. Recall from [14, Theorem 142.7] that if $A$ is an $f$-algebra with unit element
e then for all \(0 \leq x \in A\), the sequence \((x \wedge ne)_{n \in \mathbb{N}}\) converge uniformly to \(x\). The reader can consult [14] for more information about \(f\)-algebras.

Finally we recall some facts about the universally completion of a Riesz space. A Dedekind complete vector lattice is called universally complete whenever every set of pairwise disjoint positive elements has a supremum. Every vector lattice \(E\) has a universal completion \(E^u\), meaning that there exists a unique universally complete vector lattice \(E^u\) such that \(E\) can be identified with an order dense vector sublattice of \(E^u\). Moreover, if \(e\) is a weak order unit in \(E\) then \(E^u\) can be endowed with a multiplication, in such a manner that \(E^u\) becomes an \(f\)-algebra with \(e\) as identity.

3. RIESZ SUBSPACE GENERATED BY A SUBALGEBRA.

Let \(D\) be a subset of a Riesz space \(E\). The Riesz subspace generated by \(D\), that is the smallest Riesz subspace containing \(D\), will be denoted by \(R(D)\). The first theorem of this section furnishes a useful new construction, as the best of our knowledge, of the Riesz subspace generated by a vector subspace. Consider a vector subspace \(F\) of \(E\). Define a sequence \((L_n)_{n \in \mathbb{N}}\) of vector subspaces by:

\[L_1 = F\] \(L_{n+1}\) is the vector subspace generated by \(L_n\) and \(|L_n|\) for \(n = 1, 2, \ldots\)

where \(|D| = \{|x| : x \in D\}\)

for any subset \(D\) of \(E\).

**Theorem 3.1.** Let \(F\) be a vector subspace of a Riesz space \(E\), and \((L_n)_{n \in \mathbb{N}}\) defined as above. Then \(R(F) = \bigcup_{n=1}^{\infty} L_n\).

**Proof.** Since \((L_n)_{n \in \mathbb{N}}\) is an increasing sequence of vector subspaces, it follows that \(L = \bigcup_{n=1}^{\infty} L_n\) is a vector subspace of \(E\). Moreover, for all \(f \in L_n\) we have \(|f| \in L_{n+1} \subseteq L\). This implies that \(L\) is a Riesz subspace which contains \(F\). Hence \(R(F) \subseteq L\). Conversely, we prove easily by induction that \(L_n \subseteq R(F)\) for all \(n = 1, 2, \ldots\) which yields to the inclusion \(L \subseteq R(F)\). \(\Box\)

The following technical lemma will be used to prove the main theorem of this section.

**Lemma 3.2.** Let \(A\) be a semiprime \(f\)-algebra. Then the following hold

(i): For all \(a \in A^+\)
\[0 \leq a^2 \leq a + a^3.\]

(ii): For all \(a, b \in A\),
\[a^+b^+ = (ab)^+ \land ((a + a^3)^+ + (b + b^3)^+).\]

**Proof.** (i) Assume first that \(A\) has a unit element \(e\). Then
\[a^3 + a - a^2 = a(a^2 - a + e)\]
\[= a((a - 2^{-1}e)^2 + (3/4)e) \geq 0\]
as squares in $f$-algebra are positive. Since any semiprime $f$-algebra can be embedded as a Riesz subalgebra in an $f$-algebra with unit, the result follows for general case.

(ii) Let $a, b \in A$ and put $z = (a + a^3)^+ + (b + b^3)^+$. Observe that
\[ z = a^+ + (a^+)^3 + b^+ + (b^+)^3. \]
It follows from (i) that
\[ z \geq (a^+)^2 + (b^+)^2 \geq a^+b^+ \]
Moreover, since $a^-b^- \land z = 0$ we get
\[ (ab^+) \land z = (a^+b^+ + a^-b^-) \land z = a^+b^+ \land z = a^+b^+ \]
\[
\square
\]

The subalgebra generated by a Riesz subspace of an $f$-algebra $A$ fails in general to be a Riesz subspace. Take for example $A = C[0, 1]$, and $B$ the vector subspace generated by $u$ where $u(x) = x$ for all $x \in [0, 1]$. Surprising enough, it turns out that the converse situation is true. In fact, the main result of this section states that the Riesz subspace generated by a subalgebra of an $f$-algebra $A$ is indeed a subalgebra.

**Theorem 3.3.** Let $A$ be a semiprime $f$-gebra and $B$ be a subalgebra of $A$. Then the Riesz subspace $\mathcal{R}(B)$ generated by $B$ is an $f$-subalgebra of $A$.

**Proof.** We use the same construction as in Theorem 2.1 considering $B$ instead of $F$. Then $\mathcal{R}(B) = \bigcup_{n=1}^{\infty} L_n$. All that remains is to prove that $\mathcal{R}(B)$ is closed under multiplication. It suffices to show that for all $n \in \mathbb{N}$ we have the following property
\[ \mathcal{P}_n : \text{for all } f, g \in L_n, \text{ } fg \text{ and } f|g| \text{ are in } \mathcal{R}(B). \]

We will proceed by induction on $n$.

Let us show that $\mathcal{P}_1$ is true. Take $f, g \in L_1 = B$. Then $fg$ belongs to $B$ because $B$ is a subalgebra of $A$. We claim that $f|g| \in \mathcal{R}(B)$. Observe first that $(f + f^3)^+, \ (g + g^3)^+$ and $(fg)^+$ are in $\mathcal{R}(B)$. According to (ii) in Lemma 3.2, we get $f^+g^+ \in \mathcal{R}(B)$. Since $f^+g^- = (f^+)(-g^+) \in \mathcal{R}(B)$ and $f^-g^- = (-f)^+(-g)^+ \in \mathcal{R}(B)$ we derive that $f|g| \in \mathcal{R}(B)$.

Suppose now that $\mathcal{P}_n$ is true and pick two elements $f$ and $g$ in $L_{n+1}$. Then $f$ and $g$ can be written as a finite sums
\[ f = \sum_{\alpha} f_{\alpha} + \sum_{\beta} |f_{\beta}| \text{ and } g = \sum_{\gamma} g_{\gamma} + \sum_{\delta} |g_{\delta}| \]
where $f_{\alpha}, f_{\beta}, g_{\gamma}$ and $g_{\delta}$ belong to $L_n$. Applying the inductive hypothesis to these elements, a direct calculation shows therefore that $f\cdot g \in \mathcal{R}(B)$. It remains to prove that $f|g| \in \mathcal{R}(B)$. This is obvious in the case when $f$ is positive because $f|g| = |f\cdot g| \in \mathcal{R}(B)$. In the general case, using the decomposition of $f$ in (3.3), it is sufficient to show that $f_{\alpha}|g| \in \mathcal{R}(B)$ and $|f_{\beta}||g| \in \mathcal{R}(B)$. But $|f_{\beta}||g| = |f_{\beta}g| \in \mathcal{R}(B)$ and since $f_{\alpha}^+$ and $f_{\alpha}^-$ are in $L_{n+1}$ for all $\alpha$ we get by the positive case
\[ f_{\alpha} \cdot |g| = f_{\alpha}^+|g| - f_{\alpha}^-|g| \in \mathcal{R}(B) \]
The proof is now complete. \[
\square
\]
The next result is an operator-version of Theorem 3.3. In fact if we consider a Riesz homomorphism $T$ from $\mathcal{R}(B)$ to an $f$-algebra $C$ then $T$ is an algebra homomorphism if and only if its restriction to $B$ is an algebra homomorphism.

**Theorem 3.4.** Let $B$ be a subalgebra of a semiprime $f$-algebra $A$ and $C$ be an $f$-algebra. Let $T : \mathcal{R}(B) \to C$ be a Riesz homomorphism. Then, $T$ is multiplicative if and only if $T$ is multiplicative on $B$, that is $T(fg) = T(f)T(g)$ for all $f, g \in B$.

**Proof.** Only one implication requires proof. Assume then that $T(fg) = T(f)T(g)$ for all $f, g \in B$. We proceed as in proof of Theorem 3.3 by proving a similar property to $\mathcal{P}_n$ which is the following

$$\mathcal{P}_n' : \text{for all } f, g \in L_n, \ T(fg) = Tfg \text{ and } T(f |g|) = T(f)|Tg|.$$

Pick two elements $f$ and $g$ in $B$. Since $T$ is multiplicative on $B$, we get $T(fg) = T(f)T(g)$. We claim that $T(f |g|) = T(f)T(|g|)$. Using the fact that $T$ is a Riesz homomorphism and according to Lemma 3.2 we can write

$$T(f^+g^+) = T((fg)^+) \land ((T(f) + T(f^3))^+ + (T(g) + T(g^3))^+)$$

$$= (T(f)T(g))^+ \land ((T(f) + T(f^3))^+ + (T(g) + T(g^3))^+)$$

$$= T(f)^+T(g)^+.$$

Slight modifications in the above equalities yields to:

$$T(f^-g^+) = T(f)^-T(g)^+, \ T(f^-g^-) = T(f)^-T(g)^+ \text{ and } T(f^+g^-) = T(f)^+T(g)^-$$

This implies that

$$T(f |g|) = T(f)T(|g|).$$

So $\mathcal{P}_1'$ is true. It is clear now that the rest of the proof is similar to the previous one. \hfill \Box

4. Some results on bilinear maps.

In this section we will collect some proprieties of bilinear maps between $f$-algebras. Recall that for a map $S : X \rightarrow Y$, where $X$ and $Y$ are two vector spaces, the zero-set of $S$ is the set

$$N(S) = \{ x \in X : S(x) = 0 \}.$$

It is well known that if $\varphi$ and $\psi$ are two linear functionals on a vector space $E$ such that $N(\varphi) \subseteq N(\psi)$, then there exists a scalar $\lambda$ such that $\psi = \lambda \varphi$. The following lemma shows that this result can be extended to bilinear functionals.

**Lemma 4.1.** Let $E$ and $F$ be two vector spaces, $\varphi, \psi : E \times F \rightarrow \mathbb{R}$ be two bilinear functionals on $E \times F$ such that $N(\varphi) \subseteq N(\psi)$ then $\psi = \lambda \varphi$ for some $\lambda \in \mathbb{R}$.

**Proof.** If $\varphi = 0$ the result is obvious. Otherwise, pick a fixed element $e \in E$ and consider the linear functionals $\varphi_e = \varphi(e, \cdot)$ and $\psi_e = \psi(e, \cdot)$. Since $N(\varphi_e) \subseteq N(\psi_e)$, there exists some real number $\lambda(e)$ such that

$$\psi(e, h) = \lambda(e) \varphi(e, h) \text{ for all } h \in E. \quad (4.1)$$
We claim that \( \lambda(e) \) does not depend on \( e \). If \( \varphi_e = 0 \), then \( \psi_e = 0 \) and any real \( \lambda(e) \) is convenient. It is sufficient then to show that \( \lambda(e) = \lambda(f) \) for all \( e, f \) such that \( \varphi_e \) and \( \varphi_f \) are not null linear functionals. In this case there exists \( g \in F \) such that \( \varphi(e, g) \neq 0 \) and \( \varphi(f, g) \neq 0 \) (\( E \) cannot be the union of two strict subspaces). Put \( \alpha = \frac{\varphi(e, g)}{\varphi(f, g)} \) and observe that \( \varphi(e - \alpha f, g) = 0 \). So \( \psi(e - \alpha f, g) = 0 \) which implies that
\[
\psi(e, g) = \alpha \psi(f, g).
\]
We obtain
\[
\lambda(e) \varphi(e, g) = \alpha \lambda(f) \varphi(f, g) = \lambda(f) \varphi(e, g)
\]
which proves that \( \lambda(e) = \lambda(f) \) and we are done. \( \square \)

As a consequence of the previous lemma we obtain the following result.

**Proposition 4.2.** Let \( A_1 \) and \( A_2 \) be \( f \)-algebras with unit elements \( e_1 \) and \( e_2 \), respectively, and let \( T : A_1 \times A_2 \rightarrow \mathbb{R} \) be a Riesz bimorphism such that \( T(e_1, e_2) = 1 \). Then \( T \) is multiplicative.

**Proof.** Let \( a_i \in A_i \) such that \( 0 \leq a_i \leq e_i \), \( i = 1, 2 \). We define the bilinear functional \( S \) by
\[
S(x_1, x_2) = T(a_1 x_1, a_2 x_2) \quad \text{for all} \ (x_1, x_2) \in A_1 \times A_2.
\]

It follows from the following
\[
|T(a_1 x_1, a_2 x_2)| = T(|a_1 x_1|, |a_2 x_2|) \leq T(|x_1|, |x_2|) = |T(x_1, x_2)|
\]
that \( N(T) \subseteq N(S) \). According to Lemma 4.1 there exists a real number \( \lambda \) such that \( S = \lambda T \). By hypothesis,
\[
\lambda = \lambda T(e_1, e_2) = S(e_1, e_2) = T(a_1, a_2).
\]
So one has
\[
T(a_1 x_1, a_2 x_2) = T(a_1, a_2).T(x_1, x_2) \quad \text{for all} \ x_1 \in A_1, \ x_2 \in A_2 \quad (4.2)
\]

Now if \( a_1 \) and \( a_2 \) are positive, we get for all \( n = 1, 2, \ldots \) and \( x_1 \in A_1, \ x_2 \in A_2, \)
\[
T((a_1 \land n e_1)x_1, (a_2 \land n e_2)x_2) = T(a_1 \land n e_1, a_2 \land n e_2)T(x_1, x_2)
\]
The equality
\[
T(a_1 x_1, a_2 x_2) = T(a_1, a_2).T(x_1, x_2) \quad \text{for all} \ x_1 \in A_1 \text{ and } x_2 \in A_2.
\]
can be deduced by relatively uniform continuity technics and Lemma 2.1. \( \square \)

The next theorem gives the connection between Riesz bimorphisms and multiplicative bilinear maps on \( f \)-algebras. This theorem is proved in [4, Theorem 1] and we give here an alternative proof which can be adapted to multilinear maps.

**Theorem 4.3.** Let \( A_1, A_2 \) and \( B \) be \( f \)-algebras with unit elements \( e_1, e_2 \) and \( e_B \), respectively and \( T : A_1 \times A_2 \rightarrow B \) be a lattice bimorphism satisfying \( T(e_1, e_2) = e_B \). Then \( T \) is multiplicative bilinear map.
Proof. Let \( a_i, x_i \in A_i \) for \( i = 1, 2 \) and consider the the \( f \)-subalgebra \( C_i \) of \( A_i \) generated by \( \{e_i, x_i, a_i\} \). Let \( D \) be the \( f \)-subalgebra of \( B \) generated by \( T(C_1 \times C_2) \) which is finitely generated. Take now an arbitrary multiplicative positive functional \( \omega \) on \( D \). Applying Proposition 4.2 to the bilinear functional \( \omega \circ T : C_1 \times C_2 \rightarrow \mathbb{R} \), we get

\[
\omega(T(a_1x_1, a_2x_2)) = \omega(T(a_1, a_2)) \omega(T(x_1, x_2)) = \omega(T(a_1, a_2)T(x_1, x_2))
\]

Since the set of all real-valued multiplicative lattice homomorphisms on \( D \) separates the points of \( D \) (see [5, Corollary 7]) we obtain

\[
T(a_1x_1, a_2x_2) = T(a_1, a_2)T(x_1, x_2)
\]

for all \( a_1, x_1 \in A_1 \) and \( a_2, x_2 \in A_2 \).

This concludes the proof. \( \square \)

5. The Riesz tensor product of \( f \)-algebras

The tensor product of two \( f \)-algebras has not been studied before. It is natural to think to endow the Riesz tensor product \( A \overline{\otimes} B \) of \( f \)-algebras with an \( f \)-algebra structure. A first way is to extend the natural multiplication on the algebraic tensor product \( A \otimes B \) to \( A \overline{\otimes} B \). Another approach consists to embed \( A \overline{\otimes} B \) in some \( f \)-algebra \( C \) and study the stability of \( A \overline{\otimes} B \) by multiplication. Focus first on the special and important \( C(X) \)'s case.

Let \( X \) and \( Y \) be compact spaces. The algebraic tensor product \( C(X) \otimes C(Y) \) of the function-algebras \( C(X) \) and \( C(Y) \) can be viewed as the vector subspace of \( C(X \times Y) \) consisting of all functions \( h(s, t) = \sum f_i(s)g_i(t) \) where \( f_i \in C(X) \) and \( g_i \in C(Y) \). The Riesz tensor product \( C(X) \overline{\otimes} C(Y) \) is the Riesz subspace of \( C(X \times Y) \) generated by the algebra \( C(X) \otimes C(Y) \) (see for instance [11]). The following result is now an immediate consequence of Theorem 3.3.

**Theorem 5.1.** Let \( X \) and \( Y \) be compact spaces. Then the Riesz tensor product \( C(X) \overline{\otimes} C(Y) \) of \( C(X) \) and \( C(Y) \) is an \( f \)-subalgebra of \( C(X \times Y) \).

In the sequel we study the general case. Let \( A \) and \( B \) be two semiprime \( f \)-algebras. The algebraic tensor product \( A \otimes B \) can be furnished in a canonical way with an algebra product satisfying

\[(a \otimes b)(a' \otimes b') = aa' \otimes bb' \text{ for all } a, a' \in A \text{ and } b, b' \in B.\]

We start our discussion about the proprieties of the Riesz tensor product with this lemma.

**Lemma 5.2.** Let \( A \) and \( B \) be two \( f \)-algebras with unit elements \( e_A \) and \( e_B \) respectively, then \( e_A \otimes e_B \) is a weak order unit in \( A \overline{\otimes} B \).

**Proof.** Let \( 0 \leq u \) in \( A \overline{\otimes} B \) such that \( u \wedge e_A \otimes e_B = 0 \). We have to show that \( u = 0 \). We argue by contradiction and we suppose that \( u \neq 0 \). By [8, Theorem 5.8. (e)], there exist \( x \in A_+ \) and \( y \in B_+ \) such that

\[0 < x \otimes y \leq u.\]

Therefore

\[0 \leq x \wedge e_A \otimes y \wedge e_B \leq u \wedge e_A \otimes e_B = 0\]


So \(x \wedge e_A \otimes y \wedge e_B = 0\) and then \(x \wedge e_A = 0\) or \(y \wedge e_B = 0\). Hence \(x = 0\) or \(y = 0\) and we obtain
\[x \otimes y = 0\]
which is a contradiction.

It follows from Lemma 5.2 that the universal completion \((A \overline{\otimes} B)^u\) of \(A \overline{\otimes} B\) can be equipped with an \(f\)-algebra multiplication denoted by \(\ast\) such that \(e_A \otimes e_B\) is the unit element. The next result shows that \(\ast\) extends the canonical multiplication on \(A \otimes B\).

**Proposition 5.3.** Let \(A\) and \(B\) be tow \(f\)-algebras with unit \(e_A\) and \(e_B\) respectively. Then \(u \ast v = u.v\) for all \(u, v \in A \otimes B\). So \(A \otimes B\) can be considered as a subalgebra of \((A \overline{\otimes} B)^u\).

**Proof.** Let \(\sigma\) be the canonical mapping defined by:
\[
\sigma : A \times B \longrightarrow (A \overline{\otimes} B)^u
\]
\[
(x, y) \longmapsto x \otimes y
\]
Since \(\sigma\) is a Riesz bimorphism and \(\sigma(e_A, e_B) = e_A \otimes e_B\) it follows from Theorem 4.3 that \(\sigma\) is multiplicative. Consequently,
\[
\sigma((x, y).(x', y')) = \sigma(x, y) \ast \sigma(x', y')
\]
for all \(x, x' \in A\) and \(y, y' \in B\), which means that
\[
xx' \otimes yy' = (x \otimes y) \ast (x' \otimes y')
\]
This yields to
\[
(x \otimes y).(x' \otimes y') = (x \otimes y) \ast (x' \otimes y')
\]
The result follows immediately by bilinearity. \(\square\)

The previous lemma enables us to make a connection between our tow approaches mentioned in the beginning of this section and leads us to prove the main theorem of this section.

**Theorem 5.4.** Let \(A, B\) be two \(f\)-algebras with unit elements \(e_A\) and \(e_B\) respectively. Then the canonical multiplication on the algebraic tensor product \(A \otimes B\) can be extended to a multiplication on the Riesz tensor product \(A \overline{\otimes} B\), such that \(A \otimes B\) is an \(f\)-algebra with unit. Moreover, \(A \overline{\otimes} B\) satisfy the multiplicative universal property (MUP):

For all multiplicative and Riesz bimorphism \(\psi : A \times B \longrightarrow C\) where \(C\) is an \(f\)-algebra, there exists a unique algebra and Riesz homomorphism \(\psi^\otimes : A \overline{\otimes} B \longrightarrow C\) such that
\[
\psi^\otimes(f \otimes g) = \psi(f, g)\text{ for all } f \in A, g \in B.
\]

**Proof.** Observe first that \(A \overline{\otimes} B\) is the Riesz subspace \(R(A \otimes B)\) generated by \(A \otimes B\) in \((A \overline{\otimes} B)^u\). Keeping in mind that \(A \otimes B\) is a subalgebra \((A \overline{\otimes} B)^u\) (Proposition 5.3), it follows from Theorem 3.3 that \(A \overline{\otimes} B\) is an \(f\)-algebra with unit element \(e_A \otimes e_B\). It remains to prove that \(A \overline{\otimes} B\) satisfies the MUP. Let \(C\) be an \(f\)-algebra and \(\psi : A \times B \longrightarrow C\) be a multiplicative Riesz bimorphism, then there exists a lattice homomorphism \(\psi^\otimes : A \overline{\otimes} B \longrightarrow C\) such that \(\psi^\otimes(x \otimes y) = \psi(x, y)\) for all
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x ∈ A, y ∈ B. Since ψ is multiplicative then the restriction of ψ ⊗ to A ⊗ B is multiplicative, it follows from Theorem 3.4 that ψ ⊗ is multiplicative on the whole space.

□

Our last result concerns the Riesz tensor product of semiprime f-algebras. It is well known that any semiprime f-algebra C can be seen as f-subalgebra of Orth(C), where Orth(C) indicates the unital f-algebra of all orthomorphisms on C.

Theorem 5.5. Let A and B be two semiprime f-algebras. Then the Riesz tensor product A ⊗ B can be endowed with an f-algebra product, which extends the algebraic multiplication. In addition A ⊗ B satisfies the following multiplicative universal property (MUP): For all multiplicative and Riesz bimorphism ψ : A × B → C, where C is an f-algebra, there exists a unique algebra and Riesz homomorphism ψ ⊗ : A ⊗ B → C such that

ψ ⊗(f ⊗ g) = ψ(f, g) for all f ∈ A, g ∈ B.

Proof. It follows from [1, Corollary 4.5] that A ⊗ B is the Riesz subspace generated by A ⊗ B in Orth(A) ⊗ Orth(B). By Theorem 5.4 Orth(A) ⊗ Orth(B) is an f-algebra. According to Theorem 3.3, A ⊗ B is an f-subalgebra of Orth(A) ⊗ Orth(B). The proof of MUP is similar to the one of Theorem 5.4.

□

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