PARAMETRIZATION OF K-VIRTUAL DRINFELD MODULES OF RANK TWO

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Abstract. For an algebraic extension $K$ of the rational function field $\mathbb{F}_q(T)$ over a finite field, we introduce the notion of $K$-virtual Drinfeld modules as a function field analogue of $\mathbb{Q}$-curves, which are elliptic curves over $\bar{\mathbb{Q}}$ isogenous to all its Galois conjugates. Our goal in this article is to prove that all $K$-virtual Drinfeld modules of rank two with no complex multiplication are parametrized up to isogeny by $K$-rational points of a quotient curve of the Drinfeld modular curve $Y_0(n)$ with some square-free level $n$, which is an analogue of Elkies’ well-known result on $\mathbb{Q}$-curves.

1. Introduction

An elliptic curve $E$ over an algebraic closure $\bar{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ is called a $\mathbb{Q}$-curve if $E$ is isogenous to the conjugate $s^*E$ for all $s \in G_2 := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For the first time, the notion of $\mathbb{Q}$-curves was introduced by Gross [Gro93] with a much more restrictive definition, and later considered by Ribet in general setting. It is known that $\mathbb{Q}$-curves have various interesting arithmetic properties. For instance, Ribet [Rib92] characterizes $\mathbb{Q}$-curves as elliptic curves over $\bar{\mathbb{Q}}$ which are quotients of abelian varieties of “GL$_2$-type” over $\mathbb{Q}$ and shows that every $\mathbb{Q}$-curve is modular, meaning that it is a quotient of some modular Jacobian $J_1(N)$.

Clearly the condition of being $\mathbb{Q}$-curves is invariant by isogeny and so the classification of isogeny classes of $\mathbb{Q}$-curves is a natural problem. As an answer to this, Elkies proved in his unpublished paper [Elk93] that every $\mathbb{Q}$-curve with no complex multiplication is isogenous to that whose $j$-invariant arising from a $\mathbb{Q}$-rational point of the quotient curve $Y_*(N)$ of the elliptic modular curve $Y_0(N)$ of some square-free level $N$ by all Atkin-Lehner involutions. (The revised version [Elk01] of [Elk93] is available. Notice that Elkies in fact proved this for “$k$-curves”, where $k$ is an arbitrary number field.) Moreover, for any square-free $N$, it follows that any $\mathbb{Q}$-curve $E$ arising from a point of $Y_*(N)(\mathbb{Q})$ can be defined over a polyquadratic extension of $\mathbb{Q}$ (i.e., a finite abelian extension of $\mathbb{Q}$ with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^n$) and that $E$ admits an isogeny to $s^*E$ of degree dividing $N$ for any $s \in G_2$. Such $E$ is called a central $\mathbb{Q}$-curve of degree $N$. Therefore the existence of non-CM $\mathbb{Q}$-rational points of $Y_*(N)$ is equivalent to that of non-CM central $\mathbb{Q}$-curves of degree $N$. Elkies conjectured that there are no non-CM $\mathbb{Q}$-rational points of $Y_*(N)$ if $N$ is sufficiently large.

The purpose of this article is to prove a function field analogue of Elkies’ result on $\mathbb{Q}$-curves. As is well-known, there are many beautiful analogies between number fields and function fields. In 1974, Drinfeld introduced the analogue of elliptic curves under the name elliptic modules in his original paper [Dri74], which are today called Drinfeld modules. Drinfeld modules share many arithmetic properties with elliptic curves and so we may expect that there is a rich theory of a Drinfeld module analogue of $\mathbb{Q}$-curves.

Let $A = \mathbb{F}_q[T]$ be the polynomial ring over a fixed finite field $\mathbb{F}_q$ of characteristic $p$ and let $Q = \mathbb{F}_q(T)$ be the rational function field. In this paper, for any algebraic extension $K$ of $Q = \mathbb{F}_q(T)$, we introduce the notion of $K$-virtual Drinfeld $A$-modules (Definition 3.3) as an analogue of $\mathbb{Q}$-curves. Let $n$ be a non-zero ideal of $A$. In the rank-two case, we see that all Drinfeld $A$-modules arising (in the sense of Proposition 4.6) from $K$-rational points of $Y_*(n)$ are $K$-virtual if $p$ is odd, where $Y_*(n)$ is the quotient curve of the Drinfeld modular...
curve $Y_0(n)$ by all Atkin-Lehner involutions. Notice that even if $p = 2$, at least all non-CM Drinfeld $A$-modules arising from $K$-rational points of $Y_0(n)$ are $K$-virtual. Conversely, adapting Elkies’ graph-theoretic method to Drinfeld $A$-modules, we prove that all non-CM $K$-virtual Drinfeld $A$-modules of rank two are classified (up to isogeny) by $K$-rational points of some $Y_0(n)$:

**Theorem 1.1.** Let $\phi$ be a $K$-virtual Drinfeld $A$-module of rank two with no complex multiplication. Then there exists a non-zero square-free $n$, depending only on the isogeny class of $\phi$, such that

(i) $\phi$ is isogenous to a Drinfeld $A$-module arising from a $K$-rational point of $Y_0(n)$,

(ii) if an ideal $n'$ satisfies (i), then $n|n'$.

By Proposition 4.6 and Theorem 1.1 we see that every non-CM $K$-virtual Drinfeld $A$-module of rank two is isogenous to some $\phi$ arising from a point of $Y_0(n)(K)$ such that $\phi$ can be defined over a polyquadratic extension of $K$ and admits isogenies to all Galois conjugates of degree dividing $n$. Following the $\mathbb{Q}$-curve case, we call such $\phi$ a central $K$-virtual Drinfeld $A$-modules of degree $n$. Thus for any square-free $n$, the existence of non-CM $K$-rational points of $Y_0(n)$ is equivalent to that of non-CM central $K$-virtual Drinfeld $A$-modules of rank two of degree $n$.

We may expect that some higher-dimensional analogues of the above results hold. In the number field setting, as a higher-dimensional generalization of $\mathbb{Q}$-curves, the notion of abelian $k$-varieties are studied by many people. For example, using Galois cohomological method, Ribet [Rib94] and Pyle [Pyl04] show that any abelian $k$-variety with some conditions (so-called building block) can be defined up to isogeny over a polyquadratic extension of $k$. In [GM09], Guitart and Molina parametrize abelian $k$-surfaces with quaternionic multiplication by $k$-rational points of the quotient of a Shimura curve by all Atkin-Lehner involutions. In the function field setting, as a higher-dimensional generalizations of Drinfeld $A$-modules and analogues of abelian varieties, Anderson [And86] defined abelian $t$-modules and the dual notion of $t$-motives. For such objects, we can consider the “$K$-virtuality” in the same way. So we are interested in the classification of $K$-virtual abelian $t$-modules or $t$-motives up to isogeny. However, our proof in this paper depends on special properties of rank-two Drinfeld $A$-modules and so it may be difficult to extend our arguments to the higher-dimensional cases.

The organization of this article is as follows. In Section 2, we review well-known facts on Drinfeld $A$-modules and recall the definitions of degree and dual of isogenies, which are fundamental tools in our work. In Section 3, after showing basic properties of Galois conjugates of Drinfeld $A$-modules, we define $K$-virtual Drinfeld $A$-modules of arbitrary rank. We also give a non-trivial example of $Q$-virtual Drinfeld $A$-modules of rank two (Example 3.3). In the remaining sections, we restrict our attention to rank-two Drinfeld $A$-modules with no complex multiplication. Section 4 is devoted to a study of rational points of the curve $Y_0(n)$. Using moduli interpretation, we prove that if a $K$-rational point of $Y_0(n)$ satisfies some mild condition, then it gives raise to a family of $K$-virtual rank-two Drinfeld $A$-modules (Proposition 4.6). Finally, in Section 5, we give a proof of Theorem 1.1. To find the $n$ and a $K$-rational point of $Y_0(n)$ as in Theorem 1.1, we consider a Galois action on an undirected tree (so called isogeny tree) attached to a given $K$-virtual Drinfeld $A$-module.

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## 2. Isogenies of Drinfeld $A$-modules

### 2.1. Preliminary

We begin by fixing the notation in this paper. As in Section 1, let $A := \mathbb{F}_q[T]$ be the polynomial ring in one variable $T$ over the finite field $\mathbb{F}_q$ with $q$-elements of characteristic $p > 0$ and write $Q := \mathbb{F}_q(T)$ for the field of fractions of $A$. We notice that they play the same role of $\mathbb{Z}$ and $\mathbb{Q}$ respectively. Let $K/Q$ be an algebraic extension and fix an
algebraic closure $\bar{K}$ of $K$. In this paper, we regard every algebraic extension of $Q$ as a subfield of $\bar{K}$. Denote by $K^{\text{sep}} \subset \bar{K}$ the separable closure of $K$ and write $G_K := \text{Gal}(K^{\text{sep}}/K)$ for its absolute Galois group. Denote by $K\{\tau\}$ the skew polynomial ring over $K$ in one variable $\tau$ satisfying $\tau c = c^q \tau$ for any $c \in K$. Notice that $K\{\tau\}$ is isomorphic to the ring $\text{End}_{\mathbb{F}_q}((\mathbb{G}_{a,K})$ of all $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_{a,K}$ over $K$. Define the differential map

$$\partial: K\{\tau\} \to K$$

by $\partial(\sum_{i=0}^n c_i \tau^i) = c_0$. It is an $\mathbb{F}_q$-algebra homomorphism.

A Drinfeld $A$-module over $K$ is an $\mathbb{F}_q$-algebra homomorphism

$$\phi: A \to K\{\tau\}$$

such that $\partial(\phi_\alpha) = a$ for any $a \in A$ and $\phi_\alpha \neq a$ for some $a \in A$. By definition, $\phi$ is completely determined by the image of $T$: $\phi_T = T + c_1 \tau + \cdots + c_r \tau^r \in K\{\tau\}$ with $c_r \neq 0$. The integer $r$ is called the rank of $\phi$ and denoted by $\text{rk} \phi$.

**Remark 2.1.** In fact, Drinfeld modules are defined more generally: let $C$ be a smooth projective, geometrically irreducible curve over $\mathbb{F}_q$ and let $\infty \in C$ be a fixed closed point. Let $A := \Gamma(\mathcal{O}_C, C \setminus \{\infty\})$ be the ring of rational functions on $C$ regular outside $\infty$. Fix an $\mathbb{F}_q$-algebra homomorphism $\iota: A \to K$, which is not necessarily injective. Then a Drinfeld $A$-module over $(K, \iota)$ is an $\mathbb{F}_q$-algebra homomorphism

$$\phi: A \to K\{\tau\}$$

such that $\partial(\phi_\alpha) = \iota(a)$ for any $a \in A$ and $\phi_\alpha \neq \iota(a)$ for some $a \in A$. If $\iota$ is injective (resp. $\text{Ker} \iota \neq 0$), then $\phi$ is said to be of generic characteristic (resp. special characteristic). Notice that several properties of Drinfeld modules with special characteristic are different from those with generic characteristic; see [DH87] and [Gos96] for details.

In this paper, we mainly consider the case where $C = \mathbb{P}_\mathbb{F}_q^1, \infty = (1/T), A = \mathbb{F}_q[T]$, and $\iota$ is the inclusion $A \hookrightarrow Q \subset K$.

Let $\phi$ be a Drinfeld $A$-module over $K$ and let $L$ be a $K$-algebra. For any $\lambda \in L$ and $\mu = \sum_{i=0}^n c_i \lambda^i \in K\{\tau\}$, set $\mu(\lambda) := \sum_{i=0}^n c_i \lambda^i \in L$. Then $\phi$ endows the additive group $\mathbb{G}_{a,K}(L) = L$ with a new $A$-module structure by $a \cdot \lambda := \phi_\alpha(\lambda)$ for any $a \in A$ and $\lambda \in L$. Denote this new $A$-module by $\phi L$. If $L = \bar{K}$, then for a non-zero $a \in A$, we denote the module of $a$-torsion points of $\phi$ by

$$\phi[a] := \text{Ker} \left( \phi_\alpha : \bar{K} \to \bar{K} \right).$$

It is in fact a finite $A$-submodule of $\phi K^{\text{sep}}$ since $\partial(\phi_\alpha) = a \neq 0$ implies that all $\lambda$ with $\phi_\alpha(\lambda) = 0$ are separable over $K$. Hence $G_K$ acts on $\phi[a]$. For a non-zero ideal $n \subset A$, we also define the module of $n$-torsion points of $\phi$ by

$$\phi[n] := \bigcap_{0 \neq a \in n} \phi[a].$$

Clearly $\phi[n] = \phi[a]$ holds if $n = (a)$. It is known that $\phi[n]$ is a $G_K$-stable free $A/n$-module of rank equal to $\text{rk} \phi$.

Let $\phi$ and $\psi$ be Drinfeld $A$-modules over $K$. An isogeny $\mu : \phi \to \psi$ is a non-zero element $\mu \in K\{\tau\}$ such that

$$\mu \phi_\alpha = \psi_\alpha \mu$$

for any $a \in A$, and then $\phi$ is said to be isogenous to $\psi$. If $\mu \in \bar{K}^\times$, then we call it an isomorphism. Clearly $\mu \in K\{\tau\}$ yields an isogeny $\mu : \phi \to \psi$ if and only if $\mu \phi_T = \psi_T \mu$ holds. Notice that every isogeny $\mu$ is in fact contained in $K^{\text{sep}}\{\tau\}$ by Proposition 2.2 below. By definition, an isogeny $\mu : \phi \to \psi$ induces a surjective $A$-module homomorphism

$$\mu : \phi \bar{K} \to \psi \bar{K}.$$
For any field \( L/K \), an isogeny \( \mu: \phi \to \psi \) is called an \( L \)-rational isogeny (or \( L \)-isogeny, for short) if \( \mu \in L\{\tau\} \). We say that two isogenies \( \mu: \phi \to \psi \) and \( \eta: \phi' \to \psi' \) are \( L \)-equivalent if there exist \( L \)-rational isomorphisms \( \nu: \phi \to \phi' \) and \( \lambda: \psi \to \psi' \) such that \( \eta = \lambda \mu \nu^{-1} \).

We write \( \text{Hom}_L(\phi, \psi) := \{ \mu \in L\{\tau\}; \mu \phi \tau = \psi \tau \} \) and \( \text{End}_L(\phi) := \text{Hom}_L(\phi, \phi) \). Then \( \phi_a \in \text{End}_L(\phi) \) for any \( a \in A \) and so \( A \) is a subring of \( \text{End}_L(\phi) \) and \( \text{Hom}_L(\phi, \psi) \) becomes an \( A \)-module by \( a \cdot \mu := \mu \phi_a \). It is known that \( \text{End}_L(\phi) \) is a commutative \( A \)-algebra and a free \( A \)-module of rank \( \leq \text{rk} \phi \). Therefore \( \text{End}_L(\phi) \otimes_A Q \) is a finite extension of \( Q \).

The following proposition says that all isogenies are \( K^{\text{sep}} \)-rational.

**Proposition 2.2** ([Gos96 Proposition 4.7.4.]). Let \( \phi \) and \( \psi \) be Drinfeld \( A \)-modules over \( K \). For an arbitrary extension \( \Omega \) of \( K^{\text{sep}} \), the inclusion \( \text{Hom}_{K^{\text{sep}}}(\phi, \psi) \hookrightarrow \text{Hom}_\Omega(\phi, \psi) \) is an equality.

The restriction of the differential map \( \partial: K\{\tau\} \to K \) to \( \text{End}_K(\phi) \) induces an injective \( A \)-algebra homomorphism \( \partial: \text{End}_K(\phi) \hookrightarrow K \), so that its image is an \( A \)-order of a finite extension \( F/Q \) satisfying \( F \subset K \). If \( \text{rk} \phi > 1 \) and \( [F: Q] > 1 \), then we say that \( \phi \) has complex multiplication by \( F \). If in addition \( [F: Q] = \text{rk} \phi \), then we especially say that \( \phi \) has full complex multiplication. In this case, it is known that \( F \) has only one place lying above the place \( \infty = (1/T) \) of \( Q \).

**2.2. Degree and dual of isogenies.** Let \( \mu: \phi \to \psi \) be an isogeny between Drinfeld \( A \)-modules over \( K \). Since \( \text{Ker} \mu \) is a finite torsion \( A \)-submodule of \( \phi K^{\text{sep}} \), it is isomorphic to \( \bigoplus_{i=1}^n A/\mathfrak{n}_i \) for suitable \( n \) and ideals \( \mathfrak{n}_i \subset A \). Then the product

\[
\deg \mu := \prod_{i=1}^n \mathfrak{n}_i
\]

is called the degree of \( \mu \). Clearly \( \deg \mu = A \) if and only if \( \mu \) is an isomorphism. If \( \deg \mu = n \), then \( \mu \) is called an \( n \)-isogeny. If \( \text{Ker} \mu \cong A/\mathfrak{n} \), then \( \mu \) is called a cyclic \( n \)-isogeny. For example, if \( a \in A \) generates \( n \) and \( \text{rk} \phi = r \), then we have \( \text{deg} \phi_a = n^r \) since \( \text{Ker} \phi_a = \phi[\mathfrak{n}] \cong (A/\mathfrak{n})^{\otimes r} \).

Namely \( \phi_a \) is an \( n^r \)-isogeny.

A finite \( A \)-submodule \( \Lambda \subset \phi K^{\text{sep}} \) with \( \Lambda \cong A/\mathfrak{n} \) is called a cyclic \( n \)-kernel of \( \phi \). Then it follows that there is a cyclic \( n \)-isogeny \( \mu: \phi \to \psi \) with \( \text{Ker} \mu = \Lambda \) (cf. [DHST, pp. 37]) and such \( \mu \) is unique up to \( K^{\text{sep}} \)-equivalence. If \( \Lambda \) is \( G_L \)-stable for some field \( L/K \), then \( \Lambda \) is said to be \( L \)-rational. It follows that \( \Lambda \) is \( L \)-rational if and only if \( \mu \) can be taken to be \( L \)-rational.

**Lemma 2.3.** Let \( \mu: \phi_1 \to \phi_2 \) and \( \eta: \phi_2 \to \phi_3 \) be isogenies. Then \( \text{deg} \eta \mu = \text{deg} \eta \text{deg} \mu \).

**Proof.** It immediately follows from the exact sequence \( 0 \to \text{Ker} \mu \to \text{Ker} \eta \mu \to \text{Ker} \eta \to 0 \).

**Proposition 2.4.** Let \( \mu: \phi \to \psi \) be a \( K \)-rational isogeny and let \( a \in A \) be a non-zero element which annihilates \( \text{Ker} \mu \). Then there exists a unique isogeny \( \eta: \psi \to \phi \) such that \( \eta \) is \( K \)-rational and

\[
\eta \mu = \phi_a \quad \text{and} \quad \text{Ker} \eta = \mu(\phi[a]),
\]

\[
\mu \eta = \psi_a \quad \text{and} \quad \text{Ker} \mu = \eta(\psi[a]).
\]

**Proof.** The existence of \( \eta \) is well-known; see [Gos96 §4.7] for example. The uniqueness follows from the right division algorithm ([Gos96 Proposition 1.6.2]).

Let \( \mu: \phi \to \psi \) be an \( n \)-isogeny and let \( a \in A \) be the monic generator of \( n \). Then there exists a unique isogeny \( \hat{\mu}: \psi \to \phi \) such that \( \hat{\mu} \mu = \phi_a \) and \( \mu \hat{\mu} = \psi_a \) by Proposition 2.4. We call \( \hat{\mu} \) the dual isogeny of \( \mu \).

**Remark 2.5.** If \( \phi \) is of rank \( r \), then \( n^r = \text{deg} \phi_a = \text{deg} \hat{\mu} \text{deg} \mu = n \text{deg} \hat{\mu} \). Hence \( \text{deg} \hat{\mu} = n^{r-1} \). In particular, we have \( \text{deg} \mu = \text{deg} \hat{\mu} \) if \( \text{rk} \phi = 2 \).
By Proposition 2.4, any \( K \)-rational isogeny \( \mu : \phi \to \psi \) has the inverse \( \mu^{-1} \in \text{Hom}_{K}(\psi, \phi) \otimes_{A} Q \) and hence we have the isomorphisms
\[
\text{End}_{K}(\phi) \otimes_{A} Q \cong \text{Hom}_{K}(\phi, \psi) \otimes_{A} Q \cong \text{End}_{K}(\psi) \otimes_{A} Q
\]
of \( Q \)-vector spaces. This implies that \( \text{Hom}_{K}(\phi, \psi) \) is a free \( A \)-module whose rank is equal to that of \( \text{End}_{K}(\phi) \). Hence if \( \phi \) has complex multiplication, then so does \( \psi \).

For convenience, we introduce the following.

**Definition 2.6.** We say that a \( K \)-rational isogeny \( \mu : \phi \to \psi \) is primitive if there are no \( K \)-rational isogenies \( \eta : \phi \to \psi \) satisfying \( \deg \eta | \deg \mu \) and \( \deg \eta \neq \deg \mu \).

If \( \phi \) has no complex multiplication, then \( \mu \) is primitive if and only if \( \text{Hom}_{K}(\phi, \psi) \) is generated by \( \mu \) as an \( A \)-module. In this case, every \( \eta \in \text{Hom}_{K}(\phi, \psi) \) is given by \( \eta = \mu \phi_{a} \) for some \( a \in A \). Therefore we have the following.

**Proposition 2.7.** Let \( \mu_{1} \) and \( \mu_{2} \) be isogenies in \( \text{Hom}_{K}(\phi, \psi) \). Suppose that \( \phi \) has no complex multiplication. Then \( \deg \mu_{1} = \deg \mu_{2} \) if and only if \( \mu_{1} = \xi \mu_{2} \) for some \( \xi \in \mathbb{F}_{q}^{\times} \).

**Proof.** By the absence of complex multiplication, \( \mu_{1} = \mu \phi_{a_{1}} \) and \( \mu_{2} = \mu \phi_{a_{2}} \) hold for some \( a_{i} \in A \) (\( i = 1, 2 \)), where \( \mu \) is a primitive isogeny. Since \( \deg \phi_{a_{1}} = \deg \phi_{a_{2}} \) if and only if \( a_{2} = \xi a_{1} \) for some \( \xi \in \mathbb{F}_{q}^{\times} \), we obtain the conclusion. \( \square \)

## 3. \( K \)-virtual Drinfeld \( A \)-modules

In this section, we study fundamental facts on \( G_{K} \)-conjugates of Drinfeld \( A \)-modules over \( K^{\text{sep}} \) and introduce the notion of \( K \)-virtual Drinfeld \( A \)-modules as a function field analogue of elliptic \( \mathbb{Q} \)-curves.

### 3.1. Galois conjugates.

Fix an element \( s \in G_{K} \). For any \( \mu = \sum_{i=0}^{n} c_{i} \tau^{i} \in K^{\text{sep}} \{ \tau \} \), set \( *\mu := \sum_{i=0}^{n} c_{i} \tau^{i} \). Then \( K^{\text{sep}} \{ \tau \} \to K^{\text{sep}} \{ \tau \} ; \mu \mapsto *\mu \) is a ring automorphism of \( K^{\text{sep}} \{ \tau \} \).

For a Drinfeld \( A \)-module \( \phi \) over \( K^{\text{sep}} \), we define a new Drinfeld \( A \)-module \( *\phi \) by
\[
*\phi : \ A \to K^{\text{sep}} \{ \tau \}, \quad a \mapsto *\phi_{a}
\]
We call \( *\phi \) the conjugate of \( \phi \) by \( s \). Clearly we have \( \text{rk} *\phi = \text{rk} \phi \). For any isogeny \( \mu : \phi \to \psi \), we have \( *\mu *\phi_{T} = *\mu *\phi_{T} = *\psi_{T} *\mu = *\psi_{T} \mu \) and so we obtain the conjugate \( *\mu : *\phi \to *\psi \) of \( \mu \) by \( s \). Then \( \mu \mapsto *\mu \) yields an \( A \)-module isomorphism \( \text{Hom}_{K^{\text{sep}}}(\phi, \psi) \cong \text{Hom}_{K^{\text{sep}}}(\phi, \psi) \).

We see that \( \deg \phi = \deg *\mu \) for any \( s \in G_{K} \) since \( s \) induces an isomorphism \( \text{Ker} \mu \cong \text{Ker} *\mu \) as \( A \)-modules.

Considering \( G_{K} \)-conjugates of isogenies, we have the following.

**Proposition 3.1.** Let \( \phi \) and \( \psi \) be Drinfeld \( A \)-modules over \( K \). If \( \phi \) has no complex multiplication over \( K^{\text{sep}} \), then there are an element \( \lambda \in K^{\times} \) and a positive integer \( n \) with \( n \mid q - 1 \) such that any isogeny \( \mu : \phi \to \psi \) is \( K(\sqrt[n]{\lambda}) \)-rational, where \( \sqrt[n]{\lambda} \) is an \( n \)-th root of \( \lambda \).

**Proof.** It is enough to prove for a primitive isogeny \( \mu : \phi \to \psi \) in \( \text{Hom}_{K^{\text{sep}}}(\phi, \psi) \). Set \( \mu = c_{0} + c_{1} \tau + \cdots + c_{N} \tau^{N} \in K^{\text{sep}} \{ \tau \} \). Recall that \( c_{0} \neq 0 \); see §2.2. Since \( \deg \mu = \deg *\mu \) for any \( s \in G_{K} \), there is a unique \( \xi_{s} \in \mathbb{F}_{q}^{\times} \) such that \( *\mu = \xi_{s} \mu \) by Proposition 2.7. Namely we have \( *\xi_{s} = \xi_{s} *\mu \) for any \( 1 \leq i \leq N \).

Define
\[
\chi : \ G_{K} \to \mathbb{F}_{q}^{\times}, \quad s \mapsto \xi_{s}
\]
which is a group homomorphism. Denote by \( n \) the order of \( \chi \), so that \( n \mid q - 1 \). Set \( \lambda := c_{0}^{n} \). Then we have \( *\lambda = \xi_{s} \lambda = (\chi(s))^{n} \lambda = \lambda \) for any \( s \in G_{K} \) and hence \( \lambda \in K^{\times} \). Set \( c_{i} := \frac{c_{i}}{c_{0}} \) for each \( 1 \leq i \leq N \). Then we have \( c_{i} \in K \) since \( *c_{i} = \frac{c_{i}}{c_{0}} = c_{i} \) for any \( s \in G_{K} \). Consequently \( \mu \) is of the form \( \mu = c_{0} \mu' \), where \( \mu' = 1 + c_{1} \tau + \cdots + c_{N} \tau^{N} \in K \{ \tau \} \). Hence \( \mu \) is defined over \( K(c_{0}) = K(\sqrt[n]{\lambda}) \). \( \square \)
We have the following descent criterion as an analogue of Weil’s classical result \cite{Wei56} for algebraic varieties.

**Theorem 3.2.** For a Drinfeld $A$-module $\phi$ over $K^{\text{sep}}$, the following are equivalent.

1. $\phi$ has a $K$-model, that is, there exists a Drinfeld $A$-module $\psi$ over $K$ isomorphic to $\phi$.
2. There exists an isomorphism $\nu_s : \ast \phi \rightarrow \phi$ for each $s \in G_K$ such that $\ast \nu_t \cdot \nu_s = \nu_{st}$ for any $s, t \in G_K$.

**Proof.** We first prove (1) \(\Rightarrow\) (2). Let $\psi$ be a $K$-model of $\phi$ and take an isomorphism $\nu : \phi \rightarrow \psi$, so that $\nu_T = \nu_{\phi_T}^{-1}$. Since $\ast \psi = \psi$ for any $s \in G_K$, $\ast \nu$ gives rise to an isomorphism $\ast \nu : \ast \phi \rightarrow \psi$. For each $s \in G_K$, define an isomorphism $\nu_s : \ast \phi \rightarrow \phi$ by $\nu_s := \nu^{-1} \cdot \ast \nu$. Then we have $\ast \nu_t \cdot \nu_s = \nu_{st}$ for any $s, t \in G_K$.

To prove (2) \(\Rightarrow\) (1), take a family $\{ \nu_s : \ast \phi \rightarrow \phi \}_{s \in G_K}$ of isomorphisms as in (2). Since $\phi$ is actually defined over a finite extension $L$ of $Q$, we may assume that $\nu_s = \nu_t$ if $s|_L = t|_L$. Then we have a continuous map $\alpha : G_K \rightarrow K^{\text{sep}, \times} : s \mapsto \nu_s$. Here we consider the Krull topology on $G_K$ and the discrete topology on $K^{\text{sep}, \times}$. Since $\alpha$ satisfies the one-cocycle condition, there is an element $\nu \in K^{\text{sep}, \times}$ such that $\nu_s = \nu^{-1} \cdot \ast \nu$ by Hilbert’s theorem 90. Let $\psi$ be the Drinfeld $A$-module determined by $\psi_T = \nu_{\phi_T}^{-1}$. Then for any $s \in G_K$, we have

$$
\ast \psi_T = \ast \nu_s \phi_T \ast \nu^{-1} = \nu_s \ast \phi_T \nu_s^{-1}\nu^{-1} = \nu_{\phi_T}^{-1} = \psi_T.
$$

Hence $\psi$ is a $K$-model of $\phi$. \qed

### 3.2. Definition and Examples

Let $\phi$ be a Drinfeld $A$-module defined over $K^{\text{sep}}$.

**Definition 3.3.** We say that $\phi$ is $K$-virtual if $\phi$ is isogenous to $\ast \phi$ for any $s \in G_K$.

By Theorem 3.2 any $\phi$ admitting a $K$-model is trivially $K$-virtual. We also see that rank-one $\phi$ is $K$-virtual since all rank-one Drinfeld $A$-modules over $K^{\text{sep}}$ are isomorphic to each other. Therefore our interest focuses on the case where $\phi$ has no $K$-models and $\text{rk} \, \phi \geq 2$. As a non-trivial example, we can construct a rank-two $Q$-virtual Drinfeld $A$-module as follows.

**Example 3.4.** Suppose that $p \neq 2$ and fix a square root $\sqrt{T + 1} \in Q^{\text{sep}}$ of $T + 1 \in A$. Set $\mu := \sqrt{T + 1} + 1 - \tau$ and $\eta := \sqrt{T + 1} - 1 + \tau$. Then

$$
\mu \eta = (\sqrt{T + 1} + 1 - \tau)(\sqrt{T + 1} + 1 - \tau) = T + 2 + \sqrt{T + 1} - \sqrt{T + 1} = T - \sqrt{T + 1}.
$$

Denote by $\varphi$ the Drinfeld $A$-module over $Q^{\text{sep}}$ determined by $\varphi_T = \mu \eta$, so that $\text{rk} \, \varphi = 2$. Then the $\varphi$ is $Q$-virtual. Indeed, if $s \in G_Q$ fixes $\sqrt{T + 1}$, then $\ast \varphi = \varphi$. If $\ast \sqrt{T + 1} = -\sqrt{T + 1}$, then

$$
\ast \varphi_T = \ast \mu \ast \eta = (-\sqrt{T + 1} + 1 - \tau)(-\sqrt{T + 1} + 1 - \tau) = (\sqrt{T + 1} - 1 + \tau)(\sqrt{T + 1} + 1 - \tau).
$$

We have $\mu \ast \varphi_T = \mu \eta \mu = \varphi_T \mu$ and hence $\mu$ is an isogeny $\mu : \ast \varphi \rightarrow \varphi$. Here the $j$-invariant $j_\varphi$ of $\varphi$ is

$$
j_\varphi = -(2 + \sqrt{T + 1} - \sqrt{T + 1})^{q+1}.
$$

It is easy to check that $j_\varphi \notin Q$ and hence $\varphi$ has no $Q$-models by the next remark.

**Remark 3.5.** For a rank-two Drinfeld $A$-module determined by $\phi_T = T + g_\tau + \Delta \tau^2 \in \bar{K} \{ \tau \}$, its $j$-invariant is defined by

$$
j_\phi = \frac{\Delta^{q+1}}{\Delta}.
$$

It follows that $\phi$ is $\bar{K}$-isomorphic to some $\psi$ if and only if $j(\phi) = j(\psi)$. Moreover, we see that $\phi$ has a $Q(j_\phi)$-model because the Drinfeld $A$-module $\phi'$ determined by $\phi'_T = T + j_\phi \tau + j_\phi^2 \tau^2$ has the $j$-invariant $j_\phi$. Hence $\phi$ has a $K$-model if and only if $j_\phi \in K$. 

In the full complex multiplication case, using the Hayes theory \cite{Hay92} on rank-one Drinfeld modules, we have the following.

**Proposition 3.6.** Let $\phi$ be a Drinfeld $A$-module over $K^{sep}$ of rank $> 1$. If $\phi$ has full complex multiplication by a finite extension $F/Q$, then $\phi$ is isogenous to an $F$-virtual Drinfeld $A$-module. In addition, if $F^{sep} = K^{sep}$, then $\phi$ itself is $F$-virtual.

**Proof.** Recall that $F \subset K^{sep}$ and so $F^{sep} \subset K^{sep}$. Let $O_F$ be the ring of integers of $F$. By replacing $\phi$ with a suitable isogenous Drinfeld $A$-module if necessary, we may assume that $\phi$ satisfies $\text{End}_{K^{sep}}(\phi) \cong O_F$ by \cite[Proposition 4.7.19]{Gos96}. Then we obtain a Drinfeld $O_F$-module $\Phi : O_F \to K^{sep}(\tau)$ of rank one (cf. Remark \ref{rem]), satisfying $\Phi|_A = \phi$. It follows by \cite{Hay92} that $\phi$ has a model $\Psi$ defined over the Hilbert class field $H_F$ of $F$, that is, $H_F$ is the maximal unramified abelian extension of $F$ in which the place $\infty = (1/T)$ splits completely.

Take an element $s \in G_F$. Then the Chebotarev density theorem implies that there is a prime ideal $\mathfrak{p} \subset O_F$ such that $s|_{H_F} = \text{Frob}_\mathfrak{p}$, where $\text{Frob}_\mathfrak{p} \in \text{Gal}(H_F/F)$ is the Frobenius automorphism at $\mathfrak{p}$. Thus we have $s\Psi = \text{Frob}_\mathfrak{p} \Psi$. By \cite[Theorem 10.8]{Hay92}, the conjugate $\text{Frob}_\mathfrak{p} \Psi$ is isomorphic to $\Psi^s\Psi$, which is a Drinfeld $O_F$-module given by the action of ideas on $\Psi$; see \cite[pp.7]{Hay92}. It is known that $\Psi^s\Psi$ is $H_F$-isogenous to $\Psi$ and so we can take an $H_F$-rational isogeny $\mu_s : s\Psi \to \Psi$. By construction, we see that $\psi := \Psi|_A$ is a $H_F$-model of $\phi$. Since $\mu_s$ is also an isogeny $\mu_s : s\psi \to \psi$, we see that $\psi$ is $F$-virtual. If $F^{sep} = K^{sep}$, then $\phi$ is defined over $F^{sep}$ and so $\phi$ itself is $F$-virtual. \hfill $\square$

4. The modular curve $Y_s(n)$

As usual, let $K$ be an algebraic extension of $Q$ and let $n \subset A$ be a non-zero ideal. In this section, we assume that any Drinfeld $A$-module is of rank two. For any algebraic variety $Y$ over $Q$, write $Y(K)$ for the set of $K$-rational points of $Y$ and regard it as a subset of $Y(L)$ for any field $L/K$.

4.1. Atkin-Lehner involutions. We first recall some well-known facts on Drinfeld modular curves. See \cite{Gek86} \cite{GR96} \cite{Gek01} \cite{Sch97} for detail.

Denote by $Q_\infty = F(\!(1/T)\!)$ the completion of $Q$ at the place $\infty = (1/T)$ and by $C_\infty$ the completion of an algebraic closure of $Q_\infty$. Then $\Omega := C_\infty \setminus Q_\infty$ is called the Drinfeld upper half plane. The group $\text{GL}_2(A)$ acts on $\Omega$ by fractional linear transformation. For a non-zero ideal $n \subset A$ and the subgroup

$$
\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A); \ c \equiv 0 \pmod{n} \right\},
$$

the quotient $\Gamma_0(n)\backslash\Omega$ has a structure of smooth affine algebraic curve over $Q$. Denote it by $Y_0(n)$ and call the Drinfeld modular curve of $\Gamma_0(n)$-level. Let $X_0(n)$ be a unique smooth compactification of $Y_0(n)$, so that $X_0(n)$ is a smooth projective curve over $Q$ containing $Y_0(n)$ as an affine open subscheme. A point of $X_0(n) \setminus Y_0(n)$ is called a cusp. As a partial analogue of a well-known result on elliptic modular curves proved by Mazur \cite[Theorem 7.1]{Maz78}, the following is known.

**Theorem 4.1 (\cite{Pai10} Theorem 1.2).** Assume that $Q = F_2(T)$. Then the curve $Y_0(p)$ has no $Q$-rational points if $p$ is a prime ideal generated by $a_p \in A$ with $\text{deg}(a_p) \geq 3$.

Recall that $Y_0(n)$ is a coarse moduli variety parametrizing isomorphism classes of rank-two Drinfeld $A$-modules with additional structures with respect to $n$. More precisely, it is known that every $K$-rational point $x \in Y_0(n)(K)$ corresponds to a $K$-equivalence class of a $K$-rational cyclic $n$-isogeny. In other words, every $K$-rational point of $Y_0(n)$ corresponds to a $K$-isomorphism class $[\phi, \Lambda]$ of a pair $(\phi, \Lambda)$ consisting of a Drinfeld $A$-module $\phi$ over $K$ and a $K$-rational cyclic $n$-kernel $\Lambda$ of $\phi$. Here two such pairs $(\phi, \Lambda)$ and $(\phi', \Lambda')$ are said to be $K$-isomorphic if there exists an $\Lambda$-isomorphism $\nu : \phi \to \phi'$ such that $\nu \Lambda = \Lambda'$. It follows that two isogenies $\mu$ and $\eta$ are $K$-equivalent if and only if $(\phi, \text{Ker} \, \mu)$ and $(\phi', \text{Ker} \, \eta)$ are $K$-isomorphic. If $x \in Y_0(n)(K)$ is represented by a $K$-rational cyclic $n$-isogeny $\mu : \phi \to \psi$ with $\text{Ker} \, \mu = \Lambda$, then we use the notation $x = [\phi, \Lambda]$ or $x = [\mu : \phi \to \psi]$. We say that $x = [\phi, \Lambda]$ is a $\text{CM}$ point if $\phi$ has complex multiplication.
Next we review some properties of Atkin-Lehner involutions (cf. [Sch97, pp. 331–332]). For every ideal \( m \mid n \) with \((m, \frac{n}{m}) = 1\), we have an involution
\[
w_m : X_0(n) \overset{\sim}{\longrightarrow} X_0(n)
\]
defined over \( Q \). It is called the (partial) Atkin-Lehner involution with respect to \( m \) on \( X_0(n) \).

If \( m = n \), then \( w_n \) is sometimes called the full Atkin-Lehner involution. Every \( w_m \) sends cusps to cusps and so it is also an involution on \( Y_0(n) \). Denote by \( \mathcal{W}(n) \) the group consisting of all Atkin-Lehner involutions. Since
\[
w_{m_1}w_{m_2} = w_{m_3}w_{m_4} = w_{m_5}
\]
for \( m_3 = \frac{m_1m_2}{(m_1, m_2)} \), we have \( \mathcal{W}(n) \cong (\mathbb{Z}/2\mathbb{Z})^n \), where \( n \) is the number of distinct prime factors of \( n \).

For any \( w_m \in \mathcal{W}(n) \) and \( x = [\mu : \phi \to \psi] \in Y_0(n)(K) \) represented by a \( K \)-rational \( \mu \), the moduli interpretation of \( w_m x \) is as follows. If \( n = mn' \) with \((m, n') = 1\), then we have \( \text{Ker} \mu = \Lambda_m \oplus \Lambda_{m'} \) with \( \Lambda_m \cong A/m \) and \( \Lambda_{m'} \cong A/n' \). Hence \( \mu \) decomposes as \( \mu = \mu_n'\mu_m \), where \( \mu_m : \phi \to \phi_m \) is a \( K \)-rational cyclic \( m \)-isogeny with \( \text{Ker} \mu_m = \Lambda_m \) and \( \mu_n' : \phi_m \to \psi \) is a \( K \)-rational cyclic \( n' \)-isogeny with \( \text{Ker} \mu_n' = \mu_n(\text{Ker} \mu) = \mu_n(\Lambda_{m'}) \). Then it follows that
\[
w_m x = [\phi_m, \mu_m(\phi[m] \oplus \Lambda_{m'})] = [\eta : \phi_m \to \psi_m],
\]
where \( \eta : \phi_m \to \psi_m \) is a \( K \)-rational cyclic \( n \)-isogeny with \( \text{Ker} \eta = \mu_m(\phi[m] \oplus \Lambda_{m'}) \). In particular, we have \( w_m x \in Y_0(n)(K) \). In addition, the following facts hold for the above \( \mu \) and \( \eta \). Let us decompose \( \eta \) as \( \eta = \eta_n\eta_m \) with \( \eta_n : \phi_m \to \phi' \) and \( \eta_n' : \phi' \to \psi_m \) similarly as \( \mu = \mu_n'\mu_m \).

Then \( \ker \eta_n \mu_m = \phi[m] \) by construction. This implies that \( \lambda \eta_n \mu_m = \phi_{\alpha_m} \) for some \( \lambda \in K^\times \), where \( \alpha_m \) is the monic generator of \( m \).

**Lemma 4.2.** The \( \lambda \) gives rise to an isomorphism \( \lambda : \phi' \to \phi \) satisfying \( \lambda \eta_n = \lambda \).

**Proof.** Since \( \phi_T \eta_n \mu_m = \eta_n \mu_m \phi_T \) holds, the equation \( \lambda \eta_n \mu_m = \phi_{\alpha_m} \) implies
\[
\lambda \phi_T^{-1} \phi_{\alpha_m} = \lambda \phi_T \eta_n \mu_m = \lambda \eta_n \mu_m \phi_T = \phi_{\alpha_m} \phi_T = \phi_T \phi_{\alpha_m}.
\]
Hence \( \lambda \phi_T^{-1} = \phi_T \) by the right division algorithm and so \( \lambda \) is an isomorphism. The equality \( \lambda \eta_n = \lambda \) follows from the uniqueness of the dual isogeny; see Proposition 2.4. \( \square \)

Thus we obtain the following diagram.

\[
\begin{array}{ccc}
\phi & \overset{\mu}{\longrightarrow} & \psi \\
\mu_m & \searrow & \phi_m \\
\phi_m & \nearrow & \psi_m
\end{array}
\]

\begin{equation}
(4.1)
\end{equation}

If \( m = n \), then by construction \( \phi_m = \psi \) and so the image \( w_n x \) of the full Atkin-Lehner involution is represented by the dual isogeny \( \hat{\mu} : \psi \to \phi \).

**Remark 4.3.** Let \( w_m \in \mathcal{W}(n) \) be a non-trivial involution. If a \( \overline{K} \)-rational point \( x = [\phi, \Lambda] \) of \( Y_0(n) \) is fixed by \( w_m \), then \( \phi \) has complex multiplication. It is known that the number of \( w_m \)-fixed points is finite (cf. [Gek80] or [Sch97]). We have in addition the following. If \( q \) is odd, then \( \phi \) has complex multiplication by \( Q(\sqrt{a}) \) for some generator \( a \) of \( m \), and if \( q \) is even, then \( \phi \) has complex multiplication by \( Q(\sqrt{T}) \) since any square root \( \sqrt{a} \) of \( a \in m \) is contained in \( Q(\sqrt{T}) \); see [Sch97] pp. 338 for example. Hence the order of the decomposition group \( D_x := \{ w \in \mathcal{W}(n); wx = x \} \) of \( x \) is at most 2.

Denote by \( Y_0(1) \) the Drinfeld modular curve for the ideal \((1) = A \), so that any \( x \in Y_0(1)(K) \) corresponds to a \( \overline{K} \)-isomorphism class \( [\phi] \) of a Drinfeld \( A \)-module \( \phi \) over \( K \). Let \( \theta : Y_0(n) \to Y_0(1) \) be the natural map given by forgetting the level structure. Then we have
\[\theta(x) = [\phi] \text{ if } x = [\mu : \phi \rightarrow \psi] \in Y_0(n)(K).\] Define \(N_0(n)(K^{\text{sep}}) \subset Y_0(n)(K^{\text{sep}})\) to be the subset consisting of all non-CM \(K^{\text{sep}}\)-rational points of \(Y_0(n)\). Consider the map
\[
\Theta : N_0(n)(K^{\text{sep}}) \to Y_0(1)(K^{\text{sep}}) \times Y_0(1)(K^{\text{sep}})
\]
defined by \(\Theta(x) = (\theta(x), \theta(w_nx))\). The following lemma is needed in Section 5.

**Lemma 4.4.** The map \(\Theta\) is injective.

**Proof.** Take two points \(x, y \in N_0(n)(K^{\text{sep}})\) with \(x = [\mu : \phi \rightarrow \psi]\) and \(y = [\eta : \phi' \rightarrow \psi']\), where both \(\mu\) and \(\eta\) are \(K^{\text{sep}}\)-rational. Assume that \(\Theta(x) = \Theta(y)\), so that both \(\theta(x) = \theta(y)\) and \(\theta(w_nx) = \theta(w_ny)\) hold. Since \(w_nx\) and \(w_ny\) are represented by \(\mu\) and \(\eta\) respectively, we have \([\phi] = [\phi']\) and \([\psi] = [\psi']\). Thus we can take \(K^{\text{sep}}\)-isomorphisms \(\nu : \phi \rightarrow \phi'\) and \(\lambda : \psi \rightarrow \psi'\). Then \(\eta' := \lambda\mu\nu^{-1}\) yields a \(K^{\text{sep}}\)-rational cyclic \(n\)-isogeny \(\eta' : \phi \rightarrow \psi'\). Since \(\deg \eta = \deg \eta'\), Proposition \([27]\) implies that \(\eta = \xi\eta'\) for some \(\xi \in \mathbb{F}_q\). Hence \(\eta = (\xi\lambda)\mu\nu^{-1}\) and so \(\mu\) and \(\eta\) are \(K\)-equivalent. Thus we have \(x = y\).

4.2. Rational points of Atkin-Lehner quotients. Taking the quotient by the group \(W(n)\) of all Atkin-Lehner involutions, we obtain the quotient curve \(X_n(n) := X_0(n)/W(n)\). It contains the affine curve \(Y_n(n) := Y_0(n)/W(n)\). Since all elements of \(W(n)\) are defined over \(Q\), both \(X_n(n)\) and \(Y_n(n)\) are algebraic curves over \(Q\).

Denote by \(\gamma : Y_0(n) \to Y_n(n)\) the quotient map, which is defined over \(Q\). Since \(W(n)\) is a finite group of automorphisms on \(Y_n(n)\), \(\gamma\) is a finite morphism and \(W(n)\) acts transitively on the fibers of \(\gamma\) (cf. [Liu02 pp.113]). Therefore for any \(K\)-rational point \(x_n \in Y_n(n)(K)\), the pre-image \(P(x_n) \subset Y_0(n)(K)\) of \(x_n\) by \(\gamma\) is of the form \(P(x_n) = W(n)x_n\) for some \(x_n \in Y_0(n)(K)\).

We consider the following condition for \(x_n\):

\((*)\)

the set \(P(x_n)\) is contained in \(Y_0(n)(K^{\text{sep}})\).

Notice that it is equivalent to saying that \(P(x_n)\) contains at least one \(K^{\text{sep}}\)-rational point of \(Y_0(n)\). Under this condition, it follows that \(P(x_n)\) is \(G_K\)-stable since \(\gamma(x) = \gamma(x_n) = x_n\) for any \(x \in P(x_n)\) and \(s \in G_K\). The following lemma means that \(x_n\) satisfies the condition \((*)\) in almost all cases.

**Lemma 4.5.** Let \(x_n\) be a \(K\)-rational point of \(Y_n(n)\) and assume either \(q\) is odd or \(P(x_n)\) contains a non-CM point. Then \(x_n\) satisfies \((*)\).

**Proof.** Take \(x \in P(x_n)\), so that we have \(P(x_n) = W(n)x\). Denote by \(\kappa(x)\) and \(\kappa(x_n)\) the residue fields at \(x\) and \(x_n\) respectively. To check the condition \((*)\), it is enough to show that the extension \(\kappa(x)/\kappa(x_n)\) is separable. Consider the quotient \(\hat{\gamma} : Y_0(n) \to Y_0(n)/D_\xi\) by the decomposition group \(D_\xi\) of \(x\). Then the quotient map \(\gamma : Y_0(n) \to Y_n(n)\) factors as

\[
\begin{array}{ccc}
Y_0(n) & \xrightarrow{\hat{\gamma}} & Y_n(n) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
Y_0(n)/D_\xi & \xrightarrow{\kappa} & Y_n(n)/D_\xi
\end{array}
\]

such that \(Y_0(n)/D_\xi \to Y_n(n)\) is étale at \(\tilde{x} := \gamma(x)\) (cf. [Liu02 pp.147]). Hence \(\kappa(\tilde{x})/\kappa(x_n)\) is separable. If \(P(x_n)\) contains a non-CM point, then \(x\) is also a non-CM point since any element of \(W(n)\) sends non-CM points to non-CM points. Thus \(D_\xi\) is trivial by Remark \([4.3]\) and so \(\kappa(x) = \kappa(\tilde{x})\) is separable over \(\kappa(x_n)\). Suppose that \(q\) is odd. Since \(#D_\xi \leq 2\) and \(\hat{\gamma}^{-1}(\tilde{x}) = \{x\}\), we see that the degree \([\kappa(x) : \kappa(\tilde{x})]\) is at most 2. Hence \(\kappa(x)/\kappa(\tilde{x})\) is separable and so is \(\kappa(x)/\kappa(x_n)\).

The next proposition says that any point \(x_n\) satisfying \((*)\) gives raise to a family of \(K\)-virtual Drinfeld \(A\)-modules isogenous to each other.

**Proposition 4.6.** If \(x_n \in Y_n(n)(K)\) satisfies \((*)\), then any point \(x \in P(x_n)\) is represented by a pair \((\phi, \Lambda)\) such that \(\phi\) is a \(K\)-virtual Drinfeld \(A\)-module defined over a polyquadratic extension of \(K\).
Proof. Take a point \( x \in \mathcal{P}(x_s) \). Since \( x \) is \( K^{sep} \)-rational, it is represented by a \( K^{sep} \)-rational pair \((\phi, \Lambda)\). Then we see that \( x \) is represented by \((\delta \phi, \delta \Lambda)\) for any \( s \in G_K \). On the other hand, since \( \mathcal{P}(x_s) = W(n)x \) is \( G_K \)-stable, there is an involution \( w_{m_s} \in W(n) \) such that \( \delta x = w_{m_s}x \) for any \( s \in G_K \). By the moduli interpretation of \( w_{m_s}x \), we see that \( \delta \phi \) admits a cyclic \( m_s \)-isogeny to \( \phi \) and so \( \phi \) is \( K \)-virtual. Now the above correspondence \( s \mapsto w_{m_s} \) induces a well-defined group homomorphism
\[
f : G_K \to W(n)/D_x.
\]
Let \( L \subset K^{sep} \) be the fixed subfield of \( \text{Ker} f \). Then it is polyquadratic over \( K \) since \( \text{Gal}(L/K) \) injects into \( W(n)/D_x \cong (\mathbb{Z}/2\mathbb{Z})^m \) for some \( m \geq 0 \). Since \( x \) is fixed by \( G_L \), it is an \( L \)-rational point and so \( \phi \) has an \( L \)-model. \( \square \)

5. Isoezy trees associated with \( K \)-virtual Drinfeld \( A \)-modules

In this final section, we prove Theorem 5.1 saying that all non-CM \( K \)-virtual Drinfeld \( A \)-modules of rank two are isogenous to those arising from \( K \)-rational points of \( Y_s(n) \) in the manner of Proposition 4.6. For the rest of this paper, “Drinfeld \( A \)-modules” always refer to non-CM Drinfeld \( A \)-modules over \( K^{sep} \) of rank two. We use the symbol \( p \) for a non-zero prime ideal of \( A \).

5.1. Isoezyes of \( p \)-power degree. To prove Theorem 5.1 we recall some facts on isogenies \( \mu : \phi \to \psi \) of \( p \)-power degree.

Lemma 5.1. Every cyclic \( p^n \)-isogeny \( \mu : \phi \to \psi \) factors as \( \mu = \mu_n \cdots \mu_1 \) with some \( p \)-isogenies \( \mu_i \). The \( \mu_i \) are unique up to \( K^{sep} \)-equivalence.

Proof. We prove this by induction on \( n \). The case where \( n = 1 \) is trivial. Assume that \( n \geq 2 \) and any cyclic \( p^{n-1} \)-isogenies decompose unique up to \( K^{sep} \)-equivalence as a composite of \( p \)-isogenies. Suppose that \( \deg \mu = p^n \). Since \( \text{Ker} \mu \) contains a unique (cyclic) \( p \)-kernel \( \Lambda \), \( \mu \) factors as \( \mu = \mu_n \cdots \mu_1 \) such that \( \text{Ker} \mu_1 = \Lambda \) and \( \text{Ker} \mu_{n-1} = \mu(\text{Ker} \mu) \cong A/p^{n-1} \). Recall that such \( \mu_1 \) is uniquely determined up to \( K^{sep} \)-equivalence. Hence we obtain the conclusion. \( \square \)

Proposition 5.2. Let \( \mu : \phi \to \psi \) be a \( p^n \)-isogeny. Then \( \mu \) is cyclic if and only if \( \mu \) is primitive.

Proof. Let \( \mu : \phi \to \psi \) be a \( p^n \)-isogeny and take a primitive isogeny \( \eta : \phi \to \psi \). Then \( \mu = \eta \phi a \) for some non-zero \( a \in A \) since \( \phi \) has no complex multiplication. Suppose that \( \mu \) is cyclic. Considering the degree of \( \mu = \eta \phi a \), we have either \( a \in \mathfrak{p} \) or \( a \in F_q^\times \). If \( a \in \mathfrak{p} \), then \( \text{Ker} \mu = \eta^{-1}(\psi[a]) \) is not cyclic. Thus we have \( a \in F_q^\times \) and hence \( \mu \) is primitive. Conversely, if \( \mu \) is not cyclic, then \( \text{Ker} \mu \) is of the form \( \text{Ker} \mu = \Lambda_1 \oplus \Lambda_2 \) with cyclic \( p \)-power kernels \( \Lambda_i \) of \( \phi \). Hence \( \phi \) is cyclic in \( \text{particular contains \phi[p]} \). It means that \( \mu \) decomposes as \( \mu = \mu' \phi a_\mathfrak{p} \), where \( a_\mathfrak{p} \) is a generator of \( \mathfrak{p} \) and \( \mu' : \phi \to \psi \) is an isogeny. Hence \( \mu \) is not primitive. \( \square \)

Corollary 5.3. An isogeny \( \mu \) is cyclic if and only if it is primitive.

Proof. Let \( \mu \) be an isogeny with \( \deg \mu = \prod_{i=1}^n \mathfrak{p}_i^{\delta_i} \), where \( \mathfrak{p}_i \) are distinct prime ideals and \( \delta_i > 0 \). Then for each \( i \), \( \text{Ker} \mu_i \) contains a unique \( \mathfrak{p}_i^{\delta_i} \)-kernel and so \( \mu \) decomposes as \( \mu = \eta_i \mu_i \) with some isogenies satisfying \( \deg \mu_i = \mathfrak{p}_i^{\delta_i} \) and \( \deg \eta_i = \prod_{j \neq i} \mathfrak{p}_j^{\delta_j} \). If \( \mu \) is primitive, then \( \mu_i \) should be primitive for any \( i \) and so it is cyclic by Proposition 5.2. Thus we have \( \text{Ker} \mu \cong \prod_{i=1}^n \mathfrak{p}_i^{\delta_i} \cong A/\deg \mu \) and hence \( \mu \) is cyclic. On the other hand, if \( \mu \) is not primitive, then some \( \mu_i \) is not primitive and so it is not cyclic. Hence \( \mu \) itself is not cyclic. \( \square \)

Let \( \phi \) and \( \psi \) be Drinfeld \( A \)-modules isogenous to each other and take a primitive isogeny \( \mu : \phi \to \psi \). For any \( \mathfrak{p} \), define
\[
(5.1) \quad \delta_{\mathfrak{p}}(\phi, \psi) := \max\{n \in \mathbb{Z}_{\geq 0}; \deg \mu \text{ is divisible by } \mathfrak{p}^n\}.
\]
It is independent of the choice of \( \mu \) since we now consider the non-CM case. Since \( \deg \mu = \deg \hat{\mu} \) and \( \deg \hat{\mu} = \deg \mu \) for any \( s \in G_K \) hold, the following lemma is trivial.

Lemma 5.4. Let \( \phi \) and \( \psi \) be as above.
(1) $\delta_p(\phi, \psi) = \delta_p(\psi, \phi)$ for any $p$.
(2) $\delta_p(s^\phi, s^\psi) = \delta_p(\phi, \psi)$ for any $s \in G_K$ and $p$.

Remark 5.5. For any $p$, if $\delta_p(\phi, \psi) = n$, then there exists a Drinfeld $A$-module $\pi_p(\psi)$ such that any primitive isogeny $\mu : \phi \to \psi$ decomposes as $\mu = n\mu_n$, where $\mu_n : \phi \to \pi_p(\psi)$ is an $p^n$-isogeny and $\eta : \pi_p(\psi) \to \psi$ is of degree prime to $p$.

By construction, the $\pi_p(\psi)$ is unique up to isomorphisms. We see that $\phi$ is isomorphic to $\psi$ if and only if $\delta_p(\phi, \psi) = 0$ for all $p$.

5.2. Isogeny trees. We keep the above conventions. In addition, we identify all isomorphic Drinfeld $A$-modules in this subsection. Under this setting, all $K^{sep}$-equivalent isogenies are identified. Notice that the notions of degree of isogenies, primitive isogenies and dual isogenies are well-defined.

Fix a $K$-virtual Drinfeld $A$-module $\phi$. Denote by $\mathcal{I}_\phi$ the set of Drinfeld $A$-modules isogenous to $\phi$. Then every $\psi \in \mathcal{I}_\phi$ is also $K$-virtual. Hence for any $s \in G_K$, its conjugate $\psi^s$ admits an isogeny to $\phi$. Therefore $\psi^s \in \mathcal{I}_\phi$ and so $G_K$ acts on $\mathcal{I}_\phi$. Denote by

$$\delta_p : \mathcal{I}_\phi \times \mathcal{I}_\phi \to \mathbb{Z}_{\geq 0}$$

the function defined by \[\ref{5.1}], which is symmetric and $G_K$-invariant by Lemma \[5.4].

For any $p$, let $\mathcal{I}_{\phi,p}$ be the subset of $\mathcal{I}_\phi$ consisting of those admitting $p^n$-isogenies to $\phi$ for some $n \geq 0$. Then $G_K$ also acts on $\mathcal{I}_{\phi,p}$. For any $\psi \in \mathcal{I}_\phi$, an element $\pi_p(\psi) \in \mathcal{I}_{\phi,p}$ is uniquely determined as in Remark \[5.5]. Since $\pi_p(\psi) = \psi$ holds if $\psi \in \mathcal{I}_{\phi,p}$, we obtain the projection

$\pi_p : \mathcal{I}_\phi \to \mathcal{I}_{\phi,p}$.

For any $\psi \in \mathcal{I}_\phi$, it follows by construction that $\pi_p(\psi) = \phi$ for almost all $p$.

Lemma 5.6. Let $\psi, \psi_1, \psi_2 \in \mathcal{I}_\phi$. For any $p$, the projection $\pi_p$ satisfies the following.

(1) $\pi_p(\psi_1, \psi_2)$ is the unique element of $\mathcal{I}_{\phi,p}$ satisfying $\delta_p(\pi_p(\psi), \psi) = 0$.
(2) $\delta_p(\pi_p(\psi_1), \pi_p(\psi_2)) = \delta_p(\psi_1, \psi_2)$.
(3) $\pi_p(\psi_1) = \pi_p(\psi_2)$ if and only if $\delta_p(\psi_1, \psi_2) = 0$. In particular, $\psi_1 = \psi_2$ if and only if $\pi_p(\psi_1) = \pi_p(\psi_2)$ for all $p$.

Proof. (1) By construction, we have $\delta_\phi(\pi_p(\psi_1), \pi_p(\psi_2)) = 0$. Suppose that there is an element $\psi' \in \mathcal{I}_{\phi,p}$ satisfying $\delta_p(\psi', \psi) = 0$. Then we can take a primitive isogeny $\mu : \pi_p(\psi) \to \psi'$ with $\deg \mu = p^n$ for some $n \geq 0$ since $\pi_p(\psi), \psi' \in \mathcal{I}_{\phi,p}$. On the other hand, $\delta_p(\pi_p(\phi), \psi) = \delta_p(\psi', \psi) = 0$ implies that there is an isogeny $\pi_p(\psi) \to \psi'$ with degree prime to $p$. Hence $n = 0$ and so $\psi' = \pi_p(\psi)$.

(2) Set $n := \delta_p(\pi_p(\psi_1), \pi_p(\psi_2))$, so that there is a primitive $p^n$-isogeny $\mu : \pi_p(\psi_1) \to \pi_p(\psi_2)$.

To prove $n = \delta_p(\psi_1, \psi_2)$, take a primitive isogeny $\tilde{\mu} : \psi_1 \to \psi_2$. Then $p^n \pi_p(\psi_1, \psi_2)$ is the maximal $p$-power dividing $\deg \tilde{\mu}$. By the definition of $\pi_p$, there exist isogenies $\eta_i : \pi_p(\psi_i) \to \psi_i$ of degree prime to $p$ for $i = 1, 2$. Since $\mu$ and $\tilde{\mu}$ are primitive, we have $\eta_2 \eta_1 = a \cdot \mu$ and $\eta_2 \tilde{\mu} \eta_1 = b \cdot \mu$ for some $a, b \in A$. Comparing the $p$-part of the degree of them, we have $n \geq \delta_p(\psi_1, \psi_2)$ and $n \leq \delta_p(\psi_1, \psi_2)$, so that $n = \delta_p(\psi_1, \psi_2)$.

The assertion of (3) immediately follows from (2) and Remark \[5.5].

According to Lemma \[5.6], we see that the projections $\pi_p : \mathcal{I}_\phi \to \mathcal{I}_{\phi,p}$ are compatible with the $G_K$-action on $\mathcal{I}_\phi$. Indeed, for any $s \in G_K$ and $\psi \in \mathcal{I}_\phi$, we have $\delta_p(s^\phi, s^\psi) = \delta_p(s^\phi, s^\psi) = 0$ and hence $s^\pi_p(\psi) = \pi_p(s^\psi)$ by the uniqueness of $\pi_p(s^\psi)$.

Let us consider the restricted product of all $\mathcal{I}_{\phi,p}$

$$\prod_p \mathcal{I}_{\phi,p} := \left\{ (\psi_p)_p \in \prod_p \mathcal{I}_{\phi,p} : \psi_p = \phi \text{ for almost all } p \right\}$$
relative to \( \phi \), where \( p \) runs through all non-zero prime ideals of \( A \). Then we obtain a map
\[
(\pi_p)_p : I_\phi \to \prod_p I_{\phi,p},
\]
which is well-defined since \( \pi_p(\psi) = \phi \) for almost all \( p \).

**Lemma 5.7.** The map \((5.2)\) is bijective.

**Proof.** The injectivity follows from Lemma 5.6 (3). To prove the surjectivity, take \((\psi_p)_p \in \prod_p I_{\phi,p} \) and primitive isogenies \( \mu_p : \phi \to \psi_p \) for all \( p \). Since \( \text{Ker} \mu_p = 0 \) for almost all \( p \), there is an isogeny \( \mu : \phi \to \psi \) with \( \text{Ker} \mu = \oplus_p \text{Ker} \mu_p \). Then it follows that \( \psi \in I_\phi \) and \( \delta_p(\psi_p, \psi) = 0 \) for all \( p \), so that \( \pi_p(\psi) = \psi_p \). Hence \((5.2)\) maps \( \psi \) to \((\psi_p)_p \). \( \square \)

We may regard \( I_{\phi,p} \) as an infinite graph whose vertices are elements of \( I_{\phi,p} \) and edges are \( p \)-isogenies between them. Moreover, we have the following.

**Proposition 5.8.** The graph \( I_{\phi,p} \) is a directed regular tree of degree \( \#(A/p) + 1 \). For any vertices \( \pi_p(\psi_1), \pi_p(\psi_2) \in I_{\phi,p} \), the length of the path between them is fixed, namely \( \delta_p(\psi_1, \psi_2) \).

**Proof.** The undirectedness of \( I_{\phi,p} \) follows from the fact that the dual of any \( p \)-isogeny is also of degree \( p \); see Remark 2.5. For any vertex \( \pi_p(\psi) \in I_{\phi,p} \), the number of cyclic \( p \)-kernels of \( \pi_p(\psi) \) is \( \#(A/p) + 1 \). In the absence of complex multiplication, such submodules determine distinct \( p \)-isogenies, so that the degree of any vertex is \( \#(A/p) + 1 \).

Take distinct two vertices \( \pi_p(\psi_1), \pi_p(\psi_2) \in I_{\phi,p} \) and a primitive isogeny \( \mu : \pi_p(\psi_1) \to \pi_p(\psi_2) \). Then \( \deg \mu = p^n \) for some \( n > 0 \). Since \( \mu \) is cyclic by Proposition 5.2, it uniquely decomposes as \( \mu = \mu_n \mu_{n-1} \cdots \mu_1 \) with \( p \)-isogenies \( \mu_i \) by Lemma 5.1. It yields a path joining \( \pi_p(\psi_1) \) and \( \pi_p(\psi_2) \), so that \( I_{\phi,p} \) is a connected graph. In addition, since any isogeny \( \eta : \pi_p(\psi_1) \to \pi_p(\psi_2) \) is given by \( \eta = \mu \phi_a \) for some \( a \in A \), the decomposition \( \mu = \mu_n \mu_{n-1} \cdots \mu_1 \) determines the unique path joining \( \pi_p(\psi_1) \) and \( \pi_p(\psi_2) \) whose length is \( n \). Therefore \( I_{\phi,p} \) is a tree. By Lemma 5.4 (2), we have \( n = \delta_p(\psi_1, \psi_2) \). \( \square \)

By Lemma 5.4 (2), the \( GK \)-action on the set \( I_{\phi,p} \) preserves the length of paths joining any vertices. Hence \( G_K \) acts on the tree \( I_{\phi,p} \). Denote by \( \langle \phi \rangle := \{ \phi : s \in G_K \} \) the finite subset of \( I_\phi \) consisting of all \( G_K \)-conjugates of \( \phi \) and set
\[
\pi_p(\langle \phi \rangle) := \{ \pi_p(\phi) : s \in G_K \} \subset I_{\phi,p}
\]
for each \( p \). Notice that \( \pi_p(\phi) = \{ \phi \} \) for almost all \( p \).

Define \( T(\pi_p(\phi)) \) to be the minimal finite subtree of \( I_{\phi,p} \) whose vertex set contains \( \pi_p(\phi) \). Such subtree is uniquely determined. Since any terminal vertex of \( T(\pi_p(\phi)) \) belongs to \( \pi_p(\phi) \) and \( G_K \) acts on \( \pi_p(\phi) \) as permutations, the subtree \( T(\pi_p(\phi)) \) inherits a \( G_K \)-action from \( I_{\phi,p} \). Then we see that there is a unique vertex or edge of \( T(\pi_p(\phi)) \) fixed by \( G_K \), which is called the center of \( T(\pi_p(\phi)) \). Indeed, the central vertex or edge of a longest path joining two points in \( \pi_p(\phi) \) is fixed by \( G_K \). It is easy to see that such a vertex or an edge does not depend on the choice of longest paths, so that the center of \( T(\pi_p(\phi)) \) is well-defined.

For any \( \phi' \in I_\phi \), consider the finite set \( \pi_p(\langle \phi' \rangle) \) in the same way. Then for subtreess \( T(\pi_p(\phi)) \) and \( T(\pi_p(\phi')) \) of \( I_{\phi,p} \), we have the following.

**Lemma 5.9.** The center of \( T(\pi_p(\phi)) \) is an edge if and only if the center of \( T(\pi_p(\phi')) \) is an edge. In this case, the centers of \( T(\pi_p(\phi)) \) and \( T(\pi_p(\phi')) \) coincide.

**Proof.** Suppose that the center of \( T(\pi_p(\phi)) \) is an edge \( \{ \pi_p(\psi_1), \pi_p(\psi_2) \} \). Then we can take an element \( s \in G_K \) such that \( \pi_p(s \psi_1) = \pi_p(s \psi_2) \) and \( \pi_p(s \psi_2) = \pi_p(s \psi_1) \). Indeed, if not, then all of \( \pi_p(\phi) \) lie on one side of the edge \( \{ \pi_p(\psi_1), \pi_p(\psi_2) \} \), which is impossible.

Fix such \( s \in G_K \). To prove the lemma, it suffices to show that \( I_{\phi,p} \) has no \( G_K \)-fixed vertices and no \( G_K \)-fixed edges distinct from \( \{ \pi_p(\psi_1), \pi_p(\psi_2) \} \). If \( I_{\phi,p} \) has a \( G_K \)-fixed vertex \( \pi_p(\psi) \), then we see that
\[
\delta_p(\pi_p(\psi), \pi_p(\psi_1)) = \delta_p(\pi_p(s \psi), \pi_p(s \psi_1)) = \delta_p(\pi_p(\psi), \pi_p(\psi_2)),
\]
so that \( \pi_p(\psi) \) has the same distance from \( \pi_p(\psi_1) \) and \( \pi_p(\psi_2) \). But this is impossible. By the similar observation, we also see that any \( G_K \)-fixed edge of \( I_{\phi,p} \) coincides with \( \{ \pi_p(\psi_1), \pi_p(\psi_2) \} \). Hence it is the center of \( T(\pi_p(\phi')) \) for all \( \phi' \in I_\phi \). \( \square \)
Thus we readily give the proof of Theorem 1.1

**Proof of Theorem 1.1.** Since the vertex set of \( T(\pi_p(\phi)) \) is the singleton \( \{ \phi \} \) for almost all \( p \) by definition, there are only finitely many prime ideals \( p_1, p_2, \ldots, p_n \) such that the centers of \( T(\pi_p(\phi)) \) are edges. Set \( n := \prod_{i=1}^n p_i \), which depends only on the isogeny class of \( \phi \) by Lemma 5.9. For any \( 1 \leq i \leq n \), let \( \{ \psi_{p_i}, \psi'_{p_i} \} \) be the center of \( T(\pi_p(\phi)) \), so that there is a \( p_i \)-isogeny \( \psi_{p_i} \to \psi'_{p_i} \). Then by the bijection \( 5.12 \), we can take two Drinfeld \( A \)-modules \( \psi, \psi' \in \mathcal{I}_\phi \) such that

\[
\pi_p(\psi) = \begin{cases} 
\psi & \text{if } p \mid n \\
\text{the center of } T(\pi_p(\phi)) & \text{if } p \nmid n
\end{cases}
\]

and

\[
\pi_p(\psi') = \begin{cases} 
\psi' & \text{if } p \mid n \\
\text{the center of } T(\pi_p(\phi)) & \text{if } p \nmid n
\end{cases}
\]

for any \( p \). By construction, it follows that \( \delta_p(\psi, \psi') = 1 \) if \( p \mid n \), and \( \delta_p(\psi, \psi') = 0 \) if \( p \nmid n \). Hence there exists a cyclic \( n \)-isogeny \( \psi \to \psi' \). Thus we obtain a \( K^{\text{sep}} \)-rational point

\[
x := [\psi \to \psi'] \in Y_0(n)(K^{\text{sep}}).\]

Let \( \mathcal{P} := \{ w_m x ; w_m \in W(n) \} \) be the \( W(n) \)-orbit of \( x \). If \( \mathcal{P} \) is \( G_K \)-stable, then it gives rise to a \( K \)-rational point of \( Y_s(n) \) and hence (i) holds. To prove this, it suffices to show that for any \( s \in G_K \), there exists an ideal \( m_s \mid n \) such that \( s x = w_m x \). Fix \( s \in G_K \). For each \( p \mid n \), since \( \{ \pi_p(\psi), \pi_p(\psi') \} \) is the center of \( T(\pi_p(\phi)) \), we see that \( \pi_p(s \psi) \) is either \( \pi_p(\psi) \) or \( \pi_p(\psi') \). Define \( m_s \) to be the product of all prime factors \( p \mid n \) satisfying \( \pi_p(s \psi) = \pi_p(\psi) \) or \( \pi_p(s \psi) = \pi_p(\psi') \). Notice that \( \pi_p(s \psi) = \pi_p(\psi) \) and \( \pi_p(s \psi') = \pi_p(\psi') \) if \( p \nmid n \) by construction. Thus we have

\[
\pi_p(s \psi) = \begin{cases} 
\pi_p(\psi') & \text{if } p \mid m_s \\
\pi_p(\psi) & \text{if } p \nmid m_s
\end{cases}
\]

and

\[
\pi_p(s \psi') = \begin{cases} 
\pi_p(\psi) & \text{if } p \mid m_s \\
\pi_p(\psi') & \text{if } p \nmid m_s
\end{cases}
\]

for any \( p \). Let us decompose \( n \) as \( n = m_s n' \). Let \( \psi_{m_s} \to \psi'_{m_s} \) be a \( K^{\text{sep}} \)-rational cyclic \( n \)-isogeny such that \( w_{m_s} x = [\psi_{m_s} \to \psi'_{m_s}] \). Then as in §4.1, there are cyclic \( m_s \)-isogenies \( \psi \to \psi_{m_s} \) and \( \psi' \to \psi'_{m_s} \) such that the diagram

\[
\begin{array}{ccc}
\psi & \xrightarrow{\ n \ } & \psi'\\
\downarrow{m_s} & & \downarrow{m_{s'}} \\
\psi_{m_s} & \xrightarrow{\ n \ } & \psi'_{m_s}
\end{array}
\]

commutes, where the notation such as \( \xrightarrow{\ n \ } \) means a cyclic \( n \)-isogeny. From this diagram, we have

\[
\delta_p(s \psi_{m_s}, \psi) = \delta_p(\pi_p(s \psi_{m_s}), \pi_p(\psi)) = 0
\]

for any \( p \). Thus we have \( \psi_{m_s} = s \psi \). Applying similar arguments to dual isogenies of \( \psi \to \psi' \) and \( \psi_{m_s} \to \psi'_{m_s} \), we also have \( \psi'_{m_s} = s \psi' \). Consequently, we see that \( \Theta(w_{m_s} x) = \Theta(s x) \) and hence \( w_{m_s} x = s x \) by Lemma 1.3.

It remains to check the property (ii). To do this, we assume that there exists an ideal \( n' \) with \( n \nmid n' \) such that a \( K \)-rational point \( y_s \in Y_s(n')(K) \) parametrizes \( K \)-virtual Drinfeld \( A \)-modules isogenous to \( \phi \). Then there is a prime factor \( p \mid n' \). For this \( p \), we see that all \( K \)-virtual Drinfeld \( A \)-modules parametrized by \( y_s \) have the same image under \( \pi_p \), which means that the tree \( T(\pi_p(\phi)) \) has a \( G_K \)-fixed vertex. However, now the center of \( T(\pi_p(\phi)) \) is an edge and hence there are no \( G_K \)-fixed vertices of \( T(\pi_p(\phi)) \) by the proof of Lemma 5.9. This contradiction implies (ii). \( \square \)
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