Local Risk Decomposition for High-frequency Trading Systems

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Abstract

In the present work we address the problem of evaluating the historical performance of a trading strategy or a certain portfolio of assets. Common indicators such as the Sharpe ratio and the risk adjusted return have significant drawbacks. In particular, they are global indices, that is they do not preserve any local information about the performance dynamics either in time or for a particular investment horizon. This information could be fundamental for practitioners as the past performance can be affected by the non-stationarity of financial market. In order to highlight this feature, we introduce the local risk decomposition (LRD) formalism, where dynamical information about a strategy’s performance is retained. This framework, motivated by the multi-scaling techniques used in complex system theory, is particularly suitable for high-frequency trading systems and can be applied into problems of portfolio optimization.

Key words: Financial Markets; Risk; Multi-scale Systems; Complex Systems; PACS: D53

1 Introduction

Measuring the past performance of a trading system or a portfolio of assets is one of the most important issues for financial practitioners and portfolio managers. Evaluating performances heavily depends on estimating “risk”\textsuperscript{1}. In the

\textsuperscript{1} The definition of “risk” can be subjective and, in fact, it does not exist a generally accepted definition. It is often associated with the fluctuation of returns around their mean value and thus to their standard deviation. However, fluctuation towards
past different measures has been proposed but there is no general agreement 
about which one is the most robust estimator for the “quality” of a trading 
strategy ([Dacorogna et al., 2001]).

In this paper, we contribute to the risk-adjusted performance measurement 
subject by introducing a two dimensional decomposition of the profit and loss 
series, PandL, of a trading strategy. Based on this decomposition we can define 
a set of local performance indicators, where “local” refers to both time and 
investment horizon. Global indicators are then obtained via a convolution of 
the decomposed signal with user-specified kernels. The choice of the kernels, as 
well as their parameters, can highlight specific features of the trading dynamics. 
The overall idea is derived from the multiscale analysis, developed in the 
framework of complex system theory ([Bouchaud and Potters, 1999; Sornette, 
2004; Voit, 2005]).

The paper is structured as following: in the next section we briefly introduce 
some standard indicators and point out their drawbacks. In Sec. 3 we introduce 
our local risk decomposition, LRD, while in Sec. 4 we apply the method to the 
performance of different trading systems and we highlight the advantages of 
using the LRD method if compared to standard indicators. Discussions and 
conclusions are left for the last section.

2 Risk performance measures

The performance of a trading strategy are characterized by two key quantities: 
the cumulative return over time, represented by the PandL time series, and 
the risk incurred in using it. While it is intuitive to associate profitability with 
the goodness of a trading strategy, high profits can be due to lucky trades or 
temporary favorable market conditions. This is the reason why investors tend 
to monitor the performance of their trading systems in time in order to recog-
nize a possible deterioration in their strategy. The risk-adjusted performance 
measures proposed in literature, see for example [Dacorogna et al., 2001], at-
tempt to assert the quality of a trading system by assuming that an investor 
will make his/her decision based not only on the past returns but also on their 
fluctuations. Clearly the “amplitude” of fluctuations that a trader can tolerate 
depends on his/her personal appetite for risk and is thus subjective. However, 
investors tend to be risk adverse and, in practice, a trading strategy in order 
to be “acceptable” will have to display not only a good annualized profit but 
also a smooth cumulative return or PandL. In other words, the risk related to 

positive returns may not be considered a form of risk. Therefore, one sided definitions 
of standard deviation are also used by practitioners. For general references on the 
subject the reader is referred to ([Dacorogna et al., 2001; Bailey, 2005; Meucci, 2007]).
the fluctuation around the average return has to be small.

One of the most popular risk performance measures used in finance is the \textit{Sharpe ratio} \cite{Sharpe1994}, defined as

\begin{equation}
S = \frac{\langle r \rangle}{\sigma r},
\end{equation}

where $\langle r \rangle$ is the average return and $\sigma r$ is its standard deviation. The annualization factor, $A$, is $\sqrt{252}$ for daily returns or $\sqrt{12}$ for monthly. The Sharpe ratio, despite being widely used, has two notable drawbacks \cite{Dacorogna2001} among which

1. It is numerically unstable for small values of $\sigma r$,
2. It does not reveal any information about the dynamics of the returns.

The last point is of central interest in the present work. In fact, since the high-frequency dynamics of the stock market is not stationary in time \cite{Bartolozzi2006, Bartolozzi2007a}, the performance of trading systems can be subjected to similar trends.\footnote{\textbf{2} Frequently, trading strategies outperform some benchmark during a period of time by exploiting temporary inefficiencies. Once these inefficiencies are dissipated the performances of a trading strategy tend to deteriorate along.}

Another widely used performance measure is the \textit{risk adjusted return}, defined as

\begin{equation}
R_\beta = \langle r \rangle - \beta \sigma r.
\end{equation}

This indicator, derived from \textit{utility theory} \cite{Dacorogna2001, Bailey2005}, is not affected by numerical singularities. However, it depends on the subjective risk strength factor, $\beta$. Furthermore, along with the Sharpe ratio, it does not reveal any information about the evolution of the \textit{PandL}.

In the next section we introduce a multi-scale framework for estimating a risk-adjusted performance measure based on recent work in complex system theory \cite{Sornette2004}. This framework, while employing elementary block measures similar to Eqs. (1) and (2), also retains time and horizon information which can be fundamental in a the strategy selection problem.
The Local Risk Decomposition

In order to tackle the problem of non-stationarity of the performance of market strategies, we introduce the *Local Risk Decomposition* (LRD). The underlying idea of this method is to extrapolate a risk measure based on the *local* fluctuations of the *PandL*, both in time and scale (or investment horizon). The concept is similar to *detrended fluctuation analysis*, recently proposed to extract correlations from non-stationary time series in the context of DNA nucleotides sequences (Peng et al., 1994), and successively applied in finance by several authors (Cizeau et al., 1997; Liu et al., 1997; Vandewalle and Ausloos, 1997; Liu et al., 1996; Jánosi et al., 1999; Gopikrishnan et al., 2000; 2001; Muniandy et al., 2001; Matia et al., 2002; Costa and Vasconcelos, 2003; Grech and Mazur, 2004; Ivanov et al., 2004; Eisler and Kertész, 2007; Bartolozzi et al., 2007b).

The LRD method works as follows:

1. The *PandL* time series, which for high-frequency trading we can reasonably assume to be daily updated, is divided into $M = N/h$ non-overlapping boxes of equal length $h$, corresponding to different investment horizons. In our notation $x_{ih}(t)$ represents the *PandL* of the strategy under consideration over a period $h$ associated with the $i$th box.

2. For each box, first we perform a linear fit (that is, we look for the local trend) of the *PandL*, $y_{ih}(t)$, as well as the fluctuations around it,

$$
\tilde{\sigma}^i_h = \sqrt{\frac{1}{h} \sum_{t \in i^{th} \text{box}} (x_{ih}(t) - y_{ih}(t))^2}.
$$

which we take as the *local risk*. The difference between the first and last point of the fit represents the *local return*, $\tilde{r}^i_h = y_{ih}(t_2) - y_{ih}(t_1)$, at scale $h$, given $t_1$ and $t_2$ the extremes of the $i$th box.

3. The procedure of points (1) and (2) is iterated over different investment horizons $h$, in order to compare how the trading performance changes at different scales.

It is worth noting that our measures defined above, $\tilde{r}^i_h$ and $\tilde{\sigma}^i_h$, are *local* both in time and scale. Furthermore, the decision of taking the extremes of the fit as a measure of the local return is to avoid overestimating outliers of returns that may not give a fair value to the strategy under exam.

The next step involves the definition of the local performance measures. In

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3 Note that in high-frequency trading there is no reason for the *PandL* not to be updated intra-day or in a *per-trade* basis.
analogy with Eqs. (1) and (2), we define the local Sharpe ratio (LSR) as
\[
S_h^i = \frac{\tilde{r}_h^i}{\tilde{\sigma}_h^i},
\]
and the local risk adjusted return (LRA) as
\[
R_h^i = \tilde{r}_h^i - \beta \phi_h \tilde{\sigma}_h^i,
\]
where \( \beta \) is the risk aversion of the trader (equivalent to the \( \beta \) in Eq. (2)) and \( \phi_h \) is a scaling factor, defined as
\[
\phi_h = \frac{\langle \tilde{r}_h^i \rangle_M}{\langle \tilde{\sigma}_h^i \rangle_M}.
\]
Now we have two dimensional representations of performance measures that are localized both in time and investment horizon. It is important to underline at this stage that despite their similarities, the measures proposed in Eqs. (4) and (5) are not equivalent to those in Eqs. (1) and (2).

In the next section we apply our LRD to \( PnL \) curves generated by different trading strategies.

4 Local Risk Decomposition in trading systems: applications

Now we consider two examples of the LRD when applied to \( PnL \) time series generated by different strategies. In particular, the first time series, Fig. 1 (top), shows relatively stationary performance over the period under consideration, with the exception of two “bumps” in the middle of 2007 and at the beginning of 2008. These “bumps” are highlighted as a valley and a peak in the LRD, as it can be seen in the contour plots for the LRA (Fig. 1, middle-right, \( \beta = 0.75 \)) and for the LSR (Fig. 1, bottom-right). The second time series, instead, Fig. 2 (top), is more volatile if compared to the first: we have good performances up to the end of 2006 when suddenly the system starts to lose money. However, at the end of 2007 a comeback is observed. Both LRA, (Fig. 2, middle-right, \( \beta = 0.75 \)), and LSR, (Fig. 2, bottom-right), capture this dynamics very faithfully: a deep valley followed by an high peak can be observed in the last part of the time series. The LRD framework, therefore, allows the practitioner to identify and stress easily specific periods in time as well as specific investment horizons that have been particularly significant during the life (or testing) of a trading system.
Fig. 1. The two top plots are the same daily *PandL* generated by a certain trading strategy. The time series is shown twice in order to ease the comparison with the LRD contour plots reported underneath. On the right-hand side, we report the LRA (middle-right, $\beta = 0.75$) and the LSR (bottom-right). Both representations capture the “bumps” observed in 2007 and 2008. On the left-hand side, for completeness, we show the local return, $\tilde{r}_i$, (middle) and the local risk, $\tilde{\sigma}_i$, (bottom).

It is important to notice that LRA and LSR magnify differently the features of the time series presented in the former examples. This fact is due related to the investor’s particular appetite for risk, parameterized by $\beta$ and fixed to 0.75 in Figs. 1 and 2, that appears in the LRA. An aggressive trader would give more importance to the returns than to their fluctuations and, therefore, $\beta \approx 0$. By contrast, a risk adverse trader highlights the fluctuations, so to have $\beta \approx 1$. Examples of the LRA response to different sensitivities are shown in Fig. 3 for the first time series.

5 Extracting performance indices from the LRD

In the previous section, we introduced a framework to estimate local risk measures from the *PandL* of a trading strategy. The complete time/scale decomposition, despite being a faithful representation of the *PandL*’s dynamics, as well as visually appealing, can be cumbersome to use in practical applications, such as algorithms for portfolio optimization. It is, therefore, of interest to derive a single performance indicator from the information provided by the LRD.
Fig. 2. The plots are equivalent to those in Fig. 1 but for a different trading system. The performance, in this case, start being relatively volatile starting from the middle of 2006. This change in dynamics is encoded, with different emphasis, by the LRA ($\beta = 0.75$) and the LSR measures.

Fig. 3. Different contour plots of the LRA related to the PandL time series of Fig. 1 (top). The different values of $\beta$ (0.3, 0.6 and 1 from top to bottom) smooth or emphasize volatile periods according to the different appetite for risk chosen by the investor.
The advantage of having a LRD of the PandL signal lies in the possibility of customize the final indicator according to the user’s specific need. In other words, different traders may focus on different investment horizons or could be more interested in limited periods of time characterized by specific market condition: these preferences can be encoded in the integration of the LRD. In fact, for a generic local performance measure, \( f \), (LRA or LSR, for example) we define our indicator as the convolution of this quantity with time/scale kernels. In order to ease the notation, we assume a continuous decomposition for the PandL, that is \( i \to t \) and \( h \to s \), and we define an LRD indicator as

\[
\Phi^f_{\tau, \rho} = \frac{\int ds K_s(\frac{s-\rho}{\delta s}) \eta f(\tau, s)}{\int ds K_s(\frac{s-\rho}{\delta s})},
\]

where

\[
\eta f(\tau, s) = \frac{\int dt K_t(\frac{t-\tau}{\delta t}) f(t, s)}{\int dt K_t(\frac{t-\tau}{\delta t})},
\]

being \( K_s \) and \( K_t \) convolution kernels, \( \rho \) and \( \tau \) representing the “principal” investment horizon and time while \( \delta t \) and \( \delta s \) are dilatation coefficients \cite{Silverman1996}. These parameters can be tuned for different investor’s requirements, making the method particularly flexible. For example, by using hard kernels such as the Heaviside function, it is possible to cut the contribution of the performance beyond some specified look-back period. Otherwise, if it is preferred to give a weight to whole the historical performance of the trading strategy, a Gaussian kernel would be suitable. It is also important to note that Eq. (8) represents the average performance over some investment horizon \( s \) and can be used as a further proxy for specific strategy selection.

In order to underline the flexibility of our indicator \( \Phi^f_{\tau, \rho} \), we perform a numerical test on two artificial PandL time series for different choices of the parameters in Eqs. (7) and (8). We restrict the range of choices by fixing \( \tau = \max(t) \), since investors tend to give more importance on the recent performance of their strategies. The dilatation coefficients are selected according to: \( \delta s = 100 \rho \) and \( \delta t = [\max(t) - \min(t)]/4 \). The errors on the estimates have been calculated via the jacknife method \cite{Kunsch1989} and indicated between brackets as uncertainty in the last digit.

The LRD of two artificially generated PandL, each with 2000 data points, with different linear drifts as well as a different superimposed noise amplitude, is shown in Fig. 4. The first time series (blue) provides a better return at the expenses of higher volatility. The second time series (green), in contrast, exhibits a relatively stable growth. Despite the intrinsic differences, the annualized Sharpe ratio, Eq. (1), results to be the same for the two time series,
Table 1

The values for $\Phi_{LRA}^{\tau,\rho}$ and $\Phi_{LSR}^{\tau,\rho}$ along with $\eta_{LRA}(\tau, \rho)$ and $\eta_{LSR}(\tau, \rho)$ for different $\rho$. The values on the left refer to the first time series (blue) in Fig. 4, while the values on the right refer to the second time series (green). The kernel used for the integration of Eq. (7) is uniform with $K_r \equiv K_t \equiv 1$. The LRA has been normalized by its standard deviation over the time/scale boxes. This procedure is not fundamental for asserting the performances of a strategy. In brackets is the error of the last digit calculated via the jacknife method.

| $\rho$     | $\Phi_{LRA}^{\tau,\rho}$ | $\Phi_{LSR}^{\tau,\rho}$ | $\eta_{LRA}(\tau, \rho)$ | $\eta_{LSR}(\tau, \rho)$ |
|------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 50         | 0.36(1)/0.34(1)           | 2.71(7)/2.4(1)            | 0.018(6)/0.02(1)          | 0.5(2)/0.5(2)             |
| 100        | 0.36(1)/0.34(1)           | 2.71(8)/2.4(1)            | 0.040(4)/0.04(1)          | 0.64(8)/0.6(2)            |
| 250        | 0.364(9)/0.34(1)          | 2.71(6)/2.37(9)           | 0.05(1)/0.097(9)          | 0.5(1)/0.9(2)             |
| 500        | 0.36(1)/0.34(1)           | 2.71(6)/2.37(9)           | 0.23(1)/0.33(2)           | 1.53(9)/2.2(2)            |
| 1000       | 0.36(1)/0.34(1)           | 2.71(7)/2.4(1)            | 0.42(2)/0.51(2)           | 2.19(6)/2.6(1)            |

namely $S = 0.7(2)$, making them look equivalent from its prospective. On the other hand, the LRD framework gives a much broader picture regarding the performances of the two time series. The results are summarized in Table 1 and Table 2. In the first one, we report for different principal investment horizons, $\rho$, the values of $\Phi_{LRA}^{\tau,\rho}$ and $\Phi_{LSR}^{\tau,\rho}$ for the two trading systems when a uniform kernel is used, $K_r \equiv K_t \equiv 1$. In the second table, instead, we show the same results for Gaussian kernels. In the same tables we also report the values of $\eta_{LRA}(\tau, \rho)$ and $\eta_{LSR}(\tau, \rho)$, Eq. (8), at the scale of main interest, that is for $s \equiv \rho$. This estimate is, in itself, another indicator for the performance of the strategy. The results show that when we do not apply any convolution kernel, Table 1, the performance indicators $\Phi_{LRA}^{\tau,\rho}$ and $\Phi_{LSR}^{\tau,\rho}$ would pick the blue strategy, that is, the one with the highest return, as the best out of the two. However, if we consider the indicator at a specific investment horizon $\rho$, that is $\eta_{LRA}(\tau, \rho)$ and $\eta_{LSR}(\tau, \rho)$, the situation is not as clear. On the other hand, when the indicators are extracted via two Gaussian kernels centered in the last day of trade and at the horizon $\rho$, explicitly giving more importance to a particular time/scale region, the best performing system would be the green one. This result is due to the fact that the blue strategy is not performing well in the last period of the PandL series where the time kernel is centered.

This simple example highlights the flexibility of the LRD framework when compared to global quantities such as the Sharpe ratio which ignore the dynamics of the performance.

6 Discussion and conclusions

In the present paper we have introduced a local risk decomposition framework that retains dynamical information about the performance of a trading
strategy. This framework is very useful for practitioners who work at high-frequencies as it provides a map of the non-stationarity and multiscale features of the PandL time series. Moreover, from the LRD it is possible to construct a single indicator for the performance of the trading system, as shown in Sec. 2. The advantage of this indicator when compared to more traditional ones, such as the Sharpe ratio for example, lies in the fact that the user can choose to put more emphasis on some period in time or some specific investment horizons according to his/her preference. It is also important to stress the local detrending procedure in the risk estimate which we have used in order to take into account for the possible non-stationarity of the time series.
On the other hand, in order to have a reliable estimation of the dynamics at different scales, the LRD requires a reasonable amount of samples in the PandL. This drawback makes the LRD more suitable for high/medium frequency trading systems rather than log term ones.

In conclusion, the LRD framework can be a useful alternative to more traditional risk adjusted performance indicators and, consequently, it can be applied also in problems of portfolio optimization [Dacorogna et al., 2001; Bailey, 2005].

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