Excitation Spectrum and Collective Modes of Composite Fermions

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Abstract

According to the composite fermion theory, the interacting electron system at filling factor \( \nu \) is equivalent to the non-interacting composite fermion system at \( \nu^* = \nu/(1 - 2m\nu) \), which in turn is related to the non-interacting electron system at \( \nu^* \). We show that several eigenstates of non-interacting electrons at \( \nu^* \) do not have any partners for interacting electrons at \( \nu \), but, upon composite fermion transformation, these states are eliminated, and the remaining states provide a good description of the spectrum at \( \nu \). We also show that the collective mode branches of incompressible states are well described as the collective modes of composite fermions. Our results suggest that, at small wave vectors, there is a single well defined collective mode for all fractional quantum Hall states. Implications for the Chern-Simons treatment of composite fermions will be discussed.

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I. INTRODUCTION

Several spectacular phenomena have been observed in two-dimensional electron-systems (2DES) under the application of a strong transverse magnetic field \( B \). These are consequences of the formation of a new kind of particle in this system, called composite fermion (CF), which is an electron carrying an even number of vortices of the wave function \([1]\). Composite fermions are formed because, in a certain range of filling factors, electrons avoid each other most efficiently by capturing an even number of vortices of the wave function and transforming into composite fermions. The residual interaction between composite fermions is weak, and they can be treated as non-interacting to a good first approximation. The fundamental property of composite fermions is that they experience an effective field \( B^* = B - 2m\phi_0\rho \), where \( \rho \) is the 2D density of electrons, and \( \phi_0 = \hbar c/e \) is the quantum of flux. Thus, the liquid of strongly correlated electrons at \( B \) is equivalent to a gas of weakly interacting composite fermions at \( B^* \).

The wave functions of non-interacting composite fermions are constructed as follows \([1]\). Consider non-interacting electrons at filling factor \( \nu^* \), and denote their eigenstates by \( \Phi_{\nu^*,n,\alpha} \). (The filling factor is defined as \( \nu = \rho\phi_0/B \).) The spectrum of non-interacting electrons contains bands of degenerate states separated by the cyclotron energy \( \hbar \omega_c \); \( n \) is the band index, and \( \alpha \) labels the eigenstates within a band. To write the wave functions of non-interacting composite fermions, we attach \( 2m \) vortices to each electron by multiplying \( \Phi \) by \( D^m \), where the Jastrow factor

\[
D \equiv \prod_{j<k} (z_j - z_k)^2 ,
\]

where \( z_j = x_j + iy_j \) denotes the position of the \( j \)th electron as a complex number. To see that \( D \) attaches two vortices to each electron, fix all \( z_j \)'s in \( D \) except \( z_1 \). As \( z_1 \) is taken around in a closed loop around any other electron, the wave function gains a phase of \( 4\pi \), i.e., each electron sees two vortices on every other electron. The resulting CF wave function is

\[
\Phi^{CF}_{\nu^*,n,\alpha} = \mathcal{P} D^m \Phi_{\nu^*,n,\alpha} .
\]
projects the wave function onto the lowest LL of electrons to obtain states appropriate for the large $B$ limit. We will call the process of obtaining the CF state $\Phi^{CF}$ from a given electron state $\Phi$ (which involves multiplication by $D$ and then projection on to the lowest LL) “composite fermion transformation”.

The CF theory asserts that the eigenstates of interacting electrons, denoted by $\chi$, are well described by the CF wave functions, i.e.,

$$\chi_{\nu,n,\alpha} = \Phi_{\nu^*,n,\alpha}^{CF}.$$  \hfill (3)

The wave function of composite fermions on the right hand side of Eq. (3) describes an electron system at filling factor $\nu$, given by

$$\nu = \frac{\nu^*}{2m\nu^* + 1}.$$  \hfill (4)

This equation can also be written in terms of a relationship between the effective magnetic field experienced by the composite fermions, $B^* = \rho\phi_0/\nu^*$, and the external field $B = \rho\phi_0/\nu$ as:

$$B^* = B - 2m\rho\phi_o.$$  \hfill (5)

There is another way of seeing why the composite fermions experience a smaller effective magnetic field. Take a composite fermion around a loop enclosing area $A$. The total phase associated with this path is

$$2\pi(BA/\phi_0 - 2m\rho A)$$  \hfill (6)

where the first term is the usual Aharonov-Bohm phase and the second term is the phase due to the vortices bound to other composite fermions inside the loop. Equating this to $2\pi B^* A/\phi_0$, the Aharonov-Bohm phase due to an effective field $B^*$, produces Eq. (5). This further clarifies why binding of vortices to electrons leads to an effective renormalization of the magnetic field.
The principal feature of the composite fermion theory is that the strongly correlated liquid of interacting electrons at magnetic field $B$ (or filling factor $\nu$) resembles a gas of non-interacting composite fermions at magnetic field $B^*$ (or filling factor $\nu^*$). The composite fermions form quasi-LL’s [2], analogous to LL’s of electrons. There are strong experimental reasons to believe this scenario. To begin with, it provides a simple explanation of the origin of the FQHE [3]. IQHE of composite fermions ($\nu^* = n$) corresponds to electron filling factors $\nu = n/(2mn + 1)$. These are precisely the observed fractions [1], demonstrating that the FQHE of electrons is simply the IQHE of composite fermions. Many properties of the system in the FQHE regime can be understood in terms of weakly interacting composite fermions: the resistance peaks for high quality samples occur half-way between two successive quasi-LL’s (i.e., at $\nu^* = n + 1/2$) [3]; the excitation gaps are successfully modeled as quasi-cyclotron energies of composite fermions [3]; and the oscillations in resistance at relatively high temperature are well described in terms of Shubnikov-de Haas oscillations of composite fermions [4]. There is good evidence that the composite fermion description is also valid for the compressible state at $\nu = 1/2$, where they see vanishing effective field, and form a quasi-Fermi sea, investigated by Halperin, Lee, and Read [5]. The cyclotron orbit of composite fermions for small $B^*$ has been observed in several recent experiments [5].

There is also strong evidence for the validity of the CF theory from numerical studies on few electron systems [10–13]. While these numerical calculations are limited to systems with few (6-10) electrons, they have the virtue of being exact, and thereby tell us in an unprejudiced manner how much and how far the CF theory may be trusted. In detailed numerical studies on few electron systems, it has been found that there is a low-energy band in the spectrum of interacting electrons at $B$, which is remarkably similar to the low-energy band of non-interacting fermions at $B^*$, establishing the formation of composite fermions over a wide range of filling factors [11]. Furthermore, the CF wave functions, which involve no adjustable parameters, have been found to be almost identical to the exact numerical wave functions in this band. For the special case of the ground state at $\nu = 1/(2m + 1)$, the CF wave function is identical to the Laughlin wave function [14], which had already been
known to be a good representation of the exact ground state at $\nu = 1/(2m+1)$ [15,16]. These studies provide a compelling evidence for the validity of the CF theory for the low-energy physics.

This work investigates the excited states, which are also expected to form bands from the composite fermion theory. The principal motivation is experimental. For an explanation of the FQHE, which is a low-temperature phenomenon, an understanding of the states below the gap is sufficient, but several other experiments require understanding of the higher bands. Optical recombination experiments show evidence for bands in the spectrum [17]. Raman experiments probe the collective mode excitations of the FQHE system [18]. Another situation when the higher bands become relevant is when the gap disappears, which happens when the composite fermions fill a large number of quasi-LL’s, i.e., in the limit when $\nu = n/(2n+1) \rightarrow 1/2$, when composite fermions form a Fermi sea. Our study allows us to make some qualitative statements about the nature of the compressible state at $\nu = 1/2$.

Another motivation for the present study is simply to explore further the formal structure of the CF theory, and determine or extend the limits of its validity. A summary of our findings is as follows. We have studied two to three bands for several systems. We find that the CF theory continues to provide a good description of these bands. However, this happens only as a result of many magical cancellations. As will be discussed below, the system of non-interacting electrons at $\nu^*$ contains “too many states”, but just the right number of them are eliminated in the CF transformation, and the remainder provides a good description of the spectrum of interacting electrons at $\nu$.

II. THE SPHERICAL GEOMETRY

Our numerical calculations employ the spherical geometry [16] in which $N$ electrons move on the surface of a sphere under the influence of a radial magnetic field. The flux through the surface of the sphere is $N_\phi hc/e$, where $N_\phi$ is an integer. We consider spinless electrons confined to their lowest LL, as appropriate in the limit of large magnetic field. The total orbital angular momentum $L$ is analogous to the wave vector of the planar geometry; larger $L$ corresponds to larger wave vector [16].
We summarize here some basic results about the single electron wave functions, and refer the interested reader to the Appendix and two very useful papers by Wu and Yang \[19\] for more detailed information. The eigenstates in the spherical geometry are called “monopole harmonics” $Y_{S,l,m}$. For $S = 0$ they are the usual spherical harmonics $Y_{l,m}$. The quantity $l = |S|$, $|S| + 1$, ... denotes the orbital angular momentum, and $m = 0$, $\pm 1$, $\pm 2$, ... $\pm l$ is its component in the z direction. (The notation $m$ is also used for the amount of vortices attached to each electron; the meaning should be clear from the context.) Note that the minimum value of $l$ is $|S|$, and it can take either integer or half integer values.

The energy of an electron in the $l$th angular momentum shell is given by

$$E_{l,m} = \hbar^2 \frac{l(l + 1) - S^2}{2m_e R^2}$$

where $m_e$ is the electron band mass and $R$ is the radius of the sphere. Using the fact that the flux through the surface of the sphere is $2S$ flux quanta, i.e.,

$$4\pi R^2 B = 2S \left(\frac{hc}{e}\right),$$

the energy can be rewritten as

$$E_{l,m} = \frac{l(l + 1) - S^2}{2S}$$

in units of the cyclotron energy $\hbar \omega_c = \hbar eB/m_e c$. The angular momentum shells of the spherical geometry are thus analogous to the LL’s of the planar geometry. The lowest ($n = 1$) LL is given by $l = |S|$ shell, and the $n$th by the $l = |S| + n - 1$ shell. The degeneracy of the lowest ($n = 1$) LL is $2|S| + 1$ and increases by two for each successive LL.

We will assume large Zeeman energy throughout this work, and will therefore work with spinless electrons. The Coulomb interaction commutes with the total angular momentum ($L$), which will be used to label the eigenstates. All eigenstates in a given $L$ multiplet (i.e., states with different $L_z$) have the same energy. Therefore, it is sufficient to work in the subspace of smallest $L_z$, provided it is remembered that each state in this subspace denotes a multiplet. We will usually work in the subspace $L_z = 0$; one state in this subspace represents a multiplet of $2L + 1$ degenerate states of the full Hilbert space.
The Eqs.(1-5) are easily generalized to spherical geometry \[10,11\]: the Jastrow factor \(D\) in the spherical geometry is the square of the wave function of the lowest filled LL of spinless electrons, i.e., \(D = \Phi_1^2\) (which is also true in the planer geometry, aside from the exponential factors). We also find it more convenient to label the wave functions by \(S\), rather than the filling factor \(\nu\) or the magnetic field \(B\), so that the Eqs. (2) and (3) are written as (suppressing some of the subscripts)

\[ \chi_S = \Phi^{CF}_{S^*} = \mathcal{P}\Phi_1^{2m}\Phi_{S^*}. \]  

(9)

Since, for \(N\) electrons a filled LL is obtained for \(2S = N - 1\), the Eq. (9) leads to the following relation between \(S\) and \(S^*\):

\[ S = S^* + m(N - 1). \]  

(10)

Clearly, this reduces to Eq. (4) in the thermodynamic limit. An important feature of the CF theory is that the CF transformation conserves the total angular momentum \(L\). Some details of how we obtain the projected CF wave functions are given in the appendix.

III. THE BAND STRUCTURE

We start by noting that there are three systems in this problem: (i) interacting electrons at \(S\) (‘the \(\chi\) system’); (ii) non-interacting electrons at \(S^* = S + m(N - 1)\) (‘the \(\Phi\) system’); and (iii) non-interacting composite fermions at \(S^*\) (‘the \(\Phi^{CF}\) system’). We know everything about the \(\Phi\) system, and would like to learn as much as possible about the \(\chi\) system. As stated earlier, the \(\Phi\) system contains bands of degenerate states. These bands translate into bands of the \(\Phi^{CF}\) system, and eventually into bands of the \(\chi\) system. Such band structure was anticipated in Eqs. (2) and (3) in writing the subscripts. Besides predicting that there are bands in the \(\chi\) system, which is already a non-trivial accomplishment (since all states are degenerate in the absence of interactions), the composite fermion theory claims that the states in each band of the \(\chi\) system have a one-to-one mapping with the states in the corresponding band of the composite fermion (\(\Phi^{CF}\)) system, and the actual eigenstates are accurately given by the CF wave functions in Eq. (2) or (9).
An important feature of the composite fermion approach is that the Hilbert spaces of the $\Phi$ and $\chi$ systems are of drastically different sizes. This is most easily seen for a finite number of electrons: the Hilbert space of the $\Phi$ system is infinite, since all LL’s are allowed, while that of the $\chi$ system is finite, since only the lowest LL is allowed. Therefore, it is not possible, even in principle, for all bands of the $\Phi$ system to be analogous to the bands of the $\chi$ system. On the other hand, by construction, the Hilbert space of the $\Phi^{CF}$ system is identical to that of the $\chi$ system. Therefore, the analogy between the $\chi$ and the $\Phi^{CF}$ systems can be expected to be more general. It was found earlier [10], that the lowest bands of the three systems are indeed analogous in a wide range of filling factors: i.e., they have the same number of states, with identical quantum numbers. This was a remarkable result, showing that the low-energy dynamics of interacting electrons at $S$ is similar to that of non-interacting electrons at $S^*$. In the present work, we extend this result to higher bands.

We have studied four systems with $(N, S) = (6, 7.5), (7, 9), (8, 8.5), (6, 8)$. The first two represent the state at $\nu = 1/3$, the third lies somewhere between 1/3 and 2/5, and the fourth has $\nu > 1/3$. The exact (Coulomb) low-energy spectra of these systems are shown in Figs. 1-4 (a). There is a clearly defined lowest band. The second bands can also be identified rather easily. The third and higher bands are broad, mix with other bands, and are only poorly defined.

The spectra of the corresponding non-interacting electrons (the $\Phi$ system) at $(N, S^*) = (6, 2.5), (7, 3), (8, 1.5), (6, 3)$ are also shown in Figs. 1-4 (c). There are bands of degenerate states separated by the cyclotron energy; the number above each line shows the number of orthogonal multiplets at that energy. The second band is obtained by exciting one electron by one LL, and the states in the third band contain either one electron excited by two LL’s or two electrons by one LL.

The largest $L$ in each band of the $\chi$ system is correctly identified by the $\Phi$ system. The lowest bands of the two systems also match perfectly. For the 1/3 state, the lowest band contains only one state, which is the one filled LL state in the $\Phi$ system. A CF transformation of this state produces the Laughlin state, which is known to be quite close
to the ground state of the 1/3 state. However, the higher bands of the \( \chi \) and the \( \Phi \) systems do not match. (In Figs. 3 and 4, the higher bands of the \( \chi \) systems are reasonably well defined, and a lack of matching is quite clear. In Figs. 1 and 2, the third band is not very well defined. However, if we construct bands of the \( \chi \) system in analogy with the \( \Phi \) system, they would contain very high energy states at some low values of \( L \), while some low-energy states at larger values of \( L \) will be outside the bands.) The second bands in Figs. 1(c) and 2(c) have one multiplet each at \( L = 1, 2, \ldots N \), but, the corresponding bands of the \( \chi \) system in Figs. 1(a) and 2(a) do not have any multiplet at \( L = 1 \). The second bands of Figs. 3(a) and 4(a) have even less similarity to the second bands of 3(c) and 4(c). This leads us to conclude that the exact correspondence of the \( \Phi \) and the \( \chi \) systems is limited to the lowest band.

Next we consider the CF system \( \Phi^{CF} \) at \( S^* \). We construct the CF states by a CF transformation of the states of the \( \Phi \) system. Several surprising things happen.

(i) The \( L = 1 \) state of the second bands of Figs. 1(c) and 2(c) are annihilated upon projection (i.e., their projection is identically zero \([20]\)); as a result, the corresponding \( \Phi^{CF} \) systems do not have an \( L = 1 \) state in the second bands \([10]\).

(ii) Orthogonal states of the \( \Phi \) system do not, in general, produce orthogonal states upon the CF transformation. (Of course, states with different \( L \) produce orthogonal states.)

(iii) The states obtained from the second band of Figs. 3(c) and 4(c) are not even linearly independent in general. We construct an orthogonal basis using the Gramm-Schmid procedure, which is smaller than the \( \Phi \) basis we started with.

(iv) Consider states produced in this way at a given \( L \). They contain all states of the lower bands at that \( L \), i.e., the states in the lower band can be exactly expressed as a linear combination of the states thus produced. We construct a still smaller basis, which is orthogonal to the lower band CF states, and identify it with the second band of the CF system.

(v) The situation for the third bands of Figs. 1(c) and 2(c) is similar. First, the number of linearly independent states is smaller after the CF transformation. Second, the lower
band states are contained in the CF states produced from the third band of electrons. We again construct a basis which is also orthogonal to the lower band states, and identify it with the third band of the corresponding CF system.

(vi) The number of orthogonal multiplets at various $L$ in the higher bands of the $\Phi^{CF}$ systems is shown in Figs.1-4 (b). It is rather remarkable that the bands of the $\Phi^{CF}$ system exhibit a perfect one-to-one correspondence with the bands of the $\chi$ system.

(vii) In order to provide a microscopic confirmation of the CF theory, we consider a sub-basis of CF states at a given $L$ in a given band. We wish to compare it with the corresponding sub-basis of $\chi$ states. According to the CF theory, we need to diagonalize the Coulomb Hamiltonian in the CF sub-basis, determine the eigenstates, and calculate the overlap of each CF eigenstate with the corresponding eigenstate of the $\chi$ system. We define “generalized overlap” as the geometric mean of all these overlaps, given by $[\prod_j O_j]^{1/J}$, where $O_j$ is the overlap of the $j$th states of the two sub-bases, and $J$ is the number of states in the CF (or the $\chi$) sub-basis. A large generalized overlap indicates that the CF sub-basis is close to the $\chi$ sub-basis. Other than the fact that the generalized overlap gives a single measure of the similarity between two bases, it also has the advantage that its calculation does not require any diagonalization. It is given by

$$[Det < \Phi_j^{CF} | \chi_j >]^{1/J},$$

where the states $\Phi_j$ and $\chi_j$ are the properly orthonormalized states of the two sub-bases. The (generalized) overlaps are shown on the figures themselves (See Figs. 1b-4b). They are generally close to unity, indicating the microscopic validity of the CF description for higher bands.

The above discussion gives rules for constructing the bases for successive bands of interacting electrons using the CF framework. For all four systems studied above, the bases constructed in this manner provide a systematic description of the energy spectra of interacting electrons. We believe that these rules are of quite general validity in the range of filling factors where the composite fermion description is applicable for the lowest band.
We would like to emphasize that there is no \textit{a priori} reason for expecting cancellations in going from the electron system ($\Phi$) to the CF system ($\Phi^{CF}$). Of course, since the Hilbert space of the latter is smaller than that of the former, eventually, for much higher bands, such cancellations are inevitable. However, the lowest two or three bands of the systems studied above cover only a small part of the full Hilbert space of the $\Phi^{CF}$ system. Thus, it is rather non-trivial that \textit{seemingly} distinct trial wave functions, constructed from orthogonal $\Phi$ states, quite frequently turn out to be \textit{mathematically} linearly dependent. This shows that the CF trial wave functions possess certain non-trivial mathematical symmetries. A formal appreciation of these symmetries will require a better understanding of the projection operator. In order to describe the thermodynamics of the FQHE state, it would be quite useful to have some general understanding of how many states are eliminated in going from the $\Phi$ system to the $\Phi^{CF}$ system.

It should also be clear that the reduction in the number of states in the CF transformation is not a finite size effect. Even though our systems are small, the number of states in the first three bands is only a small fraction of the total number of states.

IV. COLLECTIVE MODES OF COMPOSITE FERMIONS

For incompressible states, the second band contains precisely one multiplet at each $L$ in a certain range of $L$ (see, e.g., Figs. 1 and 2). This will be called the collective mode branch (also known as the exciton or the plasmon branch). The angular momentum $L$ is roughly proportional to the momentum $q$ of the planar geometry \cite{16}.

In the usual three dimensional electron gas systems, when the density operator ($\rho_q$) acts on the ground state, it creates (for small $q$) a state whose oscillator strength is sharply peaked at some energy. It is called, for obvious reason, the “collective mode”. A single mode approximation (SMA) assumes that the peak is a delta function. Soon after Laughlin’s theory of the ground state at $\nu = 1/(2m+1)$, Girvin, MacDonald and Platzman (GMP) \cite{21} developed an SMA theory for the collective modes of the Laughlin states. Straightforwardly extending it to general fractions, the GMP wave function for the collective mode of the $\nu = n/(2n+1)$ state is constructed by applying the projected electron density operator to
the ground state as

\[ \mathcal{P} \rho^L \Phi^{CF}_n = \mathcal{P} \rho^L \mathcal{P} D \Phi_n, \quad (12) \]

where \( \Phi^{CF}_n = \mathcal{P} D \Phi_n \) is the ground state at \( \nu = n/(2n + 1) \), and

\[ \rho^L = \sum_{i=1}^{N} Y_{L,0}(\Omega_i) \quad (13) \]

is the density operator at \( L \). It has been shown that this wave function provides a good description of the low- and intermediate-\( L \) states at \( \nu = 1/3 \), as shown in Table I. It, however, fails at large \( L \). It also does not work as satisfactorily at other fractions [22,23].

The low-energy excitations of the FQHE state are described in terms of composite fermions, rather than electrons. Therefore, it might seem reasonable to consider collective modes of composite fermions. (Since an excited composite fermion is a quasielectron, this could also be called a collective mode of quasielectrons. Similar suggestions have been made in the past. See, for example [24].) In contrast to Eq. (12), which gives the collective mode of electrons, the collective mode of composite fermions of the \( \nu = n/(2n + 1) \) state is constructed by applying the CF transformation on the collective mode state at \( \nu^* = n \) [25]. Its wave function is given by

\[ \mathcal{P} D \rho^L(n \rightarrow n + 1) \Phi_n, \quad (14) \]

where \( \rho^L(n \rightarrow n + 1) \) creates the “fundamental” collective mode at \( \nu^* = n \), which involves excitation of one electron from the \( n \)th LL to the \( (n + 1) \)th LL. For the state at \( \nu^* = 1 \), the excited electron in the second LL has angular momentum \( (N + 1)/2 \) whereas the hole left behind in the lowest LL has angular momentum \( (N - 1)/2 \), so the allowed values of the total angular momentum are \( L = 1, 2, \ldots, N \). For the particle-hole excitation at \( \nu^* = 2 \), the excited electron in the third LL has angular momentum \( N/4 \) whereas the hole in the second LL has angular momentum \( (N + 4)/4 \), implying that the allowed values of the total angular momentum are \( L = 1, \ldots, (N + 2)/2 \). The wave function of the collective mode state is uniquely determined by symmetry for any given \( L \), with no fitting parameter. (In
particular, it does not depend on the cyclotron energy.) Upon CF transformation, the wave function of the collective mode of composite fermions is obtained for these values of $L$, again with no adjustable parameter.

First we note that the collective mode branch in the exact diagonalization study has maximum $L$ consistent with the prediction of the CF theory. The range of the collective mode branch is predicted correctly by the CF theory for 3/7 state as well; for spectra, see Refs. [23,26]. The electron (i.e., GMP) collective mode, on the other hand, is defined up to much higher values of $L$. Table I shows that Eq. (14) provides a good approximation of the entire collective mode branch for both 1/3 and 2/5. A good overlap guarantees that the energy of the CF state is very close to the exact energy; the Coulomb energies of the CF collective mode states are shown on the Figures themselves, and agree well with the exact energies. There is little doubt that the collective mode branches of other incompressible states can also be modeled successfully as the collective mode of composite fermions.

Eq. (14) gives the wave function of the collective mode in which one composite fermion is excited from the topmost filled quasi-LL to the lowest empty quasi-LL. One may wonder if there are more collective modes of composite fermions, corresponding to excitations of composite fermions across two or more quasi-LL’s. Analogous modes indeed exist for electrons in the IQHE regime [23]. In order to investigate this question for $\nu = 1/3$, we construct states $\mathcal{P}D\rho^{L}(1 \rightarrow 2)\Phi_{1}$, $\mathcal{P}D\rho^{L}(1 \rightarrow 3)\Phi_{1}$, and $\mathcal{P}D\rho^{L}(1 \rightarrow 4)\Phi_{1}$. We find that for $L = 2$ and 3, these are mathematically identical for systems with $N = 4$, 5, 6, and 7, and presumably for arbitrary number of electrons. (Note that there is no $L = 2$ state for the $1 \rightarrow 4$ excitation.) The total number of states is fairly large for $L = 2$ and 3, and therefore the fact that the various composite particle-hole states are identical is rather non-trivial. For $L = 4$ and 5, the various composite fermion modes are not identical, but they still have a reasonably good overlap with the $1 \rightarrow 2$ state. These results suggest that there is only one collective mode of composite fermions, despite the fact that there are several in the electron system $\Phi$. This is a striking example of the breakdown of a one-to-one correspondence in the higher bands of $\Phi$ and $\Phi^{CF}$.  

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It might be argued that our conclusion regarding the number of collective modes might not remain true in the presence of a small amount of LL mixing, always present in experiment. In this case, it would seem natural not to project the composite fermion states on to the lowest LL, so they would not be identical. However, clearly, the unprojected composite fermion states will simply provide different approximations for the same collective mode. The essential point, quite obvious on physical grounds, is that the intra-LL spectrum of interacting electrons, or the number of intra-LL collective modes, cannot change in any essential manner when a small amount of LL mixing is allowed. Therefore, our conclusion, obtained with the lowest LL approximation, should remain valid for the more realistic situation of a large though not infinite $B$.

At small $L$, the CF description of the collective mode branch of the 1/3 state is not as good as at higher $L$. The relatively poor performance of the CF theory may be attributed to the fact that the collective mode branch is coming close to the higher bands; the exact state mixes with higher band states to reduce its energy, whereas any such mixing is neglected in the CF approach. For 2/5, the description becomes worst at intermediate values of $L$, where the collective mode is the closest to the higher bands.

It is intuitively reasonable that, for small $L$, the collective mode can be thought of either as a mode of electrons or as a mode of composite fermions, since small (relative) displacements of composite fermions are indistinguishable from small displacements of electrons. This is supported by the fact that both the CF and the GMP theories provide reasonable descriptions of the true collective mode for small $L$, both for 1/3 and 2/5. (For the 1/3 state, the GMP theory does slightly better than the CF theory for small $L$. Also note that the $L = 1$ state is annihilated by the projection operator in both schemes. We note that the work of Reynolds and d’Ambrumenil [23] also finds that the SMA works quite well at small $q$ for 2/5 and 3/7.) This suggests that there is one and only one well-defined electron collective mode for all incompressible states in the limit of small $q$.

Raman experiments observe collective mode at small wave vectors, where it may be interpreted as either an electron or a composite fermion collective mode. Pinczuk et al. [18]
have reported observation of the collective mode of the 1/3 state in Raman experiments. No collective modes have yet been observed at other fractions. Recently [27], a broad secondary peak has also been observed at finite wave vectors at $\nu = 1/3$; we believe that this is an observation of the third band of composite fermions. Chen and Quinn [28] have suggested that the appearance of double peaks in photoluminescence experiments [17] is also due to the formation of bands, explained naturally within the CF framework.

V. COMPARISON WITH CHERN-SIMONS THEORY

Our few electron calculations have shown that the CF-wave-function scheme is qualitatively and quantitatively trustworthy, not only for the low-energy physics but also for the excitations. With further improvements in techniques for handling the projection operator for bigger systems, the CF-wave-function scheme should provide good quantitative approximations for various quantities in the thermodynamic limit.

The CF wave functions were motivated by a physical picture in which an even number of flux quanta are attached to each electron of $\Phi$ and then the flux is smeared to obtain electrons at a new magnetic field [1]. A Chern-Simons (CS) field-theoretical scheme has also been developed based on this picture [29,8]. In this scheme, the $\chi$ system of interacting electrons at $\nu$ is mapped on to the $\Phi$ system of non-interacting electrons at $\nu^*$ at the mean-field level. Fluctuations about the mean-field state are then treated perturbatively, usually at the random-phase-approximation (RPA) level [29,8]. In spite of the fact that both the wave-function and CS approaches are guided by the same physics, the precise relationship between them is not known. It has been asserted [29] that RPA is equivalent to multiplication by the Jastrow factor, but no proof exists.

The main implication of our results for the CF-CS scheme is that understanding of excitations in this scheme is likely to be a rather subtle and complicated issue. As stated above, the CF-CS theory proposes to approach the $\chi$ system perturbatively starting from the $\Phi$ system. However, the $\Phi$ system has the ‘correct’ number of states only in the lowest band; many states in the higher bands of this system are ‘unphysical’ in the sense that they do not have any analog in the $\chi$ system (some are annihilated upon the CF transformation, while
some produce states of the lower bands). It is not understood at present how perturbation theory starting from the Φ system will get rid of the spurious unphysical states and avoid or overcounting.

This is illustrated in the investigation of the collective modes using the Chern-Simons RPA approach \[30,26\]. It finds a series of electron collective modes for all incompressible states, including the Laughlin states. These are analogous to the electron collective modes at integer fillings. Another feature of the CF-CS approach is that the “fundamental” inter-LL mode of the \(ν^* = n\) state (i.e., the mode with energy \(ħω_c\), which corresponds to \(n \to n + 1\) electron-hole excitation) does not produce any mode at \(ν = n/(2n + 1)\); it is pushed up to the cyclotron energy \[29\]. This is actually crucial in recovering the Kohn’s theorem at \(ν\). Thus, the lowest energy electron collective mode at \(ν\) is expected to correspond to the \(2ħω_c\) collective mode of the \(ν^* = n\) state. (Lee and Wen \[31\] have also made a similar suggestion in order to explain the absence of the \(L = 1\) state of the collective mode.) These results seem to be inconsistent with our study, where several collective modes of the IQHE state are found to map into a single collective mode of the FQHE state (at least for small wave vectors), and the entire collective mode of the FQHE state is well described as the fundamental mode of composite fermions.

Our study also has consequences for the nature of the compressible state at \(ν = 1/2\). To understand this, let us approach the 1/2 state as the \(n \to ∞\) limit of the \(n/(2n + 1)\) sequence of incompressible FQHE states. Let us assume that the CF description continues to remain valid for arbitrarily large \(n\) (which is unfortunately not testable in finite system studies). Then, one would expect the ground state to correspond to \(n\) filled LL’s of electrons, but the excitations do not have a one-to-one correspondence with the excitations of the non-interacting electron state at \(ν^* = n\). In the FQHE regime, these excitations are separated by a gap, and do not have much influence on the low-energy physics of the problem. However, the gap vanishes in the limit \(n \to ∞\), as the sequence approaches \(ν = 1/2\). Two comments pertaining to this limit may be made in light of the results obtained above. (i) Since infinite filled Landau levels is nothing but a Fermi sea, the ground state of interacting electrons at
\( \nu = 1/2 \) is likely to resemble a Fermi sea of composite fermions with a well defined Fermi surface, as proposed by Halperin, Lee, and Read \[8\]. Recent experiments lend strong support to the existence of a Fermi sea at \( \nu = 1/2 \) \[9\]. (ii) However, the excitations of the 1/2 system do not have a one-to-one correspondence with the excitations of non-interacting electrons at zero magnetic field. Therefore, despite the existence of the Fermi surface, the \( \nu = 1/2 \) state is not a conventional Landau Fermi liquid in the sense of state counting.

This conclusion may seem inconsistent with that of Rezayi and Read \[13\], who recently attempted an investigation of the 1/2 state by finite size numerical study. They considered electron systems at \( S = N - 1 \), which map onto CF systems at zero effective flux (\( S^* = 0 \)), and found that the lowest band is well described in terms of the CF theory, in agreement with earlier results \[10\]. This was interpreted as a confirmation of the idea that the 1/2 state is a Fermi liquid. However, this study only deals with an ‘atom’ of composite fermions at zero flux, which is, of course, not a Fermi liquid. (Here the ‘atom’ has the special feature that electrons are constrained to move on a spherical surface.) Approaching a Fermi liquid from the atomic limit is rather subtle. To see this, take the limit \( N \to \infty \) while keeping the density fixed. The kinetic energy gap between successive angular momentum shells vanishes as \( 1/\sqrt{N} \) and the band structure is completely destroyed in this limit. Let us consider the situation when the kinetic energy gap is very small (say, compared to the interaction bandwidth) but non-zero. The low-energy excitations now consist of not only the excitations within the lowest angular momentum shell, but also excitations across several shells. The crucial point is that all these excitations are needed to produce a conventional Fermi liquid. Therefore, in order to prove the existence of a conventional Fermi liquid at \( \nu = 1/2 \), it is necessary to establish a one-to-one correspondence between all such (low-energy) excitations, which, as shown in the present work, does not exist \[32\].

While our study cautions against too literal an interpretation of the 1/2 state as a Fermi liquid, it does show that it is at least related to a Fermi liquid, and has a ground state that is likely a filled Fermi sea. Therefore, it is quite possible that eventually a Fermi-liquid-like description with renormalized parameters will prove sufficient. We believe that
the origin of the qualitatively different predictions of the CF-wave-function and the CF-CS approaches lies in the fact that no natural way is known of incorporating the large $B$ limit, or of implementing the lowest LL constraint in CS approach. This is indicated by the fact that the final formulas of the CS theory explicitly contain the band mass of the electron, which is not a parameter in a lowest LL theory.

VI. CONCLUSION AND ACKNOWLEDGEMENTS

We have investigated the excitation spectrum of interacting electrons in the FQHE regime. We find that the explanation of the excited bands is subtle, but in principle possible within the composite-fermion wave-function approach. The Hilbert space of non-interacting electrons at $\nu^*$ is much bigger than the relevant (i.e., lowest-LL) Hilbert space of interacting electrons at $\nu = \nu^*/(2\nu^* + 1)$, but the CF transformation eliminates just the right number of states from each band to bring the theory in agreement with the actual band structure of the interacting electrons at $\nu$.

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APPENDIX

Coulomb interaction

We will not need the explicit form of the monopole harmonics. Their following properties will be sufficient for our purposes.

$$Y_{S,l,m}^* = (-1)^{S+m} Y_{-S,-l,-m}$$

$$Y_{S,l,m} Y_{S',l',m'} = \sum_{l''} (-1)^{l+l'+l''+2(S''+m'')} \left( \frac{(2l + 1)(2l' + 1)}{4\pi(2l'' + 1)} \right)^{1/2}$$

$$<lm,l'm'|l''m''><lS,l'S'|l''S''> Y_{S'',m''}$$

where $S'' = S + S'$ and $m'' = m + m'$. Thus, the product of two spherical harmonics at $S$ and $S'$ produces a linear combination of spherical harmonics at $S + S'$.
\[\int d\Omega Y_{S_1l_1m_1} Y_{S_2l_2m_2} Y_{S_3l_3m_3} = (-1)^{l_1+l_2+l_3-m_3-S_3} \left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}\right)^{1/2}\]

\[< l_1 m_1, l_2 m_2 | l_3 - m_3 > < l_1 S_1, l_2 S_2 | l_3 - S_3 >\]  

(17)

provided \(S_1 + S_2 + S_3 = 0 = m_1 + m_2 + m_3\). Here, \(d\Omega\) is the integral over the angular variables.

The coulomb matrix elements can be easily evaluated using the properties of the monopole harmonics. The two-body matrix element is given by

\[\langle l_1 m_1, l_2 m_2 | V(|r_1 - r_2|) | l'_1 m'_1, l'_2 m'_2 > = \int d\Omega_1 d\Omega_2 Y_{S_1l_1m_1}^*(r_1) Y_{S_2l_2m_2}^*(r_2) \frac{e^2}{|r_1 - r_2|} Y_{S_3l_3m_3}^*(r_2) Y_{S_3l_3m_3}^*(r_1).\]  

(18)

The chord-distance between two electrons on sphere can be expressed as:

\[\frac{1}{|r_1 - r_2|} = \frac{4\pi}{R} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{0,l,m}^*(\theta_1, \phi_1) Y_{0,l,m}(\theta_2, \phi_2).\]  

(19)

With this substitution, the two integrals decouple and can be evaluated straightforwardly.

**Projection**

The projected wave functions can be expressed as

\[\mathcal{P} \Phi_1^2 \Phi_{S^*} = \sum c[\{M_j\}] |M_1, ... M_N >\]  

(20)

where \(\Phi_1\) is the Slater determinant wave function of the lowest filled LL at \(S_1 = (N - 1)/2\), \(\Phi_{S^*}\) is the Slater determinant wave function at \(S^*\), and the \(S\) of the product state is given by \(S = 2S_1 + S^*\). \(M_j\) label z-components of the single particle angular momentum \(L = S\), restricted to the range \(-S \leq M \leq S\); they satisfy the restrictions \(M_1 < M_2 < ... < M_N\) and \(\sum_j M_j = 0\), the latter because we restrict to the \(L_z = 0\) subspace. Our objective is to determine the coefficients \(c[\{M_j\}]\). The basic ingredient is the formula for projection of the product of two single particle eigenstates:

\[\mathcal{P} Y_{Slm} Y_{S'l'm'} = (-1)^{l+l'+3S''+2m''} \left(\frac{(2l + 1)(2l' + 1)}{4\pi(2S'' + 1)}\right)^{1/2}\]

\[< l'm'|S''m'' > < lS, l'S'|S''S'' > Y_{S''S''m''}\]  

(21)
with $S'' = S + S'$ and $m'' = m + m'$. In principle, the product $\Phi_1^2 \Phi_S^*$ can be fully expanded, and its projection can be obtained by repeated application of the above equation. We find it efficient first to expand $\Phi_1^2$ in terms of basis states $|m_1, ... m_N >$ at $2S_1$, and store the coefficients. (Note that since $\Phi_1^2$ is a boson wave function, $m$'s do not have to be all distinct, as is the case with $M$'s.) In order to obtain the coefficient of a given basis state $|M_1, ... M_N >$, we go over each term in the expansion of the Slater determinant $\Phi_S^*$ and search for those values of $\{m_j\}$ which will provide the correct $\{M_j\}$. Then it is a matter of complicated book-keeping to obtain the coefficient of $|M_1, ... M_N >$. In general, the eigenstates of $L$ in the $\Phi$ system are linear combinations of many Slater determinants; to obtain the projection of this state, we compute the projection for each Slater determinant as above and add the projected states with the appropriate coefficients. For further details, we refer the reader to Ref. [33].
REFERENCES

[1] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **41**, 7653 (1990); Adv. Phys. **41**, 105 (1992).

[2] The term “quasi-LL” is used for the energy levels of composite fermions to emphasize that these are different from the real LL’s of electrons. Composite fermions can occupy several quasi-LL’s even when the electrons are completely confined to their lowest real LL.

[3] D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[4] See, for example, J.K. Jain and V.J. Goldman, Phys. Rev. B **45**, 1255 (1992).

[5] V.J. Goldman, J.K. Jain, and M. Shayegan, Phys. Rev. Lett. **65**, 907 (1990); Mod. Phys. Lett. **5**, 479 (1991).

[6] R.R. Du, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. **70**, 2944 (1993).

[7] D.R. Leadley, R.J. Nicholas, C.T. Foxon, and J.J. Harris, Phys. Rev. Lett. **72**, 1906 (1994); R.R. Du, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Solid State Commun. **90**, 71 (1994).

[8] B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[9] W. Kang *et al.*, Phys. Rev. Lett. **71**, 3850 (1993); V.J. Goldman *et al.*, Phys. Rev. Lett. **72**, 2065 (1994); R.R. Du *et al.*, Phys. Rev. Lett. **70**, 2944 (1993); D.R. Leadley *et al.*, Phys. Rev. Lett. **72**, 1906 (1994); R.R. Du *et al.*, Solid State Commun. **90**, 71 (1994); R.L. Willett *et al.*, Phys. Rev. Lett. **71**, 3846 (1993).

[10] G. Dev and J.K. Jain, Phys. Rev. Lett. **69**, 2843 (1992).

[11] X.G. Wu, G. Dev, and J.K. Jain, Phys. Rev. Lett. **71**, 153 (1993).

[12] E.H. Rezayi and A.H. MacDonald, Phys. Rev. B **44**, 8395 (1991); G. Dev and J.K. Jain,
Phys. Rev. B 45, 1223 (1992); M. Kasner and W. Apel, Phys. Rev. B 48, 11435 (1993); X.C. Xie, Phys. Rev. B 49, 16833 (1994); U. Girlich and M. Hellmund, Phys. Rev. B 49, 17488 (1994); J.K. Jain and X.G. Wu, Phys. Rev. B 49, 5085 (1994); X.G. Wu and J.K. Jain, Phys. Rev. B 49, 7515 (1994); N.E. Bonestel, preprint; T. Nakajima and H. Aoki, preprint.

[13] E. Rezayi and N. Read, Phys. Rev. Lett. 72, 900 (1994).

[14] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).

[15] See, for example, G. Fano, F. Ortolani and E. Colombo, Phys. Rev. B 34, 2670 (1986).

[16] F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983), and in The Quantum Hall Effect, Eds. R.E. Prange and S.M. Girvin (Springer-Verlag, N.Y., 1987).

[17] A.J. Turberfield et al., Phys. Rev. Lett. 65, 637 (1990); B.B. Goldbert et al., Phys. Rev. Lett. 65, 641 (1990).

[18] A. Pinczuk, B.S. Dennis, L.N. Pfeiffer, and K. West, Phys. Rev. Lett. 70, 3983 (1993).

[19] T.T. Wu and C.N. Yang, Nucl. Phys. B 107, 365 (1976); Phys. Rev. D 16, 1018 (1977).

[20] The phrases ‘identically zero’, ‘exact’, or ‘mathematically identical’ refer to numerical results correct to machine accuracy (i.e., to 15-16 significant figures). There is little doubt that these results are exact.

[21] S.M. Girvin, A.H. MacDonald, and P.M. Platzmann, Phys. Rev. Lett. 54, 581 (1985); Phys. Rev. B 33, 2481 (1986).

[22] W.P. Su and Y.K. Wu, Phys. Rev. B 36, 7565 (1987).

[23] A.M. Reynolds and N.H. d’Ambrumenil, J. Phys. C 33, 5643 (1988).

[24] R.B. Laughlin, Physica 126B, 254 (1984).

[25] C. Kallin and B.I. Halperin, Phys. Rev. B 30, 5655 (1984).
[26] S.H. Simon and B.I. Halperin, Phys. Rev. B 48, 17368, (1993); ibid., B 50, 1807 (1994); S. He, S.H. Simon, and B.I. Halperin, ibid., B 50, 1823(1994).

[27] A. Pinczuk, unpublished.

[28] X.M. Chen and J.J. Quinn, Phys. Rev. Lett. 70, 2130 (1993); Phys. Rev. B 50, (1994).

[29] A. Lopez and E. Fradkin, Phys. Rev. B 44, 5246 (1991).

[30] A. Lopez and E. Fradkin, Phys. Rev. B 47, 7080 (1993).

[31] D.H. Lee and X.G. Wen, Phys. Rev. B XXX

[32] Furthermore, unlike the incompressible states, it is questionable whether meaningful information about the compressible 1/2 state can be obtained in finite size studies. For a discussion, see J.K. Jain, Phys. Rev. Lett. 73, 1051 (1994).

[33] X.G. Wu, Ph.D. Thesis.

Figure Captions

Fig.1 (a) Low-energy part of the eigenspectrum of interacting electrons in the lowest LL for the system \((N, S) = (6, 7.5)\). The energies are in units of \(\epsilon^2/\epsilon l_0\) where \(\epsilon\) is the dielectric constant of the background material and \(l_0\) is the magnetic length. The horizontal lines show the bands. The dots show the energies of the composite fermion states. (c) The low-energy spectrum of the corresponding electron system at \((N, S^*) = (6, 2.5)\). The energy separation between the bands is equal to the cyclotron energy. The integer near each dash shows the number of degenerate (orthogonal) multiplets at that energy, with \(2L + 1\) states in each multiplet. (b) The low-energy spectrum of the CF system at \((N, S^*) = (6, 2.5)\), obtained from the system in (c) as explained in the text. The separation between bands is the effective cyclotron energy of composite fermions. The integer above each dash shows the number of independent multiplets, and the number below each dash shows the (generalized) overlap of the CF basis with the exact eigenstates of (a).

Fig. 2 Same as Fig.1 for \((N, S) = (7, 9)\) and \((N, S^*) = (7, 3)\).
Fig. 3 Same as Fig.1 for \((N, S) = (8, 8.5)\) and \((N, S^*) = (8, 1.5)\).

Fig. 4 Same as Fig.1 for \((N, S) = (6, 8)\) and \((N, S^*) = (6, 3)\).

**Table Captions**

Table I. The overlap of the electron and the CF collective mode states with the exact collective mode states. (a) 1/3 state with \(N = 6\). (b) 1/3 state with \(N = 7\). (c) 2/5 state with \(N = 6\). (d) 2/5 state with \(N = 8\). Some of these results have been published before in some form, and are given here for completeness. The results for the electron (GMP) collective mode have been given (in slightly different forms) in Refs. [16,22,23]. The results for the CF collective mode (except for table b) have been published in [10].

(a) \[L\]  
|   | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| CF | 0.9484 | 0.9923 | 0.9915 | 0.9977 | 0.9800 |
| electron | 0.9796 | 0.9947 | 0.9818 | 0.8962 | 0.5435 |

(b) \[L\]  
|   | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| CF | 0.9032 | 0.9793 | 0.9970 | 0.9906 | 0.9822 | 0.9853 |
| electron | 0.9313 | 0.9860 | 0.9903 | 0.9467 | 0.7301 | 0.3984 |

(c) \[L\]  
|   | 2 | 3 | 4 |
|---|---|---|---|
| CF | 0.9999 | 0.9951 | 0.9983 |
| electron | 0.9995 | 0 | 0.9232 |

(d) \[L\]  
|   | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| CF | 0.9946 | 0.9741 | 0.9818 | 0.9971 |
| electron | 0.9941 | 0.6604 | 0.7296 | 0.8927 |