Quantitative Stability of the Brunn-Minkowski Inequality for Sets of Equal Volume

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\textit{(Dedicated to Professor Haim Brezis on the occasion of his 70th birthday)}

Abstract The authors prove a quantitative stability result for the Brunn-Minkowski inequality on sets of equal volume: If \(|A| = |B| > 0\) and \(|A + B|^\frac{1}{n} = (2 + \delta)|A|^\frac{1}{n}\) for some small \(\delta\), then, up to a translation, both \(A\) and \(B\) are close (in terms of \(\delta\)) to a convex set \(K\). Although this result was already proved by the authors in a previous paper, the present paper provides a more elementary proof that the authors believe has its own interest. Also, the result here provides a stronger estimate for the stability exponent than the previous result of the authors.

Keywords Quantitative stability, Brunn-Minkowski, Affine geometry, Convex geometry, Additive combinatorics

2010 MR Subject Classification 49Q20, 52A40, 52A20, 11P70

1 Introduction

The Brunn-Minkowski inequality is a very classical and powerful inequality in convex geometry that has found important applications in analysis, statistics, and information theory. We refer the reader to [14] for an extended exposition on the Brunn-Minkowski inequality and its relation to several other famous inequalities (see also [6–7]).

To state the inequality, we first need some basic notation. Given two subset \(A, B \subset \mathbb{R}^n\), and \(c > 0\), we define the set sum and scalar multiple by

\[ A + B := \{a + b : a \in A, b \in B\}, \quad cA := \{ca : a \in A\}. \tag{1.1} \]

We shall use \(|E|\) to denote the Lebesgue measure of a set \(E\). (If \(E\) is not measurable, \(|E|\) denotes the outer Lebesgue measure of \(E\).) The Brunn-Minkowski inequality says that, given \(A, B \subset \mathbb{R}^n\) measurable sets,

\[ |A + B|^\frac{1}{n} \geq |A|^\frac{1}{n} + |B|^\frac{1}{n}. \tag{1.2} \]
In addition, if $|A|, |B| > 0$, then equality holds if and only if there exists a convex set $K \subset \mathbb{R}^n$, $\lambda_A, \lambda_B > 0$, and $v_A, v_B \in \mathbb{R}^n$, such that

$$A \subset \lambda_A K + v_A, \quad B \subset \lambda_B K + v_B, \quad |(\lambda_A K + v_A) \setminus A| = |(\lambda_B K + v_B) \setminus B| = 0.$$ 

In other words, if equality holds in (1.2), then $A$ and $B$ are subsets of full measure in homothetic convex sets.

Because of the variety of applications of (1.2) as well as the fact one can characterize the case of equality, a natural stability question that one would like to address is the following.

Let $A, B$ be two sets for which equality in (1.2) almost holds. Is it true that, up to translations and dilations, $A$ and $B$ are close to the same convex set?

This question has a long history. First of all, when $n = 1$ and $A = B$, inequality (1.2) reduces to $|A + A| \geq 2|A|$. If one approximates sets in $\mathbb{R}$ with finite unions of intervals, then one can translate the problem to $\mathbb{Z}$, and in the discrete setting the question becomes a well studied problem in additive combinatorics. There are many results on this topic, usually called Freiman-type theorems. The precise statement in one dimension is the following.

**Theorem 1.1** Let $A \subset \mathbb{R}$ be a measurable set, and denote by $\text{co}(A)$ its convex hull. Then

$$|A + A| - 2|A| \geq \min\{|\text{co}(A) \setminus A|, |A|\},$$

or, equivalently, if $|A| > 0$, then

$$\delta(A) \geq \frac{1}{2} \min \left\{ \frac{|\text{co}(A) \setminus A|}{|A|}, 1 \right\}.$$ 

This theorem can be obtained as a corollary of a result of Freiman [12] about the structure of additive subsets of $\mathbb{Z}$ (see [13] or [17, Theorem 5.11] for a statement and a proof). However, it turns out that to prove Theorem 1.1, one only needs weaker results, and one can find an elementary self-contained proof of Theorem 1.1 in [8, Section 2].

In the case $n = 1$ but $A \neq B$, the following sharp stability result holds again as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman’s theorem in [3–4]).

**Theorem 1.2** Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A + B| < |A| + |B| + \delta$ for some $\delta \leq \min\{|A|, |B|\}$, then $|\text{co}(A) \setminus A| \leq \delta$ and $|\text{co}(B) \setminus B| \leq \delta$.

Concerning the higher dimensional case, in [1–2], Christ proved a qualitative stability result for (1.2), giving a positive answer to the stability question raised above. However, his results do not provide any quantitative control.

On the quantitative side, Diskant [5] and Groemer [15] obtained some stability results for convex sets in terms of the Hausdorff distance. More recently, in [10–11], the first author together with Maggi and Pratelli obtained a sharp stability result in terms of the $L^1$ distance, still on convex sets. Since this last result will be used later in our proofs, we state it in detail.

(Here and from now on, $E \Delta F$ denotes the symmetric difference between sets $E$ and $F$, that is, $E \Delta F = (E \setminus F) \cup (F \setminus E)$.)
Theorem 1.3 Let $A, B \subset \mathbb{R}^n$ be convex sets, and define
\[
\mathcal{A}(A, B) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|A\Delta(x_0 + \tau B)|}{|A|} : \tau = \left( \frac{|A|}{|B|} \right)^{\frac{1}{n}} \right\}, \quad \sigma(A, B) := \max \left\{ \frac{|A|}{|B|}, \frac{|B|}{|A|} \right\}.
\]
There exists a computable dimensional constant $C_0(n)$ such that
\[
|A + B|^\frac{1}{n} \geq (|A|^\frac{1}{n} + |B|^\frac{1}{n}) \left\{ 1 + \frac{\mathcal{A}(A, B)^2}{C_0(n) \sigma(A, B)^\frac{1}{n}} \right\}.
\]

More recently, in [8, Theorem 1.2 and Remark 3.2], the present authors proved a quantitative stability result when $A = B$: Given a measurable set $A \subset \mathbb{R}^n$ with $|A| > 0$, set
\[
\delta(A) := \left| \frac{1}{2}(A + A) \right| - 1 = \frac{|A + A|}{|2A|} - 1.
\]
Then, a power of $\delta(A)$ dominates the measure of the difference between $A$ and its convex hull $\text{co}(A)$.

Theorem 1.4 Let $A \subset \mathbb{R}^n$ be a measurable set of positive measure. There exist computable dimensional constants $\delta_n, c_n > 0$, such that if $\delta(A) \leq \delta_n$, then
\[
\delta(A)^{\alpha_n} \geq c_n \left| \frac{\text{co}(A) \setminus A}{|A|} \right|, \quad \alpha_n := \frac{1}{8n^{-1}((n - 1))^2}.
\]
In addition, there exists a convex set $K \subset \mathbb{R}^n$ such that
\[
\delta(A)^{n\alpha_n} \geq c_n |K \Delta A| / |A|.
\]

After that, we investigated the general case $A \neq B$. Notice that, after a dilation, one can always assume $|A| = |B| = 1$ while replacing the sum $A + B$ by a convex combination $S_t := tA + (1 - t)B$. It follows by (1.2) that $|S_t| = 1 + \delta$ for some $\delta \geq 0$. The main theorem in [9] is a quantitative version of Christ’s result. Since the proof is by induction on the dimension, it is convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of $\delta$. Here is the main result of that paper.

Theorem 1.5 Let $n \geq 2$, let $A, B \subset \mathbb{R}^n$ be measurable sets, and define $S_t := tA + (1 - t)B$ for some $t \in [\tau, 1 - \tau]$, $0 < \tau \leq \frac{1}{3}$. There are computable dimensional constants $N_n$ and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$, such that if
\[
||A| - 1| + ||B| - 1| + ||S_t| - 1| \leq \delta
\]
for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $K \subset \mathbb{R}^n$ such that, up to a translation,
\[
A, B \subset K \quad \text{and} \quad |K \setminus A| + |K \setminus B| \leq \tau^{-N_n} \delta^{\varepsilon_n(\tau)}.
\]
Explicitly, we may take
\[
M_n(\tau) = \frac{2^{3n+2} n^{3n} |\log \tau|^{3n}}{\tau^{3n}}, \quad \varepsilon_n(\tau) = \frac{\tau^{3n}}{2^{3n+1} n^{3n} |\log \tau|^{3n}}.
\]
In particular, the measure of the difference between the sets $A$ and $B$ and their convex hull is bounded by a power $\delta^\epsilon$, confirming a conjecture of Christ [1].

The result above provides a general quantitative stability for the Brunn-Minkowski inequality in arbitrary dimension. However, the exponent degenerates very quickly as the dimension increases (much faster than in Theorem 1.4), and, in addition, the argument in [9] is very long and involved. The aim of this paper is to provide a shorter and more elementary proof when $|A| = |B| > 0$, that we believe to be of independent interest.

After a dilation, one can assume with no loss of generality that $|A| = |B| = 1$. In this case, it follows by (1.2) that $\frac{1}{2}(A + B) = 1 + \delta$ for some $\delta \geq 0$, and we want to show that a power of $\delta$ controls the closeness of $A$ and $B$ to the same convex set $K$. Again, as in the previous theorem, it will be convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of $\delta$.

Here is the main result of this paper.

**Theorem 1.6** Let $A, B \subset \mathbb{R}^n$ be measurable sets, and define their semi-sum $S := \frac{1}{2}(A + B)$. There exist computable dimensional constants $\delta_n, C_n > 0$, such that if

$$||A| - 1| + ||B| - 1| + ||S| - 1| \leq \delta$$

(1.5)

for some $\delta \leq \delta_n$, then there exists a convex set $K \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset K \quad \text{and} \quad |K \setminus A| + |K \setminus B| \leq C_n \delta^{\beta_n},$$

where

$$\beta_1 := 1, \quad \beta_n := \frac{1}{2^{6n-3}3^{n-1}n!(n-1)!} \prod_{k=1}^{n} \alpha_k^2 \quad \forall n \geq 2,$$

and $\alpha_k$ is given by Theorem 1.4. (Recall that $|S|$ is the outer measure of $S$ if $S$ is not measurable.)

The proof of this theorem is specific to the case $|A|$ near $|B|$. It uses a symmetrization and other techniques introduced by Christ [2–3], Theorems 1.3–1.4, and two propositions of independent interest, Propositions 2.1–2.2 below. See Section 3 for further discussion of the strategy of the proof.

**2 Notation and Preliminary Results**

Let $\mathcal{H}^k$ denote the $k$-dimensional Hausdorff measure on $\mathbb{R}^n$. Denote by $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\pi : \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections, i.e.,

$$\pi(y, t) := y \quad \text{and} \quad \pi(y, t) := t.$$

Given a compact set $E \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, and $\lambda > 0$, we use the notation

$$E_y := E \cap \pi^{-1}(y) \subset \{y\} \times \mathbb{R}, \quad E(t) := E \cap \pi^{-1}(t) \subset \mathbb{R}^{n-1} \times \{t\},$$

$$E(\lambda) := \{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_y) > \lambda\}.$$
Following Christ [2], we consider two symmetrizations and combine them. For our purposes (see the proof of Proposition 2.1), it is convenient to use a definition of Schwarz symmetrization that is slightly different from the classical one. (In the usual definition of Schwarz symmetrization, \(E^*(t) = \emptyset\) whenever \(H^{d-1}(E(t)) = 0\).)

**Definition 2.1** Let \(E \subset \mathbb{R}^n\) be a compact set. We define the Schwarz symmetrization \(E^*\) of \(E\) as follows. For each \(t \in \mathbb{R}\),

1. If \(H^{d-1}(E(t)) > 0\), then \(E^*(t)\) is the closed disk centered at \(0 \in \mathbb{R}^{n-1}\) with the same measure.
2. If \(H^{d-1}(E(t)) = 0\) but \(E(t)\) is non-empty, then \(E^*(t) = \{0\}\).
3. If \(E(t)\) is empty, then \(E^*(t)\) is empty as well.

We define the Steiner symmetrization \(E^*\) of \(E\) so that for each \(y \in \mathbb{R}^{n-1}\), the set \(E^*_y\) is empty if \(H^1(E_y) = 0\); otherwise it is the closed interval of length \(H^1(E_y)\) centered at \(0 \in \mathbb{R}\). Finally, we define \(E^2 := (E^*)^*\).

As for instance in [2, Section 2], both the Schwarz and the Steiner symmetrization preserve the measure of sets, and the \(i\)-symmetrization preserves the measure of the sets \(E(\lambda)\). The following statement collects all these results.

**Lemma 2.1** Let \(A, B \subset \mathbb{R}^n\) be compact sets. Then \(|A| = |A^*| = |A^2|\),

\[|A^* + B^*| \leq |A + B|, \quad |A^* + B^*| \leq |A + B|, \quad |A^2 + B^2| \leq |A + B|,\]

and, for almost every \(\lambda > 0\),

\[|A \setminus \pi^{-1}(A(\lambda))| = |A^\sharp \setminus \pi^{-1}(A^\sharp(\lambda))|, \quad H^{n-1}(A(\lambda)) = H^{n-1}(A^\sharp(\lambda)),\]

where \(A(\lambda) := \{y \in \mathbb{R}^{n-1} : H^1(A_y) > \lambda\}\), \(A^\sharp(\lambda) := \{y \in \mathbb{R}^{n-1} : H^1(A^\sharp_y) > \lambda\}\).

Another important fact is that a bound on the measure of \(A + B\) in terms of the measures of \(A\) and \(B\) gives bounds relating the sizes of

\[\sup_y H^1(A_y), \quad \sup_y H^1(B_y), \quad H^{n-1}(\pi(A)), \quad H^{n-1}(\pi(B)).\]

We refer to [9, Lemma 3.2] for a proof.

**Lemma 2.2** Let \(A, B \subset \mathbb{R}^n\) be compact sets such that \(|A|, |B| \geq \frac{1}{2}\) and \(|\frac{1}{2}(A + B)| \leq 2\). There exists a dimensional constant \(M > 1\), such that

\[\frac{\sup_y H^1(A_y)}{\sup_y H^1(B_y)} \in \left(\frac{1}{M}, M\right), \quad \frac{H^{n-1}(\pi(A))}{H^{n-1}(\pi(B))} \in \left(\frac{1}{M}, M\right),\]

\[\left(\sup_y H^1(A_y)\right)H^{n-1}(\pi(A)) \in \left(\frac{1}{M}, M\right), \quad \left(\sup_y H^1(B_y)\right)H^{n-1}(\pi(B)) \in \left(\frac{1}{M}, M\right).\]

Thus, up a measure preserving affine transformation of the form \((y, t) \mapsto (\tau y, \tau^{1-n} t)\) with \(\tau > 0\), all the quantities \(\sup_y H^1(A_y), \sup_y H^1(B_y), H^{n-1}(\pi(A)), H^{n-1}(\pi(B))\) are of order one.
In particular,
\[ H^{n-1}(\pi(A)) + H^{n-1}(\pi(B)) + \sup_y H^1(A_y) + \sup_y H^1(B_y) \leq M. \] (2.3)

In this case, we say that \( A \) and \( B \) are \( M \)-normalized.

The following result of Christ [1, Lemma 4.1] shows that \( \sup_t H^{n-1}(A(t)) \) and \( \sup_t H^{n-1}(B(t)) \) are close in terms of \( \delta \).

**Lemma 2.3** Let \( A, B \subset \mathbb{R}^n \) be compact sets, define \( S := \frac{1}{2}(A + B) \), and assume that (1.5) holds for some \( \delta \leq \frac{1}{2} \). Also, suppose that \( A \) and \( B \) are \( M \)-normalized as defined in Lemma 2.2.

Then, there exists a dimensional constant \( C > 0 \) such that
\[ \frac{\sup_t H^{n-1}(A(t))}{\sup_t H^{n-1}(B(t))} \in (1 - C\delta^{\frac{2}{n}}, 1 + C\delta^{\frac{2}{n}}). \]

Two other key ingredients in our proof of Theorem 1.6 are the following propositions, whose proofs are postponed to Section 4.

**Proposition 2.1** Let \( A, B \subset \mathbb{R}^n \) be compact sets, define \( S := \frac{1}{2}(A + B) \), and assume that (1.5) holds for some \( \delta \leq \frac{1}{2} \). Also, suppose that we can find a convex set \( K \subset \mathbb{R}^n \) such that
\[ |S \Delta K| \leq C\delta^{\alpha} \]
for some \( \alpha > 0 \), where \( C > 0 \) is a dimensional constant. Then there exists a dimensional constant \( C' > 0 \) such that
\[ |\text{co}(S) \setminus S| \leq C'\delta^{\frac{2}{n}}. \]

**Proposition 2.2** Let \( A, B \subset \mathbb{R}^n \) be compact sets, define \( S := \frac{1}{2}(A + B) \), and assume that (1.5) holds for some \( \delta \leq \frac{1}{2} \). Also, suppose that
\[ |\text{co}(S) \setminus S| \leq C\delta^{\beta} \] (2.4)
for some \( \beta > 0 \), where \( C > 0 \) is a dimensional constant. Then, up to a translation,
\[ |A \Delta B| \leq C'\delta^{\frac{2}{n}}, \]
and there exists a convex set \( K \) containing both \( A \) and \( B \) such that
\[ |K \setminus A| + |K \setminus B| \leq C'\delta^{\frac{2}{n}} \]
for some dimensional constant \( C' > 0 \).

### 3 Proof of Theorem 1.6

As explained in [8], by inner approximation\(^1\) it suffices to prove the result when \( A, B \) are compact sets. Hence, let \( A \) and \( B \) be compact, define \( S := \frac{1}{2}(A + B) \), and assume that (1.5)

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\(^1\)The approximation of \( A \) (and analogously for \( B \)) is by a sequence of compact sets \( A_k \subset A \) such that \( |A_k| \to |A| \) and \( |\text{co}(A_k)| \to |\text{co}(A)| \). One way to construct such sets is to define \( A_k := A'_k \cup V_k \), where \( A'_k \subset A \) are compact sets satisfying \( |A'_k| \to |A| \), and \( V_k \subset V_{k+1} \subset A \) are finite sets satisfying \( |\text{co}(V_k)| \to |\text{co}(A)| \).
holds. We want to prove that there exists a convex set \( K \) such that, up to a translation, 
\[
A, B \subset K, \quad |K \setminus A| + |K \setminus B| \leq C_n \delta^{3n}. 
\]

Moreover, since the statement and the conclusions are invariant under measure preserving affine transformations, by Lemma 2.2, we can assume that \( A \) and \( B \) are \( M \)-normalized (see (2.3)).

Ultimately, we wish to show that, up to translation, each of \( A, B, \) and \( S \) is of nearly full measure in the same convex set. The strategy of the proof is to show first that \( S \) is close to a convex set, and then apply Propositions 2.1–2.2. To obtain the closeness of \( S \) to a convex set, we would like prove that \( \frac{1}{2}(S + S) \) is close to \( |S| \) and then apply Theorem 1.4. It is simpler, however, to construct a subset \( S \subset S \), such that \( |S \setminus S| \) is small and \( \frac{1}{2}(S + S) \) is close to \( |S| \).

To carry out our argument, one important ingredient will be to use the inductive hypothesis on the level sets \( \mathcal{A}(\lambda) \) and \( \mathcal{B}(\lambda) \) defined in (2.2). However, two difficulties arise here: First of all, to apply the inductive hypothesis, we need to know that \( \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \) and \( \mathcal{H}^{n-1}(\mathcal{B}(\lambda)) \) are close. In addition, the Brunn-Minkowski inequality does not have a natural proof by induction unless the measures of all the level sets \( \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \) and \( \mathcal{H}^{n-1}(\mathcal{B}(\lambda)) \) are the nearly same (see (3.11) below). Hence, it is important for us to have a preliminary quantitative estimate on the difference between \( \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \) and \( \mathcal{H}^{n-1}(\mathcal{B}(\lambda)) \) for most \( \lambda > 0 \). For this, we follow an approach used first in [2] and readapted in [9], in which we begin by showing our theorem in the special case of symmetrized sets \( A = A^\sharp \) and \( B = B^\sharp \) (recall Definition 2.1). Thanks to Lemma 2.1, this will give us the desired closeness between \( \mathcal{H}^{n-1}(\mathcal{A}(\lambda)) \) and \( \mathcal{H}^{n-1}(\mathcal{B}(\lambda)) \) for most \( \lambda > 0 \), which allows us to apply the strategy described above and prove the theorem in the general case.

Throughout the proof, \( C \) will denote a generic constant depending only on the dimension, which may change from line to line.

### 3.1 The case \( A = A^\sharp \) and \( B = B^\sharp \)

Let \( A, B \subset \mathbb{R}^n \) be compact sets satisfying \( A = A^\sharp, B = B^\sharp \). Since \( \pi(A(t)) \subset \pi(A(0)) = \pi(A) \) and \( \pi(B(t)) \subset \pi(B(0)) = \pi(B) \) are disks centered at the origin, applying Lemma 2.3, we deduce that 
\[
\mathcal{H}^{n-1}(\pi(A) \Delta \pi(B)) \leq C \delta^{\frac{2}{n}}. 
\]

Hence, if we define 
\[
\overline{S}_y := \bigcup_{y \in \pi(A) \cap \pi(B)} \frac{A_y + B_y}{2}, 
\]
then \( \overline{S}_y \subset S_y \) for all \( y \in \mathbb{R}^{n-1} \). In addition, using (1.5), (2.3), and (3.1), we have 
\[
1 + \delta \geq |\overline{S}| = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(S_y) \, dy \geq \int_{\pi(A) \cap \pi(B)} \mathcal{H}^1(S_y) \, dy \geq \int_{\pi(A) \cap \pi(B)} \mathcal{H}^1(\overline{S}_y) \, dy \\
\geq \frac{|A| + |B|}{2} - M \mathcal{H}^{n-1}(\pi(A) \Delta \pi(B)) \geq 1 - C \delta^{\frac{2}{n}}, 
\]
which implies (since $S \subset S$)
\[ |S \setminus S| \leq C\delta^\frac{\lambda}{2}. \] (3.2)

Furthermore, since each section $S_y$ is an interval centered at $0 \in \mathbb{R}$, for all $y', y'' \in \pi(A) \cap \pi(B)$ such that $\frac{y' + y''}{2} = y$,
\[ \frac{S_{y'}}{2} + \frac{S_{y''}}{2} = \frac{A_{y'} + B_{y'}}{2} + \frac{A_{y''} + B_{y''}}{2} \subset S_y + S_y = 2S_y, \]
which gives
\[ \frac{S + S}{2} \subset S. \] (3.3)

Recalling (1.3), by (3.2)–(3.3), we obtain that $\delta(S) \leq C\delta^\frac{\lambda}{2}$. Hence, we can apply Theorem 1.4 to $S$ to find a convex set $K$ such that
\[ |S \setminus K| \leq C\delta^\frac{\lambda}{2}. \] Hence, by (3.3),
\[ |S \setminus K| \leq C\delta^\frac{\lambda}{2}, \]
and using Propositions 2.1–2.2, we deduce that, up to a translation, there exists a convex set $K$ such that $A \cup B \subset K$ and
\[ |A \Delta B| \leq C\delta^\frac{\lambda}{2}, \quad |K \setminus A| + |K \setminus B| \leq C\delta^\frac{\lambda}{2}. \] (3.4)

Notice that, because $A = A^\natural$ and $B = B^\natural$, it is easy to check that the above properties still hold with $K^\natural$ in place of $K$. Hence, in this case, without loss of generality, one can assume that $K = K^\natural$.

3.2 The general case

Since, by Theorem 1.2, the result is true when $n = 1$, we may assume that we already proved Theorem 1.6 through $n - 1$, and we want to show its validity for $n$.

**Step 1** There exist a dimensional constant $\zeta > 0$ and $\overline{\alpha} \sim \delta^\zeta$ such that we can apply the inductive hypothesis to $A(\overline{\alpha})$ and $B(\overline{\alpha})$.

Let $A^\natural$ and $B^\natural$ be as in Definition 2.1 and denote
\[ \overline{\alpha} := \alpha_n^\natural. \] (3.5)

Thanks to Lemma 2.1, $A^\natural$ and $B^\natural$ still satisfy (1.5), so we can apply the result proved in Section 3.1 above to get (see (3.4))
\[ \int_{\mathbb{R}^{n-1}} |\mathcal{H}^{\natural}(A^\natural_y) - \mathcal{H}^{\natural}(B^\natural_y)| \, dy \leq \int_{\mathbb{R}^{n-1}} |\mathcal{H}^{\natural}(A^\natural_y \Delta B^\natural_y)| \, dy = |A^\natural \Delta B^\natural| \leq C\delta^{\overline{\alpha}} \] (3.6)
and
\[ K \supset A^\natural \cup B^\natural, \quad |K \setminus A^\natural| + |K \setminus B^\natural| \leq C\delta^{\overline{\alpha}} \] (3.7)
for some convex set $K = K^\sharp$.

In addition, because $A$ and $B$ are $M$-normalized (see (2.3)), so are $A^\sharp$ and $B^\sharp$, and by (3.7) we deduce that there exists a dimensional constant $R_n > 0$ such that

$$K \subset B_{R_n}.$$

Also, by (3.6) and Chebyshev’s inequality, we obtain that, except for a set of measure $\leq C\delta^\sharp$, 

$$|\mathcal{H}^1(A^\sharp_y) - \mathcal{H}^1(B^\sharp_y)| \leq \delta^\sharp.$$ 

Thus, recalling Lemma 2.1, for almost every $\lambda > 0$,

$$\mathcal{H}^{n-1}(A(\lambda)) = \mathcal{H}^{n-1}(A^\sharp(\lambda)) \leq \mathcal{H}^{n-1}(B^\sharp(\lambda - \delta^\sharp)) + C\delta^\sharp = \mathcal{H}^{n-1}(B(\lambda - \delta^\sharp)) + C\delta^\sharp.$$

Since, by (2.3), 

$$\int_0^M (\mathcal{H}^{n-1}(B(\lambda)) - \mathcal{H}^{n-1}(B(\lambda + \delta^\sharp))) d\lambda = \int_0^{\delta^\sharp} \mathcal{H}^{n-1}(B(\lambda)) d\lambda \leq M\delta^\sharp,$$

by Chebyshev’s inequality, we deduce that

$$\mathcal{H}^{n-1}(A(\lambda)) \leq \mathcal{H}^{n-1}(B(\lambda)) + C\delta^\sharp$$

for all $\lambda$ outside a set of measure $\leq C\delta^\sharp$. Exchanging the roles of $A$ and $B$, we obtain that there exists a set $F \subset [0, M]$, such that

$$\mathcal{H}^1(F) \leq C\delta^\sharp, \quad |\mathcal{H}^{n-1}(A(\lambda)) - \mathcal{H}^{n-1}(B(\lambda))| \leq C\delta^\sharp, \quad \forall \lambda \in [0, \infty] \setminus F. \quad (3.9)$$

Using the elementary inequality

$$\left(\frac{a + b}{2}\right)^{n-1} \geq \frac{a^{n-1} + b^{n-1}}{2} - C|a - b|^2, \quad \forall 0 \leq a, b \leq M,$$

and replacing $a$ and $b$ with $a^{\frac{1}{n-1}}$ and $b^{\frac{1}{n-1}}$, respectively, we get

$$\left(\frac{a^{\frac{1}{n-1}} + b^{\frac{1}{n-1}}}{2}\right)^{n-1} \geq \frac{a + b}{2} - C|a - b|^{\frac{2}{n-1}}, \quad \forall 0 \leq a, b \leq M \quad (3.10)$$

(notice that $|a^{\frac{1}{n-1}} - b^{\frac{1}{n-1}}| \leq |a - b|^{\frac{2}{n-1}}$). Finally, it is easy to check that

$$\frac{A(\lambda) + B(\lambda)}{2} \subset S(\lambda), \quad \forall \lambda > 0.$$

Hence, by the Brunn-Minkowski inequality (1.2) applied to $A(\lambda)$ and $B(\lambda)$, using (1.5), (2.3)
and (3.9)–(3.10), we get

\[ 1 + \delta \geq |S| = \int_0^M H^{n-1}(S(\lambda)) \, d\lambda \]

\[ \geq \frac{1}{2n-1} \int_0^M (H^{n-1}(A(\lambda)) \frac{1}{\pi r} + H^{n-1}(B(\lambda)) \frac{1}{\pi r})^{n-1} \, d\lambda \]

\[ \geq \frac{1}{2} \int_0^M (H^{n-1}(A(\lambda)) + H^{n-1}(B(\lambda))) \, d\lambda - C \int_0^M |H^{n-1}(A(\lambda)) - H^{n-1}(B(\lambda))|^{\frac{2}{n-1}} \, d\lambda \]

\[ = \frac{|A| + |B|}{2} - C\delta \frac{\pi}{n-1} \]

\[ \geq 1 - C\delta \frac{\pi}{n-1}. \]

We also observe that, since \( K = K^2 \), by Lemma 2.1, (3.8), and [2, Lemma 4.3], for almost every \( \lambda > 0 \), we have

\[ |A \setminus \pi^{-1}(A(\lambda))| = |A^2 \setminus \pi^{-1}(A^2(\lambda))| \]

\[ \leq |K \setminus \pi^{-1}(K(\lambda))| + M H^{n-1}(A^2(\lambda) \Delta K(\lambda)) \]

\[ \leq C\lambda^2 + M H^{n-1}(A^2(\lambda) \Delta K(\lambda)), \]

and analogously for \( B \). Also, by (3.7),

\[ \int_0^M (H^{n-1}(A^2(\lambda) \Delta K(\lambda)) + H^{n-1}(B^2(\lambda) \Delta K(\lambda))) \, d\lambda \leq |K \setminus A^2| + |K \setminus B^2| \leq C\delta \frac{\pi}{n}. \]

Define

\[ \eta := \min \left\{ \frac{\pi}{2(n-1)} \right\}, \]

and note that \( \eta \leq \frac{\pi}{n} \). Let \( \zeta \in (0, \eta) \) to be fixed later. Then by (3.9), (3.11)–(3.13), and by Chebyshev’s inequality, we can find a level

\[ \overline{\alpha} \in [10\delta^5, 20\delta^5], \]

such that

\[ |H^{n-1}(A(\overline{\alpha})) - H^{n-1}(B(\overline{\alpha}))| \leq C\delta^\eta, \]

\[ 2^{n-1}H^{n-1}(S(\overline{\alpha})) \leq (H^{n-1}(A(\overline{\alpha})) \frac{1}{\pi r} + H^{n-1}(B(\overline{\alpha})) \frac{1}{\pi r})^{n-1} + C\delta^{n-\zeta}, \]

\[ |A \setminus \pi^{-1}(A(\overline{\alpha}))| + |B \setminus \pi^{-1}(B(\overline{\alpha}))| \leq C(\delta^2 + \delta^{n-\zeta}), \]

In addition, from the properties \( H^{n-1}(A(\lambda)) \leq M \) for any \( \lambda > 0 \) (see (2.3)), \( \int_0^M H^{n-1}(A(\lambda)) \, d\lambda = |A| \geq 1 - \delta \), and \( s \mapsto H^{n-1}(A(\lambda)) \) is a decreasing function, we deduce that

\[ \frac{1}{2M} \leq H^{n-1}(A(\lambda)) \leq M, \quad \forall \lambda \in (0, (2M)^{-1}). \]

The same holds for \( B \) and \( S \), hence

\[ H^{n-1}(S(\overline{\alpha})), H^{n-1}(A(\overline{\alpha})), H^{n-1}(B(\overline{\alpha})) \in [(2M)^{-1}, M] \]
provided that $\delta$ is small enough. Set $\rho := \frac{1}{M^{\frac{n-1}{p^*}}(A(\lambda))^{\frac{1}{p^*}}} \in \left[ \frac{1}{M^{\frac{n-1}{p^*}}}, \frac{1}{M^{\frac{n-1}{p^*}}(2M)^{\frac{1}{p^*}}} \right]$, and define

$$A' := \rho A(\lambda), \quad B' := \rho B(\lambda), \quad S' := \rho S(\lambda).$$

By (3.16)–(3.17), we get

$$H^{n-1}(A') = 1, \quad |H^{n-1}(B') - 1| \leq C\delta^n, \quad H^{n-1}(S') \leq 1 + C\delta^{n-\zeta}.$$

while, by (1.2),

$$H^{n-1}(S')^{\frac{1}{n-\tau}} \geq \frac{H^{n-1}(A')^{\frac{1}{n-\tau}} + H^{n-1}(B')^{\frac{1}{n-\tau}}}{2} \geq 1 - C\delta^n,$$

therefore

$$|H^{n-1}(A') - 1| + |H^{n-1}(B') - 1| + |H^{n-1}(S') - 1| \leq C\delta^{n-\zeta}.$$

Thus, by the inductive hypothesis of Theorem 1.6, up to a translation there exists a $(n-1)$-dimensional convex set $\Omega'$, such that

$$\Omega' \supset A' \cup B', \quad H^{n-1}(\Omega' \setminus A') + H^{n-1}(\Omega' \setminus B') \leq C\delta^{(n-\zeta)\beta_{n-1}},$$

and defining $\Omega := \frac{\Omega'}{\rho}$ we obtain (recall that $\frac{1}{\rho} \leq M^{\frac{1}{p^*}}$)

$$\Omega \supset \mathcal{A}(\lambda) \cup \mathcal{B}(\lambda), \quad H^{n-1}(\Omega \setminus \mathcal{A}(\lambda)) + H^{n-1}(\Omega \setminus \mathcal{B}(\lambda)) \leq C\delta^{(n-\zeta)\beta_{n-1}}. \quad (3.19)$$

**Step 2** We apply Theorem 1.2 to the sets $A_y$ and $B_y$ for most $y \in \mathcal{A}(\lambda) \cap \mathcal{B}(\lambda)$.

Define $C := \mathcal{A}(\lambda) \cap \mathcal{B}(\lambda) \subset S(\lambda)$. By (3.18)–(3.19) and (2.3), we have

$$|A \setminus \pi^{-1}(C)| + |B \setminus \pi^{-1}(C)| \leq |A \setminus \pi^{-1}(\mathcal{A}(\lambda))| + |B \setminus \pi^{-1}(\mathcal{B}(\lambda))|$$

$$+ \int_{(A(\lambda)) \setminus (B(\lambda))} H^{1}(A_y) \, dy + \int_{(B(\lambda)) \setminus (A(\lambda))} H^{1}(B_y) \, dy \leq C(\delta^{2\zeta} + \delta^{n-\zeta}) + M(H^{n-1}(\Omega \setminus \mathcal{A}(\lambda)) + H^{n-1}(\Omega \setminus \mathcal{B}(\lambda)))$$

$$\leq C(\delta^{2\zeta} + \delta^{n-\zeta} + \delta^{(n-\zeta)\beta_{n-1}}) \leq C\delta^{2\zeta},$$

provided that we choose

$$\zeta := \frac{\eta\beta_{n-1}}{3}. \quad (3.21)$$

(recall that $\beta_{n-1} \leq 1$). Hence, by (1.5) and (3.20),

$$\int_{C} H^{1}(S_y \setminus \frac{A_y + B_y}{2}) \, dy \leq \int_{C} [H^{1}(S_y) - \frac{1}{2}(H^{1}(A_y) + H^{1}(B_y))] \, dy$$

$$= |S \cap \pi^{-1}(C)| - \frac{|A \cap \pi^{-1}(C)| + |B \cap \pi^{-1}(C)|}{2} \leq |S| - \frac{|A| + |B|}{2} - \frac{|A \setminus \pi^{-1}(C)| + |B \setminus \pi^{-1}(C)|}{2} \leq C\delta^{2\zeta}. \quad (3.22)$$
Write $C$ as $C_1 \cup C_2$, where

$$C_1 := \left\{ y \in C : 2\mathcal{H}^1(S_y) - \mathcal{H}^1(A_y) - \mathcal{H}^1(B_y) \leq \frac{\delta^\zeta}{2} \right\}, \quad C_2 := C \setminus C_1.$$  

By Chebyshev’s inequality and (3.22),

$$\mathcal{H}^{n-1}(C_2) \leq C\delta^\zeta,$$  

(3.23)

while, recalling (3.15),

$$\min\{\mathcal{H}^1(A_y), \mathcal{H}^1(B_y)\} \geq \frac{\zeta}{100} \geq \frac{\delta^\zeta}{2}, \quad \forall y \in C_1.$$  

Hence, by Theorem 1.2 applied to $A_y, B_y \subset \mathbb{R}$ for $y \in C_1$, we deduce that

$$\mathcal{H}^1(\text{co}(A_y) \setminus A_y) + \mathcal{H}^1(\text{co}(B_y) \setminus B_y) \leq \delta^\zeta.$$  

(3.24)

Let $\widehat{C}_1 \subset C_1$ denote the set of $y \in C_1$ such that

$$\mathcal{H}^1(\text{co}(A_y) \setminus A_y + B_y) \leq \delta^\zeta,$$  

(3.25)

and notice that, by (3.22) and Chebyshev’s inequality, $\mathcal{H}^{n-1}(C_1 \setminus \widehat{C}_1) \leq C\delta^\zeta$. Then choose a compact set $C'_1 \subset \widehat{C}_1$ such that

$$\mathcal{H}^{n-1}(C_1 \setminus C'_1) \leq C\delta^\zeta.$$  

(3.26)

**Step 3** We find $\overline{S} \subset S$, so that $|S \setminus \overline{S}|$ and $\delta(\overline{S})$ are small.

Define the compact set

$$\overline{S} := \bigcup_{y \in C'_1} \frac{A_y + B_y}{2} \subset \mathbb{R}^n.$$  

Observe, thanks to (3.20), (3.23), (3.26), (2.3) and (1.5),

$$2|\overline{S}| \geq \int_{C'_1} \mathcal{H}^1(A_y) dy + \int_{C'_1} \mathcal{H}^1(B_y) dy$$

$$\geq |A| + |B| - |A \setminus \pi^{-1}(C)| - |B \setminus \pi^{-1}(C)| - M \mathcal{H}^{n-1}(C \setminus C'_1)$$

$$\geq 2|S| - C\delta^\zeta.$$  

So, since $\overline{S} \subset S$,

$$|S \Delta \overline{S}| \leq C\delta^\zeta,$$  

(3.27)

Now, we want to estimate the measure of $\frac{1}{2}(S + \overline{S})$. First of all, since

$$S_y = \bigcup_{2y = y' + y''} \frac{A_y + B_y}{2},$$  

(3.28)

by (3.25), we get

$$\mathcal{H}^1\left( \left( \bigcup_{2y = y' + y''} \frac{A_y + B_y}{2} \right) \setminus \frac{A_y + B_y}{2} \right) \leq \delta^\zeta, \quad \forall y \in C'_1.$$  

(3.29)
Also, if we define the characteristic functions
\[ \chi^A_y(\lambda) := \begin{cases} 1, & \text{if } (y, \lambda) \in A_y, \\ 0, & \text{otherwise,} \end{cases} \quad \chi^{A^*}_y(\lambda) := \begin{cases} 1, & \text{if } (y, \lambda) \in \text{co}(A_y), \\ 0, & \text{otherwise,} \end{cases} \]
and analogously for \( B_y \), by (3.24) we have the following estimate on their convolutions:
\[ \| \chi^{A^*}_y * \chi^{B^*}_y - \chi^B_y * \chi^B_y \|_{L^\infty(\mathbb{R})} \leq \| \chi^{B^*}_y - \chi^B_y \|_{L^1(\mathbb{R})} + \| \chi^{A^*}_y - \chi^A_y \|_{L^1(\mathbb{R})} \]
\[ = \mathcal{H}^1(\text{co}(B_y^\nu) \setminus B_y^\nu + \mathcal{H}^1(\text{co}(A_y^\nu) \setminus A_y^\nu) \]
\[ \leq \delta^2 < 3\delta^2, \quad \forall y', y'' \in C'_1. \] (3.30)

Recalling that \( \pi : \mathbb{R}^n \to \mathbb{R} \) is the orthogonal projection onto the last component (that is, \( \pi(y, t) = t \)), we denote by \([a, b]\) the interval \( \pi(\text{co}(A_y^\nu) + \text{co}(B_y^\nu)) \), and notice that, since by construction
\[ \min\{\mathcal{H}^1(A_y), \mathcal{H}^1(B_y)\} \geq \overline{\lambda} \geq 10\delta^2, \quad \forall y \in C'_1 \]
(see (3.15)), this interval has length greater than \( 20\delta^2 \). Also, it is easy to check that the function \( \chi^{A^*}_y * \chi^{B^*}_y \) is supported on \([a, b]\), has slope equal to 1 (resp. \(-1\)) in \([a, a + 3\delta^2]\) (resp. \([b - 3\delta^2, b]\)), and it is greater than \( 3\delta^2 \) in \([a + 3\delta^2, b - 3\delta^2]\). Hence, since \( \pi(A_y^\nu + B_y^\nu) \) contains the set \( \{\chi^A_y * \chi^B_y > 0\} \), by (3.30), we deduce that
\[ \pi(A_y^\nu + B_y^\nu) \supset [a + 3\delta^2, b - 3\delta^2], \] (3.31)
which implies in particular that
\[ \mathcal{H}^1(\text{co}(A_y^\nu) + \text{co}(B_y^\nu)) \leq \mathcal{H}^1(A_y^\nu + B_y^\nu) + 6\delta^2, \quad \forall y', y'' \in C'_1. \] (3.32)

Also, by the same argument as in [8, Step 2-a], if we denote by
\[ [\alpha_y, \beta_y] := \pi(\text{co}(A_y) + \text{co}(B_y)), \]
using (3.25) and (3.31), we have
\[ \pi(\text{co}(A_y^\nu) + \text{co}(B_y^\nu)) \subset [\alpha_y - 16\delta^2, \beta_y + 16\delta^2], \quad \forall y', y'' \in C'_1, \] (3.33)
(Compare with [8, (3.25)].)

We now estimate the size of \( [\frac{1}{2}(|S| + |S|)]_y \). Observe that, for all \( y \in C'_1 \),
\[ \left[ \frac{1}{2}(|S| + |S|) \right]_y = \bigcup_{2y = y' + y'', y', y'' \in C'_1} \left( \frac{1}{2}(A_y^\nu + B_y^\nu) + \frac{1}{2}(A_y^\nu + B_y^\nu) \right) \]
\[ = \bigcup_{2y = y' + y'', y', y'' \in C'_1} \left( \frac{1}{2}(A_y^\nu + B_y^\nu) + \frac{1}{2}(A_y^\nu + B_y^\nu) \right) \]
\[ \subset \frac{1}{2} \left( \bigcup_{2y = y' + y'', y', y'' \in C'_1} \frac{1}{2}(A_y^\nu + B_y^\nu) + \bigcup_{2y = y' + y'', y', y'' \in C'_1} \frac{1}{2}(A_y^\nu + B_y^\nu) \right). \]
Hence, by (3.33), we deduce that each of the latter sets is contained inside the convex set \( \{y \} \times [\alpha_y - 16\delta, \beta_y + 16\delta] \), so also their semi-sum is contained in the same set, and using (3.32) with \( y' = y'' = y \), we get

\[
\mathcal{H}^1\left( \left[ \frac{\overline{S} + \overline{S}}{2} \right]_y \right) \leq \mathcal{H}^1\left( \co(A_y) + \co(B_y) \right) + 16\delta
\]

\[
\leq \mathcal{H}^1\left( \frac{A_y + B_y}{2} \right) + 22\delta
\]

\[
= \mathcal{H}^1(\overline{S}_y) + 22\delta, \quad \forall y \in C'_1.
\]

In order to estimate \( \left[ \frac{1}{2}(\overline{S} + \overline{S}) \right]_y \) when \( y \in C'_1 \setminus C'_1 \), we argue as follows. By (3.33) and the fact that \( \mathcal{H}^1(\co(A_y)) \) and \( \mathcal{H}^1(\co(B_y)) \) are universally bounded (see (2.3) and (3.24)), the following holds: If we denote by \( c_A(y) \) the barycenter of \( \co(A_y) \) (and analogously for \( B \) and \( \overline{S} \)), we have

\[
|c_A(y') + c_B(y'') - 2c_A(y)| \leq C, \quad \forall y, y', y'' \in C'_1, y = \frac{y' + y''}{2}
\]

(notice that \( \co(\overline{S}_y) = \co(A_y) + \co(B_y) \)). Exchanging the role of \( A \) and \( B \) and adding up the two inequalities, we deduce that

\[
|c_A(y') + c_B(y'') - 2c_A(y)| \leq C, \quad \forall y, y', y'' \in C'_1, y = \frac{y' + y''}{2}.
\]

As shown in [8, Step 3], this estimate combined with the fact that \( C'_1 \) is almost of full measure inside the convex set \( \Omega \) (see (3.19), (3.23) and (3.26)) proves that, up to an affine transformation of the form

\[
\mathbb{R}^{n-1} \times \mathbb{R} \ni (y, t) \mapsto (Ty, t - Ly) + (y_0, t_0)
\]

with \( T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, L : \mathbb{R}^{n-1} \to \mathbb{R}, \det(T) = 1, \) and \( (y_0, t_0) \in \mathbb{R} \), the set \( \overline{S} \) is universally bounded, say \( \overline{S} \subset B_R \) for some dimensional constant \( R \). This implies that \( \left[ \frac{1}{2}(\overline{S} + \overline{S}) \right]_y \subset [-R, R] \), so \( \mathcal{H}^1\left( \left[ \frac{1}{2}(\overline{S} + \overline{S}) \right]_y \right) \leq 2R. \)

Hence, since \( \frac{1}{2}(C'_1 + C'_1) \subset \Omega \), by (3.34), (3.19) and (3.21),

\[
\left| \frac{\overline{S} + \overline{S}}{2} \right| = \int_{\frac{1}{2}(C'_1 + C'_1) \cap C'_1} \mathcal{H}^1\left( \left[ \frac{\overline{S} + \overline{S}}{2} \right]_y \right) - \mathcal{H}^1(\overline{S}_y) \, dy
\]

\[
+ \int_{\frac{1}{2}(C'_1 + C'_1) \setminus C'_1} \mathcal{H}^1\left( \left[ \frac{\overline{S} + \overline{S}}{2} \right]_y \right) \, dy
\]

\[
\leq 22\delta \mathcal{H}^1(\mathbb{R}^{n-1}) + 2R \mathcal{H}^1(\mathbb{R}^{n-1}) \leq C \delta,
\]

that is,

\[
\delta(\overline{S}) \leq C \delta.
\]

**Step 4** Conclusion

By the previous step, we have that \( \delta(\overline{S}) \leq C \delta \). Hence, applying Theorem 1.4 to \( \overline{S} \), we find a convex set \( \overline{K} \) such that

\[
\left| \overline{S} \Delta \overline{K} \right| \leq C \delta^{\alpha \delta}.
\]
so, by (3.27),
\[ |S \Delta K| \leq C \delta^{\alpha_n} \zeta. \]

Using this estimate together with Propositions 2.1–2.2, we deduce that, up to a translation, there exists a convex set \( K \) convex such that \( A \cup B \subset K \) and
\[ |K \setminus A| + |K \setminus B| \leq C \delta^{\frac{\alpha_n}{2n}}. \]

Recalling the definition of \( \zeta \) (see (3.5), (3.14), (3.21)), we see that
\[ \beta_n := \frac{\alpha_n \zeta}{4n} = \min \left\{ \frac{1}{n-1}, \frac{1}{2} \right\} \frac{\alpha_n^2}{3 \cdot 2^n n^2} \beta_{n-1}. \]

Since \( \beta_1 = 1 \) (by Theorem 1.2), it is easy to check that
\[ \beta_n = \frac{1}{2^{6n-53n-1} n!(n-1)!} \prod_{k=1}^{n} \alpha_k^2, \quad \forall n \geq 2, \]
concluding the proof.

4 Technical Results

As in the previous section, we use \( C \) to denote a generic constant depending only on the dimension, which may change from line to line.

4.1 Proof of Proposition 2.1

Assume that
\[ |S \Delta K| \leq C \delta^\alpha \]
for some \( \alpha \in (0, 1] \). By John’s lemma (see [16]), after a volume preserving affine transformation, we can assume that \( B_{r_n} \subset K \subset B_{nr_n} \), with \( r_n = r_n(K) > 0 \) bounded above and below by positive dimensional constants. Note, however, that with this normalization, we will not be able to assume that \( A \) and \( B \) are \( M \)-normalized, since we have already chosen a different affine normalization.

We want to prove that
\[ S \subset (1 + C \delta^{\frac{\alpha}{n}}) K. \quad (4.1) \]

Let \( \overline{x}_0 \in S \setminus K \), and set \( \rho := \text{dist}(\overline{x}_0, K) = |\overline{x}_0 - \overline{x}_1| \) with \( \overline{x}_1 \in K \). Without loss of generality, we can assume that \( \overline{x}_1 = \tau e_n \), for some \( \tau > 0 \), \( \overline{x}_0 = (\tau + \rho)e_n \), and \( K \subset \{ x_n \leq \tau \} \). We need to prove that \( \rho \leq C \delta^{\frac{\alpha}{n}} \).

Let us consider the sets \( A^*, B^*, S^*, K^* \) obtained from \( A, B, S, K \) performing a Schwarz symmetrization around the \( e_n \)-axis (see Definition 2.1). Set \( S' := \frac{1}{2} (A^* + B^*) \). Since
\[ |S^* \Delta K^*| \leq |S \Delta K| \leq C \delta^\alpha, \]
and, by (1.5) (notice that \( S' \subset S^* \) and that \( |S'| \geq 1 - C \delta \) by (1.2)),
\[ |S^* \setminus S'| = |S^*| - |S'| = |S| - |S'| \leq C \delta, \]
we get that \(|S' \Delta K^*| \leq C \delta^\alpha\). In addition, \(K^* \subset \{x_n \leq \tau\}, \overline{x}_1 \in K^*\) and \(\overline{x}_0 \in S^*\). Hence, without loss of generality, we can assume from the beginning that \(A = A^*, B = B^*, S = \frac{1}{2} (A^* + B^*)\) and \(K = K^*\).

For a compact set \(E \subset \mathbb{R}^n\), recall the notation \(E(t) \subset \mathbb{R}^{n-1} \times \{t\}\) in (2.1), and define \(E[s] \subset \mathbb{R}\) by

\[
E[s] := \{t : \mathcal{H}^{n-1}(E(t)) \geq s\}. \tag{4.2}
\]

Since \(S = \frac{1}{2}(A + B)\), we have

\[
\frac{A(t) + B(t)}{2} \subset S(t), \quad \forall t \in \mathbb{R},
\]

so, by (1.2), we deduce that

\[
S[s] \supset \frac{A[s] + B[s]}{2}, \quad \forall s > 0.
\]

Hence

\[
\mathcal{H}^1(A[s]) + \mathcal{H}^1(B[s]) \leq 2\mathcal{H}^1(S[s]), \quad \forall s > 0, \tag{4.3}
\]

and integrating with respect to \(s\), by (1.5), we get

\[
4\delta \geq 2|S| - |A| - |B| = \int_0^{\infty} (2\mathcal{H}^1(S[s]) - \mathcal{H}^1(A[s]) - \mathcal{H}^1(B[s])) \, ds. \tag{4.4}
\]

Recall that \(K = K^*\), so that the canonical projection \(\pi(K)\) onto \(\mathbb{R}^{n-1}\) is a ball. We denote it \(B_R := \pi(K)\), and note that \(R \leq nr_n\), with \(r_n = r_n(K)\) given by John’s lemma at the beginning of this proof. Then, since \(|S \Delta K| \leq C \delta^\alpha\), we have

\[
C \delta^\alpha \geq |S \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^{\infty} \mathcal{H}^1(S[s]) \, ds,
\]

so, by (4.3),

\[
|A \setminus \pi^{-1}(B_R)| + |B \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^{\infty} (\mathcal{H}^1(A[s]) + \mathcal{H}^1(B[s])) \, ds \leq C \delta^\alpha. \tag{4.5}
\]

Hence, recalling that \(|A|\) and \(|B|\) are \(\geq 1 - \delta\), we deduce that

\[
\int_{\mathcal{H}^{n-1}(B_R)}^{\infty} \mathcal{H}^1(A[s]) \, ds \geq \frac{1}{2}, \quad \int_{\mathcal{H}^{n-1}(B_R)}^{\infty} \mathcal{H}^1(B[s]) \, ds \geq \frac{1}{2},
\]

and since \(R\) is universally bounded (being less than \(nr_n\)) and both functions \(s \mapsto \mathcal{H}^1(A[s]), s \mapsto \mathcal{H}^1(B[s])\) are decreasing, there exists a small dimensional constant \(c' > 0\), such that

\[
\min\{\mathcal{H}^1(A[s]), \mathcal{H}^1(B[s])\} \geq c', \quad \forall s \in (0, c'). \tag{4.6}
\]
Also, by (4.4),
\[
\int_0^c (2\mathcal{H}^1(S[s]) - \mathcal{H}^1(A[s]) - \mathcal{H}^1(B[s])) \, ds \leq 4\delta, \tag{4.7}
\]
and since \(|S\Delta K| \leq C\delta^\alpha\) and \(K \subset \{x_n \leq \tau\},\)
\[
\int_0^c \mathcal{H}^1(S[s] \setminus (-\infty, \tau]) \, ds \leq |S \setminus \{x_n \leq \tau\}| \leq C\delta^\alpha. \tag{4.8}
\]
Hence, thanks to (4.6)–(4.8), we use Theorem 1.2 and Chebyshev’s inequality to find a value
\[
\bar{s} \in [\delta\hat{\tau}, 2\delta\hat{\tau}], \tag{4.9}
\]
such that
\[
\mathcal{H}^1(\co(A[\bar{s}]) \setminus A[\bar{s}]) + \mathcal{H}^1(\co(B[\bar{s}]) \setminus B[\bar{s}]) \leq C\delta^{1-\frac{\alpha}{2}} \leq C\delta^{\frac{\alpha}{2}}
\]
(notice that \(\alpha \leq 1\) and
\[
\mathcal{H}^1(S[\bar{s}] \setminus (-\infty, \tau]) \leq C\delta^{\frac{\alpha}{2}}.
\]
Since \(\frac{1}{2}(A[\bar{s}] + B[\bar{s}]) \subset S[\bar{s}],\) this implies
\[
\frac{\co(A[\bar{s}]) + \co(B[\bar{s}])}{2} \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}].
\]
Hence, after applying opposite translations along the \(e_n\)-axis to \(A\) and \(B,\) i.e.,
\[
A \mapsto A + \ell e_n, \quad B \mapsto B - \ell e_n
\]
for some \(\ell \in \mathbb{R},\) we can assume that
\[
\co(A[\bar{s}]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}], \quad \co(B[\bar{s}]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}].
\]
Since the sets \(s \mapsto A[s], \ B[s] \) are decreasing, we deduce that
\[
\co(A[s]), \co(B[s]) \subset (-\infty, \tau + C\delta^{\frac{\alpha}{2}}], \quad \forall \, s \geq \bar{s}. \tag{4.10}
\]
We now want to bound \(\sup_{s > 0} \mathcal{H}^1(A[s]).\) (Recall that we cannot assume that \(A\) and \(B\) are \(M\)-normalized, since we already made an affine transformation to ensure that \(B_{rn} \subset K \subset B_{n_{rn}}.\))
Since \(A = A^*\), we have \(\sup_{s > 0} \mathcal{H}^1(A[s]) = \sup_{y \in \mathbb{R}^{n-1}} \mathcal{H}^1(A_y),\) so, by Lemma 2.2,
\[
\sup_{s > 0} \mathcal{H}^1(A[s]) \leq \frac{M}{\mathcal{H}^{n-1}(\pi(B))}, \quad \frac{\mathcal{H}^{n-1}(\pi(A))}{\mathcal{H}^{n-1}(\pi(B))} \in (M^{-1}, M). \tag{4.11}
\]
In addition, since \(\pi(A)\) and \(\pi(B)\) are \((n-1)\)-dimensional disks centered on the \(e_n\)-axis, \(|S\Delta K| \leq C\delta^\alpha\) and \(B_{rn} \subset K \subset B_{n_{rn}},\) we easily deduce that
\[
\frac{\pi(A) + \pi(B)}{2} = \pi(S) \supset B_{\frac{\alpha}{2}}, \tag{4.12}
\]
provided that $\delta$ is small enough. Hence, combining (4.11) and (4.12), we deduce that $\mathcal{H}^{n-1}(\pi(B))$ is bounded from away from zero by a dimensional constant, thus

$$\sup_{s>0} \mathcal{H}^1(A[s]) \leq C.$$  (4.13)

Hence, by (4.5), (4.10), (4.13) and (4.9),

$$\left| A \setminus \{x_n \leq \tau\} \right| \leq |A \setminus \pi^{-1}(B_R)| + |\pi^{-1}(B_R) \cap \{\tau \leq x_n \leq \tau + C\delta^\frac{2}{n}\}| + \int_0^\tau \mathcal{H}^1(A[s]) \, ds \leq C\delta^\alpha + C\delta^\frac{2}{n} + C\delta^\frac{2}{n} \leq C\delta^\frac{2}{n};$$

and, analogously,

$$|B \setminus \{x_n \leq \tau\}| \leq C\delta^\frac{2}{n}. \quad (4.15)$$

Now, given $r \geq 0$, let us define the sets

$$A'_r := A \cap \{x_n \leq \tau - r\}, \quad B'_r := B \cap \{x_n \leq \tau - r\}, \quad S'_r := S \cap \{x_n \leq \tau - r\}.$$  

By (4.14)–(4.15), we know that

$$|A'_0|, |B'_0| \geq 1 - C\delta^\frac{2}{n},$$

and it is immediate to check that

$$\frac{A'_0 + B'_0}{2} \subset S'_r, \quad \frac{A'_r + B'_r}{2} \subset S'_r.$$  

Also, since $K$ is a convex set satisfying $B_r \subset K \subset B_{nr}$, there exists a dimensional constant $c_n > 0$ such that

$$\left| K \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \geq c_n \min\{r^n, 1\}.$$  

Hence

$$|S'_{r/2}| \leq |S| - \left| S \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \leq |S| + |S \Delta K| - \left| K \cap \left\{ \tau - \frac{r}{2} \leq x_n \leq \tau \right\} \right| \leq 1 + C\delta^\alpha - c_n \min\{r^n, 1\},$$

and by (1.2) applied to $A'_r$ and $B'_0$, we get

$$1 - C\delta^\frac{2}{n} - C|A \cap \{\tau - r \leq x_n \leq \tau\}| \leq \frac{|A'_r|^{\frac{1}{2}} + |B'_0|^{\frac{1}{2}}}{2} \leq |S'_{r/2}|^{\frac{1}{2}} \leq 1 + C\delta^\alpha - c_n \min\{r^n, 1\},$$

which gives

$$C|A \cap \{\tau - r \leq x_n \leq \tau\}| \geq c_n \min\{r^n, 1\} - C\delta^\frac{2}{n}$$  (4.16)

(and analogously for $B$).
Since the point $\mathbf{x}_0 = (\tau + \rho)e_n$ belongs to $S = \frac{A+B}{2}$, there as to be a point $\mathbf{\bar{x}} \in A \cup B$ such that $\mathbf{\bar{x}} \cdot e_n \geq (\tau + \rho)$. Without loss of generality, assume that $\mathbf{\bar{x}} \in B$. Then, by (4.16) applied with $r = \rho$, we get

$$S \cap \{x_n \geq \tau\} \supseteq \mathbf{\bar{x}} + (A \cap \{\tau - \rho \leq x_n \leq \tau\}) \sim \frac{c_n}{C} \min\{\rho^n, 1\} - C\delta^\frac{\rho}{n},$$

which implies $\rho \leq C\delta^\frac{\rho}{n}$, proving (4.1).

Hence $\co(S) \subset (1 + C\delta^\frac{\rho}{n})K$, from which the result follows immediately.

### 4.2 Proof of Proposition 2.2

Since

$$\frac{\co(A) + \co(B)}{2} = \co(S),$$

by (1.2), (2.4)–(1.5), we have

$$|\co(A)|^\frac{\rho}{n} + |\co(B)|^\frac{\rho}{n} \leq |\co(A) + \co(B)|^\frac{\rho}{n} \leq 2|\co(S)|^\frac{\rho}{n} + 2|S|^\frac{\rho}{n} + C\delta^\beta \leq |A|^\frac{\rho}{n} + |B|^\frac{\rho}{n} + C\delta^\beta \leq |\co(A)|^\frac{\rho}{n} + |\co(B)|^\frac{\rho}{n} + C\delta^\beta,$$

from which we deduce that

$$|\co(A) \setminus A| + |\co(B) \setminus B| \leq C\delta^\beta. \quad (4.17)$$

Also, by Theorem 1.3 and the fact that $\|\co(A)| - |\co(B)|| \leq C\delta^\beta\alpha^\rho$ (see (4.17)), we obtain that, up to a translation,

$$|\co(A) \Delta \co(B)| \leq C(\delta^\frac{\rho}{n} + \delta^\beta) \leq C\delta^\frac{\rho}{n}. \quad (4.18)$$

This estimate combined with (4.17) implies that

$$|A \Delta B| \leq C\delta^\frac{\rho}{n}.$$

In addition, if we define $K := \co(A \cup B)$, then we will conclude our argument by showing that

$$|K \setminus A| + |K \setminus B| \leq C\delta^\frac{\rho}{n}. \quad (4.19)$$

Indeed, by John’s lemma (see [16]), after a volume preserving affine transformation, we can assume that $B_r \subset \co(A) \subset B_{nr}$ for some radius $r$ bounded above and below by positive dimensional constants. By (4.18) and a simple geometric argument, we easily deduce that

$$\co(B) \subset (1 + C\delta^\frac{\rho}{n})\co(A).$$

Thus

$$\co(A) \cup \co(B) \subset K \subset (1 + C\delta^\frac{\rho}{n})\co(A),$$
and (4.19) follows by (4.17)–(4.18).

Acknowledgements This work started during Alessio Figalli’s visit at MIT during the fall 2012. Alessio Figalli wishes to thank the Mathematics Department at MIT for its warm hospitality.

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