Global exponential stability analysis of anti-periodic of discontinuous BAM neural networks with time-varying delays

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Abstract. Discontinuous system is playing an increasingly important role in terms of both theory and applications. In this paper, we are concerned with discontinuous BAM (bidirectional associative memory) neural networks with time-varying delays. Under the basic framework of Filippov solution, by means of differential inclusions theory, inequality technique, fundamental solution matrix of coefficients and the non-smooth analysis theory with Lyapunov-like approach, some new sufficient criteria are given to ascertain the existence and globally exponential stability of the anti-periodic solutions for the considered BAM neural networks. Simulation results of two topical numerical examples are exploited to illustrate the improvement and advantages of the established theoretical results in comparison with some existing results. Some previous known results are extended and complemented.

1. Introduction

A bidirectional associative memory (BAM) neural network was first introduced in [11] and [12], which has crucial application perspectives to image processing, signal processing, and pattern recognition. The model is composed of neurons arranged in two layers, the U-layer and V-layer. Regarded as a network with dimension \( n+m \), it produces many nice properties due to the special structure of connection weights and its practical applications in storing paired patterns via both directions: forward and backward direction. Due to the finite switching speed of neurons and amplifiers, time delays in interactions between neurons frequently happen. In this case, delay can affect the stability of neural networks and even lead to complex dynamic behaviors, such as oscillation, chaos, and instability. In recent years, many studies have investigated the dynamical behavior of delayed BAM neural networks. See [2], [3], [14], [16], [17], [18], [20], [21], [25], [26], [29], [30], [31] and the related references therein. Moreover, we can see that the above references studying the BAM neural networks have assumed the activation functions are continuous, Lipschitz continuous or even smooth.

In recent years, nonlinear dynamical systems described by differential equations with
discontinuous right-hand sides have been extensively and successfully applied to various science and engineering fields such as mechanics, electrical engineering, automatic control, etc. Influenced by the discontinuous right-hand sides, a source of instability phenomenon for neural networks is inevitably exists. Therefore, stability analysis has become an important and necessary research theme for the neural networks with discontinuous right-hand sides. This should thank to the work of Forti and Nistri [8] in 2003. They firstly established the global stability results of a neural network modeled by a differential equation with a discontinuous right-hand side and they noted that neural network systems with discontinuous neuron activations are important and do frequently arise in the applications. After that, based on the work of Forti and Nistri [8], considerable efforts have been devoted to investigate the neural network systems with discontinuous activation functions, see [5], [10], [13] and the references therein. Moreover, it is worth noting that, compared with the BAM neural networks and discontinuous dynamical systems, little attention has been devoted to the study of the dynamic behavior of BAM neural networks with discontinuous activations so far, see [6], [27].

On the other hand, it is well known that the existence and stability of anti-periodic solutions are an important topic in nonlinear differential equation and the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations [4]. The signal transmission process of neural networks can often be described as an anti-periodic process. Over the past decades, the anti-periodic solutions of Hopfield neural networks, recurrent neural networks and cellular neural networks have actively been investigated by a large number of scholars. For details, see [19], [22], [23], [24], [28] and references therein. For example, in [28], by using the fundamental solution matrix of coefficients and Lyapunov function, the authors established the sufficient conditions for the existence and global exponential stability of the anti-periodic solution of BAM neural networks with time delays.

However, to the best of our knowledge, to date, only a few investigations have been conducted for the existence and stability of anti-periodic solutions for the BAM neural networks with discontinuous activations and time-varying delays. This may be due to the fact that the mechanism on which how the anti-periodic solution is influenced by the discontinuous activations and time-varying delays associated to the BAM neural networks is far away from clear. So, it is worthwhile to continue the investigation of the existence and stability of anti-periodic for the BAM neural networks with discontinuous activations so far, see [6], [27].

Motivated by the references listed above, in this paper, we consider a class of discontinuous BAM neural networks with time-varying delays described by the following differential equations:

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij}(t) f_j(v_j(t)) + \sum_{j=1}^{m} d_{ij}(t) f_j(v_j(t - \tau_{ij}(t))) + I_i(t), \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} p_{ji}(t) g_i(u_i(t)) + \sum_{i=1}^{n} q_{ji}(t) g_i(u_i(t - \sigma_{ji}(t))) + J_j(t),
\end{align*}
\]

where \(i = 1, 2, \ldots, n\), \(j = 1, 2, \ldots, m\), \(u_i(t), v_j(t)\) denote the potential of the cell \(i\) and \(j\) at time \(t\); \(a_i, b_j\) are positive constants, denoting the rate which the cell \(i\) and \(j\) rest their potential to the resting state when isolated from the other cells and inputs; \(c_{ij}(t), d_{ij}(t), p_{ji}(t), q_{ji}(t)\) denote the first and second-order connection weights of the neural networks; \(f_j(\cdot)\) and \(g_i(\cdot)\) denote the activation functions of the \(i\)th neurons and the \(j\)th neurons, respectively; \(I_i(t), J_j(t)\) are the \(i\)th and the \(j\)th components of an external inputs source introduced from outside the networks to the cell \(i\) and \(j\); time delays \(\tau_{ij}(t), \sigma_{ij}(t)(0 < \tau_{ij}(t) \leq \tau, 0 < \sigma_{ij}(t) \leq \sigma; \tau, \sigma\) are constants) correspond to the finite speed of axonal signal transmission.

Under the concept of Filippov solution, by applying the differential inclusions, fundamental solution matrix of coefficients, inequality technique and non-smooth analysis theory with Lyapunov-like approach, some sufficient conditions on the existence, uniqueness, global exponential stability of the anti-periodic solutions of (1) is proposed originally.

Throughout the paper, we also formulate the following assumptions:
(H1) $f_j$ and $g_i$, $i = 1, 2, ..., n$, $j = 1, ..., m$ are piecewise continuous, i.e., $f_j$ and $g_i$ are continuous in $\mathbb{R}$ except a countable set of jump discontinuous points, and in every compact set of $\mathbb{R}$, have only a finite number of jump discontinuous points.

(H2) for all $(u(t), v(t)) = (u_1(t), ..., u_n(t), v_1(t), ..., v_m(t))$, $u_i(i = i = 1, ..., n)$, $v_j(j = 1, ..., m) \in (0, +\infty)$ and $\gamma_j \in \overline{\mathbb{R}}[f_j(v_j)]$, $\eta_i \in \overline{\mathbb{R}}[g_i(u_i)]$ are bounded, monotonically non-decreasing, where

$$\overline{\mathbb{R}}[f_j(v)] = \left[ \min\{f_j(v_j^-), f_j(v_j^+)\}, \max\{f_j(v_j^-), f_j(v_j^+)\} \right],$$

$$\overline{\mathbb{R}}[g_i(x)] = \left[ \min\{g_i(u_i^-), g_i(u_i^+)\}, \max\{g_i(u_i^-), g_i(u_i^+)\} \right].$$

(H2*) for all $(u(t), v(t)) = (u_1(t), ..., u_n(t), v_1(t), ..., v_m(t))$, $u_i(i = i = 1, ..., n)$, $v_j(j = 1, ..., m) \in (0, +\infty)$ and $\gamma_j \in \overline{\mathbb{R}}[f_j(v_j)]$, $\eta_i \in \overline{\mathbb{R}}[g_i(u_i)]$ are bounded, monotonically non-increasing, where

$$\overline{\mathbb{R}}[f_j(v)] = \left[ \min\{f_j(v_j^-), f_j(v_j^+)\}, \max\{f_j(v_j^-), f_j(v_j^+)\} \right],$$

$$\overline{\mathbb{R}}[g_i(x)] = \left[ \min\{g_i(u_i^-), g_i(u_i^+)\}, \max\{g_i(u_i^-), g_i(u_i^+)\} \right].$$

(H3) $c_{ij}(t)$, $d_{ij}(t)$, $p_{ji}(t)$, $q_{ji}(t)$, $\tau_{ij}(t)$, $\sigma_{ij}(t)$, $I_i(t)$, $J_j(t) \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, and for all $t, u, v \in \mathbb{R}$, except a finite number of jump discontinuous points, it follows that

$$c_{ij}(t + \omega)f_j(v) = -c_{ij}(t)f_j(-v), \quad d_{ij}(t + \omega)f_j(v) = -d_{ij}(t)f_j(-v),$$

$$p_{ji}(t + \omega)g_i(u) = -p_{ji}(t)g_i(-u), \quad q_{ji}(t + \omega)g_i(u) = -q_{ji}(t)g_i(-u),$$

$$I_i(t + \omega) = -I_i(t), \quad J_j(t + \omega) = -J_j(t),$$

$$\tau_{ij}(t + \omega) = \tau_{ij}(t), \quad \sigma_{ij}(t + \omega) = \sigma_{ij}(t).$$

The structure of this paper is as follows. After some notations that are given in the succeeding text, we mainly give some basic definitions and important lemmas in Section 2. In Section 3, some new criteria are given to guarantee the existence, uniqueness and global exponential stability of the anti-periodic solution. In Section 4, we show the applications of the main theorem, and simulation results of two typical examples are shown to verify the obtained theoretical results. Finally, the study in this paper is concluded in 5.

2. Essential Definitions and Lemmas

In this section, we state some definitions and preliminary lemmas, which will be used throughout this paper.

Let $f(t)$ be a continuous $\omega$-antiperiodic function defined on $\mathbb{R}$. We define

$$f^M = \sup_{t \in \mathbb{R}} |f(t)|, \quad f^L = \inf_{t \in \mathbb{R}} |f(t)|.$$

Let $\varphi(s) = (\varphi_1(s), \varphi_2(s), ..., \varphi_n(s))^\top$ and $\psi(s) = (\psi_1(s), \psi_2(s), ..., \psi_m(s))^\top$ where $\varphi_i(s) \in C([-\sigma, 0], \mathbb{R})$, $i = 1, 2, ..., n$, $\psi_j(s) \in C([-\tau, 0], \mathbb{R})$, $j = 1, 2, ..., m$. Define

$$\|\varphi\| = \sup_{-\sigma \leq s \leq 0} \sum_{i=1}^n |\varphi_i(s)|, \quad \|\psi\| = \sup_{-\tau \leq s \leq 0} \sum_{i=1}^m |\psi_i(s)|.$$

Let $u(t) : \mathbb{R} \rightarrow \mathbb{R}$ be continuous in $t$. $u(t)$ is said to be $\omega$-anti-periodic on $\mathbb{R}$, if $u(t + \omega) = -u(t)$ for all $t \in R$. If a system is $\omega$-anti-periodic $(u(t + \omega) = -u(t))$, then it is $2\omega$-periodic $(u(t + 2\omega) = u(t + \omega + \omega) = -u(t + \omega) = x(t))$. 

3
Since dropping the assumption of continuity on the activation functions, we need to specify what is meant by a solution of the equation system (1) with discontinuous right-hand sides. Moreover, we need to introduce the concept of an output solution associated with a solution of (1). For this purpose, we extend the concept of the Filippov solution to the differential equation system (1) as follows.

**Definition 2.1** A function \( z(t) = (u(t), v(t))^T = (u_1(t), ..., u_n(t), v_1(t), ..., v_m(t))^T : [-\zeta, b) \to \mathbb{R}^{2n} \), \( \zeta = \max\{\tau, \sigma\} \) and \( b \in (0, +\infty] \), is a state solution of the discontinuous system (1) on \( [-\zeta, b) \) if

(1) \( z(t) = (u(t), v(t))^T \) is continuous on \( [-\zeta, b) \) and absolutely continuous on any compact interval of \( [0, b) \);

(2) there exists a measurable function \( (\eta(t), \gamma(t))^T = (\eta_1(t), ..., \eta_n(t), \gamma_1(t), ..., \gamma_m(t))^T : [-\zeta, b) \to \mathbb{R}^{2n} \), such that \( \eta_j(t) \in \mathcal{C}_{\sigma}[g_j(u(t))] \), \( \gamma_j(t) \in \mathcal{C}_{\sigma}[f_j(v_j(t))] \) for almost all for \( a.e. \ t \in [-\zeta, b) \) and

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij}(t) \gamma_j(t) + \sum_{j=1}^{m} d_{ij}(t) \gamma_j(t - \tau_{ij}(t)) + I_i(t), \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} p_{ji}(t) \eta_i(t) + \sum_{i=1}^{n} q_{ji}(t) \eta_i(t - \sigma_{ji}(t)) + J_j(t),
\end{align*}
\]

for \( a.e. \ t \in [0, b) \), \( i = 1, 2, ..., n; \ j = 1, 2, ..., m \).

Any function \( (\eta(t), \gamma(t))^T = (\eta_1(t), ..., \eta_n(t), \gamma_1(t), ..., \gamma_m(t))^T \) satisfying (2) is called an output solution associated with the state \( (u(t), v(t))^T = (u_1(t), ..., u_n(t), v_1(t), ..., v_m(t))^T \) is a solution of (1) in the sense of Filippov since it satisfies

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij}(t) \mathcal{C}_{\sigma}[f_j(v_j(t))] + \sum_{j=1}^{m} d_{ij}(t) \mathcal{C}_{\sigma}[f_j(v_j(t) - \tau_{ij}(t))]
+ I_i(t), \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} p_{ji}(t) \mathcal{C}_{\sigma}[g_i(u_i(t))] + \sum_{i=1}^{n} q_{ji}(t) \mathcal{C}_{\sigma}[g_i(u_i(t) - \sigma_{ji}(t))]
+ J_j(t),
\end{align*}
\]

for \( a.e. \ t \in [0, b) \), \( i = 1, 2, ..., n; \ j = 1, 2, ..., m \).

**Definition 2.2** [9] (IVP). For any continuous function \( (\varphi(s), \psi(s))^T = (\varphi_1(s), ..., \varphi_n(s), \psi_1(s), ..., \psi_m(s))^T : [-\zeta, b) \to \mathbb{R}^{2n} \) and any measurable selection \( (\lambda(s), \mu(s))^T = (\lambda_1(s), ..., \lambda_n(s), \mu_1(s), ..., \mu_m(s))^T : [-\zeta, b) \to \mathbb{R}^{2n} \), such that \( \lambda_j(s) \in \mathcal{C}_{\sigma}[g_j(\varphi_i(s))] \), \( \mu_j(s) \in \mathcal{C}_{\sigma}[f_j(\psi_j(s))] \), for \( a.e. \ s \in [-\zeta, b) \) by an initial value problem associated to system (1) with initial condition \( \varphi(s), \psi(s), \lambda(s), \mu(s), \) we mean the following problem: find a couple of functions \( (u(t), v(t), \eta(t), \gamma(t))^T : [-\zeta, b) \to \mathbb{R}^{2n} \times \mathbb{R}^{2n} \), such that \( (u(t), v(t))^T \) is a solution of system (1) on \( [-\zeta, b) \) for some \( b > 0 \), \( (\eta(t), \gamma(t))^T \) is an output associated to \( (u(t), v(t))^T \)

\[
\begin{align*}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij}(t) \gamma_j(t) + \sum_{j=1}^{m} d_{ij}(t) \gamma_j(t - \tau_{ij}(t)) + I_i(t), \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} p_{ji}(t) \eta_i(t) + \sum_{i=1}^{n} q_{ji}(t) \eta_i(t - \sigma_{ji}(t)) + J_j(t),
\end{align*}
\]

for \( a.e. \ t \in [0, b) \), \( i = 1, 2, ..., n; \ j = 1, 2, ..., m \).

(4)

**Definition 2.3** [27] The solution \( (u^*(t), v^*(t))^T \) of system (1) is said to be globally exponentially stable, if, for any \( (u(t), v(t))^T \) of system (1), there exist constants \( \alpha > 0 \) and \( \lambda > 0 \) such that

\[
\sum_{i=1}^{n} |u_i(t) - u^*_i(t)| + \sum_{j=1}^{m} |v_j(t) - v^*_j(t)| \leq \lambda e^{-\alpha t}.
\]
Suppose that $x(t) : [0, +\infty) \to R^n$ is absolutely continuous on any compact interval of $[0, +\infty)$. We give a chain rule for computing the time derivative of the composed function $V(x(t)) : [0, +\infty) \to R$ as follows.

**Lemma 2.4 (Chain Rule[10]).** Suppose that $V(x) : R^n \to R$ is C-regular, and that $x(t) : [0, +\infty) \to R^n$ is absolutely continuous on any compact interval of $[0, +\infty)$. Then, $x(t)$ and $V(x(t)) : [0, +\infty) \to R$ are differential for a.e. $t \in [0, +\infty)$, and we have
\[
\frac{dV(x(t))}{dt} = \left(\xi(t), \frac{dx(t)}{dt}\right), \quad \forall t \in \partial V(x(t)).
\]

**Lemma 2.5 [28] Let** $A = \begin{pmatrix} -a_{ij} & 0 \\ 0 & -b_{ij} \end{pmatrix}$, $\alpha = \min_{1 \leq i \leq n, 1 \leq j \leq m} \{a_{ij}, b_{ij}\}$, then we have
\[
\|\exp(At)\| \leq \sqrt{2e^{-at}}, \quad \text{for all } t \geq 0.
\]

### 3. Main results

In this section, we consider the existence and global exponential stability of anti-periodic solutions for neural network (1).

**Theorem 3.1** Suppose that the assumptions (H1) and (H2) hold, then for any solution $(u(t), v(t))^\top$ of system (1), there exist two positive constants
\[
M := \sqrt{2(\|\varphi\|^2 + \|\psi\|^2) + \frac{\sqrt{2}}{\alpha} \max_{1 \leq i \leq n} \sum_{j=1}^{m} \left(c_{ij}^M + d_{ij}^M\gamma_j^M + I_i^M\right)}
\]
\[
+ \frac{\sqrt{2}}{\alpha} \max_{1 \leq j \leq m} \sum_{i=1}^{n} \left(p_{ji}^M + q_{ji}^M\eta_j^M + J_j^M\right)
\]

and $\widetilde{M}$ such that
\[
|u_i(t)| \leq M, i = 1, 2, ..., n, \quad |v_j(t)| \leq M, j = 1, 2, ..., m, \quad \text{for } t \in [-\varsigma, +\infty),
\]
\[
|\eta_j(t)| \leq \widetilde{M}, i = 1, 2, ..., n, \quad |\gamma_j(t)| \leq \widetilde{M}, j = 1, 2, ..., m, \quad \text{for a.e. } t \in [-\varsigma, +\infty).
\]

**Proof:** Define the set-valued maps
\[
u_i(t) \mapsto -a_{ij}u_i(t) + \sum_{j=1}^{m} c_{ij}(t)\gamma_j(t) + \sum_{j=1}^{m} d_{ij}(t)\gamma_j(t - \tau_{ij}(t)) + I_i(t), i = 1, 2, ..., n,
\]

and
\[
u_j(t) \mapsto -b_{ij}v_j(t) + \sum_{i=1}^{n} p_{ji}(t)\eta_i(t) + \sum_{i=1}^{n} q_{ji}(t)\eta_i(t - \sigma_{ji}(t)) + J_j(t), j = 1, 2, ..., m.
\]

By (H2), one can easily see that the above set-valued maps are upper semi-continuous with nonempty compact convex values and the local existence of a solution $z(t) = (u(t), v(t))^\top$ of (3) is obviously([7], [10]). That means, the IVP of (2) has at least one solution $(u(t), v(t))^\top = (u_1, u_2, ..., u_n, v_1, v_2, ..., v_m)^\top$ on $[0, b)$ for some $b \in [0, +\infty)$ and the derivative of $u_i(t)$ and $v_j(t)$ are measurable selections from
\[
-a_{ij}u_i(t) + \sum_{j=1}^{m} c_{ij}(t)\overline{f}_j(v_j(t)) + \sum_{j=1}^{m} d_{ij}(t)\overline{f}_j(v_j(t - \tau_{ij}(t))) + I_i(t),
\]

for a.e. $t \in [0, b)$, $i = 1, 2, ..., n$. 


and 
\[-b_j v_j(t) + \sum_{i=1}^{n} p_{ij}(t) [g_i(u_i(t))] + \sum_{i=1}^{n} q_{ij}(t) [g_i(u_i(t - \sigma_{ji}(t)))] + J_j(t),\]
for a.e. \( t \in [0, b) , i = 1, 2, ..., n. \)

It follows from the Continuation Theorem [[1], Theorem 2, P78] that either \( b = +\infty \), or \( b < +\infty \) and \( \lim_{t \to b^-} \|z(t)\| = +\infty \), where \( \|z(t)\| = \sup_{t \in [0, b)} \{\sum_{i=1}^{n} |u_i(t)| + \sum_{j=1}^{m} |v_j(t)|\} \) is defined as above. Next, we will show that \( \lim_{t \to b^-} \|z(t)\| < +\infty \) if \( b < +\infty \), which means that the maximal existing interval of \( x(t) \) can be extended to \( +\infty \). From Definition 2.1, \((\eta(t), \gamma(t))^T = (\eta_1(t), ..., \eta_m(t), \gamma_1(t), ..., \gamma_m(t))^T : [-\zeta, b) \to \mathbb{R}^{2n} \), such that \( \eta_i(t) \in \overline{co}[g_i(u_i(t))] \), \( \gamma_j(t) \in \overline{co}[f_j(v_j(t))] \) for almost all for a.e. \( t \in [-\zeta, b) \) and 
\[
\begin{align*}
\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{m} c_{ij}(t) \gamma_j(t) + \sum_{j=1}^{m} d_{ij}(t) \gamma_j(t - \tau_{ij}(t)) + I_i(t), \\
\frac{dv_j(t)}{dt} &= -b_j v_j(t) + \sum_{i=1}^{n} p_{ij}(t) \eta_i(t) + \sum_{i=1}^{n} q_{ij}(t) \eta_i(t - \sigma_{ji}(t)) + J_j(t),
\end{align*}
\]
for a.e. \( t \in [0, b) , i = 1, 2, ..., n, j = 1, 2, ..., m. \)

Let
\[
z_{ij}(t) = \left( \begin{array}{c} u_i(t) \\ v_j(t) \end{array} \right), \quad A = \left( \begin{array}{cc} -a_i & 0 \\ 0 & -b_j \end{array} \right), \quad h_{ij}(t) = \left( \begin{array}{c} I_i(t) \\ J_j(t) \end{array} \right),
\]
and 
\[
F_{ij}(\eta_i(t), \gamma_j(t)) = \left( \begin{array}{c} \sum_{j=1}^{m} c_{ij}(t) \gamma_j(t) + \sum_{j=1}^{m} d_{ij}(t) \gamma_j(t - \tau_{ij}(t)) \\ \sum_{i=1}^{n} p_{ij}(t) \eta_i(t) + \sum_{i=1}^{n} q_{ij}(t) \eta_i(t - \sigma_{ji}(t)) \end{array} \right).
\]

Then the system (2) can be rewritten as 
\[
\frac{dz_{ij}(t)}{dt} = A z_{ij}(t) + F_{ij}(\eta_i(t), \gamma_j(t)) + h_{ij}(t),
\]
where \( \eta_i(t) \in \overline{co}[g_i(u_i(t))] \), \( \gamma_j(t) \in \overline{co}[f_j(v_j(t))] \) for a.e. \( t \in [0, b) , i = 1, 2, ..., n, j = 1, 2, ..., m. \)

Solving (5), it follows that 
\[
z_{ij}(t) = e^{\alpha t} z_{ij}(0) + \int_0^t e^{\alpha (t-s)} [F_{ij}(\eta_i(s), \gamma_j(s)) + h_{ij}(s)] ds.
\]

By applying Lemma 2.5 and (H2), we can have 
\[
\|z_{ij}(t)\| \leq \sqrt{2} e^{-\alpha t} \|z_{ij}(0)\| + \sqrt{2} \int_0^t e^{-\alpha (t-s)} \left\| F_{ij}(\eta_i(s), \gamma_j(s)) \right\| + \| h_{ij}(s) \| ds
\]
\[
\leq \sqrt{2} \left( \| \varphi \|^2 + \| \psi \|^2 \right) + \frac{\sqrt{2}}{\alpha} (1 - e^{-\alpha t}) \sum_{j=1}^{m} \left( (c_{ij}^M + d_{ij}^M) \gamma_j^M + I_j^M \right)
\]
\[
+ \sum_{i=1}^{n} \left( (p_{ij}^M + q_{ij}^M) \eta_i^M + J_j^M \right)
\]
\[
\leq \sqrt{2} \left( \| \varphi \|^2 + \| \psi \|^2 \right) + \frac{\sqrt{2}}{\alpha} \max_{1 \leq i \leq n} \sum_{j=1}^{m} \left( (c_{ij}^M + d_{ij}^M) \gamma_j^M + I_j^M \right)
\]
\[
+ \frac{\sqrt{2}}{\alpha} \sum_{1 \leq j \leq m} \sum_{i=1}^{n} \left( (p_{ij}^M + q_{ij}^M) \eta_i^M + J_j^M \right) \leq M.
\]
Then, we can obtain
\[ |u_i(t)| \leq M, \, i = 1, 2, ..., n, \text{ for } t \in (0, b] \]
\[ |v_j(t)| \leq M, \, j = 1, 2, ..., m, \text{ for } t \in (0, b]. \]  
(6)

Hence, \((u(t), v(t))^T\) is bounded on its existence interval \([-\varsigma, b)\). Thus, \(\lim_{t \to b^-} \|z(t)\| < +\infty\), which implies that \(b = +\infty\). Therefore, by (6), we have
\[ |u_i(t)| \leq M, \, i = 1, 2, ..., n, \, t \in [-\varsigma, +\infty), \]
\[ |v_j(t)| \leq M, \, j = 1, 2, ..., m, \, t \in [-\varsigma, +\infty). \]  
(7)

Note that, \(f_j\) has a finite number of discontinuous points on any compact interval of \(\mathbb{R}\). In particular, \(f_j\) has a finite number of discontinuous points on compact interval \([-M, M]\). Without loss of generality, let \(f_j\) discontinuous at points \(\{\rho_k^j : k = 1, 2, ..., l_j\}\) on the interval \([-M, M]\) and assume that \(-M < \rho_1^j < \rho_2^j < \cdots < \rho_{l_j}^j < M\). Let us consider a series of continuous functions:
\[
 f_j^0(x) = \begin{cases} 
 f_j(x), & x \in [-M, \rho_1^j), \\
 f_j(\rho_1^j - 0), & x = \rho_1^j; 
\end{cases} \quad f_j^{l_j}(x) = \begin{cases} 
 f_j(\rho_{l_j}^j + 0), & x = \rho_{l_j}^j, \\
 f_j(x), & x \in (\rho_{l_j}^j, M]; 
\end{cases}
\]
and
\[
 f_j^k(x) = \begin{cases} 
 f_j(\rho_k^j + 0), & x = \rho_k^j, \\
 f_j(x), & x \in (\rho_k^j, \rho_{k+1}^j), \\
 f_j(\rho_{k+1}^j - 0), & x = \rho_{k+1}^j, \\
\end{cases} \quad k = 1, 2, ..., l_j - 1.
\]

Let
\[
 M_j = \max \left\{ \max_{x \in [-M, \rho_1^j]} \left\{ f_j^0(x) \right\}, \max_{1 \leq k \leq l_j - 1} \left\{ \max_{x \in [\rho_k^j, \rho_{k+1}^j]} \left\{ f_j^k(x) \right\} \right\}, \max_{x \in [\rho_{l_j}^j, M]} \left\{ f_j^{l_j}(x) \right\} \right\},
\]
\[
 m_j = \max \left\{ \min_{x \in [-M, \rho_1^j]} \left\{ f_j^0(x) \right\}, \min_{1 \leq k \leq l_j - 1} \left\{ \min_{x \in [\rho_k^j, \rho_{k+1}^j]} \left\{ f_j^k(x) \right\} \right\}, \min_{x \in [\rho_{l_j}^j, M]} \left\{ f_j^{l_j}(x) \right\} \right\}.
\]

It is clear that
\[
 |\overline{\sigma}[f_j(x_j(t))]| \leq \max\{|M_j|, |m_j|\}, \quad j = 1, 2, ..., m.
\]
Since, \(\gamma_j(t) \in \overline{\sigma}[f_j(x_j(t))]\) for a.e. \(t \in [-\varsigma, +\infty)\) and \(j = 1, 2, ..., m\), we have
\[
 |\gamma_j(t)| \leq \max\{|M_j|, |m_j|\}, \quad \text{for a.e. } t \in [-\varsigma, +\infty), \quad j = 1, 2, ..., m.
\]  
(8)

In a similar way, let
\[
 \overline{\tilde{M}}_i = \max \left\{ \max_{x \in [-M, \rho_1^i]} \left\{ g_i^0(x) \right\}, \max_{1 \leq k \leq l_i - 1} \left\{ \max_{x \in [\rho_k^i, \rho_{k+1}^i]} \left\{ g_i^k(x) \right\} \right\}, \max_{x \in [\rho_{l_i}^i, M]} \left\{ g_i^{l_i}(x) \right\} \right\},
\]
\[
 \overline{\tilde{m}}_i = \max \left\{ \min_{x \in [-M, \rho_1^i]} \left\{ g_i^0(x) \right\}, \min_{1 \leq k \leq l_i - 1} \left\{ \min_{x \in [\rho_k^i, \rho_{k+1}^i]} \left\{ g_i^k(x) \right\} \right\}, \min_{x \in [\rho_{l_i}^i, M]} \left\{ g_i^{l_i}(x) \right\} \right\}.
\]

It is clear that
\[
 |\overline{\sigma}[g_i(x_i(t))]| \leq \max\{|\overline{\tilde{M}}_i|, |\overline{\tilde{m}}_i|\}, \quad i = 1, 2, ..., n.
\]
Then, from (8) and (9), it follows that
\[ |\eta_i(t)| \leq \max\{|\bar{M}_j|, |\bar{m}_j|\}, \text{ for a.e. } t \in [-\zeta, +\infty), \quad i = 1, 2, \ldots, n. \tag{9} \]

Let
\[ \bar{M} = \max_{1 \leq i \leq n, 1 \leq j \leq m} \left\{ |M_j|, |m_j|, |\bar{M}_j|, |\bar{m}_j| \right\}. \]

Then, from (8) and (9), it follows that
\[ |\eta_i(t)| \leq \bar{M}, \quad i = 1, 2, \ldots, n, \text{ for a.e. } t \in [-\zeta, +\infty), \]
\[ |\gamma_j(t)| \leq \bar{M}, \quad j = 1, 2, \ldots, m, \text{ for a.e. } t \in [-\zeta, +\infty). \]

Consequently, the proof is complete.

Suppose that \((u^*(t), v^*(t))^\top = (u_1^*(t), \ldots, u_n^*(t), v_1^*(t), \ldots, v_m^*(t))^\top\) is a solution of system (1) with initial conditions
\[ (u^*(s), v^*(s)) = (\varphi^*(s), \psi^*(s)), \quad \text{for a.e. } s \in [-\zeta, 0], \]

The associated output solution \((\eta^*(t), \gamma^*(t))^\top = (\eta_1^*(t), \ldots, \eta_n^*(t), \gamma_1^*(t), \ldots, \gamma_m^*(t))^\top : [-\zeta, b) \to \mathbb{R}^{2m}\) such that \(\eta_i^*(t) \in \overline{co}[g_i(u_i^*(t))]\), \(\gamma_j^*(t) \in \overline{co}[f_j(v_j^*(t))]\). Then, on the stability of system (1), we have the following results.

**Theorem 3.2** Suppose that the assumptions (H1)-(H2) hold and the following condition is satisfied:

(H4) The delays \(\tau_{ij}(t)\) and \(\sigma_{ij}(t)\) are continuously differentiable function and satisfy \(\tau_{ij}(t) \neq 1, \sigma_{ij}(t) \neq 1\), for \(i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.\) Moreover, there exist positive constants \(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m}\) and \(\rho > 0\) such that \(a_i^l \geq \rho, \ b_i^l \geq \rho\) and

\[
\limsup_{t \to +\infty} \left\{ \xi_i c_i(t) + \sum_{j=1, j \neq i}^{m} \xi_j |c_{ji}(t)| + \sum_{j=1}^{m} \xi_j \frac{e^{\rho t} |d_{ji}(\varphi_{ji}^{-1}(t))|}{1 - \tau_{ji}(\varphi_{ji}^{-1}(t))} \right\} < 0, \quad i = 1, 2, \ldots, n,
\]
\[
\limsup_{t \to +\infty} \left\{ \xi_j p_{jj}(t) + \sum_{i=1, i \neq j}^{n} \xi_i |p_{ij}(t)| + \sum_{i=1}^{n} \xi_i \frac{e^{\rho t} |q_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \sigma_{ij}(\varphi_{ij}^{-1}(t))} \right\} < 0, \quad j = 1, 2, \ldots, m.
\]

where \(\varphi_{ij}^{-1}\) is the inverse function of \(\varphi_{ij}(t) = t - \tau_{ij}(t)\) and \(\varphi_{ij}^{-1}\) is the inverse function of \(\varphi_{ij}(t) = t - \sigma_{ij}(t).\)

Then the anti-periodic solution \((u^*(t), v^*(t))^\top\) of the system (1) associated with an output \((\eta^*(t), \gamma^*(t))^\top\) is globally exponentially stable.

**Proof.** Let \((u(t), v(t))^\top\) be an arbitrary solution of the system (1) associated with outputs \((\eta(t), \gamma(t))^\top\), \([\varphi, \psi]\) is the corresponding initial values. From (H2), it follows that \(\eta_i(t) \in \overline{co}[g_i(u_i(t))], \gamma_j(t) \in \overline{co}[f_j(v_j(t))]\) for a.e. \(t \in [-\zeta, +\infty).\)

Define \(\vartheta_i(t) = \text{sign}\{u_i(t) - u_i^*(t)\}\) if \(u_i(t) \neq u_i^*(t)\); while \(\vartheta_i(t)\) can be arbitrarily choosen in \([-1, 1]\) if \(u_i(t) = u_i^*(t)\). In particular, we can choose \(\vartheta_i(t)\) as follows
\[
\vartheta_i(t) = \begin{cases} 
0, & u_i(t) - u_i^*(t) = \eta_i(t) - \eta_i^*(t) = 0, \\
\text{sign}\{\eta_i(t) - \eta_i^*(t)\}, & u_i(t) = u_i^*(t) \text{ and } \eta_i(t) \neq \eta_i^*(t), \\
\text{sign}\{u_i(t) - u_i^*(t)\}, & u_i(t) \neq u_i^*(t).
\end{cases}
\]
Obviously, \( V(t) \) can be evaluated by Lemma 2.4. Now, by applying the chain rule in Lemma 2.4, calculate the time derivative of \( V(t) \) as follows

\[
\theta_j(t) = \begin{cases} 
0, & v_j(t) - v_j^*(t) = \gamma_j(t) - \gamma_j^*(t) = 0, \\
\text{sign}\{\gamma_j(t) - \gamma_j^*(t)\}, & v_j(t) = v_j^*(t) \text{ and } \gamma_j(t) \neq \gamma_j^*(t), \\
\text{sign}\{v_j(t) - v_j^*(t)\}, & v_j(t) \neq v_j^*(t).
\end{cases}
\]

Then, we have

\[
\theta_j(t)\{v_j(t) - v_j^*(t)\} = |v_j(t) - v_j^*(t)|, \quad j = 1, 2, ..., m.
\]

Similarly, define \( \theta_j(t) = \text{sign}\{v_j(t) - v_j^*(t)\} \) if \( v_j(t) \neq v_j^*(t) \); while \( \theta_j(t) \) can be arbitrarily chosen in \([-1, 1]\) if \( v_j(t) = v_j^*(t) \). In particular, we can choose \( \theta_j(t) \) as follows

\[
\theta_j(t) = \begin{cases} 
0, & v_j(t) - v_j^*(t) = \gamma_j(t) - \gamma_j^*(t) = 0, \\
\text{sign}\{\gamma_j(t) - \gamma_j^*(t)\}, & v_j(t) = v_j^*(t) \text{ and } \gamma_j(t) \neq \gamma_j^*(t), \\
\text{sign}\{v_j(t) - v_j^*(t)\}, & v_j(t) \neq v_j^*(t).
\end{cases}
\]

Then, we have

\[
\theta_j(t)\{v_j(t) - v_j^*(t)\} = |v_j(t) - v_j^*(t)|, \quad j = 1, 2, ..., m.
\]

Let \( y_i(t) = u_i(t) - u_i^*(t) \), \( z_j(t) = v_j(t) - v_j^*(t), i = 1, 2, ..., n, j = 1, 2, ..., m \). Then, it follows from system (2) that

\[
\begin{aligned}
\frac{dy_i(t)}{dt} &= -a_i y_i(t) + \sum_{j=1}^{m} c_{ij}(t)[\gamma_j(t) - \gamma_j^*(t)] \\
&\quad + \sum_{j=1}^{m} d_{ij}(t)[\gamma_j(t - \tau_j(t)) - \gamma_j^*(t - \tau_j(t))], \\
\frac{dz_j(t)}{dt} &= -b_j z_j(t) + \sum_{i=1}^{n} p_{ji}(t)[\eta_i(t) - \eta_i^*(t)] \\
&\quad + \sum_{i=1}^{n} q_{ji}(t)[\eta_i(t - \sigma_j(t)) - \eta_i^*(t - \sigma_j(t))].
\end{aligned}
\]

Consider the following candidate Lyapunov function:

\[
V(t) = \sum_{i=1}^{n} \xi_i|y_i(t)|e^{\rho t} + \sum_{j=1}^{m} \xi_j|z_j(t)|e^{\rho t}
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \int_{t-\tau_j(t)}^{t} \frac{d_{ij}(\varphi_{ij}^{-1}(u))}{1 - \tau_{ij}'(\varphi_{ij}^{-1}(u))} |\gamma_j(u) - \gamma_j^*(u)|e^{\rho(u+\tau_{ij}^M)} du
\]

\[
+ \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_j \int_{t-\sigma_j(t)}^{t} \frac{q_{ji}(\varphi_{ji}^{-1}(u))}{1 - \sigma_{ji}'(\varphi_{ji}^{-1}(u))} |\eta_i(u) - \eta_i^*(u)|e^{\rho(u+\sigma_{ji}^M)} du.
\]

Obviously, \( V(t) \) is regular. Meanwhile, the solutions \( (u_i(t), v_j(t))^\top, (u_i^*(t), v_j^*(t))^\top \) of the system (1) are all absolutely continuous. Then, \( V(t) \) is differential for a.e. \( t \geq 0 \) and the time derivative can be evaluated by Lemma 2.4. Now, by applying the chain rule in Lemma 2.4, calculate the time derivative of \( V(t) \) along the solution trajectories of the system (1) in the sense of (2), then we can get for a.e. \( t \geq 0 \) that

\[
\frac{dV(t)}{dt} = \rho e^{\rho t} \left[ \sum_{i=1}^{n} \xi_i |y_i(t)| + \sum_{j=1}^{m} \xi_j |z_j(t)| \right] + \sum_{i=1}^{n} e^{\rho t} \xi_i \frac{dy_i(t)}{dt} + \sum_{j=1}^{m} e^{\rho t} \xi_j \frac{dz_j(t)}{dt}
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i \frac{d_{ij}(\varphi_{ij}^{-1}(t))}{1 - \tau_{ij}'(\varphi_{ij}^{-1}(t))} |\gamma_j(t) - \gamma_j^*(t)|e^{\rho(t+\tau_{ij}^M)}
\]

\[
- \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i d_{ij}(t)[\gamma_j(t - \tau_j(t)) - \gamma_j^*(t - \tau_j(t))]e^{\rho(t-\tau_j(t)+\tau_{ij}^M)} du.
\]
by straightforward computation, we further have

\[
\frac{dV(t)}{dt} = \rho e^{\rho t} \left[ \sum_{i=1}^{n} |y_i(t)| + \sum_{j=1}^{m} |z_j(t)| \right] \\
+ \sum_{i=1}^{n} e^{\rho t} \xi_i \partial_i \left\{ -a_i y_i(t) + \sum_{j=1}^{m} c_{ij}(t) [\gamma_j(t) - \gamma_j^*(t)] + \sum_{j=1}^{m} d_{ij}(t) [\gamma_j(t - \tau_{ij}(t)) - \gamma_j^*(t - \tau_{ij}(t))] \right\} \\
+ \sum_{j=1}^{m} e^{\rho t} \xi_j \theta_j \left\{ -b_j z_j(t) + \sum_{i=1}^{n} p_{ji}(t) [\eta_i(t) - \eta_i^*(t)] + \sum_{i=1}^{n} q_{ji}(t) [\eta_i(t - \sigma_{ji}(t)) - \eta_i^*(t - \sigma_{ji}(t))] \right\} \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i d_{ij}(\varphi_j^{-1}(t)) [\gamma_j(t) - \gamma_j^*(t)] e^{\rho (t + \tau_{ij})} \\
- \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_i d_{ij}(t) [\gamma_j(t - \tau_{ij}(t)) - \gamma_j^*(t - \tau_{ij}(t))] |e^{\rho (t - \tau_{ij}(t) + \tau_{ij})} du \\
+ \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_j q_{ji}(\varphi_j^{-1}(t)) [\eta_i(t) - \eta_i^*(t)] e^{\rho (t + \sigma_{ji})} \\
- \sum_{j=1}^{m} \sum_{j=1}^{m} \xi_j q_{ji}(t) [\eta_i(t - \sigma_{ji}(t)) - \eta_i^*(t - \sigma_{ji}(t))] e^{\rho (t - \sigma_{ji}(t) + \sigma_{ji})},
\]

by straightforward computation, we further have

\[
\frac{dV(t)}{dt} \leq \rho e^{\rho t} \sum_{i=1}^{n} \xi_i |y_i(t)| - \sum_{i=1}^{n} e^{\rho t} \xi_i a_i |y_i(t)| + \sum_{i=1}^{n} e^{\rho t} \xi_i c_{ii}(t) |\gamma_j(t) - \gamma_j^*(t)| \\
+ \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{m} e^{\rho t} \xi_i c_{ij}(t) |\gamma_j(t) - \gamma_j^*(t)| + \sum_{i=1}^{n} \sum_{j=1}^{m} e^{\rho t} \xi_i e^{\rho t} |d_{ij}(\varphi_j^{-1}(t))| |\gamma_j(t) - \gamma_j^*(t)| \\
+ \rho e^{\rho t} \sum_{j=1}^{m} \xi_j |z_j(t)| - \sum_{j=1}^{m} e^{\rho t} \xi_j b_j |z_j(t)| + \sum_{j=1}^{m} e^{\rho t} \xi_j p_{jj}(t) |\eta_i(t) - \eta_i^*(t)| \\
+ \sum_{j=1}^{m} \sum_{i=1,i \neq j}^{n} e^{\rho t} \xi_j p_{ji}(t) |\eta_i(t) - \eta_i^*(t)| + \sum_{j=1}^{m} \sum_{i=1}^{n} e^{\rho t} \xi_j e^{\rho t} |q_{ji}(\varphi_j^{-1}(t))| |\eta_i(t) - \eta_i^*(t)| \\
\leq -\sum_{i=1}^{n} e^{\rho t} \xi_i (\Delta_i^t - \rho) |y_i(t)| + e^{\rho t} \sum_{i=1}^{n} \left[ \xi_i c_{ii}(t) + \sum_{j=1,j \neq i}^{m} \xi_j c_{ij}(t) \right] \\
+ \sum_{j=1}^{m} e^{\rho t} |d_{ji}(\varphi_j^{-1}(t))| |\gamma_j(t) - \gamma_j^*(t)| \right].
\]
Thus, by (14), we obtain
\[
\sum_{j=1}^{m} e^{pt} \xi_j (b_j^T - \rho)|z_j(t)| + e^{pt} \sum_{j=1}^{m} \xi_j p_{jj}(t) + \sum_{i=1,i\neq j}^{n} \xi_i |p_{ij}(t)| + \sum_{i=1}^{n} \xi_i \frac{e^{pt} q_{ij} (\tilde{\sigma}_{ij}(t))}{1 - \sigma_{ij}(\tilde{\sigma}_{ij}(t))} |\eta_j(t) - \eta_j^*(t)|,
\]
which together with (H4) gives
\[
\frac{dV(t)}{dt} \leq 0, \text{ for a.e. } t \geq 0. \tag{14}
\]

Note that, by (13), we have
\[
V(t) \geq \sum_{i=1}^{n} e^{pt} \xi_i |u_i(t) - u_i^*(t)| + \sum_{j=1}^{m} e^{pt} \xi_j |v_j(t) - v_j^*(t)|
\]
Thus, by (14), we obtain
\[
\sum_{i=1}^{n} |u_i(t) - u_i^*(t)| + \sum_{j=1}^{m} |v_j(t) - v_j^*(t)| \\
\leq \frac{V(t)}{\min_{1 \leq i \leq n, 1 \leq j \leq m} \{\xi_i, \xi_j\}} \cdot e^{-pt} \leq \frac{V(0)}{\min_{1 \leq i \leq n, 1 \leq j \leq m} \{\xi_i, \xi_j\}} \cdot e^{-pt}.
\]
Since \(V(0)\) is a constant, by Definition 2.3, we can conclude that the solution \((u^*(t), v^*(t))^\top\) of system (1) is globally exponentially stable. Consequently, the proof is complete.

The following theorem is provided to guarantee the existence of anti-periodic solution of system (1).

**Theorem 3.3** Suppose that the assumptions (H1)-(H3) and (H4) hold, then system (1) admits an \(\omega\)-anti-periodic solution which is globally exponentially stable.

**Proof.** From system (1) and (H3), for any \(h \in N\), except a finite number of jump discontinuous points, we have
\[
\frac{d}{dt} \left[(-1)^{h+1} u_i(t + (h + 1)\omega) \right] = (-1)^{h+1} \frac{du_i(t + (h + 1)\omega)}{dt} \\
= (-1)^{h+1} \left\{ -a_i u_i(t + (h + 1)\omega) + \sum_{j=1}^{m} c_{ij}(t + (h + 1)\omega) f_j(v_j(t + (h + 1)\omega)) \\
+ \sum_{j=1}^{m} d_{ij}(t + (h + 1)\omega) f_j(v_j(t + (h + 1)\omega) - \tau_{ij}(t + (h + 1)\omega))) + I_i(t + (h + 1)\omega)) \right\} \\
= -a_i(-1)^{h+1} u_i(t + (h + 1)\omega) + \sum_{j=1}^{m} c_{ij}(t) f_j((-1)^{h+1} v_j(t + (h + 1)\omega)) \\
+ \sum_{j=1}^{m} d_{ij}(t) f_j((-1)^{h+1} v_j(t + (h + 1)\omega) - \tau_{ij}(t))) + I_i(t), \ i = 1, 2, ..., n.
\]
Similarly, we also have
\[
\frac{d}{dt} [(-1)^{h+1} v_j(t + (h + 1)\omega)] = (-1)^{h+1} \frac{dv_j(t + (h + 1)\omega)}{dt} \\
= -b_j(-1)^{h+1} v_j(t + (h + 1)\omega) + \sum_{i=1}^{n} p_{ji}(t) g_i((-1)^{h+1} u_i(t + (h + 1)\omega)) \\
+ \sum_{i=1}^{n} q_{ji}(t) g_i((-1)^{h+1} u_i(t + (h + 1)\omega) - \sigma_{ij}(t)) + J_j(t), \quad j = 1, 2, ..., m.
\]

Thus, for any natural number \( h \), we can see that \((u_i(t + (h + 1)\omega), v_j(t + (h + 1)\omega))\) are the solutions of system (1). Then, by Theorem 3.2, there exist constants \( N > 1 \) and \( \rho > 0 \), such that
\[
|(-1)^{h+1} u_i(t + (h + 1)\omega) - (-1)^h u_i(t + h\omega)| \\
\leq Ne^{-\rho(t+h\omega)} \sup_{s \in [-\varsigma, 0]} \sum_{i=1}^{n} |u_i(s + \omega) + u_i(s)| \\
\leq Ne^{-\rho(t+h\omega)}, \quad \forall t + h\omega > 0, \quad i = 1, 2, ..., n.
\]

Thus, for any natural number \( k \), we obtain
\[
(-1)^{k+1} u_i(t + (k + 1)\omega) = u_i(t) + \sum_{l=0}^{k} [(-1)^{l+1} u_i(t + (l + 1)\omega) - (-1)^l u_i(t + l\omega)].
\]

Then,
\[
|(-1)^{k+1} u_i(t + (k + 1)\omega)| \leq |u_i(t)| + \sum_{l=0}^{k} |(-1)^{l+1} u_i(t + (l + 1)\omega) - (-1)^l u_i(t + l\omega)|.
\]

From (15), we can choose a sufficiently large constant \( N^0 > 0 \) and a positive constant \( \lambda \) such that
\[
|(-1)^{l+1} u_i(t + (l + 1)\omega) - (-1)^l u_i(t + l\omega)| \leq \lambda(e^{-\rho\omega})^l, \quad l > N^0
\]
on any compact set of \( \mathbb{R} \). It follows from (17) and (18) that \((-1)^{k+1} u_i(t + (k + 1)\omega)\) uniformly converges to a continuous function \( u^*_i \) on any compact set of \( \mathbb{R} \). In a similar way, we can get that \((-1)^{k+1} v_j(t + (k + 1)\omega)\) uniformly converges to a continuous function \( v^*_j \) on any compact set of \( \mathbb{R} \).

Now, we will show that \((u^*_i(t), v^*_j(t))^\top\) is \( \omega \)-anti-periodic solution of system (1). Firstly, it is easy to see that \( u^*_i(t) \) and \( v^*_j(t) \) are \( \omega \)-anti-periodic, since
\[
\begin{align*}
\lim_{k \to +\infty} (-1)^k u_i(t + \omega + k\omega) \\
= - \lim_{(k+1) \to +\infty} (-1)^{k+1} u_i(t + (k + 1)\omega) \\
= - u^*_i(t), \quad i = 1, 2, ..., n.
\end{align*}
\]
and
\[
\begin{align*}
\lim_{k \to +\infty} (-1)^k v_j(t + \omega + k\omega) \\
= - \lim_{(k+1) \to +\infty} (-1)^{k+1} v_j(t + (k + 1)\omega) \\
= - v^*_j(t), \quad j = 1, 2, ..., m.
\end{align*}
\]
Next, we prove that \((u^*(t), v^*(t))^\top\) is a solution of system (1). In fact, together with the continuity of the right side of system (1) and (14) implies that \{\([-1]^{k+1}u_j(t + (k + 1)\omega)]\} and \{\([-1]^{k+1}v_j(t + (k + 1)\omega)]\} uniformly converge to continuous functions on any compact set of \(\mathbb{R}\). Thus, letting \(k \to +\infty\), we obtain
\[
\begin{align*}
\frac{du^*_j(t)}{dt} &= -a_{ij}u^*_j(t) + \sum_{j=1}^m c_{ij}(t) f_j(v^*_j(t)) + \sum_{j=1}^m d_{ij}(t) f_j(v^*_j(t - \tau_{ij}(t))) + I_i(t), \\
\frac{dv^*_j(t)}{dt} &= -b_jv^*_j(t) + \sum_{i=1}^n p_{ji}(t) g_i(u^*_i(t)) + \sum_{i=1}^n q_{ji}(t) g_i(u^*_i(t - \sigma_{ji}(t))) + J_j(t),
\end{align*}
\]
which implies that \((u^*(t), v^*(t))^\top\) is a solution of system (1). Therefore, the proof is now complete.

**Corollary 3.4** Suppose that the assumptions \((H1)\) and \((H2^*)\) hold, then for any solution \((u(t), v(t))^\top\) of system (1), there exist two positive constants
\[
M := \sqrt{2} \left( \|\varphi\|_2^2 + \|\psi\|_2^2 \right) + \frac{\sqrt{2}}{\alpha} \max_{1 \leq i \leq n} \sum_{j=1}^m \left( (c_{ij}^M + d_{ij}^M)\gamma_{ij}^M + I_i^M \right)
\]
\[
+ \frac{\sqrt{2}}{\alpha} \max_{1 \leq j \leq m} \sum_{i=1}^n \left( (p_{ji}^M + q_{ji}^M)\eta_{ji}^M + J_j^M \right)
\]
and \(\tilde{M}\) such that
\[
|u_i(t)| \leq M, i = 1, 2, ..., n, \quad |v_j(t)| \leq M, j = 1, 2, ..., m, \quad \text{for } t \in [-\varsigma, +\infty),
\]
\[
|
\eta_i(t)| \leq \tilde{M}, i = 1, 2, ..., n, \quad |\gamma_j(t)| \leq \tilde{M}, j = 1, 2, ..., m, \quad \text{for a.e. } t \in [-\varsigma, +\infty).
\]

**Proof.** The proof is similar to that of Theorem 3.1, we omit it here.

**Corollary 3.5** Suppose that the assumptions \((H1)\), \((H2^*)\) and \((H4)\) hold, then the anti-periodic solution \((u^*(t), v^*(t))^\top\) of the system (1) associated with an output \((\eta^*(t), \gamma^*(t))^\top\) is globally exponentially stable.

**Proof.** The proof is similar to that of Theorem 3.2, we omit it here.

**Corollary 3.6** Suppose that the assumptions \((H1)\), \((H2^*)\), \((H3)\) and \((H4)\) hold, then model (1) admits an \(\omega\)-anti-periodic solution.

**Proof.** The proof is similar to that of Theorem 3.3, we omit it here.

4. Numerical examples and simulations

In this section, two numerical examples will be given showing the effectiveness of the theoretical results given in the previous sections.

**Example 4.1** Consider the following discontinuous BAM neural networks with time-varying delays:
\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -a_{11}x_1(t) + c_{11}(t)f_1(v_1(t)) + c_{12}(t)f_2(v_2(t)) + d_{11}(t)f_1(v_1(t - \tau_{11}(t))) + d_{12}(t)f_2(v_2(t - \tau_{12}(t))) + I_1(t), \\
\frac{dx_2(t)}{dt} &= -a_{21}x_2(t) + c_{21}(t)f_1(v_1(t)) + c_{22}(t)f_2(v_2(t)) + d_{21}(t)f_1(v_1(t - \tau_{21}(t))) + d_{22}(t)f_2(v_2(t - \tau_{22}(t))) + I_2(t), \\
\frac{dv_1(t)}{dt} &= -b_1v_1(t) + p_{11}(t)g_1(u_1(t)) + p_{12}(t)g_2(u_2(t)) + q_{11}(t)g_1(u_1(t - \sigma_{11}(t))) + q_{12}(t)g_2(u_2(t - \sigma_{12}(t))) + J_1(t), \\
\frac{dv_2(t)}{dt} &= -b_{21}v_2(t) + p_{21}(t)g_1(u_1(t)) + p_{22}(t)g_2(u_2(t)) + q_{21}(t)g_1(u_1(t - \sigma_{11}(t))) + q_{22}(t)g_2(u_2(t - \sigma_{12}(t))) + J_2(t),
\end{align*}
\]
where \( i = 1, j = 1, \tau_{ij}(t) = 1, \sigma_{ij}(t) = 1 \) and

\[
a_1 = 0.8, \ a_2 = 0.5, \ I_1(t) = \sin(t), \ I_2(t) = \cos(t + \frac{1}{2}), \]
\[
c_{11}(t) = -0.4|\cos(t)|, \ c_{12}(t) = 0.06, \ c_{21}(t) = 0.05, \ c_{22}(t) = -0.2|\sin(t)|, \]
\[
d_{11}(t) = 0.01, \ d_{12}(t) = 0.6, \ d_{21}(t) = 0.02, \ d_{22}(t) = 0.002, \]

and

\[
b_1 = 0.6, \ b_2 = 0.7, \ J_1(t) = \cos(t), \ J_2(t) = \sin(t + \frac{1}{3}), \]
\[
p_{11}(t) = -0.3|\cos(t)|, \ p_{12}(t) = 0.05, \ p_{21}(t) = 0.01, \ p_{22}(t) = -0.5|\sin(t)|, \]
\[
q_{11}(t) = 0, \ q_{12}(t) = 0, \ q_{21}(t) = 0, \ q_{22}(t) = 0. \]

Then, for \( i, j = 1 \), we can have \( \varsigma = \max\{\tau, \delta\} = 1 \). Moreover, let

\[
f_1(x) = f_2(x) = \begin{cases} \frac{0.2x}{x^2 + 1}, & -1 < x < 1; \\ \frac{0.2x - 1}{x^2 + 1}, & x > 1, \end{cases} \quad g_1(x) = g_2(x) = \begin{cases} \arctan(x) + 1, & x < 1; \\ \arctan(x) - 1, & x > 1. \end{cases}
\]

It is easy to see that the activation function \( f_i(x) \) and \( g_j(x) \) are discontinuous, bounded, monotonically non-decreasing. This fact can be seen in Figure 1.

![Figure 1](image-url)

**Figure 1.** (a) Discontinuous activation functions \( f_i(i = 1, 2) \) for system (19); (b) Discontinuous activation functions \( g_j(j = 1, 2) \) for system (19).

Let \( \xi_1 = 5, \xi_2 = 19 \) and \( \rho = 0.1 \), we can have \( a_i^L \geq \rho, b_j^L \geq \rho \) and

\[
\lim_{t \to +\infty} \sup \left\{ \xi_1 c_{11}(t) + \xi_2 c_{21}(t) + \xi_1 e^{\rho t} |d_{11}(\varphi_{11}^{-1}(t))| \frac{e^{\rho t}}{1 - \tau_{11}(\varphi_{11}(t))} + \xi_2 e^{\rho t} |d_{21}(\varphi_{21}^{-1}(t))| \frac{e^{\rho t}}{1 - \tau_{21}(\varphi_{21}(t))} \right\} \\
\approx -0.5091 < 0,
\]
\[
\lim_{t \to +\infty} \sup \left\{ \xi_2 c_{22}(t) + \xi_1 c_{12}(t) + \xi_1 e^{\rho t} |d_{12}(\varphi_{12}^{-1}(t))| \frac{e^{\rho t}}{1 - \tau_{12}(\varphi_{12}(t))} + \xi_2 e^{\rho t} |d_{22}(\varphi_{22}^{-1}(t))| \frac{e^{\rho t}}{1 - \tau_{22}(\varphi_{22}(t))} \right\} \\
\approx -0.2545 < 0,
\]
and
\[
\limsup_{t \to +\infty} \left\{ \xi_1 p_{11}(t) + \xi_2 |p_{21}(t)| + \frac{\xi_1 e^{\rho \sigma} |q_{11}(\tilde{\varphi}_{11}^{-1}(t))|}{1 - \sigma'_{11}(\tilde{\varphi}_{11}^{-1}(t))} + \frac{\xi_2 e^{\rho \sigma} |q_{21}(\tilde{\varphi}_{21}^{-1}(t))|}{1 - \sigma'_{21}(\tilde{\varphi}_{21}^{-1}(t))} \right\} \\
\approx -1.2553 < 0,
\]
\[
\limsup_{t \to +\infty} \left\{ \xi_2 p_{22}(t) + \xi_1 |p_{12}(t)| + \frac{\xi_1 e^{\rho \sigma} |q_{12}(\tilde{\varphi}_{12}^{-1}(t))|}{1 - \sigma'_{12}(\tilde{\varphi}_{12}^{-1}(t))} + \frac{\xi_2 e^{\rho \sigma} |q_{22}(\tilde{\varphi}_{22}^{-1}(t))|}{1 - \sigma'_{22}(\tilde{\varphi}_{22}^{-1}(t))} \right\} \\
\approx -8.6435 < 0.
\]

As a result, the coefficients of system (19) satisfy all the conditions in Theorem 3.2–3.3, thus we can conclude that the system (19) possesses a unique \( \pi \)-antiperiodic solution which is globally exponentially stable. This fact can be presented in the following Figures 2, 3 and 4.

**Figure 2.** (a) Time-domain behavior of the state variables \( u_1(t) \), \( u_2(t) \), \( v_1(t) \) and \( v_2(t) \) for system (19) with random initial conditions; (b) Phase plane behavior of the state variables \( (u_1(t), u_2(t)) \) and \( (v_1(t), v_2(t)) \) for system (19).

**Figure 3.** (a) Three-dimensional trajectory of state variables \( u_1(t) \), \( u_2(t) \) and \( v_1(t) \) for system (19); (b) Three-dimensional trajectory of state variables \( u_1(t) \), \( u_2(t) \) and \( v_2(t) \) for system (19).
Then, for $i,j$, we can have $\varsigma = \max\{\tau, \delta\} = 1$. Moreover, let

$$f_1(x) = f_2(x) = \begin{cases} 
-\frac{0.2x}{x^2 + 1}, & -1 < x < 1; \\
\frac{0.2x}{x^2 + 1}, & x > 1.
\end{cases}$$

$$g_1(x) = g_2(x) = \begin{cases} 
-\arctan(x) + 1, & x < 1; \\
\arctan(x) - 1, & x > 1.
\end{cases}$$

It is easy to see that the activation function $f_i(x)$ and $g_j(x)$ are discontinuous, bounded, monotonically non-increasing. This fact can be seen in Figure 5.

Let $\xi_1 = 6$, $\xi_2 = 16$ and $\rho = 0.2$, we can have $a^i_t \geq \rho$, $b^j_t \geq \rho$ and
we can conclude that the system (20) possesses a unique \( \pi \)-antiperiodic solution which is globally exponentially stable. This fact can be presented in the following Figures 6, 7 and 8.

As a result, the coefficients of system (20) satisfy all the conditions in Corollary 3.5–3.6, thus we can conclude that the system (20) possesses a unique \( \pi \)-antiperiodic solution which is globally exponentially stable. This fact can be presented in the following Figures 6, 7 and 8.

Figure 5. (a) Discontinuous activation functions \( f_i (i = 1,2) \) for system (19); (b) Discontinuous activation functions \( g_j (j = 1,2) \) for system (19).

\[
\limsup_{t \to +\infty} \left\{ \xi_1 c_{11}(t) + \xi_2 c_{21}(t) + \xi_1 \frac{e^{\rho t}|d_{11}(\varphi_{11}^{-1}(t))|}{1 - \tau_{11}'(\varphi_{11}^{-1}(t))} + \xi_2 \frac{e^{\rho t}|d_{21}(\varphi_{21}^{-1}(t))|}{1 - \tau_{21}'(\varphi_{21}^{-1}(t))} \right\}
\approx -0.9259 < 0,
\]

\[
\limsup_{t \to +\infty} \left\{ \xi_1 c_{12}(t) + \xi_1 c_{12}(t) + \xi_1 \frac{e^{\rho t}|d_{12}(\varphi_{12}^{-1}(t))|}{1 - \tau_{12}'(\varphi_{12}^{-1}(t))} + \xi_2 \frac{e^{\rho t}|d_{22}(\varphi_{22}^{-1}(t))|}{1 - \tau_{22}'(\varphi_{22}^{-1}(t))} \right\}
\approx -0.7557 < 0,
\]

and

\[
\limsup_{t \to +\infty} \left\{ \xi_1 p_{11}(t) + \xi_2 p_{21}(t) + \xi_1 \frac{e^{\rho t}|q_{11}(\varphi_{11}^{-1}(t))|}{1 - \sigma_{11}'(\varphi_{11}^{-1}(t))} + \xi_2 \frac{e^{\rho t}|q_{21}(\varphi_{21}^{-1}(t))|}{1 - \sigma_{21}'(\varphi_{21}^{-1}(t))} \right\}
\approx -2.1836 < 0,
\]

\[
\limsup_{t \to +\infty} \left\{ \xi_2 p_{22}(t) + \xi_1 p_{12}(t) + \xi_1 \frac{e^{\rho t}|q_{12}(\varphi_{12}^{-1}(t))|}{1 - \sigma_{12}'(\varphi_{12}^{-1}(t))} + \xi_2 \frac{e^{\rho t}|q_{22}(\varphi_{22}^{-1}(t))|}{1 - \sigma_{22}'(\varphi_{22}^{-1}(t))} \right\}
\approx -0.9832 < 0.
\]

Figure 6. (a) Time-domain behavior of the state variables \( u_1(t), u_2(t), v_1(t) \) and \( v_2(t) \) for system (20) with random initial conditions; (b) Phase plane behavior of the state variables \((u_1(t), u_2(t))\) and \((v_1(t), v_2(t))\) for system (20).
Remark 4.3 Since the existence and globally exponential stability of the anti-periodic solutions of the discontinuous BAM neural networks with time-varying delays has not been studied before, it is clearly to see that all results obtained in [2], [3], [14], [16], [17], [18], [20], [21], [25], [26], [29], [30], [31] and references therein are invalid for Example 4.1 and 4.2. These imply that the results established in the present paper are essentially new. Here a novel proof is employed to establish some criteria which guarantee the existence and globally exponential stability of the anti-periodic solutions of the discontinuous BAM neural networks with time-varying delays.

5. Conclusion
This paper presents a class of discontinuous BAM neural networks with time-varying delays. Under the framework of the Filippov solution, by applying differential inclusions theory, fundamental solution matrix of coefficients, inequality technique and the non-smooth analysis theory with Lyapunov-like approach, we employ a novel argument and the easily verifiable sufficient conditions have been provided to determine the existence and global exponential stability of the anti-periodic solutions for the addressed BAM neural networks. In addition, two typical numerical examples and the corresponding simulations have been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. It should be pointed out that it is the first time to investigate the anti-periodic dynamic behavior for the discontinuous BAM neural networks with time-varying delays. Consequently, this paper shows theoretically and numerically that some related references known in the literature can be enriched and complemented.

Moreover, little attention has been devoted to the study of the anti-periodic dynamic behavior of the neural networks with discontinuous activations so far. This method affords a
possible method to analyse the existence and globally exponential stability of the anti-periodic solutions problem of other delayed BAM neural networks with discontinuous activations, such as CGBAM neural networks with discontinuous activations, Neutral BAM neural networks with discontinuous activations, Neutral CGBAM neural networks with discontinuous activations and so on. These issues will be the topic of our future research.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors read and approved the manuscript.

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