A GEOMETRIC INVARIANT THEORY COMPACTIFICATION OF 
\( M_{g,n} \) VIA THE FULTON-MACPHERSON CONFIGURATION 
SPACE

RAHUL PANDHARIPANDE

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0. Introduction

In [F-M], a compactification of the configuration space of \( n \) marked points on an algebraic variety is defined. For a nonsingular curve \( X \) of genus \( g \geq 2 \), the Fulton-MacPherson configuration space of \( X \) (quotiented by the automorphism group of \( X \)) is isomorphic to the (reduced) fiber of \( \gamma : \overline{M}_{g,n} \rightarrow \overline{M}_g \) over \([X] \in M_g\). Since the Fulton-MacPherson configuration space is defined for singular varieties, it is natural to ask whether a compactification of \( \gamma^{-1}(M_g) \) can be obtained over \( \overline{M}_g \). First, we consider the Fulton-MacPherson configuration space for families of varieties. This relative construction is then applied to the universal curve over the Hilbert scheme of 10-canonical, genus \( g \geq 2 \) curves. Following results of Gieseker, it is shown there exist linearizations of the natural \( SL \)-action on the relative configuration space of the universal curve that yield G.I.T. quotients compactifying \( \gamma^{-1}(M_g) \). These new compactifications, \( M'_{g,n} \), are described. For \( n = 1 \), \( M'_{g,1} \) and the Deligne-Mumford compactification \( \overline{M}_{g,1} \) coincide. For \( n = 2 \), \( M'_{g,2} \) and \( \overline{M}_{g,2} \) are isomorphic on open sets with codimension 2 complements. \( M'_{g,2} \) and \( \overline{M}_{g,2} \) differ essentially by the birational modification corresponding to the two minimal resolutions of an ordinary threefold double point. For higher \( n \), the compactifications \( M'_{g,n} \) and \( \overline{M}_{g,n} \) differ more substantially.

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1. Relative Fulton-MacPherson Configuration Spaces

1.1. Terminology. Let \( \mathbb{C} \) be the ground field of complex numbers. A morphism \( \mu : X \rightarrow Y \) is an immersion if \( \mu \) is an isomorphism of \( X \) onto an open subset of a

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closed subvariety of $Y$. A morphism $\gamma$ is \textit{quasi-projective} if it factors as $\gamma = \rho \circ \mu$ where $\mu$ is an open immersion and $\rho$ is projective. The only smooth morphisms considered will be smooth morphisms of relative dimension $k$ between nonsingular varieties.

1.2. Definition. We carry out the construction of Fulton and MacPherson in the relative context. Suppose $\pi : F \to B$ is a (separated) morphism of algebraic varieties. Let $n$ be a positive integer. $N = \{1, \ldots, n\}$. Wherever possible, products will be taken in the category of varieties over $B$. Define:

$$F^n_B = \prod_N F = F \times_B F \times_B \ldots \times_B F.$$  

And define:

$$(F^n_B)_0 = F^n_B \setminus \bigcup \triangle_{\{a,b\}}$$

Where $\triangle_{\{a,b\}}$ denotes the large diagonal corresponding to the indices $a, b \in N$ and the union is over all pairs $\{a, b\}$ of distinct elements of $N$. For each subset $S$ of $N$ define $F^n_S = \prod_S F$. Following the notation of [F-M], let $Bl_\triangle(F^n_S)$ denote the blow-up of $F^n_S$ along the small diagonal. There exists a natural immersion:

$$(F^n_B)_0 \subset F^n_B \times \prod_{|S| \geq 2} Bl_\triangle(F^n_S) .$$

The relative Fulton-MacPherson configuration space of $n$ marked points of $F$ over $B$, $F_B[n]$, is defined to be the closure of $(F^n_B)_0$ in the above product. When $B$ is a point, this definition coincides with that of [F-M]. Consider the composition:

$$(F_B[n] \subset F^n_B \times \prod_{|S| \geq 2} Bl_\triangle(F^n_S) \xrightarrow{\mu} F^n_B \times \prod_{|S| \geq 2} (F^n_S) \xrightarrow{\beta} F^n_B$$

where $\mu$ is the natural blow-down morphism and $\beta$ is the projection on the first factor. Since $\mu$ is a projective morphism and $F_B[n]$ is a closed subvariety,

$$\mu : F_B[n] \to \mu(F_B[n])$$

is also projective. Since $\beta : \mu(F_B[n]) \xrightarrow{\beta} F^n_B$ is an isomorphism, the morphism $\rho = \beta \circ \mu$

$$\rho : F_B[n] \to F^n_B$$

is projective. For our purposes, we shall only consider the case where $\pi : F \to B$ is a quasi-projective morphism. Also, we will be mainly interested in the case where $F$ and $F^n_B$ are irreducible varieties.
1.3. The Blow-Up Construction. Consider again the birational projective morphism
\[ \rho : F_B[n] \to F_B^N \]
It is natural to inquire whether \( \rho \) can be expressed as a composition of explicit blow-ups along canonical subvarieties. In [F-M], such a blow-up construction is given for the configuration space in case \( B \) is a point. The blow-ups in [F-M] are canonical in the following sense: if \( Y \to X \) is an immersion, the sequence of blow-ups resolving \( Y[n] \to Y^N \) is the sequence of strict transformations of \( Y^N \) under the blow-ups resolving \( X[n] \to X^N \).

The blow-up construction of Fulton and MacPherson is valid in the relative context. We now assume that \( \pi : F \to B \) is a quasi-projective morphism. In this case, there exists a factorization:
\[
\begin{array}{c}
F & \xrightarrow{i} & \mathbb{P}^r \times B \\
\downarrow \pi & & \downarrow \\
B & \xrightarrow{\text{id}} & B
\end{array}
\]
where \( i \) is an immersion. We use the notation \( \mathbb{P}^r \times B = \mathbb{P}^r_B \) and drop extra \( B \) subscripts when the meaning is clear. For example, \( (\mathbb{P}^r_B)^N \) instead of \( (\mathbb{P}^r_B)_B^N \).

We have the following commutative diagram:
\[
\begin{array}{c}
(F_B^N)_0 & \xrightarrow{i^N} & F_B^N \times \prod_{|S| \geq 2} \text{Bl}_\Delta(F_B^S) \\
\downarrow i^N & & \downarrow i^{Bl} \\
(P_B^r)_0^N & \xrightarrow{\text{id}} & (\mathbb{P}^r_B)^N \times \prod_{|S| \geq 2} \text{Bl}_\Delta((\mathbb{P}^r_B)^S)
\end{array}
\]
where \( i^N, i^{Bl} \) are immersions. We conclude from diagram (3) that \( F_B[n] \) is immersed in \( \mathbb{P}^r_B[n] \). Hence:
\[
\begin{array}{c}
F_B[n] & \xrightarrow{j} & \mathbb{P}^r_B[n] \\
\downarrow \rho & & \downarrow \eta \\
F_B^N & \xrightarrow{i^N} & (\mathbb{P}^r_B)^N
\end{array}
\]
where \( i^N, j \) are immersions. Since \( \rho \) is a projective morphism, \( j(F_B[n]) \) is closed in \( \eta^{-1}(i^N(F_B^N)) \). \( F_B[n] \) is therefore the strict transformation of \( F_B^N \) under \( \eta \). It is clear the following diagram holds:
\[
\begin{array}{c}
\mathbb{P}^r_B[n] & \xrightarrow{\text{id}} & \mathbb{P}^r[n] \times B \\
\downarrow \eta & & \downarrow \gamma \times \text{id} \\
(\mathbb{P}^r_B)^N & \xrightarrow{\text{id}} & (\mathbb{P}^r)^N \times B
\end{array}
\]
In [F-M], an explicit and canonical blow-up construction of \( \gamma \) is given. By extending each exceptional locus over the base \( B \), a blow-up construction of \( \eta \) is obtained.
see from diagram (4) that a blow-up construction of $F_B[n]$ exists by taking the strict transformation of $F_B^N$ at each blow-up of $(P_B)^N$.

1.4. Comparing $F_b[n]$ and $F_B[n]_b$. For a given $b \in B$ let $F_b$ denote the fiber of $\pi$ over $b$. From equation (1) and the definitions, it is clear there exists a natural closed immersion:

$$F_b[n] \hookrightarrow F_B[n]_b.$$ 

It is possible for $i_b$ to be a proper inclusion. Examples of this behavior will be seen in section (3). We have the following:

**Proposition 1.** If $B$ is irreducible and $\pi : F \to B$ is a smooth, quasi-projective morphism of nonsingular varieties, then for every $b \in B$, $i_b$ is an isomorphism.

*Proof.* Suppose $X$ is a fixed nonsingular algebraic variety. In [F-M], the canonical construction of $X[n]$ is given by a sequence of explicit blow-ups of $X^N$ along nonsingular centers. In the previous section, it was shown how the construction of [F-M] could be lifted to the relative context. Let $m$ be the number of blow-ups needed in the Fulton- MacPherson construction resolving $\rho : F_B[n] \to F_B^N$. Let $F_{B,j}^N$ for $0 \leq j \leq m$ denote the $j^{th}$ stage. $F_{B,0}^N = F_B^N$ and $F_{B,m}^N = F_B[n]$. Since the blow-up construction in [F-M] is canonical, for any variety $X$ similar definitions can be made. We show inductively, for each $b \in B$, the natural inclusion:

$$F_{B,j}^N \hookrightarrow (F_{B,j}^N)_b$$

is an isomorphism. For $j = 0$ the assertion is clear. The induction step rests on a simple Claim:

Suppose $S$ is an irreducible nonsingular variety, $R \to S$ is a smooth morphism, and $T \to R$ is a closed immersion smooth over $S$. Then, for any $s \in S$, the blow-up of $R_s$ along $T_s$ is naturally isomorphic to the fiber over $s$ of the blow-up of $R$ along $T$. Since all spaces are nonsingular, the assertion follows from examining normal directions of $T$ in $R$; the various smoothnesses imply all normal directions are represented in the fiber.

Assume equation (5) is an isomorphism for all $b \in B$. Since $\pi$ is smooth of relative dimension $k$, $F_b$ and $F_{B,j}^N$ are nonsingular of pure dimensions $k$, $nk$. Hence, $(F_{B,j}^N)_b$ is nonsingular of pure dimension $nk$. Also, every irreducible component of $F_B^N$ (and hence $F_{B,j}^N$) is of relative dimension $nk$. The last two facts imply the morphism:

$$\pi_{j}^N : F_{B,j}^N \to B$$

is smooth. Examination of the $(j + 1)^{th}$ center is straightforward. Because of the assumed isomorphism (5) and the knowledge that the [F-M] construction of the configuration space of a nonsingular variety over a point only involves nonsingular centers, we see that the $(j + 1)^{th}$ center is smooth over $B$. The above claim now proves the induction step. □
1.5. Universal Families. Let $X$ be a nonsingular algebraic variety. Let $\mathbf{\pi} = (x_1, \ldots, x_n)$ be $n$ ordered points of $X$. A subset $S \subset N$ is said to be coincident at $z \in X$ if $|S| \geq 2$ and for all $i \in S$, $x_i = z$. Following [F-M], for every $S$ coincident at $z$, we define a screen of $S$ at $z$ to be an equivalence class of the data $(t_i)_{i \in S}$ where:

(1) $t_i \in T_z$, the tangent space of $X$ at $z$.
(2) $\exists i, j \in S$ such that $t_i \neq t_j$.

Two data sets $(t_i)_{i \in S}$ and $(t'_i)_{i \in S}$ are equivalent if there exists $\lambda \in C^*$ and $v \in T_z$ so that for all $i \in S$, $\lambda \cdot t_i + v = t'_i$. A screen shows the tangential separation of infinitely near points. An $n$-tuple $\mathbf{\pi} = (x_1, \ldots, x_n)$ together with a screen $Q_S$ for each coincident subset $S \subset N$ constitute an $n$-pointed stable class in $X$ if the screens are compatible in the following sense. Suppose $S_1 \subset S_2$ are two subsets coincident at $z$ where $Q_{S_2}$ is represented by the data $(t_j)_{j \in S_2}$. If there exist $k, \hat{k} \in S_1$ so that $t_k \neq t_{\hat{k}}$, then $(t_j)_{j \in S_1}$ defines a screen for $S_1$. The compatibility condition requires that when this restriction of $Q_{S_2}$ is defined, it equals $Q_{S_1}$. For a nonsingular space $X$, $X[n]$ is the parameter space of $n$-pointed stable classes in $X$. Given an $n$-pointed stable class in $X$, an $n$-pointed stable degeneration of $X$ can be constructed (up to isomorphism) as follows. Let $z \in X$ occur with multiplicity in $\mathbf{\pi}$. Blow-up $X$ at $z$ and attach a $\mathbf{P}(T_z \oplus 1)$ in the natural way along the exceptional divisor at $z$. Note that $\mathbf{P}(T_z \oplus 1)$ minus the hyperplane at infinity, $\mathbf{P}(T_z)$, is naturally isomorphic to the affine space $T_z$. Let $S_z \subset N$ be the maximal subset coincident at $z$. The screen $Q_{S_z}$ associates (up to equivalence) points of $T_z$ to the indices that lie in $S_z$. Condition (2) of the screen data implies some separation of the marked points has occurred. The further screens specify in a natural way (up to equivalence of screens) the further blow-ups and markings required to separate the marked points. The final space obtained along with $n$ distinct marked points is the $n$-pointed stable degeneration associated to the given $n$-pointed stable class. See [F-M] for further details.

It is shown in [F-M] that if $X$ is a nonsingular variety, there exists a universal family of $n$-pointed stable degenerations of $X$ over $X[n]$. Let $X[n]^+$ denote this universal family. $X[n]^+$ is equipped with the following maps:

$$
\begin{array}{ccc}
X[n]^+ & \xrightarrow{\mu} & X[n] \times X \\
\downarrow{\mu_p} & & \downarrow \\
X[n] & \xrightarrow{\sigma} & X[n]
\end{array}
$$

There are $n$ sections of $\mu_p$, $\{\sigma_i\}_{i \in N}$:

$$
X[n] \xrightarrow{\sigma} X[n]^+.
$$

For any $d \in X[n]$, the fiber $\mu_p^{-1}(d)$ along with the $n$-tuple $(\sigma_1(d), \ldots, \sigma_n(d))$ is the $n$-pointed stable degeneration of $X$ associated to the $n$-pointed stable class corresponding to $d$. 

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We note here that if $C$ is a nonsingular automorphism-free curve, $n$-pointed stable classes in $C$ correspond bijectively to isomorphism classes of $n$-pointed Deligne-Mumford stable curves over $C$. Moreover, the universal family over $C[n]$ defines a map to the reduced fiber

$$\phi : C[n] \to \gamma^{-1}([C])$$

where $\gamma : \overline{M}_{g,n} \to \overline{M}_g$. Since $\phi$ is proper bijective and both spaces are normal, $\phi$ is an isomorphism. If $C$ has a finite automorphism group, $A$, we see $A$ acts on $C[n]$ and $\phi$ is $A$-invariant. Therefore $\phi$ descends to the quotient

$$\phi(C[n]/A) \to \gamma^{-1}([C]).$$

It is not hard to see that this map is proper bijective and hence an isomorphism by normality.

The map $\mu$ is a birational morphism and can be expressed as an explicit sequence of blow-ups of $X[n] \times X$ along canonical, nonsingular loci. Canonical here has the same meaning as in section (1.3) : if $Y \to X$ is an immersion of nonsingular varieties, the blow-up sequence resolving $Y[n]^+ \to Y[n] \times Y$ is the strict transform of $Y[n] \times Y$ under the blow-up sequence resolving $X[n]^+ \to X[n] \times X$. Moreover, the sections of $Y[n]^+ \to Y[n]$ are restrictions of the sections of $X[n]^+ \to X[n]$. This canonical blow-up construction is given in [F-M].

1.6. Relative Universal Families. Suppose $\pi : F \to B$ is a smooth, quasi-projective morphism of nonsingular varieties. In this case, the construction of the universal family that appears in [F-M] can be lifted to the relative context. Using the notation of section (1.3), we have an immersion:

$$F_B[n] \times_B F \to \mathbf{P}^*_B[n] \times_B \mathbf{P}^r_B.$$

Consider the diagram:

$$
\begin{array}{ccc}
\mathbf{P}^*_B[n]^+ & \longrightarrow & \mathbf{P}^r[n]^+ \times B \\
\downarrow \omega & & \downarrow \mu \times id \\
\mathbf{P}^*_B[n] \times_B \mathbf{P}^r_B & & \mathbf{P}^r[n] \times \mathbf{P}^r \times B
\end{array}
$$

For $\omega = \mu \times id$, the Fulton-MacPherson construction of the universal family can be carried out uniformly over the base by extending the centers of the blow-ups resolving $\mu$ trivially over $B$. Define $F_B[n]^+$ to be the proper transform of $F_B[n] \times_B F$ under $\omega$. We have:

$$v : F_B[n]^+ \to F_B[n] \times_B F$$

To show the space defined above, $F_B[n]^+$, has the desired geometrical properties, we argue as in the proof of Proposition 1. Let $(F_B[n] \times_B F)_j$ denote the $j^{th}$ stage of the canonical sequence of blow-ups resolving $v$. Inductively, it is shown that for each $b \in B$ there is an isomorphism:

$$v : (F_B[n] \times_B F)_j \to (F_B[n] \times_B F)_{j,b}.$$
The $j = 0$ case is established by Proposition 1. The induction step follows from the claim made in the proof of Proposition 1 and the fact that for a nonsingular variety $X$, the canonical Fulton-MacPherson resolution of $X[n]^+ \to X[n] \times X$ involves only nonsingular centers.

We conclude that fiber $F_B[n]_b^+$ over $b \in B$ is naturally isomorphic to $F_b[n]^+$. It is clear that $n$ sections $\sigma_i$ exist for

$$\omega_b : P_B^n[n]^+ \to P_B^n[n].$$

For each $b \in B$, these sections $\sigma_i$ are compatible with the $n$ natural sections of $F_b[n]^+ \to F_b[n]$. Therefore, via restriction, the $\sigma_i$ yield $n$ sections of

$$v : F_B[n]^+ \to F_B[n].$$

The fiber of $F_B[n]_\xi^+$ over $\xi \in F_B[n]$ is a $n$-pointed stable degeneration of $F_{\pi(\xi)}$. We have:

**Proposition 2.** Suppose $B$ is irreducible and $\pi : F \to B$ is a smooth, quasi-projective morphism of nonsingular varieties, then $F_B[n]^+$ along with $v$ and $\{\sigma_i\}_{i \in N}$ is a universal family of $n$-pointed stable degenerations of $F_B$ over $F_B[n]$.

**1.7. Final Note.** Suppose $\pi : G \to B$ is a projective morphism, $G$ is nonsingular, irreducible, $B$ is nonsingular, $\pi$ is flat, and for every $b \in B$ the fiber $G_b$ is reduced. In this case, let $F \subset G$ be the open set where $\pi$ is smooth. Using flatness and a tangent space calculation, we see:

$$F = \{\xi \in G | \xi \text{ is a nonsingular point of } G_{\pi(\xi)}\}$$

and $\pi : F \to B$ is a smooth, surjective morphism of nonsingular varieties. We know the space $F_B[n]$ is equipped with a universal family $F_B[n]^+$ obtained from $F_B[n] \times_B F$ by a sequence of canonical blow-ups. The problem with this universal family is that its fibers over $F_B[n]$ are $n$-pointed stable degenerations of $F_B$ not $G_B$. This problem can easily be fixed. Note there is an open inclusion:

$$F_B[n] \times_B F \subset F_B[n] \times_B G.$$

It is the case that the centers of the blow-ups resolving

$$v : F_B[n]^+ \to F_B[n] \times_B F$$

are closed in $F_B[n] \times_B G$. Using the isomorphism (6) and the explicit description of the centers of blow-ups in [F-M], this closure is not hard to check. Hence, if the sequence of blow-ups is carried out over $F_B[n] \times_B G$ the desired family of $n$-pointed stable degenerations of $G_B$ is obtained over $F_B[n]$. An $n$-pointed stable degeneration of a fiber $G_b$ is as before with the additional condition that the marked points must lie over the smooth locus of $G_b$. 

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2. The Geometric Invariant Theory Set-Up

2.1. Notation. Let $\overline{M}_g$ denote the Deligne-Mumford compactification of the moduli space of nonsingular, genus $g$, projective curves, $M_g$. Let $\overline{M}_{g,n}$ denote the Deligne-Mumford compactification of the moduli space of genus $g$ curves with $n$ marked points. There exists a natural projective morphism

$$\gamma: \overline{M}_{g,n} \to \overline{M}_g.$$ 

All these spaces are normal.

2.2. Gieseker’s construction of $\overline{M}_g$. Fix an integer $g \geq 2$. Define:

$$d = 10 \cdot (2g - 2)$$
$$R = d - g.$$ 

Define the polynomial:

$$f(m) = d \cdot m - g + 1.$$ 

Note $f(m)$ is the Hilbert polynomial of a complete, genus $g$, 10-canonical curve in $\mathbb{P}^R$. Let $H_{f,R}$ denote the Hilbert scheme of the polynomial $f$ in $\mathbb{P}^R$. If $X$ is a closed subscheme of $\mathbb{P}^R$ with Hilbert polynomial $f$, we denote the point of $H_{f,R}$ corresponding to $X$ by $[X]$. It is well known that there exists an integer $\hat{m}$ such that, for any $m \geq \hat{m}$ and any closed subscheme $X \subset \mathbb{P}^R$ corresponding to a point $[X] \in H_{f,R},$

$$h^1(I_X(m), \mathbb{P}^R) = 0$$
$$h^0(\mathcal{O}_X(m), X) = f(m).$$

Therefore, for any $m \geq \hat{m}$, there is a natural map:

$$i_m: H_{f,R} \to P(\bigwedge^{f(m)} H^0(\mathcal{O}_{\mathbb{P}^R}(m), \mathbb{P}^R)^*)_m.$$ 

Where $i_m$ is defined for each $[X] \in H_{f,R}$ as follows: by (7), there is a natural surjection

$$H^0(\mathcal{O}_{\mathbb{P}^R}(m), \mathbb{P}^R) \to H^0(\mathcal{O}_X(m), X)$$

which yields, by (8), a surjection

$$\bigwedge^{f(m)} H^0(\mathcal{O}_{\mathbb{P}^R}(m), \mathbb{P}^R) \to \bigwedge^{f(m)} H^0(\mathcal{O}_X(m), X) \cong \mathbb{C}.$$ 

The last surjection (9) is an element of $P(\bigwedge^{f(m)} H^0(\mathcal{O}_{\mathbb{P}^R}(m), \mathbb{P}^R)^*)$. The map $i_m$ is now defined on sets. That $i_m$ is an algebraic morphism of schemes can be seen by constructing (9) uniformly over $H_{f,R}$ and using the universal property of $P(\bigwedge^{f(m)} H^0(\mathcal{O}_{\mathbb{P}^R}(m), \mathbb{P}^R)^*)$. In fact, it can be shown there exists an integer $\overline{m}$ such that for every $m \geq \overline{m}$, $i_m$ is a closed immersion.
From the universal property of the Hilbert scheme, we obtain a natural \(SL_{R+1}\)-action on \(H_{f,R}\). For each \(m \geq m\), the closed immersion \(i_m\) defines a linearization of the natural \(SL_{R+1}\)-action on \(H_{f,R}\). Define the following locus \(K_g \subset H_{f,R}\): \(X \in K_g\) if and only if \(X\) is a nondegenerate, 10-canonical, genus \(g\), Deligne-Mumford stable curve in \(P_R\). \(K_g\) is a quasi-projective, \(SL_{R+1}\)-invariant subset. In [G], Gieseker shows a linearization \(i_m\) can be chosen satisfying:

(i) The stable locus of the corresponding G.I.T. quotient contains \(K_g\).
(ii) \(K_g\) is closed in the semistable locus.

From (i), we see \(K_g/SL_{R+1}\) is a geometric quotient. By (ii), \(K_g/SL_{R+1}\) is a projective variety. Since \(K_g\) is a nonsingular variety ([G]), it follows that \(K_g/SL_{R+1}\) is normal. From the definition of \(K_g\), the universal family over \(H_{f,R}\) restricted to \(K_g\) is a family of Deligne-Mumford stable curves. Therefore there exists a natural map \(\mu : K_g \rightarrow M_g\). Since \(\mu\) is \(SL\)-invariant, \(\mu\) descends to a projective morphism from the quotient \(K_g/SL_{R+1}\) to \(M_g\). Since \(\mu\) is one to one and \(M_g\) is normal, \(\mu\) is an isomorphism. Note that since \(M_g\) is irreducible, \(K_g\) is also irreducible.

2.3. The Relative \(n\)-pointed Fulton-MacPherson Configuration Space of the Universal Curve. Let \(\pi : U_H \rightarrow H_{f,R}\) be the universal family over the Hilbert scheme defined in section (2.2) where \(\pi\) is a flat, projective morphism. Let \(K_g \subset H_{f,R}\) be defined as above. Let \(U_{K_g}\) be the restriction of \(U_H\) to \(K_g\). Following the notation of section (1.2), we define \(U_{K_g}[n]\) to be the relative Fulton- MacPherson space of \(n\)-marked points on \(U_{K_g}\) over \(K_g\). From section (1.3), we see the immersion \(\zeta\):

\[
\begin{array}{ccc}
U_{K_g} & \xrightarrow{\zeta} & \mathbb{P}^R \times H_{f,R} \\
\downarrow\pi & & \downarrow\rho \\
K_g & \longrightarrow & H_{f,R}
\end{array}
\]

yields another immersion \(\zeta[n]\):

\[
\begin{array}{ccc}
U_{K_g}[n] & \xrightarrow{\zeta[n]} & \mathbb{P}^R[n] \times H_{f,R} \\
\downarrow\pi[n] & & \downarrow\rho[n] \\
K_g & \longrightarrow & H_{f,R}
\end{array}
\]

There exists a natural \(SL_{R+1}\)-action on \(\mathbb{P}^R[n]\) and therefore on \(\mathbb{P}^R[n] \times H_{f,R}\). Since \(U_{K_g}\) is invariant under the natural \(SL_{R+1}\)-action, we see \(U_{K_g}[n]\) is also \(SL_{R+1}\)-invariant. Since \(\pi\) is projective, \(U_{K_g}[n] \subset \rho[n]^{-1}(K_g)\) is a closed subset. It follows
from (i) and (ii) of section (2.2) and Propositions (7.1.1) and (7.1.2) of [P] that there exist linearizations of the natural $SL_{R+1}$-action on $P^R[n] \times H_{f,R}$ satisfying:

- (i) $U_K[n]$ is contained in the stable locus of the corresponding G.I.T. quotient.
- (ii) $(\rho[n]^{-1}(K_g))^{SS}$ is closed in the semistable locus.

From (i), (ii), and the fact that $U_K[n]$ is closed in $\rho[n]^{-1}(K_g)$, we see that $U_K[n]/SL_{R+1}$ is a geometric quotient and a projective variety. Define:

$$M_{g,n}^c = U_K[n]/SL_{R+1}.$$ 

Note there is a natural projective morphism

$$\rho : M_{g,n}^c \to \overline{M}_g$$

descending from the $SL_{R+1}$-invariant maps:

$$U_K[n] \to K_g \to \overline{M}_g.$$ 

It follows easily that $M_{g,n}^c$ is a compactification of $\gamma^{-1}(M_g)$. To see this first make the definition:

$$K_g = \{ [X] \in H_{f,R} | X \text{ is a nondegenerate, 10-canonical, nonsingular, genus } g \text{ curve} \}.$$ 

$U_K[n]$ is a dense open $SL_{R+1}$-invariant subset of $U_K[n]$. Since the morphism $\pi : U_K \to K_g$ is smooth, we see from section (1.6) that there exists a universal family of Deligne-Mumford stable $n$-pointed genus $g$ curves over $U_K[n]$. This universal family yields a canonical morphism

$$\mu : U_K[n] \to \gamma^{-1}(M_g).$$

It is easily checked that $\mu$ is $SL_{R+1}$-invariant. Therefore, $\mu$ descends to the open set, $\rho^{-1}(M_g)$, of $M_{g,n}^c$. One sees

$$\mu_d : \rho^{-1}(M_g) \to \gamma^{-1}(M_g)$$

is a bijection by Proposition (1) and the fact that, for a smooth curve $C$,

$$(C[n]/\text{automorphisms}) \cong \gamma^{-1}([C]) \subset \gamma^{-1}(M_g).$$

(See section (1.5)). Since $\rho : \rho^{-1}(M_g) \to M_g$ is projective, $\gamma : \gamma^{-1}(M_g) \to M_g$ is separated, and $\rho = \gamma \circ \mu_d$, we conclude $\mu_d$ is projective. A bijective projective morphism onto a normal variety is an isomorphism. Since $\gamma^{-1}(M_g)$ is normal, $\mu_d$ is an isomorphism.
3. A Description Of \( \mathcal{M}^c_{g,n} \)

3.1. Let \( \pi : \mathcal{U}_{K_g} \to \mathcal{K}_g \) be as above. Following section (1.7), we define \( F \subset \mathcal{U}_{K_g} \) to be the locus where \( \pi \) is smooth. \( F_{K_g}[n] \subset \mathcal{U}_{K_g}[n] \) is an open \( SL \)-invariant subset. The points of \( F_{K_g}[n] \) parameterize \( n \)-pointed stable classes on the nonsingular locus of the fibers of \( \pi \). There exists a universal family over \( F_{K_g}[n] \) which defines an \( SL \)-invariant morphism:

\[
\mu : F_{K_g}[n] \to M_{s,g,n}.
\]

Where \( M_{s,g,n} \) parameterizes \( n \)-pointed, genus \( g \), Deligne-Mumford stable curves with marked points lying over nonsingular points of the contracted stable model. Let

\[
F_{K_g}[n]/SL_{R+1} = (M^c_{g,n})^s.
\]

\( SL \)-invariance implies \( \mu \) descends to:

\[
\mu_d : (M^c_{g,n})^s \to \overline{M}^c_{g,n}.
\]

From the arguments of section (1.5), we see \( \mu_d \) is bijective. From the valuative criterion, it follows \( \mu_d \) is proper. As before, by normality, it follows that \( \mu_d \) is an isomorphism.

3.2. Points Of \( M^c_{g,n} \) Over A Singular Point. From section (3.1), it is clear only the behavior of \( \mathcal{U}_{K_g}[n] \) over a singular point of \( \mathcal{U}_{K_g} \) remains to be investigated. Since this is a local question about the smooth deformation of a node, it suffices to investigate the family:

\[
G \longrightarrow \text{Spec}(C[x,y]) \times \text{Spec}(C[t])
\]

\[
\begin{array}{ccc}
\pi & \text{Spec}(C[t]) \\
\downarrow & \downarrow \\
\text{Spec}(C[t]) & \text{Spec}(C[t])
\end{array}
\]

Where \( G \) is defined by the equation \( xy - t \). In the Fulton-MacPherson configuration space \( \text{Spec}(C[x,y])[n] \), there is a closed subset \( T_n \) corresponding to the points lying over \((0,0)\). In the notation of section (1.3),

\[
T_n = \rho^{-1}((0,0),(0,0),\ldots,(0,0))
\]

Recall the notation of section (1.2). Let \( B = \text{Spec}(C[t]) \), \( B^* = \text{Spec}(C[t]) - (0) \), and \( G^* = \pi^{-1}(B^*) \). We want to investigate the subset \( W_n \subset (T_n,0) \) that lies in the closure of \( G^*_B[n] \) in \( \text{Spec}(C[x,y])[n] \times B \).

Suppose \( \kappa \) is a family in \( (G^*_B)^N \) where all the marked points specialize to the node \( \zeta \) of \( G_0 \). After a base change, \( t \to t' \), \( \kappa \) can be defined by \( n \) sections, \( (\kappa_1, \ldots, \kappa_n) \), of \( \pi \) in a neighborhood of \( 0 \in B \). The equation of \( G \) after base change is now \( G_r = xy - t' \). Let us take \( r = 2 \). The blow-up of \( G_2 \) at \( \zeta \) is nonsingular and is
defined in an open set by the equation \( ab - 1 \) in \( \text{Spec}(C[a, b]) \times \text{Spec}(C[t]) \). The blow-down morphism is defined by the equations:

\[
\begin{align*}
x &= at \\
y &= bt.
\end{align*}
\]

Now assume that the strict transforms of the sections, \((\kappa_1, \ldots, \kappa_n)\), meet the exceptional curve \((ab = 1, t = 0)\) in distinct points \(((a_1, a^{-1}_1), \ldots, (a_n, a^{-1}_n))\), \(\forall i \ a_i \neq 0\). Then it is clear that the \(n\)-pointed stable class in \( T_n \) that is the limit of \( \kappa \) is the class in the tangent space of \( C[x, y] \) at \((0, 0)\) defined by the pairs of vectors:

\[
((a_1, a^{-1}_1), \ldots, (a_n, a^{-1}_n))
\]

in the basis \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\).

We now define a map:

\[
\theta_n : (C^{*N})_0 \to T_n
\]

Where \( \theta_n((a_1, \ldots, a_n)) \) is the \(n\)-pointed stable class defined by the tangent vectors \(((a_1, a^{-1}_1), \ldots, (a_n, a^{-1}_n))\). The preceding paragraph shows that \( \text{Image}(\theta_n) \subset W_n \).

Suppose \( n \geq 3 \). Let \( \hat{a} = (a_1, \ldots, a_n) \) and \( \hat{b} = (b_1, \ldots, b_n) \) be distinct points of \((C^{*N})_0\). Then, \( \theta_n(\hat{a}) = \theta_n(\hat{b}) \) if and only if there exists a tangent vector \((v_1, v_2)\) and an element \( \lambda \in C^* \) such that:

\[
\forall i, \quad \lambda \cdot a_i + v_1 = b_i \quad \text{and} \quad \lambda \cdot a^{-1}_i + v_2 = b^{-1}_i.
\]

These equations imply

\[
\forall i, j, \quad \lambda \cdot (a_i - a_j) = (b_i - b_j)
\]

(11)

\[
\forall i, j, \quad \lambda \cdot (a^{-1}_i - a^{-1}_j) = (b^{-1}_i - b^{-1}_j)
\]

(12)

Dividing (11) by (12) yields \( a_i \cdot a_j = b_i \cdot b_j \). For \( n \geq 3 \), we easily obtain \( \hat{a} = \pm \hat{b} \). Therefore, a component of \( W_n \) can be viewed as a compactification of \((C^{*N})_0/(\pm)\).

We note that the dimension of \( W_n \) is \( n \) for \( n \geq 3 \).

4. Comparison with \( \overline{M}_{g,n} \) for \( n = 1, 2 \)

4.1. \( n = 1 \). From the definitions, \( M_{g,1}^c \) equals \( U_{\overline{K}_g}/SL_{R+1} \). \( \pi*(U_{\overline{K}_g}) \) is a family of 1-pointed Deligne-Mumford genus \( g \) curves over \( U_{\overline{K}_g} \) via the natural diagonal section. This tautological family yields an \( SL \)-invariant morphism:

\[
\mu : U_{\overline{K}_g} \to \overline{M}_{g,1}
\]
that descends to
\[ \mu_d : M_{g,1}^c \to \overline{M}_{g,1}. \]
Since \( \mu_d \) is proper bijective and \( \overline{M}_{g,1} \) is normal, \( \mu_d \) is an isomorphism.

4.2. \( n = 2 \). Consider the family:
\[ U^2_{K_g} = U_{K_g} \times_{K_g} U_{K_g}. \]

The singular locus of \( U^2_{K_g} \), \( S \), is nonsingular of pure codimension 3 and \( SL_{R+1} \)-invariant. The singular points are pairs \((\zeta, \zeta)\) where \( \zeta \in U_{K_g} \) is a node of a fiber. Moreover, the singularities of \( U^2_{K_g} \) are étale-locally ordinary threefold double point singularities. That is, the singularities are of the form
\[ W \times Spec(\mathbb{C}[a, b, c, d]/(ab - cd)) \subset W \times Spec(\mathbb{C}[a, b, c, d]) \]

Where \( W \) is nonsingular. These assertions about the singular locus follow from the deformation theory of a Deligne-Mumford stable curve and [G].

There are three standard resolutions of the ordinary double point singularity \( Spec(\mathbb{C}[a, b, c, d]/(ab - cd)) \):

1. The blow-up along \((a, b, c, d)\).
2. For any \( \lambda \in \mathbb{C} \), the blow-up along \((a - \lambda \cdot c, \lambda \cdot b - d)\).
3. For any \( \lambda \in \mathbb{C} \), the blow-up along \((a - \lambda \cdot d, \lambda \cdot b - c)\).

Methods (2) and (3) yield the distinct small resolutions. The local description (14) implies that the blow-up of \( U^2_{K_g} \) along \( S \) is nonsingular with an exceptional divisor \( E \) that is a \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle over \( S \). Using the techniques of section (2.3), it can be shown that the natural \( SL_{R+1} \)-action on the blow-up \( Bl(S)(U^2_{K_g}) \) can be linearized so that all the points in question are stable and the quotient is projective. The diagonal embedding
\[
D : U_{K_g} \to U^2_{K_g}
\]
is divisorial except along \( S \) where it of the form of (2) and (3) in the local description (14). By definition,
\[
M^c_{g,2} = Bl_{(D)}(U^2_{K_g}) / SL_{R+1}.
\]

There is a natural blow-down map:
\[
\rho : Bl_{(S)}(U^2_{K_g}) \to Bl_{(D)}(U^2_{K_g}).
\]

Another \( SL_{R+1} \)-invariant small resolution of \( U^2_{K_g} \) can be obtain by blowing-down uniformly along the opposite ruling of \( E \) blown-down by \( \rho \). Let \( Y \) denote this other small resolution and let
\[
\overline{\rho} : Bl_{(S)}(U^2_{K_g}) \to Y
\]
be the blow-down. Linearizations can be chosen so that
\[ Y/SL_{R+1} \cong \overline{\mathcal{M}}_{g,2}. \]

There are birational morphisms
\begin{equation}
M^c_{g,2} \leftarrow \text{Bl}_{(S)}(U_{R+1}^2) /SL_{R+1} \rightarrow \overline{\mathcal{M}}_{g,2}.
\end{equation}

Consider the open loci of \( M^c_{g,2} \) and \( \overline{\mathcal{M}}_{g,2} \) where the underlying curve has no (non-trivial) automorphism. On the automorphism free loci the birational modification (15) is easy to describe. Let \( F_1 \subset M^c_{g,2} \) be the locus of 2-pointed stable classes that lie over a node in a Deligne-Mumford stable curve of genus \( g \). Similarly, let \( F_2 \subset \overline{\mathcal{M}}_{g,2} \) be the locus of 2-pointed, genus \( g \), Deligne-Mumford stable curves such that the marked points are coincident at a node in the stable contraction. On the automorphism free loci, \( M^c_{g,2} \) and \( \overline{\mathcal{M}}_{g,2} \) are the distinct small resolution of the fiber product of the universal curve with itself. Hence, on the automorphism free loci, the blow-up of \( M^c_{g,2} \) along \( F_1 \) is isomorphic to the blow-up of \( \overline{\mathcal{M}}_{g,2} \) along \( F_2 \). The modification (15) obtained by this isomorphism.

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Department of Mathematics
University of Chicago
5734 S. University Ave
Chicago, IL 60637
rahul@math.uchicago.edu