G-Graded Central Polynomials and G-Graded Posner’s Theorem

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Abstract

Let $\mathbb{F}$ be characteristic zero field, $G$ a residually finite group and $W$ a $G$-prime and PI $\mathbb{F}$-algebra. By constructing $G$-graded central polynomials for $W$, we prove the $G$-graded version of Posner’s theorem. More precisely, if $S$ denotes all non-zero degree $e$ central elements of $W$, the algebra $S^{-1}W$ is $G$-graded simple and finite dimensional over its center.

Furthermore, we show how to use this theorem in order to recapture the result of Aljadeff and Haile stating that two $G$-simple algebras of finite dimension are isomorphic iff their ideals of graded identities coincide.

1 Introduction

Let $W$ be an associative algebra over a field $\mathbb{F}$. The most basic setup of the theory of associative algebras satisfying a polynomial identity (PI in short) assumes the existence of a non-zero polynomial $f = f(x_1, ..., x_n)$ with non-commuting variables $x_1, ..., x_n$, such that $f(a_1, ..., a_n) = 0$ for every $a_1, ..., a_n \in W$. In that case we say that $W$ is a PI algebra and $f$ is a polynomial identity of $A$. The class of PI algebras includes all commutative algebras (since clearly $[x_1, x_2]$ is an identity), finite dimensional algebras, Grassmann algebras and Grassmann envelopes of finite dimensional algebras. This family is rigid in the sense of being closed to major algebraic operations. For instance, the homomorphic image of a PI algebra is easily seen to be also PI. If $A \subseteq B$ and $B$ is PI so does $A$. The direct product and even the tensor product of two PI algebras is, by a celebrated theorem of Regev [9], also PI.

PI algebras possesses properties similar to finite dimensional algebras. The Jacobson radical of an affine PI algebra is nilpotent [1]. If $A$ is a PI algebra containing no nilpotent ideals, then it also does not contain nil ideals. The primitive PI algebras are simple and finite dimensional over their center of dimension bounded by the degree of a multilinear identity. Furthermore, if $A$ is a semiprime (i.e. the intersection of its prime ideals is zero which is equivalent to $A$ having no nilpotent ideals) PI algebra, then every non-zero ideal of $A$ intersects the center of $A$ non-trivially. The last statement yields the famous theorem of Posner (see all in chapter 1.11 of [7]):

**Theorem 1.1** (Posner’s Theorem). Suppose $A$ is semiprime and PI algebra having a field as its center. Then, $A$ is a simple finite dimensional algebra over its center.

A more general version of this theorem is

**Theorem 1.2.** Suppose $A$ is semiprime and PI algebra with a center $C$. Then, every non-zero ideal of $A$ intersects non-trivially $C$. 

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The main idea of the proof involves the existence of a central polynomial for $M_n(F)$ - $n \times n$ matrices over the field $F$ and the fact, due to Amitsur, that if $A$ is semiprime, then $A[x] = A \otimes_F F[x]$ is semisimple.

In this paper we consider the group graded analogue of Posner’s Theorem. To be more precise, suppose $G$ is a residually finite group. We prove:

**Theorem 1.3 (G-graded Posner’s Theorem).** Suppose $A$ is a $G$-graded algebra over a field $F$ of characteristic zero. Suppose further that $A$ is $G$-semiprime and (ordinary) PI. Then, every $G$-graded ideal $\{0\} \neq I$ of $A$ intersects the $e$-part of the center of $A$ (denoted by $C_e$) in a non-trivial fashion.

As a result, if $C_e$ is a field, then $A$ is a $G$-simple algebra. Moreover, $A$ is finite dimensional over $C_e$.

**Remark 1.4.** In a (English translation of) paper of Balaba [4] it seems that the author have proved a version of theorem 1.3 for every $G$. However, the proof given there is valid only for finite abelian groups $G$. Indeed, the author starts his proof (Proposition 2) by using Corollary 4.5 from [5] which is valid for finite groups (it is false for infinite groups, see [6]). The end of the proof makes sense only for abelian groups. Indeed, the author claims that given an ungraded central polynomial $q(x_1, ..., x_n)$ of a $G$-graded algebra $A$, it is possible to find $a_1^0, ..., a_n^0 \in A$ - a homogenous evaluation of $q$ such that $q(a_1^0, ..., a_n^0)$ is non-zero and homogenous. If $G$ is abelian, then every monomial $a_{\sigma(1)}^0 \cdot a_{\sigma(2)}^0 \cdots a_{\sigma(n)}^0$ is of the same homogenous degree, so the above statement holds. However, if $G$ is not abelian there is no reason for it to hold.

In fact, the above difficulties are the main focus of this paper. They require a new technic to construct $G$-graded central polynomials of $G$-simple and finite dimensional algebras.

Finally, in the final section, we show how to use the above theorem to recover a result of Aljadeff and Haile [2] which states

**Theorem 1.5.** Suppose $G$ is a finite group and $A_1, A_2$ are $G$-simple algebras of finite dimension over an algebraically closed field $F$ of characteristic zero. Then, $id_G(A_1) = id_G(A_2)$ if and only if $A_1$ and $A_2$ are $G$-isomorphic over $F$.

Our technique uses very little from the structure theory of $G$-simple algebras and will be used in consequent papers in order to investigate the connection between $G$-simple structures and their graded identities.

## 2 G-graded structure theorems

Throughout this section $F$ will denote a field of any characteristic and $G$ will denote any group. The goal of this section is to prove the $G$-graded analogs of the basic structure theorems of primitive and simple PI algebras. More precisely, we will prove the $G$-graded Wedderburn’s theorem, $G$-graded Jacobson’s density theorem and Kaplansky’s theorem on primitive PI algebras. These theorems even in the $G$-graded case should be well known, however the author has failed to find a full account of their proofs in the literature.

We note that the work of Bakhturin, Segal and Zaicev (see [3]) treats the $G$-graded Wedderburn’s theorem in the case where $G$ is finite and the characteristic of $F$ is zero or coprime to $|G|$. We follow the exposition of the ungraded case in [8].

**Assumption 2.1.** Throughout the chapter $A$ denotes a $G$-graded $F$-algebra.
Definition 2.2. Let $M$ be a left $A$-module. We say that $M$ is $G$-graded $A$-module if

$$M = \bigoplus_{g \in G} M_g$$

($M_g$ is a $\mathbb{F}$-vector space) and $A_g M_h \subseteq M_{gh}$ for all $g, h \in G$. 

A $G$-graded $A$-module $M$ is called $G$-graded irreducible if it contains no non-trivial $G$-graded $A$-submodules.

Let $M$ and $N$ be $G$-graded $A$-modules. We say that the $A$-module map $\phi : M \to N$ is a $G$-graded homomorphism of degree $g \in G$, if for every $h \in G$: $\phi(M_h) \subseteq N_{gh}$.

Notation 2.3. Since $G$ and $A$ are fixed we will use the abbreviated phrases “graded module”, “graded irreducible” and “graded homomorphisms”.

Note 2.4. The kernel of a graded module homomorphism $\phi : M \to N$ is a graded module. Indeed, suppose $a = \sum_{g \in G} a_g \in M$ ($a_g \in M_g$) is mapped to zero. Since each $\phi(a_g)$ is an element of a distinct homogeneous component of $N$, the $a_g$ must be mapped to zero.

Moreover, the image of such a homomorphism is a graded submodule of $N$.

Suppose $M$ is a graded module. Let $\text{End}_A^G(M)$ denote the $\mathbb{F}$-algebra generated by all graded endomorphisms of $M$. This is a $G$-graded algebra where $\text{End}_A^G(M)_g$ consists of all graded homomorphisms of degree $g$. This indeed defines a $G$-grading on $\text{End}_A^G(M)$: It is trivial that $\text{End}_A^G(M)_g \cdot \text{End}_A^G(M)_g' \subseteq \text{End}_A^G(M)_{gg'}$.

Remark 2.5. If $G$ is a finite group, then $\text{End}_A^G(M) = \text{End}_A(M)$. Indeed, if $\phi \in \text{End}_A(M)$, write $\phi_g = \sum_{h \in G} P_{gh} \phi P_h$, where $P_h$ is the projection onto $M_h$, and notice that $\phi_h$ sits inside $\text{End}_A(M)_h$. Now,

$$\sum_{g \in G} \phi_g = \left( \sum_{h \in G} P_{gh} \phi P_h \right) \cdot \phi \cdot \left( \sum_{g \in G} P_g \right) = \phi.$$

Definition 2.6. A $G$-graded $\mathbb{F}$-algebra $D$ is called $G$-division algebra if every non-zero homogeneous element of $D$ is invertible.

Remark 2.7. It is clear that $1 \in D_e$, so in particular $D_e \neq 0$.

Lemma 2.8 (Schur’s Lemma). If $M$ is graded irreducible, then $\text{End}_A^G(M)$ is a $G$-division algebra. Moreover, if $N$ is an irreducible graded module which is non-graded isomorphic to $M$, then there are no graded homomorphisms from $M$ to $N$ but the zero homomorphism.

Proof. Let $0 \neq \phi$ be a homogeneous element in $\text{End}_A^G(M)$. Since ker $\phi$ is a graded submodule of $M$, we have ker $\phi = 0$. So $\phi$ is injective. From the same reason its image must be equal to $M$. So $\phi$ is also surjective. All in all, $\phi$ is isomorphism, thus invertible.

For the second part, let $\phi$ be a graded homomorphism between $M$ and $N$. As before, the kernel and the image must be zero or everything. If the kernel is zero and the image is $N$, we get that $\phi$ is a graded isomorphism - contradicting the assumption on $M$ and $N$. In any other case $\phi$ must be the zero homomorphism. \hfill \Box

Lemma 2.9. If $D$ is a $G$-division $\mathbb{F}$-algebra, then $D = \bigoplus_{h \in H} (D_h = Db_h)$, where $D$ is an $\mathbb{F}$-division algebra, $H$ is a subgroup of $G$ and $b_h$ is an invertible element of $D_h$. Moreover, if $\mathbb{F}$ is algebraically closed and $\dim_{\mathbb{F}} D_e < \infty$, then $D$ is a twisted group algebra. That is, $D = \mathbb{F}^\alpha H$, where $\alpha \in Z^2(H, \mathbb{F}^\times)$ (H acts trivially on $\mathbb{F}^\times$).

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Proof. Only the second part requires a proof. We first remark that we will rely heavily on the algebraic closeness of \( F \).

\( D_e \) is a division algebra over \( F \). Moreover, the center of \( D_e \) must be a finite extension of \( F \). Hence it is equal to \( F \). Thus \( D_e \) is a central simple \( F \)-algebra. So \( D_e = F \).

Let \( H \) be the subset of \( G \), such that \( D_h \neq 0 \) for \( h \in H \). If \( 0 \neq b_h \in D_h \), then multiplication by \( b_h \) from (say) the right is an \( F \)-linear homomorphism from \( D_e \) to \( D_h \). Since \( b_h \) is invertible, this homomorphism is invertible. Thus \( D_h = F b_h \).

Since the product of two invertible elements is also invertible, we deduce that \( H \) is a subgroup of \( G \).

**Definition 2.10.** A \( G \)-graded algebra \( A \) is called \( G \)-primitive if there is a \( G \)-graded module \( V \) of \( A \) such that the action of \( A \) on \( V \) is faithful and \( G \)-irreducible.

By lemma 2.9 \( \text{End}_{A}^{G}(V) = \bigoplus_{g \in H} Db_g =: D \), where \( H \) is a subgroup of \( G \), \( D \) is a division \( F \)-algebra and \( b_g \) is an invertible element of \( D \). Notice that \( H = \text{supp}D \) (the support of a \( G \)-graded algebra \( A \) is the subgroup of \( G \) generated by all \( h \in G \) for which \( A_g \neq \{0\} \)).

**Theorem 2.11.** (\( G \)-graded density). Suppose \( A \) is \( G \)-primitive \( F \)-algebra acting on the \( G \)-graded module \( V \) irreducibly and faithfully. Let \( D := \text{End}_{A}^{G}(V) \), \( D = D_e \), \( H = \text{supp}D \) and \( V_T = \bigoplus_{t \in T} V_t \), where \( T \) is a set of representatives of right cosets of \( H \) inside \( G \).

Then, for every homogeneous elements \( v_1, \ldots, v_m \in V_T \) which are \( D \)-independent and for every homogeneous \( u_1, \ldots, u_m \in V \), such that \( g_0 := (\deg u_i)^{-1} \cdot \deg v_i \) is the same for \( i = 1 \ldots m \), there is an \( a \in A \) such that \( av_i = u_i \) (\( i = 1 \ldots m \)).

**Note 2.12.** It is clear that if \( a = \sum_{g \in G} a_g \) is the decomposition of \( a \) into its homogeneous components, then \( a_g v_i = 0 \) for \( i = 1 \ldots m \) and \( g \neq g_0 \). Thus, we may replace \( a \) by \( a_{g_0} \) and thus assume that \( a \) is homogeneous.

**Proof.** We claim that it is enough to establish the following: Suppose \( U \) is a finite dimensional (over \( D \) \( G \)-graded subspace of \( V_T \) and \( v' \notin U \) is a homogeneous element of \( V_T \), then there is a homogeneous \( a \in A \) such that \( av' \neq 0 \).

To see this we proceed by induction on \( m \). Let \( U = \text{Span}_{D} \{v_1, \ldots, v_m\} \) and \( v' \notin v_{m+1} \). By induction we can find \( a_1 \in A \) such that \( a_1 v_i = u_i \) for \( i = 1 \ldots m \). Moreover, let \( a_2 \in A \) be a homogeneous element such that \( a_2 U = 0 \) and \( a_2 v' \neq 0 \). Since \( A \) acts \( G \)-irreducibly on \( V \) and \( a_2 v' \) is homogeneous, it is possible to find \( a_3 \in A \) for which \( a_3 (a_2 v') = u_{m+1} - a_1 v' \). Thus, \( a = a_3 a_2 + a_1 \) moves each \( v_i \) to \( u_i \).

We prove the claim by induction on \( m = \text{dim}_{D} U \). The case \( m = 0 \) follows from faithfulness of the action of \( A \) on \( V \). Suppose \( m > 0 \). Hence, there is a homogeneous \( v \in U \) and a subspace \( U_0 \) of \( U \) such that \( U = U_0 + D v_0 \) and \( v_0 \notin U_0 \).

By induction, the annihilator of \( V_0 \), \( \text{ann}_{A}(U_0) \), does not annihilate any element of \( V_T \) not in \( U_0 \). Moreover, \( \text{ann}_{A}(U_0) \) is a left \( G \)-graded ideal of \( A \), hence by \( G \)-irreducibility, for every homogeneous \( v \notin U_0 \), \( \text{ann}_{A}(U_0) v = V \).

Assume to the contrary that \( \text{ann}_{A}(U_0) v' = 0 \). We define a map \( \tau : V \to V \) by the following procedure: For a homogeneous \( 0 \neq v \in V \) we can always find a homogeneous \( a \in \text{ann}_{A}(U_0) \) such that \( av_0 = v \). Using \( a \) we declare \( \tau(v) = av' \). This is well defined, since if \( av_0 = 0 \), then \( a \) must be in \( \text{ann}_{A}(U) \). Hence, by our assumption, \( a \) annihilates \( v' \), that is \( av' = 0 \). Moreover, for every homogeneous \( v \), \( (\deg \tau(v))^{-1} \deg v = (\deg v')^{-1} \deg v_0 = : h \). Hence \( \tau \) is a homogeneous element of \( D \).
The map \( \tau \) is clearly \( \mathbb{F} \)-linear. Moreover, if \( b \in A \) is homogeneous, \( bab_0 = bv \), hence

\[ \tau(bv) = bab' = b\tau(v). \]

Thus, \( \tau \in D \) and therefore equals to \( ab_{h_0} \), where \( \alpha \in D \). Furthermore, for \( a \in A \):

\[ av' = \tau(av_0) = a\tau(v_0) \Rightarrow a(v' - \tau(v_0)) = 0. \]

Thus,

\[ v' = \tau(v_0) \Rightarrow v' = ab_{h_0}v_0. \]

Since both \( v_0 \) and \( v' \) are in \( V_T \), we deduce that \( v' = \alpha_v v_0 \), which is absurd. \( \Box \)

We turn to consider \( G \)-graded \( G \)-simple algebras.

**Definition 2.13.** A is \( G \)-simple if \( A^2 \neq 0 \) and every \( G \)-graded two-sided ideal of \( A \) is trivial.

It is easy to check that every \( G \)-simple algebra is also \( G \)-primitive. Indeed, Choose \( V = A \). The action is, by definition, is \( G \)-graded irreducible. Moreover, if there was a homogeneous \( a \in A \) such that \( aA = 0 \), then \( Aa \) was a two-sided \( G \)-graded ideal of \( A \). Therefore, \( aA = 0 \) or \( A \). The latter is impossible since then \( A^2 = AaA = 0 \). So \( Fa \) is a two-sided \( G \)-graded ideal of \( A \). As before, the conclusion is that \( \mathbb{F}a = 0 \Rightarrow a = 0 \). Thus, if \( a \in A \) is not necessary a homogeneous element it is impossible that \( aA = 0 \). All in all, the action is also faithful.

In order to give an example of a \( G \)-simple algebra, we define:

**Definition 2.14.** Let \( B \) be any \( G \)-graded \( \mathbb{F} \)-algebra, a natural number \( n \) and \( g = (g_1, ..., g_n) \in G^n \). Denote by \( M_g(B) \) the \( \mathbb{F} \)-algebra \( B \otimes M_n(\mathbb{F}) \), \( G \)-graded by

\[ M_g(B)_g = \text{Span}_\mathbb{F}\{b \otimes e_{i,j} | b \in B_h, g = g_i^{-1} hg_j\}. \]

**Note 2.15.** It is easy to check that if \( B \) is \( G \)-simple, then so does \( M_g(B) \).

**Definition 2.16.** Let \( A \) be a \( G \)-graded \( \mathbb{F} \)-algebra. Denote by \( A^\op \) the opposite \( G \)-graded \( \mathbb{F} \)-algebra of \( A \) defined by \( A^\op = A \) as \( \mathbb{F} \)-vector spaces; the multiplication is given by \( a_1a_2 = a_2 \cdot a_1 \), where \( \cdot \) is the multiplication in \( A \); the \( G \)-grading is given via:

\[ A^\op_g = A_g^{-1} \]

for all \( g \in G \).

**Lemma 2.17.** Let \( A \) be \( G \)-simple \( \mathbb{F} \)-algebra. Then, \( A^\op \) is also \( G \)-simple. Furthermore, if \( A \) is a \( G \)-division \( \mathbb{F} \)-algebra, so does \( A^\op \).

**Proof.** The proof is omitted. \( \Box \)

**Notation 2.18.** For an algebra \( A \), we denote by \( Z(A) \) the center of \( A \). Moreover, If \( A \) is \( G \)-graded, the center \( Z(A) \) will (usually) not be graded by \( G \), unless \( G \) is abelian. Nevertheless, we denote by \( Z(A)_e \) all elements of degree \( e \) inside \( Z(A) \).

**Definition 2.19.** A finite dimensional \( G \)-simple algebra \( A \) over \( \mathbb{F} \) is called \( G \)-central-simple algebra if \( \mathbb{F} = Z(A)_e \).
Proposition 2.20. Suppose $A$ is a $G_1$-central-simple algebra over $\mathbb{F}$. If $B$ is $G_2$-simple unital algebra over $\mathbb{F}$, then $A \otimes_{\mathbb{F}} B$ is $G_1 \times G_2$-central simple algebra over $Z(B)_e$ (here, $(A \otimes B)_{(g_1, g_2)} = A_{g_1} \otimes B_{g_2}$). In particular, if $B$ is a field (considered graded by \{e\}), then $A \otimes_{\mathbb{F}} B$ is $G$-simple algebra over $Z(A \otimes_{\mathbb{F}} B)_e = B$.

Proof. If $0 \neq I$ is a $G_1 \times G_2$-graded ideal of $A \otimes B$, and $0 \neq x = \sum_{j=1}^{t} a_j \otimes b_j \in I_{(g_1, g_2)}$ is shortest element in $I$ (that is, for every $0 \neq x' = \sum_{j=1}^{t'} a'_j \otimes b'_j \in I$, one has $t' \geq t$).

It is easy to see that $b_1, ..., b_t$ are independent over $\mathbb{F}$. Furthermore, since $A_1 A = A$, one can write $\sum_p c_p a_1 c_p'' = 1$. We can assume that $\deg c_p' \cdot g_1 \cdot \deg c_p'' = e$ for all $p$. Hence,

$$\sum_p c_p' \cdot 1 \cdot x' \cdot c_p'' \otimes 1 = 1 \otimes b_1 + d_2 \otimes b_2 + \cdots + d_t \otimes b_t,$$

where $d_j = \sum_p c_p' a_j c_p'' \in A_e$. Thus, for every $a \in A$,

$$[a, 1 \otimes b_1 + d_2 \otimes b_2 + \cdots + d_t \otimes b_t] \in I$$

is of length shorter than $t$, hence $[a, d_j] = 0$ for $j = 2, ..., t$. As a result, $d_j \in \mathbb{F}$. Hence,

$$z = 1 \otimes (b_1 + d_2 b_2 + \cdots + d_t b_t).$$

Since $b_1, ..., b_t$ are linearly independent over $\mathbb{F}$, we get that $z \neq 0$. As a result, $t = 1$ and (may assume) $x = 1 \otimes b_1$. The $G$-simplicity of $B$ and the fact that $b_1$ is homogenous forces $A \otimes B : x \cdot A \otimes B$. All in all, $I = A \otimes B$.

Next, it is clear that $Z(B)_e \subseteq Z(A \otimes B)_e$. For the other direction,

$$Z(A \otimes B)_e \ni 0 \neq z = \sum_{i=1}^{t} a_i \otimes b_i,$$

where $t$ is minimal. As before, $b_1, ..., b_t$, are independent over $\mathbb{F}$. For $a \in A$,

$$0 = [z, a] = \sum_{i=1}^{t} [a_i, a] \otimes b_i.$$ 

Thus, $[a_i, a] = 0$ for $i = 1, ..., t$. Hence, $a_i \in Z(A)_e = \mathbb{F}$. As a result, $t = 1$ and $z = 1 \otimes b_1 \in B$ As a result, $z \in Z(B)$.

Lemma 2.21. Suppose $A$ is a $G$-primitive $\mathbb{F}$-algebra acting on the $G$-graded module $V$ irreducibly and faithfully. Let $D := \text{End}_A^G(V)$ $D = B_e$, $H = \text{supp}D$ and $V_T = \bigoplus_{t \in T} V_t$, where $T$ is a set of representatives of right cosets of $H$ inside $G$. Suppose further that $V_T$ is finite dimensional over $D$.

Then, $A$ is a $G$-simple algebra over the field $\mathbb{K} = Z(A)_e$ and $\dim_\mathbb{K} A_e < \infty$. Furthermore, $A = M_\mathbb{K}(D)$ for some $g = (g_1, ..., g_n) \in G^n$ and a finite dimensional $G$-division $\mathbb{K}$-algebra $D$.

Proof. Consider the map $\phi : A^{\text{op}} \to \text{End}_D^G(V)$ given by right multiplication. It is easy to check that this is a $G$-graded homomorphism. Since the action is faithful the map is an embedding.

By lemma 2.17, $A^{\text{op}}$ is $G$-simple and thus also $G$-primitive. Moreover, suppose $v_1, ..., v_m$ is a homogeneous basis of $V_T$. Then any homogeneous $f \in \text{End}_D(V)$ is completely determined by its action on $v_1, ..., v_m$. Write $u_1 = f(v_1), ..., u_t = f(v_m)$. Then clearly $(\deg u_i)^{-1} \cdot \deg v_i$
is the same for \( i = 1...m \). Hence, by theorem 2.11 there is an \( a \in A^{op} \) such that \( \phi(a) = f \).

Therefore, \( \phi \) is a \( G \)-graded isomorphism.

It is easy to check that

\[
\text{End}_D(V) \cong D \otimes_D \text{End}_D(V_T) \cong D \otimes_D M_\mathfrak{g}(D) \cong D \otimes_{Z(D)} M_\mathfrak{g}(Z(D)),
\]

where \( \mathfrak{g} = (t_1, ..., t_1, ..., t_l, ..., t_l) \), where \{t_1, ..., t_l\} = T and each \( t_i \) appears exactly \( \dim_D V_{t_i} \) times in \( \mathfrak{g} \). Now, notice that if \( e = g^{-1}h g \) (here \( h \in H \) and \( g, h \in T \)), then \( g_i = h g_j \).

Therefore, \( g_i = g_j \) and \( h = e \) (recall that \( T \) is a transveral of \( H \) in \( G \)). As a result, \( A^{op} \subseteq D \otimes_D \text{End}_D(V_T) \). So \( \mathbb{K} = Z(A^{op})_e = Z(D) \), which is a field (since \( D \) is a division ring). All in all, \( A \) is \( G \)-isomorphic to \( M_\mathfrak{g}(D^{op}) \) which is \( G \)-simple due to lemma 2.17 and 2.15.

For the next Corollary we will need the following definition.

**Definition 2.22.** A \( G \)-graded \( \mathbb{F} \)-algebra \( A \) is called \( G \)-Artinian if every descending sequence of \( G \)-graded ideals of \( A \): \( I_1 \supseteq I_2 \supseteq \cdots \) eventually stabilizes.

**Note 2.23.** If \( \dim_\mathbb{F} A < \infty \), it is clear that \( A \) is \( G \)-Artinian.

**Theorem 2.24 (\( G \)-graded Wedderburn’s theorem).** Suppose \( A \) is \( G \)-simple and (left) \( G \)-Artinian \( \mathbb{F} \)-algebra. Then, \( A = M_\mathfrak{g}(D) \), where \( \mathfrak{g} = (g_1, ..., g_n) \in G^n \) and \( D \) is a finite dimensional \( G \)-division \( \mathbb{F} \)-algebra. If moreover \( \mathbb{K} = Z(A)_e \) is algebraically closed, then \( D = \mathbb{F}^\alpha H \), where \( H = \text{supp} D \) and \( \alpha \in Z^2(H, \mathbb{F}^\times) \).

**Proof.** We want to use [2.21] with \( V = A \) (as a \( G \)-graded left \( A \)-module). Suppose \( V_T \) is of infinite dimension over \( \mathbb{F} = D \) (see lemma 2.9). The set

\[
R_n = \{ a \in A \mid a U_n = 0 \},
\]

where \( U_n \) is an \( n \)-dimensional \( G \)-vector space inside \( V_T \), is clearly a left \( G \)-graded ideal of \( A \). By choosing \( U_1 \supseteq U_2 \supseteq \cdots \) we get \( R_1 \supseteq R_2 \supseteq \cdots \). Moreover, by irreducibility we know that these are strict inclusions. Hence, we reached a contradiction.

The statement now follows from [2.21] and lemma 2.9.

**Corollary 2.25 (\( G \)-graded Kapalansky).** If \( A \) is \( G \)-primitive \( \mathbb{F} \)-algebra and moreover satisfies an ordinary PI of degree \( d \), then the conclusion of the previous corollary holds. Furthermore, \( \mathbb{K} = Z(A)_e \) is a field and \( \dim_\mathbb{K} A_T \leq \frac{d}{2} \).

**Proof.** We use the notation of theorem 2.11. It is enough to show that \( \dim_D V_T \leq d/2 \). Indeed, otherwise there is an \( d/2 < n \)-dimensional \( G \)-graded \( D \)-subspace \( U \) of \( V_T \) and consider the subring of \( A \) given by

\[
R = \{ a \in A \mid a U \subseteq U \}.
\]

By the previous theorem there is an epimorphism from \( R \) to \( \text{End}_D(U) = M_n(D) \). However, \( M_n(\mathbb{F}) \) satisfies a PI of degree \( d < 2n \) or higher. This contradicts the assumption that \( A \) satisfies a PI of degree \( d \).

**Theorem 2.26.** Suppose \( A \) is a \( G \)-simple PI \( \mathbb{F} \)-algebra, where \( G \) is finite. Then, there is a field \( \mathbb{F} \subseteq \mathbb{K} \) and a \( G \)-graded \( \mathbb{K} \)-algebra \( B = \mathbb{K}^\alpha H \otimes M_\mathfrak{g}(\mathbb{K}) \) such that

\[
id_{G,\mathbb{F}}(A) = id_{G,\mathbb{F}}(B).
\]
Proof. By lemma 2.25, for $L = Z(A)e$, we get that $A$ is finite dimensional over $L$ and $A = M_\mathbb{F}(D)$ for a $G$-division (finite dimensional) $L$-algebra $D$. As a result, for $K = \sum L$, $K \otimes_L A \cong M_\mathbb{F}(K \otimes_L D)$ (notice that $L$ being $e$-homogeneous implies that the above isomorphism preserves the $G$-grading). Furthermore, by lemma 2.9 $K \otimes_L D$ is of the form $\mathbb{K}^\alpha H$, where $H$ is a subgroup of $G$ and $\alpha \in \mathbb{Z}^2(H, \mathbb{K}^*)$. Finally the theorem follows since

$$id_{G,\mathbb{F}}(A) = id_{G,\mathbb{F}}(K \otimes_L A).$$

□

Remark 2.27. Notice that $\mathbb{K} \otimes_\mathbb{F} id_{G,\mathbb{F}}(B) \subseteq id_{G,K}(B)$. However, the other inclusion is not assured. For instance, let $\mathbb{F} = \mathbb{Q}$, $K = \mathbb{C}$, $G = C_3 \times C_3 = \langle \sigma \rangle \times \langle \tau \rangle$ and $B = K^\alpha G$, where $u_\sigma u_\tau = \xi u_\tau u_\sigma$ (here $\xi = \sqrt{3}$) and $u_\sigma^2 = u_\tau^3 = 1$, we obtain that $x_{1,\sigma}x_{2,\tau} - \xi x_{2,\tau}x_{1,\sigma} \in id_{G,K}(B)$ but not in $\mathbb{K} \otimes_\mathbb{F} id_{G,\mathbb{F}}(B)$.

Definition 2.28. For a $G$-graded algebra $A$ we denote by $J_G(A)$ the intersection of all of its $G$-primitive ideals. We call $J_G(A)$ the $G$-graded Jacobson radical of $A$.

If $J_G(A) = \{0\}$, we say that $A$ is $G$-semisimple.

Theorem 2.29 (see [5]). If $G$ is a finite group and $\mathbb{F}$ is of zero characteristic, then $J_G(A) = J(A)$ for every $G$-graded $\mathbb{F}$-algebra $A$. In particular, $A$ is $G$-semisimple if and only if $A$ is semisimple.

Lemma 2.30. Suppose $A$ is a $G$-semisimple PI $\mathbb{F}$-algebra and $I$ is a $G$-graded ideal of $A$. Then $I$ is also $G$-semisimple algebra. Furthermore, $Z(I) \subseteq Z(A)$.

Proof. By assumption $A$ is a subdirect product of $G$-primitive algebras $A_i = A/M_i$, where $M_i$ is a $G$-primitive ideal ($i \in \mathfrak{J}$ - an index set). That is, there is a $G$-graded embedding

$$\phi : A \rightarrow \prod_{i \in \mathfrak{J}} A_i$$

such that $p_i \circ \phi$ is surjective for every $i \in \mathfrak{J}$, where $p_i : \prod_{j \in \mathfrak{J}} A_j \rightarrow A_i$ is the natural projection. By lemma 2.24 we get that each $A_i$ is, in fact, $G$-simple algebra.

Since every $A_i$ is $G$-simple, for every $i \in \mathfrak{J}$ $p_i \circ \phi(I) = 0$ or $A_i$. Denote by $\mathfrak{J}_0$ all the indexes $j \in \mathfrak{J}$ for which the result is $A_j$. Hence, the sequence

$$0 \rightarrow I \rightarrow \prod_{i \in \mathfrak{J}_0} A_i \rightarrow A_i \rightarrow 0$$

is exact. This shows that $I$ is a $G$-semisimple algebra.

If $a \in Z(I)$, then $p_i \circ \phi(a)$ is, in any case, inside $Z(A_i)$ (for every $i \in \mathfrak{J}$). As a result $a \in Z \{\prod_{i \in \mathfrak{J}} A_i\} \cap A \subseteq Z(A)$. □

3 $G$-graded PI

We will use from now on the standard notations of $G$-graded PI theory.

A $G$-graded polynomial is an element of the non-commutative free algebra $\mathbb{F}\langle X_G \rangle$ generated by the variables from $X_G = \{x_i,g| i \in \mathbb{N}, g \in G\}$. It is clear that $\mathbb{F}\langle X_G \rangle$ is itself $G$-graded, where $\mathbb{F}\langle X_G \rangle_g$ is the span of all monomials $x_{i_1,g_1} \cdots x_{i_n,g_n}$, where $g_1 \cdots g_n = g$. 
We say that a $G$-graded polynomial $f(x_{i_1,g_1}, ..., x_{i_n,g_n})$ is a $G$-graded identity of a $G$-graded algebra $A$, if $f(a_1, ..., a_n) = 0$ for all $a_1 \in A_{g_1}, ..., a_n \in A_{g_n}$. The set of all such polynomials is denoted by $id_G(A)$ and it is evident that it is an ideal of $F \langle X_G \rangle$. In fact, it is a $G$-graded $T$-ideal, that is an ideal which is closed to $G$-graded endomorphisms of $F \langle X_G \rangle$.

For $G$ finite, it is often convenient to view any ungraded polynomial as $G$-graded via the embedding

$$F \langle X \rangle \to F \langle X_G \rangle,$$

where $x_i \mapsto \sum_{g \in G} x_{i,g}$. This identification respects the $G$-graded identities, in the sense that if $f \in id(A)$ (the $T$-ideal of ungraded identities of $A$), then $f \in id_G(A)$.

As in the ungraded case, the set of $(G$-graded) multilinear identities of $A$

$$\left\{ \sum_{\sigma \in S_n} \alpha_{\sigma} x_{\sigma(1),g_{\sigma(1)}} \cdots x_{\sigma(n),g_{\sigma(n)}} \in id_G(A) | \alpha_{\sigma} \in \mathbb{F}; \ i_1, ..., i_n \text{ are all distinct} \right\}$$

$T$-generates $id_G(A)$.

Suppose $f \in F \langle X_G \rangle$, $A$ is a $G$-graded algebra and $S \subseteq A$. We write $f(S)$ for the subset of $A$ consisting of all the graded evaluations of $f$ by elements of $S$. Moreover, if $B$ is a subset of $A$ consisting of homogenous elements such that $spB = A$, then $spf(B) = A$. Thus, $f \in id_G(A)$ if and only if $f$ vanishes under graded substitutions by elements from $B$.

### 3.1 Central polynomials

In this section we introduce the main tool for proving Posner’s theorem (see section §II).

**Definition 3.1.** Let $A$ be an $\mathbb{F}$-algebra. We say that $f \in F \langle X \rangle$ is a central polynomial of $A$ if $f(A) \subseteq Z(A)$ and $f$ is not a PI of $A$.

If $A$ is also graded by a group $G$, then we say that $f \in F \langle X_G \rangle$ is a $G$-graded central polynomial of $A$ if $f(A) \subseteq Z(A)$ and $f$ is not a $G$-graded PI of $A$. However, we will most of the time use the phrase “central polynomials” also for $G$-graded central polynomials.

Suppose $\mathbb{F}$ is a field of characteristic zero. Then, the following polynomial (called the Regev polynomial)

$$F \langle X \rangle \ni L_d(x_1, ..., x_d, y_1, ..., y_d) = \sum_{\sigma, \tau \in S_d} (-1)^{\sigma \tau} x_{\sigma(1)} y_{\tau(1)} x_{\sigma(2)} y_{\tau(2)} x_{\sigma(3)} y_{\tau(3)} y_{\tau(4)} \cdots$$

$$\cdots x_{\sigma((n-1)^2+1)} \cdots x_{\sigma(d)} y_{\tau((n-1)^2+1)} \cdots y_{\tau(d)}$$

is a central polynomial of $M_n(\mathbb{F})$, where $exp(A) = d = n^2$. That is, the values of $L_d$ when evaluated on $M_n(\mathbb{F})$ are in the center of $M_n(\mathbb{F})$ and there are non-zero values (see [8]).

**Remark 3.2.** A basic property of $L_d$ polynomial is that for $a_1, ..., a_d, b_1, ..., b_d \in M_n(F)$ the value of $L_d(a_1, ..., a_d, b_1, ..., b_d)$ is non-zero if and only if each one of the sets $\{a_1, ..., a_d\}$ and $\{b_1, ..., b_d\}$ is $F$-independent.

**Lemma 3.3.** Suppose $A$ is a semiprime $\mathbb{F}$-algebra (see §II) of exponent $exp(A) = d$, where $\mathbb{F}$ is a characteristic zero field. Then, $d = n^2$ is a square and $A$ is PI equivalent to $M_n(\mathbb{F})$. Furthermore, if $f$ is a central polynomial of $A$ it is an identity of any semiprime algebra $A'$ of lower exponent $exp(A') = m^2$. 

Suppose algebra $A$ has a central polynomial of $G$-graded components. Then, if $f$ be any central polynomial of $M_n(F)$, we may view $M_n(F)$ as a subalgebra of $M_n(F)$ via the embedding $A \mapsto \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ (notice that the embedding does not preserve the unit element of $M_n(F)$), so $f$ is an identity of $M_n(F)$. □

We turn to $G$-graded polynomials.

**Definition 3.5.** A polynomial $f = f(x_1, \ldots, x_l) \in F \langle X \rangle$ is called strong central polynomial for a $G$-graded $F$-algebra $A$ (or $G$-strong central polynomial), if:

1. $f$ is a central polynomial of $A$.
2. For every $g_1, \ldots, g_l \in G$ and $a_{g_1} \in A_{g_1}, \ldots, a_{g_l} \in A_{g_l}$ such that $f_1 = f(x_{g_1}, \ldots, x_{g_l})$ is non-zero when evaluated by $\varpi_{g_1} = a_{g_1}, \ldots, \varpi_{g_l} = a_{g_l}$, then $\rho(f_1)(a_{g_1}, \ldots, a_{g_l}) \neq 0$.

The following Lemma shows that if $f$ is a strong central polynomial, then we can construct from it $e$-homogenous $G$-graded central polynomials.

**Lemma 3.6.** Suppose $f$ is $G$-graded central polynomial of a $G$-graded $F$-algebra $A$. Then, $\rho(f)$ is either central or a $G$-graded identity of $A$.

Proof. Decompose $f$ into a sum of its $G$-graded components: $f = \sum_{g \in G} \rho_g(f)$. Fix $h \in G$ and consider $y_h \in X_G$ which does not appear in $f$. Since $f$ is central, it is clear that $[y_h, f] \in id^G(A)$. Hence, $\sum_{g \in G} [y_h, \rho_g(f)] = 0$. Since $[y_h, \rho_g(f)] \in \mathbb{F} \langle X_G \rangle_h$ if and only if $g = e$ (notice that $[y_h, \rho_g(f)] \in \mathbb{F} \langle X_G \rangle_{gh} \oplus \mathbb{F} \langle X_G \rangle_{gh}$), we get $[y_h, \rho_e(f)] \in id^G(A)$. Because this holds for all $h \in G$, we conclude that $\rho(f)(A)$ lies inside the center of $A$. □

**Corollary 3.7.** Suppose $f = f(x_1, \ldots, x_l)$ is a strong central polynomial for the $G$-graded $F$-algebra $A$. Then, if $f_1 = f(x_{g_1}, \ldots, x_{g_l})$ is not an identity of $A$ (here $g_1, \ldots, g_l \in G$), $\rho(f_1)$ is a central polynomial of $A$.

We will be interested in graded central polynomials for the $G$-graded algebra $A = \mathbb{F}^n H \otimes M_g(F)$ (as in [2.14]). We introduce two methods of constructing them.

### 3.1.1 Strong central polynomials using involution.

Consider the following involution on the free algebra $\mathbb{F} \langle X \rangle$ given by: $(\alpha x_{i_1} \cdots x_{i_m})^t = \alpha x_{i_m} \cdots x_{i_1}$. We use this action to construct new (ungraded) central polynomials: Suppose $f = f(x_1, \ldots, x_m) \in \mathbb{F} \langle X \rangle$ is a central polynomial of $M_n(F)$. Since the transpose action on matrices is an involution on $M_n(F)$, we conclude that $f^t$ is also a central polynomial of $M_n(F)$. Hence, $\tilde{f} = f(x_1, \ldots, x_m)f(y_1, \ldots, y_m)^t$ is also a central polynomial.
Lemma 3.8. Consider the $G$-graded algebra $A = F^\alpha H \otimes M_\theta(F)$, where $F = \mathbb{C}$ and the values $\alpha$ takes are roots of unity. Let $f = L_d(x_1, \ldots, x_d, y_1, \ldots, y_d) \in F \langle X \rangle \subseteq \tilde{F} \langle X_G \rangle$ be the Regev polynomial, where $d = \exp(A)$. Then, for every $a_1, \ldots, a_d \in A$, the $e$ part of
\[
\check{f}(a_1, \ldots, a_d, a_1, \ldots, a_d; a_1^*, \ldots, a_d^*, a_1^*, \ldots, a_d^*)
\]
is not zero if and only if $f(a_1, \ldots, a_d, a_1, \ldots, a_d) \neq 0$, where $(cu_h \otimes M)^* = \bar{c}u^{-1}_h \otimes M^*$. (Here $\bar{c}$ denotes the complex conjugate of $c \in \mathbb{C}$ and $M^* = \mathcal{M}$ for $M \in M_m(\mathbb{C})$.)

Proof. First, we claim that $(\cdot)^*$ is an anti-automorphism of $A$. Indeed,
\[
(cu_h \otimes M \cdot c'u_{h'} \otimes M')^* = \overline{c'\alpha(h, h')}u_{h'h'}^{-1} \otimes M^*M^*
\]
whereas
\[
(cu_h \otimes M)^* (c'u_{h'} \otimes M')^* = \overline{c'u_h^{-1}} u_{h'h'}^{-1} \otimes M^*M^*.
\]
Finally,
\[
u_{h'}^{-1} u_{h'}^{-1} = (\alpha(h, h') u_{h'h'})^{-1} = \overline{\alpha(h, h')} u_{h'h'}^{-1}.
\]
The last equality follows from $|\alpha(h, h')| = 1$.

Now we are ready to prove the Lemma. If $f(a_1, \ldots, a_d, a_1, \ldots, a_d) = 0$, it is obvious that also
\[
\check{f}(a_1, \ldots, a_d, a_1, \ldots, a_d; a_1^*, \ldots, a_d^*, a_1^*, \ldots, a_d^*) = 0,
\]
in particular its $e$ component must be zero.

If, on the other hand, $f(a_1, \ldots, a_d, a_1, \ldots, a_d)$ is non-zero, since $\check{f}$ is a central polynomial of $A$, it must be equal to
\[
\sum_{g \in H} \alpha_g u_g \otimes I \in A,
\]
where at least one of the $\alpha_i \neq 0$. Since $(\cdot)^*$ is an anti-automorphism,
\[
\check{f}(a_1, \ldots, a_d, a_1, \ldots, a_d; a_1^*, \ldots, a_d^*, a_1^*, \ldots, a_d^*) = \left(\sum_{g} \alpha_g u_g\right) \left(\sum_{g} \overline{\alpha_g} u_{g^{-1}}\right) \otimes I.
\]
The $e$ part of the last expression is equal to $\sum_{g \in H} |\alpha_g|^2 > 0$. \hfill \Box

Theorem 3.9. Suppose that $F$ is any field of characteristic zero, $G$ is a finite group and $A = F^\alpha H \otimes M_\theta(F)$ is $G$-graded as in \cite{ZT}. Suppose further that $f$ is a central polynomial of $A$. Then, $\check{f}$ is a strong central polynomial of $A$.

Proof. By extending scalars, we may assume that $F$ is an algebraically closed field containing $\mathbb{C}$. Hence, since every $\alpha \in Z^2(H, F^*)$ is cohomologous to $\alpha'$ (over the algebraically closed field $\mathbb{F}$) which satisfies the assumption in the previous theorem. Moreover, by Lemma 1.3 in \cite{ZT}, the $G$-graded algebras $F^\alpha H \otimes M_\theta(F)$ and $F^{\alpha'} H \otimes M_\theta(F)$ are $G$-graded isomorphic.

As a result $A_\mathbb{C} = \mathbb{C}^\alpha H \otimes M_\theta(\mathbb{C})$ is well defined and $F \otimes_\mathbb{C} A_\mathbb{C}$. By the previous Lemma, $\check{f}$ is a strong central polynomial of $A_\mathbb{C}$, hence also if $A$. \hfill \Box
3.1.2 Strong central polynomials using projective representations of groups.

The following method is due to Eli Aljadeff.

We start with a preliminary Lemma:

Lemma 3.10. Suppose $A$ and $B$ are semisimple and finite dimensional $\mathbb{F}$-algebras. If $\phi : A \to B$ is an epimorphism, then every central minimal idempotent in $B$ can be lifted in $A$.

Proof. The kernel of $\phi$, denoted by $I$, must be of the form (as any ideal of $A$) $A_{i_1} \times \cdots \times A_{i_r}$. Without loss of generality assume that $I = A_{q+1} \times \cdots \times A_q$. Thus, the restriction $\phi|_{A_{q+1} \times \cdots \times A_q}$ is an isomorphism. From Corollary 7.50 in ???, it follows that there is $q = q'$ and there is $\sigma \in S_r$ such that $\phi(A_i) = B_{\sigma(i)}$. The Lemma is now clear.

Lemma 3.11. Assume $\mathbb{F}$ is a characteristic zero algebraically closed field and $0 \neq \rho = \sum_{g \in G} u_g \in \mathbb{F}^\rho G$ is a central idempotent. Then $\mu_e \neq 0$.

Proof. By a theorem of Schur (see Theorem 11.17 in ???) we know that there exist a finite central extension

$$0 \longrightarrow C \longrightarrow \Gamma \longrightarrow \pi \longrightarrow G \longrightarrow 0$$

and $\lambda \in \text{hom}(C, \mathbb{F}^\ast)$ such that $[\lambda \circ \beta] = [\alpha] \in H^2(G, \mathbb{F}^\ast)$, where $[\beta] \in H^2(G, C)$ corresponds to the above extension. Hence, by choosing a transversal $\pi(\gamma_1) = g_1, \ldots, \pi(\gamma_m) = g_m$ (here $G = \{e = g_1, g_2, \ldots, g_m\}$), we get an epimorphism

$$\phi : \Gamma \to \mathbb{F}^\rho G$$

given by $\phi(u_{c \gamma_i}) = \lambda(c)u_g$.

Since both algebras are finite dimensional and semisimple, by lemma[3.10] there is a central idempotent $\rho' \in \mathbb{F}G$ which projects (via $\pi$) onto $\rho$. Thus, there exists a character $\chi$ of $\Gamma$ such that:

$$\rho' = \frac{\chi(e)}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma)u_g.$$

Since $C \leq Z(\Gamma)$ and $\chi$ is a character of an irreducible representation $V$ of $\Gamma$ we know, by Schur’s Lemma, that $e$ acts on $V$ by multiplication by $\frac{1}{\dim V} \chi(e)$. Hence, $\chi(c \gamma) = \chi(c)\chi(\gamma)$.

This all boils down to:

$$\rho = \pi(\rho') = \left( \sum_{c \in C} \frac{\chi(c)\lambda(c)}{|\Gamma|} \right) \cdot \frac{\chi(e)}{|\Gamma|} \sum_{i=1}^{m} \chi(\gamma_i)u_{g_i}.$$

Since $e \neq 0$, $\chi(e) \sum_{c \in C} \chi(c)\lambda(c) \neq 0$. Hence also $\mu_e = \mu_{g_1} = \left( \sum_{c \in C} \chi(c)\lambda(c) \right) \cdot \frac{|\chi(e)|}{|\Gamma|} \neq 0$.

Theorem 3.12. Let $A = \mathbb{F}^\rho H \otimes M_q(\mathbb{F})$, where $\mathbb{F}$ is a field of zero characteristic. Suppose $F = F(x_1, \ldots, x_l)$ is a central polynomial of $A$. Then, $F$ is a strong central polynomial of $A$.

Proof. Since $\mathbb{F}^\rho H$ is semisimple (since $\mathbb{F}$ is of zero characteristic) and finite dimensional, we can write it as the product $M_{k_l}(\mathbb{F}) \times \cdots \times M_{k_1}(\mathbb{F})$, where assuming $k_1 \geq k_2 \geq \cdots \geq k_l$. So $A$ as an ungraded algebra is isomorphic to

$$(M_{k_1}(\mathbb{F}) \times \cdots \times M_{k_l}(\mathbb{F})) \otimes M_{|g|}(\mathbb{F}) = M_{n=k_1+|g|}(\mathbb{F}) \times \cdots \times M_{k_l+|g|}(\mathbb{F}).$$

As a result, $A$ is PI equivalent to $M_n(\mathbb{F})$. So, $F$ is a central polynomial of $M_n(\mathbb{F})$. Thus, $W = F \left( M_{k_1}(\mathbb{F}) \otimes M_{|g|}(\mathbb{F}) = M_{n}(\mathbb{F}) \right) = \mathbb{F} \rho \otimes \Gamma$, where $\rho = 1_{M_{k_1}(\mathbb{F})}$ is a central idempotent of $\mathbb{F}^\rho H$. By the previous Lemma, the $e$-component of $W$ is not zero, hence $f$ is a non-identity. Thus, by lemma 3.11, $f$ is a central polynomial of $B$ of degree $e$. 

\[ \square \]
3.1.3 \emph{G}-graded central polynomials for lower exponent algebras.

We start with a definition:

\begin{definition}
A \emph{G}-graded polynomial \( f \) is called \emph{\( d \)-sharp}, if for every \( G \)-simple algebra \( A \) the following holds:

\begin{itemize}
\item \( \exp(A) = d \implies f \notin id_G(A) \).
\item \( \exp(A) < d \implies f \in id_G(A) \).
\end{itemize}
\end{definition}

\begin{theorem}
Suppose \( A \) is a \( G \)-simple \( F \)-algebra, where \( F \) is a characteristic zero field, which is ungraded PI. Every ungraded central polynomial \( f \) of \( A \) is \( G \)-strong and \( \rho(f) \) is \( \exp(A) \)-sharp.
\end{theorem}

\begin{proof}
By theorem \ref{thm:2.26}, there is a field \( F \subseteq K \) and a \( G \)-graded \( K \)-algebra \( B = K^\alpha H \otimes M_\theta(K) \) such that \( id_{G,F}(A) = id_{G,F}(B) \). Hence, by theorem \ref{thm:3.12} \( f \) is \( G \)-strong central polynomial of \( A \).

The second part follows from lemma \ref{lem:3.3} since every \( G \)-simple algebra over a characteristic zero field is semisimple (hence semiprime).
\end{proof}

4 \emph{G}-graded Posner’s theorem

In order to introduce the main theorem we need the following definitions.

\begin{definition}
Let \( G \) be any group and suppose that \( A \) is a \( G \)-graded \( F \)-algebra. We say that \( A \) is a \emph{\( G \)-prime algebra} if for every two graded ideals \( I \) and \( J \) such that \( IJ \) is equal to the zero ideal, one of them is already the zero ideal. A graded ideal \( P \) of \( A \) is called \emph{\( G \)-prime} if the graded algebra \( A/P \) is \( G \)-prime.

Furthermore, \( A \) is called \emph{\( G \)-graded semiprime} if the intersection of all \( G \)-graded prime ideals of \( A \) is zero.
\end{definition}

\begin{remark}
It is a simple task to verify that every \( G \)-primitive algebra is also \( G \)-prime. Moreover, by \ref{lem:3.3}, every \( G \)-semiprime is also (ungraded) semiprime.

In the previous section we established that in the case where \( A \) is \( G \)-primitive and satisfies an ordinary PI, \( A \) is, in fact, of the form \( M_\theta(F^\alpha H) \) given that the field \( F \) is algebraically closed. The \( G \)-graded version of Posner’s theorem asserts a similar “rigidity” result for \( G \)-prime algebras (which satisfy a PI). Here is the precise statement:

\begin{conjecture}
Suppose \( G \) is any group and \( A \) is a \( G \)-prime \( F \)-algebra which also satisfies a PI, then \( S = Z(A)_e \) is a domain and \( (A \subseteq) S^{-1}A \) satisfies the conclusion of \ref{thm:2.21} for the field \( K = S^{-1}S \) (In particular, this algebra is \( G \)-simple and finite dimensional over \( K \)).
\end{conjecture}

In this section we prove this conjecture for \( G \) residually finite. The first step is to establish the claim for \( G \) finite. For this we follow the footsteps of Rowen (\cite{10}) and prove the following key theorem.

\begin{theorem}
Suppose \( A \) is a \( G \)-graded \( F \)-algebra, where \( G \) is a \emph{finite} group and \( F \) is of characteristic zero. Suppose further that \( A \) is \( G \)-semiprime and ungraded PI. Then, every \( G \)-graded ideal \( I \) intersects non-trivially \( Z(A)_e \).
\end{theorem}
Proof. By [5] \( A[x] = A \otimes_F F[x] \) is G-semisimple, where the G-grading is given by \( A[x]_g = A_g[x] \). Since \( Z(A[x]) = (Z(A))[x] \), it is suffice to prove the theorem for \( A[x] \) and \( I[x] \), so we may assume from the beginning that \( A \) is G-semisimple. By lemma 2.30 \( I \) is G-semisimple and \( Z(I) \subseteq Z(A) \). So it is enough to show that every G-semisimple PI algebra \( A \) satisfies \( Z(A)_e \neq \{0\} \). This will follow if we will show that \( A \) has a degree \( e \) central polynomial.

By the proof of lemma 2.30 \( A \) is a subdirect product of G-simple \( F \)-algebras \( A_j \), where \( j \in \mathfrak{g} \). In other words, the map

\[
\phi : A \rightarrow \prod_{j \in \mathfrak{g}} A_j
\]

is a G-graded embedding and \( \pi_i : A \rightarrow A_i \) is onto, where \( \pi_i : A \rightarrow \prod_{j \in \mathfrak{g}} A_j \rightarrow A_i \) is the natural map.

Notice that for every \( i \in \mathfrak{g} \), \( A \rightarrow A_i \) is onto, where \( \pi_i : A \rightarrow \prod_{j \in \mathfrak{g}} A_j \rightarrow A_i \) is the natural map.

Remark 4.5. The proof above shows that if \( A \) is G-semiprime (G is a finite group), then every ungraded central polynomial \( f \) of \( A \) is G-strong central polynomial of \( A \).

Remark 4.6. Notice that for \( G \) infinite, it is no longer true that \( A[x] \) is G-semisimple (take \( G = \mathbb{Z} \) and consider \( A = F[x] \) where \( \deg x^n = n \in \mathbb{Z} \)). As a result, in order to generalize the previous theorem to infinite groups one should introduce a different idea.

As a result we obtain [4.3] for finite groups. Here is the precise statement.

**Corollary 4.7.** Let \( G \) be a finite group and \( F \) a field of characteristic zero. Suppose \( A \) is G-prime over \( F \) and satisfies an ordinary PI. Then, \( S = Z(A)_e^\times \) does not contain zero divisors of \( A \). As a result \( A \) is G-embedded in \( A_1 = S^{-1}A \) which is a G-graded \( \mathbb{K} = S^{-1}Z(A)_e \)-algebra. Moreover, \( A_1 \) is finite dimensional G-simple \( \mathbb{K} \)-algebra.

Proof. Regarding the first part, it is enough to prove that no element \( c \in Z(A)_e^\times \) annihilates a non-zero homogenous element \( a \in A \). Suppose otherwise: \( ca = 0 \). Hence, \( AcA \cdot AaA \) is a product of graded ideals which is equal to \( A^2caA = 0 \). Thus, \( AcA = 0 \) or \( AaA = 0 \). It is therefore enough to show that \( AaA = 0 \) is impossible (since it is more general than \( AcA = 0 \)). Indeed, in that case \( Aa \) is a graded ideal, thus we obtain from \( AaA = 0 \) that \( Aa = 0 \) (since \( A \neq 0 \)). But now \( Fa \) is an ideal and so \( Fa = 0 \), which yields \( a = 0 \) - a contradiction.

Suppose \( I \) is a non-zero G-graded ideal of \( A_1 \). Since \( I \cap \mathbb{K} = I \cap Z(A_1) \neq 0 \), we conclude that \( I \) must be equal to \( A_1 \). By 2.25 we are done.

### 4.0.4 Grading by a quotient group

**Definition 4.8.** Let \( A \) be an \( F \)-algebra graded by \( G \) and let \( Q = G/N \) be any quotient group of \( G \). We define the **induced \( Q \)-grading** on \( A \) by setting

\[
A_{gN} = \oplus_{h \in N} A_{gh},
\]

for every \( g \in G \).
Example 4.9. Two notable special cases are when \( N = \{ e \} \) and \( N = G \). In the first case, we obtain the given \( G \)-grading on \( A \) and in the second we are in the situation of having no grading at all.

**Definition 4.10.** Let \( G \) be a group and \( Q = G/N \) be any quotient group of \( G \). Denote by \( \psi_{G,Q} \) the \( \mathbb{F} \)-algebra map \( \psi_{G,Q} : \mathbb{F} \langle X_G \rangle \to \mathbb{F} \langle X_Q \rangle \) induced by \( \psi_{G,Q}(x_{i,g}) = x_{i,gN} \).

Furthermore, \( f \in \mathbb{F} \langle X_G \rangle \) is said to be \( Q \)-stable if the diagram:

\[
\begin{array}{ccc}
\mathbb{F} \langle X_G \rangle & \xrightarrow{\psi_{G,Q}} & \mathbb{F} \langle X_Q \rangle \\
\rho_g \downarrow & & \downarrow \rho_{gN} \\
\mathbb{F} \langle X_Q \rangle_{e} & \xrightarrow{\psi_{G,Q}} & \mathbb{F} \langle X_Q \rangle_{eN}
\end{array}
\]

commutes for every \( g \in G \) on \( f \).

Example 4.11. Let \( G = C_4 = \langle \tau \rangle \) and \( N = \langle \tau^2 \rangle \). The polynomial \( f_1 = x_{2,\tau^2} \) is not \( Q \)-stable but \( f_2 = 2x_{1,e} \) is.

**4.0.5 Residually finite case.**

**Theorem 4.12.** Let \( G \) be a residually finite group and \( \mathbb{F} \) a characteristic zero field. Suppose \( A \) is \( G \)-semiprime and ungraded PI. Then, every \( G \)-graded ideal \( I \) intersects non-trivially \( Z(A)_e \).

**Proof.** By [5] \( A \) is a semiprime \( \mathbb{F} \)-algebra. Thus, \( A[x] \) is semisimple. As before we replace \( A \) and \( I \) by \( A[x] \) and \( I[x] \) what allows us to assume that \( I \) (and \( A \)) is a semisimple algebra and \( Z(I) \subseteq Z(A) \). So, it will be suffice to show that \( I \) has a strong central polynomial. In other words, we need to show that if \( A \) is \( G \)-graded and semisimple, it possess a strong central polynomial.

By lemma [3.3] \( A \) is PI equivalent to \( M_n(\mathbb{F}) \), where \( n^2 = \exp(A) \). Hence Regev’s polynomial \( F = L_{n^2}(x_1, \ldots, x_{2n^2}) \) is a central polynomial for \( A \) (in fact, we could as well chosen any other central polynomial of \( M_n(\mathbb{F}) \)).

As a result, there are \( g_1, \ldots, g_{2n^2} \in G \) for which there are \( a_{g_1}, \ldots, a_{g_{2n^2}} \in A_{g_{2n^2}} \) such that \( F(a_{g_1}, \ldots, a_{g_{2n^2}}) \neq 0 \). Denote \( F_G = F(x_{1,g_1}, \ldots, x_{2n^2,g_{2n^2}}) \) and \( f = \rho(F_G) \). We will show that \( f \) is central polynomial of \( A \).

Let \( N \) be a normal finite index subgroup of \( G \) such that all the distinct products of \( \{g_1, \ldots, g_{2n^2}\} \) of length \( 2n^2 \) remain distinct in \( Q = G/N \). Therefore, \( F_G \) is \( Q \)-stable. Regard \( A \) as a \( Q \)-graded algebra. Hence, by the proof of [13] (see also [1.5], \( \rho_{eN}(\psi_{G,Q}(F_G)) \) is \( Q \)-central polynomial of degree \( eN \) for \( A \).

By [4.4] it is clear that

\[
f(A) \subseteq (\rho_{eN}(\psi_{G,Q}(F_G)))(A) \cap A_e \subseteq Z(A)_e.
\]

Finally, \( \psi_{G,Q}(F_G)(a_{g_1}, \ldots, a_{g_{2n^2}}) = F_G(a_{g_1}, \ldots, a_{g_{2n^2}}) \neq 0 \). Hence, the fact that \( F \) is \( Q \)-strong for \( A \), forces

\[
0 \neq \rho_{eN}(\psi_{G,Q}(F_G))(a_{g_1}, \ldots, a_{g_{2n^2}}) = f(a_{g_1}, \ldots, a_{g_{2n^2}}).
\]

All in all, we shown that \( f \) is indeed a \( G \)-graded central polynomial of \( A \). \qed

**Corollary 4.13.** theorem [4.4] holds also when \( G \) is a residually finite group and \( \mathbb{F} \) is of characteristic zero.

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5 A consequence of the G-graded Posner’s Theorem.

In [2] Aljadeff and Haile proved the following theorem:

**Theorem 5.1.** Suppose $G$ is a finite group and $A_1, A_2$ are $G$-simple algebras of finite dimension over an algebraically closed field $\mathbb{F}$ of characteristic zero. Then, $id_G(A_1) = id_G(A_2)$ if and only if $A_1$ and $A_2$ are $G$-isomorphic over $\mathbb{F}$.

**Note 5.2.** It is clear that if $A_1$ is $G$-isomorphic to $A_2$, then $A_1$ and $A_2$ share the same $G$-graded identities. The other direction requires all the work.

The proof they presented is based heavily on an elaborate analysis of the structure of a $G$-simple algebra given in theorem [2,24]. Here we present an alternative proof based on the $G$-graded Posner theorem proven in the previous section.

To start, recall the construction of the ($G$-graded) generic algebra over $\mathbb{F}$. Given a finite dimensional $G$-graded $\mathbb{F}$-algebra, chose a $G$-graded $\mathbb{F}$-basis of $A$: $B = \cup_{g \in G} B_g$ ($B_g$ consists of elements of a basis of $A_g$) and consider the commutative variables

$$A = \{t_{i,g,b} | b \in B_g, g \in G, i \in \mathbb{N}\}. $$

Finally denote by $U_A$ the $\mathbb{F}$-subalgebra of $A \otimes_{\mathbb{F}} \mathbb{F}(\Lambda)$ generated by the elements $y_{i,g} = \sum_{t_{i,g,b} \in B_g} t_{i,g,b}$. It is easy to verify that the $G$-homomorphism

$$\phi : \mathbb{F}(X_G) / id_G(A) \to U$$

given by $\phi(x_{i,g}) = y_{i,g}$ is well defined and is $G$-isomorphism. As a result, we may identify $\mathbb{F}(X_G) / id_G(A)$ with $U$.

In our case we have $U_i \cong \mathbb{F}(X_G) / id_G(A_i)$ ($i = 1, 2$) and since $id_G(A_1) = id_G(A_2)$, we must have $U_1 = U_2$. As a result, we will denote this algebra (also) by $U$. Notice that $U \subseteq A_i \otimes_{\mathbb{F}} \mathbb{F}(\Lambda_i)$.

**Lemma 5.3.** $U$ is $G$-semisimple.

**Proof.** If $J_G(U) \neq 0$, then there is a $G$-graded substitution of the variables $x_{i,g} \in X_G$ by elements from $A_1$ (of course, we could as well use $A_2$) for which $J_G(U)$ is mapped to a non-zero ideal $I$ in $A_i$ and the corresponding map is surjective. However, it is well known that Jacobson radical is mapped to Jacobson radical when the map under consideration is surjective. We got a contradiction to $A_1$ being $G$-semisimple. □

**Lemma 5.4.** $Z(U_e) \subseteq \mathbb{F}(\Lambda_i) \cdot 1_{A_i} \otimes 1$ for $i = 1, 2$. As a result, $Z(U_e)$ does not contain any zero-divisors of $U$.

**Proof.** First observe that $Z(A_i)_e = \mathbb{F}1_{A_i}$. This can be verified easily using theorem [2,24]. As a result, if $f(y_{1,g_1}, ..., y_{n,g_n}) \in Z(U_e)$, then $\phi(f) \in Z(A_i \otimes \mathbb{F}(\Lambda_i))_e = Z(A_i)_e \otimes \mathbb{F}(\Lambda_i) = \mathbb{F}(\Lambda_i) \cdot 1_{A_i} \otimes 1$. □

**Remark 5.5.** It is possible to prove that $Z(A_i) = \mathbb{F}1_{A_i}$ by only using the classical Wedderburn theorem for (ungraded) semisimple algebras. Here is a sketch: Set $A = A_1$. Since the Jacobson radical is nilpotent for commutative algebras, one shows that the semisimplicity of $A$ forces $Z(A_e)$ to be also semisimple. Next, if $e_1, e_2$ are two nonzero orthogonal idempotents in $Z(A)_e$, then $Ae_i$ is $G$-graded ideal of $A$, so equals $A$. Hence, there is $a \in A$ for which $ae_1 = e_2$. But this is an absurd since $0 = e_1 \cdot e_2 = a \cdot e_2^2 = ae_2 = e_1$. Thus $Z(U_e)$ must be a field. Since surely, $\mathbb{F}1 \subseteq Z(U_e)$ and $\mathbb{F}$ is algebraically closed, we must have equality.

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Notation 5.6. \( S = Z(U)e - \{0\} \).

Due to the previous Lemma,

\[ U \subseteq S^{-1}U \subseteq A_i \otimes F(\Lambda_i). \]

Now by theorem 4.1 we know that \( S^{-1}U \) is \( G \)-simple and f.d. over \( K := Z(S^{-1}U)e \subseteq F(\Lambda_i) \) (which is a field). Furthermore, since the graded identities of a \( G \)-simple algebra are defined over an algebraically closed field (this follows from theorem 2.26 and the proof of theorem 3.9) and the fact that \( id_{G,F}(A_i) = id_{G,F}(S^{-1}U) \) forces

\[ id_{G,F}(A_i)(S^{-1}U \otimes_K F(\Lambda_i)) = id_{G,F}(A_i) (A_i \otimes F(\Lambda_i)). \]

Lemma 5.7. The \( F(\Lambda_i) \)-algebra \( S^{-1}U \otimes_K F(\Lambda_i) \) is \( G \)-simple. Furthermore, the map

\[ \nu : S^{-1}U \otimes_K F(\Lambda_i) \rightarrow S^{-1}U \cdot F(\Lambda_i) \]

is a \( G \)-graded isomorphism.

Proof. The first statement is clear by 2.20 Moreover, it is also clear that \( \nu \) is a \( G \)-graded epimorphism. Finally, the kernel of \( \nu \) (which is a \( G \)-graded ideal of \( S^{-1}U \otimes_K F(\Lambda_i) \)) is zero, thus proving that \( \nu \) is a \( G \)-graded isomorphism. \( \Box \)

Lemma 5.8. \( S^{-1}U \cdot F(\Lambda_i) = A_i \otimes F(\Lambda_i) \).

Proof. We know by now that \( S^{-1}U \cdot F(\Lambda_i) \) is a \( G \)-simple \( F(\Lambda_i) \)-subalgebra of \( A_i \otimes F(\Lambda) \) having the same \( G \)-graded identities. So

\[ \dim_{F(\Lambda_i)} S^{-1}U \cdot F(\Lambda_i) = \exp_G S^{-1}U \cdot F(\Lambda_i) = \exp_G A_i \otimes F(\Lambda) = \dim_{F(\Lambda_i)} A_i \otimes F(\Lambda). \]

Thus the statement follows. \( \Box \)

We have shown that for a field \( L \) containing both \( F(\Lambda_1) \) and \( F(\Lambda_2) \), one has

\[ A_1 \otimes_F L = A_2 \otimes_F L. \]

To conclude that \( A_1 \) and \( A_2 \) are \( G \)-isomorphic we only need to use the well known fact that if \( X \) is any algebraic variety over a field \( (F \subseteq) L \), then since \( F \) is algebraically closed, \( X(F) \) (the \( F \)-points of \( X \)) is dense in \( X \). Indeed, we take \( X \) to be \( \text{Hom}_{G,L}(A_1 \otimes_F L, A_2 \otimes_F L) \). Notice that \( X \) is defined over \( F \). The subset \( V = \text{Iso}_{G,L}(A_1 \otimes_F L, A_2 \otimes_F L) \) is open (defined by non-vanshing of a certain determinant) and non-empty. Hence, \( V \) contains a point defined over \( F \) - finishing the proof.

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