Generalized Electric Polarizability of the Proton from Skyrme Model

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(March 26, 2022)

Abstract

We calculate the electric polarizability $\alpha(q^2)$ of the proton in virtual Compton scattering using the Skyrme model. The $q^2$ dependence of the polarizability is comparable with the predictions obtained from the non-relativistic quark model and the linear sigma model. The chiral behaviors of our $\alpha(0)$ and $d^2\alpha(0)/dq^2$ agree with the results of the chiral perturbation theory. The discrepancy can be traced back to the contribution of the intermediate $\Delta$ state degenerate with the $N$ which is a characteristic of a large-$N_C$ model.
I. INTRODUCTION

Recently there has been much study on the virtual Compton scattering (VCS). Several experiments have been proposed and are being performed in order to get more information about the structure of the nucleon via the process $[1–4]$. At low energies, real Compton scattering from the nucleon are known to be described in terms of two quantities for the nucleon structure - the electric ($\alpha$) and the magnetic ($\beta$) polarizability. It is a natural thought that the virtual Compton scattering could provide even more information on the structure of the nucleon. They are called the generalized polarizabilities and are functions of the four-momentum transfer $q^2$ of the in-going virtual photon. The virtual photon could also carry a longitudinal polarization.

Theoretically Guichon, Liu and Thomas developed the formalism of the generalized polarizabilities $[4]$. They analyzed the structure-dependent part beyond the low energy theorem in terms of a multipole expansion. They also gave a first estimate for the ten generalized polarizabilities using the non-relativistic quark model and elaborated the study including the recoil effects $[5]$. Using an effective Lagrangian Vanderhaeghen $[6]$ obtained the $q^2$ behavior of the electric and the magnetic polarizabilities of the proton. The slopes of the electric and the magnetic polarizabilities were predicted by several authors with several frameworks, such as the linear sigma model $[7]$ and the heavy-baryon formulation of the chiral perturbation theory $[8]$.

In this letter, we attempt a different point of view to describe the generalized electric polarizability based on a topological soliton model (Skyrme model). Within many soliton models, only the expression and the value of $\alpha$ at $q^2 = 0$ have been obtained. The present work is the first trial to give the slope of $\alpha$ among the topological and non-topological soliton models. We also obtain the leading chiral terms of $\alpha$ and its slope and show that these agree with the results of the linear sigma model and the heavy-baryon chiral perturbation theory.

II. THE SKYRME MODEL IN THE PRESENCE OF ELECTROMAGNETIC FIELDS

Our starting point is a $U(1)$-gauged effective model. The gauged Skyrme Lagrangian is

$$\mathcal{L} = \frac{f_\pi^2}{4} Tr[D_\mu U^\dagger D^\mu U] + \frac{1}{32\epsilon^2} Tr \left[ U^\dagger D_\mu U, U^\dagger D_\nu U \right]^2 + \frac{f_\pi^2}{2} m_\pi^2 (2 - Tr[U])$$  \hspace{1cm} (2.1)

where $f_\pi$ is the pion decay constant, $\epsilon$ is the dimensionless Skyrme parameter and the $U$ is the $SU(2)$ chiral field. The covariant derivative is defined as

$$D_\mu U = \partial_\mu U + ieA_\mu U, \hspace{1cm} (2.2)$$
where $A_\mu$ is the electromagnetic field and $Q$ is the charge matrix.

$$Q = \frac{1}{6} + \frac{\tau_3}{2}. \quad (2.3)$$

The Lagrangian can be decomposed into three terms depending on the number of the gauge field.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (2.4)$$

$$\mathcal{L}_0 = \frac{f^2}{4} Tr[L_\mu L^\mu] + \frac{1}{32\epsilon^2} Tr\left[[L_\mu, L_\nu]^2\right] + \frac{f^2}{2} m^2 \pi (2 - Tr[U]), \quad (2.5)$$

$$\mathcal{L}_1 = e A_\mu J^\mu, \quad (2.6)$$

$$\mathcal{L}_2 = -\frac{1}{2} \epsilon^2 A_\mu S^{\mu\nu} A_\nu, \quad (2.7)$$

$$J_\mu = i \frac{f^2}{2} Tr\left[Q(L_\mu + R_\mu)\right] + i \frac{1}{8\epsilon^2} Tr\left[Q\left([[L_\mu, L_\nu], L''\right] + [[R_\mu, R_\nu], R''\right])\right], \quad (2.8)$$

$$S_{\mu\nu} = -g_{\mu\nu} \frac{f^2}{2} Tr[P^2] + \frac{1}{4\epsilon^2} \left[g_{\mu\nu} h_\alpha^\alpha - h_{\mu\nu}\right] \quad (2.9)$$

where

$$L_\mu = U^\dagger \partial_\mu U, \quad R_\mu = U \partial_\mu U^\dagger, \quad (2.10)$$

$$P = Q - U^\dagger QU, \quad h_{\mu\nu} = Tr[PL_\mu PL_\nu - P^2 L_\mu L_\nu]. \quad (2.11)$$

The Lagrangian $\mathcal{L}_0$ has the classical solution for the configuration called the hedgehog

$$U_h = exp[i\tau \cdot \hat{r} F(r)]. \quad (2.12)$$

The profile function $F(r)$ satisfy the following nonlinear equation

$$\frac{d^2 F}{d\tilde{r}^2} (\tilde{r}^2 + 2\sin^2 F) + 2 \frac{dF}{d\tilde{r}} \tilde{r} + \left(\frac{dF}{d\tilde{r}}\right)^2 \sin(2F) - \sin(2F) \left(1 + \frac{\sin^2 F}{\tilde{r}^2}\right)$$

$$- \left(\frac{m\pi}{f_\pi \epsilon}\right)^2 \sin(F) \tilde{r}^2 = 0 \quad (2.13)$$

where $\tilde{r} = \epsilon f_\pi r$. For the topological charge $B = 1$, the boundary condition is

$$F(0) = \pi, \quad F(\infty) = 0. \quad (2.14)$$

### III. Notations and Generalized Polarizabilities

In this article the virtual Compton scattering refers to the reaction

$$\gamma^* + p \rightarrow \gamma + p \quad (3.1)$$
where $\gamma^*$, $\gamma$ and $p$ are respectively a space-like virtual photon, a real photon and a proton. The momentum of the virtual (real) photon is represented by $q$ ($q'$).

According to the low energy theorem (LET) the leading order terms of the scattering amplitude in an expansion of $q'_0$ is completely determined by the Born amplitude. And it depends only on the ground state properties of the nucleon. Guichon et al. [4] extended LET to include the higher order terms in terms of a multipole expansion. The higher order terms rely on the excitation of the nucleon and are determined by the non-Born amplitude. The same authors developed an appropriate formalism to describe the non-Born amplitude and defined the so-called generalized polarizabilities (GPS), which are functions of $q^2$. The GPS include the electric and the magnetic polarizabilities defined through the real Compton scattering amplitude or the static energy shifts in the presence of uniform electromagnetic fields. We refer the details of the analysis of GPS to the reference [4] and use their definitions only.

A generalized polarizability (GP) is defined as

$$P^{(\rho' L', \rho L)}(q) = \frac{1}{q' L' q L} H_{NB}^{(\rho' L', \rho L)}(q', q) \mid_{q'=0}, \quad \rho', \rho = 0 \text{ or } 1,$$

where the reduced multipoles are defined by

$$H_{NB}^{(\rho' L', \rho L)}(q', q) = \frac{1}{2S+1} \sum_{\sigma' M' M} (-1)^{1/2+\sigma'+L+M}$$

$$\times \langle \frac{1}{2}, -\sigma'; \frac{1}{2}, +\sigma \mid Ss \rangle \langle L', M'; L, -M \mid Ss \rangle H_{NB}^{\rho' L' M', \rho L M}(q', q).$$

The expression for the multipole is

$$H_{NB}^{\rho' L' M', \rho L M}(q', q) = \int d\hat{q} \int d\hat{q}' V_{\mu}^{*}(\rho' L' M', \hat{q}') H_{NB}^{\mu \nu}(q', q) V_{\nu}(\rho L M, \hat{q}),$$

where the $H_{NB}^{\mu \nu}(q', q)$ is the non-Born VCS amplitude and the $V_{\mu}(\rho L M, \hat{q})$ ($\rho = 0, \ldots, 3$) are the complete 4-basis vectors defined as

$$V_{\mu}(0LM, \hat{q}) = \begin{bmatrix} Y_M^L(\hat{q}) \\ 0 \end{bmatrix}, \quad V_{\mu}(1LM, \hat{q}) = \begin{bmatrix} 0 \\ \frac{L_{q} Y_M^L(\hat{q})}{\sqrt{L(L+1)}} \end{bmatrix},$$

$$V_{\mu}(2LM, \hat{q}) = \begin{bmatrix} 0 \\ -i\hat{q} \times \frac{L_{q} Y_M^L(\hat{q})}{\sqrt{L(L+1)}} \end{bmatrix}, \quad V_{\mu}(3LM, \hat{q}) = \begin{bmatrix} 0 \\ \frac{\hat{q} Y_M^L}{L(L+1)} \end{bmatrix}.$$
\[
\alpha(q) = -\sqrt{\frac{3}{2}} e^2 P^{(01,01)}_0(q), \quad \beta(q) = -\sqrt{\frac{3}{8}} e^2 P^{(11,11)}_0(q). \tag{3.8}
\]

Using the definitions given in Eq.(3.3) we obtain
\[
\alpha(q) = \frac{1}{2} \left(\frac{e}{4\pi}\right)^2 \frac{1}{q' q} \int d\hat{q} \int d\hat{q}' \sum_{\sigma = \pm 1/2} \hat{q}' H^0_{NB}(q' \sigma', q\sigma) \hat{q}_i \big|_{q' = 0}, \tag{3.9}
\]
\[
\beta(q) = \frac{1}{2} \left(\frac{e}{4\pi}\right)^2 \frac{1}{q' q} \int d\hat{q} \int d\hat{q}' \sum_{\sigma = \pm 1/2} \epsilon_{i\alpha} \hat{q}' H^\alpha_{NB}(q' \sigma', q\sigma) \epsilon_{\beta\nu} \hat{q}_m \big|_{q' = 0}, \tag{3.10}
\]

where the integrations are over the spherical angles of \( q \) and \( q' \) and the amplitude is spin-averaged. Here we also give the formula of \( \beta(q) \) for future use.

**IV. RESULTS AND CONCLUSIONS**

As is clear from the interaction (2.4), the \( \alpha(q) \) will have two contributions; one is the seagull term and the other the dispersive term. The seagull term, \( \alpha_s \), is coming from the Lagrangian quadratic in \( A_\mu \), Eq. (2.7). The dispersive term, \( \alpha_d \), is coming from the second order perturbation theory applied to the linear interaction in \( A_\mu \), Eq. (2.6).

The computation of the \( \alpha_d \) needs the calculation of the transition matrix elements between the nucleon and the negative parity excited states.

\[
\alpha_d(q) = 2 \sqrt{\frac{3}{2}} \sum_X \langle N | d(0) | X \rangle \langle X | d(q) | N \rangle \frac{1}{m_N - m_X}, \tag{4.1}
\]

\[
d(q) = \int d^3r z J_0(r) \frac{3 j_1(qr)}{qr}, \tag{4.2}
\]

where \( j_1 \) is the 1-st order spherical Bessel function. The negative parity excited state, \( X \), most contributing to the electric dipole matrix element \( \langle X | d(q) | N \rangle \) is the bound state of the ground state nucleon and the quantum pion. There is a general belief that the \( \alpha_d(0) \) is much smaller than the \( \alpha_s(0) \) \cite{9,10}. Thus we neglect the dispersive term.

Using the Lagrangian (2.7) and the formula given in Eq. (3.3) we obtain
\[
\alpha(q) = e^2 \frac{8\pi}{9} \int_0^\infty drr^4 \sin^2 F \left[ f^2_\pi + \frac{1}{e^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right] \frac{3 j_1(qr)}{qr}. \tag{4.3}
\]

Therefore
\[
\alpha(0) = e^2 \frac{8\pi}{9} \int_0^\infty drr^4 \sin^2 F \left[ f^2_\pi + \frac{1}{e^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right], \tag{4.4}
\]
\[
\frac{d\alpha(q)}{d(q^2)} \big|_{q=0} = -e^2 \frac{4\pi}{45} \int_0^\infty drr^6 \sin^2 F \left[ f^2_\pi + \frac{1}{e^2} \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right]. \tag{4.5}
\]
For the purpose of an illustration we have performed a numerical calculation for the set of parameters of $f_{\pi} = 93$ MeV, $\epsilon = 4.26$, which gives the magnetic moments of proton and neutron as 3.2 and -2.5 $\mu_N$, respectively. Our numerical result is

$$\alpha(0) = 8.44 \times 10^{-4} \text{ fm}^3, \quad \frac{d\alpha(q)}{d(q^2)} |_{q=0} = -10.6 \times 10^{-4} \text{ fm}^5. \quad (4.6)$$

The value of the $\alpha(0)$ is comparable with the empirical value

$$\alpha(0)^{\text{expt}} = (10.9 \pm 2.2 \pm 1.4) \times 10^{-4} \text{ fm}^3. \quad (4.7)$$

The experimental value of the $\frac{d\alpha(q)}{d(q^2)} |_{q=0}$ is not yet available.

In Fig. 1, we plot the $q^2$ dependence of the generalized electric polarizability. The experiment is being performed but results are not yet available to compare with our plot. Our result is comparable with the predictions of the linear sigma model [7] and the constituent quark model [4], the comparisons with which are made in the following in addition to the comparison with CHPT [8].

Using the equation (2.13) and the Goldberger-Treiman relation [11] one can easily show that the function $F(r)$ behaves asymptotically as

$$F(r) \sim \frac{3}{8\pi} \frac{g_A}{f_\pi^2} \frac{1}{r^2} (1 + m_\pi r) e^{-m_\pi r}. \quad (4.8)$$

Thus $\alpha_s(0)$ and $\frac{d^2\alpha_s(q)}{dq^2} |_{q=0}$ diverge in the chiral limit since their integrands given in Eqs. (4.4) and (4.5) do not asymptotically vanish. We pick up the diverging chiral behaviors of the quantities which read

$$\alpha(0) \to e^2 \frac{5}{32\pi} \left( \frac{g_A}{f_\pi} \right)^2 \frac{1}{m_\pi}, \quad (4.9)$$

$$\frac{d\alpha(q)}{d(q^2)} |_{q=0} \to -e^2 \frac{1}{64\pi} \left( \frac{g_A}{f_\pi} \right)^2 \frac{1}{m_\pi^2}. \quad (4.10)$$

These agree with the results of the linear sigma model (LSM) [7] and the chiral perturbation theory (CHPT) [8] up to a numerical factor. Our leading chiral terms are three times larger than theirs. The differences can be explained by the contribution of the degenerate $\Delta$ states which were not included in LSM and CHPT. More explicitly in the limit of $m_\pi \to 0$

$$\alpha^{\text{Skyrme}} = \left[ 1 + \frac{8}{9} \left( \frac{g_{\pi N\Delta}}{g_{\pi NN}} \right)^2 \right] \alpha^{\text{CHPT}} \quad (4.11)$$

where the ratio $\frac{g_{\pi N\Delta}}{g_{\pi NN}}$ is $\frac{3}{2}$ in the Skyrme model and the number $\frac{8}{9}$ is the characteristic of a scalar-isoscalar operator. Numerically the value of the leading chiral term $\alpha^{\text{Skyrme}} (\alpha^{\text{CHPT}})$ only overestimates (underestimates) the electric polarizability. In the Skyrme model the full expression given in Eq. (3.9) is smaller than $\alpha^{\text{Skyrme}}$. In CHPT the next order chiral
terms include the contribution of the $\Delta$ state with finite $N - \Delta$ splitting via a low energy constant $[12]$. Of course the detailed analysis of the virtual Compton scattering described by the generalized polarizabilities requires high quality data which will be available in near future.

**ACKNOWLEDGMENTS**

We are thankful to Yongseok Oh for valuable discussions. This work was supported by the Korean Science and Engineering Foundation through the Center for Theoretical Physics and by Korean Ministry of Education through contract number ISBR 96-2418.
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FIG. 1. The squared 4-momentum dependence of the electric polarizability. The horizontal axis is in unit of $GeV^2$ and the vertical axis is in unit of $fm^3$. 