Geometric Ergodicity and the Spectral Gap
of Non-Reversible Markov Chains

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Abstract

We argue that the spectral theory of non-reversible Markov chains may often be more effectively cast within the framework of the naturally associated weighted-$L_\infty$ space $L^V_\infty$, instead of the usual Hilbert space $L_2 = L_2(\pi)$, where $\pi$ is the invariant measure of the chain. This observation is, in part, based on the following results. A discrete-time Markov chain with values in a general state space is geometrically ergodic if and only if its transition kernel admits a spectral gap in $L^V_\infty$. If the chain is reversible, the same equivalence holds with $L_2$ in place of $L^V_\infty$, but in the absence of reversibility it fails: There are (necessarily non-reversible, geometrically ergodic) chains that admit a spectral gap in $L^V_\infty$ but not in $L_2$. Moreover, if a chain admits a spectral gap in $L_2$, then for any $h \in L_2$ there exists a Lyapunov function $V_h \in L_1$ such that $V_h$ dominates $h$ and the chain admits a spectral gap in $L^V_\infty$. The relationship between the size of the spectral gap in $L^V_\infty$ or $L_2$, and the rate at which the chain converges to equilibrium is also briefly discussed.

Keywords: Markov chain, geometric ergodicity, spectral theory, stochastic Lyapunov function, reversibility, spectral gap.

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1 Introduction and Main Results

There is increasing interest in spectral theory and rates of convergence for Markov chains. Research is motivated by elegant mathematics as well as a range of applications. In particular, one of the most effective general methodologies used to establish bounds on the convergence rate of a geometrically ergodic chain is via an analysis of the spectrum of the chain’s transition kernel. See, e.g., [16, 22, 21, 8, 5, 12, 11, 13, 23, 19, 3, 2, 17, 14, 7, 6], and the relevant references therein.

The word spectrum naturally invites techniques grounded in a Hilbert space framework. The majority of quantitative results on rates of convergence are obtained using such methods, within the Hilbert space $L_2 = L_2(\pi)$, where $\pi$ denotes the stationary distribution of the Markov chain in question. Indeed, most successful studies have been carried out for Markov chains that are reversible, in which case a key to analysis is the fact that the transition kernel, viewed as a linear operator on $L_2$, is self-adjoint. In this paper we argue that, in the absence of reversibility, the Hilbert space framework may not be the appropriate setting for spectral analysis.

To be specific, let $X = \{X(n) : n \geq 0\}$ denote a discrete-time Markov chain with values on a general state space $X$. We assume that $X$ is equipped with a countably generated sigma-algebra $B$. The distribution of $X$ is described by its initial state $X(0) = x_0 \in X$ and the transition semigroup $\{P^n : n \geq 0\}$, where, for each $n$,

$$P^n(x, A) = \text{Pr}\{X(n) \in A \mid X(0) = x\}, \quad x \in X, A \in B.$$ 

For simplicity we write $P^1$ for the one-step kernel $P^1$. Recall that each $P^n$, like any (not necessarily probabilistic) kernel $Q(x, dy)$ acts on functions $F : X \to \mathbb{C}$ and signed measures $\nu$ on $(X, B)$, via,

$$QF(\cdot) = \int_X Q(\cdot, dy)F(y) \quad \text{and} \quad \nu Q(\cdot) = \int_X \nu(dx)Q(x, \cdot),$$

whenever the integrals exist. Throughout the paper, we assume that the chain $X = \{X(n)\}$ is $\psi$-irreducible and aperiodic; cf. [15, 18]. This means that there is a $\sigma$-finite measure $\psi$ on $(X, B)$ such that, for any $A \in B$ with $\psi(A) > 0$, and any $x \in X$,

$$P^n(x, A) > 0, \quad \text{for all } n \text{ sufficiently large}.$$ 

Moreover, we assume that $\psi$ is maximal in the sense that any other such $\psi'$ is absolutely continuous with respect to $\psi$.

1.1 Geometric ergodicity

The natural class of chains to consider in the present context is that of geometrically ergodic chains, namely, chains with the property that there exists an invariant measure $\pi$ on $(X, B)$ and functions $\rho : X \to (0, 1)$ and $C : X \to [1, \infty)$, such that,

$$\|P^n(x, \cdot) - \pi\|_{TV} \leq C(x)\rho(x)^n, \quad \text{for all } n \geq 0, \pi\text{-a.e. } x \in X,$$

where $\|\mu\|_{TV} = \sup_{A \in B} |\mu(A)|$ denotes the total variation norm on signed measures. Under $\psi$-irreducibility and aperiodicity this is equivalent [15, 19] to the seemingly stronger requirement
that there is a single constant \( \rho \in (0, 1) \), a constant \( B < \infty \) and a \( \pi \)-a.e. finite function \( V : X \to [1, \infty] \), such that,

\[
\|P^n(x, \cdot) - \pi\|_V \leq BV(x)\rho^n, \quad \text{for all } n \geq 0, \ \pi\text{-a.e. } x \in X,
\]

where \( \|\mu\|_V := \sup\{\int Fd\mu : F \in L^V_\infty\} \) denotes the \( V \)-norm on signed measures, and where \( L^V_\infty \) denotes the weighted-\( L_\infty \) space consisting of all measurable functions \( F : X \to \mathbb{C} \) with,

\[
\|F\|_V := \sup_{x \in X} \frac{|F(x)|}{V(x)} < \infty.
\]

Another equivalent and operationally simpler definition of geometric ergodicity for a \( \psi \)-irreducible, aperiodic chain \( X = \{X(n)\} \), is that it satisfies the following drift criterion [15]:

\[
\begin{align*}
\text{There is a function } V : X &\to [1, \infty], \text{ a small set } C \subset X, \\
\text{and constants } \delta > 0, \ b < \infty, \text{ such that:} \\
PV &\leq (1 - \delta)V + b\mathbb{1}_C.
\end{align*}
\]

We then say that the chain is \textit{geometrically ergodic with Lyapunov function} \( V \). In (V4) it is always assumed that the Lyapunov function \( V \) is finite for at least one \( x \) (and then it is necessarily finite \( \psi \)-a.e.). Also, recall that a set \( C \in \mathcal{B} \) is \textit{small} if there exist \( n \geq 1, \epsilon > 0 \) and a probability measure \( \nu \) on \( (X, \mathcal{B}) \) such that, \( P^n(x, A) \geq \epsilon \mathbb{1}_C(x)\nu(A) \), for all \( x \in X, A \in \mathcal{B} \).

Our first result relates geometric ergodicity to the spectral properties of the kernel \( P \). Its proof, given at the end of Section 3, is based on ideas from [12]. See Section 2 for more precise definitions.

**Proposition 1.1.** A \( \psi \)-irreducible and aperiodic Markov chain \( X = \{X(n)\} \) is geometrically ergodic with Lyapunov function \( V \) if and only if \( P \) admits a spectral gap in \( L^V_\infty \).

### 1.2 Reversibility

Recall that the chain \( X = \{X(n)\} \) is called \textit{reversible} if there is a probability measure \( \pi \) on \( (X, \mathcal{B}) \) satisfying the detailed balance conditions,

\[
\pi(dx)P(x, dy) = \pi(dy)P(y, dx).
\]

This is equivalent to saying that the linear operator \( P \) is self-adjoint on the space \( L_2 = L_2(\pi) \) of (measurable) functions \( F : X \to \mathbb{C} \) that are square-integrable under \( \pi \), endowed with the inner product \( (F, G) = \int FG^* \ d\pi \), where ‘*’ denotes the complex conjugate operation.

The following result is the natural analog of Proposition 1.1 for reversible chains. Its proof, given in Section 3, is partly based on results in [19].

**Proposition 1.2.** A reversible, \( \psi \)-irreducible and aperiodic Markov chain \( X = \{X(n)\} \) is geometrically ergodic if and only if \( P \) admits a spectral gap in \( L_2 \).
1.3 Spectral theory

The main question addressed in this paper is whether the reversibility assumption of Proposition 1.2 can be relaxed. In other words, whether the space $L^2$ can be used to characterize geometric ergodicity like $L^{V_\infty}$ was in Proposition 1.1. One direction is true without reversibility: A spectral gap in $L^2$ implies that the chain is “geometrically ergodic in $L^2$” [19][20], and this implies the existence of a Lyapunov function $V$ satisfying (V4) [20]. Therefore, the chain is geometrically ergodic in the sense of [12], where it is also shown that it must admit a central gap in $L^{V_\infty}$. A direct, explicit construction of a Lyapunov function $V_h$ is given in our first main result stated next, where quantitative information about $V_h$ is also obtained. It is proved in Section 3.

**Theorem 1.3.** Suppose that a $\psi$-irreducible, aperiodic chain $X = \{X(n)\}$ admits a spectral gap in $L^2$. Then, for any $h \in L^2$, there is $\pi$-integrable function $V_h$, such that the chain is geometrically ergodic with Lyapunov function $V_h$ and $h \in L^{V_\infty}$.

But the other direction may not hold in the absence of reversibility. Based on earlier counterexamples constructed by Häggström [9, 10] and Bradley [1], in Section 3 we prove the following:

**Theorem 1.4.** There exists a $\psi$-irreducible, aperiodic Markov chain $X = \{X(n)\}$ which is geometrically ergodic but does not admit a spectral gap in $L^2$.

1.4 Convergence rates

The existence of a spectral gap is intimately connected to the exponential convergence rate for a $\psi$-irreducible, aperiodic Markov chain. For example, if the chain is reversible, we have the following well-known, quantitative bound. See Section 2 for detailed definitions; the result follows from the results in [19], combined with Lemma 2.2 given in Section 2.

**Proposition 1.5.** Suppose that a reversible chain $X = \{X(n)\}$ is $\psi$-irreducible, aperiodic, and has initial distribution $\mu$. If the chain $X$ admits a spectral gap $\delta^2 > 0$ in $L^2$, then,

$$\|\mu P^n - \pi\|_{TV} \leq \frac{1}{2} \|\mu - \pi\|_2 (1 - \delta^2)^n, \quad n \geq 1,$$

where the $L^2$-norm on signed measures $\nu$ is defined as the $L^2(\pi)$-norm of the density $d\nu/d\pi$ if it exists, and is set equal to infinity otherwise.

In the absence of reversibility, the size of the spectral gap in $L^{V_\infty}$ precisely determines the exponential convergence rate of any geometrically ergodic chain. The result of the following proposition is stated in Lemma 2.3 in Section 2.

**Proposition 1.6.** Suppose that the chain $X = \{X(n)\}$ is $\psi$-irreducible and aperiodic. If it admits a spectral gap $\delta^V > 0$ in $L^{V_\infty}$, then, for $\pi$-a.e. $x$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|P^n(x, \cdot) - \pi\|_V = \log (1 - \delta^V).$$

In fact, the convergence is uniform in that:

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \sup_{x \in X, \|F\|_V = 1} \frac{|P^n F(x) - \int F d\pi|}{V(x)} \right) = \log (1 - \delta^V).$$
Section 2 contains precise definitions regarding the spectrum and the spectral gap of the kernel $P$ acting either on $L_2$ or the weighted-$L_\infty$ space $L_V^\infty$. Simple properties of the spectrum are also stated and proved. Section 3 contains the proofs of the first four results stated above.

2 Spectra and Geometric Ergodicity

We begin by giving precise definitions for the spectrum and spectral gap of the transition kernel $P$, viewed as a linear operator. The spectrum depends on the domain of $P$, for which we consider two possibilities:

(i) The Hilbert space $L_2 = L_2(\pi)$, equipped with the norm $\|F\|_2 = \left[\int |F|^2 d\pi\right]^{1/2}$.

(ii) The Banach space $L_V^\infty$, with norm $\|\cdot\|_V$ defined in (2).

In either case, the spectrum is defined as the set of nonzero $\lambda \in \mathbb{C}$ for which the inverse $(I_\lambda - P)^{-1}$ does not exist as a bounded linear operator on the domain of $P$. The transition kernel admits a spectral gap if there exists $\varepsilon_0 > 0$ such that $S \cap \{z : |z| \geq 1 - \varepsilon_0\}$ is finite, and contains only poles of finite multiplicity; see [12, Section 4] for more details. The spectrum is denoted $S_2$ when $P$ is viewed as a linear operator on $L_2$, and it is denoted $S_V$ when $P$ is viewed as a linear operator on $L_V^\infty$.

The induced operator norm of a linear operator $\hat{P} : L_V^\infty \to L_V^\infty$ is defined as usual via,

$$\|\hat{P}\|_V := \sup_{F \in L_V^\infty} \frac{\|\hat{P}F\|_V}{\|F\|_V},$$

where the supremum is over all $F \in L_V^\infty$ satisfying $\|F\|_V \neq 0$. An analogous definition gives the induced operator norm $\|\hat{P}\|_2$ of a linear operator $\hat{P}$ acting on $L_2$.

For a $\psi$-irreducible, aperiodic chain $X = \{X(n)\}$, geometric ergodicity expressed in the form (1) implies that $P^n$ converges to a rank-one operator, at a geometric rate: For some constants $B < \infty$, $\rho \in (0, 1)$,

$$\|P^n - 1 \otimes \pi\|_V \leq B \rho^n, \quad n \geq 0,$$

(4)

where the outer product $1 \otimes \pi$ denotes the kernel $1 \otimes \pi(x, dy) = \pi(dy)$. It follows that the inverse $[I_\lambda - P + 1 \otimes \pi]^{-1}$ exists as a bounded linear operator on $L_V^\infty$, whenever $\lambda > \rho$. This in turn implies that $P$ has a single isolated pole at $\lambda = 1$ in the set $\{\lambda \in \mathbb{C} : \lambda > \rho\}$, so that $P$ admits a spectral gap.

In Lemma 2.1 we clarify the location of poles when the chain admits a spectral gap in $L_2$ or $L_V^\infty$.

Lemma 2.1. If a $\psi$-irreducible, aperiodic Markov chain admits a spectral gap in $L_V^\infty$ or $L_2$, then the only pole on the unit circle in $\mathbb{C}$ is $\lambda = 1$, and this pole has multiplicity one.

Proof. We present the proof for $L_V^\infty$; the proof in $L_2$ is identical.

We first note that the existence of a spectral gap implies ergodicity: There is a left eigenmeasure $\mu$ corresponding to the eigenvalue $1$, satisfying $\mu P = \mu$ and $|\mu|(V) = \|\mu\|_V < \infty$. On letting $\pi(\cdot) = |\mu(\cdot)|/|\mu(X)|$ we conclude that $\pi$ is super-invariant: $\pi P \geq \pi$. Since $\pi(X) = 1$
we must have invariance. The ergodic theorem for positive recurrent Markov chains implies that \( E[G(X(n)) \mid X(0) = x] \to \int G \, d\pi \), as \( n \to \infty \), whenever \( G \in L_1(\pi) \).

Ergodicity rules out the existence of multiple eigenfunctions corresponding to \( \lambda = 1 \). Hence, if this pole has multiplicity greater than one, then there is a generalized eigenfunction \( h \in L_\infty \) satisfying,

\[
P h = h + 1.
\]

Iterating gives \( P^n h(x) = E[h(X(n)) \mid X(0) = x] = h(x) + n \) for \( n \geq 1 \). This rules out ergodicity, and proves that \( \lambda = 1 \) has multiplicity one.

We now show that if \( \lambda \in S_V \) with \( |\lambda| = 1 \), then \( \lambda = 1 \). To see this, let \( h \in L_\infty^V \) denote an eigenfunction, \( Ph = \lambda h \). Iterating, we obtain,

\[
E[h(X(n)) \mid X(0) = x] = h(x)\lambda^n.
\]

Then, letting \( n \to \infty \), the right-hand-side converges to \( \int h \, d\pi \) for a.e. \( x \), so that \( \lambda = 1 \) and \( h(x) = \int h \, d\pi \), \( \pi \)-a.e.

Therefore, for a \( \psi \)-irreducible, aperiodic chain, the existence of a spectral gap in \( L_2 \) is equivalent to the existence of a single eigenvalue \( \lambda = 1 \) on the unit circle, which has multiplicity one. The spectral gap \( \delta_2 \) is then defined as,

\[
\delta_2 = 1 - \sup\{|\lambda| : \lambda \in S_2, \lambda \neq 1\},
\]

and similarly for \( \delta_V \).

Next we state two well-known, alternative expressions for the \( L_2 \)-spectral gap \( \delta_2 \) of a reversible chain. See, e.g., [19, Theorem 2.1] and [4, Proposition VIII.1.11].

**Lemma 2.2.** Suppose \( X \) is a \( \psi \)-irreducible, aperiodic, reversible Markov chain. Then, its \( L_2 \)-spectral gap \( \delta_2 \) admits the alternative characterizations,

\[
\delta_2 = 1 - \sup\left\{ \frac{\|\nu P\|_2}{\|\nu\|_2} : \text{signed measures } \nu \text{ with } \nu(X) = 0, \|\nu\|_2 \neq 0 \right\}
\]

\[
= 1 - \lim_{n \to \infty} \left( \left\| P^n - 1 \otimes \pi \right\|_2 \right)^{1/n},
\]

where the limit is the usual spectral radius of the semigroup \( \{\hat{P}^n\} \) generated by the kernel \( \hat{P} = P - 1 \otimes \pi \), acting on functions in \( L_2(\pi) \).

A similar result holds for \( \delta_V \), even in the absence of reversibility; see, e.g., [13].

**Lemma 2.3.** Suppose \( X \) is a \( \psi \)-irreducible, aperiodic Markov chain. Then, its \( L_\infty^V \)-spectral gap \( \delta_V \) admits the following alternative characterization in terms of the spectral radius,

\[
\delta_V = 1 - \lim_{n \to \infty} \left( \left\| P^n - 1 \otimes \pi \right\|_V \right)^{1/n}.
\]
3 Proofs

First we prove Theorem 1.3. The following notation will be useful throughout this section.

For a Markov chain \( X = \{X(n)\} \), the first hitting time and first return time to a set \( C \in \mathcal{B} \) are defined, respectively, by,

\[
\sigma_C := \min \{ n \geq 0 : X(n) \in C \};
\]
\[
\tau_C := \min \{ n \geq 1 : X(n) \in C \}.
\]

(5)

Conditional on \( X(0) = x \), the expectation operator corresponding to the measure defining the distribution of the process \( X = \{X(n)\} \) is denoted \( E_x(\cdot) \), so that, for example, \( P^nF(x) = E[F(X(n)) | X(0) = x] = E_x[F(X(n))] \). For an arbitrary signed measure \( \mu \) on \( (X, \mathcal{B}) \), we write \( \mu(F) \) for \( \int F \, d\mu \), for any function \( F : X \to \mathbb{C} \) for which the integral exists.

Proof of Theorem 1.3. Since \( \pi(h^2) < \infty \), and the chain is \( \psi \)-irreducible, it follows that there exists an increasing sequence of \( h^2 \)-regular sets providing a \( \pi \)-a.e. covering of \( X \). That is, there is a sequence of sets \( \{S_r : r \in \mathbb{Z}_+\} \) such that \( \pi(S_r) \to 1 \) as \( r \to \infty \), \( S_r \subset S_{r+1} \) for each \( r \), and the following bounds hold,

\[
V_r(x) := E_x \left[ \sum_{n=0}^{\tau_{S_r}} h^2(X(n)) \right] < \infty, \quad \text{for } \pi \text{-a.e. } x
\]
\[
\sup_{x \in S_r} V_r(x) < \infty.
\]

Since the chain admits a spectral gap in \( L_2 \), combining Theorem 2.1 of [19] with Lemma 2.2 and the results of [20], we have that it is geometrically ergodic. Hence, from [15, Theorem 15.4.2] it follows that there exists a sequence of Kendall sets providing a \( \pi \)-a.e. covering of \( X \). That is, there is a sequence of sets \( \{K_r : r \in \mathbb{Z}_+\} \) and positive constants \( \{\theta_r : r \in \mathbb{Z}_+\} \) satisfying \( \pi(K_r) \to 1 \) as \( r \to \infty \), \( K_r \subset K_{r+1} \) for each \( r \), and the following bounds hold,

\[
U_r(x) := E_x \left[ \exp(\theta_r \tau_{K_r}) \right] < \infty, \quad \text{for } \pi \text{-a.e. } x
\]
\[
\sup_{x \in K_r} U_r(x) < \infty.
\]

We also define another collection of sets,

\[
C_{r,m} := \{ x \in X : U_r(x) + V_r(x) \leq m \}.
\]

For each \( r \geq 1 \), these sets are non-decreasing in \( m \), and \( \pi(C_{r,m}) \to 1 \) as \( m \to \infty \). Moreover, whenever \( C_{r,m} \in \mathcal{B}^+ \), this set is both an \( h^2 \)-regular set and a Kendall set. This follows by combining Theorems 14.2.1 and 15.2.1 of [15]. Fix \( r_0 \) and \( m_0 \) so that \( \pi(C_{r_0,m_0}) > 0 \). We henceforth denote \( C_{r_0,m_0} \) by \( C \), and let \( \theta > 0 \) denote a value satisfying the bound,

\[
E_x \left[ \exp(\theta \tau_C) \right] < \infty, \quad \text{for } \pi \text{-a.e. } x,
\]

where the expectation is uniformly bounded over the Kendall set \( C \).
The candidate Lyapunov function can now be defined as,

$$V_h(x) := E_x \left[ \sum_{n=0}^{\sigma_C} (1 + |h(X(n))|) \exp\left(\frac{1}{2} \theta n\right) \right].$$  (6)

We first obtain a bound on this function. Writing,

$$V_h(x) = E_x \left[ \sum_{n=0}^{\sigma_C} \exp\left(\frac{1}{2} \theta n\right) \right] + \sum_{n=0}^{\infty} E_x \left[ |h(X(n))| \exp\left(\frac{1}{2} \theta n\right) 1\{n \leq \sigma_C\} \right],$$

we see that the first term is finite $\pi$-a.e. by construction. The square of the second term is bounded above, using the Cauchy-Shwartz inequality, by,

$$E_x \left[ \sum_{n=0}^{\infty} |h(X(n))|^2 1\{n \leq \sigma_C\} \right] E_x \left[ \sum_{n=0}^{\infty} \exp(\theta n) 1\{n \leq \sigma_C\} \right] = U_{r_0}(x) E_x \left[ \sum_{t=0}^{\sigma_C} \exp(\theta t) \right],$$

so that $V_h$ is finite $\pi$-a.e., and we also easily see that $|h| \leq V_h$ so that $h \in L^{V_h}_\infty$.

Next we show that $V_h$ satisfies (V4): First apply $e^{\frac{1}{2} \theta P}$ to the function $V_h$ to obtain,

$$e^{\frac{1}{2} \theta PV_h}(x) = E_x \left[ \sum_{n=1}^{\tau_C} (1 + |h(X(n+1))|) \exp\left(\frac{1}{2} \theta (t+1)\right) \right]$$

(7)

We have $\tau_C = \sigma_C$ when $X(0) \in C^c$. This gives,

$$e^{\frac{1}{2} \theta PV_h}(x) = V_h(x) - (1 + |h(x)|), \quad x \in C^c.$$

If $X(0) = x \in C$, then the previous arguments imply that the right-hand-side of (7) is finite, and in fact uniformly bounded over $x \in C$. Combining these results, we conclude that there exists a constant $b_0$ such that,

$$PV_h \leq e^{\frac{1}{2} \theta V_h} + b_0 1_C$$

Regular sets are necessarily small [15, Theorem 11.3.11] so that this is a version of the drift inequality (V4).

Finally note that, by the fact that (V4) implies the weaker drift condition (V3) of [15], the function $V_h$ is $\pi$-integrable by [15, Theorem 14.0.1].

□

Theorem 1.3 states that (V4) holds for a Lyapunov function $V_h$ with $h \in L^{V_h}_\infty$. If this could be strengthened to show that for every geometrically ergodic chain and any $h \in L_2$, the chain was geometrically ergodic with a Lyapunov function $V_h$ that had $h^2 \in L^{V_h}_\infty$, then the central limit theorem would hold for the partial sums of $h(X(n))$ [15, Theorem 17.0.1]. But this is not generally possible:

**Proposition 3.1.** There exists a geometrically ergodic Markov chain on a countable state space $X$ and a function $G \in L_2$ with mean $\pi(G) = 0$, for which the central limit theorem fails in that the normalized partial sums,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} G(X(i)), \quad n \geq 1,$$

(8)

converge neither to a normal distribution nor to a point mass.
The result of the proposition appears in [9, Theorem 1.3], and an earlier counterexample in [1] yields the same conclusion. Based on these counterexamples we now show that geometric ergodicity does not imply a spectral gap in the Hilbert space setting.

Proof of Theorem 1.4. Suppose that the Markov chain \( X = \{X(n)\} \) constructed in Proposition 3.1 does admit a spectral gap in \( L_2 \). Then its autocorrelation function decays geometrically fast, for any \( h \in L_2 \): Assuming without loss of generality that \( \pi(h) = 0 \), and letting \( R_h(n) = \pi(hP^n h) \), for all \( n \), we have the bound,

\[
|R(n)| \leq \sqrt{\pi(h^2)\pi((P^n h)^2)}, \quad n \geq 1.
\]

Applying Theorem 1.3, we conclude that the right-hand-side decays geometrically fast as \( n \to \infty \). Consequently, the sequence of normalized sums,

\[
S_n := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} h(X(i)), \quad n \geq 1,
\]

is uniformly bounded in \( L_2 \), i.e.,

\[
\limsup_{n \to \infty} \mathbb{E}_\pi[S_n^2] \leq \sum_{n=-\infty}^{\infty} |R(n)|,
\]

where \( \mathbb{E}_\pi[\cdot] \) denotes the expectation operator corresponding to the stationary version of the chain. However, this is impossible for the choice of the function \( h = G \) as in Proposition 3.1: In [9, p. 81] it is shown that the corresponding normalized sums in (8) fail to define a tight sequence of probability distributions. This is a consequence of [9, Lemma 3.2].

This contradiction establishes the claim that the Markov chain of Proposition 3.1 cannot admit a spectral gap in \( L_2 \).

Finally we prove Propositions 1.1 and 1.2.

Proof of Proposition 1.1. The equivalence stated in the proposition is obtained on combining Lemma 2.1 with [12, Proposition 4.6]. To explain this, we introduce new terminology: The transition kernel is called \( V \)-uniform if \( \lambda = 1 \) is the only pole on the unit circle in \( \mathbb{C} \), and this pole has multiplicity one. Proposition 4.6 of [12] states that geometric ergodicity with respect to a Lyapunov function \( V \) is equivalent to \( V \)-uniformity of the kernel \( P \). Consequently, the direct part of the proposition holds, since \( V \)-uniformity of \( P \) implies that it admits a spectral gap in \( L_\infty^V \).

Conversely, if the chain admits a spectral gap in \( L_\infty^V \), then Lemma 2.1 states that \( P \) is \( V \)-uniform. Applying Proposition 4.6 of [12] once more, we conclude that the chain is geometrically ergodic with the same Lyapunov function \( V \).

Proof of Proposition 1.2. The forward direction of the statement of the proposition is contained in [19] and [20].

The converse again follows from Lemma 2.1 and a minor modification of the arguments used in [12, Proposition 4.6]. If the chain admits a spectral gap in \( L_2 \), then the lemma states that \( \lambda = 1 \) has multiplicity one, and that this is the only pole on the unit circle in \( \mathbb{C} \). It follows
that for some $\rho < 1$, the inverse $[zI - (P - 1 \otimes \pi)]^{-1}$ exists as a bounded linear operator on $L_2$, whenever $|z| \geq \rho$. Denote $b_\rho = \sup \{ \| [zI - (P - 1 \otimes \pi)]^{-1} \|_2 : |z| = \rho \}$, where $\| \cdot \|_2$ is the induced operator norm on $L_2$.

Following the proof of [12, Theorem 4.1], we conclude that finiteness of $b_\rho$ implies a form of geometric ergodicity: For any $g \in L_2$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\phi} [\rho e^{in\phi} I - (P - 1 \otimes \pi)]^{-1} g = \rho^{-n-1}(P^n g - \pi(g)).$$

Therefore, the $L_2$-norm of the left-hand-side is bounded by $b_\rho \|g\|_2$. This gives,

$$\|P^n g - \pi(g)\|_2 \leq b_\rho \|g\|_2 \rho^{n+1}, \quad n \geq 1.$$ 

It follows from [15, Theorem 15.4.3] that the Markov chain is geometrically ergodic. □

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