NEARLY MORITA EQUIVALENCES AND RIGID OBJECTS

ROBERT J. MARSH AND YANN PALU

Abstract. If $T$ and $T'$ are two cluster-tilting objects of an acyclic cluster category related by a mutation, their endomorphism algebras are nearly Morita equivalent (Buan et al., Cluster-tilted algebras, Trans. Amer. Math. Soc. 359(1) (2007), 323–332 (electronic)); that is, their module categories are equivalent “up to a simple module”. This result has been generalized by Yang, using a result of Plamondon, to any simple mutation of maximal rigid objects in a 2-Calabi–Yau triangulated category. In this paper, we investigate the more general case of any mutation of a (non-necessarily maximal) rigid object in a triangulated category with a Serre functor. In that setup, the endomorphism algebras might not be nearly Morita equivalent, and we obtain a weaker property that we call pseudo-Morita equivalence. Inspired by Buan and Marsh (From triangulated categories to module categories via localization II: calculus of fractions, J. Lond. Math. Soc. (2) 86(1) (2012), 152–170; From triangulated categories to module categories via localisation, Trans. Amer. Math. Soc. 365(6) (2013), 2845–2861), we also describe our result in terms of localizations.

Introduction and main results

In this paper, our aim is to prove a weak form of nearly Morita equivalence for mutations of (non-maximal) rigid objects in triangulated categories. Before recalling the case of cluster-tilting objects (see [BMR07]), we first give an example.

Let $Q$ be a linear orientation of the Dynkin diagram of type $A_3$. The Auslander–Reiten quiver of the acyclic cluster category $C_Q$, defined
in \([BMR+06]\), is as follows:

The object \(T = T_1 \oplus T_2 \oplus T_3\) is cluster-tilting. Its mutation at \(T_2\) is the cluster-tilting object \(T' = T_1 \oplus T_2^* \oplus T_3\). We write \(\Gamma\) for the cluster-tilted algebra \(\text{End}_{C}(T)^{\text{op}}\), and \(\Gamma'\) for \(\text{End}_{C}(T')^{\text{op}}\). Then, the two algebras \(\Gamma\) and \(\Gamma'\) are related as follows.

On the one hand, the functor \(C(T, -)\) induces an equivalence of categories \(C/(\Sigma T) \cong \text{mod } \Gamma\), where \(\text{mod } \Gamma\) is the category of finitely generated left modules, and the Auslander–Reiten quiver of \(\text{mod } \Gamma\) is thus

\[
\begin{array}{c}
S_2^* \\
\downarrow \\
S_2 \\
\downarrow \\
\end{array}
\]

where \(S_2 = C(T, \Sigma T_2^*)\) is the simple top of the projective indecomposable \(C(T, T_2)\).

On the other hand, the functor \(C(T', -)\) induces an equivalence of categories \(C/(\Sigma T') \cong \text{mod } \Gamma'\), and the Auslander–Reiten quiver of \(\text{mod } \Gamma'\) is thus

\[
\begin{array}{c}
S_2^* \\
\downarrow \\
S_2 \\
\downarrow \\
\end{array}
\]

where \(S_2^* = C(T', \Sigma T_2)\) is the simple top of the projective indecomposable \(C(T', T_2^*)\), where the two arrows starting at \(S_2^*\) are identified, and where dots indicate zero relations.

The two Auslander–Reiten quivers are not isomorphic; therefore, \(\Gamma\) and \(\Gamma'\) are not Morita equivalent. However, they are not very far from being so:
the difference in the Auslander–Reiten quivers comes from the simples $S_2$ and $S_2^*$. The common Auslander–Reiten quiver of the categories $\text{mod } \Gamma/(\text{add } S_2)$ and $\text{mod } \Gamma/(\text{add } S_2^*)$ is thus

![Diagram](image)

This phenomenon, proved in [BMR07], has been called “nearly Morita equivalence” by C. M. Ringel. Let us state the precise result.

Let $Q$ be an acyclic quiver, and let $T$ be a cluster-tilting object in the cluster category $\mathcal{C}_Q$. Let $T' = T/T_k \oplus T_k^*$ be the mutation of $T$ at an indecomposable summand $T_k$; then, $T'$ is also a cluster-tilting object. Let $\Gamma$ (respectively, $\Gamma'$) be the cluster-tilted algebra $\text{End}_{\mathcal{C}_Q}(T)^{\text{op}}$ (respectively, $\text{End}_{\mathcal{C}_Q}(T')^{\text{op}}$), and let $S_k$ (respectively, $S_k^*$) be the simple top of the projective indecomposable $\Gamma$-module $\mathcal{C}_Q(T, T_k)$ (respectively, the simple top of the $\Gamma'$-module $\mathcal{C}_Q(T', T_k^*)$).

Then, by a result of [BMR07], the categories $\text{mod } \Gamma/\text{add } S_k$ and $\text{mod } \Gamma'/\text{add } S_k^*$ are equivalent. By [Yan12, Corollary 4.3], nearly Morita equivalence, in the more general setup of simple, 2-periodic mutations of rigid objects (or rigid, Krull–Schmidt subcategories) in any triangulated category, follows from [Pla11, Proposition 2.7].

Our main aim in this paper is to prove an analogous result for any mutation of (non-maximal) rigid objects. Before explaining our results, let us have a look at an example that shows that one cannot expect these mutations to induce a nearly Morita equivalence in general.

Let $T = T_1 \oplus T_2 \oplus T_3$ be the rigid object of the acyclic cluster category $\mathcal{C} = \mathcal{C}_{A_4}$ given by

![Diagram](image)
and let $T' = T_1 \oplus T_2^* \oplus T_3$ be the rigid object obtained by mutating $T$ at the summand $T_2$. This means that $\Sigma T_2^*$ is the cone of a minimal right add $T/T_2$-approximation of $T_2$. In the example, there is a triangle $T_2^* \to T_1 \to T_2 \to \Sigma T_2^*$. Let $\Lambda$ (respectively, $\Lambda'$) be the algebra $\text{End}_C(T)^{\text{op}}$ (respectively, $\text{End}_C(T')^{\text{op}}$). Using results in [BM12, BM13] (see also [KR07]), we can easily compute the Auslander–Reiten quivers of mod $\Lambda$ and mod $\Lambda'$:

\[
\begin{array}{c}
S_2^* \\
\downarrow \\
\text{mod } \Lambda' \\
\downarrow \\
\text{mod } \Lambda \\
\end{array}
\]

The algebras $\Lambda$ and $\Lambda'$ are not nearly Morita equivalent. On factoring out by $S_2$ (respectively, $S_2^*$), we obtain the following Auslander–Reiten quivers:

\[
\begin{array}{c}
S_2 \\
\downarrow \\
\text{mod } \Lambda'/\text{add } S_2^* \\
\downarrow \\
\text{mod } \Lambda/\text{add } S_2 \\
\end{array}
\]

However, these algebras are not very far from being nearly Morita equivalent. Indeed, the Auslander–Reiten quivers differ by only one arrow. The corresponding morphism can be characterized in mod $\Lambda$ as being surjective with kernel in the subcategory add $S_2$.

Let $C$ be an acyclic cluster category, and let $T$ be a rigid object in $C$. Let $T' = T/T_k \oplus T_k^*$ be the mutation of $T$ at the summand $T_k$. Let $\Lambda$ (respectively, $\Lambda'$) be the algebra $\text{End}_C(T)^{\text{op}}$ (respectively, $\text{End}_C(T')^{\text{op}}$), and let $S_k$ (respectively, $S_k^*$) be the simple top of the projective indecomposable $\Lambda$-module $C(T, T_k)$ (respectively, the $\Lambda'$-module $C(T', T_k^*)$).

As suggested by the example above, let us consider the class $R$ of epimorphisms in mod $\Lambda$ with kernels in add $S_k$, and the class $R^*$ of monomorphisms in mod $\Lambda'$ with cokernels in add $S_k^*$.

**Theorem A.** There is an equivalence of categories

$$(\text{mod } \Lambda)_{R} \simeq (\text{mod } \Lambda')_{R^*}.$$
This result is not completely satisfactory since it does not resemble nearly Morita equivalence. The following remark will help in restating the theorem in a form that looks more like nearly Morita equivalence.

Let \( M \in \text{mod} \Lambda \). If there is a short exact sequence \( 0 \to S_k \to L \xrightarrow{f} M \to 0 \), the morphism \( f \) belongs to \( \mathcal{R} \). Therefore, the objects \( L \) and \( M \) become isomorphic in the localization \( (\text{mod} \Lambda)_{\mathcal{R}} \). This suggests that the objects having non-split extensions with \( S_k \) can be removed from \( \text{mod} \Lambda \) without changing the localization. We thus define \( \mathcal{E} \) to be the full subcategory of \( \text{mod} \Lambda \) whose objects \( M \) satisfy \( \text{Ext}^1_{\Lambda}(M, S_k) = 0 \). Dually, let \( \mathcal{E}' \) be the full subcategory of \( \text{mod} \Lambda' \) whose objects \( N \) satisfy \( \text{Ext}^1_{\Lambda'}(S_k^*, N) = 0 \).

It should be noted that \( \mathcal{E} \) and \( \mathcal{E}' \) are extension-closed in \( \text{mod} \Lambda \) (respectively, \( \text{mod} \Lambda' \)) and are thus exact categories.

**Theorem B.** There is an equivalence of categories

\[
(\text{mod} \Lambda)_{\mathcal{R}} \simeq \mathcal{E} / \text{add } S_k.
\]

Dually, there is an equivalence of categories

\[
(\text{mod} \Lambda')_{\mathcal{R}'} \simeq \mathcal{E}' / \text{add } S_k^*.
\]

Combining the two theorems gives the following.

**Corollary.** There is an equivalence of categories

\[
\mathcal{E} / \text{add } S_k \simeq \mathcal{E}' / \text{add } S_k^*.
\]

This resembles nearly Morita equivalence except that, unlike in the cluster-tilting case, one has to restrict to an exact subcategory before killing the simple.

Unfortunately, these statements do not specialize to a nearly Morita equivalence in the cluster-tilting case: in the setup of \([BMR07]\), we obtain a weaker statement.

The proofs of Theorems A and B are in Section 3.1 (but note that the proofs appear in reverse order to the above). In fact, we prove more general results than those mentioned above. First, we only assume the triangulated category \( \mathcal{C} \) to be Krull–Schmidt, with a Serre functor. Second, we allow mutations at non-indecomposable summands. Our results hold, in particular, in any triangulated category in the following list (whose items overlap):
• Hom-finite generalized higher cluster categories (see [Ami09, Guo11]);
• stable categories of maximal Cohen–Macaulay modules over an odd dimensional isolated hypersurface singularity (see [BIKR08]);
• cluster tubes (see [BKL08, BMV10], . . .);
• (higher) cluster categories of type $A_\infty$ (see [HJ12, HJ13]);
• the triangulated orbit categories listed in [Ami07];
• stable categories constructed from preprojective algebras in [GLS, GLS11], . . .

§1. Setup and notation

We fix a field $k$, and a Krull–Schmidt, $k$-linear, Hom-finite, triangulated category $\mathcal{C}$, with suspension functor $\Sigma$. An object $X$ in $\mathcal{C}$ is called rigid if $\text{Ext}^1_{\mathcal{C}}(X, X) = 0$, where we write $\text{Ext}^1_{\mathcal{C}}(X, Y)$ for $\mathcal{C}(X, \Sigma Y)$. We write $X^\perp$ for the right Hom-perp of $X$; that is, the subcategory of $\mathcal{C}$ on objects $Y$ such that $\mathcal{C}(X, Y) = 0$. It should be noted that this notation differs from that used in [BM13], which we often cite; here, the Hom-perpendicular categories play a key role, so we use a different notation.

Let $T \in \mathcal{C}$ be a basic rigid object. Let $R$ be a direct summand of $T$, and write $T = T \oplus R$. Let $T'$ be the object obtained from $T$ by replacing $R$ by the negative shift $R^*$ of the cone of a minimal right add $T$-approximation of $R$. We have a triangle $R^* \to B \to R \to \Sigma R^*$, with $B \in \text{add } T$, $B \to R$ a minimal right add $T$-approximation, and $T' = T \oplus R^*$. By [BMR⁺06, Lemma 6.7], $\Sigma R^* \in T^\perp$. We assume that $T'$ is rigid. More precisely, we assume that $R^*$ is rigid and that $R^*$ belongs to $\Sigma^{-1} T$. By [Y08, Proposition 2.6(1)] and [BMR⁺06, Lemma 6.5], $R$ and $R^*$ are basic and have the same number of indecomposable direct summands. We keep these assumptions throughout the paper.

Remark. We note that, if $\mathcal{C}$ is 2-Calabi–Yau, then $T'$ is automatically rigid (see [BMR⁺06, Section 6]). However, WuZhong Yang kindly warned us that this is not true in general: Example 2.20 in [YZZ15] shows that $R^*$ might not be rigid. Moreover, even when $R^*$ is rigid, it might not belong to $\Sigma^{-1} T$. An example illustrating this latter phenomenon can be found in [YZ15, Example 2.15]. In that example, $\mathcal{C}$ is the 3-cluster category of type $A_3$, which contains a cluster-tilting object $T$. If $R$ is any indecomposable summand of $T$, the (left) mutation of $T$ at $R$ gives an indecomposable rigid object $R^*$ which belongs to $\Sigma^{-1} T^\perp$, but not to $\Sigma^{-1} T$. 
In some statements, we assume additionally that \( \mathcal{C} \) has a Serre functor.

We also need some more notation. If \( X \) is an object in \( \mathcal{C} \), we write \((X)\) for the ideal of morphisms factoring through the additive subcategory \( \text{add} X \) generated by \( X \). All modules considered are left modules.

We denote by \( \mathcal{C}(T) \) the full subcategory of \( \mathcal{C} \) whose objects are the cones of morphisms \( T_1 \to T_0 \), where \( T_0, T_1 \in \text{add} T \), and by \( \overline{\mathcal{C}}(T) \) the full subcategory of \( \mathcal{C} \) whose objects are the cones of morphisms \( \overline{T_1} \to T_0 \), where \( T_0 \in \text{add} T \) and \( \overline{T_1} \in \text{add} \overline{T} \).

More generally, for any two full subcategories \( \mathcal{A} \) and \( \mathcal{B} \) of \( \mathcal{C} \), we use the notation \( \mathcal{A} * \mathcal{B} \) for the full subcategory whose objects \( X \) are extensions of an object in \( \mathcal{B} \) by an object in \( \mathcal{A} \) (i.e., \( X \) appears in a triangle \( A \to X \to B \to \Sigma A \) with \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \)). It follows from the octahedral axiom that the operation \( * \) is associative. By abuse of notation, if \( A, B \) are objects in \( \mathcal{C} \), we write \( A * B \) for \( \text{add} A * \text{add} B \).

Thus, one could also define \( \mathcal{C}(T) \) and \( \overline{\mathcal{C}}(T) \) by \( \mathcal{C}(T) = T * \Sigma T \) and \( \overline{\mathcal{C}}(T) = T * \Sigma \overline{T} \).

**Remark.** Our results hold in the more general setup of rigid subcategories. We replace \( \text{add} T \) by a rigid subcategory \( \mathcal{T} \), with the following additional assumptions: \( \mathcal{T} \) is contravariantly finite, \( \overline{\mathcal{T}} \) is functorially finite, and \( \mathcal{T}' \) is covariantly finite. This requires changing the functors of the form \( \mathcal{C}(\mathcal{T}, -) \), taking values in the category \( \text{mod} \text{End}_\mathcal{C}(T)^{\text{op}} \), into functors of the form \( \mathcal{C}(?, -)|_{\mathcal{T}} \), taking values in \( \text{mod} \mathcal{T} \), and replacing all references to [BM13] by references to [Bel13].

### §2. Pseudo-Morita equivalence

**2.1 Adjunctions**

The methods used in this subsection are inspired by [Bel13, BM13, BM12], and much resemble results in [Nak13, Section 3]. Indeed, [Nak13, Corollary 3.8] applied to the twin cotorsion pair \( (\Sigma T, T^{\perp}) \), \( (\Sigma T', T'^{\perp}) \) (where we use the notation from Section 2.2) gives the existence of a right adjoint to the fully faithful functor \( \overline{\mathcal{C}}(T)/(\Sigma T') \to \mathcal{C}/(\Sigma T') \) from which it is possible to deduce our Proposition 2.5. For the convenience of the reader, we nonetheless include a complete proof.
The subcategory $C(T)$ is known to be contravariantly finite, by [BM13, Lemmas 3.3 and 3.6]. An analogous proof gives Lemma 2.2 below. We first need a definition.

**DEFINITION 2.1.** Let $S$ be the set of morphisms $X \xrightarrow{f} Y$ in $C$ such that for any triangle $Z \to X \xrightarrow{f} Y \xrightarrow{g} \Sigma Z$ we have $Z \in T^\perp$ and $g \in (T^\perp)$.

**LEMMA 2.2.**

(a) Let $X' \xrightarrow{s} X$ be a morphism in $S$ with $X' \in \mathcal{C}(T)$. Then, $s$ is a right $\mathcal{C}(T)$-approximation of $X$.

(b) Each object $X$ in $C$ has a right $\mathcal{C}(T)$-approximation $R_0X \xrightarrow{\eta_X} X$ lying in $S$.

(c) The category $\mathcal{C}(T)$ is a contravariantly finite subcategory of $C$.

**Proof.** Suppose that $X' \xrightarrow{s} X$ is a morphism in $S$ with $X' \in \mathcal{C}(T)$. Thus, we may complete $s$ to a triangle:

$$X' \xrightarrow{s} X \xrightarrow{g} \Sigma Z \to \Sigma X',$$

where $g$ factors through $T^\perp$, and $\Sigma Z$ lies in $(\Sigma T)^\perp$.

Let $X'' \xrightarrow{u} X$ be a morphism in $C$. Assume that $X''$ lies in $\mathcal{C}(T)$, so that there is a triangle $U_0 \xrightarrow{p} X'' \to \Sigma U_1 \to \Sigma U_0$, with $U_0 \in \text{add } T$ and $U_1 \in \text{add } T$. Since $g$ factors through $T^\perp$, and $U_0 \in \text{add } T$, we have $gup = 0$, and therefore have the following commutative diagram whose rows are triangles:

Moreover, $\Sigma Z$ lies in $(\Sigma T)^\perp$, and $\Sigma U_1$ is in $\text{add } \Sigma T$, so the composition $gu = v\eta$ is zero. Thus, there is a morphism $u'$ such that $u = su'$. Part (a) is shown.

For part (b), let $X \in C$. Let $T_0X \xrightarrow{X} X$ be a minimal right add $T$-approximation of $X$. Complete it to a triangle $Y \to T_0X \to X \to \Sigma Y$. Let $T_1Y \xrightarrow{Y} Y$ be a minimal right add $T$-approximation of $Y$. Applying the
octahedral axiom, we obtain the following diagram:

\[
\begin{array}{c}
\Sigma T_1^Y \\
\downarrow \\
Y \\
\downarrow \\
Z \\
\downarrow \\
\Sigma T_1^Y
\end{array}
\begin{array}{c}
\longrightarrow \\
T_1^Y \\
\longrightarrow \\
X \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\Sigma Y \\
\longrightarrow \\
X \\
\longrightarrow \\
\Sigma Z
\end{array}
\]

Applying the functors \( C(T, -) \) and \( C(\overline{T}, -) \) to the triangles above shows that \( \Sigma Y \in T^\perp \) and \( Z \in \overline{T}^\perp \). It should be noted that \( R_0X \in \overline{C}(T) \). Then, by part (a), \( \eta_X \) is a right \( \overline{C}(T) \)-approximation of \( X \), and part (b) is shown. Part (c) follows immediately from part (b).

The following remark is stated as a lemma since it is used several times in the paper.

**Lemma 2.3.** Let \( X \to Y \) be a morphism in \( C \) with \( X \in \overline{C}(T) \), and assume that \( f \) factors through \( \overline{T} \) in \( C \). Then, \( f \) factors through \( \overline{T} \cap \overline{C}(T) \).

**Proof.** Let \( T_0 \to X \) be a minimal right \( T \)-approximation of \( X \) in \( C \). Complete the morphism \( u \) to a triangle \( \overline{T}_0 \to X \to Z \to \Sigma \overline{T}_0 \) in \( C \). As shown in [BMR+06] (apply the functor \( C(\overline{T}, -) \) to the triangle above), the cone \( Z \) belongs to \( \overline{T} \). Moreover, the composition \( fu \) vanishes since \( f \) factors through \( \overline{T} \), and it follows that the morphism \( f \) factors through \( v \). It remains to be checked that the object \( Z \) lies in \( \overline{C}(T) \). The triangle above shows that \( Z \in \overline{C}(T) \ast \Sigma \overline{T} \), and we have

\[
\overline{C}(T) \ast \Sigma \overline{T} = (\text{add } T \ast \Sigma \overline{T}) \ast \Sigma \overline{T}
\]

\[
= \text{add } T \ast (\Sigma \overline{T} \ast \Sigma \overline{T})
\]

\[
= \text{add } T \ast \Sigma \overline{T},
\]

where the last equality holds since \( \Sigma \overline{T} \) is rigid.

The following lemma, which is used in the proof of Proposition 2.5, is a particular case of [ML98, IV.1 Theorem 2(ii)].
Lemma 2.4. Let $\mathcal{B}$ be a category, and let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$. Suppose that, for any $B \in \mathcal{B}$, there is an object $G_0B \in \mathcal{A}$ and a morphism $G_0B \xrightarrow{\eta_B} B$ such that for all $A \xrightarrow{f} B$ with $A \in \mathcal{A}$, the morphism $f$ lifts uniquely through $\eta_B$. Then, the inclusion $\mathcal{A} \subseteq \mathcal{B}$ has a right adjoint $G : \mathcal{B} \to \mathcal{A}$ such that, for all $B \in \mathcal{B}$, $GB = G_0B$.

The functor $G$ of the previous lemma is defined on arrows as follows. For any $B \xrightarrow{b} B'$ in $\mathcal{B}$, $Gb$ is the unique lift through $\eta_{B'}$ of the composition $b\eta_B$:

$$
\begin{array}{ccc}
G_0B & \xrightarrow{\eta_B} & B \\
\downarrow & \swarrow & \downarrow b \\
G_0B' & \xrightarrow{\eta_{B'}} & B'
\end{array}
$$

The following proposition is inspired by [Bel13].

Proposition 2.5. The inclusion of $\mathcal{C}(T)$ into $\mathcal{C}$ induces a fully faithful functor $\mathcal{C}(T)/\overline{T} \to \mathcal{C}/\overline{T}$. Moreover, the functor $I$ admits an additive right adjoint $R$, such that, for all $X$ in $\mathcal{C}$, $RX = R_0X$, in the notation of Lemma 2.2.

Proof. The inclusion of $\mathcal{C}(T)$ in $\mathcal{C}$ induces a full functor:

$$
\begin{array}{c}
\mathcal{C}(T) \\
\downarrow Q \\
\overline{\mathcal{C}(T)}/(\overline{T} \cap \overline{\mathcal{C}(T)}) \\
\downarrow I \\
\mathcal{C}/\overline{T}
\end{array}
$$

We first check that the functor $I$ is faithful. This amounts to proving that if a morphism in $\overline{\mathcal{C}(T)}$ factors through $\overline{T}$ in $\mathcal{C}$, then it already factors through $\overline{T}$ in $\overline{\mathcal{C}(T)}$. This follows from Lemma 2.3. In what follows, we identify $\overline{\mathcal{C}(T)}/(\overline{T} \cap \overline{\mathcal{C}(T)})$ with the image of $\overline{\mathcal{C}(T)}$ in $\mathcal{C}/(\overline{T})$.

Next, we prove the existence of a right adjoint. For this, we use the particular case of [ML98, IV-1 Theorem 2(ii)], stated in Lemma 2.4.

Let $X \in \mathcal{C}$. Consider the morphism $R_0X \xrightarrow{\eta_X} X$ constructed in Lemma 2.2. We claim that $Q\eta_X$ is universal from $I$ to $X$, in the sense of MacLane; that is, any morphism in $\mathcal{C}/(\overline{T})$ from an object in $\overline{\mathcal{C}(T)}$ to
$X$ factors uniquely through $Q\eta_X$ in $\mathcal{C}/(\mathcal{T}^\perp)$. Since $\eta_X$ is a right $\mathcal{C}(T)$-approximation of $X$ in $\mathcal{C}$, its image $Q\eta_X$ is a right $\mathcal{C}(T)/(\mathcal{T}^\perp \cap \mathcal{C}(T))$-approximation of $X$ in $\mathcal{C}/(\mathcal{T}^\perp)$, so that we only have to prove uniqueness.

Let $Y \in \mathcal{C}(T)$, and let $Y \xrightarrow{u} R_0X$ be a morphism in $\mathcal{C}$ such that $Q(\eta_Xu) = 0$. Since the kernel of $Q$ is the ideal $(\mathcal{T}^\perp)$ of $\mathcal{C}$, this means that the composition $\eta_Xu$ factors through $\mathcal{T}^\perp$. Since its source belongs to $\mathcal{C}(T)$, Lemma 2.3 shows that $\eta_Xu$ factors through $\mathcal{T}^\perp \cap \mathcal{C}(T)$. Let $Y' \in \mathcal{T}^\perp \cap \mathcal{C}(T)$ be such that the square

\[
\begin{array}{ccc}
Y & \xrightarrow{a} & Y' \\
\downarrow{u} & & \downarrow{b} \\
Z & \xrightarrow{\alpha} & R_0X \xrightarrow{\eta_X} X \xrightarrow{\Sigma Z}
\end{array}
\]

commutes. Since $\eta_X$ is a right $\mathcal{C}(T)$-approximation, there exists a morphism $Y' \xrightarrow{c} R_0X$ with $b = \eta_Xc$. We have $\eta_X(u - ca) = 0$, so that the morphism $u - ca$ factors through $\alpha$. By construction, $Z \in \mathcal{T}^\perp$; therefore, we have $u \in (\mathcal{T}^\perp)$, which proves uniqueness.

Finally, we note that the functor $R$ is additive since it is the right adjoint of the additive functor $I$.

If the category admits a Serre functor $S$, then a dual version of Proposition 2.5 will be of interest to us. We first note that applying to $ST'$ the construction dual to that of $R_0$ gives, for any $X \in \mathcal{C}$, a triangle $Z \xrightarrow{\alpha} X \xrightarrow{\varepsilon_X} L_0X \xrightarrow{\Sigma Z}$, where $L_0X$ belongs to add $\Sigma^{-1}S^T * \text{add } ST'$, $\varepsilon_X$ is a minimal left add $\Sigma^{-1}S^T * \text{add } ST'$-approximation, $\alpha$ factors through $\perp(ST') = (T')^\perp$, and $\Sigma Z$ belongs to $\mathcal{T}^\perp$.

**Proposition 2.6.** Assume that the category $\mathcal{C}$ has a Serre functor $S$, and let $\mathcal{C}(T')$ be the full subcategory add $\Sigma^{-1}S^T * \text{add } ST'$ of $\mathcal{C}$. Then, the inclusion of $\mathcal{C}(T')$ into $\mathcal{C}$ induces a fully faithful functor $\mathcal{C}(T')/(\mathcal{T}^\perp) \xrightarrow{J} \mathcal{C}/(\mathcal{T}^\perp)$. Moreover, the functor $J$ admits an additive left adjoint $L$, such that $LX = L_0X$ for all $X \in \mathcal{C}$.

The only reason why we assume the existence of a Serre functor here is that it converts a left perpendicular subcategory into a right perpendicular subcategory. This allows us to view both categories in Propositions 2.5 and 2.6 as subcategories of the same category $\mathcal{C}/(\mathcal{T}^\perp)$. 
2.2 Main result

Our aim in this section is to prove that if $\mathcal{C}$ has a Serre functor then the categories $\mathcal{C}(T)/(\Sigma T')$ and $\mathcal{C}(T)/(T)$ are equivalent (Theorem 2.9). This is then used in the next section in order to compare the module categories over the endomorphism algebras of $T$ and $T'$.

We need the following key lemma, which is often used throughout the paper.

**Lemma 2.7.** We have:

(a) $\mathcal{C}(T) = T \ast \Sigma T = T \ast \Sigma T'$;
(b) $\mathcal{C}(T) \cap T^\perp = \text{add } \Sigma T'$;
(c) if $\mathcal{C}$ has a Serre functor $S$, then $(\Sigma^{-1} ST \ast ST') \cap T^\perp = \text{add } \Sigma^{-1} ST$.

**Proof.**

(a) The exchange triangle shows that $T \in T \ast \Sigma T'$. We thus have

$$T \ast \Sigma T \subseteq (T \ast \Sigma T') \ast \Sigma T$$

$$= T \ast (\Sigma T' \ast \Sigma T)$$

$$= T \ast \Sigma T'.$$

The reverse inclusion is obtained by applying this inclusion to $\Sigma T'$ (instead of $T$) in the opposite category.

(b) This immediately follows from (a).

(c) This also follows from (a):

$$(\Sigma^{-1} ST \ast ST') \cap T^\perp = (\Sigma^{-1} ST \ast ST) \cap T^\perp \ (\text{by (a)})$$

$$= \Sigma^{-1} ST.$$

Assume that $\mathcal{C}$ has a Serre functor $S$. Recall that we write $\mathcal{C}(T)$ (respectively, $\mathcal{C}(T')$) for the full subcategory $T \ast \Sigma T$ (respectively, $\Sigma^{-1} ST \ast ST'$) of $\mathcal{C}$. By Propositions 2.5, 2.6 and Lemma 2.7, we have a pair of adjoint functors $(G, H)$, where $G = LI$ and $H = RJ$. Since $I, J, L$ and $R$ are additive, so are $G$ and $H$.

\[
\begin{array}{c}
\mathcal{C}/(T^\perp) \\
\mathcal{C}(T)/(\Sigma T') \\
\mathcal{C}(T')/(\Sigma^{-1} ST)
\end{array}
\]

\[
\begin{array}{ccc}
I & \text{adjoint} & J \\
R & \text{left adjoint} & G \\
H & \text{right adjoint} & L
\end{array}
\]
Remark 2.8. We write \( \tau \) for the Auslander–Reiten translation \( \tau = SS^{-1} \) (see [RvdB02, Section I.2]). Then, by Lemma 2.7, we have that \( C(T') = \tau \overline{C}(T) \).

Theorem 2.9. Assume that \( C \) has a Serre functor \( S \). Then the functors \( G \) and \( H \) are quasi-inverse equivalences of categories. In particular, the categories \( C(T)/(\Sigma T') \) and \( C(T)/(T) \) are equivalent.

Proof. The construction would be simplified if we had that, if \( X \) belongs to \( T \ast \Sigma T \), then the left add \( \Sigma^{-1}ST \ast \) add \( ST' \)-approximation \( X \xrightarrow{\varepsilon_X} L_0X \) of \( X \) (see Proposition 2.6 and the paragraph before it) is also a minimal right \( T \ast \Sigma T \)-approximation of \( L_0X \). However, this cannot be expected to hold in general (take \( X \) to be \( \Sigma T' \), for instance).

We can modify this approach in the following way. First, since the functors \( G \) and \( H \) are additive, we may assume that \( X \) is indecomposable. This helps in proving that \( X \) is a summand of \( R_0L_0X \). Second, we add to \( X \) a minimal right add(\( \Sigma T' \))-approximation \( \Sigma T'_0 \) of \( L_0X \). This is needed in order to get a right approximation of \( L_0X \), while being harmless since the objects \( X \) and \( X \oplus \Sigma T'_0 \) are isomorphic in \( C(T)/(T') \).

Therefore, take an indecomposable object \( X \in T \ast \Sigma T \), and assume that \( X \) does not belong to add \( \Sigma T' \) (otherwise, \( X \) would be isomorphic to 0 in \( C(T)/(T') \)). Consider the triangle \( Z \xrightarrow{\alpha} X \xrightarrow{\varepsilon_X} L_0X \xrightarrow{\beta} \Sigma Z \) in \( C \), constructed in the paragraph before Proposition 2.6, where \( \alpha \in ((T')^\perp) \) and \( \Sigma Z \in \overline{T}^\perp \). Let \( \Sigma T'_0 \xrightarrow{p} L_0X \) be a minimal right add \( \Sigma T' \)-approximation of \( L_0X \) in \( C \). We claim that \( X \oplus \Sigma T'_0 \xrightarrow{[\varepsilon_X,p]} L_0X \) is a right \( T \ast \Sigma T \)-approximation of \( L_0X \) in \( C \). Let \( X' \xrightarrow{f} L_0X \) be a morphism in \( C \), with \( X' \in T \ast \Sigma T \). By assumption, there is a triangle \( T'_1 \xrightarrow{a} T_0 \xrightarrow{b} X' \xrightarrow{c} \Sigma T'_1 \) in \( C \), with \( T'_1 \in \text{add} T' \) and \( T_0 \in \text{add} \Sigma T \). Since \( T_0 \) is in add \( T \), and \( \Sigma Z \) is in \( T^\perp \), the composition \( \beta fb \) vanishes, and \( f \) induces a morphism of triangles:
Since \( \alpha \) factors through \((T')^\perp\), we have \((-\Sigma\alpha)g = 0\), and there exists \(\Sigma T'_1 \xrightarrow{u} L_0X\) such that \(g = \beta u\). This implies \(\beta(f - uc) = 0\), so that there exists \(X' \xrightarrow{v} X\) such that \(f = uc + \varepsilon_Xv\). The composition \(uc\) is in the ideal \((\Sigma T')\), and thus factors through \(p\); that is, there exists \(w\) making the following square commute:

\[
\begin{array}{ccc}
X' & \xrightarrow{c} & \Sigma T'_1 \\
\downarrow w & & \downarrow u \\
\Sigma T'_0 & \xrightarrow{p} & L_0X
\end{array}
\]

We thus have \(f = [\varepsilon_X p][\varepsilon_X w]\), and therefore the morphism \([\varepsilon_X p]\) is a right \(T \ast \Sigma T\)-approximation of \(L_0X\) in \(C\).

Since also \(R_0L_0X \xrightarrow{\eta_{L_0X}} L_0X\) is a right \(T \ast \Sigma T\)-approximation of \(L_0X\) in \(C\), we can write \(R_0L_0X\) as a direct sum \(X' \oplus X''\), and \(\eta_{L_0X} = [\eta' 0] : X' \oplus X'' \rightarrow L_0X\), where \(X' \xrightarrow{\eta'} L_0X\) is a minimal right \(T \ast \Sigma T\)-approximation. Moreover, we have \(X'' \in \text{add } \Sigma T'\), since in the triangle \(Z' \rightarrow X' \oplus X'' \rightarrow L_0X \rightarrow \text{given by Lemma 2.2, } Z' \text{ belongs to } T', \text{ and } X'' \text{ is a summand of } Z'. \text{ Thus, } X'' \text{ belongs to } \in T' \cap \mathcal{C}(T), \text{ which is add } \Sigma T' \text{ by Lemma 2.7.}\)

Now, \(X'\) is a summand of the approximation \(X \oplus \Sigma T'_0\). Moreover, \(X'\) contains \(X\) as a summand. Otherwise, we would have \(R_0L_0X \in \text{add } \Sigma T''\), which implies \(L_0X \in \overline{T}^{\perp} \cap \mathcal{C}(T') = \text{add } \Sigma^{-1}ST'\) (by applying the functor \(\mathcal{C}(T', -)\) to the triangle \(Z' \rightarrow R_0L_0X \rightarrow L_0X \rightarrow \)), which dually implies \(X \in \text{add } \Sigma T'\) (note that we assumed \(X \notin \text{add } \Sigma T''\)).

As a consequence, given a lift

\[
\begin{array}{ccc}
X & \xrightarrow{\varepsilon_X} & L_0X \\
\downarrow \tilde{\varphi}_X & & \\
R_0L_0X & \xrightarrow{\eta_{L_0X}} & L_0X
\end{array}
\]

the image \(\varphi_X\) of \(\tilde{\varphi}_X\) in \(\overline{\mathcal{C}(T)}/(\Sigma T')\), which is independent of the choice of \(\tilde{\varphi}_X\) by Proposition 2.5, is an isomorphism (and \(\varepsilon_X\) is a minimal right \(\overline{\mathcal{C}(T')}-\text{approximation of } L_0X \text{ in } \mathcal{C}/(\Sigma T'))\).

Let us check that we have defined a natural isomorphism \(\varphi : 1 \rightarrow HG\).
Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}(T)$. By construction, there is a diagram in $\mathcal{C}$

$$
\begin{array}{cccc}
X & \xrightarrow{\tilde{\varphi}_X} & R_0L_0X & \\
\downarrow & & \downarrow & \\
L_0X & \xrightarrow{Gf} & R_0L_0Y & \\
\downarrow & & \downarrow & \\
Y & \xrightarrow{\tilde{\varphi}_Y} & R_0L_0Y & \\
\end{array}
$$

where we write $\eta$ for $\eta_{L_0Y}$, and where the inner two triangles and the inner two squares commute. We thus have $\eta(HGf \circ \tilde{\varphi}_X - \tilde{\varphi}_Y \circ f) = 0$, and $HGf \circ \tilde{\varphi}_X - \tilde{\varphi}_Y \circ f$ factors through $Z \rightarrow R_0L_0Y$ in the triangle $Z \rightarrow R_0L_0Y \xrightarrow{\eta} L_0Y \rightarrow$, where $Z \in T^\perp \cap \mathcal{C}(T)$. This shows that $HGf \circ \tilde{\varphi}_X - \tilde{\varphi}_Y \circ f$ factors through $T^\perp \cap \mathcal{C}(T)$, which is add $\Sigma T'$ by Lemma 2.7. As a consequence, $\varphi$ is a natural transformation.

By duality, there is a natural isomorphism $GH \rightarrow 1$, and the functors $G$ and $H$ are quasi-inverse equivalences of categories.

2.3 A module-theoretic interpretation

In this section, we assume that $\mathcal{C}$ has a Serre functor $S$. In this case, the assumptions of functorial finiteness (see Section 1) are automatically satisfied for all rigid objects (but have to be added in the case of rigid subcategories). We write $D$ for the duality functor $\text{Hom}_k(-, k)$. Recall that $T \in \mathcal{C}$ is a basic rigid object, and $R$ is a direct summand of $T$, with $T = T_1 \oplus \cdots \oplus T_n$ and $R = T_{n+1} \oplus \cdots \oplus T_m$, where the $T_i$ are indecomposable. Recall also that $\Sigma R^*$ is the cone of a minimal right add $T$-approximation of $R$. We have $T' = T_1 \oplus R^* = T'_1 \oplus \cdots \oplus T'_m$, where $T'_i = T_i$ if $i \leq n$. Define $\Lambda$ (respectively, $\Lambda'$) to be the endomorphism algebra $\text{End}_\mathcal{C}(T)^{\text{op}}$ (respectively, $\text{End}_\mathcal{C}(T')^{\text{op}}$).

Let $S_j$ be the simple top of the indecomposable projective $\Lambda$-module $\mathcal{C}(T, T_j)$, and let $S'_j$ be the simple socle of the indecomposable injective $\Lambda'$-module $DC(\Sigma T'_j, \Sigma T')$. We consider the exact categories $\mathcal{E}$ and $\mathcal{E'}$ defined as follows. The category $\mathcal{E}$ (respectively, $\mathcal{E'}$) is the full subcategory of
mod \Lambda \) (respectively, \( \text{mod} \Lambda' \)), whose objects \( M \) (respectively, \( N \)) satisfy \( \text{Ext}^1_{\Lambda}(M, S_j) = 0 \) (respectively, \( \text{Ext}^1_{\Lambda'}(S'_j, N) = 0 \)) for all \( j > n \).

**Remark 2.10.** For each indecomposable summand \( R_i \) of \( R \), let \( R_i^* \to U_i \to R_i \to \Sigma R_i^* \) be a triangle in \( \mathcal{C} \), with \( U_i \to R_i \) a minimal right \( T \)-approximation. Then (as in [BMR+06]), the object \( R_i^* \) is indecomposable. Moreover, \( \oplus_i U_i \to \oplus_i R_i \) is a minimal right \( T \)-approximation of \( R \). As a consequence, \( R^* \) is isomorphic to \( \oplus_i R_i^* \). This shows that the basic objects \( R \) and \( R^* \) have the same number of indecomposable summands.

We can now restate Theorem 2.9 in module-theoretic terms.

**Theorem 2.11.** Suppose that \( \mathcal{C} \) has a Serre functor. Then, there is an equivalence of categories
\[
\mathcal{E}/ \text{add} \mathcal{C}(T, \Sigma R^*) \simeq \mathcal{E}'/ \text{add} D\mathcal{C}(R, \Sigma T').
\]

The proof is given later in this section. We note that, if \( \mathcal{C} \) is 2-Calabi–Yau, then the modules \( D\mathcal{C}(R, \Sigma T') \) and \( \mathcal{C}(T', \Sigma R) \) are isomorphic. We also note that although the statement of the equivalence does not need a Serre functor, the existence is needed in the proof, in order to apply Theorem 2.9.

In order to prove Theorem 2.11, we need the following two lemmas.

**Lemma 2.12.** The functor \( \mathcal{C}(T, -) \) induces a fully faithful functor
\[
\mathcal{C}(T)/(\Sigma T) \to \text{mod} \Lambda.
\]

Its essential image is \( \mathcal{E} \).

**Proof.** Let \( X \to Y \) be a morphism in \( \mathcal{C} \) factoring through \( T^\perp \). Recall that \( \mathcal{C}(T) = T \ast \Sigma T \). Assume that \( X \) belongs to \( \mathcal{C}(T) \), and let \( V_1 \to V_0 \to X \to \Sigma V_1 \) be a triangle in \( \mathcal{C} \) with \( V_0 \in \text{add} T \) and \( V_1 \in \text{add} \overline{T} \). Since \( f \) factors through \( T^\perp \), the composition \( V_0 \to X \to Y \) vanishes, and \( f \) factors through \( \Sigma V_1 \). This implies that the first part of the lemma (the fullness of \( \mathcal{C}(T, -) \) follows from [IY08, Proposition 6.2] (see also [BM13, Lemma 4.3])).

For any \( M \) in \( \text{mod} \Lambda \), let \( X \in \mathcal{C}(T) \) be such that \( X \) has no summands in \( \text{add} \Sigma T \), and \( \mathcal{C}(T, X) \simeq M \). Let \( U_\beta \to U_\alpha \to X \to \Sigma U_\beta \) be a triangle with \( U_\alpha, U_\beta \in \text{add} T \) and \( U_\alpha \to X \) right-minimal. Then, \( \mathcal{C}(T, U_\beta) \to \mathcal{C}(T, U_\alpha) \to \mathcal{C}(T, X) \to 0 \) is a minimal projective presentation of \( \mathcal{C}(T, X) \), and the dimension of \( \text{Ext}^1_{\Lambda}(M, S_j) \) is the multiplicity of \( P_j \) in \( \mathcal{C}(T, U_\beta) \).

Dually, we obtain the following.
Lemma 2.13. The functor $D\mathcal{C}(\cdot, \Sigma T')$ induces a fully faithful functor

$$\overline{\mathcal{C}}(T)/(\overline{T}) \longrightarrow \text{mod } \Lambda'.$$

Its essential image is $\mathcal{E}'$.

Proof. The proof is dual to that of Lemma 2.12. We use the description $\overline{\mathcal{C}}(T) = T \ast \Sigma T'$ from Lemma 2.7, and note that any triangle $U'_1 \rightarrow \overline{U}_0 \rightarrow X \rightarrow \Sigma U'_1$, where $U'_1$ belongs to add $T'$, and $\overline{U}_0$ belongs to add $T$, gives rise to an injective co-presentation $0 \rightarrow D\mathcal{C}(X, \Sigma T') \rightarrow D\mathcal{C}(\Sigma U'_1, \Sigma T') \rightarrow D\mathcal{C}(\Sigma \overline{U}_0, \Sigma T')$.

Proof of Theorem 2.11. By Lemma 2.12, the functor $\mathcal{C}(T, \cdot)$ induces an equivalence of categories from $\overline{\mathcal{C}}(T)/(\Sigma T)$ to $\mathcal{E}$. Since $\overline{\mathcal{C}}(T)/(\Sigma T')$ is isomorphic to $((\overline{\mathcal{C}}(T)/(\Sigma T'))/(\Sigma R^*))$, the functor $\mathcal{C}(T, \cdot)$ induces an equivalence of categories from $\overline{\mathcal{C}}(T)/\text{add } \Sigma T'$ to $\mathcal{E}/\text{add } \mathcal{C}(T, \Sigma R^*)$. Dually, one can use Lemma 2.13 to notice that the functor $D\mathcal{C}(\cdot, \Sigma T')$ induces an equivalence of categories from $(\overline{T} \ast T')/\text{add } T$ to $\mathcal{E}'/\text{add } D\mathcal{C}(R, \Sigma T')$. The statement now follows from Theorem 2.9.

There are two particular cases of Theorem 2.11 that are worth noting. They are weak forms of nearly Morita equivalences that we call pseudo-Morita equivalences. They occur in the case where $R$ is indecomposable, that is $m = n + 1$, and we make this assumption for the rest of the section. It should be noted that $R = T_m$ and $R^* = T'_m$.

Let $Q_m$ be the $\Lambda$-module $\mathcal{C}(T, \Sigma R^*) = \mathcal{C}(T, \Sigma T'_m)$ appearing in Theorem 2.11. Similarly, we have the $\Lambda'$-modules $Q'_m = D\mathcal{C}(R, \Sigma T') = D\mathcal{C}(T_m, \Sigma T')$. Then, we have the following.

Lemma 2.14. Suppose that $R$ is indecomposable. Let $e$ be the idempotent for $\Lambda$ corresponding to $T_m$. Then, we have the isomorphism

$$Q_m \simeq \Lambda/\Lambda(1 - e)\Lambda.$$

Furthermore, $Q_m$ is a simple object of $\mathcal{E}$. Dually, let $e'$ be the idempotent for $\Lambda'$ corresponding to $\Sigma T'_m$. Then, we have the isomorphism

$$Q'_m \simeq \Lambda'/\Lambda'(1 - e')\Lambda'.$$

Furthermore, $Q'_m$ is an indecomposable $\Lambda'$-module and a simple object of $\mathcal{E}'$. 

Proof. The long exact sequences associated with the exchange triangle $T'_m \to U_m \to T_m \to \Sigma T'_m$ of Remark 2.10 show that the functor $\mathcal{C}(-, \Sigma T'_m)$ vanishes on $\text{add}(\overline{T})$, and that the $\Lambda$-module $\mathcal{C}(T_m, \Sigma T'_m)$ is isomorphic to the module $\mathcal{C}/(\text{add } \overline{T})(T_m, T_m)$. We thus have an isomorphism of $\Lambda$-modules:

$$\Lambda \Lambda(1 - e) \Lambda \simeq \mathcal{C}(T, T) \simeq \mathcal{C}((\text{add } \overline{T})(T_m, T_m) \simeq \mathcal{C}(T_m, \Sigma T'_m) \simeq Q_m.$$ 

Similarly, using the exchange triangle as above, we obtain an isomorphism between the $\Lambda'$-modules $D\mathcal{C}(T_m, \Sigma T')$ and $D\mathcal{C}/(\text{add } \Sigma T')(\Sigma T', \Sigma T')$, the latter being isomorphic to $Q'_m$.

Since $Q_m$ is projective over $\Lambda/\Lambda(1 - e)\Lambda$, we have

$$\text{Ext}^1_{\Lambda}(Q_m, S_m) \simeq \text{Ext}^1_{\Lambda/\Lambda(1 - e)\Lambda}(Q_m, S_m) = 0,$$

and therefore $Q_m$ lies in $\mathcal{E}$. If $N$ is a non-trivial submodule of $Q_m$ lying in $\mathcal{E}$, then $N$ is also a $\Lambda/\Lambda(1 - e)\Lambda$-module satisfying $\text{Ext}^1_{\Lambda(1 - e)\Lambda}(N, S_m) = 0$. Since $S_m$ is the only simple $\Lambda(1 - e)\Lambda$-module, $N$ is projective over $\Lambda(1 - e)\Lambda$, so it must equal $Q_m$. It follows that $Q_m$ is a simple object of the exact category $\mathcal{E}$. Since $\mathcal{E}$ is closed under direct summands, it now follows that $Q_m$ is an indecomposable $\Lambda$-module. The proofs of the duals of these last two statements are similar.

Corollary 2.15. Suppose that $\mathcal{C}$ satisfies the assumptions in Section 1 and that it has a Serre functor. Suppose further that $R$ is indecomposable. Then, there is an equivalence of categories

$$\mathcal{E}/\text{add } Q_m \simeq \mathcal{E'}/\text{add } Q'_m.$$ 

Proof. This follows from Theorem 2.11 and Lemma 2.14.

Corollary 2.16. Suppose that the assumptions in Corollary 2.15 hold, and, in addition, that the Gabriel quiver of $\Lambda$ has no loop at the vertex corresponding to $R$. Then, there is an equivalence of categories

$$\mathcal{E}/\text{add } S_m \simeq \mathcal{E'}/\text{add } S'_m.$$ 

Proof. This is a particular case of Corollary 2.15. Indeed, by [IY08, Proposition 2.6(1)], $R^*$ is also indecomposable and has no loop. This implies that $Q_m$ and $Q'_m$ are isomorphic to $S_m$ and $S'_m$, respectively.
§3. Localization

3.1 Notation and statement of main results

We continue with the assumptions and notation from Section 1. We do not assume here that \( C \) has a Serre functor, except in Corollary 3.5. Moreover, contrary to [BM13], we do not make any skeletal smallness assumption. This is because all of the localizations that we consider are shown to be equivalent to a subquotient of \( C \). Therefore, no set-theoretic difficulties arise, and the localizations we consider are all categories without passing to a higher universe.

Recall that, by [KR07, BM13], the functor \( C(T, -) \) induces an equivalence of categories from \( C(T)/\Sigma T \) to \( \text{mod} \Lambda \). In particular, it is dense and full when restricted to \( C(T) \).

Definition 3.1. Let \( \mathcal{B} \) be the full subcategory of \( \text{mod} \Lambda \) given by the (essential) image of \( T^\perp \) under \( C(T, -) \). Let \( \mathcal{S}_{\mathcal{B},0} \) be the class of all epimorphisms \( f \in \text{mod} \Lambda \) whose kernel belongs to \( \mathcal{B} \). Dually, we let \( \mathcal{B}' \) be the full subcategory of \( \text{mod} \Lambda' \) given by the (essential) image of \( \Sigma T' \) under \( DC(-, \Sigma T') \), and set \( \mathcal{S}_{0,\mathcal{B}'} \) to be the class of all monomorphisms \( g \in \text{mod} \Lambda' \) whose cokernel belongs to \( \mathcal{B}' \).

Let \( F \) be the composition of the fully faithful functor \( \mathcal{C}(T)/\Sigma \mathcal{C} \to \mathcal{C}(T)/\Sigma T \to \text{mod} \Lambda \) and the localization functor \( \text{mod} \Lambda \to \text{mod} \Lambda \mathcal{S}_{\mathcal{B},0} \). Then, since \( \mathcal{C}(T, \Sigma R^*) \) belongs to \( \mathcal{B} \), we have that \( F(\Sigma R^*) \simeq 0 \) in \( \text{mod} \Lambda \mathcal{S}_{\mathcal{B},0} \). Hence, \( F \) induces a functor \( \overline{F} \) as in the following diagram:

\[
\begin{array}{ccc}
\mathcal{C}(T)/(\Sigma T^\perp) & \xrightarrow{\cong} & \mathcal{C}(T)/(\Sigma T) \\
\downarrow & & \downarrow \text{mod} \Lambda \\
\overline{C}(T)/(\Sigma T') & \xrightarrow{\cong} & (\text{mod} \Lambda)_{\mathcal{S}_{\mathcal{B},0}} \\
\mathcal{C}(T)/\Sigma T' & \xrightarrow{\overline{F}} & (\text{mod} \Lambda)_{\mathcal{S}_{\mathcal{B},0}}
\end{array}
\]

Our main aim in this section is to show that the following holds.

Theorem 3.2. The functor \( \overline{F} : \overline{C}(T)/(\Sigma T') \to (\text{mod} \Lambda)_{\mathcal{S}_{\mathcal{B},0}} \) is an equivalence of categories. Dually, there is an equivalence \( \overline{C}(T)/(T) \to (\text{mod} \Lambda')_{\mathcal{S}_{0,\mathcal{B}'}} \).

This has two key corollaries, which we state below, after a lemma needed in the proof of the first one.
Lemma 3.3. For any object $M \in \mathcal{B}$, there exists $X \in \mathcal{C}(T) \cap T^\perp$ such that $\mathcal{C}(T, X) \simeq M$.

Proof. For any object $M \in \mathcal{B}$, there exists an object $Y \in T^\perp$ such that $\mathcal{C}(T, Y) \simeq M$. By [BM13, Lemma 3.3], there is a triangle $Z \to X \to Y \to \Sigma Z$ in $\mathcal{C}$, where $X \in \mathcal{C}(T)$, $Z \in T^\perp$ and $\varepsilon \in (T^\perp)$. Then, we have $\mathcal{C}(T, X) \simeq \mathcal{C}(T, Y) \simeq M$, which can be seen by applying the functor $\mathcal{C}(T, -)$ to the triangle above. Moreover, $X$ belongs to $T^\perp$, since both $Z$ and $Y$ belong to $T^\perp$.

Corollary 3.4. There is an equivalence of categories

$$(\text{mod } \Lambda)_{\mathcal{B}_{0,0}} \simeq \mathcal{E} / \text{add } \mathcal{C}(T, \Sigma R^*),$$

and, dually, an equivalence of categories

$$(\text{mod } \Lambda')_{\mathcal{B}_{0,0}} \simeq \mathcal{E}' / \text{add } D\mathcal{C}(R, \Sigma T').$$

Proof. For the first statement, combine Theorem 3.2 with Lemma 2.12, and for the second statement, combine Theorem 3.2 with Lemma 2.13.

Proof of Theorem B. We set $\mathcal{C}$ to be an acyclic cluster category, and set $T$ to be a rigid object in $\mathcal{C}$. We consider the case $m = n + 1$ and $R = T_m$ is indecomposable. As in the proofs of Corollaries 2.15 and 2.16, $\mathcal{C}(T, \Sigma R^*) \simeq Q_m \simeq S_m$ in this case. In particular, there are no loops in the quiver of $\text{End}_{\mathcal{C}}(T)$ at the vertex corresponding to $S_m$.

By Lemma 3.3, we have $\mathcal{B} = \mathcal{C}(T, T^\perp) = \mathcal{C}(T, \mathcal{C}(T) \cap T^\perp)$. For $j = 1, \ldots, n + 1$, let $P_j = \mathcal{C}(T, T_j)$ be the $j$th indecomposable projective in mod $\Lambda$. Then, an object $X$ in $\mathcal{C}(T)$ lies in $T^\perp$ if and only if $\text{Hom}_{\Lambda}(P_j, \mathcal{C}(T, X)) = 0$ for $j = 1, 2, \ldots, n$, which holds if and only if $\mathcal{C}(T, X)$ lies in $\text{add}(S_m)$. It follows that $\mathcal{B} = \text{add}(S_m)$ in this case, and hence $\mathcal{S}_{\mathcal{B}_{0,0}}$ coincides with the class $\mathcal{R}$ of morphisms considered in the introduction. We see that the first statement in Theorem B follows from the first statement in Corollary 3.4. The second statement in Theorem B follows from the second statement in Corollary 3.4.

Corollary 3.5. If the category $\mathcal{C}$ admits a Serre functor, then there is an equivalence of categories

$$(\text{mod } \Lambda)_{\mathcal{B}_{0,0}} \simeq (\text{mod } \Lambda')_{\mathcal{B}_{0,0}}.$$
Proof. Combine Theorem 3.2 with Theorem 2.9.

Proof of Theorem A. Since an acyclic cluster category has a Serre functor, Theorem A follows from Corollary 3.5 and the observations in the proof of Theorem B above.

We also use Theorem 3.2 to show that the categories $C_S$ and $\tilde{C}_S$ are isomorphic (Theorem 3.19). We also remark that Lemma 3.8 may be of independent interest.

3.2 Proof of Theorem 3.2
We show the first statement of the theorem. The second statement follows from a dual argument. In order to prove that $\mathcal{F}$ is full and dense, it is enough to prove that $\mathcal{F}$ is full and dense. The functor $\mathcal{F}$ is easily seen to be dense (Proposition 3.13). Showing that it is full requires a bit more work (Lemmas 3.8 and 3.9), and in order to do so we describe, in Lemma 3.11, the category $(\text{mod } \Lambda)_S$ as a localization of $\mathcal{C}$. We then show that the functor $\text{Hom}_\Lambda(U, -)$ induces a functor $(\text{mod } \Lambda)_S \rightarrow \text{mod } \Lambda$ (Lemma 3.15). Composing $\mathcal{F}$ with this induced functor and applying results from [BM13] then gives us the faithfulness of $\mathcal{F}$ (Proposition 3.16).

Lemma 3.6. The full subcategory $\mathcal{B}$ of $\text{mod } \Lambda$ is closed under taking images and submodules.

Proof. Let $M \rightarrow N$ be a morphism in $\text{mod } \Lambda$. Then, there are objects $X, Y \in \mathcal{C}(T)$ such that $\mathcal{C}(T, X) \simeq M$ and $\mathcal{C}(T, Y) \simeq N$, and a morphism $f : X \rightarrow Y$ such that $\mathcal{C}(T, f) \simeq u$. We complete $f$ to a triangle:

$$
Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z.
$$

If $M$ lies in $\mathcal{B}$ and $u$ is an epimorphism, then, by Lemma 3.3, we may take $X$ in $\mathcal{T}^\perp$, and, by [BM13, Lemma 2.5], $h$ factors through $T^\perp$. If $N$ lies in $\mathcal{B}$ and $u$ is a monomorphism, then, by Lemma 3.3, we may take $Y$ in $\mathcal{T}^\perp$, and, by [BM13, Lemma 2.5], $f$ factors through $T^\perp$.

In either case, the result follows from applying the functor $\mathcal{C}(T, -)$ to this triangle.

Proposition 3.7. The functor $\mathcal{F}$ is dense.

Proof. For any module $M \in \text{mod } \Lambda$, let $X$ be an object in $\mathcal{C}(T)$ such that $\mathcal{C}(T, X) \simeq M$. In Lemma 2.2, we constructed a triangle $Z \rightarrow R_0X \xrightarrow{gX} X \xrightarrow{g} \Sigma Z$, with $R_0X \in \mathcal{C}(T)$, $Z \in \mathcal{T}^\perp$ and $g \in (T^\perp)$. We claim that the
morphism \( \eta_X \) is inverted in \((\text{mod } \Lambda)S_{B,0}\). There is an exact sequence in \(\text{mod } \Lambda\):

\[
C(T, Z) \to C(T, R_0X) \to C(T, X) \to \Sigma Z.
\]

Therefore, the morphism \( C(T, \eta_X) \) is surjective, and \( C(T, Z) \) surjects onto its kernel. Since \( B \) is closed under images (Lemma 3.6), we may conclude that \( C(T, \eta_X) \) belongs to \( S_{B,0} \), and the claim is shown. This shows that \( M \cong FR_0X \) in \((\text{mod } \Lambda)S_{B,0}\), and we are done. \( \square \)

**Lemma 3.8.** Let \( Z \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{\varepsilon} \Sigma Z \) be a triangle in \( \mathcal{C} \) with \( X, Y \in \mathcal{C}(T) \) and \( \varepsilon \in (T^\perp) \). Then, \( Z \) belongs to \( \mathcal{C}(T) \).

**Proof.** Let \( T^Z \xrightarrow{f} Z \) be a minimal right add \( T \)-approximation. Complete it to a triangle \( U \xrightarrow{T} T^Z \xrightarrow{\delta} \Sigma U \). We note that, since \( f \) is an approximation and \( T \) is rigid, we have \( \Sigma U \in T^\perp \), as can be seen by applying the functor \( \mathcal{C}(T, -) \) to the triangle above, in a manner similar to that of \([BMR+06, \text{Lemma 6.3}]\). (A more general version of this assertion can be found in \([Jør09, \text{Lemma 2.1}]\).)

Since \( Y \) belongs to \( \mathcal{C}(T) \), there is a triangle \( T^Y_1 \to T^Y_0 \to Y \xrightarrow{\eta} \Sigma T^Y_1 \). By assumption, the composition \( \varepsilon a \) vanishes, so that there is a morphism \( T^Y_0 \xrightarrow{b} X \) such that \( a = vb \). We thus have a morphism of triangles

\[
\begin{array}{cccccc}
T^Z & \xrightarrow{[1]} & T^Z \oplus T^Y_0 & \xrightarrow{[0 \ 1]} & T^Y_0 & \xrightarrow{0} & \Sigma T^Z \\
\downarrow{f} & & \downarrow{[uf \ b]} & & \downarrow{a} & & \downarrow{\Sigma f} \\
Z & \xrightarrow{u} & X & \xrightarrow{v} & Y & \xrightarrow{\varepsilon} & \Sigma Z 
\end{array}
\]

which we complete to a nine diagram

\[
\begin{array}{cccccc}
U & \xrightarrow{} & V & \xrightarrow{} & T^Y_1 & \xrightarrow{} & \Sigma U \\
\downarrow{f} & & \downarrow{[1]} & & \downarrow{[0 \ 1]} & & \downarrow{\Sigma f} \\
T^Z & \xrightarrow{[1]} & T^Z \oplus T^Y_0 & \xrightarrow{[0 \ 1]} & T^Y_0 & \xrightarrow{0} & \Sigma T^Z \\
\downarrow{f} & & \downarrow{[uf \ b]} & & \downarrow{a} & & \downarrow{\Sigma f} \\
Z & \xrightarrow{u} & X & \xrightarrow{v} & Y & \xrightarrow{\varepsilon} & \Sigma Z \\
\downarrow{\delta} & & \downarrow{\eta} & & \downarrow{\eta} & & \downarrow{\eta} \\
\Sigma U & \xrightarrow{} & \Sigma V & \xrightarrow{} & \Sigma T^Y_1 & & \end{array}
\]
Since \( \Sigma U \in T^\perp \), the morphism \( T^Y_1 \to \Sigma U \) vanishes, and the top triangle splits. Thus, \( U \) is a summand of \( V \), and it is enough to prove that \( V \) belongs to \( \text{add } T \). Since \( X \in \mathcal{C}(T) \), this amounts to proving that the morphism \( T^Z \oplus T^Y_0 \to X \) is an \( \text{add } T \)-approximation. Let us thus prove the latter statement. Let \( W \xrightarrow{g} X \) be a morphism in \( \mathcal{C} \) with \( W \in \text{add } T \). (The morphisms are illustrated in the diagram below.) Then, the composition \( \eta vg \) is zero, so that there is a morphism \( W \xrightarrow{c} T^Y_0 \) with \( vg = ac \). This implies \( vg = vbc \), and there is a morphism \( W \xrightarrow{d} Z \) such that \( g - bc = ud \). Now, \( f \) is an \( \text{add } T \)-approximation, so that there is a morphism \( W \xrightarrow{e} T^Z \) satisfying \( d = fe \). The last two equalities give \( g = ufe + bc = [ufb]e \), and we have shown that \([ufb]\) is an \( \text{add } T \)-approximation. 

The following diagram shows the morphisms \( g, b, c, d \) and \( e \).

\[ \begin{array}{ccc}
T^Z & \xrightarrow{[0]1} & T^Z \oplus T^Y_0 \\
\downarrow{f} & & \downarrow{[0]1} \\
Z & \xrightarrow{[ufb]} & T^Y_0 \\
\downarrow{u} & & \downarrow{0} \\
X & \xleftarrow{v} & \Sigma T^Z \\
\downarrow{g} & & \downarrow{\Sigma f} \\
Y & \xleftarrow{\varepsilon} & \Sigma Z \\
\downarrow{\eta} & & \\
\Sigma T^Y_1 & & \\
\end{array} \]

**Lemma 3.9.** Let \( Z \xrightarrow{f} X \) be a morphism in \( \mathcal{C} \) with \( Z \in \mathcal{C}(T) \). Then, \( \mathcal{C}(T, f) \) factors through \( \mathcal{B} \) if and only if \( f \) factors through \( \overline{T}^\perp \).

**Proof.** If \( f \) belongs to the ideal \( (\overline{T}^\perp) \), then \( \mathcal{C}(T, f) \) factors through \( \mathcal{B} \) by the definition of \( \mathcal{B} \). Let us prove the converse. Since \( Z \in \mathcal{C}(T) \), there is a triangle \( T_1 \to T_0 \xrightarrow{g} Z \to \Sigma T_1 \) in \( \mathcal{C} \) with \( T_0, T_1 \in \text{add } T \). Assume that \( \mathcal{C}(T, f) \) belongs to \( (\mathcal{B}) \). Then, there exists \( U \in \overline{T}^\perp \), and there exist maps \( \mathcal{C}(T, Z) \xrightarrow{b} \mathcal{C}(T, U) \) and \( \mathcal{C}(T, U) \xrightarrow{a} \mathcal{C}(T, X) \) such that \( \mathcal{C}(T, f) = a \circ b \).

We would like to lift \( a \) and \( b \) to morphisms in the category \( \mathcal{C} \). This cannot be done in general, since the functor \( \mathcal{C}(T, -) \) is not full. Fortunately, it is full when restricted to \( \mathcal{C}(T) \). We thus use [BM13, Lemma 3.3] in order to replace the object \( U \) by an object \( U' \) whose image under \( \mathcal{C}(T, -) \) is isomorphic to that of \( U \), but with the additional property that \( U' \) is in \( \mathcal{C}(T) \). Let us
therefore apply [BM13, Lemma 3.3] so as to get triangles
\[ Y_U \rightarrow U' \rightarrow U \xrightarrow{\varepsilon} \Sigma Y_U \] and
\[ Y_X \rightarrow X' \xrightarrow{\eta} X \xrightarrow{\eta} \Sigma Y_X \] in \( C \), where \( U' \), \( X' \) belong to \( C(T) \), where
\( Y_U, Y_X \) belong to \( T^\perp \), and where the morphisms \( \varepsilon \) and \( \eta \) factor through \( T^\perp \).
Since \( U \) is in \( T^\perp \) and \( Y_U \) is in \( T^\perp \), \( U' \) is in \( T^\perp \) as well.
The modules \( C(T, U) \) and \( C(T, U') \) are isomorphic, and \( C(T, u) \) is
an isomorphism, so that there are morphisms \( C(T, Z) \xrightarrow{b'} C(T, U') \) and
\( C(T, U') \xrightarrow{a'} C(T, X') \) satisfying \( C(T, u) \circ a' \circ b' = C(T, f) \).
Now, the objects \( Z, U' \) and \( X' \) all belong to \( C(T) \), so that there exist morphisms \( \alpha, \beta \) in \( C \) with
\( C(T, \alpha) = a' \) and \( C(T, \beta) = b' \). We thus have the following diagram in \( C \):

\[
\begin{array}{ccc}
T_0 & \xrightarrow{\alpha} & X' \\
\downarrow g & & \downarrow u \\
Z & \xrightarrow{f} & X \\
\downarrow \Sigma T_1 & & \\
& \Sigma T_1 &
\end{array}
\]

where the square \( f - u \alpha \beta \) commutes up to a summand in \( T^\perp \). Since \( T_0 \in \text{add } T \), the composition \( (f - u \alpha \beta)g \) vanishes, and \( f - u \alpha \beta \) factors through \( \Sigma T_1 \). This shows that \( f \) factors through \( U' \oplus \Sigma T_1 \), which belongs to \( T^\perp \), and we are done.

**Definition 3.10.** Let \( \tilde{S} \) be the class of morphisms \( X \xrightarrow{s} Y \) in \( C \) such
that, for any triangle \( Z \xrightarrow{f} X \xrightarrow{s} Y \xrightarrow{g} \Sigma Z \), we have \( f \in (T^\perp) \) and \( g \in (T^\perp) \). It should be noted that this is a weaker property than that defining \( S \) (where instead of the property \( f \in (T^\perp) \) we had \( Z \in T^\perp \)). Therefore, \( S \subseteq \tilde{S} \).

Let \( C \xrightarrow{L_{\tilde{S}}} C_{\tilde{S}} \) be the localization functor with respect to the class \( \tilde{S} \).

**Lemma 3.11.** There is a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{C(T, -)} & \text{mod } \Lambda \\
\downarrow L_{\tilde{S}} & & \downarrow L_{\tilde{S}_{G,0}} \\
C_{\tilde{S}} & \xrightarrow{C'} & (\text{mod } \Lambda)_{S_{G,0}}
\end{array}
\]

where \( G' \) is an equivalence of categories.
Proof. It is proved in [BM13] that the functor $C(T, -) : C \rightarrow \text{mod } \Lambda$ is a localization functor for the class $S_T$ of morphisms $X \rightarrow Y$ such that, when completed to a triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, we have $g, h \in (T^\perp)$. Since this class is contained in the class $\tilde{S}$, it is enough to prove $C(T, \tilde{S}) = S_{B,0}$. Let $s$ be in $\tilde{S}$. There is a triangle $Z \xrightarrow{s} X \xrightarrow{s} Y \xrightarrow{h} \Sigma Z$ in $C$ with $g \in (T^\perp)$ and $h \in (T^\perp)$. Applying the functor $C(T, -)$ gives an exact sequence in $\text{mod } \Lambda$:

$$C(T, Z) \rightarrow C(T, X) \rightarrow C(T, Y) \rightarrow 0 \rightarrow C(T, \Sigma Z),$$

where $C(T, g) : C(T, Z) \rightarrow C(T, X)$ factors through some $B \in B$. Thus, $C(T, s)$ is an epimorphism, and its kernel is isomorphic to a quotient of a submodule of $B$ (see Remark 3.12 below). By Lemma 3.6, the subcategory $B$ is stable under taking images and submodules, so that $C(T, s)$ belongs to $S_{B,0}$.

Conversely, let $0 \rightarrow B \rightarrow M \xrightarrow{f} N \rightarrow 0$ be a short exact sequence in $\text{mod } \Lambda$, with $B \in B$. There is a morphism $X \xrightarrow{s} Y$ in $C$, with $X, Y \in C(T)$ such that $C(T, s) \simeq f$. Complete it to a triangle $Z \xrightarrow{u} X \xrightarrow{s} Y \xrightarrow{v} \Sigma Z$ in $C$. Then, $v \in (T^\perp)$ since $f$ is an epimorphism, and $C(T, u)$ factors through $B$ since $su = 0$. Lemma 3.8 shows that $Z$ lies in $C(T)$, and we can apply Lemma 3.9 to conclude that $u$ factors through $T^\perp$.

Remark 3.12. Let $L \xrightarrow{g} M \xrightarrow{f} N$ be exact in an abelian category. Assume that the morphism $g$ factors as $u \circ v$ through some object $B$. Then, the kernel of $f$ is isomorphic to a quotient of a subobject of $B$.

Proof. Let $K \xrightarrow{i} M$ be a kernel for $f$. Since the sequence is exact, there is an epimorphism $L \xrightarrow{g'} K$ such that $ig' = g$. The morphism $v$ factors as in the following diagram:

$$
\begin{array}{ccc}
B' & \xrightarrow{j} & B \\
\uparrow{p} & & \uparrow{u} \\
L & \xrightarrow{g} & M \\
\downarrow{g'} & & \downarrow{f} \\
K & \xrightarrow{i} & N
\end{array}
$$

The composition $fujp = fg$ vanishes, so that $fu = 0$, and there is some $B' \xrightarrow{q} K$ so that $iq = uj$. It remains to show that $q$ is an epimorphism.
Since \( i \) is a monomorphism, the equalities \( iqp = ujp = g = ig' \) imply \( qp = g' \). Since the morphism \( g' \) is an epimorphism, \( q \) is an epimorphism also.

**Proposition 3.13.** The functor \( F \) is full.

**Proof.** Let \( X \in \mathcal{C}(T) \). Then, there is a triangle \( T_1 \xrightarrow{\beta} T_0 \xrightarrow{\alpha} X \xrightarrow{\gamma} \Sigma T_1 \) in \( \mathcal{C} \). Consider a hook diagram

\[
\begin{array}{ccc}
C(T, X) & \xrightarrow{C(T,f)} & C(T, V) \\
\downarrow & & \downarrow \\
C(T, U) & \xrightarrow{C(T,s)} & C(T, V)
\end{array}
\]

in \( \text{mod } \Lambda \), with \( U, V \in \mathcal{C}(T) \) and \( C(T, s) \in S_{B,0} \). Let us prove that the morphism \( C(T, f) \) lifts through the morphism \( C(T, s) \). The proof of Lemma 3.11 shows that \( s \) belongs to \( \tilde{S} \). We thus have a triangle \( W \xrightarrow{g} U \xrightarrow{s} V \xrightarrow{h} \Sigma U \) in \( \mathcal{C} \) with \( g \in (T^\perp) \) and \( h \in (T^\perp) \). The composition \( hf \alpha \) vanishes, so that \( f \) induces a morphism of triangles

\[
\begin{array}{ccccccc}
T_1 & \xrightarrow{\beta} & T_0 & \xrightarrow{\alpha} & X & \xrightarrow{\gamma} & \Sigma T_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}V & \xrightarrow{\beta'} & W & \xrightarrow{g} & U & \xrightarrow{s} & V & \xrightarrow{h} & \Sigma W
\end{array}
\]

The morphism \( g \) factors through \( T^\perp \), so that the composition \( ga \) is zero, giving the existence of a morphism \( b \) such that \( hb = v \). The equalities \( hf = v\gamma = hb\gamma \) imply the existence of a morphism \( c \) such that \( f = b\gamma + sc \). Therefore, \( C(T, s) \circ C(T, c) = C(T, f) \). We can conclude by induction on the number of hooks in a morphism from \( C(T, X) \) to \( C(T, Y) \).

We write \( U \) for \( C(T, T) \). Define \( \bar{\Lambda} \) to be the endomorphism algebra of \( U \) in \( \text{mod } \Lambda \). Then, \( \bar{\Lambda} \simeq \text{End}_C(T) \).

**Lemma 3.14.** The diagram

\[
\begin{array}{ccc}
\bar{C}(T)/(\Sigma T) & \xrightarrow{C(T,-)} & \text{mod } \Lambda \\
\downarrow & & \downarrow \text{Hom}_\Lambda(U,-) \\
C(T,-) & \xrightarrow{\text{mod } \bar{\Lambda}} & \text{mod } \bar{\Lambda}
\end{array}
\]

commutes up to a natural isomorphism.
Proof. For any \( X \in \mathcal{C}(T) \), define a map \( \varphi_X : \mathcal{C}(T, X) \longrightarrow \text{Hom}_\Lambda (\mathcal{C}(T, T), \mathcal{C}(T, X)) \) by \( \varphi_X(\alpha) = \mathcal{C}(T, \alpha) \). Then, \( \varphi_X \) is \( \Lambda \)-linear, since \( \mathcal{C}(T, -) \) is a covariant functor, and it is an isomorphism, since \( \mathcal{C}(T, -) : \mathcal{C}(T) / (\Sigma T) \rightarrow \text{mod } \Lambda \) is fully faithful. The transformation \( \varphi \) is easily seen to be natural.

\[ \text{Lemma 3.15. The functor } \text{Hom}(U, -) \text{ induces a functor } (\text{mod } \Lambda)_{S_{B,0}} \longrightarrow \text{mod } \Lambda. \]

Proof. Suppose that the morphism \( f = \mathcal{C}(T, s) \) lies in \( S_{B,0} \) by Lemma 3.11. In particular, \( s \) belongs to \( \tilde{S} \) by Lemma 3.11. In particular, \( s \) is part of a triangle \( (r, s, t) \) with \( r, t \in (T^\perp) \), so that \( \mathcal{C}(T, s) \) is an isomorphism (as proved in [BM13, Lemma 2.5]). Hence, by Lemma 3.14, \( \text{Hom}_\Lambda (U, f) \) is an isomorphism.

Proposition 3.16. The functor \( F \) is faithful.

Proof. Assume that \( Fu = Fv \) for some \( u, v : X \rightarrow Y \) in \( \mathcal{C}(T) \). Then, Lemmas 3.14 and 3.15 imply \( \mathcal{C}(T, u) = \mathcal{C}(T, v) \), and [BM13, Lemma 2.3] implies \( u - v \in (T^\perp) \). Since \( X \) belongs to \( \mathcal{C}(T) = T * \Sigma T' \), there is a triangle \( T_\alpha \rightarrow X \rightarrow T'_\beta \rightarrow \), with \( T_\alpha \in \text{add } T \) and \( T'_\beta \in \text{add } T' \). The composition \( (u - v)w \) vanishes, so that \( u - v \in (\Sigma T') \), and the functor \( F \) is faithful.

Proof of Theorem 3.2. By Proposition 3.7, \( F \) is dense, and by Proposition 3.13, \( F \) is full. Hence, \( F \) is also full and dense. By Proposition 3.16, \( F \) is faithful, and Theorem 3.2 follows.

3.3 More localizations

In this section, we prove, under the assumptions as in Section 1, that the localizations \( \mathcal{C}_S \) and \( \mathcal{S}_{\tilde{S}} \) are isomorphic. We note that this result does not seem to follow easily from Lemma 3.11, as one would expect by analogy with [BM13, Section 4].

Lemma 3.17. The full subcategory \( \mathcal{C}(T) \) of \( \mathcal{C} \) is stable under taking direct summands.

Proof. Let \( X, X' \in \mathcal{C} \) be so that \( X \oplus X' \) belongs to \( \mathcal{C}(T) \). Let \( U_0 \rightarrow X, V_0 \rightarrow X' \) be minimal right add \( T \)-approximations. Then, \( U_0 \oplus V_0 \rightarrow X \oplus X' \) is a minimal right add \( T \)-approximation. When completing it to a triangle
We want to show that the triangle in the second row splits, which would then imply that $U_1$ and $V_1$, being summands of $W$, belong to $\text{add} \ T$. The composition $\Sigma^{-1}X' \to V_1 \to \Sigma U_1$ is zero, so that the morphism $V_1 \to \Sigma U_1$ factors through $V_0 \to \Sigma U_0$. However, there are no non-zero morphisms from $V_0$ to $\Sigma U_1$, since $V_0 \in \text{add} \ T$, and $\Sigma U_1$ is the cone of the right $\text{add} \ T$-approximation $U_0 \to X$.

Lemma 3.18. Let $X$ and $Y$ be objects in $\overline{C}(T)$, and let $X \xrightarrow{s} Y$ be a morphism in $\tilde{S}$. Then, there exist $U \in \text{add} \ T$, and morphisms $\Sigma U \xrightarrow{c} Y$, $Y \xrightarrow{a} \Sigma U$ and $Y \xrightarrow{d} X$ in $C$ such that

1. the morphism $X \oplus \Sigma U \xrightarrow{[s \ c]} Y$ is in $S$;
2. the image in $C_S$ of the morphism $Y \xrightarrow{[d]} X \oplus \Sigma U$ is inverse to $[s \ c]$.

In particular, all morphisms in $\overline{C}(T)$ that belong to $\tilde{S}$ are inverted by the localization functor $L_S : C \to C_S$.

Proof. Let $X \xrightarrow{s} Y$ be a morphism in $\tilde{S}$, with $X$ and $Y$ in $\overline{C}(T)$. Complete it to a triangle $X \xrightarrow{s} Y \xrightarrow{v} \Sigma Z \xrightarrow{u} \Sigma X$. We first show how to define the object $\Sigma U$ and the morphisms $a, c, d$. By assumption, the morphisms $v$ and $u$ factor through $T^\perp$ and $(\Sigma T)^\perp$, respectively. There is a triangle $\overline{U} \to U \xrightarrow{\alpha} Y \xrightarrow{\delta} \Sigma \overline{U}$, with $U \in \text{add} \ T$ and $\overline{U} \in \text{add} \ T$. Since $v$ is in $(T^\perp)$, the composition $v\alpha$ vanishes, and there is a morphism $b$ as in the diagram below, such
that $v = ba$.

\[
\begin{array}{ccccccc}
U & \downarrow \alpha \\
X \xrightarrow{s} Y & \xrightarrow{v} \Sigma Z & \xrightarrow{u} \Sigma X \\
\Sigma U & \downarrow d & \downarrow a & \downarrow b \\
\end{array}
\]

The composition $ub$ also vanishes since $u$ factors through $(\Sigma T)\perp$. Therefore, there is a morphism $c$ such that $b = vc$. Moreover, there is a morphism $d$ such that $1_Y = sd + ca$. Indeed, we have the equalities $vca = ba = v$, so that $1_Y - ca$ factors through $s$. Hence, the morphism $[s, c] : X \oplus \Sigma U \rightarrow Y$ is a retraction.

Applying the octahedral axiom to the composition

\[
X \xrightarrow{[b]} X \oplus \Sigma U \xrightarrow{[s, c]} Y
\]
yields the following commutative diagram whose rows and columns are triangles in $C$.

\[
\begin{array}{ccccccc}
\Sigma^{-1}Y & \xrightarrow{f} & Z & \xrightarrow{g} & X & \xrightarrow{s} & Y \\
\downarrow k & & \downarrow h & & \downarrow 0 & & \\
\Sigma^{-1}Y & \xrightarrow{0} & Z' & \xrightarrow{X \oplus \Sigma U \xrightarrow{[s, c]} Y} & Y \\
\Sigma U & \xrightarrow{[s, c]} & \Sigma U
\end{array}
\]

Since $[s, c]$ is a retraction, the triangle in the lower row splits, and the morphism $\Sigma^{-1}Y \rightarrow Z'$ is zero. Let $p$ be a morphism from $\overline{T}$ to $Z$. Since $s \in \overline{S}$, $g$ factors through $\overline{T}^\perp$, so $gp = 0$ and therefore $p$ factors through $f$. Since $kf = 0$, $f$ factors through $h$, and therefore $p$ factors through $h$. Applying the functor $C(\overline{T}, -)$ to the triangle $\overline{U} \rightarrow Z \rightarrow Z' \rightarrow \Sigma \overline{U}$ gives $Z' \in \overline{T}^\perp$. We have thus constructed a triangle

\[
Z' \rightarrow X \oplus \Sigma U \xrightarrow{[s, c]} Y \rightarrow \Sigma Z'
\]
in $C$, where $Z'$ belongs to $\overline{T}^\perp$. This implies (1).
We now check (2). We have \([ sc ] = 1 \) in \( C \). Since \([ sc ] \) lies in \( S \) by (1), it is invertible in \( C_S \), and (2) follows.

Finally, we check the last part of the statement. Let \( \pi : X \oplus \Sigma U \to X \) be the first projection. Extending \( \pi \) to a triangle in \( C \), we have

\[
\Sigma U \to X \oplus \Sigma U \xrightarrow{\pi} X \xrightarrow{0} \Sigma^2 U.
\]

Since \( \Sigma U \in T_\perp \), and the zero map factors through \( T_\perp \), we see that \( \pi \in \mathcal{S} \). Furthermore, \( s\pi = [s 0] : X \oplus \Sigma U \to Y \), so \( s\pi - [ sc ] = [0, -c] \) factors through \( \Sigma U \), where \( \Sigma U \) lies in \( \Sigma T' \). Morphisms of the form \( A \oplus V \xrightarrow{[1 0]} A \), with \( V \in T_\perp \), lie in \( \mathfrak{S} \). Hence, as in [BM13, Lemma 3.5], \( L_S(s)L_S(\pi) = L_S([ sc ]) \). Since \( \pi, [ sc ] \) both lie in \( S \), their images under \( L_S \) are invertible in \( C_S \), and it follows that the image \( L_S(s) \) is also invertible in \( C_S \), as required.

For a morphism \( f \) that is part of a triangle \( Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z \), recall that \( f \) belongs to the collection \( \mathcal{S} \) if and only if \( Z \) belongs to \( T_\perp \), and \( h \) factors through \( T_\perp \), and that \( f \) belongs to \( \tilde{\mathcal{S}} \) if and only if \( g \) factors through \( T_\perp \), and \( h \) factors through \( T_\perp \).

**Theorem 3.19.** There is an isomorphism of categories

\[
C_S \simeq C_{\tilde{\mathcal{S}}}.
\]

*Proof.* As proved in Lemma 3.11, the categories \( C_{\tilde{\mathcal{S}}} \) and \((\text{mod } \Lambda)_{S_{B,0}}\) are equivalent. By Theorem 3.2, the category \((\text{mod } \Lambda)_{S_{B,0}}\) is equivalent to \( \overline{C}(T)/(\text{add } \Sigma T') \).

It is easy to check that any morphism of the form \( X \oplus U \xrightarrow{[1 0]} X \), with \( U \in T_\perp \), lies in \( \mathcal{S} \). Hence, arguing as in [BM13, Lemma 3.5], if \( u, v \) are any morphisms in \( C \) such that \( v \) factors through \( T_\perp \), then \( L_S(u) = L_S(u + v) \). It follows that \( L_S \) induces a functor from \( \overline{C}(T)/(\text{add } \Sigma T') \) to \( C_S \), which we also denote by \( L_S \). Since \( \tilde{\mathcal{S}} \) contains \( \mathcal{S} \), the same argument applies to \( L_{\tilde{\mathcal{S}}} \). Furthermore, by the universal property of localization, the left-hand side of the diagram

\[
\begin{array}{ccc}
\overline{C}(T)/(\text{add } \Sigma T') & \xrightarrow{\simeq} & (\text{mod } \Lambda)_{S_{B,0}} \\
\downarrow \quad & & \\
C_S & \underset{\simeq}{\longrightarrow} & C_{\tilde{\mathcal{S}}}
\end{array}
\]
commutes, where the functor $\mathcal{C}_S \to \mathcal{C}_{\tilde{S}}$ is the identity on objects. The right-hand triangle commutes by Lemma 3.11 and (3.1). It is thus enough to show that the functor $L_S : \mathcal{C}(T)/(\Sigma T') \to \mathcal{C}_S$ is an equivalence of categories.

The functor $L_S$ is dense by Lemma 2.2.

We next check that $L_S$ is full. Let $X, Y$ be objects of $\mathcal{C}(T)$, and let $s : X \to Y$ be a morphism in $S$. By part (2) of Lemma 3.18, $sd = 1_Y - ca$, where $ca$ factors through $T^\perp$. Arguing as above, we have that $L_S(s)L_S(d) = L_S(1_Y)$, so $L_S(d) = L_S(s)^{-1}$. It follows that $L_S$ (on $\mathcal{C}(T)/(\text{add } \Sigma T')$) is full.

It thus remains to prove that $L_S$ is faithful. Via the use of the functor $\mathcal{C}(T, -)$ and of the category $\text{mod } \Lambda$, this follows from results in [BM13]. Recall that we write $\Lambda$ for the endomorphism algebra of $T$ in $\mathcal{C}$. The functor $\mathcal{C}(T, -)$ from $\mathcal{C}(T)/(\Sigma T')$ to $\text{mod } \Lambda$ inverts all morphisms in $S$, as proved in [BM13, Lemma 2.4]. By the universal property of localizations, there is a (unique) functor $\mathcal{C}_S \xrightarrow{F'} \text{mod } \Lambda$ such that $\mathcal{C}(T, -) = F'L_S$. Assume that the image under $L_S$ of a morphism $f$ in $\mathcal{C}(T)$ is zero. Then, $F'L_S(f) = 0$, so that $\mathcal{C}(T, f)$ is zero in $\text{mod } \Lambda$. By [BM13, Lemma 2.3], this implies that $f$ factors through $T^\perp$. Since $X$ is in $\mathcal{C}(T)$, this implies, by Lemma 2.3, that $f$ is zero in $\mathcal{C}(T)/(\Sigma T')$, and the functor $L_S$ is faithful.

Remark 3.20. The reader might wonder why our proof makes a detour through the category $\text{mod } \Lambda$. One might think of a more direct proof as follows. Since we have an inclusion $S \subseteq \tilde{S}$, it is enough to prove that every morphism in $\tilde{S}$ is inverted in $\mathcal{C}_S$. This should easily follow from Lemma 3.18. Let $X \xrightarrow{f} Y$ be a morphism in $\tilde{S}$. Then, there is a commutative diagram

\[
\begin{array}{ccc}
R_0X & \xrightarrow{f'} & R_0Y \\
\eta_X \downarrow & & \downarrow \eta_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

where $R_0X, R_0Y$ are in $\mathcal{C}(T)$, and $\eta_X, \eta_Y$ are in $S$. It thus only remains to be checked that the morphism $f'$ can be chosen in $\mathcal{C}_S$. If so, Lemma 3.18 would imply that $f'$ is inverted by $L_S$. Since $\eta_X$ and $\eta_Y$ are in $S$, $f$ would also be inverted by $L_S$. The problem here is that even though it is easily checked that $\tilde{S}$ is stable under composition, $\tilde{S}$ does not seem to satisfy the two-out-of-three property.
§4. Examples

4.1 Mutating a cluster-tilting object at a loop

We consider the triangulated orbit category $D^b(\text{mod } A_9)/\tau^3[1]$ (see [Kel05]). Its Auslander–Reiten (AR) quiver is depicted in Figure 1. Copies of a fundamental domain are indicated by dashed lines. Let $T$ be the direct sum $a \oplus b \oplus c$. Then, $T$ is a cluster-tilting object with a loop at $c$.

Let $T'$ be the cluster-tilting object obtained by mutating $T$ at $c$. Since there is a triangle $s \to b \to c\to d\to a$, we have $T' = a \oplus b \oplus s$. The indecomposable objects lying in $\mathcal{C}(T)$ are encircled (recall that $\mathcal{C}(T) = T \ast \Sigma T = T \ast \Sigma T'$). The objects $a, b, c$ (respectively, $n, q, r$) belong to $\mathcal{C}(T)$ since they are in $\text{add } T$ (respectively $\text{add } \Sigma T'$). The remaining encircled objects are in $\mathcal{C}(T)$ since there are triangles $a \to c \to d \to b \to e \to a$, $a \to b \to g \to a \to c \to h \to a \oplus a \to c \to i \to a \oplus b \to c \to j \to b \to c \to l$ and $a \oplus b \to c \to m \to$. The other four indecomposable objects are not in $\mathcal{C}(T)$ since $s$ is the shift of $c$, and there are triangles $c \to c \to f \to a \oplus c \to c \to k \to a \oplus c \to c \to p \to$.

By Theorem 2.9, the categories $\mathcal{C}(T)/(\Sigma T')$ and $\tau \mathcal{C}(T)/(\tau T)$ are equivalent. These two categories are illustrated in Figure 2.

4.2 Mutating a rigid object at a loop

In the same category $D^b(\text{mod } A_9)/\tau^3[1]$ as in the previous example, we now consider the rigid object $T = a \oplus c$. There is a triangle $n \to a \oplus a \to c \to (\text{see Figure 3})$, so that we can choose $T'$ to be $a \oplus n$. Indecomposable objects
in $\mathcal{C}(T)$ are encircled. The shift of $T'$ is $i \oplus q$. Deleting either vertices labeled $a$ and $c$, or vertices labeled $i$ and $q$, in the encircled part of the AR quiver yields isomorphic quivers, as depicted in Figure 4.

### 4.3 Mutating at a non-indecomposable summand

In this example, we consider the triangulated orbit category $D^b(\text{mod } A_5)/\tau^{-2}[1]$ (see [Kel05]). It was shown in [BMR+06] that this category is a Krull–Schmidt, Hom-finite category with a Serre functor, and its AR quiver is depicted in Figure 5. As for the previous examples, we have not drawn the arrows. The two subquivers inside the dotted boxes have to be identified so as to match the two copies of $a, b, c$ and $d$. We choose rigid objects $T = a \oplus b \oplus c \oplus d$ and $T' = a \oplus b \oplus c' \oplus d'$. Indecomposable objects
in $\mathcal{C}(T)$ are encircled. The indecomposable objects labeled $e, f, g, h$ are the shifts of $a, b, c', d'$, respectively. As predicted by Theorem 2.9, one obtains isomorphic quivers by deleting either vertices $a, b, c, d$ or vertices $e, f, g, h$.

**Acknowledgments.** The second author would like to thank the algebra team of the university of Leeds for a pleasant atmosphere when he was a postdoc there. Both authors are indebted to Apostolos Beligiannis for a preliminary version of [Bel13] which inspired the proofs of Section 2.1, and to Karin Baur who hosted their visits to the Institute for Mathematical Research (FIM) at the ETH Zürich, where this project was initiated. Both authors would like to thank an anonymous referee for a very careful reading, and gratefully acknowledge WuZhong Yang for raising an issue in our assumptions and for illustrating it with a clever example (see the remark above).
References

[Ami07] C. Amiot, On the structure of triangulated categories with finitely many indecomposables, Bull. Soc. Math. France 135(3) (2007), 435–474.

[Ami09] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, Ann. Inst. Fourier (Grenoble) 59(6) (2009), 2525–2590.

[Bel13] A. D. Beligiannis, Rigid objects, triangulated subfactors and abelian localizations, Math. Z. 274(3) (2013), 841–883.

[BIKR08] I. Burban, O. Iyama, B. Keller and I. Reiten, Cluster tilting for one-dimensional hypersurface singularities, Adv. Math. 217(6) (2008), 2443–2484.

[BKL08] M. Barot, D. Kussin and H. Lenzing, The Grothendieck group of a cluster category, J. Pure Appl. Algebra 212(1) (2008), 33–46.

[BM12] A. B. Buan and R. J. Marsh, From triangulated categories to module categories via localization II: calculus of fractions, J. Lond. Math. Soc. (2) 86(1) (2012), 152–170.

[BM13] A. B. Buan and R. J. Marsh, From triangulated categories to module categories via localization, Trans. Amer. Math. Soc. 365(6) (2013), 2845–2861.

[BMR+06] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204(2) (2006), 572–618.

[BMR07] A. B. Buan, R. J. Marsh and I. Reiten, Cluster-tilted algebras, Trans. Amer. Math. Soc. 359(1) (2007), 323–332 (electronic).

[BMV10] A. B. Buan, R. J. Marsh and D. F. Vatne, Cluster structures from 2-Calabi–Yau categories with loops, Math. Z. 265(4) (2010), 951–970.

[GLS] C. Geiß, B. Leclerc and J. Schröer, Cluster algebra structures and semicanonical bases for unipotent groups, preprint, arXiv:math/0703039 [math.RT].

[GLS11] C. Geiß, B. Leclerc and J. Schröer, Kac–Moody groups and cluster algebras, Adv. Math. 228(1) (2011), 329–433.

[Guo11] L. Guo, Cluster tilting objects in generalized higher cluster categories, J. Pure Appl. Algebra 215(9) (2011), 2055–2071.

[HJ12] T. Holm and P. Jørgensen, On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon, Math. Z. 270(1–2) (2012), 277–295.

[HJ13] T. Holm and P. Jørgensen, Realizing higher cluster categories of Dynkin type as stable module categories, Q. J. Math. 64(2) (2013), 409–435.

[IY08] O. Iyama and Y. Yoshino, Mutation in triangulated categories and rigid Cohen–Macaulay modules, Invent. Math. 172(1) (2008), 117–168.

[Jor09] P. Jørgensen, Auslander–Reiten triangles in subcategories, J. K-Theory 3(3) (2009), 583–601.

[Kel05] B. Keller, On triangulated orbit categories, Doc. Math. 10 (2005), 551–581.

[KR07] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi–Yau, Adv. Math. 211(1) (2007), 123–151.

[ML98] S. Mac Lane, Categories for the Working Mathematician, 2nd ed., Graduate Texts in Mathematics 5, Springer, New York, 1998.

[Nak13] H. Nakaoka, General heart construction for twin torsion pairs on triangulated categories, J. Algebra 374 (2013), 195–215.

[Pla11] P.-G. Plamondon, Cluster characters for cluster categories with infinite-dimensional morphism spaces, Adv. Math. 227(1) (2011), 1–39.

[RvdB02] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15(2) (2002), 295–366.
[Yan12] D. Yang, *Endomorphism algebras of maximal rigid objects in cluster tubes*, Comm. Algebra 40(12) (2012), 4347–4371.

[YZZ15] W. Yang, J. Zhang and B. Zhu, *On cluster-tilting objects in a triangulated category with Serre duality*, Comm. Algebra (2015), to appear.

[YZ15] W. Yang and B. Zhu, *Ghost-tilting objects in triangulated categories*, preprint, arXiv:1504.00093 [math.RT].

Robert J. Marsh
*School of Mathematics*
*University of Leeds*
*Leeds*
*LS2 9JT*
*UK*

marsh@maths.leeds.ac.uk

Yann Palu
*LAMFA*
*Faculté des sciences*
*33, rue Saint-Leu*
*80039 Amiens Cedex 1*
*France*

yann.palu@u-picardie.fr