Vacuum thin shell solutions in five-dimensional Lovelock gravity

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Abstract

Junction conditions for vacuum solutions in five-dimensional Einstein-Gauss-Bonnet gravity are studied. We focus on those cases where two spherically symmetric regions of space-time are joined in such a way that the induced stress tensor on the junction surface vanishes. So a spherical vacuum shell, containing no matter, arises as a boundary between two regions of the space-time. Such solutions are a generalized kind of spherically symmetric empty space solutions, described by metric functions of the class $C^0$. New global structures arise with surprising features. In particular, we show that vacuum spherically symmetric wormholes do exist in this theory. These can be regarded as gravitational solitons, which connect two asymptotically (Anti) de-Sitter spaces with different masses and/or different effective cosmological constants. We prove the existence of both static and dynamical solutions and discuss their (in)stability under perturbations that preserve the symmetry. This leads us to discuss a new type of instability that arises in five-dimensional Lovelock theory of gravity for certain values of the coupling of the Gauss-Bonnet term.
A higher dimensional theory which has attracted much interest is Lovelock gravity \[1\]. This is because the theory, having field equations of second order in derivatives of the metric, intuitively has the right ingredients for a classical theory of gravity. In particular, the linearised perturbations about physically sensible backgrounds are well-behaved and are of the same second derivative form as in General Relativity (GR). Equivalently, the quadratic part of the perturbed Lagrangian is of the general form \(h \partial \partial h\) so there are no corrections to the propagator and no extra (ghost) fields corresponding to higher derivatives \[2, 3\].

There are however some exotic features of Lovelock gravity which certainly do not arise in GR. One such feature is the problem of (non-)determinism \[4, 5, 6\]. Given an initial data surface and a specified intrinsic metric and its first time derivative (or extrinsic curvature) one can try to integrate the Lovelock equations to evolve the metric through time. There one runs into a theoretical problem: There are solutions with spacelike surfaces on which the extrinsic curvature may be suddenly discontinuous. It can not be determined from the initial data if the extrinsic curvature will jump or if the metric will evolve smoothly. This is equivalent to the nonuniqueness problem in inverting the canonical momentum which is polynomial in the curvature \[4\]. Even for more smooth metrics there can be a problem of indeterminism, where components of the metric become arbitrary. This second kind of nondeterminism, with arbitrary functions of time appearing, only occurs in a regime where the curvature is large enough that the higher order Lovelock tensors become appreciable compared to the Einstein tensor. However the first kind of indeterminism, for metrics of class \(C^0\), is quite generic in Lovelock gravity. This means that one has to be careful in interpreting Lovelock theory as an effective theory. It is too simplistic to say that the theory is valid when the curvature is small w.r.t. a certain characteristic scale.

A natural question arises: can we look at the same phenomenon in the context of timelike surfaces. That is, discontinuities allowed in integrating the equations of motion in a spacelike direction. The nonsmooth solutions we shall present here (first found in Ref. \[7\]) are the timelike analogues of the first kind of nondeterminism. These objects are not a priori pathological objects in the theory: they can be everywhere non-spacelike (they can even be static as we shall see) and so in principle they do not violate determinism. One of the original motivations for this work was to see whether stable solitonic objects can exist in a space which at large distances looks like a positive mass solution of GR (such objects might be interpreted as branes of the Lovelock theory itself). It seems that the answer is no and the reasons why they do not exist are interesting in their own right.

The essential features can be seen in the quadratic Lovelock theory, often called Einstein-Gauss-Bonnet (EGB) theory. We shall therefore restrict ourselves to this theory and to the minimum number of dimensions, i.e. five. The action is given by the Einstein-Hilbert term, plus the Einstein cosmological term and additionally the Gauss-Bonnet combination of quadratic curvature invariants:

\[
S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R - 2\Lambda + \alpha (R^2 + R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB}) \right),
\]

where \(\kappa^2 = 8\pi G\) and \(\alpha\) represents the coupling constant of the Gauss-Bonnet term. In five dimensions, this is in fact the most general Lovelock theory since the Lovelock combination of cubic terms \(\sim O(R^3)\) identically vanishes (in \(D = 6\) they combine to a quantity which is polynomial in the curvature \[3\]). Likewise, the \(n\)th order Lovelock terms only become relevant in \(2n + 1\) or more dimensions.

The field equations associated with the action \(\square\) coupled to some matter action take the form

\[
G^A_B + \Lambda \delta^A_B + \alpha H^A_B = \kappa^2 T^A_B,
\]

where \(T^A_B\) is the stress tensor, \(G^A_B \equiv -\frac{1}{2} \delta^{ABEF} R_{EF} = R^A_B - \frac{1}{2} \delta^A_B R\) is the Einstein tensor, and

\[
H^A_B \equiv -\frac{1}{8} \delta^{AC_1...C_4} R^{D_1D_2}_{C_1C_2} R^{D_3D_4}_{C_3C_4},
\]

and where the antisymmetrized Kronecker delta is defined as \(\delta^{A_1...A_p}_{B_1...B_p} \equiv p! \delta^{A_1}_{B_1} \cdots \delta^{A_p}_{B_p}\).

The spherically symmetric solution in this theory with \(T_{AB} = 0\), i.e. the analog to the Schwarzschild black hole in Einstein’s Theory, is the Boulware-Deser solution, which reads \([9, 10, 11]\)

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_3^2, \quad f(r) = k + \frac{r^2}{4\alpha} \left( 1 + \xi \sqrt{1 + \frac{4\Lambda}{3} + \frac{16M\alpha}{r^4}} \right)
\]

\(3\).
where \( d\Omega^2 = \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 + d\chi^2 \) is the line element of the three-sphere with normalized curvature \( k = 1 \) (solutions also exist with planar and hyperbolic horizon geometry, i.e. with \( k = 0, -1 \), respectively. For simplicity, we will focus here on the spherical case \( k = 1 \) and \( \xi^2 = 1 \).

We see here a typical feature of the EGB theory: the Boulware-Deser [3] metric has two branches. The minus branch \( (\xi = -1) \) reduces to the corresponding solution of GR in the limit \( \alpha \to 0 \), as expected. However, for the plus branch \( (\xi = +1) \) this limit is ill defined. Thus, the plus branch is called the “exotic branch” of the Boulware-Deser metrics and it is usually thought of as an unstable vacuum of the theory, with ghost excitations [3, 2], and a naked singularity instead of a black hole.

Just as for Schwarzschild’s metric, \( M \) is here a constant of integration and it is associated with the mass of the solution. Let us also point out that the Boulware-Deser solution is unique only under a certain assumption about the coupling constants (in the case of 5-dimensional EGB theory the assumption is \( 4\alpha/3\Lambda \neq 1 \) discussed in Refs. [9, 2]) and also the assumption that the metric is of class \( C^2 \) [13]. It is the relaxation of this last assumption which we explore in this article.

The spherically symmetric situation gives a simple setting in which to construct some intriguing vacuum geometries which are special to Lovelock gravity: we can construct thin-shell vacuum wormholes and other objects by gluing together different Boulware-Deser metrics. In order to study these geometries we will start by discussing the junction conditions in this theory, worked out in [16, 17]. These are the analogues of the Israel conditions in GR [18]. In particular, they will be employed to join two different spherically symmetric spaces.

Let \( \Sigma \) be a timelike hypersurface separating two bulk regions of spacetime, region \( \mathcal{V}_L \) and region \( \mathcal{V}_R \) (“left” and “right”). We introduce, for convenience, the coordinates \( (t_L, r_L) \) and \( (t_R, r_R) \) and the metrics

\[
ds^2_L = -f_L dt_L^2 + \frac{dr_L^2}{f_L} + r_L^2 d\Omega_L^2,
\]

\[
ds^2_R = -f_R dt_R^2 + \frac{dr_R^2}{f_R} + r_R^2 d\Omega_R^2,
\]

in the respective regions. We are interested in the case where both \( f_L(r_L) \) and \( f_R(r_R) \) are vacuum solutions, so they will be of the form given in equation (3). In general, the mass parameter \( M_R \) will be different from \( M_L \), and \( \xi_R \) different from \( \xi_L \) so that the two different branches of the Boulware-Deser solution can be joined.

It is also convenient to parameterize the shell’s motion in the \( r - t \) plane using the proper time \( \tau \) on \( \Sigma \). In region \( \mathcal{V}_L \) we have \( r_L = a(\tau), t_L = T_L(\tau) \) and in region \( \mathcal{V}_R \) we have \( r_R = a(\tau), t_R = T_R(\tau) \). The induced metric on \( \Sigma \) induced from region \( \mathcal{V}_L \) is the same as that induced from region \( \mathcal{V}_R \), and is given by

\[
ds^2 = -dr^2 + a(\tau)^2 d\Omega_\Sigma^2.
\]

This guarantees the existence of a coordinate system where the metric is continuous (\( C^0 \)). Let us set some conventions: The hypersurface \( \Sigma \) has a single unit normal vector \( \hat{n} \) which points from left to right; and the orientation factor \( \eta \) of each bulk region is defined as follows: \( \eta = +1 \) if the radial coordinate \( r \) points from left to right, while \( \eta = -1 \) if the radial coordinate \( r \) points from right to left.

We are now in position to classify the shells according to the following definitions: \( \eta_L \eta_R > 0 \) will be called the standard orientation; \( \eta_L \eta_R < 0 \) will be called the wormhole orientation.

Integrating the field equations from left to right in an infinitesimally thin region across \( \Sigma \) one obtains the junction conditions. This relates the discontinuous change of spacetime geometry across \( \Sigma \) with the stress tensor \( S^b_a \) (see Refs. [16, 17, 19] for details).

\[
(\Omega_R)_a^b - (\Omega_L)_a^b = -\kappa^2 S^b_a,
\]

Above, the subscripts \( L, R \) signify the quantity evaluated on \( \Sigma \) induced by regions \( \mathcal{V}_L \) and \( \mathcal{V}_R \) respectively. The symmetric tensor \( \Omega^a_b \) is given by

\[
\Omega^a_b = -\delta^{ac}_{bd} K_c^d + \alpha \delta^{acde}_{bgh} \left( -K_c^f R^{gh} + \frac{2}{3} K_c^f K_c^g K_c^h \right),
\]

\(^1\)See also [13, 19] were the non-uniqueness of the solution at the point of the space parameters \( \Lambda a = -3/4 \) is analysed.

\(^2\)Notice that this geometry could correspond to joining two “exterior regions” of a spherical solution as well as two “interior regions”.

\[\text{3}\]
where $a$, $b$, ... are indices on the tangent space of the world-volume of the shell. The symbol $K^a_b$ refers to the extrinsic curvature, while the symbol $R^{ab}{}_{cd}$ appearing here corresponds to the four-dimensional intrinsic curvature (see [7] for details). Once applied to the spherically symmetric case the tensor $\Omega^b_a$ turns out to be diagonal with components

$$
\Omega^r_r = -3 a^{-3} \left( \eta^3 a^3 \sqrt{\dot{a}^2 + f} + 4 \alpha \eta \sqrt{\dot{a}^2 + f} \left( k + \frac{2}{3} \dot{a}^2 - \frac{1}{3} f \right) \right) ,
$$

$$
\Omega^\theta_\theta = \Omega^\phi_\phi = \Omega^\tau_\tau .
$$

It can be verify that the following equation is satisfied

$$
\frac{d}{d\tau} (a^3 \Omega^r_r) = \dot{a} 3 a^2 \Omega^\theta_\theta , .
$$

This equation expresses the conservation of $S^b_a$, i.e. no energy flow to the bulk, which always holds when the normal-tangential components of the energy tensor in the bulk is the same in both sides of the junction hypersurface [10, 19].

The main point here is that non-trivial solutions to (7) are possible even when $S^b_a = 0$. That is, the extrinsic curvature can be discontinuous across $\Sigma$ with no matter on the shell to serve as a source. The discontinuity is then self-supported gravitationally and this is due to non-trivial cancelations between the terms of the junction conditions. Similar configurations are impossible in Einstein gravity (in that case the junction conditions are linear in the extrinsic curvature). Since we are interested in vacuum solutions, we will consider

$$
S^b_a = 0 .
$$

From equation (11) we see that in the case $\dot{a} \neq 0$, the components of the junction condition are not independent: $(\Omega_R)_r^r - (\Omega_L)_r^r = 0 \Rightarrow (\Omega_R)_b^b - (\Omega_L)_b^b = 0$ . So it suffices to impose only the first condition, which can be factorized as follows,

$$
\left( \eta R \sqrt{\dot{a}^2 + f_R} - \eta L \sqrt{\dot{a}^2 + f_L} \right) \times
\left\{ a^2 + 4\alpha (k + \dot{a}^2) - \frac{4\alpha}{3} \left( f_R + f_L + 2 \dot{a}^2 + \eta_R \eta_L \sqrt{f_R + \dot{a}^2} \sqrt{f_L + \dot{a}^2} \right) \right\} = 0 .
$$

All the information concerning the spherically symmetric vacuum shells is contained in [13]. There exist several possibilities to be explored, corresponding to different choices in the Bolware-Deser parameters $k$, $M$ and $\xi$, combined with the two possible orientations $\eta$. This permits a very interesting catalogue of geometries which we survey later and is further explored in [7].

The first factor in (13) vanishes for the smooth metric. Thus, for non-smooth solutions we demand that the second factor vanishes. From the second factor, squaring appropriately, we obtain

$$
\dot{a}^2 = \sigma \left( f_R + f_L - 3(k + a^2/4\alpha) \right)^2 - f_R f_L = -V(a) ,
$$

This is essentially a one-dimensional problem, given by an ordinary differential equation (14), like the equation for a particle of a given energy moving radially in a spherical potential. Now, since we have squared the junction condition, we must substitute (13) back into (13) to check the consistency. When doing so we find the following restrictions

$$
- \eta_R \eta_L \left( 2 f_R + f_L - 3(k + a^2/4\alpha) \right) \left( 2 f_L + f_R - 3(k + a^2/4\alpha) \right) \geq 0 ;
$$

$$
(f_R + f_L - 2(k + a^2/4\alpha)) > 0 .
$$

So, for a dynamical vacuum shell with a timelike world-volume $\Sigma$, the scale factor of the metric (6) on $\Sigma$ is governed by (14), under the inequalities (15) and (16).

Using the inequalities we immediately obtain the following general results for dynamical or static shells:
G1) Vacuum shells with the standard orientation always involve the gluing of a plus branch ($\xi = +1$) metric with a minus branch ($\xi = -1$) metric.

G2) Vacuum shells which involve the gluing of two minus branch ($\xi = -1$) metrics exist only when the Gauss-Bonnet coupling constant $\alpha$ satisfies $\alpha < 0$. They always have the wormhole orientation.

In the analysis above it has been explicitly assumed that $\dot{\alpha} \neq 0$. It can be checked that, as expected, all the information about the constant $a$ solutions can be obtained from the dynamical case by imposing both $V(a_0) = 0$ and $V'(a_0) = 0$. Nevertheless, since the case $\dot{a} = 0$ describing static shells is of considerable interest, we shall treat it here explicitly.

So, let us now discuss the solutions for constant $a$, $a = a_0$. The bulk metric in each of the two region is assumed to be of the Boulware-Deser form (3). In this case the shell is located at fixed radius $r_L = r_R = a_0$. The proper time on the shell’s world-volume is $\tau = t_L \sqrt{f_L(a)} = t_R \sqrt{f_R(a)}$ so that the induced metric on $\Sigma$ turns out to be $d\tilde{s}^2 = -d\tau^2 + a_0^2 d\Omega_3^2$. Then, the extrinsic curvature components are $K^\tau_\tau = \eta \frac{f'}{f}$, $K^\theta_\theta = K^\phi_\phi = \eta^2 \theta \phi$ and the intrinsic curvature components are $R^\theta_\phi = k/a_0^2$, etc. The junction conditions with $S^\alpha_\beta = 0$ give:

\[
S^\tau_\tau = 0 \Rightarrow (\eta_R \sqrt{f_R} - \eta_L \sqrt{f_L}) \left( a_0^2 + \frac{4\alpha}{3} \left( 3k - f_R - f_L - \eta_L \eta_R \sqrt{f_L f_R} \right) \right) = 0 ,
\]

\[
S^\theta_\theta = 0 \Rightarrow \left( \frac{\eta_R}{\sqrt{f_R}} - \frac{\eta_L}{\sqrt{f_L}} \right) \left( k - \frac{\Lambda a_0^2}{3} - \eta_L \eta_R \sqrt{f_L f_R} \right) = 0 ,
\]

In both equations (17) and (18), the first factor vanishes if and only if the metric is smooth. Again, rejecting this as the trivial solution, we demand that the second factor vanishes in both equations (under the condition $f_L, f_R > 0$).

Let us first consider $\Lambda \neq 0$. Solving the equations we see that $f_L$ and $f_R$ obey the same quadratic equation where one $f$ has the + root of the solution and the other has the − root. We will call these solutions $f_+$ and $f_-$ respectively with corresponding parameters $\xi_+, \xi_-$ and $M_+, M_-$. Substituting the explicit expression for $f_L, f_R$, evaluated at $r = a_0$, we find

\[
1 + x \pm \sqrt{3} \sqrt{x(1 + x) \left( \frac{3}{x} + \frac{12}{\Lambda a_0^2} - 1 \right)} = 2\xi(\pm) \sqrt{1 + x + \frac{9x^2 M(\pm)}{\alpha \Lambda^2 a_0^4}} ,
\]

where we have found it convenient to define the dimensionless parameter\footnote{This parameter is important in determining the nature of the Boulware-Deser solutions. For $x < -1$ both branches are pathological, with branch singularities where the metric becomes complex. In particular there is no asymptotic region since the metric always becomes complex for $r \to \infty$. $x = -1$ is the special case, related to the Chern-Simons theory of gravity in five dimensions, where the effective cosmological constants of the two branches are the same. For $x > -1$ the $(-)$ branch solution is somewhat similar to the Schwarzschild/Schwarzschild-(A)dS black hole.}

\[x \equiv \frac{4\alpha \Lambda}{3} .\]

For a solution to exist, the square root in the l.h.s. of (19) must be real, and since we have squared the equations we must substitute back to check the consistency. So we get (19) along with the following inequalities:

\[
\frac{3}{x\Lambda a_0} (3 + x) + 2 > 0 \quad \text{ (Timelike shells)} ,
\]

\[
\frac{3}{\Lambda a_0} < 1 \quad \text{ (Standard orientation)} , \quad \frac{3}{\Lambda a_0} > 1 \quad \text{ (Wormhole orientation)}.\]

These admit solutions for a wide range of the coupling constants $\Lambda$, $\alpha$ and parameters $\xi_\pm$, $M_\pm$, which is described exhaustively in ref [7]. Here we mention some general results for static shells:

S1) Static shells with wormhole orientation only exist for $\Lambda > 0$. 
solutions also exist for $\Lambda = 0$. A class describes actual wormholes, presenting two different asymptotic regions which are connected are unstable with respect to small perturbations, as we shall see below.

S2) Static shells with wormhole orientation containing two asymptotic regions only exist for $\alpha > 0$. At least one region will be asymptotically Anti-de Sitter.

S3) Let $\Lambda \leq 0$. Then static shells exist (with standard orientation) joining (+) with (-) branches.

In S2) we used the fact that the metric is well defined as $r \to \infty$ only for $1 + 4\alpha \Lambda /3 > 0$. Also, in S1-S3 we have preemptively written the result for $\Lambda = 0$ which we now show. This is an interesting special case, in which the equations reduce to

\[
f_L + f_R = 2 + \frac{3}\alpha a^2, \quad (22)
\]

\[
\eta_L \eta_R \sqrt{f_L f_R} = 1. \quad (23)
\]

We see from the second equation that $\eta_L \eta_R$ must be $+1$, i.e. static wormholes do not exist for $\Lambda = 0$. One can also check that the consistency of the solutions leads to the condition $\alpha > 0$ as well as the $\Lambda = 0$ case of S3).

Summarizing, for the standard orientation geometries, $\eta_L \eta_R > 0$, branches are always $(\xi_{(-)}, \xi_{(+)}) = (-1, +1)$, so region $V_L$ has a different effective cosmological constant to region $V_R$, as can be seen from the expansion of the metric for large $r$. In this sense the shell is like the false vacuum bubbles studied in Refs. [20], but for a false vacuum which is of purely gravitational origin (see [2]). These kind of solutions might lead to curious implications for the global spacetime structure. For instance, we can construct a vacuum solution whose geometry, from the point of view of an external observer, would coincide with that of a black hole but, instead, would not possess a horizon. A particle in free fall would not find a horizon but rather a naked singularity as soon as it passes through the $C^0$ junction hypersurface located at $r = a > r_H$. This is depicted in Fig. 1 for the case $\Lambda = 0$ (similar solutions also exist for $\Lambda \neq 0$). However, as one would expect, such cosmic-censorship-spoiling shells are unstable with respect to small perturbations, as we shall see below.

On the other hand, there are two different classes of wormhole orientation geometries. The first class describes actual wormholes, presenting two different asymptotic regions which are connected through a throat located at radius $r_L = r_R = a$; the radius of the throat being larger than the radius where the event horizons (or naked singularities) would be. This type of geometry is an example of a vacuum spherically symmetric wormhole solution in Lovelock theory and its existence is a remarkable fact on its own. The second class of wormhole-like geometry has no asymptotic regions, and is obtained by cutting away the exterior region of both geometries and gluing the two interior regions together.

Finally, we discuss dynamical shells and the issue of stability of the static shells. In general, vacuum shells will be dynamical objects. In order to discuss their dynamics and stability let us briefly recapitulate upon the equation [14], which governs the dynamics of the shells. It takes the form:

\[
a^2 + V(a) = 0; \quad (24)
\]

Figure 1: An example of a solution with standard orientation, for $\Lambda = 0$. a) (+) branch spacetime: naked singularity, asymptotically AdS; b) (-) branch spacetime: black hole, asymptotically flat; c) By cutting out shaded regions and joining we obtain a $C^0$ vacuum solution with a “false vacuum bubble” containing a naked singularity (singularities are shown as dashed lines; the faint timelike line is the shell worldvolume).
a) \((-\) branch dS black hole

b) \((+\) branch AdS naked singularity

c) Vacuum thin shell wormhole. The naked singularity is removed.

Figure 2: diagram c) shows a static wormhole joining two asymptotic dS/AdS regions. This is a vacuum solution of Einstein-Gauss-Bonnet theory.

\[
Y = \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16Ma^4}{\alpha^4}},
\]

with which the effective potential reads

\[
V(a) = \left(1 + \frac{a^2}{4\alpha}\right) - \frac{a^2}{4\alpha} \left(\frac{3(\xi_R Y_R + \xi_L Y_L)^2 + (\xi_R Y_R - \xi_L Y_L)^2}{12(\xi_R Y_R + \xi_L Y_L)}\right).
\]

(25)

In addition to the differential equation, the solution must obey the inequalities (15) and (16).

To analyze the motion of a shell we need to know the derivatives of the potential (this is worked out in the appendix of [7]). Differentiating the potential we get the following expression for the acceleration of a moving shell,

\[
\ddot{a} = \frac{-a}{4\alpha} \left[1 - \frac{1 + 4\alpha\Lambda/3}{\xi_R Y_R + \xi_L Y_L}\right].
\]

(26)

Considering the sign of this acceleration, we can make some general observations: When \(1 + \frac{4\alpha\Lambda}{3} \geq 0\) and \(\alpha < 0\) a vacuum shell always experiences a repulsive force away from \(r = 0\); conversely when \(1 + \frac{4\alpha\Lambda}{3} \leq 0\) and \(\alpha > 0\) a vacuum shell always experiences an attractive force towards \(r = 0\). Which means that if \(\Sigma\) is a timelike shell it will either be in an (unstable) static state, or, if it is moving, will either expand or collapse, it can not be bound.

Combining with results derived from the inequalities we can state further results for dynamical shells in the regime \(1 + \frac{4\alpha\Lambda}{3} > 0\):

D1) When \(\alpha < 0\) a vacuum shell always experiences a repulsive force away from \(r = 0\).

D2) A shell joining two minus branches always experiences a repulsion away from \(r = 0\).

D3) A shell joining a minus branch with a plus branch region will either: i) be in an (unstable) static state or, after at most one bounce: ii) collapse without reexpanding or iii) expand indefinitely. It will not perform oscillations or any other bounded motion.

A corollary of D3) is that shells with standard orientation are unstable (Fig. 3).

So in summary, we have found some general results for the range of parameters \(1 + \frac{4\alpha\Lambda}{3} \geq 0\). This range is of importance as it includes the case \(|\alpha\Lambda| \ll 1\) and therefore applies when the Gauss-Bonnet
term is a small correction. Combining these results, we conclude that, in this range of parameters, all timelike vacuum shells involving the minus branch are unstable. The only vacuum shell solutions which can be static or oscillatory are wormholes which match two regions of the exotic plus branch.

Here we have focused on the case where the shell is a 3-sphere evolving though time in a spherically symmetric background. However this analysis can be straightforwardly extended to the cases of any constant curvature 3-manifold shell and to shells of either spacelike or timelike signature. This generalization and a more complete analysis of the space of solutions can be found in Ref. [7].

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