Quantum Berezinskii–Kosterlitz–Thouless transition for topological insulator

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ABSTRACT
We consider the interacting helical liquid system at the one-dimensional edge of a two-dimensional topological insulator, coupled to an external magnetic field and s-wave superconductor and map it to an XYZ spin chain system. This model undergoes quantum Berezinskii–Kosterlitz–Thouless (BKT) transition with two limiting conditions. We derive the renormalization group (RG) equations explicitly and also present the flow lines behavior. We also present the behavior of RG flow lines based on the exact solution. We observe that the physics of Majorana fermion zero modes and the gaped Ising-ferromagnetic phase, which appears in a different context. We observe that the evidence of gapless helical Luttinger liquid phase as a common non-topological quantum phase for both quantum BKT transitions. We explain analytically and physically that there is no Majorana-Ising transition. In the presence of chemical potential, the system shows the commensurate to incommensurate transition.

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1. Introduction
Berezinskii [1], in the year 1971 and Kosterlitz and Thouless [2], in the year 1973 have explained a new kind of phase transition in two-dimensional XY spin model [3] using renormalization group (RG) method [4–9]. According to Mermin–Wagner–Hohenberg theorem [10,11], continuous symmetry cannot be broken spontaneously at any finite temperature for \(d \leq 2\) (\(d=\) dimension). This is because of strong fluctuations of the goldstone modes in \(d = 1,2\) which restore the broken symmetry at long distances for finite temperature. However, the classical XY model with \(d = 2\) is found to have power-law decay in correlation function at low temperature and exponential decay in correlation function at high temperature [12]. This predicted a new kind of phase transition between them, presently known as Berezinskii–Kosterlitz–Thouless (BKT) transition.

BKT transition can successfully explain the phase transition in two-dimensional XY model by considering the topological non-trivial vortex (topological defect) configuration, where there is no requirement of spontaneous symmetry breaking. They have proposed that the disordering is facilitated by the condensation of topological defects [1,2]. The basic explanation is that, at high temperature, the correlation function decays exponentially and thermal generation of vortices is favorable for \(T \geq T_c\) (where \(T_c\) is the critical temperature of BKT transition). Thus, at a higher temperature, even number of vortices with opposite sign (i.e. vortex and anti-vortex) are produced and they are unbounded. At low temperature, i.e. at \(T < T_c\), the correlation function decays as power low and vortex and anti-vortex are bounded by forming a pair. Thus, the phase transition takes place at the critical temperature which is obtained by minimizing the free energy [13]. This transition

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was first explained in two-dimensional XY model. Therefore, the study of BKT transition is crucial in quantum many body systems since many quantum mechanical two-dimensional systems can be approximated to two-dimensional XY model [14].

The physics of low-dimensional quantum many body condensed matter system is enriched with its new and interesting emergent behavior. One-dimensional electronic systems cannot be solvable by the Fermi liquid theory due to the infrared divergence of certain vertices. An alternative theory called Tomonaga–Luttinger liquid theory has been constructed to describe the one-dimensional electronic system [15]. Hence we mention very briefly the nature of different Luttinger liquid physics to emphasis the enrich physics of helical Luttinger liquid. In this theory, the Luttinger parameter ($K$) determines the nature of interaction. $K < 1$ and $K > 1$ characterizes the repulsive and attractive interactions, respectively, where as $K = 1$ characterizes non-interacting case [16].

The physics of Luttinger liquids (LL) can be of three different forms: spinful LL, chiral LL, and helical LL. Spinful LL shows linear dispersion around the Fermi level with the difference of $2k_F$, in the momentum between left and right moving branches. Chiral LL has spin degenerated, strongly correlated electrons moving in only one direction. In helical LL one can observe the Dirac point due to the crossing of left and right moving branches, also electrons with opposite spins move in opposite directions [16]. In one dimension these helical edge states are protected by time-reversal (TR) symmetry with $T^2 = -1$. In contrast to this, spinless LL satisfies $T^2 = 1$ and chiral LL breaks TR invariance in one dimension. Spinful LL has to have an even number of TR pairs where as helical LL can have an odd number of components [17].

The realization of spinful Luttinger liquids have been observed in carbon nanotubes [18–20], GaAs/AlGaAs heterostructures [21], cleaved edge overgrowth one-dimensional channel [22], which break the TR invariance, also chiral Luttinger liquids have been observed in fractional quantum Hall edge states [23]. Quantum spin Hall insulator or topological insulator supports the helical edge states, which are realized in HgTe [24] and InAs/GaSb quantum wells [16,25–27].

Here, we consider an interacting helical liquid system at the edge of the quantum spin Hall system as our model Hamiltonian. Quantum spin Hall systems with or without Landau levels describe the helical edge states and it also describes the connection between spin and momentum. The left movers in the edge of quantum spin Hall systems are associated with down spin and right movers with up spin [28–34]. In the non-interacting case, the helical liquid is characterized by the $Z_2$ symmetry indicating that the even and odd TR components are topologically distinct [28]. In the interacting case, it is observed that helical liquid with odd number of components can not be constructed in the one-dimensional lattice [17]. Low-temperature conductance of a weakly interacting one-dimensional helical liquid without axial spin symmetry has been explored [35]. The formation of these one-dimensional states which can be controlled by the gate voltage on the topological surface has been studied and found the energy dispersion is almost linear in the momentum [36]. The impact of interaction on the helical liquid system has been studied explicitly, which results in the forming of Majorana fermion states with a high degree of stability [37]. The scattering process between fermion bands conserving momentum of helical liquid system opens a gap against interaction effect, which leads to the stabilization of Majorana fermion mode [38]. The existence of the Majorana fermion mode and the characterization of Majorana–Ising transition has also been studied extensively [39,40].

However, the renormalization group study and the physics of quantum Berezinskii–Kosterlitz–Thouless (BKT) transition have not been studied explicitly for interacting helical edge states. Quantum BKT transition is a topological quantum phase transition in the low dimensional quantum many-body system. But it has not been explored explicitly in the literature for the helical edge states or for any quantum matter [5,13]. Therefore, in this paper investigate the quantum BKT transition for the edge states of topological insulator. Quantum BKT transition happens at temperature $T = 0$. Here, we study how RG flow lines of the sine-Gordan coupling constant (i.e $B$ and $\Delta$ in the present problem) behave with the LL parameter ($K$).
2. Motivation and relevance of this study

First objective: The physics of topological state of matter is the second revolution of quantum mechanics [41]. This important concept and new important results with a high impact not only bound to the general audience of different branches of physics but also creates interest for the other branches of science. This new revolution in quantum mechanics was honored by the Nobel prize in physics in the year 2016. This is one of the fundamental motivation to study the topological state of matter for the edge state of topological insulator.

Second objective: One of the main motivations of this study is to find the quantum BKT for the one-dimensional helical edge mode of a two-dimensional topological insulator and also the limit in which it appears. We also search that what are the quantum phases appears in the quantum BKT transition, and there is any relation between the quantum phases which appear in the two different quantum BKT transition. There is no evidence of any Majorana-Ising transition for this quantum BKT transitions.

Third objective: The mathematical structure and results of the renormalization group (RG) theory are the most significant conceptual advancement in quantum field theory in the last several decades in both high-energy and condensed matter physics [42]. The need for the RG is more transparent in condensed matter physics. Therefore, in the present study, we use RG method to study the different quantum phases either topological or non-topological in character, through two quantum BKT Hamiltonians.

Fourth objective: It is very rare to find the exact solution for the problems of quantum many body condensed matter system. Here we find the exact solution of quantum BKT equation for the edge state of topological insulator. The other motivation is to find the exact solution for the RG flow line of this two quantum BKT equation. Therefore, the present study of quantum BKT provides a new perspective on topological quantum phase transition.

3. Model Hamiltonian and the derivation of quantum BKT equations

We consider the interacting helical liquid system at the one-dimensional edge of a topological insulator as our model system [17,28,43–45]. These edge states are protected by the symmetries [33,46]. Topological insulator is two-dimensional system but the physics of helical liquid at the edge of topological insulator is one-dimensional. In this edge states of helical liquid, spin and momentum are connected as the right movers are associated with the spin up and left movers are with spin down and vice versa. One can write the total fermionic field of the system as,

$$\psi(x) = e^{ikF x} \psi_{R\uparrow} + e^{-ikF x} \psi_{L\downarrow},$$  \hspace{1cm} (1)

where $\psi_{R\uparrow}$ and $\psi_{L\downarrow}$ are the field operators corresponding to right moving (spin up) and left moving (spin down) electron at the both upper and lower edges of the topological insulators.

Here we discuss the basics of this model Hamiltonian very briefly [17,37]. For the low energy collective excitation in one-dimensional system one can write the Hamiltonian as,

$$H_0 = \int \frac{dk}{2\pi} v_F |(\psi_{R\uparrow} \dagger (i\partial_\xi) \psi_{R\uparrow} - \psi_{L\downarrow} \dagger (i\partial_\xi)\psi_{L\downarrow}) + (\psi_{R\downarrow} \dagger (i\partial_\xi) \psi_{R\downarrow} - \psi_{L\uparrow} \dagger (i\partial_\xi) \psi_{L\uparrow})|,$$  \hspace{1cm} (2)

where the terms in the parenthesis represent Kramer’s pair at both edges of the system. The Hamiltonian for the non-interacting part of the one edge of the helical liquid system is,

$$H_{01} = \psi_{L\downarrow} \dagger (v_F i\partial_\xi - \mu) \psi_{L\downarrow} + \psi_{R\uparrow} \dagger (-v_F i\partial_\xi - \mu) \psi_{R\uparrow},$$  \hspace{1cm} (3)

We consider the topological insulator in the proximity of s-wave superconductor ($\Delta$) and the magnetic field (B). Thus, the additional part of the Hamiltonian is given by,

$$\delta H = \Delta \psi_{L\downarrow} \psi_{R\uparrow} + B \psi_{L\downarrow} \dagger \psi_{R\uparrow} + h.c.$$  \hspace{1cm} (4)
We will see in the present study that coupling \( \Delta \) induce the topological superconducting phase and coupling \( B \) induce the Ising-ferromagnetic phase. One can find two types of interactions which are allowed by time-reversal in helical liquid system. They are Forward and Umklapp interactions [47],

\[
H_{fw} = g_2 \psi_{L+/1} \psi_{R+/1} \psi_{R+/1}.
\]

\[
H_{um} = g_u \psi_{L+/1} \partial_x \psi_{L+/1} \psi_{R+/1} \partial_x \psi_{R+/1} + h.c.
\]

Thus, we get the total Hamiltonian as, \( H = H_{01} + H_{fw} + H_{um} + \delta H \). The authors of ref. [37] have mapped this Hamiltonian to the XYZ spin-chain model (up to a constant) i.e. \( H_{XYZ} = \sum_i H_i \), where

\[
H_i = \sum_{\alpha} J_{\alpha} S_{i+1}^\alpha S_{i+\alpha} - \mu + B(-1) S_{i}^z.
\]

This is our model Hamiltonian where, \( J_x = v_F + \Delta, J_y = v_F - \Delta \) and \( J_z = g_u \) are coupling constants. One can write the model Hamiltonian in spinless fermion form after Jordan–Wigner transformation as [40],

\[
H = -\frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + h.c) + J_z \sum_i \left( c_i^\dagger c_i - \frac{1}{2} \right) \left( c_{i+1}^\dagger c_{i+1} - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_i (c_i^\dagger c_{i}^\dagger + h.c) - \sum_i [\mu + B(-1)] \left( c_i^\dagger c_i - \frac{1}{2} \right)
\]

After the continuum field theory, one can write the Hamiltonian as [37,39,40,48,49],

\[
H = \frac{\nu}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K(\partial_x \theta(x))^2 \right] dx + \frac{B}{\pi} \int \cos (\sqrt{4\pi} \phi(x)) dx - \frac{\Delta}{\pi} \int \cos (\sqrt{4\pi} \theta(x)) dx + \frac{g_u}{2\pi^2} \int \cos (4\sqrt{\pi} \phi(x)) dx - \frac{\mu}{\sqrt{\pi}} \int \partial_x \phi(x) dx,
\]

where \( \theta(x) \) and \( \phi(x) \) are the dual fields and \( K \) is the Luttinger liquid parameter [50] of the system. The author of ref. [39] has shown explicitly the \( g_u \) has no effect on the topological state and also on the Ising-ferromagnetic state of the system. Therefore, the Hamiltonian is reduced to,

\[
H = \frac{\nu}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K(\partial_x \theta(x))^2 \right] dx + \frac{B}{\pi} \int \cos (\sqrt{4\pi} \phi(x)) dx - \frac{\Delta}{\pi} \int \cos (\sqrt{4\pi} \theta(x)) dx - \frac{\mu}{\sqrt{\pi}} \int \partial_x \phi(x) dx.
\]

4. Quantum BKT equations

The model Hamiltonians (Equations (7) and (9)) have already been studied in different context in quantum spin systems and also in condensed matter field theory [8,39,40]. In the present study, we are only interested in the physics of quantum BKT, which has not been studied in the literature. We use quantum field theoretical renormalization group method to study this problem which predicts and explains the enriched physics of quantum BKT in elegant way. The BKT equations can be derived by considering two limiting situations, i.e. one BKT equation for \( B = 0 \) and other is for \( \Delta = 0 \), in the Hamiltonian \( H \) (Equation (10)). This gives two model Hamiltonian \( H_1 \) and \( H_2 \) as
follows,

\[ H_1 = \frac{v}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K(\partial_x \theta(x))^2 \right] \, dx - \frac{\Delta}{\pi} \int \cos(\sqrt{4\pi x} \theta(x)) \, dx. \] (11)

\[ H_2 = \frac{v}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K(\partial_x \theta(x))^2 \right] \, dx + \frac{B}{\pi} \int \cos(\sqrt{4\pi x} \phi(x)) \, dx. \] (12)

At first, we set \( \mu = 0 \) for simplicity and consider its effect in later section.

Results for Hamiltonian \( H_1 \):

Here, we present the quantum BKT equations for the Hamiltonian \( H_1 \) and show that at \( T = 0 \) there exist different quantum phases with topological and non-topological properties and also the crossover between them,

\[ H_1 = \frac{v}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K(\partial_x \theta(x))^2 \right] \, dx - \frac{\Delta}{\pi} \int \cos(\sqrt{4\pi x} \theta(x)) \, dx. \] (13)

Finally, BKT equation can be derived for the Hamiltonian \( H_1 \) as (please see Appendix 1 for detailed derivation),

\[ \frac{d\Delta}{dl} = \left( 2 - \frac{1}{K} \right) \Delta, \quad \frac{dK}{dl} = \Delta^2. \] (14)

To reduce Equation (14) to standard form of BKT, we do the following transformations, \( -y_\parallel = (1 - \frac{1}{2K}) \) and finally the RG equations become,

\[ \frac{d\Delta}{dl} = -y_\parallel \Delta, \quad \frac{dy_\parallel}{dl} = -\Delta^2. \] (15)

We define the family of hyperbola parameterized by \( \alpha \),

\[ y_\parallel^2 - \Delta^2 = \alpha. \] (16)

Thus, we have,

\[ \frac{d[y_\parallel^2 - \Delta^2]}{dl} = \left[ y_\parallel \frac{dy_\parallel}{dl} + y_\parallel \frac{dy_\parallel}{dl} \right] - \left[ \frac{\Delta}{dl} \frac{d\Delta}{dl} + \Delta \frac{d\Delta}{dl} \right], \] (17)

\[ = 2y_\parallel(-\Delta^2) - 2\Delta(-y_\parallel \Delta) = 0. \]

Now we explain the different regime of the RG flow diagram. We distinguish three different regimes based on the value of \( \alpha \). In Figure 1, we can define three regions, region I (weak coupling), region II (crossover) and region III (strong coupling). We follow ref. [51] during the explanation.

(1) When \( \alpha > 0 \), parameterized hyperbolic equation is,

\[ y_\parallel = (\pm)\sqrt{\alpha \frac{1+\kappa^2}{1-\kappa^2}}, \quad \Delta = \sqrt{\alpha \frac{2\kappa}{1-\kappa}}, \quad 0 \leq \kappa < 1 \]

\[ \frac{d\kappa}{dl} = (\mp)\sqrt{\alpha} \kappa. \] (18)
This is the RG equation for parameter $\kappa$ and the solution to this is,

$$\int_{\kappa(l_0)}^{\kappa(l)} \frac{1}{\kappa(s)} ds = \int_0^l (\pm) \sqrt{\alpha} dl,$$

$$\ln \left( \frac{\kappa(l)}{\kappa(l_0)} \right) = (\pm) \sqrt{\alpha l},$$

$$\kappa(l) = \kappa(l_0)e^{(\mp)\sqrt{\alpha l}}. \hspace{1cm} (19)$$

(1.a) When $\alpha > 0$ and $\gamma_1 > 0 \ (K < \frac{1}{2})$, $\kappa(l) = \kappa(l_0)e^{-\sqrt{\alpha l}}. \hspace{1cm} (20)$

This shows the $\kappa(l)$ decreases with length scale showing the weak coupling phase. The region I is the weak coupling phase. In this phase, there is no gaped excitation, i.e. region I is in the gapless helical Luttinger liquid phase where the sine-Gordon coupling term is irrelevant. In this phase, there is no evidence of Majorana fermion mode, i.e. system is in the non-topological state.

(1.b) When $\alpha > 0$ and $\gamma_1 < 0 \ (K > \frac{1}{2})$, $\kappa(l) = \kappa(l_0)e^{\pm \sqrt{\alpha l}}. \hspace{1cm} (21)$

It is very clear from the above equation that $\kappa(l)$ increases with length scale. As a consequence of it, RG flow lines flowing off to the deep massive phase. The region III is the deep massive phase, i.e. the sine-Gordon coupling term is relevant, and the RG flows flowing off to the strong coupling regime away from the Gaussian fixed line.

(2) Now we do the analysis for crossover phase (region II). When $\alpha < 0$, parameterized hyperbolic equation is, $\gamma_1 = \sqrt{|\alpha|} \frac{2\kappa}{1 - \kappa^2}, \Delta = \sqrt{|\alpha|} \frac{1 + \kappa^2}{1 - \kappa^2}, -1 < \kappa < 1$.

$$\frac{d\kappa}{dl} = -\frac{\sqrt{|\alpha|}}{2} (1 + \kappa^2). \hspace{1cm} (22)$$

This is the RG equation for the parameter $\kappa$ and the solution to this is,

$$\int_{\kappa(l_0)}^{\kappa(l)} \frac{1}{\kappa(s) (1 + \kappa^2)} ds = -\frac{\sqrt{|\alpha|}}{2} \int_0^l dl,$$

$$\tan^{-1} (\kappa(l)) - \tan^{-1} (\kappa(l_0)) = -\frac{\sqrt{|\alpha|}}{2} (l - l_0). \hspace{1cm} (23)$$

The region II is the crossover regime. One observes the crossover from the weak coupling phase to
the strong coupling region. During this phase, the system transits from gapless phase to the proximity induced superconducting gaped phase, i.e. $\Delta \neq 0$. This is the topological superconducting phase, where the system has the Majorana fermion mode. For this situation, system transits from the non-topological state to the topological state. In practical reality, for this regime of parameter space the quantum spin Hall insulator will be in the topological state of matter with Majorana edge mode.

The difference between the region II and region III is, in region II, the field never reaches to free scalar field, i.e. ignoring the potential either at the long distance or at the short distance physics.

5. Quantum BKT equations and results for Hamiltonian $H_2$

Here we present the quantum BKT equations of the Hamiltonian $H_2$ and observe there is no topological phase,

$$H_2 = \frac{v}{2} \left[ \int \left( \frac{1}{K} (\partial_x \phi(x))^2 + K (\partial_x \theta(x))^2 \right) dx + \frac{B}{\pi} \int \cos (\sqrt{4\pi} \phi(x)) dx \right].$$

Following the same procedure of Appendix 1, we obtain the RG equations for the Hamiltonian $H_2$ as,

$$\frac{dB}{dl} = B(2 - K), \quad \frac{dK}{dl} = -B^2 K^2.$$  \hspace{1cm} (25)

We do the transformation, $y_{\parallel} = (K - 2)$ and obtain another set of RG equations in the standard form of BKT equation,

$$\frac{dB}{dl} = -y_{\parallel} B, \quad \frac{dy_{\parallel}}{dl} = -B^2.$$  \hspace{1cm} (26)

The structure of the second BKT equations (Equation (26)) is same as that of the first one (Equation (15)). Therefore, the analysis of this equation is the same as that of the first one. The only difference being, there is no evidence Majorana fermion mode in this phase. In Figure 2, the system shows only the Ising-ferromagnetic phase (region III). In practical reality, for this regime of parameter space the quantum spin Hall insulator will be in the non-topological phase. From this study, we obtain two types of BKT equations, but there is no Majorana-Ising transition. We observe Majorana-Ising transition, if we consider total RG equations of Hamiltonian $H$ (Equation (9)) [39] in which case the quantum BKT behavior will be absent.

![Figure 2](image.png)

**Figure 2.** (a) The curve is plotted for $\Delta$ with $K$. Analytical relation of $K$ and $y_{\parallel}$ is $K = \left( \frac{1}{2y_{\parallel} + 1} \right)$. (b) The curve is plotted for $B$ with $K$. Analytical relation of $K$ and $y_{\parallel}$ is $K = (y_{\parallel} + 2)$. The arrow indicates the direction of the RG flow.
6. Exact solution of the RG equations

The model Hamiltonian gives two set of RG equation which yield the quantum BKT equation. Here we derive analytical solution for these two RG equations by solving them directly. Let us consider the first set of RG equation,

\[
\frac{d\Delta}{dl} = \left(2 - \frac{1}{K}\right)\Delta, \quad \frac{dK}{dl} = \Delta^2.
\]  

(27)

We use the above two equation to derive the phase relation between the two parameters \(\Delta\) and \(K\).

\[
\frac{d\Delta}{dK} = \frac{\left(2 - \frac{1}{K}\right)\Delta}{\Delta^2} = \frac{2 - \frac{1}{K}}{\Delta}
\]  

(28)

On integration, we get \(C\) as,

\[
C = \frac{\Delta^2}{2} - 2K + \ln K
\]  

(29)

We can calculate \(C\) from the initial values \(\Delta_0, K_0\).

\[
\frac{\Delta^2}{2} = \frac{\Delta_0^2}{2} + 2(K - K_0) - \ln\left(\frac{K}{K_0}\right)
\]  

(30)

\[
\Delta = \sqrt{\Delta_0^2 + 4(K - K_0) - 2\ln\left(\frac{K}{K_0}\right)}.
\]  

(31)

Here \(\Delta_0\) and \(K_0\) are the initial value of \(K\) and \(\Delta\).

In Figure 2(a), one can observe three regions, region I corresponds weak coupling gapless helical Luttinger liquid phase, where the RG flow lines flowing off to the weak coupling phase. It is a Gaussian fixed line. Region II is the crossover regime from weak coupling to strong coupling phase. Region III corresponds to strong coupling deep massive phase, away from Gaussian fixed line, where RG flow lines are flowing off from the weak coupling phase to the strong coupling phase, i.e. the system flowing off from gapless helical LL phase to topological superconducting phase. In region III, one can observe the asymptotic nature of the system. In region III, flow lines indicate the increase in the length scale, one can observe the coupling constant increases as the length scale increases and vice versa.

Now let us consider the second set of RG equation,

\[
\frac{dB}{dl} = (2 - K)B, \quad \frac{dK}{dl} = -K^2B^2.
\]  

(32)

Now we define a quantity \(C\) as,

\[
\frac{dB}{dK} = \frac{(2 - K)B}{-K^2B^2} = \frac{2 - K}{-K^2B}
\]  

(33)

On integration, we get \(C = B^2/2 - 2/K - \ln K\).
We can calculate $C$ from the initial values $B_0$, $K_0$.

\[ \frac{B^2}{2} = \frac{B_0^2}{2} + 2 \left( \frac{1}{K} - \frac{1}{K_0} \right) + \ln \left( \frac{K}{K_0} \right) \]

\[ B = \sqrt{B_0^2 + 4 \left( \frac{1}{K} - \frac{1}{K_0} \right) + 2 \ln \left( \frac{K}{K_0} \right)} \]  

(34)

In Figure 2(b), one can observe three regions, region I corresponds weak coupling gapless helical Luttinger liquid phase, where the RG flow lines flowing off to the weak coupling phase. It is a Gaussian fixed line. Region II is the crossover regime from weak coupling to strong coupling phase, initially RG flow lines flowing off to the weak coupling regime but due to the presence of $\frac{dK}{dl}$ in the RG equation, the RG flow lines flowing off to the strong coupling phase, which is the Ising-ferromagnetic phase. Region III, corresponds to strong coupling deep massive phase, away from Gaussian fixed line, where RG flow lines flowing off from the weak coupling phase to the strong coupling phase, i.e. the system flowing off from gapless helical LL phase to Ising-ferromagnetic phase, which is non-topological in character. In region III, one can observe the asymptotic nature of the system. In region III flow lines indicate that the coupling constant increases as the length scale.

**Effect of chemical potential on RG flow lines**

We have two model Hamiltonian to study the quantum BKT. One is for the $\phi$ field and the other is for the $\theta$ field. The chemical potential ($\mu$) is related to the $\phi$ field. The effect of $\mu$ will be different for the two quantum BKT transitions. At first, we study the effect of $\mu$ for the $\phi$ field.

**Effect on the Hamiltonian $H_2$** : Here we study how the chemical potential affect the Hamiltonian $H_2$. We can absorb the $\partial_\phi \phi$ term into the quadratic Hamiltonian by the transformation of $\phi$ field, $\phi \rightarrow \phi - \frac{K \mu}{\sqrt{\pi}} x$. This gives the spatially oscillating term which modify the cosine term as, $\frac{B}{\pi} \cos (2 \sqrt{\pi} \phi + \delta_1 \phi(x))$ indicating commensurate to incommensurate transition. However, one can write another RG equation to study the effect of $\mu$, i.e. $\frac{d\delta_1}{dl} = \delta_1$ [4,39,52]. Under the condition $\delta_1(0) a \ll B(0)^{\pi / \sqrt{\pi}}$, if $B(l)$ reaches to strong coupling phase before $\delta_1(l) a \rightarrow 1$ then the system is in Ising-ferromagnetic phase. Therefore, it is clear from the above study based on the RG equation of $\delta_1$ that the different quantum phases of this system dominate in presence of chemical potential in the different regimes of the interaction space.

**Effect on the Hamiltonian $H_1$** : We consider the Hamiltonian $H_1$ in the presence of chemical potential, which we call $H_3$, and find the effect of chemical potential in RG flow diagrams. Here $\phi$ and $\theta$ fields are dual to each other (i.e. minima of the sine-Gordon coupling term for $\phi$ and $\theta$ are different), and the relation between these two fields is

\[ [\phi(x), \partial_{x'} \theta(x')] = i \delta(x - x') \].

Therefore, we are not allowed to absorb the $\phi$ field in the sine-Gordon coupling term of $\theta$. Now the Hamiltonian can be written as

\[ H_3 = \frac{v}{2} \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K (\partial_x \theta(x))^2 \right] dx - \frac{\Delta}{\pi} \int \cos (\sqrt{4 \pi} \theta(x)) dx - \frac{\mu}{\sqrt{\pi}} \int \partial_x \phi(x) dx. \]  

(35)

The quantum BKT equations of $H_3$ is given by (for detailed derivation see Appendix 2),

\[ \frac{d\Delta}{dl} = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{\sqrt{\pi}} \right) \right] \Delta, \quad \frac{dK}{dl} = \Delta^2. \]  

(36)

It is clear from the Equation (36) that in the presence of finite $\mu$, the analytical form of the equation is the same as that of $\mu = 0$, but with a modification of a factor. In Figure 3, we present
the results for finite \( \mu (\mu = 1) \), the behavior of the RG equation remain same. Therefore, it reveals from our study that the existence of Majorana fermion mode does not disappear for finite \( \mu \).

**Summary of the new and important results of the present quantum BKT study:**

We have obtained three quantum phases either topological or non-topological in character from the study of quantum BKT RG equations. One is topological, i.e. topological superconducting phase, another one is the Ising-ferromagnetic phase which is non-topological and finally we have obtained gapless helical Luttinger liquid phase, which is also non-topological in character. The region I is the helical Luttinger liquid phase where the RG flow lines flowing off the weak coupling phase. The system reaches the strong coupling phase when RG flow lines flows from region II to region III, where the sine-Gordon coupling term becomes relevant, it is topological superconducting phase or Ising-ferromagnetic phase. In region-III, field theory is asymptotically free, i.e. the system reduced to the scalar field theory by ignoring the sine-Gordon coupling term at short distance. This short distance physics of the region III, does not appear in region II.

**A comparison between the results of quantum BKT and with the results of total Hamiltonian:**

In this section, we compare the results which we have obtained from the study of two quantum BKT Hamiltonians and the total Hamiltonian of the system (Equation (10)). The total Hamiltonian of the system has already been studied in different context [4]. Very recently, the Hamiltonian in Equation (9) has also been studied in the context of topological states of matter in ref. [37,39] and [40]. We consider the Hamiltonian Equation (10) (without \( \mu \)), which yields the RG equations for \( \Delta, B \) and \( K \).

\[
H = \frac{\nu}{2} \left[ \frac{1}{K} \left( \left( \partial_x \phi(x) \right)^2 + K \left( \partial_x \theta(x) \right)^2 \right) \right] dx + \frac{B}{\pi} \int \cos \left( \sqrt{4\pi} \phi(x) \right) dx - \frac{\Delta}{\pi} \int \cos \left( \sqrt{4\pi} \theta(x) \right) dx \tag{37}
\]

The RG equations can be derived as,

\[
\frac{dB}{dl} = (2 - K)B, \tag{38}
\]

\[
\frac{d\Delta}{dl} = \left( 2 - \frac{1}{K} \right) \Delta, \tag{39}
\]

\[
\frac{dK}{dl} = \frac{1}{2\pi^2} \left( \Delta^2 - B^2 K^2 \right). \tag{40}
\]
These RG equation consist of two sine-Gordon coupling terms. One is the \( \Delta \), which induces the topological superconducting phase, and the other is \( B \), which induces the Ising-ferromagnetic phase in the system. \( K \) and \( \mu \) are the parameters of the model Hamiltonian. In the present section, we present the results.

Figure 4. Phase diagram showing, upper panel: Ising-ferromagnetic phase \((K = 0.2)\); middle panel: Majorana phase \((K = 2.4)\); lower panel: Majorana-Ising transition \((K = 0.8)\).
of the whole RG equation for the different values of $K$. $K < 1$ and $K > 1$ characterizes the repulsive and attractive interactions, respectively, where as $K = 1$ characterizes non-interacting case [4].

Figure 4 consists of three panels for different values of Luttinger liquid parameter to show the existence of different quantum phases either topological or non-topological in character and also the transition between them. The upper panel ($K = 0.2$) of the figure present the existence of Ising-ferromagnetic phase. This is because the RG flow lines flowing off to the weak coupling phase for the coupling $\Delta$, but the coupling $B$ increases. The middle panel is for $K = 2.4$. We observe that the system is in the topological state i.e. the coupling $\Delta$ increases to the strong coupling phase but there is no increase of coupling $B$. The lower panel is for $K = 0.8$. We observe Majorana-Ising transition in the system. RG flow lines increases for large initial values of $\Delta$ than $B$, thus, the system is driven to the topological state. Otherwise, the system is in the Ising-ferromagnetic phase. Therefore, we conclude that from the RG flow lines that, there is no evidence of helical LL phase for this study. But for the quantum BKT study we have not observed any Majorana-Ising transition.

In quantum BKT instead of two sine-Gordon coupling term, there is only one sine-Gordon coupling term for each Hamiltonian. The sine-Gordon coupling term gives the confining potential and the quadratic part of the Hamiltonian (Equation (37)) gives the kinetic energy contribution. Therefore, the competition between these kinetic energy (quadratic fluctuation) and sine-Gordon coupling terms finally gives winning phase of the system, instead of competition between the two sine-Gordon coupling term of the total RG equation. Therefore, in this quantum BKT, there is no Majorana-Ising transition. At the same time in the quantum BKT we present the results for RG flow diagram for a single coupling constant with $K$.

7. Conclusion

We have studied quantum BKT transition for the one-dimensional interacting edge mode of helical liquid of topological insulator and have also found two quantum BKT transition for different physical situations. We have shown the existence of topological superconducting phase, gapped Ising-ferromagnetic phase and gapless helical Luttinger liquid phase for this system through RG flow diagrams. We have found the exact solution for quantum BKT transition for the helical edge state system which appears in the quantum spin Hall system. For finite chemical potential, we also observe the presence of commensurate to incommensurate transition. We have not found any direct Majorana-Ising transition in quantum BKT transitions.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Berezinskii VL. Violation of long range order in one-dimensional and two-dimensional systems with a continuous symmetry group. Zh Eksp Teor Fiz. [Sov Phys JETP. 59(32):907–920] 1971;1970(1971):493–500.
[2] Kosterlitz JM, Thouless DJ. Ordering, metastability and phase transitions in two-dimensional systems. J Phys C: Solid State Phys. 1973;6:1181–1203.
[3] Stephen T. The two-dimensional fully frustrated XY model. 40 years of Berezinskii-Kosterlitz-Thouless theory. World Scientific Publishing Co; 2013.
[4] Giamarchi T. Quantum physics in one dimension. Oxford: Clarendon Press; 2003.
[5] Ortiz G, Cobanera E, Nussinov Z. Berezinskii-Kosterlitz-Thouless transition through the eyes of duality. 40 years of Berezinskii-Kosterlitz-Thouless theory. World Scientific Publishing Co; 2013.

[6] David S. An introduction to bosonization. Theoretical methods for strongly correlated electrons. Springer; 2004.

[7] Altland A, Simons BD. Condensed matter field theory. Cambridge: Cambridge University Press; 2010.

[8] Fradkin E. Field theories of condensed matter physics. Cambridge: Cambridge University Press; 2013.

[9] Marino EC. Quantum field theory approach to condensed matter physics. Cambridge: Cambridge University Press; 2017.

[10] Mermin ND, Wagner H. Absence of ferromagnetism or antiferromagnetism in one-or two-dimensional isotropic Heisenberg models. Phys Rev Lett. 1966;17:1133–1136.

[11] Hohenberg PC. Existence of long-range order in one and two dimensions. Phys Rev. 1967;158:383–386.

[12] Nagaosa N. Quantum field theory in condensed matter physics. Heidelberg: Springer Science & Business Media; 2013.

[13] José JV. Duality, gauge symmetries, renormalization groups and the BKT transition. Int J Modern Phys B. 2017;31:1730001.

[14] Timm C. Theory of superconductivity. Dresden: Institute of theoretical Physics; 2012.

[15] Haldane FD. 'Luttinger liquid theory' of one-dimensional quantum fluids. I. properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas. J Phys C: Solid State Phys. 1981;14:2585–2609.

[16] Li T, Wang P, Fu H, et al. Observation of a helical Luttinger liquid in InAs/GaSb quantum spin hall edges. Phys Rev Lett. 2015;115:136804.

[17] Wu C, Bernevig BA, Zhang SC. Observation of helical liquid behavior in weakly disordered quantum wires. Phys Rev Lett. 2006;96:106401.

[18] Yakob A, Stormer HL, Wingreen NS, et al. Nonuniversal conductance quantization in quantum wires. Phys Rev Lett. 1996;77:4612–4615.

[19] Chang AM, Pfeiffer LN, Wingreen NS, et al. Observation of chiral Luttinger behavior in electron tunneling into fractional quantum hall edges. Phys Rev Lett. 1996;77:2538–2541.

[20] König M, Wiedmann S, Brüne C, et al. Quantum spin hall insulator state in HgTe quantum wells. Sciences. 2007;318:766–70.

[21] Knez I, Du RR, Sullivan G. Evidence for helical edge modes in inverted InAs/GaSb quantum wells. Phys Rev Lett. 2011;107:136804.

[22] Spanton EM, Nowack KC, Du L, et al. Images of edge current in InAs/GaSb quantum wells. Phys Rev Lett. 2014;113:026804.

[23] Du L, Knez I, Sullivan G, et al. Robust helical edge transport in gated InAs/GaSb bilayers. Phys Rev Lett. 2015;114:096802.

[24] Kane CL, Mele EJ. Quantum spin Hall effect in graphene. Phys Rev Lett. 2005;95:226801.

[25] Hasan MZ, Kane CL. Colloquium: topological insulators. Rev Mod Phys. 2010;82:3045.

[26] Moore JE. The birth of topological insulators. Nature. 2010;464:194–198.

[27] Nishimori H, Ortiz G. Elements of phase transitions and critical phenomena. Oxford: OUP Oxford; 2010.

[28] Gu ZC, Wen XG. Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order. Phys Rev B. 2009;80:155131.

[29] Pollmann F, Berg E, Turner AM, et al. Symmetry protection of topological phases in one-dimensional quantum spin systems. Phys Rev B. 2012;85:075125.

[30] Chen X, Gu ZC, Wen XG. Classification of gapped symmetric phases in one-dimensional spin systems. Phys Rev B. 2011;83:035107.

[31] Schmidt TL, Rachel S, von Oppen F, et al. Inelastic electron backscattering in a generic helical edge channel. Phys Rev Lett. 2012;108:156402.

[32] Yokoyama T, Balatsky AV, Nagaosa N. Gate-controlled one-dimensional channel on the surface of a 3D topological insulator. Phys Rev Lett. 2010;104:246806.

[33] Sela E, Altland A, Rosch A. Majorana fermions in strongly interacting helical liquids. Phys Rev B. 2011;84:085114.

[34] Xu C, Moore JE. Stability of the quantum spin hall effect: effects of interactions, disorder, and $Z_2$ topology. Phys Rev B. 2006;73:045322.

[35] Sarkar S. Physics of Majorana modes in interacting helical liquid. Sci Rep. 2016;6:30569.

[36] Saha SK, Dey D, Roy MS, et al. Characterization of Majorana-Ising phase transition in a helical liquid system. J Magn Magn Mater. 2019;475:257–263.
Appendices

Appendix 1. Derivation of Quantum BKT equations for $H_1$

The Hamiltonian $H_1$ is given by,

$$H_1 = \frac{\nu}{2} \int \left[ \frac{1}{K} (\partial_x \phi(x))^2 + K (\partial_x \theta(x))^2 \right] dx - \frac{\Delta}{\pi} \int \cos(\sqrt{4\pi} \theta(x)) \, dx. \quad (A1)$$

In the Bosonized model Hamiltonian $H_1$, we rescale the fields as, $\phi \rightarrow \phi' = \phi / \sqrt{K}$ and $\theta \rightarrow \theta' = \sqrt{K} \theta$. Thus, the quadratic part of the Hamiltonian will be,

$$H_{1q}' = \frac{\nu}{2} [(\partial_x \phi')^2 + (\partial_x \theta')^2]. \quad (A2)$$

The Hamilton’s equations for the canonically conjugate fields $(\phi', \theta')$ are,

$$\partial_t \theta' = -\frac{1}{\nu} \partial_x \phi', \quad \partial_t \phi' = -\frac{1}{\nu} \partial_x \theta'. \quad (A3)$$

Thus, the Lagrangian in terms of $\theta'$ field is given by,

$$\mathcal{L}_0 = \Pi_{\theta'} \partial_t \theta' = -H_{1q}' ,$$

$$= \left( \frac{1}{\nu} \partial_t \theta' \right) \partial_x \theta' - \frac{\nu}{2} (\partial_x \theta')^2 - \frac{\nu}{2} (\partial_x \phi')^2 ,$$

$$= \frac{1}{2} [\nu^{-1}(\partial_x \theta')^2 - v(\partial_x \theta')^2]. \quad (A4)$$

The Lagrangian rewritten in the imaginary time ($\tau = it$) as,

$$\mathcal{L}_0 = -\frac{1}{4} [v^{-1}(\partial_x \theta')^2 + v(\partial_x \theta')^2]. \quad (A5)$$

The Lagrangian for interaction term will have, $\mathcal{L}_\Delta = -H_\Delta$, where,

$$\mathcal{L}_\Delta = \left( \frac{\Delta}{\pi} \right) \cos(\sqrt{4\pi} \theta(x)). \quad (A6)$$

The Euclidean action can be written as, $S_E = -\int dr \mathcal{L} = -\int dr (\mathcal{L}_0 + \mathcal{L}_\Delta)$, where $r = (\tau, x)$. Now we write the partition function in terms of Euclidean action,

$$Z = \int \mathcal{D} \theta e^{\int dr dx \left( -\frac{1}{4} [v^{-1}(\partial_x \theta')^2 + v(\partial_x \theta')^2] - \frac{\Delta}{\pi} \cos(\sqrt{4\pi} \theta(x)) \right)}. \quad (A7)$$

We write partition function in a local form using space independent fields $(\tilde{\theta})$, which describe the system at the point
The partition function can now be rewritten by substituting the above integrals as

\[ Z = \int D\theta D\tilde{\theta} e^{-S_{\theta}(\theta)} \delta(\tilde{\theta}(\tau) - \theta(\tau, 0)), \]  
(A8)

where \( \delta(\tilde{\theta}(\tau) - \theta(\tau, 0)) = \frac{1}{2\pi} \int dk_{\theta} e^{ik_{\theta}(\tilde{\theta}(\tau) - \theta(\tau, 0))}. \) We first solve for the integral

\[ I = \int d\tau dx \left( v^{-1}(\partial_1 \theta)^2 + v(\partial_1 \theta)^2 \right) \]  
(A9)

One can rewrite this integral as fourier sums, which yields

\[ I = \int d\tau dx \left[ \frac{v^{-1}}{\beta L} \sum_{q, \alpha_m} (-i\omega_n \theta^*_{q, \alpha_m}) e^{i(qx - \omega_n \tau)} \times \frac{1}{\beta L} \sum_{q, \alpha_m} (i\omega_n \theta^*_{q, \alpha_m}) e^{i(qx - \omega_n \tau)} \right] + \frac{v}{\beta L} \sum_{q, \alpha_m} (vq') \sum_{q, \alpha_m} (-i\omega_n \theta^*_{q, \alpha_m}) e^{i(qx - \omega_n \tau)} \right]

\[ = \frac{1}{\beta L} \sum_{q, \alpha_m} \left( vq^2 + v^{-1} \omega_n \theta^*_{q, \alpha_m} \right) e^{i(qx - \omega_n \tau)} \right] = \frac{1}{\beta L} \sum_{q, \alpha_m} (vq^2 + v^{-1} \omega_n \theta^*_{q, \alpha_m}) |\theta|^2. \]  
(A10)

Similarly one can solve other two integrals in the exponential of Equation (A8).

\[ i \int d\tau k_{\theta} \tilde{\theta}(\tau) = \frac{1}{\beta} \sum_{\omega_n} k(\omega_n) \tilde{\theta}(\omega_n). \]  
(A11)

\[ -i \int d\tau k_{\theta} \theta(\tau, 0) = \frac{1}{\beta} \sum_{\omega_n} k(-\omega_n) \theta^*_{\omega_n}. \]  
(A12)

Partition function can now be rewritten by substituting the above integrals as

\[ Z = \int D\theta D\tilde{\theta} e^{-\frac{1}{\beta} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] - \frac{4}{\beta} k(\omega_n) \tilde{\theta}(\omega_n) \theta(\omega_n) - \frac{1}{\beta} \sum_{\omega_n} k(\omega_n) \tilde{\theta}(\omega_n) - \int d\tau dx \mathcal{L}_0(\tilde{\theta}), \]  
(A13)

here we have transformed \( \theta \) fields back to \( \tilde{\theta} \) fields. Now we perform Gaussian integration over \( \tilde{\theta} \) field.

\[ Z \propto \int D\tilde{\theta} Dk_{\theta} e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] + \frac{1}{2} \sum_{\omega_n} k(\omega_n) \tilde{\theta}(\omega_n) - \int d\tau dx \mathcal{L}_0(\tilde{\theta}). \]  
(A14)

Taking the \( q \)-sums to the continuum limit and performing the resulting integral

\[ Z \propto \int D\tilde{\theta} Dk_{\theta} e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] + \frac{1}{2} \sum_{\omega_n} k(\omega_n) \tilde{\theta}(\omega_n) - \int d\tau dx \mathcal{L}_0(\tilde{\theta})}, \]  
(A15)

\[ \propto \int D\tilde{\theta} Dk_{\theta} e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] + \frac{1}{2} \sum_{\omega_n} k(\omega_n) \tilde{\theta}(\omega_n) - \int d\tau dx \mathcal{L}_0(\tilde{\theta})}. \]  
(A16)

Finally performing the Gaussian integration over \( k_{\theta} \) gives

\[ Z = \int D\tilde{\theta} e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] - \int d\tau dx \mathcal{L}_0(\tilde{\theta})}. \]  
(A17)

In the continuum limit of \( \omega_n \), we have

\[ Z = \int D\tilde{\theta} e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \tilde{\theta}(\omega_n) \right] - \int d\tau dx \mathcal{L}_0(\tilde{\theta})}. \]  
(A18)

Final form of the partition function can be written as,

\[ Z = \int D\theta e^{-\frac{1}{2} \sum_{\omega_n} \left[ k(\omega_n) \theta(\omega_n) \right] - \int d\tau dx \mathcal{L}_0(\theta)}. \]  
(A19)

We now separate the slow and fast fields and integrate out the fast field components. The field \( \theta = \theta_s + \theta_f \)
where,
\[ \theta_j(r) = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} e^{-i\omega r(\omega)} \quad \& \quad \theta_j(r) = \int_{\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} e^{-i\omega r(\omega)}, \] (A20)

here \( r = (x, \tau) \). Now the partition function can be written as,
\[ Z = \int D\theta_j D\theta e^{-\{S_{\Delta}(\theta_j) - S_{\Delta}(\theta_j, \theta_j)\}}, \]
\[ = \int D\theta_j e^{-\{S_{\Delta}(\theta_j)\}} e^{-\{S_{\Delta}(\theta_j, \theta_j)\}} \]
where we have used \( \langle F_f \rangle = \int D\theta e^{-S(\theta)} \). We write the effective action as,
\[ e^{-S_{\Delta}(\theta_j)} = e^{-S_{\Delta}(\theta_j)} e^{-\{S_{\Delta}(\theta_j, \theta_j)\}}. \] (A22)

Taking \( \ln \) on both side gives,
\[ S_{\Delta}(\theta_j) = S_i(\theta_j) - \ln\{e^{-S_{\Delta}(\theta_j)}\}. \] (A23)

By writing the cumulant expansion up to second order, we have
\[ S_{\Delta}(\theta_j) = S_i(\theta_j) + \{S_{\Delta}(\theta_j, \theta_j)\} - \frac{1}{2}\{(S_{\Delta}^2(\theta_j, \theta_j)) - \{S_{\Delta}(\theta_j, \theta_j)\}^2\}. \] (A24)

Now we calculate the first-order approximation \( \langle S_{\Delta}(\theta_j, \theta_j) \rangle \),
\[ \langle S_{\Delta}(\theta_j, \theta_j) \rangle = \frac{\Delta}{\pi} \int D\theta_j e^{-\{S_{\Delta}(\theta_j)\}} \int dr \{ \cos \{\sqrt{4\pi}\theta_j(r)\} \}, \]
\[ = \frac{\Delta}{\pi} \int dr \{ \cos \{\sqrt{4\pi}\theta_j(r)\} \} e^{-\{\sqrt{4\pi}\theta_j(r)\}} + H.c \}
\[ = \frac{\Delta}{\pi} \int dr \cos \{\sqrt{4\pi}\theta_j(r)\} e^{-\{\sqrt{4\pi}\theta_j(r)\}} \] (A25)

We write, \( \int_f \frac{d\omega}{\omega} = \int_{\Lambda/b}^{\Lambda/b} \frac{d\omega}{\omega} = \ln \Lambda - \ln (\Lambda/b) = \ln \left[ \frac{\Lambda/b}{\Lambda/b} \right] = \ln b. \)
\[ \langle S_{\Delta}(\theta_j, \theta_j) \rangle = \frac{\Delta}{\pi} \int dr \cos \{\sqrt{4\pi}\theta_j(r)\} e^{-\{\sqrt{4\pi}\theta_j(r)\}}, \]
\[ = b^{-\Delta} S_{\Delta}(\theta_j). \] (A26)

Thus, the effective action up to first-order cumulant expansion can be written as,
\[ S_{\Delta}(\theta_j) = S_i(\theta_j) + b^{-\Delta} S_{\Delta}(\theta_j), \]
\[ S_{\Delta}(\theta_j) = \int_{\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} |\omega| K|\theta_j(\omega)|^2 + b^{-\Delta} \int dr \left( \frac{\Delta}{\pi} \right) \cos \{\sqrt{4\pi}\theta_j(r)\}. \] (A27)

Now we rescale the parameters cut-off momentum to the original momentum by considering, \( \tilde{\Lambda} = \frac{\Lambda}{b} \), \( \tilde{\omega} = \omega b \) and \( \tilde{r} = \frac{r}{b} \). The fields will be rescaled as, \( \theta(\tilde{\omega}) = \frac{\theta(\omega)}{b} \) and we choose \( \tilde{\theta}(\tilde{r}) = \theta_j(r) \). Thus, the rescaled effective action is given by,
\[ S_{\Delta}(\theta_j) = \int_{\Lambda}^{\Lambda} \frac{d\tilde{\omega}}{2\pi b} |\tilde{\omega}| b^2 K|\tilde{\theta}(\tilde{\omega})|^2 + b^{-\Delta} \int b^2 d\tilde{r} \left( \frac{\Delta}{\pi} \right) \cos \{\sqrt{4\pi}\theta_j(r)\}. \] (A28)

Since we are working in \((1+1)\) dimensional system, we have \( d^2 r = b^2 d^2 \tilde{r} \).
\[ S_{\Delta}(\theta_j) = \int_{\Lambda}^{\Lambda} \frac{d\tilde{\omega}}{2\pi} \frac{|\tilde{\omega}| b^2 K|\tilde{\theta}(\tilde{\omega})|^2}{2} + b^{-\Delta} \int b^2 d\tilde{r} \left( \frac{\Delta}{\pi} \right) \cos \{\sqrt{4\pi}\theta_j(\tilde{r})\}, \]
\[ S_{\Delta}(\theta_j) = \int_{\Lambda}^{\Lambda} \frac{d\tilde{\omega}}{2\pi} |\tilde{\omega}| K|\tilde{\theta}(\tilde{\omega})|^2 + b^{-\Delta} \int b^2 d\tilde{r} \left( \frac{\Delta}{\pi} \right) \cos \{\sqrt{4\pi}\theta_j(\tilde{r})\}. \] (A29)

Comparing the coupling constants of rescaled effective action with the unrenormalized action one can observe that,
\[ \Delta \rightarrow \Delta b^{2-\hat{k}}. \] Thus, we write the RG flow equation as,

\[ \tilde{\Delta} = \Delta b^{2-\hat{k}}. \]

We write this equation in the differential form by setting \( b = e^{d\hat{l}} \),

\[ \tilde{\Delta} = \Delta e^{(2-\hat{k})d\hat{l}} \]

\[ \tilde{\Delta} = \Delta \left[ 1 + \left( 2 - \frac{1}{K} \right) d\hat{l} \right]. \]

Defining the differential of a parameter as \( d\Delta = \tilde{\Delta} - \Delta \) we have,

\[ \frac{d\Delta}{d\hat{l}} = \left( 2 - \frac{1}{K} \right) \Delta. \quad (A30) \]

Now, we solve for the second-order cumulant expansion,

\[ -\frac{1}{2} \left( \langle -S^2 \rangle - \langle -S^2 \rangle^2 \right) = -\frac{\Delta^2}{2\pi^2} \int \! dr \, dr' \left[ \langle -\cos \left( \sqrt{4\pi} \theta(r) \right) - \cos \left( \sqrt{4\pi} \theta(r') \right) \rangle - \left\langle \cos \left( \sqrt{4\pi} \theta(r) \right) \right\rangle - \left\langle \cos \left( \sqrt{4\pi} \theta(r') \right) \right\rangle - \right]. \quad (A31) \]

First, we calculate \( \langle S^2 \rangle \) term.

\[ \langle S^2 \rangle = \frac{\Delta^2}{\pi^2} \int \! dr \, dr' \left\langle \cos \left( \sqrt{4\pi} \theta(r) \right) \cos \left( \sqrt{4\pi} \theta(r') \right) \right\rangle, \]

\[ \langle S^2 \rangle = \frac{\Delta^2}{4\pi^2} \int \! dr \, dr' \left\langle \left( e^{\sqrt{4\pi} \theta(r)} e^{\sqrt{4\pi} \theta(r')} + H.c. \right) \left( e^{\sqrt{4\pi} \theta(r)} e^{\sqrt{4\pi} \theta(r')} + H.c. \right) \right\rangle, \]

\[ \langle S^2 \rangle = \frac{\Delta^2}{2\pi^2} \int \! dr \, dr' \left\langle \cos \sqrt{4\pi} [\theta_i(r) + \theta_i(r')] \left( e^{-2\pi \left[ \langle \phi(r) \rangle + \langle \phi(r) \rangle + 2 \langle \phi(r) \rangle \langle \phi(r') \rangle \right] \right) + \cos \sqrt{4\pi} [\theta_i(r) - \theta_i(r')] \left( e^{-2\pi \left[ \langle \phi(r) \rangle + \langle \phi(r) \rangle - 2 \langle \phi(r) \rangle \langle \phi(r') \rangle \right] \right) + H.c. \right\rangle. \quad (A32) \]

Now we calculate \( \langle S \rangle^2 \),

\[ \langle S \rangle^2 = \frac{\Delta^2}{\pi^2} \int \! dr \, dr' \left\langle \cos \left( \sqrt{4\pi} \theta(r) \right) \right\rangle \langle \cos \left( \sqrt{4\pi} \theta(r') \right) \rangle, \]

\[ \langle S \rangle^2 = \frac{\Delta^2}{4\pi^2} \int \! dr \, dr' \left\langle \left( e^{\sqrt{4\pi} \theta_i(r)} e^{\sqrt{4\pi} \theta_i(r')} + H.c. \right) \left( e^{\sqrt{4\pi} \theta_i(r)} e^{\sqrt{4\pi} \theta_i(r')} + H.c. \right) \right\rangle, \]

\[ \langle S \rangle^2 = \frac{\Delta^2}{2\pi^2} \int \! dr \, dr' \cos \sqrt{4\pi} [\theta_i(r) + \theta_i(r')] \left( e^{-2\pi \langle \phi(r) \rangle} e^{-2\pi \langle \phi(r') \rangle} \right) + \cos \sqrt{4\pi} [\theta_i(r) - \theta_i(r')] e^{-2\pi \langle \phi(r) \rangle} e^{-2\pi \langle \phi(r') \rangle} + H.c. \right\} \].
Thus, the term \((\langle S^2 \rangle - \langle S \rangle^2)\) is,

\[
\langle S^2 \rangle - \langle S \rangle^2 = \frac{\Delta^2}{2\pi^2} \int dr dr' \left[ \cos \sqrt{4\pi}[\theta_r + \theta_{r'}] \left( e^{-2\pi \left[ \langle \phi_r \rangle + \langle \phi_{r'} \rangle + \langle \phi_{r} \phi_{r'} \rangle \right]} \right) \\
+ \cos \sqrt{4\pi}[\theta_r - \theta_{r'}] \left( e^{-2\pi \left[ \langle \phi_r \rangle - \langle \phi_{r'} \rangle - 2(\langle \phi_r \phi_{r'} \rangle) \right]} + H.c \right) \\
- \frac{\Delta^2}{2\pi^2} \int dr dr' \left[ \cos \sqrt{4\pi}[\theta_r + \theta_{r'}] e^{-2\pi \langle \phi_r \rangle} e^{-2\pi \langle \phi_{r'} \rangle} \\
+ \cos \sqrt{4\pi}[\theta_r - \theta_{r'}] e^{-2\pi \langle \phi_r \rangle} e^{-2\pi \langle \phi_{r'} \rangle} + H.c \right].
\]

(A34)

\[
\langle S^2 \rangle - \langle S \rangle^2 = \frac{\Delta^2}{2\pi^2} \int dr dr' \left[ \cos \sqrt{4\pi}[\theta_r + \theta_{r'}] \left( e^{-2\pi \left[ \langle \phi_r \rangle + \langle \phi_{r'} \rangle + 2(\langle \phi_r \phi_{r'} \rangle) \right]} \right) \\
- e^{-2\pi \langle \phi_r \rangle} e^{-2\pi \langle \phi_{r'} \rangle} \\
+ \cos \sqrt{4\pi}[\theta_r + \theta_{r'}] e^{-2\pi \langle \phi_r \rangle} \left( e^{-4\pi \langle \phi_r \phi_{r'} \rangle} - 1 \right) \\
- e^{-2\pi \langle \phi_r \rangle} e^{-2\pi \langle \phi_{r'} \rangle} \right].
\]

(A35)

Thus,

\[
-\frac{1}{2} \langle S^2 \rangle - \langle S \rangle^2 = -\frac{\Delta^2}{4\pi^2} \int dr dr' \left[ \cos \sqrt{4\pi}[\theta_r + \theta_{r'}] e^{-2\pi \left[ \langle \phi_r \rangle + \langle \phi_{r'} \rangle \right]} \left( e^{-4\pi \langle \phi_r \phi_{r'} \rangle} - 1 \right) \\
+ \cos \sqrt{4\pi}[\theta_r - \theta_{r'}] e^{-2\pi \langle \phi_r \rangle} \left( e^{-4\pi \langle \phi_r \phi_{r'} \rangle} - 1 \right) \right].
\]

(A36)

The correlation function \(\langle \theta_r \theta_{r'} \rangle\) is calculated as,

\[
\langle \theta_r \theta_{r'} \rangle = \int d\theta_r e^{\left[ -\int f \frac{2\pi}{|\omega|} \theta_r (r) \right]} \theta_r (r'),
\]

(A37)

\[
\langle \theta_r \theta_{r'} \rangle = \int d\theta_r e^{\left[ -\int f \frac{2\pi}{|\omega|} \theta_r (r) \right]} \int \frac{d\omega}{2\pi} e^{-i\omega \theta_r (r)} \frac{d\omega'}{2\pi} e^{-i\omega' \theta_{r'} (r)},
\]

(A38)

\[
\langle \theta_r \theta_{r'} \rangle = \int \frac{d\omega}{2\pi} e^{-i\omega (r + r')} e^{\left[ -\int f \frac{2\pi}{|\omega| \theta_r (r) \right]} \theta_{r'} (r) \theta_{r'} (r'),
\]

(A39)

\[
\langle \theta_r \theta_{r'} \rangle \propto \int \frac{d\omega}{2\pi} e^{-i\omega (r + r')} \frac{1}{2|\omega| K} \delta (\omega + \omega'),
\]

(A40)

\[
\langle \theta_r \theta_{r'} \rangle = \frac{1}{2K} \int_{\Lambda/b < |\omega| < \Lambda} \frac{d|\omega|}{2\pi} e^{-i\omega (r - r')} |\omega|^{-1},
\]

(A41)

\[
\langle \theta_r \theta_{r'} \rangle = \frac{1}{2\pi K} \int_{\Lambda/b} \frac{d\omega}{\omega} e^{-i\omega (r - r')}.
\]

(A42)

For \(r' \to r\) we will have,

\[
\langle \theta_r \theta_{r'} \rangle \approx \langle \theta_r \rangle = \frac{1}{2\pi K} \int_{\Lambda/b} \frac{d\omega}{\omega} = \frac{1}{2\pi K} \ln b.
\]

(A43)

We introduce the relative coordinate \(s = r - r'\) and center of mass coordinate \(T = (r + r')/2\). Thus, we have,

\[
\cos \sqrt{4\pi}[\theta_r + \theta_{r'}] = \cos [4\sqrt{\pi}(\theta_T (T))].
\]
This term is RG irrelevant term. For small $s$ cosine can be approximated by,

$$
\cos \sqrt{4\pi}[\theta_1(r) - \theta_1(r')] = 1 - 2\pi(s\partial r_1(T))^2.
$$

Thus, Equation (A37) can be written as,

$$
-\frac{1}{2} \langle (S_\Delta^2) - (S_\Delta) \rangle^2 = -\frac{\Delta^2}{4\pi^2} \left( 1 - \left( \frac{1}{b} \right) \right) \int_0^{b/\lambda} ds T(1 - 2\pi(s\partial r_1(T))^2).
$$

(A44)

Here the first term turns out to be field independent term. Thus, we consider only second term,

$$
-\frac{1}{2} \langle (S_\Delta^2) - (S_\Delta) \rangle^2 = \frac{\Delta^2}{2\pi} \left( 1 - \left( \frac{1}{b} \right) \right) \int_0^{b/\lambda} ds T(2\pi(s\partial r_1(T))^2),
$$

$$
= \frac{\Delta^2}{6\pi\lambda^2} \int_0^{b/\lambda} ds \frac{d\omega}{2\pi} |K|\big(\frac{\theta(\omega)}{2}\big)^2.
$$

(A45)

After rescaling the parameters and fields the equation will have the form,

$$
-\frac{1}{2} \langle (S_\Delta^2) - (S_\Delta) \rangle^2 = \frac{\Delta^2}{6\pi\lambda^2} \int_{-\lambda/\Delta}^{\lambda/\Delta} \frac{d\omega}{2\pi} |K|\big(\frac{\theta(\omega)}{2}\big)^2.
$$

(A46)

Now the effective action can be written as,

$$
S_{\text{eff}} = \int_{-\lambda/\Delta}^{\lambda/\Delta} \frac{d\omega}{2\pi} |K|\big(\frac{\theta(\omega)}{2}\big)^2 + b^2 K \int d^2r \left( \frac{\Delta}{\pi} \right) \cos \left[ 4\pi\theta(r) \right] 
$$

$$
+ \frac{\Delta^2}{6\pi\lambda^2} \left( \left( \frac{1}{b} \right) \right) \int_{-\lambda/\Delta}^{\lambda/\Delta} \frac{d\omega}{2\pi} |K|\big(\frac{\theta(\omega)}{2}\big)^2,
$$

(A47)

$$
S_{\text{eff}} = \left[ 1 + \frac{\Delta^2}{6\pi\lambda^2} \left( \left( \frac{1}{b} \right) \right) \right] S_{\phi}(\theta) + b^2 K S_{\phi}(\theta).
$$

(A48)

Comparing the rescaled effective action with the original action we obtain RG flow equation,

$$
\tilde{K} = K \left[ 1 + \frac{\Delta^2}{6\pi\lambda^2} \left( \left( \frac{1}{b} \right) \right) \right] .
$$

(A49)

Defining the differential of a parameter as $dK = \tilde{K} - K$ and by setting $b = e^{dl}$ we get RG flow equation in the differential form,

$$
dK = \frac{K\Delta^2}{6\pi\lambda^2} \left( e^{3dl} - e^{(3-\Delta)dl} \right),
$$

$$
dK = \frac{K\Delta^2}{6\pi\lambda^2} \left[ 1 + 3dl - 1 - 3dl + \left( \frac{2}{\tilde{K}} \right) dl \right],
$$

$$
dK = \left( \frac{\Delta^2}{3\pi\lambda^2} \right).
$$

(A50)

Now we rescale $\Delta \rightarrow \Delta \sqrt{3/4\pi\lambda}$. Thus, RG flow equations of $H_1$ is given by equation,

$$
\frac{d\Delta}{dl} = \left( 2 - \frac{1}{\tilde{K}} \right) \Delta, \quad \frac{dK}{dl} = \Delta^2.
$$

(A51)

One can follow the same procedure to obtain the RG equations for the Hamiltonian $H_2$. 


Appendix 2. Derivation of Quantum BKT equations for finite $\mu$

We start with the Bosonized model Hamiltonian $H$ with $g_\alpha = 0$,

$$H = \frac{v}{2} \left[ \int K ((\partial_x \phi(x))^2 + K (\partial_x \theta(x))^2) \right] dx - \left( \frac{\mu}{\sqrt{\pi}} \right) \int \partial_x \phi(x) dx + \frac{B}{\pi} \int \cos(\sqrt{4\pi \phi(x)}) dx$$

$$- \Delta \int \cos(\sqrt{4\pi \theta(x)}) dx.$$  \hspace{1cm} (A52)

After rescaling the fields as, $\phi' = \frac{\phi}{\sqrt{K}}$ and $\theta' = \sqrt{K} \theta$, one can write,

$$H = \frac{v}{2} \left[ (\partial_x \phi')^2 + (\partial_x \theta')^2 \right] dx - \left( \frac{\mu}{\sqrt{K}} \right) \int \partial_x \phi' dx + \frac{B}{\pi} \int \cos(\sqrt{4\pi K \phi'}) dx$$

$$- \Delta \int \cos(\sqrt{4\pi K \theta'}) dx.$$ \hspace{1cm} (A53)

Writing the Lagrangian using the Hamilton’s equations, $\partial_x \theta = -\frac{1}{v} \partial_t \phi'$ and $\partial_x \phi' = -\frac{1}{v} \partial_t \theta'$ leads us to,

$$\mathcal{L}_0 = \Pi \phi \partial_t \phi' - H_0,$$

$$= \left( \frac{1}{v} \partial_t \phi' \right) \partial_x \phi' - \frac{v}{2} (\partial_x \theta')^2 - \frac{v}{2} (\partial_x \phi')^2,$$

$$= \frac{1}{2} [v^{-1} (\partial_t \phi')^2 - v(\partial_x \phi')^2].$$ \hspace{1cm} (A54)

Here the Lagrangian $\mathcal{L}_0$, is written in terms of $\phi'$ field. One can also write the $\mathcal{L}_0$ in terms of $\theta'$ field as,

$$\mathcal{L}_0 = \frac{1}{2} [v^{-1} (\partial_t \theta')^2 - v(\partial_x \theta')^2].$$ \hspace{1cm} (A55)

Putting these two together one can have $\mathcal{L}_0$ in terms of both $\phi'$ and $\theta'$,

$$\mathcal{L}_0 = \frac{1}{4} [v^{-1} (\partial_t \phi')^2 - v(\partial_x \phi')^2 + v^{-1} (\partial_t \theta')^2 - v(\partial_x \theta')^2].$$ \hspace{1cm} (A56)

Writing the $\mathcal{L}_0$ in imaginary time i.e. $\tau = it$,

$$\mathcal{L}_0 = \frac{1}{4} [v^{-1} (i \partial_t \phi')^2 + v^{-1} (i \partial_t \theta')^2 - v(\partial_x \phi')^2 - v(\partial_x \theta')^2],$$ \hspace{1cm} (A57)

$$\mathcal{L}_0 = -\frac{1}{4} [v^{-1} (\partial_t \phi')^2 + v^{-1} (\partial_t \theta')^2 + v(\partial_x \phi')^2 + v(\partial_x \theta')^2].$$ \hspace{1cm} (A58)

Lagrangian of the interaction ($\mathcal{L}_{int}$) terms can be obtained by the relation $\mathcal{L}_{int} = -H_{int}$.

$$\mathcal{L}_{int} = \mu \sqrt{\frac{K}{\pi}} \partial_x \phi' - \frac{B}{\pi} \cos(\sqrt{4\pi K \phi'}) + \frac{\Delta}{\pi} \cos(\sqrt{4\pi K \theta'}).$$ \hspace{1cm} (A59)

Writing this in imaginary time $\tau = it$,

$$\mathcal{L}_{int} = -\frac{i \mu}{v} \sqrt{\frac{K}{\pi}} \partial_t \phi' - \frac{B}{\pi} \cos(\sqrt{4\pi K \phi'}) + \frac{\Delta}{\pi} \cos(\sqrt{4\pi K \theta'}).$$ \hspace{1cm} (A61)

The Euclidean action can be written as, $S_E = -\int dr \mathcal{L} = -\int dr (\mathcal{L}_0 + \mathcal{L}_{int})$, where $r = (\tau, x)$. Thus the partition function can be derived as

$$Z = \int \mathcal{D}\phi \mathcal{D}\theta \exp \left[ -\int \lambda d\omega \frac{d\omega}{2\pi} |\omega| \left( \frac{|\phi(\omega)|^2}{2K} + \frac{K|\theta(\omega)|^2}{2} \right) \right]$$

$$- \left[ \int dr \left( \frac{i \mu}{\sqrt{\pi}} (\partial_x \phi') + \frac{B}{\pi} \cos(\sqrt{4\pi K \phi'}) - \frac{\Delta}{\pi} \cos(\sqrt{4\pi K \theta'}) \right) \right].$$ \hspace{1cm} (A62)

Now we divide the fields into slow and fast modes and integrate out the fast modes. The field $\phi$ is $\phi(r) = \phi_s(r) + \phi_f(r)$
Following the above procedure one can arrive at the following equations for

\[
\phi_i(r) = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} e^{-i\omega r} \phi(\omega) \quad \text{and} \quad \theta_j(r) = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} e^{-i\omega r} \theta(\omega),
\]

Thus, the partition function \(Z\) is,

\[
Z = \int \mathcal{D}\phi_i \mathcal{D}\phi_j \mathcal{D}\theta_i \mathcal{D}\theta_j e^{-S(\phi_i, \theta_i)} e^{-S(\phi_j, \theta_j)} e^{-S_{\text{int}}(\phi_i, \theta_i)}.
\]

Using the relation \((A)_f = \int \mathcal{D}\phi_j e^{-S(\phi_i, \theta_i)} A\), one can write,

\[
Z = \int \mathcal{D}\phi_i \mathcal{D}\theta_i e^{-S(\phi_i, \theta_i)} \{ e^{-S_{\text{int}}(\phi_i, \theta_i)} \}_f.
\]

We write the effective action as,

\[
e^{-S_{\text{eff}}(\phi_i, \theta_i)} = e^{-S(\phi_i, \theta_i)} \{ e^{-S_{\text{int}}(\phi_i, \theta_i)} \}_f.
\]

Taking \(ln\) on both side gives,

\[
S_{\text{eff}}(\phi_i, \theta_i) = S(\phi_i, \theta_i) - \ln \{ e^{-S_{\text{int}}(\phi_i, \theta_i)} \}_f.
\]

By writing the cumulant expansion up to second-order, we have

\[
S_{\text{eff}}(\phi_i, \theta_i) = S(\phi_i, \theta_i) + \{ S_{\text{int}}(\phi_i, \theta_i) \}_f - \frac{1}{2} \left( \{ S_{\text{int}}(\phi_i, \theta_i) \}_f - \{ S_{\text{int}}(\phi_i, \theta_i) \}_f^2 \right).
\]

Now we calculate the first-order cumulant expansion.

\[
\{ S_{\text{int}}(\phi_i, \theta_i) \}_f = \int \frac{d\mu}{\sqrt{\pi}} \theta_i(r) + \int \frac{d\mu}{\pi} \{ \cos(\sqrt{4\pi}\phi(r)) \}_f - \int \frac{\Delta}{\pi} \{ \cos(\sqrt{4\pi}\theta(r)) \}_f.
\]

The second term is,

\[
\int \frac{d\mu}{\pi} \{ \cos(\sqrt{4\pi}\phi(r)) \}_f = \int \frac{d\mu}{\pi} \int \mathcal{D}\phi_j e^{-S(\phi_j)} \cos(\sqrt{4\pi}\phi_j(r))
\]

\[
= \frac{1}{2\pi} \int \frac{d\mu}{\pi} \int \mathcal{D}\phi_j e^{-S(\phi_j)} \left[ e^{i\sqrt{4\pi}\phi_j(r)} \int \mathcal{D}\phi_j e^{\frac{i\mu}{\sqrt{\pi}} \phi_j(\omega) - \frac{b\mu^2}{8}} + H.c. \right],
\]

\[
= \frac{1}{2\pi} \int \frac{d\mu}{\pi} \cos(\sqrt{4\pi}\phi_j(r)) e^{-\frac{b\mu^2}{8}},
\]

\[
= \frac{1}{\pi} \int \frac{d\mu}{\pi} \cos(\sqrt{4\pi}\phi_j(r)) e^{\ln e^b},
\]

\[
= \frac{1}{\pi} (b^{-K}) \int \cos(\sqrt{4\pi}\phi_j(r)).
\]

Thus, we have,

\[
\int \frac{d\mu}{\pi} \{ \cos(\sqrt{4\pi}\phi(r)) \}_f = b^{-K} \int \frac{d\mu}{\pi} \cos(\sqrt{4\pi}\phi_j(r))
\]

Following the above procedure one can arrive at the following equations for \(\Delta\),

\[
\int \frac{d\mu}{\pi} \{ \cos(\sqrt{4\pi}\theta(r)) \}_f = b^{-K} \int \frac{d\mu}{\pi} \cos(\sqrt{4\pi}\theta_j(r)).
\]

The first-order cumulant expansion can be re-written as,

\[
\{ S_{\text{int}}(\phi, \theta) \}_f = b^{-K} \int \frac{d\mu}{\pi} \cos(\sqrt{4\pi}\phi_j(r)) - b^{-K} \int \frac{\Delta}{\pi} \cos(\sqrt{4\pi}\theta_j(r)).
\]
Now we calculate the second-order cumulant expansion which has the following terms,

\[-\frac{1}{2} ((S_{\text{int}}^2) - (S_{\text{int}})^2) = -\frac{1}{2} \int dr \, dr' \left( -\frac{\mu^2}{v^2} \right) \left( \{\partial_t \phi(r) \partial_t \phi(r') \} - \{\partial_t \phi(r) \} \{\partial_t \phi(r') \} \right) \]

\[= \frac{1}{2} \int dr \, dr' \left( \frac{B^2}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \cos (\sqrt{4\pi} \phi(r')) \right) \]

\[\left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\frac{1}{2} \int dr \, dr' \left( \frac{\Delta^2}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\frac{1}{2} \int dr \, dr' \left( \frac{i \mu B}{v \sqrt{\pi}} \right) \left( \{\partial_t \phi(r) \} \{\partial_t \phi(r') \} \right) \]

\[= \frac{1}{2} \int dr \, dr' \left( \frac{Bi \mu}{\pi v \sqrt{\pi}} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\frac{1}{2} \int dr \, dr' \left( \frac{\Delta i \mu}{v \sqrt{\pi}} \right) \left( \{\partial_t \phi(r) \} \{\partial_t \phi(r') \} \right) \]

\[= \frac{1}{2} \int dr \, dr' \left( \frac{\Delta B}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\frac{1}{2} \int dr \, dr' \left( \frac{\Delta B}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[-\frac{1}{2} \int dr \, dr' \left( \frac{\Delta^2}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \right) \left( \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

(A74)

The \(B^2\) term can be written as,

\[-\frac{1}{2} \int dr \, dr' \left( \frac{B^2}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \phi(r)) \cos (\sqrt{4\pi} \phi(r')) \right) \right) \]

\[= \frac{B^2}{4\pi^2} (1 - b^{-2K}) \int dr (\partial_t \phi)^2 - \frac{B^2}{2\pi^2} (b^{-4K} - b^{-2K}) \int dr \cos [\sqrt{16\pi} \phi(r)]. \quad \text{(A75)} \]

Similarly one can write,

\[-\frac{1}{2} \int dr \, dr' \left( \frac{\Delta^2}{\pi^2} \right) \left( \cos (\sqrt{4\pi} \theta(r)) \right) \right) \]

\[= \frac{\Delta^2}{4\pi^2} (1 - b^{-2K}) \int dr (\partial_t \theta)^2. \quad \text{(A76)} \]
Now we calculate $\Delta i\mu$ term,
\[
-\frac{1}{2} \int dr \int dr' \left( -\frac{\Delta i\mu}{\pi \sqrt{\pi}} \left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) - \left( \cos \left( \sqrt{4\pi \theta(r')} \right) \right) \right)
= -\frac{\Delta i\mu}{2 \pi \sqrt{\pi}} \int dr \int dr' \left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) \left( \cos \left( \sqrt{4\pi \theta(r')} \right) \right) \left( \partial_r \theta(r') \right)
= -\frac{\Delta i\mu}{2 \pi \sqrt{\pi}} \int dr \int dr' \left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) \left( \cos \left( \sqrt{4\pi \theta(r')} \right) \right).
\]  
(A77)

The correlation function $\left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right)$ can be written as,
\[
\left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) = -2\sqrt{\pi} \sin \left( \sqrt{4\pi \theta(r)} \right) \Psi \theta(r') e^{-2\pi \Theta(r')}
\]
Thus, we have,
\[
\left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) = -\frac{\Delta i\mu}{2 \pi \sqrt{\pi}} (1 - b_{-}) \int dr \cos \left( \sqrt{4\pi \theta(r)} \right).
\]
(A79)

Thus, combined $\Delta i\mu$ and $i\mu B$ terms gives,
\[
-\frac{1}{2} \int dr \int dr' \left( -\frac{\Delta i\mu}{\pi \sqrt{\pi}} \left( \cos \left( \sqrt{4\pi \theta(r)} \right) \partial_r \theta(r') \right) - \left( \cos \left( \sqrt{4\pi \theta(r')} \right) \right) \right)
= -\frac{\Delta i\mu}{\pi \sqrt{\pi}} (1 - b_{-}) \int dr \cos \left( \sqrt{4\pi \theta(r)} \right).
\]
(A80)

In the case of $B i\mu$, the correlation function $\left( \phi \partial_r \theta(r') \right)$ is,
\[
\left( \phi \partial_r \theta(r') \right) = \left( \phi \partial_r \right) \left( \theta \partial_r \right) = \left( \phi \partial_r \right) \left( \theta \partial_r \right)
\]
\[
= \left( \phi \partial_r \right) \left( \theta \partial_r \right) = 0
\]
(A81)

Thus, the combined $B i\mu$ and $i\mu B$ terms equals to 0.
\[
-\frac{1}{2} \int dr \int dr' \left( -\frac{B i\mu}{\pi \sqrt{\pi}} \left( \cos \left( \sqrt{4\pi \phi(r)} \right) \partial_r \phi(r') \right) - \left( \cos \left( \sqrt{4\pi \phi(r')} \right) \right) \right) = 0
\]
(A82)

Now we calculate the term $B \Delta$,
\[
-\frac{1}{2} \int dr \int dr' \left( -\frac{B \Delta}{\pi \sqrt{\pi}} \left( \cos \left( \sqrt{4\pi \phi(r)} \right) \cos \left( \sqrt{4\pi \theta(r')} \right) \right) - \left( \cos \left( \sqrt{4\pi \phi(r')} \right) \right) \right)
= \frac{B \Delta}{4 \pi \sqrt{\pi}} \int dr \int dr' \left[ \cos \left( \sqrt{4\pi \phi(r)} \right) \cos \left( \sqrt{4\pi \theta(r')} \right) \right] \left( e^{-2\pi \left( \phi \theta(r') \right)} - e^{-2\pi \left( \phi \theta(r') \right)} \right)
+ \cos \left( \sqrt{4\pi \phi(r)} \right) \cos \left( \sqrt{4\pi \theta(r')} \right) \left( e^{-2\pi \left( \phi \theta(r') \right)} - e^{-2\pi \left( \phi \theta(r') \right)} \right)
\]
(A83)

Here the correlation function is,
\[
e^{-2\pi \left( \phi \theta(r') \right)} = e^{-2\pi \left( \phi \theta(r') \right)} \left( e^{-2\pi \left( \phi \theta(r') \right)} \right)
\]
We know that $\left( \phi \theta(r') \right) = 0$. Thus, we have,
\[
e^{-2\pi \left( \phi \theta(r') \right)} = e^{-2\pi \left( \phi \theta(r') \right)} \left( \phi \theta(r') \right)
\]
(A84)
These two exponentials cancel each other making the whole term 0. Thus, the combined $B\Delta$ and $\Delta B$ term,
\[-\frac{1}{2} \int dr dr' \left( \frac{B\Delta}{\pi^2} \left( \cos \left( \sqrt{4\pi} \phi'(r) \right) \cos \left( \sqrt{4\pi} \theta'(r') \right) \right) - \left( \cos \left( \sqrt{4\pi} \phi(r) \right) \right) \left( \cos \left( \sqrt{4\pi} \theta(r') \right) \right) \right) = 0. \quad (A85)\]

Now we rescale the first and second-order cumulant expansions by replacing $r = br'$, $\omega = \frac{u}{r}$, $\phi_4(r) = \phi'(r')$ and $\phi_4(\omega) = b\phi'(\omega)$.
\[
\left< S_{int}(\phi, \theta) \right>_i = b^{2-K} \int dr' \left( \frac{B}{\pi} \cos \left( \sqrt{4\pi} \phi'(r') \right) \right) - b^{2-K} \int dr' \left( \frac{\Delta}{\pi} \cos \left( \sqrt{4\pi} \theta'(r') \right) \right), \quad (A86)\]
\[-\frac{1}{2} \left( \left< S_{int}^2 \right> - \left< S_{int} \right>^2 \right) = \frac{B^2}{4\pi} \left( b^2 - b^{2-2K} \right) \int dr' \left( \partial_r \phi' \right)^2 - \frac{B^2}{2\pi} \left( b^{2-4K} - b^{2-2K} \right) \int dr' \cos \left[ \sqrt{16\pi} \phi'(r') \right] \]
\[+ \frac{\Delta^2}{4\pi} \left( b^2 - b^{2-2} \right) \int dr' \left( \partial_r \theta' \right)^2 - \frac{\Delta \mu}{\pi^2} \left( b^2 - b^{2-2} \right) \int dr' \cos \left[ \sqrt{4\pi} \theta'(r') \right]. \quad (A87)\]

Comparison of $B$ terms,
\[B' = Bb^{2-K}.\]

We put $b = e^{dl}$ and expand the exponential up to second term, i.e. $e^{dl} = 1 + dl$. Then,
\[B' = B[1 + (2 - K) dl] = B + (2 - K)B dl.\]

We define $B' - B = dB$, thus, we have,
\[
\frac{dB}{dl} = (2 - K)B. \quad (A88)\]

Similarly on comparison of $\Delta$ terms, we have,
\[
\frac{d\Delta}{dl} = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{\nu\pi} \right) \right]. \quad (A89)\]

Comparison for $K$ terms gives,
\[
\frac{dK}{dl} = \frac{1}{2\pi^2} (\Delta^2 - B^{2-K^2}). \quad (A90)\]

Thus, we obtain RG equations,
\[
\frac{dB}{dl} = (2 - K)B, \quad (A91)\]
\[
\frac{d\Delta}{dl} = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{\nu\pi} \right) \right], \quad (A92)\]
\[
\frac{dK}{dl} = \frac{1}{2\pi^2} (\Delta^2 - B^{2-K^2}). \quad (A93)\]

Now, we consider the limits $B=0$ and $\Delta = 0$ to obtain the two BKT equations. Considering $B = 0$ we get,
\[
\frac{d\Delta}{dl} = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{\nu\pi} \right) \right], \quad \frac{dK}{dl} = \Delta^2. \quad (A94)\]

This is the RG equation for the Hamiltonian $H_3$ in Equation (35). Considering $\Delta = 0$ we get,
\[
\frac{dB}{dl} = (2 - K)B, \quad \frac{dK}{dl} = -B^2 K^2. \quad (A95)\]