Abstract: From a geometric point of view, massless spinors in 3 + 1 dimensions are composed of primary fields of weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, where the weights are defined with respect to diffeomorphisms of a sphere in momentum space. The Weyl equation thus appears as a consequence of the transformation behavior of local sections of half–canonical bundles under a change of charts. As a consequence, it is possible to impose covariant constraints on spinors of negative (positive) helicity in terms of (anti–)holomorphy conditions. Furthermore, the identification with half–differentials is employed to determine possible extensions of fermion propagators compatible with Lorentz covariance.
1 Introduction

The quest for a four-dimensional notion of analyticity and the related problem to define four-dimensional analogues of two-dimensional conformal field theories is a subject of much interest and was studied extensively in recent years. A possibility which attracted particular attention relies on the ideas of quaternionic analyticity [1], either in terms of Fueter analyticity [2], or in terms of a harmonic space approach [3, 4, 5], which is strongly connected to the twistor approach to space–time [6, 7].

On the other hand, attempts to transfer methods of 2D conformal field theory directly to 4D conformal field theories led to the discovery of structures reminiscent of Zamolodchikov’s c–theorem [8, 9], and to new results on correlation functions in 4D conformal field theories, including in particular an extension of the central charge of 2D conformal field theory to a triple of central charges in 4 dimensions [10, 11]. Of related interest are recent results on quasi-primary fields in the $O(N)$ $\sigma$–model for $2 < d < 4$ [12].

In the present paper, I will follow a different approach to transfer notions of 2D conformal field theory into 3+1 dimensions. I would like to point out that left or right handed massless spinors in 3 + 1 dimensions can be interpreted as half–differentials on spheres in momentum space. This implies the possibility to formulate covariant phase space constraints on spinors of definite helicity in terms of (anti–)meromorphy constraints (or (anti–)holomorphy constraints outside finitely many poles). More specifically, the entries of a spinor of negative helicity appear as local representations $\psi(z, \bar{z}, E)$ with respect to a conformal atlas, and transform under holomorphic transformations according to

$$\psi'(z', \bar{z}', E') = \psi(z, \bar{z}, E) \left( \frac{\partial z'}{\partial z} \right)^{-\frac{1}{2}}$$

Special cases of this transformation behavior imply the Weyl equation for massless fermions. Lorentz transformations induce via $SL(2, C)$ holomorphic transformations of spheres in momentum space, and the resulting transformation behavior of left handed spinors agrees with the equation above. Therefore, left handed spinors can be subjected to conditions

$$\frac{\partial \psi}{\partial \bar{z}} = 0$$

which are covariant under Lorentz transformations. In this sense, the identification of spinors of definite helicity with half–differentials induces notions of 2D conformal field theory in 3 + 1 dimensions.

$^1$Somewhat sloppy, I will refer to (2) as a holomorphy constraint. The degree of the divisor of a meromorphic $\lambda$–differential on a surface of genus $g$ (i.e. the sum of orders of zeros minus the sum of
To clarify the notion of half-differentials it is useful to develop a covariant primary field formalism not relying on conformal gauges [13, 14]. This will be reviewed in section 2. This section serves to explain in particular how to covariantize the factorized transformation behavior of primary fields under 2D diffeomorphisms, which translates via our construction into a factorized transformation behavior of helicity spinors under Lorentz transformations. The relation between spinors and half-differentials is then explained in detail in section 3. In section 4 the relation between half-differentials and Weyl spinors is exploited to fix the structure of massless Lorentz covariant fermion propagators. As a result we will find that the Lorentz covariant fermion correlation in the free massless limit is determined up to two functions $f_1$ and $f_2$ which depend on single, but different arguments [15]:

$$\langle \Psi(\vec{p})\overline{\Psi}(\vec{p}')\rangle =$$ (3)

$$\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) \otimes \left(\begin{array}{c} \hat{z}' \hat{z} \\ \hat{z} \hat{z}' \end{array}\right) \langle \phi(\vec{p})\phi^+(\vec{p}')\rangle + \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \otimes \left(\begin{array}{c} 1 \ -\hat{z}' \\ -z \hat{z}' \end{array}\right) \langle \psi(\vec{p})\psi^+(\vec{p}')\rangle$$

$$+ \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \otimes \left(\begin{array}{c} \hat{z} \ -\hat{z}' \\ 1 \ -\hat{z}' \end{array}\right) \langle \phi(\vec{p})\psi^+(\vec{p}')\rangle + \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right) \otimes \left(\begin{array}{c} 1 \ -z \hat{z}' \end{array}\right) \langle \psi(\vec{p})\phi^+(\vec{p}')\rangle$$

$$\langle \psi(\vec{p}_1)\phi^+(\vec{p}_2)\rangle = \langle \phi(\vec{p}_2)\phi^+(\vec{p}_1)\rangle = f_1 \left(\frac{|\vec{p}_1|}{|\vec{p}_2|}\right) \frac{1+z_1\bar{z}_2}{\sqrt{|\vec{p}_1||\vec{p}_2|}} \delta_{zz}(z_1-z_2)$$

$$\langle \psi(\vec{p}_1)\phi^+(\vec{p}_2)\rangle = \langle \phi(\vec{p}_2)\psi^+(\vec{p}_1)\rangle = f_2 \left(\frac{|\vec{p}_1|}{|\vec{p}_2|}\right) \frac{(z_1-z_2)(\bar{z}_1-\bar{z}_2)}{(1+z_1\bar{z}_1)(1+z_2\bar{z}_2)}$$

where $z(\vec{p})$ is a stereographic coordinate in momentum space:

$$z = \frac{p_+}{|\vec{p}| - p_3}$$

The full derivation of these results will be given in section 4. Of special interest are the $f_2$-terms, since these terms are the only terms which comply both with Lorentz covariance and chiral symmetry breaking. Note the consistency of this result: Since (3) provides a Lorentz covariant massless propagator, those parts of it which break chiral symmetry must also break translational invariance. This is in agreement with Eq. (5), since the right hand side of this equation cannot accommodate for a $\delta$-function in external momenta. The $f_1$-terms in turn preserve chiral symmetry: They do not contribute to a chiral condensate and anticommute with $\gamma_5$. Consistency of the result in this sector is expressed by the fact that these terms contain a $\delta$-function which restricts the correlator to parallel momenta.

pole orders) equals $2\lambda(g-1)$, so spinors satisfying (2) have at least one first order pole or two poles of order $\frac{1}{2}$ on the unit sphere in momentum space.
Indeed, one motivation for the work reported in section 4 arose from low energy QCD: Hadron spectroscopy and QCD sum rules provide evidence that chiral symmetry remains broken in the low energy sector of QCD even in the limit $m_q \to 0$ [16]. Therefore, the $f_2$-terms might shed new light on the problem of chiral symmetry breaking in low-energy QCD.

Applications of methods of 2D field theory also proved very useful in certain kinematical regions of high energy QCD, see [17, 18].

For recent investigations of confinement and chiral symmetry breaking in the framework of supersymmetric gauge theories see [19, 20] and references therein. Exploiting holomorphy constraints on superpotentials and effective couplings with respect to chiral superfields and microscopic couplings yields a whole wealth of non-perturbative results on the phases of these theories, and in particular strong evidence for the appearance of massless monopoles at certain points of the moduli space. It is very remarkable, that even modest perturbations of the superpotentials trigger monopole condensation, thus yielding confinement via the dual Meissner effect [19, 20].

Besides chiral symmetry breaking, motivation for the present work came also from the observation that quantum group symmetric Heisenberg relations break translational invariance by introducing an exponential discretization in any direction, which however cannot be interpreted naively as a lattice structure [21]. Thus quantum groups may provide a highly unusual and very attractive possibility to achieve an immanent regularization of quantum field theory with enough remnants of lattice structures to ensure finiteness, but without spoiling Lorentz symmetry.

However, I should emphasize that the results of section 4 are independent of any particular dynamics and also do not rely on any quantum group concepts. Eq. (3) essentially constitutes the result of a group theoretical investigation of massless fermion propagators in the free limit by use of the mapping between half-differentials and Weyl spinors described in section 3.

2 Covariant Primary Fields

In two-dimensional field theories two apparently different formulations of covariance existed in parallel for several years. On the one hand two-dimensional field theories can be formulated covariantly in the usual way employing tensor and spinor fields, while on the other hand it is known that in a conformal gauge primary fields can be employed to ensure covariance with respect to the conformal remnant of the diffeomorphism group [22]. This was puzzling, because there exist primary fields of half-integral order on two-manifolds, and it was not clear in what sense these could
be considered as remnants of tensor or spinor fields in a conformal gauge. Furthermore, it was unclear how half-differentials should transform under non-conformal transformations, or how they could be defined outside the realm of conformal gauge fixing. The puzzle was partially solved by the introduction of a covariant definition of primary fields, thus demonstrating that primary fields yield factorized representations of the full two-dimensional diffeomorphism group. This work also included a demonstration of isomorphy between tensor fields and covariant primary fields of integer weight. However, the exact relation between spinors in two dimensions and the covariant half-differentials of was given only recently in [14], where the formalism was further developed and applied to two-dimensional supergravity.

Initially primary fields $\Phi$ of conformal weight $(\lambda, \bar{\lambda})$ on a two-manifold $M$ are defined by their transformation behavior under a holomorphic change of charts $z \to u(z)$:

$$\Phi(u, \bar{u}) = \Phi(z, \bar{z}) \cdot \left( \frac{\partial u}{\partial z} \right)^{-\lambda} \cdot \left( \frac{\partial \bar{u}}{\partial \bar{z}} \right)^{-\bar{\lambda}}$$

where I employed the usual convention to denote the weight for the complex conjugate sector of coordinates by $\bar{\lambda}$.

The scaling dimension of the field $\Phi$ is $\Delta = \lambda + \bar{\lambda}$ and the spin is $\sigma = \lambda - \bar{\lambda}$. A cohomological investigation reveals that the spin is restricted to integer or half-integer values, while no similar restriction is imposed on the scaling dimension. We will demonstrate this in the more general setting of covariant primary fields below.

The factorized transformation behavior makes primary fields particularly convenient for the formulation of two-dimensional field theories and the investigation of short distance expansions. However, this definition of primary fields works only in a conformal gauge, i.e. in an atlas with holomorphic transition functions. This causes no problem for integer values of $\lambda$ and $\bar{\lambda}$, because the corresponding primary fields might be considered as remnants of tensor fields in the conformal gauge. However, such an interpretation is not possible for fractional conformal weights. Furthermore, if the metric of the two-manifold $M$ is considered as a dynamical degree of freedom it is very inconvenient to switch to a conformal gauge, because this implies that two degrees of freedom of the metric corresponding to the Beltrami-parameters (see below) are hidden in the holomorphic transition functions. Therefore, in a conformal gauge it is impossible to formulate the dynamics of the metric in terms of local fields.

To avoid the restriction to conformal atlases requires a generalization of equation (6) to diffeomorphisms $z \to u(z, \bar{z})$, i.e. we will define primary fields for arbitrary atlases on smooth two-manifolds, thereby introducing a covariant definition of half-

\footnote{In this section, spinor refers to 2D spinors}

\footnote{We distinguish between the spin $\sigma$ referring to rotations induced by diffeomorphisms of $M$ and the spin $s$ referring to rotations of tangent frames.}
differentials. Hence, in the sequel $z, w$ and $u$ will denote complex local coordinates, but no holomorphy conditions on transformations will be assumed any more. To define covariant primary fields it is convenient to switch to a Beltrami–parametrization of the metric:

$$(ds)^2 = \frac{2g_{zz}}{1 + \mu z \bar{z} \cdot \mu \bar{z}} \cdot |dz + \mu z \bar{z} \cdot d\bar{z}|^2$$

i.e. the Beltrami–parameters $\{\mu z \bar{z}, \mu z \}$ specify the metric modulo scaling transformations:

$$\mu z \bar{z} = \frac{g_{zz} - \sqrt{g_{zz}^2 - g_{zz} g_{\bar{z}\bar{z}}}}{g_{zz} + \sqrt{g_{zz}^2 - g_{zz} g_{\bar{z}\bar{z}}}} = \mu \bar{z}$$

$$\frac{g_{zz}}{1 + \mu z \bar{z} \cdot \mu \bar{z}} = 2 \mu z \bar{z}$$

The Beltrami–parameters satisfy $\mu z \bar{z} < 1$ and have a subtle transformation behavior under reparametrizations $z \rightarrow u(z, \bar{z})$ with $|\partial_z u| > |\partial_{\bar{z}} u|:

$$\mu u \bar{u} = \frac{\mu z \bar{z} \cdot \partial_z u - \partial_{\bar{z}} u}{\partial_z \bar{u} - \mu z \bar{z} \cdot \partial_{\bar{z}} u} = \frac{\partial_z u + \mu z \bar{z} \cdot \partial_{\bar{z}} u}{\partial_{\bar{z}} u + \mu z \bar{z} \cdot \partial_z u}$$

This transformation law implies in particular

$$\partial_z - \mu z \bar{z} \partial_{\bar{z}} = (\partial_z u - \mu z \bar{z} \partial_{\bar{u}})(\partial_{\bar{a}} - \mu u \bar{u} \partial_a) = \frac{1}{\partial_{\bar{a}} z - \mu u \bar{u} \partial_a}(\partial_{\bar{a}} - \mu u \bar{u} \partial_a)$$

This observation motivates the introduction of particular non–holonomic bases of vector fields and differentials on two–manifolds $\mathcal{M}$:

$$\mathcal{D}_z = \partial_z - \mu z \bar{z} \cdot \partial_{\bar{z}}$$

$$\mathcal{D}_z = \frac{1}{1 - \mu z \bar{z} \cdot \mu \bar{z}} (dz + \mu z \bar{z} \cdot d\bar{z})$$

$$\partial_{\bar{z}} = \frac{1}{1 - \mu z \bar{z} \cdot \mu \bar{z}} (\mathcal{D}_z + \mu z \bar{z} \cdot \mathcal{D}_z)$$

$$dz = \mathcal{D}_z - \mu z \bar{z} \cdot \mathcal{D}_z$$

These bases are distinguished by their factorized transformation properties under diffeomorphisms:

$$\mathcal{D}_u = (\mathcal{D}_u z) \mathcal{D}_z \quad \mathcal{D}_u = \mathcal{D}_z \mathcal{D}_u u \quad \mathcal{D}_u u = (\mathcal{D}_u z)^{-1}$$
thus allowing us to introduce a consistent covariant definition of primary fields:

**Definition:** A field \( \Phi \) over a two–manifold is denoted as *primary* of weight \((\lambda, \bar{\lambda})\) if its local representations \( \Phi(z, \bar{z}) \) transform under a change of coordinates \( z, \bar{z} \rightarrow u, \bar{u} \) according to

\[
\Phi(u, \bar{u}) = \Phi(z, \bar{z}) \cdot (D_z u)^{-\lambda} \cdot (D_{\bar{z}} \bar{u})^{-\bar{\lambda}}
\]

In particular any tensor representation of the diffeomorphism group factorizes into appropriate primary fields with integer weights upon expansion with respect to the non–holonomic bases \([11,12]\), but the crucial point is that fractional weights can be defined as well without conformal gauge fixing.

As we remarked before, there is a restriction on the admissible values of the weight \((\lambda, \bar{\lambda})\): In a region of three intersecting patches \( U_I, U_J, U_K \) with coordinates \( z_I, z_J, z_K, z_I = f_{IJ}(z_J, \bar{z}_J) \), etc., the product of transition functions for a roundtrip \( z_I \rightarrow z_J \rightarrow z_K \rightarrow z_I \) must yield the identity:

\[
(D_{z_K} f_{IK})^\lambda (D_{z_J} f_{KJ})^\lambda (D_{z_I} f_{JI})^\lambda (D_{\bar{z}_K} \bar{f}_{IK})^{\bar{\lambda}} (D_{\bar{z}_J} \bar{f}_{KJ})^{\bar{\lambda}} (D_{\bar{z}_I} \bar{f}_{JI})^{\bar{\lambda}} = 1
\]

For integer weights this condition is automatically fulfilled due to \( f_{KI} = f_{KJ} \circ f_{JI} \) and eq. (15). However, if \( \Delta = \frac{p}{q}, \sigma = \frac{r}{s} \) are the representations of \( \Delta \) and \( \sigma \) in terms of integers without common divisors, and if \( q \neq 1 \), then it is a non–trivial problem to fix the \( q \)–fold ambiguity in the definition of the transition functions \( (D_{z_I} f_{JI})^\lambda \cdot (D_{\bar{z}_I} \bar{f}_{JI})^{\bar{\lambda}} \) in the intersections of all patches in such a manner that the condition (17) is fulfilled.

To elaborate this further, we split the transition functions into modulus and phase according to

\[ D_{z_I} f_{IJ} = R_{IJ} \exp(i\phi_{IJ}) \]

If we now stick to the convention to choose \( R_{IJ}^\frac{1}{p} \) positive real in any intersection \( U_I \cap U_J \), then (17) reduces to

\[
\exp(i\sigma \phi_{IK}) \cdot \exp(i\sigma \phi_{KJ}) \cdot \exp(i\sigma \phi_{JI}) = 1
\]

and this defines the choice of phases as a sheaf–cohomological problem:

To clarify this define

\[
S_{IJK} \equiv \exp(i\sigma \phi_{IK}) \cdot \exp(i\sigma \phi_{KJ}) \cdot \exp(i\sigma \phi_{JI})
\]

which is an element of \( Z_q \). Consider the sheaf \( \mathcal{M} \times Z_q \) with base manifold \( \mathcal{M} \) and stalk \( Z_q \). An \( n \)–cochain is a completely antisymmetric functional of intersections of \( n + 1 \) patches with values in \( Z_q \):

\[
c(U_{I(0)} \cap U_{I(1)} \cap \ldots \cap U_{I(n)}) = c_{I(0)I(1)\ldots I(n)} = c_{I(1)I(0)\ldots I(n)}^{-1} \in Z_q
\]

\[
c(\emptyset) = 1
\]
Then there are coboundary operators $\delta_n$ in the pre–sheaf related to the cover $\{U_I\}$ mapping $n$–cochains to $(n+1)$–cochains:

$$
(\delta_0 c)_{IJ} = \frac{c_I}{c_J} \\
(\delta_1 c)_{IJK} = \frac{1}{c_{IK}}c_{JK} \\
(\delta_2 c)_{IJKL} = \frac{1}{c_{IJL}c_{IKL}}c_{JKL}
$$

and we have

$$\delta_{n+1}\delta_n = 1$$

Then $S$ as defined in (19) is a closed 2–cochain: $\delta_2 S = 1$. Unfortunately this does not imply exactness of $S$, because the phase factors $\exp(i\sigma\phi_{IJ})$ generically do not satisfy $x^q = 1$. On the other hand exactness is what we are seeking, because in this case we would have

$$S_{IJK} \equiv \exp(i\sigma\phi_{IK}) \cdot \exp(i\sigma\phi_{KJ}) \cdot \exp(i\sigma\phi_{JI})$$

$$= (\delta_1 \theta)_{IJK} = \theta_{IJ}\theta_{JK}\theta_{KI}$$

for some 1–cochain $\theta$ in $\mathcal{M} \times \mathbb{Z}_q$ and we could rescale the phase factors $\exp(i\sigma\phi_{IJ}) \to \exp(i\sigma\phi_{IJ})\theta_{JI}$ such that the condition (18) could be fulfilled. Therefore, we may admit only those values for the denominator $q$ of the spin, which correspond to a trivial cohomology group $H^2(\mathcal{M}, \mathbb{Z}_q)$. However, it is a classical result on two–manifolds that this cohomology group equals $\emptyset$ for every $\mathcal{M}$ if and only if $q = 1$ or $q = 2$ [23]. Hence, the spin of primary fields over two–manifolds is restricted to integral or half–integral values. This implies in particular that the fractional values of conformal weights appearing in the conformal grids of minimal models must be combined into the weights $(\lambda, \bar{\lambda})$ of primary fields such that $\sigma$ is half–integer or integer. This rule seems also justified empirically, because it is in agreement with the weights appearing in explicit realizations of minimal models.

Let us now take a closer look at the correspondence between tensors and spinors on the one hand and primary fields on the other hand:

As remarked before, the isomorphy between tensors and primary fields of integer weight is given by expansion with respect to the anholonomic basis $\{\mathbf{1}_1, \mathbf{1}_2\}$. More specifically, we denote a tensor $T$ with $m$ covariant and $n$ contravariant indices as a tensor of covariance $\langle m, n \rangle$. Upon expansion with respect to the primary basis $\{\mathbf{1}_1, \mathbf{1}_2\}$ a tensor of covariance $\langle m, n \rangle$ decays into $2^{m+n}$ primary fields according to the reduction formula

$$\langle m, n \rangle = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} (i - j, m - n - i + j)$$
For a tensor $T$ of covariance $\langle 2, 0 \rangle$ the transformation to the related primary fields $\mathcal{T}$ is given by:

$$T_{zz} = T_{zz} - \mu_z \bar{z} (T_{\bar{z}z} + T_{\bar{z}\bar{z}}) + \mu_z \bar{z} \mu_{\bar{z}z} T_{\bar{z}z}$$

$$T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}} - \mu_{\bar{z}} z \mu_{\bar{z}z} T_{\bar{z}z} + \mu_{\bar{z}} z \mu_{z\bar{z}} T_{\bar{z}z}$$

$$T_{z\bar{z}} = \frac{1}{(1 - \mu_z \bar{z} \mu_{\bar{z}z})^2} (T_{zz} + \mu_z \bar{z} (T_{\bar{z}z} + T_{\bar{z}\bar{z}}) + \mu_{\bar{z}} z \mu_{\bar{z}z} T_{\bar{z}z})$$

$$T_{\bar{z}z} = \frac{1}{(1 - \mu_{\bar{z}} z \mu_{z\bar{z}})^2} (T_{\bar{z}z} + \mu_{\bar{z}} z \mu_{z\bar{z}} T_{z\bar{z}} + \mu_{z} \bar{z} \mu_{z\bar{z}} T_{z\bar{z}})$$

and the conjugate formulas. A special case is the metric, where the formulas above yield $G_{zz} = G_{\bar{z}\bar{z}} = 0$ and

$$G_{zz} = g_{zz} \left(1 - \mu_z \bar{z} \mu_{\bar{z}z}\right)^2$$

In the primary field formalism we presented here the metric is represented by a real primary field $\mathcal{G}_{zz}$ and a complex Beltrami parameter $\mu_z \bar{z}$:

$$(ds)^2 = 2\mathcal{G}_{zz} Dz \cdot D\bar{z}$$

Note however that we might choose as well any other definite symmetric tensor of covariance $\langle 2, 0 \rangle$, construct the corresponding Beltrami–parameters and derive another covariant primary field formalism in exactly the same manner.

The relation between two–dimensional spinors and covariant half–differentials has been clarified by employing an appropriate zweibein formalism [14]. Therefore, consider complex orthogonal bases in the tangent frames:

$$\vec{e}_\zeta = \frac{1}{2}(\vec{e}_1 - i\vec{e}_2)$$

$$\eta_{\zeta\zeta} = 0, \quad \eta_{\zeta\bar{\zeta}} = \frac{1}{2}$$

We stick to the convention that greek indices transform under the symmetry group of the tangent bundle, while latin indices refer to transformations under diffeomorphisms. Remember that in the complex orthogonal bases rotations in the tangent bundle are diagonal:

$$\Lambda(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

For spinors we choose a Weyl basis $\gamma_1 = \sigma_1, \gamma_2 = \sigma_2$ such that the spinor representation of $SO(2)$ is diagonal as well:

$$S(\phi) = \begin{pmatrix} \exp\left(\frac{i}{2}\phi\right) & 0 \\ 0 & \exp\left(-\frac{i}{2}\phi\right) \end{pmatrix}$$
In the zweibein formalism the Beltrami parameters appear as ratios of zweibein components: Insertion of
\[ g_{zz} = e_z^ζ e_z^ζ \quad g_{z\bar{z}} = \frac{1}{2}(e_z^ζ e_{\bar{z}}^ζ + e_{\bar{z}}^ζ e_z^ζ) \]
into (8) yields
\[ e_z^ζ = \mu_z^z e_z^ζ \]
(20)
Therefore, the primary zweibein which transforms like a primary field of weight (1,0) under diffeomorphisms is
\[ e_z^ζ = e_z^ζ(1 - \mu_z^z \mu_{\bar{z}}^\bar{z}) \]
Equation (20) implies for the inverse zweibein
\[ e_{\bar{z}}^ζ = -\mu_{\bar{z}}^{\bar{z}} e_{\bar{z}}^ζ \]
and therefore the diagonal components of the inverse zweibein are primary fields of weight \((-1,0)\) and \((0,-1)\) respectively:
\[ e^ζ_ζ = e^ζ_ζ = \frac{1}{e_z^ζ} \]
Thus \(e^ζ_ζ\) transforms under factorized representations both under the diffeomorphism group and the tangent space rotations. Therefore the transformation behavior of fractional powers of \(e^ζ_ζ\) is well behaved. More specifically, \((e^ζ_ζ)^{-\lambda} (e_{\bar{z}}^\bar{z})^{-\bar{\lambda}}\) is a primary field of weight \((\lambda, \bar{\lambda})\) under diffeomorphisms and a field of spin \(s = \bar{\lambda} - \lambda\) under tangent space rotations, and we know by (18) that \(s\) is restricted to integer and half–integer values. In particular, the sought for isomorphy between covariant half–differentials \(\psi_{\sqrt{z}}\) of weight \((\frac{1}{2}, 0)\) and chiral Weyl spinors \(\psi_{\sqrt{\bar{z}}}\) is [14]
\[ \psi_{\sqrt{z}} \sqrt{e^ζ_ζ} = \psi_{\sqrt{\bar{z}}} \]
(22)
Having established equivalence between tensors and spinors on the one hand and covariant primary fields on the other hand, it is also desirable to develop a covariant primary differential calculus. Therefore, we introduce a covariant primary derivative \(D_z\) which maps primary fields of weight \((\lambda, \bar{\lambda})\) and spin \(s\) into primary fields of the same spin and weight \((\lambda + 1, \bar{\lambda})\):
\[ D_z \Phi = \mathcal{D}_z \Phi - \lambda \Gamma^z_{zz} \Phi - \bar{\lambda} \Gamma^{\bar{z}}_{\bar{z}\bar{z}} \Phi - is\Omega_z \Phi \]
(23)
Covariance of this construction with respect to diffeomorphisms \(z \to u(z, \bar{z})\) and rotations \(\bar{e}_ζ \to \bar{e}_ζ \exp(-i\phi)\) implies
\[ \Gamma^u_{uu} = (\mathcal{D}_z u)^{-1} \Gamma^z_{zz} - (\mathcal{D}_z u)^{-2} \mathcal{D}_z \mathcal{D}_z u \]
(24)
\[ \Gamma^{\bar{u}}_{\bar{u}u} = (\mathcal{D}_z u)^{-1} \Gamma^{\bar{z}}_{\bar{z}\bar{z}} - (\mathcal{D}_z u)^{-1} (\mathcal{D}_{\bar{z}} \bar{u})^{-1} \mathcal{D}_{\bar{z}} \mathcal{D}_{\bar{z}} \bar{u} \]
(25)
\[ \Omega_u = (\mathcal{D}_z u)^{-1} (\Omega_z + \mathcal{D}_z \phi) \]
(26)
In applications of this formalism in two-dimensional field theory there frequently appear the anholonomy coefficients of the primary bases (11,12), because these coefficients automatically appear as connection coefficients, if conformally gauge fixed actions like the Ising model or the bosonic string are covariantized in this formalism [13]:

\[
\left[ D_z, D_{\bar{z}} \right] = C^z_{\bar{z}z} D_z - C^z_{z\bar{z}} D_{\bar{z}}
\]

\[
dD_z = C^z_{\bar{z}z} D_z \wedge D_{\bar{z}}
\]

\[
C^z_{\bar{z}z} = \frac{1}{1 - \mu_\bar{z}^z \mu_z^\bar{z}} (D_{\bar{z}} \mu_z^\bar{z} - \mu_z^\bar{z} D_{\bar{z}} \mu_z^\bar{z})
\]

The commutator of the covariant primary derivatives is then

\[
\left[ D_z, D_{\bar{z}} \right] \Phi = (C^z_{\bar{z}z} - \Gamma^z_{\bar{z}z}) D_z \Phi - (C^z_{z\bar{z}} - \Gamma^z_{z\bar{z}}) D_{\bar{z}} \Phi - \lambda R_{z\bar{z}} \Phi + \bar{\lambda} R_{\bar{z}z} \Phi - i s F_z z \Phi
\]

(27)

with curvature and field strength

\[
R_{z\bar{z}} = D_z \Gamma^z_{z\bar{z}} - D_{\bar{z}} \Gamma^z_{z\bar{z}} - C^z_{\bar{z}z} \Gamma^z_{z\bar{z}} + C^z_{z\bar{z}} \Gamma^z_{z\bar{z}}
\]

\[
F_z z = D_z \Omega_{\bar{z}} - D_{\bar{z}} \Omega_z - C^z_{\bar{z}z} \Omega_{\bar{z}} + C^z_{z\bar{z}} \Omega_z
\]

Thus curvatures consist of primary fields of weight (1,1) in this formalism. However, due to the absence of second order terms in the connection coefficients, \( R \) is not a mere translation of the ordinary curvature tensor into the primary basis.

Similar to the tensor formalism one may impose constraints on the connection: The requirement of invariance of the metric under parallel translations implies

\[
\Gamma^z_{z\bar{z}} = D_z \ln(G_{z\bar{z}}) - \Gamma^z_{\bar{z}z}
\]

(28)

while the requirement of vanishing torsion implies

\[
\Gamma^z_{\bar{z}z} = C^z_{\bar{z}z}
\]

(29)

The consistency of the torsion constraint with (25) follows easily from the transformation behavior of the Lie bracket.

On the other hand, one may also impose a zweibein postulate:

\[
D_z e^z_\zeta = 0
\]

\[
D_{\bar{z}} e^{\bar{z}}_\zeta = 0
\]

implying

\[
i \Omega_{\bar{z}} = \Gamma^z_{z\bar{z}} + D_z \ln(e^z_\zeta)
\]

\[
= -\Gamma^z_{z\bar{z}} - D_z \ln(e^\bar{z}_\zeta)
\]

The zweibein postulate implies in particular invariance of the metric under parallel translations (28).
3 Massless Fermions and Half–Differentials

Half–differentials turn out to appear not only in two–dimensional field theories, but also in 3+1 dimensions, because space–time spinors of definite helicity define half–differentials on a sphere in momentum space and vice versa. To explain this, it is convenient to employ the Weyl representation for Dirac matrices, and to parametrize the unit sphere in momentum space in terms of stereographic coordinates:

\[ z = \frac{p_+}{|\vec{p}| - p_3} \quad (30) \]
\[ \tilde{z} = -\frac{1}{z} \quad (31) \]

For later use we also give the inversion formulas:

\[ p_1 = |\vec{p}| \frac{z + \tilde{z}}{z\tilde{z} + 1} \quad (32) \]
\[ p_2 = i|\vec{p}| \frac{\tilde{z} - z}{z\tilde{z} + 1} \quad (33) \]
\[ p_3 = |\vec{p}| \frac{z\tilde{z} - 1}{z\tilde{z} + 1} \quad (34) \]

The metric on the unit sphere reads in terms of these coordinates

\[ ds^2 = \frac{4dzd\tilde{z}}{(1 + z\tilde{z})^2} \]

and hence the zweibein of the previous section in this case is given by

\[ e_z^\zeta = \frac{2}{1 + z\tilde{z}} \]
\[ e_{\tilde{z}}^\zeta = 0 \]

Therefore, primary derivatives reduce to \( D_z = \partial_z \), while the covariant derivative of a primary field of weight \((\lambda, \bar{\lambda})\) and spin \(s\) is given by

\[ D_z \Phi = \partial_z \Phi + (2\lambda + s) \frac{\tilde{z}}{1 + z\tilde{z}} \Phi \]

The relation between the local representations \( \psi(z, \bar{z}, |\vec{p}|) \) and \( \psi(\bar{z}, \bar{\bar{z}}, |\vec{p}|) \) of a primary field of weight \((\frac{1}{2}, 0)\) is according to (16)

\[ \psi(\bar{z}, \bar{\bar{z}}, |\vec{p}|) = -z \psi(z, \bar{z}, |\vec{p}|) \quad (35) \]
where the sign ambiguity has been resolved in such a way to avoid minus signs in the expressions for Weyl spinors and Dirac spinors below. Insertion of the definition of $z$ demonstrates that this is exactly the Weyl equation for a massless spinor with opposite signs of chirality and energy:

$$
(|\vec{p}| + \vec{p} \cdot \vec{\sigma}) \begin{pmatrix}
\psi(z, \bar{z}, |\vec{p}|) \\
\psi(\bar{z}, z, |\vec{p}|)
\end{pmatrix} = 0
$$

Similarly, the relation between local representations of a primary field of weight $(0, \frac{1}{2})$

$$
\phi(\bar{z}, z, |\vec{p}|) = \bar{z} \phi(z, \bar{z}, |\vec{p}|)
$$

is the Weyl equation for a massless spinor of equal signs of energy and chirality:

$$
(|\vec{p}| - \vec{p} \cdot \vec{\sigma}) \begin{pmatrix}
\phi(\bar{z}, z, |\vec{p}|) \\
\phi(z, \bar{z}, |\vec{p}|)
\end{pmatrix} = 0
$$

Half–differentials thus yield spinor bases:

$$
\Psi^{(++)}(\vec{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \phi(z, \bar{z}, |\vec{p}|)
$$

$$
\Psi^{(-+)}(\vec{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \phi(z, \bar{z}, |\vec{p}|)
$$

$$
\Psi^{(+--)}(\vec{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -z \end{pmatrix} \psi(z, \bar{z}, |\vec{p}|)
$$

$$
\Psi^{(--)}(\vec{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -z \end{pmatrix} \psi(z, \bar{z}, |\vec{p}|)
$$

Here the first superscript denotes chirality, while the second superscript indicates the helicity. The spinors with equal signs of helicity and chirality are the positive energy solutions.

To prove covariance of this construction under Lorentz transformations, we first identify the transformation behavior of $z(\vec{p})$ to prove then from (16) that the two local representations of a half–differential transform like the components of a Weyl spinor:

Under parity or time reversal $z(\vec{p})$ goes to $z(-\vec{p}) = -\bar{z}(\vec{p})^{-1}$ and thus half–differentials of weight $(\frac{1}{2}, 0)$ become half–differentials of weight $(0, \frac{1}{2})$ and vice versa.

Under proper orthochronous Lorentz transformations $\Lambda(\omega) = \exp(\frac{i}{2} \omega_{\mu\nu} L_{\mu\nu})$, with $\omega$ the usual set of rotation and boost parameters, $z(\vec{p})$ goes to

$$
z' = z(\vec{p}') = U \circ z(\vec{p}) = \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}}
$$

$$
(41)
$$
if $E = |\vec{p}|$, and to
\[ z' = z(\vec{p}') = U^{-1T} \circ z(\vec{p}) = \frac{dz - c}{a - bz} \quad (42) \]
if $E = -|\vec{p}|$.

Here $U$ is the positive chirality spin representation of $\Lambda$:
\[
U(\omega) = \exp\left(\frac{1}{2} \omega_{\mu\nu} \sigma_{\mu\nu}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})
\]

**Proof of the transformation law (41):**

The principle of the proof will be employed repeatedly in the sequel, and works as follows: In order to prove transformation properties under the proper orthochronous Lorentz group, we first check that the proposed transformation behavior has the correct composition properties under subsequent Lorentz transformations. In order to complete the proof it is then sufficient to verify the proposed transformation properties for rotations around two different axes and boosts in a particular direction, because these transformations provide a generating set for the full proper orthochronous Lorentz group, as will be explained in Eq. (44).

First we observe that $\overline{U} \circ z = \overline{V} \circ z$ for all $z$ if and only if $U = \pm V$, and
\[
\overline{U}_2 \circ \overline{U}_1 \circ z = U_3 \circ z \quad (43)
\]
for all $z$ if and only if $U_3 = \pm U_2 \cdot U_1$. Hence the transformation law (41) provides a projective representation of $SL(2, \mathbb{C})$ or a true representation of the proper orthochronous Lorentz group $L_+^\uparrow$.

However, every proper orthochronous Lorentz transformation can be decomposed into subsequent transformations consisting of boosts in $\vec{e}_3$-directions and rotations around $\vec{e}_1$-axes and $\vec{e}_3$-axes, in the following way: Every proper orthochronous Lorentz transformation can be written as a pure rotation followed by a pure boost $B(\vec{u}) \cdot R(\vec{\phi})$ (see e.g. [24]). This can be decomposed further according to
\[
B(\vec{u}) \cdot R(\vec{\phi}) = R(-\alpha_u \vec{e}_3) \cdot R(-\beta_u \vec{e}_1) \cdot B(\vec{u} \vec{e}_3) \cdot R((\alpha_u + \gamma_\phi) \vec{e}_3) \cdot R(\beta_\phi \vec{e}_1) \cdot R(\alpha_\phi \vec{e}_3) \quad (44)
\]
where $\alpha_\phi$, $\beta_\phi$ and $\gamma_\phi$ are the Euler angles of $R(\vec{\phi})$, and
\[
\cos(\beta_u) = \vec{e}_3 \cdot \vec{u} \\
\cos(\alpha_u) \sin(\beta_u) = -\vec{e}_2 \cdot \vec{u}
\]

As a consequence of (43,44) it is sufficient to verify (41) for the set of generating transformations in order to prove that $z(\vec{p})$ as defined in (30) really satisfies (41) for $E = |\vec{p}|$. 

14
First consider a rotation around $\vec{e}_3$: The corresponding $SL(2, \mathbb{C})$–matrix is

$$U(\phi \vec{e}_3) = \begin{pmatrix} \exp\left(\frac{i}{2} \phi \right) & 0 \\ 0 & \exp\left(-\frac{i}{2} \phi \right) \end{pmatrix}$$

while $p'_+ = p_+ e^{-i\phi}$ and hence $z(p') = z(p)e^{-i\phi}$.

For a rotation around $\vec{e}_1$ the corresponding $SL(2, \mathbb{C})$–matrix is

$$U(\phi \vec{e}_1) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & i \sin\left(\frac{\phi}{2}\right) \\ i \sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{pmatrix}$$

while $z(p')$ is

$$z(p') = \frac{p_1 + i p_2 \cos(\phi) + i p_3 \sin(\phi)}{|p'| - p_3 \cos(\phi) + p_2 \sin(\phi)} = \frac{z \cos\left(\frac{\phi}{2}\right) - i \sin\left(\frac{\phi}{2}\right)}{\cos\left(\frac{\phi}{2}\right) - i z \sin\left(\frac{\phi}{2}\right)}$$

Finally, for a boost along $\vec{e}_3$ with boost parameter $u$ we have

$$U(u \vec{e}_3) = \begin{pmatrix} \exp\left(-\frac{u}{2}\right) & 0 \\ 0 & \exp\left(\frac{u}{2}\right) \end{pmatrix}$$

while $z(p') = e^{-u}z(p)$. The transformation law (42) is proved in the same way. In the last step one only has to take care that $|p'| = |p| \cosh(u) + p_3 \sinh(u)$ and $p'_3 = p_3 \cosh(u) + |p| \sinh(u)$ for $E = -|p|$.

To conclude the demonstration that half–differentials on the unit sphere in momentum space are Weyl spinors it remains to show that the two local representations of a half–differential transform like a Weyl spinor:

First assume $E = |p|$: A half–differential $\phi$ of weight $(0, \frac{1}{2})$ then transforms according to (44) into

$$\phi'(z', \bar{z}', |p'|) = (c\bar{z} + d)\phi(z, \bar{z}, |p|) \quad (45)$$

while a half-differential $\psi$ of weight $(\frac{1}{2}, 0)$ transforms according to

$$\psi'(z', \bar{z}', |p'|) = (\bar{c}z + \bar{d})\psi(z, \bar{z}, |p|) \quad (46)$$

However, due to (33,34) this is equivalent to

$$\begin{pmatrix} \phi'(z', \bar{z}', |p'|) \\ \phi'(z', \bar{z}', |p'|) \end{pmatrix} = U \cdot \begin{pmatrix} \phi(z, \bar{z}, |p|) \\ \phi(z, \bar{z}, |p|) \end{pmatrix}$$

$$\begin{pmatrix} \psi'(z', \bar{z}', |p'|) \\ \psi'(z', \bar{z}', |p'|) \end{pmatrix} = U^{-1} \cdot \begin{pmatrix} \psi(z, \bar{z}, |p|) \\ \psi(z, \bar{z}, |p|) \end{pmatrix}$$
Thus a half–differential of weight \( (0, \frac{1}{2}) \) is equivalent to a spin–\( \{ \frac{1}{2}, 0 \} \)–representation of the proper orthochronous Lorentz group \( L_{+}^{↑} \), while a half–differential of weight \( (\frac{1}{2}, 0) \) is equivalent to a spin–\( \{ 0, \frac{1}{2} \} \)–representation of the proper orthochronous Lorentz group if \( E = |\vec{p}| \).

On the other hand, if \( E = -|\vec{p}| \), then this corresponds to \( U \leftrightarrow U^{-1\dagger} \) in the equations above, and the assignment of the half–differentials \( \phi \) and \( \psi \) to representations of \( L_{+}^{↑} \) is changed.

This concludes our demonstration that half–differentials yield Weyl spinors of definite helicity in Minkowski space and vice versa.

4 Propagators

As an application of the results of the last section we now exploit the factorized transformation behavior of the half–differentials under Lorentz transformations to determine the general structure of massless fermion propagators compatible with Lorentz covariance.

The mode expansion of a massless spinor contains positive and negative frequency contributions:

\[
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{d^3 \vec{p}}{2|\vec{p}|} \left( \Psi_+(\vec{p}) \exp(ip \cdot x) + \Psi_- (\vec{p}) \exp(-ip \cdot x) \right)
\]

The components \( \Psi_{\pm} (\vec{p}) \) have expansions on helicity eigenstates, which can be expressed in terms of half–differentials employing the results of the previous section:

\[
\Psi(\vec{p}) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} \bar{z} \\ z \end{array} \right) \phi(z, \bar{z}, |\vec{p}|) + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ -z \end{array} \right) \psi(z, \bar{z}, |\vec{p}|)
\]

The expansions of both the positive and negative energy contributions contain only spinors with the same signs of chirality and helicity since \( \Psi_{c,h} (-\vec{p}) = \Psi_{c,-h}(\vec{p}) \). This is the reason for the helicity = chirality rule for massless fermions. As a consequence of this reflection in the negative energy case, the half–differentials in (17) transform according to (13, 10), irrespective of the sign of energy.

Eq. (17) yields representations of the corresponding correlation functions in terms of primary fields:

\[
\langle \Psi(\vec{p})\Psi(\vec{p}') \rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} \bar{z} \\ z \end{array} \right) \langle \phi(\vec{p})\phi^+ (\vec{p}') \rangle + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ -z \end{array} \right) \langle \psi(\vec{p})\psi^+ (\vec{p}') \rangle
\]

\[
+ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} z -zz' \\ 1 -zz' \end{array} \right) \langle \phi(\vec{p})\psi^+ (\vec{p}') \rangle + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} z' \\ -zz' -z \end{array} \right) \langle \psi(\vec{p})\phi^+ (\vec{p}') \rangle
\]
Therefore, the 2-point functions on the right hand side transform under a factorized representation of the Lorentz group. This makes this representation very convenient for the investigation of all correlations \( \langle \Psi(p)\Psi(p') \rangle \) which comply with Lorentz covariance.

This is accomplished in the following way: The transformation behavior of the 2-point functions on the right hand side of Eq. (48) is governed by (41) and by (45) or (46) respectively. Therefore, the determination of the general Lorentz covariant form of the correlators \( \langle \Psi(p)\Psi(p') \rangle \) is equivalent to the determination of the general form of correlators of half-differentials complying with their respective transformation properties. Similar to the reasoning employed in the proof of (41,42) it is sufficient to consider rotations \( R(\phi e_1) \) and \( R(\phi e_3) \), and a boost \( B(u e_3) \) and solve the covariance conditions for these transformations in order to ensure covariance with respect to the full proper orthochronous Lorentz group, since the conformal factors in (45,46) compose consistently under Lorentz transformations. Then invariance under parity \( P \) or time reversal \( T \) will impose relations between the different 2-point functions and automatically ensure invariance under charge conjugation. This will uniquely fix the propagator up to two functions \( f_1(|p|/|p'|) \) and \( f_2(-\frac{1}{2}p \cdot p') \).

**Determination of the correlation function** \( F_1(z_1, z_2, \bar{z}_1, \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|) = \langle \psi(p)\psi^+(p') \rangle \):

Eq. (46) implies invariance of \( F_1 \) under rotations \( R(\phi e_3) \), and hence

\[
F_1 = F_1(z_1, z_2, \bar{z}_1, \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|)
\]  

(49)

The other generating transformations yield more involved conditions on \( F_1 \): Under rotations \( R(\phi e_1) \) \( F_1 \) should transform according to

\[
F_1 \left( \frac{z_1 \bar{z}_1 \cos^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) + i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_1)}{z_1 \bar{z}_1 \sin^2 \left( \frac{\phi}{2} \right) + \cos^2 \left( \frac{\phi}{2} \right) - i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_1)} \right)
\]  

(50)

\[
\frac{z_2 \bar{z}_2 \cos^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) + i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_2 - \bar{z}_2)}{z_2 \bar{z}_2 \sin^2 \left( \frac{\phi}{2} \right) + \cos^2 \left( \frac{\phi}{2} \right) - i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_2 - \bar{z}_2)} \]  

\[
\frac{z_1 \bar{z}_2 \cos^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) + i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_2)}{z_1 \bar{z}_2 \sin^2 \left( \frac{\phi}{2} \right) + \cos^2 \left( \frac{\phi}{2} \right) - i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_2)} \]  

\[
F_1 \left( \frac{z_1 \bar{z}_1 \cos^2 \left( \frac{\phi}{2} \right) + \sin^2 \left( \frac{\phi}{2} \right) + i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_1)}{z_1 \bar{z}_1 \sin^2 \left( \frac{\phi}{2} \right) + \cos^2 \left( \frac{\phi}{2} \right) - i \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2} \right) (z_1 - \bar{z}_1)} \right) \]  

When I solved this equation, I did it in terms of the differential equation following from first order in \( \phi \), checking then that the general solution of that equation also
solves the global condition. However, here is a slightly more convenient solution: One may recognize that a particular solution to (50) is given by

\[ F_1 = \frac{1}{1 + z_1 \bar{z}_2} \]

This in turn implies that \( F_1 \) may differ from the particular solution at most by a factor \( G_1 \) which is invariant under rotations both around \( \vec{e}_1 \) and \( \vec{e}_3 \)-axes. However, since the conformal factors in (50) compose consistently under subsequent Lorentz transformations, this implies invariance of \( G_1 \) under the full rotation group. Therefore, the general solution of (49) and (50) is

\[ F_1 = \frac{1}{1 + z_1 \bar{z}_2} G_1 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)}, |\vec{p}_1|, |\vec{p}_2| \right) \]  

(51)

where the first argument is \( p_1 \cdot p_2 \) up to normalization.

What remains to be checked are the boost properties of \( F_1 \), and by the reasoning employed in the proof of Eq. (41) it is sufficient to check boosts along \( \vec{e}_3 \): While the behavior of the parameters \((z, \bar{z}, |p|)\) under rotations is completely specified by (41), for a boost \( B(u \vec{e}_3) \) we also have to specify the behavior of \(|p|\):

\[
\begin{align*}
  z' &= \exp(-u)z \\
  |p'| &= \frac{|p|}{z \bar{z} + 1} (\exp(-u)z \bar{z} + \exp(u))
\end{align*}
\]

Covariance of \( F_1 \) then requires

\[
G_1 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(e^u + e^{-u}z_1 \bar{z}_1)(e^u + e^{-u}z_2 \bar{z}_2)}, |\vec{p}_1|, e^u + e^{-u}z_1 \bar{z}_1, |\vec{p}_2|, e^u + e^{-u}z_2 \bar{z}_2 \right) = \frac{e^u + e^{-u}z_1 \bar{z}_2}{1 + z_1 \bar{z}_2} G_1 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)}, |\vec{p}_1|, |\vec{p}_2| \right)
\]

(52)

The limit of large boost parameter shows that this equation can be solved if and only if \( z_1 = z_2 \), and hence \( G_1 \) must contain a 2-dimensional \( \delta \)-function:

\[ G_1 = (1 + z_1 \bar{z}_2)^2 \delta_{zz}(z_1 - z_2) H_1(|\vec{p}_1|, |\vec{p}_2|) \]

Note that this result complies with Eq. (49) since the 2-dimensional \( \delta \)-function can be written in a fancy way:

\[ \delta_{zz}(z_1 - z_2) = -2z_1 \bar{z}_2 \delta(z_1 \bar{z}_1 - z_2 \bar{z}_2) \delta(z_1 \bar{z}_2 - z_2 \bar{z}_1) \]
The covariance condition under boosts then assumes the following form:

\[
H_1 \left( |\vec{p}_1| \frac{e^u + e^{-u} z \bar{z}}{1 + z \bar{z}}, |\vec{p}_2| \frac{e^u + e^{-u} z \bar{z}}{1 + z \bar{z}} \right) = \frac{1 + z \bar{z}}{e^u + e^{-u} z \bar{z}} H_1 (|\vec{p}_1|, |\vec{p}_2|)
\]

with general solution

\[
H_1 = f_1 \left( \frac{|\vec{p}_1|}{|\vec{p}_2|} \right) \frac{1}{\sqrt{|\vec{p}_1||\vec{p}_2|}}
\]

Lorentz covariance thus fixes the \((\frac{1}{2}, 0) \otimes (0, \frac{1}{2})\)-differential \(\langle \psi(\vec{p}_1) \psi^+(\vec{p}_2) \rangle\) up to a function \(f_1(|\vec{p}_1|/|\vec{p}_2|)\):

\[
\langle \psi(\vec{p}_1) \psi^+(\vec{p}_2) \rangle = f_1 \left( \frac{|\vec{p}_1|}{|\vec{p}_2|} \right) \frac{1 + z_1 \bar{z}_2}{\sqrt{|\vec{p}_1||\vec{p}_2|}} \delta_{zz}(z_1 - z_2)
\]

\(\square\)

**Determination of the correlation function** \(F_2(z_1, z_2, \bar{z}_1, \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|) = \langle \psi(\vec{p}) \psi^+(\vec{p}') \rangle\):

According to (10) and (15) the covariance conditions for rotations are for \(R(\phi_3 e_3)\):

\[
F_2(e^{-i\phi} z_1, e^{-i\phi} z_2, e^{i\phi} \bar{z}_1, e^{i\phi} \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|) = e^{i\phi} F_2(z_1, z_2, \bar{z}_1, \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|)
\]

and for \(R(\phi e_1)\):

\[
F_2(z'_1, z'_2, \bar{z}'_1, \bar{z}'_2, |\vec{p}_1|, |\vec{p}_2|) = \left( \cos \left( \frac{\phi}{2} \right) - i z_1 \sin \left( \frac{\phi}{2} \right) \right) \left( \cos \left( \frac{\phi}{2} \right) - i z_2 \sin \left( \frac{\phi}{2} \right) \right) F_2(z_1, z_2, \bar{z}_1, \bar{z}_2, |\vec{p}_1|, |\vec{p}_2|)
\]

with

\[
z' = \frac{z \cos \left( \frac{\phi}{2} \right) - i \sin \left( \frac{\phi}{2} \right)}{\cos \left( \frac{\phi}{2} \right) - i \sin \left( \frac{\phi}{2} \right)}
\]

A special solution to these equations is given by \((z_1 - z_2)^{-1}\), and hence the general solution is

\[
F_2 = \frac{1}{z_1 - z_2} G_2 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)}, |\vec{p}_1|, |\vec{p}_2| \right)
\]

Covariance with respect to boosts then requires

\[
G_2 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(e^u + e^{-u} z_1 \bar{z}_1)(e^u + e^{-u} z_2 \bar{z}_2)}, |\vec{p}_1| \frac{e^u + e^{-u} z_1 \bar{z}_1}{1 + z_1 \bar{z}_1}, |\vec{p}_2| \frac{e^u + e^{-u} z_2 \bar{z}_2}{1 + z_2 \bar{z}_2} \right) = G_2 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)}, |\vec{p}_1|, |\vec{p}_2| \right)
\]
so the dependence of the left hand side on the boost parameter must cancel identically, implying

\[ G_2 = G_2 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)} \right) \]

Therefore, the \((\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)\)–differential \(\langle \psi(\vec{p}_1)\phi^+(\vec{p}_2) \rangle\) takes the form

\[ \langle \psi(\vec{p}_1)\phi^+(\vec{p}_2) \rangle = \frac{1}{z_1 - z_2} f_2 \left( \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)} \right) \]  

(58)

Invariance under \(P\) or \(T\) then fixes the remaining 2–point functions

\[ \langle \psi(\vec{p}_1)\phi^+(\vec{p}_2) \rangle = \langle \phi(\vec{p}_2)\psi(\vec{p}_1) \rangle \]

\[ \langle \psi(\vec{p}_1)\psi^+(\vec{p}_2) \rangle = \langle \phi(\vec{p}_2)\phi^+(\vec{p}_1) \rangle \]

thus establishing the result we were seeking.

As expected, Lorentz symmetry alone complies with a bilocal propagator in momentum space, and it restricts the chiral symmetry preserving parts (58) to parallel momenta. On the other hand, chiral symmetry breaking terms must account for breaking of translational invariance. This is in agreement with (58), because these correlation functions cannot accomodate for \(\delta\)–functions preserving the direction of momentum.

The unperturbed result for the on–shell correlation

\[ \langle \psi(\vec{p})\overline{\psi}(\vec{p}') \rangle = -2p \cdot \gamma|\vec{p}|\delta(\vec{p} - \vec{p}') \]

is recovered from Eqs. (48,53,58) for \(f_1(x) = \delta(x - 1), f_2 = 0\).

Off–shell extensions of (48) can be inferred from the requirement to yield the same propagator in configuration space:

\[ S(x, x') = \frac{\Theta(t - t')}{(2\pi)^3} \int \frac{d^3\vec{p}}{2|\vec{p}|} \int \frac{d^3\vec{p}'}{2|\vec{p}'|} \exp(ip \cdot x)i\langle \Psi(\vec{p})\overline{\Psi}(\vec{p}') \rangle \exp(-ip' \cdot x') \]

\[ = \frac{\Theta(t' - t)}{(2\pi)^3} \int \frac{d^3\vec{p}}{2|\vec{p}|} \int \frac{d^3\vec{p}'}{2|\vec{p}'|} \exp(-ip \cdot x)i\langle \Psi(\vec{p})\overline{\Psi}(\vec{p}') \rangle \exp(ip' \cdot x') \]

\[ = \frac{1}{(2\pi)^4} \int d^4p \int d^4p' \exp(ip \cdot x)S(p, p') \exp(-ip' \cdot x') \]  

(59)

thus fixing the structure up to the 2 functions \(f_1, f_2\). Insertion of the unperturbed on–shell correlation \(f_1(x) = \delta(x), f_2 = 0\) yields the free Feynman propagator, of course. However, it is tempting to ask how QCD might account for the chiral symmetry breaking \(f_2\)–terms from a dynamical point of view. Furthermore, low energy QCD should also imply a modification of the \(f_1\)–terms from the standard result, since
confinement seems hardly compatible with momentum conservation on the level of single quark propagators.

**Acknowledgement:** I would like to thank Hermann Nicolai and Julius Wess for helpful discussions at various stages of this work. Support by the DFG is gratefully acknowledged.

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