Compression Limits for Random Vectors with Linearly Parameterized Second-Order Statistics

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Abstract—The class of complex random vectors whose covariance matrix is linearly parameterized by a basis of Hermitian Toeplitz (HT) matrices is considered, and the maximum compression ratios that preserve all second-order information are derived - the statistics of the uncompressed vector must be recoverable from a set of linearly compressed observations. This kind of vectors typically arises when sampling wide-sense stationary random processes and features a number of applications in signal and array processing.

Explicit guidelines to design optimal and nearly optimal schemes operating both in a periodic and non-periodic fashion are provided by considering two of the most common linear compression schemes: non-uniform sampling and random sampling. It is seen that the maximum compression ratios depend on the structure of the HT subspace where the covariance matrix of the uncompressed observations is known to be contained. Compression patterns attaining these maximum ratios are found for the case without structure as well as for the cases with circulant or banded structure.

Index Terms—Compressive Sampling, Covariance Matching, Compression Matrix Design.

I. PRELIMINARIES AND BACKGROUND

Consider the problem of estimating the second-order statistics of a zero-mean random vector \( \mathbf{x} \in \mathbb{C}^L \) from a set of \( K \) linear observations collected in the vector \( \mathbf{y} \in \mathbb{C}^K \) given by

\[
\mathbf{y} = \mathbf{\Phi x},
\]

where \( \mathbf{\Phi} \in \mathbb{C}^{K \times L} \) is a known matrix and several realizations of \( \mathbf{y} \) may be available. Depending on the context, this matrix may be referred to as the compression matrix, measurement matrix or sampling matrix, where compression is achieved since one typically has \( K \ll L \). The second-order statistics of \( \mathbf{x} \) are arranged in its covariance matrix \( \mathbf{\Sigma} \in \mathbb{C}^{L \times L} \), which is known to be a linear combination of the Hermitian Toeplitz (HT) matrices in a set \( \mathcal{S} = \{ \mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_{N-1} \} \subset \mathbb{C}^{L \times L} \), that is, there exist some scalars \( \alpha_s \) such that \( \mathbf{\Sigma} = \sum_s \alpha_s \mathbf{\Sigma}_s \).

The problem introduced above arises in those applications where some inference operation must be done over the second-order statistics of a random vector with a Toeplitz covariance matrix\(^1\). Operating on the compressed observations \( \mathbf{y} \) entails multiple advantages since the dimensions of the vectors to be manipulated may be considerably smaller. In fact, many research efforts in the last decades, especially due to the popularization of compressive sampling (also known as compressed sensing) [1], have been pointed towards designing compression methods (i.e., sampling matrices \( \mathbf{\Phi} \) and reconstruction algorithms that allow for sampling rate reductions. While most of these efforts have been focused on reconstructing \( \mathbf{x} \), there were also important advances when only the second-order statistics of this vector are of interest. This paper is concerned with problems of the second kind.

The compression ratio, defined as the quotient \( \rho = L/K \), measures how much a vector \( \mathbf{x} \) is compressed, and it is desirable that this quantity be as large as possible. The maximum value that this ratio can take on remains an open problem except for a few cases where it was computed from informal or ad-hoc criteria (see Sec. I-D). In this paper, we establish abstract criteria to define the maximum achievable compression ratio and develop a theory that provides these ratios for the most common cases of non-uniform and random sampling, operating both in a periodic or non-periodic fashion. The proofs involved in this theory are constructive, resulting in several methods for designing optimal compression matrices.

A. Covariance Matching Formulation

The prior information restricts the structure of \( \mathbf{\Sigma} \), thus determining how much \( \mathbf{x} \) can be compressed. When no such information is available, \( \mathbf{\Sigma} \) is simply constrained to be Hermitian positive semidefinite and a simple argument relying on degrees of freedom shows that no compression at all is possible. On the other hand, in the typical scenario where \( \mathbf{x} \) contains samples from a wide-sense stationary (WSS) process, the fact that \( \mathbf{\Sigma} \) is HT and positive semidefinite allows for a certain degree of compression. More generally, \( \mathbf{\Sigma} \) may be assumed to lie in the intersection of the cone of positive semidefinite matrices and the subspace determined by the span of a set of HT matrices

\[
\mathcal{S} = \{ \mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_{S-1} \} \subset \mathbb{C}^{L \times L}.
\]

This subspace, which will be throughout referred to as the covariance subspace, accounts for the prior information of the problem. Intuitively, the smaller the dimension of this subspace, the higher the compression that can be reached.

\(^1\)Recall that covariance matrices of complex vectors are always Hermitian.
Without any loss of generality, the field of scalars is restricted to \( \mathbb{R} \), that is,
\[
\Sigma = \sum_{s=0}^{S-1} \alpha_s \Sigma_s, \quad \text{with } \alpha_s \in \mathbb{R},
\]
and \( S \) is assumed to be a linearly independent set of matrices:
\[
\sum_{s=0}^{S-1} \alpha_s \Sigma_s = \sum_{s=0}^{S-1} \beta_s \Sigma_s \quad \Rightarrow \quad \alpha_s = \beta_s \quad \forall s.
\]
Therefore, \( S \) is actually a basis for the covariance subspace, which means that the decomposition in (3) is unique and, consequently, knowing the \( \alpha_s \)'s amounts to knowing \( \Sigma \). Note that, because the coefficients are real-valued and the matrices are Hermitian, it is necessary that \( S \leq 2L - 1 \) in order for \( S \) to be a linearly independent set of matrices. The second-order statistics of \( y \), arranged in \( \Sigma = \mathbb{E} \{ yy^H \} \), and those of \( x \), arranged in \( \Sigma \), are related by:
\[
\Sigma = \Phi \Sigma \Phi^H = \sum_{s=0}^{S-1} \alpha_s \Sigma_s, \quad \text{where } \Sigma_s = \Phi \Sigma_s \Phi^H.
\]
In other words, the expansion coefficients of \( \Sigma \) with respect to \( S \) are those of \( \Sigma \) with respect to the set
\[
\mathcal{S} = \{ \Sigma_0, \Sigma_1, \ldots, \Sigma_{S-1} \} \subset \mathbb{C}^{K \times K}.
\]
The matrices in \( \mathcal{S} \) are still Hermitian, but they do not generally preserve the Toeplitz structure of the matrices in \( \Sigma \). If the compression operation preserves all relevant information, then the set \( \mathcal{S} \) will still be linearly independent and the decomposition in (5) unique, which means that the knowledge of \( \Sigma \) is equivalent to the knowledge of the \( \alpha_s \)'s, which in turn amounts to knowing \( \Sigma \). Conversely, if the compression is so strong that linear independence is lost, then the decomposition in (5) is no longer unique and some second-order information about \( x \) cannot be recovered.

This paper unifies the treatment of a number of problems arising in different applications (see Sec. I-C) by noting that they can be stated as estimating a linearly parameterized covariance matrix \( \Sigma \) from the compressed observations \( y \). This is a particular instance of the well-known covariance matching problem\(^3\) [3]. For simplicity, a linear parameterization such as the one in (5) is assumed, but the results still apply to non-linear parameterizations [3] provided that the set \( \mathcal{S} \) is carefully chosen (see the discussion around Lemma 1).

B. Relation to Compressive Sampling

As in the above problem, in compressive sampling (CS) signals are acquired by linearly projecting the vector under analysis onto a particular subspace whose dimension is less than that of the observation space. This dimensionality reduction decreases the cost of the acquisition system and, sometimes, the cost of the subsequent processing. As opposed to CS, where the interest is to reduce the dimension of the observations while preserving all the information about the signal itself, the problem considered in this paper deals with performing some kind of inference operation over the statistics of \( x \), and recovering this vector is, in principle, unnecessary. For this reason, we will globally refer to this problem as compressive covariance sampling (CCS). The terms compress and sample will be used interchangeably throughout, as well as compression patterns and samplers.

An important similarity between CS and CCS lies in the compression methods utilized, which affects the nature of the matrix \( \Phi \). This paper considers two of these methods:

- **Non-uniform sampling** [4]: \( \Phi \) is composed of \( K \) different rows of the identity matrix \( I_L \), thus performing a component selection of \( x \). If \( K = \{ l_0, \ldots, l_{K-1} \} \) denotes the set containing the indices of the non-null columns of \( \Phi \), the entries of \( y = \Phi x \) are given by \( y_k = x_{l_k}, \quad l_k \in K \), where \( x = [x_0, \ldots, x_{L-1}]^T \) and \( y = [y_0, \ldots, y_{K-1}]^T \). When \( K \) is periodic (see Sec. IV-A for a formal definition), this particular sampling method is known as multi-coset sampling.

- **Random sampling**: Designing sampling matrices in CS or CCS is rather involved due to the nature of the design criteria (see e.g. [1], [5]). Fortunately, in certain cases, matrices generated at random satisfy these criteria with probability one provided that the dimensions are properly chosen. This paper considers periodic and non-periodic samplers generated by continuous distributions.

On the other hand, important differences exist between CS and CCS as to how the observations are processed:

- **CS** operates under a lossless paradigm: the dimensionality is reduced without any loss of information, i.e., \( x \) must be recoverable\(^4\) from \( y \). Since the dimension of the latter is less than the dimension of the former, certain structure must be assumed, e.g., \( x \) may be assumed to allow a sparse representation in terms of some basis of \( \mathbb{C}^L \) [1].

- In contrast, CCS is inherently lossy: since the goal of CCS is estimating the second-order statistics of a random vector, any dimensionality reduction will, in general, lead to a performance degradation. Compression is possible thanks to the redundancy in the second-order statistics, but no sparsity is required neither in \( x \) nor in \( \Sigma \).

C. Applications of CCS

We next summarize several problems that can be formulated using CCS. In each application, the covariance subspace is determined by the prior information available. The most common examples, defined in Sec. II-B, are the Toeplitz subspace, the circulant subspace and the \( d \)-banded subspace.

1) Analog-to-Information Conversion: Continuous-time signals are traditionally converted to discrete time by analog-to-digital converters (ADCs), which sample the input analog process \( x(t) \) at uniformly-spaced time instants, i.e., if we arrange the samples in the vector \( x \), then \( x_l = x(l/f_0) \), where \( f_0 \) denotes the sampling frequency. The minimum value for \( f_0 \) in the classical setting is the Nyquist frequency, which may be too demanding in some cases. For this reason, analog-to-information converters (AICs) [6]–[9] have been recently

\(^2\)This is highly convenient since the matrices are Hermitian.
\(^3\)Sometimes also known as structured covariance estimation [2].
\(^4\)Sometimes a small error may be allowed due to measurement noise.
proposed to acquire analog signals at rates below the Nyquist frequency. These signals must be somehow compressible, e.g., they can be sparse in some domain (CS approach) or a realization of a WSS process (CCS approach). AICs obtain a compressed digital representation of the input process: 
\[ x(t) \] directly observed via \( y \), without computing \( x = [x(0), x(1/f_0), \ldots, x((L-1)/f_0)]^T \). In CS, \( x \) is subsequently recovered, whereas in CCS, its statistics are estimated. Several applications of CCS using AICs are next reviewed.

- Compressive Power Spectrum Estimation: The goal is to estimate \( \Sigma \) from \( y \) with the only constraint that it must be HT and positive semidefinite, which means that the covariance subspace is the Toeplitz subspace. If the acquisition time \( L/f_0 \) is greater than the length of the autocorrelation function of \( x(t) \), then a \( d \)-banded covariance subspace may be used [10]. A circulant covariance subspace may also result by stating the problem in the frequency domain [11].

- Wideband Spectrum Sensing [12]: Applications such as dynamic spectrum sharing [13] (sometimes known as cognitive radio [14]) may require to monitor the power of different transmitters operating on wide frequency bands. An AIC may be used to sample the received signal, which is the sum of the signals received from each station. Therefore, \( y = \sum s \sigma_s y_s \), where \( \sigma_s y_s \) is the output of the AIC when only the \( s \)-th transmitter is active. If \( y_s \) is power normalized, \( \sigma_s^2 \in \mathbb{R} \) is the power of the \( s \)-th transmitter. The second-order statistics of \( y_s \), collected in \( \Sigma_s = E \{y_s y_s^H\} \), are typically known [12], [15], [16]. Writing (5) with \( \alpha_s = \sigma_s^2 \) results in a CCS problem where the covariance space is determined by the \( \Sigma_s \)’s.

- Modal analysis: AICs can be used to identify sinusoids in wideband signals [17]. The uncompressed signal samples are modeled as \( x_l = \sum_{s=1}^{S-1} A_s v_s, l + w_l \), where \( A_s \in \mathbb{C} \) is unknown with random phase, \( w_l \) is noise, and \( v_{s,l} = e^{j \omega_s l} \) is a complex exponential whose frequency \( \omega_s \) is to be estimated. Therefore, one has that \( \Sigma = \sum_{s=1}^{S-1} A_s^2 \Sigma_s \), where \( |A_s|^2 \Sigma_0 \) is the noise covariance matrix, and \( |A_s|^2 \Sigma_s, s > 0 \), is the covariance matrix of \( A_s v_{s,l} \). Since the \( \omega_s \)’s are unknown, so are the \( \Sigma_s \)’s, which means that the compression matrix \( \Phi \) must preserve the structure of any covariance subspace, i.e., \( \Phi \) must be universal (See Sec. II). Some methods to compute the \( \omega_s \)’s operate directly on the estimated \( \Sigma \) [17], but any other subspace method [20] might be applied over the reconstructed \( \Sigma \).

2) Array processing: In applications requiring estimates of the so-called angular spectrum (radar, astronomy, localization, etc.), compression may be introduced to reduce the cost of the hardware. The observations are given by \( y = \Phi x \), where

3However, it may be a convenient abstraction to think that the AIC is internally composed of a Nyquist-rate ADC that acquires \( x \) followed by a compression stage to obtain \( y \) (see e.g. [9]).

4Note that the decomposition of \( \Sigma \) in terms of a fine grid of known frequencies can be used to estimate the \( \omega_s \)’s from \( \Sigma \) or \( \Sigma_s \) using algorithms such as those in [18], [19], but cannot be used to design a sampler since the associated matrices do not constitute an independent set.

5See the discussion in Sec. VII.
the conditions for unique reconstruction of a least-squares algorithm. Suboptimal compression schemes are also provided in [10] based on the minimal sparse ruler problem, and in [44] using coprime difference bases. The case of non-periodic non-uniform sampling in circulant covariance subspaces was considered in [35] and [11], where optimal and suboptimal designs are respectively found based on specific algorithms.\footnote{The initial statement in [11], [35] uses periodic sampling, but their considerations in the frequency domain lead to non-periodic sampling.}

All the works above rely on criteria tailored to particular algorithms. Furthermore, the common CCS nature underlying these problems is not noticed from their formulation. One primary goal of the present paper is to unify the treatment of these problems under abstract criteria, i.e., irrespective of any algorithm. Our contribution is twofold: first, we set a formal framework where the maximum compression ratio is defined and establish the conditions for a compression pattern to be declared admissible. Second, we derive the maximum compression ratios and optimal compression patterns for the cases where they are unknown.

To close this section, we mention that related compression methods have been provided in other contexts, such as conventional CS, where the design hinges on completely different criteria and leads to completely different results (see e.g. [1], [5], [30], [31], [34], [36], [37], [45]).

E. Notation

If a set $A$ is finite, then $|A|$ denotes its cardinality. If $F$ is a field, then the $F$-span of a set of matrices $A$, denoted as $\text{span}_F A$, is defined as the set of all linear combinations with coefficients in $F$ of those matrices. For example, if $A$ is countable, then $\text{span}_F A = \{ A = \sum_i \alpha_i A_i : A_i \in A, \alpha_i \in F \}$. The $F$-dimension of a set $B$ is the smallest $n \in \mathbb{N}$ such that there exists some $A$ with $|A| = n$ such that $B \subset \text{span}_F A$, and it is denoted as $\dim_F B$. The image of a set $A$ through a function $\phi$ is denoted as $\phi(A)$.

As for matrix notation, lowercase is used for scalars, bold lowercase for vectors and bold capital for matrices. Superscript $T$ stands for transpose, $H$ for conjugate transpose and $\otimes$ represents the Kronecker product [46]. The $P \times Q$ matrix $A$ is indexed as:

$$\begin{bmatrix}
a_{0,0} & a_{0,1} & \cdots & a_{0,Q-1}
\vdots & \vdots & \ddots & \vdots 
a_{P-1,0} & a_{P-1,1} & \cdots & a_{P-1,Q-1}
\end{bmatrix}. \quad (7)$$

The vectorization of $A$ is the vector $\text{vec}(A) = [a_0^T, \cdots, a_{Q-1}^T]^T$, where $a_j = [a_{0,j}, \cdots, a_{P-1,j}]^T$. The $d$-th diagonal refers to the entries $(i,j)$ with $j-i=d$, where $d$ is a negative, null or positive integer. $E_{i,j}$ is a matrix with all zeros except for a 1 at the position $(i,j)$ and it is represented as $e_i$ if it has a single column.

The symbol $\bar{j}$ denotes the imaginary unit and $(x)_N$ is the remainder of the integer division of $x \in \mathbb{Z}$ by $N$, i.e., $(x)_N$ is the only element in the set $\{x+bN, b \in \mathbb{Z}\} \cap \{0, \ldots, N-1\}$.

F. Paper Structure

The rest of the paper is structured as follows. Sec. II sets the theoretical background, where maximum compression ratios and covariance samplers are defined. Sec. III presents some results to design covariance samplers, which are applied in Secs. IV and V for universal and non-universal covariance samplers, respectively. Asymptotic compression ratios are discussed in Sec. VI, whereas some final remarks and conclusions are respectively provided in Secs. VII and VIII.

II. Theoretical Framework

The definition of the maximum compression ratio requires to first determine which sampling matrices (or samplers) we are willing to accept. As explained in Sec. I, we are interested in those samplers preserving all the second-order statistical information of $x$, i.e., those samples that allow to recover the statistics of $x$ from the statistics of $y$. In order to formalize this notion, we start by associating the compression (or sampling) matrix $\Phi \in \mathbb{C}^{K \times L}$ with a function which relates the covariance matrices of $x$ and $y$ and which is defined as

$$\begin{array}{c}
\text{span}_R \Sigma \quad \xrightarrow{\phi} \quad \text{span}_R \bar{S} \\
\Sigma \quad \xrightarrow{\phi(\Sigma)} \quad \Phi \Sigma \Phi^H
\end{array} \quad (8)$$

where, recall, $S$ is a linearly independent set of $S$ HT matrices whose span, denoted as span$_R S$, is the covariance subspace. We next specify which sampling matrices are admissible:

Definition 1: A matrix $\Phi$ defines an $S$-covariance sampler\footnote{When the set $S$ is clear from the context, we will simply say that $\Phi$ defines a covariance sampler.} if the associated function $\phi$, defined in (8), is invertible.

The maximum compression ratio is the largest value of $L/K$ for which a covariance sampler $\Phi \in \mathbb{C}^{K \times L}$ can be found. Above this value, it is not possible to consistently estimate the second-order statistics of $x$, even from an arbitrarily large number of realizations of $y$, since the statistical identifiability\footnote{See [47] for a discussion of statistical identifiability in CCS.} of $S$ is lost [48]. For convenience, we will regard $L$ as given and attempt to minimize $K$. One may argue that Definition 1 does not take into account the prior knowledge that $\Sigma$ must be positive semidefinite, i.e., it may not be necessary to require that $\phi$ be invertible for all matrices in span$_R S$ but just for those which are positive semidefinite. The next lemma shows that, under certain regularity conditions satisfied in most cases, this fact can be ignored.

Lemma 1: Let $\phi$ be the function defined in (8), where $S$ is an independent set of $S$ HT matrices, let $A$ be a set of matrices such that $\dim_R [A \cap \text{span}_R S] = S$ and let $\phi|_A$ be the restriction of $\phi$ to $A \cap \text{span}_R S$, defined as:

$$\begin{array}{c}
A \cap \text{span}_R S \quad \xrightarrow{\phi|_A} \quad \phi(A \cap \text{span}_R S) \\
\Sigma \quad \xrightarrow{\phi(\Sigma)} \quad \phi(\Sigma).
\end{array} \quad (9)$$

Then, $\phi$ is invertible if and only if $\phi|_A$ is invertible.

Proof: See Appendix A.\hfill $\blacksquare$

Thus, if for $A$ the cone of positive semidefinite matrices, $S$ satisfies that $\dim_R [A \cap \text{span}_R S] = S$, then we can
disregard the fact that the covariance matrix of \( x \) is positive semidefinite. On the other hand, if \( \dim_{\mathbb{R}}[\mathcal{A} \cap \text{span}_{\mathbb{R}} \mathcal{S}] < S \), then some of the dimensions of the covariance space contain no positive semidefinite matrices, which means that it is possible to obtain a smaller basis \( S' \) of cardinality \( S' < S \) that generates all feasible covariance matrices of \( x \) and satisfies \( \dim_{\mathbb{R}}[\mathcal{A} \cap \text{span}_{\mathbb{R}} S'] = S' \). Therefore, we incur no loss of generality by assuming that \( \mathcal{S} \) satisfies this condition and dismissing the consideration that \( \Sigma \) is positive semidefinite. Moreover, it can be verified that this is actually the case for all the bases in this paper, as well as those in the applications of Sec. I-C. Similarly, other choices of \( \mathcal{A} \) allow to generalize the results in this paper to CCS problems where \( \Sigma \) is non-linearly parameterized [3].

Clearly, a matrix \( \Phi \) may define a covariance sampler for certain sets \( \mathcal{S} \) but not for others. If a matrix \( \Phi \) is a covariance sampler for any choice of \( \mathcal{S} \), we call it universal:

**Definition 2:** A sampling matrix \( \Phi \in \mathbb{C}^{K \times L} \) defines a universal covariance sampler if it is an \( \mathcal{S} \)-covariance sampler for any linearly independent set \( \mathcal{S} \) of \( L \times L \) HT matrices.

Knowing \( \mathcal{S} \) is always beneficial since \( \Phi \) may be tailored to obtain optimal compression ratios and estimation performance. Universal samplers are motivated by those cases where \( \mathcal{S} \) is unknown at the moment of designing the compression matrix.\(^{11}\) Note that the same concept has been defined in the context of exact-reconstruction of multi-band signals [36] (see [37] for a CS perspective). However, the underlying theory in that framework is completely different from the one considered here.

### A. Interpretation

Due to the definition of domain and codomain in (8), \( \phi \) clearly represents a surjective map. Therefore, the notion of invertibility actually means that \( \phi \) must be injective, that is, for any set of real coefficients \( \alpha_s \) and \( \beta_s \),

\[
\phi \left( \sum_s \alpha_s \Sigma_s \right) = \phi \left( \sum_s \beta_s \Sigma_s \right) \Rightarrow \alpha_s = \beta_s \forall s. \quad (10)
\]

This condition is, in turn, equivalent to

\[
\sum_s \alpha_s \Sigma_s = \sum_s \beta_s \Sigma_s \Rightarrow \alpha_s = \beta_s \forall s, \quad (11)
\]

which means that \( \mathcal{S} \) must be linearly independent. Thus, determining whether a given matrix \( \Phi \) defines an \( \mathcal{S} \)-covariance sampler amounts to checking whether \( \mathcal{S} = \phi(\mathcal{S}) \) is linearly independent or not. Alternatively, (11) states that no two different linear combinations of the matrices in \( \mathcal{S} \) can result in the same \( \Sigma \), which means that covariance samplers can also be defined as those samplers preserving the identifiability of the \( \alpha_s \) coefficients.

To the best of our knowledge, Definition 1 is the first attempt to formalize the design of samplers for CCS problems using abstract criteria not depending on specific algorithms. In the sequel, several results will be established to determine whether a matrix defines a covariance sampler or, in some cases, even a universal covariance sampler.

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\(^{11}\)Even the dimension of the covariance subspace may be unknown.

### B. Notable Covariance Subspaces

The results about covariance samplers derived in this paper will be particularized in Sec. V for the most common covariance subspaces, which are defined next:

1) **Toeplitz Subspace:** A matrix is Toeplitz if it is constant along its diagonals [49]. The set of all \( L \times L \) HT matrices, represented as \( \mathcal{S}_L \), is a subspace of \( \mathbb{C}^{L \times L} \) over the real scalar field,\(^{13}\) and it is the largest subspace considered in this paper. The standard basis of \( \mathcal{S}_L \) is defined as the set

\[
\mathcal{S}_T = \{I_L\} \cup \{T_1, \cdots, T_{L-1}\} \cup \{\bar{T}_1, \cdots, \bar{T}_{L-1}\}, \quad (12)
\]

where \( T_l \) denotes the HT matrix with all zeros but the entries on the diagonals \( +l \) and \( -l \), which have ones, and \( \bar{T}_l \) represents the HT matrix with all zeros but the entries on the diagonal \( +l \), which have the imaginary unit \( j \), and those on the diagonal \( -l \), which have \( -j \). Formally,

\[
T_l = J_l^T + (J_l^T)^T \quad l \geq 1 \quad (13)
\]

\[
\bar{T}_l = j J_l^T - (J_l^T)^T \quad l \geq 1, \quad (14)
\]

where \( J_l \) is the first linear shift of \( I_L \) to the right, i.e., the matrix whose \((m,n)\) element is one if \( n - m = 1 \) and zero otherwise. The basis \( \mathcal{S}_T \) shows that \( \dim_{\mathbb{R}} \mathcal{S}_T = 2L - 1 \).

2) **Circulant Subspace:** In some cases, for example when the covariance matrix of \( x \) is diagonal in the frequency domain\(^{13}\), the matrix \( \Sigma \) is circulant. A circulant matrix [49] is a matrix whose \( n \)-th row equals the \( n \)-th circular rotation of the zeroth row\(^{14}\). In other words, the element \((m,n)\) equals the element \((m',n')\) if \((m-n)_L = (m'-n')_L \). In our case, the matrices in the circulant subspace must be HT and circulant simultaneously. A possible basis for \( L \) odd is

\[
\mathcal{S}_C = \{I_L\} \cup \{C_1, \cdots, C_{(L-1)/2}\} \cup \{\bar{C}_1, \cdots, \bar{C}_{(L-1)/2}\}, \quad (15)
\]

where

\[
C_l = T_l + T_{L-l}, \quad l = 1, \ldots, [(L - 1)/2] \quad (16)
\]

\[
\bar{C}_l = T_l - T_{L-l}, \quad l = 1, \ldots, [(L - 1)/2], \quad (17)
\]

and

\[
\mathcal{S}_C = \{I_L\} \cup \{C_1, \cdots, C_{(L-1)/2}\} \cup \{\bar{C}_1, \cdots, \bar{C}_{(L-1)/2}\} \cup \{T_{(L-1)/2}\}
\]

for \( L \) even. Clearly, the dimension of this subspace equals \( L \).

3) **d-banded Subspace:** Banded covariance matrices may be considered when the autocorrelation sequence of a stationary process vanishes for large time lags \([10]\). A \( d \)-banded matrix is a matrix where all the elements above the diagonal \(+d\) and below the diagonal \(-d\) (these diagonals noninclusive) are zero. A possible basis for this subspace is given by

\[
\mathcal{S}_B^d = \{I_L\} \cup \{T_1, \cdots, T_d\} \cup \{\bar{T}_1, \cdots, \bar{T}_d\}, \quad (18)
\]

which is a subset of \( \mathcal{S}_T \). The dimension is therefore \( 2d + 1 \).

\(^{12}\)The reason is that any linear combination with real coefficients of HT matrices is also HT. This statement is false for complex coefficients.

\(^{13}\)This situation typically stems from a polyphase interpretation of the AIC in the frequency domain \([11],[35]\).

\(^{14}\)Recall the conventions introduced in Sec. I-E.
C. The Role of Periodicity

The fact that many sampling schemes operate repeatedly on a block-by-block basis leads to the concept of periodicity (see Sec. I-C). Note, however, that the subsequent processing may consider multiple blocks simultaneously. Assume that \( x \) is partitioned into \( B \) blocks of \( N = L/B \) samples\(^{15}\) as \( x = [x[0]^T, \ldots, x[B-1]^T]^T \) with \( x[b] \in \mathbb{C}^N \forall b \) and that sampling a block with \( N \) elements results in another block with \( M \) elements:

\[
y[b] = \Phi x[b], \quad b = 0, 1, \ldots, B-1,
\]

where \( y[b] \in \mathbb{C}^M \) and \( \Phi \in \mathbb{C}^{M \times N} \). The use of the term periodicity owes to the fact that the matrix \( \Phi \) does not depend on \( b \) (otherwise we could just speak about block processing).

By making \( y = [y[0]^T, \ldots, y[B-1]^T]^T \) and \( \Phi = I_B \otimes \Phi \), expression (19) results in (1). From (20), it also follows that the matrices in \( \hat{\mathcal{S}} \) are block Toeplitz with \( M \times M \) blocks.

Since \( K = MB \), the compression ratio in the periodic setting takes the form

\[
\rho = \frac{L}{K} = \frac{N}{M}.
\]

Furthermore conventions are useful when dealing with non-uniform sampling, in which case, as seen in Sec. I-B, \( \Phi \) equals a submatrix of \( I_L \) up to row permutations. For concreteness, assume that the rows of \( \Phi \) are ordered as they are in \( I_L \). The sets \( \mathcal{K} \) and \( \mathcal{M} \), which respectively contain the indices of the non-null columns of \( \Phi \) and \( \Phi \), are related by:

\[
\mathcal{K} = \{m + bN, \ m \in \mathcal{M}, b = 0, 1, \ldots, B-1\},
\]

and satisfy \( |\mathcal{K}| = K = MB \) and \( |\mathcal{M}| = M \). Loosely speaking, we say that \( \mathcal{K} \) is periodic with period \( \mathcal{M} \).

Finally, note that the periodic setting is indeed a generalization of the non-periodic one, since the non-periodic case can be retrieved just by making \( B = 1 \). For this reason, most results will be presented for periodic samplers, with occasional comments on the non-periodic setting if needed.

III. Design of Covariance Samplers

The results in this section allow to determine whether a matrix \( \Phi \) defines a covariance sampler or not, and provide useful means to design these matrices for a given \( \mathcal{S} \). They are based on the following basic result from linear algebra:

**Lemma 2:** Let \( \mathcal{S} = \{\mathcal{S}_0, \cdots, \mathcal{S}_{S-1}\} \) be a set of Hermitian matrices. If \( \mathcal{S} \) is linearly independent when considering real coefficients, that is,

\[
\sum_{s=0}^{S-1} \alpha_s \mathcal{S}_s = 0, \quad \alpha_s \in \mathbb{R} \Rightarrow \alpha_s = 0 \ \forall s,
\]

then it is also independent when considering coefficients in \( \mathbb{C} \), i.e., (23) also applies when \( \alpha_s \in \mathbb{C} \).

**Proof:** It easily follows by combining expression (23) with the fact that \( \Sigma_s = \Sigma_s^H \), \( \forall s \).

Observe that Lemma 2 would not apply if the matrices in \( \mathcal{S} \) were not Hermitian. The importance of Lemma 2 is that it allows us to focus on the complex extension of \( \phi \), defined as

\[
\begin{array}{ccc}
\text{span} \mathcal{S} & \xrightarrow{\phi_{\mathcal{C}}} & \text{span} \hat{\mathcal{S}} \\
\Sigma & \xrightarrow{\phi_{\mathcal{C}}} & \Phi \Sigma \Phi^H.
\end{array}
\]

In other words, \( \Phi \) defines a covariance sampler iff \( \phi_{\mathcal{C}} \) is an invertible function. An equivalent statement is provided by the following lemma, which is the basic tool to be used in the design of covariance samplers.

**Lemma 3:** Let \( \ker \phi_{\mathcal{C}} \) denote the set of matrices \( \Sigma \in \text{span}_{\mathcal{C}} \mathcal{S} \) satisfying \( \phi_{\mathcal{C}}(\Sigma) = 0 \). Then, a matrix \( \Phi \) defines a covariance sampler if and only if \( \ker \phi_{\mathcal{C}} = \{0\} \).

**Proof:** It is an immediate consequence of Definition 1 and Lemma 2.

A. Design of Non-Uniform Samplers

Designing covariance samplers using non-uniform sampling involves manipulating difference sets, which contain all possible distances between elements of another set:

**Definition 3:** The difference set of \( \mathcal{A} \subset \mathbb{Z} \), denoted as \( \Delta(\mathcal{A}) \), is defined as:

\[
\Delta(\mathcal{A}) = \{\delta \geq 0 : \exists a_1, a_2 \in \mathcal{A} \text{ s.t. } \delta = a_2 - a_1\}.
\]

This set operation\(^{17}\) is equivalent to a morphological dilation of \( \mathcal{A} \) by itself followed by intersection with the set of non-negative integers [50]. Note that the difference set considers no repetition of elements, i.e., every distance shows up at most once. The cardinality of \( \Delta(\mathcal{A}) \) is upper bounded by one plus the number of unordered subsets of \( \mathcal{A} \) with two elements:

\[
|\Delta(\mathcal{A})| \leq \frac{|\mathcal{A}| \cdot (|\mathcal{A}| - 1)}{2} + 1,
\]

where the +1 term accounts for the fact that 0 \( \in \Delta(\mathcal{A}) \) for any non-empty \( \mathcal{A} \). Examples of sets achieving equality can be found in [29, Theorem 1].

The correlation vector \( \sigma_s \) associated with the HT matrix \( \Sigma_s \) is defined as the first column of \( \Sigma_s \). The following theorem is a quick method to verify whether a non-uniform sampler defined by a set \( \mathcal{K} \) is a covariance sampler:

**Theorem 1:** Let \( \mathcal{S} = \{\Sigma_0, \Sigma_1, \cdots, \Sigma_{S-1}\} \) be a linearly independent set of HT matrices, let \( \{\sigma_s\}_{s=0}^{S-1} \) be the associated set of correlation vectors, and let \( \tilde{\sigma}_s \) be the vector whose entries are those of \( \sigma_s \) indexed by \( \Delta(\mathcal{K}) \). Then, \( \mathcal{K} \) defines an \( \mathcal{S} \)-covariance sampler if and only if rank \( \mathcal{R} = \text{rank} \mathcal{S} \), where

\[
\mathcal{R} = \begin{bmatrix}
\sigma_0 & \sigma_1 & \cdots & \sigma_{S-1} \\
\sigma_0^* & \sigma_1^* & \cdots & \sigma_{S-1}^*
\end{bmatrix}.
\]

**Proof:** Observe that \( \Sigma_s \) contains an element from the \( \delta \)-th diagonal of \( \Sigma_s \) iff \( |\delta| \in \Delta(\mathcal{K}) \). Now vectorize the matrices in \( \hat{\mathcal{S}} \) and arrange these vectors as columns of a matrix. By removing repeated rows and duplicating the row corresponding

\(^{15}\)The elements of \( x \) and \( y \) will also be referred to as samples, due to the connection of this problem with CS.

\(^{16}\)For simplicity, we assume that \( L \) is an integer multiple of \( B \).

\(^{17}\)The concept of difference set is similar to the difference coarray used in the context of aperture synthesis for incoherent imaging [21].
to the main diagonal we obtain $R$. Therefore, the number of linearly independent columns in $R$ equals the number of linearly independent matrices in $S$. The result follows from Lemma 3 by noting that $\ker \phi_C = \{0\}$ if $\rank R = S$. 

From (27), it is easy to conclude\(^\ddagger\) that $|\Delta(K)| - 1 \geq S$ in order for $R$ to be full column rank. Combining this expression with (26) results in the following necessary condition for $K$ to define a covariance sampler:

$$ K + (K - 1) + 1 \geq S. \quad (28) $$

### B. Design of Random Samplers

Random sampling embodies the idea of incoherence used in conventional CS [1] and allows to obtain sampling matrices with a good behavior without considering any structure of the covariance subspace other than its dimension. The next result, which is one of our major contributions, basically establishes the minimum size of a random matrix $\Phi$ to define a covariance sampler. The only requirement is that this matrix be drawn from a continuous probability distribution.

**Theorem 2:** Let $(\mathbb{C}^{M \times N}, \mathcal{F}, \mu)$ be a probability space over $\mathbb{C}^{M \times N}$, where $\mathcal{F}$ denotes the $\sigma$-field of Borel sets in $\mathbb{C}^{M \times N}$ and $\mu$ is a probability measure absolutely continuous with respect to the Lebesgue measure [51]. If $\Phi \in \mathbb{C}^{M \times N}$, with $M \leq N$, is a random matrix defined on this space, then, with probability one, the matrix $\Phi = I_B \otimes \Phi$ defines an $S$-covariance sampler if and only if $S \leq M^2(2B - 1)$, where $S$ is the cardinality of the HT basis set $S$.

**Proof:** See Appendix B. 

Note that the requirements imposed by Theorem 2 on $\mu$ are much weaker than those in CS for continuous distributions, where reconstruction results exist just for distributions satisfying certain measure concentration inequalities [5]. Moreover, Theorem 2 does not even require the elements of $\Phi$ to be i.i.d., which is a standard requirement in CS. On the other hand, Theorem 2 does not apply to the case of discrete distributions such as the Bernoulli distribution, which is widely used in CS.

### IV. Universal Covariance Samplers

After having laid the mathematical framework, we are ready to provide designs that result in covariance samplers independently of which set of HT matrices $S$ is considered. Maximum compression ratios will follow as a byproduct of this approach. The first result of this section reduces the task of checking whether a given matrix defines a covariance sampler for all possible basis sets $S$ to that of checking whether it defines a covariance sampler for just one of those sets. Before proceeding, recall from Sec. II-B that $S^L$ denotes the subspace of all $L \times L$ HT matrices.

**Lemma 4:** Let $S$ be a basis for $S^L$. Then, a sampler $\Phi$ is universal if and only if it is an $S$-covariance sampler.

**Proof:** Clearly, if $\Phi$ is universal, it is also an $S$-covariance sampler. Conversely, if $\Phi$ is an $S$-covariance sampler, it means that $\phi$ is an injective map when its domain is $\text{span}_B S = S^L$. Since $\text{span}_N S' \subset S^L \forall S'$, considering $\Phi$ along with any other set of matrices results in a function $\phi$ which is a restriction of an injective function, and therefore it is also injective. Hence, $\Phi$ is an $S'$-covariance sampler for all linearly independent sets of HT matrices $S'$.

In other words, Lemma 4 establishes the equivalence between the following two statements:

- $\Phi$ is a universal covariance sampler
- $\Phi$ is a covariance sampler for the Toeplitz subspace.

The rest of this section applies this lemma to obtain universal covariance samplers using non-uniform sampling and random sampling. The main result in the former case is that the period $M$ of $K$ must be a sparse ruler, which is a well-known mathematical object reviewed below. In the latter case, the conclusions are similar to those from Sec. III in the sense that the universality of a sampler is guaranteed with probability one if the dimensions of $\Phi$ are properly set.

#### A. Non-Uniform Sampling

The next necessary and sufficient condition for a non-uniform sampler to be universal basically states that all lags of the autocorrelation function must be identifiable from the compressed observations.

**Theorem 3:** The set $K \subset \{0, \ldots, L - 1\}$ defines a universal covariance sampler if and only if $\Delta(K) = \{0, \ldots, L - 1\}$.

**Proof:** Consider the Toeplitz covariance subspace with basis $S_T$ given by (12) and assume that $\Delta(K) = \{0, \ldots, L - 1\}$. In that case, the matrix $R$ in Theorem 1 becomes

$$ R = \begin{bmatrix} I_L & -jI_L \\ jI_L & jI_L \end{bmatrix}, \quad (29) $$

where $I_L$ is the submatrix of $I_L$ that results from removing the first column. Clearly, $R$ has rank $2L - 1$ so that, according to Theorem 1, $K$ defines an $S_T$-covariance sampler.

Now suppose that one or more elements of $\{0, 1, \ldots, L - 1\}$ are missing in $\Delta(K)$. In that case, at least two of the rows of $R$ above are missing, which, in turn, means that $\rank R < 2L - 1$. Therefore, $\rank R = 2L - 1$ if and only if $\{0, 1, \ldots, L - 1\} \subset \Delta(K)$. From this conclusion and Theorem 1, it follows that $K$ defines an $S_T$-covariance sampler iff $\Delta(K) = \{0, 1, \ldots, L - 1\}$. Applying Lemma 4 concludes the proof.

This theorem provides a very simple means to check whether $K$ is universal or not. Interestingly, this is closely related to the classical problem in number theory known as the sparse ruler problem, or as the representation of integers by difference bases (see [39], [52] and references therein). Its application to array processing dates back to the 60’s [22].

**Definition 4:** A length-$L$ (linear) sparse ruler is a set $K \subset \{0, 1, \ldots, L - 1\}$ satisfying $\Delta(K) = \{0, 1, \ldots, L - 1\}$. It is called minimal if there exists no other length-$(L - 1)$ sparse ruler with smaller cardinality.

Intuitively, we may associate this set with a classical ruler (the physical object) with some marks erased, which is still capable of measuring all integer distances between 0 and its length by considering pairs of marks. Two examples of minimal sparse rulers are shown in Fig. 1, where red dots correspond to the marks of the ruler that have not been erased. Sparse rulers exist for all $L$, although they are not necessarily

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\(^\ddagger\)Note the existence of a duplicate row in $R$. 

---
unique. For instance, two different length-10 sparse rulers are \{0, 1, 2, 3, 6, 10\} and \{0, 1, 2, 5, 7, 10\}. The most remarkable properties of a length-(L−1) sparse ruler are that the endpoints are always present, i.e., \{0, L−1\} ∈ \mathcal{K}, and that its reflection \((L−1)−\mathcal{K}=\{(L−1)−k : k ∈ \mathcal{K}\}\) is also a sparse ruler. Trivially, if \mathcal{K}\ is minimal, then \((L−1)−\mathcal{K}\) is also minimal. Therefore, (minimal) sparse rulers exist at least in pairs unless \mathcal{K}=\{(L−1)−K\}. The cardinality \(K = |\mathcal{K}|\) of a minimal sparse ruler is lower bounded as

\[
K \geq \frac{1}{2} + \sqrt{2(L−1) + \frac{1}{4}},
\]

which follows directly from (26) and is only attained for \(L−1 = 0, 1, 3\) and \(6\) (see e.g. [41]), or as (see [38], [39]):

\[
K \geq \sqrt{\tau(L−1)},
\]

where \(\tau = \max_0 2(1−\frac{\sin \theta}{\theta}) \approx 2.4345\); and upper bounded by [23]:

\[
K \leq \left[ \sqrt{3(L−1)} \right], \quad L−1 \geq 3.\tag{32}
\]

Thus, in the non-periodic case \((B=1)\), Theorem 3 reduces our design problem to finding a length-(L−1) sparse ruler, for which design algorithms abound. A trivial example is \(\{0, \ldots, L−1\}\), which clearly represents a universal sampler since in that case \(y = x\). More sophisticated constructions were discussed in [17], [23], [29], [38]–[42]. However, if the compression ratio is to be maximized, then one should look for a minimal sparse ruler, which is an exhaustive-search problem. Fortunately, there exist tables for values of \(L−1\) up to the order of 100. Although higher values of this parameter demand, in principle, intensive computation, one may resort to the designs in [23], [40], [41], which provide nearly minimal rulers despite being really simple.\(^19\)

On the other hand, it is not clear how to design sampling patterns in the periodic case \((B>1)\) since periodicity needs to be enforced on \(\mathcal{K}\). Before that, the next definition is required.

**Definition 5:** A length-(L−1) periodic sparse ruler of period \(N\), where \(N\) divides \(L\), is a set \(\mathcal{K} \subset \{0, 1, \ldots, L−1\}\) satisfying two conditions:

1. if \(k ∈ \mathcal{K}\), then \(k + bN ∈ \mathcal{K}\) for all \(b ∈ \mathbb{Z}\) such that \(0 ≤ k + bN < L\)
2. \(Δ(\mathcal{K}) = \{0, 1, \ldots, L−1\}\).

It is called minimal if there exists no other periodic sparse ruler with the same length and period but smaller cardinality.

Observe that any periodic sparse ruler is also a sparse ruler, whereas the converse need not be true. Clearly, Theorem 3 could be rephrased to say that \(\mathcal{K}\) is universal iff it is a length-(\(NB−1\)) periodic sparse ruler of period \(N\). The problem of designing non-uniform covariance samplers becomes that of designing periodic sparse rulers. The next result simplifies this task by stating that a length-(\(NB−1\)) periodic sparse ruler of period \(N\) is indeed the concatenation of \(B\) length-(\(N−1\)) sparse rulers:

**Theorem 4:** A set \(\mathcal{K}\) is a periodic sparse ruler of length \(NB−1\) and period \(N\) if and only if there exists a sparse ruler \(\mathcal{M}\) of length \(N−1\) such that

\[
\mathcal{K} = \{m + bN: \quad m ∈ \mathcal{M}, \quad b = 0, 1, \ldots, B−1\}.\tag{33}
\]

**Proof:** See Appendix D.

One of the consequences of Theorem 4 is that increasing the number of blocks in a periodic non-uniform sampler cannot improve the compression ratio. For example, concatenating two equal length-(\(N−1\)) minimal sparse rulers with \(M\) elements results in a length-(\(2N−1\)) sparse ruler with \(2M\) elements. Note, however, that the situation is different if the periodicity requirement is dropped. E.g., a minimal length-10 sparse ruler has 6 elements, whereas a length-21 minimal sparse ruler has \(8 < 6 × 2\) elements.

As a corollary of Theorem 4, we conclude that a minimal periodic sparse ruler is the concatenation of minimal sparse rulers. Thus, the problem of designing non-uniform universal covariance samplers (either periodic or non-periodic) can be reduced to designing a length-(\(N−1\)) sparse ruler \(\mathcal{M}\).

Table I illustrates the minimum value of \(M = |\mathcal{M}|\) (labeled as \(M_{UB}\)) for several values of \(N\). This enables us to obtain the exact optimum compression ratios for block lengths \(N\) up to 60, which covers most practical cases. For higher values of \(N\), one may resort to another table, to a computer program, or to the asymptotic considerations in Sec. VI. However, although there is no closed form expression for the maximum compression ratio \(\rho\) that can be achieved, the bounds in (31) and (32) show that

\[
\frac{N}{\sqrt{3(N−1)}} \leq \rho \leq \frac{N}{\sqrt{\tau(N−1)}}.\tag{34}
\]

**B. Random Sampling**

Deriving conditions for universality in the case of random sampling is simpler than in the non-uniform sampling case, since most mathematical complexity has been subsumed by Theorem 2. Moreover, the results are simpler and can be expressed in closed form.

**Theorem 5:** Let \(Φ\) be an \(M × N\) random matrix satisfying the hypotheses of Theorem 2. Then, \(Φ = I_B ⊗ Φ\) defines an universal covariance sampler with probability \(I\) if and only if

\[
M ≥ \sqrt{\frac{2NB−1}{2B−1}}.\tag{35}
\]

**Proof:** Let \(S\) denote a basis for \(\mathbb{S}_L^L\). Since the cardinality of \(S\) is given by \(2L−1 = 2NB−1\), according to Theorem 2, a sampler \(Φ\) is an \(S\)-covariance sampler iff \(2NB−1 ≤ M^2(2B−1)\), which is equivalent to (35). Universality then follows from Lemma 4.
for universal samplers, which, according to Lemma 4, are sparse rulers (relax. In particular, every period must be a Toeplitz subspace. Universal samplers represent the optimal choice when no structure exists or, simply, when it is unknown.

On the other hand, when prior information is available, the choice when no structure exists or, simply, when it is unknown. 

Theorem 6: Let \( \mathcal{A} \) be a set of integers. The \( \mathcal{A} \)-modular difference set of \( \mathcal{A} \), denoted as \( \Delta_{\mathcal{A}}(\mathcal{A}) \), is defined as

\[
\Delta_{\mathcal{A}}(\mathcal{A}) = \{ \delta \geq 0 : \exists a_1, a_2 \in \mathcal{A} \text{ s.t. } \delta = (a_2 - a_1)_{\mathcal{A}} \}. \tag{38}
\]

Clearly, for any \( \mathcal{A} \subset \{0, 1, \ldots, N-1\} \), we have that \( \Delta(\mathcal{A}) \subset \Delta_{\mathcal{A}}(\mathcal{A}) \), which means that \( |\Delta_{\mathcal{A}}(\mathcal{A})| \) is never less than \( |\Delta(\mathcal{A})| \). Actually, \( \Delta_{\mathcal{A}}(\mathcal{A}) \) will typically be larger than \( \Delta(\mathcal{A}) \) since the fact that \( \delta \) is in \( \Delta_{\mathcal{A}}(\mathcal{A}) \) implies that \( A - \delta \) is also in that set. For example, if \( \mathcal{A} = \{0, 1, 5\} \) and \( A = 10 \), then \( \Delta(\mathcal{A}) = \{0, 1, 4, 5\} \subset \Delta_{\mathcal{A}}(\mathcal{A}) = \{0, 1, 4, 5, 6, 9\} \). Finally, the cardinality of the modular difference set is upper bounded by noting that any pair of elements in a set \( \mathcal{A} \) with cardinality \( |\mathcal{A}| \) generates at most two distances in \( \Delta_{\mathcal{A}}(\mathcal{A}) \):

\[
|\Delta_{\mathcal{A}}(\mathcal{A})| \leq |\mathcal{A}| \cdot (|\mathcal{A}| - 1) + 1. \tag{39}
\]

Now that the modular difference set has been defined, it is possible to state the requirements to sample circular subspaces:

Theorem 6: Let \( \mathcal{S}_C \) be given by (15). Then, the set \( \mathcal{K} \subset \{0, \ldots, L-1\} \) is an \( \mathcal{S}_C \)-covariance sampler if and only if \( \Delta_{\mathcal{K}}(\mathcal{K}) = \{0, \ldots, L-1\} \).

Proof: See Appendix E.

Theorem 6 is therefore the dual of Theorem 3 for circular subspaces. However, in this case the conclusion does not lead to a linear sparse ruler but to a circular one:

Definition 7: A length-(\( L-1 \)) circular (or modular) sparse ruler is a set \( \mathcal{K} \subset \{0, \ldots, L-1\} \) satisfying \( \Delta_{\mathcal{K}}(\mathcal{K}) = \{0, \ldots, L-1\} \); and it is said to be minimal if no other length-(\( L-1 \)) circular sparse ruler exists with smaller cardinality.

As in the case of the linear sparse rulers from Definition 4, there is a geometric interpretation for a circular sparse ruler in terms of a physical ruler. First consider that we wrap around a conventional ruler (made of some flexible material) until the first mark and the last mark lie at unit distance, thus making a circular ruler. Now assume that some of the marks are erased, but that it is still possible to measure all distances between 0 and the length of the original ruler by considering pairs of marks. The advantage with respect to a linear ruler is that any pair of marks provides, in general, two distances, which are the lengths of the two circular segments defined by them. On the other hand, only one distance is generated by each pair of marks in linear sparse rulers. Two examples of length-20 circular sparse rulers are illustrated in Fig. 2, the one on the left being minimal. Other examples of length-(\( L-1 \)) circular sparse rulers are \( \{0, \ldots, L-1\} \) and \( \{0, \ldots, \left\lfloor \frac{L}{2} \right\rfloor \} \), which are referred to as trivial circular sparse rulers.

Circular sparse rulers, also known as difference cycles, were analyzed by the mathematical community using finite group theory and additive number theory (see [52] for an overview of the main results). Among the most remarkable properties, we note that a reflection of a circular sparse ruler is also a circular sparse ruler (see Sec. IV-A) and that any circular rotation of a circular sparse ruler \( \mathcal{K} \), defined as

\[
\mathcal{K}_{(i)} = \{(k + i)_L : \ k \in \mathcal{K}\}, \quad i \in \mathbb{Z}, \tag{40}
\]
is also a circular sparse ruler. Moreover, since $\Delta(K) \subset \Delta_L(K)$ for any $K \subset \{0, \ldots, L-1\}$, a linear sparse ruler is also a circular sparse ruler. Hence, the length of a minimal circular sparse ruler can never be greater than the length of a minimal linear sparse ruler. It is possible to go even further by noting that any length-$\lceil \frac{L}{2} \rceil$ linear sparse ruler is also a length-$(L-1)$ circular sparse ruler. For example, Fig. 2b shows a length-20 circular sparse ruler constructed with a length-10 linear sparse ruler. From this observation and (32), we obtain:

$$|K| \leq \left\lceil \sqrt{3 \frac{L}{2}} \right\rceil. \tag{41}$$

On the other hand, expression (39) yields

$$|K| \geq \frac{1}{2} + \sqrt{L - \frac{\sqrt{3}}{4}}. \tag{42}$$

This lower bound is attained at least when $L = 1 + r + r^2$ for $r$ a power of a prime number [53].

The design of a length-$(L-1)$ circular sparse ruler can be accomplished in several ways. If $L = 1 + r + r^2$ with $r$ a power of some prime number, minimal rulers can be obtained following [53]. Other cases may require exhaustive search, which motivates sub-optimal designs. An immediate choice seems to be using a length-$(L-1)$ or length-$\lceil \frac{L}{2} \rceil$ minimal linear sparse ruler [10]. In fact, the latter provides optimal solutions for most values of $L$ below 60, as we can see from Table I. Further alternatives include the coprime sampling method from [45].

The introduction of circular sparse rulers in signal and array processing seems to have been done first in [47] and later in [34], [35], [44]. Theorem 6 basically states that a covariance sampler for circulant subspaces is a length-$(L-1)$ circular sparse ruler, which gives a practical design criterion just for the non-periodic case. It is therefore the moment for periodicity to come into the picture.

**Definition 8:** A length-$(L-1)$ periodic circular sparse ruler of period $N$, where $N$ divides $L$, is a set $K \subset \{0, 1, \ldots, L-1\}$ satisfying:

1) if $k \in K$, then $k + bN \in K$ for all $b \in \mathbb{Z}$ such that $0 \leq k + bN < L$;

2) $\Delta_L(K) = \{0, 1, \ldots, L-1\}$.

It is called minimal if there is no other periodic circular sparse ruler with the same length and period but smaller cardinality.

Hence, Theorem 6 could be rephrased to say that $K$ is an $S_C$-covariance sampler iff it is a length-$(NB-1)$ periodic circular sparse ruler of period $N$. Although designing this class of rulers may seem a difficult task, the next result simplifies the search by stating that every period of such a ruler is, indeed, a circular sparse ruler.

**Theorem 7:** A set $K$ is a periodic circular sparse ruler of length $NB-1$ and period $N$ if and only if there exists a circular sparse ruler $M$ of length $N - 1$ such that

$$K = \{m + bN : m \in M, \ b = 0, 1, \ldots, B-1\} \tag{43}$$

**Proof:** See Appendix F.

Theorem 7 is the dual of Theorem 4 for circular sparse rulers. Both theorems establish an interesting relationship, which can be informally stated by saying that a periodic sparse ruler is a concatenation of sparse rulers of the same nature, that is, a periodic linear sparse ruler is the concatenation of linear sparse rulers, whereas a periodic circular sparse ruler is the concatenation of circular sparse rulers.

Table I reveals that the cardinality $M$ of a minimal circular sparse ruler is not monotone with $N$. For example, minimal length-19 circular sparse rulers have 6 elements whereas minimal length-20 circular sparse rulers have 5 elements (see [52] for a proof). Table I also illustrates the compression gain due to the knowledge that $\Sigma$ is circulant. For example, when $N = 60$, a universal sampler has a compression ratio of $\frac{N}{M} = \frac{60}{14} \approx 4.28$, whereas a covariance sampler for circulant matrices has a compression ratio of $\frac{N}{M} = \frac{60}{9} \approx 6.72$.

Although maximum compression ratios cannot be expressed in closed form, simple bounds follow from (41) and (42):

$$\frac{N}{\sqrt{3 \left\lceil \frac{N}{2} \right\rceil}} \leq \rho \leq \frac{2N}{2 + \sqrt{4N - 3}}. \tag{44}$$

2) **Random Sampling:** Similarly to universal samplers, the design using random sampling is much easier than that using non-uniform sampling. An immediate consequence of Theorem 2 follows by noting that any basis for the circulant subspace has $L = NB$ elements.

**Corollary 1:** Let $\Phi$ be an $M \times N$ random matrix satisfying the hypotheses of Theorem 2 and let $S_C$ be given by (15). Then, with probability one, the matrix $\tilde{\Phi} = I_B \otimes \Phi$ defines an $S_C$-covariance sampler if and only if

$$M \geq \sqrt{\frac{NB}{2B-1}}. \tag{45}$$

This, in turn, results in a compression ratio of

$$\rho = \frac{N}{M} \approx \sqrt{(2B - 1)N}. \tag{46}$$

For large $B$, this represents an approximate gain of $\sqrt{2}$ with respect to the universal case.
B. d-banded Covariance Subspace

1) Non-uniform sampling: The prior knowledge that \( \Sigma \) is \( d \)-banded also provides important compression benefits. In particular, we will see that, for non-uniform samplers, \( d \)-banded subspaces with \( N \leq d \leq N(B - 1) \) are compressed as circulant subspaces.

**Theorem 8:** Let \( S_B^d \) be given by (18) with \( N \leq d \leq N(B - 1) \). Then, the set

\[
K = \{ m + bn, \ m \in M, b = 0, 1, \ldots, B - 1 \}, \quad (47)
\]

where \( M \subset \{ 0, \ldots, N - 1 \} \), defines an \( S_B^d \)-covariance sampler if and only if \( M \) is a length-(\( N - 1 \)) circular sparse ruler.

**Proof:** See Appendix G.

Observe that the condition \( d \leq N(B - 1) \) is a mild assumption since we are only requiring the last \( N - 1 \) lags of the associated autocorrelation sequence to be zero.\(^{20}\) Note as well that other cases rather than \( N \leq d \leq N(B - 1) \) may be considered, resulting in completely different conclusions. For example, in the non-periodic case \((B = 1)\) it can be easily shown from Theorem 1 that the only requirement on \( K \) to define an \( S_B^d \)-covariance sampler is that \( \Delta(K) = \{ 0, \ldots, d \} \).

Combining Theorem 8 with Theorem 7, it follows that \( K \) must be a length-(\( NB - 1 \)) periodic circular sparse ruler of period \( N \), which means that the samplers for \( d \)-banded subspaces mimic those for circulant subspaces. Thus, one should apply the design and compression ratio considerations from Sec. V-A1. Interestingly, it must be observed that the latter value does not depend on \( d \) provided that this parameter remains within the aforementioned limits.

2) Random Sampling: Noting that the dimension of the \( d \)-banded subspace is \( I = 2d + 1 \) results in the following corollary of Theorem 2:

**Corollary 2:** Let \( \Phi \) be an \( M \times N \) random matrix satisfying the hypotheses of Theorem 2 and let \( S_B^d \) be given by (18). Then, with probability one, the matrix \( \Phi = I_B \otimes \Phi \) defines an \( S_B^d \)-covariance sampler if and only if

\[
M \geq \sqrt{\frac{2d + 1}{2B - 1}}. \quad (48)
\]

According to this result, the maximum compression ratio in the case of the \( d \)-banded subspace is given by

\[
\rho = \frac{N}{M} \approx \sqrt{\frac{(2B - 1)N^2}{2d + 1}}, \quad (49)
\]

which clearly improves the ratio in (37) since \( d \leq NB - 1 \).

VI. ASYMPTOTIC REGIME

We next provide the optimal compression ratios for universal random samplers and bound the optimal ratios for universal non-uniform samplers as \( M \) and \( N \) become larger.

- **Random Sampling:** The maximum compression ratio \( \rho_{RS} \) of universal random samplers is given by (37).

Asymptotically in \( N \), we have that \( \rho_{RS} \to \sqrt{\frac{2B - 1}{2B - 1} N} \), which becomes \( \rho_{RS} \to \sqrt{\frac{N}{2}} \) in the non-periodic case and \( \rho_{RS} \to \sqrt{N} \) if the number of periods \( B \) also becomes large. Alternatively, we observe that \( M \to \sqrt{\frac{2B}{2B - 1} N} \) as \( N \) becomes large, which means that \( M \to \sqrt{N} \) in the non-periodic case and \( M \to \sqrt{N} \) as \( B \to \infty \).

- **Non-uniform Sampling:** In [23], [39] it is established that the quotient \( M^2/N \) asymptotically converges to a constant \( c \), which is between \( 21/24 \) and \( 3 \), with \( M \) and \( N \) respectively denoting the cardinality and length of a minimal linear sparse ruler. Therefore, the asymptotic optimal compression ratio is given by

\[
\rho_{NS} \to \sqrt{\frac{N}{c}}. \quad (50)
\]

In terms of \( M \), this means that \( M \to \sqrt{cN} \). Interestingly, if we use nested arrays [29], [42], the maximum achievable compression we can obtain for suitable choices of the parameters is \( \rho_{NA} \to \sqrt{\frac{N}{3}} \), which is therefore suboptimal. However, they present the advantage of having a simple design. The scheme in [23], [40] allows the simple construction of sparse rulers satisfying \( M^2/N < 3 \), which entail compression ratios greater than \( \sqrt{\frac{N}{3}} \) even for finite \( M \) and \( N \).

To sum up, random sampling allows better asymptotic compression ratios than non-uniform sampling. The compression loss between both approaches is quantified by the constant \( c \), which means that between 36% and 42% compression may be lost for large \( B \) if we use non-uniform sampling instead of random sampling. Similar observations arise for non-universal samplers by using the expressions in Sec. V.

VII. DISCUSSION

In case of an arbitrary number of realizations of \( y \), the maximum compression ratio separates consistency from inconsistency in the estimation. However, the notion of consistency is not truly meaningful in case of just one realization. For those cases, the values presented here provide simple guidelines to select suitable compression ratios and a guess of the quality of the estimation, in the sense that a good performance is expected when the actual compression ratio is much lower than the maximum one and vice versa.

A covariance sampler guarantees that there is a one-to-one correspondence between the statistics of \( x \) and those of \( y \). As seen in Sec. I, the associated covariance matrices can be parameterized by the same coordinates \( \alpha_0, \ldots, \alpha_{s-1} \). Thus, it makes sense to estimate these coordinates from the observations of \( y \) and use the same coordinates to reconstruct \( \Sigma \) [25], [26]. However, this operation does not necessarily yield positive semidefinite estimates of \( \Sigma \), since the fact that \( \Sigma = \sum_s \alpha_s \Sigma_s \) is positive semidefinite for some \( \alpha_s \)'s does not mean that \( \Sigma \) is so.\(^{22}\) In cases where positive semidefiniteness is

\(^{20}\) Strictly speaking, we only need the lags \( NB - N + 1 \) through \( NB - 1 \) to be zero since the lags greater than \( NB - 1 \) are not relevant in the model.

\(^{21}\) As an informal guess, consider the length-90 minimal sparse ruler, which has 16 elements. A simple approximation yields \( c \approx 16^2/90 \approx 2.8132 \).

\(^{22}\) This problem is not present in certain applications, such as those where the matrices in \( S \) are positive semidefinite and the \( \alpha_s \)'s are non-negative [12]. In those cases, \( \Sigma \) is automatically positive semidefinite.
a concern, one may resort to the approaches in [27]–[29]. Note that this problem arises from the usage of sample statistics instead of the true ones and is therefore beyond the theoretical considerations constituting the framework of this paper.

We also remark that a number of different designs for sparse arrays in the literature do not result in covariance samplers. This is because the interest is focused on the cost of the system rather than on preserving all statistical information. For example, in some cases the focus is on the number of elements, which leads to allowing holes in the difference set. See, for instance, linear minimum hole arrays [54] and linear minimum holes-plus-redundancies arrays [55]. An example of alternative designs without missing elements is composed of linear reduced redundancy arrays [56].

VIII. Conclusions

We have derived maximum compression ratios and optimal covariance samplers for a number of cases including Toeplitz, circulant, and banded covariance subspaces. The results were derived for the general periodic case, but they can be immediately particularized to the non-periodic setting. One of the effects observed is the convenience of having long blocks.

Two common schemes were considered: non-uniform and random sampling. The design of optimal universal non-uniform samplers is related to the minimal sparse ruler problem, which is an exhaustive search problem with known near-optimal simple approximations. Some non-universal samplers, however, are connected with the circular sparse ruler problem. In both cases, periodic rulers were defined and analyzed.

In the case of random samplers, the design is much simpler since it depends solely on the size of the compression matrix relative to the dimension of the covariance subspace. Interestingly, it was observed that random sampling leads to better compression ratios than non-uniform sampling.

Appendix A

Proof of Lemma 1

Clearly, if \( \phi \) is invertible so is \( \phi|_A \). In order to prove the converse statement, it suffices to show that \( \phi \) is injective if \( \phi|_A \) is injective. This is a simple consequence of the definition of the codomains for both functions. Therefore, we need to prove that, given any two vectors \( a = [a_0, \ldots, a_{S-1}]^T \) and \( b = [b_0, \ldots, b_{S-1}]^T \) in \( \mathbb{R}^S \), the matrices

\[
\Sigma_a = \sum_s a_s \Sigma_s \quad \text{and} \quad \Sigma_b = \sum_s b_s \Sigma_s \quad (51)
\]

must satisfy that

\[
\phi(\Sigma_a) = \phi(\Sigma_b) \implies \Sigma_a = \Sigma_b \quad (52)
\]

or, equivalently, that

\[
\phi(\Sigma_a) = \phi(\Sigma_b) \implies a = b, \quad (53)
\]

since \( S \) is linearly independent. To do so, let us take \( S \) linearly independent vectors \( \alpha_0, \ldots, \alpha_{S-1} \), where \( \alpha_i = [\alpha_{i,0}, \ldots, \alpha_{i,S-1}]^T \), such that the \( S \) matrices

\[
\Sigma_{\alpha_i} = \sum_s \alpha_{i,s} \Sigma_s, \quad i = 0, \ldots, S - 1 \quad (54)
\]

are in \( A \). This operation is possible since \( \dim [\mathbb{R}^S \cap \text{span } \Sigma_S] = S \). Moreover, since \( \phi|_A \) is injective and \( \{\Sigma_{\alpha_i}\}_{i=0}^{S-1} \) is a linearly independent set of matrices, it follows that the matrices

\[
\Sigma_{\alpha_i} = \phi|_A(\Sigma_{\alpha_i}) = \phi(\Sigma_{\alpha_i}) = \sum_s \alpha_{i,s} \Sigma_s \quad (55)
\]

also form an independent set of matrices. On the other hand, since the \( S \) vectors \( \alpha_i \) constitute a basis for \( \mathbb{R}^S \), it is possible to write \( a \) and \( b \) as:

\[
a = \sum_i \tilde{a}_i \alpha_i \quad \text{and} \quad b = \sum_i \tilde{b}_i \alpha_i, \quad (56)
\]

for some \( \tilde{a}_i, \tilde{b}_i \in \mathbb{R} \), which in turn means that

\[
\Sigma_a = \sum_i \tilde{a}_i \Sigma_{\alpha_i} \quad \text{and} \quad \Sigma_b = \sum_i \tilde{b}_i \Sigma_{\alpha_i} \quad (57)
\]

or

\[
\phi(\Sigma_a) = \sum_i \tilde{a}_i \Sigma_{\alpha_i} \quad \text{and} \quad \phi(\Sigma_b) = \sum_i \tilde{b}_i \Sigma_{\alpha_i}. \quad (58)
\]

Noting that the matrices \( \Sigma_{\alpha_i} \) are linearly independent leads to the statement

\[
\phi(\Sigma_a) = \phi(\Sigma_b) \implies \tilde{a}_i = \tilde{b}_i \quad \forall i, \quad (59)
\]

which is equivalent to (53), thus concluding the proof.

Appendix B

Proof of Theorem 2

In order to show Theorem 2 we will proceed by computing the dimension of \( \ker \phi_C \), and deriving the conditions under which \( \dim \ker \phi_C = 0 \), which, in virtue of Lemma 3, are the conditions determining whether \( \Phi \) defines a covariance sampler. However, since the direct computation of \( \ker \phi_C \) is not a simple task, we perform several intermediate steps. First, we compute \( \ker \phi_C \), where \( \phi_C \) is defined as the extension of \( \phi \) to \( C^{L \times L} \):

\[
\mathbb{C}^{NB \times NB} \xrightarrow{\phi_C} \mathbb{C}^{MB \times MB} \quad (60)
\]

We later compute \( \dim \ker \phi_C \) by successive intersections as

\[
\ker \phi_C = \text{span} \, \mathbb{C} \left( T^{NB} \cap \left( \mathbb{B}^{N,B} \cap \ker \phi_C \right) \right), \quad (61)
\]

where \( T^{NB} \) represents the set of (not necessarily Hermitian) \( NB \times NB \) Toeplitz matrices and \( \mathbb{B}^{N,B} \) represents the set of \( NB \times NB \) matrices with Toeplitz \( N \times N \) blocks. The matrices in \( \mathbb{B}^{N,B} \) can thus be written as

\[
\begin{bmatrix}
A_{0,0} & \cdots & A_{0,B-1} \\
\vdots & \ddots & \vdots \\
A_{B-1,0} & \cdots & A_{B-1,B-1}
\end{bmatrix} \quad (62)
\]

where the blocks \( A_{p,p} \in \mathbb{C}^{N \times N} \) are Toeplitz. Expression (61) results from the fact that \( \ker \phi_C = \text{span} \, \mathbb{C} \cap \ker \phi_C \) and

\[
\text{span} \, \mathbb{C} \subset T^{NB} \subset \mathbb{B}^{N,B}. \quad (63)
\]

On the other hand, note that the requirement that the probability measure \( \mu \) be absolutely continuous with respect to
Lebesgue measure means that no subset\(^3\) of \(C^{L \times L}\) with zero Lebesgue measure is assigned a probability greater than zero by \(\mu\). In the rest of the proof, we will refer to this property by simply saying that \(\mu\) defines a continuous probability distribution. One of the consequences of this assumption is that the probability that any row (or column) of \(\Phi\) is in a given subspace of dimension less than \(N\) (resp. \(M\)) is zero. Another consequence is that rank \(\Phi = M \leq N\) with probability one and, as a result, the (right) null space of \(\Phi\) has dimension \(N - M\). Let us denote by \(V \subseteq N \times (N - M)\) matrix whose columns span this null space. Due to the properties of \(\Phi\), it is clear that the probability that the columns of \(V\) are contained in a given subspace of dimension less than \(N\) is zero.

We start by computing a basis for \(\ker \tilde{\Phi}\) in terms of \(V\).

**Lemma 5:** Let \(E_{i,j} \in \mathbb{C}^{B \times B}\) be the matrix with all entries set to zero but the \((i,j)\)-th entry, which is one, and let \(e_k\) denote the \(k\)-th column of the identity matrix \(I_N\). Let also \(\tilde{\Phi}\) be defined as in (60), and let the columns of \(V = [v_0, \ldots, v_{N-M-1}] \in \mathbb{C}^{N \times (N-M)}\) form a basis for the null space of \(\Phi\). Then, a basis for \(\ker \tilde{\Phi}\) is given by

\[
W = \bigcup_{i=0}^{B-1} \bigcup_{j=0}^{B-1} \left\{ E_{i,j} \otimes e_k \otimes v_l^H \right\},
\]

where

\[
W_{i,j} = \left\{ E_{i,j} \otimes e_k \otimes v_l^H, \quad \begin{array}{l} k = 0,1,\ldots,N-1, \quad l = 0,1,\ldots,N-M-1 \end{array} \right\} \cup \left\{ E_{i,j} \otimes e_k^H \otimes v_l, \quad \begin{array}{l} k = 0,1,\ldots,M-1, \quad l = 0,1,\ldots,N-M-1 \end{array} \right\}.
\]

**Proof:** See Appendix C.

Now let us evaluate the intersection \(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\), which means that we must look for the matrices in \(\ker \tilde{\Phi}\) whose \(N \times N\) blocks have a Toeplitz structure. For the sake of simplicity, let us proceed block-by-block by separately considering the subspaces generated by each \(W_{i,j}\). Clearly, the matrices in \(\spn_{\mathbb{C}} W_{i,j}\) can have, at most, a single non-null \(N \times N\) block, which is the \((i,j)\)-th block. This block is in the subspace generated by the following basis:

\[
\{ e_k \otimes v_l^H, \quad k = 0,1,\ldots,N-1, \quad l = 0,1,\ldots,N-M-1 \}
\]

\[
\cup \{ e_k^H \otimes v_l, \quad k = 0,1,\ldots,M-1, \quad l = 0,1,\ldots,N-M-1 \}.
\]

Therefore, all blocks in this subspace can be written in terms of this basis as

\[
\sum_k \sum_l \alpha_{k,l}(e_k \otimes v_l^H) + \sum_k \sum_l \beta_{k,l}(e_k^H \otimes v_l)
\]

for some \(\alpha_{k,l} \in \mathbb{C}\) and \(\beta_{k,l} \in \mathbb{C}\). The blocks with Toeplitz structure must necessarily satisfy

\[
\begin{align*}
\sum_{n=-N+1}^{N-1} \gamma_n P_n &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-M-1} \alpha_{k,l}(e_k \otimes v_l^H) \\
&+ \sum_{k=0}^{M-1} \sum_{l=0}^{N-M-1} \beta_{k,l}(e_k^H \otimes v_l)
\end{align*}
\]

for some \(\gamma_n \in \mathbb{C}\), where \(P_n\) equals \(J_n^R\) for \(n \geq 0\) and \((J_n^R)^T\) for \(n < 0\), with \(J_n\) defined in Sec. II-B.

Expression (68) represents a system of linear equations in \(\alpha_{k,l}, \beta_{k,l}\) and \(\gamma_n\), with \(N^2 - M^2 + 2N - 1\) unknowns and \(N^2 - M^2 - 1\) equations. On the other hand, since \(\Phi\), and consequently \(V\), follow a continuous distribution, it follows that there are \(\min(N^2, N^2 - M^2 + 2N - 1)\) independent matrices in (68). Consequently, if \(N^2 \geq N^2 - M^2 + 2N - 1\) the only solution is just the zero matrix, and \(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi} = \{0\}\), which in turn means that \(\ker \Phi = \{0\}\). Therefore, a sufficient condition for \(\Phi\) to define a covariance sampler (see Lemma 3) is

\[
M^2 \geq 2N - 1.
\]

Conversely, if \(N^2 < N^2 - M^2 + 2N - 1\) the subspace of solutions has dimension \(N^2 - M^2 + 2N - 1 - N^2 = 2N - M^2 - 1\). Therefore, the blocks of the matrices in \(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\) can be written as a linear combination of \(2N - M^2 - 1\) Toeplitz matrices \(M_k\). By considering all blocks, it follows that \(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\) is generated by the following basis:

\[
\{ E_{i,j} \otimes M_k, \quad i,j = 0,1,\ldots,B-1; \quad k = 0,1,\ldots,2N - M^2 - 2 \}
\]

Thus, any matrix in \(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\) can be written as

\[
\Sigma = \sum_{i,j,k} \eta_{i,j,k} E_{i,j} \otimes M_k.
\]

Now we compute the dimension of \(\mathbb{T}^{N,B} \cap \left(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\right)\). First note that \(\dim(\mathbb{B}^{N,B} \cap \ker \tilde{\Phi}) = B^2(2N - M^2 - 1)\). In order for \(\Sigma \in \mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\) to be Toeplitz, we require that \(\eta_{i,j,k}\) only depend on the difference \(i-j\), which reduces the dimension of this space to \(2N - M^2 - 1\) times the number of block diagonals, i.e., \((2N - M^2 - 1)(2B - 1)\). Moreover, since any two adjacent block diagonals share \(N-1\) diagonals, this imposes \((2B - 2)(N - 1)\) additional equations and results in

\[
\begin{align*}
\dim(\mathbb{T}^{N,B} \cap \mathbb{B}^{N,B} \cap \ker \tilde{\Phi}) &= (2N - M^2 - 1)(2B - 1) - (2B - 2)(N - 1) \\
&= (2N - M^2 - 1)(2B - 1) - (2B - 2)(N - 1).
\end{align*}
\]

At this point, note that \(\mathbb{T}^{N,B}\) is the smallest parent subspace of both \(\mathbb{T}^{N,B} \cap \mathbb{B}^{N,B} \cap \ker \tilde{\Phi}\) and \(\spn_{\mathbb{R}} S\). Since \(\Phi\) was generated according to a continuous distribution, then with probability one these two subspaces will not overlap (except for the zero matrix) unless the sum of their dimensions exceeds the dimension of the parent subspace, which is \(2NB - 1\). Therefore, \(\Phi\) defines a covariance sampler if and only if

\[
(2N - M^2 - 1)(2B - 1) - (2B - 2)(N - 1) + S \leq 2NB - 1
\]
or, equivalently
\[ S \leq M^2(2B - 1). \tag{73} \]

It remains only to show that one only needs to look at (73) in order to assess whether a matrix \( \Phi \) defines a covariance sampler, the condition in (69) being completely irrelevant. It follows from the fact that (69) implies (73). Indeed, if we multiply both sides of (69) by \((2B - 1)\), we obtain
\[ M^2(2B - 1) \geq (2N - 1)(2B - 1) \tag{74} \]
\[ = (2NB - 1) + 2(N - 1)(B - 1) \tag{75} \]
\[ \geq (2NB - 1) \geq S \tag{76} \]
where the second inequality follows from the fact that \((N - 1)(B - 1) \geq 0\) and the third one is a consequence of the linear independence of \( S \). Therefore, (69) implies (73), and \( \Phi \) defines a covariance sampler if and only if (73) holds.

**APPENDIX C**

**PROOF OF LEMMA 5**

Computing \( \ker \tilde{\phi}_C \) amounts to finding a basis for the subspace of matrices \( \Sigma \) in \( \mathbb{C}^{NB \times NB} \) satisfying \( \Phi \Sigma \Phi^H = 0 \). Vectorizing this expression (see e.g. [46]) results in the condition \((\Phi^* \otimes \Phi) z = 0\), where \( z = \text{vec} \Sigma \). Thus, \( \ker \phi_C \) is given (up to inverse vectorization) by the null space of the \((MB)^2 \times (NB)^2\) matrix \( \Phi^* \otimes \Phi \).

Since the columns of \( \tilde{V} \) constitute a basis for the null space of \( \Phi \) and since \( \Phi = I_B \otimes \Phi \), the columns of \( \tilde{V} = I_B \otimes \tilde{V} \) constitute a basis for the null-space of \( \Phi \). It can be shown that \( \ker \phi_C \) is composed of matrices of the form \( \Sigma = \tilde{V} A^H + B \tilde{V}^H \), where \( A \) and \( B \) are arbitrary matrices of the appropriate dimensions. It follows that the null space of \( \Phi^* \otimes \Phi \) is spanned by the columns of the matrix
\[ \tilde{W} = [I_{NB} \otimes \tilde{V}, \; \tilde{V}^* \otimes I_{NB}] \tag{77} \]

By the properties of the Kronecker product [46], the fact that \( \Phi \) has maximum rank implies that \( \Phi^* \otimes \Phi \) has maximum rank as well, so that its null space has dimension \((N^2 - M^2)B^2\). However, since \( \tilde{V} \) is \( NB \times (N-M)B \), it is clear that \( \tilde{W} \) has \( 2(N-M)NB^2 \) columns, which is greater than \((N^2 - M^2)B^2\). Thus, in order to obtain a basis for the null space of \( \Phi^* \otimes \Phi \) we should remove dependent columns from \( \tilde{W} \). This procedure is carried out in the following lemma:

**Lemma 6:** Let \( \tilde{V} \in \mathbb{C}^{N \times (N-M)} \), with \( M \leq N \), be a matrix whose columns generate the null space of \( \Phi \in \mathbb{C}^{M \times N} \), which follows a continuous distribution, and let \( \tilde{V} = I_B \otimes \tilde{V} \). Then, the columns of \( \tilde{W} \), defined by (77), span the same subspace as the columns of \( \tilde{W} \), which is defined as
\[ \tilde{W} = [I_{NB} \otimes \tilde{V}, \; \tilde{V}^* \otimes I_B \otimes F_M], \tag{78} \]

where \( F_M = [I_M, \; 0_{M,N-M}]^T \).

**Proof:** The procedure we follow in this proof is to remove linearly dependent columns from \( \tilde{W} \). Since the case \( B = 1 \) is quite tedious, here we only show this result for the case \( B = 1 \). The proof for the general case follows the same lines and it is easily extrapolated, but it requires an overloaded notation. For \( B = 1 \) we have that
\[ \tilde{W} = [I_N \otimes \tilde{V}, \; \tilde{V}^* \otimes I_N] \tag{79} \]

Now scale the last \( N(N-M) \) columns of \( \tilde{W} \) to obtain
\[ \tilde{W}' = [I_N \otimes V, \; G \otimes I_N], \tag{80} \]
where \( G \) is the result of scaling the columns of \( V^* \) such that the first row contains only ones24:
\[ G = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ g_{1,0} & g_{1,1} & \ldots & g_{N-1,N-M-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1,0} & g_{N-1,1} & \ldots & g_{N-1,N-M-1} \end{bmatrix} \tag{81} \]

Now consider a submatrix of \( \tilde{W}' \) obtained by retaining the first \( N(N-M) \) columns and the columns with indices \( N(N-M) + Ni, \ldots, N(N-M) + N(i+1) - 1 \), i.e.,
\[ \tilde{W}_i' = \begin{bmatrix} V & 0 & \ldots & 0 & I_N \\ 0 & V & \ldots & 0 & g_{1,i} I_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & V & g_{N-1,i} I_N \end{bmatrix}, \tag{82} \]
where \( i = 0, \ldots, N-M-1 \). Scaling the diagonal blocks on the left yields:
\[ \tilde{W}_i'' = \begin{bmatrix} V & 0 & \ldots & 0 & I_N \\ 0 & g_{1,i} V & \ldots & 0 & g_{1,i} I_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & g_{N-1,i} V & g_{N-1,i} I_N \end{bmatrix} \tag{83} \]

Now, since \( \Phi \) follows a continuous distribution, the last \( N-M \) columns of \( [V, I_N] \) can be written as linear combinations of the first \( N \) columns, which means that the last \( N-M \) columns of \( \tilde{W}_i'' \) are linearly dependent of the others. Repeating this operation for \( i = 0, \ldots, N-M-1 \) and removing from \( \tilde{W} \) the columns declared as dependent at each \( i \) gives
\[ \tilde{W} = [I_N \otimes V, \; V^* \otimes F_M], \tag{84} \]
which clearly spans the same subspace as \( \tilde{W} \). In the general case with \( B \geq 1 \) we obtain
\[ \tilde{W} = [I_{NB} \otimes V, \; V^* \otimes I_B \otimes F_M]. \tag{85} \]

Note that, indeed, the matrix defined in (78) has \((N^2 - M^2)B^2\) columns, which means that they constitute a basis for the null space of \( \Phi^* \otimes \Phi \). Upon inverse vectorization of the columns in \( \tilde{W} \) we obtain the sought basis in matrix form:
\[ \mathcal{W} = \bigcup \{ E_{i,j} \otimes e_k \otimes \varpi_l^H, \; i,j = 0,1,\ldots,B-1, \]
\[ k = 0,1,\ldots,N-1, \; \ell = 0,1,\ldots,N-M-1 \} \tag{86} \]
\[ \cup \{ E_{i,j} \otimes e_k^H \otimes \varpi_l, \; i,j = 0,1,\ldots,B-1, \]
\[ k = 0,1,\ldots,M-1, \; l = 0,1,\ldots,N-M-1 \}. \tag{87} \]

24This is always possible whenever the elements of the first row of \( V \) are all different from zero. However, it is possible with probability one to choose \( V \) such that it generates the null space of \( \Phi \) and satisfies this condition.
APPENDIX D

PROOF OF THEOREM 4

Clearly, if $\mathcal{M}$ is a length-$(N - 1)$ sparse ruler, then (33) defines a periodic sparse ruler. In order to show the converse statement, let us assume that $\mathcal{K}$ is a periodic sparse ruler and take $\mathcal{M} = \mathcal{K} \cap \{0, 1, \ldots, N - 1\}$. Therefore $\{0, \ldots, NB - 1\} \subset \Delta(\mathcal{K})$ and, in particular, $\{N(B - 1), \ldots, NB - 1\} \subset \Delta(\mathcal{K})$, which means that

$$\forall \delta \in \{N(B - 1), \ldots, NB - 1\}, \quad \exists q, p \in \mathcal{K} \text{ such that } q - p = \delta,$$

(88)

Note that, due to the periodicity of $\mathcal{K}$, any $k \in \mathcal{K}$ can be uniquely decomposed as $k = m_k + b_kN$, where $m_k \in \mathcal{M}$ and $b_k \in \{0, \ldots, B - 1\}$. Denote as $m_p$, $m_q$, $b_p$, and $b_q$ the corresponding coefficients of the decomposition of $p$ and $q$. Therefore, the condition above becomes

$$\forall \delta \in \{N(B - 1), \ldots, NB - 1\}, \exists m_p, m_q \in \mathcal{M} \quad \text{and } b_p, b_q \in \{0, \ldots, B - 1\} \text{ s.t. } m_q - m_p + (b_q - b_p)N = \delta.$$

(89)

Since $m_q - m_p \leq N - 1$ and $\delta \leq N(B - 1)$, it is clear that $b_q - b_p$ must equal $B - 1$ in order for the condition $m_q - m_p + (b_q - b_p)N = \delta$ to hold. Then, after subtracting $N(B - 1)$, the following equivalent expression arises:

$$\forall \delta \in \{0, \ldots, N - 1\}, \exists m_p, m_q \in \mathcal{M} \text{ s.t. } m_q - m_p = \delta.$$

In other words, we have that $\Delta(\mathcal{M}) = \{0, \ldots, N - 1\}$, which means that $\mathcal{M}$ is a sparse ruler.

APPENDIX E

PROOF OF THEOREM 6

Assume that $L$ is odd. The proof for $L$ even follows similar lines. If $\Delta(\mathcal{K}) = \{0, \ldots, L - 1\}$, then the matrix from Theorem 1 is given by:

$$R = \begin{bmatrix} 1 & 0^T_L & 0^T_L \\ 0_L & I_L & -jI_L \\ 0_L & K_L & jK_L \\ 1 & 0^T_L & 0^T_L \\ 0_L & I_L & jI_L \\ 0_L & K_L & -jK_L \end{bmatrix}$$

(90)

where $L = \frac{L - 1}{2}$, $0_L$ is an $L \times 1$ vector with all zeros and $K_L$ is an $L \times L$ Hankel matrix with ones in the antidiagonal and zeros elsewhere, i.e., its $(m, n)$-th element equals 1 if $m + n = L - 1$ and 0 otherwise. All the columns are linearly independent so that $\text{rank}(R) = L$ and, according to Theorem 1, $\mathcal{K}$ is an $SC$-covariance sampler.

Now consider removing elements from $\Delta(\mathcal{K})$. It can readily be seen that the rank is not maximum iff there is some $\delta \in \{0, \ldots, L\}$ such that $\delta \notin \Delta(\mathcal{K})$ and $L - \delta \notin \Delta(\mathcal{K})$. Equivalently, we can say that the rank is maximum if and only if $\Delta_L(\mathcal{K}) = \{0, \ldots, L - 1\}$.

APPENDIX F

PROOF OF THEOREM 7

Let us start by showing that if $\mathcal{M}$ is a circular sparse ruler, then $\mathcal{K}$ is a periodic circular sparse ruler or, in other words, if $\Delta_N(\mathcal{M}) = \{0, \ldots, N - 1\}$, then $\Delta_N(\mathcal{K}) = \{0, \ldots, NB - 1\} = \{N, \ldots, NB - 1\}$. Consider any $\delta \in \{0, \ldots, N - 1\}$. Since $\delta \in \Delta_N(\mathcal{M})$, at least one of the following two conditions will hold:

C1: $\exists m_1, m_2 \in \mathcal{M}, \quad m_2 = m_1$ such that

$$m_2 - m_1 = \delta$$

(91)

C2: $\exists m_1, m_2 \in \mathcal{M}, \quad m_2 < m_1$ such that

$$m_2 - m_1 = N + m_2 - m_1 = \delta$$

(92)

We next show that, in both cases, all the elements of the form $\delta + bN$, with $b = 0, \ldots, B - 1$, are in $\Delta_N(\mathcal{K})$:

- C1: consider $k_2 = m_2 + bN$ and $k_1 = m_1$ for any $b = 0, \ldots, B - 1$. Since $k_1, k_2 \in \mathcal{K}$, it follows that $(k_2 - k_1)N_B = m_2 + bN - m_1 = \delta + bN \in \Delta_N(\mathcal{K})$.

- C2: first take $k_1 = m_1$ and $k_2 = m_2 + bN$ with $b = 0, \ldots, B - 2$. Since $k_1, k_2 \in \mathcal{K}$, then $(k_2 - k_1)N_B = m_2 + bN - m_1 = \delta + bN \in \Delta_N(\mathcal{K})$. It suffices only to show that $\delta + bN \in \Delta_N(\mathcal{K})$ when $b = B - 1$. To this end, consider $k_1 = m_1$ and $k_2 = m_2 + B$, which results in $(k_2 - k_1)N_B = NB + m_2 - m_1 = N(B - 1) + N + m_2 - m_1 = N(B - 1) + \delta \in \Delta_N(\mathcal{K})$.

To sum up, we have shown that $\delta + bN \in \Delta_N(\mathcal{K})$ for any $\delta = 0, \ldots, N - 1$ and $b = 0, \ldots, B - 1$, which establishes that $\mathcal{K}$ is a circular sparse ruler.

In order to show the converse statement, we assume that $\mathcal{K}$ is a circular sparse ruler, i.e., $\Delta_N(\mathcal{K}) = \{0, \ldots, NB - 1\}$. In particular, all modular distances of the form $\delta = 0, \ldots, N - 1$ are present in $\Delta_N(\mathcal{K})$, which means that one or both of the following two conditions will be satisfied:

C1': $\exists k_1, k_2 \in \mathcal{K}, \quad k_2 \geq k_1$ such that

$$(k_2 - k_1)N_B = k_2 - k_1 = \delta$$

(93)

C2': $\exists k_1, k_2 \in \mathcal{K}, \quad k_2 < k_1$ such that

$$(k_2 - k_1)N_B = NB + k_2 - k_1 = \delta$$

(94)

It is therefore to be shown that $\delta \in \Delta_N(\mathcal{M})$ in both cases, where $\mathcal{M}$ is defined as $\mathcal{M} = \mathcal{K} \cap \{0, \ldots, N - 1\}$.

- C1': clearly, we can assume without any loss of generality that $k_1 \in \mathcal{M}$. According to whether $k_2$ is also in $\mathcal{M}$ or not, we distinguish two scenarios:

  - $k_2 \in \mathcal{M}$: in this case, it is clear that $(k_2 - k_1)N = k_2 - k_1 \in \Delta_N(\mathcal{M})$.

  - $k_2 \notin \mathcal{M}$: since $0 \leq \delta < N$, it follows that $k_2$ can be written as $k_2 = m + N$ for some $m \in \mathcal{M}$ with $m < k_1$. Therefore, $(m - k_1)N = N + m - k_1 = k_2 - k_1 = \delta \in \Delta_N(\mathcal{M})$.

- C2': since $0 \leq \delta < N$, it can be seen that $N(B - 1) < k_2 - k_1 \leq NB$, which in turn requires $k_2 \in \mathcal{M}$ and $k_1 = m + N(B - 1)$ for some $m \in \mathcal{M}$ with $m > k_2$. Now consider the circular distance between $m$ and $k_2$:

$$(k_2 - m)N = N + k_2 - m = N + k_2 - [k_1 - N(B - 1)] = k_2 - k_1 + NB = \delta \in \Delta_N(\mathcal{M}).$$

(95)
Therefore, we have shown that $\delta \in \Delta_N(\mathcal{M})$ for all $\delta = 0, \ldots, N - 1$, which means that $\mathcal{M}$ is a circular sparse ruler.

**APPENDIX G**

**Proof of Theorem 8**

If we form the matrix $R$ in Theorem 1 using the matrices from (18), we conclude that $\{0, \ldots, d\} \subset \Delta(K)$ in order for $K$ to define a covariance sampler. As we did to prove Theorem 4, we write the following necessary and sufficient condition:

$$\forall \delta \in \{0, \ldots, d\}, \exists m_1, m_2 \in \mathcal{M} \text{ and } b_1, b_2 \in \{0, \ldots, B - 1\}
\text{ such that } m_2 - m_1 + (b_2 - b_1)N = \delta. \quad (96)$$

We start by showing that if $M$ is a circular sparse ruler, then (96) holds true, i.e., $K$ is a covariance sampler. More specifically, we show that $\delta \in \Delta(K) \forall \delta \in \{0, \ldots, N(B - 1)\}$. Consider two different cases:

- **Case 0 $\leq \delta < N(B - 1)$**: Suffices to write $\delta = m_3 - b_3N$, with $m_3 \in \{0, \ldots, N - 1\}$ and $b_3 \in \{0, \ldots, B - 2\}$.

  Since $m_3 \in \Delta_N(\mathcal{M})$, then $m_3$ can be represented either as $m_3 = m_3 - m_1 + b_1N$, with $m_1, m_2 \in \mathcal{M}$ and $b_1, b_2 \in \{0, \ldots, B - 1\}$ such that $m_2 - m_1 + (b_2 - b_1)N = \delta$. In the former case make $m_2 = m_3, m_1 = m_1, b_2 = b_1 = 0$. In the latter case make $m_2 = m_3, m_1 = m_3 - 1, b_2 = b_1 + 1$ and $b_1 = 0$.

- **Case $\delta = N(B - 1)$**: This is trivial since $N(B - 1) \in \Delta(K)$ for any non-empty choice of $\mathcal{M}$.

Now, in order to show the converse theorem, we prove that if $\{0, \ldots, N - 1\} \subset \Delta(K)$, then $\{0, \ldots, N - 1\} \subset \Delta_N(\mathcal{M})$, where $\mathcal{M}$. Let us consider some $\delta \in \{0, \ldots, N - 1\}$. Since $\delta \in \Delta(K)$, it is clear that there exist $m_1, m_2 \in \mathcal{M}$ and $b_1, b_2 \in \{0, \ldots, B - 1\}$ such that $m_2 - m_1 + (b_2 - b_1)N = \delta$. In particular, $(b_2 - b_1)$ can be either 0 or 1. Therefore, for any $\delta \in \{0, \ldots, N - 1\}$, there exists $m_1, m_2 \in \mathcal{M}$ such that either $m_2 - m_1 = \delta$ or $N + m_2 - m_1 = \delta$. Noting that this condition is equivalent to the condition $\{0, \ldots, N - 1\} \subset \Delta_N(\mathcal{M})$ concludes the proof.

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