Relating spectral indices to tensor and scalar amplitudes in inflation

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Within an expansion in slow-roll inflation parameters, we derive the complete second-order expressions relating the ratio of tensor to scalar density perturbations and the spectral index of the scalar spectrum. We find that “corrections” to previously derived formulae can dominate if the tensor to scalar ratio is small. For instance, if \( V V''/(V')^2 \neq 1 \) or if \( m_P^2/(4\pi) |V'''/V'| \gtrsim 1 \), where \( V(\phi) \) is the inflaton potential and \( m_P \) is the Planck mass, then the previously used simple relations between the indices and the tensor to scalar ratio fails. This failure occurs in particular for natural inflation, Coleman–Weinberg inflation, and “chaotic” inflation.

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I. INTRODUCTION

In slow-roll inflation the energy density of the Universe is dominated by the potential energy density of some scalar field $\phi$, known as the *inflaton* field. During slow-roll inflation, scalar density perturbations and gravitational mode perturbations are produced as the inflaton field evolves. The amplitude of the scalar density perturbation as it crosses the Hubble radius after inflation is defined as

$$\left(\frac{\delta \rho}{\rho}\right)_{\text{HOR}} \equiv \frac{m}{\sqrt{2}} A_S(\lambda),$$

(1)

where the constant $m$ equals $2/5$ (or 4) if the perturbation re-enters during the matter (or radiation) dominated era. In addition to the scalar density perturbations, slow-roll inflation produces metric fluctuations, $h$, and the amplitude of the dimensionless strain on scale $\lambda$ when it crosses the Hubble radius after inflation is defined by

$$\left[k^{3/2}h\right]_{\text{HOR}} \equiv A_G(\lambda).$$

(2)

Of particular interest is the ratio of the tensor to scalar perturbations, defined as

$$R(\lambda) \equiv \frac{A_T^2(\lambda)}{m^2 A_S^2(\lambda)/2}.$$  

(3)

Both the scalar density perturbations and the tensor modes contribute to temperature fluctuations in the cosmic background radiation (CBR). On large angular scales ($\theta \gg 1^\circ$, corresponding to the horizon at the last scattering surface) CBR fluctuations are proportional to the sum of the squares of the two modes:

$$[\Delta T(\theta)/T]^2 \propto S(\theta) + T(\theta),$$

(4)

where with the normalization above,

$$S(\theta) = \frac{m^2}{2} A_S^2(\phi); \quad T(\theta) = A_G^2(\phi).$$

(5)

Here we will be interested in scales that crossed the Hubble radius during the matter-dominated epoch when $m = 2/5$.

It is convenient to parameterize the scalar and tensor spectrum by their spectral indices:

$$1 - n_S \equiv d \ln \left[ m^2 A_S^2(\lambda)/2 \right] / d \ln \lambda = d \ln [S(\lambda)] / d \ln \lambda,$$

$$n_T \equiv -d \ln [A_G^2(\lambda)] / d \ln \lambda = -d \ln [T(\lambda)] / d \ln \lambda,$$

(6)

There are three interrelated scales used to characterize sizes: $\lambda$, $\theta$, and $\phi$. The length scale $\lambda$ is related to the angular scale $\theta$ by $\lambda = \theta(34.4^\circ \Omega_0 h)^{-1}\text{Mpc}$; i.e., $\theta$ is the angle subtended on the sky today by comoving scale $\lambda$ at the surface of last scattering of the CBR. This comoving scale $\lambda$ crossed the Hubble radius during inflation when the value of the scalar field was $\phi$.\footnote{There are three interrelated scales used to characterize sizes: $\lambda$, $\theta$, and $\phi$. The length scale $\lambda$ is related to the angular scale $\theta$ by $\lambda = \theta(34.4^\circ \Omega_0 h)^{-1}\text{Mpc}$; i.e., $\theta$ is the angle subtended on the sky today by comoving scale $\lambda$ at the surface of last scattering of the CBR. This comoving scale $\lambda$ crossed the Hubble radius during inflation when the value of the scalar field was $\phi$.}
where $\lambda$ is the physical wavelength that a given scale would have today if it evolved linearly, and is given by $\lambda = [a(t_0)/a(t)]^{-H^{-1}(t)}$, where $t_0$ is the time today, and $a(t)$ and $H(t) = \dot{a}(t)/a(t)$ are the scale factor and Hubble expansion rate, respectively, when the scale left the Hubble radius at time $t$.

With the prospect of measurements of anisotropy in the cosmic microwave background at different angular scales, it may soon be possible to determine the relative contributions of scalar and tensor components to CBR fluctuations \[1\], and thus provide information about the scalar potential driving inflation \[2\]. Several attempts have already been made to develop a method to isolate the scalar and tensor components of the signal \[3\]. This work assumed a relationship between the ratio of the tensor and scalar contribution to the temperature fluctuations and the spectral indices of the form

$$T/S \simeq -7n_T \simeq 7(1 - n_S + \zeta) \quad (7)$$

$$\simeq 7(1 - n_S), \quad (8)$$

where $\zeta \equiv \left[2V'/(3H^2)\right]'$, $V$ is the inflaton potential, prime denotes $d/d\phi$, and $H$ is the expansion rate at the time when the scale $\lambda$ crossed out of the Hubble radius during inflation. It is claimed in Ref. \[3\] that $\zeta$ is small in generic models of inflation, and that confirmation of $n_T \approx n_S - 1$ would be support for inflation and provide detailed information about the first instants of the Universe. In this paper, we discuss the relationship between $R$, $n_T$ and $1 - n_S$ within an expansion of the scalar and tensor amplitudes in terms of “slow-roll” parameters\[5\]. We confirm Eq. (7) as the lowest-order result in this expansion if $|dR/d\ln \lambda| \gg |d^2R/d\ln \lambda^2|$, and we derive the next-order terms in the expansion as well. We also discuss when the approximation in Eq. (8) is accurate, i.e., what is the relative magnitude of the $\zeta$ contribution compared to $1 - n_S$.

We will now discuss the equations relating the tensor and scalar amplitudes to the spectral indices. In the Hamilton–Jacobi treatment of the field equations for inflation, the scalar field $\phi$ is used as a time variable, and the field equations are \[4\]

$$[H'(\phi)^2 - \frac{3}{2}\kappa^2 H^2(\phi) = -\frac{1}{2}\kappa^4 V(\phi); \quad \kappa^2 \dot{\phi} = -2H'(\phi), \quad (9)$$

where dot denotes a time derivative, and $\kappa^2 = 8\pi/m_{Pl}^2$ with $m_{Pl}$ the Planck mass. The scalar and tensor perturbations are calculated in an expansion in slow-roll parameters. These parameters involve combinations of derivatives of $H(\phi)$. In de Sitter space $H(\phi)$ is a constant ($\dot{\phi} = 0$). However, in slow-roll inflation $H(\phi)$ varies with time, albeit slowly. The familiar first-order result for $A_G$ and $A_S$ is

$$A_S(\phi) = -\frac{\sqrt{2}\kappa^2}{8\pi^{3/2}} \frac{H^2(\phi)}{H'(\phi)}; \quad A_G(\phi) = \frac{\kappa}{4\pi^{3/2}} H(\phi). \quad (10)$$

\[2\] We derive these relations for the density fluctuation amplitudes rather than the actual measured temperature fluctuations. This will account for the difference between 6.25 and 7 in the expressions for $1 - n_S$ and $n_T$. 

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However there are corrections to this result that depend upon the slow-roll expansion parameters $\epsilon$ and $\eta$, which are defined as:

$$
\epsilon(\phi) = \frac{2}{\kappa^2} \left[ \frac{H'(\phi)}{H(\phi)} \right]^2, \quad (11)
$$

$$
\eta(\phi) = \frac{2}{\kappa^2} \frac{H''(\phi)}{H(\phi)} = \epsilon + \frac{H(\phi)}{2H'(\phi)} \epsilon'. \quad (12)
$$

These parameters depend upon $H'(\phi)$ and $H'(\phi)$; they are second order in derivatives. In the slow-roll approximation $\epsilon$ and $\eta$ are less than one. A third parameter $\xi$, defined as:

$$
\xi(\phi) = \frac{2}{\kappa^2} \frac{H'''(\phi)}{H'(\phi)} = \eta + \frac{H(\phi)}{H'(\phi)} \eta', \quad (13)
$$

depends upon $H'''(\phi)/H'(\phi)$, and will appear in the expression for the scalar spectral index. The parameter $\xi$ can be much larger than one, even if $\epsilon$ and $\eta$ are small.

The expressions for $A_G$ and $A_S$ in Eq. (10) are correct to first order in $\{\epsilon, \eta\}$, and we will refer to this as the “first-order” results. To second order, Stewart and Lyth showed:

$$
A_S(\phi) = -\frac{\sqrt{2} \kappa^2}{8\pi^{3/2}} \frac{H^2(\phi)}{H'(\phi)} \left[ 1 - (2C + 1)\epsilon + C\eta \right] \quad (14)
$$

$$
A_G(\phi) = \frac{\kappa}{4\pi^{3/2}} \frac{H(\phi)}{H'(\phi)} \left[ 1 - (C + 1)\epsilon \right], \quad (15)
$$

where $C \equiv -2 + \ln 2 + \gamma \simeq -0.73$ and $\gamma = 0.577$ is Euler’s constant. In deriving this expression, terms of order $\epsilon^2$, $\epsilon\eta$, and $\eta^2$, etc., have been neglected in the square bracket. It is important to realize that Eq. (14) is a double expansion in $\epsilon$ and $\eta$, and that they might not be of the same order of magnitude. It makes sense to include all terms in the square brackets of Eq. (14) as long as $|\eta| \sim \epsilon$. However, because $\eta$ depends on the derivative of $\epsilon$, this is usually not the case. It is possible that the second-order terms in one expansion variable might be larger than the first-order terms in the other variable. We will see later that this occurs for many inflationary models of interest.

Let us now find the expressions for $R$, $1 - n_S$, and $n_T$ to second order in the slow-roll parameters. We can find $R$ directly from Eqs. (14) and (13). Using the relation:

$$
\frac{d \ln \lambda}{d \phi} = \frac{\kappa^2}{2} \frac{H}{H'} \left[ 1 - \epsilon \right], \quad (16)
$$

we can also find the spectral indices. The complete second-order expressions then are:

$$
R = \frac{25}{2} \frac{A_G^2}{A_S^2} \simeq \frac{25}{4} \frac{2\epsilon}{[1 - 2C(\eta - \epsilon)]} \quad (17)
$$

$$
n_T = -2\epsilon \left[ 1 + (2C + 3)\epsilon - 2(C + 1)\eta \right] \quad (18)
$$

$$
1 - n_S = 2\epsilon \left[ 2 - \frac{\eta}{\epsilon} + 4(C + 1)\epsilon - (5C + 3)\eta + C\xi(1 + 2(C + 1)\epsilon - C\eta) \right]. \quad (19)
$$

Even if $\eta$ is smaller than one, the derivative of $\eta$ can be large, resulting in $|\xi| > 1$. This turns out to be the case for a wide range of parameters in Coleman-Weinberg inflation.
As mentioned above, $\xi$ can be of order or greater than one, and is therefore not an expansion variable as are $\epsilon$ and $\eta$. We have therefore included terms of order $\epsilon^2 \xi$ and $\epsilon \eta \xi$, since they can contribute to $1 - n_S$ to second-order (i.e., they can have magnitudes similar to those terms of order $\epsilon^2$ or $\epsilon \eta$). This leads to the additional terms $4\epsilon^2 \xi C(C + 1)$ and $-2\epsilon \xi \eta C^2$ in the expression for $1 - n_S$, which were not included in Ref. [6].

Now that we have the complete expression to second order, it is easy to isolate the first-order terms:

\[
R = \frac{25}{4} 2\epsilon \\
n_T = -2\epsilon \\
1 - n_S = 2\epsilon \left[1 + \left(1 - \frac{\eta}{\epsilon} + C\xi\right)\right].
\] (20)

The term corresponding to $\zeta$ in Eq. (20) is $2\epsilon(1 - \eta/\epsilon)$. If $\eta = \epsilon$ and $|\xi| \ll 1$, then $1 - n_S = 2\epsilon = R/6.25$, and Eq. (8) is correct to first order. It is straightforward to show that $\zeta \equiv \left[2V'/\left(3H^2\right)\right]' = (6.25)^{-1}R(\eta/\epsilon - 1) - 2\epsilon \xi/3$. Clearly in terms of the slow-roll parameters, $\zeta$ can be of the same order as $1 - n_S$, $R$, or $n_T$ when $\eta \neq \epsilon$. Eq. (20) can now be rewritten as

\[
R = \frac{6.25\left(1 - n_S + \zeta\right)}{1 + \xi(C - 1/3)}.
\] (21)

We can now see that Eq. (4) holds only when $|\xi| \ll 1$.

The $\zeta$ term will contribute to the value of $1 - n_S$ if $\eta \neq \epsilon$ or if $|\xi| \gtrsim |\eta/\epsilon| \gtrsim 1$. As an example, if $|\eta/\epsilon| \gg 1$ and $|\xi| \ll 1$ (as is the case for natural inflation), then $1 - n_S \simeq -2\eta \simeq -R^{-1}dR/d\ln \lambda$, as we will see in Section III. In this case, $1 - n_S$ depends on the derivative of $R$ divided by $R$, rather than on $R$.

Finally, note that the term proportional to $\xi$ in Eq. (20) comes from the derivative of a second-order term in the expansion for $A_S$. Thus, it cannot be derived from the first-order result. However, if $A_S$ is expanded to higher orders, no higher-order derivative terms will appear. This is because there are no $\eta'$, $\eta''$, etc., terms in the expression for $A_S$ [6], since it was derived under that assumption that $\epsilon$ and $\eta$ are constant as a perturbation evolves outside the Hubble radius during inflation.

II. EXAMPLES

Let us now illustrate this formalism by a couple of examples. Most of these examples have already been worked out by the authors given in the reference section. They are

\footnote{Although $|\xi|$ can be much larger than one, it must be less than $1/\epsilon$: $|\xi| \ll 1/\epsilon$. This is because $|\eta(d\phi/dt)/(H\phi)| = |\xi\epsilon + \eta^2| \ll 1$ in order for Eq. (14) and Eq. (13) to be the correct second-order expressions. See Eq. (58) of [6].}

\footnote{We include the $\eta/\epsilon$ and $\xi$ terms in the first-order expression for $1 - n_S$ because they can be greater than or of order one.}
given here not only to illustrate the formalism developed in the last section, but also to make the case that Eq. (8) does not hold for a collection of popular inflation models.

Only in a few cases, such as power-law inflation, can one find an analytic solution for $H(\phi)$ and its derivatives. It is more useful to have an expression for the slow-roll parameters in terms of the inflaton potential and its derivatives. This is a difficult task however, since the slow-roll parameters cannot be unambiguously expressed in terms of the inflaton potential and its derivatives. This is because $V^{(5)}/V'''$ and higher-order derivatives contribute to the slow-roll parameter expressions, even though there is no restriction on their magnitudes. This point is elaborated in the Appendix.

Starting with the field equations [Eq. (9)], we can express $\epsilon$, $\eta$, and $\xi$ in terms of $V(\phi)$ and its derivatives, appearing in various combinations involving

$$\left(\frac{d^a V(\phi)}{d\phi^a}\right)^b \left(\frac{d^c V(\phi)}{d\phi^c}\right)^{-d}; \quad ab - cd = 2. \quad (22)$$

To second order in $\{\alpha, \beta, \gamma, \delta\}$ defined as

$$\kappa^2 \alpha \equiv (V'/V)^2; \quad \kappa^2 \beta \equiv V''/V; \quad \kappa^2 \gamma \equiv V'''/V'; \quad \kappa^2 \delta \equiv V''''/V''', \quad (23)$$

the expressions for $\{\epsilon, \eta, \xi\}$ when $|\xi| \ll 1$ are

$$\epsilon = \frac{1}{2} \alpha - \alpha \left[\frac{\alpha}{3} - \frac{\beta}{3}\right]$$

$$\eta = -\frac{1}{2} \alpha + \beta + \alpha \left[\frac{2\alpha}{3} - \frac{4\beta}{3} + \gamma\right] + \beta \left[\frac{\beta}{3}\right]$$

$$\xi = \frac{3}{2} \alpha - 3\beta + 2\gamma - \alpha \left[\frac{20\alpha}{3} - \frac{50\beta}{3} + \frac{13\gamma}{3}\right] - \beta \left[9\beta - 8\gamma - \frac{2\delta}{3}\right]. \quad (24)$$

These expressions are derived in the Appendix. We can see that $|\xi| \ll 1$ is satisfied when $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$. The more general solutions for $\epsilon$, $\eta$, and $\xi$ when $\gamma$ and $\delta$ are larger than or are of order one are given also in the Appendix.

Note that the first-order expressions for $\{\epsilon, \eta, \xi\}$ in the expansion in terms of $\{\alpha, \beta, \gamma, \delta\}$ can be found by ignoring all the terms in square brackets.

We can then substitute the above expressions into Eq. (17)-(19) to obtain $R$ and the spectral indices directly in terms of $\{\alpha, \beta, \gamma, \delta\}$. To second order the expressions are

$$R = \frac{25}{4} \alpha + \frac{25}{4} \alpha \left[\frac{2}{3}(3C - 1)(\alpha - \beta)\right]$$

$$n_T = -\alpha - \alpha \left[\frac{11}{6} \alpha - \frac{4}{3} \beta + 2C(\alpha - \beta)\right]$$

$$1 - n_S = 3\alpha - 2\beta + \alpha \left[\left(\frac{5}{6} + 6C\right) \alpha + (1 - 8C)\beta + \left(2C - \frac{2}{3}\right)\gamma\right] - \frac{2}{3} \beta [\beta]. \quad (25)$$

Although in this section we express results as an expansion in the parameters $\alpha$, $\beta$, $\gamma$, and $\delta$, it is important to remember that they are not the slow-roll parameters, and the slow-roll parameters cannot be expressed unambiguously in terms of them.
Again, the first order expressions can be obtained by setting the terms in the square brackets to zero. The first order expression when $|\gamma| \ll 1$ is $1 - n_S = 3\alpha - 2\beta$, as was found first in Ref. [5].

It is shown in the Appendix that if $\gamma > 1$ and $\delta$ is arbitrarily large, the corrections to $R$ and $n_T$ are small, but the corrections to the scalar spectral index, however, can be large:

$$1 - n_S = 3\alpha - 2\beta + \left(2C - \frac{2}{3}\right)\alpha\gamma + \frac{2}{3}(C - 1)\alpha\beta\delta. \quad (26)$$

Here derivatives of order $V^{(5)}/V''$ and higher have been neglected. This new expression will be important for certain regimes in Coleman-Weinberg, scale-invariant, and other models of inflation for which $|\gamma| \gg 1$ or $|\beta\delta| \gg 1$. Note that it is possible for the $\alpha\gamma$ term to be of order the $\alpha$ or $\beta$ terms, thus giving a different expression for $1 - n_S$ to lowest order. We are unable to find an inflaton potential for which this occurs, however.

The procedure we will follow is straightforward. For a given potential, we calculate $\{\alpha, \beta, \gamma, \delta\}$ using Eq. (23) and check to see if $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$. Then if we wish we can calculate the slow-roll parameters using Eq. (24), and then find $R$ and the spectral indices using Eq. (17)-(19), or we can substitute $\{\alpha, \beta, \gamma, \delta\}$ directly into Eq. (25). We now turn to the examples.

### A. Power-law inflation

As a first example, we consider the second-order corrections for power-law inflation, with potential $V(\phi) = M^4\exp(-2\phi/\phi_0)$. For power-law inflation, it is simple to show that $\alpha = \beta = \gamma = \delta = 4/(\kappa^2\phi_0^2)$. For power-law inflation the second-order terms in $\{\epsilon, \eta, \xi\}$ all vanish, and to second order $\epsilon = \eta = \xi = 2/(\kappa^2\phi_0^2)$. Therefore $|\xi| \ll 1$ and to second order

$$R = \frac{25}{2}\epsilon = \frac{25}{2}\frac{2}{\kappa^2\phi_0^2},$$

$$n_T = -2\epsilon[1 + \epsilon] = -\frac{2}{\kappa^2\phi_0^2}\left[1 + \frac{2}{\kappa^2\phi_0^2}\right],$$

$$1 - n_S = 2\epsilon[1 + \epsilon] = \frac{2}{\kappa^2\phi_0^2}\left[1 + \frac{2}{\kappa^2\phi_0^2}\right]. \quad (27)$$

So the relationship between $R$, $n_T$, and $1 - n_S$ to second order is

$$-6.25n_T = 6.25(1 - n_S) = R\left[1 + \frac{2}{25}R\right]. \quad (28)$$

For the exponential potential the second-order corrections are of order $8R^2/25$ of the first-order term. The magnitude of the second-order corrections increase with $R$, and can become important. We can also express the second-order corrections in terms of the parameters of the potential by writing $2R/25 = 2/(\kappa^2\phi_0^2) = m_{Pl}^2/(4\pi\phi_0^2)$.

Only for the exponential potential of power-law inflation is $\epsilon = \eta$. This results in $\zeta \simeq 0$. 


These results for power-law inflation are summarized in Table 1.

**B. Chaotic and hybrid inflation**

Now let’s consider a potential commonly used in chaotic inflation: $V = v\phi^p$, where $v$ and $p$ are constants. The mass dimension of $v$ is $4-p$, and $p$ is an integer. We denote as $\phi_N$ the value of $\phi$ corresponding to the value of the scalar field when the length scale of interest crossed outside the Hubble radius during inflation. The expansion parameters $\{\alpha, \beta, \gamma, \delta\}$ are easily found, giving to second order

$$\epsilon = \frac{1}{2\kappa^2\phi_N^2} \left[ p^2 - \frac{2p^3}{3\kappa^2\phi_N^2} \right],$$

$$\eta = \frac{1}{2\kappa^2\phi_N^2} \left[ p(p-2) - \frac{2p^2(p-3)}{3\kappa^2\phi_N^2} \right],$$

$$\xi = \frac{1}{2\kappa^2\phi_N^2} \left[ (p-2)(p-4) - \frac{2p^2 - 9p + 28}{3\kappa^2\phi_N^2} \right].$$

(30)

Since $\eta \approx \xi$, the condition $|\xi| \ll 1$ is satisfied. It will be convenient to define the dimensionless ratio $r_N = 1/(2\kappa^2\phi_N^2) = m_{Pl}^2/(16\pi\phi_N^2)$. The ratio $r_N$ can be related to the number of $e$-folds to the end of inflation: $r_N = 1/(4Np)$. For $p = 4$, $r_N \approx 1/(16N)$ and for $p = 2$, $r_N = 1/(8N)$. Since for the scales of interest to us $N \sim 50$, $r_N \sim 1.25 \times 10^{-3}$ for $p = 4$ and $r_N \sim 2.5 \times 10^{-3}$ for $p = 2$. The expressions to second order are

$$R = \frac{25}{4} 2r_Np^2 \left[1 - \frac{4}{3} r_N p(1 - 3C)\right]$$

$$n_T = -2r_Np^2 \left[1 + r_N p^2 + \frac{4}{3} r_N p(2 + 3C)\right]$$

$$1 - n_S = 2r_Np^2 \left[1 + \frac{2}{p} + \frac{1}{3} r_N (3p^2 + p(14 + 12C) - 12(1 - 2C))\right].$$

(31)

Note that our expansion is valid in this example, because $|\xi| \lesssim r_N \ll 1$ for $p \leq 4$. For $p = 2$, the terms in the square brackets for $R$, $n_T$ and $1 - n_S$ are $[1 - 8.5r_N]$, $[1 + 3.5r_N]$, and $[2 - 2.3r_N]$ respectively, while for $p = 4$ the terms are $[1 - 17r_N]$, $[1 + 15r_N]$ and $[1.5 + 13r_N]$. Therefore the largest second-order corrections is $17r_N \sim 2.1\%$ for $p = 4$.

Now turn to the first-order results. To first order, the relationship between $R$, $n_T$, and $1 - n_S$ is

$$R = -6.25n_T = 6.25(1 - n_S)/(1 + 2/p).$$

(32)

If we take $p = 2, 4$ and $\infty$, then $R \approx 3.1(1 - n_S)$, $4.2(1 - n_S)$ and $6.25(1 - n_S)$, respectively. Thus, the contribution of the $\zeta$ term depends upon $p$, and is negligible only for $p \gg 1$.

Note that the tensor to scalar ratio for this model need not be very small, since $R = (25/2)r_Np^2$, which for $p = 2$ and $4$ is $R = 0.125$ and $0.25$. In any case, in slightly
more complicated inflation models it is possible to have inflation during an epoch when
the potential is approximately power law, but to modify the relation between $r_N$ and $N$.

As an example of such a modified model, we examine a hybrid inflationary model \[9\] of two scalar fields with potential

$$V(\sigma, \phi) = \frac{1}{4\lambda} \left( M^2 - \lambda \sigma^2 \right)^2 + \frac{m^2}{2} \phi^2 + \frac{g^2}{2} \phi^2 \sigma^2. \quad (33)$$

When $\phi > \phi_C \equiv M/g$, the global minimum is at $\sigma = 0$. Assuming that $\sigma \simeq 0$, the effective potential during inflation is

$$V(\phi) \simeq \frac{M^4}{4\lambda} + \frac{m^2}{2} \phi^2. \quad (34)$$

Except for the addition of a constant term, the potential looks like the $p = 2$ version of
chaotic inflation.

We will assume that $M^2 > 2\lambda m^2/g^2$. During the first epoch of inflation $V \simeq m^2 \phi^2/2$, and during the second epoch $V \simeq M^4/(4\lambda)$. The value of the field at the transition
between these two epochs is given by $\phi_T = M^2/(\sqrt{2} \lambda m)$. We will also assume that
$M^2 \ll \sqrt{\lambda} m m_{Pl}^2$ (or $2\phi_T^2 \phi_C^2 \ll m_{Pl}^4$) so that inflation ends immediately after $\phi \simeq \phi_C \text{[9]}$.

In order to satisfy the slow-roll conditions during inflation, $\phi_T/m_{Pl} \gg 1/\sqrt{2\pi}$. Assuming that $\sigma \simeq 0$, the slow-roll parameters are

$$\kappa^2 \epsilon = \frac{2\phi_N^2}{(\phi_N^2 + \phi_T^2)^2} \left[ 1 - \frac{4}{3} \frac{\phi_N^2 - \phi_T^2}{\kappa^2(\phi_N^2 + \phi_T^2)^2} \right],$$

$$\kappa^2 \eta = \frac{2\phi_T^2}{(\phi_N^2 + \phi_T^2)^2} \left[ 1 + \frac{2}{3} \frac{1}{\phi_T^2} - \frac{16}{3} \frac{\phi_N^2}{\kappa^2(\phi_N^2 + \phi_T^2)^2} \right],$$

$$\kappa^2 \xi = -\frac{6\phi_T^2}{(\phi_N^2 + \phi_T^2)^2} \left[ 1 + \frac{6}{9} \frac{\phi_N^2}{\phi_T^2} - \frac{40}{27} \frac{\phi_N^2}{\phi_T^2} \frac{(\phi_N^2 + 5\phi_T^2)}{\phi_T^2 (\phi_N^2 + \phi_T^2)^2} \right]. \quad (35)$$

Note that for $\phi_N \gg \phi_T$ the results for $p = 2$ chaotic inflations obtains. We will be interested in the opposite limit, $\phi_N \ll \phi_T$. In this case

$$\epsilon = \frac{\phi_N^2}{\phi_T^2} \frac{2}{\kappa^2 \phi_T^2} \left[ 1 + \frac{4}{3\kappa^2 \phi_T^4} \right],$$

$$\eta = \frac{2}{\kappa^2 \phi_T^2} \left[ 1 + \frac{2}{3\kappa^2 \phi_T^4} \right],$$

$$\xi = -\frac{6}{\kappa^2 \phi_T^2} \left[ 1 + \frac{6}{\kappa^2 \phi_T^4} \right]. \quad (36)$$

In this case $|\epsilon| \ll \{|\eta|, |\xi|\}$ but $|\xi| \ll 1$ since $\eta \simeq -\xi/3$. The expressions to second
order are

$$R = \frac{25 \phi_N^2}{\kappa^2 \phi_T^4} \left[ 1 + \frac{4(1-3C)}{3\kappa^2 \phi_T^4} \right].$$
\[ n_T = -\frac{4}{k^2 \phi_T^2} \left[ 1 - \frac{16(2 + 3C)}{3k^2 \phi_T^2} \right] \]

\[ 1 - n_S = -\frac{4}{k^2 \phi_T^2} \left[ 1 + \frac{2}{3k^2 \phi_T^2} \right]. \quad (37) \]

To illustrate the point that it is possible to have \( R \approx 0 \) with \( 1 - n_S \) relatively large, we can work to first order in \( \{\epsilon, \eta, \zeta\} \) and first order in \( \phi_N/\phi_T \). In these limits

\[ R \sim n_T \sim 0; \quad 1 - n_S \sim -2\eta \sim -8r_T, \quad (38) \]

where we have defined \( r_T = 1/(2k^2 \phi_T^2) \) in the same manner as we have defined \( r_N \), so that the slow-roll condition is satisfied for \( r_T \ll 3/4 \). Note that \( n_S > 1 \) in this case, because \( |\eta/\epsilon| \gg 1 \), and because the second derivative of the inflaton potential is positive: \( V'' > 0 \).

The number of e-folds from the end of inflation is

\[ N = -\frac{8\pi}{m_{Pl}^2} \int \frac{V d\phi}{V'} = \frac{1}{8r_N} \left[ 1 + \frac{r_N}{r_T} \ln(r_C/r_N) - \frac{r_N}{r_C} \right], \quad (39) \]

where \( r_C \) is defined to be \( r_C \equiv 1/(2k^2 \phi_C^2) > r_T \). Since we are interested in \( N \sim 60 \), then \( r_C \gg r_N \). In addition, we normalize to the COBE results by calculating the density fluctuation amplitude,

\[ \delta_H = \frac{4\sqrt{2V}}{5\sqrt{3} m_{Pl}^2 \sqrt{\epsilon}}, \quad (40) \]

where \( V \) and \( \epsilon \) are evaluated when the scale \( \lambda \) left the Hubble radius. If \( \delta_H \) is evaluated 60 e-folds before the end of inflation, then \( \delta_H \approx 1.7 \times 10^{-5} \). It is shown in Ref. [10] that there is only a small region in parameter space allowed after imposing this constraint.

We first choose an example whereby the \( \zeta \)-term completely dominates, but for which \( R \) is very small. We take \( M = 9 \times 10^{-4} m_{Pl}, m = 5.4 \times 10^{-7} m_{Pl} \) and \( \lambda = g = 1 \). For this model, the number of e-folds in the second inflationary epoch is 100. Therefore, the observable universe would have density perturbations only from the constant potential epoch for reasonable reheat temperatures. For this model, \( r_T \approx 0.0177 \). For the scale leaving 60 e-folds before the end of inflation, \( R = 3.0 \times 10^{-3}, n_T = -4.9 \times 10^{-4} \) and \( n_S = 1.14 \) to first order. The value for \( R \) ignoring \( \zeta \) is 6.25(1 \( -n_S \)) = -0.88, which is not only negative (and therefore is not physically meaningful) but is also 294 times too large. Clearly \( R \neq 6.25(1 - n_S) \) because the \( \zeta \) term overwhelmingly dominates: \( \eta/\epsilon = r_N/r_T = 286 \). We can also find a value for \( m \) such that \( R \approx 1 \) but the \( \zeta \)-term still contributes non-negligibly. We take \( M = 2.5 \times 10^{-3} m_{Pl}, m = 4.857 \times 10^{-6} m_{Pl} \) and \( \lambda = g = 1 \). For this model, the number of e-folds in the second inflationary epoch is 61 and \( r_T = 0.024 \). In this region in parameter space, the first-order contributions vanish because \( \eta/\epsilon \approx 2 \) when \( N_\phi = 60 \). For a scale leaving 60 e-folds before the end of inflation, \( R = 0.26 \) and \( n_T = -0.042 \) to first order, and \( 1 - n_S = 1.8 \times 10^{-3} \) to second-order. Thus the value for \( R \) ignoring \( \zeta \) is 6.25(1 \( -n_S \)) = 0.011, which is a factor of 24 times too small. We see that even when \( R \) is near one, the “correction” terms can contribute nearly 100%.
C. Natural inflation

In natural inflation \cite{11} the potential is \( V(\phi) = \Lambda^4 [1 + \cos(\phi/f)] \), with \( f \sim m_{Pl} \gg \Lambda \). Inflation occurs when \(|\phi| \ll f\). The spectral indices and \( R \) are easily found to second order in \( \{\alpha, \beta, \gamma\} \). Since the expressions are unwieldy, we expand the trigonometric functions in \( \phi/f \). To lowest order in \( \phi/f \) but second order in \( \{\alpha, \beta, \gamma\} \), the expressions for \( R \) and the spectral indices are

\[
R = \frac{25}{4} \left( 1 + \frac{3C - 1}{3\kappa^2 f^2} \right) \frac{1}{4\kappa^2 f^2} \left( \phi_N^2 \right)
\]

\[
n_T = - \left( 1 + \frac{3C + 2}{3\kappa^2 f^2} \right) \frac{1}{4\kappa^2 f^2} \left( \phi_N^2 \right)
\]

\[
1 - n_S = \left( 1 - \frac{1}{6\kappa^2 f^2} \right) \frac{1}{\kappa^2 f^2}
\]

(41)

where again \( \phi_N \) is the value of \( \phi \) for the length scale of interest. Note that \( 2\beta \simeq \gamma \simeq \delta \), so that \(|\xi| \ll 1\). It is clear that since \(|\phi_N|/f \ll 1\), \( R \sim -n_T \ll 1\), but \( 1 - n_S \) can be substantial.

Now let’s consider the \( \zeta \) contribution in the first-order result. To first order

\[
\epsilon = \frac{1}{8\kappa^2 f^2} \left( \phi_N^2 \right) \left( 1 - \frac{1}{3\kappa^2 f^2} \right); \quad \eta = - \frac{1}{2\kappa^2 f^2} \left( 1 - \frac{1}{6\kappa^2 f^2} \right),
\]

(42)

so \( \eta/\epsilon \gg 1 \) is large, and in fact the \( \zeta \) contribution, proportional to \( 1 - \eta/\epsilon \), dominates. Thus, to first order, \( 1 - n_S \) is independent of \( R \), although \(-6.25n_T \sim R \) is still valid. We remind the reader that for this model \( R \ll 1 \).

D. Scale-invariant inflation

We now consider the potential found in a scale-invariant theory \cite{12}:

\[
V(\phi) = \Lambda^4 \left[ 1 + \frac{\phi - \bar{\phi}}{\phi} \exp(\phi/\bar{\phi}) \right],
\]

(43)

where \( \Lambda \) and \( \bar{\phi} > 0 \) are positive constants with mass dimension 4 and 1, respectively. This potential has a global minimum at \( \phi = 0 \), and slow-roll inflation occurs for \( \phi = \phi_N \) when \(|\phi_N/\bar{\phi}| \gg 1 \) in the regions \( \phi_N > 0 \) or \( \phi_N < 0 \). In the region of positive \( \phi \), the results resemble power-law inflation:

\[
\frac{4}{25} R = -n_T = 1 - n_S = \frac{1}{\kappa^2 \bar{\phi}^2}; \quad \phi/\bar{\phi} \gg 1
\]

(44)

Note that the results do not depend upon \( \phi_N \) (to leading order). Thus \( R, n_T, \) and \( n_S \) are truly constant.
The results in the other region of inflation ($\phi / \bar{\phi} \ll -1$) are more interesting:

$$
R = \frac{25}{4} \frac{1}{\kappa^2 \tilde{\phi}^2} \left( \frac{\phi_N}{\phi} \right)^2 \exp(-2|\phi|/\tilde{\phi})
$$

$$
n_T = -\frac{1}{\kappa^2 \tilde{\phi}^2} \left( \frac{\phi_N}{\phi} \right)^2 \exp(-2|\phi|/\tilde{\phi})
$$

$$
1 - n_S = \frac{2}{\kappa^2 \tilde{\phi}^2} \left( \frac{|\phi_N|}{\bar{\phi}} \right) \exp(-|\phi|/\tilde{\phi}).
$$

Because $\gamma \simeq \delta = 1/(\kappa^2 \tilde{\phi}^2)$, then $\kappa^2 \tilde{\phi}^2 \gg 1$ in order that Eq. (45) is valid. The second-order corrections are $1/(\kappa^2 \tilde{\phi}^2)$ for $1 - n_S$ and $-1/(\kappa^2 \tilde{\phi}^2)|\phi_N|/\phi \exp(-|\phi_N|/\bar{\phi})$ for $R$ and $n_T$. In this case $R \sim -6.25 n_T$, but $1 - n_S \propto \sqrt{R}$. In addition, note that $R$ is small. One can imagine $1 - n_S$ large enough to be detectable, but $R \sim 0$.

Because $|\alpha\gamma| < \beta$, when $\gamma > 1$ the scalar index is $1 - n_S \simeq -2\beta$ to first order, as given above in Eq. (45).

E. Coleman-Weinberg inflation

Finally, we examine the Coleman-Weinberg potential in the context of new inflation. The potential is

$$
V(\phi) = B\sigma^4/2 + B\phi^4[\ln(\phi^2/\sigma^2) - 1/2].
$$

The global minimum of this potential is at $\phi = \sigma$. Inflation occurs for $0 < \phi/\sigma \ll 1$ when the potential is nearly flat: $V \simeq B\sigma^4/2$. We can calculate the slow-roll parameters in this model with the assumption that $V \sim B\sigma^4/2$ in the denominator of $\alpha$ and $\beta$. The spectral indices and $R$ are to lowest order in $\phi_N/\sigma$

$$
R = -\frac{25}{4} n_T = \frac{25}{4} \frac{64}{\kappa^2 \sigma^2} \left( \frac{\phi_N}{\sigma} \right)^6 \left[ \ln \left( \frac{\phi_N^2}{\sigma^2} \right) \right]^2
$$

$$
1 - n_S = \frac{48}{\kappa^2 \sigma^2} \left( \frac{\phi_N}{\sigma} \right)^2 \left[ \ln \left( \frac{\phi_N^2}{\sigma^2} \right) \right] = \frac{3}{N_\phi}.
$$

Note that $\gamma = 3\delta = 6/(\kappa^2 \phi_N^2)$ in this limit, so that $\kappa^2 \phi_N^2 \gg 6$ in order for these expressions to be valid. To first order then and for any value of $\sigma < 3m_{Pl}$, $1 - n_S \propto const$ for a constant value of $N_\phi$.

Again, the fact that $n_T \neq n_S - 1$ can be traced to a large value of $\zeta$, i.e., $|\eta/\epsilon| \neq 1$. Let’s look at the slow-roll parameters with the assumption that $\phi_N/\sigma \ll 1$:

$$
\kappa^2 \epsilon = \frac{32}{\sigma^2} \left( \frac{\phi_N}{\sigma} \right)^6 \left[ \ln \left( \frac{\phi_N^2}{\sigma^2} \right) \right]^2; \quad \kappa^2 \eta = \frac{24}{\sigma^2} \left( \frac{\phi_N}{\sigma} \right)^2 \ln \left( \frac{\phi_N^2}{\sigma^2} \right)
$$

$$
\kappa^2 \xi = \frac{12}{\sigma^2} \left( \frac{\sigma}{\phi_N} \right)^2.
$$

(49)
Table 1: Magnitude of corrections to spectral relations when $|\xi| \ll 1$ ($x \equiv \phi_N/\phi$).

| Inflation Model | Second-order correction | Relative contribution of $\zeta$ term: $|1 - \eta/\epsilon|$ |
|-----------------|-------------------------|--------------------------------------------------|
| Power-law       | $8R\%$                  | 0                                                |
| Chaotic ($p \leq 4$) | $\leq 2.1\%$          | $2/p$: 100% ($p = 2$); 50% ($p = 4$)          |
| Hybrid ($r_N \ll r_T$) | $\leq 2.1\%$          | 100%                                             |
| Hybrid ($r_N \gg r_T$) | $r_T$                  | $(\phi_T/\phi_N)^2 \gg 1$                     |
| Natural         | $m_{Pl}^2/(16\pi f^2)$ | $4f^2/\phi_N^2 \gg 1$                          |
| Scale-Invariant ($x \gg 1$) | $1/(\kappa^2\phi^2)$ | 0                                                |
| Scale-Invariant ($x \ll -1$) | $\lesssim 1/(\kappa^2\phi^2)$ | $2|x|^{-1}e^{|x|} \gg 1$                     |
| Coleman-Weinberg | $\lesssim (3/(2\pi))(m_{Pl}/\phi_N)^2$ | $(3/4)(\sigma/\phi_N)^4/|\ln(\phi_N^2/\sigma^2)| \gg 1$ |

Note that $|\xi| \gg |\eta| \gg |\epsilon|$, so that the $\zeta$ contribution is dominant: $\eta/\epsilon \simeq (3/4)(\sigma/\phi_N)^4/\ln(\phi_N/\sigma)^2 \ll -1$. Therefore, for $1 - n_S$ the second-order contribution will be of order $3/(2\pi)(m_{Pl}/\phi_N)^2$, and $(3/\pi)(m_{Pl}^2\phi_N^2/\sigma^4)\ln(\phi_N^2/\sigma^2)$ for $R$ and $n_T$.

Because $|\alpha\gamma| \ll \beta$, when $\gamma > 1$ the scalar index is $1 - n_S \simeq -2\beta$ to first order, as given above in Eq. (18).

Although $R$ is very small for most values of $\sigma$, if $\sigma$ is large enough, $R$ can increase to as much as 8%. We can then numerically solve the exact expressions. Setting $N_\phi = 50$ and $\sigma = 10m_{Pl}$, we find that $R = 0.083$ and $n_S = 0.962$. Thus, if we neglect the $\zeta$ term, $6.25(1 - n_S) = 0.24$ which is almost 3 times larger than the correct value.

A summary of the results of this section is given in Table 1.

III. First-Order Expression relating $n_S$, $R$ and derivatives of $R$

Using Eqs. (11)-(13) and Eq. (16), we can express $\eta$ and $\xi$ in terms of $\epsilon$ only.

$$\eta = \epsilon + \frac{1 - \epsilon}{2\epsilon} \frac{d\epsilon}{d\ln \lambda}$$  \hspace{1cm} (50)

$$\xi = \eta + \frac{1 - \epsilon}{\epsilon} \frac{d\eta}{d\ln \lambda}$$  \hspace{1cm} (51)
\[ = \eta + \frac{1 - \epsilon}{\epsilon} \left( \frac{d\epsilon}{d\ln \lambda} - \frac{1}{2\epsilon^2} \left( \frac{d\epsilon}{d\ln \lambda} \right)^2 + \frac{1 - \epsilon}{2\epsilon} \frac{d^2\epsilon}{d\ln \lambda^2} \right). \] (52)

Using Eq. (20), we then find that the most general first-order expression relating \( n_S \), \( R \) and derivatives of \( R \) with respect to \( \ln \lambda \) is

\[ 6.25(1 - n_S) \simeq R \left[ 1 - \frac{6.25}{R^2} \left( \frac{dR}{d\ln \lambda} - C \frac{d^2R}{d\ln \lambda^2} \right) \right]. \] (53)

This shows explicitly when the old formula fails. If \( R^{-2}|dR/d\ln \lambda - C d^2R/d\ln \lambda^2| \gg 1 \), then the “correction term” dominates and 6.25(1 - \( n_S \)) \( \neq R \). Thus even if \( R(\lambda) \) changes slowly changes with scale, if \( R \ll 1 \), “corrections” to the previous formula can dominate. Note that the \( \eta/\epsilon - 1 \) and \( \xi \) terms are the \( dR/d\ln \lambda \) and \( d^2R/d\ln \lambda^2 \) terms, respectively.

IV. CONCLUSIONS

In this paper, we derive the contribution of scalar and tensor perturbations from inflation to second order in slow-roll parameters. We find that the previously derived formula fails when \( \eta \neq \epsilon \) or \( |\xi| \gtrsim 1 \). In particular, it fails for natural inflation and Coleman-Weinberg inflation, where \(|\eta| \gg \epsilon\), and for “chaotic” \( \phi^2 \) inflation, where \(|\eta| \ll \epsilon\).

For natural inflation, a type of scale-invariant inflation, and Coleman-Weinberg inflation, to first order \( 1 - n_S \simeq \text{const} \), \( 1 - n_S \propto \sqrt{R} \) and \( 1 - n_S \propto \text{const} \), respectively. Thus the relationship between \( R \) and \( 1 - n_S \) is in general not linear. We have shown that this occurs when \( V|V''|/(V')^2 = 1 \) or \( n_T^2/(4\pi) |V'''/V'| > 1 \). We have also found that the slow-roll parameters, the ratio of the tensor to the scalar amplitude, and the scalar and tensor indices cannot be expressed unambiguously in terms of the potential and its derivatives unless the higher-order derivatives of the potential are small. In addition, we calculated the general expression for \( \xi \) when \( \xi > 1 \). This alters the expression for the scalar spectral index when higher-order derivative terms are marginally large.

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Appendix

In this appendix, we derive the first-order results for $\epsilon$, $\eta$ and $\xi$ when $|\xi| \gtrsim 1$, and the second-order results when $|\xi| \ll 1$. The latter example is used in Section II of this paper.

We define the function $f \equiv H'/H$. Then using Eq. (9), which can be rewritten as

$$H^2 = \frac{\kappa^2 V}{3} \left(1 + \frac{\epsilon}{3}\right),$$

(A.1)

and $\epsilon' = 2 f (\eta - \epsilon)$, “$f$” becomes

$$f = \frac{1}{2} \frac{V'}{V} \left[1 + \frac{\eta - \epsilon}{3}\right].$$

(A.2)

The slow-roll parameters can then be determined in terms of $f$ and its derivatives: $\epsilon = 2 f^2 / \kappa^2$, $\eta = 2 (f^2 + f') / \kappa^2$ and $\xi = 2 (f^2 + 3 f' + f'' / f) / \kappa^2$. In addition, the derivatives of $\eta$ and $\sigma$ are $\eta' = f (\xi - \eta)$ and $\xi' = f (\eta / \epsilon) (\sigma - \xi)$, where

$$\sigma \equiv \frac{2 H'''}{H'}. $$

(A.3)

The mixed second-order expressions for $\epsilon$, $\eta$ and $\xi$ as a function of $\sigma$ and $\{\alpha, \beta, \gamma\}$ (as defined in Eq. (23)) are

$$\epsilon = \frac{1}{2} \alpha \left(1 + \frac{2}{3} \{\eta - \epsilon\}\right)$$

(A.4)

$$\eta = \beta \left(1 + \left\{\frac{\eta - \epsilon}{3}\right\}\right) - \frac{1}{2} \alpha \left[1 - \frac{1}{3} \xi + \left\{\eta - \frac{2}{3} \epsilon + \frac{\xi}{9} (\epsilon - \eta)\right\}\right]$$

(A.5)

$$\xi = 2 \gamma - 3 \beta \left[1 - \frac{\xi}{3} + \left\{2 \eta - \epsilon\right\}\right] + \frac{3 \alpha}{2} \left[1 + \xi \left(-\frac{2}{3} - \frac{1}{9} \eta - \frac{\xi}{27}\right) + \frac{1}{9} \sigma \eta \right]$$

$$+ \left\{2 \eta - \epsilon\right\} \left(4 \epsilon - 5 \eta - \eta \frac{\eta}{\epsilon}\right) + \frac{\sigma \eta}{27} \left(-1 + \frac{\eta}{\epsilon}\right)\right\}. $$

(A.6)

In the above equations, the first-order expressions can be obtained by setting the terms in curly brackets $\{\}$ equal to zero. Note that we cannot solve for $\xi$ (and therefore $\eta$ and $\epsilon$) until $\sigma$ is determined.

We calculate $\sigma$ from $f$ and its derivatives:

$$\frac{\kappa^2 \sigma}{2} = f^2 + 3 f' + f'' + \frac{f'}{f^2 + f} \left[2 f f' + 3 f'' + f f'' + \frac{f f''}{f^2}\right].$$

(A.7)

Because the expression for $\sigma$ to second-order is too long and lends no new insight, we calculate $\sigma$ to first-order only. We note that the second-order corrections are of order $\epsilon$, $\beta$ and $\alpha \xi$. As is similar to the mixed expressions for $\epsilon$, $\eta$ and $\xi$ in Eq. (A.4)-(A.6), several terms on the right-hand side of Eq. (A.7) will contain the factors $\sigma$ and $\sigma'$. We substitute
in the result \( \sigma' = f \left( \frac{\xi}{\eta} \right) (\tau - \sigma) \), where \( \tau = (2/\kappa^2)H^{(5)}/H'' \). Then, combining all of the \( \sigma \) terms and keeping only the largest ones, we find that

\[
\sigma = \frac{2}{\eta} \left( 1 + \frac{\alpha^2 \xi}{36\eta} \right)^{-1} \left[ \alpha \gamma [-4 + \xi] + 2\beta \gamma + \beta \delta + \frac{\alpha \xi \tau}{12} + \alpha^2 \left[ -\frac{47}{8} + \frac{\xi}{216} \left( 873 + 126\eta^2 + 27 \left( \frac{\eta}{\epsilon} \right)^2 - 9\xi \epsilon - 24 \xi \eta^2 + 3\xi^2 \right) \right] + \alpha \beta \left[ \frac{29}{2} + \frac{\xi}{216} \left( -1404 - 144\eta^2 + 84\xi \right) \right] + \beta^2 \left[ -\frac{15}{2} + \frac{3\xi}{2} \right] \right].
\]

\[\text{(A.8)}\]

The quantity \( \tau \) is the higher order derivative term mentioned at the beginning of Section II, and is of order \( \kappa^{-2}V^{(5)}/V'' \) as long as higher order derivatives are unimportant.

As an example, if \(|\xi| \ll 1\),

\[
\sigma = \frac{2}{\eta} \left[ -4\alpha \gamma + 2\beta \gamma + \beta \delta - \frac{47}{8} \alpha^2 + \frac{29}{2} \alpha \beta - \frac{15}{2} \beta^2 \right].
\]

\[\text{(A.9)}\]

Note that \( \eta \sigma \) is second order in small quantities, so that the “\( \alpha \eta \sigma / \epsilon \)” term in Eq. (A.6) is second order. We now calculate \( \epsilon, \eta \) and \( \xi \) in this limit from Eq. (A.4) - (A.6). For the second-order terms, we substitute in Eq. (A.9) and the first-order expressions for \( \epsilon, \eta \) and \( \xi \), which are \( \epsilon = \alpha/2, \eta = \beta - \alpha/2 \) and \( \xi = 2\gamma - 3\beta + 3\alpha/2 \). The final results (when \(|\xi| \ll 1\)) are given in Eq. (24).

To determine the slow-roll parameter \( \xi \) when \( 1 \ll |\xi| \ll 1/\epsilon \), we substitute Eq. (A.8) into Eq. (A.5) along with the first-order expressions

\[
\epsilon = \frac{\alpha}{2}, \quad \eta = \beta - \frac{\alpha}{2} \left( 1 - \frac{\xi}{3} \right).
\]

\[\text{(A.10)}\]

Eq. (A.6) then becomes

\[
\xi = \frac{3}{2} \alpha - 3\beta + 2\gamma - \frac{5}{6} \alpha \xi + \frac{2}{3} \beta \xi \\
+ \frac{2}{9} \alpha \left( \frac{-3\alpha + 6\beta + \alpha \xi + \alpha^2 \xi/2}{-9\alpha + 18\beta + 3\alpha \xi + \alpha^2 \xi/2} \right) \left( \beta \delta - 4\alpha \gamma + 2\beta \gamma + \alpha \gamma \xi - \frac{\alpha \xi^2}{12} + \frac{\alpha \xi \tau}{12} \right) \\
+ \alpha^2 \left[ -\frac{47}{8} + \frac{43\xi}{12} + \frac{2\xi^2}{9} - \frac{\xi^3}{108} \right] + \alpha \beta \left[ \frac{29}{2} - \frac{31\xi}{6} + \frac{\xi^2}{9} \right] \\
+ \beta^2 \left[ -\frac{15}{2} + \frac{3\xi}{3} \right].
\]

\[\text{(A.11)}\]

We now rearrange Eq. (A.11) to get an algebraic equation for \( \xi \) in terms of \( \alpha, \beta, \gamma \) and \( \delta \). In doing so, we keep only the largest terms, keeping in mind that \( \alpha \ll 1, |\beta| \ll 1 \) and \(|\alpha \xi| \ll 1\) in order that the original slow-roll expansion is valid. We do not assume
anything about $\gamma$ or $\delta$, however. Therefore, we keep the largest of all terms that include $\gamma$, $\delta$ and $\tau$. (For example, we keep terms of order $\beta$, $\gamma$ and $\xi$ and neglect terms of order $\alpha\beta$, $\alpha\gamma$, $(\alpha\xi)\gamma$, $(\alpha\xi)\xi$ and $(\alpha\xi)^2\xi$). The solution to Eq. (A.11) then is

$$\xi = \frac{1}{1 - \alpha\tau/18} \left( \frac{3}{2} \alpha - 3\beta + 2\gamma + \frac{2}{3} \beta\delta \right),$$

(A.12)

where we have implicitly assumed that $\tau$ does not depend on $\xi$. If we assume that $|\alpha\tau|/18 \ll 1$, $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$, then to first order, $\xi \simeq 3\alpha/2 - 3\beta + 2\gamma$, as found previously. Note that if $|\gamma| \gtrsim 1$, $|\beta\delta| \ll |\gamma|$ and $|\alpha\tau|/18 \ll 1$, then $\xi = 2\gamma$ to first order. In this case, $|\gamma| \ll 1/(2\alpha)$ in order that the original expansion be valid.

Because we have solved for $\xi$ to first order only in general, we can only calculate $R$ and the spectral indices to first order. Using Eq. (A.10), the slow-roll parameters are $\epsilon = \alpha/2$ and $\eta = \beta - \alpha/2 + \alpha(\gamma + \beta\delta/3)/3$ to first order. Using Eq. (20), the expressions for $R$ and the spectral indices are

$$R = 6.25\alpha = -6.25n_T$$

$$1 - n_S = 3\alpha - 2\beta + \frac{2}{3}(3C - 1)\alpha\gamma + \frac{2}{3}(C - 1)\alpha\beta\delta$$

(A.13)

to first order. This is similar to the first order result obtained when $|\xi| \ll 1$, and differs only in the appearance of the $\gamma$ and $\delta$ terms. Note that when $|\beta|/\alpha \gg |\gamma| > 1$ and $|\beta|/\alpha \gg |\beta\delta|$, the first-order results are the same as when $|\beta|/\alpha \gg 1$, $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$: $1 - n_S = -2\beta$. 

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