Topological Interactions of Non-Abelian Vortices
with Quasi-Particles in High Density QCD

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Abstract

Non-Abelian vortices are topologically stable objects in the color-flavor locked (CFL) phase of dense QCD. We derive a dual Lagrangian starting with the Ginzburg-Landau effective Lagrangian for the CFL phase, and obtain topological interactions of non-Abelian vortices with quasiparticles such as $U(1)_B$ Nambu-Goldstone bosons (phonons) and massive gluons. We find that the phonons couple to the translational zero modes of the vortices while the gluons couple to their orientational zero modes in the internal space.

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Topological defects are important in condensed matter physics since they can affect properties of matter or even the phase structure of the system. This can also be the case in the condensed matter physics of QCD \[1\]. It seems likely from theoretical studies that dense and cold quark matter is a color superconductor \[2\]. Such a state of matter can be realized in the core of neutron stars or possibly in heavy ion collisions. It is quite important to investigate the properties of a color superconductor to verify its existence experimentally and observationally. In particular, at extremely high density and low temperature, perturbative calculations are reliable, and it is believed that color-flavor locked (CFL) phase emerges where all of the three light quarks make pairs and form condensates \[2, 3\]. The original symmetry of QCD, \(G = U(1)_B \times SU(3)_c \times SU(3)_L \times SU(3)_R\), is spontaneously broken down to the color-flavor locked symmetry \(H = SU(3)_{c+L+R} \equiv SU(3)_{c+F}\) in the CFL phase, apart from discrete symmetry. It exhibits superfluidity due to the breaking of the global \(U(1)_B\) symmetry, as well as color superconductivity because of broken color symmetry.

By examining the topology of order parameter space \(G/H \simeq U(3)\) it was shown for the first time in Ref. \[4\] that topologically stable vortices exist as a consequence of the symmetry breaking in the CFL phase. The topologically stable vortices with the lowest winding number in the CFL phase are non-Abelian semisuperfluid vortices. They are called “non-Abelian” in the sense that the remaining symmetry \(H\) is a non-Abelian group \[28\]. Just as a vortex in a conventional type II superconductor carries a magnetic flux, a non-Abelian vortex in the CFL phase carries a color magnetic flux. At the same time, this object behaves as a superfluid vortex as a result of superfluid properties of the color superconductor. Therefore, when a color superconductor rotates faster than some critical velocity, these vortices are created along the axis of the rotation. When the rotation speed is further increased, the created multiple vortices should form a lattice of vortices as in ultra cold atomic superfluids. As a result, if the density in the core of a neutron star is sufficiently high for the color superconducting matter to exist, a vortex lattice should appear and it should yield some physical consequences which can give a signal. The superfluid turbulence may also occur if there is some mechanism which leads to an instability of ordered structure like the Donnelly-Glaberson instability \[8\] in atomic superfluids.

In order to find out what kind of collective structure of vortices is realized, it is essential
to determine the interaction between vortices. This can be clarified by investigating the interaction of vortices with quasiparticles, that mediate the force between vortices. The interaction between vortices and quasiparticles is also useful to study the transport properties of a color superconductor. As quasiparticles, there appear eight gluons with masses given by the Higgs mechanism, eight Nambu-Goldstone (NG) bosons (CFL mesons) associated with the chiral symmetry breaking, and one NG boson (phonon) due to the breaking of the baryon number symmetry $U(1)_B$.

One striking feature of non-Abelian semisuperfluid vortices is that they have internal degrees of freedom, which are called orientational zero modes. The existence of a vortex breaks the color-flavor locked symmetry $SU(3)_{c+F}$ down to its subgroup $SU(2) \times U(1)$ around the core of the vortex. Consequently, there appear further NG modes confined inside the core of the non-Abelian vortex \cite{9}, which parametrize the coset space

$$\mathbb{C}P^2 \simeq \frac{SU(3)_{c+F}}{SU(2) \times U(1)}.$$  

There are degenerate vortex solutions with different color magnetic fluxes, which correspond to points on the $\mathbb{C}P^2$ space. The force between largely separated non-Abelian vortices in the CFL phase is independent of internal orientations \cite{9}. This can be understood by noting that the long-range force is mediated only by massless $U(1)_B$ phonons, which are insensitive to color fluxes. However when the separation between vortices is relatively small, one expects that the force can be mediated also by massive particles which are not color singlet such as gluons, and consequently the short-range force can depend on the color fluxes or the orientational zero modes of the vortices.

In this paper, we determine the interaction between a non-Abelian vortex and quasiparticles such as gluons and phonons. To this end we derive a dual Lagrangian corresponding to the low-energy effective theory of the CFL phase by making use of a dual transformation. A dual transformation is a method widely used in various fields of physics, which relates theories that have different Lagrangians and variables with the roles of equations of motion and Bianchi identities exchanged. Under this transformation, particles and solitons typically interchange their roles, i.e. Noether currents are interchanged with topological currents, and vice-versa. Topological defects in the original theory are mapped to fundamental particles in the dually transformed description. One of the advantages of dualization is that one can deal with the interaction of topological defects by the method of ordinary field theory for parti-
cles. Dualization was first used to deal with Abelian vortices; in the case of local vortices in the Abelian Higgs model, the U(1) gauge field is mapped to massive two-form (antisymmetric tensor) fields \[10\]. On the other hand, in the case of global (or superfluid) vortices, the U(1) NG mode is mapped to massless two-form fields \[11, 12\]. In either case, vortices are mapped to strings which behave as a source of two-form fields \[11\]. Duality can be extended to non-Abelian gauge field theories; The dual transformation was applied by Seo, Okawa and Sugamoto \[13\] to SU(2) gauge theory of Higgs fields in the adjoint representation, which admits the so-called $\mathbb{Z}_2$ vortices. Gauge fields are dualized to non-Abelian generalization of two-form fields \[13, 14\] and the $\mathbb{Z}_2$ vortex appears as a source of non-Abelian two-form fields. However the $\mathbb{Z}_2$ vortices are Abelian and do not have internal orientational modes, although the gauge fields and dualized two-form fields are non-Abelian.

We apply the duality to non-Abelian vortices in the CFL phase, in which internal orientational modes reside. Whether or how these orientational zero modes couple to quasiparticles is a nontrivial question. We show that the interaction of non-Abelian vortices with quasiparticles naturally arises by a dual transformation. In the dual theory, dual gluon fields are described by massive non-Abelian two-form fields as in \[13\]. We find that non-Abelian two-form fields are coupled to non-Abelian vortices through their internal orientational zero modes given in Eq. (1). This result is quite natural because these modes correspond one-to-one to the color magnetic flux which the non-Abelian vortex carries. On the other hand, the phonons are dual to Abelian two-form fields coupled to non-Abelian vortices through their translational zero modes in the same way as for Abelian vortices, but the coupling strength is 1/3 of that for Abelian vortices. Because the interaction terms do not involve the space-time metric, they are called topological interactions. The dual Lagrangian obtained here provides a starting point to analyze a possible collective structure of vortices.

This paper is organized as follows. In Sec. \[II\] we give a Ginzburg-Landau Lagrangian of the CFL phase. In Sec. \[III\] we perform a dual transformation of the Ginzburg-Landau Lagrangian. First we take a dual of massive gluon fields to massive non-Abelian two-form fields. Then we take a dual of the $\text{U}(1)_B$ phonon to the massless two-form field. In Sec. \[IV\] we use a solution of a single non-Abelian vortex to calculate the vorticity tensor of the non-Abelian vortex explicitly. Sec. \[V\] is devoted to summary and discussion. In Appendix we will give some details of calculations omitted in the main text.
II. LOW-ENERGY EFFECTIVE THEORY OF THE CFL PHASE

In this section we introduce an effective theory of the CFL phase and discuss the symmetry of the ground state. We start with a time-dependent Ginzburg-Landau(GL) effective Lagrangian \[29\] for the CFL phase up to second order in time and spatial derivatives. The Lagrangian is written in terms of order parameters $\Phi^L$ and $\Phi^R$,

\[
[\Phi^{L(R)}]_{ai} \sim \epsilon_{abc} \epsilon_{ijk} \langle q^L_{bj} C q^L_{ck} \rangle,
\]

where $a, b, c$ and $i, j, k$ are color and flavor indices, respectively, and $q^L$ are quark fields of left (right) chirality. The crossing terms of $\Phi^L$ and $\Phi^R$ are allowed by symmetries, but are suppressed at high densities \[15\]. Here we take $\Phi^L = -\Phi^R \equiv \Phi$ so that the ground state is positive parity state. In the following we neglect the effect of U(1)$_{\text{EM}}$ electromagnetism since the mixing between broken SU(3)$_c$ color and U(1)$_{\text{EM}}$ is sufficiently small at high densities. We also consider a sufficiently high density region where the masses of three light quarks can be neglected. (We refer the reader to \[20\] for a recent discussion on the effect of a strange quark mass.)

In this case the GL Lagrangian for the CFL phase is given by \[16, 17\]

\[
\mathcal{L}(x) = \frac{\epsilon}{2} (E^a)^2 - \frac{1}{2\lambda} (B^a)^2 + K_0 \text{Tr} \left[ (D_0 \Phi)^\dagger D^0 \Phi \right] + K_1 \text{Tr} \left[ (D_i \Phi)^\dagger D^i \Phi \right] + iK'_0 \text{Tr} \left[ \Phi^\dagger D_0 \Phi \right] - V(\Phi),
\]

\[
V(\Phi) = \text{Tr} \left[ \lambda_1 (\Phi^\dagger \Phi)^2 - \lambda_2 \Phi^\dagger \Phi \right] + \lambda_3 \left( \text{Tr}[\Phi^\dagger \Phi] \right)^2,
\]

where $E^a_i = F^a_{0i}$, $B^a_i = \frac{1}{2}\epsilon_{ijk} F^a_{jk}$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $D_\mu \Phi = (\partial_\mu - ig A^a_\mu T^a) \Phi$ ($i = 1, \ldots, 3$), and $T^a$ are generators of SU($N$)$_c$ normalized as Tr[$T^a T^b$] = $\frac{1}{2} \delta^{ab}$ with color indices $a = 1, 2, \ldots, N^2 - 1$. Here the Lagrangian describes the low-energy effective theory in the CFL phase only for $N = 3$, but we have extended the order parameter field $\Phi$ to an $N$ by $N$ matrix. Coefficients $K_0$, $K'_0$, $K_1$, $\lambda_1$, $\lambda_2$, and $\lambda_3$ are GL parameters dependent on the temperature and the chemical potential of the system but are dealt with as constant parameters in this paper. $\epsilon$ and $\lambda$ are the dielectric constant and the magnetic permeability. The form of the kinetic term of gauge fields in the effective Lagrangian is deduced by requiring the gauge invariance, rotational invariance and parity conservation. The Lorentz symmetry does not have to be maintained in general since superconducting matter exists. However, there exists a modified Lorentz symmetry in which the speed of light is replaced...
by $1/\sqrt{e\lambda}$. It is always possible to restore the Lorentz invariance of the kinetic term of gauge fields by rescaling $x^0, A_0^a, K_0, K_0'$ and $K_1$. Therefore we can start with the Lagrangian in which $\epsilon$ and $\lambda$ are taken to be unity.

For notational convenience, we introduce a vector $K_\mu \equiv (K_0, K_1, K_1, K_1)^T$. Our starting point is the following GL Lagrangian

$$L(x) = -\frac{1}{4} (F_{\mu\nu})^2 + K_\mu \text{Tr} \left[ (D_\mu \Phi)^\dagger D^\mu \Phi \right] + iK_0' \text{Tr} \left[ \Phi^\dagger D_0 \Phi \right] - V(\Phi).$$

(4)

We consider the parameter region $\lambda_1 > 0, \lambda_2 > 0, \lambda_1 + N\lambda_3 > 0$. This Lagrangian includes the most general terms which are consistent with the symmetry group $G$ of generalized QCD with $N$ colors and $N$ flavors,

$$G = \frac{\text{SU}(N)_c \times \text{SU}(N)_F \times U(1)_B}{(\mathbb{Z}_N)^{c+B} \times (\mathbb{Z}_N)^{F+B}},$$

(5)

where $\text{SU}(N)_c$ is the local color symmetry, $\text{SU}(N)_F$ is the global flavor symmetry, and $U(1)_B$ is the global symmetry associated with the baryon number conservation. Under the action of the element $(V_c, V_F, e^{i\theta}) \in G$, $\Phi$ transforms as

$$\Phi \to \Phi' = e^{i\theta}V_c \Phi V_F^T,$$

(6)

where $V_c \in \text{SU}(N)_c, V_F \in \text{SU}(N)_F, e^{i\theta} \in U(1)_B$. The elements corresponding to the discrete groups in the denominator of $G$ can be written as $(z_1, z_2, z_1^{-1}z_2^{-1}) \in G$ with $z_1, z_2 \in \mathbb{Z}_N$. These transformations are removed since they do not change $\Phi$ for any value of $\Phi$.

In the CFL phase the free energy is minimized when $\Phi$ is proportional to a constant unitary matrix. By using the symmetry $G$, one can take the value of $\Phi$ without loss of generality as

$$\Phi = |\Delta| 1_N,$$

(7)

where $|\Delta|$ is a real number. By this expectation value, $\Phi$ is invariant under the restricted transformations

$$\{h|h = (Uz^{-1}, U^*, z), U \in \text{SU}(N), z \in \mathbb{Z}_N \} \subset G.$$  

(8)

Therefore the symmetry $G$ is spontaneously broken in the ground state to

$$H = \frac{\text{SU}(N)_c + F}{(\mathbb{Z}_N)^{c+F}}.$$  

(9)

The order parameter space which characterizes the degenerate ground states is given by

$$G/H \simeq \frac{U(1) \times \text{SU}(N)}{\mathbb{Z}_N} \simeq U(N).$$  

(10)
The CFL phase admits stable vortices since the first homotopy group of the order parameter space is nontrivial:

$$\pi_1 (G/H) \simeq \mathbb{Z}. \quad (11)$$

### III. THE DUAL TRANSFORMATION

In this section, we perform a dual transformation within path integral to derive a dual Lagrangian for the CFL phase. After the transformation, massive gluons are described by massive non-Abelian antisymmetric tensor fields [13] and U(1)$_B$ phonons are described by massless antisymmetric tensor fields. We show that in the dual description vortices appear as sources which can absorb or emit these particles. This is consistent with the empirical rule that a dual transformation interchanges the role of particles and solitons.

#### A. The dual transformation of massive gluons

The partition function of the CFL phase can be written as

$$Z = \int \mathcal{D} A^a_{\mu}(x) \mathcal{D} \Phi(x) \exp \left\{ i \int d^4 x \mathcal{L}(x) \right\}, \quad (12)$$

with the Lagrangian defined in Eq. (4). We shall impose the gauge fixing condition on the field $\Phi$ rather than on the gauge fields since they are integrated out in the end. The gauge fixing condition is taken care of when we consider a concrete vortex solution.

Let us introduce non-Abelian antisymmetric tensor fields $B^a_{\mu\nu}$ by a Hubbard-Stratonovich transformation

$$\exp \left[ i \int d^4 x \left\{ -\frac{1}{4} (F^a_{\mu\nu})^2 \right\} \right] \propto \int \mathcal{D} B^a_{\mu\nu} \exp \left\{ i \int d^4 x \left\{ -\frac{1}{4} \left[ m^2 (B^a_{\mu\nu})^2 - 2m \tilde{B}^a_{\mu\nu} F^a_{\mu\nu} \right] \right\}, \quad (13)$$

where $\tilde{B}^a_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} B^a_{\rho\sigma}$. The parameter $m$ introduced above is a free parameter at this stage. We will choose $m$ later so that the kinetic term of $B^a_{\mu\nu}$ is canonically normalized.

Substituting (13) into (12), we can now perform the integration over the gauge fields $A^a_{\mu}$. The degrees of freedom of gluons are expressed by $B^a_{\mu\nu}$ after this transformation. Each term
in the Lagrangian is transformed as follows:

\[ K_\mu \, \text{Tr} \{ (D_\mu \Phi)^\dagger (D^\mu \Phi) \} + iK'_0 \, \text{Tr} \left[ \Phi^\dagger D_0 \Phi \right] \]

\[ = K_\mu \, \text{Tr} \left\{ \Phi^\dagger (\tilde{\partial}_\mu + igA^a_\mu T^a) (\tilde{\partial}^\mu - igA^{b,\mu}_\mu T^b) \Phi \right\} + iK'_0 \, \text{Tr} \left[ \Phi^\dagger (\partial_0 - igA^a_\mu T^a) \Phi \right] \]

\[ = K_\mu \, \text{Tr} \{ (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) \} + iK'_0 \, \text{Tr} \left[ \Phi^\dagger \partial_0 \Phi \right] + gA^a_\mu J^{a,\mu} \]

\[ + g^2 g_{\mu\nu} \sqrt{K_\mu K_\nu} A^{a,\mu}_\mu A^{b,\nu}_\nu \, \text{Tr} \left[ \Phi^\dagger T^a \cdot T^b \Phi \right], \]  

with \( J^a_\mu \equiv -iK'_0 \, \text{Tr} \left[ \Phi^\dagger (\tilde{\partial}_\mu - \partial_\mu) T^a \Phi \right] \) and

\[ -\frac{1}{2} m\tilde{B}^a_\mu F^{a,\mu} = -\frac{1}{2} m\tilde{B}^a_\mu (2\partial_\mu A^a_\mu + gf^{abc} A^b_\mu A^c_\mu) \]

\[ = mA^a_\mu \partial_\mu \tilde{B}^a_\mu + \frac{1}{2} mgf^{abc} A^a_\mu A^b_\nu \tilde{B}^c_\mu. \]  

(15)

Performing the integration over \( A^a_\mu \), the following part of the partition function is rewritten as

\[ \int \mathcal{D}A^a_\mu \exp \left\{ i \int d^4x \left[ \frac{1}{2} g^2 A^{a,\mu} K^{ab}_\mu \, A^{b,\nu} - m \left( \partial^\nu \tilde{B}^{a,\mu} - \frac{g}{m} J^a_\mu \right) A^{a,\mu} \right] \right\} \]

\[ \propto (\det K_{\mu\nu})^{-1/2} \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \left( \frac{m}{g} \right)^2 \left( \partial_\rho \tilde{B}^{a,\rho,\mu} - \frac{g}{m} J^a_\mu \right) (K^{-1})^{ab}_{\mu\nu} \left( \partial_\sigma \tilde{B}^{b,\sigma,\nu} - \frac{g}{m} J^b_\nu \right) \right] \right\}, \]

(16)

where \( K^{ab}_\mu \) is defined by

\[ K^{ab}_\mu = \frac{1}{2} g_{\mu\nu} \sqrt{K_\mu K_\nu} \, \text{Tr} \left[ \Phi^\dagger T^a \cdot T^b \Phi \right] - \frac{m}{g} f^{abc} \tilde{B}^c_\mu \]

\[ \equiv \Phi^{ab}_\mu - \frac{m}{g} \tilde{B}^{ab}_\mu, \]  

(17)

with \( \Phi^{ab}_\mu \equiv \frac{1}{2} g_{\mu\nu} \sqrt{K_\mu K_\nu} \, \text{Tr} \left[ \Phi^\dagger T^a \cdot T^b \Phi \right] \) and \( \tilde{B}^{ab}_\mu \equiv f^{abc} \tilde{B}^c_\mu \). We define the inverse of \( K^{ab}_\mu \) by the power-series expansion in \( 1/g \)

\[ K^{-1} = \left( \Phi - \frac{m}{g} \tilde{B} \right)^{-1} = \Phi^{-1} \sum_{n=0}^{\infty} \left( \frac{m}{g} \tilde{B} \Phi^{-1} \right)^n. \]  

(18)

As a result, we obtain the following partition function

\[ Z \propto \int \mathcal{D}B^a_\mu (\det K^{ab}_\mu)^{-1/2} \exp \left\{ i \int d^4x \mathcal{L}^*_G(x) \right\}, \]

(19)

where \( \mathcal{L}^*_G \) denotes the gluonic part of the dual Lagrangian

\[ \mathcal{L}^*_G = -\frac{1}{2} \left( \frac{m}{g} \right)^2 \left( \partial_\rho \tilde{B}^{a,\rho,\mu} - \frac{g}{m} J^a_\mu \right) (K^{-1})^{ab}_{\mu\nu} \left( \partial_\sigma \tilde{B}^{b,\sigma,\nu} - \frac{g}{m} J^b_\nu \right) - \frac{1}{4} m^2 (B^a_\mu)^2. \]  

(20)
Now we define the non-Abelian vorticity tensor \( \omega^a_{\mu\nu} \) as the coefficient of the term linearly proportional to \( B^a_{\mu\nu} \). Collecting relevant terms in the above Lagrangian, the coupling between massive gluons and the vorticity is given by

\[
\mathcal{L}_G^* \supset \frac{1}{2} \frac{m}{g} \left[ \partial_\rho \tilde{B}_{a,\mu\rho} (\Phi^{-1})^{ab}_{\mu\nu} J^{b,\nu} + J^a_{\mu} (\Phi^{-1})_{\mu\nu} \partial_\rho \tilde{B}^{b,\nu\rho} \right] - \frac{1}{2} \left( \frac{m}{g} \right) J^a_{\mu} [\Phi^{-1} \tilde{B} \Phi^{-1}]^{ab}_{\mu\nu} J^{b,\nu}
\]

\[
\equiv - \frac{1}{2} \left( \frac{m}{g} \right) B^a_{\lambda\sigma} \omega^a_{\lambda\sigma},
\]

(21)

where we have defined the vorticity tensor \( \omega^a_{\mu\nu} \) as

\[
\omega^a_{\lambda\sigma} \equiv \epsilon^{\lambda\sigma\mu\nu} \left[ \partial_\nu \left\{ (\Phi^{-1})^{(ab)}_{\mu\nu} J^{b,\nu} \right\} + J^{e,\alpha}_{(ab)} (\Phi^{-1})_{\alpha\mu} f^{cda} (\Phi^{-1})^{db}_{\nu\sigma} J^{b,\sigma} \right].
\]

(22)

Here \( A^{(ab)}_{\mu\nu} \) is a symmetrized summation defined by \( A^{(ab)}_{\mu\nu} \equiv A^{ab}_{\mu\nu} + A^{ba}_{\mu\nu} \). This expression for the non-Abelian vorticity is valid for general vortex configurations. The information of vortex configuration is included in \( \Phi \) and \( J^a_{\mu} \).

**B. The dual transformation of U(1)$_B$ phonons**

In the following, we perform a dual transformation of the NG boson associated with the breaking of U(1)$_B$ symmetry. This mode corresponds to the fluctuation of the overall phase of \( \Phi \) which can be parametrized as \( \Phi(x) = e^{i\pi(x)} \psi(x) \), where \( \pi(x) \) is a real scalar field. Substituting this into the following part in the Lagrangian (11) leads to

\[
K_\mu \mathrm{Tr}\{ (\partial_\mu \Phi) \dagger (\partial^\mu \Phi) \} + iK_0' \mathrm{Tr}\{ \Phi^\dagger \partial_0 \Phi \} = K_\mu (\partial_\mu \pi)^2 M^2 - \partial^\mu \pi J_\mu^0 + K_\mu \mathrm{Tr}(\partial_\mu \psi)^2 + iK_0' \mathrm{Tr}\{ \psi^\dagger \partial_0 \psi \},
\]

(23)

with \( J_\mu^0 \equiv \delta_{\mu0} K'_0 M^2 \) and \( M^2 \equiv \mathrm{Tr} [\psi^\dagger \psi] \). We will transform the U(1)$_B$ phonon field \( \pi(x) \) into a massless two-form field \( B^0_{\mu\nu} \). Note that the field \( \pi(x) \) has a multivalued part in general; since \( \pi(x) \) is the phase degree of freedom, \( \pi(x) \) can be multivalued without violating the single-valuedness of \( \Phi(x) \). In fact the multivalued part of \( \pi(x) \) corresponds to a vortex. Let us denote the multivalued part of \( \pi(x) \) as \( \pi_{\text{MV}}(x) \).

The dual transformation of this U(1)$_B$ phonon field is essentially the same as the case of a superfluid. We basically follow the argument of [12]. Let us introduce an auxiliary field
by linearizing the kinetic term of $\pi(x)$ in the partition function as follows

$$Z \propto \int \mathcal{D}\pi \mathcal{D}\pi_{MV} \exp i \left[ \int d^4x \left( M^2 K_\mu \{\partial_\mu (\pi + \pi_{MV})\}^2 - \partial^\mu (\pi + \pi_{MV}) J_\mu^0 \right) \right] \propto \int \mathcal{D}\pi \mathcal{D}\pi_{MV} \mathcal{D}C_\mu \exp i \left[ \int d^4x \left( -\frac{C_\mu}{M^2} - 2C_\mu \sqrt{K_\mu} \partial^\mu (\pi + \pi_{MV}) - \partial^\mu (\pi + \pi_{MV}) J_\mu^0 \right) \right].$$

(24)

Integration over $\pi(x)$ gives a delta function

$$\int \mathcal{D}\pi \exp i \left[ \int d^4x \left( -2C_\mu \sqrt{K_\mu} \partial^\mu \pi + \pi \partial^\mu J_\mu^0 \right) \right] = \delta \left\{ \partial^\mu \left( 2C_\mu \sqrt{K_\mu} + J_\mu^0 \right) \right\}.$$  

(25)

Then let us introduce the dual antisymmetric tensor field $B^0_{\mu\nu}$ by

$$\int \mathcal{D}C_\mu \delta \left\{ \partial^\mu \left( 2C_\mu \sqrt{K_\mu} + J_\mu^0 \right) \right\} \cdots = \int \mathcal{D}C_\mu \mathcal{D}B^0_{\mu\nu} \delta \left( 2C_\mu \sqrt{K_\mu} + J_\mu^0 - m^0 \partial^\nu \tilde{B}^0_{\mu\nu} \right) \cdots$$

(26)

where the dots denote the rest of the integrand and $m^0$ is a parameter. By this change of variables we have introduced an infinite gauge volume, corresponding to the transformation $\delta B^0_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ with a massless vector field $\Lambda_\mu$. This can be taken care of by fixing the gauge later. There is no nontrivial Jacobian factor as the change of variables is linear.

Integrating over $C_\mu$, and transforming a resultant term in the Lagrangian as

$$m^0 \partial^\nu \tilde{B}^0_{\mu\nu} \partial^\mu \pi_{MV} = -m^0 B^{0,\rho\sigma} \epsilon^{\mu\rho\sigma} \partial^\nu \partial^\mu \pi_{MV}$$

$$\equiv -2\pi m^0 B^{0,\rho\sigma} \omega^0_{\rho\sigma},$$

(27)

where the first equality holds up to a total derivative and we have defined

$$\omega^0_{\rho\sigma} \equiv \frac{1}{2\pi} \epsilon^{\mu\rho\sigma} \partial^\nu \partial^\mu \pi_{MV}.$$  

(28)

We thus obtain the dual Lagrangian for the $U(1)_B$ phonon part

$$\mathcal{L}^\ast_{Ph} = - \left( \frac{1}{2M} \right)^2 K_\mu (m^0 \partial_\nu \tilde{B}^0_{\mu\nu} - J_\mu^0)^2 - 2\pi m^0 B^{0,\mu\nu} \omega^0_{\mu\nu}.$$  

(29)

Note that the term linear in $B^0_{\mu\nu}$ coming from the first term of (29) is a total derivative and does not contribute to the equation of motion. The partition function is proportional to

$$Z \propto \int \mathcal{D}\pi_{MV} \mathcal{D}B^0_{\mu\nu} \exp i \left[ \int d^4x \mathcal{L}^\ast_{Ph} \right].$$  

(30)

The $U(1)_B$ phonons are now described by a massless two-form field $B^0_{\mu\nu}$ and vortices appear as sources for $B^0_{\mu\nu}$.
C. The dual Lagrangian

We now summarize the results obtained so far. We have shown that the partition function $Z$ of the CFL phase is proportional to $Z^*$ with the dual Lagrangian $\mathcal{L}^*$:

$$Z \propto Z^* = \int \mathcal{D} B_{\mu\nu} \mathcal{D} \pi \mathcal{D} B_0^{\mu\nu} \mathcal{D} \psi \left( \det K_{\mu\nu} \right)^{-1/2} \exp \left\{ i \int d^4x \mathcal{L}^*(x) \right\},$$  \hspace{1cm} (31)

where

$$\mathcal{L}^* = \mathcal{L}_{G}^* + \mathcal{L}_{Ph}^* + K_\mu \text{Tr}(\partial_\mu \psi)^2 + iK'_0 \text{Tr}\{\psi^\dagger \partial_0 \psi\} - V(\psi).$$ \hspace{1cm} (32)

Here $\mathcal{L}_{G}^*$ and $\mathcal{L}_{Ph}^*$ are given in (20) and (29), respectively. We can discuss the interaction between vortices and quasiparticles in terms of the dual Lagrangian. Vortices are expected to appear as a source term for gluons and U(1)$_B$ phonons. The result above is valid for general vortex configurations.

IV. COUPLING OF NON-ABELIAN VORTEX WITH DUAL FIELDS

In the last section we have obtained the dual Lagrangian for general vortex configurations. In order to see the coupling of quasiparticles with internal orientational degrees of freedom explicitly and to discuss physical consequences, let us consider a single-vortex solution and find an expression of the vorticity tensor for this case.

A. Vortex solution

We consider a cylindrically symmetric vortex configuration along the $z$-axis. Then, the profile of a vortex solution with the lowest energy takes the form

$$\Phi_0(x) = \begin{pmatrix} f(r)e^{i\theta} & 0 \\ 0 & g(r)1_{N-1} \end{pmatrix},$$ \hspace{1cm} (33)

where $f(r)$ and $g(r)$ are functions of the radial coordinate which can be determined by solving equations of motions under their asymptotic behaviors $(f, g) = (|\Delta|, |\Delta|)$ as $r \to \infty$ and $(f, g') = (0, 0)$ as $r \to 0$.

Let us identify the $\mathbb{C}P^{N-1}$ zero mode in the background solution. To this end, it is convenient to take the singular gauge. We perform a gauge transformation on this solution
by \( V \in \text{SU}(N)_c \),
\[
V = \begin{pmatrix}
e^{-i\theta(N-1)/N} & 0 \\
0 & e^{i\theta/N} \mathbf{1}_{N-1}
\end{pmatrix},
\]
which transforms \( \Phi \) as
\[
\Phi^* = V \Phi_0
\]
\[
= e^{i\theta/N} \begin{pmatrix}
f(r) & 0 \\
0 & g(r) \mathbf{1}_{N-1}
\end{pmatrix}
\]
\[
\equiv e^{i\theta/N} \left\{ F(r) \mathbf{1}_N + G(r) \begin{pmatrix}
-N^{-1} & 0 \\
0 & \frac{1}{N} \mathbf{1}_{N-1}
\end{pmatrix}\right\},
\]
where \( F(r) \) and \( G(r) \) are functions of the radial coordinate which are related to \( f(r) \) and \( g(r) \) as
\[
f = F - \frac{N-1}{N} G, \quad g = F + \frac{1}{N} G.
\]
Under this gauge transformation, the vortex solution is physically unchanged. At this stage we fix the gauge by fixing all the local color transformation of \( \Phi \). [19]

General vortex solutions are obtained by acting color-flavor locked \( \text{SU}(N)_{c+F} \) transformations on \( \Phi^* \). Because of the existence of the vortex, the symmetry further breaks down to \( \text{SU}(N-1) \times \text{U}(1) \). We parametrize the associated \( \mathbb{C}P^{N-1} \) orientational moduli space, which is obtained by performing the \( \text{SU}(N)_{c+F} \) transformation on \( \Phi^* \) as
\[
\Phi = U \Phi^* U^{-1} = e^{i\theta/N} \left\{ F(r) \mathbf{1}_N + G(r) \left( \phi\phi^\dagger - \frac{1}{N} \mathbf{1}_N \right) \right\},
\]
where \( U \in \text{SU}(N), \) \( \phi \) is a complex \( N \)-component vector which transforms as the fundamental representation of \( \text{SU}(N)_{c+F} \) and satisfies the relation \( \phi^\dagger \phi = 1 \).

The definition of \( \phi \) has a redundancy in the overall phase. We cannot distinguish \( \phi \) and \( e^{i\alpha} \phi \) because both express the same solution, so they should identified: \( \phi \simeq e^{i\alpha} \phi \). Therefore \( \phi \) are indeed the homogeneous coordinates on \( \mathbb{C}P^{N-1} \). The low-energy excitation on the non-Abelian vortex can be described by the \( \mathbb{C}P^{N-1} \) model. It has been shown that the \( \mathbb{C}P^{N-1} \) modes are in fact normalizable and localized around the vortex [19, 20].

We shall express the vorticity tensor by orientational zero modes \( \phi \) and the profile func-
tions. $\Phi_{\mu\nu}^{ab}$ can be written as

$$
\Phi_{\mu\nu}^{ab} = \frac{1}{2} g_{\mu\nu} \sqrt{K_\mu K_\nu} \operatorname{Tr} [\Phi^\dagger T^a T^b \Phi] = \frac{1}{4} C g_{\mu\nu} \sqrt{K_\mu K_\nu} \left( \delta^{ab} + \frac{2D}{C} \phi^\dagger T^a T^b \phi \right),
$$

(38)

where we have defined $C(r)$ and $D(r)$ by

$$
C(r) \equiv (F - G/N)^2, \quad D(r) \equiv C^2 + 2G(F - G/N),
$$

(39)

whose asymptotic behaviors are $(C, D) \simeq (|\Delta|^2, 0)$ as $r \to \infty$ and $(C, D) \simeq \{(1 - 2/N)^2 |\Delta|^2, (1 - 2/N)^2 |\Delta|^2\}$ as $r \to 0$. We define the inverse of $\Phi_{\mu\nu}^{ab}$ by an expansion in the dimensionless quantity $2D/C$. The latter is a deviation of $\Phi$ from the ground state value and small except for the center of the vortex. Although we expand $(\Phi - 1)_{\mu\nu}^{ab}$ in power-series, we will sum up all the terms later. Therefore $(\Phi - 1)_{\mu\nu}^{ab}$ is calculated exactly whenever the condition $2D/C < 1$ is satisfied. This condition holds except for the vicinity of the vortex core. In matrix notation, $\Phi$ reads

$$
\Phi = \frac{C}{4} \hat{K} \left( 1 + \frac{2D}{C} \hat{\phi} \right),
$$

(40)

where the components of the matrix $\hat{\phi}$ and $\hat{K}$ are given by $\hat{\phi}^{ab} = \phi^\dagger T^a T^b \phi$ and $K_{\mu\nu} = g_{\mu\nu} \sqrt{K_\mu K_\nu}$. Then $(\Phi - 1)_{\mu\nu}^{ab}$ can be written as

$$
(\Phi - 1)_{\mu\nu}^{ab} = g_{\mu\nu} \frac{1}{\sqrt{K_\mu K_\nu}} \frac{4}{C} \left( \delta^{ab} - \frac{2D}{C} \phi^\dagger T^a T^b \phi + \left( \frac{2D}{C} \right)^2 \phi^\dagger T^a T^c \phi \phi^\dagger T^c T^b \phi + \cdots \right).
$$

(42)

More explicitly,

$$
(\Phi - 1)_{\mu\nu}^{ab} = g_{\mu\nu} \frac{1}{\sqrt{K_\mu K_\nu}} \frac{4}{C} \left( \delta^{ab} - \frac{2D}{C} \phi^\dagger T^a T^b \phi + \left( \frac{2D}{C} \right)^2 \phi^\dagger T^a T^c \phi \phi^\dagger T^c T^b \phi + \cdots \right).
$$

On the other hand, the current $J_\mu^a$ is written as

$$
J_\mu^a = -2K_\mu \operatorname{Im} \operatorname{Tr} [\Phi^\dagger \partial_\mu T^a \Phi] - \delta_{\mu0} K_0' \operatorname{Tr} \{\Phi^\dagger T^a \Phi\}
$$

$$
= -\frac{2}{N} K_\mu \left\{ \partial_\mu \theta + \frac{N K_0'}{2 K_0} \delta_{\mu0} \right\} D(r) \phi^\dagger T^a \phi - 2G^2 K_\mu \operatorname{Im} \operatorname{Tr} [\phi \phi^\dagger T^a \{\partial_\mu (\phi \phi^\dagger)\}] .
$$

(43)

B. The vorticity tensor

Let us see how the vorticity tensor can be expressed in terms of profile functions and orientational zero modes in the case of a single-vortex configuration discussed in the last
section. The Abelian component $\omega^0_{\mu\nu}$ of the vorticity tensor is readily identified as

$$\omega^0_{\mu\nu} = \frac{1}{2\pi N} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \partial^\sigma \theta. \quad (44)$$

The non-Abelian component $\omega^a_{\mu\nu}$ of the vorticity tensor is obtained by substituting $J^a_\mu$ and $\Phi^{-1}$ into (22). In order to simplify the following calculations, let us define

$$\mathcal{F}_\mu(a, b) \equiv a \phi \partial_\mu \phi^\dagger + b \partial_\mu \phi \phi^\dagger + (a - b) \phi \phi^\dagger \partial_\mu \phi \phi^\dagger, \quad a, b \in \mathbb{C}. \quad (45)$$

This quantity $\mathcal{F}$ satisfies the following relations

$$\mathcal{F}_\mu(a, b)^\dagger = \mathcal{F}_\mu(b^*, a^*), \quad \text{Tr}[\mathcal{F}_\mu(a, b)] = 0, \quad (46)$$

$$\alpha \mathcal{F}_\mu(a, b) = \mathcal{F}_\mu(\alpha a, \alpha b), \quad \mathcal{F}_\mu(a, b) + \mathcal{F}_\mu(a', b') = \mathcal{F}_\mu(a + a', b + b'), \quad (47)$$

$$\phi^\dagger \mathcal{F}_\mu(a, b) \phi = 0, \quad \phi^\dagger [\mathcal{F}_\mu(a, b), T^a] \phi = \text{Tr}(\mathcal{F}_\mu(a, -b)T^a), \quad \text{Tr} \{T^a[\mathcal{F}_\mu(a, b)^\dagger, \mathcal{F}_\nu(a, b)]\} \quad (48)$$

$$\equiv (\mathcal{F}_\mu(a, b)^\dagger, T^a \mathcal{F}_\nu(a, b)) \equiv \phi^\dagger T^a \phi \partial_\mu \phi^\dagger \phi^\dagger \phi \partial_\mu \phi \phi^\dagger + \phi^\dagger T^a \phi \phi^\dagger \phi \partial_\mu \phi + \phi^\dagger \phi \phi^\dagger T^a \phi \phi^\dagger \phi \partial_\mu \phi \phi^\dagger, \quad \text{Tr} \{T^a \mathcal{F}_\mu(-1, 1)\}, \quad (49)$$

where $[\ldots]$ denotes antisymmetrization of indices. The current $J^a_\mu$ for $a = 1, \ldots, N^2 - 1$ can be written in terms of $\mathcal{F}$ as

$$J^a_\mu = -\frac{2}{N} DK_\mu \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu 0} \right\} \phi^\dagger T^a \phi - 2G^2 K_\mu \text{Im} \text{Tr} \{\phi \phi^\dagger T^a \partial_\mu (\phi \phi^\dagger)\}$$

$$= -\frac{2}{N} DK_\mu \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu 0} \right\} \phi^\dagger T^a \phi + iG^2 K_\mu \text{Tr} \{T^a \mathcal{F}_\mu(-1, 1)\} \quad (51)$$

$$\equiv (J^a_1)_\mu + (J^a_2)_\mu.$$

As is shown in the appendix, the following quantities can be written in terms of $\mathcal{F}$ as

$$\Phi^{-1} J^a_\mu \equiv -\frac{N}{\gamma} \phi^\dagger T^a \phi \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu 0} \right\} + i\alpha \text{Tr} \{T^a \mathcal{F}_\mu(-1, 1)\} \quad (52)$$

$$\equiv (\Phi^{-1} J^a_1)_\mu + (\Phi^{-1} J^a_2)_\mu,$$

$$J \Phi^{-1} \equiv -\frac{8}{N} \gamma \phi^\dagger T^a \phi \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu 0} \right\} - i\beta \text{Tr} \{T^a \mathcal{F}_\mu(\beta, -1)\}, \quad (53)$$

where the functions $\alpha(r)$, $\beta(r)$ and $\gamma(r)$ are defined by

$$\alpha(r) \equiv \frac{4G^2}{C}, \quad \beta(r) \equiv \frac{C}{C + D}, \quad \gamma(r) \equiv \frac{D}{C + D(1 - 1/N)}. \quad (54)$$
Hence the following part in the first term of the vorticity tensor reads

\[
(J\Phi^{-1} + \Phi^{-1}J)^a_\mu = -\frac{16}{N} \gamma^\phi T^a \phi \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu0} \right\} + i\alpha(1 + \beta) \Tr [T^a \mathcal{F}_\mu (-1, 1)] ,
\]  

(55)

where we have used the linearity of \( \mathcal{F} \). The second term in (22) is also expressed by \( \mathcal{F} \), which is proportional to

\[
(J\Phi^{-1})^b f^{abc} (\Phi^{-1}J)^c_\nu = ((J_1 + J_2)\Phi^{-1})^b f^{abc} (\Phi^{-1}(J_1 + J_2))^c_\nu
\]

\[
= (J_1 \Phi^{-1} f \Phi^{-1} J_2)^a_\mu + (J_2 \Phi^{-1} f \Phi^{-1} J_1)^a_\mu + (J_2 \Phi^{-1} f \Phi^{-1} J_2)^a_\mu .
\]

(56)

Each term is calculated as follows:

\[
(J_1 \Phi^{-1} f \Phi^{-1} J_2)^a_\mu + (J_2 \Phi^{-1} f \Phi^{-1} J_1)^a_\mu
\]

\[
= -\frac{8i}{N} \gamma^\phi T^b \phi \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu0} \right\} f^{abc} \Tr (T^c \mathcal{F}_\nu (-\alpha, \alpha\beta)) + \ldots
\]

\[
= \frac{4}{N} \alpha(1 + \beta) \left\{ \partial_\mu \theta + \frac{NK'_0}{2K_0} \delta_{\mu0} \right\} \Tr \{ \mathcal{F}_\nu (1, 1) T^a \} ,
\]

(57)

\[
(J_2 \Phi^{-1})^b f^{abc} (\Phi^{-1}J)^c_\nu
\]

\[
= -\Tr [T^b \mathcal{F}_\mu (-\alpha, \beta, \alpha)] f^{abc} \Tr [T^c \mathcal{F}_\nu (-\alpha, \alpha\beta)]
\]

\[
= -\frac{i\alpha^2}{2} \Tr \{ T^a \left[ \mathcal{F}_\mu (-1, \beta)^\dagger, \mathcal{F}_\nu (-1, \beta) \right] \} ,
\]

(58)

where we have used \( f^{abc} T^c = -i[T^a, T^b] \) and traceless property of \( \mathcal{F} \). This quantity is explicitly rewritten in terms of \( \phi \) as

\[
\Tr \{ T^a \left[ \mathcal{F}_\mu (-1, \beta)^\dagger, \mathcal{F}_\nu (-1, \beta) \right] \}
\]

\[
= (1 + \beta^2) \left[ \phi^\dagger T^a \phi \partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\mu \phi^\dagger T^a \partial_\nu \phi + \phi^\dagger T^a \partial_\mu \phi \partial_\nu \phi^\dagger \phi + \partial_\mu \phi^\dagger T^a \phi \partial_\nu \phi^\dagger \phi \right] ,
\]

(59)

where we have antisymmetrized the quantity with respect to \((\mu, \nu)\), because symmetric part vanishes when contracted with a completely antisymmetric tensor \( \epsilon_{\lambda\sigma\mu\nu} \). Therefore non-Abelian components of the vorticity tensor is written in terms of \( \mathcal{F} \) as

\[
\omega^a_{\lambda\sigma} = \epsilon_{\lambda\sigma\mu\nu} \left[ \partial^\nu \left\{ -\frac{16}{N} \gamma^\phi T^a \phi \left( \partial^\mu \theta + \frac{NK'_0}{2K_0} \delta^{\mu0} \right) + i\alpha(1 + \beta) \Tr [T^a \mathcal{F}_\mu (-1, 1)] \right\}
\]

\[
+ \frac{4}{N} \alpha(1 + \beta) \left( \partial^{[\mu} \theta + \frac{NK'_0}{2K_0} \delta^{[\mu0} \right) \Tr [T^a \mathcal{F}^{\nu]}(1, 1)]
\]

\[
- \frac{i\alpha^2}{2} \Tr \{ T^a \left[ \mathcal{F}^{[\mu} (-1, \beta)^\dagger, \mathcal{F}^{\nu]} (-1, \beta) \right] \} \right] .
\]

(60)
This can be represented explicitly by orientational zero modes as
\[
\omega_{\lambda\sigma}^a = \epsilon_{\lambda\sigma\mu\nu} \left[ \partial^\nu \left\{ -\frac{16}{N} \gamma \left( \partial^\mu \theta + \frac{NK_0'}{2K_0} \delta^{\mu\nu} \right) \phi^\dagger T^a \phi \right. \right.
\]
\[
+i\alpha(1 + \beta) \left( \phi^\dagger T^a \partial^\mu \phi - \partial^\mu \phi^\dagger T^a \phi + 2\phi^\dagger T^a \phi \partial^\mu \phi^\dagger \phi \right) \left\} \right.
\]
\[
-\frac{4}{N} \alpha(1 + \beta) \left( \partial^\mu \phi^\dagger T^a \phi + \phi^\dagger T^a \partial^\mu \phi \right) \left( \partial^\nu \theta + \frac{NK_0'}{2K_0} \delta^{\nu\mu} \right)
\]
\[
-\frac{i}{2} \beta^2 (1 + \beta^2) \left( \phi^\dagger T^a \phi \partial^\mu \phi^\dagger \partial^\nu \phi + \partial^\mu \phi \phi^\dagger T^a \partial^\nu \phi + \phi^\dagger T^a \phi \partial^\mu \phi^\dagger \phi + \partial^\mu \phi \phi^\dagger T^a \phi \partial^\nu \phi^\dagger \phi \right) \right].
\]
\tag{61}

Now we estimate the relative importance of terms in \([61]\). Let us define a parameter \(\epsilon \equiv D/C\). Since \(\epsilon\) expresses the deviation of the profile function from the ground state value, \(\epsilon\) is small away from the vortex core, as noted above. We can estimate the strength of each term of the vorticity in terms of the order of \(\epsilon\): \(\alpha(r) \sim O(\epsilon^2)\), \(\beta(r) \sim O(\epsilon^0)\) and \(\gamma(r) \sim O(\epsilon)\). Therefore the leading order part of the vorticity is given by
\[
\omega_{\lambda\sigma}^a = \epsilon_{\lambda\sigma\mu\nu} \partial^\nu \left[ -\frac{16}{N} \gamma \left( \partial^\mu \theta + \frac{NK_0'}{2K_0} \delta^{\mu\nu} \right) \phi^\dagger T^a \phi \right] + O(\epsilon^2).
\]
\tag{62}

In summary, the interaction of vortices and quasiparticles, dual phonons and dual gluons, is described by the action
\[
S_{\text{int}} = S_{\text{int}}^{\text{Ph}} + S_{\text{int}}^{\text{G}},
\]
\tag{63}
respectively, in which \(S_{\text{int}}^{\text{Ph}}\) and \(S_{\text{int}}^{\text{G}}\) are defined by
\[
S_{\text{int}}^{\text{Ph}} = -2\pi m^0 \int d^4x \; B_{\mu\nu}^0 \omega_{0,\mu\nu}^a,
\]
\tag{64}
\[
S_{\text{int}}^{\text{G}} = -\frac{m}{g} \int d^4x \; B_{\mu\nu}^a \omega_{a,\mu\nu}.
\]
\tag{65}
Here \(\omega_{0,\mu\nu}^a\) and \(\omega_{a,\mu\nu}^a\) are given by \([41]\) and \([61]\), respectively. Since these interaction terms do not couple to the space-time metric, they are topological interactions.

The parameters \(m\) and \(m^0\) are chosen so that the kinetic term of two-form fields are canonically normalized. We require the kinetic terms of \(B_{\mu\nu}^a\) and \(B_{\mu\nu}^0\) to take the form
\[
\int d^4x \left[ -\frac{1}{12K_{\mu\nu\sigma}} \left( H_{\mu\nu\sigma}^a H_{0,\mu\nu\sigma}^0 + H_{\mu\nu\sigma}^0 H_{0,\mu\nu\sigma}^0 \right) \right],
\]
\tag{66}
where \(H_{\mu\nu\sigma}^a \equiv \partial_\mu B_{\nu\sigma}^a + \partial_\nu B_{\sigma\mu}^a + \partial_\sigma B_{\mu\nu}^a\), \(H_{\mu\nu\sigma}^0 \equiv \partial_\mu B_{\nu\sigma}^0 + \partial_\nu B_{\sigma\mu}^0 + \partial_\sigma B_{\mu\nu}^0\), and \(\tilde{K}_{\mu\nu\sigma} = \epsilon_{\rho\mu\nu\sigma} K^\rho\). Under this requirement, the parameters \(m\) and \(m^0\) are determined to be
\[
m = \frac{g\sqrt{C(r)}}{4}, \quad m^0 = \frac{M}{\sqrt{2}} = \sqrt{\frac{NC(r) + D(r)}{2}}.
\]
\tag{67}
The interaction is localized around the vortex while gluons and phonons propagate in the bulk. This means that the vortex actually appears as a source for these particles. Each term in (61) is proportional to $\alpha(r)$ or $\gamma(r)$ which are functions of the radial coordinate and is nonzero only in the vicinity of the vortex. The functions $\alpha(r)$ or $\gamma(r)$ decay exponentially away from the center of the vortex with the characteristic length, $\min(m_G^{-1}, m_{\chi}^{-1})$, where $m_G$ is the mass of gluons and $m_{\chi}$ is the mass of traceless part of the scalar field $\Phi$ [18]. Specific form of the profile functions $\alpha(r)$, $\beta(r)$ and $\gamma(r)$ can be determined by solving the equations of motions numerically.

Now we discuss some properties of the interaction (63). First let us look at the interaction (64) of vortices with U(1)$_B$ phonons. This part is essentially the same as a vortex in a superfluid. For general vortex configurations we can write the Abelian component of the vorticity tensor as

$$(\omega^0)^{\mu\nu}(x) = \frac{1}{N} \int d\tau d\sigma \frac{\partial (X^\mu, X^\nu)}{\partial (\tau, \sigma)} \delta^{(4)}(x - X^\mu(\tau, \sigma)), \quad (68)$$

where the the space-time position of the vortex world sheet is parametrized as $X^\mu(\tau, \sigma)$ with the world-sheet coordinates $\tau, \sigma$. The interaction of vortices with U(1)$_B$ phonons is written as

$$S_{\text{Ph int}} = -\frac{2\pi m_0}{N} \int d\sigma^{\mu\nu} B_{\mu\nu}^0, \quad (69)$$

where $d\sigma^{\mu\nu} \equiv \frac{\partial (X^\mu, X^\nu)}{\partial (\tau, \sigma)} d\tau d\sigma$ is an area element of the vortex world sheet. The magnitude of the Abelian vorticity is proportional to the winding number with respect to U(1)$_B$, so we have the factor $1/N$ which is the winding number of vortices of the lowest energy in the CFL phase. U(1)$_B$ phonons $B_{\mu\nu}^0$ do not couple to the orientational zero modes on the vortices, which is anticipated since $B_{\mu\nu}^0$ is a singlet under SU($N$)$_{c+F}$ while the orientational zero modes are in the fundamental representation.

Next, let us look at the interaction of vortices with gluons (65). As can be seen in (61), gluons actually couple to the orientational zero modes on the vortex. As a result, virtual gluons can be emitted and absorbed from orientational zero modes [31]. This will lead to physical effects. For example, when two vortices are in a small distance, the orientational zero modes residing on them can exchange virtual gluons, which results in orientation-dependent corrections to the vortex-vortex interaction. It is also possible that the orientational zero modes on a single vortex emit and absorb virtual gluons. In the case of Abelian vortices, it
is known that the emission and absorption of virtual photons from a single vortex result in the renormalization of its tension \cite{21}. The emission and absorption of virtual gluons can lead to similar effects for non-Abelian vortices.

V. SUMMARY AND DISCUSSION

We have derived a dual Lagrangian of the Ginzburg-Landau effective Lagrangian for the CFL phase, in which massive gluon fields have been dualized to massive non-Abelian two-form fields while the U(1)$_B$ phonon field has been dualized to the massless Abelian two-form field. In the dual theory, non-Abelian vortices have appeared as sources of these dual fields, and we have thus obtained the topological interactions of non-Abelian vortices with these quasiparticles. By making use of a single non-Abelian vortex solution, we have explicitly calculated the vorticity tensor of the single non-Abelian vortex, and then have found that the phonons couple to the translational zero modes of the vortices as in the Abelian vortices while the gluons couple to the internal orientational zero modes CP$^{N-1}$ of them.

Let us make comments on some applications of the dual Lagrangian obtained above.

We can investigate the nature of the force between vortices. When vortices are separated at a large distance, U(1)$_B$ phonons mediate the most part of the force between them; the force between two vortices at the distance $R$ is proportional to $1/R$ \cite{9} which can be understood as follows. The tension of individual vortex with the U(1)$_B$ winding $k$ is $E \sim k^2 \log \Lambda$ with the system size $\Lambda$. Therefore the tension of a non-Abelian vortex is $1/N^2$ of that of a U(1)$_B$ vortex. Consequently the interaction energy for two non-Abelian vortices and likewise the force between them is $1/N^2$ of those between two Abelian U(1)$_B$ vortices. This is consistent with our result \cite{69} of the coupling of U(1)$_B$ phonons to non-Abelian vortices, which is $1/N$ of the one of U(1)$_B$ phonons to Abelian vortices; the force between two non-Abelian vortices is proportional to the product of the U(1)$_B$ charges of them, which is $1/N^2$.

On the other hand, when the separation of vortices is small, exchange of virtual gluons will also contribute to the force between vortices. In that case the property of the force is dependent on the CP$^{N-1}$ internal orientations $\phi$ of the vortices because gluons (non-Abelian two-form fields) couple to the internal orientations through \cite{61}. This is natural since non-Abelian vortices carry color magnetic fluxes.

It is also interesting to apply the dual Lagrangian we have obtained to the physics of
the core of dense stars. It is well known that the rotation of ultra cold atomic superfluids leads to a vortex lattice structure. Neutron stars are also rotating rapidly, so we can expect that a lattice structure of non-Abelian vortices will be formed if the core is dense enough to accommodate the CFL matter. If the speed of rotation is high, the distance between neighboring vortices is small and internal orientation will affect the nature of the interaction of vortices. As the type of collective structures of vortices is sensitive to details of the interaction between them, it is interesting to investigate the phase diagram as a function of the speed of rotations by making use of the action we have derived. We expect that there may appear phases with vortex lattice structures different from ordinary triangular Abrikosov lattice.

The non-Abelian quantum turbulence phase where vortices are tangling and reconnecting here and there will be also possible if there is some mechanism which leads to an instability of lattice structures [32]. It is interesting to investigate the property of this turbulence since we can expect behaviors which can be considerably different from atomic superfluids or helium superfluids. The energy spectra of turbulent fluids are determined by what kind of interaction transfers the energy from lower to higher momentum region. In a helium superfluid the Kelvin wave cascade and the phonon emission are considered as the dominant mechanism of energy transfer at the length scale which is smaller than the mean vortex distance. On the other hand, in the non-Abelian quantum turbulence, not only the Kelvin wave cascade and the emission of the U(1)$_B$ phonons from the vortex but also waves of the CP$^2$ orientational zero modes can contribute to the energy cascade. The propagation of CP$^2$ waves can be affected by emission and absorption of virtual gluons. The dual Lagrangian obtained in this paper will provide a basis for the analysis of properties of the non-Abelian quantum turbulence.

Finally the method we have used here can be applied to other types of non-Abelian vortices. In the models with gauged U(1) symmetry, non-Abelian vortices also exist which are called local non-Abelian vortices. In the literature, local non-Abelian vortices in relativistic field theories with (or without) supersymmetry have been studied extensively [23] (see [24] for a review). In this case, vortices are coupled to only massive fields [33]. On the other hand, at sufficiently high density where U(1)$_A$ symmetry is effectively restored, global non-Abelian vortices [26] become topologically stable. They are expected to interact with CFL mesons topologically. Via a dual transformation, CFL mesons are described by massless
non-Abelian two-form fields with antisymmetric tensor gauge invariance, which is known as the Freedman-Townsend model [14]. Unlike the case of semisuperfluid non-Abelian vortices, interaction between two non-Abelian global vortices was shown to depend on their $\mathbb{C}P^{N-1}$ orientations [27]. This should be explained by the exchange of massless non-Abelian two-form fields. Extensions to these cases will be reported elsewhere.

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Appendix A: Derivation of (52) and (53)

Here we derive Eqs. (52) and (53). We make use of the property of Casimir operators. The generators of SU($N$) in the fundamental representation obey the following relation

$$ (T^a)^i_1(T^a)^k_l = \frac{1}{2}\delta^i_l\delta^k_1 - \frac{1}{2N}\delta^i_1\delta^k_l - \frac{1}{2}\frac{1}{N}\delta^i_j\delta^k_1.$$

(A1)

First we show that $(\Phi^{-1})^a_{\mu\nu}$ can be written as

$$ (\Phi^{-1})^a_{\mu\nu} = \frac{4g_{\mu\nu}}{C\sqrt{K_\mu K_\nu}} \left\{ 2\text{Tr}(T^a T^b) - 2(1-\beta)\phi^\dagger T^a T^b \phi + 2(1-\beta-\gamma)\phi^\dagger T^a \phi \phi^\dagger T^b \phi \right\}, \quad (A2)$$

with $\beta(r) \equiv \frac{C}{C+D}$ and $\gamma(r) \equiv \frac{D}{C+D(1-1/N)}$. Let us define the quantity

$$ X^a_{\mu\nu} \equiv \phi^\dagger T^a T^b \phi \phi^\dagger T^a T^b \phi \phi \cdots \phi^\dagger T^b \phi. \quad (A3)$$

By using the property of the Casimir operator (A1) and $\phi^\dagger \phi = 1$ it can be shown that $X_n$ satisfy the following recurrence relation

$$ X^a_{\mu\nu} = \frac{1}{2}X^a_{\mu\nu-1} - \frac{1}{2N} \left( \frac{1}{2} - \frac{1}{2N} \right)^{n-2} \phi^\dagger T^a \phi \phi^\dagger T^b \phi. \quad (A4)$$
This equation can be solved to give

\[ X_{n}^{ab} = \left( \frac{1}{2} \right)^{n-1} \left[ \phi^{\dagger} T^{a} T^{b} \phi - 2 \left\{ 1 - \left( 1 - 1/N \right)^{n-1} \right\} \phi^{\dagger} T^{a} \phi \phi^{\dagger} T^{b} \phi \right], \quad (A5) \]

with the initial condition \( X_{1}^{ab} = \phi^{\dagger} T^{a} T^{b} \phi \). The quantity \((\Phi^{-1})_{\mu\nu}^{ab}\) is defined by an expansion

\[ (\Phi^{-1})_{\mu\nu}^{ab} = g_{\mu\nu} \frac{1}{\sqrt{K_{\mu} K_{\nu} C}} \left( \delta_{ab} + \sum_{n=1}^{\infty} \left( -2D C \right)^{n} X_{n}^{ab} \right). \quad (A6) \]

Substituting Eq. (A6) into the above equation, now we perform the sum with respect to \( n \) and obtain the result (A2).

Then let us proceed to calculate \((\Phi^{-1} J)_{\mu}^{a} = (\Phi^{-1})_{\mu\nu}^{ab} J_{b\nu}^{.}\). The current \( J_{\mu}^{a} \) for \( a = 1, \cdots, N^{2} - 1 \) can be written in terms of \( F \) as Eq. (51). Note that the following relations hold

\[ \phi^{\dagger} T^{a} T^{b} \phi \phi^{\dagger} T^{b} \phi = \frac{1}{2} \left( 1 - \frac{1}{N} \right) \phi^{\dagger} T^{a} \phi, \quad (A7) \]

\[ \phi^{\dagger} T^{a} T^{b} \phi \text{ Tr} \left\{ T^{b} F_{\mu}(-1,1) \right\} = \frac{1}{2} \phi^{\dagger} T^{a} F_{\mu}(-1,1) \phi - \frac{1}{2N} \phi^{\dagger} T^{a} \phi \text{ Tr} F_{\mu}(-1,1) \]

\[ = \frac{1}{2} \text{ Tr} \left\{ T^{a} F_{\mu}(0,1) \right\}, \quad (A8) \]

\[ \phi^{\dagger} T^{b} \phi \text{ Tr} \left\{ T^{b} F_{\mu}(-1,1) \right\} = \frac{1}{2} \phi^{\dagger} F_{\mu}(-1,1) \phi - \frac{1}{2N} \text{ Tr} F_{\mu}(-1,1) \]

\[ = 0, \quad (A9) \]

where we have used Eq. (A1), \( \phi^{\dagger} \phi = 1 \) and the properties of \( F \). Applying these relations to the product of Eqs. (A2) and (51) leads to

\[ (\Phi^{-1} J)_{\mu}^{a} = -\frac{8}{N} \gamma \phi^{\dagger} T^{a} \phi \left\{ \partial_{\mu} \theta + \frac{NK_{\mu}'}{2K_{0}} \delta_{\mu\alpha} \right\} + i\alpha \text{ Tr} \left[ T^{a} F_{\mu}(-1, \beta) \right], \quad (A10) \]

with \( \alpha \equiv \frac{4G_{2}}{C} \). By taking the complex conjugate, we have

\[ (J(\Phi^{-1}))_{\mu}^{a} = -\frac{8}{N} \gamma \phi^{\dagger} T^{a} \phi \left\{ \partial_{\mu} \theta + \frac{NK_{\mu}'}{2K_{0}} \delta_{\mu\alpha} \right\} - i\alpha \text{ Tr} \left[ T^{a} F_{\mu}(\beta, -1) \right]. \quad (A11) \]

[1] K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333

[2] M. G. Alford, A. Schmitt, K. Rajagopal and T. Schafer, Rev. Mod. Phys. 80, 1455 (2008).

[3] M. G. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B 537, 443 (1999).
[4] A. P. Balachandran, S. Digal and T. Matsuura, Phys. Rev. D 73, 074009 (2006).
[5] M. M. Forbes and A. R. Zhitnitsky, Phys. Rev. D 65, 085009 (2002).
[6] K. Iida and G. Baym, Phys. Rev. D 66, 014015 (2002).
[7] K. Iida, Phys. Rev. D 71, 054011 (2005).
[8] W. I. Glaberson, W. W. Johnson and R. M. Ostermeier, Phys. Rev. Lett. 33, 1197 (1974).
[9] E. Nakano, M. Nitta, and T. Matsuura, Phys. Rev. D 78, 045002 (2008); Prog. Theor. Phys. Suppl. 174, 254 (2008).
[10] A. Sugamoto, Phys. Rev. D 19, 1820 (1979).
[11] M. Kalb and P. Ramond, Phys. Rev. D 9, 2273 (1974); Y. Nambu, Phys. Rev. D 10, 4262 (1974).
[12] K. M. Lee, Phys. Rev. D 48, 2493 (1993).
[13] K. Seo, M. Okawa and A. Sugamoto, Phys. Rev. D 19, 3744 (1979).
[14] D. Z. Freedman and P. K. Townsend, Nucl. Phys. B 177, 282 (1981).
[15] D. T. Son and M. A. Stephanov, Phys. Rev. D 61, 074012 (2000).
[16] K. Iida and G. Baym, Phys. Rev. D 63, 074018 (2001) [Erratum-ibid. D 66, 059903 (2002)].
[17] I. Giannakis and H. c. Ren, Phys. Rev. D 65, 054017 (2002).
[18] M. Eto and M. Nitta, Phys. Rev. D 80, 125007 (2009).
[19] M. Eto, E. Nakano and M. Nitta, Phys. Rev. D 80, 125011 (2009).
[20] M. Eto, M. Nitta and N. Yamamoto, Phys. Rev. Lett. 104, 161601 (2010).
[21] P. Orland, Nucl. Phys. B428, 221-232 (1994).
[22] M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi and W. Vinci, Phys. Rev. Lett. 98, 091602 (2007).
[23] A. Hanany and D. Tong, JHEP 0307, 037 (2003). R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673, 187 (2003); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 96, 161601 (2006).
[24] D. Tong, arXiv:hep-th/0509216, D. Tong, Annals Phys. 324, 30 (2009); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A 39, R315 (2006); M. Shifman and A. Yung, Rev. Mod. Phys. 79, 1139 (2007). “Supersymmetric solitons,” Cambridge, UK: Cambridge Univ. Pr. (2009) 259 p.
[25] M. Shifman and A. Yung, Phys. Rev. D 73, 125012 (2006); M. Eto et al., Phys. Rev. D 76, 105002 (2007).
[26] A. P. Balachandran and S. Digal, Phys. Rev. D 66, 034018 (2002); M. Nitta and N. Shiiki, Phys. Lett. B 658, 143 (2008); M. Eto, E. Nakano and M. Nitta, Nucl. Phys. B 821, 129 (2009).

[27] E. Nakano, M. Nitta and T. Matsunaga, Phys. Lett. B 672, 61 (2009).

[28] U(1)_B superfluid vortices found in [5, 6] have winding number one while non-Abelian semisuperfluid vortices have winding number 1/3, and consequently the former decay into three of the latter. Color magnetic flux tubes discussed in [6, 7] have triple amount of fluxes of non-Abelian semisuperfluid vortices, and are topologically and dynamically unstable.

[29] The time-dependent GL Lagrangian is valid when the temperature is close to the critical temperature and deviations from equilibrium are small.

[30] The term $\text{Tr} \left[ \partial^{\mu} \psi^{\dagger} \psi - \psi^{\dagger} \partial^{\mu} \psi \right]$ automatically vanishes since $\psi$ can be decomposed as $\psi = (\Delta + \rho) 1_N + (\chi^a + i \zeta^a) T^a$ and the modes $\zeta^a$ are absorbed by gluons.

[31] It is not possible to excite an on-shell gluon in terms of the $\mathbb{CP}^{N-1}$ effective theory. In order to describe the radiation of on-shell gluons from orientational modes, we have to include terms with higher derivatives in the $\mathbb{CP}^{N-1}$ effective theory. However, we can still discuss the scattering of on-shell gluons by vortices in terms of the Lagrangian obtained in the paper, since there is no need to excite on-shell gluons in this case.

[32] It has been shown in [22] that, in the case of local non-Abelian vortices in the model with gauged U(1)_B symmetry, collisions of two vortices always result in their reconnection even when they have different internal orientations. The same should hold for non-Abelian semisuperfluid vortices because whether the reconnection occurs or not depends on only topology.

[33] When the number of flavors is greater than the number of color, vortices are called semilocal. In this case, semilocal non-Abelian vortices are also coupled to massless Nambu-Goldstone bosons corresponding to spontaneously broken flavor symmetry [25], as well as massive particles.