DIFFUSE PLANAR PHASE BOUNDARIES IN A TWO-PHASE FLUID
WITH ONE INCOMPRESSIBLE PHASE

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Abstract. This note studies a family of Navier-Stokes-Allen-Cahn systems parameterized by temperature. Derived from an internal energy that corresponds to one incompressible and one compressible phase, this family is considered as a simple model for water. Decreasing temperature across a critical value, a transition takes place from a situation without towards one with planar diffuse phase boundaries.

In this note, we consider the Navier-Stokes-Allen-Cahn system

$$
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,
$$

$$(1) \quad \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p(\rho, c) \mathbf{I}) = \nabla \cdot \left( \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + (\lambda \nabla \mathbf{u}) \mathbf{I} - \delta \rho \nabla c \otimes \nabla c \right),$$

of evolutionary partial differential equations that model the spatiotemporal behaviour of a compressible viscous or inviscid fluid. The fluid is assumed to have a constant temperature $\theta > 0$ and to be a locally homogeneous mixture of two components such that its local state is completely described by the mass fraction $c$ of one of the components and the mass, per volume, of the mixture, $\rho$. This density $\rho$ is the reciprocal value,

$$
\rho = 1/\tau,
$$

of the fluid’s specific volume $\tau$. The behaviour of the fluid is described by a thermodynamic potential

$$
\bar{U}(\tau, c, |\nabla c|) = U(\tau, c) + 1/2 \delta |\nabla c|^2,
$$

$$
U(\tau, c) = \bar{U}(\tau, c) + W(c, \theta),
$$
in which $W(c, \theta)$ is the mixing energy and $\delta$ a positive constant. The pressure $p$ and the transformation rate $q$ derive from the potential as

$$
p(\rho, c) = \bar{p}(\tau, c) = -U_\tau(\tau, c),
$$

$$
q(\rho, c) = \bar{q}(\tau, c) = -U_c(\tau, c).
$$

System (1) was derived by Blesgen [1] and has recently been shown by Kotschote to possess strong solutions [3].

The following two theorems have been proven in [2] under certain assumptions on $\bar{U}$ and $W$.

**Theorem 1.** (Maxwell states and no-flux phase boundaries.) With $\bar{\theta} < \theta_*$ sufficiently close to a critical temperature $\theta_*$, the following holds for every $\theta \in (\bar{\theta}, \theta_*)$. There are locally uniquely determined fluid states $(\rho_0, \omega_0, \tilde{\rho}_0, \tilde{\tau}_0)$, depending continuously on $\theta$, such that (i)

$$q(\rho_0, \omega_0) = q(\tilde{\rho}_0, \tilde{\tau}_0) = 0,$$

$$p(\rho_0, \omega_0) = p(\tilde{\rho}_0, \tilde{\tau}_0)$$

with

$$(\rho_0, \omega_0) = (\tilde{\rho}_0, \tilde{\tau}_0) \quad \text{if} \quad \theta = \theta_*,$$

and (ii) if $\theta < \theta_*$, then

$$\rho_0 < \tilde{\rho}_0$$

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and system \((\mathbf{1})\) admits a no-flux \((m = 0)\) phase boundary
\[
(p(x), 0, c(x)) \quad \text{with} \quad (p(-\infty), c(-\infty)) = (\rho_0, \theta_0), \quad (p(\infty), c(\infty)) = (\rho_0, \theta_0)
\]
and (equivalently via \(x \mapsto -x\)) a no-flux phase boundary
\[
(p(x), 0, c(x)) \quad \text{with} \quad (p(-\infty), c(-\infty)) = (\rho_0, \theta_0), \quad (p(\infty), c(\infty)) = (\rho_0, \theta_0).
\]

Theorem 2. For sufficiently small mass flux \(m \neq 0\),
(i) the (left endstate, profile, right endstate) triple
\[
(p_0, 0, \theta_0), (\tilde{p}, 0, \tilde{c}), (\bar{p}_0, 0, \bar{\theta}_0)
\]
perturbs regularly to a (left endstate, profile, right endstate) triple
\[
(p_m^-, u_m^-, c_m^-), (p_m^+, u_m^+, c_m^+), (\tilde{p}_0, 0, \tilde{c}),
\]
corresponding to a traveling-wave phase boundary that is densifying if \(m > 0\) and rarefying if \(m < 0\);
(ii) the (left endstate, profile, right endstate, profile) triple
\[
(p_0, 0, \theta_0), (\tilde{p}, 0, \tilde{c}), (\rho_0, 0, \rho_0)
\]
perturbs regularly to a (left endstate, profile, right endstate) triple
\[
(p_m^-, u_m^-, c_m^-), (p_m^+, u_m^+, c_m^+),
\]
corresponding to a traveling-wave phase boundary that is rarefying if \(m > 0\) and densifying if \(m < 0\).

The present note serves to point out that Theorems 1 and 2 also hold under the following

Modelling Assumptions. (i) \(\hat{U}\) is of the form
\[
\hat{U}(\tau, c) = -(1 - c) \log \frac{\tau - c\tau_1}{1 - c},
\]
where \(\tau_1\) is a fixed value with
\[
0 < \tau_1 < 1
\]
and \(\tau\) and \(c\) range as
\[
\tau_1 < \tau < \infty \quad \text{and} \quad 0 < c < 1.
\]
(ii) With certain critical parameter values \(c_* \in (0, 1), \theta_* \in \mathbb{R},\)
\[
W(\cdot, \theta)
\]
undergoes a generic transition from convex for \(\theta > \theta_*\) to convex-concave-convex (“double-well”) for \(\theta < \theta_*\), at \(c = c_*\).

To justify assumption (i), consider first a general mixture of two non-interpenetrating phases 1 and 2 of varying mass fractions \(c, 1 - c\) and possibly varying specific volumes \(\tau_1\) and \(\tau_2\), for which
\[
\tau = c\tau_1 + (1 - c)\tau_2
\]
and
\[
\hat{U} = cU_1(\tau_1) + (1 - c)U_2(\tau_2).
\]
Then restrict attention to the case that phase 1 is perfectly incompressible and thus does not store mechanical energy,
\[
\tau_1 = \text{const} \quad \text{and} \quad U_1(\tau_1) = 0.
\]
Supposing further that phase 2 is lighter than phase 1,
\[
\tau_2 > \tau_1
\]
and that, as a simple prototypical example, its energy has the form
\[
U_2(\tau_2) = -\log(\tau_2)
\]
leads to the stated form of \(\hat{U}\) as a function of \(\tau\) and \(c\).
Noticing that $\hat{U}_\tau \hat{U}_{cc} - \hat{U}_{c\tau}^2 = 0$ and thus, in the terminology of [2],
\begin{equation}
\sgn(\Delta(\tau, c)) = \sgn(W_{cc}(c, \theta)),
\end{equation}
one readily sees that Theorem 1 follows exactly as in [2]. We illustrate this by pointing out that by Lemma 1 of [2], the proof amounts to studying the level sets of
\begin{equation}
\Gamma^0,\pi(c, y) \equiv \hat{G}(P_{\pi}(c, y), c) + W(c, \theta) + \frac{1}{2}y^2,
\end{equation}
where $y$ corresponds to $c'$,
\begin{equation}
\hat{G}(p, c) = (1 - c)(1 + \log p) + cp\tau_1
\end{equation}
is the Gibbs potential associated with $\hat{U}$ and $P_{\pi}(c, y)$ the unique positive root $p$ of
\begin{equation}
0 = (p - \pi)(c\tau_1 p + (1 - c)) + y^2 p.
\end{equation}
(The latter equation is Eq. (9) in [2].) The critical pressure is $p = p^*$, the unique solution $< 0$ of
\begin{equation}
G_c(p, c) = \tau_1 p - \log p - 1 = 0.
\end{equation}
For $(\theta, \pi)$ near $(\theta^*, p^*)$, the level landscape of $\Gamma^{0,\pi}$ undergoes a transition from one saddle (for $\theta > \theta^*$) to a saddle-maximum-saddle configuration (for $\theta < \theta^*$ and certain $\pi$). In the latter case, the two saddles are at the same level and thus connected by two heteroclinic orbits (that together surround the maximum point) if $\pi$ assumes a unique value $\pi_*(\theta)$. As in [2], Theorem 2 then follows from the transversality of the saddle-saddle connections with respect to the parameter $\pi$.

**Remark.** The choice of $-\log$ is exemplary. $U_2$ can be any function $f: (\tau_1, \infty) \to \mathbb{R}$ with $f' < 0 < f''$.

**References**

[1] T. Blesgen: A generalisation of the Navier-Stokes equations to two-phase-flows. *J. Phys. D: Appl. Phys.* 32 (1999), 1119-1123.
[2] H. Freistühler: Phase transitions and traveling waves in compressible fluids, *Arch. Rational Mech. Anal.*
[3] M. Kotschote: Strong solutions to the Navier-Stokes equations for a compressible fluid of Allen-Cahn type. *Arch. Rational Mech. Anal.* 206 (2012), 489-514.

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\(^{1}\)It is actually easier, here and in other contexts, to work directly with the Gibbs potential $G$ associated with $U$. The role of $\Delta = U_{\tau\tau}U_{cc} - U_{c\tau}^2$ is then played by the simpler quantity $G_{cc}$. In the present context, $\hat{G}_{cc} = 0$ and Eq. (2) reads $\sgn(G_{cc}(p, c)) = \sgn(W_{cc}(c, \theta))$. 

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Figure (by J. Höwing): Level lines of $\Gamma^{0,\pi}$ for $\tau_1 = 0.5$ and $W(c, \theta) = (c - 0.5)^4 + (\theta - \theta^*)(c - 0.5)^2$. Top to bottom: $\theta - \theta^* = 0.16, 0.00, -0.08$. Left to right: $\pi - p^* = -0.010, -0.001, 0.000, 0.001, 0.010$. 

The choice of $-\log$ is exemplary. $U_2$ can be any function $f: (\tau_1, \infty) \to \mathbb{R}$ with $f' < 0 < f''$.