SIDON BASIS IN POLYNOMIAL RINGS OVER
FINITE FIELDS

WENTANG KUO AND SHUNTARO YAMAGISHI

Abstract. Let $F_q[t]$ denote the ring of polynomials over $F_q$, the finite field of $q$ elements. Suppose the characteristic of $F_q$ is not 2 or 3. In this paper, we prove an $F_q[t]$-analogue of results related to the conjecture of Erdős on the existence of infinite Sidon sequence of positive integers which is an asymptotic basis of order 3. We prove that there exists a $B_2[2]$ sequence of non-zero polynomials in $F_q[t]$, which is an asymptotic basis of order 3. We also prove that for any $\varepsilon > 0$, there exists a sequence of non-zero polynomials in $F_q[t]$ such that any $n \in F_q[t]$ of sufficiently large degree can be expressed as a sum of four elements of the sequence, where one of them has a degree less than or equal to $\varepsilon \deg n$.

1. Introduction

A sequence of positive integers $\omega$ is called a Sidon sequence, if all the sums $a + a'$ $(a, a' \in \omega, a \leq a')$ are distinct. We say $\omega$ is an asymptotic basis of order $g$, if any positive integer $n$ sufficiently large can be expressed as a sum of $g$ elements of $\omega$. If $\omega$ is a Sidon sequence and also an asymptotic basis of order $g$, we say $\omega$ is a Sidon basis of order $g$. An introduction to the topic is given in [1], which we paraphrase here. It is known that there can not be a Sidon basis of order 2. The following is a conjecture of Erdős [4, 5, 6].

Conjecture 1.1. There exists a sequence of positive integers, which is a Sidon basis of order 3.

There had been progress made towards this conjecture. J.-M. Deshouilliers and A. Plagne [3] constructed a Sidon basis of order 7, and S. Kiss [10] proved the existence of a Sidon basis of order 5. S. Kiss, E. Rozgonyi and C. Sándor [11] proved that there exists a Sidon basis of order 4.

The focus of this paper is on two theorems toward this conjecture proved in [1]. We introduce some notations before we state these theorems. A sequence of positive integers $\omega$ is a $B_2[g]$ sequence if any integer $n$ has at most $g$ representations of the form $n = a + a'$ $(a, a' \in \omega, a \leq a')$. Conjecture 1.1 can be restated as follows: There exists a $B_2[1]$ sequence with $g = 1$, which is an asymptotic basis of order 3. The following theorem was proved in [1].

Theorem 1.2 (Theorem 1.2, [1]). There exists a $B_2[2]$ sequence, which is an asymptotic basis of order 3.

For any $\varepsilon$ such that $0 < \varepsilon < 1$, we say that $\omega$ is an asymptotic basis of order $g + \varepsilon$ if any positive integer $n$ sufficiently large can be represented in the following form,

$$n = a_1 + ... + a_{g+1} \ (a_1, ..., a_{g+1} \in \omega) \text{ and } \min_{1 \leq i \leq g+1} a_i \leq n^\varepsilon.$$
The following theorem was also proved in [1].

**Theorem 1.3** (Theorem 1.3, [1]). For any $\varepsilon > 0$, there exists a Sidon basis of order $3 + \varepsilon$.

Let $\mathbb{F}_q$ be the finite field of $q$ elements. In this paper, we follow the approach of [1] and prove $\mathbb{F}_q[t]$-analogue of Theorems 1.2 and 1.3. Let $\omega$ be a sequence of non-zero polynomials in $\mathbb{F}_q[t]$. We define *Sidon sequence* and $B_2[g]$ *sequence* for sequences of non-zero polynomials in $\mathbb{F}_q[t]$ in a similar manner as for sequences of positive integers. We say $\omega$ is an asymptotic basis of order $g$ if any $n \in \mathbb{F}_q[t]$ with $\deg n$ sufficiently large can be expressed as a sum of $g$ elements of $\omega$. Similarly, if $\omega$ is a Sidon sequence and also an asymptotic basis of order $g$, we say $\omega$ is a Sidon basis of order $g$. We prove the following theorem, which is an analogue of Theorem 1.2.

**Theorem 1.4.** Let $p$ be a prime, $p > 3$, and $q = p^h (h \in \mathbb{N})$. Then there exists a $B_2[2]$ sequence of non-zero polynomials in $\mathbb{F}_q[t]$, which is an asymptotic basis of order 3.

For any $\varepsilon$ such that $0 < \varepsilon < 1$, we say that $\omega$ is an asymptotic basis of order $g + \varepsilon$ if any polynomial $n \in \mathbb{F}_q[t]$ with $\deg n$ sufficiently large can be represented in the following form,

$$n = a_1 + \ldots + a_{g+1} (a_1, \ldots, a_{g+1} \in \omega) \quad \text{and} \quad \min_{1 \leq i \leq g+1} \deg a_i \leq \varepsilon \deg n.$$

We also prove the following theorem, which is an analogue of Theorem 1.3.

**Theorem 1.5.** Let $p$ be a prime, $p > 3$, and $q = p^h (h \in \mathbb{N})$. For any $\varepsilon > 0$, there exists a sequence of non-zero polynomials in $\mathbb{F}_q[t]$, which is a Sidon basis of order $3 + \varepsilon$.

In [1 Theorem 1.1], J. Cilleruelo proved Conjecture 1.1 in the setting of $\mathbb{Z}/(M\mathbb{Z})$ for $M$ sufficiently large. For each $N \in \mathbb{N}$, let $\mathcal{G}_N = \{ f \in \mathbb{F}_q[t] : \deg f < N \}$. We also prove a $\mathcal{G}_N$-analogue of Conjecture 1.1 when $N$ is a sufficiently large multiple of 4, and the characteristic of $\mathbb{F}_q$ is not 2 or 3.

**Theorem 1.6.** Let $p$ be a prime, $p > 3$, and $q = p^h (h \in \mathbb{N})$. Then for $M_0 \in \mathbb{N}$ sufficiently large, there exists a Sidon set $S = S(q, M_0)$ in $\mathcal{G}_{4M_0} \subseteq \mathbb{F}_q[t]$ such that the following holds. Given any $g \in \mathcal{G}_{4M_0}$, there exist $s_1, s_2, s_3 \in S$ with $s_i \neq s_j (i \neq j)$ such that

$$s_1 + s_2 + s_3 = g.$$

The organization of the paper is as follows. In Section 2 we prove Theorem 1.6 and its corollary, which become useful in the proof of Theorems 1.4 and 1.5. We introduce notation and results from probabilistic methods in Section 3. We then prove our main results Theorems 1.3 and 1.5 in Sections 4 and 5 respectively. We obtain several bounds in Section 6. These bounds are used in Sections 7 and 8 where we present calculations of estimates used in Sections 4 and 5 respectively. We note that this paper involves a considerable amount of computation, some of which is similar to that of [1]. In an effort to keep the paper concise, we omitted the details of some calculations, most notably in Sections 5 and 8 as they are similar to that of Sections 4 and 7 respectively. Also, we assume that the characteristic of $\mathbb{F}_q$ is not 2 or 3 for the remainder of the paper, unless it is explicitly stated otherwise.

2. Proof of Conjecture 1.1 for $\mathcal{G}_N$

Let $G$ be an abelian group. For any subset $A \subseteq G$ and $x \in G$, we denote $r_{A-A}(x)$ to be the number of representations of the form $x = a - a' (a, a' \in A)$. We say that a set $A \subseteq G$
is a Sidon set if $r_{A-A}(x) \leq 1$ whenever $x \neq 0$. This condition is equivalent to saying that the representation of elements of $G$ as a sum of two elements of $A$ is unique if it exists. In other words, if for some $a, b, c, d \in A$ we have $a + b = c + d$, then either we have $a = c, b = d$ or $a = d, b = c$. Recall from above that we defined $G_N = \{ f \in \mathbb{F}_q[t] : \deg f < N \}$, which is a group under addition. In this section, we prove Theorem 1.6 and Corollary 2.1, which are $\mathbb{F}_q[t]$-analogue of [1] Theorem 2.1 and [1] Corollary 2.1, respectively.

We recall the statement of Theorem 1.6, which confirms Conjecture 1.1 for $G_N$ when $N$ is a sufficiently large multiple of 4, and the characteristic of $\mathbb{F}_q$ is not 2 or 3.

**Theorem 1.6.** Let $p$ be a prime, $p > 3$, and $q = p^h$ ($h \in \mathbb{N}$). Then for $M_0 \in \mathbb{N}$ sufficiently large, there exists a Sidon set $S = S(q, M_0)$ in $G_{4M_0} \subseteq \mathbb{F}_q[t]$ such that the following holds. Given any $g \in G_{4M_0}$, there exist $s_1, s_2, s_3 \in S$ with $s_i \neq s_j$ ($i \neq j$) such that

$$s_1 + s_2 + s_3 = g.$$ 

**Proof.** We have the following group isomorphisms when we only consider the additive properties,

$$G_{4M_0} \cong (\mathbb{F}_q)^{4M_0} \cong (\mathbb{Z}/p\mathbb{Z})^{4hM_0} \cong \mathbb{F}_q' \times \mathbb{F}_q',$$

where $q' = p^{2hM_0}$. Therefore, if we can find a Sidon basis of order 3 in $\mathbb{F}_q' \times \mathbb{F}_q'$, then we are done.

Let $S = \{(x, x^2) : x \in \mathbb{F}_q'\}$. Then, by [2] we know that $S$ is a Sidon set in $\mathbb{F}_q' \times \mathbb{F}_q'$. For the sake of completeness, we present the proof from [2] here. We have to check that given $(0, 0) \neq (e_1, e_2) \in \mathbb{F}_q' \times \mathbb{F}_q'$, the equation $(e_1, e_2^2) - (x_1, x_2^2) = (e_1, e_2)$ uniquely determines $x_1$ and $x_2$ in $\mathbb{F}_q'$, or that it has no solution. If $e_1 = 0$, then it is clear that there do not exist $x_1$ and $x_2$ in $\mathbb{F}_q'$ that satisfy the equation. On the other hand, suppose $e_1 \neq 0$. Since $x_1 = e_1 + x_2$, we have $e_2 = (x_2 + e_1)^2 - x_2^2 = 2e_1 x_2 + e_1^2$, which uniquely determines $x_2$. Once $x_2$ is determined, there is only one choice for $x_1$. Therefore, we have shown that $r_{S-S}((e_1, e_2)) \leq 1$, and hence $S$ is a Sidon set.

Now we show $S$ is an additive basis of order 3. This is equivalent to showing that for any $(a, b) \in \mathbb{F}_q' \times \mathbb{F}_q'$, the system

$$x + y + t = a \text{ and } x^2 + y^2 + t^2 = b,$$

has a solution in $\mathbb{F}_q' \times \mathbb{F}_q' \times \mathbb{F}_q'$. We consider the polynomial

$$f(x, y) = x^2 + y^2 + (x + y - a)^2 - b = 2(x^2 + y^2 + xy - ax - ay) + a^2 - b$$

constructed from (2.1), and its homogenization

$$F(x, y, z) = 2(x^2 + y^2 + xy - axz - ayz) + (a^2 - b)z^2.$$ 

Suppose $F$ is reducible over $\overline{\mathbb{F}}_q$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$, in which case $F$ decomposes into two lines $L_1$ and $L_2$ with coefficients in $\overline{\mathbb{F}}_q$. Without loss of generality, let

$$F(x, y, z) = 2(x + \alpha_1 y + \beta_1 z)(x + \alpha_2 y + \beta_2 z),$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \overline{\mathbb{F}}_q$. By expanding out the factors, we see from the coefficients of $y^2, xy, xz$, and $yz$ that $\alpha_1 \alpha_2 = 1$, $\alpha_1 + \alpha_2 = 1$, $\beta_1 + \beta_2 = -a$, and $\alpha_1 \beta_2 + \alpha_2 \beta_1 = -a$, respectively. Since $q' = p^{2hM_0}$ and $2|(2hM_0)$, we have $\mathbb{F}_{p^2} \subseteq \mathbb{F}_q'$. From the first and the second equation, we obtain that $\alpha_1$ and $\alpha_2$ are non-zero, and

$$\alpha_1, \alpha_2 \in \mathbb{F}_{p^2} \subseteq \mathbb{F}_q'.$$
Since the characteristic of $\mathbb{F}_q$ is not 3, we also obtain $\alpha_1 \neq \alpha_2$. Then from the third and the forth equation, we can deduce that

$$\beta_1, \beta_2 \in \mathbb{F}_q.$$ 

Therefore, $F$ is in fact reducible over $\mathbb{F}_q$, and hence $f$ decomposes into two linear factors over $\mathbb{F}_q$ as follows

$$f(x, y) = F(x, y, 1) = 2(x + \alpha_1 y + \beta_1)(x + \alpha_2 y + \beta_2).$$

Thus, we see that (2.1) has at least $q'$ solutions in $\mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ in this case.

On the other hand, suppose $F$ is irreducible over $\mathbb{F}_q$. Let $V(F)$ be the hypersurface in $\mathbb{P}^2_{\mathbb{F}_q}$ defined by $F$. In this case, we may invoke a theorem by S. Lang and A. Weil [12], and obtain that $V(F)$ has $q' + O(1)$ rational points over $\mathbb{F}_q$. We know that $F(x, y, 0) = 2(x^2 + y^2 + xy)$ decomposes into two linear factors over $\mathbb{F}_q$, because it is a quadratic form in two variables. Then we can verify that $F(x, y, 0)$ has at most $O(1)$ solutions in $\mathbb{P}^1_{\mathbb{F}_q}$. Therefore, it follows that $V(F)$ contains $q' + O(1)$ points of the form $[x_0 : y_0 : 1]$ from which we deduce (2.1) has $q' + O(1)$ solutions in $\mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$.

In both cases, we have that (2.1) has at least $q' + O(1)$ solutions. Suppose $(x_1, x_2, x_3)$ is a solution to (2.1) such that $x_i = x_j$ for some $i \neq j$, without loss of generality let $i = 1$ and $j = 2$. Then, the number of such solutions is equal to the number of solutions to

$$x + x + y = a \quad \text{and} \quad x^2 + x^2 + y^2 = b.$$  

Since the equation $2x^2 + (a - 2x)^2 = b$ has at most 2 solutions in $\mathbb{F}_q$, we have that (2.2) has at most 2 solutions. Hence, the number of solutions $(x_1, x_2, x_3)$ to (2.1) such that $x_i = x_j$ for some $i \neq j$ is $O(1)$. Therefore, for each $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q$ we can find a solution $(x_1, x_2, x_3)$ to (2.1) such that $x_i \neq x_j$ ($i \neq j$), provided $q'$ is sufficiently large.

**Corollary 2.1.** Let $p$ be an odd prime, and $q = p^h$ ($h \in \mathbb{N}$). Then for $M_0 \in \mathbb{N}$ sufficiently large, there exists a Sidon set $S = S(q, M_0)$ in $\mathbb{G}_{4M_0} \subseteq \mathbb{F}_q[t]$ such that the following holds. Given any $g \in \mathbb{G}_{4M_0}$, there exist $s_1, s_2, s_3, s_4 \in S$ with $s_i \neq s_j$ ($i \neq j$) such that

$$s_1 + s_2 + s_3 + s_4 = g.$$ 

**Proof.** Let $\mathbb{F}_{q'}$ and $S \subseteq \mathbb{F}_q \times \mathbb{F}_q$ be as in the proof of Theorem 1.6. We show that $S$ satisfies the required conditions. From the proof of Theorem 1.6 we know that for any $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q$, the system

$$x + y + t + 0 = a \quad \text{and} \quad x^2 + y^2 + t^2 + 0^2 = b,$$ 

has at least $q' + O(1)$ solutions of the form $(x_1, x_2, x_3) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$, where $x_i \neq x_j$ ($i \neq j$). We observe that all of these solutions except those with at least one of $x_1, x_2, x_3$ being 0 satisfy the conditions. Without loss of generality, suppose $x_3 = 0$, then (2.3) reduces to solving

$$x + y = a \quad \text{and} \quad x^2 + y^2 = b,$$ 

which further reduces to solving a quadratic equation. Thus, it follows that the number of solutions $(x_1, x_2, x_3)$ to (2.3) with at least one of $x_1, x_2, x_3$ being 0 is $O(1)$. Therefore, it follows that there exist at least $q' + O(1)$ solutions in $\mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$, which satisfy the desired conditions. \hfill \Box
3. Preliminaries

We begin this section by introducing a result that is useful to us. The following result is known as the Borel-Cantelli lemma, which plays a crucial role in probability theory [7].

**Theorem 3.1** (The Borel-Cantelli lemma). Suppose we have a probability space \((\Omega, \mathcal{M}, \mathbb{P})\). Let \(\{E_j\}_{j \geq 1}\) be a sequence of measurable sets. If

\[
\sum_{j=1}^{\infty} \mathbb{P}(E_j) < \infty,
\]

then we have

\[
\mathbb{P}\left(\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_j\right) = 0.
\]

In other words, the Borel-Cantelli lemma states that if \(\sum_{j=1}^{\infty} \mathbb{P}(E_j) < \infty\), then with probability 1 at most a finite number of the events \(E_j\) can occur.

Throughout the paper we fix \(N\) to be a sufficiently large positive integer, and we let \(S\) be a non-empty subset of \(G_N\). Furthermore, we choose \(S\) to satisfy the conditions of Theorem 1.6 in Section 4, and we choose \(S\) to satisfy the conditions of Corollary 2.1 in Section 5. Let \(\Omega\) be the space of all sequences of polynomials in \(\mathbb{F}_q[t]\), and by \(x \equiv S(\text{mod } n_0)\), we mean \(x \equiv s(\text{mod } n_0)\) for some \(s \in S\). For each \(\gamma < 1\) and \(M \in \mathbb{N}\), we define the probability space \(\mathcal{S}_M(\gamma; S \mod n_0)\) in the following manner. We let \(\mathcal{S}_M(\gamma; S \mod n_0) = (\Omega, \mathcal{M}, \mathbb{P})\) to be the probability space of all sequences of polynomials \(\omega\), where \(\mathcal{M}\) is the appropriate \(\sigma\)-algebra, such that all the events \(x \in \omega\) are independent, and

\[
\mathbb{P}(\{x \in \omega\}) = \begin{cases} q^{-\gamma(\deg x)}, & \text{if } x \equiv S(\text{mod } n_0) \text{ and } \deg x > M, \\ 0, & \text{otherwise}. \end{cases}
\]

We refer the reader to [7] for the details on construction of such probability spaces. For simplicity, we let \(\mathbb{P}(\{0 \in \omega\}) = 0\). From here on whenever we refer to \(\mathbb{P}\) we mean this probability measure. Let \(f\) be a function from \(\mathbb{R}\) to \(\mathbb{R}\). By \(f = o_M(1)\), we mean that \(|f(M)| \to 0\) as \(M \to \infty\).

The following result is known as Janson’s inequality, see for example [1] [3] [2].

**Theorem 3.2** (Janson’s inequality). Let \(\mathcal{F}\) be a family of sets, and let \(\omega\) be a random subset. Let \(Y(\omega) = |\{\theta \in \mathcal{F} : \theta \subseteq \omega\}|\) with finite expected value \(\mu = \mathbb{E}(Y(\omega))\). Then, for \(0 \leq \varepsilon \leq 1\), we have that

\[
\mathbb{P}(\{\omega \in \Omega : Y(\omega) \leq (1-\varepsilon)\mu\}) \leq \exp(-\varepsilon^2\mu^2/(2\mu + 2\Delta(\mathcal{F}))),
\]

where

\[
\Delta(\mathcal{F}) = \sum_{\theta, \theta' \in \mathcal{F}} \mathbb{P}(\{\omega \in \Omega : \theta, \theta' \subseteq \omega\}),
\]

and \(\theta \sim \theta'\) means \(\theta \cap \theta' \neq \emptyset\) and \(\theta \neq \theta'\). In particular, if \(\Delta(\mathcal{F}) < \mu\) we have

\[
\mathbb{P}(\{\omega \in \Omega : Y(\omega) \leq \mu/2\}) \leq \exp(-\mu/16).
\]

Note in order to avoid clutter in the exposition, whenever we have a subset of \(\Omega\) of the form \(\{\omega \in \Omega : \omega \text{ satisfies } ...\}\) we simply denote it by \(\{\omega \text{ satisfies } ...\}\).

For a given vector \(\overline{y} = (y_1, ..., y_H)\), we define \(\text{Set}(\overline{y}) = \{y_1, ..., y_H\}\). We say that a collection of \(K\) distinct vectors \(\overline{x}_j\) (\(1 \leq j \leq K\)) form a disjoint set of \(K\) vectors (\(K\)-d.s.v. for short)
if \( \text{Set}(\bar{x}_j) \cap \text{Set}(\bar{x}_l) = \emptyset \) for any \( j \neq l, 1 \leq j, l \leq K \). We say that \( K \) distinct vectors with \( H \) coordinates form a \textit{vectorial sunflower of \( K \) petals}, if for some \( I \subseteq \{1, \ldots, H\} \) the following two conditions are satisfied:

i) For all \( i \in I \), all the vectors have the same \( i \)-th coordinate.

ii) The set of vectors obtained by removing the \( i \)-th coordinates, for all \( i \in I \), form a \( K \)-d.s.v.

Following the terminology of [1], we say \( F \) is a family of vectors of \( H \) coordinates if \( F \) is a subset of \( (\mathbb{F}_q[t])^H \). We have the following lemma.

**Lemma 3.3** (Vectorial sunflower lemma). Let \( F \) be a family of vectors of \( H \) coordinates. If \( F \) does not contain a vectorial sunflower of \( K \) petals, then

\[
|F| \leq H!((H^2 - H + 1)K)^H.
\]

**Proof.** This is obtained by slightly modifying the proof of [1, Lemma 3.2]. \( \square \)

Given \( F \), a family of vectors of \( H \) coordinates, we define

\[
F(\omega) = \{ \bar{x} \in F : \text{Set}(\bar{x}) \subseteq \omega \}.
\]

The following is an immediate consequence of Lemma 3.3.

**Corollary 3.4.** Let \( \{F_n\}_{n \in \mathbb{F}_q[t]} \) be a sequence of family of vectors of \( H \) coordinates. Suppose for \( \Omega(K) = \{ \omega \in \Omega : F_n(\omega) \) does not contain vectorial sunflowers of \( K \) petals for any \( n \in \mathbb{F}_q[t] \} \), we have

\[
P(\Omega(K)) = 1 - o_M(1).
\]

Then, we have

\[
P(\{|F_n(\omega)| \leq H!((H^2 - H + 1)K)^H \text{ for all } n \in \mathbb{F}_q[t] \}) \geq 1 - o_M(1).
\]

We also make use of the following proposition.

**Proposition 3.5.** Let \( \{F_n\}_{n \in \mathbb{F}_q[t]} \) be a sequence of family of vectors of \( H \) coordinates, and \( \{F_n(\omega)\}_{n \in \mathbb{F}_q[t]} \) the corresponding random family, where \( \omega \) is a random sequence in \( S_M(\gamma; S \mod n_0) \). Suppose there is \( \delta > 0 \) such that \( \mathbb{E}(|F_n(\omega)|) \ll q^{-\delta \max\{\deg n, M\}} \) for all \( n \in \mathbb{F}_q[t] \). If \( K > 1/\delta \), then

\[
P(\{F_n(\omega) \text{ contains a } K\text{-d.s.v. for some } n \in \mathbb{F}_q[t]\}) = o_M(1).
\]
Proof. By unraveling the definitions, we have the following sequence of inequalities

\[
\Pr(\{F_n(\omega) \text{ contains a } K\text{-d.s.v.}\}) \\
\ll \sum_{\tau_1, \ldots, \tau_K \in F_n \text{ form a } K\text{-d.s.v.}} \Pr(\{\text{Set } (\tau_1) \subseteq \omega\}) \cdots \Pr(\{\text{Set } (\tau_K) \subseteq \omega\}) \\
= \sum_{\tau_1, \ldots, \tau_K \in F_n \text{ form a } K\text{-d.s.v.}} \Pr(\{\text{Set } (\tau_1) \subseteq \omega\}) \cdots \Pr(\{\text{Set } (\tau_K) \subseteq \omega\}) \\
\leq \frac{1}{K!} \left( \sum_{\tau \in F_n} \Pr(\{\text{Set } (\tau) \subseteq \omega\}) \right)^K \\
= \frac{\mathbb{E}(|F_n(\omega)|)^K}{K!} \\
\ll \frac{q^{-\delta K \max\{\deg n, M\}}}{K!}.
\]

Since \((1 - \delta K) < 0\), we obtain that

\[
\Pr(\{F_n(\omega) \text{ contains a } K\text{-d.s.v. for some } n \in \mathbb{F}_q[t]\}) \\
\ll \sum_{\deg n \leq M} \Pr(\{F_n(\omega) \text{ contains a } K\text{-d.s.v.}\}) + \sum_{\deg n > M} \Pr(\{F_n(\omega) \text{ contains a } K\text{-d.s.v.}\}) \\
\ll q^{-\delta K \max\{\deg 0, M\}} + \sum_{j \leq M} (q^{j+1} - q^j)q^{-\delta KM} + \sum_{j > M} (q^{j+1} - q^j)q^{-\delta Kj} \\
\ll q^{-\delta KM} + q^{-\delta KM} \sum_{j \leq M} q^j + \sum_{j > M} q^{(1-\delta K)j} \\
= O(q^{(1-\delta K)M}) \\
= o_M(1).
\]

\[\square\]

4. Proof of Theorem 1.4

In this section, we consider the probability space \(S_M(\gamma; S \mod n_0)\), where we let \(\gamma = \frac{7}{11}\), and let \(S\) to be a non-empty subset of \(\mathbb{G}_N\) satisfying the conditions of Theorem 1.6. The basic strategy is as follows. We use the Borell-Cantelli Lemma (Theorem 3.1) to show that in the probability space \(S_M(\gamma; S \mod n_0)\), “most” of the sequences, in other words with probability 1, has “many” representations of \(n\) as a sum of three of its elements for all \(n \in \mathbb{F}_q[t]\) with \(\deg n\) sufficiently large. We then show that out of these sequences, there exists a sequence such that even after removing some of its elements to make it \(B_2[2]\), it still has at least one representation of \(n\) as a sum of three of its elements for each \(n \in \mathbb{F}_q[t]\) with \(\deg n\) sufficiently large.

For each \(n \in \mathbb{F}_q[t]\), we consider the following collection of sets

\[
\mathcal{Q}_n = \{\theta = \{x_1, x_2, x_3\} \subseteq \mathbb{F}_q[t] : x_1 + x_2 + x_3 = n, \ x_i \not\equiv x_j(\mod n_0) \text{ for } i \neq j\}.
\]

Given a sequence of polynomials \(\omega\), we let

\[
\mathcal{Q}_n(\omega) = \{\theta \in \mathcal{Q}_n : \theta \subseteq \omega\}.
\]
We define

$$\mathcal{T}_n = \{ \pi = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) : \pi \text{ satisfies } \text{Cond}(\mathcal{T}_n) \},$$

where

$$\text{Cond}(\mathcal{T}_n) = \begin{cases} 
\{x_1, x_2, x_3\} \in \mathcal{Q}_n, \\
x_1 + x_4 = x_5 + x_6 = x_7 + x_8, \ {x_1, x_4} \neq \{x_5, x_6\} \neq \{x_7, x_8\}, \\
x_1 \equiv x_5 \equiv x_7 \pmod{n_0}, \ x_4 \equiv x_6 \equiv x_8 \pmod{n_0}.
\end{cases}$$

We also let

$$\mathcal{T}_n(\omega) = \{ \pi \in \mathcal{T}_n : \text{Set}(\pi) \subseteq \omega \}.$$

The $B_2[2]$-lifting process of a sequence $\omega$ consists of removing from $\omega$ those elements $a_1 \in \omega$ such that there exist $a_2, a_3, a_4, a_5, a_6 \in \omega$ with $a_1 + a_2 = a_3 + a_4 = a_5 + a_6$ and $\{a_1, a_2\} \neq \{a_3, a_4\} \neq \{a_5, a_6\}$. We denote by $\omega_{B_2[2]}$ the resulting $B_2[2]$ sequence obtained by applying this process to $\omega$.

The quantity $|\mathcal{T}_n(\omega)|$ provides an upper bound for the number of representations of $n$ counted in $\mathcal{Q}_n(\omega)$ that are destroyed in the $B_2[2]$-lifting process of $\omega$ for the following reason. Suppose that $\theta = \{x_1, x_2, x_3\} \in \mathcal{Q}_n(\omega)$ contains an element, say $x_1$, which is removed in the $B_2[2]$-lifting process. Then, there exist $x_4, x_5, x_6, x_7, x_8 \in \omega$, which satisfy $x_1 + x_4 = x_5 + x_6 = x_7 + x_8$ and $\{x_1, x_4\} \neq \{x_5, x_6\} \neq \{x_7, x_8\}$. Since all $x_i \equiv S(\pmod{n_0})$ and $S$ is a Sidon set in $\mathbb{G}_N$, interchanging $x_5$ with $x_6$, and $x_7$ with $x_8$ if necessary, we have $x_1 \equiv x_4 \equiv x_7 \pmod{n_0}$ and $x_5 \equiv x_6 \equiv x_8 \pmod{n_0}$. Thus, we have a map from the set of $\theta \in \mathcal{Q}_n(\omega)$ destroyed in the $B_2[2]$-lifting process to $\mathcal{T}_n(\omega)$, and it is easy to see that this map is injective. Consequently, we have

$$|\mathcal{Q}_n(\omega_{B_2[2]})| \geq |\mathcal{Q}_n(\omega)| - |\mathcal{T}_n(\omega)|.$$ 

Therefore, Theorem 1.4 is established if we can prove that there exists a sequence $\omega_0$ such that for any $n \in \mathbb{F}_q[t] \setminus \{0\}$ with sufficiently large degree, we have $|\mathcal{Q}_n(\omega_0)| \gg q^{5\deg n}$ for some $\delta > 0$, and $|\mathcal{T}_n(\omega_0)| \ll 1$. We show that in some sense there are many sequences satisfying the former condition, and then we show it is also the case for the latter condition. These tasks are accomplished in Propositions 4.4 and 4.7. We then prove that there exist sequences satisfying both conditions. Before we get into the proofs of these propositions, we list three useful estimates. However, we postpone their proofs to Section 7.

Lemma 4.1. We have that

$$\mathbb{E}(|\mathcal{Q}_n(\omega)|) \gg q^{(1/11)\deg n},$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large.

Recall from (3.2), the definition of $\Delta(\cdot)$.

Proposition 4.2. We have that

$$\Delta(\mathcal{Q}_n) \ll q^{-\frac{2}{7}\deg n},$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large.

Lemma 4.3. We have that

$$\mathbb{E}(|\mathcal{T}_n(\omega)|) \ll q^{-\frac{1}{7}\max\{\deg n, M\}}.$$

We now prove the following proposition.
Proposition 4.4. We have that
\[ \mathbb{P}(\{|Q_n(\omega)| > q^{\frac{1}{11}} \deg n\}) = 1, \]
for \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large.

Proof. We apply Theorem 3.2 with \( \mathcal{F} = Q_n \) and \( Y(\omega) = |Q_n(\omega)| = |\{\theta \in Q_n : \theta \subseteq \omega\}| \), where \( \omega \) is a random sequence in \( S_M(7/11; S \mod n_0) \). We proved that \( \mu = \mu_n = \mathbb{E}(Q_n(\omega)) \gg q^{\frac{1}{11}} \deg n \) in Lemma 4.1 and \( \Delta(Q_n) \ll q^{-\frac{1}{11}} \deg n \) in Proposition 4.2. Hence for \( \deg n \) sufficiently large, we have \( \Delta(Q_n) < \mu_n \). Then, Theorem 3.2 implies that
\[ \mathbb{P}(\{|Q_n(\omega)| \leq \mu_n/2\}) \leq \exp(-\mu_n/16). \]
Therefore, we obtain that for some \( C, C' > 0 \) and \( T \in \mathbb{N} \), we can write
\[
\sum_{n \in \mathbb{F}_q[t]} \mathbb{P}(\{|Q_n(\omega)| \leq \mu_n/2\}) < C' + \sum_{j=0}^{\infty} (q^{j+1} - q^j) \exp(-Cq^{\frac{1}{11}j}) < C' + (q - 1) \sum_{j=0}^{\infty} q^j \exp(-Cq^{\frac{1}{11}j}) < \infty.
\]

Thus, Theorem 3.1 implies that with probability 1, we have \( |Q_n(\omega)| > \mu_n/2 \gg q^{\frac{1}{11}} \deg n \) for all \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large. \qed

For each \( r \in \mathbb{F}_q[t] \), we define the following families of vectors, whose expected values are bounded in Lemma 4.5.

\[
U_r = \{ \bar{x} = (x_1, x_2) : x_1 + x_2 = r, x_1 \neq x_2 \},
\]
\[
V_r = \{ \bar{x} = (x_1, x_2) : x_1 - x_2 = r, x_1 \neq x_2 \},
\]
\[
W_r = \{ \bar{x} = (x_1, x_2) : x_1 + x_2 = r, x_1 \neq x_2 \}.
\]

We prove the following lemma in Section 7.

Lemma 4.5. We have the following bounds on the expectations.

i) \( \mathbb{E}(|U_r(\omega)|) \ll q^{-\frac{1}{11}} \max\{\deg r, M\} \).

ii) \( \mathbb{E}(|V_r(\omega)|) \ll q^{-\frac{1}{11}} \max\{\deg r, M\} \).

iii) \( \mathbb{E}(|W_r(\omega)|) \ll q^{-\frac{1}{11}} \max\{\deg r, M\} \).

Lemma 4.6. Let \( \mathcal{F}_r \) be any of the three families in (4.3), then we have
\[ \mathbb{P}(\{|\mathcal{F}_r(\omega)| \text{ contains at most 12-d.s.v. for some } r \in \mathbb{F}_q[t]\}) = o_M(1). \]

Proof. For any of the three choices of \( \mathcal{F}_r \), Lemma 4.5 shows that \( \mathbb{E}(|\mathcal{F}_r(\omega)|) \ll q^{-\frac{1}{11}} \max\{\deg r, M\} \). Thus, the result follows by Proposition 3.5. \qed

We have the following proposition, which is one of the main ingredients to prove Theorem 1.4.

Proposition 4.7. We have that
\[ \mathbb{P}(\{|T_n(\omega)| \leq 10^{28} \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1 - o_M(1). \]
Proof. We claim the following statement:

Claim. With probability \(1 - o_M(1)\), \(\mathcal{T}_n(\omega)\) does not contain vectorial sunflowers of 12 petals for any \(n \in \mathbb{F}_q[t]\).

Assuming the claim holds, we can apply Corollary 3.4 to obtain
\[
\mathbb{P}(\{\mathcal{T}_n(\omega)\} \subseteq \mathbb{F}_q[t]) = 1 - o_M(1).
\]

Thus, we see that proving the above claim is sufficient to obtain our result. We prove it for distinct possible types \(I \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}\) of vectorial sunflowers in \(\mathcal{T}_n(\omega)\). We consider various cases in a similar manner as in [1, Proposition 4.2].

We let Case 1 be when \(I = \emptyset\). If two of the entries of the equation \(x_1 + x_2 + x_3 = n\) is chosen, then the third is uniquely determined. Thus there can not be a vectorial sunflower of type \(I\), where \(|I \cap \{1, 2, 3\}| = 2\). Let Case 2 be when \(|I \cap \{1, 2, 3\}| = 1\).

Let us assume \(|I \cap \{1, 2, 3\}| = 0\) or \(3\), for otherwise it is taken care of in Case 2. We split into further cases. Suppose \(I\) contains at least one of the pairs \(\{1, 4\}\), \(\{5, 6\}\) or \(\{7, 8\}\). Then we can deduce that \(|I \cap \{1, 4, 5, 6, 7, 8\}| = 2, 4\) or \(6\), because of the equation \(x_1 + x_2 + x_3 = x_5 + x_6 = x_7 + x_8\). For example, if \(\{1, 4\} \subseteq I\) and \(5 \in I\), then this forces \(6 \in I\). Suppose \(|I \cap \{1, 4, 5, 6, 7, 8\}| = 6\). Since \(1 \in I\), we have \(|I \cap \{1, 2, 3\}| = 3\), and hence, \(I = \{1, 2, 3, 4, 5, 6, 7, 8\}\). Thus, we consider the following cases, Case 3 when \(|I \cap \{1, 4, 5, 6\}| = 2\) or \(|I \cap \{5, 6, 7, 8\}| = 2\) or \(|I \cap \{1, 4, 7, 8\}| = 2\), and Case 4 when \(I = \{1, 2, 3, 4, 5, 6, 7, 8\}\). We see that the possibilities considered in this paragraph are all contained in either Case 3 or Case 4.

Suppose \(I\) does not contain any of the pairs \(\{1, 4\}\), \(\{5, 6\}\) or \(\{7, 8\}\). In this case, we have \(|I \cap \{1, 4, 5, 6, 7, 8\}| \in \{0, 1, 2, 3\}\). If \(|I \cap \{1, 4, 5, 6, 7, 8\}| = 0\), then \(1 \notin I\) and hence \(|I \cap \{1, 2, 3\}| = 0\). Therefore, \(I = \emptyset\) and this is taken care of in Case 1. We let Case 5 be when \(|I \cap \{1, 4, 5, 6, 7, 8\}| = 1\). If \(|I \cap \{1, 4, 5, 6, 7, 8\}| = 2\) or \(3\), then it is easy to see that these possibilities are taken care of in Case 3. Therefore, it is sufficient to only consider the above five cases of distinct types of \(I\).

Case 1. \(I = \emptyset\). By Lemma 4.3 we know that \(\mathbb{E}(1/\mathcal{T}_n(\omega)) \ll q^{(-1/11)\max\{\deg n, M\}}\). It then follows from Proposition 3.5 that
\[
\mathbb{P}(\{\mathcal{T}_n(\omega)\} \subseteq \mathbb{F}_q[t]) = o_M(1).
\]

Therefore, our claim holds for vectorial sunflowers of this type.

Case 2. \(|I \cap \{1, 2, 3\}| = 1\). Without loss of generality, assume \(I \cap \{1, 2, 3\} = \{1\}\). Let \(l_1\) denote the common first coordinate. If \(\mathcal{T}_n(\omega)\) contains a vectorial sunflower of 12 petals of type \(I\) for some \(n\), then there is a 12-d.s.v. \(\{x_j\}_{1 \leq j \leq 12}\), where for each \(j\) we have \(x_j = (x_{2j}, x_{3j})\), \(\text{Set}(x_j) \subseteq \omega\), and \(x_{2j} + x_{3j} = n - l_1\). Let \(r = n - l_1\). Then \(\mathcal{U}_r(\omega)\) contains a 12-d.s.v. and we obtain via Lemma 4.6 our claim for vectorial sunflowers of this type.

Case 3. \(|I \cap \{1, 4, 5, 6\}| = 2\) or \(|I \cap \{5, 6, 7, 8\}| = 2\) or \(|I \cap \{1, 4, 7, 8\}| = 2\). Suppose \(|I \cap \{1, 4, 5, 6\}| = 2\) as the other two cases are similar. We consider the following two essentially distinct subcases separately.

i) Suppose \(I \cap \{1, 4, 5, 6\} = \{1, 4\}\). Let \(l_1\) and \(l_4\) denote the common first and forth coordinates, respectively. If \(\mathcal{T}_n(\omega)\) contains a vectorial sunflower of 12 petals of type \(I\) for some \(n\), then there is a 12-d.s.v. \(\{x_j\}_{1 \leq j \leq 12}\), where for each \(j\) we have \(x_j = (x_{5j}, x_{6j})\), \(\text{Set}(x_j) \subseteq \omega\), and \(l_1 + l_4 = x_{5j} + x_{6j}\). Thus, for \(r = l_1 + l_4\), \(\mathcal{U}_r(\omega)\) contains a 12-d.s.v. and we obtain via Lemma 4.6 our claim for vectorial sunflowers of this type. We can argue in a similar manner if \(I \cap \{1, 4, 5, 6\} = \{5, 6\}\).
ii) Suppose \( I \cap \{1, 4, 5, 6\} = \{1, 5\} \). Let \( l_1 \) and \( l_5 \) denote the common first and fifth coordinates, respectively. If \( T_n(\omega) \) contains a vectorial sunflower of 12 petals of type \( I \) for some \( n \), then there is a 12-d.s.v. \( \{x_j\}_{1 \leq j \leq 12} \), where for each \( j \) we have \( x_j = (x_{4j}, x_{6j}) \), \( \text{Set}(\pi_j) \subseteq \omega \), and \( l_1 + x_{4j} = l_5 + x_{6j} \). Let \( r = l_5 - l_1 = x_{4j} - x_{6j} \). Note we have \( r \neq 0 \), because if \( l_1 = l_5 \), then the equation \( x_1 + x_4 = x_5 + x_6 \) forces \( \{x_1, x_4\} = \{x_5, x_6\} \), which is a contradiction. Thus, \( V_r(\omega) \) contains a 12-d.s.v. and we obtain via Lemma 4.6 our claim for vectorial sunflowers of this type. The remaining cases of \( |I \cap \{1, 4, 5, 6\}| = 2 \) can be treated in a similar manner.

Case 4. \( I = \{1, 2, 3, 4, 5, 6, 7, 8\} \). This is the trivial case. If two vectors have the same \( i \)-th coordinate for all \( i \in I \), then they are the same vector. Thus, in particular \( T_n(\omega) \) does not contain vectorial sunflowers of 12 petals of this type.

Case 5. \( |I \cap \{1, 4, 5, 6, 7, 8\}| = 1 \). Without loss of generality suppose that \( I \cap \{1, 4, 5, 6, 7, 8\} = 1 \). Let \( l_1 \) denote the common first coordinate. If \( T_n(\omega) \) contains a vectorial sunflower of 12 petals of type \( I \) for some \( n \), then there is a 12-d.s.v. \( \{x_j\}_{1 \leq j \leq 12} \), where for each \( j \) we have \( x_j = (x_{4j}, x_{5j}, x_{6j}, x_{7j}, x_{8j}) \), \( \text{Set}(\pi_j) \subseteq \omega \), and \( x_{5j} + x_{6j} = x_{7j} + x_{8j} = l_1 + x_{4j} \). Let \( r = l_1 \). Then \( V_r(A) \) contains a 12-d.s.v. and we obtain via Lemma 4.6 our claim for vectorial sunflowers of this type.

We remark that in order for our argument to prove Propositions 4.4 and 4.7 to work, we needed the expectation of \( Q_n \) to go to infinity as \( \deg n \to \infty \), while \( \Delta(Q_n) \) and the expectation of \( T_n \) to tend to 0, and also the expectations of \( U_r, V_r, \) and \( W_r \) to tend to 0 as \( \deg r \to \infty \). Our value of \( \gamma = \frac{7}{9} \), similarly as in [1], was chosen because it satisfies all of these conditions, and it also simplifies certain calculations. We note that it is certainly not the only possible value for the method to prove Theorem 1.4 to work.

**Proof of Theorem 1.4.** By Proposition 1.4 we have for \( \omega \in \Omega \) with probability 1 that
\[
|Q_n(\omega)| \gg q^{\gamma \deg n}
\]
for all \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large. By Proposition 4.7 we know there exists \( M \) sufficiently large such that
\[
\mathbb{P}(\{|T_n(\omega)| \leq 10^{28} \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1/2.
\]
Therefore, we deduce that there exists some \( \omega_0 \in \Omega \) such that
\[
|Q_n(\omega_0)| \gg q^{\gamma \deg n}
\]
and
\[
|T_n(\omega_0)| \leq 10^{28}
\]
for all \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large. Thus we obtain our result by the argument given in the paragraph after \( \text{[1.2]} \).

**5. Proof of Theorem 1.5**

Let \( \varepsilon > 0 \) be sufficiently small. In this section, we consider the probability space \( S_M(\gamma; S \mod n_0) \), where we let
\[
\gamma = \frac{2}{3} + \frac{\varepsilon}{9+9\varepsilon},
\]
and let \( S \) be a non-empty subset of \( \mathbb{G}_N \) satisfying the conditions of Corollary 2.1. The basic strategy is as follows. We use the Borell-Cantelli Lemma (Theorem 3.1) to show that

\[
\mathbb{P}(\{|T_n(\omega)| \leq 10^{28} \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1/2.
\]
in the probability space $S_M(\gamma; S \mod n_0)$, “most” of the sequences, in other words with probability 1, has “many” representations of $n$ as a sum of four of its elements, where one of the four elements has degree less than or equal to $(\varepsilon \deg n)$, for all $n \in \mathbb{F}_q[t]$ with deg $n$ sufficiently large. We then show that out of these sequences, there exists a sequence such that even after removing some of its elements to make it a Sidon sequence, it still has at least one of the representations of $n$ left for each $n \in \mathbb{F}_q[t]$ with deg $n$ sufficiently large.

For each $n \in \mathbb{F}_q[t]$, we consider the following collection of sets

$$\mathcal{R}_n = \{\theta = \{x_1, x_2, x_3, x_4\} : \theta \text{ satisfies Cond}(\mathcal{R}_n)\},$$

where

$$\text{Cond}(\mathcal{R}_n) = \begin{cases} x_1 + x_2 + x_3 + x_4 = n, \\
\min\{\deg x_1, \deg x_2, \deg x_3, \deg x_4\} \leq \varepsilon \deg n, \\
x_i \neq x_j \pmod{m_0} \text{ for } 1 \leq i < j \leq 4. \end{cases}$$

(5.1)

Given a sequence of polynomials $\omega$, we let

$$\mathcal{R}_n(\omega) = \{\theta \in \mathcal{R}_n : \theta \subseteq \omega\}.$$

We define

$$\mathcal{B}_n = \{\overline{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) : \overline{x} \text{ satisfies Cond}(\mathcal{B}_n)\},$$

where

$$\text{Cond}(\mathcal{B}_n) = \begin{cases} \{x_1, x_2, x_3, x_4\} \in \mathcal{R}_n, \\
x_1 + x_5 = x_6 + x_7, \quad \{x_1, x_5\} \neq \{x_6, x_7\}, \\
x_1 \equiv x_6 \pmod{m_0}, \quad x_5 \equiv x_7 \pmod{m_0}. \end{cases}$$

(5.2)

We also let

$$\mathcal{B}_n(\omega) = \{\overline{x} \in \mathcal{B}_n : \text{Set}(\overline{x}) \subseteq \omega\}.$$

The Sidon lifting process of a sequence $\omega$ consists of removing from $\omega$, those elements $a \in \omega$ such that there exist $b, c, d \in \omega$ with $a + b = c + d$ and $\{a, b\} \neq \{c, d\}$. We denote by $\omega_{\text{Sidon}}$, the resulting Sidon sequence obtained by applying this process to $\omega$.

By a similar argument as in the paragraph before (4.2), we see that $|\mathcal{B}_n(\omega)|$ is an upper bound for the number of representations counted in $\mathcal{R}_n(\omega)$ that are destroyed in the Sidon lifting process of $\omega$. Thus, we obtain

$$|\mathcal{R}_n(\omega_{\text{Sidon}})| \geq |\mathcal{R}_n(\omega)| - |\mathcal{B}_n(\omega)|.$$ (5.3)

Therefore, Theorem 1.5 is established if we can prove that there exists a sequence $\omega_0$ such that for any $n \in \mathbb{F}_q[t] \setminus \{0\}$ with deg $n$ sufficiently large, we have $|\mathcal{R}_n(\omega_0)| \gg q^{\varepsilon \deg n}$ for some $\delta > 0$, and $|\mathcal{B}_n(\omega_0)| \ll 1$. We show that in some sense there are many sequences satisfying the former condition, and then we show it is also the case for the latter condition. These tasks are accomplished in Propositions 5.4 and 5.7. We then prove that there exist sequences satisfying both conditions. Before we get into the proofs of these propositions, we list three useful estimates. However, we postpone their proofs to Section 8.

Lemma 5.1. We have that

$$\mathbb{E}(|\mathcal{R}_n(\omega)|) \gg q^{\frac{\varepsilon^2}{10\varepsilon}} \deg n,$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with deg $n$ sufficiently large.

Recall from (3.2), the definition of $\Delta(\cdot)$. 
Proposition 5.2. We have that
\[
\Delta(R_n) \ll q^{-\frac{3\epsilon+2^2}{9+3\epsilon}} \deg n,
\]
for \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large.

Lemma 5.3. We have that
\[
\mathbb{E}(|B_n(\omega)|) \ll q^{-\frac{c^2}{16}} \max\{\deg n, M\}.
\]

We now prove the following proposition.

Proposition 5.4. We have that
\[
\mathbb{P}(\{|R_n(\omega)| \gg q^{\frac{2c^2}{9+3\epsilon}} \deg n\}) = 1
\]
for \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large.

Proof. We apply Theorem 3.2 to \( F = R_n \) and \( Y = |R_n(\omega)| = \{\theta \in R_n : \theta \subseteq \omega\} \), where \( \omega \) is a random sequence in \( S_M(\gamma; S \mod n_0) \). We proved that \( \mu = \mu_n = \mathbb{E}(R_n(\omega)) \gg q^{2c^2} \) in Lemma 5.1 and \( \Delta(R_n) \ll q^{-\frac{3\epsilon+2^2}{9+3\epsilon}} \) in Proposition 5.2. Hence for \( \deg n \) sufficiently large, we have \( \Delta(R_n) < \mu_n \). Then, Theorem 3.2 implies that
\[
\mathbb{P}(\{|R_n(\omega)| \leq \mu_n/2\}) \leq \exp(-\mu_n/16).
\]
Therefore, we obtain that for some \( C, C' > 0 \) and \( T \in \mathbb{N} \), we can write
\[
\sum_{n \in \mathbb{F}_q[t]} \mathbb{P}(\{|R_n(\omega)| \leq \mu_n/2\}) < C' + \sum_{j=T}^{\infty} (q^{j+1} - q^j) \exp(-Cq^{\frac{2c^2}{9+3\epsilon}j}) \\
< C' + (q - 1) \sum_{j=T}^{\infty} q^j \exp(-Cq^{\frac{2c^2}{9+3\epsilon}j}) \\
< \infty.
\]
Thus, Theorem 3.1 implies that with probability 1, we have \( |R_n(\omega)| > \mu_n/2 \gg q^{\frac{2c^2}{9+3\epsilon}} \deg n \) for all \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large. \qed

For each \( r \in \mathbb{F}_q[t] \), we define the following families of vectors.

\begin{align*}
\mathcal{U}_r & = \{ \overline{x} = (x_1, x_2) : x_1 + x_2 = r, x_1 \neq x_2 \}, \\
\mathcal{U}_r' & = \{ \overline{x} = (x_1, x_2, x_3) : x_1 + x_2 + x_3 = r, x_i \neq x_j \ (i \neq j) \}, \\
\mathcal{V}_r & = \{ \overline{x} = (x_1, x_2) : x_1 - x_2 = r, x_1 \neq x_2 \}, \\
\mathcal{V}_r' & = \{ \overline{x} = (x_1, x_2, x_3) : x_1 + x_2 - x_3 = r, x_i \neq x_j \ (i \neq j) \}.
\end{align*}

Lemma 5.5. We have the following bounds on the expectations.

i) \( \mathbb{E}(|\mathcal{U}_r(\omega)|) \ll q^{-\frac{1}{3}} \max(\deg r, M) \).

ii) \( \mathbb{E}(|\mathcal{U}_r'(\omega)|) \ll q^{-\frac{1}{6}} \max(\deg r, M) \).

iii) \( \mathbb{E}(|\mathcal{V}_r(\omega)|) \ll q^{-\frac{1}{3}} \max(\deg r, M) \).

iv) \( \mathbb{E}(|\mathcal{V}_r'(\omega)|) \ll q^{-\frac{1}{6}} \max(\deg r, M) \).

Proof. Since the details of the proof is similar to that of Lemma 4.3 and [1, Lemma 6.6], we omit the proof here. \qed
Lemma 5.6. Let $K \in \mathbb{N}$ and $K > 18/\varepsilon^2$. Then for any of the four families $F_r$ in (5.4), we have
\[
\mathbb{P}(\{F_r(\omega) \text{ contains a } K\text{-d.s.v. for some } r \in \mathbb{F}_q[t]\}) = o_M(1).
\]

Proof. For any of the four choices of $F_r$, Lemma 5.5 shows that $\mathbb{E}(|F_r(\omega)|) \ll q^{-\frac{K}{6}} \max\{\deg r, M\}$. Thus, the result follows by Proposition 3.5. □

We prove the following proposition, which is one of the main ingredients to prove Theorem 1.5.

Proposition 5.7. Let $K \in \mathbb{N}$ and $K > 18/\varepsilon^2$. We have that
\[
\mathbb{P}(\{|B_n(\omega)| \leq 7!(7^2 - 7 + 1)K^7 \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1 - o_M(1).
\]

Proof. We claim the following statement:

Claim. Let $K$ be a positive integer such that $K > 18/\varepsilon^2$. Then with probability $1 - o_M(1)$, $B_n(\omega)$ does not contain vectorial sunflower of $K$ petals for any $n \in \mathbb{F}_q[t]$.

Assuming the claim holds, we can apply Corollary 3.4 to obtain
\[
\mathbb{P}(\{|B_n(\omega)| \leq 7!(7^2 - 7 + 1)K^7 \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1 - o_M(1).
\]

Thus, we see that proving the above claim is sufficient to obtain our result. We prove it for distinct possible types $I \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ of vectorial sunflowers in $B_n(\omega)$. We consider various cases in a similar manner as in Proposition 1.7 or [1] Proposition 5.2.

By the definition of $B_n$, it is easy to see that there cannot be a vectorial sunflower of type $I$, where $|I \cap \{1, 2, 3, 4\}| = 3$ or $|I \cap \{1, 5, 6, 7\}| = 3$. Suppose $|I \cap \{1, 2, 3, 4\}| = 4$, then we have $|I \cap \{1, 5, 6, 7\}| \geq 1$. Similarly, if $|I \cap \{1, 5, 6, 7\}| = 4$, then we have $|I \cap \{1, 2, 3, 4\}| \geq 1$. Therefore, the following six cases cover every possibility: Case 1. $I = \emptyset$, Case 2. $|I \cap \{1, 2, 3, 4\}| = 1$, Case 3. $|I \cap \{1, 2, 3, 4\}| = 2$, Case 4. $|I \cap \{1, 5, 6, 7\}| = 1$, Case 5. $|I \cap \{1, 5, 6, 7\}| = 2$, and Case 6. $I = \{1, 2, 3, 4, 5, 6, 7\}$.

The argument to show that the above claim holds for each of the six cases is similar to the argument employed in Proposition 1.7 and [1] Proposition 5.2. Thus, we omit verifying the remaining details. □

We remark that in order for our argument to prove Propositions 5.4 and 5.7 to work, we needed the expectation of $R_n$ to go to infinity as $\deg n \to \infty$, while $\Delta(R_n)$ and the expectation of $B_n$ to tend to 0, and also the expectations of $U_r$, $U'_r$, $V_r$, and $V'_r$ to tend to 0 as $\deg r \to \infty$. Our value of $\gamma = \frac{2}{3} + \frac{9 + 3\varepsilon}{9 + 3\varepsilon}$, similarly as in [1], was chosen because it satisfies all of these conditions, and it also simplifies certain calculations.

Proof of Theorem 1.5. By Proposition 5.4, we have for $\omega \in \Omega$ with probability 1 that
\[
|R_n(\omega)| \gg q^{\frac{2\varepsilon^2}{9 + 3\varepsilon}} \deg n
\]
for all $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large. By Proposition 5.7, we know there exists $M$ sufficiently large such that
\[
\mathbb{P}(\{|B_n(\omega)| \leq 7!(7^2 - 7 + 1)K^7 \text{ for all } n \in \mathbb{F}_q[t]\}) \geq 1/2.
\]

Therefore, we deduce that there exists some $\omega_0 \in \Omega$ such that
\[
|R_n(\omega_0)| \gg q^{\frac{2\varepsilon^2}{9 + 3\varepsilon}} \deg n
\]
and
\[ |\mathcal{B}_n(\omega_0)| \leq 7!((7^2 - 7 + 1)K)^7 \]
for all \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large. Thus we obtain our result by the argument given in the paragraph after (3.3).

\[ \square \]

6. **Technical Lemmas**

In this section, we calculate bounds that were used to compute estimates essential in the proof of our main results. We list the estimates used in Theorems 1.4 and 1.5 in Sections 7 and 8, respectively. We consider \( q \) to be a fixed number, and throughout these sections implicit constants in inequalities may depend on \( q \) without any further notice. Let \( \mathcal{A}, \mathcal{B} \) be two disjoint subsets of \( \mathbb{F}_q[t] \), and \( \nu_x \in \mathbb{R} (x \in \mathbb{F}_q[t]) \). In order to avoid notation clutter in the exposition, we use the following notation for summation
\[ \sum_{x \in \mathcal{A}} + \sum_{x \in \mathcal{B}} \nu_x := \sum_{x \in \mathcal{A}} \nu_x + \sum_{x \in \mathcal{B}} \nu_x. \]
We also use the notation in a similar manner when we have more than two pairwise disjoint sets. Recall from Section 3 that we have set \( \mathbb{P}(0 \in \omega) = 0 \). Thus, we use the convention throughout the remainder of the paper that for any \( \mu \in \mathbb{R} \), we let \( q^{\mu \cdot \deg 0} = 0 \). We also let \( \deg 0 := -\infty \). Finally, we note that in this section we do not require any assumption on the characteristic of \( \mathbb{F}_q \). Thus, in particular, the results of this section hold even when the characteristic of \( \mathbb{F}_q \) is 2 or 3.

For \( \alpha, \beta \in \mathbb{R} \) and \( n \in \mathbb{F}_q[t] \), we define the following quantities,
\[ \sigma_{\alpha, \beta}(n) = \sum_{x, y \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\alpha \deg x} q^{-\beta \deg y} = \sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\alpha \deg x} q^{-\beta \deg(n-x)}, \]
and
\[ \sigma_{\alpha, \beta}(n; M) = \sum_{\deg x > M} q^{-\alpha \deg x} q^{-\beta \deg(n-x)}. \]
For each \( r \in \mathbb{N} \), there are exactly \( q^{r+1} - q^r \) polynomials in \( \mathbb{F}_q[t] \) of degree \( r \). Thus given any \( \gamma > 1 \) and \( R \in \mathbb{N} \), we have
\[ (6.1) \sum_{\deg x > R} q^{-\gamma \deg x} = \sum_{r=R+1}^{\infty} (q^{r+1} - q^r) q^{-\gamma r} = (q - 1) \sum_{r=R+1}^{\infty} q^{(1-\gamma)r} \ll_{\gamma} q^{-(\gamma-1)R}. \]
Similarly for any \( \lambda \in \mathbb{R} \), \( \lambda > -1 \), and \( R \in \mathbb{N} \), we have
\[ (6.2) \sum_{\deg x < R} q^{\lambda \deg x} = \sum_{r=0}^{R-1} (q^{r+1} - q^r) q^{\lambda r} = (q - 1) \sum_{r=0}^{R-1} q^{(\lambda+1)r} = (q - 1) \frac{q^{(\lambda+1)R} - 1}{q^{\lambda+1} - 1} \ll_{\lambda} q^{(\lambda+1)R}. \]
We have the following useful lemmas.

**Lemma 6.1.** Suppose \( \alpha, \beta \in \mathbb{R} \) satisfy \( 0 < \alpha < 1, 0 < \beta < 1 \), and \( \alpha + \beta > 1 \). Then, for any \( n \in \mathbb{F}_q[t] \setminus \{0\} \), we have the following estimates:

i) \( \sigma_{\alpha, \beta}(n; M) \ll q^{-(\alpha + \beta - 1) \max\{\deg n, M\}}. \)

ii) \( \sigma_{\alpha, \beta}(n) \ll q^{-(\alpha + \beta - 1) \deg n}. \)
Here the implicit constants depend only on $\alpha$, $\beta$ and $q$. If $n = 0$, then we still have $i)$, but we have $\sigma_{\alpha,\beta}(n) \ll 1$ for $ii)$.

Proof. Since the estimate is trivial when $n = 0$, we assume $n \neq 0$. We only prove $i)$ as $ii)$ follows immediately from $i)$ by letting $M = -1$. Suppose $\deg n \leq M$. Since $\alpha + \beta > 1$, it follows by (6.1) that

$$
\sigma_{\alpha,\beta}(n; M) = \sum_{\deg x > M} q^{-\alpha \deg x} q^{-\beta \deg(n-x)} = \sum_{\deg x > M} q^{-(\alpha+\beta) \deg x} \ll q^{-(\alpha+\beta-1)M}.
$$

Suppose $N_0 = \deg n > M$. Since $-\alpha > -1$ and $\alpha + \beta > 1$, it follows by (6.1) and (6.2) that

$$
(6.3) \quad \sigma_{\alpha,\beta}(n; M) \ll \sum_{M < \deg x < N_0} q^{-\alpha \deg x} q^{-\beta \deg(n-x)} + \sum_{\deg x = N_0} q^{-\alpha \deg x} q^{-\beta \deg(n-x)} + \sum_{\deg x > N_0} q^{-(\alpha+\beta) \deg x}.
$$

We now deal with the remaining sum on the right hand side of the inequality displayed above. Given a degree $N_0$ polynomial $x$, we let $x = c'r^{N_0} + y$, where $c' \in \mathbb{F}_q \setminus \{0\}$ and $y \in \mathbb{G}_{N_0}$. If the leading coefficient of $n$ is $c' \in \mathbb{F}_q \setminus \{0\}$, then we see that $\{n + (-c')t^{N_0} + y : y \in \mathbb{G}_{N_0} \} = \mathbb{G}_{N_0}$. Thus we obtain the following bound by (6.2),

$$
\sum_{\deg x = N_0} q^{-\beta \deg(n+x)} = \sum_{\deg x = N_0} q^{-\beta \deg(n + \deg x)} + \sum_{\deg x = N_0} q^{-\beta \deg(n + (-c')t^{N_0} + y)}
$$

$$
= (q - 2) \sum_{\deg y < N_0} q^{-\beta \deg n} + (q - 2)\sum_{\deg z < N_0} q^{-\beta \deg x} \ll (q - 2)q^{N_0 - \beta N_0} + q^{(1-\beta)N_0}.
$$

Therefore, it follows from (6.3) that when $\deg n > M$, we have

$$
\sigma_{\alpha,\beta}(n; M) \ll q^{-(\alpha+\beta-1)\deg n}.
$$

Lemma 6.2. Suppose $\phi, \kappa \in \mathbb{R}$ satisfy $0 < \phi < 1, 0 < \kappa < 1$, and $\phi + \kappa > 1$. Let $r \in \mathbb{F}_q[t]$. Then we have that

$$
\sum_{\deg x > M} q^{-\phi \deg x} q^{-\kappa \deg\{\deg(r+x), M\}} \ll q^{(1-\phi-\kappa) \max\{\deg r, M\}}.
$$

Proof. We consider the two cases, $\deg r \leq M$ and $\deg r > M$, separately. Suppose $\deg r \leq M$. Then we have $\deg(r + x) = \deg x > M$. Since $\phi + \kappa > 1$, we obtain the following bound by (6.1),

$$
(6.4) \quad \sum_{\deg x > M} q^{-\phi \deg x} q^{-\kappa \deg\{\deg(r+x), M\}} \ll \sum_{\deg x > M} q^{-\phi \deg x} q^{-\kappa \deg x} \ll q^{(1-\phi-\kappa)M}.
$$
Next, suppose \( \deg r > M \). We split and simplify the sum as follows,

\[
\sum_{\deg x > M} q^{-\phi \deg x} q^{-\kappa \max(\deg(r+x),M)} \\
\leq \sum_{\deg x < \deg r} + \sum_{\deg x = \deg r} + \sum_{\deg x > \deg r} q^{-\phi \deg x} q^{-\kappa \max(\deg(r+x),M)} \\
= q^{-\kappa \deg r} \sum_{\deg x < \deg r} q^{-\phi \deg x} + \sum_{\deg x = \deg r} q^{-\phi \deg x} q^{-\kappa \max(\deg(r+x),M)} \\
+ \sum_{\deg x > \deg r} q^{(-\kappa-\phi) \deg x}.
\]

Note \(-\phi > -1\) and \(\kappa + \phi > 1\). Thus by applying (6.2) and (6.1) to the first sum and the third sum, respectively, we see that these sums are bounded by

(6.5) \(\ll q^{(1-\kappa+\phi) \deg r}\).

We now deal with the remaining second sum. Let \(N_0 = \deg x = \deg r\). Write a degree \(N_0\) polynomial \(x\) as \(x = ct^{N_0} + y\), where \(c \in \mathbb{F}_q \setminus \{0\}\) and \(y \in \mathbb{G}_{N_0}\). Given \(z \in \mathbb{F}_q[t]\) we define \(\text{lead}[z] \in \mathbb{F}_q \setminus \{0\}\) to be the coefficient of \(t^{\deg z}\) in \(z\). By separating the cases \(c \neq -\text{lead}[r]\) and \(c = -\text{lead}[r]\), we obtain the following bound for the sum in question,

(6.6) \[
\sum_{\deg x = \deg r = N_0} q^{-\phi \deg x} q^{-\kappa \max(\deg(r+x),M)} \\
= q^{-\phi N_0} \sum_{c \in \mathbb{F}_q[t] \setminus \{0\}} \sum_{\deg y < N_0} q^{-\kappa \max(\deg(r+ct^{N_0}+y),M)} \\
= (q-2)q^{-\phi N_0} \sum_{\deg y < N_0} q^{-\kappa N_0} + q^{-\phi N_0} \sum_{\deg z < N_0} q^{-\kappa \max(\deg z,M)} \\
= (q-2)q^{-\phi N_0} q^{(-1-\kappa)N_0} + q^{-\phi N_0} \sum_{\deg z \leq M} q^{-\kappa M} + q^{-\phi N_0} \sum_{M < \deg z < N_0} q^{-\kappa \deg z}.
\]

Since \(-\kappa > -1\), we can apply (6.2) to the third sum, and obtain the following bound for (6.6),

(6.7) \(\ll q^{(1-\phi-\kappa)N_0} + q^{-\phi N_0} q^{(-1-\kappa)M} \ll q^{(1-\phi-\kappa)N_0}\).

The last inequality follows, because \(1 - \kappa > 0\) and \(N_0 > M\). Therefore, we obtain that

\[
\sum_{\deg x > M} q^{-\phi \deg x} q^{-\kappa \max(\deg(r+x),M)} \ll q^{(1-\phi-\kappa) \deg r}
\]

when \(\deg r > M\). \(\square\)

We also have the following lemma, which we make use of only in Section 7.

**Lemma 6.3.** Given any polynomials \(a, b \in \mathbb{F}_q[t] \setminus \{0\}\) and \(1/2 < \gamma < 2/3\), we have

\[
\sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)} \ll q^{(1-2\gamma)(\deg a + \deg b)}.
\]

If \(a = 0\) and \(b \in \mathbb{F}_q[t] \setminus \{0\}\), then we have

\[
\sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)} \ll q^{(1-2\gamma) \deg b}.
\]
Proof. We consider the following three cases separately: \( \deg a < \deg b \), \( \deg a > \deg b \) and \( \deg a = \deg b \). We only present the details of the computation for the cases \( \deg a < \deg b \) and \( \deg a = \deg b \). The result for the case \( \deg a > \deg b \) can be verified in a similar manner as the case \( \deg a < \deg b \). First, suppose \( 0 \leq \deg a < \deg b \). We split the sum and simplify it in the following manner,

\[
\sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)}
\]

\[
= \sum_{\deg x < \deg b} + \sum_{\deg x = \deg b} + \sum_{\deg x > \deg b} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)}
\]

\[
= q^{(1-2\gamma) \deg b} \sum_{\deg x < \deg b} q^{-\gamma \deg x} - \gamma \deg(x+a) + q^{-2\gamma \deg b} \sum_{\deg x = \deg b} q^{(1-2\gamma) \deg(x+b)}
\]

\[
\leq q^{(1-2\gamma) \deg b} \sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x} - \gamma \deg(x+a) + q^{-2\gamma \deg b} \sum_{\deg x < \deg b} q^{(1-2\gamma) \deg x}
\]

Note we have \( 2\gamma > 1, 1 - 2\gamma > -1 \) and \( 4\gamma - 1 > 1 \). Thus, we apply Lemma 6.1 to the first sum, (6.2) to the second sum, and (6.1) to the third sum, in order to estimate the final expression above. Consequently, we obtain that the above expression is bounded by

\[
\ll q^{(1-2\gamma) \deg b} q^{1-2\gamma \deg a} + q^{-2\gamma \deg b} q^{2-2\gamma \deg b} + q^{2-4\gamma \deg b}
\]

\[
\ll q^{(1-2\gamma) \deg b} q^{1-2\gamma \deg a}.
\]

Our conclusion when \( a = 0 \) follows from a similar analysis as above, except that we have to use the following bound obtained via (6.1) instead of Lemma 6.1 to bound (6.8),

\[
\sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x} - \gamma \deg(x+a) \ll \sum_{x \in \mathbb{F}_q[t] \setminus \{0\}} q^{-2\gamma \deg x} \ll 1.
\]

Next, suppose \( \deg a = \deg b \). We split the sum and simplify it in the following manner,

\[
\sum_{x \in \mathbb{F}_q[t]} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)}
\]

\[
= \sum_{\deg x < \deg b} + \sum_{\deg x = \deg b} + \sum_{\deg x > \deg b} q^{-\gamma \deg x} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)}
\]

\[
= q^{(1-3\gamma) \deg b} \sum_{\deg x < \deg b} q^{-\gamma \deg x} + q^{-\gamma \deg b} \sum_{\deg x = \deg b} q^{-\gamma \deg(x+a)} q^{(1-2\gamma) \deg(x+b)}
\]

\[
\ll q^{(2-4\gamma) \deg b} = q^{(1-2\gamma)(\deg a + \deg b)}.
\]

Note we have \( -\gamma > -1 \) and \( 4\gamma - 1 > 1 \). Thus, we apply (6.2) to the first sum, and (6.1) to the third sum, and obtain that the first and the third terms of the final expression above are bounded by

\[
\ll q^{(2-4\gamma) \deg b} = q^{(1-2\gamma)(\deg a + \deg b)}.
\]
The remaining second sum requires more work to estimate. It is clear that the set of polynomials of degree (deg \( b \)) can be expressed as 

\[ \{ ct^{\deg b} + y : c \in \mathbb{F}_q \setminus \{0\}, y \in \mathbb{G}_{\deg b} \}. \]

Let \( c \in \mathbb{F}_q \setminus \{0\} \), and let lead[\( a \)] be the leading coefficient of \( a \). Then, it follows that

\begin{equation} (a + ct^{\deg b}) + \mathbb{G}_{\deg b} = \begin{cases} \mathbb{G}_{\deg b}, & \text{if } c = -\text{lead}[a], \\ (\text{lead}[a] + c)t^{\deg b} + \mathbb{G}_{\deg b}, & \text{otherwise}. \end{cases} \tag{6.10} \end{equation}

We also have a similar statement if we replace \( a \) by \( b \). We consider the second sum in the final expression of (6.9) in two separate cases, lead[\( a \)] \( \neq \) lead[\( b \)] and lead[\( a \)] = lead[\( b \)].

Suppose lead[\( a \)] \( \neq \) lead[\( b \)]. We utilize (6.10) to simplify the sum in question in the following manner,

\[
\sum_{\text{deg } x = \text{deg } b} q^{-\gamma \text{deg}(x+a)} q^{(1-2\gamma) \text{deg}(x+b)} = \sum_{c \in \mathbb{F}_q \setminus \{-\text{lead}[a], \text{lead}[b], 0\}} \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a+ct^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b+ct^{\deg b} + y)} \\
+ \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a-\text{lead}[a] t^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b-\text{lead}[a] t^{\deg b} + y)} \\
+ \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a-\text{lead}[b] t^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b-\text{lead}[b] t^{\deg b} + y)} \\
\ll q^{(2-3\gamma) \text{deg } b} + q^{(1-2\gamma) \text{deg } b} \sum_{\text{deg } z < \text{deg } b} q^{-\gamma \text{deg } z} + q^{-\gamma \text{deg } b} \sum_{\text{deg } z < \text{deg } b} q^{(1-2\gamma) \text{deg } z}.
\]

Since \(-\gamma > -1\) and \(1-2\gamma > -1\), we obtain by (6.2) that the final expression above is bounded by

\[ \ll q^{(2-\gamma) \text{deg } b}. \]

On the other hand, suppose lead[\( a \)] = lead[\( b \)]. We simplify the sum in a similar manner as in the previous case,

\begin{equation} \sum_{\text{deg } x = \text{deg } b} q^{-\gamma \text{deg}(x+a)} q^{(1-2\gamma) \text{deg}(x+b)} = \sum_{c \in \mathbb{F}_q \setminus \{-\text{lead}[b], 0\}} \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a+ct^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b+ct^{\deg b} + y)} \\
+ \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a-\text{lead}[b] t^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b-\text{lead}[b] t^{\deg b} + y)} \\
= (q-2)q^{\deg b} q^{(1-3\gamma) \text{deg } b} + \sum_{\text{deg } y < \text{deg } b} q^{-\gamma \text{deg}(a-\text{lead}[a] t^{\deg b} + y)} q^{(1-2\gamma) \text{deg}(b-\text{lead}[b] t^{\deg b} + y)}. \tag{6.11} \end{equation}

Let \( g \) be the polynomial,

\[ g = b - \text{lead}[b] t^{\deg b} - (a - \text{lead}[a] t^{\deg b}), \]
where \( \deg g < \deg b \). By the change of variable \( x = y + (a - \text{lead}[a])t^{\deg b} \), we see that the final expression in (6.11) equals to

\[
(q - 2)q^{(2 - 3\gamma)\deg b} + \sum_{\deg x < \deg b} q^{-\gamma \deg x} q^{(1 - 2\gamma) \deg (x + g)}.
\]

(6.12)

We have that if \( \deg g \leq \deg x \), then \( \deg (x + g) \leq \deg x \), and equivalently, \( -\gamma \deg (x + g) \geq -\gamma \deg x \). Since \( 1 - 2\gamma < 0 \), we bound the sum in (6.12) as follows,

\[
\sum_{\deg x < \deg b} q^{-\gamma \deg x} q^{(1 - 2\gamma) \deg (x + g)} \leq \sum_{\deg x < \deg g} q^{-\gamma \deg x} q^{(1 - 2\gamma) \deg x} + \sum_{\deg g \leq \deg x < \deg b} q^{-\gamma \deg x} q^{(1 - 2\gamma) \deg (x + g)}
\]

\[
\ll \sum_{\deg x < \deg g} q^{(1 - 3\gamma) \deg x} + \sum_{\deg z < \deg b} q^{(1 - 3\gamma) \deg z}.
\]

Since \( \gamma < 2/3 \), we have \( 1 - 3\gamma > -1 \). Thus, by (6.2) we see that the final expression above is bounded by

\[
\ll q^{(2 - 3\gamma) \deg g} + q^{(2 - 3\gamma) \deg b} \ll q^{(2 - 3\gamma) \deg b}.
\]

Consequently, (6.11) is bounded by

\[
\ll q^{(2 - 3\gamma) \deg b}.
\]

Therefore, in either case we obtain that (6.10) is bounded by

\[
\ll q^{(2 - 4\gamma) \deg b} = q^{(1 - 2\gamma)(\deg a + \deg b)}
\]

as desired. \( \square \)

7. Estimates in Theorem 1.4

Recall in Section 4 we work in the probability space \( S_M(\gamma; S \mod n_0) \), where \( \gamma = 7/11 \) and \( S \) is a non-empty subset of \( \mathbb{G}_N \) satisfying the conditions of Theorem 1.6.

Lemma 4.1. We have that

\[
\mathbb{E}(|Q_n(\omega)|) \gg q^\frac{1}{2} \deg n,
\]

for \( n \in \mathbb{F}_q[t] \setminus \{0\} \) with \( \deg n \) sufficiently large.

Proof. By the definition of \( Q_n(\omega) \), we have

\[
\mathbb{E}(|Q_n(\omega)|) = \sum_{(x_1, x_2, x_3) \in Q_n} \mathbb{P}(x_1, x_2, x_3 \in \omega) \geq q^{-3\gamma \deg n} |Q'_n|,
\]

where

\[
Q'_n = \{ (x_1, x_2, x_3) \in Q_n : x_i \equiv S \pmod{n_0}, \deg n = \deg x_i > M \}.
\]
By our choice of $S$, we know there exist distinct $s_1, s_2, s_3$ such that $n \equiv s_1 + s_2 + s_3 \pmod{n_0}$. We fix such $s_1, s_2, s_3$, and write $x_i = s_i + n_0 y_i$. Let $l \in \mathbb{F}_q[t]$ be the polynomial such that $n - s_1 - s_2 - s_3 = l n_0$. Then we have $|Q_n^*| \geq |Q_n'|$, where

$$Q_n^* = \{ y_1, y_2, y_3 : y_1 + y_2 + y_3 = l, \deg y_i = \deg n - \deg n_0 \}.$$ 

Given any element $a \in \mathbb{F}_q$, the number of solutions $(a_1, a_2, a_3) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ such that $a_1 + a_2 + a_3 = a$ and $a_i \neq 0$ is greater than or equal to $(q - 1)^2 - q$. We see this by allowing $a_1$ and $a_2$ to be any elements in $\mathbb{F}_q \setminus \{0\}$, which then uniquely determines $a_3$. This gives $(q - 1)^2$ choices, but we do not want to include cases when $a_3 = 0$, which only occurs when $a_1 + a_2 = a$. There are only $q$ combinations of $a_1$ and $a_2$ such that $a_1 + a_2 = a$. Therefore, by considering the addition coordinatewise, we obtain the following crude bound,

$$|Q_n^*| \geq \frac{1}{6} ((q - 1)^2 - q)^{\deg n_0} \geq \frac{1}{6} (q(q - 3))^{\deg n - \deg n_0} \gg q^{2 \deg n}.$$

Note the factor of $1/6$ is there to take care of possible over counting of the triplets. Hence, we obtain our result,

$$E(|Q_n(\omega)|) \geq q^{-3 \gamma \deg n} |Q_n'| \gg q^{(2 - 3 \gamma) \deg n} = q^{\frac{3}{11} \deg n}.$$ 

$\square$

**Proposition 4.2.** We have that

$$\Delta(Q_n) \ll q^{\frac{2}{11} \deg n},$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large.

**Proof.** Recall by $\theta \sim \theta'$, we mean that $\theta \cap \theta' \neq \emptyset$ and $\theta \neq \theta'$. Thus if $\theta \sim \theta'$ for $\theta, \theta' \in Q_n$, then it follows that $|\theta \cap \theta'| = 1$, for otherwise $\theta = \theta'$. Without loss of generality, let the common element be $x_1$. We bound $\Delta(Q_n)$ by applying Lemma 6.1 twice,

$$\Delta(Q_n) = \sum_{\substack{\theta, \theta' \in Q_n \\theta \sim \theta'}} P(\theta, \theta' \subseteq \omega) \ll \sum_{x_1, x_2, x_3, x_2', x_3' \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma (\deg x_1 + \deg x_2 + \deg x_3 + \deg x_2' + \deg x_3')}$$

$$\ll \sum_{x_1 \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x_1} \left( \sum_{x_2, x_3 \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma (\deg x_2 + \deg x_3)} \right)^2$$

$$\ll \sum_{x_1 \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x_1} q^{2(-4\gamma) \max(\deg(n-x_1),0)}$$

$$\ll q^{-\gamma \deg n} + \sum_{x_1 \in \mathbb{F}_q[t] \setminus \{0\}} q^{-\gamma \deg x_1} q^{(2-4\gamma) \deg(n-x_1)}$$

$$\ll q^{-\gamma \deg n} + q^{(3-5\gamma) \deg n}$$

$$\ll q^{(3-5\gamma) \deg n}.$$
Note since $0 < 4\gamma - 2 = 6/11 < 1$ and $5\gamma - 2 = 13/11 > 1$, the sum after the third last inequality satisfies the conditions required to apply Lemma 6.1. We also remark that the term $q^{-\gamma \deg n}$ in the third last inequality comes from the case $n - x_1 = 0$. 

**Lemma 4.3.** We have that
$$
\mathbb{E}(|T_n(\omega)|) \ll q^{-\frac{1}{4}\max\{\deg n, M\}}.
$$

**Proof.** Recall from Section 4 that
$$
T_n = \{\pi = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) : \pi \text{satisfies Cond}(T_n)\},
$$
where
$$
\text{Cond}(T_n) = \begin{cases}
\{x_1, x_2, x_3\} \in \mathcal{Q}_n, \\
x_1 + x_4 = x_5 + x_6 = x_7 + x_8, \ \{x_1, x_4\} \neq \{x_5, x_6\} \neq \{x_7, x_8\}, \\
x_1 \equiv x_5 \equiv x_7 \pmod{n_0}, \ x_4 \equiv x_6 \equiv x_8 \pmod{n_0}.
\end{cases}
$$

Suppose $\pi = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in T_n$. Then we know that each element of $\{x_1, x_2, x_3\}$ is distinct by the definition of $\mathcal{Q}_n$. We have that the elements of $\{x_1, x_5, x_7\}$ are distinct for the following reason. Without loss of generality, suppose $x_1 = x_5$, then the equation $x_1 + x_4 = x_5 + x_6$ forces $x_4 = x_6$. Hence, we have $\{x_1, x_4\} = \{x_5, x_6\}$, which is a contradiction. By a similar argument, the elements of $\{x_4, x_6, x_8\}$ are distinct. We classify every situation where we have a repeated element amongst $\{x_i\}_{1 \leq i \leq 8}$ by considering the following two cases separately, $x_1 \equiv x_4 \pmod{n_0}$ and $x_1 \not\equiv x_4 \pmod{n_0}$.

Case 1: $x_1 \equiv x_4 \pmod{n_0}$. If $x_1 \in \{x_6, x_8\}$, then we obtain contradiction by the following argument. Without loss of generality, suppose $x_1 = x_5$, then the equation $x_1 + x_4 = x_5 + x_6$ forces $x_4 = x_5$. Hence, we have $\{x_1, x_4\} = \{x_5, x_6\}$, which is a contradiction. Thus, it follows that the only possible repetition of $x_1$ is $x_1 = x_4$. By the definition of $\mathcal{Q}_n$, we have $x_1 \not\equiv x_2, x_3 \pmod{n_0}$. Since $x_1 \equiv x_4 \pmod{n_0}$ for $4 \leq i \leq 8$, we have $\{x_1, x_4, x_5, x_6, x_7, x_8\} \cap \{x_2, x_3\} = \emptyset$. We also know that $x_2 \not\equiv x_3$. Therefore, $x_2$ and $x_3$ do not have a repetition. If $x_1 = x_i$ for $5 \leq i \leq 8$, then the relations $x_1 + x_i = x_5 + x_6 = x_7 + x_8$ and $\{x_1, x_4\} \not\equiv \{x_5, x_6\} \not\equiv \{x_7, x_8\}$ yield contradiction. Thus, $x_4$ has no possible repetition other than $x_1 = x_4$. We continue in a similar manner and verify that the only remaining possible repetitions are $x_5 = x_6$ and $x_7 = x_8$. Note the entries of $\pi$ can not have more than one of the three possible repetitions, because otherwise it violates $\{x_1, x_4\} \not\equiv \{x_5, x_6\} \not\equiv \{x_7, x_8\}$. Therefore, the possible subcases stemming from Case 1 are $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ are distinct, and each of the following,
$$
\begin{align*}
\{x_1 = x_4 \text{ and } \{x_1, x_2, x_3, x_5, x_6, x_7, x_8\} \text{ are distinct}, \\
x_5 = x_6 \text{ and } \{x_1, x_2, x_3, x_4, x_5, x_7, x_8\} \text{ are distinct}, \\
x_7 = x_8 \text{ and } \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \text{ are distinct}.
\end{align*}
$$

Case 2: $x_1 \not\equiv x_4 \pmod{n_0}$. In this case, we have $\{x_1, x_5, x_7\} \cap \{x_4, x_6, x_8\} = \emptyset$, and consequently, $\{x_1, x_4, x_5, x_6, x_7, x_8\}$ are distinct. We know that $x_1, x_5, x_7 \not\equiv \{x_2, x_3\}$, because they are in different residue classes modulo $n_0$. Therefore, it follows that $x_1, x_5, x_7$ do not have any repetitions. Thus, we deduce that the only possible repetitions are of the form $x_j \in \{x_2, x_3\}$, where $j \in \{4, 6, 8\}$. Without loss of generality, suppose $x_4 = x_2$. Then since $x_4 = x_2 \not\equiv x_3 \pmod{n_0}$, we have that $x_3 \not\equiv x_2, x_4, x_6, x_8$. It follows that $\{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$ are distinct. We obtain similar conclusions for other cases. Therefore, the possible subcases stemming from Case 2 are $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ are distinct,
and each of the following situations, where there exists \( j \in \{4, 6, 8\} \) such that \( x_j \in \{x_2, x_3\} \) and \( \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \setminus \{x_j\} \) are distinct.

By considering all the subcases stemming from Cases 1 and 2, we can bound \( \mathbb{E}(|\mathcal{T}_n(\omega)|) \) by

\[
\mathbb{E}(|\mathcal{T}_n(\omega)|) \ll S_1 + S_2 + S_3 + S_4 + S_5 + S_6,
\]

where

\[
S_1 = \sum_{\deg x_i > M \ (i=1,\ldots,8)} q^{-\gamma(\deg x_1 + 2 \deg x_2 + 3 \deg x_3 + 4 \deg x_4 + 5 \deg x_5 + 6 \deg x_6 + 7 \deg x_7 + 8 \deg x_8)};
\]

\[
S_2 = \sum_{\deg x_i > M \ (i=1,2,3,5,6,7,8)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_5 + \deg x_6 + \deg x_7 + \deg x_8)};
\]

\[
S_3 = \sum_{\deg x_i > M \ (i=1,2,3,4,5,7,8)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_7 + \deg x_8)};
\]

\[
S_4 = \sum_{\deg x_i > M \ (i=1,2,3,4,5,6,7)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_6 + \deg x_7)};
\]

\[
S_5 = \sum_{\deg x_i > M \ (i=1,2,3,5,6,7,8)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_5 + \deg x_6 + \deg x_7 + \deg x_8)};
\]

and

\[
S_6 = \sum_{\deg x_i > M \ (i=1,2,3,4,5,7,8)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_7 + \deg x_8)}.
\]

We note that \( S_1 \) corresponds to the subcase when \( \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \) are distinct, and \( S_2, S_3, S_4 \) to the remaining subcases stemming from Case 1. There are essentially two distinct subcases amongst the remaining subcases stemming from Case 2. This is because we could relabel \( x_2 \) and \( x_3 \) as each other without affecting the analysis, since they play the same role, and similarly with \( x_6 \) and \( x_8 \). Thus, without loss of generality, it suffices to consider the situations \( x_4 = x_2 \) and \( x_6 = x_2 \), which correspond to \( S_5 \) and \( S_6 \), respectively.

We can then show that \( S_i \ll q^{(-1/11) \max\{\deg n, M\}} \) for each \( 1 \leq i \leq 6 \) (we can in fact obtain smaller bounds for \( 1 < i \leq 6 \)), from which it follows that

\[
\mathbb{E}(|\mathcal{T}_n(\omega)|) \ll q^{(-1/11) \max\{\deg n, M\}}.
\]

In an effort to keep the paper concise, we only give details for the bounds of \( S_2 \) and \( S_6 \). We note that the bound for \( S_6 \) requires the most calculation of all six. The bounds for \( S_1, S_3, S_4 \) and \( S_5 \) can be achieved in a similar manner as for \( S_2 \) and \( S_6 \).
Bound for \( S_2 \): Since the characteristic of \( \mathbb{F}_q \) is not 2, we have \( \deg(2x_1) = \deg x_1 \). By a repeated application of Lemma 6.1, we have

\[
S_2 \ll \sum_{\deg x_1 > M} q^{-\gamma \deg x_1} \sum_{x_2, x_3 \in \mathbb{F}_q \setminus \{0\}} q^{-\gamma (\deg x_2 + \deg x_3)} \left( \sum_{x_5, x_6 \in \mathbb{F}_q \setminus \{0\}} q^{-\gamma (\deg x_5 + \deg x_6)} \right)^2
\]

\[
\ll \sum_{\deg x_1 > M} q^{-\gamma \deg x_1} q^{(1-2\gamma) \max\{\deg(n-x_1),0\}} q^{(2-4\gamma) \deg (2x_1)}
\]

\[
\ll q^{(2-5\gamma) \max\{\deg n, M\}} + \sum_{\deg x_1 > M} q^{(2-5\gamma) \deg x_1} q^{(1-2\gamma) \deg (n-x_1)}.
\]

We consider the two cases \( \deg n \leq M \) and \( \deg n > M \), separately. Suppose \( \deg n \leq M \). Since \( 7\gamma - 3 = 16/11 > 1 \), we have by (6.1) that

\[
S_2 \ll q^{(2-5\gamma) \max\{\deg n, M\}} + \sum_{\deg x_1 > M} q^{(3-7\gamma) \deg x_1} \ll q^{(4-7\gamma)M}.
\]

On the other hand, suppose \( \deg n > M \). We split and simplify the final sum in (7.1) as follows,

\[
(7.2) \quad \sum_{M < \deg x_1 < \deg n} + \sum_{\deg x_1 = \deg n} + \sum_{\deg x_1 > \deg n} q^{(2-5\gamma) \deg x_1} q^{(1-2\gamma) \deg (n-x_1)}
\]

\[
\ll q^{(1-2\gamma) \deg n} \sum_{M < \deg x_1 < \deg n} q^{(2-5\gamma) \deg x_1} + q^{(2-5\gamma) \deg n} \sum_{\deg x_1 = \deg n} q^{(1-2\gamma) \deg (n-x_1)}
\]

\[
\ll q^{(1-2\gamma) \deg n} \sum_{\deg x_1 < \deg n} q^{(2-5\gamma) \deg x_1} + q^{(2-5\gamma) \deg n} \sum_{\deg x_1 > \deg n} q^{(1-2\gamma) \deg (n-x_1)}
\]

\[
\ll q^{(1-2\gamma) \deg n} \sum_{\deg x_1 < \deg n} q^{(2-5\gamma) \deg x_1} + q^{(2-5\gamma) \deg n} \sum_{\deg x_1 < \deg n} q^{(1-2\gamma) \deg (n-x_1)}
\]

\[
+ \sum_{\deg x_1 > \deg n} q^{(3-7\gamma) \deg x_1}
\]

\[
\ll q^{(2-5\gamma) \deg x_1} \sum_{\deg x_1 < \deg n}
\]

Since \( 2 - 5\gamma = -13/11 \leq -1 \), by a similar calculation as in (6.2) we see that

\[
\sum_{\deg x_1 < \deg n} q^{(2-5\gamma) \deg x_1} \ll 1.
\]

By applying this estimate to the first sum, (6.2) to the second sum, and (6.1) to the third sum in the final expression of (7.2), we obtain that

\[
S_2 \ll q^{(2-5\gamma) \max\{\deg n, M\}} + q^{(1-2\gamma) \deg n} + q^{(4-7\gamma) \deg n} \ll q^{(1-2\gamma) \deg n}
\]

when \( \deg n > M \). Therefore, by combining both cases together we obtain that

\[
S_2 \ll q^{(1-2\gamma) \max\{\deg n, M\}} \ll q^{-\frac{1}{11} \max\{\deg n, M\}}.
\]
Bound for $S_6$: In order to simplify the sum, we make the following substitutions $x_2 = n - x_3 - x_1$ and $x_5 = 2x_1 + x_3 + x_4 - n$. By Lemma 6.3 we have

(7.3)  
$$S_6 \ll \sum_{x_4 \in F_q[t] \setminus \{0\}, \deg x_1, \deg x_3 > M} q^{-\gamma (\deg x_1 + \deg(n - x_3 - x_1))} q^{-\gamma (\deg x_3 + \deg x_4)} \cdot q^{-\gamma \deg(2x_1 + x_3 + x_4 - n)} \sum_{x_7, x_8 \in F_q[t] \setminus \{0\}, x_7 + x_8 = x_1 + x_4} q^{-\gamma (\deg x_7 + \deg x_8)}$$

$$\ll \sum_{x_4 \in F_q[t] \setminus \{0\}, \deg x_1, \deg x_3 > M} q^{-\gamma (\deg x_1 + \deg(n - x_3 - x_1))} q^{-\gamma (\deg x_3 + \deg x_4)} \cdot q^{-\gamma \deg(2x_1 + x_3 + x_4 - n)} q^{(1 - 2\gamma) \max\{\deg(x_1 + x_4), 0\}}$$

$$\ll \sum_{\deg x_1, \deg x_3 > M} q^{-\gamma (\deg x_1 + \deg(n - x_3 - x_1))} q^{-\gamma \deg x_3} \cdot \sum_{x_4 \in F_q[t] \setminus \{0\}} q^{-\gamma \deg x_4} q^{-\gamma \deg(2x_1 + x_3 + x_4 - n)} q^{(1 - 2\gamma) \max\{\deg(x_1 + x_4), 0\}}.$$

For simplicity, let $a = 2x_1 + x_3 - n$. If $a \neq 0$, then we simplify the inner sum in the final expression above by Lemma 6.3.

(7.4)  
$$\sum_{x_4 \in F_q[t] \setminus \{0\}} q^{-\gamma \deg x_4} q^{-\gamma \deg(a + x_4)} q^{(1 - 2\gamma) \max\{\deg(x_1 + x_4), 0\}}$$

$$\ll q^{-\gamma \deg x_1} q^{-\gamma \deg(a - x_1)} + \sum_{x_4 \in F_q[t] \setminus \{0\}} q^{-\gamma \deg x_4} q^{-\gamma \deg(a + x_4)} q^{(1 - 2\gamma) \deg(x_1 + x_4)}$$

$$\ll q^{-\gamma \deg x_1} q^{-\gamma \deg(a - x_1)} + q^{(1 - 2\gamma)(\deg a + \deg x_1)}$$

$$\ll q^{-\gamma \deg x_1} + q^{(1 - 2\gamma)(\deg a + \deg x_1)}.$$

If $a = 0$, then we have by Lemma 6.3 that

(7.5)  
$$\sum_{x_4 \in F_q[t] \setminus \{0\}} q^{-\gamma \deg x_4} q^{-\gamma \deg(a + x_4)} q^{(1 - 2\gamma) \max\{\deg(x_1 + x_4), 0\}}$$

$$\ll q^{-2\gamma \deg x_1} + \sum_{x_4 \in F_q[t] \setminus \{0\}} q^{-\gamma \deg x_4} q^{-\gamma \deg(a + x_4)} q^{(1 - 2\gamma) \deg(x_1 + x_4)}$$

$$\ll q^{-2\gamma \deg x_1} + q^{(1 - 2\gamma) \deg x_1}$$

$$\ll q^{(1 - 2\gamma) \deg x_1}.$$
By the bounds obtained in (7.4) and (7.5), the change of variable $x_3 = n - x_1 - z$, and Lemma 6.1, we obtain from (7.3) that

\begin{align}
S_6 \ll & \sum_{x_3 \in \mathbb{F}_q[t] \setminus \{0\}, \deg x_1 > M} q^{-2\gamma \deg x_1} q^{-\gamma \deg(n-x_3-x_1)} q^{-\gamma \deg x_3} \\
& + \sum_{x_3 \in \mathbb{F}_q[t] \setminus \{0\}, \deg x_1 > M} q^{-\gamma (\deg x_1 + \deg(n-x_3-x_1))} q^{-\gamma \deg x_3} q^{(1-2\gamma)(\deg n+\deg x_1)} \\
& + \sum_{x_3, x_1, \deg x_3 > M \atop u=2x_1+x_3-n=0} q^{-\gamma (\deg x_1 + \deg(n-x_3-x_1))} q^{-\gamma \deg x_3} q^{(1-2\gamma) \deg x_1} \\
& \ll q^{-2\gamma \max\{\deg n, M\}} + \sum_{\deg x_1 > M} q^{-2\gamma \deg x_1} q^{(1-2\gamma) \deg(n-x_1)} \\
& + \sum_{\deg x_1 > M} \sum_{x_3 \in \mathbb{F}_q[t]} q^{(1-3\gamma) \deg x_1} q^{-\gamma \deg z} q^{-\gamma \deg(n-x_1-z)} q^{(1-2\gamma) \deg(x_1-z)} \\
& + \sum_{\deg x_1, \deg x_3 > M \atop 2x_1+x_3=n} q^{-\gamma \deg x_3} q^{(1-4\gamma) \deg(2x_1)}.
\end{align}

Since $2\gamma > 9/11$, we bound the first sum in the final expression above by Lemma 6.1 as follows,

\begin{align}
\sum_{\deg x_1 > M} q^{-2\gamma \deg x_1} q^{(1-2\gamma) \deg(n-x_1)} \ll \sum_{\deg x_1 > M} q^{-9/11 \deg x_1} q^{(1-2\gamma) \deg(n-x_1)} \ll q^{-\frac{9}{11} \max\{\deg n, M\}}.
\end{align}

Since $1 - 4\gamma = -17/11 < -5/11$, the third sum can be bounded by Lemma 6.1 in the following manner,

\begin{align}
\sum_{\deg x_1, \deg x_3 > M \atop 2x_1+x_3=n} q^{-\gamma \deg x_3} q^{(1-4\gamma) \deg(2x_1)} \ll \sum_{\deg x_1, \deg x_3 > M \atop 2x_1+x_3=n} q^{-\frac{9}{11} \deg x_3} q^{-\frac{7}{11} \deg(2x_1)} \ll q^{-\frac{9}{11} \max\{\deg n, M\}}.
\end{align}
Therefore, by applying Lemma 6.3 to the inner sum of the remaining second sum, we see that the final expression of (7.6) can be bounded by
\[ S_6 \ll q^{-\frac{4}{11}} \max \{ \deg n, M \} + \sum_{\deg x_1 > M, n = x_1} q^{1-3\gamma} \deg x_1 q (1-2\gamma) \deg (-x_1) + \sum_{\deg x_1 > M} q^{1-3\gamma} \deg x_1 q (1-2\gamma) (\deg(x_1 - n) + \deg(-x_1)) \ll q^{-\frac{4}{11}} \max \{ \deg n, M \} + q^{2-5\gamma} \max \{ \deg n, M \} + \sum_{\deg x_1 > M} q^{2-5\gamma} \deg x_1 q (1-2\gamma) (n-x_1).\]

Notice that the sum in the final estimate obtained above is the same as the sum in the estimate for \( S_2 \) in (7.1). Therefore, we have by the work done to bound \( S_2 \) that
\[ S_6 \ll q^{-\frac{4}{11}} \max \{ \deg n, M \} + q^{1-2\gamma} \max \{ \deg n, M \} \ll q^{-\frac{4}{11}} \max \{ \deg n, M \}. \]

\[ \square \]

**Lemma 4.5.** We have the following bounds on the expectations.

i) \( \mathbb{E}(U_{r}(\omega)) \ll q^{-\frac{3}{11}} \max \{ \deg r, M \} \).

ii) \( \mathbb{E}(V_{r}(\omega)) \ll q^{-\frac{3}{11}} \max \{ \deg r, M \} \).

iii) \( \mathbb{E}(W_{r}(\omega)) \ll q^{-\frac{3}{11}} \max \{ \deg r, M \} \).

**Proof.** By Lemma 6.1, we get that
\[ \mathbb{E}(U_{r}(\omega)) \leq \sum_{\deg x, \deg y > M} q^{-\gamma \deg x} q^{-\gamma \deg y} \ll q^{1-2\gamma} \max \{ \deg r, M \} \ll q^{-\frac{3}{11}} \max \{ \deg r, M \}, \]
and
\[ \mathbb{E}(V_{r}(\omega)) \leq \sum_{\deg x, \deg y > M} q^{-\gamma \deg x} q^{-\gamma \deg y} \ll q^{1-2\gamma} \max \{ \deg r, M \} \ll q^{-\frac{3}{11}} \max \{ \deg r, M \}. \]

Similarly, we have by Lemma 6.1 that
\[ \mathbb{E}(W_{r}(\omega)) \leq \sum_{\deg x_1 > M \atop x_3 + x_6 = r + x_4} q^{-\gamma \deg x_1} q^{-\gamma \deg x_5} q^{-\gamma \deg x_6} q^{-\gamma \deg x_7} q^{-\gamma \deg x_8} \]
\[ \leq \sum_{\deg x_4 > M \atop x_3 + x_6 = r + x_4} q^{-\gamma \deg x_4} \left( \sum_{\deg x_5, \deg x_6 > M \atop x_3 + x_6 = r + x_4} q^{-\gamma \deg x_5} q^{-\gamma \deg x_6} \right)^2 \]
\[ \ll \sum_{\deg x_4 > M} q^{-\gamma \deg x_4} q^{2-4\gamma} \max \{ \deg(r + x_4), M \}. \]

With our choice of \( \gamma = 7/11 \), we have \( 0 < 4\gamma - 2 < 1 \) and \( 5\gamma - 2 > 1 \). Therefore, by Lemma 6.2 we obtain that
\[ \mathbb{E}(W_{r}(\omega)) \ll q^{3-5\gamma} \max \{ \deg r, M \}. \]

\[ \square \]
8. Estimates in Theorem 1.5

Recall in Section 5, we work in the probability space $S_M(\gamma, S(\text{mod } n_0))$, where $\gamma = \frac{2}{3} + \frac{\varepsilon}{9+9\varepsilon}$ for some $\varepsilon > 0$ sufficiently small, and $S$ is a non-empty subset of $\mathbb{G}_N$ satisfying the conditions of Corollary 2.1. Since the computation in this section is similar to that of Section 7, we omit some of the details.

Lemma 5.1. We have that

$$\mathbb{E}(|\mathcal{R}_n(\omega)|) \gg q^{\frac{2\varepsilon^2}{1+\varepsilon} \deg n},$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large.

Proof. By the definition of $\mathcal{R}_n(\omega)$, we have

$$\mathbb{E}(|\mathcal{R}_n(\omega)|) = \sum_{\{x_1, x_2, x_3, x_4\} \in \mathcal{R}_n} \mathbb{P}(x_1, x_2, x_3, x_4 \in \omega) \geq q^{-(3+\varepsilon)\gamma \deg n} |\mathcal{R}_n'|,$$

where

$$\mathcal{R}_n' = \{\{x_1, x_2, x_3, x_4\} \in \mathcal{R}_n : x_i \equiv S \pmod{n_0}, \deg x_i = \deg n > M \ (1 \leq i \leq 3),$$

$$M < \deg x_4 \leq \varepsilon \deg n\}.$$

By our choice of $S$, we know there exist distinct $s_1, s_2, s_3, s_4$ such that $n \equiv s_1 + s_2 + s_3 + s_4 \pmod{n_0}$. We fix such $s_1, s_2, s_3, s_4$, and write $x_i = s_i + n_0y_i$. Let $l \in \mathbb{F}_q[t]$ be the polynomial such that $n - s_1 - s_2 - s_3 - s_4 = ln_0$. Then we have $|\mathcal{R}_n'| \geq |\mathcal{R}_n^*|$, where

$$\mathcal{R}_n^* = \{\{y_1, y_2, y_3, y_4\} : y_1 + y_2 + y_3 + y_4 = l, \deg y_i = \deg n - \deg n_0 \ (1 \leq i \leq 3),$$

$$\deg y_4 \leq \varepsilon \deg n - \deg n_0\}.$$

For each $y_4$ with $\deg y_4 \leq \varepsilon \deg n - \deg n_0$, we can give a lower bound to the number of polynomials $y_1, y_2, y_3$ of degree $\deg n - \deg n_0$ such that $y_1 + y_2 + y_3 = l - y_4$ in a similar manner as in Lemma 4.1 or [1, Lemma 6.7]. Therefore, we obtain that

$$|\mathcal{R}_n^*| \gg (q^2)^{\deg y_4} q^{\varepsilon (\deg n - \deg n_0)} \gg q^{(2+\varepsilon) \deg n}.$$

Thus, we have our result

$$\mathbb{E}(|\mathcal{R}_n(\omega)|) \geq q^{-(3+\varepsilon)\gamma \deg n} |\mathcal{R}_n'| \gg q^{(-\frac{3+2\varepsilon}{1+\varepsilon} - \varepsilon) \deg n} \gg q^{\frac{2\varepsilon^2}{1+\varepsilon} \deg n}.$$

\hfill \Box

Proposition 5.2. We have that

$$\Delta(\mathcal{R}_n) \ll q^{-3+\frac{2\varepsilon^2}{9+9\varepsilon} \deg n},$$

for $n \in \mathbb{F}_q[t] \setminus \{0\}$ with $\deg n$ sufficiently large.

Proof. Recall

$$\Delta(\mathcal{R}_n) = \sum_{\theta, \theta' \in \mathcal{R}_n, \theta \sim \theta'} \mathbb{P}(\theta, \theta' \subseteq \omega),$$

where by $\theta \sim \theta'$ we mean $\theta \cap \theta' \neq \emptyset$ and $\theta \neq \theta'$. By the definition of $\mathcal{R}_n$, it is clear that if $\theta \sim \theta'$, then $|\theta \cap \theta'| = 1$ or $2$. We split $\Delta(\mathcal{R}_n)$ into several sums according to $\theta \cap \theta'$ in order to estimate it. We let $\theta = \{x_1, x_2, x_3, x_4\}$ and $\theta' = \{x'_1, x'_2, x'_3, x'_4\}$. 
Case 1. \( \theta \cap \theta' = \{x_i\} \), where \( \deg x_i \leq \varepsilon \deg n \). Without loss of generality, let \( i = 1 \). Note \( 0 < \gamma < 1 \) and \( 2\gamma > 1 \). We have by Lemmas 6.1 and 6.2 that
\[
L_1 = \sum_{\substack{\deg x_i, \deg x'_j > M \ (1 \leq i \leq 4, j \leq 3) \\ x_1 + x_2 + x_3 + x_4 = n \\ x_1 + x'_2 + x'_3 + x'_4 = n \\ \deg x_i \leq \varepsilon \deg n}} q^{-\gamma (\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x'_2 + \deg x'_3 + \deg x'_4)} 
\ll q^{(4-6\gamma) \deg n} q^{(1-\gamma) \varepsilon \deg n}.
\]

Case 2. \( \theta \cap \theta' = \{x_j\} \), where \( \deg x_j > \varepsilon \deg n \). Without loss of generality, let \( j = 2 \). We have by Lemma 6.1 that
\[
L_2 = \sum_{\substack{\deg x_i, \deg x'_j > M \ (1 \leq i \leq 4, j = 3, 4) \\ \deg x_1, \deg x'_2 \leq \varepsilon \deg n \\ x_1 + x_2 + x_3 + x_4 = n \\ x_3 + x_4 = n - x_1 - x_2 \\ x'_2 + x'_3 = n - x'_1 - x'_2}} q^{-\gamma (\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x'_2 + \deg x'_3 + \deg x'_4)} 
\ll q^{(3-5\gamma) + (2-2\gamma) \varepsilon} \deg n.
\]

Case 3. \( \theta \cap \theta' = \{x_i, x_j\} \), where \( \deg x_i \leq \varepsilon \deg n \). Without loss of generality, let \( i = 1 \) and \( j = 2 \). Note we have \( 2\gamma > 1 \), \( 0 < 4\gamma - 2 < 1 \), and \( 5\gamma - 2 > 1 \). Thus, we obtain by Lemmas 6.1 and 6.2 that
\[
L_3 = \sum_{\substack{\deg x_i, \deg x'_j > M \ (1 \leq i \leq 4, j = 3, 4) \\ \deg x_1, \deg x'_2 \leq \varepsilon \deg n \\ x_1 + x_2 + x_3 + x_4 = n \\ x_3 + x_4 = n - x_1 - x_2 \\ \deg x_1 \leq \varepsilon \deg n}} q^{-\gamma (\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x'_2 + \deg x'_3 + \deg x'_4)} 
\ll q^{3-5\gamma} \deg n + (1-\gamma) \varepsilon \deg n.
\]

Case 4. \( \theta \cap \theta' = \{x_j, x_k\} \), where \( \deg x_j, \deg x_k > \varepsilon \deg n \). Without loss of generality, let \( j = 2 \) and \( k = 3 \). We obtain by Lemmas 6.1 and 6.2 that
\[
L_4 = \sum_{\substack{\deg x_i, \deg x'_j > M \ (1 \leq i \leq 4, j = 1, 4) \\ \deg x_1, \deg x'_2 \leq \varepsilon \deg n \\ x_1 + x_2 + x_3 + x_4 = n \\ x'_1 + x'_2 + x'_3 + x'_4 = n \\ \deg x_1, \deg x'_2 \leq \varepsilon \deg n}} q^{-\gamma (\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x'_2 + \deg x'_3 + \deg x'_4)} 
\ll q^{2-2\gamma} \varepsilon \deg n q^{(1-2\gamma) \deg n}.
\]

Combining all the bounds computed for \( L_1, L_2, L_3, L_4 \), we obtain that
\[
\Delta(R_n) \ll L_1 + L_2 + L_3 + L_4 
\ll q^{(4-6\gamma) \deg n} q^{(1-\gamma) \varepsilon \deg n}.
\]

Note with our choice of \( \gamma = \frac{2}{3} + \frac{\varepsilon}{9 + 9\varepsilon} \), we have
\[
(4 - 6\gamma) + (1 - \gamma) \varepsilon = -\frac{6\varepsilon}{9 + 9\varepsilon} + \left(1 - \frac{\varepsilon}{9 + 9\varepsilon}\right) \varepsilon = \frac{-3\varepsilon + 2\varepsilon^2}{9 + 9\varepsilon}.
\]
Lemma 5.3 We have that
\[ \mathbb{E}(|B_n(\omega)|) \ll q^{-\frac{2}{1+\varepsilon}\max\{\deg n, M\}}. \]

Proof. Recall from Section 5 that
\[ B_n = \{ \bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) : \bar{x} \text{ satisfies Cond}(B_n) \}, \]
where
\[ \text{Cond}(B_n) = \begin{cases} \{x_1, x_2, x_3, x_4\} \in \mathcal{R}_n, \\ x_1 + x_5 = x_6 + x_7, \quad \{x_1, x_5\} \neq \{x_6, x_7\} \\ x_1 \equiv x_6 \pmod{n_0}, \quad x_5 \equiv x_7 \pmod{n_0}. \end{cases} \]

If \( \varepsilon \deg n \leq M \), then we have \( \mathbb{E}(|B_n(\omega)|) = 0 \). Therefore, we can bound \( \mathbb{E}(|B_n(\omega)|) \ll q^{-\frac{2}{1+\varepsilon}M} \) if \( \deg n \leq M < M/\varepsilon \), and \( \mathbb{E}(|B_n(\omega)|) \ll q^{-\frac{2}{1+\varepsilon}\deg n} \) if \( M < \deg n \leq M/\varepsilon \). Thus we suppose \( \varepsilon \deg n > M \) for the remainder of the proof.

Let \( (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in B_n \). By arguing in a similar manner as in the proof of Lemma 4.3 or [1, Lemma 6.8], we can verify that the following five situations are the only cases we need to consider.

Case 1: all the \( x_i \)'s are distinct.
Case 2: \( x_6 = x_7 \) and \( \{x_1, \ldots, x_6\} \) are distinct.
Case 3: \( x_5 = x_1 \) and \( \{x_1, x_2, x_3, x_4, x_6, x_7\} \) are distinct.
Case 4: \( x_5 \in \{x_2, x_3, x_4\} \) and \( \{x_1, x_2, x_3, x_4, x_6, x_7\} \) are distinct.
Case 5: \( x_7 \in \{x_2, x_3, x_4\} \) and \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) are distinct.

By considering all of the five cases above, we obtain that \( \mathbb{E}(|B_n(\omega)|) \) can be bounded by
\[ \mathbb{E}(|B_n(\omega)|) \ll S_1 + S_2 + S_3 + S_4 + S_5, \]
where
\[ S_1 = \sum_{\deg x_i > M \ (1 \leq i \leq 7)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_6 + \deg x_7)}, \]
\[ S_2 = \sum_{\deg x_i > M \ (1 \leq i \leq 6)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_6)}, \]
\[ S_3 = \sum_{\deg x_i > M \ (i=1,2,3,4,6,7)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_6 + \deg x_7)}, \]
\[ S_4 = \sum_{\deg x_i > M \ (i=1,2,3,4,6,7)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_6 + \deg x_7)}, \]
\[ S_5 = \sum_{\deg x_i > M \ (i=1,2,3,4,6,7)} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_6 + \deg x_7)}. \]
and
\[ S_5 = \sum_{\substack{\deg x_i > M \ (1 \leq i \leq 6) \\ x_1 + x_2 + x_3 + x_4 = n \\ x_1 + x_5 = x_6 + x_2 \\ \min_{1 \leq i \leq 4} \{\deg x_i\} \leq \varepsilon \deg n}} q^{-\gamma(\deg x_1 + \deg x_2 + \deg x_3 + \deg x_4 + \deg x_5 + \deg x_6)}. \]

Note \( S_1, S_2, S_3, S_4, \) and \( S_5 \) correspond to Cases 1, 2, 3, 4, and 5, respectively.

By computing the bounds for each \( S_1, \ldots, S_5 \) in a similar manner as in Lemma 4.3 and [1, Lemma 6.8], we obtain
\[ \mathbb{E}(|B_n(\omega)|) \ll S_1 + S_2 + S_3 + S_4 + S_5 \ll q^{-\frac{2}{18} \deg n} \]
when \( \varepsilon \deg n > M. \)

\[ \square \]

**References**

[1] J. Cilleruelo, *On Sidon sets and asymptotic bases*, to appear in the Proceedings of the London Mathematical Society.

[2] J. Cilleruelo, *Combinatorial problems in finite fields and Sidon sets*, Combinatorica, 32 (2012), 497 - 511.

[3] J.-M. Deshouillers and A. Plagne, *A Sidon basis*, Acta Math. Hungar. 123(2009), 233-238.

[4] P. Erdős, *The probability method: Successes and limitations*, J. Statist. Plann. Inference, 72 (1998), 207 - 213.

[5] P. Erdős, A. Sarkozy and V.T. Sós, *On additive properties of general sequences*, Discrete Math. 136 (1994), 75 - 99.

[6] P. Erdős, A. Sarkozy and V.T. Sós, *On sum sets of Sidon sets I*, J. Number Theory 47 (1994), 329 - 347.

[7] H. Halberstam and K. F. Roth, *Sequences*, Springer-Verlag, New York, 1983.

[8] S. Janson, *Poisson approximation for large deviations*, Random Structures Algorithms 1 (1990), 221-229.

[9] S. Janson, T. Luczak and A. Rucinski, *Random graphs*, Wiley-Interscience, New York, 2000.

[10] S. Kiss, *On Sidon sets which are asymptotic basis*, Acta Math. Hungar. 128(2010), 46-58.

[11] S. Kiss, E. Rozgonyi and C. Sándor, *On Sidon sets which are asymptotic basis of order four*, arXiv:1304.5749.

[12] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math.76 (1954), 819-827.

**Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada**

_E-mail address:_ wtkuo@uwaterloo.ca

**Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada**

_E-mail address:_ syamagis@uwaterloo.ca