Abstract

In this paper, we find that the linearized collision operator $L$ of the non-cutoff Boltzmann equation with soft potential generates a strongly continuous semigroup on $H^k_n$. In the semigroup theory of Boltzmann equation with angular cutoff, the weighted $L^2$ space is fundamental. Hence for the non-cutoff theory, the weighted space $H^k_n$ should play an important role. The proof is based on pseudo-differential calculus and on our way to this result, we find that, for a general class of Weyl quantization, the $L^2$ dissipation implies $H^k_n$ dissipation. This kind of estimate is also known as the Gårding's inequality.

Keywords: Boltzmann equation, linearized collision operator, pseudo-differential operator, dissipation, strongly continuous semigroup.

1 Introduction

In this article, we are interested in proving that the linearized Boltzmann operator $L$ can generates a strongly continuous semigroup on weighted Sobolev space $H^k_n$. The main result of this paper is theorem 1.2. Previous results are on $L^2$ dissipation, but that’s not enough for generating a semigroup on $H^k_n$. The main difficulty is to prove that $L$ is dissipative on $H^k_n$ and the invertibility of $\lambda I - L$ for some $\lambda > 0$. Here we can split the linearized collision operator as $L = b^w + K$, where $b$ behave similar to an elliptic operator and $K$ is a bounded operator. So it suffices to analyze the behavior of Weyl quantization $b^w$ on $H^k_n$. Since the argument is based on pseudo-differential operator, our work can also be applied to other a more general symbol class.

1.1 Model and notations

Consider the Boltzmann equation in $d$-dimension:

$$F_t + v \cdot \nabla_x F = Q(F, F).$$

(1)
We recall some basic fact in the theory of Boltzmann equation without angular cut-off and one may refer to [2, 3, 8] for more introduction. Here $F = F(x, v, t)$ is the distribution function of particle at position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, $Q(G, F)$ is the bilinear collision operator defined for sufficiently smooth functions $F, G$ by

$$Q(G, F) := \int_{\mathbb{R}^d} \int_{S^{d-1}} B(v - v_*, \sigma)(F'G'_* - FG_*) \, d\sigma dv_* \quad (2)$$

where $F' = F(x, v', t), G'_* = G(x, v'_*, t), F = F(x, v, t), G_* = G(x, v_*, t)$, and $(v, v_*)$ are the velocities of two gas particles before collision while $(v', v'_*)$ are the velocities after collision satisfying the following conservation laws of momentum and energy,

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

We use the so-called $\sigma$-representation, that is, for $\sigma \in S^{d-1}$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

and define the angle $\theta$ in the standard way

$$\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma,$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^d$. The collision kernel cross section $B$ satisfies

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta), \quad (3)$$

for some $\gamma \in \mathbb{R}$ and function $b$. Without loss of generality, we can assume $B(v - v_*, \sigma)$ is supported on $(v - v_*) \cdot \sigma \geq 0$ which corresponds to $\theta \in [0, \pi/2]$, since $B$ can be replaced by its symmetrized form $\overline{B}(v - v_*, \sigma) = B(v - v_*, \sigma) + B(v - v_*, -\sigma)$. Moreover, we are going to work on the collision kernel without angular cut-off, which corresponds to the case of inverse power interaction laws between particles. That is,

$$b(\cos \theta) \approx \theta^{-d+1-2s} \quad \text{on } \theta \in (0, \pi/2). \quad (4)$$

Here we assume

$$s \in (0, 1), \quad \gamma \in (-d, \infty). \quad (5)$$

For Boltzmann equation without angular cut-off, the condition $\gamma + 2s \leq 0$ is called soft potential. The behavior of this kernel gives non-integrability condition

$$\int_0^{\pi/2} \sin^{d-2} \theta b(\cos \theta) \, d\theta = \infty,$$

which becomes the major difficulty in the theory of Boltzmann equation without angular cut-off.
We are looking for a solution \( f \) near the normalized equilibrium, which is the normalized global Maxwellian

\[
\mu(v) = (2\pi)^{-d/2}e^{-|v|^2/2}.
\]

Set \( F = \mu + \mu^{1/2}f \). Then the perturbation \( f \) satisfies

\[
f_t + \nu \cdot \nabla_x f = Lf + \mu^{-1/2}Q(\mu^{1/2}f, \mu^{1/2}f),
\]

where \( L \) is called the linearized Boltzmann operator defined by

\[
Lf := \mu^{-1/2}Q(\mu, \mu^{1/2}f) + \mu^{-1/2}Q(\mu^{1/2}f, \mu).
\]  

As in [9, 12], the weighted \( L^2 \) space is necessary for the analysis to Boltzmann equation with angular cut-off and soft potential, since the estimate for nonlinear term \( \mu^{-1/2}Q(\mu^{1/2}f, \mu^{1/2}f) \) in the equation is on the weighted \( L^2 \) space. While for the non-cutoff case, this kind of estimate only work in the weighted Sobolev space, since non-cutoff Boltzmann equation essentially requires derivative. We define the weighted Sobolev space \( H^k_n(\mathbb{R}^d) \) by

\[
H^k_n(\mathbb{R}^d) := \{ f \in \mathcal{S} : \| f \|_{H^k_n} < \infty \},
\]

where

\[
\| f \|_{H^k_n} := \| \langle \eta \rangle^k \mathcal{F}(\langle \cdot \rangle^n f) \|_{L^2},
\]

where \( \mathcal{F} \) is the Fourier transform on \( \mathbb{R}^d \): \( \mathcal{F}f(\eta) := \int_{\mathbb{R}^d} f(v)e^{2\pi i v \eta} dv \). For later use, we define

\[
c(v, \eta) := \langle v \rangle^n \langle \eta \rangle^k.
\]

Then \( c \) is a \( \Gamma \)-admissible weight function as well as a symbol in \( S(c) \), with \( \Gamma = |dv|^2 + |d\eta|^2 \).

One may refer to the appendix as well as [5–7, 11] for more information about pseudo-differential calculus. In the corollary 2.5 below, we can prove that

\[
\| c^w(v, D_v)f \|_{L^2} \approx \| \langle v \rangle^n \langle D_v \rangle^k f \|_{L^2} \approx \| \langle D_v \rangle^k \langle v \rangle^n f \|_{L^2}.
\]

Thus the space \( (H(c), \| \cdot \|_{H(c)}) \) is equivalent to \( (H^k_n, \| \cdot \|_{H^k_n}) \). So sometimes we don’t distinguish this two spaces below and will equip \( H^k_n \) with norm \( \| \cdot \|_{H(c)} = \| c^w(\cdot) \|_{L^2} \).

**Notations**  Throughout this article, we shall use the following notations. For any \( v \in \mathbb{R}^d \), we denote \( \langle v \rangle = (1 + |v|^2)^{1/2} \). The gradient in \( v \) is denoted by \( \partial_v \). Also we use notation \( D_v = \frac{\partial}{\partial v} \) and \( \langle D_v \rangle^k f = \mathcal{F}^{-1}(\langle \cdot \rangle^k \mathcal{F} f) \). Let \( A \subset \mathbb{R}^d \), denote \( 1_A \) to be the characteristic function that equal to 1 on \( A \) and 0 on \( \mathbb{R}^d \setminus A \).

The notation \( a \approx b \) (resp. \( a \gg b \), \( a \ll b \)) for positive real function \( a, b \) means there exists \( C > 0 \) not depending on possible free parameters such that \( C^{-1}a \leq b \leq Ca \) (resp. \( a \geq C^{-1}b, a \leq Cb \)) on their domain. \( \text{Re}(a) \) means the real part of complex number \( a \).

For pseudo-differential calculus, we write \( \Gamma = |dv|^2 + |d\eta|^2 \) to be a admissible metric. Let \( m, l \) be two \( \Gamma \)-admissible weight functions and write \( S(m) := S(m, \Gamma), H(m) := H(m, \Gamma) \). We denote \( a_{K,l} := a + Kl \).
1.2 Main results

Our first result is on general symbols. We find that the $L^2$ dissipation of Weyl quantization $a^w(v, D_v)$ can imply the $H^k$ dissipation.

**Theorem 1.1.** Let $m, l$ be two $\Gamma$-admissible weights, $\rho > 0$, $\varepsilon \in (0, 1)$. Assume $l \in S(l)$, $l \lesssim m$, $m(\eta)^{-N} \lesssim l$ for some $N > 0$ and

1. $a \in S(m)$, $\partial_\eta a \in S(\varepsilon m_{K, \ell} + \varepsilon^{-\rho} l)$ uniformly in $\varepsilon$.
2. $b^{1/2} \in S(m^{1/2})$, $\partial_\eta (b^{1/2})_{K, \ell/2} \in S(K^{-\kappa}(m^{1/2})_{K, \ell/2})$ uniformly in $K$ and $(b^{1/2})_{K, \ell/2} \gtrsim (m^{1/2})_{K, \ell/2}$. Suppose for $f \in \mathcal{S}$,

$$\text{Re}(a^w(v, D_v)f, f)_{L^2} \geq \frac{1}{C} \| (b^{1/2})^w(v, D_v)f \|^2_{L^2} - C \| (l^{1/2})^w f \|^2_{L^2},$$

(10)

for some constant $C$ independent of $f$. Then for $k, n \in \mathbb{R}$, $f \in \mathcal{S}$,

$$\text{Re}(a^w(v, D_v)f, f)_{H^k_n} \geq \frac{1}{C'} \| (b^{1/2})^w(v, D_v)c^w f \|^2_{L^2} - C_k \| (l^{1/2})^w c^w f \|^2_{L^2},$$

(11)

for some $C', C_k > 0$.

The assumption on $a$ essentially represents the smallness on $\partial_\eta a$, which can be viewed as a general version of (57). Although there are a lot of restriction on $b$, in our application to Boltzmann equation, we can choose $b = m$. Then these assumptions are trivial for checking. The real part $\text{Re}$ can be replaced by imaginary part, since they don’t have essential difference. Also the symbol $c$ can be generalized to symbol class that $\partial_\xi c$ and $\partial_\eta c$ have better decay on direction $v, \eta$ respectively.

The main idea is based on controlling the commutator $[c^w, a^w]$. Once we get this, we know the difference between $\text{Re}(a^w(v, D_v)c^w f, c^w f)_{L^2}$ and $\text{Re}(a^w(v, D_v)f, f)_{H^k_n}$. Then we can have the dissipation on $H^k_n$ from $L^2$.

As an application, we can get our result on strongly continuous semigroup. Define

$$\tilde{a}(v, \eta) := \langle v \rangle^\gamma (1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s.$$  (12)

Then $\tilde{a}$ is a $\Gamma$-admissible weight proved in [4].

**Theorem 1.2.** Assume $\gamma + 2s \leq 0$. There exists $C_1 > 1$ such that the linearized Boltzmann operator $L$ generates a strongly continuous semigroup on $H(c) = H^k_n(\mathbb{R}^d)$ with domain $D(L) := H((\tilde{a} + C_1)c)$.

The constant $C_1 > 1$ here is to ensure that $H((\tilde{a} + C_1)c) \hookrightarrow H(c)$. The linearized Boltzmann operator $L$ can be splitted as

$$L = -b^w + K.$$  (13)

So once we apply the theorem 1.1 to $b$, this main theorem 1.2 follows from the boundedness of $K$. We can prove that $K$ can be written as a pseudo-differential operator with symbol in (1), thus $K$ is bounded on $H((\tilde{a} + C_1)c)$ and hence on $H(c)$.
Organization of the article  The paper is organized as follows. In Section 2, we provide the dissipation on $H^k_n$ and have a discussion on general symbol class on $\mathbb{R}^d$. Some useful lemmas in pseudo-differential calculus are provided. In Section 3, we deal with the linearized Boltzmann operator $L = -b^w + K$ on $\mathbb{R}^3$, where Carleman representation is applied from time to time. An appendix is devoted to a short review of some useful tools used in this work such as pseudo-differential calculus and semigroup theory.

2 Dissipation on $H^k_n$

In this section, we are going to prove that the $L^2$ dissipation implies $H^k_n$ dissipation. Here, we fix $k \in \mathbb{R}$, $\kappa > 0$ and consider $\mathbb{R}^d$ to be the whole space. Let $m$, $l$ be two $\Gamma$-admissible weight functions. Recall that $a_{K,l} := a + Kl$, $m_{K,l} := m + Kl$ for $K \geq 1$. Also, we always assume in this section that

\[ l \in S(l) \quad \text{and} \quad l \lesssim m. \tag{14} \]

We should remind readers that the lemma 2.1, 2.2, 2.3, 2.4 below are valid for general $\Gamma$-admissible metric $c$, which is needed later.

Lemma 2.1. Assume $a \in S(m)$, $\partial_\eta (a_{K,l}) \in S(K^{-\kappa}m_{K,l})$ uniformly in $K$ and $|a_{K,l}| \gtrsim m_{K,l}$.

Then (1). $a^{-1}_{K,l} \in S(m_{K,l}^{-1})$, uniformly in $K$, for $K > 1$.

(2). There exists $K_0 > 1$ sufficiently large such that for all $K > K_0$, $a^w_{K,l} : H(mc) \to H(c)$ is invertible and its inverse $(a^w_{K,l})^{-1} : H(c) \to H(mc)$ satisfies

\[ (a^w_{K,l})^{-1} = G_{1,K,l}(a^{-1}_{K,l})^w = (a^{-1}_{K,l})^w G_{2,K,l}, \tag{15} \]

with $G_{1,K,l} \in L(H(mc))$, $G_{2,K,l} \in L(H(c))$, and $\|G_{1,K,l}\|_{L(H(mc))} \leq 2$, $\|G_{2,K,l}\|_{L(H(mc))} \leq 2$.

Proof. Since $l \in S(l) \subset S(m)$, we have $a_{K,l} \in S(m_{K,l}) \subset S(m)$ and so $a_{K,l}$ maps $H(mc)$ continuously into $H(c)$. By composition formula of Weyl quantization,

\[ a^w_{K,l}(a^{-1}_{K,l})^w = I + R^w_{K,l}, \tag{16} \]

where

\[ R_{K,l} = \int_0^1 (\partial_\eta a_{K,l} \partial_\eta a^{-1}_{K,l} - \partial_\eta a_{K,l} \partial_\eta a^{-1}_{K,l}) d\theta. \tag{17} \]

For any $1 \leq j \leq d$,

\[ \partial_\eta a^{-1}_{K,l} = \frac{\partial_\eta a_{K,l}}{a_{K,l}^2}, \quad |\partial_\eta a^{-1}_{K,l}| \lesssim \frac{K^{-\kappa}m_{K,l}}{m_{K,l}^2} \lesssim \frac{K^{-\kappa}}{m_{K,l}^2}, \]

Estimate on higher derivative follows from Leibniz formula. Thus $\partial_\eta a^{-1}_{K,l} \in S(K^{-\kappa}m_{K,l}^{-1})$ and $\partial_\eta a_{K,l} \in S(K^{-\kappa}m_{K,l})$ uniformly in $K$. Similarly, $\partial_\eta a^{-1}_{K,l} \in S(m_{K,l}^{-1})$ and by definition, $\partial_\eta a_{K,l} \in S(m_{K,l})$. Some useful lemmas in pseudo-differential calculus and semigroup theory.


\( S(m_{K,l}) \) uniformly in \( K \). Applying 55, for any \( N \in \mathbb{N} \), there exists \( l_N \in \mathbb{N} \) independent of \( K \) and \( \theta \) such that
\[
\| \partial_\nu a_{K,l} \# \partial_\nu a_{K,l}^{-1} \|_{N:S(1)} \leq C_N \| \partial_\nu a_{K,l} \|_{l_N:S(m_{K,l})} \| \partial_\nu a_{K,l}^{-1} \|_{l_N:S(m_{K,l})} \leq C_N' K^{-\kappa}.
\]
Similarly,
\[
\| \partial_\nu a_{K,l} \# \partial_\nu a_{K,l}^{-1} \|_{N:S(1)} \leq C_N' K^{-\kappa}.
\]
Thus \( \{ K^\kappa \partial_\nu a_{K,l} \# \partial_\nu a_{K,l}^{-1} \} \) and \( \{ K^\kappa \partial_\nu a_{K,l} \# \partial_\nu a_{K,l}^{-1} \} \) are uniformly bounded sets in \( S(1) \) with respect to \( K \) and \( \theta \). Thus by Remark 3.4 in [5], the operator \( K^\kappa R_{K,l}^w \) is linear continuous on \( H(mc) \) and \( H(c) \) with operator norm independent of \( K \). So there exists \( K_0 > 1 \) such that for \( K > K_0 \),
\[
I + K^{-\kappa}(K^\kappa R_{K,l}^w)
\]
is invertible on \( H(mc) \) and \( H(c) \) and the operator norm of inverse \( (I + R_{K,l}^w)^{-1} \) on \( H(mc) \) and \( H(c) \) are smaller than 2. Thus
\[
a_{K,l}^w(I + R_{K,l}^w)^{-1} = I \quad \text{on } H(c).
\]
Similarly, by choosing \( K_0 \) sufficiently large, we can find \( \tilde{R}_{K,l} \in S(1) \) such that \( (I + \tilde{R}_{K,l}^w)^{-1} \) is invertible on \( H(mc) \) whenever \( K > K_0 \) and
\[
(I + \tilde{R}_{K,l}^w)^{-1}(a_{K,l}^{-1})^w_{K,l} = I \quad \text{on } H(mc).
\]
Noticing \( a_{K,l}^{-1} \in S(m^{-1}) \) and \( (a_{K,l}^{-1})^w \) maps \( H(c) \) continuously into \( H(mc) \), we obtained that \( a_{K,l}^w : H(mc) \to H(c) \) has left inverse and right inverse, and hence is invertible with inverse in the form of (15).

Notice that in this lemma, the symbol \( a \) may not be real-valued. This is necessary in next section. For further application, we state a similar lemma on \( a_{K,l}^{1/2} \), which needs \( a \) to be positive.

**Lemma 2.2.** Assume \( a \in S(m) \), \( \partial_\nu(a_{K,l}) \in S(K^{-\kappa}m_{K,l}) \) uniformly in \( K \) and \( a_{K,l} \geq m_{K,l} \).
Then (1). \( a_{K,l}^{1/2} \in S(m_{K,l}^{1/2}) \), \( a_{K,l}^{-1/2} \in S(m_{K,l}^{-1/2}) \), uniformly in \( K \), for \( K > 1 \).
(2). There exists \( K_0 > 1 \) sufficiently large such that for all \( K > K_0 \), \( (a_{K,l}^{1/2})^w : H(m^{1/2}c) \to H(c) \) is invertible and its inverse \( ((a_{K,l}^{1/2})^w)^{-1} : H(c) \to H(m^{1/2}c) \) satisfies
\[
((a_{K,l}^{1/2})^w)^{-1} = F_{1,K,l}(a_{K,l}^{1/2})^w = (a_{K,l}^{-1/2})^w F_{2,K,l}.
\]
with \( F_{1,K,l} \in L(H(m^{1/2}c)) \), \( F_{2,K,l} \in L(H(c)) \), and \( \| F_{1,K,l} \|_{L(H(mc))} \leq 2 \), \( \| F_{2,K,l} \|_{L(H(mc))} \leq 2 \).

**Proof.** Firstly by assumption on \( a \) and \( l \), \( a_{K,l} \in S(m_{K,l}) \) uniformly in \( K \). Similar to lemma 2.1, we have for any \( 1 \leq j \leq d \),
\[
\partial_\nu a_{K,l} = \frac{\partial_\nu a_{K,l}}{a_{K,l}^{1/2}} \quad \text{and} \quad \frac{|\partial_\nu a_{K,l}|}{a_{K,l}^{1/2}} \leq K^{-\kappa} m_{K,l}^{1/2}.
\]
Estimate on higher derivative follows from Leibniz formula and thus $\partial_\vartheta a_{K,l}^{1/2} \in S(K^{-\kappa}m_{K,l}^{1/2})$ uniformly in $K$. Similarly, $\partial_\vartheta a_{K,l}^{-1/2} \in S(K^{-\kappa}m_{K,l}^{-1/2})$, $\partial_\vartheta a_{K,l}^{1/2} \in S(m_{K,l}^{1/2})$ and $\partial_\vartheta a_{K,l}^{-1/2} \in S(m_{K,l}^{-1/2})$ uniformly in $K$.

By composition formula of Weyl quantization,

$$(a_{K,l}^{1/2})^w(a_{K,l}^{-1/2})^w = I + R_{K,l}^w,$$

where

$$R_{K,l} = \int_0^1 (\partial_\vartheta a_{K,l}^{1/2} b \partial_\vartheta a_{K,l}^{-1/2} - \partial_\vartheta a_{K,l}^{1/2} b \partial_\vartheta a_{K,l}^{-1/2}) d\theta.$$ 

Thus the following argument is exactly the same as lemma 2.1 and we omit them.

**Lemma 2.3.** Let $m$ be $\Gamma$-admissible weight such that $a \in S(m)$. Assume $a^w : H(mc) \to H(c)$ is invertible. If $b \in S(m)$, then there exists $C > 0$, depending only on $a$ and the seminorms of $b$, such that for $f \in H(mc)$,

$$\|b(v, D_v)f\|_{H(c)} + \|b^w(v, D_v)f\|_{H(c)} \leq C\|a^w(v, D_v)f\|_{H(c)}.$$  (19)

**Proof.** Applying Corollary 2.6.28 in [11], we know that there exists $a_{-1} \in S(m^{-1})$ such that $a\#b = b\#a = 1$. Thus $a_{-1}^w a^w = I$ on $H(mc)$. Since $b \in S(m)$, we have $b\#a_{-1} \in S(1)$ and hence $b^w a_{-1}^w$ is a linear bounded operator on $H(c)$. Thus

$$b^w = b^w a_{-1}^w a^w \text{ on } H(mc),$$

and so for $f \in H(mc)$,

$$\|b^w(v, D_v)f\|_{H(c)} \leq C_{K,l}\|a^w(v, D_v)f\|_{H(c)}.$$ 

On the other hand, $b(v, D_v) = (J^{-1/2}b)^w$ and $J^{-1/2}b \in S(m)$, thus $b(v, D_v)$ has the same bound as $b^w(v, D_v)$.

In lemma 2.1, we obtained that $(a_{K+\varepsilon^2}^{(1+\varepsilon)}l)^w$ is invertible for sufficiently large $K$. Hence the following corollary is a similar result to lemma 2.3 but the proof is slightly different, since $b$ belongs to a different symbol class and we need the constant to be independent of $\varepsilon$.

**Lemma 2.4.** Assume $a \in S(m)$, $\partial_\vartheta (a_{K,l}) \in S(K^{-\kappa}m_{K,l})$ uniformly in $K$ and $a_{K,l} \gtrsim m_{K,l}$. Let $\rho > 0$ and $b \in S(\varepsilon m_{K,l} + \varepsilon^{-\rho}l)$, uniformly in $\varepsilon \in (0, 1)$. Then there exists $K_0 > 0$, such that for $f \in H(mc)$, $\varepsilon \in (0, 1)$,

$$\|b(v, D_v)f\|_{H(c)} + \|b^w(v, D_v)f\|_{H(c)} \leq C_{K,l,d}(\varepsilon\|a^w(v, D_v)f\|_{H(c)} + \varepsilon^{-\kappa}\|l^w f\|_{H(c)}).$$  (20)
Proof. From lemma 2.1, we have \( a_{K,l}^{-1} \in S(m_{K,l}^{-1}) \) for \( K > 1 \), and there exists \( K_0 > 1 \) such that for \( K > K_0 \),

\[
(a_{K,l}^w)^{-1} = (a_{K,l}^{-1})^w G_{2,K,l}, \tag{21}
\]

with \( G_{2,K,l} \in L(H(c)) \). Since \( b \in S(\varepsilon m_{K+\varepsilon^{-1}l}) \), we have \( \varepsilon^{-1}b \# a_{K+\varepsilon^{-1}l}^{-1} \in S(1) \), uniformly in \( \varepsilon \).

Write

\[
b^w = \varepsilon^{-1}b^w (a_{K+\varepsilon^{-1}l}^{-1})^w G_{2,K,l} \varepsilon (a_{K+\varepsilon^{-1}l})^w, \tag{22}
\]

then

\[
\|b^w(v, D_v)f\|_{H(c)} \leq C_{K,l,d} \|a_{K+\varepsilon^{-1}l}^{-1}(v, D_v)f\|_{H(c)}. \tag{23}
\]

Similar to the previous lemma, \( b(v, D_v) = (J^{-1/2}b)^w \) and \( J^{-1/2}b \in S(\varepsilon m_{K,l} + \varepsilon^{-\kappa}l) \), thus \( b(v, D_v) \) has the same bound as \( b^w \).

For \( k, n \in \mathbb{R} \), it’s trivial to obtain that \( \langle v \rangle^n \) and \( \langle D_v \rangle^k \), as Weyl quantization, are invertible, since \( \langle v \rangle^n \) is only a multiplication while \( \langle D_v \rangle^k \) is a multiplier. We define

\[
c = \langle \eta \rangle^k \langle v \rangle^n. \tag{24}
\]

Then \( c \in S(\langle \eta \rangle^k \langle v \rangle^n) \), \( \partial_v c \in S(\langle \eta \rangle^k \langle v \rangle^{n-1}) \), \( \partial_{\eta} c \in S(\langle \eta \rangle^{k-1} \langle v \rangle^n) \) and we have the following useful corollary. There are many ways to prove this corollary, here we provide one by applying the above lemmas.

**Corollary 2.5.** Let \( k, n \in \mathbb{R} \), then we have the equivalence

\[
\|c^w(v, D_v)f\|_{L^2} \approx \|\langle v \rangle^n\langle D_v \rangle^k f\|_{L^2} \approx \|\langle D_v \rangle^k \langle v \rangle^n f\|_{L^2}, \tag{25}
\]

and hence this two norms are equivalent on \( H_n^k \).

**Proof.** The symbols of \( \langle v \rangle^n\langle D_v \rangle^k\langle v \rangle^{-n} \) and \( c^w \langle v \rangle^{-n} \) belong to \( S(\langle \eta \rangle^k) \). Letting \( m = \langle \eta \rangle^k \) in lemma 2.3, we find that for \( f \in H(\langle \eta \rangle^k) \),

\[
\|\langle v \rangle^n\langle D_v \rangle^k\langle v \rangle^{-n} f\|_{L^2} \lesssim \|\langle D_v \rangle^k f\|_{L^2},
\]

\[
\|c^w(v, D_v)\langle v \rangle^{-n} f\|_{L^2} \lesssim \|\langle D_v \rangle^k f\|_{L^2}.
\]

So for any \( f \) such that \( \|\langle D_v \rangle^k\langle v \rangle^n f\|_{L^2} < \infty \),

\[
\|\langle v \rangle^n\langle D_v \rangle^k f\|_{L^2} \lesssim \|\langle D_v \rangle^k\langle v \rangle^n f\|_{L^2},
\]

\[
\|c^w(v, D_v)f\|_{L^2} \lesssim \|\langle D_v \rangle^k\langle v \rangle^n f\|_{L^2}.
\]

Similarly, the symbol of \( \langle D_v \rangle^k\langle v \rangle^n\langle D_v \rangle^{-k} \) belongs to \( S(\langle v \rangle^n) \). Letting \( m = \langle v \rangle^n \) in lemma 2.3, we find that for any \( f \in L^2 \),

\[
\|\langle D_v \rangle^k\langle v \rangle^n f\|_{L^2} \lesssim \|\langle v \rangle^n f\|_{L^2},
\]

\[
\|\langle v \rangle^n\langle D_v \rangle^{-k} f\|_{L^2} \lesssim \|\langle v \rangle^n f\|_{L^2}.
\]
and for $f$ such that $\|\langle v \rangle^n (D_v)^k f \|_{L^2} < \infty$, 

$$\|\langle v \rangle^n (D_v)^k f \|_{L^2} \lesssim \|\langle v \rangle^n (D_v)^k f \|_{L^2}.$$ 

So we proved the second equivalence. Besides, the symbol $c$ satisfies lemma 2.1 with $m = l = \langle \eta \rangle^k \langle v \rangle^n$. Thus there exists $K > 1$ such that $c_{K,l}^w = (K + 1)c^w : H(\langle \eta \rangle^k \langle v \rangle^n) \rightarrow L^2$ is invertible in the form of (15). Thus applying lemma 2.3 and $\langle D_v \rangle^k \langle v \rangle^n \in S(\langle \eta \rangle^k \langle v \rangle^n)$, we have

$$\|\langle D_v \rangle^k \langle v \rangle^n f \|_{L^2} \lesssim \|c^w(v, D_v)f\|_{L^2}.$$ 

Here we give a version of Gårding’s inequality, which is needed in the next section.

**Theorem 2.6.** Assume $a \in S(m)$, $\partial_\theta(a_{K,l}) \in S(K^{-\kappa}m_{K,l})$ uniformly in $K$ and $a_{K,l} \gtrsim m_{K,l}$. Then there exists $K_0 > 1$ such that for $K > K_0$, $f \in \mathcal{S}$, we have

$$Re(a_{K,l}^w(v, D_v)f, f)_{L^2} \approx \|a_{K,l}^{1/2} f\|_{L^2}^2,$$  

(26)

If in addition, $b^{1/2} \in S(m^{1/2})$, then

$$Re(a^w(v, D_v)f, f)_{L^2} \geq \frac{1}{C}\|(b^{1/2})^w(v, D_v)f\|_{L^2}^2 - C\|f^w(v, D_v)f\|_{L^2},$$

(27)

for some constant $C$ independent of $f$.

**Proof.** Notice $a$ satisfies the assumption of lemma 2.2, thus there exists $K_0 > 1$ such that for $K > K_0$, $(a_{K,l}^{1/2})^w : H(m^{1/2}) \rightarrow L^2$ is invertible with formula (18). Hence by lemma 2.3 and $b^{1/2} \in S(m^{1/2})$, we have for $f \in H(m^{1/2})$,

$$\|(b^{1/2})^w f\|_{L^2} \leq C_K \|a_{K,l}^{1/2}f\|_{L^2}$$

(28)

On the other hand,

$$(a_{K,l}^{1/2})^w(a_{K,l}^{1/2})^w = a_{K,l}^w + R_{K,l}^w,$$

(29)

with

$$R_{K,l}^w = \int_0^1 (\partial_\theta a_{K,l}^{1/2} \partial_\theta a_{K,l}^{1/2} - \partial_\theta a_{K,l}^{1/2} \partial_\theta a_{K,l}^{1/2}) d\theta.$$

(30)

Similar to the proof in lemma 2.1, since $\partial_\theta a_{K,l}^{1/2} \in S(m_{K,l}^{1/2})$ and $\partial_\theta a_{K,l}^{1/2} \in S(K^{-\kappa}m_{K,l}^{1/2})$ uniformly in $K$, we have $\partial_\theta a_{K,l}^{1/2} \partial_\theta a_{K,l}^{1/2}$ and $\partial_\theta a_{K,l}^{1/2} \partial_\theta a_{K,l}^{1/2}$ belong to $S(K^{-\kappa}m_{K,l})$ uniformly in $K$ and $\theta$. Hence $R_{K,l} \in S(K^{-\kappa}m_{K,l})$ uniformly in $K$. Write

$$R_{K,l}^w = K^{-\kappa}(a_{K,l}^{1/2})^w F_{1,K,l} \left( K^{\kappa}(a_{K,l}^{-1/2})^w R_{K,l}^w (a_{K,l}^{-1/2})^w \right) F_{2,K,l}(a_{K,l}^{-1/2})^w,$$

(31)
where \( g^w := K^\kappa (a^{-1/2})^w R^w_{K,l}(a^{-1/2})^w \) has symbol in \( S(1) \) uniformly in \( K \), hence is bounded on \( L^2 \).
Thus for \( f \in \mathscr{F} \),

\[
(R^w_{K,l}f, f)_{L^2} = K^{-\kappa} \left( F_{1,K,l}g^w F_{2,K,l}(a^{1/2})^w f, (a^{1/2})^w f \right)_{L^2},
\]

\[
\| (R^w_{K,l}f, f)_{L^2} \| \leq K^{-\kappa} C \| (a^{1/2})^w f \|_{L^2}^2.
\]

Recall that the norm of operators \( F_{1,K,l} \) and \( F_{2,K,l} \) are smaller than 2. We choose \( K_0 \) sufficiently large such that for \( K > K_0 \),

\[
\| (R^w_{K,l}f, f)_{L^2} \| \leq \frac{1}{2} \| (a^{1/2})^w f \|_{L^2}^2.
\]

Then for \( f \in \mathscr{F} \),

\[
\| (a^{1/2})^w f \|_{L^2}^2 = (a^w_{K,l}f, f)_{L^2} + (R^w_{K,l}f, f)_{L^2},
\]

\[
\| (a^{1/2})^w f \|_{L^2}^2 \approx Re(a^w_{K,l}f, f)_{L^2}.
\]

Together with (28), we get (27).

Now we come to the main theorem.

**Theorem 2.7.** Let \( \rho > 0, \varepsilon \in (0, 1) \). Assume \( l \in S(l), l \lesssim m, m(\eta)^{-N} \lesssim l \) for some \( N > 0 \) and (1). \( a \in S(m) \), \( \partial^\alpha a \in S(\varepsilon m K l \pm \varepsilon^\rho l) \) uniformly in \( \varepsilon \).
(2). \( b^{1/2} \in S(m^{1/2}), \partial^\alpha b^{1/2} \in S(K^{-\kappa}(m^{1/2})_{K,l}^{1/2}) \) uniformly in \( K \) and \( (b^{1/2})_{K,l}^{1/2} \gtrsim (m^{1/2})_{K,l}^{1/2} \).
Suppose for \( f \in \mathscr{F}, \nabla \)

\[
Re(a^w(v, D_v)f, f)_{L^2} \geq \frac{1}{C} \| (b^{1/2})^w(v, D_v)f \|_{L^2}^2 - C \| (l^{1/2})^w f \|_{L^2}^2,
\]

for some constant \( C \) independent of \( f \). Then for \( k \in \mathbb{R}, f \in \mathscr{F}, \nabla \)

\[
Re(a^w(v, D_v)f, f)_{H^k_{\kappa}} \geq \frac{1}{C'} \| (b^{1/2})^w(v, D_v)c^w f \|_{L^2}^2 - C_k \| (l^{1/2})^w c^w f \|_{L^2}^2,
\]

for some \( C', C_k > 0 \).

**Proof.** We claim that for any \( k, n \in \mathbb{R} \), there exists constant \( C_{k,n} > 0 \) such that for \( \varepsilon > 0, f \in \mathscr{F}, \nabla \)

\[
\| (a^w f, f)_{H^k_{\kappa}} - (a^w c^w f, c^w f)_{L^2} \| \leq \varepsilon \| (b^{1/2})^w c^w f \|_{L^2}^2 + C_{k,\varepsilon} \| (l^{1/2})^w c^w f \|_{L^2}^2.
\]

Then letting \( \varepsilon \) small, we have for \( f \in \mathscr{F}, \nabla \)

\[
Re(a^w f, f)_{H^k_{\kappa}} \geq Re(a^w c^w f, c^w f)_{L^2} - (\varepsilon \| (b^{1/2})^w c^w f \|_{L^2}^2 + C_{k,\varepsilon} \| (l^{1/2})^w c^w f \|_{L^2}^2)
\]

\[
\geq \frac{1}{C} \| (b^{1/2})^w c^w f \|_{L^2}^2 - \varepsilon \| (b^{1/2})^w c^w f \|_{L^2}^2 - C_{k,\varepsilon} \| (l^{1/2})^w c^w f \|_{L^2}^2
\]

\[
\geq \frac{1}{C'} \| (b^{1/2})^w c^w f \|_{L^2}^2 - C_k \| (l^{1/2})^w c^w f \|_{L^2}^2.
\]
So it suffices to control the last term. Since $b^{1/2}$ satisfies the assumptions of lemma 2.1, there exists $K_0 > 1$ such that for $K > K_0$, $(b^{1/2})_{K,l/2} \in S((m^{1/2})_{K,l/2})$ is invertible in the form of (15) and $((b^{1/2})_{K,l/2})^{-1} \in S((m^{1/2})_{K,l/2}^{-1})$. Noticing $c = \langle \eta \rangle^k \langle v \rangle^n$, $c^{-1} \in S(\langle \eta \rangle^{-k} \langle v \rangle^{-n})$, $\partial_0 a \in S(\varepsilon m_{K,l} + \varepsilon^{-\rho} l)$, we have, for any $0 < \varepsilon < 1$,

$$[c^w(v, D_v), a^w(v, D_v)] \in Op(\varepsilon m_{K,l} + \varepsilon^{-\delta} l) \langle \eta \rangle^k \langle v \rangle^n,$$

$$[c^w(v, D_v), a^w(v, D_v)](c^{-1})^w \in Op(\varepsilon m_{K,l} + \varepsilon^{-\delta} l),$$

for some $\delta > 0$. Thus fixing $K > K_0$,

$$g^w := ((b^{1/2})_{(K+\varepsilon^{-1-\delta})^{1/2},l/2}^{-1})^w [c^w, a^w](c^{-1})^w \in Op(\varepsilon m_{1/2} + (K + \varepsilon^{-1-\delta})^{1/2} l^{1/2})$$

uniformly in $\varepsilon$ and hence by lemma 2.4, for $f \in \mathcal{S}$,

$$\|g^w(v, D_v)f\|_{L^2} \leq C_{k,K} \varepsilon\|((b^{1/2})_{(K+\varepsilon^{-1-\delta})^{1/2},l/2})^w f\|_{L^2}. \quad (36)$$

Also, in the proof of corollary 2.5, we have shown that $c^w$ is invertible in the form of (15). Thus for $f \in \mathcal{S}$,

$$\|([c^w, a^w]f, c^w f)_{L^2}\|
= \|([c^w, a^w](c^w)^{-1} c^w f, c^w f)_{L^2}\|
= \bigg|\bigg(((b^{1/2})_{(K+\varepsilon^{-1-\delta})^{1/2},l/2})^{-1} [c^w, a^w](c^w)^{-1} c^w f, (b^{1/2})_{(K+\varepsilon^{-1-\delta})^{1/2},l/2} c^w f\bigg)_{L^2}\bigg|
\leq \varepsilon C_{k,K} \|((b^{1/2})_{(K+\varepsilon^{-1-\delta})^{1/2},l/2})^w c^w f\|_{L^2}^2.$$  

By choosing $\varepsilon > 0$ sufficiently small and fixing $K > 2K_0$, we proved the claim. \hfill \Box

## 3 Semigroup of linearized Boltzmann operator

In this section, we will prove the main result 1.2. Now we consider functions in $\mathbb{R}^3$. To obtain a pseudo-differential form of the linearized Boltzmann operator, we will follow the argument in [4]. Fix $0 < \delta \leq 1$. Let $\varphi(t)$ be a positive radial function that equal to 1 when $|t| \leq 1/4$ and 0 when $|t| \geq 1$. Let $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$ and $\tilde{\varphi}_\delta(v) = 1 - \varphi_\delta(v)$. Then $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$ equal to 0 for $|v| \geq \delta$ and 1 for $|v| \leq \delta/2$. Also, [4] has shown that

$$L = -a^w(v, D_v) - \left(-L_2 - \widetilde{L}_{1,\delta,a} - L_{1,3,\delta} - L_{1,4,\delta} + a_s(v, D_v) + (a(v, D_v) - a^w(v, D_v))\right), \quad (37)$$
where

\[
L_2 f := \int \int B(\mu_\ast)^{1/2} ((\mu')^{1/2} f'_\ast - \mu^{1/2} f_\ast) \ dv_\ast d\sigma,
\]

\[
\tilde{L}_{1, \delta, a} f := \int \int B\tilde{\varphi}_\delta(v' - v)(\mu_\ast)^{1/2}(\mu'_\ast)^{1/2} f' \ dv_\ast d\sigma,
\]

\[
L_{1, 3, \delta} f := f(v) \int \int B\varphi_\delta(v' - v)(\mu'_\ast - \mu_\ast) \ dv_\ast d\sigma,
\]

\[
L_{1, 4, \delta} f := f(v) \int \int B\varphi_\delta(v' - v)(\mu'_\ast)^{1/2}((\mu_\ast)^{1/2} - (\mu'_\ast)^{1/2}) \ dv_\ast d\sigma,
\]

\[
a_s(v, D_v)f := - \int \int B\varphi_\delta(v' - v)(\mu'_\ast)^{1/2}(f' - f)((\mu_\ast)^{1/2} - (\mu'_\ast)^{1/2}) \ dv_\ast d\sigma
\]

\[
a(v, D_v)f := - \int \int B\varphi_\delta(v' - v)\mu'_\ast(f' - f) \ dv_\ast d\sigma
\]

\[+ f(v) \int \int B\tilde{\varphi}_\delta(v' - v)\mu_\ast \ dv_\ast d\sigma,\]

where \(a_s\) and \(a\) can be written in the form of Carleman representation.

\[
a_s(v, \eta) := - \int \int_{E_0, h} \tilde{b}_1_{|\alpha| \geq |h|}\varphi_\delta(h) \frac{|\alpha + h|^{1+\gamma + 2s}}{|h|^{3+2s}} \mu^{1/2}(v + \alpha)
\]

\[
(\varepsilon^{-2\pi i h \cdot \eta} - 1)(\mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha)) \ d\alpha dh
\]

\[
a(v, \eta) := \int \int_{E_0, h} \tilde{b}_1_{|\alpha| \geq |h|}\varphi_\delta(h) \frac{|\alpha + h|^{1+\gamma + 2s}}{|h|^{3+2s}} \mu(v + \alpha)(1 - \cos(2\pi \eta \cdot h)) \ d\alpha dh
\]

\[+ \int \int_{E_0, h} \tilde{b}_1_{|\alpha| \geq |h|}\tilde{\varphi}_\delta(h) \frac{|\alpha + h|^{1+\gamma + 2s}}{|h|^{3+2s}} \mu(v + \alpha - h) \ d\alpha dh.\]

Recall that

\[
\tilde{a}(v, \eta) := \langle v \rangle^\gamma(1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s. \tag{38}
\]

Proposition 3.7 in [4] shows that \(\tilde{a}\) is a \(\Gamma\)-admissible weight, \(\tilde{a} \approx a\) and

\[
a, \tilde{a} \in S(\tilde{a}), \ \partial_\eta a, \partial_\eta \tilde{a} \in S(\varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{\gamma + 2s}), \tag{39}
\]

uniformly in \(\varepsilon\).

We define a symbol \(b\) by

\[
b^w(v, D_v) := a^w(v, D_v) + a_s(v, D_v) + (a(v, D_v) - a^w(v, D_v)). \tag{40}
\]

Then

\[
L = -b^w(v, D_v) + \left( L_2 + \tilde{L}_{1, \delta, a} + L_{1, 3, \delta} + L_{1, 4, \delta} \right). \tag{41}
\]

Firstly, we analyze the pseudo part \(b^w\). To apply the theorem 1.1, we let

\[
l(v) := \langle v \rangle^{\gamma + 2s}. \tag{42}
\]

Then \(l \in S(l)\) and \(l \leq \tilde{a}\). Please notice that \(a\) is positive while \(b\) may not.
Theorem 3.1. Assume $\gamma + 2s \leq 0$. There exists $C_1 > 0$ such that $-(C_1 + b)w : H(\tilde{a}_1 c) \to H(c)$ generates a contraction semigroup on $H(c)$, with $\tilde{a}_1 := \tilde{a} + C_1$. Consequently, $-b^w : H(\tilde{a}_1 c) \to H(c)$ generates a strongly continuous semigroup on $H(c)$.

Proof. For any $C_1 > 1$, write $b_1 := C_1 + b$ and $a_1 := C_1 + a$. By semigroup theory 4.4, it suffices to show that there exists $C_1 > 1$ such that $((-b^w_1, D(b^w_1)))$ is dissipative on Hilbert space $H^w_{\eta}$ with $D(b^w_1) := H(\tilde{a}_1 c)$ and $\lambda I + b^w_1 : H(\tilde{a}_1 c) \to H(c)$ is invertible for some $\lambda > 0$. Notice here $\tilde{a}_1 \geq 1$, hence the identity operator maps $H(\tilde{a}_1 c)$ into $H(\tilde{a}_1 c) \subset H(c)$ and so $\lambda I + b^w_1$ is well-defined.

To prove $b^w_1$ is dissipative on $H(c)$, we shall verify the assumptions in theorem 1.1. Let $a_d$ to be the symbol of $a^w(v, D_v) - a(v, D_v)$ as a Weyl quantization. Then by lemma 4.4 in [4], we have

$$a_s, a_d, \partial_\eta \tilde{a}, \partial_\eta a \in S(\varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{\gamma+2s}),$$

(43) uniformly in $\varepsilon$. Thus $b = a + a_s + a_d \in S(\tilde{a})$ and $\partial_\eta b \in S(\varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{\gamma+2s})$ uniformly in $\varepsilon$. So $b_{K,l} \in S(\tilde{a}_{K,l})$ and if we choose $\varepsilon = K^{-1/2}$, then we have

$$\partial_\eta (b_{K,l}) \in S(K^{-1/2} \tilde{a}_{K,l})$$

(44) uniformly in $K$. Thus $b$ satisfies assumption (1) in theorem 1.1 with $m = \tilde{a}$ and it’s trivial that $\tilde{a}$ satisfies assumption (2) in theorem 1.1 with $m = \tilde{a}$ by using (39).

On the other hand, $a, \tilde{a} \in S(\tilde{a})$. Choosing $\varepsilon = K^{-1/2}$ in (39), we find that $\partial_\eta a, \partial_\eta \tilde{a} \in S(K^{-1/2} \tilde{a}_{K,l})$. Since $a \approx \tilde{a}$, we know $a_{K,l} \gtrsim \tilde{a}_{K,l}$ and hence $a$ satisfies theorem 2.6 with $m = \tilde{a}$. Thus there exists $C > 0$ such that for $f \in \mathcal{S}$,

$$\text{Re}(a^w(v, D_v) f, f)_{L^2} \geq \frac{1}{C} \|\tilde{a}^{1/2}(v, D_v) f\|^2_{L^2} - C \|\langle v \rangle^{\gamma/2+s} f\|^2_{L^2}.\tag{45}$$

Since $\partial_\eta \tilde{a} \in S(\varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{\gamma+2s})$ uniformly in $\varepsilon \in (0, 1)$ and $\tilde{a} \geq \langle v \rangle^{\gamma+2s}$, we have $\partial_\eta \tilde{a}_{K,l} \in S(\varepsilon \tilde{a}^{1/2} + \varepsilon^{-1} \langle v \rangle^{\gamma/2+s})$ uniformly in $\varepsilon$. Hence choosing $\varepsilon = K^{-1/2}$, we know that $\tilde{a}^{1/2}$ satisfies lemma 2.1 with $m = \tilde{a}$ and $l = \langle v \rangle^{\gamma/2+s}$ therein. Thus there exists $K_0 > 1$ such that for $K > K_0$, $\tilde{a}_{K,l} : H(\tilde{a}) \to L^2$ is invertible in the form of (15) and $\langle \tilde{a}^{1/2} \rangle^{(K,l)}_{K,l}^- \in S((\tilde{a}^{1/2})_{K,l}^-)^{-1}$.

As in [4], we let $a_{\text{pseudo}} := a_s + a_d \in S(\varepsilon \tilde{a}_{K+\varepsilon^{-2} l})$ uniformly in $\varepsilon$. Noticing

$$((\tilde{a}^{1/2})_{K+\varepsilon^{-2},l/2})^{-1} a_{\text{pseudo}} \in S \left( \frac{\varepsilon (\tilde{a} + (K + \varepsilon^{-2}) l)}{\tilde{a}^2 + (K + \varepsilon^{-2}) l^2} \right) \subset S(\varepsilon \tilde{a}^{1/2} + \varepsilon^{-1} \langle v \rangle^{\gamma/2+s}),$$

uniformly in $\varepsilon$, we can apply lemma 2.4 to get

$$|\langle a_{\text{pseudo}} w f, f \rangle_{L^2}| = |\langle ((\tilde{a}^{1/2})_{K,l/2}^-)^{-1} a_{\text{pseudo}} w (\tilde{a}^{1/2})_{K,l/2}^- f \rangle_{L^2}| \leq C(\varepsilon \|\tilde{a}^{1/2} w (v, D_v) f\|^2_{L^2} + C(K, \varepsilon) \|\langle v \rangle^{\gamma/2+s} f\|^2_{L^2}).$$

Then picking $\varepsilon$ small, we have for $f \in \mathcal{S}$,

$$\text{Re}(b^w(v, D_v) f, f)_{L^2} = \text{Re}(a^w(v, D_v) f, f)_{L^2} + \text{Re}(a_{\text{pseudo}} w f, f)_{L^2} \geq \frac{1}{C'} \|\tilde{a}^{1/2} w (v, D_v) f\|^2_{L^2} - C \|\langle v \rangle^{\gamma/2+s} f\|^2_{L^2}.$$
Now, all the assumptions in theorem 1.1 are fulfilled. Hence there exists $C_0 \geq 1$ such that for any $f \in \mathcal{S}$,
\[
\Re(b^w(v, D_v)f, f)_{H_0^k} \geq \frac{1}{C_0} \|(\tilde{a}^{1/2})^w(v, D_v)e^w f\|_{L^2}^2 - C_0 \|(v)^{\gamma/2+s} e^w f\|_{L^2}^2 \quad (46)
\]
\[
\geq \frac{1}{C_0} \|(\tilde{a}^{1/2})^w(v, D_v)\langle D_v \rangle^k f\|_{L^2}^2 - C_0 \|f\|_{H_0^k}, \quad (47)
\]
since $\gamma + 2s \leq 0$. Thus whenever $C_1 > C_0 \geq 1$, for $f \in \mathcal{S}$,
\[
\Re((b_1)^w(v, D_v)f, f)_{H_0^k} \geq 0. \quad (48)
\]
Recall that the domain of $b_1$ is $H(\tilde{a}_1 c) \hookrightarrow H_0^k$. Thus the above inequality is valid for $f \in D(b_1)$, since $\mathcal{S}$ is dense in $H(\tilde{a}_1 c)$.

Now we let $l_1 \leq \tilde{a}_1$, $b_1 \in S(\tilde{a}_1)$, $\partial_{\gamma}((b_1)_{K,l_1}) \in S(K^{-1/2}(\tilde{a}_1)_{K,l_1})$ uniformly in $K$.
Since $a_s, a_d \in S(\varepsilon \tilde{a} + \varepsilon^{-1}(v)^{\gamma+2s})$, we choose $\varepsilon$ small enough, then
\[
|C_1 + b(v, \eta) + K| \geq K + C_1 + a(v, \eta) - |a_s(v, \eta)| - |a_d(v, \eta)|
\geq K + C_1 + \tilde{a}(v, \eta) - \frac{1}{2} \tilde{a} - C(v)^{\gamma+2s} \geq K + C_1 + \tilde{a},
\]
if $C_1 > 2C$. Thus fixing $C_1$ sufficiently large, $b_1(v, \eta)$ satisfies the assumption of lemma 2.1 with $m = \tilde{a}_1$ and $l = l_1$ therein. Hence there exists sufficiently large $K_0$ such that for $\lambda > K_0, \lambda I + C_1 I + b^w(v, D_v) : H(\tilde{a}_1 c) \rightarrow H(c)$ is invertible, hence is surjective.

Thus by 4.4, $(-(b_1)^w, D(b_1))$ generates a contraction semigroup on $H(c)$. But $C_1 I$ is a bounded perturbation on $H(c)$, hence $(-b^w(v, D_v), D(b_1))$ generates a strongly continuous semigroup on $H(c)$. \hfill \square

Now it suffices to prove the operator inside the parentheses of (41) is bounded on $H(c)$, then our proof of 1.2 is completed. So next we will show that the operator in the parentheses is actually Weyl quantization with symbol in $S(1)$. The idea here is to use Carleman representation.

**Theorem 3.2.** Assume $\gamma + 2s \leq 0$, the operators $L_{1,3,\delta}, \tilde{L}_{1,3,\delta}, L_{1,4,\delta}$ and $L_2$ are all Weyl quantization with symbols in $S((v)^{\gamma+2s}) \subset S(1)$. Hence they are bounded on $H(c)$.

**Proof.** Set $f \in \mathcal{S}$. For the part $L_{1,3,\delta}$, by lemma 2.3 in [4],
\[
L_{1,3,\delta} f = S \ast_{v^*} \mu(v) f(v), \quad (49)
\]
with $S(z) = S_1(z) + S_2(z)$ satisfying
\[
|S_1(z)| \leq C|z|^{\gamma}, \quad |S_2(z)| \leq C\delta^{2-2s}|z|^{\gamma+2s-2}.
\]
So by (58),
\[
|\partial_{v^*}^a (S \ast_{v^*} \mu(v))| = |S \ast_{v^*} (\partial_{v^*}^a) \mu(v)| \leq C((v)^{\gamma} + \delta^{2-2s}(v)^{\gamma+2s-2}).
\]

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Thus fixing $\delta > 0$, the symbol of $L_{1,3,\delta}$ belongs to $Op(1)$.

Now we turn to the non-singular part $L_{1,\delta,a}$.

$$\tilde{L}_{1,\delta,a}f = \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \tilde{\varphi}(h) \mu^{1/2}(v + \alpha - h) \mu^{1/2}(v + \alpha) f(v - h) d\alpha dh$$

$$=: \tilde{a}_{1,\delta,a}(v, D_v)f,$$

with

$$\tilde{a}_{1,\delta,a}(v, \eta) := \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \tilde{\varphi}(h) \mu^{1/2}(v + \alpha - h) \mu^{1/2}(v + \alpha)e^{2\pi ih \cdot \eta} d\alpha dh.$$

Thus

$$\partial^\beta_v \partial^\gamma \tilde{a}_{1,\delta,a} = \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} 1_{|\alpha| \geq |\eta|/2} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \tilde{\varphi}(h)$$

$$\partial^\beta_v (\mu^{1/2}(v + \alpha - h) \mu^{1/2}(v + \alpha)) \partial^\gamma e^{2\pi ih \cdot \eta} d\alpha dh,$$

$$|\partial^\beta_v \partial^\gamma \tilde{a}_{1,\delta,a}| \leq C_{\beta,\beta,d,s,\gamma} \int_{\mathbb{R}^d} \int_{E_{0,h}} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \tilde{\varphi}(h)$$

$$\mu^{1/4}(v + \alpha - h) \mu^{1/4}(v + \alpha)|h|^{|\beta|} d\alpha dh$$

$$\leq C_{\beta,\beta,d,s,\gamma} \int_{\mathbb{R}^d} \int_{E_{0,h}} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \tilde{\varphi}(h) \mu^{1/8}(v + \alpha - h) d\alpha dh$$

$$\leq C(v)^{\gamma+2s}.$$

The last inequality follows from the argument of Proposition 3.5 in [4], since it's the same as equation (38) therein.

For $L_{1,4,\delta}$, by Lemma 2.5 in [4], we have

$$L_{1,4,\delta}f = -\frac{1}{2} L_{1,3,\delta} f - D(v)f,$$

where

$$D(v) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{E_{0,h}} b(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \varphi(h) \left( \mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha) \right)^2 d\alpha dh.$$

Hence by lemma 3.4 below,

$$|\partial^\beta_v D(v)| \leq \frac{1}{2} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \varphi(h) \left| \partial^\beta_v \left( \mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha) \right)^2 \right| d\alpha dh,$$

$$\leq C \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \varphi(h) \mu^{1/8}(v + \alpha)|h|^2 d\alpha dh,$$

$$\leq C\delta^{2-2s}(v)^{\gamma+2s},$$

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where the last step follows from Lemma 2.5 in [4].

Now we deal with the last term $L_2$.

$$L_2f = \int \int B(\mu^*)^{1/2} ((\mu')^{1/2} f' - \mu^{1/2} f) \, dv_\sigma d\sigma$$

$$= \int \int B ((\mu')^{1/2} f'_* - (\mu')^{1/2} f)_* \, dv_\sigma d\sigma + \int \int B(\mu')^{1/2} ((\mu^*)^{1/2} - (\mu^*)^{1/2}) f'_* \, dv_\sigma d\sigma$$

$$= \int \int B(\mu^{1/2} f')' ((\mu')^{1/2} - \mu^{1/2}) \, dv_\sigma d\sigma$$

$$+ \mu^{1/2} \int \int B ((\mu')^{1/2} f'_* - (\mu')^{1/2} f)_* \, dv_\sigma d\sigma$$

$$+ \mu^{1/2} \int \int B ((\mu^*)^{1/2} - (\mu^*)^{1/2}) f'_* \, dv_\sigma d\sigma$$

$$= L_{2,r}f + L_{2,ca}f + L_{2,c}f + L_{2,d}f.$$

We will investigate these four parts separately. For $L_{2,ca}$, by Cancellation Lemma, there exists constant $C$ depending only on $B$ hence only on $s$ such that

$$L_{2,ca}f = C\mu^{1/2} \int_{\mathbb{R}^d} |v - v_\gamma| (\mu^{1/2} f)_* \, dv_*$$

$$= C\mu^{1/2} \int_{\mathbb{R}^d} |v_\gamma| \mu^{1/2} (v_\gamma + v) f(v_\gamma + v) \, dv_*$$

$$= C\mu^{1/2} \int_{\mathbb{R}^d} |v_\gamma| \mu^{1/2} (v_\gamma + v) \int_{\mathbb{R}^d} \hat{f}(\eta) e^{2\pi i (v_\gamma + v) \cdot \eta} \, d\eta \, dv_*$$

$$= a_{2,ca}(v, D_v)f,$$

with

$$a_{2,ca}(v, \eta) = C\mu^{1/2} \int_{\mathbb{R}^d} |v_\gamma| \mu^{1/2} (v_\gamma + v) e^{2\pi i v_\gamma \cdot \eta} \, dv_*.$$

Then

$$|\partial^{\alpha}_v \partial^\beta_\eta a_{2,ca}(v, \eta)| \leq C_{\alpha,\beta} \mu^{1/4} (v) \int_{\mathbb{R}^d} |v_\gamma| \mu^{1/4} (v_\gamma + v) |v_\gamma|^{1/2} \, dv_*$$

$$\leq C_{\alpha,\beta} \mu^{1/4} (v) (v_\gamma)^{\gamma + |\beta|}$$

$$\leq C_{\alpha,\beta} \mu^{1/8} (v) \leq C_{\alpha,\beta,b}.$$

Thus $a_{2,ca} \in S(1)$. For $L_{2,c}$, by using Carleman representation,

$$L_{2,c} = \mu^{1/2} \int \int B ((\mu^*)^{1/2} - (\mu^*)^{1/2}) f'_* \, dv_\sigma d\sigma$$

$$= \int \int E_{0,h} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{3 + 2s}} \mu^{1/2} (v_\gamma + v - h - \mu^{1/2} (v + \alpha)) f(v + \alpha) \, d\alpha dh$$

$$= a_{2,c}(v, D_v)f,$$

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with

\[ a_{2,c}(v, \eta) := \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^\gamma + 1 + 2s}{|h|^{3+2s}} \mu^{1/2}(v) \left( \mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha) \right) e^{2\pi i \alpha \cdot \eta} d\alpha dh \]

\[ = \frac{1}{2} \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^\gamma + 1 + 2s}{|h|^{3+2s}} \mu^{1/2}(v) \left( \mu^{1/2}(v + \alpha - h) + \mu^{1/2}(v + \alpha + h) - 2\mu^{1/2}(v + \alpha) \right) e^{2\pi i \alpha \cdot \eta} d\alpha dh. \]

We split this integral into two parts: \( 1_{|\alpha| \geq 1} \) and \( 1_{|\alpha| \leq 1} \), the non-singular part and singular part. Then for any multi-index \( \beta, \beta \in \mathbb{N}^d \),

\[ |\partial_\alpha^\beta \partial_\eta^\beta a_{2,c, non-singular}| \]

\[ \leq C_{\beta, \beta} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} 1_{|\alpha| \geq 1} \frac{|\alpha + h|^\gamma + 1 + 2s}{|h|^{3+2s}} \mu^{1/2} \left( \mu^{1/2}(v + \alpha - h) + \mu^{1/2}(v + \alpha) \right) |\alpha|^{\beta} d\alpha dh \]

\[ \leq C_{\beta, \beta} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} 1_{|\alpha| \geq 1} \frac{1}{|h|^{3+2s}} \left( |\alpha + h|^\gamma + 1 + 2s + |\beta| \right) \mu^{1/4} \left( \mu^{1/4}(v + \alpha - h) + \mu^{1/4}(v + \alpha) \right) |\alpha|^{\beta} d\alpha dh \]

\[ \leq C_{\beta, \beta} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} 1_{|\alpha| \geq 1} \frac{1}{|h|^{3+2s}} \mu^{1/4} \left( \mu^{1/8}(v + \alpha - h) + \mu^{1/8}(v + \alpha) \right) d\alpha dh \]

\[ \leq C_{\beta, \beta} \mu^{1/72}(v) \leq C_{\beta, \beta}, \]

by using the lemma 3.3 below. For the singular part, applying lemma 3.4 below, we have

\[ |\partial_\alpha^\beta \partial_\eta^\beta a_{2,c, singular}| \]

\[ = |\partial_\alpha^\beta \partial_\eta^\beta \frac{1}{2} \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} 1_{|\alpha| \leq 1} \frac{|\alpha + h|^\gamma + 1 + 2s}{|h|^{3+2s}} \mu^{1/2}(v) \]

\[ \left( \mu^{1/2}(v + \alpha - h) + \mu^{1/2}(v + \alpha + h) - 2\mu^{1/2}(v + \alpha) \right) e^{2\pi i \alpha \cdot \eta} d\alpha dh \]

\[ \leq C_{\beta, \beta} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} 1_{|\alpha| \leq 1} \frac{1}{|h|^{3+2s}} \mu^{1/4}(v) \mu^{1/16}(v + \alpha)|\alpha|^{\beta} d\alpha dh, \]

\[ \leq C_{\beta, \beta} \int_{\mathbb{R}^d} \int_{E_{0,h}} 1_{|\alpha| \geq |h|} 1_{|\alpha| \leq 1} \frac{1}{|h|^{3+2s}} \mu^{1/32}(v) \mu^{1/32}(v + \alpha) d\alpha dh, \]

\[ \leq C_{\beta, \beta} \mu^{1/32}(v). \]
Thus $a_{2,c} \in S(1)$. For the part $L_{2,r}$, the argument is similar.

\[
L_{2,r}f = \int \int B(\mu^{1/2}f)'((\mu^{1/2})' - \mu^{1/2}) \, dv, d\sigma \\
= \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \mu^{1/2}(v + \alpha)\left(\mu^{1/2}(v - h) - \mu^{1/2}(v)\right) \, d\alpha d\sigma \\
=: a_{2,r}(v, D_v)f,
\]

with

\[
a_{2,r}(v, \eta) := \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \mu^{1/2}(v + \alpha)\left(\mu^{1/2}(v - h) - \mu^{1/2}(v)\right) e^{2\pi i \alpha \cdot \eta} \, d\alpha d\sigma \\
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \mu^{1/2}(v + \alpha) \left(\mu^{1/2}(v - h) + \mu^{1/2}(v + h) - 2\mu^{1/2}(v)\right) e^{2\pi i \alpha \cdot \eta} \, d\alpha d\sigma.
\]

We split the integral into singular and non-singular part, and then the argument is similar to the part $L_{2,c}$ and we can obtain $a_{2,r} \in S(1)$. It remains to study $L_{2,d}$ which is

\[
L_{2,d} = \int \int B((\mu')^{1/2} - \mu^{1/2}) ((\mu_s')^{1/2} - (\mu_s)^{1/2}) f' \, dv, d\sigma \\
= \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \left(\mu^{1/2}(v - h) - \mu^{1/2}(v)\right) \left(\mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha)\right) f(v + \alpha) \, d\alpha d\sigma \\
=: a_{2,d}(v, D_v)f,
\]

with

\[
a_{2,d}(v, \eta) := \int_{\mathbb{R}^d} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} \left(\mu^{1/2}(v - h) - \mu^{1/2}(v)\right) \left(\mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha)\right) e^{2\pi i \alpha \cdot \eta} \, d\alpha d\sigma.
\]

Now using the identity $a^2 - b^2 = (a + b)(a - b)$ and lemma 3.3, we can split the Gaussian function into

\[
\left(\mu^{1/2}(v - h) - \mu^{1/2}(v)\right) \left(\mu^{1/2}(v + \alpha - h) - \mu^{1/2}(v + \alpha)\right) \\
= \mu^{1/80}(v)\mu^{1/80}(v + \alpha) \left(\mu^{1/4}(v - h) - \mu^{1/4}(v)\right) \left(\mu^{1/4}(v + \alpha - h) - \mu^{1/4}(v + \alpha)\right).
\]

Then the remaining analysis is exactly the same as before. That is to split the integral into singular and non-singular parts. The terms inside the parentheses will cancel the singularity on $h$ and then we can have $a_{2,d} \in S(1)$.  

Here we list two short lemmas used in the proof.
Lemma 3.3. If $|\alpha| \geq |h|$, $\alpha \cdot h = 0$ then
\[
\mu(v-h)\mu(v+\alpha) = \mu(v)\mu(v+\alpha-h) \leq \mu^{1/9}(v)\mu^{1/9}(v+\alpha),
\]
\[
\mu(v-h)\mu(v+\alpha-h) \leq \mu^{1/20}(v)\mu^{1/20}(v+\alpha).
\]

Proof. Since $\alpha \cdot h = 0$, we have $|v-h|^2 + |v+\alpha|^2 = |v|^2 + |v+\alpha-h|^2$ and the first equality if proved. Notice $|v+\alpha| \leq |v-h| + |\alpha + h| \leq |v-h| + \sqrt{2}|\alpha| \leq (1 + \sqrt{2})|v-h| + \sqrt{2}|v+\alpha-h|$ and $|v| \leq |v-h| + |h| \leq |v-h| + |\alpha| \leq 2|v-h| + |v+\alpha-h|$, we have
\[
|v|^2 + |v+\alpha|^2 \leq 20(|v-h|^2 + |v+\alpha-h|^2),
\]
and the second inequality is proved. Similarly, $|v+\alpha| \leq |v| + |\alpha - h| \leq 2|v| + |v+\alpha-h|$ and hence
\[
|v|^2 + |v+\alpha|^2 \leq |v|^2 + 8|v|^2 + 2|v+\alpha-h|^2 \leq 9(|v|^2 + |v+\alpha-h|^2),
\]
and hence the first inequality is proved.

Lemma 3.4. If $|h| \leq 1$, for $\beta \in \mathbb{N}^3$, there exists $C_{\beta} > 0$ such that for $v \in \mathbb{R}^3$,
\[
|\partial^\beta_{\nu} (\mu^{1/2}(v-h) - \mu^{1/2}(v))| \leq C_{\beta}|h|\mu^{1/16}(v),
\]
\[
|\partial^\beta_{\nu} (\mu^{1/2}(v-h) + \mu^{1/2}(v+h) - 2\mu^{1/2}(v))| \leq C_{\beta}|h|^2\mu^{1/16}(v).
\]

Proof. We recall the definition $\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$. Firstly, by mean value theorem, we have for some $\delta \in (0,1)$,
\[
\left| (\mu^{1/2}(v-h) - \mu^{1/2}(v)) \right| \leq |h|\partial_{\nu}(\mu^{1/2})(v-\delta h) \leq C|h|\mu^{1/4}.
\]
Thus
\[
\left| e^{-|v-h|^2/4} - e^{-|v|^2/4} \right| \leq C|h|e^{-|v|^2/8},
\]
\[
\left| e^{v-h/2-|h|^2/4} - 1 \right| \leq C|h|e^{v/2},
\]
Now notice $\partial^\beta_{\nu} (\mu^{1/2}(v-h) - \mu^{1/2}(v)) = C\partial^\beta_{\nu}(e^{-|v|^2/4}(e^{v-h/2-|h|^2/4} - 1))$, then the first estimate follows from Leibniz formula. The second inequality follows similarly.

4 Appendix

Pseudo-differential calculus We recall some notation and theorem of pseudo-differential calculus. For details, one may refer to Chapter 2 in the book [11], Proposition 1.1 in [6] and [5,7] for details. As above, we set $\Gamma = |dv|^2 + |dh|^2$, but notice that the following are also valid for general admissible metric. Let $M$ be an $\Gamma$-admissible weight function. That is, $M : \mathbb{R}^{2d} \to (0, +\infty)$
satisfies the following conditions:

(a) (slowly varying) there exists $\delta > 0$ such that for any $X, Y \in \mathbb{R}^{2d}$, $|X - Y| \leq \delta$ implies

$$M(X) \approx M(Y);$$

(b) (temperance) there exists $C > 0$, $N \in \mathbb{R}$, such that for $X, Y \in \mathbb{R}^{2d}$,

$$\frac{M(X)}{M(Y)} \leq C(X - Y)^N. \quad (52)$$

A direct result is that if $M_1, M_2$ are two $\Gamma$-admissible weight, then so is $M_1 + M_2$ and $M_1M_2$. Consider symbols $a(v, \eta, \xi)$ as a function of $(v, \eta)$ with parameters $\xi$. We say that $a \in S(M, \Gamma)$ uniformly in $\xi$, if for $\alpha, \beta \in \mathbb{N}^d$, $v, \eta \in \mathbb{R}^d$,}

$$|\partial^\alpha_v \partial^\beta_\eta a(v, \eta, \xi)| \leq C_{\alpha, \beta}M, \quad (53)$$

with $C_{\alpha, \beta}$ a constant depending only on $\alpha$ and $\beta$, but independent of $\xi$. The space $S(M, \Gamma)$ endowed with the seminorms

$$\|a\|_{k; S(M, \Gamma)} = \max_{0 \leq |\alpha| + |\beta| \leq k} \sup_{(v, \eta, \xi) \in \mathbb{R}^{2d}} |M(v, \eta)|^{-1}[\partial^\alpha_v \partial^\beta_\eta a(v, \eta, \xi)|, \quad (54)$$

becomes a Fréchet space. Sometimes we write $\partial_v a \in S(M, \Gamma)$ to mean that $\partial_\eta a \in S(M, \Gamma)$ ($1 \leq j \leq d$) equipped with the same seminorms. We formally define the pseudo-differential operator by

$$(op_v a)u(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi} a((1-t)x + ty, \xi)u(y) \, dy \, d\xi,$$

for $t \in \mathbb{R}$, $f \in \mathcal{S}$. In particular, denote $a(v, D_v) = op_v a$ to be the standard pseudo-differential operator and $a^w(v, D_v) = op_{1/2}\partial_\eta a$ to be the Weyl quantization of symbol $a$. We write $A \in Op(M, \Gamma)$ to represent that $A$ is a Weyl quantization with symbol belongs to class $S(M, \Gamma)$. One important property for Weyl quantization of a real-valued symbol is the formal self-adjointness on $L^2$. Here, formal means the equation for self-adjointness is valid once they are well-defined.

Let $a_1(v, \eta) \in S(M_1, \Gamma), a_2(v, \eta) \in S(M_2, \Gamma)$, then $a_1^w a_2^w = (a_1 \# a_2)^w, a_1 \# a_2 \in S(M_1M_2, \Gamma)$ with

$$a_1 \# a_2(v, \eta) = a_1(v, \eta)a_2(v, \eta) + \int_0^1 (\partial_\eta a_1 \# \partial_\eta a_2 - \partial_v a_1 \# \partial_v a_2) \, d\theta,$$

$$g \# h(Y) := \frac{1}{2i} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\pi i\theta(Y - Y_1, Y - Y_2)} g(Y_1) \cdot h(Y_2) \, dY_1 \, dY_2,$$

with $Y = (v, \eta)$. For any non-negative integer $k$, there exists $l, C$ independent of $\theta \in [0, 1]$ such that

$$\|g \# h\|_{k; S(M_1M_2, \Gamma)} \leq C\|g\|_{l, S(M_1, \Gamma)}\|h\|_{l, S(M_2, \Gamma)}.$$
Thus if $\partial_\eta a_1, \partial_\eta a_2 \in S(M'_1, \Gamma)$ and $\partial_\eta a_1, \partial_\eta a_2 \in S(M'_2, \Gamma)$, then $[a_1, a_2] \in S(M'_1 M'_2, \Gamma)$, where $[\cdot, \cdot]$ is the commutator defined by $[A, B] := AB - BA$.

We can define a Hilbert space $H(M, g) := \{ u \in \mathcal{S}': \| u \|_{H(M, g)} < \infty \}$, where

$$\| u \|_{H(M, g)} := \int M(Y)^2 \| \varphi_Y u \|_{L^2}^2 |g_Y|^{1/2} dY < \infty,$$

and $(\varphi_Y)_{Y \in \mathbb{R}^{2d}}$ is any uniformly confined family of symbols which is a partition of unity. If $a \in S(M)$ is an isomorphism from $H(M')$ to $H(M'M^{-1})$, then $(a^w u, a^w v)$ is an equivalent Hilbertian structure on $H(M)$. Moreover, the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $H(M)$.

For $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, $m \in \mathbb{R}$, the metric $g_{\rho, \delta} := (\xi)^{2\delta}|dx|^2 + (\xi)^{-2\rho}|d\xi|^2$ is admissible and

$$H^m = H((\xi)^m, g_{1,0}) = H((\xi)^m, g_{\rho, \delta}).$$

This can be proved by using the technique in corollary 2.5.

Let $a \in S(M, g)$, then $a^w : H(M_1, g) \to H(M_1/M, g)$ is linear continuous, in the sense of unique bounded extension from $\mathcal{S}$ to $H(M_1, \Gamma)$. Also the existence of $b \in S(M^{-1}, \Gamma)$ such that $b#a = a#b = 1$ is equivalent to the invertibility of $a^w$ as an operator from $H(MM_1, \Gamma)$ onto $H(M_1, \Gamma)$ for some $\Gamma$-admissible weight function $M_1$.

For the metric $\Gamma = |dv|^2 + |d\eta|^2$, the map $J^t = \exp(2\pi i D_v \cdot D_\eta)$ is an isomorphism of the Fréchet space $S(M, \Gamma)$, with polynomial bounds in the real variable $t$, where $D_v = \partial_v/i$, $D_\eta = \partial_\eta/i$. Moreover, $a(x, D_v) = (J^{-1/2}a)^w$.

**Carleman representation and cancellation lemma** Now we have a short review of some useful facts in the theory of Boltzmann equation. One may refer to [1, 4] for details. The first one is the so called Carleman representation. For measurable function $F(v, v_s, v'_s)$, if any sides of the following equation is well-defined, then

$$\int_{\mathbb{R}^3} \int_{S^2} b(\cos \theta)|v - v_s|^\gamma F(v, v_s, v'_s) d\sigma dv_s$$

$$= \int_{\mathbb{R}^3} \int_{E_{0,h}} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{\alpha + h|\gamma + 1 + 2s}{|h|^{\gamma + 2s}} F(v, v + \alpha - h, v - h, v + \alpha) d\sigma dh,$$

where $\tilde{b}(\alpha, h)$ is bounded from below and above by positive constants, and $\tilde{b}(\alpha, h) = \tilde{b}(\pm \alpha, \pm h)$, $E_{0,h}$ is the hyper-plane orthogonal to $h$ containing the origin. The second is the cancellation lemma. Consider a measurable function $G(|v - v_s|, |v - v'|)$, then for $f \in \mathcal{S}$,

$$\int_{\mathbb{R}^3} \int_{S^2} G(|v - v_s|, |v - v'|) b(\cos \theta)(f'_s - f_s) d\sigma dv_s = S *_{v_s} f(v),$$

where $S$ is defined by, for $z \in \mathbb{R}^3$,

$$S(z) = 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( G\left(\frac{|z|}{\cos \theta/2}, \frac{|z| \sin \theta/2}{\cos \theta/2}\right) - G(|z|, |z| \sin(\theta/2)) \right) d\theta.$$
Semigroup theory Here we write some well-known result from semigroup theory. One may refer to [10] for more details.

**Definition 4.1.** A linear operator \((A, D(A))\) on a Banach space \(X\) is called dissipative if \(\|(\lambda I - A)x\| \geq \lambda\|x\|\) for all \(\lambda > 0\) and \(x \in D(A)\).

**Proposition 4.2.** An operator \((A, D(A))\) is dissipative if and only if for every \(x \in D(A)\) there exists \(j(x) \in \{x' \in X' : \langle x, x' \rangle = \|x\|^2 = \|x'\|^2\}\) such that

\[ \text{Re}(Ax, j(x)) \leq 0. \quad (56) \]

**Theorem 4.3.** For a densely defined, dissipative operator \((A, D(A))\) on a Banach space \(X\) the following statements are equivalent.

(a) The closure \(\overline{A}\) of \(A\) generates a contraction semigroup.

(b) \(\text{Im}(\lambda I - A)\) is dense in \(X\) for some (hence all) \(\lambda > 0\).

**Corollary 4.4.** Let \((A, D(A))\) be a dissipative operator on a reflexive Banach space such that \(\lambda I - A\) is surjective for some \(\lambda > 0\). Then \(A\) is densely defined and generates a contraction semigroup.

**Theorem 4.5.** Let \((A, D(A))\) be the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) satisfying \(\|T(t)\| \leq M e^{\omega t}\) for all \(t \geq 0\) and some \(\omega \in \mathbb{R}, M \geq 1\). If \(B \in L(X)\), then \(C \coloneqq A + B\) with \(D(C) \coloneqq D(A)\) generates a strongly continuous semigroup \((S(t))_{t \geq 0}\) satisfying \(\|S(t)\| \leq M e^{(\omega + M\|B\|)t}\) for all \(t \geq 0\).

In the end, we write two useful inequality: for any \(n_1 < n_2 < n_3\),

\[ \langle v \rangle^{n_2} \leq \varepsilon \langle v \rangle^{n_3} + C_{n_1, n_2, n_3} \varepsilon^{-\frac{n_2-n_1}{n_3-n_2}} \langle v \rangle^{n_1}. \quad (57) \]

For \(\rho > 0, \delta \in \mathbb{R}, \alpha > -d, \beta \in \mathbb{R}\), we have

\[ \int_{\mathbb{R}^d} |v|^\alpha \langle v \rangle^\beta \langle v + u \rangle^\delta e^{-\rho |v+u|^2} \, dv \approx \langle u \rangle^{\alpha+\beta}, \quad (58) \]

where constants may depend on the parameters. The first one is a version of Young’s inequality while the second is lemma 2.5 in [2].

**References**

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