Abstract

In this paper we present generalisations of Paley-Wiener type theorems to Mellin and (Laplace-)Fourier transforms of rapidly decreasing smooth functions with positive support and log-polyhomogeneous asymptotic expansion at zero. This article is based on the thesis [CdC] uses results borrowed from [RS], [RS81], [P79], [FGD], [P79], [SZ].

Introduction

A classical theorem due to R. Paley and N. Wiener [PW] provides a necessary and sufficient condition to build a holomorphic extension of the Fourier transform \( \hat{f} \) of a function \( f \in L^2(\mathbb{R}) \) with exponential growth \( |\hat{f}(\zeta)| \leq C e^{a|\zeta|} \) and supported on \([-a, a]\). This theorem was generalized to distributions by L. Schwartz in [Schw66], and since then this type of results are known as Paley-Wiener theorems.

The main goal of this paper is to provide Paley-Wiener type theorems for the Mellin and Fourier transform of rapidly decreasing functions with positive support and log-polyhomogeneous asymptotic behavior at zero. To the author’s knowledge, in that degree of generalization, the results concerning the Fourier transform included in Section 4.2 are new. As a particular case we recover a Paley-Wiener type theorem for rapidly decreasing functions which are smooth up to the boundary. This result plays an important role in Boutet de Monvel’s pseudo-differential calculus [BdM], and provides a fundamental tool for
the traciality of the non-commutative residue on BdM’s algebra done by B.V. Fedosov, F. Golse, E. Leichtnam and E. Schrohe in [FGLS]. The tools presented here are used in an essential way in [CdC] to prove the traciality of the so called canonical trace for log-polyhomogeneous pseudo-differential operators (see [KV] and [Lesch99]) on manifolds with boundary. We hope that this short survey can also be of use for whom might want to get acquainted with Paley-Wiener type theorems, independently of their applications in geometric analysis.

In order to make this document self-contained we also recall well-known Paley-Wiener type theorems. As we go along we compare our results with existing results in the literature, in particular, we compare a Paley-Wiener result involving the Mellin tranform of rapidly decreasing functions with positive support and log-polyhomogeneous asymptotic behavior at zero (cf. Theorem 2, see also [RS], [RS81]) with the respective Fourier transform (cf. Theorem 1).

This document is organized as follows: In Section 3.2, we recall basic properties of and Paley-Wiener type results for the Mellin transform of smooth compactly supported functions \( C_0^\infty(\mathbb{R}^+) \) as well as of compactly supported functions in \( L^2(\mathbb{R}^+) \) (see, [RS81], [P79], [FGD]). Finally, we give similar results for \( S(\mathbb{R}^+) \) (cf. Proposition 6). In Section 3.3, Using results of [FGD], we deduce a Paley-Wiener type theorem for functions in \( \mathcal{S}_p(\mathbb{R}^+) \) (cf. Theorem 1). In Section 3.4, following [P79] we recall a Paley-Wiener type theorem of the Mellin transform of extendable compactly supported distribution \( E'_c(\mathbb{R}^+) \) (cf. Proposition 11).

In Section 4.1, we derive an explicit expression for the Fourier transform of log-homogeneous distributions, restricting ourselves to distributions of order \( a \in \mathbb{C} \setminus \{\ldots, -2, -1\} \) and log-type \( k \in \mathbb{Z}_{\geq 0} \). In Section 4.2, we combine the results obtained in Section 4.1 to derive a Paley-Wiener type theorem for the Fourier transform of functions in \( \mathcal{S}_p(\mathbb{R}^+) \). More precisely, we relate the Fourier transform of a function in \( \mathcal{S}_p(\mathbb{R}^+) \) with a holomorphic function in \( \mathbb{C}^- := \{\zeta \in \mathbb{C} \mid \text{Im}\, \zeta < 0\} \) with log-polyhomogeneous asymptotic behavior at infinity, which is continuous up to the boundary in \( \mathcal{S}'(\mathbb{R}^+) \) (cf. Theorem 2). In particular, this result can be applied to functions in \( S(\mathbb{R}^+) \) (cf. Proposition 8). Finally, in Section 4.3 we recall a Paley-Wiener type result of the Laplace-Fourier transform of tempered distribution with compact support, see [SZ], similar to the one described in Section 3.4 for the Mellin transform.

1 Notations and preliminary definitions

We introduce the necessary notations for the subsequent sections and the definitions to follow.

Notations:

- \( \mathbb{R}^+ := \{x \mid x > 0\} \) and \( \mathbb{R}^- := \{x \mid x < 0\} \) denote the real half-spaces without and with
boundary, respectively. Similarly, 
\[ C_+ := \{ \zeta \in \mathbb{C} \mid \text{Im} \zeta > 0 \} \quad \text{(resp.} \quad C_- := \{ \zeta \in \mathbb{C} \mid \text{Im} \zeta < 0 \} \) and 
\[ \overline{C}_+ := \{ \zeta \in \mathbb{C} \mid \text{Im} \zeta \geq 0 \} \quad \text{(resp.} \quad \overline{C}_+ := \{ \zeta \in \mathbb{C} \mid \text{Im} \zeta \leq 0 \} \) denotes the complex half-plane without and with boundary, respectively.

- By a cut-off function \( \omega \) around zero we mean a smooth compactly supported function which is nonnegative, decreasing, and equal to 1 near zero. By an excision function \( \chi \) we mean a smooth function which is nonnegative and vanishes near zero and is equal to 1 outside of a neighborhood of zero. Note that \( 1 - \chi \) is a cut-off function.

- We denote by \( \xi \mapsto [\xi] \) the strictly positive function for which \( [\xi] = |\xi| \) for \( |\xi| \geq 1 \), and by \( O(|\xi|^{-\infty}) \) we mean a rapidly decreasing function, i.e. a function that decreases faster than any polynomial.

- Let \( U \) be an open subset of \( \mathbb{R} \). Let \( C_c^\infty(U) \) be the set of all compact support smooth functions on \( \mathbb{R} \). Let \( S(\mathbb{R}) \) be the set of all smooth rapidly decreasing function on \( \mathbb{R} \), i.e., \( u \) lies in \( S(\mathbb{R}) \) iff \( u \) satisfies that for any \( \alpha, \beta \in \{0, 1, 2, \ldots \} \), there exists a positive constant \( C_{\alpha, \beta} \) such that
\[
\sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta_x u(x)| \leq C_{\alpha, \beta}.
\]

- Let \( S(\mathbb{R}_+) \) be the set of Schwartz functions on \( \mathbb{R}_+ \) which are smooth up to the boundary, i.e. \( u \) lies in \( S(\mathbb{R}_+) \) if there exists a function \( \tilde{u} \in S(\mathbb{R}) \) such that its restriction \( \tilde{u}(x)|_{x>0} \) coincides with \( u(x) \). Let \( S^\prime(\mathbb{R}) \) be the space of tempered distributions.

- Let \( U \subset \mathbb{R} \) be an open subset. A distribution \( f \) in \( U \) is a linear form on \( C^\infty(U) \) such that for every compact set \( K \subset U \) there exist constants \( C \) and \( k \) such
\[
|f(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha_x \phi|, \quad \forall \phi \in C_c^\infty(K). \tag{1}
\]
The set of all distributions on \( U \) is denoted by \( \mathcal{D}^\prime(U) \). If the same integer \( k \) can be used in (1) for every compact \( K \) we say that \( f \) is of order \( \leq k \) and we denote the set of such distributions by \( \mathcal{D}^{\prime,k}(U) \).

- We say \( f \in \mathcal{D}^\prime(U) \) vanishes on \( V \subset U \) if \( \langle f, \phi \rangle = 0 \) for all \( \phi \in C_c^\infty(U) \) with support in \( V \). The maximal open set \( V \) on which the distribution \( f \) vanishes is called the support of \( f \).

- The set of compactly supported distributions is denoted by \( \mathcal{E}^\prime(\mathbb{R}) \), and \( \mathcal{E}^\prime_c(\mathbb{R}_+) \) denotes the set of positive compact supported distribution in \( \mathcal{E}^\prime(\mathbb{R}) \) which can be extended to a compact support distribution in \( \mathbb{R} \).
We are now ready to introduce some basic definitions (see e.g. [Rud], [PW], [Schw52], [SB]).

**Definition 1.** Let $U \subset \mathbb{R}$ be an open subset. Denote by $C_c^\infty(U)$ the set of all smooth functions with compact support in $U$.

- **The Mellin Transform:** The Mellin Transform is defined as a map $\mathcal{M}: C_c^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$ given by
  \[
  \mathcal{M}[f](s) := \int_0^\infty t^s f(t) \frac{dt}{t}, \quad s \in \mathbb{C},
  \]
  where $\mathcal{A}(\mathbb{C})$ denotes the set of all holomorphic functions on $\mathbb{C}$. The Mellin map $f \mapsto \mathcal{M}[f]$ is continuous, and has a continuous inverse $\mathcal{M}^{-1}$ given by
  \[
  f(t) = \int_{-\infty}^{\infty} t^{-(\xi+i\eta)} \mathcal{M}[f](\xi + i\eta) d\eta \quad \text{for arbitrary } \xi.
  \]
  which defines a continuous map.

- **The Fourier transform:** Let $S(\mathbb{R})$ denote the set of all smooth rapidly decreasing functions, also called the set of Schwartz functions. The Fourier transform $\mathcal{F}: S(\mathbb{R}) \to S(\mathbb{R})$ is a continuous bijection with continuous inverse given by
  \[
  \mathcal{F}[u](\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}[u](x) := \int_{\mathbb{R}} e^{ix\xi} u(\xi) \, d\xi,
  \]
  where $dx := \frac{1}{(2\pi)^{1/2}} dx$, $\mathcal{F}[u](\xi)$ is also denote by $\hat{u}$.

The following proposition relate the Mellin and the Fourier transforms, a proof follows from straight way computation.

**Proposition 1.** The Mellin transform in terms of a Fourier transform: For $\xi \in \mathbb{R}$, let $m_\xi: C_c^\infty(\mathbb{R}_+) \to C_c^\infty(\mathbb{R})$ be the isomorphism of vector spaces given by $f \mapsto e^{-\xi t} f(e^{-t})$. Then, for $s = \xi + i\eta$ we have
  \[
  \mathcal{M}[f](s) = \mathcal{F} \circ m_\xi[f](\eta) = \mathcal{F}[e^{-\xi t} f(e^{-t})](\eta), \quad \forall f \in C_c^\infty(\mathbb{R}_+).
  \]
  If $\chi: \mathbb{R} \to \mathbb{R}_+$ denotes the diffeomorphism of groups $\chi(x) = e^{-x}$ then, its pullback $\chi^*: C_c^\infty(\mathbb{R}_+) \to C_c^\infty(\mathbb{R})$, given by $(\chi^*[f])(x) = f(e^{-x})$, coincides with $m_0$. In particular, for $s = i\eta$, we have
  \[
  \mathcal{M}[f](i\eta) = \int_0^\infty t^{i\eta} f(t) \frac{dt}{t} = \int_{-\infty}^{\infty} e^{-ix\eta} f(e^{-x}) dx = \mathcal{F} \circ \chi^*[f](\eta).
  \]
  As we shall see later, both the Mellin and Fourier transforms extend to more general spaces which we introduce below.

The space of Fourier transforms of functions in $S(\mathbb{R}_+)$ plays an important
role in Boutet de Monvel’s pseudo-differential boundary calculus \cite{BdM}. The
smoothness up to the boundary yields the Taylor expansion around $x = 0$,
$u(x) \sim \sum_{j \geq 0} \frac{1}{j!} u^{(j)}(0)x^j$. We consider more general smooth rapidly decreasing
functions allowing for log-polyhomogeneous asymptotic behavior at zero. To
the set $p = \{(p_j, m_j) \in \mathbb{C} \times \mathbb{N} \mid \text{for } j \in \mathbb{N}\}$ which prescribes the type of polyho-
mogeneous singularity at zero, we assign the space $S_p(\mathbb{R}_+)$ of smooth functions
$u$ on $\mathbb{R}_+$ which are rapidly decreasing at infinity and such that, as $x \to 0^+$,
$$u \sim \sum_{j=0}^{\infty} m_j \sum_{k=0}^{m_j} c_{jk} x^{p_j} \ln^k x.$$ 
These spaces were considered by many authors, e.g. H. Kegel, B-W. Schulze, S.
Rempel \cite{RS, RS81, RS82}, G. Grubb \cite{G05}, J. Seiler, E. Schrohe and many
others.

Let us also mention that Paley-Wiener type theorems for compact support
functions in the spaces $L^2(\mathbb{R})$, $S(\mathbb{R})$ and $S'(\mathbb{R})$ are considered in \cite{SB}, The-
orems 11.1.1.-11.1.4. Here we focus in Paley-Wiener theorems for functions
with positive support.

**Acknowledgements.** It is a pleasure to thank Sylvie Paycha, Carolina
Neira and Elmar Schrohe for their encouragements, suggestions and contribu-
tions to improve this paper. I also thank Alexander Cardona for helpful com-
ments on an earlier draft of this paper. Part of the research on which this paper
is based was carried out during visits to Potsdam University which the author
thanks for its hospitality. Let me also thank Sylvie Paycha and Bert-Wolfgang
Schulze for the scientific advice they gave me during my stays. This research
has been supported by the \textit{Vicerrectoría de Investigaciones} and the \textit{Faculty of
Sciences} of the Universidad de los Andes.

## 2 Functions with log-polyhomogeneous asymptotic behavior at zero

In this section we study smooth rapidly decreasing functions with positive sup-
port and log-polyhomogeneous asymptotic behavior at zero.

We use a similar notation to the one used in \cite{RS}. We denote by $\mathbb{P}$ the set of
all a sequences $p = \{(p_j, m_j) \in \mathbb{C} \times \mathbb{N} \mid 0 \leq m_j \leq m \text{ for } j \in \mathbb{N}\}$ with
$$\text{Re} p_j \to \infty \text{ as } j \to \infty, \text{ Re} p_j \leq \text{Re} p_{j+1} \text{ for } j \in \mathbb{N}.$$ 
In particular, denote by $p_0$ the sequence
$$p_0 = \{(p_j, m_j) \in \mathbb{C} \times \mathbb{N} \mid p_j = j, m_j = 0, \text{ for } j \in \mathbb{N}\},$$
corresponding to the power set associated to the usual Taylor series type ex-
pansion around $x = 0$ for functions in $S(\mathbb{R}_+)$. 
Definition 2. Let $p = \{(p_j, m_j)\} \subset \mathbb{P}$. Consider the subset $\mathcal{S}_p(\mathbb{R}_+^*) \subset \mathcal{S}(\mathbb{R}_+)$ of functions $u$ with the following log-polyhomogeneous asymptotic expansion around zero

$$u(x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x,$$

as $x \to 0^+$. In other words, a function $u$ lies in $\mathcal{S}_p(\mathbb{R}_+^*)$ if and only if for any cut-off function $\omega$ there exists a sequence $\{a_{jk} = a_{jk}(u) \in \mathbb{C} \mid 0 \leq k \leq m_j, \text{ for } j \in \mathbb{N}\}$ and $N \in \mathbb{N}$ such that for

$$u_0(x) := \omega(x) \left( u(x) - \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x \right),$$

we have $\partial_x^k u_0(x) = O(x^{\text{Re} p_{j+1} - k}) \forall k \in \mathbb{N}$, and such that $e_+ (1 - \omega) u \in \mathcal{S}(\mathbb{R})$. In this case, we say that $u$ has a log-polyhomogeneous asymptotic behavior at zero of type $p$. This kind of singularity is also called conormal singularity at zero of type $p$.

Similarly, denote by $\mathbb{P}$ the set of all sequences $p = \{(p_j, m_j)\} \subset \mathbb{P}$ for which

$$\text{Re} p_j \to -\infty \text{ as } j \to \infty, \text{ Re} p_j \geq \text{Re} p_{j+1} \text{ } j \in \mathbb{N}.$$

Definition 3. Let $p = \{(p_j, m_j)\} \in \mathbb{P}$. The subset $C^\infty_\sigma(\mathbb{R}) \subset C^\infty(\mathbb{R})$ denotes the set of functions $u$ with the following an asymptotic expansion

$$u(x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x,$$

as $|x| \to \infty$. In other words, a function $u \in C^\infty_\sigma(\mathbb{R})$ if and only if $u$ is a smooth function on $\mathbb{R}$ and, for any cut-off function $\omega$, there exists a sequence $\{a_{jk} = a_{jk}(u) \in \mathbb{C} \mid 0 \leq k \leq m_j, \text{ for } j \in \mathbb{N}\}$ and $N \in \mathbb{N}$ such that

$$u_\infty(x) := (1 - \omega(x)) \left( u(x) - \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x \right),$$

then $\partial_x^k u_\infty(x) = O(x^{\text{Re} p_{j+1} - k}) \forall k \in \mathbb{N}$. In this case, we say that $u$ has a log-polyhomogeneous asymptotic behavior at infinity of type $p$. This kind of singularity is also called conormal singularity at infinity of type $p$.

For $\alpha \in \mathbb{C}$ and $p \in \mathbb{P}$ (or $p \in \mathbb{P}$), we set $T^\alpha p = \{ (p_j + \alpha, m_j) \mid j \in \mathbb{N} \}$, which we will call the translation of $p$ by $\alpha$.

Remark 1. In Section 3.1, we will consider homogeneity properties of $x_\pm^a$, and we will be advocated to restrict ourselves to non-negative integer powers of the type $x_\pm^a$ (with $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, cf. (20)). This lead us to the following definition.
Definition 4. We call a power set \( p = \{ (p_j, m_j) \in C \times N \mid j \in N \} \in \mathbb{P} \) an appropriate power set if \( p \) contains \( p_0 = \{ (j, 0) \mid j \in N \} \) as a subset and, if for all \( j \in N \), \( p_j \) is not a negative integer.

Example 1. Let \( u(x) = x^{-1/2}e^{-x^2} \) defined for \( x \in \mathbb{R}_+ \), then \( u(x) \sim \sum_{j \geq 0} \frac{x^{j-\frac{1}{2}}}{j!} \), so that \( u(x) \) lies in \( S_{T^{-\frac{1}{2}}p_0}(\mathbb{R}_+) \) with \( T^{-\frac{1}{2}}p_0 = \{ (j - 1/2, 0) \mid j \in N \} \).

Two inclusion \( p \) and \( p' \) between two appropriate power sets \( p \) and \( p' \) induces the following inclusion

\[ S_p(\mathbb{R}_+) \subset S_{p}(\mathbb{R}_+) \subset S(\mathbb{R}_+) \]

Lemma 1. For any \( p \) an appropriate power sets we have

\[ S(\mathbb{R}_+) \subset S_p(\mathbb{R}_+) \subset S(\mathbb{R}_+) \]

Moreover, the conormal singularity given by \( p_0 = \{ (j, 0) \mid j \in N \} \) corresponds to the Taylor expansion of a smooth function up to the boundary, i.e.

\[ S_{p_0}(\mathbb{R}_+) = S(\mathbb{R}_+) \]

Proof. The first assertion follows from the definition of \( S_p(\mathbb{R}_+) \) and the fact that \( p \) is an appropriate power sets. Let us prove the second assertion. First, \( S_{p_0}(\mathbb{R}_+) \subset S(\mathbb{R}_+) \) follows from Taylor’s asymptotic expansion at \( x = 0^+ \). To see \( S_{p_0}(\mathbb{R}_+) \subset S(\mathbb{R}_+) \), let \( u \in S_{p_0}(\mathbb{R}_+) \). This assumption implies that \( u \) is a rapidly decreasing function and smooth for \( x \gg 0 \). The existence of \( \tilde{u} \in S(\mathbb{R}) \) such that \( \tilde{u}|_{\mathbb{R}_+} = u \) then directly follows from the main result in \([Se]\) where a smooth function defined in a half space, all of whose derivatives have continuous limits at the boundary, is extended to a \( C^\infty \)-function in the whole space.

Finally, let us remark that in \([RS]\), Section 1.2, the authors show that \( S_p(\mathbb{R}_+) \) may be equipped with a Fréchet topology derived from the \( L^2 \)-scalar product, however, such topology is outside from the aims of this document.

3 Mellin transform and Paley-Wiener theorems

Following \([FGD]\) and \([P79]\), we now recall Paley-Wiener type theorems for the Mellin transform. We begin with a brief summary of the main properties of the Mellin transform we will need.

3.1 The Mellin transform and some of its properties

Basic properties of the Mellin transform. We follow \([FGD]\) to recall some properties of the Mellin transform we will use throughout this chapter.

Fundamental strip: Let \( f(t) \) be a continuous function on \( \mathbb{R}_+ \) such that

\[ f(t) = \begin{cases} O(t^{-\alpha}) & \text{for } t \to 0, \\ O(t^{-\beta}) & \text{for } t \to \infty. \end{cases} \]
for some real numbers $\alpha < \beta$. The convergence of $\mathcal{M}[f]$ follows from the inequality

$$|\mathcal{M}[f](s)| \leq C \int_0^1 t^{\operatorname{Re}s-1+\alpha} dt + c \int_1^\infty t^{\operatorname{Re}s-1+\beta} dt,$$

where $c$ and $C$ denote some constants. Then, $\mathcal{M}[f](s)$ exists for any $s \in \mathbb{C}$ in the strip $\{ \alpha < \operatorname{Re}s < \beta \}$. The largest open strip in which the integral converges is called the fundamental strip for $f$. By the Cauchy-Riemann equations, $\mathcal{M}[f](s)$ is analytic inside its fundamental strip.

By means of the change of variables $t = e^{-z}$, and the formula for inverse of the Fourier transform, we can obtain a formula for the inverse of the Mellin transform.

**Proposition 2.** (FGD, Theorem 2.) Let $f(t)$ be a continuous integrable function with fundamental strip $(a,b)$. If there exists a real number $c \in (a,b)$ such that $\mathcal{M}[f](c + it)$ is integrable. Then, for $s = c + i\eta$, the following equality holds

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-(c+i\eta)} \mathcal{M}[f](c + i\eta) d\eta.$$

The Mellin and Fourier transform have similar properties due to relation (2). Some of these properties are enunciated in the following Proposition. A proof of these properties follows by straightforward calculations (see FGD).

**Proposition 3.** Let $f$ be a function whose Mellin transform admits the fundamental strip $(a,b)$, and let $p$ be a non zero real number, $r, q$ positive real numbers. We have:

(i) $\mathcal{M}[t^z f(t)](s) = \mathcal{M}[f](s + z)$, on $(a - \operatorname{Re} z, b - \operatorname{Re} z)$;

(ii) $\mathcal{M}[f(t^p)](s) = \frac{1}{p} \mathcal{M}[f](\frac{s}{p})$, on $(pa, pb)$;

(iii) $\mathcal{M}\left[ \frac{d}{dt} f(t) \right](s) = -(s-1)\mathcal{M}[f](s-1)$, on $(a-1, b-1)$;

(iv) $\mathcal{M}\left[ \ln tf(t) \right](s) = \frac{d}{ds} \mathcal{M}[f](s)$, on $(a, b)$;

(v) $\mathcal{M}\left[ \int_0^t f(x) dx \right](s) = -s \mathcal{M}[f](s)$;

(vi) $\mathcal{M}\left[ \int_0^t f(x) dx \right](s) = -\frac{1}{s} \mathcal{M}[f(t)](s)$.

Finally, we denote by $H$ the Heaviside function on $[0,1]$ then

(vii) $\mathcal{M}[H(t)](s) = \frac{1}{s}$ (resp. $\mathcal{M}[1-H(t)] = -\frac{1}{s}$), on $\langle 0, \infty \rangle$ (resp. on $\langle -\infty, 0 \rangle$);

(viii) $\mathcal{M}[H(t) t^d \ln^k t](s) = \frac{(-1)^k k!}{(s+d)^{k+1}}$ on $(-d, \infty)$;

(ix) $\mathcal{M}[(1-H(t)) t^d \ln^k t](s) = -\frac{(-1)^k k!}{(s+d)^{k+1}}$ on $(-\infty, d)$. 
3.2 A Paley-Wiener theorem for Schwartz functions smooth up to the boundary with positive support

Let $\beta \in \mathbb{C}$, in the following $\Gamma_\beta$ will denote the vertical complex line \( \{ s \in \mathbb{C} \mid \Re s = \beta \} \).

**Proposition 4.** ([P79], Theorem 3.1.) A complex function $h$ on $\mathbb{C}$. Then $h$ is a Mellin transform of some $u \in C_c^\infty(\mathbb{R}_+)$ if and only if $h$ is analytic, and for any positive integer $m$ there exist constants $C_m$ and $a > 0$ such that

\[
|h(s)| \leq C_m(s)^{-m}e^{a|\Re s|}.
\]

In this case, we have $\text{supp } u \subset [e^{-a}, e^a]$.

**Lemma 2.** The Mellin transform $M$ maps $C_c^\infty(\mathbb{R}_+)$ to $L^2(\Gamma_{1/2})$, and extends to $M : L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2})$, which is an isometric isomorphism, where the norm on $L^2(\Gamma_{1/2})$ is given by

\[
\|M[f]\|_{L^2(\Gamma_{1/2})} = \int_{-\infty}^{\infty} \left| M[f]\left(\frac{1}{2} + i\eta\right) \right|^2 d\eta.
\]

We recall a a Paley-Wiener type theorem.

**Proposition 5.** ([P79], Theorem 3.2.) Let $u \in L^2(\mathbb{R}_+)$, and assume $\text{supp } u \subset (0, e^a]$ for some $a > 0$. Then the Mellin transform $h(s) := M[u](s)$ extends from $\Gamma_{1/2}$ to an analytic function $h(s)$ in $\{ \Re s > 1/2 \}$ such that

(i) $h_\xi(\eta) := h(\xi + 1/2 + i\eta)$ is in $L^2(\mathbb{R}_\eta)$ for every $\xi > 0$, and satisfies

\[
h_\xi(\eta) \to M[u](1/2 + i\eta) \text{ in } L^2(\mathbb{R}_\eta) \text{ as } \xi \to 0^+;
\]

(ii) there exists a constant $C > 0$ such that

\[
\|h_\xi\|_{L^2(\mathbb{R}_\eta)} \leq Ce^{a\xi}.
\]

Conversely, if $h$ is an analytic function in $\{ \Re s > 1/2 \}$ satisfying the estimates (8) for some constants $C, a > 0$, then $h$ is the Mellin transform of a function $u \in L^2(\mathbb{R}_+)$ with $\text{supp } u \subset (0, e^a]$.

We may restrict the domain of the map $M : L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2})$ to $\mathcal{S}(\mathbb{R}_+)$.

**Proposition 6** can be compared with Theorem 17, concerning the Fourier transform. The following result which we refer to as the Paley-Wiener theorem for functions smooth up to the boundary, yields a singular expansion:

Let us recall a useful definition (see, e.g. [FGD]): We call a singular expansion of a meromorphic function $h(s)$ in $\Omega \subset \mathbb{C}$, denoted by $\Xi$, a formal sum of singular elements of $h(s)$ at each pole of $h$ in $\Omega$. It is basically a sum of Laurent expansions around all poles truncated to the $O(1)$. For instance,

\[
\frac{1}{s^2(s+1)} \cong \left[ \frac{1}{s+1} + 2 \right]_{s=-1} + \left[ \frac{1}{s^2} - \frac{1}{s} \right]_{s=0} + \left[ \frac{1}{2} \right]_{s=1} \text{ for } s \in (-2, 2).
\]
Proposition 6. The Mellin transform $\mathcal{M}$ maps functions $u$ in $\mathcal{S}(\mathbb{R}^+)$ to holomorphic functions $\mathcal{M}[u](s)$ on the strip $(0, \infty)$ which can be extended to meromorphic functions on $\mathbb{C}$ with simple poles at $s = -j$ for $j \in \mathbb{N}$, i.e. $\mathcal{M}[u](s) \cong \sum_{j=0}^{\infty} a_j (s+j)^{-j}$, and such that for any excision function $\chi$ for the set $\{-j\}_{j \in \mathbb{N}}$, we have

$$\chi(s)\mathcal{M}[u](s)|_{\Gamma_{\chi}} \in \mathcal{S}(\Gamma_{\chi}).$$

A proof can be done by using the Taylor expansion of $u(x)$ around zero; Proposition 4 and some properties of the Mellin transform in Proposition 3. However, we will give a proof of this fact in Section 3.3 using results involving Theorem 1, below.

Remark 2. Notice that if we set $\tilde{M}[u](s) := \frac{1}{\Gamma(s)} \mathcal{M}[u](s)$, for $u \in \mathcal{S}(\mathbb{R}^+)$, where $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ is the Gamma function, integration by parts shows $\Gamma$ has simple poles at $-j, j \in \mathbb{N}$. It therefore follows from Proposition 6 that $\tilde{M}$ maps functions in $\mathcal{S}(\mathbb{R}^+)$ to holomorphic functions in $\mathbb{C}$. We call $\tilde{M}[u](s)$ the normalized Mellin transform.

3.3 A Paley-Wiener theorem for functions with log-polynomial asymptotic behavior at zero and positive support

In this section, using known Paley-Wiener type theorems, which are recalling here, we characterize the image under the Mellin transform of spaces $\mathcal{S}_p(\mathbb{R}^+)$ (cf. Theorem 1).

We first prove a result relative the Mellin transform of functions of the type $\omega(t)t^d\ln^k t$, for $\omega$ a cut-off function.

Lemma 3. For any a cut-off function $\omega$, the Mellin transform of

$$\omega_{dk}(t) := \omega(t)t^d\ln^k t \quad \text{and} \quad \tilde{\omega}_{dk}(t) := (1-\omega)(t)t^d\ln^k t$$

are meromorphic functions on $\mathbb{C}$ and

$$\mathcal{M}[\omega_{dk}](s) = -\mathcal{M}[\tilde{\omega}_{dk}](s) = \frac{(-1)^k k!}{(s+d)^{k+1}} + h_{dk}(s),$$

where $h_{dk}$ is an entire function which is rapidly decreasing in $t$ along parallel lines to $c + it$ with $c \in (-d, \infty)$.

Proof. We prove the result for $\omega_{dk}$, for $\tilde{\omega}_{dk}$ it is enough to observe that $\mathcal{M}[f(\frac{1}{t})](s) = -\mathcal{M}[f(t)](s)$. Let $\chi(x) = e^{-x}$ $x \in \mathbb{R}$. For $f(x) := (\chi^*\omega_{dk})(x)$ and $\omega'_{dk}(x) := (\chi^*\omega_{dk})(x)$, we have

$$f(x) = \omega'_{dk}(x)x^{-k}e^{-dx}.$$
We write \( \omega'_{dk}(x) = H(x) + (\omega'_{dk}(x) - H(x)) \), since the difference \( \omega'_{dk} - H \) is a compactly supported function we have \( h_{dk}(s) = \mathcal{M}[\omega'_{dk} - H](s) \) is an analytic function on \( \mathbb{C} \). Applying (5) to \( f \) we have that

\[
\mathcal{M}[\omega'_{dk}](\xi + i\eta) = \mathcal{F}^{-1}[H(x)e^{-x\xi}f(x)](\eta) + h_{dk}(s).
\]

Finally, setting \( s = \xi + i\eta \) and using that \( \mathcal{F}[x^n f(x)] = i^n \partial_x^n \mathcal{F}[f](\xi) \) and \( \mathcal{F}[H(x)e^{-ax}] = (a + i\xi)^{-1} \), we obtain

\[
\mathcal{M}[\omega'_{dk}](s) = \int_{-\infty}^{\infty} e^{i\eta(\xi + i\eta)}H(x)x^{-k}e^{-xdx} + h_{dk}(s)
\]

\[
= \partial_{\xi + i\eta}[(\xi + i\eta) + d)^{-1}] + h_{dk}(s)
\]

\[
= \frac{(-1)^k k!}{(s + d)^{k+1}} + h_{dk}(s).
\]

The following Theorem gives us a singular expansion for the Mellin transform of functions with log-polyhomogeneous behaviour around zero and infinity. We now recall two important results in [FDG], called there Direct Mapping Theorem and Converse Mapping Theorem.

**Proposition 7.** [Direct Mapping Theorem] Let \( f(t) \) be a continuous function with Mellin transform \( \mathcal{M}[f](s) \) defined in non-empty strip \( (a,b) \).

1. Assume that \( f(t) \) admits the following asymptotic expansion for \( t \to 0 \):

\[
f(t) = \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{p_j} \ln^k t + O(t^M),
\]

where \(-M < -Re p_j \leq a \) with \( M \in \mathbb{Z} \). Then \( \mathcal{M}[f](s) \) has a meromorphic continuation to the strip \((-M, b)\), and for \( s \in (-M, b) \):

\[
\mathcal{M}[f](s) \approx \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} \frac{(-1)^k k!}{(s + p_j)^{k+1}}.
\]

2. Let \( f(t) \) have the following asymptotic expansion for \( t \to \infty \):

\[
f(t) = \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{-p_j} \ln^k t + O(t^{-M}),
\]

where \( b \leq -Re p_j < M \) with \( M \in \mathbb{Z} \). Then \( \mathcal{M}[f](s) \) has a meromorphic continuation to the strip \((a, M)\), and for \( s \in (a, M) \):

\[
\mathcal{M}[f](s) \approx -\sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} \frac{(-1)^k k!}{(s + p_j)^{k+1}}.
\]
Proposition 8. [Converse Mapping Theorem] Let $f(t)$ be continuous on $(0, \infty)$ with Mellin transform $\mathcal{M}[f](s)$ with non-empty fundamental strip $(a, b)$.

1. • Assume that $h(s)$ admits a meromorphic continuation in the strip $(-M, b)$ for some $-M < a$ with a finite number of poles in $(-M, a)$, and is analytic on $\text{Re } s = -M$. Moreover we assume that it admits the following singular expansion:

$$\mathcal{M}[f](s) \cong \sum_{j=0}^{M} \sum_{k=0}^{m_j} a_{jk}^k \frac{(-1)^k k!}{(s + p_j)^{k+1}}, \text{ for } s \in (-M, a),$$ (10)

• and that there exists a real number $c$ in $(a, b)$ satisfying the estimate

$$\exists r > 1, \quad \mathcal{M}[f](s) = O(|s|^{-r})$$

when $|s| \to \infty$ in $-M \leq \text{Re } s \leq c$.

Then $f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} h(s) ds$ has the following asymptotic expansion at $0$,

$$f(t) = \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{p_j} \ln^k t + O(t^M).$$

2. • Assume that $h(s)$ admits a meromorphic continuation to the strip $(a, M)$ for some $M > b$ with a finite numbers of poles $(a, M)$, and is analytic on $\text{Re } s = M$. Moreover, we assume that it admits the singular expansion (10) for $s \in (c, M)$.

• We also assume the existence of real number $c$ in $(a, b)$ such that such that for some $r > 1$, satisfies the estimate (11) when $|s| \to \infty$ in $c \leq \text{Re } s \leq M$.

then $f(t) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} h(s) ds$ and it has the following asymptotic expansion at $\infty$,

$$f(t) = - \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{p_j} \ln^k t + O(t^{-M}).$$

Let us now characterize the image under the Mellin transform of the spaces $S_p(\mathbb{R}^+)$ (see, [2]).

Theorem 1. Let $a \in \mathbb{C}$. An analytic function $h(s)$ on the strip $(a, \infty)$ is a Mellin transform of a function $f$ in $S_p(\mathbb{R}^+)$ with asymptotic expansion for $t \to 0$ of the form

$$f(t) = \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{p_j} \ln^k t + O(t^M),$$ (12)

where $-M < -\text{Re } p_j \leq a$, if and only if
(i) $h(s)$ is defined for $\text{Re } s > -\min_{j \in \mathbb{N}} \text{Re } p_j$ and has a meromorphic continuation in the strip $(-M, \infty)$ with singular expansion

$$h(s) \cong \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} \frac{(-1)^k k!}{(s + p_j)^{k+1}}$$

for $s \in (M, a)$, and it is analytic on $\text{Re } s = -M$.

(ii) there exists a real number $c \in (a, \infty)$ such that for some $r > 1$,

$$h(s) = O(|s|^{-r})$$

when $|s| \to \infty$ in $-M \leq \text{Re } s \leq c$.

Proof. Let $p = \{(p_j, m_j)\}_{j \in \mathbb{N}}$. Let $f \in S_p(\mathbb{R}_+)$ with asymptotic expansion as before and a cut-off function $\omega$. Then

$$f(t) = \omega(t) \left( \sum_{j=0}^{M-1} \sum_{k=0}^{m_j} a_{jk} t^{p_j} \ln^k t + O(t^M) \right) + (1 - \omega(t))f(t).$$

Notice that the result is independent of the choice of the cut-off function $\omega$ from the fact that $\omega - \omega'$ has compact support for any other cut-off function $\omega'$, and the Mellin transform of a function with compact support is an analytic function. We know from Proposition 6 that the Mellin transform of $(1 - \omega(t))f(t)$ is an analytic function defined in the whole $\mathbb{C}$. From Proposition 7, $h(s) = M[f](s)$ admits a meromorphic continuation in the strip $(-M, \infty)$ with singular expansion (13).

The estimate $h(s) = O(|s|^{-r})$ comes from Proposition 8.

Let us now prove the converse statement. Let $h$ an meromorphic function satisfying the conditions of Theorem 1. Let $c \in (a, b)$, and set

$$f(t) = \int_{c-i\infty}^{c+i\infty} t^{-s}h(s)ds,$$

it follows from Theorem 8 that $f(t)$ satisfies (12). It now remains to show that $f$ is a rapidly decreasing smooth function. The smooth condition follows from Proposition 11 and the equality

$$\frac{d^k}{dt^k}f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s-1) \cdots (s-k)t^{-s}h(s)ds.$$

The rapidly decreasing condition follows from the fact that, for every $l, k \in \mathbb{N}$,

$$|t^l \frac{d^k}{dt^k}f(t)| \leq C_m \int_{c-i\infty}^{c+i\infty} |t^{-s+l}h(s)|ds \leq C_m \int_{c-i\infty}^{c+i\infty} t^l(s)^{k-m}ds < \infty,$$

for $m$ chosen large enough.

We finally come to the proof of the Proposition 6.
Proof of Proposition 6. Let \( u \in \mathcal{S}(\mathbb{R}_+) \). We have that for any integer \( M > 0 \), the following asymptotic expansion

\[
u(t) = \sum_{j=0}^{M-1} a_j t^j + O(t^M)\]

It follows from Theorem 1, \( \mathcal{M}[u](s) \) is an analytic function on \( \text{Re } s > 0 \) and has a meromorphic continuation in \( \mathbb{C} \) with singular expansion

\[h(s) \sim \sum_{j=0}^{M-1} a_j \frac{1}{s + p_j}\] (14)

for \( s \in (-\infty, 0) \).

Now, for \( s = \xi + i\eta \) and fixed \( \xi \), for any \( \alpha, \beta \in \mathbb{N} \) and any excision function \( \chi \) for the set \( \{ j \} \), since (cf. 5), there exists a positive constant \( C_{\alpha, \beta} \) such that

\[\sup_{\eta \in \Gamma_\eta} |\eta^\alpha \partial_\eta^\beta \chi \mathcal{M}[u](s)| = \sup_{\eta \in \Gamma_\eta} |\eta^\alpha \partial_\eta^\beta \mathcal{F}[e^{-\xi t} u(e^{-t})](\eta)| \leq C_{\alpha, \beta},\]

the last inequality follows from well-known properties of the Fourier transform and the fact that \( u \) lies in \( \mathcal{S}(\mathbb{R}_+) \). Therefore, \( \chi(s)\mathcal{M}[u](s) \in \mathcal{S}(\Gamma_\eta) \).

3.4 A Paley-Wiener theorem for extendable distributions with positive support

In this section we follow [P79] in order to recall the main properties of the Mellin transform acting on distributions we are interested in.

The space \( \mathcal{E}'(\mathbb{R}_+) \). Let \( \mathcal{E}'(\mathbb{R}_+) \) denote the subset of extendable distributions in \( \mathcal{D}'(\mathbb{R}_+) \) with compact support in \( \mathbb{R} \), i.e. \( U \in \mathcal{D}'(\mathbb{R}_+) \) if and only if there exists \( V \in \mathcal{D}'(\mathbb{R}) \) with compact support such that \( r_+ V = U \), here \( r_+ \) denotes the restriction map to \( \mathbb{R}_+ \). The space \( \mathcal{E}'(\mathbb{R}_+) \) may be equipped with the following algebraic structure: Let \( U_i, i = 1, 2 \), be distribution in \( \mathcal{E}'(\mathbb{R}_+) \). The convolution product \( U_1 * U_2 \) of \( U_1 \) and \( U_2 \) given by

\[
(U_1 * U_2, \phi) := \langle U_1(s) \otimes U_2(t), \phi(st) \rangle, \quad \forall \phi \in \mathcal{D}'(\mathbb{R}_+).
\]

Let \( \Omega \) be an open subset of \( \mathbb{R} \). The structure theorem for distributions shows that for any \( U \in \mathcal{D}'(\Omega) \) and any compact set \( K \subset \Omega \), there exists a continuous function \( F \in C^0(K) \) and a multi index \( \alpha \), such that \( U = \mathcal{D}_\alpha F \) on \( K \). In this case, \( \alpha \) is called the order of \( U \) (see, e.g. [H.I]).

Proposition 9. ([Schw66], Theorem 24) For any distribution \( V \) with compact support in \( \mathbb{R} \) which extends some \( U \in \mathcal{E}'(\mathbb{R}_+) \), there exists \( m \in \mathbb{N} \), such that \( V \) is a distribution of order \( \leq m \) and the function

\[
s \mapsto \mathcal{M}[U](s) = \langle V, r_+ t^{s-1} \rangle\]
is well defined as holomorphic function for \( \Re(s) > m + 1 \) and independent of the choice of extension \( V \) of \( U \). The function \( M[U](s) \) is called the Mellin transform of \( U \).

The space \( \mathcal{H}'_+ \). Let \( \mathcal{H}'_+ \) be the space of holomorphic functions \( h \) defined on half-planes \( \Re s > r \) (\( r \in \mathbb{R} \)), identifying two such functions if they coincide in some half-plane, and satisfy

\[
|h(s)| \leq C(s)^m e^{a \Re s}, \text{ for } \Re s > r,
\]

for some constants \( C > 0 \), \( m \in \mathbb{Z} \), \( a > 0 \) and \( r \in \mathbb{R} \), which can depend on \( h \). The space \( \mathcal{H}'_+ \) may be equipped with the following algebraic structure: Let \( h_i, i = 1, 2 \), be functions in \( \mathcal{H}'_+ \). The product \( h_1 h_2 \) of \( h_1 \) and \( h_2 \) is given by

\[
s \mapsto h_1(s)h_2(s).
\]

Lemma 4. The Mellin transform \( M[U](s) \) of any \( U \in \mathcal{E}'_+(\mathbb{R}^+) \) lies in \( \mathcal{H}'_+ \).

Proof. For any \( U \in \mathcal{E}'_+(\mathbb{R}^+) \) with support in \((0, a)\), there exists \( f \in C^0(\mathbb{R}^+) \) and \( m \in \mathbb{N} \) such that \( \text{supp } f \subset (0, a) \) and \( U = D^m f \). Then, applying properties of the Mellin transform we obtain

\[
M[U](s) = (-1)^m(s-1)(s-2) \cdots (s-m) \int_0^a t^{s-m-1} f(t) dt,
\]

and therefore \( M[U](s) \) therefore satisfies the estimate (15) for \( r > m \).

For \( h \in \mathcal{H}'_+ \) satisfying the estimate (15) and \( c > r \), we define

\[
\mathcal{R}[h](t) := D' \lim_{A \to \infty} \frac{1}{2\pi i} \int_{c-iA}^{c+iA} t^{-s}h(s)ds.
\]

As it is described in the next Theorem, the map \( \mathcal{R} \) defines a linear map from \( \mathcal{H}'_+ \) to \( \mathcal{E}'_+(\mathbb{R}^+) \), and the distribution \( \mathcal{R}[h](t) \) is called the inverse Mellin transform of \( h \).

Proposition 10. (\cite{P79}, Theorem 2.3.) The Mellin transform \( M \) is an algebra isomorphism from \( \mathcal{E}'_+(\mathbb{R}^+) \) to \( \mathcal{H}'_+ \), with inverse \( \mathcal{R} \). Moreover, if \( U \in \mathcal{E}'_+(\mathbb{R}^+) \) and \( h(s) = M[U](s) \in \mathcal{H}'_+ \), the support of \( U \) is contained in \((0, c)\) if and only if \( h \) verifies (15) with \( a = c \).

For any \( h \in \mathcal{H}'_+ \), the distribution \( \mathcal{R}[h](t) \) is called the inverse Mellin transform of \( h \). The next Theorem which concerns Mellin transforms of extendable compactly supported distributions, will be later stated in the language of Fourier transforms of tempered distributions with positive support in section 4.3, Proposition 18.

Proposition 11. (\cite{P79}, Theorem 3.1.) The function \( h \in \mathcal{H}'_+ \) is the Mellin transform \( h(s) = M[U](s) \) of a distribution \( U \in \mathcal{E}'_+(\mathbb{R}^+) \) with support \([e^{-a}, e^a]\),
a > 0 if and only if \( h(s) \) is an entire function and the following estimate holds for some \( m \in \mathbb{N} \),

\[
|h(s)| \leq \langle s \rangle^m e^{a|\text{Re} s|}, \ s \in \mathbb{C}.
\]

Moreover, \( h \in \mathcal{H}' \) is the Mellin transform \( h(s) = \mathcal{M}[U](s) \) of a \( C^\infty \)-function \( U \) with support \([e^{-a}, e^a]\), \( a > 0 \) if and only if \( h \) is an entire function and for all \( m \in \mathbb{N} \) there exits \( C_m > 0 \) such that

\[
|h(s)| \leq C_m \langle s \rangle^{-m} e^{a|\text{Re} s|}, \ s \in \mathbb{C}.
\]

4 Fourier transform and Paley-Wiener theorems

In this section we discuss Paley-Wiener type results for the Fourier transform, similar to those derived in Section 3 for the Mellin transform. We relate the Fourier transform of functions with positive/ negative support and log-polyhomogeneous asymptotic expansion at zero with analytical functions in \( \mathcal{A}(\mathbb{C}_+) \) having an asymptotic expansion at infinity (cf. Theorem 2).

4.1 Homogeneous and log-homogeneous distributions and their the Fourier transform

In this section we summarize known results on homogeneous and log-homogeneous distributions (see also \[Esk\], \[H.I\], section 3; \[vG\], \[FG\]) and we comment along the way, additionally, we compute explicitly the Fourier (and the inverse) transform of the mentioned distributions, see Proposition 13, 14 and Corollary 1.

For \( x \in \mathbb{R} \) and \( a \in \mathbb{C} \), with \( \text{Re} \ a > -1 \), let \( x_\pm^a \) be a homogeneous tempered distributions defined by the local integrable functions

\[
x_\pm^a := \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \text{and} \quad x_\pm := \begin{cases} |x|^a & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}.
\]

For any \( x \in \mathbb{R} \), we have

\[
xx_\pm^a = x_\pm^{a+1}, \text{ if } \text{Re} \ a > -1,
\]

and, using the distributional extension of a derivative (Theorem 3.1.3. in \[H.I\]), we have

\[
\frac{d}{dx} x_\pm^a = ax_\pm^{a-1}, \text{ if } \text{Re} \ a > 0.
\]

If \( a = 0 \) then the l.h.s. of (17) is the Heaviside function \( H'(x) = \delta_0 \) and the r.h.s. is zero.

Two methods are presented in \[H.I\] to extend \( x_\pm^a \) to all \( a \in \mathbb{C} \) as a tempered distribution, preserving when possible properties (16) and (17). We illustrate one of them, for \( x_+^a \), called Riesz’s Method. For fixed \( \phi \in C^\infty_c(\mathbb{R}) \), set

\[
a \mapsto I_a(\phi) := (x_+^a, \phi) = \int_0^\infty x^a \phi(x) dx.
\]
Integration by parts $k$ times yields

$$I_a(\phi) = \frac{(-1)^k}{(a+1)\cdots(a+k)} I_{a+k}(\phi^{(k)}).$$

The right-hand side is analytic for $\Re a > -k -1$ outside a set of simple poles $\{-1, -2, -3, \ldots, -k\}$. Then, for $a \notin \mathbb{Z}_{<0}$, $\Re a + k > -1$ we can define $x^a_+$ as the tempered distributions

$$x^a_+ = \frac{\partial^k x^{a+k}}{(a+1)\cdots(a+k)}$$

and

$$x^a_- = \frac{(-1)^k \partial^k x^{a+k}}{(a+1)\cdots(a+k)}.$$ 

Moreover, the residue of $a \mapsto I_a(\phi)$ at $a = -k$, for $k \in \mathbb{Z}_{>0}$ is given by

$$\lim_{a \to -k} (a+k) I_a(\phi) = \lim_{a \to -k} \frac{(-1)^k I_a(\phi)}{(a+1)\cdots(a+k)} = \phi^{(k-1)}(0)/(k-1)!,$$

i.e. $\text{Res}_{a=-k} x^a_+ = \frac{(\pi i)^{k-1}}{(k-1)!} \delta^{(k-1)}$, so subtracting the singular part, we obtain for $x^a_+$

$$x^{-k}_+(\phi) := \lim_{\epsilon \to 0} I_a(\phi) - \frac{\phi^{(k-1)}(0)}{\epsilon(k-1)!} = -\int_0^\infty \ln x \phi^{(k)} dx + \left(\sum_{j=1}^{k-1} \frac{1}{j}\right) \phi^{(k-1)}(0)/(k-1)!.$$

**Remark 3.** This extension is unique as a result of the uniqueness of the analytic continuation.

For $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we have

$$\frac{d}{dx} x^a = ax^{a-1} \quad \text{and} \quad (x^a, \phi) = t^a(x^a, \phi_t),$$

where $\phi_t(x) = \phi(tx)$. However, for $k \in \mathbb{Z}_{<0}$, we have

$$\frac{d}{dx} x^{-k}_+ = -kx^{-k-1}_+ + \frac{(-1)^k}{k!} \delta_0^{(k)},$$

and

$$\langle x^{-k}_+, \phi \rangle = t^{-k} \langle x^{-k}_+, \phi \rangle + \frac{(-1)^{k-1}}{(k-1)!} \phi^{(k-1)}(0) \ln t,$$

so the homogeneity is partly lost.

For $\Re a > -1$ and $k \in \mathbb{N}$ the functions $x \mapsto x^a t^k x$, defined on $\mathbb{R}$ by

$$x^a t^k x := \begin{cases} x^a \ln x^k & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad x^a t^k x := \begin{cases} |x|^a \ln |x|^k & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases},$$

are locally integrable, and hence they can be extended as tempered distributions on $\mathbb{R}$ for any $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. We can build $x^a t^k x$ from differentiating $x^a_+$ with respect to $a$. For $\Re a > -k$, the map $a \mapsto x^a_+$ is analytic hence we have

$$x^a t^k x = \partial^k_a x^a_+ = \partial_a^k \left( x^a_+ \right).$$

(21)
Remark 4. The extensions are unique by the uniqueness of the analytic continuation.

Proposition 12. ([H.I], Theorem 3.1.11.) Let $I$ be an open interval on $\mathbb{R}$ and let

$$Z = \{ z \in \mathbb{C} \mid \text{Re}(z) \in I, \ 0 < \text{Im} \ z < \gamma \}$$

be a one sided complex neighborhood. For an analytic function $f$ in $Z$ such that for a non-negative integer $N$,

$$|f(z)| \leq C (\text{Im} \ z)^{-N}, \ z \in Z,$$

then $f(\cdot + iy)$ has a limit $f_0 \in D'(N+1)(I)$ as $y \to 0$, that is,

$$\lim_{y \to 0^+} \int f(x + iy) \phi(x) \, dx = \langle f_0, \phi \rangle, \ \phi \in C_0^N(I).$$

The following equations which contain double signs in either side, are to be understood as double equations: one equation holding for the upper signs and the other holding for the lower signs. By Proposition 12, the function $z^a$, defined in $\mathbb{C} \setminus \mathbb{R}^-$ as $e^{a \ln z}$, where $z \in \mathbb{R}^+$, has distributional boundary values

$$(\xi \pm i0)^a := D^\ell \lim_{\eta \to 0^+} (\xi \pm i\eta)^a,$$

on the real axis from the upper and lower half planes. Now, for any test function $\phi$, the function $a \mapsto \langle (\xi \pm i0)^a, \phi \rangle$ is the limit of entire analytic functions, so it is an entire analytic function. Additionally, on the one hand, for $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we have

$$(\xi \pm i0)^a = \xi_+^a + e^{\pm a \pi i} \xi_-^a, \quad \xi(\xi \pm i0)^a = \xi_-^{a+1} + e^{\pm a \pi i} \xi_+^{a+1}. \quad (22)$$

Furthermore, if $a = -k$ where $k$ is a positive integer, we have

$$(\xi \pm i0)^{-k} = \xi_+^{-k} + (-1)^k \xi_-^{-k} + \frac{\pi i(-1)^k}{(k-1)!} \delta_{\xi}^{k-1}.$$

On the other hand, if we set $\xi^{-k} := ((\xi + i0)^{-k} + (\xi - i0)^{-k})/2$, we have

$$\xi^{-k} = \xi_+^{-k} + (-1)^k \xi_-^{-k}, \quad \xi \xi^{-k} = \xi^{1-k}.$$

Finally, for all $\phi \in C_0^1(\mathbb{R})$, we have $\xi^{-1}(\phi) = (P.V. \frac{1}{\xi})(\phi) := \lim_{\epsilon \to 0} \int_{|\xi| > \epsilon} \frac{\delta(\xi)}{\xi} \, d\xi,$

which is usually called the principal value of $\frac{1}{\xi}$.

The Fourier transform of log-homogeneous distribution: Now, we describe some results related with Fourier transform of distributions in (19) and (21), in some case we give proofs for such results since they are not easy to found in the literature.
The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ can be extended can be extended to an isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ given by $\langle \mathcal{F}[f], u \rangle := \langle f, \mathcal{F}[u] \rangle$ for $u \in \mathcal{S}(\mathbb{R})$. The following definition of the Laplace transform is due to L. Schwartz [Schw52]. For a compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^n)$, the Laplace-Fourier transform, by abuse of notation we also denote by $\mathcal{F}[u]$, is defined by a $C^\infty$-function given by

$$\xi \mapsto \mathcal{F}[u](\xi) := \langle u, e^{-ix\xi} \rangle,$$

here $\langle \cdot, \cdot \rangle$ is the dual pair. The Laplace-Fourier transform may be extended to tempered distribution $u$ in $\mathcal{S}'(\mathbb{R})$ whose support is bounded at the left, then in this case

$$\mathcal{F}[u](\zeta) := \langle u, e^{-ix\zeta} \rangle_{\mathcal{S}', \mathcal{S}},$$

is well defined on $\Re \zeta < 0$. Moreover, setting $\zeta = \xi + i\eta$ and fixed $\xi$, the Laplace-Fourier transform $\mathcal{F}[u](\zeta)$ coincides with the Fourier transform of the function $\hat{u}(\eta)$.

**Lemma 5.** ([H.I], Theorem 7.1.16.) If $u \in \mathcal{S}'(\mathbb{R})$ is a homogeneous distribution of degree $a$, then its Fourier transform $\mathcal{F}[u](\phi) := \langle u, \mathcal{F}[\phi] \rangle$ is a homogeneous distribution of degree $-a - 1$.

**Remark 5.** Let $\omega_1$ and $\omega_2$ be cut-off functions. We have that $\omega_1 - \omega_2$ has compact support contained in a ring. Thus, the Fourier transform $(\omega_1 - \omega_2)x_+^a$ is a smooth function on $\mathbb{R}$ and

$$\mathcal{F}[(\omega_1 - \omega_2)x_+^a] = O([\xi]^{-\infty}).$$

Similarly for the inverse Fourier transform.

Consequently, the subsequent statements of Proposition 13, Proposition 14 and Corollary 1 are independent of the chosen cut-off function $\omega$ modulo smooth function of order $O([\xi]^{-\infty})$.

**Proposition 13.** Let $\omega$ be a cut-off function and $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$.

1. The Fourier transform of $x_+^a$ is positively homogeneous of degree $-a - 1$ and

$$\mathcal{F}[x_+^a](\xi) = \frac{\Gamma(a+1)}{\sqrt{2\pi}} e^{\mp i\pi(a+1)/2} (\xi \mp i0)^{-a-1},$$

and from Equation (22) it follows that $\mathcal{F}[(x \pm i0)^a](\xi) = \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{\pm i\pi(a/2)} \xi_\pm^{-a-1}$.

2. The Fourier transform of $(1 - \omega)x_+^a$ is a smooth function on $\mathbb{R}$ and we have

$$\mathcal{F}[(1 - \omega)x_+^a](\xi) = O([\xi]^{-\infty}) \text{ as } |\xi| \to \infty.$$

3. Since $\mathcal{F}[x_+^a] = \mathcal{F}[\omega x_+^a] + \mathcal{F}[(1 - \omega)x_+^a]$ it follows that

$$\mathcal{F}[\omega x_+^a](\xi) = \frac{\Gamma(a+1)}{\sqrt{2\pi}} e^{\mp i\pi(a+1)/2} (\xi \mp i0)^{-a-1} + O([\xi]^{-\infty}).$$
4. Finally,
\[
\frac{d}{d\xi} \mathcal{F}[x^a_\pm](\xi) = -i \mathcal{F}[x(x^a_\pm)](\xi) = -i \mathcal{F}[x^{a+1}_\pm](\xi).
\]

Proof. 1. Observe that when \( \eta > 0 \) and \( \text{Re} \, a > -1 \) the Fourier transform of the rapidly decreasing function \( e^{-\eta x^a_+} \) is
\[
\xi \mapsto \Gamma(a + 1) \int_0^\infty x^a_+ e^{-x^a_+(\eta + i\xi)} \, dx
\]
where the last integral is taken on the ray generated by \( \eta + i\xi \) and \( z^a_+ \) is defined in \( \mathbb{C} \) slit along \( \mathbb{R}_- \) (so \( 1^a_+ = 1 \)). From the Cauchy integral formula it follows that the integral can be taken along \( \mathbb{R}_+ \). Therefore the Fourier transform reads
\[
\xi \mapsto \Gamma(a + 1)(\eta + i\xi)^{-a-1} = \Gamma(a + 1) e^{-\pi(a+1)/2(\xi - i\eta)^{-a-1}}.
\]
When \( \eta \to 0 \), the Fourier transform of \( x^a_+ \) has the form
\[
\langle \xi^k, \hat{\phi} \rangle = \Gamma(a + 1) e^{-\pi(a+1)/2(\xi - i0)^{-a-1}} \langle \phi \rangle, \quad \phi \in \mathcal{S}, \quad \text{Re}(a) > -1.
\]
Both sides are entire analytic functions of \( a \) so the identity extends to all \( a \in \mathbb{C} \). The second statement follows by the well known identity
\[
\Gamma(-a) \Gamma(a + 1) = \frac{\pi}{\sin(\pi a)}.
\]
2. Since \( \omega(x)x^a_+ \) is a compactly supported distribution, it follows from Lemma 9 that the Fourier transform of \( \omega(x)x^a_+ \) is a smooth function on \( \mathbb{R} \). This proves the first part of this item. For the second part, let \( \phi \in \mathcal{S}(\mathbb{R}) \).
\[
\langle \xi^k, \hat{\phi} \rangle = \langle \mathcal{F}[\omega(x)x^a_+], \xi^k \phi \rangle
\]
\[
= \langle (1 - \omega)x^a_+, \mathcal{F}[\xi^k \phi] \rangle
\]
\[
= \langle (1 - \omega)x^a_+, (-D^k)\mathcal{F}[\phi] \rangle
\]
\[
= \langle \mathcal{F}[D^k((1 - \omega)x^a_+)], \phi \rangle.
\]
If \( k < \text{Re} \, a + 1 \), then \( D^k((1 - \omega)x^a_+) \) is an integrable function and its Fourier transform is therefore well-defined and bounded. Thus we see that \( \mathcal{F}[((1 - \omega)x^a_+)](\xi) = O(|\xi|^{-k}) \) for each \( k \in \mathbb{N} \).

3. Item (3) in Proposition 13 is immediate from (1) and (2).

4. It follows from the definition of the derivative of a distribution.

Proposition 14. Let \( \omega \) be a cut-off function and \( a \in \mathbb{C} \).

1. The inverse Fourier transform of \( (\xi \pm i0)^a \) is
\[
\mathcal{F}^{-1}[(\xi \pm i0)^a] = \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{\pm i\pi(a/2)} x^a_\pm \cdot
\]
In particular, \( \mathcal{F}^{-1}[\mathcal{F}[x^a_\pm]] = x^a_\pm \).
2. The inverse Fourier transform of the tempered distribution $\omega(\xi \mp i0)^a$ is a smooth function and we have
\[ \mathcal{F}^{-1}[\omega(\xi \mp i0)^a](x) = O(|x|^{-\infty}). \]

3. Since $\mathcal{F}^{-1}[(\xi \mp i0)^a] = \mathcal{F}^{-1}[\omega(\xi \mp i0)^a] + \mathcal{F}^{-1}[(1 - \omega)(\xi \mp i0)^a]$ it follows
\[ \mathcal{F}^{-1}[(1 - \omega)(\xi \mp i0)^a](x) = \frac{\sqrt{2\pi}}{\Gamma(a + 1)} e^{\pm i\pi a/2} x^{a-1} + O(|x|^{-\infty}). \]

4. Finally,
\[ \frac{d}{dx} \mathcal{F}^{-1}[(\xi \mp i0)^a](x) = -i \mathcal{F}^{-1}[(\xi \mp i0)^{a+1}](x). \]

**Proof.** The first statement follows from the first item in the above Proposition by the Fourier transform inversion formula $\mathcal{F}^{-1} = \mathcal{F}^3$:
\[
\mathcal{F}^2[(\xi \pm i0)^a] = \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{\pm i\pi a/2} \mathcal{F}[x^{-a-1}] = \frac{\sqrt{2\pi}}{\Gamma(-a)} \frac{\Gamma(-a)}{\sqrt{2\pi}} \mathcal{F}[(\xi \pm i0)^a] = e^{\pm i\pi a} (\xi \mp i0)^a.
\]
Then $\mathcal{F}^3[(\xi \pm i0)^a] = e^{\pm i\pi a} \mathcal{F}[(\xi \mp i0)^a] = \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{\pm i\pi a/2} x^{a-1}$. The others statement follow in a similar way. \(\square\)

As a consequence of Proposition 13 and 21 we have the following result.

**Corollary 1.** Let $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $k \in \mathbb{N}$ and let $\omega$ be a cut-off function.

1. There exists some $k \in \mathbb{Z}$ such that $\text{Re} a + k > -1$ and
\[ \mathcal{F}[x^a \ln^l x^\pm](\xi) = i^k \xi^k \partial_a^l \left( \frac{\mathcal{F}[x^a \ln^l x^\pm](\xi)}{(a + 1) \cdots (a + k)} \right) \] (25)

is independent of the choice of integer $k$. In particular, $\mathcal{F}[x^a \ln^k x^\pm](\xi)$ is log-polyhomogeneous of homogeneity degree $-a - 1$ and logarithmic degree $l$, i.e. for any $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $l \in \mathbb{N}$, there exist constants $c_{ij}^+, c_{ij}^-$, $j = 0, 1, \ldots, l$ (depending of $a, l$) such that
\[ \mathcal{F}[x^a \ln^l x^\pm](\xi) = \sum_{j=0}^l c_{ij}^+ \xi^{-a-1} \ln^{l-j} \xi + \sum_{j=0}^l e^{\pi(-a-1)\pi i} c_{ij}^- \xi^{-a-1} \ln^{l-j} \xi. \]

In particular for $l = 0$, we have
\[ \mathcal{F}[x^a_+](\xi) = c^+ \xi^{-a-1} + e^{i(a+1)\pi} c^- \xi^{-a-1}. \]
2. The Fourier transform of \((1 - \omega) x^a_\pm \ln^l x\) is a smooth function on \(\mathbb{R}\) and as \(|\xi| \to \infty\)
\[ \mathcal{F}[(1 - \omega) x^a_\pm \ln^l x_\pm](\xi) = O(|\xi|^{-\infty}). \]

3. Finally, there exist constants \(c^+_j, c^-_j, j = 0, 1, \cdots l\) (depending on \(a, l\)) such that
\[ \mathcal{F}[\omega x^a_\pm \ln^l x_\pm](\xi) = \sum_{j=0}^l c^+_j \xi^{a-1} \ln^{l-j} \xi + \sum_{j=0}^l c^-_j \xi^{-a-1} \ln^{l-j} \xi + O(|\xi|^{-\infty}). \]

**Proof.** 1. For \(\text{Re } a > -k\), it follows that
\[ \mathcal{F}[x^a_\pm \ln^l x](\xi) = \partial^k_a \left( \mathcal{F}[\partial^l_a x^{a+k}_\pm](\xi) \right) = i^k \xi^k \partial^l_a \left( \mathcal{F}[x^{a+k}_\pm](\xi) \right). \]
The independence of the choice of \(k\) comes from the independence of \(k\) in the definition \(x^a_\pm\) in (15). It follows from Proposition 13 that \(\mathcal{F}[x^a_\pm]\) is positively homogeneous of degree \(-a - 1\). Thus, \(\mathcal{F}[x^a_\pm \ln^l x_\pm](\xi)\) is log-polyhomogeneous of homogeneity degree \(-(a + k) - 1 + k = -a - 1\) and logarithmic degree \(l\). More precisely, by a straightforward computation we obtain
\[
\mathcal{F}[x^a_\pm \ln^l x] = i^k \xi^k \partial^l_a \left( \frac{\mathcal{F}[x^{a+k}_\pm](\xi)}{(a + 1) \cdots (a + k)} \right) = (2\pi)^{l/2} i^k \xi^k \partial^l_a \left( \frac{\Gamma(a + k + 1) e^{-i \pi(a+k+1)/2}}{(a + 1) \cdots (a + k)} \right) \]
\[
= \frac{(2\pi)^{l/2} i^k \Gamma(a + k + 1) e^{-i \pi(a+k+1)/2}}{(a + 1) \cdots (a + k)} \xi^{-a-1} \times \left( -\sum_{j=1}^k \frac{1}{a + j} \frac{\Gamma'(a + k + 1)}{\Gamma(a + k + 1)} + \frac{i \pi}{2} + \ln(\xi - i 0) \right). \]
The assertion comes from \((\xi \pm i 0)^a = \xi^a_e e^{\pm i \pi a} e^{i \pi a} \), see (22). The general assertion for \(\mathcal{F}[x^a_\pm \ln^l x]\) follows from \(\mathcal{F}[x^a_\pm \ln^l x] = \partial^{-1}_a \mathcal{F}[x^a_\pm \ln x]\).

2. Follows from item 2. in Proposition 13 and equation (25).

3. The assertion follows from item 1. and 2. in Corollary 11 and
\[ x^a_\pm \ln^l x = \omega x^a_\pm \ln^l x + (1 - \omega)x^a_\pm \ln^l x. \]
4.2 A Paley-Wiener theorem for functions with log\nobreakdash-polyhomogeneous asymptotic behavior at zero and positive support

We establish a Paley-Wiener type theorem, for the Fourier transform, of functions with log\nobreakdash-polyhomogeneous asymptotic behavior at zero (cf. Definition 2).

Notice that, for \( u \in L^2(\mathbb{R}_\pm) \), the extension by zero of \( e^\pm u \) lies in \( L^2(\mathbb{R}) \). Similarly, we can extend the operators \( e^\pm \) to functions \( u \in S_p(\mathbb{R}_+) \) for any appropriate power set \( p \). The following Proposition is due to C. Neira, E. Schrohe and S. Paycha.

**Proposition 15.** Let \( p \in \mathcal{P} \) be an appropriate power set. Any \( u \) in \( S_p(\mathbb{R}_+) \) admits an extension \( e^+_u \) in \( S'(\mathbb{R}_+) \), where \( S'(\mathbb{R}_+) \) denotes the set of tempered distributions on \( \mathbb{R} \) with support in \( \mathbb{R}_+ \), and we have, as \( x \to 0^+ \),

\[
e^+_u(x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x.
\]  

**Proof.** Given a cut-off function \( \omega \), there is a sequence \( \{a_{jk}\} \) of complex numbers such that \( \exists N \in \mathbb{N}, \omega(x)(u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_{jk} x^{p_j} \ln^k x) = O(x^{\text{Re} N + 1}) \).

may be extended by zero for \( N \) large enough. Moreover, equation (18) yields the extension \( x^a \ln x \) of \( x^a \ln x \) to \( -x \) by zero. Finally, we obtain

\[
e^+_u(x) = \omega(x) \left( \sum_{j=0}^{N} c_j x^{d+j} + O(x^{\text{Re} N + 1}) \right) + (1 - \omega)(x)u(x).
\]

\( \square \)

Let \( p \) be an appropriate power sets. For \( u \in S_p(\mathbb{R}_+) \), \( e^+_u \) defines a functions on \( \mathbb{R} \) with support in \( \mathbb{R}_+ \). Applying Proposition 15 we can set the following definition.

**Definition 5.** Let \( p \in \mathcal{P} \) be an appropriate power sets and let

\[
\mathcal{H}_p^\pm = \{ \mathcal{F}[e^+_u(\xi \mp i0)] \mid u \in S_p(\mathbb{R}_\pm) \}.
\]

Let us further set \( \mathcal{H}_p = \mathcal{H}_p^+ \oplus \mathcal{H}_p^- \).

**Lemma 6.** Taking Fourier transform in Lemma 1 it follows that for any appropriate power sets \( p = \{(p_j, m_j)\} \in \mathcal{P} \), we have

\[
\mathcal{H}_p^\pm = \mathcal{H}_{p_0}^\pm \subset \mathcal{H}_p^\pm \subset \mathcal{H}^\pm.
\]
where \( p_0 = \{(j, 0) \mid j = 0, 1, \cdots \} \). Moreover, we have
\[
\mathcal{H}_p^\pm \subset C^\infty_p(\mathbb{R})
\]
where \( p' \in \mathbb{N} \) is a convenient power sets, which contains \( \{(-p_j - 1, m_j - i) \mid i, j = 0, 1, \cdots \} \) as subset.

Proof. It just remain to show \( u \in \mathcal{S}_p(\mathbb{R}_+^+) \implies \mathcal{F}[e_\pm u] \in C^\infty_p(\mathbb{R}) \) for some power sets \( p' \) containing \( \{(-p_j - 1, m_j - i) \mid i, j \in \mathbb{N} \} \) as subset. Let \( p \) be an appropriate power sets and let \( \omega \) be a cut-off function and \( u \in \mathcal{S}_p(\mathbb{R}_+) \). There exists \( N \in \mathbb{N} \) such that
\[
u(x) = \omega \left( \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_{j,k} x_p^j \ln^k |x| + (1 - \omega)u(x) + O(Re_p^{N+1}) \right).
\]
Therefore, the Fourier transform of \( u \) is
\[
\mathcal{F}[e_\pm u](\xi) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_{j,k} \mathcal{F}[\omega x_p^j \ln^k |x|] + \mathcal{F}[(1 - \omega)u(x)] + \mathcal{F}[O(Re_p^{N+1})].
\]
The conclusion follows for combining the following facts:

1. The term \( \mathcal{F}[(1 - \omega)u] \) is a continuous and rapidly decreasing function since \( (1 - \omega)u \) is a Schwartz function.
2. \( \mathcal{F}[x_p^j \ln^k |x|] = \mathcal{F}[e_\pm x_p^j \ln^k |x|] \) is well-defined since \( p_j \notin \mathbb{Z}_{<0} \ j \in \mathbb{N} \).
3. From Corollary \[1\] we have
\[
\mathcal{F}[\omega x_p^j \ln^k |x|](\xi) = \sum_{j=0}^{k} c_j^+ \xi^{-p_j - 1} \ln^{k-j} |\xi| + \sum_{j=0}^{k} c_j^- \xi^{-p_j - 1} \ln^{k-j} |\xi| + O(|\xi|^{-\infty}). \quad (27)
\]
4. Finally, the Fourier transform of the term \( O(Re_p^{N+1}) \) is a function of the form \( O(|\xi|^{-Re_p^{N+1} - 1}) \).

From remark \[3\] \( \mathcal{F}[\omega x_p^j \ln^k |x|] \) is actually independent of the choice of the cut-off function \( \omega \) modulo continuous rapidly decreasing functions. \( \square \)

**Example 2.** Let \( d \in \mathbb{C} \setminus \mathbb{Z}_{<0} \). If \( u(x) \) denotes the function defined as \( x^d e^{-x} \) if \( x > 0 \) and 0 if \( x \leq 0 \) we have \( u \in \mathcal{S}_p(\mathbb{R}_+) \) for \( p = \{(d + j, 0) \mid j \in \mathbb{N} \} \), more precisely,
\[
u(x) \sim \sum_{j=0}^{\infty} (-1)^j x^{d+j} \text{ as } x \to 0^+.\]

Hence there exists a sequence a complex number \( a_0, a_1, \cdots \), such that
\[
\mathcal{F}[u](\xi) \sim \sum_{j=0}^{\infty} a_j (\xi - i0)^{-d-j-1} \text{ as } |\xi| \to \infty.
\]
Proposition 16. Any function in $\mathcal{H}_p^\pm$ for an appropriate power set $p \in \mathbb{P}$, can be represented by an analytic function on $\mathbb{C}_+$ that extends continuously to $\mathbb{C}_+$ in $\mathcal{S}'(\mathbb{R})$.

Proof. Let $p$ be an appropriate power sets and $\omega$ be a cut-off function and $u \in S_p(\mathbb{R}_\pm)$. There exists $N \in \mathbb{N}$ such that

$$\mathcal{F}[e_\pm u](\xi) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} a_j \mathcal{F}[\omega x_j^{p_j} \ln^k x] + \mathcal{F}[O(x^{\Re p_{N+1}})] + \mathcal{F}[(1 - \omega) u(x)]. \quad (28)$$

We want to prove that each term in (28) can be represented by an analytic function on $\mathbb{C}_+$. We first note that from Remark 5 it follows that the assertion is independent (modulo analytic function) of the cut-off function used in the proof. Next, we see that the proof follows from the following observations:

1. The assertion is valid for term $\mathcal{F}[\omega x_j^{p_j} \ln^k x]$ since $e_\pm(x^{p_j} \ln x) = x_j^{p_j} \ln x \in \mathcal{S}'(\mathbb{R})$ has support in $[0, \infty)$ (resp.$(-\infty, 0]$) and Equation (24).

2. The assertion for the second and third terms follows from Proposition 17 and the fact that the Fourier transform of $(1 - \omega)(x)u(x)$ has order $O(|\xi|^{-\infty})$, respectively.

$$\square$$

We are now ready to characterize functions in $\mathcal{H}_p^\pm$, for appropriate power sets $p \in \mathbb{P}$. Recall $\mathcal{H}_\pm = \mathcal{H}_p^\pm \subset \mathcal{H}_p^\pm \subset \mathcal{H}_p^{\pm, \pm}$, where $p_0 = \{(j, 0) \mid j = 0, 1, \ldots \}.$

Theorem 2. Let $p \in \mathbb{P}$ be an appropriate power set. A function $h(\xi)$ lies in $\mathcal{H}_p^+$ (resp. $\mathcal{H}_p^-$) if and only if

- it can be represented by an analytic function $h$ on $\mathbb{C}_-$ (resp. on $\mathbb{C}_+$) that extends in $\mathcal{S}'(\mathbb{R})$ continuously to $\mathbb{C}_-$ (resp. $\mathcal{C}_+$) by $h(\xi - i0)$ (resp. $h(\xi + i0)$),

- $h$ has the following growth at infinity $\forall \zeta = \xi + i\eta \in \mathbb{C}_+, \exists m \in \mathbb{N}, \exists C > 0$ such that

$$|h(\zeta)| \leq C_n(\zeta)^m$$

Moreover, if $p$ does not contain logarithmic indices, i.e. $p$ has the form $\{(p_j, 0) \mid j \in \mathbb{N}\}$, then $h(\xi - i0)$ has an asymptotic expansion as $|\xi| \to \infty$

$$h(\xi - i0) \sim \sum_{j=0}^{\infty} c_j(\xi - i0)^{-p_j - 1} = \sum_{i=0}^{\infty} c_j^+ \xi_+^{-p_j - 1} + c_j^- \xi_+^{-p_j - 1}, \quad (29)$$

where $c_j$ are described in (20), $c_j^- = e^{i(p_j+1)}c_j^+$ for $j = 0, 1, \ldots$, and

$$h(\xi + i0) \sim \sum_{j=0}^{\infty} c_j(\xi + i0)^{-p_j - 1} = \sum_{i=0}^{\infty} c_j^+ \xi_+^{-p_j - 1} + c_j^- \xi_+^{-p_j - 1},$$
with \( c_j^- = e^{-i(p_j+1)\pi} c_j^+ \) for \( j = 0, 1, \ldots \). More generally, allowing logarithmic powers \( p = \{(p_j, m_j)\} \), the asymptotic expansion of \( h(\xi \mp i0) \) reads

\[
h(\xi \mp i0) \sim \sum_{i,j=0}^{\infty} \sum_{k=0}^{m_j} a_{ijk}(\xi \mp i0)^{-p_j-1} \ln^{k-i}(\xi \mp i0)
= \sum_{i,j=0}^{\infty} \sum_{k=0}^{m_j} a_{ijk}^+ \xi_+^{-p_j-1} \ln^{k-i}(\xi \mp i0) + b_{ijk}^+ \xi_-^{-p_j-1} \ln^{k-i}(\xi \mp i0)
\]

for some complex numbers \( b_{ijk}^+ \) and \( b_{ijk}^- \) for \( i, j = 0, 1, \ldots \) and \( k = 0, 1, \ldots m_j \).

**Proof.** For \( u \in \mathcal{S}_p(\mathbb{R}_+) \) set \( H = \mathcal{F}[e^\pm u] \in \mathcal{H}_p^\pm \). We know from (24) \( H \) can be represented as an analytic function \( h \) on \( \mathbb{C}_+ \), and that extends continuously to \( \mathbb{C}_i \). By the maximum principle it follows that \( |h(\zeta)| \leq C_u|\zeta|^m \) for \( |\zeta| > 1 \) and \( \text{Im}\zeta \geq 0 \). First, let us consider asymptotic expansion of \( u \) with not log terms. Now, to prove that \( H \) has an asymptotic expansion at infinity (29), let \( u \sim \sum a_j x_j^{p_j} \in \mathcal{S}_p(\mathbb{R}_+) \), i.e. Let \( \omega \) be a cut-off function and \( u \in \mathcal{S}_p(\mathbb{R}_+) \). There exists \( N \in \mathbb{N} \) such that

\[
u(x) = \omega(\sum_{j=0}^{N} \sum_{k=0}^{m_j} a_j x^{p_j} + O(x^{\text{Re}pN+1})) + (1-\omega)u(x).
\]

It follows from Proposition 13 and \( (\xi \pm i0)^a = \xi_+^a + e^{\pm \alpha \pi i} \xi_-^a \) that

\[
\mathcal{F}[e^\pm u](\xi) = \sum_{j=0}^{N} a_j \mathcal{F}[\omega x_j^{p_j}] + \mathcal{F}[O(x^{\text{Re}pN+1})] + \mathcal{F}[(1-\omega)u(x)]
\]

\[
= \sum_{j=0}^{N} a_j \frac{\Gamma(p_j+1)}{\sqrt{2\pi}} e^{-i\pi(p_j+1)/2} (\xi - i0)^{-p_j-1}
+ O(x^{-\text{Re}pN+1}) + O(|\xi|^{-\infty})
\]

\[
\sim \sum_{j=0}^{N} a_j \frac{\Gamma(p_j+1)}{\sqrt{2\pi}} e^{-i\pi(p_j+1)/2} (\xi_+^{-p_j-1} + e^{i(p_j+1)\pi} \xi_-^{-p_j-1}).
\]

Let us prove the converse assertion. Let \( h \) be an analytic function which satisfies the enumerated properties in Theorem 2. From Proposition 14 we have \( H \) lies in \( \mathcal{H}_p^\pm \) for some \( p \in \mathbb{R}_p \). Now, we consider the following asymptotic expansion

\[
u(x) = \omega(\sum_{j=0}^{N} \sum_{k=0}^{m_j} a_j x^{p_j} \ln^{k} x + O(x^{\text{Re}pN+1})) + (1-\omega)u(x).
\]

The corresponding assertion for logarithmic terms follows from (27). \( \square \)

Now, from Theorem 2 we deduce well-known Paley-Wiener type theorems concerning to the Fourier transform of rapidly decreasing smooth function with
The range $H^+$ of semi-norms given by $\sup_{x \in \mathbb{R}^+} |x^k \partial^m u(x)|$. If we set

$$H^+ := H^+_\mathbb{R} = \{ F[e_{\pm} u](\xi) \mid u \in S(\mathbb{R}^+) \}.$$ 

The range $H^+$ equipped with the image topology of $S(\mathbb{R}^+)$, i.e. the countable family of semi-norms given by

$$\| h_+ \|_{m,k} = \| \xi^k \partial^m h_+(\xi) \|_{L^2}, \text{ for } m, k \in \mathbb{N},$$

is a Fréchet space, and we have the following result.

**Lemma 7.** The map $F : e_+ S(\mathbb{R}^+) \to H^+$ is an isomorphism of Fréchet spaces. We denote by $F^{-1}$ its inverse.

**Proof.** The bijectivity follows from bijectivity of the Fourier transform acting over the space $L^2$. To see that $F$ is continuous it suffices to show that $h_n(\xi) \to h(\xi)$ as $n \to \infty$ in the Fréchet topology of $H^+$ with $h_n(\xi) := F[e_{+} u_n](\xi)$, and $h(\xi) := F[e_{+} u](\xi)$ where $(u_n)$ is a sequence in $S(\mathbb{R}^+)$ which converges to $u$ in the Fréchet topology of $S(\mathbb{R}^+)$. We have, for any $m, k \in \mathbb{N}$, that as $n \to \infty$,

$$\| h_n - h \|_{m,k} = \| \xi^m \partial^m (F[u_n] - F[u](\xi)) \|_{L^2} = \| \partial^m_x (x^k (u_n(x) - u(x))) \|_{L^2} \to 0.$$

\[ \square \]

As a consequence of the dominated convergence theorem, we have that for any integrable function $f(s, x)$ defined for $(s, x) \in I \times \mathbb{R}$, where $I$ is an open interval of $\mathbb{R}$, and there exists $\partial f / \partial s$ and $g$ integrable such that $|\partial f(s, x) / \partial x| \leq g(x)$. Then $F(s) = \int f(s, x)dx$ is differentiable and $dF/ds = \int df/\partial sdx$.

**Lemma 8.** If $u \in S(\mathbb{R}^+)$ (resp. $u \in S(\mathbb{R}_-)$) then $F[e_{\pm} u]$ is a smooth function which can be represented by an entire function $h(\zeta)$ in $\mathbb{C}_-$ (resp. $\mathbb{C}_+$) which extends continuously to $\mathbb{C}_-$ (resp. $\mathbb{C}_+$). By abuse of notation we shall set $h(\zeta) = F[e_{\pm} u](\zeta)$.

**Proof.** Let $u \in S(\mathbb{R}_+)$. For $\zeta = \xi - i\eta$ with $\xi \in \mathbb{R}$, $\eta < 0$. The integral (cf. (221))

$$h(\zeta) = \int_{-\infty}^{\infty} e^{ix(\xi - i\eta)} u(x)dx = \int_{-\infty}^{\infty} e^{ix\xi} (e^{-i\eta} u(x)) dx$$

converges absolutely. Since $u$ is a rapidly decreasing function, it follows $x \to xu(x)$ is $L^1$-integrable, and using the above comment we can differentiate in the integral. The function $h$ is therefore an analytic function as a consequence of the Cauchy-Riemann equations. \[ \square \]

**Proposition 17.** The space $H^\pm$ consists precisely of smooth functions $h$ defined on $\mathbb{R}$ which
(i) can be represented by an analytic function $h$ on $\mathbb{C}_\pm$ that extends continuously to $\mathbb{C}_\pm$,

(ii) and such $h$ has the following asymptotic expansion at infinity

$$h(\zeta) \sim \sum_{k=0}^{\infty} a_k \zeta^{-k-1} \text{ for } |\zeta| \to \infty, \ z \in \mathbb{C}_\pm,$$

(31)

which can be differentiated formally.

Moreover, setting $H := H^+ \oplus H^- \oplus \mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials, we have

(iii) The space $H^\pm$ and $H$ are algebras with respect to the product of complex valued functions.

(iv) $H$ consists precisely of all functions $h \in C^\infty(\mathbb{R})$ which have an expansion

$$h \sim \sum_{k=0}^{\infty} c_k \xi^{-k-1} \text{ for } |\xi| \to \infty \text{ in } \mathbb{R},$$

which can be differentiated formally.

Proof. The first part follows as a consequence of Proposition 8. The second part is proved as follows, for $u \in S(\mathbb{R}^+)$, integration by parts we get

$$\int e^{-ix\zeta} u(x) \, dx \sim \sum_{j=0}^{\infty} u^{(j)}(0) / (i\zeta)^{j+1},$$

and it can be extended on $\mathbb{C}_-$ setting $\zeta = \xi + i\eta$. For the last part concerning the symbolic property of $h(\xi \mp i0)$, let $k \in \mathbb{N}$ and $h = \mathcal{F}[e_x u]$. Then $x \to x^k$ also lies in $S(\mathbb{R}^+)$ and we have

$$(-i)^k \mathcal{F}[e_x x^k u](\xi) = \partial_{\xi}^k \mathcal{F}[e_x u](\xi) = \partial_{\xi}^k h(\xi).$$

Hence

$$|\mathcal{F}[e_x x^k u]| = |\partial_{\xi}^k h(\xi)| \leq C_0 (\xi)^{-1-k} \forall \eta \in \mathbb{R},$$

so that $h$ defines a symbol of order at most $-1$. The item (iii) is obvious. It follows immediately from Theorem 2 item (iv). \hfill \square

4.3 A Paley-Wiener theorem for (tempered) distributions with positive support

The previous results extend to a similar results for tempered distributions. Before to describe these, let us first give a Paley-Wiener results concerning to compact supported distribution in $\mathbb{R}$, see Equation (23).

Lemma 9. (Rudin, Theorem 7.2. and 7.23) For any compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ its (Laplace-)Fourier transform $\mathcal{F}[u]$ can be extended to an analytical function with $\zeta = \xi + i\eta$ in the complex plane $\mathbb{C}$, cf. Equation (23).

By abuse of notation we denote by $\mathcal{F}[u](\zeta)$ the analytical extension. Moreover, there exists constants $C, M > 0$ and an $n_0$ such that, for every $\zeta \in \mathbb{C}^n$,

$$|\mathcal{F}[u](\zeta)| \leq C(\zeta)^{n_0} e^{M|\text{Im } \zeta|}.$$  

(32)

Conversely, any entire function $h$ satisfying $\mathcal{F}[u](\zeta)$ in $\mathbb{C}$, there exists $u \in \mathcal{E}'(\mathbb{R})$ such that $\mathcal{F}[u](\zeta) = h(\zeta)$ for every $\zeta \in \mathbb{C}$. 

\hfill \square
Let us set
\[ \mathcal{H}^{\prime, \pm} = \{ F[U](\xi) \mid U \in \mathcal{S}'(\mathbb{R}_\pm) \}, \]
where \( \mathcal{S}'(\mathbb{R}_\pm) \) stands for the set of tempered distributions with positive/negative support. Set
\[ \mathcal{H}' = \mathcal{H}^{\prime, +} \oplus \mathcal{H}^{\prime, -}. \]

Before we state the following Theorem, let us recall a well-known fact [RS], which can give the flavor of the following result: A function \( H \in L^2(\mathbb{R}_\xi) \) is a Fourier transform of a function \( u \) in \( L^2(\mathbb{R}_+) \) if and only if \( H \) can be represented by an analytic function \( h(\zeta) \) on \( \mathbb{C}_\varepsilon \) that extends continuously to \( \mathbb{C}_\mp \) by \( h(\xi + i0) \) in \( L^2 \), i.e.
\[ h(\xi \pm i\eta) \rightarrow H_\xi \text{ in } L^2(\mathbb{R}), \text{ as } \eta \rightarrow 0^+. \]
Moreover, by imposing the condition that \( h \) satisfies the estimative \(|h(\zeta)| \leq C e^{|\zeta|}\) for some \( C, a > 0 \), we have \( \text{supp } u \subset [-a, a] \). For tempered distributions in \( \mathcal{H}' \), we find:

**Proposition 18.** ([SZ], Theorem 1.) A tempered distribution \( H \) on \( \mathbb{R} \) lies in \( \mathcal{H}' \) if and only if

1. \( h \) can be represented by an analytic function \( h \) on \( \mathbb{C}_\varepsilon \), that extends continuously to \( \mathbb{C}_\mp \) by \( h(\xi \mp i0) \) in \( \mathcal{S}'(\mathbb{R}) \),
2. and \( h \) has the following growth at infinity \( \forall \zeta = \xi + i\eta \in \mathbb{C}_\varepsilon, \exists m, n \in \mathbb{N}, \exists C_n > 0, \) such that
\[ |h(\zeta)| \leq C_n (|\zeta|^m h_n(\eta), \quad (33) \]
where \( h_n(\eta) \) is the map equal to one for \(|\eta| > 1 \) and \(|\eta|^{-n} \) for \(|\eta| \leq 1 \).

In this case, the analytic representation \( h \) of \( H \) is given by \( h = F[U] \) the Laplace-Fourier transform.

From Proposition [RS] one can equip a product to the space \( \mathcal{H}'^{\prime, +} \) with an algebraic structure, see [SZ].

**Definition 6.** Let \( U \) and \( V \) be distributions in \( \mathcal{S}'(\mathbb{R}) \) which can be extended to analytic functions \( u \) and \( v \) defined on \( \mathbb{C}_\varepsilon \) (cf. Proposition [18]). We set
\[ (UV)(\xi) := \lim_{\eta \rightarrow 0^\mp} u(\xi + i\eta)v(\xi + i\eta) \text{ in } \mathcal{S}'(\mathbb{R}). \]

**Example 3.** For \( m \in \mathbb{N} \) we denote by \( H^m(x) \) the function \( x^m \), if \( x \geq 0 \) and 0, if \( x < 0 \). Its Fourier transform
\[ \mathcal{F}[H^m](\xi) = m!( -i )^m \xi^{-m-1} - i^m \pi \delta^{(m)}, \]
extends analytically to \( h^m(\zeta) = ( -i )^{k+1} k! \zeta^{-k-1} \) exists by Proposition [18]. For \( m, n \in \mathbb{N} \) we have
\[ (\mathcal{F}[H^m]\mathcal{F}[H^n])(\xi) = \lim_{\eta \rightarrow 0^-} h^m(\xi + i\eta)h^n(\xi + i\eta) = m!n!(-i)^{m+n+1}( -i \xi^{-m-n-2} - \pi \delta^{(m+n)}). \]

\(^1 h(\xi \mp i0) \) denotes the limit \( h(\xi + i\eta) \rightarrow H_\xi \) in \( \mathcal{S}'(\mathbb{R}) \), as \( \eta \rightarrow 0^\mp \).
Finally, from Proposition 12 and Proposition 18 we obtain a characterization of elements in $\mathcal{H'}$.

**Corollary 2.** A tempered distribution $H$ on $\mathbb{R}$ lies in $\mathcal{H'}$ if and only if it can be represented by an analytic function $h$ on $\mathbb{C}_+ \cup \mathbb{C}_-$ and it defines a distribution in $\mathcal{S}'(\mathbb{C})$ given by

$$H(\varphi) = \int \int h(\xi + i\eta)\varphi(\xi, \eta)d\xi d\eta, \, \varphi \in \mathcal{S}(\mathbb{R}^2),$$

such that

$$\langle \frac{\partial H}{\partial \zeta}, \varphi \rangle = i\frac{1}{2} (h(\cdot + i0) - h(\cdot - i0), \varphi(\cdot, 0)), \, \varphi \in \mathcal{S}(\mathbb{R}^2),$$

and we have $H = h$ in $\mathcal{S}(\mathbb{C} \setminus \mathbb{R})$.

**References**

[BdM] Boutet de Monvel, L. *Boundary problems for pseudo-differential operators*. Acta Math. **126**, Number 1, pp. 11–51, 1971.

[CdC] Del Corral, C. *Canonical Trace and Pseudo-differential Operators on Manifolds with Boudary*. Ph.D. Thesis, Universidad de los Andes, 2016.

[Esk] Éskin, G. *Boundary values problems for Elliptic pseudodifferential Equations*. Translations of Mathematical Monographs. American Mathematical Society. Vol. 52, 1980. Translated from the Russian by S. Smith.

[FGLS] Fedosov, B.V., Golse, F., Leichtnam, E. and Schrohe, E. *The Non-commutative Residue for Manifolds with Boundary*. Journal of Functional Analysis, **142**, pp. 1–31, 1996.

[FGD] Flajolet, Ph., Gourdon, X. and Dumas, Ph. *Mellin transforms and asymptotics: Harmonic sums*. Theoretical Computer Science **144**, pp. 3–58, 1995.

[FG] Franssens, G.R. *One-dimensional associated homogeneous distributions*. Mathematical Methods in the Applied Sciences; Published in Wiley Online Library, 2012.

[G05] Grubb, G. *On the logarithm component in trace defect formulas*. Communications in Partial Differential Equations, **30**, pp. 1671–1716, 2005.

[H.I] Hörmander, L. *The analysis of linear partial differential operators I*. Springer-Verlag, Berlin, Heidelberg, 1983.

[KV] Kontsevich, M. and Vishik, S. *Geometry of determinants of elliptic operators*. Functional Analysis on the Eve of the 21st Century (Rutgers Conference in honor of I. M. Gelfand 1993). Vol. I, edited by S. Gindikin et al. Progr. Math. **131**, Birkhauser, Boston, pp. 173–197, 1995.
[Lesch99] Lesch, M. *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols*. Annals of Global Analysis and Geometry 17, no. 2, pp. 151-187, 1999.

[PW] Paley, R.; Wiener, N. *Fourier Transforms in the Complex Domain*. Reprint of the 1934 original, American Mathematical Society Colloquium Publications, Am. Math. Soc. Providence, RI, 1987.

[P79] Pierre, J. *Transformation de Mellin et Développements Asymptotiques*. L’Enseignement Mathématique 25, pp. 285–308, 1979.

[RS81] Rempel, S. and Schulze, B-W. *Parametrices and boundary symbolic Calculus for elliptic boundary problems without the transmission property*. Math. Nachr. 105, pp.45-149, 1981.

[RS] Rempel, S. and Schulze, B.-W. *Index theory of elliptic boundary value problems*. Springer Verlag, 1982.

[RS82] Rempel, S. and Schulze. B.-W. *Complex powers for pseudo-differential problems, I*. Math. Nachr. 111, pp.41-109, 1982.

[Rud] Rudin, W. *Functional Analysis*. Tata McGraw-Hill, New delhi, 1974.

[Schw52] Schwartz, L. *Transformation de Laplace des distributions*. Comm. Sém. Math. Univ. Lund , pp. 196–206, 1952.

[Schw66] Schwartz, L. *Theorie des distributions*. Hermann, Paris, 1966.

[SB] Simon, B. *Basic Complex Analysis, A Comprehensive Course in Analysis, Part 2A*. American Mathematical Society, 2015.

[SZ] Shambayati, R. and Zielezny, Z. *On Fourier transforms of distributions with one-sided bounded support*. Proc. Amer. Math. Soc., Volume 88, Number 2, pp. 237–243, 1983.

[Se] Seeley, R.T. *Extension of C^\infty functions defined in a half space*. Journal: Proc. Amer. Math. Soc. 15, pp. 625–626. MSC: Primary 46.38, 1964.

[vG] Von Grudzinski, O. *Quasihomogeneous Distributions*. North-Holland Mathematics studies (Continuation of the Notas de Matematica); 165, 2004.