Radiative and leptonic decays of the pseudoscalar charmonium state $\eta_c$

Mao-Zhi Yang

Department of Physics, Nankai University, Tianjin, 300071, China and Kavli Institute for Theoretical Physics China, CAS, Beijing, 100190, China

The radiative and leptonic decays of $\eta_c \to \gamma\gamma$ and $\eta_c \to l^+l^-$ are studied. For $\eta_c \to \gamma\gamma$ decay, the second-order electromagnetic tree-level diagram gives the leading contribution. The decay rate of $\eta_c \to \gamma\gamma$ is calculated, the prediction is in good agreement with the experimental data. For $\eta_c \to l^+l^-$, both the tree and loop diagrams are calculated. The analysis shows that the loop contribution dominates, the contribution of tree diagram with $Z^0$ intermediate state can only modifies the decay rate by less than 1%. The prediction of the branching ratios of $\eta_c \to e^+e^-$ and $\mu^+\mu^-$ are very tiny within the standard model. The smallness of these predictions within the standard model makes the leptonic decays of $\eta_c$ sensitive to physics beyond the standard model. Measurement of the leptonic decay may give information of new physics.

PACS numbers: 13.20.Jf

The radiative and leptonic decays of $\eta_c \to \gamma\gamma$ and $\eta_c \to l^+l^-$ involve electromagnetic and weak interactions. The decay rates are determined not only by electroweak interaction, but also by strong interaction, which binds the quark-antiquark pair $c\bar{c}$ in $\eta_c$ together. The decay amplitudes of these decay processes are generally convolutions of the wave function of $\eta_c$ and the electroweak transition amplitude of $c\bar{c} \to \gamma\gamma$ and/or $c\bar{c} \to l^+l^-$. The decay process of $\eta_c \to \gamma\gamma$ can be used to test the decay constant and the wave function of $c\bar{c}$ in this meson state. The decay rate of $\eta_c \to \gamma\gamma$ has been measured in experiment $[1]$. Within the framework of the standard model, the leptonic decay $\eta_c \to l^+l^-$ is contributed by fourth-order electromagnetic transition and tree-level weak transition induced by $Z^0$. The low probability of a fourth-order electromagnetic and weak transition makes the leptonic decays sensitive to hypothetical interactions arising from physics beyond the standard model, such as the existence of a light pseudoscalar Higgs boson in the next-to-minimal supersymmetric standard model $[2, 3, 4, 5, 6]$ or leptoquark bosons that carry both quark and lepton flavors $[7, 8, 9, 10]$, both of which can enhance the leptonic decays of $\eta_c$ with appropriate values of new physics parameters in these models.

In the literature there have been many theoretical works on the calculations of two-photon decays of pseudoscalar heavy quarkonia $\eta_c$, etc., based on relativistic quark model or potential model $[11, 12, 13]$, Bethe-Salpeter equation $[14, 15]$, heavy-quark spin symmetry $[16]$, and Lattice QCD $[17]$. However the decay rates of leptonic decays of pseudoscalar heavy quarkonia have not been known yet.

In this work the radiative and leptonic decays of $\eta_c \to \gamma\gamma$ and $\eta_c \to l^+l^-$ are studied consistently within the framework of the standard model, by using a method different from those used in the literature. The effective Hamiltonians in quark level for these decays are calculated at first. Then the factorization formula is derived, the decay amplitudes are expressed as convolutions of meson wave function and the hard transition amplitudes, where the wave function is controlled by nonperturbative QCD, for which I use the result calculated in QCD sum rule $[18]$, while the hard transition amplitudes are calculated with perturbation theory. For the process $\eta_c \to l^+l^-$, the loop diagrams are calculated analytically. The infrared divergence is analyzed, the possible information of new physics is also briefly discussed.

The diagram for the decay of $\eta_c \to \gamma\gamma$ is depicted in Fig.1. The transition matrix element relevant to Fig.1(a) can be written in coordinate space as

$$T_1 = \langle \gamma\gamma | \int d^4x d^4y \bar{c}(x)iQ_{\epsilon}e\gamma_{\mu} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}iQ_{\epsilon}e\gamma_{\mu}c(y)A^\mu(x)A^\nu(y) | \eta_c \rangle,$$

where $e$ is the absolute value of the charge of electron, $Q_{\epsilon}$ the charge of the $c$ quark in unit of $e$, $A^\mu(x)$ and $A^\nu(y)$ are the electromagnetic fields for the two photons. Contracting the creation operators of the two photons with that in the electromagnetic fields, then the amplitude $T_1$
\[
T_1 = \int d^4x \, d^4y \, \int d^4k \, \frac{i}{(2\pi)^4} \, e^{-ik(x-y)} \langle \bar{c} \gamma_\mu c \rangle_{\alpha \beta_0} \epsilon_1^{\mu} e^{ik_1 x} e^{ik_2 y} \, \epsilon_2^{\nu} \]  

(2)

where \(\alpha\) and \(\beta\) are the Dirac spinor indices. It is understood that the repeated indices are summed. In both eqs. (1) and (2), summations over color degrees of freedom and a factor \(\text{PExp}\{i \int dz\, T^n A_0^\mu(z)\}\) between \(\bar{c}(x)\) and \(c(y)\) that makes the amplitude gauge-invariant are indicated.

The leading-twist wave function for \(\eta_c\) meson can be defined through the matrix element \(\langle 0|\bar{c}(x)_\alpha c(y)_\beta|\eta_c\rangle\) \[19, 20\], which will be further discussed in Appendix A,

\[
\langle 0|\bar{c}(x)_\alpha c(y)_\beta|\eta_c\rangle = \frac{i}{4} f_{\eta_c} \int du e^{-i(up+x+\bar{u}p-y)} \left\{ \bar{\epsilon}_\rho \gamma_5 \beta_0 \phi(u, \mu) \right\} \]  

(3)

where \(u = 1 - \mu\) is an energy scale, and \(f_{\eta_c}\) is the decay constant of \(\eta_c\), which is defined as

\[
\langle 0|\bar{c}(x)\gamma_5 c|\eta_c\rangle = i f_{\eta_c} p_\mu. \]  

(4)

here \(p_\mu\) is the four-momentum of the meson.

With the matrix element given in eq. (3), the transition matrix element \(T_1\) becomes

\[
T_1 = -\frac{1}{4} f_{\eta_c} Q_{\bar{c}c}^2 e^2 \int_0^1 du \, \int d^4x \, d^4y \, \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(x-y)} \epsilon_2^{\nu} \]  

\[
\epsilon_1^{\mu} \epsilon_2^{\nu} Tr[\bar{\epsilon}\gamma_5 \gamma_\mu (k + m_c) \gamma_\nu]. \]  

(5)

It is not difficult to perform the integration over the coordinates \(x, y\) and the momentum \(k\). After these manipulations, one can obtain

\[
T_1 = i f_{\eta_c} Q_{\bar{c}c}^2 e^2 \int_0^1 du \, \frac{\phi(u, \mu)}{(uk_1 - uk_2)^2 - m_c^2} \]  

\[
\epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon^{\rho\sigma\mu\nu} (2\pi)^4 \delta^4(k_1 + k_2 - p). \]  

(6)

Remove the overall four-momentum conservation factor \((2\pi)^4 \delta^4(k_1 + k_2 - p)\), one can obtain the decay amplitude contributed by Fig[1](a)

\[
A_1 = i f_{\eta_c} Q_{\bar{c}c}^2 e^2 \int_0^1 du \, \frac{\phi(u, \mu)}{(uk_1 - uk_2)^2 - m_c^2} \]  

\[
\epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon^{\rho\sigma\mu\nu}. \]  

(7)

The contribution of Fig[1](b) can be obtained by making the exchange of \(k_1 \leftrightarrow k_2\) and \(\epsilon_1 \leftrightarrow \epsilon_2\) in \(A_1\)

\[
A_2 = A_1 \left( \epsilon_1 \leftrightarrow \epsilon_2 \right). \]  

(8)

Then the total contribution of Fig[1](a) and (b) is

\[
A = A_1 + A_2. \]  

(9)

Using \(k_1^2 = k_2^2 = 0\) and \(2k_1 \cdot k_2 = m_{\eta_c}^2\), one can simplify the amplitude \(A\) to be

\[
A = 2i f_{\eta_c} Q_{\bar{c}c}^2 e^2 \int_0^1 du \, \frac{\phi(u, \mu)}{u \tilde{m}_{\eta_c}^2 + m_c^2} \]  

\[
\epsilon_1^{\mu} \epsilon_2^{\nu} \epsilon^{\rho\sigma\mu\nu}. \]  

(10)

For simplicity in the following calculation, one can define a new quantity \(M\) as

\[
M = 2i f_{\eta_c} Q_{\bar{c}c}^2 e^2 \int_0^1 du \, \frac{\phi(u, \mu)}{u \tilde{m}_{\eta_c}^2 + m_c^2}, \]  

(11)

then we can obtain the square of the total amplitude

\[
|A|^2 = \frac{1}{2} m_{\eta_c}^4 |M|^2. \]  

(12)

The decay width can be calculated by the following formula

\[
\Gamma(\eta_c \rightarrow \gamma \gamma) = \frac{1}{2! \, 8\pi} |A|^2 \frac{\bar{k}}{m_{\eta_c}^3}, \]  

(13)

where \(\bar{k}\) is the statistic factor for two identical particles, \(\bar{k}\) is the three-momentum of one of the photons in the rest frame of \(\eta_c\).

Next let us discuss the leptonic decay \(\eta_c \rightarrow l^+l^-\). In the standard model, the Feynman diagrams for this process include photon-Fermion loop diagrams and \(Z^0\) tree diagram, which are depicted in Fig[2]

\[\text{FIG. 2: Diagrams for } \eta_c \rightarrow l^+l^-\]. (a) and (b) QED contribution where two virtual photons as intermediate state, (c) weak \(Z^0\) contribution.

The effective Hamiltonian describing the transition of \(\bar{c}c \rightarrow l^+l^-\) can be calculated with the help of the Feynman diagram in Fig[2]. The contribution of Fig[2] (a) is
calculated to be

\[ H_1 = Q^2e^4 \int \frac{d^4k}{(2\pi)^4} \]

\[ \frac{\bar{c}\gamma^\mu(k + m_l)\gamma^\nu l \bar{c}\gamma_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu c}{[(k^2 - m_l^2 + ie)[(k - k_1)^2 + ie][(k + k_2)^2 + ie]}
\times[(k - k_1 + p_e)^2 - m_e^2 + ie)], \]

where \( m_l \) is the lepton mass, \( p_e \) the four-momentum of the quark. The quark fields \( c \) and \( \bar{c} \) are in the momentum space. They are related to the field operators in the coordinate space by

\[ c = \int d^4x \bar{c}(x)e^{ip_e \cdot x}, \]

\[ \bar{c} = \int d^4y c(y)e^{ip_e \cdot y}. \]

Then the effective Hamiltonian \( H_1 \) can be reexpressed in terms of quark fields in the coordinate space

\[ H_1 = \int d^4x \bar{c}^e e^{ip_e \cdot x} \int d^4y c e^{ip_e \cdot y} Q^2e^4 \int \frac{d^4k}{(2\pi)^4} \]

\[ \frac{\bar{c} \gamma^\mu(k + m_l) \gamma^\nu l \bar{c}\gamma_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu c(y)}{a_k}, \]

where \( a_k \) is defined as

\[ a_k \equiv [(k^2 - m_l^2 + ie)[(k - k_1)^2 + ie] \times[(k + k_2)^2 + ie][(k - k_1 + p_e)^2 - m_e^2 + ie]]. \]

With the effective Hamiltonian \( H_1 \), the amplitude contributed by Fig.2 (a) is

\[ A_1 = \int \frac{d^4p_c}{(2\pi)^4} \frac{d^4p_e}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} (\bar{u}(k_1)\gamma^\mu(k + m_l)\gamma^\nu v(k_2) \]

\[ \times \langle 0|\bar{c}(x)\gamma_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu c(y)\rangle \frac{1}{a_k}, \]

The matrix element in the above equation can be treated as

\[ \langle 0|\bar{c}(x)\gamma_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu c(y)\rangle \]

\[ = \langle 0|\bar{c}(x)\gamma_\nu(c(y)\beta_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu |\eta_c\rangle. \]

With the help of eq. (20) and the wave function of \( \eta_c \) defined in eq. (4), the decay amplitude \( A_1 \) can be expressed as

\[ A_1 = Q^2e^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}
\]

\[ \frac{\bar{u}(k_1)\gamma^\mu(k + m_l)\gamma^\nu v(k_2)}{a_k} \times \frac{-i}{4} f_{\eta_c} \int \phi(u, \mu)Tr[\bar{u}(k_1)\gamma^\nu(k - k_1 + \bar{u}p + m_e)\gamma_\mu |\eta_c\rangle \frac{1}{a_k}. \]

Perform the trace operation and using the identity of the gamma matrices

\[ \gamma_a \gamma_b \gamma_c = g_{a\beta} \gamma_\alpha + g_{a\beta} \gamma_\alpha - g_{\alpha \beta} \gamma_a - i\epsilon_{\alpha \beta \gamma} \gamma_\mu \gamma_5, \]

the amplitude \( A_1 \) can be further calculated to be

\[ A_1 = iQ^2e^4 \int \phi(u, \mu)\int \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}
\]

\[ \times \bar{u}(k_1)\gamma^\mu \int \bar{u}(k_1)\sigma^{\mu \nu} v(k_2) \]

\[ \cdot \langle 0|\bar{c}(x)\gamma_\nu(k - k_1 + \beta_e + m_e)\gamma_\mu c(y)\rangle \frac{1}{a_k}, \]

where

\[ a_{1\alpha} = 2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}
\]

\[ \times \frac{p_{\alpha}k \cdot (k - k_1 + \bar{u}p - (k - k_1 + \bar{u}p)_{\alpha}p \cdot k \]

\[ a_{2\sigma} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}
\]

Next one needs to perform the loop integrations in the coefficients \( a_{1\alpha} \) and \( a_{2\sigma} \).

Using \( p = k_1 + k_2 \) and a few steps of algebra manipulation, one can cancel one propagator in the coefficient \( a_{1\alpha} \), then the four-point loop integration of \( a_{1\alpha} \) can be reduced to a sum of a few three-point loop integrals

\[ a_{1\alpha} = \int \frac{d^4k}{(2\pi)^4} \left\{ p_{\alpha} \times \left[ \frac{1}{[(k - k_1)^2 + ie][(k + k_2)^2 + ie][(k - k_1 + \bar{u}p)^2 - m_e^2 + ie]}
\right.
\]

\[ + \frac{\bar{u}}{(k^2 - m_l^2 + ie)[(k + k_2)^2 + ie][(k - k_1 + \bar{u}p)^2 - m_e^2 + ie]}
\]

\[ - \frac{\bar{u}}{(k - k_1 + \bar{u}k_2)^2_{\alpha}} \}

With the Feynman parameterization, the integration over the loop momentum in the above equation can be
performed, then the coefficient $a_{1\alpha}$ can be expressed as integrals over Feynman parameters

$$ a_{1\alpha} = \frac{-i}{16\pi^2} \int_0^1 dx \int_0^1 dy \left\{ \frac{1}{a_1} + \frac{u}{a_2} \right\} \left( \frac{\bar{u}}{a_3} - \frac{y + u(x - 1)}{k_{1\alpha} + \bar{u}k_{2\alpha}} \right) + \frac{-u\bar{x}k_{1\alpha} + (\bar{u} - \bar{u}x - y)k_{2\alpha}}{a_2} \right\}, \quad (27) $$

with

$$ a_1 = m_{\eta c}^2(\bar{u}x + y)(\bar{u}x + y - 1) + xu\bar{u} + xm_c^2 - i\epsilon, $$

$$ a_2 = [1 - 2x - 2y + (x + y)^2]m_{\eta c}^4 + xu(\bar{x}x - y)m_{\eta c}^2 - i\epsilon, $$

$$ a_3 = [1 - 2x - 2y + (x + y)^2]m_{\eta c}^2 + x\bar{x}u - y)m_{\eta c}^2 - i\epsilon. $$

While the four-point loop integration of the coefficient $a_2^e$ in eq. (28) cannot be reduced, it can be expressed as a 3-fold integral of Feynman parameters

$$ a_2^e = \frac{i}{16\pi^2} \int_0^1 dx \int_0^1 dy \int_0^1 z dy dz \left( \frac{(ux + z - u)}{a_4^4} k_1^2 - (\bar{u}x + y - \bar{u})k_2^2 \right), \quad (29) $$

where

$$ a_4 = (1 - x - y - z)^2 m_{\eta c}^4 + [xu\bar{u} - (ux + z)(\bar{u}x + y)]m_{\eta c}^2 + xm_c^2 - i\epsilon. $$

Substitute $a_{1\alpha}$ and $a_2^e$ in eqs. (27) and (29) into eq. (23), the amplitude $A_1$ can be reduced to

$$ A_1 = \frac{Q^2 e^4 f_{\eta c}}{16\pi^2} \int_0^1 d\phi(u, \mu) \int_0^1 dx \int_0^{1-x} dym_l \left\{ \frac{2}{a_1} + \frac{1 + (2u - 1)x - y}{a_2} + \frac{1 + (1 - 2u)x - y}{a_3} \right\} \left( \bar{u}(k_1)\gamma_5 v(k_2) \right) + \int_0^{1-x-y} dz \left[ \frac{1 - x - y - z}{a_4^4} \bar{u}(k_1)\sigma_{\mu\nu} v(k_2)\epsilon_{\nu\rho\sigma} k_1^2 k_2^2 \right]. \quad (31) $$

The contribution of Fig. 2b can be calculated in the same way. It can be finally shown that the contribution of Fig. 2b $A_2$ is the same as that of Fig. 2a if the condition $\phi(u, \mu) = \phi(\bar{u}, \mu)$ maintained. This condition is well satisfied by the wave function of $\eta_c$.

Then one can obtain the total contribution of the loop-diagrams

$$ A_1 = A_1 + A_2 = c_1 \bar{u}(k_1)\gamma_5 v(k_2) + c_2 \bar{u}(k_1)\sigma_{\mu\nu} v(k_2)\epsilon_{\nu\rho\sigma} k_1^2 k_2^2, \quad (32) $$

with

$$ c_1 = \frac{Q^2 e^4 f_{\eta c}}{8\pi^2} m_l \int_0^1 d\phi(u, \mu) \int_0^1 dx \int_0^{1-x} dym_l \left\{ \frac{2}{a_1} + \frac{1 + (2u - 1)x - y}{a_2} + \frac{1 + (1 - 2u)x - y}{a_3} \right\}, \quad (33) $$

$$ c_2 = \frac{Q^2 e^4 f_{\eta c}}{8\pi^2} m_l \int_0^1 d\phi(u, \mu) \int_0^1 dx \int_0^{1-x-y} dz \left( \frac{1 - x - y - z}{a_4^4} \right), \quad (34) $$

where $a_1 \sim a_4$ are given in eqs. (23) and (30).

The coefficients $c_1$ and $c_2$ are calculated analytically. Before performing the integrations over Feynman parameters, let us make the phase convention in the integrals. The imaginary angles $\theta$’s for any imaginary quantities are restricted in the range $-\pi < \theta < \pi$. Then the logarithms appeared during the integration have a cut along the negative real axis. For any two imaginary quantities $a$ and $b$, the following relations about the logarithm are held [21]

\[ \ln(ab) = \ln(a) + \ln(b) + \eta(a, b), \quad (35) \]

where \[ \eta(a, b) = 2\pi i \{ \theta(-\text{Im}a)\theta(-\text{Im}b)\theta(\text{Im}ab) - \theta(\text{Im}a)\theta(\text{Im}b)\theta(-\text{Im}ab) \}, \]

\[ \theta(x) \] is the unit step function.

The analytical results for the Feynman parameter integration are obtained. To express the results conveniently, some quantities are defined as $r_c = m_{\eta c}^2/m_{\eta c}^2$, $\rho_l = m_l^2/m_{\eta c}^2$, $\alpha = \frac{2\rho_l}{\sqrt{1 - 4\rho_l}}$, and $\beta = \sqrt{1 - 4\rho_l}$. The result for $c_1$ is

$$ c_1 = \frac{Q^2 e^4 f_{\eta c}}{8\pi^2} m_l \left[ f_1(u) + f_2(u) + f_3(u) \right], \quad (36) $$

where the functions $f_1(u)$, $f_2(u)$ and $f_3(u)$ are

\[ f_1(u) = \frac{1}{(1 - u)u(r_c + (1 - u)u)} \left[ 2r_c \ln(r_c) + u(1 - 2r_c - 3u + 2u^2) \ln(r_c - (1 - u)^2 - i\epsilon) + (1 - u) \right. \]

\[ \times (u \ln(-r_c + (1 - u)^2 + i\epsilon) - (2r_c + u - 2u^2) \ln(r_c - u^2 - i\epsilon) + u \ln(-r_c + u^2 + i\epsilon)), \]
f_2(u) = \left\{ \begin{array}{l}
\frac{\beta}{(1-4r_l)^2} \{ 2(-1 + 4r_l) \ln(2) \ln(\frac{1-\beta}{1+\beta}) - \ln(2) \ln(\frac{1+\beta}{1-\beta}) + 2Li_2(\frac{\beta}{1+\beta}) - Li_2(\frac{2\beta}{1-\beta}) \\
- 2Li_2(\frac{\beta}{1+\beta}) + Li_2(\frac{2\beta}{1+\beta}) \} + \frac{\beta^2}{-1+4r_l^2} \beta \ln(16) - 2\beta \ln(1 + \frac{1}{\beta}) + 2\beta \ln(\frac{1}{\beta}) + 2\beta \ln(\frac{1}{\beta}) \\
- 2\beta \ln(\frac{1-\beta}{1+\beta}) - \ln(2) \ln(\frac{1-\beta}{1+\beta}) + \beta^2 \ln(2) \ln(\frac{1-\beta}{1+\beta}) + \ln(2) \ln(\frac{1+\beta}{1-\beta}) - \beta^2 \ln(2) \ln(\frac{1+\beta}{1-\beta}) \\
+ 2(-1 + \beta^2 )Li_2(\frac{\beta}{-1+\beta}) + (1-\beta^2) Li_2(\frac{2\beta}{1+\beta}) + 2Li_2(\frac{\beta}{1+\beta}) - 2\beta^2 Li_2(\frac{\beta}{1+\beta}) - Li_2(\frac{2\beta}{1+\beta}) \\
+ \beta^2 Li_2(\frac{2\beta}{1+\beta}) \} \right\} + \frac{-1 + 4r_l}{4r_l} \left\{ \frac{1}{1-4r_l} \left[ -t_1 \ln(t_1) + t'_1 \ln(t'_1) - t_2 \ln(t_2) + t'_2 \ln(t'_2) \right] \\
+ (-1 + t_1 - \beta) \ln(t_1 - t_2) + (1 - t'_1 + \beta) \ln(t_1 - t'_1) + (1 + t_2 - \beta) \ln(t_2 - t_1) + (1 - t'_2 + \beta) \ln(t'_2 - t_1), \\
+ (1 - t'_2 + \beta) \ln(1 - t'_2 + \beta) + \frac{4r_l^2}{(1-4r_l)^2} \left[ -t_1 \ln(t_1) + t'_1 \ln(t'_1) - t_2 \ln(t_2) + t'_2 \ln(t'_2) \right] \\
- \ln(t_1 + \alpha) + \ln(t'_1 + \alpha) - \ln(t_2 + \alpha) + \ln(t'_2 + \alpha) + \frac{t_1}{(t_1 + \alpha)(t_1 + \alpha)} \ln(1 + t_1 + \beta) + \frac{t_2}{(t_2 + \alpha)(t_2 + \alpha)} \ln(1 + t_2 + \beta) \\
- \ln(1 + t_1 + \beta) - \ln(1 + t_2 + \beta) + \ln(t_1 + \alpha) + \ln(t_2 + \alpha) + \frac{t_1}{t_1 + \alpha} \ln(t_1 + \alpha) + \frac{t_2}{t_2 + \alpha} \ln(t_2 + \alpha) \\
- \ln(-t'_1) + \frac{\alpha}{t'_1 + \alpha} \ln(\frac{1 + \alpha + \beta}{t'_1 + \alpha}) + \ln(-t'_2) + \frac{\alpha}{t'_2 + \alpha} \ln(\frac{1 + \alpha + \beta}{t'_2 + \alpha}) \\
- \ln(1 - t_2 + \beta) + \ln(1 - t_2p + \beta) + \frac{t_1}{t_2 + \alpha} \ln(1 + \alpha + \beta) + \frac{t_2}{t_2 + \alpha} \ln(1 + \alpha + \beta) + Li_2(t_1 + \alpha) - Li_2(t'_1 + \alpha) \\
+ Li_2(t_2 + \alpha) - Li_2(t'_2 + \alpha) - Li_2\left(\frac{-1 + t_1 - \beta}{t_1 + \alpha}\right) + Li_2\left(\frac{-1 + t'_1 - \beta}{t'_1 + \alpha}\right) - Li_2\left(\frac{-1 + t_2 - \beta}{t_2 + \alpha}\right) \\
+ I_{s_{1Z}}\left(\frac{-i}{t_1 + \alpha}\right) \ln(-t_1) + I_{s_{1Z}}\left(\frac{-i}{t'_1 + \alpha}\right) \ln(-t'_1) + I_{s_{1Z}}\left(\frac{-i}{t_2 + \alpha}\right) \ln(-t_2) + I_{s_{1Z}}\left(\frac{-i}{t'_2 + \alpha}\right) \ln(-t'_2), \\
+ (r_1 + (-1 + u)u) \ln((1 - u)u) - r_1 \ln((-1 + u)u) - (r_1 - u + u^2) \ln(u^2) \right\} \right\},
\end{array} \right. \\
\text{where} \\
\begin{align}
t_1 &= \frac{2r_1(\sqrt{1 - 4r_l} + u) + 2r_1(1 - u)}{2r_1 - (1 + \sqrt{1 - 4r_l})u} + i\epsilon, \\
t_2 &= \frac{2r_1(\sqrt{1 - 4r_l} + u) - 2r_1(1 - u)}{2r_1 - (1 + \sqrt{1 - 4r_l})u} - i\epsilon, \end{align}
\begin{align}
t'_1 &= \frac{2r_1(\sqrt{1 - 4r_l} - u) - 2r_1(1 - u)}{2r_1 - (1 + \sqrt{1 - 4r_l})u} + i\epsilon, \\
t'_2 &= \frac{2r_1(\sqrt{1 - 4r_l} - u) + 2r_1(1 - u)}{2r_1 - (1 + \sqrt{1 - 4r_l})u} - i\epsilon.
\end{align}

The function \(Li_2(z)\) is the polylogarithm function, which is defined as \(Li_2(z) = \int_0^1 dt \frac{\ln(1-t)}{1-t} \). Finally, the third function is
\[f_3(u) = f_2(1 - u).
\]

The analytical result of the 3-fold Feynman parameter integration in \(c_2\) (eq. (44)) is tedious, it is not presented here. Some formulas used in the procedure of integration are given in Appendix B.

It is not difficult to calculate the contribution of weak interaction, where the virtual \(Z^0\) acts as the intermediate state (Fig 2c). The contribution of Fig 2c to the amplitude of \(\eta_c \to l^+l^-\bar{c}\) is
\[A_{l\gamma \bar{c}} = \langle l^+l^- | \bar{V}_{luZ} l \frac{-i}{p^2 - m^2_Z + i\epsilon} e^{i\phi} \bar{V}_{cZ} | \eta_c \rangle,
\]
where \(p = k_1 + k_2\), and \(V_{luZ}\) and \(V_{cZ}\) are the coupling vertices of \(\bar{u}Z\) and \(\bar{c}Z\), respectively,
\[V_{luZ} = i\epsilon\gamma^\mu \left( -\frac{4\sin^2 \theta_W + 1}{4 \sin \theta_W \cos \theta_W} \right) - \frac{1}{4 \sin \theta_W \cos \theta_W \gamma_5},
\][V_{cZ} = i\epsilon\gamma^\mu \left( -\frac{8\sin^2 \theta_W - 3}{12 \sin \theta_W \cos \theta_W} + \frac{1}{4 \sin \theta_W \cos \theta_W \gamma_5}.\right)\]
with $\theta_W$ being the Weinberg angle.

Contracting the leptonic final state with the leptonic fields operator and using the definition of the decay constant of $\eta_c$, one can finally obtain the amplitude

$$A_{z^0} = \frac{2m_\ell e^2}{(4\sin\theta_W \cos\theta_W)^2} \frac{f_{\eta_c}}{p^2 - m_Z^2} \bar{u}(k_1) \gamma_5 v(k_2). \quad (45)$$

Then the amplitude of the leptonic decay $\eta_c \rightarrow l^+l^-$ is the sum of the loop and tree diagrams

$$A(\eta_c \rightarrow l^+l^-) = A_l + A_{z^0}. \quad (46)$$

The decay width of this decay is

$$\Gamma(\eta_c \rightarrow l^+l^-) = \frac{1}{8\pi} |A(\eta_c \rightarrow l^+l^-)|^2 \frac{2|k_2|}{m_{\eta_c}}, \quad (47)$$

where $k_2$ is the three-momentum of one of the leptons in the rest frame of $\eta_c$. The branching ratio of the decay is defined by

$$Br(\eta_c \rightarrow l^+l^-) = \frac{\Gamma(\eta_c \rightarrow l^+l^-)}{\Gamma_{tot}}, \quad (48)$$

where $\Gamma_{tot}$ are the total decay width of the $\eta_c$ meson.

In the numerical calculation, the wave function of $\eta_c$ is taken to be $[18]

$$\phi(u, \mu \sim m_\ell) = N 4u(1-u)\exp\left(-\frac{\beta}{4u(1-u)}\right), \quad (49)$$

where $N$ is the normalization factor, the parameter $\beta = 3.8 \pm 0.7$. The value of the decay constant of $\eta_c$ is $f_{\eta_c} = 0.346 \pm 0.017$ GeV $[18]$, the mass of $c$ quark $m_c = 1.4$ GeV, the total decay width of $\eta_c$ is $\Gamma_{tot} = 26.7 \pm 3.0$ MeV $[1]$.

With the parameter inputs and the wave function of $\eta_c$ given above, the prediction to the branching ratio of $\eta_c \rightarrow \gamma\gamma$ is

$$B(\eta_c \rightarrow \gamma\gamma) = 2.43^{+0.39}_{-0.34} \times 10^{-4}. \quad (50)$$

where the uncertainty comes from the uncertainty of the total decay width of $\eta_c$, the uncertainty of the decay constant $f_{\eta_c}$ and the uncertainty of the parameter $\beta$ in the wave function. The main contribution comes from the uncertainties of the total decay width and decay constant of $\eta_c$, which are about 10% of the central value, while the error caused by the parameter $\beta$ is small, which is only about 1%.

The prediction in eq. (50) is in good agreement with the experimental data

$$B^{exp}(\eta_c \rightarrow \gamma\gamma) = 2.4^{+1.1}_{-0.9} \times 10^{-4}. \quad (51)$$

Next I will go on to the numerical discussion on $\eta_c \rightarrow l^+l^-$. The loop integral is infrared divergent if the lepton mass is zero. This can be shown by taking the asymptotic limit of $r_l \rightarrow 0$ for the analytical results of loop-integral-functions $f_1(u), f_2(u)$ and $f_3(u)$ in eqs. (37), (38) and (39). The most singular term behaves like $\frac{1}{r_l \ln(r_l)}$. Considering the decay amplitude in eq. (31) is proportional to $m_l \sim \sqrt{r_l}$ due to the helicity suppression, the most singular term in the decay amplitude behaves like $\frac{1}{\sqrt{r_l} \ln(r_l)}$. The curves for the coefficient $c_1$ as the lepton mass approaching zero are shown in Fig. 3. Only when the lepton mass $m_l < 0.2 m_e$, here $m_e$ is the mass of electron, does the numerical result begin to be severely affected by the infrared divergence. As the lepton mass not less than 0.2$m_e$, the numerical value is reliable. The situation for $c_2$ is similar to $c_1$. Therefore, for the cases of $m_l = m_e$ and $m_l = m_\mu$ ($m_\mu$ the muon mass), the loop integral is not affected by the infrared divergence.

The values of $c_1$ and $c_2$ are presented in Table I.

![Behavior of the coefficient $c_1$ as $m_l \rightarrow 0$. (a) the real part of $c_1$; (b) the imaginary part of $c_1$. The horizontal axis is for the ratio of $m_l/m_e$.](image)
The branching ratios are given in Table I. The leptonic decays of $\eta_c$ are dominated by the two-photon loop diagrams, the contribution of $Z^0$ tree diagram modifies the decay rates by less than 1%.

The branching ratios of $\eta_c \to l^+l^-$ are very tiny within the frame work of the standard model. Charm physics including leptonic, semileptonic and hadronic charm decays will be studied at BESIII [22, 23]. About $10 \times 10^9 J/\psi$ events will be accumulated with one year designed luminosity. Considering the branching ratio of $J/\psi \to \gamma \eta_c$ being $(1.3 \pm 0.4)\%$ [1], about $1.3 \times 10^8 \eta_c$ mesons can be produced through the radiative decays of $J/\psi \to \gamma \eta_c$. Assuming the detection efficiency is about 20%, then the sensitivity of the measurement of $\eta_c \to l^+l^-$ can reach $3.8 \times 10^{-8}$. The standard model prediction to the leptonic decay of $\eta_c \to l^+l^-$ is just below the sensitivity of BESIII. However, the low decay rate within the standard model makes the leptonic decay of $\eta_c \to l^+l^-$ sensitive to physics beyond the standard model, such as the existence of leptoquark bosons, Fig. 4(a), or the light pseudoscalar Higgs boson, Fig. 4(b). Especially when the mass of the light pseudoscalar Higgs lies near the mass of $\eta_c$ meson, Fig. 4(b) can significantly enhance the leptonic decay rate of $\eta_c$. Measurement of $\eta_c \to l^+l^-$ decay, especially for $\eta_c \to \mu^+\mu^-$ channel, can give information of physics beyond the standard model.

In summary, the radiative and leptonic decays of $\eta_c \to \gamma\gamma$ and $\eta_c \to l^+l^-$ have been studied consistently within the framework of the standard model. The theoretical prediction of the decay rate of $\eta_c \to \gamma\gamma$ is in well agreement with the experimental measurement. For the leptonic decays of $\eta_c \to \mu^+\mu^-$ and $e^+e^-$, within the framework of the standard model, the loop diagrams of electromagnetic interaction dominate, the tree diagram of weak interaction involving $Z^0$ propagator can only modifies the decay rates by less than 1%. The decay rates of $\eta_c \to \mu^+\mu^-$ and $e^+e^-$ are tiny, which makes them sensitive to some new physics beyond the standard model. Measurement of these decay rates may shed light on the existence of new physics.

I thank H.B. Li for helpful discussions, and be grateful to Kavli Institute for Theoretical Physics China, where part of this work was done. This work is supported in part by the National Natural Science Foundation of China under contracts Nos. 10675108, 10735080.

### Appendix A

The wave function of a meson $M$ composed of a quark-antiquark pair $q\bar{q}$ can be defined through the matrix element $\langle 0|\bar{q}(x)\gamma\alpha q(y)\gamma\beta M\rangle$, where summation over color degrees of freedom and a gauge invariant factor $\text{Pexp}\{i\int dz^{\mu} T^\alpha A^{\mu}_\alpha(z)\}$ between $\bar{q}(x)$ and $q(y)$ are indicated.

Using the equality of Fietz transformation

\begin{equation}
\delta_{ij}\delta_{lk} = \frac{1}{4}\delta_{ij}\delta_{lk} + \frac{1}{4}\langle\gamma_5\rangle_{ij}\langle\gamma_5\rangle_{lk} + \frac{1}{4}\langle\gamma_\mu\rangle_{ij}\langle\gamma_\mu\rangle_{lk} - \frac{1}{4}\langle\gamma_\mu\gamma_5\rangle_{ij}\langle\gamma_\mu\gamma_5\rangle_{lk} + \frac{1}{8}\langle\sigma^{\mu\nu}\gamma_5\rangle_{ij}\langle\sigma^{\mu\nu}\gamma_5\rangle_{lk},
\end{equation}
For a pseudoscalar meson $M$, the matrix elements of the scalar and vector current are all zero. For the nonzero matrix element the wave function can be defined as

$$\langle 0 | \bar{q}_1(x) \gamma_\mu q_2(y) | M \rangle = i f M p_\mu \int_0^1 du e^{-i(u p \cdot x + \bar{u} p \cdot y)} \phi(u, \mu) ,$$

(53)

$$= \langle 0 | \bar{q}_1(x) q_2(y) | M \rangle \delta_{\rho \sigma} \delta_{\beta \alpha} + \frac{1}{4} (\gamma_5)_{\rho \sigma} (\gamma_5)_{\beta \alpha}$$

$$+ \frac{1}{4} (\gamma_\mu)_{\rho \sigma} (\gamma_5)_{\beta \alpha} - \frac{1}{4} (\gamma_\mu \gamma_5)_{\rho \sigma} (\gamma_5)_{\beta \alpha}$$

$$+ \frac{1}{8} (\sigma^\mu \gamma_5)_{\rho \sigma} (\sigma^{\mu \nu} \gamma_5)_{\beta \alpha} .$$

For a pseudoscalar meson $M$, the matrix element can be written as

$$\langle 0 | \bar{q}_1(x) q_2(y) | M \rangle = \langle 0 | \bar{q}_1(x) \gamma_\mu q_2(y) | M \rangle \delta_{\rho \sigma} \delta_{\beta \alpha} + \frac{1}{4} (\gamma_5)_{\rho \sigma} (\gamma_5)_{\beta \alpha}$$

(54)

$$= \frac{1}{4} \langle 0 | \bar{q}_1(x) q_2(y) | M \rangle \delta_{\beta \alpha} + \frac{1}{4} \langle 0 | \bar{q}_1(x) \gamma_\mu q_2(y) | M \rangle (\gamma_5)_{\beta \alpha}$$

$$+ \frac{1}{4} \langle 0 | \bar{q}_1(x) \gamma_5 q_2(y) | M \rangle (\gamma_\mu)_{\beta \alpha}$$

$$+ \frac{1}{4} \langle 0 | \bar{q}_1(x) \sigma^{\mu \nu} q_2(y) | M \rangle (\sigma^{\mu \nu} \gamma_5)_{\beta \alpha} .$$

For the decay processes of $\eta$, only the leading-twist wave function is considered. The contributions of the wave functions of twist-3 are suppressed due to their spin structure. Therefore they are safely dropped.

Appendix B

Some formulas used in the calculation of the 3-fold Feynman parameter integration are

$$\int dx \frac{1}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln(2ax + b - \sqrt{b^2 - 4ac}) - \ln(2ax + b + \sqrt{b^2 - 4ac}) ,$$

(58)

$$\int dx \frac{x}{(ax^2 + bx + c)^2} = \frac{bx + 2c}{(b^2 - 4ac)(ax^2 + bx + c)}$$

(59)

$$+ \frac{b}{b^2 - 4ac} \int dx \frac{1}{ax^2 + bx + c} .$$

For the integration of the type

$$\int dy \frac{n_1 y + n_2}{(a_1 y^2 + b_1 y + c_1)(a_2 y^2 + b_2 y + c_2)} ,$$

(60)

the integrand can be decomposed as

$$\frac{n_1 y + n_2}{(a_1 y^2 + b_1 y + c_1)(a_2 y^2 + b_2 y + c_2)} = \frac{k_1 y + l_1}{a_1 y^2 + b_1 y + c_1} + \frac{k_2 y + l_2}{a_2 y^2 + b_2 y + c_2} ,$$

(61)

where the coefficients $k_{1,2}$ and $l_{1,2}$ can be solved within the above equation. Then the integration of eq. (60) can be reduced into a simpler form.

For the integral like

$$\int dx \frac{k x + l}{a x^2 + b x + c \sqrt{e x} + f} \ln(g x + b \pm \sqrt{e x} + f) .$$

(62)

the integration can be performed by making the variable transformation $t = \sqrt{e x} + f$. While for

$$\int dx \frac{k x + l}{a x^2 + b x + c_1} \frac{1}{\sqrt{a_2 x^2 + b_2 x + c_2}}$$

$$\times \ln(g x + b \pm \sqrt{a_2 x^2 + b_2 x + c_2}) ,$$

(63)

with $a_2 > 0$, the variable transformation shall be $\sqrt{a_2 x^2 + b_2 x + c_2} = t - \sqrt{a_2 x}$.

$$\int dt \frac{\ln(t - a)}{t - b} = \ln(t - a) \ln \left( \frac{t - a}{a - b} + 1 \right) + \text{Li}_2 \left( \frac{t - a}{a - b} \right) ,$$

(64)

where $\text{Li}_2(z)$ is the polylogarithm function, which is defined as $\text{Li}_2(z) = \int \frac{\ln(1 - t)}{t} dt$. 

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