A BOUND ON THE NORM OF OVERCONVERGENT $p$-ADIC MULTIPLE POLYLOGARITHMS

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Abstract. We generalize the definition of overconvergent $p$-adic multiple polylogarithms and of $p$-adic cyclotomic multiple zeta values and we prove a bound on their norm. A byproduct of the proof is a characterization of these objects in terms of certain regularized $p$-adic iterated integrals. The generalization of the definition consists in replacing the underlying Frobenius structure by its iterations. The bound on the norms of overconvergent $p$-adic multiple polylogarithms that we obtain is a prerequisite for our subsequent papers on $p$-adic cyclotomic multiple zeta values. This is Part I-1 of $p$-adic cyclotomic multiple zeta values and $p$-adic pro-unipotent harmonic actions.

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0. Introduction

0.1. Cyclotomic multiple zeta values and multiple polylogarithms. Cyclotomic multiple zeta values are the following complex numbers, where $(n_i)_{d} = (n_1, \cdots, n_d)$ is a tuple of positive integers, and
\( \zeta((n_i)_d; (\xi)_d) = \sum_{0 < m_1 < \ldots < m_d} \frac{(\xi_1)^{m_1} \cdots (\xi_d)^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}} \) 

(0.1)

Denoting by \( n = n_d + \cdots + n_1 \) and \( (\epsilon_n, \ldots, \epsilon_1) = (0, \ldots, 0, \xi_d, \ldots, 0, \ldots, 0, \xi_1) \), we have

(0.2)

\[
\zeta((n_i)_d; (\xi)_d) = (-1)^d \int_{t_n = 0}^{1} \frac{dt_n}{t_n - \epsilon_n} \int_{t_{n-1} = 0}^{t_n} \cdots \int_{t_1 = 0}^{t_2} \frac{dt_1}{t_1 - \epsilon_1}
\]

Equation (0.2) shows that cyclotomic multiple zeta values are periods of the comparison between the Betti and De Rham realizations of the pro-unipotent fundamental groupoid \( \pi^\text{un}_1 \) of \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \) (DG, §5.16).

Multiple polylogarithms \( (\mathbb{C}) \) are the following power series, which are convergent for \( z \in \mathbb{C} \) such that \( |z| < 1 \):

(0.3)

\[
\operatorname{Li}((n_i)_d; (\xi)_d)(z) = (-1)^d \sum_{0 < m_1 < \ldots < m_d} \frac{(\xi_1)^{m_1} \cdots (\xi_d)^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}}
\]

They are solutions to the KZ equation on \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \):

(0.4)

\[
d \operatorname{Li}((n_i)_d; (\xi)_d)(z) = \left\{ \begin{array}{ll}
\frac{dz}{z} \operatorname{Li}((n_i)_{d-1}, n_d - 1; (\xi)_d) & \text{if } n_d \geq 2 \\
\frac{dz}{z - \xi_d} \operatorname{Li}((n_i)_{d-1}; (\xi)_d) & \text{if } n_d = 1
\end{array} \right.
\]

0.2. \( p \)-adic cyclotomic multiple zeta values, \( p \)-adic multiple polylogarithms and their overconvergent variants. Let \( p \) be a prime number which does not divide \( N \). Let \( K \) be a totally unramified finite extension of \( \mathbb{Q}_p \) containing the \( N \)-th roots of unity in \( \mathbb{Q}_p \). The KZ equation (0.3) has a Frobenius structure over \( K \), and this defines the crystalline pro-unipotent fundamental groupoid of \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \) (D §11, [C], [S], [S2]).

The theory of Coleman integration ([C], [B], [Y]) enables, by means of this Frobenius structure, to define \( p \)-adic analogues of the integrals (0.2) and (0.3) : this defines \( p \)-adic cyclotomic multiple zeta values \( \zeta(p)_d((n_i)_d; (\xi)_d) \in K \) and \( p \)-adic multiple polylogarithms \( \operatorname{Li}^p_d((n_i)_d; (\xi)_d) \) \((\mathbb{P}^1, \mathbb{F}) \) for \( N = 1 \), \( \mathbb{Y} \) for any \( N \) (from now on we drop the assumption \( (n_d, \xi_d) \neq (1, 1) \) used above). The power series expansion of \( \operatorname{Li}_p^d((n_i)_d; (\xi)_d) \) at \( 0 \) is convergent for \( z \in K \) such that \( |z| < 1 \) and is identical to (0.3).

Our goal in this paper is to give the foundations of an explicit theory of \( p \)-adic cyclotomic multiple zeta values. For computations and explicit formulas, it is more convenient to consider another type of \( p \)-adic analogues of the integrals (0.2) and (0.3). Let \( \alpha \) be a positive integer. We consider the "Frobenius of the KZ equation iterated \( \alpha \) times" (Definition [LZ]), and we define the associated \( p \)-adic cyclotomic multiple zeta values (Definition [LZ]), as numbers

(0.6)

\[
\zeta_{p, \alpha}( (n_i)_d; (\xi)_d) \in K
\]

The simple relation between \( \zeta_{p, \alpha} \) and \( \zeta^p \) is explained in [LZ] (see also [Y], Proposition 3.10 and [F2], Theorem 2.8) for particular cases. We also define (Definition [LZ]) the overconvergent \( p \)-adic multiple polylogarithms

(0.7)

\[
\operatorname{Li}^p_{d, \alpha}( (n_i)_d; (\xi)_d)
\]

which are overconvergent rigid analytic functions on a certain convenient affinoid subspace of \( \mathbb{P}^{1, an}/K \). The definitions of \( \zeta_{p, \alpha} \) and \( \operatorname{Li}^p_{d, \alpha} \) generalize definitions in [DG], [U1] for \( N = 1 \), and in [Y], [U2] for any \( N \), which correspond to particular values of \( \alpha \).
0.3. **Summary of the paper.** The setting and definitions are established in §1. An expression of $\text{Li}_{p,\alpha}^\dagger$ in terms of $L_p^{\text{KZ}}$ and $\zeta_{p,\alpha}$, equation (2.1.3), is established and studied in §2. We introduce and study a notion of regularized $p$-adic iterated integrals (§3), and, using §2 and §3, we characterize $\text{Li}_{p,\alpha}^\dagger$ and $\zeta_{p,\alpha}$ in terms of regularized iterated integrals (§4). This characterization gives our main result, which is a prerequisite for our explicit theory of $p$-adic cyclotomic multiple zeta values, established in [J.I-2], [J.I-3] and subsequent papers.

In the statement below, the norm $\| \|$ used is the norm on the $K$-algebra of global rigid analytic functions on the affinoid $U^{an} = \mathbb{P}^1, an \backslash \mathbb{U}_\zeta \mu_n(K) B(\xi, 1)$ over $K$ defined in terms of the power series expansion at 0 as $\| \sum_{n \in \mathbb{N}} a_n z^n \| = \sup_{n \in \mathbb{N}} |a_n|_p$. Here $B(\xi, 1) = \{ z \in K \mid |z - \xi|_p < 1 \}$.

A word of depth $d$ and weight $n$ is a sequence $w = ((n_i)_{d}; (\xi_i)_{d})$ as above, with $n_1 + \cdots + n_d = n$.

**Theorem.** For any $d \in \mathbb{N}^*$, we have $\max_{w, \text{ word}} \max_{\text{weight}(w) = n, \text{depth}(w) = d} || \text{Li}_{p,\alpha}^\dagger (w) || n \to \infty 0$.

An explicit upper bound for any $|| \text{Li}_{p,\alpha}^\dagger (w) ||$ can be obtained by our proof; however, what we need for [J.I-2] and subsequent papers is only the above information. Our method also gives bounds on the norms $|\zeta_{p,\alpha}(w)|_p$ (§4.2), but this is not needed for our subsequent papers. In the $N = 1$ case, a more precise bound on $|\zeta_{p,\alpha}(w)|_p$ can be deduced from the works of Akagi-Hirose-Yasuda [AHY] and Chatais-tamating [Cha], who use completely different methods and work with lifts of Frobenius-invariant paths.

Appendix A is a description of the $K$-algebra of global rigid analytic function of the space $U^{an}$ mentioned above, which arises as a reformulation of a classical theorem of Mahler [M] in terms of the rigid analytic space $U^{an}$. It is a prerequisite for §§3 and §4. We will investigate its further meaning in terms of more general rigid analytic spaces in a subsequent paper.

Appendix B is a byproduct of the proof: considering all $p$’s and $\alpha$’s at the same time, and we define a certain explicit space of adelic sums of series, which plays the role of a $p$-adic analogue of the type of series appearing in [L.I] and enables to express the overconvergent $p$-adic multiple polylogarithms, resp. the $p$-adic cyclotomic multiple zeta values; it will reappear in all our explicit theory of $p$-adic cyclotomic multiple zeta values.

The strategy of the paper is inspired by a strategy for computing $p$-adic cyclotomic multiple zeta values, suggested by Deligne in depth one in his foundational paper [D], §19.6, and then used by Ünver in [U1], [U2], in depth one and two. Ünver’s paper [U3], which deals with the $N = 1$ case, was written at the same time with the present paper, and [U4] appeared after the present paper; the main result of [U1] [U2] [U3] [U4] is an algorithm to compute $p$-adic cyclotomic multiple zeta values (Theorem 1.1 of [U4]), which is implied by the result of the present paper. The explicit formulas for $p$-adic cyclotomic multiple zeta values which we can deduce from our proof, as well as those in [J.I-2] and [J.I-3], are different from Ünver’s.

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1. **Definitions**

We review some of the pro-unipotent fundamental groupoid of $\mathcal{P}^1 \setminus \{0, \mu_N, \infty\}$ (§1.1), and we generalize the definition of $p$-adic cyclotomic multiple zeta values (§1.2) and of overconvergent $p$-adic multiple polylogarithms (§1.3) by replacing the Frobenius by the "iterated Frobenius", which depends on the number of iterations $\alpha$ in $\mathbb{N}^* \cup \{-N^*\}$.

1.1. **Preliminaries on the pro-unipotent fundamental groupoid of $\mathcal{P}^1 \setminus \{0, \mu_N, \infty\}$**.
1.1. Non-commutative formal power series and shuffle Hopf algebras. For any \( \mathbb{Q} \)-algebra \( C \), and any set of formal variables \( a = \{ a_1, \ldots, a_r \} \), let \( C\langle a_1, \ldots, a_r \rangle \), resp. \( C\langle\langle a_1, \ldots, a_r \rangle\rangle \), be the non-commutative \( C \)-algebra of polynomials, resp. of formal power series over the non-commuting variables \( a_1, \ldots, a_r \), with coefficients in \( C \). Let \( \text{Wd}(a) \) be the set of words over the alphabet \( a = \{ a_1, \ldots, a_r \} \), including the empty word \( \emptyset \). It is a basis of the free \( C \)-module \( C\langle a_1, \ldots, a_r \rangle \).

**Notation 1.1.1.** For \( f = f(a_1, \ldots, a_r) \in C\langle a_1, \ldots, a_r \rangle \) and \( w \in \text{Wd}(a) \), let \( f[w] \in C \) be the coefficient of \( w \) in \( f \); i.e. we have \( f[w] = \sum_{w \in \text{Wd}(a)} f[w]w \). The notation \( f[w] \) extends to linear combinations of words by linearity.

The shuffle Hopf algebra \( C^{m,a} \) is the \( \mathbb{Q} \)-vector space \( \mathbb{Q} \langle a_1, \ldots, a_r \rangle \), graded by the length of words over \( a \), endowed with the shuffle product \( \cdot \) defined by \( (a_{n+1} \ldots a_{n+i}) \cdot (a_{n} \ldots a_{i-1}) \) where the sum is over permutations \( \sigma \) of \( \{ 1, \ldots, n+n' \} \) such that \( \sigma(n) > \ldots > \sigma(1) \) and \( \sigma(n+n') > \ldots > \sigma(n+1) \); the deconcatenation coproduct \( \Delta : a_{n} \ldots a_{i} \mapsto \sum_{n'=0}^{n} a_{n} \ldots a_{n'+1} \otimes a_{n'+1} \ldots a_{i} \); the counit \( \epsilon \) equal to the augmentation map; the antipode \( S : a_{n} \ldots a_{i} \mapsto (-1)^{n-1} a_{i} \ldots a_{n} \). (We order the words from the right to the left in order to follow the conventions in the literature on \( p \)-adic multiple zeta values.) We have

\[
\text{Spec}(C^{m,a})(C) = \{ f \in C\langle a_1, \ldots, a_r \rangle \mid \forall w, w' \in \text{Wd}(a), f[w \cdot w'] = f[w]f[w'], \text{ and } f[\emptyset] = 1 \}
\]

1.1.2. The groupoid \( \pi_1^{\text{un,DR}}(\mathbb{P}^1 \setminus \{ 0, \mu_N, \infty \}) \) and the canonical base-point \( \omega_{\text{DR}} \). The De Rham pro-unipotent fundamental groupoid \( \pi_1^{\text{un,DR}} \) of smooth algebraic varieties over a field of characteristic zero is defined in \( \text{[D]} \), §10.27, §10.30,(ii). Let \( X \) be \( \mathbb{P}^1 \setminus \{ 0, \mu_N, \infty \} \) over a field \( K \) of characteristic zero which contains a primitive \( N \)-th root of unity. The De Rham pro-unipotent fundamental groupoid of \( X \) is a groupoid in pro-affine schemes over \( X \), with the following base-points: the points of \( X \), the points of punctured tangent spaces \( T_x - \{ 0 \} \), \( x \in \{ 0, \infty \} \cup \mu_N(K) \) called tangential base-points (\( \text{[D]} \), §15), and the canonical base-point \( \omega_{\text{DR}} \) (\( \text{[D]} \), (12.4.1)).

Namely, if \( x, y \) are two base-points, we have a pro-affine scheme over \( X \) denoted \( \pi_1^{\text{un,DR}}(X, y, x) \), whose points are called the pro-unipotent De Rham paths from \( x \) to \( y \); we denote by \( \pi_1^{\text{un,DR}}(X, x, x) \) a pro-unipotent group scheme; if \( x, y, z \) are three base-points, we have a groupoid multiplication which is a morphism of schemes \( \pi_1^{\text{un,DR}}(X, z, y) \times \pi_1^{\text{un,DR}}(X, y, x) \to \pi_1^{\text{un,DR}}(X, z, x) \); the groupoid multiplication makes each \( \pi_1^{\text{un,DR}}(X, y, x) \) into a bi-torsor under \( \pi_1^{\text{un,DR}}(X, x, x) \).

(Following the convention of \( \text{[D]} \) and \( \text{[DG]} \), we denote the groupoid multiplication from the right to the left.)

By \( \text{[D]} \), §12, we have canonical isomorphisms of schemes \( \pi_1^{\text{un,DR}}(X, y, x) \cong \pi_1^{\text{un,DR}}(X, \omega_{\text{DR}}) \), which are compatible with the groupoid structure, and, canonical paths \( y 1_x \in \pi_1^{\text{un,DR}}(X, y, x)(K) \), which are compatible with the groupoid structure.

This reduces the description of the groupoid \( \pi_1^{\text{un,DR}}(X) \) to the one of the group scheme \( \pi_1^{\text{un,DR}}(X, \omega_{\text{DR}}) \).

By \( \text{[D]} \), §12.9, the affine group scheme \( \Pi = \pi_1^{\text{un,DR}}(X, \omega_{\text{DR}}) \) is canonically isomorphic to the exponential of the completed free Lie algebra over the generators \( e_x, x \in \{ 0 \} \cup \mu_N(K) \). This is also \( \text{Spec}(\mathcal{O}^{x,\epsilon_0,\mu_N}) \) where \( \epsilon_{0,\mu_N} \) is the alphabet \( \{ e_x \mid x \in \{ 0 \} \cup \mu_N(K) \} \). This object is described explicitly by §1.1.1.

1.1.3. The KZ connection and its formal canonical solution. By \( \text{[D]} \), (12.5.5), (12.12.1), one has the following canonical connection on \( \pi_1^{\text{un,DR}}(X, \omega_{\text{DR}}) \times X \) called the Knizhnik-Zamolodchikov connection and denoted by \( \nabla_{\text{KZ}} \), which appeared in equation (1.1.3):

\[
\nabla_{\text{KZ}} f := df - \left( e_0 \frac{dz}{z} + \sum_{\xi \in \epsilon_0,\mu_N(K)} e_\xi \frac{dz}{z - \xi} \right) f
\]

It has a canonical formal solution \( \text{Li} \in K[[z]][\log(z)][\langle e_0, \epsilon_0, \mu_N \rangle] \), whose coefficient \( \text{Li}(e_0^{n_0-1} e_{\xi_1} \cdots e_0^{n_1-1} e_{\xi_1}) \) is the power series (1.1.3), and whose coefficient \( \text{Li}(e_0) = \log(z) \).

The cyclotomic multiple harmonic sums are the numbers which arise as coefficients in the power series (1.1.3):

\[
\mathfrak{h}_m((n_i)_d; (\xi_i)_{d+1}) = m^{n_1 + \cdots + n_d} \sum_{0 < m_1 < \cdots < m_d < m} \left( \frac{\xi_2}{\xi_1} \right)^{m_1} \cdots \left( \frac{\xi_{d+1}}{\xi_d} \right)^{m_d} \left( \frac{1}{\xi_{d+1}} \right)^{m}
\]
where \( m, d \) and the \( n_i \)'s are positive integers and the \( \xi_i \)'s are \( N \)-th roots of unity.

1.1.4. Iterations of the crystalline Frobenius of \( \pi_{1, \text{un,DR}}^{(p^n)}(X_K) \). The notion of crystalline \( \pi_{1, \text{un}} \) is first defined in [D], \S 11, \S 13.6, as an enrichment by a crystalline Frobenius structure of the notion of \( \pi_{1, \text{un,DR}} \). Variants of this notion, defined more directly and in Tannakian terms, are given in [CL] and in [S1, S2]. We follow the point of view of [D] which is the most elementary one.

Let \( p \) be a prime number which does not divide \( N \). Let \( k = \mathbb{F}_q \), a finite field of characteristic \( p \), which contains a primitive \( N \)-th root of unity. Let \( R = W(k) \) its ring of Witt vectors ; it is the ring generated by \( \mathbb{Z}_p \) and the roots of unity of order prime to \( p \) whose reduction is in \( \mathbb{F}_q \). Let \( K \) be the field of fractions of \( R \), equal to the field generated by \( \mathbb{Q}_p \), and the same roots of unity. Let \( X_k, X_R \) and \( X_K \) be \( \mathbb{P}^1 \setminus \{0, \mu_N, \infty\} \) over, respectively, \( k, R \) and \( K \). Let \( \sigma : R \to R \) be the Frobenius automorphism of \( R \), which lifts the \( p \)-th power Frobenius of \( k \). We denote also by \( \sigma : K \to K \) its extension to \( K \). It is an automorphism of \( K \) which is an isometry for the \( p \)-adic metric.

Let \( \alpha \) be a positive integer, and \( \sigma^\alpha = \sigma \circ \cdots \circ \sigma \). For any \( \xi \in \mu_N(K) \), we have \( \sigma^\alpha(\xi) = \xi^{p^\alpha} \). Let \( X^{(p^n)}_R \) be the pull-back of \( X_R \) by \( \sigma^\alpha \) and \( X^{(p^n)}_K = X^{(p^n)}_R \times_{\text{Spec } R} \text{Spec } K \). By functoriality of the De Rham pro-unipotent fundamental groupoid, the description of \( \pi_{1, \text{un,DR}}(X_K) \) given in \S 1.1.1 provides a similar description of \( \pi_{1, \text{un,DR}}^{(p^n)}(X^{(p^n)}_K) \). We denote by \( \omega^{(p^n)}_{\text{DR}} \) the pull-back of the canonical base-point \( \omega_{\text{DR}} \) by \( \sigma^\alpha \). We denote by \( \mathcal{O}^{\text{uni,}\omega^{(p^n)}_{\text{DR}}}(\pi_{1, \text{un,DR}}^{(p^n)}(X^{(p^n)}_K), \omega^{(p^n)}_{\text{DR}}) \) : this is the shuffle Hopf algebra over the alphabet \( e_{0, \mu_N}^{(p^n)} = \{0\} \cup \{e_{\alpha} | \xi \in \mu_N(K) \} \) (we write \( e_{\alpha} \) and not \( \xi^{p^\alpha} \) in order to keep track of the fact that we are working on \( X^{(p^n)}_K \) and not on \( X_K \) ; this will be useful : see Definition 2.2.1). We denote by \( \nabla^{(p^n)}_{\text{KZ}} \) the pull-back of the \( \text{KZ} \) equation by \( \sigma^\alpha \), namely, the connection on \( \pi_{1, \text{un,DR}}^{(p^n)}(X^{(p^n)}_K), \omega^{(p^n)}_{\text{DR}} \times X^{(p^n)}_K \) defined by

\[
    f \mapsto df - \left( \frac{d\zeta'}{\zeta'} + \sum_{\xi \in \mu_N(K)} e_{\xi} \frac{dz'}{z' - \xi^{p^n}} \right) f
\]

By [D], \S 13.6, the crystalline Frobenius of \( \pi_{1, \text{un,DR}}^{(p^n)}(X_K) \) is a canonical \( \sigma \)-linear isomorphism of groupoids

\[
    \phi : \pi_{1, \text{un,DR}}^{(p^n)}(X^{(p^n)}_K) \simeq \pi_{1, \text{un,DR}}^{(p^n)}(X^K)
\]

equal to the inverse of the Frobenius \( F_{X/K*} \) defined in [D] \S 11.11 (11.11.2), (11.11.3). We propose to study, more generally, the following :

Definition 1.1.2. For \( \alpha \) a positive integer, let \( \phi_\alpha = \phi \circ \sigma^\alpha \circ \cdots \circ (\sigma^{\alpha-1})^* \). For \( \phi_\alpha \) is a canonical \( \sigma^\alpha \)-linear isomorphism \( \pi_{1, \text{un,DR}}^{(p^n)}(X^{(p^n)}_K) \simeq \pi_{1, \text{un,DR}}^{(p^n)}(X^K) \).

For \( \alpha \) divisible by the integer \( \frac{\log(\alpha)}{\log(p)} \), i.e. \( \alpha \) such that \( p^\alpha \) is a power of \( q \), we have \( \sigma^\alpha = \text{id} \) ; in particular, the source and target of \( \phi_\alpha \) are equal, and \( \phi_\alpha \) and is an iteration of \( \phi_{\frac{\alpha}{\log(p)}} \). More generally, for any positive integer \( \alpha \), we will abusively call \( \phi_{\pm \alpha} \) an "iteration of \( \phi_{\pm 1} \)."

1.2. \( p \)-adic cyclotomic multiple zeta values, \( p \)-adic multiple polylogarithms and their overconvergent variants : definitions. In the definitions, we use the canonical isomorphisms \( \pi_{1, \text{un,DR}}^{(p^n)}(X, y, x) \simeq \pi_{1, \text{un,DR}}^{(p^n)}(X_K, \omega_{\text{DR}}) = \text{Spec}(\mathcal{O}^{(p^n)}_{\text{uni}, \mu_N}) \) mentioned in \S 1.1.2.

1.2.1. \( p \)-adic cyclotomic multiple zeta values. We now generalize the notion of \( p \)-adic cyclotomic multiple zeta values, by replacing the Frobenius by the iterated Frobenius. Below, \( \nabla \) means the tangent vector \( \nabla \) at \( x \) ; it is a tangential base-point of \( \pi_{1, \text{un,DR}}^{(p^n)}(X_K) \) in the sense reviewed in \S 1.1.2.

Notation 1.2.1. (i) [DG], \S 5. For all \( x \in \{0\} \cup \mu_N(K) \), let \( \Pi_{x, \mu_N} = \pi_{1, \text{un,DR}}^{(p^n)}(X_K, \nabla, \nabla_0)(K) \)

(ii) For all \( x \in \{0\} \cup \mu_N(K) \), let \( \Pi_{x}^{(p^n)} = \pi_{1, \text{un,DR}}^{(p^n)}(X_K, \nabla, \nabla_0)(K) \)

(iii) [DG], \S 5.16. Let \( \tau : \mathbb{G}_m(\mathbb{Q}) \times K \langle \langle e_{0, \mu_N} \rangle \rangle \to K \langle \langle e_{0, \mu_N} \rangle \rangle \) be the group action defined by \( (\lambda, f)(\{e_{x} \}_{x \in \{0\} \cup \mu_N(K)}) \mapsto f(\{\lambda e_{x} \}_{x \in \{0\} \cup \mu_N(K)}) \).

The map \( \tau \) multiplies the coefficient of a word \( w \) in a power series \( f \in K \langle \langle e_{0, \mu_N} \rangle \rangle \) by \( \lambda^{\text{weight}(w)} \), where the weight of a word is its number of letters.

The \( p \)-adic cyclotomic multiple zeta values can be defined as coefficients of the unique element of
\[ \Pi_{1,0}(K) \text{ which is invariant by } \phi_{\log(p)} \text{ (} \mathbb{F}1 \text{ Definition 2.17, } \mathbb{F}2 \text{ Theorem 2.5, for } N = 1 \text{ ; } \mathbb{V} \text{ Definition 2.4 and §3.3 for any } N) \]. That point of view follows the theory of Coleman integration \[ \mathbb{C}0, \mathbb{B}3, \mathbb{V} ; \] we will study that notion and relate it to the one below in \[ \mathbb{I}1,3 \].

Alternatively, \( p \)-adic cyclotomic multiple zeta values can be defined as coefficients of the image of the Frobenius of the canonical path in \( \Pi_{1,0}(K) \) in the sense introduced in §1.1.2. This has been defined for \( N \in \{1,2\} \), \(\alpha = 1\) (\[ D\mathbb{C} \] §5.28) ; \( N = 1 \), \(\alpha = -1\) (\[ U\mathbb{I} \], §1) ; any \( N \) and \(\alpha = \frac{\log(q)}{\log(p)}\), (\[ Y \], Definition 3.1) ; any \( N \) and \(\alpha = -1\) (\[ U\mathbb{I} \], §2.2.3). We generalize these definitions. Below, the notation \( y_{1,x} \) refers to the canonical path from \( x \) to \( y \).

**Definition 1.2.2.** For any positive integer \(\alpha\), let

\[
\Phi_{p,\alpha} = (\tau(p^\alpha) \circ \phi_\alpha)(1_11_{\alpha}) \in \Pi_{1,0}(K)
\]

\[
\Phi_{p,-\alpha} = \phi_{-\alpha}(1_11_{\alpha}) \in \Pi(p^\alpha)(K)
\]

Let \( p \)-adic cyclotomic multiple zeta values be the following numbers, where the \( n_i \)'s are positive integers and the \( \xi_i \)'s are \( N \)-th roots of unity :

\[
\zeta_{p,\alpha}((n_i)d) = (-1)^d \Phi_{p,\alpha}[e_0^{n_d-1}e_{\xi_d} \ldots e_0^{n_1-1}e_{\xi_1}] \in K
\]

\[
\zeta_{p,-\alpha}((n_i)d) = (-1)^d \Phi_{p,-\alpha}[e_0^{n_d-1}e_{(\xi_d)^\alpha} \ldots e_0^{n_1-1}e_{(\xi_1)^\alpha}] \in K
\]

Here, the presence of the factor \( \tau(p^\alpha) \) is our convention which is meant to be adapted to the computations in our subsequent papers.

We generalize to our \( p \)-adic cyclotomic multiple zeta values the conjectural description of their algebraic relations stated for \(\alpha = 1\), \( N = 1 \), \[ D\mathbb{C} \], and which is implicit in \[ Y \] for any \( N \) and \(\alpha = \frac{\log(q)}{\log(p)}\).

**Conjecture 1.2.3.** For any positive integer \(\alpha\) the correspondence \(\zeta_{p,\alpha}(w) \mapsto \zeta(w)\) defines an isomorphism of algebras over the \( N \)-th cyclotomic field, from the algebra generated by the \( p \)-adic cyclotomic multiple zeta values \(\zeta_{p,\alpha}(w)\), to the algebra generated by cyclotomic multiple zeta values \(\mathbb{U}1\) moded out by the ideal generated by \(\zeta(2)\).

1.2.2. \( p \)-adic multiple polylogarithms. The \( p \)-adic multiple polylogarithms are defined as analogues of Frobenius-invariant paths, namely, i.e. they are Coleman integrals, as are the numbers \( \zeta_{p}^{K\mathbb{Z}} \) evoked in §1.2. Let us fix a determination \(\log_p\) of the \( p \)-adic logarithm. Let \( L_{p,X_K}^{K\mathbb{Z}} \),resp. \( L_{p,X_K}^{p,X_K} \), be the unique non-commutative generating series of Coleman functions on \( X_K \), resp. \( X_K^{p,X_K} \), relatively to the chosen determination of \(\log_p\), which is a horizontal section of \( \nabla_{K\mathbb{Z}} \) resp. \( \nabla_{p,X_K} \) and satisfies the asymptotics

\[
\text{Li}_{p,X_K}^{K\mathbb{Z}}(z) \sim e_0^\log_p(z), \text{ resp. } \text{Li}_{p,X_K}^{p,X_K}(z) \sim e_0^{\log_p(z)}.
\]

The coefficients of these formal power series are called \( p \)-adic multiple polylogarithms (for \( N = 1 \) and depth 1, \[ C\mathbb{O} \] ; for \( N = 1 \), \[ F\mathbb{I} \] Theorem 3.3 ; for any \( N \), \[ Y \], Proposition 2.6).

For \( z \in K \) such that \( |z|_p < 1 \), one has the following power series expansion (compare with \[ E\mathbb{X} \] and see also equation \[ I1.I3 \]) :

\[
L_{p,X_K}^{K\mathbb{Z}}[e_0^{n_d-1}e_{\xi_d} \ldots e_0^{n_1-1}e_{\xi_1}](z) = (-1)^d \sum_{0 < m_1 < \ldots < m_d} \frac{\left(\frac{\xi_d}{\xi_1}\right)^{n_1} \ldots \left(\frac{\xi_{d-1}}{\xi_1}\right)^{n_d-1} \left(\frac{\xi_d}{\xi_0}\right)^{n_0}}{m_1^{n_1} \ldots m_d^{n_d}}
\]

\[
L_{p,X_K}^{p,X_K}[e_0^{n_d-1}e_{(\xi_d)^\alpha} \ldots e_0^{n_1-1}e_{(\xi_1)^\alpha}](z) = (-1)^d \sum_{0 < m_1 < \ldots < m_d} \frac{\left(\frac{\xi_d^{(\alpha)}}{\xi_1^{(\alpha)}}\right)^{n_1} \ldots \left(\frac{\xi_{d-1}^{(\alpha)}}{\xi_1^{(\alpha)}}\right)^{n_d-1} \left(\frac{\xi_d}{\xi_0}\right)^{n_0}}{m_1^{n_1} \ldots m_d^{n_d}}
\]

The expression of \( L_{p,X_K}^{K\mathbb{Z}} \) and \( L_{p,X_K}^{p,X_K} \) in terms of Frobenius-invariant paths can be found in \[ F\mathbb{F} \], Theorem 2.3 for \( N = 1 \). We note that, in this paper, \( L_{p,X_K}^{K\mathbb{Z}} \) and \( L_{p,X_K}^{p,X_K} \) are the only objects which depend on a choice of determination of the \( p \)-adic logarithm.
1.2.3. Overconvergent variants of $p$-adic multiple polylogarithms associated with the iterated Frobenius.

The overconvergent variants of $p$-adic multiple polylogarithms are defined by means of the images of the canonical De Rham paths by the Frobenius, like the numbers $\zeta_{p,\alpha}$ (Definition 1.2.2), but at variable base-points. This requires to choose an affinoid subspace of $\mathbb{P}^{1,an}/K$.

**Notation 1.2.4.** Let $U^{an}$ be the affinoid rigid analytic space $(\mathbb{P}^{1,an} - \cup_{\xi \in \mu_N(K)} B(\xi, 1))/K$, where $B(\xi, 1)$ is the disk of points whose reduction modulo $p$ is $\xi$.

Let $A(U^{an})$ resp. $A^1(U^{an})$ be the $K$-algebra of global rigid analytic functions resp. overconvergent global rigid analytic functions over $U^{an}$. They are Banach $K$-algebras with the norm defined in terms of the power series expansion at $0$ by $\| \sum c_m z^m \| = \sup_{m \in \mathbb{N}} |c_m|_p$.

Let $F$ be the $\sigma$-linear lift of Frobenius on $A(U^{an})$ defined by $f(z) \mapsto f(z^p)$.

The space $U^{an}$ is considered first in [D], §19.6 for $N = 1$, and in [Y] for any $N$, and used later in Furusho’s and Ünver’s papers on $pMZV_{\mu_N}$’s.

In the next definition, the left multiplication by a canonical path below is a convention will be practical for the computations in §2.1. This definition generalizes particular cases in [D], §19.6, [U1], [U2] and [Y].

**Definition 1.2.5.** For $\alpha$ a positive integer, for $z \in U^{an}(K)$,

$$\text{Li}_{p,\alpha}(z) = \xi_{0}^{I} \cdot z^{\pi^\infty}. \varphi^{\alpha}(z \xi_{0}) \in \Pi_{0,0}(K)$$

$$\text{Li}_{p,\alpha}^{\dagger}(z) = \xi_{0}^{I} \cdot z^{\pi^\infty}. \varphi^{-\alpha}(z \xi_{0}) \in \Pi_{0,0}(K)$$

The overconvergent $p$-adic multi-polylogarithms are the following functions, where the $n_i$’s are positive integers and the $\xi_i$'s are $N$-th roots of unity :

$$\text{Li}_{p,\alpha}^{\dagger}((n_i)_{d}; (\xi_i)) = (-1)^{d} \text{Li}_{p,\alpha}^{\dagger}(\varepsilon_{0}^{n_{d} - 1} \varepsilon_{\xi_{d}} \cdots \varepsilon_{0}^{n_{1} - 1} \varepsilon_{\xi_{1}}) \in A^{1}(U^{an})$$

$$\text{Li}_{p,\alpha}((n_i)_{d}; (\xi_i)) = (-1)^{d} \text{Li}_{p,\alpha}^{\dagger}(\varepsilon_{0}^{n_{d} - 1} \varepsilon_{\xi_{d}}^{(p)} \cdots \varepsilon_{0}^{n_{1} - 1} \varepsilon_{\xi_{1}}^{(p)}) \in A^{1}(U^{an})$$

**Remark 1.2.6.** In the case of $\mathbb{P}^{1} \setminus \{0, 1, \infty\}$, i.e. $N = 1$, the space $U^{an}$ is equal to $\mathbb{P}^{1,an} \setminus B(1, 1)$, and there exists another natural choice of affinoid subspace, namely $\mathbb{P}^{1,an} \setminus B(\infty, 1)$, equal to the analytic unit disk $\mathbb{Z}^{an}_{p}$. Those two spaces differ by the homography $z \mapsto \frac{1}{z - 1}$, which sends $(0, 1, \infty) \mapsto (0, \infty, 1)$. The reason for choosing $U^{an}$ instead of $\mathbb{Z}^{an}_{p}$ is that working with $\mathbb{Z}^{an}_{p}$ would require to replace $f(z) \mapsto f(z^p)$ in Notation 1.2.3 by $f(z) \mapsto f(\frac{z}{z - 1})$, i.e. the conjugation of $f(z) \mapsto f(z^p)$ by $f(z) \mapsto f(\frac{z}{z - 1})$, which would make more complicated the computations in the next sections.

In the rest of this work, we focus of the objects defined through positive number of iterations of the Frobenius, i.e. $\zeta_{p,\alpha}$ and $\text{Li}_{p,\alpha}^{\dagger}$ with $\alpha$ a positive integer. One has similar results and proofs for $\zeta_{p,\alpha}$ and $\text{Li}_{p,\alpha}^{\dagger}$, which are left to the reader. We will refer to $\zeta_{p,\alpha}^{\dagger}$ and $\text{Li}_{p,\alpha}^{\dagger}$ in [J.I-3], where they will appear naturally.

2. Overconvergent $p$-adic multiple polylogarithms and iterated integrals

The formal properties of the Frobenius imply an expression of the overconvergent $p$-adic multiple polylogarithms in terms of $p$-adic cyclotomic multiple zeta values and $p$-adic multiple polylogarithms (Proposition 2.1.3) ; we encode it as an expression of overconvergent $p$-adic multiple polylogarithms in terms of certain $p$-adic iterated integrals, which is inductive with respect to the weight (Proposition 2.2.3).

**Notation 2.0.1.** $\omega_{z_0}(z) = \frac{z - z_0}{z^{\infty} - z_0}$ for all $z_0 \in \{0\} \cup \mu_N(K)$.

2.1. The differential equation satisfied by overconvergent $p$-adic multiple polylogarithms.

By [D], §11.11, the Frobenius $\varphi$ is characterized by the fact that it commutes with the connections $\nabla_{KZ}$ and $\nabla^{(p)^\infty}_{KZ}$. In this paragraph, we make this commutation property explicit. According to [D], §11.9, this amounts to write a collection of differential equations satisfied by the Frobenius, one for each element of an open affinoid covering of $\mathbb{P}^{1,an}/K$, and gluing them. For our purposes, it will actually be enough to consider the affinoid subspace $U^{an}$ of §1.3, and only the values of the Frobenius at canonical paths, namely, the overconvergent $p$-adic multiple polylogarithms.

We will view the KZ connection as a connection on $\Pi_{0,0} \times X$. Thus, let us write it the Frobenius on $\Pi_{0,0}(K)$, in terms of $p$-adic cyclotomic multiple zeta values.
Notation 2.1.1. For any $\xi \in \mu_N(K)$, let $f \mapsto f^{(\xi)}$ the map $\Pi_{1,0}(K) \to \Pi_{\xi,0}(K)$ induced by the functoriality of $\pi_1^{un,DR}(X_K)$ with respect to the automorphism $z \mapsto \xi z$ of $X_K$.

Lemma 2.1.2. The map $\tau(p^\alpha) \phi^{\alpha}$ sends $e_0 \in \text{Lie}(\Pi_{0,0}^{(p^\alpha)})$ to $e_0$ and, for all $\xi \in \mu_N(K)$, it sends $e_{\xi(p^\alpha)} \in \text{Lie}(\Pi_{0,0}^{(p^\alpha)})$ to $\text{Ad}_{\Phi_{\gamma,\beta}^{(\xi)}}(e_\xi) = (\Phi_{\gamma,\beta}^{(\xi)})^{-1}e_{\xi}\Phi_{\gamma,\beta}^{(\xi)}$.

(We denote $(\Phi_{\gamma,\beta}^{(\xi)})^{-1}e_{\xi}\Phi_{\gamma,\beta}^{(\xi)}$ by $\text{Ad}_{\Phi_{\gamma,\beta}^{(\xi)}}(e_\xi)$ and not by $\text{Ad}_{(\Phi_{\gamma,\beta}^{(\xi)})^{-1}}(e_\xi)$ in order to be coherent with the fact that we read the groupoid multiplication from the right to the left.)

Proof. Let $x \in \{0\} \cup \mu_N(K)$ and $T_x$ be the tangent space to $\mathbb{P}^1$ at $x$. We have $T_x \setminus \{0\} \simeq \mathbb{G}_m$, and thus by [2.1.4] $\pi_1^{un,DR}(T_x \setminus \{0\})$ admits a canonical base-point $\omega_{\text{DR}}$ satisfying all the properties of §1.1 such that we have $\pi_1^{un,DR}(T_x \setminus \{0\}, \omega_{\text{DR}}) = \text{Spec}(\mathcal{O}^{\text{un}}(e_\alpha))$. It is endowed with the canonical connection on $\pi_1^{un,DR}(T_x \setminus \{0\}, \omega_{\text{DR}}) \times T_x \setminus \{0\}$ defined by $f \mapsto \frac{d}{dz}e_x f$, whose crystalline Frobenius structure is the trivial one given by $e_x \mapsto \frac{1}{p}e_x$.

By the compatibility between the Frobenius and the tangential base-points ([1], §15), for each $\xi \in \mu_N(K)$, we have

$$\begin{equation}
\phi^{\alpha} : e_{\xi(p^\alpha)} \in \text{Lie}(\Pi_{\xi,0}^{(p^\alpha)}) \mapsto \frac{1}{p^\alpha}e_\xi \in \text{Lie}(\Pi_{\xi,0}^{(p^\alpha)})
\end{equation}$$

By the compatibility between canonical paths and the groupoid structure of $\pi_1^{un,DR}(X_K)$, we have the following equality, where, in the left-hand side $e_{\xi(p^\alpha)} \in \text{Lie}(\Pi_{0,0})$ and, in the right-hand side, $e_{\xi(p^\alpha)} \in \text{Lie}(\Pi_{\xi,0}^{(p^\alpha),\xi(p^\alpha)})$:

$$\begin{equation}
e_{\xi(p^\alpha)} = (a_{1\xi(p^\alpha)})e_{\xi(p^\alpha)}(\xi(p^\alpha)1)0
\end{equation}$$

We apply $\tau(p^\alpha) \phi^{\alpha}$ to equation (2.1.2) : by the compatibility between the Frobenius with the groupoid structure of $\pi_1^{un,DR}(X_K^{(p^\alpha)})$, this gives $(\tau(p^\alpha) \phi^{\alpha})(e_{\xi(p^\alpha)}) = (\tau(p^\alpha) \phi^{\alpha})(a_{1\xi(p^\alpha)})$.

By Definition 2.1.2 and equation (2.1.1), we obtain the result. \hspace{1cm} \Box

The above Lemma generalizes a known fact ($N \in \{1, 2\}$ and $\alpha = 1$, [DG] §5.28 ; $N = 1$ and $\alpha = -1$, [U1], §4.3 ; any $N$ and $\alpha = \log_{(p)}(\gamma)$ ; [Y] ; any $N$ and $\alpha = -1$, [U2], equation (2.2.6)).

The differential equation which characterizes the Frobenius can be formulated as a differential equation satisfied by $\mathcal{L}_{1, \alpha}^{\dagger}$, or as a functional relation involving $p$-adic multiple polylogarithms and their overconvergent variants. We use Notation 2.1.

Proposition 2.1.3. We have :

$$\begin{equation}
d\mathcal{L}_{i, \alpha}^{\dagger}(z) = \left(p^\alpha \omega_0(z)e_0 + \sum_{\xi \in \mu_N(K)} p^\alpha \omega_\xi(z)e_\xi \right) - \mathcal{L}_{i, \alpha}^{\dagger}(z) = \mathcal{L}_{i, \alpha}^{\dagger}(z) - \mathcal{L}_{i, \alpha}^{\dagger}(z)
\end{equation}$$

Equivalently,

$$\begin{equation}
\mathcal{L}_{i, \alpha}^{\dagger}(z) = \mathcal{L}_{i, \alpha}^{\dagger}(z) - \mathcal{L}_{i, \alpha}^{\dagger}(z)
\end{equation}$$

Proof. By [2.1.4], §7.30.28, we view the KZ connection as the connection on the trivial bundle $\Pi_{0,0} \times X_K$ with values in $\text{Lie}\Pi_{0,0} \otimes \mathcal{O}^1(X_K)$ given by

$$\begin{equation}
f \mapsto f^{-1}(df - (e_0\omega_0 + \sum_{\xi \in \mu_N(K)} e_\xi\omega_\xi)f)
\end{equation}$$

By [2.1.4.1], the Frobenius $\phi$ commutes with the connections, which gives the commutativity of the following diagram (where $F$ is as in Notation 1.1.3)

$$\begin{equation}
\begin{array}{c}
(F_{\gamma})^*(\Pi_{0,0}^{(p^\alpha)}(K) \times X_K) \\
\downarrow \phi^{\alpha} \\
\Pi_{0,0}^{(p^\alpha)}(K) \times A(U^{an})
\end{array}
\end{equation}$$

We apply the arrows of (2.1.6) to the canonical De Rham path $z_{1, \alpha}$:

(a) The canonical De Rham path $z_{1, \alpha}$ is mapped by $(F_{\gamma})^*\nabla_{\text{KZ}}^{(p^\alpha)}$ to $-\omega_0(z^{p^\alpha})e_0 - \sum_{\xi \in \mu_N(K)} \omega_\xi(z^{p^\alpha})e_{\xi(p^\alpha)}$. 

[1] David Jarossay

[2] David Jarossay

[3] David Jarossay

[4] David Jarossay

[5] David Jarossay

[6] David Jarossay

[7] David Jarossay

[8] David Jarossay

[9] David Jarossay

[10] David Jarossay

[11] David Jarossay

[12] David Jarossay

[13] David Jarossay
(by equation (2.1.3)), which is mapped by $\phi^\alpha \otimes \text{id}$ to $\frac{1}{p^\alpha} \omega_0(z) e_0 - \sum_{\xi \in \mu(N)(K)} \omega_\xi(z) \frac{1}{p^\alpha} Ad_{\tau(p^{-\alpha})\Phi^{(\alpha)}_{p}(\epsilon_\xi)}$ (by Lemma 2.2.5).

(b) The canonical De Rham path $z L_{p,\alpha}$ is mapped by $\phi^\alpha$ to $\tau(p^{-\alpha}) L_{p,\alpha}^j$ (by Definition 1.2.6), which is mapped by $\nabla_{KZ}$ to $(\tau(p^{-\alpha}) L_{p,\alpha}^j)^{-1} \left( d\tau(p^{-\alpha}) L_{p,\alpha}^j - (\omega_0(z) e_0 + \sum_{\xi \in \mu(N)(K)} \omega_\xi(z) e_\xi) \tau(p^{-\alpha}) L_{p,\alpha}^j \right)$ (by equation (2.1.5)).

The results of (a) and (b) above are the same by the commutativity of (2.1.6); applying $\tau(p^\alpha)$ to (2.1.6) we obtain equation (2.1.7).

Now, by the definition of $p$-adic multiple polylogarithms (§1.3.1) we have $\nabla_{KZ} L_{p,\alpha}^{pK} = 0$ and $\nabla_{KZ} L_{p,\alpha}^{pK(\alpha)} = 0$, whence the right-hand side of (2.1.4) is, like $L_{p,\alpha}^j$, a solution to the following differential equation on $L$:

\[
(2.1.7) \quad dL = \left( p^\alpha \omega_0(z) e_0 + \sum_{\xi \in \mu(N)(K)} p^\alpha \omega_\xi(z) e_\xi \right) L - L \left( \omega_0(z p^\alpha) e_0 + \sum_{\xi \in \mu(N)(K)} \omega_\xi(p^\alpha(z) e_\xi) \right)
\]

By Definition 1.2.5 we have $L_{p,\alpha}^j(0) = 1$, and by the definition of $L_{p,\alpha}^{pK}$ and $L_{p,\alpha}^{pK(\alpha)}$ (§1.3.1) the right-hand side of (2.1.4) is equivalent when $z \rightarrow 0$ to $e^{p^\alpha \log(z) e_0} e^{-p^\alpha \log(z p^\alpha) e_\xi} = 1$, where we have used $\log(z p^\alpha) = p^\alpha \log(p z)$. The equation (2.1.7) being pro-unipotent, it has a unique solution $L \in K[[[\epsilon_0,\mu(N)]]]$ such that $L(0) = 1$ (for any word $w$, equation (2.1.7) plus $L(0) = 1$ determine $L[w]$ by induction on the weight of $w$).

Variants of equation (2.1.4) for $N = 1$, $\alpha = -1$ are written, [13] §19.6, equation (19.6.2), [14] §5.2 Proposition 1; any $N$, $\alpha = -1$, [15] equation (2.2.9); the particular case for any $N$, $\alpha = 0$ [16] Remark 3.2. Equation (2.1.3) is written in the case $N = 1$, $\alpha = 1$, in [17] Theorem 2.14.

2.2. Decomposition of overconvergent $p$-adic multiple polylogarithms in terms of iterated integrals. The goal of this paragraph is to rewrite (2.1.3), as a "decomposition" of $L_{p,\alpha}$ in terms of certain iterated integrals. It arises initially as given by an induction on the weight. We will turn this induction into an induction on the depth in §4.

Definition 2.2.1. Let $e_{0,\mu(N) \cup \mu(N)^{p\alpha}}$ be the alphabet $\{ e_0, e_{\xi_1}, \ldots, e_{\xi_N}, e_{\xi_N^{p\alpha}}, \ldots, e_{\xi_N^{n\alpha}} \}$. The weight, resp. depth of a word over the alphabet $e_{0,\mu(N) \cup \mu(N)^{p\alpha}}$ is its number of letters, resp. its number of letters distinct from $e_0$.

Identifying the alphabet $e_{0,\mu(N) \cup \mu(N)^{p\alpha}}$ to $\{ p^\alpha \omega_0(z) \} \cup \{ p^\alpha \omega_\xi(z) \ | \ \xi \in \mu(N)(K) \} \cup \{ \omega_\xi(z p^\alpha) \ | \ \xi \in \mu(N)(K) \}$ in the natural way, we regard the right-hand side of equation (2.1.4) as a map $\mathcal{O}^{\mu e_{0,\mu(N) \cup \mu(N)^{p\alpha}}} \rightarrow K[[\epsilon]]$, which we are going to factorise by certain iterated integrals (Definition 2.2.3) and a certain map of "decomposition" (Definition 2.2.2). In order to define the map of decomposition, let us write explicitly the coefficients of equation (2.1.3): for any $w = e_0^{n_0-1} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1}$, where the $n_i$’s are positive integers and the $\xi_i$’s are $N$-th roots of unity, we have

\[
(2.2.1) \quad dL_{p,\alpha}^j(e_0^{n_0-1} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1}) = \sum_{w_1, w_2} \sum_{\text{words } w_1, w_2} \left( \text{Ad}_{\Phi^{(\alpha)}_{p}(\epsilon_\xi)}(e_\xi)[w_2] \right) \omega_{p^\alpha(z)} L_{p,\alpha}^j[e_0^{n_0-1} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1}]
\]

where the restriction to $w_2 \neq \emptyset$ in the first line of (2.2.1) above is due to the fact that the term indexed by $(w_1, w_2) = (w, \emptyset)$ is 0: the coefficient of the empty word in $\text{Ad}_{\Phi^{(\alpha)}_{p}(\epsilon_\xi)}(e_\xi)$ is zero because $\text{Ad}_{\Phi^{(\alpha)}_{p}(\epsilon_\xi)}(e_\xi)$ is obtained by multiplying two non-commutative formal power series $e_\xi$, which is of weight 1.
Definition 2.2.2. Let $\text{dec}_{p,\alpha} : \mathcal{O}^{m, e_{n,\mu; N}} \to \mathcal{O}^{m, e_{n,\mu; N} \cup (p^\alpha)} \otimes \mathbb{Q} K$ be defined by induction on the weight, as follows: $\text{dec}_{p,\alpha}(0) = 1$; for all $n \in \mathbb{N}^*$, $\text{dec}_{p,\alpha}(e_0^n) = e_0 \text{dec}_{p,\alpha}(e_0^{n-1}) - e_0 \text{dec}_{p,\alpha}(e_0^{n-1}) = 0$, and for any $w = e_0^{n_1} e_{\xi_1} \ldots e_0^{n_{k-1}} e_{\xi_{k-1}} e_0^{n_k-1}$ word over $e_{0,\mu; N}$, where the $n_i$'s are positive integers and the $\xi_i$'s are $N$-th roots of unity ($d \geq 1$),

\[(2.2) \quad \text{dec}_{p,\alpha}(e_0^{n_1} e_{\xi_1} \ldots e_0^{n_{k-1}} e_{\xi_{k-1}} e_0^{n_k-1}) = \sum_{\text{words } w_i \text{ with } \text{depth}(w_i) \geq 1 \text{ or } w_i = \emptyset} \sum_{\xi \in \mu; N(K)} \left( \text{Ad}_{\Phi_{\xi,\alpha}}(e_\xi)[w_2]\right) e_\xi(\mu\alpha) \text{ dec}_{p,\alpha}(w_1) + \]


Given that $\text{Li}^i_{p,\alpha}$ has no pole at 0 by Definition 1.2.5, we can consider only iterated integrals which have no pole at 0:

Definition 2.2.3. Let $\mathcal{O}_{\text{conv}}^{m, e_{n,\mu; N} \cup (p^\alpha)} \subset \mathcal{O}^{m, e_{n,\mu; N} \cup (p^\alpha)}$ be the vector subspace generated by words whose rightmost letter is not $e_0$. Let

\[
\int_{p,\alpha} : \mathcal{O}_{\text{conv}}^{m, e_{n,\mu; N} \cup (p^\alpha)} \to K[[z]]
\]

be the linear map defined by induction by $\int 0 = 1$ and the following variant of the KZ equation (1.3):

\[
(2.3) \quad d \int_{p,\alpha} e_0^{n_d-1} e_{\xi_d} \ldots e_0^{n_1-1} e_{\xi_1} u_1 = \begin{cases} p^\alpha \frac{dz}{z} \int_{p,\alpha} e_0^{n_d-2} e_{\xi_d} \ldots e_0^{n_1-1} e_{\xi_1} u_1 & \text{if } n_d \geq 2 \\ p^\alpha \frac{dz}{z} \int_{p,\alpha} e_0^{n_d-1} e_{\xi_d} \ldots e_0^{n_1-1} e_{\xi_1} u_1 & \text{if } n_d = 1 \text{ and } u_d = 1 \\ e_0^{n_d-1} e_{\xi_d} \ldots e_0^{n_2-1} e_{\xi_2} u_2 & \text{if } u_1 = 1 \text{ and } u_2 = (p^\alpha) \end{cases}
\]

and the condition that $\int_{p,\alpha} e_0^{n_d-1} e_{\xi_d} \ldots e_0^{n_1-1} e_{\xi_1} u_1$ has constant coefficient equal to 0.

Explicitly, these iterated integrals are given as the power series expansions, which are convergent on $\{z \in K \mid |z|_p < 1\}$; this is a variant of (1.3):

\[
(2.4) \quad \int_{p,\alpha} e_0^{n_d-1} e_{\xi_d} \ldots e_0^{n_1-1} e_{\xi_1} u_1 = (p^\alpha) \frac{dz}{z} \sum_{n_1 + \ldots + n_d = 0} \frac{(\xi_1)_{n_1} \ldots (\xi_d)_{n_d}}{m_1^{n_1} \ldots m_d^{n_d}} \in K[[z]]
\]

We can now reformulate equation (2.3): below we view $\text{Li}^i_{p,\alpha}$ as a map $\mathcal{O}^{m, e_{n,\mu; N}} \to K[[z]]$, which is enabled by the fact that each $\text{Li}^i_{p,\alpha}[w]$, being an element of $A^i(U^m)$, is characterized by its power series expansion at 0.

Proposition 2.2.4. The map $\text{dec}_{p,\alpha}$ take values in $\mathcal{O}_{\text{conv}}^{m, e_{n,\mu; N} \cup (p^\alpha)}$ and we have

\[
(2.5) \quad \text{Li}^i_{p,\alpha} = \int_{p,\alpha} \circ \text{dec}_{p,\alpha}
\]

Proof. The differential equation (2.2.1) together with $\text{Li}^i_{p,\alpha}(z = 0) = 1$ determine $\text{Li}^i_{p,\alpha}[w]$ for all words $w$ by induction on the weight of $w$. By equations (1.1.2) and (2.2.4), $\int_{p,\alpha} \circ \text{dec}_{p,\alpha}$ satisfies the same differential equation than (2.2.1). Moreover, by Definition 1.2.4 and Definition 2.2.1, for any word $w$, \((\int_{p,\alpha} \circ \text{dec}_{p,\alpha})(w)(z = 0) = 0\) if $w$ is non-empty and 1 if $w$ is the empty word. This concludes the proof. \(\square\)
Remark 2.2.5. One can also write another formula for $\text{dec}_{p,\alpha}$, by using operations in the shuffle Hopf algebra $O^{\text{m},c_{\mu}^{\alpha}\mu_{\nu}(K)}$ arising as duals of the product, inversion and composition of non-commutative formal power series which appear in equation (2.1.4).

3. Regularized $p$-adic iterated integrals and bounds on their norms

Let the affinoid space $f_{\text{an}} = B_{\text{an}} \setminus \{ \xi \in K \}$ and $A(U^{\text{an}})$ its $K$-algebra of global rigid analytic functions (Notation 1.2.2). We define in §3.2 regularized $p$-adic iterated integrals (Definition 3.2.3), which lie in a specific subalgebra of $\{ f \in A(U^{\text{an}}) \mid f(0) = 0 \}$ defined in §3.1 (Definition 3.1.1); we show that they can be computed by induction on the depth (§3.3, Proposition 3.2.6) and we prove bounds on their norms (§3.4, Proposition 3.3.1).

3.1. A subalgebra of the algebra of rigid analytic functions on a subspace of $B_{\text{an}}/K$.

By Proposition 3.0.1, the map "coefficients at 0", $\text{Ci}_0 : \left( f = \sum_{m \in \mathbb{N}} c_m z^m \right) \mapsto \left( m \in \mathbb{N} \mapsto c_m \in K \right)$ induces an isometric isomorphism of Banach spaces over $K$ where the target is the space of continuous functions on $A_{\text{an}}$ (Notation 1.2.4). We define in the absolutely convergent series expansion:

$$\sum_{\ell \in \mathbb{N}} \sum_{\xi \in \mu_{\mu}(K)} c_{\ell, \xi} |m - r_0|^l \in \mathbb{N}$$

Any element of $L_{\alpha_{\mu}}(Z_{\mu}(N), K)$ is locally analytic, by the fact that $N^* = \text{dense in } Z_{\mu}(N)$ and because for any $\xi \in \mu_{\mu}(K)$, the map $m \mapsto \xi^m$ is constant on each class of congruence modulo $N$. We have a natural isomorphism of vector spaces $L_{\alpha_{\mu}}(Z_{\mu}(N), K) \simeq S_{\alpha_{\mu}}^{N^0}$ defined by $c \mapsto \left( c_{\ell, \xi}(r_0) \right)_{\ell \in \mathbb{N}} \mapsto \xi_{\mu}(K)$, $c_{\ell, \xi}(r_0) = \sum_{l = 0}^{N^0} c_{\ell, \xi} |r_0|^l$.

Example 3.1.2. For any $\xi_0 \in \mu_{\mu}(K)$, $l \in \mathbb{N}$ and $\xi \in \mu_{\mu}(K)$,

(i) $\text{Ci}_0 \left( \frac{z^{\mu}}{z - \xi_0} \right)_{\ell = 0} = \sum_{m \in \mathbb{N}} \left( \frac{z^{\mu}}{\xi_0} \right)^{|m|} \in L_{\alpha_{\mu}}(Z_{\mu}(N), K)$ and $(\text{Ci}_0 \frac{z - \xi_0}{z - \xi_0})^{(l, \xi)}(r_0) = \left\{ \begin{array}{ll} -\rho^\alpha & \text{if } (l, \xi) = (0, \xi_0), \\
\text{otherwise} & \end{array} \right.$

(ii) $\text{Ci}_0 \left( \frac{z^{\rho^\alpha}}{z^{\rho^\alpha} - \xi_0} \right)_{\ell = 0} = \sum_{m \in \mathbb{N}} \left( \frac{z^{\rho^\alpha}}{\xi_0} \right)^{|m|} \in L_{\alpha_{\mu}}(Z_{\mu}(N), K)$ and $(\text{Ci}_0 \frac{z - \xi_0^{\rho^\alpha}}{z^{\rho^\alpha} - \xi_0^{\rho^\alpha}})^{(l, \xi)}(r_0) = \left\{ \begin{array}{ll} -\rho^\alpha & \text{if } (l, \xi, r) = (0, \xi_0, 0) \\
\text{otherwise} & \end{array} \right.$

Lemma 3.1.3. Let $l, l' \in \mathbb{N}$, $\xi \in \mu_{\mu}(K)$.

(i) The map sending $u \in \mathbb{N}^*$ to, respectively, $\sum_{u_1 = 1}^{u - 1} u_1 (u - u_1)^l \xi^{u_1}$, $\sum_{u_1 = 1}^{u_1} u_1 (u - u_1)^l \xi^{u_1}$, $\sum_{u_1 = 0}^{u - 1} u_1 (u - u_1)^l \xi^{u_1}$, $\sum_{u_1 = 0}^{u_1} u_1 (u - u_1)^l \xi^{u_1}$ is a $K$-linear combination of the functions $u \mapsto u^l \xi^u$, $l_0 \in \{0, \cdots, l + l' + 1\}$.

(ii) If we denote, respectively, by $B_{t_0}^{(l, l'), L}$, $B_{t_0}^{(l, l'), L}$, $B_{t_0}^{(l, l'), L}$ the coefficients of $u^{l_0} \xi^u$ in the above
expression, then we have

\[
\min \left( v_p(B_l^{(l', \xi)}), v_p(B_0^{(l', \xi)}), v_p(B_0^{-A_0(l', \xi)}), v_p(B_0^{-A_0(l', \xi)}) \right) \geq -1 - \frac{\log(1 + l + l')}{\log(p)}
\]

Proof. Let us prove the result for \( u^{-1} \sum_{u_1=1}^{u-1} u_1^j (u-u_1)^{l} \xi^{u_1} \), the other cases being similar. By writing \((u-u_1)^j = \sum_{l'=0}^{j} \binom{j}{l'} u^{l'} u_1^{j-l'} \), we are reduced to the case \( l' = 0 \). We distinguish two cases:

- If \( \xi = 1 \), we can write \( \sum_{u_1=1}^{u} u_1^j (u-u_1)^{l} \xi^{u_1} = \sum_{u_1=1}^{u} u_1^j \xi^{-u_1} - \sum_{u_1=1}^{u} u_1^j u^{-u_1} \xi^{-u_1} \). By Von Staudt-Clausen's theorem, we have \( v_p(B_t^{-u_1}) \geq 1 \) and, given that \( p^{n(l+1)} \leq 1 + 1 \) thus \( v_p(B_t^{-u_1}) = -v_p(l+1) \geq -1 - \frac{\log(1 + l + l')}{\log(p)} \), we have \( v_p(B_t^{-u_1}) B_t^{-u_1} \geq 1 - \frac{\log(1 + l + l')}{\log(p)} \).

- If \( \xi \neq 1 \), we consider the equation \( \sum_{u_1=1}^{u} T^{u_1} = \frac{1}{t} - T^{u_1} - 1 \), where \( T \) is a formal variable, we apply to it \( T \frac{1}{t-1} \), and we substitute \( \xi \) to \( T \) : this gives, by induction on \( \xi \), the existence of a polynomial \( P_l \), with coefficients in \( \mathbb{Z}[\xi, \frac{1}{t}, \frac{1}{\xi}] \subset \mathbb{Z}_p \), of degree \( l \), such that \( \sum_{u_1=1}^{u} u_1^j \xi^{u_1} = P(l)(u_1) \xi^{u_1} \).

We now prove that \( C_{f_0}^{-1}(L_{a \xi}(\mathbb{Z}_p(N), K)) \) (where \( C_{f_0} \) is the "map coefficients of the power series expansion at 0", see Proposition 3.1.4) is a subalgebra of \( A(U^a) \), with an explicit formula for the multiplication.

Proposition 3.1.4. Let \( f_1, f_2 \in A(U^a) \) with \( f_1(0) = f_2(0) = 0 \), such that \( c_1 = C_{f_0}(f_1) \) and \( c_2 = C_{f_0}(f_2) \) are elements of \( L_{a \xi}(\mathbb{Z}_p(N), K) \). Then the map \( c = C_{f_0}(f_1 f_2) \) is the element of \( L_{a \xi}(\mathbb{Z}_p(N), K) \) determined as follows: for all \( L \in \mathbb{N} \) and \( \xi \in \mathbb{N}_K(K) \),

\[
(3.1.2) \quad c^{(l, \xi)}(0) = \sum_{l', \xi \in \mu_N} \sum_{L \in \mathbb{N}} (p^{l\xi} + L) \sum_{\xi' \in \mu_N} \left( B_{L}^{(l', \xi') \rho_0} c_1^{(l, \xi)}(0) c_2^{(l', \xi')}(0) + \sum_{r=1}^{p^{l-1} - l r} B_{L}^{(l', \xi') \rho_0} (\xi')^{r} c_1^{(l, \xi)}(r) c_2^{(l', \xi')}(p^{l-1} - l r) \right)
\]

and, for all \( r_0 \in \{1, \ldots, p^{l-1} - 1\} \),

\[
(3.1.3) \quad c^{(l, \xi)}(r_0) = \sum_{l', \xi \in \mu_N} \sum_{L \in \mathbb{N}} (p^{l\xi} + L) \sum_{\xi' \in \mu_N} \left( \left( \frac{\xi'}{\xi} \right)^{(r_0 - r_0)} c_1^{(l, \xi)}(r) c_2^{(l', \xi')}(r_0) + \sum_{r=1}^{p^{l-1} - l r} B_{L}^{(l', \xi') \rho_0} (\xi')^{r} c_1^{(l, \xi)}(r) c_2^{(l', \xi')}(p^{l-1} - l r) \right)
\]

Proof. Let us prove first the part of result concerning the power series expansion of \( c \) at \( r_0 = 0 \). For any \( u \in \mathbb{N}^a \), we have

\[
c(p^a u) = \sum_{u=A}^{u=1} c_1(p^a u) c_2(p^a u - m) = \sum_{u=1}^{u=A} c_1(p^a u) c_2(p^a (u-u')) + \sum_{u=1}^{u=A} c_1(p^a u') c_2(p^a (u-u') + p^{a+1} - 1)
\]

Using the hypothesis that \( c_1 \) and \( c_2 \) are in \( L_{a \xi}(\mathbb{Z}_p(N), K) \), we obtain, with the notations of Definition 3.1.1

\[
(3.1.4) \quad c(p^a u) = \sum_{u=1}^{u=A} \sum_{l', \xi \in \mu_N} \sum_{l' \in \mathbb{N}} c_1^{(l, \xi)}(0) \xi^{-p^a u'} (p^a u') \xi^{(l', \xi')}(0) \xi^{-p^a (u-u')} (p^a (u-u'))^{l'}
\]

\[
+ \sum_{u=1}^{u=A} \sum_{r=1}^{p^{a+1} - 1} \sum_{l', \xi \in \mu_N} \sum_{l' \in \mathbb{N}} c_1^{(l, \xi)}(r) \xi^{-p^a u'} (p^a u') \xi^{(l', \xi')}(p^a - r) \xi^{-p^a (u-1-u')} (p^a (u-u'))^{l'}
\]
Expressing the sums \( \sum_{u=1}^{u-1} \left( \frac{\xi}{\xi} \right)^{p^u u^u} u^u (u - u)'' \) and \( \sum_{u=0}^{u-1} \left( \frac{\xi}{\xi} \right)^{p^u u^u} u^u (u - 1 - u)'' \), which appear in (3.1.4), by means of Lemma 3.1.3 and inverting a double absolutely convergent series, we obtain

\[
c(p^a u) = \sum_{L \in \mathbb{N}} \sum_{\xi(p)} c(L,\xi)(0) \xi^{-p^a u} (p^a u)^L
\]

where \( c(L,\xi)(0) \) is as in equation (3.1.2).

Let us prove that the sequences \( (c(L,\xi)(0))_{L \in \mathbb{N}} \) of equation (3.1.2) are in \( \mathcal{S}_\alpha \). For convenience, let us characterize \( \mathcal{S}_\alpha \subset K^N \) as the subspace of the sequences \( b = (b_i)_{i \in \mathbb{N}} \) such that there exist \( \kappa_i, \kappa''_i \in \mathbb{R}_+^* \) such that, for all \( l \in \mathbb{N} \), we have : \( v_p(b_i) \geq -\kappa_i - \frac{\kappa''_i}{\log(p)} \log(l + \kappa''_i) - (\alpha - 1)l \). By the hypothesis that \( c_1 \) and \( c_2 \) are in \( \text{L}_A S \), \( \xi \), \( K \), we can find, for \( i \in \{1, 2\} \), \( \kappa_i, \kappa''_i \in \mathbb{R}_+^* \) such that the inequality \( v_p(c_i(L,\xi)(r)) \geq -\kappa_i - \frac{\kappa''_i}{\log(p)} \log(l + l_i) - (\alpha - 1)l \) is satisfied for all \( \xi \in \mu_N \) and all \( r \in \{0, \ldots, p^a - 1\} \). Then, by equation (3.1.2),

\[
v_p(c(L,\xi)(0)) \geq \inf_{l' \in \mathbb{N}} \left( \inf_{l' + 1 \geq L} \right) \left( \alpha(l + l' - L) - (\alpha - 1)(l + l') \right) - (1 + \frac{\log(l + l')}{\log(p)}) + \kappa_i + \frac{\kappa''_i}{\log(p)} \log(l + \kappa''_i) + \kappa_i + \frac{\kappa''_i}{\log(p)} \log(l' + \kappa''_i) \right)
\]

where we have set \( \delta = l + l' - L \). The map \( \delta \mapsto -\frac{1 + \kappa_i + \kappa''_i}{\log(p)} \log(\delta + \max(1, \kappa_i, \kappa''_i)) - (\alpha - 1)L \) is increasing on \([1 + \frac{\log(p)}{\log(p)}, \log(p)]\) and all \( \delta \in [-1, +\infty) \); thus

\[
(3.1.5) \quad v_p(c(L,\xi)(0)) \geq - (2 + \kappa_i + \kappa''_i) - \frac{1 + \kappa_i + \kappa''_i}{\log(p)} \log(1 + \max(1, \kappa''_i)) - (\alpha - 1)L
\]

Since the inequality (3.1.5) is true for \( L \in \mathbb{N} \) outside the set \([0, \frac{1 + \kappa_i + \kappa''_i}{\log(p)}] \cap \mathbb{N} \) which is finite and independent of \( L \), we deduce that \((c(L,\xi)(0)) \in \mathcal{S}_\alpha \).

The proof of the analogous results concerning the power series expansion of \( c(r_0 + p^a u) \) is entirely similar, starting with the following equation : for any \( u \in \mathbb{N} \) and \( r_0 \in \{1, \ldots, p^a - 1\} \),

\[
c(r_0 + p^a u) = \sum_{u=1}^{u} \sum_{r_0=0}^{r_0-1} c_1(p^a u + r) c_2(p^a(u - u') + r_0)
\]

3.2. Regularized \( p \)-adic iterated integrals ; definition and computation by induction on the depth. Let the operator "unique primitive which vanishes at 0" on power series with coefficients in \( K \) :

\[
\int_{0}^{z} : f = \sum_{m \in \mathbb{N}} c_m z^m \in K[[z]] \mapsto \sum_{m \in \mathbb{N}} c_m \frac{z^{m+1}}{m+1} \in K[[z]]
\]

The variant of the KZ equation (3.1.3) satisfied by the iterated integrals of Definition 3.1.3 can be reformulated as

\[
\int_{p,\alpha} \xi^{n_{a-1}} \xi_{u}^{n_{a-1}} \cdots \xi_{0}^{n_{a-1}} e_{\xi_1^{n_{a-1}}} = \left\{ \begin{array}{ll}
\int_{0}^{z} p^a \frac{dz}{z} \left( \int_{p,\alpha} e_{0}^{n_{a-1}} e_{\xi^{n_{a-1}}} \cdots e_{0}^{n_{a-1}} e_{\xi_1^{n_{a-1}}} \right) & \text{if } n_d \geq 2 \\
\int_{0}^{z} p^a \frac{dz}{z} \left( \int_{p,\alpha} e_{0}^{n_{d-1}} e_{\xi_{d-1}^{n_{d-1}}} \cdots e_{0}^{n_{d-1}} e_{\xi_1^{n_{d-1}}} \right) & \text{if } n_d = 1 \text{ and } u_d = 1 \\
\int_{0}^{z} p^a \frac{dz}{z} \left( \int_{p,\alpha} e_{0}^{n_{d-1}} e_{\xi_{d-1}^{n_{d-1}}} \cdots e_{0}^{n_{d-1}} e_{\xi_1^{n_{d-1}}} \right) & \text{if } n_d = 1 \text{ and } u_d = (p^a)
\end{array} \right.
\]
We are going to define a regularized variant of the above system of equations whose solution lies in the subalgebra introduced in §3.1. We start by defining a variant of the space $LA_{S_o}(\mathbb{Z}_p(N), K)$ with poles.

**Definition 3.2.1.** Let $LA_{S_o}(\mathbb{Z}_p(N), K)$ be the set of functions $\mathbb{Z}_p \setminus \{0\} \to K$ satisfying series expansion as in (3.1.1), such that, if $r_0 \in \{0, \ldots, p^\alpha - 1\}$, we have a power series expansion, $|m - r_0|_p \leq p^{-\alpha}$, of the following form

$$c(m) = \frac{\sum \sum c((l, \xi))(r_0) \xi^{-m(m - r_0)}I}{\sum \sum c((l, \xi))(0) \xi^{-m(m - r_0)}I + \sum \sum c((l, \xi))(0) \xi^{-m(m - r_0)}I},$$

where $(c((l, \xi))(r_0))_{\ell \in \mathbb{N}}$ is a sequence in $S_o$ for any $r_0 \in \{0, \ldots, p^\alpha - 1\}$ and any $\xi \in \mu_N(K)$.

**Example 3.2.2.** For any $\zeta_0 \in \mu_N(K)$, $C_{\zeta_0} \left( \int_0^1 \omega_{\zeta_0}^\alpha(z^{p^\mu}) = -p^\alpha \sum_{m \in \mathbb{N}} \frac{z^{p^\mu}}{p^\mu m_c^{p^\mu}} \right) \in LA_{S_o}(\mathbb{Z}_p(N), K)$ and $C_{\zeta_0} \left( \int_0^1 \omega_{\zeta_0}^\alpha(z^{p^\mu})((l, \xi))_{(r_0)} = \begin{cases} -p^\alpha & \text{if } (l, \xi, r_0) = (-1, \zeta_0, 0) \\ 0 & \text{otherwise} \end{cases} \right)$.

Now we decompose $LA_{S_o}(\mathbb{Z}_p(N), K)$ into a regular part and a “pure pole” part.

**Lemma 3.2.3.** We have $LA_{S_o}(\mathbb{Z}_p(N), K) = LA_{S_o}(\mathbb{Z}_p(N), K) \bigoplus \left( \bigoplus_{\xi \in \mu_N(K)} K \int_0^1 \omega_{\zeta_0}^\alpha(z^{p^\mu}) \right)$

**Proof.** Let us denote by $\xi_0$ a primitive $N$-th root of unity in $K$. Let $V_\xi = \{(\xi_0)_{\ell \leq \ell} \in \mu_N(\mathbb{Q}(\xi)) \mid \text{ Vandermonde matrix associated with the sequence } (\xi^1, \ldots, \xi^N) \}$.

An element $c \in LA_{S_o}(\mathbb{Z}_p(N), K) \cap \text{Vect} \left( \int_{p, \alpha} \omega_{\zeta_0}^\alpha(z^{p^\mu}), \mid \xi \in \mu_N(K) \right)$ is in particular continuous when $|m|_p$ tends to 0 and $m$ remains in a given congruence class modulo $N$. Thus, $\lim_{\substack{|m|_p \to 0 \\ m \equiv m_0 \mod N}} \frac{mc(m)}{mc(0)} = V_\xi \left( \begin{array}{c} \cdots \\ \xi_0 \end{array} \right)$ and $V_\xi$ is invertible. This proves that $LA_{S_o}(\mathbb{Z}_p(N), K) \cap \text{Vect} \left( \int_{p, \alpha} \omega_{\zeta_0}^\alpha(z^{p^\mu}), \mid \xi \in \mu_N(K) \right) = \{0\}$. The rest of the statement follows easily, using Example 3.2.2. \qed

**Lemma 3.2.4.** Let $f \in A(U^{an})$ such that $f(0) = 0$. If $f \in C_{\zeta_0}^{-1} LA_{S_o}(\mathbb{Z}_p(N), K)$, then $\int_0^z f_\omega \in C_{\zeta_0}^{-1} LA_{S_o}(\mathbb{Z}_p(N), K)$ (where $\int_0^z$ is defined in equation (3.2.1)). Moreover, $C_{\zeta_0} \left( \int_0^1 f_\omega \right)^{(-1, \xi)}(0) = C_{\zeta_0}(f)^{(-1, \xi)}(0)$.

**Proof.** Let $c = C_{\zeta_0}(f)$. We have $f(z) = \sum m \in \mathbb{N} c(m) z^m$, and $\int_0^z f_\omega = \sum m \in \mathbb{N} c(m) z^m$. By the assumption, there exists, for any $r_0 \in \{0, 1, \ldots, p^\alpha - 1\}$ and $\xi \in \mu_N(K)$, there exist sequences $(c((l, \xi))(r_0))_{\ell \in \mathbb{N}} \in S_o$, such that, for $|m - r_0|_p \leq p^{-\alpha}$, $c(m) = \sum_{l \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} c((l, \xi))(r_0) \xi^{-m} (m - r_0)^{\ell}$. By $c((l, \xi))(r_0))_{\ell \in \mathbb{N}} \in S_o$, there exists a polynomial $P_{l, r_0}$ such that, for all $l \in \mathbb{N}$, $|c((l, \xi))(r_0)|_p \leq P_{l, r_0}(l)^{(\alpha - 1)^l}$. Then:

(a) For $|m|_p \leq p^{-\alpha}$, we have

$$\frac{c(m)}{m} = \sum_{l \geq -1, \xi \in \mu_N(K)} c((l+1, \xi))(r_0) \xi^{-m} m^l$$

and the sequences $(c((l+1, \xi))(r_0))_{\ell \in \mathbb{N}}$ are clearly in $S_o$.

(b) For $|m - r_0|_p \leq p^{-\alpha}$ with $r_0 \in \{0, 1, \ldots, p^\alpha - 1\}$, we have

$$\frac{c(m)}{m} = \sum_{l \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} c((l, \xi))(r_0) \xi^{-m} (m - r_0)^l \left( \sum_{l \in \mathbb{N}} \frac{(m - r_0)^l_{r_0}}{r_0^{l+1}} \right) = \sum_{l \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} c((l, \xi))(r_0) \xi^{-m} (m-r_0)^l$$

for $|m|_p \leq p^{-\alpha}$.
Moreover, we have \(|\frac{1}{r_0}|b < p\). Thus, with \(P_{\xi,r_0}\) defined above,

\[
\left| \sum_{l=0}^{\nu} \frac{c(l,\xi)(r_0)}{r_0^l} \right|_p \leq \sum_{l=0}^{\nu} P_{\xi,r_0}(l)p^{(a-1)(l+\nu-l)} = \sum_{l=0}^{\nu} P_{\xi,r_0}(l)p^{(a-1)l'}
\]

and there exists a polynomial \(Q_{\xi,r_0}\) such that for all \(l' \in \mathbb{N}\), \(\sum_{l=0}^{\nu} P_{\xi,r_0}(l) = Q_{\xi,r_0}(l')\).

\[\square\]

**Definition 3.2.5.** Let \(\text{reg}^{\mathcal{L}A} : \text{LA}^{\mathbb{p} \mathcal{A}}(\mathbb{Z}_p^N)(K) \rightarrow \text{LA}_{\mathcal{S}_n}^{\mathcal{L}A}(\mathbb{Z}_p^N)(K)\) be the projection associated with the direct sum decomposition of Lemma \(3.1.3\), we omit \(l'\) when it is equal to 0. We note that the definition implies \(\text{B}_{\mathcal{S}B}^{(l',\xi)} = \text{B}_{\mathcal{S}B}^{(l',\xi)} - \delta_{l,l_0}\).

**Proposition 3.2.6.** Let \(w_d = e_0^{n_d-1}e_{\xi_d}^{n_{\xi_d}}\ldots e_0^{n_1-1}e_{\xi_1}^{n_{\xi_1}} \in \mathcal{O}_{\mathcal{S}B}^{\mathcal{L}A}(\mathbb{Z}_p^N)\) and let \(c_d = \text{Cl}_0\text{Reg}_{l_p\alpha} w_d\).

Let \(w_{d-1} = e_0^{n_d-1}e_{\xi_d-1}^{n_{\xi_d}}\ldots e_0^{n_1-1}e_{\xi_1}^{n_{\xi_1}}\) and \(c_{d-1} = \text{Cl}_0\text{Reg}_{l_p\alpha} w_{d-1}\). For any \(l \in \mathbb{N}\), \(\xi \in \mu_N(K)\) and \(r_0 \in \{1, \ldots, p^n\}\), we have

(i) if \(u_d = 1\), then

\[
(3.2.5) \quad c_{d-1}(\xi,0) = -(p^n)^a \sum_{L \geq 1} (p^n)^L \left[ B_{\xi_d}^{(l+n_d+L)}(\xi) c_{d-1}(l+n_d+L,\xi)(0) + \sum_{r=1}^{p^n-1} B_{\xi_d}^{(l+n_d+L)}(\xi) c_{d-1}(l+n_d+L,\xi)(r) \right]
\]

(ii) if \(u_d = (p^n)\), then

\[
(3.2.6) \quad c_{d-1}(\xi,0) = -(p^n)^a \sum_{L \geq 1} (p^n)^L \left[ B_{\xi_d}^{(l+n_d+L)}(\xi) c_{d-1}(l+n_d+L,\xi)(0) \right.
\]

\[
+ \sum_{r=1}^{p^n-1} B_{\xi_d}^{(l+n_d+L)}(\xi) c_{d-1}(l+n_d+L,\xi)(r) - \sum_{r=1}^{p^n-1} \left( \xi_d(\xi) \right)^{r_0} c_{d-1}(l+n_d+L,\xi)(r) \]

\[
(3.2.7) \quad c_{d-1}(\xi,0) = -(p^n)^a \sum_{L \geq 1} (p^n)^L \left[ \sum_{r=1}^{p^n-1} \left( \xi_d(\xi) \right)^{r_0} c_{d-1}(l+n_d+L,\xi)(0) \right]
\]

(iii) if \(u_d = (p^n)\), then

\[
(3.2.8) \quad c_{d-1}(\xi,0) = -(p^n)^a \sum_{L \geq 1} (p^n)^L \left[ \sum_{r=1}^{p^n-1} \left( \xi_d(\xi) \right)^{r_0} c_{d-1}(l+n_d+L,\xi)(r) \right]
\]
Proof. (a) We apply Proposition 3.1.4 in the particular case \( (f_1, f_2) = (f, \frac{z}{z - \xi_d}) \), using Example 3.1.2.

(i) This gives that \( c = \text{Cl}_0(f, \frac{z}{z - \xi_d}) \) is in \( \text{LA}_{S_n}(z^{(N)}_p, K) \) and satisfies, for all \( l \in \mathbb{N}, \xi \in \mu_N(K), r_0 \in \{1, \ldots, p^n - 1\} \),

\[
\tag{3.2.9} c(l, \xi)(0) = - \sum_{L \geq 1} (p^n)^L \left[ B^{l+L}(\xi_d)^{\rho_n} c(l, \xi)(0) + \sum_{r=1}^{p^n-1} B^{l+L} \xi_d^{-(\xi_d)} \xi^r \right] c(l+L, \xi)(r)
\]

\[
\tag{3.2.10} c(l, \xi)(r_0) = - \sum_{L \geq 1} (p^n)^L \left[ B^{l+L}(\xi_d)^{\rho_n} \left( \xi_d \right)^{r_0} c(l+L, \xi)(0) + \sum_{r=1}^{p^n-1} \xi_d^{-(\xi_d)} \left( \xi_d \right)^{(r_0-r)} c(l+L, \xi)(r) \right]
\]

(b) We apply Proposition 3.1.4 in the particular case \( (f_1, f_2) = \left( f, \frac{z^{p^n}}{z^{p^n} - \xi_d} \right) \), using Example 3.1.2.

This gives that \( c = \text{Cl}_0(f, \frac{z^{p^n}}{z^{p^n} - \xi_d}) \) is in \( \text{LA}_{S_n}(z^{(N)}_p, K) \) and satisfies, for all \( l \in \mathbb{N}, \xi \in \mu_N(K), r_0 \in \{1, \ldots, p^n - 1\} \),

\[
\tag{3.2.11} c(l, \xi)(0) = - \sum_{L \geq 1} (p^n)^L B^{l+L}(\xi_d)^{\rho_n} c(l, \xi)(0)
\]

\[
\tag{3.2.12} c(l, \xi)(r_0) = - \sum_{L \geq 1} (p^n)^L B^{l+L}(\xi_d)^{\rho_n} c(l, \xi)(r_0)
\]

(c) Then we apply the following statement to the results of (a) and (b): let \( w \in \mathcal{O}_{\text{conv}}^{(m, c(\mu_N, \mu_N^{(p^n)}))}, n \in \mathbb{N}, c = \text{Cl}_0 \text{Reg} \int_{p, \alpha} w \) and \( \tilde{c}_n = \text{Cl}_0 \text{Reg} \int_{p, \alpha} e^0 w \). Then \( \tilde{c} \) is the element of \( \text{LA}_{S_n}(z^{(N)}_p, K) \) determined by, for all \( l, \xi \) and for all \( r_0 \in \{1, \ldots, p^n - 1\} \),

\[
\tag{3.2.13} \tilde{c}(l, \xi)(0) = (p^n)^n c(l+n, \xi)(0)
\]

\[
\tag{3.2.14} \tilde{c}(l, \xi)(r_0) = (p^n)^n \sum_{l=-n}^l \frac{1}{p^{l-n} r_0^{l-n}} c(l, \xi)(r_0)
\]

Indeed, (3.2.13) is deduced from (3.2.2) and the definition of the regularization (Definition 3.2.5), by induction on \( n \); (3.2.11) is obtained like (3.2.3) with \( \frac{c(m)}{m^n} \) instead of \( \frac{c(m)}{m} \), given that the regularization affects only the coefficients at \( r_0 = 0 \).

\[\square\]

### 3.3. Bounds on the norms of regularized \( p \)-adic iterated integrals

We deduce from §3.1 and §3.2 bounds on the norms of regularized \( p \)-adic iterated integrals. The norm is the one of Notation 1.2.3.

**Proposition 3.3.1.** For any word \( w = c^{n_1-1}_1 \varepsilon^{n_2}_2 \cdots c^{n_l-1}_l \varepsilon^{n_1}_1 \) in \( \mathcal{O}_{\text{conv}}^{(m, c(\mu_N, \mu_N^{(p^n)}))} \), letting \( n = \sum_{i=1}^d n_i \) be its weight, we have

\[
\left\| \text{Reg} \int_{p, \alpha} w \right\| \leq p^{-\left(n-2d\frac{\log(p)}{m(p)} - \frac{\log(p)}{m(p)} \log(n+3d)\right)}
\]

**Proof.** We proceed in three steps.

(a) We prove by induction of the depth that for each \( r_0 \in \{0, \ldots, p^n - 1\} \), \( l \in \mathbb{N}, \xi \in \mu_N(K) \):

\[
\tag{3.3.1} V_p \left( \text{Cl}_0 \text{Reg} \int_{p, \alpha} w \right)^{(l, \xi)}(r_0) \geq \inf_{L_d \geq 2} \left[ \alpha \left( \sum_{i=1}^d n_i + L_d \right) - d \left( 1 + \log \left( l + \sum_{i=1}^d n_i + L_d + d \right) \right) \right] - (\alpha - 1) \left( l + L_d + \sum_{i=1}^d n_i \right)
\]
This follows by induction on the depth by applying the result to the word \( w_{d-1} = e_0^{n_{d-1}} e_{\xi d-1}^{n_{d-1}} \cdots e_0^{n_1} e_{\xi_1}^{n_1} \) by Proposition 3.2.4 and the bounds on the valuations of the coefficients \( B \) in Lemma 3.1.3.

The second line in equation (3.3.1) is also equal to

\[
\inf_{L_d \in \mathbb{Z}} \left[ \sum_{i=1}^{d} n_i + L_d - d \left( 1 + \frac{\log(l + \sum_{i=1}^{d} n_i + L_d + d)}{\log(p)} \right) - (\alpha - 1)l \right]
\]

(b) We bound the inf in equation (3.3.2). Let the function \( f : L \in (-l - \sum_{i=1}^{d} n_i - d, +\infty) \mapsto L - \frac{d}{\log(p)} \log(L + l + \sum_{i=1}^{d} n_i + d) \in \mathbb{R} \); it is increasing over \([t_0, +\infty)\), where \( t_0 = \frac{d}{\log(p)} (-l - \sum_{i=1}^{d} n_i + d), \) and it reaches its minimum at \( t_0 \), which is \( f(t_0) = t_0 - \frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right) \). We distinguish two cases:
- If \( t_0 \geq -d \), then \( f(t_0) = -d - \frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right) \).
- If \( t_0 \leq -d \), then \( f \) is increasing in \([-d, +\infty)\) and \( \inf_{[-d, +\infty)} f \) is \( f(-d) = -d - \frac{d}{\log(p)} \log(l + \sum_{i=1}^{d} n_i + 1) \).

We deduce

\[
\inf_{[-d, +\infty)} f \geq \min \left( -d - \frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right), -d - \frac{d}{\log(p)} \log(l + \sum_{i=1}^{d} n_i + 1) \right)
\]

We deduce the following bound on the inf in equation (3.3.2)

\[
\inf_{L_d \in \mathbb{Z}} \left[ \sum_{i=1}^{d} n_i + L_d - d \left( 1 + \frac{\log(l + \sum_{i=1}^{d} n_i + L_d + d)}{\log(p)} \right) - (\alpha - 1)l \right]
\]

(c) For all \( r_0 \in \{0, \ldots, p^\alpha - 1\} \), \( m \in \mathbb{N}^\ast \) such that \( |m - r_0|_p \leq p^{-\alpha} \) (i.e. \( v_p((m - r_0)^t) \geq \alpha t \)), we have \( \langle \text{Cf}_0, \text{Reg} \int_{p,\alpha} w(m) = \sum_{\ell \in \mathbb{N} \xi \in \mu_N(K)} \sum_{i=1}^{d} c^{(i,\ell)}(m - r_0)^t \rangle \). By (3.3.3), we deduce

\[
v_p(c(m)) \geq \inf_{l \in \mathbb{N}} \left[ l + \sum_{i=1}^{d} n_i - 2d - \frac{d}{\log(p)} \left( \log(l + \sum_{i=1}^{d} n_i + 1) + \log \left( \frac{d}{\log(p)} \right) \right) \right]
\]

The function \( g : l \mapsto - \frac{d}{\log(p)} \log(l + \sum_{i=1}^{d} n_i + 1) \) is increasing on \([t_1, +\infty)\) with \( t_1 = \frac{d}{\log(p)} \sum_{i=1}^{d} n_i - 1 \), and reaches its minimum at \( t_1 \), which is equal to \( g(t_1) = t_1 - \frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right) \). Since we have \( n_i \geq 1 \) for all \( i \), we have \( t_1 \leq \frac{d}{\log(p)} - d - 1 \leq 4d - d - 1 = 3d - 1 \). We again distinguish two cases:
- If \( t_1 \geq 0 \), then since \( t_1 \leq 3d - 1 \) we have \( \inf_{l \in \mathbb{N}} g(l) \geq \min_{l \in [0, 3d - 1] \cap \mathbb{N}} \ g(l) \geq 0 - \frac{d}{\log(p)} \log(3d + \sum_{i=1}^{d} n_i). \n\]
- If \( t_1 \leq 0 \) then \( \inf_{l \in \mathbb{N}} g(l) = g(0) = - \frac{d}{\log(p)} \log(\sum_{i=1}^{d} n_i + 1). \n\)

In both cases, we have \( \inf_{l \in \mathbb{N}} g(l) \geq - \frac{d}{\log(p)} \log(\sum_{i=1}^{d} n_i + 3d). \) Thus equation (3.3.3) becomes

\[
v_p \left( \langle \text{Cf}_0, \text{Reg} \int_{p,\alpha} w(m) \rangle \right) \geq \sum_{i=1}^{d} n_i - 2d - \frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right) - \frac{d}{\log(p)} \log(\sum_{i=1}^{d} n_i + 3d)
\]

This implies the result, since the map \( \text{Cf}_0 \) is an isometry (Proposition A.0.4).

**Remark 3.3.2.** The bound in Proposition 3.3.1 becomes better when \( p \) becomes large.

4. **Regularity of overconvergent \( p \)-adic multiple polylogarithms and end of the proof**

We prove (Proposition 4.1.3) that, in the decomposition of overconvergent \( p \)-adic multiple polylogarithms of Proposition 2.2.3, the iterated integrals of Definition 2.2.3 can be replaced by their regularized variants of Definition 3.2.5 and that the decomposition map of Definition 2.2.2 can be replaced by a "regularized" variant (Definition 4.1.2). This is a consequence of the analytic nature of the overconvergent \( p \)-adic multiple polylogarithms. This gives a characterization of \( \text{Li}_{p,\alpha} \) and \( \text{Ad}_{\psi^{(i)}_{p,\alpha}}(\xi) \) (\( \xi \in \mu_N(K) \)) in
terms of regularized iterated integrals. This characterization can be written by an induction on the depth (Proposition 4.2.2). Combining this characterization and the bounds on the norms of regularized iterated integrals (Proposition 4.3.1) this enables to finish the proof of the main theorem (Proposition 4.3.1).

4.1. Regularized decomposition of overconvergent $p$-adic multiple polylogarithms in terms of regularized iterated integrals.

**Lemma 4.1.1.** We have $\text{LA}_{\mathbb{S}_n}^\text{polar}(\mathbb{Z}_p(N), K) \cap \mathcal{A}_n^\prime(\mathbb{Z}_p(N), K) = \text{LA}_{\mathbb{S}_n}(\mathbb{Z}_p(N), K)$.

**Proof.** This is a byproduct of the proof of Lemma 3.2.2 and the formula in Example 3.2.2. □

The differential equation of $\text{Li}_{p,\alpha}^{1,2}$ amounts to (we are again using (equation 3.2.1))

\[
\text{Li}_{p,\alpha}^{1,2} = \sum_{w_1, w_2 \text{ words}} \sum_{\xi \in \mu_n(K)} \left( \text{Ad}_{\phi(\xi)}(w_2) \right) \int_0^1 \omega_{\xi} (\omega^p) \text{Li}_{p,\alpha}^{1,2} [w_1] + \left( \int_0^1 p^n \omega_0 (z) \text{Li}_{p,\alpha}^{1,2} [e_{w_1} \cdot e_{w_2}] d\omega_0 (z) \right) \text{Li}_{p,\alpha}^{1,2} [w_1] - \left( \int_0^1 p^n \omega_0 (z) \text{Li}_{p,\alpha}^{1,2} [w_1] d\omega_0 (z) \right) \text{Li}_{p,\alpha}^{1,2} [w_1]
\]

We note that the first line of (4.1.1) contains the term $\sum_{\xi \in \mu_n(K)} \left( \text{Ad}_{\phi(\xi)}(w) \right) \omega_{\xi} (\omega^p)$, which corresponds to $(w_1, w_2) = (0, w)$. By Example 3.2.2 the integral $\int_0^1 \sum_{\xi \in \mu_n(K)} \left( \text{Ad}_{\phi(\xi)}(w) \right) \omega_{\xi} (\omega^p) \in \text{LA}_{\mathbb{S}_n}^\text{polar}(\mathbb{Z}_p(N), K)$ and is a "pure pole" in the sense of the decomposition of Lemma 3.2.3. We remove it in the next definition.

**Definition 4.1.2.** Let $\text{Reg} \text{dec}_{p,\alpha}: \mathcal{O}^{m, e_{\mu_n}, \omega_{\xi}^p} \to \mathcal{O}^{m, e_{\mu_n}, \omega_{\xi}^p} \otimes \mathbb{Q}$ be defined by induction on the weight, as follows: $\text{Reg} \text{dec}_{p,\alpha}(0) = 1$; for all $n \in \mathbb{N}^*$, $\text{Reg} \text{dec}_{p,\alpha}(e_0^n) = e_0 \text{Reg} \text{dec}_{p,\alpha}(e_0^{n-1}) - e_0 \text{Reg} \text{dec}_{p,\alpha}(e_0^{-1})(0)$, and for any $w = e_0^{n_1} e_0^{n_2} \cdots e_0^{n_k}$, where the $n_i$'s are positive integers and the $\xi_i$'s are $N$-th roots of unity,

\[
\text{Reg} \text{dec}_{p,\alpha}(e_0^{n_1} - e_0^{n_k}) = \sum_{w_1, w_2 \text{ words}} \sum_{\xi \in \mu_n(K)} \left( \text{Ad}_{\phi(\xi)}(w_2) \right) e_{\xi^p} (\text{Reg} \text{dec}_{p,\alpha}(w_1) + \left( \int_0^1 p^n \omega_0 (z) e_{\xi} \text{Reg} \text{dec}_{p,\alpha}(w_1) d\omega_0 (z) \right) - \left( \int_0^1 p^n \omega_0 (z) \text{Reg} \text{dec}_{p,\alpha}(w_1) d\omega_0 (z) \right) e_{\xi^p})
\]

The next Proposition is a regularized variant of the statement $\text{Li}_{p,\alpha}^{1,2} = \int_{p,\alpha} \circ \text{dec}_{p,\alpha}$ (Proposition 2.2.4) and, at the same time, a characterization of $\text{Li}_{p,\alpha}^{1,2}$ and $\text{Ad}_{\phi(\xi)}(w)$ for any $\xi \in \mu_n(K)$ in terms of regularized iterated integrals. As in Proposition 2.2.4 we regard $\text{Li}_{p,\alpha}^{1,2}$ as a function on $\mathcal{O}^{m, e_{\mu_n}, \omega_{\xi}^p}$.

**Proposition 4.1.3.** We have:

\[
\text{Li}_{p,\alpha}^{1,2} = \text{Reg} \int_{p,\alpha} \circ \text{dec}_{p,\alpha}
\]
and, for any word \( w = e_0^{n_0-1} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1} \), with the \( n_i \)'s positive integers and the \( \xi_i \)'s roots of unity, for all \( \xi \in \mu_N(K) \),

\[
(4.1.4) \quad \text{Ad}_{\Phi_{p,\alpha}}(\xi)[w] = \sum_{w_1, w_2 \text{ words with } w_1 \geq w_2} \sum_{\xi \in \mu_N(K)} \left( \text{Ad}_{\Phi_{p,\alpha}}(\xi)[w_2] \right) \text{Cf}_0 \left( \frac{z^p}{\xi^p - z} \right) ^{0,\xi}(0) + \]

\[
\left\{ \begin{array}{ll}
p^\alpha(\text{Cf}_0 \text{Li}_{p,\alpha}^\dagger[e_0^{n_0-2} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1}])^{0,\xi}(0)
& \text{if } n_d \geq 2, \quad n_0 \geq 2 \\
p^\alpha(\text{Li}_{p,\alpha}^\dagger[e_0^{n_0-2} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1}])^{0,\xi}(0)
& \text{if } n_d \geq 2, \quad n_0 = 1 \\
p^\alpha(\text{Li}_{p,\alpha}^\dagger[e_0^{n_0-2} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1}])^{0,\xi}(0)
& \text{if } n_d = 1, \quad n_0 \geq 2 \\
p^\alpha(\text{Li}_{p,\alpha}^\dagger[e_0^{n_0-2} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1}])^{0,\xi}(0)
& \text{if } n_d = 1, \quad n_0 = 1 
\end{array} \right.
\]

**Proof.** In weight 0, we have \( \text{Li}_{p,\alpha}(0) = 1 \) and the result is clear. Assume that the result holds in weight up to \( 0, \ldots, n-1 \). Let \( w \) be a word on \( e_0, \ldots, e_n \) of weight \( n \). For any \( \xi \in \mu_N(K) \), \( \text{Cf}_0(\frac{z^p}{z - \xi}) \) and \( \text{Cf}_0(\frac{z^p}{\xi^p - z}) \) are in \( \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \) (Example 3.1.2). By Proposition 3.1.3 and the induction hypothesis, all the products of the form \( \text{Li}_{p,\alpha}(w') \frac{z^p}{z - \xi} \) and \( \text{Li}_{p,\alpha}(w') \frac{z^p}{\xi^p - z} \) with \( w' \) a word of weight strictly lower than the weight of \( w \), are in \( \text{Cf}_0^{-1} \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \). Moreover, for all \( \xi \in \mu_N(K) \), we have \( \omega(\xi) = \frac{z}{z - \xi} \) and \( \omega(\xi) = \frac{z^p}{\xi^p - z} \). By Lemma 3.2.4 we deduce that \( \text{Cf}_0(\text{Li}_{p,\alpha}(w)) \) is in \( \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \).

On the other hand, since \( \text{Li}_{p,\alpha}(w) \subset \mathbb{A}^{1}(\mathbb{A}^n) \subset \mathbb{A}^{1}(\mathbb{A}^n) \), by Proposition 4.0.1 \( \text{Cf}_0(\text{Li}_{p,\alpha}(w)) \) is in \( \mathcal{C}(\mathbb{Z}_p^N, K) \). Thus, \( \text{Cf}_0(\text{Li}_{p,\alpha}(w')) \in \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \cap \mathcal{C}(\mathbb{Z}_p^N, K) \). By Lemma 4.1.1 we deduce \( \text{Cf}_0(\text{Li}_{p,\alpha}(w)) \in \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \).

By Lemma 3.2.8 this implies that the polar part, in the sense of the direct sum decomposition of \( \text{LA}_{S_n}(\mathbb{Z}_p^N, K) \), vanishes. By Lemma 3.2.4 the vanishing of this polar part is equation 4.1.1. \( \square \)

### 4.2. Computation of the regularized decomposition by induction on the depth

We write a formula for \( \text{Reg dec}_{p,\alpha} \) which is inductive with respect to the depth (Proposition 4.2.2).

**Lemma 4.2.1.** For all \( n \in \mathbb{N}^* \), we have \( \text{Li}_{p,\alpha}^\dagger[e_0^n] = 0 \), \( \text{Reg dec}_{p,\alpha}(e_0^n) = \text{dec}_{p,\alpha}(e_0^n) = 0 \), and \( \text{Ad}_{\Phi_{p,\alpha}}(\xi)[e_0^n] = 0 \) for all \( \xi \in \mu_N(K) \).

**Proof.** The fact that \( \text{Reg dec}_{p,\alpha}(e_0^n) = \text{dec}_{p,\alpha}(e_0^n) = 0 \) for all \( n \in \mathbb{N}^* \) has already been observed as an immediate byproduct of the definitions (Definition 3.2.2 and Definition 4.1.2). It implies \( \text{Li}_{p,\alpha}^\dagger[e_0^n] = 0 \) for all \( n \in \mathbb{N}^* \) by Proposition 2.2.3. Finally, \( \text{Ad}_{\Phi_{p,\alpha}}(\xi)[e_0^n] = \{ \Phi_{p,\alpha}^{\dagger} \}^{-1} e_0^n \Phi_{p,\alpha}^{\dagger} \) is the product of some formal power series and \( e_0^n \) which is of depth 1, and it thus contains only terms of depth \( \geq 1 \) ; thus \( \text{Ad}_{\Phi_{p,\alpha}}(\xi)[e_0^n] = 0 \) for all \( n \in \mathbb{N}^* \) (and for \( n = 0 \), which we already used since equation (2.2.1)). \( \square \)

**Proposition 4.2.2.** We have, for any word \( w = e_0^{n_0-1} e_{\xi_0} \ldots e_0^{n_1-1} e_{\xi_1} e_0^{n_0-1} \), with the \( n_i \)'s positive integers and the \( \xi_i \)'s roots of unity,

\[
(4.2.1) \quad \text{Reg dec}_{p,\alpha}(w) =
\sum_{0 \leq l_0 \leq n_0 - 1} \sum_{0 \leq l_1 \leq n_1 - 1} \sum_{\text{words } w_1, w_2 \text{ with depth}(w_1) \geq 1, \text{depth}(w_2) \geq 1} \text{Ad}_{\Phi_{p,\alpha}}(\xi)[w_1] \text{Ad}_{\Phi_{p,\alpha}}(\xi)[w_2] (-1)^{l_0 + l_1 - 1} \left( e_0^{l_0 + l_1} e_{\xi_0} \right) \text{Reg dec}_{p,\alpha}(w_2)
\]

\[
- \sum_{0 \leq l_0 \leq n_0 - 1} (-1)^{l_0 + l_1 - 1} \left( e_0^{l_0 + l_1} e_{\xi_0} \right) \text{Reg dec}_{p,\alpha}(e_0^{n_0-1} \ldots e_0^{n_1-1} e_{\xi_1})
\]
Proof. In the first line of equation (4.1.2), the terms such that depth\((w_2) = 0\) of depth\((w_1) = 0\) vanish by Lemma 4.2.1. Thus, the first line of equation (4.1.2) can be replaced by

\[ \sum_{\omega_1,\omega_2 \text{ words}} \sum_{\xi \in \mu_N(K)} \left( A_{\phi_{\xi}^p}(\xi)[w_2] \right) e_{\xi(i^+)\cdot \text{Reg} \text{ dec}_{p,\alpha}(w_1)} \]

The result is then obtained by induction on \((n, d, n_0)\) for the lexicographical order on \(N^2\), noting that if \((n_0, n_d) = (1, 1)\) the formula for \(\text{Reg} \text{ dec}_{p,\alpha}\) is already inductive with respect to the depth and that \((n_0, n_d)\) decreases when applying \(\text{Reg} \text{ dec}_{p,\alpha}\) in the other cases.

4.3. End of the proof of the main theorem. Given that \(A_{\phi_{\xi}^p}(\xi) = \{\phi_{\xi}^p\}_{\xi \in \mu(N)}\), for any word \(w\) of weight \(n\) and depth \(d\), \(A_{\phi_{\xi}^p}(\xi)[w]\) is a \(Z\)-linear combination of \(p\)-adic cyclotomic multiple zeta values of weight \(n - 1\) and depth \(d - 1\); more precisely, \(A_{\phi_{\xi}^p}(\xi)[w] = \sum_{w_1, w_2 \text{ words}} \Phi_{p,\xi}[S(w_1) \cdot w_2]\) (by the formula for the antipode and product of shuffle Hopf algebras, §1.1.1). This is the reason for the shift in the weight and the depth below.

Corollary 4.3.1. (i) Each \(L_{1,\alpha}^d[w]\) is a \(Z\)-linear combination of products \(\left( \prod_{i=1}^{r} A_{\phi_{\xi}^p}(\xi_i)[w_i] \right) \text{Reg} f_{p,\alpha}(\vec{w})\), where the \(\xi_i\)'s are in \(\mu_N(K)\), the \(w_i\)'s are words on \(e_{0,\mu_N}\) of depth \(d\), \(r \in N\), and \(\vec{w}\) is a word over \(e_{0,\mu_N,\mu_N}(\alpha)\) such that \(\sum_{i=1}^{r} (\text{depth}(w_i - 1) + \text{depth}(\vec{w})) \leq \text{depth}(w)\) and \(\sum_{i=1}^{r} (\text{weight}(w_i - 1) + \text{weight}(\vec{w})) = \text{weight}(w)\).

(ii) Each \(A_{\phi_{\xi}^p}(\xi)[w]\) is a \(Z\)-linear combination of products \(\left( \prod_{i=1}^{r} A_{\phi_{\xi}^p}(\xi_i)[w_i] \right) C_{0,\alpha}(e_0^{\frac{\alpha}{\infty}} \text{Reg} f_{p,\alpha}(\vec{w})\) and \(\left( \prod_{i'=1}^{r'} A_{\phi_{\xi'}^p}(\xi'_i)[w'_i] \right) C_{0,\alpha}(e_0^{\frac{\alpha}{\infty}} \text{Reg} f_{p,\alpha}(\vec{w}'))\) where the \(\xi_i\)'s and \(\xi_i'\)'s are in \(\mu_N(K)\), the \(w_i\)'s and \(w'_i\)'s are words on \(e_{0,\mu_N,\mu_N}(\alpha)\) of depth \(d\), \(r, r' \in N\), and \(\vec{w}\) is a word over \(e_{0,\mu_N,\mu_N}(\alpha)\) such that \(\sum_{i=1}^{r} (\text{depth}(w_i) - 1) + \text{depth}(\vec{w}) \leq \text{depth}(w)\) and \(\sum_{i=1}^{r} (\text{weight}(w_i) - 1) + \text{weight}(\vec{w}) = \text{weight}(w)\).

Proof. (i) follows by induction on the depth from Proposition 4.3.1 and Proposition 4.3.2 noting that equation (4.2.1) is homogeneous for the weight and depth, in a sense which agrees with the shift explained before the statement : in the first line of (4.2.1), we have, for all \((w_1, w_2)\), weight\((w) = (\text{weight}(w_1) - 1) + \text{weight}(w_2) + \text{weight}(e_0^{\alpha} e_{\xi(w_i)})\) and depth\((w) = (\text{depth}(w_1) - 1) + \text{depth}(w_2) + \text{depth}(e_0^{\alpha} e_{\xi(w_i)})\) and, in the second line of (4.2.1), weight\((w) = \text{weight}(e_0^{\alpha} e_{\xi(w_i)}) + \text{weight}(e_0^{\alpha} e_{\xi(w_i)}) + \text{depth}(e_0^{\alpha} e_{\xi(w_i)})\) and \((\sum_{i=1}^{r} (\text{depth}(w_i) - 1) + \text{depth}(\vec{w})) \leq \text{depth}(w)\).

Proof. (ii) follows from (i) by induction on the weight, using equation (4.1.2).

Corollary 4.3.2. For any \(d\), there exists \(\kappa_d, \kappa_d', \kappa_d'' \in \mathbb{R}_+\) such that, for any word \(w\) of weight \(n\) and depth \(d\) over \(e_{0,\mu_N}\), we have \(v_{A(U^{\alpha})}(L_{1,\alpha}^d[w]) \geq n - \kappa_d - \kappa_d' \log(n + \kappa_d'')\) and \(v_p(A_{\phi_{\xi}^p}(\xi)[w]) \geq n - \kappa_d - \kappa_d' \log(n + \kappa_d'')\) where, for \(f \in A(U^{\alpha})\), \(v_{A(U^{\alpha})}(f)\) is defined by \(\|f\| = p^{-v_{A(U^{\alpha})}(f)}\).

Proof. This follows by induction on \(d\) from Corollary 4.3.1 in which we note that \(r \leq \text{depth}(w)\), Proposition 3.3.1 equations (3.2.9), (3.2.10), (3.2.11), (3.2.12) which describe the multiplication by \(C_{0,\alpha}(e_0^{\frac{\alpha}{\infty}})\) in \(LA_\infty(\mathbb{Z}_p^N, K)\), and the bounds on the valuations of the coefficients \(B\), which appear in those equations, in Lemma 3.1.3.

The Corollary 4.3.2 implies the theorem, knowing that, at \(d\) fixed,

\[ n - \kappa_d - \kappa_d' \log(n + \kappa_d'') \rightarrow \infty \]

APPENDIX A. RECasting A THEOREM OF MAHLER AS A CHARACTERIZATION OF AN AFFINNO SUBSPACE OF \(\mathbb{P}^1_{\text{an}}/K\), PRELIMINARY TO §3, §4

This section is an interpretation of a classical theorem of Mahler in terms of rigid analytic spaces. The result (Proposition A.0.3) is a characterization of the \(K\)-Banach space \(A(U^{\alpha})\) (Notation 1.2.4) in terms of the power series expansion at 0 of its elements.

Let \(L\) be a \(\mathbb{Q}_p\)-Banach space. The sequence of Mahler coefficients of a sequence \((c_m)_{m \in \mathbb{N}} \in L^N\),
or more generally, of a function \( c : \mathbb{Z}_p \to L \) for which we denote by \( c_m = c(m) \), is the sequence 
\[
\left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} c_{m-j} \right)_{m \in \mathbb{N}} \in L^N. \]
The following statement is a classical theorem of Mahler \cite{M}; the formulation which we reproduce can be found, as well as a simple proof, in \cite{CZ}, Theorem 2.8: The map 
\( L^N \to L^N \) which sends a sequence to its Mahler coefficients induces an isometry between \( C(\mathbb{Z}_p, L) \) equipped with the norm of uniform convergence and the space \( l^0_\infty = \{ (c'_m) \in L^N \mid c'_m \to 0 \} \) of sequences tending to 0, equipped with the norm \( \| \|_\infty \). In particular, a map \( \mathbb{N}^* \to L \) can be interpolated by an element of \( C(\mathbb{Z}_p, L) \) if and only if its sequence of Mahler coefficients tends to 0.

**Notation A.0.1.** For any \( L \) a \( K \)-Banach space and \( M \) a metric space we denote by \( C(M, L) \) the \( K \)-vector space of continuous functions \( M \to L \).

**Notation A.0.2.** For any \( f \in A(U^{an}) \), and \( m \in \mathbb{N} \), the coefficient of degree \( m \) in the power series expansion of \( f \) at 0 is denoted by \( f[z^m] \).

We explain a relation between Mahler’s theorem, rigid analytic functions and \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \). Let the operator \( (z \mapsto \frac{1}{z-1}) : L^N \to L^N \) defined as the conjugation of the automorphism \( \sum_{m \in \mathbb{N}} c_m z^m \to \sum_{m \in \mathbb{N}} c_m (\frac{1}{z-1})^m \) of \( L[[z]] \) by the natural isomorphism \( L[[z]] \approx L^N \). A simple computation shows the following:

**Lemma A.0.3.** For all \( (c_m) \in L^N \), \((z \mapsto \frac{1}{z-1})c_m = \left\{ \begin{array}{ll} c_0 & \text{if } m = 0 \\ (-1)^m \sum_{m' = 0}^{m} (-1)^{m'} (m'_{m-1}) c_{m'} & \text{if } m > 0 \end{array} \right. \)

The Proposition below is a new formulation and a new proof of Proposition 1 in \cite{IK}, §5. We use Notation A.0.2. We recall that \( A(U^{an}) \) is equipped with the norm defined by \( \| \sum a_m z^m \| = \sup_{m \in \mathbb{N}} |a_m|_p \).

Let \( \mathbb{Z}_p^N = \lim_{L \to \infty} \mathbb{Z}/Lp^L \mathbb{Z} \subset \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}_p \), this isomorphism being an homeomorphism where \( \mathbb{Z}/N\mathbb{Z} \) is equipped with the discrete topology. Thus, \( \mathbb{Z}_p^N \) is the disjoint union of \( N \) copies \( \mathbb{Z}_p \), regarded as the closures of \( m_0 + N\mathbb{N}, m_0 \in \{ 1, \ldots, N \} \), respectively.

Following \cite{Ko}, §2, let, for \( z \in U^{an}(K) \), for all \( u \in \mathbb{N}^* : \mu_{z,N}(n + p^n N\mathbb{Z}_p) = \frac{z^n}{1 - z^{p^n N}} \), \( n \in \{ 1, \ldots, p^n N \} \) (when \( z = \infty \), this is \( -1 \) if \( n = p^n N \) and 0 otherwise). This defines a measure on \( \mathbb{Z}_p^N \).

The integral of \( c \in C(\mathbb{Z}_p^N, L) \) with respect to \( \mu_{z,N} \) is \( \int_{\mathbb{Z}_p^N} c \mu_{z,N} = \lim_{n \to \infty} \sum_{r=1}^{p^n N} c(r) \mu_{z,N}(r + p^n N\mathbb{Z}_p) \). The statement below gives a new formulation and proof to Proposition 3.0.3 of \cite{D}, §3.

**Proposition A.0.4.** For any complete complete field extension \( L \) of \( K \), we have an isometric isomorphism of Banach spaces over \( K \)
\[
\{ f \in A(U^{an}) \otimes_K L \mid f(0) = 0 \} \xrightarrow{\sim} C(\mathbb{Z}_p^N, L)
\]
defined by
\[
\text{Cf}_0 : f \mapsto \text{the interpolation of } (m \in \mathbb{N}^* \mapsto [f(z^m)] \in L)
\]
whose inverse is
\[
\text{Cf}_0^{-1} : c \mapsto \text{the map } (z \mapsto \int_{\mathbb{Z}_p^N} c \mu_{z,N})
\]

**Proof.** (a) By definition, we have \( \{ f \in A(\mathbb{Z}_p^N) \otimes_{\mathbb{Q}_p} L \mid f(0) = 0 \} = \{ \sum_{m \in \mathbb{N}} c_m z^m \mid c_m \in L^N, c_m \to 0 \} \). We apply to this equality the map \( z \mapsto -\frac{1}{z} \).

On the one hand, since \( z \mapsto -\frac{1}{z} \) is the unique homography that sends \( (0, 1, \infty) \to (0, \infty, 1) \), it induces an involutive isomorphism between \( U^{an} = (\mathbb{P}^1^{an} - B(1, 0)/\mathbb{Q}_p) \) and the rigid analytic unit disk \( \mathbb{D}^an = (\mathbb{P}^1^{an} - B(\infty, 1))/\mathbb{Q}_p \), and induces an isomorphism between \( \{ f \in A(\mathbb{D}^an) \otimes_{\mathbb{Q}_p} L \mid f(0) = 0 \} \) and \( \{ f \in A(U^{an}) \otimes_{\mathbb{Q}_p} L \mid f(0) = 0 \} \), and the formula in Lemma A.0.3 implies easily that this map is an isometry.

On the other hand, by Lemma A.0.3 and Mahler’s theorem, the same map induces an isometric isomorphism between \( \{ \sum_{m \in \mathbb{N}} c_m z^m \mid c_m \in L^N, c_m \to 0 \} \) and \( \{ \sum_{m \in \mathbb{N}} c_m z^m \mid c_m \in L^N, m \to c_m \in \} \).
(b) We first prove that \( f \in A(U^{an}) \otimes_K L \), has power series expansion \( \sum c_m z^m \) at 0, then the map \( m \mapsto c_m \) interpolates to an element of \( C(Z_p^N, L) \).

The elements of \( A(U^{an}) \otimes_K L \) are the uniform limits of sequences of rational fractions over \( \mathbb{P}^1 \) whose poles are in \( \cup \xi \in \nu \mathcal{X} \); moreover, the uniform norm \( \|f\|_p = \sup |f(z)|_p \) is equivalent to our chosen norm \( \|\sum a_m z^m\| = \sup_n |a_m|_p \). Thus, by density, it is enough to prove that the statement is true for these rational fractions. Without loss of generality, we can assume \( L \) algebraically closed.

We are reduced to prove the statement for a function \( f(z) = \frac{1}{(z - z_0)^\nu} \) with \( z_0 \in B(\xi, 1) \) for a certain \( \xi \in \nu \mathcal{X}(K) \) and \( \nu \in \mathbb{N}^* \). Indeed, the polynomial part of a partial fraction decomposition of a rational function in \( A(U^{an}) \otimes_K L \) is necessarily constant, because such a rational fraction is bounded at \( \infty \). We have \( \xi^{-1} z_0 \in B(\xi, 1) \) and \( \frac{1}{(z - z_0)^\nu} = \xi^{-\delta}(z) \) where \( f \in A(\mathbb{P}^{1, an} \setminus B(1,1)) \otimes_K L \).

By (a), letting \( f = \sum c_m z^m \) be the power series expansion of \( f \) at 0, the map \( m \in \mathbb{N}^* \mapsto c_m \in L \) defines an element of \( C(Z_p, L) \). We have \( f(\xi^{-1} z) = \sum c_m \xi^{-m} z^m \) and, since \( m \mapsto \xi^{-m} \) is constant on each class of congruence modulo \( N \), we deduce the map \( m \in \mathbb{N}^* \mapsto c_m \xi^{-m} \in L \) defines an element of \( C(Z_p^N, L) \).

This proves that the map defined in (A.0.1) sends \( A(U^{an}) \otimes_K L \mid f(0) = 0 \) to \( C(Z_p^N, L) \). The fact that it is an isometry its injectivity follow from its definition.

(c) Let \( c \in C(Z_p^N, L) \). For all \( z \in U^{an}(L) \). We have \( \int_{z_p} c d\mu_{z,N} = \lim_{u \rightarrow \infty} \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m = \lim_{u \rightarrow \infty} \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m \in \mathbb{N}^* \) and, for each \( m \), \( c(m \mod N^pu) \) is the unique integer in \( \{1, \ldots, N^pu\} \) congruent to \( m \mod N^pu \) and the limit is in \( L \).

Since \( Z_p^N \) is compact, \( c \) is uniformly continuous; this implies that the sequence \( \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m \) is Cauchy for the norm \( |||| \) on the bounded elements of \( L^\infty \). Moreover, \( \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m = \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m \) is a rational function over \( \mathbb{P}^1 \) whose poles are in \( \cup \xi \in \nu \mathcal{X}(K) \).

Given that \( A(U^{an}) \) is the space of uniform limits of rational functions over \( \mathbb{P}^1 \) whose poles are in \( \cup \xi \in \nu \mathcal{X}(K) \), and that the uniform norm \( ||f||_\infty = \sup |f(z)|_p \) is equivalent to our chosen norm on \( A(U^{an}) \), the sequence \( \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m \in \mathbb{N}^* \), sequence converges to an element of \( A(U^{an}) \otimes_K L \).

Moreover, the limit in question is the element defined by \( \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m = \sum_{m=1}^{+\infty} c(m \mod N^pu) z^m \).

This implies that the map defined in (A.0.2) is a map \( C(Z_p, L) \rightarrow \{ f \in A(U^{an}) \otimes_K L \mid f(0) = 0 \} \), inverse of the map defined in (A.0.1).

\[ \square \]

**Remark A.0.5.** By considering quotients, Proposition (A.0.4) gives a statement on Tate algebras in one variable. Further, by gluing affinoid curves, it gives a surprising alternative description of rigid analytic curves over \( \mathbb{Q}_p \). We will develop this remark as well as its generalization to more general rigid analytic spaces in another paper.

**Appendix B. An adelic interpretation of the computation**

Our proof of the main theorem also gives, aside bounds on norms, a inductive way to compute over-convergent \( p \)-adic multiple polycyclic and \( p \)-adic cyclotomic multiple zeta values. This appendix answers to the question of characterizing which type of formula we obtain and, at the same time, to the question of finding a \( p \)-adic analogue of the type of series appearing in equation (0.1).

The following numbers will play a central role in our explicit theory of \( p \)-adic cyclotomic multiple zeta values.

**Definition B.0.1.** We call primary weighted cyclotomic multiple harmonic sums the numbers

\[ \text{har}_{\rho^p} ((n_1)_d; (\xi)_{d+1}) = (p^\rho)^{n_1 + \cdots + n_d} \text{hp}_{\rho^p} ((n_1)_d; (\xi)_{d+1}) \]

The primary weighted cyclotomic multiple harmonic sums satisfy an inequality "valuation \( \geq \) weight":

**Lemma B.0.2.** We have \( v_{p}(\text{har}_{\rho^p} ((n_1)_d; (\xi)_{d+1})) \geq \sum_{i=1}^{d} n_i \).

**Proof.** We split the domain of summation \( \{ (m_1)_d \in \mathbb{N}^d \mid 0 < m_1 < \cdots < m_d < p^\rho \} \) of multiple harmonic sums (equation (1.23)) into the disjoint union of two subsets: first, the subset characterized by \( p^{\rho-1} | n_i \),
for all $i$, whose contribution to the multiple harmonic sum is exactly $\text{har}_p ((n_i)_{d}; (\xi_i^{p^{-1}})_{d+1})$ which has valuation $\geq \sum_{i=1}^{d} n_i$, and, secondly, its complement, whose contribution has higher $p$-adic valuation.

The formulas obtained by our proof can be written in terms of prime weighted multiple harmonic sums. However, since they are only elements of the $N$-th cyclotomic field which plays the role of a field of coefficients, in order to be able to talk about "linear combinations" of them without having them absorbed in the field of coefficients, we consider all values of $p$ and $\alpha$ at the same time. Let us denote by $K = K_p$, and let $\mathcal{P}_N$ be the set of prime numbers which do not divide $N$.

We note that the constants $\kappa_d, \kappa'_d, \kappa''_d$ of Corollary 4.3.2 can be chosen independent of $p$ (this follows from the fact that $-\frac{d}{\log(p)}$ and $-\frac{d}{\log(p)} \log \left( \frac{d}{\log(p)} \right)$, which appear in Proposition 4.3.1 are lower bounded uniformly in $p$). Let $C_N$ be the $N$-th cyclotomic field, which we regard as included in all $K_p$, $p \in \mathcal{P}_N$.

We denote by $O_{\text{Bound}}(d)$ the set of formal infinite sums $\sum_{n \in \mathbb{N}} w_n$, where $w_n$ is a $C_N$-linear combination of words of weight $n$ and depth $\leq d$ with coefficients in $\{ x \in C_N | \forall p \in \mathcal{P}_N, v_p(x) \geq -\kappa_d - \kappa'_d \log(n + \kappa''_d) \}$.

**Definition B.0.3.** We fix a positive integer $d$. Let $\overline{\text{har}}_{p,N}^{\leq d}$ be the $\mathbb{Z}$-module defined as the image of the map $O_{\text{Bound}}(d) \rightarrow \prod_{(p,\alpha) \in \mathcal{P}_N \times N^*} K_p$ which sends $\sum_{n \geq 0} w_n \mapsto (\sum_{n \geq 0} \text{har}_p^{\alpha}(w_n))_{(p,\alpha) \in \mathcal{P}_N \times N^*}$.

**Lemma B.0.4.** The series $\sum_{n \geq 0} \text{har}_p^{\alpha}(w_n)$ as above are convergent in $K_p$, uniformly with respect to $p$, $\alpha$, and $\sum_{n \geq 0} w_n$ in $\mathcal{P}_N$, $N^*$ and $O_{\text{Bound}}(d)$ respectively. In particular, for any $n_0 \in \mathbb{N}$, the reduction of $\overline{\text{har}}_{p,N}^{\leq d}$ modulo $(p^{n_0})_{(p,\alpha) \in \mathcal{P}_N \times N^*}$ is a finitely generated $\mathbb{Z}$-module.

**Proof.** The first part of the statement follows from Lemma B.0.2. It implies that, for any $n_0 \in N^*$, we have $\sum_{n \geq 0} \text{har}_p^{\alpha}(w_n) \equiv \sum_{n=0}^{n_1} \text{har}_p^{\alpha}(w_n) \pmod{p^{n_0}}$ where $n_1$ can be chosen uniformly in $p$, $\alpha$ and $\sum_{n \geq 0} w_n$.

We will state a notion formalizing the sequences $(\text{har}_p^{\alpha}(w))_{(p,\alpha) \in \mathcal{P}_N \times N^*}$ in $\mathcal{J}$. The kernel of the map in Definition B.0.3 will be studied and formalized in $\mathcal{J}$. The next Proposition, we take the convention that, for $r_0 \in N^*$, $n \in N^*$, $\xi \in \mu_N(K)$, $\text{har}_r^{\alpha}(n, \xi) = \xi^{n_0} \overline{\text{har}}_{p,N}^{\leq d}$ (this is compatible with equation 1.13) which involves only indices of the type $((n_i)_d, (\xi_i)_{d+1})$.

**Proposition B.0.5.** (i) Any sequence $\left( \text{Ad}_{\Phi_{p,\alpha}(\xi)}([w]) \right)_{(p,\alpha) \in \mathcal{P}_N \times N^*}$ with $w$ a word over $e_{0,\mu_N}$ of depth $d$, $\xi \in \mu_N(K)$, is an element of $\overline{\text{har}}_{p,N}^{\leq d}$.

(ii) Let a word $w \in O_{\text{Bound}}(d)$, $l \in \mathbb{N}$ and $\xi \in \mu_N(K)$. We have $\left( (\text{Ci}_0 \text{Li}_{p,\alpha}(\xi)_{(l,\xi)})_{(p,\alpha) \in \mathcal{P}_N \times N^*} \in \overline{\text{har}}_{p,N}^{\leq d} \right)$ and there exist words $w' \in O_{\text{Bound}}(d)$ of depth $\leq d$ (in a finite set $l$) and elements $h_{w,l,\xi,\ell} \in \overline{\text{har}}_{p,N}^{\leq d}$ such that, for all $r_0 \in N^*$, $\left( (\text{Ci}_0 \text{Li}_{p,\alpha}(\xi)_{(l,\xi,\ell)})_{(p,\alpha) \in \mathcal{P}_N \times N^*} \pmod{p^{r_0}} \right)$ is the image of $h_{w,l,\xi,\ell} + \sum_{i \in \mathbb{N}} h_{w,l,\xi,\ell,\xi,\ell}(\pmod{p^{r_0}})$ by the canonical projection $\prod_{(p,\alpha) \in \mathcal{P}_N \times N^*} K_p \mapsto \prod_{p^e > r_0} K_p$.

**Proof.** The statement (ii) is actually true for the regularized iterated integrals of $\mathcal{J}$, by induction on the depth using Proposition 3.2.8. This implies (i) and (ii) by Proposition 1.1.3.

We note that we have not encountered in our computation the $p$-adic cyclotomic multiple zeta values, which are coefficients of $\Phi_{p,\alpha}$, but, instead, the coefficients of $\text{Ad}_{\Phi_{p,\alpha}(\xi)}$, $\xi \in \mu_N(K)$. This goes back to Lemma 2.1.2. Actually, by the formal properties of the Frobenius, the coefficients $\text{Ad}_{\Phi_{p,\alpha}(\xi)}$, $\xi \in \mu_N(K)$ are the versions of $p$-adic cyclotomic multiple zeta values that arise naturally if we want to make an explicit theory. We will formulate all our subsequent papers, including $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$, as an explicit theory of the numbers $\text{Ad}_{\Phi_{p,\alpha}(\xi)}$, $\xi \in \mu_N(K)$, which will be formalized in $\mathcal{J}$. The simple correspondence between the coefficients of $\Phi_{p,\alpha}$ and those of $\text{Ad}_{\Phi_{p,\alpha}(\xi)}$, $\xi \in \mu_N(K)$ is explained in a separated paper $\mathcal{J}$. From the point of view of the theory of associators. Let us just give the simplest example, which corresponds to the ordinary $p$-adic zeta values, of both this correspondence and our computation.

**Example B.0.6.** (Depth one (d=1) and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (N=1)).

For $n \in N^*$, we have $\text{dec}_{p,\alpha}(e_0^{n-1} e_1) = e_0^{n-1} (e_1 - e_{1(p^{n-1})})$: this follows from Definition 3.2.8 Lemma
Let \( \zeta \) note that the pole at \( s = 1 \) of the \( \xi \) (last fact is proved for \( \alpha = -1 \) in [U2], equation (4.1.2), and the proof works for any \( \alpha \in \mathbb{N}^* \)). This implies by Proposition A.1.3 that, for all \( n \in \mathbb{N}^* \), \( \text{Li}_{p,\alpha}(e_0^{n-1}e_1)(z) = (p^n)^n \sum_{m \geq 1} \frac{z^m}{m^n} \), and, by Proposition A.0.4:

\[
\text{Li}_{p,\alpha}^{\dagger}(e_0^{n-1}e_1)(z) = (p^n)^n \int_{\mathbb{Z}_p} \frac{1}{m^n} \text{d}\mu_x(m)
\]

where \( 1_{\mathbb{Z}_p} \) is the characteristic function of the set \( \{ m \in \mathbb{Z}_p | v_p(m) < \alpha \} \). This generalizes Koblić’s formula \(- \frac{1}{p} \log_\ast \left( \frac{1 - z}{1 - z^p} \right) = \int_{\mathbb{Z}_p} \frac{\text{d}\mu_x(x)}{x} \) (Ko §5, Lemma 1) and Coleman’s formula ([Co], Lemma 7.2 p. 202) which is, with our notations, \( \text{Li}_{p,\alpha}^{\dagger}(e_0^{n-1}e_1) = p^n \int_{\mathbb{Z}_p} \frac{\text{d}\mu_x(x)}{x^n} \).

If \( n \geq 2 \), we also have \( \Phi_{p,\alpha}(\xi_\text{sh}) = 0 \) (this follows from \( \Phi_{p,\alpha}(e_0^{n-1}e_1) = 0 \)). For \( l, n \in \mathbb{N}^* \), we have:

\[
\Phi_{p,\alpha}(e_1e_0^{n-1}e_1) = \Phi_{p,\alpha}(e_0e_1^{n-1}e_0) = (1)^{n+1} \Phi_{p,\alpha}(e_0^{n-1}e_1) = (1)^{n} \Phi_{p,\alpha}(e_0^{n-1}e_1).
\]

One can prove that \( \text{dec}(e_1e_0^{n-1}e_1) = e_1e_0^{n-1}e_1 - e_1(e_0^{n-1}e_1) \) and \( \text{dec}(e_0e_1^{n-1}e_0) = (1)^{n} \text{dec}(e_0^{n-1}e_1) \), and deduce that:

\[
\zeta_{p,\alpha}(n) = \frac{1}{n} - \sum_{l \in \mathbb{N}} \left( \frac{1}{l} \right) B_l \text{har}_p(n + l - 1)
\]

Let \( L_p \) be the \( p \)-adic \( L \)-function of Kubota-Leopoldt and \( \omega \) be the Teichmüller character. The equation (4) p. 173 of [Ca] is equivalent to \( \zeta_{p,\alpha}(n) = p^n L_p(n, \omega^{1-n}) \) for all \( n \in \mathbb{N}^* \), via [F2], Example 2.10 (1). Replacing \( \zeta_{p,\alpha}(n) \) by \( p^n L_p(n, \omega^{1-n}) \) in the \( \alpha = 1 \) case of (B.0.1), we recover the Theorem 6.6 of [R]. We note that the pole at \( s = 1 \) of the \( p \)-adic \( L \)-function of Kubota-Leopoldt is visible on that formula.

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