SUPER-TEICHMÜLLER SPACES AND RELATED STRUCTURES

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Abstract. This short note provides an overview of some theorems and conjectures obtained by the author and his collaborators. It is an extended abstract for the Oberwolfach workshop “New Trends in Teichmüller Theory and Mapping Class Groups”, 2 September - 8 September 2018.

1. Introduction.

The study of the superstring theory has drawn the attention to the very important generalizations of Riemann surfaces, known as super-Riemann surfaces (SRS) that could be viewed as $(1|N)$-dimensional complex supermanifolds with extra structures [1], [2]. Moduli spaces of $N = 1$ SRS are of special importance (see e.g. [11] for a review).

One of the ways to look at the corresponding Teichmüller spaces $ST(F)$ (here $F$ is the underlying Riemann surface of genus $g$ with $s$ punctures), is the view through the prism of the higher Teichmüller theory. They are defined in a manner analogous to that for the standard pure even case, $ST(F) = Hom'(\pi_1(F) \rightarrow G)/G$, where instead of $PSL(2,\mathbb{R})$, $G$ is a supergroup $OSp(1|2)$. Here $\pi_1(F)$ is the fundamental group of the underlying Riemann surface with punctures, and Hom’ stands for the homomorphisms that map the elements of $\pi_1(F)$, corresponding to small loops around the punctures, to parabolic elements of $OSp(1|2)$, which means that their natural projections to $PSL(2,\mathbb{R})$ are parabolic elements.

The image of the fundamental group under Hom’ produces a generalization of the standard Fuchsian group $\Gamma$, which acts on a super-analogue of the upper half-plane $H^+$ producing $N = 1$ super-Riemann surfaces as a factor $H^+/\Gamma$ [1]. It is necessary to build the super analogues of known objects in Teichmüller theory for the successful study of such spaces. In this note I will give an overview of the results from [9], [5], [6] concerning the generalization of the Penner coordinates [7], [8] for the super-Teichmüller space.

The Penner coordinates are the coordinates on $\mathbb{R}_+^s$-bundle $\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$ over the Teichmüller space of $T(F)$ of $s$-punctured surfaces with negative Euler characteristics.

The construction is based on the ideal triangulation of $F$ (i.e. vertices of triangulation are the punctures) and the assignment of a positive number
to every edge of the triangulation. An important feature of these coordinates
is that under the elementary changes of triangulation, known as Whitehead
moves, or flips generating the mapping class group, the change of coordinates
is rational, described by the so-called Ptolemy relations. Therefore, it is
making the mapping class group action rational.

The difficulty in constructing an analogue of these coordinates for $S\bar{T}(F) =
\mathbb{R}^s_+ \times ST(F)$ is that $ST(F)$ has many connected components enumerated by
spin structures. Thus, to proceed further it is necessary to have a suitable
combinatorial description of the spin structures.

2. Fatgraphs and spin structures

Consider the trivalent fatgraph $\tau$, corresponding to an $s$-punctured ($s > 0$)
Riemann surface $F$ of the negative Euler characteristic (i.e. a graph
with trivalent vertices), which is homotopically equivalent to $F$, with cyclic
orderings on half-edges for every vertex [8] induced by the orientation of the
surface.

There is a one-to-one correspondence between the ideal triangulations and
trivalent fatgraphs. Let $\omega$ be an orientation on the edges of $\tau$. As in [9],
we define a fatgraph reflection at a vertex $v$ of $(\tau, \omega)$ as a reversal of the
orientations of $\omega$ on every edge of $\tau$ incident to $v$. Let us define the $O(\tau)$
to be the equivalence classes of orientations on a trivalent fatgraph $\tau$ of $F$,
where $\omega_1 \sim \omega_2$ if and only if $\omega_1$ and $\omega_2$ differ by a finite number of fatgraph
reflections. In [9],[5], we identified such classes of orientations on fatgraphs
with the spin structures on $F$. The paths corresponding to the boundary
cycles on the fatgraph (i.e. the punctures of $F$) are divided into two classes
depending on the parity of number $k$ – the number of edges with orientation
opposite to the canonical orientation of $\gamma$.

The punctures are called Ramond (R) when $k$ is even, and Neveu-Schwarz
(NS) [11] when $k$ is odd. In [9], we have also proved that under the flip
transformations the orientations change in the generic situation as in Figure
1, where $\varepsilon_i$ stand for orientations on edges, and extra minus sign stands for

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$\varepsilon_1$};
\node (2) at (1,0) {$\varepsilon_3$};
\node (3) at (2,0) {$\varepsilon_2$};
\node (4) at (3,0) {$\varepsilon_4$};
\node (5) at (1,1) {$\varepsilon_2$};
\node (6) at (1,-1) {$\varepsilon_4$};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (1) -- (5);
\draw (1) -- (6);
\end{tikzpicture}
\end{center}

Figure 1. Spin graph evolution in the generic situation

the orientation reversal.

3. Main Result.

In [9], we have proved the following Theorem.
Theorem. i) The components of $\tilde{ST}(F)$ are determined by the space of spin structures on $F$. For each component $C$ of $\tilde{ST}(F)$, there are global affine coordinates on $C$ given by assigning to a triangulation $\Delta$ of $F$:
- one even coordinate called $l$-length for each edge;
- one odd coordinate called $\mu$-invariant for each triangle, taken modulo an overall change of sign.

In particular we have a real-analytic homeomorphism:
$$C \rightarrow \mathbb{R}_{>0}^{6g-6+3s+4g-4+2s} / \mathbb{Z}_2.$$

ii) The super Ptolemy transformations [9] provide the analytic relations between coordinates assigned to different choice of triangulation $\Delta'$ of $F$, namely upon flip transformation. Explicitly (see Figure 2), when all $a, b, c, d$ are different edges of the triangulations of $F$, the Ptolemy transformations are as follows:

$ef = (ac + bd)\left(1 + \frac{\sigma \theta \sqrt{\chi}}{1 + \chi}\right)$, $\nu = \frac{\sigma - \theta \sqrt{\chi}}{\sqrt{1 + \chi}}$, $\mu = \frac{\theta + \sigma \sqrt{\chi}}{\sqrt{1 + \chi}}$,

where $\chi = \frac{ac}{bd}$, so that the evolution of arrows is as in Figure 1.

To every edge $e$ (see the picture above) we can associate the shear coordinate $z_e = \log(\frac{ef}{\nu \mu})$. These parameters satisfy a linear relation for every puncture, and together with odd variables they form a set of coordinates on the $ST(F)$, thus producing the removal of the decoration, as it was in the pure even case. Here we mention that there is a physically and algebro-geometrically interesting refinement of $ST(F)$ studied in [6], corresponding to the removal of certain odd degrees of freedom associated with $R$-punctures.

4. $N = 2$ Super-Teichmüller space and beyond.

Replacing $OSp(1|2)$ in the definition of $ST(F)$ by $OSP(2|2)$, one obtains the super-Teichmüller space of punctured $N = 2$ SRS. It has been investigated in [5], and the analogue of Penner coordinates was constructed there. Unlike $N = 1$ case, the resulting $N = 2$ SRS, obtained by uniformization, correspond to a certain subspace in the moduli space of $N = 2$ SRS [2]. We
also mention that according to [2] \( N = 2 \) SRS are in one-to-one correspondence with \((1|1)\)-dimensional supermanifolds.

An important problem [10] will be to see explicitly how to glue \( N = 1 \) and \( N = 2 \) super-Riemann surfaces using the fatgraph data following the analogue with the Strebel theory.

Another important task is to understand the (complexified) version of the results of [9], [5] in the context of spectral networks and the abelianization construction of Gaiotto, Moore and Neitzke [3]. In the super case it looks that only quasi-abelianization seems to work: one should be able to describe our constructions via the moduli space of \( GL(1|1) \) local systems.

Finally, the super-Ptolemy transformations from the Theorem above, discovered in [9] (see also [5] for \( N = 2 \) case) should lead to new interesting generalizations of cluster algebras.

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