Matched witness for multipartite entanglement

Xiao-yu Chen¹ · Li-zhen Jiang¹ · Zhu-an Xu²

Abstract Entanglement criteria for multipartite entangled states are obtained by matching witnesses to multipartite entangled states. The necessary and sufficient criterion of separability for three qubit X states is given as an example to illustrate the procedure of finding a criterion. The result is utilized to obtain the noise tolerance of W state. The necessary and sufficient criteria of three partite separability and full separability for four qubit noisy cluster states, three partite separability for four qubit noisy GHZ states are obtained.

Keywords Entanglement witness · Separable criterion · Cluster state · GHZ state

1 Introduction

Quantum entanglement is considered as the central resource in the manipulation of quantum information. It plays an important role in quantum communication at the present day and quantum computation in the future. A state is called separable if it can be written as the probability mixture of product states [1], otherwise it is entangled. To determine a given state is entangled or not is still very difficult in general. A lot of progress has been achieved for the criteria of entanglement or separability. Among them are the Peres–Horodecki criterion [2,3], the computable cross norm [4] or realignment criterion [5], the entropy criterion, the uncertainty criterion [6] and so on [7,8]. All these criteria are necessary conditions for separability, and
violation each of them implies entanglement. Peres–Horodecki criterion is also sufficient for states in Hilbert spaces of dimension $2 \times 2$, $2 \times 3$ and some Gaussian states. The criteria are mainly applied to bipartite states, and only a few of them are applicable for multipartite entanglement [9–14]. The classification of multipartite entanglement is given in terms of partial and genuine multipartite entanglement, for an introduction see, e.g., [15,16]. A prominent multipartite entangled state is the GHZ state [17], which possesses the maximal amount of genuine multipartite entanglement and all its reduced density matrices are separable. Another example is the $W$ state [18], and it contains a certain amount of genuine multipartite entanglement and retains maximally bipartite entanglement when any one of the three qubits is traced out. They are not equivalent under stochastic local operations and classical communication [18].

Entanglement witnesses are Hermitian operators for detecting entanglement [19–21]. An entanglement witness has nonnegative expectations for all separable states and has negative expectation for at least one entangled state. The latter condition offers the possibility of experimental detection of entanglement via the measurement of the witness. An entanglement witness is called optimal if it has zero mean on some separable states. Denote the optimal entanglement witness as $\hat{W}$, it is always possible to write it in the form of [21]

$$\hat{W} = B(\hat{M})\mathbb{I} - \hat{M},$$

where $\mathbb{I}$ is the identity operator of the system, $\hat{M}$ is a general Hermitian operator, and the function $B(\hat{M})$ denotes the maximally achievable mean for separable states:

$$B(\hat{M}) = \sup\{tr(\rho_{\text{sep}}\hat{M})\}$$

The supremum is taken over all separable states $\rho_{\text{sep}}$ (in fact pure product states suffice) which are defined with respect to the given partitions. To find $B(\hat{M})$ of a given $\hat{M}$, the so-called separability eigenvalue equations were developed [21] recently. However, the previous works are on the separable states and are not closely related to the entangled state under investigation. The method of constructing the operator $\hat{M}$ is by no means systematic. It is rather ad hoc. We find that the parameters of $\hat{M}$ can be gradually changed to fit with the entangled state. Thus, there are two steps in finding separable criteria. The first step is to find the optimal witness $\hat{W}$ for a given Hermitian operator $\hat{M}$, and this gives rise to a necessary criterion of separability. The second step is to adjust the parameters of $\hat{M}$ to match with the entangled state while keeping the witness optimal, this possibly leads to a sufficient criterion of separability.

In this paper, we will study the optimal and matched entanglement witnesses for multipartite entangled states with emphasis on graph diagonal states. With the matched entanglement witness method, we find that the process of finding the necessary criteria of separability for some multipartite states may also give rise to the sufficient criteria.
2 Entanglement witness in characteristic form

A multipartite state \( \rho \) is separable when it can be written as \([1]\)

\[
\rho = \sum_i p_i \rho_i^{A_1} \otimes \rho_i^{A_2} \otimes \ldots \otimes \rho_i^{A_N}.
\]

where \( \rho_i^{A_j} \) is the state (it is always possible to assume it to be pure) of \( A_j \) part, \( p_i \) form a probability distribution. The method of partition of a system into \( N \) parts may change with the index \( i \) to cope with the concepts of biseparable, \( k \)—separable and fully separable. In the following of this section, we will consider the three qubit system to simplify the notations. The notations can be easily extended to multipartite qubit system. Firstly, let us consider the full separability problem of a three qubit state \( \rho \) whose characteristic function is

\[
R_{ijk} = \text{tr}[(\rho \otimes \sigma_i \otimes \sigma_j \otimes \sigma_k)],
\]

where \( i, j, k = 0, 1, 2, 3; \sigma_i \) are Pauli matrices for \( i = 1, 2, 3, \) and \( \sigma_0 = I \) is the \( 2 \times 2 \) identity matrix. Let the characteristic function of a Hermitian operator \( \hat{M} \) be \( M_{ijk} \). For simplicity, we set \( M_{ijk} \) to be zero when \( R_{ijk} \) is zero. A state \( \rho \) is entangled when

\[
\text{tr}(\rho \hat{W}) = 8 \left[ B(\hat{M}) - \sum_{ijk} R_{ijk} M_{ijk} \right] < 0,
\]

while for all pure product states \( |\psi\rangle = |\psi_x\rangle |\psi_y\rangle |\psi_z\rangle \) we have

\[
\langle \psi | \hat{W} | \psi \rangle = 8 \left[ B(\hat{M}) - \sum_{ijk} R_{ijk}^s M_{ijk} \right] \geq 0.
\]

where \( R_{ijk}^s = \langle \psi | \sigma_i \otimes \sigma_j \otimes \sigma_k | \psi \rangle = x_i y_j z_k \). We have denoted \( |\psi_x\rangle \langle \psi_x| = \frac{1}{2} \sum_{i=0}^3 x_i \sigma_i \) with Bloch representation. Thus, \( x = (x_1, x_2, x_3) \) is a unit vector, namely \( |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 \) and \( x_0 = 1, (y_j,z_k) \) are similarly defined). Without loss of generality, we assume \( M_{000} = 0 \) (\( \hat{M} \) is traceless), then

\[
B(\hat{M}) = \max_{x,y,z} \sum_{ijk} M_{ijk} x_i y_j z_k
\]

Denote vectors \( \mathbf{M} = (M_{001}, M_{002}, M_{003}, \ldots, M_{333}) \) and \( \mathbf{R} = (R_{001}, R_{002}, R_{003}, \ldots, R_{333}) \). The inner product of vectors \( \mathbf{M} \) and \( \mathbf{R} \) is \( \mathbf{M} \cdot \mathbf{R} = \sum_{i,j,k=0; (i,j,k) \neq (0,0,0)} M_{ijk} R_{ijk} \); then, the necessary condition of full separability is that \( \mathbf{M} \cdot \mathbf{R} \) is upper bounded by \( B(\hat{M}) \),

\[
\mathbf{M} \cdot \mathbf{R} \leq B(\hat{M}).
\]

Violation of it implies entanglement.
Fig. 1 $W_1$ is an optimal witness which is tangent to the separable state set, any witness can be parallelly moved to become optimal. $W_2$ is an optimal and matched witness, rotation is needed to transform $W_1$ to $W_2$.

For an entangled state $\rho$ with characteristic function $R_{ijk}$, we consider a state $\rho' = p\rho + (1-p)\mathbb{I}/8$, the mixture of $\rho$ with white noise $\mathbb{I}/8$. The state $\rho'$ is entangled if $\text{tr}(\rho'\hat{W}) = B(\hat{M}) - p\mathbf{M} \cdot \mathbf{R} < 0$. We may properly choose $\mathbf{M}$ such that $\mathbf{M} \cdot \mathbf{R}$ is positive. We calculate the following minimization

$$ P(R) = \min_{\mathbf{M}} \frac{B(\hat{M})}{\mathbf{M} \cdot \mathbf{R}}. $$

(6)

If $p > P(R)$, the state $\rho'$ is entangled. Hence, $P(R)$ is the fraction of $\rho$ such that the mixture just becomes fully separable. Or we say that $1 - P(R)$ is the white noise tolerance of $\rho$. The optimization of an entanglement witness $\hat{W} = c\mathbb{I} - \hat{M}$ is to optimize $c$. We may consider $c$ as a parameter of $\hat{W}$. To find the matched witness, we optimize $\mathbf{M}$ in (6). The vector $\mathbf{M}$ can be seen as all the other parameters of $\hat{W}$.

In Fig. 1, we use a rotation to represent the matching process while optimization is represented by parallel displacement.

For the problem of biseparability of a three qubit state, we first consider the characteristic function of a pure two qubit state $|\psi\rangle = \sum_{m,n=0}^{1} \alpha_{mn} |mn\rangle$. The characteristic function of $|\psi\rangle$ is denoted as $T_{ij} = \langle \psi | \sigma_i \otimes \sigma_j | \psi \rangle$. A state $\rho^{bs}$ is biseparable if

$$\rho^{bs} = \sum_{l} \left( p_{1l} \rho_{l}^{A_1} \otimes \rho_{l}^{A_2A_3} + p_{2l} \rho_{l}^{A_2} \otimes \rho_{l}^{A_1A_3} + p_{3l} \rho_{l}^{A_3} \otimes \rho_{l}^{A_1A_2} \right),$$

(7)

where $\rho_{l}^{A_mA_n}$ is a pure two qubit state, $\rho_{l}^{A_m}$ is a pure qubit state. An optimal genuine entanglement witness $\hat{W} = B(\hat{M})\mathbb{I} - \hat{M}$ should fulfill the condition

$$\text{tr}(\rho^{bs}\hat{W}) = 8 \left[ B(\hat{M}) - \sum_{ijk} R_{ijk}^{bs} M_{ijk} \right] \geq 0.$$
for any biseparable state $\rho^{bs}$. Let
\[
M_x = \max_{x, T^{(23)}} \sum_{ijk} M_{ijk} x_i T_{jk}^{(23)},
\]
and
\[
B(\hat{M}) = \max\{M_x, M_y, M_z\},
\]
where $M_y, M_z$ are defined similarly as $M_x$. Here $x$ is a unit vector, and $T^{(23)}$ is a matrix with entries $T_{jk}^{(23)}$. The necessary condition of biseparability is still described by inequality (5), with $B(\hat{M})$ defined in (8). We may also find matched witness with (6).

3 Three qubit $X$ states

The density matrix of a three qubit $X$ state is a $8 \times 8$ matrix with diagonal entries, anti-diagonal entries, and all the other entries are zero. The possible nonzero entries are denoted as $\rho_{00}, \rho_{11}, \ldots, \rho_{77}$ for diagonal elements and $\rho_{01}, \rho_{16}, \ldots, \rho_{70}$ for anti-diagonal elements. The decimal subscript $l$ is equivalent to the three bit binary string $l_1l_2l_3$ such that $l = 4l_1 + 2l_2 + l_3$. Hence, we may write $\rho_{16}$ as $\rho_{001}$, $\rho_{110}$ in three bit binary subscripts. The fully separable problem of three qubit $X$ states was proposed and partly solved in [11], and we will give a strict and systematic solution to this problem. A simpler problem is the full separability of three qubit $GHZ$ diagonal states, the sufficient condition of which was given in [12], and the necessary condition was studied in [13]. For a three qubit $X$ state, the nonzero elements of the characteristic function are $R_{ijk}$ with the subscripts $i, j, k = 0, 3$ or $i, j, k = 1, 2$. The unit vector $x$ can be written as
\[
x = (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1)
\]
with $\theta_1 \in [0, \pi], \varphi_1 \in [0, 2\pi]$. Similarly, $y, z$ are expressed with $\theta_2, \varphi_2$ and $\theta_3, \varphi_3$, respectively. Then $B(\hat{M}) = \max_{\theta, \varphi} F(\theta, \varphi)$, with
\[
F(\theta, \varphi) = f(\theta) + s_1s_2s_3g(\varphi),
\]
and $\theta = (\theta_1, \theta_2, \theta_3), \varphi = (\varphi_1, \varphi_2, \varphi_3)$, where
\[
f(\theta) = M_{003}c_3 + M_{030}c_2 + M_{300}c_1 + M_{033}c_2c_3
+ M_{303}c_1c_3 + M_{330}c_1c_2 + M_{333}c_3c_2c_1,
\]
we have denoted $\cos \theta_i = c_i, \sin \theta_i = s_i$ for short, and
\[
g(\varphi) = M_{111}c_1'c_2'c_3' + M_{122}c_1's_2's_3'
+ M_{212}s_1'c_2's_3' + M_{221}s_1's_2'c_3',
\]
for the case of real anti-diagonal entries of an X state. Where \( c_i' = \cos \varphi_i, s_i' = \sin \varphi_i \).

Before we deal with the general situations, let us consider the case of using entanglement witness to obtain Peres–Horodecki criterion for the full separability of \( GHZ \) diagonal states, which are X states with \( \rho_{ii} = \rho_{7-i,7-i} \) and real anti-diagonal entries [12,13]. We assume that \( M_{111} = 1, M_{122} = M_{212} = M_{221} = -1 \). Then \( g(\varphi) = \cos(\varphi_1 + \varphi_2 + \varphi_3) \leq 1 \). For a \( GHZ \) diagonal state, we have \( R_{003} = R_{030} = R_{300} = R_{333} = 0 \). We can assume \( M_{003} = M_{030} = M_{300} = M_{333} = 0 \). We may further assume \( M_{033} = M_{303} = M_{330} = -1 \), then

\[
F(\theta, \varphi) = -c_1c_3 - c_1c_2 + g(\varphi)s_1s_2s_3 \\
\leq -c_1c_3 - c_1c_2 + |s_1s_2s_3| \\
\leq -c_1c_2 + \sqrt{s_1^2s_2^2 + (c_1 + c_2)^2} \\
= -c_1c_2 + 1 + c_1c_2 = 1.
\]

The separable condition (5) now is \( M \cdot R = R_{111} - R_{122} - R_{212} - R_{221} - R_{033} - R_{330} - R_{303} \leq 1 \), which is

\[
\rho_{00} \geq \rho_{07}. \tag{10}
\]

If we reverse the signs of \( M_{111}, M_{122}, M_{212}, M_{221} \), we will reverse the sign of \( \rho_{07} \) in \( (10) \); thus, we have

\[
\rho_{00} \geq |\rho_{07}|. \tag{11}
\]

If we only reverse the signs of \( M_{122}, M_{212}, \) we have \( g(\varphi) = \cos(\varphi_1 + \varphi_2 - \varphi_3) \leq 1 \). Then \( |\rho_{07}| \) at the right-hand side of \( (11) \) could be replaced with \( |\rho_{16}| \). Similarly, \( |\rho_{07}| \) can be replaced with \( |\rho_{25}| \) and \( |\rho_{34}| \). By reversing the signs of two of \( M_{033}, M_{303}, M_{330} \) to +1, we can replace the left-hand side of \( (11) \) with \( \rho_{11}, \rho_{22} \) and \( \rho_{33} \). Hence, Peres–Horodecki criterion

\[
\min_{i=0,\ldots,3} \{\rho_{ii}\} \geq \max_{j=0,\ldots,3} \{|\rho_{j,7-j}|\} \tag{12}
\]

for the full separability of \( GHZ \) diagonal states is recovered using entanglement witness method.

On the other hand, if we assume \( M_{111} = M_{122} = M_{212} = M_{221} = 1 \) in (9), then

\[
g(\varphi) = c_1'c_2'c_3' + c_1's_2's_3' + s_1'c_2's_3' + s_1's_2'c_3' \leq \sqrt{\cos^2(\varphi_1 - \varphi_2) + \sin^2(\varphi_1 + \varphi_2)} \leq \sqrt{2}.
\]

We may assume \( M_{033} = M_{303} = M_{330} = -\sqrt{2} \), hence \( F(\theta, \varphi) \leq \sqrt{2} \). The separable condition then is \( M \cdot R = R_{111} + R_{122} + R_{212} + R_{221} - (R_{033} + R_{330} + R_{303}) \sqrt{2} \leq \sqrt{2} \), which is

\[
\rho_{00} \geq \frac{1}{2\sqrt{2}} (\rho_{07} + \rho_{16} + \rho_{25} + \rho_{34}).
\]
Let $\rho_{16} = \rho_{25} = \rho_{34} = -\rho_{07}$, the separable condition is

$$\rho_{00} \geq \sqrt{2} \rho_{16}.$$  

Which detects entangled state that can not be detected by $\rho_{00} \geq \rho_{16}$ derived from Peres–Horodecki criterion.

The parameters $M_{111}, M_{122}, M_{212}, M_{221}$ in $g(\varphi)$ are essentially classified as two types represented by the two cases discussed above. The details are summarized in Lemma 1 in the following. The fully separable criteria of three qubit $X$ states based on Lemma 1 are described in Theorem 1 and Theorem 2.

**Lemma 1** Denote the maximum of $g(\varphi)$ in (9) with respect to $\varphi$ as $g_m$, then

$$g_m = \begin{cases} \sqrt{\frac{(\delta \alpha + \beta \gamma)(\delta \beta + \alpha \gamma)(\delta \gamma + \alpha \beta)}{\delta \alpha \beta \gamma}}, & \text{for } \delta \alpha \beta \gamma > 0 \text{ and } q \geq 0; \\ \max\{|M_{111}|, |M_{122}|, |M_{212}|, |M_{221}|\}, & \text{otherwise}. \end{cases}$$

Where $(\delta, \alpha, \beta, \gamma) = \frac{1}{4}(M_{111}, M_{122}, M_{212}, M_{221})\Gamma$, here $\Gamma$ is a $4 \times 4$ matrix with all of its diagonal entries being $-1$ and off-diagonal entries being $+1$. $q = q_0q_1q_2q_3$ with $(q_0, q_1, q_2, q_3) = (\alpha \beta \gamma, \delta \beta \gamma, \delta \alpha \gamma, \delta \alpha \beta)$.

The proof can be found in Appendix A.

**Theorem 1** A three qubit $X$ state with identical diagonal entries and real anti-diagonal entries is fully separable iff

$$R \leq 1,$$

where

$$R = \begin{cases} \sqrt{\frac{(R_0R_1+R_2R_3)(R_0R_2+R_1R_3)(R_0R_3+R_1R_2)}{R_0R_1R_2R_3}}, & \text{for } Q > 0 \text{ and } r \geq 0; \\ \max\{|\rho_{07}|, |\rho_{16}|, |\rho_{25}|, |\rho_{34}|\}, & \text{otherwise}. \end{cases}$$

(14)

with $R_0 = R_{111}, R_1 = R_{122}, R_2 = R_{212}, R_3 = R_{221}$ and $Q = R_0R_1R_2R_3$. Here $r = r_0r_1r_2r_3$ and vector $(r_0, r_1, r_2, r_3) = (R_1R_2R_3, R_0R_2R_3, R_0R_1R_3, R_0R_1R_2)\Gamma$.

The proof is given in Appendix B.

**Theorem 2** A three qubit $X$ state with real anti-diagonal entries is fully separable iff

$$\min_{i=0,\ldots,3} (\sqrt{\rho_{00} \rho_{33} \rho_{55} \rho_{66}}, \sqrt{\rho_{11} \rho_{22} \rho_{44} \rho_{77}}, \sqrt{\rho_{ii} \rho_{77-i,7-i}}) \geq \frac{1}{8}. \tag{15}$$

The proof is given in Appendix C.

As an application of Lemma 1, let us consider the noise tolerance of $W$ state. The mixture of $W$ state with white noise is

$$\rho_W = p |W\rangle \langle W| + \frac{1-p}{8} \mathbb{I}, \tag{16}$$

$\mathbb{I}$.
where \( p \in (0, 1] \) and \( I \) is the \( 8 \times 8 \) identity matrix of the tensor product space. We choose the witness operator as

\[
\hat{W} = I + \sigma_3 \sigma_3 \sigma_3 - \sigma_3 \sigma_1 \sigma_1 - \sigma_1 \sigma_3 \sigma_1 - \sigma_1 \sigma_1 \sigma_3.
\]

Here, \( \sigma_i \sigma_j \sigma_k = \sigma_i \otimes \sigma_j \otimes \sigma_k \), the tensor product symbol is omitted for simplicity and we will omit it in the following when there is no confusion. Then for fully separable pure state \( \rho_s = \bigotimes_{i=1}^3 \varrho_i \) with

\[
\varrho_i = \frac{1}{2} (I + \sin \theta_i \cos \phi_i \sigma_1 + \sin \theta_i \sin \phi_i \sigma_2 + \cos \theta_i \sigma_3),
\]

we have \( Tr \hat{W} \rho_s = 1 - L \), with

\[
L = - \cos \theta_1 \cos \theta_2 \cos \theta_3 + L_1 \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
+ L_2 \sin \theta_1 \cos \theta_2 \sin \theta_3 + L_3 \sin \theta_1 \sin \theta_2 \cos \theta_3,
\]

where \( L_1 = \cos \phi_2 \cos \phi_3, L_2 = \cos \phi_1 \cos \phi_3, L_3 = \cos \phi_1 \cos \phi_2 \). Comparing (17) with (9), the maximum of \( L \) then is equal to the right-hand side of (13) with \((M_{111}, M_{122}, M_{212}, M_{221}) = (-1, L_1, L_2, L_3)\). We will prove that \( \delta \alpha \beta \gamma \leq 0 \) (see Appendix E), thus the maximum of \( L \) is determined by the second line of the right-hand side of (13). It is \( L \leq \max\{1, |L_1|, |L_2|, |L_3|\} = 1 \) since \( |L_i| \leq 1 \) for \( i = 1, 2, 3 \). Hence \( Tr \hat{W} \rho_s \geq 0 \) and \( \hat{W} \) is optimal for \( L = 1 \) is achievable. The noisy \( W \) state (16) is entangled if \( Tr \hat{W} \rho_W < 0 \). We have \( Tr \hat{W} \rho_W = 1 - 3p \). Thus, \( \rho \) is entangled if

\[
p > \frac{1}{3}.
\]

The condition (18) detects more entangled states than the condition \( p > \frac{23}{63} \) obtained in [21]. It is a necessary condition of full separability.

4 Tripartite separability of noisy four qubit \( GHZ \) state

The four qubit \( GHZ \) state \( |GHZ_4\rangle \) is a graph state characterized by its four stabilizer generators \( \sigma_1 \sigma_3 \sigma_3 \sigma_3, \sigma_3 \sigma_1 \sigma_1, \sigma_3 \sigma_1 \sigma_1, \sigma_3 \sigma_1 \sigma_1 \sigma_1 \). We may apply Hadamard transformations on all the qubits except the first one, and the generators then become \( K_1 = \sigma_1 \sigma_3 \sigma_1 \sigma_1, K_2 = \sigma_3 \sigma_1 \sigma_1 \sigma_1, K_3 = \sigma_3 \sigma_1 \sigma_3 \sigma_1, K_4 = \sigma_3 \sigma_1 \sigma_3 \sigma_1 \). The noisy \( GHZ \) state is \( \rho = p |GHZ_4\rangle \langle GHZ_4| + \frac{1-p}{16} I \) with \( I = I_{1111} \) for four qubit systems. The biseparability and full separability of \( \rho \) are known [9]. For the tri-separability, we find the matched witness with nonzero parameters \( M_{3333} = 2, M_{1111} = M_{2222} = 1, M_{2211} = M_{2121} = M_{2112} = M_{1221} = M_{1212} = M_{1122} = -1 \) (We omit all those parameters which are zeros). In the following, we will show that \( B(\hat{M}) = 2 \). The critical value of \( p \) for tri-separable is \( P(R) = \frac{1}{3} \). The state \( \rho \) is a mixture of three part product states if \( p \leq \frac{1}{3} \) and can not be three partite separable for \( p > \frac{1}{3} \).
4.1 Necessary condition

Consider the qubits 1, 2, 3, 4, we first classify the qubits into three parts with the first two qubits in a part, and the third and the fourth are the other two parts. We denote the partition as 12|3|4. For the given $M_{klmn}$, we have $B(\hat{M}) = \max f$, where

$$f = 2T_{33}z_3z_4 + (T_{11} + T_{22})(x_3x_4 + y_3y_4)$$

$$+(T_{21} + T_{12})(y_3x_4 + x_3y_4),$$

subject to $x_i^2 + y_i^2 + z_i^2 = 1$ ($i = 3, 4$) and $T_{ij} = \langle \psi | \sigma_i \otimes \sigma_j | \psi \rangle$, where $|\psi\rangle$ is a two qubit pure state. Denote $a = z_3z_4$, $b = x_3x_4 + y_3y_4$, $c = y_3x_4 + x_3y_4$. Then $f = \langle \psi | \mathcal{M} | \psi \rangle$, where $\mathcal{M}$ is a 4 × 4 matrix with eigenvalues $a \pm 2c = 2z_3z_4 \pm 2(y_3x_4 + x_3y_4) \leq 2$ and $-a \pm 2c = -2z_3z_4 \pm 2(x_3x_4 + y_3y_4) \leq 2$. The maximum of $f$ is equal to the largest eigenvalue of $\mathcal{M}$ which is 2. Thus, $f \leq 2$ for 12|3|4 partition. Due to the symmetry of the problem, we have $f \leq 2$ for all six kinds of partitions. Thus, we have $B(\hat{M}) = 2$, and the necessary condition for tri-separability is obtained as $p \leq \frac{1}{5}$.

4.2 Sufficient condition

The noisy four qubit GHZ state with $p = \frac{1}{5}$ can be written as

$$\frac{1}{16} \left[ \frac{1}{5} (\mathbb{1} + \sigma_1 \sigma_1 \sigma_1 \sigma_1 + \sigma_3 \sigma_3 I I - \sigma_2 \sigma_2 \sigma_1 \sigma_1) \right. $$

$$+ \frac{1}{5} (\mathbb{1} + \sigma_2 \sigma_2 \sigma_2 \sigma_2 - \sigma_2 \sigma_1 \sigma_2 \sigma_1 + I \sigma_3 I \sigma_3) \right. $$

$$+ \frac{1}{5} (\mathbb{1} - \sigma_2 \sigma_1 \sigma_2 \sigma_1 - \sigma_1 \sigma_1 \sigma_2 \sigma_2 + \sigma_3 I \sigma_3 I ) \right. $$

$$+ \frac{1}{5} (\mathbb{1} - \sigma_1 \sigma_2 \sigma_2 \sigma_1 - \sigma_1 \sigma_2 \sigma_1 \sigma_2 + I I \sigma_3 \sigma_3) \right. $$

$$\left. + \frac{1}{5} (\mathbb{1} + \sigma_3 I I \sigma_3 + I \sigma_3 \sigma_3 I + \sigma_3 \sigma_3 \sigma_3 \sigma_3) \right].$$

Each round bracket in the above expression is tri-separable. The first four round brackets are tri-separable in partitions 12|3|4, 13|2|4, 1|3|24, 1|2|34, respectively. The last one is fully separable. For example, the first round bracket is

$$(\mathbb{1} + \sigma_1 \sigma_1 \sigma_1 \sigma_1 + \sigma_3 \sigma_3 I I - \sigma_2 \sigma_2 \sigma_1 \sigma_1)$$

$$= \frac{1}{2} (II + \sigma_3 \sigma_3 + \sigma_1 \sigma_1 - \sigma_2 \sigma_2) \otimes (II + \sigma_1 \sigma_1)$$

$$+ \frac{1}{2} (II + \sigma_3 \sigma_3 - \sigma_1 \sigma_1 + \sigma_2 \sigma_2) \otimes (II - \sigma_1 \sigma_1).$$

The components $(II \pm \sigma_1 \sigma_1) = \frac{1}{2} (I + \sigma_1) \otimes (I \pm \sigma_1) + \frac{1}{2} (I - \sigma_1) \otimes (I \mp \sigma_1)$ are separable for the third and the fourth qubits. The components $[II + \sigma_3 \sigma_3 \pm (\sigma_1 \sigma_1 - \sigma_2 \sigma_2)]$ are
unnormalized two qubit states for the first two qubits. Thus, $\mathbb{I} + \sigma_1 \sigma_1 \sigma_1 + \sigma_3 \sigma_3 I I - \sigma_2 \sigma_2 \sigma_1 \sigma_1$ is tri-separable for partition 12|3|4.

5 Full separability of noisy four qubit cluster state

The four qubit cluster state $|Cl_4\rangle$ is a graph state characterized by its four stabilizer generators $K_1 = \sigma_1 \sigma_3 I I$, $K_2 = \sigma_3 \sigma_1 I I$, $K_3 = I \sigma_3 \sigma_1 \sigma_3$, $K_4 = I I \sigma_3 \sigma_1$. The noisy cluster state is

$$\rho = p |Cl_4\rangle \langle Cl_4| + \frac{1 - p}{16} \mathbb{I}. \quad (19)$$

The biseparability of the state is known [9]. We will consider the full separability of the state in this section and the three partite separability in the next section.

For the full separability, we find that the witness with the following nonzero parameters is a matched witness, $M_{3130} = M_{0313} = M_{3101} = M_{1013} = M_{2230} = M_{0322} = M_{2201} = M_{1022} = 1, M_{3223} = M_{1331} = M_{2112} = 2, M_{2123} = M_{3212} = -2$. In the following, we will show that $B(\hat{M}) = 2$. Hence, the noise tolerance of full separability for $|Cl_4\rangle$ is $p_{tol} = 1 - P(R)$ with $P(R) = \frac{1}{9}$. The state $\rho$ is fully separable for $p \leq \frac{1}{9}$ and is entangled for $p > \frac{1}{9}$.

5.1 The necessary condition

For the given $M_{klmn}$, denote $B(\hat{M}) = \max f$, where

$$f = (z_1 x_2 + y_1 y_2)(z_3 + x_4) + (z_2 + x_1) \times (x_3 z_4 + y_3 y_4) + 2(z_1 y_2 y_3 z_4 + x_1 z_2 z_3 x_4) + y_1 y_2 x_3 y_4 - y_1 y_2 y_3 z_4 - z_1 y_2 x_3 y_4),$$

subject to $x_i^2 + y_i^2 + z_i^2 = 1$ ($i = 1, \ldots, 4$). Using $x_4^2 + y_4^2 + z_4^2 = 1$, the maximization over $(x_4, y_4, z_4)$ leads to $f \leq f_1$ where

$$f_1 = b \cos \theta + \sqrt{(b + d \cos \theta)^2 + e^2 \sin^2 \theta}.$$

With $b = z_1 x_2 + y_1 y_2$, $d = 2x_1 z_2$, $e = \sqrt{(z_2 + x_1)^2 + 4(z_1 y_2 - y_1 x_2)^2}$, and $z_3 = \cos \theta$ is assumed. There are two solutions to the maximization of $f_1$ over $\theta$. The first is $\sin \theta = 0$, thus $f_1 = \pm b + \pm b + d \leq 2$. The inequality comes from $f_1 = \pm 2b + d = 2[\pm (z_1 x_2 + y_1 y_2) + x_1 z_2] \leq 2$ when $\pm b + d \geq 0$ and $f_1 = -d \leq 2$ when $\pm b + d < 0$. The second solution is $\cos \theta = \frac{b}{e - d}$ subject to $-1 \leq \frac{b}{e - d} \leq 1$. Thus, $f_1 = \frac{b^2}{e - d} + e$. A simple numeric calculation shows that $f_1 \leq 2$. Hence $B(\hat{M}) = 2$. The necessary condition of full separability is $p \leq \frac{1}{9}$.
5.2 The sufficient condition

The maximization of $f$ in the above subsection hints the process of decomposing a separable state into its explicit separable expression. If the maximal $f_1 = 2$ is achieved by one of the terms, say $2z_1y_2y_3z_4$, we have $2z_1y_2y_3z_4 = 2$. The solutions of $z_1y_2y_3z_4 = 1$ are that $z_1, y_2, y_3, z_4$ should be equal to $\pm1$, and the number of $-1$ should be even. Then there are 8 solutions. Each solution corresponds to a product state; for example, $z_1 = y_2 = y_3 = z_4 = 1$ corresponds to a product state proportional to $(I + \sigma_3)(I + \sigma_2)(I + \sigma_2)(I + \sigma_3)$. Summing up all the 8 product states gives rise to unnormalized fully separable state $I + \sigma_3\sigma_2\sigma_2\sigma_3$. Similarly, all the other solutions of $f = 2$ can be utilized to obtain the product states. Thus, the mixture of the product states will compose the noisy cluster state if the noise is under some threshold.

The noisy cluster state with $p = \frac{1}{9}$ can be written as

$$
\rho = \frac{1}{16} \left[ \frac{1}{9} (I + \sigma_3\sigma_2\sigma_2\sigma_3) + \frac{1}{9} (I + \sigma_2\sigma_1\sigma_1\sigma_2) + \frac{1}{9} (I - \sigma_2\sigma_1\sigma_2\sigma_3) + \frac{1}{9} (I - \sigma_3\sigma_2\sigma_1\sigma_3) + \frac{1}{9} (I + \sigma_3\sigma_1\sigma_3I + \sigma_3\sigma_1I\sigma_1 + I\sigma_3\sigma_1) + \frac{1}{9} (I + \sigma_2\sigma_2\sigma_2\sigma_3 + \sigma_2\sigma_2I\sigma_1 + I\sigma_3\sigma_1) + \frac{1}{9} (I + \sigma_3\sigma_2\sigma_2I + \sigma_1I\sigma_2\sigma_2 + \sigma_1\sigma_3II) + \frac{1}{9} (I - \sigma_3\sigma_1\sigma_3I - \sigma_1\sigma_3II + \sigma_1\sigma_3\sigma_3\sigma_1) \right].
$$

Each round bracket in the above expression is fully separable.

6 Tripartite separability of four qubit cluster state in white noise

For the tri-separability of noisy cluster state (19), we find the witness to be $\widehat{W} = B(\hat{M})I - \hat{M}$, where

$$
\hat{M} = -\sigma_1\sigma_3II - I\sigma_3\sigma_1 + 3\sigma_1\sigma_3\sigma_3\sigma_1
+ \sigma_3\sigma_1\sigma_3I + \sigma_1I\sigma_2\sigma_2 + 3\sigma_2\sigma_1\sigma_1\sigma_2
+ \sigma_2\sigma_2\sigma_3I + \sigma_1I\sigma_1\sigma_3 + 3\sigma_3\sigma_2\sigma_2\sigma_3
+ \sigma_3\sigma_1I\sigma_1 + I\sigma_3\sigma_1\sigma_3 - 3\sigma_3\sigma_2\sigma_1\sigma_2
+ \sigma_2\sigma_2I\sigma_1 + I\sigma_3\sigma_2\sigma_2 - 3\sigma_3\sigma_2\sigma_1\sigma_2,
$$

and $B(\hat{M}) = 5$. The entanglement is detected if $p > \frac{5}{21}$. Hence, the state $\rho$ is a mixture of three partite product states if $p \leq \frac{5}{21}$ and can not be three partite separable for $p > \frac{5}{21}$. 
6.1 Necessary condition

Consider the partition 12|3|4. The tripartite separable state is a mixture of pure product state \( |\psi_{12}\rangle |\psi_3\rangle |\psi_4\rangle \). We have

\[
B_{12}(\hat{M}) = \max \langle \psi_{12} | \mathcal{M} | \psi_{12} \rangle,
\]

where the matrix \( \mathcal{M} = \langle \psi_3 | (\psi_4 | \hat{M} | \psi_3) | \psi_4 \rangle \) is the partial trace of \( \hat{M} \) over the third and the fourth qubits. The maximization is taken over all product state \( |\psi_{12}\rangle |\psi_3\rangle |\psi_4\rangle \). Applying the Hadamard transform \( H_2 \) on the first qubit, we obtain the matrix \( \mathcal{M}' = (H_2 \otimes I)\mathcal{M}(H_2 \otimes I) \), and the eigenvalues do not change since the Hadamard transform is unitary. Hence, \( B_{12}(\hat{M}) \) is equal to the largest eigenvalue of \( \mathcal{M}' \). Denote the Bloch vectors of \( |\psi_i\rangle \) as \( (x_i, y_i, z_i) \) with \( i = 1, \ldots, 4 \), and it follows that,

\[
\mathcal{M}' = (3z_3x_4 - 1)\sigma_3 \sigma_3 - z_3 x_4 I + (x_3 z_4 + y_3 y_4) \\
\times (\sigma_3 I + I \sigma_3) + (z_3 + x_4)(\sigma_1 \sigma_1 - \sigma_2 \sigma_2) \\
+ 3( y_3 z_4 - x_3 y_4)(\sigma_1 \sigma_2 + \sigma_2 \sigma_1).
\]

The eigenvalues of \( \mathcal{M}' \) are \( \lambda_{1,2} = 2z_3x_4 - 1 \pm 2[(x_3 z_4 + y_3 y_4)^2 + (z_3 + x_4)^2 + 9(y_3 z_4 - x_3 y_4)^2]^{1/2} \), and \( \lambda_{3,4} = 1 - 4z_3x_4 \). The maximum of \( \lambda_1 \) is 5, and it is achieved when

\[
z_3 = x_4 = 0, y_3z_4 - x_3y_4 = \pm 1; \quad (21)
\]

or

\[
z_3 = x_4 = \pm 1. \quad (22)
\]

The maximums of \( \lambda_3 \) and \( \lambda_4 \) are 5, and they are achieved when

\[
z_3 = -x_4 = \pm 1. \quad (23)
\]

Hence, the eigenvalues of \( \mathcal{M}' \) are tight upper bounded by 5. We thus arrives \( B_{12}(\hat{M}) = 5 \) for the partition 12|3|4.

For the partition 1|23|4, we have \( B_{23}(\hat{M}) = \max_{|\psi_{23}\rangle} \langle \psi_{23} | \mathcal{M} | \psi_{23} \rangle \), with matrix \( \mathcal{M} = \langle \psi_1 | (\psi_4 | \hat{M} | \psi_1) | \psi_4 \rangle \). Using Hadamard transform \( H_2 \) and unitary transform \( U = \text{diag}\{1, e^{i\theta_4}, e^{i\theta_1}, e^{i(\theta_1 + \theta_4)}\} \), with \( \theta_1 = \tan^{-1}(y_1/z_1) \), \( \theta_4 = \tan^{-1}(y_4/z_4) \), the matrix \( \mathcal{M} \) is transformed to \( \mathcal{M}' = U(H_2 \otimes I)\mathcal{M}(H_2 \otimes I)U^\dagger \). The eigenequation of matrix \( \mathcal{M}' \) can be factorized. The maximal eigenvalue of \( \mathcal{M}' \) can be proved to be 5, hence \( B(\hat{M}) = 5 \), which is achieved at

\[
x_1 = \pm 1, \quad x_4 = \pm 1; \quad (24)
\]

or

\[
x_1 = 0, \quad x_4 = 0. \quad (25)
\]

A similar calculation shows that \( B_{13}(\hat{M}) = B_{14}(\hat{M}) = B_{34}(\hat{M}) = B_{24}(\hat{M}) = 5 \) for the partitions 13|2|4, 14|2|3, 1|2|34 and 1|3|24, respectively. Thus for all six
partitions, we have $B_{ij}(\hat{M}) = 5$ with $ij = 12, 13, 14, 23, 24, 34$. Hence,

$$B(\hat{M}) = \max_{ij=12,13,12,23,24,34} B_{ij}(\hat{M}) = 5.$$  

### 6.2 Sufficient condition

For $p = P(\mathbf{R}) = \frac{5}{27}$, we will prove explicitly that the noisy cluster state (19) is tripartite separable. Let us consider partition $1|23|4$, the maximal eigenvalue of $\mathcal{M}$ is achieved under the conditions (24) or (25). The condition $x_1 = 1, x_4 = 1$ corresponds to the state $\frac{1}{2}(I + \sigma_1)$ for the first qubit and $\frac{1}{2}(I + \sigma_1)$ for the fourth qubit. The condition $x_1 = 1, x_4 = 1$ also leads to a diagonal $\mathcal{M}$. We have $\mathcal{M} = \text{diag}\{1, -3, -3, 5\}$. Hence, the maximal eigenvalue is achieved by $|\psi_{23}\rangle = |11\rangle$. Thus, the pure state achieves $B(\hat{M}) = 5$ is $\frac{1}{2}(I + \sigma_1) \otimes |11\rangle \langle 11| \otimes (I + \sigma_1) = \frac{1}{16}(I + \sigma_1) \otimes (I - \sigma_3) \otimes (I + \sigma_1)$. The other three cases of (24) lead to three similar separable states. Averaging on all these four states, we arrive at the separable state

$$\rho_0 = \frac{1}{16}(I - \sigma_1\sigma_3 I I - I I \sigma_3\sigma_1 + \sigma_1\sigma_3\sigma_3\sigma_1).$$  

(26)

The condition (25) corresponds to states $\frac{1}{2}(I + \sin \theta_1\sigma_2 + \cos \theta_1\sigma_3)$ for the first ($i = 1$) and the fourth ($i = 4$) qubits. With this condition, the eigenvector corresponds to the maximal eigenvalue is $|\psi_{23}\rangle = \frac{1}{4}(|00\rangle + e^{i\theta_1}|01\rangle + e^{i\theta_1}|10\rangle - e^{i\theta_1+i\theta_4}|11\rangle)$. We may write $|\psi_{23}\rangle \langle \psi_{23}| = \frac{1}{4}(I I + \cos \theta_1\sigma_1\sigma_3 + \sin \theta_1\sin \theta_4\sigma_1\sigma_1 + \sin \theta_1\sigma_1\sigma_3 + \cos \theta_1\cos \theta_4\sigma_2\sigma_2 + \cos \theta_4\sigma_3\sigma_1 - \cos \theta_1\sin \theta_4\sigma_2\sigma_1 + \sin \theta_4\sigma_3\sigma_2 - \sin \theta_1\cos \theta_4\sigma_1\sigma_2$).

Denote the tripartite separable state as $\varrho_1(\theta_1, \theta_4) = \frac{1}{4}(I + \sin \theta_1\sigma_2 + \cos \theta_1\sigma_3) \otimes |\psi_{23}\rangle \langle \psi_{23}| \otimes (I + \sin \theta_4\sigma_2 + \cos \theta_4\sigma_3)$. Let $\overline{\varrho_1} = \frac{1}{16} \sum_{j,k=0}^{3} \theta_1(\frac{2j+1}{4}\pi, \frac{2k+1}{4}\pi)$. Thus, we have the tripartite separable state

$$\overline{\varrho_1} = \xi + \frac{1}{32}(\sigma_3\sigma_1\sigma_3 + I \sigma_3\sigma_2\sigma_2 + \sigma_2\sigma_2\sigma_3 I + \sigma_3\sigma_1\sigma_3 I),$$  

(27)

where $\xi = \frac{1}{16}I + \frac{1}{64}(\sigma_2\sigma_1\sigma_1\sigma_2 - \sigma_2\sigma_1\sigma_2\sigma_3 + \sigma_3\sigma_2\sigma_2\sigma_3 - \sigma_3\sigma_2\sigma_1\sigma_2)$. Similarly, we have the tripartite separable states

$$\overline{\varrho_2} = \xi + \frac{1}{32}(\sigma_1 I \sigma_1\sigma_3 + \sigma_1 I \sigma_2\sigma_2 + \sigma_2\sigma_2 I \sigma_1 + \sigma_3 I \sigma_1 I),$$  

(28)

$$\overline{\varrho_3} = \xi + \frac{1}{32}(\sigma_3\sigma_1 I I + \sigma_1 I \sigma_2\sigma_2 + \sigma_2\sigma_2 I \sigma_3 I + \sigma_1 I \sigma_1 I),$$  

(29)

$$\overline{\varrho_4} = \xi + \frac{1}{32}(I \sigma_3\sigma_1 I I + \sigma_1 I \sigma_2\sigma_2 + \sigma_2\sigma_2 I \sigma_1 + \sigma_3 I \sigma_1 I),$$  

(30)

for partitions $14|2|3, 13|2|4, 1|3|24$, respectively. We have a tripartite separable state

$$\rho_1 = \frac{1}{2}(\overline{\varrho_1} + \overline{\varrho_2}) = \frac{1}{2}(\overline{\varrho_3} + \overline{\varrho_4}).$$  

(31)
For the partition 12|3|4, the maximal eigenvalue of $\mathcal{M}'$ is achieved at the condition of either (21) or (22), or (23). When $z_3 = x_4 = 1$, the third and the fourth qubits are in the states $\frac{1}{2}(I + \sigma_3)$ and $\frac{1}{2}(I + \sigma_1)$, respectively. The $\mathcal{M}'$ is reduced to $2\sigma_3\sigma_3 - II + 2(\sigma_1\sigma_1 - \sigma_2\sigma_2)$ with eigenvector $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ for its largest eigenvalue $\lambda_1 = 5$. The corresponding eigenvector for $\mathcal{M}$ is $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$. Hence, the tripartite separable state that achieves the condition $B(\widehat{M}) = 5$ is $\frac{1}{\sqrt{2}}|\psi_{12}\rangle \otimes (I + \sigma_3) \otimes (I + \sigma_1) = \frac{1}{16}(II + \sigma_1\sigma_3 + \sigma_3\sigma_1 + \sigma_2\sigma_2) \otimes (I + \sigma_3) \otimes (I + \sigma_1)$. Similarly, the tripartite separable state that achieves the condition $B(\widehat{M}) = 5$ is $\frac{1}{16}(II + \sigma_1\sigma_3 - \sigma_3\sigma_1 - \sigma_2\sigma_2) \otimes (I - \sigma_3) \otimes (I - \sigma_1)$ for the case $z_3 = x_4 = -1$. The average of these two states gives rise to the tripartite separable state

$$\varrho_5 = \frac{1}{16}((I + \sigma_1\sigma_3 II + II\sigma_3\sigma_1 + \sigma_1\sigma_3\sigma_3\sigma_1 + \sigma_3\sigma_1\sigma_3 I + \sigma_2\sigma_3\sigma_1 + \sigma_3\sigma_1\sigma_1 + \sigma_2\sigma_2 I\sigma_1).$$

Due to the symmetry, we have the tripartite separable state

$$\varrho_6 = \frac{1}{16}((I + \sigma_1\sigma_3 II + II\sigma_3\sigma_1 + \sigma_1\sigma_3\sigma_3\sigma_1 + I\sigma_3\sigma_1\sigma_3 + I\sigma_3\sigma_2\sigma_2 + \sigma_1 I\sigma_1\sigma_3 + \sigma_1 I\sigma_2\sigma_2)$$

for the partition 1|2|34. We thus have a tripartite state

$$\rho_2 = \frac{1}{2} (\varrho_5 + \varrho_6). \quad (32)$$

For the case of (21), $\mathcal{M}'$ is reduced to $-\sigma_3\sigma_3 \pm 3(\sigma_1\sigma_2 + \sigma_2\sigma_1)$, where the sign $\pm$ are for the cases $y_3z_4 - x_3y_4 = \pm 1$, respectively. The eigenvector for the largest eigenvalue $\lambda_1 = 5$ is $\frac{1}{\sqrt{2}}(|00\rangle \pm i|11\rangle)$. The corresponding eigenvector of $\mathcal{M}$ is $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm i|01\rangle + |10\rangle \mp i|11\rangle)$. Hence, $|\psi_{12}\rangle \langle \psi_{12} \rangle = \frac{1}{4}[II + \sigma_1\sigma_3 \pm (\sigma_2\sigma_1 - \sigma_3\sigma_2)]$. The third and the fourth qubits are $\frac{1}{2}(I + \cos \theta_3 \sigma_1 + \sin \theta_3 \sigma_2)$ and $\frac{1}{2}(I + \sin \theta_3 \sigma_2 + \cos \theta_3 \sigma_3)$ with $\theta_4 = \theta_3 \pm \frac{\pi}{2}$, respectively. The product state of the third and the fourth qubits then is $\frac{1}{2}(I + \cos \theta_3 \sigma_1 + \sin \theta_3 \sigma_2) \otimes (I \mp \sin \theta_3 \sigma_3 \pm \cos \theta_3 \sigma_2)$. The average on $\theta_3 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ leads to the state $\frac{1}{2}(II \pm \frac{1}{2}(\sigma_1\sigma_2 - \sigma_2\sigma_3))$. The tripartite separable states are $\frac{1}{16}[II + \sigma_1\sigma_3 \pm (\sigma_2\sigma_1 - \sigma_3\sigma_2)] \otimes [II \pm \frac{1}{2}(\sigma_1\sigma_2 - \sigma_2\sigma_3)]$. Averaging on the $\pm$ states gives rise to a tripartite separable state

$$\varrho_7 = \frac{1}{16}(II + \sigma_1\sigma_3) \otimes II \frac{1}{32}(\sigma_2\sigma_1 - \sigma_3\sigma_2) \otimes (\sigma_1\sigma_2 - \sigma_2\sigma_3).$$
By the symmetry, for the partition 1|2|34, we have tripartite separable state

\[ \rho_8 = \frac{1}{16} II \otimes (II + \sigma_3 \sigma_1) \]
\[ + \frac{1}{32} (\sigma_2 \sigma_1 - \sigma_3 \sigma_2) \otimes (\sigma_1 \sigma_2 - \sigma_2 \sigma_3). \]

Thus, we have a tripartite separable state

\[ \rho_3 = \frac{1}{2} (\rho_7 + \rho_8). \] (33)

At last, we can compose the tripartite separable state as

\[ \rho = \frac{1}{21} (\rho_0 + 12 \rho_1 + 4 \rho_2 + 4 \rho_3) \]
\[ = \frac{5}{21} |Cl_4\rangle \langle Cl_4| + \frac{1}{21} \mathbb{I}. \] (34)

7 Conclusion

We have utilized the characteristic function of the witness operator to investigate the multipartite separability of multipartite quantum states. The necessary condition of separability can be obtained for any given characteristic function of witness operator as far as the algebraic maximization can be worked out. The sufficient criterion is obtained by matching the characteristic function to the given quantum state whose separability is to be researched. We use the three qubit X state to illustrate the process of finding the necessary and sufficient criterion of full separability with matched witness method. The result can be used to derive the noise tolerance of W state, although the condition derived is only necessary for the full separability of noisy W state, it detects more entangled states than previous one in the literature. We have obtained the necessary and sufficient conditions for tripartite separability and full separability of four qubit cluster state in white noise, the necessary and sufficient conditions for the tripartite separability of four qubit GHZ state in white noise. The noise tolerances of the tripartite separability and full separability of four qubit cluster state are \( \frac{16}{21} \) and \( \frac{8}{9} \), respectively. The noise tolerance of the tripartite separability of four qubit GHZ is \( \frac{4}{5} \). These conditions are necessary and sufficient. We also explicitly construct the separable states for these four qubit noisy states with given parties. The matched witness method is suitable in finding the multipartite separable criterion for quantum states with less characteristic variables.

It is anticipated that the matched witness method should also be applied to multipartite qudit states and even continuous variable states. For multipartite qudit system, the canonical choice for a complete basis of observables is usually given by the so-called generalized Gell-Mann matrices, generators of the special unitary group SU(d). However, it was suggested very recently that a Hermitian generalization of Pauli matrices to higher dimensions which is based on Heisenberg–Weyl operators is more efficient for
entanglement detection and with a smooth transition to infinite dimensions [22]. For continuous variable system, the characteristic function is well defined for any given continuous variable witness $\hat{M}$. The question is how to obtain the maximal value $f(\hat{M})$. For a bipartite system, this turns out to be the eigenvalue problem of integral equation with the kernel being the characteristic function.

**Acknowledgements** Supported by the National Natural Science Foundation of China (Grant No. 11375152) and (partially) supported by National Basic Research Program of China (Grant No. 2014CB921203) are gratefully acknowledged.

**Appendix A: Proof of Lemma 1**

*Proof* It is not difficult to eliminate two angles in $g(\varphi)$, say, $\varphi_1$ and $\varphi_2$ by maximization. We have

$$g(\varphi) \leq g_1(\varphi_3) = \sqrt{A + \sqrt{B}},$$

with $A = \delta^2 + \gamma^2 - 2\delta\gamma \cos 2\varphi_3$, $B = \alpha^2 + \beta^2 + 2\alpha\beta \cos 2\varphi_3$. The solutions of

$$\frac{dg_1(\varphi_3)}{d\varphi_3} = 0$$

are (1) $\sin 2\varphi_3 = 0$, it gives rise to the second line of (13), and (2)

$$\cos 2\varphi_3 = \frac{\alpha^2 \beta^2 (\delta^2 + \gamma^2) - \gamma^2 \delta^2 (\alpha^2 + \beta^2)}{2\alpha\beta \gamma \delta (\gamma \delta + \alpha\beta)},$$

it gives rise to the first line of (13). The condition for the existence of the solution (2) is $|\cos 2\varphi_3| \leq 1$, which leads to $q \geq 0$. The condition $\delta\alpha\beta\gamma > 0$ comes from $q_0^2 = (\delta\alpha + \beta\gamma)(\delta\beta + \alpha\gamma)(\delta\gamma + \alpha\beta) - \delta\alpha\beta\gamma M_{111}^2 \geq 0$. Thus, when $\delta\alpha\beta\gamma > 0$, the first line of (13) is larger than the second line. □

**Appendix B: Proof of Theorem 1**

*Proof* “Only if”: At present case, we have $B(\hat{M}) = g_m$ by assuming all the $M_{ijk}$ be 0 except $M_{111}, M_{122}, M_{212}, M_{221}$.

In (13), if $g_m = \max\{|M_{111}|, |M_{122}|, |M_{212}|, |M_{221}|\}$, let $|M_{111}|$ be the biggest w.l.o.g. The necessary condition (5) is

$$L \equiv M_{111}R_0 + M_{122}R_1 + M_{212}R_2 + M_{221}R_3 \leq |M_{111}|.$$  \hspace{1cm} (37)

We may choose the signs of $M_{111}, M_{122}, M_{212}, M_{221}$ to match the signs of $R_i$ ($i = 0, \ldots, 3$) in order to make $L$ larger. If $Q < 0$, then one or three $R_i$ are negative. Let $R_0$ be negative, and the other $R_i$ be positive for definite. We may further require $|M_{122}| = |M_{212}| = |M_{221}| = |M_{111}|$ in order to make the left side of inequality (37) even larger. This is possible if we choose $\alpha = \beta = \gamma = 0$. We then obtain $\delta (-R_0 + R_1 + R_2 + R_3) \leq |\delta|$. Thus, $8|\rho_{07}| \leq 1$, and there are similar inequalities for the other anti-diagonal entries. We then have $8\max\{|\rho_{07}|, |\rho_{16}|, |\rho_{25}|, |\rho_{34}|\} \leq 1$. When $\delta\alpha\beta\gamma < 0$, let $|\rho_{07}|$ be the largest one among $\{|\rho_{07}|, |\rho_{16}|, |\rho_{25}|, |\rho_{34}|\}$. If $\rho_{07} > 0$, we may choose $\delta < 0$ and $\alpha, \beta, \gamma > 0$, we have $\frac{L}{g_m} - 8\rho_{07} \leq 0$, and
the equality is achieved when $|\delta| \to \infty$. If $\rho_{07} < 0$, we may choose $\delta > 0$ and $\alpha, \beta, \gamma < 0$, we have $\frac{L}{g_m} + 8\rho_{07} \leq 0$, the equality is achieved when $\delta \to \infty$. Thus, we have proved that $R = \max_{\mathbf{Q}} \frac{L}{g_m} = 8 \max(|\rho_{07}|, |\rho_{16}|, |\rho_{25}|, |\rho_{34}|)$ for the cases of $Q < 0$ or $\delta \alpha \beta \gamma < 0$.

If $g_m = \sqrt{(\delta \alpha + \beta \gamma)(\delta \beta + \alpha \gamma)(\delta \gamma + \alpha \beta)/(\delta \alpha \beta \gamma)}$, we may rewrite equation (6) as $P(\mathbf{R}) = \min_{\delta, \alpha, \beta, \gamma} \frac{g_m}{L}$, where $L = 8(-\delta \rho_{07} + \gamma \rho_{16} + \beta \rho_{25} + \alpha \rho_{34})$ is assumed to be positive. Then $\frac{dP(\mathbf{R})}{d\delta} = 0$ leads to $\frac{\partial \ln g_m}{\partial \delta} = \frac{\partial \ln L}{\partial \delta}$, which is $-8\rho_{07} = \frac{L}{2}(-\delta \alpha + \beta \gamma + \beta \delta + \alpha \gamma + \alpha \delta + \delta \beta - 1)$. Similarly, $\rho_{16}, \rho_{25}, \rho_{34}$ can also be obtained. Then after some algebra, we have $R_0 = GLq_1q_2q_3$, $R_1 = GLq_0q_2q_3$, $R_2 = GLq_0q_1q_3$, $R_3 = GLq_0q_1q_2$, with $G = \frac{1}{8 g_m (3 \delta \alpha \beta \gamma)^2}$. The first line of the right-hand side of (14) can be written as

\[
R = GL\sqrt{(q_0 q_1 + q_2 q_3)(q_0 q_2 + q_1 q_3)(q_0 q_3 + q_1 q_2)},
\]

Note that $q_0 q_1 + q_2 q_3 = 4 \delta \alpha \beta \gamma (\delta \alpha + \beta \gamma)$, we arrive at $R = \frac{L}{g_m}$. Thus,

\[
P(\mathbf{R}) = \frac{1}{R}.
\]

The state is fully separable when $R \leq 1$. The condition $Q > 0$ guarantees the validation of the first line of (14). Since $Q = G^4 L^4 q^3$, so $Q \geq 0$ is equivalent to $q \geq 0$. Also, we have $r = (4 q^2 G^3 L^3)^4 (\delta \alpha \beta \gamma)^3$; hence, $r > 0$ is equivalent to $\delta \alpha \beta \gamma > 0$.

"If": In the case of $Q > 0$ (and $r > 0$), that the state is fully separable when $R \leq 1$ is shown in [11–13] [where $R$ is the first line of (14)]. In the case of $Q < 0$, the Peres–Horodecki criterion is the condition of full separability [12], namely, $R \leq 1$ [where $R$ is the second line of (14)]. What left is the case of $Q > 0$ (and $r < 0$). We can prove that the intersection of $R = 1$ [where $R$ is the first line of (14)] and $\rho_{07} = \pm \frac{1}{\sqrt{8}}$ is $r = 0$ with some algebra. Thus, the states at $\rho_{07} = \pm \frac{1}{\sqrt{8}}$ and $r = 0$ are fully separable. The convex property of fully separable state set then leads to the conclusion that the states with $Q > 0$, $r < 0$ are fully separable when $R \leq 1$ [where $R$ is the second line of (14)].

In the coordinate of $(\rho_{07}, \rho_{16}, \rho_{25}, \rho_{34})$, the shape of the fully separable state set is as follow: consider a four-dimensional hypercube centered at origin and with side length $1/4$, and the 16 vertices are located at $|\rho_{07}| = |\rho_{16}| = |\rho_{25}| = |\rho_{34}| = \frac{1}{8}$. The vertex $(\rho_{07}, \rho_{16}, \rho_{25}, \rho_{34}) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$ corresponds to the state $\frac{1}{8}(III + \sigma_1\sigma_1\sigma_1)$, which is fully separable. The vertices with two or four anti-diagonal elements being $-\frac{1}{8}$ correspond to states which are local equivalent to the fully separable state $\frac{1}{8}(III + \sigma_1\sigma_1\sigma_1)$. The vertex $(\rho_{07}, \rho_{16}, \rho_{25}, \rho_{34}) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}\right)$ corresponds to an entangled state. The corner that contains this vertex is cut by the hypersurface $R = 1$ where $R$ is the first line of (14). Similarly, the other 7 corners with vertices which have odd number of $-\frac{1}{8}$ coordinate components are also cut. Thus, the fully separable state set is a four-dimensional hypercube with 8 of its corners cut. The 8 vertex states with even number of $-\frac{1}{8}$ coordinate components, and the states in the cut surfaces are fully
separable. This guarantees that all the states corresponding to the inner and surface points in the cut hypercube is fully separable.

\[ \square \]

Appendix C: Proof of Theorem 2

Proof The “only if” : Denote \( F_1(\theta) = \max_\varphi F(\theta, \varphi) = f(\theta) + g_m s_1 s_2 s_3 \), and let the maximum of \( F_1(\theta) \) with respect to \( \theta_3 \) as \( F_2(\theta_1, \theta_2) \), then

\[
F_2(\theta_1, \theta_2) = a + bc_2 + \sqrt{(c + dc_2)^2 + e^2 s_2^2}. \tag{39}
\]

Where \( a = M_{300} c_1, b = M_{030} + M_{330} c_1, c = M_{003} + M_{303} c_1, d = M_{033} + M_{333} c_1, e = g_m s_1 \).

We consider two cases. Case (1), let us assume \( e^2 = d^2 - c^2 \), which requires \( M_{033} = M_{330}, M_{030} = M_{333} \) (or \( M_{033} = -M_{330}, M_{030} = -M_{333} \)), \( M_{033}^2 = M_{030} + g_m^2 \). Then (39) is

\[
F_2(\theta_1, \theta_2) = a + bc_2 + |d + cc_2|. \]

Assuming \( d + cc_2 < 0 \), then \( F_2(\theta_1, \theta_2) = a + bc_2 - d - cc_2 \). Let \( b = c \), that is \( M_{030} = M_{003}, M_{330} = M_{303} \), then

\[
F_2(\theta_1, \theta_2) = a - d. \]

If we further assume \( M_{300} = M_{333} \), then \( F_2(\theta_1, \theta_2) = -M_{033} \) is a constant. Hence, if we choose the set of parameters as \( M_{030} = M_{333} = M_{300} = M_{003} = \sin \eta, g_m = \cos \eta, M_{033} = M_{330} = M_{303} = -1, \) then the assumptions \( e^2 = d^2 - c^2, b = c, d + cc_2 < 0 \) can be fulfilled. The necessary condition of full separability (5) reads

\[
4 \sin \eta (\rho_{00} - \rho_{77}) + R \cos \eta \leq 4(\rho_{00} + \rho_{77}) \tag{40}
\]

for any \( \eta \). We have used \( R_{003} + R_{030} + R_{300} + R_{333} = 4(\rho_{00} - \rho_{77}) \) and \( 1 + R_{033} + R_{303} + R_{330} = 4(\rho_{00} + \rho_{77}) \). The necessary condition can be refined to

\[
\sqrt{16(\rho_{00} - \rho_{77})^2 + R^2} \leq 4(\rho_{00} + \rho_{77}),
\]

which is

\[
\frac{1}{8} R \leq \sqrt{\rho_{00} \rho_{77}}.
\]

Similarly, we obtain \( \frac{1}{8} R \leq \sqrt{\rho_{333} \rho_{44}} \) by choosing \( M_{033} = -M_{330}, M_{030} = -M_{333} \) instead. If we properly choose the parameters such that \( e^2 = d^2 - c^2, b = -c, d + cc_2 > 0 \), then we arrive at the necessary conditions of \( \frac{1}{8} R \leq \sqrt{\rho_{22} \rho_{55}} \) and \( \frac{1}{8} R \leq \sqrt{\rho_{22} \rho_{55}} \).
\[ \sqrt{\rho_{11}\rho_{66}} \]. Hence, we have the necessary condition
\[
\frac{1}{8} R \leq \min_{i=0, \ldots, 3} \sqrt{\rho_{ii}\rho_{7-i,7-i}}. \tag{41}
\]

Case (2), let us assume \( M_{033} = M_{300} = M_1, M_{030} = M_{200} = M_2, M_{003} = M_{303} = M_3, M_{333} = M_0 \) and \( g_m = \sqrt{m_0m_1m_2m_3} \), with the vector \((m_0, m_1, m_2, m_3) = -\frac{1}{2}H_4(M_0, M_2, M_1, M_3)\). Here \( H_4 \) is the \( 4 \times 4 \) Hadamard matrix. Let \( M_0 \) be negative and \(-M_0\) be sufficiently large. Then, we can prove that
\[
B(\tilde{M}) = -M_{333} = -M_0, \tag{42}
\]
see Appendix D for details. Hence, the necessary condition of full separability is
\[
\frac{1}{8} g_m R \leq m_0\rho_{00} + m_1\rho_{33} + m_2\rho_{55} + m_3\rho_{66}, \tag{43}
\]
which is true for all possible choices of \( m_i \) \((i = 0, \ldots, 3)\). Notice that \( m_0\rho_{00} + m_1\rho_{33} + m_2\rho_{55} + m_3\rho_{66} \geq 4\sqrt{m_0m_1m_2m_3}\sqrt{\rho_{00}\rho_{33}\rho_{55}\rho_{66}} \), the identity is achieved when
\[
m_0\rho_{00} = m_1\rho_{33} = m_2\rho_{55} = m_3\rho_{66}. \tag{44}
\]
Thus, the condition (43) can be refined as \( \sqrt{\rho_{00}\rho_{33}\rho_{55}\rho_{66}} \geq \frac{1}{8} R \). Similarly we have \( \sqrt{\rho_{11}\rho_{22}\rho_{44}\rho_{77}} \geq \frac{1}{8} R \). Hence, the necessary condition of full separability is refined as
\[
\min(\sqrt{\rho_{00}\rho_{33}\rho_{55}\rho_{66}}, \sqrt{\rho_{11}\rho_{22}\rho_{44}\rho_{77}}) \geq \frac{1}{8} R. \tag{45}
\]
Here the role of matched witness is clearly shown by (44).

For the "if" part, consider the operator identity
\[
A_1 \otimes A_2 \otimes A_3 + B_1 \otimes B_2 \otimes B_3
= \frac{1}{4} [(A_1 + B_1) \otimes (A_2 + B_2) \otimes (A_3 + B_3) \\
+ (A_1 + B_1) \otimes (A_2 - B_2) \otimes (A_3 - B_3) \\
+ (A_1 - B_1) \otimes (A_2 + B_2) \otimes (A_3 - B_3) \\
+ (A_1 - B_1) \otimes (A_2 - B_2) \otimes (A_3 + B_3)].
\]
Let \( A_i = I + \cos \theta_i\sigma_3, B_i = \sin \theta_i(\cos \varphi_i\sigma_1 + \sin \varphi_i\sigma_2) \), then the state \( \frac{1}{\sqrt{8}} (A_1 \otimes A_2 \otimes A_3 + B_1 \otimes B_2 \otimes B_3) \) is fully separable for any \( \theta_i \) and \( \varphi_i \). Let \( B'_i = \sin \theta_i(\cos \varphi_i\sigma_1 - \sin \varphi_i\sigma_2) \), then the state
\[
\varphi = \frac{1}{16} (2A_1 \otimes A_2 \otimes A_3 + B_1 \otimes B_2 \otimes B_3 \\
+ B'_1 \otimes B'_2 \otimes B'_3)
\]
is a fully separable $X$ state with real anti-diagonal entries. The anti-diagonal entries have been treated in Theorem 1; we have $R = |\sin \theta_1 \sin \theta_2 \sin \theta_3|$ for the fully separable state $\rho$. In the case that all the terms in the bracket of the left-hand side of (15) are equal, it is always possible to choose proper $\theta_i$ such that the state $\rho$ in (15) is equal to the fully separable state $\rho$. If some of the terms in bracket at the left-hand side of (15) are not equal with each other, then we have $\rho = (1 - \kappa)\rho_d + \kappa \varrho$, with $0 \leq \kappa < 1$, and $\rho_d$ is a diagonal state in computational basis thus fully separable.

Appendix D: Proof of (42) in Appendix C

We start from Eq. (39). The aim is to maximize $F_2(\theta_1, \theta_2)$ when $a = M_1 c_1$, $b = M_2 + M_3 c_1$, $c = M_3 + M_2 c_1$, $d = M_1 + M_0 c_1$, $e = g_m s_1$ with $c_1 = \cos \theta_1$, $s_1 = \sin \theta_1$ and $g_m$ is related with $M_i$ as in the main text. The maximization of $F_2$ with respect to $\theta_2$ leads to two solutions. The first solution is $\sin \theta_2 = 0$; thus for $F_3(\theta_1) = \max_{\theta_2} F_2(\theta_1, \theta_2)$, we have

$$F_3(\theta_1) = a + b + |c + d|,$$

which is a linear function of $c_1$. The maximal value of $F_3(\theta_1)$ is $F_4 = -M_0$ when $-M_0$ is positive and large enough. The second solution is

$$\cos \theta_2 = \frac{1}{e^2 - d^2} \left( cd + b \sqrt{\frac{e^2(c^2 + e^2 - d^2)}{(b^2 + e^2 - d^2)}} \right).$$

Notice that the second solution does not exist if the absolute of right-hand side of (47) exceeds 1. The maximum of $F_2(\theta_1, \theta_2)$ with respect to $\theta_2$ is

$$F_3(\theta_1) = a + \frac{1}{e^2 - d^2} (bcd + \text{sign}(b^2 + e^2 - d^2) \times \sqrt{e^2(c^2 + e^2 - d^2)(b^2 + e^2 - d^2)})$$

The equation can be rewritten as

$$e^4 + (c^2 + b^2 - d^2 - h^2)e^2 + (hd - bc)^2 = 0.$$  

where $h = a - F_3(\theta_1)$. Suppose $F_3(\theta_1)|_{\theta_1 = \theta_0} = -M_0$ for some $\theta_0$, then the solution of (49) is $\cos \theta_1|_{\theta_1 = \theta_0} = -\frac{A + \sqrt{A^2 - B^2}}{B}$, where $A = M_0^2 + M_1^2 - M_2^2 - M_3^2$, $B = 2(M_0 M_1 - M_2 M_3)$. Thus, $-M_0$ is an achievable value of function $F_3(\theta_1)$. The derivative of Eq. (49) with respect to $x = \cos \theta_1$ gives rise to $\frac{dh}{dx} \mid_{x = x_0} = M_1$ and $\frac{d^2h}{dx^2} \mid_{x = x_0} = -\frac{A^2 - B^2}{e^4 h} \mid_{\theta = \theta_0}$, where $x_0 = \cos \theta_0$. Hence, we arrive at
\[
\left. \frac{dF_3}{dx} \right|_{x=x_0} = 0, \quad (50)
\]

\[
\left. \frac{d^2F_3}{dx^2} \right|_{x=x_0} = \frac{A^2 - B^2}{e^2h} \bigg|_{\theta=\theta_0} < 0. \quad (51)
\]

The inequality comes from the fact that \( h(\theta_0) = M_0 + M_1 \cos \theta_0 < 0 \) if we choose \( |M_0| > |M_1| \) and \( M_0 \) is negative. In order to make each of \( m_i \) \((i = 0, \ldots, 3)\) positive, we have to choose \( M_0 \) with such a property. Combining all the solutions together, we conclude that the maximum of \( F_2(\theta_1, \theta_2) \) at case (2) is \(-M_0\) for sufficiently large and positive \(-M_0\).

**Appendix E: Proof of the condition for W state**

The definition of the vector \((\delta, \alpha, \beta, \gamma)\) now is \((\delta, \alpha, \beta, \gamma) = (-1, L_1, L_2, L_3)\Gamma\), which is

\[
\delta = \frac{1}{4}(1 + L_1 + L_2 + L_3),
\]

\[
-\alpha = \frac{1}{4}(1 + L_1 - L_2 - L_3),
\]

\[
-\beta = \frac{1}{4}(1 - L_1 + L_2 - L_3),
\]

\[
-\gamma = \frac{1}{4}(1 - L_1 - L_2 + L_3).
\]

Notice that if we apply a transformation to \(\phi_1\) such that \(\cos \phi_1 \rightarrow -\cos \phi_1\), then \((L_1, L_2, L_3) \rightarrow (L_1, -L_2, -L_3)\), then \(\delta \leftrightarrow -\alpha, \beta \leftrightarrow \gamma\). The product \(\delta \alpha \beta \gamma\) is invariant. Similarly, the transformation \(\cos \phi_i \rightarrow -\cos \phi_i\) \((i = 2, 3)\) also keeps \(\delta \alpha \beta \gamma\) invariant. Thus, we may assume \(\cos \phi_i \geq 0\) for \(i = 1, 2, 3\). Then

\[
-4\alpha = 1 + \cos \phi_2 \cos \phi_3 - \cos \phi_1 (\cos \phi_2 + \cos \phi_3)
\]

\[
\geq 1 + \cos \phi_2 \cos \phi_3 - \cos \phi_2 - \cos \phi_3
\]

\[
\geq 0.
\]

So \(\alpha \leq 0\). Similarly, we have \(\beta \leq 0, \gamma \leq 0\). While \(\delta > 0\). Thus, \(\delta \alpha \beta \gamma \leq 0\).

**References**

1. Werner, R.F.: Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277–4281 (1989)
2. Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77, 1413–1415 (1996)
3. Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A 223, 1–8 (1996)
4. Rudolph, O.: Computable cross-norm criterion for separability. Lett. Math. Phys. 70, 57–64 (2004)
5. Chen, K., Wu, L.A.: A matrix realignment method for recognizing entanglement. Quant. Inf. Comput. 3, 193–202 (2003)
6. Gühne, O.: Characterizing entanglement via uncertainty relations. Phys. Rev. Lett. 92, 117903 (2004)
7. Doherty, A.C., Parrilo, P.A., Spedalieri, F.M.: Distinguishing separable and entangled states. Phys. Rev. Lett. 88, 187904 (2002)
8. Li, M., Wang, J., Fei, S.-M., Li-Jost, X.: Quantum separability criteria for arbitrary-dimensional multipartite states. Phys. Rev. A 89, 022325 (2014)
9. Gühne, O., Seevinck, M.: Separability criteria for genuine multiparticle entanglement. New J. Phys. 12, 053002 (2010)
10. Jungnitsch, B., Moroder, T., Gühne, O.: Taming multiparticle entanglement. Phys. Rev. Lett. 106, 190502 (2011)
11. Gühne, O.: Entanglement criteria and full separability of multi-qubit quantum states. Phys. Lett. A 375, 406–410 (2011)
12. Kay, A.: Optimal detection of entanglement in Greenberger–Horne–Zeilinger states. Phys. Rev. A 83, 020303(R) (2011)
13. Chen, X.Y., Jiang, L.Z., Yu, P., Tian, M.: Necessary and sufficient fully separable criterion and entanglement of three-qubit Greenberger-Horne-Zeilinger diagonal states. Quantum Inf. Process 14, 2463–2476 (2015)
14. Huber, M., Mintert, F., Gabriel, A., Hiesmayr, B.C.: Detection of high-dimensional genuine multipartite entanglement of mixed states. Phys. Rev. Lett. 104, 210501 (2010)
15. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865–942 (2009)
16. Gühne, O., Tóth, G.: Entanglement detection. Phys. Rep. 474, 1–75 (2009)
17. Greenberger, D.M., Horne, M.A., Shimony, A., Zeilinger, A.: Bell's theorem without inequalities. Am. J. Phys. 58, 1131–C1143 (1990)
18. Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A 62, 062314 (2000)
19. Terhal, B.M.: Bell inequalities and the separability criterion. Phys. Lett. A 271, 319–326 (2000)
20. Bourennane, M., Eibl, M., Kurtsiefer, C., Gaertner, S., Weinfurter, H., Gühne, O., Hyllus, P., Bruß, D., Lewenstein, M., Sanpera, A.: Experimental detection of multipartite entanglement using witness operators. Phys. Rev. Lett. 92, 087902 (2004)
21. Sperling, J., Vogel, W.: Multipartite entanglement witnesses. Phys. Rev. Lett. 111, 110503 (2013)
22. Asadian, A., Erker, P., Huber, M., Klöckl, C.: Heisenberg-Weyl observables: bloch vectors in phase space. Phys. Rev. A 94, 010304(R) (2016)