CENTRALIZER CONSTRUCTION FOR TWISTED YANGIANS

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Abstract
For each of the classical Lie algebras $\mathfrak{g}(n) = \mathfrak{o}(2n + 1), \mathfrak{sp}(2n), \mathfrak{o}(2n)$ of type $B$, $C$, $D$ we consider the centralizer of the subalgebra $\mathfrak{g}(n - m)$ in the universal enveloping algebra $U(\mathfrak{g}(n))$. We show that the $n$th centralizer algebra can be naturally projected onto the $(n - 1)$th one, so that one can form the projective limit of the centralizer algebras as $n \to \infty$ with $m$ fixed. The main result of the paper is a precise description of this limit (or stable) centralizer algebra, denoted by $A_m$. We explicitly construct an algebra isomorphism $A_m = Z \otimes Y_m$, where $Z$ is a commutative algebra and $Y_m$ is the so-called twisted Yangian associated to the rank $m$ classical Lie algebra of type $B$, $C$, or $D$. The algebra $Z$ may be viewed as the algebra of virtual Laplace operators; it is isomorphic to the algebra of polynomials with countably many indeterminates. The twisted Yangian $Y_m$ (and hence the algebra $A_m$) can be described in terms of a system of generators with quadratic and linear defining relations which are conveniently presented in $R$-matrix form involving the so-called reflection equation. This extends the earlier work on the type $A$ case by the second author.

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0. Introduction

Let $\mathfrak{g}$ be a complex reductive Lie algebra, let $\mathfrak{g}' \subset \mathfrak{g}$ be a reductive subalgebra, and let $Z(\mathfrak{g}, \mathfrak{g}')$ denote the centralizer of $\mathfrak{g}'$ in the universal enveloping algebra $U(\mathfrak{g})$. The centralizer algebra $Z(\mathfrak{g}, \mathfrak{g}')$ appears, for example, in the following situation. Assume $V$ is an irreducible $\mathfrak{g}$-module which decomposes (under the action of $\mathfrak{g}'$) into a direct sum of irreducible finite-dimensional $\mathfrak{g}'$-modules $W_\lambda$ with certain multiplicities $\text{mult}_\lambda$ (here $\{\lambda\}$ is the set of dominant highest weights for $\mathfrak{g}'$ that occur in $V$); then this decomposition can be written as

$$V \simeq \sum_{\lambda} U_\lambda \otimes W_\lambda, \quad \dim U_\lambda = \text{mult}_\lambda,$$

where, for each $\lambda$,

$$U_\lambda = \text{Hom}_{\mathfrak{g}'}(W_\lambda, V)$$

is an irreducible $Z(\mathfrak{g}, \mathfrak{g}')$-module; see, e.g., Dixmier [D], Section 9.1.

For some special couples $(\mathfrak{g}, \mathfrak{g}')$ the centralizer algebra turns out to be commutative and so $\text{mult}_\lambda \equiv 1$. This holds, for example, for the couples $(\mathfrak{gl}(N), \mathfrak{gl}(N - 1))$ and $(\mathfrak{o}(N), \mathfrak{o}(N - 1))$, but even for the allied couple $(\mathfrak{sp}(2n), \mathfrak{sp}(2n - 2))$ the centralizer algebra is noncommutative and its structure seems to be complicated. It is believable that an understanding of the algebra $Z(\mathfrak{sp}(2n), \mathfrak{sp}(2n - 2))$ and its representations could lead to a solution of an old problem – the construction of an orthonormal Gelfand–Tsetlin-type basis for representations of $\mathfrak{sp}(2n)$.

In the present paper we investigate certain series of couples $(\mathfrak{g}, \mathfrak{g}')$ which are indexed by two parameters $m < n$:

| type A  | type B  | type C  | type D  |
|--------|--------|--------|--------|
| $\mathfrak{gl}(n)$ | $\mathfrak{o}(2n + 1)$ | $\mathfrak{sp}(2n)$ | $\mathfrak{o}(2n)$ |
| $\mathfrak{gl}(n - m)$ | $\mathfrak{o}(2(n - m) + 1)$ | $\mathfrak{sp}(2(n - m))$ | $\mathfrak{o}(2(n - m))$ |

and we study the ‘stable structure’ of the centralizer $Z(\mathfrak{g}, \mathfrak{g}')$ as $n \to \infty$ with $m$ fixed. In more detail, write $\mathfrak{g} = \mathfrak{g}(n)$, $\mathfrak{g}' = \mathfrak{g}_m(n)$ and abbreviate

$$A_m(n) = Z(\mathfrak{g}(n), \mathfrak{g}_m(n)).$$

It turns out that for any fixed $m$ there exist natural projections

$$\pi_n : A_m(n) \to A_m(n - 1)$$

which are algebra morphisms and which preserve filtration (induced by the standard filtration of enveloping algebras), so that one can form the projective limit

$$A_m = \lim_{\leftarrow} (A_m(n), \pi_n), \quad n \to \infty$$
in the category of filtered algebras. We call this the centralizer construction.

The main result of this paper is that each of the limit algebras $A_m$ (in contrast to the centralizers $A_m(n)$) admits a very precise description. Namely, we find for $A_m$ a system of generators with quadratic and linear defining relations which are conveniently written using the $R$-matrix formalism. Moreover, this kind of presentation of the algebra $A_m$ makes it possible to study its representations (and hence representations of the centralizer algebras).

For couples $(\mathfrak{g}, \mathfrak{g}')$ of type $A$ this was done earlier in Olshanski [O1], [O2]. In that case

$$A_m = A_0 \otimes Y(\mathfrak{gl}(m)),$$

where $A_0$ is a commutative algebra (isomorphic to the ring of polynomials with countably many indeterminates) and $Y(\mathfrak{gl}(m))$ is the Yangian for the Lie algebra $\mathfrak{gl}(m)$. The algebra $Y(\mathfrak{gl}(m))$ first appeared in the works of L. D. Faddeev’s school on the Yang–Baxter equation; see, e.g., [TF]. One starts with the ‘ternary relation’

$$R(u - v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R(u - v), \quad (0.2)$$

where $u, v$ are formal parameters, $R(u)$ is Yang’s $R$-matrix,

$$R(u) = 1 \otimes 1 - u^{-1} \sum_{i,j=1}^{m} E_{ij} \otimes E_{ji} \in \text{End}(\mathbb{C}^m \otimes \mathbb{C}^m)(u),$$

the $E_{ij}$ are matrix units, and $T(u)$ is a matrix–valued formal series in $u^{-1}$:

$$T(u) = (t_{ij}(u)), \quad t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} t^{(k)}_{ij} u^{-k}, \quad 1 \leq i, j \leq m.$$

Then the relation (0.2) implies a system of quadratic relations on the symbols $t^{(k)}_{ij}$ which is just the system of defining relations for $Y(\mathfrak{gl}(m))$. Note that $Y(\mathfrak{gl}(m))$ is a Hopf algebra and that it can be viewed as a deformation of the enveloping algebra $U(\mathfrak{gl}(m)[x])$, where

$$\mathfrak{gl}(m)[x] = \mathfrak{gl}(m) + \mathfrak{gl}(m)x + \mathfrak{gl}(m)x^2 + \ldots$$

is the Lie algebra of $\mathfrak{gl}(m)$-valued polynomials (a polynomial current Lie algebra).

V. G. Drinfeld defined Yangians corresponding to arbitrary simple Lie algebras [D1], [D2]. The Yangians form an important family of quantum groups and have a rich representation theory; see [CP1], [CP2], [C2], [D2].

One could expect the algebras $A_m$ associated with type $B, C, D$ couples $(\mathfrak{g}, \mathfrak{g}')$ in (0.1) to be connected with orthogonal or symplectic Drinfeld’s Yangians. However, this is not at all true, and we have the following result (announced in [O3]).
Main Theorem. For the couples \((g, g')\) of type B, C, D one has

\[ A_m = Z \otimes Y^\pm(M), \]

where \(Z\) is a commutative algebra, \(M = 2m\) or \(M = 2m + 1\) (for types C, D or B, respectively), and \(Y^\pm(M)\) is a certain ‘Yangian-type’ quadratic algebra which can be realized as a one-sided Hopf ideal in the Yangian \(Y(\mathfrak{gl}(M))\).

It should be emphasized that the algebras \(Y^\pm(M)\) are not Hopf algebras. We call these new objects twisted Yangians (the sign ‘+’ is taken in the orthogonal case, the sign ‘−’ is taken in the symplectic case).

Like the Yangian \(Y(\mathfrak{gl}(m))\), the algebra \(Y^\pm(M)\) is a deformation of an enveloping algebra; the corresponding Lie algebra is an involutive subalgebra of \(\mathfrak{gl}(M)[x]\):

\[ \{ f \in \mathfrak{gl}(M)[x] \mid f(-x) = -(f(x))^t \}, \]

where \(t = t_{\pm}\) stands for the matrix transposition which corresponds to a symmetric or alternating form in \(\mathbb{C}^M\). This is a ‘twisted’ polynomial current Lie algebra, which explains the term ‘twisted Yangian’.

There are two kinds of defining relations for generators of \(Y^\pm(M)\): quadratic and linear, and both are conveniently written in an \(R\)-matrix form. The presentation of the quadratic relations is similar to (0.2) but more complicated:

\[ R(u - v)(S(u) \otimes 1)R^t(-u - v)(1 \otimes S(v)) = (1 \otimes S(v))R^t(-u - v)(S(u) \otimes 1)R(u - v), \]

where

\[ R^t(u) = (\text{id} \otimes t)(R(u)) = (t \otimes \text{id})(R(u)). \]

As with the algebra \(Y(\mathfrak{gl}(m))\), such a modification of the relation (0.2) also first appeared in mathematical physics (see Cherednik [C1] and Sklyanin [S]); nowadays it is called the reflection equation, see, e.g., [KK], [KS], [KJC].

The system of linear relations for the generators of \(Y^\pm(M)\) is written as the following symmetry relation on the matrix \(S\):

\[ S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}. \]

The explanation of this relation is rather simple: it reflects the fact that the matrices \(X \in \mathfrak{g}(m)\) are antisymmetric with respect to the transposition \(t\).

Let us describe one more aspect of the centralizer construction. Existence of natural embeddings \(\mathfrak{g}(n) \to \mathfrak{g}(n + 1)\) allows us to form the inductive limit Lie algebra

\[ \mathfrak{g}(\infty) = \lim_{\to \infty} \mathfrak{g}(n), \quad n \to \infty. \]
which is one of the algebras

\begin{align*}
\text{type A} & : \mathfrak{gl}(\infty) \\
\text{type B} & : \mathfrak{o}(2\infty + 1) \\
\text{type C} & : \mathfrak{sp}(2\infty) \\
\text{type D} & : \mathfrak{o}(2\infty).
\end{align*}

(Note that an appropriate modification of the centralizer construction can be also applied to another version of the algebra \(\mathfrak{gl}(\infty)\), denoted by \(\mathfrak{gl}(2\infty + 1)\), see Remark 2.23.)

By its very definition, \(A_m(n)\) is a subalgebra of \(A_{m+1}(n)\), which leads to an algebra embedding \(A_m \to A_{m+1}\), so that we can form the inductive limit algebra

\[ A = \lim_{\to} A_m, \quad m \to \infty \]

and it turns out that there exists a natural embedding \(U(\mathfrak{g}(\infty)) \to A\). Finally, we show that an irreducible highest weight representation \(L_\lambda\) of the Lie algebra \(\mathfrak{g}(\infty)\) can be canonically extended to the algebra \(A\) provided the highest weight \(\lambda\) satisfies a stability condition described below; see (0.5).

In contrast to the enveloping algebra \(U(\mathfrak{g}(\infty))\), the algebra \(A\) has a large center \(Z\), which coincides with the subalgebra \(A_0\) or \(A_{-1}\) for the series \(C, D\) or \(B\), respectively, while the center of \(U(\mathfrak{g}(\infty))\) is trivial. Thus, \(A\) is more like the enveloping algebras \(U(\mathfrak{g}(n))\) than \(U(\mathfrak{g}(\infty))\). This idea is developed in the papers [O1], [O2] which deal with the type \(A\) case. In particular, it is explained there how one can realize elements of \(A\) as left invariant differential operators on a certain infinite-dimensional classical group; then elements of the center \(Z\) become biinvariant (or Laplace) operators.

The stability condition on \(\lambda\) mentioned above takes the following form: assume (to simplify the discussion) that \(\mathfrak{g}(\infty)\) is of type \(C\) or \(D\) and choose as a Cartan subalgebra the set of doubly infinite diagonal matrices of the form

\[ a = \text{diag}(\ldots, 0, 0, \ldots, a_2, a_1, -a_1, -a_2, \ldots, 0, 0, \ldots); \]

then

\[ \langle \lambda, a \rangle = \sum_{i=1}^{\infty} \lambda_i a_i, \]

where

\[ \lambda_i = \lambda_{i+1} = \cdots = c, \quad i \gg 1, \quad (0.5) \]

which means that \(\lambda\) has only a finite number of nonzero labels on the (infinite) Dynkin diagram corresponding to the algebra \(\mathfrak{g}(\infty)\). In particular, among the modules \(L_\lambda\) satisfying the stability condition there are analogs of finite-dimensional modules. It is worth mentioning that the stable value \(c\) of the coordinates of \(\lambda\) is involved as a parameter in the centralizer construction.
This paper is organized as follows. In Sections 1 and 2 we generalize slightly
the construction of the algebra $A \supset U(\mathfrak{gl}(\infty))$ from [O2]. Here we make it depend
upon a parameter. Although the arguments do not differ substantially from those
of [O2] we present them in detail, to facilitate the reader’s understanding of the
more involved proof in the case of type $B,C,D$ algebras.

Section 3 is parallel to Section 1. Here we introduce a category $\Omega(c)$ of $\mathfrak{g}(\infty)$-modules, where $\mathfrak{g}$ is of type $B,C$ or $D$, and $c$ is a parameter. The category $\Omega(c)$ is similar to the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand. We also introduce the algebra $Z$ of ‘virtual Laplace operators’ (the future center of $A$) which operates in
modules belonging to $\Omega(c)$.

The main results are concentrated in Section 4. We define the algebra $A$ (which
depends on $c$), then show that it extends the action of $U(\mathfrak{g}(\infty))$ in modules from the
category $\Omega(c)$, and prove the Main Theorem (Theorem 4.17) about the structure
of $A$.

The results make it possible to study representations of the centralizer algebras
$A_m(n)$ by using the techniques developed for twisted Yangians [M4].

The final Section 5 contains a few comments and open questions.

To conclude, let us mention some papers related to our subject. A detailed expo-
sition of the structure theory of the Yangian $Y(\mathfrak{gl}(m))$ and the twisted Yangians is
given in [MNO]; this is our basic reference. Representations of $Y(\mathfrak{gl}(m))$ are studied
by Cherednik [C2], Kirillov and Reshetikhin [KR], Molev [M1], Nazarov and
Tarasov [NT1], [NT2]. Applications of the twisted Yangians to explicit construc-
tions of central elements in classical enveloping algebras and to Capelli identities
are given in [M3] and [MN]. Families of Bethe-type commutative subalgebras in
$Y^{\pm}(M)$ were constructed in [NO].

A remark is necessary in regard to our formula numbering: we enumerate the
formulas in any subsection independently and use triple numbering when referring
them back in other subsections.

1. Highest weight modules and virtual Laplace operators for the Lie
algebra $\mathfrak{gl}(\infty)$

In this section we introduce a family of categories $\Omega(c), c \in \mathbb{C}$ of modules over
the Lie algebra $\mathfrak{gl}(\infty)$. This category is similar to the well-known category $\mathcal{O}$ of
Bernstein–Gelfand–Gelfand. We also construct the algebras of ‘virtual Laplace
operators’ for this Lie algebra.

1.1. We denote by $\mathfrak{gl}(n)$ the Lie algebra of complex $n \times n$-matrices. For any $n > 1$
one has the natural inclusion $\mathfrak{gl}(n-1) \subset \mathfrak{gl}(n)$, where the subalgebra $\mathfrak{gl}(n-1)$ is
spanned by the matrix units $E_{ij}$ with $1 \leq i, j \leq n-1$. We denote by $\mathfrak{gl}(\infty)$ the
corresponding inductive limit of the Lie algebras $\mathfrak{gl}(n)$ as $n \to \infty$:

$$\mathfrak{gl}(\infty) = \bigcup_{n=1}^{\infty} \mathfrak{gl}(n).$$
In other words, $\mathfrak{gl}(\infty)$ is the Lie algebra of all complex matrices $A = (A_{ij})$, where $i$ and $j$ run over the set $\{1, 2, \ldots\}$, such that the number of nonzero entries $A_{ij}$ is finite. The matrix units $E_{ij}$, $1 \leq i, j < \infty$ form a basis of $\mathfrak{gl}(\infty)$.

We let $\mathfrak{h}$ denote the Cartan subalgebra of diagonal matrices and $\mathfrak{n}_+$ (respectively, $\mathfrak{n}_-$) the subalgebra of upper (respectively, lower) triangular matrices in $\mathfrak{gl}(\infty)$. One has the triangular decomposition

$$\mathfrak{gl}(\infty) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$  

For a linear functional $\lambda \in \mathfrak{h}^*$ set $\lambda_i = \lambda(E_{ii})$. We shall identify $\lambda$ with the sequence $(\lambda_1, \lambda_2, \ldots)$.

1.2. A $\mathfrak{gl}(\infty)$-module $V$ is said to be highest weight if it has a cyclic vector $v$ satisfying $\mathfrak{n}_+ v = \{0\}$ and there exists $\lambda \in \mathfrak{h}^*$ such that $hv = \lambda(h)v$ for any $h \in \mathfrak{h}$. The functional $\lambda$ is the highest weight of $V$, and $v$ is the highest weight vector of $V$; it is unique up to scalar multiples. Sometimes $V$ is also referred to as the module with the highest weight $\lambda$. The universal $\mathfrak{gl}(\infty)$-module $M_\lambda$ with the highest weight $\lambda \in \mathfrak{h}^*$ (the Verma module) may be defined as the quotient of the universal enveloping algebra $U(\mathfrak{gl}(\infty))$ by the left ideal generated by $\mathfrak{n}_+$ and the elements $h - \lambda(h)$, $h \in \mathfrak{h}$. Denote by $L_\lambda$ the unique nontrivial irreducible quotient of $M_\lambda$.

Let $n$ be a positive integer. For a $\mathfrak{gl}(\infty)$-module $V$ and a functional $\mu = (\mu_1, \mu_2, \ldots) \in \mathfrak{h}^*$ set

$$V_n(\mu) = \{v \in V| E_{ii}v = \mu_i v \text{ for } i > n \text{ and } E_{ij}v = 0 \text{ for } 1 \leq i < j; \; j > n\}.$$  

Clearly, the subspaces $V_n(\mu)$ form an ascending chain of subspaces in $V$. Set

$$V_\infty(\mu) = \bigcup_{n=1}^{\infty} V_n(\mu)$$

and note that $V_\infty(\mu)$ is a submodule of $V$ since $V_n(\mu)$ is $\mathfrak{gl}(n)$-invariant for every $n$. We shall write $\mu \sim \mu'$ if $\mu_i = \mu'_i$ for $i \gg 0$. Then $\mu \sim \mu'$ implies $V_\infty(\mu) = V_\infty(\mu')$.

1.3. Definition. Fix a number $c \in \mathbb{C}$ and consider the functional $\lambda^c := (c, c, \ldots) \in \mathfrak{h}^*$. Then $\Omega(c)$ is defined to be the category of $\mathfrak{gl}(\infty)$-modules $V$ satisfying the condition $V_\infty(\lambda^c) = V$.

1.4. Proposition. Let $V$ be a $\mathfrak{gl}(\infty)$-module with the highest weight $\lambda$, where $\lambda \sim \lambda^c$. Then $V$ belongs to $\Omega(c)$. In particular, $\Omega(c)$ contains the modules $M_\lambda$ and $L_\lambda$ with $\lambda \sim \lambda^c$.

Proof. Suppose that $\lambda_i = c$ for $i \geq n + 1$. Then the highest weight vector $v$ lies in the subspace $V_n(\lambda^c)$. Since $v$ is a cyclic vector of $V$, the submodule $V_\infty(\lambda^c)$ coincides with $V$. □
1.5. Remark. In the category $\Omega(c)$, there is a distinguished module, namely, the one-dimensional module $L_{(c,c,\ldots)} = L_{\lambda c}$. For any complex numbers $c$ and $d$, the mapping $V \mapsto V \otimes L_{(d-c,d-c,\ldots)}$ establishes a category isomorphism $\Omega(c) \to \Omega(d)$ such that $L_{(c,c,\ldots)} \mapsto L_{(d,d,\ldots)}$. Thus, all the categories $\Omega(c)$ are canonically isomorphic, and we could assume, with no real loss of generality, that $c = 0$. However, we prefer to keep the parameter $c$ in the formulas below in order to emphasize a similarity with the case of Lie algebras $\mathfrak{so}(2\infty)$, $\mathfrak{so}(2\infty+1)$, and $\mathfrak{sp}(2\infty)$.

1.6. Let $\lambda \in \mathfrak{h}^*$ be arbitrary. For $n = 1, 2, \ldots$ denote by $L^{(n)}_{\lambda}$ the cyclic $\mathfrak{gl}(n)$-submodule of $L_\lambda$ generated by the highest weight vector $v \in L_\lambda$. Set $\mathfrak{n}^{(n)}_+ = \mathfrak{n}_+ \cap \mathfrak{gl}(n)$. Then $v$ is annihilated by $\mathfrak{n}^{(n)}_+$, so that $L^{(n)}_{\lambda}$ is a highest weight module over $\mathfrak{gl}(n)$.

**Proposition.** The $\mathfrak{gl}(n)$-module $L^{(n)}_{\lambda}$ is irreducible for each $n$.

**Proof.** Consider the decomposition $\mathfrak{n}_+ = \mathfrak{n}^{(n)}_+ \oplus \overline{\mathfrak{n}}^{(n)}_+$, where

$$\overline{\mathfrak{n}}^{(n)}_+ := \text{span of } \{E_{ij} \mid i < j; j \geq n + 1\}.$$  

Note that $\overline{\mathfrak{n}}^{(n)}_+$ is normalized by $\mathfrak{n}^{(n)}_- := \mathfrak{n}_- \cap \mathfrak{gl}(n)$, that is, $[\mathfrak{n}^{(n)}_-, \overline{\mathfrak{n}}^{(n)}_+] \subseteq \overline{\mathfrak{n}}^{(n)}_+$. Since $L^{(n)}_{\lambda}$ coincides with $\mathfrak{U}(\mathfrak{n}^{(n)}_-)v$ and $v$ is annihilated by $\overline{\mathfrak{n}}^{(n)}_+ \subset \mathfrak{n}_+$, this implies that $L^{(n)}_{\lambda} \subset L_\lambda$ is annihilated by $\overline{\mathfrak{n}}^{(n)}_+$.

Now assume we are given a vector $w \in L^{(n)}_{\lambda}$ that is annihilated by $\mathfrak{n}^{(n)}_+$. Then, by the above argument, $w$ is annihilated by the whole algebra $\mathfrak{n}_+$. Since $L_\lambda$ is irreducible, $w$ must lie in $Cv$ which implies that $L^{(n)}_{\lambda}$ is irreducible. $\square$

The proposition means that $L_\lambda$ may be regarded as the inductive limit $\lim_{n \to \infty} L^{(n)}_{\lambda}$ as $n \to \infty$ of irreducible highest weight $\mathfrak{gl}(n)$-modules.

1.7. A functional $\lambda \in \mathfrak{h}^*$ will be called a *dominant weight for $\mathfrak{gl}(\infty)$* if $\lambda_i = \lambda_{i+1} \in \mathbb{Z}_+$ for $i = 1, 2, \ldots$. This is equivalent to say that for any $n$ the restriction of $\lambda$ to $\mathfrak{h}^{(n)} := \mathfrak{h} \cap \mathfrak{gl}(n)$ is a dominant weight for $\mathfrak{gl}(n)$ in the usual sense.

It follows from Proposition 1.6 that any module $L_\lambda$ corresponding to a dominant weight $\lambda$ is the inductive limit of irreducible finite-dimensional $\mathfrak{gl}(n)$-modules.

The dominant weights $\lambda \sim \lambda^0$ may be identified with Young diagrams, and the corresponding $\mathfrak{gl}(\infty)$-modules $L_\lambda$ may be viewed as infinite-dimensional analogs of the irreducible polynomial $\mathfrak{gl}(n)$-modules. These modules $L_\lambda$ are closely related to the so-called tame irreducible representations of the group $U(\infty)$, see Olshanski [O2].

Let us associate to $\mathfrak{gl}(\infty)$ the semi-infinite Dynkin diagram of type $A$. Then there is a bijective correspondence between the dominant weights $\lambda \sim \lambda^c$ and the labelings of the nodes of that diagram by nonnegative integers where only a finite number of them is nonzero:
1.8. Here we recall the well-known construction of the Harish–Chandra isomorphism (see, e.g., Dixmier [D], Section 7.4).

Let \( \tilde{\mathfrak{g}} \) be a reductive Lie algebra and \( \tilde{\mathfrak{h}} \) a Cartan subalgebra of \( \tilde{\mathfrak{g}} \). Choose a system of positive roots for \((\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})\) and consider the corresponding triangular decomposition \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+ \) where \( \tilde{\mathfrak{n}}_- \) and \( \tilde{\mathfrak{n}}_+ \) are spanned by the negative and positive root vectors, respectively. Then one has a decomposition of the universal enveloping algebra

\[
U(\tilde{\mathfrak{g}}) = (\tilde{\mathfrak{n}}_- U(\mathfrak{g}) + U(\mathfrak{g}) \tilde{\mathfrak{n}}_+) \oplus U(\mathfrak{h}).
\]

Denote by \( U(\mathfrak{g})^\mathfrak{h} \) the centralizer of \( \mathfrak{h} \) in \( U(\mathfrak{g}) \). Then the projection onto the second component in (1) gives an algebra homomorphism \( \omega : U(\mathfrak{g})^\mathfrak{h} \to U(\mathfrak{h}) \) called the Harish–Chandra homomorphism.

Further, let \( W \) be the Weyl group of \((\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})\) and let \( \rho \in \tilde{\mathfrak{h}}^* \) be the half–sum of positive roots. The following affine transformations of \( \tilde{\mathfrak{h}}^* \)

\[
\lambda \mapsto w'(\lambda) := w(\lambda + \rho) - \rho, \quad \lambda \in \tilde{\mathfrak{h}}^*, \quad w \in W
\]

form a finite group \( W' \) isomorphic to \( W \). The group \( W' \) operates on the algebra \( P(\tilde{\mathfrak{h}}^*) \) of polynomial functions on \( \tilde{\mathfrak{h}}^* \), which can be identified with \( U(\mathfrak{h}) \).

Let \( Z(\mathfrak{g}) \) denote the center of \( U(\mathfrak{g}) \). Since \( Z(\mathfrak{g}) \subset U(\mathfrak{g})^\mathfrak{h} \), we may restrict \( \omega \) to \( Z(\mathfrak{g}) \), and it turns out that \( \omega : Z(\mathfrak{g}) \to U(\mathfrak{h}) = P(\mathfrak{h}^*) \) is an isomorphism of the algebra \( Z(\mathfrak{g}) \) onto the algebra \( P(\mathfrak{h}^*)^{W'} \) of \( W' \)-invariants in \( P(\mathfrak{h}^*) \). This isomorphism

\[
\omega : Z(\mathfrak{g}) \to P(\mathfrak{h}^*)^{W'}
\]

is called the Harish–Chandra isomorphism.

The Harish–Chandra isomorphism (3) has a natural interpretation in terms of eigenvalues of central elements. Namely, let \( a \) be an arbitrary element of \( Z(\mathfrak{g}) \) and let \( f_a = \omega(a) \in P(\mathfrak{h}^*) \) be the corresponding \( W' \)-invariant polynomial. Then \( a \) operates in any highest weight \( \mathfrak{g} \)-module \( V \) as the scalar operator \( f_a(\lambda) \cdot 1 \), where \( \lambda \) stands for the highest weight of \( V \).

1.9. Now we apply the above construction to \( \tilde{\mathfrak{g}} = \mathfrak{gl}(n) \). Take \( \tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \cap \mathfrak{gl}(n) \), \( \tilde{\mathfrak{h}} = \mathfrak{h}_\mathfrak{n} = \mathfrak{h} \cap \mathfrak{gl}(n) \) and identify \( P(\mathfrak{h}^*) = U(\mathfrak{h}) \) with \( \mathbb{C}[\lambda_1, \ldots, \lambda_n] \), where the coordinates \( \lambda_1, \ldots, \lambda_n \) in \( \mathfrak{h}_\mathfrak{n} \) correspond to the basis \( E_{11}, \ldots, E_{nn} \). Then

\[
\rho = \left( \frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{n-1}{2} \right) = \left( \frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2} \right) + (-1, -2, \ldots, -n).
\]
The Weyl group coincides with the symmetric group of degree \( n \) permuting the coordinates \( \lambda_1, \ldots, \lambda_n \). Since the vector \( \left( \frac{n+1}{2}, \ldots, \frac{n+1}{2} \right) \) is invariant under any permutation of coordinates, we may replace in (1.8.2) the vector \( \rho \) by the vector \((-1,-2,\ldots,-n)\). This implies that a polynomial in \( \lambda_1, \ldots, \lambda_n \) is \( W' \)-invariant if and only if it is symmetric with respect to the new variables \( \lambda_1-1, \lambda_2-2, \ldots, \lambda_n-n \). The last property will be referred to as the **shifted symmetry**, and we will denote by \( \Lambda^*_n \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n] \) the subalgebra of all shifted symmetric polynomials. Thus, the Harish-Chandra isomorphism for the Lie algebra \( \mathfrak{gl}(n) \) is an algebra isomorphism \( \omega : Z(\mathfrak{gl}(n)) \to \Lambda^*_n \). Note that this isomorphism preserves the filtration: that of \( Z(\mathfrak{gl}(n)) \) is inherited from \( U(\mathfrak{gl}(n)) \) and that of \( \Lambda^*(n) \) is determined by the usual degree of polynomials.

**1.10.** Fix \( c \in \mathbb{C} \) and consider morphisms

\[
\pi_{n,c} : \Lambda^*_n \to \Lambda^*_{n-1}, \quad n = 1, 2, \ldots
\]  

(1) defined by the formula

\[
(\pi_{n,c}(f))(\lambda_1, \ldots, \lambda_{n-1}) = f(\lambda_1, \ldots, \lambda_{n-1}, c), \quad f \in \Lambda^*_n.
\]

Since \( \pi_{n,c} \) preserves the filtration, we can define the projective limit in the category of filtered algebras:

\[
\Lambda^*_c = \lim_{\leftarrow} \Lambda^*_n, \quad n \to \infty.
\]

In other words, an element \( f \in \Lambda^*_c \) is a sequence \((f_n)\) such that

\[
f_n \in \Lambda^*_n \quad \text{for} \quad n = 1, 2, \ldots, \quad \pi_{n,c}(f_n) = f_{n-1} \quad \text{for} \quad n \geq 2
\]

and

\[
\deg f := \sup_n \deg f_n < \infty.
\]

We call \( \Lambda^*_c \) the **algebra of shifted symmetric functions** (with parameter \( c \)); cf. [OO1].

Note that elements of \( \Lambda^*_c \) are well-defined functions on the set of all sequences \( \lambda = (\lambda_1, \lambda_2, \ldots) \sim \lambda^c \). Let us emphasize that the definition of the algebra \( \Lambda^*_c \), contrary to that of \( \Lambda^*(n) \), depends on the value of the parameter \( c \).

The construction of the algebra \( \Lambda^*_c \) of shifted symmetric functions is quite similar to that of the algebra \( \Lambda \) of symmetric functions (see Macdonald [M]), and both algebras are indeed closely related to each other. Namely, it is easily seen that \( \Lambda \) is naturally isomorphic to the graded algebra \( \text{gr} \Lambda^*_c \) associated with the filtered algebra \( \Lambda^*_c \).
In the algebra $\Lambda^*_c$, one can define analogs of power sums, elementary symmetric functions, and complete symmetric functions:

$$p_m(\lambda) = \sum_{k=1}^{\infty} (\lambda_k - k)^m - (c - k)^m, \quad m = 1, 2, \ldots,$$

$$1 + \sum_{m=1}^{\infty} e_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 + (\lambda_k - k)t}{1 + (c - k)t},$$

$$1 + \sum_{m=1}^{\infty} h_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 - (c - k)t}{1 - (\lambda_k - k)t},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots) \sim \lambda^c$.

The interrelations between these functions are exactly the same as in the case of symmetric functions; see Macdonald [M], Chapter I, (2.6), (2.10), (2.10'). As for the algebra $\Lambda$, each of the families $\{p_m\}, \{e_m\}, \{h_m\}$ can be taken as a system of algebraically independent generators of the algebra $\Lambda^*_c$. Thus we may write

$$\Lambda^*_c = \mathbb{C}[p_1, p_2, \ldots] = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[h_1, h_2, \ldots].$$

It should be noted that there are other ways to define the shifted symmetric functions $p_m$, $e_m$, and $h_m$ which can be more suitable for certain reasons; see [OO1]. However, the above ‘naive’ definition will be sufficient for the purposes of this article.

1.11. Consider the morphisms

$$\pi_{n,c} : Z(\mathfrak{gl}(n)) \rightarrow Z(\mathfrak{gl}(n - 1)), \quad n = 1, 2, \ldots,$$

which correspond to morphisms (1.10.1) under the identification $Z(\mathfrak{gl}(n)) \simeq \Lambda^*(n)$.

**Definition.** The *algebra of virtual Laplace operators*, denoted as $Z_c = Z_c(\mathfrak{gl}(\infty))$, is defined as the projective limit of the sequence of the filtered algebras $Z(\mathfrak{gl}(n))$:

$$\mathbb{C} \leftarrow Z(\mathfrak{gl}(1)) \leftarrow Z(\mathfrak{gl}(2)) \leftarrow Z(\mathfrak{gl}(3)) \leftarrow \ldots.$$

Clearly, the algebra $Z_c$ can be identified with the algebra $\Lambda^*_c$ of shifted symmetric functions.

2. The centralizer construction for $\mathfrak{gl}(\infty)$

Here we introduce an algebra $A$ associated with the Lie algebra $\mathfrak{gl}(\infty)$. Then we prove that $A$ is isomorphic to the tensor product of the algebra of virtual Laplace
operators $Z_c$, defined in Section 1, and the Yangian $Y(\infty)$. The latter algebra is the inductive limit of the Yangians $Y(n) = Y(\mathfrak{gl}(n))$ (see Definition 2.14).

2.1. As in the previous section fix $c \in \mathbb{C}$ and suppose that $m \in \{0, 1, 2, \ldots\}$. We shall assume that $n \geq m$. Let us introduce the following notation:

- $\mathfrak{g}_m(n)$ is the subalgebra in $\mathfrak{gl}(n)$ spanned by the matrix units $E_{ij}$ subject to the condition $m + 1 \leq i, j \leq n$;
- $A(n)$ is the universal enveloping algebra of $\mathfrak{gl}(n)$;
- $A_m(n)$ is the centralizer of $\mathfrak{g}_m(n)$ in $A(n)$; in particular, $A_0(n)$ is the center of $A(n)$;
- $G_m(n)$ is the subgroup in $\text{GL}(n)$ consisting of matrices stabilizing the basis vectors $e_i$, $1 \leq i \leq m$ in $\mathbb{C}^n$ and the space spanned by the remaining vectors;
- $I(n)$ is the left ideal in $A(n)$ generated by the elements $E_{in} - \delta_{in} c$, $i = 1, \ldots, n$;
- $J(n)$ is the right ideal in $A(n)$ generated by the elements $E_{ni} - \delta_{ni} c$, $i = 1, \ldots, n$;
- $A(n)^0$ is the centralizer of $E_{nn}$ in $A(n)$.

2.2. Proposition. Let $L = I(n) \cap A(n)^0$. Then

(i) $L = J(n) \cap A(n)^0$, so that $L$ is a two-sided ideal of the algebra $A(n)^0$;

(ii) the following decomposition holds

$$A(n)^0 = L \oplus A(n - 1).$$

Proof. By the Poincaré–Birkhoff–Witt theorem, any element $a$ of the algebra $A(n)$ can be uniquely written as the sum of elements of the form

$$E_{n1}^{p_1} \cdots E_{n,n-1}^{p_{n-1}} x E_{1n}^{q_1} \cdots E_{n-1,n}^{q_{n-1}} (E_{nn} - c)^r, \quad x \in A(n - 1).$$

Note that

$$[E_{nn}, E_{ni}] = E_{ni}, \quad [E_{nn}, E_{in}] = -E_{in}, \quad 1 \leq i < n,$n

and

$$[E_{nn}, x] = 0, \quad x \in A(n - 1).$$

This implies that any element of the form (1) is an eigenvector for $\text{ad}(E_{nn})$ with the eigenvalue

$$(p_1 + \cdots + p_{n-1}) - (q_1 + \cdots + q_{n-1}).$$

Thus, $a$ belongs to $A(n)^0$ if and only if for any component (1) in the decomposition of $a$ with $x \neq 0$ the corresponding expression (2) vanishes.

Now let $a \in A(n)^0$. Then $a$ is the sum of components of the form (1) with

$$p_1 + \cdots + p_{n-1} = q_1 + \cdots + q_{n-1}.$$ 

Clearly, any such component belongs to $I(n) \cap J(n)$ provided either the expression (3) or the number $r$ is nonzero. Further, there is at most one (nonzero) component for which both the sum (3) and the number $r$ vanish: such a component is an element $x \in A(n - 1)$. □

2.3. Let $\pi_{n,c}$ stand for the projection of the algebra $A(n)^0$ onto $A(n - 1)$ with the kernel $L$. By Proposition 2.2, $\pi_{n,c}$ is an algebra homomorphism (cf. this definition of $\pi_{n,c}$ with that of the Harish-Chandra homomorphism in 1.8).
Proposition. Let \( m < n \). Then the restriction of \( \pi_{n,c} \) to \( A_m(n) \) defines an algebra homomorphism

\[
\pi_{n,c} : A_m(n) \to A_m(n - 1).
\]

Proof. Indeed, the ideal \( I(n) \) is clearly invariant under the adjoint action of \( \mathfrak{gl}(n-1) \) and, in particular, under that of the subalgebra \( \mathfrak{g}_m(n-1) \). Therefore the image \( \pi_{n,c}(a) \) of any element \( a \in A_m(n) \) lies in \( A_m(n - 1) \). □

2.4. Consider the sequence of homomorphisms defined in Proposition 2.3:

\[
A_m(m) \xleftarrow{\pi_{m+1,c}} A_m(m+1) \leftarrow \cdots \xleftarrow{\pi_{n,c}} A_m(n) \leftarrow \cdots. \tag{1}
\]

It follows from the proof of Proposition 2.3 that these morphisms preserve the canonical filtrations of the universal enveloping algebras.

Definition. The algebra \( A_m \) is defined to be the projective limit of the sequence (1) of the algebras \( A_m(n) \), where the limit is taken in the category of filtered associative algebras. (Let us emphasize that this construction depends on the parameter \( c \), which is omitted only to simplify the notation.)

In other words, an element of the algebra \( A_m \) is a sequence of the form \( a = (a_m, a_{m+1}, \ldots, a_n, \ldots) \) where \( a_n \in A_m(n) \), \( \pi_{n,c}(a_n) = a_{n-1} \) for \( n > m \), and

\[
\deg a := \sup_{n \geq m} \deg a_n < \infty,
\]

where \( \deg a_n \) denotes the degree of \( a_n \) in the universal enveloping algebra \( A(n) \).

2.5. Note that the homomorphisms \( \pi_{n,c} \) are compatible with the natural embeddings \( A_m(n) \hookrightarrow A_{m+1}(n) \), that is, the following diagram is commutative:

\[
\begin{array}{ccc}
A_m(n) & \to & A_{m+1}(n) \\
\pi_{n,c} \downarrow & & \pi_{n,c} \downarrow \\
A_m(n-1) & \to & A_{m+1}(n-1).
\end{array}
\]

Therefore we can define an embedding \( A_m \hookrightarrow A_{m+1} \) as follows:

\[
(a_m, a_{m+1}, a_{m+2}, \ldots) \mapsto (a_{m+1}, a_{m+2}, \ldots).
\]

Definition. The algebra \( A \) is the inductive limit of the associative filtered algebras:

\[
A = \bigcup_m A_m.
\]

The canonical embedding of the universal enveloping algebra \( A(\infty) \) into \( A \) is defined as follows: to an element \( x \in A(\infty) \) we associate the sequence \( (x, x, \ldots) \). This embedding is well defined since \( x \) belongs to some \( A(m) \) and hence to \( A_m(n) \) for all \( n > m \).
2.6. Proposition. The center of the algebra $A$ coincides with $A_0$.

Proof. It is clear that $A_0$ is contained in the center. Conversely, suppose that $a = (a_m, a_{m+1}, \ldots) \in A$ is a central element. Then it commutes with $\mathfrak{gl}(\infty) \subset A$ and so $a_n$ lies in the center of $A(n)$ for all $n \geq m$ which implies that $a \in A_0$. □

Note that the morphism $\pi_{n,c} : A_0(n) \to A_0(n - 1)$ defined in Proposition 2.3, coincides with (1.11.1). Therefore, $A_0$ is naturally isomorphic to the algebra $Z_c$ of virtual Laplace operators.

In the next few subsections we describe the structure of the graded algebras $\text{gr } A_m$ and $\text{gr } A$ with respect to the filtration given by (2.4.2).

2.7. We shall use the following notation:

- $P(n)$ is the symmetric algebra of the space $\mathfrak{gl}(n)$;
- $P_m(n)$ is the subalgebra of the elements of $P(n)$, which are invariant under the adjoint action of the subalgebra $\mathfrak{g}_m(n)$;
- $I'(n)$ is the ideal in $P(n)$ generated by the elements $E_{in}$, $i = 1, \ldots, n$;
- $J'(n)$ is the ideal in $P(n)$ generated by the elements $E_{ni}$, $i = 1, \ldots, n$.

We can repeat the constructions from 2.1–2.6, replacing $A_m(n)$ by $P_m(n)$, $I(n)$ by $I'(n)$, and $J(n)$ by $J'(n)$. As a result we obtain a commutative graded algebra $P_m$ which is the projective limit of the commutative graded algebras $P_m(n)$. Then we can define the algebra $P$ as the inductive limit of the algebras $P_m$.

Note that the algebra $P$ contains the symmetric algebra of the Lie algebra $\mathfrak{gl}(\infty)$, and the algebra $P_m$ coincides with the subalgebra of $\mathfrak{g}_m(\infty)$-invariants in $P$.

Here the role of the algebra $\Lambda^*_c$ is played by the algebra of symmetric functions. In more detail, $P_0(n)$ coincides with the algebra of invariants of $n \times n$ matrices (with respect to the adjoint action) and may be identified with the algebra of symmetric polynomials in $n$ variables $x_1, \ldots, x_n$. Then $P_0$ may be identified with the algebra of symmetric functions in infinitely many variables $x_1, x_2, \ldots$.

2.8. Proposition. The canonical isomorphisms $\text{gr } A(n) \to P(n)$ induce isomorphisms $\text{gr } A_m \to P_m$ and $\text{gr } A \to P$.

Proof. Let $\left( A^k(n) \right)$, $k = 0, 1, 2, \ldots$ stand for the canonical filtration of the universal enveloping algebra $A(n)$, and let $P(n) = \bigoplus P^k(n)$ stand for the canonical gradation of the symmetric algebra $P(n)$. Set

\[ A^k_m(n) = A_m(n) \cap A^k(n), \quad P^k_m(n) = P_m(n) \cap P^k(n). \]

The canonical isomorphism

\[ A^k(n)/A^{k-1}(n) \to P^k(n) \] (1)
commutes with the adjoint action of $\mathfrak{gl}(n)$ and hence with that of $\mathfrak{g}_m(n)$. Since all the spaces in (1) are semisimple $\mathfrak{g}_m(n)$-modules, we obtain from (1) the isomorphisms

$$A^k_m(n)/A^k_{m-1} \rightarrow P^k_m(n),$$

which allow us to identify $\text{gr } A_m(n)$ with $P_m(n)$.

Further, let $k$ be fixed. Then for any $n > m$ the diagram

$$\begin{array}{ccc}
A^k_m(n) & \longrightarrow & P^k_m(n) \\
\downarrow & & \downarrow \\
A^k_m(n-1) & \longrightarrow & P^k_m(n-1)
\end{array}$$

is commutative, which follows immediately from the definition of the ideals $I(n)$ and $I'(n)$. This provides us with isomorphisms $\text{gr } A_m \rightarrow P_m$ for any $m$, hence, with an isomorphism $\text{gr } A \rightarrow P$. □

2.9. Let us identify $\mathfrak{gl}(n)$ with its dual space using the bilinear form $(X,Y) \mapsto \text{tr } (XY^\sigma)$ on $\mathfrak{gl}(n)$, where $Y \mapsto Y^\sigma$ denotes the standard matrix transposition: $(E_{ij})^\sigma = E_{ji}$. Then we may identify the algebra $P(n)$ with the algebra of polynomial functions $\phi(x)$ on $\mathfrak{gl}(n)$. Under this identification a matrix unit $E_{ij} \in \mathfrak{gl}(n)$ becomes the function $x \mapsto x_{ij}$, where $x$ stands for a matrix of size $n \times n$. (This is the reason why we have preferred to use the form $\text{tr } (XY^\sigma)$ instead of the invariant form $\text{tr } (XY)$).

**Proposition.** The algebra $P_m(n)$ is generated by the polynomials

$$p^{(M)}_i(x) = \text{tr } (x^M) \quad (1)$$

and

$$p^{(M)}_{i|j}(x) = (x^M)_{ij}, \quad (2)$$

where $i, j \leq m$ and $M = 1, 2, \ldots$.

**Proof.** For $m = 0$ there are no elements of type (2) and the claim is well known (see, e.g., [D], Section 7.3), so we shall assume $m \geq 1$. Write $x$ as a block matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

according to the decomposition $n = m + (n - m)$. It suffices to prove that any polynomial $\phi(x)$ satisfying the $G_m(n)$-invariance condition

$$\phi(x) \equiv \phi(a, b, c, d) = \phi(a, bg^{-1}, gc, gdg^{-1}),$$

where $g$ is an arbitrary invertible matrix of the same shape as $d$, is a polynomial function of the invariants $\text{tr } (x^M)$ and $(x^M)_{ij}$ with $i, j \leq m$.

Let us note first that these invariants can be replaced by the invariants of the form $\text{tr } (d^M), \; M \geq 1, \; (bd^{M-2}c)_{ij}, \; M \geq 2$, and $a_{ij}$. Indeed, we have $x_{ij} = a_{ij}$,
and for $M \geq 2$ the invariant $(x^M)_{ij} - (bd^{M-2}c)_{ij}$ can be expressed in terms of the invariants $a_{kl}$ and $(bd^{N-2}c)_{kl}$ with $N < M$. Hence, instead of $(x^M)_{ij}$ we can take $(bd^{M-2}c)_{ij}$ for $M \geq 2$. Moreover, $\text{tr}(x^M) - \text{tr}(d^M)$ can be expressed in terms of the invariants of the form $a_{kl}$, $\text{tr}(d^N)$ with $N < M$, and $(bd^{N-2}c)_{kl}$ with $N \leq M$. Hence, instead of $\text{tr}(x^M)$ we can take $\text{tr}(d^M)$.

To simplify the notation, we shall assume below that $m = 1$ (the generalization to the case $m > 1$ will be obvious). We can ignore the element $a$, so we have to show that any polynomial invariant $\phi(b, c, d)$, where $c$ is an element of the $\text{GL}(n-1)$-module $V$ of column vectors of the length $n-1$, $b$ is an element of the dual module $V^*$ of row vectors, and $d \in V \otimes V^*$, can be expressed in terms of $\text{tr}(d^M)$ and $bd^{M-2}c$. Any such invariant can be decomposed into a sum of expressions of the form

$$\psi(b_1, \ldots, b_p, c_1, \ldots, c_q, d_1, \ldots, d_r),$$

where $\psi$ is a multilinear invariant. In its turn, $\psi$ is determined by a multilinear invariant of the form

$$\chi(b_1, \ldots, b_p, c_1, \ldots, c_q, u_1, \ldots, u_r, v_1, \ldots, v_r),$$

where $b_1, \ldots, b_p, u_1, \ldots, u_r \in V^*$ and $c_1, \ldots, c_q, v_1, \ldots, v_r \in V$.

The multilinear invariants for the general linear group are described by the classical invariant theory. Let us recall some basic facts of this theory [W]. First of all, nonzero invariants exist only when the number of vector arguments is equal to the number of covector arguments. So in (4) we have $p = q$. Next, any invariant can be uniquely written as a polynomial in ‘elementary invariants’ that come from the canonical pairing $V \otimes V^* \to \mathbb{C}$. In our notation, this means that $\chi$ may be represented as a linear combination of monomials in bilinear invariants $b_ic_j$, $b_iv_j$, $u_ic_j$, $u_iv_j$ such that in each of the monomials each letter appears exactly once. Since the letters $u$ and $v$ are related with the block $d$, we must keep an eye on the position of these letters with equal numbers. In each monomial we can meet, first, closed chains of the form

$$(u_{k_1}v_{k_2})(u_{k_2}v_{k_3})\ldots(u_{k_M}v_{k_1}),$$

and, second, open chains of the form

$$(b_kv_{m_1})(u_{m_1}v_{m_2})\ldots(u_{m_{M-1}}c_l).$$

Looking again at formula (3), we see that under the passage $\chi \mapsto \psi \mapsto \phi$, each chain (5) gives rise to the invariant $\text{tr}(d^M)$, and each chain (6) corresponds to an invariant of the form $bd^{M-2}c$. □

2.10. Fix numbers $K$ and $m$ such that $K \in \{1, 2, \ldots\}$ and $m \in \{0, 1, \ldots\}$. Assume that the indices $i, j, M$ satisfy the conditions $i, j \leq m$ and $1 \leq M \leq K$. 
**Proposition.** For a sufficiently large $n$ the elements $p^{(M)}_{ij|n}$, $p^{(M)}_n$ of the algebra $P_m(n)$ with the indices satisfying the above conditions are algebraically independent.

**Proof.** For any triple $(i, j, M)$ satisfying our assumptions we choose a subset

$$\Omega_{ijM} \subset \{m + 1, m + 2, \ldots\}$$

of cardinality $M - 1$ in such a way that all these subsets be disjoint. Let $n$ be so large that all of them belong to $\{m + 1, m + 2, \ldots, n - K\}$. Let

$$y_i, \ 1 \leq i \leq K, \quad \text{and} \quad z_{ijM}, \ 1 \leq i, j \leq m, \ 1 \leq M \leq K,$$

be complex parameters. Given $i, j, M$ we introduce a linear operator $x_{ijM}$ in $C^n$ depending on the parameter $z_{ijM}$. Let $a_1 < \cdots < a_{M-1}$ be all the elements of $\Omega_{ijM}$. Then $x_{ijM}$ transforms the canonical basis vectors of $C^n$ as follows:

$$e_j \mapsto z_{ijM}e_{a_{M-1}},$$

$$e_{a_{M-1}} \mapsto e_{a_{M-2}}, \ldots, e_{a_1} \mapsto e_i,$$

$$e_k \mapsto 0 \quad \text{for} \quad k \notin \{j\} \cup \Omega_{ijM}.$$

Next we define a linear operator $x$ in $C^n$ depending on all the parameters,

$$xe_k = \begin{cases} 
\sum_{i,j,M} x_{ijM}e_k, & k \leq n - K, \\
y_{k-(n-K)}e_k, & k = n - K + 1, \ldots, n.
\end{cases} \quad (1)$$

Regard $x$ as a $n \times n$ matrix. Then we have

$$p^{(M)}(x) = y_1^M + \cdots + y_K^M + \phi_M(\ldots, z_{klL}, \ldots),$$

$$p^{(M)}_{ij|n}(x) = z_{ijM} + \psi_{ijM}(\ldots, z_{klL}, \ldots), \quad L < M,$$

where $\phi_M$ does not depend on $y_1, \ldots, y_K$, and $\psi_{ijM}$ only depends on the parameters $z_{klL}$ with $L < M$. This implies that our polynomials are algebraically independent even if they are restricted to the affine subspace of matrices of the form (1). $\square$

**2.11. Proposition.** We have

$$p^{(M)}_n \in p^{(M)}_{n-1} + I'(n), \quad (1)$$

$$p^{(M)}_{ij|n} \in p^{(M)}_{ij|n-1} + I'(n), \quad i, j \leq n - 1. \quad (2)$$

**Proof.** By (2.9.1),

$$p^{(M)}_n(x) = \text{tr}(x^M) = \sum_{i_1, \ldots, i_M=1}^n x_{i_1i_2}x_{i_2i_3}\cdots x_{i_Mi_1}. \quad (3)$$
Here \( p_n^{(M)} \) is viewed as a polynomial function on the space of \( n \times n \) matrices. Using the identification \( x_{ij} \leftrightarrow E_{ij} \) (see 2.9) we may rewrite (3) as follows

\[
p_n^{(M)} = \sum_{i_1, \ldots, i_M = 1} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_M i_1},
\]

regarding \( p_n^{(M)} \) as an element of \( \mathbb{P}(n) \).

Now let us split the sum (4) into two parts. The first part will consist of all monomials with \( i_1 \leq n - 1, \ldots, i_M \leq n - 1 \), and the second part will consist of all remaining monomials. Then the first part of the sum clearly equals \( p_n^{(M)} - 1 \). In the second part of the sum each monomial contains a letter of the form \( E_{jn} \) and so lies in the ideal \( I'(n) \). This proves (1).

To verify (2), we write \( p_{ij|n}^{(M)} \) as

\[
p_{ij|n}^{(M)} = \sum_{k_1, \ldots, k_{M-1}} E_{ik_1} E_{k_1 k_2} \cdots E_{k_{M-1} j},
\]

where the indices range over \( \{1, \ldots, n\} \) and repeat the previous argument. \( \square \)

**2.12.** Given \( M = 1, 2, \ldots \), consider the sequence

\[
p^{(M)} := (p_n^{(M)}), \quad n \geq 1.
\]

We claim that \( p^{(M)} \) is an element of \( \mathbb{P}_0 \). Indeed, \( p_n^{(M)} \) is a homogeneous element of \( \mathbb{P}_0(n) \) of degree \( M \), and by Proposition 2.11 the difference \( p_n^{(M)} - p_{n-1}^{(M)} \) lies in the ideal \( I'(n) \). Similarly, fix \( M, i, j, m \) such that \( 1 \leq i, j \leq m \). Then the sequence

\[
p_{ij}^{(M)} := (p_{ij|n}^{(M)}|n \geq m)
\]

is an element of \( \mathbb{P}_m \).

**Proposition.** For any fixed \( m \geq 0 \) the elements \( p^{(M)} \) and \( p_{ij}^{(M)} \) with \( M = 1, 2, \ldots \) and \( 1 \leq i, j \leq m \), are algebraically independent generators of the algebra \( \mathbb{P}_m \).

**Proof.** Let \( p = (p_n| n \geq m) \in \mathbb{P}_m \). By Proposition 2.9, for any \( n \geq m \) the element \( p_n \in \mathbb{P}_m(n) \) can be represented as a polynomial \( \phi_n \) in the variables \( p_n^{(M)}, p_{ij|n}^{(M)} \), where \( M \leq \deg p \) and \( i, j \leq m \). By Proposition 2.10, \( \phi_n \) does not depend on \( n \) for sufficiently large \( n \). Hence, the elements \( p^{(M)} \) and \( p_{ij}^{(M)} \) are generators. Their algebraic independence is clear from Proposition 2.10. \( \square \)
2.13. **Corollary.** The algebra $P$ is isomorphic to the algebra of polynomials in countably many variables $p^{(M)}, p^{(M)}_{ij}$, where $M = 1, 2, \ldots$ and $i, j \in \{1, 2, \ldots\}$. □

Now we need some preparations to prove the main result of this section, Theorem 2.18.

2.14. Recall the definition of the Yangian $Y(n) = Y(gl(n))$ (for more details see [MNO], Section 1). It is a complex associative algebra with countably many generators

\[ t^{(M)}_{ij}, \quad 1 \leq i, j \leq n, \quad M = 1, 2, \ldots \]  

(1)

and the quadratic defining relations which can be written as follows. Combine the generators (1) into series in $u^{-1}$:

\[ t_{ij}(u) := \delta_{ij} + \sum_{M=1}^{\infty} t^{(M)}_{ij} u^{-M}, \quad 1 \leq i, j \leq n. \]  

(2)

Then the defining relations take the form:

\[ [t_{ij}(u), t_{kl}(v)] = \frac{1}{u - v} (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)). \]  

(3)

Next, the series (2) are combined into a single $n \times n$-matrix

\[ T(u) = (t_{ij}(u))_{i,j=1}^{n} = \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij}. \]  

(4)

In terms of the $T$-matrix (4), the relations (3) can be written as a single ‘ternary relation’ (0.2); see Introduction.

The quantum determinant $\text{qdet} \, T(u)$ of the matrix $T(u)$ is a formal series in $u^{-1}$ with coefficients from $Y(n)$ defined as follows:

\[ \text{qdet} \, T(u) = \sum_{p \in S_n} \text{sgn} (p) \, t_{p(1),1}(u) \cdots t_{p(n),n}(u - n + 1), \]  

(5)

where $S_n$ is the group of permutations of the indices $\{1, \ldots, n\}$. The coefficients of the quantum determinant $\text{qdet} \, T(u)$ are algebraically independent generators of the center of the algebra $Y(n)$.

2.15. Given $n$, let $E$ stand for the $A(n)$-valued matrix of order $n$ whose $(i, j)$-entry is $E_{ij} \in gl(n) \subset A(n)$. (We use the symbol $E_{ij}$ to denote both the generators of $gl(n)$ and the auxiliary matrix units).
Consider the mapping \( \eta \) which takes the generators \( t_{ij}^{(M)} \) of the Yangian \( Y(n) \) to elements of \( A(n) \) as follows:

\[
\eta : T(u) \mapsto \left( 1 - \frac{E}{u} \right)^{-1},
\]

or, in more detail,

\[
\eta(t_{ij}^{(M)}) = (E^M)_{ij} = \sum_{k_1, \ldots, k_{M-1}} E_{ik_1} E_{k_1k_2} \cdots E_{k_{M-1}j} \in A(n),
\]

where \( k_1, \ldots, k_{M-1} \) range over \( \{1, \ldots, n\} \). It was proved in [MNO], Subsection 1.19 that \( \eta \) defines an algebra homomorphism \( Y(n) \to A(n) \).

2.16. **Proposition.** For any \( c \in \mathbb{C} \) the mapping

\[
\varphi_n : T(u) \mapsto \frac{u + n - c}{u + n} \left( 1 - \frac{E}{u + n} \right)^{-1}
\]

defines an algebra homomorphism \( Y(n) \to A(n) \).

**Proof.** It was shown in [MNO], Proposition 1.12 that the following mappings define automorphisms of the algebra \( Y(n) \): shift of the formal parameter \( u \) by a constant, \( T(u) \mapsto T(u + a) \); multiplication by a formal series, \( T(u) \mapsto f(u)T(u) \), where \( f(u) = 1 + f_1u^{-1} + f_2u^{-2} + \cdots, f_i \in \mathbb{C} \). Therefore, by 2.15 the mapping

\[
T(u) \mapsto f(u + a) \left( 1 - \frac{E}{u + a} \right)^{-1}
\]

defines an algebra homomorphism \( Y(n) \to A(n) \). It remains to take \( f(u) = 1 - cu^{-1} \) and \( a = n \). \( \square \)

2.17. The defining relations (2.14.3) and the Poincaré–Birkhoff–Witt theorem for the Yangian (see [MNO], Corollary 1.23) imply that for any \( m \geq 1 \) one has a natural inclusion

\[
Y(m) \hookrightarrow Y(m + 1).
\]

So, for any \( m \leq n \) we can regard the Yangian \( Y(m) \) as a subalgebra in \( Y(n) \).

**Proposition.** The image of the restriction of the homomorphism \( \varphi_n \) to the subalgebra \( Y(m) \) is contained in the centralizer \( A_m(n) \).

**Proof.** It follows from the defining relations (2.14.3) that

\[
[t_{k_l}^{(1)}, t_{ij}(u)] = \delta_{il} t_{kj}(u) - \delta_{kj} t_{il}(u).
\]

(2)
In particular,

\[ [t^{(1)}_{kl}, t_{ij}(u)] = 0 \quad (3) \]

for \( i, j \leq m < k, l \). On the other hand, it is easy to see from (2.16.1) that

\[ \varphi_n(t^{(1)}_{kl}) = E_{kl} - \delta_{kl} c. \quad (4) \]

Together with (3) this implies that \([E_{kl}, \varphi_n(t_{ij}(u))] = 0\) provided that the indices \( i, j, k, l \) satisfy the above restrictions. By definition of \( A_m(n) \) this means that \( \varphi_n(t^{(m)}_{ij}) \in A_m(n) \). □

2.18. Fix an arbitrary \( m \geq 1 \) and assume that \( n \) varies from \( m \) to infinity. By Proposition 2.17, for any \( n \) we have a homomorphism of the algebra \( Y(m) \) to \( A_m(n) \), defined by \( \varphi_n \).

**Theorem.** For any fixed \( m \geq 1 \) the sequence \( (\varphi_n|n \geq m) \) defines an algebra embedding \( \varphi: Y(m) \hookrightarrow A_m \). Moreover, one has the isomorphism

\[ A_m = A_0 \otimes Y(m), \quad (1) \]

where the Yangian is identified with its image under the embedding \( \varphi \).

**Proof.** We shall prove the theorem in several steps.

**Step 1.** To prove that the sequence of homomorphisms \( (\varphi_n|n \geq m) \) defines an algebra homomorphism \( \varphi: Y(m) \rightarrow A_m \) we need to verify that the following diagram is commutative:

\[
\begin{array}{ccc}
Y(m) & \rightarrow & Y(m) & \rightarrow & \cdots & \rightarrow & Y(m) & \rightarrow & \cdots \\
\downarrow \varphi_m & & \downarrow \varphi_{m+1} & & \cdots & & \downarrow \varphi_n & & \\
A_m(m) & \leftarrow & A_m(m+1) & \leftarrow & \cdots & \leftarrow & A_m(n) & \leftarrow & \cdots .
\end{array}
\]

Denote the image of \( t_{ij}(u) \) under the homomorphism (2.16.1) by \( \tau_{ij|n}(u) \) and prove by induction on \( M \) that for the coefficients of this series one has (we use notation 2.1):

\[ \begin{align*}
(i) & \quad \tau^{(M)}_{in|n} \in I(n), \quad 1 \leq i \leq n, \quad M \geq 1; \\
(ii) & \quad \tau^{(M)}_{ij|n} - \tau^{(M)}_{ij|n-1} \in I(n), \quad 1 \leq i, j \leq n-1, \quad M \geq 1.
\end{align*} \quad (2) \]

Consider the matrix \( T(u) = (\tau_{ij|n}(u))_{i,j=1}^n \). By (2.16.1),

\[ T(u)(u + n - E) = u + n - c. \]

Hence

\[ uT(u) = u + n - c + T(u)(E - n). \quad (3) \]
Write $\mathcal{T}(u) = \mathcal{T}^{(0)} + \mathcal{T}^{(1)}u^{-1} + \ldots$. Then (3) implies that $\mathcal{T}^{(0)} = 1$ and

$$\mathcal{T}^{(1)} = E - c,$$

$$\mathcal{T}^{(M)} = \mathcal{T}^{(M-1)}(E - n), \quad M \geq 2. \tag{4}$$

By (4) we have

$$\tau_{in|n}^{(1)} = E_{in} - \delta_{in}c \in I(n), \quad 1 \leq i \leq n,$$

and

$$\tau_{ij|n}^{(1)} - \tau_{ij|n-1}^{(1)} = 0, \quad 1 \leq i, j \leq n - 1.$$  

So, we have verified (2) for $M = 1$. For $M > 1$ we obtain from (5) that

$$\tau_{in|n}^{(M)} = \sum_{a=1}^{n} \tau_{ia|n}^{(M-1)}(E_{an} - \delta_{an}n)$$

$$= \sum_{a=1}^{n-1} \tau_{ia|n}^{(M-1)}E_{an} + \tau_{in|n}^{(M-1)}(E_{nn} - c) + (c - n)\tau_{in|n}^{(M-1)}.$$  

By the induction hypotheses, this expression lies in $I(n)$, which proves (i) in (2).

Again using (5) we obtain for $1 \leq i, j \leq n - 1$ that

$$\tau_{ij|n}^{(M)} - \tau_{ij|n-1}^{(M)} = \sum_{a=1}^{n} \tau_{ia|n}^{(M-1)}(E_{aj} - \delta_{aj}n) - \sum_{a=1}^{n-1} \tau_{ia|n-1}^{(M-1)}(E_{aj} - \delta_{aj}(n-1)),$$

which can be rewritten as

$$\sum_{a=1}^{n-1} (\tau_{ia|n}^{(M-1)} - \tau_{ia|n-1}^{(M-1)})E_{aj} - (n-1)(\tau_{ij|n}^{(M-1)} - \tau_{ij|n-1}^{(M-1)}) + \tau_{in|n}^{(M-1)}E_{nj} - \tau_{ij|n}^{(M-1)}.$$  

Since $\varphi_n$ is an algebra homomorphism, we obtain from (2.17.2) and (2.17.4) that

$$(\tau_{ia|n}^{(M-1)} - \tau_{ia|n-1}^{(M-1)})E_{aj} = E_{aj}(\tau_{ia|n}^{(M-1)} - \tau_{ia|n-1}^{(M-1)})$$

$$+ \tau_{ij|n}^{(M-1)} - \tau_{ij|n-1}^{(M-1)} - \delta_{ij}(\tau_{aa|n}^{(M-1)} - \tau_{aa|n-1}^{(M-1)}),$$

while

$$\tau_{in|n}^{(M-1)}E_{nj} - \tau_{ij|n}^{(M-1)} = E_{nj}\tau_{in|n}^{(M-1)} + [\tau_{in|n}^{(M-1)}, E_{nj}] - \tau_{ij|n}^{(M-1)}$$

$$= E_{nj}\tau_{in|n}^{(M-1)} - \delta_{ij}\tau_{nn|n}^{(M-1)}.$$  

Using (i) and the induction hypotheses, we complete the proof of (2). In particular, we have proved that for $i, j \leq m$ the sequence $(\tau_{ij|n|n}^{(M)}, n \geq m)$ is an element of the algebra $A_m$, and so, the homomorphism $\varphi$ is well defined.
Step 2. Here we verify that $\varphi$ is an embedding, that is, its kernel is trivial. Recall that $A_m$ is a filtered algebra with $\text{gr} A_m = P_m$; see Proposition 2.8. Consider the highest order term of the sequence $\tau_{ij}^{(M)} := (\tau_{ij}^{(M)}| n \geq m)$. By definition,

$$\tau_{ij|n}(u) = \frac{u + n - c}{u + n} \left(1 - \frac{E}{u + n}\right)_i^j.$$ 

Hence $\tau_{ij|n}$ has the form

$$\tau_{ij}^{(M)} = (E^M)_i^j + \sum_{k \geq 1} a_k (E^{M-k})_i^j,$$

where $a_k \in \mathbb{C}$. This proves that the image of $\tau_{ij|n}$ in the $M$-th component of the graded algebra $P(n) = \text{gr} A(n)$ is $p_{ij|n}^{(M)}$. So, the highest order term of $\tau_{ij}$ coincides with $p_{ij}^{(M)}$.

Since the algebra $\text{gr} A_m = P_m$ is commutative, we obtain from Proposition 2.12 that the elements $\tau_{ij}^{(M)}$ of the algebra $A_m$ satisfy a Poincaré–Birkhoff–Witt-type condition: given an arbitrary ordering of the elements $\tau_{ij}^{(M)}$, any element of the subalgebra in $A_m$ generated by the $\tau_{ij}^{(M)}$ has a unique representation as a polynomial in the $\tau_{ij}^{(M)}$. Using the Poincaré–Birkhoff–Witt theorem for the Yangian (see [MNO], Corollary 1.23), we conclude that the homomorphism $\varphi$ taking $\tau_{ij}^{(M)}$ to $\tau_{ij}^{(M)}$, has zero kernel.

Step 3. It remains to prove the decomposition (1). The arguments of Step 2 show that the graded algebra $\text{gr} Y(m)$ can be identified with the subalgebra of $\text{gr} A_m = P_m$ generated by the elements $p_{ij}^{(M)}$, $M = 1, 2, \ldots$. So, the required statement follows from Proposition 2.12. □

2.19. By making use of the inclusion (2.17.1) we define the algebra $Y(\infty)$ as the corresponding inductive limit as $m \to \infty$. The following corollary is immediate from Theorem 2.18.

**Corollary.** One has the isomorphism $A = A_0 \otimes Y(\infty)$. □

2.20. **Remark.** The Yangian $Y(m)$ has a nontrivial center generated by the coefficients of the quantum determinant $q\text{det} T(u)$; see 2.14 and [MNO], Theorem 2.13. Therefore, the center of the algebra $A_m$ strictly contains $A_0$. However, the center of the algebra $Y(\infty)$ is trivial, since by Proposition 2.6, the center of the algebra $A$ coincides with $A_0$.

Note also that in contrast to the algebra $Y(m)$, the ‘infinite’ Yangian $Y(\infty)$ apparently does not possess a Hopf algebra structure.

2.21. The following proposition shows that the algebra $A$ may be regarded as a ‘true’ analog of the universal enveloping algebra $A(\infty)$ for the Lie algebra $\mathfrak{gl}(\infty)$. 
Remark 1.5), the category $\Omega(\mathfrak{c})$ algebra

The results of Sections 1 and 2 can be carried out to the Lie algebra $A$, the same is true for all vectors of the module $V_{algebra A}$, the same is true for all vectors of the module $V$. In contrast to the category $\Omega(A)$ for a finite number of the parameter, namely, the difference $-\lambda_j, c_+$ for a sufficiently large $n$, and so, $abv = (ab)v$. This action of $A$ clearly extends the action of its subalgebra $A(\infty)$. □

2.22. Proposition. Let $V$ be a $\mathfrak{gl}(\infty)$-module with the highest weight $\lambda$. Then every element $a \in A_0$ acts on $V$ as the scalar operator $f_a(\lambda) \cdot 1$, where $f_a \in \Lambda_c^*$ is the shifted symmetric function which corresponds to $a$ under the identification $A_0 = \Lambda^*_c$ from 1.11.

Proof. Let $v$ be the highest weight vector of the module $V$ and $a = (a_n)$ be an arbitrary element of $A_0$. The definitions of the Harish-Chandra isomorphism (see 1.8) and the isomorphism $A_0 \rightarrow \Lambda_c^*$ imply that $a_n v = f_a(\lambda)v$ for a sufficiently large $n$. Hence $av = f_a(\lambda)v$. Since $v$ is a cyclic vector and $a$ belongs to the center of the algebra $A$, the same is true for all vectors of the module $V$. □

2.23. Remark. The results of Sections 1 and 2 can be carried out to the Lie algebra $\mathfrak{gl}(2\infty + 1)$ of all complex matrices $A = (A_{ij})$ where $i, j \in \mathbb{Z}$ and only a finite number of the $A_{ij}$ is nonzero. In particular, here $\Omega(\mathfrak{c})$ is replaced with the category $\Omega(c_-, c_+)$ of modules which ‘stabilize’ in both negative and positive directions (cf. 1.3, and 3.3 below). In contrast to the category $\Omega(\mathfrak{c})$ for $\mathfrak{gl}(\infty)$ (see Remark 1.5), the category $\Omega(c_-, c_+)$ for $\mathfrak{gl}(2\infty + 1)$ involves a substantial continuous parameter, namely, the difference $c_- - c_+$.

3. Highest weight modules and virtual Laplace operators for the Lie algebras $\mathfrak{o}(2\infty)$, $\mathfrak{o}(2\infty + 1)$ and $\mathfrak{sp}(2\infty)$

In this section we introduce analogs of the category $\Omega(\mathfrak{c})$ for the infinite orthogonal and symplectic Lie algebras. Then we construct the algebras of ‘virtual Laplace operators’ for these Lie algebras.

3.1. Let us introduce the Lie algebra $\mathfrak{gl}(2\infty)$ of all complex matrices $A = (A_{ij})$ where $i$ and $j$ run through the set $\mathbb{Z} \setminus \{0\}$ and the number of nonzero $A_{ij}$ is finite. The standard matrix units $E_{ij}$, $i, j \in \mathbb{Z} \setminus \{0\}$ form a basis of this algebra. Let us denote by $\mathfrak{g}$ any of the following Lie algebras:

$$\mathfrak{o}(2\infty) = \{A \in \mathfrak{gl}(2\infty) \mid A_{-j, -i} = -A_{ij}\},$$

$$\mathfrak{o}(2\infty + 1) = \{A \in \mathfrak{gl}(2\infty + 1) \mid A_{-j, -i} = -A_{ij}\},$$

$$\mathfrak{sp}(2\infty) = \{A \in \mathfrak{gl}(2\infty) \mid A_{-j, -i} = -\text{sgn} i \text{ sgn} j \cdot A_{ij}\}.$$
The Lie algebra $g$ is spanned by the elements

$$F_{ij} := E_{ij} - \theta_{ij} E_{-j,-i},$$

where $\theta_{ij} \equiv 1$ for the orthogonal Lie algebras, and $\theta_{ij} = \text{sgn } i \text{ sgn } j$ for the symplectic Lie algebra.

We let $h$ denote the Cartan subalgebra of diagonal matrices, $n_+$ (respectively, $n_-$) the subalgebra of upper (respectively, lower) triangular matrices in $g$, and $b := h \oplus n_+$ the Borel subalgebra in $g$. One has the triangular decomposition

$$g = n_- \oplus h \oplus n_+.$$

For a linear functional $\lambda \in h^*$ set $\lambda_i = \lambda(F_{ii})$. Then we have $\lambda_{-i} = -\lambda_i$, so $\lambda$ is uniquely determined by the sequence $(\ldots, \lambda_{-2}, \lambda_{-1})$.

3.2. A $g$-module $V$ is said to be highest weight if it has a cyclic vector $v$ satisfying $n_+ v = \{0\}$ and there exists $\lambda \in h^*$ such that $hv = \lambda(h)v$ for any $h \in h$. The functional $\lambda$ is the highest weight of $V$, and $v$ is the highest weight vector of $V$; it is unique up to scalar multiples. Sometimes $V$ is also referred to as the module with the highest weight $\lambda$. The universal $g$-module $M_\lambda$ with the highest weight $\lambda \in h^*$ (the Verma module) is defined as the quotient of the universal enveloping algebra $U(g)$ by the left ideal generated by $n_+$ and the elements $h - \lambda(h)$, $h \in h$. Denote by $L_\lambda$ the unique nontrivial irreducible quotient of $M_\lambda$.

For $n = 1, 2, \ldots$ set

$$g(n) = \text{span of } \{F_{ij} \mid -n \leq i, j \leq n\} \subset g,$$

$$b_n = \{A \in b \mid A_{ij} = 0 \text{ for } -n \leq i, j \leq n\} \subset b.$$

Then $g(n)$ is isomorphic to $\mathfrak{o}(2n)$, $\mathfrak{o}(2n + 1)$, or $\mathfrak{sp}(2n)$, respectively, and $g(n) + b_n$ is a parabolic subalgebra of $g$.

For a $g$-module $V$ and a functional $\mu \in h^*$ define an ascending chain of subspaces

$$V_n(\mu) = \{v \in V \mid F_{ii} v = \mu_i v \text{ and } F_{ij} v = 0, \ i < j, \ for \ F_{ii}, F_{ij} \in b_n\}.$$

Set

$$V_\infty(\mu) = \bigcup_{n=1}^{\infty} V_n(\mu)$$

and note that $V_\infty(\mu)$ is a submodule of $V$, since $V_n(\mu)$ is $g(n)$-invariant for every $n$. We shall write $\mu \sim \mu'$ if $\mu_i = \mu'_i$ for $|i| \gg 0$. Then $\mu \sim \mu'$ implies $V_\infty(\mu) = V_\infty(\mu')$.

3.3. Definition. Fix a number $c \in \mathbb{C}$ and consider the functional $\lambda^c \in h^*$ such that $(\lambda^c)_i = c$ for $i = -1, -2, \ldots$. We define the category $\Omega(c)$ as the category of those $g$-modules $V$ for which $V_\infty(\lambda^c) = V$. 
3.4. One easily checks that any $\mathfrak{g}$-module with the highest weight $\lambda \sim \lambda^c$ belongs to $\Omega(c)$. In particular, $\Omega(c)$ contains the modules $M_\lambda$ and $L_\lambda$ with $\lambda \sim \lambda^c$ (cf. Proposition 1.4).

3.5. For an arbitrary $\lambda \in \mathfrak{h}^*$ denote by $L_\lambda^{(n)}$ the cyclic $\mathfrak{g}(n)$-submodule of $L_\lambda$ generated by the highest weight vector $v \in L_\lambda$. Then $L_\lambda^{(n)}$ is clearly a highest weight module over $\mathfrak{g}(n)$. Exactly as in the $\mathfrak{gl}(n)$-case (see Proposition 1.6), one proves that all $\mathfrak{g}(n)$-modules $L_\lambda^{(n)}$ are irreducible, so that $L_\lambda$ can be viewed as the inductive limit $\lim_{\rightarrow} L_\lambda^{(n)}$ as $n \to \infty$.

Now we suppose that $\lambda \in \mathfrak{h}^*$ satisfies the conditions

(i) $\lambda \sim \lambda^c$,
where $c \in \frac{1}{2}\mathbb{Z}_+$ for $\mathfrak{g} = \mathfrak{o}(2\infty)$ or $\mathfrak{g} = \mathfrak{o}(2\infty + 1)$,
and $c \in \mathbb{Z}_+$ for $\mathfrak{g} = \mathfrak{sp}(2\infty)$;

(ii) $\lambda_{-(i+1)} - \lambda_{-i} \in \mathbb{Z}_+$ for $i = 1, 2, \ldots$,
and in addition $\lambda_{-2} \geq |\lambda_{-1}|$ for $\mathfrak{g} = \mathfrak{o}(2\infty)$,
and $\lambda_{-1} \geq 0$ for $\mathfrak{g} = \mathfrak{o}(2\infty + 1)$ and $\mathfrak{g} = \mathfrak{sp}(2\infty)$.

Then $L_\lambda^{(n)}$ is a finite-dimensional irreducible module with the dominant highest weight obtained by restricting $\lambda$ to $\mathfrak{h}^{(n)} := \mathfrak{h} \cap \mathfrak{g}(n)$. So, the module $L_\lambda$ is the inductive limit of finite-dimensional irreducible $\mathfrak{g}(n)$-modules.

Let us associate a semi-infinite Dynkin diagram with the Lie algebra $\mathfrak{g}$. Just as in the finite-dimensional case we can represent any $\lambda \in \mathfrak{h}^*$ as a collection of labels attached to the nodes of this diagram as follows:

in the case of $\mathfrak{g} = \mathfrak{o}(2\infty)$;

in the case of $\mathfrak{g} = \mathfrak{o}(2\infty + 1)$;

in the case of $\mathfrak{g} = \mathfrak{sp}(2\infty)$. Then the conditions (i) and (ii) express the fact that all the labels are nonnegative integers and the number of nonzero labels is finite.
3.6. Let \( \rho \in \mathfrak{h}^\ast \) denote the half sum of positive roots of \( \mathfrak{g} \) relative to \( \mathfrak{b} \), or which is the same, \( \rho|_{\mathfrak{h}(n)} \) is the half sum of positive roots of \( \mathfrak{g}(n) \) for any \( n \). Then for \( i = 1, 2, \ldots \) we have

\[
\rho - i = -\rho_i = \begin{cases} 
  i - 1, & \text{for } \mathfrak{g} = \mathfrak{o}(2\infty), \\
  i - \frac{1}{2}, & \text{for } \mathfrak{g} = \mathfrak{o}(2\infty + 1), \\
  i, & \text{for } \mathfrak{g} = \mathfrak{sp}(2\infty).
\end{cases}
\]

Denote by \( Z(n) \) the center of the universal enveloping algebra \( U(\mathfrak{g}(n)) \). Using the Harish-Chandra isomorphism (see 1.8), we identify \( Z(n) \) with the algebra \( M^\ast(n) \) of the polynomial functions \( f(\lambda - n, \ldots, \lambda - 1) \) on \( \mathfrak{h}(n)^\ast \) which are invariant under the ‘shifted’ action (1.8.2) of the Weyl group \( W \). More precisely, set \( l_i = \lambda_i + \rho_i \) and consider \( f \) as a function in the variables \( l_{-n}, \ldots, l_{-1} \). Then \( f \) must be invariant with respect to all permutations of the variables and all transformations \( l_i \mapsto \pm l_i \), where in the case of \( \mathfrak{g}(n) = \mathfrak{o}(2n) \) the number of ‘-’ has to be even. Note that both canonical filtrations of \( U(\mathfrak{g}(n)) \) and \( \mathbb{C}[\lambda - n, \ldots, \lambda - 1] \) induce the same filtration on \( Z(n) \).

3.7. Let us fix \( c \in \mathbb{C} \) and consider morphisms \( \pi_{n,c} : Z(n) \to Z(n - 1) \) defined as follows:

\[
(\pi_{n,c}(f))(\lambda_{n+1}, \ldots, \lambda_{-1}) = f(c, \lambda_{n+1}, \ldots, \lambda_{-1}),
\]

where \( f \in Z(n) \) is identified with an element of \( M^\ast(n) \).

Definition. The algebra \( Z_c = Z_c(\mathfrak{g}) \) of virtual Laplace operators is defined as the projective limit of the sequence of the filtered algebras \( Z(n) \) as \( n \to \infty \).

3.8. Let \( M_c^\ast \) denote the algebra generated by the family of functions \( p_m(\lambda) \), \( m = 1, 2, \ldots \), where

\[
p_m(\lambda) = \sum_{i=-1,-2,\ldots} (l_i^{2m} - (l^c_i)^{2m}),
\]

\( l = \lambda + \rho \), \( l^c = \lambda^c + \rho \), and \( \lambda \sim \lambda^c \).

Proposition. The algebras \( Z_c \) and \( M_c^\ast \) are isomorphic to each other.

Proof. Denote by \( Z_k(n) \) the subspace of elements of the algebra \( Z(n) \) of degree \( \leq k \) and by \( Z_c^k \) the space of all sequences of the form \( z = (z_1, z_2, \ldots) \) such that \( z_n \in Z_k(n) \) and \( \pi_{n,c}(z_n) = z_{n-1} \). Then by Definition 3.7,

\[
Z_c = \bigcup_{k \geq 1} Z_c^k.
\]

It follows from the well-known description of the polynomial \( W \)-invariants (see, e.g., Želobenko [Ž]) that the algebra \( M^\ast(n) \) is generated by the polynomials

\[
p_{m|n}(\lambda) = \sum_{i=-n}^{-1} (l_i^{2m} - (l^c_i)^{2m}), \quad m = 1, 2, \ldots n,
\]
and in the case of $\mathfrak{o}(2n)$ also by the polynomial $p'_n(\lambda) = l_{-1} \cdots l_{-n}$.

Let $z = (z_n) \in Z^k_c$. Identify $z_n$ with an element of $M^*(n)$. Then for $n > k$ this element can be uniquely represented as a polynomial $P_n$ in the $p_{m|n}$ with $2m \leq k$. (The element $p'_n$ in the case of $\mathfrak{o}(2n)$ is eliminated, because $\deg p'_n > \deg z_n$). Since the polynomials $p_{m|n}$ are algebraically independent and $\pi_{n,c}(p_{m|n}) = p_{m|n-1}$, we have $\pi_{n,c}(P_n) = P_{n-1}$ for $n > k + 1$. This means that $z$ is represented as a polynomial in the elements $p_m := (p_{m|n} | n \geq m)$. □

### 3.9.

The algebra $M^*_c$ can be regarded as an analog of the algebra of shifted symmetric functions $\Lambda^*_c$ (see 1.10) for the case of $B, C, D$ series. As with $\Lambda^*_c$ one can introduce analogs of the elementary and complete symmetric functions by the formulas

$$1 + \sum_{m=1}^{\infty} e_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 + l^2_k t}{1 + (l^c_k)^2 t},$$
$$1 + \sum_{m=1}^{\infty} h_m(\lambda)t^m = \prod_{k=1}^{\infty} \frac{1 - (l^c_k)^2 t}{1 - l^2_k t}.$$ 

Each of the families $\{p_m\}, \{e_m\}, \{h_m\}$ can be taken as a system of algebraically independent generators of the algebra $M^*_c$, so that

$$M^*_c = \mathbb{C}[p_1, p_2, \ldots] = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[h_1, h_2, \ldots].$$

### 4. The centralizer construction for the orthogonal and symplectic Lie algebras

Here we carry over the construction of Section 2 to each of the Lie algebras $\mathfrak{g} = \mathfrak{o}(2\infty), \mathfrak{o}(2\infty + 1), \mathfrak{sp}(2\infty)$. We construct an algebra $A = A(\mathfrak{g})$ and prove that $A$ is isomorphic to the tensor product of the algebra of virtual Laplace operators $Z_c$ and the algebra $Y^\pm(\infty)$. The latter algebra is the inductive limit of the twisted Yangians $Y^\pm(N)$ as $N \to \infty$; see 4.12 for the definition of $Y^\pm(N)$. The key calculations are contained in 4.14 and 4.16, and the result is stated in Theorem 4.17. The original proof of the theorem, as outlined in [O3], Section 5, was simplified thanks to application of the automorphism (4.14.7) of the twisted Yangian, which was introduced in [M4].

#### 4.1.

Fix a number $c \in \mathbb{C}$ and suppose that $m \in \{0, 1, 2, \ldots\}$ in the case of $\mathfrak{g} = \mathfrak{o}(2\infty), \mathfrak{sp}(2\infty)$ and $m \in \{-1, 0, 1, \ldots\}$ in the case of $\mathfrak{g} = \mathfrak{o}(2\infty + 1)$. Assume that $n \geq m$. We shall use the following notation:

$\mathfrak{g}_m(n)$ is the subalgebra in $\mathfrak{g}(n)$ spanned by the elements $F_{ij}$ subject to the condition $m + 1 \leq |i|, |j| \leq n;$
A(n) is the universal enveloping algebra of \( g(n) \);  
Z(n) is the center of \( A(n) \);  
\( A_m(n) \) is the centralizer of \( g_m(n) \) in \( A(n) \); in particular, \( Z(n) \) coincides with \( A_0(n) \) or \( A_{-1}(n) \) in the case of \( g = o(2\infty) \), \( sp(2\infty) \) or \( g = o(2\infty + 1) \), respectively;  
\( G(n) \) is the classical Lie group corresponding to the Lie algebra \( g(n) \), namely \( G(n) = SO(2n), SO(2n + 1), \) and \( Sp(2n) \) for \( g(n) = o(2n), o(2n + 1), \) and \( sp(2n) \), respectively;  
\( G_m(n) \) is the subgroup in \( G(n) \) consisting of matrices stabilizing the basis vectors \( e_i, |i| \leq m \) in the coordinate space and the subspace spanned by the remaining basis vectors;  
\( I(n) \) is the left ideal in \( A(n) \) generated by the elements \( F_{in} + \delta_{ic}, i = -n, \ldots, n \);  
\( J(n) \) is the right ideal in \( A(n) \) generated by the elements \( F_{ni} + \delta_{ic}, i = -n, \ldots, n \);  
\( A(n)^0 \) is the centralizer of \( F_{nn} \) in \( A(n) \) (note that \( A_m(n) \subset A(n)^0, m < n \)).

4.2. Proposition. Let \( L = I(n) \cap A(n)^0 \). Then  
(i) \( L = J(n) \cap A(n)^0 \) and \( L \) is a two-sided ideal of the algebra \( A(n)^0 \);  
(ii) one has the decomposition  
\[
A(n)^0 = L \oplus A(n - 1).  
\]  

Proof. By the Poincaré–Birkhoff–Witt theorem, any element \( a \) of the algebra \( A(n) \) can be uniquely written as a sum of the components of the form  
\[
F_{n,-n+1}^{p_{n-1}} \cdots \cdot F_{n,-n+1}^{p_{n-1}} x_{n+1}^{q_{n-1}} \cdots F_{n,n+1}^{q_{n-1}} (F_{nn} + c)^r, \quad x \in A(n - 1),  
\]  
in the orthogonal case; and  
\[
F_{n,-n}^{p_{n}} \cdots \cdot F_{n,-n}^{p_{n}} x_{n+1}^{q_{n}} \cdots F_{n,n}^{q_{n}} (F_{nn} + c)^r, \quad x \in A(n - 1),  
\]  
in the symplectic case. The condition \( a \in A(n)^0 \) means that \([a, F_{nn}] = 0\). So, if \( x \neq 0 \) then  
\[
p_{n-1} + \cdots + p_{n-1} = q_{n-1} + \cdots + q_{n-1},  
\]  
in the orthogonal case; and  
\[
2p_{-n} + p_{-n+1} + \cdots + p_{n-1} = 2q_{-n} + q_{n-1} + \cdots + q_{n-1},  
\]  
in the symplectic case. Hence the component (2) (respectively, (3)) belongs to \( I(n) \cap J(n) \) if either \( r \neq 0 \) or the sum (4) (respectively, (5)) does not vanish. This proves (i). Furthermore, the component (2) (respectively, (3)) belongs to \( A(n - 1) \) if and only if \( r = 0 \) and the sum (4) (respectively, (5)) vanishes, that is, \( p_i = q_i = 0 \) for all \( i \), which implies (1). \( \square \)

4.3. By Proposition 4.2, the projection \( \pi_{n,c} \) of the algebra \( A(n)^0 \) onto \( A(n - 1) \) with the kernel \( L \) is an algebra homomorphism.
**Proposition.** Let $m < n$. The restriction of $\pi_{n,c}$ to $A_m(n)$ defines an algebra homomorphism

$$\pi_{n,c} : A_m(n) \to A_m(n-1).$$

**Proof.** Indeed, the ideal $I(n)$ is clearly invariant under the adjoint action of $g(n-1)$ and, in particular, under that of the subalgebra $g_m(n-1)$. Therefore the image $\pi_{n,c}(a)$ of any element $a \in A_m(n)$ lies in $A_m(n-1)$. \[\square\]

**4.4.** Consider the sequence of morphisms defined in Proposition 4.3:

$$A_m(m) \xrightarrow{\pi_{m+1,c}} A_m(m+1) \leftarrow \cdots \xrightarrow{\pi_{n,c}} A_m(n) \leftarrow \cdots . \quad (1)$$

It follows from the proof of Proposition 4.3 that these morphisms preserve the canonical filtrations of universal enveloping algebras.

**Definition.** The associative algebra $A_m$ is defined as the projective limit of the sequence (1) of the algebras $A_m(n)$ in the category of filtered algebras.

An element of the algebra $A_m$ is a sequence $a = (a_m, a_{m+1}, \ldots, a_n, \ldots)$, where $a_n \in A_m(n)$ and $\pi_{n,c}(a_n) = a_{n-1}$ for each $n$. Moreover, $\sup \deg a_n < \infty$. Here $\deg a_n$ denote the degree of the element $a_n$ in the universal enveloping algebra $A(n)$. For $a \in A_m$ set

$$\deg a := \sup_{n \geq m} \deg a_n. \quad (2)$$

**4.5.** Note that the morphisms $\pi_{n,c}$ are compatible with the embeddings $A_m(n) \hookrightarrow A_{m+1}(n)$ (cf. 2.5) and so, we can define an embedding $A_m \hookrightarrow A_{m+1}$ as follows:

$$(a_m, a_{m+1}, a_{m+2}, \ldots) \mapsto (a_{m+1}, a_{m+2}, \ldots).$$

**Definition.** The algebra $A$ is the inductive limit of associative filtered algebras: $A = \bigcup_m A_m$.

The canonical embedding of the universal enveloping algebra $A(\infty)$ into $A$ is defined as follows: to an element $x \in A(\infty)$ we associate the sequence $(x, x, \ldots)$. This embedding is well defined since $x$ belongs to some $A(m)$ and hence to $A_m(n)$ for all $n > m$.

**4.6.** The same argument as in 2.6 proves that the center of the algebra $A$ coincides with $A_0$ in the case of $g = \mathfrak{so}(2\infty), \mathfrak{sp}(2\infty)$ and with $A_{-1}$ in the case of $g = \mathfrak{so}(2\infty+1)$.

Note that the morphism $\pi_{n,c} : A_0(n) \to A_0(n-1)$ (respectively, $\pi_{n,c} : A_{-1}(n) \to A_{-1}(n-1)$) defined in Proposition 4.3 coincides with the morphism (3.7.1). Therefore, the algebra $A_0$ (respectively, $A_{-1}$) is naturally isomorphic to the algebra of virtual Laplace operators $Z_c = Z_c(\mathfrak{g})$. 
In the next few subsections we describe the structure of the graded algebras \( \text{gr} A_m \) and \( \text{gr} A \) with respect to the filtration given by (4.4.2).

4.7. We shall use the following notation:

- \( P(n) \) is the symmetric algebra of the space \( g(n) \);
- \( P_m(n) \) is the subalgebra of the elements of \( P(n) \), which are invariant under the adjoint action of the subalgebra \( g_m(n) \);
- \( I'(n) \) is the ideal in \( P(n) \) generated by the elements \( F_{in}, i = -n, \ldots, n \);
- \( J'(n) \) is the ideal in \( P(n) \) generated by the elements \( F_{ni}, i = -n, \ldots, n \).

We can repeat the constructions from 4.1–4.6, replacing \( A_m(n) \) with \( P_m(n) \), \( I(n) \) with \( I'(n) \), and \( J(n) \) with \( J'(n) \), thus obtaining a commutative graded algebra \( P_m \) which is the projective limit of the commutative graded algebras \( P_m(n) \). Then we can define the algebra \( P \) as the inductive limit of the algebras \( P_m \).

Note that the algebra \( P \) contains the symmetric algebra of the Lie algebra \( g(\infty) \), and the algebra \( P_m \) coincides with the subalgebra of \( g_m(\infty) \)-invariants in \( P \).

4.8. Proposition. The canonical isomorphisms \( \text{gr} A(n) \to P(n) \) define isomorphisms \( \text{gr} A_m \to P_m \) and \( \text{gr} A \to P \).

Proof. The proof is the same as that of Proposition 2.8. □

4.9. Let us identify \( P(n) \) with the algebra of polynomial functions in the matrix elements of a matrix \( x = (x_{ij})_{i,j=-n}^n \) such that \( x^t = -x \), where \( (x^t)_{ij} = \theta_{ij} x_{-j,-i} \); see 3.1.

Proposition. Any element \( \phi \) of the algebra \( P_m(n) \) such that \( \deg \phi < n - m \) can be represented as a polynomial in the functions

\[
\begin{align*}
\quad & p^{(M)}_n(x) = \text{tr} (x^M), \quad (1) \\
\quad & p^{(M)}_{ij|n}(x) = (x^M)_{ij} \quad (2)
\end{align*}
\]

with \(|i|, |j| \leq m\).

Proof. Let us introduce submatrices \( a, b, c, d \) of the matrix \( x \) according to the decomposition \( n = m + (n - m) \). Namely, \( a \) and \( d \) are the square matrices whose rows and columns are enumerated by the indices \( i, j \in \{-m, \ldots, m\} \) and \( i, j \notin \{-m, \ldots, m\} \), respectively, while \( b \) corresponds to the subsets of indices \( i \in \{-m, \ldots, m\} \) and \( j \notin \{-m, \ldots, m\} \), and \( c = -b^t \). (In other words, to obtain \( a, b, c, d \) we represent the set of indices \( \{-n, \ldots, n\} \) as the union \( \{-m, \ldots, m\} \cup \{-n, \ldots, -m - 1, m + 1, \ldots, n\} \) and take the corresponding decomposition of the matrix \( x \) in blocks).

We have to prove that any polynomial \( \phi(x) \) of degree \( < n - m \) satisfying the invariance condition

\[ \phi(x) = \phi(a, b, c, d) = \phi(a, bg^{-1}, gc, gdg^{-1}), \quad g \in G_m(n) \]
is a polynomial function of the invariants \( \text{tr}(x^M) \) and \((x^M)_{ij}\); here we regard elements \( g \in \text{G}_m(n) \cong \text{G}(n - m) \) as matrices of the same shape as \( d \). Repeating the arguments of the proof of Proposition 2.9 we see that these invariants can be replaced by invariants of the form \( a_{ij}, \text{tr}(d^M) \) with \( M \geq 1 \), and \((bd^{M-2}c)_{ij}\) with \( M \geq 2 \).

Since \( b = -c^t \) our task is reduced to describing the polynomial invariants \( \phi(c, d) \), where \( c \) lies in the direct sum of several copies of the \text{G}_m(n)-module \( V \) of column vectors of the length \( 2(n - m) \), and \( d \) is an element of the adjoint module \( \text{g}_m(n) \). Any such invariant can be decomposed into a sum of expressions of the form

\[
\psi(c, \ldots, c, d, \ldots, d),
\]

where \( \psi \) is a multilinear invariant. That is, \( \psi \) is a morphism of \( \text{G}_m(n) \)-modules

\[
\psi : V \otimes \cdots \otimes V \otimes \text{g}_m(n) \otimes \cdots \otimes \text{g}_m(n) \to \mathbb{C},
\]

for some \( q \) and \( r \), where we consider \( \mathbb{C} \) as the trivial module. Note that the adjoint module \( \text{g}_m(n) \) is isomorphic to \( \Lambda^2(V) \) in the orthogonal case, and to \( S^2(V) \) in the symplectic case. Let us identify the exterior or symmetric square of the module \( V \) with a submodule in the tensor product \( V \otimes V \). Then any morphism (3) can be obtained as the restriction of a morphism of \( \text{G}_m(n) \)-modules of the form

\[
\chi : V \otimes \cdots \otimes V \otimes V \otimes \cdots \otimes V \to \mathbb{C}.
\]

Consider the invariant nondegenerate bilinear form on the space \( V \) which is preserved by the action of \( \text{G}_m(n) \):

\[
\langle e_i, e_j \rangle = \delta_{i,-j} \quad \text{in the orthogonal case},
\]

\[
\langle e_i, e_j \rangle = \text{sgn } i \cdot \delta_{i,-j} \quad \text{in the symplectic case},
\]

where \( \{e_i\} \) is the canonical basis of \( V \cong \mathbb{C}^{2(n-m)}, i = -n, \ldots, -m-1, m+1, \ldots, n. \)

Using the classical invariant theory for the orthogonal and symplectic groups (see H. Weyl [W], Theorems 2.9.A and 6.1.A), we obtain that the invariant

\[
\chi(c_1, \ldots, c_q, u_1, v_1, \ldots, u_r, v_r), \quad c_i, u_i, v_i \in V
\]

can be represented as a sum of monomials in bilinear invariants \( \langle c_i, c_j \rangle, \langle c_i, u_j \rangle, \langle c_i, v_j \rangle, \langle u_i, u_j \rangle, \langle u_i, v_j \rangle, \langle v_i, v_j \rangle \) such that in each of the monomials each letter appears exactly once. Here we have used the assumption \( \deg \phi < n - m \), which implies that \( q + r < n - m \) and hence \( q + 2r < 2(n - m) \). This has enabled us to eliminate the invariants of the form \( \det [w_1, \ldots, w_{2(n-m)}] \), \( w_i \in V \), which could occur in the orthogonal case. (In the symplectic case this assumption is
not essential). Note that if for some \( i \) we permute the letters \( u_i \) and \( v_i \) in such a monomial, then its restriction will still determine, up to a sign, the same invariant of the form (3). Therefore we may only consider the monomials which are products of submonomials of the form (4).

We shall regard the matrix
\[
\langle v_{s_1}, u_{s_2} \rangle \langle v_{s_2}, u_{s_3} \rangle \cdots \langle v_{s_k}, u_{s_1} \rangle
\]
and
\[
\langle c_\alpha, u_{s_1} \rangle \langle v_{s_1}, u_{s_2} \rangle \cdots \langle v_{s_k}, c_\beta \rangle \quad \text{(in particular, } \langle c_\alpha, c_\beta \rangle).\]

Let us check that the monomials (4) and (5) determine the invariants of the form \( \text{tr} (d^k) \) and \( (c^i d^k c)_{ij} \), respectively.

Consider first the orthogonal case and calculate \( \chi(d) \), where the invariant \( \chi \) is determined by a monomial of the form (4). We can obviously assume that \((s_1, \ldots, s_k) = (1, \ldots, k)\). That is,
\[
\chi : V^{\otimes 2k} \to \mathbb{C}, \quad \chi : u_1 \otimes v_1 \otimes \cdots \otimes u_k \otimes v_k \mapsto \langle v_1, u_2 \rangle \langle v_2, u_3 \rangle \cdots \langle v_k, u_1 \rangle.
\]

We shall regard the matrix \( d \) as an element of \( V \otimes V \) using the vector space isomorphism \( g_m(n) \to \Lambda^2(V) \) defined by the formula \( F_{ij} \mapsto e_i \otimes e_{-j} - e_{-j} \otimes e_i \). We have \( d = \sum d_{ij} F_{ij} = \frac{1}{2} \sum d_{ij} F_{ij} \), because \( d_{ij} = -d_{-j,-i} \). Therefore,
\[
d \otimes \cdots \otimes d = \frac{1}{2^k} \sum_{k} d_{i_1 j_1} \cdots d_{i_k j_k} F_{i_1 j_1} \otimes \cdots \otimes F_{i_k j_k} \mapsto \\
\frac{1}{2^k} \sum d_{i_1 j_1} \cdots d_{i_k j_k} (e_{i_1} \otimes e_{-j_1} - e_{-j_1} \otimes e_{i_1}) \otimes \cdots \otimes (e_{i_k} \otimes e_{-j_k} - e_{-j_k} \otimes e_{i_k}) \\
= \sum d_{i_1 j_1} \cdots d_{i_k j_k} (e_{i_1} \otimes e_{-j_1}) \otimes \cdots \otimes (e_{i_k} \otimes e_{-j_k}).
\]

So, by (6)
\[
\chi(d) = \sum d_{i_1 j_1} \cdots d_{i_k j_k} \langle e_{-j_1}, e_{i_2} \rangle \langle e_{-j_2}, e_{i_3} \rangle \cdots \langle e_{-j_k}, e_{i_1} \rangle \\
= \sum d_{i_1 j_1} \cdots d_{i_k j_k} \delta_{j_1 i_2} \delta_{j_2 i_3} \cdots \delta_{j_k i_1} = \sum d_{i_1 i_2} d_{i_2 i_3} \cdots d_{i_k i_1} = \text{tr} (d^k).
\]

In the symplectic case the vector space isomorphism \( g_m(n) \to S^2(V) \) can be defined by the formula \( F_{ij} \mapsto \text{sgn} j (e_i \otimes e_{-j} + e_{-j} \otimes e_i) \). An analogous calculation shows that here for an invariant \( \chi \) of the form (4) one has \( \chi(d) = (-1)^k \text{tr} (d^k) \).

For the monomials of the form (5) the calculation is quite similar and will be omitted. □

4.10. Since \( x^t = -x \), it follows from (4.9.1) and (4.9.2) that
\[
p_n^{(M)}(x) = (-1)^M p_n^{(M)}(x), \quad p_{ij | n}^{(M)}(x) = (-1)^M \theta_{ij} p_{-j, -i | n}^{(M)}(x).
\]
So, $p_n^{(M)}$ vanishes for odd $M$, whereas for the elements $p_{ij|M}^{(M)}$ we may impose the following restrictions on $i, j, M$:

- in the orthogonal case: $i + j < 0$ for $M$ odd, $i + j \leq 0$ for $M$ even;
- in the symplectic case: $i + j < 0$ for $M$ even, $i + j \leq 0$ for $M$ odd.

Fix $K \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$ in the case of $\mathfrak{g}(n) = \mathfrak{o}(2n)$, $\mathfrak{sp}(2n)$ and $m \in \{-1, 0, 1, \ldots\}$ in the case of $\mathfrak{g}(n) = \mathfrak{o}(2n + 1)$.

**Proposition.** Let $|i|, |j| \leq m$ and $1 \leq M \leq K$. Suppose $n$ is large enough. Then the elements $p_n^{(M)}$ with $M$ even and the elements $p_{ij|M}^{(M)}$, where $i, j, M$ satisfy the above restrictions, are algebraically independent.

**Proof.** We shall only consider the orthogonal case. The proof in the symplectic case can be obtained by an obvious adjustment.

For any triple $(i, j, M)$ satisfying our assumptions we choose a subset

$$\Omega_{ijM} \subset \{m + 1, m + 2, \ldots\}$$

of cardinality $M - 1$ in such a way that all these subsets be disjoint. Let $n$ be so large that all of them belong to $\{m + 1, m + 2, \ldots, n - K\}$. Introduce complex parameters

$$y_1, \ldots, y_K \quad \text{and} \quad z_{ijM},$$

where $|i|, |j| \leq m$ and $M = 1, \ldots, K$.

Let us now define a linear operator $x_{ijM}$ in $\mathbb{C}^N$ depending on $z_{ijM}$ as follows. Let $e_{-n}, \ldots, e_n$ be the canonical basis in $\mathbb{C}^N$ and $a_1 < \cdots < a_{M-1}$ be all the elements of $\Omega_{ijM}$. Then for $i + j < 0$

$$x_{ijM} : \quad e_j \mapsto z_{ijM}e_{a_{M-1}}, \quad e_{a_{M-1}} \mapsto e_{a_{M-2}}, \ldots, \quad e_{a_1} \mapsto e_i, \quad e_{-i} \mapsto -e_{-a_1}, \quad e_{-a_1} \mapsto -e_{-a_2}, \ldots, \quad e_{-a_{M-1}} \mapsto -z_{ijM}e_j,$$

$$e_k \mapsto 0 \quad \text{for} \quad k \notin \{-i\} \cup \{j\} \cup \pm \Omega_{ijM};$$

while for $i + j = 0$

$$x_{i,-i,M} : \quad e_{-i} \mapsto z_{i,-i,M}e_{a_{M-1}} - \frac{1}{2}e_{-a_1}, \quad e_{a_{M-1}} \mapsto e_{a_{M-2}}, \ldots, \quad e_{a_1} \mapsto \frac{1}{2}e_i, \quad e_{-a_1} \mapsto -e_{-a_2}, \ldots, \quad e_{-a_{M-1}} \mapsto -z_{i,-i,M}e_i,$$

$$e_k \mapsto 0 \quad \text{for} \quad k \notin \{\pm i\} \cup \pm \Omega_{i,-i,M}.$$

Note that $(x_{ijM})^t = -x_{ijM}$. Moreover, for the $(i, -i)$-matrix element of the $M$th power of the operator $x_{i,-i,M}$ we have

$$((x_{i,-i,M})^M)_{i,-i} = \begin{cases} \frac{1}{2}z_{i,-i,M}, & \text{if } M \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Now define a linear operator $x$ in $\mathbb{C}^N$ depending on all the variables by setting
\[ xe_a = \begin{cases} \sum_{i,j,M} x_{ij} M e_a, & \text{for } |a| \leq n - K, \\ \text{sgn} a \cdot y_{|a|-(n-K)} e_a, & \text{for } n - K < |a| \leq n. \end{cases} \] (1)

Then for any matrix \( x \) of the form (1) we have

\[ p_n^{(M)}(x) = 2(y_1^M + \cdots + y_K^M) + \phi(\ldots, z_{abL}, \ldots); \]
\[ p_{ij|n}^{(M)}(x) = z_{ij} M + \psi(\ldots, z_{abL}, \ldots), \quad L < M; \]

where the indices \( i, j, M \) satisfy the assumptions of the proposition (\( \phi \) and \( \psi \) do not depend on \( y_1, \ldots, y_K \)). Thus our polynomials are algebraically independent even if they are restricted to the affine subspace of matrices of the form (1). \( \square \)

4.11. It follows from (4.9.1) and (4.9.2) that

\[ p_n^{(M)} \in p_n^{(M)} + I'(n), \quad p_{ij|n}^{(M)} \in p_{ij|n-1}^{(M)} + I'(n); \]

cf. 2.11. Hence, the sequence

\[ p^{(M)} := (p_n^{(M)} | n \geq 0) \]

is a well-defined element of the algebra \( P_0 \) in the case of \( g(n) = o(2n) \), \( sp(2n) \) and of the algebra \( P_{-1} \) in the case of \( g(n) = o(2n+1) \), while the sequence

\[ p_{ij}^{(M)} := (p_{ij|n}^{(M)} | n \geq m) \]

is a well-defined element of the algebra \( P_m \).

**Proposition.** The following elements are algebraically independent generators of the commutative algebra \( P_m \) (below \( |i|, |j| \leq m \)).

(i) In the orthogonal case:

\[ p^{(M)}, \quad \text{where } \ M = 2, 4, 6, \ldots, \]
\[ p_{ij}^{(M)}, \quad \text{where } \ i + j < 0 \quad \text{for } \ M \text{ odd}, \quad \text{and } \ i + j \leq 0 \quad \text{for } \ M \text{ even}. \]

(ii) In the symplectic case:

\[ p^{(M)}, \quad \text{where } \ M = 2, 4, 6, \ldots, \]
\[ p_{ij}^{(M)}, \quad \text{where } \ i + j < 0 \quad \text{for } \ M \text{ even}, \quad \text{and } \ i + j \leq 0 \quad \text{for } \ M \text{ odd}. \]

Dropping the restriction \( |i|, |j| \leq m \), we obtain algebraically independent generators of the commutative algebra \( P \).

**Proof.** Let \( p = (p_n | n \geq m) \in P_m \). By Proposition 4.9, for any \( n \) such that \( \deg p < n - m \) the element \( p_n \in P_m(n) \) can be represented as a polynomial \( \phi_n \) in
the variables $p_n^{(M)}$, $p_{ij}^{(M)}$ with $|i|, |j| \leq m$, $M \leq \deg p$. By Proposition 4.10, $\phi_n$ does not depend on $n$ for sufficiently large $n$. Hence, the elements $p^{(M)}$ and $p_{ij}^{(M)}$ are generators. Their algebraic independence is clear from Proposition 4.10. □

Next few subsections contain preliminaries for the proof of the main theorem of this section, Theorem 4.17.

4.12. Let us recall the definition of the twisted Yangian (see [MNO], Section 3 for further details). This is an associative algebra naturally associated with the orthogonal or symplectic Lie algebra and denoted by $Y^+(2n)$, $Y^+(2n + 1)$, and $Y^-(2n)$ for the cases of $\mathfrak{o}(2n)$, $\mathfrak{o}(2n + 1)$, and $\mathfrak{sp}(2n)$, respectively. We shall consider all the three cases simultaneously. Consider the Yangian $Y(N)$ (see 2.14), where we assume that the generators $t_{ij}^{(M)}$ are enumerated by the indices $i, j$ running through the set $\{-n, \ldots, -1, 0, 1, \ldots n\}$ if $N = 2n + 1$ and through the set $\{-n, \ldots, -1, 1, \ldots n\}$ if $N = 2n$.

Let us introduce the $S$-matrix $S(u) = (s_{ij}(u))$ by setting $S(u) := T(u)T^t(-u)$, or, in terms of matrix elements,

$$s_{ij}(u) = \sum_a \theta_{aj}t_{ia}(u)t_{-j,-a}(-u).$$

Write

$$s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)}u^{-1} + s_{ij}^{(2)}u^{-2} + \cdots.$$

The twisted Yangian $Y^\pm(N)$ is the subalgebra of $Y(N)$ generated by the elements $s_{ij}^{(1)}, s_{ij}^{(2)}, \ldots$, where $-n \leq i, j \leq n$.

One can show that the $S$-matrix satisfies the ‘quaternary relation’ (0.3) and the ‘symmetry relation’ (0.4); see [MNO], Section 3 for the proof. They can be rewritten in terms of the generating series $s_{ij}(u)$ as follows:

$$[s_{ij}(u), s_{kl}(v)] = \frac{1}{u-v}(s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u))$$

$$- \frac{1}{u+v}(\theta_{k,-j}s_{i,-k}(u)s_{-j,l}(v) - \theta_{i,-l}s_{k,-i}(v)s_{-l,j}(u)) 
+ \frac{1}{u^2-v^2}(\theta_{i,-j}s_{k,-i}(u)s_{-j,l}(v) - \theta_{i,-j}s_{k,-i}(v)s_{-j,l}(u))$$

and

$$\theta_{ij}s_{-j,-i}(-u) = s_{ij}(u) \pm \frac{s_{ij}(u) - s_{ij}(-u)}{2u}.$$ 

These are defining relations for the twisted Yangian $Y^\pm(N)$, so that one can regard $Y^\pm(N)$ either as a subalgebra in $Y(N)$ or as an abstract algebra with the generators $s_{ij}^{(M)}$ and the relations (1) and (2).
There is an analog of the quantum determinant for the twisted Yangians. It is denoted by \( \text{sdet} S(u) \) and is called the Sklyanin determinant. This is a formal series in \( u^{-1} \) and its coefficients generate the center of the algebra \( Y^\pm(N) \). The Sklyanin determinant is related to the quantum determinant by the formula:

\[
\text{sdet} S(u) = \gamma_n(u) \text{qdet} T(u) \text{qdet} T(-u + N - 1),
\]

where

\[
\gamma_n(u) = \begin{cases} 
1 & \text{for } Y^+(N), \\
\frac{2u + 1}{2u - 2n + 1} & \text{for } Y^-(2n).
\end{cases}
\] (3)

An explicit determinant-type expression analogous to (2.14.5) for \( \text{sdet} S(u) \) in terms of the generators \( s_{ij}(u) \) was given in [M2].

4.13. Denote by \( F \) the \( N \times N \)-matrix formed by the elements \( F_{ij}, -n \leq i, j \leq n, \)

\[
F = \sum_{i,j} F_{ij} \otimes E_{ij} \in A(n) \otimes \text{End} (\mathbb{C}^N).
\]

It was proved in [MNO], Proposition 3.11 that the mapping

\[
\eta : S(u) \mapsto 1 + \frac{F}{u \pm 1/2}
\] (1)

defines an algebra homomorphism \( Y^\pm(N) \to A(n) \). This implies that the coefficients of the series \( \eta(\text{sdet} S(u)) \) belong to the center \( Z(n) \) of the algebra \( A(n) \). The images of these coefficients under the Harish-Chandra isomorphism \( \omega : Z(n) \to M^*(n) \) (see 1.8) were found in different ways in [M2], Section 5 and [MN], Section 6. The result can be written as follows:

\[
\omega : \gamma_n(u) \eta(\text{sdet} S(-u + \frac{N}{2} - 1)) \mapsto \prod_{i=1}^{n} \frac{(u + 1/2)^2 - l_i^2}{(u + 1/2)^2 - \rho_i^2},
\] (2)

where \( \gamma_n(u) \) is defined in (4.12.3).

4.14. Introduce the following series in \( u^{-1} \) with coefficients in \( M^*(n) \):

\[
\chi_n(u) = \frac{u + \rho_1 - c + 1/2}{u + \rho_1 + 1/2} \prod_{i=1}^{n} \frac{(u + 1/2)^2 - l_i^2}{(u + 1/2)^2 - (\rho_i - c)^2}.
\] (1)

Note that \((\rho_i - c)^2\) in the denominator is equal to \((l_i^c)^2\). Therefore, using definition (3.7.1) we immediately obtain that

\[
\pi_{n,c} : \chi_n(u) \mapsto \chi_{n-1}(u),
\] (2)

and so, the sequence \( \chi(u) = (\chi_n(u)) \) is an element of \( Z_c[[u^{-1}]] \). This fact will be used later.

In the next proposition we regard \( \chi_n(u) \) as an element of \( Z(n)[[u^{-1}]] \) identifying \( M^*(n) \) with \( Z(n) \) via the Harish-Chandra isomorphism \( \omega \). Set

\[
\kappa_n := \frac{N \mp 1}{2}.
\] (3)
Proposition. The mapping
\[ \varphi_n : S(u) \mapsto \frac{u + c + \kappa_n}{u + \kappa_n} \chi_n(u) \left(1 - \frac{F}{u + \kappa_n}\right)^{-1} \]  \hspace{1cm} (4)
defines an algebra homomorphism
\[ \varphi_n : Y^\pm(N) \to A(n). \]  \hspace{1cm} (5)

Proof. Define the matrix \( \hat{S}(u) \) by the following formula:
\[ \text{sdet } S(u) = \hat{S}(u)S(u - N + 1). \]  \hspace{1cm} (6)
It was proved in [M4], Proposition 1.1 that the mapping
\[ S(u) \mapsto \gamma_n(u) \hat{S}(-u + \frac{N}{2} - 1) \]  \hspace{1cm} (7)
defines an automorphism of the algebra \( Y^\pm(N) \). Furthermore, it is obvious from the defining relations (4.12.1) and (4.12.2) that any invertible even formal series \( g(u) \in \mathbb{C}[u^{-2}] \) also defines an automorphism of the algebra \( Y^\pm(N) \) given by \( S(u) \mapsto g(u)S(u) \).

Fix the following series
\[ g_n(u) = \prod_{i=1}^{n-1} \left(1 - \frac{(\rho_{i+1} - 1/2)^2}{u^2 - (\rho_i - c - 1/2)^2}\right). \]
Taking the composition of these automorphisms with the homomorphism \( \eta \) (see (4.13.1)) we obtain another algebra homomorphism \( Y^\pm(N) \to A(n) \) such that
\[ S(u) \mapsto g_n(u) \gamma_n(u) \eta(\hat{S}(-u + \frac{N}{2} - 1)). \]
Let us check that this homomorphism coincides with \( \varphi_n \). Indeed, by definition (6),
\[ \hat{S}(-u + \frac{N}{2} - 1) = \text{sdet } S(-u + \frac{N}{2} - 1) \left(S(-u - \frac{N}{2})\right)^{-1}. \]
We have
\[ \eta : \left(S(-u - \frac{N}{2})\right)^{-1} \mapsto \left(1 - \frac{F}{u + \kappa_n}\right)^{-1}. \]
A direct calculation with the use of (4.13.2) and the relations \( \rho_{i+1} = \rho_i - 1, \kappa_n = -\rho_n + 1/2 \) shows that
\[ g_n(u) \gamma_n(u) \eta(\text{sdet } S(-u + \frac{N}{2} - 1)) = \frac{u + c + \kappa_n}{u + \kappa_n} \chi_n(u). \]  \hspace{1cm} \( \square \)

4.15. If follows from the defining relations (4.12.1), (4.12.2) and the Poincaré–Birkhoff–Witt theorem for the twisted Yangian (see [MNO], Remark 3.14) that for any \( N \geq 3 \) one has natural inclusions
\[ Y^\pm(N - 2) \hookrightarrow Y^\pm(N). \]  \hspace{1cm} (1)
Assume that \( 0 \leq m \leq n \) in the case of \( \mathfrak{g} = \mathfrak{o}(2n), \mathfrak{sp}(2n) \), and \( -1 \leq m \leq n \) in the case of \( \mathfrak{g} = \mathfrak{o}(2n + 1) \) and set \( M = 2m \) and \( M = 2m + 1 \), respectively. Then using (1) we can regard \( Y^\pm(M) \) as a subalgebra in \( Y^\pm(N) \).
Proposition. The image of the restriction of the homomorphism \( \varphi_n \) to the subalgebra \( Y^\pm(M) \) is contained in the centralizer \( A_m(n) \).

Proof. It follows from (4.12.1) that
\[
[s_{kl}^{(1)} , s_{ij}(u)] = \delta_{il}s_{kj}(u) - \delta_{kj}s_{il}(u) - \theta_{i,-l}\delta_{k,-i}s_{-l,j}(u) + \theta_{k,-j}\delta_{-j,i}s_{i,-k}(u). \tag{2}
\]
In particular,
\[
[s_{kl}^{(1)} , s_{ij}(u)] = 0 \quad \text{for} \quad |i|, |j| \leq m < |k|, |l|. \tag{3}
\]
On the other hand, it is easy to verify that the image of \( s_{kl}^{(1)} \) under the homomorphism \( \varphi_n \) coincides with \( F_{kl} \). So, (3) implies that \([F_{kl} , \varphi_n(s_{ij}(u))] = 0. \Box\)

4.16. Proposition. The sequence of homomorphisms \( (\varphi_n) \ n \geq m \) defines a homomorphism
\[
\varphi : Y^\pm(M) \to A_m. \tag{1}
\]

Proof. We have to verify that the homomorphisms \( \varphi_n \) are compatible with the sequence of morphisms (4.4.1), that is, the following diagram is commutative:
\[
\begin{array}{c}
Y^\pm(M) \quad Y^\pm(M) \quad \cdots \quad Y^\pm(M) \quad \cdots \\
\varphi_m \downarrow \quad \varphi_{m+1} \downarrow \quad \cdots \quad \varphi_n \downarrow \\
A_m(m) \quad A_m(m+1) \quad \cdots \quad A_m(n) \quad \cdots \\
\end{array}
\]

Let us set
\[
\mathcal{S}(u) = \frac{u + c + \kappa_n}{u + \kappa_n} \left(1 - \frac{F}{u + \kappa_n}\right)^{-1} \tag{2}
\]
and denote the matrix elements of \( \mathcal{S}(u) \) by \( \sigma_{ij|n}(u) \). We need to prove that
\[
\chi_n(u)\sigma_{ij|n}(u) - \chi_{n-1}(u)\sigma_{ij|n-1}(u) \in I(n)[[u^{-1}]], \quad |i|, |j| \leq n - 1. \tag{3}
\]
However, by (4.14.2)
\[
\chi_n(u) - \chi_{n-1}(u) \in I(n)[[u^{-1}]]. \tag{4}
\]
Hence, since the coefficients of \( \chi_n(u) \) belong to \( M^*(n) \simeq Z(n) \), to prove (3) we only need to show that
\[
\sigma_{ij|n}(u) - \sigma_{ij|n-1}(u) \in I(n)[[u^{-1}]], \quad |i|, |j| \leq n - 1. \tag{5}
\]
We shall prove by induction on \( k \) that for the coefficients of the series \( \sigma_{ij|n}(u) \) one has:
\[
\begin{align*}
\text{(i)} & \quad \sigma_{in|n}^{(k)} \in I(n), \quad -n \leq i \leq n, \quad k \geq 1; \\
\text{(ii)} & \quad \sigma_{ij|n}^{(k)} - \sigma_{ij|n-1}^{(k)} \in I(n), \quad -n + 1 \leq i, j \leq n - 1, \quad k \geq 1. \tag{6}
\end{align*}
\]
By definition of $\mathfrak{S}(u)$,

$$\mathfrak{S}(u)(u + \kappa_n - F) = u + c + \kappa_n.$$ 

Hence

$$u \mathfrak{S}(u) = u + c + \kappa_n + \mathfrak{S}(u)(F - \kappa_n).$$  

(7)

Write $\mathfrak{S}(u) = \mathfrak{S}(0) + \mathfrak{S}(1)u^{-1} + \ldots$. Then (7) implies that $\mathfrak{S}(0) = 1$ and

$$\mathfrak{S}(1) = F + c,$$  

(8)

$$\mathfrak{S}(k) = \mathfrak{S}(k-1)(F - \kappa_n), \quad k \geq 2.$$  

(9)

By (8) we have

$$\sigma_{in|n}^{(1)} = F_{in} + \delta_{in}c \in I(n), \quad -n \leq i \leq n,$$

and

$$\sigma_{ij|n}^{(1)} - \sigma_{ij|n-1}^{(1)} = 0, \quad -n + 1 \leq i, j \leq n - 1.$$

So, we have verified (6) for $k = 1$. For $k > 1$ we obtain from (9) that

$$\sigma_{in|n}^{(k)} = \sum_{a=-n}^{n} \sigma_{ia|n}^{(k-1)}(F_{an} - \delta_{an}\kappa_n)$$

$$= \sum_{a=-n}^{n-1} \sigma_{ia|n}^{(k-1)} F_{an} + \sigma_{in|n}^{(k-1)}(F_{nn} + c) - (c + \kappa_n)\sigma_{in|n}^{(k-1)}.$$

By the induction hypotheses, this expression lies in $I(n)$, which proves (i) in (6). Again using (9) we obtain for $-n + 1 \leq i, j \leq n - 1$ that

$$\sigma_{ij|n}^{(k)} - \sigma_{ij|n-1}^{(k)} = \sum_{a=-n}^{n} \sigma_{ia|n}^{(k-1)}(F_{aj} - \delta_{aj}\kappa_n) - \sum_{a=-n+1}^{n-1} \sigma_{ia|n-1}^{(k-1)}(F_{aj} - \delta_{aj}\kappa_{n-1}),$$

which can be rewritten as

$$\sigma_{ij|n}^{(k)} - \sigma_{ij|n-1}^{(k)} = \sum_{a=-n+1}^{n-1} (\sigma_{ia|n}^{(k-1)} - \sigma_{ia|n-1}^{(k-1)})F_{aj}$$

$$- \kappa_n \sigma_{ij|n}^{(k-1)} + \kappa_{n-1} \sigma_{ij|n-1}^{(k-1)} + \sigma_{in|n}^{(k-1)} F_{nj}$$

$$+ \sigma_{i,-n|n}^{(k-1)} F_{-n,j}.$$  

(10)

(11)

(12)

Let us consider the summands on the right hand side separately. By Proposition 4.14 the relations (4.12.1) are satisfied by the matrix elements of the matrix
\( \chi_n(u) \mathcal{G}(u). \) Then this is also true for the matrix elements of \( \mathcal{G}(u) \) because the coefficients of \( \chi_n(u) \) are central in \( A(n). \) So, using (4.15.2) and (8) we obtain for a summand in (10):

\[
(s^{(k-1)}_{ia|n} - s^{(k-1)}_{ia|n-1}) F_{aj} = F_{aj}(s^{(k-1)}_{ia|n} - s^{(k-1)}_{ia|n-1}) + s^{(k-1)}_{ij|n} - s^{(k-1)}_{ij|n-1} - \delta_{ij} (s^{(k-1)}_{aa|n} - s^{(k-1)}_{aa|n-1}) - \theta_{a,-a} \delta_{ij} \sigma^{(k-1)}_{a,-a,j|n-1} + \theta_{i,-j} \delta_{i,j} \sigma^{(k-1)}_{a,-i|n-1}.
\]

Note that \( \kappa_n - \kappa_{n-1} = 1. \) Hence, for (11) we have:

\[
-\kappa_n s^{(k-1)}_{ij|n} + \kappa_{n-1} s^{(k-1)}_{ij|n-1} + s^{(k-1)}_{in|n} F_{nj}
\]

\[
= -\kappa_{n-1} (s^{(k-1)}_{ij|n} - s^{(k-1)}_{ij|n-1}) + s^{(k-1)}_{in|n} F_{nj} - s^{(k-1)}_{ij|n}
\]

\[
= -\kappa_{n-1} (s^{(k-1)}_{ij|n} - s^{(k-1)}_{ij|n-1}) + s^{(k-1)}_{in|n} F_{nj} - \delta_{ij} s^{(k-1)}_{nn|n}.
\]

Finally, (12) looks as

\[
s^{(k-1)}_{i,-n|n} F_{-n,j} = -\theta_{-j,n} s^{(k-1)}_{i,-n|n} F_{-j,n}.
\]

Using (i), we find by the induction hypotheses that all the elements (10), (11) and (12) belong to the ideal \( I(n), \) which completes the proof of (6). \( \square \)

### 4.17

The following theorem is our main result. We consider the homomorphism \( \varphi = (\varphi_n| n \geq m) \) defined in Proposition 4.16. We use notation (4.14.1), (4.14.3), and (4.14.4).

**Theorem.** The homomorphism \( \varphi \) is an embedding of the twisted Yangian \( Y^\pm(M) \)

\( \text{into the algebra } A_m. \) Moreover, one has the decomposition

\[
A_m = \mathbb{Z}_c \otimes Y^\pm(M),
\]

where the twisted Yangian is identified with its image under the embedding \( \varphi, \) and \( \mathbb{Z}_c = \mathbb{Z}_c(\mathfrak{g}) \) is the algebra of virtual Laplace operators.

**Proof.** To prove the first claim of the theorem we have to verify that the kernel of the homomorphism \( \varphi \) is trivial. Recall that \( A_m \) is a filtered algebra with \( \text{gr } A_m = \mathbb{P}_m; \) see Proposition 4.8. Let us find the highest order term \( \varphi(s^{(k)}_{ij}) \) of the sequence \( \varphi(s^{(k)}_{ij}) = (\varphi_n(s^{(k)}_{ij})| n \geq m). \) By definition,

\[
\varphi_n(s_{ij}(u)) = \frac{u + c + \kappa_n}{u + \kappa_n} \chi_n(u) \left(1 - \frac{F}{u + \kappa_n}\right)^{-1}_{ij},
\]

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where \( \chi_n(u) \) is defined by formula (4.14.1). Denote by \( \varphi_n(s_{ij}^{(k)}) \) and \( \chi_n^{(k)} \) the images of \( \varphi_n(s_{ij}^{(k)}) \) and \( \chi_n^{(k)} \) in the \( k \)th component of \( \text{gr} \, A(n) = P(n) \). Then by (2),

\[
\varphi_n(s_{ij}^{(k)}) = (F^k)_{ij} + (F^{k-1})_{ij} \chi_n^{(1)} + \cdots + \chi_n^{(k)} = p_{ij|n}^{(k)} + p_{ij|n}^{(k-1)} \chi_n^{(1)} + \cdots + \chi_n^{(k)}.
\]

Hence,

\[
\varphi(s_{ij}^{(k)}) = p_{ij}^{(k)} + p_{ij}^{(k-1)} \chi^{(1)} + \cdots + \chi^{(k)},
\]

where \( \chi^{(k)} \) denotes the highest order term of the sequence \( \chi^{(k)} = (\chi_n^{(k)}) \). The sequence \( \chi(u) = (\chi_n(u)) \) is an element of \( Z_c[[u^{-1}]] \); see (4.14.2). So, \( \chi^{(k)} \) belongs to the algebra \( P_0 \) or \( P_{-1} \) depending on whether \( g(n) = \mathfrak{o}(2n) \), \( \mathfrak{sp}(2n) \) or \( \mathfrak{o}(2n + 1) \). Proposition 4.11 implies that the elements of the algebra \( P_m 

\begin{align*}
p_{ij}^{(2k)}, & \quad i + j \leq 0; \\
p_{ij}^{(2k-1)}, & \quad i + j < 0; \\
k = 1, 2, \ldots,
\end{align*}

in the orthogonal case, and

\begin{align*}
p_{ij}^{(2k)}, & \quad i + j < 0; \\
p_{ij}^{(2k-1)}, & \quad i + j \leq 0; \\
k = 1, 2, \ldots,
\end{align*}

in the symplectic case, are algebraically independent over the subalgebra \( P_0 \) or \( P_{-1} \). So, by (3), the same is true for the elements

\begin{align*}
\varphi(s_{ij}^{(2k)}), & \quad i + j \leq 0; \\
\varphi(s_{ij}^{(2k-1)}), & \quad i + j < 0; \\
k = 1, 2, \ldots,
\end{align*}

in the orthogonal case, and

\begin{align*}
\varphi(s_{ij}^{(2k)}), & \quad i + j < 0; \\
\varphi(s_{ij}^{(2k-1)}), & \quad i + j \leq 0; \\
k = 1, 2, \ldots,
\end{align*}

in the symplectic case. Now, the Poincaré–Birkhoff–Witt theorem for \( Y^\pm(M) \) (see [MNO], Remark 3.14) implies that the kernel of \( \varphi \) is trivial.

To prove the decomposition (1) it suffices to note that the graded algebra \( \text{gr} \, Y^\pm(M) \) can be identified with the subalgebra of \( \text{gr} \, A_m = P_m \) generated by the elements \( \varphi(s_{ij}^{(k)}), \ k = 1, 2, \ldots \). \( \square \)

4.18. Using the inclusions (4.15.1) we can define the algebra \( Y^\pm(\infty) \) as the corresponding inductive limit as \( N \to \infty \). Theorem 4.17 immediately implies the following corollary.

**Corollary.** One has the isomorphism \( A = Z_c \otimes Y^\pm(\infty) \). \( \square \)

The following are analogs of Theorem 2.21 and Proposition 2.22 for the Lie algebra \( \mathfrak{g} = \mathfrak{o}(2\infty), \mathfrak{sp}(2\infty), \mathfrak{o}(2\infty + 1) \). They are proved by the same argument.
4.19. **Theorem.** Any $g$-module $V \in \Omega(c)$ has a natural $A$-module structure. □

4.20. **Proposition.** Let $V$ be a $g$-module with the highest weight $\lambda$. Then every element $a \in A_0$ in the case of $g = o(2\infty)$, $sp(2\infty)$, and $a \in A_{-1}$ in the case of $g = o(2\infty+1)$ acts on $V$ as the scalar operator $f_a(\lambda) \cdot 1$, where $f_a \in M^*_c$ corresponds to $a$ under the identification of $A_0$ or $A_{-1}$ with $M^*_c$ from (3.8). □

5. **Comments**

5.1. At least, two aspects of the centralizer construction seem to be nontrivial and rather surprising:

- first, the existence of the projections $A_m(n) \to A_m(n-1)$ which makes it possible to arrange the centralizer algebras into a projective chain (2.3 and 4.3 above);
- second, the fact that the Yangians $Y(m), Y^\pm(M)$ appear in the description of the limit centralizer algebra $A_m$. Indeed, the Yangians are certain quantized (or deformed) enveloping algebras, while in the definition of $A_m$ there is no indication on deformations!

An explanation of this phenomenon is that the commutation relations for the classical Lie algebras can be expressed in an $R$-matrix form (which, for the $B,C,D$ cases, involves a reflection equation), see [MNO].

5.2. Thus, the centralizer construction shows that the Yangians $Y(m)$ and $Y^\pm(M)$ are deeply connected with the classical Lie algebras. One could even say that these Yangians are implicitly contained in the enveloping algebras of the infinite rank Lie algebras of type $A, B, C, D$. Note that unlike the twisted Yangians $Y^\pm(M)$, Drinfeld’s Yangians of type $B, C, D$ are not related to the corresponding Lie algebras in this way: they cannot be projected onto the universal enveloping algebras (see Drinfeld [D1]).

5.3. We believe that the centralizer construction can be applied to certain Lie superalgebras. (About the Yangians corresponding to the ‘strange’ Lie superalgebras see Nazarov [N1], [N2]). There is a version of the centralizer construction for the symmetric groups; see [O1].

5.4. As was mentioned in the Introduction, irreducible finite-dimensional representations of the centralizer algebra $A_m(n)$ can be lifted to the corresponding Yangian $Y(m)$ or $Y^\pm(M)$. It would be interesting to further study this construction. For instance, to understand, especially for the $B, C, D$ case, what representations appear as result of such a lifting.

5.5. In the present paper we have substantially exploited certain stability effects. For instance, our main result can be viewed as a description of the “stable structure” (or the “stable part” of the defining relations) for the centralizer algebras.
A somewhat different kind of stabilization is employed in the definition of the category $\Omega(c)$ of modules over the infinite rank classical Lie algebra $\mathfrak{g}(\infty) = \operatorname{inj}\lim\mathfrak{g}(n)$ of type $B, C, D$. Note that any irreducible finite-dimensional $\mathfrak{g}(n)$-module can be viewed as a fragment of an irreducible $\mathfrak{g}(\infty)$-module which belongs to $\Omega(c)$ with an appropriate choice of the parameter $c$. (Note also that the same fact holds for the doubly infinite version of $\mathfrak{gl}$, see 2.23). This stability effect can be related to R. Howe’s theory of reductive dual pairs [H]; especially, its fermionic part. It would be interesting to develop this idea and relate it to other stability effects in representation theory; see, e.g., Benkart–Britten–Lemire [BBL], Brylinski [B], Stanley [S1].

5.6. In [O2], Remark 2.1.20 one can find a description of a set of generators for the virtual center $Z_c$ (case of $\mathfrak{gl}(\infty)$, $c = 0$; see 1.10–1.11 above) together with a description of their images under the isomorphism $Z_c = \Lambda_c^*$. A somewhat different construction is proposed in Gould–Stoilova [GS].

For the $B, C, D$ case, using an appropriate modification of the ideas of [OO2], one can construct a linear basis in the algebra $M_c^*$, introduced in 3.8.

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