Giant Magnons in $AdS_4 \times CP^3$: Embeddings, Charges and a Hamiltonian

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This paper studies giant magnons in $CP^3$, which in all known cases are old solutions from $S^5$ placed into two- and three-dimensional subspaces of $CP^3$, namely $CP^1$, $RP^2$ and $RP^3$. We clarify some points about these subspaces, and other potentially interesting three- and four-dimensional subspaces. After confirming that $\Delta - (J_1 - J_4)/2$ is a Hamiltonian for small fluctuations of the relevant ‘vacuum’ point particle solution, we use it to calculate the dispersion relation of each of the inequivalent giant magnons. We comment on the embedding of finite-$J$ solutions, and use these to compare string solutions to giant magnons in the algebraic curve.

1 Introduction

Classical string solutions in $AdS_5 \times S^5$ have played an important role in the study of the duality to $\mathcal{N} = 4$ SYM. [1–3] It seems that this pattern is being repeated in the new $\mathcal{N} = 6$ duality [4], in which planar superconformal Chern–Simons theory is dual to string theory on $AdS_4 \times CP^3$.

Some of the most interesting recent papers study strings moving in an $AdS_2 \times S^1$ subspace, where although the classical solutions are identical to those long used in the $\mathcal{N} = 4$ case, the quantum properties are different. The results from semiclassical quantisation [5–8] can be compared to those from the asymptotic Bethe ansatz, and at present there appear to be some difficulties. [9]

This paper is instead about string solutions exploring primarily the $CP^3$ factor. One would expect to find analogues of the giant magnons [3] here, which in the $\mathcal{N} = 4$ case live in an $S^2 \subset S^5$. And indeed, it turns out that the same solutions exist in $CP^3$. [10,11] There are two inequivalent ways to embed the basic $S^2$ magnon, into either $CP^1 = S^2$ or $RP^2 = S^2/\mathbb{Z}_2$, [10] both two-dimensional subspaces of $CP^3$.

In either theory, the anomalous dimension can be calculated as the Hamiltonian of some spin chain. [12–14,10] The giant magnons are dual to the elementary excitations of this spin chain, and have a periodic dispersion relation $\Delta - J = \sqrt{1 + f^2(\lambda) \sin^2(p/2)}$ which on the gauge side is an symptom of the discrete spatial dimension of the spin chain, and on the string side
arises from $p$ being an angle along an equator. The conformal dimension $\Delta$ and the R-charge $J$ are mapped by AdS/CFT to energy and angular momentum of the string state. For the state dual to the (ferromagnetic) vacuum of the spin chain, which is a point particle, $\Delta$ are mapped by AdS/CFT to energy and angular momentum of the string state. For the state arises from $p$ conjectured to be true for all $\lambda$ is $h$ instead)

An important difference between the old $\mathcal{N} = 4$ case and the new $\mathcal{N} = 6$ case is the behaviour of the function $f(\lambda)$, the only part of the dispersion relation not fixed by supersymmetry. [15, 3] In the old case, calculations of $f(\lambda)$ at both large and small $\lambda$ give $f(\lambda) = \sqrt{\lambda}/\pi$, and this is conjectured to be true for all $\lambda$. In the new case, however, the function (often called $h$ instead) is $h(\lambda) = \lambda$ at small $\lambda$ but $h(\lambda) \sim \lambda^{1/2}$ at large $\lambda$. Our knowledge of this function at large $\lambda$ comes (in both cases) from studying classical string theory, and so depends on the correct identification of the relevant string solutions.

Dyonic giant magnons are those with more than one large angular momentum, dual to a large condensate of impurities on the spin chain. These are string solutions in $S^3$, and they can at least sometimes be embedded into $CP^3$ in much the same way as the basic magnon, generalising the $RP^2$ magnons and living in an $RP^3$ subspace. [16, 17] There is room for dyonic solutions with other angular momenta, truly exploring $CP^3$, including those generalising the $CP^1$ magnon. While we have not been able to find such solutions, we discuss where they might live. The subspace frequently called $S^2 \times S^2$ in the literature is in fact just $RP^2$, and while there is a genuine $S^2 \times S^2$ subspace, one cannot place arbitrary $S^2$ string solutions into each factor, because the equations of motion couple the two factors. Likewise the $S^2 \times S^1$ subspace studied by [17] has extra constraints limiting what solutions can exist there.

Contents

In section 2 we write down a few relevant facts about ABJM theory and its spin-chain description, and in section 3 we look at its string dual in $AdS_4 \times CP^3$. In section 4 we calculate fluctuations about the point particle solution corresponding to the spin chain vacuum, showing that $\Delta - (J_1 - J_4)/2$ is a Hamiltonian for these.

Section 5 is a catalogue of existing giant magnon solutions in various subspaces of $CP^3$, single-spin magnons in $CP^1$ and $RP^2$, and dyonic magnons in $RP^3$. Section 6 looks at other subspaces of potential interest, including the four-dimensional spaces $S^2 \times S^2$ and $CP^2$, and also $S^2 \times S^1$. Section 7 is a brief discussion of finite-$J$ solutions, which can be embedded in the same way, and their dispersion relations.

We discuss and conclude in section 8. Extra details of the geometry, and how to analyse strings in it using Lagrange multipliers, are discussed in two appendices.

2 Groups in ABJM theory

The $\mathcal{N} = 6$ superconformal Chern–Simons-matter theory of ABJM [4] of interest here has gauge symmetry $U(N) \times U(N)$. We will only study its scalars $A_i, B_i$. The fields $A_1, A_2$ are matrices in the $(N, \bar{N})$ representation of this (one fundamental index, one anti-fundamental), and the fields $B_1, B_2$ in the $(\bar{N}, N)$. There is a manifest $SU(2)_A$ R-symmetry in which the $A$s form a doublet, and $SU(2)_B$ acting on the $B$s. There is also the conformal group $SO(2, 3)$, since we are in 2+1 dimensions. Taking spacetime to be $R \times S^2$, we restrict attention to fields in the lowest

1Section 5 and the discussion of finite $J$ in section 8 are new in version 2 of this paper.

2These of theories were discovered after the explorations of 3-dimensional superconformal theories with non-Lie-algebra gauge symmetry by BLG, [18] and build on earlier work on Chern–Simons-matter theories by [19].
Kaluza-Klein mode on this $S^2$, i.e. in the singlet representation of $SO(3)_r$, which is the spatial part of the conformal group.

In [20] it was proven that the full R-symmetry is in fact $SU(4)$, with the following vector in the fundamental representation:

$$Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger) \quad (1)$$

and $Y_4^\dagger$ in the anti-fundamental. If we keep only $(Y^1, Y^4) = (A_1, B_1^\dagger)$ then we have a subgroup called $SU(2)_{G'}$, and if we keep only $(Y^2, Y^3) = (A_2, B_1^\dagger)$ then we have the subgroup $SU(2)_G$.

This theory is dual to membranes on $AdS_4 \times S^7/\mathbb{Z}_k$, where $(k, -k)$ are the level numbers of the two Chern–Simons terms. The 't Hooft limit $N \to \infty$ with $\lambda = N/k$ fixed sends $k \to \infty$, and reduces the dual theory to type IIA strings on $AdS_4 \times CP^3$.

To find a spin-chain description, [13, 10, 14] study gauge invariant operators of length $2L$ of the form

$$\mathcal{O} = \chi_{\alpha_1}^{B_1^\dagger} \chi_{\alpha_2}^{B_2^\dagger} \cdots \chi_{\alpha_L}^{B_L^\dagger} Y^{-A_1}_{B_1^\dagger} Y^{-A_2}_{B_2^\dagger} \cdots Y^{-A_L}_{B_L^\dagger} Y^{Y_1}_{Y_4^\dagger}.$$

When $\chi$ is fully symmetric (in the $A$s, and in the $B$s) and traceless, $\mathcal{O}$ is a chiral primary, thus protected, and has scaling dimension $\Delta = L$. In this case the anomalous dimension, defined $D = \Delta - L$, will be zero.

The $SU(2)_A \times SU(2)_B$ sector refers to operators $\mathcal{O}$ in which only $Y^1, Y^2$ and $Y_3^\dagger, Y_4^\dagger$ appear. (That is, only fields $A_1, A_2, B_1$ and $B_2$. The two factors in the name are $SU(2)_A$ and $SU(2)_B$.) The $SU(3)$ sector allows operators with $Y^1, Y^2, Y^3$ and $Y_4^\dagger$. For both of these, the vacuum is taken to be

$$\mathcal{O}_{\text{vac}} = \text{tr} \left( Y^{Y_1}_{Y_4^\dagger} \right)^L. \quad (2)$$

This has $\Delta = L$, and $J = L$, where $J$ is the Cartan generator in $SU(2)_{G'}$: $J(Y^1) = \frac{1}{2}$ and $J(Y^4) = -\frac{1}{2}$, thus $J(Y^1) = +\frac{1}{2}$.

In the $SU(2)_A \times SU(2)_B$ sector, the two-loop anomalous scaling dimension is computed by the sum of the Hamiltonians of two independent Heisenberg $XXX$ spin chains, for the even and odd sites. The momentum constraint (from the $U(N)$ trace $tr$) is that the sum of their momenta be zero. (This is slightly weaker than the $N = 4$ case, [12] where there is one total momentum which must be zero.)

### 3 The geometry of $CP^3$

The string dual of ABJM theory (in the 't Hooft limit) lives in the 10-dimensional space $AdS_4 \times CP^3$, with sizes specified by the metric

$$ds^2 = \frac{R^2}{4} ds^2_{AdS_4} + R^2 ds^2_{CP^3} \quad (3)$$

where $R^2 = 2^{5/2}\pi \sqrt{\lambda}$. The large-$\lambda$ limit gives strongly coupled gauge theory, dual to classical strings. In addition to this (string-frame) metric, there is a dilaton and RR forms, given by [4], which do not influence the motion of classical strings.

The metric for $CP^3$ is given in [4] as

$$ds^2_{CP^3} = \frac{dz_i d\bar{z}_i}{\rho^2} - \frac{|z_i d\bar{z}_i|^2}{\rho^4}, \quad \text{where} \quad \rho^2 = z_i \bar{z}_i \quad (4)$$

These subscripts are the notation of [10], except that they have $B_1$ and $B_2$ the other way around: their spin chain vacuum is $\text{tr}(A_i B_i^\dagger)^L$ rather than the $\text{tr}(Y^1 Y_4^\dagger)^L$ of [13] which we use. [3]
in terms of the homogeneous co-ordinates \( z \in \mathbb{C}^4 \), where \( z \sim \lambda z \) for any complex \( \lambda \). The SU(4) isometry symmetry is manifest here, with \( z \) in the fundamental representation. AdS/CFT identifies this isometry group with the SU(4) R-symmetry group, so it is natural to take \( z \) to be in the same basis as the fields \( Y^A \) in (1) above.

There are two angular parameterisations commonly used. One set of angles was given by [21]:

\[
ds_{CP^3}^2 = d\mu^2 + \frac{1}{4} \sin^2 \mu \cos^2 \mu \left[ d\chi + \sin^2 \alpha (d\psi + \cos \theta \, d\phi) \right]^2 + \sin^2 \mu \left[ d\alpha^2 + \frac{1}{4} \sin^2 \alpha \left( d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \alpha (d\psi + \cos \theta \, d\phi)^2 \right) \right]
\]

with ranges \( \alpha, \mu \in [0, \frac{\pi}{2}] \), \( \theta \in [0, \pi] \), \( \phi \in [0, 2\pi] \) and \( \psi, \chi \in [0, 4\pi] \). Another was given by [22]:

\[
ds_{CP^3}^2 = d\xi^2 + \frac{1}{4} \sin^2 2\xi \left( d\eta + \frac{1}{2} \cos \vartheta_1 \, d\varphi_1 - \frac{1}{2} \cos \vartheta_2 \, d\varphi_2 \right)^2 + \frac{1}{4} \cos^2 \xi \left( d\vartheta_1^2 + \sin^2 \vartheta_1 \, d\varphi_1^2 \right) + \frac{1}{4} \sin^2 \xi \left( d\vartheta_2^2 + \sin^2 \vartheta_2 \, d\varphi_2^2 \right)
\]

where \( \xi \in [0, \frac{\pi}{2}] \), \( \vartheta_1, \vartheta_2 \in [0, \pi] \), \( \varphi_1, \varphi_2 \in [0, 2\pi] \) and \( \eta \in [0, 4\pi] \). (This can be obtained by building \( S^7 \) from \( S^3 \times S^3 \) with the seventh co-ordinate \( \xi \) controlling their relative sizes.) In appendix A we give the maps between these angles and the homogeneous co-ordinates.

The Penrose limit describes the geometry very near to a null geodesic [23] and has been very important in AdS/CFT. [24] This has been studied in AdS\(_4 \times CP^3 \) by [10], where the particle travels along \( \chi = 4t \) with \( \alpha = 0, \mu = \pi/4 \) in terms of the angles in (5), and by [25,11], who use co-ordinates (6), expanding near \( \vartheta_1 = \vartheta_2 = 0, \xi = \pi/4 \) with distance along the line \( \tilde{\psi} = \eta + (\varphi_1 - \varphi_2)/2 = -2t \). In all cases, the test particle moves along the path

\[ z = \frac{\sqrt{2}}{2} (e^{it}, 0, 0, e^{-it}). \]

This has large angular momentum in opposite directions on the \( z_1 \) and \( z_4 \) planes, as one would expect for the state dual to the operator (2). This led [13] to write this state down as the string state dual to the vacuum \( \mathcal{O}_{\text{vac}} \).

4 Fluctuation Hamiltonian for the point particle

In the AdS\(_5 \times S^5 \) case, the string state dual to the spin chain vacuum \( \text{tr}(\Phi_1 + i\Phi_2)^L \) is a point particle with \( X = (\cos t, \sin t, 0, 0, 0, 0) \). This state has large angular momentum in the 1-2 plane, \( J = \Delta \). By studying small fluctuations of this state, viewed as a string solution, one can show that \( \Delta - J \) is a Hamiltonian for the physical modes. [2] Semiclassical quantisation treats these modes as quantum fields with energy \( \Delta - J \). Giant magnons are excitations above this vacuum, and so their semiclassical quantisation involves calculating quantum corrections to this energy. [26]

In the present AdS\(_4 \times CP^3 \) case, given the point particle state (7) and the vacuum (2), it is reasonable to guess that \( \Delta - (J_1 - J_2)/2 \) will play the same role. Here we confirm this, by explicitly deriving the fluctuation Hamiltonian.

\(^{4}\)We stress that there are not different Penrose limits for the different giant magnon sectors. To get precisely this path \( z \), using our conventions given in (22) and (23), we fix in addition \( \theta = \pi \) (in the first case) and \( \varphi_1 = \varphi_2 \) (in the second), and also swap \( z_2 \leftrightarrow z_4 \) in the second case.
Write the metric for the $AdS_4$ factor in the form

$$ds^2_{AdS_4} = - \left( 1 + \frac{r^2}{1 - r^2} \right)^2 d\tau^2 + \frac{4}{(1 - r^2)^2} dr^2$$

(8)

where $r = r_i$, $i = 1, 2, 3$ are zero at the centre of $AdS$, and $\tau$ is $AdS$ time. (In our notation worldsheet space and time are $x, t$.) For the $CP^3$ sector we use yet another set of co-ordinates, which are convenient for this calculation.\footnote{The advantage of these co-ordinates (as opposed to the angles) is that the identification of the charges $J_i$ here with those for the magmons in section 5 and those for the gauge theory in section 2 is transparent. To cover the whole space with these co-ordinates we need $\beta \in [0, \pi]$ and $\epsilon \in [-1, 1]$. This is clearly seen in terms of the inhomogeneous co-ordinates $z_1/z_4 = e^{i\beta}((1 + \epsilon)/(1 - \epsilon)$ and $z_2/z_4, z_3/z_4$. (Similar, but not identical, co-ordinates were used by [6].)} We write

$$Z = \left( e^{i\beta} \frac{1 + \epsilon}{\sqrt{2}}, y_1 + iy_2, y_3 + iy_4, e^{-i\beta} \frac{1 - \epsilon}{\sqrt{2}} \right)$$

(9)
in terms of which $\rho^2 = \bar{z}_iz_i = 1 + \epsilon^2 + y^2$ (where $y^2 = y_jy_j$). The metric (4) then becomes

$$ds^2_{CP^3} = \frac{(1 + \epsilon^2) d\beta^2 + dy^2}{1 + \epsilon^2 + y^2}$$

$$+ \frac{(\epsilon dy + y \cdot dy)^2 + (2\epsilon d\beta + y_1dy_2 - y_2dy_1 + y_3dy_4 - y_4dy_3)^2}{(1 + \epsilon^2 + y^2)^2}.$$

On the second line here we expand near $r = y = 0, \epsilon = 0$ and present only the terms that we will need. The point particle travels on the line $\tau = 2t, \beta = t$, and we define perturbations about this as follows:

$$\tau = 2t + \frac{1}{\lambda^{1/4}} \tilde{\tau}$$

$$\beta = t + \frac{1}{\lambda^{1/4}} \tilde{\beta}$$

$$\epsilon = \frac{1}{\lambda^{1/4}} \tilde{\epsilon}$$

$$y = \frac{1}{\lambda} \tilde{y}.$$

(11)

The perturbations $\tilde{\tau}$ and $\tilde{\beta}$ will lead to modes which are pure gauge, but are needed for now to maintain conformal gauge.

The Lagrangian is $L = \frac{1}{2} (-\gamma_{00} + \gamma_{11})$ and the Virasoro constraints are $\gamma_{00} + \gamma_{11} = 0$ and $\gamma_{01} = 0$, in terms of the induced metric $\gamma_{ab}$. The components we need are:

$$\gamma_{00} = G_{\mu\nu} \partial_{\tau} X^\mu \partial_{\tau} X^\nu$$

$$= \frac{1}{\lambda^{1/4}} \left[ - \partial_{\tau} \tilde{\tau}^2 + 2\partial_{\tau} \tilde{\beta}^2 \right]$$

$$+ \frac{1}{\sqrt{\lambda}} \left[ \frac{(\partial_{\tau} \tilde{\tau})^2}{4} + (\partial_{\tau} \tilde{\beta})^2 + (\partial_{\tau} \tilde{\epsilon})^2 + (\partial_{\tau} \tilde{y})^2 - 4\tilde{\tau}^2 - 4\tilde{\beta}^2 - 4\tilde{\epsilon}^2 - 4\tilde{y}^2 \right]$$

$$+ \frac{1}{\lambda^{3/4}} \left[ -4\tilde{\tau}^2 \partial_{\tau} \tilde{\beta} + \partial_{\tau} \tilde{\beta} (\ldots) + \partial_{\tau} \tilde{y} (\ldots) \right] + o \left( \frac{1}{\lambda} \right)$$
where \( \ldots \) indicates terms not needed for this calculation, and

\[
\gamma_{11} = G_{\mu\nu} \partial_x X^\mu \partial_x X^\nu
\]

\[
= \frac{1}{\sqrt{\lambda}} \left[ -\frac{(\partial_x \tilde{r})^2}{4} + (\partial_x \tilde{r})^2 + (\partial_x \tilde{\beta})^2 + (\partial_x \tilde{\epsilon})^2 + (\partial_x \tilde{y})^2 \right] + o(\frac{1}{\lambda}).
\]

Next we define the string’s conserved charges. \( \Delta \) is the charge generated by time translation:

\[
\Delta = 2\sqrt{2} \lambda \int dx \frac{\partial L}{\partial \partial_t} \left[ \gamma, r, \beta, \epsilon, y \right]
\]

\[
= 2\sqrt{2} \lambda^{3/4} \int dx \frac{\partial L}{\partial \partial_t} \left[ \gamma, \tilde{r}, \tilde{\beta}, \tilde{\epsilon}, \tilde{y} \right]
\]

and \( J_i \) is the charge generated by rotation of the \( z_i \) complex plane:

\[
J_1 = 2\sqrt{2} \lambda \int dx \frac{\partial L}{\partial \partial_t(\arg Z_1)}
\]

\[
= 2\sqrt{2} \lambda \int dx \left[ \frac{\text{Im}(Z_1 \partial_t Z_1)}{\rho^2} - \left| Z_1 \right|^2 \sum_i \text{Im}(Z_i \partial_t Z_i) \right]
\]

\[
J_4 = 2\sqrt{2} \lambda \int dx \left[ \frac{\text{Im}(Z_4 \partial_t Z_4)}{\rho^2} - \left| Z_4 \right|^2 \sum_i \text{Im}(Z_i \partial_t Z_i) \right].
\]

Substituting in the above mode definitions, we get

\[
\Delta = \sqrt{2} \int dx \left[ \sqrt{\lambda} + \frac{\lambda^{1/4}}{2} \partial_t \tilde{r} + 4\tilde{r}^2 + o\left(\frac{1}{\lambda^{1/4}}\right) \right]
\]

\[
J_1 = \sqrt{2} \int dx \left[ \sqrt{\lambda} - \lambda^{1/4} \partial_t \tilde{\beta} - 4\tilde{\epsilon}^2 - \tilde{y}^2 + (\tilde{y}_2 \partial_t \tilde{y}_1 - \tilde{y}_1 \partial_t \tilde{y}_2 + \tilde{y}_4 \partial_t \tilde{y}_3 - \tilde{y}_3 \partial_t \tilde{y}_4) + o\left(\frac{1}{\lambda^{1/4}}\right) \right]
\]

\[
J_4 = \sqrt{2} \int dx \left[ -\sqrt{\lambda} - \lambda^{1/4} \partial_t \tilde{\beta} + 4\tilde{\epsilon}^2 + \tilde{y}^2 + (\tilde{y}_2 \partial_t \tilde{y}_1 - \tilde{y}_1 \partial_t \tilde{y}_2 + \tilde{y}_4 \partial_t \tilde{y}_3 - \tilde{y}_3 \partial_t \tilde{y}_4) + o\left(\frac{1}{\lambda^{1/4}}\right) \right].
\]

These diverge as \( \lambda \to \infty \), but for the linear combination used below, the \( o(\sqrt{\lambda}) \) terms cancel. The \( o(\lambda^{1/4}) \) terms, linear in the fluctuations, can be re-written as quadratic \( o(1) \) terms using the Virasoro constraint \( \gamma_{00} + \gamma_{11} = 0 \). This leads to

\[
\Delta - \frac{J_1 - J_4}{2} = \frac{\sqrt{2}}{2} \int dx \left[ (\partial_t \tilde{r})^2 + (\partial_x \tilde{r})^2 + 4\tilde{r}^2 + (\partial_t \tilde{\beta})^2 + (\partial_x \tilde{\epsilon})^2 + 4\tilde{\epsilon}^2 + (\partial_t \tilde{\beta})^2 + (\partial_x \tilde{\epsilon})^2 + 4\tilde{\epsilon}^2 + (\partial_t \tilde{y})^2 + (\partial_x \tilde{y})^2 + \tilde{y}^2 \right]
\]

\[
- \frac{(\partial_t \tilde{r})^2}{4} - \frac{(\partial_x \tilde{r})^2}{4} + (\partial_t \tilde{\beta})^2 + (\partial_x \tilde{\epsilon})^2 + 4\tilde{\epsilon}^2 + (\partial_t \tilde{y})^2 + (\partial_x \tilde{y})^2 + \tilde{y}^2 + o\left(\frac{1}{\lambda^{1/4}}\right).
\]

The terms on the last line are the gauge modes, generating infinitesimal reparameterisations, so would not be included in semiclassical quantisation. After dropping these, we are left with the

\[\text{Note that in deriving these charges we treat } Z_1, \ldots, Z_4 \text{ as independent fields, even though they are in fact related through } Z \sim \lambda Z, \text{ which defines } CP^3 \text{ from } C^4. \text{ Therefore, we do this before adopting the parametrisation } \Omega, \text{ in which we have fixed some of this gauge freedom by writing only six (not eight) real co-ordinates.} \]
Hamiltonian\[8\] \(\Delta - \frac{J_1 - J_4}{2} = \sqrt{2} \int dx \mathcal{H}\), where\[8\]

\[
\mathcal{H} = \frac{1}{2} \left[ (\partial_t \tilde{r})^2 + (\partial_x \tilde{r})^2 + 4 \tilde{r}^2 + (\partial_t \tilde{\epsilon})^2 + (\partial_x \tilde{\epsilon})^2 + 4 \tilde{\epsilon}^2 + (\partial_t \tilde{y})^2 + (\partial_x \tilde{y})^2 + \tilde{y}^2 \right].
\]

This describes eight massive modes: the three \(\tilde{r}_i\) in \(AdS_4\), plus \(\tilde{\epsilon}\) and the four \(\tilde{y}_i\) in \(CP^3\). As was noted by [6], one of the \(CP^3\) modes, \(\tilde{\epsilon}\), has reached across the aisle to have the same mass as the \(AdS\) modes \(\tilde{r}\). The same list of masses was also found by [25,10,11] when studying the Penrose limit, and by [5,6] for modes of spinning strings in the \(AdS_2 \times S^1\) subspace.

5 Placing giant magnons into \(CP^3\)

Recall that the Hoffman–Maldacena giant magnon [3] is a rigidly rotating classical string solution in \(\mathbb{R} \times S^2\), given in timelike conformal gauge by

\[
\cos \theta_{\text{mag}} = \sin \frac{p}{2} \text{sech} \, u \\
\tan (\phi_{\text{mag}} - t) = \tan \frac{p}{2} \text{tanh} \, u
\]

where \(u = (x - t \cos \frac{\theta}{2})/\sin \frac{\theta}{2}\) is the boosted spatial co-ordinate for a soliton with worldsheet velocity \(\cos(p/2)\). The spacetime is \(ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta \, ds^2\) — by timelike gauge we mean that the target-space time is also worldsheet time\[4\].

We define conserved charges here as follows:

\[
\Delta = \sqrt{2\lambda} \int dx \, 1 \\
J_{\text{sphere}} = \sqrt{2\lambda} \int dx \, \text{Im} \left( W_1 \partial_t W_1 \right).
\]

This \(\Delta\) matches [13] used above when the \(AdS\) fluctuations \(\tilde{r}\) and \(\tilde{r}\) are turned off. Note that we keep the same prefactor \(\sqrt{2\lambda}\) here, which is not the one we would use in the \(AdS_5 \times S^5\) case. Finally, we write the complex embedding co-ordinates \(W_1 = e^{i\phi_{\text{mag}}} \sin \theta_{\text{mag}}\) and \(W_2 = \cos \theta_{\text{mag}}\).[10]

Both \(\Delta\) and \(J_{\text{sphere}}\) are infinite for the solution [14], but their difference is finite:

\[
\Delta - J_{\text{sphere}} = 2\sqrt{2\lambda} \sin \left( \frac{p}{2} \right).
\]

The parameter \(p\) is the (absolute value of the) momentum of the spin chain excitation in the dual gauge theory, which is why this is called a dispersion relation. It is also equal to the opening angle \(\Delta\phi_{\text{mag}}\) of the string solution on the equator of \(\theta_{\text{mag}}\).

We now turn to solutions in \(\mathbb{R} \times CP^3\), with metric \(ds^2 = -dt^2 + ds^2_{CP^3}\). All solutions will be in conformal gauge, and with worldsheet time \(t\) related to \(AdS\) time \(\tau\) by \(\tau = 2t\), so we will...
continue to use the definition of $\Delta$ from (15), although for $J$ we must now use (12). We will also continue to use the parameter $p \in [0, 2\pi]$ in all the cases below, and while this should still be a momentum in the dual theory, we make no comment here on the precise factors involved.

5.1 The subspace $CP^1$

If we set $z_2 = z_3 = 0$, or in terms of angles (5), $\alpha = 0$, then we obtain the space $CP^1 = S^2$ with metric
\[ ds^2 = \frac{1}{4} \left[ d(2\mu)^2 + \sin^2(2\mu) d\left( \frac{X}{2} \right)^2 \right]. \tag{17} \]

This is a sphere of radius $\frac{1}{2}$, so to place the magnon solution (14) here (as was done by [10]) maintaining conformal gauge we need to set
\[ 2\mu = \theta_{\text{mag}}(2x, 2t) \]
\[ \frac{X}{2} = \phi_{\text{mag}}(2x, 2t). \tag{18} \]

Using the map (33), given in appendix A and choosing $\theta = \pi$, we obtain
\[ Z(x, t) = \frac{1}{\sqrt{2}} \left( e^{i\phi_{\text{mag}}(2x, 2t)} \sqrt{1 - \cos \theta_{\text{mag}}(2x, 2t)}, 0, 0, e^{-i\phi_{\text{mag}}} \sqrt{1 + \cos \theta_{\text{mag}}} \right). \tag{19} \]

Calculating charges for this solution, using definitions (12) for $J$ and (15) for $\Delta$, we recover the dispersion relation
\[ \Delta - J_1 - J_4 = \sqrt{2\lambda} \sin \left( \frac{p}{2} \right). \tag{20} \]

We should check that this subspace is a legal one, meaning that solutions found here are guaranteed to be solutions in the full space. This can be done by finding the conformal gauge equations of motion coming from the Polyakov action with the metric (5), and confirming that $\alpha$’s equation is solved by $\alpha = 0$.[12] But in this case it is easier to note that $z_2 = z_3 = 0$ trivially solves their equations of motion, [35], which we derive in appendix B.

5.2 The subspace $RP^2$

A second embedding of the $S^2$ solution was first used by [11][13]
\[ Z(x, t) = \frac{1}{\sqrt{2}} \left( e^{i\phi_{\text{mag}}(x, t)} \sin \theta_{\text{mag}}(x, t), \cos \theta_{\text{mag}}, \cos \theta_{\text{mag}}, e^{-i\phi_{\text{mag}}} \sin \theta_{\text{mag}} \right). \tag{21} \]

This solution lives in an $RP^2$ subspace, as can be seen by simply rotating some of the planes

---

[11] Note that if you were to omit the second term in (12) when calculating $J$, thus effectively using (16) appropriate for the sphere, you would get instead $\Delta - (J_1 - J_4)/2 = \sqrt{2\lambda} \sin \left( \frac{p}{2} \right)$. In the $RP^2$ and $RP^3$ subspaces discussed below, this second term vanishes.

[12] In addition to solving the conformal gauge equations of motion, a string solution must be in conformal gauge, i.e. must solve the Virasoro constraints. If the solution on the subspace is in conformal gauge, then it follows trivially that the solution in the full space is too: the induced metric $\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ is influenced only by those directions the solution explores, and in these directions the metric $G_{\mu\nu}$ is the same in both the full space and the subspace.

[13] We discuss the equations of motion used by [11] for strings in $CP^3$ in appendix B.
in \( \mathbb{C}^4 = \mathbb{R}^8 \) by \( \frac{\pi}{2} \); in terms of new co-ordinates \( w \) defined by

\[
\begin{align*}
  w_1 &= \frac{1}{\sqrt{2}} (z_1 + \bar{z}_1) \\
  w_2 &= \frac{1}{\sqrt{2}} (z_2 + \bar{z}_3) \\
  w_3 &= \frac{1}{\sqrt{2}} (z_1 - \bar{z}_4) \\
  w_4 &= \frac{1}{\sqrt{2}} (z_2 - \bar{z}_3),
\end{align*}
\]

(22)

this solution has \( w_3 = w_4 = 0 \) and is precisely the original giant magnon in the other two co-ordinates:

\[
(W_1, W_2) = (e^{i\phi_{\text{mag}}} \sin \theta_{\text{mag}}, \cos \theta_{\text{mag}}).
\]

The reason this is \( RP^2 \) rather than \( S^2 \) is that sending \( (w_1, w_2) \rightarrow -(w_1, w_2) \) gives an overall sign change on \( z \), and these two points are identified in \( CP^3 \).

The subspace which this magnon explores can also be obtained from the metric (6), by fixing \( \vartheta_1 = \frac{\pi}{2}, \vartheta_2 = \frac{\pi}{2}, \varphi_1 = 0 \) and \( \eta = 0 \). The metric then becomes

\[
ds^2 = d\xi^2 + \sin^2 \xi \left( \frac{2\varphi}{2} \right)^2
\]

and the magnon (21) is simply \( \xi = \theta_{\text{mag}}(x, t), \varphi_2 = 2\phi_{\text{mag}}(x, t) \). This can be checked to be a legal restriction from the equations of motion for the four angles fixed.

This subspace is sometimes, rather misleadingly, referred to as \( S^2 \times S^2 \). It is true that \( |z_1|^2 + |z_2|^2 = \frac{1}{2} \) and \( |z_3|^2 + |z_4|^2 = \frac{1}{2} \), and \( \text{Im} \ z_2 = 0 = \text{Im} \ z_3 \). These restrictions alone would describe a subspace of \( \mathbb{C}^4 \), namely \( S^2 \times S^2 \subset \mathbb{C}^2 \times \mathbb{C}^2 \). But we are in \( CP^3 \), not \( C^4 \), and the space described by \( \theta, \phi \) (or by \( \xi, \varphi_2 \)) has only two dimensions — these two \( S^2 \) factors are not independent. In section 6.2 below we discuss a genuine four-dimensional \( S^2 \times S^2 \) subspace.

The charges of this solution are very simply related to those of the magnon on the sphere, since the extra term in the \( CP^3 \) angular momentum (12) compared to the that for the sphere vanishes: \( J_{\text{sphere}} = J_1 = \frac{1}{4}(J_1 - J_4) \), and we get simply

\[
\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2} \lambda \sin \left( \frac{p}{2} \right).
\]

One difference from the magnon on \( S^2 \) is that when \( p = \pi \), the magnon becomes a single closed string. Its cusps, at opposite points on the equator of \( S^2 \), are in fact at the same point in \( RP^2 \). In general the magnon connects two points a distance \( \Delta \varphi_2 = 2\Delta \phi_{\text{mag}} = 2p \) apart on the equator, but \( \varphi \sim \varphi + 2\pi \) so \( p = \delta \) and \( p = \pi + \delta \) both connect the same two points. As was noted by [10], this can be viewed as giving rise to a second class of magnons, with

\[
\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2} \lambda \sin \left( \frac{\pi + \delta}{2} \right) = 2\sqrt{2} \lambda \cos \left( \frac{\delta}{2} \right).
\]

Figure 4 shows two magnons on \( S^2 \) and then on \( RP^2 \), one with \( p = \frac{1}{2} \) and another with \( p = \pi - \frac{1}{2} \). In the \( RP^2 \) case they have opposite opening angles \( \delta = \pm \frac{1}{2} \), thus form a single closed string, while in the \( S^2 \) case the total opening angle is \( \pi \).

---

14 In \( S^2 \), the standard co-ordinates have ranges \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), and changing \( \theta \rightarrow \pi - \theta \) and \( \phi \rightarrow \phi + \pi \) simultaneously moves you to the antipodal point on \( S^2 \). But performing this change in the subspace of \( CP^3 \) parameterised by (21) changes \( z \rightarrow -z \), and these two points are identified by the definition of \( CP^3 \). This is what makes the subspace \( RP^2 = S^2 / \mathbb{Z}_2 \) instead of \( S^2 \). To obtain co-ordinates which cover this subspace only once, we can shorten the range of either \( \theta \) or \( \phi \), and in figure 4 we choose to restrict to \( \phi \in [0, \pi] \) while keeping \( \theta \in [0, \pi] \).
Figure 1: Two giant magnons are shown (in red) on the unit sphere $S^2$ (left), on $RP^2$ (centre, drawn here as half a sphere) and on $CP^1$, a sphere of radius $\frac{1}{2}$ (right). In all cases they have $p_1 = \frac{1}{2}$ and $p_2 = \pi - \frac{1}{2}$, which leads to a closed string in the $RP^2$ case, but not in the $S^2$ or $CP^1$ cases.

In both the $RP^2$ and $CP^1$ cases, the equator is of length $\pi$, and we parameterise it by $\beta \in [0, \pi]$. The magnon with $p_1 = \frac{1}{2}$ spans $\Delta \beta = \frac{1}{2}$ in the $RP^2$ case, but only $\Delta \beta = \frac{1}{4}$ in the $CP^1$ case. On $CP^1$ we have also drawn a third magnon (in blue) with $p_3 = 1$, which spans the same length of equator $\Delta \beta = \frac{1}{2}$ as does the $p_1$ magnon on $RP^2$.

5.3 The subspace $RP^3$

In the $AdS_5 \times S^5$ case, Dorey’s giant magnons with a second large angular momentum $J' \sim \sqrt{\lambda}$ allow one to see that the dispersion relation is $\Delta - J_{\text{sphere}} = \sqrt{J'^2 + \frac{\lambda}{\pi^2} \sin^2(p/2)}$. [27] These necessarily live in $S^3$ rather than $S^2$. They are called dyonic magnons, and (embedding $S^3 \subset \mathbb{C}^2$) can be written

$$W_1 = e^{it} \left( \cos \frac{p}{2} + i \sin \frac{p}{2} \tanh U \right)$$
$$W_2 = e^{iV} \sin \frac{p}{2} \text{sech} U$$

where

$$U = (x \cosh \beta - t \sinh \beta) \cos \alpha$$
$$V = (t \cosh \beta - x \sinh \beta) \sin \alpha$$
$$\cot \alpha = \frac{2r}{1 - r^2} \sin \frac{p}{2}$$
$$\tanh \beta = \frac{2r}{1 + r^2} \cos \frac{p}{2}.$$ 

The parameter $p$ is still the opening angle along the equator in the $W_1$ plane, although $\cos(p/2)$ is clearly no longer the worldsheet velocity. Sending the new parameter $r \to 1$ reproduces the original giant magnon.

The second method of embedding $S^2$ solutions into $CP^3$, given by (21), points out a way to embed $S^3$ solutions:

$$Z = \frac{1}{\sqrt{2}} (W_1, W_2, \bar{W}_2, \bar{W}_1).$$

(24)

As before, this is in fact a subspace $RP^3$ rather than $S^3$, thanks to the identification of $(w_1, w_2) \sim -(w_1, w_2)$ implied.\(^{15}\)

\(^{15}\)Note that the rotation from $z$ to $w$ given by (22) is not an isometry, and in particular that the identification $z \sim \lambda z$ which defines $CP^3$ does not apply afterwards: $w \propto \lambda w$ for complex $\lambda$. If $w_3 = w_4 = 0$, as is implied by (23), then the phases of $w_1$ and $w_2$ are both physical. (Which is good if we’re claiming that the dyonic magnon has momentum along both of them.)

However, the relation $w \sim \lambda w$ is true for real $\lambda$, and since we have fixed $w_1^2 + w_2^2 = 1$ by starting with a string solution on $S^2$, the identification $(w_1, w_2) \sim -(w_1, w_2)$ is all that survives.
Embedding a dyonic giant magnon in this way gives a $CP^3$ solution with charges\[^{15}\]

\[
\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2} \lambda \frac{1 + r^2}{2r} \sin \left(\frac{p}{2}\right) \\
\frac{J_2 - J_3}{2} = 2\sqrt{2} \lambda \frac{1 - r^2}{2r} \sin \left(\frac{p}{2}\right).
\]

These satisfy the relation

\[
\Delta - \frac{J_1 - J_4}{2} = \sqrt{\left(\frac{J_2 - J_3}{2}\right)^2 + 8\lambda \sin^2 \left(\frac{p}{2}\right)}.
\]

Notice that the second angular momentum here is that carried by $Y^2$ and $Y_3^1$, which are the impurities we insert into the vacuum \(^{(2)}\) to make magnons in the $SU(2) \times SU(2)$ sector.

This subspace can also be obtained from \((6)\), by fixing $\vartheta_1 = \frac{\pi}{2}$, $\vartheta_2 = \frac{\pi}{2}$ and $\eta = 0$. The metric becomes

\[
ds^2 = d\xi^2 + \sin^2 \xi \left(d\varphi_2^2 + \vartheta_1 d\varphi_2^2\right) + \cos^2 \xi \left(d\eta + \frac{1}{2} \cos \vartheta_1 d\varphi_1\right)^2.
\]

This restriction can be checked to be a legal one from the equations of motion for the angles $\vartheta_1$, $\vartheta_2$ and $\eta$. The dyonic giant magnon in this space was re-derived by \[[17]\], using exactly these angles. It was also re-derived by \[[16]\] using co-ordinates $z$.

Like the $RP^2$ magnons above, at $p = \pi$ these form single closed strings, and beyond this $(\pi < p < 2\pi)$ give a second class of magnons connecting the same two points on the equator as the magnon with $\tilde{p} = p - \pi$.

### 6 Some larger subspaces

All of the solutions we have discussed so far are known from the $AdS_5 \times S^5$ case, and explore only subspaces $S^2$ or $S^3 \subset S^5$. In this section look at two subspaces of $CP^3$ on which new solutions might exist: $CP^2$ and $S^2 \times S^2$.

We also study restrictions of this $S^2 \times S^2$ down to three or two dimensions (in sections \[6.3\] and \[6.4\]) since the resulting spaces have been used in the literature.

#### 6.1 The subspace $CP^2$

The first larger nontrivial subspace we can find is $CP^2$, obtained by setting $z_3 = 0$. In terms of the angles \((5)\), the restriction is $\vartheta_2 = 0$ (and $\varphi_2 = 0$, since this is now redundant) and the metric becomes

\[
ds^2 = d\xi^2 + \frac{1}{4} \cos^2 \xi \left(d\varphi_1^2 + \sin^2 \vartheta_1 d\varphi_1^2\right) + \frac{1}{4} \sin^2 2\xi \left(d\eta + \frac{1}{2} \cos \vartheta_1 d\varphi_1\right)^2.
\]

The two manifest isometries here are along $\varphi_1$ and $\eta$. When $\xi = 0$ this is an $S^2$ equivalent to \((17)\) (exchange $z_2 \leftrightarrow z_4$ to align them perfectly). Perhaps allowing $\xi \neq 0$ will allow new dyonic solutions here, generalising the $CP^1$ solution \((19)\) just as the dyonic $RP^3$ solution generalises the $RP^2$ solution.

Note that this is certainly a legal subspace, for the same reason as given for $CP^1$: setting $z_3 = 0$ certainly solves the $z_3$ equation of motion.

\[^{15}\]In calculating these charges from \((12)\), the same cancellation of the second term happens here as happened in the previous section. Thus using the charges one would expect for $S^7 \subset C^4$ gives the right answer here. This does not work in the $CP^1$ case, see footnote \[17\].
6.2 The subspace $S^2 \times S^2$

If we set $\varphi_1 = \varphi_2$ and $\vartheta_1 = \vartheta_2$ in metric (6), we get the four-dimensional space

$$ds^2 = \frac{1}{4} [d(2\xi)^2 + \sin^2(2\xi) \, d\eta^2] + \frac{1}{4} [d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2]$$

which is $S^2 \times S^2$ (possibly up to co-ordinate ranges), and of course the new angles are defined $\vartheta \equiv (\vartheta_1 + \vartheta_2)/2$ and $\varphi \equiv (\varphi_1 + \varphi_2)/2$.

On such a product space, the Polyakov action splits into two terms, giving two non-interacting sets of target-space co-ordinates. Any two $S^2$ string solutions can be placed onto the same worldsheet, completely independently. Choosing giant magnon solutions, worldsheet scattering between these sectors would be trivial, just as it would be on two decoupled Heisenberg spin chains.

The restrictions needed to obtain this space are that $\vartheta_+ = \vartheta_1 - \vartheta_2 = 0$ and $\varphi_+ = \varphi_1 - \varphi_2 = 0$, and unfortunately the equations of motion for $\vartheta_-$ and $\varphi_-$ are not automatically solved by this choice: instead they give complicated relations between the other co-ordinates. The equation for $\vartheta_-$ reads

$$0 = -\partial_t (\cos 2\xi \partial_t \vartheta) + \partial_x (\cos 2\xi \partial_x \vartheta) + \frac{1}{2} \cos 2\xi \sin 2\theta \left( \partial_t^2 \varphi - \partial_x^2 \varphi \right) - \sin^2 2\xi \sin \vartheta \left( \partial_t \eta \partial_t \varphi - \partial_x \eta \partial_x \varphi \right)$$

and that for $\varphi_-$ reads

$$0 = -\partial_t \left( \sin^2 2\xi \cos \vartheta \partial_t \eta + \cos 2\xi \sin^2 \vartheta \partial_t \varphi \right) + \partial_x \left( \sin^2 2\xi \cos \vartheta \partial_x \eta + \cos 2\xi \sin^2 \vartheta \partial_x \varphi \right).$$

These constraints do not of course rule out the existence of solutions on this subspace. But placing an arbitrary $S^2$ solution into each of the factors is unlikely to produce a solution, because of these equations coupling $\xi, \eta$ to $\vartheta, \varphi$.

6.3 The subspace $S^2 \times S^1$

If we further restrict the above subspace by holding one of the angles fixed, we will get $S^2 \times S^1$ (again up to identifications). Setting $\vartheta = \frac{\pi}{2}$ gives the space studied by [17], with metric

$$ds^2 = \frac{1}{4} [d(2\xi)^2 + \sin^2(2\xi) \, d\eta^2 + d\varphi^2] .$$

The equation of motion for $\vartheta$ is solved by $\vartheta = \frac{\pi}{2}$, and the constraints imposed by $\vartheta_- = 0$ and $\varphi_- = 0$ above simplify to

$$0 = -\partial_t \vartheta + \partial_x \vartheta$$

$$0 = -\partial_t \left( \cos 2\xi \partial_t \varphi \right) + \partial_x \left( \cos 2\xi \partial_x \varphi \right).$$

These constraints were not taken into account by [17], who sets $\vartheta_- = 0$ before calculating the equation of motion for $\vartheta$ (which is indeed solved) but without ever calculating the equation of motion for $\vartheta_-$. The magnon ansatz used there sets $\eta = \omega t + f(u)$, $\varphi = \nu t$ and $\xi = g(u)$, in terms of boosted $u = \beta t + \alpha x$. The first constraint then implies $\beta f'(u) = -\omega$, while for a...

\footnote{The constraint \((26)\) can also be obtained without using $\vartheta_-$, by simply setting $\vartheta_1 = \frac{\pi}{2}$ and $\vartheta_2 = \frac{\pi}{2}$ in their equations of motion.}
magnon solution one typically has $f(u) \propto \tanh u$. The second constraint (27) implies $\beta = 0$, so together they imply $\omega = 0$.

This problem does not arise in the other case studied by [17], where the $\vartheta_-$ equation is solved by $\eta = 0$, and $\varphi_1 \neq \varphi_2$ so there is no $\varphi_-$ constraint. The resulting subspace is the $RP^3$ discussed in section 5.3.

6.4 The subspace $CP^1$, again

Finally, we can restrict the subspace $S^2 \times S^2$ of (25) by holding both of the angles in one factor constant, to obtain $S^2$. Setting $\xi$ and $\eta$ to be constants leaves the space

$$ds^2 = \frac{1}{4} [d\vartheta^2 + \sin^2 \vartheta d\varphi^2]$$

which is, like our $CP^1$ of section 5.1, a sphere of radius $\frac{1}{2}$. This is a legal subspace, as the equations of motion for $\xi$ and $\eta$ are automatically solved (because a stationary particle anywhere on the sphere is a solution) and the constraints arising from $\vartheta_- = 0$ and from $\varphi_- = 0$ become simply the equations of motion for $\vartheta$ and $\varphi$.

When $\xi = \frac{\pi}{2}$, and using the conventions given in appendix A, this space is embedded by

$$z = \left( e^{i\varphi/2} \cos \frac{\vartheta}{2}, 0, 0, e^{-i\varphi/2} \sin \frac{\vartheta}{2} \right).$$

This is precisely the same subspace $CP^1$ as in (17), although we obtained it there by fixing $\alpha = 0$ in the other set of angles (4). Fixing $\xi$ to some other value will simply rotate the 1-2 and 3-4 planes, but in all cases the space is $S^2 = CP^1$. Like the subspace $RP^2$ discussed in section 5.2 this one is sometimes referred to as $S^2 \times S^2$ in the literature.

These co-ordinates were used by [28] to study finite-$J$ effects on the $CP^1$ giant magnon. We give their results in (29) below.

7 Finite-$J$ corrections

All of the giant magnons we have written down so far have both infinite energy and infinite angular momentum. As can be seen from (15), this corresponds to infinite worldsheet length in the timelike conformal gauge we are using.

The first treatment of giant magnons $AdS_5 \times S^5$ at finite $J$ was by [29], who worked in uniform lightcone gauge, in which the worldsheet density of $J$, rather than of $\Delta$, is constant. Their gauge has a parameter $a \in [0, 1]$, and at $a = 0$ (and in conformal gauge) they obtained the following correction to the dispersion relation:

$$\varepsilon = \Delta - J = \frac{\sqrt{\lambda}}{\pi} \sin \left( \frac{P}{2} \right) \left[ 1 - 4 \frac{\varepsilon^2}{\lambda} \sin^2 \left( \frac{P}{2} \right) e^{-2J/\varepsilon} + o(e^{-4J/\varepsilon}) \right]$$

$$= \frac{\sqrt{\lambda}}{\pi} \sin \left( \frac{P}{2} \right) \left[ 1 - 4 \sin^2 \left( \frac{P}{2} \right) e^{-2\Delta/\varepsilon} + \ldots \right]$$

Exact solutions at any $J$ were studied by [30], where it was shown that they are connected by the Pohlmeyer map to kink-train solutions of sine-gordon theory. The apparent gauge-dependence of the results of [29] was resolved by [31], using the fact that the solutions are periodic both on the worldsheet and in the azimuthal angle on the sphere, and so can be viewed as wound strings on $S^2/\mathbb{Z}_n$. [31, 32] The scattering of finite-$J$ magnons was studied by [33], using the connection to sine-gordon theory in finite volume.
The finite-$J$ generalisations of the basic giant magnon are still solutions moving on $S^2$, and so one can place them into $\mathbb{C}P^3$ using either of the maps presented in sections 5.1 and 5.2 above. For the $RP^2$ giant magnon, the corrected dispersion relation was derived by [34] to be

$$\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2}\lambda \sin \left(\frac{p}{2}\right) \left[ 1 - 4\sin^2 \left(\frac{p}{2}\right) e^{-2\Delta/\sqrt{2}\lambda \sin \left(\frac{\theta}{2}\right)} + \ldots \right].$$  \hspace{1cm} (28)

For the $CP^1$ giant magnon, [28] give the result\footnote{Here is brief note about deriving these two results from the original $S^2$ case. The integrals defining the charges are now over a finite length $-L < x < L$, so write $J(L)$ and $\Delta(L)$. Note that $\Delta(2L) = 2\Delta(L)$. To get the charges for one magnon, we must integrate from one cusp to the next: choose $L$ such that $\theta_{\text{mag}}(x = \pm L, t = 0)$ are at the first cusps. For the $RP^2$ case, the relationship we used before $J_{\text{sphere}}(L) = J_1(L) = (J_1(L) - J_4(L))/2$ still holds, leading to (29). We wrote the $S^2$ result above using the prefactor appropriate for $AdS_5 \times S^5$, so to get this result for the $AdS_4 \times CP^3$ theory have replaced $\sqrt{\lambda}$ by $2\sqrt{2}\lambda$. For the $CP^1$ case, the cusp at $\theta_{\text{mag}}(L, 0)$ is at $Z_{CP^1}(\frac{\theta}{2}, 0)$, thanks to the scaling [13]. The relationship between charges is that $J_1(\frac{\theta}{2}) - J_4(\frac{\theta}{2}) = \frac{1}{2} J_{\text{sphere}}(L)$. Thus $\Delta(\frac{\theta}{2}) - (J_1(\frac{\theta}{2}) - J_4(\frac{\theta}{2}))/2 = \Delta(\frac{\theta}{2}) - \frac{1}{2} J_{\text{sphere}}(L) = \frac{1}{2} \left( \Delta(L) - J_{\text{sphere}}(L) \right)$. In the result (29), it is the energy for one magnon $\Delta(\frac{\theta}{2})$ which appears both on the left hand side and in the exponent.}

\begin{equation}
\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2}\lambda \sin \left(\frac{p}{2}\right) \left[ 1 - 4\sin^2 \left(\frac{p}{2}\right) e^{-2\Delta/\sqrt{2}\lambda \sin \left(\frac{\theta}{2}\right)} + \ldots \right]. \hspace{1cm} (29)
\end{equation}

We observe that, even at finite $J$, two $CP^1$ magnons have the same dispersion relation as one $RP^2$ magnon, provided all three have the same value of the parameter $p$\footnote{Note that that essentially all the properties of the two $CP^1$ magnons add up to give those of the single $RP^2$ magnon: energy $\Delta$, angular momentum ($J_1 - J_4$)/2, worldsheet length $L$ and opening angle along the equator (which we call $\Delta\beta$ in the next section).}

Dyonic giant magnons can also be studied at finite $J$; this has been done for those in $S^5$ from this string sigma-model perspective by [30, 35], and for those in $RP^3 \subset CP^3$ by [16, 36].

In the $AdS_5 \times S^5$ case these corrections can also be calculated using algebraic curves [37] or using the Lüscher formula [38], and these agree with the string sigma-model result presented above. For calculations on the gauge theory side of the correspondence see [39]. In $AdS_4 \times CP^3$ the same list of methods is possible, and we discuss these further in section 8.3 below.

8 Discussion and conclusion

In this paper we have only discussed giant magnon solutions known from $AdS_5 \times S^5$, but have been careful about how these are placed into $CP^3$. Here we summarise these results, comment on more general solutions, and comment on connections to approaches other than the classical string sigma-model.

8.1 Single-charge giant magnons

In sections 5.1 and 5.2 we looked at two different ways to embed the basic single-charge giant magnon [13], into either $CP^1$ or $RP^2$. [10, 11] This $CP^1$ is a two-sphere of radius $\frac{1}{2}$, while $RP^2$ is half a two-sphere, so both have an equator of length $\pi$. We lined up the embeddings into $\mathbb{C}^4$ such that, in both cases, the equator is the line

$$z = \frac{1}{\sqrt{2}} \left( e^{i\beta}, 0, 0, e^{-i\beta} \right)$$

18Here is brief note about deriving these two results from the original $S^2$ case. The integrals defining the charges are now over a finite length $-L < x < L$, so write $J(L)$ and $\Delta(L)$. Note that $\Delta(2L) = 2\Delta(L)$. To get the charges for one magnon, we must integrate from one cusp to the next: choose $L$ such that $\theta_{\text{mag}}(x = \pm L, t = 0)$ are at the first cusps.

For the $RP^2$ case, the relationship we used before $J_{\text{sphere}}(L) = J_1(L) = (J_1(L) - J_4(L))/2$ still holds, leading to (29). We wrote the $S^2$ result above using the prefactor appropriate for $AdS_5 \times S^5$, so to get this result for the $AdS_4 \times CP^3$ theory have replaced $\sqrt{\lambda}/\pi \rightarrow 2\sqrt{2}\lambda$. For the $CP^1$ case, the cusp at $\theta_{\text{mag}}(L, 0)$ is at $Z_{CP^1}(\frac{\theta}{2}, 0)$, thanks to the scaling [13]. The relationship between charges is that $J_1(\frac{\theta}{2}) - J_4(\frac{\theta}{2}) = \frac{1}{2} J_{\text{sphere}}(L)$. Thus $\Delta(\frac{\theta}{2}) - (J_1(\frac{\theta}{2}) - J_4(\frac{\theta}{2}))/2 = \Delta(\frac{\theta}{2}) - \frac{1}{2} J_{\text{sphere}}(L) = \frac{1}{2} \left( \Delta(L) - J_{\text{sphere}}(L) \right)$. In the result (29), it is the energy for one magnon $\Delta(\frac{\theta}{2})$ which appears both on the left hand side and in the exponent.
where we name the angle \( \beta \in [0, \pi] \), as in \( \ref{0} \) above, to avoid confusion.

Since the basic magnon \( \ref{1} \) has opening angle \( \Delta \phi_{\text{mag}} = p \), these two solutions have

\[
\begin{align*}
CP^1 : & \quad \beta = \chi/4 = \phi_{\text{mag}}/2 \quad \Rightarrow \quad \Delta \beta = p/2 \\
RP^2 : & \quad \beta = \varphi_2/2 = \phi_{\text{mag}} \quad \Rightarrow \quad \Delta \beta = p'
\end{align*}
\]

(where we now write \( p' \) for the parameter of the \( RP^2 \) magnon, to distinguish it from the \( CP^1 \) case’s \( p \)). A single giant magnon is not a closed string solution, one must join a set of them together at their endpoints on the equator. The condition for a set \( p_i \) of \( CP^1 \) magnons or \( p'_j \) of \( RP^2 \) magnons to close is that the total opening angle \( \Delta \beta \) should be a multiple of \( \pi \), that is,

\[
\begin{align*}
CP^1 : & \quad \sum_i p_i = 2\pi n \\
RP^2 : & \quad \sum_j 2p'_j = 2\pi n, \quad n \in \mathbb{Z}.
\end{align*}
\]

The point particle \( \ref{7} \) moves along the same equator too, and by calculating fluctuations of this solution, we checked in section 4 that \( \Delta - J_1 - J_4 \) is indeed a Hamiltonian for them, just as \( \Delta - J \) is in the \( S^5 \) case. Calculating the same difference of charges for the two magnon embeddings, we obtained dispersion relations \( \ref{20} \) and \( \ref{23} \), which we now write also in terms of the opening angle \( \Delta \beta \):

\[
\begin{align*}
CP^1 : & \quad \Delta - J_1 - J_4 = \sqrt{2\lambda} \sin \left( \frac{p}{2} \right) = \sqrt{2\lambda} \sin \left( \Delta \beta \right) \\
RP^2 : & \quad \Delta - J_1 - J_4 = 2\sqrt{2\lambda} \sin \left( \frac{p'}{2} \right) = 2\sqrt{2\lambda} \sin \left( \frac{\Delta \beta}{2} \right).
\end{align*}
\]

Notice that these agree at small \( \Delta \beta \). The limit \( p \to 0 \) takes you from giant magnons to the Penrose limit (via the interpolating case of \( \ref{40} \), studied here by \( \ref{41} \)). Finite-\( J \) effects in the Penrose limit were studied by \( \ref{42} \).

As noted in section \( \ref{5.2} \), there is also a second magnon on \( RP^2 \) for any given opening angle \( \Delta \beta \), which has charges \( \ref{10} \)

\[
RP^{2'} : \quad \Delta - J_1 - J_4 = 2\sqrt{2\lambda} \cos \left( \frac{\Delta \beta}{2} \right).
\]

For small \( \Delta \beta \) this is almost a circular string, with its ends slightly offset along the equator — see figure \( \ref{1} \) on page \( \ref{10} \) above.

### 8.2 More solutions!

While we used the giant magnon on \( S^2 \) \( \ref{14} \) as an example, the subspaces we have described exist independently of it, and any other string solution moving on \( S^2 \) can be placed into either of these subspaces of \( CP^3 \) in the same way. Thus not only finite-\( J \) magnons (as discussed in section \( \ref{7} \) above) but also scattering solutions \( \ref{43} \) and single spikes \( \ref{44–46} \) all exist in both the \( CP^1 \) and \( RP^2 \) subspaces. The equations of motion do not notice the global identification \( (w_1, w_2) \sim -(w_1, w_2) \) which distinguishes \( RP^2 \) from \( S^2 \), and the fact that \( CP^1 \) is a sphere of radius \( \frac{1}{2} \) can be dealt with by the same scaling \( \ref{18} \) that we used for the basic magnon.

---

20Single-spike solutions of all kinds can be easily obtained from their giant magnon partners by the \( x \leftrightarrow t \) exchange discussed in \( \ref{44, 45} \). As in \( R \times S^5 \), this exchange (keeping \( X^0 = t \) is a symmetry of the equations of motion \( \ref{45} \) and the Virasoro constraints for \( R \times CP^3 \). Thus the classical solutions have no properties which cannot be read off from the corresponding magnon solution. However, the quantum properties are quite different. \( \ref{45} \)
Many papers interpret the magnon on $RP^2$ (and also that on $RP^3$) as being two magnons, one in each half of the embedding space $\mathbb{C}^2 \times \mathbb{C}^2$. [11, 16] It is then tempting to identify these two halves with the even- and odd-site spin chains in the dual description’s $SU(2) \times SU(2)$ sector. For the known solutions, however, these two halves are not independent: in fact they are always locked together, and by a trivial change of co-ordinates we can write them as a single $RP^2 = S^2/\mathbb{Z}_2$ space. This does not rule out the existence of two independent magnon sectors, such that a pair of magnons of the same parameter $p$ gives us again the known $RP^2$ solution. But at present individual solutions in these two sectors are not known.

The single-parameter giant magnon on $S^3$ has a two-parameter dyonic generalisation on $S^3$, and in section 5.3 we looked at how to map this into $RP^3 \subset CP^3$, where it generalises the $RP^2$ solution. The dyonic generalisation of the $CP^1$ solution is not known, but it might lie in the $CP^2$ subspace we discussed in section 5.1.

It would be very interesting to find some indication among the magnon solutions of the weaker momentum constraint: the momentum in just the even-site or just the odd-site spin chain need not vanish, only the total. Combining the two closure conditions to give $\sum_i p_i + \sum_j 2p_j' = 2\pi n$ cannot be the answer, because these two classes of magnons are certainly inequivalent solutions, while the even- and odd-site spin chains are related by an $SU(4)$ rotation.

8.3 Beyond the classical sigma-model

The classical string solutions we have discussed are well-known from the $S^5$ case, and explore only $S^2$ or $S^3$-like subspaces of $CP^3$. Their classical properties (and indeed those of solutions we have not discussed, such as scattering solutions) are not strongly affected by being transplanted to the new space. However, their quantum properties will certainly depend on the whole space, as was the case for spinning string solutions in $AdS_2 \times S^1$ studied by [5]. The relevant supersymmetric sigma model (for strings on $AdS_4 \times CP^3$) was first studied by [6, 7]. Using this one would like to perform a calculation like that done for magnons in $AdS_5 \times S^5$ by [26].

Like the equations of motion, the Pohlmeyer map [47] to the sine-gordon field $\alpha$ (given by $\cos \alpha = -\partial_i W_i \partial_i W_i + \partial_\lambda W_i \partial_\lambda W_i$ in the $S^2$ case) depends only locally on the target-space co-ordinates. Thus strings on either $CP^1$ or $RP^2$ will be classically equivalent to the sine-gordon model. The condition that the string closes $\sum \Delta \beta \sim 0$ plays no role in the sine-gordon model, thus the second class of magnons, which we called $RP^{2\nu}$ above, has no special meaning in sine-gordon theory. As quantum systems, strings on $\mathbb{R} \times S^2$ are quite different to the sine-gordon model, thanks to the different notion of energy, and this complicates the translation of the $n$-body description of solitons in sine-gordon theory to this case. [48, 3, 49] The Pohlmeyer reduction has been extended to the full superstring on $AdS_5 \times S^5$, [50] and also to strings moving on $CP^3$. [51].

Classical strings in $AdS_4 \times CP^3$ can also be studied using the algebraic curve, in which the 10 eigenvalues $q_a$ of the monodromy matrix $\Omega$ are analytic functions of the spectral parameter, and their various poles and branch points control the solution. [52] Giant magnons in this picture were studied by [53], and are of two distinct kinds, ‘small’ and ‘big’. Their dispersion relations are as follows:

- small GM: $\varepsilon = \sqrt{\frac{1}{4} + 2\lambda \sin^2 \left(\frac{p}{2}\right)} \rightarrow \sqrt{2\lambda} \sin \left(\frac{p}{2}\right)$ when $\sqrt{\lambda} \gg 1$
- big GM: $\varepsilon = \sqrt{\frac{1}{4} + 8\lambda \sin^2 \left(\frac{p}{4}\right)} \rightarrow 2\sqrt{2\lambda} \sin \left(\frac{p}{4}\right)$.

It would seem natural to identify these with the $CP^1$ and $RP^2$ magnons of the string sigma-model, presumably with $p' = p/2 = \Delta \beta$. There are two ‘small GM’ sectors, together often called...
the $SU(2) \times SU(2)$ sector.

However, the study of finite-$J$ corrections to these paints a different picture. According to [54], two 'small GM’s in the two sectors, both with the same momentum $p$, have a correction $\delta \varepsilon$ matching the $RP^2$ string result (28). This does seems to point to the interpretation of the $RP^2$ string solution as two giant magnons, as was originally claimed by [11]. However, the same paper’s result for one ‘small GM’ does not match any of the string calculations, apparently leaving open the identification both of the string state for this, and of the algebraic curve corresponding to the $CP^1$ string. Finite-$J$ corrections have also been studied using the Lüscher formula by [55, 54], and the results agree with those from the algebraic curve.

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A More about $CP^3$’s geometry

The complex projective space $CP^3$ is defined to be

$$CP^3 = \frac{\mathbb{C}^4}{z \sim \lambda z}$$

where $z = z_a$ are called homogeneous co-ordinates. We can split this identification into $z \sim rz$ and $z \sim e^{i\phi}z$ (for any $r, \phi \in \mathbb{R}$) and then replace the first one with the condition $|z|^2 = 1$, to obtain a sphere with one identification

$$CP^3 = \frac{S^7}{z \sim e^{i\phi}z} = \frac{S^7}{U(1)}.$$

The isometry group is $SU(4)$, acting in the natural way on $z$. Since the stabiliser group of (say) the point $z_4 = 1$ is $U(3)$, we can also write

$$CP^3 = \frac{SU(4)}{U(3)}.$$

The infinitesimal form of the standard Fubini–Study metric for this is

$$ds^2_{CP^3} = \frac{dz_i d\bar{z}_i}{\rho^2} - \frac{|z_i d\bar{z}_i|^2}{\rho^4} = ds^2_{\text{sphere}} - d\gamma^2$$

$$= ds^2_{\text{flat}} - dp^2 - d\gamma^2$$

where $\rho^2 = z_i \bar{z}_i$. (Note that in some conventions the metric is 4 times this, [56, 10] making $CP^1$ (17) a unit sphere.) In the second and third lines above, $ds^2_{\text{flat}} = dz_i d\bar{z}_i$ is the Euclidean metric.
for \( \mathbb{C}^4 \), and \( ds^2_{\text{sphere}} \) is a metric for \( S^7 \) in terms of these embedding co-ordinates. Instead of fixing \( \rho = 1 \), this way of treating the sphere subtracts off the component coming from radial motion (and scales the rest appropriately). In turn, \( CP^3 \) can be obtained from the sphere by fixing the total phase \( \gamma = \arg \prod_i z_i \), or instead by subtracting the total phase component. These two pieces are

\[
d\rho = \frac{1}{2\rho^2} (z_id\bar{z}_i + \bar{z}_idz_i) = \frac{1}{\rho} \Re (\bar{z}_idz_i) \]

\[
d\gamma = \frac{i}{2\rho^2} (z_id\bar{z}_i - \bar{z}_idz_i) = \frac{1}{\rho^2} \Im (\bar{z}_idz_i) .
\]

We now present the maps between the homogeneous co-ordinates and the two sets of angles we have used. These are taken from [25] and [21], although we have shuffled the \( z_i \). For the metric (6) (whose \( \eta \) is often called \( \psi \))

\[
d s^2_{CP^3} = d\mu^2 + \frac{1}{4} \sin^2 \mu \left[ d\chi + \sin^2 \alpha \left( d\psi + \cos \theta \ d\phi \right) \right]^2
\]

\[
+ \frac{1}{4} \cos^2 \mu \left( d\theta_1^2 + \sin^2 \theta \ d\phi_1^2 \right) + \frac{1}{4} \sin^2 \alpha \left( d\psi_2^2 + \sin^2 \theta \ d\phi_2^2 \right)
\]

the relationship is:

\[
z_1 = \sin \chi \cos(\theta_2/2) e^{-i\eta/2} e^{i\xi/2}
\]

\[
z_2 = \cos \chi \cos(\theta_1/2) e^{i\eta/2} e^{i\xi/2}
\]

\[
z_3 = \cos \chi \cos(\theta_2/2) e^{-i\eta/2} e^{-i\xi/2}
\]

\[
z_4 = \sin \chi \sin(\theta_1/2) e^{-i\eta/2} e^{-i\xi/2}.
\]

For the other set of angular variables (5)

\[
d s^2_{CP^3} = d\mu^2 + \frac{1}{4} \sin^2 \mu \cos^2 \mu \left[ d\chi + \sin^2 \alpha \left( d\psi + \cos \theta \ d\phi \right) \right]^2
\]

\[
+ \sin^2 \mu \left[ d\alpha^2 + \frac{1}{4} \sin^2 \alpha \left( d\theta^2 + \sin^2 \theta \ d\phi^2 + \cos^2 \alpha \left( d\psi + \cos \theta \ d\phi \right) \right) \right]
\]

the map is specified by

\[
z_1/z_4 = \tan \mu \cos \alpha \ e^{i\chi/2}
\]

\[
z_2/z_4 = \tan \mu \sin \alpha \ \sin(\theta/2) \ e^{i\chi/2} \ e^{i(\psi-\phi)/2}
\]

\[
z_3/z_4 = \tan \mu \cos \alpha \ \cos(\theta/2) \ e^{i\chi/2} \ e^{i(\psi+\phi)/2}.
\]

These ratios \( z_i/z_4 \) are called inhomogeneous co-ordinates, and cover the patch \( z_4 \neq 0 \) with no identifications. [56] With the ranges given, the trigonometric functions controlling the amplitudes are always positive in both of these cases. From the phases of the inhomogeneous co-ordinates of \( z_i/z_4 \) it is easy to see that ranges of the remaining angles are correct.

\section*{B Strings in homogeneous co-ordinates}

To study bosonic string theory in \( S^n \), it is often convenient to use embedding co-ordinates for \( \mathbb{R}^{n+1} \) and then constrain the radius to 1. This avoids all the trigonometric functions needed for angular co-ordinates, and (in AdS/CFT) also gives a simple correspondence between the
R-symmetry generators and the rotations of this space. We can do the same for \( CP^3 \), using homogeneous co-ordinates \( z \). We will need two constraints, \( \rho^2 = 1 \) and \( \gamma = 0 \).

### B.1 Using Lagrange multipliers

Begin by writing the metric for \( \mathbb{R} \times CP^3 \) as

\[
 ds^2 = - (dX^0)^2 + d\bar{z}_i G_{ij} dz_j \quad \text{with} \quad G_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{z_i \bar{z}_j}{\rho^4} 
\]

In conformal gauge, and with \( X^0 = \kappa t \), the Polyakov action is

\[
 S = \int \frac{dx dt}{2\pi} R^2 \mathcal{L} = 2\sqrt{2\lambda} \int dx dt \mathcal{L} 
\]

\[
 2\mathcal{L} = \kappa^2 + \partial^a \bar{Z}_i G_{ij} \partial_a Z_j + \Lambda_\rho (\bar{Z}_i Z_i - 1) + i\Lambda_\gamma (Z_1 Z_2 Z_3 Z_4 - \bar{Z}_1 \bar{Z}_2 \bar{Z}_3 \bar{Z}_4). 
\]

Note that \( \Lambda_\gamma \in \mathbb{R} \), since the piece in brackets is proportional to \( 2i \sin \gamma \). In calculating Euler–Lagrange equations for this, we set \( \rho = 1 \) immediately, simplifying \( \partial G_{ij} / \partial Z_i \) etc. greatly. The Lagrange multipliers can be read off from the parallel component of the equations (i.e. \( \bar{Z}_i \) times \( Z_i \)’s equation of motion) which is:

\[
 \Lambda_\rho - 4i (Z_1 Z_2 Z_3 Z_4) \Lambda_\gamma = \partial_t \bar{Z}_i \partial_t Z_i - 2 |\bar{Z}_i \partial_t Z_i|^2 - \partial_x \bar{Z}_i \partial_x Z_i + 2 |\bar{Z}_i \partial_x Z_i|^2.
\]

(This 4 is the number of complex embedding co-ordinates.) The right-hand side here is real, which implies \( \Lambda_\gamma = 0 \). Using this, we find the equation of motion for \( Z_i \) to be

\[
 - \partial_t (G_{ij} \partial_t Z_j) + \partial_x (G_{ij} \partial_x Z_j) = Z_i \Lambda_\rho - (\bar{Z}_j \partial_t Z_j) \partial_t Z_i + (\bar{Z}_j \partial_x Z_j) \partial_x Z_i.
\]

The Virasoro constraints are

\[
 - \kappa^2 + \partial_t \bar{Z}_i G_{ij} \partial_t Z_j + \partial_x \bar{Z}_i G_{ij} \partial_x Z_j = 0 \\
 \Re \left( \partial_t \bar{Z}_i G_{ij} \partial_x Z_j \right) = 0.
\]

The result that \( \Lambda_\gamma = 0 \) deserves a little explanation. If we were to analyse strings on the sphere using a similar metric (in fact exactly \( ds^2_{\text{sphere}} \) from (31) above):

\[
 2\mathcal{L} = 1 + \partial^a X_i \partial_a X_j g_{ij} + \Lambda (X^2 - 1), \quad \text{with} \quad g_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{X_i X_j}{\rho^4} 
\]

then we would also find \( \Lambda = 0 \), although the equations of motion are the same as are obtained with \( g_{ij} = \delta_{ij} \) (i.e. using \( ds^2_{\text{flat}} \)). In some sense the metric is enforcing the constraint for us. The reason we had \( \Lambda_\rho \neq 0 \) in the \( CP^3 \) case above was that we set \( \rho = 1 \) at an early stage of the calculation.
B.2 Constraining $S^7$ solutions

The approach of [11] (and others) to strings on $CP^3$ is to find solutions on the sphere $S^7 \in \mathbb{C}^4$, and then further demand that the two Noether charges from $\partial_\gamma$ vanish:

$$0 = C_0 \equiv \sum_{i=1}^{4} \text{Im} (\bar{Z}_i \partial_t Z_i) , \quad 0 = C_1 \equiv \sum_{i=1}^{4} \text{Im} (\bar{Z}_i \partial_x Z_i) .$$

This is true for the $RP^2$ solution (21) given by [11], and more generally, for any solution on the larger $RP^3$ subspace of section 5.3. In terms of the co-ordinates $w$ from (22), the condition $w_3 = w_4 = 0$ which defines this subspace implies $C_0 = C_1 = 0$, and also reduces the equations of motion to those for the sphere $S^3$ embedded in $(w_1, w_2)$.

But more general solutions, such as the $CP^1$ solution (19), do not solve these constraints, nor do they solve the equations of motion for $S^7 \subset \mathbb{C}^4$. So these conditions (solution on $S^7$, and $C_0 = C_1 = 0$) are certainly not necessary for a solution. Whether they are sufficient is not entirely clear to us.

Finally, we note that in terms of charges $J_i$ we used throughout, something like the constraint $C_0 = 0$ does hold: $\sum_{i=1}^{4} J_i = 0$ follows trivially from the definition (12).

References

[1] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, A semi-classical limit of the gauge/string correspondence, Nucl. Phys. B636 (2002) 99–114 [arXiv:hep-th/0204051].

[2] S. Frolov and A. A. Tseytlin, Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$, JHEP 06 (2002) 007 [arXiv:hep-th/0204226].

[3] D. M. Hofman and J. M. Maldacena, Giant magnons, J. Phys. A39 (2006) 13095–13118 [arXiv:hep-th/0604135].

[4] O. Aharony, O. Bergman, D. L. Jafferis and J. M. Maldacena, $\mathcal{N} = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals, JHEP 10 (2008) 091 [arXiv:0806.1218].

[5] T. McLoughlin and R. Roiban, Spinning strings at one-loop in $AdS_4 \times CP^3$, arXiv:0807.3965.

[6] G. Arutyunov and S. Frolov, Superstrings on $AdS_4 \times CP^3$ as a coset sigma-model, JHEP 09 (2008) 129 [arXiv:0806.4940].

[7] B. Stefanski, Green–Schwarz action for type IIA strings on $AdS_4 \times CP^3$, Nucl. Phys. B808 (2009) 80–87 [arXiv:0806.4948].

\[21\] A similar approach to strings on the sphere is to find solutions in flat (embedding) space and then reject all those which do not have $\rho = 1$. In this case solving the flat space equations and having $\rho = 1$ is sufficient to find a solution, but not necessary.

For example, when studying loops of string rotating in $S^3$, there is one critical speed at which they are solutions in unconstrained $\mathbb{R}^4$ too. [57] But faster and slower motions are possible on the sphere, with extreme cases of a point particle and a stationary hoop, which are not solutions in $\mathbb{R}^4$. 

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[8] B. Chen and J.-B. Wu, Semi-classical strings in $AdS_4 \times CP^3$, JHEP 09 (2008) 096 [arXiv:0807.0802].

[9] N. Gromov and P. Vieira, The all loop $AdS_4/CFT_3$ Bethe ansatz, [arXiv:0807.0777].
C. Ahn and R. I. Nepomechie, $\mathcal{N} = 6$ super Chern–Simons theory S-matrix and all-loop Bethe ansatz equations, JHEP 09 (2008) 010 [arXiv:0807.1924].
N. Gromov and V. Mikhaylov, Comment on the scaling function in $AdS_4 \times CP^3$, arXiv:0807.4897.
T. McLoughlin, R. Roiban and A. A. Tseytlin, Quantum spinning strings in $AdS_4 \times CP^3$: testing the Bethe ansatz proposal, JHEP 11 (2008) 069 [arXiv:0809.4038].
C. Ahn and R. I. Nepomechie, An alternative S-matrix for $\mathcal{N} = 6$ Chern–Simons theory\? arXiv:0810.1915.

[10] D. Gaiotto, S. Giombi and X. Yin, Spin chains in $\mathcal{N} = 6$ superconformal Chern–Simons-matter theory, [arXiv:0806.4589].

[11] G. Grignani, T. Harmark and M. Orselli, The $SU(2) \times SU(2)$ sector in the string dual of $\mathcal{N} = 6$ superconformal Chern–Simons theory, arXiv:0806.4959.

[12] J. A. Minahan and K. Zarembo, The Bethe-ansatz for $\mathcal{N} = 4$ super Yang–Mills, JHEP 03 (2003) 013 [arXiv:hep-th/0212208].

[13] J. A. Minahan and K. Zarembo, The Bethe ansatz for superconformal Chern–Simons, JHEP 09 (2008) 040 [arXiv:0806.3951].

[14] D. Bak and S.-J. Rey, Integrable spin chain in superconformal Chern–Simons theory, JHEP 10 (2008) 053 [arXiv:0807.2063].

[15] N. Beisert, The $su(2|2)$ dynamic S-matrix, [arXiv:hep-th/0511092].

[16] C. Ahn, P. Bozhilov and R. C. Rashkov, Neumann–Rosochatius integrable system for strings on $AdS_4 \times CP^3$, JHEP 09 (2008) 017 [arXiv:0807.3134].

[17] S. Ryang, Giant magnon and spike solutions with two spins in $AdS_4 \times CP^3$, JHEP 11 (2008) 084 [arXiv:0809.5106].

[18] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D75 (2007) 045020 [arXiv:hep-th/0611108].
A. Gustavsson, Algebraic structures on parallel M2-branes, [arXiv:0709.1260].
J. Bagger and N. Lambert, Comments on multiple M2-branes, JHEP 02 (2008) 105 [arXiv:0712.3738].
M. Van Raamsdonk, Comments on the Bagger–Lambert theory and multiple M2-branes, JHEP 05 (2008) 015 [arXiv:0803.3803].
A. Gustavsson, One-loop corrections to Bagger–Lambert theory, Nucl. Phys. B807 (2009) 315–333 [arXiv:0805.4443].

[19] J. H. Schwarz, Superconformal Chern–Simons theories, JHEP 11 (2004) 078 [arXiv:hep-th/0411077].
D. Gaiotto and X. Yin, Notes on superconformal Chern–Simons-matter theories, JHEP 08 (2007) 056 [arXiv:0704.3740].
D. Gaiotto and E. Witten, Supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang–Mills theory, [arXiv:0804.2992].
K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $\mathcal{N} = 4$ superconformal Chern–Simons theories with hyper and twisted hyper multiplets, JHEP 07 (2008) 091 [arXiv:0805.3662].

[20] M. Benna, I. R. Klebanov, T. Klose and M. Smedback, Superconformal Chern–Simons theories and $AdS_4/CFT_3$ correspondence, JHEP 09 (2008) 072 [arXiv:0806.1519].

[21] C. N. Pope and N. P. Warner, An $su(4)$ invariant compactification of $d = 11$ supergravity on a stretched seven sphere, Phys. Lett. B150 (1985) 352.
M. Cvetić, H. Lu and C. N. Pope, *Consistent warped-space Kaluza–Klein reductions, half-maximal gauged supergravities and $CP^n$ constructions*, Nucl. Phys. **B597** (2001) 172–196 [arXiv:hep-th/0007109].

G. W. Gibbons and C. N. Pope, *$CP^2$ as a gravitational instanton*, Commun. Math. Phys. **61** (1978) 239.

R. Penrose, *Differential geometry and relativity*. Reidel, Dordrecht, 1976.

D. E. Berenstein, J. M. Maldacena and H. S. Nastase, *Strings in flat space and pp waves from $\mathcal{N} = 4$ super yang mills*, JHEP **04** (2002) 013 [arXiv:hep-th/0202021].

T. Nishioka and T. Takayanagi, *On type IIA penrose limit and $\mathcal{N} = 6$ Chern–Simons theories*, JHEP **08** (2008) 001 [arXiv:0806.3391].

J. A. Minahan, *Zero modes for the giant magnon*, JHEP **02** (2007) 048 [arXiv:hep-th/0701005].

G. Papathanasiou and M. Spradlin, *Semiclassical quantization of the giant magnon*, JHEP **06** (2007) 032 [arXiv:0704.2389].

N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, J. Phys. **A39** (2006) 13119–13128 [arXiv:hep-th/0604175].

H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, JHEP **09** (2006) 024 [arXiv:hep-th/0605155].

B.-H. Lee, K. L. Panigrahi and C. Park, *Spiky strings on AdS$_4 \times CP^3$*, JHEP **11** (2008) 066 [arXiv:0807.2559v3].

G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, Nucl. Phys. **B778** (2007) 1–35 [arXiv:hep-th/0606126].

O. Linden and G. Semenoff, *Finite size giant magnon*, arXiv:0803.4028.

B. Vicedo, *Giant magnons and singular curves*, JHEP **12** (2007) 078 [arXiv:hep-th/0703180].

N. Gromov and P. Vieira, *The AdS$_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, Nucl. Phys. **B789** (2008) 175–208 [arXiv:hep-th/0703191].
[38] M. Lüscher, *Volume dependence of the energy spectrum in massive quantum field theories. 1. stable particle states*, Commun. Math. Phys. **104** (1986) 177.

[39] D. Serban and M. Staudacher, *Planar $\mathcal{N} = 4$ gauge theory and the inozemtsev long range spin chain*, JHEP **06** (2004) 001 [arXiv:hep-th/0401057].

[40] J. M. Maldacena and I. Swanson, *Connecting giant magnons to the pp-wave: An interpolating limit of AdS$_5$ × S$^5$*, Phys. Rev. **D76** (2007) 026002 [arXiv:hep-th/0612079].
H. Hayashi, K. Okamura, R. Suzuki and B. Vicedo, Large winding sector of AdS/CFT, JHEP 11 (2007) 033 [arXiv:0709.4033].

C. Ahn and P. Bozhilov, Finite-size effects for single spike, JHEP 07 (2008) 105 [arXiv:0806.1085].

S. Jain and K. L. Panigrahi, Spiky strings in $\text{AdS}_4 \times \text{CP}^3$ with Neveu–Schwarz flux, [arXiv:0810.3516].

[47] K. Pohlmeyer, Integrable Hamiltonian systems and interactions through quadratic constraints, Commun. Math. Phys. 46 (1976) 207–221.

F. Lund and T. Regge, Unified approach to strings and vortices with soliton solutions, Phys. Rev. D14 (1976) 1524.

[48] A. Mikhailov, A nonlocal poisson bracket of the sine-gordon model, [arXiv:hep-th/0511069].

[49] S. N. M. Ruijsenaars and H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. 170 (1986) 370–405.

O. Babelon and D. Bernard, The sine-gordon solitons as a n body problem, Phys. Lett. B317 (1993) 363–368 [arXiv:hep-th/9309154].

I. V. Aniceto and A. Jevicki, N-body dynamics of giant magnons in $R \times S^2$, [arXiv:0810.4548]

[50] M. Grigoriev and A. A. Tseytlin, Pohlmeyer reduction of $\text{AdS}_5 \times S^5$ superstring sigma model, Nucl. Phys. B800 (2008) 450–501 [arXiv:0711.0155].

A. Mikhailov and S. Schäfer-Nameki, Sine-gordon-like action for the superstring in $\text{AdS}_2 \times S^5$, JHEP 05 (2008) 075 [arXiv:0711.0195].

M. Grigoriev and A. A. Tseytlin, On reduced models for superstrings on $\text{AdS}_n \times S^n$, Int. J. Mod. Phys. A23 (2008) 2107–2117 [arXiv:0806.2623].

[51] H. Eichenherr and J. Honerkamp, Reduction of the $\text{CP}^N$ nonlinear sigma model, J. Math. Phys. 22 (1981) 374.

R. C. Rashkov, A note on the reduction of the $\text{AdS}_4 \times \text{CP}^3$ string sigma model, Phys. Rev. D78 (2008) 106012 [arXiv:0808.3057].

J. L. Miramontes, Pohlmeyer reduction revisited, JHEP 10 (2008) 087 [arXiv:0808.3365].

[52] N. Gromov and P. Vieira, The $\text{AdS}_4/\text{CFT}_3$ algebraic curve, [arXiv:0807.0437]

[53] I. Shenderovich, Giant magnons in $\text{AdS}_4/\text{CFT}_3$: dispersion, quantization and finite-size corrections, [arXiv:0807.2861].

[54] T. Lukowski and O. Ohlsson Sax, Finite size giant magnons in the $\text{SU}(2) \times \text{SU}(2)$ sector of $\text{AdS}_3 \times \text{CP}^3$, [arXiv:0810.1246].

[55] D. Bombardelli and D. Fioravanti, Finite-size corrections of the $\text{CP}^3$ giant magnons: the Lüscher terms, [arXiv:0810.0704].

[56] S. Kobayashi and K. Nomizu, Foundations of differential geometry. Interscience Publishers, New York, 1963.

[57] R. Roiban, A. Tirziu and A. A. Tseytlin, Slow-string limit and ‘antiferromagnetic’ state in $\text{AdS}_4/\text{CFT}$, Phys. Rev. D73 (2006) 066003 [arXiv:hep-th/0601074].