The Quiver Matrix Model
and
2d – 4d Conformal Connection

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Abstract

We review the quiver matrix model (the ITEP model) in the light of the recent progress on 2d-4d connection of conformal field theories, in particular, on the relation between Toda field theories and a class of quiver superconformal gauge theories. On the basis of the CFT representation of the $\beta$ deformation of the model, a quantum spectral curve is introduced as $\langle \langle \det(x - ig_s \partial \phi(z)) \rangle \rangle = 0$ at finite $N$ and for $\beta \neq 1$. The planar loop equation in the large $N$ limit follows with the aid of $W_n$ constraints. Residue analysis is provided both for the curve of the matrix model with the “multi-log” potential and for the Seiberg-Witten curve in the case of $SU(n)$ with $2n$ flavors, leading to the matching of the mass parameters. The isomorphism of the two curves is made manifest.

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1 Introduction

Matrix models have had successes in several stages of the developments in string theory, gauge theory and related studies of integrable systems. A list of those in the last twenty years include 2d gravity, exact evaluation of gluino condensate prepotential, topological strings, etc.

Recent progress has been triggered by the construction of a large class of $\mathcal{N} = 2$ superconformal $SU(n)$ “generalized quiver” gauge theories in four dimensions by Gaiotto [1]. (See also [2, 3, 4, 5, 6]). Subsequently an interesting conjecture has been made by Alday, Gaiotto and Tachikawa (AGT) [7] (and its $SU(n)$ generalization by [8]) on the equivalence of the Nekrasov partition function [9, 10, 11] and the 2d conformal block of the Toda field theory. These are followed by a number of extensive checks and pieces of supporting evidence [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Very recently, Dijkgraaf and Vafa [24] have suggested an explanation of this phenomenon by the so-called quiver matrix model [37, 38, 39, 40, 41]. (In this paper, it is occasionally referred to as the ITEP model [37, 38]). Their reasoning is based on the matrix model realization of type IIB topological strings on a local Calabi-Yau with a local $\mathcal{A}_{n-1}$ singularity which geometrically engineers the gauge theory. By choosing “multi-Penner” potentials [42] for the $\mathcal{A}_{n-1}$ quiver matrix model, they argued that the spectral curve of the matrix model at large $N$ (the size of the matrix) can be understood as the Seiberg-Witten curve of the attendant $SU(n)$ generalized quiver superconformal gauge theory (SCFT).

The AGT conjecture is regarded as a more concrete realization of the folklore connection between 4d gauge theories and the attendant 2d sigma models (see [43] for instance), that led to the study of two dimensional quantum integrable field theories in late seventies ([44] for instance). In the light of potential importance of this subject, we find it useful to devote the substantial part of the present paper in reviewing and reformulating the basic structure of the quiver matrix model [45].

The quiver matrix model associated with Lie algebra $\mathfrak{g}$ of ADE type with rank $r$ is obtained as a solution to extended Virasoro constraints, i.e., $W(\mathfrak{g})$ constraints [46, 47, 48] at finite $N$ [37, 38]. This fact that the model automatically implements the $W(\mathfrak{g})$ constraints at finite $N$ is an advantage over more traditional two- and multi-matrix models where finite $N$ Schwinger-Dyson equations are typically $w_{1+\infty}$ type and are more involved [50, 51, 52, 53]. The model is defined by using $r$ independent free massless chiral bosons in two dimensional CFT with the central charge $c = r$ and the final form of the partition function is formulated as the integrations over eigenvalues of the matrices. A key ingredient of its construction is a set

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1 This kind of reasoning is seen in the construction of other eigenvalue models [49].
of screening charges of the $c = r$ CFT. By construction, the partition function respects the $W(g)$, the extended Casimir algebra generated by higher spin currents which commute with the screening charges. The partition function can be reformulated as integrations over the matrices in the adjoint and bi-fundamental representations.

In the context of the AGT conjecture for more general Toda CFTs, it is natural to consider “$\beta$-ensembles” of ADE quiver matrix models. In this paper, using the $\beta$-ensemble of the $A_{n-1}$ quiver matrix model at finite $N$, we define a non-commutative Calabi-Yau threefold with the quantum deformed $A_{n-1}$ singularity as

$$uv + \langle\langle \det(x - ig_s \partial \phi(z))\rangle\rangle = 0, \quad [x, z] = -iQg_s. \quad (1.1)$$

For $n = 3$, it takes the following form

$$uv + (x - t_3(z))(x - t_2(z))(x - t_1(z))$$

$$- (x - t_3(z))f_1(z) - f_2(z)(x - t_1(z)) - g_1(z) + g_2(z) = 0, \quad (1.2)$$

with the non-commutativity $[x, z] = -iQg_s = \epsilon_1 + \epsilon_2$.

In order to bring these analyses in the more recent context, we take a simple example, namely, $\mathcal{N} = 2$ SU(3) SQCD with 6 flavors to provide a residue analysis of the quiver matrix model curve and that of the Seiberg-Witten curve from type IIA and M-theory consideration. This leads to the matching of the mass parameters.

A main result of this paper is to establish the isomorphism of the spectral curve of the quiver matrix model ($A_n$ type and beta deformed in general) in the planar limit and the corresponding Seiberg-Witten curve in the Witten-Gaiotto form. This has become possible as we have reformulated the matrix model curve, starting from the $W_n$ constraints at finite $N$, proposing the curve as a form of the characteristic equation, and finally using the singlet factorization in the planar limit to write it in a form where the isomorphism is rather manifest. To the best of our knowledge, such a systematic investigation has not been attempted and only clumsy expressions in the planar limit have been available. (See references in subsequent sections.)

This paper is organized as follows. In section 2, after giving several punchlines, we review known facts on the one-matrix model, using the CFT notation and the Virasoro constraints at finite $N$ [54]. In section 3, ordinary ADE quiver matrix models are reviewed in the same spirit and their “$\beta$-ensemble” is introduced. For the case of $A_{n-1}$ quiver matrix model, we show that the quantum spectral curve

$$\langle\langle \det(x - ig_s \partial \phi(z))\rangle\rangle = 0, \quad [x, z] = -iQg_s, \quad (1.3)$$
is well-defined within the matrix model integral. The partition function obeys the $W_n$ constraints at finite $N$ which are the properties of the original matrix integrals. The planar loop equation follows from these structures together with the large $N$ factorization. In section 4, we adopt Dijkgraaf-Vafa’s recipe to treat $\mathcal{N} = 2$ superconformal gauge theory by using the quiver matrix model with Penner like action. We concentrate on the $A_2$ quiver matrix model and consider the spectral curve. Section 5 treats the Seiberg-Witten curve for $SU(n)$ gauge theory with $N_f = 2n$ flavors. We see that this curve enjoys the same properties as those of the spectral curve. In particular, we give a matching of the mass parameters first for the case of $n = 3$. In order to render the isomorphism of the two curves clearer, a subsection is devoted to establishing this point. A matching of the mass parameters for general $n$ is readily given.

The major reason that we emphasized starting with the finite $N$ spectral curve is already stated above: through this procedure and the large $N$ factorization of the singlet operators, the isomorphism has been established in this paper. The continuous flow, beginning with the construction at finite $N$, and ending with the singlet factorization in the planar limit has been indispensable in order for this paper to be legible and self-contained.

2 One-matrix model and the $\beta$-ensemble

In the case of $A_1$, the quiver matrix model corresponds to the Hermitian one-matrix model \[55\] (see also \[56, 57\]). The associated CFT is a single free boson with $c = 1$. On the other hand, the Liouville CFT which appears in AGT conjecture for $SU(2)$ has the central charge $c = 1 + 6Q^2$ with $Q = b + (1/b)$. It is known that there is a one-matrix model which has a connection with the CFT with $c = 1 + 6Q^2$. It is the $\beta$-ensemble of one-matrix model with $\beta = -b^2$. It is easy to deal with the Liouville CFT by introducing the Feign-Fuchs background charge in the CFT notation. The CFT notation works well for the $\beta$-ensemble of the one-matrix model and the appearance of the CFT with $c = 1 + 6Q^2$ in the matrix model is a built-in result. The case of $\beta = 1$ corresponds to the ordinary Hermitian one-matrix model. The AGT conjecture implies that these deformation parameters are related to Nekrasov’s deformation parameters $\epsilon_1$ and $\epsilon_2$ by \[24\]

$$\epsilon_1 = -ibgs, \quad \epsilon_2 = -\frac{ig}{b}. \quad (2.1)$$

Note that $\epsilon_1 \epsilon_2 = -g_s^2$ and $\epsilon_1/\epsilon_2 = b^2$.

It is known that the $\beta$-ensemble of one-matrix model at finite $N$ is related to a quantum

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\[2\]In AGT \[7\], a different parametrization with $\epsilon_1 = b$ and $\epsilon_2 = 1/b$ is used.
(non-commutative) spectral curve of the form
\[
\left( x + \frac{1}{2} W'(z) \right) \left( x - \frac{1}{2} W'(z) \right) - f(z) = 0,
\]
whose non-commutativity is given by
\[
[x, z] = -i Q g_s = \epsilon_1 + \epsilon_2.
\]
Using the $\beta$-ensemble matrix model, it is possible to define a non-commutative local Calabi-Yau threefold with a quantum deformed $A_1$ singularity
\[
uv + \left( x + \frac{1}{2} W'(z) \right) \left( x - \frac{1}{2} W'(z) \right) - f(z) = 0.
\]
It can be written as the form
\[
uv + x^2 - g^2 \langle \langle T(z) \rangle \rangle = 0,
\]
where $\langle \langle \rangle \rangle$ is one-matrix model average over the $\beta$-ensemble and $T(z)$ is the energy momentum tensor of $c = 1 + 6Q^2$ CFT expressed as the collective field of the matrix eigenvalues. Later, we will give exact definitions of these quantities.

### 2.1 Hermitian one-matrix model: undeformed case

In this case the relevant matrix model is the Hermitian one-matrix model. The partition function takes the form
\[
Z = \int [dM] \exp \left( \frac{1}{g_s} \text{Tr} W(M) \right).
\]
Here $M$ is an $N \times N$ Hermitian matrix. Correlation function of this matrix model is defined by
\[
\langle \langle \mathcal{O} \rangle \rangle := \frac{1}{Z} \int [dM] \mathcal{O} \exp \left( \frac{1}{g_s} \text{Tr} W(M) \right).
\]
Here $\mathcal{O} = \mathcal{O}(M)$ is a function of the Hermitian matrix.

The partition function (2.6) can be written in terms of the eigenvalues $\lambda_I$ of the matrix $M$:
\[
Z = \int d^N \lambda \Delta(\lambda)^2 \exp \left( \sum_{I=1}^{N} \frac{1}{g_s} W(\lambda_I) \right),
\]
where $\Delta(\lambda)$ is the Vandermonde determinant
\[
\Delta(\lambda) = \prod_{1 \leq I < J \leq N} (\lambda_I - \lambda_J).
\]
It is well-known that the Hermitian matrix model has a close connection with the $c = 1$ free chiral boson. The partition function can be rewritten in terms of CFT operators. The mode expansion of the chiral boson $\phi(z)$ is chosen as follows:

$$\phi(z) = \phi_0 - i a_0 \log z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n}, \quad (2.10)$$

and the non-trivial commutation relations are given by

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [\phi_0, a_0] = i. \quad (2.11)$$

Hence, our normalization of the correlator is given by

$$\langle \phi(z) \phi(w) \rangle = -\log(z - w). \quad (2.12)$$

The energy momentum tensor with the central charge $c = 1$ is given by

$$T(z) = -\frac{1}{2} : (\partial \phi(z))^2 := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \quad (2.13)$$

The screening charges which commute with the Virasoro generators $L_n$ are given by

$$Q_\pm = \int dz : e^{\pm i \sqrt{2} \phi(z)} :, \quad (2.14)$$

with a certain integration contour.

The Fock vacuum is given by

$$a_n |0\rangle = 0, \quad \langle 0| a_{-n} = 0, \quad n \geq 0. \quad (2.15)$$

Let

$$\langle N \rangle := \langle 0| e^{-i \sqrt{2} N \phi_0}. \quad (2.16)$$

Then, the partition function (2.6) of the Hermitian matrix model can be rewritten in terms of the free chiral boson as follows

$$Z = \langle N \rangle \exp \left( \frac{1}{2\sqrt{2\pi g_s}} \oint dz W(z) \partial \phi(z) \right) (Q_+)^N |0\rangle. \quad (2.17)$$

Associated with this expression, for an operator $\mathcal{O}$ constructed from the boson oscillators, we use the following notation

$$\langle \mathcal{O} \rangle_{\text{CFT}} := \frac{1}{Z} \langle N \rangle \exp \left( \frac{1}{2\sqrt{2\pi g_s}} \oint dz W(z) \partial \phi(z) \right) \mathcal{O} (Q_+)^N |0\rangle. \quad (2.18)$$
Within the normal ordering, correlators consisting of the chiral boson $\phi (z)$ in CFT have their counterparts in the matrix model correlators (2.7):

$$\langle : O(\phi) \cdots : \rangle_{\text{CFT}} = \langle \langle O(\phi) \cdots \rangle \rangle.$$  (2.19)

In the matrix model correlator, the chiral boson is realized as a collective field of the eigenvalues:

$$i\phi (z) = \frac{1}{\sqrt{2g_s}} W(z) + \sqrt{2} \text{Tr} \log(z - M).$$  (2.20)

It is known that the partition function of the Hermitian matrix model at finite $N$ obeys the Virasoro constraints [54]. In the CFT language, it follows from the commutativity of the Virasoro generators $L_n$ with the screening charge $Q_+$ and

$$L_n|0\rangle = 0, \quad n \geq -1.$$  (2.21)

In the ITEP construction, the Virasoro constraints manifestly hold at finite $N$ as

$$\langle L_n \rangle_{\text{CFT}} = 0, \quad n \geq -1.$$  (2.22)

They are equivalent to the regularity of the correlator of the energy momentum tensor

$$\langle T(z) \rangle_{\text{CFT}} = \langle \langle T(z) \rangle \rangle$$  (2.23)

at $z = 0$.

Now, using the help of the Hermitian matrix model correlator, a local Calabi-Yau threefold with $A_1$ singularity over a Riemann surface $\Sigma$ can be defined by

$$uv + x^2 - \frac{1}{4} W'(z)^2 - f(z) = 0.$$  (2.24)

Using the collective field expression

$$T(z) = -\frac{1}{2} (\partial \phi(z))^2, \quad i\partial \phi(z) = \frac{1}{\sqrt{2g_s}} W'(z) + \sqrt{2} \text{Tr} \frac{1}{z - M},$$  (2.25)

we have (for the derivation, see the next subsection, below (2.38))

$$g_s^2 \langle \langle T(z) \rangle \rangle = \frac{1}{4} W'(z)^2 + f(z),$$  (2.26)

where

$$f(z) = \left\langle g_s \sum_{I=1}^N W'(z) - W'(\lambda_I) \right\rangle \frac{z - \lambda_I}{z - \lambda_I}.$$  (2.27)

Hence, the local Calabi-Yau threefold is a surface in $(u, v, x, z) \in \mathbb{C}^4$ defined by

$$uv + x^2 - \frac{1}{4} W'(z)^2 - f(z) = 0.$$  (2.28)
At $uv = 0$, it describes some algebraic curve $\Sigma$ in $(x, z) \in \mathbb{C}^2$:

$$x^2 - \frac{1}{4} W'(z)^2 - f(z) = 0. \quad (2.29)$$

Note that this algebraic curve is well-defined for finite $N$ due to the Virasoro constraints of the matrix model.

### 2.2 β-ensemble

Nekrasov’s deformation corresponds to the modification of the energy-momentum tensor (2.13) by the introduction of the background charge à la Feign-Fuchs:

$$T(z) = -\frac{1}{2} : \partial \phi(z)^2 : + \frac{Q}{\sqrt{2}} \partial^2 \phi(z), \quad Q = b + \frac{1}{b}. \quad (2.30)$$

This energy momentum tensor has the central charge $c = 1 + 6Q^2$. Undeformed case is recovered at $b = i$, $Q = 0$ and $\epsilon_1 = -\epsilon_2 = g_s$.

Screening charges for this energy momentum tensor are given by

$$Q_+(b) = \int dz : e^{\sqrt{2b} \phi(z)} :, \quad Q_-(b) = \int dz : e^{\sqrt{2b}^{-1} \phi(z)} :. \quad (2.31)$$

Let

$$\langle N; b \rangle := \langle 0 | e^{-\sqrt{2b}N\phi_0}. \quad (2.32)$$

It is natural to consider the following deformation of the partition function (2.17):

$$Z := \langle N; b \rangle \exp \left( \frac{1}{2\sqrt{2} \pi g_s} \int dz W(z) \partial \phi(z) \right) (Q_+(b))^N |0\rangle$$

$$= \int d^N \lambda (\Delta(\lambda))^{-2b^2} \exp \left( -\frac{ib}{g_s} \sum_{I=1}^{N} W(\lambda_I) \right). \quad (2.33)$$

This matrix model is known as the β-ensemble \cite{57, 58, 59, 60} with $\sqrt{\beta} = -ib$. For $\beta = 1/2, 1, 2$, it corresponds to the integrations over an orthogonal, hermitian and symplectic matrix respectively.

Instead of the screening charge $Q_+(b)$, we can use $Q_-(b)$ to express the partition of the β-ensemble model. The corresponding expressions are obtained by replacing $b$ with $b^{-1}$.

It is known that this partition function is related to a non-commutative (or quantum) spectral curve \cite{59}. In this case, the non-commutativity is given by

$$[x, z] = -iQg_s = -i \left( b + \frac{1}{b} \right) g_s = \epsilon_1 + \epsilon_2. \quad (2.34)$$
For $Q \neq 0$, $x$ can be realized as a differential operator $x = -iQg_s \partial/\partial z$.

Note that the energy-momentum tensor (2.30) can be defined by the Miura transformation:

\[ x = \left( x + \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) \left( x - \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) := x^2 - g_s^2 T(z). \tag{2.35} \]

The collective field expression of the chiral boson $\phi(z)$ now becomes

\[ i\phi(z) = \frac{1}{\sqrt{2}g_s} W(z) - ib\sqrt{2} \text{Tr} \text{Log}(z - M). \tag{2.36} \]

Note that

\[ J(z) = i \partial \phi(z) = \frac{1}{\sqrt{2}g_s} W'(z) - ib\sqrt{2} \frac{1}{z - M}. \tag{2.37} \]

Using the collective field expression, we see that

\[ g_s^2 T(z) = \frac{1}{4} (W'(z))^2 - \frac{i}{2} Q g_s W''(z) - ibg_s \sum_{I=1}^{N} \frac{W'(z) - W'(\lambda_I)}{z - \lambda_I} \]

\[ - ibg_s \sum_{I=1}^{N} \frac{W'(\lambda_I)}{z - \lambda_I} + \sum_{I=1}^{N} \frac{g_s^2}{(z - \lambda_I)^2} - 2b^2 g_s^2 \sum_{I=1}^{N} \frac{1}{z - \lambda_I} \sum_{J \neq I} \frac{1}{\lambda_I - \lambda_J}. \tag{2.38} \]

The terms in the second line of (2.38) correspond to the “singular” part of the energy-momentum tensor and we can see that

\[ T(z) \bigg|_{\text{singular part}} \times e^{-S_{\text{eff}}} = \sum_{I=1}^{N} \frac{\partial}{\partial \lambda_I} \left( \frac{1}{z - \lambda_I} e^{-S_{\text{eff}}} \right), \tag{2.39} \]

where

\[ e^{-S_{\text{eff}}} = (\Delta(\lambda))^{-2b^2} \exp \left( -\frac{ib}{g_s} \sum_{I=1}^{N} W(\lambda_I) \right). \tag{2.40} \]

Therefore the Virasoro constraints for the deformed one-matrix model imply that

\[ \langle \langle g_s^2 T(z) \rangle \rangle = \frac{1}{4} W'(z)^2 - \frac{i}{2} Q g_s W''(z) + f(z), \tag{2.41} \]

where

\[ f(z) := \left\langle -ibg_s \sum_{I=1}^{N} \frac{W'(z) - W'(\lambda_I)}{z - \lambda_I} \right\rangle. \tag{2.42} \]

Here the matrix model average is defined as in the undeformed case. Explicitly for some function $O(\lambda)$ of the eigenvalues, we have

\[ \langle \langle O(\lambda) \rangle \rangle = \frac{1}{Z} \int d^N \lambda \langle\langle O(\lambda) \rangle \rangle \exp \left( -\frac{ib}{g_s} \sum_{I=1}^{N} W(\lambda_I) \right). \tag{2.43} \]

\[ \text{Later, we consider the case in which } W(z) \text{ has logarithmic singularities.} \]
Hence the quantum spectral curve related to the $\beta$-ensemble is defined by
\[
\left\langle \left( x + \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) \left( x - \frac{ig_s}{\sqrt{2}} \partial \phi(z) \right) \right\rangle = x^2 - g_s^2 \left\langle T(z) \right\rangle = 0. \tag{2.44}
\]

Explicitly, it is given by
\[
x^2 - \frac{1}{4} W''(z)^2 + \frac{i}{2} Q g_s W''(z) - f(z) = 0. \tag{2.45}
\]

Note that this quantum spectral curve can be rewritten as
\[
\left( x + \frac{1}{2} W'(z) \right) \left( x - \frac{1}{2} W'(z) \right) - f(z) = 0. \tag{2.46}
\]

Therefore, the associated local Calabi-Yau threefold also becomes a non-commutative surface:
\[
uv + \left( x + \frac{1}{2} W'(z) \right) \left( x - \frac{1}{2} W'(z) \right) - f(z) = 0, \quad [x, z] = -iQ g_s. \tag{2.47}
\]

Strictly speaking, for $Q \neq 0$, $x$ is a differential operator and the quantum spectral curve is a differential equation for some “wave function.” This equation has a close connection with $W$-gravity and Hitchin systems. For recent discussion on this point in the light of the AGT conjecture, see [14, 28, 35] and references therein. From the point of view of the $\mathcal{D}$-module, see [61]. In string theory, the non-commutativity corresponds to turn on a constant NS two-form.

Our main concern is the large $N$ string/gauge duality. In the planar limit (a large $N$ limit with the ’t Hooft coupling $S = g_s N$ kept finite), $g_s \to 0$ and thus the non-commutativity vanished:\[ [x, z] \to 0. \]

\section{2.3 Large $N$ limit}

The partition function (2.33) has a topological expansion
\[
Z = \exp \left( \sum_{k=0}^{\infty} g_s^{k-2} F_{k/2} \right). \tag{2.48}
\]

In the large $N$ limit with the ’t Hooft coupling $S = g_s N$ kept finite, leading contribution comes from the planar part $F_0$ in (2.48) and can be evaluated by the saddle point method. For simplicity, we assume that the parameter $b$ is pure imaginary and $\beta = -b^2 > 0$. In this case Nekrasov’s deformation parameters $\epsilon_1$ and $\epsilon_2$ are real.

\textsuperscript{4}The Nekrasov function in a different limit such that $\epsilon_2 \to 0$ while keeping $\epsilon_1$ finite is investigated in [62, 82, 83]. In this limit, the non-commutativity remains finite.
The stationary conditions $\partial S_{\text{eff}}/\partial \lambda_I = 0$ yield

$$W'(\lambda_I) - 2ib \sum_{I=1}^{N} \frac{1}{\lambda_I - \lambda_J} = 0, \quad I = 1, 2, \ldots, N. \quad (2.49)$$

Since we assume that $b$ is pure imaginary, these stationary equations have real solutions $\lambda_I$.

We evaluate the partition function around a classical solution with certain filling fractions $\nu_i = N_i/N$ around the local extrema $W'(a_i) = 0$.

In the planar limit, the large $N$ factorization yields

$$\lim_{g_s \to 0} \langle \langle x + i g_s \sqrt{2} \partial \phi(z) \rangle \rangle = (x + y(z))(x - y(z)) = x^2 - y(z)^2, \quad (2.50)$$

where

$$y(z) := \lim_{g_s \to 0} \frac{g_s}{\sqrt{2}} \langle \langle i \partial \phi(z) \rangle \rangle = \frac{1}{2} W'(z) - ib \int \frac{\rho(\lambda)}{z - \lambda} d\lambda. \quad (2.51)$$

Here $\rho(\lambda)$ is the density function of the solution to the stationary conditions $\rho(\lambda) = \lim_{N \to \infty} \frac{1}{N} \sum_{I=1}^{N} \delta(\lambda - \lambda_I). \quad (2.52)$

The stationary conditions in the large $N$ limit go to

$$W'(\lambda) - 2ibSP \int \frac{\rho(\lambda')}{\lambda - \lambda'} d\lambda' = 0. \quad (2.53)$$

Here $P$ denotes the principal value. Note that in the stationary conditions and in the definition of $y(z)$, the parameter $b$ always appears in the combination $\tilde{S} = -ibS$. Therefore, if we replace the 't Hooft coupling $S = g_s N$ by $\tilde{S} = -ibS = -ibg_s N$, we can use the large $N$ formulas of the undeformed Hermitian one-matrix model. We will call $\tilde{S}$ deformed 't Hooft coupling.

Hence, in the large $N$ limit, the local Calabi-Yau is deformation of $A_1 \times \mathbb{C}$

$$uv + x^2 - y^2(z) = 0, \quad (2.54)$$

and the algebraic curve $\Sigma$ becomes

$$x^2 - y^2(z) = 0, \quad (2.55)$$

and the points $(x, z)$ on $\Sigma \subset \mathbb{C}^2$ can be covered by two sheets $(x, z) = (\pm y(z), z)$.

In the large $N$ limit, the Virasoro constraints become an algebraic equation

$$y^2(z) = \frac{1}{4} W'(z)^2 + f(z). \quad (2.56)$$
3 ADE quiver matrix models and their “β-ensemble”

In this section, we first briefly review the ADE quiver matrix models \[37, 38, 39, 40, 41\]. An excellent review for the undeformed case can be found in \[63\]. Then, we introduce the “β-ensemble” or deformed ADE quiver matrix models. For \(A_{n-1}\) cases, they can be found in \[64\].

Using the β-deformed \(A_{n-1}\) quiver matrix model, we introduce a non-commutative local Calabi-Yau threefold related to deformations of \(A_{n-1}\) singularities.

3.1 ADE quiver matrix models and CFT with \(c = r\)

Let \(g\) be a finite dimensional Lie algebra of ADE type with rank \(r\), \(h\) the Cartan subalgebra of \(g\), and \(h^*\) its dual. We sometimes denote the natural pairings between \(h\) and \(h^*\) by \(\langle \cdot, \cdot \rangle\):

\[
\alpha(h) = \langle \alpha, h \rangle, \quad \alpha \in h^*, \quad h \in h.
\]  
(3.1)

Let \(\alpha_a \in h^* (a = 1, 2, \ldots, r)\) be simple roots of \(g\) and \(\langle \cdot, \cdot \rangle\) is the inner product on \(h^*\). Our normalization is chosen as \((\alpha_a, \alpha_a) = 2\). The fundamental weights are denoted by \(\Lambda^a (a = 1, 2, \ldots, r)\)

\[
(\Lambda^a, \alpha_a^\vee) = \delta_a^b, \quad \alpha_a^\vee = \frac{2\alpha_a}{(\alpha_a, \alpha_a)}.
\]  
(3.2)

In the Dynkin diagram of \(g\) we associate \(N_a \times N_a\) Hermitian matrices \(M_a\) with vertices \(a\) for simple roots \(\alpha_a\), and complex \(N_a \times N_b\) matrices \(Q_{ab}\) and their Hermitian conjugate \(Q_{ba} = Q_{ab}^\dagger\) with links connecting vertices \(a\) and \(b\). We label links of the Dynkin diagram by pairs of node label \((a, b)\) with an ordering \(a < b\). Let \(E\) and \(A\) be the set of “edges” \((a, b)\) (with \(a < b\)) and the set of “arrows” \((a, b)\) respectively:

\[
E := \{(a, b) \mid 1 \leq a < b \leq r, \ (\alpha_a, \alpha_b) = -1\},
\]  
(3.3)

\[
A := \{(a, b) \mid 1 \leq a, b \leq r, \ (\alpha_a, \alpha_b) = -1\}.
\]  
(3.4)

The partition function of the quiver matrix model \[37, 38, 39, 40, 41\] associated with \(g\) is given by

\[
Z = \int \prod_{a=1}^r [dM_a] \prod_{(a,b) \in A} [dQ_{ab}] \exp \left( \frac{1}{g_s} W(M, Q) \right),
\]  
(3.5)

where

\[
W(M, Q) = i \sum_{(a,b) \in A} s_{ab} \text{Tr} Q_{ba} M_a Q_{ab} + \sum_{a=1}^r \text{Tr} W_a(M_a),
\]  
(3.6)
with real constants \( s_{ab} \) obeying the conditions \( s_{ab} = -s_{ba} \). Note that
\[
\prod_{(a,b) \in A} |dQ_{ab}| = \prod_{(a,b) \in E} |dQ_{ba}dQ_{ab}|, \tag{3.7}
\]
\[
\sum_{(a,b) \in A} s_{ab} \text{Tr} Q_{ba} M_a Q_{ab} = \sum_{(a,b) \in E} s_{ab} \left( \text{Tr} Q_{ba} M_a Q_{ab} - \text{Tr} Q_{ab} M_b Q_{ba} \right). \tag{3.8}
\]
The integration measures \([dM_a]\) and \([dQ_{ba}dQ_{ab}]\) are defined by using the metrics \(\text{Tr}(dM_a)^2\) and \(\text{Tr}(dQ_{ba}dQ_{ab})\) respectively.

Integrations over \(Q_{ab}\) are Gauss-Fresnel type and are easily performed:
\[
\int [dQ_{ba}dQ_{ab}] \exp \left( \frac{i s_{ab}}{g_s} \left( \text{Tr} Q_{ba} M_a Q_{ab} - \text{Tr} Q_{ab} M_b Q_{ba} \right) \right) = \det \left( M_{a} \otimes 1_{N_b} - 1_{N_a} \otimes M_{b}^T \right)^{-1}, \tag{3.9}
\]
where \(1_n\) is the \(n \times n\) identity matrix and \(T\) denotes transposition. For simplicity we have chosen the normalization of the measure \([dQ_{ba}dQ_{ab}]\) to set the proportional constant in the right-handed side of (3.9) to be unity.

Now the integrand depends only on the eigenvalues of \(r\) Hermitian matrices \(M_a\). Let us denote them by \(\lambda^I_{a}(a = 1, 2, \ldots, r\) and \(I = 1, 2, \ldots, N_a)\). The partition function of the quiver matrix model (ITEP model) now reduces to the form of integrations over the eigenvalues of \(M_a\) with real constants \(s_{ab}\)
\[
Z = \int \prod_{a=1}^{r} \left( \prod_{I=1}^{N_a} d\lambda^I_a \right) \Delta_{g}(\lambda) \exp \left( \sum_{a=1}^{r} \sum_{I=1}^{N_a} \frac{1}{g_s} W_a(\lambda^I_a) \right), \tag{3.10}
\]
where \(W_a\) is a potential and
\[
\Delta_{g}(\lambda) = \prod_{a=1}^{r} \prod_{1 \leq i < j \leq N_a} (\lambda^I_a - \lambda^J_a)^2 \prod_{1 \leq a < b \leq r} \prod_{I=1}^{N_a} \prod_{J=1}^{N_b} (\lambda^I_a - \lambda^J_b)^{s_{ab}}. \tag{3.11}
\]
The partition function (3.10) can be rewritten in terms of CFT operators. Let \(\phi(z)\) be \(\mathfrak{h}\)-valued massless chiral field and \(\phi_{a}(z) := \langle \alpha_{a}, \phi(z) \rangle\). Their correlators are given by
\[
\langle \phi_{a}(z)\phi_{b}(w) \rangle = -(\alpha_{a},\alpha_{b}) \log(z - w), \quad a, b = 1, 2, \ldots, r. \tag{3.12}
\]
The modes
\[
\phi(z) = \phi_{0} - i a_{0} \log z + i \sum_{n \neq 0} \frac{a_{n}}{n} z^{-n} \in \mathfrak{h} \tag{3.13}
\]
obey the commutation relations
\[
[\langle \alpha, a_{n} \rangle, \langle \beta, a_{m} \rangle] = n \delta_{n+m,0}(\alpha, \beta), \quad [\langle \alpha, \phi_{0} \rangle, \langle \beta, a_{0} \rangle] = i(\alpha, \beta), \quad \alpha, \beta \in \mathfrak{h}^{*}. \tag{3.14}
\]
The Fock vacuum is given by
\[
\alpha(a_{n})|0\rangle = 0, \quad \langle 0|\alpha(a_{-n}) = 0, \quad n \geq 0, \quad \alpha \in \mathfrak{h}^{*}. \tag{3.15}
\]
Let
\[ \langle \{ N_a \} \rangle := \langle 0 \rangle \exp \left( -i \sum_{a=1}^{N} N_a \alpha_a(\phi_0) \right). \] (3.16)

It is convenient to introduce the $\mathfrak{h}^*$-valued potential $W(z)$ by
\[ W(z) := \sum_{a=1}^{r} W_a(z) \Lambda^a \in \mathfrak{h}^*. \] (3.17)

Note that $W_a(z) = (\alpha_a, W(z))$.

The energy-momentum tensor is given by
\[ T(z) = - \frac{1}{2} : K(\partial \phi(z), \partial \phi(z)) :, \] (3.18)
where $K$ is the Killing form. Let $H^i$ ($i = 1, 2, \ldots, r$) be an orthonormal basis of the Cartan subalgebra $\mathfrak{h}$ with respect to the Killing form: $K(H^i, H^j) = \delta^{ij}$. In this basis, the components of the $\mathfrak{h}$-valued chiral boson are just $r$ independent free chiral bosons:
\[ \phi(z) = \sum_{i=1}^{r} H^i \phi_i(z), \quad \langle \phi_i(z) \phi_j(w) \rangle = -\delta_{ij} \log(z-w), \] (3.19)

and the energy-momentum tensor in this basis is given by
\[ T(z) = - \frac{1}{2} \sum_{i=1}^{r} : (\partial \phi_i(z))^2 :, \] (3.20)

The central charge is $c = r$.

Note that for a root $\alpha$, $[H^i, E_a] = \alpha^i E_a$ with $\alpha^i = \alpha(H^i) = \langle \alpha, H^i \rangle$. Then, the bosons $\phi_a(z)$ associated with the simple roots $\alpha_a$ are expressed in this basis as follows:
\[ \phi_a(z) = \langle \alpha_a, \phi(z) \rangle = \sum_{i=1}^{r} \alpha_a^i \phi_i(z) \equiv \alpha_a \cdot \phi(z), \quad a = 1, 2, \ldots, r. \] (3.21)

For roots $\alpha$ and $\beta$, the inner product on the root space is expressed in their components as $\langle \alpha, \beta \rangle = \sum_{i=1}^{r} \alpha^i \beta_i$. Here $\alpha^i = \alpha(H^i)$ and $\beta^i = \beta(H^i)$.

The screening charges associated with the simple roots are defined by
\[ Q_a := \int \frac{dz}{2\pi g_s} : e^{i \phi_a(z)} :, \quad a = 1, 2, \ldots, r, \] (3.22)
with an appropriate contour integration.

Using these definitions, the partition function (3.10) can be written as follows
\[ Z = \langle \{ N_a \} \rangle \exp \left( \frac{1}{2\pi g_s} \int_{0}^{\infty} dz \langle W(z), \partial \phi(z) \rangle \right) (Q_1)^{N_1} \cdots (Q_r)^{N_r} \langle 0 \rangle. \] (3.23)
The chiral scalar field appears in the matrix model as the collective field of the eigenvalues

$$i\langle \alpha, \phi(z) \rangle = (\alpha, \frac{1}{g_s} W(z)) + \sum_{a=1}^{r} (\alpha, \alpha_a) \log \det(z - M_a), \quad \alpha \in \mathfrak{h}^*.$$  \hspace{1cm} (3.24)

In particular, for \( \alpha = \alpha_a \),

$$i\phi_a(z) = \frac{1}{g_s} W_a(z) + \sum_{b=1}^{r} (\alpha_a, \alpha_b) \log \det(z - M_b). \hspace{1cm} (3.25)$$

### 3.2 \( \beta \)-ensemble of ADE quiver matrix model

Inspired by the recent AGT conjecture, we are interested in the conformal Toda field theory based on a finite-dimensional Lie algebra \( \mathfrak{g} \) of ADE type. For the conformal Toda field theories, the energy momentum tensor is given by

$$T(z) = -\frac{1}{2} : K(\partial\phi(z), \partial\phi(z)) : + Q \langle \rho, \partial^2 \phi(z) \rangle,$$  \hspace{1cm} (3.26)

where \( Q = b + (1/b) \) and \( \rho \) is the Weyl vector of \( \mathfrak{g} \), half the sum of the positive roots. In the orthonormal basis, it takes the form

$$T(z) = -\frac{1}{2} \sum_{i=1}^{r} : (\partial\phi_i(z))^2 : + Q \sum_{i=1}^{r} \rho_i \partial^2 \phi_i(z) = -\frac{1}{2} : \partial\phi(z) \cdot \partial\phi(z) : + Q \rho \cdot \partial^2 \phi(z). \hspace{1cm} (3.27)$$

The central charge is given by

$$c = r + 12Q^2(\rho, \rho) = r \left\{ 1 + h(h + 1)Q^2 \right\}. \hspace{1cm} (3.28)$$

Here \( h \) is the Coxeter number of the simply-laced Lie algebra \( \mathfrak{g} \) whose rank is \( r \). Explicitly, \( h_{A_n-1} = n \) (with \( r = n - 1 \)), \( h_{D_r} = 2r - 2 \), \( h_{E_6} = 12 \), \( h_{E_7} = 18 \) and \( h_{E_8} = 30 \).

The partition function of the corresponding \( \beta \)-ensemble quiver matrix model (with \( \beta = -b^2 \)) is the following deformation of the quiver matrix model (3.10)

$$Z := \int \prod_{a=1}^{r} \left\{ \prod_{I=1}^{N_a} d\lambda^{(a)}_I \right\} \left( \Delta_{\mathfrak{g}}(\lambda) \right)^{-b^2} \exp \left( -\frac{ib}{g_s} \sum_{a=1}^{r} \sum_{I=1}^{N_a} W_a(\lambda^{(a)}_I) \right). \hspace{1cm} (3.29)$$

At \( b = i \), it reduces to the original quiver matrix model (3.10).

The corresponding collective field realization of chiral scalars is given by

$$i\langle \alpha, \phi(z) \rangle = (\alpha, \frac{1}{g_s} W(z)) - ib \sum_{a=1}^{r} (\alpha, \alpha_a) \log \det(z - M_a), \quad \alpha \in \mathfrak{h}^*.$$  \hspace{1cm} (3.30)

Now we require the non-commutativity

$$[x, z] = -iQ g_s, \quad Q = b + \frac{1}{b}. \hspace{1cm} (3.31)$$
If $Q \neq 0$, $x$ can be realized as $x = -iQ g_s \partial$. Here $\partial = \partial/\partial z$.

Note that in the simple root basis $\phi_a(z) = \alpha_a \cdot \phi(z)$, the collective fields are given by

$$i\phi_a(z) = \frac{1}{g_s} W_a(z) - ib \sum_{a'=1}^{r} C_{aa'} \log(z - M_a), \quad C_{aa'} := (\alpha_a, \alpha_{a'}). \quad (3.32)$$

The energy momentum tensor in this basis has the form

$$T(z) = \sum_{a,a'=1}^{r} (C^{-1})_{aa'} \left( -\frac{1}{2} : \partial \phi_a(z) \partial \phi_{a'}(z) : + Q \partial^2 \phi_{a'}(z) \right). \quad (3.33)$$

Here we have used $\rho = \sum_{a=1}^{r} \Lambda^a$. Using (3.32) and (3.33), we can check that the partition function (3.29) obey the Virasoro constraints.

### 3.3 $A_{n-1}$ quiver matrix model and non-commutative spectral curve

In this subsection, we consider the case of $g = A_{n-1} = \mathfrak{su}(n)$ with its rank $r = n - 1$. The generators of $\mathfrak{su}(n)$ algebra in the defining representation are $n \times n$ traceless Hermitian matrices. The generators of the Cartan subalgebra $\mathfrak{h}$ can be chosen as diagonal ones. Using the defining representation, the Killing form can be chosen as the trace of $n \times n$ matrices: $K(X, Y) = \text{Tr}(XY)$ for $X, Y \in \mathfrak{g}$.

In the case of $A_{n-1}$ quiver matrix model, it is convenient to denote the chiral boson as follows:

$$\phi(z) = \text{diag}(\varphi_1(z), \varphi_2(z), \ldots, \varphi_n(z)) \in \mathfrak{h}, \quad \text{Tr} \phi(z) = 0. \quad (3.34)$$

Let $\varepsilon_i$ be a linear map from $n \times n$ diagonal matrices to $\mathbb{C}$ such that

$$\varepsilon_i(\text{diag}(x_1, x_2, \ldots, x_n)) = x_i, \quad i = 1, 2, \ldots, n. \quad (3.35)$$

The simple roots $\alpha_a$ and the fundamental weights $\Lambda^a$ of $A_{n-1}$ algebra are given by

$$\alpha_a = \varepsilon_a - \varepsilon_{a+1}, \quad \Lambda^a = \sum_{i=1}^{a} \varepsilon_i - \frac{a}{n} \sum_{i=1}^{n} \varepsilon_i, \quad a = 1, 2, \ldots, n - 1. \quad (3.36)$$

With $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, we have the (symmetrized) Cartan matrix of $A_{n-1}$:

$$C_{aa'} = (\alpha_a, \alpha_{a'}) = 2\delta_{aa'} - \delta_{a,a'+1} - \delta_{a+1,a'}. \quad (3.37)$$

It follows that $(\Lambda^a, \Lambda^{a'}) = (C^{-1})_{aa'} = (1/n) \min(a, a') \{n - \max(a, a')\}$. The explicit form of the Weyl vector is given by

$$\rho = \frac{1}{2} \sum_{a>0} \alpha = \frac{1}{2} \sum_{i=1}^{n} (n - 2i + 1) \varepsilon_i. \quad (3.38)$$
Let \( v_i \in h^* \) \((i = 1, 2, \ldots, n)\) be the \( n \) weights in the defining representation with the highest weight \( \Lambda^1 \):
\[
v_i = \Lambda^1 - \sum_{a=1}^{i-1} \alpha_a = \varepsilon_i - \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j, \quad i = 1, 2, \ldots, n. \tag{3.39}
\]

Note that \((v_i, v_j) = \delta_{ij} - (1/n)\). Using these weights, the components of the chiral boson \( \phi \) can be obtained by \( \phi \) associated with the simple roots \( \alpha \) as collective fields of eigenvalues, given by \( \langle \alpha, \phi(z) \rangle = \varphi_a(z) \). The relation between the \( n - 1 \) bosons \( \varphi_a(z) \) associated with the simple roots \( \alpha_a \) and these \( n \) bosons \( \varphi_i(z) \) are given by \( \langle \alpha, \phi(z) \rangle = \varphi_a(z) - \varphi_{a+1}(z) \). Their correlators are given by
\[
\langle \varphi_a(z) \varphi_{a'}(w) \rangle = - (\alpha_a, \alpha_{a'}) \log(z - w), \quad a, a' = 1, 2, \ldots, n - 1, \quad \tag{3.40}
\]
\[
\langle \varphi_i(z) \varphi_j(w) \rangle = -(v_i, v_j) \log(z - w) = - \left( \delta_{ij} - \frac{1}{n} \right) \log(z - w), \quad i, j = 1, 2, \ldots, n. \quad \tag{3.41}
\]

Let us introduce \( n \) spin-1 currents with one constraint as follows
\[
J_i(z) := i \partial \varphi_i(z), \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} J_i(z) = 0. \tag{3.42}
\]

Applying \( (3.30) \) to the weights \( v_i \) \( (3.39) \), we can see that the spin-1 currents \( J_i(z) \) \( (3.42) \) are, as collective fields of eigenvalues, given by
\[
J_i(z) = \frac{1}{g_s} t_i(z) - i b \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) \text{Tr} \frac{1}{z - M_a}, \tag{3.43}
\]
where
\[
t_i(z) := (v_i, W'(z)) = \sum_{a=i}^{n-1} W'_a(z) - \frac{1}{n} \sum_{a=1}^{n-1} a W'_a(z), \quad i = 1, 2, \ldots, n. \tag{3.44}
\]

The partition function of the \( A_{n-1} \) quiver matrix models obeys the \( W_n = W(A_{n-1}) \) constraints. The generators of \( W_n \) algebra are currents \( W^{(s)}(z) \) with spin \( s \) \((s = 2, 3, \ldots, n)\). It can be constructed from the spin 1-currents \( J_i(z) = i \partial \varphi_i(z) \) by the Miura transformation \[47\]
\[
: \det (x - ig_s \partial \phi(z)) : = : \prod_{1 \leq i \leq n} (x - g_s J_i(z)) : = x^n + \sum_{k=2}^{n} (-1)^k g_s^k W^{(k)}(z) x^{n-k}. \tag{3.45}
\]

Here \([x, z] = -iQg_s\) and we use the following ordering of product for non-commuting objects \( A_i \):
\[
\prod_{i_0 \leq i \leq i_1} A_i := A_{i_1} A_{i_1-1} \cdots A_{i_0+1} A_{i_0}. \tag{3.46}
\]
In particular, the current $W^{(2)}(z)$ is proportional to the energy momentum tensor $T(z)$

$$T(z) = -W^{(2)}(z)$$

$$= \sum_{1 \leq i < j \leq n} : \partial \varphi_i(z) \partial \varphi_j(z) : + Q \sum_{i=1}^{n} (n - i) \partial^2 \varphi_i(z)$$

$$= -\frac{1}{2} \sum_{i=1}^{n} : (\partial \varphi_i(z))^2 : + \frac{Q}{2} \sum_{i=1}^{n} (n - 2i + 1) \partial^2 \varphi_i(z)$$

$$= -\frac{1}{2} : \text{Tr}(\partial \varphi(z))^2 : + Q \langle \rho, \partial^2 \phi(z) \rangle.$$  

(3.47)

In (3.47), to go to the third line and the last, we used the traceless condition $\sum_i \varphi_i = 0$ and the explicit form of the Weyl vector $\rho$ (3.38), respectively. The central charge of this energy momentum tensor is $c = (n - 1)(1 + n(n + 1)Q^2)$.

The $W_n$ constraints are equivalent to the regularity of the correlation function

$$\langle \langle \det(x - ig_s \partial \phi(z)) \rangle \rangle.$$  

(3.48)

Let us see this correlator is well-defined at finite $N_a$. The collective field current $J_i(z)$ (3.43) has simple poles at $z = \lambda^{(i)}$ and at $z = \lambda^{(i-1)}$. As a function of $z$,

$$\det(x - ig_s \partial \phi(z)) = \prod_{1 \leq i < n} (x - g_s J_i(z))$$  

(3.49)

has poles at $z = \lambda^{(a)}_I$ which may cause singularity of its correlator $\langle \langle \det(x - i \partial \phi(z)) \rangle \rangle$ at real $z$. First, let us examine singularity at $z = \lambda^{(1)}_I$. Note that from the form of the collective field current (3.43), the singularities only come from the factor $(x - g_s J_2(z))(x - g_s J_3(z))$. The other factor $(x - g_s J_n(z)) \cdots (x - g_s J_3(z))$ is independent of $\lambda^{(1)}_I$. It is not difficult to check that near $z \to \lambda^{(1)}_I$,

$$\prod_{1 \leq i \leq n} (x - g_s J_j(z)) = -g_s^2 \frac{\partial}{\partial \lambda^{(1)}_I} \left( \prod_{3 \leq i \leq n} (x - g_s J_j(z)) \frac{1}{z - \lambda^{(1)}_I} e^{-S_{\text{eff}}} \right) e^{S_{\text{eff}}} + O(1),$$  

(3.50)

where the effective action $S_{\text{eff}}$ is defined by

$$e^{-S_{\text{eff}}} := (\Delta_{A_{n-1}}(\lambda))^{-b^2} \exp \left( -\frac{i b}{g_s} \sum_{a=1}^{n-1} \sum_{I=1}^{N_a} W_a(\lambda^{(a)}_I) \right).$$  

(3.51)

Similar relations hold near other points $z \to \lambda^{(a)}_I$. Therefore, the singular part of $\det(x - ig_s \partial \phi(z))$ is given by

$$\det(x - ig_s \partial \phi(z)) \bigg|_{\text{singular}} \times e^{-S_{\text{eff}}}$$

$$= -g_s^2 \sum_{a=1}^{n-1} \sum_{I=1}^{N_a} \frac{\partial}{\partial \lambda^{(a)}_I} \left( \prod_{i+2 \leq j \leq n} (x - g_s J_j(z)) \frac{1}{z - \lambda^{(a)}_I} \prod_{1 \leq k \leq i-1} (x - g_s J_k(z)) e^{-S_{\text{eff}}} \right).$$  

(3.52)
Hence we have the $W_n$ constraints:

$$\left\langle \left\langle \det(x - i g_s \partial \phi(z)) \right\rangle \right\rangle_{\text{singular}} = 0. \quad (3.53)$$

Therefore, we can use this correlator to define a non-commutative Calabi-Yau threefold as follows:

$$uv + \left\langle \left\langle \det(x - i g_s \partial \phi(z)) \right\rangle \right\rangle = 0,$$  

and the associated non-commutative spectral curve $\Sigma$ can be defined by

$$\left\langle \left\langle \det(x - i g_s \partial \phi(z)) \right\rangle \right\rangle = 0, \quad (3.55)$$

with the non-commutativity $[x, z] = -iQg_s$. These are well-defined at finite $N_a$.

Note that the correlator takes the form

$$\left\langle \left\langle \det(x - i g_s \partial \phi(z)) \right\rangle \right\rangle = x^n + \sum_{k=2}^n (-1)^k P_k(z) x^{n-k}, \quad P_k(z) = g_s^k \left\langle \left\langle W^{(k)}(z) \right\rangle \right\rangle. \quad (3.56)$$

In principle, the explicit form of $P_k(z)$ can be determined by examining regular part of $\det(x - i g_s \partial \phi(z))$. But actual calculation is tedious for general $n$. In general, the non-commutative geometry takes the form

$$uv + \prod_{1 \leq i \leq n} (x - t_i(z)) + \cdots = 0. \quad (3.57)$$

In the next subsection, we give the explicit form of the spectral curve for $n = 3$.

Before going to the case of $n = 3$, let us consider the large $N$ limit. As in the one-matrix model case, we take $g_s \to 0$ limit with the deformed 't Hooft couplings $\tilde{S}_a := -i b g_s N_a$ fixed. In the large $N$, the saddle points are determined by the stationary condition $\partial S_{\text{eff}} / \partial \lambda^{(a)}_I = 0$. In the $g_s \to 0$ limit, the non-commutativity is lost and the large $N$ factorization gives

$$\lim_{g_s \to 0} \left\langle \left\langle \det(x - i g_s \partial \phi(z)) \right\rangle \right\rangle = \prod_{i=1}^n (x - y_i(z)), \quad (3.58)$$

where

$$y_i(z) := \lim_{g_s \to 0} i g_s \left\langle \left\langle \partial \varphi_i(z) \right\rangle \right\rangle = t_i(z) + \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) \tilde{S}_a \int \frac{\rho_a(\lambda)}{z - \lambda} d\lambda. \quad (3.59)$$

Here the density functions $\rho_a(\lambda)$ are defined by

$$\rho_a(\lambda) := \lim_{N_a \to \infty} \frac{1}{N_a} \sum_{I=1}^{N_a} \delta(\lambda - \lambda^{(a)}_I), \quad a = 1, 2, \ldots, n - 1. \quad (3.60)$$

In the deformed $A_{n-1}$ quiver matrix model, the deformed geometry takes the form

$$uv + \prod_{i=1}^n (x - y_i(z)) = uv + \prod_{i=1}^n (x - t_i(z)) + \cdots = 0. \quad (3.61)$$
Therefore, the deformed spectral curve $\Sigma$ is covered by $n$ Riemann sheets and the points on the $i$-th sheet can be parametrized by $(x, z) = (y_i(z), z)$. Hence they are fibered over the complex $z$-plane.

Generalization to $D_r$ and $E_r$ cases is straightforward at least in the large $N$ limit. Replace the 't Hooft couplings $S_a = g_s N_a$ by the deformed ones $\tilde{S}_a = -ibg_s N_a = \epsilon_1 N_a$. The finite non-commutative form of the spectral curve would be fixed by requiring the consistency with the symmetry of the partition function (3.29).

### 3.4 Deformed $A_2$ quiver model and non-commutative curve

For $\beta$-deformed $A_2$ quiver matrix model, the explicit forms of the currents are given by

$$
\begin{align*}
g_s J_1(z) &= t_1(z) - ibg_s \operatorname{Tr} \frac{1}{z - M_1}, \\
g_s J_2(z) &= t_2(z) + ibg_s \operatorname{Tr} \frac{1}{z - M_1} - ibg_s \operatorname{Tr} \frac{1}{z - M_2}, \\
g_s J_3(z) &= t_3(z) + ibg_s \operatorname{Tr} \frac{1}{z - M_2},
\end{align*}
$$

(3.62)

where

$$
t_1(z) = \frac{1}{3}(2W'_1(z) + W'_2(z)), \quad t_3(z) = -\frac{1}{3}(W'_1(z) + 2W'_2(z)),
$$

(3.63)

and $t_2 = -(t_1 + t_3)$. With some work, we can see that at finite $N_a$ (with $[x, z] = -ig_s a$), the regular part of the “spectral determinant” is given by

$$
\begin{align*}
\prod_{1 \leq i \leq 3} (x - g_s J_i(z)) &\bigg|_{\text{regular}} \\
= \prod_{1 \leq i \leq 3} (x - t_i(z)) \\
- (x - t_3(z))(-ibg_s) &\sum_{I=1}^{N_1} \frac{W'_1(z) - W'_1(\lambda^{(1)}_I)}{z - \lambda^{(1)}_I} - (-ibg_s) \sum_{J=1}^{N_2} \frac{W'_2(z) - W'_2(\lambda^{(2)}_J)}{z - \lambda^{(2)}_J} (x - t_1(z)) \\
- (-ibg_s)^2 &\sum_{I=1}^{N_1} \sum_{J=1}^{N_2} \frac{W'_1(z) - W'_1(\lambda^{(1)}_I)}{(z - \lambda^{(1)}_I)(\lambda^{(1)}_I - \lambda^{(2)}_J)} + (-ibg_s)^2 \sum_{I=1}^{N_1} \sum_{J=1}^{N_2} \frac{W'_2(z) - W'_2(\lambda^{(2)}_J)}{(z - \lambda^{(2)}_J)(\lambda^{(2)}_J - \lambda^{(1)}_I)} \\
+ g_s^2 &\sum_{I=1}^{N_1} \frac{\partial}{\partial \lambda^{(1)}_I} \left( \frac{t_3(z) - t_3(\lambda^{(1)}_I)}{z - \lambda^{(1)}_I} e^{-S_{\text{eff}}} \right) e^{S_{\text{eff}}} + g_s^2 \sum_{J=1}^{N_2} \frac{\partial}{\partial \lambda^{(2)}_J} \left( \frac{t_1(z) - t_1(\lambda^{(2)}_J)}{z - \lambda^{(2)}_J} e^{-S_{\text{eff}}} \right) e^{S_{\text{eff}}}.
\end{align*}
$$

(3.64)
The last two terms in the right-handed side of (3.64) do not contribute to the correlator. Therefore, we have the explicit form of the finite $N$ spectral curve mentioned in the introduction:

\[
0 = \langle \langle \det(x - ig_s \partial \phi(z)) \rangle \rangle \\
= \prod_{1 \leq i \leq 3} (x - t_i(z)) - (x - t_3(z)) f_1(z) - f_2(z)(x - t_1(z)) - g_1(z) + g_2(z),
\]

where

\[
f_a(z) := \left\langle \left\langle \left( -ibg_s \sum_{I=1}^{N_a} W'_a(z) - W'_a(\lambda^{(a)}_I) \right) \right\rangle \right\rangle,
\]

\[
g_a(z) := \left\langle \left\langle \left( -ibg_s \right)^2 \sum_{I=1}^{N_a} \sum_{J=1}^{N_a} \frac{W'_a(z) - W'_a(\lambda^{(a)}_I)}{(z - \lambda^{(a)}_I)(\lambda^{(a)}_I - \lambda^{(a)}_J)} \right\rangle \right\rangle,
\]

with $\bar{a} = 3 - a$. At the commutative point $[x, z] = 0 (b = i)$, the algebraic curve of this form (in the large $N$ limit) is already known [41, 65, 66, 67, 68, 69].

In the large $N$ limit with general $b$, spectral curve factorizes into

\[
0 = \prod_{i=1}^{3} (x - t_i(z)) - f_1(z)(x - t_3(z)) - f_2(z)(x - t_1(z)) - g_1(z) + g_2(z)
\]

\[
= \prod_{i=1}^{3} (x - y_i(z)),
\]

where

\[
y_1(z) = t_1(z) + \tilde{S}_1 \omega_1(z), \quad y_2(z) = t_2(z) - \tilde{S}_1 \omega_1(z) + \tilde{S}_2 \omega_2(z), \quad y_3(z) = t_3(z) - \tilde{S}_2 \omega_2(z).
\]

Here $\tilde{S}_a = -ibg_s N_a$ and

\[
\omega_a(z) = \lim_{N_a \to \infty} \frac{1}{N_a} \left\langle \left\langle \frac{1}{\text{Tr} \left\langle \frac{1}{z - M_a} \right\rangle} \right\rangle \right\rangle = \int \frac{\rho_a(\lambda)}{z - \lambda} d\lambda.
\]

### 4 Dijkgraaf-Vafa’s proposal: multi-Penner potential

In [7, 8], it was pointed out that the correlation function of $A_{n-1}$ Toda field theory (Liouville theory for $n = 2$) in two dimensions can be associated with Nekrasov’s partition function [9, 10] of $SU(n)$ quiver gauge theory. For $SU(n)$ gauge theory with $2n$ hypermultiplets its Nekrasov partition function is identified with the (chiral) four point function of the Toda theory on a sphere:

\[
\langle V_{\mu_0}(z_0)V_{\mu_1}(z_1)V_{\mu_2}(z_2)V_{\mu_3}(z_3) \rangle,
\]

(4.1)
under the identification (2.1) between the deformation parameters and the parameter $b$ appearing in the central charge (3.28) of the Toda theory.

The positions of the vertex operators correspond to punctures of the sphere. Three of them can be chosen at $z = 0, 1, \infty$. As we will see later, this sphere can be identified with the one on which $n$ M5-branes wrap. The vectors $\hat{\mu}_p$ in the dual space $\mathfrak{h}^*$ of the Cartan subalgebra are expanded as $\hat{\mu}_p = \sum_a \mu_{p,a} \Lambda^a$. In general, it is possible to consider many types of punctures in the sphere on which M5-branes wrap. This variety corresponds to the flavor symmetries of the gauge theory [1]. In the Toda theory, this corresponds to choice of the vectors $\hat{\mu}_p$. The vertex operator $V_{\hat{\mu}}(z)$ corresponding to the “simple” puncture associated with the $U(1)$ flavor symmetry is \[ \hat{\mu} \propto \Lambda^1 \text{ or } \Lambda^{n-1} \] (4.2)

where $\Lambda^1 (\Lambda^{n-1})$ is the highest weight of the (anti-)fundamental representation. The other types of punctures correspond to more generic choices of $\hat{\mu}$. The $SU(n)$ gauge theory with $2n$ hypermultiplets has $SU(n)^2 \times U(1)^2$ flavor symmetry as a subgroup of $U(2n)$. Therefore, for this gauge theory, the corresponding correlation function is such that two vertex operators are of “simple” type like (4.2) while the other two are of generic “$SU(n)$” type.

As discussed in [24], by using the relation between $A_{n-1}$ Toda field theory and the quiver matrix model [37, 38], we reach the matrix model which describes $\mathcal{N} = 2$ $SU(n)$ quiver gauge theory. The prescription introduced in [24] is that the original matrix model action is set to zero. Under zero action, we consider the correlation functions of the vertex operators considered above. In the correlation function, the vertex operator can be written in terms of the matrices by using (3.30) as

\[ V_{\hat{\mu}}(z) = : e^{i(\hat{\mu}, \phi(z))} : \prod_{a=1}^{n-1} \det(z - M_a)^{-ib(\hat{\mu}, \alpha_a)} \] (4.3)

With zero action ($W_a(z) = 0$), we define the chiral four-point correlation function which corresponds to $SU(n)$ gauge theory with $2n$ hypermultiplets by

\[ \left< : e^{i(\hat{\mu}_0, \phi(q_0))} : \prod_{p=1}^{3} e^{i(\hat{\mu}_p, \phi(q_p))} : \right> \equiv \langle 0 | : e^{i(\hat{\mu}_0, \phi(q_0))} : \prod_{p=1}^{3} e^{i(\hat{\mu}_p, \phi(q_p))} : Q_1^{N_1} Q_2^{N_2} \ldots Q_{n-1}^{N_{n-1}} | 0 \rangle, \] (4.4)

where $Q_a = \int d\lambda : e^{b\phi_a(\lambda)} :$. For later convenience, we set $\mu_p := g_s \hat{\mu}_p$ ($p = 0, 1, 2, 3$). The momentum conservation condition is required

\[ \mu_0 + \sum_{p=1}^{3} \mu_p + \sum_{a=1}^{n-1} S_a \alpha_a = 0. \] (4.5)
Using this four-point function, we define the partition function of the deformed $A_{n-1}$ quiver matrix model by

$$Z := \lim_{q_0 \to \infty} q_0^{\langle \hat{\mu}_0, \hat{\nu}_0 \rangle} \left( : e^{i \langle \hat{\mu}_0, \phi(q_0) \rangle} : \prod_{p=1}^{3} e^{i \langle \hat{\mu}_p, \phi(q_p) \rangle} : \right)$$

$$= \int \prod_{a=1}^{n-1} \left\{ \prod_{l=1}^{N_a} d\lambda^{(a)}_l \right\} \left( \Delta_{A_{n-1}}(\lambda) \right)^{-b^2} \exp \left( -\frac{ib}{g_s} \sum_{a=1}^{n-1} \sum_{l=1}^{N_a} W_a(\lambda^{(a)}_l) \right),$$

(4.6)

where the determinantal form of the vertex operators (4.3) at punctures $q_p$ is converted into a set of new matrix model potentials $W_a(z)$ with multi-Penner type interaction:

$$W_a(z) = \sum_{p=1}^{3} (\mu_p, \alpha_a) \log(q_p - z).$$

Later we will set $q_1 = 0$, $q_2 = 1$ and $q_3 = q$.

We denote the components of $\mu_p$ ($p = 0, 1, 2, 3$) in the fundamental weight basis by $\mu_{p,a}$: $\mu_p = \sum_a \mu_{p,a} \Lambda^a$. As explained above, we let the operators at $z = q_2, q_3$ be of the simple type. That is, we set

$$\mu_2 = \mu_{U(1)_1} \Lambda^1, \quad \mu_3 = \mu_{U(1)_2} \Lambda^{n-1}.$$  

(4.8)

The operators at $z = q_1, \infty$ are of the generic type.

### 4.1 $A_2$ quiver matrix model corresponding to $SU(3)$ gauge theory

In this subsection, we consider $A_2$ quiver matrix model with the action (4.7). As seen above, the Nekrasov partition function for $\mathcal{N} = 2$ supersymmetric $SU(3)$ gauge theory with six massive flavors is expected to be described by $A_2$ quiver matrix model with the action (4.7) with $n = 3$.

We have explicitly seen the spectral curve of $A_2$ quiver matrix model in the large $N_a$ limit in the subsection 3.4. For convenience, we rewrite (3.68) in the following form:

$$x^3 = p(z)x + q(z),$$

(4.9)

where

$$p(z) = t_1(z)^2 + t_3(z)^2 + t_1(z)t_3(z) + f_1(z) + f_2(z),$$

$$q(z) = -t_1(z)t_3(z)(t_1(z) + t_3(z)) - f_1(z)t_3(z) - f_2(z)t_1(z) + g_1(z) - g_2(z).$$

(4.10)

By substituting the action (4.7) and (4.8), the explicit forms of $t_i(z)$ (3.63) are given by

$$t_1(z) = \frac{1}{3} \sum_{p=1}^{3} \frac{(\mu_p, 2\alpha_1 + \alpha_2)}{z - q_p} = \frac{1}{3} \left( \frac{2\mu_{1,1} + \mu_{1,2}}{z} + \frac{2\mu_{U(1)_1}}{z - 1} + \frac{\mu_{U(1)_2}}{z - q} \right),$$

$$t_3(z) = -\frac{1}{3} \sum_{p=1}^{3} \frac{(\mu_p, \alpha_1 + 2\alpha_2)}{z - q_p} = -\frac{1}{3} \left( \frac{\mu_{1,1} + 2\mu_{1,2}}{z} + \frac{\mu_{U(1)_1}}{z - 1} + \frac{2\mu_{U(1)_2}}{z - q} \right),$$

(4.11)
and $t_2(z) = -t_1(z) - t_3(z)$. Here $\mu_{1,a}$ are components of $\mu_1$ in the fundamental weight basis: $\mu_1 = \mu_{1,1}\Lambda_1 + \mu_{1,2}\Lambda_2$.

The form of $f_a$ (3.66) and that of $g_a$ (3.67) are

$$f_a(z) = \sum_{p=1}^{3} \frac{f_p^{(a)}}{z - q_p}, \quad g_a(z) = \sum_{p=1}^{3} \frac{g_p^{(a)}}{z - q_p},$$

(4.12)

where $f_p^{(a)}$ and $g_p^{(a)}$ are constants. As seen above, the coordinate $z$ parametrizes the sphere. The spectral curve is a triple cover of this sphere and each value of $x$ at each sheet is given by $y_i(z)$.

We consider a one-form $\lambda_m := y_i(z)\,dz$ ($i = 1, 2, 3$) with $y_i(z)$ given by (3.69). Later on this will be identified with the Seiberg-Witten one-form on the $i$-th sheet. From the form of the spectral curve, we can see that the one-form has simple poles at $z = q_p$ and at $z = \infty$.

The residues of $y_i(z)\,dz$ ($i = 1, 2, 3$) at the simple punctures $z = 1$ and $z = q$ are given by

$$\text{res}_{z=1}(y_i(z)\,dz) = \frac{1}{3} \left( 2\mu_{U(1)_1}, -\mu_{U(1)_1}, -\mu_{U(1)_1} \right),$$

$$\text{res}_{z=q}(y_i(z)\,dz) = \frac{1}{3} \left( \mu_{U(1)_2}, \mu_{U(1)_2}, -2\mu_{U(1)_2} \right).$$

(4.13)

(4.14)

These will be compared with the mass parameter associated with the $U(1)$ flavor symmetry in the gauge theory. The residues at $z = 0$ are of generic type:

$$\text{res}_{z=0}(y_i(z)\,dz) = \frac{1}{3} \left( 2\mu_{1,1} + \mu_{1,2}, -\mu_{1,1} + \mu_{1,2}, -\mu_{1,1} - 2\mu_{1,2} \right).$$

(4.15)

This takes the same form as the mass parameters associated with the $SU(3)$ flavor symmetry, which will be analyzed in the next section.

Using the momentum conservation (4.5), the residues at $z = \infty$ are found to be

$$\text{res}_{z=\infty}(y_i(z)\,dz) = \frac{1}{3} \left( 2\mu_{0,1} + \mu_{0,2}, -\mu_{0,1} + \mu_{0,2}, -\mu_{0,1} - 2\mu_{0,2} \right).$$

(4.16)

This also takes the same form as the $SU(3)$ mass parameters.

5 **$\mathcal{N}=2$ gauge theory and Seiberg-Witten curve**

We consider $\mathcal{N}=2$ $SU(n)$ gauge theory with $N_f = 2n$. The low energy effective theory (and the quantum Coulomb moduli space) of $\mathcal{N}=2$ supersymmetric gauge theory is described by the Seiberg-Witten curve and the meromorphic one-form called Seiberg-Witten one-form [70, 71, 72, 73, 74]. We will see that the Seiberg-Witten curve of this theory enjoys the same properties as the spectral curve considered above for the $SU(n)$ ($A_{n-1}$) case.
5.1 $SU(3)$ gauge theory with $N_f = 6$

We first consider the $SU(3)$ gauge theory with six massive hypermultiplets. This theory has an exactly marginal coupling which is the microscopic gauge coupling constant:

$$\tau_{UV} = \frac{\theta_{UV}}{\pi} + \frac{8\pi i}{g_{UV}^2}. \tag{5.1}$$

The type IIA brane construction and its M-theory lift lead to the Seiberg-Witten curve which is a hypersurface in 2-complex dimensional space $(t, v)$ [73 II]

$$(v - m_1)(v - m_2)(v - m_3)t^2 - (1 + q_{UV})(v^3 + P v^2 + Q v + R)t$$
$$+ q_{UV}(v - m_4)(v - m_5)(v - m_6) = 0, \tag{5.2}$$

where $m_i$ ($i = 1, \ldots, 6$) are the mass parameters of the hypermultiplets and $q_{UV} = e^{2\pi i \tau_{UV}}$. We can see that the coordinate $v$ has mass dimension one and therefore the constants $P$, $Q$ and $R$ have mass dimension one, two and three respectively. In principle, the constants depend on the mass parameters and the Coulomb moduli parameters $u^{(2)}$ and $u^{(3)}$ which correspond to $\langle \text{Tr} \phi^2 \rangle$ and $\langle \text{Tr} \phi^3 \rangle$ at weak coupling. The dimensional analysis and the regularity constraint in the massless limit show that $Q$ and $R$ are, respectively, linear in $u^{(2)}$ and $u^{(3)}$ and also include the terms $m^2$ and $m^3$. Also, $P$ depends only on the mass parameters. The curve is translated into the following form

$$(t - 1)(t - q_{UV})v^3 = M_2(t)v^2 + V_2^{(2)}(t)v + V_2^{(3)}(t), \tag{5.3}$$

where $M_2$, $V_2^{(2)}$ and $V_2^{(3)}$ are degree two polynomials in $t$:

$$M_2(t) = \left(\sum_{i=1}^{3} m_i \right) t^2 + P(1 + q_{UV})t + q_{UV} \left(\sum_{j=4}^{6} m_j \right),$$
$$V_2^{(2)}(t) = -\left(\sum_{1 \leq i < j \leq 3} m_i m_j \right) t^2 + Q(1 + q_{UV})t - q_{UV} \left(\sum_{4 \leq i < j \leq 6} m_i m_j \right), \tag{5.4}$$
$$V_2^{(3)}(t) = m_1 m_2 m_3 t^2 + R(1 + q_{UV})t + m_4 m_5 m_6 q_{UV}.$$

As in [II], by shifting the coordinate $v$ appropriately, we can eliminate the quadratic term in $v$. Then, by changing the coordinate $v = xt$, we obtain [II]

$$x^3 = \frac{P_4^{(2)}(t)}{(t(t - 1)(t - q_{UV}))^2}x + \frac{P_6^{(3)}(t)}{(t(t - 1)(t - q_{UV}))^3}, \tag{5.5}$$

where $P_4^{(2)}(t)$ and $P_6^{(3)}(t)$ are the degree 4 and 6 polynomials respectively and can be written in terms of $M_2$, $V_2^{(2)}$ and $V_2^{(3)}$

$$P_4^{(2)}(t) = \frac{M_2(t)^2}{3} + V_2^{(2)}(t)(t - 1)(t - q_{UV}),$$
$$P_6^{(3)}(t) = \frac{2M_2(t)^3}{27} + \frac{1}{3} M_2(t) V_2^{(2)}(t)(t - 1)(t - q_{UV}) + V_2^{(3)}(t)(t - 1)^2(t - q_{UV})^2. \tag{5.6}$$
In this coordinate, the Seiberg-Witten one-form is $\lambda_{SW} = x \, dt$. We denote by $t$ a coordinate in the sphere on which three M5-branes wrap. Therefore, $(t, x)$ are local coordinates in the cotangent bundle of this sphere. We identify the sphere with the one that appeared in the quiver matrix model and Toda theory. This implies $q = q_{UV}$.

From the expression (5.5), we can see that the Seiberg-Witten one-form has poles at $t = 0, 1, q_{UV}$ and $\infty$. As found in [1] and we will see below, the structure of those poles corresponds to the flavor symmetry of the quiver gauge theory, because the residues of the Seiberg-Witten one-form are the mass parameters associated with the flavor symmetries.

The residues of the Seiberg-Witten one-form $x \, dt$ at $t = 1$ and $t = q_{UV}$ are given by

\begin{align*}
\text{res}_{t=1}(x \, dt) &= \frac{M_2(1)}{3(1-q_{UV})} (2, -1, -1), \\
\text{res}_{t=q_{UV}}(x \, dt) &= \frac{M_2(q_{UV})}{3q_{UV}(1-q_{UV})} (1, 1, -2),
\end{align*}

where each value denotes the residue at the $i$-th sheet of the Riemann surface.

The residues at $t = 0$ are found to be

\begin{equation}
\text{res}_{t=0}(x \, dt) = \frac{1}{3} (2m_4 - m_5 - m_6, -m_4 + 2m_5 - m_6, -m_4 - m_5 + 2m_6).
\end{equation}

The flavor symmetry $U(1) \times SU(3)$ associated with the punctures at $t = 1, \infty$ are the symmetry of three hypermultiplets whose masses are $m_i (i = 1, 2, 3)$. Since the mass associated with $U(1)$ is $\sum_{i=1}^{3} m_i/3$, the residue at $t = 1$ should be of the form

\begin{equation}
\frac{1}{3} \left( 2 \sum_{i=1}^{3} m_i, - \sum_{i=1}^{3} m_i, - \sum_{i=1}^{3} m_i \right).
\end{equation}

Also, the residue at $t = q_{UV}$ should be of the form

\begin{equation}
\frac{1}{3} \left( \sum_{j=4}^{6} m_j, \sum_{j=4}^{6} m_j, -2 \sum_{j=4}^{6} m_j \right).
\end{equation}

These determine $M_2(t)$ completely and lead to

\begin{equation}
P = -q_{UV} \frac{1 + q_{UV}}{1 + q_{UV}} \sum_{i=1}^{6} m_i.
\end{equation}

From this, we can compute the residues at $t = \infty$ as

\begin{equation}
\frac{1}{3} (-2m_1 + m_2 + m_3, m_1 - 2m_2 + m_3, m_1 + m_2 - 2m_3),
\end{equation}

which corresponds to the Cartan part of $SU(3)$. At $t = 0$, we also find the similar form as above. From the residue analysis alone, we cannot determine the constants $Q$ and $R$ in (5.2).
5.1.1 Matching of the matrix model parameters

Obviously, the coordinate \( t \) is identified with \( z \) and \( x \, dt \) on the \( i \)-th sheet with \( y_i \, dz \). Comparing the residues (4.13), (4.14) with (5.7), we find that the parameters in \( A_2 \) quiver matrix model must be chosen as

\[
\mu_{U(1)}^{(1)} = \sum_{i=1}^{3} m_i, \quad \mu_{U(1)}^{(2)} = \sum_{j=4}^{6} m_j, \tag{5.13}
\]

in order to yield the Seiberg-Witten curve of the gauge theory. Also, by comparing (4.16) with (5.12), and (4.15) with (5.8), we have

\[
\mu_{0,1} = -m_1 + m_2, \quad \mu_{0,2} = -m_2 + m_3, \quad \mu_{1,1} = m_4 - m_5, \quad \mu_{1,2} = m_5 - m_6. \tag{5.14}
\]

To summarize, we have determined the “weights” of the vertex operators \( V(\mu_p/g_s)(q_p) \) as follows:

\[
\begin{align*}
\mu_0 &= (-m_1 + m_2)\Lambda^1 + (-m_2 + m_3)\Lambda^2, \\
\mu_1 &= (m_4 - m_5)\Lambda^1 + (m_5 - m_6)\Lambda^2, \\
\mu_2 &= (m_1 + m_2 + m_3)\Lambda^1, \\
\mu_3 &= (m_4 + m_5 + m_6)\Lambda^2. \tag{5.15}
\end{align*}
\]

The matrix model potential \( W_a(z) \) with the multi-log interaction are finally fixed as (up to constants)

\[
\begin{align*}
W_1(z) &= (m_4 - m_5) \log z + \left( \sum_{i=1}^{3} m_i \right) \log(1 - z), \\
W_2(z) &= (m_5 - m_6) \log z + \left( \sum_{j=4}^{6} m_j \right) \log(q_{UV} - z). \tag{5.16}
\end{align*}
\]

From the momentum conservation (4.5), the deformed 't Hooft couplings \( \tilde{S}_a = -i g_s N_a \) are related to the mass parameters of the gauge theory as follows:

\[
\begin{align*}
\tilde{S}_1 &= -m_2 - m_3 - m_4, \\
\tilde{S}_2 &= -m_3 - m_4 - m_5. \tag{5.17}
\end{align*}
\]

5.2 Isomorphism between the matrix model curve and the Seiberg-Witten curve for general \( n \)

It is straightforward to generalize the analysis for \( n = 3 \) to general \( n \). The spectral curve (3.55) of the \( A_{n-1} \) quiver matrix model has the form

\[
x^n = \sum_{k=2}^{n} (-1)^{k-1} P_k(z) x^{n-k}, \tag{5.18}
\]
where
\[ P_k(z) = g_s^k \langle \mathcal{W}^{(k)}(z) \rangle. \] (5.19)

With the choice of the multi-Penner potential [4.7], \( P_k(z) \) have the following form of the singularities:
\[ P_k(z) = \frac{Q_k(z)}{(z(z-1)(z-q))^k}, \] (5.20)
for some polynomials \( Q_k(z) \) in \( z \).

On the other hand, The Seiberg-Witten curve [75, 1] for the \( SU(n) \) gauge theory with \( N_f = 2n \) massive hypermultiplets is given by
\[
\prod_{i=1}^{n} (v - m_i) t^2 - (1 + q_{UV}) \left( v^n + P v^{n-1} + \sum_{\ell=2}^{n} Q_{\ell} v^{n-\ell} \right) t + q_{UV} \prod_{j=1}^{n} (v - \tilde{m}_j) = 0. \] (5.21)

The mass parameters of the hypermultiplets are denoted by \( m_i (i = 1, 2, \ldots, 2n) \), and here for simplicity, we have set \( \tilde{m}_j := m_{n+j} (j = 1, 2, \ldots, n) \). Also \( q_{UV} = e^{\pi i \tau_{UV}} \). This curve can be rewritten as
\[
(t - 1)(t - q_{UV})v^n = M_2(t) v^{n-1} + \sum_{\ell=2}^{n} V_{2}^{(\ell)}(t) v^{n-\ell}, \] (5.22)
where
\[
M_2(t) = \sigma_1(m) t^2 + (1 + q_{UV}) P t + q_{UV} \sigma_1(\tilde{m}), \]
\[
V_{2}^{(\ell)}(t) = (-1)^{\ell-1} \sigma_\ell(m) t^2 + (1 + q_{UV}) Q_{\ell} t + (-1)^{\ell-1} \sigma_\ell(\tilde{m}). \] (5.23)

Here \( \sigma_\ell(m) \) (resp. \( \sigma_\ell(\tilde{m}) \)) is the \( \ell \)-th elementary symmetric polynomial of \( \{m_1, \ldots, m_n\} \) (resp. of \( \{\tilde{m}_1, \ldots, \tilde{m}_n\} \)).

By a change of variable
\[
v = xt + \frac{M_2(t)}{n(t-1)(t-q_{UV})}, \]
the Seiberg-Witten curve [5, 22] turns into the Gaiotto form:
\[
x^n = \frac{P_{4}^{(2)}(t) x^{n-2}}{(t(t-1)(t-q_{UV}))^2} + \frac{P_{6}^{(3)}(t) x^{n-3}}{(t(t-1)(t-q_{UV}))^3} + \cdots + \frac{P_{2n}^{(n)}(t)}{(t(t-1)(t-q_{UV}))^n}, \] (5.24)
where \( P_{2k}^{(k)}(t) \) are the following degree 2\( k \) polynomials in \( t \):
\[
P_{2k}^{(k)}(t) = \frac{n!(k-1)!}{k!(n-k)!} \left( \frac{M_2(t)}{n} \right)^k \]
\[+ \sum_{\ell=2}^{k} \frac{(n-\ell)!}{(n-k)!(k-\ell)!} \left( \frac{M_2(t)}{n} \right)^{k-\ell} V_{2}^{(\ell)}(t) \left( (t-1)(t-q_{UV}) \right)^{\ell-1}. \] (5.26)
By evaluating the residue at $t = 1$ and at $t = q$ as in the case of $n = 3$, the constant $P$, which enters in $M_2(t)$ (5.23), is fixed as

$$P = -\frac{q_{UV}}{1 + q_{UV}} \sum_{i=1}^{n} (m_i + \bar{m}_i).$$  \hspace{1cm} (5.28)

Now, we can clearly see the similarity between the matrix model curve (5.18) in the planar limit and the Seiberg-Witten curve (5.26).

Let us compare the residues of one-forms $y_i(z) \, dz$ and $x \, dt$. In the $A_{n-1}$ quiver matrix model, the curve (5.18) in the planar limit factorizes into

$$\prod_{i=1}^{n} (x - y_i(z)) = 0,$$  \hspace{1cm} (5.29)

where $y_i(z)$ is defined by (3.59). The residues of $y_i(z) \, dz$ ($i = 1, 2, \ldots, n$) at the simple punctures $z = 1$ and $z = q$ are given by

$$\text{res}_{z=1}(y_i(z) \, dz) = \frac{\mu_{U(1)_i}}{n} \left( n - 1, -1, \ldots, -1 \right),$$  \hspace{1cm} (5.30)

$$\text{res}_{z=q}(y_i(z) \, dz) = \frac{\mu_{U(1)_i}}{n} \left( 1, \ldots, 1, -(n-1) \right).$$  \hspace{1cm} (5.31)

The residues at generic punctures $z = 0$ and $z = \infty$ are found to be

$$\text{res}_{z=0}(y_i(z) \, dz) = \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) (\mu_1, \Lambda^a),$$  \hspace{1cm} (5.32)

$$\text{res}_{z=\infty}(y_i(z) \, dz) = \sum_{a=1}^{n-1} (\delta_{i,a} - \delta_{i,a+1}) (\mu_0, \Lambda^a),$$  \hspace{1cm} (5.33)

for $i = 1, 2, \ldots, n$. While the residues of the Seiberg-Witten one-form $x \, dt$ at $t = 1$ and $t = q_{UV}$ are given by

$$\text{res}_{t=1}(x \, dt) = \frac{M_2(1)}{n(1 - q_{UV})} \left( n - 1, -1, \ldots, -1 \right) = \frac{1}{n} \sigma_1(m) \left( n - 1, -1, \ldots, -1 \right),$$

$$\text{res}_{t=q_{UV}}(x \, dt) = \frac{M_2(q_{UV})}{n q_{UV}(1 - q_{UV})} \left( 1, \ldots, 1, -(n-1) \right) = \frac{1}{n} \sigma_1(\bar{m}) \left( 1, \ldots, 1, -(n-1) \right),$$  \hspace{1cm} (5.34)

where each value denotes the residue at the $i$-th sheet of the Riemann surface. The residues at $t = 0$ and at $t = \infty$ on the $i$-th sheet ($i = 1, 2, \ldots, n$) are given by

$$\text{res}_{t=0}(x \, dt) = \bar{m}_i - \frac{1}{n} \left( \sum_{j=1}^{n} \bar{m}_j \right),$$

$$\text{res}_{t=\infty}(x \, dt) = \frac{1}{n} \left( \sum_{j=1}^{n} m_j \right) - m_i.$$  \hspace{1cm} (5.35)
For general $n$, these residues match if the weights of the vertex operators are identified with the mass parameters of the gauge theory as follows

$$
\mu_0 = \sum_{a=1}^{n-1} (-m_a + m_{a+1}) \Lambda^a, \quad \mu_1 = \sum_{a=1}^{n-1} (\tilde{m}_a - \tilde{m}_{a+1}) \Lambda^a,
$$

$$
\mu_2 = \left( \sum_{i=1}^{n} m_i \right) \Lambda^1, \quad \mu_3 = \left( \sum_{i=1}^{n} \tilde{m}_i \right) \Lambda^{n-1}.
$$

The matrix model potentials $W_a(z)$ ($a = 1, 2, \ldots, n-1$) are fixed as

$$
W_a(z) = (\tilde{m}_a - \tilde{m}_{a+1}) \log z + \delta_{a,1} \left( \sum_{i=1}^{n} m_i \right) \log(1-z) + \delta_{a,n-1} \left( \sum_{i=1}^{n} \tilde{m}_i \right) \log(q_{UV} - z).
$$

With this choice of the multi-log potentials, the $A_{n-1}$ quiver matrix model curve (5.18) in the planar limit coincides with the $SU(n)$ Seiberg-Witten curve (5.26) with $N_f = 2n$ massive hypermultiplets.

It is rather simple to provide the counting of the 0d-parameters and 4d-parameters and the matching of the numbers of parameters. With the rule concerning with the types (simple versus generic) of the punctures [8] explained before, the four of the vertex operators contain $2(n-1) + 2 = 2n$ parameters. This matches the number of mass parameters in the gauge theory side. As for the number of the Coulomb moduli, let us first note that, in the planar limit, the $A_{n-1}$ quiver matrix model develops an array(or ladder) of $n-1$ species of two-cut eigenvalue distributions. The Riemann surface developed is genus $n-1$ and is depicted as a linear array of $n$ $P_1$s with adjacent two (say, $P_i$ and $P_{i+1}$) connected by two pipes consisting of the $i$-th eigenvalue distribution. Each of the $n-1$ Coulomb moduli is obtained simply by integrating the matrix model differential over one of the two pipes and of course represent the filling fraction of the $i$-th eigenvalue distribution. This is an obvious generalization of $n = 1$ case noted in [24].

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