Research Article

Half-Logistic Xgamma Distribution: Properties and Estimation under Censored Samples

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This paper proposed a new probability distribution, namely, the half-logistic xgamma (HLXG) distribution. Various statistical properties, such as, moments, incomplete moments, mean residual life, and stochastic ordering of the proposed distribution, are discussed. Parameter estimation of the half-logistic xgamma distribution is approached by the maximum likelihood method based on complete and censored samples. Asymptotic confidence intervals of model parameters are provided. A simulation study is conducted to illustrate the theoretical results. Moreover, the model parameters of the HLXG distribution are estimated by using the maximum likelihood, least square, maximum product spacing, percentile, and Cramer–von Mises (CVM) methods. Superiority of the new model over some existing distributions is illustrated through three real data sets.

1. Introduction

In the recent years, generalization and extension of existing distributions, to enhance flexibility in modeling a variety of data, have received attention of many researchers. Marshall and Olkin [1] proposed a general method of adding a parameter to a family of distributions which is called the Marshall–Olkin-G (MO-G) class. The new parameter gives more flexibility and extends several well-known distributions. For any continuous baseline with the cumulative distribution function (cdf) say \( G(x) \), and the probability density function (pdf), say \( g(x) \), the MO-G family is defined as follows:

\[
f_{\text{MO-G}}(x; \delta) = \frac{\delta g(x)}{[1 - \delta G(x)]^2}, \quad (1)
\]

where \( \delta > 0, \) \( \delta = 1 - \delta \), and \( G(x) = 1 - G(x) \) is the survival function (sf). Lately, some of the notable new families of distributions include beta-G [2, 3], gamma-G [4], Kumaraswamy-G [5], transformed-transformer [6], exponentiated generalized [7], Weibull-G [8], exponentiated half-logistic-G [9], Kumaraswamy–Weibull-G [10], exponentiated Weibull-G [11], additive Weibull-G [12], type I half-logistic-G [13], type II half-logistic-G (TIIHL-G) [10], generalized additive Weibull-G [14], odd exponentiated half-logistic-G [15], generalized odd log-logistic-G [16], inverse Weibull-G [17], power Lindley-G [18], and type II generalized Topp–Leone-G [19].

Hassan et al. [20] proposed a class of probability distribution generated by a half-logistic random variable with the following cdf:

\[
F_{\text{TIIHL-G}}(x; \lambda, \zeta) = \frac{2[G(x; \zeta)]^{\lambda}}{1 + [G(x; \zeta)]^\lambda}, \quad \lambda > 0, \ x \in \mathbb{R}, \quad (2)
\]

where \( \lambda \) is the shape parameter and \( G(x; \zeta) \) is a baseline cdf, which relies on a parameter vector \( \zeta \). The pdf corresponding to (2) is given by

\[
f_{\text{TIIHL-G}}(x; \lambda, \zeta) = \frac{2\lambda g(x; \zeta)[G(x; \zeta)]^{\lambda-1}}{[1 + [G(x; \zeta)]^\lambda]^2}, \quad \lambda > 0. \quad (3)
\]

For \( \lambda = 1 \) in (3) and \( \delta = 0.5 \) in (1), pdf (3) provides the MO-G family. The pdf (3) provides a number of known
distributions as particular cases with more flexibility in their skewness and kurtosis.

Recently, Sen et al. [21] proposed the xgamma (XG) distribution, following the idea of Lindley distribution. The XG model is a special finite mixture from exponential with a scale parameter ($\theta$) and gamma with a scale parameter ($\theta$) and shape parameter ($\beta$). The pdf and cdf of the XG distribution are, respectively, given by

$$G(x; \theta) = 1 - \left(1 + \theta + \theta x + (\theta^2 x^2/2)/ (1 + \theta)\right) e^{-\theta x}, \quad x > 0, \theta > 0,$$

and

$$g(x; \theta) = \frac{\theta^2}{1 + \theta} \left(1 + \theta x^2/2\right) e^{-\theta x}, \quad x > 0, \theta > 0.$$

Sen et al. [21] investigated the structural properties of the XG distribution, and they have found that in many cases that the XG distribution has more flexibility than the exponential distribution. The extensions of XG distribution were studied in [22, 23] and [24], among others.

Our objective here is two folds: First, based on the TI11HL-G class, we propose a new distribution related to xgamma distribution. We call the new model as the half-logistic xgamma (HLXG) distribution, and we study its several statistical properties. Second, the maximum likelihood (ML) method is employed to estimate the model parameters of the HLXG based on complete and censored samples. Further, (1 − $\nu$)% asymptotic confidence intervals (CIs) of the model parameters are constructed. Simulation and application issues are considered. The rest of the paper is organized as follows. Sections 2 and 3 provide the pdf, cdf, hazard rate function (hrf), and structure properties of the HLXG model. In Section 4, point and approximate CIs of model parameters are derived under complete, type I censoring (TIC) and type II censoring (TIIIC). Also, in the same section, the behavior of the estimates is studied via a simulation study. Real data sets are analyzed to demonstrate the flexibility of the HLXG distribution over some known distributions in Section 5. The article ends with some concluding remarks.

2. The Half-Logistic Xgamma Distribution

In this section, we provide a more flexible model by adding one extra shape parameter to the XG model for improving its goodness-of-fit to real data. The motivations of the HLXG distribution are (i) to obtain a more flexible pdf with right skewed, unimodal, and reversed $J$-shape; (ii) to be capable of modeling decreasing, increasing, and semimatbathazard rate shapes; and (iii) to provide more flexibility to model the various types of data.

**Definition 1.** A random variable $X$ is said to have a HLXG distribution, if its cdf is given by

$$F(x; \theta, \lambda) = 2 \left[ 1 - A(\theta, x) \right]^\lambda \left[ 1 + [1 - A(\theta, x)]^{\lambda - 1} \right], \quad x > 0,$$

where $A(\theta, x) = (1 + \theta + \theta x + (\theta^2 x^2)/2) e^{-\theta x}$.

The corresponding pdf is given as follows:

$$f(x; \lambda, \theta) = 2\lambda \theta^2 (1 + \theta x^2/2) e^{-\theta x} \left[ 1 - A(\theta, x) \right]^{\lambda - 1} / \left[ 1 + [1 - A(\theta, x)]^{\lambda - 1} \right], \quad x > 0, \theta > 0, \lambda > 0.$$

(7)

A random variable $X$ with pdf (7) will be denoted by HLXG$(\lambda, \theta)$. For $\lambda = 1$ in (7), we obtain the MO extended xgamma distribution with $\delta = 0.5$ as a special new model. Further, the sf and hrf of the HLXG are given, respectively, by

$$\bar{F}(x; \lambda, \theta) = 1 - [1 - A(\theta, x)]^{\lambda-1} / \left[ 1 + [1 - A(\theta, x)]^{\lambda-1} \right]$$

and

$$h(x; \lambda, \theta) = 2\lambda \theta^2 (1 + \theta x^2/2) e^{-\theta x} \left[ 1 - A(\theta, x) \right]^{\lambda - 1} / \left[ 1 + [1 - A(\theta, x)]^{\lambda - 1} \right].$$

The pdf and hrf plots for the HLXG are displayed in Figure 1 for some given values of parameters.

Figure 1 reveals that the pdf of $X$ is quite flexible and can take asymmetric forms, among others. Also, the hrf can be increasing, decreasing, and semimatbathtub shapes. In general, they reinforce the importance of the HLXG model to fit real lifetime data.

3. Main Properties

In this section, we give some important properties of the HLXG distribution such as ordinary and incomplete moments, stochastic ordering, mean residual life, and mean waiting time, among others.

3.1. Expansion. This section provides a useful expansion of the HLXG pdf due its complicated form to obtain its structure properties. Since the generalized binomial series is

$$(1 + z)^{-c} = \sum_{i=0}^{\infty} (-1)^i \binom{c + i - 1}{i} z^i, \quad |z| < 1, c > 0,$$

then, by applying (8) in (7), we have

$$f(x; \lambda, \theta) = \sum_{c=0}^{\infty} (-1)^c (i + 1) \lambda \theta^2 \left[ 1 + \theta x^2/2 \right] e^{-\theta x} \left[ 1 - A(\theta, x) \right]^{\lambda - 1}.$$  

Employing the binomial expansion in the last term of (9), we have

$$f(x; \lambda, \theta) = \sum_{c=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \binom{i+j+1}{i} \lambda \theta^2 \left[ 1 + \theta x^2/2 \right] e^{-\theta x} \left[ 1 - A(\theta, x) \right]^{\lambda - 1}.$$

Again, we use the binomial expansion more than one time; then, we have

$$f(x; \lambda, \theta) = \sum_{c=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \lambda \theta^2 m (1 + \theta)^{m+1} \left[ 1 - A(\theta, x) \right]^{\lambda - 1}.$$

(10)
3.2. Moments and Related Statistics. In this section, we obtain moments and some related measures of the HLXG distribution with pdf defined in (10). The $s^{th}$ moment about the origin of the random variable $X$ has the HLXG distribution obtained as follows:

$$
\mu'_s = \sum_{i,j=0}^{\infty} \eta_{i,j,m,k} \Gamma(m + k + s + 1) / (\theta(j + 1))^{m+k+s+1} + \psi_{i,j,m,k} \Gamma(m + k + s + 3) / (\theta(j + 1))^{m+k+s+3},
$$

(12)

where $\Gamma(\cdot)$ is the gamma function. Furthermore, the $s^{th}$ central moment of a given random variable $X$ is defined by

$$
\mu_s = E(X - \mu)^s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (\mu'_i) \mu_{s-i}.
$$

(13)

Table 1 gives the numerical values for certain parameter values of mean ($\mu'_i$), variance ($\sigma^2$), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the HLXG distribution. It is evident from Table 1 that skewness and kurtosis heavily depend on the value of the parameters of the new distribution.

Additionally, the $s^{th}$ lower incomplete moment, say $\pi_s(t)$, of the HLXG distribution is given by

$$
\pi_s(t) = \sum_{i,j=0}^{\infty} \eta_{i,j,m,k} \Gamma(m + k + s + 1) / (\theta(j + 1))^{m+k+s+1} + \psi_{i,j,m,k} \Gamma(m + k + s + 3) / (\theta(j + 1))^{m+k+s+3} / (\theta(j + 1))^{m+k+s+3},
$$

(15)

where $\gamma(\cdot,t)$ is the lower incomplete gamma function. The first incomplete moment, say $\pi_1$, for $s = 1$ in (15) is obtained. The Lorenz and Bonferroni curves are the essential applications of the first incomplete moment. The Lorenz curve, say $LO(t)$, and the Bonferroni curve, say $BO(t)$, of the HLXG are obtained, respectively, as follows:

$$
LO(t) = \frac{\pi_1(t)}{E(T)} = \frac{\sum_{i,j=0}^{\infty} \eta_{i,j,m,k} \gamma(m + k + 2, \theta(j + 1)t)}{\sum_{i,j=0}^{\infty} \eta_{i,j,m,k} \Gamma(m + k + 2) / (\theta(j + 1))^{m+k+2}} + \psi_{i,j,m,k} \gamma(m + k + 4, \theta(j + 1)t) / (\theta(j + 1))^{m+k+4}.
$$

(16)
Table 1: Some moments of the HXLG distribution.

| \( \mu_i \) | \( (\lambda = 0.5, \theta = 2) \) | \( (\lambda = 1, \theta = 2) \) | \( (\lambda = 2, \theta = 2) \) | \( (\lambda = 3, \theta = 2) \) | \( (\lambda = 0.5, \theta = 0.5) \) | \( (\lambda = 1, \theta = 0.5) \) |
|----------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \mu_1 \) | 0.315 | 0.559 | 0.897 | 1.124 | 2.192 | 4.014 |
| \( \sigma^2 \) | 0.296 | 0.451 | 0.606 | 0.691 | 7.405 | 8.322 |
| CS | 3.119 | 2.22 | 1.59 | 1.275 | 2.118 | 1.775 |
| CK | 15.912 | 9.586 | 6.537 | 5.495 | 9.05 | 7.273 |

Table 2: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HXLG model under TIIC for \((\lambda = 0.5 \text{ and } \theta = 0.5)\).

| \( n \) | \( r \) | Parameters | ML | Bias | MSE | 90% LB | 90% UB | 90% AL | 95% LB | 95% UB | 95% AL |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | \( \lambda \) | 0.5510 | 0.0069 | 0.0432 | 0.2250 | 0.3309 | 0.3957 | 0.3580 | 0.4730 |
| | \( \theta \) | 0.5803 | 0.0094 | 0.0643 | 0.2620 | 0.3720 | 0.4379 | 0.4000 | 0.5250 |
| 30 | \( \lambda \) | 0.5299 | 0.0073 | 0.0457 | 0.2720 | 0.3770 | 0.4429 | 0.4050 | 0.5300 |
| | \( \theta \) | 0.5593 | 0.0107 | 0.0667 | 0.3120 | 0.4220 | 0.4879 | 0.4500 | 0.5750 |
| 50 | \( \lambda \) | 0.5217 | 0.0103 | 0.0577 | 0.2920 | 0.4020 | 0.4679 | 0.4300 | 0.5550 |
| | \( \theta \) | 0.5433 | 0.0119 | 0.0687 | 0.3320 | 0.4420 | 0.5079 | 0.4700 | 0.6050 |
| 70 | \( \lambda \) | 0.5071 | 0.0089 | 0.0432 | 0.2480 | 0.3530 | 0.4189 | 0.3800 | 0.5050 |
| | \( \theta \) | 0.5555 | 0.0114 | 0.0643 | 0.2880 | 0.3980 | 0.4639 | 0.4200 | 0.5450 |
| 100 | \( \lambda \) | 0.5050 | 0.0079 | 0.0402 | 0.2280 | 0.3330 | 0.3989 | 0.3600 | 0.4850 |
| | \( \theta \) | 0.5017 | 0.0104 | 0.0612 | 0.2680 | 0.3780 | 0.4439 | 0.4000 | 0.5250 |

Table 3: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HXLG model under complete and TIIC for \((\lambda = 0.5 \text{ and } \theta = 1.2)\).

| \( n \) | \( r \) | Parameters | ML | Bias | MSE | 90% LB | 90% UB | 90% AL | 95% LB | 95% UB | 95% AL |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | \( \lambda \) | 0.5416 | 0.0141 | 0.0229 | 0.3455 | 0.4555 | 0.5105 | 0.4300 | 0.6350 |
| | \( \theta \) | 1.5025 | 0.0325 | 0.0847 | 0.4767 | 0.5867 | 0.6417 | 0.5612 | 0.7662 |
| 30 | \( \lambda \) | 0.5344 | 0.0244 | 0.0144 | 0.3641 | 0.4741 | 0.5291 | 0.4586 | 0.6641 |
| | \( \theta \) | 1.4224 | 0.0224 | 0.0190 | 0.3707 | 0.4807 | 0.5357 | 0.4652 | 0.6707 |
| 50 | \( \lambda \) | 0.5311 | 0.0231 | 0.0141 | 0.3704 | 0.4804 | 0.5354 | 0.4649 | 0.6754 |
| | \( \theta \) | 1.3807 | 0.0187 | 0.0136 | 0.3749 | 0.4849 | 0.5399 | 0.4694 | 0.6849 |
| 70 | \( \lambda \) | 0.5261 | 0.0261 | 0.0052 | 0.4225 | 0.5325 | 0.6025 | 0.5220 | 0.7270 |
| | \( \theta \) | 1.3005 | 0.0105 | 0.0104 | 0.4224 | 0.5324 | 0.6024 | 0.5220 | 0.7270 |
| 100 | \( \lambda \) | 0.5223 | 0.0223 | 0.0033 | 0.4363 | 0.5523 | 0.6323 | 0.5418 | 0.7563 |
| | \( \theta \) | 1.2722 | 0.0172 | 0.0119 | 0.4419 | 0.5519 | 0.6319 | 0.5414 | 0.7559 |
Table 4: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HXLG model under complete and TIC for $(\lambda = 1.7$ and $\theta = 0.9$).

| n  | $r$ | Parameters | ML Bias MSE | 90% LB UB AL | 95% LB UB AL |
|----|----|------------|-------------|--------------|--------------|
| 21 | 0.1 | $\lambda$ | 1.9508 0.2508 0.4006 1.0597 2.8420 1.7823 0.8890 | 3.0126 2.1236 |
| 30 | 0.3 | $\theta$ | 0.9795 0.0795 0.0445 0.6481 1.3110 0.6630 0.5846 | 1.3745 0.7899 |
| 27 | 0.5 | $\lambda$ | 1.8748 0.1748 0.2036 1.1078 2.6419 1.5341 0.9609 | 2.7887 1.8278 |
| 30 | 0.5 | $\theta$ | 0.9549 0.0540 0.0332 0.6768 1.2311 0.5543 0.6237 | 1.2842 0.6605 |
| 35 | 0.5 | $\lambda$ | 1.8608 0.1608 0.1895 1.1304 2.5912 1.4608 0.9905 | 2.7311 1.7405 |
| 45 | 0.5 | $\theta$ | 0.9501 0.0501 0.0278 0.6911 1.2091 0.5180 0.6415 | 1.2587 0.6172 |
| 50 | 0.5 | $\lambda$ | 1.8334 0.1334 0.1636 1.1994 2.4675 1.2681 1.0780 | 2.5889 1.5109 |
| 50 | 0.5 | $\theta$ | 0.9632 0.0632 0.0217 0.7077 1.2187 0.5110 0.6588 | 1.2676 0.6088 |
| 70 | 0.5 | $\lambda$ | 1.8020 0.1020 0.1246 1.2391 2.3649 1.1257 1.1313 | 2.4727 1.3413 |
| 90 | 0.5 | $\theta$ | 0.9494 0.0494 0.0158 0.7344 1.1645 0.4301 0.6932 | 1.2057 0.5125 |
| 100| 0.5 | $\lambda$ | 1.7859 0.0859 0.1112 1.2500 2.3219 1.0718 1.1474 | 2.4245 1.2771 |
| 100| 0.5 | $\theta$ | 0.9416 0.0416 0.0139 0.7415 1.1417 0.4002 0.7032 | 1.1800 0.4768 |

Table 5: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HXLG model under complete and TIC for $(\lambda = 0.5$ and $\theta = 0.5$).

| n  | $T$ | Parameters | ML Bias MSE | 90% LB UB AL | 95% LB UB AL |
|----|----|------------|-------------|--------------|--------------|
| 30 | 2.5 | $\lambda$ | 0.5234 0.0234 0.0128 | 0.3389 0.3049 | 0.3969 0.3639 |
| 4  | 2.5 | $\theta$ | 0.5466 0.0466 0.0415 | 0.2456 0.2325 | 0.3645 0.3396 |
| 50 | 2.5 | $\lambda$ | 0.5069 0.0069 0.0058 | 0.3692 0.3642 | 0.4646 0.4596 |
| 4  | 2.5 | $\theta$ | 0.5151 0.0151 0.0158 | 0.2899 0.2849 | 0.7403 0.7353 |
| 100| 2.5 | $\lambda$ | 0.4993 0.0007 0.0045 | 0.4040 0.4090 | 0.5946 0.5996 |
| 4  | 2.5 | $\theta$ | 0.5067 0.0067 0.0089 | 0.3493 0.3543 | 0.6464 0.6514 |
| 30 | 4  | $\lambda$ | 0.5221 0.0221 0.0122 | 0.3282 0.3242 | 0.7161 0.7011 |
| 4  | 4  | $\theta$ | 0.4990 0.0010 0.0040 | 0.4098 0.4158 | 0.5882 0.5932 |

Table 6: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HXLG under complete and TIC for $(\lambda = 0.5$ and $\theta = 1.2$).

| n  | $T$ | Parameters | ML Bias MSE | 90% LB UB AL | 95% LB UB AL |
|----|----|------------|-------------|--------------|--------------|
| 30 | 2.5 | $\lambda$ | 0.5270 0.0270 0.0121 | 0.3630 0.3680 | 0.6910 0.6960 |
| 4  | 2.5 | $\theta$ | 1.3309 0.1309 0.1940 | 0.7125 0.7275 | 1.9493 1.9643 |
| 50 | 2.5 | $\lambda$ | 0.5299 0.0299 0.0114 | 0.3687 0.3737 | 0.6911 0.6961 |
| 4  | 2.5 | $\theta$ | 1.3492 0.1492 0.1850 | 0.7610 0.7660 | 1.9374 1.9424 |
| 100| 2.5 | $\lambda$ | 0.5220 0.0220 0.0074 | 0.3963 0.3913 | 0.6476 0.6526 |
| 4  | 2.5 | $\theta$ | 1.2486 0.0486 0.1130 | 0.7989 0.8039 | 1.6983 1.7033 |
| 30 | 4  | $\lambda$ | 0.5219 0.0219 0.0072 | 0.3989 0.3939 | 0.6449 0.6499 |
| 4  | 4  | $\theta$ | 1.2468 0.0468 0.1042 | 0.8242 0.8292 | 1.6693 1.6743 |
| 50 | 4  | $\lambda$ | 0.5001 0.0001 0.0024 | 0.4157 0.4207 | 0.5844 0.5894 |
| 4  | 4  | $\theta$ | 1.2348 0.0348 0.0406 | 0.9149 0.9249 | 1.5547 1.5697 |
| 100| 4  | $\lambda$ | 0.4995 0.0005 0.0021 | 0.4170 0.4210 | 0.5821 0.5871 |
| 4  | 4  | $\theta$ | 1.2310 0.0310 0.0328 | 0.9313 0.9363 | 1.5308 1.5358 |
Table 7: ML estimates, biases, MSES, LBs, UBs, and ALs of the HLGX under complete and TIC for ($\lambda = 1.7$ and $\theta = 0.9$).

| $n$ | $T$ | Parameters | ML  | Bias | MSE | 90%  | 95%  | LB  | UB  | AL  | LB  | UB  | AL  |
|-----|-----|------------|-----|------|-----|------|------|-----|-----|-----|-----|-----|-----|
| 2.5 | 30  | $\lambda$ | 1.9974 | 0.2974 | 0.5593 | 1.0347 | 2.9601 | 1.9254 | 0.8503 | 3.1445 | 2.2942 |
|     |     | $\theta$  | 0.9956 | 0.0956 | 0.0612 | 0.6387 | 1.3525 | 0.7137 | 0.5704 | 1.4208 | 0.8504 |
| 4   | 50  | $\lambda$ | 1.9080 | 0.2080 | 0.3347 | 1.0963 | 2.7196 | 1.6233 | 0.9409 | 2.8750 | 1.9341 |
|     |     | $\theta$  | 0.9626 | 0.0626 | 0.0389 | 0.6704 | 1.2547 | 0.5842 | 0.6145 | 1.3106 | 0.6961 |

and

$$BO(t) = \frac{LO(t)}{F(t; \lambda, \theta)} = \frac{\sum_{i,j=0}^{\infty} \left[ \sum_{i,j=0}^{\infty} \eta_{i,j} \gamma_{i,j} \left( m + k + 2, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+2} \right] + \psi_{i,j} \gamma_{i,j} \left( m + k + 4, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+4}}{F(t; \lambda, \theta) \sum_{i,j=0}^{\infty} \left[ \sum_{i,j=0}^{\infty} \eta_{i,j} \gamma_{i,j} \left( m + k + 2, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+2} \right] + \psi_{i,j} \gamma_{i,j} \left( m + k + 4, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+4}}.$$  

3.3. Mean Residual Life and Mean Waiting Time. The mean residual life (MRL) function at age $t$ measures the expected remaining lifetime of an individual of age $t$. The MRL of $X$ is given by

$$M(t) = \frac{1}{F(t; \lambda, \theta)} \left[ E(T) - \pi_1(T) \right] - t.$$  

Hence, the MRL of $X$ can be obtained as follows:

$$M(t) = \frac{1}{F(t; \lambda, \theta)} \left[ \sum_{i,j=0}^{\infty} \eta_{i,j} \gamma_{i,j} \left( m + k + 2, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+2} \right] + \psi_{i,j} \gamma_{i,j} \left( m + k + 4, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+4}.$$  

The mean waiting time (MWT) of an item failed in an interval $[0, t]$ is defined as

$$\overline{M}(t) = t - \frac{\pi_1(t)}{F(t)}.$$  

Hence, the MWT of the HLGX distribution is determined as follows:

$$\overline{M}(t) = t - \frac{1}{F(t; \lambda, \alpha, \beta)} \left[ \sum_{i,j=0}^{\infty} \eta_{i,j} \gamma_{i,j} \left( m + k + 2, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+2} \right] + \psi_{i,j} \gamma_{i,j} \left( m + k + 4, \theta(j + 1)t \right) / \left( \theta(j + 1) \right)^{m+k+4}.$$  

3.4. Stochastic Ordering. Shaked and Shanthikumar [25] stated that for independent random variables $X$ and $Y$ with cdfs $F_X$ and $F_Y$, respectively, $X$ is said to be smaller than $Y$ in the following ordering, if the following is available:

(i) Stochastic order ($X \leq_s Y$) if $F_X(x) \geq F_Y(x)$ for all $x$

(ii) Likelihood ratio order ($X \leq_r Y$) if $f_X(x)/f_Y(x)$ is decreasing in $x$

(iii) Hazard rate order ($X \leq_h Y$) if $h_X(x) \geq h_Y(x)$ for all $x$
Table 8: The sf and hrf estimates at $t_0 \in \{3, 5, 10\}$ of the HLGX distribution under TIIC.

| Parameters | $\lambda$ | $\theta$ | $n$ | $r$ | Exact values | $\hat{F}(3)$ | $\hat{F}(5)$ | $\hat{F}(10)$ | $\hat{h}(3)$ | $\hat{h}(5)$ | $\hat{h}(10)$ |
|------------|-----------|----------|-----|-----|--------------|------------|------------|------------|------------|------------|-----------|
|            | 0.5       | 0.5      | 50  | 45  | $F(50) = 0.0074909$ | 0.213266  | 0.105614   | 0.016006   | 0.346226   | 0.357975   | 0.397136   |
|            | 0.5       | 1.2      | 50  | 45  | $F(50) = 0.000601$ | 0.028081  | 0.003789   | 0.000004   | 1.051110   | 1.061410   | 1.072880   |
|            | 1.7       | 0.9      | 50  | 45  | $F(50) = 0.0028425$ | 0.268432  | 0.074945   | 0.000004   | 0.951551   | 0.961405   | 0.981762   |

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| Parameters | $\theta$ | $n$ | $T$ | Exact values | $\tilde{F}(3)$ | $\tilde{F}(5)$ | $\tilde{F}(10)$ | $\tilde{h}(3)$ | $\tilde{h}(5)$ | $\tilde{h}(10)$ |
|------------|---------|-----|-----|--------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\lambda$  | $\theta$ |     |     |              |                |                |                |                |                |                |
| 0.5        | 0.5     | 30  | 2.5 | $\bar{F}(3) = 0.233346$ | 0.216432 | 0.107133 | 0.016198 | 0.346273 | 0.358332 | 0.397679 |
|            |         |     | 4   | $\bar{F}(5) = 0.122910$ | 0.217353 | 0.107951 | 0.016484 | 0.344680 | 0.356563 | 0.395511 |
|            |         |     | 2.5 | $\bar{F}(10) = 0.022299$ | 0.227453 | 0.117434 | 0.020097 | 0.326357 | 0.336167 | 0.370517 |
|            |         | 50  | 4   | $h(3) = 0.316873$     | 0.224381 | 0.114758 | 0.019105 | 0.330856 | 0.341117 | 0.376559 |
|            |         |     | 2.5 | $h(5) = 0.325682$     | 0.229037 | 0.119549 | 0.02132  | 0.321265 | 0.330391 | 0.363401 |
|            |         | 100 | 4   | $h(10) = 0.357710$    | 0.229797 | 0.120191 | 0.02137  | 0.32089  | 0.329326 | 0.362107 |
|            | 1.2     | 30  | 2.5 | $\bar{F}(3) = 0.0413473$ | 0.031630 | 0.004572 | 0.000020 | 0.919579 | 1.012360 | 1.148190 |
|            |         |     | 4   | $\bar{F}(5) = 0.0074909$ | 0.030409 | 0.004257 | 0.00007  | 0.934418 | 1.026280 | 1.166240 |
|            |         |     | 2.5 | $\bar{F}(10) = 0.0000601$ | 0.038328 | 0.006391 | 0.000041 | 0.853517 | 0.937665 | 1.067000 |
|            |         | 50  | 4   | $h(3) = 0.8152270$    | 0.038497 | 0.006439 | 0.000041 | 0.852064 | 0.935098 | 1.065200 |
|            |         |     | 2.5 | $h(5) = 0.8926640$    | 0.037983 | 0.006485 | 0.000044 | 0.842627 | 0.924811 | 1.053380 |
|            |         | 100 | 4   | $h(10) = 1.0191500$   | 0.038291 | 0.006580 | 0.000046 | 0.839676 | 0.920789 | 1.049710 |
| 1.7        | 0.9     | 30  | 2.5 | $\bar{F}(3) = 0.2849660$ | 0.265858 | 0.068976 | 0.00462  | 0.629726 | 0.710327 | 0.818858 |
|            |         |     | 4   | $\bar{F}(5) = 0.0861552$ | 0.274869 | 0.075262 | 0.00860  | 0.604664 | 0.681868 | 0.786724 |
|            |         |     | 2.5 | $\bar{F}(10) = 0.0028425$ | 0.301250 | 0.098725 | 0.003734 | 0.533615 | 0.603083 | 0.696883 |
|            |         | 50  | 4   | $h(3) = 0.5597790$    | 0.290238 | 0.088247 | 0.002953 | 0.556904 | 0.626226 | 0.723173 |
|            |         |     | 2.5 | $h(5) = 0.6288880$    | 0.293989 | 0.090596 | 0.003149 | 0.550078 | 0.619223 | 0.715160 |
|            |         | 100 | 4   | $h(10) = 0.7261120$   | 0.284586 | 0.085431 | 0.002760 | 0.562942 | 0.632723 | 0.730538 |
(iv) Mean residual life order \((X \leq_{ml} Y)\) if \(m_X(x) \geq m_Y(x)\) for all \(x\).

We have the following chain of implications among the various partial orderings discussed above:

\[
X \leq_{hr} Y \implies X \perp Y \implies X \leq_{ml} Y. \quad (22)
\]

**Theorem 1.** Let \(X-HLXG(\lambda_1, \theta_1)\) and \(Y-HLXG(\lambda_2, \theta_2)\). If \(\lambda_1 \geq \lambda_2\) and \(\theta_1 \geq \theta_2\), then \(X \leq_{hr} Y, X \leq_{lr} Y, X \leq_{ml} Y, \) and \(X \leq_{nu} Y\).

**Proof.** It is sufficient to show that \(f_X(x)/f_Y(x)\) is a decreasing function of \(x\); the likelihood ratio is

\[
f_X(x)/f_Y(x) = \left(1 + \theta_1 x^2/2\right)e^{-\theta_1 x} \left[1 - A(\theta_1, x)\right]^{1/2} \left(1 + \theta_2 x^2/2\right)e^{-\theta_2 x} \left[1 - A(\theta_2, x)\right]^{1/2}.
\]

Therefore,

\[
\frac{d}{dx} \log f_X(x) = \frac{\theta_1 x}{1 + \theta_1 x^2} - \frac{\lambda_1}{1 - A(\theta_1, x)} - \frac{2\lambda_2 [1 - A(\theta_2, x)]^{1/2} - 1}{1 + [1 - A(\theta_2, x)]^{1/2}} - \frac{\theta_2 x}{1 + \theta_2 x^2} + \frac{\lambda_2}{1 - A(\theta_2, x)} + \frac{2\lambda_1 [1 - A(\theta_1, x)]^{1/2} - 1}{1 + [1 - A(\theta_1, x)]^{1/2}} < 0,
\]

where \(A'(\theta_i, x) = -e^{-\theta_i x} [\theta_i^2 + 2\theta_i x^2/1 + \theta_i], i = 1, 2\). Thus, \(f_X(x)/f_Y(x)\) is decreasing in \(x\), and hence \(X \leq_{hr} Y\). Similarly, we can conclude for \(X \leq_{hr} Y, X \leq_{ml} Y, \) and \(X \leq_{nu} Y\).

**4. Parameter Estimation under Censored Samples**

In survival analysis and the industrial life testing model, it is necessary to minimize the cost and/or the duration of a life-testing experiment, so one may choose to terminate the test early, which results in the so-called censored sampling scheme. TIC and TIIC are the most popular types of censoring. The life testing experiment is ended at a predetermined time, say \(T\), for the TIC, while for the TIIC, the test is ended when a predetermined number of failures, say \(r\), is reached.

In the following sections, the maximum likelihood (ML) estimators of the model parameters of the HLXG model based on complete, TIC, and TIIC are provided. Approximate CIs are constructed. Further, estimators of survival function and hazard rate function for different mission times are given.

**4.1. Maximum Likelihood Estimators via TIIC.** Let \(X_{(1)} < X_{(2)} < \cdots < X_{(n)}\) be a TIIC of size \(r\) from a life test on \(n\) items whose lifetimes have the HLXG model with parameters \(\lambda\) and \(\theta\). The log-likelihood function, denoted by \(\ln \ell_i\), of \(r\) failures and \((n - r)\) censored values is given by

\[
\ln \ell_i = \ln r + 2 \lambda + 2 \theta \ln \theta + \sum_{i=1}^{r} \left[\ln \left(1 + \frac{\theta x_{(i)}^2}{2}\right) - \frac{r}{r} \theta x_{(i)}^2\right] + (n - r) \left[\ln \left(1 - A(\theta, x_{(i)})\right) - r \ln (1 + \theta) - 2 \sum_{i=1}^{r} \left(1 - A(\theta, x_{(i)})\right) + (n - r) \left[\ln \left(1 - A(\theta, x_{(i)})\right)\right] - \theta x_{(i)}^2\right],
\]

where \(x_{(i)} = Q(\theta)\). The uniroutefunction of the R software can be used to solve the nonlinear equation (25), numerically, and then the HLXG random variable \(X\) can be generated, where \(u\) has the uniform distribution on the interval \((0, 1)\).
where \( c = n!/(r-1)! \) \((n-r)\)! , \( A(\theta, x_{(i)}) = (1 + \theta + \theta x_{(i)} + \theta^2 x_{(i)}^2/2 + 1 + \theta) e^{-\theta x_{(i)}} \), and \( A(\theta, x_{(r)}) = (1 + \theta + \theta x_{(r)} + \theta^2 x_{(r)}^2/2 + 1 + \theta) e^{-\theta x_{(r)}} \).

The first partial derivatives of the log-likelihood function with respect to \( \lambda \) and \( \theta \) are obtained as follows:

\[
\frac{\partial \ln \ell_1}{\partial \lambda} = \frac{r}{\lambda} + \sum_{i=1}^{r} \ln \left[ 1 - A(\theta, x_{(i)}) \right] - \sum_{i=1}^{r} \frac{2 \ln \left[ 1 - A(\theta, x_{(i)}) \right]}{1 - A(\theta, x_{(i)})} + 1 \]

\[
\frac{\partial \ln \ell_1}{\partial \theta} = \frac{r (2 + \theta)}{\theta (1 + \theta)} + \sum_{i=1}^{r} \frac{x_{(i)}^2}{2 + \theta x_{(i)} - \sum_{i=1}^{r} x_{(i)}} - \sum_{i=1}^{r} \frac{(\lambda - 1) \partial A(\theta, x_{(i)})}{\partial \theta} \left[ 1 - A(\theta, x_{(i)}) \right] \]

\[
+ \sum_{i=1}^{r} \frac{2 \lambda [1 - A(\theta, x_{(i)})]^\lambda - 1 \partial A(\theta, x_{(i)})}{1 - \left[ 1 - A(\theta, x_{(i)}) \right]} + 1 \frac{1}{\left[ 1 - A(\theta, x_{(i)}) \right]} \]

\[
\frac{\partial \ln \ell_1}{\partial \theta} = \frac{r (2 + \theta)}{\theta (1 + \theta)} + \sum_{i=1}^{r} \frac{x_{(i)}^2}{2 + \theta x_{(i)} - \sum_{i=1}^{r} x_{(i)}} - \sum_{i=1}^{r} \frac{(\lambda - 1) \partial A(\theta, x_{(i)})}{\partial \theta} \left[ 1 - A(\theta, x_{(i)}) \right] \]

\[
+ \sum_{i=1}^{r} \frac{2 \lambda [1 - A(\theta, x_{(i)})]^\lambda - 1 \partial A(\theta, x_{(i)})}{1 - \left[ 1 - A(\theta, x_{(i)}) \right]} + 1 \frac{1}{\left[ 1 - A(\theta, x_{(i)}) \right]} \]

where

\[
\frac{\partial A(\theta, x_{(i)})}{\partial \theta} = \frac{-2(2\theta + \theta^2) x_{(i)} - \theta^2 x_{(i)}^2 - (\theta^2 + \theta^3) x_{(i)}^3}{2(1 + \theta)^2} e^{-\theta x_{(i)}} \]

\[
\frac{\partial A(\theta, x_{(r)})}{\partial \theta} = \frac{-2(2\theta + \theta^2) x_{(r)} - \theta^2 x_{(r)}^2 - (\theta^2 + \theta^3) x_{(r)}^3}{2(1 + \theta)^2} e^{-\theta x_{(r)}} \]

Putting \( \partial \ln \ell_1/\partial \lambda \) and \( \partial \ln \ell_1/\partial \theta \) equal to zero and solving these equations numerically provide the ML estimator of \( \lambda \) and \( \theta \), respectively. Note that for \( r = n \), we obtain the ML of the model parameters in case of a complete sample.

4.2. Maximum Likelihood Estimators via TIC. Suppose that \( n \) items, whose lifetimes have the HLGX distribution, are placed on a life test, and the test is terminated at a specified time \( T \) before all \( n \) items have failed. The number of failures \( r \) and all failure times are random variables. The log-likelihood function, based on TIC, is given by

\[
\ln \ell_2 = \ln c + r \ln 2 + 2 r \ln \theta + \sum_{i=1}^{r} \ln \left( 1 + \frac{\theta x_{(i)}^2}{2} \right) - \sum_{i=1}^{r} \theta x_{(i)}
\]

\[
+ (\lambda - 1) \sum_{i=1}^{r} \left[ 1 - A(\theta, x_{(i)}) \right] - r \ln (1 + \theta)
\]

\[
- 2 \sum_{i=1}^{r} \ln \left[ 1 + \left[ 1 - A(\theta, x_{(i)}) \right] \right] + (n - r)
\]

\[
\langle \ln \left[ 1 - [1 - A(\theta, T)] \right] - \ln \left[ 1 + [1 - A(\theta, T)] \right] \rangle,
\]

\[
(31)
\]

where \( A(\theta, T) = (1 + \theta + \theta T + \theta^2 T^2/2 + 1 + \theta) e^{-\theta T} \).

The first partial derivatives of the log-likelihood function with respect to \( \lambda \) and \( \theta \) are obtained as follows:

\[
\frac{\partial \ln \ell_2}{\partial \lambda} = \frac{r}{\lambda} + \sum_{i=1}^{r} \ln \left[ 1 - A(\theta, x_{(i)}) \right] - \sum_{i=1}^{r} \frac{2 \ln \left[ 1 - A(\theta, x_{(i)}) \right]}{1 - A(\theta, x_{(i)})} + 1 \]

\[
- 2(\lambda - 1) \sum_{i=1}^{r} \left[ 1 - A(\theta, x_{(i)}) \right] - r \ln (1 + \theta)
\]

\[
- 2 \sum_{i=1}^{r} \ln \left[ 1 + \left[ 1 - A(\theta, x_{(i)}) \right] \right] + (n - r)
\]

\[
\langle \ln \left[ 1 - [1 - A(\theta, T)] \right] - \ln \left[ 1 + [1 - A(\theta, T)] \right] \rangle,
\]

\[
(32)
\]

where \( \partial A(\theta, T) = -2(2\theta + \theta^2) T - 2 \theta T^2 - (\theta^2 + \theta^3) T^3 e^{-\theta T} \).

Putting \( \partial \ln \ell_2/\partial \lambda \) and \( \partial \ln \ell_2/\partial \theta \) equal to zero and solving these equations numerically provide the ML estimators of \( \lambda \) and \( \theta \).

4.3. Approximate Confidence Intervals. In this section, approximate confidence intervals (CIs) of the model parameters for the HLGX distribution are obtained. The asymptotic variances and the covariance matrix of the ML estimates are given by the elements of the inverse of the Fisher information matrix \( I_{ij}(\theta) = E \left[ \cdot \partial \ln \ell_i/\partial \lambda \partial \theta \right] \). Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the Fisher information matrix is
given by $I_{ij}(\theta) = \left\{ -\partial^2 \ln c / \partial \lambda \partial \bar{\theta} \right\}$, which is obtained by dropping the expectation on operation $E$ and replacing $\lambda$ and $\bar{\theta}$ with $\hat{\lambda}$ and $\hat{\bar{\theta}}$, respectively (see [26]). The asymptotic variance-covariance matrix $F$ for the maximum likelihood estimates can be written as follows:

$$F = I_{ij}(\hat{\lambda}, \hat{\bar{\theta}}) = -\frac{\partial^2 \ln c_i}{\partial \ln \lambda \partial \ln \bar{\theta}}_{\lambda=\hat{\lambda}, \bar{\theta}=\hat{\bar{\theta}}}.$$ (35)

Hence, the approximate 100 $(1 - v)$% two-sided CIs for $\lambda$ and $\bar{\theta}$ are, respectively, given by

$$\hat{\lambda} \pm Z_{n/2} \sqrt{\text{var}(\lambda)}, \quad \hat{\bar{\theta}} \pm Z_{n/2} \sqrt{\text{var}(\bar{\theta})},$$ (36)

where $Z_{n/2}$ is the upper $v/2^{th}$ percentile of the standard normal distribution. By a similar way, we can find the approximate 100 $(1 - v)$% two-sided CIs for $\lambda$ and $\bar{\theta}$ under TIIC and a complete sample.

### 4.4. Simulation Study

This section provides a simulation study to assess the behavior of the estimators in case of complete, TIC, and TIIC. Mean square errors (MSEs), biases, lower bounds (LBs) of the CIs, upper bounds (UBs) of the CIs, and average lengths (ALs) of 90% and 95% are calculated via Mathematica 9. The following algorithm is designed as follows:

(i) 10000 random samples of size $n = 30, 50, 100$ are generated from HLXG distribution. The corresponding number of failures is selected as $r = (21, 27, 30), (35, 45, 50), (70, 90, 100)$.

(ii) Values of true parameters $(\lambda, \bar{\theta})$ are taken as $(0.5, 0.5), (0.5, \bar{\theta} = 1.2), (1.7, 0.9)$.

(iii) The termination time is selected as $T = 2.5$ and 4 in case of TIC.

(vi) The ML estimates, MSEs, biases, LBs, UBs, and ALs for selected values of parameters are calculated.

(v) Numerical outcomes are listed in Tables 2–4 based on complete and TIIC, while Tables 5–7 contain simulation results based on TIC.

(vi) ML estimates of $sf, h, h(t_0)$, at different mission time $t_0$ where $t_0 \in [3, 5, 10]$ of the HLXG distribution, are calculated. Observed values under TIIC and complete samples are recorded in Table 8, while Table 9 contains numerical results based on TIC.

From Tables 2–9, we observed the following:

The biases, MSEs, and average length of $\lambda$ and $\bar{\theta}$ decrease as $n$ increases

As the censoring level time values, $T$, increase, the biases, MSEs, and ALs of $\lambda$ and $\bar{\theta}$ decrease

The biases, MSEs, and ALs of $\lambda$ and $\bar{\theta}$ decrease as the number of failures $r$ increases

The lengths of the CIs become narrower as $n$ increases

The ALs of the CIs increase as the confidence levels increase from 90% to 95%. Also, as the number of failures $r$ increases, the AL of CIs decreases

As the mission time increases, the estimates of $sf$ decrease while the $hrf$ estimates increase for all sampling schemes

As the sample size increases, the $sf$ and the $hrf$ estimates are increasing

The MSEs, biases, and ALs are increasing as $\theta$ increases from 0.5 to 1.2 at $\lambda = 0.5$

### 5. Data Analysis

In this section, we fit our proposed HLXG model to the following three real data sets. Furthermore, parameter estimation of the three real data sets by using the ML, least square (LS), maximum product spacing (MPS), percentile (PE), and Cramer–von Mises (CVM) methods is provided.

The first data consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett [27]. The data are given as follows:

$3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 65, 2.12, 1.8, 0.85, 4.38, 2, 1.18, 1.71, 1.17, 2.17, 0.39, 2.79, 1.08, 2.73, 0.98, 1.73, 1.59, 1.92, 2.38, 3.56, 2.55, 3.22, 3.39, 4.9, 1.69, 3.11, 3.6, 2.05, 1.61, 2.03, 2.48, 1.25, 2.48, 1.12, 2.88, 2.87, 3.19, 1.87, 2.95, 2.67, 4.2, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 3.22, 3.15, 1.47, 5.56, 1.84, 1.36, 2.59, 2.83, 2.56, 3.33, 2.93, and 2.97.

The second data set (gauge lengths of 10 mm) is obtained by Kundu and Raqab [28]. This data set consists of 63 observations and are listed as follows:

$1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.324, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.562, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, and 5.020.

The third data set is censored data discussed by Sickle-Santangelo et al. [29] and given by Klein and Moeschberger [30]. The data consist of death times (in weeks) of patients with cancer of the tongue with an aneuploid DNA profile. Data under study were based on the effects of ploidy on the prognosis of patients with cancers of the mouth. Patients were selected in whom a paraffin-embedded sample of the cancerous tissue had been taken, and the times to re-infection for patients with sexually transmitted diseases were calculated. Follow-up survival data were obtained on each patient. The tissue samples were examined using a flow cytometer to determine if the tumor had an aneuploid (abnormal) or diploid (normal) DNA profile using a technique discussed by Sickle-Santangelo et al. [29].

The observations are 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61, 65, 67, 70, 72, 73, 74, 77, 79, 80, 81, 87, 87, 88, 89, 91, 93, 93, 96, 97, 100, 101, 104, 104, 108, 109, 120, 131, 150, 157, 167, 231, 240, and 400.
### Table 10: Analytical results of the HLXG model and the other competing models for the first data.

| Model | ML estimates (SE) | $-2\ln L$ | AIC | CAIC | BIC | HQIC |
|-------|-------------------|-----------|-----|------|-----|------|
| HLXG  | $\hat{\lambda} = 6.023 (1.018)$ $\hat{\theta} = 1.371 (0.099)$ | 288.519 | 292.519 | 292.642 | 292.519 | 294.627 |
| GO    | $\hat{\alpha} = 0.789 (0.0774)$ $\hat{\lambda} = 0.099 (0.031)$ | 297.868 | 301.868 | 301.992 | 301.868 | 303.977 |
| EBXII | $\hat{\alpha} = 2.118 (0.76539)$ $\hat{\beta} = 13.687 (8.346)$ $\hat{\lambda} = 1.358 (0.38)$ | 316.051 | 322.051 | 322.301 | 322.051 | 325.214 |
| MOIL  | $\hat{\alpha} = 6.501 (0.87256)$ $\hat{\beta} = 3.151 (0.909)$ $\hat{\lambda} = 0.0046 (0.00323)$ | 304.48 | 310.48 | 310.73 | 310.48 | 313.643 |

### Table 11: Goodness-of-fit measures of the HLXG model and the other competing models for the first data.

| Model | $A^*$ | $P$ value | $K$–$S$ | $P$ value |
|-------|-------|-----------|---------|-----------|
| HLXG  | 1.0953 | 0.0072 | 0.101 | 0.26 |
| GO    | 1.25295 | 0.0029 | 0.1026 | 0.243 |
| EBXII | 2.77675 | 5.489 $\times 10^{-7}$ | 0.14053 | 0.039 |
| MOIL  | 2.30358 | 7.821 $\times 10^{-6}$ | 0.11066 | 0.173 |

### Table 12: Analytical results of the HLXG model and the other competing models for the second data.

| Model | ML estimates (SE) | $-2\ln L$ | AIC | CAIC | BIC | HQIC |
|-------|-------------------|-----------|-----|------|-----|------|
| HLXG  | $\hat{\lambda} = 73.168 (30.67447)$ $\hat{\theta} = 2.193 (0.189)$ | 113.683 | 117.683 | 117.887 | 117.282 | 119.369 |
| GO    | $\hat{\alpha} = 1.399 (0.12793)$ $\hat{\lambda} = 0.008978 (0.000448)$ | 139.296 | 142.296 | 142.496 | 141.895 | 143.982 |
| EBXII | $\hat{\alpha} = 3.926 (2.7811)$ $\hat{\beta} = 1320 (270.9)$ $\hat{\lambda} = 1.646 (0.937)$ | 116.228 | 122.228 | 122.634 | 121.626 | 124.756 |
| MOIL  | $\hat{\alpha} = 6.943 (0.85665)$ $\hat{\beta} = 3.544 (0.608)$ $\hat{\lambda} = 0.0043 (0.00378)$ | 162.595 | 168.595 | 169.002 | 167.993 | 171.124 |

### Table 13: Goodness-of-fit measures of the HLXG model and the other competing models for the second data.

| Model | $A^*$ | $P$ value | $K$–$S$ | $P$ value |
|-------|-------|-----------|---------|-----------|
| HLXG  | 0.4434 | 0.286 | 0.0941 | 0.632 |
| GO    | 1.6283 | 3.52 $\times 10^{-4}$ | 0.139 | 0.175 |
| EBXII | 0.6169 | 0.108 | 0.096 | 0.607 |
| MOIL  | 22.3882 | 0 | 0.2076 | 0.0088 |

### Table 14: Analytical results of the HLXG model and the other competing models for the third data.

| Model | ML estimates (SE) | $-2\ln L$ | AIC | CAIC | BIC | HQIC |
|-------|-------------------|-----------|-----|------|-----|------|
| HLXG  | $\hat{\lambda} = 0.619 (0.09717)$ $\hat{\theta} = 0.021 (0.00326)$ | 559.128 | 563.128 | 563.252 | 562.56 | 564.624 |
| GO    | $\hat{\alpha} = 0.00194 (0.000169)$ $\hat{\lambda} = 5.489 (5.569)$ | 559.789 | 563.789 | 563.912 | 563.221 | 565.285 |
| EBXII | $\hat{\alpha} = 2.997 (1.60698)$ $\hat{\beta} = 62.12 (76.415)$ $\hat{\lambda} = 0.324 (0.136)$ | 591.406 | 597.406 | 597.656 | 596.554 | 599.65 |
| MOIL  | $\hat{\alpha} = 1.703 (0.24084)$ $\hat{\beta} = 393.894 (284.417)$ $\hat{\lambda} = 0.032 (0.038)$ | 573.349 | 579.349 | 579.599 | 578.497 | 581.593 |
Table 15: Goodness-of-fit measures of the HLXG model and the other competing models for the third data.

| Model | $A^*$ | $P$ value | $K-S$ | $P$ value |
|-------|-------|-----------|-------|-----------|
| HLXG  | 1.0383| 0.00991   | 0.1386| 0.27      |
| GO    | 1.0829| 0.0077    | 0.1572| 0.153     |
| EBXII | 3.8372| $1.468 \times 10^{-9}$ | 0.2181| 0.014     |
| MOIL  | 2.4612| $3.225 \times 10^{-6}$ | 0.171 | 0.095     |

Figure 2: Log-likelihood for the first data.

Figure 3: Log-likelihood for the second data.

Figure 4: Log-likelihood for the third data.
First, we want to compare the fits of the HLXG model with the Gompertz (GO), exponentiated Burr XII (EBXII), and MO inverse Lomax (MOIL) distributions. We estimate the model parameters by using the ML method. We compare the goodness-of-fit of the models using $-2\ln L$ where $L$ denotes the log-likelihood function evaluated at the ML estimates, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), and Bayesian information criterion (BIC). We also reported the Kolmogorov–Smirnov (K–S) and Anderson–Darling ($A^*$) statistics and their corresponding $P$ values. The pdfs of the IL, GIL, EIL, MOIL, and IW models are, respectively, given by

\begin{align*}
  f_{\text{Go}}(x; \alpha, \lambda) &= \alpha \lambda e^{\lambda x} e^{-\lambda e^x}, \quad x, \alpha, \lambda > 0, \\
  f_{\text{EBXII}}(x; \alpha, \beta, \lambda) &= \alpha \beta \lambda x^{\beta-1} (1 + x^\lambda)^{-(\beta-1)}, \quad x, \alpha, \beta, \lambda > 0, \\
  f_{\text{MOIL}}(x; \alpha, \beta, \lambda) &= \frac{\alpha \beta \lambda (1 + \beta x)^{-\alpha-1}}{x^2 (1 - (1 - \lambda) [1 - (1 + \beta x)^{-\alpha}])^2}, \quad x, \alpha, \beta, \lambda > 0.
\end{align*}

Figure 5: The empirical pdf, cdf, sf, and pp plots for the first data.
The ML estimates (and the corresponding standard errors in parentheses) of parameters and values of $-2 \ln L$, AIC, BIC, CAIC, and HQIC and the values of $K$–$S$ and $A^*$ statistics and their corresponding $P$ values for the first data set are displayed in Tables 10 and 11, respectively, for the second dataset in Tables 12 and 13, respectively, and for the third data set, in Tables 14 and 15, respectively. The log-likelihood for the first, second, and third data sets are given in Figures 2–4, respectively.

It is observed from Tables 10–15 that the HLXG distribution has the smallest values of $-2 \ln L$, AIC, CAIC, BIC, HQIC, $K$–$S$, and $A^*$ but the largest $P$ value among all other competitive models. Hence, it is evidenced that our proposed model performed the best for all three data sets. The relative histogram and the fitted densities and the plots of the fitted empirical cdf and the empirical sf and $p$–$p$ plots of the three data sets are displayed in Figures 5–7, respectively. These figures supported the above conclusions to some extent.

Tables 16–18 provide the ML, LS, MPS, PC, and CV estimates of the unknown parameters of the HLXG distribution based on four methods of estimation. Also, values of $-2 \ln L$, AIC, BIC HQIC, and CAIC are listed. From these tables, we conclude that all estimation methods performed well.
Figure 7: The empirical pdf, cdf, sf, and pp plots for the third data.

Table 16: The parameter estimates under various methods and the goodness-of-fit statistics for (Nicholas) data.

| Method | $\hat{\theta}$ | $\hat{\lambda}$ | $-2\ln L$ | AIC | BIC | HQIC | CAIC |
|--------|----------------|-----------------|-----------|-----|-----|------|------|
| ML     | 1.371          | 6.023           | 288.519   | 292.519 | 292.519 | 294.627 | 292.642 |
| LS     | 1.375          | 6.477           | 289.157   | 293.157 | 293.157 | 295.265 | 293.28  |
| MPS    | 0.015          | 0.118           | 774.822   | 778.822 | 778.822 | 780.931 | 778.946 |
| PC     | 1.063          | 3.7             | 298.925   | 302.925 | 302.925 | 305.034 | 303.049 |
| CV     | 1.392          | 6.723           | 289.437   | 293.437 | 293.437 | 295.545 | 293.56  |

Table 17: The parameter estimates under various methods and the goodness-of-fit statistics for the second (Kundu) data.

| Method | $\hat{\theta}$ | $\hat{\lambda}$ | $-2\ln L$ | AIC | BIC | HQIC | CAIC |
|--------|----------------|-----------------|-----------|-----|-----|------|------|
| ML     | 2.193          | 73.168          | 113.683   | 117.683 | 117.282 | 119.369 | 117.887 |
| LS     | 1.996          | 46.235          | 114.934   | 118.934 | 118.533 | 120.62  | 119.134 |
| MPS    | 0.027          | 0.143           | 489.716   | 493.716 | 493.315 | 495.402 | 493.916 |
| PC     | 2.006          | 47.873          | 114.748   | 118.748 | 118.347 | 120.434 | 118.948 |
| CV     | 2.031          | 50.444          | 114.496   | 118.496 | 118.095 | 120.182 | 118.696 |
6. Some Concluding Remarks

A new probability distribution related to xgamma, called the half-logistic xgamma distribution, is proposed in this paper. Some important statistical properties, including moments, incomplete moments, mean residual life, mean waiting time, quantile function, and stochastic ordering of the suggested model, are derived. Parameter estimation of model parameters is discussed via the maximum likelihood method from complete and censored samples. Further, survival and hazard rate function estimates are obtained for different sample sizes. Also, asymptotic confidence intervals of parameters are constructed. Good performance and accuracy of the maximum likelihood estimators of the parameters are examined through a simulation study. Based on maximum likelihood, least square, maximum product spacing, percentile, and Cramer–von Mises (CV) methods, we estimate the model parameters of the HLXG distribution using three real data sets. Furthermore, three real data sets are analyzed to demonstrate the superiority of the proposed half-logistic xgamma distribution compared to some existing distributions.

Data Availability

In order to obtain the numerical data set used to carry out the analysis reported in the manuscript, please contact Mahmoud Elsehetry at ma_sehetry@hotmail.com.

Conflicts of Interest

The authors declare no conflicts of interest.

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