A Gribov-Zwanziger type action invariant under background gauge transformations

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We propose a Gribov-Zwanziger type action for the Landau-DeWitt gauge that preserves, for any gauge group, the invariance under background gauge transformations. At zero temperature, and to one-loop accuracy, the model can be related to the Gribov no-pole condition. We apply the model to the deconfinement transition in SU(2) and SU(3) Yang-Mills theories and compare the predictions obtained with a single or with various (color dependent) Gribov parameters that can be introduced in the action without jeopardizing its background gauge invariance. The Gribov parameters associated to color directions orthogonal to the background can become negative, while keeping the background effective potential real. In some cases, the proper analysis of the transition requires the potential to be resolved in those regions.

I. INTRODUCTION

Much progress has been achieved lately in the continuum description of the non-perturbative dynamics at play in the deconfinement transition of pure Yang-Mills theories. First, a good handle on the related center symmetry was possible thanks to the use of background field methods [1, 2] which allow for the definition of order parameters equivalent to the Polyakov loop, but simpler to compute in practice [3]. Second, relevant non-perturbative dynamics could be captured thanks to the use of sophisticated functional methods such as the functional renormalization group [3–5], the infinite tower of Dyson-Schwinger equations [6–10] or variational approaches [11–14].

On top of these achievements, more phenomenological approaches [15–18] seem to indicate that, in the Landau gauge (and in its background extension, the so-called Landau-DeWitt gauge), a pivotal part of the non-perturbative dynamics may originate in the gauge-fixing procedure and the proper handling of the associated Gribov copy problem [20]. In fact, according to these studies, once such a non-perturbative gauge-fixing is implemented, at least in some approximate form, the perturbative expansion becomes viable at low energies [16–19], while it breaks down in the more standard Faddeev-Popov gauge-fixing. This is an interesting perspective that could open the way to the perturbative evaluation of quantities that are usually considered as genuinely non-perturbative. Although speculative, the idea certainly deserves to be further investigated and tested.

For instance, in a series of recent works, the Curci-Ferrari (CF) action [21] has been proposed as a model for a non-perturbative gauge-fixing in the Landau gauge [15–16, 22]. The underlying conjecture of these studies is that a CF gluon mass term may arise after the Gribov copies have been accounted for by means of an uneven averaging procedure [22]. Although no rigorous mechanism for the generation of such a CF mass has been identified in the Landau gauge, a similar mass term could be generated in a non-linear version of the Landau gauge [23]. Moreover and interestingly, relatively simple one-loop calculations of zero-temperature correlation functions in the CF model [15–16, 24] agree quite well with first principle lattice simulations of Yang-Mills correlation functions in the Landau gauge [27–31]. The model has also been extended to finite temperature in the Landau-DeWitt gauge where it gives a good description of center-symmetry breaking in pure Yang-Mills theories [17]. In this case, two-loop corrections could also be computed [33–34], showing some sign of apparent convergence and supporting the idea that perturbation theory may indeed be applicable once the Gribov problem has been properly handled. Moreover, matter fields can also be included in the analysis, see Refs. [35–38].

Another possible way to deal with the Gribov problem in the Landau gauge is the so-called Gribov-Zwanziger approach [20–24, 40]. The idea in that case is to restrict the domain of the functional integral to a region that contains less Gribov copies, in practice the so-called first Gribov region, defined by the positivity of the Faddeev-Popov operator $\partial D$. With the price of introducing some auxiliary fields, a formulation of this restriction was constructed in terms of a local and renormalizable quantum field theory [39]. It has since then known various refinements in order to match lattice results at zero-temperature [41–42].

At finite temperature, the situation is less clear. Although many interesting works apply the Gribov-Zwanziger approach to thermal scenarios [13–38], they all rely on the implicit assumption that the output of the Gribov-Zwanziger construction in such cases is given by the zero temperature Gribov-Zwanziger action taken over a compact time interval of length $\beta = 1/T$. However, this assumption is far from obvious and possibly incorrect. In fact, as recently discussed in Refs. [49–50].
the presence of the compact time direction and the related periodic boundary conditions lift the degeneracy of the lowest, non-zero eigenvalues of the free Faddeev-Popov operator. This, in turn, leads to a modification of the Faddeev-Popov action which is not just the usual zero-temperature modification taken over a compact time interval. In particular, the so-obtained action is not $O(4)$ Euclidean invariant in the zero-temperature limit, unless the Gribov parameter goes to zero. Even though it raises some conceptual issues, such as the question of the renormalizability of the corresponding action,ootnote{Another issue is that, for the approach to correspond to a bona-fide gauge-fixing, the $O(4)$ breaking terms in the zero-temperature limit should not affect the physical observables. This question deserves further investigation and probably requires the identification of the appropriate BRST (Becchi-Rouet-Stora-Tyutin) symmetry.} this is certainly a good path to be followed in view of a proper discussion of the Gribov-Zwanziger approach at finite temperature in the Landau gauge.

In the case of the Landau-DeWitt gauge, the situation is similar to that of the Landau gauge prior to the results of Refs. [39] [50]. There is to date no first principle derivation of the associated Gribov-Zwanziger action, only models that try to incorporate the effect of restricting the functional integral to the corresponding first Gribov region. In particular, in Ref. [18] (see also Ref. [53]), a Gribov-Zwanziger type action for the Landau-DeWitt gauge has been proposed – independently of whether it corresponds to a faithful implementation of the Gribov restriction – and applied to the study of center-symmetry breaking in SU(2) Yang-Mills theory. This action has the convenient property that it reproduces the usual, renormalizable, $O(4)$ invariant, Landau gauge Gribov-Zwanziger action in the zero-temperature and zero-background limits. However, as it was pointed out in Ref. [51], it is not invariant under background gauge transformations, a property that plays a crucial role when dealing with center symmetry in a gauge-fixed setting. Surprisingly, the one-loop background effective potential obtained in Ref. [18] displays background gauge invariance but, as it was also clarified in Ref. [51], this is due to a missing term in the evaluation of the potential. In this latter reference, a new model action was also put forward, based on a construction that preserves both BRST symmetry and background gauge invariance with the price of introducing a Stueckelberg type field, not so easy to deal with, specially at finite temperature.

In this article, we follow a slightly different route than that of Ref. [51]. We first revisit the model of Ref. [18] and show how it can be very simply upgraded into a fully background gauge invariant one, that in addition correctly generates the one-loop results of that reference. This opens the way to the evaluation of higher order corrections in a manifestly background gauge invariant setting. We also try to discuss to which extent the model can be seen as a faithful implementation of the Gribov restriction for the Landau-DeWitt gauge.

In Sec. [11] we introduce the model as a minimal, background gauge invariant modification of the action used in Ref. [18]. In Sec. [11] we compute the corresponding one-loop background effective potential for any gauge group and, in Sec. [15] we use it to investigate the deconfinement phase transition in SU(2) and SU(3) Yang-Mills theories. In particular, we study the impact on the transition temperatures of the use of color dependent Gribov parameters, as allowed by the model. Finally, in Sec. [1] we provide a further motivation for the model by showing, at zero temperature and at leading order, how it is connected to the Gribov no-pole condition applied to the Landau-DeWitt gauge. We also discuss some of the difficulties that occur at finite temperature (similar to the ones discussed in Refs. [39] [50] for the Landau gauge), when trying to interpret the model as arising from a faithful implementation of the Gribov restriction. The various formulae needed for our analysis, in particular in the case where certain Gribov parameters become negative, are gathered in App. [A].

II. A BACKGROUND GAUGE INVARIANT GRIBOV-ZWANZIGER TYPE ACTION

A. The problem

In the original proposal of Ref. [18], the action reads

\[
S = \int_x \left\{ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i h^a D_{\mu}^a D_{\mu}^a + D_{\mu}^a \gamma_5 D_{\mu}^a \gamma_5 + g \gamma^{1/2} f^{abc} a^a_{\mu} (\tilde{\varphi}^b_{\mu} - \tilde{\varphi}^c_{\mu}) - \gamma d d_G \right\},
\]

with $D_{\mu}^a \equiv \partial_{\mu}^a \delta^{ab} - g f^{abc} A_{\mu}^c$ the covariant derivative in the presence of the background field configuration $A_{\mu}$. Here, as compared to Ref. [18], we consider a general group of dimension $d_G$ and we take the gauge-fixing parameter to zero by introducing a Nakanishi-Lautrup field $h$. We also redefine some of the fields for convenience as well as the Gribov parameter $\gamma$, and we use a slightly different notation, more in line with the conventions of Ref. [31]: the total gauge field is written $A_{\mu}$, whereas the fluctuation about the background field $\bar{A}_{\mu}$ is written $a_{\mu} \equiv A_{\mu} - \bar{A}_{\mu}$.

The first line of Eq. (1) is nothing but the Faddeev-Popov action associated to the Landau-DeWitt gauge $D_{\mu} a_{\mu} = 0$, which reduces to the usual Landau gauge $\partial_{\mu} A_{\mu} = 0$ in the limit of a vanishing background. In this latter case, the fields $\varphi_{\mu}$, $\bar{\varphi}_{\mu}$, $\omega_{\mu}$ and $\bar{\omega}_{\mu}$ and the Gribov
parameter $\gamma$ appearing in the second line of Eq. (1) implement the restriction (in gauge field space) to the first Gribov region, defined by the two conditions $\partial_\mu A_\mu = 0$ and $D_\mu D_\mu > 0$.

The generalization of the first Gribov region in the presence of the background $A_\mu$ is defined by the two conditions $\overline{D}_\mu a_\mu = 0$ and $\overline{D}_\mu D_\mu > 0$. Since the latter are invariant under the background gauge transformations,

$$\overline{(A_\mu^\dagger)^a t^a} = U A_\mu^a t^a U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger, \quad (2)$$

$$\overline{(a_\mu^\dagger)^a t^a} = U a_\mu^a t^a U^\dagger, \quad (3)$$

it follows that any action that models the restriction to the first Gribov region associated to the Landau-DeWitt gauge should be invariant under such transformations. This is indeed the case for the Faddeev-Popov part of the action (1), provided that one transforms the fields $h$, $c$ and $\bar{c}$ similarly to $a_\mu$. However, as it was recently pointed out in Ref. [51], the rest of the action is not invariant: if one assumes the fields $\varphi_\mu$, $\bar{\varphi}_\mu$, $\omega_\mu$ and $\bar{\omega}_\mu$ transform as tensor products of two $a_\mu$'s, contractions such as $D^{ac}_{\mu\nu} c^e_{\mu\nu}$ do not transform covariantly. One can make these contractions covariant by assuming that $\varphi_\mu$, $\bar{\varphi}_\mu$, $\omega_\mu$ and $\bar{\omega}_\mu$ transform only with respect to their first color index (the one upon which the covariant derivatives act) but then terms such as $f^{abc}_{\mu\nu} c^e_{\mu\nu}$ are not invariant anymore. Despite this fact, and as already mentioned in the Introduction, the one-loop background effective potential obtained in Ref. [18] appears to be background gauge invariant. As was later observed in Ref. [51], this potential obtained in Ref. [18] appears to be background gauge invariant. As was later observed in Ref. [51], this is due to the omission of some terms in the evaluation of the one-loop background effective potential that derives from the action (1).

To overcome these problems, a new action was put forward in Ref. [51], based on a BRS compatible model for the Gribov restriction, that automatically ensured the invariance under background gauge transformations. This construction is however not so easy to implement in practice because it requires the introduction of a $SU(N)$-valued field $Y$ and $\bar{Y}$, which complicates the analysis. Moreover, at finite temperature, in order to preserve center symmetry, one needs a priori to integrate over fields $h$ that are periodic up to an element of the center of the gauge group, that is over topologically distinct sectors. How to achieve this in practice in terms of the Stuckelberg field is not completely clear.

Here, a different route is followed: we choose to sacrifice BRST symmetry with the benefit of obtaining a background gauge invariant setting that is easy to implement at finite temperature. We show that the action (1) can be very simply upgraded into a background gauge invariant one and that the latter leads exactly to the same one-loop background effective potential as the one that was obtained in Ref. [18]. In fact our results will be slightly more general since our analysis will also reveal that it is possible to introduce color-dependent Gribov parameters without jeopardizing the background gauge invariance. We shall investigate this possibility in the application of the model to the deconfinement transition.

### B. A background gauge invariant model

In what follows, we assume for simplicity that the background is temporal, constant and in the diagonal part of the algebra of the gauge group (the Cartan subalgebra) spanned by a set of generators that we denote $\{t^i\}$.

$$\beta g A_\mu (x) = \delta_{\mu 0} r^j t^j.$$  

Correspondingly, we restrict to background gauge transformations that preserve this structure. We shall recall their explicit form below. The inverse temperature $\beta = 1/T$ has been introduced in Eq. (4) in order to express the background in terms of the dimensionless components $r^j$.

In order to understand how to restore the background gauge invariance to the action (1), it is convenient to change to a Cartan-Weyl basis. This corresponds to an orthonormal change of basis $it^i \rightarrow it^\alpha$ in the complexified Lie algebra of the group, see for instance Ref. [51] (and also Ref. [52] for details on the conventions and properties used from here on). In particular, an invariant quantity $X^\alpha Y^\alpha$ appears in the new basis as $(X^\alpha) Y^\alpha$, where $X^\alpha$ denotes complex conjugation, and the action (1) rewrites

$$S = \int d^4 x \left\{ \frac{1}{4} (F_{\mu\nu})^a F^a_{\mu\nu} + i (h^\alpha) \overline{D}_\mu c^\alpha \overline{c}^\mu \right\} \delta_{\mu 0} r^j t^j + \gamma d d_G \right\},$$

with $F_{\mu\nu} = \partial_\mu A^\nu_\alpha - \partial_\nu A^\mu_\alpha + g f^{\kappa \lambda \eta} A^\lambda_\mu A^\eta_\nu$, $D^\alpha = \partial_\mu \delta^\alpha - g f^{\kappa \lambda \eta} A^\lambda_\mu$ and $[t^\lambda, t^\eta] = i f^{\kappa \lambda \eta} t^\kappa$. In a Cartan-Weyl basis,

$$^{5}$$

and $\varphi_\mu$ are of the numerical type. Because they are contracted with the structure constant tensor, they should be taken anti-symmetric with respect to their color indices.

$^3$ For gauge field configurations satisfying the Landau gauge condition, the Faddeev-Popov operator $D_\mu D_\mu$ is hermitian. Then, it makes sense to look for gauge field configurations such that this operator is, in addition, positive definite.

$^4$ In fact, the spectrum of the Faddeev-Popov operator is globally invariant under these transformations.

$^5$ In the ideal scenario where one would select one Gribov copy per obit, we expect BRST symmetry to be broken. We note however that, in the GZ scenario, a local BRST symmetry could be identified $^{52}$.

$^6$ This is usually sufficient to investigate the deconfinement transition.
the color indices $\kappa$ are vectors in a space isomorphic to the Cartan subalgebra (the components of $\kappa$ will be denoted $\kappa^j$ from here on). They can take two types of values: either $\kappa = 0^{(j)}$ is “a zero” or $\kappa = \alpha$ is a root of the algebra of the gauge group. Because the roots always appear in pairs $(\alpha, -\alpha)$, one can redefine the structure constants as $f^{\alpha \lambda \tau} = -if^{-\kappa \lambda \tau}$ with the benefit that $f^{\lambda \tau \kappa}$ is anti-symmetric and color conserving (meaning that the background covariant derivatives become diagonal: $\bar{\nabla}_\tau = (\bar{\nabla}_\kappa \lambda \xi)^* \bar{\nabla}_\kappa \phi^\lambda \xi$).

Next, we recall that background gauge transformations $\delta \phi^\lambda \xi = ig f^{\lambda \kappa \tau} \phi_{\kappa}^\tau$ and the fact that the roots always appear in pairs $(\alpha, -\alpha)$ and the various zeros. It would be natural to define the transformation of the fields $\phi_{\mu \nu}^\lambda \omega^\kappa$, $\bar{\phi}_{\mu \nu}^\lambda \omega^\kappa$, $\omega^\mu$, $\bar{\omega}^\mu$ as

$$X^{\kappa \lambda \chi}(x) = e^{i \tilde{\tau} \phi_{\mu \nu}^\lambda \chi \kappa \lambda \chi(x) \tau}.$$  

However the second and third lines of Eq. (7) are not invariant under this transformation because objects such as $\bar{D}_\mu^\kappa X^{\kappa \lambda}$ do not transform in a covariant way. The reason being that the covariant derivative does not contract on the second color index of $X^{\kappa \lambda}$. We can however make this index harmless by noticing that the objects $\tilde{X}^{\kappa \lambda}(x) \equiv X^{\kappa \lambda}(x) e^{-i \bar{\tau} r^\lambda \tau}$ transform as

$$\tilde{X}^{\kappa \lambda}(x) = e^{i \tilde{\tau} \phi_{\mu \nu}^\lambda \chi \kappa \lambda \chi(x) \tau}.$$  

It follows that the upgraded action

$$S_{\text{new}} = \int x \left\{ \frac{1}{4} (F_{\mu \nu})^* F_{\mu \nu}^\kappa + i (h_{\mu})^* D_{\mu \nu}^\kappa a_\mu^\kappa \right. \right.$$  

with $F_{\mu \nu}^\kappa = \partial_{\mu} A_{\nu}^\kappa - \partial_{\nu} A_{\mu}^\kappa - ig f^{(\kappa \lambda \tau)} A_{\lambda}^\nu A_{\tau}^\mu$, $D_{\mu}^\kappa = \partial_{\mu} + \frac{1}{2} \gamma^{\nu} \gamma_{\mu} \epsilon^{\nu \lambda \tau} f_{\lambda \tau}$.

Another important property of the Cartan-Weyl bases (when the $\tilde{\tau}^j$'s are chosen equal to the $\tilde{r}^j$'s in Eq. (4)) is that the background covariant derivatives become diagonal: $D_{\mu}^{\kappa \lambda} = \delta^{\kappa \lambda} D_{\mu}^\kappa$ with $D_{\mu}^\kappa = \partial_{\mu} - ir^\gamma \gamma T_\delta x^0$ and $r^\gamma = r^\gamma \gamma^j$. Therefore, the action (6) can be simplified as

$$S = \int x \left\{ \frac{1}{4} (F_{\mu \nu})^* F_{\mu \nu}^\kappa + i (h_{\mu})^* D_{\mu \nu}^\kappa a_\mu^\kappa \right.$$  

where, for later convenience, we have separated the covariant derivatives as $D_{\mu}^{\kappa \lambda} = \delta^{\kappa \lambda} D_{\mu}^\kappa + ig f^{(-\kappa \lambda \tau)} a_{\mu \tau}^\kappa$.

Next, we recall that background gauge transformations that preserve the form of the background (4) read

$$r^{\alpha j} = r^{\alpha j} + \alpha^{\alpha \tau},$$  

where the $\alpha^{\alpha i}$ are certain vectors that we do not need to specify further here, see for instance Ref. [34] for more details. The Faddeev-Popov part of the action (7) is invariant under these transformations provided one transforms $a$, $h$, $c$ and $\tilde{c}$ according to

$$X^{\kappa \chi}(x) = e^{i \tilde{\tau} \phi_{\mu \nu}^\lambda \chi \kappa \chi(x) \tau},$$  

as one can easily check using the property $\bar{D}_\mu^\kappa X^{\kappa \chi}(x) = e^{i \tilde{\tau} \phi_{\mu \nu}^\lambda \chi \kappa \chi(x) \tau}$ and the fact that $f^{\kappa \lambda \eta}$ conserves color.

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7 Below, we shall recall the zeros and the roots for the SU(2) and SU(3) groups. Note that there are as many zeros as there are dimensions in the Cartan subalgebra, hence the label $(j)$ to denote the various zeros.

8 Again, one could modify the transformation rule of $X^{\kappa \chi}$ such that the derivatives $\bar{D}_\mu^\kappa X^{\kappa \chi}$ transform covariantly but this would jeopardize the invariance of the last line of Eq. (7).

9 We should mention, however, that it is far from obvious that our proposal or the one in Ref. [34] correspond to faithful implementations of the Gribov restriction at finite temperature. We briefly discuss this issue in Sec. VII.
C. Color-dependent Gribov parameters

Before closing this section, it should be mentioned that the model can, and will, be extended by introducing color-dependent Gribov parameters \( \gamma_\mu \) without affecting the background gauge invariance.\(^8\) -\(^10\)\(^11\)

\[
S_{\text{new}} = \int \left\{ \frac{1}{4} (F_{\mu\nu}^\gamma)^* F_{\mu\nu}^\gamma + i (h^K)^* D_{\mu}^\gamma a_\mu^K \\
+ (D_{\mu}^\gamma c^K c^K + i g D_{\mu}^\gamma c^K f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma c^{\lambda} \\
+ (D_{\mu}^\gamma c^{\lambda} \xi c^K \xi^K + i g D_{\mu}^\gamma c^{\lambda} \xi f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma \xi^{\lambda_1} \\
+ (D_{\mu}^\gamma \xi c^K \xi^K + i g D_{\mu}^\gamma \xi f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma \xi^{\lambda_1} \\
- ig \gamma_\mu^{1/2} f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma (\bar{\varphi}_\mu^{\lambda_1} - \bar{\varphi}_\mu^{\lambda}) + d \sum_\gamma \gamma_\mu \right\}. \quad (14)
\]

We will see below that the Gribov parameters are all degenerate at zero temperature. At finite temperature however, there is no reason for them to remain equal and, therefore, it will be interesting to compare the situation where a unique Gribov parameter is attributed to all color modes with the one where Gribov parameters are allowed to depend on color.

Our main focus being the study of the deconfinement transition it is however of crucial importance to preserve the invariance under so-called Weyl transformations, because only then the background field, as obtained from the minimization of the background effective potential is an order parameter for center symmetry.\(^33\) -\(^35\) Since the Weyl transformations typically connect certain roots \( \alpha \) and \( \beta \) with each other, a simple way to ensure Weyl symmetry is to impose that \( \gamma_\alpha = \gamma_\beta \) for such roots. If one also wants to preserve invariance under charge conjugation, one possibility is to impose that \( \gamma_\alpha = \gamma_{-\alpha} \). In what follows, we shall consider groups where Weyl transformations and charge conjugation allow to connect all roots with each other and therefore we introduce a single Gribov parameter \( \chi_\gamma \) for all these “charged” modes. In contrast for each “neutral” mode\(^36\) corresponding to \( \kappa = (0, j) \), we can a priori introduce a different Gribov parameter \( \chi_\gamma(j) \). In this case, the most general action reads

\[
S_{\text{new}} = \int \left\{ \frac{1}{4} (F_{\mu\nu}^\gamma)^* F_{\mu\nu}^\gamma + i (h^K)^* D_{\mu}^\gamma a_\mu^K \\
+ (D_{\mu}^\gamma c^K c^K + i g D_{\mu}^\gamma c^K f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma c^{\lambda} \\
+ (D_{\mu}^\gamma c^{\lambda} \xi c^K \xi^K + i g D_{\mu}^\gamma c^{\lambda} \xi f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma \xi^{\lambda_1} \\
+ (D_{\mu}^\gamma \xi c^K \xi^K + i g D_{\mu}^\gamma \xi f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma \xi^{\lambda_1} \\
- ig \gamma_\mu^{1/2} f^{\lambda\lambda_1} a_{\mu\lambda_1}^\gamma (\bar{\varphi}_\mu^{\lambda_1} - \bar{\varphi}_\mu^{\lambda}) + d \sum_\gamma \gamma_\mu \right\}, \quad (15)
\]

where \( d_G \) is the dimension the gauge group and \( d_C \) the dimension of the Cartan subalgebra.

In fact, this choice of Gribov parameters is just a sufficient condition to ensure Weyl symmetry but it is not necessary. Weyl symmetry is more generally preserved in the following sense: the action \((14)\) is invariant under a Weyl transformation that exchanges \( \alpha \) and \( \beta \) provided one also performs the transformation \( \gamma_\alpha \leftrightarrow \gamma_\beta \). This symmetry is trivially inherited by the background effective potential due to the extremization needed to determine the Gribov parameters, which are then promoted to functions of the background. I.e., when action \((14)\) is evaluated for the values of the \( \gamma \)'s obtained through this process, Weyl invariance is guaranteed in the usual sense. The same remarks apply to charge conjugation.

In summary, we shall study three different scenarios, all compatible with background gauge invariance, including Weyl invariance:

- **Degenerate case:** all \( \gamma_\mu \)'s taken equal.
- **Partially degenerate:** all \( \gamma_\mu \)'s taken equal.
- **Non-degenerate case:** all \( \gamma_\mu \)'s taken different.

III. THE MODEL AT ONE-LOOP

In this section, we evaluate the background effective potential and the corresponding gap equation(s) at one-loop order, for any gauge group.

A. Background effective potential

The quadratic part of the action in Fourier space writes

\[
\frac{1}{2} \int_p X^*(p) M(p) X(p) + \int_p \bar{Y}^*(p) N(p) Y(p), \quad (16)
\]

with \( X^t = (a^\gamma_\mu, h^\lambda, \varphi^\xi, \varphi^\xi) \) and \( Y^t = (c^\gamma, \omega^\xi, \bar{\varphi}^\xi) \). The one-loop background effective potential reads

\[
V(\bar{A}, \{\gamma_\kappa\}) = -d \sum_\kappa \gamma_\kappa + \frac{1}{2} \ln \det M - \ln \det N. \quad (17)
\]

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\(^{10}\) The reason why the color label of \( \gamma \) is the one associated to \( a_\mu \) is that the fields \( \varphi_\mu \) are auxiliary fields that help localizing the action.

\(^{11}\) These are finite color rotations that leave the Cartan subalgebra globally invariant.

\(^{12}\) The terminology “charged” and “neutral” arises from the fact that \( \kappa \cdot rT \) can be seen as a color-dependent imaginary chemical potential.
We have
\[
M = \begin{pmatrix}
Q^2 \delta_{\kappa\kappa'} & -Q^2 \delta_{\kappa\lambda'} & -ig\gamma^\nu \gamma^\mu f^{(-\kappa)\eta^\prime}\delta_{\mu\nu} & ig\gamma^\nu \gamma^\mu f^{(-\kappa)\eta^\prime}\delta_{\mu\nu} \\
-Q^2 \delta_{\kappa\lambda'} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-ig\gamma^\nu \gamma^\mu f^{(-\kappa)\eta^\prime}\delta_{\mu\nu} & 0 & 0 & Q^2 \delta_{\eta^\prime\lambda'}
\end{pmatrix},
\]
where we have introduced the shifted momenta $Q^\nu \equiv Q_\mu + r \cdot \kappa T \delta_{\mu0}$ and the subscript $\ast$ on $f^{\kappa\lambda\tau}$ denotes complex conjugation. In order to compute the determinant of $M$, we consider it as a block matrix of the form $(A B | C D)$, with $A$ and $D$ invertible, and use
\[
det M = \det D \times \det \left( A - BD^{-1}C \right) = \det A \times \det \left( D - CA^{-1}B \right).
\]
A simple calculation shows that
\[
A - BD^{-1}C = \left( Q^2 \right) \delta_{\kappa\kappa'} + 2g^2 \gamma_\mu f^{(-\kappa)\eta^\prime} f^{(-\kappa)\eta^\prime} Q^2 \delta_{\mu\nu} - Q^2 \delta_{\kappa\lambda'},
\]
where a summation over $\eta$ and $\xi$ is implied in the first element. We next use that that the structure constants conserve color, to write $f^{(-\kappa)\eta^\prime} f^{(-\kappa)\eta^\prime} Q^2 = f^{(-\kappa)\eta^\prime} Q^2$, where $C_{ad}$ denotes the Casimir of the adjoint representation. We obtain
\[
A - BD^{-1}C = \left( \prod_\kappa \left( Q^4 \delta_{\kappa\kappa'} + (Q^2 + m^4) Q^2 \delta_{\mu\nu} - Q^2 \delta_{\kappa\lambda'} \right) \right),
\]
where we have defined $m^4 \equiv 2g^2 C_{ad} \gamma_\kappa$. Using the second form of Eq. (19), we find
\[
\det \left( A - BD^{-1}C \right) = \prod_\kappa \left( Q^4 \delta_{\kappa\kappa'} + (Q^2 + m^4) Q^2 \delta_{\mu\nu} - Q^2 \delta_{\kappa\lambda'} \right) \]
and then
\[
det M = \det D \times \prod_\kappa \left( Q^4 \delta_{\kappa\kappa'} + (Q^2 + m^4) Q^2 \delta_{\mu\nu} - Q^2 \delta_{\kappa\lambda'} \right)
\]
On the other hand, it is trivially shown that
\[
det N = \left( \det D \right)^{1/2} \times \prod_\kappa Q^2 \delta_{\kappa\kappa'}.
\]
Therefore
\[
V(r, \{ m^4 \}) = -d \sum_\kappa \frac{m^4}{2g^2 C_{ad}} + \frac{d - 1}{2} \sum_\kappa \int_Q \ln \frac{Q^4 + m^4}{Q^2} \\
- \frac{1}{2} \sum_\kappa \int_Q \ln Q^2,
\]
where we have introduced the notations
\[
\int_Q \equiv \mu^{d-2} T \sum_q \int_q \text{ and } \int_q \equiv \int_q \frac{d-1}{(2\pi)^{d-1}},
\]
with $d = 4 - 2\epsilon$.

In the SU(2) case, and assuming that all the Gribov parameters $\gamma_\kappa \propto m^4$ are equal, the expression in Eq. (25) is exactly the one-loop potential obtained in Ref. [18].

B. Gribov parameters

The Gribov parameters are usually obtained from a saddle-point approximation, which boils down to extremizing the potential, not only with respect to the background but also with respect to the Gribov parameters themselves. It is important to realize that, even though the Gribov parameters will, at least in the present setting, always be found real, some of them could – and will – become negative. For the various cases studied, we find the following gap equations:

Degenerate case:
\[
0 = \sum_\kappa \left[ \frac{d}{d - 1} \frac{1}{g^2 C_{ad}} - \hat{J}_\kappa (m^4) \right],
\]
Partially degenerate case:
\[
\forall j, \quad 0 = \frac{d}{d-1} \frac{1}{g^2 C_{ad}} - \hat{J}_{(j)}(m_{(j)}^4),
\]
\[
0 = \sum_{\alpha} \left[ \frac{d}{d-1} \frac{1}{g^2 C_{ad}} - \hat{J}_{\alpha}(m_{\alpha}^4) \right],
\] (28)

Non-degenerate case:
\[
\forall j, \quad 0 = \frac{d}{d-1} \frac{1}{g^2 C_{ad}} - \hat{J}_{(j)}(m_{(j)}^4),
\]
\[
\forall \alpha, \quad 0 = \frac{d}{d-1} \frac{1}{g^2 C_{ad}} - \hat{J}_{\alpha}(m_{\alpha}^4),
\] (29)

where we have introduced the sum-integral
\[
\hat{J}_\kappa(m^4) = \int_Q^T \frac{1}{Q_{\kappa}^4 + m^4}.
\] (30)

1. Zero temperature limit

In the zero-temperature limit, because the shifted momenta \(Q_{\kappa}\) can always be shifted back to \(Q\), all Gribov parameters obey the same equation
\[
0 = \frac{d}{d-1} \frac{1}{g^2 C_{ad}} - \hat{J}(m_{\text{vac}}^4),
\] (31)
with
\[
\hat{J}(m^4) = \int_Q^T \frac{1}{Q^4 + m^4}.
\] (32)

This integral, if restricted to real Gribov parameters, is defined only for \(m^4 > 0\); its evaluation is recalled in App. A. We then arrive at the well known zero-temperature gap equation [22]
\[
0 = 1 - \frac{3g^2 C_{ad}}{64\pi^2} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\mu^4}{m_{\text{vac}}^4} + \frac{5}{6} \right],
\] (33)
which can be renormalized by setting (minimal subtraction scheme)
\[
\frac{1}{g^2 C_{ad}} = \frac{1}{g^2(\bar{\mu}) C_{ad}} + \frac{3}{64\pi^2} \frac{1}{\epsilon}.
\] (34)

The renormalized equation reads
\[
0 = 1 - \frac{3g^2(\bar{\mu}) C_{ad}}{128\pi^2} \left[ \ln \frac{\bar{\mu}^4}{m_{\text{vac}}^4} + \frac{5}{3} \right],
\] (35)
and is solved as
\[
m_{\text{vac}}^4 = \bar{\mu}^4 \exp \left( \frac{5}{3} - \frac{128\pi^2}{3g^2(\bar{\mu}) C_{ad}} \right).
\] (36)

From Eq. (34), we find that the renormalized coupling runs with the beta function
\[
\beta_g^4 \equiv \frac{d g^4}{d \bar{\mu}} = -g^4 \bar{\mu} \frac{d(1/g^2)}{d \bar{\mu}} = -\frac{3g^4 C_{ad}}{32\pi^2}.
\] (37)

The sign is compatible with asymptotic freedom but the coefficient is not the expected one at order \(g^4\). This happens due to other \(g^4\) contributions arising from the two-loop corrections to the background effective potential. The two-loop gap equation has been determined and renormalized at zero temperature in Ref. [57]. At this order the Gribov parameter also receives a renormalization. We expect the same renormalization factors to renormalize the finite temperature two-loop gap equation. We shall consider this equation in a subsequent work together with the two-loop corrections to the background effective potential.

In principle we could use Eq. (36) to fix the scale \(m_{\text{vac}}^4\) in terms of the known value of \(g(\bar{\mu})\) in the minimal subtraction scheme at some scale \(\bar{\mu} = \bar{\mu}_0\). However, since the running of \(g(\bar{\mu})\) does not coincide, not even at order \(g^4\), with the true running, we expect large errors in the scale setting. We therefore postpone this question to a forthcoming two-loop study – where the running coupling should be exact at leading order. Thus, in what follows, we express all our results in units of \(m_{\text{vac}}^4\), what also allows for an easy comparison with Ref. [18]. We finally mention that the solution \(m_{\text{vac}}^4\) is unique, given the renormalized coupling at the scale \(\bar{\mu}\). This means that not only the Gribov parameters all obey the same equation at zero-temperature but also that they all become equal, as announced above.

2. Finite temperature case

Following Ref. [18], we can always parametrize the gap equations at finite temperature in terms of the solution \(m_{\text{vac}}^4\) at zero temperature. Subtracting the zero temperature equations from the finite temperature ones we find the following gap equations

Degenerate case:
\[
0 = \sum_{\kappa} \Delta \hat{J}_{\kappa}(m_\kappa^4; m_{\text{vac}}^4),
\] (38)

Partially degenerate:
\[
\forall j, \quad 0 = \Delta \hat{J}_{(j)}(m_{(j)}^4; m_{\text{vac}}^4),
\]
\[
0 = \sum_{\alpha} \Delta \hat{J}_{\alpha}(m_{\alpha}^4; m_{\text{vac}}^4),
\] (39)

Non-degenerate case:
\[
\forall j, \quad 0 = \forall j, \Delta \hat{J}_{(j)}(m_{(j)}^4; m_{\text{vac}}^4),
\]
\[
\forall \alpha, \quad 0 = \Delta \hat{J}_{\alpha}(m_{\alpha}^4; m_{\text{vac}}^4),
\] (40)

where we have introduced the UV finite difference
\[
\Delta \hat{J}_{\kappa}(m^4; m_{\text{vac}}^4) \equiv \hat{J}_{\kappa}(m^4) - \hat{J}(m_{\text{vac}}^4)
\]
\[
= \int_Q^T \frac{1}{(\omega_n + T r \cdot \kappa)^2 + q^2)^2 + m^4} - \int_Q^T \frac{1}{Q^4 + m_{\text{vac}}^4}. \] (41)
Some useful remarks are in order here. First of all, $\Delta J_{0(\alpha)}(m^4; m^4_{\text{vac}})$ is a strictly decreasing function over the interval $m^4 \in [0, +\infty]$ that diverges positively as $m^4 \to 0^+$ and becomes negative as $m^4 \to +\infty$. This implies that the gap equation for the neutral Gribov parameter $m^4_{(0(\alpha))}$ has a unique solution and therefore that all neutral Gribov parameters coincide. We shall denote their common value $m^4_N$ in the following. The same behavior holds for the function $\sum_k \Delta J_k(m^4; m^4_{\text{vac}})$ and then the gap equation for the degenerate Gribov parameter $m^4$ has a unique solution. It also follows that $m^4_N$ and $m^4$ are strictly positive.

Similar conclusions hold for $m^4_\alpha$ and $m^4_{\text{ch}}$ with the noticeable difference that these parameters can become negative. Indeed, it is easily checked that $\Delta J_\alpha(m^4; m^4_{\text{vac}})$ is a strictly decreasing function over the interval $m^4 \in [-M^4_{\alpha}, +\infty]$, with $M^4_{\alpha} \equiv \min_{\beta} (2\pi n + r \cdot \alpha)^2 T^4$. It diverges positively as $m^4 \to -M^4_{\alpha}$ and becomes negative as $m^4 \to \infty$. From this it follows that the gap equation for $m^4_\alpha$ has a unique solution for given values of the temperature and the background but this solution can become negative since the only constraint is that it should remain strictly larger than $-M^4_{\alpha}$. In fact, we can determine at which temperature $m^4_\alpha$ may vanish. We just need to enforce a zero solution in the corresponding equation, namely

$$0 = \Delta \hat{J}_\alpha(0; m^4_{\text{vac}}). \quad (42)$$

Similar considerations apply to $m^4_{\text{ch}}$, but now the function $\sum J_\alpha(m^4; m^4_{\text{vac}})$ diverges as $m^4 \to -\min M^4_{\alpha}$. Again the temperature at which $m^4_{\text{ch}}$ may vanish can be obtained by solving the equation

$$0 = \sum_\alpha \Delta \hat{J}_\alpha(0; m^4_{\text{vac}}). \quad (43)$$

In practice, when evaluating $\Delta \hat{J}_\alpha(m^4; m^4_{\text{vac}})$, we need to distinguish the case where $m^4 > 0$ and the case where $-M^4_{\alpha} < m^4 < 0$. These two cases are discussed in App. A. For $m^4 > 0$, we find

$$\Delta \hat{J}_\alpha(m^4; m^4_{\text{vac}}) = \frac{1}{32\pi^2} \ln \frac{m^4_{\text{vac}}}{m^4} \left[ \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 + im^2}} \right]$$

$$+ \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 + im^2}} \right]$$

$$- \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 - im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 - im^2}} \right]$$

$$= \frac{1}{32\pi^2} \ln \frac{m^4_{\text{vac}}}{m^4} \left[ \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 + im^2}} \right]$$

$$- \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 - im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 - im^2}} \right]$$

$$- \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 + im^2}} \right]$$

$$= \frac{1}{32\pi^2} \ln \frac{m^4_{\text{vac}}}{m^4} \left[ \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 + im^2}} \right]$$

$$- \frac{1}{4m^2} \int_q \frac{1}{\sqrt{q^2 - im^2}} \cos(r \cdot \kappa) - e^{-\beta \sqrt{q^2 - im^2}} \right]$$

For $-M^4_{\alpha} < m^4 < -M^4 < 0$, we find instead

$$\Delta \hat{J}_\alpha(m^4; m^4_{\text{vac}}) = \frac{1}{32\pi^2} \ln \frac{m^4_{\text{vac}}}{M^4} \left[ \frac{1}{4M^2} \int_q \frac{1}{\sqrt{q^2 + M^2 - q^2} \cos(r \cdot \kappa) - \cos(\beta \sqrt{M^2 - q^2})} \right]$$

$$+ \frac{1}{4M^2} \int_q \frac{1}{\sqrt{q^2 + M^2 - q^2} \cos(r \cdot \kappa) - \cos(\beta \sqrt{M^2 - q^2})} \right]$$

$$+ \frac{1}{4M^2} \int_q \frac{1}{\sqrt{q^2 + M^2 - q^2} \cos(r \cdot \kappa) - \cos(\beta \sqrt{M^2 - q^2})} \right]. \quad (44)$$

For practical purposes, it is convenient to absorb the integrable singularity (in the second integral) as $q \to M$ using the change of variables $u = \sqrt{q^2 + M^2}$. For consistency, we apply similar changes of variables to the other two integrals.

C. Finite form of the effective potential

Finally, the integrals that enter the one-loop potential are also well known and recalled in Appendix A. Using Eq. (51), we find $V(r, \{m^2\}) = \sum_k V_k(r, m^2)$ with

$$V_k(r, m^4) = \frac{d-1}{2} \Delta \hat{K}_k(m^4, m^4_{\text{vac}}) - \frac{d}{4} \Delta \hat{K}_0(m^4_{\text{vac}}) \quad (46)$$

and

$$\Delta \hat{K}_k(m^4, m^4_{\text{vac}}) = \ln(Q_k^2 + m^4) - m^4 \int_Q \frac{1}{Q_k^2 + m^4_{\text{vac}}} \quad (47)$$

It is easily checked that this expression is UV finite, up to a quartic divergence that vanishes in dimensional regularization. More precisely, in the case where $m^4$ is positive, we find (see App. A)

$$\Delta \hat{K}_k(m^4, m^4_{\text{vac}}) = \frac{m^4}{32\pi^2} \left[ \ln \frac{m^4_{\text{vac}}}{m^4} + 1 \right]$$

$$+ T \int_q \ln(e^{-2\beta \sqrt{q^2 + im^2}} - 2e^{-\beta \sqrt{q^2 + im^2}} \cos(r \cdot \kappa) + 1)$$

$$+ T \int_q \ln(e^{-2\beta \sqrt{q^2 - im^2}} - 2e^{-\beta \sqrt{q^2 - im^2}} \cos(r \cdot \kappa) + 1) \quad (48)$$

For $-M^4 < m^4 = -M^4 < 0$, we find instead

$$\Delta \hat{K}_k(m^4, m^4_{\text{vac}}) = -\frac{M^4}{32\pi^2} \left( 1 + \ln \frac{m^4_{\text{vac}}}{M^4} \right)$$

$$+ T \int_{q < M} \ln(2 \cos(\beta \sqrt{M^2 - q^2}) - 2 \cos(r \cdot \kappa))$$

$$+ T \int_{q > M} \ln(e^{-2\beta \sqrt{q^2 - M^2}} - 2e^{-\beta \sqrt{q^2 - M^2}} \cos(r \cdot \kappa) + 1)$$

$$+ T \int_q \ln(e^{-2\beta \sqrt{q^2 + M^2}} - 2e^{-\beta \sqrt{q^2 + M^2}} \cos(r \cdot \kappa) + 1) \quad (49)$$
FIG. 1. Top: Degenerate vs non-degenerate Gribov parameters (in units of $m_{\text{vac}}$) for a confining background. These correspond to the actual Gribov parameters up to $T_c \equiv T_c/m_{\text{vac}} \sim 0.402$ and $T_c \sim 0.324$ respectively. Bottom: Gribov parameters at the minimum of the background effective potential.

IV. APPLICATION TO THE DECONFINEMENT TRANSITION

In what follows we use the previous formalism to study the deconfinement transition in SU(2) and SU(3) Yang-Mills theories. We minimize the background effective potential with respect to the order parameter $r$, taking into account the $r$-dependence of the Gribov parameter(s) via the gap equation(s), that is by minimizing $V(r, \{m^2_n(r)\})$. We first revisit the SU(2) results of Ref. [18] by including the possibility of color dependent Gribov parameters and then extend our analysis to the SU(3) case.

A. SU(2) case

In this case $\kappa \in \{-1,0,+1\}$ and the confining point corresponds to $r = \pi$. The partially degenerate and non-degenerate cases coincide.

1. Critical temperature

Since we expect the transition to be second order, we can evaluate $T_c$ by requiring that (we illustrate the degenerate case here but the same discussion holds for the non-degenerate one)

$$\frac{d^2}{dr^2} V(r, m^2(r)) \bigg|_{r=\pi} = 0.$$  \hspace{1cm} (50)

Since $\partial V/\partial m^2 |_{r,m^2(r)} = 0$, we have

$$\frac{d}{dr} V(r, m^2(r)) = \frac{\partial}{\partial r} V(r, m^2(r))$$  \hspace{1cm} (51)

and then

$$\frac{d^2}{dr^2} V(r, m^2(r)) = \frac{\partial^2}{\partial r^2} V(r, m^2(r)) + \frac{\partial^2}{\partial r \partial m^2} V(r, m^2(r)) \frac{dm^2(r)}{dr}.$$  \hspace{1cm} (52)

Finally, it is easily shown that $dm^2(r)/dr |_{r=\pi} = 0$\footnote{This is because $dm(r)/dr |_{r=\pi} = 0$ is proportional to $\sum_\alpha \int_0^T Q^2_\alpha Q^2_\beta m^4 = \int_0^T (\omega_n + \pi T)(\omega_n + \pi T)^2 + q^2 - \int_0^T (\omega_n - \pi T)(\omega_n - \pi T)^2 + q^2)$}, obtained from solving Eq. (42) which takes here the form

$$\frac{1}{8\pi^2} = \int_q \left[ \frac{1 - 2 f_q}{4 q^3} + \frac{1}{m^2_{\text{vac}}} \frac{\text{Im} \frac{1}{2 \sqrt{q^2 + im^2_{\text{vac}}}}}{} \right].$$  \hspace{1cm} (55)

The non-degenerate case is obtained upon making the replacement $m^2(\pi) \rightarrow m_{\text{Ch}}^2(\pi)$.

In order to find the transition temperatures in each case, we need to determine the temperature dependence of $m^2(\pi)$ and $m_{\text{Ch}}^2(\pi)$. This is shown in Fig. 1 together with the temperature dependence of $m_{\text{Ch}}^2(\pi)$ for completeness. We observe that $m_{\text{Ch}}^2(\pi)$ decreases rapidly and even changes sign (as already anticipated in the previous section) at a temperature $T/m_{\text{vac}} \sim 0.344$, obtained from solving Eq. (42) which takes here the form

$$\sum_\alpha \int_0^T Q^2_\alpha Q^2_\beta m^4 = \int_0^T (\omega_n + \pi T)(\omega_n + \pi T)^2 + q^2 - \int_0^T (\omega_n - \pi T)(\omega_n - \pi T)^2 + q^2).$$

Using the changes of variables $\omega_n \rightarrow -\omega_n - 2\pi T$ and $\omega_n \rightarrow -\omega_n + 2\pi T$, we find that the sum-integrals are both zero.
The decrease of $m_{\text{ch}}^4(\pi)$ with the temperature has the effect of lowering the transition temperature as compared to the degenerate case. We find

$$\frac{T_{c}\text{non-deg}}{m_{\text{vac}}} \sim 0.324,$$

which should be compared to the result of Ref. 18

$$\frac{T_{c}\text{deg}}{m_{\text{vac}}} \sim 0.402.$$

This represents a change of the transition temperature by 20\%-25%.

2. Effective potential

In order to compute the potential as a function of $r$, we first need to determine, for each temperature, the $r$-dependence of the Gribov parameters. This dependence is shown in Fig. 2. Above $T/m_{\text{vac}} \sim 0.344$, a gap opens in the values of $r$, over which $m_{\text{ch}}^4$ becomes negative. At each temperature, the boundaries of this interval can be determined by solving

$$\frac{1}{8\pi^2} = \int_q \left[ \frac{1 + 2\text{Re}n_q - ir\cdot\kappa T}{4q^3} + \frac{1}{m_{\text{vac}}^2} \text{Im} \frac{1}{2\sqrt{q^2 + im_{\text{vac}}^2}} \right].$$

We stress that, despite $m_{\text{ch}}^4$ becoming negative, the potential remains real. The results for the potential are shown in Fig. 3. We verify that the transition is second order and that the transition temperatures agree with the estimates given above. We also note that the minimum never enters the region of negative $m_{\text{ch}}^4$, as can also be seen in Fig. 1 (bottom), where we show the Gribov parameters at the minimum of the potential.

FIG. 2. $r$-dependence of the Gribov parameters for various temperatures (in units of $m_{\text{vac}}$). Top: degenerate case. Middle: non-degenerate case, neutral mode. Bottom: non-degenerate case, charged mode.

FIG. 3. SU(2) background effective potentials for various temperatures (in units of $m_{\text{vac}}$). Top: degenerate case. Bottom: non-degenerate case.
We shall rename the minimum of the background effective potential.

FIG. 4. Top: Degenerate vs partially or non-degenerate Gribov parameters (in units of $m_{\text{vac}}$) for a confining background. Middle: Degenerate vs partially degenerate Gribov parameters at the minimum of the background effective potential. Bottom: Degenerate vs non-degenerate Gribov parameters at the minimum of the background effective potential.

**B. SU(3) case**

We can repeat a similar analysis for the SU(3) gauge group. In this case there are two neutral modes $\kappa = 0^{(3)}$ and $\kappa = 0^{(8)}$, and six roots $\kappa = \alpha$, with $\alpha \in \{\pm(1,0), \pm(1/2, \sqrt{3}/2), \pm(1/2, -\sqrt{3}/2)\}$. The confining point is $r = (4\pi/3,0)$. Moreover, due to charge conjugation invariance, we can restrict the analysis to $r = (r_3,0)$. We shall rename $r_3$ as $r$ in what follows. We also mention that

$$m_{(1,0)}^4 = m_{(-1,0)}^4$$

and

$$m_{(1/2, \sqrt{3}/2)}^4 = m_{(-1/2, \sqrt{3}/2)}^4 = m_{(1/2, -\sqrt{3}/2)}^4 = m_{(-1/2, -\sqrt{3}/2)}^4.$$  

Therefore, in the non-degenerate case, we only need to introduce two charged Gribov parameters, denoted $m_{\text{Ch},1}^4$ and $m_{\text{Ch},2}^4$ respectively. As it is easily checked, at the confining point they both coincide with the charged Gribov parameter $m_{\text{Ch}}^4(r)$ of the partially degenerate case, and in general, $m_{\text{Ch},2}^4(r) = m_{\text{Ch},1}^4(r/2).

1. Highest spinodal

In the SU(3) case, we expect the transition to be first order so we cannot determine the transition temperature so simply as above. However we expect the spinodal temperatures to be quite close to the transition temperature. The highest spinodal can be determined using the same method as above because it occurs at $r = 4\pi/3$. We first evaluate the curvature at $r = 4\pi/3$. To this purpose, we notice that Eqs. (51) and (52) are still valid. Moreover, both in the degenerate and the partially degenerate cases, it is easily shown that $dm(r)/dr\big|_{r=4\pi/3} = 0$.

It follows that

$$\frac{d^2}{dr^2} V(r,m(r)) \bigg|_{r=4\pi/3} = \frac{\partial^2}{\partial r^2} V(r,m(r)) \bigg|_{r=4\pi/3}.$$  

In the degenerate case, the condition for a vanishing curvature reads then

$$3\text{Re} \int_q e^{-3\beta\sqrt{q^2 + im^2}} + 4e^{-2\beta\sqrt{q^2 + im^2}} + e^{-\beta\sqrt{q^2 + im^2}} \cdot (e^{-2\beta\sqrt{q^2 + im^2}} + e^{-\beta\sqrt{q^2 + im^2}} + 1)^2
- 2\text{Re} \int_q e^{-3\beta q} + 4e^{-2\beta q} + e^{-\beta q}
= 0.$$  

The partially degenerate case is obtained upon making the replacement $m^2(4\pi/3) \rightarrow m_{\text{Ch}}^2(4\pi/3)$. The non-degenerate case cannot be treated in this way. The corresponding transition temperature will be determined in the next section.

14 This is because $dm(r)/dr\big|_{r=4\pi/3} = 0$ is proportional to

$$\sum_{\alpha}^{\alpha = 3} \int_Q Q^2_\alpha Q^2_\alpha \cdot m_{\text{Ch}}^2 = 2 \left[ \int_Q \left( \omega_\chi + 4\pi/3T \omega_\chi + \frac{q^2}{4} \right) \left( \omega_\chi + 4\pi/3T \omega_\chi + \frac{q^2}{4} \right) + m_{\text{Ch}}^2 \right] + \int_Q \left( \omega_\chi + 2\pi/3T \omega_\chi + \frac{q^2}{4} \right) \left( \omega_\chi + 2\pi/3T \omega_\chi + \frac{q^2}{4} \right) + m_{\text{Ch}}^2.$$

Using the change of variables $\omega_\chi \rightarrow -\omega_\chi - 2\pi T$ in the second integral, we find that the bracket is zero.
FIG. 5. $r$-dependence of the Gribov parameters for various temperatures (in units of $m_{\text{vac}}$). We show only the degenerate and partially degenerate cases. The non-degenerate charged Gribov parameters are obtained in terms of the SU(2) one respectively as $m_{\text{Ch},1}^4(r) = m_{\text{Ch},SU(2)}^4(r)$ and $m_{\text{Ch},1}^4(r) = m_{\text{Ch},SU(2)}^4(r/2)$.

The temperature dependence of the Gribov parameters at the minimum is shown in Fig. 4. Using this temperature dependence, we can determine the spinodal temperatures. We find

$$\frac{T_{c}^{\text{part-deg}}}{m_{\text{vac}}} \sim \frac{T_{c}^{\text{part-deg}}}{m_{\text{vac}}} \sim 0.409,$$

so again a 20% difference.

2. Effective potential

Once again, to compute the potential we need to know the background dependence of the Gribov parameters.

as compared to the result of Ref. [18]

$$\frac{T_{c}^{\text{deg}}}{m_{\text{vac}}} \sim \frac{T_{c}^{\text{deg}}}{m_{\text{vac}}} \sim 0.512,$$

so again a 20% difference.
This is shown in Fig. 5 where one sees that the charged ones can become negative. For the degenerate and partially degenerate cases, we find transition temperatures very close to the higher spinodal temperatures determined above. For the completely non-degenerate case, we find
\[ T_{\text{non-deg}}^{\text{vac}} \sim 0.48, \] (65)
which represents a 6\% difference with respect to the degenerate case. We mention that, as compared to the degenerate and partially degenerated cases, it was crucial in the non-degenerate case to be able to resolve the potential in the region where the Gribov parameters become negative because the minimum lies in this region just before the transition occurs, as can be seen in the bottom plot of Fig. 4.

C. Comparison with the Curci-Ferrari model

We finally compare our model at one-loop with a similar calculation in the CF model. To this purpose we show the Polyakov loops in Fig. 7. We observe that the growth of the order parameter above \( T_c \) is slower in the Gribov-Zwanziger approach than in the CF model. This is more qualitatively in line with the behavior observed on the lattice.

V. RELATION WITH THE GRIBOV RESTRICTION

In this section, we investigate the relation between the model (13) and the restriction of the functional integral to the first Gribov region. We first show that, at zero temperature and to one-loop accuracy, the model can be related Gribov no-pole condition applied to the Landau-DeWitt gauge. We then argue that the result is not so surprising since, at zero temperature, there is a trivial mapping between the Landau and Landau-DeWitt gauges. Finally, we investigate the extension to the finite temperature case, emphasizing similar difficulties than the ones discussed in Refs. [49, 50].

A. Relation with the Gribov no-pole condition at zero temperature

We first recall how the no-pole condition is constructed at one-loop order in the Landau gauge at zero temperature\[15\] and then extend it to the Landau-DeWitt gauge.

Consider the ghost propagator \( G_{ab}(K,P,A) \) in the presence of a gauge field configuration \( A \). If we evaluate this propagator for \( P = K \) and \( b = a \), we obtain
\[ G_{aa}(K,K,A) = \int x \int y (e^{iKx} \delta_{ac})^* (\partial D)_{cd}^{-1}(x,y)(e^{iKy} \delta_{ad}). \] (66)
If \( A \) belongs to the first Gribov region, it follows by construction that \( G_{aa}(K,K,A) > 0, \forall a \) and \( \forall K \). In other words, by imposing these inequalities, one restricts \( A \) to lie in a domain that still contains the Gribov region. Moreover, if starting from inside the Gribov region (say from \( A = 0 \)), we approach its boundary (the so-called Gribov horizon), at least one of the \( G_{aa}(K,K,A) \)'s diverges and changes sign. This means that the Gribov horizon lies inside the boundary of the region defined by the conditions \( G_{aa}(K,K,A) > 0, \forall a \) and \( \forall K \).

In practice, it is not simple to impose the conditions for all \( a \)'s separately and instead one imposes \( \text{tr} \overline{G}(K,K,A) > 0, \forall K \), where
\[ \overline{G}(K,A) \equiv \frac{1}{\text{Vol}O(4)} \int_{\Lambda \in O(4)} O(\Lambda K, A). \] (67)
This defines a priori a larger domain in \( A \)-space but again, when approaching the Gribov horizon from inside the Gribov region, at least one of the \( \text{tr} \overline{G}(K,K,A) \)'s

\[ \text{Deg.} \quad \text{Par. deg.} \quad \text{Non-deg.} \quad \text{CF} \]

\[ \text{T/T}_c \]

\[ t \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

FIG. 7. Polyakov loops. Top: SU(2). Bottom: SU(3).
has to change sign and the Gribov horizon lies inside the boundary of the region defined by $\text{tr} \mathcal{G}(K, K, A) > 0, \forall K$. Let us also mention that, for the practical evaluation of $\text{tr} \mathcal{G}(K, K, A)$, one can always assume that $A$ is transverse.

\begin{equation}
G_{ab}(K, P; A) = \frac{(2\pi)^d}{K^2} \delta_{ab} \delta^{(d)}(K - P) + ig f_{bda} \frac{P^\mu}{K^2} A^d_\mu(K - P) + (ig)^2 f_{cda} f_{bec} \frac{1}{K^2} \int Q \frac{(K - Q)_\mu}{(K - Q)^2} P^\nu_{\rho} A^d_\rho(Q) A^a_\nu(K - Q - P),
\end{equation}

and therefore

\begin{equation}
\frac{1}{V_d d_G} \text{tr} \mathcal{G}(K, K; A) = \frac{1}{K^2} \left[ 1 + \sigma(K^2, A) \right],
\end{equation}

with

\begin{equation}
\sigma(K^2, A) = \frac{1}{V_d (d - 1)} \frac{g^2 C_{ad} P^\mu_{\rho}(K)}{d_G} \times \int Q \frac{P^\mu_{\rho}(Q)}{(K - Q)^2} A^\alpha_\rho(Q) A^\alpha_\rho(-Q),
\end{equation}

where the labels $\alpha$ and $\rho$ are summed over. In deriving this expression, we have used that $A$ can be taken transverse, and, by using appropriate changes of variables, we have traded the average over $O(4)$ Euclidean rotations of $K$ by the average

\begin{equation}
\frac{A^\alpha_\rho(Q) A^\alpha_\rho(-Q)}{\text{Vol} O(4)} \int_{\Lambda \in O(4)} A^\alpha_\Lambda(Q) A^\alpha_\Lambda(-\Lambda Q).
\end{equation}

The previous formula corresponds to the strict expansion of a propagator to order $g^2$. To this order, this is equivalent to

\begin{equation}
\frac{1}{V_d d_G} \text{tr} \mathcal{G}(K, K, A) = \frac{1}{K^2} \frac{1}{1 - \sigma(K^2, A)},
\end{equation}

In this 1PI-resummed form, the result is expected to be more accurate.

The Gribov no-pole condition corresponds a priori to the infinite set of conditions

\begin{equation}
\forall K, 1 - \sigma(K^2, A) > 0.
\end{equation}

However, it is usually argued that it is enough to impose the no-pole condition in the form

\begin{equation}
1 - \sigma(0, A) > 0.
\end{equation}

This is because $\sigma(K^2, A)$ is a decreasing function of $K^2$. In fact, because $A^\alpha_\rho(Q) A^\alpha_\rho(-Q)$ depends only on $Q^2$, the dependence with respect to $K$ originates only from the angular integral

\begin{equation}
\Omega_d(K^2/Q^2) = \int_0^\pi \frac{d\theta}{K^2/Q^2 + 1 - 2 K/Q \cos \theta}.
\end{equation}

Given these preliminary remarks, at order $g^2$, one finds

\begin{equation}
\Theta(1 - \sigma(0, A)) \propto \int_{-\infty + i \epsilon}^{+\infty + i \epsilon} \frac{d\beta}{2\pi i} e^{\beta(1 - \sigma(0, A))}.
\end{equation}

The partition function becomes

\begin{equation}
Z = \int \mathcal{D} \text{Acc}\, \Theta(1 - \sigma(0, A)) e^{-S_{FP}[A, c, \xi, h]}
= \int_{-\infty + i \epsilon}^{+\infty + i \epsilon} \frac{d\beta}{2\pi} e^{\beta - \ln \beta - V_d f(\beta)}.
\end{equation}

Given that, in the gluonic sector, the quadratic part of the action in Fourier space becomes (we introduce a gauge-fixing parameter $\xi$ that we will send to zero at the end)

\begin{equation}
K^\mu_{\nu\rho}(Q) = \delta^{ab} \left[ Q^2 P^\mu_{\rho}(Q) + \frac{Q^2}{\xi} P^\mu_{\rho}(Q) + \frac{m^4(\beta)}{Q^2} \delta_{\mu\nu} \right],
\end{equation}

In Fig. 8, we show that $\Omega_d(x)$ is a decreasing function of $x > 0$, for $d \geq 2$ and, since $A^\alpha_\rho(Q) A^\alpha_\rho(-Q)$ is positive, it follows that $\sigma(K^2, A)$ decreases indeed with $K^2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The function $\Omega_d(x)$ for $d \geq 2$ and $x > 0$.}
\end{figure}

In the limit $K \to 0$, one finds

\begin{equation}
\sigma(0, A) = \frac{1}{V_d d_G} \int Q \frac{A^\alpha_\rho(Q) A^\alpha_\rho(-Q)}{Q^2},
\end{equation}

where we have used that $\int_Q f(Q^2) A^\alpha_\rho(Q) A^\alpha_\rho(-Q) = \int d^d Q f(Q^2) A^\alpha_\rho(Q) A^\alpha_\rho(Q)$. In order to implement the constraint \cite{12}, one then writes

\begin{equation}
\Theta(1 - \sigma(0, A)) \propto \int_{-\infty + i \epsilon}^{+\infty + i \epsilon} \frac{d\beta}{2\pi i} e^{\beta(1 - \sigma(0, A))}.
\end{equation}

The partition function becomes

\begin{equation}
Z = \int \mathcal{D} \text{Acc}\, \Theta(1 - \sigma(0, A)) e^{-S_{FP}[A, c, \xi, h]}
= \int_{-\infty + i \epsilon}^{+\infty + i \epsilon} \frac{d\beta}{2\pi} e^{\beta - \ln \beta - V_d f(\beta)}.
\end{equation}

Given that, in the gluonic sector, the quadratic part of the action in Fourier space becomes (we introduce a gauge-fixing parameter $\xi$ that we will send to zero at the end)
with
\[
m^4(\beta) \equiv \frac{2}{d} g^2 C_{ad} \frac{\beta}{dG} V_d,
\]
(80)
one obtains, at one-loop order
\[
f(\beta) = d_G \left[ \frac{d - 1}{2} \int_Q \ln Q^2 + m^4(\beta) \right] Q^2 \\
+ \frac{1}{2} \int_Q \ln Q^2 + \xi m^4(\beta) - \int_Q \ln Q^2 \right],
\]
(81)
where the last term is the ghost contribution. One can evaluate the integral over \( \beta \) using a saddle-point approximation. One finds \( \ln Z \sim \beta_0 + \ln \beta_0 - V_d f(\beta_0) \), with
\[
0 = 1 - \frac{1}{\beta_0} - \frac{d - 1}{d} g^2 C_{ad} \int_Q \frac{1}{Q^2 + m^4(\beta_0)}.
\]
(82)
If we assume \( m^4(\beta_0) \) to have a non-trivial infinite volume limit, \( \beta_0 \) has to diverge linearly with \( V_d \) and we arrive at a free-energy density that coincides with the zero-temperature and zero-background limit of Eq. (25) with
\[
0 = 1 - \frac{d - 1}{d} g^2 C_{ad} \int_Q \frac{1}{Q^2 + m^4(\beta_0)}.
\]
(83)

The extension to the Landau-DeWitt gauge is rather straightforward: one switches to a Cartan-Weyl basis (which implies in particular replacing \( i f^{abc} \) by \( f^{\kappa \lambda \gamma} \), replaces \( \Lambda_\mu \) by \( a_\mu \) and each momentum by its appropriately shifted version. One then considers the ghost propagator
\[
G_{\kappa \lambda}(K, P; a; \bar{A})
\]
in the presence of a gauge-field configuration \( a \) and a background \( \bar{A} \), and evaluates
\[
\begin{align*}
G_{\kappa \lambda}(K, K, a; \bar{A}) &= \int \int (e^{iK_{\kappa - \lambda} \delta_{\kappa \lambda}}) (\bar{D}D)_{\eta}^{-1}(x, y) (e^{iK_{\kappa - \lambda} \delta_{\kappa \lambda}}).
\end{align*}
\]
(84)
Again, if \( a \) belongs to the first Gribov region, we have \( G_{\kappa \lambda}(K, K, a; \bar{A}) > 0, \forall a \) and \( \forall K \). Similarly to the Landau gauge case, we shall impose instead
\[
\sum_{\kappa} G_{\kappa \lambda}(K, K, a; \bar{A}) > 0, \forall K,
\]
(85)
and where we can assume that \( a \) is transverse in a background covariant way. At order \( g^2 \), We find

\[
G_{\kappa \lambda}(K, P; a; \bar{A}) = \frac{(2\pi)^d}{K^2} \delta_{\kappa \lambda} \delta^{(d)}(K - P) + g f_{\lambda \rho} \frac{P_\rho}{K^2} a_\lambda^\eta(K - P) \\
+ g^2 f_{\eta(-\kappa)(-\kappa)} \frac{1}{K^2} \int_Q \frac{(K - Q)_{\mu}}{(K - Q)^2} f_{\lambda \mu}^\eta a_\lambda^\mu(K - Q - P).
\]
(86)

and therefore
\[
\frac{1}{V_d} \sum_{\kappa} G_{\kappa \lambda}(K, K, a; \bar{A}) = \frac{1}{K^2} \int_Q \frac{1}{1 - \sigma(K^2, a; \bar{A})}.
\]
(87)
with
\[
\sigma(K^2, a; \bar{A}) = \frac{1}{V_d} \frac{g^2 C_{ad} | P|^\beta(K)}{dG} \\
\times \int_Q \frac{P_{\lambda \mu}^\beta(Q)}{(K - Q)^2} a_\lambda^\mu(Q - \eta) a_\mu^\eta(-Q - \eta).
\]
(88)

In deriving these expressions, before taking the average over \( \Lambda \)-transformations, we have used that, at zero temperature, one can always shift the integration momentum \( Q_\eta \) to \( Q \). Then, by appropriate changes of variables, we have traded the average over \( O(4) \) Euclidean rotations of \( K \) by the average
\[
a^\eta_\rho(Q - \eta) a_\rho^\eta(-Q - \eta)
\]
and
\[
a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta)
\]
It is easily checked that \( a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta) \) is positive and depends only on \( Q^2 \). Therefore we are in a similar situation as above, with \( \sigma(K^2, a; \bar{A}) < \sigma(0, a; \bar{A}) \).

\[
\sigma(0, a; \bar{A}) = \frac{1}{V_d} g^2 C_{ad} \int Q \frac{a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta)}{Q^2} \\
= \frac{1}{V_d} g^2 C_{ad} \int Q \frac{a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta)}{Q^2}.
\]
(89)

where we made use of
\[
\int Q f(Q^2) a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta) = \int Q f(Q^2) a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta)
\]
and we changed the integration variable back to \( Q_\eta \).

After introducing a parameter \( \beta \) to impose the no-pole condition, we arrive at \( \ln Z = \beta_0 - \ln \beta_0 - V_d \sum_{\kappa} f_\kappa(\beta) \) with
\[
f_\kappa(\beta) = \frac{d - 1}{d} \int Q \ln Q^2 + m^4(\beta) \]
(89)

\[
= \frac{1}{V_d} g^2 C_{ad} \int Q \frac{a_\rho^\eta(Q - \eta) a_\rho^\eta(-Q - \eta)}{Q^2}.
\]
(90)
+ 1 \int_{Q} \ln \frac{Q_{\kappa}^{4} + \xi m^4(\beta)}{\xi Q_{\kappa}^{2}} - \int_{Q} \ln Q_{\kappa}^{2}, \quad (91)

and

m^4(\beta) = \frac{2 g^2 C_{\text{ad}}}{d} \frac{\beta}{dG} \frac{1}{d}. \quad (92)

The parameter $m^4$ is fixed through the saddle-point equation

$$1 = \frac{d - 1}{d} \frac{g^2 C_{\text{ad}}}{dG} \sum_{\kappa} \int_{Q} \frac{1}{Q_{\kappa}^{2} + m^4}. \quad (93)$$

This is nothing but the gap equation obtained with the model $[13]$. Of course at zero-temperature, one can always shift the momenta $Q_{\kappa}$ back to $Q$, in which case the free-energy density and the gap equations coincide trivially with the ones obtained in the Landau gauge.

### B. Mapping to the Landau gauge

The previous results are not surprising because, at zero temperature, the expression for the partition function in the Landau-DeWitt gauge can be related to the one in the Landau gauge through a trivial transformation of the fields, namely $[16]$.

$$(X^U)_{\mu}^{\kappa}(x) = e^{i g A^\kappa - \phi} X^{\kappa}(x) \quad (94)$$

First, using the property

$$\bar{D}^\mu_{\kappa}(X^U)^{\kappa}(x) = e^{i g A^\kappa - \phi} \partial_{\mu} X^{\kappa}(x), \quad (95)$$

it is easily checked that, upon this change of variables, the Faddeev-Popov action for the Landau-DeWitt gauge becomes the Faddeev-Popov action for the Landau gauge, after one renames $a_{\mu}$ into $A_{\mu}$. It is then easily checked that if one starts from the Gribov-Zwanziger action for the Landau gauge and apply the change of variables (after renaming $A_{\mu}$ into $a_{\mu}$)

$$(X^U)_{\mu}^{\kappa}(x) = e^{-i g A^\kappa - \phi} X^{\kappa}(x), \quad (96)$$

$$(X^U)^{\kappa\lambda}_{\mu}(x) = e^{-i g A^\kappa A^\lambda + \phi} X^{\kappa\lambda}(x), \quad (97)$$

one obtains the action $[12]$.

We should mention however that this mapping crucially relies on the fact that the boundary conditions are not important at zero temperature, at least in the Faddeev-Popov framework. To check this, consider Yang-Mills fields on a compact time interval of length $L$ (which will eventually be sent to $\infty$) with $\theta$ obeying the boundary conditions of the form

$bc1$: $a_{\mu}^\kappa(\tau + L, \bar{x}) = e^{igB^\kappa} a_{\mu}^\kappa(\tau, \bar{x}), \quad (98)$

with $B$ a constant vector in a space isomorphic to the Cartan subalgebra. For the partition function to be invariant under gauge transformations, the latter should be chosen to preserve the boundary $[98]$. This means that the Faddeev-Popov procedure applied to the Landau-DeWitt gauge leads to the usual action but with the peculiarity that all fields obey the boundary conditions $[98]$.

Consider now a two-point function (this could be any correlation function, including the partition function) $G_{\kappa\lambda}^{bc1}(x, y; \bar{A})$, computed within this particular gauge-fixing. We will now show that, in the “zero temperature” limit ($L \to \infty$) it coincides with the same correlation function computed within the same gauge, but with periodic boundary conditions

$bc2$: $a_{\mu}^\kappa(\tau + (L \to \infty), \bar{x}) = a_{\mu}^\kappa(\tau, \bar{x}), \quad (99)$

To show this, we first apply the change of variables (93) with $A$ replaced by $B$. This turns the boundary conditions of all fields into periodic ones, while changing the background from $A$ to $A + B$ and multiplying all correlation functions by appropriate phase factors:

$$G_{\kappa\lambda}^{L,bc1}(x, y; \bar{A}) = e^{-i(\tau + \phi L)} \frac{B}{g L} G_{\kappa\lambda}^{L,bc2}(x, y; \bar{A} + \bar{B}). \quad (100)$$

Next, one applies a background gauge transformations to obtain

$$G_{\kappa\lambda}^{L,bc1}(x, y; \bar{A}) = e^{-i(\tau + \phi L)} \frac{B}{g L} G_{\kappa\lambda}^{L,bc2}(x, y; \bar{A} + \bar{B})', \quad (101)$$

with $B' = B - n\bar{a}/(g L)$, for any $\bar{a}$ that maintains the $1/L$-periodicity of the fields and for any $n \in \mathbb{Z}$. Taking the “zero temperature” limit as $L \to \infty$ with $u$ any real number and $n \to \infty$, we arrive at

$$G_{\kappa\lambda}^{\infty,bc1}(x, y; \bar{A}) = e^{-i(\bar{A} + \phi L)} \frac{B}{g L} G_{\kappa\lambda}^{\infty,bc2}(x, y; \bar{A} + \bar{B}'), \quad (102)$$

with $B' = B - u\bar{a}$. Since the $\bar{a}$'s form a basis of the Cartan subalgebra, repeated use of the previous formula leads to

$$G_{\kappa\lambda}^{\infty,bc1}(x, y; \bar{A}) = G_{\kappa\lambda}^{\infty,bc2}(x, y; \bar{A}). \quad (103)$$

As announced, the zero-temperature correlation functions in the Faddeev-Popov gauge-fixing are the same for the two sets of boundary conditions.

It is however not clear how these remarks extend to the Gribov gauge-fixing. In particular, we should notice that in the above derivation, the use of shifted momenta in $[84]$ implicitly restricts the search for eigenstates of

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16 Of course this does not mean that the two gauges are identical because the correlations functions are not the same. However they are related by trivial identities, see for instance $[53]$.

17 Under an infinitesimal transformation we have $\delta a_{\mu}^\kappa = \partial_{\mu} \theta^\kappa - i g f^{\kappa,\lambda\eta} \theta a_{\mu}^\eta$. If $\theta^\kappa$ obeys the boundary conditions $[98]$, then, using that $f^{\kappa,\lambda\eta}$ is color conserving, one finds that $\delta a_{\mu}^\kappa$ also obeys the boundary conditions $[98]$. 

the Faddeev-Popov operator to eigenstates with certain boundary conditions, those that are precisely mapped to the periodic eigenstates in the Landau gauge. It is not clear to us whether this is what should be done or how taking into account other boundary conditions would affect the result.

C. Extension to finite temperature?

The problem with the boundary conditions is even more visible at finite temperature. First of all, in this case, there is no change of variables that allows to get rid of the background, since the allowed transformations are constrained by the periodicity of the fields. Moreover, as it has been discussed in Refs. [49, 50], the periodic boundary conditions directly affect the implementation of the Gribov gauge-fixing via the Gribov-Zwanziger construction. Let us here summarize the argument in the case of the Landau gauge and then briefly speculate on the consequences for the Landau-DeWitt gauge. A more detailed discussion is postponed for a future investigation.

The Gribov-Zwanziger construction is based on the perturbative evaluation of the lowest non-zero eigenvalues of the Faddeev-Popov operator, starting from the lowest non-zero (degenerate) eigenvalue of the free Faddeev-Popov operator. At zero temperature, working in a box of volume $L^4$ with periodic boundary conditions, the eigenstates of the free Faddeev-Popov operator are of the form $\exp(i(2\pi/L)(n_0 \tau + \vec{n} \cdot \vec{x}))$, with $n_\mu \in \mathbb{Z}$, $\forall \mu$, and the corresponding eigenvalues are $(2\pi/L)^2(n_0^2 + ||\vec{n}||^2)$. Therefore, the lowest non-zero eigenvalue corresponds to states with $n_0^2 + ||\vec{n}||^2 = 1$. In contrast, at finite temperature, where the system is in a box of size $\beta L^3$, the periodic eigenstates are rather $\exp(i(2\pi/\beta)n_0 \tau + (2\pi/L)\vec{n} \cdot \vec{x})$ and the corresponding eigenvalues are $(2\pi/\beta)^2n_0^2 + (2\pi/L)^2||\vec{n}||^2$. Therefore, in this case, the smallest, non-zero eigenvalue corresponds to states with $n_0 = 0$ and $||\vec{n}||^2 = 1$. This has a direct imprint on the Gribov-Zwanziger construction and leads to an action that is not simply the zero-temperature Gribov-Zwanziger action taken over a compact time interval, see Refs. [49, 50] for more details.

We mention here that, even though this asymmetrical treatment of the temporal and spatial components is to be expected at finite temperature, it leads to some unexpected features. In particular, in the zero-temperature limit, one does not recover the usual Gribov-Zwanziger action but rather an action that explicitly breaks the Euclidean $O(4)$ invariance of the vacuum theory. This raises some conceptual issues, in particular concerning the renormalizability of the action or the potential contamination of the zero-temperature observables by these $O(4)$-breaking terms. Of course, if the Gribov-Zwanziger construction corresponds to a bona-fide gauge-fixing, we expect the $O(4)$ breaking terms to be restricted to the gauge-fixing sector and not to affect the $O(4)$-invariance or the UV finiteness of the zero-temperature observables. However, since the Gribov restriction is never implemented exactly in practice, these issues deserve a careful investigation.

We leave this interesting questions for a future work and end this section by speculating on the implications of the previous remarks for the Landau-DeWitt gauge. In the Landau-DeWitt gauge at finite temperature, the role of the free Faddeev-Popov operator is played by $D^2$ but the fields remain periodic. Therefore, the eigenstates are still of the form $\exp(i(2\pi/\beta)n_0 \tau + (2\pi/L)\vec{n} \cdot \vec{x})$ but the eigenvalues become $(2\pi n_0 + r \cdot \kappa)^2/\beta^2 + (2\pi/L)^2||\vec{n}||^2$. It follows that, for generic backgrounds such that $r \cdot \kappa$ is not a multiple of $2\pi$, the lowest non-zero eigenvalues correspond to $\kappa = 0$, $n_0 = 0$ and $||\vec{n}||^2 = 1$. So not only would the Gribov-Zwanziger procedure affect only the spatial components of the gauge field but only those color components that are aligned with the background. In this case the order parameter for the deconfinement transition – the Polyakov loop or the background $A$ at the minimum of the background effective potential – would not interact with the Gribov region at one-loop order, in contrast to what happens in the present work or in [18]: the search for possible effects on the deconfinement transition would necessarily start at two-loop order.

VI. CONCLUSIONS AND OUTLOOK

We have put forward a Gribov-Zwanziger type action for the Landau-DeWitt gauge that remains invariant under background gauge transformations. At zero-temperature and to one-loop accuracy, our model can be related to the Gribov no-pole condition applied to the Landau-DeWitt gauge. Moreover, in contrast to other recent proposals, our model does not require the introduction of a Stueckelberg field.

Without spoiling the background gauge invariance, our approach allows for color dependent Gribov parameters, a possibility which we have investigated together with its impact on the deconfinement transition. We have observed variations of the transition temperature up to 20%. We have also observed that certain Gribov parameters can become negative while maintaining a real

\[18\] We shall not discuss it here but the implementation of the Gribov no-pole condition is also substantially modified.
effective potential. In fact, in some cases, the transition
is only properly accounted for if \( m^4 \) is allowed to become negative.

Our model allows for the evaluation of higher corrections in a manifestly background gauge invariant way.
We are currently evaluating the two-loop background effective potential and the corresponding finite temperature two-loop gap equations for the Gribov parameters.

Finally, it is important to mention that, at finite temperature, none of the existing proposals, including ours, can be understood so far as faithful implementations of the Gribov-Zwanziger restriction for the Landau-DeWitt gauge. In this respect, it would be important to generalize the considerations of Refs. [49, 50] to the Landau-DeWitt gauge, along the lines of the discussion that we have initiated in Sec. V C.

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Appendix A: Formulae

In what follows, we derive various formulae used in the main text. It will be important to allow for negative values of the Gribov parameter \( m^4 \) in those sum-integrals where the frequency is shifted by \( r \cdot \kappa \). In fact the parameter \( m^4 \) can take values down to \(-M^4_\kappa \) with \( M^4_\kappa \equiv \min_{n \in \mathbb{Z}} (2\pi n + r \cdot \kappa)^4 T^4 \).

1. Sum-integral entering the gap equation

The gap equation involves the sum-integral

\[
\hat{J}_\kappa(m^4) \equiv \int_Q T^4 \frac{1}{Q^4 + m^4}.
\] (A1)

At zero temperature, it does not depend on the background since the latter can be shifted away by a change of variables. In that case, the Gribov parameter \( m^4 \) should be taken positive (without loss of generality, we can assume that \( m^2 > 0 \)). We can then use

\[
\frac{1}{Q^4 + m^4} = -\frac{1}{m^2} \text{Im} \left( \frac{1}{Q^2 + im^2} \right),
\] (A2)

together with the formula

\[
\int_Q \frac{1}{Q^2 + M^2} = -\frac{M^2}{16\pi^2} \left[ \frac{1}{\epsilon} + \ln \frac{\tilde{\mu}^2}{M^2} + 1 \right],
\] (A3)

valid for any non-negative (possibly complex) \( M^2 \), to arrive at

\[
\hat{J}(m^4) \equiv \int_Q \frac{1}{Q^4 + m^4} = \frac{1}{16\pi^2} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\tilde{\mu}^4}{m^4} + 1 \right].
\] (A4)

We can proceed similarly at finite temperature, but this time we need to distinguish the cases \( m^4 > 0 \) and \(-M^4_\kappa < m^4 < 0 \). If \( m^4 > 0 \), we use again (A2) and the usual formula for the tadpole sum-integral at finite temperature. We find

\[
\hat{J}_\kappa(m^4) = -\frac{1}{m^2} \int_q \text{Im} \left[ \frac{1 + n \sqrt{q^2 + im^2 - iT \cdot \kappa} + n \sqrt{q^2 + im^2 + iT \cdot \kappa}}{2\sqrt{q^2 + im^2}} \right].
\] (A5)

Because \( m^2 \) is real, the contribution 1 in the numerator leads to the zero temperature limit (A4). Rewriting also the finite temperature contribution in a simpler way, we arrive at

\[
\Delta \hat{J}_\kappa(m^4; m^4_{\text{vac}}) \equiv I_\kappa(m^4) - I(m^4_{\text{vac}})
= \frac{1}{32\pi^2} \ln \frac{m^4_{\text{vac}}}{m^4} - \frac{1}{m^2} \int_q \text{Im} \left[ \frac{1}{\sqrt{q^2 + im^2}} \frac{e^{i\sqrt{q^2 + im^2}/T \cos(r \cdot \kappa)} - 1}{e^{2\sqrt{q^2 + im^2}/T} - 2e^{i\sqrt{q^2 + im^2}/T \cos(r \cdot \kappa)} + 1} \right],
\] (A6)
which we also rewrite for later convenience as

\[
\Delta \hat{J}_\kappa(m^4, m^4_{\text{vac}}) = \frac{1}{32\pi^2} \ln \frac{m_{\text{vac}}^4}{m^4} + \frac{1}{2im^2} \int_{q < M} \frac{1}{2\sqrt{q^2 + im^2}} \cos(r \cdot \kappa) - e^{-\sqrt{q^2 + im^2}/T} \cos(r \cdot \kappa) - e^{-\sqrt{q^4-m^4}/T} \frac{1}{2im^2} \int_{q > M} \frac{1}{2\sqrt{q^2 - M^2}} \cos(r \cdot \kappa) - e^{-\sqrt{q^2-M^2}/T} \cdot (A7)
\]

If \(-M_{r,\kappa} < m^4 < 0\), we write \(m^2 = iM^2\) (we can assume that \(M^2 > 0\)) and use again \(A2\) but rather as a difference. We find

\[
\hat{J}_\kappa(m^4) = \frac{1}{2M^2} \int_{q < M} \frac{1 + n \sqrt{M^2-q^2-iTr_{\kappa}} + n \sqrt{M^2-q^2+iTr_{\kappa}}}{2i\sqrt{M^2 - q^2}} \frac{1 + n \sqrt{q^2-M^2-iTr_{\kappa}} + n \sqrt{q^2-M^2+iTr_{\kappa}}}{2\sqrt{q^2-M^2}} - \frac{1}{2M^2} \int_{q > M} \frac{1 + n \sqrt{q^2+iTr_{\kappa}} + n \sqrt{q^2-iTr_{\kappa}}}{2\sqrt{q^2+M^2}} \cdot (A8)
\]

where we have conveniently separated the first two integrals. We note that the integrands are regular when \(q \to M\). Moreover the first integrand does not have singularities arising from the Bose-Einstein distributions because, by assumption, \(0 < M < M_{r,\kappa}\) and we have \(M_{r,\kappa} < \pi T\). We also note that all the integrals that enter the above formula are real. For the first integral this is shown using

\[
1 + n_{ia} + n_{ib} = n_{ia} - n_{ib} = \frac{e^{-ib} - e^{ia}}{(e^{ia} - 1)(e^{-ib} - 1)} = \frac{\sin((a + b)/2)}{2i \sin(a/2) \sin(b/2)} = \frac{\sin((a + b)/2)}{i \cos((a - b)/2) - \cos((a + b)/2)} \cdot (A9)
\]

Contrary to the previous case, not all the 1’s in \(A8\) lead to the zero temperature contribution, so we cannot use the same trick as above to compute \(\Delta \hat{J}_\kappa(m^4, m^4_{\text{vac}})\). However, since the latter is finite, we can compute it using any regulator. With a 3d cut-off, we have

\[
\hat{J}(m^4) = \frac{1}{4\pi^2 m^2} \text{Im} \int_0^\Lambda dq \frac{q^2}{\sqrt{q^2 - im^2}} \cdot (A10)
\]

and then, after some calculation,

\[
\Delta \hat{J}_\kappa(m^4) = \frac{1}{32\pi^2} \ln \frac{m_{\text{vac}}^4}{M^4} - \frac{1}{2M^2} \int_{q < M} \frac{1}{2\sqrt{M^2 - q^2}} \cos(r \cdot \kappa) - e^{-\sqrt{M^2 - q^2}/T} \frac{1}{2M^2} \int_{q > M} \frac{1}{\sqrt{q^2 - M^2}} e^{\sqrt{q^2-M^2}/T} \cos(r \cdot \kappa) - 1 \frac{1}{2M^2} \int_{q} \frac{1}{\sqrt{q^2 + M^2}} e^{\sqrt{q^2+M^2}/T} \cos(r \cdot \kappa) - 1 \cdot (A11)
\]

or equivalently

\[
\hat{J}_\kappa(m^4) = \frac{1}{32\pi^2} \ln \frac{m_{\text{vac}}^4}{M^4} - \frac{1}{2M^2} \int_{q < M} \frac{1}{\sqrt{M^2 - q^2}} \cos(r \cdot \kappa) - e^{-\sqrt{M^2 - q^2}/T} \frac{1}{2M^2} \int_{q > M} \frac{1}{\sqrt{q^2 - M^2}} \cos(r \cdot \kappa) - e^{-\sqrt{q^2-M^2}/T} \frac{1}{2M^2} \int_{q} \frac{1}{\sqrt{q^2 + M^2}} \cos(r \cdot \kappa) - e^{-\sqrt{q^2+M^2}/T} \cdot (A12)
\]

Finally, we will also need \(J_{\kappa}(m^4) m^4 \to 0\) (which exists for \(r \cdot \kappa \notin 2\pi Z\)). Using \(A13\), we find

\[
\hat{J}_\kappa(m^4 \to 0) = -\int_q \frac{d}{dq^2} \frac{1}{2q} (1 - n_{q-iTr_{\kappa}} + n_{q+iTr_{\kappa}}) - \frac{1}{8\pi^2} + \int_q \frac{1}{4q^3} (A13)
\]
2. Sum-integral entering the potential

The same discussion can be applied to the sum-integral

\[ \hat{K}_\kappa(m^4) \equiv \int_Q T \ln(Q^4 + m^4) \] (A14)

that appears in the effective potential. At zero temperature, the integral is defined only for \( m^4 > 0 \) (again if we restrict to real values of \( m^4 \)). We then use

\[ \ln(Q^4 + m^4) = 2 \Re \ln(Q^2 + im^2), \] (A15)

together with

\[ \int_Q \ln(Q^2 + M^2) = -\frac{M^4}{32\pi^2} \left[ \frac{1}{\epsilon} + \ln \frac{\mu^2}{M^2} + \frac{3}{2} \right], \] (A16)

valid for any non-negative \( M^2 \). We find

\[ \hat{K}(m^4) \equiv \int_Q \ln(Q^4 + m^4) = \frac{m^4}{16\pi^2} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\mu^4}{m^4} + \frac{3}{2} \right]. \] (A17)

Similarly, at finite temperature, we have

\[ \hat{K}_r(m^4) = \int_q 2\Re \left[ \sqrt{q^2 + im^2} + T \ln(e^{-2\sqrt{q^2+im^2}/T} - 2e^{-\sqrt{q^2+im^2}/T} \cos(r \cdot \kappa) + 1) \right]. \] (A18)

for \( m^4 > 0 \). In this case the first term inside the bracket corresponds to the zero temperature contribution and can be replaced by the explicit formula (A17). Then

\[ \Delta \hat{K}_\kappa(m^4; m^4_{vac}) \equiv \hat{K}_\kappa(m^4) - m^4 \hat{f}(m^4_{vac}) \]

\[ = \frac{m^4}{32\pi^2} \left[ \ln \frac{m^4_{vac}}{m^4} + 1 \right] \]

\[ + T \int_q \ln(e^{-2\sqrt{q^2+im^2}/T} - 2e^{-\sqrt{q^2+im^2}/T} \cos(r \cdot \kappa) + 1) \]

\[ + T \int_q \ln(e^{-2\sqrt{q^2+im^2}/T} - 2e^{-\sqrt{q^2+im^2}/T} \cos(r \cdot \kappa) + 1). \] (A19)

Instead, if \(-M^4_{\kappa} < m^4 = -M^2 < 0\), we find

\[ \hat{K}_\kappa(m^4) = \int_{q<M} \left[ i\sqrt{M^2 - q^2} + T \ln(e^{-2i\sqrt{M^2-q^2}/T} - 2e^{-i\sqrt{M^2-q^2}/T} \cos(r \cdot \kappa) + 1) \right] \]

\[ + \int_{q>M} \left[ \sqrt{q^2 - M^2} + T \ln(e^{-2\sqrt{q^2-M^2}/T} - 2e^{-\sqrt{q^2-M^2}/T} \cos(r \cdot \kappa) + 1) \right] \]

\[ + \int_q \left[ \sqrt{q^2 + M^2} + T \ln(e^{-2\sqrt{q^2+M^2}/T} - 2e^{-\sqrt{q^2+M^2}/T} \cos(r \cdot \kappa) + 1) \right]. \] (A20)

We have

\[ e^{-2i\sqrt{M^2-q^2}/T} - 2e^{-i\sqrt{M^2-q^2}/T} \cos(r \cdot \kappa) + 1 = 2e^{-i\sqrt{M^2-q^2}/T} \cos(\sqrt{M^2 - q^2}/T) - \cos(r \cdot \kappa). \] (A21)

Since \( 0 < M/T < \pi \), we can apply the formula \( \ln(ab) = \ln a + \ln b \) and then

\[ \hat{K}_\kappa(m^4) = \int_{q<M} T \ln(2 \cos(\sqrt{M^2 - q^2}/T) - 2 \cos(r \cdot \kappa)) \]

\[ + \int_{q>M} \left[ \sqrt{q^2 - M^2} + T \ln(e^{-2\sqrt{q^2-M^2}/T} - 2e^{-\sqrt{q^2-M^2}/T} \cos(r \cdot \kappa) + 1) \right] \]

\[ + \int_q \left[ \sqrt{q^2 + M^2} + T \ln(e^{-2\sqrt{q^2+M^2}/T} - 2e^{-\sqrt{q^2+M^2}/T} \cos(r \cdot \kappa) + 1) \right]. \] (A22)
where each integral is real. Once again, in this case, the zero-temperature contribution is not so easily extracted and we cannot use the same trick as above to compute $\Delta \hat{K}_\kappa(m^4, m^2_{\text{vac}})$. However, up to quartic divergence (that does not depend on $T$ or $r$), we can compute it using any regulator. We use a 3d cut-off and find

\[
\begin{align*}
\Delta \hat{K}_\kappa(m^4, m^2_{\text{vac}}) & = -\frac{M^4}{32\pi^2} \left( 1 + \ln \left( \frac{m^4_{\text{vac}}}{M^4} \right) \right) + \int_{q < M} T \ln(2 \cos(\sqrt{M^2 - q^2}/T) - 2 \cos(r \cdot \kappa)) \\
& + \int_{q > M} T \ln(e^{-2\sqrt{q^2 - M^2}/T} - 2e^{-\sqrt{q^2 - M^2}/T} \cos(r \cdot \kappa) + 1) \\
& + \int q T \ln(e^{-2\sqrt{q^2 + M^2}/T} - 2e^{-\sqrt{q^2 + M^2}/T} \cos(r \cdot \kappa) + 1).
\end{align*}
\]

(A23)

We check that the derivative with respect to $M^4$ gives $-\Delta \hat{J}_\kappa(m^4, m^2_{\text{vac}})$, as it should.
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