Modular invariants and singularity indices of hyperelliptic fibrations

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Abstract The modular invariants of a family of semistable curves are the degrees of the corresponding divisors on the image of the moduli map. The singularity indices were introduced by G. Xiao to classify singular fibers of hyperelliptic fibrations and to compute global invariants locally. In the semistable case, we show that the modular invariants corresponding with the boundary classes are just the singularity indices. As an application, we show that the formula of Xiao for relative Chern numbers is the same as that of Cornalba-Harris in the semistable case.

Keywords Modular invariants, singularity indices, moduli space of curves

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1 Introduction

The modular invariants of a family of curves were introduced by Tan ([10]). They are the degrees of the corresponding divisors on the image of the moduli map. In the language of arithmetic algebraic geometry, a modular invariant is a certain height of arithmetic curves, for example, Faltings height is the modular invariant corresponding to Hodge class. Modular invariants can be used to describe the lower bound for effective Bogomolov conjecture which is about the finiteness of algebraic points of small height ([15, 16]). More recently, Prof. Tan found that the modular invariants are invariants of differential equations, which were expected by mathematicians in 19th century to study the qualitative properties of differential equations ([11]).

Historically, the study of fibred surfaces is started by Kodaira ([6]), who gave a complete classification theory for elliptic fibrations. This combinatoric classification of elliptic fibers is used in the computation of the modular invariants. But such a classification is too complicated for the case when the genus \( g \geq 2 \). There are more than one hundred classes of singular fibers of genus 2 ([8, 9]), and the number of classes of singular fibers increases quickly as the genus becomes bigger. Horikawa ([5]) classified the singular fibers of genus \( g = 2 \) into 5 classes from a different point of view. Based on Horikawa's work, Xiao ([13, 14]) introduced the singularity indices (see Definition 2.11) to classify singular fibers for hyperelliptic fibrations, furthermore, he obtained the local-global formulas, and determined the fundamental group from his classification.

In what follows, we will prove that these two basic invariants, the modular invariants corresponding to boundary classes and the singularity indices, coincide with each other for semistable fibrations.

Before starting this result, we explain our notations and assumptions.

A family of curves of genus \( g \) is a fibration \( f : S \to C \) whose general fibers \( F \) are smooth curves of genus \( g \), where \( S \) is a complex smooth projective surface, and \( C \) is a smooth curve of genus \( b \). The family is called semistable if all the singular fibers are semistable curves. (Recall that a semistable curve \( F \) is a reduced connected curve that

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has only nodes as singularities and every smooth rational components of \( F \) meets the other components at no less than 2 points.) If all the smooth fibers are hyperelliptic, we say that the family is hyperelliptic. We always assume that \( f \) is relatively minimal, i.e., there is no \((-1)\)-curve in any singular fiber.

If \( r \) is a non-negative real number, we denote by \([r]\) the integral part of \( r \). Hence when \( m \) is a positive integer, \( m - 2\lfloor m/2 \rfloor \) is zero if \( m \) is even, or 1 otherwise.

For a fibration \( f : S \to C \), we have three fundamental relative invariants which are non-negative,

\[
K^2_f = K^2_{S/C} = K^2_S - 8(g - 1)(b - 1),
\]

\[
e_f = \chi_{\text{top}}(S) - 4(g - 1)(b - 1),
\]

\[
\chi_f = \deg f_* \omega_{S/C} = \chi(O_S) - (g - 1)(b - 1).
\]

Let \( f \) be a locally non-trivial fibration, the slope of \( f \) is defined as

\[
\lambda_f = K^2_f/\chi_f.
\]

For \( g \geq 2 \), let the moduli map induced by a semistable family \( f \) be

\[
J : C \to \overline{M}_g,
\]

which is a holomorphic map from \( C \) to the moduli space \( \overline{M}_g \) of semistable curves of genus \( g \). For each \( \mathbb{Q} \)-divisor class \( \eta \) of the moduli space \( \overline{M}_g \), we can define an invariant \( \eta(f) = \deg J^* \eta \) which satisfies the base change property, i.e., if \( \tilde{f} : \tilde{X} \to \tilde{C} \) is the pullback fibration of \( f \) under a base change \( \pi : \tilde{C} \to C \) of degree \( d \), then \( \eta(\tilde{f}) = d \cdot \eta(f) \) (see [10]). Consequently, for a non-semistable family \( f \), we have

\[
\eta(f) = \frac{\eta(\tilde{f})}{d},
\]

where \( \tilde{f} \) is the semistable model of \( f \) corresponding to a base change of degree \( d \).

We call the invariant \( \eta(f) \) of the family \( f \) the modular invariant corresponding to \( \eta \).

Let \( \Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor} \) be the boundary divisors of \( \overline{M}_g \), and \( \delta_i(f) \) be the modular invariant corresponding to the divisor class \( \delta_i = [\Delta_i] \) in Pic(\( \overline{M}_g \)) \( \otimes \mathbb{Q} \), \( i = 0, 1, \ldots, \lfloor g/2 \rfloor \). Let \( \lambda \in \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \) be the Hodge class, \( \delta = \delta_0 + \ldots + \delta_{\lfloor g/2 \rfloor} \), and \( \kappa = 12\lambda - \delta \).

For these classes, we have the modular invariants \( \lambda(f), \delta(f) \) and \( \kappa(f) \) of \( f \). If \( f \) is semistable, then

\[
\lambda(f) = \chi_f, \quad \delta(f) = e_f, \quad \kappa(f) = K^2_f.
\]

We say that a singularity \( p \) in a semistable curve \( F \) is a node of type \( i \) if its partial normalization at \( p \) consists of two connected components of arithmetic genera \( i \) and \( g - i \geq i \), for \( i > 0 \), and is connected for \( i = 0 \). The node of the semistable curve corresponding to a general point of \( \Delta_0 \) is \( \alpha \)-type, i.e., an ordinary double point of an irreducible curve, hence it is a node of type 0. For a general point in \( \Delta_i \), the corresponding node is of type \( i \) \((i \geq 1)\) (see the following figure).

Figure 1: Node of type \( i \) \((i \geq 1)\)
Denote by $\delta_i(F)$ the number of nodes of type $i$ ($i \geq 0$).

The general point in the intersection $\Delta_1 \cap \cdots \cap \Delta_k$ of $k$ distinct boundary divisors corresponds to a semistable curve with $k$ nodes which are of types $i_1, \ldots, i_k$ respectively.

For the moduli space $\mathcal{H}/g$ of semistable hyperelliptic curves, the intersection of $\Delta_0$ with $\mathcal{H}/g$ breaks up into $\Xi_0, \Xi_1, \ldots, \Xi_{(g-1)/2}$. We denote by $\Theta_i$ the restriction of $\Delta_i$ ($i \geq 1$) on $\mathcal{H}/g$. Suppose $F$ is a semistable hyperelliptic curve with hyperelliptic involution $\sigma$, and $p \in F$ is a node of type 0. If $p = \sigma(p)$, then we set $k = 0$; if $p \neq \sigma(p)$, and the partial normalization of $F$ at $p$ and $\sigma(p)$ consists of two connected components of arithmetic genera $k$ and $g - k - 1 \geq k$, then the node $p$ (resp. nodal pair $\{p, \sigma(p)\}$) is called a node (resp. nodal pair) of type $(0, k)$. Then the nodes of semistable curves corresponding to a general point of $\Xi_k$ are of type $(0, k)$ (see the following figure).

![Nodes of type (0, k) (k ≥ 0)](Figure 2: Nodes of type (0, k) (k ≥ 0)

A semistable hyperelliptic curve is a double cover of a tree of rational curves branched over $2g + 2$ points (see X.3 in [1]), which is induced by the involution map. Since the points $p$ and $\sigma(p)$ map to the same point in some $\mathbb{P}^1$, we treat them together as a nodal pair $\{p, \sigma(p)\}$.

Let

$$\begin{align*}
\mathcal{N}_{2,1}(F) &= \{p \in F : p \text{ is a node of type } (0, 0), p = \sigma(p)\}, \\
\mathcal{N}_{2,2}(F) &= \{(p, \sigma(p)) \in F : \{p, \sigma(p)\} \text{ is a nodal pair of type } (0, 0), p \neq \sigma(p)\}.
\end{align*}$$

Denote by $\mathcal{N}_{2k+2}(F)$ (resp. $\mathcal{N}_{2k+1}(F)$) the set of all the nodal pairs $\{p, \sigma(p)\}$ of type $(0, k)$ (resp. nodes $p$ of type $k$) ($k > 0$). Then we define

$$\begin{align*}
\xi_0(F) &= |\mathcal{N}_{2,1}(F)| + 2|\mathcal{N}_{2,2}(F)|, \\
\xi_k(F) &= |\mathcal{N}_{2k+2}(F)|, \quad \delta_k(F) := |\mathcal{N}_{2k+1}(F)|, \quad k \geq 1. \tag{1.4}
\end{align*}$$

From now on, we assume that $f : S \to C$ is hyperelliptic. Suppose $f$ is semistable, let $\delta_k(f)$ (resp. $\xi_k(f)$) be the modular invariants corresponding to the boundary divisors $\Theta_k$ (resp. $\Xi_k$). Then (cf. [4])

$$\begin{align*}
\delta_k(f) &= \sum_{i=1}^s \delta_k(F_i) \quad (k \geq 1), \\
\xi_k(f) &= \sum_{i=1}^s \xi_k(F_i) \quad (k \geq 0), \tag{1.5}
\end{align*}$$

where $F_1, \ldots, F_s$ are all singular fibers of $f$, and

$$\delta_0(f) = \sum_{k \geq 0} \xi_k(f). \tag{1.6}$$

It’s proved that in [3], if $f$ is a semistable fibration, then

$$\begin{align*}
(8g + 4)\lambda(f) &= g\xi_0(f) + \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} 2(k + 1)(g - k)\xi_k(f) + \sum_{k=1}^{\lfloor g/2 \rfloor} 4k(g - k)\delta_k(f), \\
\delta(f) &= \xi_0(f) + \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} 2\xi_k(f) + \sum_{k=1}^{\lfloor g/2 \rfloor} \delta_k(f). \tag{1.7}
\end{align*}$$

On the other hand, for a hyperelliptic fibration $f : S \to C$, the relative canonical map $\Phi : S \dashrightarrow \text{Proj}(f_*\omega_S/C)$ induced by $f_*\omega_S/C$ is a generic double cover. Then we
can choose a reasonable double cover which is determined by genus \( g \) datum \((P, R, \delta)\), where \( P \) is a geometric ruled surface \( \varphi : P \to C \), \( R \) is the branch locus, and \( \delta \) is the square root of \( R \) (see Section 2.1). Thus there is a map \( \varphi_{R} : R \to C \) induced by \( \varphi \). Xiao introduced the singularity indices \( s_{2}(f), s_{3}(f), \ldots, s_{g+2}(f) \) (see Definition 2.11), to describe the contribution of the singular points of \( R \), the smooth ramified points of \( \varphi_{R} \) and the vertical components of \( R \) to the relative invariants \( K_{2}^{2}, \chi_{f} \) and \( e_{f} \). He obtained the following local-global formulas using these singularity indices \( s_{k}(f) \)'s (see Theorem 2.14).

\[
(8g + 4) \chi_{f} = g(s_{2}(f) - 2s_{g+2}(f)) + \sum_{k=2}^{|s_{2}+1|} 2(k+1)(g-k)s_{2k+2}(f) + \sum_{k=1}^{|s_{2}|} 4k(g-k)s_{2k+1}(f),
\]

\[
e_{f} = s_{2}(f) - 3s_{g+2}(f) + \sum_{k=1}^{|s_{2}|} 2s_{2k+2}(f) + \sum_{k=1}^{|s_{2}|} s_{2k+1}(f).
\]

(1.8)

Note that Xiao’s equations do not need the semistable condition. If \( f \) is semistable, then \( s_{g+2}(f) = 0 \) (see Corollary 2.5). Comparing equations (1.7) with (1.8), it is natural to build up the relation between modular invariants with singularity indices.

A double point \( p \) of a semistable curve \( F \) is called separable if \( F \) becomes disconnected when normalize \( F \) locally at \( p \); otherwise, \( p \) is called inseparable. Xiao showed that for each semistable fibration \( f \) of genus 2, \( s_{2}(f) \) (resp. \( s_{3}(f) \)) is the number of inseparable (resp. separable) double points of all singular fibers of \( f \) (14), i.e.,

\[
\xi_{0}(f) = s_{2}(f), \quad \delta_{1}(f) = s_{3}(f).
\]

(1.9)

If we subdivide the inseparable nodal points into nodes of type \((0, k) \) \((k \geq 0)\), and subdivide the separable nodes into nodes of type \( i \) \((i \geq 1)\), then we can get that the modular invariants \( \delta_{i}(f) \), \( \xi_{j}(f) \) are the same as the singularity indices \( s_{k}(f) \) for each \( g \geq 2 \):

**Theorem 1.1.** Suppose \( f \) is a semistable hyperelliptic fibration of genus \( g \), then

\[
\delta_{i}(f) = s_{2i+1}(f) \quad (k \geq 1), \quad \xi_{i}(f) = s_{2i+2}(f) \quad (k \geq 0).
\]

(1.10)

Hence

\[
\delta_{0}(f) = s_{2}(f) + 2s_{4}(f) + \cdots + 2s_{2[(g-1)/2]2+2}(f).
\]

(1.11)

Considering the equations in (1.3) and (1.10), it is likely that there exists a more general correspondence between modular invariants and relative invariants. Precisely, we expect that if \( \mathcal{M} \) is any kind of moduli space, and \( \eta \) is a divisor class of \( \mathcal{M} \), especially the generator of Pic(\( \mathcal{M} \)), there is a reasonable relative invariant which coincides with the modular invariant \( \eta(f) \) corresponding to \( \eta \) for each semistable family \( f \) of curves in \( \mathcal{M} \). Recently, there is another such corresponding showed in [3].

In §2, we recall Xiao’s study of hyperelliptic fibration, including the reason for starting from genus \( g \) datum, the definition of singularity indices, and the local-global formulas. In §3, we repeat the work [12] of Yuping Tu on semistable criterion firstly, which concerns the sufficient and necessary conditions of branch locus such that the fibration is semistable. From these conditions, we prove our result locally by constructing bijective maps between sets of singularities \( \mathcal{R}_{s} \) with sets of nodes (or nodal pairs) \( \mathcal{N}_{s} \).
2 Singularity indices

2.1 Genus $g$ data

For the reader’s convenience, we recall the notions of double cover and minimal even resolution firstly.

Let $P$ be a smooth surface, and $R$ a reduced even divisor (the image of $R$ in $\text{Pic}(P)$ is divisible by 2) on $P$. Let $\delta$ be an invertible sheaf such that $\mathcal{O}_P(R) = \delta^{\otimes 2}$, and we call $\delta$ the square root of $R$ for convenience. In fact, a reduced even divisor $R$ on $P$ and an invertible sheaf $\delta$ with $\mathcal{O}_P(R) = \delta^{\otimes 2}$ determine a unique double cover $\pi : S \to P$ branched along $R$ (see I.7 in [2]). Thus $(R, \delta)$ is called a double cover datum. If $R$ is reduced smooth, then $S$ is smooth.

If $\psi_1 : P_1 \to P$ is a blowing-up of $P$ centered at a point $x$ of $R$ of order $m$, set

$$R_1 := \psi_1^* (R) - 2[m/2]E, \quad \delta_1 := \psi_1^* \delta - [m/2]E,$$

where $E$ is the exceptional (-1)-curve of $\psi_1$. Then $(R_1, \delta_1)$ is called a reduced even inverse image of $(R, \delta)$ under $\psi_1$. In what follows, we call $R_1$ a reduced even inverse image of $R$ briefly, since $\delta_1$ is determined by $(R, \delta)$ and $R_1$.

Definition 2.1. An even resolution of $R$ is a sequence of blowing-ups $\tilde{\psi} = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_r : \tilde{P} \to P$

$$\tilde{\psi} : (\tilde{P}, \tilde{R}) = (P_r, R_r) \xrightarrow{\psi_r} \cdots \xrightarrow{\psi_2} (P_2, R_2) \xrightarrow{\psi_1} (P_1, R_1) \xrightarrow{\psi_0} (P_0, R_0) = (P, R),$$

satisfying the following conditions:

(i). $\tilde{R}$ is a smooth reduced even divisor,

(ii). $R_i$ is the reduced even inverse image of $R_{i-1}$ under $\psi_i$.

Furthermore, $\tilde{\psi}$ is called the minimal even resolution of the singularities of $R$ if

(iii). $\psi_i$ is the blowing-up of $P_{i-1}$ centered at a singular point $x_i$ of $R_{i-1}$ for any $1 \leq i \leq r$.

If the even resolution of $\tilde{\psi} : \tilde{P} \to P$ of $R$ is minimal, then for any even resolution $\psi' : P' \to P$, there exists a morphism $\alpha : P' \to \tilde{P}$ such that

$$\alpha(R') = \tilde{R}, \quad \alpha(\delta') = \tilde{\delta}.$$  

Here $\alpha(\delta') = \tilde{\delta}$ means that there exists a divisor $D' \in \text{Pic}(P')$ with $\delta' \cong \mathcal{O}_{P'}(D')$ such that $\tilde{\delta} \cong \mathcal{O}_{\tilde{P}}(\alpha(D'))$.

Note that the minimal even resolution is unique.

If $x_i \in P_{i-1}$ lies in $E_j$ ($j < i$), that is, $\psi_j \circ \cdots \circ \psi_{i-1}(x_i) = x_j$, we say that $x_i$ is infinitely near $x_j$.

Let $x_i$ be a singularity of $R$ of order $\text{ord}_{x_i}(R) = m_i$. If $m_i \leq 3$ and for any $x_j$ infinitely near $x_i$ ($j > i$) we have $m_j \leq 3$, then $x_i$ is called a negligible singularity, since such a singularity does not change the invariants $K^2, \chi_f$ (see (2) in [13]).

Unless stated otherwise, the singularities (resp. the smooth points) of $R$ include all the infinitely near singularities (resp. the smooth points) of $R_i$ in $P_i$ for $1 \leq i \leq r$. If we want to specify a singularity (resp. a smooth point) $p$ of $R$, we will point out the surface which $p$ lies in.

Now we want to introduce the genus $g$ datum associated to a hyperelliptic fibration $f : S \to C$, according to Xiao’s approach in [13] [14].

Since the generic fiber $F$ of $f$ is hyperelliptic, we glue the involution $\sigma F$ of $F$ together, and then we get a real map $\sigma : S \to S$. The map $\sigma$ is in fact a morphism, because $f$ is assumed to be relatively minimal. Let $\rho : \tilde{S} \to S$ be the minimal composition of blowing-ups of $S$ centered at all the isolated fixed points of $\sigma$, and $\tilde{\sigma} : \tilde{S} \to \tilde{S}$
be the induced map of \( \sigma \) on \( \tilde{S} \). Then \( \tilde{P} = \tilde{S}/\langle \tilde{\sigma} \rangle \) is smooth. Let \( \tilde{\theta} : \tilde{S} \to \tilde{P} \) be the corresponding double cover branched along a smooth reduced divisor \( \tilde{R} \) in \( \tilde{S} \). Then \( \theta_{\ast}(\mathcal{O}_{\tilde{S}}) \cong \mathcal{O}_{\tilde{P}} \oplus \tilde{\delta}' \) where \( \delta' \) is an invertible sheaf with \( \tilde{\delta}'^{\otimes 2} \cong \mathcal{O}_{\tilde{S}}(\tilde{R}) \).

Let \( \Phi_K : S \to \text{Proj}(f_\ast\omega_{S/C}) \) be the relative canonical map. \( \Phi_K \) is a generic double cover, for its restriction on a generic fiber \( F \) of \( f \) is the double cover induced by the involution of \( F \). Let \( \rho : \tilde{S} \to S \) be the minimal composition of blowing-ups centered at all base points of \( \Phi_K \) and all isolated fixed points. Then the birational morphism \( \tilde{S} \to \tilde{S} \) is an isomorphism because of the minimality of \( \rho \). Hence \( \rho = \rho \) and \( \tilde{S} \cong \tilde{S} \).

This gives another process to get the double cover \( \tilde{\theta} : \tilde{S} \to \tilde{P} \) and the branch locus \( \check{R} \).

The morphism \( \tilde{\varphi} : \tilde{P} \to C \) induced by \( f \) is a birational ruling (a fibration whose general fibers are rational curves). There are many choices to give a birational morphism \( \tilde{\varphi} : \tilde{P} \to P \) from \( \tilde{P} \) to a geometric ruled surface \( \varphi : P \to C \) over \( C \) which induces a reduced divisor \( R = \tilde{\varphi}(\check{R}) \) in \( P \). All such geometric ruled surfaces differ by elementary transforms. We want to choose one such that \( R^2 \) is the smallest.

We mean by a curve \( D \) on \( S \) a nonzero effective divisor.

**Definition 2.2.** Let \( D \) be an irreducible curve on a fibred surface \( S \) with fibration \( f : S \to C \). If \( f(D) \) is a point, we call \( D \) a vertical curve.

**Lemma 2.3** (Lemma 6 in [13]). There is a birational morphism \( \tilde{\varphi} : \tilde{P} \to P \) over \( C \), where every fiber of the induced morphism \( \varphi : P \to C \) is a \( \mathbb{P}^1 \), such that:

Let \( \delta \) be the image of \( \delta \) in \( P \), and \( R_{\delta} \) be the sum of the non-vertical irreducible components of \( R \). Then \( R^2 \) is the smallest among all such choices, and the singularities of \( R_{\delta} \) are at most of order \( g + 1 \). Therefore as \( R \) is reduced, the singularities of \( R \) are of order at most \( g + 2 \), and if \( p \) is a singular point of order \( g + 2 \), \( R \) contains the fiber of \( \varphi \) passing through \( p \).

**Definition 2.4.** Let \( P \) be a geometric ruled surface over \( C \), and \((R, \delta)\) be a double cover datum on \( P \). If \((R, \delta)\) satisfies that the intersection number of \( R \) with a general fiber \( \Gamma \) of \( \varphi : P \to C \) is \( R\Gamma = 2g + 2 \), and the order of any singularity of the non-vertical part \( R_{\delta} \) of \( R \) is at most \( g + 1 \), we call \((P, R, \delta)\) a genus \( g \) datum.

We have shown that there is a genus \( g \) datum \((P, R, \delta)\) in Lemma 2.3 associated to a given hyperelliptic fibration \( f \) in the above. On the other hand, let \((P, R, \delta)\) be a genus \( g \) datum over a smooth curve \( C \), \( \tilde{\varphi} : \tilde{P} \to P \) be the minimal even resolution of \((P, R)\), and let \( \check{\theta} : \check{S} \to \check{P} \) be the double cover determined by \((\check{R}, \check{\delta})\). Then \( \check{S} \) is smooth. Let \( \rho : \check{S} \to S \) be the morphism of contracting all the vertical \((-1)\)-curves. Then we get a hyperelliptic fibration \( f : S \to C \).

Hence we need to study the vertical \((-1)\)-curves in \( \tilde{S} \).

**Lemma 2.5** ([14]). Let \((P, R, \delta)\) be a genus \( g \) datum, and \( \Gamma \) be any fiber of \( P \to C \), whose inverse image in \( \tilde{S} \) is a \((-1)\)-curve. In other words, the strict transform of \( \Gamma \) in \( \tilde{P} \) is a \((-2)\)-curve contained in \( \tilde{R} \). If \( g \) is even, then one of the following two cases is satisfied:

1. \( R_{\delta} \) intersects with \( \Gamma \) at two distinct points \( x, y \), \( m_x(R_{\delta}) = m_y(R_{\delta}) = g + 1 \); or
2. \( R_{\delta} \) intersects with \( \Gamma \) at one point, and the point is a singularity of type \((g + 1 \to g + 1)\), which is tangent to \( \Gamma \).
If \( g \) is odd, then \( R_h \) intersects with \( \Gamma \) at one point, and it is a singularity of type \((g + 2 \to g + 2)\), which is tangent to \( \Gamma \).

**Lemma 2.6** ([13]). Suppose \( E \) is a vertical \((-1)\)-curve in \( \tilde{S} \), then the image \( \tilde{E} \) of \( E \) in \( \tilde{P} \) is an isolated \((-2)\)-curve contained in \( \tilde{R} \), and \( \tilde{E} \) either comes from a blow-up of a singularity of \( R \) with odd order, or is a strict transform of a fiber in Lemma 2.5. Conversely, for any singularity of \( R \) with odd order or any fiber in Lemma 2.5, there is a corresponding vertical \((-2)\)-curve.

The above two lemmas are easy (see [13]), and we omit their proofs.

**Remark 2.7.** As stated in [13], if we start from a hyperelliptic fibration \( f : S \to C \), we can choose a genus \( g \) datum over a smooth curve \( C \), such that \( R^2 \) is the smallest, and then the case (1) in Lemma 2.5 doesn’t occur. Accordingly, Lemma 2.6 turns to be Lemma 7 in [13]. In what follows, we always assume that the genus \( g \) datum associated with \( f \) satisfies that \( R^2 \) is the smallest.

Consequently, in order to study hyperelliptic fibrations we only need to consider genus \( g \) data.

### 2.2 Singularity indices

Based on this preparation, we are able to define the singularity indices.

Let \((P,R,\delta)\) be a genus \( g \) datum over a smooth curve \( C \), and \( \psi \) in expression (2.2) be the minimal even resolution of \((P,R)\). We decompose \( \psi \) into \( \psi' : \tilde{P} \to P \) followed by \( \tilde{\psi} : \tilde{P} \to \tilde{P} \), where \( \psi' \) and \( \tilde{\psi} \) are composed respectively of negligible and non-negligible blowing-ups. We may assume \( \tilde{\psi} = \psi_{1} \circ \cdots \circ \psi_{t} \), for \( t \leq r \). And denote by \((\tilde{R},\tilde{\delta})\) the reduced even inverse image of \((R,\delta)\) in \( \tilde{P} \).

**Definition 2.8.** Let \( x_{i} \) be a singularity of \( R_{i-1} \) of odd order \( 2k+1 \) \((1 \leq k \leq [(g+1)/2])\). If \( R_{i} \) has a unique singularity on the inverse image of \( x_{i} \), say \( x_{i+1} \), with order \( 2k+2 \), then we call \( x_{i} \) a singularity of \( R_{i-1} \) of type \((2k+1 \to 2k+2)\).

**Definition 2.9.** Let \( f : S \to C \) be a fibration and \( D \) a reduced curve on \( S \). Let \( \phi : D \to C \) be the natural projection induced by \( f \). Let \( \nu : \tilde{D} \to D \) be the normalization of \( D \), \( D_h \) be the union of all the irreducible components of \( \tilde{D} \) which maps projectively onto \( C \), and \( \nu_h : D_h \to D \) be the induced map. The ramification index \( r(D) \) of \( \phi \) is defined as follows:

- If \( q \in D_h \) is a ramification point of \( \phi \circ \nu_h \), then the ramification index \( r_q(D) \) is defined as usual;
- If \( p \) is a singularity of \( D \) with order \( m_p \), then the ramification index is \( r_p(D) = m_p(m_p - 1) \);
- If \( E \) is an isolated vertical curve of \( \tilde{D} \), we define the ramification index to be \( r_E(D) = \chi_{\text{top}}(E) \);

Furthermore, we define

\[
r(D) := \sum_{q \in D_h} r_q(D) + \sum_{p \in D} m_p(m_p - 1) - \sum_{E \subseteq \tilde{D} \text{ isolated vertical curve}} \chi_{\text{top}}(E). \tag{2.3}
\]

**Remark 2.10.** It is easy to see that

\[
r(D) = D^2 + DK_{S/C}, \tag{2.4}
\]

from the adjoint formula \( K_{S}D + D^2 = -2\chi(O(D)) \) (see [14]).

When we consider singular fiber \( F \) of \( f \), the singularities and ramification points of branch locus are those over \( f(F) \) without confusion.
Definition 2.11 ([13 14]). Let \( f : S \to C \) be a hyperelliptic fibration, and \((P, R, \delta)\) be the corresponding genus \( g \) datum. Suppose \( \mathcal{F} \) is any fiber of \( f \), we denote by \( \Gamma \) the fiber of \( P \to C \) over \( f(\mathcal{F}) \). The singularity indices \( s_k(\mathcal{F}) \) \( (2 \leq k \leq g + 2) \) are defined as following.

(1). Let \( E_1, \ldots, E_k \) be all the isolated vertical \((-2)\)-curves in \( \hat{R} \). Let \( \hat{R}_p = \hat{R} - E_1 - \cdots - E_k \), then \( s_2(\mathcal{F}) \) is defined to be the ramification index of \( \hat{R}_p \) over the point \( f(\mathcal{F}) \). Concisely, if we denote by \( \mathcal{R}_{2,1}(\mathcal{F}) \) the set of all ramification points of \( \hat{R} \) over \( f(\mathcal{F}) \), by \( \mathcal{R}_{2,2}(\mathcal{F}) \) the set of all singularities of \( \hat{R}_p \), and by \( \mathcal{R}_{2,-}(\mathcal{F}) \) the set of all vertical components in \( \hat{R}_p \), then

\[
s_2(\mathcal{F}) = \sum_{q \in \mathcal{R}_{2,1}(\mathcal{F})} (|\Gamma, \hat{R}_p| - 1) + \sum_{q \in \mathcal{R}_{2,2}(\mathcal{F})} m_q(m_q - 1) - 2|\mathcal{R}_{2,-}(\mathcal{F})|. \tag{2.5}
\]

(2). If \( k \) is odd, denote by \( \mathcal{R}_k(\mathcal{F}) \) the set of all singularities of \( R \) of type \((k \to k)\), then \( s_k(\mathcal{F}) := |\mathcal{R}_k(\mathcal{F})| \).

(3). If \( k \geq 4 \) is even, denote by \( \mathcal{R}_k(\mathcal{F}) \) the set of all singularities of \( R \) of order \( k \), not belonging to a singularity of type \((k + 1 \to k + 1)\) or \((k - 1 \to k - 1)\), then \( s_k(\mathcal{F}) := |\mathcal{R}_k(\mathcal{F})| \).

Define

\[
s_k(\mathcal{F}) = \sum_{i=1}^{s} s_k(F_i), \tag{2.6}
\]

where \( F_1, \ldots, F_s \) are all the singular fibers of \( f \).

Remark 2.12. Xiao introduced the singularity indices in order to compute the contribution of singular fibers to the invariants \( K_f^2, \chi_f \). It is convenient to put \( x_i, x_{i+1} \) in Definition 2.8 together, and regard the pair \( \{x_i, x_{i+1}\} \) of points as one singularity of type \((2k + 1 \to 2k + 1)\), that is, the total contribution of \( x_i \) and \( x_{i+1} \) to singularity indices adds one to \( s_{2k+1} \) only.

Example 2.13. Let \((x, t)\) be the local coordinate of \( \mathbb{P}^1 \times \Delta \), where \( \Delta \) is the open unit disc of \( \mathbb{C} \). Let

\[
h(x, t) = (x + t) ((x - a_0)^2 + t^2) \cdot (x - a_2)^2 + t^2) \cdot (x - a_3)^3 + t^3), \tag{2.7}
\]

where \( a_i \)'s are distinct nonzero complex numbers. Let \( f : S_\Delta \to \Delta \) be the local hyperelliptic fibration of genus \( g \) defined by local equation

\[
y^2 = h(x, t). \tag{2.8}
\]

Let \( F = f^{-1}(0) \) be the central fiber of \( f \) over the origin, \( \Gamma \) the fiber of \( \mathbb{P}^1 \times \Delta \to \Delta \) over the origin.

![Figure 3](image-url)
Furthermore, the singularity indices are genus (Theorem 5.1.7, [14])

Theorem 2.14

Let $p$ fiber of $\phi$ \Corollary 2.15

Moreover, the left equality holds if and only if

$s_3$ Modular invariants in semistable case

Using the singularity indices, Xiao obtained the following local-global formula.

Using the singularity indices, Xiao obtained the following local-global formula.

**Theorem 2.14** (Theorem 5.1.7, [14]). Let $f : S \to C$ be a hyperelliptic fibration of genus $g$, then

\[
(8g + 4)\chi_f = g(s_2(f) - 2s_{g+2}(f)) + \sum_{k=2}^{\left\lceil \frac{g+4}{2} \right\rceil} 2(k + 1)(g - k)s_{2k+2}(f)
\]

\[
+ \sum_{k=1}^{\left\lceil \frac{g+4}{2} \right\rceil} 4k(g - k)s_{2k+1}(f),
\]

\[
e_f = s_2(f) - 3s_{g+2}(f) + \sum_{k=1}^{\left\lceil \frac{g+4}{2} \right\rceil} 2s_{2k+2}(f) + \sum_{k=1}^{\left\lceil \frac{g+4}{2} \right\rceil} s_{2k+1}(f),
\]

\[
(2g + 1)K_f^2 = (g - 1)s_2(f) + 3s_{g+2}(f) + \sum_{k=1}^{\left\lceil \frac{g+4}{2} \right\rceil} a_k s_{2k+2}(f) + \sum_{k=1}^{\left\lceil \frac{g+4}{2} \right\rceil} b_k s_{2k+1}(f),
\]

where $a_k = 6((k + 1)(g - k) - 4g - 2)$, and $b_k = 12k(g - k) - 2g - 1$.

As a corollary, he proved that

**Corollary 2.15** ([14]). Suppose $f$ is hyperelliptic, then the slope of $f$

\[
\frac{4g - 4}{g} \leq \lambda_f \leq \begin{cases} 12 - \frac{8g+4}{g+1}, & \text{if } g \text{ is even}, \\ 12 - \frac{8g+4}{g+1}, & \text{if } g \text{ is odd}, \end{cases}
\]

Moreover, the left equality holds if and only if $s_2(f) \neq 0, s_k = 0 (k > 2)$, and the right equality holds if and only if $s_{2g/2+2} \neq 0$ and other singularity indices are all zero.

### 3 Modular invariants in semistable case

At the beginning of this section, we fix notations firstly.

Let $(P, R, \delta)$ be a genus $g$ datum over a smooth curve $C$, and $\tilde{\psi}$ in (2.20) be the minimal even resolution. Let $f : S \to C$ be the fibration determined by the datum, and $F$ be any singular fiber of $f$. Denote by $\tilde{F}$ the total transform of $F$ by $\rho : S \to S$, which is a birational morphism contracting all the vertical ($-1$)-curves. Let $\Gamma$ be the fiber of $\varphi : P \to C$ over $t = f(F)$, and we call $\Gamma$ the image of $F$ in $P$ briefly. Let $\tilde{\Gamma} = \psi^*(\Gamma)$ be the total transform of $\Gamma$ by the minimal even resolution $\tilde{\psi} : P \to P$ of
To keep it simple, we also denote by $R$ (resp. $\Gamma$) the strict transform of $R$ (resp. $\Gamma$) under the even resolution $\psi$.

$$
\tilde{F} \subset \tilde{S} \xrightarrow{\delta} \tilde{P} \supset \Gamma
$$

Denote by $B = \tilde{\theta}^{-1}(\Gamma)$ the inverse image of $\Gamma$ in $\tilde{F}$, and by $B_i = \tilde{\theta}^{-1}(E_i)$ the inverse image of the exceptional curve $E_i$. Then $B$ (resp. $B_i$) may be composed by two irreducible curves $B'$ and $B''$ (resp. $B'_i$ and $B''_i$). Let

$$
\tilde{\Gamma} = \Gamma + \sum_{i=1}^{r} m_i E_i,
$$

then

$$
\tilde{F} = \tilde{\theta}^*(\tilde{\Gamma}) = \tilde{\theta}^*(\Gamma) + \sum_{i=1}^{r} m_i \tilde{\theta}^*(E_i) = n B + \sum_{i=1}^{r} n_i B_i,
$$

where $n = 1, 2$ and $n_1 = m_i$ or $n_1 = 2m_i$. Therefore, $F = \rho(\tilde{F})$ is obtained by contracting $(-1)$-curves in $\tilde{F}$.

**Definition 3.1.** An even resolution at point $p$ of $R$ is a sequence of blowing-ups $\psi_p = \psi_1 \circ \psi_2 \circ \ldots \circ \psi_l : \hat{P} \rightarrow P$

$$(\hat{P}, \hat{R}) = (P_1, R_1) \xrightarrow{\psi_1} \ldots \xrightarrow{\psi_l} (P_1, R_1) = (P, R),$$

satisfying the following conditions:

(i). all the points of $\hat{R}$ infinitely near $p$, including $p$, are smooth,

(ii). $R_i$ is the reduced even inverse image of $R_{i-1}$ under $\psi_i$.

Furthermore, $\psi_p$ is called the minimal even resolution at $p$ of $R$ if

(iii). $\psi_i$ is the blowing-up of $P_{i-1}$ centered at a singular point $p_i$ of $R_{i-1}$ which is infinitely near $p$ for any $1 \leq i \leq l$.

If the resolution $\psi_p$ is minimal, we call the desired number $l$ of blowing-ups the length of the minimal even resolution $\psi_p$ at $p$ of $R$, and we denote the length $l$ by $l_p$. The exceptional curves $E_i$'s $(1 \leq i \leq l)$ in $R$ are called exceptional curves from $p$ briefly.

For example, if $p$ is an ordinary singularity of even order, then $l_p = 1$. If $p$ is a singularity of type $(3 \rightarrow 3)$, then $l_p \geq 2$.

Let $p$ be any singularity of $R$, and $E_1, \ldots, E_p$ be all the exceptional curves from $p$ in $P_p$. Set

$$
E_p := m_1 E_1 + \ldots + m_p E_p, \quad B_p := \tilde{\theta}^*(E_p), \quad m_i = \text{mult}_{E_i}(\tilde{\theta}) \quad (\text{See } \text{(3.1)}).
$$

Then we call $E_p$ the block of $\tilde{\Gamma}$ from $p$, and call

$$
F_p := \rho(B_p).
$$

the block of $F$ from $p$.

Assume that $\Gamma$ is not contained in $R$. Let $p_1, \ldots, p_e$ be all the singularities of $R$ on $\Gamma$ in $P$, and $B_{p_0} = \tilde{\theta}^*(\Gamma)$, then we can decompose $F$ into finite blocks

$$
F = F_{p_1} + F_{p_1} + \ldots + F_{p_e},
$$

and we call it the modular decomposition of $F$.
Example 3.2. [Continuation of Example 2.13] Let \( f : S_{\Delta} \to \Delta \) be the local fibration in Example 2.13. Then \( l_{p_1} = 1, \ l_{p_2} = 1, \ l_{p_3} = 2 \). The blocks of \( \Gamma \) are

\[
\mathcal{E}_{p_1} = E_1, \ \mathcal{E}_{p_2} = E_2, \ \mathcal{E}_{p_3} = E_{31} + E_{32}.
\]

Here \( E_1, E_2, E_{32} \) are not contained in \( \tilde{R} \), and \( E_{31} \) is contained in \( \tilde{R} \). Then the blocks of \( F \) are

\[
F_{p_0} = B, \quad F_{p_1} = B_1, \quad F_{p_2} = B_2', \quad F_{p_3} = B_3', \quad F_{p_3} = B_3.
\]

In the above equation, \( B \) is a rational curve with a node \( q_0 \); \( B_1 \) is \( \mathbb{P}^1 \) meeting \( B \) at two points \( q_{11}, q_{12} \); \( B_2', B_2'' \) are both \( \mathbb{P}^1 \) meeting with \( B \) at \( q_{21}, q_{22} \) respectively and with each other at two points \( q_{23}, q_{24} \); and \( B_{31} \) is a smooth elliptic curve meeting with \( B \) at \( q_3 \). Then \( F \) is semistable, and the modular decomposition of \( F \) is

\[
F = \sum_{i=0}^{2} F_{p_i} = B + B_1 + B_2' + B_2'' + B_{32}.
\]

### 3.1 Semistable criterion

There is a criterion for semistable hyperelliptic fiber given by Tu \[12\]. We rewrite the result and proof here, for the reference is in Chinese.

**Lemma 3.3** \[12\]. Suppose \( F \) is a semistable fiber of a hyperelliptic fibration \( f : S \to C \). Then we have the following.

1. If \( g \) is odd, then \( \Gamma \) is not contained in \( R \); if \( g \) is even and \( \Gamma \) is contained in \( R \), then \( \Gamma \) is the fiber in Lemma 2.3.

2. Suppose \( p \) is a smooth point of \( R \) in \( P \), then the intersection number of \( R \) with \( \Gamma \) at \( p \) is \((R, \Gamma)_p \leq 2 \).

3. If \( p \) is a singularity of \( R \) in \( P \), then we have \((R, \Gamma)_p = \text{ord}_p(R) \).

4. Let \( q \) and \( E \) be the same as above. If \( \text{ord}_q(R) = 1 \) is even, then \( E \) is not contained in the branch locus \( \tilde{R} \). If \( \text{ord}_q(R) = 1 \) is odd, then either \( E \) is contained in \( \tilde{R} \), and thus \( E \) is from a singularity of type \((k \to k) \) \((k \text{ is odd, } k \geq l) \); or \( E \) is not contained in \( \tilde{R} \), and thus \( q \) is a singularity of type \((k \to k) \).

5. Let \( q \in R \) \((\text{for some } i \geq 1) \) be an infinitely near singularity, then there is exactly one exceptional curve \( E_q \) in \( P_i \) passing through \( q \), and \((R, E_q)_q = \text{ord}_q(R) \).

6. Let \( q \) be an infinitely near smooth point, and \( E \) be the same as above, then \((R, E)_q \leq 2 \), and \( E \) is not contained in \( \tilde{R} \).

**Proof.** (1). Suppose \( \Gamma \subset \tilde{R} \), then \( B \) is a component of \( \tilde{F} \) with multiplicity 2, for \( \pi^*(\Gamma) = 2B \), furthermore, \( B^2 = \Gamma^2/2 \). If \( \Gamma^2 \leq -4 \), then \( B^2 \leq -2 \), hence \( B \) is a multiple component in \( \tilde{F} \) which can not be contracted, contradicting with the assumption that \( F \) is semistable. Thus we get that if \( \Gamma \subset \tilde{R} \), then \( \Gamma \) is a \((-2)\)-curve in \( \tilde{P} \).

By Lemma 2.3 we know that if \( g \) is even then \( \Gamma \) is as in (1) in Lemma 2.3. And if \( g \) is odd, then any singularity of \( R \) is of type \((g + 2 \to g + 2) \), and we need twice blow-up so that the intersection point of \( \Gamma \) with the exceptional curve is a smooth point of \( R \). Hence there is a \((-1)\)-curve, say \( E_2 \), with multiplicity 2 in \( \tilde{F} \). It’s easy to see that \( E_2 \) is
not contained in $\tilde{R}$, and $\pi^*(2E_2) = 2B_2$ in $F$ is irreducible with $B_2^2 \leq -2$, therefore $B_2$ is an un-contractile multiple component in semistable curve $\tilde{F}$, which is impossible. In a word, when $g$ is odd, $\Gamma$ is not contained in $R$.

(2). For what follows, we assume that $\Gamma$ is not contained in $R$ since (1). Let $n = (R, \Gamma)_p$, we take the local coordinate $(x, t)$ of $p$ such that the local equations of $\Gamma$ and $R$ at $p$ are $t = 0$ and $t + x^n = 0$ respectively. Then the local equation of $F$ in $S$ is $y^2 - x^n = 0$. If $n \geq 3$, it is a singularity of type $A_{n-1}$ on $F$, and then $F$ is not semistable.

(3). Suppose not, thus $(R, \Gamma)_p > \ord_p(R)$. Let $\psi_1$ be a blow-up at $p$, and $E_1$ be the exceptional curve. Then the intersection point $p'$ of $\Gamma$ with $E_1$ is still on $R$. Let $\psi_2$ be the successive blow-up centered at $p_1$, and $E_2$ the exceptional curve. Then the total transform of $\Gamma$ by $\psi_1 \circ \psi_2$ is

$$\tilde{\Gamma}_2 = \Gamma + 2E_2 + E_1,$$

and $B_2$ is with multiplicity at least 2 in $F$. Hence $E_2$ must be a $(-1)$-curve in $\tilde{S}$, $E_2$ a $(-2)$-curve in $\tilde{R}$, and $p_1$ must be a singularity of type $(k \to k)$ ($k$ is odd) (Lemma 2.5). Furthermore, there is a singularity $p_2$ on $E_2$ of order $k + 1$. Let $\psi_3$ be the blow-up centered at $p_2$ with exceptional curve $E_3$. Then

$$\tilde{\Gamma}_3 = \Gamma + 2E_3 + 2E_2 + E_1,$$

$E_3$ is not contained in $\tilde{R}$, and $B_3$ is an un-contractile multiple component in $\tilde{F}$.

(4). Suppose $\ord_q(R)$ is even and $E$ is contained in $\tilde{R}$, then $\ord_q(R_i)$ is odd. Let $\psi : \tilde{P}_{i+1} \to \tilde{P}_i$ be the blow-up centered $q$ with exceptional curve $E'$ lying in branch locus, then the intersection point $E' \cap E$ is a singularity of $\tilde{R}_{i+1}$, and $E'^2 \leq -2$ in $\tilde{P}_{i+1}$. Hence $E^2 \leq -4$ in $\tilde{P}$, and then $B$ is an un-contractile multiple component in $F$. Consequently, we proved the first part of (4).

The second part of (4) is a direct corollary of Lemma 2.6.

(5). Suppose $E_1$ and $E_2$ are both through $q$. When $\ord_q(R)$ is even, then the exceptional curve $E_3$ of the blow-up at $q$ is of multiplicity at least 2, and $E_3$ is not contained in $\tilde{R}$. So $B_3$ is an un-contractile multiple component in $F$. When $\ord_q(R) = k$ is odd, then $q$ should be of type $(k \to k)$, and $E_3$ is contained in $\tilde{R}$ and of multiplicity at least 2. Blowing up the infinitely near singularity $q'$ of $q$, then the exceptional curve $E_4$ is not contained in $\tilde{R}$ of multiplicity at least 2, which is impossible. The proof of the second part of (5) is analogous to that of (3).

(6). The proof of the second part is the same as that of (1), and the rest is the same as that of (2). We omit the detail. □

Remark 3.4. Let $F$ be a semistable fiber of $f$, and $p$ be a singularity of $R$. Then there is exactly one curve $E_p$ passing through $p$ (Lemma 3.3 (3)-(5)). We call $E_p$ the exceptional curve through $p$. Note that $E_p$ is either $\Gamma$ or an exceptional curve.

Corollary 3.5. If $F$ is a semistable hyperelliptic fiber of genus $g$, then $s_{g+2}(F) = 0$.

Proof. From Lemma 3.3 (1), we know that if $g$ is odd, then $s_{g+2}(F) = 0$; if $g$ is even, then $\Gamma$ is the fiber in Lemma 2.6 but it is impossible by Lemma 3.3 (3), and then $s_{g+2}(F) = 0$. □

3.2 Proof of Theorem 1.1

We first consider the effect of the smooth points of $R$ to the arithmetic genus.

Lemma 3.6. Let $F$ be a semistable fiber of $f$. Assume that the image $\Gamma$ in $P$ of $F$ is not contained in $R$. Suppose all the intersection points $p_1, \ldots, p_k$, $q_1, \ldots, q_k$ of $R$ on $\Gamma$ are smooth, where $(\Gamma, R)_{p_i} = 2$ and $(\Gamma, R)_{q_i} = 1$. 

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1). If $k_2 \neq 0$, then $F$ is an irreducible curve with $k_1$ nodes corresponding to $p_i$, the geometric genus of $F$ is $[(k_2 - 1)/2]$, and

$$p_a(F) = [(\Gamma R - 1)/2] = [(2k_1 + k_2 - 1)/2] = k_1 + [(k_2 - 1)/2].$$

2). If $k_2 = 0$, then $F$ is composed of two smooth rational curves meeting with each other at $k_1$ distinct points, thus

$$F = \tilde{\Theta}^*(\Gamma) + \sum_{i=1}^{k_1} \tilde{\Theta}^*(E_i) = (B' + B'') + \sum_{i=1}^{k_1} B_i,$$

where every irreducible component is a smooth rational curve, $B_i$ meets $B'$ and $B''$ normally at one point respectively for each $1 \leq i \leq k_1$, and there is no other intersection. Hence

$$p_a(F) = [(\Gamma R - 1)/2] = k_1 - 1.$$

**Proof.** The proof is obvious, and we omit it. □

Then we consider the effect of the singularities.

**Lemma 3.7.** Suppose $F$ is a semistable fiber of $f$, and the image $\Gamma$ in $P$ of $F$ is not contained in $R$. If $p$ is a singularity of $R$ such that the exceptional curve $E_p$ through $p$ is not contained in the branch locus. Then the arithmetic genus of the block of $F$ from $p$ is

$$p_a(F_p) = \left\lfloor \frac{(E_p, R)_p - 1}{2} \right\rfloor. \quad (3.7)$$

**Proof.** We use induction on the length $l_p$ of the minimal even resolution $\tilde{\psi}_p$ of $R$ at $p$. Note that $\left\lfloor ((2k + 1) - 1)/2 \right\rfloor = \left\lfloor ((2k + 1) - 1)/2 \right\rfloor = k$, and $\text{ord}_p(R) = (R, E_p)_p$ for any singularity of $R$ on $E_p$ from Lemma 3.3. We may assume $E_p$ is $\Gamma$, since the proof for exceptional curves is similar.

If $l_p = 1$, then $\text{ord}_p(R) = 2k + 2$ is even and $p$ is an ordinary singularity. The exceptional curve $E_1$ from $p$ is not contained in $R$, and $E_1$ meets $R$ in $P_1$ transversely at $2k + 2$ distinct points. Hence $\mathcal{E}_p = E_1$, and $B_p = B_1$ with $p_a(B_1) = k$.

If $l_p = 2$ and $\text{ord}_p(R) = 2k + 2$ is even, then there is exactly one infinitely near singularity $p_1$ of $R$ in $P_1$, which is an ordinary singularity of even order, say $2k_2$. Hence $E_1, E_2$ are not contained in $R$, $\mathcal{E}_p = E_1 + E_2$, and $\mathcal{E}_p$ meets $R$ in $P_2$ transversely at $2k + 2$ distinct points. Let $E_1 R = 2k_1 + 2$, then $k_1 + k_2 = k$. Thus $p_a(B_1) = k_1$, $p_a(B_2) = k_2 - 1$, and $B_1$ intersects with $B_2$ at two points transversely.

$$p_a(F_p) = p_a(B_p) = p_a(B_1) + p_a(B_2) + 1 = k.$$

If $l_p = 2$ and $\text{ord}_p(R) = 2k + 1$ is odd, then $p$ is a singularity of $(2k + 1 \to 2k + 1)$. $E_1$ is contained in $R$, $E_2$ is not contained in $R$, and $\mathcal{E}_p = E_1 + E_2$, where $E_2$ meets $R$ in $P_2$ transversely at $2k + 2$ distinct points. It’s easy to see that $B_1$ is a $(-1)$-curve and $B_2$ is a smooth curve with genus $k$. Hence

$$p_a(F_p) = p_a(B_2) = k.$$

Assume that (3.7) holds for any non-negative integer $l < l_p$. We want to prove (3.7) holds for $l_p$.

If $\text{ord}_p(R) = 2k + 1$ is odd, let $\psi_1 : P_1 \to P$ be the blowing-up at $p$. Then there is exactly one infinitely near singularity $q$ of $R$ in $P_1$, and $(R, E_1)_q = 2k + 1$, $\text{ord}_q(R_1) = 2k + 2$. Let $\psi_2 : P_2 \to P_1$ be the successive blowing-up at $q$. It is clear that $E_1$ is contained in $\mathcal{R}$, but $E_2$ is not.
Let \( q_1, \ldots, q_\alpha \) be all the infinitely near singularities of \( q \) in \( P_2 \). Hence \( l_q < l_p \) for \( 1 \leq i \leq \alpha \). Suppose \( q_1, \ldots, q_\beta (\beta \leq \alpha) \) are all the singularities with even order. Let \( (R, E_2)_q = 2k_i + 2 \) for \( 1 \leq i \leq \beta \), and let \( (R, E_2)_q = 2k_j + 1 \) for \( \beta + 1 \leq j \leq \alpha \). Let the total intersection number of \( R \) with \( E_2 \) at all the smooth points of \( R \) in \( P_2 \) be \( (R, E_2)_{sm} \). Then

\[
2k + 2 = \sum_{i=1}^{\beta} (2k_i + 2) + \sum_{j=\beta+1}^{\alpha} (2k_j + 1) + (R, E_2)_{sm} + 1
\]

\[
= 2(k_1 + \cdots + k_\beta) + 2\beta + 2(k_{\beta+1} + \cdots + k_\alpha) + (\alpha - \beta) + (R, E_2)_{sm} + 1
\]

\[
= 2(k_1 + \cdots + k_\alpha) + (R, E_2)_{sm} + (\alpha + \beta) + 1.
\]

Hence

\[
k = \left( \sum_{i=1}^{\alpha} k_i \right) + \frac{(R, E_2)_{sm} + \alpha + \beta - 1}{2}.
\]

(3.8)

It is easy to see that in \( \tilde{P} \),

\[
\tilde{R}E_2 = (R, E_2)_{sm} + (\alpha - \beta) + 1.
\]

By Lemma 3.6

\[
p_a(B_2) = \left[ \frac{(R, E_2)_{sm} + (\alpha - \beta) + 1 - 1}{2} \right] = \frac{(R, E_2)_{sm} + \alpha - \beta - 1}{2}.
\]

(3.9)

The block of \( \tilde{\Gamma} \) from \( p \) is

\[
\mathcal{E}_p = E_1 + E_2 + \sum_{i=1}^{\alpha} \mathcal{E}_{q_i}.
\]

Combining the equations (3.8) and (3.9), then

\[
p_a(F_p) = p_a(B_p - 2B_1) = p_a(B_2) + \left( \sum_{i=1}^{\alpha} p_a(F_{q_i}) \right) + \beta
\]

\[
= \frac{(R, E_2)_{sm} + \alpha - \beta - 1}{2} + \left( \sum_{i=1}^{\alpha} k_i \right) + \beta
\]

(3.10)

where the block \( F_{q_i} \) intersects with \( B_2 \) at two points, and adds one to the arithmetic genus for each \( 1 \leq i \leq \beta \). Here we used the induction assumption.

If \( \text{ord}_p(R) = 2k + 2 \) is even, take \( \psi_i : P_1 \rightarrow P \) the blow-up at \( p \). Let \( q_1, \ldots, q_\alpha \) be all the infinitely near singularities of \( p \) on \( P_1 \). Then the rest of the proof is the same as the odd case above.

Now we can prove the identities between singularity indices (Definition 2.11) with modular invariants \( \delta_1(F), \xi_1(F) \) (see [1.4] - [1.6]).

**Theorem 3.8.** Let \( f : S \rightarrow C \) be a semistable hyperelliptic fibration of genus \( g \), and \( F \) be any singular fiber of \( f \), then

\[
s_{2k+1}(F) = \delta_k(F) \; (k \geq 1), \quad s_{2k+2}(F) = \xi_k(F) \; (k \geq 0).
\]

(3.11)

**Proof.** (1). Proof of \( s_{2k+1}(F) = \delta_k(F) \), \( k \geq 1 \).

We define a bijective map

\[
\alpha_{2k+1} : \mathcal{R}_{2k+1}(F) \rightarrow \mathcal{N}_{2k+1}(F)
\]

(3.12)
between sets as follows:

If \( p \in \mathcal{R}_{2k+1}(F) \), then \( E_p \) (the exceptional curve through \( p \)) is not contained in \( \hat{R} \).
Let \( \hat{\Gamma} = \mathcal{E}_p + \mathcal{E}_{\hat{p}} \), where \( \mathcal{E}_p \) is the block of \( \hat{\Gamma} \) from \( p \). Then the decomposition of \( F \) is
\[
F = F^p + F^\hat{p},
\]
where \( p_*(F^p) = \left( ((R, E_p)p - 1)/2 \right) = k \), and \( F^\hat{p} \) intersects with \( F^p \) at a point, say \( q \), which is a node of type \( k \). We define \( \alpha_{2k+1}(p) = q \in \mathcal{N}_{2k+1}(F) \), then \( \alpha_{2k+1} \) is well-defined.

On the other hand, if \( q \in \mathcal{N}_{2k+1}(F) \), then \( F \) consists of a genus \( k \) curve \( F_q \) and a genus \( g-k \) curve \( F_q^* \), and they intersect with each other at two points transversely. Then \( q \) is an isolated fixed point of the hyperelliptic involution \( \sigma \). Thus the inverse image of \( q \) in \( \hat{F} \) under \( \rho : \hat{S} \to S \) is a \((-1)\)-curve \( B \). Hence \( \hat{\theta}(B) \) is a \((-2)\)-curve contained in \( \hat{R} \), which is from a singularity, say \( p \), of type \((2k'+1 \to 2k'+1)\) (cf. Lemma 2.6 and (4) in Lemma 3.3). Since \( \hat{\theta}^*(\mathcal{E}_p) = \rho^*(F_q) \), the arithmetic genus of the block of \( F \) from \( p \) is
\[
k' = p_*(\hat{\theta}^*(\mathcal{E}_p)) = p_*(F_q) = k.
\]
Thus \( p \in \mathcal{R}_{2k+1}(F) \), and \( \alpha_{2k+1}(p) = q \). Hence it is clear that \( \alpha_{2k+1} \) is surjective and injective.

Therefore,
\[
s_{2k+1}(F) = |\mathcal{R}_{2k+1}(F)| = |\mathcal{N}_{2k+1}(F)| = \delta_k(F).
\]

(2). Similar proof of \( s_{2k+2}(F) = \xi_k(F) \), \( k \geq 1 \).
We define a bijective map
\[
\alpha_{2k+2} : \mathcal{R}_{2k+2}(F) \to \mathcal{N}_{2k+2}(F)
\]
(3.13)

between sets as follows:

If \( p \in \mathcal{R}_{2k+2}(F) \), then \( E_p \) is not contained in \( \hat{R} \), and the exceptional curve \( E_1 \) of the blowing-up at \( p \) is not in \( \hat{R} \) either. Hence \( \hat{\theta}^{-1}(p) \) consists of two points \( q, \sigma(q) \). Let \( \hat{\Gamma} = \mathcal{E}_p + \mathcal{E}_{\hat{p}} \), then \( F = F_q + F_q^* \), \( p_*(F_q) = k \) and \( F_q^* \) meets \( F_q^* \) at \( q, \sigma(q) \) transversely. So the nodal pair \( \{q, \sigma(q)\} \in \mathcal{N}_{2k+2}(F) \). Hence we are able to define \( \alpha_{2k+1}(p) = \{q, \sigma(q)\} \).

On the other hand, if \( \{q, \sigma(q)\} \in \mathcal{N}_{2k+2}(F) \), then \( F = F_q + F_q^* \), where \( p_*(F_q) = k \), and they intersect with each other at two points \( q, \sigma(q) \) transversely. We may assume that \( F = F_q + F_q^* \), which meet at \( q \) and \( \sigma(q) \). Then \( \hat{\theta}(q) = \hat{\theta}(\sigma(q)) \), say \( p \), is an intersection point of two curves not in \( \hat{R} \). Hence we can decompose \( \hat{\Gamma} = \mathcal{E}_p + \mathcal{E}_{\hat{p}} \), where \( R\mathcal{E}_p = 2k + 2 \) and \( \mathcal{E}_p \) meets \( \mathcal{E}_{\hat{p}} \) at \( p \) only. The curve \( \mathcal{E}_p \) is from a singularity of order \( \text{ord}_p(R) = R\mathcal{E}_q = 2k + 2 \). Therefore, \( p \) is the inverse image of \( \{q, \sigma(q)\} \) under \( \alpha_{2k+2} \), and \( \alpha_{2k+2} \) is bijective.

So
\[
s_{2k+2}(F) = |\mathcal{R}_{2k+2}(F)| = |\mathcal{N}_{2k+2}(F)| = \xi_k(F).
\]

(3). Proof of \( s_2(F) = \xi_0(F) \).

If \( E \) is a vertical component of \( \hat{R} \), then \( B = \hat{\theta}^*(E) \) is a multiple component of \( \hat{F} \), so \( B \) is a \((-1)\)-curve for \( E \) is semistable, and then we know that \( E \) is a \((-2)\)-curve in \( \hat{R} \). Hence \( R\mathcal{E}_p \) is the strict transform of \( R \) in \( \hat{R} \), and \( |\mathcal{R}_{2-2}(F)| = 0 \).

If \( p \in \mathcal{R}_{2,1}(F) \), then \( p \) is a smooth point of \( R \), \( (R, E_p)p = 2, r_p(R) = 1 \), and \( \hat{\theta}^{-1}(p) \) is an \( \alpha \)-type node \( q \). Conversely, each \( \alpha \)-type node \( q \) is a singularity \( p \) of type \( A_1 \) whose local equation is \( t + x^2 = 0 \). So we get a bijective map
\[
\alpha_{2,1} : \mathcal{R}_{2,1}(F) \to \mathcal{N}_{2,1}(F).
\]

(3.14)

If \( p \in \mathcal{R}_{2,2}(F) \), then \( p \) is an ordinary double point, and \( r_p(R) = 2 \). By the same discussion in (2), we can obtain a bijective map
\[
\alpha_{2,2} : \mathcal{R}_{2,2}(F) \to \mathcal{N}_{2,2}(F),
\]

(3.15)
Hence, we have that
\[ s_2(F) = |R_{2,1}(F)| + 2|R_{2,2}(F)| = |N_{2,1}(F)| + 2|N_{2,2}(F)| = \xi_0(F). \]

Theorem 1.1 is a corollary of the above theorem.

**Remark 3.9.** From the Corollary 2.15 and Theorem 1.1, we know that a family \( f : S \rightarrow C \) of hyperelliptic semistable curves with lowest slope if and only if the image \([f]\) of \( f \) by the moduli map \( J \) intersects with \( \Xi_0 \) only, and \( f \) with highest slope if and only if \([f]\) intersects with \( \Delta_{|g/2|} \) only. See [7] for families with highest slope.

**Example 3.10** (Continuation of Example 3.2). From the analysis of the blocks of \( F \) in Example 3.2 we can easy to know that
\[ p_a(F_{p_1}) = 1, \quad p_a(F_{p_2}) = 1, \quad p_a(F_{p_3}) = 1, \] and the sets of nodes are
\[ N_{2,1}(F) = \{q_0, q_{23}, q_{24}\}, \quad N_{2,2}(F) = \{(q_{11}, q_{12})\}, \]
\[ N_3(F) = \{q_3\}, \quad N_4(F) = \{(q_{21}, q_{22})\}. \]

Hence the numbers of nodes on \( F \) are
\[ (\xi_0(F), \xi_1(F), \xi_2(F)) = (5, 1, 0), \]
\[ (\delta_1(F), \delta_2(F)) = (1, 0). \]

Comparing these two equations with (2.9) and (2.10), we give an example for the above theorem.

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