Priorities in tock-CSP

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Abstract
The tock-CSP encoding embeds a rich and flexible approach to modelling discrete timed behaviours in CSP where the event tock is interpreted to mark the passage of time. The model checker FDR provides tailored support for tock-CSP, including a prioritisation operator that has typically been used to ensure maximal progress, where time only advances after internal activity has stabilised. Prioritisation may also be used on its own right as a modelling construct. Its operational semantics, however, is only congruent over the most discriminating semantic model of CSP; the finite-linear model. To enable sound and compositional reasoning in a tock-CSP setting, we calculate a denotational definition for prioritisation. For that we establish a Galois connection between a specialisation of the finite-linear model, with tock and ✓, that signals termination, as special events, and ✓-tock-CSP, a model for tock-CSP that captures termination, deadlines, and is adequate for reasoning about timed refinement. Our results are mechanised using Isabelle/HOL.

Keywords: Semantics, Galois connections, process algebras, time, priorities

1. Introduction
Process algebras like CSP [1] enable modelling of reactive systems via named events that correspond to atomic and instantaneous interactions of interest. To specify budgets and deadlines, and to reason about liveness and safety over time, an explicit notion of time is required. Roscoe introduced the tock-CSP encoding, where an event tock is used to mark the passage of discrete time. Extensive use of tock-CSP has been reported [2–6].

The CSP model-checker FDR [7] has operators tailored for tock-CSP. In addition, it implements a Pri≤(P) operator that can be used to prioritise events according to a partial order ≤. The behaviour is that of P, but changed so that whenever events a and b are available, then if b is of strictly higher priority than a, that is, a < b, then a, and the behaviour following a, is pruned.

Example 1. R = a → Skip □ b → Skip. Process R offers an external choice (□) between behaving as a prefixing (→) on a or b, followed by immediate termination (Skip). Prioritising R, with a < b yields the process b → Skip.

Prioritisation can be used in FDR to enforce maximal progress, that is, that time can only advance after internal activity has stabilised, by prioritising τ, the internal action, and ✓, which signals termination, over tock. Pri≤ also endows CSP with extra expressivity [8], and has been applied also in abstraction techniques [9], reducing refinement in different CSP semantics to traces refinement [10], or as a modelling construct on its own [5, 6].

The operator Pri≤ has an intuitive operational semantics. However, for Pri≤ to be congruent over denotational models of CSP, namely the finite-linear (FL [1]) and refusal testing (RT [11, 12]) models, the partial order needs to be constrained [8]. Thus, FDR actually implements a constrained form of Pri≤, where, for example, ✓ and τ are maximal in the order. However, this is insufficient for Pri≤ to be congruent over weaker models such as ✓-tock [13] and the stable-failures (F [1]).

Example 2.

S = a → Skip □ b → Skip ⊓ (Wait 1 ; S)
T = (a → Skip □ b → Skip) ⊓ (Wait 1 ; T)
Process $S$ makes an internal choice ($\triangleright$) between offering events $a$, or $b$, and terminating, or waiting a time unit ($\text{Wait} 1$) and then behaving as $S$ again, specified using sequential composition ($;.$). Process $T$ also makes an internal choice, but the choice between $a$ and $b$ is external. Although it may seem that $T$ is more deterministic than $S$, deadlocking for a time unit, before making an internal choice again that may lead to the refusal of $a$ and $b$, is a possibility for $T$. The $\triangleright$-tock model does not distinguish $S$ and $T$, just like the $F$ model does not distinguish $a \rightarrow \text{Skip} \sqcap b \rightarrow \text{Skip} \sqcap \text{Stop}$ and $(a \rightarrow \text{Skip} \sqcap b \rightarrow \text{Skip}) \sqcap \text{Stop}$.

However, prioritising $S$ and $T$, with $a < b$, assuming $\triangleright$ and tock are maximal in the order and that tock is not prioritised over $b$, yields different results. $T$ becomes equal to a process $\text{pri}T = b \rightarrow \text{Skip} \sqcap (\text{Wait} 1; \text{pri}T)$, whereas the prioritisation of $S$ bears no effect. The incongruence arises as a result of $\text{Pri}_<\unit$ being defined over $F$, where it distributes over internal choice. To reason about priorities in other models we need different definitions for $\text{Pri}_\preceq$.

In this paper we calculate a definition of $\text{Pri}_\preceq$ for tock-CSP via a stepwise Galois connection between the $FL$ model and the $\triangleright$-tock model [13], the only sound model for tock-CSP that can be used to reason about timed refinement, and that captures deadlines, termination and erroneous Zeno behaviour. It is a faithful account of the tock-CSP dialect and is mechanised in Isabelle/HOL [14]. Included in this work is a mechanisation of the $FL$ model, that handles termination, via the special event $\triangleright$, and a specialisation that includes tock.

This paper is structured as follows. In Section 2.1 we describe $\triangleright$-tock and $FL$. In Section 3 we formally define $FL$. In Section 4 we define a Galois connection between a specialisation of $FL$, which includes tock as event, and calculate the induced definition of $\text{Pri}_\preceq$. We conclude in Section 5.

2. Models

Here we describe the $\triangleright$-tock and $FL$ models by summarizing material from [13] and [1].

2.1. $\triangleright$-tock

The $\triangleright$-tock model is defined in terms of a set $\Sigma$ of events, not including $\triangleright$ and tock. The complete set is defined as $\Sigma_{tock}$. The semantics of processes is a set of sequences of observations of type $\text{Ob}_s$, defined below. An observation is either an event in $\Sigma_{tock}$, or the refusal of some subset of $\Sigma_{tock}$, so $\text{Ob}_s$ has two constructor functions $\text{evt}$ and $\text{ref}$.

Definition 1. $\text{Ob}_s := \text{evt} (\langle \Sigma_{tock} \rangle) \cup \text{ref} (\langle \mathbb{P} \Sigma_{tock} \rangle)$

The type of valid traces is defined as $TT$, which is the set of all sequences $t$ with elements of type $\text{Ob}_s$ that satisfy three conjuncts, defined below.

Definition 2. $TT := \{ t : \text{seq} \text{ Ob}_s \mid \forall i : \text{dom} t \bullet (i < \#t \Rightarrow t(i) \neq \text{evt} \triangleright \land (i < \#t \land t(i) \in \text{ran ref}) \Rightarrow t(i + 1) = \text{evt tock} \land t(i) = \text{evt tock} \Rightarrow (i > 1 \land t(i - 1) \in \text{ran ref}) \}$

The first conjunct ensures that $\triangleright$ can only appear as the last event of $t$. The second conjunct requires that every refusal before the last ($i < \#t$) is followed by a tock. Finally, the third conjunct ensures that every tock is preceded by a refusal.

The healthiness conditions of $\triangleright$-tock [13], whose composition is named $TT$, ensure properties of the standard models of CSP in the context of $TT$ traces. Namely, the empty trace is an observation of every process; prefix closure and subset closure of refusals; and an event that cannot be performed is refused. Finally, whenever there is a stable refusal in a trace, then there is always at least one (other) trace where the refusal includes $\triangleright$.

2.2. Finite-linear

The $FL$ model is characterised in [1] by finite sequences $\langle A_0, e_0, \ldots, A_i, e_i, A_{i+1} \rangle$ of alternating acceptances $A_i$ and events $e_i$. An acceptance is either a set, recording the events being offered, or null ($\bullet$) indicating the impossibility to observe such a set because of instability. An event $e_i$ necessarily belongs to $A_i$ if $A_i$ is not $\bullet$. Valid sequences end in an acceptance, or $\bullet$ followed by $\triangleright$. The healthiness conditions of $FL$ ensure prefix closure, and require that, whenever an acceptance $A_i$ is observed, then any events in $A_i$ can also be performed.

The prefix relation for sequences in this model allows $\bullet$ to precede a stable acceptance set, so that $\langle \bullet \rangle$ and $\langle \bullet, e_0, \bullet \rangle$ are prefixes of $\langle A_0, e_0, A_1 \rangle$, for example. Crucially, and unlike other CSP models, there is no upward-closure of acceptance sets.

3. Formalising the $FL$ model

In Section 3.1 we define a recursive data type to capture $FL$ traces. In Section 3.2 we formalise
the healthiness conditions. Finally, in Section 3.3 we formally define $\text{Pri}_\prec$. While Roscoe [1] characterises the $\mathcal{FL}$ model and studies in depth its relationship to other CSP models, here we define its data model and healthiness conditions with the level of formality required to mechanise it (as an Isabelle theory as presented in [15], for example).

3.1. Model

An acceptance is either a null acceptance or a set of events. It is defined by the type $\text{Acc}$, which has two constructors functions $\bullet$ and $\text{aset}$, respectively, where $\langle \_ \rangle$ is the Z [16] syntax for constructors.

Definition 3: $\text{Acc} ::= \bullet \mid \text{aset} \langle \_ \_ \rangle$

We also define: $e \in \mathcal{FL} A$ to be true exactly when $A$ is the result of applying $\text{aset} B$ for some set $B$ and $e \in B$; and a prefix order on $\text{Acc}$, where $\bullet$ is the least element, and $\text{aset} A \leq \text{aset} A$, which can be lifted to define the prefix order on $\mathcal{FL}$ traces.

An acceptance followed by an event is a pair of type $\text{Aev}$ associating elements of $\text{Acc}$ to $\Sigma_{\text{tick}}$.

Definition 4 (Acceptance-event).

$$\text{Aev} : \text{Acc} \times \Sigma_{\text{tick}}$$

$$\forall B : \mathcal{P} \Sigma_{\text{tick}}; (\text{aset} B, e) : \text{Aev} \bullet e \in B$$

An acceptance-event pair $(\text{aset} B, e)$ is valid exactly when $e$ is a member of set $B$. A pair $(\bullet, e)$ imposes no condition on $e$. Next we define the type of non-empty traces for $\mathcal{FL}$ as $\mathcal{FL}$, a recursive data type with two constructors, $\text{acc}$ and $\text{aev}$.

Definition 5. $\mathcal{FL} ::= \text{acc} \langle \text{Acc} \_ \rangle \mid \text{aev} \langle \text{Aev} \times \mathcal{FL} \_ \rangle$

The function $\text{acc}$ takes a single acceptance while $\text{aev}$ takes an acceptance-event pair and an $\mathcal{FL}$ trace. Processes in $\mathcal{FL}$ are defined by a set of $\mathcal{FL}$ traces, effectively finite non-empty sequences ending in an acceptance. Unlike the original presentation of $\mathcal{FL}$, we consider $\checkmark$ as a regular event. This simplifies the type definition of $\mathcal{FL}$ and allows us to give a general account of $\text{Pri}_\prec$ where $\checkmark$ can be prioritised.

To facilitate presentation we abbreviate $\text{acc} A$ as $\langle A \rangle_{\mathcal{FL}}$, and $\text{aev} \langle A, e \rangle, \rho$ as $\langle A, e \rangle \# \rho$. Furthermore a recursive application of $\#$ a number of times, such as $\langle A_0, e_0 \rangle \# (\ldots \# A_i)$ is abbreviated as $\langle (A_0, e_0), \ldots, A_i \rangle_{\mathcal{FL}}$. This is a typical approach to encoding finite lists via recursive data types.

3.2. Healthiness conditions

The healthiness conditions are listed in Table 1.

The first, $\mathcal{FL}_0$, although not listed in [1, p.257], is required to ensure that every process has some behaviour. Together with $\mathcal{FL}_1$ (adopted from [1]), which ensures prefix closure, $\mathcal{FL}_0$ ensures that every process has at least the trace $\langle \_ \rangle_{\mathcal{FL}}$.

The next condition $\mathcal{FL}_2$ from [1] is restated using $\in \mathcal{FL}$ and a concatenation operator $\langle A \rangle_{\mathcal{FL}}$ that is closed under $\mathcal{FL}$, unlike the general sequence concatenation operator. $\mathcal{FL}_2$ states that whenever a trace $\rho$ concatenated with an acceptance $A$ is in $P$, then for every event $e$ in $A$ there must be a trace in $P$ that performs $e$. It is the result of concatenating $\rho$ with the trace consisting of the acceptance-event pair $(A, e)$ followed by $\bullet$.

To ensure $\checkmark$ is only possible after $\bullet$ as the very last event of a trace, $\mathcal{FL}_3$ requires valid traces to be in a set $\mathcal{FL}_\checkmark$. It consists of $\mathcal{FL}$ traces where $\checkmark$ is not offered in any acceptance and, if $\checkmark$ appears, it is as the last event followed by $\bullet$.

The $\mathcal{FL}$ model forms a complete lattice under the refinement order defined by subset inclusion. The top is the process $\text{div}$ whose only observation is the trace $\langle \_ \rangle_{\mathcal{FL}_\checkmark}$, while the bottom is $\mathcal{FL}_\checkmark$, the set of all possible observations: the process $\text{Chaos}(\Sigma)$.

3.3. Prioritise

The semantics of $\text{Pri}_\prec(P)$ is defined in $\mathcal{FL}$ pointwise over the $\mathcal{FL}$ traces of $P$ as follows.

Definition 6. $\text{Pri}_\prec(P) \equiv \{\rho, \sigma : \mathcal{FL} \mid \rho \text{pri}_\prec \sigma \wedge \sigma \in P \bullet \rho\}$

It is the set of traces $\rho$ related by $\text{pri}_\prec$ to a trace $\sigma$ drawn from $P$, that is, a trace $\rho$ is a possible prioritisation of $\sigma$. The relation $\text{pri}_\prec$ is a fully specified version of that presented in [8] generalised to cater for any event, including $\checkmark$, and defined inductively in Definition 7 as required for mechanical reasoning.
Observe that $\text{pri}_p$ is parametric over an arbitrary partial order $p$ on events. Before explaining the definition, we illustrate the calculation of traces $\rho$, as related by $\text{pri}_\leq$, from traces $\sigma$ of process $R$ (of Example 1), assuming $a < b$ and that $\checkmark$ is maximal in the order. The set of traces of $R$ is presented below, as well as those of $\text{Pri}_\leq(R)$, where we omit the $\mathcal{F}_\mathcal{L}$ subscripts. We observe that the result is independent of whether $b$ is maximal in the order.

**Example 3.**

$$\text{traces}(R) = \{ (\{ \},), \{ (a, b), \}, \{ (a, b), (\checkmark), \}, \{ (a, b), (\checkmark), \}, \}$$

We have traces: $\langle a, b \rangle$ recording that both events $a$ and $b$ are stably offered, and traces such as $\langle (a, b), (a, \checkmark) \rangle$ recording that after stably offering both events, and performing event $a$, then termination, encoded by $\checkmark$ happens unstably, and similarly for event $b$; and finally traces that complete the prefix closure as required by $\mathcal{F}_\mathcal{L}$.

$$\text{traces}(\text{Pri}_\leq(R)) = \{ (\{ \},), \{ (b), \}, \{ (b), (\checkmark), \}, \{ (b), (\checkmark), \}, \}$$

The traces of $\text{Pri}_\leq(R)$ are those of $b \rightarrow \text{Skip}$. □

Prioritisation of $R$ with $a < b$, as previously discussed in Section 4, leads to pruning the behaviours of the prefixing on $a$. The traces of $\text{Pri}_\leq(R)$ are those related to the traces of $R$ by $\text{pri}_\leq$.

The trace $\langle \{ \} \rangle$ is related to itself as it is a valid observation of every process in $\mathcal{F}_\mathcal{L}$. The trace $\langle \{ b \} \rangle$ is related to $\langle \{ a, b \} \rangle$, as $\{ b \}$ is the set obtained by eliminating events in $\{ a, b \}$ for which an event of strictly higher priority is also in the set, namely $a$ is eliminated as $b$ is of higher priority. Similarly, the trace $\langle \{ a, b \}, b \rangle$ is related to $\langle \{ a, b \}, a \rangle$, and $\langle \{ a, b \}, (\checkmark), b \rangle$ is related to $\langle \{ a, b \}, (\checkmark), a \rangle$ as, in addition, $\checkmark$ is maximal. The trace $\langle \{ b \}, a \rangle$ is related to $\langle \{ a, b \}, b \rangle$ as $\{ b \}$ is a prefix of $\{ b \}$, and, as discussed, $\{ b \}$ is the set obtained from $\{ a, b \}$ according to the priority order. Similar observation applies to $\langle \{ b \}, (\checkmark), b \rangle$, which is related to $\langle \{ a, b \}, (\checkmark), a \rangle$. Traces where the event $a$ appears are not related by $\text{pri}_\leq$ and thus are eliminated by $\text{Pri}_\leq(R)$. Below we define $\text{pri}_p$.

**Definition 7.**

$$\text{pri}_{\leq}(\sigma) = \langle \Sigma_{\text{lock}} \leftrightarrow \Sigma'_{\text{lock}} \rightarrow (\mathcal{F} \leftrightarrow \mathcal{FL}) \rangle$$

$$\forall A, Z \in \text{Acc}; \ p : \Sigma'_{\text{lock}} \leftrightarrow \Sigma_{\text{lock}} \bullet \langle A \rangle \text{pri}_{p}(\langle Z \rangle) = \langle A = \text{priacc}_{p}(\langle Z \rangle) \rangle$$

$\langle (A, e) \notin \rho \rangle \text{pri}_{p}(\langle (Z, e) \notin \sigma \rangle) \Rightarrow \langle A \leq \text{priacc}_{p}(\langle Z \rangle) \land p \text{pri}_{p} \sigma \land \neg \max(p, e) \Rightarrow e \in \mathcal{F}_{\mathcal{L}} \text{priacc}_{p}(\langle Z \rangle) \rangle$

In general, a trace $\langle A \rangle$ is related to $\langle Z \rangle$ by $\text{pri}_{p}$, whenever $A$ is $\text{priacc}_{p}(\langle Z \rangle)$, specified below, that defines the result of prioritising an acceptance $Z$. In Example 3 above, we have that $\text{priacc}_{p}(\langle \{ a, b \} \rangle) = \langle \{ b \} \rangle$.

**Definition 8.**

$$\text{priacc}_{p}(\langle Z \rangle) = \langle Z \cap \{ e \mid \neg (\exists b \cdot b \in Z \land e <_{p} b) \} \rangle$$

Formally, $\text{priacc}_{p}(\langle \cdot \rangle)$ is $\langle \cdot \rangle$, whereas for an acceptance set $Z$, $\text{priacc}_{p}(\langle Z \rangle)$ is the acceptance set of events in $Z$ for which no event $b$ of strictly higher priority, according to the order $p$, exists in $Z$.

A trace $\langle A, e \rangle \notin \rho$ is related to $\langle (Z, e) \notin \sigma \rangle$ by $\text{pri}_{p}$, exactly when $A$ is a prefix of the acceptance permitted by prioritisation, as defined by $\text{priacc}_{p}(\langle Z \rangle)$, $\rho$ is related to $\sigma$ by $\text{pri}_{p}$, and if the event $e$ is not maximal in the partial order $p$, where $\max(p, e) = \neg \exists x \cdot e <_{p} x$, that is, no other event has strictly higher priority than $e$, then $e$ must be in the acceptance permitted by prioritisation. In other words, events of maximal priority are never eliminated, whereas those for which an event of higher priority exists need to be in the resulting acceptance set obtained via the intersection. In particular, if $Z$ is $\langle \cdot \rangle$ and $e$ is not maximal, then it is not related by $\text{pri}_{p}$. Although our definition of $\text{pri}_{p}$ is rather concise, we have established using our mechanisation that it is equivalent to that of Roscoe [8].

A formal definition for $\text{pri}_{\leq}$ suitable for mechanical reasoning enables key results about $\text{Pri}_{\leq}$ to be established, namely that it is monotonic and closed under the healthiness conditions. Using our mechanisation we have established the following novel results where the subscript $\mathcal{F}_{\mathcal{L}}$ indicates that these are operators of the $\mathcal{F}_{\mathcal{L}}$ model.
Lemma 1

\[ \text{Pri}_\leq \circ \text{Pri}_\leq (P) = \text{Pri}_\leq (P) \]
\[ \text{Pri}_\leq (P \cap_{FL} Q) = \text{Pri}_\leq (P) \cap_{FL} \text{Pri}_\leq (Q) \]
\[ \text{Pri}_\leq (a \rightarrow_{FL} P) = a \rightarrow_{FL} \text{Pri}_\leq (P) \]

The first result in Lemma 1 establishes that \( \text{Pri}_\leq \) is idempotent; the second that it distributes through internal choice; and the third that it distributes through prefixing. It also distributes through sequential composition, as established next.

Lemma 2 Provided \( \checkmark \) has maximal priority, \( \text{Pri}_\leq (P ; Q) = \text{Pri}_\leq (P) ; \text{Pri}_\leq (Q) \).

In this case \( \checkmark \) must be maximal in the order, as otherwise the possibility for \( P \) to terminate could be eliminated. More interestingly, the prioritisation of an external choice between prefixings on events \( a \) and \( b \), where \( a < b \), is the prioritisation of the prefixing on \( b \), with any behaviour following on from \( a \) pruned, as established by the lemma.

Lemma 3 Provided \( a < b \),
\[ \text{Pri}_\leq (b \rightarrow_{FL} P \sqcup_{FL} a \rightarrow_{FL} Q) = \text{Pri}_\leq (b \rightarrow_{FL} P) \]

Proof of this, and other key results, is available\(^1\).

Our formalisation of \( \mathcal{FL} \) and \( \text{Pri}_\leq \) are used next to calculate a definition for \( \text{Pri}_\leq \) in \( \checkmark\text{-}\text{tick} \).

4. Galois connection

The key to defining a Galois connection between \( \mathcal{FL} \) and \( \checkmark\text{-}\text{tick} \) is relating \( FL \) traces, which record acceptances before every event, and \( TT \) traces, which instead record subset-closed refusals, and only before \( tick \) events or at the end of a trace, as previously discussed in Section 2.1. We define the Galois connection stepwise to simplify proofs. In Section 4.1 we first consider a Galois connection between \( \mathcal{FL} \) and a variant of \( \checkmark\text{-}\text{tick} \) where refusals sets are not subset closed; instead they are maximal: they record exactly what is refused. In the following Section 4.2 we define a Galois connection with full \( \checkmark\text{-}\text{tick} \) by completing the subset closure of refusals. Fig. 1 which is explained as we discuss each step, provides a depiction of the connections. Finally, in Section 4.3, we present the result of calculating the induced definition of \( \text{Pri}_\leq \) for \( \checkmark\text{-}\text{tick} \).

\( ^1 \)https://github.com/robo-star/tick-tack-CSP

4.1. From \( \mathcal{FL}\text{-}\text{tick} \) to maximal \( \checkmark\text{-}\text{tick} \)

The pair of functions \( \text{fl2ttm} \) and \( \text{ttm2fl} \), mapping between \( \mathcal{FL}\text{-}\text{tick} \) and the maximal variant of \( \checkmark\text{-}\text{tick} \), labelled as \( \text{TTM} \) in Fig. 1 is defined below.

Definition 9.

\[ \text{fl2ttm}(P) \equiv \{ \rho : \mathcal{FL} \mid \rho \in P \bullet \text{fl2ttobs}(\rho) \} \]
\[ \text{ttm2fl}(P) \equiv \bigcup \{ Q \mid \text{fl2ttm}(Q) \subseteq P \} \]

The function \( \text{fl2ttm} \) is defined as the set of traces obtained by applying \( \text{fl2ttobs} \), a total non-injective function from \( \mathcal{FL} \) to \( TT \) traces, defined next, to every trace \( \rho \) in \( P \). The inverse mapping \( \text{ttm2fl} \) is uniquely defined in terms of \( \text{fl2ttm} \), and is the distributed union over the set of \( \mathcal{FL} \) processes that can be mapped via \( \text{fl2ttm} \) to be a subset of \( P \). This construction is standard for Galois connections where one adjoint uniquely defines the other.

The function \( \text{fl2ttobs} \) is defined as follows.

Definition 10.

\[ \text{fl2ttobs} : \mathcal{FL} \rightarrow TT \]
\[ \forall a : \text{Acc} ; \; e : \Sigma^* ; \; A : \mathcal{P}\Sigma_{\text{tick}} ; \bullet \]
\[ \text{fl2ttobs}((\bullet)_{\mathcal{FL}}) = \emptyset \]
\[ \text{fl2ttobs}\langle a \rangle_{\mathcal{FL}} = \langle \text{ref} \{ z \mid z \notin A \} \rangle \]
\[ \text{fl2ttobs}\langle (a, e) \neq \rho \rangle = \langle \text{ev} e \rangle \cap \text{fl2ttobs}(\rho) \]
\[ \text{fl2ttobs}\langle a, \text{ev} e \rangle = \langle \text{ev} e \rangle \cap \text{fl2ttobs}(\rho) \]
\[ \langle \text{ref} \{ z \mid z \notin A \} , \text{ev} e \rangle = \langle \text{ev} e \rangle \cap \text{fl2ttobs}(\rho) \]

It maps: the null trace \((\bullet)_{\mathcal{FL}}\) to the empty trace \(\emptyset\), since \(\checkmark\text{-}\text{tick}\) does not record instability; and every trace whose only element is an acceptance set \( a \) to a singleton trace consisting of a refusal obtained
as the set complement of \( A \). The mapping of an acceptance-event pair followed by a trace \( \rho \) is split into three cases: (1) acceptances \( a \) preceding regular events \( e \), other than \( \text{tock} \), are mapped to a trace \( \langle \text{ev} e \rangle \) whose only element is the event \( e \), concatenated with the result of applying \( \text{fl2ttob} \) to \( \rho \), since \( \checkmark\text{-tock} \) does not record refusals before such events; (2) for the same reason, null acceptances preceding \( \text{tock} \) are mapped to the empty trace; (3) acceptance sets preceding \( \text{tock} \) are mapped to a trace consisting of a refusal set, obtained as the complement of \( A \), followed by the event \( \text{tock} \), concatenated with the application of \( \text{fl2ttob} \) to \( \rho \).

The functional application of \( \text{fl2ttob} \) to a healthy process \( P \) satisfies nearly all the healthiness conditions of \( \checkmark\text{-tock} \), as established by Lemma 2. It also satisfies extra healthiness conditions, defined below. They are relevant to obtain the final Galois connection in the next section.

**Lemma 4.** Provided \( P \) satisfies FL0-3, \( \text{fl2ttob}(P) \) satisfies TT0, TT1w, TT2-4 and TTM1-3.

Namely, \( \text{fl2ttob}(P) \) satisfies TT0 and TT2-4, but not TT1 [13], which requires subset closure of refusals. This is expected, as \( \text{fl2ttob} \) is a function. (The subset closure of refusals is completed by the connection in Section 4.2.) Instead, we have prefix-closure of sequences, ensured by a healthiness condition we call TT1w, a weaker form of TT1 that does not enforce subset closure of refusals.

In addition, refusals are maximal, and so \( \text{fl2ttob}(P) \) satisfies the additional conditions TTM1-3 listed in Table 2. TT1 requires that every event \( e \) that is not that can be performed. TTM2 is similar, but requires \( \text{tock} \) to happen after the refusal. TT3 requires that \( P \) is a subset of \( \text{TT}_\checkmark \), the set of all traces where every refusal contains \( \checkmark \). The three conditions, together named TT0 (see Fig. 1), characterise the set of \( \checkmark\text{-tock} \) processes whose refusal sets contain exactly the events refused.

Since \( \text{fl2ttob} \) is monotonic with respect to the refinement order, and \( \text{ttm2fl} \) is closed under the healthiness conditions of FL, we have a Galois connection between FL-toc and maximal \( \checkmark\text{-tock} \).

Next we focus on the subset closure of refusals to complete the Galois connection.

### 4.2. Subset-closure of refusals leading to full \( \checkmark\text{-tock} \)

Completing the subset closure of refusal sets in \( \checkmark\text{-tock} \) is achieved by the function mkTT1.

| Name | Definition |
|------|------------|
| \( \text{TTM}1(P) \) | \( \rho \cap \langle \text{ref} X \rangle \in P \land e \notin X \land e \neq \text{tock} \Rightarrow \rho \cap \langle \text{ev} e \rangle \in P \) |
| \( \text{TTM}2(P) \) | \( \rho \cap \langle \text{ref} X \rangle \in P \land \text{tock} \notin X \Rightarrow \rho \cap \langle \text{ref} X, \text{ev} \text{tock} \rangle \in P \) |
| \( \text{TTM}3(P) \) | \( P \subseteq \text{TT}_\checkmark \) |

Table 2: The healthiness conditions of maximal \( \checkmark\text{-tock} \).

**Definition 11.**

\[
\text{mkTT1}(P) = \{ \rho, \sigma : \text{TT} \mid \rho \subseteq \sigma \land \sigma \in P \bullet \rho \}
\]

\[
\text{unTT1}(P) = \bigcup \{ Q \mid \text{mkTT1}(Q) \subseteq P \}
\]

With \( \text{mkTT1}(P) \), we have the union of \( P \) with a set of traces \( \rho \) related by \( \subseteq \), the prefix relation on \( \checkmark\text{-tock} \) traces defined in [13], to \( \sigma \) drawn from \( P \). The other adjoint is \( \text{unTT1} \) and is uniquely defined in terms of \( \text{mkTT1} \). It is the distributed union over processes \( Q \) (where \( Q \) satisfies TT1 and TT1w) and whose mapping via \( \text{mkTT1}(Q) \) is a subset of \( P \). In other words, \( \text{unTT1} \) undoes the subset closure of refusals resulting from \( \text{mkTT1}\), yielding a process whose refusals are maximal, as defined by TT0, and whose traces are prefix-closed under TT1w.

Importantly, as stated below, \( \text{mkTT1} \) preserves healthiness of processes that satisfy TT0 and TT2-4, and is \( \text{TT1} \)-healthy as \( \text{mkTT1}(P) = P \) if, and only if, TT1(P).

**Lemma 5 (Closure).** Provided \( P \) satisfies TT0, TT2-4, TT1w and TTM1-3, then \( \text{mkTT1}(P) \) is a \( \checkmark\text{-tock} \) process, that is, it satisfies TT.

Thus, we have a Galois connection between maximal \( \checkmark\text{-tock} \), labelled as TT0 in Fig. 1, and full \( \checkmark\text{-tock} \). Next, we compose both connections to obtain a Galois connection between FL and \( \checkmark\text{-tock} \), depicted in Fig. 1 by the dashed arrows.

**Definition 12.**

\[
\text{fl2ttb}(P) = \text{mkTT1} \circ \text{fl2ttob}(P)
\]

\[
\text{tt2fl}(P) = \text{ttm2fl} \circ \text{unTT1}(P)
\]

The function \( \text{fl2ttb}(P) \) maps FL processes to \( \checkmark\text{-tock} \). In the opposite direction we have \( \text{tt2fl} \).

4.3. Prioritise for \( \checkmark\text{-tock} \)

Similar to the definition of \( \text{Pri}_E \), which is pointwise over FL traces using a relation \( \text{pri} \), we have a similar characterisation for \( \text{Pri}_{\text{TT}_\checkmark} \) as defined next.
Definition 13.

\( \text{Pr}^\text{TT} \leq (P) = \{ \rho, \sigma | \rho \text{ pr}^\text{TT}[\leq (P)] \sigma \land \sigma \in P \bullet \rho \} \)

The traces of \( \text{Pr}^\text{TT} \leq \) is the set of \( TT \) traces \( \rho \) related to a trace \( \sigma \), from \( P \), by a parametric relation \( \text{pr}^\text{TT}[\leq (P)] \), specified in Definition 13 that takes three parameters: the partial order \( \leq \) over events in \( \Sigma^\text{event} \); the empty sequence \( \{\} \), and \( P \).

For a trace \( \sigma \), prioritisation needs to take into account, at each point in \( \sigma \) where an event is present, the refusal sets that can be observed just before it, as well as whether other events can be performed. So the second parameter of \( \text{pr}^\text{TT} \) is the trace considered so far in the process specified as the third parameter. In the initial case, when considering \( \sigma \), the trace considered so far is the empty sequence.

Example 4. We first enumerate traces of \( T \), with \( \Sigma = \{ a, b \} \).

\[
\text{traces}(T) = \begin{cases}
\{\} & , \langle \text{ref} \{\}\rangle, \\
\langle a\rangle & , \langle \text{evt} a\rangle, \\
\langle a, \text{evt } b\rangle & , \langle \text{evt } a, \text{evt } b\rangle, \\
\langle \text{ref} \{a, b\}\rangle & , \langle \text{ref} \{a, b, \text{evt } b\}\rangle, \langle \text{evt } b, \text{evt } tock\rangle, \ldots
\end{cases}
\]

In addition to the empty trace \( \{\} \), we focus on traces containing maximal refusal sets. The trace \( \langle \text{ref} \{\\}\rangle \) stems from the timed external choice, in that every event not in the refusal set, such as \( a, b \) and \( tock \) is possible. The traces \( \langle \text{evt } a\rangle \) and \( \langle \text{evt } b\rangle \) capture the possibility to perform either event, and the traces \( \langle \text{evt } a, \text{evt } b\rangle \) and \( \langle \text{evt } b, \text{evt } tock\rangle \) capture the termination that follows after each event. The possibility to keep delaying the choice is captured by traces starting with \( \langle \text{ref} \{a, b, \text{evt } b\}\rangle, \langle \text{evt } b, \text{evt } tock\rangle \), where every event other than \( tock \) is observed to be refused, followed by \( a \) and possibly one of the illustrated traces above. This refusal set is what distinguishes process \( T \) from \( R \) in \( \text{evt } b, \text{evt } tock \), as in the case of \( R \) it is not possible to refuse both events \( a \) and \( b \) at any time. (\( R \) in Example 1 refines \( T \), by resolving the internal choice in \( T \)).

The traces of \( \text{Pr}^\text{TT} \leq (T) \) are those related to the traces of \( T \) by \( \text{Pr}^\text{TT}[\leq (T)] \), as follows.

The trace \( \{\} \) is related to itself as it is a valid trace of every process (see 1 in Definition 13). The trace \( \langle \text{ref} \{\\}\rangle \) is related to itself, and every trace whose refusal set is a subset of \( \{\\} \) is related to it because of subset closure (see 2). Similarly, \( \langle \text{ref} \{a, b\}\rangle \) is related to itself, and to traces whose refusal set is a subset of \( \{a, b, \text{evt } b\}\). The trace \( \langle \text{evt } a\rangle \) is related to itself as it is prioritised over \( a \), so can only happen unstably, and \( tock \) is assumed not to be prioritised over \( b \). The traces \( \langle \text{evt } b, \text{evt } \rangle \) and \( \langle \text{evt } a, \text{evt } \rangle \) are similarly related to themselves, given that \( \text{evt } \) has maximal priority. Finally, because \( tock \) is maximal, \( \langle \text{ref} \{a, b, \text{evt } b\}\rangle \) is related to itself (see 3).

Because \( b \) may be refused by \( T \) if the choice is delayed as a result of the internal choice being resolved to \( \text{Wait } 1 \); \( T \), no traces of \( T \) are pruned by the application of \( \text{Pr}^\text{TT} \leq (T) \) (as they are by the application of \( \text{Pr}^\leq (T) \) as explained in Example 1).

Central to the definition of \( \text{pr}^\text{TT}[\leq (p, \sigma, Q)] \) is a function \( \text{pr}^\text{ref}(p, \sigma, Q, S) \), which considers the effect of prioritisation over a refusal set \( S \) (see cases 2 to 6 of Definition 14). A trace \( \langle \text{ref } S \rangle \) is related to \( \langle \text{ref } S \rangle \) by \( \text{pr}^\text{TT}[\leq (p, \sigma, Q)] \) whenever \( R \) is a subset of...
**Definition 15.**

\[ \text{priref}(p, \sigma, Q, S) = \]
\[ S \cup \{ \varepsilon \mid \sigma \vdash \langle \text{ref } S, \text{ ext } \text{tock} \rangle \in Q \land \varepsilon <_p \text{tock} \}
\]
\[ \cup \{ \varepsilon \mid \exists b \cdot b \notin S \land \sigma \vdash (\text{ext } b) \in Q \land \text{and } b \neq \text{tock} \land b \neq \varepsilon \land \varepsilon <_p b \} \]

The function \text{priref} is defined as the union of \( S \) and two sets: the set of events \( e \) of strictly lower priority than \text{tock}, according to the order \( p \), if the trace \( \sigma \) concatenated with the trace where the refusal \( S \) followed by \text{tock} is in \( Q \); and the set of events \( e \) such that there is some event \( b \) of strictly higher priority than \( e \), which is not \text{tock} nor \( \checkmark \) and is not in \( S \), but that can be performed in \( Q \) after trace \( \sigma \) as \( \sigma \vdash \langle \text{ext } b \rangle \). Thus, to ascertain whether an event \( e \) can be refused because of prioritisation, it is necessary to determine whether an event of higher priority, other than \( \checkmark \), that is not refused in \( S \) can be performed after trace \( \sigma \). The reason \( \checkmark \) is excluded from the comparison is because it is an unstable event, and so it is never in a genuine offer to the environment. To illustrate this point, we consider the process in the following example.

**Example 5.** \( K = \text{Skip} \sqcap f \to \text{Skip} \). In addition to \( e \) being offered stably, we have the possibility that process \( K \) terminates. Assuming \( f <_p \checkmark \), applying \text{priref}(p, \cdot, K, \cdot) would result in the set \{ \} if \( f \) could be compared with \( \checkmark \) in \text{priref}. Because \( \checkmark \) is never a genuine option for the environment, however, it is incorrect to compare it with events of lower priority. Moreover, as illustrated in this example, the set of traces induced by such a definition of \text{priref}(p, \cdot, K, \cdot) would be unhealthy, as no refusal set \( f \), \( \checkmark \) could be obtained by applying \text{priref} to an initial refusal of \( K \) (see Section 2.7).

Similarly, a trace \( \langle \text{ref } R, \text{ ext } \text{tock} \rangle \vdash \rho_0 \) is related to \( \langle \text{ref } S, \text{ ext } \text{tock} \rangle \vdash \rho_1 \) by \( \text{PriTT} \vdash \text{PriTT} \) (see 3) whenever \( R \) is a subset of \text{priref}(p, \sigma, Q, S), and, in addition, \text{tock} is not refused as the result of applying \text{priref}, and the traces \( \rho_0 \) and \( \rho_1 \) are related by \( \text{PriTT} \vdash \text{PriTT} \) (where the second parameter is \( \sigma \) concatenated with \( \sigma \vdash \langle \text{ref } S, \text{ ext } \text{tock} \rangle \)).

For example, assuming \( \text{tock} <_p \rho \), then the result of applying \text{priref}(p, \cdot, T, \cdot) would be \( \{ a, b, \checkmark \} \) because the trace \( \langle \text{ext } a \rangle \) is in \( T \), and thus the trace \( \langle \{ a, b, \checkmark \}, \text{ ext } \text{tock} \rangle \) would be pruned by \( \text{PriTT} \).

The final case 3 in Definition 15 specifies that a trace \( \langle \text{ext } e \rangle \vdash \rho_0 \) is related to \( \langle \text{ext } e \rangle \vdash \rho_1 \) by \( \text{PriTT} \vdash \text{PriTT} \) and, if \( e \) is not maximal in the priority order \( p \), then it is not \( \checkmark \) and there is some refusal \( Z \) observable after \( \sigma \) in \( Q \), such that application of \text{priref}(p, \sigma, Q, Z) does not lead to \( e \) being refused. For example, trace \( \langle \text{ext } a \rangle \) from process \( T \) is related to itself when considering \( a <_p b \), as although \( a \) is not maximal there is a trace \( \langle \text{ref } b \rangle \) in \( T \), by subset closure, where the application of \text{priref}(p, \cdot, T, \{ b \}) yields \{ b \}. In words, if \( e \) is maximal then the trace is related, and in particular, if \( \checkmark \) appears in a trace it must be maximal. Otherwise, if \( e \) is neither maximal nor \( \checkmark \), then it must be the case that application of \text{priref} to \( Q \) indicates that \( e \) can be chosen; not refused.

Thus, events that are not maximal are not discarded by prioritisation if, at the same time, events of higher priority can be refused. We consider, an alternative presentation of \( T \), by distributing the internal choice over the external choice.

\[
T = \left( \begin{array}{c}
(a \to \text{Skip} \sqcap (\text{Wait } 1 ; T)) \\
(b \to \text{Skip} \sqcap (\text{Wait } 1 ; T))
\end{array} \right)
\]

This clearly shows the possibility for \( a \) to be stably offered for a time unit, if the first internal choice is decided in favour of \( a \), while the second internal choice is made in favour of a delay. Thus, before the first time unit has elapsed, there is the possibility that no \( b \) is on offer, and thus prioritisation is ineffective at that point. Another presentation of \( T \) is exactly \( S \), which, as discussed in Section 4, has the same semantics as \( T \) in \( \checkmark \)-tock, but not in \( FL \). This is inherently related to algebraic properties enjoyed by \( \checkmark \)-tock processes, which are consistent with the \( F \) model within a time unit.

Our main result is that the \( \text{PriTT} \vdash \) operator of \( \checkmark \)-tock is exactly that induced by the Galois connection with \( FL \) as established next.

**Theorem 1.** Provided \( P \) is TT-healthy,

\[ \text{PriTT} \vdash (P) \equiv fl2tt \circ \text{Pri} \vdash \circ tt2fl(P) \]

For a \( \checkmark \)-tock process \( P \), \( \text{PriTT} \vdash (P) \) can be obtained by mapping \( P \) into \( FL \), via \( tt2fl \), then ap-
plying the prioritisation operator of $\mathcal{F}L$, and finally mapping the result into $\check{\mu}$-tick via $\mu 2 \nu$.

5. Conclusions

To endow $\check{\mu}$-tick with a prioritise operator consistent with $\mathcal{F}L$ of $\mathcal{F}L$, we have established a Galois connection between $\mathcal{F}L$ and $\check{\mu}$-tick, and calculated an induced definition. With that, we have a definition that is by construction monotonic and healthy. It is strictly not pointwise because prioritisation of events in a trace needs to take into account other refusal sets reachable at the same point.

The formalisation of $\mathcal{F}L$, although presented here in terms of a set $\Sigma_{\check{\mu} \tau}$ is fully parametric in our mechanisation in Isabelle/HOL, and thus depends only on $\Sigma$. The mechanisation of the $\mathcal{F}L$ model is layered. We have a faithful mechanisation of $\mathcal{F}L$, built stepwise to consider termination, and discrete time, by including $\check{\mu}$ and $\check{\nu}$ incrementally.

The algebraic laws from Section 3 for $\mathcal{F}L$ are novel, and more importantly are applicable to $\mathcal{RT}$ as a consequence of Roscoe’s results on the hierarchy of CSP semantic models. Congruence of the operational semantics of $\mathcal{F}L$ over that model, however, requires further restrictions to the order $\preceq$, including that $\check{\mu}$ is maximal, enforced, for example, when model-checking with FDR.

Effectively, $\mathcal{F}L$ can look into nondeterministic choices unlike other CSP operators. This presents challenges for its implementation in FDR. It is known that $\mathcal{F}L$ cannot be specified via combinators, as used in FDR, and instead an extended theory of combinators is required [8]. Thus, $\mathcal{F}L$ is actually implemented in FDR as a function that operates over an LTS, rather than via combinator semantics. A practical strategy for implementing $\mathcal{F}L$ in FDR, for example, may require some form of bisimulation, or search, to examine refusals reachable by sequences of $\tau$ transitions from each state being prioritised in the LTS.

A denotational definition for $\mathcal{F}L$ enables algebraic properties to be explored. It is in our plans to propose laws in support of a refinement approach to verification of robotic simulations [1], where prioritisation is used as part of capturing the assumptions routinely made by roboticists in simulations.

Finally, the semantic model $\check{\mu}$-tick is somewhere between $\mathcal{F}$ and $\mathcal{RT}$ in terms of expressivity, as indicated by the way refusals are recorded. Our results are likely to be enlightening in the calculation of a counterpart of $\mathcal{F}L$ for $\mathcal{F}$ as well.

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