Entropy, Inequality and Statistical Complexity for the Modified Pöschl-Teller Potential in Non-Extensive Formalism

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Abstract. In this study we have obtained non-extensive entropies and have showed that these entropies satisfy the non-extensive uncertainty relation for the wave function corresponding to the modified Pöschl-Teller potential. We have also computed statistical complexity depending on entropic index $q$ for the same wave function and we have showed that complexity decreases by increasing of $q$.

1. Introduction

According to the Heisenberg inequality any two non-commuting conjugate observables cannot be simultaneously measured with arbitrary precision [1, 2]. Considering two Hermitian operators $\hat{x}$ and $\hat{p}_x$ corresponding to the observables $x$ and $p_x$ in a finite $N$-dimensional Hilbert space, the uncertainty principle in the Robertson form [3] reads $\Delta \hat{x} \Delta \hat{p}_x \geq \frac{1}{2} |\langle \psi | [\hat{x}, \hat{p}_x] |\psi \rangle|$ where $\Delta \hat{x}$ and $\Delta \hat{p}_x$ denote the standard deviation of the distributions. These deviations can be obtained from given wave functions. However, this form of the uncertainty relation faces several main obstacles in practice. First one is due to the state dependence of the right hand side of the above equation, since this prevents one from obtaining a fixed lower bound [4, 5, 6]. The second obstacle is coming from the using of the standard deviations on its left hand side, because the standard deviation relies heavily on the second term, and it cannot be used when the second term diverges [7]. To overcome the mentioned difficulties, it was proposed to that entropic uncertainty relations are based on information entropies that are more appropriate uncertainty measures such as Shannon, Fisher, Reyni and Tsallis. For example, Bialynicki-Birula and Mycielski (BBM) inequality [8] is given in terms of Shannon information entropies as $S_q + S_p \geq N(1 + \ln \pi)$ where $S_q = - \int |\psi(q)|^2 \ln(|\psi(q)|^2 \, dq$ is the position space information entropy, $S_p = - \int |\psi(p)|^2 \ln(|\psi(p)|^2 \, dp$ is the momentum space information entropy, and $N$ is dimension of the space. The entropic uncertainty relations derived in terms of the Shannon entropy remove the first aforementioned obstacle, namely, the dependence on the state when the uncertainty is assessed. However, the second obstacle is remaining of the divergent second moment. The resolution of both obstacles is possible by using the non-additive Tsallis information entropy as shown in Ref. [9].
In this study we discuss non-extensive uncertainty inequality and complexity for the modified Pöschl-Teller (MPT) potential in non-extensive formalism. We have obtained Tsallis entropies for the MPT potential and have showed that these entropies satisfy non-extensive uncertainty inequality. We also discuss complexity for this potential.

2. Entropy, Inequality and Statistical Complexity

2.1. Non-extensive Entropy

A generalized non-extensive form of entropy was introduced in 1988 by Tsallis [10]

\[
S_q = \frac{1}{q - 1} \left( 1 - \sum_{i=1}^{W} p_i^q \right), \quad \sum_{i=1}^{W} p_i = 1, \quad k > 0
\]

where \( q \) is a real parameter sometimes called simply entropic index. This expression is reduced to Boltzmann-Gibbs prescription in the limit of \( q \to 1 \). The expression also provides a link between the microscopic dynamics and macroscopic thermodynamics. The cases \( q < 1, q > 1 \) and \( q = 1 \) correspond to superadditivity, subadditivity and additivity, respectively [10]. In the literature there are some researches [11]-[12] about obtaining insight onto the mathematical and physical meaning of \( q \) parameter and its associated non-extensive formalism. Also there are some researches that calculated the value of the entropic index \( q \) which is the measure of complexity [13].

2.2. Entropic Inequality

The probability densities in \( X \)-space are \( |\psi(r)|^2 \) and that in the \( P \)-space (momentum) is \( |\tilde{\psi}(k)|^2 \), both of which are normalized to unity. The corresponding Tsallis entropies for continuous form are defined as [14, 15]

\[
S_p(X) = \frac{1}{p - 1} \left( 1 - \int d^N r |\psi(r)|^{2p} \right) \quad S_q(P) = \frac{1}{q - 1} \left( 1 - \int d^N k |\tilde{\psi}(k)|^{2q} \right)
\]

with condition \( 1/p + 1/q = 2 \). The inequality for non-extensive entropies is given by (See: Ref. [15])

\[
[1 + (1 - q) S_q(P)]^{1/2q} \leq \left( \frac{\pi}{q} \right)^{N/4q} \left( \frac{\pi}{p} \right)^{-N/4p} [1 + (1 - p) S_p(X)]^{1/2p}
\]

in the limit of \( p \) and \( q \) going to unity, Eq. 3 reduce to BBM inequality.

2.3. Statistical Complexity

A simple measure for statistical complexity was proposed by LMC (Lopez-Ruiz-Mancini-Calbet) [16]

\[
C^{LMC} = S.D
\]

where \( S \) is the information entropy and \( D \) is the disequilibrium can be taken as some kind of distance to an equiprobability. In non-extensive formalism [13], LMC complexity can be written as

\[
C_q(\rho) = S_q(P).D(\rho(r))
\]

where \( S_q(P) \) is the Tsallis entropy and \( D \) is the disequilibrium given in a simple form \( D = \int \rho^2(r)dr \) where \( \rho \) is the probability density [13, 17].
2.4. MPT Potential

The MPT is given by \( V(r) = -\frac{V_0}{\cosh^2(\alpha r)} \) where \( \alpha \) is a parameter and \( V_0 \) is the potential depth. An approximate solution of Schrödinger equation in the spherical coordinates for MPT potential is given by Agboola [18] as

\[
\Psi_{nlm}(r, \Omega) = r^{N-1} R_{nl}(r) Y_{lm}^m(\Omega) \tag{6}
\]

\( Y_{lm}^m(\Omega) \) is the spherical part and \( R_{nl} \) is the radial part of the wave function. The radial part of the wave function is

\[
R_{nl}(s) = C_{n_r} s^{2\nu}(1-s)^{\frac{1}{2}} P_{n_r}^{\nu,\epsilon}(1-2s) \tag{7}
\]

where \( P_{n_r}^{\alpha,\epsilon}(s) \) is Jacobi polynomials [19]

\[
P_{n}^{\alpha,b}(s) = \frac{(-1)^n}{n!2^n(1-s)^{\alpha}(1+s)^{b}} \frac{d^n}{ds^n}((1-s)^{\alpha+n}(1+s)^{b+n}) \quad \text{with} \quad s = \tanh^2(\alpha r) \tag{8}
\]

and \( C_{n_r} \) is the normalization constant

\[
C_{n_r} = \sum_{k=0}^{n_r} \frac{(-n_r)_k (\nu + \epsilon + n_r + 1)_k (4\nu + \frac{1}{2})_k}{(\epsilon + 4\nu + \frac{1}{2})_k (\nu + 1)_k k!} \times \sum_{j=0}^{n_r} \frac{(-n_r)_j (\nu + \epsilon + n_r + 1)_j (4\nu + k + \frac{1}{2})_j}{(\epsilon + 4\nu + k + \frac{1}{2})_j (\nu + 1)_j j!} = \frac{\alpha}{B\left(4\nu + \frac{1}{2}, \epsilon\right)} \tag{9}
\]

where \( \nu, \alpha, \epsilon \) are constants of wave function, \((...)_{k,j}\) are Poschammer functions and \( n_r = 0, 1, 2, \ldots \) for the ground state and excited states.

3. Numerical Results

We deal with only radial part of wave function (6) and we set \( x = r \). In order to see the form of radial wave function for MPT potential, the probability densities in position and momentum spaces (i.e., \( |\psi(x)|^2 \) and \( |\tilde{\psi}(k)|^2 \) ) are given in Figure 1 for different parameters.

![Figure 1: The probability density in position and momentum space (red: \( n = 0, \alpha = 1 \), blue: \( n = 1, \alpha = 2 \), green: \( n = 2, \alpha = 3 \)).](image-url)
As it can be seen from Figure 1 (a) that position-space probability density of the wave function has non-Gaussian form. This indicates that the wave function has also non-Gaussian form. Therefore to cope with main obstacles mentioned above we can prefer to use a suitable entropic uncertainty relation instead of Heisenberg one. In Figure 2, non-extensive entropies \( S_p(X) \) and \( S_q(P) \) for ground state \( n = 0 \) (\( \alpha = \nu = \epsilon = 1 \)) have been plotted versus \( p \) and \( q \), respectively. As it can be seen clearly from Figure 2 (a) and (b) for small \( p \) and \( q \) values, non-extensive entropies take maximum values and they decrease by increasing \( p \) and \( q \) values.

In the limit of \( p,q \to 1 \) Tsallis entropies reduce to Shannon entropies. For \( p < 1 \) and \( q < 1 \), these potentials satisfy non-extensive uncertainty inequality given Eq. 3. In Table 1, the results are given for different values of \( p \) and \( q \) with condition \( 1/p + 1/q = 2 \) [15].

Table 1: The table for non-extensive inequality for the Modified Posch Teller Potential

| \( p \) | \( q \) | \( \left[ 1+(1-p)S_p(X) \right]^{1/2p} \) | \( \left[ 1+(1-q)S_q(P) \right]^{1/2q} \) | \( N/4q \) | \( \pi/2p \) | \( \left( \pi/2p \right)^{N/4p} \) |
|-------|-------|----------------|----------------|----------|---------|---------------|
| \( 3/4 \) | \( 3/2 \) | 1.42513 | \( \geq \) | 0.164791 |
| \( 5/6 \) | \( 5/4 \) | 1.24473 | \( \geq \) | 0.169184 |
| \( 8/7 \) | \( 6/7 \) | 1.7092 | \( \geq \) | 0.876745 |

Furthermore, we have obtained statistical complexity for radial wave function in Eq. (7) by using statistical complexity measure of \( C_q \) in Eq. (5). Non-extensive statistical complexity \( C_q \) for first excited state is given in Figure 3. As it can be seen in the Figure 3 that the non-extensive complexity decreases by increasing of the \( q \) values. For \( q = 0 \) complexity has maximum value, however, in the limit \( q \to 1 \), complexity goes to zero. One can see that Tsallis’s \( q \) parameter plays important role on the complexity since equiprobable states in the system and information entropies change depending on the \( q \) values.

4. Conclusion
In this study we briefly introduce entropy, uncertainty inequality and statistical complexity in the non-extensive formalism. Later we have obtained the non-extensive entropies for the wave function corresponding to MPT potential and have showed that these entropies satisfy the
Figure 3: Complexity as a function of $q$ for the first excited state ($n = 1$), ($0 < q < 1$)

non-extensive uncertainty relation. We have also computed statistical complexity depending on Tsallis’s index $q$ for the same wave function and we have showed that complexity decreases by increasing of $q$. We conclude that using of the non-extensive entropies and uncertainty for non-Gaussian wave function such as MPT wave function is appropriate to overcome main obstacles mentioned in the introduction.

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References
[1] Heisenberg, W., ”Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik”, Zeitschrift fur Physik, 43, 3, 172 (1927).
[2] Kennard, E. H., ”Zur Quantenmechanik einfacher Bewegungstypen”, Zeitschrift fr Physik, 44, 4, 326 (1927).
[3] Robertson, H. P., ”The uncertainty principle”, Phys. Rev. 34, 163 (1929).
[4] Maassen, H. and Uffink, J. B. M., ”Generalized entropic uncertainty relations” Phys. Rev. Lett. 60, 1103 (1988).
[5] Deutsch, D., Phys. Rev. Lett.”Uncertainty in Quantum Measurements” 50, 631 (1983).
[6] Uffink, J. B. M. and Hilgevoord,”Uncertainty principle and uncertainty relations” J., Found. Phys. 15 925 (1985).
[7] Lillo, F. and Mantegna, R. N., ”Anomalous spreading of power-law quantum wave packets” Phys. Rev. Lett. 84, 1061 (2000).
[8] Bialynicki-Birula, I., Mycielski, J., ”Uncertainty relations for information entropy in wave mechanics”, Comm. Math. Phys., 44, 2, 129 (1975).
[9] Abe, S., Martinez, S., Pennini, F. and Plastino, A., ”Nonextensive thermodynamic relations” Phys. Lett. A 295, 74 (2002).
[10] Tsallis, C., ”Possible generalization of Boltzmann-Gibbs statistics”, Journal of Statistical Physics, 52, 479487 (1988).
[11] Tsallis, C., ”I. Nonextensive Statistical Mechanics and Thermodynamics: Historical Background and Present Status”, Springer, Nonextensive Statistical Mechanics and Its Applications, 560, 3-98 (2001).
[12] Zelenyi, L. M., Milovanov, A. V., ”Fractal Properties of Sunspots”, Soviet Astronomy Letters, 17, 425 (1991).
[13] Abe, S., Landsberg, P. T., Plastino, A. R., and Yamano T., ”Nonadditive statistical measure of complexity and values of the entropic index $q$”, arXiv:cond-mat/0402217v1 (2004).
[14] C. Tsallis, in: New Trends in Magnetism, Magnetic Materials and their Applications, eds. J.L.Morán-Lopez and J.M. Sánchez (Plenum Press, New York, p.451, 1994).
[15] Rajagopal, A. K., ”The Sobolev inequality and the Tsallis entropic uncertainty relation”, Physics Letters A, 205, 1, 32 (1995).
[16] Lopez-Ruiz, R., Mancini, H., Calbet, X., ”A statistical measure of complexity”, Phys. Lett. A, 209, 5-6, p. 321-326 (1995).
[17] Yamano, T., "A statistical complexity measure with nonextensive entropy and quasi-multiplicativity", J. Math. Phys. 45, 1974 (2004).

[18] Agboola, D., "Solutions to the Modified Pöschl-Teller Potential in D-dimensions", Chin. Phys. Lett., 27, 4, 040301 (2010).

[19] Askey, R. and James, A. W., "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials", American Mathematical Soc., 319 (1985).