The acoustic radiation force: a gravitation-like field.

Pierre-Yves Gires\textsuperscript{1}, Jerome Duplat\textsuperscript{2}, Aurélien Drezet\textsuperscript{3} & Cedric Poulain\textsuperscript{1,3}
\textsuperscript{(1)} Univ. Grenoble Alpes, CEA, LETI, F-38000 Grenoble, France
\textsuperscript{(2)} Univ. Grenoble Alpes, INAC-SBT, F-38000 Grenoble, France and
\textsuperscript{(3)} Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Neel, F-38000 Grenoble, France

In this letter, we propose an expression for the instantaneous acoustic radiation force acting on a compressible sphere when it is immersed in a sound field with a wavelength much larger than the particle size (Rayleigh scattering regime). We show that the leading term of the radiation force can alternatively be expressed as the time average of a fluctuating gravitation-like force. In other words, the effect of the acoustic pressure gradient is to generate a local acceleration field encompassing the sphere, which gives rise to an apparent buoyant force, making the object move in the incoming field. When averaging over time, we recover the celebrated Gor’kov expression and emphasize that two terms appear, one local and one convective, which identify to the well-known monopolar and dipolar contributions.

Since Rayleigh’s pioneering work on sound waves \cite{1}, later followed by Langevin and Brillouin, among others \cite{2–4}, it is known that like its electromagnetic cousin, acoustic waves can transfer linear momentum to a particle, even in a perfect fluid. The mean effect is referred to as the radiation or pressure force, with its associated tension or radiation stress tensor \cite{4}. Upon reflection, one might wonder how an elemental sound wave, with its harmonic pressure-velocity oscillation (in time and space), could exert a non-zero average force. Like with electromagnetic waves, the answer lies in the second order effect arising for sound waves: the particle pulsates in volume while moving back and forth in the acoustic oscillating flow, yielding a small hysteretic displacement. These incremental displacements accumulate over millions of cycles per second (at the sound frequency), and may be interpreted as the result of an average or macroscopic force on the particle.

Such an average force was first calculated by King for a hard sphere in a perfect fluid \cite{5}, and generalized by Yosioka \cite{6} for compressible objects. It still remains an active subject of research with increasingly heavy mathematics addressing complex objects, wave fields, and more realistic effects \cite{7,8}. Likewise, impressive applications of the radiation force to microfluidics have been published in the last ten years. This has given rise to the new discipline of acoustofluidics \cite{9}, in which the ability of standing waves to arrange, trap and sort living cells has been demonstrated both in propagative \cite{10} and evanescent fields \cite{11}.

Despite this great success in applications, very few papers address fundamental questions such as the effect of added mass or compressibility at short time when a sphere is accelerated by the sound field and becomes non-buoyant in the fluid.

In order to gain insight into the physics at play, here we will focus on the instantaneous (i.e., not averaged) dynamics of the particle in the incoming field. By following a Lagrangian approach and comparing with the celebrated Gor’kov expression of the radiation force \cite{12}, we will show that the radiation force can also be interpreted as the average of a fluctuating buoyant force, shedding new light on its physical origin.

Let us consider a non-moving, infinite and non viscous compressible fluid in which a compressible sphere of radius $a$ is immersed, in the absence of gravity. The whole system is excited by a time-harmonic acoustic wave characterized by its incident Eulerian velocity, pressure and density fields, respectively ($\mathbf{v}_\text{in}$, $p_\text{in}$, $\rho_\text{in}$). The acoustic wavelength $\lambda_f$ in the fluid is supposed much larger than the sphere radius (Rayleigh regime): $a \ll \lambda_f$. Assuming that after a certain time a steady state can be reached \cite{13}, we define the radiation force $\mathbf{F}_{\text{rad}}(\mathbf{r})$ as the time-averaged force exerted on any particle positioned at position $\mathbf{r}$.

$$\mathbf{F}_{\text{rad}}(\mathbf{r}) = \langle \mathbf{F}(\mathbf{r}_p(t)) \rangle,$$

where $\mathbf{F}(\mathbf{r}_p(t))$ is the instantaneous force exerted upon the small particle when the medium is insonified and $\mathbf{r}_p(t)$ the position of the particle at time $t$.

In an inviscid fluid, only pressure forces can apply on the particle, so that

$$\mathbf{F}(\mathbf{r}_p(t)) = \int_{S_p(t)} p \mathbf{dS},$$

$p$ being the total pressure field (incident and scattered), the normal being oriented towards the particle surface $S_p(t)$. For concision, $\mathbf{F}(\mathbf{r}_p(t))$ will be hereafter denoted $\mathbf{F}(t)$, keeping in mind that it refers to the moving particle located at position $\mathbf{r}_p(t)$. In general, the instantaneous force $\mathbf{F}(t)$ is not known, only its averaged value.

By using an asymptotic approach detailed below, we are going to derive the expression for an equivalent instantaneous radiation force $\mathbf{F}_{\text{rad}}(t)$, so that at the leading order in the Mach number, $\mathbf{F}(t) = \mathbf{F}_{\text{rad}}(t) + \mathbf{f}(t)$, with $\mathbf{f}$ a zero-mean function.

The (small) acoustic Mach number $\varepsilon$ is: $\varepsilon = \frac{v_m}{c_{f_0}}$, $c_{f_0} = 1/\sqrt{\gamma_f0\kappa_f0}$ being the fluid sound velocity, with respectively $\gamma_f0$ and $\kappa_f0$ its equilibrium density and compressibility, while $v_{m,c}$ is the characteristic fluid velocity.
The subscripts $p$ and $f$ refer respectively to the particle and the fluid, while the 0 subscript denotes quantities at rest, i.e., in the absence of acoustic perturbation. Following Xie et al. [14], we also describe the scaling of the particle size with an exponent $\alpha$, so that the particle radius $a$ over the wavelength $\lambda$ is such that

$$
\frac{a}{\lambda} = O(\varepsilon^\alpha).
$$

(3)

In this framework, a well known approach for estimating the mean radiation force $\bar{F}_\text{rad}$ was undertaken by Gor’kov [12] and was subsequently detailed by Bruus [15, 10]. It consists in transforming the integral over the time varying surface particle $S_p(t)$ into an integral over a fixed remote surface where the far-field approximation allows to find the leading term.

We begin by recalling the Gor’kov approach, on which our analysis is partly based. Some implicit assumptions of the theory are discussed in the Supplemental Material (SM) 1 [17]. Introducing $\Pi$ the total momentum density flux tensor defined as $\Pi_{ij} = \rho \delta_{ij} + \rho v_i v_j$, a momentum balance shows that the radiation force can be approximated by the mean momentum flux through any fixed surrounding surface $S$

$$
\bar{F}_\text{rad} \simeq -\int_S (\Pi) \cdot dS,
$$

(4)

with $dS$ oriented outward the surface $S$. Here, as done by Gor’kov and [12], the additional contribution $\left( \frac{4}{c^3} \int V(t) \rho \partial V dV \right)$ associated with the rate of linear momentum of the fluid located between the surfaces $S$ and $S_p(t)$ has been omitted (see SM 1 [17] for a discussion). Then, choosing $S$ in the far field region, recognizing that the leading term in $\langle \Pi \rangle$ only depends on the particle monopolar and dipolar contributions, Gor’kov shows that for a standing incident field the leading term of $\bar{F}_\text{rad}$ is $p c^2 a^2 O(\varepsilon^{2+\alpha})$ and derives from the acoustic potential $U(r)$ so that

$$
\bar{F}_\text{rad} \simeq -\nabla U(r),
$$

(5)

with

$$
U(r) = V_{p0} \left( \frac{f_1}{2} \kappa_{f0} \rho_p^2 \langle \rho_p^2 \rangle - \frac{3f_2}{4} \rho_{f0} \langle \rho_f^2 \rangle \right)
$$

(6)

$$
f_1 = 1 - \hat{\kappa},
$$

(7)

$$
f_2 = \frac{2(\hat{\varrho} - 1)}{\hat{\varrho} + 1},
$$

(8)

($\hat{\kappa} = \frac{\kappa_{p0}}{\kappa_{f0}}, \hat{\varrho} = \frac{\varrho_{p0}}{\varrho_{f0}}$) being respectively the equilibrium compressibility and density ratios of the particle over the fluid, while $V_{p0}$ is the particle rest volume.

With this in mind let us now go back to the definition of the scattered pressure field $p_s$ as the correction of the incident field required to account for the presence of the particle:

$$
p = p_{in} + p_s.
$$

(9)

From Eqs. 11 and 12 it is thus always possible to calculate $\bar{F}_\text{rad}$ as $\bar{F}_\text{rad}(r) = \langle \int_{S_{p,in}(t)} p_{in} \partial S \rangle$.

The subscripts $p$ and $f$ refer respectively to the particle and the fluid, while the 0 subscript denotes quantities at rest, i.e., in the absence of acoustic perturbation. Following Xie et al. [14], we also describe the scaling of the particle size with an exponent $\alpha$, so that the particle radius $a$ over the wavelength $\lambda$ is such that

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$$
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$$

(3)
We identify both a local and convective contributions, respectively denoted \( \mathbf{F}_{a}^{\text{loc}}(t) \) and \( \mathbf{F}_{a}^{\text{conv}}(t) \), so that

\[
\mathbf{F}_{a}(t) \approx \mathbf{F}_{a}^{\text{loc}}(t) + \mathbf{F}_{a}^{\text{conv}}(t),
\]

with

\[
\mathbf{F}_{a}^{\text{loc}}(t) = \tilde{\mathbf{F}}_{a}(\mathbf{r}, t),
\]

and

\[
\mathbf{F}_{a}^{\text{conv}}(t) = \int_{0}^{t} \mathbf{v}_{p}(t')dt' \cdot \nabla \tilde{\mathbf{F}}_{a}(\mathbf{r}, t).
\]

The local force can be expressed, keeping terms up to order \( O(\varepsilon^{2+\alpha}) \), as

\[
\mathbf{F}_{a}^{\text{loc}}(t) = -V_{f0} \left[ -3 \frac{f_{2}}{2} f_{1} \nabla p_{in} + \kappa f_{0} \frac{f_{1}}{2} \nabla p_{in}^{2} \right] (\mathbf{r}, t),
\]

where use of Euler’s equation has been made as detailed in SM 2.

Averaging the above expression over time then yields

\[
\langle \mathbf{F}_{a}^{\text{loc}}(\mathbf{r}_{p}(t)) \rangle = -\nabla \left[ V_{f0} \kappa f_{0} \frac{f_{1}}{2} (\tilde{p}_{in}^{2}) \right].
\]

It is noteworthy that the term in brackets corresponds to the first term (monopolar contribution) in the Gor’kov force potential expressed in Eq. 6.

In order to obtain the mean convective term contribution, we start expressing the first order particle velocity \( \mathbf{v}_{p}(t) \) as a function of the incident acoustic velocity field \( \mathbf{v}_{in} \). The particle being accelerated in the incident sound field of velocity \( \mathbf{v}_{in}(\mathbf{r}, t) \), the surrounding fluid inertia leads to an added mass effect (see [18]), which first appears at the order \( O(\varepsilon) \) of this letter. At this order, Batchelor [19] shows that \( \mathbf{v}_{p} \) can be expressed as

\[
\dot{\mathbf{r}}_{p}(t) = \mathbf{v}_{p}(t) = (1 - f_{2}) \mathbf{v}_{in}(\mathbf{r}_{p}(t)).
\]

Now, inserting expressions [18] and [20] in Eq. 17 we get

\[
\mathbf{F}_{a}^{\text{conv}}(t) = V_{f0} \frac{3}{2} f_{2} \int_{0}^{t} \mathbf{v}_{in}(\mathbf{r}, t') dt' \cdot \nabla (\nabla p_{in}(\mathbf{r}, t)) .
\]

After some calculations detailed in SM 3, involving the linearized Euler equation and an integration by parts, it comes

\[
\langle \mathbf{F}_{a}^{\text{conv}}(\mathbf{r}_{p}(t)) \rangle = -\nabla \left[ -V_{f0} \frac{3 f_{2}}{4} \tilde{\varrho}_{f0} (\tilde{p}_{in}^{2}) \right] .
\]

Likewise, the term in brackets is the dipolar contribution of the Gor’kov potential expressed in Eq. 6.

Combining the mean local and convective contributions given in Eqs. 19 and 22 we obtain, at the leading order,

\[
\mathbf{F}_{\text{rad}} \simeq \left\langle \mathbf{F}_{a}(\mathbf{r}_{p}(t)) \right\rangle.
\]

\( \mathbf{F}_{a}(t) \) can thus be identified to the instantaneous radiation force \( \mathbf{F}_{\text{rad}}(t) \) defined in Eq. 5 in complete agreement with Gor’kov’s results. This Lagrangian derivation of the radiation force constitutes the first main finding of this letter. The second important result, that we will now discuss, concerns the physical interpretation of this radiation force.

For this purpose we introduce an effective gravitation field \( \mathbf{g}_{in} = \nabla \tilde{p}_{in} \) and a relative density \( \Delta \varrho = \varrho_{p} - \varrho_{f} \) in Eq. 12 and Eq. 23 which leads to

\[
\mathbf{F}_{\text{rad}} = \langle \Delta \varrho \mathbf{V}_{p} \mathbf{g}_{in} \rangle.
\]

The radiation force can thus be understood as the leading term in the mean force which would result from a gravitational field modulation equal to \( \mathbf{g}_{in}(t) \). In other words, \( \mathbf{F}_{a}(t) \) can be seen as a fluctuating apparent weight, resulting from the combination of two oscillating quantities:

- a forcing effect: an incident acoustic gravity-like acceleration field \( \mathbf{g}_{in}(\mathbf{r}, t) \)
- a response effect: owing to their compressibility, both the particle volume and the fluid densities oscillate at the forcing frequency, rendered by the relative density term \( \Delta \varrho \mathbf{V}_{p} = \frac{\Delta \varrho}{\varrho_{f}} \mathbf{m}_{p} \) (\( \mathbf{m}_{p} \): constant particle mass)

By analogy with the buoyancy force (i.e. Archimede’s law) arising when a particle has a density or compressibility different from the fluid in which it is immersed, the oscillating force \( \mathbf{F}_{a}(t) \) is equivalent to a rapidly fluctuating ‘acoustic gravitational force’. As we have shown, the time-average of this force, taking both the temporal and spatial structure of the field into account, leads to the classic radiation force expressed by Gor’kov for standing waves. Our conclusion is also in agreement with Gor’kov’s work for progressive wave for which the radiation force is expected to be zero at this order as well.

We will now try to give a comprehensive picture of the physics at play in the gravitation-like force, considering a particle in a plane standing wave \( p_{in} = -p_{0} \cos(\omega t) \cos(kx) \), with \( k = 2\pi/\lambda \) and \( \omega = kc_{f0} \) respectively the wavenumber and angular frequency of the wave. As explained above, the acoustic gravitational effect results from two contributions: a local one, associated with the oscillation of the particle apparent mass \( m_{p}(1 - \frac{\varrho_{f}}{\varrho_{p}}) \) in the acoustic gravitation field, and a convective one, linked to the local exploration of the field by the oscillating particle. Let us now separate both contributions by considering two limit cases for a particle initially located between a pressure antinode (at \( x = 0 \)) and the nearest node in the \( x > 0 \) region (see Fig. 1).

**Case a: a neutrally buoyant but compressible particle**
We first consider a particle both neutrally buoyant (\( \varrho_{p0} = \varrho_{f0} \)) and more compressible than the fluid (\( \kappa_{p0} > \kappa_{f0} \), i.e.
$f_1 < 0$. In this case, only the local contribution remains as the convective term vanishes ($f_2 = 0$). Let us figure out the particle movement over a time period $T$. The sinusoidal green dashed line on Fig. 1a) represents the movement of a fluid particle in the sound wave, which is also, at leading order, the particle movement since $f_2 = 0$ so that $v_p = (1 - f_2)v_{in} = v_{in}$ (no added mass effect comes into play in this case).

At time $t = 0$, the pressure gradient is such that $g_{in}$ is oriented downward [20]. The particle is compressed and hence denser than the hosting fluid at its location, so that it plunges downwards. At $t = T/4$, the pressure is zero, the particle thus following the non perturbed trajectory (no radiation force). At $t = T/2$, the pressure gravity reverses but the particle also expands, so that it is ‘lighter’ and thus keeps sinking. At $t = 3T/4$ it follows the same trajectory (no force). Overall, the acoustic gravity is always out of phase with the density shift: for the whole period, the particle keeps ‘falling’ towards the pressure antinode at $x = 0$ and the instantaneous radiation force maintains the same orientation.

Case b: an iso-compressible particle, denser than the fluid. We now consider a particle iso-compressible ($f_1 = 0$) but denser than the hosting fluid ($f_2 > 0$), also located between $x = 0$ and $x = \pi/k$, as shown on Fig. 1b). To the green fluid particle trajectory is now added the flattened magenta dashed line, which represents the trajectory the particle would follow if only added-mass effect would apply. First, the particle dynamic is such that it plunges between $t = 0$ and $t = T/4$ since it is denser than the fluid in a downward gravity field. However, at $t = T/2$, the gravity field reverses so that the particle rises. Here, the point is that the gravity field (or acoustic pressure gradient) being greater at the location where the particle is at $T/2$ than at $t = 0$, the radiation force over the cycle does not balance symmetrically. On average, a net upward radiation force (toward the nearest node) is exerted upon the particle.

It is noteworthy that in this case, the effect of the added mass is to alter (with a factor $(1 - f_2)$) the particle trajectory obtained by integrating $v_p = (1 - f_2)v_{in}$. In other words, the ‘landscape’ explored by the particle determines the hysteretic values of the gravity field that the particle will encounter at extremal positions and henceforth contributes to its amplitude. In the convective (or ‘landscape’) effect, we note that the radiation force reverses within a cycle while it is not the case in the local one, where it maintains the same orientation.

In conclusion, the time-resolved Lagrangian approach is another way to illustrate both the time-averaged scattered contributions (monopolar and dipolar scattering terms) appearing in the Gor’kov potential of the averaged radiation force. Eq. 24 gives a clear interpretation (see Fig. 1) of the acoustically induced buoyancy effect, from which both terms are derived: (i) a local term, originating from the compressibility ratio, a neutrally buoyant particle alternatively sinking and floating within the time varying acoustic field, and (ii) a convective one, a denser but iso-compressible particle encountering a stronger instantaneous force as it approaches pressure nodes where the pressure gradient is the largest. Overall, it shows that the radiation force has the structure of an inertial force in an equivalent gravity acceleration field created by the acoustic pressure gradients in the fluid. Therefore, it is possible to consider the acoustic radiation force as a linear centrifugal force. This idea is indeed reinforced by
the ability of the acoustic radiation force to sort particles according to their size, density and compressibility ratios (acoustophoresis [9]), which is reminiscent of centrifugal forces. Correlatively, our findings may also shed new light on the ability of the radiation force to universally deform interfaces [21], or to separate miscible fluids of different densities, as recently evidenced [22, 23]. Beyond acoustics, we wonder how this approach could be transposed to other wave natures for which a radiation force also arises. We hope this paper will stimulate future work in this direction.

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Supplenental Material

SM1. MEAN MOMENTUM BALANCE AND THE GOR’KOV APPROXIMATION

In this section, we are going to discuss the applicability of the Gor’kov formulation (see [1]) in the context of a free-to-move (not suspended) particle. It is to our knowledge an issue which has never been discussed in details in the literature so far although it is mostly employed in this context.

Let us consider a compressible particle of varying outer surface $S_p(t)$ immersed in a sound field, and define a much larger surface $S$ in the far-field region. $V(t)$ is the control volume delimited externally by $S$ and internally by $S_p(t)$, as sketched on Fig. S1.

In the general case, the momentum change rate of the fluid volume $V$ writes

$$
\frac{d}{dt}\int_{V(t)} \rho \mathbf{v} dV = - \int_S \Pi \cdot dS - \int_{S_p} \rho dS. \quad (S1)
$$

Averaging Eq. S1 over time, we then get

$$
\bar{F}_{\text{rad}} = -\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV \right \} \int_S \Pi \cdot dS \quad (S2)
$$

$$
= - \int_S \Pi \cdot dS - \frac{P(T) - P(0)}{T}, \quad (S3)
$$

with

$$
P(t) = \int_{V(t)} (\rho \mathbf{v})(t) dV. \quad (S4)
$$

In order to simplify the problem, let us first assume that the movement of the particle mass center in the sound flow is perfectly periodic in space and time so that the particle mean displacement is exactly zero. This can be achieved by means of an additional external and time independent force hereafter denoted $\mathbf{F}^*$. Hence, we have

$$
\mathbf{F}^* + \int_{S_p(t)} \rho dS = m_p \Gamma(t), \quad (S5)
$$

with $\Gamma$ and $m_p$ respectively the particle acceleration and mass.

By averaging over time, it is clear that the force $\mathbf{F}^*$ must exactly compensate the mean radiation force $\bar{F}_{\text{rad}} = \int_{S_p} \rho \mathbf{d}S$ to ensure a zero-mean displacement (or acceleration).

Thanks to the additional force, the movement is perfectly periodic in time, so that the term $\langle \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV \rangle$ cancels. With this assumption of a suspended particle, we recover the expression for the radiation force as used by Gor’kov (first equation of [1]), also in agreement with the assumption of perfect stationarity in the more detailed paper of Settnes and Bruus (see equations A1a to A1f of ref. [2]).

Now, in the more general case of a particle free to move (i.e. when $\mathbf{F}^* = 0$) the particle movement is no longer exactly periodic in time so that a tiny incremental displacement $\delta(t)$ accumulate at every sound cycle. Consequently, the fluid momentum is no longer periodic and the momentum term $\langle \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} dV \rangle$ is a priori not exactly zero and the total radiation force differs from the Gor’kov expression. Let us estimate the order of magnitude of $\delta$ and subsequently of the associated correction onto the force.

If we denote by $r_p(t)$ and $r_p^*(t)$ the particle displacement respectively for a non suspended and a suspended particle (in the presence of $\mathbf{F}^*$), we can write:

$$
r_p(t) = r_p^*(t) + \delta(t). \quad (S6)
$$

Let us first estimate $\delta$. When pushed by the radiation force, the force acting on the particle is of the order $\rho c^2 a^2 O(\varepsilon^2 + \alpha)$. At large times (quasi stationary regime), viscosity can no longer be neglected so that the particle will reach a drift velocity of the order $F_{\text{rad}}/(\mu a) \sim cO(\varepsilon^{2+\alpha})$ (Stokes drag). Therefore, the difference $\delta$ between a suspended particle and a free one is of the order $\lambda O(\varepsilon^{2+\alpha})$. The force on the particle writes (following the notations introduced in the manuscript)

$$
\mathbf{F}(r_p(t)) = \mathbf{F}(r_p^*(t) + \delta(t)) = \mathbf{F}_{\text{loc}} + \nabla(\bar{\mathbf{F}}_a - (r_p^* + \delta)) \quad (S7)
$$

In terms of forces, the difference between the zero-mean displacement particle and the free-to-move particle yields a correction $\delta \mathbf{F}$:

$$
\delta \mathbf{F} = \nabla(\bar{\mathbf{F}}_a) \delta \quad (S8)
$$

which is of higher order.

In other words, the estimation of $\langle \mathbf{F}_a \rangle$ assuming periodicity does not significantly differ from its estimation for a free particle. Nevertheless, for a free particle, $\langle \mathbf{F}_a \rangle = -\nabla U$ does not represent the total radiation force $\bar{F}_{\text{rad}}$.

FIG. S1. Notations used for the momentum balance.
SM2. LOCAL TERM EXPANSION

In order to derive the $\varepsilon$ expansion of the local term $F^{\text{loc}}_a$, we first insert the constant particle mass $m_p = \theta_p(t) V_{p,\text{in}}(t)$ into Eq. (12) of the main paper:

$$F^{\text{loc}}_a(t) = \frac{(\theta_p - \theta_f)}{\theta_f} \nabla p_{\text{in}} V_p$$  \hspace{1cm} (S9)

$$= m_p (1 - \frac{\theta_f}{\theta_p}) \nabla p_{\text{in}} V_p. \hspace{1cm} (S10)$$

We then successively get

$$F^{\text{loc}}_a(t) = m_p \left[ 1 - \frac{\theta_f}{\theta_p} \frac{\nabla p_{\text{in}}}{\theta_f} \right] (r, t) \hspace{1cm} (S11)$$

$$= m_p \left[ 1 - \frac{\theta_f}{\theta_p} - \frac{\theta_f}{\theta_f} (\kappa_f - \kappa_p) p_{\text{in}} \right] \frac{\nabla p_{\text{in}}}{\theta_p} (r, t) \hspace{1cm} (S12)$$

$$= -V_{\theta_0} \left[ \frac{3}{2} f_2 \nabla p_{\text{in}} + \kappa_f \frac{f_1}{2} \nabla p_{\text{in}}^2 \right] (r, t), \hspace{1cm} (S13)$$

passing from Eq. S12 to S13 using $(f_1, f_2)$ definition at Eqs. (7) and (8).

Moreover, we emphasize that between Eq. S11 and Eq. S12 we used the postulate about the volume $V_{p,\text{in}}$ and $S_{p,\text{in}}$ as discussed in the main article. More precisely, writing $\theta_f \simeq \theta_f(1 + \kappa_f p_{\text{in}})$ and similarly $\theta_p \simeq \theta_p(1 + \kappa_p p_{\text{in}})$ we consider explicitly the effect of the incident pressure, i.e., omitting the effect of $p_s$.

SM3. CONVECTIVE TERM EXPANSION

Starting from Eq. (21), we first have

$$F_{\text{conv}}(t) = V_{\theta_0} \frac{3}{2} f_2 \int_0^t v_{\text{in}}(r, t') dt' \cdot \nabla (\nabla p_{\text{in}}(r, t)) \hspace{1cm} (S14)$$

$$= V_{\theta_0} \frac{3}{2} f_2 \int_0^t v_{\text{in}}(r, t') dt' \cdot \nabla \left( -\frac{\partial v_{\text{in}}}{\partial t} (r, t) \right), \hspace{1cm} (S15)$$

where we passed from Eq. S14 to S15 using the linearized Euler’s equation.

Then, with respect to the mean contribution of the convective term, we successively get

$$\langle F_{\text{conv}} \rangle = \frac{1}{T} \int_0^T F_{\text{conv}}(t) dt$$ \hspace{1cm} (S16)

$$= -V_{\theta_0} \theta_f(\kappa_f - \kappa_p) \int_0^t v_{\text{in}}(r, t') dt' \cdot \partial_t \nabla v_{\text{in}}(r, t)$$ \hspace{1cm} (S17)

$$= -V_{\theta_0} \theta_f(\kappa_f - \kappa_p) \left[ \int_0^t v_{\text{in}}(r, t') dt' \cdot \nabla v_{\text{in}}(r, t) \right]$$

$$\hspace{1cm} - \int_0^T (v_{\text{in}} \cdot \nabla v_{\text{in}})(r, t') dt' \hspace{1cm} (S18)$$

$$= V_{\theta_0} \theta_f(\kappa_f - \kappa_p) \int_0^T \left( v_{\text{in}} \cdot \nabla v_{\text{in}} \right)(r, t') dt' \hspace{1cm} (S19)$$

$$= V_{\theta_0} \theta_f(\kappa_f - \kappa_p) \int_0^T \left( \frac{1}{T} \int_0^T v_{\text{in}}^2(r, t') dt' \right)$$

$$\hspace{1cm} = V_{\theta_0} \theta_f(\kappa_f - \kappa_p) \int_0^T \left( \frac{3}{4} f_2 \nabla v_{\text{in}}^2(r) \right), \hspace{1cm} (S20)$$

where Eq. S14 transforms to S17 from the gradient and time derivative linearities. Eq. S17 to S18 after an integration by parts, Eq. S18 to S19 as the incoming wave is such that $\frac{1}{T} \int_0^T v_{\text{in}}(r, t') dt' = 0$, and finally Eq. S19 to S20 through the gradient and integration operators linearity and taking into account $\nabla \times v_{\text{in}} = 0$.

We emphasize that by applying a similar expansion as obtained for $F_a(t)$ we can also calculate

$$F_b(t) = \langle \int_{S_{p,\text{in}}(t)} p_{\text{in}} dS \rangle \hspace{1cm} (S22)$$

which after a separation $F_b(t) \simeq F^{\text{loc}}_b(t) + F^{\text{conv}}_b(t)$ similar to Eqs. 14-17 leads to

$$\langle F_b(t) \rangle \simeq V_{\theta_0} \left( \frac{\kappa_f}{2} \nabla (p_{\text{in}}^2) - \frac{1 - f_2}{2} \frac{\partial v_{\text{in}}}{\partial t} \right). \hspace{1cm} (S23)$$

This formula is important since it shows that even in the case of $f_1 = f_2 = 0$, for a standing wave, the contribution $\langle \int_{S_{p,\text{in}}(t)} p_{\text{in}} dS \rangle$ doesn’t vanish. More precisely in this regime we have

$$\langle F_b(t) \rangle \simeq V_{\theta_0} \left( \frac{\kappa_f}{2} \nabla (p_{\text{in}}^2) - \frac{1 - f_2}{2} \frac{\partial v_{\text{in}}}{\partial t} \right). \hspace{1cm} (S24)$$

where appears the second order acoustic pressure $p_{\text{in}}^{(2)} = \frac{\kappa_f}{2} p_{\text{in}}^2 - \frac{1}{2} \frac{\partial v_{\text{in}}}{\partial t} v_{\text{in}}$ in the fluid.

For a standing wave with $p_{\text{in}} = A \cos(\omega t) \cos(\omega x/c_f)$, $v_{\text{in}} = \frac{A}{c_f} \sin(\omega t) \sin(\omega x/c_f)$ we have in particular

$$\langle p_{\text{in}}^{(2)} \rangle = \frac{\kappa_f A^2}{4} \cos(2\omega x/c_f)$$

and thus

$$\langle F_b(t) \rangle \simeq -\frac{\hat{x}}{c_f} V_{\theta_0} \frac{2\omega \kappa_f A^2}{4} \sin(2\omega x/c_f). \hspace{1cm} (S25)$$
In the main text this motivated the introduction of the coefficient $\beta(t)$ in Eq. 10 and all the subsequent developments.

[1] L. Gor’kov, Soviet Physics Doklady 6, 773 (1962).
[2] M. Settnes and H. Bruus, Physical Review E 85, 016327 (2012).