On two-fermion BMN operators

Burkhard Eden

Dipartimento di Fisica, Università di Roma “Tor Vergata”
I.N.F.N. - Sezione di Roma “Tor Vergata”
Via della Ricerca Scientifica, 1
00133 Roma, ITALY
e-mail: burkhard.eden@roma2.infn.it

Abstract

We show how to determine the lowest order mixing of all scalar with two-fermion two impurity BMN operators in the antisymmetric representation of $SO(4)$. Differentiation on harmonic superspace allows one to derive two-loop anomalous dimensions of gauge invariant operators from this knowledge: the value for the second anomalous correction to the dimension is essentially the square of the two-fermion admixture. The method effectively increases the loop order by one. For low $J$ we find agreement to all orders in $N$ with results obtained upon diagonalisation of the $\mathcal{N} = 4$ dilation operator.

We give a formula for the generalised Konishi anomaly and display its role in the mixing. For $J = 2$ we resolve the mixing up to order $g^2$ in the singlet representation. The sum of the anomaly and the naive variation of the leading two-fermion admixtures to the singlets is exactly equal to the two-fermion terms in the antisymmetric descendants.
1 Introduction

The AdS/CFT correspondence [1] in its strong form has the drawback that the string side of the duality is virtually inaccessible to calculation. More recently, a special limit of the underlying geometry has been considered [2], in which the string theory becomes solvable [3]. The article [4] established a field theory dual. String states are related to composites of very many copies of a given scalar field of $\mathcal{N} = 4$ SYM with a few other elementary fields, commonly termed “impurities”.

This work addresses once again the set of operators with two impurities: we focus on the mixing between operators made out of only scalar fields and scalar operators with two-fermion impurities. In the BMN proposal the $SU(4)$ R-symmetry of the $\mathcal{N} = 4$ theory is broken to $U(1)_J \otimes SO(4)$. The chosen scalar field $Z$, say the field $\phi_1$ in the $\mathcal{N} = 1$ formulation of the theory, is charged under $U(1)_J$ but does not transform under the $SO(4)$ factor. The impurities are neutral but rotate under $SO(4)$. Two impurities may carry a singlet, an antisymmetric and a symmetric traceless representation of $SO(4)$.

We do not enter into the subtleties of the actual limit, although the study is certainly motivated by the BMN proposal. In the full $\mathcal{N} = 4$ theory with unbroken $SU(4)$ R symmetry the two impurity BMN operators are highest weights of an $[0, J, 0]$, a $[2, J, 0]$ and a $[2, J, 2]$ irrep of $SU(4)$, respectively. The operators are made out of many elementary fields, which can be arranged into traces of the associated gauge group generators in many different ways. Since all these objects have the same naive conformal dimension one has to disentangle the operator mixing, i.e. to find operators with well-defined conformal dimension in the quantum theory.

Operator mixing in $\mathcal{N} = 4$ is difficult to solve exactly even at the lowest order in the coupling constant [5, 6]. The large $N$ limit provides a natural simplification. The original work [4] gives a solution for this case. Later on much effort has been dedicated to determining subleading (in $N$) corrections to the one-loop anomalous dimensions [7, 8] and there is even a two-loop calculation [9]. Degeneracy of the anomalous dimensions of various types of operators has led to the conjecture that they belong to supermultiplets, see [10] for vector operators and a comment on two-fermion operators, and [11] for the all-scalar composites. According to the latter article the highest weight states are the singlets, the antisymmetric and symmetric operators are descendants.
For $J = 2$, this had first been observed in [5]. A part of this article is dedicated to demonstrating how the structure of the mixing problem in the $J = 2$ example generalises to the whole class of operators.

From an $\mathcal{N} = 4$ perspective we are dealing with scalar multiplets of $SU(2, 2|4)$ which carry an $SU(4)$ representation $[0, J, 0]$ and have naive scaling weight $\Delta = J + 2$. Such operators are semishort [12] in free field theory and may become long in the interacting case.

Interestingly, the two-impurity all-scalar BMN operators are the highest weights of four multiplets that are separate in free field theory ($[0, J, 0], \Delta = J + 2$; $[2, J, 0], \Delta = J + 3$ and the conjugate representation, $[2, J, 2], \Delta = J + 4$), but that merge if interactions are switched on [13]: the descendant structure [11] indeed derives from the commutator term in the supersymmetry transformations, which comes with the Yang-Mills coupling constant.

But there is a second effect, which we illustrate by an example: the two operators $\mathcal{K}_1 = (\phi_I \bar{\phi}^I)$ and the lowest component of the stress-energy tensor $Q_{20}$ are orthogonal, because they are in different representations. Under classical supersymmetry they have descendants in the same representation, namely

$$\mathcal{K}_{10} = g(Z[\phi_2, \phi_3]) \quad Q_{10} = (\lambda^\alpha \lambda_\alpha) + 4g(Z[\phi_2, \phi_3])$$

(up to scaling) which are clearly not orthogonal to order $g^2$. The explanation is that we have omitted an “anomalous” part of the supersymmetry variation of $\mathcal{K}_1$: the correct descendant is

$$\mathcal{K}_{10} = g(Z[\phi_2, \phi_3]) + \frac{g^2 N}{32\pi^2} (\lambda^\alpha \lambda_\alpha) .$$

It was shown in [14] how the missing piece of the descendant can be derived in a graph calculation. In point splitting regularisation one must insert the gauge connection between two elementary fields that are not at the same point. It is the supersymmetry variation of the connection that accounts for the two-fermion part of the descendant. The two-fermion piece became known as the “Konishi anomaly”.

In a separate paper we will present a graph calculation concerning the analogous anomaly for $[0, J, 0]$ operators with weight $\Delta = J + 2$. In BMN inspired notation the highest weight of such an operator can be a combination of

$$\mathcal{O}_I = Z^J \phi_\alpha \bar{\phi}^\alpha, \quad \mathcal{O}_{III} = Z^{J+1} \bar{Z}$$

(3)
(a = 2, 3) in some arrangement of the fields into gauge group traces. We preempt the result of the exercise: the anomaly of the BMN singlets is correctly reproduced by the functional differential operator

$$\mathcal{F}_K = -\frac{1}{16} \frac{g^2}{4\pi^2} \left( \left[ \lambda^\alpha, \frac{\delta}{\delta \phi_I} \right] \frac{\delta}{\delta \bar{\phi}^I} \right) + (\phi \leftrightarrow \bar{\phi})$$

(4)

where $I = 1, 2, 3$ (so the operator does act on $Z, \bar{Z}$). Supersymmetry and orthogonality considerations much like in the example above suffice to fix the lowest order two-fermion admixtures to the all scalar BMN’s. Order $g^2$ orthogonality then yields the $g^2$ mixing, too.

Next, superspace two-point functions of primary operators and descendants have different normalisations, because the descendant is usually obtained by a differentiation which brings out factors depending on the dimension of the operator \[16\]. We consider the standard gauge invariant BMN operators as opposed to \[17\] which is concerned with gauge non-invariant composites. It turns out that the differentiation trick when applied to the BMN operators relates the two-loop anomalous dimensions rather directly with the two-fermion admixtures of the antisymmetric descendants. For $J = 0, 1, 2$ and gauge group $SU(N)$ we work out the two-loop anomalous dimensions from a one-loop calculation. We agree to all orders in $N$ with results obtained from the two-loop dilation operator of \[18\]. On the other hand the two-loop dilation operator approach reproduces the results of the $g^4$ graph calculation \[9\].

The idea of the dilation operator grew out of the recent work about spin chains realised by $\mathcal{N} = 4$ operators \[19\], and references therein. Integrability of the spin chain enables the authors of \[18\] to predict anomalous dimensions up to three and four loops. The main aim of this article is to advocate the superspace differentiation method as a possible way to check these claims, since it effectively increases the loop order by one.

Finally, we demonstrate the validity of (4) by matching the double-fermion admixture in the antisymmetric representation with the sum of the anomaly and the classical variation of a two-fermion addition in the singlet. In particular, the protected weight four double trace operator $\mathcal{D}_{20}$ \[20\] \[21\] \[22\] has a non-zero anomaly. The multiplet can be short only because there is a cancellation against another term.

\[1\] M. Bianchi independently derived a more general formula \[15\].
1.1 Plan of the paper

In Section 2 we address the diagonalisation of BMN operators with two scalar impurities in each of the three possible SO(4) representations (singlet, antisymmetric and symmetric traceless). Although we do not aspire to resolve the mixing for arbitrary N, we give bases for one-loop protected and unprotected operators in each of the three representations and demonstrate the descendant structure for general N and J. We show that the one-loop mixing matrix of the singlet operators equals the tree-level mixing of their antisymmetric descendants. The situation persists between the antisymmetric operators and their symmetric descendants.

Section 3 explains the aforementioned equivalence of mixing matrices on the basis of differentiation of abstract two-point functions on $\mathcal{N} = 4$ harmonic superspace. The absence of descendants for the one-loop protected operators is shown to take the form of a shortening condition of the “semishort” type \[12\].

In Section 4 we discuss mixing with operators involving fermion impurities in the antisymmetric representation.

In Section 5 we fix the operator mixing up to order $g^2$ in the antisymmetric and symmetric representations, for $J = 0, 1, 2$ and $SU(N)$ gauge group. We derive the two-loop anomalous dimensions by differentiation on superspace.

In Section 6, we check consistency of our results with the diagonalisation of the two-loop dilation operator.

Finally, in Section 7 we fix the operator mixing through $g^2$ in the singlet. We check our formula for the anomaly and re-derive our equation for the two-loop anomalous dimensions.

\[2\] To be more precise, we should talk about the tree-level and single logarithm parts of “the matrix of two-point functions”. Throughout the paper we avoid this more correct but rather clumsy nomenclature in favour of “mixing matrix”.

The formula for the generalised Konishi anomaly is interesting in its own right. It would be fascinating to make contact with \[23\] which derives an anomaly for a similar set of operators in an $\mathcal{N} = 1$ setting.
In two appendices we discuss the $\mathcal{N} = 4$ supersymmetry transformations and the $SU(4) \to SO(4) \times U(1)$ branching relevant in the BMN limit, and give technical details of the calculations of Section 2.

2 BMN operators with two impurities

Throughout the paper it is assumed that for each value of $J$ the rank of the gauge group $N$ is high enough for all operators to be independent.

We distinguish two classes of operators: type I has both impurities in the same gauge trace, type II has the impurities in different gauge traces. We shall study the tree-level and one-loop mixing of the charge $J$ objects

$$O_{I,ab}^{(J_0,J_1|J_2\cdots J_k)} = (\phi_a Z^{J_0} \phi_b Z^{J_1-J_0}) \prod_{i=2}^{n} (Z^{J_i})$$

$$O_{II,ab}^{(J_0,J_1|J_2\cdots J_k)} = (\phi_a Z^{J_1-J_0}) (\phi_b Z^{J_0}) \prod_{i=2}^{n} (Z^{J_i})$$

with total $U(1)_J$ charge $J = \sum_{i=1}^{n} J_i$ and $J_0 \leq J_1$ ($J_0 \neq 0$ for type II in $SU(N)$). To save space we have denoted traces with parantheses ($\phi, \phi^c$). The impurities $\phi_a, \phi_b$ can be any of $\phi_2, \phi_3, \phi^c_2, \phi^c_3$. Operators involving fermion or gauge field impurities decouple from these at order $g^2$, where $g$ is the YM coupling.

The one-loop combinatorics has the surprising feature of not touching upon the factor $\prod_i Z^{J_i}$. We will therefore often avoid writing out the product and rather use the abbreviation $\Pi Z$ in most formulae.

Since resolving the mixing has proven to be much more intricate than originally expected we will mainly focus on identifying protected operators and/or decoupling of classes of operators from other classes. To distinguish the representations, we use $st$ to denote symmetric tensors, $as$ to denote antisymmetric tensors and we put $sin$ for singlets.

We use the $\mathcal{N} = 1$ formalism in Euclidean signature. The tree-level two-point
functions of charge $J$ singlet operators have coordinate dependence

$$
\langle O_{\text{sin}} O_{\text{sin}}^\dagger \rangle_{g^0} = \frac{1}{(4\pi^2 x_{12}^2)^{J+2}}.
$$

(7)

What is more, there is only one one-loop superspace integral. Its $\theta, \bar{\theta} = 0$ component yields

$$
\langle O_{\text{sin}} O_{\text{sin}}^\dagger \rangle_{g^2} = \frac{1}{(4\pi^2 x_{12}^2)^{J}} \frac{g^2}{(4\pi^2)^4} \int \frac{d^4x_5}{x_{15}^2 x_{25}^2} = \frac{1}{(4\pi^2 x_{12}^2)^{J+2}} \frac{g^2}{4\pi^2} \frac{1}{2} \left( \ln(x_{12}^2) + \alpha \right).
$$

(8)

We have not indicated the divergence in the integral. It has to be cancelled by renormalisation of the correctly diagonalised operators which introduces the scheme dependent constant $\alpha$ behind the logarithm.

The difficulty lies in the combinatorics for $U(N)$ or $SU(N)$ gauge group. In this section we do not explicitly evaluate the combinatorics, but rather present proofs based on only a few Wick contractions, employing the rules

$$
(T^a A)(T^a B) = (AB) - \frac{c_0}{N} (A)(B),
$$

(9)

$$
(T^a A T^a B) = (AB) - \frac{c_0}{N} (AB),
$$

(10)

where () denotes a trace and $c_0 = 0, 1$ in $U(N)$ and $SU(N)$, respectively.

The scalar fields are to be contracted using the “propagator” $\langle \phi_1^a \phi_1^{J+b} \rangle = \delta^{ab} \delta^J_f$. The idea is to contract the impurities and leave $Z$ and $\bar{Z}$ untouched [8]. Chain sums occurring in the one-loop calculations are collected into correlators with one more occurrence of $Z, \bar{Z}$. In the one-loop calculations all trace terms with $c_0$ actually cancel; the formulae hold for $U(N)$ as well as $SU(N)$.

We will move part of the discussion to Appendix (11) in order to avoid blurring the conclusions in the main text.

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3There is a caveat here: the $\mathcal{N} = 1$ formalism needs a connection between chiral and antichiral fields. For the singlets there is a class of graphs involving a YM line emanating from the connection. These always seem to combine with another class of YM exchange to make the above statement true. We rely on this assumption.


2.1 The antisymmetric representation

The tree-level mixing between a type I and a type II object (so far without explicit symmetrisation) is given in (148). It is invariant under both \( J_0 \leftrightarrow J_1 - J_0 \) and \( \bar{J}_0 \leftrightarrow \bar{J}_1 - \bar{J}_0 \). Consequently, the mixing vanishes if one of the two operators is antisymmetrised w.r.t. exchange of the two impurities. Thus the antisymmetric type I operators decouple at tree-level from type II.

The type I / type I and type II / type II one-loop mixing is given in formulae (150) and (151), respectively. From the symmetry under exchange of the impurities we deduce that antisymmetric type II operators have vanishing one-loop mixing with anything else. They are one-loop protected.

In the antisymmetric representation there is therefore a very clear-cut criterion: the potentially unprotected operators are of type I. They are automatically tree-orthogonal to the (one-loop) protected operators. Note that any antisymmetric type I operator is a sum of commutators and can be obtained as a supersymmetry variation of some singlet.

2.2 The symmetric traceless representation

Most conveniently we choose a representative with twice the same chiral impurity field, say \( \phi_2 \). The only non-vanishing graphs involve a matter exchange corresponding to the effective vertex [8]:

\[
: ([\bar{Z}, \bar{\phi}_2^2], [Z, \phi_2]) : .
\]  

(11)

When looking for protected operators we can restrict our attention to one “half” of the graphs: the contraction of the antichiral matter vertex on the operator has an open \( \bar{\phi}_3 \) leg. The chiral matter vertex contracted with the conjugate operator at the other end of the two-point function similarly has an open \( \phi_3 \) leg, which is its only connection to the first half. The vanishing of the first half of the two-point function is a sufficient condition for an operator to be one-loop protected.
We are therefore led to consider linear combinations

\[ \mathcal{L} = \sum_f c_f \mathcal{O}_{st, I}^f + \sum_h d_h \mathcal{O}_{st, II}^h \]  \hspace{1cm} (12)

of type I and type II symmetric operators, whose contraction on the antichiral matter vertex vanishes. Explicit calculations for low \( J \) show that these exhaust more than one half of the total space of operators.

Let us sharpen this statement. It is a quick calculation to check that all terms in the contraction of the antichiral matter vertex on any type I or II symmetric operator have the gauge trace structure of antisymmetric type I operators of charge \( J - 1 \), with impurities \( \phi_2 \) and \( \bar{\phi}^3 \). Since the contraction operation is linear, its kernel — the protected operators arising in this way — has as its dimension the number of all symmetric charge \( J \) operators minus the dimension of the image. The remaining, potentially unprotected operators \( \mathcal{O}_u \) must be tree-level orthogonal to this set of protected linear combinations. The dimension of this space is exactly that of the image of the contraction operation. It is therefore smaller or equal to the number of antisymmetric charge \( J - 1 \) operators.

The tree-orthogonality condition is:

\[ \langle \mathcal{L} \mathcal{O}_u^\dagger \rangle = 0 \]  \hspace{1cm} (13)

We will now prove that descendants of charge \( J - 1 \) antisymmetric operators have this property. Take a representative

\[ \mathcal{O}_{as, I} = \Pi_Z (\phi_2 Z^{J_0} \bar{\phi}^3 Z^{J_1 - J_0 - 1}) - (J_0 \leftrightarrow J_1 - J_0 - 1) \]  \hspace{1cm} (14)

and apply the supersymmetry variation \(^4\)

\[ (\bar{\delta}^1)^2 \bar{\phi}^I \to \frac{1}{2} \epsilon^{IJK} [\phi_J, \phi_K], \quad (\delta^1)^2 \phi_I = 0. \]  \hspace{1cm} (15)

The symbol refers to the double application of a certain supercharge of the \( \mathcal{N} = 4 \) theory, see Appendix (10). We find

\[ (\bar{\delta}^1)^2 \mathcal{O}_{as, I} = 2 \Pi_Z (\phi_2 Z^{J_0 + 1} \phi_2 Z^{J_1 - J_0 - 1}) - 2 \Pi_Z (\phi_2 Z^{J_1 - J_0} \bar{\phi}^3 Z^{J_0}), \]  \hspace{1cm} (16)

\(^4\) The commutator term in the supersymmetry transformations \( \text{153b} \) comes with a factor of \( g \), which is omitted here for reasons of simplicity.
a difference of symmetric charge \( J \) type I operators. We can write the double supersymmetry transformation as a contraction on a chiral vertex. Then the tree-orthogonality condition becomes
\[
\langle \mathcal{L} \left( [\bar{\phi}^2, \bar{Z}] \phi^3 \right) \mathcal{O}_{as, 1}^\dagger \rangle = 0
\]
and is automatically fulfilled since the contraction of the antichiral vertex on \( L \) is zero by definition.

Now, any charge \( J-1 \) antisymmetric type I operator has such a descendant, whereas \((\bar{\delta}^1)^2\) yields zero when acting on a type II operator. Thus we have shown that a basis for the symmetric charge \( J \) operators carrying one-loop anomalous dimension is given by the descendants of the antisymmetric charge \( J-1 \) type I operators. In particular, these are simply differences of symmetric type I operators.

On the other hand, the coefficients in the protected linear combinations are \( N \) dependent and we did not obtain them in closed form.

### 2.3 The singlets

For technical convenience we introduce \( SO(4) \) singlets: let us define
\[
\mathcal{O}_{\text{sin}, I}^{(J_0, J_1, J_2 \ldots J_k)} = (\phi_a Z^{J_0} \bar{\phi}^a Z^{J_1-J_0}) \prod_{i=2}^{k} (Z^{J_i}) + (J_0 \leftrightarrow J_1 - J_0),
\]
\[
\mathcal{O}_{\text{sin}, II}^{(J_0, J_1, J_2 \ldots J_k)} = (\phi_a Z^{J_0} (\bar{\phi}^a Z^{J_1-J_0}) \prod_{i=2}^{k} (Z^{J_i}) + (J_0 \leftrightarrow J_1 - J_0),
\]
\[
\mathcal{O}_{\text{sin}, III}^{(J_1, J_2 \ldots J_k)} = (\bar{Z} Z^{J_1}) \prod_{i=2}^{k} (Z^{J_i}),
\]
with \( a = 2, 3 \) in the first two operators. Tree-level orthogonalisation w.r.t. one-loop protected operators organises the \( SO(4) \) singlets into components of various \( SU(4) \) representations, see below.

In order to identify protected operators we consider directly the one-loop mixing matrix of the singlets. Details of the calculation are given in Appendix (11). We
find that type II operators are protected. The non-vanishing pieces of the mixing matrix are type I / I, I / III, and III / III and are given in (152), (153) and (154), respectively. The matrix is singular and it is not hard to find the zero eigenvectors. For each type III operator there is a protected linear combination with a set of type I operators with the same \( \Pi_i(Z^J) \). As a basis for the unprotected operators we may choose type I. The tree-level orthogonalisation w.r.t. the protected structures involves coefficients with rather non-trivial \( N \) dependence and we did not obtain a result in closed form.

The expressions for the one-loop mixing of the various operators are exactly the negative of the tree-level mixing of their descendants under \((\bar{\delta}^1)^2\). This observation extends to the type II operators: they vanish under the supersymmetry variation and correspondingly the one-loop contribution to their two-point function with any other operator is zero.

Let us now return to the \( SU(4) \) picture. The operators pick up a \( Z\bar{Z} \) piece:

\[
\mathcal{O}_{\text{sin}, I}^{(J_0,J_1|\ldots)} = \Pi Z \left( (\phi_2 Z_0 \phi^2 Z^{J_1-J_0}) + (\phi_3 Z_0 \phi^3 Z^{J_1-J_0}) + (Z^{J_1+1} \bar{Z}) \right) + (J_0 \leftrightarrow J_1 - J_0),
\]

\[
\mathcal{O}_{\text{sin}, II}^{(J_0,J_1|\ldots)} = \Pi Z \left( (\phi_2 Z_0)(\phi^2 Z^{J_1-J_0}) + (\phi_3 Z_0)(\phi^3 Z^{J_1-J_0}) + (Z^{J_0+1})(Z^{J_1-J_0} \bar{Z}) \right) + (J_0 \leftrightarrow J_1 - J_0).
\]

The property of protectedness of the type II operators is now lost. As a guideline for identifying the one-loop protected operators we use the criterion that they be annihilated by \((\bar{\delta}^1)^2\). We introduce a change of basis

\[
\tilde{\mathcal{O}}_{\text{sin}, I}^{(J_0,J_1|\ldots)} = \sum_{f=0}^{J_0} \mathcal{O}_{\text{sin}, I}^{(f,J_1|\ldots)} - \frac{J_0 + 2}{J_1 + 3} \sum_{f=0}^{J_1} \mathcal{O}_{\text{sin}, I}^{(f,J_1|\ldots)},
\]

\[
\tilde{\mathcal{O}}_{\text{sin}, II}^{(J_0,J_1|\ldots)} = \mathcal{O}_{\text{sin}, II}^{(J_0,J_1|\ldots)} - \frac{1}{2(J_1 - J_0 + 2)} \sum_{f=0}^{J_1-J_0-1} \mathcal{O}_{\text{sin}, I}^{(f,J_1|\ldots)} - \frac{1}{2(J_0 + 2)} \sum_{f=0}^{J_0-1} \mathcal{O}_{\text{sin}, I}^{(f,J_0-1|\ldots)}.
\]
Under \((\tilde{\delta}^1)^2\) these operators behave like
\[
\begin{align*}
\hat{O}_{\text{sin}, I}^{(J_0, J_1 | \ldots)} &\rightarrow -2\Pi_Z((\phi_2 Z^{J_0+1}\phi_3 Z^{J_1-J_0}) - (\phi_3 Z^{J_0+1}\phi_2 Z^{J_1-J_0})), \\
\hat{O}_{\text{sin}, II}^{(J_0, J_1 | \ldots)} &\rightarrow 0.
\end{align*}
\]

Hence all antisymmetric type I operators of charge \(J + 1\) are descendants of the \(\hat{O}_{\text{sin}, I}\). Further, one may check from (152), (153) and (154) that the redefined type II operators are one-loop protected and that the one-loop mixing matrix of the new type I operators is
\[
\langle \hat{O}_{\text{sin}, I}^{(J_0, J_1 | \ldots)} | \hat{O}_{\text{sin}, I}^{(\bar{J}_0, \bar{J}_1 | \ldots)} \rangle g^2 = \]
\[
8 \Pi_Z \Pi_Z ((Z^{J_0+1}Z^{J_1-J_0})(Z^{J_1-J_0}Z^{J_0+1}) - (Z^{J_0+1}Z^{J_0+1})(Z^{J_1-J_0}Z^{J_1-J_0})),
\]
i.e. the negative of the tree-level mixing of their descendants.

### 2.4 Summary

We have confirmed the intuition that unprotected operators are essentially type I objects: in the singlet type I orthogonalised w.r.t. protected operators, in the antisymmetric representation exactly type I and in the symmetric differences of type I operators. Amongst the unprotected operators, the symmetric ones are descendants of antisymmetric operators, and the antisymmetric operators are descendants of singlets. This generalises exactly the structure observed in the \(J = 2\) example discussed in [3].

Further, the one-loop mixing of the singlet operators is proportional to the tree-level mixing of their symmetric descendants from which it follows by unitarity that there is no one-loop protected singlet operator other than the ones we identified. Likewise, the one-loop mixing of the antisymmetric type I operators is
\[
\langle \Pi_Z ((\phi_2 Z^{J_0} \phi_3 Z^{J_1-J_0}) - (J_0 \leftrightarrow J_1 - J_0)) \Pi_Z ((\phi_2 Z^{J_0} \phi_3 Z^{J_1-J_0}) - (J_0 \leftrightarrow J_1 - J_0)) \rangle g^2 = \]
\[
4 \Pi_Z \Pi_Z \left( (Z^{J_0}Z^{J_0}) (Z^{J_1-J_0+1}Z^{J_1-J_0+1}) + (Z^{J_0+1}Z^{J_0+1}) (Z^{J_1-J_0}Z^{J_1-J_0}) \right)
\]
\[
-(Z^{J_0+1}Z^{J_0}) (Z^{J_1-J_0}Z^{J_1-J_0+1}) - (Z^{J_0}Z^{J_0+1}) (Z^{J_1-J_0}Z^{J_1-J_0})
\]
\[
-(J_0 \leftrightarrow J_1 - J_0)
\],

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which equals the negative of the tree-level mixing matrix of their symmetric descend-
dants under the $\mathcal{N} = 4$ supersymmetry variation $(\delta_4)^2$, see Appendix (10).

The diagonalisation of a set of operators in $\mathcal{N} = 4$ SYM at lowest order in the gauge
coupling constant requires orthogonalisation of the tree-level and the one-loop mix-
ing matrices $[5, 6]$. The change of basis has to coincide in the three representations
as they are connected by supersymmetry. Let $V_{\sin}, V_{\text{as}}, V_{\text{st}}$ denote the tree-level
mixing of the singlets, the antisymmetric and the symmetric BMN operators, re-
respectively. Further we define $Z$ to be the change of basis which diagonalises the
singlets and $\Gamma_1$ the diagonal matrix of first anomalous dimensions. By reinstating
the $g$ in the commutator terms in the supersymmetry transformations and putting
in the normalisation factors from (7) and (8) we obtain,$^{5, 6}$

\begin{align*}
V_{\sin} &= Z^{-1} Z^\dagger^{-1}, \\
V_{\text{as}} &= 2 Z^{-1} \Gamma_1 Z^\dagger^{-1}, \\
V_{\text{st}} &= 4 Z^{-1} (\Gamma_1)^2 Z^\dagger^{-1}.
\end{align*}

### 3 Superspace two-point functions

The analysis in Section (2) was largely based on the $SO(4)$ decomposition suitable for
the BMN limit. In this section we need the $\mathcal{N} = 4$ harmonic superspace of $[24, 25, 12]$
and the shortening conditions of $[12]$, i.e. representation theory of $SU(4)$. Indeed the
two pictures are not so different: at low $J$ one can check explicitly that the tree-level
orthogonalisation w.r.t. the protected operators rearranges the $SO(4)$ singlets into
$SU(4)$ representations. We illustrate this by the example of the Konishi operator
and the stress energy tensor multiplet $Q_{20}$: for $J = 0$ we have the two $SO(4)$ singlets
$O_1 = (\phi_\alpha \bar{\phi}^\alpha), O_2 = (Z \bar{Z})$. The operator $Q_{20} = O_1 - 2 O_2$ is protected. Tree-level
orthogonalisation of $O_1$ w.r.t. $Q_{20}$ gives $K_1 = O_1 - 1/3 O_2 = 2/3 (\bar{\phi}_I \phi^I)$, which is
the usual lowest component of the Konishi operator, barring a normalisation factor.
For $J = 1, 2$ this works in a strictly analogous way: the operators that can pick up
a one-loop anomalous dimension are in the $[0, J, 0]$ representation of $SU(4)$. They

\footnote{This requires scaling (27) up by a factor of 4 in order to match the operator normalisation in
the chain of descendants singlet/antisymmetric/symmetric representation.}

\footnote{We thank Y.S.Stanev for a discussion leading to equation (28).}
have naive scaling dimension $\Delta = J + 2$. In [11] this was elaborated for general $J$ but to leading order in $N$.

Next, observe that the supersymmetry transformations (138) are invariant under multiplication by the $SU(4)$ harmonics $u^A_A$ of the $\mathcal{N} = 4$ harmonic superspace of [12], because the $u$ variables do not transform under $Q$-supersymmetry. We associate the supersymmetry transformations with superspace covariant derivatives as follows:

$$\delta^A_A \leftrightarrow D^A \quad \bar{\delta}^A \leftrightarrow \bar{D}^A$$

We will use the double derivative $(\bar{D^2})^2$ to go from the singlets (or rather the $[0, J, 0]$ ground states) to the antisymmetric descendants and the double derivative $(D^4)^2$ to pass to the symmetric descendants. Note that there can be no field redefinitions due to the $(\theta \sigma^\mu)\partial_\mu$ (or c.c.) part of the covariant derivatives, since $\{\bar{D}, D^4\} = 0$ and we put $\theta, \bar{\theta} = 0$ after differentiation.

According to [12] the field content of the highest weight state of the $[0, J, 0]$ multiplets with scaling weight $\Delta$ is correctly reproduced by the product

$$O^\Delta_{[0, J, 0]} = W^J_{12}(\Psi \bar{\Psi})^{(\Delta - J)/2},$$

where $W_{12}$ is the $\mathcal{N} = 4$ on-shell YM multiplet and $\Psi$ is the $\mathcal{N} = 4$ chiral multiplet. If and only if the operator has exactly $\Delta = J + 2$, the multiplet will obey the shortening conditions

$$(\bar{D^2})^{(rs)} O^{[0, J, 0]}_{[0, J, 0]} = 0, \quad r, s \in \{1, 2\},$$

$$(D^2)^{(\hat{r} \hat{s})} O^{[0, J, 0]}_{[0, J, 0]} = 0, \quad \hat{r}, \hat{s} \in \{3, 4\}. $$

The multiplet is “semishort” in this case. An example is the operator $D^2_{20}$ from [20] at $J = 2$. In Section (2) we had indeed concluded that the one-loop protected operators had vanishing descendants in the antisymmetric $6$, which via (146) coincides with the shortening conditions of the last equation.

The superspace form of the operators (30) suggests a similar factorisation of their two-point functions:

$$\langle O^\Delta_{[0, J, 0]}(z_1) \bar{O}^\Delta_{[0, J, 0]}(z_2) \rangle = \langle W_{12}(1) \bar{W}^{12}(2) \rangle^J \left( \langle \Psi(1) \bar{\Psi}(2) \rangle \langle \bar{\Psi}(1) \Psi(2) \rangle \right)^{(\Delta - J)/2}$$

(32)
This is indeed the unique superspace two-point function with the correct transformation properties under the full superconformal group $SU(2,2|4)$, because the three pieces transform locally at both ends like the constituents $W, \Psi, \bar{\Psi}$ and because $Q$ and $S$ supersymmetry fix the complete dependence on the spinor coordinates. For details of the construction see [21, 26]. The most elegant way of constructing such two- or three-point functions is probably superconformal inversion [27].

For later use we note that the $\theta, \bar{\theta} = 0$ component of the function is [21]

$$\langle O(x_1) \bar{O}(x_2) \rangle|_{\theta, \bar{\theta} = 0} = \frac{(u_1^A(1)u_A^1(2)u_2^B(1)u_B^2(2))^J}{(x^2_{12})^{2\Delta}}. \quad (33)$$

If the harmonics are stripped off, the factor in the numerator will become the combination of Kronecker deltas typical for a two-point function of operators in the $[0, J, 0]$ irrep of $SU(4)$.

In order to go to the descendant two-point function we apply the differential operator $(\bar{D}_1)^2|_1 (D^1)^2|_2$. The field $W_{12}$ is Grassmann analytic: the operator $\bar{D}_1$ annihilates it. Likewise, the field $\bar{W}^{12}$ at the second point obeys the complex conjugate shortening conditions and is annihilated by $D^1$. The second factor of the two-point function is also not seen by the differentiation due to its chirality. Hence the differential operator goes through to the third factor in (32):

$$\langle O(z_1) \bar{O}(z_2) \rangle = \langle W_{12}(1) \bar{W}^{12}(2) \rangle^J \langle \Psi(1) \bar{\Psi}(2) \rangle^{(\Delta - J)/2} \langle \bar{\Psi}(1) \Psi(2) \rangle^{(\Delta - J)/2} \quad (34)$$

Now, the antichiral/chiral two-point function is simply

$$\langle \bar{\Psi}(1) \Psi(2) \rangle^{(\Delta - J)/2} = e^{-2(\bar{\theta}^{(1)} \sigma^a \theta_{12})} \partial_{x_1} \left( \frac{1}{(x_{1}^R - x_2^L)^2} \right)^{(\Delta - J)/2}, \quad (35)$$

where the labels 1,2 indicate the points. In the $x$ difference there is an antichiral $x^R$ at point 1 and a chiral $x^L$ at point 2. The differential operator with the outer thetas put to zero will now act on the exponential only. It produces a contraction of harmonics times a box operator. The lowest component of the descendant two-point function is therefore:

$$\left( (D_1)^2|_1 (D^1)^2|_2 \langle O(z_1) \bar{O}(z_2) \rangle \right)|_{\theta, \bar{\theta} = 0} = \quad (36)$$

$$c_1(\Delta - J - 2)(\Delta - J) \frac{(u_1^A(1)u_A^1(2)u_2^B(1)u_B^2(2))^J(u_1^C(1)u_C^1(2))^2}{(x^2_{12})^{\Delta+1}},$$

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where the constant $c_1$ is some even power of 2 and depends on how we scale the covariant derivatives on superspace relative to the $\mathcal{N} = 4$ supersymmetry transformations acting on the elementary fields.

The harmonic projector indicates that the descendant at point 1 carries the $SU(4)$ irrep $[2, J, 0]$. We may verify this also directly from the fact that the all-scalar descendants are type I operators: the raising operators of $SU(4)$ may be chosen such that $\phi_3 = \phi_{14} \rightarrow \phi_2 = \phi_{13} \rightarrow Z = \phi_{12}$. These operations annihilate the antisymmetric type I operators with impurities $\{\phi_2, \phi_3\}$, whereby the latter are highest weight states. If we put all boxes with label 1 into the first row of a Young tableau, all boxes with label 2 in the second row etc. we see that there is a complete column, which is to be deleted. The Dynkin labels of the representation are then once again found to be $[2, J, 0]$.

The central observation of this section is another one, though: while the two-point function of the descendants has the expected $x$-dependence, it comes with a factor

$$ (\Delta - J)(\Delta - J - 2) = 2\left(\gamma_1 \frac{g^2}{4\pi^2} + \left(\frac{\gamma_2}{2} + \gamma_2\right)\frac{g^4}{(4\pi^2)^2} + \ldots\right). $$

Hence in an orthogonal basis the descendant two-point function has to lowest order an additional factor $2\gamma_1 g^2/(4\pi^2)$ relative to the ground state two-point function. This explains our observation of the equality of the matrix of one-loop two-point functions of the singlets with the tree-level two-point functions of the antisymmetric descendants. More is true: the second correction to the anomalous dimension $\gamma_2$ is contained in the constant at order $g^4$. In the following we will calculate this number for the $J = 0, 1, 2$ singlet operators for gauge group $SU(N)$.

To go from the antisymmetric operators to the symmetric operators we use the differential operator $(D^4)^2|_1 (\bar{D}_4)^2|_2$. By chirality and Grassmann analyticity this acts only on the second factor of (32). Once again, we obtain a harmonic projector and a box operator. The $\theta, \bar{\theta} = 0$ component of this descendant two-point correlator looks like

$$ (D^4)^2|_1 (D_4)^2|_1 (D^4)^2|_2 (D^1)^2|_2 \langle O(z_1) \bar{O}(z_2) \rangle \big|_{\theta, \bar{\theta} = 0} = c_1^2 (\Delta - J - 2)^2 $n \big|_{\theta, \bar{\theta} = 0} = c_1^2 (\Delta - J - 2)^2 \left(\frac{u_1^A(1)u_3^A(2)u_2^B(1)u_B^B(2)}{(x_{12}^2)^{\Delta + 2}} + \frac{J(u_1^C(1)u_3^C(2))^2(u_4^C(1)u_4^C(2))^2}{(x_{12}^2)^{\Delta + 2}} \right). $$

To read off the representation of the descendant at point 1 we dualise the upper 4 into a lower antisymmetrised $\{1, 2, 3\}$. On putting equal labels into the rows of a
Young tableau, we find the Dynkin labels \([2, J, 2]\). On the other hand the operators able to carry one-loop anomalous dimension were differences of type I objects. The sequence \((D^4)^2(\bar{D}_1)^2\) takes the singlets to symmetric operators with two \(\phi_2 = \phi_{13}\) impurities. Differences of these are indeed annihilated by the raising operations defined above. Determining the representation from its highest weight we again fall upon \([2, J, 2]\).

Last, the occurrence of the second factor \((37)\) in \((38)\) explains the equality of the one-loop matrix of two-point functions of the antisymmetric operators with the tree-level two-point correlators of their symmetric descendants.

4 Operators with fermion impurities

So far we have only considered operators constructed from scalar fields. In the \([0, J, 0]\) irrep of \(SU(4)\) at naive scaling weight \(\Delta = J + 2\) there are also the operators (we give the highest weight state)

\[
\begin{align*}
\mathcal{Y}_{[0,J,0]} &= Z^{(J-1)}\bar{\psi}_1\psi_2, \\
\tilde{\mathcal{Y}}_{[0,J,0]} &= Z^{(J-1)}\bar{\psi}_3\bar{\psi}_4, \\
\mathcal{X}_{[0,J,0]} &= Z^{(J-2)}(D^\mu Z)(D_\mu Z)
\end{align*}
\]

with some arrangement of the elementary fields in traces of the gauge group generators.

Likewise, in the \([2, J, 0]\) irrep with naive scaling weight \(\Delta = J + 3\) we can write

\[
\mathcal{Y}_{[2,J,0]} = Z^J \bar{\psi}_1\psi_1.
\]

In the \([2, J, 2]\) with classical dimension \(\Delta = J + 4\) there is no Lorentz scalar involving fermion impurities. These are the “pure operators” of \([18]\).

There should be mixing of the all-scalar operators with their two fermion counterparts. The double-fermion operators have one elementary field less than their
all-scalar partners. On general field theory grounds we expect mixing like

\[ O_{\sin} + g Y_{[0, J, 0]} + g \tilde{Y}_{[0, J, 0]} + g^2 \mathcal{O}_{\sin} + g^2 X_{[0, 2, 0]} + \ldots, \]

and similarly in the \([2, J, 0]\) irrep.

Consider the operators \(Y_{[2, J, 0]}\). We can again distinguish type I and II operators where the fermions play the role of the impurities:

\[
Y_{[2, J, 0], I} = \Pi_Z (\psi_1^a Z^J_0 \psi_1^a Z^{(J_1-J_0)}),
\]

\[
Y_{[2, J, 0], II} = \Pi_Z (\psi_1^a Z^J_0) (\psi_1^a Z^{(J_1-J_0)}).
\]

These operators have the descendants

\[
(D^4)^2 Y_{[2, J, 0], I} = 4g^2 \Pi_Z ([Z, \phi_2] Z^J_0 [Z, \phi_2] Z^{(J_1-J_0)}),
\]

\[
(D^4)^2 Y_{[2, J, 0], II} = 0.
\]

Hence the descendants of the type I double-fermion terms are differences of type I symmetric operators, quite like the descendants of the antisymmetric all-scalar operators. We now have more operators in the \([2, J, 0]\) irrep than descendants in the \([2, J, 2]\). Operators with equal descendants must be part of the same multiplets. Moreover, they have the same naive scaling weight and \(SU(4)\) and spin assignments so that they compete for the same slot in the multiplets, in a pictorial manner of speaking.

Recall that the derivative of the all-scalar operators comes with a single power of \(g\). We can therefore avoid the problem by defining the order \(g\) all-scalar addition to the \(Y_{[2, J, 0]}\) such that the overall descendant is zero:

\[
\hat{Y}_{[2, J, 0], I}^{(J_0, J_1)} = Y_{[2, J, 0], I}^{(J_0, J_1)} + 2g(C_{as, I}^{(J_0+1, J_1+1)} - C_{as, I}^{(J_0, J_1+1)}),
\]

\[
\hat{Y}_{[2, J, 0], II}^{(J_0, J_1)} = Y_{[2, J, 0], II}^{(J_0, J_1)}.
\]

Here \(C_{as, I}^{(J_0, J_1+1)} = \Pi_i (Z_i^J) (\phi_2 Z^J_0 \phi_3 Z^{(J_1-J_0+1)}) - (\phi_2 \leftrightarrow \phi_3)\) and the \(J_i, i > 1\) are equal for \(Y, O_{as}\). Order \(g\) admixtures of type II operators are not determined by this criterion.
It would be impossible to cancel the descendants of the all-scalar antisymmetric type I BMN’s: they pick up a single power of the coupling constant $g$ under the supercovariant derivative, whereas their order $g$ two-fermion addition goes to $g^3$ times a symmetric descendant.

Lowest order orthogonalisation of the $\hat{Y}$ w.r.t. the antisymmetric all-scalar BMN operators surprisingly fixes the latter’s two-fermion admixture:

$$0 = \langle (O_{a,I}^f + g B_{h}^{f} Y^h) (Y^{\dagger \bar{f}} + g \bar{A}_{h}^{\dagger \bar{f}} O_{a,I}^{\dagger \bar{h}}) \rangle_{g^1} \quad (47)$$

$$= g \bar{A}_{h}^{\dagger \bar{f}} \langle O_{a,I}^{f} O_{a,I}^{\dagger \bar{h}} \rangle_{g^0} + g B_{h}^{f} \langle Y^{h} Y^{\dagger \bar{f}} \rangle_{g^0}$$

is a non-singular\(^7\) linear system for the matrix $B$ in terms of $A$, which is in turn defined by (46). Note that the equations are not altered by admixtures to the $\hat{Y}$ of type II all-scalar operators; the latter are tree-orthogonal to type I, see Section (2).

The “anomaly coefficients” $B$ found in the worked examples below are of very simple form. This matrix can perhaps be found in closed form by the techniques of Section (2).

5 Operator mixing and anomalous dimensions for $J=0,1,2$

In this section we restrict to gauge group $SU(N)$ with the obvious motivation of reducing the number of operators. We wish to achieve an explicit solution of the mixing, not relying on any particular limit.

The case $J=2$ is the most interesting of the three since there is non-trivial operator mixing, which we will fix up to $g^2$ for the antisymmetric and the symmetric BMN operators. The discussion of the fermion mixing in the singlet is postponed to Section (7).

\(^{7}\)Unitarity of the theory requires the matrix of two-point functions of the $Y_{[2,J,0]}$ to be non-singular at tree-level.
5.1 On the singlets

We have the six $SO(4)$ all-scalar singlets

\[
\begin{align*}
\mathcal{O}_{\sin,1} &= 2(ZZ)(\phi_r \bar{\phi}^r), \\
\mathcal{O}_{\sin,2} &= (\phi_r \bar{\phi}^r ZZ) + (\bar{\phi}^r \phi_r ZZ), \\
\mathcal{O}_{\sin,3} &= 2(\phi_r Z \bar{\phi}^r Z), \\
\mathcal{O}_{\sin,4} &= 2(\phi_r Z \bar{\phi}^r), \\
\mathcal{O}_{\sin,5} &= 4(\phi_r Z)(\bar{\phi}^r Z) + 2(ZZ)(\phi_r \bar{\phi}^r) - 4(ZZ)(\bar{Z}Z), \\
\mathcal{O}_{\sin,6} &= 2(\phi_r Z \bar{\phi}^r Z) + 2\left((\phi_r \bar{\phi}^r ZZ) + (\bar{\phi}^r \phi_r ZZ)\right) - 4(\bar{Z}ZZZ),
\end{align*}
\]

with $r \in \{2, 3\}$. The last two operators can be obtained from the 1/2 BPS highest weights $(ZZ)(ZZ)$ and $(ZZZZZ)$, respectively, via the $SU(4)$ lowering operator $\partial_4^1 \partial_3^2$.\textsuperscript{8} They are protected. The fourth operator is of type II and is also (one-loop) protected. The matrix

\[
\mathcal{R} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

orthogonalises the first three operators w.r.t. all the protected operators and the fourth one w.r.t. the 1/2 BPS components. The first four operators now contain the full $SU(4)$ traces $\phi_a \bar{\phi}^a + Z \bar{Z}$, so that they are seen to belong to the $[0, 2, 0]$ of $SU(4)$. A second transformation

\[
\mathcal{M} =
\begin{pmatrix}
\frac{3}{2N} & 0 & 0 & 0 \\
\frac{2(N^2-2)}{3N^2-2} & -\frac{1}{2} & 0 & 0 \\
\frac{N^2-4}{N(3N^2-2)} & \frac{1}{2} & 0 & 0 \\
\frac{5(N^2-3)}{4N(N^2+1)} & \frac{5}{2} & 0 & 0 \\
\frac{1}{2} & \frac{5}{2} & 0 & 0 \\
\frac{5}{2} & \frac{5}{2} & 0 & 0
\end{pmatrix}
\]

rotates to the basis of $\mathfrak{g}$. The fourth operator is the semishort $D_{20}$.

\textsuperscript{8}The derivative $\partial_4^I$ replaces a lower $J$ by a lower $I$. 

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In [5] the diagonalisation of the first three operators was worked out explicitly. Let us define

\[ \tilde{\mathcal{O}}_{\text{sin}}^f = \mathcal{R}_h^f \mathcal{O}_{\text{sin}}^h. \]  

(51)

The tree-level and the one-loop logarithm of the two-point functions of the first three operators are diagonalised by the further transformation

\[ Z = V M \]  

(52)

(with the appropriate restriction of \( M \)) and

\[ V = \begin{pmatrix} m_1 & 1 & l_1 \\ m_2 & 1 & l_2 \\ m_3 & 1 & l_3 \end{pmatrix}, \]  

(53)

where

\[ m_f = \frac{\xi_f}{(N^2 + 1)(N^2 - 2)}, \]  

(54)

\[ l_f = -\frac{24}{5} m_f^2 \left( \frac{(N^2 + 1)^3}{(3N^2 - 2)(N^2 - 3)(N^2 - 9)} \right) + \frac{3}{5} m_f \left( \frac{(N^4 + 4)(N^4 - 19N^2 + 10)(N^2 + 1)}{N(3N^2 - 2)(N^2 - 2)(N^2 - 4)(N^2 - 9)} \right) + \frac{4}{5} \frac{N^2(N^2 + 1)^2}{(N^2 - 2)^2(N^2 - 9)} \]

and the \( \xi_f \) are the three roots of the polynomial equation

\[ 0 = 8N x^3 + (-N^6 + 2N^4 + 68N^2 - 40)x^2 + (-3N^9 - 16N^7 + 132N^5 - 80N^3)x + (9N^12 - 84N^10 + 244N^8 - 224N^6 + 64N^4). \]  

(55)

Let

\[ \tilde{\mathcal{O}}_{\text{sin}}^f = \mathcal{Z}_h^f \tilde{\mathcal{O}}_{\text{sin}}^h. \]  

(56)

We find

\[ \langle \tilde{\mathcal{O}}_{\text{sin}}^f \tilde{\mathcal{O}}_{\text{sin}}^{f \dagger} \rangle_0 = \frac{9(N^2 - 1)}{(N^2 - 9)(N^2 - 2)(3N^2 - 2)} \left[ -6(N^2 - 6)\xi_f^2 + \frac{N(N^6 - 23N^4 + 112N^2 - 44)\xi_f + 2N^2(N^2 - 4)(3N^2 - 2)(N^4 - 8N^2 + 6)}{N(3N^2 - 2)(N^2 - 2)(3N^2 - 2)} \right] \]  

(57)

and

\[ \gamma_{1.f} = \frac{1}{N(3N^2 - 2)(N^2 - 2)(3N^2 - 2)} \left[ \frac{8N\xi_f^2}{(N^6 - 2N^4 - 68N^2 + 40)\xi_f + 2N^3(N^2 - 4)(N^2 - 6)(3N^2 - 2)} \right]. \]  

(58)
5.2 The antisymmetric and symmetric representations

The classical supersymmetry variation (138) of the singlets (48) yields
\[ O_{as,1} = 4(ZZ)((\phi_2\phi_3Z) - (\phi_3\phi_2Z)), \]
\[ O_{as,2} = 2((\phi_2\phi_3ZZZ) - (\phi_3\phi_2ZZZ)) - 2((\phi_2Z\phi_3ZZ) - (\phi_3Z\phi_2ZZ)), \]
\[ O_{as,3} = 4((\phi_2Z\phi_3ZZ) - (\phi_3Z\phi_2ZZ)), \]
\[ O_{as,4} = 0, \]
\[ O_{as,5} = 0, \]
\[ O_{as,6} = 0. \]

Multiplication with \( R \) acts as the identity because the protected operators have vanishing descendants. The transformation to the orthogonal basis is therefore simply \( Z \).

Next, we have the double-fermion operators
\[ Y_1 = (\psi_1^\alpha \psi_{1\alpha} ZZ), \]
\[ Y_2 = (\psi_1^\alpha Z \psi_{1\alpha} Z), \]
\[ Y_3 = (\psi_1^\alpha \psi_{1\alpha})(ZZ), \]
\[ Y_4 = (\psi_1^\alpha Z)(\psi_{1\alpha} Z), \]
from which we define hatted operators without symmetric descendants:
\[ \hat{Y}^f = Y^f + gA^f_{as} O^h_{as} \]
with the \( 4 \times 3 \) matrix
\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

We can now use equation (17) to determine the two-fermion term behind the \( O_{as} \).
We define a \( 3 \times 4 \) matrix of “anomaly coefficients”:
\[ \hat{O}_{as}^f = O_{as}^f + \frac{g}{4\pi^2} B^f_{h} Y^h \]

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Before we give the actual solution let us switch a comment on the fermion propagator. From (138) we see that \( \{ Z, \psi_2 \} \) can be put into an \( \mathcal{N} = 1 \) chiral multiplet:

\[
\Phi_1 = Z(x_L) - \theta^\alpha \psi_{2\alpha}(x_L) + \ldots
\]

The \( \mathcal{N} = 1 \) directions are \( \delta_1, \bar{\delta}_1 \). On expanding \( \langle \bar{\Phi}_1 \Phi_1 \rangle \) according to (35) we find for the fermion propagator:

\[
\langle \bar{\psi}^{2\dot{a}}(1) \psi^{\dot{a}}_2(2) \rangle = 2 \left( \sigma^\mu \right)^{\dot{a}a} \partial_{1\mu} \frac{1}{4\pi^2 x_{12}^2}
\]

We can now proceed to solving (47):

\[
\mathcal{B} = \frac{1}{4} \begin{pmatrix}
-4 & 4 & -2N & 0 \\
2N & 2N & 0 & 2 \\
-2N & N & -1 & 0 \\
-2N & N & -1 & 0
\end{pmatrix}
\]

The hatted operators are not diagonal to order \( g^2 \) after the change of basis by \( Z \): the tree-level and simple log parts remain diagonal, of course. But the tree-level two-point functions of the two-fermion part contribute a non-diagonal mixing at order \( g^2 \). In order to cancel this we introduce a \( g^2 \) addition into the hatted operators:

\[
\hat{O}_{as} = O_{as} + \frac{g^2}{4\pi^2} B^a \gamma^b + \frac{g^2}{4\pi^2} C^a \gamma^b O_{as}
\]

The matrix \( C \) can be uniquely determined by the following three criteria:

- After changing basis via \( Z \) we do not wish \( C \) to have diagonal components, because these correspond to trivial rescalings of the operators. In more mathematical terms:

\[
\left[ Z C \hat{Z}^{-1} \right]_{ff} = 0, \quad \forall f \quad (\text{no sum}) .
\]

- After the change of basis we want the operators to be orthogonal. Define \( \mathcal{H}_{ff} = \langle \gamma_f \gamma_f^\dagger \rangle_{g^0} \) and \( \mathcal{G}_{hh} = \langle O_{as,h} \hat{O}_{as,h}^\dagger \rangle_{g^0} \). The \( g^2 \) constant contribution is

\[
S = \frac{g^2}{4\pi^2} Z \left( \frac{1}{4\pi^2} \mathcal{B} \mathcal{H} \mathcal{B}^\dagger + \mathcal{C} \mathcal{G} + \mathcal{G} \mathcal{C}^\dagger \right) Z^\dagger
\]

and we impose

\[
S_{fh} = 0, \quad f \neq h .
\]
Third, we want the \((D^4)^2\) descendants of the operators to stay orthogonal. Let

\[
O_{st,f} = (D^4)^2 O_{as,f}, \quad \mathcal{P}_{ff} = (O_{st,f} O_{st,f}^\dagger) g^2. \tag{71}
\]

(The descendant correlator has lowest order \(g^2\).) The \(g^2\) subleading constant contribution to the two-point functions of the descendants is

\[
\mathcal{T} = \frac{g^2}{4\pi^2} \mathcal{Z} \left( (-\mathcal{B} A + \mathcal{C}) \mathcal{P} + \mathcal{P}(\mathcal{C}^\dagger - A^\dagger B^\dagger) \right) \mathcal{Z}^\dagger. \tag{72}
\]

Orthogonality means

\[
\mathcal{T}_{fh} = 0 \quad f \neq h. \tag{73}
\]

Note that \(\mathcal{C}\) does not contribute to the diagonal elements of \(\mathcal{S}, \mathcal{T}\) because we required it to have vanishing diagonal in the orthogonal basis.

All in all we have nine linear equations for the nine elements of the matrix \(\mathcal{C}\). In the case at hand the solution is unique if somewhat complicated:

\[
\mathcal{C} = \frac{1}{40(800 - 1180N^2 + 116N^4 + N^6)} \begin{pmatrix}
  c_{11} & c_{12} & c_{13} \\
  c_{21} & c_{22} & c_{23} \\
  c_{31} & c_{32} & c_{33}
\end{pmatrix}, \tag{74}
\]

\[
\begin{align*}
c_{11} & = -40N^3(-24 + 11N^2), \\
c_{12} & = 40(-120 + 212N^2 - 22N^4 + 11N^6), \\
c_{13} & = -40(-120 + 152N^2 - 90N^4 + 3N^6), \\
c_{21} & = -5(280 - 204N^2 - 206N^4 + 23N^6), \\
c_{22} & = -2N(-920 + 1916N^2 - 138N^4 + N^6), \\
c_{23} & = -N(3240 - 2812N^2 + 326N^4 + 3N^6), \\
c_{31} & = 10(-440 + 812N^2 - 270N^4 + 13N^6), \\
c_{32} & = 8N(-320 + 376N^2 + 72N^4 + N^6), \\
c_{33} & = 2N(-920 + 1436N^2 + 82N^4 + N^6).
\end{align*} \tag{75}
\]
Let us conclude this section with two observations about the structure of the problem:

- All equations we have solved here are linear. The only quadratic problem is the original one of \[5\], i.e. to fix the \(g^0\) mixing. Whereas this is expected for the matrix \(C\) it comes as a surprise for \(B\).

- In the orthogonal basis the only contribution to the \(g^2\) subleading constant comes from the fermion admixture and its derivative in the antisymmetric representation and the symmetric representation, respectively. Below we will calculate the second anomalous dimension from exactly this contribution. This is in sharp contrast to \[17\], where fermions were not considered.

### 5.3 How to extract \(\gamma_2\)

Let \(\hat{O}\) denote the diagonalised operators as before. We factor the \(x\)-dependence out of the two-point functions. To this end we multiply the two-point functions of singlets by \(X_{\sin} = (4\pi^2 x_{12}^2)^{j+2}\), the two-point functions of their antisymmetric descendants by \(x_{12}^2 X_{\sin}\) and those of the symmetric descendants by \(x_{12}^4 X_{\sin}\). Since we need to keep track of the powers of \(g\) in the supersymmetry transformation from singlet to antisymmetric representation we also need to scale the \(O_{\alpha\beta}\) by \(g\).

We define therefore \(\tilde{S}_{ff} = (g^2 x_{12}^2 X_{\sin}) S_{ff}\) and \(\tilde{T}_{ff} = (g^2 x_{12}^4 X_{\sin}) T_{ff}\). Let the tree-level normalisation constant of the singlet two-point functions be denoted as \(a_0\).

What remains of the two-point functions in the antisymmetric representation is:

\[
\langle \hat{O}_{\alpha\beta} \hat{O}_{\alpha\beta}^\dagger \rangle = 2 \left( a_{0,f} + a_{2,f} g^2 \right) \left( \gamma_{1,f} g^2 \frac{g^2}{4\pi^2} - \gamma_{1,f} g^4 \frac{1}{(4\pi^2)^2} (\ln(x_{12}^2) + \alpha) \right) \\
+ \tilde{S}_{ff} + O(g^6) 
\]

Similarly:

\[
\langle \hat{O}_{\alpha\beta} \hat{O}_{\alpha\beta}^\dagger \rangle = 4 \left( a_{0,f} + a_{2,f} g^2 \right) \left( \gamma_{2,f} g^4 \frac{1}{(4\pi^2)^2} - \gamma_{1,f} g^6 \frac{1}{(4\pi^2)^3} (\ln(x_{12}^2) + \alpha) \right) \\
+ \tilde{T}_{ff} + O(g^8)
\]
Recall that the one-loop graph calculations lead in both cases to only one type of divergent $x$-space integral, see (8). The constant $\alpha$ behind the logarithm is therefore the same in both correlators for any well-defined regularisation scheme. The other constant $a_2$ can arise from the derivative of an order $g$ double-fermion addition $\tilde{\gamma}_{[0,t,0]}$ to the singlet operators. After all, hidden in $\mathcal{T}_{ff}$ the correct two-point function of the symmetric descendants contains a rescaling stemming from the derivative of the anomaly of the antisymmetric operators. Even if generally $a_2 \neq 0$, as an overall normalisation it is not affected by the differentiation $(D^4)^2$ which leads from the first to the second correlator.

Next,
\[
\tilde{S}_{ff} = \frac{g^4}{(4\pi^2)^2} \left[ \frac{1}{2} \frac{9(N^2 - 1)}{(N^2 - 9)(N^2 - 2)(3N^2 - 2)} \right] \\
N \left( 20N(N^2 - 10)\xi_f^2 + (1000 - 2028N^2 + 498N^4 - 25N^6 - N^8)\xi_f + \\
N(N^2 - 4)(3N^2 - 2)(84 + 96N^2 - 35N^4 + 3N^6) \right) 
\]
and one finds
\[
\tilde{T}_{ff} = 4\gamma_{1,f} \frac{g^2}{4\pi^2} \tilde{S}_{ff}.
\]

The difference (77) $- 2\gamma_{1,f}g^2/(4\pi^2)$ (76) is independent of $a_{2,f}, \alpha$.

We want to match it with the corresponding difference of abstract superspace functions (35) $- 2\gamma_{1,f}g^2/(4\pi^2)$ (36). The arbitrary normalisation $c_1$ in those equations can be fixed by individually matching the lowest order of (36) and (76). We also take into account the tree-level normalisation $a_{0,f}$. The resulting equation is
\[
\tilde{S}_{ff} = a_{0,f} \left( \gamma_{1,f} + 2\gamma_{2,f} \right) \frac{g^4}{(4\pi^2)^2}.
\]
Explicitly:

\[
\gamma_{2,f} = -\frac{1}{4N(N^2 - 4)(N^2 - 2)^2(3N^2 - 2)(800 - 1180N^2 + 116N^4 + N^6)}
\left[8N(-13680 + 24976N^2 - 9016N^4 + 668N^6 + 5N^8)\xi_f^2 + (547200 - 1995840N^2 + 2250272N^4 - 791920N^6 + 61640N^8 + 6068N^{10} - 634N^{12} - 5N^{14})\xi_f + 2N(N^2 - 4)(3N^2 - 2)(20800 + 4320N^2 - 72048N^4 + 42112N^6 - 6980N^8 + 338N^{10} + 3N^{12})\right]
\]

(The dimension of the singlets was defined as \(\Delta = J + 2 + \gamma_1 \frac{g^2}{4\pi^2} + \gamma_2 \frac{g^4}{(4\pi^2)^2} + \ldots\))

5.4 \(J = 1\)

The \(SO(4)\) singlets are

\[
\begin{align*}
\mathcal{O}_{\text{sin},1} &= (\phi_a \bar{\phi}^a Z) + (\bar{\phi}^a \phi_a Z), \\
\mathcal{O}_{\text{sin},2} &= (\phi_a \bar{\phi}^a Z) + (\bar{\phi}^a \phi_a Z) - 2(\bar{\phi} \phi Z) .
\end{align*}
\]

The second operator equals \(\frac{2}{3} \partial_1^I \partial_3^{J I} (ZZZ)\), whence it is a component of a 1/2 BPS state. Tree-level orthogonalisation gives

\[
\hat{\mathcal{O}}_{\text{sin}} = \mathcal{O}_{\text{sin},1} - \frac{1}{2} \mathcal{O}_{\text{sin},2} = \frac{1}{2} \left((\phi_1 \bar{\phi}_I Z) + (\bar{\phi}_I \phi_1 Z)\right), \quad I \in \{1, 2, 3\},
\]

which has tree-level two-point function

\[
\langle \hat{\mathcal{O}}_{\text{sin}} \hat{\mathcal{O}}_{\text{sin}}^\dagger \rangle = \frac{2(N^2 - 1)(N^2 - 4)}{N}
\]

and first anomalous dimension

\[
\gamma_1 = 2N .
\]

In the antisymmetric representation we have

\[
\begin{align*}
\mathcal{O}_{\text{as}} &= 2\left(\phi_2 \phi_3 ZZ - (\phi_3 \phi_2 ZZ)\right), \\
\mathcal{V} &= (\psi_1^a \psi_1 Z),
\end{align*}
\]

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which receive the order $g$ corrections

$$\hat{\mathcal{Y}} = \mathcal{Y} + g \mathcal{O}_{as},$$  \hfill (88) $$
$$
$$\hat{\mathcal{O}}_{as} = \mathcal{O}_a + \frac{g}{4\pi^2} N \mathcal{Y}.$$  \hfill (89) 

The matrix $C$ is absent. Repeating the same steps as above we find

$$\gamma_2 = -\frac{3N^2}{2}.$$  \hfill (90) 

### 5.5 $J = 0$

The $SO(4)$ singlets are

$$\mathcal{O}_{sin,1} = 2(\phi_a \bar{\phi}^a),$$  \hfill (91) $$
$$\mathcal{O}_{sin,2} = 2(\phi_a \bar{\phi}^a) - 4(\bar{Z}Z).$$  \hfill (92) 

The second operator is a component of the stress-energy tensor multiplet $\mathcal{O}_{20}$, as mentioned above. Tree-level orthogonalisation completes $\mathcal{O}_{sin,1}$ to the Konishi operator

$$\tilde{\mathcal{O}}_{sin} = \mathcal{O}_{sin,1} - \frac{1}{3} \mathcal{O}_{sin,2} = \frac{4}{3}(\phi_I \bar{\phi}^I),$$  \hfill (92) 

with tree-level two-point function

$$\langle \tilde{\mathcal{O}}_{sin} \tilde{\mathcal{O}}_{sin}^I \rangle = \frac{16(N^2 - 1)}{3}.$$  \hfill (93) 

Its first anomalous dimension is

$$\gamma_1 = 3N.$$  \hfill (94) 

In the antisymmetric representation we have

$$\mathcal{O}_{as} = 4(Z[\phi_2, \phi_3]),$$  \hfill (95) $$
$$\mathcal{Y} = (\psi_1^a \bar{\psi}_{1a}),$$  \hfill (96) 

27
which receive the order $g$ corrections

\begin{align}
\hat{Y} &= Y + g \mathcal{O}_{as}, \\
\hat{O}_{as} &= \mathcal{O}_{as} + \frac{g}{4\pi^2} N \frac{Y}{2}.
\end{align}

(97) (98)

For the second correction to the anomalous dimension we find

$$\gamma_2 = -3N^2.$$ 

(99)

6 Test against the two-loop dilation operator

According to [18] the correctly orthogonalised operators and their dimensions can be obtained from the eigenvalue problem of the dilation operator represented as a functional differentiation on a given operator basis. The two-loop differential operator adapted to the symmetric two impurity BMN operators is

$$\Delta = \Delta_0 + \frac{g^2}{16\pi^2} \Delta_2 + \frac{g^4}{(16\pi^2)^2} \Delta_4 + \ldots$$

(100)

with

\begin{align}
\Delta_0 &= (Z\tilde{Z}) + (\phi_2\tilde{\phi}_2), \\
\Delta_2 &= -2([[\phi_2, Z][\tilde{\phi}_2, \tilde{Z}]]), \\
\Delta_4 &= -2 : ([[\phi_2, Z], [\tilde{\phi}_2, \tilde{Z}], Z]) : - : ([[\phi_2, Z], [\tilde{\phi}_2, \tilde{Z}], \phi_2]) : -2 : ([[\phi_2, Z], T^a][\tilde{\phi}_2, \tilde{Z}], T^a) :.
\end{align}

(101)

In this formula $\tilde{Z} = \delta/\delta Z$ etc. and the normal ordering means that the functional derivatives do not act within the operator itself.

A first consistency check with our material is the zero eigenspace of $\Delta_2$ and $\Delta_4$. For $J = 0, 1, 2$ with gauge group $SU(N)$ we find agreement: the orthogonal complement of the zero eigenspace is given by differences of type I symmetric operators.

Let us restrict our attention to such operators. The dilation operators $\Delta_{0,2,4}$ send the space into itself, so that we may associate matrices with them:

\begin{align}
\Delta_0 &\cong (J + 4) I \\
\Delta_2 &\cong D_2, \\
\Delta_4 &\cong D_4.
\end{align}

(102)
Let our operators compose a vector $\mathcal{O}$. The operators as well as the eigenvalues (viz the dimensions) have an expansion in $g^2$ like the dilation operator:

\begin{align}
\mathcal{O} &= \mathcal{O}_0 + \frac{g^2}{4\pi^2} \mathcal{O}_2 + \ldots \\
\lambda &= \lambda_0 + \frac{g^2}{4\pi^2} \lambda_2 + \frac{g^4}{(4\pi^2)^2} \lambda_4.
\end{align}

We arrange the eigenvalues into a diagonal matrix $\Lambda$. The eigenvalue problem

\begin{equation}
\Delta \mathcal{O} = \Lambda \mathcal{O}
\end{equation}

expanded through $g^2$ gives the equation

\begin{equation}
D_2 \mathcal{O}_0 = \Lambda_2 \mathcal{O}_0.
\end{equation}

Suppose we solve this by going to a diagonal basis (Above we denoted this by $\hat{\mathcal{O}}$. For simplicity we do not change all the symbols.) The next order of the eigenvalue problem is

\begin{equation}
D_2 \mathcal{O}_2 + D_4 \mathcal{O}_0 = \Lambda_2 \mathcal{O}_2 + \Lambda_4 \mathcal{O}_0.
\end{equation}

As before, we introduce the $g^2$ mixing $C'$

\begin{equation}
\mathcal{O}_3 = C' \mathcal{O}_0
\end{equation}

to obtain

\begin{equation}
D_4 = (\Lambda_2 C' - C' \Lambda_2) + \Lambda_4.
\end{equation}

In this equation the first term on the r.h.s. has zero diagonal, because $\Lambda_2$ is diagonal. As a consequence, in the basis $\hat{\mathcal{O}}$ the diagonal of the matrix $D_4$ contains the $g^4$ contribution to the eigenvalues and the off-diagonal part has to be cancelled by the $g^2$ operator admixture defined by $C'$. Note that the matrix $C'$ has no influence on $\Lambda_4$, quite like in our calculation above. If we require the absence of trivial $g^2$ rescalings, $C'$ is once again uniquely determined.

For $J = 0, 1$ agreement with our method is immediate. For $J = 2$ we used 150 digits precision numerics under Mathematica to calculate in the dilation operator method. For the $g^2$ operator mixing we must have

\begin{equation}
\left[ Z(-BA + C)Z^{-1} - C' \right]_{fh} = 0, \quad f \neq h,
\end{equation}

while the values for $\gamma_2$ can be directly compared. To the given accuracy there are no deviations.
7 Operator Mixing in the Singlet

Recall that the $SO(4)$ singlet contains the operators $\mathcal{O}_{\sin}, \mathcal{Y}_{[0,J,0]}, \mathcal{Y}_{[0,J,0]}$ and $\mathcal{X} = Z^{J-2}(D^\mu Z)(D_\mu Z)$ in some gauge trace arrangement. We will not dedicate much attention to the $\mathcal{X}$ objects. Mixing between the all-scalar and two-fermion operators certainly exists: the author learned in discussion$^9$ that for example $\mathcal{D}_{20}$ at $J = 2$ has subleading corrections.

The framework of this article gives an elegant way of determining the lowest order terms: as in the antisymmetric representation we can fix the order $g$ correction to the $\mathcal{Y}$ by supersymmetry. One can then like in equation (47) use lowest order orthogonalisation to establish the leading correction to the $\mathcal{O}_{\sin}$, too.

For $J = 2$ there are only two Yukawa structures $^{[22]}$:

$$\hat{\mathcal{K}}_{20}^+ = (Z\psi_{[1\alpha]}^2 \psi_{[2\beta]}^\alpha) + g \hat{A}_f \mathcal{O}_{\sin}^f + \frac{gN}{32\pi^2}((\partial^\mu Z)(\partial_\mu Z)) + \ldots$$

$$\hat{\mathcal{K}}_{20}^- = (Z\psi_{[3\alpha]}^4 \psi_{[4\beta]}^\alpha) + g \hat{A}_f \mathcal{O}_{\sin}^f + \frac{gN}{32\pi^2}((\partial^\mu Z)(\partial_\mu Z)) + \ldots$$

with

$$\hat{A} = \{0, -1, 1, 0, 0, 0\}.$$  \hspace{1cm} (112)

Both of these are level four descendants of the Konishi scalar $\mathcal{K}_1$. In the same way, all the operators $\mathcal{Y}, \mathcal{Y}$ are at least level two descendants: the naive supersymmetry variation $(\hat{\delta}^2)^2$ acts on antisymmetric all-scalar BMN operators with impurities $\{\phi_2, \phi_3\}$ like

$$(\hat{D}_2)^2 \quad \Pi_Z((\phi_2 Z^p \phi_3 Z^{k-p}) - (p \leftrightarrow k - p)) =$$

$$- \quad \Pi_Z((\psi_\alpha^3 Z^p \psi_\beta^\alpha Z^{k-p}))$$

$$- \quad g \Pi_Z((\phi_I Z^{p+1} \phi^I Z^{k-p}) + (\phi^I Z^{p+1} \phi_I Z^{k-p})))$$

$$+ \quad g \Pi_Z((\phi_I Z^p \phi^I Z^{k-p+1}) + (\phi^I Z^p \phi_I Z^{k-p+1})) ,$$

$$(\hat{D}_2)^2 \quad \Pi_Z((\phi_2 Z^p)(\phi_3 Z^{k-p}) - (p \leftrightarrow k - p)) =$$

$$- \quad \Pi_Z((\psi_\alpha^3 Z^p)(\psi_\beta^\alpha Z^{k-p})).$$  \hspace{1cm} (114)

Up to order $g^2$ there should be no anomaly. The supersymmetry transformation $(D^3)^2$ acting on $\mathcal{O}_{\sin}$ with impurities $\{\phi_2, \phi^3\}$ gives analogous formulae with $\mathcal{Y}$ replaced by $\mathcal{Y}$ but remarkably an identical $\mathcal{O}_{\sin}$ part.

$^9$We are grateful to M.Bianchi, G.Rossi and Y.S.Stanev.
Similarly, the $\mathcal{X}$ operators are descendants of the $\mathcal{Y}_{[2,J,0]}$: 

\[
(\bar{D}_2)^2 \Pi_Z(\psi_1^\alpha Z^p \psi_1 \alpha Z^{k-p}) = \Pi_Z(\Pi_{(D^\mu Z)Z^p(D_\mu Z)Z^{k-p}}) - g\Pi_Z(\psi_1^\alpha Z^{p+1} \psi_2 \alpha Z^{k-p}) + g\Pi_Z(\psi_1^\alpha Z^p \psi_2 \alpha Z^{k-p+1}),
\]

\[
(\bar{D}_2)^2 \Pi_Z(\psi_1^\alpha Z^p)(\psi_1 \alpha Z^{k-p}) = -\Pi_Z((D^\mu Z)Z^p)((D_\mu Z)Z^{k-p}).
\]

In Section [4] we had defined the operators $\hat{O}_f^f = \mathcal{O}_f^f + \frac{g}{4\pi^2} \mathcal{B}_f^f \mathcal{Y}_{[2,J,0]}^h$. From the formulae above their descendants under $(\bar{D}_2)^2$ are of the form

\[
(\bar{D}_2)^2 \hat{O}_{as,J-1} = \bar{\mathcal{Y}}_{[0,J,0]} + g\mathcal{O}_{sin,J} + \frac{g}{4\pi^2} Z^{J-2}(\partial^\mu Z)(\partial_\mu Z) + \ldots
\]

where the dots denote terms of order $g^2$ and higher.

On the other hand, for the $\mathcal{O}_{sin}$ we expect mixing like

\[
\hat{O}_{sin}^d = \mathcal{O}_{sin}^d + \frac{g}{4\pi^2} \bar{\mathcal{B}}_c^d (\mathcal{Y}^c + \bar{\mathcal{Y}}^c) + \frac{g^2}{4\pi^2} \bar{\mathcal{C}}_f^d \mathcal{O}_{sin}^f + \frac{g^2}{4\pi^2} \bar{\mathcal{D}}_d^d \mathcal{X}^h + \ldots
\]

These operators are apparently primary. They ought to be orthogonal to the descendants (117) which belong to multiplets with highest weights at lower $J$. Order $g$ orthogonalisation fixes the matrix $\bar{\mathcal{B}}$ just as in eq. (47) in the antisymmetric representation. The resulting linear system of equations is once again guaranteed to be non-singular due to the unitarity of the theory. Note that the coefficients for $\mathcal{Y}$ and $\mathcal{Y}$ always come out equal.

Let us carry out this programme for the case $J = 2$. For the $\mathcal{O}_{sin}$ we use the basis (118). The Yukawa like operators are (110), (111) from above. Order $g$ orthogonality w.r.t. the $\hat{O}_{sin}$ determines

\[
\bar{\mathcal{B}} = \{1, \frac{N}{4}, -\frac{N}{2}, -\frac{1}{2}, 0, 0\}.
\]

The double derivative $(\bar{D}_1)^2$ replaces both fermions in $\bar{\mathcal{Y}}$ by commutators of scalars, whereas we find

\[
(\bar{D}_1)^2 (Z\psi_1^\alpha \psi_2 \alpha) = -g (\psi_1^\alpha \psi_1, Z) Z, \quad (120)
\]

because the first transformation sends $\psi_2 \alpha$ to a Yang-Mills covariant derivative $D = \partial + g[A, \cdot]$ on $Z$ and the second variation converts the vector field in the derivative into another fermion.
Second, \((\bar{D}_1)^2\) when acting on \(O_{\sin}\) produces not only the naive descendant but also the generalised Konishi anomaly. We repeat formula (4)

\[
F_K = -\frac{1}{16} \frac{g^2}{4\pi^2} \left( \psi^\alpha_1 \left[ \psi^\alpha_{1\alpha}, \frac{\delta}{\delta \phi_I} \frac{\delta}{\delta \bar{\phi}^I} \right] + (\phi \leftrightarrow \bar{\phi}) \right)
\]

for the anomalous part of the supersymmetry. The generalised Konishi anomaly does not lead to order \(g^3\) all-scalar admixtures and it annihilates the 1/2 BPS states. Remarkably, the operator \(F_K\) changes the gauge trace structures whereas the naive supersymmetry transformations never do so.

Let us define the coefficient matrix \(B_1\) by

\[
F_K O_{\sin,f} = \frac{g^2}{4\pi^2} B^h_{1f} \mathcal{Y}_h
\]

with the four \(\mathcal{Y}_{[2,2,0]}\) from equation (60). Similarly, we arrange the double-fermion terms from the naive \((\bar{D}_1)^2\) acting on \(Y_{[0,2,0]}\) in each \(\hat{O}_{\sin}^f\) into a form \(\frac{g^2}{4\pi^2} B^h_{2f} \mathcal{Y}_h\). We find

\[
B_1 + B_2 = \frac{1}{4} \begin{pmatrix}
0 & 0 & -2N & 0 \\
-N & 0 & -1 & 0 \\
0 & 0 & 0 & 2 \\
-2 & 2 & 0 & 0
\end{pmatrix} + \frac{1}{4} \begin{pmatrix}
-4 & 4 & 0 & 0 \\
-N & N & 0 & 0 \\
2N & -2N & 0 & 0 \\
2 & -2 & 0 & 0
\end{pmatrix}.
\]  

The two empty lines relating to \(O_{\sin,5}, O_{\sin,6}\) have been omitted. In the fourth line the two contributions cancel: the absence of the descendant is necessary for \(\mathcal{D}_{20}\) to be semishort. For the first three operators the sum of the two matrices exactly reproduces \(B\) from equation (66).

In the cases \(J = 0,1\) there are no \(\mathcal{Y}, \tilde{\mathcal{Y}}\) operators in the singlet (also no \(X\)). Correspondingly, the generalised Konishi anomaly accounts for the whole double fermion admixture to the antisymmetric descendants of the long operators.

Let us proceed by fixing the order \(g^2\) additions to \(\mathcal{D}_{20}\). The two Konishi descendants are \(\hat{\mathcal{K}}_{20}^+ = (D^3)^2(D^4)^2 \hat{\mathcal{K}}_1\) and \(\hat{\mathcal{K}}_{20}^- = (\bar{D}_1)^2(\bar{D}_2)^2 \hat{\mathcal{K}}_1\). Clearly, the first operator is annihilated by \(\{D^3, D^4\}\). We conclude that to order \(g^2\) \(\{(D^3)^2, (D^3 D^4), (D^4)^2\}\) take its Yukawa part into the negative of the derivatives of the \(O_{\sin}\) admixture, and similarly for \(\{(\bar{D}_1)^2, (\bar{D}_1 \bar{D}_2), (\bar{D}_2)^2\}\) acting on \(\hat{\mathcal{K}}_{20}^-\). The sum

\[
\hat{O}_{\sin}^4 = O_{\sin}^4 - \frac{1}{2} \frac{g^2}{4\pi^2} \left( (Z\psi^{\alpha}_{[1}\psi^{\alpha}_{2]}) + (Z\psi^{[3}_{\alpha}\psi^{4]}_{\alpha}) \right) - \frac{1}{2} \frac{g^2}{4\pi^2} \tilde{A}_f O_{\sin}^f
\]

(123)
has vanishing descendants (up to order $g^3$) in all six components of the antisymmetric representation, because under each of \{($\bar{D}_1$)$^2$, \ldots, ($D^3$)$^2$, \ldots\} the derivative of one Yukawa term cancels the generalised Konishi anomaly and that of the other compensates the variation of the $g^2$ scalar remixing.

In the basis
\[ \tilde{O}^f_{\sin} = R^f_h O^h_{\sin} \] (124)
the fourth operator is $D_{20}$. The vector $\tilde{A}$ goes into
\[ \tilde{A} = \{0, -1, 1, \frac{5N}{3N^2 - 2}, 0, 0\} . \] (125)
Thus $D_{20}$ picks up a $g^2$ rescaling which can omit. The result is:
\[ \tilde{D}_{20} = D_{20} - \frac{1}{2} \frac{g}{4\pi^2} ((Z\psi_3^{\alpha}\psi_2^{[\alpha}) + (Z\psi_3^{[\alpha}\psi_4^{4[i]}) + \frac{1}{2} \frac{g^2}{4\pi^2} (O^2_{\sin} - O^3_{\sin}) \] (126)
with
\[ D_{20} = \frac{2}{5} (3(\phi^f Z)(\bar{\phi}^f Z) - (\phi^f \bar{\phi}^f)(ZZ)) . \] (127)

An addition of $g^2 ((\partial Z)(\partial Z))$ remains undetermined.

In the new basis let us write (for simplicity we omit the tilde on all symbols)
\[ \hat{O}^f_{\sin} = O^f_{\sin} + \frac{g}{4\pi^2} \hat{B}^f (Y + \bar{Y}) + \frac{g^2}{4\pi^2} \hat{C}^f_h O^h_{\sin} + \frac{g^2}{4\pi^2} \hat{C}^f_4 D_{20} , \] (128)
for $f, h \in \{1, 2, 3\}$. The condition $\langle \hat{O}^f_{\sin} \hat{D}^\dagger_{20} \rangle g^2 = 0$ gives
\[ \hat{C}^f_4 = \left\{ \frac{10N(N^2 + 1)}{(3N^2 - 2)^2}, \frac{5(3N^4 - 8)}{4(3N^2 - 2)^2}, \frac{-5(N^2 - 2)}{2(3N^2 - 2)} \right\} . \] (129)

For completeness we mention that in this basis
\[ \hat{B} = \left\{ \frac{2(N^2 + 1)}{3N^2 - 2}, \frac{3N^4 - 8}{4N(3N^2 - 2)}, \frac{-N^2 - 2}{2N} \right\} . \] (130)

The matrix $\hat{C}^f_h$ can be fixed as follows:
• After changing to the tree- and one-loop logarithm orthogonal basis it should not have diagonal elements.

• In the orthogonal basis we want it to cancel $g^2$ off-diagonal contributions introduced by the Yukawa admixtures. The resulting equation is like (59) in the antisymmetric representation. Let $\mathcal{H} = \langle \mathcal{Y} \mathcal{Y}^\dagger \rangle g^0 + \langle \mathcal{Y} \mathcal{Y}^\dagger \rangle g^0$ and $\mathcal{G}_{h\bar{h}} = \langle \mathcal{O}_{\text{sin},h} \mathcal{O}_{\text{sin},\bar{h}}^\dagger \rangle g^0$. The order $g^2$ constant part in the mixing is

$$\mathcal{S} = \frac{g^2}{4\pi^2} \mathcal{Z} (\mathcal{B} \otimes \mathcal{B}^\dagger \mathcal{C} \mathcal{G} + \mathcal{G} \mathcal{C}^\dagger) \mathcal{Z}^\dagger$$

and we demand $\mathcal{S}_{fh} = 0$, $f \neq h$ as before.

• The descendants in the antisymmetric representation must be as determined in the preceding sections. (We have already checked the double-fermion terms.) The $g^2$ subleading all-scalar contribution in the antisymmetric descendants is

$$\left(\mathcal{D}_1\right)^2 \frac{\mathcal{O}_{\text{sin}}^f}{g^3} = \frac{g^3}{4\pi^2} (-\mathcal{B} \otimes \mathcal{A}^\dagger + \mathcal{C}) \mathcal{O}_{\text{as}}^f,$$

but we have to take care of the fact that the first term in the bracket actually introduces diagonal rescalings of the antisymmetric operators in the orthogonal basis; we had banned these in the above. We can therefore only impose

$$\left(\mathcal{Z} (-\mathcal{B} \otimes \mathcal{A}^\dagger + \mathcal{C} - \mathcal{C}) \mathcal{Z}^{-1}\right)_{fh} = 0, \quad f \neq h.$$

These are all in all 12 equations on 9 matrix elements, which constitutes a stringent consistency check. A solution does indeed exist:

$$\mathcal{C} = \frac{1}{20N(3N^2 - 2)(800 - 1180N^2 + 116N^4 + N^6)} \begin{pmatrix}
\hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\
\hat{c}_{21} & \hat{c}_{22} & \hat{c}_{23} \\
\hat{c}_{31} & \hat{c}_{32} & \hat{c}_{33}
\end{pmatrix},$$

(134)
with
\[
\begin{align*}
\dot{c}_{11} &= 20N^2(-1200 + 2252N^2 - 1096N^4 + 77N^6), \\
\dot{c}_{12} &= 40N(-680 + 428N^2 + 762N^4 - 195N^6 + 26N^8), \\
\dot{c}_{13} &= -20N(-1360 + 1376N^2 + 1520N^4 - 428N^6 + 13N^8), \\
\dot{c}_{21} &= -10N(-140 - 548N^2 + 893N^4 - 190N^6 + 9N^8), \\
\dot{c}_{22} &= 4(8000 - 17460N^2 + 11188N^4 - 2741N^6 + 114N^8 + 3N^{10}), \\
\dot{c}_{23} &= -32000 + 55240N^2 - 3232N^4 - 8306N^6 + 794N^8 + 3N^{10}, \\
\dot{c}_{31} &= 10N(-2 + 3N^2)(-220 + 626N^2 - 126N^4 + 3N^6), \\
\dot{c}_{32} &= 4(-2 + 3N^2)(-4000 + 7180N^2 - 2094N^4 + 252N^6 + N^8), \\
\dot{c}_{33} &= -4(-2 + 3N^2)(-4000 + 5730N^2 - 2629N^4 + 167N^6 + N^8).
\end{align*}
\] (135)

As an illustration of our differentiation method we re-derive the second anomalous dimensions in going from the singlet to the antisymmetric representation: the singlet operators in our definition have a $g^2$ constant contribution to their two-point functions arising from the square of the Yukawa additions:
\[
\hat{S}_{ff} = \frac{g^2}{4\pi^2} \frac{9N(N^2 - 1)}{2(N^2 - 2)^2(3N^2 - 2)^2} (8N - 14N^3 + 3N^5 - 4\xi f)^2 (136)
\]

On the other hand, the diagonal rescalings of the descendant operators turn out to be
\[
2\gamma_{1,f} a_{2,f} \left( \frac{g^4}{4\pi^2} \right) = \frac{g^4}{(4\pi^2)^2} \left( \hat{Z} (\hat{B} \otimes \hat{A}^\dagger) \hat{G} - \hat{G} (\hat{A} \otimes \hat{B}^\dagger) \hat{Z}^\dagger \right)_{ff} = 2\gamma_{1,f} \frac{g^2}{4\pi^2} \hat{S}_{ff}. (137)
\]

(We have taken out the $x$-dependence and a factor of $(4\pi^2)^{-(J+2)}$.) On forming a difference of the descendant and $2\gamma_{1,f}g^2/(4\pi^2)$ times the singlet two-point functions these two constants cancel. Matching with the abstract prediction of harmonic superspace immediately reproduces equation (80).

8 Conclusions

We have clarified how to compute two-loop anomalous dimensions of gauge invariant BMN operators using a one-loop calculation supplemented by differentiation on
superspace. The method requires determining the lowest order mixing of all-scalar and two-fermion operators, which can be deduced by a set of linear equations. The second anomalous dimension is found from the square of the two fermion admixtures. The first problem one encounters in going to (moderately) higher $J$ is in solving the linear equations. Luckily, the coefficients in the matrices $B, B_1, B_2$ in (66), (122) appear to be either $O(N^0)$ or $O(N^1)$. It should be possible to find an analytic solution. Second, higher $J$ is made difficult by the exponential increase of Wick contractions. Third, to fix the order $g^0$ mixing is a non-linear problem that cannot be solved explicitly in the general case. The differentiation and the dilation operator methods share the last feature, of course.

Since the differentiation idea gives one additional loop order for free it is an obvious avenue of research to try and push the method to the next loop order at least for low $J$. We hope to obtain information relevant to the integrability of the spin chain picture.

We note that the values for $\gamma_2$ seem to bear no simple relation to $\gamma_1$, see also [18]. The hope to find universal formulae for the whole class of operators even at finite $J$ and $N$ is therefore slim.

Our arguments do certainly rely on a crucial assumption — namely that the $\mathcal{N}=4$ supersymmetry transformations can more or less be taken at face value. It is of course a long standing problem how to justify this: supersymmetry is essentially not compatible with any known regulator and has to be enforced step by step using Ward identities.\textsuperscript{10} The most striking manifestation of the problem relevant to this work is the occurrence of the Yang-Mills covariant derivative in the supersymmetry transformation of the spinor fields: supersymmetry itself is disjoint from gauge symmetry. In a manifest quantum formulation like $\mathcal{N}=1$ one fixes the Wess-Zumino gauge to eliminate the unphysical fields. Supersymmetry has to be accompanied by a compensating gauge transformation in order to conserve the gauge choice [25], which will eventually lead to the covariantisation of the derivative. We rather take the point of view that the covariant derivative is the only possible outcome of the procedure since otherwise the variation of a gauge singlet would not be another gauge invariant operator.

The cancellation of the two-fermion part of the $[2,2,0]$ descendant of $D_{20}$ is a nec-

\textsuperscript{10}We thank M.Bianchi and G.Rossi for a clarifying discussion on this point.
ecessary condition for it to be semishort. We observed that the operator suffers the general Konishi anomaly and only the combination with a second term coming from the variation of the covariant derivative in question will allow the multiplet to be short. The two terms are in fact of very similar origin since it is parallel transport by the Yang-Mills covariantised derivative that causes the anomaly.

The supersymmetry variation can act on the connection in a point splitting regularisation, which is the mechanism originally displayed by Konishi. It is presumably true that this sort of effect is always subleading w.r.t. the classical supersymmetry variations, since it is an essential manner a quantum feature. We will discuss the generalisation of Konishi’s argument to the BMN operators in a forthcoming publication. When transforming all-scalar operators, the graphs leading to the one-loop anomaly will cancel in the antisymmetric representation, while they are present in the BMN singlet. The resulting $g^2$ shift can be traded for a double-fermion admixture. In the literature it is affirmed that this is an operator identity [29]. In particular, there should not be order $g^2$ reshufflings of the scalars due to the generalised anomaly. It is perhaps worth investigating whether other anomalous contributions can arise in the supersymmetry variations of the two-fermion terms. Careful investigation of the steps of our calculation seems to exclude this to the given order in the coupling.

The combination of the two contributions in the variation of the singlets into the two-fermion terms in the antisymmetric descendants gives a splendid confirmation of our formula for the generalised Konishi anomaly. We have checked the phenomenon for $J = 0 \ldots 5$ and $SU(N)$ gauge group.

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The $\mathcal{N}=4$ SYM theory has the fields $\{\phi_{AB}, \psi_{A\alpha}, \tilde{\psi}_{\dot{\alpha}}^A, A_\mu\}$ transforming under $SU(4)$, i.e. $A, B \in \{1 \ldots 4\}$. The transformation rules are\(^{11}\)

\[
\begin{align*}
\delta \phi_{AB} &= \eta_{\alpha}^A \psi_{B\alpha} + \frac{1}{2} \epsilon_{ABCD} \eta^{C\tilde{\alpha}} \tilde{\psi}^{D\tilde{\alpha}}, \\
\delta \psi_{A\alpha} &= \eta_{\alpha}^B F_{(\alpha \beta)} + g \eta_{\beta} \left[ \phi_{AC}, \phi_{BC} \right] + \eta^{B\tilde{\alpha}} D_{\alpha \beta} \phi_{AB}, \\
\delta \tilde{\psi}_{\dot{\alpha}}^A &= \eta^{A\tilde{\beta}} \tilde{F}_{(\tilde{\alpha} \tilde{\beta})} + g \eta_{\tilde{\beta}}^{\dot{\alpha}} \left[ \tilde{\phi}_{BC}, \tilde{\phi}_{AC} \right] + \eta^{B\dot{\alpha}} D_{\tilde{\alpha} \tilde{\beta}} \tilde{\phi}_{AB}, \\
\delta A_\mu &= \psi_{1a} \left( \sigma_\mu \right)^{\alpha \dot{\alpha}} \eta_a^{\alpha} + \eta_{ia} \left( \sigma_\mu \right)_{\alpha \dot{\alpha}} \tilde{\psi}^{\alpha \dot{\alpha}}. 
\end{align*}
\]

The scalar fields obey the reality constraint\(^{(139)}\)

\[
(\phi_{AB}) = \phi^{AB} = -\frac{1}{2} \epsilon_{ABCD} \phi_{CD}. 
\]

In order to make contact with the BMN limit it is convenient to decompose $SU(4)$ into $SO(4) \times U(1)_J$. We use $Z = \phi_{12}$ for the charged singlet and $\phi_2 = \phi_{13}, \phi_3 = \phi_{14}$ to denote the remaining two complex scalars.

The fermions decompose according to

\[
\psi_A^\alpha \rightarrow \psi_r^{\alpha (+1/2)} \cdot \psi_{\dot{r}}^{\alpha (-1/2)}\quad (140)
\]

with $A \in \{1 \ldots 4\}, r \in \{1, 2\}, \dot{r} \in \{3, 4\}$. There is a similar decomposition for the hermitean conjugate.

So there are the four complex spinors $\psi_r^{\alpha (+1/2)}$ and $\tilde{\psi}_{\dot{r}}^{\alpha (+1/2)}$ with $\Delta = 3/2$ and $J = 1/2$, or simply $\Delta - J = 1$ and also four complex spinors $\psi_r^{\alpha (-1/2)}$ and $\tilde{\psi}_{\dot{r}}^{\alpha (-1/2)}$ with $\Delta = 3/2$ and $J = -1/2$, or simply $\Delta - J = 2$.

\(^{11}\)In the first line we mean antisymmetrisation with weight 1.
The supersymmetry charges undergo a similar decomposition and it turns out to be very convenient to separate them into sets that do and do not annihilate $Z$. To this end, define

\[ \delta_A : \eta_B = 0, B \neq A; \quad \bar{\eta} = 0, \]
\[ \bar{\delta}^A : \bar{\eta}^B = 0, B \neq A; \quad \eta = 0. \]

From (138) it easy to see that the variations

\[ \bar{\delta}^r : r \in \{1, 2\}, \quad \delta^r : \dot{r} \in \{3, 4\} \]

preserve $Z$.

Consider first the transformation $\bar{\delta}^1$:

\[ \bar{\delta}^1 Z^I (\phi_2 \bar{\phi}_2 + \phi_3 \bar{\phi}_3 + Z \bar{Z}) = -\bar{\eta}^1 \bar{\alpha} (\phi_2 \bar{\psi}_3 + \phi_3 \bar{\psi}_4 + Z \bar{\psi}_2) \]

A second application of the same transformation will now act only on the fermions. Furthermore, the field strength tensor cannot be generated since $(\bar{\eta}^1)^2$ has no spin $(0, 1)$ part. We find that $(\bar{\delta}^1)^2$ acts on the elementary fields in the singlet operators like (the transformation parameter and a factor 2 have been omitted)

\[ Z \to 0, \quad Z \to g [\phi_2, \phi_3], \]
\[ \phi_2 \to 0, \quad \bar{\phi}_2 \to -g [Z, \phi_3], \]
\[ \phi_3 \to 0, \quad \bar{\phi}_3 \to g [Z, \phi_2]. \]

This is the transformation mainly used in the paper, namely

\[ \phi_I \to 0, \quad \bar{\phi}^I \to \frac{g}{2} \epsilon^{IJK} [\phi_J, \phi_K] \]

where now $I \in \{1, 2, 3\}$.

More generally, the six components of the antisymmetric $SO(4)$ representation are obtained from the singlets by the scalar part of

\[ \{\bar{\delta}^r, \bar{\delta}^s\}, \quad \{\delta^r, \delta^s\} \]

and the nine components of the symmetric traceless representation are found using

\[ \{\bar{\delta}^r, \bar{\delta}^s\} \{\delta^r, \delta^s\} \]

(scalar part in both anticommutators).
11 Technicalities

In the antisymmetric and symmetric representations let us choose \( \phi_2 \) and \( \phi_3 \) as impurities. The \( N = 1 \) super Feynman rules give non-vanishing correlations only with operators of the conjugate type involving the fields \( \bar{Z}, \bar{\phi}^2, \bar{\phi}^3 \). We remark that a symmetric representative with two equal impurities picks up an extra combinatorical factor of two.

The tree-level mixing between a type I and a type II object is

\[
\Pi_Z(\phi_2 Z^J_0 \phi_3 Z^{J_1-J_0}) \Pi_{\bar{Z}}(\bar{\phi}^2 \bar{Z}^{J_1-J_0})(\bar{\phi}^3 \bar{Z}^J_0) = \Pi_Z(\bar{Z}^J_0 \bar{Z}^{J_1-J_0} Z^{J_1-J_0}) + \frac{c_0}{N^2} \Pi_Z(Z^{J_1})(\bar{Z}^J_0)(\bar{Z}^{J_1-J_0}) - \frac{c_0}{N} \Pi_Z(Z^{J_1} \bar{Z}^{J_1-J_0})(\bar{Z}^J_0).
\]

(148)

It is invariant under \( J_0 \leftrightarrow J_1 - J_0 \) and \( \bar{J}_0 \leftrightarrow \bar{J}_1 - \bar{J}_0 \).

Let us consider the one-loop Feynman diagrammes. Only matter exchange graphs contribute. In these a chiral and an antichiral matter vertex are connected on one leg, leading to the following three effective vertices:

\[
: ([\bar{\phi}^2, \bar{\phi}^3], [\phi_2, \phi_3]) : = : ([\bar{Z}, \bar{\phi}^2], [Z, \phi_2]) : = : ([\bar{Z}, \bar{\phi}^3], [Z, \phi_3]) : \quad (149)
\]

The contraction of the first vertex above on the type II object gives a commutator \( ...[\bar{Z}^J_0, \bar{Z}^{J_1-J_0}]... = 0 \). When dealing with the second vertex it is best to contract only the \( \phi_2, \phi_3 \) fields (and c.c.), but to leave the \( \bar{Z}, Z \) from the vertex untouched. Following \[9\] the normal ordering can be respected by explicitly subtracting out a contraction between these fields. After some simplifications:\[12\]

\[
\Pi_Z(\phi_2 Z^J_0 \phi_3 Z^{J_1-J_0}) \Pi_{\bar{Z}}(\bar{\phi}^2 \bar{Z}^{J_1-J_0})(\bar{\phi}^3 \bar{Z}^J_0) = ([\bar{Z}, \bar{\phi}^2], [Z, \phi_2]) : = \Pi_Z \left( (Z^J_0 \bar{Z}^J_0)(Z^{J_1-J_0} \bar{Z}^{J_1-J_0}) + (Z^J_0 \bar{Z}^{J_1-J_0})(Z^{J_1-J_0} \bar{Z}^J_0) \right.
\]

\[
- (Z^J_0)(Z^{J_1-J_0} \bar{Z}^{J_1}) - (Z^{J_1-J_0})(Z^J_0 \bar{Z}^{J_1}) \right).
\]

(150)

\[12\text{In expressions with } ...Z^I \bar{Z} \bar{Z}^J ... \text{ we contract, say, the single } \bar{Z} \text{ field and collect the sums into the terms given above. During the process contractions onto } \Pi_Z \text{ do occur but they drop out in the end.}\]
The result is symmetric under $J_0 \leftrightarrow J_1 - J_0$ and $\bar{J}_0 \leftrightarrow \bar{J}_1 - \bar{J}_0$ separately. The remaining effective vertex yields the expression above with both index exchanges, an equal contribution.

Next, the one-loop mixing between two type II operators shares the feature that the first effective vertex does not contribute. The second vertex yields

$$\Pi_Z(\phi_2 Z^{J_0})(\phi_3 Z^{J_1 - J_0}) \Pi_Z(\bar{\phi}^2 \bar{Z}^{\bar{J}_1 - \bar{J}_0})(\bar{\phi}^3 \bar{Z}^{\bar{J}_0}) : ([\bar{Z}, \phi^2], [Z, \phi_2]) : \quad (151)$$

This expression is again symmetric under both $J_0 \leftrightarrow J_1 - J_0$ and $\bar{J}_0 \leftrightarrow \bar{J}_1 - \bar{J}_0$. The third vertex gives an equal contribution.

Let us now turn to the one-loop mixing in the singlet sector. We start by discussing the mixing of type I with type II operators. As above, in non-vanishing one-loop graphs the interaction must involve the impurities. We may divide the calculation into two sectors: first, one impurity of each of the operators in the two-point function is involved. The total contribution of such graphs is 16 times the r.h.s. of equation (150). Second, the interaction is only between the impurities. The contribution from this sector is equal but of opposite sign, hence there is exact cancellation.

This pattern is repeated in the mixing of type II with type II operators: we find 16 times the r.h.s. of equation (151) from the first sector and its negative from the second. It is easy to check that type II operators do not mix with the type III either. We arrive at the conclusion that type II singlet operators are one-loop protected.

Next we address the non-vanishing two-point functions. The one-loop mixing of a type III with another type III is via a Yang-Mills exchange. We find:

$$\langle \Pi_Z(Z^{J_1} \bar{Z}) \Pi_Z(\bar{Z}^{\bar{J}_1} Z) \rangle_{g^2} = 2\Pi_Z \Pi_Z \left( (Z^{J_1} \bar{Z}^{\bar{J}_1}) - (Z^{J_0} \bar{Z}^{\bar{J}_1} J_0 \bar{Z}^{\bar{J}_0}) \right) \quad (152)$$

For the mixing of a type I operator with a type III we find only one sort of matter exchange diagramme, here calculated for the first term of the type I singlet:

$$\Pi_Z(\phi_2 Z^{J_0} \bar{\phi}^2 \bar{Z}^{\bar{J}_1 - J_0}) \Pi_Z(\bar{Z}^{\bar{J}_1} Z) : ([\phi_2, Z][\bar{\phi}^2 \bar{Z}]) : \quad (153)$$

$$= \Pi_Z \Pi_Z \left( (Z^{J_0+1})^{(J_1 - J_0) Z^{\bar{J}_1}} + (Z^{J_1 - J_0 + 1})^{(J_0) Z^{\bar{J}_1}} \right)$$

$$- (Z^{J_0})^{(J_1 - J_0 + 1) \bar{Z}^{\bar{J}_1}} - (Z^{J_1 - J_0})^{(J_0 + 1 \bar{Z}^{\bar{J}_1})} \right)$$
This is symmetric under $J_0 \leftrightarrow J_1 - J_0$ so that the other three terms of the type I singlet all add an equal contribution.

Third, the type I / type I mixing matrix is

$$\langle \Pi_Z((\phi_2 Z_{J_0} \bar{\phi}^2 Z_{J_1 - J_0}) + \ldots) \Pi_{\bar{Z}}((\phi_2 \bar{Z}_{J_0} \bar{\phi}^2 \bar{Z}_{J_1 - J_0}) + \ldots) \rangle_{g^2}$$

$$= 8 \Pi_Z \Pi_{\bar{Z}} \left( (Z_{J_0+1} \bar{Z}_{J_0})(Z_{J_1-J_0} \bar{Z}_{J_1-J_0+1}) + (Z_{J_0} \bar{Z}_{J_0+1})(Z_{J_1-J_0+1} \bar{Z}_{J_1-J_0}) 
- (Z_{J_0} \bar{Z}_{J_0})(Z_{J_1-J_0+1} \bar{Z}_{J_1-J_0+1}) - (Z_{J_0+1} \bar{Z}_{J_0+1})(Z_{J_1-J_0+1} \bar{Z}_{J_1-J_0}) 
+ (\bar{J}_0 \leftrightarrow \bar{J}_1 - \bar{J}_0) \right) .$$

References

[1] J. Maldacena, “The large $N$ limit of superconformal field theories and supergravity”, *Adv. Theor. Math. Phys.* 2 (1998) 231, [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200), G.G. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge theory correlators from non-critical string theory”, *Phys. Lett.* B428 (1998) 105, [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109), E. Witten, “Anti de Sitter space and holography”, *Adv. Theor. Math. Phys.* 2 (1998) 253, [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150).

[2] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory”, *JHEP* 0201 (2002) 047, [hep-th/0110242](https://arxiv.org/abs/hep-th/0110242), M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “Penrose limits and maximal supersymmetry”, *Class. Quant. Grav.* 19 (2002) L87, [hep-th/0201081](https://arxiv.org/abs/hep-th/0201081), M. Blau, J. Figueroa-O’Farrill and G. Papadopoulos, “Penrose limits, supergravity and brane dynamics”, [hep-th/0202111](https://arxiv.org/abs/hep-th/0202111).

[3] R.R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background”, *Nucl. Phys.* B625 (2002) 70, [hep-th/0112044](https://arxiv.org/abs/hep-th/0112044), R.R. Metsaev and A.A. Tseytlin, “Exactly solvable model of superstring in Ramond-Ramond plane wave background”, *Phys. Rev.* D65 (2002) 126004, [hep-th/0202109](https://arxiv.org/abs/hep-th/0202109).

[4] D. Berenstein, J.M. Maldacena and H. Nastase, “Strings in flat space and pp-waves from $\mathcal{N} = 4$ super Yang Mills”, *JHEP* 0204 (2002) 013, [hep-th/0202021](https://arxiv.org/abs/hep-th/0202021).
[5] M. Bianchi, B. Eden, G. Rossi and Y.S. Stanev, “On operator mixing $\mathcal{N} = 4$ SYM”, *Nucl. Phys.* **B646** (2002) 69, hep-th/0205321.

[6] G. Arutyunov, S. Penati, A.C. Petkou, A. Santambrogio and E. Sokatchev, “Nonprotected operators in $\mathcal{N} = 4$ SYM and multiparticle states of AdS(5) sugra”, *Nucl. Phys.* **B643** (2002) 49, hep-th/0206020.

[7] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, “pp-wave string interactions from perturbative Yang-Mills theory”, *JHEP* **0207** (2002) 017, hep-th/0205089. N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, “Operator mixing and the BMN correspondence”, *JHEP* **0210** (2002) 068, hep-th/0209002.

[8] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, “A new double-scaling limit of $\mathcal{N} = 4$ super Yang-Mills theory and pp-wave strings”, *Nucl. Phys.* **B643** (2002) 3, hep-th/0205033. N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, “BMN correlators and operator mixing in $\mathcal{N} = 4$ Yang-Mills theory”, *Nucl. Phys.* **B650** (2003) 125, hep-th/0208178.

[9] D.J. Gross, A. Mikhailov and R. Roiban, “Operators with large R charge in $\mathcal{N} = 4$ Yang-Mills theory”, *Annals Phys.* **301** (2002) 31, hep-th/0205066.

[10] U. Gürsoy, “Vector operators in the BMN correspondence”, hep-th/0208041.

[11] N. Beisert, “BMN operators and superconformal symmetry”, hep-th/0211032.

[12] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, “Shortening of primary operators in $\mathcal{N}$-extended SCFT$_4$ and harmonic superspace analyticity”, *Adv. Theor. Math. Phys.* **3** (1999) 1149, hep-th/9912007. S. Ferrara and E. Sokatchev, “Short representations of $SU(2,2|\mathcal{N})$ and harmonic superspace analyticity”, *Lett. Math. Phys.* **52** (2000) 247, hep-th/9912168. “Superconformal interpretation of BPS states in AdS geometries”, *Int. J. Theor. Phys.* **40** (2001) 935, hep-th/0005151.

[13] P.S. Howe, K.S. Stelle and P.K. Townsend, “Supercurrents”, *Nucl. Phys.* **B192** (1981) 332; L. Andrianopoli and S. Ferrara, “On short and long multiplets in the AdS/CFT correspondence”, *Lett. Math. Phys.* **48** (1999) 145, hep-th/9812067. P.J. Heslop and P.S. Howe, “A note on composite operators in $\mathcal{N} = 4$ SYM”, *Phys. Lett.* **B516** (2001) 367, hep-th/0106238.
Dolan and H. Osborn, “On short and semi-short representations for four-dimensional superconformal symmetry”, hep-th/0209056; G. Arutyunov and E. Sokatchev, “A note on the perturbative properties of BPS operators”, Class. Quant. Grav. 20 (2003) L123, hep-th/0209103; E. D’Hoker, P.J. Heslop, P.S. Howe and A.V. Ryzhov, “Systematics of quarter BPS operators in \( \mathcal{N} = 4 \) SYM”, JHEP 0304 (2003) 038, hep-th/0301104.

[14] K. Konishi, “Anomalous supersymmetry transformation of some composite operators in SQCD”, Phys. Lett. B135 (1984) 195; K. Konishi and K. Shizuya, “Functional integral approach to chiral anomalies in supersymmetric gauge theories”, Nuovo Cim. A90 (1985) 111.

[15] M. Bianchi, private communication

[16] D. Anselmi, “The \( \mathcal{N} = 4 \) quantum conformal algebra”, Nucl. Phys. B541 (1999) 369, hep-th/9809192

[17] A. Santambrogio and D. Zanon, “Exact anomalous dimensions of \( \mathcal{N} = 4 \) Yang-Mills operators with large \( R \) charge”, Phys. Lett. B545 (2002) 425, hep-th/0206079

[18] N. Beisert, C. Kristjansen and M. Staudacher, “The dilation operator of conformal \( \mathcal{N} = 4 \) super Yang-Mills theory”, hep-th/0303060

[19] J.A. Minahan and K. Zarembo, “The Bethe ansatz for \( \mathcal{N} = 4 \) super Yang-Mills”, JHEP 0303 (2003) 013, hep-th/0212208; N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings”, hep-th/0306139; N. Beisert, “The complete one-loop dilation operator of \( \mathcal{N} = 4 \) super Yang-Mills theory”, hep-th/0307015; N. Beisert and M. Staudacher, “The \( \mathcal{N} = 4 \) SYM integrable super spin chain” hep-th/0307042

[20] G. Arutyunov, S. Frolov and A.C. Petkou, “Operator product expansion of the lowest weight CPOs in \( \mathcal{N} = 4 \) SYM(4) at strong coupling”, Nucl. Phys. B586 (2000) 547, Erratum-ibid. B609 (2001) 539, hep-th/0005182; “Perturbative and instanton corrections to the OPE os CPOs in \( \mathcal{N} = 4 \) SYM(4)”, Nucl. Phys. B602 (2001) 238, Erratum-ibid. B609 (2001) 540, hep-th/0010137

[21] G. Arutyunov, B. Eden, A.C. Petkou and E. Sokatchev, “Exceptional non-renormalisation properties and OPE analysis of chiral four point functions In
\[ \mathcal{N} = 4 \text{ SYM}(4). \] Nucl. Phys. B620 (2002) 380, \texttt{hep-th/0103230}.

G. Arutyunov, B. Eden and E. Sokatchev, “On non-renormalization and OPE in superconformal field theories”, Nucl. Phys. B619 (2001) 359, \texttt{hep-th/0105254}.

B. Eden and E. Sokatchev, “On the OPE of 1/2 BPS short operators in \[ \mathcal{N} = 4 \text{ SCFT}(4) \]”, Nucl. Phys. B618 (2001) 259, \texttt{hep-th/0106249}.

S. Ferrara and E. Sokatchev, “Universal properties of superconformal OPEs for 1/2 BPS operators in \[ 3 \leq D \leq 6 \]”, New Jour. Phys. 4 (2002) 2, \texttt{hep-th/0110174}.

P.J. Heslop and P.S. Howe, “OPEs and three-point correlators of protected operators in \[ \mathcal{N} = 4 \text{ SYM} \]”, Nucl. Phys. B626 (2002) 265, \texttt{hep-th/0107212}.

[22] M. Bianchi, S. Kovacs, G.C. Rossi and Y.S. Stanev, “Properties of the Konishi multiplet in \[ \mathcal{N} = 4 \text{ SYM theory} \]”, JHEP 0105 (2001) 042, \texttt{hep-th/0104016}.

[23] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, “Chiral rings and anomalies in supersymmetric gauge theory”, JHEP 0212 (2002) 071, \texttt{hep-th/0211170}.

[24] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, “Harmonic superspace”, University Press (Cambridge - UK, 2001) 306 pages.

[25] G.G. Hartwell and P.S. Howe, “(N, P, Q) harmonic superspace”, Int. J. Mod. Phys. A10 (1995) 3901, \texttt{hep-th/9412147}.

“A superspace survey”, Class. Quant. Grav. 12 (1995) 1823.

[26] S.M. Kuzenko and S. Theisen, “Correlation functions of conserved currents in \[ \mathcal{N} = 2 \text{ superconformal field theory} \]”, Class. Quant. Grav. 17 (2000) 665, \texttt{hep-th/9907107}.

[27] E.A. Ivanov, private communication to E. Sokatchev.

[28] J. Wess and J. Bagger, “Supersymmetry and supergravity”, Univ. Press (Princeton - USA, 1992) 259 pages.

[29] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, “Nonperturbative aspects in supersymmetric gauge theories”, Phys. Rept. 162 (1988) 169.