Ghost Cohomologies and New Discrete States in Supersymmetric $c = 1$ Model

Masters Thesis

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Abstract

As of today, string theory appears to be one of the most promising physical models unifying the fundamental interactions in nature, such as electromagnetic (gauge) interactions and the gravity. It is also one of the best candidates for constructing the consistent theory of quantum gravity. While the perturbative theory of strings appears to be well explored by now, we still lack an adequate formulation of string-theoretic formalism in the non-perturbative, or strongly coupled regime. One of the approaches, allowing us to explore the non-perturbative dynamics of strings (as well as of other physical theories with gauge degrees of freedom) is the formalism of ghost cohomologies, studied in this thesis. This approach is based on the fact that virtually all the crucial information on non-perturbative physics of gauge theories (including superstring theory with its reparametrizational gauge symmetry and local worldsheet supersymmetry) is carried by physical operators, which are physical (BRST invariant and non-trivial), but not manifestly gauge-invariant. Typically, these operators belong to a very special sector of the Hilbert space of gauge theories, where the matter and the ghost degrees of freedom are mixed. These physical operators are defined as elements of ghost cohomologies, studied in this thesis. In this work, we explore the formalism of ghost cohomologies on the example of supersymmetric $c = 1$ model (or, equivalently, two-dimensional supergravity) which is one of the simplist models of superstrings, with elegant and transparent structure of the spectrum of physical states (vertex operators). We show how the presence of the ghost cohomologies enlarges the spectrum of states and leads to new intriguing symmetries of the theory and points to possible nontrivial relations of two-dimensional supergravity to physical theories.
in higher dimensions. We also develop general prescription for constructing BRST-invariant and nontrivial vertex operators.
Chapter 1

Introduction

The outline of this thesis work is the following. In the first section, we review the basic concepts of BRST quantization and vertex operator formalism in perturbative Ramond-Neveu-Schwarz (RNS) superstring theory, describing strings with the local worldsheet gauge supersymmetry [1]. We conclude the section by studying the global space-time symmetries of this theory and observe its action and show that the generator of this symmetry is given by a very special type of the worldsheet current (vertex operator) which violate the principle of ghost picture equivalence [8]. In the sections 1.1 and 1.2, we explore the general property of special vertex operators of this type and show their relevance to non-perturbative physics of strings, on the examples of critical string theory and supersymmetric $c = 1$ model [4, 5, 6]. We present geometrical reasons for the appearance of picture-dependent operators and introduce the notion of ghost cohomologies that classifies the vertices with the ghost-matter mixing (section 2.1) [7]. The section 2.3 of this work particularly explores the question of BRST invariance of vertex operators from the ghost cohomologies of positive and negative ghost numbers. One particular problem with these vertex operators is that they are not manifestly BRST-invariant as they don’t commute with the supercurrent terms of the BRST charge. We propose a general prescription that allows to restore BRST invariance of these states by adding $b - c$ ghost dependent terms to the ex-
pressions of these operators and demonstrate how this scheme works on the example of the ghost-matter mixing five-form state of critical NSR superstring theory in ten dimensions. We also show an explicit calculation of the correction terms restoring the BRST invariance of the $T_{12}$ current. In the concluding section, we discuss physical implications of our results and directions for the future work.

1.1 Ramond-Neveu-Schwarz (RNS) Model of Superstring Theory

A string (open or closed) is a one-dimensional extended object embedded in $d$ space-time dimensions. Such an object is a generalization of the concept of relativistic particle, which is a zero-dimensional (point-like) object embedded in $d$ dimensions [2, 1]. Objects like strings were historically introduced as people attempted to find a solution for the problem of quark confinement. It is well-known that quarks exist in the bound state only as the interaction force between them grows with distance [9]. Therefore the natural model of confinement involves a combination of quarks connected with “springs” which tensions grow with displacement. The role of such a spring, connecting a pair of quarks, is played by the Faraday lines of gluon fields, mediating the strong interaction in the standard model of particle physics. Unlike, for example, usual electromagnetic field lines of electric charges which fill the space, the gluon lines are supposedly confined to thin flux tubes. So in order to understand the mechanism of quark confinement, one has to study the dynamics of such thin flux tubes propagating in space time, i.e. of the objects identical to open strings. So far, string theory has not been able to fully explain the quark confinement mechanism because it turned out to be hard to find an open string model, which partition function would exactly reproduce the expectation value of the Wilson’s loop in Quantum Chromodynamics ($QCD$) [9]. Nevertheless, a significant progress has been made in string-theoretic approach to $QCD$ in the recent years and by now string theory
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appears to be one of the most successful models to describe the strong interaction, even though such a description is still incomplete [10, 11, 12]. In the meantime, however, it has been realized that possible applications of string theory go far beyond QCD and theory of strong interaction. That is, the study of the string excitation spectrum shows that various excitations of strings (interpreted as various elementary particles existing in nature) include photons and gravitons. As a result, string theory appears to be the candidate for a model unifying electromagnetic and gravitational fundamental interactions and, up to date, the best known framework to construct consistent theory of quantum gravity. We start with the review of basic known facts about string dynamics and then generalize it to the supersymmetric case. Just like the classical action of a relativistic particle, moving between two given points is given by the Lorenz relativistic length (interval ) of its trajectory (the worldline) between these points, the natural generalization of the action for a string is the area of its two-dimensional trajectory (the worldsheet), restricted by the initial and the final positions of a string. In this sense, string theory is equivalent to the dynamics of random 2-dimensional surfaces, embedded in d-dimensional space-time. Let \( \xi_1, \xi_2 \) be the local coordinates parametrizing the worldsheet and \( X_m, \ (m = 0, 1, ..., d-1) \) the d-dimensional space-time coordinates. Then the string action, the straightforward generalization of the action for the relativistic particle (as explained above) is given by [13]:

\[
S_{\text{string}} = -\frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \gamma^{ab} \partial_a X^m \partial_b X^n (\xi_1, \xi_2) \eta_{mn}
\] (1.1)

where \( \gamma^{ab}(\xi_1, \xi_2)(a, b = 1, 2) \) is the induced worldsheet metric and \( \eta^{mn} \) s Minkowski metric in space-time. Since the area of the surface does not depend on its 3 parametrization, the action (1) is obviously symmetric under the reparametrizations of the local worldsheet coordinates:
\( \xi_1 \to f_1 = f_1(\xi_1, \xi_2) \)
\( \xi_2 \to f_2 = f_2(\xi_1, \xi_2) \) 
with \( f_1, f_2 \) being the new choice for the local coordinates. The reparametrization symmetry (2) is the local bosonic gauge symmetry and it is of crucial importance in the string-theoretic formalism. Since the total number of gauge symmetries (2) is 2, one can use them to eliminate 2 out of 3 independent components of the metric tensor \( \gamma^{ab} \). In particular, the gauge transformations (2) can be used to bring it to the conformally flat form

\[ \gamma^{ab} \to e^\varphi(\xi_1, \xi_2) \delta^{ab} \]  
(1.3)

where \( \varphi \) is some function of the the worldsheet coordinates. Furthermore, the action (1) is invariant under the Weyl rescalings of the metric

\[ \gamma^{ab} \to e^\sigma(\xi_1, \xi_2) \gamma^{ab} \]  
(1.4)

since the determinant of the metric transforms as

\[ \gamma \equiv \text{det}||\gamma_{ab}|| \to e^{-\sigma} \gamma \]  
(1.5)

under (3). By using the rescaling (4), one can eliminate (on the classical level) the scale factor of \( e^\varphi \) in the worldsheet metric, thus reducing the action (1) to the one with the flat metric (or equivalently, the metric on the sphere). It must be realized, however, that the symmetry of the action (1) under the rescaling (3) is classical and, in general, it is violated on the quantum level due to the Liouville anomaly. That is, while the action (1) is invariant under (4), the measure of functional integration in 4 the worldsheet metric is not. Namely, under the combination of reparametrizations (2) and the Weyl rescaling (4) the integration measure transforms as [13]:
\[ D[\gamma^{ab}] \rightarrow D[\varphi] e^{-S_{\text{Liouville}}} D[b] D[c] e^{-S_{b-c}} \] (1.6)

where the functional integration measure \( D[F] \) for any field can be formally defined as \( D[F] = \prod_{z, \bar{z}} dF(z, \bar{z}) \). The action for the fermionic \( b-c \) reparametrization ghost fields:

\[ S_{bc} = \frac{1}{4\pi} \int d^2z (b \partial \bar{c} + \bar{b} \partial c) \] (1.7)

is such that, upon integrating over \( b \) and \( c \) its exponent gives the Faddeev-Popov determinant corresponding to the fixing of the conformal gauge \([1, 13, 14]\). Upon the reparametrization and the fixing of the flat metric, it is convenient to bosonize the \( b \) and \( c \) ghost fields in terms of single free bosonic field according to \([15]\)

\[ b = e^{-\sigma} \]
\[ c = e^{\sigma} \] (1.8)

where the Green’s function of the \( \sigma \) field:

\[ <\sigma(z)\sigma(w)> = \ln(z - w) \] (1.9)

ensures that the bosonization relations (9) reproduce all the correlators involving \( b \) and \( c \). The action for the Liouville field (the scale factor) \( \varphi \):

\[ S_{\text{Liouville}} = \frac{D - 26}{36\pi} \int d^2z (\partial \varphi \bar{\partial} \varphi + 2\mu_0 b e^{b \varphi}) \] (1.10)

reflects the anomaly of the functional integration measure \( D[\varphi] \) under the Weyl rescaling \([13]\). Here \( \mu_0 \) is the cosmological constant and the constant \( b \) is related to the Liouville field’s background charge: \( Q = b + \frac{1}{b} = \sqrt{25 - \frac{d}{3}} \) chosen so that the total central charge of the system is zero \([1, 2, 13]\).
\[ c_X + c_{b-c} + c_\varphi = 0 \] (1.11)

Such a condition is necessary for the nilpotence of the BRST charge of the theory, which ensures in turn that the gauge symmetry of the reparametrizations (2) is preserved on the quantum level (see below). In conformal field theory, the central charges can be determined from the two-point correlation functions of the stress-energy tensors of the appropriate fields [16, 17]:

\[ <T(z)T(w)> = \frac{c}{(z-w)^4} \] (1.12)

where the stress-energy tensor is defined as \( T \equiv T_{zz} \); \( T_{ab} = 2\pi(\gamma)\frac{dS}{\delta\gamma_{ab}} \). Using the expressions for the stress-tensors of the \( X, \varphi \) and the ghost fields:

\[
T_x = -\frac{1}{2} \partial X_m \partial X^m \\
T_\varphi = -\frac{1}{2} (\partial \varphi)^2 + \frac{Q}{2} \partial^2 \varphi \\
T_{bc} = \frac{1}{2} (\partial \sigma)^2 + \frac{3}{2} \partial^2 \sigma
\] (1.13)

and the operator products:

\[
X^m(z)X^n(w) \sim -\eta^{mn} \ln(z-w) \\
\varphi(z)\varphi(w) \sim -\ln(z-w) \\
\sigma(z)\sigma(w) \sim \ln(z-w)
\] (1.14)

which follow directly from the actions (1),(7) and (10). It is straightforward to show that \( c_X = d, \ c_\varphi = 1 + 3Q^2, \ c_{b-c} = -26 \). Thus the full matter+ghost action of a string in the conformal gauge is given by

\[
S = \frac{-1}{4\pi} \int d^2z \partial X_m \partial X^m + S_{\text{Liouville}} + S_{b-c}
\] (1.15)

The string model with such an action, however, has two drawbacks. First of all, as various excitations of a string are interpreted as elementary particles, the string-
theoretic model (1,15) has an obvious problem of not being able to generate fermions. Secondly, it can be shown that the excitation spectrum of the bosonic string (15) contains a tachyonic physical state, leading to the instability of the vacuum. In order to resolve these two problems, the fermionic degrees of freedom are introduced in the theory through the procedure of supersymmetrization. The obtained model is called superstring theory. The supersymmetrization particularly eliminates the tachyonic mode from the spectrum. Essentially, in string theory, there are two different models introducing the supersymmetry. The first model, called Ramond-Neveu-Schwarz (RNS) superstring theory, introduces the supersymmetry on the worldsheet, as will be explained below. Such a model has a manifest local supersymmetry on the worldsheet, which is the local fermionic gauge symmetry - the superpartner of the bosonic gauge symmetry of the reparametrizations (see below). Although the action of RNS model has no manifest supersymmetry in space-time, it can be demonstrated, by direct construction of the space-time supercharge (spin operator) that the spectrum of RNS model is supersymmetric and thus the space-time symmetry is present. Alternatively, one can introduce the supersymmetry directly in space-time, without manifest supersymmetrization of the worldsheet degrees of freedom. Such a model is called Green-Schwarz (GS) superstring theory. The drawback of the GS model is that its covariant quantization is problematic and therefore usually it is quantized in the non-covariant light-cone gauge, at which the physical spectrum obtained is incomplete. Although there exist many interesting isomorphisms between RNS and GS superstring models, they are not the subject of this work. Below and elsewhere, we shall restrict our discussion to the RNS model only. In the RNS model, the supersymmetrization procedure is the following. First, one elevates the two-dimensional worldsheet surface to the superspace, that is, one extends the worldsheet coordinates (z, ¯z) with the pair of anticommuting Grassmann coordinates θ and ¯θ satisfying by definition 18.
\[ \theta^2 = \bar{\theta}^2 = 0 \]
\[ \theta \theta = - \bar{\theta} \bar{\theta} \]
\[ \int d\theta = 0, \quad \int d\theta \theta = 1 \]  

(1.16)

Then, one replaces all the fields of the theory \( \Phi(z, \bar{z}) \) with the appropriate superfields \( \Phi(z, \bar{z}, \theta, \bar{\theta}) \) and expands them in \( \theta \) and \( \bar{\theta} \), using the fact that this expansion is finite.

The fields appearing as the coefficients in this expansion are called the superpartners.

Finally, to supersymmetrize the action (15) one replaces the worlds heet integration with the superspace integral

\[ \int d^2 z \to \int d^2 z d^2 \theta \]  

(1.17)

while replacing the derivatives in \( z \) and \( \bar{z} \) with the covariant counterparts:

\[ \partial_z \to D_z = \partial_{\theta} + \theta \partial_z \]
\[ \partial_{\bar{z}} \to D_{\bar{z}} = \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}} \]  

(1.18)

The expansions of the appropriate superfields are given by:

\[ X^m(z, \bar{z}, \theta, \bar{\theta}) = X^m(z, \bar{z}) + \theta \psi^m(z, \bar{z}) + \bar{\theta} \bar{\psi}^m(z, \bar{z}) + \theta \bar{\theta} H^m(z, \bar{z}) \]
\[ \varphi(z, \bar{z}, \theta, \bar{\theta}) = \varphi(z, \bar{z}) + \theta \lambda(z, \bar{z}) + \bar{\theta} \bar{\lambda}(z, \bar{z}) + \theta \bar{\theta} F(z, \bar{z}) \]
\[ C(z, \theta) = c(z) + \theta \gamma(z) \]
\[ \bar{C}(\bar{z}, \bar{\theta}) = \bar{c}(\bar{z}) + \bar{\theta} \bar{\gamma}(\bar{z}) \]
\[ B(z, \theta) = \beta(z) + \theta b(z) \]
\[ \bar{B}(\bar{z}, \bar{\theta}) = \bar{\beta}(\bar{z}) + \bar{\theta} \bar{b}(\bar{z}) \]  

(1.19)

Note that the bosonic ghosts \( \beta \) and \( \gamma \), the superpartners of \( b \) and \( c \) are those that can be equivalently obtained by supersymmetrizing the action (1) first and choosing the superconformal gauge to fix the local worldsheet supersymmetry (fermionic gauge symmetry). To simplify things, however, we have fixed the conformal gauge first and then have supersymmetrized the theory. Integrating out \( \theta \) and \( \bar{\theta} \), it is straightforward
to show that the full ghost+matter action of the RNS superstring theory in the superconformal gauge is given by [19]:

\[ S_{RNS} = -\frac{1}{4\pi} \int d^2z(\partial X_m \bar{\partial} X^m + \psi_m \bar{\partial} \psi^m + \bar{\psi}_m \partial \bar{\psi}^m) + S_{\text{ghost}} + S_{\text{Liouville}} \]

\[ S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z(b \partial \bar{c} + b \bar{\partial} \bar{c} + \beta \partial \bar{\gamma} + \bar{\beta} \partial \gamma) \]

\[ S_{\text{Liouville}} = \frac{d-10}{36\pi} \int d^2z(\partial \varphi \bar{\partial} \bar{\varphi} + \lambda \partial \lambda + \bar{\lambda} \partial \bar{\lambda} - F^2 + 2\mu_0 b e^{b \varphi}(ib\lambda \bar{\lambda} - F)) \]

\[ Q = b + \frac{1}{b} = \sqrt{\frac{9-d}{2}} \]

As in the $b-c$ case, it is convenient to bosonize the superconformal $\beta$ and $\gamma$ ghosts in terms of the pair of free 2d scalar bosons $\phi$ and $\chi$. The bosonization relations are given by [15]:

\[ \gamma = e^{\phi-\chi} \]

\[ \beta = e^{\chi-\phi} \partial \chi \]

\[ <\chi(z)\chi(w)> = - <\phi(z)\phi(w)> = \ln(z-w) \]

and the full matter+ghost stress-energy tensor is

\[ T_{\text{matter}} = -\frac{1}{2} \partial X_m \partial X^m - \frac{1}{2} \partial \psi_m \bar{\partial} \psi^m \]

\[ T_{\text{ghost}} = \frac{1}{2}(\partial \sigma)^2 + \frac{3}{2} \partial^2 \sigma + \frac{1}{2}(\partial \chi)^2 + \frac{1}{2} \partial^2 \chi - \frac{1}{2}(\partial \phi)^2 - \partial^2 \phi \]

\[ T_{\text{Liouville}} = -\frac{1}{2}(\partial \varphi)^2 + \frac{\varphi}{2} \partial^2 \varphi \]

As in the bosonic case, the background charge $Q$ is chosen so that

\[ c_{\text{matter}} + c_{\text{ghost}} + c_{\text{Liouville}} = 0 \]

Note that in the supersymmetric case the critical dimension of the space-time is 10.

Now that we have described the framework of the RNS superstring theory, the next step is to describe its physical states, which are described by vertex operators.
1.2 Vertex operators and picture equivalence

As has been already mentioned above, the basic concept of string theory implies that various oscillation modes of a string are interpreted as elementary particles (such as a photon or a graviton). More generally, such oscillations may also correspond to non-perturbative objects, such as solitons, black holes or D-branes. In string theoretic formalism, these oscillations are described in terms of vertex operators [1, 2, 20]. Typically, these operators are the objects of the form

\[ V = P(\partial X^m, \partial^2 X^m, ..., \psi^m, \partial \psi^m, \text{ghosts}...)e^{ik_m X^m} \] (1.23)

where \( P \) is some polynomial in the fields and their derivatives, entering the action (20) and \( k_m \) corresponds to the momentum of the particle. In order to be physical (that is, in order to describe an emission of a physical object by a string), the vertex operators must satisfy all the underlying gauge symmetries of string theory (such as reparametrizations and local supersymmetry). In particular, it implies their conformal invariance. Generally speaking, the operator is physical if and only if it is the element of the BRST cohomology which definition is as follows. The crucial property of all the theories with gauge symmetries is that any local gauge symmetry of the theory automatically entails its invariance under another symmetry transformations, called BRST (Becchi-Rouet-Stora-Tiutin) symmetry [20, 21]. That is, if an action of the theory is locally invariant under some local gauge symmetry with some parameter (bosonic or fermionic), it turns out that it is also automatically invariant under the transformations, which form is the same as that of those gauge transformations, but with the local gauge parameter replaced with the appropriate Faddeev-Popov ghost of the opposite statistics (for example, the parameter of the reparametrization symmetry is replaced by the fermionic ghost \( c \) while the fermionic parameter of the local worldsheet supersymmetry is replaced by the bosonic superconformal ghost field \( \gamma \)). The crucial property of the BRST charge \( Q_{brst} \), generating these transformations
is that it is nilpotent, that is, $Q_{brst}^2 = 0$. For the RNS model [1 2], the precise expression for the BRST charge is given by [15]

$$Q_{brst} = \int \frac{dz}{2i\pi} \{cT - bcdc - \frac{1}{2} \gamma \psi_m \partial X^m - \frac{1}{4} \gamma^2 b\}$$  

(1.24)

In view of what has been explained above, the physical vertex operators of the RNS model must be BRST-invariant (BRST-closed), that is, to satisfy

$$\{Q_{brst}, V\} = 0$$  

(1.25)

One should exclude, however, the trivial subclass of BRST-exact operators satisfying

$$V = \{Q_{brst}, W\}$$  

(1.26)

for some W. Such operators are automatically BRST-invariant due to nilpotence of $Q_{brst}$ but have no physical significance since any correlation function of $V$ satisfying 11 (25,26) with any set of BRST-invariant operators $V_1, ..., V_N$ is zero. Indeed, one has

$$<VV_1...V_N> = <\{Q_{brst}, W\}V_1...V_N> = - \sum_{i=1}^{N} <WV_1...V_{i-1}\{Q_{brst}, V_i\}V_{i+1}...V_N> = 0$$  

(1.27)

since $\{Q_{brst}, V_i\} = 0$ by definition. Thus the vertex operators are physical if they are BRST-closed but not BRST-exact. As the nilpotence of the BRST charge is reminiscent of the analogous properties of the exterior derivative of differential forms, physical operators are called the elements of BRST-cohomology. The concrete form of the vertex operator depends on the physical properties of the particle it describes. For example, to construct the vertex operator of a photon (which is the massless vector boson) one has to look for the BRST-invariant and non-trivial object of the type (23) which is in the vector representation of the space-time Lorentz group; the
masslessness condition \( \kappa^2 = 0 \) and the transversality condition, reproducing the well-known properties of photons, must follow from the \( BRST \)-invariance condition (25) in this approach. Analogously, if one is able to construct a massless \( BRST \)-invariant and non-trivial vertex operator in the symmetric rank 2 tensor representation of the Lorenz group, such an object is to be identified with the graviton, etc. Below, we shall discuss some particular examples of such a construction. First of all, let us try to understand some general properties, satisfied by vertex operators. In string or superstring theory, there are two basic types of vertex operators, corresponding to open and closed strings. Open strings typically have configuration of a line with open ends; the simplest worldsheets topology of such a string is that of a disc. Closed strings are those having the form of loops; their simplest worldsheets topology is that of a sphere. Accordingly, open string vertex operators are those that are integrated over the boundary of the disc (contour integrals); closed string vertex operators, on the other hand, are integrated over the entire worldsheet surface (two-dimensional integrals). Let us start with open string vertex operators. It is easy to see that such operators, in order to be physical, must be primary fields of dimension 1 for open strings and (1, 1) for closed strings. In conformal field theory, the primary fields of dimension \((h, \bar{h})\) are the observables \( \varphi^{h, \bar{h}} \) that transform under the conformal transformations: \( z \to f(z); \bar{z} \to \bar{f}(\bar{z}) \) according to [16, 17]

\[
\varphi^{h, \bar{h}}(z, \bar{z}) \to \left( \frac{df}{dz} \right)^h \left( \frac{d\bar{f}}{d\bar{z}} \right)^{\bar{h}} \left( f(z), \bar{f}(\bar{z}) \right)
\]

(1.28)

The operator product expansion (\( OPE \)) of the stress energy tensor with primary fields has a remarkably simple form:

\[
T(z) \varphi^{h, \bar{h}}(w, \bar{w}) = \frac{h \varphi^{h, \bar{h}}(w, \bar{w})}{(z - w)^2} + \frac{\partial \varphi^{h, \bar{h}}(w, \bar{w})}{z - w} + O(z - w)^0
\]

(1.29)

and analogously for \( \bar{T} \).
Chapter 2

Ghost Cohomologies

2.1 Ghost Cohomologies in RNS Model

In RNS superstring theory the physical states are described by BRST-invariant and nontrivial vertex operators, corresponding to various string excitations. Typically, these operators are defined up to transformations by the picture-changing operator \( \Gamma = [Q_{\text{brst}}, \xi] \) and its inverse \( \Gamma^{-1} = c \partial_e e^{-2\phi} \) where \( \xi = e^\chi \) and \( \phi, \chi \) is the pair of the bosonized superconformal ghosts. Acting with \( \Gamma \) or \( \Gamma^{-1} \) changes the ghost number of the operator by 1 unit, therefore each perturbative string excitation (such as a photon or a graviton) can be described by an infinite set of physically equivalent operators differing by their ghost numbers, or the ghost pictures. Typically, for a picture n operator one has [7, 8, 15]

\[
: \Gamma V^{(n)} := V^{(n+1)} + \{Q_{\text{brst}}, \ldots\}
\]

and

\[
: \Gamma^{-1} V^{(n)} := V^{(n-1)} + \{Q_{\text{brst}}, \ldots\}
\]

The inverse and direct picture-changing operators satisfy the OPE identity
\[\Gamma(z)\Gamma^{-1}(w) = 1 + \{Q_{\text{brst}}, \Lambda(z, w)\}\]  
(2.1)

\[\Lambda(z, w) =: \Gamma^{-1}(z) (\xi(w) - \xi(z)) :\]

The standard perturbative superstring vertices can thus be represented at any integer (positive or negative) ghost picture. Such a discrete picture changing symmetry is the consequence of the discrete automorphism symmetry in the space of the supermoduli\[22\]. Varying the location of picture-changing operator (or equivalently varying the super Beltrami basis) inside correlation functions changes them by the full derivative in the supermoduli space. This ensures their picture invariance after the appropriate moduli integration, if the supermoduli space has no boundaries or global singularities. The global singularities, however, do appear in case if the correlation function contains vertex operators \(V(z_n)\) such that the supermoduli coordinates diverge faster than \((z - z_n)^{-2}\) when they approach the insertion points on the worldsheet\[22\]. Should this be the case, the moduli integration of the full derivative term would result in the nonzero boundary contribution and the correlation function would be picture-dependent. The example of such a vertex operator in critical \(RNS\) superstring theory is the five-form state \[8\]

\[V_5(k) = H(k) \oint \frac{dz}{2i\pi} U_5(z) \equiv H_{m_1...m_5}(k) \oint \frac{dz}{2i\pi} e^{-3\phi} \psi^{m_1}...\psi^{m_5} e^{ikX}(z)\]  
(2.2)

with the five-form \(H\) subject to the \(BRST\) non-triviality condition

\[k_{[m_6H_{m_1...m_5]}}(k) \neq 0\]  
(2.3)

or simply \(dH \neq 0\). That is, if the condition (32) isn’t satisfied, there exists an operator

\[S_5 = H_{m_1...m_5}(k) \oint \frac{dz}{2i\pi} \partial\xi e^{-4\phi} \psi^{m_1}...\psi^{m_5} (\psi^m \partial X_m) e^{ikX}\]  
(2.4)

which commutator with \(Q_{\text{brst}}\) is \(V_5\), i.e. the five-form (31) is trivial. Indeed, if the \(H\)-form is closed, the \(S_5\) operator (33) is the primary field and its commutator with
γψₘ∂ₘₓ X term of the BRST charge gives \( V₅ \). If, however, the condition (32) is satisfied, \( S₅ \) doesn’t commute with the stress tensor \( eT \) term of \( Q_{\text{brst}} \) and therefore \( V₅ \) cannot be represented as a commutator of \( Q \) with \( S₅ \) or any other operator, as we will show by explicit computation of its correlators.

This \( V₅ \) operator is annihilated by the picture-changing transformation and the supermoduli approaching \( U(z) \) diverge as \((z - w)^{-4}\) which can be read off the OPE of \( U \) with the worldsheet supercurrent:

\[
G(w)U₅(z) \sim (z - w)^{-4}cξe^{-iφψᵐ₁...ψᵐ₅}e^{ikX} + ...
\] (2.5)

As it will be shown below, the correlation functions involving the operators of the \( V₅ \)-type are picture-dependent and in fact this is one of the direct consequences of (34). The \( V₅ \)-operator of (31) is an example of the ghost-matter mixing operator as its dependence on the superconformal ghost degrees of freedom can’t be removed by the picture-changing and therefore isn’t just an artifact of the gauge. This operator exists at all the negative pictures below \(-3\) but has no version at pictures \(-2\) or higher [8]. At the first glance the existence of such a physical operator seems to contradict our standard understanding of the picture-changing, implying that all the vertices must exist at all equivalent pictures. So the first suspicion is that the \( V₅ \)-state is BRST-trivial, as the relation (30) implies that all the BRST-invariant operators annihilated by picture-changing can be written as \([Q_{\text{brst}}, U]\) for some \( U \). Indeed, if \( V₅ \) is annihilated by \( Γ \), using (30) and the invariance of \( V₅ \), we have

\[
0 \equiv \lim_{w \to z} \Gamma^{-1}(u)Γ(w)V₅(z) = V₅(z) + \lim_{w \to z} \{Q_{\text{brst}}, Λ(u, w)V₅(z)\}
\] (2.6)

and hence \( V₅ \) is the BRST commutator:

\[
V₅(z) = -\lim_{w \to z} \{Q_{\text{brst}}, Λ(u, w)V₅(z)\}
\] (2.7)

This commutator can be written as
\[
\lim_{w \to z} \{Q_{brst}, \Lambda(u, w)V_5(z)\} = \lim_{w \to z} \{Q_{brst}, \Gamma^{-1}(u) (\xi(w) - \xi(u)) V_5(z)\} \\
= \lim_{w \to z} \Gamma^{-1}(u) (\Gamma(w) - \Gamma(u)) V_5(z)
\]

(2.8)

where we used \(\{Q_{brst}, \xi\} = \Gamma\) and \([Q_{brst}, \Gamma^{-1}] = 0\). But since \(V_5\) is annihilated by \(\Gamma\) at coincident points

\[
\lim_{w \to z} \Gamma^{-1}(u) \Gamma(w) V_5(z) = 0
\]

(2.9)

and the commutator (36) is given by

\[
V_5(z) = \{Q_{brst}, \xi \Gamma^{-1}(u)V_5(z)\}
\]

(2.10)

Thus \(V_5\) is the \(BRST\) commutator with an operator outside the small Hilbert space(15) and therefore (35,36) do not by themselves imply its triviality. This point is actually quite a conceptual one. The very possibility of having the \(BRST\) non-trivial and invariant operators annihilated by \(\Gamma\) (or similarly, by \(\Gamma^{-1}\)) follows from the fact that the bi-local \(\Lambda(u, w)\)-operator (which, by itself, is in the small Hilbert space) vanishes at coincident points. Had this not been the case, (36) would have meant that any operator \(V\) , annihilated by \(\Gamma\), would have been given by the commutator \(V(z) = -\{Q_{brst}, \Lambda(z, \bar{z})V\}\), i.e. would have been \(BRST\) exact. But since \(\Lambda(z, \bar{z}) = 0\) the \(OPE\) (30) only leads to the trivial identity of the type \(V = V\) , as it is clear from (39). The picture-dependent \(V5\) operator is actually \(BRST\) non-trivial and invariant, i.e. it is a physical state. In fact, the \(V_5\)-operator is not related to any point-like perturbative string excitation but to the D-brane dynamics, and its non-perturbative character is somehow encrypted in its non-trivial picture dependence. While the questions of its non-triviality have been discussed previously\[8\] below we shall review the standard arguments leading to the standard concept of picture equivalence of physical states (which indeed is true for the usual perturbative
point-like string excitations like a graviton or a photon) and show where precisely these arguments fail for the case of $V_5$. So let us further analyze the question of the nontriviality of picture-dependent operators. The next standard argument for the picture-equivalence stems from the fact that one is able to freely move the picture-changing operators inside the correlators. Indeed, since the derivatives of $\Gamma$ are all BRST-trivial: $\partial^n \Gamma = \{Q_{\text{brst}}, \partial^n \xi \}$; $n = 1, 2, ...$ one can write

$$\Gamma(w) = \Gamma(z) + \sum_n \frac{(w - z)^n}{n!} \partial^n \xi$$  \hspace{1cm} (2.11)

Next, using the identity $1 =: \Gamma \Gamma^{-1}$: following directly from (30) one can write for any correlator including $V_5$ and some other physical operators $U_i(z_i)$, $i = 1, ..., n$

$$< V_5(z) U_1(z_1) ... U_n(z_n) > = < \Gamma^{-1}(w) V_5(z) U_1 ... U_n >$$  \hspace{1cm} (2.12)

for any point $w$. Using (40) one can move $\Gamma$ from $w$ to $z$ but, since $: \Gamma U_5 : (z) = 0$ this would imply that the correlator (41) is zero, bringing about the suspicion for the BRST triviality of $V_5$. However, this argument only implies the vanishing of some (but not all) of the correlation functions of $V_5$ but, strictly speaking, says nothing about its BRST triviality. That this argument does not guarantee the vanishing of all of the correlators of $V_5$ can be shown from the following. The non-singular OPE of $\Gamma$ with $U_5$ is given by

$$\Gamma(z) U_5(w) \sim (z - w)^2 e^{-2\phi_\mu m_1 ... \psi_{m_4} (i(k\psi)\psi_{m_5} + \partial X_{m_5}) e^{ikX}}$$  \hspace{1cm} (2.13)

(here we ignore the $c \partial \xi$ term of $\Gamma$ which is irrelevant for integrated vertices). This operator product vanishes for $z = w$. Imagine, however, that the correlation function involving $U_5$ also contains another integrated physical operator, $\oint \frac{du}{2\pi i} W_5(u)$ such that the OPE of $\Gamma$ and $W_5$ is singular and the singularity order is greater, or equal to $(z - u)^{-2}$. Suppose such a singularity cannot be removed by the picture-changing so it is not the artifact of a picture choice (as this is the case for e.g. the photon at
picture +1) Then integral over \( u \) particularly includes the vicinity of \( z \), the location of \( U_5 \). If \( \lim_{u \to w} \Gamma(z) W_5(u) \sim (z-w)^{-n}; n \geq 2 \), moving the location of \( \Gamma \) from \( z \) to \( w \) can no longer annihilate the correlation function and hence corresponding amplitude will be nonzero. Thus, the first important conclusion we draw is that (40) does not imply the vanishing of all of the amplitudes of \( V_5 \) but only of those not containing the operators having singular OPE’s with \( \Gamma \). For example, all of the corelators of the type \( < V_5(z_1)...V_5(z_m)Z_1(u_1)...Z_n(u_n) > \) where \( Z_i(u_i) \) are perturbative superstring vertices (e.g. a photon) must certainly vanish. However, this does not imply the triviality of \( V_5 \). Consider an operator \( W_5 = H_{m_1...m_5} \oint_{2\pi} e^{\phi \psi_{m_1}...\psi_{m_5}} e^{ikX}(z) \) which is similar to the five-form (31) with \( e^{-3\phi} \) replaced by the operator \( e \) of the identical conformal dimension \( -\frac{3}{2} \). Though this operator isn’t BRST-invariant below we shall demonstrate how its invariance can be restored by adding the \( b-c \) ghost dependent terms. Its \( OPE \) with \( \Gamma \) is given by

\[
\Gamma(z)e^{\phi \psi_{m_1}...\psi_{m_5}} e^{ikX}(w) \sim (z-w)^{-2} e^{2\phi \psi_{m_1}...\psi_{m_5}} (i(k\psi)\psi_{m_5} + \partial X_{m_5}) e^{ikX}(w) + ...
\]

(2.14)

i.e. is precisely what we are looking for. So our goal now is to derive the correction terms restoring the BRST-invariance of \( W_5 \) and to demonstrate the nonzero correlator involving the \( V_5 \) and \( W_5 \) operators. This would be a sufficient proof of the non-triviality of both \( V_5 \) and \( W_5 \). We start with the BRST invariance restoration. The strategy is the following. Consider the BRST charge given by

\[
Q_{brst} = \oint \frac{dz}{2i\pi} (cT - bc\partial c - \frac{1}{2} \gamma \psi_m \partial X^m - \frac{1}{4} \gamma^2 b)
\]

(2.15)

Introduce an operator

\[
L(z) = -4ce^{2\gamma - 2\phi} \equiv: \xi \Gamma^{-1}:
\]

(2.16)
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satisfying \( \{Q_{\text{brst}}, L\} = 1 \) Consider a non-invariant operator \( V \) satisfying \( \{Q_{\text{brst}}, V\} = W \) for some \( W \). Then, as \( W \) is BRST-exact, clearly the transformation \( V \to V_{\text{inv}} = V - LW \) restores BRST-invariance. Applying this scheme for \( W_5 \), we have

\[
\{Q_{\text{brst}}, W_5\} = H_{m_1...m_5}(k) \oint \frac{dz}{2i\pi} e^{2\phi - \chi + ikX} R^{m_1...m_5}_1(z) + be^{3\phi - 2\chi + ikX} R^{m_1...m_5}_2(z) \quad (2.17)
\]

where

\[
R^{m_1...m_5}_1(z) = -\frac{1}{2} \psi^{m_1}...\psi^{m_5}(\psi \partial X) - \frac{1}{2} \psi^{[m_1}...\psi^{m_4]}(\partial^2 X^{m_5]) + \partial X^{m_5]}(\partial \phi - \partial \chi)) - \frac{i}{2} \psi^{m_1}...\psi^{m_5}(k\psi)(\partial \phi - \partial \chi) + (k \partial \psi) \quad (2.18)
\]

and

\[
R^{m_1...m_5}_1(z) = -\frac{1}{4}(2\partial \phi - 2\partial \chi \partial \sigma) \psi^{m_1}...\psi^{m_5} \quad (2.19)
\]

The next step is to cast this commutator as

\[
\{Q_{\text{brst}}, W_5\} = \frac{1}{2} H_{m_1...m_5}(k) \oint \frac{dz}{2i\pi} (z - w)^2 \partial^2 R^{m_1...m_5}_1 + e^{3\phi - 2\chi} R^{m_1...m_5}_2(z) \quad (2.20)
\]

where \( w \) is some worldsheet point; the expression (49) can obviously be brought to the form (46) by partial integration. The contour integral is taken around \( w \); as the choice of \( w \) is arbitrary, any dependence on it in shall disappear in the end in correlation functions. The next step is to insert the \( L \)-operator (45) inside the integral (49).

Evaluating the \( OPE \) of \( L \) with the integrand of (49) we obtain

\[
\{Q_{\text{brst}}, W_5\} = \frac{1}{2} H_{m_1...m_5}(k) \oint \frac{dz}{2i\pi} (z - w)^2 \partial^2 (e^{2\phi - \chi} R^{m_1...m_5}_1 + e^{3\phi - 2\chi} R^{m_1...m_5}_2(z)) \quad (2.20)
\]
\[ W_{5\text{inv}}(k, w) = H_{m_1...m_5}(k) \{ \oint \frac{dz}{2\pi i} e^{i\phi} \psi^{m_1}...\psi^{m_5} e^{ikX} \]
\[ -\frac{1}{2} \oint w \frac{dz}{2\pi i} (z - w)^2 : L\partial_z^2(e^{2\phi - \gamma R_1^{m_1...m_5}} + e^{3\phi - 2\gamma R_2^{m_1...m_5}}(z)) : \} \]
\[ = H_{m_1...m_5}(k) \{ \oint \frac{dz}{2\pi i} e^{i\phi} \psi^{m_1}...\psi^{m_5} e^{ikX} - 2 \oint w \frac{dz}{2\pi i} (z - w)^2 ce^{\gamma R_1^{m_1...m_5}}(k, z) \} \] (2.21)

This concludes the construction of the BRST-invariant 5-form state at picture +1. However one still has to check if \( W_{5}\)-operator is non-trivial. This is the legitimate concern since the BRST-invariant integrand of (50) is BRST-invariant and has conformal dimension 3 and it is well-known that invariant operators of conformal dimension other than 0 can always be expressed as BRST commutators. Indeed, the commutator of any operator of dimension \( h \) with the zero mode of \( T \) satisfies

\[ [T_0, V_h] = hV_h \] (2.22)

where \( T_0 = \oint \frac{dz}{2\pi i} zT(z) \). Since \( T_0 = \{ Q_{\text{brst}}, b_0 \} \), for any invariant \( V_h \) one has

\[ V_h = \frac{1}{h} \{ Q_{\text{brst}}, b_0 V_h \} \] (2.23)

Acting on (50) with \( b_0 \) gives

\[ b_0 W_{5\text{inv}} = \frac{2}{3} \oint w \frac{dz}{2\pi i} (z - w)^3 e^{\gamma R_1^{m_1...m_5}}(k, z) \] (2.24)

accordingly \( W_{5\text{inv}} \) can be represented as commutator

\[ W_{5\text{inv}} = \{ Q_{\text{brst}}, \frac{1}{6} \oint w \frac{dz}{2\pi i} (z - w)^2 \xi R_1^{m_1...m_5}(k, z) \} \] (2.25)

i.e. it is the BRST commutator with the operator outside the small Hilbert space. For this reason, the operator \( W_{5\text{inv}} \) is BRST nontrivial. The commutator (52) also insures that the \( W_{5\text{inv}} \) operator is in the small Hilbert space despite its manifest dependence on since \( \{ Q_{\text{brst}}, \xi \} \) is just the picture-changing operator. An example
of non-zero correlation function is a 3-point correlator of 2 five-form states and one photon. As it is clear from the above, one five-form should be taken at a picture +1 and another at a picture -3, as any 3-point correlator containing the same-picture five-forms would vanish. Technically, this means that a photon state appears in the operator product of two five-forms of positive and negative pictures, but not in the $OPE$ of the same picture five-forms. The non-zero correlator involves one unintegrated picture +1 photon \[15\]:

\[
V_{ph}(q, z) = A_m(q) : \Gamma\{c(\partial X^m + i(kq)\psi^m) + \gamma\psi^m\}e^{iqX} : (z) \tag{2.26}
\]

one unintegrated picture $-3$ five-form:

\[
V_5(k, z) = H_{m_1...m_5}(k) : ce^{-3\phi}\psi^{m_1}\psi^{m_5}e^{ikX} : (z) \tag{2.27}
\]

and one integrated five-form $W_{5\text{inv}}(p)$ of (50) at picture $+1$. Such a combination of pictures ensures the correct ghost number balance of the correlator necessary to cancel the background ghost charges - that is, the $\phi$ ghost number $-2$, $\chi$ ghost number $+1$ and $b-c$ ghost number $-3$, due to the contribution of the correction term $c\xi R^m_1...m_5(k, z)$ of $W_5$. Such a three-point function is in some way unusual as the usual perturbative three-point correlators in string theory normally involve the unintegrated vertices only. Evaluating the correlator and evaluating the contour integral around the insertion point $z_2$ of $W_{5\text{inv}}$ we find the result to given by

\[
<V_5(k, z_1)W_{5\text{inv}}(p, z_2)V_{ph}(q, z_3)> = H_{m_1...m_5}(k)H^{m_1...m_5}(p)((kq)(pA) - (pq)(kA))
\tag{2.28}
\]

After the Fourier transform, this correlator leads to the low-energy effective action term given by
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\[ S_{\text{eff}} \sim \int d^{10}x (dH)^m_{l_1...l_5} (dH)^{l_1...l_5 m} F_{mn} \]  

(2.29)

Remarkably, this term, stemming from the worldsheet correlator of two five-forms with a photon on a sphere is identical to the one obtained from the disc amplitude of a photon with two Ramond-Ramond operators of the 6-form field strengths \( dH \). Thus the \( V_5 \) open string five-form state is identified as a source of the Ramond-Ramond 5-form charge, i.e. a D4-brane. This calculation also illustrates the BRST non-triviality of \( V_5 \) and \( W_5 \) states. Having illustrated the appearance of the picture-dependent physical vertex operators, we are now prepared to give a formal definition of ghost cohomologies [7, 8].

1) The positive ghost number \( N \) cohomology \( H_N (N > 0) \) is the set of physical (BRST invariant and non-trivial) vertex operators existing at positive superconformal ghost pictures \( n \geq N \), that are annihilated by the inverse picture-changing operator \( \Gamma^{-1} = c \partial \xi e^{-2\phi} \) at the picture \( N \). \( \Gamma^{-1}V^{(N)} := 0 \) This means that the picture \( N \) is the minimal positive picture at which the operators \( V \subset H_N \) can exist.

2) The negative ghost number \( -N \) cohomology \( H_{-N} \) consists of the physical vertex operators that existing at negative superconformal pictures \( n \leq N \) (\( N > 0 \)) and are annihilated by the direct picture changing at picture \( -N \) : \( \Gamma V^{(-N)} := 0 \).

3) The operators existing at all pictures, including picture zero, at which they decouple from superconformal ghosts, are by definition the elements of the zero ghost cohomology \( H_0 \). The standard string perturbation theory thus involves the elements of \( H_0 \), such as a photon. The picture \( -3 \) and picture \( +1 \) five-forms considered above are the elements of \( H_{-3} \) and \( H_1 \) respectively.

4) Generically, there is an isomorphism between the positive and negative ghost cohomologies: \( H_{-N-2} \sim H_N \); \( N \geq 1 \) as the conformal dimensions of the operators \( e^{-(N+2)\phi}(z) \) and \( e^{N\phi} \) are equal and given by \( -\frac{N^2}{2} - N \). That is, to any element of \( H_{-N-2} \) there corresponds an element from \( H_N \), obtained by replacing \( e^{-(N+2)\phi} \rightarrow e^{N\phi} \) and adding the appropriate \( b-c \) ghost terms in order to restore the BRST-invariance,
using the prescription mentioned before. For this reason, we shall refer to the cohomologies $H_{-N-2}$ and $H_N$ as dual. Note that the cohomologies $H_{-1}$ and $H_{-2}$ are empty, as any operator existing at pictures $-2$ or $-1$ is either trivial or can always be brought to picture zero by $\Gamma$.

The important property of the ghost cohomologies is that typically, the elements of $H_{-N}$ have singular operator products with the inverse picture-changing $\Gamma^{-1}$ while the elements of $H_N$ have singular OPE’s with the direct picture-changing $\gamma$. As has been explained above, it is this property that ensures that correlators containing the vertices annihilated by direct and inverse picture-changing, do not vanish despite (30), provided that the correlator contains at least two vertices from the dual cohomologies of opposite signs. At least one of these vertices must be taken in the integrated form, for the reason we have pointed out above. In critical strings, the only nontrivial cohomologies are $H_{-3}$ and its dual $H_{1}$, i.e. the five-form states (31). In non-critical strings, because of the Liouville dressing, cohomologies of higher ghost numbers may appear as well. In the next section we will show that the dimension 1 currents from the higher ghost cohomologies enhance the symmetry algebra of the target space and can be regarded as the generators of hidden space-time symmetries originating from extra dimensions.

2.2 Extended Discrete States in $c = 1$ Supersymmetric Model

Another important physical example of how the ghost cohomologies appear is that of the $c = 1$ supersymmetric model (or equivalently, one-dimensional non-critical strings) which is the subject of this work. This model is known for its elegance and simplicity which makes it an excellent playground. The spectrum of physical states of non-critical one-dimensional string theory (or, equivalently, of critical string theory in two dimensions) is known to contain the “discrete states” of non-standard $b - c$ ghost
numbers 0 and 2 (while the standard unintegrated vertex operators always carry the ghost number 1 \[4, 5, 6\]). These states appear at the special (integer or half integer) values of the momentum \(p\) but are absent for generic \(p\). The appearance of these states is closely related to the \(SU(2)\) symmetry, emerging when the theory is compactified on a circle with the self-dual radius \[4, 6\]. The \(SU(2)\) symmetry is generated by 3 currents of conformal dimension 1:

\[
\begin{align*}
T_3 &= \oint \frac{dz}{2\pi} \partial X(z) \\
T_+ &= \oint \frac{dz}{2\pi} e^{iX\sqrt{2}} \\
T_- &= \oint \frac{dz}{2\pi} e^{-iX\sqrt{2}}
\end{align*}
\]  

(2.30)

The tachyonic operators \(W_s = e^{iX\sqrt{2}}\) with non-negative integer \(s\) are the highest weight vectors in the representation of the “angular momentum” \(s\) (here \(X\) is the \(c = 1\) matter field). The discrete primaries are then constructed by repeatedly acting on \(W_s\) with the lowering operator \(T_-\) of \(SU(2)\). For example, acting on \(W_s\) \(n\) times with \(T_-\) one generally produces the primary fields of the form \(W_{s,n} = P(\partial X, \partial^2 X, ...) e^{i\sqrt{2}X - n\sqrt{2}X}\) with \(n < s\), where \(P\) is some polynomial in the derivatives of \(X\), straightforward but generally hard to compute. After the Liouville dressing and the multiplication by the ghost field \(c\), \(W_{s,n}\) becomes the dimension 0 primary field \(cW_{s,n}\), i.e. the BRST-invariant operator for a physical state. The set of operators \(cW_{s,n}\) thus consists of the multiplet states of \(SU(2)\) which \(OPE\) structure constants can be shown to form the enveloping of \(SU(2)\) \[4\]. The discrete operators of non-standard adjacent ghost numbers 0 and 2, \(Y_{s,n}^0\) and \(Y_{s,n}^{+2}\) can be obtained from \(cW_{s,n}\) simply by considering the \(BRST\) commutators \[7\]: \(\{Q_{BRST}, Y_{s,n}^0\} = cW_{s,n}\) and \(\{Q_{BRST}, cW_{s,n}\} = Y_{s,n}^{+2}\).

One has to evaluate these commutators for arbitrary \(s\) first and then take \(s\) to be the appropriate integer number. For arbitrary \(s\) the ghost number 0 and 2 \(Y\)-operators are respectively \(BRST\) non-invariant and exact, but for the integer values of \(s\) they become independent physical states, defining the BRST cohomologies with non-trivial \(b - c\) ghost numbers. At the first glance, things naively could seem to be the same
in case when the $c = 1$ theory is supersymmetrized on the worldsheet. However, in this paper we will show that the interaction of the $c = 1$ theory with the system of ghosts dramatically extends the spectrum of the physical discrete states. Consider the $c = 1$ model supersymmetrized on the worldsheet coupled to the super Liouville field. The worldsheet action of the system in the conformal gauge is given by [8]:

\[
S = S_{\chi-\psi} + S_L + S_{b-c} + S_{\beta-\gamma},
\]

\[
S_{\chi-\psi} = \frac{1}{4\pi} \int d^2 z \left\{ \partial X \bar{\partial} X + \psi \bar{\partial} \psi + \bar{\psi} \partial \psi \right\},
\]

\[
S_L = \frac{1}{4\pi} \int d^2 z \left\{ \partial \varphi \bar{\partial} \varphi + \lambda \bar{\partial} \lambda + \bar{\lambda} \partial \lambda - F^2 + 2\mu_0 b e^{b\varphi} (ib\lambda \bar{\lambda} - F) \right\},
\]

\[
S_{b-c} + S_{\beta-\gamma} = \frac{1}{4\pi} \int d^2 z \left\{ b \bar{\partial} c + b \bar{\partial} \bar{c} + \beta \partial \gamma + \bar{\beta} \partial \bar{\gamma} \right\}
\]

with $Q \equiv b + b^{-1}$ being the background charge. The stress tensors of the matter and the ghost systems and the standard bosonization relations for the ghosts are given by

\[
T_m = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} \partial \psi \bar{\partial} \psi - \frac{1}{2} (\partial \varphi)^2 + \frac{Q}{2} \partial^2 \varphi
\]

\[
T_{gh} = \frac{1}{2} (\partial \sigma)^2 + \frac{3}{2} \partial^2 \sigma + \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} \partial^2 \chi - \frac{1}{2} (\partial \varphi)^2 - \partial^2 \varphi
\]

\[
c = e^\sigma, \quad b = e^{-\sigma}
\]

\[
\gamma = e^{\phi - \chi}, \quad \beta = e^{\chi - \phi} \partial \chi
\]

The central charge of the super Liouville system is equal to $c_L = 1 + 2Q^2$ and, since $c_{\chi-\psi} + c_L + c_{b-c} + c_{\beta-\gamma} = 0$, we have $Q^2 = 4$ for one-dimensional non-critical superstrings.

The $SU(2)$ algebra is now generated by the dimension 1 currents [7]:

\[
T_{0,0} = \oint \frac{dz}{2\pi} \partial X
\]

\[
T_{0,1} = \oint \frac{dz}{2\pi} e^{iX} \psi
\]

\[
T_{0,-1} = \oint \frac{dz}{2\pi} e^{-iX} \psi
\]

Here and elsewhere the first lower index refers to the superconformal ghost number and the second to the momentum in the $X$ direction. In comparison with the bosonic case, a crucial novelty emerges due to the interaction of the matter with the $\beta - \gamma$
system of superconformal ghosts. That is, we will show that, apart from the dimension one currents (62) generating $SU(2)$, there is also a set of $BRST$-nontrivial dimension 1 currents mixed with $\beta - \gamma$ ghosts, i.e. those existing at nonzero $\beta - \gamma$ ghost pictures and not reducible to (62) by any picture-changing transformation. These extra currents turn out to enhance the actual underlying symmetry of the theory from $SU(2)$ two $SU(n)$ with $n \geq 3$ being the highest order ghost number cohomology of the operators in the current algebra (see the discussion below). For example, the currents of the form

$$T_{-n,n-1} = \oint \frac{dz}{2i\pi} e^{-n\phi+X_\psi(z)}$$

(2.34)

with ghost numbers $n \leq -3$ are $BRST$-nontrivial, even though they are annihilated by picture-changing transformation and no picture zero version of these currents exists. The possibility of existence of $BRST$-nontrivial operators annihilated by picture-changing has been discussed in case of $n = -3$ for the critical $D = 10$ superstrings. In this paper we shall also present a proof of the $BRST$ non-triviality of the dimension 1 currents with arbitrary negative ghost numbers $n \leq -3$ in two-dimensional critical superstring theory. Next, acting on $T_{-n,n-1}$ repeatedly with $T_-$ of (62) one obtains the currents of the form

$$T_{-n,m} = \oint \frac{dz}{2i\pi} P_{-n,m}(\partial X_\psi, \partial^2 X_\psi, \ldots, \partial \psi, \partial^2 \psi, \ldots) e^{-n\phi+i \partial X}$$

(2.35)

$$|m| \leq n - 1$$

which generally are the $BRST$ non-trivial Virasoro primaries of ghost number $-n$ annihilated by picture-changing. Here $P_{-n,m}$ are the polynomials in $\partial X_\psi$ and their derivatives, with the conformal weight $h = \frac{1}{2}(n^2 - m^2) + n + 1$ so that the overall dimension of the integrands in (64) is equal to 1. In this paper we will derive the precise expressions for these polynomials for the cases $n = -3$ and $n = -4$. So if one considers all the picture-inequivalent currents with ghost numbers $p = 0, -3, \ldots, -n$, including those of (62), (all the dimension 1 currents with ghost numbers -1 and -2
are equivalent to the $SU(2)$ generators (62) by picture-changing) one has the total number $n^2-1$ of T-generators (64). In this paper we will show (precisely for $|n| \leq 4$ and conjecture for larger values of $|n|$) that the algebra of operators $T_{p,m}, |m| \leq |p-1|, p = -3, ..., -n$ combined with 3 $SU(2)$ generators of (62) is simply $SU(n)$ with the generators $T_{0,m}$ ($m = 0, -3, -4, ..., -n$) being in the Cartan subalgebra of $SU(n)$.

The next step is easy to guess. For each given $n$ one starts with the Liouville dressed tachyonic Virasoro primaries

$$\oint \frac{dz}{2i\pi} V_l = \oint \frac{dz}{2i\pi} e^{ilX+(l-1)\varphi}(l\bar{\psi} - i(l-1)\bar{\varphi})$$

with integer $l$ and acts on them with various combinations of lowering T-operators (i.e. those having singular OPEs with integrands of $V_l$). The obtained operators will be the multiplets of $SU(n)$, including the operators of BRST cohomologies with non-trivial ghost dependence (not removable by the picture changing). These operators are generally of the form

$$V_{q,n,m,l} = \oint \frac{dz}{2i\pi} e^{(l-1)\varphi - q\phi + imX} P_{q,n,m,l}(\partial X, \partial^2 X, ..., \psi, \bar{\psi}, ..., \partial \phi, \partial^2 \phi, ...) : (z)$$

$$|m| \leq l$$

(2.36)

Of course, the usual well-known discrete states form the operator subalgebra of ghost cohomology number zero in the space of $\{V_{q,n,m,l}\}$. Here and elsewhere the term “ghost cohomology” refers to the factorization over all the picture-equivalent states; thus the ghost cohomology of number $-q (q > 0)$ consists of $BRST$ invariant and non-trivial operators with the ghost numbers $-r \leq -q$ existing at the maximal picture $-q$, so that their picture $-q$ expressions are annihilated by the operation of the picture-changing; conversely, the operators of a positive ghost cohomology number $q > 0$ are the physical operators annihilated by the inverse picture changing at the picture $q$, so that they don’t exist at pictures below $q$. Of course, the ghost cohomologies are not
cohomologies in the literal sense, since the picture-changing operator is not nilpotent. In the rest of the thesis we will particularly demonstrate the above construction by precise computations, deriving expressions for the new vertex operators of the ghost-dependent discrete states and computing their structure constants.

2.3 Extended Current Algebra and Ghost-dependent Discrete States

As usual, in the supersymmetric case three $SU(2)$ currents (62) can be taken at different ghost pictures. For example, the picture -1 expressions for the currents (62): $e^{-\phi}\psi, e^{-\phi\pm iX}$ are the only dimension 1 generators at this superconformal ghost number; similarly, all the ghost number -2 dimension 1 operators: $e^{-2\phi}\partial X, e^{-2\phi\pm iX}\psi$ are just the the $SU(2)$ generators (62) at picture -2. However, at ghost pictures of -3 and below, the new dimension 1 generators, not reducible to (62) by picture-changing, appear. The first example is the generator given by the worldsheet integral

$$T_{-3,2} = \oint dz \frac{dz}{2i\pi} e^{-3\phi+2iX}\psi(z)$$ (2.37)

where as usual $\phi$ is a bosonized superconformal ghost with the stress tensor $T_{\phi} = -\frac{1}{2}(\partial\phi)^2 - \partial^2\phi$. This operator is annihilated by the picture-changing transformation, and we leave the proof of its BRST non-triviality until the next section. Given the generator (66), it is now straightforward to construct the dimension 1 currents of the ghost number −3 with the momenta 0, ±1 and −2, which are the Virasoro primaries, not related to (62) by picture-changing. For instance this can be done by taking the lowering operator $T_{0,-1} = \oint dz \frac{dz}{2i\pi} e^{-iX}\psi(z)$ of $SU(2)$ and acting on (66). Performing this simple calculation, we obtain the following extra five generators in the ghost number −3 cohomology:
\[ T_{-3,2} = \oint \frac{dz}{2\pi i} e^{-3\phi + 2iX} \psi(z) \]
\[ T_{-3,1} = \oint \frac{dz}{2\pi i} e^{-3\phi + iX} (\partial \psi \psi + \frac{i}{2}(\partial X)^2 + \frac{i}{2}\partial^2 X)(z) \]
\[ T_{-3,-1} = \oint \frac{dz}{2\pi i} e^{-3\phi - iX} (\partial \psi \psi + \frac{i}{2}(\partial X)^2 - \frac{i}{2}\partial^2 X)(z) \]
\[ T_{-3,0} = \oint \frac{dz}{2\pi i} e^{-3\phi} (\partial^2 X \psi - 2\partial X \partial \psi)(z) \]
\[ T_{-3,-2} = \oint \frac{dz}{2\pi i} e^{-3\phi - 2iX} \psi(z) \]

It is straightforward to check that all these generators are the primary fields commuting with the BRST charge. The next step is to show that the operators (67) taken with 3 standard SU(2) generators \( T_{0,0}, T_{0,1} \) and \( T_{0,-1} \) of (62) combine into 8 generators of SU(3) (up to picture-changing transformations), with \( T_{0,0} \) and \( T_{-3,0} \) generating the Cartan subalgebra of SU(3). Here an important remark should be made. Straightforward computation of the commutators of some of the generators (67) can be quite cumbersome due to high order singularities in the OPE’s involving the exponential operators with negative ghost numbers as \( e^{\alpha \phi}(z) e^{\beta \phi}(w) \sim (z - w)^{-\alpha \beta} e^{-(\alpha + \beta) \phi}(w) + ... \)

For instance, the computation of the commutator of \( T_{-3,2} \) with \( T_{-3,-2} \) would involve the OPE \( e^{-3\phi + 2iX}(z) e^{-3\phi - 2iX}(w) \sim (z - w)^{-13} e^{-6\phi}(w) + ... \) so the OPE coefficient in front of the single pole would be cumbersome to compute. In addition, as the ghost picture of the r.h.s. of the commutator is equal to -6 one would need an additional picture-changing transformation to relate it to the T-operators (62), (67). Things, however, can be simplified if we note that the operators \( T_{-n,m} \) of ghost cohomology \( -n \) \((n = 3, 4, ...)\) are picture-equivalent to the appropriate operators from the positive ghost number \( n - 2 \) cohomologies. That is, if in the expressions for any of these operators one replaces \( e^{-n \phi} \) with \( e^{(n-2)\phi} \) (which has the same conformal dimension as \( e^{-n \phi} \)) and keeps the matter part unchanged, one gets an equivalent BRST-invariant and nontrivial operator, in the sense that replacing one operator with another inside any correlator does not change the amplitude. Such an equivalence is generally up to certain b-c ghost terms needed to preserve the BRST-invariance of the operators with positive superconformal ghost numbers; however, these terms, generally do not
contribute to correlators due to the $b - c$ ghost number conservation conditions.

Thus, for example, one can replace

$$T_{-3,2} = \oint \frac{dz}{2i\pi} e^{-3\phi + 2iX} \psi(z) \rightarrow T_{+1,2} = \oint \frac{dz}{2i\pi} e^{\phi + 2iX} \psi(z)$$

(2.39)

(up to the $b - c$ ghost terms). The picture-equivalence of these operators means that, even though they are not straightforwardly related by picture-changing transformations (as $T_{-3,2}$ and $T_{+1,2}$ are respectively annihilated by direct and inverse picture-changings), any correlation functions involving these vertex operators are equivalent under the replacement (68). The precise relation between these operators is given by

$$T_{+1,2} = Z(\Gamma^4 : ce^{-3\phi + 2iX} : )$$

(2.40)

where $\Gamma^4 :$ is the normally ordered fourth power of the usual picture-changing operator $\Gamma$ while the $Z$-transformation mapping the local vertices to integrated is the analogue of $\Gamma$ for the $b - c$ ghosts [1]. The $Z$-transformation can be performed by using the $BRST$-invariant non-local $Z$-operator of $b - c$ ghost number $-1$; the precise expression for this operator was derived in another paper [1] and is given by

$$Z = \oint_w \frac{du}{2i\pi} (u - w)^3 (bT + 4ce^{2\phi - 2iX}T^2)(u)$$

where $T$ is the full matter + ghost stress-energy tensor; the integral is taken over the worldsheet boundary and acts on local operators at $w$. Similarly, for any ghost number $-n$ generators one can replace

$$T_{-n,m} \rightarrow T_{n-2,m}; \; n \leq -3$$

(2.41)

Since the Liouville and the $\beta - \gamma$ stress tensors both have the same $Q^2$, the equivalence relations (68), (70) can formally be thought of as the special case of the reflection identities in $d = 2$ super Liouville theory in the limit of zero cosmological constant


\[ L = \frac{i}{2}T_{0,0}, \quad H = \frac{i}{3\sqrt{2}}T_{-3,0} \]

\[ G_+ = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,1} + T_{-3,1}), \quad G_- = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,1} - T_{-3,1}) \]

\[ F_+ = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,-1} + T_{-3,-1}) \]

\[ F_- = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,-1} - T_{-3,-1}) \]

\[ G_3 = \frac{1}{\sqrt{2}}T_{-3,2}, \quad F_3 = \frac{1}{\sqrt{2}}T_{-3,-2} \]

Then the commutators of the operators \( L \) and \( H \) with each other and with the rest of (71) are given by

\[ [L, H] = 0 \]

\[ [L, G_+] = \frac{1}{2}G_+; \quad [L, G_-] = \frac{1}{2}G_-; \quad [L, F_+] = -\frac{1}{2}F_+; \quad [L, F_-] = -\frac{1}{2}F_- \]

\[ [L, G_3] = G_3; \quad [L, F_3] = -F_3 \]

\[ [H, G_+] = -G_+; \quad [H, G_-] = G_-; \quad [H, F_+] = F_+; \quad [H, F_-] = -F_- \]

\[ [H, G_3] = [H, F_3] = 0 \]

i.e. \( L \) and \( H \) indeed are in the Cartan subalgebra; the remaining commutators of the currents (71) are then easily computed to give

\[ [G_3, G_+] = 0; \quad [G_3, G_-] = 0; \quad [G_3, F_+] = G_-; \quad [G_3, F_-] = -G_+ \]

\[ [F_3, F_+] = 0; \quad [F_3, F_-] = 0; \quad [F_3, G_+] = F_-; \quad [F_3, G_-] = -F_+ \]

\[ [F_-, F_+] = -F_3; \quad [G_-, G_+] = -G_3; \quad [G_-, F_+] = [F_-, G_+] = 0 \]

\[ [F_+, G_+] = L - \frac{1}{2}H; \quad [F_-, G_-] = L + \frac{1}{2}H; \quad [G_3, F_3] = 2L \]

Thus the operators \( L, \quad H, \quad F_\pm, \quad G_\pm, \quad F_3 \) and \( G_3 \) of (13) simply define the Cartan-Weyl basis of \( SU(3) \). Their \( BRST \) invariance ensures that this \( SU(3) \) algebra intertwines with the superconformal symmetry of the theory. Therefore, just like in the case of usual \( SU(2) \) discrete states of two-dimensional supergravity, we can generate the
extended set of discrete super Virasoro primaries by taking a dressed tachyonic exponential operator

\[ V = \int \frac{dz}{2i\pi} e^{d X + (l-1)\varphi(I\psi - i(l-1)\lambda)} \]

at integer values of \( l \) and acting with the generators of the lowering subalgebra of \( SU(3) \) (\( F_\pm \) and \( F_3 \)). The obtained physical operators are then the multiplets of \( SU(3) \) and include the new discrete physical states of non-trivial ghost cohomologies, not reducible to the usual \( SU(2) \) primaries by any picture-changing transformations. Such a construction will be demonstrated explicitly in the next section. The scheme explained above can be easily generalized to include the currents of higher values of ghost numbers. For example, to include the currents of ghost numbers up to -4 (or up to +2, given the equivalence relations (70)) one has to start with the generator \( T_{-4,3} = \int \frac{dz}{2i\pi} e^{-4\phi+3iX} \psi \). The conformal dimension of the integrand is equal to 1. This generator is not related to any of the operators (62), (67) by the picture-changing and therefore is the part of the ghost number -4 cohomology. This cohomology contains 7 new currents \( T_{-4,m}, |m| \leq 3 \), which are the BRST-invariant super Virasoro primaries. As previously, these currents can be generated from \( T_{-4,3} \) by acting on it repeatedly with \( T_{0,-1} \) of (62). The resulting operators are given by (2.45):
Here $P^{(n)}_{\pm iX}$; $n = 2, 3, 4$ are the conformal weight $n$ polynomials in the derivatives of $X$ defined as

$$P^{(n)}_{f(X(z))} = e^{-f(X(z))} \frac{\partial^n}{\partial z^n} e^{f(X(z))}$$

(2.46)

for any given function $f(X)$. For example, taking $f = iX$ one has $P^{(1)}_{iX} = i \partial X$, $P^{(2)}_{iX} = i \partial^2 X - (\partial X)^2$ etc. The special case $f(X(z)) = X(z) = \frac{z^2}{2}$ gives the usual Hermite polynomials in $z$. The definition (75) can be straightforwardly extended to the functions of $n$ variables $f \equiv f(X_1(z), ... X_n(z))$. The direct although lengthy computation of the commutators (using the picture-equivalence relations (70)) shows that seven $T_{-4,n}$-generators of the ghost number -4 cohomology (16) taken with 8 generators of $SU(3)$ of (71) combine into 15 generators of $SU(4)$. As in the case of $SU(3)$ the Cartan subalgebra of $SU(4)$ is generated by the zero momentum currents $L$, $H$ and $T_{-4,0}$ of (71) and (74). As previously, this $SU(4)$ algebra intertwines with the conformal symmetry of the theory. The repeated applications of the lowering subalgebra of $SU(4)$ to the dressed tachyonic vertex lead to the extended set of the ghost-dependent discrete states which are the multiplets of $SU(4)$. It is natural to assume that this construction can be further generalized to include the generators of higher ghost numbers. The total number of the $BRST$-invariant generators $T_{-n,m}$ with the ghost numbers $-N \leq -n \leq -3$ and the momenta $-n + 1 \leq m \leq n - 1$ for each $n$, combined with three standard $SU(2)$ generators, is equal to $N^2 - 1$. It is natural to conjecture that altogether they generate $SU(N)$, although in this paper we leave this fact without a proof. As before, the Cartan subalgebra of $SU(N)$ is generated by the commuting zero momentum generators $T_{-n,0}; n = 0; 3, 4, ..., N$. Applying repeatedly the lowering $SU(N)$ subalgebra (i.e. the $T_{-n,m}$'s with $m \leq 0$) to the dressed tachyon operator would then generate the extended set of the physical ghost-dependent discrete states - the multiplets of $SU(N)$. In the next section we will address the question of $BRST$-nontriviality of the $T_{-n,m}$-generators. Finally, we will demonstrate the explicit construction of new ghost-dependent discrete states from
2.4 **BRST Nontriviality of The $T_{-n,m}$-Currents**

The BRST charge of the one-dimensional NSR superstring theory is given by the usual worldsheet integral [15]

$$Q_{\text{brst}} = \oint \frac{dz}{2i\pi} \left\{ cT - b\partial c + \gamma G_{\text{matter}} - \frac{1}{4} b\gamma^2 \right\} \quad (2.47)$$

where $G_{\text{matter}}$ is the full matter ($c = 1 + \text{Liouville}$) supercurrent. The BRST-invariance of the $T_{-n,m}$-currents is easy to verify by simple calculation of their commutators with $Q_{\text{brst}}$. Indeed, as the primaries of dimension one and $b-c$ ghost number zero they commute with the stress-tensor part of $Q_{\text{brst}}$ and their operator products with the supercurrent terms of $Q_{\text{brst}}$ are all non-singular. The BRST-invariance of the picture-equivalent currents $T_{n-2,m} = Z(: \Gamma^{2n-2}cS_{-n,m} :)$, [22] where $S_{-n,m}$ are the integrands of $T_{-n,m}$ and $Z$ is the picture changing operators for the $b-c$ ghost terms [15], simply follow from the invariance of $\Gamma$ and $Z$. The BRST-nontriviality of these operators is less transparent and needs separate proof. In principle, the BRST non-triviality of any of the operators can be proven if one shows that they produce non-vanishing correlators, which is what will be done precisely in the next section for the case of $n = 3$. In this section, however, we will present the proof for general values of $n$, without computing the correlators. For each $n$, it is sufficient to prove the non-triviality of $T_{-n,n-1}$-operators, as the currents with the lower momenta are obtained from $T_{-n,n-1}$ by repeated applications of the lowering generator of $SU(2)$. The BRST non-triviality of new discrete states the multiplets of $SU(n)$ then automatically follows from the non-triviality of the $T$-currents. In other words, we need to show that for each $n$ there are no operators $W_n$ in the small Hilbert space such that $[Q_{\text{brst}}, W_n] = T_{-n,n-1}$. As it is clear from the form of the BRST charge (76) there are only two possible expressions for $W_n$ (up to total derivatives that drop out after the
worldsheet integration) which commutator with the BRST charge may produce the T-currents:

\[ W_n = W^{(1)}_n + W^{(2)}_n \]

\[ W^{(1)}_n = \sum_{k=1}^{n-1} \alpha_k \oint \frac{dz}{2\pi i} e^{-(n+1)\phi + i(n-1)X} \partial^{(k)} \xi \partial^{(n-k)} X \]

\[ W^{(2)}_n = \sum_{k,l=1,k\neq l}^{k+l \leq 2n} \alpha_{kl} \oint \frac{dz}{2\pi i} e^{-(n+2)\phi + i(n-1)X} \psi \partial^{(k)} \xi \partial^{(2n-k-l)} c \]

with \( \alpha_k \) and \( \alpha_{kl} \) being some coefficients and \( \xi = e^x \). Generically, the \( W \)-operators may also contain the worldsheet derivatives of the exponents of \( \phi \) (corresponding to the derivatives of delta-functions of the ghost fields), but one can always bring these operators to the form (77) by partial integration. Clearly, the operators \( W^{(1)}_n \) and \( W^{(2)}_n \) are the conformal dimension one operators satisfying the relations

\[ [\oint \frac{dz}{2\pi i} \gamma^2 \psi \partial X, W^{(1)}_n] \sim T_{-n,n+1} \]

\[ [\oint \frac{dz}{2\pi i} \gamma^2 b, W^{(2)}_n] \sim T_{-n,n-1} \]

\[ [\oint \frac{dz}{2\pi i} \gamma^2 \psi \partial X, W^{(2)}_n] = [\oint \frac{dz}{2\pi i} \gamma^2 b, W^{(2)}_n] = 0 \]

Therefore the T-currents are BRST-trivial if and only if there exists at least one combination of the coefficients \( \alpha_k \) or \( \alpha_{kl} \) such that \( W_n \) commutes with the stress tensor part of \( Q_{brst} \), that is, either

\[ [\oint \frac{dz}{2\pi i} (cT - b\partial c), W^{(1)}_n] = 0 \]

\[ (2.50) \]

or

\[ [\oint \frac{dz}{2\pi i} (cT - b\partial c), W^{(2)}_n] = 0 \]

\[ (2.51) \]

So our goal is to show that no such combinations exist. We start with \( W^{(1)}_n \). Simple computation of the commutator, combined with the partial integration (to get rid of terms with the derivatives of \( \phi \)) gives
\[
\left[ \oint \frac{dz}{2i\pi}, W^{(1)}_n \right] = e^{-(n+1)\phi + i(n-1)X} \sum_{k=1}^{n-1} \sum_{a=1}^{k} \frac{kl}{n(k-a)} \partial^{(a)} c \{ \partial^{(k-a+1)} \xi \partial^{(n-k)} X \alpha_k \\
+ \partial^{(n-k)} \xi \partial^{(k-n+1)} X \alpha_{n-k} \} - \alpha_k \{ n \partial c \partial^{(k)} \xi \partial^{(n-k)} X + c \partial^{(k)} \xi \partial^{(n-k)} X \}
\] (2.52)

The operator \( W^{(1)}_n \) is \( BRST \)-trivial if for any combination of the \( \alpha_k \) coefficients this commutator vanishes. The condition \( [Q_{brst}, W^{(1)}_n] = 0 \) gives the system of linear constraints on \( \alpha_k \). That is, it's easy to see that the right hand side of (81) consists of the terms of the form

\[
\sim e^{-(n+1)\phi + i(n-1)X} \partial^{(a)} c \partial^{(b)} \xi \partial^{(n-a-b+1)} X
\]

with \( a, b \geq 1, a + b \leq n \). The \( T_{-n,n-1} \)-operator is \( BRST \)-trivial if and only if the coefficients (each of them given by some linear combination of \( \alpha_k \)) in front of the terms in the right hand side of (81) vanish for each independent combination of \( a \) and \( b \). The number of independent combinations of \( a \) and \( b \) is given by \( \frac{1}{2} n(n+1) \) and is equal to the number of constraints on \( \alpha_k \). The number of \( \alpha_k \)'s is obviously equal to \( n - 1 \) that is, for \( n \leq -3 \) the number of the constraints is bigger than the number of \( \alpha \)'s. This means that the system of equations on \( \alpha_k \) has no solutions but the trivial one \( \alpha_k = 0; k = 1, \ldots, n - 1 \) and no operators of the \( W^{(1)}_n \)-type satisfying (79) exist. Therefore there is no threat of the \( BRST \)-triviality of the \( T_{-n,n+1} \)-operator from this side. Next, let's consider the case of \( W^{(2)}_n \) and show that there is no combination of the coefficients \( \alpha_{kl} \) for which (80) is satisfied. The proof is similar to the case of \( W^{(1)}_n \).

The number of independent coefficients \( \alpha_{kl} \) (such that \( k, l \geq 1; k < l \) and \( k + l \leq 2n \)) is equal to \( N_1 = n^2 - n \). The commutator \( \left[ \oint \frac{dz}{2i\pi} (cT - bc\partial c), W^{(2)}_n \right] = 0 \) leads to the terms of the form

\[
\sim \int \frac{dz}{2i\pi} e^{-(n+2)\phi + i(n-1)X} \psi^{(k)} \xi \partial^{(m)} \xi \partial^{(n+1-k-l-m)} c
\]

with all integer \( k, l, m \) satisfying \( k, l \geq 1, m \geq 0; k < l, m < 2n + 1 - k - l - m; \)
$k+l+m \leq 2n+1$. The total number of these independent terms, multiplied by various linear combinations of $kl$, is given by the sum $N_2 = 2 \sum_{p=1}^{n-1} p(n-p)$ and is equal to the number of the linear constraints on $\alpha_{kl}$. Since for $n \leq -3$ one always has $N_2 > N_1$, there are no non-zero solutions for $\alpha_{kl}$, and therefore no operators of the $W_n^{(2)}$-type exist. This concludes the proof of the BRST-nontriviality of the $T$-currents. Our proof implied that, generically, the constraints on $\alpha$’s are linearly independent. In fact, such an independence can be demonstrated straightforwardly by the numerical analysis of the systems of linear equations implied by $[\oint \frac{dz}{2\pi i} (cT - bc\partial c), W_n^{(1),(2)}] = 0$.

2.5 \textit{SU}(n) Multiplets and Their Structure Constants

In this section, we will demonstrate the straightforward construction of the $SU(N)$ multiplets of the ghost-dependent discrete states and compute their structure constants in the case of $N = 3$. The discrete states generated by the lowering operators of the current algebra realise various (generically, reducible) representations of the $SU(3)$. We start with the decomposition of the current algebra \cite{23}

$$SU(3) = N_+ \oplus N_0 \oplus N_- \quad (2.53)$$

with the operators $L$ and $H$ being in the Cartan subalgebra $N_0$, the subalgebra $N_+$ consisting of 3 operators $G_\pm$ and $G_3$ with the unit positive momentum and with 3 lowering operators $F_\pm$ and $F_3$ with the unit negative momentum being in $N_-$. This corresponds to the Gauss decomposition of $SL(3, C)$ which compact real form is isomorphic to $SU(3)$. Then the full set of the $SU(3)$ multiplet states can be obtained simply by the various combinations of the $N_-$-operators acting on the set of the highest weight vector states. So our goal now is to specify the highest weight vectors. In fact, it’s easy to check that, just as in the $SU(2)$ case, the highest weight vectors are simply the dressed tachyonic operators.
where $l$ is integer valued, however there is one important subtlety. It is clear, for example, that the tachyon with $l = 1$ cannot be the highest weight vector since 

$$[H, \oint \frac{dz}{2i\pi} V_l] \equiv [H, T_{0,1}] = \frac{1}{\sqrt{2}} T_{-3,1},$$

therefore $\oint \frac{dz}{2i\pi} V_l$ is not an eigenvector of $N_0$. In addition, $\oint \frac{dz}{2i\pi} V_l$ isn’t annihilated by $N_+$ since $[T_{-3,1}, \oint \frac{dz}{2i\pi} V_l] \sim T_{-3,2}$. Therefore we have to examine carefully how the operators of $N_0$ and $N_+$ act on generic $\oint \frac{dz}{2i\pi} V_l$. First of all, it is clear that all the $V_l$’s with $l \geq 2$ are annihilated by $N_+$ since their OPE’s with the integrands of $G_\pm$ and $G_3$ are non-singular. Also, all these tachyons are the eigenvectors of $L$ of $N_0$ with the weight $\frac{1}{2}$ i.e. one half of the isospin value (the actual isospin value must be taken equal to $l$ since our conventions involve the factor of $-\frac{1}{2}$ in the stress-energy tensor and hence our normalization of the $<XX>$-propagator is one half of the one used in literature [4]; thus the $L$-operator corresponds to one half of the appropriate Gell-Mann matrix). So we only need to check how these operators are acted on by the hypercharge generator $H$ of $N_0$. Simple calculation gives

$$[H, \oint \frac{dz}{2i\pi} V_l] = \left[ \frac{i}{\sqrt{2}} \oint \frac{dw}{2i\pi} e^{-3\phi} (\partial^2 X \psi - 2 \partial X \partial \psi), \oint \frac{dw}{2i\pi} e^{ilX} (1 \psi - i(l - 1) \lambda)(w) \right]$$

$$= \frac{i}{\sqrt{2}} \oint \frac{dw}{2i\pi} e^{-3\phi + ilX + (l-1)\varphi} \left\{ 3il^2 \partial \psi \partial \overline{\psi} - 3l \partial^2 X + \frac{1}{2} il^2 P_{-3\phi}^2 + (6l \partial X - 3l(l - 1) \psi \lambda) \partial \phi + 3l(l - 1) \partial \psi \lambda \right\}$$

It is convenient to integrate by parts the terms containing the derivatives of $\phi$. Performing the partial integration we get

$$[H, \oint \frac{dz}{2i\pi} V_l] = \frac{i}{\sqrt{2}} \oint \frac{dz}{2i\pi} e^{-3\phi + ilX + (l-1)\varphi} \left\{ 3il^2 \partial \psi \partial \overline{\psi} - l(1 + \frac{l^2}{2}) \partial^2 X + il^2(l - 1) \partial^2 \varphi + l(l - 1)(2 \partial \psi \lambda - \psi \partial \lambda) \right\}$$

$$+ (il \partial X + (l - 1) \partial \varphi) (2l \partial X - l(l - 1) \psi \lambda) + \frac{il^2}{2} (il \partial X + (l - 1) \partial \varphi)^2$$

The operator on the right-hand side of (85) is just the dressed tachyon at the picture.
3, up to the hypercharge related numerical factor. To compute the value of the hypercharge we have to picture transform this operator three times to bring it to the original zero picture. Since we are working with the integrated vertices, the $c$-ghost term of the local picture-changing operator $\sim c\partial\xi$ doesn’t act on the integrand (this can also be seen straightforwardly from the $[Q_{\text{brst}}; \xi...]$-representation of the picture-changing), while the $b$-ghost term annihilates the right-hand side of (85). Therefore it is sufficient to consider the matter part of the picture-changing operator

$$
\Gamma =: \delta(\beta)G_{\text{matter}} := -\frac{i}{\sqrt{2}}e^{\phi}(\psi \partial X + \lambda \partial \varphi + \partial \lambda)
$$

(2.57)

Again, the factor of $\frac{1}{\sqrt{2}}$ in the matter supercurrent $G_{\text{matter}}$ appears because our normalization of the $<XX>$-propagator differs from paper [4] by the factor of $\frac{1}{\sqrt[3]{2}}$.

Applying the picture-changing operator to the right-hand side of (86) gives

$$
\Gamma : [H, \oint \frac{dz}{2i\pi} V_i] := -\frac{i}{\sqrt{2}}\frac{i}{3\sqrt{2}} \oint \frac{dz}{2i\pi} e^{-2\phi+ilX} \psi(z) \times (2-2l)
$$

(2.58)

i.e. the tachyon at the picture -2. Finally, applying the picture-changing to the right-hand side of (29) two times more is elementary and we obtain [7]

$$
\Gamma^3 : [H, \oint \frac{dz}{2i\pi} V_i(z)] = \frac{l(l-1)}{6} \oint \frac{dz}{2i\pi} V_i
$$

(2.59)

i.e. we have proven that the tachyons with the integer momenta $l \geq 2$ are the highest weight vectors of $SU(3)$. The coefficient in front of $V_i$ gives the value of the tachyon’s $SU(3)$ hypercharge

$$
s(l) = \frac{l(l-1)}{6}
$$

(2.60)

Since either $l$ or $l-1$ is even, this guarantees that possible values of the hypercharge are always the multiples of $\frac{1}{3}$. Having determined the highest weight vectors we can now easily obtain the spectrum of the physical states - the multiplets of $SU(3)$. The
CHAPTER 2. GHOST COHOMOLOGIES

vertex operators are simply given by

\[ \oint \frac{dz}{2i\pi} V_{l;p_1,p_2,p_3} = \ell \prod F_{p_1}^{p_1} F_{p_2}^{p_2} F_{p_3}^{p_3} \oint \frac{dz}{2i\pi} V_l(z) \]  

(2.61)

with all possible integer values of \( p_1, p_2 \) and \( p_3 \) such that \( p_1 + p_2 + 2p_3 \leq 2l \) This construction is in fact isomorphic to the Gelfand-Zetlin basis of the irreps of \( SU(3) \), or to the tensor representations of \( SU(3) \) on the \( T^P \)-spaces generated by the polynomials of degree \( P = p_1 + p_2 + p_3 \) in three variables \( x_1, x_2, x_3 \) spanned by \( x_1^{p_1} x_2^{p_2} x_3^{p_3} \), with \( x_{1,2,3} \) being the covariant vectors under \( SU(3) \). In particular, the values of \( P \) can be used to label the irreducible representations. In our case, the values of \( p_1, p_2, p_3 \) and \( P \) can be easily related to the isospin projection \( m \), the hypercharge \( s \) and the ghost cohomology numbers \( N \) of the vertex operators (90). Applying \( L \) and \( H \) to the operators (90) and using the commutation relations (72) we get

\[ s = p_1 - p_2 + \frac{l(l-1)}{6} \]
\[ m = l - p_1 - p_2 - 2p_3 \]
\[ N_{\text{max}} = -3P = -3(p_1 + p_2 + p_3) \]  

(2.62)

where \( N_{\text{max}} = -3P \) is the biggest (in terms of the absolute value) ghost cohomology number of the operators appearing in the expression for \( V_{l;p_1,p_2,p_3} \). Conversely,

\[ p_1 = -\frac{1}{2}(m - l + s) - \frac{1}{3} N_{\text{max}} - \frac{1}{12}(l - 1) \]
\[ p_2 = \frac{1}{2}(m - k - s) - \frac{1}{2} N_{\text{max}} + \frac{1}{12}(l - 1) \]
\[ p_3 = l - m + \frac{1}{3} N_{\text{max}} \]  

(2.63)

This concludes the demonstration how the states from higher Ghost Cohomologies extend the current algebra of the physical States.
Chapter 3

Correction terms of $T_{1,2}$

In this chapter we will show an explicit computation of the correction terms that restore the BRST invariance for $T_{1,2}$ which is the current of ghost cohomology +1, the dual current of $T_{-3,2}$ existing at ghost cohomology -3. Using the prescription $V \rightarrow V_{\text{inv}} = V - LW$, where $W = \{Q_{BRST}, V\}$. We start by calculating

$$
\{Q_{brst}, T_{1,2}\} = \oint \frac{dz}{2i\pi} \left\{ -\frac{1}{2}e^{\phi - \chi} \partial X - \frac{1}{4}e^{2\phi - 2\chi} b \right\} \oint \frac{dw}{2i\pi} e^{\phi + 2iX} \\
= -\frac{1}{2} \oint \frac{dw}{2i\pi} e^{2\phi - \chi + 2iX} (\partial^2 X + \partial X (\partial \phi \partial \chi)) - \frac{1}{2} \oint \frac{dw}{2i\pi} e^{3\phi - 2\chi + 2iX} \psi P_{(\phi)}^{(1)} P_{(\phi)}^{(2)}
$$

where $P_{(n)}^{(f(z))}$ was defined in equation (75)

$$
= W^{(1)} + W^{(2)}
$$

with

$$
W^{(1)} = -\frac{1}{2} \oint \frac{dz}{2i\pi} e^{2\phi - \chi + iX} \left\{ \partial^2 X + \partial X (\partial \phi - \partial \chi) \right\} \tag{3.1}
$$

$$
W^{(2)} = -\frac{1}{4} \oint \frac{dz}{2i\pi} e^{3\phi - 2\chi + 2iX} \psi \left\{ 2\partial \phi - 2\partial \chi - \partial \sigma \right\} \tag{3.2}
$$

As proved before, there exists an operator
CHAPTER 3. CORRECTION TERMS OF $T_{1,2}$

$$L(z) = -4c e^{2\chi - 2\phi}$$

satisfying $\{Q_{BRST}, L\} = 1$.

Making the transformation: $T_{1,2}^{inv} = T_{1,2} - LW$ it is straightforward to notice that $\{Q, T_{1,2}^{inv}\} = 0$. Now: $L(z)W(w) := L(z)W^{(1)}(w) + L(z)W^{(2)}(w)$:

The process for calculating the correction terms goes as follows:

1- Take: $L(z)W(w)$ and expand around the midpoint $\frac{z + w}{2}$ up to the $(z - w)^2$ terms.

2- Calculate $\lim_{z \to w} \left\{ \frac{1}{2} \oint \frac{dz}{2\pi i} (z - u)^2 \partial_z^2 L(z)U(z) \right\}$

3- Finally we integrate the obtained answer by parts to get an answer of the form:

$$a \oint \frac{dz}{2\pi i} e^{\phi + 2iX} \psi + \oint \frac{dz}{2\pi i} e^{\phi + 2iX} \psi(z-u)(extra\ terms) + \oint \frac{dz}{2\pi i} e^{\phi + 2iX} \psi(z-u)^2(extra\ terms)$$

These terms are the correction terms of $T_{1,2}$.

This is an explicit calculation of the mentioned terms:

$$: L(z)W^{(1)}(w) := \lim_{z \to w} 2 \oint e^{2\chi - 2\phi + \sigma}(z) e^{2\phi - \chi + 2iX}(\partial X + \partial X(\partial \phi - \partial \chi))(w) \frac{dz}{2\pi i}$$

$$= \lim_{z \to w} 2 \oint (z - w)^2 c e^{\chi + 2iX}(\partial^2 X + \partial X(\partial \phi - \partial \chi)) \frac{dz}{2\pi i}$$

$$= \lim_{z \to w} 2 \oint (z - u)^2 \partial_z^2 \left\{ (z - w)^2 c e^{\chi + 2iX}(\partial^2 X + \partial X(\partial \phi - \partial \chi)) \right\} \frac{dz}{2\pi i}$$

$$= 2 \oint (z - u)^2 c e^{\chi + 2iX}(\partial^2 X + \partial X(\partial \phi - \partial \chi)) \frac{dz}{2\pi i}$$

and

$$: L(z)W^{(2)}(w) := \lim_{z \to w} \oint e^{2\chi - 2\phi + \sigma}(z) e^{3\phi - 2\chi + 2iX} \psi(2\partial \phi - 2\partial \chi - \partial \sigma)(w) \frac{dz}{2\pi i}$$

Contracting relevant terms from the first and the second expressions, and expanding around the middle point $\frac{z + w}{2}$ and keeping the terms that are first and second order.
powers in $z - w$, i.e. the $(z - w)$ and the $(z - w)^2$ terms in the expansion, we get the following:

**$z - w$ terms:**

$$\lim_{z \to w} \oint \frac{dz}{2\pi i} e^{\phi + 2iX} (z - w) \left\{ \psi(2\partial \phi - 2\partial \chi - \partial \sigma) + \psi(\partial \chi - \partial \phi + \frac{\partial \sigma}{2} - \frac{3}{2}\partial \phi + \partial \chi - iX + \frac{\partial \sigma}{2}) - \frac{\partial \psi}{2} \right\}$$

(3.5)

The first term comes from extracting the exponentials together and keeping the other terms, the second term and the third term come from contracting the partial derivatives of $\phi, \chi, \text{ and } \sigma$ functions of $w$ with the first exponential function of $z$, and then expanding the exponentials then $\psi$ around the $\frac{(z+w)}{2}$ term to the first order.

Now summing up the terms we get:

$$\lim_{z \to w} \oint e^{\phi + 2iX} \left\{ (z - w) \left\{ \psi(2\partial \phi - 2\partial \chi - \partial \sigma + 2\partial \chi - \frac{3}{2}\partial \phi + \partial \sigma - iX) - \frac{\partial \psi}{2} \right\} \right\} \frac{dz}{2\pi}$$

$$= \lim_{z \to w} \oint e^{\phi + 2iX} (z - w) \left\{ \psi(- \frac{\partial \phi}{2} - iX) - \frac{\partial \psi}{2} \right\} \frac{dz}{2\pi}$$

The second order partial derivative of the first order $z - w$ terms as a function of $w$ will give us in the limit $z \to w$:

$$\lim_{z \to w} \partial_z^2 \left\{ \oint e^{\phi + 2iX} (z - w) \left\{ \psi(- \frac{\partial \phi}{2} - iX) - \frac{\partial \psi}{2} \right\} \frac{dz}{2\pi} \frac{dw}{2\pi} \right\}$$

$$= \lim_{z \to w} - \partial_z \left\{ \oint e^{\phi + 2iX} \left\{ \psi(- \frac{\partial \phi}{2} - iX) - \frac{\partial \psi}{2} \right\} \frac{dz}{2\pi} \right\}$$

In this equation, the factor of half coming from the differentiation of the function of $\frac{(z+w)}{2}$ by $\partial_w$ is multiplied by the factor $-2$ coming from the second order derivative of the whole function so we only retain a minus sign outside. So we get:

$$\lim_{z \to w} - \oint \frac{(z - w)^2}{2} \partial_z \left\{ e^{\phi + 2iX} \left( \psi(- \frac{\partial \phi}{2} - iX) - \frac{\partial \psi}{2} \right) \right\} \frac{dz}{2i\pi}$$
CHAPTER 3. CORRECTION TERMS OF $T_{1,2}$

\[
\lim_{z \to w} - \oint \frac{(z-u)^2}{2} e^{\phi + 2iX} \{ \partial \psi \left( -\frac{\partial \phi}{2} - iX \right) - \psi \left( -\frac{\partial^2 \phi}{2} - i\partial^2 X \right) \frac{\partial^2 \psi}{2} \\
+ \left( \partial \phi + 2i\partial X \right) \left( -\frac{\partial \phi}{2} - iX \right) - \frac{\partial \psi}{2} \left( \partial \phi + 2i\partial X \right) \frac{\partial \psi}{2} \}
\]

\[
= \oint \frac{dz}{2i\pi} \frac{(z-u)^2}{4} e^{\phi + 2iX} \left\{ \psi \left( \frac{\partial^2 \phi}{2} + i\partial^2 X - (\partial \phi + 2i\partial X) \left( -\frac{\partial \phi}{2} - i\partial X \right) \right) - \partial \psi \left( -\partial \phi - 2iX + \frac{\partial^2 \psi}{2} \right) \right\}
\]

(3.6)

\((z-w)^2\) terms: The \((z-w)^2\) are obtained as

\[
\lim_{z \to w} \oint \frac{dz}{2i\pi} e^{\phi + 2iX} (z-w)^2 \left\{ -\frac{1}{2} (2\partial^2 \phi - 2\partial^2 \chi - \partial^2 \sigma) \psi - \frac{\partial \psi}{2} (2\partial \phi - 2\partial \chi - \partial \sigma) + \psi (2\partial \phi - 2\partial \chi - \partial \sigma) \right\} \\
+ \psi (2\partial \phi - 2\partial \chi - \partial \sigma) (\partial \chi - \partial \phi + \frac{\partial \sigma}{2} - \frac{3}{2} \partial \phi + \partial \chi - iX + \frac{\partial \sigma}{2}) - \frac{\partial \psi}{2} + 1 \left\{ (2\partial^2 \chi - 2\partial^2 \phi + \partial^2 \sigma) + (2\partial \chi - 2\partial \phi + \partial \sigma)^2 \right\} \psi + \frac{1}{8} \left\{ (3\partial^2 \phi - 2\partial^2 \chi + 2i\partial^2 X - \partial^2 \sigma) + (3\partial \phi - 2\partial \chi + 2i\partial X - \partial \sigma) \partial \psi + \frac{1}{8} \partial^2 \psi \right\} \\
- \frac{1}{4} (2\partial \chi - 2\partial \phi + \partial \sigma) (3\partial \phi - 2\partial \chi + 2i\partial X - \partial \sigma) \psi - \frac{1}{4} (2\partial \chi - 2\partial \phi + \partial \sigma) \partial \psi \right\}
\]

(3.7)

he first and the second terms come from contracting the exponentials then expanding the partial derivative term then the \(\psi\) term around \(\frac{(z+w)}{2}\) to the first order, then contracting the partial derivative term with the first exponential and taking the second order expansion of the first expression, the product of the first expansion orders of the first expression and the second expression then finally taking the second order expansion term of the second expression lead to the other terms consecutively.

So we get:
\begin{align*}
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\int \frac{dz}{2\pi} e^{\phi + 2iX(z-w)^2} \{ \psi \left( -\partial^2 \phi + \partial^2 X + \frac{\partial^2 \sigma}{2} + (2\partial_X - \frac{3}{2}\phi - i\partial_X - \partial \sigma)(2\partial \phi - 2\partial_X - \partial \sigma) \right) \\
& \quad + \frac{1}{8}(2\partial_X - 2\partial \phi + \partial \sigma)^2 + \frac{3}{8}(3\partial \phi - 2\partial_X + 2i\partial_X - \partial \sigma)^2 \\
& \quad + \frac{3}{4} \partial_X \partial^2 \phi + \frac{3}{8} \partial^2 \sigma + \frac{i}{4} \partial^2 X - \frac{1}{8} \partial^2 \sigma \\
& \quad - \frac{1}{4}(2\partial_X - 2\partial \phi + \partial \sigma)(3\partial \phi - 2\partial_X + 2i\partial_X - \partial \sigma) \}
& \quad + \partial \psi \left\{ -\partial \phi + \partial_X + \frac{\partial \sigma}{2} + \frac{i}{2} \partial_X - \frac{\partial \alpha}{4} - \frac{\partial \phi}{2} - \frac{\partial \gamma}{4} \right\} + \frac{1}{8} \partial^2 \psi \}

\int \frac{dz}{2\pi} e^{\phi + 2iX(z-w)^2} \{ \psi \left( -\frac{3}{4} \partial^2 \phi + \frac{3}{2} \partial^2 X + \frac{\partial^2 \sigma}{2} + (2\partial \phi - 2\partial_X - \partial \sigma) \right) \\
& \quad + (2\partial \phi - 2\partial_X - \partial \sigma)(\frac{7}{4} \partial X - \frac{3}{2} \partial \phi + \frac{i}{4} \partial \sigma - 2i\partial_X) \\
& \quad + (3\partial \phi - 2\partial_X + 2i\partial_X - \partial \sigma)(-\frac{3}{4} \partial_X + \frac{3}{2} \partial \phi - \frac{3}{8} \partial \sigma + \frac{i}{4} \partial X) \}
& \quad + \partial \psi \left\{ \frac{3}{4} \partial \phi + \frac{i}{2} \partial X \right\} + \frac{1}{8} \partial^2 \psi \}

\text{Now using the fact that: } \lim_{z \to w} (z-w)^2 f\left( \frac{z+w}{2} \right) = 2f(w) = 2f(z), \text{ our equation becomes upon differentiating twice w.r.t. } w \text{ then taking the limit as } z \to w \text{ we get:}

\int \frac{dz}{2\pi} e^{\phi + 2iX(z-w)^2} \{ \psi \left( -\frac{3}{4} \partial^2 \phi + \frac{3}{2} \partial^2 X + \frac{\partial^2 \sigma}{2} + (2\partial \phi - 2\partial_X - \partial \sigma) \right) \\
& \quad + (2\partial \phi - 2\partial_X - \partial \sigma)(\frac{7}{4} \partial X - \frac{3}{2} \partial \phi + \frac{i}{4} \partial \sigma - 2i\partial_X) \\
& \quad + (3\partial \phi - 2\partial_X + 2i\partial_X - \partial \sigma)(-\frac{3}{4} \partial_X + \frac{3}{2} \partial \phi - \frac{3}{8} \partial \sigma + \frac{i}{4} \partial X) \}
& \quad + \partial \psi \left\{ \frac{3}{4} \partial \phi + \frac{i}{2} \partial X \right\} + \frac{1}{8} \partial^2 \psi \}

\text{Collecting all terms from equations (98) and (101), and integrating the obtained functions multiplied by } \frac{(z-w)}{2} \text{ over a closed loop as defined in the second step of the process we get the following expression to integrate:}

\int \frac{dz}{2\pi} e^{\phi + 2iX(z-w)^2} \{ \psi \left( -\partial^2 \phi + \frac{9}{4} \partial^2 \chi + \frac{25}{2} \partial^2 X + \partial^2 \sigma + \frac{1}{2}(\partial \phi + 2i\partial X)^2 \right) \\
& \quad + (2\partial \phi - 2\partial_X - \partial \sigma)(\frac{7}{4} \partial X - \frac{3}{2} \partial \phi + \frac{i}{4} \partial \sigma - 2i\partial_X) \\
& \quad + (3\partial \phi - 2\partial_X + 2i\partial_X - \partial \sigma)(-\frac{3}{4} \partial_X + \frac{3}{2} \partial \phi - \frac{3}{8} \partial \sigma + \frac{i}{4} \partial X) \}
& \quad + \partial \psi \left\{ \frac{3}{4} \partial \phi + \frac{3i}{4} \partial X \right\} + \frac{1}{4} \partial^2 \psi \}
CHAPTER 3. CORRECTION TERMS OF $T_{1,2}$

Now taking the term involving $\partial^2 \psi$ and integrating by parts twice, $\partial \psi$ and integrating by parts once, and the $\psi$ terms we obtain:

$\partial^2 \psi$ term:

$$\frac{3}{8} \oint \frac{dz}{2\pi i} (z - u)^2 e^{\phi + 2iX} \partial^2 \psi$$

$$= -\frac{3}{8} \oint \frac{dz}{2\pi} e^{\phi + 2iX} \{2(z - u) + (z - u)^2 (\partial \phi + 2i\partial X)\} \partial \psi$$

$$= \frac{3}{8} \oint \frac{dz}{2\pi} e^{\phi + 2iX} \{2 + 4(z - u)(\partial \phi + 2i\partial X) + (z - u)^2 (\partial^2 \phi + 2i\partial^2 X + (\partial \phi + 2i\partial X)^2)\} \psi$$

(3.11)

$\partial \psi$ term:

$$\frac{1}{2} \oint \frac{dz}{2\pi i} (z - u)^2 e^{\phi + 2iX} \partial \psi \{\frac{5}{2} \partial \phi + 3i \partial X\}$$

$$= -\frac{1}{2} \oint \frac{dz}{2\pi} e^{\phi + 2iX} \{\psi(z - u)^2 \left(\frac{5}{2} \partial^2 \phi + 3i\partial^2 X + (\frac{5}{2} \partial \phi + 3i \partial X)(\partial \phi + 2i\partial X)\right) + 2(z - u)(\frac{5}{2} \partial \phi + 3i \partial X)\}$$

(3.12)

$\psi$ term:

$$\frac{1}{2} \oint \frac{dz}{2\pi i} (z - u)^2 e^{\phi + 2iX} \{\psi\{-\partial^2 \phi + \frac{9}{4} \partial^2 \chi + \frac{3}{2} i \partial^2 X + \partial^2 \sigma + \frac{1}{2} (\partial \phi + 2i \partial X)^2 \}

+ (2\partial \phi - 2\partial \chi - \partial \sigma)(\frac{7}{4} \partial \chi - \frac{9}{2} \partial \phi + \frac{7}{4} \partial \sigma - 2i \partial X)$$

$$+ (3\partial \phi - 2\partial \chi + 2i \partial X - \partial \sigma)(-\frac{1}{2} \partial \chi + \frac{7}{4} \partial \phi - \frac{3}{4} \partial \sigma + \frac{1}{2} \partial X)$$

(3.13)

Collecting the terms from equations (103), (104) and (105) we get:
CHAPTER 3. CORRECTION TERMS OF $T_{1,2}$

\[
\frac{3}{4} \oint \frac{dz}{2\pi i} e^{\psi + 2iX} \psi + \oint \frac{dz}{2\pi i} (z - u) e^{\psi + 2iX} \psi \{ (\frac{3}{2} \partial \phi + 3i \partial X) - \frac{5}{2} \partial \phi - 3i \partial X \}
\]
\[
\oint \frac{dz}{2\pi i} (z - u)^2 e^{\psi + 2iX} \psi \{ \frac{3}{4} \partial^2 \phi + \frac{3}{2} i \partial^2 X + \frac{3}{4} (\partial \phi + 2i \partial X)^2 - \frac{5}{2} \partial^2 \phi - 3i \partial^2 X
\]
\[
- (\frac{5}{2} \partial \phi + 3i \partial X)(\partial \phi + 2i \partial X) - \partial^2 \phi + \frac{9}{4} \partial^2 \chi + \frac{3}{2} i \partial^2 X + \partial^2 \sigma + \frac{1}{2} (\partial \phi + 2i \partial X)^2
\]
\[
+ (2 \partial \phi - 2 \partial \chi - 2 \partial X)(\frac{5}{4} \partial \phi - \frac{9}{2} \partial \phi + \frac{7}{4} \partial \sigma - 2i \partial X)
\]
\[
+ (3 \partial \phi - 2 \partial \chi + 2i \partial X - \partial \sigma)(-\frac{3}{2} \partial \chi + \frac{5}{2} \partial \phi - \frac{3}{2} \partial \sigma + \frac{5}{2} \partial X)
\}
\]

(3.14)

so we finally get

\[
\frac{3}{4} \oint \frac{dz}{2\pi i} e^{\psi + 2iX} \psi + \oint \frac{dz}{2\pi i} (z - u) e^{\psi + 2iX} \psi (-\partial \phi)
\]
\[
+ \frac{1}{2} \oint \frac{dz}{2\pi i} (z - u)^2 e^{\psi + 2iX} \psi \{ -\frac{11}{4} \partial^2 \phi + \frac{9}{4} \partial^2 \chi + \partial^2 \sigma
\]
\[
- (\partial \phi + i \partial X)(\partial \phi + 2i \partial X) + (2 \partial \phi - 2 \partial \chi - \partial \sigma)(\frac{7}{4} \partial \chi - \frac{9}{2} \partial \phi + \frac{7}{4} \partial \sigma - 2i \partial X)
\]
\[
+ (3 \partial \phi - 2 \partial \chi + 2i \partial X - \partial \sigma)(-\frac{3}{2} \partial \chi + \frac{5}{2} \partial \phi - \frac{3}{2} \partial \sigma + \frac{5}{2} \partial X)
\}
\]

(3.15)

Thus the correction terms of $T_{1,2}$ are the terms in equation (95) and (107) which restore the $BRST$ invariance of the current. This procedure can be applied similarly to all current cohomologies and it is a general prescription to find the additional terms that restore the $BRST$ invariance of any non-invariant current carrying the ghost cohomology at a given picture.
Chapter 4

Conclusions

In this work we have demonstrated that the appearance of the new physical ghost-dependent generators from the first non-trivial ghost cohomology $H_{-3} \sim H_1$ in non-critical RNS superstring theories leads to the enhancement of the current algebra of space-time generators. These generators correspond to new hidden symmetries of the supersymmetric $c = 1$ model, associated with the new ghost-dependent vertex operators. We have demonstrated that the currents from the $H_{-3} \sim H_1$ ghost cohomology extend the current algebra from $SU(2)$ to $SU(3)$. Just as the $SU(2)$ current algebra is isomorphic to area preserving diffeomorphisms in two-dimensions, the $SU(2) \sim SL(3,R)$ algebra of currents is isomorphic to the volume preserving diffeomorphisms in $d = 3$. This result implies the holographic relations between two-dimensional supergravity and field-theoretic degrees of freedom in three dimensions. The extra dimension is effectively generated by the superconformal ghost degrees of freedom. Physically, the symmetry of the volume preserving diffeomorphism correspond to the motion of incompressible liquid in three dimensions. The interesting question therefore is whether the above mentioned holographic relation could be used to reformulate the problems of hydrodynamics in higher space-time dimensions on the string-theoretic language. In particular, such a correspondence could shed the light on the problem of turbulence, especially the question of the transition from laminar to
turbulent flows in incompressible liquids. The natural and important generalization of the above results would be to consider the ghost cohomologies of higher numbers, for example $H_{-4} \sim H_2$ and to explore their influence on the internal symmetries of the theory. The basic conjecture discussed in this work is that including the currents from the cohomologies of ghost number up to $N : H_{-N-2} \sim H_N$ extends the current algebra to $SU(N + 2)$ corresponding to volume preserving diffeomorphisms (motion of incompressible liquid) in $d = N + 2$ dimensions. Thus it appears that in $c = 1$ supersymmetric model each new cohomology effectively corresponds to opening up a theory to a new hidden space-time dimension. We have proven this fact for $N = 1$ and conjectured for higher $N$’s. From the technical point of view, the $c > 1$ case appears to be more complicated, but it is tempting to assume that qualitatively the same scenario is still true for $c > 1$: that is, the extra space-time generators $H_{-N-2} \sim H_N$ ; $N = 2, 3...$ correspond to yet unknown space-time symmetries of the theory, related to yet unknown hidden space-time dimensions. Technically, the space-time symmetry generators from $H_{-N-2} \sim H_N$ should inherit their structure from the Cartan generators of $SU(N + 2)$ of the $c = 1$ case, precisely as we have shown it for the $N = 1$ case. Unfortunately, for $N \geq 2$ the expressions for the Cartan generators become quite cumbersome and we leave the $N \geq 2$ case for future work. Another important result of this work is the explicit construction of the correction terms for the currents with higher ghost cohomologies which provide us with explicit isomorphism between positive and negative ghost cohomologies. Since the commutators in the current algebra typically involve the operators from the ghost cohomologies of both signs, this isomorphism undoubtedly will be of crucial technical importance for the calculations involving the vertex operators of higher order cohomologies from $H_{-N-2} \sim H_N$; $N \geq 2$. Hopefully, these calculations will demonstrate the existence of new holographic relations between field theories in higher dimensions, and will lead to discovery of new intriguing hidden symmetries in space-time.
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