Treatment of an accelerating observer
via the special theory of relativity

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Abstract
When treated in isolation, the acceleration of an observer $\mathcal{B}$ in a spaceship is felt as a gravitational force (characterized by the spacetime metric) in system $S'$. Acceleration of the spaceship can also be defined in terms of a change in velocity $v(t)$ measured in an inertial system $S$ by an observer $\mathcal{A}$. In that case, the clock time $t'$ of $\mathcal{B}$ simultaneous with measured time $t$ in system $S$ is $t' = t\sqrt{1 - v^2(t)/c^2}$. The converse case, simultaneous time $t'$ on $\mathcal{A}$'s clock when $\mathcal{B}$ measures time $t'$ in system $S'$, is more difficult because $S'$ is not an inertial system.

This work covers several matters related to the 'twin paradox': (i) It gives a special-relativity (STR) formula for $t(t')$; (ii) it then applies that correctly to the standard twin paradox (where $\mathcal{B}$ travels at constant velocity throughout); (iii) subsequently it applies it to a twin-paradox situation with a realistic velocity profile to show that there is no discontinuity in $t(t')$, and (iv) finally the new formulation is applied to the case in which $v(t)$ is unknown but must be inferred from the acceleration observed by $\mathcal{B}$. The dependence $t(t')$ for that case was previously obtained only by means of the general theory of relativity (GTR) and the new STR result entirely agrees with it.

Keywords: relativity, relativistic velocity, acceleration, twin paradox

I. INTRODUCTION

The subject of this work lies within the 'twin paradox', first formulated by Einstein as part of his special theory of Relativity\(^1\). This deals with the high-velocity trip of a space traveler, $\mathcal{B}$ (male), to a distant star and back. Upon comparing his/her clock to that of the Earthbound twin, $\mathcal{A}$ (female), it is found that $\mathcal{B}$ has aged less than $\mathcal{A}$. Many aspects of this problem have been dealt with previously\(^2\)–\(^{17}\). Even so, there are some aspects that require further analysis. This work also aims to provide a clearer picture of the relativistic peculiarities for those less familiar with the 'paradox'.

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In section II, a corrected formula in the special theory of relativity (STR) is given for the time $t$ on $\mathcal{A}$'s clock that is simultaneous with the time $t$ on $\mathcal{B}$'s clock. This formula appears to cover at least one case previously thought to be understood properly only in the general theory (GTR).

In the standard version, briefly discussed in section III (in which the Earth is an inertial frame and velocity is constant during the entire trip, except for a reversal of direction at the star), $\mathcal{A}$'s realization of the time on $\mathcal{B}$'s clock differs from his realization of the time on $\mathcal{A}$'s clock. Stated in relativistic terms, $\mathcal{A}$'s proper time for $\mathcal{B}$'s motion differs from $\mathcal{B}$'s proper time for $\mathcal{A}$'s apparent motion. While $\mathcal{A}$'s proper clock time (time $t'$ on $\mathcal{B}$'s clock) continuously lags behind $\mathcal{A}$'s own clock, $\mathcal{B}$'s proper clock time (time $t$ on $\mathcal{A}$'s clock) also appears to lag continuously behind his own clock, except at the point of velocity reversal at which point $\mathcal{B}$'s proper clock time makes a sudden and discontinuous jump forwards in time so that $\mathcal{B}$'s clock has nevertheless lagged behind $\mathcal{A}$'s upon comparison after $\mathcal{B}$'s return to Earth. See Fig. 2 for a preview of the effect. We introduce a more realistic velocity profile in section IV for $\mathcal{B}$'s motion (as measured by $\mathcal{A}$) to demonstrate that the actual proper time of $\mathcal{B}$ changes smoothly and continuously instead of making a sudden discontinuous jump.

In section V, it is demonstrated that $\mathcal{B}$ can deduce the simultaneous time on $\mathcal{A}$'s clock from the acceleration experienced as a gravitational effect. It will reproduce results hitherto obtained only via GTR$^{18-22}$. Where deemed helpful, appendices provide mathematical details.

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II. GENERAL PROPER-TIME $t(t')$ FORMULA

Figure 1 illustrates the trajectory of $\mathcal{B}$ as viewed in the inertial frame of $\mathcal{A}$ (who is earthbound; nevertheless all effects of Earth's gravity will be ignored here). Figure 1a is for $0 < t < T$ whereas Fig. 1b is for $T < t < 2T$ (return trip), but the left-hand figure suffices for the analysis. The importance of Fig. 1b is discussed in section III. The destination is reached at time $T$ and the symmetric return to Earth at time $2T$. 
Fig. 1a  

Fig. 1b  

Fig. 1. Trajectory of $\mathcal{B}$ in $\mathcal{A}'$'s coordinate frame $S$ with Minkowski coordinates at point $B$.

Minkowski coordinates for the non-inertial system of $\mathcal{B}$ are indicated by $ct'$, $x'$ axes at location $B$. These coordinates help illustrate the differences in what each observer considers to be a simultaneous event. Time $t' = t_B$ is associated with two times on the $ct$ axis: $t = t_A$ and $t = t_C$. Points $B$ and $C$ lie on a $ct = \text{constant}$ line and they represent locations that $\mathcal{A}$ regards as being simultaneous, whereas points $B$ and $A$ lie on a $ct' = \text{constant}$ line which $\mathcal{B}$ therefore regards as being simultaneous. This distinction is crucial in interpreting what a clock will see at point $B$.

The simplest case is a calculation of the time $t'$ on $\mathcal{B}'$'s clock that is simultaneous with $\mathcal{A}'$'s clock time $t$:

$$t' = \int_0^t dt_i (1 - v^2(t)/c^2)$$

(1)

given that $c$ is the velocity of light in vacuo, and $v$ the velocity of $\mathcal{B}$ as noted by $\mathcal{A}$\textsuperscript{23}. However, a similar formula for $\mathcal{A}'$'s clock time simultaneous with $\mathcal{B}'$'s, namely $t(t')$, is more complicated because $\mathcal{B}'$'s trajectory does not lie in an inertial frame. To deal with that situation, consider the angle $\psi$, which changes as point $B$ moves along the trajectory of $\mathcal{B}$. It is given by $\tanh \psi = \beta$ where $\beta = v/c$ is the normalized velocity at point $B$ as observed by $\mathcal{A}$. That change is discontinuous when $OB$ and $BF$ are straight lines (as in the standard TP). The hyperbolic tangent is needed because the Minkowski diagram at $B$ represents a hyperbolic geometry\textsuperscript{24},
which then yields for the proper time of $\mathfrak{B}$ (with respect to system $S$) the well-known time-dilation formula\textsuperscript{23} of Eq. 1. In terms of Fig. 1, $t \equiv t_C$ and $t' \equiv t_B$.

The evaluation of $t_A(t_B) = t(t')$, i.e. the clock time of $\mathfrak{A}$ simultaneous with $\mathfrak{B}$'s observed clock time $t_B = t'$, is the main emphasis of this section. A reminder: $A$ and $B$ are simultaneous for observer $\mathfrak{B}$. The hyperbolic geometry in Fig. 1a for $0 < t' < T'/2$ shows that

$$
cdt_A \equiv d(OA) = d(OC) - d(AC) = cdt_C - d[(BC)\tanh \psi].
$$

$$
= cdt_B(1 - \beta^2)^{-1/2} - d[\beta(OD)] = cdt_B(1 - \beta^2)^{-1/2} - d(\beta x_D)
$$

(2)

Since all terms are expressed as infinitesimals it follows that

$$
c t_A = c \int_0^{t_C} dt' [1 - \beta^2(t')]^{-1/2} - \beta(t_B)x_D
$$

(3)

There is an apparent inconsistency here in using $\beta$ as given by $\mathfrak{A}$, but expressed as a function of $t'$, which will be dealt with below. Distance $x_D$ has been covered by $\mathfrak{A}$ in time $t_C$ and is therefore equal to

$$
x_D = \int_0^{t_C} dt v(t) = c \int_0^{t_C} dt \beta(t) = c \int_0^{t_C} dt' [1 - \beta(t')^{-1/2}]
$$

(4)

It is necessary in (4) to have replaced $dt$ by $dt' \cosh \psi$ in the integrand, and then $t_C$ in the upper bound by $t_B \equiv t'$ (which is simultaneous with $t_C$ in $\mathfrak{A}$'s frame, hence is identical with it). Velocity $\beta(t') = \beta(t)$ where it is understood that the value at point $C$ is needed. The result\textsuperscript{25} is

$$
t = \int_0^{t'} dt_1 [1 - \beta^2(t_1)]^{-1/2} - \beta(t') \int_0^{t'} dt_1 \beta(t_1) \frac{\beta(t_1)}{\sqrt{1 - \beta(t_1)}}
$$

(5)

When $t' > T'/2$, $d(OC) - d(AC)$ becomes $d(OC) + d(AC)$ in (2) because $\beta(t') < 1$. Equation (5) suffices also in this case due to the sign change in $\beta(t')$, which indicates that the situation of Fig. 1b applies. Lest it be thought that $\mathfrak{B}$ needs to know $v(t)$ as registered by $\mathfrak{A}$, let it be clarified here that $\beta(t_1) = \tanh(g t_1/c)$ as can be verified from Eqs. (A2) and (A3), and time $t'_1$ as well as acceleration $g$ are therefore directly measurable by $\mathfrak{B}$. Finally, this relationship between $t$ and $t'$ holds for an arbitrary velocity distribution of the accelerating observer.

### III. STANDARD TWIN-PARADOX TIME DILATION RESULT
It is trivially verifiable that the standard twin-paradox (TP) results\textsuperscript{2} when $\beta(t) = \beta$ (a constant) for $t < T$ and $\beta(t) = -\beta$ for $T < t < 2T$. Let $\gamma = 1/\sqrt{1-\beta^2}$. Then Eq. (5) yields

\begin{align*}
t &= t' \sqrt{1 - \beta^2} \quad \text{for} \quad t' < T' \\
t &= t'(1 + \beta^2)/\sqrt{1 - \beta^2} \quad \text{for} \quad T' < t' < 2T' \\
T &= T'/\sqrt{1 - \beta^2} \quad (6)
\end{align*}

\(\mathfrak{B}\)'s\ proper\ time\ for\ \(\mathfrak{A}\)'s\ apparent\ motion,\ \(t_\mathfrak{A} = t\),\ makes\ a\ discontinuous\ jump\ at\ \(\mathfrak{B}\)'s\ measured\ time\ \(t' = T'\)\ from\ \(t = T(1 - \beta^2)\)\ to\ \(t = T(1 + \beta^2)\),\ and\ \(t = T\)\ is\ the\ midpoint\ of\ the\ discontinuity\ in\ \(t\).\ This\ jump\ is\ also\ suggested\ in\ Figs.\ 1\ after\ replacing\ the\ curved\ trajectories\ by\ straight\ ones\ (in\ which\ case\ point\ \(A\)\ jumps\ from\ below\ to\ above\ point\ \(C\)\ as\ \(\mathfrak{B}\)\ crosses\ point\ \(B\)).\ It\ is\ shown\ as\ calculated\ from\ (6)\ in\ Fig.\ 2.\ Also\ obvious\ from\ the\ graph,\ \(\mathfrak{A}\)'s\ clock\ advances\ more\ slowly\ than\ \(\mathfrak{B}\)'s\ everywhere\ except\ at\ \(t' = T'\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{\(\mathfrak{A}\)'s\ aging\ time\ as\ predicted\ by\ \(\mathfrak{B}\)\ for\ the\ standard\ TP\ ($\beta = 0.8$);\ dashed\ curve\ shows\ \(t/T = t'/T'\)\ as\ a\ reference.}
\end{figure}

In\ any\ more\ realistic\ version\ of\ the\ TP,\ the\ transition\ from\ \(v = +\beta\)\ to\ \(-\beta\)\ is\ not\ discontinuous;\ it\ should\ be\ smooth\ and\ continuous.\ This\ will\ be\ shown\ explicitly\ for\ an\ improved\ sinusoidal\ velocity\ distribution\ (see\ section\ IV).\ While\ the\ time\ \(t(t')\)\ is\ \(\mathfrak{A}\)'s\ actual\ clock\ time\ at
time $t'$ for $\mathcal{B}$ in system $S'$, $\mathcal{A}$ does not experience such a discontinuity and $\mathcal{A}$ predicts simultaneous time $t'$ on $\mathcal{B}$'s clock to be

$$t'(t) = \int_0^t dt_1 \sqrt{1 - \beta^2(t)}$$

(7)

which would show, for $t' = T'$ to $t' = 2T'$ in Fig. 2, as a dotted-line continuation. Plots for other values of $\beta$ are similar.

**IV. A SINUSOIDAL VELOCITY PROFILE WITH ZERO END VELOCITIES AND ACCELERATIONS**

The following velocity profile for a 'twin-paradox' trip is somewhat more realistic in that both velocity and acceleration are zero at $t = 0$, $T$, and $2T$, although maintenance of an exactly sinusoidal velocity profile is an over-idealization:

$$\beta_1(t) = \frac{1}{2} \beta_0 [1 - \cos(2\pi t/T)] \quad \text{for } 0 < t < T$$

$$\beta_2(t) = -\frac{1}{2} \beta_0 [1 - \cos(2\pi t/T)] = -\beta_1(t) \quad \text{for } T < t < 2T$$

(8)

and it is shown in Fig. 3 for $\beta_0 = 0.8$ (similar profiles for other values of $\beta$):

![Velocity profile for sinusoidal model ($\beta = 0.8$).](image_url)

Fig. 3. Velocity profile for sinusoidal model ($\beta = 0.8$).
The proper-time interval of $\mathfrak{B}$ in terms of $dt$ is $dt' = dt\sqrt{1 - \beta^2(t')}$. Upon formal integration via (1) one obtains an elliptic function:

$$t' = \int_0^T dt_1 [1 - (\beta_0^2 \sin^2(\pi t/T))]^{1/2} = E \left( \frac{\pi}{T} \arcsin \beta_0 \right)$$

(9)

as given by Abramowitz & Stegun\textsuperscript{26}. The results of numerical integration are plotted below, for $\beta_0 = 0.8$, 0.6, and 0.4 in Fig. 4, and they show a monotonic lag of $\mathfrak{B}$'s simultaneous time $t'$ behind the measured time $t$ of $\mathfrak{A}$.

![Fig. 4. Proper time $t'(t)/T'$ (solid curve) vs. $t/T$ for $\beta_0 = 0.8$, 0.6, and 0.4.](image)

Now consider the situation from the point of view of traveler $\mathfrak{B}$. First apply Eq. (5) for $t' < T'$. For $T' < t' < 2T'$, one needs to work out (5) using $\beta(t_1) < 0$ whenever $t_1 > T'$. The result is:

$$t = \int_0^{T'} dt_1 [1 - \beta_1^2(t_1)]^{-1/2} + |\beta_1(t')|\int_0^{T'} dt_1 \frac{\beta(t_1)}{\sqrt{1 - \beta^2(t_1)}}$$

$$+ \int_T^{T'} dt_1 [1 - \beta_1^2(t_1)]^{-1/2} - \beta_1(t') \int_T^{T'} dt_1 \frac{\beta(t_1)}{\sqrt{1 - \beta^2(t_1)}}$$

(10)

Below are plots of Eq. (10), for $\beta = 0.8$, 0.6, and 0.4 in Fig. 5, in which $\mathfrak{A}$'s registered time $t$, simultaneous with the time $t'$ on $\mathfrak{B}$'s clock, is plotted as a function of $t'/T'$. The transition around the time $t' = T'$ now is gradual instead of abrupt as in Fig. 5. The same data are shown in Fig. 6 with $t - t'$ plotted as a function of $t'/T'$. 

![Figures showing plots of Eq. 10](image)
Fig. 5. $t/T$ (solid curve) as a function of $t'/T'$ for $\beta_0 = 0.8$, 0.6, and 0.4.

Fig. 6. $(t - t')/T$ (solid curve) as a function of $t'/T'$ for $\beta_0 = 0.8$, 0.6, and 0.4.

In fact, the symmetry of the velocity profile ensures that the return trip is an inverted-mirror image of the trip to the star but starting from a value of $t(T')$ instead of from $t(0) = 0$.

V. UNIFORM ACCELERATION TREATED IN STR

Another version of the TP will provide an important test of Eq. (5). It is inspired by a problem in a number of articles\textsuperscript{18-22}; one in which $\mathfrak{B}$ experiences uniform acceleration, e.g. $g \approx 9.8 \, \text{m/s}^2$ for $0 < t < T/2$, starting from $\beta(t = 0) = 0$, followed by $-g$ for $T/2 < t < T$ at which time $\mathfrak{B}$ arrives at the destination star with zero velocity and acceleration. The return trip is symmetric for $T < t < 2T$. If indeed the coordinate frame of $t'$ moves with $\mathfrak{B}$, then $t' = 0$, and because the acceleration is then equivalent with gravitationally-induced space curvature experienced by $\mathfrak{B}$ one might conclude that this problem requires general relativity (GTR). Here, we wish to show that a result for $t(t')$ follows directly from the STR Eq. (5). It will then be demonstrated in section VI that the same result follows from GTR, e.g. as implied (but not shown explicitly) by Perrin\textsuperscript{18} and shown explicitly in a different form [namely as $t'(t)$] by Blecher\textsuperscript{20,27}.

The first problem is that application of (5) requires knowledge of $\beta(t)$ as experienced by $\mathfrak{A}$ in point $C$ of Figs. 1, which has to be obtained indirectly from the acceleration experienced by $\mathfrak{B}$ (this is worked out in Appendix 1)

$$\beta(t) = \frac{gt/c}{\sqrt{1 + (gt/c)^2}} \quad \text{or} \quad \beta(t) = \tanh(gt'/c) \quad \text{for} \quad t < T/2$$

$$\beta(t) = \frac{g(T-t)/c}{\sqrt{1 + [g(T-t)/c]^2}} \quad \text{or} \quad \beta(t) = \tanh[g(T' - t')/c] \quad \text{for} \quad T/2 < t < T$$

(11)
The length of the trip is determined by the magnitude of \( T \); large values of \( T \) also indicate velocities close to \( c \) for some interval around time \( T/2 \).

The dependence of simultaneous \( \mathcal{B} \)'s time \( t' \) upon time \( t \) of \( \mathcal{A} \) is easily established (see Appendix 2):

\[
t' = \frac{\varrho}{g} \text{arcsinh} \left( \frac{\varrho T}{c} \right) \quad \text{for} \quad 0 < t < T/2
\]

\[
\delta t' = \frac{\varrho}{g} [\text{arcsinh} \left( \frac{\varrho T}{2c} \right) + \text{arcsinh} \left( \frac{\varrho(t-T)}{c} \right)] \quad \text{for} \quad T/2 < t < T
\]  

(12)

Hence at the star, \( t = T \) and \( \delta t' = \frac{\varrho}{g} \text{arcsinh} \left( \frac{\varrho T}{2c} \right) \) which results in \( t' = 2 \frac{\varrho}{g} \text{arcsinh} \left( \frac{\varrho T}{2c} \right) \) upon return to Earth (in agreement with other work\(^2\)).

It requires appreciably more work to establish (see Appendix 3) the converse: the time predicted by \( \mathcal{B} \) for \( \mathcal{A} \)'s clock from (5) as

\[
t = \frac{\varrho}{g} \text{tanh} \left( \frac{\varrho t' c}{c} \right) \quad \text{for} \quad t' < T'
\]

\[
t = \frac{\varrho}{g} \left\{ 2 \text{sinh} \left( \frac{\varrho T}{2c} \right) - \text{tanh} \left( \frac{\varrho(T-t')}{c} \right) \left[ 2 \text{cosh} \left( \frac{\varrho T}{2c} \right) - 1 \right] \right\} \quad \text{for} \quad T'/2 < t' < T'
\]  

(13a)

The two expressions are continuous at \( t' = T'/2 \). Upon arrival at the star (\( t' = T' \)), the second of these equations becomes \( t = T = 2 \frac{\varrho}{g} \text{sinh} \left( \frac{\varrho T}{2c} \right) \), same as in (12). The prediction must agree with \( \mathcal{A} \)'s clock (as it indeed does) because \( \mathcal{A} \)'s clock can be synchronized with another on the star (which is in the same inertial system), and that other clock is then at the same location as \( \mathcal{B} \). The return trip is symmetric and therefore adds (13a) to \( t = 2 \frac{\varrho}{g} \text{sinh} \left( \frac{\varrho T}{2c} \right) \) for \( T' < t' < 3T'/2 \) (for the first line) and for \( 3T'/2 < t' < 2T' \) (for the second line). Therefore \( t = 2T = 4 \frac{\varrho}{g} \text{sinh} \left( \frac{\varrho T}{2c} \right) \) upon \( \mathcal{B} \)'s return to Earth\(^2\). Hence

\[
t = 2 \frac{\varrho}{g} \text{sinh} \left( \frac{\varrho T}{2c} \right) + \frac{\varrho}{g} \text{tanh} \left[ \frac{\varrho(T-t')}{c} \right] \quad \text{for} \quad T' < t' < 3T'/2
\]

\[
t = \frac{\varrho}{g} \left\{ 4 \text{sinh} \left( \frac{\varrho T}{2c} \right) - \text{tanh} \left( \frac{\varrho(T-T')}{c} \right) \left[ 2 \text{cosh} \left( \frac{\varrho T}{2c} \right) - 1 \right] \right\} \quad \text{for} \quad 3T'/2 < t' < 2T'
\]  

(13b)

Figures 7 show plots of \( t/T' \) vs. \( t/T' \) for \( T' = 2, 3, \) and \( 4 \) (top row) and \( T' = 5, 8, \) and \( 10 \) (bottom row). Results for \( T < t < 2T \) need not be shown, because the return trip is symmetric and only adds on to the maximal value of \( t(T') \).
Fig. 7. Ά's estimate of Α's clock time $t/T'$ vs. $t'/T'$ for $T' = 2, 3, \text{ and } 4$ y (upper three) and $T' = 5, 8, \text{ and } 10$ y (lower three).

Whereas plots of $t(t')$ for the standard TP show a vertical jump in $t$ at $t' = T'$, the situation here is quite different; there appears to be an almost constant value of $t$ for a stretch of $t'$ around $t' = T'/2$ (when $T'$ is significantly larger than 1). This appears to be due to the fact that $β(t')$, as given in (5), has an almost constant value near $β = 1$ in that stretch. The behavior of Ά's clock, as perceived by Β, can vary in different ways, depending upon the profile of $β(t')$.

VI. UNIFORM ACCELERATION TREATED IN GTR

In the previous section, the description of the changes in velocity is in terms of the traveler Β, and specifically in terms of the (equivalent) gravitational force Β experiences. Previous work has been largely confined to a treatment by means of GTR^{18-22}. Perrin^{18} treats this version of the TP from the point of view of a non-flat metric (i.e. as prescribed by GTR) and Müller, et al.^{22} treat this problem partly from the point of view of STR but these authors do not supply an equivalent of Eqs. (13). Blecher^{27} has supplied a comparison of Ά's clock time $t$ as predicted by Β when Β's own clock time is $t'$, but his expression is in the form $t'(t)$ whereas a comparison with the result in section IV requires $t(t')$. His expression for $t'(t)$ is
\[ t' = T' - \frac{c}{2g} \ln \left\{ \frac{1 - \beta_m}{1 + \beta_m} \frac{B + (gT/c - \beta)}{A + (gT/c - \beta)} \right\}, \quad \beta_m = \tanh \left( \frac{gT'}{2c} \right) \]

\[ A = -2 \sqrt{\frac{1 - \beta_m}{1 + \beta_m}} + (1 - \beta_m), \quad B = 2 \sqrt{\frac{1 + \beta_m}{1 - \beta_m}} - (1 + \beta_m) \]  

(14)

Inversion of this equation yields

\[ t = \frac{\xi}{g} \left[ 1 + 2 \frac{Q(\xi-1) - \xi}{\xi + \xi^2} \right] \quad \text{given that} \quad \xi \equiv \sqrt{\frac{1 + \beta}{1 - \beta}}, \quad Q \equiv e^{g(t' - T')/c} \]  

(15)

Equation (15) can be shown (see Appendix 4) to be identical with (13a), and therefore confirms that the methods of sections II and IV lead to identical results for \( t(t') \).

VII. CONCLUSION

The fact that the proper time \( t'(t) \) of an accelerating particle can be treated within STR was noted, among others, by Hartle. That the effect is real has been verified by experiment. What is new in this work is a generic equation for the time \( t(t') \) that an accelerating observer predicts at his clock time \( t' \) for the clock time \( t \) in an inertial system, valid for any velocity distribution \( v(t) \).

A picture of the aging effect of \( \mathcal{A} \) also can be found for the standard twin paradox (no acceleration except at the initial, turn-around, and final points) in Müller. But there are authors who dispute the existence of a twin paradox; however that is not an opinion shared by most others.

APPENDIX 1. DERIVATION OF EQS. (11)

Traveler \( \mathcal{B} \) has velocity \( v(t) \) as observed in \( \mathcal{A}'s system \( S \). \( \mathcal{B}'s acceleration \( g \) is what \( \mathcal{B} \) experiences as gravitationally induced spacetime curvature. In \( \mathcal{A}'s frame \( S \): 

4-vector \( \mathbf{u} = \gamma(c, v, 0, 0) \) with \( \gamma = (1 - v^2/c^2)^{-1/2} \equiv (1 - \beta^2)^{-1/2} \)

4-vector \( \mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{dv}{dt} \frac{d\mathbf{u}}{dt} = \gamma \frac{d}{dt} [\gamma(c, v, 0, 0)] = (c\gamma \frac{dx}{dt}, \gamma \frac{d(\gamma v)}{dt}, 0, 0) \)

This yields: \( \dot{a}^0 = c\gamma \frac{dx}{dt} = c\gamma^4 \beta \frac{d\beta}{dt} \) and \( \dot{a}^1 = c\gamma^4 \frac{d\beta}{dt} \)  

(A1)
Now do the Lorentz transform of the 4-acceleration components using $a'^1 = g$ in $\mathcal{B}'$'s frame for $0 < t < T/2$: With $a'^1 = \gamma(a^1 - \beta a^0)$ one obtains

\[
g = c \gamma^3 \frac{d\beta}{d\tau}, \quad \frac{d\beta}{d\tau} = \frac{\zeta}{g} \gamma^{-3} \quad \text{leads to} \quad \int dt = \frac{\zeta}{g} (1 - \beta^2)^{-3/2} d\beta = \frac{\zeta}{g} \left[ (1 - \beta^2)^{-1/2} - (1 - \beta_0^2)^{-1/2} \right] \]

and therefore

\[
t - t_0 = \frac{\zeta}{g} \left[ (1 - \beta_0^2)^{-1/2} - (1 - \beta^2)^{-1/2} \right]
\]

It is given that $\beta_0 = 0$, $t_0 = 0$:

\[
t = \frac{\zeta}{g} \beta (1 - \beta^2)^{-1/2}, \quad \beta(t) = \frac{gt/c}{\sqrt{1 + (gt/c)^2}} \tag{A2}
\]

Furthermore, Eq. (1) yields

\[
t' = \int_0^t dt_1 \left\{ 1 - \frac{(gt/c)^2}{1 + (gt/c)^2} \right\}^{1/2} = \int_0^t dt_1 \left[ 1 + (gt/c)^2 \right]^{-1/2} = \frac{\zeta}{g} \text{arcsinh} \frac{gt}{c} \quad \text{resulting in}
\]

\[
t = \frac{\zeta}{g} \sinh \frac{gt}{c} \tag{A3}
\]

(as also found by Müller, et al.\textsuperscript{28}). Thus (A3) shows in (A2) that velocity $v(t)$ can be deduced by $\mathcal{B}$ from his clock time $t'$ and $g$. Now assume $a'^1 = -g$ for $T/2 < t < T$ to obtain

\[
f^T_t dt_1 = \frac{\zeta}{g} \int_0^\beta d\beta \left[ \beta_1(1 - \beta_1^2)^{-1/2} \right] \quad \text{resulting in}
\]

\[
T - t = \frac{\zeta}{g} \left[ \beta f(1 - \beta_f^2)^{-1/2} - \beta(1 - \beta^2)^{-1/2} \right] \quad \text{and we may set} \quad \beta_f = \beta(T) = 0
\]

\[
T - t = -\frac{\zeta}{g} \beta(1 - \beta^2)^{-1/2} \quad \text{so that} \quad \beta(t) = \frac{g(T-t)/c}{\sqrt{1 + g(T-t)/c}} \tag{A4}
\]

which gives the correct (and continuous) result at $t = T/2$.

In addition to these results, the time $t'$ of $\mathcal{B}$ simultaneous with $\mathcal{A}$'s clock time $t$ (the proper time of $\mathcal{A}$) is then

\[
t' = \int_t^T dt_1 \left\{ 1 - \frac{[g(T-t)/c]^2}{1 + [g(T-t)/c]^2} \right\}^{-1/2} = \int_t^T dt_1 \left[ 1 + (gt/T)^2 \right]^{-1/2}
\]

\[
= -\int_{T-t}^0 dt_1 \left[ 1 + (gt/T)^2 \right]^{-1/2} = \frac{\zeta}{g} \text{arcsinh} \frac{g(T-t)}{c}
\]

This then yields:

\[
t = \frac{\zeta}{g} \sinh \frac{g(T-t)}{c} \tag{A5}
\]

**APPENDIX 2. DERIVATION OF EQS. 12**
The time $t'$ of $\mathbb{B}$ (simultaneous with $\mathfrak{A}$'s time $t$) follows directly from the usual time-dilation formula and from (A2) and (A4) in Appendix 1:

$$
t' = \int_0^t dt_1 [1 - \beta^2(t_1)]^{1/2} \quad \text{and} \quad dt_1 = \frac{\xi}{g} (1 - \beta^2)^{-3/2} d\beta \quad \Rightarrow \quad t' = \frac{\xi}{g} \int_0^\beta d\beta_1 (1 - \beta_1^2)^{-1}
$$

$$
t' = \frac{\xi}{g} \text{arctanh} \beta(t) = \frac{\xi}{g} \text{arctanh} \left( \frac{g t/c}{\sqrt{1 + (g t/c)^2}} \right) = \frac{\xi}{g} \text{arcsinh} \left( \frac{gt}{c} \right) \quad \text{for} \quad 0 < t < T \quad (A6)
$$

For the 2nd leg, $T/2 < t < T$ one has acceleration $- g$ with added time $\delta t'$:

$$
\delta t' = \int_{t/2}^t dt_1 [1 - \beta^2(t_1)]^{1/2} \quad \text{and} \quad dt_1 = - \frac{\xi}{g} (1 - \beta^2)^{-3/2} d\beta \quad \text{from which is obtained}
$$

$$
\delta t' = - \frac{\xi}{g} \int_{\beta(T/2)}^{\beta(T)} d\beta_1 (1 - \beta_1^2)^{-1} = \frac{\xi}{g} [\text{arctanh} \beta(T/2) - \text{arctanh} \beta(t)]
$$

$$
= \frac{\xi}{g} \text{arcsinh} \left( \frac{g t}{2c} \right) + \frac{\xi}{g} \text{arcsinh} \left( \frac{g(T-t)}{c} \right) \quad \text{for} \quad T/2 < t < T \quad (A7)
$$

APPENDIX 3. DERIVATION OF EQS. 13A

For $t' < T'/2$ one finds from Eqs. (A2) and (A3)

$$
[1 - \beta^2(t_1)]^{1/2} = \left\{ 1 - \left[ \frac{g t/c}{\sqrt{1 + (g t/c)^2}} \right]^2 \right\}^{-1/2} = \left[ 1 + \sinh^2 \frac{gt}{c} \right]^{1/2} = \cosh \left( \frac{gt}{c} \right)
$$

$$
\int_0^{t'} dt_1 [1 - \beta^2(t_1)]^{1/2} = \int_0^{t'} dt_1 \cosh \left( \frac{gt}{c} \right) = \frac{\xi}{g} \sinh \left( \frac{gt}{c} \right) \quad (A8)
$$

$$
\frac{\beta(t_1)}{\sqrt{1 - \beta^2(t_1)}} = \tanh \left( \frac{gt_1}{c} \right) \cosh \left( \frac{gt_1}{c} \right) = \sinh \left( \frac{gt_1}{c} \right)
$$

$$
\beta(t') \int_0^{t'} dt_1 \frac{\beta(t_1)}{\sqrt{1 - \beta^2(t_1)}} = \tanh \left( \frac{gt'}{c} \right) \int_0^{t'} dt_1 \sinh \left( \frac{gt}{c} \right)
$$

$$
= \frac{\xi}{g} \tanh \left( \frac{gt'}{c} \right) \left[ \sinh \left( \frac{gt'}{c} \right) - 1 \right] = \frac{\xi}{g} \left[ \sinh \left( \frac{gt'}{c} \right) - \tanh \left( \frac{gt'}{c} \right) \right] \quad (A9)
$$

This, when applied to (5) for $0 < t' < T'/2$, results in the first of Eqs. (13a).

When the second of Eqs. (5) is applied for $T'/2 < t' < T'$ one finds similarly

$$
[1 - \beta^2(t_1)]^{-1/2} = \left[ 1 + \sinh \left( \frac{g(T-t_1)}{c} \right) \right]^{-1/2} = \cosh \left( \frac{g(T-t_1)}{c} \right)
$$

$$
\int_0^{t'} dt' [1 - \beta^2(t_1)]^{-1/2} = \int_0^{T/2} dt_1 \cosh \left( \frac{gt_1}{c} \right) + \int_{T/2}^{t'} dt_1 \cosh \left( \frac{g(t_1 - T)}{c} \right)
$$

$$
= \frac{\xi}{g} \left[ \sinh \left( \frac{gt'}{2c} \right) + \sinh \left( \frac{g(t'-2T)}{2c} \right) - \sinh \left( \frac{gT}{2c} \right) \right] = \frac{\xi}{g} \left[ 2 \sinh \left( \frac{gt'}{2c} \right) - \sinh \left( \frac{g(T-t')}{c} \right) \right] \quad (A10)
$$
\[ \int_0^t dt_1 \beta(t_1)[1 - \beta^2(t_1)]^{-1/2} \]
\[= \int_0^{T/2} dt_1 \tanh \left( \frac{\beta(t_1)}{c} \right) \cosh \left( \frac{\beta(t_1)}{c} \right) + \int_{T/2}^t dt_1 \tanh \left( \frac{\beta(t_1)}{c} \right) \cosh \left( \frac{g(T-t_1)}{c} \right) \]
\[= \frac{\epsilon}{g} \left[ \cosh \left( \frac{gT}{2c} \right) - 1 \right] + \frac{\epsilon}{g} \left\{ \cosh \left( \frac{gT}{2c} \right) - \cosh \left( \frac{g(T-t')}{c} \right) \right\} \]
\[= \frac{\epsilon}{g} \left[ 2 \cosh \left( \frac{gT}{2c} \right) - \cosh \left( \frac{g(T-t')}{c} \right) - 1 \right] \quad (A11) \]

\[ \beta(t') \int_0^t dt_1 \beta(t_1)[1 - \beta^2(t_1)]^{-1/2} \]
\[= \frac{\epsilon}{g} \tanh \left( \frac{g(T-t')}{c} \right) \left[ 2 \cosh \left( \frac{gT}{2c} \right) - \cosh \left( \frac{g(T-t')}{c} \right) - 1 \right] \]
\[= \frac{\epsilon}{g} \left[ 2 \cosh \left( \frac{gT}{2c} \right) - 1 \right] \tanh \left( \frac{g(T-t')}{c} \right) + \frac{\epsilon}{g} \sinh \left( \frac{g(T-t')}{c} \right) \quad (A12) \]

Together this yields the desired result.

\[ t = \frac{\epsilon}{g} \left[ 2 \sinh \left( \frac{gT}{2c} \right) - \sinh \left( \frac{g(T-t')}{c} \right) \right] \]
\[- \frac{\epsilon}{g} \left[ 2 \cosh \left( \frac{gT}{2c} \right) - 1 \right] \tanh \left( \frac{g(T-t')}{c} \right) + \frac{\epsilon}{g} \sinh \left( \frac{g(T-t')}{c} \right) \]
\[= \frac{\epsilon}{g} \left\{ 2 \sinh \left( \frac{gT}{2c} \right) - \tanh \left( \frac{g(T-t')}{c} \right) \left[ 2 \cosh \left( \frac{gT}{2c} \right) - 1 \right] \right\} \quad (A13) \]

The derivation for \( T' < t' < 2T' \) is essentially the same, except for minor details.

**APPENDIX 4. EQUIVALENCE OF GTR RESULT**

Here, it is demonstrated briefly that Eqs. (15) and (13a) are identical. Starting from (13a), the following algebraic changes lead to the desired result:

\[ \frac{g}{c} = \left\{ 2 \sinh \left( \frac{gT}{2c} \right) - \tanh \left( \frac{g(T-t')}{c} \right) \right\} \left[ 2 \cosh \left( \frac{gT}{2c} \right) - 1 \right] \]
\[= \left( e^{gT/2c} - e^{-gT/2c} \right) - \frac{e^{g(T-t')/c} - e^{-g(T-t')/c}}{e^{gT/2c} + e^{-gT/2c} + 1} \]
\[= \left( e^{gT/2c} - e^{-gT/2c} \right) - \frac{e^{g(T-t')/c} - e^{-g(T-t')/c}}{e^{gT/2c} + e^{-gT/2c} + 1} \quad (A14) \]

Define \( \xi \equiv \sqrt{1+\beta^2} = e^{gT/2c} \) [which follows from (A2) and (A3)] and \( Q = e^{(t'-T')/c} \), and the above becomes

\[ \frac{g}{c} = \left( \xi - \frac{1}{\xi} \right) - \frac{\xi^2 - Q}{\xi + Q} \left( \xi + \frac{1}{\xi} - 1 \right) = \frac{(\xi^2 + Q)(\xi - \frac{1}{\xi}) - (\xi^2 - Q)(\xi + \frac{1}{\xi} - 1)}{\xi^2 + Q} \quad (A15) \]
This then yields \[ \frac{\mathbf{g}}{c} = \frac{2g(Q-1)+(\xi^2-Q)}{\xi^2+Q} = 1 + \frac{2g(Q-1)+((\xi^2-Q)-(\xi^2+Q))}{\xi^2+Q} = 1 + \frac{2(Q-1)\xi}{\xi^2+Q} \] which is Eq. (15).

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