On The Discrete Morse Functions for Hypergraphs

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Abstract. A hypergraph can be obtained from a simplicial complex by deleting some non-maximal simplices. In this paper, we study the embedded homology as well as the homology of the (lower-)associated simplicial complexes for hypergraphs. We generalize the discrete Morse functions on simplicial complexes. We study the discrete Morse functions on hypergraphs as well as the discrete Morse functions on the (lower-)associated simplicial complexes of the hypergraphs.

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1 Introduction

Hypergraph is an important model for complex networks, for example, the collaboration network. In mathematics,

- a hypergraph can be either a simplicial complex or a "non-closed simplicial complex"\(^1\) with some non-maximal faces missing;
- a simplicial complex can be regarded as a special hypergraph with no non-maximal faces missing.

\(^1\) first authors.

\(^2\) A simplicial complex \(K\) is closed in the following sense:

(a). combinatorically closed: for any simplex \(\sigma \in K\) and any non-empty subset \(\tau \subseteq \sigma\), we have \(\tau \in K\);
(b). topologically closed: the geometric realization \(|K|\) (cf. [20]) of \(K\) is a closed subset of some Euclidean space \(\mathbb{R}^N\).

Note that (a) \(\iff\) (b).

Definition. Let \(\mathcal{H}\) be a hypergraph (cf. Definition \([3]\)). Let \(K_{\mathcal{H}}\) be an (abstract) simplicial complex such that each hyperedge (cf. Definition \([2]\)) of \(\mathcal{H}\) is a simplex of \(K_{\mathcal{H}}\). Let \(|K_{\mathcal{H}}|\) be the geometric realization of \(K_{\mathcal{H}}\) in \(\mathbb{R}^N\). For each \(\sigma \in K\), let \(|\sigma|\) be the geometric simplex in \(|K|\).

We define the geometric realization \(|\mathcal{H}|\) of \(\mathcal{H}\) to be the subset of \(|K_{\mathcal{H}}|\) given by

\(|\mathcal{H}| = \{ x \in \mathbb{R}^N \mid \text{there exists } \sigma \in \mathcal{H} \text{ such that } x \in \text{Int}(|\sigma|) \}\).

A hypergraph \(\mathcal{H}\) can be non-closed in the following sense:

(a). combinatorically non-closed: there exists an hyperedge \(\sigma \in \mathcal{H}\) and an non-empty subset \(\tau \subseteq \sigma\) such that \(\tau \notin \mathcal{H}\);
(b). topologically non-closed: the geometric realization \(|\mathcal{H}|\) of \(\mathcal{H}\) is not a closed subset of some Euclidean space \(\mathbb{R}^N\).

Note that (a) \(\iff\) (b).
It is natural to generalize the discrete Morse functions on simplicial complexes (cf. [7] - [14]) and consider discrete Morse functions on hypergraphs.

Firstly, We review some backgrounds in (a), (b) and (c).

(a). Hypergraphs and Simplicial Complexes. Let $V$ be a totally-ordered finite set whose elements are called vertices. Let $2^V$ denote the power-set of $V$. Let $\emptyset$ denote the empty set. A hypergraph $\mathcal{H}$ on $V$ is a subset of $2^V \setminus \{\emptyset\}$ (cf. [5, 22]). An element of $\mathcal{H}$ is called a hyperedge. For any $\sigma \in \mathcal{H}$, if $\sigma$ consists of $n + 1$ vertices in $V$, then we say that $\sigma$ is of dimension $n$ and write $\sigma$ as $\sigma(n)$.

An (abstract) simplicial complex is a hypergraph satisfying the following condition (cf. [15, p. 107], [26, Section 1.3], [6]): for any $\sigma \in \mathcal{H}$ and any non-empty subset $\tau \subseteq \sigma$, we have that $\tau$ must be a hyperedge in $\mathcal{H}$. A hyperedge of a simplicial complex is called a simplex.

(b). The Usual Homology of Simplicial Complexes and The Embedded Homology of Hypergraphs. The homology theory for simplicial complexes (cf. [15, Chapter 2]) is well-known in algebraic topology. Let $K$ be a simplicial complex. Let $R$ be a commutative ring with unit. For each $n \geq 0$, by taking all the (formal) $R$-linear combinations of the $n$-simplices in $K$, we have a free $R$-module $C_n(K; R)$. The $n$-th boundary map $\partial_n : C_n(K; R) \rightarrow C_{n-1}(K; R)$ is an $R$-linear map such that

$$\partial_n(v_0v_1\ldots v_n) = \sum_{i=0}^{n} (-1)^i v_0 \ldots \hat{v}_i \ldots v_n$$

for any $n$-simplex $v_0v_1\ldots v_n \in K$ with $v_0 \prec v_1 \prec \cdots \prec v_n$. It can be verified that $\partial_n \circ \partial_{n+1} = 0$ for each $n \geq 0$. The $n$-th homology of $K$, with coefficients in $R$, is defined to be the quotient $R$-module

$$H_n(K; R) = \text{Ker} \partial_n / \text{Im} \partial_{n+1}.$$  

In 2019, S. Bressan, J. Li, S. Ren and J. Wu [6] defined an embedded homology for hypergraphs as an algebraic generalization of the usual homology of simplicial complexes. Let $\mathcal{H}$ be a hypergraph. Let $K_{\mathcal{H}}$ be an arbitrary simplicial complex such that each hyperedge of $\mathcal{H}$ is a simplex of $K_{\mathcal{H}}$. For each $n \geq 0$, let $\partial_n$ be the $n$-th boundary map of $K_{\mathcal{H}}$. Let $R_n(\mathcal{H})$ be the free $R$-module consisting of all the (formal) $R$-linear combinations of the $n$-hyperedges in $\mathcal{H}$. The infimum chain complex of $\mathcal{H}$ is defined as

$$\text{Inf}_n(\mathcal{H}) = R_n(\mathcal{H}) \cap \partial_n^{-1}(R_{n-1}(\mathcal{H})), \quad n \geq 0$$

and the supremum chain complex of $\mathcal{H}$ is defined as

$$\text{Sup}_n(\mathcal{H}) = R_n(\mathcal{H}) + \partial_n(R_{n+1}(\mathcal{H})), \quad n \geq 0.$$  

It is proved in [6] that the homology of the infimum chain complex and the homology of the supremum chain complex are isomorphic. This homology is called the embedded homology of $\mathcal{H}$. Particularly, if $\mathcal{H}$ is an (abstract) simplicial complex, then its embedded homology coincides with the usual homology.

(c). The Discrete Morse Theories for Simplicial Complexes, Graphs and Digraphs. During the 1990s and the 2000s, R. Forman [7] - [13] has developed a discrete
Morse theory for simplicial complexes. A discrete Morse function $f$ on a simplicial complex $K$ was defined by assigning a real number $f(\sigma)$ to each simplex $\sigma \in K$ such that for any $n \geq 0$ and any $n$-simplex $\alpha^{(n)} \in K$, there exist at most one $\beta^{(n+1)} > \alpha^{(n)}$ in $K$ with $f(\beta) \leq f(\alpha)$ and at most one $\gamma^{(n-1)} < \alpha^{(n)}$ in $K$ with $f(\gamma) \geq f(\alpha)$. The discrete gradient vector field $\text{grad} f$ on $K$ was defined by assigning arrows from $\alpha$ to $\beta$ whenever $\alpha^{(n)} < \beta^{(n+1)}$ and $f(\alpha) \geq f(\beta)$, for any $n \geq 0$ and any $\alpha^{(n)}, \beta^{(n+1)} \in K$. The critical simplices were defined as the simplices which are neither the heads nor the tails of any arrows in $\text{grad} f$. A chain complex consisting of the formal linear combinations of the critical simplices was constructed and the homology of this chain complex is proved to be isomorphic to the homology of the original simplicial complex. The Discrete Morse theory has applications in both pure mathematics (cf. [16, 17, 23, 25]) and data technologies (cf. [18, 21, 24, 25]).

Inspired by the discrete Morse theory in [7] - [14], people have studied the discrete Morse theories for graphs and digraphs. During the 2000s, R. Ayala, L.M. Fernández, D. Fernández-Ternero, A. Quintero and J.A. Vilches [1] - [4] gave a discrete Morse theory for graphs and studied related topics. In 2021, Yong Lin, Chong Wang and Shing-Tung Yau [19] gave a discrete Morse theory for digraphs.

Throughout the discrete Morse theory for simplicial complexes in [7] - [14], the discrete Morse theory for graphs in [1] - [4] and the discrete Morse theory for digraphs in [19], the homology groups, the discrete Morse functions, the discrete gradient vector fields, and the critical simplices play important and fundamental roles.

Secondly, we introduce the motivation of this paper. The paper is motivated by the following question:

**Question.** Whether a discrete Morse theory for hypergraphs and their embedded homology can be developed as a generalization of the discrete Morse theory for simplicial complexes and their homology (cf. [7] - [14])?

In this paper, we make a first step towards the answer. As a generalization of the homology theory for simplicial complexes, we explore the embedded homology of hypergraphs in Section 2 and study the homomorphisms between the embedded homology of hypergraphs in Theorem 2.11. Moreover, as a generalization of the discrete Morse theory for simplicial complexes, we study the discrete Morse functions on hypergraphs as well as the corresponding discrete gradient vector fields and critical hyperedges, from Section 3 to Section 6.

Thirdly, we summarize the outline of this paper. Let $\mathcal{H}$ be a hypergraph. In Section 2 we review the definition of the associated simplicial complex $\Delta \mathcal{H}$ (cf. [22]) which the smallest simplicial complex such that each hyperedge of $\mathcal{H}$ is a simplex of $\Delta \mathcal{H}$. We define the lower-associated simplicial complex $\delta \mathcal{H}$ to be the largest simplicial complex such that each simplex of $\delta \mathcal{H}$ is a hyperedge of $\mathcal{H}$. We prove in Theorem 2.11 that a morphism of hypergraphs induces a homomorphism of the embedded homology, a homomorphism of the homology of the associated simplicial complexes, and a homomorphism of the homology of the lower-associated simplicial complexes. From Section 3 to Section 6 we generalize the discrete Morse functions on simplicial complexes and define the discrete Morse functions on hypergraphs. We study the discrete gradient vector fields of the discrete Morse functions on hypergraphs as well as the corresponding critical hyperedges. We characterize the critical hyperedges in Theorem 5.4.

\[3\]In fact, in [7] - [14], R. Forman has developed a discrete Morse theory for general cell complexes. In particular, the discrete Morse theory in [7] - [14] is applicable for simplicial complexes.
2 The Associated Simplicial Complexes and The Embedded Homology for Hypergraphs

In this section, we review some definitions as well as some basic properties about hypergraphs, their (lower-)associated simplicial complexes, and the embedded homology. We also give some examples and show that the homology of the associated simplicial complexes, the homology of the lower-associated simplicial complexes, and the embedded homology of the hypergraphs detect the topology of hypergraphs from different aspects.

2.1 Hypergraphs, The Associated Simplicial Complexes, and The Lower-Associated Simplicial Complexes

Let \( V \) be a finite set with a total order \( \prec \).

**Definition 1.** A hyperedge \( \sigma \) on \( V \) is a non-empty subset of \( V \).

For any hyperedge \( \sigma \) on \( V \), we can write \( \sigma \) uniquely as a subset
\[
\sigma = \{v_0, v_1, \ldots, v_n\}
\] (2.1)
of \( V \); or equivalently, we can also write \( \sigma \) uniquely in the form of a sequence
\[
\sigma = v_0v_1\ldots v_n
\]
for some \( n \geq 0 \) where \( v_0, v_1, \ldots, v_n \in V \) and \( v_0 \prec v_1 \prec \cdots \prec v_n \). Throughout this paper, we adopt the former notation in (2.1) for an \( n \)-simplex.

**Definition 2.** We say that \( \sigma \) given by (2.1) is an \( n \)-hyperedge and call \( n \) the dimension of \( \sigma \).

**Definition 3** (cf. [5, 6, 22]). A hypergraph \( H \) on \( V \) is a collection of hyperedges on \( V \).

**Definition 4** (cf. [6, 15]). An (abstract) simplicial complex \( K \) on \( V \) is a hypergraph on \( V \) such that for any \( \sigma \in K \) and any non-empty subset \( \tau \subseteq \sigma \), we always have \( \tau \in K \).

A hyperedge of a simplicial complex is also called a simplex.

**Definition 5.** Let \( H \) and \( H' \) be two hypergraphs. We say that \( H \) can be embedded in \( H' \) and write \( H \subseteq H' \) if for any hyperedge \( \sigma \in H \), we always have that \( \sigma \in H' \).

Let \( H \) be a hypergraph on \( V \).

**Definition 6** (cf. [22]). The associated simplicial complex \( \Delta H \) of \( H \) is the smallest simplicial complex that \( H \) can be embedded in.

Let \( \sigma = v_0v_1\ldots v_n \) be an \( n \)-hyperedge on \( V \). The next lemma is straight-forward from Definition 3 and Definition 6.

**Lemma 2.1.** (cf. [22] Lemma 8). The associated simplicial complex \( \Delta \sigma \) of \( \sigma \) is the collection of all the nonempty subsets of \( \sigma \)
\[
\Delta \sigma = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n \text{ and } 0 \leq k \leq n\}.
\] (2.2)
The next lemma is straightforward from Definition 5 and Definition 6.

**Lemma 2.2.** Let \( H \) and \( H' \) be two hypergraphs such that \( H \subseteq H' \). Then \( \Delta H \subseteq \Delta H' \).

The next lemma follows from the above.

**Lemma 2.3.** For any hypergraph \( H \), its associated simplicial complex \( \Delta H \) has its set of simplices as the union of the \( \Delta \sigma \)'s for all \( \sigma \in H \), i.e.
\[
\Delta H = \{ \tau \in \Delta \sigma \mid \sigma \in H \}.
\]

**Proof.** Firstly, the right-hand side of (2.3) is a simplicial complex whose set of simplices contains all the hyperedges of \( H \). Thus by Definition 6,
\[
\Delta H \subseteq \{ \tau \in \Delta \sigma \mid \sigma \in H \}.
\]

Secondly, we take any hyperedge \( \sigma \in H \) and let \( \{ \sigma \} \) be the hypergraph with the single hyperedge \( \sigma \). Since \( \{ \sigma \} \subseteq H \), it follows from Lemma 2.2 that \( \Delta \sigma \subseteq \Delta H \). Letting \( \sigma \) run over all the hyperedges in \( H \), it follows that
\[
\{ \tau \in \Delta \sigma \mid \sigma \in H \} = \cup_{\sigma \in H} \Delta \sigma \subseteq \Delta H.
\]

Summarizing (2.4) and (2.5), we obtain (2.3). □

**Definition 7.** The lower-associated simplicial complex \( \delta H \) of \( H \) is the largest simplicial complex that can be embedded in \( H \).

The next lemma follows from Definition 7.

**Lemma 2.4.** Then the set of simplices of \( \delta H \) consists of the hyperedges \( \sigma \in H \) whose associated simplicial complexes \( \Delta \sigma \) are subsets of \( H \). In other words,
\[
\delta H = \{ \sigma \in H \mid \Delta \sigma \subseteq H \}
\]
\[
= \{ \tau \in \Delta \sigma \mid \Delta \sigma \subseteq H \}.
\]

**Proof.** Firstly, for any \( \sigma \in H \), if \( \Delta \sigma \subseteq H \), then by Definition 7 we have
\[
\Delta \sigma \subseteq \delta H.
\]

Letting \( \sigma \) run over all the hyperedges of \( H \), it follows that
\[
\{ \tau \in \Delta \sigma \mid \Delta \sigma \subseteq H \} = \cup_{\Delta \sigma \subseteq H} \Delta \sigma \subseteq \delta H.
\]

Secondly, let \( \sigma \) be any simplex of \( \delta H \). Then by Definition 7 we have \( \Delta \sigma \subseteq H \). Letting \( \sigma \) run over all the hyperedges of \( H \), it follows that
\[
\delta H \subseteq \{ \sigma \in H \mid \Delta \sigma \subseteq H \}.
\]

Thirdly, since \( \sigma \in \Delta \sigma \), we have
\[
\{ \sigma \in H \mid \Delta \sigma \subseteq H \} \subseteq \{ \tau \in \Delta \sigma \mid \Delta \sigma \subseteq H \}.
\]

Summarizing all the three points, we obtain (2.6). □
The next proposition follows from Lemma 2.3 and Lemma 2.4.

**Proposition 2.5.** As hypergraphs,
\[
\delta H \subseteq H \subseteq \Delta H.
\]  
(2.7)

Moreover, one of the equalities holds iff. both of the equalities hold iff. \( H \) is a simplicial complex.

**Proof.** The relations (2.7) follow from Definition 6 and Definition 7 directly. Alternatively, (2.7) also follow from Lemma 2.3 and Lemma 2.4. In the following, we will prove the second assertion. We divide the proof into two steps.

**Step 1.** \( \delta H = H \iff H \) is a simplicial complex.

(\( \Leftarrow \)): Obvious.

(\( \Rightarrow \)): Suppose \( \delta H = H \). Then by Lemma 2.4, it follows that for any \( \sigma \in H \), we always have \( \Delta \sigma \subseteq H \). Hence \( H \) is a simplicial complex.

**Step 2.** \( \Delta H = H \iff H \) is a simplicial complex.

(\( \Leftarrow \)): Obvious.

(\( \Rightarrow \)): Suppose \( \Delta H = H \). Then by Lemma 2.3 it follows that for any \( \sigma \in H \) and any non-empty subset \( \tau \subseteq \sigma \), we always have \( \tau \in H \). Hence \( H \) is a simplicial complex.

Summarizing both Step 1 and Step 2, we finish the proof. \( \Box \)

### 2.2 Chain Complexes and Homology Groups for Hypergraphs

Let \( R \) be a commutative ring with unit. Recall that any simplicial complex \( K \) gives a chain complex \( C_*(K; R) \). Let \( H \) be any hypergraph on \( V \). The associated simplicial complex \( \Delta H \) gives a chain complex \( C_*(\Delta H; R) \). We use
\[
\partial_n : C_n(\Delta H; R) \rightarrow C_{n-1}(\Delta H; R), \quad n = 0, 1, 2, \ldots,
\]
to denote the boundary maps of this chain complex. For simplicity, we sometimes denote \( \partial_n \) as \( \partial \) and omit the dimension \( n \).

By Definition 6, Definition 7 and Proposition 2.5, the lower-associated simplicial complex \( \delta H \) is a simplicial sub-complex of \( \Delta H \). Consequently, \( \delta H \) gives a sub-chain complex \( C_*(\delta H; R) \) of \( C_*(\Delta H; R) \). The boundary map of \( C_*(\delta H; R) \) is the restriction of \( \partial_n \) to \( C_*(\delta H; R) \), i.e.
\[
\partial_n |_{C_*(\delta H; R)} : C_n(\delta H; R) \rightarrow C_{n-1}(\delta H; R), \quad n = 0, 1, 2, \ldots.
\]

Let \( D_+ \) be a graded sub-\( R \)-module of \( C_*(\Delta H; R) \).

**Definition 8.** (cf. [6, Section 2]). The *infimum chain complex* \( \{ \text{Inf}_n(D_+, C_*(\Delta H; R)) \}_{n \geq 0} \) is the largest sub-chain complex of \( C_*(\Delta H; R) \) contained in \( D_+ \) as graded \( R \)-modules.

**Lemma 2.6.** (cf. [6, Section 2]). The *infimum chain complex* can be expressed explicitly as
\[
\text{Inf}_n(D_+, C_*(\Delta H; R)) = D_n \cap \partial_n^{-1}(D_{n-1}), \quad n \geq 0.
\]

\( \Box \)
Definition 9. (cf. [6 Section 2]). The supremum chain complex \( \{ \text{Sup}_n(D_\ast, C_\ast(\Delta H; R)) \}_{n \geq 0} \) is the smallest sub-chain complex of \( C_\ast(\Delta H; R) \) containing \( D_\ast \) as graded \( R \)-modules.

Lemma 2.7. (cf. [6 Section 2]). The supremum chain complex can be expressed explicitly as

\[
\text{Sup}_n(D_\ast, C_\ast(\Delta H; R)) = D_n + \partial_{n+1}(D_{n+1}), \quad n \geq 0,
\]

\[\square\]

It is obvious that as chain complexes, we have

\[
\{ \text{Inf}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Inf}_n(D_\ast)} \}_{n \geq 0} \subseteq \{ \text{Sup}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Sup}_n(D_\ast)} \}_{n \geq 0} \subseteq \{ C_n(\Delta H; R), \partial_n \}_{n \geq 0}.
\]

(2.8)

Consider the canonical inclusion

\[\iota : \text{Inf}_n(D_\ast, C_\ast(\Delta H; R)) \rightarrow \text{Sup}_n(D_\ast, C_\ast(\Delta H; R)), \quad n \geq 0.\]

We have the next lemma.

Lemma 2.8. The canonical inclusion \( \iota \) induces an isomorphism of homology groups

\[
\iota_* : H_k(\{ \text{Inf}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Inf}_n(D_\ast)} \}_{n \geq 0}) \cong H_k(\{ \text{Sup}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Sup}_n(D_\ast)} \}_{n \geq 0}).
\]

(2.9)

Proof. Let \( k \geq 0 \) be an integer. By the definition of homology groups, we have

\[
H_k(\text{Sup}_n(D_\ast, C_\ast(\Delta H; R))) = \ker(\partial_k|_{D_k + \partial_{k+1}D_{k+1}})/\text{im}(\partial_{k+1}|_{D_{k+1} + \partial_{k+2}D_{k+2}})
\]

and

\[
H_k(\text{Inf}_n(D_\ast, C_\ast(\Delta H; R))) = \ker(\partial_k|_{D_k + \partial_{k+1}D_{k+1}})/\text{im}(\partial_{k+1}|_{D_{k+1} + \partial_{k+2}D_{k+2}}).
\]

Moreover, by the proof of [6] Proposition 2.4 or by a direct calculation, we have

\[
\ker(\partial_k|_{D_k + \partial_{k+1}D_{k+1}}) = \partial_k D_k + \ker(\partial_k|_{D_k}),
\]

\[
\text{im}(\partial_{k+1}|_{D_{k+1} + \partial_{k+2}D_{k+2}}) = \partial_{k+1} D_{k+1},
\]

\[
\ker(\partial_k|_{D_k + \partial_{k+1}D_{k+1}}) = \ker(\partial_k|_{D_k}),
\]

\[
\text{im}(\partial_{k+1}|_{D_{k+1} + \partial_{k+2}D_{k+2}}) = \ker(\partial_n|_{D_k}) \cap \partial_{k+1} D_{k+1}.
\]

For each \( k \geq 0 \), the canonical inclusion \( \iota \) induces a homomorphism

\[
\iota_* : H_k(\{ \text{Inf}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Inf}_n(D_\ast)} \}_{n \geq 0}) \rightarrow H_k(\{ \text{Sup}_n(D_\ast, C_\ast(\Delta H; R)), \partial_n |_{\text{Sup}_n(D_\ast)} \}_{n \geq 0})
\]

of homology groups. Precisely, \( \iota_* \) sends an element

\[
d_k + \ker(\partial_n|_{D_k}) \cap \partial_{k+1} D_{k+1},
\]
where \( d_k \in D_k \) such that \( \partial_k d_k = 0 \), in the quotient \( R \)-module 
\[
\text{Ker}(\partial_k|_{D_k})/(\text{Ker}(\partial_n|_{D_k}) \cap \partial_{k+1}D_{k+1})
\]
to the element
\[
d_k + \partial_{k+1}D_{k+1}
\]
in the quotient \( R \)-module
\[
(\partial_{k+1}D_{k+1} + \text{Ker}(\partial_k|_{D_k}))/\partial_{k+1}D_{k+1}.
\]
By the isomorphism theorem of modules, we have that \( \iota_* \) is an isomorphism. \( \square \)

Remark 1: In \([6, Proposition 2.4]\), it is proved that for each \( k \geq 0 \), the homology groups
\[
H_k(\{\text{Inf}_n(D_*, C_*(\Delta \mathcal{H}; R)), \partial_n|_{\text{Inf}_n(D_*)}\}_{n \geq 0})
\]
and
\[
H_\ast(\{\text{Sup}_n(D_*, C_*(\Delta \mathcal{H}; R)), \partial_n|_{\text{Sup}_n(D_*)}\}_{n \geq 0})
\]
are isomorphic. Here in Lemma 2.8 we strengthen \([6, Proposition 2.4]\) and prove that the canonical inclusion induces such an isomorphism.

With the help of Lemma 2.8, the embedded homology groups of \( D_\ast \) can be defined:

**Definition 10.** (cf. \([6, Section 2]\)). We call the homology groups in (2.9) the embedded homology of \( D_\ast \) and denote the homology groups as \( H_n(D_\ast, C_*(\Delta \mathcal{H}; R)), n \geq 0 \).

For each \( n \geq 0 \), we use \( R(\mathcal{H})_n \) to denote the free \( R \)-module generated by all the \( n \)-hyperedges of \( \mathcal{H} \). We consider the specific graded sub-\( R \)-module
\[
D_n = R(\mathcal{H})_n, \quad n \geq 0
\]
of \( C_*(\Delta \mathcal{H}; R) \). As a special case of Definition 10, the embedded homology groups of \( \mathcal{H} \) can be defined:

**Definition 11.** (cf. \([6, Subsection 3.2]\)). We call the homology groups
\[
H_n(\mathcal{H}; R) = H_n(R(\mathcal{H})_+, C_*(\Delta \mathcal{H}; R)), \quad n \geq 0
\]
the embedded homology of \( \mathcal{H} \) with coefficients in \( R \) and denote the homology groups as \( H_n(\mathcal{H}; R), n \geq 0 \).

2.3 Homomorphisms Among The Homology Groups for Hypergraphs

We have the next lemma.

**Lemma 2.9.** The canonical inclusions
\[
\delta \mathcal{H} \overset{\delta}{\longrightarrow} \mathcal{H} \overset{\iota_\text{\Delta}}{\longrightarrow} \Delta \mathcal{H}
\]
induce homomorphisms
\[
H_\ast(\delta \mathcal{H}; R) \overset{(\iota_\text{\Delta})}{\longrightarrow} H_\ast(\mathcal{H}; R) \overset{(\iota_\text{\Delta})}{\longrightarrow} H_\ast(\Delta \mathcal{H}; R)
\]
of homology groups.
By Definition 7, there exists some \( \phi \) such that \( \phi \) and \( \Delta \) is an \( n \)-hyperedge of \( H \).

Let \( \phi \) be a hypergraph on \( V \). Since \( \phi \) is a morphism of hypergraphs, we have that whenever \( \sigma \) is a \( k \)-hyperedge of \( H \), we always have that

\[
\phi(\sigma) = \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_k)\}
\]

(2.10)
is an \( l \)-hyperedge of \( H' \) for some \( 0 \leq l \leq k \), where \( l \) is the number of distinct vertices among \( \varphi(v_0), \varphi(v_1), \ldots, \varphi(v_k) \).

Let \( \varphi : H \to H' \) be a morphism of hypergraphs. By an argument similar to [6] Section 3.1, we have the next lemma.

**Lemma 2.10.** A morphism \( \varphi : H \to H' \) of hypergraphs induces two simplicial maps

\[
\delta \varphi : \delta H \to \delta H'
\]

and

\[
\Delta \varphi : \Delta H \to \Delta H'
\]
such that \( \varphi = (\Delta \varphi) |_H \) and \( \delta \varphi = \varphi |_{\delta H} \).

**Proof.** Let \( n \geq 0 \). Let \( \{v_0, v_1, \ldots, v_n\} \) (these \( n+1 \) vertices are distinct) be an \( n \)-hyperedge of \( H \). Since \( \varphi \) is a morphism of hypergraphs, we have that \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)\} \) (these \( n+1 \) vertices may not be distinct) is an \( m \)-hyperedge of \( H' \) for some \( 0 \leq m \leq n \).

For any \( k \geq 0 \) and any \( k \)-simplex \( \{u_0, u_1, \ldots, u_k\} \) of \( \delta H \), we define

\[
(\delta \varphi)(\{u_0, u_1, \ldots, u_k\}) = \{\varphi(u_0), \varphi(u_1), \ldots, \varphi(u_k)\}.
\]

(2.11)

By Definition [7] there exists some \( n \geq k \) and some \( n \)-hyperedge \( \{v_0, v_1, \ldots, v_n\} \) of \( H \) such that \( \{u_0, u_1, \ldots, u_k\} \) is a subset of \( \{v_0, v_1, \ldots, v_n\} \). Since \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)\} \) is an \( m \)-hyperedge of \( H' \) and \( \{\varphi(u_0), \varphi(u_1), \ldots, \varphi(u_k)\} \) is a subset of \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)\} \), it
follows that \( \{\varphi(v_0), \varphi(u_1), \ldots, \varphi(u_k)\} \) is an \( l \)-simplex of \( \delta H' \) for some \( 0 \leq l \leq k \). Therefore, \( \delta \varphi \) is a simplicial map.

For any \( k \geq 0 \) and any \( k \)-simplex \( \{w_0, w_1, \ldots, w_k\} \) of \( \Delta H \), we define

\[
(\Delta \varphi)(\{w_0, w_1, \ldots, w_k\}) = \{\varphi(w_0), \varphi(w_1), \ldots, \varphi(w_k)\}.
\]

By Definition 6, for any \( 0 \leq l \leq k \) and any subset \( \{v_0, v_1, \ldots, v_l\} \) of \( \{w_0, w_1, \ldots, w_k\} \), we always have that \( \{v_0, v_1, \ldots, v_l\} \) is an \( l \)-hyperedge of \( H \). Hence there exists some \( 0 \leq m \leq l \) such that \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_l)\} \) is an \( m \)-hyperedge of \( H' \). This implies that for any subset \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_l)\} \) of \( \{\varphi(w_0), \varphi(w_1), \ldots, \varphi(w_k)\} \), we always have that \( \{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_l)\} \) is an \( m \)-hyperedge of \( H' \) for some \( 0 \leq m \leq l \). Consequently, \( \{\varphi(w_0), \varphi(w_1), \ldots, \varphi(w_k)\} \) is a simplex of \( \Delta H' \). Therefore, \( \Delta \varphi \) is a simplicial map.

Finally, it is obvious that \( \varphi = (\Delta \varphi) \vert_H \) and \( \delta \varphi = \varphi \vert_{\delta H} \).

The next theorem follows from Proposition 2.6, Lemma 2.9, Lemma 2.10 and [6, Proposition 3.7].

**Theorem 2.11.** A morphism \( \varphi : H \rightarrow H' \) of hypergraphs induces homomorphisms between the homology groups

\[
(\delta \varphi)_* : H_*(\delta H) \rightarrow H_*(\delta H'),
\]

\[
(\Delta \varphi)_* : H_*(\Delta H) \rightarrow H_*(\Delta H'),
\]

\[
\varphi_* : H_*(H) \rightarrow H_*(H')
\]

such that the following diagram commutes

\[
\begin{array}{ccc}
H_*(\delta H; R) & (i_\Delta)_* & H_*(\Delta H; R) \\
\downarrow (\delta \varphi)_* & & \downarrow (\Delta \varphi)_* \\
H_*(\delta H'; R) & (i_\Delta)_* & H_*(\Delta H'; R)
\end{array}
\]

In addition, if both \( H \) and \( H' \) are simplicial complexes, then \( \varphi \) is a simplicial map and the three homomorphisms \( (\delta \varphi)_*, (\Delta \varphi)_* \) and \( \varphi_* \) in (2.13), (2.14) and (2.15) are the same.

**Proof.** The simplicial maps \( \delta \varphi \) and \( \Delta \varphi \) induce homomorphisms between the homology groups (2.13) and (2.14) respectively. Moreover, by [6, Proposition 3.7], we have an induced homomorphism between the embedded homology (2.15). By a direct diagram chasing, we can prove that the diagram in Theorem 2.11 commutes.

Suppose in addition that both \( H \) and \( H' \) are simplicial complexes. Then by Proposition 2.5 we have \( \Delta H = \delta H = H \) and \( \Delta H' = \delta H' = H' \). It follows from the commutative diagram that all the homomorphisms \( (i_\Delta)_*, (i_\delta)_* \) and \( (i_\delta)_* \) are isomorphisms. It also follows from (2.10), (2.11) and (2.12) that the three homomorphisms \( (\delta \varphi)_*, (\Delta \varphi)_* \) and \( \varphi_* \) in (2.13), (2.14) and (2.15) are the same.

**2.4 Examples**

The associated simplicial complex, the lower-associated simplicial complex and the embedded homology detect the topology of hypergraphs from different aspects. Given two hypergraphs \( H \) and \( H' \), we consider the following conditions:
Example 2.12. Let
\[ \mathcal{H} = \{\{v_0, v_1, v_2, v_3\}, \{v_0\}\}, \]
\[ \mathcal{H}' = \{\{v_0, v_1, v_2, v_3\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_0\}\}. \]

Then \(\delta \mathcal{H} = \delta \mathcal{H}' = \{\{v_0\}\}.\) Moreover, both \(\Delta \mathcal{H}\) and \(\Delta \mathcal{H}'\) are the tetrahedron. Furthermore, \(H_1(\mathcal{H}) = 0\) and \(H_1(\mathcal{H}') = \mathbb{Z}^{\oplus 3}.\)

The next example shows that (1) and (2) cannot imply (3).

Example 2.13. (see Figure 1.) Let
\[ \mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \]
\[ \{v_0, v_1, v_3\}, \{v_1, v_2, v_4\}, \{v_3, v_4, v_5\}\}, \]
\[ \mathcal{H}' = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \]
\[ \{v_0, v_1, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_3, v_4, v_5\}\}. \]

Then \(\delta \mathcal{H} = \delta \mathcal{H}' = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}.\) Moreover, \(H_n(\mathcal{H}) = H_n(\mathcal{H}') = 0\) for \(n \geq 1\) and \(H_0(\mathcal{H}) = H_0(\mathcal{H}') = \mathbb{Z}^{\oplus 6}.\) Furthermore, \(H_1(\Delta \mathcal{H}) = \mathbb{Z},\) and \(H_1(\Delta \mathcal{H}') = 0.\)

\[ \text{Figure 1: Example 2.13} \]

For general hypergraphs \(\mathcal{H}\) and \(\mathcal{H}',\) the following example shows that the homomorphisms (2.13), (2.14) and (2.15) of the homology groups can be distinct.

Example 2.14. Let \(\mathcal{H} = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}\}.\) Let \(\sigma = \{v_0, v_1, v_2\}.\) Let \(\mathcal{H}' = \mathcal{H} \cup \{\sigma\}.\) Let \(\varphi\) be the canonical inclusion of \(\mathcal{H}\) into \(\mathcal{H}'.\) Then

(a) \(H_0(\mathcal{H}) = H_0(\mathcal{H}') = 0,\) \(H_1(\mathcal{H}) = \mathbb{Z},\) and \(H_1(\mathcal{H}') = 0.\) Moreover, \(\varphi_*\) is

- the identity map from zero to zero for the homology groups of dimension 0,
the zero map on $\mathbb{Z}$ for the homology groups of dimension 1;
(b). $\delta \mathcal{H} = \delta \mathcal{H}' = \emptyset$. Moreover, $(\delta \varphi)_*$ is the identity map from zero to zero for the homology groups of all dimensions;
(c). $\mathcal{H} \simeq S^1$ and $\mathcal{H}' \simeq \ast$. Moreover, $(\Delta \varphi)_*$ is
- the identity map on $\mathbb{Z}$ for the homology groups of dimension 0,
- the zero map on $\mathbb{Z}$ for the homology groups of dimension 1.

3 Discrete Morse Functions on Hypergraphs and Critical Hyperedges

Let $\mathcal{H}$ be a hypergraph on $V$. In this section, we define the discrete Morse functions on $\mathcal{H}$ and their critical hyperedges. We study the extensions of discrete Morse functions on hypergraphs to discrete Morse functions on the associated simplicial complexes.

3.1 Discrete Morse Functions on Hypergraphs

Definition 13. A function $f : \mathcal{H} \rightarrow \mathbb{R}$ is called a discrete Morse function on $\mathcal{H}$ if for every $n \geq 0$ and every $\alpha^{(n)} \in \mathcal{H}$, both of the two conditions hold:

(i). $\# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f(\beta) \leq f(\alpha), \beta \in \mathcal{H} \} \leq 1$;
(ii). $\# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f(\gamma) \geq f(\alpha), \gamma \in \mathcal{H} \} \leq 1$.

Remark 2: Note that Definition 13 is a generalization of [9, Definition 2.1]. In particular, we take $\mathcal{H}$ to be a simplicial complex in Definition 13. Then the discrete Morse functions defined in Definition 13 are the same as the discrete Morse functions defined in [9, Definition 2.1].

The next lemma follows from Definition 13.

Lemma 3.1. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs such that $\mathcal{H}' \subseteq \mathcal{H}$. Suppose $f : \mathcal{H} \rightarrow \mathbb{R}$ is a discrete Morse function on $\mathcal{H}$. Let $f' = f |_{\mathcal{H}'}$ be the restriction of $f$ to $\mathcal{H}'$. Then $f'$ is a discrete Morse function on $\mathcal{H}'$.

Proof. For every $\alpha^{(n)} \in \mathcal{H}$, since $\mathcal{H}' \subseteq \mathcal{H}$,

$\{ \beta^{(n+1)} > \alpha^{(n)} \mid f'(\beta) \leq f'(\alpha), \beta \in \mathcal{H}' \} \subseteq \{ \beta^{(n+1)} > \alpha^{(n)} \mid f(\beta) \leq f(\alpha), \beta \in \mathcal{H} \}$.

Since $f$ is a discrete Morse function on $\mathcal{H}$,

$\# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f'(\beta) \leq f'(\alpha), \beta \in \mathcal{H}' \} \leq \# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f(\beta) \leq f(\alpha), \beta \in \mathcal{H} \} \leq 1$.

Similarly,

$\# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f'(\gamma) \geq f'(\alpha), \gamma \in \mathcal{H}' \} \leq \# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f(\gamma) \geq f(\alpha), \gamma \in \mathcal{H} \} \leq 1$.

Hence $f'$ is a discrete Morse function on $\mathcal{H}'$. \\[\square\]

Remark 3: In particular, if $\mathcal{H}$ is a simplicial complex, then Lemma 3.1 is reduced to [9] Lemma 2.1.
The next proposition follows from Lemma 3.1 immediately.

**Proposition 3.2.** (i). Let \( \overline{f} : \Delta H \rightarrow \mathbb{R} \) be a discrete Morse function on \( \Delta H \). Then \( f = \overline{f} |_H \) is a discrete Morse function on \( H \);

(ii). Let \( f : H \rightarrow \mathbb{R} \) be a discrete Morse function on \( H \). Then \( f = \overline{f} |_{\delta H} \) is a discrete Morse function on \( \delta H \).

**Proof.** We apply Lemma 3.1 to the pair \( H \subseteq \Delta H \). We obtain (i). We apply Lemma 3.1 to the pair \( \delta H \subseteq H \). We obtain (ii). \( \square \)

The next corollary is a consequence of \([9]\) and Proposition 3.2.

**Corollary 3.3.** For any hypergraph \( H \), there exist discrete Morse functions \( \overline{f} \) on \( \Delta H \), \( f \) on \( H \), and \( \underline{f} \) on \( \delta H \) such that \( \overline{f} = f |_H \) and \( \underline{f} = f |_{\delta H} = \overline{f} |_{\delta H} \).

**Proof.** By \([9, \text{Section 4}]\), there exists a discrete Morse function \( \overline{f} \) on \( \Delta H \) (in the sense of \([9, \text{Definition 2.1}]\)). By Proposition 3.2, we obtain a discrete Morse function \( f \) on \( H \) (in the sense of Definition 13) and a discrete Morse function \( \underline{f} \) on \( \delta H \) (in the sense of \([9, \text{Definition 2.1}]\)) such that \( f = \overline{f} |_H \) and \( \underline{f} = f |_{\delta H} = \overline{f} |_{\delta H} \). \( \square \)

### 3.2 Critical Hyperedges

Let \( f \) be a discrete Morse function on \( H \).

**Definition 14.** A hyperedge \( \alpha^{(n)} \in H \) is called **critical** if both of the following two conditions hold:

(i). \( \# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f(\beta) \leq f(\alpha), \beta \in H \} = 0; \)

(ii). \( \# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f(\gamma) \geq f(\alpha), \gamma \in H \} = 0. \)

**Remark 4:** In particular, if \( H \) is a simplicial complex, then the critical hyperedges defined in Definition 14 are the same as the critical simplices defined in \([9, \text{Definition 2.2}]\).

**Definition 15.** We use \( M(f, H) \) to denote the set of all critical hyperedges.

The next lemma is equivalent to Definition 14.

**Lemma 3.4.** For any \( n \geq 0 \) and any \( \alpha^{(n)} \in H \), we have that \( \alpha \notin M(f, H) \) if at least one of the following conditions hold:

(A). there exists \( \beta^{(n+1)} > \alpha^{(n)}, \beta \in H \), such that \( f(\beta) \leq f(\alpha) \);

(B). there exists \( \gamma^{(n-1)} < \alpha^{(n)}, \gamma \in H \), such that \( f(\gamma) \geq f(\alpha) \).

**Proof.** The lemma is the contrapositive statement of Definition 14. \( \square \)

**Lemma 3.5.** Let \( H \) and \( H' \) be two hyperedges such that \( H' \subseteq H \). Let \( f : H \rightarrow \mathbb{R} \) be a discrete Morse function on \( H \) and let \( f' = f |_{H'} \). Then

\[ M(f, H) \cap H' \subseteq M(f', H'). \]
Proof. The lemma follows from a straightforward verification by using Definition 14. Let \( \alpha^{(n)} \in M(f, H) \cap H' \). Then since \( \alpha^{(n)} \in M(f, H) \), by Definition 14,
\[
\# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f(\beta) \leq f(\alpha), \beta \in H \} = 0
\]
and
\[
\# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f(\gamma) \geq f(\alpha), \gamma \in H \} = 0.
\]
Since \( H' \subseteq H \), it follows that
\[
\# \{ \beta^{(n+1)} > \alpha^{(n)} \mid f'(\beta) \leq f'(\alpha), \beta \in H' \} = 0
\]
and
\[
\# \{ \gamma^{(n-1)} < \alpha^{(n)} \mid f'(\gamma) \geq f'(\alpha), \gamma \in H' \} = 0.
\]
Therefore, we obtain \( \alpha \in M(f', H') \). The lemma is proved.

The next proposition follows from Lemma 3.5 immediately.

**Proposition 3.6.** Let \( \overline{f} : \Delta H \rightarrow \mathbb{R} \) be a discrete Morse function on \( \Delta H \). Let \( f : H \rightarrow \mathbb{R} \) and \( f : \delta H \rightarrow \mathbb{R} \) be the discrete Morse functions induced from \( \overline{f} \). Then
\[
M(f, \Delta H) \cap H \subseteq M(f, H),
\]
\[
M(f, \Delta H) \cap \delta H \subseteq M(f, \delta H),
\]
\[
M(f, H) \cap \delta H \subseteq M(f, \delta H).
\]

Proof. We apply Lemma 3.5 to the pair \( H \subseteq \Delta H \). Then we obtain (3.1). Similarly, we apply Lemma 3.5 to the pairs \( \delta H \subseteq \Delta H \) and \( \delta H \subseteq H \). Then we obtain the other two subset relations.

### 3.3 Extensions of Discrete Morse Functions

The next lemma is proved in [9].

**Lemma 3.7.** (cf. [9, Lemma 2.5]). Let \( K \) be a simplicial complex. Let \( g \) be a discrete Morse function on \( K \). Let \( \alpha \) be a simplex of \( K \). Then the conditions
(A). there exists \( \beta^{(n+1)} > \alpha^{(n)}, \beta \in K \), such that \( g(\beta) \leq g(\alpha) \);
(B). there exists \( \gamma^{(n-1)} < \alpha^{(n)}, \gamma \in K \), such that \( g(\gamma) \geq g(\alpha) \)
cannot both be true.

The next lemma is a consequence of Lemma 3.7.

**Lemma 3.8.** Let \( H \) be a hypergraph. Let \( \overline{f} \) be a discrete Morse function on \( \Delta H \). Let \( \alpha \in \Delta H \). Then the conditions
(A). there exists \( \beta^{(n+1)} > \alpha^{(n)}, \beta \in \Delta H \), such that \( \overline{f}(\beta) \leq \overline{f}(\alpha) \);
(B). there exists \( \gamma^{(n-1)} < \alpha^{(n)}, \gamma \in \Delta H \), such that \( \overline{f}(\gamma) \geq \overline{f}(\alpha) \)
cannot both be true.
Proof. In Lemma 3.7 we let $K$ be $\Delta H$ and let $g$ be $f$. The lemma follows. \[\square\]

The next proposition follows from Lemma 3.8.

Proposition 3.9 (Obstructions for The Extensions of Discrete Morse Functions). Let $\mathcal{H}$ be a hypergraph. Let $f$ be a discrete Morse function on $\mathcal{H}$. If there exists $\alpha \in \mathcal{H}$ such that both (A) and (B) in Lemma 3.4 hold for $\alpha$, then $f$ cannot be extended to be a discrete Morse function $\tilde{f}$ on $\Delta \mathcal{H}$.

Proof. Suppose to the contrary that $f$ can be extended to be a discrete Morse function $\tilde{f}$ on $\Delta \mathcal{H}$. Then by Lemma 3.8 for any $\alpha \in \Delta \mathcal{H}$, the conditions (A) and (B) in Lemma 3.8 cannot both be true. Note that for any $\alpha \in \mathcal{H}$, we have $f(\alpha) = \tilde{f}(\alpha)$. Thus for any $\alpha \in \mathcal{H}$, the conditions (A) and (B) in Lemma 3.4 cannot both be true. This contradicts with our assumption that there exists $\alpha \in \mathcal{H}$ such that both (A) and (B) in Lemma 3.4 hold for $\alpha$. Therefore, $f$ cannot be extended to be a discrete Morse function $\tilde{f}$ on $\Delta \mathcal{H}$. \[\square\]

The next is an example for Proposition 3.9.

Example 3.10. Let $\mathcal{H} = \{\{v_0\}, \{v_0, v_1\}, \{v_0, v_1, v_2\}\}$. Let $f(\{v_0\}) = 2$, $f(\{v_0, v_1\}) = 1$, $f(\{v_0, v_1, v_2\}) = 0$.

Then $f$ is a discrete Morse function on $\mathcal{H}$, and $\{v_0, v_1\}$ is not critical.

(i). The hyperedge $\{v_0, v_1\}$ satisfies both of the conditions (A) and (B);

(ii). The discrete Morse function $f$ cannot be extended to a discrete Morse function on $\Delta \mathcal{H}$.

The next proposition proves that under certain conditions, the obstruction for the extensions of discrete Morse functions given in Proposition 3.9 would fail.

Proposition 3.11. Suppose $\mathcal{H}$ satisfies the following condition

\textbf{Condition (C).} for any $n \geq 1$ and any hyperedges $\beta^{(n+1)} > \alpha^{(n)} > \gamma^{(n-1)}$ of $\mathcal{H}$, there exists $\hat{\alpha}^{(n)} \in \mathcal{H}$, $\hat{\alpha} \neq \alpha$, such that $\beta > \hat{\alpha} > \gamma$.

Then the conditions (A) and (B) in Lemma 3.4 cannot both be true.

Proof. The proof is similar with the proof of [9, Lemma 2.5]. Let $\mathcal{H}$ be a hypergraph satisfying the condition (C). Let $f$ be a discrete Morse function on $\mathcal{H}$. Suppose (A) in Lemma 3.4 is true. Then by Definition 3.3 we have

\[ f(\beta) > f(\hat{\alpha}) \tag{3.2} \]

for any $\hat{\alpha}^{(n)} \in \mathcal{H}$ with $\hat{\alpha} \neq \alpha$ and $\beta > \hat{\alpha} > \gamma$. By (3.2) and (A) in Lemma 3.4 we have

\[ f(\hat{\alpha}) < f(\alpha). \tag{3.3} \]

Now suppose (B) in Lemma 3.4 is true. Then by Definition 3.3 we have

\[ f(\gamma) < f(\hat{\alpha}) \tag{3.4} \]

for any $\hat{\alpha}^{(n)} \in \mathcal{H}$ with $\hat{\alpha} \neq \alpha$ and $\beta > \hat{\alpha} > \gamma$. By (3.4) and (B) in Lemma 3.4 we have

\[ f(\hat{\alpha}) > f(\alpha). \tag{3.5} \]

Note that (3.3) and (3.5) contradict with each other. Therefore, the conditions (A) and (B) cannot both be true. \[\square\]
There do exist hypergraphs $\mathcal{H}$ and discrete Morse functions $f$ on $\mathcal{H}$ such that the condition (C) in Proposition 3.11 does not hold while the conditions (A) and (B) in Lemma 3.3 cannot both be true. We consider the next example.

**Example 3.12.** Let $\mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1, v_2\}\}$. Let $f(\{v_0\}) = f(\{v_1\}) = f(\{v_2\}) = 2$, $f(\{v_0, v_1, v_2\}) = 0$. Then $f$ is a discrete Morse function on $\mathcal{H}$. All the hyperedges are critical. The condition (C) in Proposition 3.11 does not hold while for any $\alpha \in \mathcal{H}$, the conditions (A) and (B) in Lemma 3.3 cannot both be true.

**Proposition 3.13.** In Example 3.12, $f$ cannot be extended to be a discrete Morse function on $\Delta \mathcal{H}$.

*Proof.* Suppose to the contrary, $\mathcal{F} : \Delta \mathcal{H} \rightarrow \mathbb{R}$ is a discrete Morse function such that $f = \mathcal{F} |_{\mathcal{H}}$. Then there exists at least two edges among $\{v_0, v_1\}, \{v_1, v_2\}$ and $\{v_0, v_2\}$, say $\{v_0, v_1\}$ and $\{v_1, v_2\}$, such that

$$\mathcal{F}(\{v_0, v_1\}) < \mathcal{F}(\{v_0, v_1, v_2\}), \quad \mathcal{F}(\{v_1, v_2\}) < \mathcal{F}(\{v_0, v_1, v_2\}).$$

Moreover, we have both of the followings

- $\mathcal{F}(u) < \mathcal{F}(\{v_0, v_1\})$ for $u = v_0$ or $v_1$;
- $\mathcal{F}(w) < \mathcal{F}(\{v_1, v_2\})$ for $w = v_1$ or $v_2$.

Hence there exists at least one 0-hyperedge $\{x\}$ among $\{v_0\}, \{v_1\}$ and $\{v_2\}$ where $\mathcal{F}(\{x\})$ is smaller than $\mathcal{F}(\{v_0, v_1, v_2\})$. This contradicts that $\mathcal{F}$ is the extension of $f$. \(\square\)

### 4 Discrete Gradient Vector Fields on Hypergraphs

Let $\mathcal{H}$ be a hypergraph. In this section, we study the discrete gradient vector fields on $\mathcal{H}$.

#### 4.1 Abstract Discrete Gradient Vector Fields on Hypergraphs

**Definition 16.** An (abstract) discrete gradient vector field $\mathcal{V}$ on $\mathcal{H}$ is a collection of pairs $\{\alpha_i^{(n)} < \beta_i^{(n+1)}\}$ of hyperedges of $\mathcal{H}$, $n \geq 0$, such that there is no non-trivial closed paths of the form $\alpha_0^{(n)}, \beta_0^{(n+1)}, \alpha_1^{(n)}, \beta_1^{(n+1)}, \alpha_2^{(n)}, \ldots, \alpha_r^{(n)}, \beta_r^{(n+1)}, \alpha_{r+1}^{(n)} = \alpha_0^{(n)}$,

where for each $0 \leq i \leq r$, we have $\{\alpha_i^{(n)} < \beta_i^{(n+1)}\} \in \mathcal{V}$, $\{\alpha_i^{(n)} < \beta_i^{(n+1)}\} \in \mathcal{V}$ and $\alpha_i \neq \alpha_{i+1}$.

**Definition 17.** Let $\mathcal{V}$ be a discrete gradient vector field on $\mathcal{H}$. If there does not exist any $n \geq 1$ and any triple $(\gamma^{(n-1)}, \alpha^{(n)}, \beta^{(n+1)})$ of hyperedges in $\mathcal{H}$ such that

$$\gamma^{(n-1)} < \alpha^{(n)} < \beta^{(n+1)}$$

and

$$\{\gamma < \alpha\} \in \mathcal{V} \quad \text{and} \quad \{\alpha < \beta\} \in \mathcal{V},$$

then we call $\mathcal{V}$ a semi-proper discrete gradient vector field.
Definition 18. Let $V$ be a discrete gradient vector field on $H$. If each hyperedge of $H$ is in at most one pair in $V$, then we call $V$ a proper discrete gradient vector field.

Remark 5: By Definition 17 and Definition 18, it is direct to see that a proper discrete gradient vector field is semi-proper while semi-proper discrete gradient vector field may not be proper.

Let $V$ be a discrete gradient vector field on $H$.

Definition 19. We define an $R$-linear map

$$R(V): R(H)_n \rightarrow R(H)_{n+1} \tag{4.1}$$

induced from $V$ where for each $\alpha^{(n)} \in H$, the $R$-linear map $R(V)$ sends $\alpha$ to

- $-(\partial\beta, \alpha)\beta$ if $\{\alpha^{(n)} < \beta^{(n+1)}\} \in V$,
- $0$ if $\{\alpha^{(n)} < \beta^{(n+1)}\} \notin V$.

Remark 6: In Definition 19, the notion $(\partial\beta, \alpha)$ is the incidence number of $\beta$ and $\alpha$ in the chain complex $C_\ast(\Delta H; R)$ (cf. [9, p. 98]). Note that $(\partial\beta, \alpha)$ takes the values $\pm 1$ where $1$ refers to the multiplicative unity of $R$.

Lemma 4.1. Let $V$ be a discrete gradient vector field on $H$. Then $V$ is semi-proper if and only if the induced $R$-linear map $R(V)$ satisfies

$$R(V) \circ R(V) = 0. \tag{4.2}$$

Proof. Let $V$ be a discrete gradient vector field on $H$. Let $R(V)$ be the induced $R$-linear map.

$(\Rightarrow)$: Suppose $V$ is semi-proper. In order to prove (4.2), we take an arbitrary $\alpha^{(n)} \in H$. By the linear property of $R(V) \circ R(V)$, it suffices to prove

$$R(V) \circ R(V)(\gamma^{(n)}) = 0. \tag{4.3}$$

Case 1. $R(V)(\gamma^{(n)}) = 0$.

Then (4.3) follows immediately.

Case 2. $R(V)(\gamma^{(n)}) \neq 0$.

Suppose $\alpha^{(n+1)} \in H$ such that $\{\gamma^{(n)} < \alpha^{(n+1)}\} \in V$ and $\langle \partial\alpha, \gamma \rangle \neq 0$. Since $V$ is semi-proper and $\{\gamma^{(n)} < \alpha^{(n+1)}\} \in V$, we have that $\alpha$ is not in any other pairs of the form $\{\alpha^{(n+1)} < \beta^{(n+2)}\}$ in $V$. Thus we have $R(V)(\alpha) = 0$. Consequently,

$$R(V) \circ R(V)(\gamma^{(n)}) = R(V) \left( \sum_{\gamma^{(n)} < \alpha^{(n+1)}, \langle \partial\alpha, \gamma \rangle \neq 0} -\langle \partial\alpha, \gamma \rangle \alpha \right) = \sum_{\gamma^{(n)} < \alpha^{(n+1)}, \langle \partial\alpha, \gamma \rangle \neq 0} -\langle \partial\alpha, \gamma \rangle R(V)(\alpha) = 0.$$

Summarizing both Case 1 and Case 2, we obtain (4.3). This implies (4.2).

$(\Leftarrow)$: Suppose $R(V)$ satisfies (4.2). In order to prove that $V$ is semi-proper, we suppose to the contrary that $V$ is not semi-proper. Then by Definition 17, there exist $n \geq 1$ and a triple triple $(\gamma^{(n-1)}, \alpha^{(n)}, \beta^{(n+1)})$ of hyperedges in $H$ such that

$$\gamma^{(n-1)} < \alpha^{(n)} < \beta^{(n+1)}$$
and \[ \{ \gamma < \alpha \} \in V \quad \text{and} \quad \{ \alpha < \beta \} \in V. \]

Consequently,
\[
\langle R(V) \circ R(V)(\gamma^{(n-1)}), \beta^{(n+1)} \rangle = -\langle \partial \alpha, \gamma \rangle \langle R(V)(\alpha^{(n)}), \beta^{(n+1)} \rangle \\
= \langle \partial \alpha, \gamma \rangle \langle \partial \beta, \alpha \rangle \langle \beta^{(n+1)}, \beta^{(n+1)} \rangle \\
= \langle \partial \alpha, \gamma \rangle \langle \partial \beta, \alpha \rangle = \pm 1.
\]

This contradicts with our assumption that \( R(V) \) satisfies (4.2). Therefore, \( V \) is semi-proper.

\[ \square \]

Remark 7: In particular, if \( H \) is a simplicial complex, then the \( R \)-linear map \( \partial f \) is defined in [9, Definition 6.1].

### 4.2 Discrete Gradient Vector Fields of Discrete Morse Functions on Hypergraphs

Let \( f \) be a discrete Morse function on \( H \).

**Definition 20.** The discrete gradient vector field of \( f \), denoted as \( \text{grad} f \), is the collection of all the pairs \( \{ \alpha^{(n)} < \beta^{(n+1)} \} \) of the hyperedges in \( H \) such that \( f(\beta) \leq f(\alpha) \).

**Lemma 4.2.** Let \( f \) be a discrete Morse function on \( H \). Then \( \text{grad} f \) is a discrete gradient vector field on \( H \).

**Proof.** By Definition [10] to prove that \( \text{grad} f \) is a discrete gradient vector field on \( H \), we only need to prove that there is no nontrivial closed paths of the form

\[ \alpha^{(n)}_0, \beta^{(n+1)}_0, \alpha^{(n)}_1, \beta^{(n+1)}_1, \alpha^{(n)}_2, \beta^{(n+1)}_2, \ldots, \alpha^{(n)}_r, \beta^{(n+1)}_r, \alpha^{(n)}_{r+1} = \alpha^{(n)}_0, \]

where for each \( 0 \leq i \leq r \), we have \( \{ \alpha^{(n)}_i < \beta^{(n+1)}_i \} \in \text{grad} f \), \( \{ \alpha^{(n)}_{i+1} < \beta^{(n+1)}_i \} \in \text{grad} f \) and \( \alpha_i \neq \alpha_{i+1} \). Suppose to the contrary, there exists such a closed path. Then by Definition [20] we have

\[ f(\beta_i) \leq f(\alpha_i) \quad \text{and} \quad f(\beta_i) \leq f(\alpha_{i+1}). \]

This contradicts with Definition [13]. Therefore, there is no such closed paths, which implies that \( \text{grad} f \) is a discrete gradient vector field on \( H \).

\[ \square \]

**Lemma 4.3.** Let \( f \) be a discrete Morse function on \( H \). Then for any \( n \geq 0 \) and any \( \alpha^{(n)} \in H \), the induced \( R \)-linear map \( R(\text{grad} f) \) of \( \text{grad} f \) is given as follows:

(i). If there exists \( \beta^{(n+1)} > \alpha^{(n)} \), \( \beta \in H \), such that \( f(\beta) \leq f(\alpha) \), then
\[
R(\text{grad} f)(\alpha) = -\langle \partial \beta, \alpha \rangle \beta;
\]

(ii). If there does not exist any \( \beta^{(n+1)} > \alpha^{(n)} \), \( \beta \in H \), such that \( f(\beta) \leq f(\alpha) \), then
\[
R(\text{grad} f)(\alpha) = 0.
\]
Theorem. The lemma follows directly from Definition 19.

Example 4.4. Consider the hypergraph \( H \) and the discrete Morse function \( f \) given in Example 3.10. Let \( \text{grad} f \) be the discrete gradient vector field of \( f \). Then

\[
\begin{align*}
(\text{grad} f)(\{v_0\}) &= \{v_0, v_1\}, \\
(\text{grad} f)(\{v_0, v_1\}) &= -\{v_0, v_1, v_2\}.
\end{align*}
\]

Hence \( (\text{grad} f) \circ (\text{grad} f)(\{v_0\}) = -\{v_0, v_1, v_2\} \neq 0 \).

Nevertheless, the next proposition shows that by imposing the condition (C) in Proposition 3.11 on the hypergraphs, the discrete gradient vector fields of discrete Morse functions must be proper.

Proposition 4.5. Let \( H \) be a hypergraph satisfying the condition (C) in Proposition 3.11. Let \( f \) be a discrete Morse function on \( H \). Then \( \text{grad} f \) is a proper discrete gradient vector field.

Proof. Firstly, we prove that \( \text{grad} f \) is semi-proper. By Lemma 4.1, it is sufficient to prove that

\[
\text{R}(\text{grad} f) \circ \text{R}(\text{grad} f) = 0. \tag{4.4}
\]

The proof is similar with the proof of \([9, \text{Theorem 6.3 (1)}]\). Suppose \( \gamma^{(n-1)}, \alpha^{(n)} \in H \) and

\[
\text{R}(\text{grad} f)(\gamma^{(n-1)}) = \pm \alpha^{(n)}.
\]

Then \( \gamma^{(n-1)} < \alpha^{(n)} \) and \( f(\gamma^{(n-1)}) \geq f(\alpha^{(n)}) \). Since \( H \) satisfies the condition (C), by Proposition 3.11 we see that there does not exist any \( \beta^{(n+1)} \in H \) such that \( \alpha^{(n)} < \beta^{(n+1)} \) and \( f(\beta^{(n+1)}) \leq f(\alpha^{(n)}) \). Thus

\[
\text{R}(\text{grad} f)(\alpha^{(n)}) = 0,
\]

which implies

\[
\text{R}(\text{grad} f) \circ \text{R}(\text{grad} f)(\gamma^{(n-1)}) = 0.
\]

Therefore, we obtain (4.4).

Secondly, we prove that \( \text{grad} f \) is proper. Suppose to the contrary, \( \text{grad} f \) is not proper. Then since \( \text{grad} f \) is semi-proper, it follows from Definition 18 and Definition 17 that at least one of the followings is satisfied:

(a). There exist \( \gamma_1^{(n-1)}, \gamma_2^{(n-1)}, \alpha^{(n)} \in H \) with \( \gamma_1 \neq \gamma_2 \) such that

\[
\{ \gamma_1 < \alpha \} \in \text{grad} f \quad \text{and} \quad \{ \gamma_2 < \alpha \} \in \text{grad} f;
\]

(b). There exist \( \beta_1^{(n+1)}, \beta_2^{(n+1)}, \alpha^{(n)} \in H \) with \( \beta_1 \neq \beta_2 \) such that

\[
\{ \alpha < \beta_1 \} \in \text{grad} f \quad \text{and} \quad \{ \alpha < \beta_2 \} \in \text{grad} f.
\]

Equivalently, we can re-state (a) and (b) respectively as (a)’ and (b)’.
(a)’. There exist $\gamma_1^{(n-1)}, \gamma_2^{(n-1)}, \alpha^{(n)} \in \mathcal{H}$ with $\gamma_1 \neq \gamma_2$ such that
$$R(\grad f)(\gamma_1^{(n-1)}) = \pm \alpha^{(n)} \quad \text{and} \quad R(\grad f)(\gamma_2^{(n-1)}) = \pm \alpha^{(n)}.$$  

(b)’. There exist $\beta_1^{(n+1)}, \beta_2^{(n+1)}, \alpha^{(n)} \in \mathcal{H}$ with $\beta_1 \neq \beta_2$ such that
$$R(\grad f)(\alpha^{(n)}) = \pm \beta_1^{(n+1)} \quad \text{and} \quad R(\grad f)(\alpha^{(n)}) = \pm \beta_2^{(n+1)}.$$  

Since $R(\grad f)$ is an $R$-linear map, we see that (b)’ is impossible. On the other hand, (a)’ implies that $\gamma_i < \alpha$ and $f(\alpha) \leq f(\gamma_i)$ for $i = 1, 2$. This contradicts that $f$ is a discrete Morse function on $\mathcal{H}$. Therefore, $\grad f$ must be proper.  

\subsection{4.3 Restrictions and Extensions of Discrete Gradient Vector Fields}

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs such that $\mathcal{H}' \subseteq \mathcal{H}$.

\textbf{Lemma 4.6.} Let $\mathcal{V}$ be a discrete gradient vector field on $\mathcal{H}'$. Then $\mathcal{V}$ is also a discrete gradient vector field on $\mathcal{H}$.

\textbf{Proof.} Let $\mathcal{V}$ be a discrete gradient vector field on $\mathcal{H}'$. We generalize the Hasse diagram construction (cf. [12, Section 6]) of simplicial complexes and apply a similar argument to hypergraphs. Specifically,

- We construct a digraph $D_H$ (resp. $D_{H'}$) as follows:
  - THE VERTICES OF $D_H$ (resp. $D_{H'}$): the vertices of $D_H$ (resp. $D_{H'}$) are in 1-1 correspondence with the hyperedges of $\mathcal{H}$ (resp. $\mathcal{H'}$);
  - THE DIRECTED EDGES OF $D_H$ (resp. $D_{H'}$): for any hyperedges $\alpha$ and $\beta$ of $\mathcal{H}$ (resp. of $\mathcal{H'}$), we assign a directed edge in $D_H$ (resp. in $D_{H'}$) from $\beta$ to $\alpha$, denoted as $\beta \rightarrow \alpha$, iff. $\alpha^{(n)} < \beta^{(n+1)}$ for some $n \geq 0$.

- We construct a digraph $D_{H',\mathcal{V}}$ as follows:
  - THE VERTICES OF $D_{H',\mathcal{V}}$: the vertices of $D_{H',\mathcal{V}}$ are in 1-1 correspondence with the hyperedges of $\mathcal{H'}$;
  - THE DIRECTED EDGES OF $D_{H',\mathcal{V}}$: for any directed edge $\beta^{(n+1)} \rightarrow \alpha^{(n)}$ in $D_{H'}$,
    * if $\{\alpha^{(n)}, \beta^{(n+1)}\} \in \mathcal{V}$, then we reverse the direction of $\beta \rightarrow \alpha$ in $D_{H'}$ and obtain a new directed edge $\alpha \rightarrow \beta$. We assign $\alpha \rightarrow \beta$ as a directed edge of $D_{H',\mathcal{V}}$;
    * if $\{\alpha^{(n)}, \beta^{(n+1)}\} \notin \mathcal{V}$, then we assign $\beta \rightarrow \alpha$ as a directed edge of $D_{H',\mathcal{V}}$.

- We construct a digraph $D_{H,\mathcal{V}}$ as follows:
  - THE VERTICES OF $D_{H,\mathcal{V}}$: the vertices of $D_{H,\mathcal{V}}$ are in 1-1 correspondence with the hyperedges of $\mathcal{H}$;
  - THE DIRECTED EDGES OF $D_{H,\mathcal{V}}$: Let $E(-)$ denotes the set of directed edges of a digraph and let
    $$E(D_{H,\mathcal{V}}) = E(D_{H',\mathcal{V}}) \cup (E(D_H) \setminus E(D_{H'})).$$
    We notice that for any directed edge $\beta \rightarrow \alpha$ in $E(D_H) \setminus E(D_{H'})$, at least one of $\alpha$ and $\beta$ is not a vertex of $D_{H'}$ (otherwise the directed edge $\beta \rightarrow \alpha$ would be in $E(D_{H'})$). Hence the union in (4.5) is a disjoint union.
We note that the following two statements are equivalent:

(I). there is no non-trivial closed paths on $H$ (resp. $H'$) of the form in Definition 16;

(II). there is no non-trivial closed paths in the underlying graph of $D_{H,V}$ (resp. $D_{H',V}$) of the form

$$\alpha_0^{(n)}, \beta_0^{(n+1)}, \alpha_1^{(n)}, \beta_1^{(n+1)}, \cdots, \alpha_r^{(n)}, \beta_r^{(n+1)}, \alpha_{r+1}^{(n)} = \alpha_0^{(n)},$$

such that for each $0 \leq i \leq r$, all of the three conditions are satisfied:

(a). $\{\alpha_i^{(n)} < \beta_i^{(n+1)}\} \in V$,
(b). $\{\alpha_{i+1}^{(n)} < \beta_{i+1}^{(n+1)}\} \in V$,
(c). $\alpha_i \neq \alpha_{i+1}$.

Since $V$ is a discrete gradient vector field on $H'$, it follows from the definition that there is no non-trivial closed paths on $H'$ of the form in Definition 16. Thus there is no non-trivial closed paths in the underlying graph of $D_{H',V}$ satisfying the conditions (a), (b) and (c) in (II). By the construction of $D_{H,V}$, it follows that there is no nontrivial closed paths in the underlying graph of $D_{H,V}$ satisfying the conditions (a), (b) and (c) in (II) as well. Consequently, there is no non-trivial closed paths on $H$ of the form in Definition 16. Therefore, we have that $V$ is a discrete gradient vector field on $H$.

Remark 8: The last paragraph in proof of Lemma 4.6 is a similar argument of [13, Theorem 6.2].

The next corollary follows from Lemma 4.6

Corollary 4.7. Let $V$ be a proper discrete gradient vector field on $H'$. Then $V$ is also a proper discrete gradient vector field on $H$.

Proof. Let $V$ be a proper discrete gradient vector field on $H'$. By Lemma 4.6 $V$ is a discrete gradient vector field on $H$. Since $V$ is proper on $H'$, each hyperedge of $H'$ appears in at most one pair in $V$. Let $\alpha \in H$. Then we have the following cases:

Case 1: $\alpha \notin H'$.
Then $\alpha$ does not appear in any pair in $V$.

Case 2: $\alpha \in H'$.
Then $\alpha$ appears in at most one pair in $V$.

Summarizing both cases, it follows that each hyperedge of $H$ appears in at most one pair in $V$. Hence $V$ is proper on $H$ as well.

The next corollary is a consequence of Lemma 4.6

Corollary 4.8. Let $V$ be a proper discrete gradient vector field on $H$. Then $V$ extends to a proper discrete gradient vector field $\nabla$ on $\Delta H$ such that $\nabla|_{H} = V$ and $\nabla|_{\Delta H \setminus H} = 0$.

Proof. We substitute the pair $H' \subseteq H$ of hypergraphs in Corollary 4.7 with the pair $H \subseteq \Delta H$. Then from the proper discrete gradient vector field $V$ on $H$ we obtain a proper discrete gradient vector field $\nabla$ on $\Delta H$ such that $\nabla|_{H} = V$ and $\nabla|_{\Delta H \setminus H} = 0$.

Remark 9: In Corollary 4.8 we use the following notations

- $\nabla|_{H} = V$ for $R(\nabla)|_{R(H)} = R(V)$;
- $\nabla|_{\Delta H \setminus H} = 0$ for $R(\nabla)|_{R(\Delta H \setminus H)} = 0$.
5 Discrete Morse Functions and Their Gradients on Hypergraphs

Let $H$ be a hypergraph. Let $\Delta H$ be the associated simplicial complex and $\delta H$ be the lower-associated simplicial complex. Let $\overline{f}$ be a discrete Morse function on $\Delta H$. Let $f$ be the restriction of $\overline{f}$ on $H$ and $\underline{f}$ be the restriction of $\overline{f}$ on $\delta H$. Then $f$ is a discrete Morse function on $H$ and $\underline{f}$ is a discrete Morse function on $\delta H$. In this section, we study the discrete gradient vector field of $f$ on $\Delta H$, the discrete gradient vector field of $f$ on $H$, and the discrete gradient vector field of $\underline{f}$ on $\delta H$.

5.1 Restrictions and Extensions of The Discrete Gradient Vector Fields

Let $H$ and $H'$ be hypergraphs such that $H' \subseteq H$. Let $f$ and $f'$ be discrete Morse functions on $H$ and $H'$ respectively such that $f' = f |_{H'}$. We consider the discrete gradient vector fields $\text{grad } f'$ on $H'$ and $\text{grad } f$ on $H$.

Lemma 5.1. Let $\pi(H, H')$ be the canonical projection from $R(H)_*$ to $R(H')_*$ sending an $R$-linear combination of hyperedges $\sum_{\sigma_i \in H \setminus H', x_i \in R} y_j \tau_j$ to the $R$-linear combination of hyperedges $\sum_{\sigma_i \in H \setminus H', x_i \in R} y_j \tau_j$. Then

$$R(\text{grad } f') = \pi(H, H') \circ R(\text{grad } f).$$

Proof. Let $\alpha^{(n)}, \beta^{(n+1)} \in H$. We divide the proof into the following steps.

Step 1. Suppose $\alpha, \beta \in H'$. Then

$$R(\text{grad } f')(\alpha) = -(\partial \beta, \alpha) \beta$$

if and only if both of the followings are satisfied:

- $\alpha < \beta$;
- $f'(\beta) \leq f'(\alpha)$

if and only if both of the followings are satisfied:

- $\alpha < \beta$;
- $f(\beta) \leq f(\alpha)$

if and only if

$$R(\text{grad } f)(\alpha) = -(\partial \beta, \alpha) \beta.$$

Step 2. Suppose $\alpha \in H'$. Then

$$R(\text{grad } f')(\alpha) = 0$$

if and only if there does not exist any $\beta \in H'$ such that both of the followings are satisfied:
• \( \alpha < \beta \);
• \( f'(\beta) \leq f'(\alpha) \)

if and only if there does not exist any \( \beta \in \mathcal{H}' \) such that both of the followings are satisfied:
• \( \alpha < \beta \);
• \( f(\beta) \leq f(\alpha) \)

if and only if either of the followings is satisfied:
• there does not exist any \( \gamma \in \mathcal{H} \) such that both of the followings are satisfied:
  - \( \alpha < \gamma \);
  - \( f(\gamma) \leq f(\alpha) \)
• there exists \( \gamma \in \mathcal{H} \setminus \mathcal{H}' \) such that both of the followings are satisfied:
  - \( \alpha < \gamma \);
  - \( f(\gamma) \leq f(\alpha) \)

if and only if either of the followings is satisfied:
• \( R(\text{grad} f)(\alpha) = 0 \);
• there exists \( \gamma \in \mathcal{H} \setminus \mathcal{H}' \) such that \( R(\text{grad} f)(\alpha) = -\langle \partial \gamma, \alpha \rangle \gamma \).

**Step 3.** Suppose \( \alpha \in \mathcal{H}' \). Then either of the followings is satisfied:
• \( R(\text{grad} f')(\alpha) = 0 \);
• \( R(\text{grad} f')(\alpha) = -\langle \partial \beta, \alpha \rangle \beta \) for some \( \beta \in \mathcal{H}' \).

**Step 4.** Suppose \( \alpha \in \mathcal{H}' \). We consider the following cases:
CASE 1. \( R(\text{grad} f')(\alpha) = 0 \).
SUBCASE 1.1. \( R(\text{grad} f)(\alpha) = 0 \).
Then \( R(\text{grad} f)(\alpha) = \pi(\mathcal{H}, \mathcal{H}') \circ R(\text{grad} f)(\alpha) = 0 \).
SUBCASE 1.2. \( R(\text{grad} f)(\alpha) \neq 0 \).
Then by the last line in Step 2, we have that
\[
R(\text{grad} f)(\alpha) = -\langle \partial \gamma, \alpha \rangle \gamma
\]
for some \( \gamma \in \mathcal{H} \setminus \mathcal{H}' \). Since
\[
\pi(\mathcal{H}, \mathcal{H}')(-\langle \partial \gamma, \alpha \rangle \gamma) = 0,
\]
it follows that
\[
\pi(\mathcal{H}, \mathcal{H}') \circ R(\text{grad} f)(\alpha) = 0 = R(\text{grad} f')(\alpha).
\]
CASE 2. \( R(\text{grad} f')(\alpha) \neq 0 \).
Then by Step 3, we have that \( R(\text{grad} f')(\alpha) = -\langle \partial \beta, \alpha \rangle \beta \) for some \( \beta \in \mathcal{H}' \). By Step 1, we have
\[
\pi(\mathcal{H}, \mathcal{H}') \circ R(\text{grad} f)(\alpha) = \pi(\mathcal{H}, \mathcal{H}')(-\langle \partial \beta, \alpha \rangle \beta)
= -\langle \partial \beta, \alpha \rangle \beta
= R(\text{grad} f')(\alpha).
\]
Summarizing the cases, it follows that for any $\alpha \in H'$ we always have 
\[ R(\text{grad } f')(\alpha) = \pi(H, H') \circ R(\text{grad } f)(\alpha). \]

**Step 5.** By the last line in Step 4 as well as the linear property of $R(\text{grad } f)$, $R(\text{grad } f')$ and $\pi(H, H')$, we obtain (5.1) and finish the proof. \[\square\]

Let $H$ be a hypergraph. The next proposition follows with the help of Corollary 4.8.

**Proposition 5.2.** Suppose $H$ is a hypergraph satisfying the condition (C) (cf. Proposition [3.11]). Let $g$ be a discrete Morse function on $H$. Then there exists a discrete Morse function $\overrightarrow{f}$ on $\Delta H$ such that both of the followings are satisfied:

(i). $\text{grad } f = \text{grad } g$ where $f = \overrightarrow{f} |_H$;

(ii). $(\text{grad } f) |_{\Delta H \setminus H} = 0$.

**Proof.** Suppose $H$ is a hypergraph satisfying the condition (C). Let $g$ be a discrete Morse function on $H$. Then by Definition [14] Definition [15] and Proposition [3.11] we have that for any $\alpha \in H$, the conditions (A) and (B) in Lemma [3.4] cannot both be true. It follows that $\text{grad } g$ is a proper discrete gradient vector field on $H$. By Corollary 4.8 $\text{grad } g$ extends to a proper discrete gradient vector field $\overrightarrow{\text{grad } g}$ on $\Delta H$ such that

\[ \overrightarrow{\text{grad } g} |_H = \text{grad } g \]

and

\[ \overrightarrow{\text{grad } g} |_{\Delta H \setminus H} = 0. \]

By [13] Theorem 3.5], there exists a discrete Morse function $\overrightarrow{f}$ on $\Delta H$ such that

\[ \overrightarrow{\text{grad } g} = \overrightarrow{\text{grad } f}. \]

Let $f = \overrightarrow{f} |_H$ be the restriction of $\overrightarrow{f}$ to $H$. Then

\[ \text{grad } f = \text{grad}(\overrightarrow{f} |_H) = (\text{grad } \overrightarrow{f}) |_H = \overrightarrow{\text{grad } g} |_H = \text{grad } g \]

and

\[ \overrightarrow{\text{grad } f} |_{\Delta H \setminus H} = \overrightarrow{\text{grad } g} |_{\Delta H \setminus H} = 0. \]

We finish the proof. \[\square\]

**Remark 10:** By Proposition 5.2 in order to study the discrete gradient vector fields of discrete Morse functions on the hypergraphs satisfying the condition (C), it is sufficient to study the discrete gradient vector fields of discrete Morse functions on the associated simplicial complexes as well as the restrictions of the discrete gradient vector fields to the hypergraphs.

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5.2 Discrete Gradient Vector Fields and Critical Hyperedges

By Corollary 3.3, we let \( \mathcal{f} \) and \( f \) be the discrete Morse functions on \( \Delta \mathcal{H} \) and \( \delta \mathcal{H} \) respectively such that \( f = \mathcal{f} |_{\delta \mathcal{H}} \). Let \( f = f |_{\mathcal{H}} \) be the discrete Morse function on \( \mathcal{H} \). We consider the discrete gradient vector fields

(i). \( V = \text{grad} f \) on \( \mathcal{H} \),
(ii). \( \nabla = \text{grad} f \) on \( \Delta \mathcal{H} \),
(iii). \( \nabla = \text{grad} f \) on \( \delta \mathcal{H} \).

Let \( R(V), R(\nabla) \) and \( R(\nabla) \) be the \( R \)-linear maps induced by \( V, \nabla \) and \( \nabla \) respectively.

The next lemma follows.

Lemma 5.3. (i). \( R(\nabla) \circ R(\nabla) = 0 \);
(ii). \( M(f, \mathcal{H}) = \{ \sigma \in \mathcal{H} \mid \sigma \notin \text{Im}(R(\nabla)) \text{ and } R(\nabla)(\sigma) = 0 \} \);
(iii). \#\{\gamma^{(n-1)} \in \mathcal{H} \mid R(\nabla)(\gamma^{(n-1)}) = \pm \alpha \} \leq 1 \text{ for any } \alpha^{(n)} \in \mathcal{H} ;
(iv). The following diagram commutes
\[ \begin{array}{ccc}
C_n(\delta \mathcal{H}; R) & \xrightarrow{\text{inclusion}} & R(\mathcal{H})_n \\
\downarrow R(\nabla) & & \downarrow R(\nabla) \\
C_{n+1}(\delta \mathcal{H}; R) & \xleftarrow{\text{inclusion}} & R(\mathcal{H})_{n+1}
\end{array} \]

\[ \begin{array}{ccc}
\pi(\mathcal{H}, \delta \mathcal{H}) & \xrightarrow{\pi(\mathcal{H}, \delta \mathcal{H})} & \pi(\Delta \mathcal{H}, \mathcal{H}) \\
\downarrow R(\nabla) & & \downarrow R(\nabla) \\
C_{n+1}(\Delta \mathcal{H}; R) & \xleftarrow{\text{inclusion}} & C_{n+1}(\Delta \mathcal{H}; R)
\end{array} \]

Proof. The proofs of (i), (ii), (iii) are similar with the proofs of (1), (2), (3) of [9, Theorem 6.3] respectively. The proof of (iv) follows from Lemma 5.1.

With the help of Proposition 3.6 and Lemma 5.3, the next theorem follows.

Theorem 5.4. Let \( \sigma \in \mathcal{H} \). Then
\[ \sigma \in M(f, \mathcal{H}) \setminus (M(\overline{\mathcal{f}}, \Delta \mathcal{H}) \cap \mathcal{H}) \quad (5.2) \]
if and only if one of the followings holds
(i). \( R(\nabla)(\sigma) = \pm \eta \) for some \( \eta \in \Delta \mathcal{H} \setminus \mathcal{H} \) and \( R(\nabla)(\tau) \neq \pm \sigma \) for any \( \tau \in \Delta \mathcal{H} \);
(ii). \( R(\nabla)(\sigma) = \pm \eta \) for some \( \eta \in \Delta \mathcal{H} \setminus \mathcal{H} \) and there exists \( \tau \in \Delta \mathcal{H} \setminus \mathcal{H} \) such that \( R(\nabla)(\tau) = \pm \sigma \);
(iii). \( R(\nabla)(\sigma) = 0 \) and there exists \( \tau \in \Delta \mathcal{H} \setminus \mathcal{H} \) such that \( R(\nabla)(\tau) = \pm \sigma \).

Proof. We divide the proof into the following four steps.

Step 1. Let \( \sigma \in \mathcal{H} \). We apply Lemma 5.3 (ii).

(a). By applying Lemma 5.3 (ii) on \( \overline{\mathcal{f}} \) and \( \Delta \mathcal{H} \), we have that \( \sigma \in M(\overline{\mathcal{f}}, \Delta \mathcal{H}) \setminus \mathcal{H} \) if and only if both of the followings are satisfied:

- \( R(\nabla)(\sigma) = 0 \);
- \( R(\nabla)(\tau) \neq \pm \sigma \) for any \( \tau \in \Delta \mathcal{H} \).
By taking the contrapositive statement, we have that
\[ \sigma \notin M(f, \Delta H) \cap \mathcal{H} \]
if and only if either of the followings are satisfied:
- \( R(\nabla)(\sigma) \neq 0; \)
- there exists \( \tau \in \Delta H \) such that \( R(\nabla)(\tau) = \pm \sigma. \)

(b). By applying Lemma 5.3 (ii) to \( f \) and \( \mathcal{H} \), we have that
\[ \sigma \in M(f, \mathcal{H}) \]
if and only if both of the followings are satisfied:
- \( R(\nabla)(\sigma) = 0; \)
- \( R(\nabla)(\tau) \neq \pm \sigma \) for any \( \tau \in \mathcal{H}. \)

**Step 2.** We apply Lemma 5.3 (iv).

(a). For any \( \sigma \in \mathcal{H} \) we have that
\[ R(\nabla)(\sigma) = 0 \]
if and only if
\[ \pi(\Delta H, \mathcal{H}) \circ R(\nabla)(\sigma) = 0 \]
if and only if either of the followings are satisfied:
- \( R(\nabla)(\sigma) = 0; \)
- \( R(\nabla)(\sigma) = \pm \eta \) for some \( \eta \in \Delta H \setminus \mathcal{H}. \)

(b). For any \( \sigma, \tau \in \mathcal{H} \) we have that
\[ R(\nabla)(\tau) \neq \pm \sigma \]
if and only if
\[ \pi(\Delta H, \mathcal{H}) \circ R(\nabla)(\tau) \neq \pm \sigma \]
if and only if
\[ R(\nabla)(\tau) \neq \pm \sigma \]
if and only if either of the followings are satisfied:
- \( R(\nabla)(\tau) = 0; \)
- \( R(\nabla)(\tau) = \pm \eta \) for some \( \eta \in \Delta H \setminus \mathcal{H}. \)

**Step 3.** Summarizing Step 1 and Step 2, for any \( \sigma \in \mathcal{H} \) we have that (5.2) holds if and only if both of the followings are satisfied:
- \( \sigma \in M(f, \mathcal{H}); \)
• σ /∈ (M(\overline{f}, \Delta H) \cap H)
which happens if and only if both of the followings are satisfied:
• both of the followings are satisfied:
  – either of the followings are satisfied:
    * R(\overline{V})(\sigma) = 0;
    * R(\overline{V})(\sigma) = \pm \eta \text{ for some } \eta \in \Delta H \setminus H;
  – R(\overline{V})(\tau) \neq \pm \sigma \text{ for any } \tau \in H;
• either of the followings are satisfied:
  – R(\overline{V})(\sigma) \neq 0;
  – there exists \tau \in \Delta H \text{ such that } R(\overline{V})(\tau) = \pm \sigma.

**Step 4.** Let σ \in M(f, H) \setminus (M(\overline{f}, \Delta H) \cap H). We consider the following cases.

**Case 1.** R(\overline{V})(\sigma) \neq 0.
Then by line 9 in Step 3, we have that R(\overline{V})(\sigma) = \pm \eta \text{ for some } \eta \in \Delta H \setminus H. By the last three lines in Step 3, we have the following subcases.

**Subcase 1.1.** For any τ \in \Delta H, we always have R(\overline{V})(\tau) \neq \pm \sigma.
Then we obtain Theorem 5.4 (i).

**Subcase 1.2.** There exists τ \in \Delta H \text{ such that } R(\overline{V})(\tau) = \pm \sigma.
Then by line 10 of Step 3, we have τ \in \Delta H \setminus H. Thus we obtain Theorem 5.4 (ii).

**Case 2.** R(\overline{V})(\sigma) = 0.
Then by the last line in Step 3, there exists τ \in \Delta H \text{ such that } R(\overline{V})(\tau) = \pm \sigma. By line 10 in Step 3, we have τ \in \Delta H \setminus H. We obtain Theorem 5.4 (iii).

Summarizing the above cases, we finish the proof. \qed

The next corollary is a consequence of Theorem 5.4

**Corollary 5.5.** Suppose R(\overline{V})(\alpha) = R(\overline{V})(\alpha) \text{ for } \alpha \in H \text{ and } R(\overline{V})(\alpha) = 0 \text{ for } \alpha \in \Delta H \setminus H. Then

M(f, H) = M(f, \Delta H) \cap H.

**Proof.** It suffices to prove that there does not exist any σ \in H such that (5.2) holds. Since we assume R(\overline{V})(\alpha) = R(\overline{V})(\alpha) \text{ for } \alpha \in H \text{ and } R(\overline{V})(\alpha) = 0 \text{ for } \alpha \in \Delta H \setminus H, \text{ it is direct that there does not exist any } \sigma \in H \text{ such that at least one of (i), (ii), or (iii) in Theorem 5.4 holds. Consequently, by Theorem 5.4 there does not exist any } \sigma \in H \text{ satisfying (5.2).} \qed

The next corollary follows from Proposition 5.2 and Corollary 5.5

**Corollary 5.6.** Suppose H satisfies the condition (C) (cf. Proposition 3.11). Let g be a discrete Morse function on H, \overline{f} a discrete Morse function on \Delta H given in Proposition 5.2, and f = \overline{f} |_H. Then

M(g, H) = M(f, H) = M(\overline{f}, \Delta H) \cap H.
Proof. By Proposition 5.2 (i), we have \( \text{grad} f = \text{grad} g \). Thus \( M(g, \mathcal{H}) = M(f, \mathcal{H}) \). By Proposition 5.2 (i) and (ii), we have
\[
R(\text{grad} f)(\alpha) = R(\text{grad} f)(\alpha)
\]
for \( \alpha \in \mathcal{H} \) and
\[
R(\text{grad} f)(\alpha) = 0
\]
for \( \alpha \in \Delta \mathcal{H} \setminus \mathcal{H} \). Thus by Corollary 5.5, we have \( M(f, \mathcal{H}) = M(\overline{f}, \Delta \mathcal{H} \cap \mathcal{H}) \). We finish the proof.

6 An Example

Example 6.1 (cf. Figure 2). Consider a hypergraph
\[
\mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_1\}, \{v_0, v_3\}, \{v_1, v_3\}, \{v_0, v_1, v_2\}\}.
\]

(i). The infimum chain complex is
\[
\text{Inf}_0(\mathcal{H}) = R(\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}),
\text{Inf}_1(\mathcal{H}) = R(\{v_0, v_1\}, \{v_0, v_3\}, \{v_1, v_3\}),
\text{Inf}_2(\mathcal{H}) = 0
\]
and the supremum chain complex is
\[
\text{Sup}_0(\mathcal{H}) = R(\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}),
\text{Sup}_1(\mathcal{H}) = R(\{v_0, v_1\}, \{v_0, v_3\}, \{v_1, v_3\}, \{v_1, v_2\} - \{v_0, v_2\}),
\text{Sup}_2(\mathcal{H}) = R(\{v_0, v_1, v_2\}).
\]

(ii). By a direct calculation, the embedded homology is
\[
H_0(\mathcal{H}; R) = R^2, \quad H_1(\mathcal{H}; R) = R, \quad H_2(\mathcal{H}; R) = 0.
\]

(iii). The associated simplicial complex \( \Delta \mathcal{H} \) is the simplicial complex obtained by adding the 1-simplices
\[
\{v_0, v_2\}, \{v_1, v_2\}
\]
to \( \mathcal{H} \).

(iv). The lower-associated simplicial complex \( \delta \mathcal{H} \) is the discrete set of the vertices \( \{v_0, v_1, v_2, v_3\} \).

(v). Let \( \overline{f} \) be a discrete Morse function on \( \Delta \mathcal{H} \) given by
\[
\overline{f}(\{v_0\}) = 1,
\overline{f}(\{v_1\}) = \overline{f}(\{v_2\}) = \overline{f}(\{v_3\}) = 0,
\overline{f}(\{v_0, v_1\}) = \overline{f}(\{v_1, v_2\}) = \overline{f}(\{v_1, v_3\}) = 1,
\overline{f}(\{v_0, v_2\}) = \overline{f}(\{v_0, v_3\}) = 2,
\overline{f}(\{v_0, v_1, v_2\}) = 2.
\]
Then the set of the critical simplices of $\overline{f}$ in $\Delta H$ is

$$M(\overline{f}, \Delta H) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}\}.$$  

Let $\nabla = \text{grad} \overline{f}$ be the discrete gradient vector field on $\Delta H$. Then

$$\nabla(\{v_0\}) = \{v_0, v_1\},$$
$$\nabla(\{v_0, v_2\}) = \{v_0, v_1, v_2\},$$
$$\nabla(\sigma) = 0 \text{ for any } \sigma \in \Delta H \setminus \{\{v_0\}, \{v_0, v_2\}\}.$$  

(vi). Let $f$ be the restriction of $\overline{f}$ on $H$. Then $f$ is given by

$$f(\{v_0\}) = 1, \quad f(\{v_1\}) = f(\{v_2\}) = f(\{v_3\}) = 0, \quad f(\{v_0, v_1\}) = f(\{v_1, v_3\}) = 1, \quad f(\{v_0, v_3\}) = 2, \quad f(\{v_0, v_1, v_2\}) = 2.$$  

Moreover, the set of the critical hyperedges of $f$ in $H$ is

$$M(f, H) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_0, v_3\}, \{v_1, v_3\}, \{v_0, v_1, v_2\}\}.$$  

Let $\nabla = \text{grad} f$ be the discrete gradient vector field on $H$. Then

$$\nabla(\{v_0\}) = \{v_0, v_1\},$$
$$\nabla(\{v_0, v_2\}) = \{v_0, v_1, v_2\},$$
$$\nabla(\sigma) = 0 \text{ for any } \sigma \in H \setminus \{\{v_0\}, \{v_0, v_2\}\}.$$  

(vii). Let $\underline{f}$ be the restriction of $f$ on $\delta H$. Then $\underline{f}$ is given by

$$f(\{v_0\}) = 1, \quad f(\{v_1\}) = f(\{v_2\}) = f(\{v_3\}) = 0.$$  

Moreover, the set of the critical simplices of $\underline{f}$ in $\delta H$ is

$$M(\underline{f}, \delta H) = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}\}.$$  

Let $\nabla = \text{grad} \underline{f}$ be the discrete gradient vector field on $\delta H$. Then

$$\nabla(\{v_i\}) = 0, \quad i = 0, 1, 2, 3.$$  

Figure 2: Example 6.1.
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