KOLYVAGIN’S TRACE RELATIONS FOR SIEGEL SIXFOLDS

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Abstract. In his earlier preprints the author offered a program of generalization of Kolyvagin’s result of finiteness of SH to the case of some motives which are quotients of cohomology motives of Shimura and Drinfeld varieties. The present paper is devoted to the first step of this program — finding of an analog of Kolyvagin’s trace relations. We solve it for Siegel sixfolds and for the Hecke correspondences related to the matrices diag(1, 1, 1, p, p, p) and diag(p, 1, 1, p, p^2, p^2). This is the first non-trivial case for Shimura varieties. Some results for other types of Siegel varieties and Hecke correspondences are obtained.

Ideas and methods of the present paper open a large new area of research: results given here constitute a tiny part of what can be done. Particularly, maybe it is possible to realise for Drinfeld varieties of any even rank the program of generalization of Kolyvagin’s result.

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1. Introduction.

1.1. Statement of result, and justification of the subject.

Kolyvagin ([K89] and subsequent papers) proved finiteness of Tate-Shafarevich group and of group of rational points of modular\(^1\) elliptic curves over \(\mathbb{Q}\) of analytic rank 0. There is a natural problem

\((*)\) To generalize this result to the case of some submotives of high-dimensional Shimura varieties and/or Drinfeld modular varieties.

The present paper is devoted to generalization of the first step of Kolyvagin’s proof — trace relations for Heegner points (see (1.2.1)) — to the case of Siegel varieties of genus 3 (some results are valid for any \(g\)).

Remark. The author offers in preprint [L04.2] a program of proof of (*) for Shimura varieties, he solves in this paper some further steps of the proof and indicates the obstacles. The results of the present paper and of [L04.2] give some evidence that maybe we shall be able to extend Kolyvagin’s result to the functional case.

The subject of the present paper is the following. Let \(X\) be an irreducible component of a Shimura variety of a fixed level, \(V \subset X\) its Shimura subvariety and \(\mathfrak{T}_p\) a \(p\)-Hecke correspondence on \(X\) (\(p\) is a prime fixed throughout the paper). We get some information on the structure of \(\mathfrak{T}_p(V)\) (which a priori is a cycle on \(X\)) for the case when \(X\) is a Siegel variety of genus \(g\), \(V\) its subvariety corresponding to a reductive group \(GU(\mathfrak{r},\mathfrak{s})\), \(\mathfrak{r} + \mathfrak{s} = g\) (points of \(V\) parametrize abelian \(g\)-folds with multiplication by an imaginary quadratic field \(K\)) and \(\mathfrak{T}_p = T_p\) or \(T_p, i\) (see (1.3.1) for the definition). Namely, we find (roughly speaking) the set of irreducible components of \(\mathfrak{T}_p(V)\), their fields of definition and Galois action on them. The results are complete for the case \(\mathfrak{T}_p = T_p, g = 3\) and near to complete for any \(g\). For the case \(\mathfrak{T}_p = T_p, 1\) only the general components are described completely.

The answers are given in terms of geometry of the finite set \(\mathfrak{T}_p(t)\), where \(t \in V\) is a generic point. \(\mathfrak{T}_p(t)\) is some kind of Grassmann variety, see (1.3.4), (1.3.6). We introduce 2 partitions \(D\) and \(I\) of \((=\) equivalence relations on\) \(\mathfrak{T}_p(t)\): roughly speaking, two points of \(\mathfrak{T}_p(t)\) are equivalent iff they belong to the same irreducible component of \(\mathfrak{T}_p(V)\), see 1.4, 2.8 for the exact definitions. We get a simple description of these partitions \(D\) and \(I\) in terms of geometry of \(\mathfrak{T}_p(t)\). Namely, we

\(^1\)After 1999 we can omit the word “modular” but in 1989 the Taniyama–Shimura conjecture was not proved yet. Modularity of the elliptic curve is used essentially in Kolyvagin’s proof.
introduce in 1.5 elementary “geometric” partitions $\mathcal{D}, \mathcal{H}$ on $\mathcal{T}_p(t)$, and we describe $\mathcal{D}, \mathcal{I}$ and the Galois actions on the set of parts of $\mathcal{I}$ in terms of $\mathcal{D}, \mathcal{H}$. Results for the case $\mathcal{T}_p = T_p$ are given in subsection 1.5 (theorems 3.3.1, 3.3.3, 1.5.3, conjectures 1.5.4, 1.5.6, 1.5.7), and for the case $\mathcal{T}_p = T_{p,i}$ in subsection 1.6 (conjecture 1.6.3, theorems 4.2.15, 4.2.19, conjecture 4.2.20).

The structure of calculations shows that we have a deep and beautiful theory. It is easy to get an algebraic criterion of equivalence of points of $\mathcal{T}_p(t)$ (Theorem 2.16; see also (3.3.1.3) for its explicit form), but it is not clear how to solve the corresponding equations. Paper [L01] contains a solution of this problem for Siegel varieties of genus 2. Its method is based on explicit formulas for the image of $GU(1,1)$ in $GSp_4$. These formulas do not exist for $g > 2$.

Nevertheless, application of the crucial formula (3.3.1.4) permits us to solve the problem for the simplest Hecke correspondence $\mathcal{T}_p = T_p$ uniformly for all $g$ (for a generic equivalence class) much easier than it was made in [L01] for $g = 2$.

Situation for other Hecke correspondences $\mathcal{T}_p = T_{p,i}$ is more complicated. Formulas (4.2.3) (analogs of (3.3.1.4) for $T_{p,i}$) permit us to reduce the problem of finding of a generic equivalence class to solving of equations (4.2.15.10 – 4.2.15.12) which look rather hopeless from the first sight.

But here we get the second wonder: the equations can be greatly simplified! The process of their simplification reminds assembling a puzzle when different parts unexpectedly and beautifully fit together. For example, sometimes in the process of proof it was seen that if some polynomial $P$ belonged to an ideal $\mathcal{J}$ of $\mathbb{Q}[X_1, ..., X_n]$ then the calculations would be much simpler. Since in this situation nature is benevolent for us, I concluded that $P$ must belong to $\mathcal{J}$. It was absolutely not seen beforehand. I used a computer program; yes, computer shows that $P \in \mathcal{J}$. Later it was possible to prove (without computer) that really in all cases $P \in \mathcal{J}$.

The reader can tell that if a problem has a simple answer then formulas that appear in its solution do simplify. But

We do not know beforehand that our problem has a simple solution!

Really, when we consider a problem and try to generalize it (increasing dimension of objects for example) it can happen that formulas become more and more complicated, and no general idea is seen. So, the most important information obtained in the present paper is the following: the problem of finding of generic components of $\mathcal{T}_p(V)$ is solvable (to find a solution — if we know that it exists — is an easier task).

And what happens with non-generic components? I do not know. For the case $\mathcal{T}_p = T_p$ there is Conjecture 1.5.4 which gives us the complete answer. Without doubt, for any fixed genus $g$ and the rank $j$ of some auxiliary matrix, there exists a proof of conjecture 1.5.4 similar to the proof of the theorem 3.3.4 (case $g = 3, j = 2$). But does exist a uniform proof for all $g, j$?

If $\mathcal{T}_p = T_{p,i}$ then the situation is worse. Practically, for non-generic components of $\mathcal{T}_p(V)$ I can work only “modulo $p$” but not “modulo $p^2$”, see Section 5.3. Does a simple description of equivalence classes for this case exist? If not, what is possible to tell about asymptotics of the quantity of irreducible components, etc.?
It is interesting that situations for different Hecke correspondences are quite un-
similar. Because of applications to (*), we are interested to consider irreducible
components of $\mathfrak{T}_p(V)$ which are defined over the $p$-ring class field of $K$ (good com-
ponents) while components which are defined over $K$ (bad components) generate
obstacles. For the case $\mathfrak{T}_p = T_p$ the generic components are good and the special
ones are bad, for $\mathfrak{T}_p = T_{p,1}$ ($g = 3$) the situation is (roughly speaking) inverse, and
for $\mathfrak{T}_p = T_{p,2}$ there is no good components at all. What is a non-formal expla-
nation of this phenomenon?

Finally, the subjects of [L04.1], [L04.2] and of the present paper open a large
area of investigation, see introduction to [L04.2] for a dis-
cussion of possibilities of
investigation of high level, and Subsection 1.8 of the prese-
ent paper. It would be
very important to generalize these results to the functional case.

The rest of Section 1 contains definitions (subsections 1.3, 1.4) and statements
of the main theorems and conjectures (subsections 1.5, 1.6).

1.2. Kolyvagin’s trace relations.

Let us give definitions for the original Kolyvagin’s case. Let $N$ be a level,
$X_0(N) = \Gamma_0(N) \backslash \mathcal{H}$ the compactification of the modular curve of level $N$, $K = \mathbb{Q}(\sqrt{-\Delta})$ an imaginary quadratic field. In order to simplify the notations and
proofs, we consider only the case when $h(K) = 1$, although this restriction is not
essential. Sometimes when we use notation $K^1$ — the Hilbert class field of $K$ —
this means that the corresponding result is valid for any $K$.

Recall the definition of Heegner point (see for example [GZ86], [K89] for the
details). Points $t$ of the open part of $X_0(N)$ are in one-to-one correspondence with
the isogenies of elliptic curves $\psi_t : A_t \rightarrow A'_t$ such that $\text{Ker } \psi_t = \mathbb{Z}/N\mathbb{Z}$. A point
$t \in X_0(N)$ is called a Heegner point with respect to $K$ if both $A_t, A'_t$ have complex
multiplication by the same order of $K$. A Heegner divisor is a Galois orbit of a
Heegner point; Heegner divisors are exactly 0-dimensional Shimura subvarieties of
$X_0(N)$ in the sense of Deligne ([D71]). If all prime factors of $N$ split in $K$ then for
a given $K$ there exists a “principal” Heegner point $x_1 \in X_0(N)(K^1)$.

The main object of the present paper is a prime $p$. We restrict ourselves to
the case $p \neq 2$ is inert in $K$, i.e. $\left(\frac{-\Delta}{p}\right) = -1$. The case $p$ splits in $K$ is much
more complicated technically, see for example [L01] where for $g = 2$ it is treated
completely. Further, we consider only $p$ such that $p$ does not divide $N$. Recall
that $K^p$ — the ring class field of $K$ of conductor $p$ — is an abelian extension of $K$
corresponding to some subgroup in the idele group $I_K$ of $K$, see for example [K89]
for the exact definition. We have: $K^p/K$ is ramified only at $p$, and $\text{Gal } (K^p/K^1) = \mathbb{Z}/(p + 1)$.

There are 2 objects associated to $p$: a Heegner point $x_p \in X_0(N)(K^p)$ and the
$p$-Hecke correspondence $T_p$ on $X_0(N)$. We have a formula (equality of divisors on
$X$, case $h(K) = 1$):

$$T_p(x_1) = \text{Tr}_{K^p/K}(x_p)$$ (1.2.1)

(Kolyvagin’s trace relation for $X_0(N)$ ([K89])).

We can consider (1.2.1) as a description of the action of $T_p$ on $x_1$ where $x_1$ is an
irreducible component of a 0-dimensional Shimura subvariety of $X$. So, for a
triple $X, V, \mathfrak{T}_p$ of (1.1) a high-dimensional analog of (1.2.1) is the description of
the cycle \( \Sigma_p(V) \). Namely, this cycle is a formal sum of irreducible components of some other Shimura subvarieties of \( X \). The problem is to describe the structure of \( \Sigma_p(V) \), particularly its representation as a formal sum of Shimura subvarieties, their irreducible components and Galois action on them.

### 1.3. Definitions related to Hecke correspondences.

We consider the case \( X \) is a Siegel variety of genus \( g \) of any level \( N \), so the corresponding algebraic group \( G \) is \( GSp_{2g}(\mathbb{Q}) \). The algebra of \( p \)-Hecke correspondences on \( X \) is the ring of polynomials with \( g \) generators denoted by \( T_p, T_{p,1}, \ldots, T_{p,g-1} \). They are double cosets corresponding to the diagonal matrices \( \tau_p, \tau_{p,1}, \ldots, \tau_{p,g-1} \) respectively, where

\[
\tau_p = \text{diag} \left( \underbrace{1, \ldots, 1}_g, p, \ldots, p \right)
\]

and \( \tau_{p,i} = \text{diag} \left( \underbrace{p, \ldots, p}_{i \text{ times}}, \underbrace{1, \ldots, 1}_{g-i \text{ times}}, \underbrace{p, \ldots, p}_{i \text{ times}}, \underbrace{p^2, \ldots, p^2}_{g-i \text{ times}} \right) \quad (1.3.1)

\( i = 0, \ldots, g \). (Caution: here and below notations of the present paper are other than in [L04.1], [L04.2]). We have: \( T_{p,0} \) is the trivial correspondence, and \( T_{p,0} \in \mathbb{Z}[T_p, T_{p,1}, \ldots, T_{p,g-1}] \).

Let us recall various interpretations of \( T_p, T_{p,i} \). Let us consider the double cosets \( \Gamma \tau_p \Gamma, \Gamma \tau_{p,i} \Gamma \) (where \( \Gamma = GSp_{2g}(\mathbb{Z}_p) \)) and their decomposition as a union of ordinary cosets

\[
\Gamma \tau_p \Gamma = \bigcup_{j \in \mathcal{S}(g)} \Gamma \sigma_j, \quad \Gamma \tau_{p,i} \Gamma = \bigcup_{j \in \mathcal{S}(g,i)} \Gamma \sigma_j \quad (1.3.2)
\]

where \( \mathcal{S}(g), \mathcal{S}(g,i) \) are the (abstract) sets of indices and \( \sigma_j \in GSp_{2g}(\mathbb{Q}) \cap M_{2g}(\mathbb{Z}) \) are representatives. We use for our calculations the explicit sets of these representatives described in [L04.1], Section 2.3 for \( \Sigma_p = T_p \) and in Section 2.4 for \( \Sigma_p = T_{p,i} \). These sets will be denoted by \( S(g), S(g,i) \) respectively, i.e. \( j \mapsto \sigma_j \) is an isomorphism \( \mathcal{S}(g) \to S(g), \mathcal{S}(g,i) \to S(g,i) \).

For a generic \( t \in X \) the set \( T_p(t) \) (resp. \( T_{p,i}(t) \)) is in \( 1-1 \) correspondence with \( \mathcal{S}(g) \) (resp. \( \mathcal{S}(g,i) \)). Let \( A_t \) be the abelian \( g \)-fold corresponding to \( t \). \( (A_t)_p \) — the group of \( p \)-torsion points of \( A_t \) — is a \( 2g \)-dimensional vector space over \( \mathbb{F}_p \) endowed with a non-degenerated skew form \( B \) coming from a Riemann form on \( A_t \).

\( (1.3.3) \) The set \( T_p(t) \) (and hence \( \mathcal{S}(g) \)) is in \( 1-1 \) correspondence with the set of \( B \)-isotropic \( g \)-dimensional subspaces in \( (A_t)_p \). We denote this variety by \( G_I(g,2g)(\mathbb{F}_p) = G_I(g, (A_t)_p) \) (\( G_I \) means the isotropic Grassmannian). So, for the case \( \Sigma_p = T_p \) we have identifications

\[
T_p(t) = \mathcal{S}(g) = G_I(g,2g)(\mathbb{F}_p) = G_I(g, (A_t)_p) \quad (1.3.4)
\]

For \( \sigma \in S(g) \) we denote by

\[
W_\sigma \subset (A_t)_p \quad (1.3.5)
\]

the corresponding isotropic \( \mathbb{F}_p \)-subspace.
Analogously, for $t \in X$, $\mathfrak{F}_p = T_{p,i}$, the set $T_{p,i}(t)$ is isomorphic to a generalized Grassmannian $G_I(i,g,2g)$ over $\mathbb{F}_p$ (see 1.6.1 for the definition); it is a scheme over $\mathbb{Z}/p^2\mathbb{Z}$. The analog of (1.3.4) is

$$T_{p,i}(t) = \mathfrak{S}(g,i) = S(g,i) = G_I(i,g,2g) \quad (1.3.6)$$

1.4. Definitions related to partitions.

We consider a generic point $t \in V$. In this case $\mathfrak{F}_p(t) \subset \mathfrak{F}_p(V)$, and the structure of $\mathfrak{F}_p(V)$ as a union of Shimura subvarieties defines various partitions of ( = equivalence relations on) $\mathfrak{F}_p(t)$ and isomorphic sets given in (1.3.4), (1.3.6). Roughly speaking, 2 points of $\mathfrak{F}_p(t)$ belong to the same partition set ( = equivalent) iff they belong to the same Shimura subvariety. So, we must describe these partitions. It turns out that (roughly speaking) partition sets are subschemes of $\mathfrak{F}_p(t)$.

The problem of finding partitions on $\mathfrak{F}_p(t)$ can be formulated in terms of algebraic groups $G, G_V$ over $\mathbb{Q}$ associated to $X, V$ respectively. Definitions of 3 different types of partitions are given in [L01], Section 4 in terms of equivalence relations on $\mathfrak{F}_p(t)$ ($S$ of [L01] is $\mathfrak{F}_p(t)$). We shall treat in the present paper only 2 partitions:

(a) the strong equivalence of [L01]: 2 points of $\mathfrak{F}_p(t)$ are strongly equivalent ( $\iff$ belong to the same partition set of strong partition) iff they belong to the same irreducible component of $\mathfrak{F}_p(V)$. See [L01], 4.1 for the explicit formulas. For brevity, we call strong partition by I-partition (I of irreducible).

(b) D-equivalence: 2 points of $\mathfrak{F}_p(t)$ are D-equivalent ( $\iff$ belong to the same partition set of D-partition) iff (roughly speaking) there exists a Shimura subvariety (in the sense of [D71]) of $\mathfrak{F}_p(V)$ containing both these points (see 2.8 for the exact definition). This partition was not considered in [L01].

Theorem 2.16 gives us a criterion of D-equivalence.

(1.4.1) We introduce some notations related to I- (resp. D-)partition of $S(g)$ (resp. $S(g,i)$). To avoid overuse of the word “respectively”, we give these notations only for the Hecke correspondence $T_p$; notations for $T_{p,i}$ are similar. The set of parts of the partition will be denoted by $S_I(g)$ (resp. $S_D(g)$), it is the quotient set of $S(g)$ by the I- (resp. D-)equivalence relation. For $\sigma \in S(g)$ we denote the corresponding element of the quotient set by $I(\sigma)$ (resp. $D(\sigma)$). For $k \in S_I(g)$ (resp. $k \in S_D(g)$) we denote the corresponding subset of $S(g)$ by $S(g)_I(\sigma)$ (resp. $S(g)_D(\sigma)$) is the set of elements of $S(g)$ which are I- (resp. D-)equivalent to $\sigma$. The irreducible component of $T_p(V)$ (resp. the Shimura subvariety of $T_p(V)$, see Section 2 for the exact description of this object) that corresponds to $k$ will be denoted by $T_p(V)_k$. Particularly, for $\sigma \in S(g)$ the irreducible component of $T_p(V)$ (resp. the Shimura subvariety of $T_p(V)$) that corresponds to $\sigma$ will be denoted by $T_p(V)_I(\sigma)$ (resp. $T_p(V)_D(\sigma)$).

(1.4.2) Clearly I-partition is stronger than D-partition. For $k \in S_D(g)$ we denote by $k_1$ the set of parts of I-partition that are contained in $S(g)_k$. Let $k_1 \subset S_I(g)$ be an element, $L_{k_1}$ the field of definition of $T_p(V)_{k_1}$ and $L_k$ the reflex field of $T_p(V)_k$. According [D71], Gal $(L_k)$ acts simply transitively on $k_1$ (this action comes from the action of Gal $(L_k)$ on the set of irreducible components of $T_p(V)_k$), and Gal $(L_{k_1})$ is the stabiliser of $k_1$. 

6
We shall study mainly the D-partition, and only 3 conjectures (1.5.6, 1.5.7 and 4.2.20) are concerned to I-partition and the above Galois action.

Because of applications to [L04.2], we shall be interested mainly by the case of $T_p$ for any $g$ and of $T_{p,1}$ for $g = 3$.

1.5. Contents of Section 3: Case $\mathcal{S}_p = T_p$.

Here we formulate theorems and conjectures for the case $\mathcal{S}_p = T_p$. We introduce a $\mathcal{D}$-partition of the sets of (1.3.4) ($\mathcal{D}$ because of dimension) which has a simple geometric definition, and we describe relations between $\mathcal{D}$-partition and I- and D-partitions.

The set of parts of $\mathcal{D}$-partition is indexed by even numbers $j$, $g \leq j \leq 2g$. The definition of $\mathcal{D}_j$ — the $j$-th part of $\mathcal{D}$-partition of $S(g) = G_1(g, (A_t)_p)$ is the following. Recall that $V$ is a Shimura variety of PEL-type parametrizing abelian $g$-folds with multiplication by $K$. The exact definition of $V$ is given in Subsection 3.1. Since $p$ inert in $K$, we have $O_K/p = \mathbb{F}_{p^2}$. Let $t$ be a generic point of $V$. Action of $O_K$ on $A_t$ endows $(A_t)_p$ by the structure of $\mathbb{F}_{p^2}$-module. Let $\sigma \in S(g)$ be any element and $W_\sigma \subset (A_t)_p$ from 1.3.5. $\mathbb{F}_{p^2}W_\sigma$ is a $\mathbb{F}_{p^2}$-subspace of $(A_t)_p$ whose $\mathbb{F}_p$-dimension can take any even value from $g$ to $2g$. We define

$$\mathcal{D}_j = \{\sigma \in S(g) \mid \dim_{\mathbb{F}_p}(\mathbb{F}_{p^2}W_\sigma) = j\}$$ (1.5.1)

Most of below theorems are proved only on the open part of $G_1(g, (A_t)_p)$ (see (3.2.1)) denoted by $G_1(g, (A_t)_p)^{open}$; the corresponding open subset of $S(g)$ is denoted by $S(g)^{open}$. The exact statement of what is proved is given in the body of the paper; here we give the statements of these theorems in their complete form. We mark these theorems with (*) and we leave the number of this theorem in the body of the paper. Further, we give here statements of some theorems whose proof is not given at all, because it is elementary. We mark these theorems with (**).

Remark 1.5.2. Since the below theorems are not proved completely, formally they should be formulated as conjectures. Absolute evidence that they are true (and moreover that their complete proof is a routine work) comes from consideration of some particular cases. For example, I made complete calculations for the case $g = 2$; they are so large and so elementary, that there were no meaning to include them in the text of [L01] (see Proposition 6.3.7, (d)).

So, the problem of complete proof of the theorems (*) and (**) can be considered as an exercise for those who will continue these investigation. See Remark 3.3.2 and 1.8, (1).

Let $\sigma \in S(g)^{open}$. We can associate it a matrix $T(\sigma) \in M_g(\mathbb{Z})$ (see 3.2.2) and $T(\tilde{\sigma}) \in M_g(\mathbb{F}_p)$ — its reduction modulo $p$. It has the following property:

(*) Lemma 3.2.3. If $\sigma \in \mathcal{D}_j$ then rank $T(\tilde{\sigma}) = j - g$.

(*) Theorems 3.3.1, 3.3.3. If $g$ is odd then $\mathcal{D}_{2g}$ is a D-partition set. If $g$ is even $\neq 2$ then $\mathcal{D}_{2g}$ is a union of two D-partition sets: 2 elements $\sigma_i, \sigma_j \in \mathcal{D}_{2g}$ are D-equivalent iff the ratio $\det T(\tilde{\sigma}_i)/\det T(\tilde{\sigma}_j)$ is a square in $\mathbb{F}_p^*$.

Theorem 3.3.5. If $g$ is odd then all components of $T_p(V)$ do not coincide with $V$ itself.
Theorem 1.5.3. If $g$ is even then $D_g$ is a part of $D$-partition, and moreover it is the only part such that the corresponding component of $T_p(V)$ is $V$ itself.

For $j \neq g, 2g$ we have only a

Conjecture 1.5.4. $D$-partition of $S(g)$ is a subpartition of its $D$-partition. 2 elements $\sigma_i, \sigma_j \in D_j$ are $D$-equivalent iff quadratic forms over $\mathbb{F}_p$ defined by the matrices $T(\tilde{\sigma}_i), T(\tilde{\sigma}_j)$ are isomorphic up to multiplication by a scalar.

Remark. The theory of quadratic forms over finite fields shows that Conjecture 1.5.4 implies that if $g$ is odd then $D$-partition coincide with $D$-partition, while if $g$ is even $\neq 2$ then each $D_j$ consists of two parts of $D$-partition.

Parts of $D$-partition that are contained in $D_{2g}$ are called the good parts, the only part consisting of $D_g$ is called the trivial part, and parts that are contained in $D_j, j \neq g, 2g$, are called the bad parts. The corresponding components of $T_p(V)$ have the same names.

(*) Theorem 3.3.4. Conjecture 1.5.4 is true for $g = 3, (r, s) = (2, 1)$.

(*) Corollary. For $g = 3, (r, s) = (2, 1)$ there exist 1 good component of $T_p(V)$, 1 bad component and no trivial components (i.e. equal to $V$ itself).

(*) Theorem 3.3.6. For any $g$, any $r \neq s$ the field of definition of any irreducible component of a good component of $T_p(V)$ is $K^p$.

(**) Theorem 1.5.5. For all $r, s$ the restriction of I-partition on all bad parts of $D$-partition is trivial (i.e. consists of one part). If $r = s$ then the same is true for good parts as well. The field of definition of all mentioned components is $K^1$.

Now let us formulate conjectures on I-partition. If an irreducible component of $T_p(V)$ is defined over $K^1$ then the corresponding part of I-partition is also a part of $D$-partition. So, we need only to describe I-partition on the set $D_{2g}$.

There exists the Plücker embedding $\mathcal{P}: G_1(g, 2g)(\mathbb{F}_p) \hookrightarrow G(g, 2g)(\mathbb{F}_p) \hookrightarrow P^m(\mathbb{F}_p)$ where $m = \binom{2g}{g} - 1$. $\mathcal{P}(D_i) \subset \mathcal{P}(S(g)) \subset \mathcal{P}(G_1(g, 2g)(\mathbb{F}_p)) \subset P^m$ are some kind of determinantal varieties in $P^m$.

Remark 3.2.5. $\mathcal{P}(\cup_{j=g}^{2g-2} D_j)$ — the union of all bad parts (and the trivial part for even $g$) — is the intersection of $\mathcal{P}(G_1(g, 2g))$ with a codimension 2 linear subspace $P_2$ of $P^m$.

This intersection is an analog of the conic line $C$ of the case $g = 2$ (see [L01], Theorem 0.7; Section 3, Figure 1).

We consider the set of hyperplanes in $P^m$ containing $P_2$; this set is isomorphic to $P^1(\mathbb{F}_p)$. Intersections of $D_{2g}$ with these hyperplanes form a partition of $D_{2g}$ which we denote by $\mathcal{H}$ (partition of hyperplane sections).

Conjecture 1.5.6. The restriction of I-partition on $D_{2g}$ is the intersection of $\mathcal{H}$-partition with the restriction of $D$-partition on $D_{2g}$.

Particularly, for $g \neq 2$ any part of $D$-partition in $D_{2g}$ contains $p + 1$ parts of I-partition which are indexed by elements of $P^1(\mathbb{F}_p)$. 


Conjecture 1.5.7. The action of Galois group \( \text{Gal} \left( K^p/K^1 \right) = \mathbb{F}_{p^2}^*/\mathbb{F}_p^* \) on the set of parts of I-partition that are contained in any fixed part of D-partition on \( D_{2g} \) coincides with the natural action of \( \text{Gal} \left( K^p/K^1 \right) \) on \( P^1(\mathbb{F}_p) \).

Remark 1.5.8. I think that the proof of Conjectures 1.5.6, 1.5.7 can be got easily using the same ideas as the ones used in proofs of other theorems of the present paper. For the conjecture 1.5.6 this opinion is based on the analogy with the case \( g = 2 \) (see [L01]). Lemma 3.2.8 and Remark 3.2.9 give an argument in favor of Conjecture 1.5.7 for odd \( g \). I think that the conjecture 1.5.7 is true also for even \( g \) because we cannot imagine any other Galois action.

(1.5.9) To formulate the remaining theorems of Section 3, we need to quote some definitions of [L01], Section 3 (slightly changing notations):

A restriction of a Hecke correspondence \( \mathfrak{T}_p \) to \( V \) can be considered as a linear combination of correspondences \( \mathfrak{T}_{V,i} = \mathfrak{T}_{p,V,i} \) between \( V \) and subvarieties \( V_i \subset X \):

\[
\mathfrak{T}_p|_V = \sum_i \rho_i \mathfrak{T}_{V,i}
\]

where \( i \) runs over the set of classes of I-equivalence, \( \rho_i \) are multiplicities.

Some subvarieties \( V_i \) can coincide but all \( \mathfrak{T}_{V,i} \) are, by definition, different. We denote the graph of \( \mathfrak{T}_{V,i} \) by \( \Gamma_{V,i} \subset V \times V_i \) and projections \( \Gamma_{V,i} \rightarrow V \):

by \( \pi_1 = \pi_{V,i,1}, \pi_2 = \pi_{V,i,2} \).

The degree of \( \pi_2 \) is an important invariant of the corresponding component of \( \mathfrak{T}_p(V) \), because it enters as the multiplicity in the Abel-Jacobi map. For \( \mathfrak{T}_p = T_p \) we prove:

(*) Proposition 3.4.1. For any \( g \) the degree of \( \pi_2 \) for any good component is 1.

Proposition 3.4.3. For \( g = 3 \) the degree of \( \pi_2 \) for the bad component is \( p + 1 \).

1.6. Contents of Section 4: Case \( \mathfrak{T}_p = T_{p,i} \).

Let \( W \) be a module over \( \mathbb{Z}/p^2 \), \( W_p \) its submodule of \( p \)-torsion. \( W \) has invariants \( d_1 = \dim W/W_p, d_2 = \dim W_p \).

(1.6.1) The analog of (1.3.3) for the present case is the following. The set \( T_{p,i}(t) \) is isomorphic to the generalized Grassmannian \( G_1(i, g, 2g) \), i.e. the set of isotropic \( W \subset (A^2)^{p^2} \) such that \( d_1(W) = g - i, d_2(W) = g + i \). Like in Subsection 1.5, we consider only the open part \( S(g,i)_{\text{open}} \) of \( S(g,i) \).

Let \( \sigma \in S(g,i)_{\text{open}} \). The analog of \( T(\sigma) \) of 3.2.2 is a pair of matrices \( \mu_1(\sigma), \mu_2(\sigma) \in M_g(K) \) (see 4.1.2.5). The analog of \( D \)-partition for this case is defined in 4.2.9, 4.2.10 (see Remark 4.2.11 explaining why this definition is distinct from its analog for \( \mathfrak{T}_p = T_p \)). Namely, we have a disjoint union

\[
S(g,i) = \mathcal{D}_s \cup \left( \bigcup_{j=0}^{i} \mathcal{D}_j \right)
\]

(elements of \( \mathcal{D}_s \) are the most special elements, elements of \( \mathcal{D}_i \) are the most general elements).
Conjecture 1.6.3. D-partition of $S(g, i)$ is a subpartition of $\mathcal{D}$-partition.

We know the answer for the general part $\mathcal{D}_1$.

(*) Theorem 4.2.15. Let $g, i$ be arbitrary. If $g - i$ is odd then $\mathcal{D}_1$ is a D-partition set. If $g - i$ is even then $\mathcal{D}_1$ is a union of two D-partition sets.

These sets are described as follows. Let $\sigma_i, \sigma_j \in S(g, i)^{\text{open}}$ be 2 elements. We associate to them $(g - i) \times (g - i)$-matrices $G_{22i}, G_{22j}$ with entries in $\mathbb{Z}$ (see 4.1.5 and the above formulas). $\sigma_i, \sigma_j$ are D-equivalent iff the ratio $\det \tilde{G}_{22i} / \det \tilde{G}_{22j}$ is a square in $\mathbb{F}_p^*$. 

From now we consider the case $g = 3, i = 1$.

(*) Theorems 4.2.16, 4.2.17. Conjecture 1.6.3 is true for $g = 3, i = 1$.

We refer to $\mathcal{D}_*, \mathcal{D}_0, \mathcal{D}_1$ as of special, intermediate, general type respectively. Further, there exists a disjoint union $\mathcal{D}_* = \mathcal{D}_{0,*} \cup \mathcal{D}_{1,*}$ defined in Theorem 1.6.5, b, c.

(*) Theorem 4.2.18. $\mathcal{D}_{0,*}$ is a part of D-partition, and moreover it is the only part such that the corresponding component of $T_p(V)$ is $V$ itself.

Theorems 4.2.15 — 4.2.18 show that the D-partition of $S(3, 1)$ is a subpartition of the partition $S(3, 1) = \mathcal{D}_{0,*} \cup \mathcal{D}_{1,*} \cup \mathcal{D}_0 \cup \mathcal{D}_1$. I do not know what is the restriction of D-partition on $\mathcal{D}_0$ and $\mathcal{D}_{1,*}$. See Section 5.3 for propositions which are the first step to the solution of problem of finding of this partition. Computer calculations for the case $p = 3$ show that there are more than one part of D-partition sets in $\mathcal{D}_{1,*}$. Apparently, if $p$ tends to infinity, the quantity of these parts should be one.

Theorem 4.2.19. Irreducible components of Shimura subvarieties of $T_{p,1}(V)$ that correspond to the parts of D-partition that are contained in $\mathcal{D}_0$ are defined over $K^p$.

Remark. All other irreducible components of $T_{p,1}(V)$ are defined over “small” extensions of $K^1$, see 4.2.21. Unlike the case of $\mathcal{T}_p = T_p$, for $g = 3$, $\mathcal{T}_p = T_{p,1}$ the Galois action is non-trivial for the intermediate case.

(1.6.4) Analog of Conjectures 1.5.6, 1.5.7 (i.e. the description of I-partition on $\mathcal{D}_0$ and the Galois action on the set of these parts) is given in 4.2.20. It is necessary to emphasize that we have 2 natural options for the partition; I do not know which of these 2 options holds.

Let us give now an analog of the Lemma 3.2.3 — the interpretation of the sets $\mathcal{D}_*$ in terms of $d_1, d_2$ of $\mathbb{F}_p W_\sigma$. Their possible values are the following (dimensions over $\mathbb{F}_p$): $(4, 4), (4, 6), (2, 4), (2, 6)$.

(**) Theorem 1.6.5. a) $\sigma \in \mathcal{D}_0 \cup \mathcal{D}_1 \iff d_1, d_2(\mathbb{F}_p W_\sigma) = (4, 4)$ or $(4, 6)$; b) $\sigma \in \mathcal{D}_{1,*} \iff d_1, d_2(\mathbb{F}_p W_\sigma) = (2, 6)$; c) $\sigma \in \mathcal{D}_{0,*} \iff d_1, d_2(\mathbb{F}_p W_\sigma) = (2, 4)$. □

Finally, we can consider the following geometric construction. Factorization of $W_\sigma$ by $pW_\sigma$ is a projection $pr$ from $G_I(i, g, 2g)$ to $G_I(g - i, 2g)$ (with a fibre $\mathbb{F}_p^{(g - i)(g - i + 1)/2}$). The projection of $\mathcal{D}$-partition on $G_I(2, 6)$ is described as follows:

(**) Theorem 1.6.6. a) $pr(\mathcal{D}_*)$ is an intersection of $G_I(2, 6)$ (in its Plücker embedding) and a codimension 4 linear subspace.
b) $pr(\mathcal{D}_* \cup \mathcal{D}_0)$ is a section of $G_I(2,6)$ by a hypersurface.

1.7. Contents of Section 5: Miscellaneous.

Section 5 contains some separate results that form neither a complete theory nor a logical chain, and at the moment they have no direct application to the program of generalization of Kolyvagin’s theorem, although maybe will acquire this application in future. These results can be treated as an introduction to the future investigations. Some of them are not proved completely and are formulated as conjectures.

Section 5.1. Case of the Hecke correspondence $T_{p,2}$. This Hecke correspondence is not as interesting for us as $T_{p,1}$, because no one irreducible component of $T_{p,2}(V)$ is defined over $K^p$ (see (5.1.3), (5.1.4)), and hence we cannot apply this correspondence in order to generalize Kolyvagin’s method. We find an explicit description of $\mathcal{D}_*$ (Theorem 5.1.1; it turns out that it is non-empty only if $p \equiv 3 \mod 4$) and show that $\mathcal{D}_0 = \emptyset$ (Lemma 5.3.5.4). Further, we get evidence that Conjecture 1.6.3 is true for this case (Theorems 4.2.16, 5.1.2).

Section 5.2. Here we investigate the action of $T_p$ on Shimura subvarieties of $T_p(V)$. This could be useful for the program of generalization of Kolyvagin’s theorem, because maybe consideration of the action of $T_p$ on various Shimura subvarieties of $T_p(V)$ will permit us to eliminate by elementary way the unpleasant coefficient $\tau_p$ of [L04.2], (1.8), (1.9).

Section 5.3. The restriction of D-partition to some sets $\mathcal{D}_i$ is unknown. Nevertheless, for $g = 3$ it is possible to prove some weakened conditions of “equivalence” of $\sigma_1, \sigma_2 \in \mathcal{D}_j$ for the cases $i = 1, j = *$ (Theorem 5.3.2) and $j = 0$ (Theorem 5.3.1), and for the cases $i = 2, j = *$ (Theorem 5.3.3) and $j = 1$ (Conjecture 5.3.5).

Section 5.4. Here it is proved that there is no coincidences between irreducible components of $T_{p,i}(V)$ for different $i$.

1.8. Conjectures and possibilities of further investigation.

Let us give some possibilities of further investigation related to the subject of the present paper in order of increasing complexity. See also possibilities of further investigation in [L04.2].

1. The reader sees that most theorems of the present paper (they are marked by (*) are not proved in all generality. So, the first problem is to prove them completely.

The main reason why most theorems are not proved in all generality is the necessity to treat all open charts of the Grassmannian corresponding to $\mathcal{X}_p(t)$, and not only the one such chart as it is done in the present paper. The reader can compare the length of the proof of the Proposition 3.4.3 where the consideration of all open charts is inevitable, with the length of proofs of other theorems, and evaluate the length of the complete proofs of all theorems for the case of arbitrary $g$. Apparently, it is necessary to invent a new method of proof if we do not want to have a several hundred-page paper. See Remark 3.3.2.

Some theorems marked by (**) are not proved at all. These theorems, as well as conjectures 1.5.6, 1.5.7, 4.2.20, can be proved easily using the same methods.
2. What is the degree of the hypersurface of Theorem 1.6.6, b? What is the interpretation of parts of I-partition of Conjecture 4.2.20 in terms of the geometry of $G_I(2, 6)$ (see Theorems 1.6.5, 1.6.6)? It is interesting to find the degree of $\pi_2$ (see 1.5.9) for the bad components for $g = 4, j = 2$. Is it a multiple of $p + 1$?

3. Prove Conjecture 1.5.4. We can expect that for any fixed pair $(g, j)$ (notations of (1.5)) there exists a proof of 1.5.4 for these $(g, j)$ similar to the proof of 3.3.4 (case $g = 3, j = 2$). Does exist a uniform proof for all $(g, j)$?

4. Investigate the case $p$ splits in $K$. According [L01], this case is much more complicated. Maybe it will be possible to find a good description of this situation.

5. To study more complicated types of pairs (and more generally, $n$-tuples) of Shimura varieties $X, V$. For example, if the reflex field of $V$ is any CM-field, then we get more interesting examples of Galois action.

6. Investigate in more details the case of $T_{p,i}$. I think that the formula (4.2.9) for $D_*$ is too rough. In order to get a correct definition of $D$-partition it is necessary to take into consideration the rank of $\mu_i(\sigma)$.

7. To find D-partition of the intermediate and special cases for $g = 3, i = 1$. If it is impossible to prove a general theorem, then it would be interesting to find partitions for some small $p$ by means of computer calculations.

8. To prove analogous results for the functional case (Drinfeld modules and more generally abelian T-motives of Anderson). We know that the functional case is usually easier. Maybe in some cases the phenomenon of Section 4.4 of [L04.2] (the pseudo-Euler systems for the case $g = 3, \mathcal{T}_p = T_p$ are identically 0) does not occur, or it will be possible to use the methods of [Z85] (see [L04.2], appendix 3) in order to find the Abel-Jacobi image of bad components?

Section 2. D-equivalence.

Let $X_c$ be a connected component of a Shimura variety of level $N$. Recall that a $p$-Hecke correspondence $\mathcal{T}_p$ on $X_c$ is a diagram

$$
\begin{array}{ccc}
& X_{p,c} & \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X_c & & X_c
\end{array}
$$

where $X_{p,c}$ is a Shimura variety of (roughly speaking) level $pN$. In this section $X_{p,c}$ and all subsequent objects with subscript $p$ depend on $\mathcal{T}_p$ — the type of Hecke correspondence. For $t \in X_c$ we have by definition: $\mathcal{T}_p(t) = \pi_2(\pi_1^{-1}(t))$. It turns out that instead of considering equivalences on the set $\mathcal{T}_p(t)$, it is easier to consider equivalences on the set $\pi_1^{-1}(t)$ and then to apply $\pi_2$ to this equivalence relation.

A non-formal definition of $D$-equivalence on the set $\mathcal{T}_p(t)$ is given in the introduction; its definition on the set $\pi_1^{-1}(t)$ is analogous. For the exact definition see 2.8.

We use notations of [D71]. Let us consider Deligne data that define a Shimura variety $X$ of a fixed level, namely, a reductive group $G$ over $\mathbb{Q}$, a map $h : \mathbb{S} \to G$ over $\mathbb{R}$ (where $S = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_m)$), and a level $N$ subgroup $K \subset G(\mathbb{A}_f)$. We denote by $\mathcal{D}$ the Hermitean symmetric domain that corresponds to $G, h$. So,
\[ X = [D \times K \backslash G(\mathbb{A}_f)]/G(\mathbb{Q}) \]

Let \( C(X) \) be the set of irreducible components of \( X \); it is isomorphic to the set of double cosets \( K \backslash G(\mathbb{A}_f)/G(\mathbb{Q}) \). The irreducible component that corresponds to the trivial double coset \( K \cdot 1 \cdot G(\mathbb{Q}) \) will be denoted by \( X_c \). Further, we denote \( \Gamma = K \cap G(\mathbb{Q}) \), so \( X_c = D/\Gamma \) (the action of some elements of \( G(\mathbb{Q}) \), and hence of \( \Gamma \), on \( D \) can be trivial).

We use the same notations for \( X_p \rightarrow X \) and \( V \leftarrow X \) using subscripts \( p, V \) respectively. Let \( K_p \subseteq K \) be a level \( p \) subgroup,

\[ X_p = [D \times K_p \backslash G(\mathbb{A}_f)]/G(\mathbb{Q}) \]

the corresponding Shimura variety, \( \pi = \pi_1 : X_p \rightarrow X \) and \( C(\pi) : C(X_p) \rightarrow C(X) \) the natural projection maps, and \( \Gamma_p = K_p \cap G(\mathbb{Q}) \). Let

\[ K = \bigcup_{i \in R} K_pr_i \tag{2.1} \]

be a coset decomposition, here \( R \) is the set of these cosets. There is an equivalence relation on \( R \) (we call it 1-equivalence and denote by \( \sim \)):

\[ r_i \sim r_j \iff K_pr_iG(\mathbb{Q}) = K_pr_jG(\mathbb{Q}) \tag{2.2} \]

We denote the quotient set of \( R \) by 1-equivalence by \( \Omega \).

**Remark.** (2.2) can be written in the form \( K_pr_i\Gamma = K_pr_j\Gamma \), because \( r_i \sim r_j \iff r_j \in K_pr_iG(\mathbb{Q}) \), i.e. \( r_j = kr_i g, k \in K_p, g \in G(\mathbb{Q}) \), \( g \in \Gamma \).

**Proposition 2.3.** \( C(\pi)^{-1}(X_c) \) [i.e. the set of irreducible components of \( \pi^{-1}(X_c) \)] is \( \Omega \).

**Proof.** Since \( C(X_p) = K_p \backslash G(\mathbb{A}_f)/G(\mathbb{Q}) \), we have \( C(\pi)^{-1}(X_c) = K_p \backslash KG(\mathbb{Q})/G(\mathbb{Q}) \). When we choose a set of representatives of these double cosets, we can choose them in \( K \); hence we can choose a subset of \( \{r_i\}_{i \in R} \). It is clear that \( r_i \sim r_j \) is exactly 1-equivalence. \( \Box \)

We denote by \( X_{p,c} \) the irreducible component of \( X_p \) that corresponds to the trivial double coset \( K_p \cdot 1 \cdot G(\mathbb{Q}) \). Obviously \( X_{p,c} = D/\Gamma_p, X_{p,c} \subseteq \pi^{-1}(X_c) \), and we denote by \( \pi_c : X_{p,c} \rightarrow X_c \) the restriction of \( \pi \) to \( X_{p,c} \). Further we consider for simplicity only the following case:

If \( \gamma \in \Gamma \) acts trivially on \( D \) then \( \gamma \in \Gamma_p \).

In this case \( \Gamma/\Gamma_p \hookrightarrow K/K_p = R \). We denote by \( R_c \) the image of \( \Gamma/\Gamma_p \) in \( R \). Obviously, \( \deg \pi_c : X_{p,c} \rightarrow X_c \) is \( \#(\Gamma/\Gamma_p) \). Further, for \( t \in D \) we have a formula

\[ \pi_c^{-1}(\bar{t}) = \{(tr_i \times r_i)\}_{i \in R_c} \tag{2.4} \]

where \( \bar{t} \in X_c \) is the projection of \( t \times 1 \), and \( (tr_i \times r_i) \in D \times G(\mathbb{A}_f) \).

Now let us consider \( V \rightarrow X \). Let \( G_V \subset G \) be an inclusion of groups commuting with \( h_V : S \rightarrow G_V \); \( \mathcal{K}_V = G_V(\mathbb{A}_f) \cap \mathcal{K} \) a level subgroup; \( V \) the Shimura subvariety of \( X \) corresponding to \( (h_V, G_V, \mathcal{K}_V) \) and \( V_c \) the irreducible component of \( V \) that corresponds to the trivial double coset \( \mathcal{K}_V \cdot 1 \cdot G_V(\mathbb{Q}) \).
**Proposition 2.5.** The set of irreducible components of $\pi_c^{-1}(V_c)$ is the quotient set of $\{r_i\}_{i \in R_c}$ by the equivalence relation:

$$r_i \sim r_j \iff \Gamma_p r_i G_V = \Gamma_p r_j G_V$$

(2.6)

**Proof.** A version of this proposition for Hecke correspondences is [L01], Proposition 4.4, (1). The proof of the present version of this proposition is completely analogous. □

This equivalence relation is the I-equivalence. Slightly changing notations of (1.4.1), we denote the quotient set of $\Gamma / \Gamma_p$ by this equivalence relation by $(\Gamma / \Gamma_p)_I$.

The above varieties form the diagram:

$$
\begin{array}{ccc}
\pi_c^{-1}(V_c) & \rightarrow & X_{p,c} \\
\downarrow & & \downarrow \\
V_c & \rightarrow & X_p \\
\downarrow & & \downarrow \\
V & \rightarrow & X
\end{array}
$$

(2.7)

Now we must “unify” (in partition sets) those irreducible components of $\pi_c^{-1}(V_c)$, which belong to a Shimura subvariety. We choose and fix $i \in R_c$, we choose the corresponding representative $r_i$ in $\Gamma$ and we denote it by $s$. For any group $G$ we denote the conjugate group $s G s^{-1}$ by $G_s$. We define a Shimura variety $V_{p,s}$ that corresponds to the subgroup $K_{V,p,s} = K_V \cap K_p^{-1} = G_V(\mathbb{A}_f) \cap K_p^{-1}$. The projection $V_{p,s} \rightarrow V$ is denoted by $\pi_{V,s}$. There exists an $s$-conjugate variety $V_{p,s}^s = s(D_V) \times K_V^s \backslash G_V(\mathbb{A}_f)^s / G_V(\mathbb{Q})^s$ which enters in the diagram (2.7) as follows:

$$
\begin{array}{ccc}
V_{p,s}^s & \xrightarrow{i_{p,s}} & X_p \\
\downarrow & & \downarrow \\
V & \rightarrow & X
\end{array}
$$

where $i_{p,s}$ is induced by identical inclusions of corresponding subgroups of $G(\mathbb{A}_f)$, and the left vertical arrow (denoted by $\pi_{V,s}^s$) is induced by the map $t \mapsto s^{-1}(t)$ for $t \in s(D_V)$, $s^{-1}(t) \in D_V$.

**Definition 2.8.** 2 elements $\bar{s}_1$, $\bar{s}_2$ of $(\Gamma / \Gamma_p)_I$ are called D-equivalent if there exists $s \in \Gamma$ such that both irreducible components of $\pi_c^{-1}(V_c)$ that correspond to $\bar{s}_1$, $\bar{s}_2$ are contained in $V_{p,s}^s$. 2 elements $s_1$, $s_2$ of $\Gamma / \Gamma_p$ are called D-equivalent if their images in $(\Gamma / \Gamma_p)_I$ are D-equivalent.

Now let us find an algebraic criterion of D-equivalence. Let

$$\mathcal{K}_V = \bigcup_{i \in R_{V,s}} \mathcal{K}_{V,p,s} \mathcal{K}_{V,i,s}$$

be a coset decomposition. There is the relation of 1-equivalence on $R_{V,s}$ defined like in (2.2), and the corresponding quotient set $\Omega_{V,s}$. Let $J_{V,s}$ be a subset of $R_{V,s}$ defined as follows:

$$j \in J_{V,s} \iff sr_{V,j,s}^{-1} \in \mathcal{K}_p G(\mathbb{Q})$$

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It is obvious that $J_{V,s}$ is a union of partition sets of 1-equivalence on $R_{V,s}$; we denote the corresponding quotient set by $Q_{V,0,s} \subseteq Q_{V,s}$. It is easy to see that there exists an inclusion $\alpha_s : Q_{V,0,s} \to (\Gamma/\Gamma_p)_I$ is defined at the level of representatives as follows. Let $r_{V,i,s} \in Q_{V,0,s}$, $sr_{V,i,s}s^{-1} = kg$, where $k \in K_p$, $g \in G(Q)$. Then $g \in \Gamma$. By definition, $gs$ is a representative of $\alpha_s(r_{V,i,s})$. Particularly, $\alpha_s(1) = s$.

**Proposition 2.9.** $Q_{V,0,s}$ is isomorphic to the set of irreducible components of $(\pi_{V,s})^{-1}(V_c) \cap \mathcal{X}_{p,c} = \pi_{c}^{-1}(V_c) \cap \mathcal{V}_{p,s}$ (both intersections are in $X_p$). Moreover, the corresponding inclusion is exactly $\alpha_s$.

**Proof.** An easy calculation in groups and cosets, with use of (2.4). □

**Proposition 2.10.** 2 elements $r_1$, $r_2 \in \Gamma/\Gamma_p$ are $D$-equivalent iff the double cosets $K_pr_iK_V$ coincide ($i = 1, 2$), i.e. iff

$$r_2 \in K_pr_1K_V$$

(2.11)

**Proof.** We see that 2 elements $r_1$, $r_2 \in \Gamma/\Gamma_p$ are D-equivalent iff $r_2 \in \text{im}(\alpha_{r_1})$. This means that $\exists i \in R_{V,r_1}$ such that $r_1 r_{V,s} r_i r_1^{-1} = kg$, where $k \in K_p$, $g \in G(Q)$, and such that $r_2 = gr_1$. This is equivalent to (2.11). □

Now we must replace the coset decomposition (2.1) by the coset decomposition (1.3.2). In order to simplify notations, until the end of this section $\tau_p$ (resp. $S(g)$) will mean either $\tau_p$ (resp. $S(g)$ ) itself (for $\mathfrak{T}_p = T_p$) or $\tau_{p,i}$ (resp. $S(g,i)$) of (1.3.1) (for $\mathfrak{T}_p = T_{p,i}$). In both cases we have $K_{p} \tau_p \Gamma \tau_p^{-1} \cap K$, $\Gamma_p = \tau_p \Gamma \tau_p^{-1} \cap \Gamma$. We need a lemma on relation between 2 sets of representatives of cosets.

**Lemma 2.12.** Let $\Gamma \tau_p \Gamma = \bigcup_{j \in \Theta(g)} \Gamma \sigma'_j$ be a coset decomposition such that all representatives $\sigma'_j$ satisfy

$$\sigma'_j \in \tau_p \Gamma$$

(2.13)

We denote

$$r_j = \tau_p \sigma'^{\prime-1}_j$$

(2.14)

Then $\Gamma = \bigcup_j \Gamma r_j$, and vice versa. □

**Remark.** We use notation $\sigma'_j$ instead of $\sigma_j$, because not all $\sigma \in S(g)$ satisfy (2.13). Namely,

$$\sigma \in S(g) \text{ satisfy (2.13) iff } \sigma \in S(g)^{open}.$$  

(2.15)

Substituting (2.14) in (2.11) we get immediately the following

**Theorem 2.16.** 2 elements $\sigma_1$, $\sigma_2 \in S(g)$ are D-equivalent iff the double cosets $K \sigma_1 K_V$ coincide ($i = 1, 2$). This condition can be rewritten as follows:

$$\exists g \in K_V \text{ such that } \sigma_1 \alpha(g) \sigma_2^{-1} \in K$$

(2.17)

**Remark 2.18.** We need to consider $\pi_2$-images of irreducible components of the above subvarieties of $\mathcal{V}_{p,c}$. In principle, $\pi_2$ can glue some of them, see [L01] for the detailed consideration of this phenomenon. But if $p$ is split in $K$ then it is easy to check that (for the case of $V$, $X$ defined in 3.1) there is no such gluing. So, Theorem 2.16 describes us also $D$-equivalence on the set $\mathfrak{T}_p(t)$, and not only on the set $\pi_1^{-1}(t)$.
Now we need a lemma that describes $L_{k_1}$ of (1.4.2) — the field of definition of irreducible components of $T_p(V)_k$. Let $s \in \Gamma$ be an element such that $s \in S(g)_{k_1}$, i.e. $T_p(V)_k \subset V^s_{p,s}$. For simplicity, we formulate it only for the case of $V$, $X$ defined in 3.1, and moreover for the case when $r \neq s$, i.e. when the reflex field of $V$ is $K$:

**Lemma 2.19.** $L_{k_1}$ is the abelian extension of $K$ that corresponds (by the global class field theory) to the group $\det \mathcal{K}_{V,p,s}$, where $\det : G_V(\mathbb{A}_Q) \to \text{Res}_{K/F}(G_m)(\mathbb{A}_Q) = I_K$ is the determinant map ($G_m$ is the multiplicative group, $I_K$ is the idele group). $\square$

**Section 3.** Case $\mathfrak{T}_p = T_p$.

**Subsection 3.1.** Definition of $V$.

Here we define the inclusion $V \subset X$. $X$ is the Siegel variety of genus $g$ and level $N$, i.e. it corresponds to the following Deligne data:

1) $G = GSp_{2g}(\mathbb{Q})$;

2) The map $h_X : \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \to G$ and the subgroup $\mathcal{K} = \mathcal{K}_N \subset G(\mathbb{A}_f)$ of level $N$ are the standard ones: they are defined like in one-dimensional case, but instead numbers we have in mind $g \times g$-blocks of matrices.

Recall the explicit formulas for the unitary group $G_V = GU(\tau, \mathfrak{s})$ where $\tau + \mathfrak{s} = g$, $K = \mathbb{Q}(\sqrt{-\Delta})$. If $L$ is a field such that $\sqrt{-\Delta} \notin L$ (we shall consider only cases $L = \mathbb{F}_p, \mathbb{Q}, \mathbb{Q}_p$) and $g \in G_V(L)$ then $g = h + k\sqrt{-\Delta}$, where $g$ satisfy

$$gE_{\tau\mathfrak{s}}g^t = \lambda E_{\tau\mathfrak{s}}$$

(3.1.1) i.e. $h, k \in M_g(L)$ satisfy

$$hE_{\tau\mathfrak{s}}h^t + \Delta kE_{\tau\mathfrak{s}}k^t = \lambda E_{\tau\mathfrak{s}}$$

(3.1.2)

$$kE_{\tau\mathfrak{s}}h^t = hE_{\tau\mathfrak{s}}k^t$$

(3.1.3)

(here $E_{\tau\mathfrak{s}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where sizes of diagonal blocks are $r, s$).

There exists an inclusion $\alpha : G_V \to G$ defined as follows:

$$\alpha(g) = \begin{pmatrix} h & kE_{\tau\mathfrak{s}} \\ -\Delta E_{\tau\mathfrak{s}}k & E_{\tau\mathfrak{s}}hE_{\tau\mathfrak{s}} \end{pmatrix}$$

(3.1.4)

Let all prime divisors of $N$ split in $K$. The trivial $\mathbb{Q}$-structure on $G_V$, the standard map $h_V : \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \to GU(\tau, \mathfrak{s})$ and the level subgroup $\mathcal{K}_V = \mathcal{K} \cap \alpha(G_V(\mathbb{A}_f))$ are Deligne data for $G_V$. We denote the corresponding Shimura variety by $V$. The inclusion $\alpha$ defines an inclusion $V \subset X$.

It is well-known that $\dim V = \mathfrak{r}\mathfrak{s}$, the reflex field of $V$ is $K$ if $\tau \neq \mathfrak{s}$ and $\mathbb{Q}$ otherwise, and if $\tau \neq \mathfrak{s}$ then the field of definition of irreducible components of $V$ is $K^1$ — the Hilbert class field of $K$. We shall be interested mainly in the first non-trivial case $g = 3$, $\mathfrak{r} = 2$, $\mathfrak{s} = 1$. In this case $V$ is a Picard modular surface.

**Subsection 3.2.** Preliminary notations and lemmas.

Recall that a description of $S(g)$ can be found in [L04.1], Section 2.3. We find D-equivalence only on the open part of $S(g)$. This open part $S(g)^{\text{open}}$ corresponds
to the case $I = \{1, \ldots, g\}$ in notations of [L04.1]. Elements of this open part have the form

$$\sigma = \sigma(s) = \begin{pmatrix} E_g & s \\ 0 & pE_g \end{pmatrix}$$

(3.2.1)

where $s \in M_g(\mathbb{Z})^{sym}$ and its entries belong to a fixed system of residues modulo $p$ (see Remark 3.3.2 for the case when $\sigma \notin S(g)^{\text{open}}$).

Since $p$ is inert in $K$, we have: $K \otimes \mathbb{Q}_p = \mathbb{Q}_p^2$. We denote the residue map by $\text{res}: \mathbb{Z}_p^2 \to \mathbb{F}_p^2$ or simply by tilde, and the non-trivial automorphism of $\mathbb{Z}_p^2, \mathbb{F}_p^2$ by bar.

We need the following objects related to $\sigma = \sigma(s)$ ($E_g = 1_g = 1$ is the unit matrix):

$$t = t(\sigma) = E_g - \sqrt{-\Delta} E_{rs}, \quad T = T(\sigma) = \bar{t}(\sigma) E_{rs} t(\sigma)^t = E_{rs} + \Delta s E_{rs}$$

(3.2.2)

Recall that $W_\sigma \subset (A_t)_p$ is defined in (1.3.5).

**Lemma 3.2.3.** $\text{rank } T(\tilde{\sigma}) = \text{dim}_{\mathbb{F}_p}(\mathbb{F}_p^2 W_\sigma) - g$.

**Proof.** Straightforward. Let $\{e_i\}$ be an $O_K/p$-basis of $(A_t)_p$, so $\{e_i, \sqrt{-\Delta} e_i\}$ form an $\mathbb{F}_p$-basis of $(A_t)_p$. $W_\sigma$ has an $\mathbb{F}_p$-basis whose matrix in $\{e_i, \sqrt{-\Delta} e_i\}$ is $(E_g, \bar{s})$. The matrix of an $\mathbb{F}_p$-basis of $\sqrt{-\Delta} W_\sigma$ is $(-\Delta \bar{s} E_{rs}, E_{rs})$. So, $\text{dim}_{\mathbb{F}_p}(\mathbb{F}_p^2 W_\sigma)$ is the rank of $\begin{pmatrix} E_g & \bar{s} \\ -\Delta \bar{s} E_{rs} & E_{rs} \end{pmatrix}$. Elementary transformations transform it to $\begin{pmatrix} E_{rs} & \bar{s} \\ 0 & E_{rs} + \Delta \bar{s} E_{rs} \end{pmatrix} = \begin{pmatrix} E_g & \bar{s} \\ 0 & T(\tilde{\sigma}) \end{pmatrix}$. □

Recall that the $\mathcal{D}$-partition is defined by (1.5.1). We have a

**Corollary 3.2.4.** $\sigma_i$ is $\mathcal{D}$-equivalent to $\sigma_j \iff \text{ranks of } T(\tilde{\sigma}_i), T(\tilde{\sigma}_j)$ are equal. Particularly, $\sigma \in \mathcal{D}_g \iff T(\tilde{\sigma}) = 0$ (for even $g$), and $\sigma \in \mathcal{D}_2g \iff t(\tilde{\sigma}), T(\tilde{\sigma})$ are invertible. □

**Remark 3.2.5.** $\mathcal{D}_2g$ and its complement in $S(g)$ have a geometric interpretation in the Plücker embedding of $S(g) = G_1(g, 2g)(\mathbb{F}_p)$. To simplify notations, we consider the case $g = 3$. We denote coordinates in the space $P^m(\mathbb{F}_p)$ ($m = \binom{2g}{g} - 1$) by $v_{i_1i_2i_3}$ where $i_j \in \{1, 2, 3, 1', 2', 3'\}$ (for example, $v_{123}$-th coordinate of any $\sigma$ is 1 and $v_{1'2'3'}$-th coordinate of any $\sigma$ is $\det \bar{s}$).

It is clear that

$$\det \tilde{t} = f_1(\bar{s}) + \sqrt{-\Delta} f_2(\bar{s})$$

(3.2.6)

where

$$f_1(\bar{s}) = \varepsilon_{1'23}v_{1'23} + \varepsilon_{12'3}v_{12'3} + \varepsilon_{123'}v_{123'} + \varepsilon_{1'2'3'}v_{1'2'3'}$$

$$f_2(\bar{s}) = \varepsilon_{123}v_{123} + \varepsilon_{1'2'3}v_{1'2'3} + \varepsilon_{123'}v_{123'} + \varepsilon_{1'2'3'}v_{1'2'3'}$$

(3.2.7)

and $\varepsilon_{***}$ are some constants ($\pm$ some power of $\Delta$). The same formulas hold for any $g$: indices run over all subsets of the set of elements of the matrix $\begin{pmatrix} 1, & \cdots, & g' \\ 1', & \cdots, & g' \end{pmatrix}$ such that each column contains 1 element of this subset; indices of the first (resp. second) formula of (3.2.7) correspond to subsets containing even (resp. odd) elements of the first line of this matrix.
So, $S(g) - \mathcal{D}_{2g} = \bigcup_{j=g}^{2g-2} \mathcal{D}_j$ is the intersection of the Plücker embedding of $\mathfrak{P}(G_1(g, 2g))$ with a codimension 2 linear subspace $P_2$ of $P^m$. $P_2$ is given by equations $f_i(s) = 0$, $i = 1, 2$. This is an analog of the conic line $C$ of the case $g = 2$ (see [L01], Theorem 0.7; Section 3, Figure 1).

Recall that $\mathcal{H}$ is the partition of hyperplane sections of Conjecture 1.5.6. $\sigma_1, \sigma_2$ belong to the same part of $\mathcal{H}$-partition iff $(f_1(\hat{s}_1) : f_2(\hat{s}_1)) = (f_1(\hat{s}_2) : f_2(\hat{s}_2))$. Action of $\mathbb{F}_{p^2}^*$ on $(A_{ij})_p$ induces the action of $\mathbb{F}_{p^2}^*/\mathbb{F}_p^*$ on $S(g) = G_1(g, 2g)(\mathbb{F}_p)$. We call it $\mathcal{F}$-action. We have the following lemma which for odd $g$ gives evidence in favor of Conjecture 1.5.7:

**Lemma 3.2.8.** Let $\sigma_1 \in \mathcal{D}_{2g}^{open}$. If $g$ is even then all points of the $\mathcal{F}$-orbit of $\sigma_1$ belong to the same part of $\mathcal{H}$-partition. If $g$ is odd then there is exactly 1 point of the $\mathcal{F}$-orbit of $\sigma_1$ in each part of $\mathcal{H}$-partition.

**Proof.** Let $\sigma_2 = \sqrt{-\Delta}(\sigma_1)$ respectively the $\mathcal{F}$-action. We use notations of Lemma 3.2.3 omitting for simplicity non-essential factors $E_{rs}$. We have:

$$\det(E_g - \sqrt{-\Delta}\hat{s}_1) = f_1(\hat{s}_1) + \sqrt{-\Delta}f_2(\hat{s}_1)$$

$$\det(\Delta\hat{s}_1 - \sqrt{-\Delta}E_g) = f_1(\hat{s}_2) + \sqrt{-\Delta}f_2(\hat{s}_2)$$

hence $f_1(\hat{s}_2) + \sqrt{-\Delta}f_2(\hat{s}_2) = (\sqrt{-\Delta})^g(f_1(\hat{s}_1) + \sqrt{-\Delta}f_2(\hat{s}_1))$. The lemma follows immediately from this formula. $\square$

**Remark 3.2.9.** The meaning of the Lemma 3.2.8 is the following. We can only guess what is the action of Galois group on the set of parts of $I$-partition in a given good part of $D$-partition. There are 2 natural actions: $\mathcal{F}$-action and the action of $\mathbb{Z}/(p+1)$ on the set of parts of $\mathcal{H}$-partition. Lemma 3.2.8 shows that they coincide for odd $g$, hence most likely it is the desired Galois action.

**Subsection 3.3.** Finding of $D$-equivalence.

Let $i = 1, 2$, $s_i \in M_g(\mathbb{Z})^{symm}$, $\sigma_i = \sigma(s_i) \in S(g)^{open}$ be as in (3.2.1). We denote $t(\sigma_i)$, $T(\sigma_i)$ simply by $t_i$, $T_i$.

**Theorem 3.3.1.** $\sigma_1, \sigma_2 \in \mathcal{D}_{2g} \cap S(g)^{open}$ are $D$-equivalent iff the following condition (3.3.1.1) holds:

(3.3.1.1) Either $g$ is odd, or $g$ is even and the ratio $\det T(\sigma_i)/\det T(\sigma_j)$ is a square in $\mathbb{F}_p^*$.

**Proof.** We consider $g$ of (2.17), $g = h + k\sqrt{-\Delta}$, $h, k \in M_g(\mathbb{Z}_p)$. The explicit calculation shows that the $g \times g$-block structure of $\sigma_1 \alpha(g)\sigma_2^{-1}$ is the following:

$$\sigma_1 \alpha(g)\sigma_2^{-1} = \begin{pmatrix} A_{11} & p^{-1}A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

(3.3.1.2)

where entries of all $A_{ij} = A_{ij}(h, k, s_1, s_2)$ are some polynomials with integer coefficients in entries of $h, k, s_1, s_2$ and hence $\in M_g(\mathbb{Z}_p)$. This means that

$\sigma_1$ is $D$-equivalent to $\sigma_2$ $\iff$ $\exists g \in G_V(\mathbb{Z}_p)$ such that $A_{12} = 0$.

The proof is based on the following crucial formula:
\[ \tilde{t}_1 \tilde{g} \tilde{t}_2 = B + A_{12} E_{ts} \sqrt{-\Delta} \]  

(3.3.1.3)

where \( B \in M_g(\mathbb{Z}_p) \) is some matrix. This means that we can rewrite (2.17) as follows:

\[ \sigma_1 \text{ is } D\text{-equivalent to } \sigma_2 \iff \exists g \in K_V \text{ such that} \]

\[ \tilde{t}_1 \tilde{g} \tilde{t}_2 \in M_g(\mathbb{F}_p) \]  

(3.3.1.4)

We shall prove:

A. If \( \exists g \in G_V(\mathbb{Z}_p) \) such that (3.3.1.4) holds (i.e. \( \tilde{t}_1 = 0 \)) then (3.3.1.1) holds;

B. If (3.3.1.1) holds then \( \exists g \in G_V(\mathbb{Z}_p) \) such that \( A_{12} = 0 \).

To prove (A) we denote \( \tilde{t}_1 \tilde{g} \tilde{t}_2 \) by \( \tilde{B} \) (i.e. we reduce (3.3.1.3) modulo \( p \)). Since both \( \tilde{t}_1 \) are invertible we get \( \tilde{\sigma} = \tilde{t}_1^{-1} \tilde{B} \tilde{t}_2^{-1} \). Substituting this value of \( \tilde{\sigma} \) in (3.1.1), we get

\[ \tilde{B}(E_{ts} \tilde{T}_2 E_{ts})^{-1} \tilde{B}^t = \lambda \tilde{T}_1 \]  

(3.3.1.5)

The theory of quadratic forms over \( \mathbb{F}_p \) shows that (3.3.1.5) implies (3.3.1.1).

Inversely, if (3.3.1.1) holds, then \( \exists B \in GL_g(\mathbb{Z}_p) \) such that \( B(E_{ts} T_2 E_{ts})^{-1} B^t = \lambda T_1 \). So, \( g = \tilde{t}_1^{-1} \tilde{B} \tilde{t}_2^{-1} \in G_V(\mathbb{Z}_p) \), and \( A_{12} = 0 \).

Finally, we must check that all possible types of \( T(\tilde{\sigma}) \) can be realized. This is obvious (it is sufficient to consider diagonal matrices \( s \)). \( \square \)

**Remark 3.3.2.** The first step of reformulation of the Theorem 3.3.1 for the case of any \( \sigma \in \mathcal{D}_{2g} \) (not necessarily \( \sigma \in S(g)^{open} \)) is the following. If \( \sigma = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \)

where \( A, B, D \) are \( g \times g \)-matrices described in [L04.1], Section 2.3, (2), then we can set \( t(\sigma) = A + \sqrt{-\Delta}(B + D) \) up to possible factors \(-1\) and/or \( E_{ts} \).

**Theorem 3.3.3.** If \( \sigma_1, \sigma_2 \in S(g)^{open}, \sigma_1 \in \mathcal{D}_{2g} \) and \( \sigma_2 \not\in \mathcal{D}_{2g} \) then \( \sigma_1 \) and \( \sigma_2 \) are not D-equivalent.

**Proof.** We consider here only the case when \( \sigma_1 \) is D-equivalent to the matrix \( \begin{pmatrix} E_g & 0 \\ 0 & pE_g \end{pmatrix} \), i.e. the signature of \( \tilde{T}_1 \) is trivial (proofs for the cases of another signatures of \( \tilde{T}_1 \) are analogous). So, we can take \( \sigma_1 = \begin{pmatrix} E_g & 0 \\ 0 & pE_g \end{pmatrix}, s_1 = 0 \) and hence \( t_1 = E_g \). Assume that there exists \( g \) from (2.17). Then (3.3.1.3) implies that \( \tilde{g} \tilde{t}_2 \in M_g(\mathbb{F}_p) \). This means that \( \tilde{k} = \tilde{h} \tilde{s}_2 E_{ts} \). Substituting this formula in (3.1.2) we get \( \tilde{h} \tilde{T}_2 \tilde{k}^t = \lambda E_{ts} \). Since det \( \tilde{T}_2 = 0 \), this is contradicts to the condition \( \lambda \in \mathbb{Z}_p^* \).

\( \square \)

**Theorem 3.3.4.** If \( g = 3, r = 2, s = 1 \) and 2 elements \( \sigma_1, \sigma_2 \in \mathcal{D}_4 \cap S(g)^{open} \) then \( \sigma_1 \) and \( \sigma_2 \) are D-equivalent.

**Proof.** \( \det \tilde{t}_i = 0 \) implies that 2 eigenvalues of \( \tilde{s}_i E_{21} \) are \( \pm \frac{1}{\sqrt{\Delta}} \) and another one — denoted by \( r_i \) — is in \( \mathbb{F}_p \). Let \( w_i = (w_{i1}, w_{i2}, w_{i3}) \in (\mathbb{F}_p)^3, v_i = (v_{i1}, v_{i2}, v_{i3}) \in \mathbb{F}_p^3 \). Hence, we can take \( \tilde{t}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{pmatrix} \) and:

\( \det \tilde{t}_i = 0 \) implies that 2 eigenvalues of \( \tilde{s}_i E_{21} \) are \( \pm \frac{1}{\sqrt{\Delta}} \) and another one — denoted by \( r_i \) — is in \( \mathbb{F}_p \). Let \( w_i = (w_{i1}, w_{i2}, w_{i3}) \in (\mathbb{F}_p)^3, v_i = (v_{i1}, v_{i2}, v_{i3}) \in \mathbb{F}_p^3 \). Hence, we can take \( \tilde{t}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{pmatrix} \) and:

\[ \det \tilde{t}_i = 0 \]
$\tilde{v}_i = (\tilde{v}_{i1}, \tilde{v}_{i2}, \tilde{v}_{i3})$ be eigenvectors of $\tilde{s}_i E_{21}$ with eigenvalues $r_i, \frac{1}{\sqrt{-\Delta}}$ and $-\frac{1}{\sqrt{-\Delta}}$ respectively. We denote $L_i = \begin{pmatrix} w_{i1} & v_{i1} & \tilde{v}_{i1} \\ w_{i2} & v_{i2} & \tilde{v}_{i2} \\ w_{i3} & v_{i3} & \tilde{v}_{i3} \end{pmatrix}$, so $\det L_i \neq 0$ and

$$\tilde{t}_i = L_i D_i L_i^{-1} \quad (3.3.4.1)$$

where $D_i = \text{diag} \left( 1 - r_i \sqrt{-\Delta}, 0, 2 \right)$. We assume existence of $g$ satisfying (2.17), and we denote as earlier $\tilde{t}_i g \tilde{t}_2$. (3.3.1.3) implies $B \in M_3(\mathbb{F}_p)$.

Taking $\tilde{t}_i$ from (3.3.4.1), we get:

$$\bar{D}_1 \bar{L}_i^{-1} \tilde{g} L_2 D_2 = \bar{L}_i^{-1} B L_2 \quad (3.3.4.2)$$

For any matrix $X$ we have: the second line (resp. column) of $\bar{D}_1 X$ (resp. $XD_2$) is 0. Hence, (3.3.4.2) implies that both second line and column of $\bar{L}_i^{-1} B L_2$ are 0. Further, we denote by $T_{132}$ the permutation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

so we have equalities

$$\bar{L}_i = L_i T_{132} \quad (3.3.4.3)$$

and (because $\bar{B} = B$)

$$\bar{L}_i^{-1} B L_2 = L_i^{-1} B \bar{L}_2 = T_{132} \bar{L}_i^{-1} L_2 T_{132} \quad (3.3.4.4)$$

(3.3.4.4) implies that the third line and column of $\bar{L}_i^{-1} B L_2$ are also 0. So,

$$\bar{D}_1 \bar{L}_i^{-1} \tilde{g} L_2 D_2 = b \cdot \text{diag} \left( 1, 0, 0 \right) \quad (3.3.4.5)$$

where $b \in \mathbb{F}_p$.

Solving the equation $XD_2 = b \cdot \text{diag} \left( 1, 0, 0 \right)$ we see that $\bar{D}_1 \bar{L}_i^{-1} \tilde{g} L_2 = \begin{pmatrix} b & 0 \\ 0 & 0 & \Delta \end{pmatrix}$. Analogously, solving the equation $\bar{D}_1 X = \begin{pmatrix} b & 0 \\ 0 & 0 & \Delta \end{pmatrix}$ we see that

$$\bar{L}_i^{-1} \tilde{g} L_2 = \begin{pmatrix} \frac{b}{(1-r_2 \sqrt{-\Delta})(1+r_1 \sqrt{-\Delta})} & 0 \\ * & * \\ 0 & * \end{pmatrix} \quad (3.3.4.6)$$

All these considerations are invertible, i.e. (3.3.4.6) is equivalent to (3.3.1.4). We denote $\bar{L}_i^{-1} \tilde{g} L_2$ by $g_0$. Now we shall show that (3.3.1.4) and

$$\tilde{g} E_{21} \tilde{t}_i = \lambda E_{21} \quad (3.3.4.7)$$

(the $\mathbb{F}_{p^2}$-residue of (3.1.1)) can be satisfied simultaneously for any $s_1, s_2$ of the statement of the theorem. Really, substituting $g_0$ to (3.3.4.7) we get
\[ g_0(L_2^{-1}E_{21}(L_2^{-1})^t)g_0^t = \lambda(L_1^{-1}E_{21}(L_1^{-1})^t) \]  

(3.3.4.8)

Analog for the present case of a theorem that eigenvectors of a symmetric matrix are orthogonal is the following. We denote

\[ D_{0i} \overset{\text{def}}{=} L_i^{-1}s_iE_{21}L_i = \text{diag} \left( r_i, \frac{1}{\sqrt{-\Delta}}, -\frac{1}{\sqrt{-\Delta}} \right) \]

so we have:

\[ s_i = L_iD_{0i}L_i^{-1}E_{21} = s_i^t = E_{21}(L_i^{-1})^tD_{0i}L_i^t \]

i.e.

\[ (L_i^{-1}E_{21}(L_i^{-1})^t)D_{0i}(L_i^tE_{21}L_i) = D_{0i} \]

Since all diagonal entries of \( D_{0i} \) are different we get that \( L_i^tE_{21}L_i \) is a diagonal matrix. It is invertible, we denote entries of the inverse matrix as follows:

\[ (L_i^tE_{21}L_i)^{-1} = \text{diag}(d_{i1}, d_{i2}, \bar{d}_{i2}) \]  

(3.3.4.9)

where \( d_{i1} \in \mathbb{F}_p, d_{i2} \in \mathbb{F}_p^2 \). Taking into consideration (3.3.4.3), (3.3.4.6) and (3.3.4.8) give us the following equations:

\[
\begin{pmatrix}
 b' & x_{12} & 0 \\
 x_{21} & x_{22} & x_{23} \\
 0 & x_{32} & 0
\end{pmatrix} \begin{pmatrix}
 d_{21} & 0 & 0 \\
 0 & 0 & d_{22} \\
 0 & \bar{d}_{22} & 0
\end{pmatrix} \begin{pmatrix}
 \bar{b}' & \bar{x}_{21} & 0 \\
 \bar{x}_{12} & \bar{x}_{22} & \bar{x}_{32} \\
 0 & \bar{x}_{23} & 0
\end{pmatrix} = \lambda \begin{pmatrix}
 d_{11} & 0 & 0 \\
 0 & 0 & d_{12} \\
 0 & \bar{d}_{12} & 0
\end{pmatrix}
\]

where \( b' = \frac{b}{(1-r_2\sqrt{-\Delta})(1+r_1\sqrt{-\Delta})} \). For any \( s_1, s_2 \) a solution can be found immediately, for example \( x_{12} = x_{21} = x_{22} = 0, b \in \mathbb{F}_p^* \) arbitrary, \( \lambda = \frac{b'b'd_{21}}{d_{11}}, x_{23} = 1, x_{32} = \frac{\lambda\bar{d}_{22}}{d_{22}} \). So, we have got \( \bar{g} \) satisfying (3.3.1.4), (3.3.4.7). Further, we have

**Lemma 3.3.4.10.** The reduction map \( G_V(\mathbb{Z}_p) \to G_V(\mathbb{F}_p) \) is surjective. □

Application of this lemma gives us the theorem. □

**Theorem 3.3.5.** If \( g \) is odd then all components of \( T_p(V) \) do not coincide with \( V \) itself.

**Proof.** Obvious. Really, we have the following criterion of coincidence ([L01], Section 4a, Proposition 4.4, (4)):

\[ \Gamma_V\sigma^{-1} \cap \Gamma \neq \emptyset \]  

(3.3.5.1)

(the inclusion \( G_V \to G \) is \( \alpha \)). If \( g \in \Gamma_V \) satisfies \( \alpha(g)\sigma^{-1} \in \Gamma \) then \( \lambda(g) = p \).

This is impossible by a trivial reason (in this case we have \( N_{K/\mathbb{Q}}(\det g) = p^g \) which for odd \( g \) contradicts to a condition \( p \) inert in \( K \)). □

**Theorem 3.3.6.** For any \( g \), any \( \mathfrak{r} \neq \mathfrak{s} \) the field of definition of any good component is \( K^p \).
Proof. We fix \( \sigma \in S(g)^{\text{open}} \). According 2.15, we can take \( r = \tau_p \sigma^{t-1} \) (see 2.14). We apply Lemma 2.19 (s of 2.19 is our \( r \)). It is easy to see that for points \( I \) of \( K \), \( I \neq p \), the \( t \)-component of \( \det K_{V,p,r} \) contains \( O_{K_I}^t \), so we must find only the \( p \)-component of \( \det K_{V,p,r} \). Let \( g = h + k\sqrt{-\Delta} \in G_V(\mathbb{Z}_p) \) be as earlier \((G_V(\mathbb{Z}_p) \) is \( p \)-component of \( K_V \)). The \( p \)-component of \( K_p \) is the set of matrices whose \((2,1)\)-block is \( \equiv 0 \mod p \).

3.3.6.1. The following 3 conditions are equivalent:

\[ g \text{ belongs to the } p \text{-component of } K_{V,p,r} \quad (3.3.6.2) \]

\[ \iff r\alpha(g)r^{-1} \text{ belongs to the } p \text{-component of } K_p \quad (3.3.6.3) \]

\[ \iff \text{the } (2,1)\text{-block of } r\alpha(g)r^{-1} \text{ is } \equiv 0 \mod p \quad (3.3.6.4) \]

We denote the \((2,1)\)-block of \( r\alpha(g)r^{-1} \) by \( \mathfrak{B}_{21} \). There is a formula for it (practically, this is 3.3.1.3):

\[ \tilde{t} E_{a \varepsilon} g t' = \mathfrak{A} - \mathfrak{B}_{21} \sqrt{-\Delta} \quad (3.3.6.5) \]

where \( t = t(\sigma) \) and \( \mathfrak{A} \in M_g(\mathbb{Z}_p) \). If \( \det \tilde{t} \neq 0 \) (i.e. \( \sigma \in \mathcal{D}_{2g} \)) then \( \mathfrak{B}_{21} = 0 \implies \det g \in \mathbb{Z}_p + p\sqrt{-\Delta} \mathbb{Z}_p \). It is easy to see that \( \det g \) can take any values in \( \mathbb{Z}_p^* + p\sqrt{-\Delta} \mathbb{Z}_p \), i.e. the field of definition of the good component is \( K^p \). \( \square \)

Analogously, we get the following

Theorem 3.3.7. The field of definition of all bad components is \( K \). \( \square \)

Subsection 3.4. Finding of the degree of \( \pi_2 \).

We use notations of \( 1.5.9 \).

Proposition 3.4.1. For any \( g \) the degree of \( \pi_2 \) for any good component is 1.

Proof. We restrict ourselves by the case of the good component with thivial signature of \( T(\tilde{\sigma}) \); proof for other components is analogous. So, we can take \( s = 0 \) and \( \sigma = \left( \begin{array}{cc} E_g & 0 \\ 0 & pE_g \end{array} \right) \). We apply formula 4.10 of [L01] for the degree of \( \pi_2 \). We set \( g = h + k\sqrt{-\Delta} \) like above \((g \text{ of } [L01], 4.10 \text{ is } \alpha(g) \text{ in the notations of the present paper}) \). So, the degree of \( \pi_2 \) is the quantity of \( \sigma' \in S(g) \) such that there exists \( g \) such that

\[ \sigma' \sigma \alpha(g) \in p \cdot GSp_{2g}(\mathbb{Z}) \quad (3.4.1.1) \]

Particularly, for \( g \) satisfying (3.4.1.1) we have

\[ \text{ord}_p(\det g) = 0 \quad (3.4.1.2) \]

We use description of \( S(g) \) from [L04.1], Section 2.3 \((\sigma' \text{ is denoted in } [L04.1], \text{ Section 2.3 by } \gamma) \). Let us fix \( \sigma' \). Attached to \( \sigma' \) is a subset \( I \) of \( \{1, \ldots, g\} \).

Let \( j \in \{1, \ldots, g\}, j \not\in I \). Multiplying the \( j \)-th line of \( \sigma' \sigma \) by the \( r \)-th (resp. \( g + r \)-th) column of \( \alpha(g) \) we get that (3.4.1.1) implies \( h_{jr} \in \mathbb{Z} \) (resp. \( k_{jr} \in \mathbb{Z} \)). Analogously, for \( i \in I \), multiplying the \( g + i \)-th line of \( \sigma' \sigma \) by columns of \( \alpha(g) \),
we get that \( h_{ir}, k_{ir} \in \frac{1}{p} \mathbb{Z} \). Further, multiplying the \( i \)-th line of \( \sigma'\sigma \) by columns of \( \alpha(g) \), we see that all terms of the corresponding scalar product — except one — are integer, hence this only remaining term — which is \( h_{ir} \) or \( k_{ir} \) — is also integer.

Let us consider the case \( I \neq \emptyset \), and let \( i \) be the minimal element of \( I \). Multiplying the \( i \)-th line of \( \sigma'\sigma \) by all columns of \( \alpha(g) \) we get that (3.4.1.1) implies that the first \( i \) lines of \( g \) are linearly dependent mod \( p \). This contradicts to (3.4.1.2). So, we get that if for the given \( \sigma' \) there exists \( g \) such that (3.4.1.1) holds then \( I = \emptyset \). There exists only one such \( \sigma' = \begin{pmatrix} pE_g & 0 \\ 0 & E_g \end{pmatrix} \), and we can take \( g = 1. \)

Now let us consider the case of the bad component for \( g = 3 \).

**Remark 3.4.2.** To use the formula 4.10 of [L01], we need to check all elements of \( S(g) \), not only of its open part. This explains why the proof of the following Proposition 3.4.3 is so long: I am forced to treat all 8 open components of \( S(g) \) separately. I do not see a method to find a uniform method of theating. For higher \( g \) and a partition set contained in \( \mathcal{D}_i \) it is possible to treat some cases uniformly, so the quantity of cases is a polynomial of \( g, i \).

We shall see (Remark 3.4.4) that the set of \( \sigma' \in S(3) \) satisfying (3.4.1.1) is an irreducible subvariety of \( S(3) = G_1(3, 6) \). We can expect that this is true for all \( g, i \). Proof of this fact will reduce the quantity of cases that we shall have to consider in order to find the degree of \( \pi_2 \).

**Proposition 3.4.3.** For \( g = 3 \) the degree of \( \pi_2 \) for the bad component is \( p + 1 \).

**Proof.** We use the same formula 4.10 of [L01]. We fix \( \sigma = \begin{pmatrix} E_g & W \\ 0 & pE_g \end{pmatrix} \), where \( s = W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & w_1 & w_2 \\ 0 & w_2 & w_1 \end{pmatrix} \), \( w_1, w_2 \in \mathbb{Z} \) satisfy \( w_1^2 - w_2^2 = -\frac{1}{\Delta} \) mod \( p^2 \) (this implies that \( s \) corresponds to the bad component). It is more convenient to take \( g = \frac{1}{p}(h + k\sqrt{-\Delta}) \). Let \( \sigma' = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \) and \( I \) be as in [L04.1], Section 2.3. We denote \( P = \sigma'\sigma\alpha(g) \). Although we prove this theorem for \( g = 3 \), some arguments are valid for any \( g \), so we shall use sometimes \( g \) instead of 3.

The entries of \( i \)-th line of \( \sigma'\sigma \) are multiples of \( p \) for \( i \not\in I \), \( i \in \{1, \ldots, g\} \), and for all \( i \in \{g + 1, \ldots, 2g\} \). For this set of \( i \)'s conditions \( P_{ij} \in p\mathbb{Z} \) can be treated as linear congruences modulo \( p \) on elements \( h_{\alpha j}, k_{\beta j} \):

\[
P_{ij} \in p\mathbb{Z} \iff \sum_{\alpha = 1}^{g} (C_{11})_{i\alpha} h_{\alpha j} + \sum_{\alpha = 1}^{g} (C_{12})_{i\alpha} k_{\alpha j} \equiv 0 \text{ (for } i \not\in I, i \in \{1, \ldots, g\}\text{)}
\]

\[
P_{ij} \in p\mathbb{Z} \iff \sum_{\alpha = 1}^{g} (C_{21})_{i\alpha} h_{\alpha j} + \sum_{\alpha = 1}^{g} (C_{22})_{i\alpha} k_{\alpha j} \equiv 0 \text{ (for } i \in \{g + 1, \ldots, 2g\}\text{)}
\]

\[
P_{i,g+j} \in p\mathbb{Z} \iff \sum_{\alpha = 1}^{g} (C_{31})_{i\alpha} h_{\alpha j} + \sum_{\alpha = 1}^{g} (C_{32})_{i\alpha} k_{\alpha j} \equiv 0 \text{ (for } i \not\in I, i \in \{1, \ldots, g\}\text{)}
\]
$P_{i,g+j} \in p\mathbb{Z} \iff \sum_{\alpha=1}^{g} (C_{41})_{i\alpha} h_{\alpha j} + \sum_{\alpha=1}^{g} (C_{42})_{i\alpha} k_{\alpha j} \equiv 0 \text{ (for } i \in \{g+1, \ldots, 2g\})$

where $j \in \{1, \ldots, g\}$ and the coefficient matrix $C$ has the $4 \times 2$-block structure, $(C_{uv})_{xy}$ is the $(x, y)$-th element of the $(u, v)$-block of $C$. It is easy — but tedious — to write down expressions for all 8 blocks of $C$ (for example, $C_{21} = C_{42} = 0$, $C_{22} = -\Delta DE_{rs}$, $C_{41} = DE_{rs}$ etc).

Looking at the second (resp. fourth) block line of $C$ we get immediately that the corresponding congruences imply that all entries of $k$ (resp. $h$) are $p$-integer. This means that

**3.4.3.1** If the rank of $C$ modulo $p$ is $2g$ then all entries of $h$, $k$ are $0$ modulo $p$.

**Case** $I = \{1, 2, 3\}$. $\sigma' = \begin{pmatrix} 1 & B_3 \\ 0 & p \end{pmatrix}$, where $s = B_3 = \{b_{ij}\}$ is any symmetric $3 \times 3$-matrix. We shall use also the (1,2)-block partitions of $B_3$, $W$, $h$, $k$: $B_3 = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$, $W = \begin{pmatrix} 0 & 0 \\ 0 & W_2 \end{pmatrix}$, $h = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and analogously for $k$.

Calculating explicitly entries of $P$ we see that the condition (3.4.1.1) gives us immediately congruences:

3.4.3.2) $h \equiv \Delta(W + pB_3)E_{21}k \mod p^2$
3.4.3.3) $k \equiv -(W + pB_3)E_{21}h \mod p^2$

Substituting $h$ from (3.4.3.2) to (3.4.3.3) we get

3.4.3.4) $[1 + \Delta W E_{21} W E_{21} + \Delta p(B_3 E_{21} W + W E_{21} B_3) E_{21}] k \equiv 0 \mod p^2$ which is in the above notations

3.4.3.5) $t_{11} + \Delta p B_{12} E_{11} W E_{11} t_{21} \equiv 0 \mod p^2$
3.4.3.6) $t_{12} + \Delta p B_{12} E_{11} W E_{11} t_{22} \equiv 0 \mod p^2$
3.4.3.7) $\Delta p W_2 E_{11} B_{12}^t t_{11} + \Delta p (B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) E_{11} t_{21} \equiv 0 \mod p^2$
3.4.3.8) $\Delta p W_2 E_{11} B_{12}^t t_{12} + \Delta p (B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) E_{11} t_{22} \equiv 0 \mod p^2$

Substituting value of $t_{11}$ from (3.4.3.5) to (3.4.3.7) (resp. $t_{12}$ from (3.4.3.6) to (3.4.3.8)), we get that (3.4.3.7), (3.4.3.8) become

3.4.3.9) $(B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) E_{11} t_{2i} \equiv 0 \mod p$, $i = 1, 2$.

If $\det(B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) \not\equiv 0 \mod p$, then (3.4.3.9) imply that $t_{2i} \equiv 0 \mod p$. Further, (3.4.3.5), (3.4.3.6) imply that $t_{1i} \equiv 0 \mod p^2$, (3.4.3.2) implies that $b_{1i} \equiv 0 \mod p^2$, $h \equiv 0 \mod p$. This contradicts to (3.4.1.1).

Now we need a

**Lemma 3.4.3.10.** If $\det(B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) \equiv 0 \mod p$ then $B_{22} E_{11} W_2 + W_2 E_{11} B_{22} \equiv 0 \mod p$ and $B_{22} = \gamma \begin{pmatrix} w_2 & w_1 \\ w_2 & w_1 \end{pmatrix}$ where $\gamma$ is a scalar factor.

**Proof.** Direct calculation. Really, condition $\det(B_{22} E_{11} W_2 + W_2 E_{11} B_{22}) \equiv 0 \mod p$ in terms of $b_{ij}$ is
\[-\Delta w_2^2b_{22} + (2\Delta w_1w_2 + 2w_2\sqrt{-\Delta})b_{23} + (\Delta w_1^2 - 2\Delta w_2^2 - 2w_1\sqrt{-\Delta})b_{33}] \]

\[-\Delta w_2^2b_{22} + (2\Delta w_1w_2 - 2w_2\sqrt{-\Delta})b_{23} + (\Delta w_1^2 - 2\Delta w_2^2 + 2w_1\sqrt{-\Delta})b_{33}] = 0

this is the union of 2 imaginary lines on the projective plane (b_{22} : b_{23} : b_{33}) whose real intersection point corresponds to the case \(B_{22}E_{11}W_2 + W_2E_{11}B_{22} = 0 \mod p\). Solving the corresponding system, we get the above expression for \(B_{22}\). \(\Box\)

In this case (3.4.3.7), (3.4.3.8) are always satisfied, substituting values of \(\xi_{1i}\) from (3.4.3.5), (3.4.3.6) to (3.4.3.2) we get (congruences are modulo \(p^2\)):

\[h \equiv \left( \frac{\Delta pB_{12}E_{11}\xi_{21}}{\Delta (W_2 + pB_{22})E_{11}\xi_{21}}, \frac{\Delta pB_{12}E_{11}\xi_{22}}{\Delta (W_2 + pB_{22})E_{11}\xi_{22}} \right) \text{ and hence}
\]

\[pg \equiv \left( \frac{\Delta pB_{12}E_{11}(1 - \sqrt{-\Delta}W_2E_{11})\xi_{21}}{\Delta (W_2 + pB_{22})E_{11} + \sqrt{-\Delta} \cdot 1_2}, \frac{\Delta pB_{12}E_{11}(1 - \sqrt{-\Delta}W_2E_{11})\xi_{22}}{\Delta (W_2 + pB_{22})E_{11} + \sqrt{-\Delta} \cdot 1_2} \right) \]

(3.4.3.11)

Now we check the condition \((pg)E_{21}(pg^t) = \lambda E_{21}\), where \(\text{ord}_p(\lambda) = 2\). Recall that \(b_{ij}, k_{ij}\) are entries of \(B_3, k\) respectively. Diagonal entries of \((pg)E_{21}(pg^t)\) are elements of the ring of polynomials \(\mathbb{Z}[w_1, w_2, \Delta, b_{ij}, k_{ij}]\) factored by \(\Delta(w_1^2 - w_2^2) + 1 = 0\). A calculation in this ring shows that

if \(B_{22} = 0\) then \(((pg)E_{21}(pg^t))_{11} = Qp^2((pg)E_{21}(pg^t))_{33}\) \((3.4.11')\)

for some \(Q \in \mathbb{Z}[w_1, w_2, \Delta, b_{ij}, k_{ij}]/(\Delta(w_1^2 - w_2^2) + 1)\).

Since \(B_{22}\) enters in (3.4.3.11) with a coefficient \(p\), we see that the condition \(\text{ord}_p((pg)E_{21}(pg^t)_{11}) = 2\) together with (3.4.11') contradicts to the condition \(\text{ord}_p((pg)E_{21}(pg^t)_{33}) = 2\). This means that for \(\sigma'\) of type \(I = \{1, 2, 3\}\) there is no \(g\) satisfying (3.4.1.1).

**Case** \(I = \{2, 3\}\). \(\sigma' = \left( \begin{array}{ccc}
p & 0 & 0 \\
-D^t & 1 & 0 \\
0 & 0 & 1 & D \\
0 & 0 & 0 & p \end{array} \right)\) where \(D, B\) are respectively \(1 \times 2, 2 \times 2\)-matrices, \(B\) is symmetric. The condition that the entries of the first and the third block lines of \(\sigma'\sigma(\alpha(g))\) are in \(p\mathbb{Z}\) gives us immediately that entries of \(\xi_{1i}, \xi_{1i}\) are in \(p\mathbb{Z}\) (i = 1, 2).

**Subcase 1.** \(D = 0\). The condition that the entries of the second block line of \(\sigma'\sigma(\alpha(g))\) are in \(p\mathbb{Z}\) becomes

\[h_{2i} \equiv \Delta(W_2 + pB)E_{11}\xi_{2i} \mod p^2 \quad (3.4.3.12)\]

\[\xi_{2i} \equiv -(W_2 + pB)E_{11}h_{2i} \mod p^2 \quad (3.4.3.13)\]

or \([E_{11} + \Delta W_2E_{11}W_2 + p\Delta(W_2E_{11}B + BE_{11}W_2)]E_{11}h_{2i} \equiv 0 \mod p^2\), i.e.

\[(W_2E_{11}B + BE_{11}W_2)E_{11}h_{2i} \equiv 0 \mod p \quad (3.4.3.13')\]
**Subcase 1a.**  $B \neq \gamma \begin{pmatrix} w_2 & w_1 \\ w_1 & w_2 \end{pmatrix}$. Using Lemma 3.4.3.10 we get that (3.4.3.13') implies $b_{2i}, t_{2i} \equiv 0 \mod p$. (3.4.3.12), (3.4.3.13) become $t_{2i} \equiv -W_2E_{11}b_{2i}, \ b_{2i} \equiv \Delta W_2E_{11}t_{2i} \mod p^2$. This means that $(1_2 + W_2E_{11}\Delta)g_{2i} \equiv 0 \mod p$, i.e. that the second and the third lines of $g$ are linearly dependent. This contradicts to (3.4.1.1).

**Subcase 1b.**  $B = \gamma \begin{pmatrix} w_2 & w_1 \\ w_1 & w_2 \end{pmatrix}$. For these $\sigma'$ there exist $g$ satisfying (3.4.1.1). Really, if we take any $\mathbf{h}_{1i} \equiv \mathbf{t}_{1i} \equiv \mathbf{h}_{2i} \equiv \mathbf{t}_{2i} \equiv 0 \mod p\mathbb{Z}$, any $b_{2i} \equiv t_{2i} \equiv 0 \mod p\mathbb{Z}$. Now we must find $\mathbf{h}_{1i}, \mathbf{t}_{1i}, \mathbf{b}_{2i}$ such that $g$ satisfies (3.1.1). This can be done by many ways. For example, we can take $g_{12} = g_{21} = 0$ and $\mathbf{h}_{12} = \mathbf{t}_{22} = \mu E_{11}$ for some $\mu$. Simple calculations show that for all $\gamma$ (3.1.1) can be satisfied.

**Subcase 2.**  $D \neq 0$. As earlier, elementary transformations of congruences $(\sigma'\sigma(g))_{ij} \equiv 0 \mod p$ give us the desired. Firstly, the condition that the entries of the third block line of $\sigma'\sigma(g)$ are in $p\mathbb{Z}$ implies that

$$DE_{11}b_{2i} \equiv 0 \mod p, \ DE_{11}t_{2i} \equiv 0 \mod p$$  \hspace{1cm} (3.4.3.14)

These conditions together with the condition that the entries of the second block line of $\sigma'\sigma(g)$ are in $p\mathbb{Z}$ implies that

$$DE_{11}W_2E_{11}b_{2i} \equiv 0 \mod p, \ DE_{11}W_2E_{11}t_{2i} \equiv 0 \mod p$$  \hspace{1cm} (3.4.3.15)

It is easy to check that $W_2E_{11}$ has no eigenvectors in $\mathbb{F}_p$ (because $\left(\frac{-\Delta}{p}\right) = -1$), so (3.4.3.14), (3.4.3.15) imply that entries of $b_{2i}, t_{2i}$ are in $p\mathbb{Z}$.

Now the condition that the entries of the third block line of $\sigma'\sigma(g)$ are in $p\mathbb{Z}$ becomes

$$-D^t\mathbf{h}_{1i} + \mathbf{h}_{2i} - \Delta W_2E_{11}t_{2i} \equiv 0 \mod p^2$$  \hspace{1cm} (3.4.3.16)

$$D^t\mathbf{t}_{1i} - \mathbf{t}_{2i} - W_2E_{11}b_{2i} \equiv 0 \mod p^2$$  \hspace{1cm} (3.4.3.17)

Elementary transformations reduce this system to the system

$$\begin{pmatrix} -w_1d_1 + w_2d_2 & d_1 \\ -w_2d_1 + w_1d_2 & d_2 \end{pmatrix} \begin{pmatrix} h_{1i} \\ \mathbf{t}_{1i} \end{pmatrix} \equiv 0 \mod p^2$$

(here $d_i$ are entries of $D$).

Since $\det \begin{pmatrix} -w_1d_1 + w_2d_2 & d_1 \\ -w_2d_1 + w_1d_2 & d_2 \end{pmatrix}$ is never 0, we get that $h_{1i}, \mathbf{t}_{1i} \equiv 0 \mod p^2$ — a contradiction to (3.4.1.2). So, for this case there is no $g$ satisfying (3.4.1.1).

**Case I** = \{1\}. $\sigma' = \begin{pmatrix} 1 & 0 & 0 & b_{11} & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. (3.4.1.1) gives us immediately that the second and the third lines of $h, k$ are $\equiv 0 \mod p$, and their first lines are $\equiv 0 \mod p^2$. This contradicts to (3.4.1.2).
Case $I = \{2\}$. $\sigma' = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 \\ -d_{12} & 1 & 0 & 0 & b_{22} & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & d_{12} & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Writing explicitly the matrix $C'$ for this case, we see immediately that its rank is 6, i.e. (3.4.3.1) implies that $h, k \equiv 0 \mod p$. Further, the condition $[\sigma'\sigma\alpha(g)]_{2i} \in p\mathbb{Z}$ implies that

$$(d_{12} - 1 - w_1\sqrt{-\Delta} \ w_2\sqrt{-\Delta})g \equiv 0 \mod p$$

This contradicts to (3.4.1.2).

Case $I = \{3\}$. $\sigma' = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 \\ -d_{13} & -d_{23} & 1 & 0 & 0 & b_{33} \\ 0 & 0 & 0 & 1 & 0 & d_{13} \\ 0 & 0 & 0 & 0 & 1 & d_{23} \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}$. The non-0 lines of $C'$ modulo $p$ form the following matrix $C'$:

$$C' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\Delta & w_1 \\ 0 & 0 & 0 & -\Delta & 0 & \Delta d_{13} \\ 0 & 0 & 0 & 0 & -\Delta & \Delta d_{23} \\ 0 & w_1 & -w_2 & 0 & 1 & 0 \\ 1 & 0 & -d_{13} & 0 & 0 & 0 \end{pmatrix}$$

If $d_{13} \not\equiv 0 \mod p$ then it is seen immediately that the rank of $C'$ is 6; in any case, the determinant of the submatrix of $C'$ formed by removing of its third and seventh lines is $\pm w_2(w_2d_{23}^2 + 2w_1d_{23} - w_2)$ which is never 0 because of $w_1^2 - w_2^2 \equiv \frac{-1}{\Delta} \mod p^2$ and because $p$ is inert in $K$.

We denote the third line of $(1,1)$-block (resp. of $(1,2)$-block) of $\sigma'\sigma$ by $l_1$ (resp. by $l_2$). It is easy to check that the condition $[\sigma'\sigma\alpha(g)]_{3i} \in p\mathbb{Z}$ implies that

$$(l_1 + \sqrt{-\Delta}l_2)E_{21}g \equiv 0 \mod p$$

(like in the case $I = \{2\}$). This contradicts to (3.4.1.2).

Cases $I = \{1, 3\}$ and $I = \{1, 2\}$. $\sigma' = \begin{pmatrix} 1 & 0 & 0 & b_{11} & 0 & b_{13} \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & -d_{23} & 1 & b_{13} & 0 & b_{33} \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & d_{23} \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}$ and

$$\sigma' = \begin{pmatrix} 1 & 0 & 0 & b_{11} & b_{12} & 0 \\ 0 & 1 & 0 & b_{12} & b_{22} & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively. The non-0 lines of $C$ are
uniquely by the relation
$$\mu \equiv p \mod Z$$

of representatives of $Z$ in $Z$, i.e. $h_{2i}, h_{3i}, k_{2i}, k_{3i}$ are in $pZ$. Considering the condition that the $(1, i)$-th and the $(1, i + g)$-th element of $\sigma' \sigma \alpha(g)$ are integer, we get immediately that $h_{1i}, k_{1i}$ are in $pZ$. Considering again the same condition, we get that moreover $h_{1i}, k_{1i}$ are in $p^2Z$. This is sufficient to get a contradiction.

**Case** $I = \emptyset$. $\sigma' = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. This is the trivial case, $g = 1$ satisfies (3.4.1.1).

We see that in $S(3)$ there are $p$ matrices of type $I = \{2, 3\}$, Subcase 1b, and 1 matrix of type $I = \emptyset$ such that there exists $g$ satisfying (3.4.1.1). This gives us our result. \(\square\)

**Remark 3.4.4.** It is easy to check that the above $p+1$ matrices form a projective line in $S(3) = G_1(3,6) \subset P^{10}$.

**Section 4. Case of Hecke correspondence $I_\rho = T_{p,1}$.**

**Subsection 4.1. Some matrix equalities.**

Here we formulate and prove some matrix equalities that will be necessary for the next subsection. The open part $S(g, i)^{open} \subset S(g, i)$ is the set of the following block matrices (diagonal blocks have sizes $i, g-i, i, g-i$):

$$
\sigma = \begin{pmatrix}
p & 0 & 0 & pC^t \\
-A & 1 & C & U \\
0 & 0 & p & pA^t \\
0 & 0 & 0 & p^2
\end{pmatrix},
$$

(4.1.1)

where $A, C$ are $i \times (g-i)$-matrices, $U$ is a $(g-i) \times (g-i)$-matrix with entries in $Z$. Moreover, entries of $A, C$ run over a fixed system of representatives of $Z$ modulo $pZ$, diagonal and upper-triangular entries of $U$ run over a fixed system of representatives of $Z$ modulo $p^2Z$, and lower-triangular entries of $U$ are defined uniquely by the relation

$$
CA^t - AC^t = U^t - U
$$

(4.1.2)

which is equivalent to the condition $\sigma \in GSp_{2g}$.

Let us define some objects associated to $\sigma$. Firstly, we define $\mu_1 = \mu_1(\sigma)$, $\mu_2 = \mu_2(\sigma) \in M_g(\mathbb{Z}(\sqrt{-\Delta}))) \hookrightarrow M_g(\mathbb{Z}_{p^2})$ (here and below diagonal blocks have sizes $i, g-i$):

$$
\mu_1 = \begin{pmatrix}
-1 \\
-A + C\sqrt{-\Delta} & -U\sqrt{-\Delta}
\end{pmatrix}, \mu_2 = \begin{pmatrix}
\sqrt{-\Delta} \\
-A + C\sqrt{-\Delta} & -U\sqrt{-\Delta}
\end{pmatrix}
$$

(4.1.2.5)
To simplify notations, we consider only the case \((r, s) = (i, g - i)\), and we denote \(E_{rs} = E\). Further, we define \(G = \mu_1 E \tilde{\mu}_1^t\). Let the block structure of \(G\) be 
\[
\begin{pmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{pmatrix}
\]
Finally, if \(\det \mu_1(\sigma)\) is invertible in \(\mathbb{Z}_{p^2}\) then we define matrices \(X_1, X_2\) by the formula 
\[
\mu_2 = \begin{pmatrix} X_1 & X_2 \\ 0 & 1 \end{pmatrix} \mu_1
\]
(4.1.3)
and \(F = G^{-1} = (\mu_1 E \tilde{\mu}_1^t)^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\).

**Remark 4.1.4.** Matrices \(\mu_1, \mu_2\) play symmetric roles in the contents of the present subsection, so it is possible to rewrite it for the case \(\det \mu_2(\sigma)\) is invertible in \(\mathbb{Z}_{p^2}\). Recall that if \(\sigma \notin \mathcal{D}_*\) (see (4.2.9)) then either \(\det \mu_1(\tilde{\sigma})\), or \(\det \mu_2(\tilde{\sigma})\), or both of them are \(\neq 0\). So, when we shall consider in the sequel an element \(\sigma \notin \mathcal{D}_*\), we shall assume always that \(\det \mu_1(\tilde{\sigma}) \neq 0\). If not, then we interchange roles of \(\mu_1(\sigma), \mu_2(\sigma)\) and we get the proof for the case \(\det \mu_1(\tilde{\sigma}) = 0, \det \mu_2(\tilde{\sigma}) \neq 0\).

We shall apply terminology "real", "imaginary" etc. to the extensions \(\mathbb{Q}_p \hookrightarrow \mathbb{Q}_{p^2}, \mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2}\) in the obvious sense.

**Proposition 4.1.5.** \(G\) is real (and hence symmetric) matrix.

**Proof.** Follows immediately from the definition of \(G\) and (4.1.2). \(\square\)

Clearly the same is true for \(F\).

**Proposition 4.1.6.** \(\text{im } X_1 = F_{11}, \text{im } X_2 = F_{12}\) (and hence \(F_{21} = \text{im } X_2^t\)).

**Proof.** Writing down the real and the imaginary parts of the (1,1)- and (1,2)-block entries of the equality (4.1.3) we get 4 equalities:

\[
\begin{align*}
-r \text{e } X_1 + \text{r } X_2 A - \Delta \text{im } X_2 C &= 0 \quad (4.1.6.1r) \\
-\text{i } X_1 + \text{r } X_2 C + \text{i } X_2 A &= 1 \quad (4.1.6.1i) \\
\Delta \text{im } X_1 C^t - \text{r } X_2 + \Delta \text{im } X_2 U &= 0 \quad (4.1.6.2r) \\
-r \text{e } X_1 C^t - \text{r } X_2 U - \text{i } X_2 &= -A^t \quad (4.1.6.2i)
\end{align*}
\]

where the last 2 symbols \(r\) (resp. \(i\)); \(1\) (resp. \(2\)) in the number of the equality mean that it comes from the equality for the real (resp. imaginary) part of (1,1) (resp. (1,2))-blocks of the equality (4.1.3).

Now we eliminate \(\text{r } X_1, \text{r } X_2\) from these equalities (firstly we find \(\text{r } X_2\) from (4.1.6.2r) and secondly \(\text{r } X_1\) from (4.1.6.1r)); we get

\[
\text{im } X_1(-1 + \Delta C^t C) + \text{im } X_2(A + \Delta U C) = 1 \quad (4.1.6.3)
\]

\[
\text{im } X_1(\Delta C^t U - \Delta C^t AC) + \text{im } X_2(-1 - \Delta U AC^t + \Delta CC^t - \Delta U^2) = -A^t \quad (4.1.6.4)
\]

It is sufficient to check that
\[ \text{im } X_1 \ G_{11} + \text{im } X_2 \ G_{21} = 1 \quad (4.1.6.5) \]
\[ \text{im } X_1 \ G_{12} + \text{im } X_2 \ G_{22} = 0 \quad (4.1.6.6) \]

Writing down explicitly \( G_{jk} \) we get immediately that (4.1.6.5) = (4.1.6.3) and (4.1.6.6) = (4.1.6.3) \( \cdot A^t + \) (4.1.6.4). □

**Corollary 4.1.7.** \( \det \text{im } \tilde{X}_1 \neq 0 \iff \det \tilde{G}_{22} \neq 0. \) □

**4.1.8.** If \( \det \text{im } \tilde{X}_1 \neq 0 \) then we define \( W = -F_{11}^{-1}F_{12} = -(\text{im } X_1)^{-1}\text{im } X_2. \)

**Corollary 4.1.9.** In this case we have for any \( g, i: \)
\[ F_{11}W + F_{12} = 0; \quad (4.1.9.1) \]
\[ W G_{22} = G_{12} \quad (4.1.9.2) \]
\[ F_{21}W + F_{22} = G_{22}^{-1} \quad (4.1.9.3) \]
\[ \det G = \det G_{22} \det (G_{11} - WG_{22}W^t) \quad (4.1.9.4) \]

**Proof.** (4.1.6.6) (resp. (4.1.6)) and the definition of \( W \) imply immediately (4.1.9.2) (resp. (4.1.9.1)). Further, (4.1.9.3) follows from a general matrix formula:

let \( \mathcal{F} = \begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix}, \mathcal{G} = \mathcal{F}^{-1} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{pmatrix} \) be arbitrary block matrices, \( \mathcal{F}_{11} \) is invertible and \( \mathcal{W} = -\mathcal{F}_{11}^{-1}\mathcal{F}_{12}. \) So, \( \mathcal{F}_{21}\mathcal{W} + \mathcal{F}_{22} = \mathcal{G}_{22}^{-1}. \) Analogously, (4.1.9.4) also holds for any \( \mathcal{F}, \mathcal{G} = \mathcal{F}^{-1}. \) □

Now we consider the case when \( \det \text{im } \tilde{X}_1 = 0. \)

**Lemma 4.1.10.** Let \( D \neq 0 \) be a matrix with entries in \( \mathbb{F}_p \) such that \( \text{Dim } \tilde{X}_1 = 0. \) Then \( \text{Dim } \tilde{X}_2 \neq 0. \) Particularly, \( \det \text{im } \tilde{X}_1 = 0 \) implies \( \text{im } \tilde{X}_2 \neq 0. \)

**Proof.** We reduce (4.1.6.1i), (4.1.6.2r) modulo \( p, \) multiply them by \( D \) from the left and eliminate \( D \) \( \text{re } \tilde{X}_2. \) \( \text{Dim } \tilde{X}_1 = 0 \) implies \( \text{Dim } \tilde{X}_2(A + \Delta \tilde{U} \tilde{C}) = D, \) this contradicts to conditions \( \text{Dim } \tilde{X}_2 = 0, \ D \neq 0. \) □

For a vector row \( X = (x_1, x_2) \) we denote by \( X^O \) the orthogonal vector \((-x_2, x_1),\) and for a vector column \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) \( X^O \) will mean \( \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \)

**Proposition 4.1.11.** Let \( g = 3, i = 1, \tilde{F}_{11} = 0. \) Then
\[ \tilde{F}_{12}^O \tilde{F}_{22} \tilde{F}_{12}^{Ot} = -\det \tilde{F} \quad (4.1.11.1) \]

(we identify a \( 1 \times 1 \)-matrix with a number);
\[ \tilde{F}_{21}^O \tilde{F}_{21}^{Ot} = \frac{-1}{\det \tilde{F}} \tilde{G}_{22} \quad (4.1.11.2) \]
\[ \tilde{F}_{22} \tilde{F}_{12}^{Ot} = -\det \tilde{F} \tilde{G}_{21}^O \quad (4.1.11.3) \]
The determinant of the matrix formed by vectors $\tilde{F}_{21}$, $\tilde{G}_{21}^O$ is 1. (4.1.11.4)

**Proof.** All these equalities hold for any symmetric invertible $3 \times 3$-matrix $F$ having $F_{11} = 0$. □

Now we need a lemma for the case when both $\det \tilde{\mu}_1 = \det \tilde{\mu}_2 = 0$. The following simple proof of this lemma is due to A. Zelevinskij.

**Lemma 4.1.12.** Let $g$ is arbitrary, $i = 1$. If $\det \tilde{\mu}_1 = \det \tilde{\mu}_2 = 0$ then the $(g - 1) \times g$-matrix $\left( \hat{A} + \tilde{C} \sqrt{-\Delta} \quad -1 - \tilde{U} \sqrt{-\Delta} \right)$ has rank $\leq g - 2$.

**Proof.** We consider the following $(g + 1) \times g$-matrix $\mu$:

\[
\begin{pmatrix}
-1 & -\tilde{C}^t \sqrt{-\Delta} \\
\sqrt{-\Delta} & -\tilde{A}^t \sqrt{-\Delta} \\
\hat{A} + \tilde{C} \sqrt{-\Delta} & -1 - \tilde{U} \sqrt{-\Delta}
\end{pmatrix}
\]

For each $i = 1, \ldots, g - 1$ we consider 2 minors $m_{i1}, m_{i2}$ of this matrix:

$m_{i1}$ is formed by all lines of $\mu$, except the $i$-th line of the block $(\hat{A} + \tilde{C} \sqrt{-\Delta} \quad -1 - \tilde{U} \sqrt{-\Delta})$, and by all columns of $\mu$;

$m_{i2}$ is formed by all lines of the block $(\hat{A} + \tilde{C} \sqrt{-\Delta} \quad -1 - \tilde{U} \sqrt{-\Delta})$, and by all columns of $\mu$, except the $i$-th column of the block $-1 - \tilde{U} \sqrt{-\Delta}$.

Elementary transformations show that (taking into consideration 4.1.2) for all $i$ we have $\det m_{i1} = \det m_{i2}$. This implies immediately the lemma. □

**Subsection 4.2.** Finding of D-equivalences of some $\sigma_k$.

Let $\sigma_1, \sigma_2 \in S(g, i)^{open}$, i.e. they are matrices of type (4.1.1). Let us consider objects associated to $\sigma$ defined in Section 4.1, namely $A, U, C, \mu_i, G, G_{ij}, F, F_{ij}, X_i$ $(i, j = 1, 2)$. These objects for the above $\sigma_k$ ($k = 1, 2$) will be denoted by $A_k, U_k, C_k, \mu_{ik}, G_k, G_{ijk}, F_k, F_{ijk}, X_{ik}$ respectively. Recall that $\sigma_1, \sigma_2$ are D-equivalent iff

$\exists g = h + k \sqrt{-\Delta} \in G_V(\mathbb{Z}_p)$ such that $\sigma_1 \alpha(g) \sigma_2^{-1}$ has integer entries.

Analog of (3.3.1.2) is the following:

$$
\sigma_1 \alpha(g) \sigma_2^{-1} = \begin{pmatrix}
* & * & p^{-1}A_{14} \\
p^{-1}A_{21} & * & * \\
p^{-1}A_{23} & * & * \\
* & * & * \\
p^{-1}A_{34} & * & *
\end{pmatrix}
$$

(4.2.1)

where *’s and $A_{**}$ are some polynomials with integer coefficients in $h, k, A_i, U_i, C_i$ $(i = 1, 2)$ and hence $\in M_q(\mathbb{Z}_p)$.

Analog of $t(\sigma)$ of (3.2.2) is the pair $\mu_1(\sigma), \mu_2(\sigma)$. For $i = 1, 2$ we denote

$$
B_i = B_i(\sigma_1, \sigma_2) = \mu_{i1} g E \mu_{i2} E^\alpha
$$

(4.2.2)

where $\alpha = 0$ for $i = 1$ and $\alpha = 1$ for $i = 2$ (this factor $E^\alpha$ is not important).

Analogs of (3.3.1.3) are

$$
im B_1 = \begin{pmatrix}
* & A_{14} \\
A_{23} & A_{24}
\end{pmatrix};
im B_2 = \begin{pmatrix}
* & A_{34} \\
A_{21} & A_{24}
\end{pmatrix}
$$

(4.2.3)

These equalities show that $B_1, B_2$ have the form

$$
\begin{pmatrix}
\mathbb{Z}_p + \sqrt{-\Delta} \mathbb{Z}_p & \mathbb{Z}_p + p \sqrt{-\Delta} \mathbb{Z}_p \\
\mathbb{Z}_p + p \sqrt{-\Delta} \mathbb{Z}_p & \mathbb{Z}_p + p^2 \sqrt{-\Delta} \mathbb{Z}_p
\end{pmatrix}
$$

(4.2.4)
(sets mean that the corresponding entry of $B_1, B_2$ belongs to this set).

(4.2.5) Now we can formulate an analog of (3.3.1.4):

\[ \sigma_1 \text{ is } D\text{-equivalent to } \sigma_2 \iff \exists g \in G_V(\mathbb{Z}_p) \text{ such that } B_1, B_2 \text{ satisfy } (4.2.4). \]

In some cases I cannot find classes of D-equivalence. Nevertheless, as a first step of the future investigations in many cases it is possible to prove that some $\sigma_1, \sigma_2 \in S(g, i)$ satisfy the following condition that is weaker than (4.2.5):

**Condition 4.2.6.** $\exists g \in G_V(\mathbb{Z}_p)$ such that

\[ B_1, B_2 \in \left( \begin{array}{cc} \mathbb{Z}_p + \sqrt{-\Delta} \mathbb{Z}_p & \mathbb{Z}_p + p \sqrt{-\Delta} \mathbb{Z}_p \\ \mathbb{Z}_p + p \sqrt{-\Delta} \mathbb{Z}_p & \mathbb{Z}_p + p \sqrt{-\Delta} \mathbb{Z}_p \end{array} \right) \]  

(4.2.7)

(the only difference is the first power of $p$ in (2,2)-block) which is equivalent to the condition

\[ \tilde{B}_1, \tilde{B}_2 \in \left( \begin{array}{cc} \mathbb{F}_p + \sqrt{-\Delta} \mathbb{F}_p & \mathbb{F}_p \\ \mathbb{F}_p & \mathbb{F}_p \end{array} \right) \]  

(4.2.8)

Now we define the $D$-partition of $S(g, i)$ (see 1.6.2). We give formulas only for $\sigma \in S(g, i)^{open}$ leaving the definitions for other $\sigma$ as a subject of further investigation, see Remark 4.2.11. Firstly,

\[ \sigma \in D_* \iff \det \mu_1(\tilde{\sigma}) = \det \mu_2(\tilde{\sigma}) = 0 \]  

(4.2.9)

Further, if $\det \mu_1(\tilde{\sigma}) \neq 0$ then for $j = 0, \ldots, i$ we define

\[ \sigma \in D_j \iff \text{rank } im \tilde{X}_1 = j \]  

(4.2.10)

**Remark 4.2.11.** We have the following table of the corresponding formulas for $D$-partition for the cases $\Sigma_p = T_p$ and $T_{p,i}$:

| $\Sigma_p$ | $T_p$ | $T_{p,i}$ |
|------------|-------|-----------|
| Lemma 3.2.3 (property) | Formulas 4.2.9, 4.2.10 (definition) | Theorems 1.6.5, 1.6.6 (properties) |
| Formula 1.5.1 (definition) | Theorems 1.6.5, 1.6.6 (properties) |

The reason of this non-coincidence is the following: theorems 1.6.5, 1.6.6 do not permit us to distinguish between all possible cases, so I cannot take their formulas as definitions. See also 1.8.6.

Particularly, I do not know the exact definition of $D$-partition for the complement of $S(g, i)^{open}$, hence all subsequent theorems are proved only for $S(g, i)^{open}$.

**Remark 4.2.12.** If $\det \mu_1(\tilde{\sigma}) = 0, \det \mu_2(\tilde{\sigma}) \neq 0$ then (see Remark 4.1.4) we can define $D_j, j = 0, \ldots, i$, interchanging the roles of $\mu_1, \mu_2$ in (4.2.10). Since for any invertible complex matrix $X$ we have $\text{rank } im X = \text{rank } im (X^{-1})$ the formula (4.2.10) is invariant respectively the permutation of $\mu_1, \mu_2$.

**Remark 4.2.13.** For $q = 3, i = 1$ this definition is equivalent to the following one. We define a $2 \times 2$-matrix $\mathcal{M} = \mathcal{M}(\sigma)$ with entries in $\mathbb{F}_p$ whose $i$-th line $(r_{i1}, r_{i2})$ is the line of coordinates of $\det \mu_i(\tilde{\sigma})$ in the basis $(1, \sqrt{-\Delta})$:

\[ r_{i1} + r_{i2} \sqrt{-\Delta} = \det \mu_i(\tilde{\sigma}), \quad r_{ij} \in \mathbb{F}_p, \quad i = 1, 2 \]
We have: \( (k = 0, 1) \):

\[ \mathfrak{D}_* = \{ \sigma \in S(3, 1)^{\text{open}} | \text{rank } (\mathfrak{K}(\sigma)) = 0 \} \quad (4.2.13.1) \]

\[ \mathfrak{D}_k = \{ \sigma \in S(3, 1)^{\text{open}} | \text{rank } (\mathfrak{K}(\sigma)) = k + 1 \} \quad (4.2.13.2) \]

**Remark 4.2.14.** For \( g = 3, i = 2 \) Lemma 5.3.5.4 shows that \( \mathfrak{D}_0 = \emptyset \).

**Theorem 4.2.15.** For any \( g, i \) we have: if \( g - i \) is odd then \( \sigma_1, \sigma_2 \in \mathfrak{D}_i \) are \( \mathfrak{D} \)-equivalent. If \( g - i \) is even then \( \sigma_1 \) is \( \mathfrak{D} \)-equivalent to \( \sigma_2 \) iff the ratio \( \det \tilde{G}_{221}/\det \tilde{G}_{222} \) is a square in \( \mathbb{F}^*_p \).

**Proof.** We have \( \det \mu_{1i} \neq 0, i = 1, 2. \) (4.2.2) implies that

\[ gE = \mu_{i1}^{-1} B_1 \mu_{12}^{-1} \quad (4.2.15.1) \]

\[ B_2 = \begin{pmatrix} X_{11} & X_{21} \\ 0 & 1 \end{pmatrix} B_1 \begin{pmatrix} X_{12} & X_{22} \\ 0 & 1 \end{pmatrix} t E \quad (4.2.15.2) \]

When we shall \( \mathfrak{D} \)-prove equivalence of \( \sigma_1, \sigma_2 \) in this and subsequent theorems, we shall prove that there exist \( B_1, B_2 \) satisfying the following condition:

\[ B_1, B_2 \in \left( \mathbb{Z}_p + \sqrt{-\Delta} \mathbb{Z}_p \right) \left( \mathbb{Z}_p \right) \quad (4.2.15.2') \]

which is stronger than (4.2.4), and when we shall prove non-\( \mathfrak{D} \)-equivalence, we shall prove that do not exist \( B_1, B_2 \) satisfying (4.2.8). We denote \( B_1 = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{pmatrix} \).

Let us assume that (4.2.15.2') for \( B_1 \) is satisfied. (4.2.15.2) gives us

\[ B_2 = \begin{pmatrix} \mathfrak{B}_{21} X_{12}^{t} + \mathfrak{B}_{22} X_{22}^{t} \\ \mathfrak{B}_{22} \end{pmatrix} X_{11} \mathfrak{B}_{12} + X_{21} \mathfrak{B}_{22} \quad (4.2.15.3) \]

So, if

\[ \text{im } X_{11} \mathfrak{B}_{12} + \text{im } X_{21} \mathfrak{B}_{22} = 0 \quad (4.2.15.4) \]

\[ \text{im } X_{12} \mathfrak{B}_{21}^{t} + \text{im } X_{22} \mathfrak{B}_{22}^{t} = 0 \quad (4.2.15.5) \]

then (4.2.15.2') for \( B_2 \) is satisfied.

Condition \( \sigma_i \in \mathfrak{D}_i \) is equivalent to the condition \( \det \text{im } X_{1i} \in \mathbb{Z}_p^*, i = 1, 2. \) So, if we set

\[ \mathfrak{B}_{12} = W_i \mathfrak{B}_{22}, \quad \mathfrak{B}_{21} = \mathfrak{B}_{22} W_i^{t} \quad (4.2.15.7) \]

(\( W_i \) are defined in (4.1.8)) then (4.2.15.4), (4.2.15.5) will be satisfied. Substituting the expression for \( g \) from (4.2.15.1) to (3.1.1) we get that \( B_1 \) must satisfy

\[ B_1 F_2 B_1^{t} = \lambda G_1 \quad (4.2.15.9) \]

Writing down (4.2.15.9) in block form and taking into consideration (4.2.15.7), we get that (3.1.1) is equivalent to the following 3 equalities:
\[ \mathcal{B}_{11} F_{112} \tilde{\mathcal{B}}_t^{11} + W_1 \mathcal{B}_{22} F_{212} \mathcal{B}_t^{11} + \mathcal{B}_{11} F_{122} \mathcal{B}_t^{22} W_1^t + W_1 \mathcal{B}_{22} F_{222} \mathcal{B}_t^{22} W_1^t = \lambda G_{111} \] (4.2.15.10)

\[ \mathcal{B}_{11} (F_{112} W_2 + F_{122}) \mathcal{B}_t^{22} + W_1 \mathcal{B}_{22} (F_{212} W_2 + F_{222}) \mathcal{B}_t^{22} = \lambda G_{121} \] (4.2.15.11)

\[ \mathcal{B}_{22} (W_2' (F_{112} W_2 + F_{122}) + (F_{212} W_2 + F_{222})) \mathcal{B}_t^{22} = \lambda G_{221} \] (4.2.15.12)

For those who does not know Corollary 4.1.9 these formulas look rather complicated, does not it? But using this corollary we get that (4.2.15.12) becomes

\[ \mathcal{B}_{22} G_{222}^{-1} \mathcal{B}_t^{22} = \lambda G_{221} \] (4.2.15.13)

and (4.2.15.11) follows from (4.2.15.12). The theory of quadratic forms over \( \mathbb{F}_p \) shows that the statement of the theorem is necessary, and if it is satisfied, then (4.2.15.11), (4.2.15.12) are satisfied.

Finally, (4.2.15.10) has the form \( \mathcal{B}_{11} P \mathcal{B}_t^{11} + Q^t \mathcal{B}_t^{11} Q + R = 0 \) where \( P, Q, R \) are some real matrices, \( P, R \) are symmetric and \( \det P \neq 0 \). The substitution \( \mathcal{B}_{11} = \mathcal{B}_t^{11} - Q^t P^{-1} \) eliminates the linear terms: we get \( \mathcal{B}_t^{11} P \mathcal{B}_t^{11} + R' = 0 \), \( R' = R - Q^t P^{-1} Q \). It is easy to prove that always \( \det R' \neq 0 \) (if \( \sigma_2 = \left( \begin{array}{cc} E_g & 0 \\ 0 & pE_g \end{array} \right) \) (i.e. if \( A_2, U_2, C_2 = 0 \) then this follows immediately from (4.1.9.4); the proof for other \( \sigma_2 \) is similar). So, the theory of Hermitian forms over \( \mathbb{Z}_p \) shows that we can choose \( \mathcal{B}_{11} \) such that (4.2.15.10) can be satisfied. \( \square \)

**Proposition 4.2.16.** For the case \( g = 3 \) we have: if \( i = 1, \sigma_1 \in \mathcal{D}_0, \sigma_2 \in \mathcal{D}_1 \) or \( i = 2, \sigma_1 \in \mathcal{D}_1, \sigma_2 \in \mathcal{D}_2 \) then \( \sigma_1 \) is not D-equivalent to \( \sigma_2 \).

**Proof.** According Remark 4.1.4, we consider only the case when \( \det \tilde{\mu}_{11} \neq 0 \). We have: \( \det \tilde{X}_{11} = 0, \det \tilde{X}_{12} \neq 0 \). Let us assume existence of \( B_1 \).

Case \( i = 1 \). Equalities (4.2.15.1) - (4.2.15.3) and the reductions of (4.2.15.4) - (4.2.15.5) continue to be true, so (4.2.15.7) becomes

\[ \tilde{\mathcal{B}}_{21} = \tilde{\mathcal{B}}_{22} \tilde{W}_2^t \] (4.2.16.1)

hence \( \det \tilde{X}_{21} \tilde{\mathcal{B}}_{22} = 0 \). (4.2.16.1) implies that \( \det \tilde{X}_{21} \tilde{\mathcal{B}}_{21} = 0 \), i.e. \( \det \tilde{X}_{21} \), \( \tilde{\mathcal{B}}_{21}, \tilde{\mathcal{B}}_{22} \) = 0. According Lemma 4.1.10, we see that \( B_1 \) is degenerate. This contradicts to (4.2.15.1).

Case \( i = 2 \). Equalities (4.2.15.1) - (4.2.15.3), the reductions of (4.2.15.4) - (4.2.15.5), and (4.2.16.1) continue to be true. There exists a \( 1 \times 2 \)-matrix \( D \neq 0 \) such that \( D \) im \( \tilde{X}_{11} = 0 \), hence the reduction of (4.2.15.4) implies that \( D \) im \( \tilde{X}_{21} \tilde{\mathcal{B}}_{22} = 0 \), Lemma 4.1.10 implies that \( \tilde{\mathcal{B}}_{22} = 0 \) and (4.2.16.1) implies that \( \tilde{\mathcal{B}}_{21} = 0 \), hence \( \det \tilde{B}_1 = 0 \). This contradicts to (4.2.15.1). \( \square \)

**Proposition 4.2.17.** For the case \( g = 3, i = 1 \) we have: if \( \sigma_1 \in \mathcal{D}_0 \cup \mathcal{D}_1, \sigma_2 \in \mathcal{D}_* \) then \( \sigma_1 \) and \( \sigma_2 \) are not D-equivalent.
Proof. \( \sigma_1 \in \mathcal{D}_0 \cup \mathcal{D}_1 \) implies that there exists \( i = 1 \) or \( 2 \) such that \( \det \mu_{i1} \neq 0 \mod p \). For a matrix \( A \) we denote by \( A_{(j)} \) the \( j \)-th column of \( A \). According Lemma 4.1.12, \( (\hat{\mu}_{i2}^t)_3 = \gamma (\hat{\mu}_{i2}^t)_2 \mod p \) where the coefficient of proportionally \( \gamma \in \mathbb{F}_p (\sqrt{\Delta}) \) obviously is not real. This implies that \( (B_i)_3 = \gamma (B_i)_2 \mod p \), hence both \( (B_i)_2 = (B_i)_3 = 0 \mod p \). Since \( (B_i)_2 = \mu_{i1} E (\hat{\mu}_{i2}^t)_2 \) and \( (\hat{\mu}_{i2}^t)_2 \neq 0 \mod p \), we get that the columns of \( \mu_{i1} E \) are linearly dependent modulo \( p \) — a contradiction to the condition \( \det \mu_{i1}, \det \mu \neq 0 \mod p \). \( \square \)

Proposition 4.2.18. Let \( \sigma \in \mathcal{D}_* \). Then \( T_{\sigma_1}(V)_{I(\sigma)} = V \iff \sigma \in \mathcal{D}_{0,*} \) (see 1.4.1 for the notations). Particularly, \( \mathcal{D}_{0,*} \) is a part of \( \mathcal{D} \)-partition.

Proof. To simplify calculations, we consider only the case of \( \sigma \) such that (in notations of 4.1.1) \( A, C = 0 \), and \( U \in M_2(\mathbb{Z}) \) is any symmetric matrix. Calculations for the remaining \( \sigma \) are similar but more complicated. Like in the proof of the theorem 3.3.5, we use 3.3.5.1.

This means that \( T_{\sigma_1}(V)_{I(\sigma)} = V \iff \) there exists \( \sigma \in \mathcal{D}_* \) such that \( \det \sigma = 0 \) and entries of \( \alpha(\sigma) \sigma^{-1} \) are integer.

We write \( \sigma = h + k \sqrt{-\Delta} \) and we denote block entries of \( \sigma, h, k \) by \( \sigma_{ij}, h_{ij}, k_{ij} \) respectively. Writing down the entries of \( \alpha(\sigma) \sigma^{-1} \) we get that they are integer iff

\[(4.2.18.0) \] Entries of \( h_{11}, h_{21}, k_{11}, k_{21} \) are \( 0 \mod p \)
and

\[-h_{12} U + k_{12} \equiv 0 \mod p^2 \\
\Delta k_{12} U + h_{12} \equiv 0 \mod p^2 \tag{4.2.18.1} \]

\[-h_{22} U + k_{22} \equiv 0 \mod p^2 \\
\Delta k_{22} U + h_{22} \equiv 0 \mod p^2 \tag{4.2.18.2} \]

(4.2.18.1) (resp. (4.2.18.2)) implies that \( (1 + \Delta U^2) h_{12} \) (resp. \( (1 + \Delta U^2) h_{22} \)) \( 0 \mod p^2 \).

Further, we have:

\[ \text{ord } p(\det(1 + U \sqrt{-\Delta})) = 0 \iff \text{ord } p(\det(1 + \Delta U^2)) = 0; \]

\[ \text{ord } p(\det(1 + U \sqrt{-\Delta})) = 1 \iff 1 + \Delta U^2 \equiv 0 \mod p, \neq 0 \mod p^2, \]
and in this case \( \text{ord } p\left(\frac{1+\Delta U^2}{p}\right) = 0; \)

\[ \text{ord } p(\det(1 + U \sqrt{-\Delta})) \geq 2 \iff 1 + \Delta U^2 \equiv 0 \mod p^2 \]

This means that if \( \text{ord } p(\det(1 + U \sqrt{-\Delta})) = 0 \) then \( h_{ij} = k_{ij} = 0 \mod p^2 \); together with (4.2.18.0) this means that \( \text{ord } p(\det \sigma) \geq 5 \), i.e. \( T_{\sigma_1}(V)_{I(\sigma)} \neq V \).

If \( \text{ord } p(\det(1 + U \sqrt{-\Delta})) = 1 \) then (4.2.18.0) — (4.2.18.2) imply that \( h_{ij} = ph_{ij}', k_{ij} = pk_{ij}' \) with integer \( h_{ij}', k_{ij}' \) and that \( h_{ij}', k_{ij}' \) satisfy

\[-h_{12}' U + k_{12}' \equiv 0 \mod p \\
\Delta k_{12}' U + h_{12}' \equiv 0 \mod p \]

\[-h_{22}' U + k_{22}' \equiv 0 \mod p \\
\Delta k_{22}' U + h_{22}' \equiv 0 \mod p \]
These conditions mean that \((h'_{12} + \sqrt{-\Delta} h'_{12})(1 - \sqrt{-\Delta} U) = 0\), i.e. the second and the third columns of \(\frac{1}{p}g\) are linearly dependent. This means that \(\text{ord}_p(\det g) > 3\), i.e. \(T_{p,1}(V)_{I(\sigma)} \neq V\).

Finally, if \(\text{ord}_p(\det(1 + U\sqrt{-\Delta})) \geq 2\) then we can take \(g_{11} = p, g_{12} = g_{21} = 0, g_{22} = h_{22}(1 + U\sqrt{-\Delta})\), and the condition \(\lambda(g) = p^2\) becomes \(\frac{1 + \Delta U}{p} = h_{22}^{-1}h_{22}^{-1}t\). This equation can be solved for any \(U\) such that the corresponding \(\sigma\) is in \(D_{0,*}\).

**Theorem 4.2.19.** For \(g = 3, i = 1\) the field of definition of any intermediate irreducible component of \(T_{p,1}(V)\) is \(K^p\).

**Proof** is similar to the proof of the theorem 3.3.6. In the present case \(\sigma \in S(g,1)^{\text{open}}\) is given by (4.1.1). It satisfies the analog of (2.13), and we can take \(r = r_{\tau,\sigma}t^{-1}\). The \(p\)-component of \(K_p\) is the set of matrices \(g \in GSp_{2g}(\mathbb{Z}_p)\) having the block structure \(\gamma_{ij}, i, j = 1, \ldots, 4\) such that

\[
\gamma_{12}, \gamma_{32}, \gamma_{41}, \gamma_{43} \equiv 0 \mod p, \quad \gamma_{42} \equiv 0 \mod p^2 \tag{4.2.19.1}
\]

Analogs of (3.3.6.1) - (3.3.6.3) hold for the present situation, and (3.3.6.4) is changed (according to (4.5.2.1)) as follows:

\[
(4.2.19.2) \ g \ \text{belongs to the } p\text{-component of } K_{V, p, r} \ \text{iff} \ \ [r\alpha(g)r^{-1}]_{12}, [r\alpha(g)r^{-1}]_{32}, [r\alpha(g)r^{-1}]_{41}, [r\alpha(g)r^{-1}]_{43} \ \text{are } \equiv 0 \ \text{mod } p,
\]

and \([r\alpha(g)r^{-1}]_{42}\) is \(\equiv 0 \mod p^2\).

To formulate an analog of (3.3.6.5), we denote \(B_i = B_i(\sigma,\sigma) \ (i = 1, 2)\) like in (4.2.2). The analog of (3.3.6.5) is the following:

\[
\text{im } B_1 = \begin{pmatrix}
* & [r\alpha(g)r^{-1}]_{41} \\
[r\alpha(g)r^{-1}]_{32} & [r\alpha(g)r^{-1}]_{42}
\end{pmatrix}, \quad \text{im } B_2 = \begin{pmatrix}
* & [r\alpha(g)r^{-1}]_{43} \\
[r\alpha(g)r^{-1}]_{12} & [r\alpha(g)r^{-1}]_{42}
\end{pmatrix} \tag{4.2.19.3}
\]

According Remark 4.1.4, we can restrict ourselves by the case when \(\det \mu_1(\sigma) \neq 0\). Let us take any \(g \in K_{V, p, r}\). Since \(X_1\) is real, the reduction of (4.2.15.4) implies that \((\text{im } \tilde{X}_2)\tilde{B}_{22} = 0\), and lemma 4.1.10 implies that \(\det \tilde{B}_{22} = 0\). (4.2.19.2), (4.2.19.3) imply that \(\tilde{B}_1 = \begin{pmatrix}
* + \sqrt{-\Delta} * \\
* & *
\end{pmatrix}\)

(entries of *’s are in \(F_p\)). Since sizes of blocks are \((1, 2)\), this means that \(\text{im } \tilde{B}_1 = 0\), i.e. \(\det B_1 \in \mathbb{Z}_p + p\sqrt{-\Delta}\mathbb{Z}_p\), and hence \(\det g \in \mathbb{Z}_p + p\sqrt{-\Delta}\mathbb{Z}_p\).

For any \(\sigma \in D_0\), any \(\alpha \in \mathbb{Z}_p + p\sqrt{-\Delta}\mathbb{Z}_p\) it is easy to construct examples of matrices \(g \in K_{V, p, r}\) such that \(\det g = \alpha\). This means that the field of definition of this component is \(K^p\). □

To formulate the conjecture on the restriction of I-partition on \(D_0\) and on the Galois action on the corresponding set of irreducible components, we must recall that the projectivization of the set of 2×2-matrices of rank 1 is \(P^1 \times P^1\). Recall that for \(\sigma \in D_0\) the matrix \(\mathcal{R}(\sigma)\) (defined in 4.2.13) is of rank 1. Let \(P(\mathcal{R}(\sigma)) \in P^1(F_p) \times P^1(F_p)\) be its projectivization. So, there exists a map \(D_0 \rightarrow P^1(F_p) \times P^1(F_p)\) defined on \(\sigma \in D_0\) as follows: \(\sigma \mapsto P(\mathcal{R}(\sigma))\). Let \(\pi_i : D_0 \rightarrow P^1(F_p) \ (i = 1, 2)\) be the composition of this map with the projection of \(P^1(F_p) \times P^1(F_p)\) to the \(i\)-th
factor. Let $\sigma \in \mathcal{D}_0$. We denote the restriction of $\pi_i$ to $S(3,1)_{D(\sigma)}$ by $\pi_{\sigma,i}$ (we use notations of (1.4.1), (1.4.2)).

**Conjecture 4.2.20.** There exists $i = 1$ or $2$ such that parts of I-partition restricted to $S(3,1)_{D(\sigma)}$ coincide with the fibers of $\pi_{\sigma,i}$. Particularly, $D(\sigma)_I$ — the set of these parts — is indexed by elements of $P^1(F_p)$, and the action of Galois group $\text{Gal}(K^p/K^1) = F^*_p/\overline{F}^*_p$ on $D(\sigma)_I$ coincides with the natural action of $\text{Gal}(K^p/K^1)$ on $P^1(F_p)$.

I do not know whether $i$ is $1$ or $2$.

**Remark 4.2.21.** For $g = 3$, $i = 1$ there exists a constant $c$ (which does not depend on $p$) and an extension $L/K^1$ of degree $c$ such that the field of definition of any special component of $T_{p,1}(V)$ is a subfield of $L$.

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**Idea of the proof.** We use notations of Theorem 4.2.19. We consider only one $\sigma$ given by (4.1.1) with $A, C = 0$, $U = \begin{pmatrix} u_2 & u_3 \\ u_3 & -u_2 \end{pmatrix}$ where $u_2$, $u_3$ satisfy $\Delta(u_2^2 + u_3^2) \equiv -1 \mod p$. We denote by $\nu$ the coefficient of proportionality between the second and the third lines of $\bar{\mu}_1(\sigma)$. Obviously $\nu$ is not real, hence (4.2.2), (4.2.19.2), (4.2.19.3) imply that

$$\tilde{B}_1 = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4.2.21.1)

where $* \in F^*_p$. Moreover, the expression for $\mu_2$ shows that (4.2.21.1) is equivalent to (4.2.8) for $B_2$.

A direct calculation shows that (4.2.21.1) is equivalent to the following relations between entries $g_{ij}$ of $g$:

$$g_{22} - g_{33} - \nu g_{23} - \bar{\nu} g_{32} = 0$$

(4.2.21.2)

$$-\nu g_{12} + g_{13} = 0$$

(4.2.21.3)

$$\nu g_{21} + g_{31} = 0$$

(4.2.21.4)

Even diagonal matrices $g$ satisfying (4.2.21.2)–(4.2.21.4) can have (almost) any determinant in $F^*_p$ after reduction; really, their determinants form a subgroup of index $\leq 3$ in $F^*_p$. This implies the desired. □

**Section 5. Miscellaneous.**

**Subsection 5.1. Case $g = 3$, $i = 2$.**

We use notations of Section 4.1 (particularly of 4.1.1), so $E$ will mean $E_{12}$. Let us describe $\mathcal{D}_*$. We denote $A = (a_1 a_2)$, $C = (c_1 c_2)$ (the $1 \times 2$-matrices), $U = (u)$ (the $1 \times 1$-matrix). Further, if $p \equiv 3 \mod 4$ then we denote $1/\sqrt{\Delta}$ by $r$ (we have $r \in F_p$) and $r\sqrt{-\Delta}$ by $i$ (we have $i \in F^*_p$ is purely imaginary and $i^2 = -1$).
Theorem 5.1.1. $\mathcal{D}_*$ is non-empty iff $p \equiv 3 \mod 4$. In this case $\mathcal{D}_* \cap S(3,2)^{\text{open}}$ is the set of matrices of form 4.1.1 such that

$$a_1^2 + a_2^2 \equiv 1, \ U \equiv 0, \ C \equiv (\varepsilon ra_2, -\varepsilon ra_1) \mod p$$

(5.1.1.1)

where $\varepsilon = \pm 1$.

Proof. Direct calculations. Conditions $\det \mu_1(\tilde{\sigma}) = \det \mu_2(\tilde{\sigma}) = 0$ are the following (equalities in $\mathbb{F}_p$):

$$u + a_1 c_1 + a_2 c_2 = 0$$

$$u - a_1 c_1 - a_2 c_2 = 0$$

$$a_1^2 + a_2^2 = 1$$

$$c_1^2 + c_2^2 = 1/\Delta$$

Solving this system we get the desired. □

Remark. I am very astonished by this result, because this is the only case (in [L01] and in the present paper) when something depends on residue of $p$ modulo 4.

Theorem 5.1.2. There exist $\sigma_1 \in \mathcal{D}_1, \sigma_2 \in \mathcal{D}_*$ which are not D-equivalent.

Remark. Conjecturally, any $\sigma_1 \in \mathcal{D}_1, \sigma_2 \in \mathcal{D}_*$ are not D-equivalent.

Proof. We choose $\sigma_1, \sigma_2$ having respectively (in notations of 4.1.1)

$$A = (1\ 0), \ U = 0, \ C = (0\ 0)$$

(5.1.2.1)

$$A = (1\ 0), \ U = 0, \ C = (0\ r)$$

(5.1.2.2)

(i.e. for $\sigma_2$ we have $a_1 = 1, a_2 = 0, \varepsilon = -1$ of 5.1.1.1). We shall prove that the condition 4.2.6 is not satisfied for these $\sigma_i$. We shall make all calculations in $\mathbb{F}_p$ without indicating tilde, i.e. all elements are reduced. We have

$$\mu_{12} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -i \\ 1 & i & -1 \end{pmatrix},$$

$$\mu_{22} = \begin{pmatrix} \sqrt{-\Delta} & 0 & -\sqrt{-\Delta} \\ 0 & \sqrt{-\Delta} & 0 \\ 1 & i & -1 \end{pmatrix}.$$  

We denote the third line of $\mu_{11}$ by $l$. Conditions (4.2.8) for $B_1$ have the following form:

5.1.2.3. $\mu_{11} E \tilde{l}$ is real (condition of reality of $(B_1)_{s3}$);

5.1.2.4. $(\mu_{11} E)_{31}$ is real (condition of reality of $(B_1)_{31}$).

Reality of $(B_1)_{32}$ follows from these conditions, because the second line of $\mu_{12}$ is a lineal combination with real coefficients of its first and third lines.

Conditions (4.2.8) for $B_2$ have the following form:
5.1.2.5. \( \mu_2 \vec{g} E \vec{l} \) is real (condition of reality of \((B_2)_{33}\));

5.1.2.6. \((\mu_2 \vec{g} E)_{32} \) is imaginary (condition of reality of \((B_2)_{32}\));

5.1.2.7. \(-r(B_2)_{32} + (B_2)_{33} = 0\) (because the first line of \(\mu_{22}\) is the linear combination of its second and third lines with coefficients \(-i, \sqrt{-\Delta}\)).

Since

\[
\mu_{21} = \begin{pmatrix} X_{11} & X_{21} \\ 0 & 1 \end{pmatrix} \mu_{11}
\] (5.1.2.8)

we see that (5.1.2.3), (5.1.2.5) are equivalent to the condition

5.1.2.9. \( \mu_1 \vec{g} E \vec{l} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \) where \(*\) is real. So, (5.1.2.7) implies

\((B_2)_{32} = 0 \implies (\mu_2 \vec{g} E)_{32} = 0 \implies (\mu_1 \vec{g} E)_{32} = 0\) (because of 5.1.2.8).

We see that 5.1.2.7, 5.1.2.9 are equivalent to the condition

\[
\mu_{11} \vec{g} = \begin{pmatrix} v_{11} & v_{12} & v_{11} - iv_{12} + z_1 \\ v_{21} & v_{22} & v_{21} - iv_{22} \\ z_2 & 0 & z_2 \end{pmatrix}
\] (5.1.2.10)

where \(v_{ij} \in \mathbb{F}_p^2\), \(z_i \in \mathbb{F}_p\).

Now we take \( Z = \begin{pmatrix} \bar{t} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \), so \( \vec{g} \overset{\text{def}}{=} \mu_{11} \vec{g} Z = \begin{pmatrix} z_1 & v_{12} & v_{11} \\ 0 & v_{22} & v_{21} \\ 0 & 0 & z_2 \end{pmatrix} \). In terms of \( \vec{g}^{-1} \), the condition (3.1.1) has the form

\[
Z^{-1} E \bar{Z} t^{-1} = \lambda \vec{g}^{-1} G_1 \vec{g} t^{-1}
\] (5.1.2.11)

\((G_1 = \mu_{11} E \bar{\mu}_{11}^t \text{ from 4.1}). \) We have: \( Z^{-1} E \bar{Z} t^{-1} = \begin{pmatrix} -1 & -i & -1 \\ i & 0 & i \\ -1 & -i & 0 \end{pmatrix} \) and \( \vec{g}^{-1} \)

has entries similar to the entries of \( \vec{g} \). Writing explicitly the (2,2)-entry of the right hand side of (5.1.2.11), we get a contradiction. \(\square\)

**Proposition 5.1.3.** The field of definition of the intermediate component that corresponds to \(\sigma\) given by (5.1.2.1) is \(K\).

**Proof.** We use notations of Theorem 4.2.19. We denote the \((i, j)\)-th element (not block!) of \(B_1\) by \(b_{ij}\), \(1 \leq i, j \leq 3\). We get that the left hand sides of (4.2.15.4), (4.2.15.5) are \( \begin{pmatrix} b_{33} \\ -b_{23} \end{pmatrix} \), \( \begin{pmatrix} \bar{b}_{33} \\ -\bar{b}_{23} \end{pmatrix} \), hence the condition that \(B_2\) satisfies (4.2.4) is

\[
\bar{B}_1 = \begin{pmatrix} * & * & b_{13} \\ * & * & 0 \\ b_{31} & 0 \end{pmatrix}
\] where \(b_{13}, b_{31} \in \mathbb{F}_p\) and \(* \in \mathbb{F}_p^2\). Writing down explicitly condition (3.1.1)

we get immediately that \(\det B_1\) — and hence \(\det \vec{g}\) — can take any value. So, the field of definition of the corresponding component is \(K\). \(\square\)

**Remark 5.1.4.** Conjecturally, all special components of \(T_{p,2}(V)\) are defined over some field \(L\) of Remark 4.2.21.
Subsection 5.2. Action of $T_p$ on components of $T_p(V)$.

We use notations of (1.4.1). Let $\sigma \in S(g)$ be an element. For $i = 0, \ldots, g$ we denote

$$S(g, i, \sigma) = \{\sigma' \in S(g) | \sigma' \sigma \in \Gamma_{T_p,i}\Gamma\}$$

Therefore $S(g) = \bigcup_{i=0}^{g} S(g, i, \sigma)$ (all unions in this subsection are disjoint), and for $k \in S_D(g, i)$ we denote

$$S(g, i, \sigma)_k = \{\sigma' \in S(g, i, \sigma) | \sigma' \sigma = \bigcup_{\sigma_j \in S(g, i)_k} \Gamma\sigma_j\}.$$ 

It is easy to check that if $\sigma_1, \sigma_2 \in S(g)$ are D-equivalent then generally $S(g, i, \sigma_1)_k \neq S(g, i, \sigma_2)_k$, but

$$S(g, i, \sigma_1)_k \neq \emptyset \iff S(g, i, \sigma_2)_k \neq \emptyset \quad (5.2.1)$$

Let us fix some $t \in S_D(g)$ (i.e. a class of D-equivalence in $S(g) = a$ Shimura subvariety $T_p(V)_t$ of $T_p(V)$). The answer on the problem of description of $T_p(T_p(V)_t)$ is given by the following two propositions:

**Proposition 5.2.2.** Let $i \in \{0, \ldots, g\}$, $k \in S_D(g, i)$. Then $T_p(V)_k$ is a component of $T_p(T_p(V)_t)$ iff for $\sigma \in S(g)_t$ we have: $S(g, i, \sigma)_k \neq \emptyset$ (according to (5.2.1), this condition does not depend on a choice of $\sigma$). $\square$

Now let us find multiplicities. We have a formula $T_p^2 = \sum_{i=0}^{g} W_p(i) T_p,i$ where $W_p(i) = \prod_{j=1}^{i} (p^j + 1)$. Let us fix $\sigma'' \in S(g, i)_k$. We can represent the coset $\Gamma\sigma''$ by $W_p(i)$ ways as the following product:

$$\Gamma\sigma'' = \Gamma\sigma_j^s \sigma_j$$

where $\sigma_j^s, \sigma_j \in S(g)$, and $j = 1, \ldots, W_p(i)$ is the number of the way of representation as the product.

**Proposition 5.2.4.** The multiplicity of $T_p(V)_k$ in $T_p(T_p(V)_t)$ is the quantity of $j$ in (5.2.3) such that we have $\sigma_j^s \in S(g)_t$. $\square$

Now let us apply propositions 5.2.2, 5.2.4 to some $i, t, k$. We denote 2 good elements of $S_D(3)$ by $k_{g1}, k_{g2}$ and the bad element by $k_b$. Analogously, we denote 2 general elements of $S_D(3,1)$ by $k_{1g1}, k_{1g2}$, the intermediate elements by $k_{1ia}$ ($\alpha$ runs over the set of components of $T_p(V)$ of intermediate type), the special element corresponding to $D_{0,*}$ by $k_{1s0}$ and other special elements by $k_{1sa}$ ($\alpha$ runs over the set of components of $T_p(1)(V)$ of special type).

5.2.5. Case $i = 1$, $t = k_{g1}$. We take $\sigma = \begin{pmatrix} E_3 & 0 \\ 0 & pE_3 \end{pmatrix} \in S(3)_{k_{g1}}$, and let $\sigma' \in S(3)$ be a generic element such that $\sigma' \sigma \in S(3, 1)$. This $\sigma'$ corresponds to the case $I = \{2, 3\}$, where $I$ is from [L04.1], Section 2.3. We take $\sigma'$ from the proof of the Proposition 3.4.3, Case $I = \{2, 3\}$, $D = (d_1 \ d_2)$. In these notations $\sigma' \sigma$ is a matrix described in (4.1.1), with $A = D^t$, $C = 0$, and $U$ of (4.1.1) is $\equiv 0 \mod p$. This means that $\mu_1(\sigma' \sigma) = \begin{pmatrix} -E_1 & 0 \\ D^t & -E_2 \end{pmatrix}$, $\mu_2(\sigma' \sigma) = \begin{pmatrix} \sqrt{-\Delta}E_1 & -D\sqrt{-\Delta} \\ D^t & -E_2 \end{pmatrix}$.
So, $\mathfrak{R}(\sigma'\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 - d_1^2 - d_2^2 \end{pmatrix}$ ($\mathfrak{R}$ is defined in 4.2.13). This means that $\sigma'\sigma$ never belongs to $D_*$, and belongs to $D_0$ iff $d_1^2 + d_2^2 = 1$. So, we have got a 

**Proposition 5.2.6.** $T_p(T_p(V)_{k_{g2}})$ (i.e. $T_p$ of a good component of $T_p(V)$) contains components of $T_p(V)$ of general and intermediate types. $\Box$

**Remark 5.2.7.** (a) Consideration of other (non-generic) components of $S(3)$, and of $k_{g2}$, shows that $T_p$ of any good component of $T_p(V)$ does not contain components of $T_p(V)$ of special type.

(b) In terms of the Plücker coordinates on $S(3)$ (see notations of Remark 3.2.5) the condition $d_1^2 + d_2^2 = 1$ is equivalent to the condition $v_{123}(\sigma') - v_{123}(\sigma') - v_{123}(\sigma') = 0$.

5.2.8. Case $i = 1$, $t = k_b$. We take $\sigma$ from the proof of the Proposition 3.4.3 (see the first line of the proof), and $\sigma'$ as in 5.2.5. We have

$$\mathfrak{R}(\sigma'\sigma) = \begin{pmatrix} -\Delta w_1(-d_1^2w_1 + 2d_1w_2 - w_1) & -w_1(1 - d_1^2) \\ \Delta(-d_1^2w_1 + 2d_1w_2 - w_1) & 1 - d_1^2 \end{pmatrix} \quad (5.2.9)$$

We have $\det\mathfrak{R}(\sigma'\sigma) = 0$ computer, file logachev19, calculations on pages 21.1.2, 45.1-3]. So, we have got a 

**Proposition 5.2.10.** $T_p$ of the bad component of $T_p(V)$ contains the component of $T_p(V)$ of intermediate type. $\Box$

As earlier we have a 

**Remark 5.2.11.** Consideration of other (non-generic) components of $S(3)$ shows that $T_p$ of the bad component of $T_p(V)$ does not contain components of $T_p(V)$ of general type, and does contain components of special type (although the above matrices (5.2.9) are all non-0).

Now we use Proposition 5.2.4 in order to find multiplicity for the case when $k = k_{111}$ corresponds to a component of intermediate type. We take the simplest value of $\sigma''$ defined by (4.1.1) with $U = 0$, $C = 0$, $A^t = (1 \ 0)$, this $\sigma''$ belongs to $S(3,1)_{k_{111}}$. To describe $\sigma_j$, $\sigma_j'$ ($j = 0, \ldots, p$; $W_p(1) = p + 1$) we need the following matrices:

$$M = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix},$$

$$M_j = \begin{pmatrix} 1 & 0 & 0 & j & j & 0 \\ 0 & 1 & 0 & j & j & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \quad j = 0, \ldots, p - 1.$$

We have $\sigma_j = M$, $\sigma_j' = M_j$, $j = 0, \ldots, p - 1$ and $\sigma_p = M_0$, $\sigma_p' = M$. Using for example (3.2.6), (3.2.7) we get that $M \in S(3)_{k_b}$ and all $M_j \in S(3)_{k_{g1}} \cup S(3)_{k_{g2}}$. So, we get
Proposition 5.2.12. (a) The multiplicity of the simplest intermediate component of $T_{p,1}(V)$ in $T_p(T_p(V)_{k_p})$ is $1$.

(b) Let $m_i$ ($i = 1, 2$) be the multiplicity of the simplest intermediate component of $T_{p,1}(V)$ in $T_p(T_p(V)_{k_p})$. Then $m_i$ is the quantity of $j \in \{0, \ldots, p - 1\}$ such that $M_j \in S(3)_{k_p}$ (i.e. the signature of $\sigma(M_j)$ is of type $i$). Particularly, $m_1 + m_2 = p$.

Remark 5.2.12'. We can expect that are similar results hold for any intermediate component of $T_{p,1}(V)$.

5.2.13. Case $i = 2$, $t = k_{g1}$. We follow 5.2.5. Namely, we take the same $\sigma$ as in 5.2.5, and let $\sigma' \in S(3)$ be a generic element such that $\sigma' \sigma \in S(3, 2)$. This $\sigma'$ corresponds to the case $I = \{3\}$, where $I$ is from [L04.1], Section 2.3. We take $\sigma'$ from the proof of the Proposition 3.4.3. Case $I = \{3\}$, $D^t = (d_{13}, d_{23})$. In these notations $\sigma' \sigma$ is a matrix described in (4.1.1), with $A = D^t$, $C = 0$, and $U$ of (4.1.1) is $\equiv 0 \mod p$.

We have $\mu_1(\sigma's) = \left(\begin{array}{cc} -E_2 & 0 \\ A & -E_1 \end{array}\right)$, $\mu_2(\sigma's) = \left(\begin{array}{cc} \sqrt{-\Delta}E_2 & -A^t\sqrt{-\Delta} \\ A & -E_1 \end{array}\right)$. We get that always $\det \mu_1 \neq 0$, $X_1 = \sqrt{-\Delta} \left(1 + d_{13}^2 d_{23} \quad d_{13}d_{23} \quad 1 + d_{23}^2 \right)$, $\det \im X_1 = 1 + d_{13}^2 + d_{23}^2$.

So, we have got a

Proposition 5.2.14. $T_p(T_p(V)_{k_p})$ (i.e. $T_p$ of a good component of $T_p(V)$) contains components of $T_{p,2}(V)$ of general and intermediate types. □

Remark 5.2.15. Apparently, $T_p$ of any good component of $T_p(V)$ does not contain components of $T_{p,2}(V)$ of special type. To check it carefully, we must consider $\sigma'$ of non-generic type.

5.2.16. Case $i = 2$, $t = k_b$. It is easy to check that $T_p$ of the bad component of $T_p(V)$ contains all components of $T_{p,2}(V)$.

Subsection 5.3. A theorem on a weak “equivalence” of components of $\mathcal{D}_{1,*}$.

Here we prove that in many cases the condition that $\sigma_1$, $\sigma_2$ are $\mathcal{D}$-equivalent implies that $\sigma_1$, $\sigma_2$ satisfy 4.2.6.

Throughout all this subsection all calculations will be made in $\mathbb{F}_p$, i.e. we shall consider reductions of all objects. To simplify notations we shall not indicate tilde.

We use here notations of the beginning of Subsection 4.2.

Proposition 5.3.1. For the case $g = 3$, $i = 1$ we have: any $\sigma_1$, $\sigma_2 \in \mathcal{D}_0 \cap S(3,1)^{open}$ satisfy 4.2.6.

Proof. $\sigma \in \mathcal{D}_0 \iff \{\det \mu_1(\sigma) = 0 \text{ and } \det \mu_2(\sigma) \neq 0\}$ or $\{\det \mu_1(\sigma) \neq 0 \text{ and } \det \im X_1 = 0\}$.

Case 1. Both $\sigma_1$, $\sigma_2$ satisfy $\{\det \mu_1 \neq 0 \text{ and } \im X_1 = 0\}$.

According (4.1.6), $F_{11i} = 0$. (4.2.15.4), (4.2.15.5) and (4.1.6) give us:

$$F_{121}\mathcal{B}_{22} = F_{122}\mathcal{B}_{22}^t = 0 \quad (5.3.1.1)$$

This means that if

$$\mathcal{B}_{22} = kF_{211}F_{122}^O \quad (5.3.1.2)$$
where $k \in \mathbb{F}_p^*$ (recall that all $F, G$ are real symmetric) then $B_2$ satisfy (4.2.8).  
Analogs of (4.2.15.10) - (4.2.15.12) for the present case (taking into consideration (5.3.1.1)) are

\[
\mathfrak{B}_{12} F_{212} \mathfrak{B}_{11}^t + \mathfrak{B}_{11} F_{122} \mathfrak{B}_{12}^t + \mathfrak{B}_{12} F_{222} \mathfrak{B}_{12}^t = \lambda G_{111} \tag{5.3.1.3}
\]

\[
\mathfrak{B}_{12} (F_{212} \mathfrak{B}_{21}^t + F_{222} \mathfrak{B}_{22}^t) = \lambda G_{121} \tag{5.3.1.4}
\]

\[
\mathfrak{B}_{22} F_{222} \mathfrak{B}_{22}^t = \lambda G_{221} \tag{5.3.1.5}
\]

Substituting (5.3.1.2) to (5.3.1.5) we get

\[
k^2 F_{211}^O (F_{122}^O F_{222}^O F_{122}^O) F_{211}^O = \lambda G_{221}
\]

Applying (4.1.11.1) for $F_2$ and (4.1.11.2) for $F_1$ we get

\[
k^2 \frac{\det F_2}{\det F_1} G_{221} = \lambda G_{221}
\]

i.e. if we take any $k \in \mathbb{Z}_p^*$, $\lambda = k^2 \frac{\det F_2}{\det F_1}$ and $\mathfrak{B}_{22}$ from (5.3.1.2) then (5.3.1.5) will be satisfied.

Substituting (5.3.1.2) to (5.3.1.4) we get (taking into consideration (4.1.11.3) for $F_2$)

\[
\mathfrak{B}_{12} Z = \lambda G_{121} \tag{5.3.1.6}
\]

where $Z = F_{212} \mathfrak{B}_{21}^t - k(\det F_2) G_{212} F_{212}^O$.

Since vectors $F_{212}, G_{212}^O$ are never linearly dependent (because of (4.1.11.4)) and $F_{121} \neq 0$, we get that for almost all $\mathfrak{B}_{21} \det Z \neq 0$. We choose any such $\mathfrak{B}_{21}$, so for $\mathfrak{B}_{12} = \lambda G_{121} Z^{-1}$ (5.3.1.4) is satisfied.

Finally, in order to show that (5.3.1.3) has a solution respectively $\mathfrak{B}_{11}$ it is sufficient to check that $\mathfrak{B}_{12} F_{212} \neq 0$. Let us assume the contrary: $\mathfrak{B}_{12} F_{212} = 0$. This means that $\mathfrak{B}_{12} = \alpha F_{122}^O$. Substituting this value to (5.3.1.6) we get $\alpha(k(\det F_2) F_{122}^O G_{212} F_{212}^O = \lambda G_{121}$. This implies that vectors $F_{121}^O, G_{121}$ are linearly dependent. But their determinant is 1 (because of (4.1.11.4)) — a contradiction.

**Case 2.** $\sigma_1$ satisfy $\det \mu_1(\sigma_1) \neq 0$, $\det \mu_2(\sigma_1) = 0$, and $\sigma_2$ satisfy $\det \mu_1(\sigma_2) = 0$, $\det \mu_2(\sigma_2) \neq 0$.

Here we change slightly the definitions of the previous cases. Namely, we set:

$\mu_{21} = \begin{pmatrix} 0 & X_{21} \\ 0 & 1 \end{pmatrix}, \mu_{11}, \mu_{12} = \begin{pmatrix} 0 & Y_{22} \\ 0 & 1 \end{pmatrix}, \mu_{22}, B_0 = \mu_{11} g \bar{\mu}_{22}^t, B_0 = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} \end{pmatrix}.$

Further, let $G_1, F_1$ be as earlier, and $G_2 = F_2^{-1} = \mu_{22} E \bar{\mu}_{22}^t$, with the same partition on blocks as earlier. As earlier we have: $\text{im} X_{21}$ is proportional to $F_{121}$, $\text{im} Y_{22}$ is proportional to $F_{122}$, and all are non-zero.

Further, we have

\[
B_2 = \begin{pmatrix} 0 & X_{21} \\ 0 & 1 \end{pmatrix} B_0, \quad B_1 = B_0 \begin{pmatrix} 0 & 0 \\ \bar{Y}_{22}^t & 1 \end{pmatrix} \tag{5.3.1.7}
\]
(4.2.7) and (5.3.1.7) imply that \( \text{im} X_{21} B_{22} = \text{im} Y_{22} = 0 \). Substituting the expression for \( g \): \( g = \mu_{11}^{-1} B_0 \mu_{22}^{-1} \) in (3.1.1) we get formulas analogous to (5.3.1.3) - (5.3.1.5). The end of the proof is similar to the one of the Case 1.

Proof for the remaining cases when for one or both \( \sigma_1, \sigma_2 \) satisfy \( \{ \det \mu_1 = 0 \) and \( \det \mu_2 \neq 0 \} \) is analogous (we use Remark 4.1.4). □

**Proposition 5.3.2.** For the case \( g = 3, i = 1 \) we have: if \( \sigma_1, \sigma_2 \in D_\ast \cap S(3, 1)^{\text{open}} \) and \( \sigma_2 \) satisfy the condition \( A_2 = C_2 = 0 \) then \( \sigma_1, \sigma_2 \) satisfy 4.2.6.

**Remark.** I think that this is true for any \( \sigma_1, \sigma_2 \in D_\ast \), without the restriction \( A_2 = C_2 = 0 \).

**Proof.** We denote the scalar product of vectors by \( < ., . > \). In order to simplify notations, sometimes we do not distinguish between the row and column vectors.

We denote \( A_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \), \( C_1 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, U_i = \begin{pmatrix} u_{22i} \\ u_{32i} \end{pmatrix} \). We assume the existence of \( g \) satisfying (4.2.6), transform the corresponding formulas and show that these transformations are invertible. We shall use notations \( g_i \) for the \( i \)-th column of \( g \), (4.1.2.5), Lemma 4.1.12, condition \( \sigma_1, \sigma_2 \in D_\ast \cap S(3, 1)^{\text{open}} \) and (4.2.7) imply that

\[
\begin{pmatrix}
-1 & -c_1 \sqrt{-\Delta} & -c_2 \sqrt{-\Delta} \\
-1 & a_1 & -a_2
\end{pmatrix}
\begin{pmatrix}
g_2 | g_3
\end{pmatrix}
\begin{pmatrix}
-1 + u_{222} \sqrt{-\Delta} \\
u_{322} \sqrt{-\Delta}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]  
(5.3.2.1)

\[
(a_1 + c_1 \sqrt{-\Delta} & -1 - u_{221} \sqrt{-\Delta} \quad -u_{231} \sqrt{-\Delta}) g_1 = 0
\]  
(5.3.2.2)

We denote \( M = \begin{pmatrix}
-1 & -c_1 \sqrt{-\Delta} & -c_2 \sqrt{-\Delta} \\
-1 & a_1 & -a_2
\end{pmatrix} \). A direct calculation shows that

\[
\det \mu_{11} = \det \mu_{21} = 0 \iff \det M = 0
\]  
(5.3.2.3)

If \( A = C = 0 \) then the same arguments as in the proof of 4.2.18 show us that the proposition holds. If \( A \neq 0 \) or \( C \neq 0 \) then the second and the third lines of \( M \) are linearly independent and in this case (5.3.2.3) means that the first line of \( M \) is a linear combination of its second and third line.

Let us find a solution to the above equations. We denote the vector product of the lines of

\[
\begin{pmatrix}
-1 & -c_1 \sqrt{-\Delta} & -c_2 \sqrt{-\Delta} \\
-1 & a_1 & -a_2
\end{pmatrix}
\]  

by \( \mathfrak{A} \), \( \mathfrak{A} = (c_1 a_2 - c_2 a_1) \sqrt{-\Delta} - a_2 - c_2 \sqrt{-\Delta}, a_1 + c_1 \sqrt{-\Delta} \). (5.3.2.1) implies that \( (g_2 | g_3) \begin{pmatrix}
-1 + u_{222} \sqrt{-\Delta} \\
u_{322} \sqrt{-\Delta}
\end{pmatrix} \) is the \( \eta' \mathfrak{A} \) for some coefficient \( \eta' \). We have:

\[
(-1 + u_{222} \sqrt{-\Delta}) g_2 + u_{322} \sqrt{-\Delta} g_3 = \eta' \mathfrak{A}, \text{ or}
\]

\[
g_3 = \eta \mathfrak{A} + \gamma g_2
\]  
(5.3.2.4)

where \( \gamma = \frac{1 - u_{222} \sqrt{-\Delta}}{u_{322} \sqrt{-\Delta}} \left( \frac{-\Delta}{p} \right) = -1 \) implies \( u_{232} \neq 0 \) and \( \eta \) is a coefficient.

Conditions \( < E_{21} g_2, \mathfrak{g}_1 > = 0, < E_{21} g_3, \mathfrak{g}_1 > = 0 \) and (5.3.2.4) imply that \(< E_{21} \mathfrak{A}, \mathfrak{g}_1 > = 0 \). Condition \( < E_{21} g_3, \mathfrak{g}_2 > = 0 \) and (5.3.2.4) imply that
Further, Lemma 4.1.12 implies that \( < E_{21} \mathfrak{A}, \mathfrak{g}_2 > = 0 \). Condition \( \sigma_2 \in \mathfrak{D}_* \) implies that \( \gamma \gamma = -1 \). So, we get that (5.3.2.4), (5.3.2.5) imply that always \( < E_{21} \mathfrak{g}_2, \mathfrak{g}_2 > = < E_{21} \mathfrak{g}_3, \mathfrak{g}_3 > \). Very well. Further, \( \mathfrak{g}_1 \) must be orthogonal to both vectors \( a_1 + c_1 \sqrt{-\Delta}, -1 - u_{221} \sqrt{-\Delta}, -u_{231} \sqrt{-\Delta} \) (condition (5.3.2.2)) and \( \mathfrak{A} \). But again Lemma 4.1.12 implies that they are proportional.

All above considerations are invertible, so in order to find \( \mathfrak{g} \) it is enough to choose a vector \( \mathfrak{g}_1 \) orthogonal to \( \mathfrak{A} \) such that \( < E_{21} \mathfrak{g}_1, \mathfrak{g}_1 > = 1 \), and a vector \( \mathfrak{g}_2 \) orthogonal to \( E_{21} \mathfrak{g}_1 \) such that \( < E_{21} \mathfrak{g}_2, \mathfrak{g}_2 > = 1 \) and such that \( < E_{21} \mathfrak{a}, \mathfrak{g}_2 > \neq 0 \). It is clear that it is possible. Finally, we take \( \mathfrak{g}_3 \) from 5.3.2.4, and all conditions are satisfied. □

**Theorem 5.3.3.** For the case \( g = 3, i = 2 \) we have: there exists \( \sigma_1 \in \mathfrak{D}_* \cap S(3, 2)^{open} \) such that for any \( \sigma_2 \in \mathfrak{D}_* \cap S(3, 2)^{open} \) we have: \( \sigma_1, \sigma_2 \) satisfy 4.2.6.

**Proof.** We use the notations of the beginning of 5.1, we take \( \sigma_1 \) given by (5.1.2.2), (in 5.1.2 this element is denoted by \( \sigma_2 \)), and \( \sigma_2 \) is given by (5.1.1.1) with any \( a_1, a_2, \varepsilon \). Formula for all \( \mathfrak{g} \) satisfying (4.2.8) is the following:

\[
\mathfrak{g} = k \begin{pmatrix} \omega + \frac{1}{2} (\varepsilon a_2 - \rho) & \frac{1}{2} (a_2 + \varepsilon \rho) + i (\omega - a_1) & 0 \\ \frac{1}{2} (\varepsilon a_2 + \rho) + i (\omega - a_1) & -\varepsilon \omega + \frac{1}{2} (-a_2 + \varepsilon \rho) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

(5.3.3.1)

where \( k, \omega \) are parameters in \( \mathbb{F}_p^* \) such that \( \rho = \sqrt{1 - (a_1 - 2 \omega)^2} \in \mathbb{F}_p^* \) (clearly this condition can be satisfied for many values of \( \omega \)). □

**Remark 5.3.4.** Method of finding of the formula (5.3.3.1).

We denote by \( l_i \) the third line of \( \mu_{11} \) (\( i = 1, 2 \)). We can treat the free 3-dimensional module \( \mathbb{F}_p^3 \), as the free 6-dimensional module \( \mathbb{F}_p^6 \), and the imaginary part of the Hermitian form \( H(v_1, v_2) \overset{\text{def}}{=} v_1 \bar{v}_2^t \) as a bilinear form on \( \mathbb{F}_p^6 \). More exactly, there exists an isomorphism \( \iota : \mathbb{F}_p^3 \rightarrow \mathbb{F}_p^6 \) such that

\[
\text{im } H(v_1, v_2) = < \iota(v_1), \iota(v_2) >
\]  

(5.3.4.1)

where \( < ., . > \) is the ordinary scalar product on \( \mathbb{F}_p^6 \) (finding of formulas for \( \iota \) is an elementary exercise).

Conditions (4.2.8) are corollaries of the following conditions:

\[
\text{im } H(l_1 \mathfrak{g} E, m_i') = 0, \quad \text{im } H(m_i, l_2 \bar{E}^t) = 0
\]  

(5.3.4.2)

where \( m_i \) (resp. \( m_i' \), \( i = 1, \ldots, 5 \), are 5 non-equal lines of matrices \( \mu_{11}, \mu_{21} \) (resp. \( \mu_{12}, \mu_{22} \)). (5.3.4.1), (5.3.4.2) imply that \( \iota(l_1 \mathfrak{g} E_{21}) = k' v', \iota(l_2 \bar{E}^t_{21}) = k v \), where \( v, v' \) are 6-vectors whose coordinates are 5-minors of matrices formed by \( \iota(m_i), \iota(m_i') \) respectively, \( i = 1, \ldots, 5 \), and \( k, k' \in \mathbb{F}_p^* \). An easy calculation shows that \( v = \iota(l_1), v' = \iota(l_2) \), hence (4.2.8) is a corollary of the conditions

\[
l_1 \mathfrak{g} E_{21} = k' l_2
\]  

(5.3.4.3)
\[ gE_2l_2^t = kl_1^t \]  

Further, multiplying (5.3.4.3) by \( \bar{l}_2 \) from the right and (5.3.4.4) by \( \bar{l}_1 \) from the left, taking into consideration that \( H(l_1, l_1) = H(l_2, l_2) = 3 \) (this follows from the explicit formulas for them, see below), we get that if \( p > 3 \) then \( k = k' \).

Now we represent \( gE_2 \) in the block form: \( g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \) with sizes of diagonal blocks 2,1. We denote \( l_i = (l_{i1}, -1) \) (\( i = 1, 2 \)), where \( l_{i1} \) is a 2-vector formed by 2 first coordinates of \( l_i \), and \(-1\) is the third coordinate of both \( l_1, l_2 \). In these notations formulas (5.3.4.3), (5.3.4.4) are equivalent to the following expressions for \( g_{ij} \) in terms of \( g_{11} \):

\[
\begin{align*}
    g_{12} &= -kl_{11} + g_{11}l_{21}^t \\
    g_{21} &= -kl_{21} + l_{11}g_{11} \\
    g_{22} &= -k + l_{11}g_{11}l_{21}^t
\end{align*}
\]

Now we substitute these formulas in (3.1.1). We get:

\[
\begin{align*}
    g_{11}A^t_{11}g_{11}^t + g_{11}B + \bar{B}^t\bar{g}_{11}^t = C \quad &\text{(5.3.4.5)} \\
    (g_{11}A^t_{11} + B^t\bar{g}_{11}^t - k^2E_2)\bar{l}_{11}^t = 0 \quad &\text{(5.3.4.6)} \\
    l_{11}g_{11}A^t_{11}l_{11}^t = -\lambda - k^2 \quad &\text{(5.3.4.7)}
\end{align*}
\]

where \( A = E_2 - l_{21}^t l_{21}, B = kl_{21}l_{11}, C = \lambda E_2 - \bar{l}_{11}^t l_{11} = \begin{pmatrix} \lambda + k^2 & ik^2 \\ -ik^2 & \lambda + k^2 \end{pmatrix} \).

Now we make a change of variables in order to eliminate the linear terms in (5.3.4.5): \( g_{11} = \gamma - \bar{B}^tA^{-1} \), and we substitute the square part of (5.3.4.5) to (5.3.4.6), (5.3.4.7). We get:

\[
\begin{align*}
    \gamma A^t \gamma^t &= C + \bar{B}^tA^{-1}B \\
    [(\lambda + k^2)E_2 - \gamma B + \bar{B}^tA^{-1}B]\bar{l}_{11}^t &= 0 \\
    -l_{11}\gamma B\bar{l}_{11}^t - l_{11}\bar{B}^t\gamma\bar{l}_{11}^t + l_{11}(C + 2\bar{B}^tA^{-1}B)\bar{l}_{11}^t + (\lambda + k^2) &= 0
\end{align*}
\]

Substituting values of all objects in the above equations, we get that (5.3.4.6), (5.3.4.7) become respectively

\[
\begin{align*}
    -2k\gamma l_{21}^t + (\lambda - 3k^2)l_{11}^t &= 0 \quad &\text{(5.3.4.8)} \\
    -2kl_{11}\gamma l_{21}^t - 2kl_{21}\gamma l_{11}^t + 3\lambda - 11k^2 &= 0 \quad &\text{(5.3.4.9)}
\end{align*}
\]

Substituting the expression for \( \gamma l_{21}^t \) from (5.3.4.8) to (5.3.4.9) we get that (5.3.4.9) becomes \( \lambda = k^2 \) and (5.3.4.8) becomes
\[ \gamma_1^2 + k \gamma_2 = 0 \] (5.3.4.10)

Now we write \( \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix} \) and \( \gamma_j = x_j + iy_j \). (5.3.4.10) gives us expressions of \( \gamma_j \) in terms of \( x_j, y_j \). Substituting in (3.1.1) we get a system of equations with unknowns \( x_j, y_j \). Formula (5.3.3.1) comes directly from these solutions.

**Conjecture 5.3.5.** For the case \( g = 3, i = 2 \) we have: any \( \sigma_1, \sigma_2 \in \mathcal{D}_1 \cap S(3, 2)^{open} \) satisfy 4.2.6.

**Idea of the proof.** We use notations of Subsection 4.1. According Remark 4.1.4, we consider only the case \( \det m_{11} \neq 0, \det m_{12} \neq 0 \). We have \( G_{22i} = 0 \) (Corollary 4.1.7) and hence
\[ \text{im } X_{1i}G_{12i} = 0 \] (5.3.5.1)
(from 4.1.6.6). Further, \( F_{11i}G_{12i} = 0 \) (definition of \( F, G \)). Further,
\[ \mathcal{B}_{22} = 0 \] (5.3.5.2)
— like in the Proposition 4.2.16, case \( i = 2 \), hence (4.2.15.5) implies that
\[ \text{im } X_{12} \mathcal{B}_{21}^t = 0 \] (5.3.5.3)

**Lemma 5.3.5.4.** \( \text{im } X_{1i} \neq 0 \) (recall that all equalities are in \( \mathbb{F}_p \)).

**Proof.** If not, then eliminating \( \text{re } X_{2i} \) from (4.1.6.1i), (4.1.6.2r), we get
\[ \text{im } X_{2i}(\Delta B_iC_i + A_i) = 1 \] — a contradiction, because \( \text{im } X_{2i}(\Delta B_iC_i + A_i) \) is a rank 1 matrix. \( \square \)

So, (5.3.5.1) and (5.3.5.3) imply that
\[ \mathcal{B}_{21} = kG_{212} \] (5.3.5.5)

Analogs of (5.3.1.3) - (5.3.1.5) for the present case (taking into consideration (5.3.5.2)) are

\[ \mathcal{B}_{11} F_{112} \mathcal{B}_{11}^t + \mathcal{B}_{12} F_{212} \mathcal{B}_{11}^t + \mathcal{B}_{11} F_{122} \mathcal{B}_{12}^t + \mathcal{B}_{12} F_{222} \mathcal{B}_{12}^t = \lambda G_{111} \] (5.3.5.6)

\[ \mathcal{B}_{21} F_{112} \mathcal{B}_{11}^t + \mathcal{B}_{21} F_{122} \mathcal{B}_{12}^t = \lambda G_{211} \] (5.3.5.7)

\[ \mathcal{B}_{21} F_{112} \mathcal{B}_{21}^t = \lambda G_{221} \] (5.3.5.8)

Since \( G_{21i}, F_{11i} = 0 \) and \( G_{21i}, F_{12i} = 1 \) (definition of \( F, G \)), we get from (5.3.5.5) that (5.3.5.8) is always satisfied and (5.3.5.7) becomes \( \mathcal{B}_{12} = \frac{1}{k} G_{121} \). So, we must only to substitute these values in (5.3.5.6) and to find \( \mathcal{B}_{11} \). I think that this is always possible.

**Subsection 5.4.** Non-coincidence of components of \( T_\ast(V) \) for different types of Hecke correspondences.

Arguments like in 3.3.5 show that components of \( T_p(V) \) do not coincide with components of \( T_{p,i}(V) \) for any \( i \).
For the case $T_p = T_{p,0}$ the theory is similar to the one of Section 3. Analog of (3.2.1) is $\sigma = \sigma(s) = \left( \begin{array}{cc} E_g & s \\ 0 & p^2E_g \end{array} \right)$ where $s \in M_g(\mathbb{Z})^{symm}$ and its entries belong to a fixed system of residues modulo $p^2$, $T = T(\sigma)$ is defined like in (3.2.2).

Components of $T_{p,0}(V)$ are defined over $K^{p^2}, K^p$ and $K^1$. For $g = 3$ the component $T_{p,0}(V)_I(\sigma)$ corresponding to $\sigma$ is defined over $K^1$ iff $\det T \equiv 0 \mod p^2$. We call it the special component.

**Proposition 5.4.1.** For $g = 3$ the special component of $T_{p,0}(V)$ does not coincide with $V$ itself.

**Proof.** We use 3.3.5.1. Really, we shall prove that even the condition

$$G_V(Q_p) \sigma^{-1} \cap G(Q_p) \neq \emptyset$$

is not satisfied. This condition is equivalent to the following one: there exists $g = h + k\sqrt{-\Delta} \in G_V(Q_p), h, k \in M_3(\mathbb{Z}_p)$, such that

$$\alpha(g)\sigma^{-1} \text{ has integer entries}$$

and $\text{ord}_p(\det(g)) = 3$.

We have: $\alpha(g)\sigma^{-1} = \left( \begin{array}{cc} h & p^{-2}(-hs + kE_{21}) \\ -\Delta E_{21}k & p^{-2}(-\Delta E_{21}ks + E_{21}hE_{21}) \end{array} \right)$ hence (5.4.1.1) is equivalent to the following congruences:

$$k \equiv hsE_{21} \mod p^2 \quad (5.4.1.3)$$

$$\Delta ks + hE_{21} \equiv 0 \mod p^2 \quad (5.4.1.4)$$

Substituting $k$ from (5.4.1.3) to (5.4.1.4) we get:

$$hT \equiv 0 \mod p^2 \quad (5.4.1.5)$$

(5.4.1.6) To simplify calculations, we consider a particular value of $\sigma$, namely $\sigma = \sigma(s)$, where $s = W$ is from the beginning of the proof of 3.4.3, $w_1, w_2 \in \mathbb{Z}$ satisfy $w_1^2 - w_2^2 \equiv \frac{-1}{\Delta} \mod p^4$. We have $T \equiv \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \mod p^4$, hence (5.4.1.5) is equivalent to

$$h \equiv \left( \begin{array}{ccc} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{array} \right) \mod p^2 \quad (5.4.1.7)$$

According (5.4.1.3), $k$ must satisfy $k = hsE_{21} + p^2A$ with $A \in M_3(\mathbb{Z}_p)$. Condition (3.1.1) is the following:

$$hTh^t + p^2\{\sqrt{-\Delta}(-hE_{21}A^t + AE_{21}h^t) + \Delta(Ah^t + hsA^t)\} + p^4\Delta AE_{21}A = \lambda E_{21}$$

(5.4.1.8) implies $hTh^t \equiv 0 \mod p^4$. Dividing (5.4.1.8) by $p^2$ and considering it modulo $p$, we get equations in $\mathbb{F}_p$ (tildes are omitted):

$$hE_{21}A^t = (hE_{21}A^t)^t; \quad (5.4.1.9)$$
$Ash^t + hsA^t = \lambda^t E_21,$ \hfill (5.4.1.10)

where $\lambda' = \lambda/p^2 \neq 0$ in $\mathbb{F}_p$.

We denote by $h'$ the $3 \times 2$-matrix obtained by rejecting the first (zero) column of $h$. If the rank of $h'$ is $\leq 1$ then rank of $Ash^t + hsA^t \leq 2$: a contradiction. If the rank of $h'$ is 2 then the set of $X$ such that $Xh^t = hX^t$ is the set ($\ast h'^t \alpha$), where $\ast$ means an arbitrary $3 \times 1$-matrix, $|\ast$ means the operation of gluing of matrices and $\alpha$ runs over the set of symmetric $2 \times 2$-matrices. (5.4.1.9) implies an equality $AE_{21} = (\ast h'^t \alpha)$ which gives us $A = (\ast h'^t \alpha E_{11})$.

For this $A$ we have $Ash^t = (\ast h'^t \alpha E_{11}) sh^t = h'^t \alpha E_{11} W_2 h'^t$ and $hsA^t = h'W_2 E_{11} \alpha h'^t$ (here $W_2 = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_1 \end{pmatrix}$), hence the left hand side of (5.4.1.10) is $h'(\alpha E_{11} W_2 + W_2 E_{11} \alpha) h'^t$

which is of rank at most 2 - a contradiction to (5.4.1.10). \hfill $\square$

**Proposition 5.4.2.** The special component of $T_{p,0}(V)$ does not coincide with (a) one of general components of $T_{p,1}(V)$; (b) the general component of $T_{p,2}(V)$; (c) one of the intermediate components of $T_{p,2}(V)$.

**Proof.** As usually, we denote $g = h + k\sqrt{-\Delta}$. Obviously, Theorem 2.16 can be generalized to the case of different $S(g,i)$ as follows:

**5.4.2.0** Let $\sigma_1 \in S(g,i_1)$, $\sigma_2 \in S(g,i_2)$. Then $T_{p,i_1}(V)_{D(\sigma_1)} = T_{p,i_2}(V)_{D(\sigma_2)}$ iff (2.17) holds.

As earlier we can consider only the $p$-component of adeles, i.e. (2.17) means that there exists $g \in G_V$ such that $\sigma_1 \alpha(g) \sigma_2^{-1} \in G_X(\mathbb{Z}_p)$. If $\sigma_1$, $\sigma_2$ are fixed then $4g^2$ entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ are linear forms of $2g^2$ variables $\sigma_{i,j}, \kappa_{i,j}$. We denote by $L_{\lambda,\mu} = L_{\lambda,\mu}(h_{i,j}, k_{i,j})$ the linear form that is the $(\lambda,\mu)$-th entry of $\sigma_1 \alpha(g) \sigma_2^{-1}$. The set of variables $\sigma_{i,j}, \kappa_{i,j}$ is $\mathbb{Q}_{p}^{2g^2}$, and the condition that all entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ are integer ( $\iff$ all $L_{\lambda,\mu}(h_{i,j}, k_{i,j})$ are integer) give us a $\mathbb{Z}_p$-linear subspace in $\mathbb{Q}_{p}^{2g^2}$. We denote this subspace by $R$; we shall see that $R$ is a lattice. $T_{p,i_1}(V)_{D(\sigma_1)} = T_{p,i_2}(V)_{D(\sigma_2)}$ iff there exists $(h_{i,j}, k_{i,j})$ such that $\text{ord}_p \text{det} g = 0$.

Since we are proving the proposition not for any $g$ but only for $g = 3$, I recommend to the reader to get the $4g^2 \times 2g^2$-matrix of coefficients of $L_{\lambda,\mu}$ by means of computer calculations. We denote this matrix by $M$. To prove non-equivalence, we shall consider submatrices of $M$; these submatrices will define us upper bounds of $R$.

For any component under consideration we consider a fixed value of $\sigma = \sigma_1$ or $\sigma_2$ such that $T_{p,i}(V)_{l(\sigma)}$ is equal to this component. For the special component of $T_{p,0}(V)$ we take $\sigma_1$ from 5.4.1.6.

**Case A.** A general component of $T_{p,1}(V)$. We take $\sigma_2$ from (4.1.1) with $A = U = C = 0$ (for other general component of $T_{p,1}(V)$ the proof is analogous). Immediately from the condition that all entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ are integer we get the following table of inferior bounds of $p$-orders of $h_{i,j}, k_{i,j}$ ($p$-order is $\geq$ of the number in the table):
(2,1), (2,4), (3,1), (3,4)-entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ are linear forms of only 4 variables $h_{i1}, k_{i1}, i = 2, 3$. The corresponding $4 \times 4$-submatrix of $M$ (i.e. the $4 \times 4$-matrix of coefficients of these forms) is $\frac{1}{p} M'$ where entries of $M'$ are in $\mathbb{Z}_p$ and $\det M' \neq 0$. This means that $\text{ord}_p(h_{i1}, k_{i1}) \geq 1$. This implies that $\text{ord}_p(\det g) \geq 1$, i.e. there is no equivalence in this case.

**Case B.** The general component of $T_{p,2}(V)$. We take $\sigma_2$ from (4.1.1) with $A = U = C = 0$. Analog of (5.4.2.1) for this case is the following:

\[
\begin{array}{cccc}
& h & k \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 0 & 1 & 1 & 2 & 1 & 1 \\
2 & 0 & -1 & -1 & -2 & -1 & -1 \\
3 & 0 & -1 & -1 & -2 & -1 & -1 \\
\end{array}
\]

(2,2), (2,5), (3,2), (3,5)-entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ are linear forms of only 4 variables $h_{i2}, k_{i2}, i = 2, 3$, and the $4 \times 4$-matrix of coefficients of these forms has $\det \neq 0 \mod p$, so $\text{ord}_p(h_{i2}, k_{i2}) \geq 1$.

The same is true for variables $h_{i3}, k_{i3}, i = 2, 3$ ((2,3), (2,6), (3,3), (3,6)-entries of $M$).

Finally, treating the (2,1), (2,4), (3,1), (3,4)-entries of $M$ and variables $h_{i1}, k_{i1}, i = 2, 3$, we get that again the $4 \times 4$-matrix of coefficients of these forms has $\det \neq 0 \mod p$.

Using table 5.4.2.2 we get the non-equivalence for this case.

**Case C.** The intermediate component of $T_{p,2}(V)$. We take $\sigma_2$ from (4.1.1) with $A = (1 \ 0), U = C = 0$. Analog of (5.4.2.1) for this case is the following:

\[
\begin{array}{cccc}
& h & k \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 0 & 0 & 1 & 1 & 1 \\
2 & -1 & -1 & -2 & -2 & -1 \\
3 & -1 & -1 & -2 & -2 & -1 \\
\end{array}
\]

12 entries of $\sigma_1 \alpha(g) \sigma_2^{-1}$ (namely, (2,*), (3,*)-entries) are linear forms of only 12 variables (all except $h_{1*}, k_{1*}$) and the determinant of the corresponding $12 \times 12$-submatrix of $M$ is $\neq 0 \mod p$. This implies the non-equivalence for this case.

**Remark.** I think that for other components also there is no coincidence.

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