Regularization of central forces with damping in two and three dimensions

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Abstract Regularization of damped motion under central forces in two and three dimensions is investigated and equivalent; undamped systems are obtained. The dynamics of a particle moving in $\frac{1}{r}$ potential and subjected to a damping force is shown to be regularized using Levi-Civita transformation. We then generalize this regularization mapping to the case of damped motion in the potential $r^{-\frac{2N}{N+1}}$. Further equation of motion of a damped Kepler motion in three dimensions is mapped to an oscillator with inverted sextic potential and couplings, in four dimensions using Kustaanheimo–Stiefel regularization method. It is shown that the strength of the sextic potential is given by the damping coefficient of the Kepler motion. Using homogeneous Hamiltonian formalism, we establish the mapping between the Hamiltonian of these two models. Both in two and three dimensions, we show that the regularized equation is nonlinear, in contrast to undamped cases. Mapping of a particle moving in a harmonic potential subjected to damping to an undamped system with shifted frequency is then derived using Bohlin–Sundman transformation.

1 Introduction

Dissipation being unavoidable in natural systems is of intrinsic interest. The investigation of such systems has a long history and continues to be an active area of research \cite{1–3}. Damped harmonic oscillator is one of the systems that has been studied vigorously as a prototype of dissipative systems. Various approaches such as (i) coupling the system to a heat bath and (ii) use of Bateman–Caldirola–Kanai (BCK) \cite{4–6} model which uses a time-dependent Lagrangian/Hamiltonian have been developed and adopted for studying different aspects of dissipative systems. Each of these methods though having unique advantages has some unsatisfactory features \cite{7–10}.

BCK model has been shown to be plagued by difficulties in interpretation even at the classical level, as the time-dependent BCK–Lagrangian/Hamiltonian leading to correct damped equation of motion is shown to describe a variable mass system rather than a truly damped system.
oscillator [7,9]. It has been argued that the equivalence between BCK model and damped harmonic oscillator is not valid globally (i.e., not for all times) and they are equivalent only for finite time scales [10].

Generalization of action principle which allows to include non-conservative systems in its ambit was presented long ago [11] and this method can be used to study dissipative systems. Use of contact manifolds to study dissipative systems is a new upsurge of interests among mathematicians [12–14]. Regularization is considered to be a tool for converting singularities of a system of differential equations into regular ones. This is carried out by changing the dependent and/or independent variables of a system of (singular) differential equations so that the new system is free of singularities [15]. A well-developed theory of regularization in celestial mechanics might be attributed to Bohlin, Sundman and Levi-Civita [16,17]. The notion of Levi-Civita regularization is now used for the regularization of the binary collisions in the planar (two dimensional) Kepler problem [17]. Generalization of these approach to three dimensions was obtained by Kustaanheimo and Stiefel [18]. These regularization schemes also lead to linear differential equations. There are other schemes of regularization such as those developed by Moser [19] and Ligon–Schaaf [20]. Moser showed that the flow of the \( n \)-dimensional Kepler problem on the surface of constant negative energy is conjugate to the geodesic flow on the unit tangent bundle of \( S^n \). The Ligon–Schaaf regularization procedure tackles together all the surfaces of negative energies.

Kepler problem is the most studied system which has singularity. Keplerian orbit where the separation of particle from the central mass goes to zero is called a collision orbit. Detailed analysis of such orbits are necessary for artificial satellite missions, at the starting point as well as at the final landing point in moon/planet/exoplanet, the separation distance vanishes. Another situation that requires regularization is the low energy orbits where there is no collision, but the probe passes very close to moon/planet to achieve boost (known as slingshot effect) where the separation is negligibly small compared to other length scales involved [21]. Regularization also becomes handy in dealing with perturbations to Kepler problem due to effects such as presence of a third body, change in the shape of bodies from spheri cal one assumed. Motion under the influence of central force in the presence of friction arise in systems such as Rydberg atom and binary stars and has been of interest. Effect of friction on the shape of orbits has been studied [22–24]. Kepler problem with drag term, which is linear in velocity and inversely proportional to the square of the separation distance was studied and analytical solutions were obtained [25–27]. In these studies, existence of a conserved quantity which is related to angular momentum was exploited in deriving the solutions. Further, for a specific choice of coefficient of the drag term, this equation was shown to be related to that of harmonic oscillator [26]. Kepler equations augmented with drag term was used to model the orbits of artificial satellites landing on other planets/moon, for orbits of re-entry in to earth’s atmosphere as well as for analyzing motion of low orbit satellites. In this paper, we study regularization of collision orbits of particle moving under the influence of inverse power law potentials and also subjected to velocity-dependent damping, in two and three dimensions.

In this paper, we use the approaches of Levi-Civita and Kustaanheimo and Stiefel to regularize problem of a particle moving in central potentials and subjected to damping in two and three dimensions. In particular, we investigate the construction of equivalent model corresponding to systems with damping, viz: (i) particle in two and three dimensions, moving under the action of the \( r^{-\frac{2N}{N+1}} \) potential subjected to damping, and (ii) particle in two dimensions moving under the influence of a potential of the form \( r^{-2N/N+1} \) where \( N \in \mathbb{Z} \), subjected to damping. We then study regularization of collision orbits of the particle in these situations. The equations of motion describing the damped motion, as well as the corresponding
Lagrangian/Hamiltonian in the above cases have explicit time dependence. Thus, for these systems, energy is not a conserved quantity and this hinders the direct application of Levi-Civita/K–S-transformations to these systems with damping. Also it is of interest to see how these equations transform under the re-parametrization of time, which is an integral part of these regularizations. The re-parametrization of time which makes the motion near the collision point “slow” does affect the form of the equations of motion in terms of the new coordinates, through velocity and acceleration. Since in the case of damped systems we study, not only Lagrangian and Hamiltonian, but the equations of motion also depend explicitly on the time, it is of interest to see how this explicit time dependence affects the regularization. In our analysis, we start with a Lagrangian (which can be related to a BCK type, time-dependent Lagrangian) whose equation of motion describes a particle subjected to Kepler potential, in addition to a velocity-dependent damping. We first map these equations using a time-dependent point transformation such that the transformed equations follow as Euler–Lagrange equations from a time-independent Lagrangian. This allows us to construct conserved energy, which in turn allows the implementation of the mapping of dynamics on a constant energy surface. We apply Levi-Civita map/K–S transformation to these equations, after re-expressing them in terms of complex coordinates/quaternions. The equation in terms of the new complex coordinates/quaternions and time parameter is shown to describe a harmonic oscillator augmented by an inverted sextic potential in two dimensions/four dimensions.

The regularization scheme known as Levi-Civita transformation for two-dimensional Kepler problem and its generalization to three dimensions known as Kustaanheimo–Stiefel (K–S) transformation, do linearize the differential equations describing the motion of a particle under the influence of gravitational force exerted by a central body and also removes the singularity of the equations when the separation of these two bodies vanishes. The procedure consists of three steps;

1. In the first step, one implements a re-parametrization of time so that the velocity defined in terms of “new” time variable, do not diverge as the separation distance approaches zero. One now uses chain rule and re-expresses the time derivatives of the position coordinates appearing in the differential equation in terms of the “new” time variable.

2. In the second step, one applies Levi-Civita conformal squaring (of the complex coordinates) for two-dimensional case and K–S transformation which relates coordinates of three-dimensional space to square of quaternions. Thus, the equations of motion are now re-expressed in terms of new coordinates and their derivatives with respect to “new” time parameter. Thus, obtained equation is regular but nonlinear.

3. In the third or final step, one starts with the conserved energy associated with the initial system, written in terms of kinetic and potential energies. One re-expresses the velocities appearing in kinetic part in terms of the derivative of new coordinates with respect to “new” time parameter and coordinates in potential energy are rewritten in terms of new coordinates. This conserved quantity is then used to re-express the equations of motion obtained in the second step above. Straightforward calculations then convert the equations of motion to be regular and linear one. This equation is then solved and by fixing the numerical value of the conserved quantity, one maps orbits of fixed energy to regular solutions of the linear equation.

In this paper, we apply these regularization methods to Kepler problem in two and three dimensions as well as to a generic inverse power law potential, all subjected to velocity-dependent damping. We show that the mapping leads to regular, but coupled equations. Further, the coupling terms all have the damping parameter as coefficient. We also study the
application of Bohlin–Sundman transformation to the equation describing damped harmonic oscillator in two dimensions. After applying a time-dependent coordinate transformation, we map the damped harmonic motion to that of a shifted harmonic oscillator. As earlier, this allows us to define conserved energy which facilitates the mapping of dynamics from a constant energy surface. This equation, after re-expressing in terms of complex coordinates, is mapped to that of 2-dim Kepler problem by Bohlin–Sundman transformation [16].

This paper is organized as follows. In the next section, we show that the Levi-Civita transformation maps the Kepler’s equations with damping in two dimensions to the equations of motion of a harmonic oscillator with an additional, inverted, sextic potential. This is done by first mapping the damped Kepler equations to an equivalent set of equations which do not have explicit time dependence. These equations come from a time-independent Lagrangian and thus the corresponding energy is a conserved quantity. These equations are mapped by Levi-Civita regularization map to equations describing a harmonic oscillator, augmented with inverted, sextic potential. In Sect. 3, we use Levi-Civita map to obtain a regularized equation corresponding to equation describing a particle moving, in the presence of friction, under the generic inverse power law potential of the form \( r^{-2N/N+1}, 0 \leq 2N/N+1 < 2 \). In obtaining this mapping, we follow the same steps as we have used in Sect. 2, in deriving equivalence between two-dimensional Kepler motion with drag to that of harmonic oscillator in the presence of inverted sextic potential. In both these cases, we express the conformal squaring of the coordinates (discussed above) as a matrix equation and further show that this equation can be expressed using a generic matrix connecting old and new coordinate. This matrix is written in terms of two permutation operators which do commute among themselves. In Sect. 4, we study the Kepler problem in three dimensions in the presence of damping. After a brief summary of quaternions, in Sect. 4.1, we show that the K–S transformation maps the equations of motion of three-dimensional Kepler problem with damping to that of a four-dimensional harmonic oscillator with the inverted sextic potential. In Sect. 4.2, we show the mapping of these two systems at the level of Hamiltonians, using homogeneous Hamiltonian formalism. Our concluding remarks are given in Sect. 5. In Appendix A, we show the mapping of damped harmonic motion to that of Kepler problem, using Bohlin–Sundman transformation.

2 Kepler problem in the presence of damping in two dimensions: Levi-Civita map

In this section, we apply Levi-Civita map to damped Kepler problem in two dimensions and obtain an equivalent but regularized formulation. We begin with a BCK type, time-dependent Lagrangian whose equation of motion describes a particle moving in Kepler potential and also subjected to velocity-dependent damping. We start by mapping these equations using a time-dependent point transformation, so that the modified equations are Euler–Lagrange equations from a time-independent Lagrangian. This permits us to build conserved energy, which allows us to apply the mapping of dynamics on a constant energy surface. After re-expressing these equations in terms of complex coordinates, we use the Levi-Civita map to get equations for a harmonic oscillator with inverted sextic potential and interactions.

We note that the equation of motion following from a BCK-type Lagrangian

\[
L = e^{\lambda t} \left[ \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{ke^{-\frac{\lambda t}{r}}}{r} \right]
\]  

(2.1)
where \( \tilde{r} = \sqrt{x_1^2 + x_2^2} \), \( m = \frac{m_1 + m_2}{m_1 m_2} \) and \( k = G m_1 m_2 \) is

\[
m \dddot{x}_i + \lambda m \ddot{x}_i + \frac{k e^{-\frac{3 \dot{x}_i}{\tilde{r}}}}{\tilde{r}^3} x_i = 0, \quad i = 1, 2,
\]

(2.2)

where \( \dot{x}_i = \frac{dx_i}{dt} \) and \( \ddot{x}_i = \frac{d^2 x_i}{dt^2} \). These equations describe motion of a particle of mass \( m_1 \) under the influence of a gravitational potential of another body of mass \( m_2 \) as well as a damping force, in two dimensions. Here, when \( \lambda \rightarrow 0 \), Eq. 2.2 becomes equation of motion for well-known Kepler problem in two dimensions. Note that this equation has explicit time dependence and the re-parametrization of time applied in the first step of regularization, will not remove this explicit dependence on the “old” time. Also, the energy of this system is not conserved and thus implementing the third step of regularization is not possible. In order to avoid these complications, we now rewrite these equation of motion in terms of a new set of coordinates \((X_1, X_2)\) which are related to old coordinates through time-dependent transformation given by

\[
X_1 = x_1 e^{\frac{\dot{x}_1}{2}},
\]

\[
X_2 = x_2 e^{\frac{\dot{x}_2}{2}}.
\]

(2.3)

In terms of these coordinates, Eq. (2.2) become

\[
m \dddot{X}_1 - \frac{m \lambda^2}{4} X_1 + \frac{k X_1}{(X_1^2 + X_2^2)^{\frac{3}{2}}} = 0,
\]

(2.4)

\[
m \dddot{X}_2 - \frac{m \lambda^2}{4} X_2 + \frac{k X_2}{(X_1^2 + X_2^2)^{\frac{3}{2}}} = 0.
\]

(2.5)

These equations do not have explicit time dependence and first difficulty in implementing the regularization is now circumvented.

It is easy to see that these are Euler–Lagrange equations following from the Lagrangian

\[
L = \frac{m}{2} (\dot{X}_1^2 + \dot{X}_2^2) + \frac{m \lambda^2}{8} (X_1^2 + X_2^2) - \frac{m \lambda}{2} (X_1 \dot{X}_1 + X_2 \dot{X}_2) + \frac{k}{r}.
\]

(2.6)

Note that the third term can be rewritten as \( -\frac{m \lambda}{r} \frac{d}{dt} (X_1^2 + X_2^2) \), a total derivative and thus will not contribute to equations of motion. Following the standard procedure, we calculate the canonical conjugate momenta corresponding to \( X_1 \) and \( X_2 \) as

\[
P_{X_1} = m \dot{X}_1 - \frac{\lambda}{2} m X_1,
\]

(2.7)

\[
P_{X_2} = m \dot{X}_2 - \frac{\lambda}{2} m X_2
\]

(2.8)

and using these derive the Hamiltonian as

\[
H = \frac{(P_{X_1}^2 + P_{X_2}^2)}{2m} + \frac{\lambda}{2} (X_1 P_{X_1} + X_2 P_{X_2}) - \frac{k}{r},
\]

(2.9)

where we now use \( r = \sqrt{X_1^2 + X_2^2} \). This Hamiltonian does not have any explicit time dependence and is invariant under time translations and hence total energy of this system is conserved. Thus, the second difficulty mentioned above in implementing regularization is also avoided.
We note that the angular momentum $L = X_1 P_{X_2} - X_2 P_{X_1}$ is also conserved, as in the usual Kepler problem. For later purposes, we express (negative of) this Hamiltonian, which is a conserved quantity, in terms of velocities and coordinates, viz:

$$-\mathcal{E} = \frac{m}{2} (\dot{X}_1^2 + \dot{X}_2^2) - \frac{\lambda^2}{8} m (X_1^2 + X_2^2) - \frac{k}{r}. \quad (2.10)$$

We re-express Eqs. (2.4) and (2.5) in terms of complex variable $Z$ given by $Z = X_1 + iX_2$ as

$$m \ddot{Z} + \frac{k}{|Z|^2} Z = \frac{m\lambda^2}{4} Z, \quad (2.11)$$

where $|Z| = \sqrt{X_1^2 + X_2^2} = r$. Note that the derivative with respect to the time variable $t$ is represented by “overdot,” in the above equations.

We now apply a re-parametrization of time and demand\(^1\)

$$\frac{d}{dt} = \frac{c}{r} \frac{d}{d\tau}. \quad (2.12)$$

Note here that the $c$ appearing in the above equation is a proportionality constant (and not the velocity of light). Applying this re-parametrization, we find

$$\frac{dZ}{dr} = \frac{c}{r} \frac{dZ}{d\tau}, \quad (2.13)$$

$$\frac{d^2Z}{dr^2} = \frac{c^2}{r^2} \frac{d^2Z}{d\tau^2} - \frac{c^2}{r^3} \frac{dr}{d\tau} \frac{dZ}{d\tau}. \quad (2.14)$$

Using these, we re-express Eq. (2.11) as

$$c^2 (r \frac{d^2Z}{d\tau^2} - \frac{dr}{d\tau} \frac{dZ}{d\tau}) + \mu Z = \frac{\lambda^2}{4} r^3 Z, \quad (2.15)$$

where $\mu = k/m$.

Next, we apply a coordinate transformation (Levi-Civita regularization) and rewrite above equation in terms new complex coordinate $U = U_1 + iU_2$ which is related to $Z$ as

$$Z = \gamma U^2 \quad (2.16)$$

which we can rewrite in matrix representation as following,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \gamma A(U) = \gamma \hat{U}_1 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad (2.17)$$

where

$$A(U) = \hat{U}_1 = \begin{pmatrix} U_1 & -U_2 \\ U_2 & U_1 \end{pmatrix}. \quad (2.18)$$

Which we can rewrite as follows,

$$A(U) = \left( \mathcal{P}^{(1)} U, \mathcal{P}^{(2)} U \right), \quad (2.19)$$

where we have considered following set of permutations,

$$\mathcal{P}^{(1)} U = (U_1, U_2)^T, \quad \mathcal{P}^{(2)} U = (-U_2, U_1)^T. \quad (2.20)$$

---

\(^1\) Conserved angular momentum in damped Kepler problem and perturbed harmonic oscillator (see Eqs. (2.6), (2.30)) allows us to equate these numbers and this leads to the relation in Eq. (2.12).
Which have properties like,
\[ p^{(i)} p^{(j)} = p^{(j)} p^{(i)} \quad \text{and} \quad (p^i)^T p^i = \delta_{ij}. \]  

(2.21)

In Eq. (2.16), \( \gamma \) is constant having dimensions of inverse length (without lose of generality, we set this \( \gamma \) to be one from now onward). This sets \( \gamma = U \bar{U} = \vert U \vert^2 \). \( r^2 = \bar{Z} Z \). We also find \( \frac{dZ}{d\tau} = 2U \frac{dU}{d\tau}, \frac{d^2Z}{d\tau^2} = 2(\frac{dU}{d\tau})^2 + 2U \frac{d^2U}{d\tau^2} \) and \( \frac{d}{d\tau} = \bar{U} \frac{dU}{d\tau} + \frac{d\bar{U}}{d\tau} \). Using this, Eq. (2.15) becomes

\[
2c^2 U \bar{U} \left( U \frac{d^2U}{d\tau^2} + \left( \frac{dU}{d\tau} \right)^2 \right) - c^2 \left( \bar{U} \frac{dU}{d\tau} + \frac{d\bar{U}}{d\tau} \right) 2U \frac{dU}{d\tau} + \mu U^2
\]

\[
= \frac{\lambda^2}{4} U^2 (\bar{U} U)^3. \quad (2.22)
\]

This equation, after straightforward algebra, gives

\[
2r \frac{d^2U}{d\tau^2} - 2 \left| \frac{dU}{d\tau} \right|^2 U - \frac{\lambda^2}{4c^2} r^3 U + \frac{\mu}{c^2} U = 0. \quad (2.23)
\]

Next, we express the conserved quantity \(-E\) in terms of \( \frac{dU}{d\tau} \). For this, we first re-express \(-E\) in terms of \( Z \) and \( \frac{dZ}{d\tau} \) as

\[
- \mathcal{E} = m \frac{d\bar{Z}}{d\tau} \frac{dZ}{d\tau} - \frac{m \lambda^2}{8} \bar{Z} Z - \frac{k}{|Z|} \quad (2.24)
\]

and apply re-parametrization to replace derivative with respect to \( t \) with derivative with respect to \( \tau \) and finally, re-express these terms using the derivative of \( U \). This gives

\[
- \mathcal{E} = \frac{m c^2}{r} \left( \frac{dU}{d\tau} \right)^2 - \frac{m \lambda^2}{8} r^2 - \frac{k}{r} \quad (2.25)
\]

\[
\Rightarrow 2mc^2 \left( \frac{dU}{d\tau} \right)^2 - \frac{m \lambda^2}{8} r^2 - \frac{k}{r}. \quad (2.26)
\]

Using this, we rewrite \( \left| \frac{dU}{d\tau} \right|^2 \) in Eq. (2.23) and find

\[
\frac{d^2U}{d\tau^2} + \left( \frac{\mathcal{E}}{2mc^2} - \frac{3 \lambda^2}{16c^2} r^2 \right) U = 0, \quad (2.27)
\]

where \( r^2 = \vert U \vert^4 \). This equation describes a perturbed harmonic oscillator for \( \lambda \ll 1 \). To see this clearly, we rewrite the above equation in \( U_1 \) and \( U_2 \) as

\[
U_1'' + \frac{1}{2c^2} \left( \frac{\mathcal{E}}{m} - \frac{3 \lambda^2}{8} (U_1^2 + U_2^2)^2 \right) U_1 = 0 \quad (2.28)
\]

\[
U_2'' + \frac{1}{2c^2} \left( \frac{\mathcal{E}}{m} - \frac{3 \lambda^2}{8} (U_1^2 + U_2^2)^2 \right) U_2 = 0 \quad (2.29)
\]

where a “prime” over \( U \) stands for derivative with respect to the new time variable, \( \tau \). It is easy to see that these equations follow from the Lagrangian

\[
\mathcal{L} = \frac{m}{2}(U_1^2 + U_2^2) - \frac{\mathcal{E}}{2c^2(U_1^2 + U_2^2)} + \frac{1}{32} \frac{m \lambda^2}{c^2} (U_1^2 + U_2^2)^3. \quad (2.30)
\]
We now see that the system described by the above Lagrangian is oscillator in two dimensions with inverted sextic potential (i.e., with coefficients of $U_1^2$ and $U_2^2$ are negative) and couplings. Note that, for small $\lambda$, we can treat this system as uncoupled harmonic oscillators with perturbations involving couplings and inverted sextic potential.\(^2\)

The Hamiltonian following from the above Lagrangian is given by

$$H = \frac{p_{U_1}^2}{2m} + \frac{p_{U_2}^2}{2m} + \frac{E}{4c^2}(U_1^2 + U_2^2) - \frac{1}{32} \frac{m\lambda^2}{c^2} (U_1^2 + U_2^2)^3. \quad (2.31)$$

Note that the $\lambda$-dependent term is with negative coefficient.

Thus, we have shown that the equations of motion following from the Lagrangian in Eq. (2.6) are mapped by Levi-Civita map to equations following from the Lagrangian in Eq. (2.30). This generalizes the equivalence of the Kepler problem to the harmonic oscillator in two dimensions to the equivalence of the Kepler problem in the presence of a damping force, to a perturbed harmonic oscillator. This is shown by first mapping Eq. (2.2) describing the damped Kepler equations in two dimensions to Eqs. (2.4, 2.5) (which follow from the Lagrangian given in Eq. (2.6)), which are then mapped to that of harmonic oscillator in two dimensions with specific couplings and inverted sextic potential. This mapping is obtained for the surface defined by the constant value of $E$. From Eq. (2.24), we note that this conserved quantity reduces to energy of the Kepler system in the limit $\lambda \to 0$ and thus, in the limit $\lambda \to 0$, we get back the well-known equivalence between Kepler motion and harmonic oscillations in two dimensions, on the constant energy surface.

### 3 Mapping of motion in $\frac{1}{r^{N+1}}, \ 0 \leq 2N/N+1 < 2$ potential with damping in two dimensions

In this section, we show that the Levi-Civita mapping regularizes the damped motion in generic inverse power law potential given by $\frac{1}{r^{N+1}}, \ 0 \leq 2N/N+1 < 2$. In the undamped case, such a generalization was shown in [28]. After expressing the transformations between old and new coordinates as a matrix equation, in [28], this matrix was written using a pair of permutation operators. The commuting nature of these permutation operators was crucial in proving various identities which in turn were used for this generalization of Levi-Civita transformation. Here, we apply this generalization to the system moving under the influence of the potential $\frac{1}{r^{N+1}}, \ 0 \leq 2N/N+1 < 2$ and subjected to damping.

We start from a BCK-type Lagrangian given by

$$L = e^{\lambda t} \left[ \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{ke^{-\frac{2N+1}{N+1} \lambda t}}{\tilde{r}^{2N}} \right]. \quad (3.1)$$

where $\tilde{r} = \sqrt{x_1^2 + x_2^2}$, $m = \frac{m_1 + m_2}{m_1 m_2}$ and $k = Gm_1 m_2$. The equations of motion following from this Lagrangian are

$$m\ddot{x}_i + \lambda m\dot{x}_i + \left( \frac{2N}{N+1} \right) \frac{ke^{-\frac{2N+1}{N+1} \lambda t}}{\tilde{r}^{4N+2}} \dot{x}_i = 0, \quad i = 1, 2, \quad (3.2)$$

\(^2\) Inserting $\gamma$ back into above equation, we find the coefficient of the last term in the Lagrangian is $m\lambda^2 \gamma^2 \frac{1}{32c^2}$. 

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where $\dot{x}_i = \frac{dx_i}{dt}$ and $\ddot{x}_i = \frac{d^2x_i}{dt^2}$. Here, when $N = 1$, we get back equation of motions for perturbed Kepler problem given in Eq. 2.2 and additionally when $\lambda \rightarrow 0$, Eq. 3.2 becomes equation of motion for well-known Kepler problem in two dimensions.

We now rewrite these equation of motion in terms a new set of coordinates $(X_1, X_2)$ which are related to old coordinates through time-dependent transformation given by

$$
X_1 = x_1 e^{\frac{\lambda t}{2}}, \\
X_2 = x_2 e^{\frac{\lambda t}{2}}.
$$

In terms of these coordinates, Eq. (3.2) becomes

$$
m\ddot{X}_i - m\frac{\lambda^2}{4} X_i + \left(\frac{2N}{N+1}\right) \frac{kX_i}{r^{4N+2}} = 0, \quad i = 1, 2 \tag{3.4}
$$

where $r = \sqrt{X_1^2 + X_2^2}$. Here, we can see that these equations following from the Lagrangian

$$
L = \frac{m}{2} (\dot{X}_1^2 + \dot{X}_2^2) + \frac{m\lambda^2}{8} (X_1^2 + X_2^2) - \frac{m\lambda}{2} (X_1 \dot{X}_1 + X_2 \dot{X}_2) + \frac{k}{r^{2N+1}},
$$

which has no explicit time dependence. Note that here $N = 1$ leads to Lagrangian of perturbed Kepler problem given in Eq. (2.6) and third term as a total derivative term will not contribute to equation of motions. Following the standard procedure, we calculate the canonical conjugate momenta corresponding to $X_1$ and $X_2$ as

$$
P_{X_1} = m \dot{X}_1 - \frac{\lambda}{2} m X_1 \tag{3.6}$$
$$
P_{X_2} = m \dot{X}_2 - \frac{\lambda}{2} m X_2 \tag{3.7}
$$

and using these we derive the Hamiltonian as

$$
\mathcal{H} = \frac{(P_{X_1}^2 + P_{X_2}^2)}{2m} + \frac{\lambda}{2} (X_1 P_{X_1} + X_2 P_{X_2}) - \frac{k}{r^{2N+1}},
$$

where $r = \sqrt{X_1^2 + X_2^2}$ and at $N = 1$ above Hamiltonian leads to Eq. (2.9). This Hamiltonian does not have any explicit time dependence and is invariant under time translations and hence total energy of this system is conserved. For later purposes, we express (negative of) this Hamiltonian, which is a conserved quantity, in terms of velocities and coordinates, viz:

$$
-\mathcal{E} = -\frac{E}{m} = \frac{1}{2} (\dot{X}_1^2 + \dot{X}_2^2) - \frac{\lambda^2}{8} (X_1^2 + X_2^2) - \frac{\mu}{r^{2N+1}},
$$

where $\frac{k}{m} = \mu$

We re-express Eq. (3.4) in terms of complex variable $Z$ given by $Z = X_1 + iX_2$ as

$$
m\dddot{Z} + \left(\frac{2N}{N+1}\right) \frac{k}{|Z|^{4N+2}} Z = \frac{m\lambda^2}{4} Z, \tag{3.10}
$$

where $|Z| = \sqrt{X_1^2 + X_2^2} = r$. Note that the derivative with respect to the time variable $t$ is represented by “overdot,” in the above equations.
We now apply a re-parametrization of time to avoid presence of singularity from the equations of motion and demand
\[
\frac{\text{d}}{\text{d}t} = \frac{1}{g(r)} \frac{\text{d}}{\text{d}\tau}, \quad \text{where } g(r) = (N + 1)^2 r^{\frac{2N}{N+1}} \tag{3.11}
\]

Note here that the \(N = 1\) leads to Eq. (2.12), time re-parametrization used for perturbed Kepler problem. Applying this re-parametrization, we find

\[
\frac{\text{d}Z}{\text{d}\tau} = \left(\frac{r^{\frac{2N}{N+1}}}{{(N + 1)^2}}\right)^2 \frac{\text{d}Z}{\text{d}\tau} + \left(\frac{2N}{(N + 1)^2}\right) \frac{1}{r^{\frac{2N}{N+1}}} \frac{\text{d}r}{\text{d}\tau} \frac{\text{d}Z}{\text{d}\tau}. \tag{3.12}
\]

Using these, we re-express Eq. (3.10) as

\[
\frac{1}{(N + 1)^4 r^{\frac{4N}{N+1}}} \frac{\text{d}^2Z}{\text{d}\tau^2} - \frac{2N}{(N + 1)^5} \frac{1}{r^{\frac{2N+1}{N+1}}} \frac{\text{d}r}{\text{d}\tau} \frac{\text{d}Z}{\text{d}\tau} + \left(\frac{2N}{N + 1}\right) \frac{1}{r^{\frac{4N+2}{N+1}}} Z = \frac{\lambda^2}{4} Z, \tag{3.13}
\]

where \(\mu = k/m\).

Next, we apply a coordinate transformation (Levi-Civita regularization) and consider following relations [28],

\[
Z = \hat{U}_N U \tag{3.15}
\]

where,

\[
U^{(N)} = \hat{U}_{N-1} U \quad \text{and} \quad \hat{U}_N = \left(\mathcal{P}^{(1)} U^{(N)}, \mathcal{P}^{(2)} U^{(N)}\right), \quad N \geq 1 \tag{3.16}
\]

with \(\hat{U}_0 = I\) and \(\hat{U}_1 = A(U)\) as explained in Eq. (2.18). Here, we consider same definition of permutation matrices as discussed previously in Eq. (2.20), i.e.,

\[
\mathcal{P}^{(1)} U = (U_1, U_2)^T, \quad \mathcal{P}^{(2)} U = (-U_2, U_1)^T, \tag{3.17}
\]

which have following properties like,

\[
\mathcal{P}^{(i)} \mathcal{P}^{(j)} = \mathcal{P}^{(j)} \mathcal{P}^{(i)} \quad \text{and} \quad (\mathcal{P}^{i})^T \mathcal{P}^j = \delta_{ij}. \tag{3.18}
\]

We also consider following relations [28],

\[
\hat{U}_N^T \hat{U}_N = \hat{U}_N \hat{U}_N^T = R^{2N} I_{2 \times 2} \quad \text{and} \quad (U^{(N+1)})' = (N + 1) \hat{U}_N U', \tag{3.19}
\]

where \((')\) denotes time derivative with respect to fictitious time \(\tau\).

These set \(r = R^{N+1}, r' = (N + 1)R^N R'\). We also find (in matrix representation) \(Z' = (N + 1) \hat{U}_N U', Z'' = (N + 1) \hat{U}_N^T U' + (N + 1) \hat{U}_N U''\), using which, we can rewrite Eq. (3.14) in matrix representation as,

\[
(N + 1) \hat{U}_N^T U' + (N + 1) \hat{U}_N U'' - \frac{1}{R} (2N)(N + 1) R' (\hat{U}_N U') - \frac{\lambda^2}{4} (N + 1)^4 R^{4N} \hat{U}_N U + \frac{\mu (2N)(N + 1)^3}{R^2} \hat{U}_N U = 0 \tag{3.20}
\]
This equation, after straightforward algebra, gives

\[ U'' - \frac{\lambda^2}{4} (N + 1)^3 R^{4N} U + \left\{ \frac{\mu(2N)(N + 1)^2}{R^2} - \frac{N \left( \sum (U_i')^2 \right)}{R^2} \right\} U = 0. \]  

(3.21)

Here, for \( N = 1 \), we get back Eq. (2.22) with \( c \) being \( \frac{1}{4} \). Next, we express the conserved quantity \(-\mathcal{E}\) in terms of new-coordinates. For this, we first re-express \(-\mathcal{E}\) in terms of \( Z \) and \( \frac{dZ}{dt} \) as

\[ -\mathcal{E} = \frac{1}{2} \frac{1}{(N + 1)^2 R^{2N}} \left( \sum (U_i')^2 \right) - \frac{\lambda^2}{8} R^{2N+2} - \frac{\mu}{R^{2N}} \]  

(3.22)

and apply re-parametrization of time, Eq. (3.11) and finally, re-express these terms of matrix representation using the derivative of \( U \). This gives

\[ -\mathcal{E} = \frac{1}{2} \frac{1}{(N + 1)^2 R^{2N}} \left( \sum (U_i')^2 \right) - \frac{\lambda^2}{8} R^{2N+2} - \frac{\mu}{R^{2N}} \]  

(3.23)

Using this, in Eq. (3.21) we get,

\[ U'' - \frac{\lambda^2}{4} (2N + 1)(N + 1)^2 R^{4N} U + \mathcal{E} R^{2N-2}(2N)(N + 1)^2 U = 0, \]  

(3.24)

where \( R^2 = U_1^2 + U_2^2 \). We rewrite the above equation in \( U_1 \) and \( U_2 \) as

\[ U_i'' - \frac{\lambda^2}{4} (2N + 1)(N + 1)^2 (U_1^2 + U_2^2)^{2N} U_i + \mathcal{E} (U_1^2 + U_2^2)^{N-1} (2N)(N + 1)^2 U_i = 0, \]

\[ i = 1, 2 \]  

(3.25)

where a “prime” over \( U \) stands for derivative with respect to the new time variable, \( \tau \). These equations are obvious extensions of the Lagrangian

\[ \mathcal{L} = \frac{1}{2} m(U_1'^2 + U_2'^2) - \mathcal{E} m(N + 1)^2 (U_1^2 + U_2^2)^N \]

\[ + \frac{\lambda^2}{8} m(N + 1)^2 (U_1^2 + U_2^2)^{2N+1} \]  

(3.26)

Note that for \( N = 1 \), these equations of motions lead to Eqs. (2.28) and (2.29), i.e., equation of motions for perturbed harmonic oscillator problem and in addition \( \lambda \to 0 \), these describe motion of usual harmonic oscillator. Note that the \( \lambda^2 \)-dependent terms are nonlinear and these terms come due to the damping present in the original system.

4 Kepler problem in three dimensions in the presence of damping: Kustaanheimo–Stiefel transformation

The Kustaanheimo–Stiefel (KS) transformation, which is a three-dimensional extension of the Levi-Civita transformation, maps the singular equations of motion of the three-dimensional Kepler problem to the linear and regular equations of a four-dimensional harmonic oscillator. Here, we investigate the regularization of damped Kepler problem in three
dimensions. We first give a brief summary of the definitions and identities involving quaternions, which are defined as
\[
U = U_0 + iU_1 + jU_2 + kU_3, \quad \text{with } i^2 = j^2 = k^2 = -1. \tag{4.1}
\]
\[
i \cdot j = -j \cdot i = k; \quad j \cdot k = -k \cdot j = i; \quad k \cdot i = -i \cdot k = j. \tag{4.2}
\]
We define the complex conjugate as
\[
\bar{U} = U_0 - iU_1 - jU_2 - kU_3 \tag{4.3}
\]
and a $*$-operation as
\[
U^* = U_0 + iU_1 + jU_2 - kU_3. \tag{4.4}
\]
We note
\[
\bar{U}U = |U| = \sum_{i=0}^{3} U_i^2 = \bar{U}^*U^* = |U^*|^2. \tag{4.5}
\]
We also define a three vector
\[
X = X_0 + iX_1 + jX_2 \tag{4.6}
\]
which is given in terms of the quaternions using the KS transformation as
\[
X = U \bar{U}^*. \tag{4.7}
\]
Since $\bar{X} = X$, we find $(U \bar{U}^*) = X$. One may write the above equation in the matrix form (with $X_3 = 0$) as
\[
\begin{pmatrix}
X_0 \\
X_1 \\
X_2 \\
X_3
\end{pmatrix} = A(U) \begin{pmatrix}
U_0 \\
U_1 \\
U_2 \\
U_3
\end{pmatrix}, \tag{4.8}
\]
where
\[
A(U) = \begin{pmatrix}
U_0 - U_1 - U_2 & U_3 \\
U_1 & U_0 - U_3 - U_2 \\
U_2 & U_3 & U_0 & U_1 \\
U_3 - U_2 & U_1 & -U_0
\end{pmatrix}. \tag{4.9}
\]
This we can rewrite as follows,
\[
A(U) = \begin{pmatrix}
\mathcal{P}^{(0)}U & \mathcal{P}^{(1)}U & \mathcal{P}^{(2)}U & \mathcal{P}^{(3)}U
\end{pmatrix}, \tag{4.10}
\]
where we have considered following set of permutations,
\[
\mathcal{P}^{(0)}U = (U_0, U_1, U_2, U_3)^T, \quad \mathcal{P}^{(1)}U = (-U_1, U_0, U_3, -U_2)^T \\
\mathcal{P}^{(2)}U = (-U_2, -U_3, U_0, U_1)^T, \quad \mathcal{P}^{(3)}U = (U_3, -U_2, U_1, -U_0)^T. \tag{4.11}
\]
Here, above used permutation matrices has some properties like,
\[
\mathcal{P}^{(i)}\mathcal{P}^{(j)} \neq \mathcal{P}^{(j)}\mathcal{P}^{(i)} \quad \text{and} \quad (\mathcal{P}^{(i)})^T\mathcal{P}^{(j)} = \delta_{ij}. \tag{4.12}
\]
We note that $X_3 = 0$, which naturally comes from the requirement $U \bar{U}^* = (U \bar{U}^*)^*$. We also note that $A(U)$ satisfy
\[
A(U)^T A(U) = rI. \tag{4.13}
\]
Defining \( r = \sqrt{\sum_{i=0}^{2} X_i^2} \), we find
\[
r = \sqrt{|X|^2} = |U|^2 = |U^*|^2.
\] (4.14)

Defining derivative as \( \frac{dX}{d\tau} = X' \), we find
\[
U dU^* = dU U^*
\] (4.15)
\[
X' = U U'^* + U' U^* = 2U U'^*
\] (4.16)
\[
X'' = 2U' U'^*;
\] (4.17)
\[
r' = U' \bar{U} + \bar{U} U'.
\] (4.18)

We also find
\[
dX = 2A(U)dU.
\] (4.19)

Defining the momenta as
\[
\frac{dX_i}{d\tau} = P_i, \quad \frac{dU_i}{d\tau} = \bar{P}_i, \; i = 0, 1, 2, 3
\] (4.20)
shows that the compatibility with Eq. (4.19) sets
\[
\frac{d\tau}{dr} = \frac{1}{4r}.
\] (4.21)

It can be verified easily that the anzats
\[
P = \frac{1}{2r} A(U) \bar{p}
\] (4.22)
guarantee that the transformation \( (X_i, P_i) \rightarrow (U_i, \bar{P}_i) \) is a canonical transformation.

This is manifesting the generalization of Levi-Civita’s conformal squaring. Hence the KS transformation maps \( U = (U_0, U_1, U_2, U_3) \in \mathbb{R}^4 \) to \( X = (X_0, X_1, X_2+) \in \mathbb{R}^3 \). Since the mapping does not preserve the dimension, its inverse in the usual sense does not exist.

Now, instead of considering the squaring of complex numbers as in the case of Levi-Civita regularization, we define mapping in (4.7). The image \( U \mapsto X := U U^* \) is the set of quaternions with vanishing \( k \) components, which can be identified with \( \mathbb{R}^3 \). By direct computation, (4.7) yields
\[
X_0 = U_0^2 - U_1^2 - U_2^2 + U_3^2, \quad X_1 = 2(U_0 U_1 - U_2 U_3), \quad X_2 = 2(U_0 U_2 + U_1 U_3),
\] (4.23)
follows directly from \( X := U U^* \), thus \( X = X^* \). This is exactly the K–S transformation in its classical form. Denoting the space coordinates by \( (X_0, X_1, X_2) \) with conjugate momenta \( (P_0, P_1, P_2) \), the symplectic form becomes
\[
\omega = dX_0 \wedge dP_0 + dX_1 \wedge dP_1 + dX_2 \wedge dP_2
\]
hence we can write \( \{X_i, P_n\} = \delta_{in} \) for \( i, n = 0, 1, 2 \). We must recall that \( \omega \) generates the folded symplectic structure [30], if we choose \( U_3 = 0 \), then
\[
\omega = 2(U_0 dU_0 - U_1 dU_1 - U_2 dU_2) \wedge dP_0 + (U_0 dU_1 + U_1 dU_0) \wedge dP_1 + (U_0 dU_2 + U_2 dU_0) \wedge dP_2.
\]
which yields \( \omega \wedge \omega \wedge \omega = U_0(U_0^2 - U_1^2 - U_2^2) dU_0 \wedge dP_0 \wedge dU_1 \wedge dP_1 \wedge dU_2 \wedge dP_2 \), hence this is a hyperbolic like \( m \)-folded symplectic structure. A folded symplectic structure is a closed 2-form which is nondegenerate except on a hypersurface.

By direct computation one can check

\[
dX = d(U^* U) = dU U^* + U dU^* = 2dU U^* = 2 dU dU^*,
\]

(4.24)

provided the “k” component vanishes, which yields a bilinear relation \([15,18,29]\),

\[
U_0 dU_3 - U_3 dU_0 + U_2 dU_1 - U_1 dU_2 = 0.
\]

(4.25)

This condition appears since KS transformation is a mapping from \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \), it therefore, leaves one degree of freedom in the parametric space undetermined. By imposing this bilinear relation (4.25), the tangential map in (4.23) change into a linear map and also this yields bilinear constraint \([15,18,29]\),

\[
U_0 \tilde{p}_0 - U_2 \tilde{p}_1 + U_1 \tilde{p}_2 - U_0 \tilde{p}_3 = 0, \text{ which makes } P_3 = 0.
\]

Again by taking Poisson bracket operation between coordinates, we get,

\[
\{ U_i, \tilde{p}_j \} = \delta_{ij} \text{ for } i, j = 0, 1, 2, 3.
\]

(4.26)

These relations implies that the mapping in (4.7) is canonical in nature.

### 4.1 Damped Kepler problem

In this subsection, we study the regularization of collision orbits of a particle’s motion in \( \frac{1}{r} \) potential that is also exposed to a damping force in three dimensions, which follow from a time-dependent Lagrangian. We first use a time-dependent point transformation to turn these equations into Euler–Lagrange equations derived from this time-independent Lagrangian. This helps us to construct conserved energy, which then allows us pursuance of the mapping of dynamics on the constant energy surface. After re-expressing these equations in terms of quaternions, we apply the K–S map to them. The equations are shown to represent a harmonic oscillator with inverted sextic potential and interactions.

We start with the Lagrangian describing Damped Kepler problem in three dimensions

\[
L = e^{\lambda t} \left( \frac{m}{2} \sum_{i=0}^{2} \dot{x}_i^2 + \frac{k}{r^2} e^{-\frac{3\lambda t}{2}} \right),
\]

(4.27)

where \( r = \sqrt{\sum_{i=0}^{2} x_i^2} \) and \( \dot{x}_i = \frac{dx_i}{dt} \). Here and below, we take \( i = 0, 1 \) and 2, rather than usual values 1, 2 and 3 for later convenience. Euler–Lagrange equation following from this Lagrangian is

\[
m\ddot{x}_i + \lambda m \dot{x}_i + \frac{k}{r^3} e^{-\frac{3\lambda t}{2}} = 0, \quad i = 0, 1, 2
\]

(4.28)

Applying the transformations

\[
x_i \rightarrow X_i = x_i e^{\frac{\lambda t}{2}}, \quad i = 0, 1, 2
\]

(4.29)

we rewrite the above Lagrangian in terms of new coordinates \( X_i \) as

\[
L = \frac{m}{2} \left( \sum_{i=0}^{2} \dot{X}_i^2 + \frac{\lambda^2}{4} \sum_{i=0}^{2} X_i^2 - \lambda \sum_{i=0}^{2} X_i \dot{X}_i \right) + \frac{k}{r},
\]

(4.30)
here $r = \sqrt{\sum_{i=0}^{2} X_i^2}$. Note that the damping parameter $\lambda$ in Eqs. (4.27), (4.28) is now appearing as the coefficient of $X_i^2$ and $X_i \dot{X}_i$ terms.

Equations of motion following from this Lagrangian are

$$m \ddot{X}_i - \frac{m \lambda^2}{4} X_i + \frac{k}{r^3} X_i = 0, \quad i = 0, 1, 2. \tag{4.31}$$

We next derive the corresponding Hamiltonian corresponding to this Lagrangian. Calculating the conjugate momentum using

$$P_i = \frac{\partial L}{\partial \dot{X}_i} = m \dot{X}_i - \frac{m \lambda^2}{2} X_i, \quad i = 0, 1, 2 \tag{4.32}$$

and using this we obtain

$$H = \frac{2}{m} \sum_{i=0}^{2} P_i^2 + \frac{\lambda}{2} \sum_{i=0}^{2} X_i P_i - \frac{k}{r}, \tag{4.33}$$

(Here and below, we have used summation convention) which is a constant of motion. For later purposes, we express the total Hamiltonian in terms of the velocities as

$$E = \frac{m}{2} \sum_{i=0}^{2} \dot{X}_i^2 - \frac{m \lambda^2}{8} \sum_{i=0}^{2} X_i^2 - \frac{k}{r} \tag{4.34}$$

and define

$$-\mathcal{E} = -\frac{E}{m} = \sum_{i=0}^{2} \dot{X}_i^2 - \frac{\lambda^2}{8} \sum_{i=0}^{2} X_i^2 - \frac{\mu}{r} \tag{4.35}$$

which is a constant of motion.

Next, we define a re-parametrization of time through the relation

$$\frac{d}{dr} = \frac{1}{4r} \frac{d}{d\tau} \tag{4.36}$$

and thus we find

$$\dot{X}_i = \frac{1}{4r} X'_i; \quad \ddot{X}_i = \frac{1}{16r^2} X''_i - \frac{1}{16r^3} r' X'_i. \tag{4.37}$$

Using the above re-parametrization, we re-express the equations of motion (4.31) as

$$\frac{X''_i}{r^2} - \frac{1}{r^3} r' X'_i - 4 \lambda^2 X_i + \frac{16 \mu}{r^3} X_i = 0, \quad i = 0, 1, 2. \tag{4.38}$$

where $\mu = \frac{k}{m}$.

Now, using Eqs. (4.16, 4.17, 4.18), we re-express Eq. (4.38) as

$$2r U U''* - 2 |U'|^2 U U^* + 16 \mu U U^* - 4r^3 \lambda^2 U U^* = 0. \tag{4.39}$$

Next, we apply re-parametrization to the conserved quantity, $-E$ in Eq. (4.35) and re-express it as

$$-\mathcal{E} = \frac{1}{32r^2} |X'|^2 - \frac{\lambda^2}{8} \frac{r^2}{r^2} - \frac{\mu}{r} \tag{4.40}$$
which in terms of quaternions become

\[- \mathcal{E} = \frac{1}{8r} |U'|^2 - \frac{\lambda^2}{8} r^2 - \frac{\mu}{r} \text{ and we find} \]

\[-16\mathcal{E} r = 2 |U'|^2 - 2\lambda^2 r^3 - 16\mu. \]  (4.41)

Using this, we obtain from Eq. (4.39)

\[U'' + (8\mathcal{E} - 3\lambda^2 r^2) U' = 0. \]  (4.42)

Since \( r = |U|^2 \), we have \( r^2 = |U|^4 \) and using this, we re-express the above equation, explicitly, in components, as

\[U''_i + (8\mathcal{E} - 3\lambda^2 |U|^4) U_i = 0, \quad i = 0, 1, 2, 3. \]  (4.43)

These equations are the Euler–Lagrange equations following from the Lagrangian

\[L_{SO} = \frac{1}{2} m \sum_{i=0}^{3} (U'_i)^2 - 4m\mathcal{E} \sum_{i=0}^{3} U_i^2 + \frac{m\lambda^2}{2} \left( \sum_{i=0}^{3} U_i^2 \right)^3, \quad i = 0, 1, 2, 3. \]  (4.44)

This Lagrangian, describes a harmonic oscillator with inverted sextic potential and couplings in four dimensions. The Hamiltonian following from this is given by

\[H = \sum_{i=0}^{3} \frac{\tilde{P}_i^2}{2m} + 4m\mathcal{E} \left( \sum_{i=0}^{3} U_i^2 \right) - \frac{m\lambda^2}{2} \left( \sum_{i=0}^{3} U_i^2 \right)^3. \]  (4.45)

Note that the equations of perturbed Kepler problem in three dimensions given in Eq. (4.31) is mapped under K–S transformation and re-parametrization of time variable, to equations of Harmonic oscillator with inverted sextic potential and couplings. Here too, we see that the regularized equation is nonlinear and all the nonlinear terms have \( \lambda \) as the coefficient, indicating that their origin is due to the damping in the original system.

4.2 Homogeneous Hamiltonian

We now establish the equivalence between the damped Kepler problem in three dimensions to harmonic oscillator with inverted sextic potential and interactions in four dimensions at the level of Hamiltonians. This is shown using homogeneous Hamiltonian formalism. Homogeneous Hamiltonian formalism [15,31–33] treats both space coordinates and time on an equal footing. This is achieved (i) by treating the time parameter \( t \) as a function of another parameter, say \( s \) (then, \( x(t) \equiv x(s) \)), and (ii) by introducing a conjugate momentum for the time \( t(s) \). This approach has been shown to be suitable to derive relativistic wave equations, using the usual replacement of momentum operators with the derivatives with respect to conjugate variables [33]. This approach provides a natural way to study the quantization of systems with time-dependent constraints [32]. This approach naturally fits into the canonical treatment of regularization transformations. In both Levi-Civita transformation and K–S transformation, time is treated as a function of new parameter. Using this framework, one can easily show that the regularization transformations are canonical transformations [15]. Homogeneous formalism of non-relativistic H-atom was used to show its equivalence to harmonic oscillator in four dimensions [34].

We have seen that the damped Kepler problem in three dimensions is equivalent to the system described by the Lagrangian in Eq. (4.30) and here we show that homogeneous Hamiltonian corresponding to this model is equivalent to the Hamiltonian for four-dimensional oscillator with inverted sextic potential and interactions. Note that the Lagrangian obtained
by time-dependent point transformation of damped Kepler problem in three dimensions given in Eq. (4.30) is equivalent to

$$\mathcal{L} = \frac{m}{2} \left( \sum_{i=0}^{2} \dot{X}_{i}^{2} + \frac{\lambda^2}{4} \sum_{i=0}^{2} X_{i}^{2} \right) + \frac{k}{r} \quad (4.46)$$

as they differ only by a total derivative term and thus both lead to same equations of motion. The Hamiltonian derived from the above Lagrangian is

$$\mathcal{H} = \frac{1}{2m} \sum_{i=0}^{2} P_{i}^{2} - \frac{\lambda^2 m^2}{8} \sum_{i=0}^{2} X_{i}^{2} - \frac{k}{r} \quad (4.47)$$

Starting from this Hamiltonian, we derive the Hamiltonian describing harmonic oscillator with inverted sextic potential and interactions. For this, we first define the corresponding homogeneous Hamiltonian as

$$\bar{H}^H = 4r \left( \mathcal{H} + P_{t} \right) = \frac{2r}{m} \sum_{i=0}^{2} P_{i}^{2} - \frac{r \lambda^2 m^2}{2} \sum_{i=0}^{2} X_{i}^{2} - 4k + 4P_{t}r. \quad (4.48)$$

We now re-express this Hamiltonian in terms of $\tilde{p}_{i}$ and $U_{i}$, where we also use $r = \sum_{i=0}^{3} U_{i}^{2}$ and $\sum_{i=0}^{2} X_{i}^{2} = \left( \sum_{i=0}^{3} U_{i}^{2} \right)^{2}$, to obtain

$$\tilde{H}^H = \sum_{i=0}^{3} \frac{\tilde{p}_{i}^{2}}{8m} + 4E \left( \sum_{i=0}^{3} U_{i}^{2} \right) - \frac{\lambda^2 m^2}{8} \left( \sum_{i=0}^{3} U_{i}^{2} \right)^{3} + p_{s}. \quad (4.49)$$

In the above, we have identified $P_{t}$ with $4E$ and $4k$ with $-p_{s}$ which is the conjugate of the new time parameter $\tau$. Note that the above homogeneous Hamiltonian $\bar{H}^H$ is defined in four dimensions. The corresponding non-homogeneous Hamiltonian is

$$\tilde{H} = \sum_{i=0}^{3} \frac{\tilde{p}_{i}^{2}}{2m} + 4E \left( \sum_{i=0}^{3} U_{i}^{2} \right) - \frac{\lambda^2 m^2}{2} \left( \sum_{i=0}^{3} U_{i}^{2} \right)^{3} \quad (4.50)$$

and it describes inverted sextic oscillator with couplings in four dimensions. We see that this Hamiltonian in Eq. (4.50) is exactly same as one obtained in Eq. (4.45) with identification,

$$\tilde{p}_{i} = \tilde{P}_{i}, \quad E = m\mathcal{E} \quad (4.51)$$

Note that $U_{i}^{2}$-dependent term in the Hamiltonian in Eqs. (4.45) and (4.50) is in the form of harmonic oscillator potential with identification of relation between energy of perturbed Kepler problem and strength (angular frequency) of harmonic oscillator, i.e., $4E = \frac{1}{2}m\omega_{0}^{2}$. Here, in Eqs. (4.45) and (4.50), damping parameter $\lambda$ appear as the coefficient of inverted sextic potential and couplings.

5 Conclusion

In this paper, we have studied the regularization of central force systems with damping, both in two and three dimensions. The systems we have analyzed are (i) Kepler problem in two dimensions with friction included, (ii) particle moving under the influence of generic power
law potential of the form \( r^{-2N/N+1}, 0 \leq 2N/N + 1 < 2 \) subjected also to a force linear in velocities and (iii) damped Kepler problem in three dimensions. We have shown that the Levi-Civita transformation does regularize the equations of motion of first two systems while K–S transformations do the same for third system. In the first and third cases, the regularized equations describe harmonic oscillator with inverted sextic potential and interactions, in two and four dimensions, respectively. In the second case, regularized equation is that of a harmonic oscillator with potential and interaction that depend on the actual value of \( N, 0 \leq N < 2 \). In all the three cases, we note that the nonlinear terms in the regularized equations have \( \lambda \) as the coefficient where \( \lambda \) is the damping parameter in the original models.

The explicit time dependence of damped systems studied here is reflected in the fact that the corresponding Lagrangians/Hamiltonians also have explicit time dependence and thus breaks time translation invariance that the undamped systems enjoy. Thus, in these undamped systems, we do not have conserved energy as expected for dissipative systems. Existence of conserved quantity is essential for implementing Levi-Civita and/or K–S transformations. To overcome this obstacle, we have mapped the time-dependent equations (equivalently, Lagrangians) describing the damped systems to equivalent systems without explicit time dependence (see Eqs. (2.6, 3.5, 4.30). These systems do have conserved energies and we use them in applying regularization transformations, in three steps as discussed in the introduction. The regularized equations obtained here are nonlinear, unlike their undamped counter-parts. Our results reduced to the well-known, undamped cases in the limit \( \lambda \to 0 \), as expected.

We note that in the case of generic potential \( r^{-2N/N+1}, 0 \leq 2N/N + 1 < 2 \) with damping in two dimensions, the permutation operators (see Eq. (3.18)) were still commuting as in the undamped case. But in the case of three dimensions, we note that the permutation operators (see Eq. (4.12)) are not commuting. This feature of permutation operators mutually commuting is not due to damping and even when there is no damping, these operators in the three dimensions would be non-commuting. Thus, the identities satisfied by the transformation of coordinates in two dimensions shown in [28] will not be valid for three-dimensional Kepler problem. Thus, generalization of K–S transformation to the case generic potential \( r^{-2N/(N+1)} \) as this was done in two dimensions fails.

One could actually construct conserved quantities from the time-dependent Lagrangians/Hamiltonians of the damped systems considered here by adapting/generalizing the method used in [35]) for deriving such a conserved quantity in one dimension. These conserved quantities will reduce to the familiar conserved energy when the damping parameter \( \lambda \) vanishes. But using these conserved quantities in applying regularization transformations will lead to equations that depend on both new and old time parameters.

In an interesting paper, Andrade et al. [36] studied Levi-Civita regularization problem of the Kepler problem on surfaces of constant curvature. We would also like to study the regularization problem of the Kepler equation with a drag force on surfaces of constant curvature, both positive and negative, \( \mathbb{S}^2 \) and \( \mathbb{H}^2 \), respectively.

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**Data Availability Statement** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.
Appendix: A motion in $r^2$ potential with damping in two dimensions: Bohlin–Sundman map

In this Appendix, we study the application of Bohlin–Sundman mapping to the equations of motion describing a damped harmonic oscillator. After mapping these equations to that of a shifted harmonic oscillator by a time-dependent point transformation, we re-express the equations in terms of complex coordinates. Then by applying a re-parametrization of time followed by a coordinate change, we map these equations describing a dynamics on a constant energy surface to that of a particle moving in $\frac{1}{r}$ potential.

We start from the equations of motions

$$q_i'' + \lambda q_i' + \Omega^2 q_i, \ i = 1, 2 \tag{A.1}$$

describing damped harmonic motion in two dimensions. Here $q_i' = \frac{dq_i}{d\tau}$ and $q_i'' = \frac{d^2q_i}{d\tau^2}$. We now apply the time-dependent coordinate transformation

$$x_i = q_i e^{\frac{\lambda \tau}{2}}, i = 1, 2 \tag{A.2}$$

and rewrite the above equations of motion as

$$x_1'' + \tilde{\Omega}^2 x_1 = 0, \tag{A.3}$$

$$x_2'' + \tilde{\Omega}^2 x_2 = 0, \tag{A.4}$$

where $\tilde{\Omega}^2 = \Omega^2 - \frac{\lambda^2}{4}$. These are Euler–Lagrange equations following from the Lagrangian

$$L = \frac{m}{2} \left( x_1'^2 + x_2'^2 \right) - \frac{m}{2} \tilde{\Omega}^2 \left( x_1^2 + x_2^2 \right) - \frac{m\lambda}{2} (x_1x_1' + x_2x_2'). \tag{A.5}$$

We re-express these equations using complex coordinate

$$\omega = x_1 + ix_2 \tag{A.6}$$

as

$$\omega'' + \tilde{\Omega}^2 \omega = 0. \tag{A.7}$$

We now apply Bohlin–Sundman transformation

$$\omega \rightarrow Z = \omega^2 \tag{A.8}$$

and also implement re-parametrization of time$^4$ using

$$\dot{Z} \frac{dZ}{d\tau} = \frac{\bar{\omega}}{2} \frac{d\omega}{d\tau} \tag{A.9}$$

Using Eq. (A.8) in Eq. (A.9), we get

$$\frac{d}{d\tau} = \frac{1}{4\bar{\omega} \omega} \frac{d}{d\tau} \tag{A.10}$$

---

$^3$ These equations follow from the BCK–Lagrangian $L = \frac{1}{2m} \left( \dot{q}_i^2 - \Omega^2 q_i^2 \right) e^{\lambda \tau}, i = 1, 2.$

$^4$ Bohlin–Sundman transformation has been applied to derive the mapping between two-dimension harmonic oscillator to two-dimension Kepler problem. Here, angular momentum is conserved in both systems and demanding these constants of motion are proportional to each other, results in the relation between time parameters of these two systems. In the present case too, angular momentum is conserved for damped harmonic oscillator as well as (damped) Kepler problem in two dimensions.
and using this we find
\[
\dot{Z} = \frac{dZ}{dt} = \frac{1}{2\bar{\omega}} \frac{d\omega}{d\tau} \quad \text{(A.11)}
\]
\[
\ddot{Z} = \frac{d^2Z}{dt^2} = \frac{1}{8\bar{\omega}^2} \left[ \frac{d^2\omega}{d\tau^2} - \frac{1}{\bar{\omega}^2} \left( \frac{d\omega}{d\tau} \right) \left( \frac{d\omega}{d\tau} \right) \right] \quad \text{(A.12)}
\]

From Eq. (A.7), we have \( \omega'' = -\tilde{\Omega}^2 \omega \) and using this, we rewrite the second equation in the above as
\[
\ddot{Z} = -\frac{1}{8\bar{\omega}^2} \left[ \frac{1}{\omega} \bar{\Omega}^2 \omega + \frac{1}{\omega^2} \left( \frac{d\omega}{d\tau} \right) \left( \frac{d\omega}{d\tau} \right) \right] \quad \text{(A.13)}
\]
\[
= -\frac{Z}{8|Z|^3} \left[ \left( \frac{d\tilde{\omega}}{d\tau} \right) \left( \frac{d\omega}{d\tau} \right) + \bar{\Omega}^2 \tilde{\omega} \omega \right] \quad \text{(A.14)}
\]

We now derive the conserved “energy” associated with the Lagrangian in Eq. (A.5) and re-express the terms in the \([\ ]\) appearing in the above equation. To this end, we first obtain the conjugate momenta corresponding to \( x_i \) as
\[
p_i = mx'_i - \frac{m\lambda}{2} x_i, \ i = 1, 2 \quad \text{(A.15)}
\]
and construct the Hamiltonian in terms of velocities as
\[
H = \frac{m}{2} \left( x_1'^2 + x_2'^2 \right) + \frac{m\bar{\Omega}^2}{2} \left( x_1^2 + x_2^2 \right) \quad \text{(A.16)}
\]
\[
= \frac{m}{2} \left[ \tilde{\omega}' \omega' + \bar{\Omega}^2 \tilde{\omega} \omega \right] \equiv E. \quad \text{(A.17)}
\]
Since \( E \) above is a constant, we use it to re-express Eq. (A.14) as
\[
\ddot{Z} = -\frac{E}{4m} \frac{Z}{|Z|^3}. \quad \text{(A.18)}
\]

We note that with the identification of conserved \( E \) with \( 4k = m\tilde{\Omega}^2 \), the strength of Kepler potential,
\[
\frac{E}{4} = k, \quad \text{(A.19)}
\]

Eq. (A.18) is the Kepler’s equation in two dimensions, written in the complex coordinate \( Z = X_1 + iX_2 \).

Thus, we have mapped the equation of “damped” harmonic oscillator in two dimensions, to the equation of motion corresponding to the (undamped) Kepler problem in two dimensions.

We now start with the expression for energy of 2-dim Kepler system, described in terms of \( \tilde{Z} \) and \( Z \) and re-express it in terms of \( \tilde{\omega} \) and \( \omega \), i.e.,
\[
E_{\text{Kepler}} = \frac{m}{2} \frac{d\tilde{Z}}{dt} \frac{dZ}{dt} - \frac{k}{|Z|} \quad \text{(A.20)}
\]
\[
= \frac{m}{8} \left[ \frac{1}{\tilde{\omega} \omega} \left( \frac{d\tilde{\omega}}{d\tau} \right) \left( \frac{d\omega}{d\tau} \right) \right] - \frac{k}{\tilde{\omega} \omega}. \quad \text{(A.21)}
\]

Using above equation and Eq. (A.19), we get
\[
- E_{\text{Kepler}} = \frac{m}{8} \left( \Omega^2 - \frac{\lambda^2}{4} \right). \quad \text{(A.22)}
\]
Thus, we find

1. The equations of motion of two-dimensional damped Harmonic oscillator in Eq. (A.1) are first mapped to equations of a shifted harmonic oscillator given in Eq. (A.4) which are then mapped to that of (undamped) Kepler problem in two dimensions.

2. The strength of the Kepler potential is related to the conserved energy of the shifted harmonic oscillator.

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