The Garrison–Wong quantum phase operator revisited

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We revisit the quantum phase operator $\Phi$ introduced by Garrison and Wong. Denoting by $N$ the number operator, we provide a detailed proof of the Heisenberg commutation relation $\Phi N - N \Phi = iI$ on the natural maximal domain $D(\Phi N) \cap D(N \Phi)$ as well as the failure of the Weyl commutation relations, and discuss some further interesting properties of this pair.

I. INTRODUCTION

A Heisenberg pair is an ordered pair $(A,B)$ of (possibly unbounded) self-adjoint operators, acting on the same Hilbert space $H$, such that for all $h \in D(AB) \cap D(BA)$ we have $ABh - BAh = ih$.

Here, $D(AB) = \{ h \in D(B) : Bh \in D(A) \}$ and similarly the other way around. A Weyl pair is an ordered pair $(A,B)$ of (possibly unbounded) self-adjoint operators, acting on the same Hilbert space $H$, such that for all $t,s \in \mathbb{R}$ we have the operator identity

$$e^{iA} e^{iB} = e^{-ist} e^{iB} e^{iA}.$$ 

Here, $(e^{iA})_{t \in \mathbb{R}}$ and $(e^{iB})_{t \in \mathbb{R}}$ are the strongly continuous one-parameter groups generated by $iA$ and $iB$ in the sense of Stone’s theorem. By straightforward differentiation (see Ref.10) every Weyl pair is seen to be a Heisenberg pair, but the converse is false unless $A^2 + B^2$ is essentially self-adjoint (this is the Rellich–Dixmier theorem, see Ref.22 (Theorem 4.6.1) for a precise statement).

The standard textbook example (see Ref.14 (Section 12.2), Ref.22 (Section 2.11)) of a Heisenberg pair that is not a Weyl pair is the pair $(A,B)$ on $H = L^2(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle in the complex plane) given by

$$Af(\theta) = \theta f(\theta), \quad Bf(\theta) = \frac{1}{i} f'(\theta).$$

This is a variation of the standard position-momentum pair.

The aim of this short note is to revisit, from a mathematician’s point of view, some well known facts about the number-phase pair $(N,\Phi)$ on the Hilbert space $H^2(\mathbb{D})$ (where $\mathbb{D}$ is the open unit disc in the complex plane; the relevant definitions are given below). The interest of this pair derives from it being another example of a Heisenberg pair that is not a Weyl pair. This fact is well known and contained in Ref.14, except for some details concerning domains which we provide here. We also point out some further interesting features of this pair, providing along the way a rigorous justification of some observations in Ref.14. As such this note does not contain new results, but we hope that it could be of some use to the more mathematically inclined reader interested in the subject.

II. QUANTUM PHASE AND NUMBER

The problem of defining quantum phase operators has been considered by many authors2,6–8,11,17,19,20,25 and has been reviewed in several places3,18,21. It has recently found application in the context of quantum computing and quantum error correction codes12,13,23. The proposal by Garrison and Wong11 is particularly attractive from a mathematical perspective. On the Hilbert space

$$H := H^2(\mathbb{D})$$

whose elements consist of the holomorphic functions $f(z) = \sum_{n \in \mathbb{N}} c_n z^n$ on the unit disc $\mathbb{D}$ for which

$$\| f \|^2 := \sum_{n \in \mathbb{N}} |c_n|^2$$

is finite, they consider the bounded self-adjoint Toeplitz operator $\Phi$ with symbol $\arg(z)$, defined for functions $f, g \in H$ by the relation

$$\langle \Phi f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta \, f(e^{i\theta}) g(e^{i\theta}) \, d\theta.$$ (1)

Here we identify the function $f(z) = \sum_{n \in \mathbb{N}} c_n z^n$ with the Fourier series $f(e^{i\theta}) = \sum_{n \in \mathbb{N}} c_n e^{in\theta}$ and similarly for $g$. The choice of $\Phi$ as the quantum phase operator has been critically evaluated on physical grounds by several authors2,6,10.

The number operator is the unbounded self-adjoint operator $N$ in $H$ given by

$$N f(z) = z f'(z)$$

on its maximal domain

$$D(N) = \left\{ f = \sum_{n \in \mathbb{N}} c_n e_n \in H : \sum_{n \in \mathbb{N}} n^2 |c_n|^2 < \infty \right\}.$$ 

The spectrum of $N$ is given by $\sigma(N) = \mathbb{N} := \{0, 1, 2, \ldots\}$ and

$$N e_n = ne_n, \quad n \in \mathbb{N},$$

where the functions

$$e_n(z) := z^n, \quad n \in \mathbb{N},$$

form an orthonormal basis of eigenvectors in $H$. 

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In what follows we write \( [\Phi, N] := \Phi N - N\Phi \) for the commutator of \( \Phi \) and \( N \), which we view as an operator defined on its maximal domain
\[
D([\Phi, N]) := D(\Phi) \cap D(N\Phi) = D(N) \cap D(\Phi),
\]
where
\[
D(\Phi N) := \{ f \in D(N) : Nf \in D(\Phi) = H \} = D(N),
D(N\Phi) := \{ f \in D(\Phi) : \Phi f \in D(N) \}.
\]

III. MAIN RESULT

It was shown by Garrison and Wong\(^{11}\) that the Heisenberg commutation relation
\[
[\Phi, N] f := (\Phi N - N\Phi) f = if
\]
holds for all functions \( f \) in a suitable subspace \( Y \), introduced in Lemma\(^{5}\) which is dense in \( H \) and contained in the domain of the commutator \( [\Phi, N] \). This fact, which we take for granted for the moment, self-improves as follows.

Proposition 1 For all \( f \in D([\Phi, N]) \) one has \( [\Phi, N] f = if \).

Proof. Let us denote by \( A \) and \( B \) the operator \( [\Phi, N] \) with domains \( D(A) = Y \) and \( D(B) = D([\Phi, N]) \). Then both \( A \) and \( B \) are densely defined and we have \( A \subseteq B \). By Lemma\(^{4}\) \( A \) is simply the restriction of the bounded operator \( iB \). This operator is closable and since \( Y \) is dense its closure equals \( \overline{A} = i\mathcal{H} \) with domain \( D(A) = H \).

The self-adjointness of \( \Phi \) and \( N \) immediately implies that
\[
(i[\Phi, N] f | g) = i((Nf | \Phi g) - (\Phi f | N g)) = (f | i[\Phi, N] g)
\]
for all \( f, g \in D([\Phi, N]) \). This means that \( iB \) is symmetric. In particular, \( iB \) (and hence \( B \)) is closable, a closed extension being given by its adjoints.

It now follows that \( iA = \overline{A} \subseteq B \) and therefore we must have \( D(B) = H \). As a result, \( \overline{B} = A = i\mathcal{H} \), and the asserted result follows. \( \square \)

In the terminology introduces earlier, the proposition says that \( (\Phi, N) \) is a Heisenberg pair. That it is not a Weyl pair can be seen by checking against the conditions of the Rellich–Dixmier theorem (as in Ref\(^{11}\)) or by noting that the Stone–von Neumann uniqueness theorem (see Ref\(^{14}\) (Chapter 14)) implies that both operators in a Weyl pair must be unbounded.

Remark 2 We could generalise the definition of a Heisenberg pair by insisting only that the commutation relation \( ABh - BAh = ih \) hold for all \( h \in Y \), where \( Y \) is some given dense subspace of \( H \) contained in \( D(AB) \cap D(BA) \). The above proof can be repeated verbatim to show that this definition, which is the one used in Ref\(^{11}\), is equivalent to the one given in the Introduction.

Remark 3 The following observation serves to justify our approach of interpreting the commutator \( [\Phi, N] \) in terms of its maximal domain: Let \( iN \) be the generator of a bounded \( C_0 \)-group on a Banach space \( X \). There does not exists a bounded linear operator \( T \) on \( X \) with the following two properties:

(i) for all \( x \in D(N) \) one has \( Tx \in D(N) \);

(ii) the identity \( TNx - NTx = ix \) holds for all \( x \in D(N) \).

Indeed, this is an immediate consequence of the second part of Ref\(^{16}\) (Theorem 3) to \( A = B = iN \) and \( C = I \). In our setting where \( N \) is the number operator, the arguments in the preceding remark imply that the operator \( \Phi \) fails property (i) for the function \( x = 1 \), the constant-one function.

Let us now give a detailed derivation of the Garrison–Wong result, filling in some domain issues along the way. We split the result into two lemmas, Lemmas\(^{4}\) and\(^{5}\). The starting point is the following explicit representation for \( \Phi \), which follows readily from (1):

\[
(\Phi_{e_n}| e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{i(m-n)\theta} \, d\theta = -i \frac{(-1)^{m-n}}{m-n} \delta_{m\neq n}.
\]

Since \( Ne_n = ne_n \), this gives

\[
(N\Phi_{e_n}| e_n) - (\Phi_{e_n}| Ne_n)
= -i \frac{(-1)^{m-n}}{m-n} \delta_{m\neq n}
= -i \frac{(-1)^{m-n}}{m-n} \delta_{m\neq n}.
\]

It follows that if \( f, g \in D(N) \) are finite sums of the form \( f = \sum_{j=0}^{d} c_je_j \) and \( g = \sum_{j=0}^{d} d_je_j \), then

\[
(Nf|\Phi g) - (\Phi f | Ng)
= -i \sum_{j,k=0}^{d} (-1)^{j+k} c_j d_k \delta_{j\neq k}
= i(f|g) - i \sum_{j=0}^{d} c_j d_j - i \sum_{j=0}^{d} (-1)^{j+k} c_j d_j \delta_{k\neq k}
= i(f|g) - i \sum_{j=0}^{d} (-1)^{j+k} c_j \left( \sum_{k=0}^{d} (-1)^k d_k \right) - i \sum_{k=0}^{d} (-1)^k d_k.
\]

For arbitrary \( f = \sum_{j=0}^{d} c_je_j \) and \( g = \sum_{j=0}^{d} d_je_j \) in \( D(N) \) (with convergence of the sums in \( H \)) we consider the truncations \( f_t = \sum_{j=0}^{11} c_je_j \) and \( g_t = \sum_{j=0}^{11} d_je_j \), which satisfy \( f_t, g_t \in D(N) \) and \( f_t \to f \) and \( g_t \to g \) in the graph norm of \( D(N) \). In combination with the boundedness of \( \Phi \) this gives

\[
(Nf|\Phi g) - (\Phi f | Ng)
= \lim_{t \to \infty} ((Nf_t|\Phi g_t) - (\Phi f_t | Ng_t))
= i(f|g) - i \sum_{j=0}^{d} (1)^{j+k} c_j \left( \sum_{k=0}^{d} (-1)^k d_k \right),
\]

where the limits in the last step exist by the absolute summability

\[
\sum_{j=0}^{d} |c_j| \leq \left( \sum_{j=0}^{11} \frac{1}{j+1} \right)^{1/2} \left( \sum_{j=0}^{11} (j+1)^2 |c_j|^2 \right)^{1/2}
\]

using the Cauchy–Schwarz inequality. Both terms in the right-hand side product are finite, the second because we are assuming that \( f \in D(N) \). In particular, if \( f = \sum_{j=0}^{d} c_je_j \) belongs to...
D(N), then the series defining \( f \) converges absolutely on \( \mathbb{D} \), and therefore such functions extend continuously to \( \mathbb{D} \).

The following lemma gives a necessary and sufficient condition for functions \( f \in D(N) \) to satisfy the Heisenberg commutation relation. It provides some details for Ref. 12 (Eq. (4.8)) as well as a converse to it.

**Lemma 4** For a function \( f = \sum_{j \in \mathbb{N}} c_j e_j \) in \( D(N) \) the following assertions are equivalent:

1. \( f \in D([\Phi, N]) \) and \( [\Phi, N] f = if \);
2. \( \sum_{j \in \mathbb{N}} (-1)^j c_j = 0 \);
3. \( f(-1) = 0 \).

**Proof.** The equivalence (2) ⇔ (3) is clear by the preceding observations.

(1)⇒(2): In the converse direction, if \( f \in D(N) \) belongs to \( D([\Phi, N]) \) and \( [\Phi, N] f = if \), then the above computation gives

\[
|\langle f | f \rangle | = \left| \sum_{j \in \mathbb{N}} (-1)^j c_j \right|^2
\]

and therefore \( \sum_{j \in \mathbb{N}} (-1)^j c_j = 0 \).

(2)⇒(1): Let \( f, g \in D(N) \) and suppose that \( f = \sum_{j \in \mathbb{N}} c_j e_j \) with \( \sum_{j \in \mathbb{N}} (-1)^j c_j = 0 \). The above computation then gives

\[
\langle Nf | \Phi g \rangle - \langle \Phi f | Ng \rangle = i \langle f | g \rangle
\]

and therefore

\[
|\langle \Phi f | Ng \rangle| \leq (||\Phi N f|| + ||f|| ||g||).
\]

This bound shows that \( \Phi f \in D(N^*) = D(N) \), which subsequently gives \( f \in D([\Phi, N]) \) and

\[
\langle [\Phi, N] f | g \rangle = (\Phi N f) g - (N \Phi f) g = i \langle f | g \rangle.
\]

This being true for all \( g \) in the dense subspace \( D(N) \), it follows that \( [\Phi, N] f = if \). \( \square \)

These results imply the following curious cancellation result: If a sequence of complex scalars \( (c_n)_{n \in \mathbb{N}} \) satisfies

(i) \( \sum_{n \in \mathbb{N}} n^2 |c_n|^2 < \infty \);

(ii) \( \sum_{n \in \mathbb{N}} n^2 \left( \sum_{m \in \mathbb{N}, m \neq n} \left| \frac{(-1)^{m-n}}{m-n} c_m \right| \right)^2 < \infty \),

then

\[
\sum_{n \in \mathbb{N}} (-1)^n c_n = 0.
\]

To see this, note that by (2), for functions \( f = \sum_{n \in \mathbb{N}} c_n e_n \) we have

\[
\Phi f(z) = \sum_{m \in \mathbb{N}} c_m \Phi e_m(z) = -i \sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{N}, n \neq m} \frac{(-1)^{m-n}}{m-n} c_n \right) z^n
\]

after changing the order of summation. Thus (i) and (ii) say that \( f \in D(N) \) and \( f \in D(N[\Phi]) \), respectively, so together they say that \( f \in D([\Phi, N]) \). For such functions, Proposition 11 asserts that the Heisenberg commutation relation holds, and therefore the stated conclusion holds by virtue of Lemma 4.

The next lemma from Ref. 13 implies that \( D([\Phi, N]) \) is dense in \( H \). The simple proof is included for the sake of completeness.

**Lemma 5** The subspace \( Y \) of \( H \) consisting of all functions \( f \in D(N) \) satisfying the equivalent conditions of Lemma 4 is dense in \( H \).

**Proof.** By the lemma 4 for all integers \( k \geq 1 \) the function \( f_k := e_0 + \sum_{j=0}^{k-1} \frac{1}{j+1} e_{2j+1} \) belongs to \( Y \). Moreover we have \( \|e_0 - f_k\| = 1/k \). As a result, \( e_0 \) belongs to the closure \( \bar{Y} \) of \( Y \) in \( H \). Again by the lemma, for all \( n \in \mathbb{N} \) we have \( e_n + e_{n+1} \in Y \). This implies that \( e_n \in \bar{Y} \) for all \( n \in \mathbb{N} \), and therefore \( Y \) is dense in \( H \).

It follows from Lemma 5 that \( D([\Phi, N]) \) is dense in \( H \). In the light of this, the following negative result is perhaps somewhat surprising.

**Proposition 6** The domain \( D([\Phi, N]) \) is not dense in \( D(N) \) with respect to the graph norm of the latter.

**Proof.** We begin by observing that

\[
\Phi e_m(z) = -i \sum_{n \in \mathbb{N}, n \neq m} (-1)^{m-n} \frac{m-n}{m-n} z^n.
\]

By taking \( m = 0 \), for \( e_0 = 1 \) this gives

\[
\Phi 1(z) = i \sum_{n=1}^{\infty} (-1)^n n z^n = -i \log(1 + z).
\]

For all \( g \in D([\Phi, N]) \) we have, using that \( N^* 1 = N 1 = 0 \),

\[
(1|\Phi N |g) = (1|\Phi Ng) = (\Phi 1|Ng).
\]

By the definition of adjoint operators, we have \( 1 \in D([\Phi, N]^*) \) if and only if there exists a constant \( C \) such that for all \( g \in D([\Phi, N]) \) we can estimate \( |(1|\Phi N |g)| \leq C ||g|| \). If that is the case, we also obtain that

\[
|(\Phi 1|Ng)| \leq C ||g||, \quad g \in D([\Phi, N]).
\]

Suppose now, for a contradiction, that \( D([\Phi, N]) \) is dense in \( D(N) \) with respect to the graph norm. Then, by density, (4) implies the stronger statement

\[
|(\Phi 1|Ng)| \leq C ||g||, \quad g \in D(N).
\]

But this is equivalent to asserting that \( \Phi 1 \in D(N^*) = D(N) \). But in that case \( \Phi 1(z) = -i \log(1 + z) \) (cf. (3)) would extend continuously to \( \mathbb{D} \), which is not the case.

Since \( D([\Phi, N]) \) is dense in \( H \) the adjoint operator \( [\Phi, N]^* \) is well defined, and the preceding argument proves that if
D([Φ, N]) is dense in D(N), then 1 /∈ D([Φ, N]*). But then we arrive at the contradiction

1 /∈ D([Φ, N]*) = D([Φ, N]* ) = D((iI)* ) = H,

using Ref.24 (Theorem 1.8) to justify the first equality. □

Let V be the contraction on H defined by the left shift

Ve_0 := 0, Ve_n := e_{n−1}, n ⩾ 1.

Identifying the functions e_n with elements of L^2(𝕋), the two-sided left shift on L^2(𝕋) is a unitary extension of V and is therefore given by a unique projection-valued measure P on 𝕋. Compressing P to H produces a positive operator-valued measure (POVM) Q on H such that

V^k = ∫︁_𝕋 λ^k dQ(λ), k = 0, 1, 2, ...

and this property uniquely characterises Q as a POVM. For the details the reader is referred to Ref.24; see also15.

As implicitly observed on page 87 of Ref.2, the Garrison–Wong operator Φ can be characterised in terms of Q as follows.

**Proposition 7** Φ = ∫︁_𝕋 arg(λ) dQ(λ).

The rigorous interpretation of the integral on the right-hand side is as follows. Denoting by B_b(𝕋) the Banach space of all bounded Borel measurable functions on 𝕋, one uses the boundedness of the Borel calculus of the projection-valued measure P associated with V to obtain that there exists a unique linear mapping Ψ : B_b(𝕋) → L(H), the space of bounded operators on H, satisfying

Ψ(1_B) = Q_B, B ⊆ 𝕋 Borel,

and

∥Ψ(f)∥ ⩽ ∥f∥_∞, f ∈ B_b(𝕋).

It further satisfies

Ψ(f*) = Ψ(λ̄), f ∈ B_b(𝕋).

We now define

∫︁_𝕋 f dQ := Ψ(f), f ∈ B_b(𝕋).

For all f_1, f_2 ∈ A(𝕋), the uniform closure of the trigonometric polynomials in C(𝕋), we have

Ψ(f_1)Ψ(f_2) = Ψ(f_1 f_2),

but this property does not extend to general functions f_1, f_2 ∈ B_b(𝕋).

**Proof of Proposition 7** We equipartition 𝕋 = (−π, π] into k subintervals of length 2π/k by setting I_j := (−π + 2π(j − 1)/k, −π + 2π j/k] for j = 1, . . . , k. Then, by the continuity of Ψ : f → ∫︁_𝕋 f dQ.

\[
\int_𝕋 \text{arg}(λ) dQ(λ) = \lim_{k → ∞} \int_𝕋 \text{arg}_k(λ) dQ(λ) = \lim_{k → ∞} \sum_{j=1}^{k} \frac{2π j}{k} Q(I_j),
\]

where arg_k(λ) := \sum_{j=1}^{N} \frac{2π j}{k} 1_{I_j}(λ). To compute Q(I_j) we use that Q is the compression to H of the projection-valued measure P associated with the two-sided shift U on L^2(𝕋). The latter is given by P(I_n) f = 1_n f for f ∈ L^2(𝕋). Accordingly, if we denote the inclusion mapping H → L^2(𝕋) by J, then for all f ∈ H we have

Q(I_j) f = J^* P(I_n) J f = J^* 1_n f.

It follows that

\[
\int_𝕋 \text{arg}(λ) dQ(λ) = \lim_{k → ∞} \sum_{j=1}^{k} \frac{2π j}{k} J^* 1_n f,
\]

identifying 1_n with the multiplication operator f → 1_n f from H to L^2(𝕋). On the other hand, by the definition of the operator Φ,

(Φ f | g) = ∫_𝕋 arg(λ) f(λ) g(λ) dλ, f, g ∈ H,

we have

Φ f = J^* (arg(·) f(·)) = \lim_{k → ∞} J^* (arg_k(·) f(·)) = \lim_{k → ∞} \sum_{j=1}^{k} \frac{2π j}{k} J^* 1_n f.

This completes the proof. □

**Remark 8** The arguments used in the proof imply that Q_B = 0 if the Borel set B has measure 0. It follows that the integral ∫_𝕋 f dQ is well defined for functions f ∈ L^∞(𝕋). With essentially the same proof as above one shows that for any φ ∈ L^∞(𝕋) the bounded Toeplitz operator T_φ on H with symbol φ is given by

T_φ = ∫_𝕋 f dQ.

As observed in Ref.2, the POVM Q obeys the following “covariance property”. For the reader’s convenience we include the simple proof.

**Proposition 9** For all t ∈ ℝ and Borel sets B ⊆ 𝕋 we have

e^{i t N} Q(B) e^{-i t N} = Q(e^{it} B),

where (e^{it} N)_{t≥0} is the unitary C_0-group on H generated by iN and e^{it} B = \{e^{it} λ : λ ∈ B\} is the rotation of B over angle t.
Proof. The properties of the projection-valued measure $P$ used in the proof of Proposition [7] imply that for the trigonometric functions $e_n$, $k \in \mathbb{N}$, we have

$$Q(B) e^{-\alpha N} e_n |e_m\rangle = (P(B) e^{-\alpha N} e_n |e_m\rangle) = e^{-\text{int}} (1_B e_n |e_m\rangle)$$

while at the same time

$$(e^{-\alpha N} Q(e^{N}B) e_n |e_m\rangle = (P(e^{N}B) e^{-\alpha N} e_n |e_m\rangle) = e^{-\text{int}} (1_B e_n |e_m\rangle)$$

$$= e^{-\text{int}} \int_{B} \lambda^{-m-n} d\lambda = e^{-\text{int}} \int_{B} (e^{-\alpha N})^{-m-n} d\mu$$

$$= e^{-\text{int}} \int_{B} \mu^{-m-n} d\mu = e^{-\text{int}} (1_B e_n |e_m\rangle).$$

Since the span of the trigonometric functions is dense in $H$, this completes the proof. \(\square\)

This contrasts with the failure of the Weyl commutation relations

$$\phi^N e^{i\alpha\Phi} e^{-\alpha N} = e^{-i\beta} e^{i\alpha\Phi}, \quad s, t \in \mathbb{R}. \quad (5)$$

This failure is usually demonstrated by showing that (5) would imply the identity $a = e^{-i\alpha} N^{1/2}$, where $a$ is the annihilation operator associated with $N$ (so that $a^* a = N$); this identity is subsequently shown to be impossible if at the same time $\Phi$ is to be self-adjoint (see Ref.\(^2_{11,25}\)).

Here, by elementary methods, we will give a direct proof of the more precise result that the Weyl relation (5) fails for every fixed $s \neq 0$:

**Proposition 10** Let $T$ be an arbitrary bounded operator on $H$. If $s \in \mathbb{R}$ is such that for all $t \in \mathbb{R}$ one has

$$e^{i\alpha N} e^{i\beta T} e^{-i\alpha N} = e^{-\text{int}} e^{i\beta T}, \quad (6)$$

then $s = 0$. The same conclusion holds if we assume that $T$ is a (possibly unbounded) self-adjoint operator on $H$.

**Proof.** Suppose that $s \in \mathbb{R}$ is such that (6) holds for all $t \in \mathbb{R}$. Choose $m, n \in \mathbb{N}$ so that $(e^{i\beta T} e_n |e_m\rangle \neq 0$. Applying (6) to $e_n$ and taking inner products with $e_m$, we obtain

$$e^{i(m-n)} (e^{i\beta T} e_n |e_m\rangle = e^{-\text{int}} (e^{i\beta T} e_n |e_m\rangle).$$

This can hold for all $t \in \mathbb{R}$ only if $e^{i(m-n)} = e^{-i\text{int}}$ for all $t \in \mathbb{R}$, forcing $s = n - m \in \mathbb{Z}$.

Suppose next that $s = k \in \mathbb{Z}$ is such that (6) holds for all $t \in \mathbb{R}$. If $k \geq 1$, the above argument shows that we must have $(e^{i\beta T} e_n |e_m\rangle = 0$ unless $n - m = k$, which implies that $e^{i\beta T} e_n$ is a multiple of $e_{n-k}$ if $n \geq k$ and $e^{i\beta T} e_n = 0$ if $0 \leq n \leq k - 1$. Given a fixed $n \in \mathbb{N}$, it follows that $e^{i\beta T} e_n = 0$ for all sufficiently large $m \in \mathbb{N}$. But this is impossible as it would lead to the contradiction

$$e_n = e^{-i\text{int}} e^{i\beta T} e_n = 0.$$