CONSTRUCTING INVOLUTIVE TABLEAUX WITH GUILLEMIN NORMAL FORM

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Abstract. Involutivity is the formal algebraic property that guarantees solutions to an analytic and torsion-free exterior differential system or partial differential equation via the Cartan–Kähler theorem. Guillemin normal form characterizes involutive systems by an almost-commuting property of the symbol map on certain subspaces of the prolongation’s tableau. This article examines Guillemin normal form in detail, aiming at a more systematic approach to classifying involutive systems.

The main result is an explicit quadratic condition for involutivity of the type suggested but not completed in Chapter IV §5 of the book Exterior Differential Systems by Bryant, Chern, Gardner, Goldschmidt, and Griffiths. To show the utility of this condition, several classical facts regarding the involutive tableaux and the characteristic variety are re-proven explicitly using elementary techniques.

It is hoped that this approach will make the subject accessible to a wider population of researchers, including undergraduates.

Contents

1. Background 2
2. Endovolutivity 4
3. Involutivity 8
4. The Characteristic Variety 12
5. Proof of Theorem 3.2 14
6. Discussion 24
References 25
1. Background

Fix complex vector spaces (or bundles, etc.) $W$ and $V$ of dimension $r$ and $n$, respectively. A **tableau** is a vector space $A$ with an exact sequence

\[ 0 \to A \to W \otimes V^* \xrightarrow{\sigma} H^1(A) \to 0. \]

A homomorphism $\sigma$ with kernel $A$ is called a **symbol**, taking values in the cokernel $H^1(A) = A^\perp$.

Given a tableau $A$, consider the tableau $A^{(1)} \to A \otimes V^*$ given as the kernel of the map $\delta_\sigma : A \otimes V^* \to W \otimes \wedge^2 V^*$ by $\delta_\sigma = (1_W \otimes \delta) \circ (\sigma \otimes 1_{V^*})$ where $\delta$ is the skewing map $\delta : V^* \otimes V^* \to V^* \wedge V^*$. This $A^{(1)}$ is called the (first) **prolonged tableau**, and the map $\sigma \delta$ is the (first) **prolonged symbol**. The cokernel is written $H^2(A)$. Proceeding in this way, we obtain the Spencer exact sequences:

\[ 0 \to W \xrightarrow{1} H^0 \to 0 \]

\[ 0 \to A \xrightarrow{\sigma} W \otimes V^* \to H^1 \to 0 \]

\[ 0 \to A^{(1)} \xrightarrow{\delta_\sigma} A \otimes V^* \xrightarrow{\delta_\sigma} W \otimes \wedge^2 V^* \to H^2 \to 0 \]

\[ 0 \to A^{(2)} \xrightarrow{\delta_\sigma} A^{(1)} \otimes V^* \xrightarrow{\delta_\sigma} W \otimes \wedge^3 V^* \to H^3 \to 0 \]

\[ \vdots \]

\[ 0 \to A^{(n-1)} \xrightarrow{\delta_\sigma} A^{(n-2)} \otimes V^* \xrightarrow{\delta_\sigma} W \otimes \wedge^n V^* \to H^n \to 0 \]

The spaces $H^{\rho+1}(A)$ are called the Spencer cohomology groups associated to the tableau $A$.

Tableaux, symbols, and prolongations arise from the following situation: Let $M$ denote a $C^\omega$ manifold, and let $I$ be a finitely generated, differentially closed ideal of the exterior algebra $(\Omega^\ast(M), d)$, interpreted as a $C^\omega(M)$-module. The data $(M, I)$ is called an exterior differential system, and in particular any (analytic) system of partial differential equations can be expressed as an exterior differential system. The exterior differential systems perspective is particularly useful for finding solutions of overdetermined systems. (See [BCG+90] for additional context.) Let $M^{(1)}$ be a smooth, connected component of the subvariety $V_n(I) \subset \text{Gr}_n(TM)$ comprised of maximal ordinary integral elements $e \subset T_p M$ of $I$. (We let $n$ denote this maximal dimension.) For every $e \in M^{(1)}$, let $V = e \otimes \mathbb{C}$ and $W = e^\perp \otimes \mathbb{C}$, so that $V$ and $W$ are vector bundles over $M^{(1)}$. Let $r = \dim W$. For any $e \in M^{(1)}$ over $p \in M$, let $A = TM^{(1)} \otimes \mathbb{C}$, and recall that $T\text{Gr}_n(T_p M) \otimes \mathbb{C}$
is canonically identified with $W \otimes V^*$. Then $A$ is a subbundle of $W \otimes V^*$ over $M^{(1)}$, and the symbol $\sigma$ is the corresponding annihilator of the subspace $TM^{(1)} \subset T\text{Gr}_n(T_pM)$. In this situation, one may also interpret $V$, $W$, and $A$ to be fixed vector spaces and see their elements as 1-forms $TM^{(1)} \to V \oplus W \oplus A$ describing a linear Pfaffian system differentially generated by $W$ on $M^{(1)}$. However, we will not pursue that viewpoint here, restricting our attention to the algebraic aspects.

Given bases of $W$ and $V^*$, $A$ is comprised of $r \times n$ matrices\(^1\), and the generators of $A$ appear in the first $s_1$ entries of the first columns, the first $s_2$ entries of the second columns, and so on to the first $s_n$ entries of the last column. These numbers, called Cartan characters, are constant over a dense open subset of bases (called the generic bases), and those constant values satisfy $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq 0$. Let $\ell$ denote the index of the last non-zero $s_\ell$. Hence, $\dim A = s_1 + s_2 + \cdots + s_\ell$. It is a standard exercise to show $\dim A^{(1)} \leq s_1 + 2s_2 + \cdots + \ell s_\ell$.

The most interesting property to study for a tableau is

**Definition 1.3 (Involutivity.).** A tableau is called involutive if and only if equality holds in the relation $\dim A^{(1)} = s_1 + 2s_2 + \cdots + ns_n$.

Verification of this equality is called Cartan’s test. Via Cartan’s test and the Cartan–Kähler theorem, involutivity guarantees that the exterior differential system $(M, \mathcal{I})$ admits solutions to the general Cauchy initial-value problem.\(^2\) Moreover, those solutions occur in families parametrized by $r$ constants, $s_1$ functions of 1 variable, $s_2$ functions of 2 variables, \ldots, $s_\ell$ functions of $\ell$ variables. That is, the Cartan characters describe the Hilbert scheme of solutions.

In [Gui68], Guillemin characterized involutive tableaux in terms of a “normal form” for the symbol map, using Quillen’s results on the exactness of Spencer cohomology from [Qui64]. Guillemin normal form was reconsidered for exterior differential systems in [Yan87] and [BCG+90]. To aide future computational and theoretical applications of Guillemin normal form, this article\(^3\) is intended to clarify explicitly some details of Guillemin normal form that were vague or omitted in those references. The main result is Theorem 3.2. All the other lemmas and theorems appear in some form in the references; however, the proofs offered here remain notable because they should be accessible to an undergraduate who understands linear algebra.

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\(^1\)For an exterior differential system, those are matrix-valued 1-forms over $C^\infty(M^{(1)}, \mathbb{C})$.

\(^2\)\ldots on a submanifold where torsion vanishes. We do not discuss torsion here.

\(^3\)This article began as an appendix to [Smi14], so there is some overlap in the presentation. However, be aware that some indices—such as $i, j$—and some notations—such as $Y^*$—differ between the two articles.
Figure 1. A tableau in coordinates, with Cartan characters $s_1 \geq s_2 \geq \cdots \geq s_\ell$. The upper-left shaded entries are independent generators. The lower-right entries depend on them via $\pi_i^a = B_{i,b}^{a,\lambda} \pi_\lambda^b$, summed as in (2.1).

2. Endovolutivity

Permanently reserve the index ranges $i, j, k, l \in \{1, \ldots, n\}$ and $\lambda, \mu \in \{1, \ldots, \ell\}$ and $\varrho, \varsigma \in \{\ell + 1, \ldots, n\}$ and $a, b, c, d \in \{1, \ldots, r\}$. Let $(u_i)$ and $(w_a)$ denote bases of $V^*$ and $W$, respectively, so that an element $\pi \in W \otimes V^*$ is written as a matrix $\pi = \pi_i^a (w_a \otimes u^i)$. For a dense, open subset of these bases, the generators of the subspace $A$ appear in the matrix $\pi$ according to the Cartan characters, in the first $s_1$ entries of column 1, the first $s_2$ entries of column 2, and so on up to the first $s_\ell$ entries of column $\ell$. Set $s_\varrho = 0$ for $\varrho > \ell$. The basis is called generic in this case, where the sequence $(s_1, s_2, \ldots, s_n)$ is lexicographically maximized. Let $U^*$ denote the $\ell$-dimensional subspace spanned by $u^1, \ldots, u^\ell$. Let $Y^*$ denote the complementary subspace spanned by $u^1, \ldots, u^n$. Using this basis, we sometimes use the dual spaces $Y = (U^*)^\perp = \langle u_{\ell+1}, \ldots, u_n \rangle$ and $U = (Y^*)^\perp = \langle u_1, \ldots, u_\ell \rangle$.

Using a generic basis, the symbol $\sigma$ can be expressed as a minimal system of equations of the form

$$\left\{ 0 = \pi_i^a - B_{i,b}^{a,\lambda} \pi_\lambda^b \right\}_{s_i < a}$$

where $B_{i,b}^{a,\lambda} = 0$ unless $\lambda \leq i$ and $b \leq s_\lambda$ and $s_i < a$. See Figure 1. Using the coefficients $B_{i,b}^{a,\lambda}$, we define an element of

$$V^* \otimes V \otimes W \otimes W^* \cong \text{End}(V^*) \otimes \text{End}(W)$$
by
\begin{equation}
\sum_{a \leq s_i} \delta^a_b (w_a \otimes w^b) \otimes (u^i \otimes u_\lambda) + \sum_{a > s_i} B^{a,\lambda}_{i,b} (w_a \otimes w^b) \otimes (u^i \otimes u_\lambda).
\end{equation}

Then, for each $\phi = \varphi; u^i \in V^*$, there is a homomorphism $B(\phi) : V \to \text{End}(W)$ defined by (2.2) as
\begin{equation}
B(\varphi)(v) = \sum_{a \leq s_\lambda} \varphi_\lambda v^\lambda \delta^a_b (w_a \otimes w^b) + \sum_{a > s_i} \varphi_\lambda B^{a,\lambda}_{i,b} v^i (w_a \otimes w^b).
\end{equation}

where $v^i = u^i(v)$ or $v = v^i u_i$ using the dual basis for $V$. Note that $B(\varphi) = B(\xi)$ if $\varphi_\lambda = \xi_\lambda$ for all $\lambda$, so (2.2) is really an element of $V^* \otimes U \otimes \text{End}(W)$. We write $B^\lambda_i$ for $B(u^\lambda)(v_i)$, but note that $(B^\lambda_i)^a_b$ is not quite the same as $B^{a,\lambda}_{i,b}$ due to the identity term in (2.2); in particular,
\begin{equation}
(B^\lambda_i)^a_b = \begin{cases} 
\delta^a_b, & \text{if } \lambda = i \text{ and } a \leq s_\lambda, \\
B^{a,\lambda}_{i,b}, & \text{if } a > s_i.
\end{cases}
\end{equation}

For each $i$, let
\begin{equation}
W_i^- = \{ z = w_a z^a : z^a = 0 \forall a \leq s_i \} \\
W_i^+ = \{ z = w_a z^a : z^a = 0 \forall a > s_i \}
\end{equation}

So that $W = W_i^- \oplus W_i^+$ and $W_1^- \supset W_2^- \supset \cdots \supset W_n^-$ according to $s_1 \geq s_2 \cdots \geq s_n$. Of course, for $\rho > \ell$, we have $W_\rho^- = \emptyset$. For each $\lambda$, consider also the subspace
\begin{equation}
A_\lambda^- = \{ \pi = B(u^\lambda)(\cdot) z, \ z \in W_\lambda^- \} \subset A
\end{equation}

The symbol relations (2.1) say that the coefficients $\pi_\lambda^a$ of $\pi \in A_\lambda^-$ are determined by the choice of $z \in W_\lambda^-$, so $A_\lambda^-$ and $W_\lambda^-$ are isomorphic via the projection onto first $s_\lambda$ entries in column $\lambda$. Using this basis and isomorphism, there is a decomposition
\begin{equation}
A = \bigoplus_{\lambda=1}^\ell A_\lambda^- \cong \bigoplus_{\lambda=1}^\ell W_\lambda^-.
\end{equation}

Specifically, if $\pi = \pi_\lambda^a (w_a \otimes u^i) \in A$, then let
\begin{equation}
z_\lambda = \sum_a \pi_\lambda^a w_a \in W, \ \text{for } \pi_\lambda^a = \begin{cases} 
\pi_\lambda^a, & a \leq s_\lambda \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

So, the decomposition (2.7) yields
\begin{equation}
\pi = \sum_\lambda \pi_\lambda = \sum_\lambda B(u^\lambda)(\cdot) z_\lambda \in W \otimes V^*
\end{equation}

Since $\dim W_\lambda^- = s_\lambda$, this is a more precise version of the statement that, for a generic flag, the tableau matrix has $s_1$ generators in the first column, $s_2$ in the second column, and so on until the final $s_\ell$ generators in the $\ell$ column.
For any $\varphi = \varphi_i u^i \in V^*$, define the subspaces
\[
W^- (\varphi) = W^-_{\min\{i : \varphi_i \neq 0\}} \\
A^- (\varphi) = \{ \pi = B(\varphi)(\cdot) z, \ z \in W^- (\varphi) \}.
\]

**Definition 2.11 (Endovolutivity).** A tableau expressed in a generic basis as (2.1) is called endovolutive if and only if $B_{a,\lambda}^{i,b} = 0$ for all $a > s_\lambda$ using that basis. See Figure 2.

The property of endovolutivity is discussed (but not named) on Page 147 (Page 127 in the online version) of [BCG+90] and in Section 1.2 of [Yan87]. Endovolutivity is so-named here because of the following lemma.

**Lemma 2.12.** Fix a basis for $V^*$ and for $W$. The tableau $A$ is endovolutive if and only if $B(\varphi)(v)$ is an endomorphism of $W^- (\varphi)$ for all $\varphi \in V^*$ and $v \in V$.

**Proof.** This follows easily from our definition of $W^- (\varphi)$. If $i' = \min\{i : \varphi_i \neq 0\}$, then endovolutivity in a generic basis implies $B^\lambda_j$ preserves $W^-_\gamma$ for all $\lambda \geq i'$, and $B(\varphi)(u_j) = \varphi_\lambda B^\lambda_j$. Conversely, each $u^\lambda$ is a particular choice of $\varphi$, and $W^- (u^\lambda) = W^-_\lambda$, so $B(u^\lambda)(v)$ is an endomorphism of $W^- (u^\lambda)$ if and only if $B_{a,\lambda}^{i,b} = 0$ for all $a > s_\lambda$.

The interesting part of the converse is the observation that the property of endovolutivity does not depend on the generic basis of $V^*$. If we have two generic bases, $(\tilde{u}^i)$ and $(u^j)$ related by $\tilde{u}^j = g_j^i u^i$, then with respect to the $(u^j)$ basis, $W^- (\tilde{u}^j) = W^-_{\min\{i : g_j^i \neq 0\}}$, which is generally $W^-_1$, so $B_{a,1}^{i,b} = 0$ for $a > s_1$. But, consider $\varphi = \tilde{u}^j - g_j^i u^i$. Then $\min\{i : \varphi_i \neq 0\} \geq 2$, so
we can establish $B^{n,2}_{k,b} = 0$ for $a > s_2$. Proceed by finding pivots of $g$ as in LU-decomposition. □

When considering endovolutive tableaux, it useful to arrange the symbol endomorphisms as an $\ell \times n$ array of $r \times r$ matrices:

$$\vec{B} = \begin{bmatrix} I_{s_1} & B_{1}^1 & B_{2}^1 & B_{3}^1 & \cdots & B_{s_1}^1 \\ 0 & I_{s_2} & B_{1}^2 & B_{2}^2 & B_{3}^2 & \cdots & B_{s_2}^2 \\ 0 & 0 & I_{s_3} & B_{1}^3 & B_{2}^3 & \cdots & B_{s_3}^3 \\ 0 & 0 & 0 & I_{s_4} & \cdots & B_{1}^4 & \cdots & B_{s_4}^4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & B_{1}^{\lambda} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & I_{s_\ell} & \cdots & B_{1}^{\ell_\ell} \end{bmatrix}$$

In (2.13), each $r \times r$ matrix in row $\lambda$ is 0 outside the upper-left $s_\lambda \times s_\lambda$ part. Note that a change of basis on $V$, like taking $\tilde{u}^1 = u^1 + \varphi_2 u^2 + \cdots + \varphi_\ell u^\ell$ as in the proof of Lemma 2.12, causes (2.13) to change by a block-wise conjugation:

$$\vec{B} \mapsto \begin{bmatrix} 1 & \varphi_2 & \cdots & \varphi_\ell \\ 1 & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \vdots & \ddots & 1 \end{bmatrix} \vec{B} \begin{bmatrix} 1 & -\varphi_2 & \cdots & -\varphi_\ell & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

For example, the symbol of the endovolutive tableau

$$\pi = \begin{bmatrix} \eta^1 & \eta^4 & \eta^6 \\ \eta^2 & \eta^5 & Q_4 \eta^4 + Q_5 \eta^5 + T_1 \eta^1 + T_2 \eta^2 + T_3 \eta^3 \\ \eta^3 & P_1 \eta^1 + P_2 \eta^2 + P_3 \eta^3 & R_1 \eta^1 + R_2 \eta^2 + R_3 \eta^3 \end{bmatrix},$$

with $(s_1, s_2, s_3) = (3, 2, 1)$ will be arranged as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ P_1 & P_2 & P_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ T_1 & T_2 & T_3 \\ R_1 & R_2 & R_3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ Q_4 & Q_5 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}.$$
of Guillemin normal form of the type suggested but not completed in Chapter IV §5 of [BCG+90].

3. Involutivity

**Lemma 3.1** (Linear Involutivity Criteria). Let $A$ denote a tableau given in a generic basis by symbol relations (2.1), as in Figure 1. If $A$ is involutive, then $A$ admits a basis of $W$ in which it is endovolutive, as in Figure 2.

**Theorem 3.2** (Quadratic Involutivity Criteria). Let $A$ denote an endovolutive tableau given in a generic basis by symbol relations (2.1), as in Figure 1. The tableau $A$ is involutive if and only if for all $b$, all $\lambda < i < j$ and $\lambda \leq \mu < j$, and all $a > s_i$, we have

$$\left( B^\lambda_i B^\mu_j - B^\lambda_j B^\mu_i \right)_b^a = 0.$$

In particular, this implies $B(u^\lambda)(v)$ is an endomorphism of $W^-(u^\lambda)$ such that for all $v, \tilde{v} \in (U^*)^\perp$,

$$[B(u^\lambda)(v), B(u^\lambda)(\tilde{v})] = 0.$$

A proof of Lemma 3.1 appears on Pages 145–147 (Pages 126–127 in the online version) of [BCG+90] and it is implicit in Section 1.2 of [Yan87]. Theorem 3.2 is proven in Section 5, which is modeled on the approach of [Yan87].

**Corollary 3.3** (Quillen, Guillemin). If $A$ is involutive, then $A|_U$ is involutive, and the natural map between prolongations $A^{(1)} \rightarrow (A|_U)^{(1)}$ is bijective.

**Proof.** This is Theorem A in [Gui68], where it is proven with a large diagram chase using Quillen’s exactness theorem from [Qui64]. But, using Theorem 3.2, this is immediate, as the quadratic condition still holds if the range of indices $\lambda, \mu, i, j$ is truncated at $\ell$ (or greater). In particular, the generators $(z^\alpha_{\mu})_{a \leq s_\mu}$ of $A$ are preserved. As explored in Section 5, the contact relation $z^\alpha_{\mu} = Z^\alpha_{\mu,i} u^i$ gives coordinates $Z^\alpha_{\mu,i}$ to the prolongation $A^{(1)} \subset A \otimes V^*$, and the $s_1 + 2s_2 + \cdots + \ell s_\ell$ independent generators are $Z^\alpha_{\mu,\lambda}$ with $a \leq s_\mu$ and $\lambda \leq \mu$. These generators remain independent under restriction to $U$, too. $\Box$

Another important subspace is the part of $A$ that is rank-one in $W \otimes U^*$; that is, consider

$$(3.4) \quad A^1(\varphi) = A \cap \{ z \otimes \varphi + J \text{ for some } z \in W, J \in W \otimes Y^* \}.$$
The image of $A^1(\varphi)$ under the projection $(W \otimes V^*) \to W \otimes U^*$ is comprised of rank-one homomorphisms, so the projection $A^1(\varphi) \to W$ is well-defined, with image

$W^1(\varphi) = \{ z \in W : z \otimes \varphi + J \in A \text{ for some } J \in W \otimes Y^* \}$.  

The spaces $W^-(\varphi)$ and $W^1(\varphi)$ are distinct, and they play different roles, but their relationship is clear:

**Lemma 3.6.** Suppose that $A$ is an endovolutive tableau. For any $\lambda$,

$W^1(u^\lambda) = \{ z \in W^- : B_\mu^\lambda z = \delta^\lambda_\mu z \forall \mu \leq \ell \}$.  

More generally, for any $\varphi \in U^*$,

$W^1(\varphi) = \{ z \in W^-(\varphi) : \left( \sum_\lambda \varphi_\lambda B_\mu^\lambda - \varphi_\mu I \right) z = 0 \forall \mu \leq \ell \}$.  

There is nothing to prove; Lemma 3.6 merely states the condition that the $W \otimes U^*$ part of $\pi = B(\varphi)(\cdot)z$ is rank-one.

**Corollary 3.7.** Suppose that $A$ is an endovolutive tableau. For generic $\varphi$, $\dim W^1(\varphi) = s_\ell$.

**Proof.** For generic $\varphi \in U^*$, we have $\min \{ i : \varphi_i \neq 0 \} = 1$ and $\max \{ i : \varphi_i \neq 0 \} = \ell$. So, $W^-(\varphi) = W^-_1$. For each $\mu = 1, \ldots, \ell$, the condition $(\varphi_\lambda B_\mu^\lambda - \varphi_\mu I) z = 0$ has no $W^-_\mu$ component, but the $W^+_\mu$ component has rank up to $s_{\mu - 1} - s_\mu$. The rank falls if and only if $\varphi_\mu$ is an eigenvalue of $\sum_{\lambda < \mu} \varphi_\lambda B_\mu^\lambda$. By fixing $\varphi_1, \varphi_2, \ldots, \varphi_{\mu - 1}$ and varying $\varphi_\mu$, we can see that this condition achieves its maximum rank for a Zariski-open set of values of $\varphi$. Then $\dim W^1(\varphi) = s_1 - (s_1 - s_2) - (s_2 - s_3) - \cdots - (s_{\ell - 1} - s_\ell) = s_\ell$. \hfill \Box

Note that, unlike with $W^-(\varphi)$, the definition of $W^1(\varphi)$ does not rely on the basis; it requires only a splitting of $0 \to U^* \to V^* \to (U^*)^\perp \to 0$ to decide where $J$ takes values. Probably for this reason, it is the space studied in the more algebraic references. The next theorem\footnote{In those references, the domain is restricted to $v \in (U^*)^\perp$, but that limitation is artificial.} is called Lemma 4.1 in [Gui68] and Proposition 6.3 in Chapter VIII of [BCG+90].

**Theorem 3.8** (Guillemin). Suppose that $A$ is involutive. For every $\varphi \in U^*$ and $v \in V$, the restricted homomorphism $B(\varphi)(v) \mid W^1(\varphi)$ is an endomorphism of $W^1(\varphi)$. Moreover, for all $v, \bar{v} \in V$,

$[B(\varphi)(v), B(\varphi)(\bar{v})] \mid W^1(\varphi) = 0$.  

\[\text{dim} W^1(\varphi) = s_\ell.\]
Theorem 3.8 is important because it reveals the intimate relationship between involutivity and the characteristic variety, discussed in Section 4. This theorem is an easy consequence of Theorem 3.2 and Corollary 3.3. The structure of the proof is identical to the original proof in [Gui68], but whereas Guillemin uses the subtle exactness result from [Qui64], we can rely on the much easier Lemma 3.10. Compare this lemma to the exact sequence (3.4) in [Gui68].

**Lemma 3.10.** For \( A \) involutive, this sequence is exact:

\[
0 \to W \otimes S^2 Y^* \to H^1 \otimes Y^* \xrightarrow{\delta} H^2
\]

**Proof.** This proof is just an explicit description of the maps in a generic basis and an application of Corollary 3.3.

The sequence makes sense because we can split (1.1) as \( W \otimes V^* = A \oplus H^1 \) by identifying the space \( H^1 \) with \( \{ \sum_{a > s_i} \pi_i^a(w_a \otimes u^i) \} \subset W \otimes V^* \), which is the space spanned by the unshaded entries in Figure 1. Using this identification, two elements \( \sum_{a > s_i} \pi_i^a(w_a \otimes u^i) \) and \( \sum_{a > s_i} \hat{\pi}_i^a(w_a \otimes u^i) \) of \( W \otimes V^* \) are equivalent in \( H^1 \) if and only if \( \pi_i^a - \hat{\pi}_i^a = \sum_{b < s_i} B_{i,b}^a \lambda B_{i,b}^b \) for some \( \{ z_i^a : a \leq s_i \} \), the shaded entries in Figure 1. In other words, the projection \( W \otimes V^* \to H^1 \) is defined by (2.1), and the projection \( W \otimes V^* \to A \) is defined by (2.8).

Since \( s_\varrho = 0 \) for all \( \varrho > \ell \), the inclusion \( W \otimes Y^* \subset W \otimes V^* \) is an inclusion \( W \otimes Y^* \subset H^1 \). Hence, the inclusion is understood as

\[
W \otimes S^2 Y^* \subset (W \otimes Y^*) \otimes Y^* \subset H^1 \otimes Y^*.
\]

An element of \( H^1 \otimes Y^* \) is written in \( W \otimes V^* \otimes Y^* \) as

\[
P = \sum_{a > s_\lambda} P_{a,\lambda}(w_a \otimes u^\lambda \otimes u^c) + \sum_{a > 0} P_{a,\varrho}(w_a \otimes u^\rho \otimes u^c).
\]

The image \( \delta(H^1 \otimes Y^*) \) in \( H^2 \) is

\[
\delta(H^1 \otimes Y^*) \subset \delta(W \otimes V^* \otimes Y^*) \subset W \otimes \wedge^2 V^*,
\]

so \( \delta P \in W \otimes \wedge^2 V^* \) is of the form

\[
\delta P = \sum_{a > s_\lambda} P_{a,\lambda}(w_a \otimes u^\lambda \otimes u^c) + \sum_{a > 0} \frac{1}{2} \left( P_{a,\varrho}^a - P_{a,\varrho}^b \right)(w_a \otimes u^\rho \otimes u^c).
\]

Recall that \( H^2 = \frac{W \otimes \wedge^2 V^*}{\delta_{(A \otimes V^*)}} \). So, \( \delta P \equiv 0 \) in \( H^2 \) if and only if there is some \( T \in A \otimes V^* \) such that \( \delta_{(A \otimes V^*)}(T) = \delta(P) \) in \( W \otimes \wedge^2 V^* \). Looking at (3.14), it is apparent that such \( T \) must have \( \delta_{(A \otimes V^*)}(T|_U) = 0 \), as \( \delta(P) \) has no \( U^* \) terms. By involutivity and Corollary 3.3, we consider the involutive tableau

\[
0 \to A|_U \to W \otimes U^* \xrightarrow{\sigma|_U} H^1_U \to 0
\]
with prolongation
\[(3.16) \quad 0 \to (A|_U)^{(1)} \to A|_U \otimes U^* \overset{\delta_{\sigma|_U}}{\to} W \otimes \wedge^2 U^* \to H^2_U \to 0.\]

Therefore, \( T|_U \in A|_U \otimes U^* \) lies in the kernel of \( \delta_{\sigma|_U} \), so \( T|_U \in (A|_U)^{(1)} \). Therefore, Corollary 3.3 tells us \( T \in A^{(1)} \). That is, \( \delta(P) \equiv 0 \in H^2 \) if and only if \( \delta(P) = \delta_{\sigma}(T) = 0 \).

Therefore, \( \delta(P) \equiv 0 \in H^2 \) if and only if \( P^a_{\lambda,\varsigma} = 0 \) and \( P^a_{\epsilon,\varsigma} = P^a_{\varsigma,\epsilon} \) on these index ranges. This occurs if and only if \( P = P^a_{\epsilon,\varsigma}(w_a \otimes u^\epsilon \otimes u^\varsigma) \), meaning \( P \in W \otimes S^2 Y^* \).

\[\Box\]

\textbf{Proof of Theorem 3.8.} The structure of this proof is identical to the original proof in [Gui68]. It is reproduced here as an application of of Theorem 3.2 via Corollary 3.3.

Suppose that \( z \in W^1(\varphi) \), so that \( \pi = B(\varphi)(\cdot)z = z \otimes \varphi + J \) for some \( J \in W \otimes Y^* \) with \( J_\varphi = J_\varphi \omega_a \in W^*(\varphi) \) for all \( \varphi \). First, we must show that the span of the columns \( J_\varphi \) of \( J \) lies in \( W^1(\varphi) \).

Consider the element \( -J \otimes \varphi = -J_\varphi \varphi_\lambda(w_a \otimes u^\lambda \otimes u^\epsilon) \in H^1 \otimes Y^* \). Because \( z \otimes \varphi + J \in A \), it must be that \( z \otimes \varphi \otimes \varphi \in W \otimes V^* \otimes V^* \) represents the same point in \( H^1 \otimes Y^* \). So, we can compute
\[(3.17) \quad -J_\varphi \varphi_\lambda(w_a \otimes u^\lambda \otimes u^\epsilon) \equiv z \otimes \varphi \wedge \varphi = 0 \in H^2 \]

By Corollary 3.3, there exists \( Q = Q^a_{\varphi,\varsigma}(w_a \otimes u^\varsigma \otimes u^\epsilon) \in W \otimes S^2 Y^* \) such that \( -J \otimes \varphi - Q \in A \otimes Y^* \). That is, writing \( Q_\varphi = Q^a_{\varphi,\varsigma}(w_a \otimes u^\varphi) \in W \otimes U^* \), we have \( J_\varphi \otimes \varphi + Q_\varphi \in A \) for all \( \varphi \), meaning \( J_\varphi \in W^1(\varphi) \) for all \( \varphi \). Therefore, for any \( v \in V \), we have \( B(\varphi)(v)z = \varphi(v)z + J(v) \in W^1(\varphi) \).

Now, mapping again, \( B(\varphi)(\cdot)J_\varphi = J_\varphi \otimes \varphi + Q_\varphi \), so \( B(\varphi)(v_\varsigma)J_\varphi = Q_\varphi \), which is already known to be symmetric in \( \varphi, \varsigma \). Therefore,
\[(3.18) \quad B(\varphi)(v)B(\varphi)(v_\varsigma)z = B(\varphi)(\tilde{v}) (\varphi(v)z + J(v)) = \varphi(v)B(\varphi)(\tilde{v})z + u^\epsilon(v)B(\varphi)(\tilde{v})J_\varphi \]
\[= \varphi(v)(\varphi(\tilde{v})z + J(\tilde{v})) + u^\epsilon(v)(\varphi(\tilde{v})J_\varphi + Q_\varphi(\tilde{v})) \]
\[= \varphi(v)(\varphi(\tilde{v})z + J(\tilde{v}))(\varphi(v)(\varphi(\tilde{v}) + Q_\varphi(\tilde{v})) + (\varphi(v)(\varphi(\tilde{v}) + Q_\varphi(\tilde{v})) + (\varphi(v)(\varphi(\tilde{v}) + Q_\varphi(\tilde{v}))) \]

This is symmetric in \( v, \tilde{v} \), giving the commutativity condition (3.9) \( \Box \).

It is useful to see a proof that relies on Theorem 3.2 more directly.

\textbf{Alternate Proof of Theorem 3.8.} Fix \( \varphi \in U^* \). Let \( \kappa = \min\{i : \varphi_i \neq 0\} \). Rescale \( \varphi \) so that \( \varphi_\kappa = 1 \). Then replace the basis \( u^1, \ldots, u^\kappa, \ldots, u^n \) with \( \tilde{u}^1, \ldots, \tilde{u}^\kappa, \ldots, \tilde{u}^n \) where \( \tilde{u}^\kappa = \varphi \). By Lemma 2.12, the tableau and symbol are still endovolutive, and the involutivity criteria from Theorem 3.2 still hold, though the matrices \( B^\lambda_i \) have been replaced by linear combinations as in (2.14).

Suppose that \( z \in W^1(\varphi) = W^1(\tilde{u}^\kappa) \). Recall that the definition of \( W^1(\varphi) \) does not depend on the basis, so we may apply Lemma 3.6 in the new basis:
$B_\mu^\kappa z = \delta_\mu^\kappa z$. Write $B(\varphi)(\cdot)z = z \otimes \tilde{u}^\kappa + J$, and examine Lemma 3.6 on a column $J_\varrho$ of $J$. For all $\mu = \kappa + 1, \ldots, \ell$, compute:

$$
\begin{align*}
(B_\mu^\kappa - \delta_\mu^\kappa I) J_\varrho &= (B_\mu^\kappa - \delta_\mu^\kappa I) B_\varrho^\kappa z \\
&= B_\mu^\kappa B_\varrho^\kappa z - \delta_\mu^\kappa B_\varrho^\kappa z \\
&= B_\varrho^\kappa (B_\mu^\kappa - \delta_\mu^\kappa I) z \\
&= 0.
\end{align*}
$$

The exchange of $\varrho$ and $\mu$ is allowed because of the commutativity condition in Theorem 3.2. This establishes that $W^1(\varphi)$ is preserved, and commutativity follows using the quadratic condition in this basis. \hfill \square

### 4. The Characteristic Variety

Since $A$ lies in $W \otimes V^*$, a space of matrices, there is a homogeneous ideal defining rank-one elements. Write its variety as

$$
C = A \cap \text{Var} \left\{ \pi_i^a \pi_j^b - \pi_j^a \pi_i^b \right\} = A \cap \left\{ w \otimes \xi : w \in W, \xi \in V^* \right\}.
$$

As a set, the characteristic variety $\Xi$ is the projection of $C$ to $V^*$. More precisely, $\Xi$ is defined by the characteristic ideal $\mathcal{M}$ on $V^*$ that is obtained from the rank-one ideal on $W \otimes V^*$ in the following way: For any $\xi \in V^*$, define $\sigma_\xi : W \to H^1$ by $\sigma_\xi(z) = \sigma(z \otimes \xi)$. Then $C$ is the incidence correspondence of $\Xi$ for the symbol map $\sigma_\xi$. See Figure 3.

This relationship is rephrased in Lemma 4.2

**Lemma 4.2.** If $\xi \in \Xi$, $v \in V$, and $z \in \ker \sigma_\xi \subset W$, then

$$
B(\xi)(v)z = \xi(v)z.
$$

In particular, $z$ is an eigenvector of $B(\xi)(v)$ for all $v$.

**Proof.** Set $\pi = z \otimes \xi \in C \subset A$, so $\pi_i^a = z^a \xi_i$ for all $a, i$, and this $\pi$ must satisfy the symbol relations (2.1). In particular, $z^a \xi_i = B_{k,b}^{a,\lambda} z^b \xi_\lambda$ for $a > s_i$. 

---

**Figure 3.** The rank-one variety $C$ is the incidence correspondence for the characteristic variety $\Xi$. 

Therefore

\[ B(\xi)(v)z = \sum_{a \leq s_i} \xi_i v^i z^a w_a + \sum_{a > s_i} B_{i,b}^a \lambda \xi^b v^i w_a \]

(4.4)

\[ = \sum_{a \leq s_i} \xi_i v^i z^a w_a + \sum_{a > s_i} \xi_i v^i z^a w_a \]

\[ = \sum_{a,i} \xi_i v^i z^a w_a. \]

(Here we see the utility of including the first summand in Equation (2.2).) \(\square\)

**Lemma 4.5.** Suppose that \( A \) is an endovolutive tableau. Fix \( \varphi \in U^* \) and suppose that \( z \in W^-(\varphi) \) such that \( z \) is an eigenvector of \( B(\varphi)(v) \) for every \( v \in V \). Then there is a \( \xi \in \Xi \) over \( \varphi \in U^* \) such that \( z \in W^1(\varphi) \), so \( z \otimes \xi \in A \).

**Proof.** For each \( v \in V \), let \( \xi(v) \) denote the eigenvalue corresponding to \( v \), so that \( \xi(v)z = B(\varphi)(v)z \). Because \( B(\varphi)(v)z \) is linear in \( v \), so is \( \xi(v) \). Then \( \xi = \xi_i u^i \in V^* \). Therefore, \( B(\varphi)(\cdot)z = z \otimes \xi \). In particular, the rank-one condition implies that

\[ \sum_{\lambda \leq \mu} \varphi_\lambda B^\lambda_\mu z = \xi_\mu z = \sum_{\lambda \leq \mu} \xi_\lambda B^\lambda_\mu z, \quad \forall \mu \leq \ell. \]

(4.6)

This is the same expression as in Lemma 3.6, so by comparing recursively over \( \mu = 1, 2, \ldots, \ell \), we see that \( \xi_\lambda = \varphi_\lambda \) for all \( \lambda \), so \( z \in W^1(\varphi) \subset W^-(\varphi) \). \(\square\)

Lemma 4.5 deserves a warning: There may be multiple \( \xi \) over the same \( \varphi \), for perhaps there are different \( z \in W^-(\varphi) \) admitting different sequences of eigenvalues \( \xi_\varrho \), for \( \varrho > \ell \), associated to the same \( \varphi \). However, Lemma 4.7 shows that this is at most a finite multiplicity.

**Lemma 4.7.** Suppose that \( A \) is an endovolutive tableau. Then the map of projective varieties induced by \( \Xi \rightarrow U^* \) is a finite branched cover. In particular, we have \( \text{dim } \Xi = \text{dim } U^* = \ell \).

**Proof.** Fix \( \varphi \in U^* \). If it were true that the set of \( \xi \) projecting to a particular \( \varphi \) were infinite, then the parameter \( \xi_i \) would take infinitely many values in some expression of the form

\[ \text{det} \left( \sum_{\lambda} \varphi_\lambda B^\lambda_i - \xi_i I \right) = 0. \]

(4.8)

But, the matrix \( \sum_{\lambda} \varphi_\lambda B^\lambda_i \in \text{End}(W_1^-) \) can have at most \( s_1 \) eigenvalues. \(\square\)

The companion article [Smi14] measures how much of \( Y^* \) is spanned by \( \Xi/U^* \), and Lemma 4.5 plays a major role there.

**Corollary 4.9.** If \( A \) is involutive, then for any \( \varphi \in U^* \), there exists some \( z \) such that the hypotheses of Lemma 4.5 hold.
Proof. Since we are working over $\mathbb{C}$, the commutativity condition (3.9) guarantees that common eigenvectors exist for $\{B(\varphi)(v) : v \in V\}$. □

**Theorem 4.10.** If $A$ is involutive, then $\dim \Xi = \ell$ and $\deg \Xi = s_\ell$.

**Proof.** We work in endovolutive coordinates. From Lemma 4.7, we already know that $\dim \Xi = \ell$.

Fix a generic point $\xi \in \Xi$ over $\varphi \in U^*$. We must determine the degree of the condition $C_\xi \neq 0$. Note that $C_\xi$ must be a subvariety of $W^1(\varphi) \otimes \xi$, and $W^1(\varphi)$ is a linear subspace of $W$, so the degree of $\Xi$ is the degree of some condition on $W^1(\varphi)$.

By Lemma 4.2 and (4.1), the condition that $C_\xi$ is nontrivial is precisely the condition that

$$
(4.11) \quad \det \left( \sum_{\lambda} \xi_\lambda B_\lambda^\varphi - \xi I \right) = 0, \forall i.
$$

Since we may restrict our attention to $W^1(\varphi) \otimes \xi$, only these terms contribute to the non-linear part of the ideal:

$$
(4.12) \quad \det \left( \sum_{\lambda} \xi_\lambda B_\lambda^\varphi - \xi_\varphi I \right) = 0, \forall \varphi > \ell.
$$

or, without coordinates,

$$
(4.13) \quad \det (B(\xi)(v) - \xi(v)I) = 0, \forall v \in (U^*)^\perp.
$$

For a particular $v$, this is the characteristic polynomial of $B(\xi)(v)$ as an endomorphism of $W^1(\varphi)$. By involutivity and Theorem 3.8, all $B(\xi)(v)$ for $v \in (U^*)^\perp$ admit the same factorization type for their respective characteristic polynomials, so it does not matter which $v$ we consider. By definition, the characteristic polynomial of $B(\xi)(v)|_{W^1(\varphi)}$ has degree $\dim W^1(\varphi)$. Therefore, $\deg \Xi = s_\ell$ follows from Corollary 3.7. □

5. **Proof of Theorem 3.2**

Choose generic bases for $W$ and $V^*$. As a subspace, $A \subset W \otimes V^*$ is defined by a minimal set of equations

$$
(2.1 \ bis) \quad \pi^a_i = \sum_{b \leq s_\lambda} B^{a,\lambda}_{i,b} a^b_\lambda, \quad \forall a > s_i,
$$

where $B^{a,\lambda}_{i,b} = 0$ unless $\lambda \leq i$ and $b \leq s_\lambda$. Moreover, we assume that the basis of $W$ is endovolutive, so that $B^{a,\lambda}_{i,b} = 0$ if $a > s_\lambda$. 

Let \( \{z^a_i, a \leq s_i\} \) be a basis for the abstract vector space \( A \). Define a monomorphism \( A \rightarrow W \otimes V^* \) by

\[
\begin{align*}
\pi^a_1 &= B^{a,1}_{1,b} z^b_1, \\
\pi^a_2 &= B^{a,1}_{2,b} z^b_1 + B^{a,2}_{2,b} z^b_2, \\
\pi^a_3 &= B^{a,1}_{3,b} z^b_1 + B^{a,2}_{3,b} z^b_2 + B^{a,3}_{3,b} z^b_3, \\
&\vdots \\
\pi^a_n &= B^{a,1}_{n,b} z^b_1 + B^{a,2}_{n,b} z^b_2 + \cdots + B^{a,n}_{n,b} z^n_b.
\end{align*}
\]

We set \( B^{a,\lambda}_{\lambda,b} = \delta^a_b \) for \( a \leq s_\lambda \), so that (5.1) satisfies (2.1).

The prolongation \( A^{(1)} \subset A \otimes V^* \) is given by coefficients \( \{Z^a_{i,j}, a \leq s_i\} \) with the “contact” system taking the form \( z^a_i - Z^a_{i,j} u^j \). By Cartan’s test, the condition of involutivity means that exactly \( s_1 + 2s_2 + \cdots + ns_n \) of these coefficients are independent functions on \( A^{(1)} \).

This proof is based on Section 1.1 of [Yan87], where it is shown that a tableau is involutive precisely when, in a generic coframe, the usual 2-form condition

\[
0 \equiv \pi^a_1 \wedge u^1 + \pi^a_2 \wedge u^2 + \cdots + \pi^a_n \wedge u^n
\]

is equivalent to the sequence of conditions

\[
\begin{align*}
0 &\equiv \pi^a_1 \wedge u^1, \mod u^2, \ldots, u^n \\
0 &\equiv \pi^a_1 \wedge u^1 + \pi^a_2 \wedge u^2, \mod u^3, \ldots, u^n \\
0 &\equiv \pi^a_1 \wedge u^1 + \pi^a_2 \wedge u^2 + \pi^a_3 \wedge u^3, \mod u^4, \ldots, u^n \\
&\vdots \\
0 &\equiv \pi^a_1 \wedge u^1 + \pi^a_2 \wedge u^2 + \cdots + \pi^a_k \wedge u^k, \mod u^{k+1}, \ldots, u^n \\
&\vdots \\
0 &\equiv \pi^a_1 \wedge u^1 + \pi^a_2 \wedge u^2 + \cdots + \pi^a_n \wedge u^n.
\end{align*}
\]

The argument shows that this sequence of conditions forces \( Z^a_{i,j} \) with \( a \leq s_i \) \( j \leq i \) to be a complete set of independent generators of \( A^{(1)} \), providing it with a dimension of \( s_1 + 2s_2 + 3s_3 + \cdots + ns_n \). To emphasize these terms in the following computations, we underline them.

The proof proceeds by induction, sequentially verifying that each term of each row of (5.3) yields a condition of the desired form. One may interpret this as induction on \( n \) of Cartan’s test for all tableaux of size \( r \otimes n \).
Row 1 of (5.3) prolongs to

\[ 0 = \pi^a \land u^1 \]
\[ = B^{a,1}_{1,b} \land u^1 \]
\[ = B^{a,1}_{1,b} Z^b_{1,i} \land u^1 \]
\[ \equiv B^{a,1}_{1,b} Z^b_{1,1} \land u^1, \quad \text{mod } u^2, \ldots, u^n. \]

(5.4)

which is trivial. Every tableau with \( n = 1 \) is involutive. The \( Z^b_{1,1} \) terms account for \( s_1 \) generators of \( A^{(1)} \).

Row 2 of (5.3) prolongs to

\[ 0 = \pi^a \land u^1 + \pi^a_2 \land u^2 \]
\[ = \left(B^{a,1}_{1,b} \land u^1 + \left(B^{a,1}_{2,b} + B^{a,2}_{2,b} \land u^2 \right) \right) \land u^2 \]
\[ = \left(B^{a,1}_{1,b} Z^b_{1,i} \right) \land u^1 + \left(B^{a,1}_{2,b} Z^b_{1,i} + B^{a,2}_{2,b} Z^b_{2,i} \right) \land u^2 \]
\[ \equiv \left(B^{a,1}_{2,b} Z^b_{1,1} + B^{a,2}_{2,b} Z^b_{2,1} - B^{a,1}_{1,b} Z^b_{1,2} \right) \land u^1 \land u^2, \quad \text{mod } u^3, \ldots, u^n. \]

(5.5)

Endovolutivity implies this is trivial when projected to \( W^+_1 \) (meaning “for \( a > s_1 \)”). But, when considering the projection to \( W^-_1 \), we see the condition

\[ \forall a > s_1 \]
\[ Z^a_{1,2} = B^{a,1}_{2,b} Z^b_{1,1} + B^{a,2}_{2,b} Z^b_{2,1}, \quad \forall a > s_1 \]

(5.6)

The \( Z^b_{2,1} \) and \( Z^b_{2,2} \) terms account for \( 2s_2 \) new generators of \( A^{(1)} \). So far, there is no quadratic condition on \( B^a_i \); therefore all endovolutive tableaux with \( n = 2 \) are involutive.

Notation! It is clear we must confront a proliferation of indices \( a, b, c, \ldots \) covering \( W \). Henceforth, we suppress these indices and work directly on \( W \)-valued objects. Instead of saying “\( \forall a > s_i \)” we say “on \( W^-_i \)” and “on \( W^-_1 \)” Note that this always refers to projection on the range of the expression, not a restriction of its domain; by our definition of \( B^a_i \), we may assume the domain is always \( W \).
Row 3 of (5.3) prolongs to

\[(5.7)\]
\[
0 = \pi_1 \wedge u^1 + \pi_2 \wedge u^2 + \pi_3 \wedge u^3 \\
= (B^1_1 z_1) \wedge u^1 + (B^2_2 z_2) \wedge u^2 + (B^3_3 z_3) \wedge u^3 \\
= (B^1_1 Z_{1,1} + B^2_2 Z_{1,2} + B^3_3 Z_{1,3}) u^1 + u^2 \\
+ (B^1_1 Z_{2,1} + B^2_2 Z_{2,2} + B^3_3 Z_{2,3}) u^1 + u^2 \\
+ (B^1_1 Z_{3,1} + B^2_2 Z_{3,2} + B^3_3 Z_{3,3}) u^1 + u^2
\]

For this to vanish each component \(u^i \wedge u^j\) with \(i < j\) must vanish separately. The \(u^1 \wedge u^2\) term repeats conditions already seen in row 2, namely

\[(5.8)\]
\[
Z_{1,2} = B^1_2 Z_{1,1}, \text{ on } W_1^-. \]

The \(u^1 \wedge u^3\) term is similar,

\[(5.9)\]
\[
Z_{1,3} = B^1_3 Z_{1,1}, \text{ on } W_1^-. \]

The \(u^2 \wedge u^3\) term is more interesting, because it requires expansion using the previous relations:

\[(5.10)\]
\[
Z_{2,3} = B^1_3 Z_{1,2} + B^2_3 Z_{2,2} + B^3_3 Z_{3,2} - B^1_1 Z_{1,3} \\
= B^1_3 \left( B^2_2 Z_{1,1} + B^2_2 Z_{2,1} \right) + B^2_3 Z_{2,2} + B^3_3 Z_{3,2} \\
- B^1_1 \left( B^1_3 Z_{1,1} + B^2_3 Z_{2,1} + B^3_3 Z_{3,1} \right) \\
= (B^1_1 B^2_2 - B^1_3 B^3_3) Z_{1,1} + (B^1_1 B^2_2 - B^1_3 B^3_3) Z_{2,1} \\
+ (B^1_1 B^2_2 - B^1_3 B^3_3) Z_{3,1} + B^2_3 Z_{2,2} + B^3_3 Z_{3,2}, \text{ on } W_1^-.
\]

On \(W_2^+\) Equation (5.10) merely shows how \(Z_{2,3}\) depends on the previous coordinates on \(A^{(1)}\). Note that the \(Z_{3,1}, Z_{3,2}\) and \(Z_{3,3}\) terms contribute another 3s3 generators of \(A^{(1)}\). Cartan’s test fails if and only if other relations appear among the generators \(Z_{i,j}, i \geq j\). However, on \(W_2^+\), many of the terms vanish by definition, and the rest impose a new quadratic condition:

\[(5.11)\]
\[
0 = \left( B^1_1 B^1_2 - B^1_3 B^3_3 \right) Z_{1,1} + (B^1_1 B^2_2 - B^1_3 B^3_3) Z_{2,1} + (B^1_1 B^3_3 - B^1_3 B^3_3) Z_{3,1}, \text{ on } W_2^+.
\]
Therefore an endovolutive tableau with \( n = 3 \) is involutive if and only if each term of (5.11) holds on \( W_2^+ \).

It is useful to see another case, where things become more interesting.

Row 4 of (5.3) prolongs to

\[
0 = \pi_1 \land u^1 + \pi_2 \land u^2 + \pi_3 \land u^3 + \pi_4 \land u^4 \\
= (B_1^1 z_1) \land u^1 \\
+ (B_1^2 z_1 + B_2^2 z_2) \land u^2 \\
+ (B_1^3 z_1 + B_3^2 z_2 + B_3^3 z_3) \land u^3 \\
+ (B_1^4 z_1 + B_3^2 z_2 + B_3^4 z_3 + B_4^1 z_4) \land u^4 \\
\] (5.12)

This term imposes no quadratic conditions.

From the \( u^1 \land u^1 \) term:

\[
Z_{1,4} = B_1^1 Z_{1,1} + B_2^2 Z_{2,1} + B_3^3 Z_{3,1} + B_4^4 Z_{4,1}. \\
\] (5.13)

After expanding these terms, modulo \( u^5, \ldots, u^n \), the several conditions are found. From the \( u^1 \land u^1 \) term:

\[
Z_{2,4} = B_1^1 Z_{1,2} + B_2^2 Z_{2,2} + B_3^3 Z_{3,2} + B_4^4 Z_{4,2} - B_1^1 Z_{1,4} \\
= B_1^1 \left( B_2^1 Z_{1,1} + B_2^2 Z_{2,1} \right) + B_2^2 Z_{2,2} + B_3^3 Z_{3,2} + B_4^4 Z_{4,2} \\
- B_1^1 \left( B_1^1 Z_{1,1} + B_1^2 Z_{2,1} + B_3^3 Z_{3,1} + B_4^4 Z_{4,1} \right). \\
\] (5.14)

Equation (5.14) becomes a new quadratic condition when projected on \( W_2^+ \):

\[
0 = B_1^1 \left( B_2^1 Z_{1,1} + B_2^2 Z_{2,1} \right) \\
- B_2^2 \left( B_1^1 Z_{1,1} + B_1^2 Z_{2,1} + B_3^3 Z_{3,1} + B_4^4 Z_{4,1} \right) \\
= (B_1^1 B_2^1 - B_1^1 B_1^2) Z_{1,1} \\
+ (B_1^1 B_2^2 - B_1^1 B_1^3) Z_{2,1} \\
+ (B_1^1 B_3^3 - B_1^1 B_1^4) Z_{3,1} \\
+ (B_1^1 B_4^4 - B_1^1 B_1^4) Z_{4,1}, \text{ on } W_2^+. \\
\] (5.15)

When reading (5.15), recall that \( B_1^4 = 0 \) if \( i < \lambda \).
The \( u^3 \wedge u^4 \) term becomes

\[
Z_{3,4} = B_1^1 Z_{1,3} + B_4^2 Z_{2,3} + B_4^3 Z_{3,3} + B_4^4 Z_{4,3} - B_3^1 Z_{1,4} - B_3^2 Z_{2,4} \\
= B_1^1 \left( B_3^1 Z_{1,1} + B_3^2 Z_{2,1} + B_3^3 Z_{3,1} \right) \\
+ B_4^2 \left( B_3^1 \left( B_3^2 Z_{1,1} + B_3^2 Z_{2,1} \right) + B_3^3 Z_{2,2} + B_3^3 Z_{3,2} \right) \\
- B_2^1 \left( B_3^1 Z_{1,1} + B_3^2 Z_{2,1} + B_3^3 Z_{3,1} \right) \\
+ B_3^4 Z_{3,3} + B_4^4 Z_{4,3} \\
- B_3^4 \left( B_4^1 Z_{1,1} + B_4^2 Z_{2,1} + B_4^3 Z_{3,1} + B_4^4 Z_{4,1} \right) \\
- B_3^4 \left( B_4^1 \left( B_4^2 Z_{1,1} + B_4^2 Z_{2,1} \right) + B_4^3 Z_{2,2} + B_4^3 Z_{3,2} + B_4^4 Z_{4,2} \right) \\
- B_2^1 \left( B_4^1 Z_{1,1} + B_4^2 Z_{2,1} + B_4^3 Z_{3,1} + B_4^4 Z_{4,1} \right)
\]

Equation (5.16) becomes a new quadratic condition when projected on \( W_3^+ \):

\[
0 = B_1^1 \left( B_3^1 Z_{1,1} + B_3^2 Z_{2,1} + B_3^3 Z_{3,1} \right) \\
+ B_4^2 \left( B_3^1 \left( B_3^2 Z_{1,1} + B_3^2 Z_{2,1} \right) + B_3^3 Z_{2,2} + B_3^3 Z_{3,2} \right) \\
- B_2^1 \left( B_3^1 Z_{1,1} + B_3^2 Z_{2,1} + B_3^3 Z_{3,1} \right) \\
- B_3^4 \left( B_4^1 Z_{1,1} + B_4^2 Z_{2,1} + B_4^3 Z_{3,1} + B_4^4 Z_{4,1} \right) \\
- B_3^4 \left( B_4^1 \left( B_4^2 Z_{1,1} + B_4^2 Z_{2,1} \right) + B_4^3 Z_{2,2} + B_4^3 Z_{3,2} + B_4^4 Z_{4,2} \right) \\
- B_2^1 \left( B_4^1 Z_{1,1} + B_4^2 Z_{2,1} + B_4^3 Z_{3,1} + B_4^4 Z_{4,1} \right), \text{ on } W_3^+.
\]

This looks like a mess, but collecting terms reveals a pattern:

\[
0 = \left( B_1^1 B_3^3 - B_1^3 B_1^3 + B_3^2 \left( B_3^3 B_4^1 - B_4^4 B_1^3 \right) - B_2^3 \left( B_4^4 B_1^3 - B_1^1 B_3^3 \right) \right) Z_{1,1} \\
+ \left( B_1^1 B_3^3 - B_1^3 B_1^3 + B_3^2 \left( B_3^3 B_4^1 - B_4^4 B_1^3 \right) - B_2^3 \left( B_4^4 B_1^3 - B_1^1 B_3^3 \right) \right) Z_{2,1} \\
+ \left( B_1^1 B_3^3 - B_1^3 B_1^3 + B_3^2 \left( B_3^3 B_4^1 - B_4^4 B_1^3 \right) - B_2^3 \left( B_4^4 B_1^3 - B_1^1 B_3^3 \right) \right) Z_{3,1} \\
+ \left( B_1^1 B_3^3 - B_1^3 B_1^3 + B_3^2 \left( B_3^3 B_4^1 - B_4^4 B_1^3 \right) - B_2^3 \left( B_4^4 B_1^3 - B_1^1 B_3^3 \right) \right) Z_{4,1}
\]

on \( W_3^+ \).
Notice that, using the quadratic relations already discovered for \( n = 2 \) and \( n = 3 \), we obtain

\[
0 = (B^1_1 B^1_3 - B^1_3 B^1_4) Z_{1,1} + (B^1_1 B^2_3 - B^1_3 B^2_4) Z_{2,1} + (B^2_3 B^2_3 - B^2_3 B^2_7) Z_{2,2} + (B^1_1 B^3_3 - B^1_3 B^3_4) Z_{3,1} + (B^2_3 B^3_3 - B^2_3 B^3_4) Z_{3,2} + (B^1_1 B^4_3 - B^1_3 B^4_4) Z_{4,1} + (B^2_3 B^4_3 - B^2_3 B^4_4) Z_{4,2}, \text{ on } W^+_3. \tag{5.19}
\]

The generators \( Z_{4,1}, Z_{4,2}, Z_{4,3}, \) and \( Z_{4,4} \) provide \( A^{(1)} \) with another 4s dimensions. Cartan’s test fails if and only if other relations appear among the generators \( Z_{i,j}, i \geq j \). An endovolutive tableau with \( n = 4 \) is involutive if and only if each term of (5.11) and (5.15) holds on \( W^+_2 \) and each term of (5.19) holds on \( W^+_3 \).

**Inductive Hypothesis.**

Fix \( k \) and \( l < k \). Assume for induction that the following are equivalent

(i) The first \( k - 1 \) rows of the 2-form condition (5.3) are satisfied.

(ii) The \( s_1 + 2s_2 + \cdots + (k - 1)s_{k-1} \) elements \( Z_{a,i}^\alpha \) of \( A^{(1)} \) with \( a \leq s_j \) and \( i \leq j < k \) are independent.

More precisely, examining each row of (5.3) in detail, assume for induction that the following are equivalent

(i) The first \( k - 1 \) rows of the 2-form condition (5.3) are satisfied, and the \( u^i \wedge u^k \) terms vanish in the \( k \)th row for all \( i < l \).

(ii) For all \( (j,i) < (k,l) \) in lexicographic ordering on pairs \( \{(j,i), i \leq j\} \), we have

\[
Z_{i,j} - \sum_{\mu=i}^{j} B^\mu_j Z_{\mu,i} = \sum_{\lambda=1}^{i-1} \left( B^\lambda_j Z_{\lambda,i} - B^\lambda_i Z_{\lambda,j} \right), \text{ on } W^-_i. \tag{5.20}
\]

and for all \( \lambda < i < j \) and \( \lambda \leq \mu \leq j \), we have

\[
B^\lambda_i B^\mu_j - B^\lambda_j B^\mu_i = 0, \text{ on } W^+_i. \tag{5.21}
\]
To perform the inductive step, we compute the \( u^i \land u^k \) terms in the \( k \)th row of (5.3):

\[
0 = \pi_1 \land u^1 + \cdots + \pi_l \land u^l + \cdots + \pi_k \land u^k
= \cdots + \left( \sum_{\lambda \leq \ell} B_{\ell}^\lambda z_\lambda \right) \land u^l + \cdots + \left( \sum_{\mu \leq k} B_{k}^\mu z_\mu \right) \land u^k
\]

(5.22)

\[
\equiv \left( \sum_{\lambda \leq \ell} B_{\ell}^\lambda Z_{\lambda,k} \right) u^k \land u^l + \left( \sum_{\mu \leq k} B_{k}^\mu Z_{\mu,l} \right) u^l \land u^k,
\]

\[
\mod u^{k+1}, \ldots, u^n.
\]

What follows is a tedious expansion and reduction of (5.22) using (5.20) and (5.21), along the lines of what was performed for (5.10), (5.16) and (5.14) above. The goal is to expand (5.22) in terms of the elements \( Z_{j,i} \) for \( j \geq i \) that remain independent if and only if Cartan’s test holds.

For a little bit of sanity in the expansion that follows, we break up these sums using the index ranges \( \lambda_0 = 1, \ldots, l - 1 \), and \( \mu_0 = l, \ldots, k \). Moreover, for every \( p \geq 0 \), we have index ranges \( \lambda_{p+1} = 1, \ldots, \lambda_p - 1 \) and \( \mu_{p+1} = \lambda_p, \ldots, k \).

The vanishing of the \( u^i \land u^k \) term of the \( k \)th row of (5.3) is equivalent to

\[
0 = B_{k}^{\lambda_0} Z_{\lambda_0,l} + B_{k}^{\mu_0} Z_{\mu_0,l} - B_{l}^{\lambda_0} Z_{\lambda_0,k} - B_{l}^{\lambda_0} Z_{\lambda_0,k}.
\]

(5.23)

Rearranging terms,

\[
Z_{l,k} - B_{k}^{\mu_0} Z_{\mu_0,l} = B_{k}^{\lambda_0} Z_{\lambda_0,l} - B_{l}^{\lambda_0} Z_{\lambda_0,k}
\]

(5.24)

and expanding the right-hand-side by the inductive hypothesis,

\[
= B_{k}^{\lambda_0} \left( B_{l}^{\lambda_1} Z_{\lambda_1,\lambda_0} - B_{l}^{\lambda_1} Z_{\lambda_1,l} + B_{l}^{\mu_1} Z_{\mu_1,\lambda_0} \right)
- B_{l}^{\lambda_0} \left( B_{k}^{\lambda_1} Z_{\lambda_1,\lambda_0} - B_{k}^{\lambda_0} Z_{\lambda_0,k} + B_{k}^{\mu_1} Z_{\mu_1,\lambda_0} \right)
\]

(5.25)

and expanding again by inductive hypothesis,

\[
= \left( B_{k}^{\lambda_0} B_{l}^{\mu_1} - B_{l}^{\lambda_0} B_{k}^{\mu_1} \right) Z_{\mu_1,\lambda_0}
+ B_{k}^{\lambda_0} \left( B_{l}^{\lambda_1} (B_{k}^{\lambda_0} Z_{\lambda_2,\lambda_1} - B_{k}^{\lambda_2} Z_{\lambda_2,\lambda_0} + B_{k}^{\mu_2} Z_{\mu_2,\lambda_1})
- B_{l}^{\lambda_1} (B_{k}^{\lambda_2} Z_{\lambda_2,\lambda_1} - B_{k}^{\lambda_2} Z_{\lambda_2,l} + B_{k}^{\mu_2} Z_{\mu_2,\lambda_1})) \right)
- B_{l}^{\lambda_0} \left( B_{k}^{\lambda_1} (B_{k}^{\lambda_2} Z_{\lambda_2,\lambda_1} - B_{k}^{\lambda_2} Z_{\lambda_2,\lambda_0} + B_{k}^{\mu_2} Z_{\mu_2,\lambda_1})
- B_{k}^{\lambda_1} (B_{k}^{\lambda_2} Z_{\lambda_2,\lambda_1} - B_{k}^{\lambda_2} Z_{\lambda_2,k} + B_{k}^{\mu_2} Z_{\mu_2,\lambda_1})) \right)
\]

(5.26)
and rearranging,

$$
(5.27) \quad = \left( B_k^{\lambda_0} B_{l_1}^{\mu_1} - B_k^{\lambda_0} B_{l_0}^{\mu_1} \right) Z_{\mu_1, \lambda_0}
+ \left[ B_k^{\lambda_0} \left( B_{l_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{l_1}^{\lambda_2} \right) - B_l^{\lambda_0} \left( B_{k_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{k_1}^{\lambda_2} \right) \right] Z_{\lambda_2, \lambda_1}
+ \left[ B_l^{\lambda_0} B_{k_1}^{\lambda_1} - B_k^{\lambda_0} B_{l_1}^{\lambda_1} \right] B_{l_2}^{\lambda_2} Z_{\lambda_2, \lambda_0}
+ B_k^{\lambda_0} B_{\lambda_0}^{\lambda_1} B_{l_1}^{\lambda_2} Z_{\lambda_2, l}
+ B_l^{\lambda_0} B_{\lambda_0}^{\lambda_1} B_{l_2}^{\lambda_2} Z_{\lambda_2, k}
+ \left[ B_k^{\lambda_0} \left( B_{l_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{l_1}^{\lambda_2} \right) - B_l^{\lambda_0} \left( B_{k_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{k_1}^{\lambda_2} \right) \right] Z_{\mu_2, \lambda_1}.
$$

and canceling the $Z_{\mu_2, \lambda_1}$ terms and expanding the others by the inductive hypothesis,

$$
(5.28) \quad = \left( B_k^{\lambda_0} B_{l_1}^{\mu_1} - B_k^{\lambda_0} B_{l_0}^{\mu_1} \right) Z_{\mu_1, \lambda_0}
+ \left[ B_k^{\lambda_0} \left( B_{l_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{l_1}^{\lambda_2} \right) - B_l^{\lambda_0} \left( B_{k_1}^{\lambda_1} B_{\lambda_0}^{\lambda_2} - B_{\lambda_0}^{\lambda_1} B_{k_1}^{\lambda_2} \right) \right] Z_{\lambda_2, \lambda_1}
+ \left[ B_l^{\lambda_0} B_{k_1}^{\lambda_1} - B_k^{\lambda_0} B_{l_1}^{\lambda_1} \right] B_{\lambda_2}^{\lambda_2}
\cdot \left( B_{\lambda_1}^{\lambda_3} Z_{\lambda_3, \lambda_2} - B_{\lambda_2}^{\lambda_3} Z_{\lambda_3, \lambda_1} + B_{\mu_3}^{\lambda_3} Z_{\mu_3, \lambda_2} \right)
+ B_k^{\lambda_0} B_{\lambda_0}^{\lambda_1} B_{l_1}^{\lambda_2} \left( B_{l_1}^{\lambda_3} Z_{\lambda_3, \lambda_2} - B_{\lambda_2}^{\lambda_3} Z_{\lambda_3, l} + B_l^{\mu_3} Z_{\mu_3, \lambda_2} \right)
- B_l^{\lambda_0} B_{\lambda_0}^{\lambda_1} B_{l_2}^{\lambda_2} \left( B_{l_2}^{\lambda_3} Z_{\lambda_3, \lambda_2} - B_{\lambda_2}^{\lambda_3} Z_{\lambda_3, k} + B_k^{\mu_3} Z_{\mu_3, \lambda_2} \right)
$$
and rearranging,

\begin{equation}
(5.29)
= \left( B_k^{\lambda_0} B_i^{\mu_1} - B_i^{\lambda_0} B_k^{\mu_1} \right) Z_{\mu_1, \lambda_0}
+ \left[ B_k^{\lambda_0} B_i^{\lambda_1} \left( B_{\lambda_0}^{\lambda_2} B_{\lambda_1}^{\lambda_3} - B_{\lambda_1}^{\lambda_2} B_{\lambda_0}^{\lambda_3} \right) - B_i^{\lambda_0} B_k^{\lambda_1} \left( B_{\lambda_0}^{\lambda_2} B_{\lambda_1}^{\lambda_3} - B_{\lambda_1}^{\lambda_2} B_{\lambda_0}^{\lambda_3} \right)
+ B_i^{\lambda_0} B_k^{\lambda_1} \left( B_{\lambda_0}^{\lambda_2} B_{\lambda_1}^{\lambda_3} - B_{\lambda_1}^{\lambda_2} B_{\lambda_0}^{\lambda_3} \right) + \lambda_0 \lambda_1 \left( B_k^{\lambda_0} B_{\lambda_0}^{\lambda_1} - B_{\lambda_0}^{\lambda_1} B_k^{\lambda_0} \right) \right] \lambda_{\lambda_3, \lambda_2}.
\end{equation}

The $Z_{\mu_3, \lambda_2}$ terms cancel by the inductive hypothesis.

Comparing (5.29) to (5.27), it is apparent that this pattern continues as we expand by the inductive hypothesis; in particular, notice that the upper indices on $Z_{\lambda_p, \lambda_q}$ or $Z_{\mu_p, \lambda_p-1}$ always appear as $\lambda_0, \lambda_1, \ldots, \lambda_p$ (or $\mu_p$), while the lower indices vary through signed permutations of $(l, k, \lambda_0, \ldots, \lambda_q, \ldots, \lambda_p)$ that end in $\lambda_p, \lambda_q$. Because these indices satisfy $1 \leq \lambda_p < \lambda_{p-1} < \ldots < \lambda_0 < l$, eventually every $Z_{\lambda_p, \lambda_q}$ term will reduce by repeated application of (5.20) to terms of the form $Z_{\mu_p, \lambda_p-1}$. Therefore, by pairing the lower-index permutations by transposition in the third-to-last and fourth-to-last slots, the $Z_{\mu_p, \lambda_p-1}$ terms always appear as

\begin{equation}
(5.30)
\cdots \left( B_i^{\lambda_p-1} B_j^{\mu_1} - B_j^{\lambda_p-1} B_i^{\mu_1} \right) Z_{\mu_p, \lambda_p-1}, \text{ with } \mu_p \geq \lambda_{p-1}, \quad (j, i) < (k, l),
\end{equation}

which vanishes by the inductive hypothesis.

Therefore, Equation (5.23) reduces by induction to

\begin{equation}
(5.31)
Z_{l, k} - B_k^{\lambda_0} Z_{\mu_0, l} = \left( B_k^{\lambda_0} B_i^{\mu_1} - B_i^{\lambda_0} B_k^{\mu_1} \right) Z_{\mu_1, \lambda_0}.
\end{equation}

On $W^+_l$, the left-hand side vanishes, so the independence of the $s_1 + 2s_2 + \cdots + ks_k$ elements $Z_{\mu_1, \lambda_0}$ required by Cartan’s test is equivalent to the condition

\begin{equation}
(5.32)
\left( B_i^{\lambda_0} B_k^{\mu_1} - B_k^{\lambda_0} B_i^{\mu_1} \right) = 0, \text{ projected to } W^+_l
\end{equation}

for all $\lambda_0 \leq \mu_1 \leq k$ and $\lambda_0 < l$. \qed
6. DISCUSSION

When reading Theorem 3.2, seasoned experts are forgiven for pausing to wonder “Didn’t I know that before? It’s the whole idea behind involutivity!” Perhaps these computations have been intuitively “in the air” for many years, but an explicit description of Guillemin normal form has not appeared in the literature. Because of this omission, Guillemin normal form has been difficult to apply in practice, as indicated by the shockingly low number of citations of \[ \text{Gui68} \]—almost 50 years later, AMS MathSciNet shows only four references to this major result.

I believe this sort of constructive translation has been ignored in the literature because the vast majority of work on exterior differential systems has followed two styles: One style applies involutivity to specific examples of interest to other fields of mathematics; the application at hand presents a particular tableau, and Cartan’s test is sufficient to proceed. The other style verifies that the overall approach to overdetermined systems and Lie pseudogroups via prolongation is sensible; the sophisticated language of commutative algebra builds a formal theory in a coordinate-independent manner.

One notable exception to this dichotomy is \[ \text{Yan87} \], which identifies a hyperbolicity condition whereby involutivity guarantees solutions in the \( C^\infty \) category. This condition relies on the micro-local geometry of the characteristic variety, which, as demonstrated in Section 4, is controlled by the symbol coefficients \( B^{a,\lambda}_{k,b} \) that define the tableau. Perhaps the lack of a mechanical interpretation of Guillemin normal form has forestalled the discovery of other structures amid the jungle of involutive tableaux. There are probably other subcategories of involutive systems which allow unusually loose regularity for the Cauchy problem.

On the theoretical side, it would be interesting to see how many of the hard classical theorems in the subject can be re-proven with elementary techniques. (Existing references such as \[ \text{BCG}^+90 \] present elementary proofs only in the case of rectangular tableaux.) Specifically, Lemma 3.10 is very close to Quillen’s theorem on the exactness of \( H^0 \to H^1 \to H^2 \to \cdots \). The other hard theorem is the integrability of the characteristic variety, and a proof of that theorem using Guillemin’s original formulation is the subject of \[ \text{GQS70} \]. That result was applied immediately to study primitive Lie pseudogroups.

Particularly because it is easy to program into computer algebra systems, Theorem 3.2 should allow more experimentation with and exploration of
these special subcategories by mathematicians of all ages. I hope it provides future researchers of exterior differential systems and Lie pseudogroups an easier way to understand and apply the profound results of [Qui64] and [Gui68].

Finally, note that Theorem 3.2 is the first step to answering a very fundamental open question, which is expressed in Footnote 7 in Chapter IV of [BCG+90]: “What is the dimension of the variety of involutive tableaux with certain fixed Cartan characters?” The recursive nature of the proof of Theorem 3.2 seems to say that Theorem 3.2 is a minimal set of quadratic conditions for involutivity, but even so it is not immediately clear how to account for general coordinate changes or in what sense the endovolutive expression of $B^A_i$ is unique for a given abstract tableau.

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8I am writing an open-source package called Symbol for Sage using this approach. You can help at https://bitbucket.org/curieux/symbol_sage.