CENTRAL SPLITTING OF MANIFOLDS WITH NO CONJUGATE POINTS

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ABSTRACT. Each compact Riemannian manifold with no conjugate points admits a family of functions whose integrals vanish exactly when central Busemann functions split linearly. These functions vanish when all central Busemann functions are sub- or superharmonic. When central Busemann functions are convex or concave, they must be totally geodesic. These yield generalizations of the splitting theorems of O’Sullivan and Eberlein for manifolds with no focal points and, respectively, nonpositive curvature.

1. Introduction

A key insight in Burago–Ivanov’s proof that each Riemannian torus with no conjugate points is flat is that the asymptotic norm of its fundamental group \( \pi_1(\mathbb{T}^k) \cong \mathbb{Z}^k \) is Riemannian, in the sense that it is generated by an inner product \( \langle , \rangle \). For a more general compact Riemannian manifold \( N \) whose fundamental group has center \( Z(\pi_1(N)) \) of rank \( k \), O’Sullivan [17] showed that, when \( N \) has no focal points, it must be foliated by totally geodesic and flat \( k \)-dimensional toruses and, moreover, be covered by an isometric product with a flat \( \mathbb{T}^k \). This generalized a theorem of Wolf [18] about manifolds with nonpositive sectional curvature. It was later shown by Eberlein [9] that each compact manifold with nonpositive sectional curvature and \( Z(\pi_1(N)) \) of rank \( k \) is finitely covered by a diffeomorphic product with \( \mathbb{T}^k \). The aim of this paper is to explore conditions, between having no conjugate points and no focal points, that allow for variations on these results.

The asymptotic norm of \( Z(\pi_1(N)) \) will be shown to be Riemannian under conditions on \( N \) that ensure a stronger property, namely, that central Busemann functions on the universal covering space \( \hat{N} \) split linearly. Associated to each \( z_0, z_1 \in Z(\pi_1(N)) \) is a function \( F_{z_0 z_1} : N \to \mathbb{R} \) defined by (3.1); the linear splitting of central Busemann functions is one of a number of conditions equivalent to the vanishing of all \( \int_N F_{z_0 z_1} \, d\text{vol}_N \).

**Theorem 1.1.** Let \( N \) be a compact Riemannian manifold with no conjugate points and \( Z \) a subgroup of \( Z(\pi_1(N)) \). Then the following are equivalent:

(i) \( \int_N F_{z_0 z_1} \, d\text{vol}_N = 0 \) for all \( z_0, z_1 \in Z \);
(ii) \( B(z_0, z_1) = \frac{1}{\text{vol}(N)} \int_N h(\omega_{z_0}, \omega_{z_1}) \, d\text{vol}_N \) for all \( z_0, z_1 \in Z \);
(iii) \( h(\omega_{z_0}(x), \omega_{z_1}(x)) = B(z_0, z_1) \) for all \( x \in N \) and all \( z_0, z_1 \in Z \);
(iv) \( \omega \sum_i m_i z_i = \sum_i m_i \omega z_i \) for all \( z_i \in Z \) and all \( m_i \in \mathbb{Z} \).

The notation that appears in Theorem 1.1 is defined in the next two sections. The proof is largely an application of Green’s identity. When the above conditions
are satisfied, one may define two integral formulations of an inner product that generates the asymptotic norm of $Z$. Moreover, a number of topological properties, which are known in the case of no focal points, hold (cf. \[17\]).

**Theorem 1.2.** Let $N$ be a compact $n$-dimensional Riemannian manifold with no conjugate points and $Z$ a subgroup of $Z(\pi_1(N))$ for which statements (i)-(iv) in Theorem \[17\] hold. Then so do the following:

(a) $Z \cong \mathbb{Z}^k$ for some $0 \leq k \leq n$;
(b) The asymptotic norm $\| \cdot \|_{\infty}$ of $Z$ with respect to any isomorphism $Z \to \mathbb{Z}^k$ is Riemannian;
(c) If $w_1, \ldots, w_k$ generate $Z$ and $H_1, \ldots, H_k$ are corresponding horospheres, then $H = \cap_{i=1}^k H_i$ is a simply connected submanifold of $\hat{N}$;
(d) $\hat{N}$ is diffeomorphic to $\hat{H} \times \mathbb{R}^k$;
(e) There exists a sequence of normal covering maps

$$
\hat{H} \times \mathbb{R}^k \xrightarrow{\psi_k} N_0 \times \mathbb{T}^k \xrightarrow{\phi_k} N,
$$

with respective deck transformation groups $\pi_1(N_0) \times Z$ and $\Gamma$, such that $\psi_k$ is a product map, $N_0$ is orientable, $\pi_1(N_0)$ is a normal subgroup of $\pi_1(N)$ containing the commutator subgroup $[\pi_1(N), \pi_1(N)]$, and the sequences

$$
0 \to \pi_1(N_0) \times Z \to \pi_1(N) \to \Gamma \to 0
$$

and

$$
0 \to (\pi_1(N_0)/[\pi_1(N), \pi_1(N)]) \times Z \to H_1(N, Z) \to \Gamma \to 0
$$

are exact.

Because $[\pi_1(N), \pi_1(N)] \subseteq \pi_1(N_0)$, the covering map $\phi_0$ is Abelian.

It is clear from \[3.1\] that the integrals $\int_N F_{\rho_{\pi}} d\text{vol}_N$ vanish when central Busemann functions are harmonic. Lemma \[5.1\] states that sub- or superharmonic central Busemann functions must be harmonic, which implies the following.

**Corollary 1.3.** Let $N$ be a compact $n$-dimensional Riemannian manifold with no conjugate points and $Z$ a subgroup of $Z(\pi_1(N))$. If each Busemann function associated with $Z$ is sub- or superharmonic, then statements (i)-(iv) in Theorem \[17\] and the conclusions of Theorem \[1.2\] hold.

Since they must be harmonic, concave or convex central Busemann functions must in fact be totally geodesic. It is well known that Busemann functions are convex when $N$ has no focal points or, more narrowly, nonpositive sectional curvature. Thus the following generalizes the splitting theorems of O’Sullivan and Eberlein.

**Theorem 1.4.** Let $N$ be a compact $n$-dimensional Riemannian manifold with no conjugate points and $Z$ a subgroup of $Z(\pi_1(N))$. If each Busemann function associated with $Z$ is convex or concave, then, in addition to the conclusions of Theorem \[1.2\], the following hold:

(a) $\hat{N}$ is isometric to $\hat{H} \times \mathbb{R}^k$, where $Z$ acts on each $\mathbb{R}^k$-fiber by translations;
(b) $N$ is foliated by totally geodesic and flat $k$-dimensional toruses;
(c) There exists a sequence of Riemannian covering maps

$$
\hat{H} \times \mathbb{R}^k \xrightarrow{\psi_1} N_1 \times \mathbb{T}^k \xrightarrow{\phi_1} N,
$$

where $N_1$ is orientable, $\psi_1$ is a product, $\phi_1$ has finitely many sheets, and each map restricts on each $\mathbb{R}^k$- or $\mathbb{T}^k$-fiber to a totally geodesic and locally isometric immersion onto a leaf of the $\mathbb{T}^k$-foliation below.
Theorem 1.5) cannot be strengthened to the isometric splitting of a finite cover, as there are examples in \([6, 7, 8]\) of compact \(N_1 \times \mathbb{T}^k\) with nonpositive curvature that are not finitely covered by isometric products with flat \(k\)-dimensional toruses.

Eberlein additionally proved that every compact manifold having nonpositive sectional curvature and fundamental group with nontrivial center is of a canonical form. This result also generalizes, provided one modifies the definition of a canonical manifold in \([9]\) appropriately. Let \(H\) be a complete and simply connected manifold with no conjugate points, \(\Gamma_0\) a properly discontinuous group of isometries of \(H\), where \(\hat{H}/\Gamma_0\) is compact, and \(\rho : \Gamma_0 \to \mathbb{T}^k\) a homomorphism whose kernel contains no nontrivial elements with fixed points. Define an action of \(\Gamma_0\) on \(\hat{H} \times \mathbb{T}^k\) by \(\Phi(\xi, \hat{x}) = (\Phi(\hat{x}), \rho(\Phi) \cdot \xi)\) for all \(\Phi \in \Gamma_0, \xi \in \mathbb{T}^k\), and \(\hat{x} \in \hat{H}\). Then the quotient \((\hat{H} \times \mathbb{T}^k)/\Gamma_0\) is called a \(k\)-canonical manifold.

**Theorem 1.5.** Let \(N\) be a compact \(n\)-dimensional Riemannian manifold with no conjugate points and \(Z\) a subgroup of \(Z(\pi_1(N))\). If each Busemann function associated with \(Z\) is convex or concave, then \(N\) is a \(k\)-canonical manifold for \(k = \text{rank } Z\). When \(Z = Z(\pi_1(N))\), \(\Gamma_0\) is centerless.

The proof of Theorem 1.5 will be omitted, as it exactly follows Eberlein’s argument, using Theorem 1.4(a) in the place of Lemma 1 of \([9]\).

### 2. Preliminaries

Throughout this paper, \((N, h)\) will denote a compact, connected, and \(C^r\) Riemannian manifold for \(r \geq 2\). The Riemannian universal covering map will be denoted by \(\tilde{\pi} : \tilde{N} \to N\) and the covering metric by \(\tilde{h}\). The metric \(h\) will have no conjugate points, which by definition means that the exponential map on each tangent space is nonsingular. It is well known that a complete manifold with nonpositive sectional curvature has no focal points, that a complete manifold with non focal points has no conjugate points, and that neither of these implications is reversible within the space of compact manifolds of any fixed dimension at least two \([12]\). Note that a complete manifold has no conjugate points if and only if, whenever \(p, q \in \tilde{N}\), there exists a unique unit-speed geodesic in \(\tilde{N}\) from \(p\) to \(q\). In particular, \(\tilde{h}\) also has no conjugate points.

For any tangent vector \(v\), denote by \(\gamma_v\) the geodesic \(t \mapsto \exp(tv)\). Corresponding to each unit vector \(v \in T\tilde{N}\) is the Busemann function \(b_v : \tilde{N} \to \mathbb{R}\) defined by \(b_v(x) = \lim_{t \to \infty} \left[t - d(\gamma_v(t), x)\right]\). It will be convenient to generalize the idea of a Busemann function to arbitrary tangent vectors by setting

\[
b_v = \begin{cases} 
\|v\|b_v/\|v\| & \text{if } v \neq 0 \\
0 & \text{if } v = 0
\end{cases}
\]

for any \(v \in T\tilde{N}\). It was essentially shown by Busemann \([8]\) that associated to each \(z \in Z(\pi_1(N))\) is a unique constant-length vector field \(\omega_z\) on \(N\) with the property that each \(\gamma_{\omega_z(x)}\) on \(N\) is a closed geodesic representing \(z\) in \(\pi_1(N)\). The inverse function theorem implies that \(\omega_z\) is \(C^{r-1}\) (see \([6]\) for details). Denote by \(\hat{\omega}_z\) the lift of \(\omega_z\) to \(\tilde{N}\). Following the arguments in \([2]\) and \([13]\), one finds that, for each \(p \in \tilde{N}\), \(b_{\hat{\omega}_z(p)}\) has gradient field \(\hat{\omega}_z\).

For each \(v \in T\tilde{N}\), a horosphere of \(v\) is, by definition, a level set of the Busemann function \(b_v\). When \(z \in Z(\pi_1(N))\), the horospheres of \(\hat{\omega}_z(\hat{x})\) are the leaves of the normal distribution to \(\hat{\omega}_z\) for any \(\hat{x} \in \tilde{N}\) and, consequently, form a \(C^r\) foliation of
\[ T = \text{inj} \] are, they may in some cases be used to construct fundamental domains of \( \pi \). A preliminary lemma is first presented.

**Lemma 2.1.** If \( z \in Z(\pi_1(N)) \) is primitive, in the sense that \( z \neq mz' \) for all \( |m| > 1 \) and \( z' \in \pi_1(N) \), then, for each \( \dot{x} \in \dot{N} \), \( \pi \circ \gamma_{\dot{z}}(\dot{x}) \) is injective on \( [0, \|z\|_\infty) \).

**Proof.** Assume that \( \pi \circ \gamma_{\dot{z}}(t_0) = \pi \circ \gamma_{\dot{z}}(t_1) \) for \( 0 \leq t_0 < t_1 < \|z\|_\infty \). Write \( T = t_1 - t_0 \). Without loss of generality, replace \( x \) with \( \gamma_{\dot{z}}(t_0) \), so that

\[ \pi \circ \gamma_{\dot{z}}(0) = \pi \circ \gamma_{\dot{z}}(T) = \pi \circ \gamma_{\dot{z}}(\|z\|_\infty). \]

Since \( \text{inj}(\pi(x)) > 0 \), \( T = (a/b)\|z\|_\infty \) for some relatively prime \( a, b \in \mathbb{Z} \) with \( |b| > 1 \). Note that \( ma = nb + 1 \) for some \( m, n \in \mathbb{Z} \). Since \( \pi \circ \gamma_{\dot{z}}(0) = \pi \circ \gamma_{\dot{z}}((ma/b)\|z\|_\infty) = \pi \circ \gamma_{\dot{z}}(0), \gamma_{\dot{z}}(\pi(0)), |b, (b/a)|\|z\|_\infty | \) is a closed geodesic that represents an element \( \omega \) of \( \pi_1(N) \) with \( z = \omega w \), which is a contradiction. \( \square \)

**Lemma 2.2.** Let \( z \in Z(\pi_1(N)) \) be primitive and \( H \) a horosphere of \( z \). Then there exist disjoint open subsets \( U_1, \ldots, U_K \) of \( H \) such that, for \( V_i = U_i \times [0, \|z\|_\infty) \) and \( V = \bigcup_{i=1}^K V_i \), \( \pi \) is injective on \( V \) and \( \text{vol}_h(\dot{N} \setminus \pi(V)) = 0 \).

**Proof.** It follows from \( \|z\|_\infty \)-periodicity that there exists \( \varepsilon > 0 \) such that for each \( x \in H \), the exponential map is injective on \( B(x, \varepsilon) \times [0, \|z_0\|_\infty) \subseteq NH \). By compactness, one may choose a finite subset \( \{W_1, \ldots, W_K\} \) of \( \{B(x, \varepsilon) \mid x \in H\} \) with the property that \( \{\pi(W_i \times [0, \|z_0\|_\infty])\} \) is an open cover of \( N \). Let \( U_i = W_i \). The remaining \( U_i \) are defined inductively: Suppose \( U_1, \ldots, U_{i-1} \) have been defined and satisfy \( U_j \subseteq W_j \). For each \( 1 \leq j \leq i-1 \), there exist at most finitely many \( \alpha_{j,1}, \ldots, \alpha_{j,M_j} \in \pi_1(N) \) such that \( \alpha_{j,m}(U_j \times \|z\|_\infty) = U_i \). Let

\[ i-1 \quad M_j \]

\[ U_i = W_i \setminus \bigcup_{j=1}^{i-1} \bigcup_{m=1}^{M_j} \alpha_{j,m}(U_j \times \|z\|_\infty). \]

The sets \( U_1, \ldots, U_K \) constructed in this way have the desired properties. \( \square \)

The action of \( \pi_1(N) \) on \( \dot{N} \) by deck transformations will be denoted by \( (\alpha, x) \mapsto \alpha(x) \). The following is a special case of an important lemma of Ivanov–Kapovich. However, as they point out, the methods of Croke–Schroeder suffice to prove it for central elements. In particular, the narrow statement here requires only that the metric be \( C^2 \). The more general statement in [15] requires \( C^k \) regularity for some \( k \) depending on \( n \).

**Lemma 2.3.** Let \( \alpha \in \pi_1(N) \) and \( z \in Z(\pi_1(N)) \). If \( \gamma_\alpha \) is an axis of \( z \), then \( b_\alpha(\gamma_\alpha(\dot{x})) - b_\alpha(\dot{x}) \) is independent of \( \dot{x} \in \dot{N} \).

For any choices of \( p, q \in \dot{N} \), \( b_{\gamma_\alpha(p)} \) and \( b_{\gamma_\alpha(q)} \) differ by a constant. Thus the function \( B(z_0, z_1) = b_{\gamma_\alpha(p)}(z_0(x)) - b_{\gamma_\alpha(p)}(\dot{x}) \) is also independent of the choice of \( p \). Loosely speaking, \( B(z_0, z_1) \) is the change in the Busemann functions of \( z_1 \) in the direction of \( z_0 \).

Lemma 2.3 implies, by the argument in [15], the virtual splitting of cyclic subgroups of \( Z(\pi_1(N)) \).
Theorem 2.4 (Ivanov–Kapovitch). For each nontrivial \( z \in Z(\pi_1(N)) \), there exists a finite-index subgroup \( G \) of \( \pi_1(N) \) isomorphic to a direct product \( G' \times \mathbb{Z} \), under which identification \( z \) corresponds to a generator of the \( \mathbb{Z} \)-factor.

A simple consequence is that, when \( Z(\pi_1(N)) \) has rank at least \( k \), a finite-index subgroup of \( \pi_1(N) \) splits as a product with \( \mathbb{Z}^k \).

Corollary 2.5. If \( z_1, \ldots, z_k \) are independent elements of \( Z(\pi_1(N)) \), then there exists a finite-index subgroup \( G \) of \( \pi_1(N) \) isomorphic to a direct product \( G' \times \mathbb{Z}^k \), under which identification the \( \mathbb{Z}^k \)-factor is generated by elements of the form \( z_1^{m_1}, \ldots, z_k^{m_k} \) for \( m_i \geq 1 \).

Proof. Let \( \varphi_i = (\pi_i, \sigma_i) : G_i \to G'_i \times \mathbb{Z} \) be the isomorphisms guaranteed by Theorem 2.4 where each \( G'_i \) is taken to be a subgroup of \( G_i \) on which \( \pi_i \) is the identity. Let \( G = \bigcap_{i=1}^k G_i \), \( G' = \bigcap_{i=1}^k G'_i \), and \( \varphi' = \pi_k \circ \cdots \circ \pi_1 \). Then \( \varphi = (\pi'_i, \sigma_{i_1}, \ldots, \sigma_k) : G \to G' \times \mathbb{Z}^k \) is a homomorphism. Note that

\[
\ker \varphi \subseteq \bigcap_{i=1}^k \ker \sigma_i \subseteq \bigcap_{i=1}^k G_i' = G'.
\]

Since each \( \pi_i \) is the identity on \( G'_i \), \( \varphi \) must be one-to-one. Since \( G \) has finite index, there exist smallest \( m_i \geq 1 \) such that \( z_i^{m_i} \in G \). It follows that \( \varphi(G) \) is isomorphic to \( G' \times \mathbb{Z}^k \) in such a way that the \( z_i^{m_i} \) generate the \( \mathbb{Z}^k \)-factor. \( \square \)

Because \( \hat{N} \) contains no nontrivial closed geodesics, \( Z(\pi_1(N)) \) must be torsion-free [17] (more generally, because \( N \) is aspherical, \( \pi_1(N) \) is torsion-free [14]). Let \( Z \) be a subgroup of \( Z(\pi_1(N)) \) of rank \( k \). Then \( Z \cong \mathbb{Z}^k \). Following [2], one would like to define an inner product on \( \mathbb{R}^k \) that extends the restriction of \( B \) to \( Z \times Z \) with respect to a fixed isomorphism \( Z \to \mathbb{Z}^k \). In fact, elementary arguments show that \( B \) satisfies some of the properties of an inner product: \( B(mz_0, z_1) = mB(z_0, z_1) = B(z_0, mz_1) \) for all \( m \in \mathbb{Z} \), the expression \( \|z\|_\infty = B(z, z)^{1/2} \) agrees with the asymptotic norm of the orbit metric on \( Z \) obtained from its action on \( \hat{N} \) (see [4]), and, significantly, \( B \) satisfies the Cauchy–Schwarz inequality,

\[
\|B(z_0, z_1)\| \leq \|z_0\|_\infty \|z_1\|_\infty,
\]

with equality if and only if \( z_0 \) and \( z_1 \) are rationally related, in the sense that \( mz_0 = \ell z_1 \) for some \( m, \ell \in \mathbb{Z} \) not both zero. However, it is not clear that, in general, \( B \) is symmetric and additive in each argument.

3. Integral formulations of \( B \)

Let \( z_0, z_1 \in Z(\pi_1(N)) \). The function \( F_{z_0, z_1} : N \to \mathbb{R} \) in Theorem 1.2 will be defined by an equivariant construction on \( \hat{N} \). Fix \( \hat{p} \in \hat{N} \) and \( v_i = \hat{\omega}_{z_i}(\hat{p}) \). For \( x \in N \), let \( \hat{x} \in \pi^{-1}(x) \) and \( v = \hat{\omega}_{z_0}(\hat{x}) \), and define \( F_{z_0, z_1}(x) \) to be the average value of \( b_{v_i} \Delta b_{v_0} \) along \( \gamma_x \) with respect to arclength:

\[
F_{z_0, z_1}(x) = \int_{\gamma_x(0, 1]} b_{v_i} \Delta b_{v_0} ds.
\]

More specifically, \( F_{z_0, z_1} = 0 \) if \( z_0 \) is the identity, and

\[
F_{z_0, z_1}(x) = \frac{1}{\|z_0\|_\infty} \int_{\gamma_x(0, 1]} b_{v_i} \Delta b_{v_0} ds
\]

otherwise. To show that \( F_{z_0, z_1} \) is well defined, a preliminary lemma is needed.
Lemma 3.1. Let \( z \in Z(\pi_1(N)) \) be nontrivial. Let \( \hat{x} \in \hat{N}, v = \hat{\omega}_z(\hat{x}) \), and \( T \in \mathbb{R} \). For each \( t \geq T \), denote by \( H_t \) the horosphere of \( z \) along which \( b_0 = t \). Let \( U \subset H_T \) be any open set with \( C^r \) boundary and compact closure, and let \( U_t = U \times \{ T + t \} \) with respect to the splitting \( \hat{N} \cong H_T \times \mathbb{R} \). Then

\[
\frac{d}{dt} \text{vol}_{H_{T+t}}(U \times \{ T + t \}) = \frac{1}{\|z\|\infty} \int_{U \times \{ T + t \}} \Delta b_v \text{dvol}_{H_{T+t}}.
\]

Proof. Denote by \( \nu \) the outward-pointing unit normal vector field along the \( C^r \) portion of \( \partial U_t \), so that, except at corners, \( \nu = -\frac{\hat{\omega}_z}{\|\hat{\omega}_z\|\infty} \) along \( U \times \{ T \} \), \( \nu = \frac{\hat{\omega}_z}{\|\hat{\omega}_z\|\infty} \) along \( U \times \{ T + t \} \), and \( \hat{h}(\nu, \hat{\omega}_z) = 0 \) along \( \partial U \times [0, \|z\|\infty] \). By Green’s identity,

\[
\int_{U_t} \Delta b_v \text{dvol}_N = \int_{\partial U_t} \hat{h}(\hat{\omega}_z, \nu) \text{dvol}_{\partial U_t},
\]

\[
= \|z\|\infty[\text{vol}_{H_{T+t}}(U \times \{ T + t \}) - \text{vol}_{H_T}(U \times \{ T \})].
\]

It follows from the coarea formula that

\[
\int_{U_t} \Delta b_v \text{dvol}_N = \int_T^{T+t} \int_{U \times \{ T + s \}} \Delta b_v \text{dvol}_{H_{T+t}} \text{d}s.
\]

The result follows immediately. \( \square \)

If \( \hat{x}' \in \pi^{-1}(x) \) and \( v' = \hat{\omega}_{z_0}(\hat{x}') \), then \( \hat{x}' = \alpha(x) \) for some \( \alpha \in \pi_1(N) \). Note that, for all \( t \), \( \Delta b_{v_0}(\gamma_{\nu}(t)) = \Delta b_{v_0}(\gamma_{\nu'}(t)) \) and, by Lemma 2.2, \( b_{v_0}(\gamma_{\nu}(t)) - b_{v_0}(\gamma_{\nu'}(t)) = B(\alpha, z_1) \in \mathbb{R} \). Moreover, by Lemma 3.1 with \( T = b_{v_0}(\hat{x})/\|z_0\|\infty^2 \) and \( U^z = B(\gamma_{\nu}(-T), \varepsilon) \subseteq H_0 \),

\[
\int_{\gamma_{v_0}[0,1]} \Delta b_{v_0} \text{d}t = \lim_{\varepsilon \to 0} \int_{U^z} \Delta b_{v_0} \text{dvol}_N,
\]

\[
= \lim_{\varepsilon \to 0} \|z_0\|\infty[\text{vol}_{H_{T+\|z_0\|\infty}}(U^z \times \{ T + \|z_0\|\infty \}) - \text{vol}_{H_T}(U^z \times \{ T \})]
\]

\[
= 0.
\]

Thus

\[
\int_{\gamma_{v_0}[0,1]} b_{v_1} \Delta b_{v_0} \text{d}t = \int_{\gamma_{v_0}[0,1]} [b_{v_1} + B(\alpha, z_1)] \Delta b_{v_0} \text{d}s
\]

\[
= \int_{\gamma_{v_0}[0,1]} b_{v_1} \Delta b_{v_0} \text{d}s + B(\alpha, z_1) \int_{\gamma_{v_0}[0,1]} \Delta b_{v_0} \text{d}s
\]

\[
= \int_{\gamma_{v_0}[0,1]} b_{v_1} \Delta b_{v_0} \text{d}s.
\]

So \( F_{z_0, z_1} \) is well defined. Similarly, \( F_{z_0, z_1} \) is independent of the choice of \( \hat{p} \), as replacing \( \hat{p} \) with \( \hat{q} \) induces a constant change in \( b_{v_1} \) of \( b_{v_1}(\hat{q}) - b_{v_1}(\hat{p}) \) while leaving \( \Delta b_{v_0} \) unchanged.

For any \( \hat{p} \) as above, let \( S \) be a fundamental domain of \( \pi \) constructed using a horosphere \( H_T \) of \( v_0 \) as in Lemma 2.2 and write \( S_t = S \times [t, t + \|z_0\|\infty] \) for each
Let $t \in [T, T + \|z_0\|_\infty]$. Then
\[
\int_{\mathcal{N}} F_{z_0 z_i} \, d\text{vol}_N = \int_{\mathcal{S}_T} \int_{\gamma_{z_0 t}(x)} b_{v_i} \Delta b_{v_0} \, ds \, d\text{vol}_N
\]
(3.2)
\[
= \int_{T}^{T + \|z_0\|_\infty} \int_{\mathcal{S} \times \{t\}} \int_{\gamma_{z_0 t}(x)} b_{v_i} \Delta b_{v_0} \, ds \, d\text{vol}_H, \, dt
\]
\[
= \int_{T}^{T + \|z_0\|_\infty} \int_{\mathcal{S}_T} b_{v_i} \Delta b_{v_0} \, d\text{vol}_N \, dt.
\]

**Theorem 3.2.** Let $z_0, z_1 \in Z(\pi_1(N))$. Then
\[
B(z_0, z_1) = \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} [h(\omega_{z_0}, \omega_{z_1}) + F_{z_0 z_1}] \, d\text{vol}_N.
\]

**Proof.** By Theorem 2.4, there exist a finite covering map $\psi : \mathcal{N} \to N$, with covering metric $\hat{h}$, and a primitive $\hat{z}_0 \in Z(\pi_1(N))$ such that $\psi_{*}(\hat{z}_0) = z_0$. For each $i = 1, 2$, denote by $\hat{\omega}_i$ the lift of $\omega_i$ to $\mathcal{N}$. Then
\[
\frac{1}{\text{vol}(N)} \int_{\mathcal{N}} h(\omega_{z_0}, \omega_{z_1}) \, d\text{vol}_N = \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} \hat{h}(\hat{\omega}_0, \hat{\omega}_1) \, d\text{vol}_N.
\]
Fix $\hat{x} \in \mathcal{N}$ and $v_i = \hat{\omega}_i(\hat{x})$. Let $T \in \mathbb{R}$. For the horospheres $H_t$ of $z_0$, let $U_1, \ldots, U_K$ be the open subsets of $H_T$ obtained by applying Lemma 2.2 to the covering $\mathcal{N} \to \mathcal{N}$, and write $S = \bigcup_{i=1}^{K} U_i$ and $S_t = S \times \{t, t + \|z_0\|_\infty\}$. If each $U_i$ has $C^r$ boundary, then Green’s identity shows that
\[
\int_{S_t} \hat{h}(\hat{\omega}_0, \hat{\omega}_1) \, d\text{vol}_N = \int_{\partial S_t} b_{v_i} \hat{h}(\hat{\omega}_0, \nu) \, d\text{vol}_{\partial S_t} - \int_{S_t} b_{v_i} \Delta b_{v_0} \, d\text{vol}_N.
\]
Applying Lemma 2.3 one has that
\[
\int_{\partial S_t} b_{v_i} \hat{h}(\hat{\omega}_0, \nu) \, d\text{vol}_{\partial S_t} = \|z_0\|_\infty \left[ \int_{S \times \{t + \|z_0\|_\infty\}} b_{v_i} \, d\text{vol}_{H_{t + \|z_0\|_\infty}} - \int_{S \times \{t\}} b_{v_i} \, d\text{vol}_{H_t} \right]
\]
\[
= \|z_0\|_\infty B(z_0, z_1) \text{vol}(S \times \{t\}).
\]
Suppose that $z_0$ is not the identity, as the result holds otherwise. Substituting the above into equation (3.3), integrating both sides from $T$ to $T + \|z_0\|_\infty$, and using the fact that $\int_{S_t} \hat{h}(\hat{\omega}_0, \hat{\omega}_1) \, d\text{vol}_N = \int_{\mathcal{N}} \hat{h}(\hat{\omega}_0, \hat{\omega}_1) \, d\text{vol}_N$ yields
\[
B(z_0, z_1) = \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} h(\omega_{z_0}, \omega_{z_1}) \, d\text{vol}_N
\]
(3.4)
\[
+ \frac{1}{\|z_0\|_\infty \text{vol}(N)} \int_{T}^{T + \|z_0\|_\infty} \int_{S_t} b_{v_i} \Delta b_{v_0} \, d\text{vol}_N \, dt
\]
\[
= \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} h(\omega_{z_0}, \omega_{z_1}) \, d\text{vol}_N + \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} F_{z_0 z_1} \, d\text{vol}_N,
\]
the latter equality following from equation (3.2). In the general case, one may apply a similar argument to a union of open sets $U_{i,m} \subseteq U_i$ that have $C^r$ boundary and whose measures converge to that of $U_i$ as $m \to \infty$.

**Corollary 3.3.** Let $z_0, z_1 \in Z(\pi_1(N))$. Then the following hold:
(a) $B(z_0, z_1) = B(z_1, z_0)$ if and only if $\int_{\mathcal{N}} F_{z_0 z_1} \, d\text{vol}_N = \int_{\mathcal{N}} F_{z_1 z_0} \, d\text{vol}_N$;
(b) $B(z_0, z_1) = \frac{1}{\text{vol}(N)} \int_{\mathcal{N}} h(\omega_{z_0}, \omega_{z_1}) \, d\text{vol}_N$ if and only if $\int_{\mathcal{N}} F_{z_0 z_1} \, d\text{vol}_N = 0$. 
\[ \square \]
Let $Z$ be any rank $k$ subgroup of $Z(\pi_1(N))$, and fix an isomorphism $D : Z \to \mathbb{Z}^k$. Denote by $e_1, \ldots, e_k$ the standard basis for $\mathbb{R}^k$, and let $w_i = D^{-1}(e_i)$. Let $f : T^k \to N$ be any fixed $C^1$ map satisfying $f_*(\alpha_i) = w_i$, where $\alpha_1, \ldots, \alpha_k$ are generators for $\pi_1(T^k)$. General theory guarantees the existence of such maps, although in this case they may be constructed explicitly by iteratively exponentiating around loops (see [3]).

**Theorem 3.4.** Let $z_0, z_1 \in Z$ and $\beta_1 = f^*(\omega_{z_1})$ (i.e., $\beta_1(v) = h(\omega_{z_1}, f_*(v))$ for all $v \in T^k$). Then $B(z_0, z_1) = \int_{T^k} \beta_1 \circ D(z_0) d\text{vol}_g$ for the standard flat metric $g$ on $T^k$.

**Proof.** Suppose $k \geq 2$, as otherwise the result is clear. Denote by $\phi : \mathbb{R}^k \to T^k$ the covering map that quotients by $\mathbb{Z}^k$. Let $V = D(z_0) = \sum_{i=1}^m v_i e_i$ be a constant vector field on $T^k$. Suppose $V \neq 0$, so that some $v_i \neq 0$. Let $u_i = V$ and, for each $j \neq i$, let $u_j$ be the vector in $\mathbb{R}^k$ whose only nonzero entries are $-v_j$ in the $i$-th component and $v_i$ in the $j$-th component. Let $P$ be the parallelepiped in $\mathbb{R}^k$ determined by $u_1, \ldots, u_k$. Then $P$ is the union of $v_i^{k-2} \sum_{j=1}^k v_j^2 = v_i^{k-2} \|V\|_{\mathbb{R}^k}$ fundamental domains of $\phi$. Denote by $Q$ the face of $P$ that contains the origin but not $u_i$, and, for each $p \in Q$, denote by $\alpha_p$ the geodesic $t \mapsto p + tV/\|V\|_{\mathbb{R}^k}$.

The map $f$ lifts to a map $F : \mathbb{R}^k \to N$ such that $f \circ \phi = \pi \circ F$. Let $\bar{V} = \phi^{-1}(V)$, so that $h(\omega_{z_1}, f_*(\bar{V})) = \bar{h}(\omega_{z_1}, F_*(\alpha'_p(t)))$.

The coarea formula implies that

$$\int_T h(\omega_{z_1}, f_*(\bar{V})) d\text{vol}_g = \frac{1}{v_i^{k-2} \|V\|_{\mathbb{R}^k}} \int_Q \left[ \int_0^{\|V\|_{\mathbb{R}^k}} \bar{h}(\omega_{z_1}, F_*(\alpha'_p(t))) dt \right] d\text{vol}_Q$$

$$= \frac{1}{v_i^{k-2} \|V\|_{\mathbb{R}^k}} \int_Q \left[ b_{z_1} \circ F(x + V) - b_{z_1} \circ F(x) \right] d\text{vol}_Q$$

$$= \frac{1}{v_i^{k-2} \|V\|_{\mathbb{R}^k}} \int_Q \left[ b_{z_1}(z_0(F(x))) - b_{z_1}(F(x)) \right] d\text{vol}_Q$$

$$= \frac{1}{v_i^{k-2} \|V\|_{\mathbb{R}^k}} \int_Q B(z_0, z_1) d\text{vol}_Q$$

$$= B(z_0, z_1).$$

If $V = 0$, then $z_0$ is the identity, and one obtains the same equality. \qed

**4. PROOF OF THEOREMS 1.1 AND 1.2**

Because the proof of Theorem 1.1 is simplest when $Z$ has finite rank, the equivalence of (i)-(iv) will first be proved under the additional assumption that $Z \cong \mathbb{Z}^k$. It then be shown that this assumption is superfluous. As before, $D : Z \to \mathbb{Z}^k$ will be an isomorphism and $w_i = D^{-1}(e_i)$, so that $w_1, \ldots, w_k$ generate $Z$.

Write $\mathcal{B}_0 = \{\omega_z \mid z \in Z\}$, and denote by $\mathcal{B}$ the vector space of finite formal linear combinations of elements of $\mathcal{B}_0$ with real coefficients. (For each $\hat{x} \in \hat{N}$, $\mathcal{B}$ may be identified with the space of finite formal combinations of Busemann functions in $Z$ that vanish at $\hat{x}$.) Roughly speaking, one may extend $D$ to a linear direction at infinity $\mathcal{D} : \mathcal{B} \to \mathbb{R}^k$ by setting $\mathcal{D}(\sum_i a_i \omega_{z_i}) = \sum_i a_i D(z_i)$. In the other direction, there is the inclusion $\iota : \mathbb{Z}^k \to \mathcal{B}$ defined by $\iota(\sum_i m_i e_i) = \omega \sum_i m_i w_i$. 
For each $\zeta_1, \zeta_2 \in \mathcal{B}$ (i.e., $\zeta_i = \sum_j a_{ij} \omega_{z_j}$ for $z_j \in Z$ and $a_{ij} \in \mathbb{R}$), let $\beta_i = f^*(\zeta_i)$. Define
\[
G(\zeta_1, \zeta_2) = \frac{1}{2} \int_{\mathbb{R}^k} [\beta_1 \circ \mathcal{D}(\zeta_2) + \beta_2 \circ \mathcal{D}(\zeta_1)]d\mu_g.
\]
It is clear that $G$ is symmetric and bilinear. By Corollary 3.3(a) and Theorem 3.4 when (i) holds,
\[
G(\omega_{z_0}, \omega_{z_1}) = B(z_0, z_1) = B(z_1, z_0).
\]
At the same time, one may define a semi-inner product $H$ on $\mathcal{B}$ by
\[
H(\zeta_1, \zeta_2) = \frac{1}{\text{vol}(N)} \int_N h(\zeta_1, \zeta_2)d\text{vol}_N.
\]
Again assuming (i), Corollary 3.3(b) implies that
\[
H(\omega_{z_0}, \omega_{z_1}) = B(z_0, z_1) = B(z_1, z_0).
\]
In this case, since they agree on $\mathcal{B}_0 \times \mathcal{B}_0$, $G$ and $H$ define the same semi-inner product on $\mathcal{B}$ and, consequently, the same semi-norm. Overloading notation, write $\| \cdot \|_\infty = \| \cdot \|_G = \| \cdot \|_H$.

**Lemma 4.1.** Let $Z$ be a subgroup of $Z(\pi_1(N))$ such that $\omega_{\sum_i m_i z_i} = \sum_i m_i \omega_{z_i}$ for all $z_i \in Z$ and all $m_i \in \mathbb{Z}$. Then, for all $z_0, z_1 \in Z$, the following hold:
(a) $[\omega_{z_0}, \omega_{z_1}] = 2\nabla_{\omega_{z_0}} \omega_{z_1}$ (i.e., $\nabla_{\omega_{z_0}} \omega_{z_1} = -\nabla_{\omega_{z_1}} \omega_{z_0}$);
(b) $h([\omega_{z_0}, \omega_{z_1}], \omega_{z_0}) = h([\omega_{z_0}, \omega_{z_1}], \omega_{z_1}) = 0$;
(c) $h(\omega_{z_0}(x), \omega_{z_1}(x)) = B(z_0, z_1)$ for all $x \in N$.

**Proof.** One has that
\[
0 = \nabla_{\omega_{z_0+z_1}} \omega_{z_0+z_1} = \nabla_{\omega_{z_0}} \omega_{z_0} + \nabla_{\omega_{z_0}} \omega_{z_1} + \nabla_{\omega_{z_1}} \omega_{z_0} + \nabla_{\omega_{z_1}} \omega_{z_1}
= \nabla_{\omega_{z_0}} \omega_{z_1} + \nabla_{\omega_{z_1}} \omega_{z_0},
\]
so $\nabla_{\omega_{z_0}} \omega_{z_1} = -\nabla_{\omega_{z_1}} \omega_{z_0}$. This proves (a). Moreover,
\[
0 = \omega_{z_1} [h(\omega_{z_0}, \omega_{z_0})] = 2h(\nabla_{\omega_{z_1}} \omega_{z_0}, \omega_{z_0}) = -2h(\nabla_{\omega_{z_0}} \omega_{z_1}, \omega_{z_0}) = -2\omega_{z_0} [h(\omega_{z_1}, \omega_{z_0})].
\]
One may deduce (b) from the first three equalities above. Let $\hat{x} \in \hat{N}$, and write $\gamma = \gamma_{\omega_{z_0}}(\hat{x})$. Since $h(\hat{\omega}_{z_1}, \hat{\omega}_{z_0})$ is constant along $\gamma$, one finds that
\[
B(z_0, z_1) = b_{z_1}(z_0(x)) - b_{z_1}(x)
= \int_0^1 h(\hat{\omega}_{z_0} \circ \gamma)(t), (\hat{\omega}_{z_1} \circ \gamma)(t)\,dt
= h(\hat{\omega}_{z_0}(x), \hat{\omega}_{z_1}(x)),
\]
which proves (c). \(\square\)

It is now possible to prove Theorem 1.1 when $Z \cong \mathbb{Z}^k$. Statements (i) and (ii) are equivalent by Corollary 3.3. By Lemma 4.1(c), (iv) implies (iii), and it is clear that (iii) implies (ii). Therefore, the proof can be completed by showing that (i) implies (iv).
Suppose (i) holds. Write \( \alpha = \omega \sum_i m_i z_i \), \( \beta = \sum_i m_i \omega z_i \), and \( \zeta = \alpha - \beta \). Then

\[
\|\zeta\|_2^2 = G(\zeta, \zeta) = \int_T \tilde{h}(\zeta, D(\zeta)) \, \text{dvol}_g = \int_T \tilde{h}(\zeta, 0) \, \text{dvol}_g = 0,
\]

which means that

\[
0 = \text{vol}(N) H(\zeta, \zeta) = \int_N h(\zeta, \zeta) \, \text{dvol}_h = \int_N \left( \|\alpha\|_h^2 + \|\beta\|_h^2 - 2 h(\alpha, \beta) \right) \, \text{dvol}_h.
\]

Thus

\[
(4.1) \quad \int_N h(\alpha, \beta) \, \text{dvol}_h = \int_N \frac{1}{2} (\|\alpha\|_h^2 + \|\beta\|_h^2) \, \text{dvol}_h.
\]

By the Cauchy–Schwarz inequality,

\[
h(\alpha, \beta) \leq \|\alpha\|_h \|\beta\|_h,
\]

so

\[
\int_N (\|\alpha\|_h^2 + \|\beta\|_h^2) \, \text{dvol}_h \leq \int_N 2 \|\alpha\|_h \|\beta\|_h \, \text{dvol}_h.
\]

At the same time, \( 2 \|\alpha\|_h \|\beta\|_h \leq \|\alpha\|_h^2 + \|\beta\|_h^2 \). Thus

\[
\int_N 2 \|\alpha\|_h \|\beta\|_h \, \text{dvol}_h = \int_N (\|\alpha\|_h^2 + \|\beta\|_h^2) \, \text{dvol}_h,
\]

which implies that \( \|\alpha\|_h = \|\beta\|_h \) on \( N \). Substituting into equation (4.1), one finds that

\[
\int_N h(\alpha, \beta) \, \text{dvol}_h = \int_N \|\alpha\|_h \|\beta\|_h \, \text{dvol}_h
\]

and, consequently, that \( h(\alpha, \beta) = \|\alpha\|_h \|\beta\|_h \) on \( N \). It follows that \( \alpha = \beta \), which is (iv).

**Lemma 4.2.** Let \( Z \cong \mathbb{Z}^k \) be a subgroup of \( \mathbb{Z}(\pi_1(N)) \) for which statements (i)-(iv) of Theorem 1.1 hold. Then, for each \( \hat{x} \in \hat{N} \), the set \( \{\hat{\omega}_{w_1}(\hat{x}), \ldots, \hat{\omega}_{w_k}(\hat{x})\} \) is linearly independent.
Proof. Suppose that, for some $c_1, \ldots, c_k \in \mathbb{R}$, $\sum_{i=1}^{k} c_i  \omega_{w_i}(x) = 0$. Then

$$0 = \left\| \sum_{i=1}^{k} c_i  \omega_{w_i}(x) \right\|_h = \sum_{i,j=1}^{k} c_i c_j h(\omega_{w_i}(x), \omega_{w_j}(x)) = \sum_{i,j=1}^{k} c_i c_j h(\omega_{w_i}(y), \omega_{w_j}(y)) = \left\| \sum_{i=1}^{k} c_i  \omega_{w_i}(y) \right\|_h$$

for all $y \in N$. Thus $\sum_{i=1}^{k} c_i  \omega_{w_i} = 0$ on $N$. Therefore, $0 = \| \sum_{i=1}^{k} c_i  \omega_{w_i} \|_H = \| \sum_{i=1}^{k} c_i  \omega_{w_i} \|_G$ and, consequently, $0 = \mathcal{D}(\sum_{i=1}^{k} c_i  \omega_{w_i}) = \sum_{i=1}^{k} c_i e_i$. This forces $c_i = 0$ for all $i$. \hfill $\square$

It follows that any subgroup of $Z(\pi_1(N))$ for which any of (i)-(iv) holds has rank at most $n$, which completes the proof of Theorem [1.1] as well as Theorem [1.2](a).

Theorem [1.2] b) is a consequence of the following lemma and the fact that, when (i)-(iv) hold, $G$ agrees with $H$.

**Lemma 4.3.** Let $Z$ be a subgroup of $Z(\pi_1(N))$ of rank $k$. If $G$ is positive semi-definite, then

$$G(\iota(z_0), \iota(z_1)) = G(\omega_{z_0}, \omega_{z_1}) = \frac{1}{2} [B(z_0, z_1) + B(z_1, z_0)]$$

extends to an inner product on $\mathbb{R}^k$ that induces the asymptotic norm $\| \cdot \|_{\infty}$ of $Z$ with respect to the isomorphism $D : Z \to \mathbb{Z}^k$.

**Proof.** Since $G$ is positive semi-definite, it is a semi-inner product and induces a semi-norm $\| \cdot \|_G$ on $\mathcal{B}$. If $\zeta \in \text{Ker}(\mathcal{D})$, then it follows from the definition of $G$ that $\| \zeta \|_G = 0$. Conversely, suppose $\| \zeta \|_G = 0$. Since $\| \iota(\cdot) \|_G = \| \cdot \|_{\infty}$, there exists $c > 0$ such that $\| \iota(x) \|_G \geq c \| x \|_{\mathbb{R}^k}$ for all $x \in \mathbb{Z}^k$. Write $W = \text{span} \{ \omega_{w_1}, \ldots, \omega_{w_k} \}$. Since $\mathcal{D}|_W : W \to \mathbb{R}^k$ is an invertible linear map, there exists $C > 0$ such that $\| w \|_{\mathcal{D}} \leq C \| \mathcal{D}(w) \|_{\mathbb{R}^k}$ for all $w \in W$. There exist $K_i \to \infty$ and $v_i \in W$ with $\| \mathcal{D}(v_i) \|_{\mathbb{R}^k} \leq 1$ such that $\mathcal{D}(K_i \zeta + v_i) \in \mathbb{Z}^k$. Since $\mathcal{D} \circ \iota$ is the identity map on $\mathbb{Z}^k$, one has that

$$\mathcal{D}(K_i \zeta + v_i - \iota \circ \mathcal{D}(K_i \zeta + v_i)) = 0,$$

and, consequently,

$$\| K_i \zeta + v_i \|_{\mathcal{D}} - \| \iota \circ \mathcal{D}(K_i \zeta + v_i) \|_{\mathcal{D}} \leq \| K_i \zeta + v_i - \iota \circ \mathcal{D}(K_i \zeta + v_i) \|_{\mathcal{D}} = 0.$$

It follows that

$$c \| \mathcal{D}(K_i \zeta + v_i) \|_{\mathbb{R}^k} \leq \| \iota \circ \mathcal{D}(K_i \zeta + v_i) \|_{\mathcal{D}} = \| K_i \zeta + v_i \|_{\mathcal{D}} \leq \| K_i \zeta \|_{\mathcal{D}} + \| v_i \|_{\mathcal{D}} \leq C.$$

Therefore, $\| \mathcal{D}(\zeta) \|_{\mathbb{R}^k} \leq (1 + C/c)/K_i \to 0$ as $i \to \infty$ and, consequently, $\mathcal{D}(\zeta) = 0$. Thus $\text{Ker}(\mathcal{D}) = \{ \zeta \in \mathcal{B} \ | \ \| \zeta \|_G = 0 \}$.

Let $\mathcal{B}$ be the vector space obtained as the quotient of $\mathcal{B}$ by $\text{Ker}(\mathcal{D})$, $\tilde{G}$ the corresponding inner product on $\mathcal{B}$, and $\| \cdot \|_{\tilde{G}}$ the induced norm. The map $\mathcal{D}$ descends to a map $\tilde{\mathcal{D}}$ on $\mathcal{B}$ with trivial kernel. By construction, $\tilde{\mathcal{D}}(\omega_{w_i}) = e_i$.
for each $i$, so $\hat{\mathcal{G}}$ is surjective. Thus $\hat{\mathcal{G}}$ is a linear isomorphism, and $\hat{\mathcal{G}}^{-1}(\mathbb{Z}^k) = \{ \sum_i m_i [\omega_i] | m_i \in \mathbb{Z} \} = \{ [\omega_i] | z \in \mathbb{Z} \}$ projects to a dense subset of the unit sphere in $\hat{\mathcal{H}}$. Since
\[
\| [\omega_z] \|_2^2 = \hat{G}([\omega_z], [\omega_z]) = G(\omega_z, \omega_z) = B(z, z) = \| z \|^2 = \| \hat{\mathcal{G}}([\omega_z]) \|_2^2
\]
for all $z \in \mathbb{Z}$, it follows by continuity that $\hat{\mathcal{G}} : (\hat{\mathcal{Z}}, \| \cdot \|_2) \to (\mathbb{R}^k, \| \cdot \|_\infty)$ is an isomorphism of normed spaces. Consequently, $\hat{\mathcal{G}}(\hat{\mathcal{H}})$ is an inner product on $\mathbb{R}^k$ that induces $\| \cdot \|_\infty$. On $\mathbb{Z}^k$, $\hat{\mathcal{G}}(\hat{\mathcal{H}})$ is proper. □

By Lemma 4.2 the vector fields $\omega_{w_1}, \ldots, \omega_{w_k}$ are everywhere linearly independent. Thus the involutive $C^{r-1}$ distribution $\cap_{i=1}^k \omega_{w_i}$ has constant dimension $n-k$, and by Frobenius’s theorem it foliates $\hat{\mathcal{N}}$ by $C^r (n-k)$-dimensional submanifolds, each of which is contained in an intersection of the form $\cap_{i=1}^k H_i$ for horospheres $H_1, \ldots, H_k$ of $w_1, \ldots, w_k$, respectively. The following lemma implies that each such intersection is contained in one leaf of the foliation and, as a consequence, its leaves are exactly those intersections.

**Lemma 4.4.** Let $Z$ be a subgroup of $Z(\pi_1(N))$ for which statements (i)-(iv) in Theorem 1.7 hold. Fix $\hat{x} \in \hat{\mathcal{N}}$, and, for each $i$, denote by $H_i$ the horosphere of $w_i$ through $\hat{x}$. Then $\hat{H} = \cap_{i=1}^k H_i$ is connected.

**Proof.** The proof is by induction. Write $v_j = \omega_{w_j}(\hat{x})$. Since $b_{v_j}$ has nonzero gradient, the projection $\hat{\mathcal{N}} \to H_{s_1}$ along the integral curves of $b_{v_j}$ is a continuous surjection, which implies that $H_1$ is connected. If the result holds for $\cap_{i=1}^j H_i$, then, by Corollary 4.3(c), the restriction of $b_{v_{j+1}}$ to $\cap_{i=1}^j H_i$ has nonzero gradient, so $\cap_{i=1}^{j+1} H_i$ is similarly connected. □

Define a map $\Psi : \hat{H} \times \mathbb{R}^k \to \hat{\mathcal{N}}$ in the following way: For each $(\hat{y}, s_1, \ldots, s_k) \in \hat{H} \times \mathbb{R}^k$, let $\hat{x}_0 = \hat{y}$, and inductively define $\hat{x}_{i+1} = \gamma_{\omega_{v_{i+1}}(\hat{x}_i)}(s_{i+1}) = \exp_{\hat{x}_i}(s_{i+1}\omega_{w_{i+1}}(\hat{x}_i))$. Let $\Psi(\hat{y}, s_1, \ldots, s_k) = \hat{x}_k$. The splittings $\hat{\mathcal{N}} \cong H_i \times \mathbb{R}$ ensure that $D\Psi$ is nonsingular, and, consequently, the inverse function theorem implies that $\Psi$ is a local diffeomorphism.

**Lemma 4.5.** $\Psi$ is proper.

**Proof.** It follows from Corollary 3.3 that $[B(w_i, w_j)]$ is a positive-definite symmetric matrix, so $\sum_{i=1}^k |b_i \circ \Psi(\hat{y}, m_1, \ldots, m_k)| \to \infty$ uniformly as $\sum_{i=1}^k |m_i| \to \infty$ for $m_i \in \mathbb{Z}$. By periodicity, there exists a Lipschitz constant, uniform in $\hat{y} \in \hat{\mathcal{N}}$, for all maps of the form $\Psi(\hat{y}, \cdot)$. Thus $\sum_{i=1}^k |b_i \circ \Psi(\hat{y}, s_1, \ldots, s_k)| \to \infty$ uniformly as $\sum_{i=1}^k |s_i| \to \infty$. Let $X \subset \hat{\mathcal{N}}$ be compact. Then $\sum_{i=1}^k |b_i|$ is bounded on $X$, which implies that the projection of $\Psi^{-1}(X)$ onto the $\mathbb{R}^k$-factor is compact. At the same time, the projection of $\Psi^{-1}(X)$ onto the $\hat{H}$-factor is contained within a closed ball around $\hat{x}$, so it too is compact. Thus $\Psi^{-1}(X)$ is compact. □

Hadamard’s global inverse function theorem implies that $\Psi$ is a diffeomorphism, which proves parts (d) and, in turn, (c) of Theorem 1.7.

For a fixed $\hat{x} \in \hat{\mathcal{N}}$, let $\hat{H}$ be the intersection of horospheres $\cap_{i=1}^k H_i$ containing $\hat{x}$. Following the argument in [17], set $G_0 = \{ g \in \pi_1(N) | g(\hat{x}) \in H \}$. By Lemma 2.3, $G_0$ is normal, contains the commutator subgroup $[\pi_1(N), \pi_1(N)]$, and acts freely and properly discontinuously on $\hat{H}$ by isometries. Note that the subgroup $G'_0$ of $G_0$
consisting of orientation-preserving elements has all of those same properties, and that the quotient space \( N_0 = \hat{H}/G'_0 \) is orientable. The subgroup \( G \) generated by \( G'_0 \) and \( Z \) is isomorphic to \( G'_0 \times Z \), and the quotient space \( \hat{H}/G \) is diffeomorphic to \( N_0 \times \mathbb{T}^k \). One obtains normal covering maps
\[
\hat{H} \times \mathbb{R}^k \xrightarrow{\psi} N_0 \times \mathbb{T}^k \to N.
\]
Note that \( \psi \) may be assumed a product and that \( G'_0 = \pi_1(N_0) \). Let \( \Gamma \) denote the quotient group \( \pi_1(N)/G \). By construction,
\[
0 \to \pi_1(N_0) \times Z \to \pi_1(N) \to \Gamma \to 0
\]
is a short exact sequence, which implies that
\[
0 \to \pi_1(N_0) \to \pi_1(N) \to \Gamma \to 0
\]
is as well. This proves Theorem 1.1(e).

5. Proof of Theorem 1.4

A straightforward volume argument shows that, whenever a central Busemann function is everywhere subharmonic or everywhere superharmonic, it must be harmonic.

**Lemma 5.1.** Let \( N \) be a compact Riemannian manifold with no conjugate points and \( b_v \) a central Busemann function on \( N \). If \( b_v \) is either sub- or superharmonic, then it is harmonic.

**Proof.** If \( b_v \) vanishes identically, the result is clear. Suppose \( b_v \) corresponds to a nontrivial element of \( Z(\pi_1(N)) \). Let \( T, U \), and, for each \( t \geq T \), \( H_t \) be as in Lemma 3.1. Then
\[
\frac{d}{dt} \nu_{H_{T+t}}(U \times \{T+t\}) = \frac{1}{\|z\|_\infty} \int_{U \times \{T+t\}} \Delta b_v d\nu_{H_{T+t}}.
\]
Since the right-hand side is either nonnegative or nonpositive, \( \nu_{H_{T+t}}(U \times \{T+t\}) \) is \( \|z\|_\infty \)-periodic, and \( U \) and \( T \) are arbitrary, \( \Delta b_v \) must vanish identically. \( \square \)

If \( b_v \) is a convex or concave central Busemann function, then it is subharmonic or, respectively, superharmonic. By Lemma 5.1 it is therefore harmonic. Since every convex or concave harmonic function is totally geodesic, one obtains the following.

**Lemma 5.2.** Let \( b_v \) be a central Busemann function on \( N \). If \( b_v \) is convex or concave, then it is totally geodesic.

For the remainder of this section, the hypotheses of Theorem 1.4 will be assumed. In this case, the conclusions of Theorem 1.1 hold. In particular, \( Z \cong \mathbb{Z}^k \) for some \( 0 \leq k \leq n \). As before, let \( w_1, \ldots, w_k \) generate \( Z \).

Fix \( \hat{x} \in N \), and, for each \( i \), let \( v_i = \hat{\omega}_{w_i}(\hat{x}) \). Since \( b_{v_i} \) is totally geodesic, each of its horospheres is a totally geodesic submanifold of \( N \). This and the fact that \( \nabla \hat{\omega}_{w_i} \hat{\omega}_{w_i} = 0 \) imply that \( \hat{\omega}_{w_i} \) is parallel. Thus the \( k \)-dimensional distribution \( \mathcal{S} = \text{span} \{\hat{\omega}_{w_1}, \ldots, \hat{\omega}_{w_k}\} \) is involutive and, by Frobenius’s theorem, foliates \( N \) by \( k \)-dimensional submanifolds. Because \( \mathcal{S} \) restricts on each leaf of this foliation to a
globally parallel orthonormal frame, its leaves are flat and totally geodesic Euclidean spaces. It follows from de Rham’s splitting theorem [5] that $\tilde{N}$ is isometric to $\tilde{H} \times \mathbb{R}^k$ for any intersection of horospheres $\tilde{H}$ as in Theorem 1.4. Note that $Z$ acts on each $\mathbb{R}^k$-fiber by translations, which completes the proof of Theorem 1.4(a). The proof of part (b) follows exactly as in [17]. The distribution $\mathcal{D}$ projects to a parallel distribution on $N$, the leaves of which are compact, flat, totally geodesic, and without holonomy; consequently, they must be tori.

Lemma 3 of [9] is not known to generalize to the case of no conjugate points, but Theorem 1.4(c) may still be proved using Theorem 2.4 and the argument in Proposition 5 of [10]. This work was largely motivated by the question of whether the center theorem of Wolf [18] and O’Sullivan [17] generalizes to the case of no conjugate points.

6. Additional Results and Questions

This work was largely motivated by the question of whether the center theorem of Wolf [18] and O’Sullivan [17] generalizes to the case of no conjugate points.

**Question 6.1.** Let $N$ be a compact Riemannian manifold with no conjugate points such that $Z(\pi_1(N)) \cong \mathbb{Z}^k$.

(a) Is $N$ foliated by totally geodesic and flat $k$-tori?

(b) Does the universal covering space $\tilde{N}$ split isometrically as $\tilde{H} \times \mathbb{R}^k$?

As a starting point, one might try to show that the asymptotic norm on $Z(\pi_1(N))$ is Riemannian. Even more fundamentally, it is not a priori clear that subgroups of $Z(\pi_1(N))$ must have finite rank.

It is natural to look for geometric conditions, weaker than having no focal points, that ensure the linear splitting of central Busemann functions in Theorem 1.1. The following argument shows that it suffices to control the asymptotic geometry of distance spheres on $\tilde{N}$: Fix a nontrivial $z \in Z(\pi_1(N))$. The $C^{r-1}$ regularity of $\omega_z$ guarantees that the stable Jacobi tensor along each $\gamma_\omega(z)$ is bounded. The argument in Proposition 5 of [10] shows that, at each point $\hat{x} \in \tilde{N}$, the second fundamental forms of the distance spheres $d_B(\gamma_\omega(z), t)$, for $v = \hat{\omega}_z(\hat{x})/\|\hat{\omega}_z(\hat{x})\|$, converge to that of the horosphere of $z$ through $\hat{x}$.

For any unit vector $v \in T_{\hat{x}}\tilde{N}$, define the distance function $\rho_{\hat{x}}(\cdot) = d_{\tilde{N}}(\cdot, \hat{x})$ to be asymptotically subharmonic in the direction of $v$ (respectively, superharmonic) if $\liminf_{t \to \infty} \Delta \rho_{\hat{x}}(\gamma_{\omega}(t)) \geq 0$ (respectively, $\limsup_{t \to \infty} \Delta \rho_{\hat{x}}(\gamma_{\omega}(t)) \leq 0$).
On $\hat{N} \setminus \{\hat{x}\}$, let $\kappa^*_\hat{x}$ and $\kappa^+_\hat{x}$ be the functions equal to the smallest and, respectively, largest eigenvalues of $\text{Hess}\rho_{\hat{x}}$, and similarly define $\rho_{\hat{x}}$ to be asymptotically convex in the direction of $v$ (respectively, concave) if $\liminf_{t \to \infty} \kappa^*_\hat{x}(\gamma_v(t)) \geq 0$ (respectively, $\limsup_{t \to \infty} \kappa^+_\hat{x}(\gamma_v(t)) \leq 0$). Lemmas 5.1 and 5.2 imply the following.

**Proposition 6.2.** Let $z \in Z(\pi_1(N))$. Then the following hold:

(a) If, for all $\hat{x} \in \hat{N}$, $\rho_{\hat{x}}$ is asymptotically subharmonic in the direction of $v = \hat{\omega}_{\hat{x}}(\hat{x})$, then, for each such $v$, $b_v$ is harmonic;

(b) If, for all $\hat{x} \in \hat{N}$, $\rho_{\hat{x}}$ is asymptotically convex in the direction of $v = \hat{\omega}_{\hat{x}}(\hat{x})$, then, for each such $v$, $b_v$ is totally geodesic.

Similar results hold in the case of asymptotically superharmonic or asymptotically convex Busemann functions.

**Question 6.3.** Are there natural geometric conditions, other than having no focal points, that ensure central Busemann functions are asymptotically subharmonic or asymptotically convex?

Natural conditions to consider are Ricci curvature bounds, the effects and limitations of which are discussed thoroughly in [11].

It’s also unclear whether the Heber foliation of the unit sphere bundle of a torus with no conjugate points [13] generalizes to the case where Busemann functions in $Z$ split linearly.

**Question 6.4.** Let $N$ be a compact Riemannian manifold with no conjugate points and $Z$ a subgroup of $Z(\pi_1(N))$ such that statements (i)-(iv) in Theorem 1.1 hold. Write $\mathcal{W} = \{\sum_{i=1}^k a_i \hat{\omega}_{w_i} \mid a_i \in \mathbb{R}\}$, and, for any $\hat{x} \in \hat{N}$, write $\mathcal{V}_{\hat{x}} = \{\nabla b_v \mid v = \sum_{i=1}^k a_i \hat{\omega}_{w_i} (\hat{x}) \text{ for } a_i \in \mathbb{R}\}$. Is $\mathcal{V}_{\hat{x}} = \mathcal{W}$?

An elementary argument shows that, for each nonzero $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, there is a unique $C^r$ horofunction $h_a$ such that any sequence of rational directions that converge to $a$ induces a sequence of Busemann functions that vanish at $\hat{x}$ and converge uniformly on compact sets to $h_a$. Moreover, the gradient flow of $h_a$ is through geodesics, and its level sets are the integral submanifolds of the codimension one involutive distribution $(\sum_{i=1}^k a_i \hat{\omega}_{w_i})^\perp$. However, without the linear divergence of geodesics in [13], it’s not clear that, when $a$ is irrational, $h_a = b_v$ for $v = \sum_{i=1}^k a_i \hat{\omega}_{w_i} (\hat{x})$.

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