Minor crossing number is additive over arbitrary cuts

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Abstract

We prove that if $G$ is a graph with a minimal edge cut $F$ of size three and $G_1$, $G_2$ are the two (augmented) components of $G - F$, then the crossing number of $G$ is equal to the sum of crossing numbers of $G_1$ and $G_2$. Combining with known results, this implies that crossing number is additive over edge-cuts of size $d$ for $d \in \{0, 1, 2, 3\}$, whereas there are counterexamples for every $d \geq 4$. The techniques generalize to show that minor crossing number is additive over edge cuts of arbitrary size, as well as to provide bounds for crossing number additivity in arbitrary surfaces. We point out several applications to exact crossing number computation and crossing critical graphs, as well as provide a very general lower bound for the minor crossing number of the Cartesian product of an arbitrary graph with a tree.
1 Introduction

We consider the problem of finding, or at least bounding, the crossing number of a graph $G$ based on the crossing numbers of its components when decomposing $G$ via small edge cuts. We assume that the reader is familiar with the concept of crossing numbers of graphs in surfaces: each crossing of non-adjacent edges in a drawing counts. Let $G$ be a graph and $\Sigma$ a surface, then $\text{cr}_\Sigma(G)$ denotes the minimum number of crossings of some drawing of $G$ in $\Sigma$. We further consider a related concept, minor crossing numbers, to which our techniques also apply. For a graph $G$ and a surface $\Sigma$, the minor crossing number of $G$ in $\Sigma$ is the minimum crossing number in $\Sigma$ over all graphs that have $G$ as a minor: $\text{mcr}_\Sigma(G) = \min_{G \preceq H} \text{cr}_\Sigma(H)$. A graph $H$ yielding equality in this definition is said to be a realizing graph of $G$, its optimal drawing is a realizing drawing. Intuitively, this concept allows for further minimization of the number of crossings in a drawing of $G$ by replacing each vertex of $G$ with a tree. More can be found in [6], where the concept was introduced, or in [5], where an embedding method, sharing some intuitive background with our methods, is presented in the context of the minor crossing number. For both crossing number concepts, we may omit the subscript when considering the sphere or, equivalently, the plane.

Let $G = (V, E)$ be a connected graph and $F \subseteq E(G)$ a cut in $G$ of size $d = |F|$. Let $H_1$ and $H_2$ be the two components of $G - F$. When studying a graph invariant, it is natural to ask, how does the value of that invariant on $G$ depend on the values on $H_1$ and $H_2$. When considering this question for crossing numbers, we need to define auxiliary graphs $G_i = G / H_{3-i}$, obtained from $G$ by contracting $H_{3-i}$, for $i = 1, 2$. Note that $G_i$ is also obtained from $H_i$ by adding a new vertex and connecting it to all the endvertices of $F$ in $H_i$.

We can view such cuts also in an inverse way, leading to the technically stronger concept of zip products of graphs. Introduced in [2], we consider it here in a version generalized from simple to arbitrary (multi)graphs. For $i = 1, 2$, let $G_i$ be a graph with a vertex $v_i$ of degree $d$, whose adjacent edges in $G_i$ form the set $F_i$. Let $\sigma : F_1 \to F_2$ be any bijection, and let $G$ be the graph obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges $vv$ for each $vv_1 \in F_1$ and corresponding $vv_2 = \sigma(vv_1) \in F_2$. We may denote these new edges as $F$. We say $G$ is the zip product of $G_1$ and $G_2$ at $v_1$ and $v_2$, respectively, for bijection $\sigma$. For the rest of the paper, we refer to the edges and vertices of $G$ belonging to the subgraph $G_1 - v_1$ ($G_2 - v_2$) as green (red, respectively), and to the edges $F$ as blue.

A bundle $B$ of a vertex $v$ in $G$ is a union of $d_G(v)$ pairwise edge disjoint
paths in \( G - v \), where \( d_G(v) \) denotes the degree of \( v \) in \( G \): all these paths start in the neighborhood \( N_G(v) \) of \( v \) and end at some (common) fixed vertex in \( G \), denoted as the sink of \( B \). In particular, the number of paths starting at any \( u \in N_G(v) \) equals the number of edges between \( u \) and \( v \) in \( G \). Two bundles of \( v \) in \( G \) are coherent, if they have distinct sinks and are edge disjoint. We can observe that the edges \( F \) arising from the zip product of \( G_1 \) and \( G_2 \) are a minimum cut separating \( w_1 \) from \( w_2 \) whenever, for \( i = 1, 2 \), \( G_i \) has a bundle of \( v_i \) with sink \( w_i \). Inversely, any minimum cut \( F \) gives rise to a corresponding zip product, and, whenever \(|F| \leq 3\), there always exists at least one corresponding bundle in each component. Our main result is the following:

**Theorem 1** Let \( \Sigma \) be an arbitrary surface and let \( G \) be a zip product of \( G_1 \) and \( G_2 \) at \( v_1 \) and \( v_2 \), respectively. If \( d_{v_i}(G_i) \leq 3 \), or if each of \( v_1 \) and \( v_2 \) has two coherent bundles in \( G_1 \) and \( G_2 \), respectively, then \( cr_\Sigma(G) \geq cr_\Sigma(G_1) + cr_\Sigma(G_2) \).

Theorem 1 generalizes the following result, as well as removes the two-bundle condition for small cuts:

**Theorem 2** ([2]) Let \( G \) be a zip product of \( G_1 \) and \( G_2 \) at \( v_1 \) and \( v_2 \) with \( d(v_1) = d(v_2) \). If each of \( v_1 \) and \( v_2 \) has two coherent bundles in \( G_1 \) and \( G_2 \), respectively, then \( cr(G) \geq cr(G_1) + cr(G_2) \).

As the counterexamples to the claim of Theorem 1 in the presence of just one bundle at each \( v_i \) are exhibited in [1] for any \( d_{v_i}(G_i) \geq 4 \), our result closes the question of additivity of crossing numbers over cuts with at most one bundle at each vertex. Furthermore, our approach gives an alternative proof of Theorem 2 that allows for generalization into higher surfaces. Our methods also generalize to the minor crossing number, establishing the following:

**Theorem 3** Let \( \Sigma \) be an arbitrary surface and let \( G \) be a zip product of \( G_1 \) and \( G_2 \) at \( v_1 \) and \( v_2 \), respectively. Then, \( mcr_\Sigma(G) \geq mcr_\Sigma(G_1) + mcr_\Sigma(G_2) \).

Note that for the minor crossing number, no bundles are required for the additivity of lower bounds. Furthermore, additivity of minor crossing number over blocks of a graph is established in [6], and Theorem 3 is a generalization of that result. Also, relationships of minor crossing number and bisection width have been studied in [5]; the major difference here is that the crossing number is estimated in terms of the minor crossing number.
of the two graphs resulting from the cut, but in the bisection width method, the lower bound is given in terms of the size of the smallest cut splitting the graphs into roughly equal parts. However, the embedding method from the same paper does give the bound in terms of the crossing number of the embedded graph, and our result could be considered a refinement of that method. In the proof of Theorem 3 we essentially find a specific embedding of the disjoint union of $G_1$ and $G_2$ into $G$, yielding the desired lower bound.

2 Auxiliary lemmata

We first state some key ingredients needed in our proofs of Theorems 1 and 3. If $(e, f)$ is a crossing of $e, f \in E(G)$ in some drawing of $G$, then we denote by $G^x$ the graph obtained by subdividing $e$ and $f$ and identifying the two new vertices.

**Lemma 4** Let $\Sigma$ be an arbitrary surface and let $G^{(e,f)}$ be obtained from $G = (V, E)$ by subdividing two distinct edges $e, f \in E(G)$ and identifying the new vertices into a vertex $x$. Then $cr_{\Sigma}(G^{(e,f)}) \geq cr_{\Sigma}(G) - 1$. Moreover, if $e$ and $f$ cross in some optimal drawing $D$ of $G$ in $\Sigma$, then we have equality.

**Proof.** Suppose not. So there would be a drawing of $G^{(e,f)}$ with at most $cr_{\Sigma}(G) - 2$ crossings. Then we could reintroduce the crossing instead of the vertex $x$ to obtain a drawing of $G$ with at most $cr_{\Sigma}(G) - 1$ crossings, a contradiction.

Now if $D$ is an optimal drawing of $G$ in $\Sigma$, we can place $x$ at the same point as the crossing between $e$ and $f$ and obtain a drawing of $G^{(e,f)}$ with $cr(G) - 1$ crossings, yielding the lower bound. \( \square \)

Recall that $\Sigma = \Sigma_1 \# \Sigma_2$ denotes the connected sum of two surfaces, and that if $\Sigma$ is a sphere, then so are $\Sigma_1$ and $\Sigma_2$. The following lemma will help us establishing a (hypothetical) minimum counterexample to our main theorems.

**Lemma 5** Let $\Sigma$ be a surface, let $G$ be a zip product of $G_1$ and $G_2$ at vertices of degree $d$, such that (i) $cr_{\Sigma}(G) < \min_{\Sigma = \Sigma_1 \# \Sigma_2} cr_{\Sigma_1}(G_1) + cr_{\Sigma_2}(G_2)$ and (ii) $G$ has the smallest crossing number among the graphs with these properties. If $D$ is an optimal drawing of $G$ in $\Sigma$, then any crossing in $D$ is a red-green crossing (i.e., a crossing between a red and a green edge).
**Proof.** Assume that $D$ has a crossing $\times$ not of the type red-green. In each of the following cases, we find an alternative to graph $G$ with smaller crossing number, a contradiction required to establish the claim.

First, assume that $\times$ is a green-green crossing. By Lemma 4, $cr_{\Sigma}(G^x) = cr_{\Sigma}(G) - 1$ and $cr_{\Sigma_1}(G^x_1) \geq cr_{\Sigma_1}(G_1) - 1$. As $G^x$ is a zip product of $G^x_1$ and $G^x_2$, we have $cr_{\Sigma}(G^x) = cr_{\Sigma}(G) - 1 \leq cr_{\Sigma_1}(G_1) - 1 + cr_{\Sigma_2}(G_2) \leq cr_{\Sigma_1}(G^x_1) + cr_{\Sigma_2}(G_2)$. Since the argument applies to arbitrary $\Sigma = \Sigma_1 \# \Sigma_2$, $G^x$ contradicts the choice of $G$. Similarly, we can show that $D$ has no crossings of type red-red.

Second, assume that $\times$ is a crossing between a green edge and a blue edge with green endvertex $v$. Now, $G^x$ is a zip product of $G^x_1$ and $G^x_2$, and a similar contradiction as before applies. Similarly, we can show that $D$ has no crossings of type blue-red.

Third, let $\times$ be a crossing of two blue edges with green endvertices $v$ and $w$ (note that, by the optimality of $D$, $v \neq w$). The graph $G^x_1$ has a double edge $xv_1$, where $x$ is the new vertex. The graph $G^x$ is a zip product of $G^x_1$ and $G^x_2$, which has crossing number equal to $cr(G) - 1$ by Lemma 4, a final contradiction to the choice of $G$.

**Lemma 6** Let $H_1 = (V, E)$ be a graph embedded in some surface $\Sigma$, let $H_2$ be its dual in $\Sigma$, and, for $i = 1, 2$, let $T_i \leq H_i$ be an arbitrary tree. Then, $|E(T_1)| + |E(T_2)| \leq |E(H_1)|$.

**Proof.** Let $f, m,$ and $n$ be the number of faces, edges, and vertices, respectively, of $H_1$. Due to duality, $H_2$ has $f$ vertices, $m$ edges, and $n$ faces. By Euler’s formula, $m = n + f - 2 + g$, where $g$ is the genus of $\Sigma$. As $T_1$ and $T_2$ live in different graphs, they are totally disjoint. If $k$ is their total number of vertices, then $k \leq n + f$ and they have $k - 2 \leq n + f - 2 \leq m = |E(H_1)|$ edges.

**Lemma 7** Let $\Sigma$ be a surface, assume that $\Sigma = \Sigma_1 \# \Sigma_2$, and let $G$ be a zip product of $G_1$ and $G_2$ at vertices of degree at most three w.r.t. some bijection $\sigma$. Then, (i) $cr_{\Sigma}(G) \leq cr_{\Sigma_1}(G_1) + cr_{\Sigma_2}(G_2)$ and (ii) $mcr_{\Sigma}(G) \leq mcr_{\Sigma_1}(G_1) + mcr_{\Sigma_2}(G_2)$.

**Proof.** First we prove (i). For $i = 1, 2$, let $D_i$ be an optimal drawing of $G_i$ in $\Sigma_i$. Let $N_i$ be a small disk around $v_i$, such that $D_i \cap N_i$ is a star. We can obtain a drawing of $G$ in $\Sigma$ with $cr_{\Sigma_1}(G_1) + cr_{\Sigma_2}(G_2)$ crossings by identifying the surfaces $\Sigma_i \setminus N_i$ along the boundaries of $\partial N_i$ such that the
edges originally adjacent to \( v_1 \) or \( v_2 \) match up according to \( \sigma \). Note that we may need to mirror \( D_2 \) to match the vertex rotation of \( v_2 \) with the one of \( v_1 \).

For (ii), observe that any realizing graph of \( G_i \) has a cubic vertex \( v'_i \) in the tree representing \( v_i \). A drawing of a graph with \( G \) minor that establishes the claimed upper bound can thus be obtained from arbitrary realizing drawings \( D_i \) of \( G_i \) in \( \Sigma_i \) following the same steps as in the proof of (i).

\[ \square \]

3 Additivity theorems and consequences

In this section we prove Theorems 1 and 3. Although the proof of the former could follow the same steps as the proof of the latter, we provide independent proofs for clarity.

Proof of Theorem 1. For \( d = 0, 1 \), the statement is trivial. Although the following arguments also apply for \( d = 2 \), this case has been known before [8]. Therefore, we may assume \( d = 3 \), or \( d \geq 4 \) and each of \( v_1 \) and \( v_2 \) has two coherent bundles. Let \( G \) be a counterexample with smallest crossing number and let \( D \) be an optimal drawing of \( G \). By Lemma 5, each crossing in \( D \) is a crossing of a green and a red edge. Let \( D' \) be the drawing obtained from \( D \) by (i) adding some uncrossed dotted green (red) edges in the interior of the faces of the red (green) drawing, so that the green (red) graph, induced by a red (green) face is connected, (ii) contracting all green and red edges that do not cross (note that all dotted edges are now contracted) and (iii) subdividing every edge of \( D \) that is crossed several times. Hence every edge in \( D' \) is crossed precisely once, and every crossing is still of type green-red. Then \( D' \) induces two graphs, \( H_1 \) and \( H_2 \), spanned by green and red edges, respectively, and embedded in \( \Sigma \). They are duals of each other, hence have the same number of edges, and the number of crossings of \( D \) and \( D' \) is equal to this number of edges. Furthermore, the possible two coherent bundles in \( G_i \) contract to coherent bundles in \( H_i \). Let \( S_1 \) and \( S_2 \) be the set of green and red endvertices of blue edges \( F \), respectively.

For \( d \leq 3 \) and \( i = 1, 2 \), let \( T_i \) be a tree in \( H_i \) containing all the vertices of \( S_i \). For \( i = 1, 2 \), let \( D_i \) be the subdrawing of \( D \) spanned by \( (G_i - v_i) \cup F \) and merged with the subdrawing of \( D' \) spanned by \( T_{3-i} \). The total number of crossings in \( D_1 \) and \( D_2 \) equals the number of edges in \( T_1 \cup T_2 \). By Lemma 6 this is at most \( |E(H_1)| = |E(H_2)| \), which is equal to \( cr_\Sigma(D') = cr_\Sigma(D) = cr_\Sigma(G) \).
In $D_i$, for $i = 1, 2$, we can contract the nodes $S_{3-i}$ along $T_{3-i}$ into a single vertex $v_i$ to obtain a drawing $D'_i$ of $G_i$ with $\text{cr}_\Sigma(D_i) = \text{cr}_\Sigma(D'_i)$. As $\text{cr}_\Sigma(G) = \text{cr}_\Sigma(D) \geq \text{cr}_\Sigma(D'_1) + \text{cr}_\Sigma(D'_2) \geq \text{cr}_\Sigma(G_1) + \text{cr}_\Sigma(G_2)$, $G$ is not a counterexample, a contradiction establishing the claim.

For $d \geq 4$ and $i = 1, 2$, the graph $H_i$ has two coherent (i.e., edge disjoint) bundles starting at the vertices of $S_i$. Let $B_i$ be one with less than half of the edges of $H_i$. Let $D_i$ be the subdrawing of $D$ spanned by $(G_i - v_i) \cup F$ and merged with the subdrawing of $D'$ spanned by $B_{3-i}$. We may assume that $D_i[B_{3-i}]$ is a drawing of a $d$-star, as otherwise we can split each vertex of $B_{3-i}$ that is common to two of the bundle paths in its small neighborhood and route the edges of the paths properly to satisfy the assumption. The total number of crossings in $D_1$ and $D_2$ equals the number of edges in $B_1 \cup B_2$. As each bundle has at most half of the edges of $H_i$ and $|E(H_1)| = |E(H_2)|$, the drawings $D_1$ and $D_2$ have together at most $|E(H_1)| = \text{cr}_\Sigma(D') = \text{cr}_\Sigma(D) = \text{cr}_\Sigma(G)$ crossings.

By contracting the edges of $B_{3-i}$ into a single vertex $v_i$ in $D_i$, for $i = 1, 2$, we obtain drawings $D'_i$ of $G_i$ with $\text{cr}_\Sigma(D_i) = \text{cr}_\Sigma(D'_i)$. As $\text{cr}_\Sigma(G) = \text{cr}_\Sigma(D) \geq \text{cr}_\Sigma(D'_1) + \text{cr}_\Sigma(D'_2) \geq \text{cr}_\Sigma(G_1) + \text{cr}_\Sigma(G_2)$, $G$ is not a counterexample, a contradiction establishing the claim.

Following essentially similar ideas as in the proof of Theorem 1, we can prove Theorem 3. The major difference is that we substitute the minimum counterexample argument by a more technical treatment of the crossings involving blue edges. This could also be done in the previous proof, but at the expense of its clarity.

**Proof of Theorem 3.** For $d = 0, 1$, the statement is established in [3]. By induction on $d$, we may assume that $G_i - v_i$ is connected. Let $G$ be as in the statement. Let $G'$ be its realizing graph, and let $D$ be an optimal drawing of $G'$ in $\Sigma$, i.e. an realizing drawing of $G$. Let $F$ be the set of blue edges, and let $b_1$ (respectively, $b_2$) be the number of crossings in $G'$ that involve a blue and a green (respectively, red) edge. There is a natural extension of the red and green colors from $G$ to $D$: any vertex or edge of $G'$ corresponding to a vertex or edge of $G_1$ is green, those corresponding to $G_2$ are red, and any crossing of two green (red) edges is green (red, respectively).

To obtain $D'$ from $D$, we first remove the blue edges, introduce vertices at all monochromatic crossings, contract all green and red edges that in the subsequent drawing are not crossed, and properly subdivide every edge of $D$ that is crossed several times. Hence every edge in $D'$ is crossed precisely once, and every crossing is of a green and a red edge. Then, $D'$ induces two graphs embedded in $\Sigma$, $H_1$ and $H_2$, spanned by green and red edges,
respectively. These graphs are duals of each other, have the same number of edges, and the number of crossings of $D'$ is equal to this number of edges.

Let $S_1$ and $S_2$ be the set of green and red endvertices of blue edges, respectively, and let $T_i$ be a tree in $H_i$ containing all the vertices of $S_i$. For $i = 1, 2$, let $D'_i$ be the subdrawing of $D'$ spanned by $H_i \cup T_{3-i}$ and augmented as follows: (a) we add the $F$-segments from the drawing $D$, (b) we split any crossing of two $F$-edges by rerouting the crossing paths (preserving the fact that $F \cup T_i$ is connected), and (c) in $D'_1$ (respectively, $D'_2$), we only maintain the segment of the blue edge connecting its green (respectively, red) endvertex with the first red (green) point in the drawing (which is either a crossing with a red (green) edge or the red (green) endvertex; thus all green (red) endvertices of blue edges are connected to the tree, but the blue edges never cross the red). The total number of crossings in $D'_1$ and $D'_2$ equals the number of edges in $T_1 \cup T_2$, increased by $b_1 + b_2$. By Lemma 6 this is at most $|E(H_1)| + b_1 + b_2 \leq cr(D') + b_1 + b_2$. Let $D_i$ be obtained by uncontracting the previously contracted subdrawings of $D$ within a small neighborhood of their corresponding vertices in $D'_i$. Any crossing in $D_i$ but not in $D'_i$ exists in $D$ but not in $D'$ and let $r$ be the number of such crossings.

Then, $D'_i$ is a (not necessarily optimal) drawing of a graph that has $G_i$ as a minor. Therefore, $\text{mcr}_\Sigma(G_i) \leq cr(D'_i)$ and the claim follows.

We state some corollaries that easily follow from the above theorems.

**Corollary 8** Let $G$ be a graph, and let $F \subseteq E(G)$ be a minimal edge cut of $G$. Let $G_i$, $i = 1, 2$, be obtained from the two components $H_i$ of $G - F$ by adding to each of them a new vertex $v_i$ and connecting it to the endvertices of $F$ in $H_i$. If $|F| \leq 3$, then

$$cr_\Sigma(G_1) + cr_\Sigma(G_2) \leq cr_\Sigma(G) \leq \min_{\Sigma=\Sigma_1 \neq \Sigma_2} (cr_{\Sigma_1}(G_1) + cr_{\Sigma_2}(G_2)),$$

$$\text{mcr}_\Sigma(G_1) + \text{mcr}_\Sigma(G_2) \leq \text{mcr}_\Sigma(G) \leq \min_{\Sigma=\Sigma_1 \neq \Sigma_2} (\text{mcr}_{\Sigma_1}(G_1) + \text{mcr}_{\Sigma_2}(G_2)).$$

**Proof.** Combine the lower bounds of Theorems 1 and 3 with the upper bound in Lemma 7.

**Corollary 9** Let $G$ be a graph, and let $F \subseteq E(G)$ be a minimal edge cut of $G$. Let $G_i$, $i = 1, 2$, be obtained from the two components $H_i$ of $G - F$ by
adding to each of them a new vertex $v_i$ and connecting it to the endvertices of $F$ in $H_i$. If $|F| \leq 3$, then
\[
\text{cr}(G) = \text{cr}(G_1) + \text{cr}(G_2), \\
\text{mcr}(G) = \text{mcr}(G_1) + \text{mcr}(G_2).
\]

**Proof.** Combine Corollary 8 with the observation, that whenever $\Sigma$ is the sphere and $\Sigma = \Sigma_1 \# \Sigma_2$, then both $\Sigma_1$ and $\Sigma_2$ are spheres.

The reader will easily see that Corollary 9 implies the desired crossing number of $G$ in Corollaries 10 and 11. Arguments from [4], which we do not repeat here, establish the criticality of $G$: the crucial fact is that zipping of a critical graph makes the edges involved in the zip product crossing critical.

**Corollary 10** For $i = 1, 2$, let $G_i$ be a $k_i$-crossing critical graph and $v_i \in V(G_i)$ such that $d_{G_1}(v_1) = d_{G_2}(v_2) \leq 3$. If $G$ is any zip product of $G_1$ and $G_2$ at $v_1$ and $v_2$, then $G$ is a $k_1 + k_2$-critical graph.

**Corollary 11** Let $G$ be any graph and let $S \subset V(G)$ be a vertex cover of $G$ containing only vertices of degree 2 and 3. For each $v \in S$, let $G_v$ be a $k_v$-critical graph with a vertex $u_v \in V(G_v)$ of degree $d_G(v)$. Let $G^S$ be the graph obtained from $G$ by iteratively zipping the graphs $G_v$ with $G$ at vertices $v$ and $u_v$. Then, $G^S$ is a $k$-critical graph for $k = \text{cr}(G) + \sum_{v \in S} k_v$.

**Corollary 12** Let $G$ be any graph with $m$ edges and crossing number $r$. For any $k \geq m + r + 1$, there exists an infinite family of $k$-crossing-critical graphs that all contain $G$ as a subdivision.

**Proof.** Let $G'$ be obtained from $G$ by subdividing every edge. The new vertices all have degree two and form a vertex cover $S$ of $G'$. For a selected vertex $w \in S$, let $G_w$ be any graph from the infinite family of 2-crossing-critical graphs, constructed by Kochol in [7], with one of its edges subdivided by a vertex $u_w$. For any $v \in S \setminus \{w\}$, let $G_v$ be a $K_{3,3}$ with one edge subdivided by a vertex $u_v$. By Corollary 11, $G^S$ is a $r + m + 1$-crossing-critical graph. For $l = k - r - m - 1 > 0$, zipping $l$ copies of $K_{3,3}$ to $G^S$ establishes the claim.

**Corollary 13** Let $G$ be a 3-edge-connected crossing critical graph and let $F \subseteq E(G)$ be a minimal edge cut of $G$ of size 3. Let $H_1$ and $H_2$ be the two components of $G - F$ and, for $i = 1, 2$, let $G_i := G/H_{3-i}$. Then there exists a crossing critical graph $J_i$ with $H_i \subseteq J_i \subseteq G_i$. 

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Proof. It is easy to see that it is sufficient to show that, for \(i = 1, 2\), the following is true: if \(e \in E(H_i)\), then \(\text{cr}(G_i - e) < \text{cr}(G_i)\).

For a fixed \(i \in \{1, 2\}\), let \(e\) any fixed edge of \(H_i\). Since \(G\) is 3-edge-connected, \(H_i - e\) is connected. We can apply Theorem [1] to each of the graphs \(G\) and \(G - e\) and obtain that \(\text{cr}(G) = \text{cr}(G_i) + \text{cr}(G_{3-i})\) and \(\text{cr}(G - e) = \text{cr}(G_i - e) + \text{cr}(G_{3-i})\). Now, note that these equations and the fact \(\text{cr}(G - e) < \text{cr}(G)\), imply \(\text{cr}(G_i - e) < \text{cr}(G_i)\), as desired.

The following result was established in [8]:

**Theorem 14 ([8])** Let \(G\) be a connected crossing-critical graph with minimum degree at least 3. Then there is a collection \(J_1, J_2, \ldots, J_\ell\) of 3-edge-connected crossing critical graphs, each of which is contained as a subdivision in \(G\), such that \(\text{cr}(G) = \sum_{i=1}^{\ell} \text{cr}(J_i)\).

An appropriately repeated application of Corollaries 14 and 13 yields the following.

**Corollary 15** Let \(G\) be a connected crossing-critical graph with minimum degree at least 3. Then there is a collection \(I_1, I_2, \ldots, I_{\ell'}\) of internally 4-edge-connected crossing critical graphs, each of which is contained as a subdivision in \(G\), such that \(\text{cr}(G) = \sum_{i=1}^{\ell'} \text{cr}(I_i)\).

We conclude with a lower bound for the minor crossing number of the Cartesian product of an arbitrary graph with an arbitrary tree. Arguments of [3] establish that \(G \Box T\) can be obtained as a zip product of graphs \(\{G^{(d_G(v))} \mid v \in V(T)\}\), where \(G^{(i)}\) denotes the join of \(G\) with an independent set of \(i\) vertices. Then, Theorem [3] establishes the following lower bound:

**Corollary 16** Let \(T\) be any tree and \(G\) any graph. Then,

\[
\text{mcr}_\Sigma(T \Box G) \geq \sum_{v \in V(T)} \text{mcr}_\Sigma(G^{(d_T(v))}).
\]

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