Continuous Time-Delay Estimation From Sampled Measurements

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Abstract: An algorithm for continuous time-delay estimation from sampled output data and a known input of finite energy is presented. The continuous time-delay modeling allows for the estimation of subsample delays. The proposed estimation algorithm consists of two steps. First, the continuous Laguerre spectrum of the output (delayed) signal is estimated from discrete-time (sampled) noisy measurements. Second, an estimate of the delay value is obtained via a Laguerre domain model using a continuous-time description of the input. The second step of the algorithm is shown to be intrinsically biased, the bias sources are established, and the bias itself is modeled. The proposed delay estimation approach is compared in a Monte-Carlo simulation with state-of-the-art methods implemented in time, frequency, and Laguerre domain demonstrating comparable or higher accuracy in the considered scenario.

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1. INTRODUCTION

The problem of (pure) time-delay estimation has been repeatedly addressed over the years but still cannot be seen as solved completely. It has numerous application areas that require particular properties of the estimator regarding its accuracy, computational complexity, robustness to a certain class of disturbances and channel attenuation. There are also significant differences between time-delay estimation methods intended for controller design (Fridman, 2004) and signal processing applications. In modeling for control, an overestimate of the delay is preferred to an underestimate due to concerns regarding closed-loop stability. Besides, by enforcing controller robustness, the sensitivity of the closed-loop system to delay uncertainty can be effectively minimized. In signal processing, time-delay estimation (time-of-arrival) constitutes the basis of imaging and distance measurement technology. Radars, sonars, and lidars (Minkoff, 1992), as well as ultrasound applications (Svilainis, 2019), very much rely on the accuracy of the time-delay estimator, and the sampling rate of the involved signals is a clear bottleneck in enhancing performance. Measuring the time-delay in sampling time is therefore often insufficient and estimating subsample delays becomes a necessity.

Several approaches to time-delay estimation have been covered in the literature. Perhaps the most common ones are the generalized cross-correlation approach (see e.g., Carter (1987); Jacovitti and Scarano (1993)) and the two-step parameter estimation approach (see e.g., Chan et al. (1981); Reed et al. (1981)). The generalized cross-correlation method is a conventional method where the estimator is given by the maximizer of the cross-correlation function between the reference (input) signal and its delayed version (output). To obtain subsample delay estimates, interpolation in time or frequency domain is used (Cespedes et al., 1995; Svilainis, 2019). The two-step parameter estimation method (a.k.a. maximum-likelihood approach to interpolation) is based on estimating a non-causal FIR model of the delay operator in time domain, and then using sinc interpolation to interpolate the tap coefficients and derive a subsample estimate; see e.g., Wen et al. (2015).

Orthogonal functional bases, e.g. Laguerre and Kautz, are extensively used in systems theory for parsimonious representation of dynamics and signals (Heuberger et al., 2005). The goal of their use is generally to approximate infinite-dimensional dynamics; see e.g. Mäkilä and Partington (1999). The concept of (continuous) time-delay estimation based on Laguerre spectra of the input and output has been introduced in Fischer and Medvedev (1999). The estimation algorithm presented there is essentially least squares exploiting the shift structure of the regressor in Laguerre domain. A time-delay estimation approach combining Padé approximation of the delay operator with a Laguerre basis representation of the model is proposed in Isaksson et al. (2001). In a benchmark comparison of delay estimation methods reported in Björklund and Ljung (2003), the algorithms based on Laguerre functions were commended for their robustness.

This paper continues along the path initiated in Abdalmoaty and Medvedev (2022) where the problem of estimating a discrete delay, in a stochastic framework, based on Laguerre spectra of the input and output was addressed. The main contributions of the present paper are as follows: i) An algorithm for estimating the continuous Laguerre spectrum from sampled measurements of the signal is proposed ii) A time-delay estimator making use of the estimated Laguerre spectrum of the output is devised. It is...
shown to be intrinsically biased and the sources of this bias are identified and analysed iii) A Monte-Carlo numerical performance analysis of the proposed algorithm suggests comparable or higher accuracy compared to alternative state-of-the-art approaches. Despite a demonstrated potential, the proposed method is not meant to replace existing ones, but rather to offer a suitable alternative in some applications.

The rest of the paper is organized as follows. First, background on continuous Laguerre functions is provided. Then, the delay estimation problem is formulated in a stochastic framework. Further, the proposed method is explained in detail and related to some available approaches in the literature. An extensive simulation study illustrates the accuracy of the delay estimate and the impact of the user-selected parameters in the algorithm. Finally, conclusions are drawn.

2. BACKGROUND

2.1 Laguerre domain

The Laplace transform of the $k$-th continuous Laguerre function is given by

$$\ell_k(s) = \frac{\sqrt{2p}}{s} T^k(s), \quad T(s) = \frac{s - p}{s + p},$$

for $s \in \mathbb{C}$, $k \in \mathbb{N}_0$, where $p > 0$ is the Laguerre parameter, and $T(s)$ is the continuous Laguerre shift operator. Let $\mathbb{H}^2_c$ be the Hardy space of functions analytic in the open left half-plane. The set $\{\ell_k\}_{k \in \mathbb{N}_0}$ is an orthonormal complete basis in $\mathbb{H}^2_c$ with respect to the inner product

$$\langle W, V \rangle = \frac{1}{2\pi i} \int_{-\infty}^{\infty} W(s)V(-s) \, ds.$$  

Any function $W \in \mathbb{H}^2_c$ can be represented as a Laguerre series

$$W(s) = \sum_{k=0}^{\infty} w_k \ell_k(s), \quad w_j = \langle W, \ell_j \rangle,$$

and $\{w_j\}_{j \in \mathbb{N}_0}$ is then referred to as the continuous Laguerre spectrum of $W$, or simply the Laguerre spectrum of $W$.

2.2 Time-delay in Laguerre domain

The well-known associated Laguerre polynomials (see e.g. Szegő (1939)) are explicitly given by

$$L_m(\xi; \alpha) = \sum_{n=0}^{m} \frac{1}{n!} \left(\frac{\alpha + \xi}{m - n}\right)^n (-\xi)^n, \quad \forall m \in \mathbb{N}, \quad \xi \in \mathbb{R}.$$  

In what follows, only the polynomials with a particular value of $\alpha$ are utilized and the shorthand notation $L_m(\xi) \equiv L_m(\xi; \alpha)|_{\alpha=-1}$ is introduced. The associated Laguerre polynomials obey the following three-term relationship (Rainville, 1960)

$$L_{m+1}(\xi) = \frac{1}{m+1}(\xi + 2m)L_m(\xi) - \frac{m-1}{m+1}L_{m-1}(\xi).$$

Consider a signal $u(t) \in L_2[0, \infty)$ given by its Laguerre spectrum $\{u_j\}_{j \in \mathbb{N}_0}$, passed through a pure delay block $y(t) = u(t-\tau), \ \tau \geq 0, \ t \in [0, \infty)$, resulting in the output $y(t) \in L_2[0, \infty)$ with the spectrum $\{y_j\}_{j \in \mathbb{N}_0}$. Then, according to Hidayat and Medvedev (2012), the following relation holds between the spectra

$$y_j = \sum_{k=0}^{j-1} h_{j-k}(\kappa) u_k + h_0(\kappa) u_j, \ \forall j \in \mathbb{N}_0,$$

where $h_k(\kappa) = e^{-\kappa \frac{\pi}{4}} L_k(\kappa)$ and $\kappa = 2\rho \tau$. Notice that $L_0(\kappa) = 1$ and, therefore, $h_0(\kappa) = e^{-\kappa \frac{\pi}{4}}$. The polynomials $h_i, i = 0, 1, \ldots$, by an analogy with linear time-invariant systems, can be interpreted as the Laguerre-domain Markov parameters of delay operator (6). Since the operator is infinite dimensional, a Hankel matrix of the Markov parameters always has full rank.

3. DELAY ESTIMATION PROBLEM

Consider a case where the output signal measurements of (6) are sampled with the sampling time $\Delta > 0$ and corrupted by noise. The observations at the discrete-time instances $t_n = n\Delta, n = 0, 1, \ldots, N-1$ are thus given by

$$z_n = y(t_n) + \epsilon_n = u(t_n - \tau) + \epsilon_n,$$

where $\epsilon_n$ is a discrete-time measurement noise with zero mean and known variance $\lambda$. Further assumptions on the noise will be introduced later.

Given the data triplet $D_N = \{(z_n)_{n=0}^{N-1}, \{u_k\}_{j=0}^{J}, p\}$, where $\{u_j\}_{j=0}^{J}$ for some finite positive integer $I$, is the finite Laguerre spectrum of the input with respect to a fixed Laguerre parameter value $p$ in (1), the problem considered in this paper is to devise an estimator of the delay $\tau \in \mathbb{R}_+: D_N \mapsto \hat{\tau}$.

Assume that $u(t)$ is non-zero on the time interval $[0, T_u]$. Let $T_{\text{max}}$ be an upper bound on $\tau$. Then the measurement interval is chosen as $[0, T]$ with $T = T_u + T_{\text{max}}$.

The Laguerre spectrum of $y$ defined by (3) can, in time domain, be evaluated by calculating the projections

$$y_j = \int_0^{\infty} y(t) \ell_j(t) \, dt.$$  

Since the continuous function $y(t)$ is not available, it has to be reconstructed using $\{z_n\}_{n=0}^{N-1}$. A straightforward way of doing this is to apply (e.g., polynomial) interpolation and then numerically evaluate integral (9). This approach disregards the presence of noise. Another approach is to assume a continuous signal model of the output as a linear combination of the Laguerre functions (can be written as a linear time-invariant (LTI) system), and estimate the Laguerre coefficients by implementing a sampled-data algorithm. Then a stochastic noise model can be exploited to minimize the estimate variance.

4. MAXIMUM-LIKELIHOOD, FREQUENCY INTERPOLATION, AND THE CRLB

4.1 Maximum-Likelihood Estimation

**Laguerre domain:** Suppose that the measurement noise is Gaussian with a constant variance $\lambda$. Then $z_n$ is a realization of a Gaussian process with a mean function

$$\mu_n(\tau) = u(t_n - \tau) = y(t_n) = \sum_{j=0}^{\infty} y_j \ell_j(t_n) = \sum_{j=0}^{\infty} \ell_j(t_n) \sum_{k=0}^{j-1} (h_{j-k}(\kappa) u_k + h_0(\kappa) u_j).$$
The negative log-likelihood function of \( \tau \) is such that
\[
- \log p(Z; \tau) \propto \Delta \sum_{n=0}^{N-1} (z_n - \mu_n(\tau))^2, \tag{10}
\]
where \( Z \triangleq [z_0 \ldots z_{N-1}]^T \), and hence the Maximum-Likelihood (ML) estimator of \( \tau \) is defined as the minimizer of the mean square error between the observations and the mean function. Recall that \( \kappa = 2p\tau \). Then, from the definition of \( h_k(\cdot) \) and (5), one has
\[
\partial_\tau h_k(\tau) = -pe^{-p\tau}L_k(2p\tau) + e^{-p\tau}\partial_\tau L_k(2p\tau),
\]
\[
\partial_\tau L_k(2p\tau) = \frac{2(2p\tau + k - 1)}{k} \partial_\tau L_{k-1}(2p\tau) + \frac{2p}{k} \partial_\tau L_{k-1}(2p\tau) - \frac{k - 2}{k} \partial_\tau L_{k-2}(2p\tau),
\]
where \( \partial_\tau \) denotes the partial derivative w.r.t. \( \tau \). These expressions can be used to evaluate the gradient of (10).

**Time domain:** Because the input signal is assumed to be constructed in Laguerre domain and possess a finite Laguerre spectrum by design, it holds that \( u(t) = \sum_{k=0}^I u_j \ell_j(t) \) and, consequently, the mean function of the observations and its gradient \( \partial_\tau \mu_n(\tau) \) can be evaluated in closed form. It holds that
\[
\partial_\tau u(t_n - \tau) = \sum_{j=0}^I u_j \partial_\tau \ell_j(t_n - \tau),
\]
where \( \partial_\tau \ell_j(t_n - \tau) = 0 \) \( \forall j \in \mathbb{N} \). Otherwise, when \( t_n - \tau \geq 0 \),
\[
\partial_\tau \ell_0(t_n - \tau) = p\sqrt{2pe^{-p(t_n - \tau)}},
\]
\[
\partial_\tau \ell_1(t_n - \tau) = p\sqrt{2pe^{-p(t_n - \tau)}(2p(t_n - \tau) - 1)} \\
\vdots \\
- 2p\sqrt{2pe^{-p(t_n - \tau)}},
\]
etc. Therefore, the ML estimate of the time-delay \( \tau \) can be computed as a root of the equation
\[
\sum_{n=0}^{N-1} \left[ \begin{array}{c} z_n - \sum_{j=0}^I u_j \ell_j(t_n - \tau) \\ \sum_{j=0}^I u_j \partial_\tau \ell_j(t_n - \tau) \end{array} \right] = 0.
\]
Observe that such an estimator is generally biased, and does not come with any finite-sample guarantees. However, when certain conditions hold, it is expected to possess optimal asymptotic statistical properties: consistency and asymptotic normality, with asymptotic covariance equal to the asymptotic Cramér-Rao lower bound.

### 4.2 The Cramér-Rao lower bound
Using the log-likelihood function, the CRLB of unbiased estimators of \( \tau \) given \( D_N \) is evaluated to
\[
\lambda \sum_{n=0}^{N-1} (\partial_\tau \mu_n(\tau))^2 = \frac{\sum_{n=0}^{N-1} (\partial_\tau u(n\Delta - \tau))^2}{\lambda}, \tag{11}
\]
where \( [\cdot, \cdot] \) mean rounding to the nearest integer above and below, respectively.

### 4.3 Estimation via frequency interpolation
Time-delay estimation from discrete (sampled) data is usually performed by evaluating an integer multiple of the sampling time \( k\Delta \) that maximizes the cross-correlation function between \( z_n \) and \( u(t_n) \):
\[
r(k) \triangleq \sum_{n=0}^{N-1} z_{n-k}u(t_n),
\]
\[
k^* \triangleq \arg\max_{k \in \{0, 1, \ldots, N-1\}} r(k).
\]
To estimate a subsample delay, the cross-correlation function \( r \) is interpolated in time or frequency domain (Cespedes et al., 1995). Assuming that the initial delay \( k^*\Delta \) is removed from \( r \) by phase shifting, and letting \( R_m \) be the discrete Fourier transform obtained after this removal, the subsample delay is evaluated as \( k^*\Delta + \delta\tau \), where \( \delta\tau \) satisfies \( \arg(R_m) = \omega_m\delta\tau \) and can be estimated by weighted averaging over multiple frequencies \( \omega_m \). This can be done, e.g., by minimizing a power-spectrum weighted 2-norm \( \sum_{k=1}^N \delta \tau - (\arg(R_k)/\omega_m))^2 \) as suggested in (Svilainis, 2019, Sec. II, equation 12).

### 5. PROPOSED APPROACH

#### 5.1 Estimating the Laguerre spectrum of delay output
Because \( y(t) \in \mathbb{L}_2[0, \infty) \), it holds that \( y(t) = \sum_{j=0}^{\infty} y_j \ell_j(t) \) where \( \ell_j(t) = L^{-1}[\ell_j(s)] \) is the inverse Laplace transform (impulse response) of \( \ell_j(s) \). The approach developed here is based on an alternative representation of \( y(t) \) that does not depend explicitly on \( \tau \). For a given fixed positive integer \( K \), we can decompose the output signal as follows
\[
y(t) = \sum_{k=0}^{K} y_k \ell_k(t) + \tilde{y}(t). \tag{12}
\]
Substituting for \( y(t_n) \) in (8) yields
\[
z_n = \sum_{k=0}^{K} y_k \ell_k(t_n) + \tilde{y}(t_n) + e_n, \tag{13}
\]
for \( n = 0, \ldots, N-1 \). The observation process can also be expressed using model (7) that explicitly depends on \( \tau \)
\[
z_n = \sum_{k=0}^{K} \sum_{m=0}^{K} (h_{j-m}(\kappa) u_m) \ell_k(t_n) + \tilde{y}(t_n) + e_n. \tag{14}
\]
These two models constitute the starting point of the approach. Model (13) can be written in vector form as
\[
Z = \Phi Y + \Phi \beta + E, \nonumber
\]
\[
Z = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_K \end{bmatrix}, \quad E = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{bmatrix}, \nonumber
\]
\[
\Phi = \begin{bmatrix} \ell_0(t_0) & \ell_1(t_0) & \cdots & \ell_K(t_0) \\ \ell_0(t_1) & \ell_1(t_1) & \cdots & \ell_K(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_0(t_{N-1}) & \ell_1(t_{N-1}) & \cdots & \ell_K(t_{N-1}) \end{bmatrix}, \nonumber
\]
and, similarly, for \( \Phi \) that is a matrix with infinite number of columns, and \( \beta \) that is an infinite-dimensional vector whose entries represent the spectrum tail. In order to construct an estimator of \( Y \), we use the following assumption.

**Assumption:** \( K \) is sufficiently large, and \( \Delta \) is sufficiently small so that \( \tilde{y}(t_n) \approx 0 \), \( \forall n \).
An estimate of the first part of the output spectrum \( Y \) can then be obtained via linear least-squares as
\[
\hat{Y} = \arg \min_Y \| Z - \Phi Y \|_2^2.
\]
Assuming that \( \Phi \) is well-conditioned (can be ensured prior to signal measurement by selecting \( p \)) the estimator is given as \( \hat{Y} = (\Phi^T\Phi)^{-1}\Phi^T Z \). Due to the sampling process and the truncation in the Laguerre sum, this estimator will have a small, but non-zero bias.

**Proposition 1.** Assume that \( u(t) \in L_2[0,\infty), \mathbb{E}[\epsilon_k] = 0 \) \( \forall k \), and the sampling time \( \Delta > 0 \), \( p \), as well as the measurement time interval \( (N-1)\Delta \) are such that \( \Phi \) is full rank. Then, if \( \beta \neq 0 \), the estimator \( \hat{Y} \) of \( Y \) is biased. As \( \Delta \to 0 \), the bias becomes negligible.

**Proof.** Notice that \( y(t) \in L_2[0,\infty) \), and
\[
\mathbb{E}[Y] = Y + (\Phi^T\Phi)^{-1}\Phi^T \beta + \mathbb{E}[E].
\]
Now observe that \( \Phi^T\Phi \neq 0 \), but \( \mathbb{E}[E] = 0 \). As \( \Delta \to 0 \), and taking into account the orthonormality of the Laguerre functions, the bias becomes "small".

The sampled Laguerre functions used to construct \( \Phi \) can be easily computed using an impulse invariant transform of a continuous-time LTI state-space model of order \( K+1 \), whose state \( x_j(t) = \ell_j(t) \). Using the definition of the Laguerre function in (1), this LTI model is found to have the state-space matrices
\[
A_c = -2p \begin{bmatrix} 0.5 & 0 & 0 & \ldots & 0 \\ 1 & 0.5 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0.5 \end{bmatrix}, B_c = \sqrt{2p} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix},
\]
and \( C_c \) given by an identity matrix. A discrete-time state-space model whose impulse response matches the sought values at the sampling times is defined by the matrices
\[
A_d = e^{h\Delta}, \quad B_d = e^{h\Delta}B, \quad C_d = C_c, \quad \text{and} \quad D_d = B_c.
\]
The \((k+1)\)-th row of \( \Phi^T \) is then given as
\[
[\Phi^T]_{(k+1)} = \begin{cases} D_d, & k = 0 \\ C_d A_d^{k-1} D_d, & 0 < k \leq K \end{cases}
\]

### 5.2 Delay estimation in Laguerre domain

Given the relation between the time-delay Markov parameters in (7) and the associated Laguerre polynomials (4), it is possible to obtain a closed-form expression for \( \tau \) in terms of any three consecutive Markov parameters. The expressions for the continuous-time case are similar in structure to those derived for the discrete-time case in Abdalmoaty and Medvedev (2022).

Solving (5), when \( \xi = \kappa \), for \( \kappa \) and substituting the Markov parameters \( h_m(\kappa) = e^{-\kappa L_m(\kappa)} \), leads to the expression
\[
\tau = \frac{(m+1)h_{m+1} + (m-1)h_{m-1} - 2mh_m}{h_m},
\]
for any \( m \geq 1 \).

Suppose that the first \( M \) Markov parameters are available, namely \( h_0, \ldots, h_{M-1} \). Define the following two vectors
\[
A \triangleq \begin{bmatrix} h_0 & \vdots & 0 \\ h_{M-2} & \vdots & 0 \\ h_{M-1} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} h_0 \\ \vdots \\ h_{M-2} \end{bmatrix},
\]
and the tridiagonal matrix \( \Omega \in \mathbb{R}^{M-1 \times M-1} \)
\[
\Omega \triangleq \begin{bmatrix} 0 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 2 & -2 & \ldots & 0 & 0 \\ 0 & -1 & 4 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -(M-3) & 2(M-1) \end{bmatrix}.
\]
Then, the time-delay satisfies the exact linear vector equation \( A = 2pB \tau \) and can be computed as
\[
\tau = \frac{1}{2p} B^\top A.
\]
To estimate \( \tau \) using this formula, we need to estimate the vectors \( A \) and \( B \). Since the Laguerre spectrum of \( u(t) \) is known and an estimate of the Laguerre spectrum of the output signal is available, we can devise a linear least-squares estimate of the Markov parameters as follows.

From model (7), we have \( Y = T(U)H \),
\[
Y = \begin{bmatrix} y_0 \\ \vdots \\ y_k \end{bmatrix}, \quad T(U) = \begin{bmatrix} u_0 & 0 & \ldots & 0 \\ u_1 & u_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_k & u_{K-1} & \ldots & u_0 \end{bmatrix}, \quad H = \begin{bmatrix} h_0 \\ \vdots \\ h_K \end{bmatrix}
\]
and \( T(U) \) is non-singular whenever \( u_0 \neq 0 \). While \( U \) is known, the spectrum \( Y \) is unknown but is estimated; we replace the left hand side by its estimate plus the bias and the random error
\[
\hat{Y} = Y + (Y - E[\hat{Y}]) - \frac{\xi}{\text{bias}} = T(U)H.
\]
Neglecting the bias term and rewriting the expression gives \( \hat{Y} = T(U)H + \xi \). Then the least-squares estimate of \( H \) is
\[
\hat{H} = T^{-1}(U)\hat{Y}.
\]
This estimate is unbiased only if \( \hat{Y} \) is, and it can be used to estimate \( A \) and \( B \), and construct an estimator of \( \tau \).

### 5.3 Accuracy analysis of the estimators

To summarize the exposition in the previous section; the models in time and Laguerre domain lead to the following relations:
\[
Z = \Phi Y + \hat{\Phi} \beta + E, \quad Y = T(U)H,
\]
\[
\tau = \frac{1}{2p} B^\top A.
\]

where the known values are \( \Phi, T(U), p \), whereas \( A \) and \( B \) are functions of the unknown \( H \). These relations are then "inverted" to obtain \( \hat{\tau} \) as follows
\[
\hat{H} = T^{-1}(U)(\Phi^\top \hat{\Phi}^{-1} \hat{\Phi}^\top Z), \quad \hat{\tau} = \frac{1}{2p} B(H)^\top A(\hat{H}).
\]
In particular,
\[
\hat{H} = H + T^{-1}(U)(\Phi^\top \hat{\Phi}^{-1} \hat{\Phi}^\top \beta + T^{-1}(U)(\Phi^\top \Phi)^{-1} \Phi^\top E),
\]

The second term of the last equation represents the bias (deterministic error) which depends on the true unknown delay via \( \beta \), and the third term gives the random error. The covariance of \( \hat{H} \) is given by
\[
\text{cov}(\hat{H}) = \lambda T^{-1}(U)(\Phi^\top \Phi)^{-1} T^{-1}(U),
\]
and can be computed as
\[
\text{MSE}(\hat{H}) = \| T^{-1}(U)(\Phi^\top \Phi)^{-1} \hat{\Phi}^\top \beta \|^2 + \lambda \text{tr} (T^{-1}(U)(\Phi^\top \Phi)^{-1} T^{-1}(U)).
\]
The estimator of $\tau$ in (17) is thus biased even when $\hat{H}$ is unbiased. We clarify this by showing that we have an errors-in-variables case (see Söderström (2018)), and find an expression for the bias. Let $1_{M-1} \triangleq [0 \ 0 \ \ldots \ 1]^T \in \mathbb{R}^{M-1}$

$$
\hat{B} = B(\hat{H}) = [\hat{H}]_{1:M-1} = \begin{bmatrix}
\hat{h}_0 \\
\vdots \\
\hat{h}_{M-2}
\end{bmatrix} = B + \begin{bmatrix}
\hat{h}_0 \\
\vdots \\
\hat{h}_{M-2}
\end{bmatrix},
$$

$$
\hat{A} = A(\hat{H}) = \Omega \hat{B} - (M-1)\hat{h}_{M-2} 1_{M-1} = A + \Omega E_{B} - (M-1)\hat{h}_{M-2} 1_{M-1},
$$

and notice that we have the following errors-in-variables model

$$
A = 2pB \tau, \quad \hat{A} = A + E_A, \quad \hat{B} = B + E_B,
$$

where the “measurements” are $\hat{A}$ and $\hat{B}$, and the noises $E_A$ and $E_B$ are correlated.

**Theorem 2.** The bias of $\hat{\tau}$ in (17) is given by

$$
\frac{1}{2p} \mathbb{E} \left[ \begin{bmatrix} \varepsilon_1 \\ B^\top B + \varepsilon_2 \end{bmatrix} - \tau \mathbb{E} \left[ \begin{bmatrix} \varepsilon_2 \\ B^\top B + \varepsilon_2 \end{bmatrix} \right] \right],
$$

where

$$
\varepsilon_1 = E_{B^\top A} + E_{A^\top B} + E_{B^\top E_B},
$$

$$
\varepsilon_2 = 2E_{B^\top B} + E_{B^\top E_B}.
$$

**Proof.** The proof is by straightforward algebraic manipulations and omitted due to space limitation.

As expected, the bias depends on the unknown true delay value in a complex way via the true Markov parameters. Notice that $B$ depends on both $\tau$ and $p$, and thus the dependence of the bias on $p$ comes in a non-trivial form as well. Yet, the bias expression is meaningful and could potentially be used for bias correction.

### 5.4 Experiment design

In order to pose a well-defined problem, both the spectrum of the input and the Laguerre parameter $p$ need to be tuned a priori to the chosen experimental settings. This can be done once the sampling time $\Delta$ and the number of samples $N - 1$ are decided. Then, we assume that the number of Laguerre functions used for signal approximation is fixed to some finite positive integer, and that a rough estimate $\hat{\tau}$ of the time-delay is available.

Since the final objective is to estimate $\tau$, an ideal experiment design would minimize the MSE of $\hat{\tau}$. However, computing the MSE of $\hat{\tau}$ in (17) is not an easy task. Here, we will be content with maximizing the accuracy of the Markov parameters estimator $\hat{H}$ (minimize the MSE of $\hat{H}$), and save the study of more advanced designs for a future contribution. The signal is then obtained by solving the following experiment design problem

$$
(p^*, \{u_k\}_{k=0}^I) = \arg \min_{p, u_0, \ldots, u_I} \text{MSE} \left( \hat{H}(p, \{u_k\}_{k=0}^I) \mid \hat{\tau} \right),
$$

subject to $p > 0$, $u_0 > 0$, $\sum_{k=0}^I u_k \leq \eta$,

$$
u_k \geq 0 \quad \text{for odd } k,
$$

$$
u_k = -\nu_{k-1} \quad \text{for even } k,
$$

for an odd value $I$, and a given upper bound $\eta$ on the input energy and initial estimate/guess $\hat{\tau}$. Notice that $\sum_{k=0}^I u_k^2$ is the squared 2-norm of $u(t)$ by Parseval’s theorem. We would like to stress here that the above experiment design problem is to be solved offline and iterative numerical solvers can be employed. Alternatively, when $I$ is a small number (say $3$ – then total number of unknowns $= 3$), the problem can be handled efficiently by gridding.

The MSE in the objective functions is that of $\hat{H}$ defined in (18). The variance term in the MSE is easily evaluated; the bias term, however, cannot be evaluated using the expression given by (18) (recall that $\beta$ is infinite dimensional). Nevertheless, for a given $\hat{\tau}$, the bias can be evaluated numerically, e.g., by simulating a noise-free output trajectory $y(t)$, finding its projection $\hat{Y}$, computing $T^{-1}(U)\hat{Y}$, and subtracting it from the vector $H$ constructed using (4).

A remark on the constraints on $\{u_k\}$ is in place here: The proposed constraints ensure that $u(0) = 0$, and thus the continuity of the output signal. The reason for this is to avoid the Gibbs phenomenon and reduce the number of parameters used to tune the input.

### 6. NUMERICAL EXPERIMENTS

#### 6.1 Noise-free simulations

To study the bias, we ran noise-free simulations with the following settings:

- time resolution for continuous time simulation $1 \mu s$,
- $\text{SNR} = 46.0205 \text{ dB}$; noise variance $= 0.01$, with constraints on the input spectrum as described above with $I = 3$ and $\eta = 2$,
- $T = 0.5 s$ and $K = 6$,

and solved the optimal design for each of the following sampling times $\Delta \in \{1 \times 10^{-4}, \ 8 \times 10^{-5}, \ 6 \times 10^{-5}\}$ using the initial guess $\hat{\tau} = \Delta$. The considered true time delays are $\tau \in \{3 \times 10^{-5}, \ 2 \times 10^{-5}, \ 1 \times 10^{-5}\}$. All values are in seconds. Thus, we have the case where $\hat{\tau}$ overestimates $\tau$, which lies within the first sampling interval. Then we evaluate the bias in $\hat{\tau}$.

The obtained results are presented in Fig. 1. As expected, the bias in $\hat{\tau}$ decreases with decreasing sampling time, and smaller bias is achieved for the more accurate initial delay estimates, as shown in Fig. 1.
The signal is then obtained by solving
\[ H(\tau) = \text{fixed to some finite positive integer, and that a} \]

\[ t \]

tuned a priori to the chosen experimental settings. This

dependence of the bias on \( p \)

Notice that

As expected, the bias depends on the unknown true delay

The estimator of \( \tau \),

Since the final objective is to estimate

Fig. 2. The signal \( u(t) \) used in the Monte-Carlo study

6.2 Monte-Carlo Simulations

We also ran a Monte-Carlo simulation study where we

compared the performance of the considered algorithm to

i) time-domain maximum-likelihood estimator derived in

Section 4, ii) Laguerre-domain \( \hat{\tau} \) estimator where \( \{y_t\} \)

are estimated by approximating the defining integral (9) using

cubic splines, iii) Frequency-domain interpolation method

as described in Section 4.3 - we used the best reported

method in Svilainis (2019) (weighted \( L_2 \) norm), and the

MATLAB code given there. Our experiment setup is as

follows: Time resolution for continuous-time simulation =

10\( \mu \)s, true time-delay \( \tau = 0.00133, T = 0.5\)s, \( \Delta = 0.0003s \),

so that \( N = 1667 \), and the discrete-time noise variance is

0.01. The number of Laguerre functions used to design

the input is 4 (\( I = 3 \)), and \( \eta = 2 \) (SNR = 200).

The initial guess of the time-delay in the experiment design

\( \hat{\tau} = \Delta \) (which is quite far from the true value).

The used signal \( u(t) \) is shown in Fig. 2 where the Laguerre parameter

\( p \approx 50 \). Finally, the number of Laguerre functions used for

the output signal model is set to 12 (\( K = 11 \)).

The experiment is performed over 10000 independent
data sets; the results in terms of the bias, variance and

(normalized) MSE are summarized in Table 1.

| (discrete-time) | Bias  | Var    | MSE     |
|-----------------|-------|--------|---------|
| CRLB; see (11)  | 1.011 \( \times 10^{-9} \) |         |         |
| ML              | 4.016 \( \times 10^{-7} \) | 1.001 \( \times 10^{-9} \) | 4.090 \( \times 10^{-8} \) |
| Proposed        | 5.507 \( \times 10^{-8} \) | 3.592 \( \times 10^{-9} \) | 1.466 \( \times 10^{-7} \) |
| Interp. Lag.    | 1.306 \( \times 10^{-5} \) | 4.121 \( \times 10^{-9} \) | 1.752 \( \times 10^{-7} \) |
| Interp. Freq.   | 1.985 \( \times 10^{-6} \) | 1.660 \( \times 10^{-8} \) | 6.781 \( \times 10^{-7} \) |

Table 1. Bias, variance and normalized MSE
(by a factor \( \sqrt{N} \)) of \( \hat{\tau} \) for the compared methods.

7. CONCLUSION

An approach to the estimation of the continuous time

delay from sampled output measurements is proposed. It

is based on the Laguerre-domain relationship between

the spectra of the input and output and consists of two con-

secutive steps: First the continuous Laguerre spectrum of

the output is estimated from sampled noisy measurements.

Then the delay value is estimated from the spectra of

the input and the output using the concept of Markov

parameters of the delay operator. Although the developed

approach demands elaborate tuning, it is shown in a nu-

merical simulation example to exhibit high performance in

comparison with state-of-the-art continuous delay estima-

tion algorithms, for the particular kind of signals used.

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