Greedoids on Vertex Sets of Unicycle Graphs

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Abstract

A maximum stable set in a graph $G$ is a stable set of maximum size. $S$ is a local maximum stable set of $G$, and we write $S \in \Psi(G)$, if $S$ is a maximum stable set of the subgraph spanned by $S \cup N(S)$, where $N(S)$ is the neighborhood of $S$. $G$ is a unicycle graph if it owns only one cycle. In [10] we have shown that the family $\Psi(T)$ of a forest $T$ forms a greedoid on its vertex set. Bipartite, triangle-free, and well-covered graphs $G$ whose $\Psi(G)$ form greedoids were analyzed in [11, 12, 16], respectively.

In this paper we characterize the unicycle graphs whose families of local maximum stable sets form greedoids.

Keywords: unicycle graph, tree, bipartite graph, König-Egerváry graph, local maximum stable set, greedoid, uniquely restricted maximum matching.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ induced by $X$, and by $G - W$ we mean the subgraph $G[V - W]$, where $W \subseteq V(G)$. The graph $G$ is unicycle if it owns only one cycle. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V, vw \in E\}$. If $|N(v)| = 1$, then $v$ is a pendant vertex. We denote the neighborhood of the set $A \subseteq V$ by $N_G(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and its closed neighborhood by $N_G[A] = A \cup N_G(A)$, or shortly, $N(A)$ and $N[A]$, if there is no ambiguity.
By $K_n, C_n$ we mean the complete graph on $n \geq 1$ vertices, and the chordless cycle on $n \geq 3$ vertices, respectively.

A stable set in $G$ is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a maximum stable set of $G$, and the stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. By $\Omega(G)$ we denote the family of all maximum stable sets of the graph $G$.

A set $A \subseteq V(G)$ is a local maximum stable set of $G$ if $A$ is a maximum stable set in the subgraph induced by $N[A]$, i.e., $A \in \Omega(G[N[A])]$, [10]. Let $\Psi(G)$ stand for the family of all local maximum stable sets of $G$. For instance, any set $S$ consisting of only pendant vertices belongs to $\Psi(G)$, while the converse is not generally true; e.g., the set $\{e, g\} \in \Psi(G)$ contains no pendant vertex, where $G$ is the graph in Figure 1.

![Figure 1: A graph having various local maximum stable sets.](image)

Figure 1: A graph having various local maximum stable sets.

Clearly, not any stable set of a graph $G$ is included in some maximum stable set of $G$. For example, there is no $S \in \Omega(G)$ such that $\{b, d, h\} \subset S$, where $G$ is the graph presented in Figure 1. In [13], Nemhauser and Trotter Jr. showed that every local maximum stable set of a graph can be enlarged to one of its maximum stable sets.

A matching in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of $M$ share a common vertex. We denote the size of a maximum matching (a matching of maximum cardinality) by $\mu(G)$. Let us recall that $G$ is a Kőnig-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$, [2, 19]. It is known that every bipartite graph is a Kőnig-Egerváry graph [3, 6].

A greedoid, [1, 5], is a pair $(V, F)$, where $F \subseteq 2^V$ is a non-empty set system satisfying the following conditions:

- **accessibility**: for every non-empty $X \in F$, there is an $x \in X$ such that $X - \{x\} \in F$;
- **exchange**: for $X, Y \in F, |X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in F$.

The following theorem shows that it is enough to prove that $\Psi(G)$ satisfies the accessibility property, in order to validate that $\Psi(G)$ is a greedoid.

**Theorem 1.1** [17] If the family $\Psi(G)$ satisfies the accessibility property, then it satisfies the exchange property as well.

Clearly, $\Omega(G) \subseteq \Psi(G)$ holds for any graph $G$. 

If \( S \in \Psi(G) \), \( |S| = k \geq 2 \), then sometimes there exists a chain
\[
\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, ..., x_{k-1}\} \subset \{x_1, ..., x_{k-1}, x_k\} = S
\]
such that \( \{x_1, x_2, ..., x_j\} \in \Psi(G) \), for all \( j \in \{1, ..., k - 1\} \); such a chain is called an accessibility chain for \( S \). It is evident that \( x_1 \) must be a simplicial vertex, i.e., a vertex whose closed neighborhood induces a complete graph in \( G \) (in particular, any pendant vertex is also simplicial).

![Graphs](image)

Figure 2: \( G_1 \) and \( G_2 \) are unicycle graphs, but only \( \Psi(G_2) \) is a greedoid.

For instance, \( S_1 = \{a, b, d\} \) and \( S_2 = \{b, c, d\} \) belong to \( \Psi(G_1) \), where \( G_1 \) is the graph in Figure 2 but only \( S_1 \) has an accessibility chain, namely, \( \{a\} \subset \{a, d\} \subset S_1 \). Nevertheless, having a simplicial vertex is a necessary but not a sufficient condition for a stable set to admit an accessibility chain; e.g., \( S_2 = \{b, c, d\} \in \Psi(G_1) \) has a pendant vertex and no accessibility chain. However, there exist graphs where every maximum stable set has an accessibility chain, e.g., the graph \( G_2 \) from Figure 2.

Evidently, if \( \Psi(G) \) has the accessibility property, then every \( S \in \Psi(G) \), \( |S| = k \geq 2 \), has an accessibility chain.

In this paper we characterize the unicycle graphs whose family of local maximum stable sets are greedoids. Namely, we demonstrate that if \( C_k \) is the unique cycle of \( G \), then the family \( \Psi(G) \) is a greedoid for \( k = 3 \), while for \( k \geq 4 \), \( \Psi(G) \) is a greedoid if and only if either (a) \( k \) is an even number and all maximum matchings of \( G \) are uniquely restricted, or (b) \( k \) is an odd number and the closed neighborhood of every local maximum stable set of \( G \) induces a König-Egerváry graph.

2 Results

Let \( C \) be the unique cycle of a graph \( G \). Clearly, for every \( e \in E(C) \), the resulting graph \( G - e \) is a forest.

**Theorem 2.1** [10] For any forest \( T \), \( \Psi(T) \) is a greedoid on its vertex set.

This assertion fails for general graphs, and even for unicycle graphs is not always true (e.g., see the graph \( G \) in Figure 3 whose family \( \Psi(G) \) is not a greedoid).

**Proposition 2.2** If \( \Psi(G) \) is a greedoid, then \( \Omega(C_k) \cap \Psi(G) = \emptyset \) for its every induced cycle \( C_k \) of size \( k \geq 4 \).
Proof. Suppose that there exists $S \in \Omega(C_k) \cap \Psi(G)$ in $G$ for some $k \geq 4$. Since $\Psi(G)$ is a greedoid, there is a chain of local maximum stable sets
\[
\{x_1\} \subset \{x_1, x_2\} \subset \ldots \subset \{x_1, x_2, \ldots, x_{k-1}\} \subset \{x_1, x_2, \ldots, x_q\} = S,
\]
where $q = |S| \geq 2$. Hence, $x_1$ must be a pendant vertex in $G$, contradicting the fact that $x_1$ belongs to $V(C_k)$.

The graph $G$ from Figure 3 satisfies the condition that $\Omega(C_k) \cap \Psi(G) = \emptyset$ for its every cycle $C_k$ of size $k \geq 4$. Nevertheless, $\Psi(G)$ is not a greedoid, since $S = \{a, d, g\} \in \Psi(G)$, while $S$ admits no accessibility chain. In other words, the converse of Proposition 2.2 is not true.

Figure 3: $S = \{a, d, g\} \in \Psi(G)$, but $S$ admits no accessibility chain in $G$.

In the sequel, we distinguish between the following cases: $C = C_3$ and $C = C_k$, $k \geq 4$.

Theorem 2.3 If $G$ is a graph that has a $C_3$ as its unique cycle, then $\Psi(G)$ is a greedoid.

Proof. Let $V(C_3) = \{x_i : 1 \leq i \leq 3\}$.

If $V(G) = V(C_3)$, then it is easy to see that $\Psi(G)$ is a greedoid.

Let $V(G) \neq V(C_3)$. According to Theorem 1.1, it is sufficient to show that $\Psi(G)$ satisfies the accessibility property. In other words, we have to build an accessibility chain for any $S \in \Psi(G)$.

Let $T_{ij}, i \in \{1, 2, 3\}, j \in \{0, 1, \ldots, n_i\}$ be subtrees of $G$ such that: $T_{i0} = (\{x_i\}, \emptyset)$, while for $j \geq 1$, $T_{ij}$ is joined by an edge to $x_i$, $i \in \{1, 2, 3\}$, respectively (whenever such a subtree exists).

- **Case 1.** $S \cap V(C_3) = \emptyset$.

Then $S \in \Psi(G - \{x_1, x_2, x_3\})$ and $T = G - \{x_1, x_2, x_3\}$ is a forest. Therefore, by Theorem 1.1, there is an accessibility chain for $S$ in $T$. This is an accessibility chain for $S$ in $G$, as well, because the neighborhoods of the sets belonging to the chain are the same in $T$ and $G$.

- **Case 2.** $S \cap V(C_3) \neq \emptyset$, e.g. $S \cap V(C_3) = \{x_1\}$.

Let us denote
\[
G_1 = G[\cup\{V(T_{ij}) : 0 \leq j \leq n_1\}], \quad S_1 = S \cap V(G_1),
\]
\[
G_2 = G[\cup\{V(T_{2j}) : 1 \leq j \leq n_2\}], \quad G_3 = G[\cup\{V(T_{3j}) : 1 \leq j \leq n_3\}],
\]
\[
S_{ij} = S \cap V(T_{ij}), 1 \leq j \leq n_i, \quad S_i = \cup\{S_{ij} : 1 \leq j \leq n_i\}, i = 2, 3.
\]
Therefore, either $x \in S$. Consequently, $2.1)$, there is $v < |W|$. Since $N_{G_1}[S_2] \subseteq N_G[S_2]$ and $x_1 \notin N_G[S_2]$, it follows that $S_1 \cup W \cup S_3$ is a stable set in $N_G[S]$, but larger than $S$, in contradiction to the choice $S \in \Psi(G)$.

Claim 2. $S_1 \in \Psi(G_1)$.

Otherwise, there must exist some stable set $W \subseteq N_G[S_1]$ with $|S_1| < |W|$. Hence, $W$ is also stable in $G$, and since $N_{G_1}[S_1] \subseteq N_G[S_1]$, it follows that $W \cup S_2 \cup S_3$ is a stable set in $N_G[S]$, but larger than $S$, in contradiction to the choice $S \in \Psi(G)$.

Claim 3. There is an accessibility chain of $S$ in $G$.

We distinguish between the following two cases.

Case 3.1. $S_1 - \{x_1\} \in \Psi(G_1)$.

Hence, we infer that $S_1 - \{x_1\} \in \Psi(G_1 - \{x_1\})$, which together with Claim 1 imply

$$(S_1 - \{x_1\}) \cup S_2 \cup S_3 \in \Psi(G - \{x_1, x_2, x_3\}).$$

Therefore, according to Theorem 2.1, there exists an accessibility chain for the local maximum stable set $(S_1 - \{x_1\}) \cup S_2 \cup S_3$ in $T = G - \{x_1, x_2, x_3\}$, because $T$ is a forest. This is an accessibility chain for $(S_1 - \{x_1\}) \cup S_2 \cup S_3$ in $G$, as well, because the neighborhoods of the sets belonging to the chain are the same in $T$ and $G$. Clearly, this gives rise to an accessibility chain for $S_1 \cup S_2 \cup S_3$.

Case 3.2. $S_1 - \{x_1\} \notin \Psi(G_1)$.

Since $S_1 \in \Psi(G_1)$ (by Claim 2) and $\Psi(G_1)$ is a greedoid (by Theorem 2.1), there is $v \in S_1$, such that $S_1 - \{v\} \notin \Psi(G_1)$.

We assert that $(S_1 - \{v\}) \cup S_2 \cup S_3 \in \Psi(G)$.

Otherwise, there is a stable set $A \subseteq N_G[(S_1 - \{v\}) \cup S_2 \cup S_3]$ with

$$|A| > |S_1| - 1 + |S_2| + |S_3|.$$

Therefore, either $x_2 \in A$ or $x_3 \in A$. Without lack of generality suppose that $x_2 \in A$. Hence, there exists a stable set $B \subseteq N_G[S_2]$ in $G_2$ such that $|B| = |S_2|$ and $B \cup \{x_2\}$ is stable in $G$. Since $S_1 - \{x_1\} \notin \Psi(G_1)$, there exists $S_0 \subseteq N[S_1 - \{x_1\}]$, such that $|S_0| > |S_1 - \{x_1\}|$. Clearly, $x_1 \notin S_0 \subseteq N[S_1]$. Consequently, $S_0 \cup B \cup \{x_2\} \cup S_3$ is a stable set in $N_G[S]$ of size greater than $|S|$, and that contradicts the choice of $S \in \Psi(G)$. 

Claim 1. $S_2, S_3$ are local maximum stable sets in $G_2, G_3$, respectively.
Thus, we obtain

\[ S, (S_1 - \{v\}) \cup S_2 \cup S_3 \in \Psi(G), \]

\[ x_1 \in S_1 - \{v\} \in \Psi(G_1), \quad S_2 \in \Psi(G_2), \quad S_3 \in \Psi(G_3), \]

and

\[ S = S_1 \cup S_2 \cup S_3 \supset (S_1 - \{v\}) \cup S_2 \cup S_3. \]

Now, if \((S_1 - \{v\}) - \{x_1\} \in \Psi(G_1)\), we continue as in Case 1, which leads immediately to an accessibility chain.

Otherwise, we find some vertex \(v' \in S_1 - \{v\} \in \Psi(G_1), v' \neq x_1\), such that \((S_1 - \{v\}) - \{v'\}\) belongs to \(\Psi(G_1)\), and we can continue, as in Case 2, to increase the size of the chain of local maximum stable sets we are building one by one.

Since the set \(S\) is finite, at the end of the above procedure we obtain an accessibility chain of \(S\) in \(G\). ■

Let us remark that the graph \(G\) in Figure 3 is a unicycle bipartite graph, and as we mentioned before, its family \(\Psi(G)\) is not a greedoid. However, there exist bipartite graphs whose families \(\Psi(G)\) are greedoids.

Trying to characterize these bipartite graphs, we found out an interesting connection between their local maximum stable sets of a graph and their matchings, but of some special kind \([9, 10, 11, 12]\).

A perfect matching is a matching saturating all the vertices of the graph. A matching \(M\) of a graph \(G\) is called a uniquely restricted matching if \(M\) is the unique perfect matching of the subgraph induced by the vertices it saturates \([4]\).

Recall that a cycle \(C\) is alternating with respect to a matching \(M\) if for any two incident edges of \(C\) exactly one of them belongs to \(M\), \([7]\). It is clear that an \(M\)-alternating cycle should be of even size.

**Theorem 2.4** \([4]\) A matching \(M\) in a graph \(G\) is uniquely restricted if and only if \(G\) does not contain an alternating cycle with respect to \(M\).

Notice that the graph \(G\) in Figure 3 has maximum matchings that are not uniquely restricted. It turns out that the existence of such matchings is the real reason why \(\Psi(G)\) is not a greedoid.

**Theorem 2.5** \([11]\) For a bipartite graph \(G\), the family \(\Psi(G)\) is a greedoid if and only all maximum matchings of \(G\) are uniquely restricted.

According to Theorem 2.4, all maximum matchings of a unicycle non-bipartite graph \(G\) are uniquely restricted.
Nevertheless, it is not sufficient for \( \Psi(G) \) to be a greedoid. For example, the graphs \( G_1, G_2 \) in Figure 4 are unicycle non-bipartite graphs, \( \Psi(G_2) \) is a greedoid, while \( \Psi(G_1) \) is not a greedoid, because \( \{u,v\} \in \Psi(G_1) \), but \( \{u\}, \{v\} \notin \Psi(G_1) \).

\[ u \quad v \]
\[ G_1 \quad G_2 \]

Figure 4: Non-bipartite triangle-free graphs with unique perfect matchings.

A triangle-free graph is a graph having no induced subgraph isomorphic to \( C_3 \). The graphs in Figure 4 are triangle-free König-Egerváry graphs, while \( G_1[\{u,v\}] \) is not a König-Egerváry graph. The existence of such a subgraph is the reason for \( \Psi(G_1) \) not to be a greedoid.

**Theorem 2.6** [12] If \( G \) is a triangle-free graph, then \( \Psi(G) \) is a greedoid if and only if all its maximum matchings are uniquely restricted and the closed neighborhood of each local maximum stable set of \( G \) induces a König-Egerváry graph.

Now, using the fact that, by Theorem 2.4, all the maximum matchings of a graph without even cycles must be uniquely restricted, and combining Theorems 2.3, 2.5, 2.6, we conclude with the following.

**Theorem 2.7** Let \( C_k \) be the unique cycle of the graph \( G \). Then, the following statements are true:

(i) if \( k = 3 \), then \( \Psi(G) \) is a greedoid;

(ii) if \( k = 2q \geq 4 \), then \( \Psi(G) \) is a greedoid if and only if all maximum matchings of \( G \) are uniquely restricted;

(iii) if \( k = 2q + 1 \geq 5 \), then \( \Psi(G) \) is a greedoid if and only if the closed neighborhood of every local maximum stable set of \( G \) induces a König-Egerváry graph.

### 3 Conclusions

In this paper we have completely characterized the unicycle graphs whose families of local maximum stable sets form greedoids on their vertex sets. In [9] we showed that even unicycle graphs whose families of local maximum stable sets are greedoids can be recognized in polynomial time. The key question that remains open is whether there exists a polynomial time recognition algorithm for odd unicycle graphs whose families of local maximum stable sets are greedoids.
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