Affine $ADE$ bundles over complex surfaces with $p_g = 0$

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Abstract

We study simply-laced simple affine Lie algebra bundles over complex surfaces $X$. Given any Kodaira curve $C$ in $X$, we construct such a bundle over $X$. After deformations, it becomes trivial on every irreducible component of $C$ provided that $p_g(X) = 0$.

When $X$ is a blowup of $\mathbb{P}^2$ at nine points, there is a canonical $\hat{E}_8$-bundle $\mathcal{E}_{\hat{E}_8}$ over $X$. We show that the geometry of $X$ can be reflected by the deformability of $\mathcal{E}_{\hat{E}_8}$.

1 Introduction

Given a complex surface $X$ and a sublattice $\Lambda \subset \text{Pic}(X)$, if $\Lambda$ is isomorphic to the root lattice $\Lambda_g$ of a simple Lie algebra $g$, then we have a root system $\Phi$ of $g$ and we can associate a Lie algebra bundle $\mathcal{E}_g^0$ over $X$ \cite{[8][13][14]},

$$\mathcal{E}_g^0 := O_X^{\oplus r} \oplus \bigoplus_{\alpha \in \Phi} O_X(\alpha).$$

This can be generalized to affine Lie algebras $\hat{g}$ \cite{13}.

There are many instances that this happens: when $X_n$ is a del Pezzo surface, namely a blowup of $\mathbb{P}^2$ at $n \leq 8$ points in general position (or $\mathbb{P}^1 \times \mathbb{P}^1$), $\langle K_{X_n} \rangle \subset \text{Pic}(X_n)$ is isomorphic to $\Lambda_{E_n}$. Thus we have an $E_n$-bundle over $X_n$. By restriction, we have an $E_n$-bundle over any anti-canonical curve $\Sigma$ in $X_n$. Notice that $\Sigma$ is always an elliptic curve. For a fixed elliptic curve $\Sigma$, the above construction gives a bijection between del Pezzo surfaces containing $\Sigma$ and $E_n$-bundles over $\Sigma$ \cite{14}[5][9][13][16][24]. Such an identification was predicted by the F-theory/string duality in physics \cite{9}. This was generalized to all simple Lie algebras in \cite{14,15}. When $n = 9$, $X_9$ is not Fano and $E_9 = \hat{E}_8$ is an affine Lie algebra. Corresponding results for the $\hat{E}_8$-bundle over $X_9$ are obtained in \cite{13}.

When $X$ is the canonical resolution of a surface $X'$ with a rational singularity, the exceptional curve $C = \cup C_i$ is an $ADE$ curve of type $g$. Therefore $C_i$’s span a sublattice of $\text{Pic}(X)$ which is isomorphic to $\Lambda_g$, thus giving a $g$-bundle $\mathcal{E}_g^0$ over $X$. When $p_g(X) = 0$, there exists a deformation $\mathcal{E}_g^\varphi$ of $\mathcal{E}_g^0$ such that $\mathcal{E}_g^\varphi$ is trivial on each $C_i$, thus it descends to the singular surface $X'$ \cite{[8][3].
When $X$ is a relatively minimal elliptic surface, Kodaira classified all possible singular fibers (see e.g. [1]) and we call such a curve $C = \cup C_i$ a Kodaira curve. Its irreducible components $C_i$’s span a sublattice of $\text{Pic}(X)$ which is isomorphic to the root lattice of an affine root system $\Phi_{\hat{g}}$ and therefore we can construct an affine Lie algebra bundle $E_{\hat{g}}^8$ over $X$.

**Theorem 1** (Lemma 14, Proposition 19 and Theorem 23) Given any complex surface $X$ with $p_g = 0$. If $X$ has a Kodaira curve $C = \cup_{i=0}^{g} C_i$ of type $\hat{g}$, then

(i) given any $(\varphi_{C_i})'_{i=0} \in \Omega^{0,1}(X, \bigoplus_{i=0}^{g} O(C_i))$ with $\varphi_{C_i} = 0$ for every $i$, it can be extended to $\varphi = (\varphi_{a_i})_{a_i \in \Phi^{+}_{\hat{g}}} \in \Omega^{0,1}(X, \bigoplus_{a_i \in \Phi^{+}_{\hat{g}}} O(a))$ such that $\overline{\varphi}_C := \overline{\varphi} + \text{ad}(\varphi)$ is a holomorphic structure on $E_{\hat{g}}^8$. We denote the new bundle as $E_{\hat{g}}^8$. (ii) $\overline{\varphi}_C$ is compatible with the Lie algebra structure on $E_{\hat{g}}^8$. (iii) $E_{\hat{g}}^8$ is trivial on $C_i$ if and only if $[\varphi_{C_i}] \not\equiv 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$. (iv) There exists $[\varphi_{C_i}] \in H^1(X, O(C_i))$ such that $[\varphi_{C_i}] \not\equiv 0$.

In the second half of this paper, we explain how the geometry of $X_9$, a blowup of $\mathbb{P}^2$ at nine points, can be reflected by the deformability of the $E_{\hat{g}}^8$-bundle $E_{\hat{g}}^8$ over it. Among other things, we obtained the following results.

**Theorem 2** (Theorem 25) $E_{\hat{g}}^8$ is totally non-deformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position.

**Theorem 3** (Theorem 26) Suppose $-K_{X_9}$ is nef, then

(i) $X_9$ admits an elliptic fibration with a multiple fiber of multiplicity $m$ $(m \geq 1)$ if and only if $E_{\hat{g}}^8$ is deformable in $(-mK)$-direction but not in $(-m + 1)K$-direction.

(ii) $X_9$ has a (maximal) ADE curve $C$ of type $\hat{g}$ if and only if $E_{\hat{g}}^8$ is (maximal) $\hat{g}$-deformable.

(iii) $X_9$ has a (maximal) Kodaira curve $C$ of type $\hat{g}$ if and only if $E_{\hat{g}}^8$ is (maximal) $\hat{g}$-deformable.

The organization of this paper is as follows. Section 2 gives the construction of the (affine) ADE Lie algebra bundles directly from (affine) ADE curves. In section 3, we assume $p_g(X) = 0$. We construct deformations of the holomorphic structures on these bundles such that the new bundles are trivial over irreducible components of the curve. We will consider the $E_n$-bundle over a blowup of $\mathbb{P}^2$ at $n \leq 9$ points in section 4 and show how the deformability of this bundle can reflect the geometry of the underlying surface. In the appendix, we review the basic construction of affine Lie algebras.

**Notations.** For a holomorphic bundle $(E_0, \overline{\varphi}_0)$ with $E_0 = \bigoplus_{i} O(D_i)$, $\overline{\varphi}_0$ means the $\overline{\varphi}$-operator for the direct sum holomorphic structure. If we construct a new holomorphic structure $\overline{\varphi}_\varphi$ on $E_0$, we denote the resulting bundle as $E_{\varphi}$.

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2 Affine $ADE$ bundles from affine $ADE$ curves

2.1 $ADE$ and affine $ADE$ curves

Definition 4 A curve $C = \bigcup C_i$ in a surface $X$ is called an $ADE$ (resp. affine $ADE$) curve of type $g$ (resp. $\hat{g}$) if each $C_i$ is a smooth $(-2)$-curve in $X$ and the dual graph of $C$ is a Dynkin diagram of the corresponding type.

It is known that $C$ is an $ADE$ curve if and only if $C$ can be contracted to a rational singularity. In this case, the intersection matrix $(C_i \cdot C_j) < 0$ \cite{11}.

If $C$ is an affine $ADE$ curve, then the intersection matrix $(C_i \cdot C_j) \leq 0$ and there exists unique $n_i$'s up to overall scalings such that $F := \sum n_i C_i$ satisfies $F \cdot F = 0$. Dynkin diagrams of affine $ADE$ types are drawn as follows and the corresponding $n_i C_i$'s are labelled in the pictures. $ADE$ Dynkin diagrams can be obtained by removing the node corresponding to $C_0$.

![Dynkin diagrams of affine $ADE$ types](image)

Figure 1. Dynkin diagrams of affine $ADE$ types
Remark 5 We will also call a nodal or cuspidal rational curve with trivial normal bundle an $A_0$ curve.

Remark 6 By Kodaira’s classification of fibers of relative minimal elliptic surfaces, every singular fiber is an affine $ADE$ curve unless it is rational with a cusp, tacnode or triplepoint (corresponding to type $II$ or $III(A_1)$ or $VI(A_2)$ in Kodaira’s notations), which can also be regarded as a degenerated affine $ADE$ curve of type $\hat{A}_0$, $\hat{A}_1$ or $\hat{A}_2$ respectively. In this paper, we will not distinguish affine $ADE$ curves from their degenerated forms since they have the same intersection matrices. We also call the affine $ADE$ curves as Kodaira curves.

Definition 7 A bundle $E$ is called an $ADE$ (resp. affine $ADE$) bundle of type $g$ (resp. $\hat{g}$) if $E$ has a fiberwise Lie algebra structure of the corresponding type.

In the following two subsections, we will recall an explicit construction of the Lie algebra $g$-bundles, loop Lie algebra $Lg$-bundles and the affine Lie algebra $\hat{g}$-bundles from (affine) $ADE$ curves in $X$.

2.2 $ADE$ bundles

Suppose $C = \bigcup_{i=1}^{r} C_i$ is an $ADE$ curve of type $g$ in $X$, we will construct the corresponding $ADE$ bundle $E_{0}^{g}$ over $X$ as follows [3].

Note the rank $r$ of $g$ equals the number of $C_i$’s, we denote $\Phi := \{\alpha = \langle \sum_{i=1}^{r} a_i C_i \rangle \in H^2(X, \mathbb{Z}) | \alpha^2 = -2\}$, then $\Phi$ is a simply-laced root system of $g$ with a base $\Delta := \{[C_i] | i = 1, 2, \ldots, r\}$. We have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. We define a bundle $E_{0}^{g,(\Phi)}$ over $X$ as follows:

$$E_{0}^{g,(\Phi)} := O^{\oplus r} \mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha).$$

Here $O(\alpha) = O(\sum_{i=1}^{r} a_i C_i)$ where $\alpha = \langle \sum_{i=1}^{r} a_i C_i \rangle$. There is an inner product $\langle , \rangle$ on $\Phi$ defined by $\langle \alpha, \beta \rangle := -\alpha \cdot \beta$, negative of the intersection form.

For every open chart $U$ of $X$, we take $x_{\alpha}^{U}$ to be a nonvanishing section of $O_{U}(\alpha)$ and $h_{\alpha}^{U}$ (1 ≤ $i$ ≤ $r$) nonvanishing sections of $O_{0}^{\oplus r}$. Define a Lie algebra structure $[\ , \ ]_{\Phi}$ on $E_{0}^{g,(\Phi)}$ such that $\{x_{\alpha}^{U}’s, h_{\alpha}^{U}’s\}$ is the Chevalley basis [11], i.e.

(a) $[h_{\alpha}^{U}, h_{\beta}^{U}]_{\Phi} = 0$, 1 ≤ $i$, $j$ ≤ $r$.

(b) $[h_{\alpha}^{U}, x_{\beta}^{U}]_{\Phi} = \langle \alpha, C_i \rangle x_{\alpha}^{U}$, 1 ≤ $i$ ≤ $r$, $\alpha \in \Phi$.

(c) $[x_{\alpha}^{U}, x_{-\alpha}^{U}]_{\Phi} = h_{\alpha}^{U}$ is a $\mathbb{Z}$-linear combination of $h_{\beta}^{U}$.

(d) If $\alpha$, $\beta$ are independent roots, and $\beta - p\alpha$, $\cdots$, $\beta + q\alpha$ is the $\alpha$-string through $\beta$, then $[x_{\alpha}^{U}, x_{\beta}^{U}]_{\Phi} = 0$ if $q = 0$, otherwise $[x_{\alpha}^{U}, x_{\beta}^{U}]_{\Phi} = \pm (p+1)x_{\alpha+\beta}^{U}$.

Since $g$ is a simply-laced Lie algebra, all the roots for $g$ have the same length, we have any $\alpha$-string through $\beta$ is of length at most 2. So (d) can be written as $[x_{\alpha}^{U}, x_{\beta}^{U}]_{\Phi} = n_{\alpha,\beta} x_{\alpha+\beta}^{U}$, where $n_{\alpha,\beta} = \pm 1$ if $\alpha + \beta \in \Phi$, otherwise $n_{\alpha,\beta} = 0$. It is easy to check that these Lie algebra structures are compatible with different trivializations of $E_{0}^{g,(\Phi)}$. Hence $E_{0}^{g,(\Phi)}$ is a Lie algebra bundle of type $g$ over $X$. 

4
2.3 Affine ADE bundles

Suppose \( C = \bigcup_{i=0} C_i \) is an affine ADE curve of type \( \hat{g} \) in \( X \), we will construct the corresponding affine ADE bundle \( E_0^g \) of type \( \hat{g} \) over \( X \) as follows.

First, we choose an extended root of \( \hat{g} \), say \( C_0 \), then \( g \) is corresponding to the Dynkin diagram consists of those \( C_i \) with \( i \neq 0 \), i.e., \( \Phi := \{ \alpha = \sum_{i \neq 0} a_i C_i \in H^2(X, \mathbb{Z}) | \alpha^2 = -2 \} \) is the root system of \( g \). As above, we have a \( g \)-bundle \( E_0^g(a, \Phi) = O(\omega) \oplus O(\alpha) \). We define

\[
E_0^{Lg, \Phi} := \bigoplus_{n \in \mathbb{Z}} (E_0^{g, \Phi} \otimes O(nF)) \text{ and } E_0^{\hat{g}, \Phi} := \bigoplus_{n \in \mathbb{Z}} (E_0^{g, \Phi} \otimes O(nF)) \oplus O.
\]

We know \( \Phi = \{ \alpha + nF | \alpha \in \Phi, n \in \mathbb{Z} \} \cup \{ nF | n \in \mathbb{Z}, n \neq 0 \} \) is an affine root system and it decomposes into union of positive and negative roots, i.e. \( \Phi = \Phi^+ \cup \Phi^- \), where \( \Phi^+ = \{ \sum a_i C_i \in \Phi^\circ | a_i \geq 0 \text{ for all } i \} = \{ \alpha + nF | \alpha \in \Phi^+, n \in \mathbb{Z}_{\geq 0} \} \cup \{ \alpha + nF | \alpha \in \Phi^-, n \in \mathbb{Z}_{\geq 1} \} \cup \{ nF | n \in \mathbb{Z}_{\geq 1} \} \) and \( \Phi^- = -\Phi^+ \).

To describe the Lie algebra structures, we proceed as before, for every open chart \( U \) of \( X \), we take a local basis \( e^U \) of \( E_0^{g, \Phi} \) \( | U \) (\( e^U \) is just \( h^U \) or \( x^U \) as above), \( e^U_{nF} \) of \( O(nF)| U \), \( e^U_{c} \) of \( O| U \), compatible with the tensor product, for example, \( e^U_{nF} \otimes e^U_{mF} = e^U_{(n+m)F} \). Then define

\[
[e^U_i, e^U_j, e^U_k]_{Lg, \Phi} := [e^U_i, e^U_j]_{\Phi} e^U_k e^U_{(n+m)F}, \quad (1)
\]

\[
[e^U_i, e^U_j, e^U_k + \lambda e^U_c, e^U_{nF} + \mu e^U_{c} + \nu e^U_{(n+m)F}]_{h_\Phi} := [e^U_i, e^U_j]_{\Phi} e^U_k e^U_{(n+m)F} + n\delta_{n+m,0} k(e^U_i, e^U_j) e^U_c. \quad (2)
\]

Here \( [\ , \ ] \Phi \) is the Lie bracket on \( E_0^{g, \Phi} \) and \( k(x, y) = Tr(adx \ ady) \) is the Killing form on \( g \).

Lemma 8 (1) (resp. (2)) defines a fiberwise loop (resp. affine) Lie algebra structure which is compatible with any trivialization of \( E_0^{Lg, \Phi} \) (resp. \( E_0^{\hat{g}, \Phi} \)).

Proof. See Proposition 23 of [13]. [ ]

From the above lemma, we have the following result.

Proposition 9 If \( C \) is an affine ADE curve of type \( \hat{g} \) in \( X \), then \( E_0^{Lg, \Phi} \) (resp. \( E_0^{\hat{g}, \Phi} \)) is a loop (resp. affine) Lie algebra bundle of type \( Lg \) (resp. \( \hat{g} \)) over \( X \).

Note any \( C_i \) with \( n_i = 1 \) can be chosen as the extended root (Appendix).

Proposition 10 The loop Lie algebra bundle \( (E_0^{Lg, \Phi}, [\ , \ ]_{Lg, \Phi}) \) does not depend on the choice of the extended root.

Proof. Suppose \( C_k \) (\( k \neq 0 \)) is another root with \( n_k = 1 \), we denote \( \Psi = \{ \beta = | \sum_{i \neq k} h_i C_i | \in H^2(X, \mathbb{Z}) | \beta^2 = -2 \} \), then \( \Psi \) is a root system of \( g \). As before, we construct the Lie algebra bundle \( E_0^{g, \Phi} \) and \( E_0^{(Lg, \Phi)} \) from \( \Psi \).

We denote \( \alpha_0 := \sum_{i \neq 0} n_i C_i = F - C_0 \), the longest root in \( \Phi \). For any \( \alpha = \sum_{i \neq 0} a_i C_i \in \Phi \), \( a_k(\alpha) \) can only be 0, \pm 1. Hence there is a bijection
between $\Phi$ and $\Psi$ given by $\alpha \mapsto \beta = \alpha - a_k(\alpha)F$. Then from the definitions of $c_0^L(\Phi, \Psi)$ and $c_0^L(\Psi, \Phi)$, we know they are the same as holomorphic vector bundles.

We compare the Lie brackets on them. We choose a local basis of $\mathcal{C}_0^L(\Phi, \Psi)$ compatible with those of $c_0^L(\Phi, \Psi)$ and define $[\ , \ ]_{L\Phi, \Psi}$ similarly as $[\ , \ ]_{L\Phi, \Phi}$, i.e.

(i) when $\beta = \alpha \in \Phi \cap \Psi$, we take $x_\beta = x_\alpha$;
(ii) when $\beta = \alpha + F \in \Psi^+ \setminus \Phi$, we take $x_\beta = x_\alpha e_F$;
(iii) when $\beta = \alpha - F \in \Psi^- \setminus \Phi$, we take $x_\beta = x_\alpha e_F$;
(iv) take $h_i (i \neq k)$ as before, take $h_0 = -h_\alpha_b$ as we want $[x_{C_0}, x_{-C_0}]_{L\Phi, \Psi} = [x_{-\alpha_0 + F}, x_{\alpha_0 - F}]_{L\Phi, \Psi}$.

It is obvious $[\ , \ ]_{L\Phi, \Psi} = [\ , \ ]_{L\Phi, \Phi}$ on $\mathcal{C}_0^L(\Phi, \Psi) \cong c_0^L(\Phi, \Phi)$. ■

For the affine case, we recall that the Killing form of $g$ is the symmetric bilinear map $k : g \times g \rightarrow \mathbb{C}$ defined by $k(x, y) = Tr(ad x ad y)$. It is ad-invariant, that is for $x, y, z \in g$, $k([x, y], z) = k(x, [y, z])$.

**Lemma 11** For any simple simply-laced Lie algebra $g$ with a Chavelly basis $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq r\}$ and $m^*(g)$ the dual Coxeter number of $g$, we have

(i) $k(h_i, x_\alpha) = 0$ for any $i$ and $\alpha$;
(ii) $k(x_\alpha, x_\beta) = 0$ for any $\alpha + \beta \neq 0$;
(iii) $k(h_i, h_j) = 2m^*(g) \langle C_i, C_j \rangle$;
(iv) $k(x_\alpha, x_{-\alpha}) = 2m^*(g)$ for any $\alpha$.

**Proof.** Directly from the Killing form $k$ being ad-invariant or see [17]. ■

**Proposition 12** The affine Lie algebra bundle $(\mathcal{C}_0^{\tilde{g}, \Phi}, [\ , \ ]_{\tilde{g}, \Phi})$ does not depend on the choice of the extended root.

**Proof.** Follow the notations in Proposition 10 but we will take $h_0 = -h_{\alpha_0} + 2m^*(g)e_c$. We will check that $[\ , \ ]_{\tilde{g}, \Phi} = [\ , \ ]_{\tilde{g}, \Phi}$ on $\mathcal{C}_0^{\tilde{g}, \Phi} = \mathcal{C}_0^L(\tilde{g}, \Phi)$:

(a) when $\beta_1 = \alpha_1 + F, \beta_2 = \alpha_2 - F \in \Psi^+ \setminus \Phi$, $\alpha_1, \alpha_2 \in \Phi \setminus \Psi$ we have

$$[h_{\beta_1}e_{nF}, h_{\beta_2}e_{mF}]_{\tilde{g}, \Phi} = n\delta_{n+m, 0}k(h_{\beta_1}, h_{\beta_2})e_c,$$

which is the same with

$$[h_{-\alpha_1}e_{nF}, h_{-\alpha_2}e_{mF}]_{\tilde{g}, \Phi} = n\delta_{n+m, 0}k(h_{\alpha_1}, h_{\alpha_2})e_c,$$

since $k(h_{\beta_1}, h_{\beta_2}) = 2m^*(g) \langle \beta_1, \beta_2 \rangle = 2m^*(g) \langle F - \alpha_1, F - \alpha_2 \rangle = k(h_{\alpha_1}, h_{\alpha_2})$.

(b) For $[h_{\alpha}e_{nF}, x_\alpha e_{mF}]_{\tilde{g}, \Phi}$, automatically from $k(h_i, x_\alpha) = 0$ and loop case.

(c) When $\beta = \alpha + F \in \Psi^+ \setminus \Phi, \alpha \in \Phi^+ \setminus \Psi$,

$$[x_\beta e_{nF}, x_{-\beta} e_{mF}]_{\tilde{g}, \Phi} = h_\beta e_{(n+m)F} + n\delta_{n+m, 0}k(x_{\beta}, x_{-\beta})e_c,$$

which is the same with

$$[x_{-\alpha} e_{(n+1)F}, x_{\alpha} e_{(m-1)F}]_{\tilde{g}, \Phi} = -h_{\alpha} e_{(n+m)F} + (n + 1)\delta_{n+m, 0}k(x_{\alpha}, x_{-\alpha})e_c,$$

by considering $m + n = 0$ and $m + n \neq 0$ separately.

(d) For $[x_\alpha e_{nF}, x_{\alpha_2} e_{mF}]_{\tilde{g}, \Phi}$, with $\alpha_1 + \alpha_2 \neq 0$, automatically from $k(x_{\alpha_1}, x_{\alpha_2}) = 0$ and loop case. ■

For simplicity, we will omit $\Phi$ in $(g, \tilde{g}), (Lg, \Phi)$ and $(\tilde{g}, \Phi)$ when there is no confusion.
3 Trivialization of $\mathcal{E}_0^g$ over $C_i$'s after deformations

If $C = \cup C_i$ is an affine ADE curve in $X$, then the corresponding $F = \sum n_i C_i$ satisfies $F \cdot F = 0$, i.e. $O_F(F)$ is a topologically trivial bundle. If $O_F(F)$ is trivial holomorphically and $q(X) = 0$, then from the long exact sequence of cohomologies induced by $0 \to O_X \to O_X(F) \to O_F(F) \to 0$, we know $H^0(X, O_X(F)) \cong \mathbb{C}^2$. Hence $F$ is a fiber of an elliptic fibration on $X$.

Suppose $X$ is an elliptic surface, i.e. there is a smooth curve $B$ and a surjective morphism $\pi : X \to B$ whose generic fiber $F_b$ ($b \in B$) is an elliptic curve. Assume $\pi$ is singular at $b_0 \in B$ and $F_{b_0} = \sum n_i C_i$ is a singular fiber of type $\tilde{g}$. Hence, we have a $\tilde{g}$-bundle $\mathcal{E}_0^g$ over $X$. The restriction of $\mathcal{E}_0^g$ to any fiber $F_b$, other than $F_{b_0}$, is trivial because $F_b \cap C_i = \emptyset$ for any $i$. However, $\mathcal{E}_0^g|_{F_{b_0}}$ is not trivial, for instance $O(-C_i)|_{C_i} \cong O_{\mathbb{P}^1}(2)$. Nevertheless, we will show that after deformations of holomorphic structures, $\mathcal{E}_0^g$ will become trivial on every irreducible component of $F_{b_0}$.

3.1 Review of ADE cases

In our earlier paper [3], we showed how to take successive extensions to make the $g$-bundle $\mathcal{E}_0^g$ trivial on every component $C_i$ of the ADE curve $C = \cup_{i=1}^r C_i$.

Definition 13 Given any $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi} O(\alpha))$, we define $\nabla_{\varphi} : \Omega^{0,1}(X, \mathcal{E}_0^g) \to \Omega^{0,1}(X, \mathcal{E}_0^g)$ by

$$\nabla_{\varphi} := \nabla_0 + ad(\varphi) := \nabla_0 + \sum_{\alpha \in \Phi^+} ad(\varphi_\alpha),$$

More explicitly, if we write $\varphi_\alpha = c_\alpha^\nu x_\alpha^\nu$ locally for some one form $c_\alpha^\nu$, then $ad(\varphi_\alpha) = c_\alpha^\nu ad(x_\alpha^\nu)$. It is easy to check that $\nabla_{\varphi}$ is well-defined and compatible with the Lie algebra structure, i.e. $\nabla_{\varphi} |_{\bigoplus_{\Phi}} = 0$. For $\nabla_{\varphi}$ to define a holomorphic structure, we need

$$0 = \nabla_{\varphi}^2 = \sum_{\alpha \in \Phi^+} (\nabla_0 c_\alpha^\nu + \sum_{\beta+\gamma=\alpha} (n_{\beta,\gamma} c_\beta^\nu c_\gamma^\nu)ad(x_\alpha^\nu),$$

that is $\nabla_0 \varphi_\alpha + \sum_{\beta+\gamma=\alpha} (n_{\beta,\gamma} \varphi_\beta \varphi_\gamma) = 0$ for any $\alpha \in \Phi^+$. Explicitly:

$$\begin{cases}
\nabla_0 \varphi_{C_i} = 0 & i = 1, 2, \cdots, r \\
\nabla_0 \varphi_{C_i + C_j} = n_{C_i, C_j} \varphi_{C_i} \varphi_{C_j} & \text{if } C_i + C_j \in \Phi^+ \\
\vdots
\end{cases}$$

Recall $\{C_i\}_{i=1}^r \subset \Phi^+$ is a base.

Proposition 14 Given any $(\varphi_{C_i})_{i=1}^r \in \Omega^{0,1}(X, \bigoplus_{i=1}^r O(C_i))$ with $\nabla_{\varphi} \varphi_{C_i} = 0$ for any $i$, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$ satisfying $\nabla_{\varphi}^2 = 0$. Namely we have a holomorphic $g$-bundle $\mathcal{E}_0^g$ over $X$. 

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Lemma 15 If \( p_{\delta}(X) = 0 \), then

(i) for any \( \alpha \in \Phi^+ \), \( H^2(X, O(\alpha)) = 0 \).

(ii) the restriction homomorphism \( H^1(X, O_X(C_i)) \to H^1(X, O_{C_i}(C_i)) \) is surjective.

Theorem 16 For any given \( i \), the holomorphic \( g \)-bundle \( E^g_\varphi \) over \( X \) is trivial on \( C_i \) if and only if \( [\varphi_{C_i}|_{C_i}] \neq 0 \).

Note that part (ii) of Lemma 15 says that such \( \varphi_{C_i} \)'s can always be found.

3.2 Trivializations in loop ADE cases

Definition 17 Given any \( \varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi^+} O(\alpha)) \), we define

\[ \overline{\partial}_{(\varphi, \Phi)} : \Omega^{0,0}(X, E^L_0) \to \Omega^{0,1}(X, E^L_0) \]

by

\[ \overline{\partial}_{(\varphi, \Phi)} := \overline{\partial}_0 + ad(\varphi). \]

More explicitly, similarly as explained in §3.1, we have

\[ \overline{\partial}_{(\varphi, \Phi)} := \overline{\partial}_0 + \sum_{n \geq 0} \sum_{\alpha \in \Phi^+} (c_{\alpha+nF} e_{nF} ad(x_{\alpha}) + c_{-\alpha+(n+1)F} e_{(n+1)F} ad(x_{-\alpha})) \]

\[ + \sum_{n \geq 0} \sum_{i=1}^{r} c_{(n+1)F} e_{(n+1)F} ad(h_i), \]

Proposition 18 \( \overline{\partial}_{(\varphi, \Phi)} \) is compatible with the Lie algebra structure on \( E^L_0 \).

Proof. \( \overline{\partial}_{(\varphi, \Phi)} \) is compatible with the Lie algebra structure on \( E^L_0 \).

For \( \overline{\partial}_{(\varphi, \Phi)} \) to define a holomorphic structure, we need \( \overline{\partial}_{(\varphi, \Phi)}^2 = 0 \), which is equivalent to the following equations:

\[
\left\{ \begin{array}{ll}
\overline{\partial}_0 \varphi^n_{\alpha+F} &= \sum_{p+q=n} \sum_{\alpha \in \Phi^+} \pm a_i(h_\alpha) \varphi^{\alpha+pF}_{\alpha}\varphi^{\alpha+qF}_{-}\varphi \\
\overline{\partial}_0 \varphi^{\alpha+nF} &= \sum_{p+q=n} (\sum_{\alpha_1+\alpha_2=\alpha} \varphi^{\alpha_1}_{\alpha_1+pF}\varphi^{\alpha_2+qF}_{\alpha_2} + \sum_{r=1}^{r} (\alpha, C_i) \varphi^{\alpha+pF}_{\alpha+qF}) \\
\overline{\partial}_0 \varphi^{\alpha+nF} &= \sum_{p+q=n} (\sum_{\alpha_2+\alpha_1=\alpha} \varphi^{\alpha_2}_{\alpha_1+pF}\varphi^{\alpha_2+qF}_{\alpha_2} + \sum_{r=1}^{r} (-\alpha, C_i) \varphi^{\alpha-pF}_{\alpha-\alpha+qF}),
\end{array} \right.
\]

where \( a_i(h_\alpha) \) is the coefficient of \( h_i \) in \( h_\alpha \).

Proposition 19 Given any \( (\varphi_i)_{i=0} \in \Omega^{0,1}(X, \bigoplus_{i=0}^{r} O(C_i)) \) with \( \overline{\partial}_{\varphi} \varphi_{C_i} = 0 \) for every \( i \), it can be extended to \( \varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi^+} O(\alpha)) \) satisfying

\( \overline{\partial}_{\varphi}^2 = 0 \). Namely we have a holomorphic \( Lg \)-bundle \( E^L_0 \) over \( X \).

In order to prove this proposition, we need the following lemma.
Lemma 20 If $p_g(X) = 0$, then for any $\alpha \in \Phi^+$, $n \in \mathbb{Z}_{\geq 0}$, $H^2(X, O(nF))$, $H^2(X, O(\alpha + nF))$ and $H^2(X, O(-\alpha + (n + 1)F))$ are zeros.

Proof. Since $F$ is an effective divisor and $H^0(X, K_X) = 0$, we have for any $n \geq 0$, $H^0(X, K_X(-nF)) = 0$. This is equivalent to $H^2(X, O(nF)) = 0$ by Serre duality. Similarly, $H^2(X, O(\alpha + nF)) = 0$ follows from $H^0(X, K_X(-\alpha)) \cong H^2(X, O(\alpha)) = 0$ (Lemma 17). The proof of $H^2(X, O(-\alpha + (n + 1)F)) = 0$ uses the fact that $F - \alpha$ is an effective divisor for any $\alpha \in \Phi^+$. ■

Proof. (of Proposition 14) the equation $\overline{\partial}_{(\varphi, \Phi)}^2 = 0$ can be rewritten as follows:

$$\begin{aligned}
&\frac{\partial_0}{\partial_0} \varphi_{\alpha_i} = 0 \text{ for } i = 1, 2, \ldots, r, \\
&\frac{\partial_0}{\partial_0} \varphi_{\alpha} = \sum_{a_1, a_2 = a} (\pm \varphi_{\alpha_i} \varphi_{\alpha_2}), \\
&\frac{\partial_0}{\partial_0} \varphi_{-\alpha + F} = \sum_{a_1 = a} (\pm \varphi_{\alpha_i} \varphi_{-\alpha + F}), \\
&\frac{\partial_0}{\partial_0} \varphi_{-\alpha + F} = \sum_{a_1 = a} (\pm a(h_\alpha) \varphi_{\alpha_i} \varphi_{-\alpha + F}), \\
&\ldots
\end{aligned}$$

where $\alpha_0 = F - C_0$ is the longest root in $\Phi$.

Firstly, we can solve for all the $\varphi_{\alpha_i}$'s, $\alpha \in \Phi^+$ from $H^2(X, O(\alpha)) = 0$ (Proposition 14). Secondly, we get all the $\varphi_{-\alpha + F}$'s, $\alpha \in \Phi^+$ from $H^2(X, O(-\alpha + F)) = 0$. Thirdly, since we have all the $\varphi_{\alpha_i}$'s and $\varphi_{-\alpha + F}$'s, we can solve for all the $\varphi_{F}$'s for $1 \leq i \leq r$ from $H^2(X, O(F)) = 0$. Do this process for $\varphi_{\alpha_{1+nF}}$, $\varphi_{-\alpha+(n+1)F}$ and $\varphi_{(n+1)F}$ inductively on $n$. ■

By Lemma 15 there always exists $\varphi_{C_i} \in H^0(C_i) \cong \mathbb{C}$ such that $0 \neq [\varphi_{C_i}] \in H^1(X, O(C_i)) \cong \mathbb{C}$ for each $i = 0, 1, \ldots, r$.

Theorem 21 For any given $i$, the holomorphic $Lg$-bundle $E^Lg_{\varphi}$ over $X$ is trivial on $C_i$ if and only if $[\varphi_{C_i}] \neq 0$.

Proof. The proof will be given in §3.4 and §3.5. In §3.4, we deal with all the loop ADE cases except loop $E_8$ case which will be analyzed in §3.5. ■

3.3 Trivializations in loop ADE cases

Follow the notations in §3.2, we define $\overline{\partial}_{(\varphi, \Phi)} := \overline{\partial}_0 + ad(\varphi)$ on $E^g_{\varphi}$, note the adjoint action here is defined using the affine Lie bracket.

Proposition 22 $\overline{\partial}_{(\varphi, \Phi)}$ is compatible with the Lie algebra structure on $E^g_{\varphi}$.

Proof. $\overline{\partial}_{(\varphi, \Phi)}(\cdot, \cdot)_{\Phi, \Phi} = 0$ follows directly from the Jacobi identity and the Killing from being invariant under the adjoint action. ■

It is easy to see that $\overline{\partial}_{(\varphi, \Phi)} = 0$ in the affine case is equivalent to $\overline{\partial}_{(\varphi, \Phi)} = 0$ in the loop case. Hence we have a new holomorphic structure $\overline{\partial}_{(\varphi, \Phi)}$ on $E^g_{\varphi}$.

Theorem 23 For any given $i$, the holomorphic $\mathfrak{g}$-bundle $E^g_{\varphi}$ over $X$ is trivial on $C_i$ if and only if $[\varphi_{C_i}], C_i] \neq 0$.

Proof. This follows from Theorem 22 $0 \rightarrow O \rightarrow E^g_{\varphi} \rightarrow E^Lg_{\varphi} \rightarrow 0$ and Ext $(O, O) = 0$. ■
3.4 Proof (except the loop $E_8$ case)

In this subsection, we use the symmetry of the affine $ADE$ Dynkin diagram (except $E_8$) to show that $\mathcal{E}_0^{L,\theta}$ is trivial on $C_i$ if and only if $|\check{\varphi}_{C_i},|C_i| \neq 0$.

Recall, topologically, $\mathcal{E}_0^{L,\theta}$ is $\mathcal{E}_0^{L,\theta} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{E}_n^{(g,\theta)} \otimes O(nF))$, but with a holomorphic structure $\overline{\partial}_{(\varphi, \Psi)}$ of the following upper triangular block shape:

\[
\overline{\partial}_{\varphi} = \begin{pmatrix}
\left. \overline{\partial}_{\mathcal{E}_n^{(g,\theta)} \otimes O(n+1)F} \right|_{nF}^* & * \\
0 & \left. \overline{\partial}_{\mathcal{E}_n^{(g,\theta)} \otimes O(nF)} \right|_{nF}^* & * \\
0 & 0 & \left. \overline{\partial}_{\mathcal{E}_n^{(g,\theta)} \otimes O((n-1)F)} \right|_{nF}^* & \\
\end{pmatrix}
\]

i.e. $\mathcal{E}_n^{L,\theta}$ is constructed from successive extensions of $\mathcal{E}_n^{(g,\theta)} \otimes O(nF)$'s.

Note $\overline{\partial}_{(\varphi, \Psi)}|_{\mathcal{E}_0^{(g,\theta)}} = \overline{\partial}_0 + \sum_{\alpha \in \Phi^+} \text{ad}(\varphi, \alpha)$. By Theorem 10, for every $i \neq 0$, $\mathcal{E}_n^{(g,\theta)}$ is trivial on $C_i$ if and only if $|\check{\varphi}_{C_i},|C_i| \neq 0$. We also know $O(F)|C_i$ is trivial for every $i$ because $F \cdot C_i = 0$. Thus, when $i \neq 0$, $\mathcal{E}_n^{L,\theta}|C_i$ is constructed from successive extensions of trivial vector bundles over $C_i \cong \mathbb{P}^1$. This implies that $\mathcal{E}_n^{L,\theta}|C_i$ is trivial if and only if $|\check{\varphi}_{C_i},|C_i| \neq 0$ as $\text{Ext}^1_{\mathcal{O}_i}(O, O) = 0$.

Now we consider $i = 0$. Since $\mathfrak{g} \neq \mathfrak{E}_8$, the affine Dynkin diagram always admits a diagram automorphism, that means we can write $\mathcal{E}_0^{L,\theta}$ as $\bigoplus_{n \in \mathbb{Z}} (\mathcal{E}_n^{(g,\theta)} \otimes O(nF))$ (see Proposition 10). Suppose the extended root corresponding to $\Psi$ is $C_0$, and the longest root in $\Psi$ is $\beta_0$.

We will rewrite the holomorphic structure $\overline{\partial}_{(\varphi, \Psi)}$ in terms of the $\Psi$ root system. Note $\overline{\partial}_{(\varphi, \Psi)}$ is determined by the loop Lie algebra structure which is independent of the choice of the extended root. We choose a local base of $\mathcal{E}_0^{(g,\Psi)}$ as in Proposition 10 and define $\overline{\partial}_{(\varphi, \Psi)}$ to be the same with $\overline{\partial}_{(\varphi, \Psi)}$, then obviously $\psi_D = \varphi_D$ when $D \neq nF$.

Because $(\mathcal{E}_n^{(L,\Psi)}, \overline{\partial}_{(\varphi, \Psi)}) = (\mathcal{E}_n^{(L,\Psi)}, \overline{\partial}_{(\varphi, \Psi)})$ as a holomorphic vector bundle, similar to the arguments in $(\mathcal{E}_n^{(L,\Psi)}, \overline{\partial}_{(\varphi, \Psi)})$ case, we have when $i \neq k$, $\mathcal{E}_n^{L,\theta}$ is trivial on $C_i$ if and only if $|\check{\varphi}_{C_i},|C_i| \neq 0$. Note $\psi_{C_0} = \varphi_{-\alpha_0} = \varphi_{C_0}$. So we have Theorem 21 when $\mathfrak{g} \neq \mathfrak{E}_8$.

3.5 Proof for the loop $E_8$ case

Similar to the above subsection, we have when $i = 1, 2, \cdots, 8$, $\mathcal{E}_n^{L,\mathfrak{E}_8}$ is trivial on $C_i$ if and only if $|\check{\varphi}_{C_i},|C_i| \neq 0$. The question is what about $C_0$?

We recall $\mathcal{E}_0^{\mathfrak{E}_8} := O^{\mathfrak{E}_8} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha)$. For any $\alpha \in \Phi$, we write $a_1(\alpha)$ as the coefficient of $C_1$ in $\alpha$, then $O(\alpha)|_{C_0} \cong O_{\mathfrak{g}_0}(a_1(\alpha))$. Among $\Phi^+$, there are 63 roots with $a_1(\alpha) = 0$, corresponding to the positive roots of the Lie sub-algebra $E_7$; 56 roots with $a_1(\alpha) = 1$, corresponding to weights of the standard representation.
of $E_7$; 1 root with $a_1(\alpha) = 2$, which is just the longest root $\alpha_0 = F - C_0$. We denote $\mathcal{E}_0^{E_7} \triangleq \bigoplus_{\alpha \in \Phi, a_1(\alpha) = 0} O(\alpha)$, $V_0^+ \triangleq \bigoplus_{\alpha \in \Phi, a_1(\alpha) = 1} O(\alpha)$ and $V_0^- \triangleq \bigoplus_{\alpha \in \Phi, a_1(\alpha) = -1} O(\alpha)$, then

$$\mathcal{E}_0^{E_8} = \mathcal{E}_0^{E_7} \oplus O \oplus V_0^+ \oplus V_0^- \oplus O(\alpha_0) \oplus O(-\alpha_0).$$

When $O(\alpha)$ is a summand of $V_0^+$, i.e. $O(\alpha)|c_0 \cong O_{p_1}(1)$, we have $O(\alpha + C_0) \cong O_{p_2}(-1)$ and $\alpha + C_0 = F - (\alpha_0 - \alpha)$ with $(\alpha_0 - \alpha) \in \Phi^+$, that is $O(\alpha + C_0)$ is a summand of $V_0^-(F)$. Since $F = \alpha_0 + C_0$ satisfies $F \cdot F = 0$, we have $O(F)|c_0 \cong O_{p_1}, O(\alpha_0)|c_0 \cong O_{p_1}(2)$ and $O(2F - \alpha_0)|c_0 \cong O_{p_1}(-2)$.

For the loop $E_8$-bundle, we have

$$\mathcal{E}_0^{L_{E_8}} = \bigoplus_{n \in \mathbb{Z}} ([\mathcal{E}_0^{E_8} \otimes O(nF))
= \bigoplus_{n \in \mathbb{Z}} ([\mathcal{E}_0^{E_7} \otimes O \oplus V_0^+ \oplus V_0^- \oplus O(\alpha_0) \oplus O(-\alpha_0)) \otimes O(nF))
= \bigoplus_{n \in \mathbb{Z}} ([\mathcal{E}_0^{E_7} \otimes O \oplus V_0^+ \oplus V_0^- (F) \oplus O(\alpha_0 - F) \oplus O(F - \alpha_0)) \otimes O(nF)).$$

We denote $L_{0}^{248} \triangleq \mathcal{E}_0^{E_7} \otimes O \oplus V_0^+ \oplus V_0^- (F) \oplus O(\alpha_0 - F) \oplus O(F - \alpha_0)$. From definition of $\overline{\partial}_\varphi$, $\mathcal{E}_0^{L_{E_8}}$ is built from successive extensions of $L_{0}^{248} \otimes O(nF)$'s, i.e.

$$\overline{\partial}_\varphi = \begin{pmatrix}
\mathcal{E}_0^{L_{E_8}}|_{L_{248}}^\otimes & & & & \\
& \mathcal{E}_0^{L_{E_8}}|_{L_{248}}^\otimes & & & \\
& & \mathcal{E}_0^{L_{E_8}}|_{L_{248}}^\otimes & & \\
& & & \mathcal{E}_0^{L_{E_8}}|_{L_{248}}^\otimes & \\
& & & & \mathcal{E}_0^{L_{E_8}}|_{L_{248}}^\otimes
\end{pmatrix}.$$
is nonzero only if \( D' - D \geq 0 \)

\[
\overline{\partial}_\varphi |_{L^{248}} = \begin{pmatrix}
\overline{\partial}_{V_{-}}(F) & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\
0 & \partial_{O(F-\alpha_0)} - A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\
0 & 0 & \partial_{V_{+}} & A_{34} & A_{35} & A_{36} & A_{37} \\
0 & 0 & 0 & \overline{\partial}_{\mathcal{E}_{+}} & A_{45} & A_{46} & A_{47} \\
0 & 0 & 0 & 0 & \partial_{O} & A_{56} & A_{57} \\
0 & 0 & 0 & 0 & 0 & \partial_{O(\alpha_0-F)} & 0 \\
\end{pmatrix}.
\]

Now we restrict this to \( C_0 \), the 56 pairs \( \{O_{p^1}(-1), O_{p^1}(1)\} \)'s are in \( V_0^{-}(F)|_{C_0} \oplus V_0^{+}(C_0) \). Since \( A_{23} = (0, 0, \cdots, 0)_{56 \times 1} \) and

\[
A_{13} = \begin{pmatrix}
\pm \varphi_{C_0} & * & \cdots & * \\
0 & \pm \varphi_{C_0} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \pm \varphi_{C_0} \\
\end{pmatrix}_{56 \times 56},
\]

if \( [\varphi_{C_0}|_{C_0}] \neq 0 \), then we have a trivialization of the 56 pairs \( \{O_{p^1}(-1), O_{p^1}(1)\} \)'s over \( C_0 \) by Lemma 32 in [3].

For the triple \( \{O_{p^1}(-2), O_{p^1}, O_{p^1}(2)\} \), we review the trivialization of \( A_{1} \) Lie algebra bundle. In \( A_{1} \) case, we have an \( A_{1} \)-bundle \( \mathcal{E}_{\varphi}^{A_{1}} \), which topologically is \( \mathcal{E}_{0}^{A_{1}} = O \oplus O(C) \oplus O(-C) \), but with a holomorphic structure as follows:

\[
\overline{\partial}_\varphi = \begin{pmatrix}
\overline{\partial}_0 & \pm \varphi_{C} & 0 \\
0 & \overline{\partial}_0 & \pm \varphi_{C} \\
0 & 0 & \overline{\partial}_0 \\
\end{pmatrix},
\]

where \( \varphi_{C} \in H^{0,1}(X, O(C)) \). From [3], we know if \( [\varphi_{C}|_{C}] \neq 0 \), then \( \mathcal{E}_{\varphi}^{A_{1}} \) is trivial on \( C \). Back to our case, the triple \( \{O_{p^1}(-2), O_{p^1}, O_{p^1}(2)\} \) has the corresponding submatrices \( A_{25} = (\varphi_{C_0})_{1 \times 1}, A_{57} = (\varphi_{C_0})_{1 \times 1} \) and \( A_{27} = (0)_{1 \times 1} \). Since \( A_{23}, A_{24}, A_{26}, A_{47} \) and \( A_{47} \) are all zero matrices, from the trivialization of \( A_{1} \) Lie algebra bundle, we know if \( [\varphi_{C_0}|_{C_0}] \neq 0 \), then we have a trivialization of the triple \( \{O_{p^1}(-2), O_{p^1}, O_{p^1}(2)\} \) over \( C_0 \).

Hence if \( [\varphi_{C_0}|_{C_0}] \neq 0 \), then \( \{L^{248}, \overline{\partial}_\varphi |_{L^{248}}\} \) is trivial on \( C_0 \), which implies \( (\mathcal{E}_{\varphi}^{L_{E_8}}, \overline{\partial}_\varphi) \) is also trivial on \( C_0 \). Hence, we have Theorem 24 for \( L_{E_8} \) case.

### 4 \( E_{n} \)-bundle over \( X_{n} \) with \( n \leq 9 \)

When \( X = X_{n} \) is a blowup of \( \mathbb{P}^2 \) at \( n \) points \( x_1, \cdots, x_n \) with \( n \leq 9 \), there is a canonical (affine) Lie algebra bundle \( \mathcal{E}_{0}^{E_{n}} \) over it, where \( E_{0} \) is the affine \( E_8 \). In this section, we will give a detail study of the relationship between the geometry of \( X_{n} \) and the deformability of \( \mathcal{E}_{0}^{E_{n}} \).
4.1 $E_n$-bundle over $X_n$ with $n \leq 9$

The Picard group $\text{Pic}(X_n) \cong H^2(X_n, \mathbb{Z})$ is a rank $n + 1$ lattice with generators $h, l_1, \ldots, l_n$, where $h$ is the class of lines in $\mathbb{P}^2$ and $l_i$ is the exceptional class of the blow-up at $x_i$. So $h^2 = 1 = -l_i^2$ and $h \cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_n, \mathbb{Z}) \cong \mathbb{Z}^{1,n}$. The canonical class is $K_{X_n} = -3h + l_1 + \cdots + l_n$. Denote

$$\Phi_n := \{ \alpha \in H^2(X_n, \mathbb{Z}) | \alpha^2 = -2, \alpha \cdot K = 0 \}.$$

Then $\Phi_n$ is a root system of type $E_n$ when $n \leq 8$ and $\Phi_8$ is an affine real root system of $\hat{E}_8$ (also denoted as $E_8$). More explicitly, $\Phi_{\hat{E}_8} := \Phi_9 \cup \{ mK_{X_9} | m \neq 0, m \in \mathbb{Z} \}$ forms a root system of (untwisted) affine $E_8$-type (that is, $\hat{E}_8$-type) with $\Phi^{re}_{\hat{E}_8} := \Phi_9$ the set of real roots and $\Phi^{im}_{\hat{E}_8} := \{ mK_{X_9} | m \neq 0, m \in \mathbb{Z} \}$ the set of imaginary roots (see [10] or [13]). We have an $\hat{E}_8$-bundle $\mathcal{E}_0^{\hat{E}_8}$ over $X_9$:

$$\mathcal{E}_0^{\hat{E}_8} = O \oplus \bigoplus_{\alpha \in \Phi^{re}_{\hat{E}_8}} O(\alpha) \oplus \bigoplus_{\beta \in \Phi^{im}_{\hat{E}_8}} O(\beta).$$

The Lie algebra structure on $\mathcal{E}_0^{\hat{E}_8}$ is explained in [13]. When $n \leq 8$, $\mathcal{E}_0^{E_n} = O \oplus \bigoplus_{\alpha \in \Phi_n} O(\alpha)$ is an $E_n$-bundle over $X_n$.

Suppose $C = \cup C_i$ is an (affine) $ADE$ curve of type $g$ in $X_n$, then $C_i$’s generates a subroot system $\Phi$ inside $\Phi_n$ since $C_i \cdot K = 0$ for every $i$. Therefore the corresponding bundle $\mathcal{E}_0^g$ is a Lie algebra subbundle of $\mathcal{E}_0^{E_n}$.

Suppose $\mathcal{E}_0^g$ is a $g$-bundle over a surface $X$ corresponding to a root system $\Lambda_0 \subset \text{Pic}(X)$ of type $g$.

**Definition 24** A Lie algebra sub-bundle $\mathcal{F}$ of $\mathcal{E}_0^g$ is called strict if there exists a sub-root lattice $\Lambda$ of $\Lambda_0$ such that $\mathcal{F}$ is a direct sum of line bundles corresponding to the roots in $\Lambda$.

In order to describe $\mathcal{E}_0^{\hat{E}_8}$ as a central extension of a loop Lie algebra bundle over $X_9$, we pick any smooth $(-1)$-curve $l$ in $X_9$, then we have

$$\mathcal{E}_0^{\hat{E}_8} \cong \mathcal{E}_0^{E_8} \otimes (\bigoplus_{n \in \mathbb{Z}} O(nK_{X_9})) \oplus O,$$

where $\mathcal{E}_0^{E_8}$ is the pull-back of the $E_8$-bundle over $X_8$ via $\pi : X_9 \to X_8$, the blow down map of $l$. The next proposition describes the converse.

**Proposition 25** When $\mathcal{E}_0^{\hat{E}_8}$ is a central extension of a loop $E_8$-sub-bundle over $X$ for some strict $E_8$-bundle $\mathcal{F}_0^{\hat{E}_8}$ over $X_9$, i.e.

$$\mathcal{E}_0^{\hat{E}_8} \cong \mathcal{F}_0^{\hat{E}_8} \otimes (\bigoplus_{n \in \mathbb{Z}} O(nK_{X_9})) \oplus O,$$

as a Lie algebra bundle isomorphism, then there is a unique (possibly reducible) $(-1)$-curve $l$ in $X$ such that $\mathcal{F}_0^{\hat{E}_8}$ is constructed from those $\alpha \in \Lambda^{re}$ satisfying $\alpha \cdot l = 0$. 

13
\textbf{Proof.} Denote $\Delta_{E_8} = \{\alpha_1, \cdots , \alpha_8\}$ as a root base of the corresponding $E_8$ Lie algebra from $\mathcal{F}^E_0$, we need to find a unique $(-1)$-curve $l$ in $X$ such that $l \cdot \alpha_i = 0$ for any $\alpha_i$ in $\Delta_{E_8}$. Since $\{\pm 1\} \times W(E_8)$ acts on the set of all root bases of $E_8$, simply transitively \cite{12} and $W(E_8)$ acts on the set of $(-1)$-curves \cite{13}, we only need to find $l$ for one particular root base of any $E_8$ in $E_8$ and show that such a $l$ is unique. For example, if we take $\alpha_1 = h - l_1 - l_2 - l_3, \alpha_k = l_{k-1} - l_k$ for $k = 2, \cdots , 8$, then we can take $l = l_9$ and by the condition that $l \cdot \alpha_i = 0$, $l^2 = -1 = l \cdot K$, we know such a $l$ is unique. \rightline{\blacksquare}

4.2 Deformability of such $E_0^{E_8}$

In this subsection, we will describe relationships between the geometry of $X_9$ and the deformability of $E_0^{E_8}$. Similar results for $X_n$ and $E_0^{E_n}$ with $n \leq 8$ can be easily deduced from this case.

Recall when $Pic(X)$ contains a lattice $\Lambda$ isomorphic to a root lattice $\Lambda_g$, then we have a $g$-bundle $E$ over $X$ \cite{9, 14, 15, 13}.

$$E := O(\mathbb{P}^6) \oplus \bigoplus_{\alpha \in \Phi} O(\alpha).$$

Infinitesimal deformations of holomorphic structures on $E$ are parametrized by $H^1(X, \text{End}(E))$, and those which also preserve the Lie algebra structure are parametrized by $H^1(X, ad(E)) = H^1(X, E)$ since $g$ is simple. Hence we introduce the following definitions.

\textbf{Definition 26} (i) $E$ is called fully deformable if there exists a base $\Delta \subset \Phi$ such that $H^1(X, O(\alpha)) \neq 0$ for any $\alpha \in \Delta$.

(ii) $E$ is called $\mathfrak{h}$-deformable if there exists a strict $\mathfrak{h}$ Lie algebra sub-bundle $\mathfrak{E}^\mathfrak{h} \subset E$ which is fully deformable.

(iii) $E$ is called deformable in $\alpha$-direction for $\alpha \in \Phi$ if $H^1(X, O(\alpha)) \neq 0$.

(iv) $E$ is called totally non-deformable if $H^1(X, O(\alpha)) = 0$ for any $\alpha \in \Phi$.

Recall the holomorphic structure $\mathcal{J}_\phi$ or $\mathcal{J}_{(\phi, \Phi)}$ defined in §3.1 and §3.2 on $E$ admits a filtration determined by the height of the roots (if the root base $\Delta = \{\alpha_1, \alpha_2, \cdots , \alpha_r\}$, then for any $\alpha \in \Phi$, we have $\alpha = \sum a_i \alpha_i$ and the height of $\alpha$ is defined to be $ht(\alpha) := \sum a_i$).

\textbf{Remark 27} When $E$ is fully deformable and if for every simple root $\alpha \in \Delta$, $O(\alpha) = O(C_\alpha)$ for some smooth irreducible curve $C_\alpha$, then $C = \cup_{\alpha \in \Delta} C_\alpha$ is an ADE or affine ADE curve in $X$. In this case, we can show that $H^2(X, O(\alpha)) = 0$ for any $\alpha \in \Phi$ and the $g$ or $\tilde{g}$ bundle $E$ admits a deformation into a filtrated bundle which is trivial on every $C_\alpha$ (see section 3). When $E$ is totally non-deformable, $\mathcal{J}_\phi$ can only be $\mathcal{J}_{0}$.

The main results of this section are the followings.

\textbf{Theorem 28} $E_0^{E_8}$ over $X_9$ is totally non-deformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position.
Let us recall some facts about elliptic fibrations on $X_9$. Any elliptic fibration on $X_9$ must be relatively minimal, i.e. there is no $(-1)$-curves in any of its fibrations, as there is no elliptic fibration on $X_8$, this is because the Euler characteristic of any elliptic surface is a multiple of 12 and also $\chi(X_9) = 12$. There is at most one multiple fiber, say of multiplicity $m$. This happens precisely when there exists an irreducible pencil of degree $3m$ in $\mathbb{P}^2$ with $9$ base points, each of multiplicity $m$ and $X_9$ is the blow up of $\mathbb{P}^2$ at these $9$ points. We can characterize the existence of such an elliptic fibration on $X_9$ in terms of deformability of $\mathcal{E}^{E_8}_0$ along imaginary root directions. For instance, $X_9$ with $-K_{X_9}$ nef admits an elliptic fibration (without multiple fiber) if and only if $\mathcal{E}^{E_8}_0$ is deformable in $(mK)$-direction for some $m \in \mathbb{N}$ (with $m = 1$). Deformability of $\mathcal{E}^{E_8}_0$ can also detect the existence of $ADE$ or Kodaira curves in $X$.

**Theorem 29** Suppose $-K_{X_9}$ is nef, then

(i) $X_9$ admits an elliptic fibration with a multiple fiber of multiplicity $m$ ($m \geq 1$) if and only if $\mathcal{E}^{E_8}_0$ is deformable in $(-mK)$-direction but not in $(-m+1)K$-direction.

(ii) $X_9$ has an (maximal) $ADE$ curve $C$ of type $\mathfrak{g}$ if and only if $\mathcal{E}^{E_8}_0$ is (maximal) $\mathfrak{g}$-deformable.

(iii) $X_9$ has a (maximal) Kodaira curve $C$ of type $\mathfrak{g}$ if and only if $\mathcal{E}^{E_8}_0$ is (maximal) $\mathfrak{g}$-deformable.

Here we say an $ADE$ or Kodaira curve $C$ is maximal if it is not proper contained in another $ADE$ or Kodaira curve. We say $\mathcal{E}^{E_8}_0$ is maximal $\mathfrak{g}$ (or $\mathfrak{g}$) deformable if there does not exist another fully deformable (affine) Lie algebra sub-bundle of $\mathcal{E}^{E_8}_0$ containing this $\mathfrak{g}$ (or $\mathfrak{g}$) bundle.

### 4.3 Negative curves in $X_9$

In this subsection, we study negative rational curves in $X_9$. We can get corresponding results for $X_n$ with $n \leq 8$ from this $n = 9$ case.

A divisor $D$ in $X$ is called a $(-m)$-class if $D \cdot D = -m$ and $D \cdot K = m - 2$. An effective $(-m)$-class is called a $(-m)$-curve. Note when $D = \sum n_i C_i$ is a $(-m)$-curve, we will also denote the corresponding curve $\cup C_i$ as $D$.

Use the notations in the above subsection, every effective divisor $D = ah - \sum_{i=1}^9 a_i l_i \in Pic(X_9)$ must have $a = D \cdot h \geq 0$. It is well-known that all $(-1)$-classes are effective, and there are infinite number of them in $X_9$. There are also infinite number of $(-2)$-classes, but whether they are effective or not depends on the positions of the 9 blow-up points.

**Definition 30** Let $x_1, \cdots, x_n$ be $n$ distinct points in $\mathbb{P}^2$. These $n$ points are said to be non-special with respect to Cremona transformations if for any Cremona transformation $T$ with centers within $x_i$’s, the points $y_1, \cdots, y_n$ corresponding to $x_i$’s under $T$ are distinct points such that no three points among $y_1, \cdots, y_n$ are collinear.
Definition 31 ([13]) Let \( x_1, \cdots, x_9 \) be 9 points in \( \mathbb{P}^2 \), we say they are in general position if they satisfy the following three conditions:

(i) they are distinct points in \( \mathbb{P}^2 \);
(ii) they are non-special with respect to Cremona transformations;
(iii) there is a unique cubic curve passing through all of them.

The conditions (i) and (ii) mean that any 8 of these 9 points are in general position. That is, no lines pass through three of them, no conics pass through six of them, and no cubic curves pass through eight of them with one of the eight points being a double point.

If the 9 blowing up points are in general position, then there is no effective \((-2\)-class in \( X_9 \) [13]. In general, there are at most finite number of \((-m\)-curves with \( m \geq 3 \).

Lemma 32 Let \( D = ah - \sum_{i=1}^{9} a_i l_i \) be a \((-m\)-curve in \( X_9 \) with \( m \geq 3 \), then

(i) \( m \leq 9 \);
(ii) \( 0 \leq a \leq 3 \);
(iii) \(-1 \leq a_i \leq 2 \) for all \( i \), and there exists some \( j \) with \( a_j = 1 \);
(iv) there are finite number of such curves.

Proof. (i) Since \( D \) is a \((-m\)-curve, \( D \cdot D = -m \) and \( D \cdot K = m - 2 \), i.e.

\[
\sum a_i^2 = a^2 + m \quad \text{and} \quad \sum a_i = 3a + m - 2.
\]

From the above two equations, we have

\[
(3a + m - 2)^2 = (\sum a_i)^2 \leq 9(\sum a_i^2) = 9(a^2 + m).
\]

Thus, \( a \leq \frac{m^2 + 13m - 4}{6(m-2)} \), also \( a \geq 0 \) since \( D \) is effective, hence \( m \leq 12 \).

When \( m \geq 10 \), we must have \( a = 0 \), that means \( \sum a_i^2 = m \) and \( \sum a_i = m - 2 \), hence \( \sum a_i^2 - \sum a_i = 2 \), which implies every \( a_i \) satisfies \( |a_i| \leq 1 \) and there exists exactly one \( a_i \) with \( a_i = -1 \). But we also have \( \sum a_i = m - 2 \geq 8 \), which is impossible since we only have nine \( a_i \)'s.

(ii) When \( m \geq 4 \), \( a \leq \frac{m^2 + 13m - 4}{6(m-2)} \leq \frac{8}{3} < 3 \). When \( m = 3 \), \( a \leq \frac{m^2 + 13m - 4}{6(m-2)} = \frac{13}{3} < 5 \). Hence we only need to prove there is no \((-3\)-curve with \( a = 4 \).

Suppose not, then there exists \( a_i \)'s such that \( \sum a_i^2 = 19 \) and \( \sum a_i = 13 \). From \( \sum a_i^2 - \sum a_i = 6 \), we know \(-2 \leq a_i \leq 3 \). If there is any \( a_i \) with \( a_i = 3 \), then the other \( a_i \)'s can only be \( 0 \) or \( 1 \), but we have \( \sum a_i = 13 \) and there is only nine \( a_i \)'s, which is impossible. Hence \(-2 \leq a_i \leq 2 \), from \( \sum a_i^2 - \sum a_i = 6 \), we can have at most three \( a_i \)'s equal to 2, which is also impossible since \( \sum a_i = 13 \). (iii) From \( \sum a_i^2 = a^2 + m \), \( \sum a_i = 3a + m - 2 \) and \( 0 \leq a \leq 3 \), we have

\[
\sum a_i = 3a + m - 2 \geq a^2 + m - 2 = \sum a_i^2 - 2.
\]

Hence \(-1 \leq a_i \leq 2 \). And there are three cases:

Case 1, one \( a_i \) equal to 2, the others equal to 0 or 1;
Case 2, one \( a_i \) equal to \(-1 \), the others equal to 0 or 1;
Case 3, all $a_i$’s are equal to $0$ or $1$.

By $\sum a_i = 3a + m - 2 \geq 1$, we know in case 2 and case 3, there must exist some $a_i$ with $a_i = 1$. In case 1, if there is no $a_i$ with $a_i = 1$, then $D = ah - 2l_j$. From $\sum a_i = a^2 + m$, $\sum a_i = 3a + m - 2$, we have $a = 0$, $m = 4$, hence $D = -2l_j$, which is not an effective divisor.

(iv) It is obvious from the above results.

From this lemma, we can easily obtain the following as a corollary.

**Corollary 33** If there exists a $(-m)$-curve in $X_9$ with $m \geq 3$, then there also exists a $(-m+1)$-curve in $X_9$.

**Proof.** If $D \in \{ah - \sum a_i l_i\}$ is a $(-m)$-curve in $X_9$ with $m \geq 3$, then there exists $j$ with $a_j = 1$ by (iii) of Lemma 32. It is easy to check that $D + l_j$ is a $(-m+1)$-curve in $X_9$.

If the 9 blowing up points are in general position, then there is no $(-2)$-curve in $X_9$, as a consequence, there is also no $(-m)$-curve in $X_9$ with $m \geq 3$. The following result shows that this happens exactly when $X_9$ is almost Fano. We include a proof here as we could not find it in the literatures.

**Lemma 34** $X_9$ has no $(-m)$-curve with $m \geq 3$ if and only if $-K_{X_9}$ is nef.

**Proof.** If $-K$ is nef, then from $C \cdot K^{-1} = 2 - m \geq 0$ for any $(-m)$-curve $C$, we know $m \leq 2$.

Conversely, assume $X_9$ has no $(-m)$-curve with $m \geq 3$. Since $X_9$ is a blowup of $\mathbb{P}^2$ at nine points $\{x_i\}_{i=1}^9$, we have an effective anti-canonical divisor $D$. Recall when $D \cdot \Sigma < 0$ for any irreducible curve $\Sigma$ in $X$, $\Sigma$ must be a component of $D$. So if $D$ is an irreducible curve or a Kodaira curve, then $D$ is nef. We denote the image of $D$ in $\mathbb{P}^2$ as $C$, which is a cubic curve passing through these 9 blowing up points.

(i) If $C$ is smooth, then we are done as $D \cong C$ and therefore irreducible.

(ii) If $C$ is reduced and irreducible, then it must be a nodal or cuspidal cubic. If $\{x_i\}_{i=1}^9 \cap \text{sing}(C) = \emptyset$ (\text{sing}(C) means the set of singular points on $C$), then $D \cong C$ and we are done. Otherwise, say $x_1 \in \text{sing}(C)$ and we write the strict and proper transformations of $C$ in $\text{Bl}_{x_1}(\mathbb{P}^2)$ as $C_1$ and $C_1 + E$ respectively. Then the remaining $x_i$’s must have exactly 1 point (resp. 7 points) lying on $E$ (resp. $C_1$) in order to avoid having $(-m)$-curve with $m \geq 3$. Thus $D$ is a Kodaira curve of type $\widehat{A}_1$ or $\text{III} \widehat{(A}_1)$ for $C$ being a nodal or cuspidal respectively.

(iii) If $C$ is reduced and reducible, then $C = B \cup H_0$ or $H_1 \cup H_2 \cup H_3$ with $B$ and $H_j$’s are conic and distinct lines in $\mathbb{P}^2$. As before, we must have exactly 6 $x_i$’s on $B$ and 3 $x_i$’s on each $H_j$ and none on $\text{sing}(C)$. Thus $D \cong C$ is a Kodaira curve of type $\widehat{A}_1$, $\widehat{A}_2$, $\text{III} \widehat{(A}_1)$ or $\text{VI} \widehat{(A}_2)$.

(iv) If $C$ is non-reduced, $C = 3H$, $D$ must have a $(-m)$-curve with $m \geq 3$. Hence $D$ is an irreducible curve or a Kodaira curve, and we are done.

In the following two lemmas, we will use Lemma 2.21 in [1] to give a criteria of a curve in $X_9$, being an $ADE$ or affine $ADE$ curve. Lemma 2.21 can be reformulated as follows: if $C = \cup_{i=1}^r C_i$ is a connected curve in a surface $X$
satisfying: (i) \( C_i^2 = -2 \) and \( C_i \cdot K_X = 0 \) for any \( i \); (ii) \( C_i \cdot C_j \leq 1 \) for any \( i \neq j \); (iii) \((C_i \cdot C_j)_{rr} \leq 0 \). Then when \((C_i \cdot C_j)_{rr} < 0\), \( C \) is an ADE curve, otherwise, it is an affine ADE curve.

**Lemma 35** Suppose \(-K_{X_n} \ (n \leq 8)\) is nef. Let \( C = \cup C_i \) be a connected curve in \( X_n \). If \( C \cdot K_{X_n} = 0 \), then \( C \) is an ADE curve.

**Proof.** Since \(-K_{X_n} \) is nef, \( C \cdot K_{X_n} = 0 \) implies \( C_i \cdot K_{X_n} = 0 \) for each \( i \), i.e. \( [C_i] \in \langle K \rangle^\perp \cong \Lambda_{E_n} \). We have \( C_i^2 < 0 \) and \((C_i + C_j)^2 < 0 \) for any \( i \) and \( j \).

Together with the genus formula, we have \( C_i^2 = -2 \) and \( C_i \cdot C_j \leq 1 \) for \( i \neq j \).

By Lemma 2.21 in [1], we know \( C \) is an ADE curve. \( \blacksquare \)

For \( n = 9 \) case, we have the following lemma.

**Lemma 36** Suppose \(-K_{X_9} \) is nef. Let \( C = \cup C_i \) be a connected curve in \( X_9 \). If \( C \cdot K_{X_9} = 0 \) and \( C_i + K_{X_9} \) is not effective for each \( i \), then \( C \) is a smooth elliptic curve, an ADE curve or an affine ADE curve.

**Proof.** Since \(-K_{X_9} \) is nef, \( C \cdot K_{X_9} = 0 \) implies \( C_i \cdot K_{X_9} = 0 \) for each \( i \), i.e. \([C_i] \in \langle K \rangle^\perp \cong \Lambda_{E_9} \). We have \( C_i^2 \leq 0 \) and \((C_i + C_j)^2 \leq 0 \) for any \( i \) and \( j \). Moreover, for any effective divisor \( D \in \langle K_{X_9} \rangle^\perp \), if \( D^2 = 0 \), then \( D \in \langle mK_{X_9} \rangle \) for some non-zero integer \( m \). From \( C_i^2 \leq 0 \) and genus formula, we have \( C_i^2 = -2 \) or 0.

If there exists \( C_i \) such that \( C_i^2 = 0 \), then \( C_i \in \langle mK \rangle \) for some non-zero integer \( m \). Since \( C_i + K_{X_9} \) is not effective, we know \( m = -1 \), i.e. \( C_i \in \langle -K \rangle \). If \( C \) is not irreducible, there exists \( C_i \) which intersects \( C_1 \), which is impossible. So \( C = C_i \in \langle -K \rangle \) is an elliptic curve or an affine \( A_9 \) curve by Lemma 24.

If \( C_i^2 = -2 \) for any \( i \), then \( C_i \cdot C_j \leq 2 \) for any \( i \neq j \). If there exist \( C_i \) and \( C_j \) such that \( C_i \cdot C_j = 2 \), then \((C_i + C_j)^2 = 0 \), \( C_i + C_j \in \langle mK \rangle \) for some integer \( m \). Hence \( C = C_i \cup C_j \) is an affine \( A_1 \) curve, this is because if \( C_k \) is another irreducible component of \( C \) and assume it intersects with \( C_i \), then it must be an irreducible component of \( C_j \), which contradicts to \( C_j \) being irreducible. Otherwise, we will have \( C_i^2 = -2 \) for each \( i \) and \( C_i \cdot C_j \leq 1 \) for \( i \neq j \). By Lemma 2.21 of [1], we know \( C \) is an ADE or affine ADE curve. \( \blacksquare \)

### 4.4 Proof of theorems [28] and [29]

**Proof.** (of Theorem [28]) If the nine blowup points in \( \mathbb{P}^2 \) are in general position, then for any \( \alpha \in \Phi_9 \), we have \( h^0(X, O(\alpha)) = 0 \) [13]. Since \( K \cdot K = 0 \), we also have \( K - \alpha \in \Phi_9 \) and therefore \( h^2(X, O(\alpha)) = 0 \) by Serre duality. However the Riemann-Roch formula gives \( \chi(X, O(\alpha)) = 1 + \frac{a^2 - aK}{2} = 0 \) and therefore \( h^1(X, O(\alpha)) = 0 \). For the imaginary roots \( mK \)'s, from Lemma 4 and Proposition 11 in [13], we have \( h^0(X, O(mK)) = 0 \) and \( h^0(X, O(-mK)) = 1 \) for \( m \geq 1 \). By Serre duality and Riemann-Roch formula, we have \( h^1(X, O(mK)) = 0 \) for any imaginary root \( mK \). Hence \( \xi^E_0 \) is totally non-deformable.

Conversely, if \( \xi^E_0 \) is totally non-deformable, then \( X \) has no (possibly reducible) \((-2)\)-curve, hence no \((-n)\)-curve with \( n \geq 2 \). By Proposition 10 in [19], this implies the nine blowup points are non-special with respect to Cremona
transformations. Also from $h^1(X, O(mK)) = 0$ for any imaginary root $mK$, we get $h^0(X, O(-mK)) = 1$, we have a unique cubic curve in $\mathbb{P}^2$ passing through all of the blow-up points. Hence, the nine blow-up points in $\mathbb{P}^2$ are in general position.

**Proof.** (of Theorem 29) (i) We have $h^1(X, O(-mK)) = h^0(X, O(-mK)) - 1$ for any $m$ by Riemann-Roch formula. So $\mathcal{E}_0$ is deformable in $(-mK)$-direction if and only if $h^0(X, O(-mK)) = 2$.

Let $F_0 \in [-K]$, then by Proposition 2.2 of [2], $X$ admits an elliptic fibration with a multiple fiber of multiplicity $m$ if and only if $O_{F_0}(F_0)$ is of order $m$ in $\text{Pic}(F_0)$. But $O_{F_0}(mF_0) \cong O_{F_0}$ if and only if $h^0(O_{F_0}(mF_0)) = 1$ as $O_{F_0}(mF_0)$ is topologically trivial. By the exact sequence

$$0 \rightarrow O_X \rightarrow O_X(mF_0) \rightarrow O_{F_0}(mF_0) \rightarrow 0$$

together with $h^1(X, O_X) = 0$, we know $h^0(O_X(mF_0)) = h^0(O_{F_0}(mF_0))$. So $m = \min(n : h^0(O_{F_0}(mF_0)) = 1) = \min(n : h^0(X, O(-mK)) = 2)$.

(ii) If $X$ has an ADE curve $C$ of type $\mathfrak{g}$, we can use it to construct a fully deformable $\mathfrak{g}$-subbundle of $\mathcal{E}_0$ as in §3.2. When $C$ is maximal, then this $\mathfrak{g}$-subbundle is not contained in any other fully deformable Lie algebra subbundle of $\mathcal{E}_0$.

Conversely, if $\mathcal{E}_0$ is maximal $\mathfrak{g}$-deformable, then we can find a base $\Delta \subset \Phi$ such that $h^1(X, O(\alpha)) \neq 0$ for every $\alpha \in \Delta$. Since $\chi(\mathcal{O}(\alpha)) = 1 + \frac{\alpha^2 - \mathcal{O} K}{2} = 0$, we must have $h^0(\mathcal{O}(\alpha)) = h^2(\mathcal{O}(\alpha)) = 0$, that is neither $\alpha$ nor $K - \alpha$ is effective. Hence, there must exist some integers $m$'s such that $\alpha + mK$ is effective because $-K$ is effective, we denote the largest such $m$ as $m_\alpha$.

We claim that for every $\alpha \in \Delta$, $C_\alpha = |\alpha + m_\alpha K|$ is an irreducible $(-2)$-curve. If so, then $C = \cup_{\alpha \in \Delta} C_\alpha$ is a maximal ADE curve of type $\mathfrak{g}$. If there exists reducible $C_\alpha$, we write $C_\alpha = \cup D_i$. Then each $D_i$ is perpendicular to $K$ as $-K$ is nef and $C_\alpha \cdot K = 0$. Since $C_\alpha + K$ is not effective, every $D_i + K$ is also not effective and $D_i \notin |-K|$. Hence $D_i^2 = -2$ for any $i$ as $D_i^2 = 0$ will imply $D_i \in |-K|$. We know $C_\alpha$ is connected, this is because if $C_\alpha$ is not connected, then one of its connected component must have self-intersection zero from $C_\alpha^2 = -2$, which contradicts to $C_\alpha + K$ is not effective. Hence $C = \cup_{\alpha \in \Delta} C_\alpha$ is an (affine) ADE curve by Lemma 29. It is obvious that this curve strictly contains a $\mathfrak{g}$-curve, which contradicts to $\mathcal{E}_0$ being maximal $\mathfrak{g}$-deformable.

(iii) The proof is similar to (ii).

**Remark 37** If $X_9$ admits an elliptic fibration, then we can find $m$ such that $h^1(X_9, O(-mK)) \neq 0$. Conversely, if $h^1(X_9, O(-mK)) \neq 0$, we need to add the condition of $-K$ being nef to show that $X$ admits an elliptic fibration. To see this, we take $x_1, \ldots, x_5$ to be 5 points on a line $l \subset \mathbb{P}^2$, and another 4 generic points (not on $l$) $x_6, \ldots, x_9$ in $\mathbb{P}^2$. Then we have an one parameter family of conics $C_t$’s passing through these 4 points. If we blow up $\mathbb{P}^2$ at these 9 points
and denote the strict transforms of \( l \) and \( C_t \) with same notations, then \( l^2 = -4 \), \( C_t^2 = 0 \). Moreover \( C_t + l \in | -K | \) and \( h^0(X, O(-K)) = 2 \). But \( -K \) is not nef as \( (-K) \cdot l = -2 \), which implies that \( X \) is not elliptic.

From the above, we can easily deduce similar results for the \( E_n \)-bundle \( E_n^E \) over \( X_n \) when \( n \leq 8 \), namely

(i) \( E_n^E \) is totally non-deformable if and only if the \( n \) blowup points in \( \mathbb{P}^2 \) are in general position.

(ii) When \( -K_{X_n} \) nef, \( E_n^E \) is maximal \( g \)-deformable if and only if \( X_n \) has a maximal \( g \) curve.

5 Appendix

In this appendix, we recall some results on affine Lie algebras \[\{12,13\}\]. If \((g, [\cdot, \cdot])\) is a finite dimensional simple Lie algebra, then the corresponding loop Lie algebra is \( Lg := g \otimes \mathbb{C}[t, t^{-1}] \), with the Lie bracket defined by 
\[
[a \otimes t^n, b \otimes t^m]_{Lg} = [a, b] \otimes t^{m+n},
\]
where \( a, b \in g \), \( m, n \in \mathbb{Z} \).

The corresponding untwisted affine Lie algebra \( \hat{g} \) is constructed as a central extension of \( Lg \), with one-dimensional center \( \mathbb{C}c \), i.e. \( \hat{g} = Lg \oplus \mathbb{C}c \). The Lie bracket on \( \hat{g} \) is defined by the formula 
\[
[a \otimes t^n + \lambda c, b \otimes t^m + \mu c]_{Lg} = [a, b] \otimes t^{m+n} + n\delta_{n+m,0}k(a, b)c,
\]
where \( \lambda, \mu \in \mathbb{C} \) and \( k \) is the Killing form on \( g \).

We can obtain the affine Dynkin diagram of \( \hat{g} \) from the Dynkin diagram of \( g \) by adding one node to it, corresponding to the extended root and labelling as \( C_0 \). But in the affine \( ADE \) except affine \( E_8 \) case, from the symmetry of the affine Dynkin diagrams, we have different choices of labelling the extended root.

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