Refinement of the Parshin symbol for surfaces

I. Horozov *

Mathematisches Institut,
Universität Tübingen, Auf der Morgenstelle 10,
72076 Tübingen, Germany

Abstract

On an algebraic curve there are Tate symbols, which satisfy Weil reciprocity law. The analogues in higher dimensions are the Parshin symbols, which satisfy Kato-Parshin reciprocity laws. We give a refinement of the Parshin symbol for surfaces, using iterated integrals in the sense of Chen. The product of the refined symbol over the cyclic permutations of the functions recovers the Parshin symbol. Also, we construct a logarithmic version of the Parshin symbol. We prove reciprocity laws for both the refined symbol and a logarithm of the Parshin symbol.

Contents

0 Introduction 2
1 Background on iterated integrals 10
  1.1 Definition of iterated integrals over a path ........................................ 10
  1.2 Differential equation and generating series of iterated integrals. One-dimensional case ................................................................. 10
  1.3 Multiplication formulas ................................................................. 11
  1.4 Two-dimensional iterated integral over a surface ................................ 11
  1.5 Slicing a membrane ........................................................................ 12
  1.6 Composition of a path and a 2-dimensional region ........................... 14
  1.7 Three-dimensional iterated integrals .................................................. 14
2 Abstract reciprocity law 16
  2.1 Definitions ...................................................................................... 16
  2.2 Construction of an abstract reciprocity law ....................................... 21
3 Construction of a new symbol and a reciprocity law 27
  3.1 Local properties. Integrating over the torus $T_{jl}'$ ........................... 27
  3.2 Semi-local behavior. Integrating over the torus $T_{jl}$ .......................... 29
  3.3 Differential equation ....................................................................... 34
  3.4 The commutators $[\alpha_i, \beta_i]$ ....................................................... 35

*E-mail: ivan.horozov@uni-tuebingen.de
3.5 Extra terms .......................................................... 37
3.6 New symbol ........................................................... 40
3.7 Reciprocity law for the logarithm of the new symbol ............. 40

4 Logarithmic version of the Parshin symbol ......................... 41
  4.1 Integration over a torus revisited .................................. 41
  4.2 Logarithmic symbol .................................................. 41
  4.3 Vanishing of the commutator terms ................................ 42
  4.4 Vanishing of the extra terms ....................................... 44
  4.5 Reciprocity law ...................................................... 45

5 Refinement of the Parshin symbol ...................................... 46
  5.1 Logarithmic version of a refinement of the Parshin symbol .......................... 46
  5.2 Logarithmic reciprocity law ........................................ 46
  5.3 Refinement of the Parshin symbol and a reciprocity law ........... 47
  5.4 Example ............................................................. 47

0 Introduction

This paper is the second one from a series of papers on reciprocity laws on complex varieties. Instead of using a homology or a cohomology theory, we use properties of certain fundamental groups. We capture structures of these fundamental groups by examining iterated integrals. In this way, we have proven various reciprocity laws on a Riemann surface in [H]. We use some of these reciprocity laws here. But more importantly, we develop further the use of iterated integrals in establishing new reciprocity laws.

Parshin has considered iterated integrals [P3] at the same time as Chen [Ch]. However, the ones by Chen are more general and we use some of his constructions.

Let us recall the Weil reciprocity for the Tate symbol. Let $f_1$ and $f_2$ be two non-zero rational functions on a Riemann surface $C$. At a point $P$ on $C$, let $x$ be a rational function on $C$, which has a zero of order 1 at $P$. Let $n_k$ be the (vanishing) order of $f_k$ at $P$. Define $g_k$ as

$$g_k = x^{-n_k} f_k,$$

for $k = 1, 2$. Note that $g_k$ is a rational function, depending on the choices of $P$ and $x$, which has no zero and no pole at the point $P$. Then the Tate symbol is defined as

$$\{f_1, f_2\}_P = (-1)^{n_1 n_2} \left( \frac{f_1^{n_2}}{f_2^{n_1}} \right) (P) = (-1)^{n_1 n_2} \frac{g_1(P)^{n_2}}{g_2(P)^{n_1}}.$$

The Weil reciprocity for the Tate symbol is

$$\prod_{P \in C} \{f_1, f_2\}_P = 1.$$

We are going to use a local coordinate, which is a rational function $x$, as opposed to a uniformizer in the corresponding local field.

Let us explain what is the relation between a local coordinate and a uniformizer. By a local coordinate at a point $P$ on a curve $C$ we mean a rational function $x$ on $C$, which has a zero of order 1 at $P$. The rational function $x$ might have other zeros or poles; but
this will not play an essential role. Let \( \mathcal{O}_C \) be the structure sheaf on \( C \). Let \( U \) be a Zariski open set in \( C \), where \( x \) is defined. Denote also by \( \mathcal{O}_{C,P} \) the local ring of stalks at \( P \). Define \( m_P \) to be the maximal ideal in \( \mathcal{O}_{C,P} \). And let

\[
\hat{\mathcal{O}}_{C,P} = \lim_\leftarrow \mathcal{O}_{C,P}/m_P^N
\]

be the completion of the local ring \( \mathcal{O}_{C,P} \). Then

\[
x \in \mathcal{O}(U) \subset \mathcal{O}_{C,P} \subset \hat{\mathcal{O}}_{C,P}.
\]

Thus, we can think of the rational function \( x \) as a local uniformizer in the complete local ring \( \hat{\mathcal{O}}_{C,P} \).

There is analogous statement for surfaces. We recall the Parshin symbol for a surface. Then we give a definition in terms of rational functions as opposed to uniformizers, which is not as general as the algebraic definition but it allows us to use integrals. We need the new definition in order to construct a refinement of the Parshin symbol together with several logarithmic symbols.

Let \( f_1, f_2, f_3 \) be three non-zero rational functions on a smooth complex surface \( X \). Let \( \mathcal{O}_X \) be the structure sheaf of the surface \( X \). Let \( C \) be a non-singular curve in \( X \) and \( P \) be a point on \( C \). Let \( \mathcal{I}_C \) be the sheaf of ideals, defining the curve \( C \). Let \( \widehat{(\mathcal{O}_X)}_C \) be the completion of the structure sheaf \( \mathcal{O}_X \) with respect to the sheaf of ideals \( \mathcal{I}_C \). Let \( x \) be an element of \( \widehat{(\mathcal{O}_X)}_C/\mathcal{I}_C = \mathcal{O}_C \) such that

\[
\bar{h}_k \equiv x^{-m_k}f_k \mod \mathcal{I}_C.
\]

Let \( y \) be a uniformizer at \( P \) in the completion \( \widehat{(\mathcal{O}_C)}_P \). Let \( m_P \) be the maximal ideal in \( \widehat{(\mathcal{O}_C)}_P \), defining \( P \), and let

\[
n_k = ord_P(\bar{h}_k) = ord_P(x^{-m_k}f_k \mod \mathcal{O}_C).
\]

Define \( \tilde{g}_k \in \mathbb{C} \) so that

\[
\tilde{g}_k \equiv y^{-n_k}\bar{h}_k \mod m_P \equiv y^{-n_k}(x^{-m_k}f_k \mod \mathcal{O}_C) \mod m_P.
\]

Then the Parshin symbol (computed by Fesenko and Vostokov [FV]) is defined as

\[
\{f_1, f_2, f_3\}_{C,P} = (-1)^K \tilde{g}_1^{D_1} \tilde{g}_2^{D_2} \tilde{g}_3^{D_3},
\]

where

\[
D_1 = \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}, D_2 = \begin{vmatrix} m_3 & n_3 \\ m_1 & n_1 \end{vmatrix}, D_3 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}
\]

and

\[
K = n_1n_2m_3 + n_2n_3m_1 + n_3n_1m_2 - m_1m_2n_3 - m_2m_3n_1 - m_3m_1n_2.
\]

One of the Kato-Parshin reciprocity laws is

\[
\prod_P \{f_1, f_2, f_3\}_{C,P} = 1,
\]
where the product is over all points $P$ on $C$.

We need an alternative definition of the Parshin symbol, for which we can apply analytic techniques. Given three non-zero functions on $X$, $f_1$, $f_2$ and $f_3$, we want the decomposition $f_k = x^{k_1}y^{k_2}g_k$, for $k = 1, 2, 3$, so that $g_k$ is well defined and non-zero at $P$ and $x$ and $y$ are rational functions on $X$, not just uniformizers. This is not always possible. So we make assumptions on the divisors of $f_1$, $f_2$ and $f_3$. We assume that the divisors of the functions $f_1$, $f_2$ and $f_3$ have normal crossings (that is, at intersection point only two components meet transversally). Also, we assume that each component of these divisors is a non-singular curve. These properties can be achieved for any non-zero functions $f_1$, $f_2$ and $f_3$, after successive blow-ups of the surface $X$.

Now we consider carefully the algebraic definition of the Parshin symbol. Denote by $C_i$ for $i = 0, 1, \ldots, N$ the components of the divisors of $f_1$, $f_2$ and $f_3$. Instead of choosing $x \in (O_X)_{C_0}I_{C_0} - (O_X)_{C_0}I_{C_0}^2$, defining $C$ in a neighborhood of $P$, we choose a rational function $x$, such that $ord_{C_0}x = 1$ and $x$ defines $C$ in a neighborhood of $P$. In particular, for such a rational function, we have $x \in (O_X)_{C_0}I_{C_0} - (O_X)_{C_0}I_{C_0}^2$. Now we use the divisors of $f_1$, $f_2$ and $f_3$ have smooth normal crossing. Let $P \in C_0 \cap C_j$. Since $x$ defines $C$ in a neighborhood of $P$, we have that $ord_{C_j}x = 0$. Define $m_k = ord_{C_0}f_k$. Similarly we define a rational function $y$ such that $ord_{C_j}y = 1$ and $ord_{C_0}y = 0$. Let $n_k = ord_{C_j}f_k$. Let $h_k = x^{-m_k}f_k$. Then

$$h_k \sim h_k \mod I_{C_0}.$$ Define

$$\bar{y} \sim y \mod I_{C_0}$$
on C_0. Let $\bar{g}_k = \bar{y}^{-n_k}\bar{h}_k$ on $C_0$. Then similar to the one dimensional case, we have

$$\bar{g}_k \sim \bar{g}_k \mod m_P,$$

where $m_P$ is the maximal ideal in the local ring $O_{C_0,P}$. Moreover, $\bar{g}_k = \bar{g}_k(Q) = g_k(Q)$.

Let

$$\text{div}(f_k) = \sum_{i=0}^{N} n_{ki}C_i.$$ Let $x_i$ be a rational function on $X$, which has zero along $C_i$ of order 1 (this is the order of vanishing of $x_i$ at the generic point), and has no zeroes or poles along $C_j$ for $j \neq i$. That is,

$$ord_{C_j}x_i = \delta_{ij}.$$ The rational functions $x_i$ might have other zeros or poles. However, this poses only a technical consideration and it does not contribute to the symbols that we consider.

Note that in the algebraic definition, the Parshin symbol

$$\{f_1, f_2, f_3\}_{C,P} = 1,$$

if $C$ is not a divisor of any of the functions $f_1$, $f_2$, $f_3$, or if $P$ is not an intersection of two curves from the divisors of the functions. Thus, it is enough to consider only the case, when $C$ is a curve from the divisors of the functions $f_1$, $f_2$ and $f_3$ and $P$ is a point of intersection of two divisors.
Let \( C = C_0 \) and let \( P \in C_0 \cap C_j \) for \( j > 0 \). We are going to define \( \{f_1, f_2, f_3\}_{C_0, P} \), using the rational functions \( x_0 \) and \( x_j \). Let \( x = x_0 \) and \( y = x_j \). For their corresponding exponents, define \( m_k = n_{k0} = \text{ord}_{C_0} f_k \) and \( n_k = n_{kj} = \text{ord}_{C_j} f_k \). Let
\[
f_k = x^{m_k} y^{n_k} g_k.
\]
Note that \( g_k \) is a rational function, which is well defined and non-zero at \( P \).

The definition in terms of rational function \( g_k \), which we are going to use, is
\[
\{f_1, f_2, f_3\}_{C_0, P} = (-1)^K g_1(P)^{D_1} g_2(P)^{D_2} g_3(P)^{D_3},
\]
where \( D_1, D_2, D_3 \) and \( K \) are defined as above in terms of \( m_k \) and \( n_k \) for \( k = 1, 2, 3 \).

Now we can define a refinement of the Parshin symbol \( \{f_1, f_2, f_3\}_{C_0, P} \). First, we need one coherence condition: For each point \( P \in C_0 \cap C_j \) and for each curve \( C_j \) we are going to use the same rational function \( x = x_0 \), which has order 1 at \( C_0 \). The refinement of the Parshin symbol will be invariant with respect to choices of the rest of the rational functions \( x_i \) for \( i > 1 \).

**Definition 0.1** With the above notation, we define a refinement of the Parshin symbol
\[
(f_1, f_2, f_3)_{C_0, P}^{x_0} = (-1)^{n_1 n_3 m_2 - m_1 m_3 n_2} \left( \frac{g_1(P)^{n_3}}{g_3(P)^{n_1}} \right)^{m_2}.
\]
The new symbol resembles a power of the Tate symbol for curves. However, there is a simple relation among the Parshin symbol and the refinement of the Parshin symbol.

**Theorem 0.2** (The Parshin symbol in terms of the refinement of the Parshin symbol)
\[
\{f_1, f_2, f_3\}_{C_0, P} = \prod_{\text{cycd}} (f_1, f_2, f_3)_{C_0, P}^{x_0},
\]
where the product is taken over cyclic permutations of the indexes of \( f_1, f_2 \) and \( f_3 \).

Moreover, we have a reciprocity law for the refinement of the Parshin symbol.

**Theorem 0.3** A reciprocity law for the refinement of the Parshin symbol is
\[
\prod_P (f_1, f_2, f_3)_{C_0, P}^{x_0} = 1,
\]
where the product is taken over all points \( P \) on \( C_0 \).

As we mentioned in the beginning of the introduction, we use iterated integrals of differential 1-forms with logarithmic poles. An example of such differential form is \( df_1/f_1 \). Its integral is \( \log(f_1) \). With this approach, first we obtain logarithmic symbols with additive reciprocity laws. After exponentiating, we obtain the above symbols and the corresponding multiplicative reciprocity laws.

In order to define the logarithmic symbols we need to integrate over certain paths \( \gamma_i \).
Definition 0.4 (Points, loops and paths on $C_0$) Let $\{P_1, \ldots, P_M\} = C_0 \cap (\bigcup_j C_j)$ be all intersection points of $C_0$ with the rest of the curves from the divisors of $f_1, f_2, f_3$. Let $Q \in C_0$ be a base point, different from $P_1, \ldots, P_M$. Let $R_i$ be a point on $C_0$, which is within an $\epsilon$-neighborhood of the intersection point $P_i$. Let $\sigma_i$ be a simple loop on $C_0$ around $P_i$, based at $Q$. Let $\gamma_i$ be a path from $Q$ to $R_i$. Define also $\delta_i$ to be a (small) simple loop around $P_i$, based at $R_i$, so that

$$\sigma_i = \gamma_i \delta_i^{-1} \gamma_i^{-1}.$$ 

Choose the paths $\gamma_i$ so that the loop $\delta$, defined by

$$\delta = \left( \prod_{i=1}^{g} (\alpha_i, \beta_i) \right) \left( \prod_{i=1}^{N} \sigma_i \right),$$

is homotopic to the trivial loop at $Q$, where $g$ is the genus of the curve $C_0$ and $[\alpha_i, \beta_i]$ is the commutator of loops $\alpha_i$ and $\beta_i$ around the handles of $C_0$.

At the point $P_i$ define $g_k$ so that

$$f_k = x_0^{m_k} x_j^{n_k} g_k$$

on the surface $X$, using the rational functions $x_0$ and $x_j$ for some $j$ as local coordinates as opposed to uniformizers. Then $g_k$ is well-defined and non-zero at $P_i$.

Definition 0.5 Logarithm of the refinement of the Parshin symbol is defined by

$$\text{Log} (f_1, f_2, f_3)^{x_0, \gamma_i}_{C_0, P_i} = (2\pi i)^2 \left( \pi i (m_2 n_1 n_3 - n_2 m_1 m_3) + m_2 n_3 \int_{\gamma_i} \frac{dg_1}{g_1} - m_2 n_1 \int_{\gamma_i} \frac{dg_3}{g_3} \right).$$

Let $n_{kj} = \text{ord}_{C_j} f_k$. Let also $L_j$ be the number of intersection points of $C_0$ with $C_j$. Define $h_k$ by $h_k = x_0^{-m_k} f_k$. Define also

$$D_1(j) = \begin{vmatrix} m_2 & n_{2j} \\ m_3 & n_{3j} \end{vmatrix}, \quad D_2 = \begin{vmatrix} m_3 & n_{3j} \\ m_1 & n_{1j} \end{vmatrix}, \quad D_3 = \begin{vmatrix} m_1 & n_{1j} \\ m_2 & n_{2j} \end{vmatrix}$$

Theorem 0.6 A reciprocity law for the logarithm of the refinement of the Parshin symbol is

$$\sum_i \text{Log} (f_1, f_2, f_3)^{x_0, \gamma_i}_{C_0, P_i} = (2\pi i)^3 (M + N),$$

where

$$M = \sum_{j_1 < j_2} (n_{1j_1} D_1(j_2) - n_{3j_1} D_3(j_2)) L_{j_1} L_{j_2} + \sum_{j_2 = 1}^{N} (m_{1j_2} D_1(j_2) - n_{3j_2} D_3(j_2)) \frac{1}{2} L_{j_2} (L_{j_2} - 1)$$

and

$$N = (2\pi i)^{-2} m_1 \sum_{j=1}^{g} \left( \int_{\alpha_j} \frac{dh_1}{h_1} \int_{\beta_j} \frac{dh_2}{h_2} - \int_{\beta_j} \frac{dh_1}{h_1} \int_{\alpha_j} \frac{dh_2}{h_2} \right) + (2\pi i)^{-2} m_3 \sum_{j=1}^{g} \left( \int_{\alpha_j} \frac{dh_1}{h_1} \int_{\beta_j} \frac{dh_3}{h_3} - \int_{\beta_j} \frac{dh_1}{h_1} \int_{\alpha_j} \frac{dh_3}{h_3} \right),$$

6
The relation between the refinement of the Parshin symbol and the logarithm of the refinement of the Parshin symbol is given by the following theorem.

**Theorem 0.7**

\[
(f_1, f_2, f_3)_{C_0, P_i}^{x_0} = \exp((2\pi i)^{-2}\Log(f_1, f_2, f_3)_{C_0, P_i}^{x_0}) g_1(Q)^{m_2n_3} g_2(Q)^{-m_2n_1},
\]

where \(Q\) is the base point.

**Definition 0.8** We define logarithm of the Parshin symbol as

\[
\Log\{f_1, f_2, f_3\}_{C_0, P_i}^{x_0} = \prod_{\text{cyc}} \Log(f_1, f_2, f_3)_{C_0, P_i}^{x_0},
\]

where the product is taken over cyclic permutations of the indexes of \(f_1, f_2\) and \(f_3\).

**Theorem 0.9** The relation to the Parshin symbol is

\[
\{f_1, f_2, f_3\}_{C_0, P_i}^{x_0} = \exp((2\pi i)^{-2}\Log\{f_1, f_2, f_3\}_{C_0, P_i}^{x_0}) g_1(Q)^{D_1} g_2(Q)^{D_2} g_3(Q)^{D_3},
\]

where \(Q\) is the base point.

A reciprocity law for the logarithm of the Parshin symbol is given by the following theorem.

**Theorem 0.10**

\[
\sum_i \Log\{f_1, f_2, f_3\}_{C_0, P_i}^{x_0} = 0.
\]

It is interesting to compare the logarithmic symbols on a complex surface \(\Log\{f_1, f_2, f_3\}_{C_0, P_i}^{x_0}\) and \(\Log(f_1, f_2, f_3)_{C_0, P_i}^{x_0}\) with logarithmic symbol on a complex curve. We call the logarithmic symbol on a complex curve a logarithm of the Tate symbol.

**Definition 0.11** (Logarithm of the Tate symbol) Let \(f_1\) and \(f_2\) be two non-zero rational functions on \(C\). Let \(P_1, \ldots, P_M\) be the points in the divisors of \(f_1\) and \(f_2\). Denote by \(m_i\) the order of \(f_1\) at \(P_i\) and by \(n_i\) the order of \(f_2\) at \(P_i\).

Near \(P_i\), let \(x\) be a rational function on \(C\), which has order 1 at \(P_i\). Define \(g_1\) and \(g_2\) by

\[
f_1 = x^{m_i} g_1 \quad \text{and} \quad f_2 = x^{n_i} g_2.
\]

Let \(Q \in C\) be a base point, different from \(P_1, \ldots, P_M\). Let \(R_i\) be a point on \(C\), which is within an \(\epsilon\)-neighborhood of the intersection point \(P_i\). Denote by \(\sigma_i\) a simple loop on \(C\) around \(P_i\), based at \(Q\). Let \(\gamma_i\) be a path from \(Q\) to \(R_i\). Define also \(\sigma_i^0\) to be a (small) simple loop around \(P_i\), based at \(R_i\), so that

\[
\sigma_i = \gamma_i \sigma_i^0 \gamma_i^{-1}.
\]

Choose the paths \(\gamma_i\) so that the loop \(\delta\), defined by

\[
\delta = \left( \prod_{i=1}^g [\alpha_i, \beta_i] \right) \left( \prod_{i=1}^N \sigma_i \right),
\]

is a loop going around all \(P_i\) and \(R_i\).
is homotopic to the trivial loop at \( Q \), where \( g \) is the genus of the curve \( C \) and \([\alpha_i, \beta_i]\) is the commutator of loops \( \alpha_i \) and \( \beta_i \) around the handles of \( C \). We define the logarithmic Tate symbol by

\[
\log\{f_1, f_2\}^\gamma_{P_i} = (2\pi i) \left( \pi i m_i n_i + n_i \int_{\gamma_i} \frac{dg_1}{g_1} - m_i \int_{\gamma_i} \frac{dg_2}{g_2} \right).
\]

**Theorem 0.12** The relation to the Parshin symbol is

\[
\{f_1, f_2\}_{P_i} = \exp((2\pi i)^{-1} \log\{f_1, f_2\}^\gamma_{P_i}) g_1(Q)^{m_i} g_2(Q)^{n_i},
\]

where \( Q \) is the base point.

A reciprocity law for the logarithm of the Parshin symbol is given by the following theorem.

**Theorem 0.13**

\[
\sum_i \log\{f_1, f_2\}^\gamma_{P_i} = (2\pi i)^2 (M + N),
\]

where

\[
M = \sum_i m_i n_i, \\
N = \sum_{i=1}^g \left( \int_{\alpha_i} \frac{dg_1}{g_1} \int_{\beta_i} \frac{dg_2}{g_2} - \int_{\beta_i} \frac{dg_1}{g_1} \int_{\alpha_i} \frac{dg_2}{g_2} \right).
\]

Note that the reciprocity law for the logarithm of the Tate symbol (Theorem 0.13) resembles the reciprocity law for the logarithm of the refinement of the Parshin symbol (Theorem 0.6), when we look at the right hand side of the corresponding equalities. However, in the reciprocity law for the logarithm of the Parshin symbol (Theorem 0.10) the right hand side is simply zero, which is different from the reciprocity law for the logarithm of the Tate symbol.

The refinement of the Parshin symbol can be defined over any field \( K \). However, our proof of the reciprocity law works only over a field \( K \) of characteristic zero. Let us explain how we relate the field \( K \) to the field of complex numbers \( \mathbb{C} \). First, the rational numbers \( \mathbb{Q} \) can be embedded in \( K \). The surface \( X \) over \( K \) is defined by finitely many polynomials over \( K \). We can adjoin all the coefficient of these polynomials to \( \mathbb{Q} \). Then we obtain a finitely generated algebra over \( \mathbb{Q} \). Denote its field of fractions by \( K_0 \). Note that \( X \) can be defined over \( K_0 \). Since the field \( K_0 \) is of finite transcendence degree over \( \mathbb{Q} \), we can embed \( K_0 \) in \( \mathbb{C} \). Let \( \bar{K}_0 \) be the algebraic closure of \( K_0 \). Since \( \mathbb{C} \) is algebraically closed, we have that \( \bar{K}_0 \subset \mathbb{C} \). Now we can use integration over the complex numbers to define the refinement. After we have defined the refinement of the Parshin symbol, we notice that its values are in \( \bar{K}_0 \). The reciprocity law is a product over all points of intersection of \( C_0 \) with the remaining components of the divisors. If we consider first the product over the conjugate points with respect to the Galois group \( \text{Gal}(\bar{K}_0/K_0) \), we obtain a symbol which is defined over \( K_0 \).

We have a higher dimensional analogue of the new symbol and a reciprocity law for it. It will appear in another paper. Also, my student Zhenbin Luo [L] has generalized the new symbol on a surface over a nilpotent extension of the complex numbers. Luo gives a generalization of Contou-Carrere symbol to surfaces. An alternative generalization of Contou-Carrere symbol to surfaces is given by Romo [R].
An alternative approach to the Tate symbol and the Parshin symbol come from logarithmic functionals (see [Kh]). The direction we take is closer to the one in the papers by Deligne [D] and Brylinski and McLaughlin [BrMc1]. The common point is that certain connection, in geometric sense, is the key structure. The difference in our situation is that the connection is not flat.

It will be interesting to find characteristic classes that give the refinement of the Parshin symbol. For analogues of the Parshin symbol in terms of characteristic classes (see [BrMc2]). The use of such more conceptual approach might give a proof for the reciprocity law of the new symbol for a surface over a finite field. Note that the refinement of the Parshin symbol can be defined in the same way for a surface over a finite field. There must be a version of the new symbol in the arithmetic case for curves over a number ring.

Now let us say a few words about the structure of the paper. In Section 1 we recall basic properties of iterated integrals in the sense of Chen. The section ends with analogues of the Stokes formula for iterated integrals in dimension 2 and 3 (Theorems 1.5 and 1.9).

In Section 2, we make the key geometric construction, which is a foundation for the rest of the paper. Besides the geometric construction, Subsection 2.1 contains most of the definitions needed for the rest of the paper. In Subsection 2.2, we construct an abstract reciprocity law, which we use in the later sections in order to prove the explicit reciprocity laws, which we stated in the introduction.

In Section 3, we combine the abstract reciprocity from Section 2 and the properties of iterated integrals, written in Section 1, in order to obtain a new logarithmic symbol $\text{Log}(f_1, f_2, f_3)^{x_0, \gamma}_{C_0, P}$. This is the first logarithmic symbol that we construct. The other symbols are constructed later in the paper. Also, we prove a reciprocity law for the logarithm of the new symbol. This logarithmic symbol is obtained as a limit of an iterated integral over a particular loop. A key part of this construction is a differential equation written in Subsection 3.3. This differential equation has no local solutions. The way we should solve the differential equation is by restricting it to a path. One may think of it as a connection, which is not flat. A solution of this differential equation is a generating series of iterated integrals. However, we do not need the whole generating series. We need only specific term of it, which correspond to one iterated integral. This is where the analogy with a non-flat connection breaks. For that reason we call it a differential equation.

In Section 4, we construct a logarithm of the Parshin symbol and prove a reciprocity law for it. Besides many computations of iterated integrals based on the definitions in Subsection 2.4, we use one of the differential equations from subsection 3.3. In that differential equation we put an equivalence among the formal non-commuting variables. This technical condition corresponds to considering a linear combination of iterated integrals as opposed to one iterated integral. A particular linear combination of iterated integrals gives us the logarithm of the Parshin symbol $\text{Log}(f_1, f_2, f_3)^{\gamma}_{C_0, P}$, which we discussed earlier in the introduction. From the logarithm of the Parshin symbol, we can recover the Parshin symbol by exponentiation.

In Section 5, we define the logarithm of the refinement of the Parshin symbol as a difference

$$
\text{Log}(f_1, f_2, f_3)^{x_0, \gamma}_{C_0, P} = \text{Log}(f_1, f_2, f_3)^{x_0, \gamma}_{C_0, P} - \text{Log}(f_1, f_2, f_3)^{\gamma}_{C_0, P}.
$$
Using the reciprocity laws for the other two symbols, we obtain a reciprocity law for the logarithm of the refinement of the Parshin symbol. After exponentiation of the logarithm of the refinement of the Parshin symbol, we obtain a refinement of the Parshin symbol, which is roughly speaking $1/3$ of the Parshin symbol. We end the paper with an example of the reciprocity law for the (multiplicative) refinement of the Parshin symbol.

Acknowledgments. I would like to thank Alexey Parshin and Pierre Deligne for the useful commentaries on the earlier version of the paper. I would like to thank Ivan Fesenko for fruitful conversation on tame symbols. Also, I would like to thank Zhenbin Luo for lengthy discussions on my approach as well as to thank Anton Deitmar for the interest in this paper.

I would like to thank the University of Durham for the kind hospitality and also to thank the Arithmetic Algebraic Geometry Marie Curie Network for the financial support. I would like to thank both Brandeis University, where this work was developed and Universität Tübingen, where this work was finished.

1 Background on iterated integrals

In this section we recall known properties of iterated integrals, which we are going to use heavily in the rest of the paper. One can look at [Ch] and [G] for more properties of iterated integrals. We include this section, because it is essential for the rest of the paper. This section establishes both the notation and the main properties of iterated integrals, which we are going to use throughout the paper.

1.1 Definition of iterated integrals over a path

Definition 1.1 Let $\omega_1, \ldots, \omega_n$ be holomorphic $1$-forms on a simply connected open subset $U$ of the complex plane $\mathbb{C}$. Let

$$
\gamma : [0, 1] \to U
$$

be a path. We define an iterated integral of the forms $\omega_1, \ldots, \omega_n$ over the path $\gamma$ to be

$$
\int_{\gamma} \omega_1 \circ \ldots \circ \omega_n = \int_{0}^{1} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \ldots \wedge \gamma^* \omega_n(t_n).
$$

It is called iterated because it can be defined inductively by

$$
\int_{\gamma} \omega_1 \circ \ldots \circ \omega_n = \int_{0}^{1} (\int_{[0, t]} \omega_1 \circ \ldots \circ \omega_{n-1}) \gamma^* \omega_n(t).
$$

1.2 Differential equation and generating series of iterated integrals. One-dimensional case

When we consider an iterated integral, we can let the end point vary in a small neighborhood. Then the iterated integral becomes an analytic function.

Let $\omega_1, \ldots, \omega_n$ be differentials of 3rd kind on a Riemann surface $X$. Following the idea of Manin [M], we consider the differential equation

$$
dF = F \sum_{i=1}^{n} A_i \omega_i,
$$

where $A_i$ are constants and $\omega_i$ are differentials of 3rd kind.
where $A_1, \ldots, A_n$ are non-commuting formal variables, which commute with the differentiation $d$. Let $P$ be a point of $X$ such that none of the differential forms has a pole at $P$. It is easy to check that the function

$$F(z) = 1 + \sum_i (A_i \int_P^z \omega_i) + \sum_{i,j} A_i A_j \int_P^z \omega_i \circ \omega_j + \sum_{i,j,k} A_i A_j A_k \int_P^z \omega_i \circ \omega_j \circ \omega_k + \ldots.$$  

is a solution to the differential equation with initial condition $F(P) = 1$ at the point $P$. The summation continues so that every iterated integral of the given $n$ 1-forms is present in the summation. Note that

$$d \int_P^z \omega_{i_1} \circ \ldots \circ \omega_{i_k-1} \circ \omega_{i_k} = \left( \int_P^z \omega_{i_1} \circ \ldots \circ \omega_{i_k-1} \right) \omega_{i_k}.$$  

Note that the coefficient of the above integrals in the solution $F$ is

$$A_{i_1} \ldots A_{i_k-1} A_{i_k},$$

whose indices enumerate the order of iteration of the differential forms. Each of the indices $i_1, \ldots, i_k$ is an integer among $\{1, 2, \ldots, n\}$ and repetitions of indices is allowed.

### 1.3 Multiplication formulas

We can take a path $\gamma$ from $P$ to $z$. We denote the solution of the differential equation by $F_\gamma$. If $\gamma_1$ is a path that ends at $Q$ and $\gamma_2$ is a path that starts at $Q$ we can compose them. Denote the composition by $\gamma_1 \gamma_2$.

**Theorem 1.2 (Composition of paths)** With the above notation, we have

$$F_{\gamma_1} F_{\gamma_2} = F_{\gamma_1 \gamma_2}.$$  

**Corollary 1.3 (Composition of paths)** On the level of iterated integrals we have

$$\int_{\gamma_1 \gamma_2} \omega_{i_1} \circ \ldots \circ \omega_{i_n} = \sum_{i=0}^n \int_{\gamma_1} \omega_{i_1} \circ \ldots \circ \omega_{i_i} \int_{\gamma_2} \omega_{i_{i+1}} \circ \ldots \circ \omega_{i_n},$$  

where for $i = 0$ we define $\int_{\gamma_1} \omega_{i_1} \circ \ldots \circ \omega_{i} = 1$, and similarly, for $i = n$ we define $\int_{\gamma_2} \omega_{i_{i+1}} \circ \ldots \circ \omega_{i_n} = 1$.

### 1.4 Two-dimensional iterated integral over a surface

In this subsection we recall certain type of Chen’s iterated integrals [Ch], which we call 2-dimensional iterated integrals. These integrals are defined by integration over a (real) 2-dimensional region of a variety. After we recall the definition and give notation, we state an analogue of Stokes theorem for 2-dimensional iterated integrals, which expresses 1-dimensional iterated integrals over the boundary of a two dimensional region in terms of 2-dimensional iterated integrals over the same region.
Let $X$ be a smooth complex manifold of dimension at least 2. Let $\omega_1, \ldots, \omega_n$ be closed holomorphic differential 1-forms on $X$. Let $\omega^{(2)}$ be a closed holomorphic differential 2-form on $X$. Let

$$
\gamma_\bullet : [0,1] \times [0,1] \to X
$$

be a homotopy of the path

$$
\gamma_0 : [0,1] \to X
$$

and

$$
\gamma_1 : [0,1] \to X,
$$

which fixes the end points, such that $\gamma_\bullet(t, 0) = \gamma_0(t)$ and $\gamma_\bullet(t, 1) = \gamma_1(t)$. Define also a domain

$$
\Delta^n = \{(t_1, \cdots, t_n, s) \in [0,1]^{n+1} | 0 \leq t_1 \leq \cdots \leq t_n, 0 \leq s \leq 1\}.
$$

**Definition 1.4** A 2-dimensional iterated integral over $\gamma_\bullet$ is defined by

$$
I^i_\bullet = \int \int_{\gamma_\bullet} \omega_1 \circ \cdots \circ \omega_{i-1} \circ \omega^{(2)} \circ \omega_{i+1} \circ \cdots \circ \omega_n = 
$$

$$
= \int_{\Delta^n} \gamma^*_\bullet \omega_1(t_1, s) \wedge \cdots \wedge \gamma^*_\bullet \omega_{i-1}(t_{i-1}, s) \wedge \gamma^*_\bullet \omega^{(2)}(t_i, s) \wedge \gamma^*_\bullet \omega_{i+1}(t_{i+1}, s) \wedge \cdots \wedge \gamma^*_\bullet \omega_n(t_n, s).
$$

For fixed $s$ let

$$
I_s = \int_{\gamma_s} \omega_1 \circ \cdots \omega_n.
$$

Now let us recall Stokes theorem for two dimensional iterated integrals.

**Theorem 1.5** Let

$$
\tilde{\omega}^{(2)}_i = \omega_i \wedge \omega_{i+1}.
$$

The 2-form $\tilde{\omega}^{(2)}_i$ will be used in the definition of $I^i_\bullet$. Then

$$
I_0 - I_1 = \sum_{i=1}^{n-1} I^i_\bullet.
$$

1.5 Slicing a membrane

In this subsection, we examine what happens to the 2-dimensional iterated integrals, when the 2-dimensional region is cut into two pieces. We write an analogue of Theorem 1.2 and Corollary 1.3 for 1-dimensional iterated integrals, when the path is cut in two pieces. Both of these formulas are given on the level of iterated integrals and on the level of generating series of iterated integrals.

Consider the square $[0,1] \times [0,1]$. We will separate it into two domains in the following way.

Let $\gamma_0$ be the path that follows the lower edge and the right edge of the square. Namely,

$$
\gamma_0 : [0,2] \to [0,1]^2
$$

$$
\gamma_0(t) = \begin{cases} 
(t, 0) & \text{for } 0 \leq t \leq 1 \\
(1, t-1) & \text{for } 1 \leq t \leq 2
\end{cases}
$$
Let \( \gamma_1 \) be the path that follows the left edge and the upper edge of the square. Namely,

\[
\gamma_1 : [0, 2] \to [0, 1]^2
\]

\[
\gamma_1(t) = \begin{cases} 
(0, t) & \text{for } 0 \leq t \leq 1 \\
(t - 1, 1) & \text{for } 1 \leq t \leq 2 
\end{cases}
\]

We want to consider a homotopy \( \gamma_s \) between the two paths \( \gamma_0 \) and \( \gamma_1 \). Let

\[
\gamma_s(t) = \begin{cases} 
(0, t) & \text{for } 0 \leq t \leq s \\
(t - s, s) & \text{for } s \leq t \leq 1 + s \\
(1, t - 1) & \text{for } 1 + s \leq t \leq 2
\end{cases}
\]

Consider the two rectangles \( S_0 = [0, 1/2] \times [0, 2] \) and \( S_1 = [1/2, 1] \times [0, 2] \), where \( s \) varies in the first interval and \( t \) varies in the second interval. Let \( \sigma_0(t) = (t, 0) \) for \( 0 \leq t \leq 1/2 \) and \( \sigma_1(t) = (t, 1) \) for \( 1/2 \leq t \leq 1 \). Then \( \gamma_s \) for \( 0 \leq s \leq 1/2 \) has domain \( S_0 \cup \sigma_1 \) and \( \gamma_s \) for \( 1/2 \leq s \leq 1 \) has domain \( \sigma_0 \cup S_1 \). We are going to write \( S_0 \sigma_1 \) instead of \( S_0 \cup \sigma_1 \), when the end of the paths in \( S_0 \) is the beginning of the path \( \sigma_1 \).

Let

\[
\gamma_\bullet(s, t) = \gamma_s(t).
\]

Then we can slice the square \( [0, 1] \times [0, 1] \) into the following two domains:

\[
\gamma_\bullet|S_0 \sigma_1
\]

and

\[
\gamma_\bullet|\sigma_0 S_1.
\]

Let also

\[
S = S_0 \sigma_1 \cup \sigma_0 S_1.
\]

As always we denote by \( s \) the variable, which parameterizes the variation of paths. Note that in the 2-dimensional iterated integrals the variables \( t_1, t_2, \cdots \) cannot be permuted. However, in direction of increasing \( s \), we have “commutativity” in the sense that we can integrate first \( s \) in the interval \([1/2, 1]\) and then add this to the integral of \( s \) in the interval \([0, 1/2]\), where in both integrals we have the same domain for \( t_1, t_2, \cdots \). Let

\[
\tau_\bullet : S \to X
\]

be a variation of paths on \( X \) and let \( \omega_1, \cdots, \omega_n \) be closed 1-forms on \( X \). Let

\[
f_i dt = \tau_\bullet^* \omega_i
\]

on \( S \). Then we have the following lemma.

**Lemma 1.6**

\[
\left( \int_S - \int_{S_0 \sigma_1} - \int_{\sigma_0 S_1} \right) f_1 dt_1 \wedge \cdots \wedge f_{i-1} dt_{i-1} \wedge (f_i f_{i+1} ds \wedge t_i) \wedge f_i dt_{i+2} \wedge \cdots \wedge f_n dt_n = 0.
\]
1.6 Composition of a path and a 2-dimensional region

Let $\gamma_\bullet$ be a homotopy of paths, fixing the end points. The 2-dimensional region is the one covered by $\gamma_\bullet$. Let $\sigma$ be a path, whose starting point is the ending point of the paths in $\gamma_\bullet$. Let $\gamma_\bullet \sigma$ be the composition of a path from $\gamma_\bullet$ with the path $\sigma$. We want to express a 2-dimensional iterated integral over $\gamma_\bullet \sigma$ in terms of iterated integrals over $\sigma$ and 2-dimensional iterated integrals over $\gamma_\bullet$.

These integrals are related by the following theorem.

**Theorem 1.7** Let $\omega_1, \ldots, \omega_n$ be closed 1-forms. Then

$$\int_{\gamma_\bullet \sigma} \omega_1 \circ \cdots \circ (\omega_i \wedge \omega_{i+1}) \circ \cdots \circ \omega_n = \sum_{j=i+1}^n \int_{\gamma_\bullet} \omega_1 \circ \cdots \circ (\omega_i \wedge \omega_{i+1}) \circ \cdots \circ \omega_j \int_{\sigma} \omega_{j+1} \circ \cdots \circ \omega_n.$$  

**Proof.** The proof is the same as in the case of 1-dimensional iterated integrals. The only difference is that if $j < i$ then the domain for the first integral will have dimension $j + 1$, which is larger than the number of differential 1-forms, which is $j$. Thus, the first integral will be zero.

1.7 Three-dimensional iterated integrals

In this section we recall certain type of Chen’s iterated integrals, which we call 3-dimensional iterated integrals. These integrals are defined by integration over a (real) 3-dimensional region of a variety. After we recall the definition and give notation, we state an analogue of Stokes theorem for 3-dimensional iterated integrals, which expresses 2-dimensional iterated integrals over the boundary of a 3-dimensional region in terms of 2-dimensional iterated integrals over the same region.

Let $X$ be a smooth complex manifold of dimension at least 3. Let $\omega_1, \ldots, \omega_n$ be closed holomorphic differential 1-forms on $X$. Let $\omega_1^{(2)}, \ldots, \omega_n^{(2)}$ be closed holomorphic differential 2-forms on $X$. Let $\omega^{(3)}$ be a closed holomorphic differential 2-form on $X$. Let $\gamma_\bullet_0 : [0, 1] \times [0, 1] \to X$ and $\gamma_\bullet_1 : [0, 1] \times [0, 1] \to X$ be two homotopies of paths, which fix the end points. Let $\gamma_\bullet_\bullet : [0, 1]^3 \to X$ be a homotopy of the two homotopies of the path $\gamma_\bullet_0 : [0, 1]^2 \to X$ and $\gamma_\bullet_1 : [0, 1]^2 \to X$, which fixes the end points, such that $\gamma_\bullet_\bullet(t, s_1, 0) = \gamma_\bullet_0(t, s_1)$ and $\gamma_\bullet_\bullet(t, s_1, 1) = \gamma_\bullet_1(t, s_1)$. Define also the following homotopies of paths $\gamma_\bullet_0(t, s_2) = \gamma_\bullet(t, 0, s_2)$
and
\[ \gamma_1(t, s_2) = \gamma_1(t, s_2). \]

Define also a domain
\[ \Delta^n_{\bullet \bullet} = \{(t_1, \ldots, t_n, s_1, s_2) \in [0,1]^{n+2} | 0 \leq t_1 \leq \cdots \leq t_n, 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\}. \]

We will omit the variables \( s_1 \) and \( s_2 \) from the notation of \( \gamma_1 \omega_i(s_1, s_2, t_i), \gamma_2 \omega_i^{(2)}(s_1, s_2, t_i) \) and \( \gamma_3 \omega_i^{(3)}(s_1, s_2, t_i) \). We will write \( \gamma_1 \omega_i(t_i), \gamma_2 \omega_i^{(2)}(t_i) \) and \( \gamma_3 \omega_i^{(3)}(t_i) \), respectively, in their place.

**Definition 1.8** A 3-dimensional iterated integral over \( \gamma_{\bullet \bullet} \) can be of two types. One of the types is defined by
\[
I_{\bullet \bullet}^{i,i+1} = \int \int \omega_1 \cdots \omega_{i-1} \circ \omega^{(3)}(i) \circ \omega_{i+1} \cdots \omega_n =
\]
\[
= \int_{\Delta^n_{\bullet \bullet}} \gamma_1 \omega_1(t_1) \land \cdots \land \gamma_2 \omega_{i-1}(t_{i-1}) \land \gamma_3 \omega^{(3)}(t_i) \land \gamma_4 \omega_{i+1}(t_{i+1}) \land \cdots \land \gamma_3 \omega_n(t_n).
\]

And the other type is defined for \( i < j - 1 \)
\[
I_{\bullet \bullet}^{i,j} = \int \int \omega_1 \circ \cdots \circ \omega_{i-1} \circ \omega^{(2)}(i) \circ \omega_{i+1} \circ \cdots \circ \omega_{j-1} \circ \omega^{(2)}(j) \circ \omega_{j+1} \circ \cdots \circ \omega_n =
\]
\[
= \int_{\Delta^n_{\bullet \bullet}} \gamma_1 \omega_1(t_1) \land \cdots \land \gamma_2 \omega_{i-1}(t_{i-1}) \land \gamma_3 \omega^{(2)}(i) \land \gamma_4 \omega_{i+1}(t_{i+1}) \land \cdots \land \gamma_2 \omega_{j-1}(t_{j-1}) \land \gamma_3 \omega^{(2)}(j) \land \gamma_4 \omega_{j+1}(t_{j+1}) \land \cdots \land \gamma_3 \omega_n(t_n).
\]

And set
\[ I_{\bullet \bullet}^{j,i} = I_{\bullet \bullet}^{i,j}. \]

Now let us recall Stokes theorem for 3-dimensional iterated integrals. Let
\[ \bar{\omega}_j^{(2)} = \omega_j \land \omega_{j+1}. \]

Let
\[ \bar{\omega}_i^{(3)} = \omega_i^{(2)} \land \omega_{i+1} \]

and let
\[ \bar{\omega}_{i-1}^{(3)} = \omega_{i-1} \land \omega_i^{(2)} \]

**Theorem 1.9** The forms \( \bar{\omega}_j^{(2)}, \bar{\omega}_i^{(3)} \) and \( \bar{\omega}_{i-1}^{(3)} \) will enter in the definition of the 3-dimensional iterated integrals. The 2-form \( \omega_i^{(2)} \) will enter in the definition of the 2-dimensional iterated integrals. Then
\[
I_{\bullet \bullet}^i - I_{1 \bullet}^i + I_{\bullet 0}^i - I_{1 1}^i = 2 \sum_{j \neq i} I_{\bullet \bullet}^{j,i}.
\]

We will use a particular case of this theorem, which we will formulate in a corollary.

**Corollary 1.10** Over a two dimensional complex manifold, we have
\[
I_{0 \bullet}^i - I_{1 \bullet}^i + I_{\bullet 0}^i - I_{1 1}^i = 0.
\]
2 Abstract reciprocity law

2.1 Definitions

Consider $X$ a projective smooth surface over the complex numbers. Let $f_k$ for $k = 1, 2, 3$ be rational functions on $X$.

**Definition 2.1 (Divisors $C_i$ and $D_{ij}$)** Denote by $C_i$ for $i = 0, 1, \ldots, N$ the components of the divisors of $f_1$, $f_2$ and $f_3$. Assume that the curves $C_i$ are non-singular and that they intersect properly. Let

$$div(f_k) = \sum_{i=0}^{N} n_{ki}C_i.$$

Let $x_i$ be a rational function on $X$, which has zero along $C_i$ of order 1, and has no zeroes or poles along $C_j$ for $j \neq i$. Let

$$div(x_i) = C_j + \sum_{j=1}^{M_i} m_{ij}D_{ij}.$$

We choose $x_i$ for $i = 0, 1, \ldots, M$ so that $D_{ij} \neq D_{i'j'}$ for $(i, j) \neq (i', j')$. Let $y_{ij}$ be a rational function on $X$, which has zero along $D_{ij}$ of order 1, and has no zeroes or poles along $C_{i'}$ for $i' = 0, 1, \ldots, M$ and along $D_{i'j'}$ for $(i', j') \neq (i, j)$.

We will examine what happens along the curve $C_0$. Choose a small positive number $\epsilon$. Let $0 < \epsilon_0 < \epsilon$ and $0 < \epsilon_1 < \epsilon$. We will define a tubular neighborhood of radius $\epsilon_0$ around the following space: the curve $C_0$ minus $\epsilon_1$ neighborhoods of the intersections of $C_0$ with $C_j$ for $j > 0$ and with $D_{ij}$ for all $i$ and $j$. Near the intersection points we will define tori. We are going to make precise what means $\epsilon_0$ or $\epsilon_1$ neighborhood. We need this 2-dimensional region in order to examine iterated integrals over it. Afterwards, we are going to cut this 2-dimensional region so that topologically it looks like a rectangle, where each side is a path on $X$, which lies on an algebraic curve.

**Definition 2.2 (Intersection points $P_{jl}$ and $P_{1jl}$)** Let

$$\bigcup_{j=1}^{N} P_{jl} = C_1 \cap C_j$$

and let

$$\bigcup_{j=1}^{M_1} P_{1jl} = C_1 \cap D_{1j}.$$

We will construct compact neighborhoods, which separate the points $P_{jl}$ and $P_{1jl}$.

**Definition 2.3 (Neighborhoods $\tilde{T}_{jl}$ and $\tilde{T}_{1jl}$ of the points $P_{jl}$ and $P_{1jl}$)** Let

$$E_j = \{P \in X ||x_0(P)|| < \epsilon_0 \text{ and } |x_j(P)| < \epsilon_1 \}.$$  

Define

$$E_{1j} = \{P \in X \mid \left| \frac{x_0(P)}{y_{1j}(P)} \right| < \frac{\epsilon_0}{m_{ij}} \text{ and } |y_{1j}(P)| < \epsilon_1 \}.$$  

Denote by $\tilde{T}_{jl}$ the connected component in $E_j$ around the point $P_{jl}$. Also, denote by $\tilde{T}_{1jl}$ the connected component in $E_{1j}$ around the point $P_{1jl}$.
We will find relations between the exponents \(m_{1j}\) and the small constants \(\epsilon_{1j}\) so that the set \(E_{1j}\) consists of small compact sets.

If \(m_{1j} > 0\) then we want \(\frac{m_{1j}}{\epsilon_{1j}} < \epsilon\) and \(\epsilon_{1j} < \epsilon\). This can be achieved by taking

\[
\epsilon_0 < \epsilon^{1+m_{1j}}
\]

and

\[
\epsilon_{1j}^{m_{1j}} < \min(\epsilon^{m_{1j}}, \frac{\epsilon_0}{\epsilon}).
\]

If \(m_{1j} < 0\) then we want \(\frac{\epsilon_0}{\epsilon_{1j}} < \epsilon\) and \(\epsilon_{1j} < \epsilon\). This can be achieved by taking

\[
\epsilon_0 < \epsilon
\]

and

\[
\epsilon_{1j} < \epsilon.
\]

Globally, we can choose \(\epsilon_0\) so that \(\epsilon_0 < \epsilon^{1+\max m_{1j}}\), where the maximum is taken over all \(j\)'s such that \(m_{1j} > 0\). If all \(m_{1j} < 0\) then choose \(\epsilon_0 < \epsilon\). For \(\epsilon_{1j}\), we choose \(\epsilon_{1j} < \epsilon\) if \(m_{1j} < 0\) and \(\epsilon_{1j}^{m_{1j}} < \min(\epsilon^{m_{1j}}, \frac{\epsilon_0}{\epsilon})\).

For fixed \(j\), we can choose \(\epsilon\) small enough so that the compact sets \(E_j\) is a disjoint union of neighborhoods - one for each point \(P_{jl}\). Similarly, for fixed \(j\), we can choose \(\epsilon\) small enough so that the compact sets \(E_{1j}\) is a disjoint union of neighborhoods - one for each point \(P_{1jl}\). Pick the smallest values of \(\epsilon\) among the above choices. We can choose \(\epsilon\) even smaller, so that the compact neighborhoods \(\tilde{T}_{jl}\) for all \(j\) and \(l\) and the compact neighborhoods \(\tilde{T}_{1jl}\) for all \(j\) and \(l\) are disjoint.

**Definition 2.4** *(Tubular neighborhood)* Let

\[
Tb = \{P \in X | x_1(P) = u\} - \bigcup_{j,l} \tilde{T}_{jl} - \bigcup_{j,l} \tilde{T}_{1jl}.
\]

**Definition 2.5** *(Foliation of the tubular neighborhood)* Let \(u\) be a complex number such that \(|u| \leq \epsilon_0\). Define

\[
Tb_u = \{P \in X | x_1(P) = u\}.
\]

Note that \(Tb_0 \subset C_0\).

**Definition 2.6** Define

\[
D_{\epsilon_0} = \{u \in \mathbb{C} | |u| < \epsilon_0\}.
\]

**Lemma 2.7** *(fibration)* For all \(u < \epsilon_0\), the sequence of topological space

\[
Tb_u \to Tb \to D_{\epsilon_0}
\]

is a fibration, which splits, since \(D_{\epsilon_0}\) is contractible.

**Proof.** The map

\[
x_0 : X \to \mathbb{C}
\]

is a restriction of the map

\[
\bar{x}_0 : Tb \to D_{\epsilon_0}
\]
which is singular only at finitely many points in the range $D_{\epsilon_0}$. Thus, by decreasing $\epsilon_0$, we can assume that $\bar{x}_0$ is singular at most at one point of $D_{\epsilon_0}$. If it turns out to be non-singular then we have a fibration.

Assume the map $\bar{x}_0$ has one singular point $u_0$ in $D_{\epsilon_0}$. If $u_0 \neq 0$ then we can decrease $\epsilon_0$ so that $\epsilon_0 < u_0$. Then $\bar{x}_0$ will not have a singular point.

Assume that $\bar{x}_0$ is singular at $0 \in D_{\epsilon_0}$. Consider the boundaries of $Tb_0$ and $Tb_u$. Note that the connected components of their boundaries are circles. Also, each of the two spaces have the same number of connected components of their boundaries. We claim that small neighborhoods of their boundaries are homotopic. Let $\epsilon'$ be a small positive real number. Then connected components of the domains

$$\{P \in Tb_u | \epsilon_1 < \|x_j(P)\| < (1 + \epsilon')\epsilon_1\}$$

and

$$\{P \in Tb_0 | \epsilon_1 < \|x_j(P)\| < (1 + \epsilon')\epsilon_1\}$$

are cylinders, corresponding to the points $P_{jl}$ for admissible values of $l$. Similarly, connected components of the domains

$$\{P \in Tb_u | \epsilon_{1j} < \|y_{1j}(P)\| < (1 + \epsilon')\epsilon_{1j}\}$$

and

$$\{P \in Tb_0 | \epsilon_{1j} < \|y_{1j}(P)\| < (1 + \epsilon')\epsilon_{1j}\}$$

are cylinders, corresponding to the points $P_{1jl}$ for admissible values of $l$. By decreasing $u$ we obtain a degeneration of the topological surface $Tb_u$ to $Tb_0$. Now one can use Riemann-Hurwitz Theorem. Since the number of connected components of the boundaries of $Tb_0$ and $Tb_u$ are the same and also small neighborhoods of the boundaries are homotopic, we obtain that the topological surfaces $Tb_0$ and $Tb_u$ have the same genus. Thus, $Tb_0$ and $Tb_u$ are homotopic. This proves that we have a fibration map $\bar{x}_0$. A direct consequence of the above lemma is the following corollary.

**Corollary 2.8** *(homotopy)* For all $u < \epsilon_0$ there is a homotopy map

$$h : D_{\epsilon_0} \times Tb_0 \to Tb,$$

such that

$$h(u, Tb_0) = Tb_u.$$

**Definition 2.9** *(Points, loops and paths on $Tb_0$)* Let $Q \in Tb_0$ be an interior point. Let $R_{jl}$ and $R_{1jl}$ be base points on each of the connected components of the boundary of $Tb_0$. We choose $R_{jl}$ and $R_{1jl}$ so that they are close to the intersection points $P_{jl}$ and $P_{1jl}$.

Let $\sigma_{jl}^0$ be a loop on $Tb_0$ defined by

$$\sigma_{jl}^0 : [0, 1] \to T_{jl}$$

$$t \mapsto (x_0, x_j)^{-1}(0, \epsilon_1 e^{2\pi it}),$$

starting at $R_{jl} = (x_0, x_j)^{-1}(0, \epsilon_1)$. Let also $\sigma_{1jl}^0$ be a loop on $Tb_0$ defined by

$$\sigma_{1jl}^0 : [0, 1] \to T_{1jl}$$
\[ t \mapsto (x_0, y_{1j})^{-1}(0, \epsilon_{1j}e^{2\pi it}), \]

starting at \( R_{1jl} = (x_0, x_j)^{-1}(0, \epsilon_{1j}) \).

Let \( \gamma_{jl} \) be a path from \( Q \) to \( R_{jl} \) and let \( \gamma_{1jl} \) be a path from \( Q \) to \( R_{1jl} \).

Denote by

\[ \sigma_{jl} = \gamma_{jl}\sigma_0^{jl}
\]

and

\[ \sigma_{1jl} = \gamma_{1jl}\sigma_0^{1jl}. \]

Choose the paths \( \gamma_{jl} \) and \( \gamma_{1jl} \) so that the loop \( \delta \), defined by

\[ \delta = \left( \prod_{i=1}^{g}[\alpha_i, \beta_i] \right) \left( \prod_{j=1}^{N}\sigma_{jl} \right) \left( \prod_{j=1}^{M_1}\sigma_{1jl} \right), \]

is homotopic to the trivial loop at \( Q \), where \( g \) is the genus of the curve \( Tb_0 \) and \([\alpha_i, \beta_i]\) is the commutator of loops \( \alpha_i \) and \( \beta_i \) around the handles of \( Tb_0 \). Note that the genus of the topological surface \( Tb_0 \) is the same as the genus of the curve \( C_0 \).

**Definition 2.10 (Points, loops and paths on \( Tb_u \))** We lift the points \( Q, R_{jl}, R_{1jl} \), and the loops and paths \( \gamma_{jl}, \gamma_{1jl}, \sigma_{jl}, \sigma_{1jl}, \sigma_0^{jl}, \sigma_0^{1jl}, \alpha_i \) and \( \beta_i \) from \( Tb_0 \) to \( Tb_u \), using the homotopy \( h \) from Corollary 2.8. Let

\[ \tilde{Q} = h(u, Q) \]

\[ \tilde{R}_{jl} = h(u, R_{jl}), \quad \tilde{R}_{1jl} = h(u, R_{1jl}), \]

\[ \tilde{\gamma}_{jl} = h(u, \gamma_{jl}), \quad \tilde{\gamma}_{1jl} = h(u, \gamma_{1jl}), \]

\[ \tilde{\sigma}_{jl} = h(u, \sigma_{jl}), \quad \tilde{\sigma}_{1jl} = h(u, \sigma_{1jl}), \]

\[ \tilde{\alpha}_i = h(u, \alpha_i), \quad \tilde{\beta}_i = h(u, \beta_i) \]

on \( Tb_u \).

**Definition 2.11 (Loops around the curve \( C_0 \))** For \( u = \epsilon_0e^{2\pi it} \), we define the loops

\[ \tau(t) = h(\epsilon_0e^{2\pi it}, Q), \quad \text{starting at } \tilde{Q} \]

\[ \tau_{jl}(t) = h(\epsilon_0e^{2\pi it}, R_{jl}), \quad \text{starting at } \tilde{R}_{jl} \]

\[ \tau_{1jl}(t) = h(\epsilon_0e^{2\pi it}, R_{1jl}), \quad \text{starting at } \tilde{R}_{1jl} \]

on \( Tb \) around \( C_0 \).
**Definition 2.12** (Tori $T'_{jl}$, $T'_{1jl}$, $T_{jl}$, $T_{1jl}$ and $T_i$) For $u = \epsilon_0 e^{2\pi it}$, we define the tori $T_{jl}$ and $T_{1jl}$

\[
T'_{jl} = \{ h(\epsilon_0 e^{2\pi it}, \sigma'_{jl}(s)) | 0 \neq s \neq 1, 0 \neq t \neq 1 \}, \\
T'_{1jl} = \{ h(\epsilon_0 e^{2\pi it}, \sigma'_{1jl}(s)) | 0 \neq s \neq 1, 0 \neq t \neq 1 \}, \\
T_{jl} = \{ h(\epsilon_0 e^{2\pi it}, \sigma_{jl}(s)) | 0 \neq s \neq 1, 0 \neq t \neq 1 \}, \\
T_{1jl} = \{ h(\epsilon_0 e^{2\pi it}, \sigma_{1jl}(s)) | 0 \neq s \neq 1, 0 \neq t \neq 1 \}, \\
T_i = \{ h(\epsilon_0 e^{2\pi it}, [\alpha_i, \beta_i](s)) | 0 \neq s \neq 1, 0 \neq t \neq 1 \},
\]

where $T'_{jl}$ and $T'_{1jl}$ are tori near the points $P_{jl}$ and near $P_{1jl}$, respectively.

**Definition 2.13** ('boundary' of the Tori $T_{jl}$, $T_{1jl}$ and $T_i$) By a boundary of any of the above tori, we mean the loop on the torus, corresponding to the boundary of the defining region. More precisely, consider the two loops $\tilde{\sigma}'_{jl}$ and $\tau_{jl}$. They are generators of the torus $T_{jl}$. Cut the torus $T_{jl}$ along both loops $\tilde{\sigma}'_{jl}$ and $\tau_{jl}$. We obtain a square. By a boundary of the torus $T_{jl}$, we mean the boundary of this square. Algebraically, this means that the boundary is a commutator. Explicitly,

\[
\partial T'_{jl} = [\tilde{\sigma}'_{jl}, \tau_{jl}], \text{ starting at } \tilde{R}_{jl}, \\
\partial T'_{1jl} = [\tilde{\sigma}'_{1jl}, \tau_{jl}], \text{ starting at } \tilde{R}_{1jl}, \\
\partial T_{jl} = [\tilde{\sigma}_{jl}, \tau_{jl}], \text{ starting at } \tilde{Q}, \\
\partial T_{1jl} = [\tilde{\sigma}_{1jl}, \tau_{jl}], \text{ starting at } \tilde{Q}, \\
\partial T_i = [[\tilde{\alpha}_i, \tilde{\beta}_i], \tau], \text{ starting at } \tilde{Q}.
\]

**Definition 2.14** For fixed $u$ let $\mu$ be a loop on $T_{b\mu}$, starting at the point $\tilde{Q}$, defined as a map

\[
\mu : [0, 1] \to T_{b\mu}.
\]

Consider the torus $T$ as the image of $[0, 1]^2$ to $X$

\[
T : [0, 1]^2 \to X,
\]

\[
T(s_0, s_1) = h(\epsilon_0 e^{2\pi is_0}, \mu(s_1)),
\]

where $h$ is the homotopy defined in Corollary 2.8. Note that we have define two generators of the torus $T$, namely, $\mu$ and $\tau$. Similarly to Definition 2.13, we define a boundary of $T$, denoted by $\partial T$, by a commutator

\[
\partial T = [\mu, \tau].
\]
Definition 2.15 We are going to define a new homotopy of paths that parameterizes the torus \( T \) from the previous definition. For \( s \in [0,1] \) we define

\[
\gamma'_s(t) = \begin{cases} 
(st,0) & t \in [0,1] \\
(s,t-1) & t \in [1,2] \\
(s(3-t),1) & t \in [2,3] \\
(0,4-t) & t \in [3,4] 
\end{cases}
\]

Note that for fixed \( s \in (0,1) \) the path \( \gamma'_s \) starts at \((0,0)\) and continues along the boundary of the rectangle with sides \( s \) and \( 1 \). In the limit \( s = 1 \) the path \( \gamma'_1 \) becomes the boundary of the square \([0,1]^2\). And for \( s = 0 \) the path \( \gamma'_0 \) becomes just one edge in direction \( x_2 \) with length 1. Let \( \gamma(s,t) = \gamma'_s(t) \). The parametrization of the torus \( T \) is given by \( \gamma = T \circ \gamma'_s \), where \( \circ \) denotes a composition of functions. Define also \( \gamma_s = T \circ \gamma'_s \).

2.2 Construction of an abstract reciprocity law

For the torus \( T \), we have the following non-commutative Stokes Theorem (Theorem 1.5).

Theorem 2.16

\[
\begin{align*}
(a) \int_{\partial T} \frac{df_1 \circ df_2}{f_1} &= \int_T \frac{df_1}{f_1} \wedge \frac{df_2}{f_2}; \\
(b) \int_{\partial T} \frac{df_1 \circ df_2 \circ df_3}{f_1} &= \int_T \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) + \int_T \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \circ \frac{df_3}{f_3}.
\end{align*}
\]

Proof. The integrals on the left hands side are defined in Definition 1.1, using the path \( \gamma_1 = T \circ \gamma'_1 : t \mapsto X \)

constructed in Definition 2.15. The integrals on the righthand side are defined in Definition 1.4, using the homotopy of paths \( \gamma = T \circ \gamma_1 \), constructed in Definition 2.15. Applying Theorem 1.5 to \( \gamma \), we obtain the above theorem.

Definition 2.17 (relation in the fundamental group) Let \( \delta = \left( \prod_{i=1}^g [\alpha_i,\beta_i] \right) \left( \prod_{j=1}^N \sigma_{jl} \right) \left( \prod_{j=1}^{M_1} \sigma_{1jl} \right) \), be homotopic to the trivial loop at \( Q \) in \( T_b \). Let \( \delta_{s_2} \), for \( 0 \leq s_2 \leq 1 \) be a homotopy between the loop \( \delta \) and the trivial loop at \( Q \), where the homotopy is with fixed starting point \( Q \). So that for \( s_2 = 1 \) we have \( \delta_1 = \delta \).
and for \( s_2 = 0 \) we have
\[ \delta_0 \equiv Q. \]

For each value of \( s_2 \) and for fixed \( u \) consider the loop
\[ \tilde{\delta}_{s_2} = h(u, \delta_{s_2}) \]
on \( Tb_u \), where \( h \) is the homotopy defined in Corollary 2.8. Note that \( \tilde{\delta}_{s_2} \) gives a homotopy sitting in \( Tb_u \) between \( \tilde{\delta}_1 \) and the trivial loop at \( Q \).

**Theorem 2.18** Using Definition 2.17 of \( \tilde{\delta}_1 \), by taking \( s_2 = 1 \), and Definition 2.11 of \( \tau \), we have
\[ \int_{[\tilde{\delta}_1, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 0. \]

**Remark 2.19** Theorem 2.18 will be interpreted later as the sum of the logarithmic symbols is an integer multiple of \((2\pi i)^3\). Where are the logarithmic symbols? The logarithmic symbols will be
\[ \int_{[\tilde{\sigma}_{jl}, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}. \]
The integrals
\[ \int_{[\tilde{\sigma}'_{jl}, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 0 \]
where we use \( \tilde{\sigma}'_{jl} \) instead of \( \tilde{\sigma}_{jl} \). And finally, the integer multiple of \((2\pi i)^3\) will come from
\[ \int_{[[\tilde{\alpha}_i, \tilde{\beta}_i], \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}. \]
Actually there is one other source of integer multiples of \((2\pi i)^3\), which will be called 'extra terms'.

We are going to prove this theorem using Theorem 1.6 and Non-commutative Stokes Theorem for iterated integrals stated in Theorem 1.9. Before doing that we have to make several definitions.

**Definition 2.20** (For 3-dimensional non-commutative Stokes Theorem) For each value of \( s_2 \) and for fixed \( u \) consider the loop
\[ \tilde{\delta}_{s_2} = h(u, \delta_{s_2}) \]
on \( Tb_u \), where \( h \) is the homotopy defined in Corollary 2.8. Now use Definition 1.14. Instead of the loop \( \mu \) use \( \tilde{\delta}_{s_2} \), in order to define a torus. Denote the corresponding torus by \( T_{s_2} \). Now use Definition 1.15 to define parametrization of the torus \( T_{s_2} \). For fixed \( s_2 \), let
\[ \gamma_{s,s_2} = T_{s_2} \circ \gamma'. \]
Similarly, for fixed \( s \) and \( s_2 \) let
\[ \gamma_{s,s_2} = T_{s_2} \circ \gamma'. \]
Now instead of the variable $s$ write the variable $s_1$. When $s_1$ denote

$$\gamma_{s_1, \bullet}(s_2, t) = \gamma_{s_1, s_2}(t).$$

Let also

$$\gamma_{\bullet, \bullet}(s_1, s_2, t) = \gamma_{s_1, s_2}(t).$$

**Proof.** (of Theorem 1.18) In the notation of Definition 1.19, the domain of integration is $\partial T_1$. It is enough to show that

$$\int \int_{T_1} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = 0$$

and

$$\int \int_{T_1} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 0.$$

Using the notation from Definition 1.19 and from subsection 1.7, we have

$$\int \int_{T_1} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \int \int_{\gamma_{s_1, \bullet}} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right).$$

From Theorem 1.9, we have

$$2 \int \int \int_{\gamma_{\bullet, \bullet}} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = \left( \int \int_{\gamma_{s_1, \bullet}} - \int \int_{\gamma_{s_0, \bullet}} + \int \int_{\gamma_{\bullet, 0}} - \int \int_{\gamma_{s_1, 0}} \right) \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right).$$

We have identified the last integral. Note also that

$$\frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = 0,$$

because $X$ is a 2-dimensional variety. The domain $\gamma_{s_0, 0}$ comes from $\tilde{s}_2$ for $s_2 = 0$, which is the trivial loop at $\tilde{Q}$. Then $\gamma_{s_0, 0}$ is a 1-dimensional region. Therefore,

$$\int \int_{\gamma_{s_0, 0}} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = 0.$$

For fixed $s_1$ the domain of integration $\gamma_{s_1, \bullet}$ lies inside $Tb_u$. Since $Tb_u$ is a subset of the algebraic curve on $X$ given by the equation $x_0 = u$, we obtain that

$$\frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = 0.$$

Therefore,

$$\int \int_{\gamma_{s_0, \bullet}} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = 0$$

and

$$\int \int_{\gamma_{s_1, \bullet}} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = 0.$$
Thus, the integral
\[ \int \int_{\gamma_1} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = 0. \]

Similarly, we can prove that
\[ \int \int_{T_1} \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \circ \frac{df_3}{f_3} = \int \int_{\gamma_1} \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \circ \frac{df_3}{f_3} = 0. \]

Now we use Theorem 2.16, and we obtain
\[ \int \int_{\partial T_1} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 0. \]

**Definition 2.21** \((\mu_i)\) Define loops \(\mu_i\) so that \(\mu_i\) is either \(\tilde{\alpha}_j, \tilde{\beta}_j\) or \(\tilde{\sigma}_{jl}\), or \(\tilde{\sigma}_{1jl}\). Let \(K\) be the number of all the above different loops. The integer \(K\) is chosen so that for the ordered product we have
\[ K \prod_{i=1}^{K} \mu_i = \tilde{\delta}_1. \]

**Definition 2.22** \((\pi_j)\) Define loops \(\pi_i\) by
\[ \pi_j = \prod_{i=1}^{j} \mu_i. \]

Note that \(\pi_1 = \mu_1\) and \(\pi_K = \tilde{\delta}_1\), which is homotopic to the trivial loop at \(\tilde{Q}\).

**Lemma 2.23** (a) \([\pi_{j+1}, \tau] = \pi_j[\mu_{j+1}, \tau] \pi_j^{-1}[\pi_j, \tau] \]
(b) \([\pi_K, \tau] = \prod_{i=1}^{K} (\pi_{K-i}[\mu_{K-i+1}, \tau] \pi_{K-i}^{-1}). \]

**Proof.** It follows by direct computation of commutators.

**Lemma 2.24**
\[ 0 = \int_{[\pi_K, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = \sum_{i=1}^{K} \int_{[\pi_{K-i}[\mu_{K-i+1}, \tau] \pi_{K-i}^{-1}]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}. \]

**Proof.** The first equality is Theorem 2.18. For the second equality, we use composition of paths in iterated integrals stated in Corollary 1.3. By induction, we can decompose the path \([\pi_K, \tau]\) into two paths, then into three paths and finally into \(K\) paths \(\pi_{K-i}[\mu_{K-i+1}, \tau] \pi_{K-i}^{-1}\) for \(i = 1, \ldots, K\). Using Corollary 1.3, we can expand the iterated integral over \([\pi_K, \tau]\) in terms of sum of products of iterated integrals over \(\pi_{K-i}[\mu_{K-i+1}, \tau] \pi_{K-i}^{-1}\) for \(i = 1, \ldots, K\). Note that
\[ \int_{[\pi_{K-i}[\mu_{K-i+1}, \tau] \pi_{K-i}^{-1}]} \omega_1 = 0 \]
for any differential 1-form \(\omega\). Therefore, when we have a product of iterated integrals in the expansion, we must have at least one integral of one differential 1-form.
That gives zero contribution. Therefore, there are no products in the expansion over the smaller paths, but only a sum of triple iterations over each of the smaller paths $\pi_{K-i}^\mu_{K-i+1, \tau} \pi_{K-i}^{-1}$. This proves the lemma. Now we are going to examine each of the components $\int_{\pi_i^\mu_{i+1, \tau} [\mu_{i+1, \tau}^{-1} f_1 \circ f_2 \circ f_3].$

Lemma 2.25

$$\int_{\pi_i^\mu_{i+1, \tau} [\mu_{i+1, \tau}^{-1} f_1 \circ f_2 \circ f_3 = \int_{[\mu_{i+1, \tau}] f_1 \circ f_2 \circ f_3}
+ \int_{\pi_i^\mu_{i+1, \tau} [\mu_{i+1, \tau}^{-1} f_1 \circ f_2 \circ f_3 + \int_{[\mu_{i+1, \tau}] f_1 \circ f_2 \circ f_3}.}

Proof. Decompose the path $\pi_i^\mu_{i+1, \tau} [\mu_{i+1, \tau}^{-1}$ into $\pi_i, [\mu_{i+1, \tau}$ and $\pi_i^{-1}$. Then use Corollary 1.3. Finally, use that

$$\int_{[\mu_{i+1, \tau}] \omega = 0$$

for any differential 1-form $\omega$.

Lemma 2.26 (a) For $i \leq g$, where $g$ is the genus of $C_0$, we have $\mu_i = [\tilde{\alpha}_i, \tilde{\alpha}_i]$. Then

$$\int_{\mu_i} \omega = 0$$

and consequently,

$$\int_{\pi_i} \omega = 0$$

for a 1 form $\omega$. (b) When $\mu_i = \tilde{\sigma}'_{jl}$, we have

$$\int_{[\mu_i, \tau]} f_1 \circ f_2 = 0$$

and

$$\int_{[\mu_i, \tau]} f_1 \circ f_2 \circ f_3 = 0.$$

Proof. Part (a) is straight forward. Part (b) will be proven in subsection 3.1.

Definition 2.27 When $\mu_i = \tilde{\sigma}'_{jl}$, we define a new logarithmic symbol as

$$\text{Log} f_1, f_2, f_3_{[\mu_i, \tau]} = \lim_{\epsilon \to 0} \int_{[\mu_i, \tau]} f_1 \circ f_2 \circ f_3.$$

Definition 2.28 For $i \leq g$, where $g$ is the genus of $C_0$, we have $\mu_i = [\tilde{\alpha}_i, \tilde{\beta}_i]$. Then we call the integral

$$\int_{[\mu_i, \tau]} f_1 \circ f_2 \circ f_3$$

the commutator $[\alpha, \beta]$-terms.
Definition 2.29 Consider the terms
\[ \int_{\pi_i} df_1 \int_{[\mu_{i+1}, \tau]} \frac{df_2}{f_2} \circ \frac{df_3}{f_3} \]
and
\[ \int_{[\mu_{i+1}, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_{\pi_i^{-1}} \frac{df_3}{f_3} \]
in Lemma 2.26. The only case when these terms do not vanish is when \( i > g \) and \( \mu_{i+1} = \tilde{\sigma}_{jl} \). In this case we call the above two integrals extra terms.

Remark 2.30 We are going to show that a commutator \([\alpha, \beta]\)-term gives an integer multiple of \((2\pi i)^3\) in Lemma 3.25 and also that the extra terms give an integer multiple of \((2\pi i)^3\) in Theorem 3.30.

Theorem 2.31 (Abstract reciprocity law)
\[ \sum_{P_{jl}} \log[f_1, f_2, f_3]_{C_0, f_{jl}}^{2\pi i Q} = - \sum (\text{commutator terms}) - \sum (\text{extra terms}). \]

Note that from the previous remark the right hand side of the above equation is an integer multiple of \((2\pi i)^3\).

Proof. For \( i \) such that \( \mu_i = \tilde{\sigma}_{jl} \), by Lemma 2.25, we have that the sum of the Log-symbols and the extra terms gives the sum of
\[ \int_{\pi_i[\mu_{i+1}, \tau] \pi_i^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}, \]
over the above type of indexes \( i \). For \( i \leq g \) we have \( \mu_i = [\tilde{\alpha}_i, \tilde{\alpha}_i] \). Then by Lemma 2.6 (a), we have
\[ \int_{[\mu, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = \int_{\pi_i-1[\mu, \tau] \pi_i^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}. \]
For \( i \) such that \( \mu_i = \tilde{\sigma}_{jl} \), by Lemma 2.26(b), we have that
\[ \int_{\pi_i-1[\mu, \tau] \pi_i^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 0. \]
We have to show that the sum of the above three integrals is zero. By Lemma 2.24, we have that this sum is equal to
\[ \int_{[\pi_K, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}, \]
where \( \pi_K = \tilde{\delta}_1 \). Now from Theorem 2.18, the above integral is zero.
3 Construction of a new symbol and a reciprocity law

3.1 Local properties. Integrating over the torus $T'_{jl}$

In this subsection, we use the notation from subsection 2.4. Consider the torus $T'_{jl}$ from Definition 2.12. We use Definition 2.14 and 2.15 in order to define iterated integrals over $T'_{jl}$ and over $\partial T'_{jl}$. Let $x_i$ be as in Definition 2. Then near the point $P_{jl}$ we have local coordinates $x_0$ and $x_j$. For $k = 1, 2, 3$ let

$$f_k = x_0^{n_{k0}} x_j^{n_{kj}} g_k,$$

where $g_k$ are holomorphic functions near $(x_0, x_j) = (0, 0)$. We are going to simplify the notation.

**Definition 3.1 (Integers $m_k$ and $n_k$, function $x$, $y$ and $g_k$ and torus $T_0$)**

We are going to use the integers $m_k$ instead of $n_{k0}$, $n_k$ instead of $n_{kj}$, and the functions $x$ instead of $x_0$, $y$ instead of $x_j$, (see Definition 2.1 for $n_{k0}$ and for $n_{kj}$). Also we are going to use $T_0$ instead of $T'_{jl}$, (see Definition 2.12 for $T'_{jl}$). We define $g_k$ for $k = 1, 2, 3$ by

$$f_k = x^{m_k} y^{n_k} g_k.$$ 

Note that $g_k$ is non-zero holomorphic function near $(x, y) = (0, 0)$.

**Lemma 3.2**

(a) $\int \int_{\partial T_0} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} = \int \int_{T_0} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} = -(m_1 n_2 - m_2 n_1) (2\pi i)^2$;

(b) $\int \int_{T_0} \frac{df_1}{f_1} \wedge \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = -(m_1 + n_1)(m_2 n_3 - m_3 n_2) \frac{(2\pi i)^3}{2} + O(\epsilon)$;

(c) $\int \int_{T_0} \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \wedge \frac{df_3}{f_3} = (m_3 + n_3)(m_1 n_2 - m_2 n_1) \frac{(2\pi i)^3}{2} + O(\epsilon)$.

**Proof.** Denote by $O(x, y)$ all the terms of degree at least 1 in the variable $x$ or $y$. Denote by $O(1)$ all the terms of degree at least 0 or higher. Note that on $T_0$ we have $|x| < \epsilon$ and $|y| < \epsilon$. For part (a), we have

$$\frac{df_1}{f_1} = m_1 \frac{dx}{x} + n_1 \frac{dy}{y} + \frac{dg_1}{g_1} = m_1 \frac{dx}{x} + n_1 \frac{dy}{y} + dO(x, y),$$

since $g_1$ has no zeroes or poles at $x = 0$ or $y = 0$. Similarly,

$$\frac{df_2}{f_2} = m_2 \frac{dx}{x} + n_2 \frac{dy}{y} + dO(x, y).$$
We obtain the system. Then we have to integrate the first 1-form up to \((x, t)\) up to a point \((x, t)\). Then the first 1-form in the 2-dimensional iterated integral, must be integrated over \(\gamma\) iterated integral, we must have the 2-form integrated over \(T\)

\[
\int_{T_0} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} = (m_1 n_2 - m_2 n_1) \int_{T_0} \frac{dx_0}{x_0} \wedge \frac{dx_j}{x_j} = -(m_1 n_2 - m_2 n_1) (2\pi i)^2.
\]

Since \(x\) and \(y\) vary over loops, we only pick the residues, when we compute the integral. Therefore,

\[
\int_{T_0} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} = (m_1 n_2 - m_2 n_1) \int_{T_0} \frac{dx_0}{x_0} \wedge \frac{dx_j}{x_j} = -(m_1 n_2 - m_2 n_1) (2\pi i)^2.
\]

For part (b) denote by \(\gamma (s, t) = T_0 \circ \gamma'(t)\), as in Definition 2.15. Note that for \(t \notin [0, 1]\), the image of \(\gamma\) is one dimensional. In order to have a non-zero 2-dimensional iterated integral, we must have the 2-form integrated over \(\gamma\) restricted to \((s, t) \in [0, 1] \times [1, 2]\). All other intervals for \(t\) will have no contribution, since \(\gamma\) has 1-dimensional image. Then the first 1-form in the 2-dimensional iterated integral, must be integrated over \(\gamma_s\) up to a point \((s, t) \in [0, 1] \times [1, 2]\). Consider the image of \(\gamma_s\) in the \((x_0, x_j)\)-coordinate system. Then we have to integrate the first 1-form up to \((x'_0, x'_j)\) along \(\gamma_s\) for some \(s\). We obtain

\[
\int_{T_0} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \int_{[0, 1]^2} \left( \int_{(x'_0, x'_j) = (0, 0)} \frac{df_1}{f_1} (x'_1, x'_2) \right) \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) (x_1, x_2).
\]

We have

\[
\int_{(x'_1, x'_2) = (0, 0)} \frac{df_1}{f_1} (x'_1, x'_2) = m_1 \frac{dx_0}{x_0} + n_1 \frac{dx_j}{x_j} + O(\epsilon).
\]

Therefore,

\[
\int_{T_0} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \int_{T_0} \left( m_1 \frac{dx_0}{x_0} + n_1 \frac{dx_j}{x_j} \right) \circ \left( (m_2 n_3 - m_3 n_2) \frac{dx_0}{x_0} \wedge \frac{dx_j}{x_j} + O(\epsilon) \right) = -(m_1 + n_1) (m_2 n_3 - m_3 n_2) \frac{(2\pi i)^3}{2} + O(\epsilon).
\]

Part (c) can be proven in a similar way as part (b).

**Corollary 3.3** For the torus \(\partial T_{ij}\), we have (a) \(\int_{\partial T_{ij}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{T_0} \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} = 0\);

(b) \(\int_{\partial T_{ij}} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = O(\epsilon)\);

(c) \(\int_{\partial T_{ij}} \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right) \circ \frac{df_3}{f_3} = O(\epsilon)\).

**Proof.** We are going to use the curves \(D_{ij}\) from Definition 2.1. Since the differential forms \(\frac{df_k}{f_k}\) have no residue along \(D_{ij}\), we obtain that the corollary as a consequence of Lemma 3.2.
3.2 Semi-local behavior. Integrating over the torus $T_{jl}$.

We are going to use the construction of the domain of integration from the subsection 2.4.

**Definition 3.4** (Paths $\gamma_0, \gamma$, loops $\sigma_0, \sigma, \tau_0$) We are going to use

- $\gamma_0$ instead of $\gamma_{jl}$,
- $\gamma$ instead of $\tilde{\gamma}_{jl}$,
- $\sigma_0$ instead of $\tilde{\sigma}_{jl}$,
- $\sigma$ instead of $\tilde{\sigma}_{jl}$,
- $\tau_0$ instead of $\tau_{jl}$.

See Definition 2.9 for $\gamma_{jl}$. See Definition 2.10 for $\tilde{\gamma}_{jl}$, $\tilde{\sigma}_{jl}$, and $\tilde{\sigma}_{jl}$. See Definition 2.11 for $\tau_{jl}$.

**Definition 3.5** ($\tau'$) Let $\tau'$ be a loop on $T_{b_u}$ starting at $\tilde{Q}$, defined in the following way.

First define a constant $C$ by

$$(x, y)(\tilde{Q}) = (\epsilon_1, C).$$

For $t \in [0,1]$ define the loop $\tau'$ so that

$$\tau'(0) = \tilde{Q}$$

and

$$\tau'(t) \subset (x, y)^{-1}(\epsilon_1 e^{2\pi it}, C).$$

Note that $\tau'$ does not vary in direction of $y$, while the loop $\tau$ might vary in that direction. See Definition 2.11 for $\tau$.

**Definition 3.6** ($T$ and $T'$) We are going to use

$$T$$ instead of $T_{jl}$.

Let $T'$ denote a torus with generators $\tau'$ and $\sigma$.

**Definition 3.7** ($\lambda'$) For fixed $t$, define also $\Lambda^t$ to be a segment of the straight line, in the complex plane $x = \epsilon_1 e^{2\pi it}$ in the coordinate system $(x, y)$, joining $(x, y)(\tau(t))$ with $(x, y)(\tau_j(t))$. Let

$$\lambda^t \subset (x, y)^{-1}L^t$$

be the corresponding path, joining $\tau(t)$ with $\tau'(t)$.

**Proposition 3.8** The following two iterated integrals are equal in the limit $\epsilon \to 0$. That is,

$$\lim_{\epsilon \to 0} \int_{[\sigma, \tau']} \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} - \int_{[\sigma, \tau]} \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_3}}{x_{i_3}} = 0,$$

where the commutator $[\sigma, \tau']$ in the first integral is the boundary of $T'$ and the commutator $[\sigma, \tau]$ in the second integral is the boundary of $T$. 29
Remark 3.9 It is easier to do explicit computations, using the first integral, involving \( \tau' \).

Recall that \( \tau_0 \) and \( \sigma_0 \) are the two loops on \( T_0 \) (see Definitions 2.13 and 3.3).

**Definition 3.10** \((S, S^{-1}, S' \text{ and } S'^{-1})\) Let \( S \) be the region bounded by

\[
\partial S = \gamma \tau_0 \gamma^{-1} \tau^{-1}.
\]

Let \( S^{-1} \) be the region bounded by

\[
\partial S^{-1} = \gamma^{-1} \tau \gamma \tau^{-1}.
\]

Let \( S' \) be the region bounded by

\[
\partial S' = \gamma \tau_0 \gamma^{-1} \tau'^{-1}.
\]

Let also \( S'^{-1} \) be the region bounded by

\[
\partial S'^{-1} = \gamma^{-1} \tau \gamma \tau'^{-1}.
\]

**Lemma 3.11**

(a) \[
\int_{[\sigma, \tau']} \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} = \int_{\partial T_0} \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} + \\
+ \int_{\sigma_0} \frac{dx_{i_1}}{x_{i_1}} \int \int_{S'^{-1}} \frac{dx_{i_2}}{x_{i_2}} \wedge \frac{dx_{i_3}}{x_{i_3}},
\]

(b) \[
\int_{[\sigma, \tau]} \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} = \int_{\partial T_0} \frac{dx_{i_1}}{x_{i_1}} \circ \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} + \\
+ \int_{\sigma_0} \frac{dx_{i_1}}{x_{i_1}} \int \int_{S^{-1}} \frac{dx_{i_2}}{x_{i_2}} \wedge \frac{dx_{i_3}}{x_{i_3}}.
\]

**Proof.** Consider the domain \( S \cup \gamma S^{-1} \gamma^{-1} \).

Note that \( S \) and \( \gamma S^{-1} \gamma^{-1} \) parameterize the same domain but with opposite orientations. Then

\[
\int_{S \cup \gamma S^{-1} \gamma^{-1}} \frac{dx_{i_1}}{x_{i_1}} \circ \left( \frac{dx_{i_2}}{x_{i_2}} \wedge \frac{dx_{i_3}}{x_{i_3}} \right) = 0.
\]

and

\[
\int_{S \cup \gamma S^{-1} \gamma^{-1}} \left( \frac{dx_{i_1}}{x_{i_1}} \wedge \frac{dx_{i_2}}{x_{i_2}} \right) \circ \frac{dx_{i_3}}{x_{i_3}} = 0.
\]

From Lemma 1.6 for composition of domains of 2-dimensional iterated integrals for the domain \( T \), we have
Lemma 3.12

\[
\int T \frac{dx_{i1}}{x_{i1}} \circ \left( \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} \right) = \int \int S \frac{dx_{i1}}{x_{i1}} \circ \left( \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} \right) + \int \int S^{-1} \frac{dx_{i1}}{x_{i1}} \circ \left( \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} \right) + \\
+ \int_{\gamma} \frac{dx_{i1}}{x_{i1}} \int \int \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} + \\
+ \int \int T_0 \frac{dx_{i1}}{x_{i1}} \circ \left( \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} \right) + \int_{\gamma} \frac{dx_{i1}}{x_{i1}} \int \int T_0 \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} + \\
+ \int_{\gamma} \frac{dx_{i1}}{x_{i1}} \int \int S^{-1} \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}}
\]

Using the above relation for the domain \(S \cup \gamma S^{-1} \gamma^{-1}\), we obtain part (b) of the Lemma. Part (a) can be done in a similar way, when we write \(S'\) instead of \(S\) and \(\tau'\) instead of \(\tau\). Denote by \(S'^{-1} - S^{-1}\) the region bounded by \(\tau_j \tau^{-1}\). Then we have the following lemma.

Lemma 3.13

\[
\int \int [\sigma, \tau'] \frac{dx_{i1}}{x_{i1}} \circ \frac{dx_{i2}}{x_{i2}} \circ \frac{dx_{i3}}{x_{i3}} - \int \int [\sigma, \tau] \frac{dx_{i1}}{x_{i1}} \circ \frac{dx_{i2}}{x_{i2}} \circ \frac{dx_{i3}}{x_{i3}} = \\
= \int_{\sigma_0} \frac{dx_{i1}}{x_{i1}} \int \int S'^{-1} - S^{-1} \frac{dx_{i2}}{x_{i2}} \land \frac{dx_{i3}}{x_{i3}} + \\
+ \int \int S'^{-1} - S^{-1} \frac{dx_{i1}}{x_{i1}} \land \frac{dx_{i2}}{x_{i2}} \int_{\sigma_0} \frac{dx_{i3}}{x_{i3}}
\]

For the region \(S'^{-1} - S^{-1}\), we have the following lemma.

Lemma 3.13

\[
\int \int S'^{-1} - S^{-1} \frac{dx_{i1}}{x_{i1}} \land \frac{dx_{i2}}{x_{i2}} = \int \int \tau'^{-1} - \tau^{-1} \frac{dx_{i1}}{x_{i1}} \circ \frac{dx_{i2}}{x_{i2}}.
\]

Proof. It follows from non-commutative Stokes theorem for iterated integrals (Theorem 1.5).

Proof. (of Proposition 3.8) In order to prove the proposition it remains to show that

\[
\lim_{\epsilon \to 0} \int_{\tau' \tau^{-1}} \frac{dx_{i1}}{x_{i1}} \circ \frac{dx_{i2}}{x_{i2}} = 0.
\]

Afterwards, we can use the above three lemmas.

If \(i_1 \neq 0\) or \(i_2 \neq 0\) then the above integral becomes 0 in the limit \(\epsilon \to 0\). If \(i_1 = i_2 = 0\) then \(\frac{dx_{i1}}{x_{i1}} \land \frac{dx_{i2}}{x_{i2}} = 0\). And by Lemma 3.6, we have that

\[
\int_{\tau' \tau^{-1}} \frac{dx_{i1}}{x_{i1}} \circ \frac{dx_{i2}}{x_{i2}} = 0.
\]

Denote by

\[
\omega_k = \frac{df_k}{f_k}
\]

for \(k = 1, 2, 3\). Using Stokes formula for 1- and 2-dimensional iterated integrals (Theorem 1.5), we obtain the following lemma.
Lemma 3.14

\[ \int_{\partial T'} \omega_1 \circ \omega_2 \circ \omega_3 = \int \int_{T'} \omega_1 \circ (\omega_2 \wedge \omega_3) + \int \int_{T_{jl}} (\omega_1 \wedge \omega_2) \circ \omega_3. \]

Using Lemma 1.6 for composition of domains of 2-dimensional iterated integrals, we obtain the following lemma.

Lemma 3.15

(a) \[ \int \int_{T'} \omega_1 \circ (\omega_2 \wedge \omega_3) = \frac{1}{(2\pi i)^3} \int \int_{T_0} \omega_1 \circ (\omega_2 \wedge \omega_3) + \int \int_{\gamma_0} \omega_1 \frac{dg_1}{g_1} + \int \int_{\sigma_0} \omega_1 \frac{dg_3}{g_3} + O(\epsilon); \]

(b) \[ \int \int_{T'} (\omega_1 \wedge \omega_2) \circ \omega_3 = \frac{1}{(2\pi i)^3} \int \int_{T_0} (\omega_1 \wedge \omega_2) \circ \omega_3 + \int \int_{\gamma_0} \omega_1 \frac{df_1}{f_1} + O(\epsilon). \]

Proof. It is the same as the proof of Lemma 3.7.

Lemma 3.16

(a) \[ \frac{1}{(2\pi i)^3} \int \int_{T'} \omega_1 \circ (\omega_2 \wedge \omega_3) = -(m_1 + n_1)(m_2n_3 - m_3n_2)\pi i + \]

\[ -(m_2n_3 - m_3n_2) \int \int_{\gamma_0} \omega_1 \frac{dg_1}{g_1} + \]

\[ -m_3n_1 \int \int_{\gamma_0} \omega_1 \frac{dg_3}{g_3} - \]

\[ +m_2n_1 \int \int_{\gamma_0} \omega_1 \frac{dg_3}{g_3} + O(\epsilon); \]

(b) \[ \frac{1}{(2\pi i)^3} \int \int_{T'} (\omega_1 \wedge \omega_2) \circ \omega_3 = (m_3 + n_3)(m_1n_2 - m_2n_1)\pi i + \]

\[ -(m_1n_2 - m_2n_1) \int \int_{\gamma_0} \omega_1 \frac{dg_3}{g_3} - \]

\[ +m_1n_3 \int \int_{\gamma_0} \omega_1 \frac{dg_2}{g_2} + \]

\[ -m_1n_3 \int \int_{\gamma_0} \omega_1 \frac{dg_1}{g_1} + O(\epsilon). \]

Proof. For part (a) we have

\[ \int \int_{T_0} \omega_1 \circ (\omega_2 \wedge \omega_3) = \int \int_{T_0} \omega_1 \circ (\omega_2 \wedge \omega_3) + \]

\[ + \int \int_{\gamma_0} \omega_1 \int \int_{T_0} \omega_2 \wedge \omega_3 + \int \int_{\sigma_0} \omega_1 \int \int_{S_{jl-1}} \omega_2 \wedge \omega_3 \]

From Lemma 4.3 part (b), we have

\[ \int \int_{T_0} \omega_1 \circ (\omega_2 \wedge \omega_3) = \frac{1}{2} (2\pi i)^3 (m_1 + n_1)(m_2n_3 - m_3n_2). \]

This takes care of the first summand in this lemma. Consider the functions \( f_k \) and \( g_k \) in terms of the local coordinates \( x \) and \( y \) near the point \( P_{jl} \). Using Lemma 4.3 part (a), we obtain

\[ \int \int_{\gamma_0} \omega_1 \int \int_{T_0} \omega_2 \wedge \omega_3 = -(m_2n_3 - m_3n_2) (2\pi i)^2 \int \int \omega_1 \frac{df_1}{f_1}. \]

32
When we express $f_1$ in terms of $g_1$ and powers of $x$ and $y$, we obtain
\[
\frac{1}{(2\pi i)^2} \int_\gamma \omega_1 \int_{T_0} \omega_2 \wedge \omega_3 = -(m_2 n_3 - m_3 n_2) \int_\gamma \frac{d(x^{m_1} y^{n_1} g_1)}{x^{m_1} y^{n_1}} = -(m_2 n_3 - m_3 n_2) \left( n_1 \int_\gamma \frac{dy}{y} + \int_\gamma \frac{dg_1}{g_1} \right).
\]

Now we have to compute
\[
\int_\sigma \omega_1 \int \int_{S'_{-1}} \omega_2 \wedge \omega_3.
\]
We have
\[
\int_\sigma \omega_1 = m_1 (2\pi i).
\]
We can simplify the double integral by using that $S'_{-1} = \gamma^{-1} \times \tau'$. Also, we can express $f_k$ as $x^{m_k} y^{n_k} g_k$ and then use that under the integral we must have $dx/x$, since we integrate the variable $x$ over the loop $T_0$. For the differential 2-form under the integral, we have
\[
\omega_2 \wedge \omega_3 = \left( m_2 \frac{dx}{x} + n_2 \frac{dy}{y} + \frac{dg_2}{g_2} \right) \wedge \left( m_3 \frac{dx}{x} + n_3 \frac{dy}{y} + \frac{dg_3}{g_3} \right).
\]

When we pick only the terms that involve $dx/x$, we obtain the 2-form
\[
(m_2 n_3 - m_3 n_2) \frac{dx}{x} \wedge \frac{dy}{y} + m_2 \frac{dx}{x} \wedge \frac{dg_3}{g_3} - m_3 \frac{dx}{x} \wedge \frac{dg_2}{g_2}.
\]
The term $-(m_2 n_3 - m_3 n_2) \frac{dx}{x} \wedge \frac{dy}{y}$ gives
\[
\int \int_{S'_{-1}} -(m_2 n_3 - m_3 n_2) \frac{dx}{x} \wedge \frac{dy}{y} = -(m_2 n_3 - m_3 n_2)(2\pi i) \int_\gamma \frac{dy}{y}.
\]
This term cancels with the term
\[
\lim_{\epsilon \to 0} -(m_2 n_3 - m_3 n_2) n_1 \int_\gamma \frac{dy}{y},
\]
coming from
\[
\int_\gamma \omega_1 \int \int_{T_0} \omega_2 \wedge \omega_3.
\]

The 2-form
\[
m_2 \frac{dx}{x} \wedge \frac{dg_3}{g_3}
\]
gives
\[
\int \int_{S'_{-1}} m_2 \frac{dx}{x} \wedge \frac{dg_3}{g_3} = -m_2 (2\pi i) \int_\gamma \frac{dg_3}{g_3}.
\]

Finally, the 2-form
\[
m_3 \frac{dx}{x} \wedge \frac{dg_2}{g_2}
\]
gives
\[
\int \int_{S'_{-1}} m_3 \frac{dx}{x} \wedge \frac{dg_2}{g_2} = m_3 (2\pi i) \int_\gamma \frac{dg_2}{g_2}.
\]
This proves part (a) of the lemma.
Part (b) can be proven in the same way.
3.3 Differential equation

We will define a partial differential equation, which has no solution locally, but has a solution only over a fixed path, when it is reduced to an ordinary differential equation. Let $A_i$ for $i = 1, \ldots, n$ be indeterminant constants which do not commute. One can think of them as constant square matrices. Let $h_i$ for $i = 1, \ldots, n$ be rational functions on $X$. Consider the differential equation

$$dF = F \sum_{i=1}^{n} A_i \frac{dh_i}{h_i},$$

where the multiplication is matrix multiplication (or multiplication of indeterminant constants). When this partial differential equation is restricted to a path, we obtain a solution, given by a generating series of iterated integrals.

From Lemma 2.6 it seems that the differential $\frac{df_k}{f_k}$ should anti-commute. Thus, our first attempt to relate a differential equation to a logarithmic version of the Parshin symbol is the following

$$dF = F \sum_{k=1}^{3} A_k \frac{df_k}{f_k},$$

where the indeterminant $A_k$ for $k = 1, 2, 3$ anti-commute.

Why this attempt is not good? The factor $x_1$ from $f_1$ and $x_1$ from $f_2$ lead to summands $\frac{dx_1}{f_1}$ in $\frac{df_1}{f_1}$ and $\frac{dx_1}{f_1}$ in $\frac{df_2}{f_2}$. These two copies of $\frac{dx_1}{f_1}$ commute, when iterated. However, because the constants $A_1$ and $A_2$ anti-commute in the above differential equation, we obtain that the two copies of $\frac{dx_1}{f_1}$ - one coming from $f_1$ and the other from $f_2$ - should anti-commute. Thus, the both commute and anti-commute, when we use the above differential equation.

Instead of considering the functions $f_k$ for $k = 1, 2, 3$, we must decompose the three functions into factors. We want these factors to enter in the choice of local coordinates. (Later we will show independence of the choices of local coordinates.) Also, we want similar factors coming from different $f_k$’s to commute and different factors coming from different functions $f_k$ to anti-commute. In order to do that we must distinguish between similar factors, coming from different functions $f_k$’s.

**Definition 3.17** (factorization of the functions $f_k$ for $k = 1, 2, 3$.) We are going to use the definitions of $x_i$ and the corresponding powers $n_{ki}$ for $i = 1, \ldots, N$ from Definition 2.1. Define $x_{N+k}$ for $k = 1, 2, 3$ by

$$x_{N+k} = f_k \prod_{i=1}^{N} x_i^{-m_{ki}}.$$

Define also the constants $m_{N+l,k}$ for $k, l = 1, 2, 3$ by

$$m_{N+l,k} = \delta_{kl},$$

where $\delta_{kl}$ is the Kroneker delta function. We are going to use the factorizations

$$f_k = \prod_{i=1}^{N+3} x_i^{m_{ki}}.$$
Then we have
\[
\frac{df_k}{f_k} = \sum_{i=1}^{N+3} m_{k_i} \frac{dx_i}{x_i}.
\]

**Definition 3.18** Consider the differential equation
\[
dF = F \sum_{i=1}^{N+3} A_{ki} m_{k_i} \frac{dx_i}{x_i}.
\]
We are going to use solutions of this differential equation along a path $\gamma$, which we will denote by $F_\gamma$.

**Definition 3.19** (Variable $A$ for the purpose of the new symbol) We set an equivalence among certain monomials in the variables $A_{ki}$. Let
\[
A \sim A_{1,j_1} A_{2,j_2} A_{3,j_3}
\]
for all values of $j_1$, $j_2$ and $j_3$.

**Definition 3.20** (Variable $B$ for the purpose of the logarithm of the Parshin symbol) Let $\sigma$ be a permutation of $\{1, 2, 3\}$. Define an equivalence
\[
B \sim A_{1,j_1} A_{2,j_2} A_{3,j_3} \sim A_{\sigma(1),j_1} A_{\sigma(2),j_2} A_{\sigma(3),j_3}.
\]
For $j_1 \neq j_2 \neq j_3 \neq j_1$, let
\[
A_{1,j_1(1)} A_{2,j_2(2)} A_{3,j_3(3)} \sim \text{sign}(\sigma) A_{1,j_1} A_{2,j_2} A_{3,j_3}.
\]
For $j_1 = j_2 \neq j_3$ and for a permutation $\sigma$ of $\{1, 2\}$, define the equivalence
\[
A_{1,j_1(1)} A_{2,j_2(1)} A_{3,j_3(2)} \sim \text{sign}(\sigma) A_{1,j_1} A_{2,j_1} A_{3,j_2}.
\]
For $j_1 = 0$ and $j_2 \neq 0$ define
\[
B \sim A_{1,j_1} A_{2,j_1} A_{3,j_2}.
\]

### 3.4 The commutators $[\alpha_i, \beta_i]$

In order to prove a reciprocity law for the logarithm of the new symbol $\log([f_1, f_2, f_3]_{a_1, b_1})$, we have to consider the loops on $C_0$, given by the commutator $[\alpha_j, \beta_j]$.

**Definition 3.21** Define a torus
\[
T'_j = h(u, [\alpha_j, \beta_j]),
\]
for $|u| = \epsilon_1$.

**Lemma 3.22** Let $\alpha$ be any loop on $X$ and let $f$ be a rational function on $X$. Then
\[
\int_{\alpha} \frac{df}{f} = 2\pi in
\]
for some integer $n$. 

35
Proof. Consider $f$ as a function from the variety $X$ to $\mathbb{C}P^1 - \{0, \infty\}$. On $\mathbb{C}P^1 - \{0, \infty\}$ we have the differential form $dz/z$. Then

$$\int_{\alpha} \frac{df}{f} = \int_{\alpha} f^* \frac{dz}{z} = \int_{f(\alpha)} \frac{dz}{z} = 2\pi in$$

for some integer $n$.

We are going to use the following lemma for commutators.

**Lemma 3.23** (a) $\int_{[\alpha, \beta]} \omega = 0$;
(b) $\int_{[\alpha, \beta]} \omega_1 \circ \omega_2 = \int_\alpha \omega_1 \int_\beta \omega_2 - \int_\beta \omega_1 \int_\alpha \omega_2$;
(c) $\int_{[\alpha, \beta]} \omega_1 \circ \omega_2 \circ \omega_3 = \int_\alpha \omega_1 \circ \omega_2 \int_\beta \omega_3 - \int_\beta \omega_1 \circ \omega_2 \int_\alpha \omega_3 +$

$$+ \int_\alpha \omega_3 \circ \omega_2 \int_\beta \omega_1 - \int_\beta \omega_3 \circ \omega_2 \int_\alpha \omega_1 -$$

$$- \int_\alpha \omega_1 \int_\beta \omega_2 \int_\alpha \omega_3 + \int_\beta \omega_1 \int_\alpha \omega_2 \int_\beta \omega_3.$$

**Proof.** Part (a) is trivial. Part (b) is Theorem 3.1 in [H]. Part (c) is Theorem 4. in [H].

**Proposition 3.24** The 2-dimensional iterated integrals

$$\int \int_{\partial T'_j} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}$$

is an integer times $(2\pi i)^3$.

**Proof.** Let $\sigma$ be the loop in the fiber of the torus $T'_j$. First we prove the following lemma.

**Lemma 3.25**

$$(2\pi i)^{-3} \int_{[[\alpha, \beta], \sigma]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}$$

is an integer.

**Proof.** Using part (c) of the Lemma 5.7 we obtain

$$\int_{[[\alpha, \beta], \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = \int_{[\alpha, \beta]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_\tau \frac{df_3}{f_3} - \int_\tau \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_{[\alpha, \beta]} \frac{df_3}{f_3} +$$

$$+ \int_{[\alpha, \beta]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_\tau \frac{df_3}{f_3} - \int_\tau \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_{[\alpha, \beta]} \frac{df_3}{f_3} -$$

$$- \int_{[\alpha, \beta]} \frac{df_1}{f_1} \int_\tau \frac{df_2}{f_2} \int_{[\alpha, \beta]} \frac{df_3}{f_3} + \int_\tau \frac{df_1}{f_1} \int_{[\alpha, \beta]} \frac{df_2}{f_2} \int_\tau \frac{df_3}{f_3}.$$

From part (a) of Lemma 5.7, we have that an integral of a 1-form over a commutator is zero. Therefore,

$$\int_{[[\alpha, \beta], \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = \int_{[\alpha, \beta]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_\tau \frac{df_3}{f_3} + \int_{[\alpha, \beta]} \frac{df_3}{f_3} \circ \frac{df_2}{f_2} \int_\tau \frac{df_1}{f_1}.$$
Now we use part (b) of Lemma 5.7. And we obtain
\[
\int_{[\alpha, \beta], \tau} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = \left( \int_{\alpha} \frac{df_1}{f_1} \int_{\beta} \frac{df_2}{f_2} - \int_{\beta} \frac{df_1}{f_1} \int_{\alpha} \frac{df_2}{f_2} \right) \int_{\tau} \frac{df_3}{f_3} +
+ \left( \int_{\alpha} \frac{df_3}{f_3} \int_{\beta} \frac{df_2}{f_2} - \int_{\beta} \frac{df_3}{f_3} \int_{\alpha} \frac{df_2}{f_2} \right) \int_{\tau} \frac{df_1}{f_1}
\]
By Lemma 5.6, all of the integrals of 1-forms are integer multiples of $2\pi i$. This proves the lemma.

**Proof.** (of Proposition 5.15) The boundary of the torus $T'$ is $[\sigma[\alpha, \beta]]$. By the above lemma the integral is an integer multiple of $(2\pi i)^3$.

**Definition 3.26** $(h_k)$ We have that $m_k$ is the order of the function $f_k$ along the curve $C_0$. Also we have that $x$ is a rational function on $X$ that has zero of order 1 along $C_0$. Define functions $h_k$ by
\[
f_k = x^{m_k} h_k .
\]

**Corollary 3.27** For fixed $j$ the contribution from the commutator $[\alpha_j, \beta_j]$ is the iterated integrals
\[
\int \int_{\partial T_j} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3} = 2\pi i m_1 \left( \int_{\alpha_j} \frac{dh_1}{h_1} \int_{\beta_j} \frac{dh_2}{h_2} - \int_{\beta_j} \frac{dh_1}{h_1} \int_{\alpha_j} \frac{dh_2}{h_2} \right) +
+ 2\pi i m_3 \left( \int_{\alpha_j} \frac{dh_1}{h_1} \int_{\beta_j} \frac{dh_3}{h_3} - \int_{\beta_j} \frac{dh_1}{h_1} \int_{\alpha_j} \frac{dh_3}{h_3} \right).
\]

### 3.5 Extra terms

The source of the extra terms are the integrals
\[
\int_{\pi_i} \frac{df_1}{f_1} \int_{[\mu_{i+1}, \tau]} \frac{df_2}{f_2} \circ \frac{df_3}{f_3}
\]
and
\[
\int_{[\mu_{i+1}, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \int_{\pi_i^{-1}} \frac{df_3}{f_3}
\]
from Definition 2.29. Note that their difference is the same as the sum of the coefficients equivalent to $A$ in $F_{\pi_i[\mu_{i+1}, \tau]\pi_i^{-1}} - F_{[\mu_{i+1}, \tau]}$.

In terms of the differential equation, an extra term is the difference between the coefficients of $A_{1i}, A_{2i}, A_{3i}$ in $F_{\pi_i[\mu_{i+1}, \tau]\pi_i^{-1}}$ and in $F_{[\mu_{i+1}, \tau]}$.

**Lemma 3.28** The difference between the coefficients of $A_{1j}, A_{2j}, A_{3j}$ in $F_{\pi_i[\mu_{i+1}, \tau]\pi_i^{-1}}$ and in $F_{[\mu_{i+1}, \tau]}$ is
\[
(2\pi i)^2 n_{1j} n_{2j} n_{3j} \left( \int_{\pi_i} \frac{dx_{j1}}{x_{j1}} \int_{[\mu_{i+1}, \tau]} \frac{dx_{j2}}{x_{j2}} \circ \frac{dx_{j3}}{x_{j3}} - \int_{\pi_i} \frac{dx_{j3}}{x_{j3}} \int_{[\mu_{i+1}, \tau]} \frac{dx_{j1}}{x_{j1}} \circ \frac{dx_{j2}}{x_{j2}} \right).
\]

**Proof.** Cut the loop $\pi_i[\mu_{i+1}, \tau]\pi_i^{-1}$ into 3 loops $\pi_i$, $[\mu_{i+1}, \tau]$ and $\pi_i^{-1}$. Then use Corollary 1.3 for composition of paths.

Recall that $L_j$ is the number of intersection points of $C_0$ and $C_j$.
Definition 3.29 For $k = 1, 2, 3$ consider the orders of vanishing $m_k$ and $m_{kj}$ of $x$ and $x_j$, respectively. (see Definitions 3.23 and 3.25.) Let

$$D_1(j) = \left| \begin{array}{cc} m_2 & n_{2j} \\ m_3 & n_{3j} \end{array} \right|, \quad D_2(i) = \left| \begin{array}{cc} m_3 & n_{3j} \\ m_1 & n_{1j} \end{array} \right|, \quad D_3(i) = \left| \begin{array}{cc} m_1 & n_{1j} \\ m_2 & n_{2j} \end{array} \right|. $$

Theorem 3.30 The sum of the coefficients equivalent to $A$ in

$$\sum_i (F_{\pi[i,\mu+1,\tau]} \pi_i^{-1} - F_{[\mu+1,\tau]})$$

is

$$\sum_{j_1 < j_2} (n_{1j_1}D_1(j_2) - n_{3j_1}D_3(j_2))L_{j_1}L_{j_2} + \frac{1}{2} \sum_{j_2=1}^N (n_{1j_2}D_1(j_2) - n_{3j_2}D_3(j_2))L_{j_2}(L_{j_2} - 1).$$

Proof. Fix $j_1, j_2$ and $j_3$. We are going to use the above lemma. We consider the sum

$$\sum_i \int_{\pi[i]} \frac{dx_{j_1}}{x_{j_1}} \int_{[\mu+1,\tau]} \frac{dx_{j_2}}{x_{j_2}} \int_{x_{j_3}} \frac{dx_{j_3}}{x_{j_3}}.$$

The second integral is zero if $j_2 \neq 0$ and $j_3 \neq 0$. Let $j_3 = 0$. Again the second integral is zero if $\mu_{i+1} \neq \sigma_{j_2,t}$ for all $l = 1, \ldots, L_{j_2}$. Suppose that $\mu_{i+1} = \sigma_{j_2,t}$. Then

$$\int_{[\mu+1,\tau]} \frac{dx_{j_2}}{x_{j_2}} \int_{x_{j_2}} \frac{dx_{j_2}}{x_{j_2}} = (2\pi i)^2.$$

Let $l > 1$ so that $\mu_{i+1} = \sigma_{j_2,t}$. Then

$$\pi_i = \pi_{j_2,t-1} = \left( \prod_{j'=1}^{j_2-1} \left( \prod_{l'=1}^{L_{j'}} \tilde{\sigma}_{j' \mu'} \right) \right) \prod_{l'=1}^{l-1} \tilde{\sigma}_{j_2 l'}.$$

Then we have

$$(2\pi i)^{-1} \int_{\pi_i} \frac{dx_{j_1}}{x_{j_1}} = \begin{cases} L_{j_1} & \text{for } j_1 < j_2 \\ l - 1 & \text{for } j_1 = j_2 \\ 0 & \text{for } j_1 > j_2. \end{cases}$$

Therefore,

$$(2\pi i)^{-3} \sum_{l=2}^{L_{j_2}} \int_{\pi_{j_2,t-1}} \frac{dx_{j_1}}{x_{j_1}} \int_{[\mu_{j_2,t},\tau]} \frac{dx_{j_2}}{x_{j_2}} \int_{x_{j_2}} \frac{dx_{j_2}}{x_{j_2}} \int_{x_{j_2}} \frac{dx_{j_2}}{x_{j_2}} = \begin{cases} L_{j_1}(L_{j_2} - 1) & \text{for } j_1 < j_2 \\ \frac{1}{2} L_{j_2}(L_{j_2} - 1) & \text{for } j_1 = j_2 \\ 0 & \text{for } j_1 > j_2. \end{cases}$$

If $l = 1$ in $\mu_{i+1} = \sigma_{j_2,t} = \sigma_{j_2,1}$. Then

$$\pi_i = \pi_{j_2-1,L_{j_2}-1} = \prod_{j'=1}^{j_2-1} \left( \prod_{l'=1}^{L_{j'}} \tilde{\sigma}_{j' \mu'} \right).$$
The contribution of the extra term for \( l = 1 \) is
\[
(2\pi i)^{-3} \int_{\pi_{j_2, l-1}} \frac{dx_{j_1}}{x_{j_1}} \int_{[\mu_{j_2, l}, \tau]} \frac{dx_{j_2}}{x_{j_2}} \circ \frac{dx_0}{x_0} = \begin{cases} 
L_{j_1} & \text{for } j_1 < j_2 \\
0 & \text{for } j_1 \geq j_2.
\end{cases}
\]
Therefore,
\[
(2\pi i)^{-3} \sum_i \int_{\pi_i} \frac{dx_{j_1}}{x_{j_1}} \int_{[\mu_i, \tau]} \frac{dx_{j_2}}{x_{j_2}} \circ \frac{dx_0}{x_0} = \begin{cases} 
L_{j_1} L_{j_2} & \text{for } j_1 < j_2 \\
\frac{1}{2} L_{j_2} (L_{j_2} - 1) & \text{for } j_1 = j_2 \\
0 & \text{for } j_1 > j_2.
\end{cases}
\]
Therefore, the coefficient of \( A_{1j_1} A_{2j_2} A_{30} \) in
\[
\sum_i (F_{\pi_i}[\mu_{i+1}, \tau])
\]
is
\[
\begin{align*}
&n_{1j_1} n_{2j_2} m_3 L_{j_1} L_{j_2} & \text{for } j_1 < j_2 \\
&\frac{1}{2} n_{1j_2} n_{2j_2} m_3 L_{j_2} (L_{j_2} - 1) & \text{for } j_1 = j_2 \\
&0 & \text{for } j_1 > j_2.
\end{align*}
\]
Similarly, the coefficient of \( A_{1j_1} A_{20} A_{3j_2} \) in
\[
\sum_i (F_{\pi_i}[\mu_{i+1}, \tau])
\]
is
\[
\begin{align*}
&-n_{1j_1} n_{3j_2} m_2 L_{j_1} L_{j_2} & \text{for } j_1 < j_2 \\
&-\frac{1}{2} n_{1j_2} n_{3j_2} m_2 L_{j_2} (L_{j_2} - 1) & \text{for } j_1 = j_2 \\
&0 & \text{for } j_1 > j_2.
\end{align*}
\]
Therefore, the coefficient of \( A_{1j_1} A_{2j_2} A_{30} \) plus the coefficient of \( A_{1j_1} A_{20} A_{3j_2} \) in
\[
\sum_i (F_{\pi_i}[\mu_{i+1}, \tau])
\]
is
\[
\begin{align*}
&n_{1j_1} D_1(j_2) L_{j_1} L_{j_2} & \text{for } j_1 < j_2 \\
&\frac{1}{2} n_{1j_2} D_1(j_2) L_{j_2} (L_{j_2} - 1) & \text{for } j_1 = j_2 \\
&0 & \text{for } j_1 > j_2.
\end{align*}
\]
Note that the coefficient of \( A_{10} A_{2j_2} A_{3j_1} \) in
\[
\sum_i (F_{\pi_i}[\mu_{i+1}, \tau])
\]

39
is zero.

Similarly, the coefficient of $A_{10}A_{j_2}A_{j_1}$ plus the coefficient of $A_{1j_2}A_{20}A_{j_1}$ in

$$\sum_i (F_{[\mu_{i+1}\tau]} \pi^{-1})$$

is

$$-n_{3j_1} D_3(j_2) L_{j_1} L_{j_2} \quad \text{for } j_1 < j_2$$

$$-\frac{1}{2} n_{3j_2} D_3(j_2) L_{j_2} (L_{j_2} - 1) \quad \text{for } j_1 = j_2$$

$$0 \quad \text{for } j_3 > j_2.$$

The contribution of all the extra terms is

$$\sum_{j_1 < j_2} (n_{1j_1} D_1(j_2) - n_{3j_1} D_3(j_2)) L_{j_1} L_{j_2} + \frac{1}{2} \sum_{j_2 = 1}^N (n_{1j_2} D_1(j_2) - n_{3j_2} D_3(j_2)) L_{j_2} (L_{j_2} - 1)$$

**Corollary 3.31** Each of the extra terms is an integer multiple of $(2\pi i)^3$.

### 3.6 New Symbol

**Definition 3.32** (Logarithmic version of the new symbol) For $k = 1, 2, 3$ define $m_k = n_{k1}$, $n_k = n_{kj}$ and $g_{kj} = x_1^{-m_k} x_j^{-n_k} f_k = \prod_{i \neq 1, j} x_i^{n_{ki}}$

Define a logarithm of the new symbol as

$$\text{Log}[f_1, f_2, f_3]_{\gamma_{jl}}^{\gamma_{jl}} C_{0, P_{jl}} = \frac{1}{2} (2\pi i)^3 ((m_1 + n_1) D_1 - (m_3 + n_3) D_3) +$$

$$(D_1 + m_2 n_3) \int_{g_{j_1}} \frac{dg_{j_1}}{g_{j_1}} + D_2 \int_{g_{j_2}} \frac{dg_{j_2}}{g_{j_2}} + (D_3 - m_2 n_1) \int_{g_{j_3}} \frac{dg_{j_3}}{g_{j_3}},$$

where

$$D_1 = \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} m_3 & n_3 \\ m_1 & n_1 \end{vmatrix}, \quad \text{and } D_3 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}.$$

Using Lemma 3.16 and Lemma 3.14, we obtain the following corollary.

**Corollary 3.33** In terms of iterated integrals, the logarithm of the new symbol is

$$\text{Log}[f_1, f_2, f_3]_{\gamma_{jl}}^{\gamma_{jl}} C_{0, P_{jl}} = \int_{[\sigma_{j_1}, \tau]} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \circ \frac{df_3}{f_3}.$$
where
\[ M = \sum_{j_1 < j_2} (n_{1j_1} D_1(j_2) - n_{3j_1} D_3(j_2)) L_{j_1} L_{j_2} \]
\[ + \frac{1}{2} \sum_{j_2=1}^{N} (n_{1j_2} D_1(j_2) - n_{3j_2} D_3(j_2)) L_{j_2} (L_{j_2} - 1), \]
\[ N = (2\pi i)^{-2} m_1 \left( \int_{\alpha_j} \frac{dh_3}{h_3} \int_{\beta_j} \frac{dh_2}{h_2} - \int_{\beta_j} \frac{dh_3}{h_3} \int_{\alpha_j} \frac{dh_2}{h_2} \right) + \]
\[ + (2\pi i)^{-2} m_3 \left( \int_{\alpha_j} \frac{dh_1}{h_1} \int_{\beta_j} \frac{dh_2}{h_2} - \int_{\beta_j} \frac{dh_1}{h_1} \int_{\alpha_j} \frac{dh_2}{h_2} \right), \]

**Proof.** From Theorem 2.31, we have that the sum of the Log-symbols is equal to the sum of the extra terms \( N \) and the sum of the commutator terms \( M \). The sum of the commutator terms is given by Corollary 3.27. And the sum of the extra terms is given by Theorem 3.30. This proves the theorem.

### 4 Logarithmic version of the Parshin symbol

#### 4.1 Integration over a torus revisited

We are going to use the differential equation from Definition 3.18 and the equivalence from Definition 3.20.

**Proposition 4.1** The sum of the coefficients of the monomials equivalent to \( B \) in \( F_{[\sigma_j, \tau]} \) is
\[ -4(m_1 m_2 n_3 + m_1 n_2 m_3 + n_1 m_2 m_3 - n_1 n_2 m_3 - n_1 m_2 n_3 - m_1 n_2 n_3) (2\pi i)^3 \]
\[ \times \frac{1}{2}. \]

**Proof.** Using Definition 3.20 and Lemma 3.16, we have that the term \( A_{1,0} A_{2,0} A_{3,j} \) contributes \(-m_1 m_2 n_3 (2\pi i)^3/2\). Similarly, the term \( A_{1,j} A_{2,j} A_{3,0} \) contributes \(+n_1 n_2 m_3 (2\pi i)^3/2\).

When we consider all possible permutations of the indexes, we obtain the above proposition.

**Remark 4.2** When we divide the coefficient in the above lemma by \(-4(2\pi i)^2\) and then exponentiate, we obtain the sign in the Parshin symbol.

#### 4.2 Logarithmic symbol

Let \( Q \) be a base point on \( C_1 \). Consider the loop \( \sigma_{jl} \) around the point \( P_{jl} \).

**Definition 4.3** For \( k = 1, 2, 3 \) define \( m_k = n_{k1}, n_k = n_{kj} \) and \( g_{kj} = \frac{f_j}{x_j^{\gamma_j}}, x_j^{\nu_{jk}} = \prod_{i \neq j, k} x_i^{\nu_{ki}} \).

Define a logarithm of the Parshin symbol as
\[ \text{Log}(f_1, f_2, f_3)_{C_1, P_{jl}} = (2\pi i)^2 \left( \pi i K + D_1 \int_{\gamma_{jl}} \frac{dg_{1j}}{g_{1j}} + D_2 \int_{\gamma_{jl}} \frac{dg_{2j}}{g_{2j}} + D_3 \int_{\gamma_{jl}} \frac{dg_{3j}}{g_{3j}} \right), \]

where
\[ K_j = m_1 m_2 n_3 - m_1 n_2 m_3 + n_1 m_2 m_3 - n_1 n_2 m_3 + n_1 m_2 n_3 - m_1 n_2 n_3, \]
\[ D_1 = \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} m_3 & n_3 \\ m_1 & n_1 \end{vmatrix}, \quad \text{and} \quad D_3 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}. \]

**Theorem 4.4** The coefficient of \(-\frac{1}{4}B\) for the integral \(F\) along the loop, based at \(P_{j_1}\), is \(\Log\{f_1, f_2, f_3\}^Q_{C_1, P_{j_1}}\).

**Proof.** The portion \(K_j\) from the symbol was considered in subsection 4.1. The remaining portion is anti-symmetrization of the indexes of Lemma 3.15. That gives precisely the sum of the coefficients equivalent to \(-\frac{1}{4}B\).

**Corollary 4.5** The Parshin symbol of the functions \(f_1, f_2, f_3\) at \(C_1, P_{j_1}\) is

\[ \{f_1, f_2, f_3\}_{C_1, P_{j_1}} = \left( f_1^{D_1} f_2^{D_2} f_3^{D_3} \right) (Q) \cdot \exp \left( (2\pi i)^{-2} \left( \Log\{f_1, f_2, f_3\}_{C_1, P_{j_1}} \right) \right). \]

**Proof.** It follows by direct computation.

### 4.3 Vanishing of the commutator terms

**Definition 4.6** Let

\[ N_{ijk} = (2\pi i)^{-3} \int_{[\alpha, \beta, \sigma]} \frac{dx_i}{x_i} \circ \frac{dx_j}{x_j} \circ \frac{dx_k}{x_k} \]

be the integer from the above lemma.

**Lemma 4.7** For \(i \neq j \neq k \neq i\) we have

\[ N_{ijk} + N_{jki} + N_{kij} = 0. \]

**Proof.** It follows by direct computation from the formula for \(N_{ijk}\) from Lemma 3.18(c).

For \(i \neq j\), the coefficient of \(A_{1i}A_{2j}A_{3j}\) is \(n_{1j}n_{2j}n_{3j}N_{iji}\). Note that

\[ A_{1i}A_{2i}A_{3j} \sim -A_{1i}A_{3j}A_{2i}. \]

The coefficient of \(A_{1i}A_{2i}A_{3j}\) is \(n_{1i}n_{2j}n_{3j}N_{iji}\). And the coefficient of \(A_{3j}A_{1i}A_{2i}\) is \(n_{1i}n_{2i}n_{3j}N_{jii}\). Note that

\[ A_{3j}A_{1i}A_{2i} \sim A_{1i}A_{3j}A_{2i}. \]

**Lemma 4.8** For \(i \neq j\), we have

\[ N_{iij} - N_{ijj} + N_{jii} = 4 \left( \int_{\alpha} \frac{dx_j}{x_j} \int_{\beta} \frac{dx_i}{x_i} - \int_{\beta} \frac{dx_j}{x_j} \int_{\alpha} \frac{dx_i}{x_i} \right) \int_{\sigma} \frac{dx_i}{x_i}. \]

**Proof.** For each of the three summands, we have

\[ N_{iij} = \int_{[\alpha, \beta, \sigma]} \frac{dx_i}{x_i} \circ \frac{dx_j}{x_j} \circ \frac{dx_i}{x_i} = \]

\[ = \left( \int_{\alpha} \frac{dx_i}{x_i} \int_{\beta} \frac{dx_j}{x_j} - \int_{\beta} \frac{dx_j}{x_j} \int_{\alpha} \frac{dx_i}{x_i} \right) \int_{\sigma} \frac{dx_i}{x_i} + \]

\[ + \left( \int_{\alpha} \frac{dx_j}{x_j} \int_{\beta} \frac{dx_i}{x_i} - \int_{\beta} \frac{dx_i}{x_i} \int_{\alpha} \frac{dx_j}{x_j} \right) \int_{\sigma} \frac{dx_i}{x_i} = \]

\[ = \left( \int_{\alpha} \frac{dx_i}{x_i} \int_{\beta} \frac{dx_j}{x_j} - \int_{\beta} \frac{dx_j}{x_j} \int_{\alpha} \frac{dx_i}{x_i} \right) \int_{\sigma} \frac{dx_i}{x_i}. \]
Thus, 

\[ N_{iji} - N_{iji} + N_{jii} = 4 \int_{\sigma} \frac{dx_i}{x_i} \left( \int_{\alpha} \frac{dx_j}{x_j} \int_{\beta} \frac{dx_i}{x_i} - \int_{\beta} \frac{dx_j}{x_j} \int_{\alpha} \frac{dx_i}{x_i} \right). \]

Obviously, when three of the indexes coincide then \( N_{iii} = 0 \).

**Proposition 4.9** The sum of the coefficients equivalent to \( B \) in \( F[[\alpha_i^u,\beta_i^u]\r] \) is zero.

**Proof.** From Lemma 3.18, we obtain that the only contributions to the coefficient of \( B \) come from \( A_{ij}A_{jk}A_{ik} \), when only two of the indexes \( i, j, k \) coincide. From Lemma 3.19, it follows that the contributions are integer combinations of integrals of the type

\[ \int_{\sigma} \frac{dx_i}{x_i} \left( \int_{\alpha} \frac{dx_j}{x_j} \int_{\beta} \frac{dx_i}{x_i} - \int_{\beta} \frac{dx_j}{x_j} \int_{\alpha} \frac{dx_i}{x_i} \right). \]

The integral

\[ \int_{\sigma} \frac{dx_i}{x_i} \]

is not zero only when \( i = 1 \). However, for \( i = 1 \) the variable \( x_1 \) becomes the constant \( u \) along the loops \( \alpha_i^u \) and \( \beta_i^u \), because \( \alpha_i^u \) and \( \beta_i^u \) are defined on the curve

\[ T b^u = \{ Y \in T b|x_1(Y) = u \}. \]

Therefore,

\[ \int_{\alpha_i^u} \frac{dx_1}{x_1} = 0 \]

and

\[ \int_{\beta_i^u} \frac{dx_1}{x_1} = 0. \]
Thus,
\[
\int_{\sigma} \frac{dx_i}{x_i} \left( \int_{\alpha_i} \frac{dx_j}{x_j} \int_{\beta_i} \frac{dx_i}{x_i} - \int_{\beta_i} \frac{dx_j}{x_j} \int_{\alpha_i} \frac{dx_i}{x_i} \right) = 0.
\]

4.4 Vanishing of the extra terms

The sum of the extra terms is the same as the coefficients equivalent to \( B \) in \( F_{\pi, [\mu_i+1, \tau]} \).

**Proposition 4.10** The coefficient of \( B \) in

\[ \sum_1 F_{\pi, [\mu_i+1, \tau]} + F_{[\mu_i+1, \tau]} \]

is zero.

We need two lemmas in order to compute the coefficient of \( A_{1i_1} A_{2i_2} A_{3i_3} \)

**Lemma 4.11** The coefficient of \( A_{1i_1} A_{2i_2} A_{3i_3} \) in \( F_{\pi, [\mu_i+1, \tau]} \) is

\[ (2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i}, \int_{\mu_i+1, \tau} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i}, \int_{\mu_i+1, \tau} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} \right) = 0. \]

**Proof.** Cut the loop \( \pi, [\mu_i+1, \tau] \) into 3 loops \( \pi, [\mu_i+1, \tau] \) and \( \pi, \tau \). Then use Corollary 1.3 for composition of paths.

**Lemma 4.12** We have the following relation

(a) \( \int_{[\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} = - \int_{[\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} \).

(b) \( \int_{[\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} = 0. \)

**Proof.** By Lemma 3.18 part (b), we have that

\[ \int_{[\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} = \int_{\mu_i+1} \frac{dx_i}{x_i} \int_{\tau} \frac{dx_i}{x_i} \int_{[\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i}. \]

The right hand side is anti-symmetric on the indexes \( i_1 \) and \( i_2 \). This proves part (a). For part (b), use the above equality when \( i_1 = i_2 \). 

**Proof.** (Proposition 4.7) Let \( i_1 \neq i_2 \neq i_3 \neq i_1 \). By Lemmas 4.8 and 4.9, we have that the coefficient of \( 3i_2 A_{2i_2} A_{1i_1} \) along the path \( \pi, [\mu_i+1, \tau] \) is

\[ (2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i}, \int_{\mu_i+1, \tau} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i}, \int_{\mu_i+1, \tau} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} \right) = \]

\[ (2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( - \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} + \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} \right) = \]

\[ (2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( + \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} - \int_{\pi, [\mu_i+1, \tau]} \frac{dx_i}{x_i} \circ \frac{dx_i}{x_i} \right). \]
Note that this is the same as the coefficient of $A_{1i_1}A_{2i_2}A_{3i_3}$. However, $A_{3i_3}A_{2i_2}A_{1i_1} \sim -A_{1i_1}A_{2i_2}A_{3i_3}$. Therefore the two coefficients cancel in the equivalence.

By Lemmas 4.8 and 4.9(b), when three of the indexes coincide, that is $i_1 = i_2 = i_3$, the corresponding of the extra terms are zero.

It remain to examine what happens when two of the indexes coincide. Compare the coefficients of $A_{1i_1}A_{2i_2}A_{3i_2}$, $A_{1i_1}A_{3i_2}A_{2i_2}$ and $A_{3i_2}A_{1i_1}A_{2i_2}$. By Lemma 4.8, the coefficient of $A_{1i_1}A_{2i_2}A_{3i_2}$ in $F_{\pi_1[\mu_{i+1},\tau_1]_{\pi_1^{-1}}}$ is

$$(2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( \int_{\pi_1} \frac{dx_{i_1}}{x_{i_1}} \int_{[\mu_{i+1},\tau_1]} \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} \right).$$

The coefficient of $A_{1i_1}A_{3i_2}A_{2i_2}$ in $F_{\pi_1[\mu_{i+1},\tau_1]_{\pi_1^{-1}}}$ is

$$(2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( \int_{\pi_1} \frac{dx_{i_1}}{x_{i_1}} \int_{[\mu_{i+1},\tau_1]} \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} \right).$$

And finally, the coefficient of $A_{3i_2}A_{1i_1}A_{2i_2}$ in $F_{\pi_1[\mu_{i+1},\tau_1]_{\pi_1^{-1}}}$ is

$$(2\pi i)^2 n_{i_1} n_{i_2} n_{i_3} \left( \int_{\pi_1} \frac{dx_{i_1}}{x_{i_1}} \int_{[\mu_{i+1},\tau_1]} \frac{dx_{i_2}}{x_{i_2}} \circ \frac{dx_{i_3}}{x_{i_3}} \right).$$

The only case when the above integrals are not zero is when $i_2 = 0$. Set $i_1 = j$. Note that

$$\int_{[\mu_{i+1},\tau_1]} \frac{dx_{j}}{x_{j}} \circ \frac{dx_{0}}{x_{0}} = (2\pi i)^2$$

for $\mu_{i+1} = \tilde{\sigma}_{j\ell}$. Then $\int_{\pi_1} \frac{dx_{j}}{x_{j}} = 2\pi i l$.

However, the sum of the extra terms coming from $A_{1i_1}A_{2i_2}A_{3i_2}$, $A_{1i_1}A_{2i_2}A_{3i_1}$ and $A_{1i_2}A_{2i_1}A_{3i_1}$ is zero. Thus, there is no contribution from the extra terms in the logarithm of the Parshin symbol.

4.5 Reciprocity law

**Theorem 4.13** For the logarithm of the Parshin symbol we have the following reciprocity

$$\sum_{f,l} \Log\{f_1, f_2, f_3\}_{\Pi_{f,l}} = 0.$$

**Proof.** Form Theorem 2.31, we have that the sum of the Log-symbols applied to an iteration

$$\frac{dx_i}{x_i} \circ \frac{dx_j}{x_j} \circ \frac{dx_k}{x_k}$$

45
is equal to the sum of the extra terms and the sum of the commutator terms. We consider
the sum of the coefficients equivalent to $\frac{-1}{4B}$. The sum of the commutator terms is
zero by Proposition 4.9. And the sum of the extra terms is zero by Proposition 4.10.
This proves the theorem.

5 Refinement of the Parshin symbol

5.1 Logarithmic version of a refinement of the Parshin symbol

Definition 5.1 Define a logarithm of the refinement of the Parshin symbol as the dif-
ference

$$\text{Log}(f_1, f_2, f_3)_{C_0, P_{j_1}} = \text{Log}[f_1, f_2, f_3]_{C_0, P_{j_1}} - \text{Log}\{f_1, f_2, f_3\}_{C_0, P_{j_1}}.$$

Theorem 5.2

$$\text{Log}(f_1, f_2, f_3)_{C_0, P_{j_1}} = (2\pi i)^2 \left( \pi i (m_2 n_1 n_3 - n_2 m_1 m_3) + m_2 n_3 \int_{\gamma_{j_1}} \frac{d\gamma_{j_1}}{g_{j_1}} - m_2 n_1 \int_{\gamma_{j_1}} \frac{d\gamma_{j_1}}{g_{j_1}} \right).$$

Proof. It follows directly from Definitions 3.28 and 4.3. As a direct consequence we
obtain a refinement of the logarithm of the Parshin symbol.

Corollary 5.3

$$\text{Log}(f_1, f_2, f_3)_{C_0, P_{j_1}} + \text{Log}(f_2, f_3, f_1)_{C_0, P_{j_1}} + \text{Log}(f_3, f_1, f_2)_{C_0, P_{j_1}} = \text{Log}\{f_1, f_2, f_3\}_{C_0, P_{j_1}}.$$

5.2 Logarithmic reciprocity law

Theorem 5.4 We have the following reciprocity law

$$\sum_{j_1} \text{Log}(f_1, f_2, f_3)_{C_0, P_{j_1}} = (2\pi i)^3 (M + N),$$

where

$$M = (2\pi i)^3 \sum_{j_1 < j_2} (n_{1j_1} D_1(j_2) - n_{3j_1} D_3(j_2)) L_{j_1} L_{j_2}$$

$$+ (2\pi i)^3 \sum_{j_2 = 1}^N (n_{1j_2} D_1(j_2) - n_{3j_2} D_3(j_2)) \frac{1}{2} L_{j_2} (L_{j_2} - 1)$$

$$N = (2\pi i)^{-2} m_1 \left( \int_{\alpha_1} \frac{d\alpha_1}{h_1} \int_{\beta_1} \frac{d\beta_1}{h_1} - \int_{\beta_1} \frac{d\beta_1}{h_1} \int_{\alpha_1} \frac{d\alpha_1}{h_1} \right) +$$

$$+ (2\pi i)^{-2} m_3 \left( \int_{\alpha_3} \frac{d\alpha_3}{h_3} \int_{\beta_3} \frac{d\beta_3}{h_3} - \int_{\beta_3} \frac{d\beta_3}{h_3} \int_{\alpha_3} \frac{d\alpha_3}{h_3} \right).$$

Proof. It follows from the reciprocity laws for $\text{Log}(f_1, f_2, f_3)_{C_0, P_{j_1}}$ and $\text{Log}\{f_1, f_2, f_3\}_{C_0, P_{j_1}}$,
stmtated in Theorems 3.34 and 4.13, respectively.
5.3 Refinement of the Parshin symbol and a reciprocity law

Definition 5.5 Define a refinement of the Parshin symbol as

\[(f_1, f_2, f_3)_{C_0, P_{jl}} = \frac{g_{1j}(Q)^{m_2 n_3}}{g_{3j}(Q)^{m_2 n_1}} \exp \left( (2\pi i)^{-2} \Log(f_1, f_2, f_3)_{C_0, P_{jl}} \right) \]

Theorem 5.6 Define a refinement of the Parshin symbol as

\[(f_1, f_2, f_3)_{C_0, P_{jl}} = (-1)^{m_2 n_1 n_3 - n_2 m_1 m_3} \frac{g_{1j}(P_{jl})^{m_2 n_3}}{g_{3j}(P_{jl})^{m_2 n_1}} \]

Remark 5.7 For the local coordinates at \(P_{jl}\) we use the variables \(x_0\) and \(x_j\). The symbol is invariant of the choices of \(x_j\). However, it is dependent on \(x_0\). Let us explain to what extent the refinement of the Parshin symbol depends on the choice of \(x_0\). The variable \(x_0\) vanishes along the curve \(C_0\). And we use the same \(x_0\) for each of the points \(P_{jl}\), where we compute the symbol. If we have a different choice of \(x_0\) then we obtain a different symbol. But the new choice of \(x_0\) is the same for each point \(P_{jl}\).

Theorem 5.8 Using the above definition, we have

\[\prod_{jl} (f_1, f_2, f_3)_{C_0, P_{jl}}^{x_0} = 1.\]

Proof. It follows by exponentiating the reciprocity law from Theorem 4.5.

5.4 Example

Let \(X = \mathbb{C}P^1 \times \mathbb{C}P^1\) with projective coordinates \((x_0 : x_1) \times (y_0, y_1)\). For \(n = 1, 2, 3\) let

\[f_n = (x_1 - ax_0)^i_n (x_1 - bx_0)^j_n (x_1 - cx_0)^k_n \left( \frac{y_1}{y_0} \right)^l_n,\]

where \(i_n + j_n + k_n = 0\) for \(n = 1, 2, 3\). Let \(C\) be the variety \(y_1 = 0\). Consider the functions \(f_n\) in affine coordinate system by setting \(x_0 = 1\) \(y_0 = 1\). Let

\[f_n = (x_1 - a)^i_n (x_1 - b)^j_n (x_1 - c)^k_n y_1^l_n.\]

The points which will have non-zero symbol are \(P_a = (a, 0)\), \(P_b = (b, 0)\) and \(P_c = (c, 0)\). Let the base point \(Q\) be with coordinates \(Q = (0, 0)\).

First we compute the symbol \((f_1, f_2, f_3)_{C, P_a}\). Let

\[z = z(x_1) = \frac{x_1 - a}{x_1 - b}.\]

Then

\[x_1 = \frac{bz - a}{z - 1}.\]

Also

\[x_1 - b = \frac{b - a}{z - 1}\]
and
\[ x_1 - c = \frac{(b - c)z + c - a}{z - 1}. \]

Then
\[ f_n(z, y_1) = z^n y_1^{i_n} \left( \frac{b - a}{z - 1} \right)^{i_n + j_n} \left( \frac{(b - c)z + c - a}{z - 1} \right)^{k_n}. \]

Then at \( P_a \), we have
\[ g_n(z, y_1) = \left( \frac{b - a}{z - 1} \right)^{i_n + j_n} \left( \frac{(b - c)z + c - a}{z - 1} \right)^{k_n}. \]

Using that \( i_n + j_n + k_n = 0 \), we obtain
\[ g_n(z, y_1) = \left( \frac{b - a}{z - 1} \right)^{-k_n} \left( \frac{(b - c)z + c - a}{z - 1} \right)^{k_n}. \]

At the point \( P_a \), which is with coordinates \((z, y_1) = (0, 0)\), we have
\[ g_n(P_a) = \left( \frac{a - c}{a - b} \right)^{k_n}. \]

Therefore,
\[ g_3(P_a)^{i_1 l_2} g_1(P_a)^{-i_3 l_2} = \left( \frac{a - b}{a - c} \right)^{l_2} \begin{vmatrix} i_1 & k_1 \\ i_3 & k_3 \end{vmatrix}. \]

For the sign of the symbol, we have \((-1)^{i_1 i_3 l_2 - i_3 i_1 l_2}\). Finally, the symbol at \( P_a \) is
\[ (f_1, f_2, f_3)_{C,P_a} = (-1)^{i_1 i_3 l_2 - i_1 i_3 l_2} \left( \frac{a - b}{a - c} \right)^{l_2} \begin{vmatrix} i_1 & k_1 \\ i_3 & k_3 \end{vmatrix}. \]

For the symbol at the points \( P_b \) and \( P_c \), we have to permute cyclicly \( a, b, c \) and use the same cyclic permutation for the powers \( i_n, j_n, k_n \). We obtain
\[ (f_1, f_2, f_3)_{C,P_b} = (-1)^{j_1 j_3 l_2 - j_1 j_3 l_2} \left( \frac{b - c}{b - a} \right)^{l_2} \begin{vmatrix} j_1 & i_1 \\ j_3 & i_3 \end{vmatrix}. \]

and
\[ (f_1, f_2, f_3)_{C,P_c} = (-1)^{k_1 k_3 l_2 - k_1 k_3 l_2} \left( \frac{c - a}{c - b} \right)^{l_2} \begin{vmatrix} k_1 & j_1 \\ k_3 & j_3 \end{vmatrix}. \]

Let
\[ A = \frac{a - b}{a - c} \]

and
\[ B = \frac{b - c}{b - a}. \]
which enter in the first two symbols. Then for the fraction in the last symbol, we have

\[ \frac{c-a}{c-b} = -(AB)^{-1}. \]

The product of the three symbols is

\[ (f_1, f_2, f_3)_{C, P_a} (f_1, f_2, f_3)_{C, P_b} (f_1, f_2, f_3)_{C, P_c} = \]

\[ = (-1)^{(i_1 i_3 + j_1 j_3 + k_1 k_3) l_2 - (i_2 + j_2 + k_2) l_1 l_3} (-A B) \]

Combining the power of \( A \) and using that \( i_n + j_n + k_n = 0 \), we obtain

\[ l_2 \begin{vmatrix} i_1 & k_1 \\ i_3 & k_3 \end{vmatrix} - l_2 \begin{vmatrix} k_1 & j_1 \\ k_3 & j_3 \end{vmatrix} = l_2 \begin{vmatrix} -j_1 & k_1 \\ -j_3 & k_3 \end{vmatrix} - l_2 \begin{vmatrix} k_1 & j_1 \\ k_3 & j_3 \end{vmatrix} = 0. \]

Similarly, the powers of \( B \) cancel. For the sign we have

\[ (-1)^{(i_1 i_3 + j_1 j_3 + k_1 k_3) l_2 - (i_2 + j_2 + k_2) l_1 l_3} (-1)^{-l_2} \begin{vmatrix} k_1 & j_1 \\ k_3 & j_3 \end{vmatrix} \]

For the power of \((-1)\) modulo 2, we have

\[ (i_1 i_3 + j_1 j_3 + k_1 k_3) l_2 - (i_2 + j_2 + k_2) l_1 l_3 + l_2 (k_1 j_3 + j_1 k_3) = \]

\[ l_2 (i_1 i_3 + (j_1 + k_1)(j_3 + k_3)) - (i_2 + j_2 + k_2) l_1 l_3 \]

\[ l_2 (i_1 i_3 + (-i_1)(-i_3)) + 0 \cdot l_1 l_3 = \]

\[ = 0 \mod 2 \]

Therefore,

\[ (f_1, f_2, f_3)_{C, P_a} (f_1, f_2, f_3)_{C, P_b} (f_1, f_2, f_3)_{C, P_c} = 1. \]

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