WIGGLE ISLAND

DANNY CALEGARI

ABSTRACT. A wiggle is an embedded curve in the plane that is the attractor of an iterated function system associated to a complex parameter $z$. We show the space of wiggles is disconnected — i.e. there is a wiggle island.

1. Squiggles and wiggles

A squiggle is a continuous map $w_z : I \to \mathbb{C}$ where $I := [0, 1]$, depending on a complex parameter $z$ with $|z| < 1$ and $|1 - z| < 1$ in the following way.

If we define the two maps

$$f_z : x \to -zx + z, \quad g_z : x \to (z - 1)x + 1$$

and inductively define a sequence of maps $w^i_z : I \to \mathbb{C}$ by

1. $w^0_z : [0, 1] \to \mathbb{C}$ is the identity map; and
2. $w^i_z$ is obtained by concatenating the two maps $(g_z w^{i-1}_z(z))$ and $(f_z w^{i-1}_z(z))$, and reversing the orientation (i.e. precomposing with $t \to 1 - t$);

then $w_z$ is the limit of $w^i_z$ as $i \to \infty$. This limit exists because when $|z| < 1$ and $|1 - z| < 1$ both $f_z$ and $g_z$ are uniformly contracting. See Figure 1 illustrating $w^i_z$ for $z = 0.4 + 0.4i$ and $i = 0, 1, 2, 3$.

![Figure 1. $w^i_z$ for $z = 0.4 + 0.4i$ and $i = 0, 1, 2, 3$](image)

A wiggle is a squiggle which is an embedding. The set of wiggles $w_z$ is parameterized by an open subset $W := \{z \text{ such that } w_z \text{ is a wiggle}\}$ of $\mathbb{C}$. A wiggle with parameter $z$ has Hausdorff dimension $d$ where $|z|^d + |1 - z|^d = 1$; thus $W$ is a subset of the open disk of radius $1/2$ centered at $1/2$ (because a squiggle might intersect itself its Hausdorff dimension only satisfies the inequality $|z|^d + |1 - z|^d \geq 1$). Denote the complement of $W$ in this disk by $R$. Figure 2 depicts $W$ (in white) as a subset of this disk (note how $W$ very nearly fills the entire disk!)

The set $W$ contains one big component, the connected component of $1/2$; this big component contains the real interval $(0, 1)$ and the imaginary interval $(-0.5i, 0.5i)$. However, the big component is not all of $W$.

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Figure 2. $W$ (in white) as a subset of $\{z : |z - 1/2| < 1/2\}$.

Theorem 1.1 (Wiggle Island). $W$ is not connected.

Figure 3 depicts a wiggle $w_z$ for $z$ in an ‘island’ component of $W$ centered at approximately $z = 0.3409 + 0.43486i$. The island is invisible at the resolution of Figure 2; a zoomed in image of the island is Figure 6.

The wiggle $w_z$ for $z$ as above may not be deformed through wiggles to $w_{1/2}$ (i.e. the unit interval).

Figure 3. An approximation to $w_z$ for $z = 0.3409 + 0.43486i$; this is an embedded arc.
Remark 1.2 (The Carpenter’s Rule Problem). The Carpenter’s Rule Problem, first posed by Stephen Schanuel and George Bergman in the early 1970’s, asks whether every embedded planar polygonal arc may be straightened (i.e. moved through embeddings to a straight arc by changing the angles but not the lengths of the segments). Connelly, Demaine and Rote showed in 2000 [2] that the answer to the Carpenter’s Rule Problem is yes.

The Wiggle Problem asks analogously whether every wiggle may be straightened through a family of wiggles; Theorem 1.1 says that the answer to the Wiggle Problem is no.

2. Certifying in and out

In this section we give two stable numerical criteria to certify that $z \in \mathcal{W}$ resp. $z \in \mathcal{R}$.

2.1. Certifying $z \in \mathcal{W}$. Fix $z$ and write $\gamma := w_z(I)$ and abbreviate $f_z$ and $g_z$ by $f$ and $g$. Observe that $\gamma = f\gamma \cup g\gamma$, and by induction $z \in \mathcal{W}$ if and only if $f\gamma \cap g\gamma = \{z\}$. Let $S_n$ denote the set of words of length $n$ in the alphabet $\{f, g\}$ and let $S = \cup_n S_n$. By abuse of notation we think of $u \in S_n$ as a map, obtained by composing $f$ or $g$ according to the letters of $u$. Let $fS_{n-1}$ resp. $gS_{n-1}$ represent words of length $n$ beginning with $f$ and $g$ respectively.

Lemma 2.1. Define

$$R = \max \left( \frac{|z-1|}{2|1-|z||}, \frac{|z|}{2|1-|1-z||} \right)$$

and let $B$ be the ball of radius $R$ about $1/2$. Then $\gamma$ is contained in $B$.

Proof. Both $f$ and $g$ take $B$ inside itself. \hfill \Box

Thus to show $z \in \mathcal{W}$ it would suffice to show, for any $n$, that $uB \cap vB = \emptyset$ for all $u \in fS_n$ and $v \in gS_n$ except for one specific pair for which $uB \cap vB = \{z\}$. Unfortunately this is impossible; necessarily $z$ itself is in the interior of some $uB \cap vB$.

The key observation is that $ffgf^{-1} = ggf^{-1}$; in particular, $ffg\gamma \cup ggf\gamma$ is a dilated copy of $\gamma$ itself. Thus if $ffg\gamma \cap ggf\gamma$ contains a point other than $z$, then $f\gamma \cap g\gamma$ contains a point other than $z$ which is not in $ffg\gamma \cap ggf\gamma$. We therefore obtain the following algorithm which, if it terminates, certifies that $z$ is in the interior of $\mathcal{W}$:

Initialize $L$ to the set of pairs $(u, v) \in fS_2 \times gS_2 - (ffg, ggf)$.

while $L \neq \emptyset$ do
for all $(u, v) \in L$ do
if $uB \cap vB = \emptyset$ then
remove $(u, v)$ from $L$
else
replace $(u, v)$ with $\{(uf, vf), (uf, vg), (ug, vf), (ug, vg)\}$
end
end
end
2.2. Certifying \( z \in \mathcal{R} \). We now show how to modify the algorithm from the previous subsection to certify (numerically) that \( z \in \mathcal{R} \). Actually our modified algorithm certifies that \( z \) is contained in the interior of \( \mathcal{R} \), and therefore fails for \( z \) in the frontier of \( \mathcal{R} \). However one consequence of the nature of the algorithm is that it implies that the interior of \( \mathcal{R} \) is dense.

The idea is a modification of the method of traps, introduced in [1] to prove Bandt’s Conjecture on interior points in \( \mathcal{M} \), the connectivity locus for another 1-parameter family of complex 1-dimensional IFSs (and the analog to \( \mathcal{R} \) for this family).

The idea is very simple. Let \( u \in fS \) and \( v \in gS \) and choose some finite \( n \). Let \( uS_nB \) denote the union of translates of \( B \) for all elements of \( uS_n \) and likewise \( vS_nB \).

**Definition 2.2** (Stable Crossing). The triple \((u, v, n)\) is a stable crossing if there are disjoint proper rays \( r^{\pm} \), \( s^{\pm} \) from points \( p^{\pm} \) and \( q^{\pm} \) in \( u\gamma, v\gamma \) to infinity so that \( r^{\pm} \) are disjoint from \( vS_nB \), so that \( s^{\pm} \) are disjoint from \( uS_nB \), and so that \( r^{\pm} \) link \( s^{\pm} \) at infinity.

The existence of a stable crossing is, in fact, stable in \( z \); and given \((u, v, n)\) stable for \( z \) one may easily estimate a lower bound on the radius of a ball around \( z \) for which this triple continues to be stable.

Furthermore, the existence of a stable crossing certifies \( z \in \mathcal{R} \) for elementary topological reasons.

**Lemma 2.3.** If \((u, v, n)\) is a stable crossing for \( z \) then \( z \in \mathcal{R} \).

**Proof.** Let \( p^{\pm} \in u\gamma \) and \( q^{\pm} \in v\gamma \) be the finite endpoints of \( r^{\pm} \) and \( s^{\pm} \) respectively. Since \( u\gamma \) and \( v\gamma \) are path connected, there are arcs \( \alpha \subset u\gamma \) and \( \beta \subset v\gamma \) joining \( p^{\pm} \) and \( q^{\pm} \). Then \( r^{\pm} \cup \alpha \cup r^{-} \) and \( s^{\pm} \cup \beta \cup s^{-} \) are properly immersed lines in \( \mathbb{C} \) that are embedded and disjoint at infinity where their endpoints are linked. Thus their algebraic intersection number (rel. their ends) is odd, so they must intersect. But by hypothesis the only place they might intersect is \( \alpha \) with \( \beta \).

To search for stable crossings we enumerate pairs \((u, v)\) by the algorithm from the previous section, then for each we compute \( uS_nB \) and \( vS_nB \) for some finite \( n \) and look for rays \( r^{\pm}, s^{\pm} \) and points \( p^{\pm}, q^{\pm} \) forming a stable crossing. If we find one we certify \( z \) (and some ball of computable radius about it) as lying in \( \mathcal{R} \).

Let’s say more generally that two compact path-connected subsets \( K, L \subset \mathbb{C} \) have a stable crossing if there is some \( \epsilon \) and disjoint rays \( r^{\pm}, s^{\pm} \) from points \( p^{\pm}, q^{\pm} \) in \( K \) and \( L \) so that \( r^{\pm} \) is disjoint from the \( \epsilon \)-neighborhood of \( L \) and \( s^{\pm} \) is disjoint from the epsilon-neighborhood of \( K \), and \( r^{\pm}, s^{\pm} \) link at infinity. Thus \( u\gamma, v\gamma \) have a stable crossing if and only if \((u, v, n)\) is a stable crossing for some \( n \). See Figure 5 for some relevant examples.

As an application of stable crossings we prove that the interior of \( \mathcal{R} \) is dense in \( \mathcal{R} \). Both the statement and the proof are very similar to Theorem 7.2.7 from [1].

**Theorem 2.4** (Interior is Dense). The interior of \( \mathcal{R} \) is dense in \( \mathcal{R} \).

**Proof.** It suffices to find a stable crossing arbitrarily close to any \( z_0 \in \mathcal{R} \). Let \( z_0 \in \mathcal{R} \) so that \( u\gamma \) intersects \( v\gamma \) for some finite words \( u, v \) which do not start with the prefix \( ffg, ggf \). Without loss of generality, we may assume \( u, v \) both have length \( n \gg 1 \). There are complex numbers \( \alpha, \beta \) so that \( \gamma = \alpha u^{-1} u\gamma + \beta \), and by padding \( u \) or \( v \) or both with additional letters
Lemma 2.5
Proof. If \( g \) is not convex, there is some complex number \( z = 1/\beta \) with nonempty interior intersect \( v\gamma \).

As we vary \( z \) we get \( \gamma(z) = \alpha(z) v^{-1} u\gamma(z) + \beta(z) \) for suitable analytic functions \( \alpha \) and \( \beta \). Now, \( \beta(z) \) is not constant; one way to see this is to observe that it is much much bigger at \( z = 1/2 \) than at \( z_0 \). If \( n \) is big enough, \( u\gamma \) and \( v\gamma \) have diameter extremely small compared to the first nonvanishing derivative of \( \beta \) at \( z_0 \). Thus for any \( \epsilon \), there is an \( n \) so that if we choose \( n \) as above, and \( U \) is the disk of radius \( \epsilon \) about \( z_0 \), then \( \{\beta(z), z \in U\} \) contains the disk of radius \( 1/\epsilon \) about zero, while \( \alpha(z) \) stays essentially constant. Thus we would be done if we could show that there is some \( \mu \leq 1/\epsilon \) so that \( \gamma \) and \( \alpha\gamma + \mu \) have a stable crossing. This is proved in Lemma 2.5 and Lemma 2.6.

Thus Theorem 2.4 is reduced to the following two lemmas:

Lemma 2.5 (Translation Lemma). Let \( K \) be a compact full subset of \( \mathbb{C} \) and suppose \( K \) is not convex. There is some complex number \( \mu \) of order \( \text{diam}(K) \) so that \( K \) and \( K + \mu \) have a stable crossing.

This is Lemma 7.2.2 from [1]. Finally we need to understand the set of \( z \) for which \( \gamma(z) \) is convex.

Lemma 2.6 (\( \gamma \) not convex). For \( |z - 1/2| < 1/2 \) and \( z \) not real, \( \gamma(z) \) is not convex.

Of course if \( |z - 1/2| < 1/2 \) and \( z \) is real then \( \gamma(z) = [0, 1] \) and \( z \in \mathcal{W} \).

Proof. If \( \gamma(z) \) is convex but not real it encloses a full subset \( X \subset \mathbb{C} \) with nonempty interior (in particular \( X \) has Hausdorff dimension 2). We claim that actually \( \gamma(z) = X \). To see this, let’s let \( p \in X \) be a point which realizes the maximal distance \( \epsilon \) to \( \gamma(z) \). Because \( X \) is convex, and \( f(X), g(X) \) intersect, it follows that \( X = f(X) \cup g(X) \) (without convexity there might be some omitted region ‘trapped’ between \( f(X) \) and \( g(X) \)). But then \( p \in f(X) \) or \( g(X) \); thus the distance from \( p \) to \( \gamma(z) \) is at most \( \max(|z|\epsilon, |1 - z|\epsilon) \). Since \( |z| \) and \( |1 - z| \) are both strictly less than 1 it follows that \( \epsilon = 0 \) and therefore \( \gamma(z) = X \) and has Hausdorff dimension \( d = 2 \). But then \( |z|^2 + |1 - z|^2 \geq 1 \) so \( |z - 1/2| \geq 1/2 \) and we are done.

Compare to Lemma 7.2.3 from [1]. This completes the proof of Theorem 2.4. We end this subsection with a conjecture:

Conjecture 2.7. Every interior point of \( \mathcal{R} \) is certified by some stable crossing \( (u, v, n) \).

2.3. Proof of Theorem 1.1. Using the techniques of the previous two sections, we may numerically certify some \( z \in \mathcal{W} \) and numerically check that a polygonal loop separating \( z \) from 0 is contained in \( \mathcal{R} \). To make this computationally feasible we pursue the following strategy:
(1) find an apparent island by some experimentation, in a small region $U \subset \mathbb{C}$;
(2) compute a region $O \subset \mathbb{C}^* \times \mathbb{C}$ so that $\gamma(z)$ and $\alpha \gamma(z) + \beta$ have a stable crossing for all $(\alpha, \beta) \in O$ and all $z \in U$;
(3) run the algorithm on a narrowly spaced grid of $z \in U$ to generate crossing pairs $(u, v)$ and for each compute $(\alpha(z), \beta(z))$ such that $\gamma(z) = \alpha(z)v^{-1}u\gamma(z) + \beta(z)$;
(4) if $(\alpha(z), \beta(z))$ is in $O$, compute an $\epsilon$ so that $(\alpha(z'), \beta(z')) \in O$ when $|z' - z| < \epsilon$; and finally
(5) when we have generated enough $\epsilon$-balls centered at $z$ in a fine enough grid to surround a point in $W$, we have proved the theorem.

We elaborate on these points.

![Figure 4](image1.png)

**Figure 4.** The disk $\Gamma$ encloses $\gamma(z)$ for all $z \in U$. The vertices 0 and 1 of $\gamma(z)$ are distinguished.

The region $U$ is an open square centered at $0.3409 + 0.43486i$ with width 0.00005. We may then readily determine the vertices of a polyhedral disk $\Gamma$ that contains a thin neighborhood of $\gamma(z)$ for all $z \in U$; see Figure 4.

![Figure 5](image2.png)

**Figure 5.** Stable crossings of $\Gamma$ and $\alpha \Gamma + \beta$.

Next we compute some open polydisks in $\mathbb{C}^* \times \mathbb{C}$ parameterizing $(\alpha, \beta)$ for which $\Gamma$ and $\alpha \Gamma + \beta$ have a suitable stable crossing (with rays landing at 0, 1 and $\beta$, $\alpha + \beta$ respectively, which are always in $\gamma(z)$ and $\alpha \gamma(z) + \beta$); see Figure 5.

The program wiggle implements the algorithm from the previous section, and rigorously finds $\epsilon$-balls (actually polygons) centered at points $z$ near $W$ with a stable crossing for some fixed $(u, v, n)$. The result of the output is Figure 6. The blocks of solid color in the figure
correspond to specific \((u, v, n)\). These blocks surround Wiggle Island (in the north of the figure), completing the proof of Theorem 1.1. The isthmus to the southeast is part of the big component of \(W\).

\[\text{Figure 6. Stable crossings in } \mathcal{R} \text{ surround Wiggle Island.}\]

Intermediate between Wiggle Island and the mainland there is apparent in the Figure a smaller island, and a speck (which on magnification turns out too be an island too). It seems likely that there are infinitely many islands, arranged in an asymptotically geometric spiral converging to an algebraic point \(z_c \sim 0.340922 + 0.43481i\). One could give a rigorous numerical proof of this by a modification of the argument proving Theorem 1.1, applied to the tangent cone to \(W\) centered at \(z_c\), though we have not pursued this. An exactly analogous argument for \(M\) certifying the existence of a spiral of islands is proved in § 9 of [1] and we refer the interested reader to that paper for details.

We end with a conjecture, parallel to Conjecture 9.2.7 from [1]:

**Conjecture 2.8.** Every point in the frontier of \(\mathcal{R}\) is the limit of islands in \(W\) of diameter going to zero.

**Remark 2.9.** In fact the existence of Wiggle Island has been well-known to Europeans as least since the early 17th century, and is well-documented in the literature [3].

**References**

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[2] R. Connelly, E. Demaine and G. Rote, *Straightening polygonal arcs and convexifying polygonal cycles*, Proc. 41st Annual Symposium on Foundations of Computer Science, 432–442, Redondo Beach, California, November 2000.  
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University of Chicago, Chicago, Ill 60637 USA
Email address: dannyc@math.uchicago.edu