Application of bipolar coordinates to the analysis of the structure of viscous fluid flow between two rotating cylinders

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Abstract. The motion of a viscous fluid between two circular rotating cylinders is studied. The transition to the bipolar coordinates is applied to determine the general analytical expressions for stream function. The structure of the stream lines is investigated in the cases when one of the cylinders is motionless and when the both cylinders rotate.

1. Introduction
The problem of viscous fluid flow between two rotating circular cylinders has important applications: the motion of fluid lubricating oils in the gap between the rotating shaft and the bearing cushion, the problems of theory of hydrodynamic stability, the problem of mixing. The simplest solution can be obtained in the case of axisymmetric rotation of the cylinders with constant angular velocities [1]. In 1906 N. E. Joukowski and S. A. Chaplygin have considered a friction of a lubricating layer between the spike and the bearing from a mathematical point of view [2]. By its hydrodynamic essence, this is the problem of research of a viscous fluid between two eccentrically arranged circular cylinders, one of which (inner cylinder) rotates with a constant angular velocity and the second one (outer cylinder) is motionless [2]. In this work we will consider the rotation of both inner and outer cylinders. Mathematical model of the problem is described by boundary value problem for biharmonic equation in eccentric ring. For solving it we will use bipolar coordinates in the same way as N. E. Joukowski and S. A. Chaplygin. The Reynolds number is assumed to be small and the equations of motion are solved in the linear Stokes approximation. This paper is a continuation of the short report [3].

2. Mathematical model and bipolar coordinates
Let us consider the motion of a confined fluid in the case when the inertial forces can be neglected (the Stokes approximation). For the plane problem, Lagrange introduced the stream function \( \Psi(x, y) \) (see [4]) and expressed the velocity components in terms of it in the incompressibility assumption:

\[
\begin{align*}
    v_x &= \frac{\partial \Psi}{\partial y}, & v_y &= -\frac{\partial \Psi}{\partial x}.
\end{align*}
\]  

(1)

The stream function satisfies the biharmonic equation \( \Delta^2 \Psi = 0 \) [4]. This observation later became common practice and is widely used in classical literature on viscous fluid dynamics [4–6] and also when numerical solutions were constructed [7–9].

Fluid flow between the cylinders is plane-parallel. On figure 1 the transverse cross-section of the system of two cylinders is presented. It is the domain between two circles. The radii of outer and inner
circles are \( \rho_1 \) and \( \rho_2 \), respectively, and the distance between their centers is \( \Delta x \). We introduce a Cartesian coordinate system whose origin is located at the center of the outer circle and the \( x \) axis passes through the line of centers of the circles. So, in solving the problem we will use the bipolar coordinates which are somewhat different from those used in [2] (figure 1):

\[
\xi = \angle MFA - \angle MF'A, \quad \eta = \ln \left( \frac{FM'}{FM} \right).
\]  

The relation between the bipolar coordinates (2) and the Cartesian coordinates is

\[
\eta = \frac{1}{2} \ln \left( \frac{(x-c+a)^2 + y^2}{(x-c-a)^2 + y^2} \right), \quad \xi = \arctg \frac{x-c+a}{y} - \arctg \frac{x-c-a}{y},
\]  

where \( 2a \) is the distance between the poles \( F \) and \( F' \) and \( c = -a \cosh \eta_1 \).

\[\text{Figure 1. Bipolar coordinates (1) and lines } \eta = \text{const}.\]

The lines \( \xi = \text{const} \) are circles containing chord \( FF' \) and the lines \( \eta = \text{const} \) are orthogonal circles (figure 1). The problem reduces to determination of the biharmonic stream function \( \Psi \) inside the domain between two circles: the outer circle \( \eta = \eta_1 \) of radius \( \rho_1 \) and the inner circle \( \eta = \eta_0 \) of radius \( \rho_0 \). From (3) there follow the equalities:

\[
a = \frac{1}{2\Delta x} \sqrt{(\Delta x^2 - \rho_1^2 - \rho_0^2)^2 - 4\rho_1^2 \rho_0^2}, \quad \eta_1 = \ln \frac{\sqrt{a^2 + \rho_1^2} + a}{\rho_1}, \quad \eta_0 = \ln \frac{\sqrt{a^2 + \rho_0^2} + a}{\rho_0}.
\]

### 3. Analytical solution of the problem

In [3] it was shown that the function \( \Psi \) can be represented in the form:

\[
\Psi = N(\eta) + M(\eta) / \left( \cosh \eta - \cos \xi \right), \quad N(\eta) = A\eta - F \cosh 2\eta - G \sinh 2\eta,
\]

\[
M(\eta) = B \sinh \eta + C \cosh \eta + E \eta \sinh \eta + F \eta \cosh 2\eta + G \cosh \eta \cosh 2\eta.
\]

**The boundary conditions of the problem:** the inner and the outer cylinders rotate with angular velocities \( \omega_0 \) and \( \omega_1 \), respectively; then on their surfaces the velocities are equal to \( U_0 = \omega_0 \rho_0 \) and \( U_1 = \omega_1 \rho_1 \). The stream function \( \Psi \) must be constant on both surfaces. Hence there follow the conditions for the functions \( M(\eta) \) and \( N(\eta) \) [3]:

\[
M(\eta_0) = 0, \quad M(\eta_1) = 0, \quad \frac{dN}{d\eta} \bigg|_{\eta=\eta_0} = 0, \quad \frac{dM}{d\eta} \bigg|_{\eta=\eta_0} = -U_0 a, \quad \frac{dN}{d\eta} \bigg|_{\eta=\eta_1} = 0, \quad \frac{dM}{d\eta} \bigg|_{\eta=\eta_1} = -U_1 a.
\]

Conditions (6) give system of equations to determine six coefficients of the stream function (5). By solving the obtained system we determine six unknown coefficients of the function \( \Psi \):
\[ A = a \cosh \beta \left( \frac{U_1 \cosh \eta_1 - U_0 \cosh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} - \sinh \beta (\cosh \alpha - \cosh \beta) \right) \]
\[ E = a \left( \frac{U_1 \sinh \eta_1 - U_0 \sinh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} + \frac{U_1 \cosh \eta_1 - U_0 \cosh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} \right) \]
\[ C = 0.5 \sinh \eta_0 \left( E \sinh \eta_0 + A \cosh \eta_0 + aU_0 \right) + 0.5 \sinh \eta_1 \left( E \sinh \eta_1 + A \cosh \eta_1 + aU'_1 \right) \]
\[ B = -E\eta_0 - \cosh \eta_0 \left( E \cosh \eta_0 + A \sinh \eta_0 + aU_0 \right), \quad F = -0.5 A \sinh \alpha / \cosh \beta, \quad G = 0.5 A \cosh \alpha / \cosh \beta. \]

where the notations \( \alpha = \eta_0 + \eta_1 \) and \( \beta = \eta_1 - \eta_0 \) are introduced.

Thus, we obtain the stream function \( \Psi(\zeta, \eta) \) in which the bipolar coordinates \( \zeta, \eta \) can be expressed in terms of the Cartesian coordinates by the formulas (3).

4. The analysis of the flow structure

4.1. The types of fixed points

Now we analyze the structure of the fluid flow. It is determined by the presence of fixed points on the axis \( x \). So in figure 2 the streamlines \( \Psi = \text{const} \) are presented when \( \rho_1 = 1, \rho_0 = 0.3, \Delta x = 0.35, \omega_1 = -1, \omega_0 = 4 \) (cylinders rotate in different directions), there are two fixed points \( K_0 \) and \( K_1 \). Point \( K_0 \) is the point of hyperbolic type: in its small neighborhood the streamlines are hyperbolas. Point \( K_1 \) is the point of elliptic type: in its small neighborhood the streamlines are ellipses. Near this point a vortex is formed. The point of elliptic type corresponds to a stable equilibrium position. For any sufficiently small deviation from the equilibrium position the particle of the fluid constantly moves in a small neighborhood of this point. Hyperbolic fixed point corresponds to an unstable equilibrium position. For any arbitrarily small deviation from the equilibrium position the particle of the fluid deviates from the equilibrium position by a finite distance in a sufficiently long time.

\[ \begin{align*}
A &= a \cosh \beta \left( \frac{U_1 \cosh \eta_1 - U_0 \cosh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} - \sinh \beta (\cosh \alpha - \cosh \beta) \right) \\
E &= a \left( \frac{U_1 \sinh \eta_1 - U_0 \sinh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} + \frac{U_1 \cosh \eta_1 - U_0 \cosh \eta_0}{\beta (\cosh \alpha \cosh \beta - 1)} \right) \\
C &= 0.5 \sinh \eta_0 \left( E \sinh \eta_0 + A \cosh \eta_0 + aU_0 \right) + 0.5 \sinh \eta_1 \left( E \sinh \eta_1 + A \cosh \eta_1 + aU'_1 \right) \\
B &= -E\eta_0 - \cosh \eta_0 \left( E \cosh \eta_0 + A \sinh \eta_0 + aU_0 \right), \quad F = -0.5 A \sinh \alpha / \cosh \beta, \quad G = 0.5 A \cosh \alpha / \cosh \beta.
\end{align*} \]

where the notations \( \alpha = \eta_0 + \eta_1 \) and \( \beta = \eta_1 - \eta_0 \) are introduced.

Thus, we obtain the stream function \( \Psi(\zeta, \eta) \) in which the bipolar coordinates \( \zeta, \eta \) can be expressed in terms of the Cartesian coordinates by the formulas (3).

4.2. Flow structure when one of the cylinders is motionless

We first investigate this question for the case when one of the cylinders is motionless. Then the stream function will be proportional to the angular velocity of rotation of the other cylinder, and the location of fixed points does not depend on it. Without loss of generality, we assume that \( \rho_1 = 1 \) and \( \Delta x > 0 \). Then the coordinates \( x \) of the fixed points will depend on \( \rho_0 \) and \( \Delta x \). The fixed point is found from the condition that the velocity is equal to 0.

**Figure 2.** The scheme of the fluid flow between two cylinders rotating in opposite directions.
It can be shown that since the function $\Psi(x, \eta)$ is monotonic on a segment $\eta_1 < \eta < \eta_0$, in the case when one of the cylinders is motionless, there are no fixed points on the segment $AB$ (see figure 2). On $CD$, the fixed point is found from equation

$$V(\eta) = d\Psi(0, \eta)/d\eta = 0 \quad (\eta_1 < \eta < \eta_0).$$

We define the limiting dependence $\Delta x^*(\rho_0)$ so that when $\Delta x < \Delta x^*(\rho_0)$ there are no fixed points on $CD$, and when $\Delta x > \Delta x^*(\rho_0)$ there is a fixed point on $CD$. In [8] it was shown that the limiting condition with motionless outer cylinder has the form $V(\eta_1) = 0$, and with motionless inner cylinder has the form $V(\eta_0) = 0$. It divides two cases: the presence and absence of a fixed point on the segment $CD$. On figure 3 the limit curves $\Delta x^*(\rho_0)$ obtained from these considerations for the cases when one of the cylinders is motionless are shown. These results are consistent with the limit curves obtained in [10], where the dependence $\Delta x/(1-\rho_0)$ is given.

![Figure 3](image1.png)

**Figure 3.** Limit curve $\Delta x^*(\rho_0)$ for cases when one cylinder is motionless: a– $\omega_1 = 0$; b – $\omega_0 = 0$.

On figure 4 the schemes of fluid flow for some different values of $\Delta x$ are shown when the inner cylinder is motionless ($\rho_1 = 1$, $\rho_0 = 0.4$, $\omega_0 = 0$, $\omega_1 = 1$). On figure 5 the schemes of fluid flow for some different values of $\Delta x$ are presented when the outer cylinder is motionless ($\rho_1 = 1$, $\rho_0 = 0.5$, $\omega_0 = 0$, $\omega_1 = 1$). As one can see from these figures, if there is a fixed point on the segment $CD$, it is the point of elliptic type.

![Figure 4](image2.png)

**Figure 4.** The schemes of fluid flow for different values of $\Delta x$ ($\rho_1 = 1$, $\rho_0 = 0.4$, $\omega_1 = 0$, $\omega_1 = 1$): a– $\Delta x = 0.125$; b– $\Delta x = 0.175$; c– $\Delta x = 0.5$. 

![Figure 5](image3.png)
Figure 5. The schemes of fluid flow for different values of $\Delta x$ ($\rho_1 = 1$, $\rho_0 = 0.5$, $\omega_1 = 0$, $\omega_0 = 1$):

a – $\Delta x = 0.125$; b – $\Delta x = 0.225$; c – $\Delta x = 0.4$.

4.3. Flow structure when the both cylinders rotate

If the both cylinders rotate, then two fixed points can appear on the axis $x$. Moreover, if there are two fixed points, then if the cylinders rotate in the same direction, the both these points lie on the segment $CD$, and if the cylinders rotate in the opposite directions, one of them will lie on the segment $CD$, and another will lie on the segment $AB$ (figure 2). If we assume again that $\rho_1 = 1$ and $\Delta x > 0$, then in this case the coordinates $x$ of the fixed points will depend not only on $\rho_0$ and $\Delta x$, but also on angular velocities ratio.

On figure 6 the schemes of fluid flow for some different values of $\omega_0$ and $\Delta x$ are shown when the cylinders rotate in the same directions ($\rho_1 = 1$, $\rho_0 = 0.3$, $\omega_1 = 1$, $\omega_0 > 0$). As one can see from the figure 6, three cases are possible:

- there are no any fixed points in the flow domain (figure 6 (a));
- there is one fixed point of elliptic type on the segment $CD$ of $x$-axis (figure 6 (b));
- there are two fixed points of elliptic type on the segment $CD$ (figure 6 (c)).

Figure 6. The schemes of fluid flow for different values of $\Delta x$ and $\omega_0$ ($\rho_1 = 1$, $\rho_0 = 0.3$, $\omega_1 = 1$):

a – $\Delta x = 0.35$, $\omega_0 = 20$; b – $\Delta x = 0.475$, $\omega_0 = 24$; c – $\Delta x = 0.525$, $\omega_0 = 12$.

On figure 7 the schemes of fluid flow for some different values of $\omega_0$ and $\Delta x$ are presented when the cylinders rotate in the opposite directions ($\rho_1 = 1$, $\rho_0 = 0.2$, $\omega_1 = -1$, $\omega_0 > 0$). As one can see from the figure 7, if the cylinders rotate in the opposite directions, even for small value of $\Delta x$, there are
always two fixed points on $x$-axis: one of them is the point of hyperbolic type and it is located on the segment $AB$; another is the point of elliptic type and it is located on the segment $CD$.

![Figure 7. The schemes of fluid flow for different values of $\Delta x$ and $\omega_0$](image)

5. Conclusion
Generalization of the Joukowski-Chaplygin solution of the hydrodynamic problem of the motion of a viscous fluid between two eccentrically arranged circular cylinders is obtained. The problem is solved for the case of rotation of the both cylinders. For solving the problem the Stokes approximation and bipolar coordinates are used.

The structures of stationary flows are analyzed. In the case of rotation of one of the cylinders on the Ox axis only one fixed point can occur. In the case of rotation of the both cylinders: when rotating in different directions, two fixed points are located on the $x$-axis on two sides of the inner cylinder; when rotating in one direction in the case of two fixed points, both are located on a larger segment of the $x$-axis.

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References
[1] Petrov N P 1883 Friction in machines and the effect of lubricating fluid (St. Peterb.: Inzh. Zh.)
[2] Joukowski N E and Chaplygin S A 1906 Tr. Otdeleniya Fiz. Nauk O-va Lyubitelei Estestvoznaniya 13(1) 24-33
[3] Kazakova A O 2019 ANS Conference Series: Scientific Heritage of Sergey A. Chaplygin (Nonholonomic Mechanics, Vortex Structures and Hydrodynamics) 82-4
[4] SlezkinN A 1955 Viscous incompressible fluid dynamics (Moscow: Gostekhizdat)
[5] Lamb H 1945 Hydrodynamics (New York: Dover)
[6] Loitsyanskii L G 1996 Mechanics of liquids and gases (New York: Begell House)
[7] Terentiev A G and Terentiev A A 2002 Izvestiya NANI ChR 2 44-62
[8] Kazakova A O and Petrov A G 2016 Fluid Dynamics 51(3) 311-20
[9] Terentiev A G, Kazakova A O and Mikishanina E A 2018 Proc. of the 6th All-Russian Scientific Conference “Information Technologies for Intelligent Decision Making Support” 1 34-42
[10] Ballal B Y and Rivlin R S 1976 Arch. Rational Mech. Anal. 62 237-94