Atiyah-Singer Index Theorem in an \(SO(3)\) Yang-Mills-Higgs System and Derivation of a Charge Quantization Condition

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(Received May 23, 2007)

The Atiyah-Singer index theorem is generalized to a two-dimensional \(SO(3)\) Yang-Mills-Higgs (YMH) system. The generalized theorem is proven by using the heat kernel method and a nonlinear realization of \(SU(2)\) gauge symmetry. This theorem is applied to the problem of deriving a charge quantization condition in the four-dimensional \(SO(3)\) YMH system with non-Abelian monopoles. The resulting quantization condition, \(eg = n (n \in \mathbb{Z})\), for an electric charge \(e\) and a magnetic charge \(g\) is consistent with that found by Arafune, Freund and Goebel. It is shown that the integer \(n\) is half of the index of a Dirac operator.

§1. Introduction

The theory of magnetic monopoles has been studied by many people from various physical and mathematical points of view.\(^1\)–\(^{13}\) Although there is yet no experimental evidence of the existence of magnetic monopoles, it is believed that monopoles play crucial roles in long-standing problems in theoretical physics, such as the grand unification of forces and the confinement problem in quantum chromodynamics.

One of the most important consequences of the theory of monopoles is that electric charges are quantized in units that are inversely proportional to the magnetic charge of the monopole. Such an interesting role of monopoles was discovered by Dirac within the framework of quantum mechanics.\(^1\) In the natural units such that \(\hbar = c = 1\), the quantization condition shown by Dirac reads

\[
eg g = \frac{n}{2}, \quad n \in \mathbb{Z},
\]

where \(e\) and \(g\) denote the electric and magnetic charges, respectively. Another charge quantization condition was discovered by Schwinger in his study of relativistic quantum electrodynamics with a magnetic charge.\(^2\) Schwinger’s quantization condition reads

\[
eg g = n, \quad n \in \mathbb{Z}.
\]

The difference between these two conditions is essentially due to a difference in the string singularities of the gauge potentials adopted in the Dirac and Schwinger formalisms.\(^3\)

Recently, both the conditions (1·1) and (1·2) were derived in a unified manner by utilizing the Atiyah-Singer index theorem in two dimensions.\(^4\) (For the Atiyah-Singer index theorem in any even number of dimensions, see, e.g., Refs. 16–23.)

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The approach taken there is a second quantized approach in the sense that all the eigenfunctions of the Dirac operator are taken into account simultaneously. Unlike previous approaches, this approach requires neither classical notion for paths around a string singularity nor the concepts of patches and sections. To derive Eqs. (1.1) and (1.2), it is only necessary to solve a simple Dirac equation in two dimensions and to formally count the number of zero-modes of the Dirac operator. Each of Eqs. (1.1) and (1.2) can be regarded as the necessary and sufficient condition that the Atiyah-Singer index theorem in two dimensions be valid for the $U(1)$ gauge theory with a monopole background.

The above-mentioned conditions are concerned with Abelian monopoles. In addition to Abelian monopoles, there also exist so-called non-Abelian monopoles. They are realized in some non-Abelian gauge theories as solitonic objects.\(^5\)\(^-\)\(^13\) Non-Abelian monopoles have been studied since 't Hooft and Polyakov independently discovered that a simultaneous system of field equations in the $SO(3)$ Yang-Mills-Higgs (YMH) theory admits a static solution representing monopoles with unit magnetic charge.\(^5\),\(^6\) Arafune, Freund and Goebel clarified the geometric origin of the conserved magnetic charge defined in the $SO(3)$ YMH theory, and found the charge quantization condition valid in this theory: \(^7\)

$$eg = n, \quad n \in \mathbb{Z}.$$  \hspace{1cm} (1.3)

Here, $e$ is the Yang-Mills electric charge, and $g$ denotes the conserved magnetic charge. Arafune et al. demonstrated this condition by considering the homotopy class of a triplet of Higgs fields. Configurations of monopoles with non-minimal magnetic charge, $g = n/e \ (|n| \geq 2)$, were constructed by Bais using a method in which the Dirac monopole potential is embedded in the $SU(2)$ Lie algebra.\(^8\)

In this paper, we study an alternative approach to deriving Eq. (1.3) using a generalization of the Atiyah-Singer index theorem in two dimensions. This approach is a non-Abelian analog of the approach taken in Ref. 4) to derive Eqs. (1.1) and (1.2). In order to treat $SO(3)$ non-Abelian monopoles, we first consider a generalization of the Atiyah-Singer index theorem to a two-dimensional $SO(3)$ YMH system. Although the generalized theorem itself is valid for any two-dimensional manifold with spin structure, here we choose a sphere embedded in four-dimensional spacetime as the two-dimensional manifold. Then, taking the radius of the sphere to be infinitely large, we actually solve a massless Dirac equation on this sphere to find the zero-modes of the Dirac operator included in this equation. After examining the number of chirality zero-modes of the Dirac operator, it is shown that the generalized Atiyah-Singer index theorem leads to the charge quantization condition (1.3). In this argument, the integer $n$ is defined as half of the index of the Dirac operator.

This paper is organized as follows. Section 2 presents the Atiyah-Singer index theorem generalized to a two-dimensional $SO(3)$ YMH system. A simple proof of this theorem is also given there. In §3, the generalized theorem is applied to deriving a charge quantization condition, which is shown to be identical to Eq. (1.3). Section 4 is devoted to a summary and discussion. The appendix gives concrete forms of the $SO(3)$ monopole configurations and, with these configurations, illustrates the charge quantization condition derived in §3.
§2. Atiyah-Singer index theorem in an $SO(3)$ YMH system

Let $\mathcal{M}$ be a compact, oriented, two-dimensional Riemannian manifold without boundary. Let $A$ be a hermitian Yang-Mills connection on $\mathcal{M}$ that takes values in the Lie algebra $\mathfrak{su}(2)$ of the group $SU(2)$, and let $\Phi$ be a hermitian scalar field on $\mathcal{M}$ that also takes values in $\mathfrak{su}(2)$. Then $A$ and $\Phi$ can be expanded as $A = A^I \tau_I$ ($I = 1, 2, 3$) and $\Phi = \Phi^I \tau_I$ in terms of the Pauli matrices $\tau_I$. The component fields $(A^I, \Phi^I)$ constitute an $SO(3)$ YMH system. Here, we impose the normalization condition $\text{tr}(\Phi \Phi) = 2$, or equivalently $\sum_{I=1}^{3} (\Phi^I)^2 = 1$, without destroying the $SO(3)$ symmetry of the system; accordingly, $(\Phi^I)$ is treated as a normalized triplet of Higgs fields. Under this condition, it is possible to diagonalize $\Phi$ in such a way that $v^\dagger \Phi v = \tau_3$, with $v \in SU(2)$. Thus, $\Phi$ can be represented as

$$\Phi = v \tau_3 v^\dagger.$$  \hfill (2.1)

From $A$, $\Phi$ and $\Psi_i \equiv v \tau_i v^\dagger$ $(i = 1, 2)$, we define the new connection

$$A^\perp \equiv A - \frac{1}{2e} \epsilon_{ij3} \text{tr}(\Psi_i D\Phi) \Psi_j,$$  \hfill (2.2)

where $D\Phi \equiv d\Phi - i(e/2)[A, \Phi]$, and $e$ is an electric charge. Obviously, $A^\perp$ takes values in $\mathfrak{su}(2)$.

Assuming that $\mathcal{M}$ possesses a spin structure, we consider a self-adjoint Dirac operator $i\overline{D}^\perp$ containing $A^\perp$ instead of $A$. In terms of local coordinates $(q^\alpha)$ $(\alpha = 1, 2)$ on $\mathcal{M}$, the Dirac operator $i\overline{D}^\perp$ can be expressed as

$$i\overline{D}^\perp \equiv i(\tau_0 \otimes \sigma_a) e_a^\alpha D^\perp_\alpha,$$  \hfill (2.3)

with

$$D^\perp_\alpha \equiv (\tau_0 \otimes \sigma_0) \partial_\alpha + \frac{i}{2} \tau_0 \otimes (\omega_\alpha \sigma_3) - i\frac{e}{2} A^\perp_\alpha \otimes \sigma_0.$$  \hfill (2.4)

Here $\partial_\alpha \equiv \partial/\partial q^\alpha$, $e_a^\alpha$ $(a = 1, 2)$ is an inverse zweibein on $\mathcal{M}$, and $\omega_\alpha$ is a spin connection in two dimensions. Both $\tau_0$ and $\sigma_0$ denote the $2 \times 2$ unit matrices, while $\sigma_a$ and $\sigma_3$ denote the Pauli matrices which are understood as Dirac matrices in two dimensions. The symbol $\otimes$ stands for the tensor product of the $\tau$-matrices and the $\sigma$-matrices. It is obvious that the operator $i\overline{D}^\perp$ can be represented as a $4 \times 4$ matrix.

Let $\varphi_{t,s}^{l,s}(t, s = +, - ; \nu_t,s = 1, \ldots, n_{t,s})$ be chirality zero-modes of $i\overline{D}^\perp$ characterized by

$$i\overline{D}^\perp \varphi_{t,s}^{l,s} = 0,$$  \hfill (2.5)

$$(\Phi \otimes \sigma_0) \varphi_{t,s}^{l,s} = t \varphi_{t,s}^{l,s},$$  \hfill (2.6)

$$(\tau_0 \otimes \sigma_3) \varphi_{t,s}^{l,s} = s \varphi_{t,s}^{l,s}.$$  \hfill (2.7)

*) The matrix $v = v(\Phi)$ is completely determined from $\Phi$ up to a phase factor. Under the left action of $g \in SU(2)$, the matrix $v(\Phi)$ transforms as $gv(\Phi) = v(\Phi')h$, with $h \in U(1)$. In this way, the $SU(2)$ gauge symmetry can be realized in a nonlinear manner with the aid of $v(\Phi)$.\textsuperscript{14, 15)} The matrices \{v(\Phi)\} that correspond to the possible values of $\Phi$ are sometimes referred to as coset representatives of the coset space $SU(2)/U(1)$.\textsuperscript{14, 15)}
Here, $n_{t,s}$ denotes the number of chirality zero-modes specified by $(t,s)$. With Eqs. (2.1) and (2.3), we can verify that $[iD^\perp, \Phi \otimes \sigma_0] = (v \otimes \sigma_0)(iD^\perp, \tau_3 \otimes \sigma_0)(v^\dagger \otimes \sigma_0) = 0$, where $iD^\perp$ is defined by replacing $A_\alpha^\perp$ contained in Eq. (2.3) by $\tilde{A}_\alpha^3 \tau_3$, the third component of $A_\alpha \equiv v^\dagger A_\alpha v + (2t/e)v^\dagger \partial_\alpha v = \tilde{A}_\alpha^I \tau_I$. Hence, Eqs. (2.5) and (2.6) can be satisfied simultaneously. Having defined the chirality zero-modes by Eqs. (2.5)–(2.7), we can state the Atiyah-Singer index theorem generalized to a two-dimensional $SO(3)$ YMH system:

$$n_{++} - n_{+-} - n_{-+} + n_{--} = \frac{e}{4\pi} \int_M d^2q \varepsilon^{\alpha\beta} F_{\alpha\beta}.$$  \hspace{1cm} (2.8)

Here, $\varepsilon^{\alpha\beta}$ ($\varepsilon^{12} = 1$) is the contravariant Levi-Civita tensor density in two dimensions, and $F_{\alpha\beta}$ is the 't Hooft tensor in two dimensions,

$$F_{\alpha\beta} = \frac{1}{2} \text{tr} \left[ \Phi F_{\alpha\beta} + \frac{i}{2e} \Phi (D_\alpha \Phi D_\beta \Phi - D_\beta \Phi D_\alpha \Phi) \right].$$  \hspace{1cm} (2.9)

with $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - i(e/2)[A_\alpha, A_\beta].$ The left-hand side of Eq. (2.8) is referred to as the Atiyah-Singer index of the Dirac operator $iD^\perp$. In the remaining part of this section, we prove the Atiyah-Singer index theorem (2.8).

To this end, we first consider the eigenvalue equation

$$iD^\perp \varphi_N(q) = \lambda_n \varphi_N(q),$$  \hspace{1cm} (2.10)

with an eigenvalue $\lambda_n$ and an eigenfunction $\varphi_N$. Here, $N$ is a collective index, $N = (n, \nu)$, where $\nu$ is a label that distinguishes between the degenerate eigenfunctions corresponding to $\lambda_n$. The eigenfunction $\varphi_N$ is assumed to be sufficiently smooth that $[\partial_\alpha, \partial_\beta] \varphi_N = 0$ holds. Because $iD^\perp$ is self-adjoint, the eigenvalue $\lambda_n$ is purely real, and the eigenfunctions $\{\varphi_N\}$ can be assumed to form a complete orthonormal set. Now we evaluate the function

$$A_{\text{reg}}(q) \equiv \lim_{\zeta \to 0} \sum_N \varphi_N^\dagger(q)(\Phi \otimes \sigma_3) e^{-\zeta \lambda_n^2} \varphi_N(q)$$

$$= \lim_{\zeta \to 0} \sum_N \varphi_N^\dagger(q)(\Phi \otimes \sigma_3) \exp\left[ -\zeta (iD^\perp)^2 \right] \varphi_N(q)$$

$$= \lim_{\zeta \to 0} \lim_{q' \to q} \text{Tr}\left( (\Phi \otimes \sigma_3) G^\perp(q, q', \zeta) \right),$$  \hspace{1cm} (2.11)

with

$$G^\perp(q, q', \zeta) \equiv \sum_N \left\{ \exp\left[ -\zeta (iD^\perp)^2 \right] \varphi_N(q) \right\} \varphi_N^\dagger(q').$$  \hspace{1cm} (2.12)

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*) In component form, $F_{\alpha\beta}$ and $D_\alpha \Phi$ can be expressed as $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + e \epsilon^{IJK} A_\alpha^I A_\beta^J A_\gamma^K$ and $D_\alpha \Phi = \partial_\alpha \Phi + e \epsilon^{IJK} A_\alpha^I \Phi^K$. From these, we see that the fundamental electric charge, or the gauge coupling constant, in the present $SO(3)$ YMH system is $e$, not $e/2$. In terms of the component fields, the 't Hooft tensor (2.9) is written $F_{\alpha\beta} = \Phi^I F_{\alpha\beta}^I - (1/e) \epsilon_{IJK} \Phi^I D_\alpha \Phi^J D_\beta \Phi^K$. This often appears in the literature on non-Abelian monopoles.}
Here, “Tr” represents the trace taken over both the $τ$- and $σ$-matrices. Following the procedure used in Ref. 4), which is based on the heat kernel method, we can calculate the two-point function $G^\perp(q, q', ς)$ in the limit $q' \to q$. From this calculation, we obtain, for $0 < ς \ll 1$,

$$
\lim_{q' \to q} G^\perp(q, q', ς)
= \frac{1}{4πς} \tau_0 \otimes σ_0 - \frac{1}{48π} \tau_0 \otimes (Rσ_0) + \frac{e^{αβ}F_{αβ}^\perp}{16π} σ_3 + O(ς),
$$

where $e^{αβ} ≡ \det(e_α^β)|e^{αβ}$, $F_{αβ}^\perp ≡ ∂_α A_β^\perp - ∂_β A_α^\perp - i(e/2)[A_α^\perp, A_β^\perp]$, and $R$ is the scalar curvature of $M$. Inserting (2.13) into Eq. (2.11) and evaluating the trace over the $σ$-matrices lead to

$$
A_{reg}(q) = \frac{e}{8π} \text{tr}(ϕ^αβ F_{αβ}^\perp),
$$

where the trace over the $τ$-matrices remains.

Next, we consider a (generalized) chiral decomposition of $ϕ_N$,

$$
ϕ^{t,s}_N ≡ \frac{1}{4} \{(τ_0 + tΦ) \otimes (σ_0 + sσ_3)\} N\varphi_N,
$$

where $t, s = +, −$. Because $Φ^2 = (vτ_3v)^2 = τ_0$ and $σ_3^2 = σ_0$, it is easy to see that the components $ϕ^{t,s}_N$ satisfy the eigenvalue equations

$$
(Φ \otimes σ_0) ϕ^{t,s}_N = tϕ^{t,s}_N,
$$

$$
(τ_0 \otimes σ_3) ϕ^{t,s}_N = sϕ^{t,s}_N.
$$

Furthermore, $ϕ^{t,s}_N$ satisfy the orthogonality relations $ϕ^{t,s}_N ϕ^{−t,s'}_N = ϕ^{t,s}_N ϕ^{t',−s}_N = 0$. In terms of $ϕ^{t,s}_N$, Eq. (2.10) can be written

$$
iD^\perp ϕ^{t,s}_N(q) = λ_n φ^{t,s}_N(q).
$$

Here we assume that $λ_0 = 0$. Thereby, the corresponding eigenfunctions $ϕ^{t,s}_{0,νt,s}$ are treated as the chirality zero-modes of $iD^\perp$, and Eqs. (2.5), (2.6) and (2.7) are understood as Eqs. (2.18), (2.16) and (2.17) in the case $n = 0$, respectively. Equation (2.18) shows that when $n \neq 0$, there is a one-to-one correspondence between $ϕ^{t,s}_N$ and $ϕ^{t,s}_{N−}$. Consequently, it follows that the number of elements of $\{ϕ^{t,s}_N\}_{n\neq0}$ is equal to the number of elements of $\{ϕ^{t,s}_N\}_{n\neq0}$. Also, when $n \neq 0$, it can be proved using Eq. (2.18) that

$$
\frac{d^2q}{4π^{1/2}} \sqrt{|g(q)|} \varphi^{t,s}_N(q) \varphi^{t,s'}_N(q) = \frac{d^2q}{4π^{1/2}} \sqrt{|g(q)|} \varphi^{t,s}_N(q) \varphi^{t,s'}_N(q),
$$

with $\sqrt{|g|} ≡ |\det(e_α^β)|^{−1}$. Since the zero-modes $ϕ^{t,s}_{0,νt,s}$ are eigenfunctions of $iD^\perp$, the set $\{ϕ^{t,s}_{0,νt,s}\}$ satisfying the orthonormality condition

$$
\frac{d^2q}{4π^{1/2}} \sqrt{|g(q)|} ϕ^{t,s}_{0,νt,s}(q) ϕ^{t',s'}_{0,νt',s'}(q) = δ_{tt'}δ_{ss'}δ_{νt,s}δ_{νt',s'},
$$

where $D^\perp ϕ^{t,s}_N(q) = ϕ^{t,s}_N(q)$.
Combining Eqs. (2.14) and (2.21) yields

\[
\int_{\mathcal{M}} d^2q \sqrt{g(q)} A_{\text{reg}}(q) = \lim_{\delta \to 0} \sum_{N} e^{-qN} \sum_{t,s} \int_{\mathcal{M}} d^2q \sqrt{g(q)} \varphi_{N}^t s \varphi_{N}^t s(q) = \sum_{t,s} t \| s n_{t,s}.
\]

Combining Eqs. (2.14) and (2.21) yields

\[
n_{++} - n_{+-} - n_{-+} + n_{--} = \frac{e}{8\pi} \int_{\mathcal{M}} d^2q \text{tr} (\Phi e^{\alpha \beta} F_{\alpha \beta}^\perp) .
\]

Now, with Eq. (2.1), it is easy to show \( v^\dagger (D\Phi) v = e \epsilon_{3ij} \tau_i \tilde{A}^j \), or

\[
D\Phi = e \epsilon_{3ij} \psi_i \tilde{A}^j,
\]

where

\[
\tilde{A} \equiv v^\dagger A v + \frac{2i}{e} v^\dagger dv = \tilde{A}^I \tau_I .
\]

Substituting Eq. (2.23) into Eq. (2.2), we can express \( A^\perp \) as

\[
A^\perp = v \tilde{A}^3 \tau_3 v^\dagger + (2i/e) v dv^\dagger.
\]

Since this is just a gauge transformation of \( \tilde{A}^3 \tau_3 \), the field strength \( F_{\alpha \beta}^\perp \) can be written as \( F_{\alpha \beta}^\perp = v F_{\alpha \beta}^\perp \tau_3 v^\dagger \), with \( F_{\alpha \beta}^\perp = \partial_{\alpha} \tilde{A}_\beta - \partial_{\beta} \tilde{A}_\alpha \). Then, it follows that

\[
\text{tr} (\Phi F_{\alpha \beta}^\perp) = 2 F_{\alpha \beta}^\perp .
\]

Using Eq. (2.23), it is readily shown that \( \text{tr} (\Phi D\Phi D\Phi) = e^2 \text{tr} (\tau_3 \tilde{A}^2) \). Also, expressing the field strength \( F_{\alpha \beta} \) as \( F_{\alpha \beta} = \tilde{F}_{\alpha \beta} v^\dagger \), with \( \tilde{F}_{\alpha \beta} = \partial_{\alpha} \tilde{A}_\beta - \partial_{\beta} \tilde{A}_\alpha - i(e/2) [A_\alpha, \tilde{A}_\beta] \), we see that \( \text{tr} (\Phi F_{\alpha \beta}) = \text{tr} (\tau_3 \tilde{F}_{\alpha \beta}) \). Then, we can write the 't Hooft tensor (2.9) as

\[
\mathcal{F}_{\alpha \beta} = \frac{1}{2} \text{tr} \left[ \tau_3 \tilde{F}_{\alpha \beta} + \frac{i}{2e} e^2 \tau_3 (\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \right] = \frac{1}{2} \text{tr} \left[ \tau_3 (\partial_{\alpha} \tilde{A}_\beta - \partial_{\beta} \tilde{A}_\alpha) \right] = F_{\alpha \beta}^\perp,
\]

which makes it clear that the 't Hooft tensor is indeed an Abelian field strength. Combining Eqs. (2.25) and (2.26) yields \( \text{tr} (\Phi F_{\alpha \beta}^\perp) = 2 \mathcal{F}_{\alpha \beta} \). Inserting this into Eq. (2.22) leads to Eq. (2.8). Thus, the Atiyah-Singer index theorem in an \( SO(3) \) YMH system, Eq. (2.8), is proved.
§3. Derivation of a charge quantization condition

In this section, we derive a charge quantization condition in the static SO(3) YMH system in four-dimensional spacetime, $M^4$, by utilizing the Atiyah-Singer index theorem (2.8). For this purpose, we choose a sphere $S^2_R$ of radius $R$ embedded in $M^4$ as the two-dimensional manifold $M$. To derive the correct charge quantization condition using Eq. (2.8), we need to examine its left-hand side in detail, showing relations valid among the numbers $n_{t,s}$ $(t, s = +, -)$. These relations are beyond the Atiyah-Singer index theorem and can be found only by solving the Dirac equation (2.5) in the case $M = S^2_R$ with $R \to \infty$. For this reason, we actually solve it in this section by carrying out appropriate gauge transformations so that Eq. (2.5) takes a simple form. Then we show relations valid among $n_{t,s}$ and derive a charge quantization condition using these relations and the Atiyah-Singer index theorem.

Having chosen the sphere $S^2_R$ as $M$, it is natural for us to proceed with the study using spherical coordinates. In terms of spherical coordinates, $(q^1, q^2) = (\theta, \phi)$ $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$, on $S^2_R$, the diagonalized inverse zweibein $e_{a}^\alpha$ takes the form $(e_{a}^\alpha) = \text{diag}(R^{-1}, R^{-1} \sin^{-1} \theta)$. The associated spin connection $\omega_{\alpha}$ is found to be $\omega_{\alpha} = -\delta_{\alpha2} \cos \theta$.$^{4}$ We can regard the 't Hooft tensor $F_{\alpha\beta}$ as the radial component of the magnetic field at $S^2_R$. In accordance with the theory of non-Abelian monopoles, the conserved magnetic charge in the SO(3) YMH system is given by$^{7,13}$

$$
g = \frac{1}{4\pi} \int_{S^2_R} d^2 q \frac{1}{2} e^{\alpha\beta} F_{\alpha\beta} = \frac{1}{4\pi} \int_{S^2_R} F, \tag{3.1}$$

where the integral is evaluated in the limit $R \to \infty$. In this limit, the right-hand side of Eq. (2.8) with $M = S^2_R$ becomes $2eg$.

In order to make the left-hand side of Eq. (2.8) clearer in the case $M = S^2_R$ with $R \to \infty$, we first rewrite Eq. (2.5) as

$$i\tilde{D}^\perp \varphi_{0}^{t,s} = 0, \tag{3.2}$$

where

$$\varphi_{0}^{t,s} \equiv (v^\dagger \otimes \sigma_0)\varphi_{0}^{t,s}, \tag{3.3}$$

and $i\tilde{D}^\perp$ is defined by replacing $A_{\alpha}^\perp$ contained in Eq. (2.3) by $\tilde{A}_{\alpha}^3 \tau_3$. (Here, the label $\nu_{t,s}$ is omitted for conciseness.) The transformation $(\varphi_{0}^{t,s}, A_{\alpha}^\perp) \to (\varphi_{0}^{t,s}, \tilde{A}_{\alpha}^3 \tau_3)$ is simply a gauge transformation. The zero-mode $\varphi_{0}^{t,s}$ satisfies Eq. (2.7) and $(\tau_3 \otimes \sigma_0)\varphi_{0}^{t,s} = t\varphi_{0}^{t,s}$ instead of Eq. (2.6). This implies that only the $(t, s)$-component of the four-column vector $\varphi_{0}^{t,s}$ remains non-vanishing, and thus $\varphi_{0}^{t,s}$ is expressed in component form as

$$\varphi_{0}^{t,s} \nu_{t,s'} = \delta_{t}^t \delta_{s}^{s'} \tilde{u}^{t,s}. \tag{3.4}$$

Here, $\tilde{u}^{t,s}$ is a function of $(\theta, \phi)$. In terms of $\tilde{u}^{t,s}$, the Dirac equation (3.2) is written

$$\left[ \frac{\partial}{\partial \theta} - it e A_{1}^3 + \frac{1}{2} \cot \theta + \frac{is}{\sin \theta} \left( \frac{\partial}{\partial \phi} - it e A_{2}^3 \right) \right] \tilde{u}^{t,s} = 0. \tag{3.5}$$
Now, consider the gauge transformation

\[ \tilde{u}^t,s \longrightarrow \hat{u}^t,s \equiv \exp \left[ -it^e \frac{e}{2} \int_0^\theta \dot{A}_1^3(\theta', \phi) d\theta' \right] \tilde{u}^t,s, \quad (3.6a) \]
\[ \dot{A}_1^3 \rightarrow \dot{A}_1^3 \equiv 0, \quad (3.6b) \]
\[ \dot{A}_2^3 \rightarrow \dot{A}_2^3 \equiv \dot{A}_2^3 - \frac{\partial}{\partial \phi} \int_0^\theta \dot{A}_1^3(\theta', \phi) d\theta', \quad (3.6c) \]

which, of course, leaves \( \tilde{F}_{12}^{13} = \partial_1 \dot{A}_2^3 - \partial_2 \dot{A}_1^3 \) invariant. Applying the gauge transformation (3.6) to Eq. (3.5), we can simplify it to

\[ \left[ \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta + \frac{is}{\sin \theta} \left( \frac{\partial}{\partial \phi} - it^e \frac{e}{2} \dot{A}_2^3 \right) \right] \hat{u}^t,s = 0. \quad (3.7) \]

It should be noted that Eq. (3.6a) is merely a regular phase transformation, because, unlike the azimuthal angle \( \phi \), the polar angle \( \theta \) is unrelated to winding of a closed path around an axis. For this reason, there is no essential difference between Eqs. (3.5) and (3.7), and the number of regular solutions of Eq. (3.5) is equal to that of Eq. (3.7). In the following, we treat Eq. (3.7) to examine the number of chirality zero-modes of \( i\tilde{\Phi} \). Because \( \hat{\varphi}_{0}^t,s \) is a spinor field, it has to change sign under a \( 2\pi \) rotation in \( \phi \). This condition and the single-valuedness of \( \dot{A}_1^3 \) under a \( 2\pi \) rotation in \( \phi \) lead to the anti-periodicity condition \( \hat{u}^t,s(\theta, \phi + 2\pi) = -\hat{u}^t,s(\theta, \phi) \) via Eqs. (3.4) and (3.6a). Accordingly, \( \hat{u}^t,s(\theta, \phi) \) can be expressed as the Fourier series

\[ \hat{u}^t,s(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z} + \frac{1}{2}} \hat{v}^t_m(\theta)e^{im\phi}. \quad (3.8) \]

Substituting Eq. (3.8) into Eq. (3.7) and using the orthonormality relation

\[ \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(m-m')\phi} = \delta_{m,m'}, \quad m, m' \in \mathbb{Z} + \frac{1}{2}, \quad (3.9) \]

we obtain

\[ \left( \frac{d}{d\theta} + \frac{1}{2} \cot \theta - \frac{sm}{\sin \theta} \right) \hat{v}^t_m + \frac{ste}{2\sin \theta} \sum_{m' \in \mathbb{Z} + \frac{1}{2}} \int_0^{2\pi} \frac{d\phi}{2\pi} \dot{A}_2^3 e^{i(m-m')\phi} \hat{v}^t_{m'} = 0. \quad (3.10) \]

Assume here that \( \dot{A}_2^3 \) is independent of \( \phi \). Then, noting that \( \tilde{F}_{12}^{13} \) can be written as \( \tilde{F}_{12}^{13} = \partial_1 \dot{A}_2^3 \) by using Eq. (3.6c), we see that \( \tilde{F}_{12}^{13} \) is also independent of \( \phi \) and depends only on \( \theta \). This condition is actually realized on the sphere \( S_R^2 \) in the limit \( R \to \infty \), because the magnetic field at \( S_R^2 \) becomes spherically symmetric as \( R \) increases to infinity. (The condition that \( \tilde{F}_{12}^{13} \) depends only on \( \theta \) holds in more general situations in which the magnetic field at \( S_R^2 \) is axially symmetric.) Equation (3.10) now reads

\[ \left( \frac{d}{d\theta} + \frac{1}{2} \cot \theta - \frac{sm}{\sin \theta} + \frac{ste}{2\sin \theta} \dot{A}_2^3(\theta) \right) \hat{v}^t_m = 0, \quad (3.11) \]
whose solution is readily found to be

\[
\hat{v}^{t,s}(\theta) = c^{t,s}_{m} \left( \sin \frac{\theta}{2} \right)^{sm+\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^{-sm-\frac{1}{2}} \exp \left[ -ste \int_{0}^{\theta} d\theta' \frac{\hat{A}_{3}^{2}(\theta')}{2\sin \theta'} \right],
\]

with \( c^{t,s}_{m} \) being an appropriate constant. Here, we choose \( c^{t,s}_{m} \) to be a normalization constant, if \( \hat{v}^{t,s}(\theta) \) is regular, and hence normalizable, on the interval \( 0 \leq \theta \leq \pi \).

Suppose that \( st \) in Eq. (3.12) is fixed, for instance, as \( st = + \). Then, \( \hat{v}^{+,-}_{m} = \hat{v}^{-,-}_{m} \) is valid with the choice \( c^{+,-}_{m} = c^{-,-}_{m} \), and hence the number of the regular solutions \( \{ \hat{v}^{+,-}_{m} \}_{m \in \mathbb{Z}'+1/2} \) is equal to that of the regular solutions \( \{ \hat{v}^{-,-}_{m} \}_{m \in \mathbb{Z}'+1/2} \).

We simply express this fact as \( \# \{ \hat{v}^{+,-}_{m} \}_{m \in \mathbb{Z}'+1/2} = \# \{ \hat{v}^{-,-}_{m} \}_{m \in \mathbb{Z}'+1/2} \). Here, \( \mathbb{Z}' \) denotes an appropriate subset of the set of integers \( \mathbb{Z} \), which is found by examining the regularity of the solution (3.12). (The symbol \( \# \{ * \} \) stands for the number of elements contained in \( \{ * \} \).) The fundamental set of solutions for Eq. (3.7) is given by \( \{ \hat{u}^{t,s}_{m} \}_{m \in \mathbb{Z}'+1/2} \), with \( \hat{u}^{t,s}_{m}(\theta, \phi) \equiv (2\pi)^{-1/2} \hat{v}^{t,s}_{m}(\theta) e^{i m \phi} \), and it is obvious that \( \# \{ \hat{u}^{+,-}_{m} \}_{m \in \mathbb{Z}'+1/2} = \# \{ \hat{v}^{-,-}_{m} \}_{m \in \mathbb{Z}'+1/2} \). The set \( \{ \hat{u}^{t,s}_{m} \}_{m \in \mathbb{Z}'+1/2} \) yields the fundamental set of solutions for Eq. (3.5), i.e. \( \{ \hat{v}^{t,s}_{m} \}_{m \in \mathbb{Z}'+1/2} \) with \( \hat{u}^{t,s}_{m} \equiv \exp \left[ it(e/2) \int_{0}^{\theta} \hat{A}_{3}^{2} d\theta' \right] \hat{u}^{t,s}_{m} \), via Eq. (3.6a). Because Eq. (3.6a) is a regular phase transformation, as mentioned above, it is guaranteed that \( \# \{ \hat{u}^{+,-}_{m} \}_{m \in \mathbb{Z}'+1/2} = \# \{ \hat{v}^{+,-}_{m} \}_{m \in \mathbb{Z}'+1/2} \). Recalling that the regular solutions of Eq. (3.5) lead to the zero-modes of \( i \hat{D} \) in such a way that \( \varphi^{t,s}_{0,m} = \delta^{t} \delta^{s} \hat{u}^{t,s}_{m} \), we see that \( \# \{ \varphi^{+,-}_{0,m} \}_{m \in \mathbb{Z}'+1/2} = \# \{ \varphi^{-,-}_{0,m} \}_{m \in \mathbb{Z}'+1/2} \). The zero-modes of \( i \hat{D} \) are connected with those of \( i \hat{D} \) by the unitary transformation (3.3):

\( \varphi^{t,s}_{0,m} = (v \otimes \sigma_{0}) \varphi^{t,s}_{m} \). Hence, it follows that \( \# \{ \varphi^{+,-}_{0,m} \}_{m \in \mathbb{Z}'+1/2} = \# \{ \varphi^{-,-}_{0,m} \}_{m \in \mathbb{Z}'+1/2} \), or simply

\[
n_{++} = n_{--},
\]

where \( n_{t,s} \) for \( ts = + \) is given by \( n_{t,s} = \# \{ \varphi^{t,s}_{0,m} \}_{m \in \mathbb{Z}'+1/2} \). If \( st \) in Eq. (3.12) is fixed as \( st = - \), then \( \hat{v}^{+,-}_{m} = \hat{v}^{-,-}_{m} \) is valid with the choice \( c^{+,-}_{m} = c^{-,-}_{m} \). Following the same procedure as in the case \( st = + \), we can show that \( \# \{ \varphi^{+,-}_{0,m} \}_{m \in \mathbb{Z}'''+1/2} = \# \{ \varphi^{-,-}_{0,m} \}_{m \in \mathbb{Z}'''+1/2} \). Here, \( \mathbb{Z}'' \) denotes an appropriate subset of \( \mathbb{Z} \), which is different from \( \mathbb{Z}' \) in general. This relation for the numbers of chirality zero-modes is simply written

\[
n_{--} = n_{++},
\]

where \( n_{t,s} \) for \( ts = - \) is given by \( n_{t,s} = \# \{ \varphi^{t,s}_{0,m} \}_{m \in \mathbb{Z}'''+1/2} \). Using Eqs. (3.13) and (3.14), the left-hand side of Eq. (2.8) with \( \mathcal{M} = S_{R}^{2} \) can be written as \( 2(n_{++} - n_{--}) \), at least in the limit \( R \to \infty \).

As mentioned under Eq. (3.1), the right-hand side of Eq. (2.8) with \( \mathcal{M} = S_{R}^{2} \) becomes \( 2eg \) in the limit \( R \to \infty \). Thus, in the present case, Eq. (2.8) reduces to

\[
n_{++} - n_{--} = eg.
\]

The left-hand side of Eq. (3.15) is just the difference between the numbers of positive \((s = +)\) and negative \((s = -)\) chirality zero-modes having the common signature \( t =
The difference \( n_{++} - n_{+-} \) is, of course, an integer, and by setting \( n = n_{++} - n_{+-} \), Eq. (3.15) can be expressed as

\[
eg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\n
This is precisely the charge quantization condition (1.3), although the integer \( n \) here has its own meaning. Thus, we have derived a correct charge quantization condition by utilizing the Atiyah-Singer index theorem (2.8).

§4. Summary and discussion

In this paper, we first considered a generalization of the Atiyah-Singer index theorem to a two-dimensional \( SO(3) \) YMH system. The generalized theorem (2.8) was proven by using the heat kernel method and a nonlinear realization of the \( SU(2) \) gauge symmetry.

Using the Atiyah-Singer index theorem (2.8) and the relations (3.13) and (3.14), we have derived the charge quantization condition (3.16). This is identical to the charge quantization condition (1.3) found by Arafune et al.\(^7\) They showed Eq. (1.3) by considering continuous mappings from \( S^2_R \) into the unit sphere, \( S^2_\Phi \), defined by \( \sum_{I=1}^3 (\phi^I)^2 = 1 \). According to their analysis, the integer \( n \) can be geometrically interpreted as both the Kronecker index and the Brouwer degree of the mapping \( f_\Phi: S^2_R \rightarrow S^2_\Phi \). The integer \( n \) is also equal to the sum of the Poincaré-Hopf indices associated with \( f_\Phi \). Furthermore, \( n \) can be understood as an integer characterizing the homotopy class of \( f_\Phi \), or as an element of the homotopy group \( \pi_2(S^2_R) = \mathbb{Z} \). Arafune et al. stated that the Kronecker index, the Brouwer degree, the sum of the Poincaré-Hopf indices and the homotopy class are all equivalent ways of characterizing \( f_\Phi \).

By contrast, we have shown Eq. (1.3) without referring to the mapping \( f_\Phi \). In fact, no mappings like \( f_\Phi \) were considered in the proof of Eq. (2.8). Also, what we did to derive Eq. (3.16) from Eq. (2.8) was only to examine the number of chirality zero-modes of the Dirac operator (2.3). In this sense, our approach to showing Eq. (1.3) is new, and as a result, another interpretation of \( n \) turns out to be possible: The integer \( n \) can be interpreted as half of the index \( 2(n_{++} - n_{+-}) \) of the Dirac operator (2.3). [In fact, we set \( n = n_{++} - n_{+-} \) above Eq. (3.16)]. If we take Eq. (1.3) as given by other approaches, the argument presented in this paper can be understood as an illustration or verification of the Atiyah-Singer index theorem in a two-dimensional \( SO(3) \) YMH system.

In the \( U(1) \) gauge theory with a monopole background, the condition corresponding to Eq. (3.15) is found from the Atiyah-Singer index theorem in two dimensions to be\(^4\)\(^1\)

\[
 n_{++} - n_{+-} = 2eg .
\]  
(4.1)

Here, \( n_{++} \) (\( n_{+-} \)) denotes the number of positive (negative) chirality zero-modes of the Dirac operator in the \( U(1) \) gauge theory. It should be noted that the right-hand side of the above equation is

\[^{1}\] In Ref. 4), it was shown that Eq. (4.1) reduces to the Dirac quantization condition \( eg = n/2 \) (\( n \in \mathbb{Z} \)) or the Schwinger quantization condition \( eg = \frac{n}{2} \) (\( n \in \mathbb{Z} \)), according to the choice of the gauge potential.
side of this condition is twice that of Eq. (3.15). This is due to the fact that in the SO(3) YMH theory, there exist twice as many chirality zero-modes as in the U(1) gauge theory, as may be seen from Eqs. (3.13) and (3.14). The difference between Eqs. (3.15) and (4.1) explains, in terms of the Atiyah-Singer index theorem, why the charge quantization condition in the SO(3) YMH theory, \( eg = n \) (\( n \in \mathbb{Z} \)), is different from the Dirac quantization condition, \( eg = n/2 \) (\( n \in \mathbb{Z} \)), by a factor of two.

In the appendix, we see that the SO(3) monopole configurations can be reduced to either Abelian monopole configurations of Dirac type or those of Schwinger type, depending on the choice of a constant contained in the monopole potential. Along the line of the present argument, these two types of configurations can be treated in a unified manner without a careful treatment of the string singularity in the monopole potential. Actually, as can be seen in the appendix, both the Dirac and Schwinger charge quantization conditions for the SO(3) YMH system can be derived by formally counting the number of zero-modes of the Dirac operator.

We have applied the Atiyah-Singer index theorem (2.8) to only a particular case in which \( S^2_R \) is chosen as \( \mathcal{M} \) and SO(3) non-Abelian monopoles are assumed to exist in the system. As a future subject of study, it would be interesting to consider applications of the theorem (2.8) to other physical systems in which \( \mathcal{M} \) has a non-trivial topology. In these applications, new relations other than Eq. (3.16) may be found from Eq. (2.8). It is also of interest to generalize the Atiyah-Singer index theorem to the SU(\( N \)) YMH system in two dimensions. In the presence of SU(\( N \)) non-Abelian monopoles,\(^9\)–\(^{13}\) such a generalized theorem should provide an analog of the condition (3.16). We hope to address these issues in the future.

Acknowledgements

The author would like to thank Prof. K. Fujikawa for his encouragement and useful comments. This work was supported in part by the Nihon University Research Grant (No. 06-069).

Appendix

This appendix gives an illustration of the charge quantization condition (3-16) using concrete forms of the SO(3) monopole configurations.

Let us consider the Yang-Mills connection \( A \) and the scalar field \( \Phi \) defined by

\[
A = k(\sin n\phi \, d\theta + n \sin \theta \cos \theta \cos n\phi \, d\phi)\tau_1 \\
+ k(\cos n\phi \, d\theta - n \sin \theta \cos \theta \sin n\phi \, d\phi)\tau_2 \\
- kn \sin^2 \theta \, d\phi \tau_3 ,
\]

(\( A \cdot 1 \))

\[
\Phi = \sin \theta \cos n\phi \tau_1 - \sin \theta \sin n\phi \tau_2 + \cos \theta \tau_3 ,
\]

(\( A \cdot 2 \))

where \( k \) is a real constant and \( n \) must be an integer to insure the single-valuedness of \( A \) and \( \Phi \) under a \( 2\pi \) rotation in \( \phi \). Up to \( k \), the expressions (\( A \cdot 1 \)) and (\( A \cdot 2 \)) are essentially the same as those given by Bais.\(^8\),\(^{13}\)
The curvature two-form of $A$ is found from Eq. (A.1) to be

$$F = \frac{1}{2} F_{\alpha\beta} dq^\alpha dq^\beta = dA - i\frac{e}{2} AA$$

$$= -2kn\left(1 + \frac{k e}{2}\right)\Phi \sin \theta d\theta d\phi. \quad (A.3)$$

From Eqs. (A.1) and (A.2), covariant differentiation of $\Phi$ is obtained as

$$D\Phi = d\Phi - i\frac{e}{2}[A, \Phi] = (1 + k e) d\Phi, \quad (A.4)$$

which leads to

$$D\Phi D\Phi = -2kn(1 + k e)^2 \Phi \sin \theta d\theta d\phi. \quad (A.5)$$

From the combination of Eqs. (A.3) and (A.5) given by

$$F + \frac{i}{2e} D\Phi D\Phi = \frac{n}{e} \Phi \sin \theta d\theta d\phi, \quad (A.6)$$

the 't Hooft tensor written in terms of differential forms is found to be

$$\mathcal{F} = \frac{1}{2} F_{\alpha\beta} dq^\alpha dq^\beta = \frac{1}{2} \text{tr}\left(\Phi F + \frac{i}{2e} \Phi D\Phi D\Phi\right)$$

$$= \frac{n}{e} \sin \theta d\theta d\phi. \quad (A.7)$$

Consequently, the conserved magnetic charge (3.1) is obtained as

$$g = \frac{1}{4\pi} \int_{S^2_R} \mathcal{F} = \frac{n}{e}. \quad (A.8)$$

Thus, Eqs. (A.1) and (A.2) are shown to be configurations of $SO(3)$ monopoles with magnetic charge $n/e \ (n \in \mathbb{Z})$. Equation (A.8) reduces to Eq. (1.3) with the identification $n = n$. Although each of Eqs. (A.3) and (A.5) depends on $k$, a particular combination (A.6), and hence Eqs. (A.7) and (A.8), are independent of $k$. This remarkable fact implies that the 't Hooft tensor and the magnetic charge obtained here are actually determined only by the scalar field (A.2); they are independent of the Yang-Mills connection (A.1). In Ref. 7), Arafune et al. showed that the magnetic charge in the $SO(3)$ YMH theory is completely specified in terms of a triplet of Higgs fields. The result of our analysis is thus consistent with their statement.

From the scalar field (A.2), the matrix $v$ satisfying Eq. (2.1) is determined to be

$$v(\theta, \phi) = \begin{pmatrix}
\frac{e^{i(n+\bar{n})\phi}}{2} \cos \frac{\theta}{2} & -\frac{e^{-i\bar{n}\phi}}{2} \sin \frac{\theta}{2} \\
\frac{e^{i\bar{n}\phi}}{2} \sin \frac{\theta}{2} & \frac{e^{-i(n+\bar{n})\phi}}{2} \cos \frac{\theta}{2}
\end{pmatrix}, \quad (A.9)$$

where $\bar{n}$ is a real constant. Let us recall Eq. (3.3). Because $\varphi_0^{t,s}$ and $\tilde{\varphi}_0^{t,s}$ are spinor fields, they change sign under a $2\pi$ rotation in $\phi$. Accordingly, as seen from Eq.
From Eq. (A.16), it follows that
\[ A = - \left( k + \frac{1}{e} \right) \left( \sin \phi d\theta - n \sin \theta \cos \phi d\phi \right) \tau_1 \]
\[ + \left( k + \frac{1}{e} \right) \left( \cos \phi d\theta + n \sin \theta \sin \phi d\phi \right) \tau_2 \]
\[ - \frac{1}{e} (l + n \cos \theta) d\phi \tau_3, \quad (A.10) \]
where \( l \equiv n + 2\bar{n}. \) The third component, \( \vec{A}^3, \) is immediately read from (A-10) as
\[ \vec{A}^3 = -e^{-1}(l + n \cos \theta) d\phi = \vec{A}^3 dq^\alpha, \]
with
\[ \vec{A}^3_1 = 0, \quad (A.11a) \]
\[ \vec{A}^3_2 = -\frac{1}{e} (l + n \cos \theta). \quad (A.11b) \]
Then, from Eq. (3-6c), it follows that \( \vec{A}^3_2 = -e^{-1}(l + n \cos \theta), \) and hence Eq. (3-11) takes the form
\[ \left[ \frac{d}{d\theta} + \frac{1}{2} (1 - stn) \cot \theta - s \left( m + \frac{tl}{2} \right) \csc \theta \right] \hat{v}_{m}^{t,s} = 0. \quad (A.12) \]
This equation can readily be solved as
\[ \hat{v}_{m}^{t,s} (\theta) = \hat{c}_{m}^{t,s} \left( \sin \frac{\theta}{2} \right)^{p_{m}^{t,s}} \left( \cos \frac{\theta}{2} \right)^{q_{m}^{t,s}} \quad (A.13) \]
where \( \hat{c}_{m}^{t,s} \) is an appropriate constant, and
\[ p_{m}^{t,s} \equiv sm + \frac{1}{2} \left\{ st(n + l) - 1 \right\}, \quad (A.14) \]
\[ q_{m}^{t,s} \equiv -sm + \frac{1}{2} \left\{ st(n - l) - 1 \right\}. \quad (A.15) \]
The solution \( \hat{v}_{m}^{t,s} \) diverges at neither \( \theta = 0 \) nor \( \pi \) if and only if \( p_{m}^{t,s}, q_{m}^{t,s} \geq 0. \) In this case, \( \hat{v}_{m}^{t,s} \) is normalizable with respect to the usual \( L^2 \) norm. This fact enables us to choose \( \hat{c}_{m}^{t,s} \) in the case \( p_{m}^{t,s}, q_{m}^{t,s} \geq 0 \) to be a normalization constant of \( \hat{v}_{m}^{t,s}. \) The conditions \( p_{m}^{t,s}, q_{m}^{t,s} \geq 0 \) necessary for \( \hat{v}_{m}^{t,s} \) to be regular can together be written as
\[ -\frac{1}{2} \left\{ st(n + l) - 1 \right\} \leq sm \leq \frac{1}{2} \left\{ st(n - l) - 1 \right\}. \quad (A.16) \]
From Eq. (A.16), it follows that \( -\{ st(n + l) - 1 \} \leq \{ st(n - l) - 1 \}, \) which can be simplified as \( stn \geq 1. \) This condition implies that if \( n \) is a positive integer, \( n \in \mathbb{Z}^+, \) then \( st = +. \) Hence, when \( n \in \mathbb{Z}^+, \) there exist no regular solutions \( \hat{v}_{m}^{t,+} \) or \( \hat{v}_{m}^{-}, \) and thus \( n_{+} = n_{-} = 0. \) Similarly, if \( n \) is a negative integer, \( n \in \mathbb{Z}^-, \) then the condition \( stn \geq 1 \) implies \( st = - \). Hence, when \( n \in \mathbb{Z}^-, \) there exist no regular solutions \( \hat{v}_{m}^{t,-} \) or \( \hat{v}_{m}^{t,-} \)
or \( \hat{v}_{m-} \), and thus that \( n_{++} = n_{--} = 0 \). It is now obvious that when \( n = 0 \), there exist no regular solutions, and hence \( n_{t,s} = 0 \) \((t, s = +, -)\). This illustrates the Lichnerowicz vanishing theorem.\(^{28}\)

In the following, we consider the two cases \( l = n \) and \( l = 0 \) in particular, because in these cases, it is easy to count the number of regular solutions \( \hat{v}_{m+} \) and \( \hat{v}_{m-} \) for \( n \in \mathbb{Z}^+ \) and the number of regular solutions \( \hat{v}_{m+} \) and \( \hat{v}_{m-} \) for \( n \in \mathbb{Z}^- \).

A.1. The case \( l = n \)

In the case \( l = n \), \( \hat{A}_2^3 \) reduces to a monopole potential of Dirac type, \( \hat{A}_2^3 = -e^{-1}n(1 + \cos \theta) \), and Eq. (A.16) reads

\[
-stn + \frac{1}{2} \leq sm \leq -\frac{1}{2}.
\]

(A-17)

First, suppose that \( n \in \mathbb{Z}^+ \), or equivalently \( st = + \). Then Eq. (A.17) becomes \(-(2n - 1)/2 \leq sm \leq -1/2 \). Because \( m \) takes half-integer values, the allowed values of \( sm \) are seen to be \( sm = -1/2, -3/2, \ldots, -(2n - 1)/2 \). This implies that the number of regular solutions \( \hat{v}_{m+} \) and the number of regular solutions \( \hat{v}_{m-} \) are both \( n \), and it follows that \( n_{++} = n_{--} = n \). As a result, taking into account the fact that \( n_{+-} = n_{-+} = 0 \) for \( n \in \mathbb{Z}^+ \), we have

\[
n = n_{++} - n_{--} = n, \quad n \in \mathbb{Z}^+.
\]

(A-18)

Next, suppose that \( n \in \mathbb{Z}^- \), or equivalently \( st = - \). Then Eq. (A.17) becomes \((2n + 1)/2 \leq sm \leq -1/2 \), and the allowed values of \( sm \) are found to be \( sm = -1/2, -3/2, \ldots, (2n + 1)/2 \). This implies that the number of regular solutions \( \hat{v}_{m+} \) and the number of regular solutions \( \hat{v}_{m-} \) are both \(-n \), and it follows that \( n_{++} = n_{--} = -n \). Recalling that \( n_{+-} = n_{-+} = 0 \) for \( n \in \mathbb{Z}^- \), we have

\[
n = n_{++} - n_{--} = n, \quad n \in \mathbb{Z}^-.
\]

(A-19)

Equations (A-18) and (A-19), together with the fact that \( n_{t,s} = 0 \) for \( n = 0 \), are brought together in the form \( n = n \) with \( n \in \mathbb{Z} \). Combining this with Eq. (A-8) leads to \( eg = n \), with \( n \in \mathbb{Z} \). Thus, the charge quantization condition (3-16) is illustrated with the monopole configurations (A-1) and (A-2).

If \( l = -n \), \( \hat{A}_2^3 \) reduces to another monopole potential of Dirac type, \( \hat{A}_2^3 = e^{-1}n(1 - \cos \theta) \). This is merely a mirror image of the potential in the case \( l = n \). Following the same procedure as in the case \( l = n \), we again obtain the condition \( eg = n \) with \( n \in \mathbb{Z} \).

A.2. The case \( l = 0 \)

In the case \( l = 0 \), \( \hat{A}_2^3 \) reduces to a monopole potential of Schwinger type, \( \hat{A}_2^3 = -e^{-1}n \cos \theta \), and Eq. (A-16) reads

\[
-\frac{1}{2} stn + \frac{1}{2} \leq sm \leq \frac{1}{2} stn - \frac{1}{2}.
\]

(A-20)

We should note that \( n \) here takes only even values, because \( l = 0 \) implies \( n = -2n \) and \( \bar{n} \) takes integer values.
First, suppose that \( n \) is a positive even integer, \( n \in 2\mathbb{Z}^+ \), or equivalently \( st = + \). Then Eq. (A.20) becomes \(-(n-1)/2 \leq sm \leq -(n-1)/2\). Because \( m \) takes half-integer values, the allowed values of \( sm \) are seen to be \( sm = \pm 1/2, \pm 3/2, \ldots, \pm (n-1)/2 \). This implies that the number of regular solutions \( \hat{v}_m^+ \) and the number of regular solutions \( \hat{v}_m^- \) are both \( n \), and it follows that \( n_{++} = n_{--} = n \). As a result, taking into account the fact that \( n_{+-} = n_{-+} = 0 \) for \( n \in 2\mathbb{Z}^+ \), we have

\[
n = n_{++} - n_{+-} = n, \quad n \in 2\mathbb{Z}^+.
\]  

(A.21)

Next, suppose that \( n \) is a negative even integer, \( n \in 2\mathbb{Z}^- \), or equivalently \( st = - \). Then Eq. (A.20) becomes \((n+1)/2 \leq sm \leq -(n+1)/2\), and the allowed values of \( sm \) are found to be \( sm = \pm 1/2, \pm 3/2, \ldots, \pm (n+1)/2 \). This implies that the number of regular solutions \( \hat{v}_m^{+-} \) and the number of regular solutions \( \hat{v}_m^{-+} \) are both \(-n\), and it follows that \( n_{+-} = n_{-+} = -n \). Noting that \( n_{++} = n_{--} = 0 \) for \( n \in 2\mathbb{Z}^- \), we have

\[
n = n_{++} - n_{+-} = n, \quad n \in 2\mathbb{Z}^-.
\]  

(A.22)

Equations (A.21) and (A.22), together with the fact that \( n_{t,s} = 0 \) for \( n = 0 \), are brought together in the form \( n = n \), where \( n \) is an even integer, \( n \in 2\mathbb{Z} \). Combining this with Eq. (A.8) leads to \( eg = n \) with \( n \in 2\mathbb{Z} \). Thus, the charge quantization condition (3-16) is illustrated also in the case \( l = 0 \) with the configurations (A.1) and (A.2), though \( n \) here is restricted to even integers.

A.3. Comments

It has been seen that when \( l = n \) (or \( l = -n \)), \( n \) can take all integer values, while when \( l = 0 \), \( n \) can take only even values. This fact implies the following: If it is assumed that \( n \) in Eqs. (A.1) and (A.2) can take all integer values, then it is possible to make gauge transformations that yield the Dirac potentials \( A^3_\pm = \pm e^{-1}n(1 \pm \cos \theta) \); in this case, the gauge transformation that gives the Schwinger potential \( A^3_\pm = -e^{-1}n \cos \theta \) is not allowed. By contrast, if \( n \) in Eqs. (A.1) and (A.2) is assumed to take only even values, the gauge transformation that gives the Schwinger potential is allowed. In this case, it is, of course, possible to make the gauge transformations that give the Dirac potentials. In this way, the allowed gauge transformations are determined by the integer values that \( n \) takes.

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