A FROBENIUS MANIFOLD FOR $\ell$-KRONECKER QUIVER

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Abstract. We construct a Frobenius structure whose intersection form coincides with the generalized Cartan matrix of the $\ell$-Kronecker quiver $K_\ell$ and underlying complex manifold is isomorphic to the space of stability conditions for the bounded derived category of finitely generated modules over the path algebra $\mathbb{C}K_\ell$.

1. Introduction

A Frobenius manifold is a complex manifold whose tangent bundle is a flat family of Frobenius algebras with grading operator called the Euler vector field satisfying certain conditions (see Definition 2.1). The notion of a Frobenius manifold is formulated by B. Dubrovin [Dub96] in his study of differential equations and integrable systems (e.g., Painlevé equations) related to 2-dimensional topological field theories. However, this structure (except for the potentiality) was firstly discovered by K. Saito and his collaborators [Sai93, SYS80] in their study of the invariant ring of a finite reflection group and its orbit space. Its regular subspace coincides with the domain for period mappings of a certain differential form on cycles in the Milnor fiber of the corresponding singularity. Later, inspired by the above construction, a Frobenius structure on the deformation space of an isolated hyper-surface singularity is constructed via the choice of a special differential form called a primitive form $\zeta$ in the de Rham cohomology group of the total space twisted by the differential of the singularity (see [Sai86]). The existence of a primitive form is proved by M. Saito [Sai89] (for tame holomorphic functions by Douai–Sabbah [DS03, DS04]). In particular, the Gelfand–Leray form $\zeta$ of a primitive form gives the differential form for the period mappings mentioned above.

T. Bridgeland introduced the notion of a space of stability conditions for a triangulated category (constructed from a symplectic manifold, a tame function with isolated critical points etc). Roughly speaking, this space is a complex moduli manifold consisting of tuples $(Z, P)$ of a group homomorphism called a central charge $Z$ from the Grothendieck group to $\mathbb{C}$ and an $\mathbb{R}$-graded family of full additive categories $P$ satisfying certain conditions ([Bri07]). The importance of the space of stability conditions is that this complex manifold is conjectured to be the moduli space of deformations of a mathematical object
mentioned inside the bracket above (e.g. see [Tak05, KST07, KST09]). Indeed, this conjecture is verified for some cases concerning the derived categories of (resp. Calabi–Yau completions of) Fukaya–Seidel categories for tame functions with isolated critical points, equivalently, the derived categories of modules over the mirror path algebras. Namely, in [BQS14, HKK17] (resp. [Ike14, Ike17, Wan19]), their central charges are given by the oscillatory integrals (resp. period integrals of Gelfand–Leray forms) for primitive forms and the spaces of stability conditions are isomorphic to the (resp. universal cover of regular) orbit spaces of corresponding Weyl group invariant theories. More precisely, the following commutative diagrams summarize their results for $A_2$-singularity $F_t(z) = z^3 + t^2z + t^4$:

\[
\begin{array}{cccc}
\mathfrak{h}/W & \cong & \text{Stab}(D^b(A_2)) \\
\Pi & & \downarrow \zeta \\
T_p(\mathfrak{h}/W) & \cong & \text{Hom}(K_0(D^b(A_2)), \mathbb{C})
\end{array}
\]

\[
\begin{array}{cccc}
\mathfrak{h}_{\text{reg}}/W & \cong & \text{Stab}(D_N(A_2)) \\
\Pi_N & & \downarrow \zeta \\
T_p(\mathfrak{h}_{\text{reg}}/W) & \cong & \text{Hom}(K_0(D_N(A_2)), \mathbb{C})
\end{array}
\]

Here we denote by $\mathfrak{h}$ (resp. $\mathfrak{h}_{\text{reg}}$) the Cartan subalgebra (resp. without reflection hyperplanes), by $D^b(A_2)$ (resp. $D_N(A_2)$) the bounded derived category of finitely generated modules over the path algebra $\mathbb{C}A_2$ (resp. of finite total dimension dg-modules over the Calabi–Yau $N$-completion of $\mathbb{C}A_2$), by $Z$ (resp. $Z_N$) the local homeomorphism $(Z, \mathcal{P}) \mapsto Z$, by $\Sigma_N F_t$ the $N$-th suspension $F_t + \sum_{i=1}^{N} x_i^2$ and the tilde stands for the universal cover.

Note that the integral dual lattice $\mathfrak{h}_Z^*$ is naturally isomorphic to the relative homology group of Lefschetz thimbles (resp. the middle dimensional homology group of vanishing cycles) and integrations are taken after analytic continuations from the reference point $p$ to $t$ (see also [Sai86, Section 5 (5.7)]).

Frobenius structures on the orbit spaces of the Weyl groups are constructed for finite Coxeter groups [SYS80, Dub99a], extended affine Weyl groups [DZ98, DSZZ15], elliptic Weyl groups and Jacobi groups [Ber00a, Ber00b, Sat10] (see also [KaMaSe15, KMS18] for Saito structures without metrics). The purpose of the present paper is to construct the
Frobenius manifold structure from the invariant theory of the Weyl group associated to $\ell$-Kronecker quiver $K_\ell$.

**Theorem 1.1** (Theorem 4.1). Let $W$ be the Weyl group of $K_\ell$ and $\tilde{X}$ be the universal cover of a certain subspace $X$ of $\mathfrak{h}$ (see Definition 3.4). There exists a unique Frobenius structure of rank 2 and dimension $1 - \frac{2}{\rho}$ on $\tilde{X} \mathcal{L} W$ such that $e = \frac{\partial}{\partial h^1}$, $E = h^1 \frac{\partial}{\partial h^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}$ for the flat coordinates $(t^1, t^2)$ in Subsection 4.1 and the intersection form coincides with the generalized Cartan matrix $A$ of $\ell$-Kronecker quiver $K_\ell$.

Here $h := \frac{2\pi \sqrt{-1}}{\log \rho}$ where $\rho > 1$ is the eigenvalue of the Coxeter transformation. Since $\rho = \exp \left( \frac{2\pi \sqrt{-1}}{h} \right)$ tautologically, the number $h$ can be regard as a generalization of the Coxeter number. Inspired by the conjectural relation between Frobenius structure and spaces of stability conditions explained above, the covering space of regular orbit subspace $\tilde{X}_{\text{reg}} / W (\subset \tilde{X} / W)$ is obtained as the space of stability conditions for the bounded derived category of nilpotent modules over the preprojective algebra $\Pi_2(K_\ell)$ associated to $K_\ell$. Moreover, due to [DK16], the space $\tilde{X} / W$ is isomorphic to the space of stability conditions for the bounded derived category of finitely generated modules over the path algebra $\mathbb{C}K_\ell$ (see Proposition 3.18). We expect stronger results that the deformed flat coordinates on the Frobenius manifold $\tilde{X} / W$ coincide with the central charges on $\text{Stab}(\mathcal{D}^{\mathcal{L}}(K_\ell))$ (see Conjecture 4.10).

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2. **Frobenius manifolds and periods**

We recall the definition of the Frobenius manifold and related properties.

### 2.1. Frobenius manifolds.

The original definition and their basic properties are given by B. Dubrovin [Dub96]. For the notations, we use the following definition:

**Definition 2.1** ([ST08, Definition 2.1]). Let $M = (M, \mathcal{O}_M)$ be a connected complex manifold or a formal manifold over $\mathbb{C}$ of dimension $\mu$ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M, \Omega^1_M$ respectively and $d$ a complex number. A *Frobenius structure of rank $\mu$ and dimension $d$ on $M$* is a tuple $(\eta, \circ, e, E)$, where $\eta$ is a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, $\circ$ is $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$, etc.
defining an associative and commutative $\mathcal{O}_M$-algebra structure with the unit $e$, and $E$ is a holomorphic vector field on $M$, called the Euler vector field, which are subject to the following axioms:

(i) The product $\circ$ is self-adjoint with respect to $\eta$: that is,
$$\eta(\delta \circ \delta', \delta''') = \eta(\delta, \delta' \circ \delta'''), \quad \delta, \delta', \delta''' \in \mathcal{T}_M.$$

(ii) The Levi–Civita connection $\nabla: \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$ with respect to $\eta$ is flat: that is,
$$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in \mathcal{T}_M.$$

(iii) The tensor $C: \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{T}_M \to \mathcal{T}_M$ defined by $C_{\delta \delta'} := \delta \circ \delta'$, $(\delta, \delta' \in \mathcal{T}_M)$ is flat: that is,
$$\nabla C = 0.$$

(iv) The unit element $e$ of the $\circ$-algebra is a $\nabla$-flat holomorphic vector field: that is,
$$\nabla e = 0.$$

(v) The bilinear form $\eta$ and the product $\circ$ are homogeneous of degree $2 - d$ ($d \in \mathbb{C}$), 1 respectively with respect to Lie derivative Lie$_E$ of Euler vector field $E$: that is,
$$\text{Lie}_E(\eta) = (2 - d)\eta, \quad \text{Lie}_E(\circ) = \circ.$$

Next we expose some basic properties of the Frobenius manifold without their proofs. Let us consider the space of horizontal sections of the connection $\nabla$:
$$\mathcal{T}_M^{\parallel} := \{ \delta \in \mathcal{T}_M \mid \nabla_{\delta'} \delta = 0 \text{ for all } \delta' \in \mathcal{T}_M \}$$
which is a local system of rank $\mu$ on $M$ such that the metric $\eta$ takes constant value on $\mathcal{T}_M^{\parallel}$. Namely, we have
$$\eta(\delta, \delta') \in \mathbb{C}, \quad \delta, \delta' \in \mathcal{T}_M^{\parallel}.$$

**Proposition 2.2** ([ST08, Definition 2.2]). At each point of $M$, there exist local coordinates $(t^1, \ldots, t^\mu)$, called flat coordinates, such that $e = \partial_1$, $\mathcal{T}_M^{\parallel}$ is spanned by $\partial_1, \ldots, \partial_\mu$ and $\eta(\partial_i, \partial_j) \in \mathbb{C}$ for all $i, j = 1, \ldots, \mu$, where we denote $\partial / \partial t^i$ by $\partial_i$.

The axiom $\nabla C = 0$, implies the following:

**Proposition 2.3** ([ST08, Proposition 2.4]). At each point of $M$, there exist the local holomorphic function $\mathcal{F}$, called Frobenius potential, satisfying
$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}, \quad i, j, k = 1, \ldots, \mu,$$
for any system of flat coordinates. In particular, one has
$$\eta_{ij} := \eta(\partial_i, \partial_j) = \partial_i \partial_j \mathcal{F}.$$
Denote by $\eta^{ij}$ the $(i, j)$-entry of the matrix $(\eta_{ij})^{-1}$.

**Example 2.4** ([Dub96, Example 1.1]). Let $M$ be a Frobenius manifold of rank 2 and dimension $d$ whose flat coordinates are $(t^1, t^2)$. If $d \neq -1, 1, 3$, then the Frobenius potential $\mathcal{F}$ of $M$ is given as follows:

$$\mathcal{F}(t^1, t^2) = \frac{1}{2} \eta_{12} (t^1)^2 t^2 + c(t^2)^{\frac{4-d}{3-d}},$$

where $\eta_{12} \in \mathbb{C}\backslash\{0\}$, $c \in \mathbb{C}$.

### 2.2. The intersection form and the second structure connection.

Let $M$ be a Frobenius manifold of rank $\mu$ and dimension $d$. We recall the intersection form of the Frobenius manifold $M$.

**Definition 2.5** ([Dub96, Appendix G (G.1)]). Define $\Delta$ as the determinant of the $\mathcal{O}_M$-endomorphism $C_E \in \text{End}_{\mathcal{O}_M}(T_M)$:

$$\Delta := \det(C_E) \in \mathcal{O}_M,$$

called the discriminant. Set $\mathcal{D} := \{t \in M \mid \Delta(t) = 0\}$ (called the discriminant locus) and $M_{\text{reg}} := M \backslash \mathcal{D}$ (called the regular subspace).

The intersection form of the Frobenius manifold $M$ is defined as follows:

**Definition 2.6** ([Dub96, Lecture 3 (3.13)]). Let $(s^1, \ldots, s^\mu)$ be local coordinates of $M$. Set the $\mathcal{O}_M$-symmetric bilinear form $g : \Omega^1_M \times \Omega^1_M \to \mathcal{O}_M$ as

$$g(\omega, \omega') := \sum_{a, b=1}^\mu \langle \frac{\partial}{\partial s^a}, \omega \rangle \cdot \eta (ds^a, ds^b) \cdot \langle C_E \frac{\partial}{\partial s^b}, \omega' \rangle,$$

where $\langle -,- \rangle$ is the contraction. The $\mathcal{O}_M$-bilinear form $g$ is called the intersection form of the Frobenius manifold $M$.

The intersection form $g$ is independent of a choice of local coordinates. The intersection form with respect to the flat coordinates $(t^1, \ldots, t^\mu)$ is given by

$$g = \sum_{a, b=1}^\mu \eta^{ab} E \frac{\partial^2 \mathcal{F}}{\partial t^a \partial t^b}.$$

Denote by $\nabla$ the Levi–Civita connection of $g$. The connection $\nabla$ is often called the Second structure connection or the Dubrovin connection.

**Proposition 2.7** ([Her02, Theorem 9.4]). The second structure connection $\nabla$ is a flat connection.
Definition 2.8 ([Dub96, Definition G.1]). Actions of the fundamental group $\pi_1(M_{\text{reg}})$ on the universal covering of $M_{\text{reg}}$ can be lifted to isometries in $\text{Aut}(\mathbb{C}^\mu, I)$ on $\mathbb{C}^\mu \cong T^*_p M_{\text{reg}}$ with respect to the bilinear form $I$ induced by the intersection form $g$. Denote by $\Theta$ the associated representation:

$$\Theta : \pi_1(M_{\text{reg}}, p) \longrightarrow \text{Aut}(\mathbb{C}^\mu, I).$$

The group

$$W_M := \text{Im } \Theta$$

is called the monodromy group of the Frobenius manifold.

Definition 2.9 ([Dub04, Definition 6]). A function $x \in \mathcal{O}_{M_{\text{reg}}}$ is called a period if the 1-form $dx \in \Omega^1_{M_{\text{reg}}}$ is flat with respect to the second structure connection $\nabla$, i.e.,

$$\nabla dx = 0.$$

Proposition 2.10. Let $(x^1,\ldots,x^\mu)$ be local flat coordinates on $M_{\text{reg}}$ with respect to $g$. If $d \neq 1$, then the following function:

$$t(x^1, \ldots, x^\mu) := \sum_{i=1}^{\mu} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) x^ix^j$$

is a flat function, namely, $\nabla dt = 0$.

Proof. The statement follows from similar arguments in [Sai86, Section 3.3, 3)] and [Sai86, Section 5.3)].

The following Lemma 2.11 will be the key lemma which enables us to reconstruct structure coefficients of the product $\circ$ from contravariant components of the Levi–Civita connection for the intersection form $g$ (see Subsection 4.3):

Lemma 2.11 ([Dub96, Lemma 3.4]). Let $g$ be the intersection form of a Frobenius structure $M$ of rank $\mu$ and dimension $d$. The following equality holds:

$$g(dt^i, \nabla_{\frac{\partial}{\partial x^k}} dt^j) = \left(\frac{d-1}{2} + d_j\right) \sum_{a,b=1}^{\mu} \eta^{ia}\eta^{jb}\partial_a\partial_b F,$$

where $d_j$ is the degree of the flat coordinate $t_j$ with respect to $E$, i.e., $\nabla_{\frac{\partial}{\partial t^j}} E = d_j \frac{\partial}{\partial t^j}$.
3. The Generalized Root System Associated to $\ell$-Kronecker Quiver

Throughout this paper, we assume that $\ell \in \mathbb{Z}$ and $\ell \geq 3$. We summarize results for the generalized root system associated to $\ell$-Kronecker quiver.

**Definition 3.1.** The $\ell$-Kronecker quiver $K_\ell$ consists of vertices $\{1, 2\}$ and $\ell$ edges from 1 to 2:

$$K_\ell : \begin{array}{lll}
1 & \overset{\alpha_1}{\rightarrow} & 2 \\
\alpha_\ell & \overset{\alpha_2}{\rightarrow} & \end{array}$$

(i) The *generalized Cartan matrix* associated to $\ell$-Kronecker quiver is defined as

$$A := \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}2 & -\ell \\
-\ell & 2
\end{pmatrix},$$

where $a_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji})$, $\delta_{ij}$ is the Kronecker delta and $q_{ij}$ is the number of arrows from $i$ to $j$.

(ii) The *root lattice* $L$ associated to $K_\ell$ is the following free abelian group

$$L := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$$

with generators $\alpha_1$ and $\alpha_2$, called *simple roots*.

(iii) Define the symmetric bilinear form as follows:

$$I : L \times L \rightarrow \mathbb{Z}; \quad I(\alpha_i, \alpha_j) := a_{ij}.$$

Define the *reflection* with respect to the simple root $\alpha_i$ as the following isometric homomorphism $r_i : L \rightarrow L$ with respect to $I$:

$$r_i(\lambda) := \lambda - I(\lambda, \alpha_i)\alpha_i, \quad \lambda \in L.$$

(iv) The group $W := \langle r_1, r_2 \rangle \subset \text{Aut}(L, I)$ is called the *Weyl group*. Define the set of *real roots* $\Delta^\text{re}$ as

$$\Delta^\text{re} := \{w(\alpha_i) \in L \mid w \in W, \ i = 1, 2\},$$

(v) Define the *Coxeter transformation* $c$ as $c := r_1r_2 \in W$.

Let $\mathcal{D}^b(K_\ell)$ be the bounded derived category of finitely generated modules over the path algebra associated to $K_\ell$, $\chi$ the Euler form on the Grothendieck group $K_0(\mathcal{D}^b(K_\ell))$, $S_i$ the simple module corresponding to the vertex $i$, $\Delta^\text{re}(\mathcal{D}^b(K_\ell)) := \langle r_{[S_1]}, r_{[S_2]} \rangle \{[S_1], [S_2]\}$ where $r_{[S_i]}(\lambda) := \lambda - (\chi + \chi^T)(\lambda, [S_i])[S_i]$ for $\lambda, [S_i] \in K_0(\mathcal{D}^b(K_\ell))$ and $[S[1]]$ the automorphism on $K_0(\mathcal{D}^b(K_\ell))$ induced by the Serre functor $S$ shifted by $[1]$. 
Lemma 3.2. The tuples \((L, I, \Delta^r, c)\) and \((K_0(D^b(K_\ell)), \chi + \chi^T, \Delta^r(D^b(K_\ell)), [S[1]])\) form generalized root systems in the sense of [STW] Definition 2.1. Moreover, these tuples coincide with each other under the identification \([S_i] = \alpha_i\) \((i = 1, 2)\).

Proof. For the former assertion, see [STW] Section 2.2. The latter assertion follows from straightforward calculations.

The matrix representations of \(r_i\) and \(c\) (denote them by same symbols with \(r_i\) and \(c\)) with respect to the basis \((\alpha_1, \alpha_2)\) are given by

\[
r_1 = \begin{pmatrix} -1 & \ell \\ 0 & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 0 \\ \ell & -1 \end{pmatrix}, \quad c = \begin{pmatrix} \ell^2 - 1 & -\ell \\ \ell & -1 \end{pmatrix}.
\]

Let \(\rho, \rho^{-1}\) be eigenvalues (spectral radius) of the Coxeter transformation \(c:\)

\[
\rho := \frac{\ell^2 - 2 + \sqrt{\ell^4 - 4\ell^2}}{2}, \quad \rho^{-1} = \frac{\ell^2 - 2 - \sqrt{\ell^4 - 4\ell^2}}{2}.
\]

3.1. The space \(X\).

Definition 3.3. Let the group homomorphism \(\alpha_i^* : L \to \mathbb{C}\) be the dual of \(\alpha_i\), i.e., \(\langle \alpha_i^*, \alpha_j \rangle := \alpha_i^*(\alpha_j) = \delta_{ij}\). Set

\[
\mathfrak{h}_\mathbb{R} := \text{Hom}_\mathbb{Z}(L, \mathbb{R}) \cong \mathbb{R}\alpha_1^* \oplus \mathbb{R}\alpha_2^*,
\]

\[
\mathfrak{h} := \text{Hom}_\mathbb{Z}(L, \mathbb{C}) \cong \mathbb{C}\alpha_1^* \oplus \mathbb{C}\alpha_2^*,
\]

and call \(\mathfrak{h}\) the Cartan subalgebra. Set the dual spaces

\[
\mathfrak{h}_\mathbb{R}^* := L \otimes \mathbb{Z} \mathbb{R} \cong \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2
\]

\[
\mathfrak{h}^* := L \otimes \mathbb{Z} \mathbb{C} \cong \mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2
\]

The \(W\)-action on \(L\) can be extended linearly to \(\mathfrak{h}^*\) and \(\mathfrak{h}_\mathbb{R}^*\). Define the \(W\)-action on \(\mathfrak{h}\) and \(\mathfrak{h}_\mathbb{R}\) by

\[
\langle w(Z), \lambda \rangle := \langle Z, w^{-1}(\lambda) \rangle \quad \text{for} \ Z \in \mathfrak{h} \text{ and } \lambda \in \mathfrak{h}_\mathbb{C}^*.
\]

The vector spaces \(\mathfrak{h}, \mathfrak{h}_\mathbb{R}, \mathfrak{h}^*, \mathfrak{h}_\mathbb{R}^*\) are equipped with the Euclidean topology for finite dimensional ones.

Definition 3.4 (cf. [Kac90] section 5.1]). Set

\[
L_+ := \sum_{i=1}^2 \mathbb{Z}_{\geq 0} \alpha_i, \quad L_- := -L_+ = \sum_{i=1}^2 \mathbb{Z}_{\leq 0} \alpha_i.
\]

(i) Define the set of positive real roots \(\Delta^r_+\) and the one of negative real roots \(\Delta^r_-\) as

\[
\Delta^r_+ := \Delta^r \cap L_+, \quad \Delta^r_- := \Delta^r \cap L_-.
\]
Theorem 3.9. There is a covering map

$$\pi : \text{Stab}^\circ (D^b(\Pi_2(K_\ell))) \to X_{\text{reg}}/W$$

and the subgroup $\mathbb{Z}[2] \times \text{Br}(D^b(\Pi_2(K_\ell))) \subset \text{Aut}(D^b(\Pi_2(K_\ell)))$ acts as the group of deck transformations.

(ii) Let $K := \{\lambda \in L_+ \setminus \{0\} \mid I(\lambda, \alpha_i) \leq 0, \ i = 1, 2\}$. The set of imaginary roots $\Delta^\text{im}$ is defined as $\Delta^\text{im} := \Delta^\text{im}_+ \cup \Delta^\text{im}_-$, where the set of positive imaginary roots $\Delta^\text{im}_+$ and the one of negative imaginary roots $\Delta^\text{im}_-$ are defined as

$$\Delta^\text{im}_+ := W(K) = \{w(\lambda) \mid w \in W, \ \lambda \in K\}, \quad \Delta^\text{im}_- := -\Delta^\text{im}_+.$$ 

Definition 3.5. The imaginary cone $\mathcal{I}$ is defined as the closure of the convex hull of $\Delta^\text{im}_+ \cup \{0\}$:

$$\mathcal{I} := \overline{\text{Conv}(\Delta^\text{im}_+ \cup \{0\})} \subset \mathfrak{h}_+^\ast.$$ 

Set $\mathcal{I}_0 := \mathcal{I} \setminus \{0\}$ and call it the blunt imaginary cone.

Let $\nu := \frac{\ell + \sqrt{\ell^2 - 4}}{2}$ and $\nu^{-1} = \frac{\ell - \sqrt{\ell^2 - 4}}{2}$. Obviously $\nu^2 = \rho$. We have

$$\mathcal{I} = \left\{ z_1 \alpha_1 + z_2 \alpha_2 \in \mathfrak{h}_+^\ast \mid z_1, z_2 \geq 0, \sum_{i,j=1}^2 z_i z_j I(\alpha_i, \alpha_j) \leq 0 \right\}$$

$$= \left\{ z_1 \alpha_1 + z_2 \alpha_2 \in \mathfrak{h}_R^+ \mid z_1, z_2 \geq 0, \ 2(z_1^2 - \ell z_1 z_2 + z_2^2) \leq 0 \right\}$$

$$= \left\{ z_1 \alpha_1 + z_2 \alpha_2 \in \mathfrak{h}_R^+ \mid z_1, z_2 \geq 0, \ (z_1 - \nu z_2)(z_1 - \nu^{-1} z_2) \leq 0 \right\}.$$ 

Definition 3.6 ([Ike14, Definition 2.7]). For $\lambda \in \mathfrak{h}_+^\ast$, set $H_\lambda := \{Z \in \mathfrak{h} \mid Z(\lambda) = 0\}$. Define $X \subset \mathfrak{h}$ as

$$X := \mathfrak{h} \setminus \bigcup_{\lambda \in \mathcal{I}_0} H_\lambda,$$ 

and define $X_{\text{reg}} \subset X$ as

$$X_{\text{reg}} := X \setminus \bigcup_{\alpha \in \Delta^\text{re}_+} H_\alpha.$$ 

Lemma 3.7 ([Ike14, Lemma 2.9]). The sets $X$ and $X_{\text{reg}}$ are open subsets of $\mathfrak{h}$.

Proposition 3.8 ([Ike14, Proposition 2.17]). The $W$-action can be restricted to $X$ and $X_{\text{reg}}$. This $W$-action is properly discontinuous on $X$, in particular, is free on $X_{\text{reg}}$.

Let $D^b(\Pi_2(K_\ell))$ be the bounded derived category of nilpotent modules over the pre-projective algebra $\Pi_2(K_\ell)$ associated to $K_\ell$, $\mathbb{Z}[2] \subset \text{Aut}(D^b(\Pi_2(K_\ell)))$ the subgroup generated by the shift functor [2]. Denote by $\text{Br}(D^b(\Pi_2(K_\ell)))$ the subgroup of $\text{Aut}(D^b(\Pi_2(K_\ell)))$ generated by spherical twists.

Theorem 3.9 ([Ike14, Theorem 1.1]). There is a covering map

$$\pi : \text{Stab}^\circ (D^b(\Pi_2(K_\ell))) \to X_{\text{reg}}/W$$

and the subgroup $\mathbb{Z}[2] \times \text{Br}(D^b(\Pi_2(K_\ell))) \subset \text{Aut}(D^b(\Pi_2(K_\ell)))$ acts as the group of deck transformations.
Lemma 3.10. Define the new basis \((\beta_1, \beta_2) := (\alpha_1, \alpha_2)P\) where
\[
P := \begin{pmatrix}
\frac{\nu}{\sqrt{(\ell^2-4)\nu}} & \frac{1}{\sqrt{(\ell^2-4)\nu}} \\
\frac{1}{\sqrt{(\ell^2-4)\nu}} & \frac{\nu}{\sqrt{(\ell^2-4)\nu}} \\
\end{pmatrix}.
\]

(i) The matrix representations \(R_i\) of reflections \(r_i\) and the one of \(I\) on \(h_C^*\) with respect to the basis \((\beta_1, \beta_2)\) are given by
\[
R_1 := P^{-1} r_1 P = \begin{pmatrix}
0 & \nu \\
\nu^{-1} & 0 \\
\end{pmatrix}, \\
R_2 := P^{-1} r_2 P = \begin{pmatrix}
0 & \nu^{-1} \\
\nu & 0 \\
\end{pmatrix}, \\
P^T A P = \begin{pmatrix}
0 & -1 \\
-1 & 0 \\
\end{pmatrix}.
\]
We also have \(P^{-1} c P = P^{-1} r_1 r_2 P = R_1 R_2\) and use the same symbol \(c\) for \(P^{-1} c P\).

(ii) Let \((\beta_1^*, \beta_2^*)\) be the dual basis for \((\beta_1, \beta_2)\). Then \((\beta_1^*, \beta_2^*) = (\alpha_1^*, \alpha_2^*)(P^T)^{-1}\).

(iii) The matrix representations of reflections \(r_i\) \((i = 1, 2)\) on \(h\) with respect to the basis \((\beta_1^*, \beta_2^*)\) are given by \((R_i^T)^{-1}\) and \((R_i^T)^{-1}\) respectively.

Denote by \((x^1, x^2)\) and \((x_1, x_2)\) the linear coordinates with respect to the basis \((\beta_1^*, \beta_2^*)\) of the Cartan subalgebra \(h\) and the basis \((\beta_1, \beta_2)\) of the dual \(\mathbb{C}\)-vector space \(h_C^*\) respectively.

(iv) We have
\[
R_1 \cdot (x^1, x^2) = (\nu^{-1} x^2, \nu x^1), \quad R_2 \cdot (x^1, x^2) = (\nu x^2, \nu^{-1} x^1).
\]

(v) We have
\[
\mathcal{I} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0, \ x_1 x_2 \leq 0\},
\]
and hence
\[
X = \mathbb{C}^2 \setminus \bigcup_{0 \leq a \leq 1} \{(x^1, x^2) \in \mathbb{C}^2 \mid ax^1 + (1-a)x^2 = 0\} = \mathbb{C}^2 \setminus \bigcup_{0 \leq \lambda \leq \infty} \{(x^1, x^2) \in \mathbb{C}^2 \mid x^1 = -\lambda x^2\}.
\]

Proof. The statements (i) to (iv) follow from straightforward calculations. The statement (v) follows from [Ike14, Lemma 2.8].

Lemma 3.11. Define the cycle \(\gamma : [0, 1] \to X\) as
\[
\gamma(t) := \left(e^{2\pi\sqrt{-1}t}, e^{2\pi\sqrt{-1}t}\right) \in X.
\]
Then \(\gamma\) is a generator of the fundamental group \(\pi_1(X)\) of \(X\) and hence \(\pi_1(X) = \langle \gamma \rangle \cong \mathbb{Z}\).

Proof. By Lemma 3.10 (v), there exists an isomorphism of complex manifolds:
\[
X \xrightarrow{\cong} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \mathbb{R}_{\leq 0}); \quad (x^1, x^2) \mapsto \left(x^1, \frac{x^1}{x^2}\right).
\]
The cycle \(\{(e^{2n\sqrt{-1}}, 1) \mid t \in [0, 1]\} \subset (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \mathbb{R}_{\leq 0})\) is an generator of the fundamental group \(\pi_1((\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \mathbb{R}_{\leq 0})) \cong \pi_1(\mathbb{C} \setminus \{0\}) \times \pi_1(\mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cong \mathbb{Z}\) and the image of \(\gamma\) by the above isomorphism.

3.2. Weyl group invariant theory. Set
\[
\tilde{X} := \{(y^1, y^2) \in \mathbb{C}^2 \mid |\text{Im}(y^1 - y^2)| < \pi\} \subset \mathbb{C}^2.
\]

**Proposition 3.12.** The space \(\tilde{X}\) is the universal covering space of \(X\). The covering map is given by
\[
\pi_X : \tilde{X} \rightarrow X, \quad (y^1, y^2) \mapsto (e^{y^1}, e^{y^2})
\]

**Proof.** For \((y^1, y^2) \in \tilde{X}\), set \(\lambda := e^{y^1-y^2} = e^{\text{Re}(y^1-y^2)}e^{\sqrt{-1}\text{Im}(y^1-y^2)}\). Since \(\lambda\) is not a negative real number since \(|\text{Im}(y^1 - y^2)| < \pi\) and \(e^{y^1} = e^{y^1-y^2+y^2} = \lambda e^{y^2}\), the map \(\pi_X\) is well-defined. The statement follows from Lemma 3.11. □

**Remark 3.13.** Fix the isomorphism \(\mathfrak{h} \cong \mathbb{C}^2\) via the basis \((\beta_1^*, \beta_2^*)\). Define the map \(\pi_{\mathbb{C}^2} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \cong \mathfrak{h}\) as \((y^1, y^2) \mapsto (e^{y^1}, e^{y^2})\). The boundary \(\partial \tilde{X}\) of \(\tilde{X}\) is given by
\[
\partial \tilde{X} = \{(y^1, y^2) \in \mathbb{C}^2 \mid |\text{Im}(y^1 - y^2)| = \pi\}.
\]
We have \(\pi_{\mathbb{C}^2}(\partial \tilde{X}) = \mathfrak{h}\backslash X = \bigcup_{\lambda \in \mathbb{Z}_0} H_\lambda\).

**Remark 3.14.** If we choose the following domain as \(\tilde{X}\):
\[
\{(y^1, y^2) \in \mathbb{C}^2 \mid (2n-1)\pi < \text{Im}(y^1 - y^2) < (2n+1)\pi\} \quad (n \neq 0),
\]
the monodromy group of the resulting Frobenius manifold does not coincide with the Weyl group for \(K_\ell\). See also Proposition 4.8.

We can lift the Weyl group action on \(\tilde{X}\) as follows:
\[
R_1 \cdot (y^1, y^2) = (y^2 - \log \nu, \ y^1 + \log \nu)
\]
\[
R_2 \cdot (y^1, y^2) = (y^2 + \log \nu, \ y^1 - \log \nu).
\]
The \(W\)-action above is equivariant to \(\pi_X\). The fundamental group \(\pi_1(X)\) acts on the universal covering space \(\tilde{X}\). This \(\pi_1(X)\)-action is given by
\[
\gamma \cdot (y^1, y^2) = (y^1 + 2\pi \sqrt{-1}, \ y^2 + 2\pi \sqrt{-1})
\]
for the generator \(\gamma \in \pi_1(X)\). These two actions on \(\tilde{X}\) can be extended naturally on \(\mathbb{C}^2 \cap \tilde{X}\). Note that these actions on \(\mathbb{C}^2\) are properly discontinuous.
Lemma 3.15. Set
\[
\tilde{U} := \left\{ (y^1, y^2) \in \tilde{X} \mid \left| \text{Re} (y^1 - y^2) \right| \leq \log \nu \right\} \subset \tilde{X},
\]
\[
U := \pi_X (\tilde{U}) = \left\{ (x^1, x^2) \in X \mid \nu^{-1} \leq \frac{|x^1|}{|x^2|} \leq \nu \right\} \subset X.
\]
Then the subset \( \tilde{U} \) is a fundamental domain of the \( W \)-action in (3.2) and (3.3) on \( \tilde{X} \). The subset \( U \) is a fundamental domain of the \( W \)-action in Lemma 3.10 (iv) on \( X \).

Proof. The statement follows from explicit presentations of actions (3.2) and (3.3). \( \square \)

Definition 3.16. Let \( W \rightacts \mathbb{C}^2 \supset \tilde{X} \) be the \( W \)-action in (3.2) and (3.3). Define the complex analytic space \( \mathbb{C}^2//W \) as follow:

- Its underlying space is \( \mathbb{C}^2/W \), the quotient space of \( \mathbb{C}^2 \) by \( W \).
- Let \( \pi : \mathbb{C}^2 \to \mathbb{C}^2/W \) be the quotient map. Denote by \( O_{\mathbb{C}^2}^W \) the \( W \)-invariant subsheaf of \( O_{\mathbb{C}^2} \) the sheaf of germs of holomorphic functions for \( \mathbb{C}^2 \). Define the structure sheaf \( O_{\mathbb{C}^2//W} := \pi_* O_{\mathbb{C}^2}^W \).

Define the complex analytic space \( \tilde{X}///W \) as follows:

- Its underlying space is \( \tilde{X}/W \), the quotient space of \( \tilde{X} \) by \( W \).
- Let \( \tilde{\pi} : \tilde{X} \to \tilde{X}/W \) be the quotient map. Denote by \( O_{\tilde{X}}^W \) the \( W \)-invariant subsheaf of \( O_{\tilde{X}} \), the sheaf of germs of holomorphic functions for \( \tilde{X} \). Define the structure sheaf \( O_{\tilde{X}///W} := \tilde{\pi}_* O_{\tilde{X}}^W \).

Proposition 3.17. Define the map \( \varphi : \mathbb{C}^2 \to \mathbb{C}^2; (y^1, y^2) \mapsto (s^1, s^2) \) where
\[
s^1 = e^{\frac{h}{2}(y^1-y^2)} - e^{\frac{h}{2}(y^2-y^1)},
\]
\[
s^2 = y^1 + y^2.
\]
Then the map \( \varphi \) induces an isomorphism between complex analytic spaces
\[
\varphi : \mathbb{C}^2//W \xrightarrow{\simeq} \mathbb{C}^2, \quad [(y^1, y^2)] \mapsto (s^1, s^2).
\]
In particular, the complex analytic space \( \mathbb{C}^2//W \) is a complex manifold.

Proof. We have \( \varphi \circ \pi = \varphi \) and \( \varphi_* O_{\mathbb{C}^2//W} = \varphi_* \pi_* O_{\mathbb{C}^2}^W \cong \varphi_* O_{\mathbb{C}^2}^W \). We shall show that \( \varphi_p : O_{\mathbb{C}^2, \varphi(p)} \to O_{\mathbb{C}^2, p}^W \) is isomorphic for any \( p \in \mathbb{C}^2 \). This morphism \( \varphi_p \) is given by
\[
\varphi_p(f)(y^1, y^2) = f \circ \varphi(y^1, y^2) = f \left( e^{\frac{h}{2}(y^1-y^2)} - e^{\frac{h}{2}(y^2-y^1)}, y^1 + y^2 \right) \in O_{\mathbb{C}^2, p}^W
\]
for \( f(s^1, s^2) \in O_{\mathbb{C}^2, \varphi(p)} \).
We construct the inverse map \( \psi_p : \mathcal{O}_{\mathbb{C}^2, p}^W \to \mathcal{O}_{\mathbb{C}^2, \varphi(p)} \) of \( \varphi_p \). For every \( n_1, n_2 \in \mathbb{Z}_{\geq 0} \), set the differential \( D_{n_1, n_2} : \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_{\mathbb{C}^2} \) as

\[
D_{n_1, n_2} := \sum_{i=0}^{n_1+n_2} \frac{(n_1+n_2)!}{i!(n_1+n_2-i)!} \frac{1}{2n_2! n_1!} \left( e^{\frac{h}{2} (y^1 - y^2)} + e^{\frac{h}{2} (y^2 - y^1)} \right)_{n_1} \left( \frac{\partial}{\partial y^1} \right)^i \left( \frac{\partial}{\partial y^2} \right)^{n_1+n_2-i}.
\]

Then \( \psi_p \) is given by

\[
\psi_p(g)(s^1, s^2) := \sum_{n_1, n_2 \in \mathbb{Z}_{\geq 0}} \frac{1}{n_1! n_2!} \left. D_{n_1, n_2}(g) \right|_{(y^1, y^2) = p} (s^1 - s^1(p))^{n_1} (s^2 - s^2(p))^{n_2}
\]

for \( g(y^1, y^2) \in \mathcal{O}_{\mathbb{C}^2, p}^W \). We have

\[
\left( \frac{\partial}{\partial s^1} \right)^{n_1} \left( \frac{\partial}{\partial s^2} \right)^{n_2} \left( \frac{\partial y^1}{\partial s^1} \frac{\partial}{\partial y^2} + \frac{\partial y^2}{\partial s^1} \frac{\partial}{\partial y^1} \right)^{n_1} \left( \frac{\partial y^1}{\partial s^2} \frac{\partial}{\partial y^2} + \frac{\partial y^2}{\partial s^2} \frac{\partial}{\partial y^1} \right)^{n_2} = D_{n_1, n_2}.
\]

The value of \( \left. D_{n_1, n_2}(g) \right|_{(y^1, y^2) = p} \) is determined uniquely by \( (s^1, s^2) \) since \( g \) is a \( W \)-invariant function. Hence, we have \( \psi_p = \varphi^{-1}_p \) by the Taylor expansion. \( \square \)

By Definition 3.16 and Proposition 3.17 we have the following commutative diagram of complex manifolds:

\[
\begin{array}{cc}
\tilde{X} & \mathbb{C}^2 \\
\downarrow & \downarrow \\
\tilde{X}/W & \mathbb{C}^2/W.
\end{array}
\]

**Proposition 3.18.** Set

\[
E := \left\{ z \in \mathbb{C} \left| \frac{(\text{Re } z)^2}{\exp \left( \frac{\pi^2}{\log \rho} \right) - \exp \left( - \frac{\pi^2}{\log \rho} \right)} + \frac{(\text{Im } z)^2}{\exp \left( \frac{\pi^2}{\log \rho} \right) + \exp \left( - \frac{\pi^2}{\log \rho} \right)} < 1 \right\} \subset \mathbb{C}.
\]

Then the morphism given by (3.4) induces an isomorphism between complex manifolds:

\[
\tilde{X}/W \cong E \times \mathbb{C}, \quad [(y^1, y^2)] \mapsto (s^1, s^2).
\]

In particular, the complex manifold \( \tilde{X}/W \) is isomorphic to the space of stability conditions \( \text{Stab}(D^b(K_\ell)) \) of \( D^b(K_\ell) \).
Proof. It is easily shown that the complex manifold $E \times \mathbb{C}$ is the image of $\tilde{X} \mod W$ by $\varphi$. Latter statement follows from $\text{Stab}(\mathcal{D}^b(K_\ell)) \cong \mathbb{H} \times \mathbb{C}$ in [DK16, Theorem 1.5] and an isomorphism $\mathbb{H} \cong E$ as complex manifolds. □

Remark 3.19. The functions $e^{s^1}$ and $s^2$ are single-valued functions on $X$:

$$s^1 = \left( \frac{x^1}{x^2} \right)^\frac{1}{2} - \left( \frac{x^2}{x^1} \right)^\frac{1}{2},$$

$$e^{s^2} = x^1x^2.$$

Note that $s^1$ is an infinitely multi-valued function on $X$.

4. A Frobenius structure for $\ell$–Kronecker quiver

Set $h := \frac{2\pi\sqrt{-1}}{\log \rho}$. The eigenvalues $\rho$ and $\rho^{-1}$ of the Coxeter transformation $c$ can be expressed as follows:

$$\rho = \exp \left( \frac{2\pi\sqrt{-1}}{h} \right), \quad \rho^{-1} = \exp \left( -\frac{2\pi\sqrt{-1}}{h} \right).$$

The value $h$ can be regarded as a generalization of the Coxeter number.

We shall construct a Frobenius structure of rank 2 and dimension $1 - \frac{2}{h}$ on the complex manifold $\tilde{X} \mod W$ whose intersection form coincides with the generalized Cartan matrix $A$ for $K_\ell$.

Theorem 4.1. There exists a unique Frobenius structure of rank 2 and dimension $1 - \frac{2}{h}$ on $\tilde{X} \mod W$ such that $e = \frac{\partial}{\partial t^1}$, $E = t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}$ for the flat coordinates $(t^1, t^2)$ in Subsection 4.1 and the intersection form coincides with the generalized Cartan matrix $A$ of $\ell$-Kronecker quiver $K_\ell$.

Remark 4.2. Since the Frobenius structure is of rank 2 and dimension $1 - \frac{2}{h}$, the Euler vector field $E$ is given by $E = t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}$ with respect to the flat coordinates $(t^1, t^2)$. In particular, it is easily shown that the Frobenius potential $F$ must be

$$F(t^1, t^2) = \frac{1}{2} \eta_2 (t^1)^2 t^2 + c(t^2)^{h+1}, \quad c \in \mathbb{C} \setminus \{0\},$$

and hence the Frobenius structure on $\tilde{X} \mod W$ should be unique if it exists (see also Example 2.4).
4.1. **The unit vector field and the Euler vector field.** Let \((s^1, s^2)\) be the coordinates of \(\tilde{X}/W\) in Proposition 3.18. Set \(t^1 := e^{\frac{h}{2} s^2} \cdot s^1, \quad t^2 := e^{s^2}\).

The map \(\mathbb{C} \times E \to \mathbb{C} \times \mathbb{C}^*; \quad (s^1, s^2) \mapsto (t^1, t^2)\) is a local homeomorphism. The coordinates \((t^1, t^2)\) are locally equal to
\[
t^1 = (x^1)^h - (x^2)^h, \quad t^2 = x^1 x^2\]
where \((x^1, x^2)\) are the coordinates of \(X\) in Definition 3.10. Define degrees of \(t^1\) and \(t^2\) as \(\text{deg} t^1 := h\) and \(\text{deg} t^2 := 2\) due to Proposition 2.10 and local expressions of \(t_1\) and \(t_2\) above. Set
\[
e := \frac{\partial}{\partial t^1}, \quad E := \frac{\text{deg} t^1}{h} t^1 \frac{\partial}{\partial t^1} + \frac{\text{deg} t^2}{h} t^2 \frac{\partial}{\partial t^2} = t^1 \frac{\partial}{\partial t^1} + 2 \frac{t^2}{h} \frac{\partial}{\partial t^2}.
\]
It will be shown later that \((t^1, t^2)\) are flat coordinates and \(e\) and \(E\) are the unit vector field and the Euler vector field.

**Remark 4.3.** Note that the Euler vector field \(E\) is equal to \(\frac{2}{h} \frac{\partial}{\partial s^2}\). This observation can be regarded as an analogy of [DZ98, Theorem 2.1 (ii)].

4.2. **The non-degenerate \(\mathcal{O}_{\tilde{X}/W}\)-symmetric bilinear form and flat coordinates.** Let \(g : \Omega^1 X \times \Omega^1 X \to \mathcal{O}_X\) be the bilinear form on the cotangent sheaf of \(X\) induced by the generalized Cartan matrix of \(K_\ell\) via the natural isomorphism \(T^*_p \mathfrak{h} \cong \mathfrak{h}^*\) and the restriction to \(X\):
\[
g(dx^i, dx^j) := I(\beta^*_i, \beta^*_j) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]
The bilinear form \(g\) induces the one on the cotangent sheaf of \(\tilde{X}\) (denote the induced bilinear form on \(\Omega^1 \tilde{X}\) by the same symbol \(g\) for simplicity). With respect to the coordinates \((y^1, y^2)\) of \(\tilde{X}\) in (3.1), we have
\[
g(dy^i, dy^j) = \sum_{a,b=1}^2 \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} g(dx^a, dx^b)
= \begin{pmatrix} 0 & -\frac{1}{e^{y^1+y^2}} \\ -\frac{1}{e^{y^1+y^2}} & 0 \end{pmatrix}.
\]
The bilinear form \( g \) on \( \Omega^1_{\widetilde{X}} \) induces the bilinear form on \( \Omega^1_{\widetilde{X}/W} \) (denote this induced bilinear form by the same symbol \( g \) again). By the definition of the coordinates \((s^1, s^2)\) in Proposition \ref{prop:coordinates}, the induced bilinear form \( g : \Omega^1_{\widetilde{X}/W} \times \Omega^1_{\widetilde{X}/W} \to \Theta_{\widetilde{X}/W} \) is given by

\[
g(ds^i, ds^j) = \sum_{a,b=1}^2 \frac{\partial s^i}{\partial y^a} \frac{\partial s^j}{\partial y^b} g(dy^a, dy^b) = \begin{pmatrix} \frac{2}{e^{s^2}} & 0 \\ 0 & \frac{h^2}{2e^{s^2}}(4 + (s^1)^2) \end{pmatrix}.
\]

**Proposition 4.4.** Define \( \eta : \Omega^1_{\widetilde{X}/W} \times \Omega^1_{\widetilde{X}/W} \to \Theta_{\widetilde{X}/W} \) as \( \eta := \text{Lie}_e g \).

Then the bilinear form \( \eta \) defines a non-degenerate and flat \( \Theta_{\widetilde{X}/W} \)-symmetric bilinear form on \( T_{\widetilde{X}/W} \) (denote this bilinear form on \( T_{\widetilde{X}/W} \) by the same symbol \( \eta \)).

**Proof.** With respect to the local coordinates \((t^1, t^2)\), we have

\[
\eta(dt^i, dt^j) = \frac{\partial}{\partial t^1} \left( g(dt^i, dt^j) \right) - g \left( \text{Lie}_{\frac{\partial}{\partial t^1}} dt^i, dt^j \right) - g \left( dt^i, \text{Lie}_{\frac{\partial}{\partial t^1}} dt^j \right) = \frac{\partial}{\partial t^1} \left( g(dt^i, dt^j) \right).
\]

By the equation (4.1), we have

\[
(g(dt^i, dt^j)) = \begin{pmatrix} 2h^2(t^2)^{h-1} & -ht^1 \\ -ht^1 & -2t^2 \end{pmatrix}
\]

and hence

\[
(\eta(dt^i, dt^j)) = \begin{pmatrix} 0 & -h \\ -h & 0 \end{pmatrix}, \quad \text{i.e.,} \quad \left( \eta \left( \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \right) \right) = \begin{pmatrix} 0 & -\frac{1}{h} \\ -\frac{1}{h} & 0 \end{pmatrix}.
\]

Therefore, the bilinear form \( \eta \) is a non-degenerate and flat bilinear form whose flat coordinates are \((t^1, t^2)\). \( \square \)

4.3. **The product structure.** Recall Lemma \ref{lem:product}. Set \( \Gamma_k^{ij} := g \left( dt^i, \nabla_{\frac{\partial}{\partial x^k}} dt^j \right) \) where \( \nabla \) is the Levi–Civita connection of \( g \) on \( \Omega^1_{\widetilde{X}/W} \). Define \( C_k^{ij} \) as follows:

\[
C_k^{ij} := \frac{h}{\deg t^k - 1} \sum_{a=1}^2 \eta_a \cdot \Gamma_a^{jk}, \quad i, j, k \in \{1, 2\}
\]

where \( \eta := \eta \left( \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \right) \) in Proposition \ref{prop:eta}. 

Lemma 4.5. Set
\[ F := -\frac{1}{2h} (t^1)^2 t^2 + \frac{1}{h^2 - 1} (t^2)^{h+1}. \]
Then we have
\[ C_{ij}^k = \sum_{a=1}^2 \eta^{ka} \frac{\partial^3 F}{\partial t^a \partial t^i \partial t^j}. \]

Proof. The statement follows from straightforward calculations. □

The following proposition follows from Lemma 4.5:

Proposition 4.6. Define the \( O_{\tilde{X}/W} \)-linear map \( \circ : \mathcal{T}_{\tilde{X}/W} \times \mathcal{T}_{\tilde{X}/W} \to \mathcal{T}_{\tilde{X}/W} \) by
\[ \frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j} := \sum_{k=1}^2 C_{ij}^k \frac{\partial}{\partial t^k}, \quad i, j = 1, 2. \]
Then the product \( \circ \) is commutative and associative. Moreover, \( e = \frac{\partial}{\partial t^1} \) is the unit vector field for \( \circ \).

4.4. The discriminant locus and the monodromy group. Denote by \( D_{\tilde{X}/W} \) the zeros of the discriminant \( \Delta_{\tilde{X}/W} \) of the Frobenius manifold \( (\tilde{X}/W, \eta, \circ, e, E) \). The discriminant locus \( D_{\tilde{X}/W} \) is given by
\[ D_{\tilde{X}/W} = \{(s^1, s^2) \in \tilde{X}/W \mid s^1 = \pm 2\sqrt{-1}\} \]
with respect to the global coordinates \( s^1, s^2 \) by the equation (4.1). With respect to the flat coordinates \( (t^1, t^2) \), we have
\[ D_{\tilde{X}/W} = \{(t^1, t^2) \in \tilde{X}/W \mid (t^1)^2 + 4(t^2)^h = 0\}. \]

Proposition 4.7. Set
\[ \tilde{X}_{\text{reg}} := \pi^{-1}_X (X_{\text{reg}}) \subset \tilde{X}. \]
Then we have
\[ \tilde{X}_{\text{reg}}/W \cong (\tilde{X}/W) \setminus D_{\tilde{X}/W}. \]

Proof. Recall Proposition 3.8. Since \( \bigcup_{\alpha \in \Delta_+^n} H_\alpha = \{(x^1, x^2) \mid x^2 = \nu^{2k+1} x^1 \ (k \in \mathbb{Z})\} \subset X \), we have
\[ \pi^{-1}_{\tilde{X}} \left( \bigcup_{\alpha \in \Delta_+^n} H_\alpha \right) = \{(y^1, y^2) \in \tilde{X} \mid y^1 = y^2 + (2k + 1) \log \nu, \ k \in \mathbb{Z}\} \subset \tilde{X} \]
and hence
\[ \tilde{X} \setminus \pi^{-1}_{\tilde{X}} \left( \bigcup_{\alpha \in \Delta_+^n} H_\alpha \right) = \tilde{X}_{\text{reg}}. \]
By the definition of \((s^1, s^2)\), the image of \((y^1, y^2) \in \pi^{-1}_X \left( \bigcup_{\alpha \in \Delta^+_e} H_\alpha \right)\) by \(\varphi\) are

\[
s^1 = \exp \left( -\frac{h}{2} (2k + 1) \log \nu \right) - \exp \left( \frac{h}{2} (2k + 1) \log \nu \right)
= \exp \left( - \left( k + \frac{1}{2} \right) \pi \sqrt{-1} \right) - \exp \left( \left( k + \frac{1}{2} \right) \pi \sqrt{-1} \right)
= \begin{cases} 
-2\sqrt{-1} & \text{if } k \text{ is even} \\
2\sqrt{-1} & \text{if } k \text{ is odd},
\end{cases}
\]

\[
s^2 = 2y^1 + (2k + 1) \log \nu.
\]

Therefore the isomorphism \((4.2)\) holds. \(\square\)

**Proposition 4.8.** The monodromy group \(W_{\tilde{X}/W}\) of \((\tilde{X}/W, \eta, \circ, e, E)\) coincides with the Weyl group \(W\).

**Proof.** By the construction, the coordinates \(x^1\) and \(x^2\) are periods of the Frobenius manifold \((\tilde{X}/W, \eta, \circ, e, E)\):

\[
x^1 = e^{\frac{s^2}{2}} \left( \frac{s^1 + \sqrt{(s^1)^2 + 4}}{2} \right)^{\frac{1}{\nu}},
\]

\[
x^2 = e^{\frac{s^2}{2}} \left( \frac{-s^1 + \sqrt{(s^1)^2 + 4}}{2} \right)^{\frac{1}{\nu}}.
\]

Fix a point \(* \in \tilde{X}_{\text{reg}}\). For \(i = 1, 2\), let \(\tilde{\gamma}_i\) be a path on \(\tilde{X}\) from \(*\) to \(r_i(\ast\ast)\). Denote by \(\gamma_i := \varphi(\tilde{\gamma}_i)\). Then \(\gamma_1\) (resp. \(\gamma_2\)) is a loop around \(2\sqrt{-1}\) (resp. \(-2\sqrt{-1}\)) on \((\tilde{X}/W)_{\text{reg}}\) and these loops are generators of the fundamental group \(\pi_1((\tilde{X}/W)_{\text{reg}}, \varphi(\ast\ast))\). Note that the branch cut of \(\sqrt{(s^1)^2 + 4}\) is the line between \(2\sqrt{-1}\) and \(-2\sqrt{-1}\). The monodromy representation \(M_{\gamma_i}\) of \(\gamma_i\) is given by

\[
M_{\gamma_1} = \begin{pmatrix} 0 & \nu \\ \nu^{-1} & 0 \end{pmatrix}, \quad M_{\gamma_2} = \begin{pmatrix} 0 & \nu^{-1} \\ \nu & 0 \end{pmatrix}
\]

for periods \((x^1, x^2)\). Therefore, we have \(W_{\tilde{X}/W} \cong W\). \(\square\)

Finally, we show that the Frobenius manifold \((\tilde{X}/W, \eta, \circ, e, E)\) is semi-simple in the sense of [Dub96, Lecture 3]:

**Proposition 4.9.** The Frobenius manifold \((\tilde{X}/W, \eta, \circ, e, E)\) is semi-simple.
Proof. Set

\[ u^1 := e^{hs^2} \left( s^1 + 2\sqrt{-1} \right), \quad u^2 := e^{hs^2} \left( s^1 - 2\sqrt{-1} \right). \]

We can easily check that \((u^1, u^2)\) form canonical coordinates. \qed

4.5. Perspectives. In the case of the root system of type \(A_2\), there exists an isomorphism between the Frobenius manifold \(h/W\) and \(\text{Stab}(\mathcal{D}^b(A_2))\) such that the central charge map is identified with the oscillatory integrals on \(h/W\) (BQS14, HKK17). Based on the results and Proposition 3.18, we expect the following

**Conjecture 4.10.** There should exist an isomorphism \(\varphi : \tilde{X}/W \to \text{Stab}(\mathcal{D}^b(K_\ell))\) such that \(\varphi\) is compatible with a deformed flat coordinates (see Dub99a) and the central charge map. That is, the following diagram commutes;

\[
\begin{array}{ccc}
\tilde{X}/W & \xrightarrow{\varphi} & \text{Stab}(\mathcal{D}^b(K_\ell)) \\
(\tilde{t}_1|_{u=1}, \tilde{t}_2|_{u=1}) & \searrow & \mathbb{C}^2 \\
& & \searrow \varphi \\
& & \text{Stab}(\mathcal{D}^b(K_\ell))
\end{array}
\]

In particular, we have

\[
\widehat{\nabla}|_{u=1} = \varphi^*d,
\]

where \(d\) is the trivial connection on

\[
T\text{Stab}(\mathcal{D}^b(K_\ell)) \cong \text{Stab}(\mathcal{D}^b(K_\ell)) \times \text{Hom}_\mathbb{Z}(K_0(\mathcal{D}^b(K_\ell)), \mathbb{C})
\]

and \(\widehat{\nabla}|_{u=1}\) is the restriction of the first structure connection on \(T(\tilde{X}/W)\), which is defined by

\[
\widehat{\nabla}\delta' := \nabla_{\delta} \delta + \frac{1}{u} \delta \circ \delta'.
\]

Through the present work, we reach to partial results that exponents, duality among them and the non-degenerate symmetric bilinear form can be obtained via eigenvalues and eigenvectors of the Coxeter transformation for a generalized root system whose Cartan matrix is non-degenerate and symmetric (e.g., see Lemma 3.10).

The present paper will be the first of a series of our attempts to construct Frobenius structures from the Weyl group invariant theories associated to such generalized root systems, moreover on the spaces of stability conditions.

**References**

[Ber00a] M. Bertola. *Frobenius manifold structure on orbit space of Jacobi groups; Part I*, Differential Geometry and its Applications, Volume 13, Issue 1, July 2000, Pages 19–41
[Ber00b] M. Bertola, *Frobenius manifold structure on orbit space of Jacobi groups; Part II*, Differential Geometry and its Applications, Volume 13, Issue 3, November 2000, Pages 213–233.

[Bri07] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) 166 (2007), no. 2, 317345.

[BQS14] T. Bridgeland, Y. Qiu and T. Sutherland, *Stability conditions and $A_2$-quiver*, Advances in Mathematics, Volume 365, 13 May 2020, 107049.

[DK16] G. Dimitrov and L. Katzarkov, *Some new categorical invariants*, arXiv:1602.09117v3.

[Dub96] B. Dubrovin, *Geometry of 2d topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348.

[DS03] A. Douai and C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures (I)*, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) 53 (2003), no. 4, 1055–1116.

[DS04] A. Douai and C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures (II)*, Frobenius manifolds (Quantum cohomology and singularities), Hertling, C. and Marcolli, M. eds, Aspects of Mathematics, vol. E36, Vieweg, 2004, pp. 1–18.

[DZ98] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*, Compositio Math. 111 (1998), no. 2, 167219.

[DSZZ15] B. Dubrovin, I. Strachan, Y. Zhang and D. Zuo, *Extended affine Weyl groups of BCD type, Frobenius manifolds and their Landau - Ginzburg superpotentials*, Advances in Mathematics 351 (2019) 8979.

[HKK17] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes tudes Sci. 126 (2017), 247318.

[Her02] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, Cambridge University Press, Spring 2002.

[Ike14] A. Ikeda, *Stability conditions for preprojective algebras and root systems of Kac–Moody Lie algebras*, arXiv:1402.1392.

[Ike17] A. Ikeda, *Stability conditions on CY_N categories associated to An-quivers and period maps*, Math. Ann. (2017) 367:1-49.

[Kac90] V.G. Kac, *Infinite Dimensional Lie Algebras*, third edition, Cambridge Univ. Press, Cambridge, 1990.
[KST07] H. Kajiura, K. Saito and A. Takahashi, Matrix factorizations and representations of
quivers II: Type ADE case, Advances in Mathematics 211(1) (2007) 327–362.
[KST09] H. Kajiura, K. Saito and A. Takahashi, Triangulated categories of matrix factorizations
for regular systems of weights with $\epsilon = -1$, Advances in Mathematics 220(5) (2009) 1602–
1654.
[KMS18] Y. Konishi, S. Minabe and Y. Shiraishi, Almost duality for Saito structure and complex
reflection groups, Journal of Integrable Systems 2018(3) 1-48.
[KaMaSe15] M. Kato, T. Mano and J. Sekiguchi, Flat structure on the space of isomonodromic
deformations, arXiv:1511.01608.
[Kai86] K. Saito, Period mapping associated to a primitive form, Publ. Res. Inst. Math. Sci. 19
(1983), no. 3, pp. 1231–1264.
[Kai93] K. Saito, On a linear structure of the quotient variety by a finite reflection group, Publ.
RIMS 1993 Volume 29 Issue 4 Pages 535–579.
[Kai89] M. Saito On the structure of Brieskorn lattice Annales de l’Institut Fourier, Volume 39
(1989) no. 1, pp. 27–72.
[ST08] K. Saito and A. Takahashi, From Primitive Forms to Frobenius manifolds, Proceedings
of Symposia in Pure Mathematics, 78 (2008) 31-48.
[STW] Y. Shiraishi, A. Takahashi and K. Wada, On Weyl Groups and Artin Groups Associated
to Orbifold Projective Lines, Journal of Algebra 453(1) 249–290, 5 (2016).
[SYS80] K. Saito, T. Yano and J. Sekiguchi, On a certain generator system of the ring of
invariants of a finite reflection group, Comm. Algebra 8 (1980), no. 4, 373408.
[Sat10] I Satake, Frobenius manifolds for elliptic root systems, Osaka J. Math. 47 (2010), no. 1,
301330.
[Tak05] A. Takahashi, Matrix Factorizations and Representations of Quivers I,
arXiv:math/0506347
[Wan19] C-H. Wang, Stability conditions and braid group actions on affine $A_n$ quivers,
arXiv:1902.05315v1.

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