The Unimodality of the Crank on Overpartitions

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Abstract. Let $N(m, n)$ denote the number of partitions of $n$ with rank $m$, and let $M(m, n)$ denote the number of partitions of $n$ with crank $m$. Chan and Mao proved that for any nonnegative integers $m$ and $n$, $N(m, n) \geq N(m + 2, n)$ and for any nonnegative integers $m$ and $n$ such that $n \geq 12$, $n \neq m + 2$, $N(m, n) \geq N(m, n - 1)$. Recently, Ji and Zang showed that for $n \geq 44$ and $1 \leq m \leq n - 1$, $M(m - 1, n) \geq M(m, n)$ and for $n \geq 14$ and $0 \leq m \leq n - 2$, $M(m, n) \geq M(m, n - 1)$. In this paper, we analogue the result of Ji and Zang to overpartitions. Note that Bringmann, Lovejoy and Osburn introduced two type of cranks on overpartitions, namely the first residue crank and the second residue crank. Consequently, for the first residue crank $\overline{M}(m, n)$, we show that $\overline{M}(m - 1, n) \geq \overline{M}(m, n)$ for $m \geq 1$ and $n \geq 3$ and $\overline{M}(m, n) \geq \overline{M}(m, n + 1)$ for $m \geq 0$ and $n \geq 1$. For the second residue crank $\overline{M}_2(m, n)$, we show that $\overline{M}_2(m - 1, n) \geq \overline{M}_2(m, n)$ for $m \geq 1$ and $n \geq 0$ and $\overline{M}_2(m, n) \geq \overline{M}_2(m, n + 1)$ for $m \geq 0$ and $n \geq 1$. Moreover, let $M_k(m, n)$ denote the number of $k$-colored partitions of $n$ with $k$-crank $m$, which was defined by Fu and Tang. They conjectured that when $k \geq 2$, $M_k(m - 1, n) \geq M_k(m, n)$ except for $k = 2$ and $n = 1$. With the aid of the inequality $\overline{M}(m - 1, n) \geq \overline{M}(m, n)$ for $m \geq 1$ and $n \geq 3$, we confirm this conjecture.

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1 Introduction

The first residue crank and the second residue crank on overpartitions are two important statistics in the theory of partitions. The main objective of this paper is to investigate the distribution of the first residue and the second residue on overpartitions. Recall that the rank of a partition was defined by Dyson [16] as the largest part minus the number of parts. The crank of a partition was introduced by Andrews and Garvan [1] as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones.
Let $p(n)$ denote the number of partitions of $n$. Dyson [16] first conjectured and later confirmed by Atkin and Swinnerton-Dyer [7] that the rank can provide combinatorial interpretations to the first two Ramanujan’s congruences $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$. Andrews and Garvan [4] showed that the crank may combinatorially interpret the Ramanujan’s congruences $p(5n + 4) \equiv 0 \pmod{5}$, $p(7n + 5) \equiv 0 \pmod{7}$ and $p(11n + 6) \equiv 0 \pmod{11}$. Relations between ranks and cranks of partitions have been studied by several authors, for example, Andrews and Lewis [5], Garvan [18], Lewis [24, 25].

Let $N(m, n)$ denote the number of partitions of $n$ with rank $m$, and for $n > 1$, let $M(m, n)$ denote the number of partitions of $n$ with crank $m$, and for $n = 1$, set

$$M(0, 1) = -1, \quad M(-1, 1) = M(1, 1) = 1,$$

and $M(m, 1) = 0$ for $m \neq 0, -1, 1$.

Andrews and Garvan [4] derived the following generating function of $M(m, n)$:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m, n)z^m q^n = \frac{(q; q)_\infty}{(zq; q)_\infty(q/z; q)_\infty}. \quad (1.1)$$

Here and throughout the rest of this paper, we adopt the common $q$-series notation [11]:

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$ 

Chan and Mao [10] discovered the inequality on $N(m, n)$ as stated below.

**Theorem 1.1** (Chan and Mao). For any nonnegative integers $m$ and $n$,

$$N(m, n) \geq N(m + 2, n),$$

and for any nonnegative integers $m$ and $n$ such that $n \geq 12$, $n \neq m + 2$,

$$N(m, n) \geq N(m, n - 1).$$

Recently, Ji and Zang [23] proved the following unimodal property of the crank on ordinary partitions.

**Theorem 1.2** (Ji and Zang). For $n \geq 44$ and $1 \leq m \leq n - 1$, we have

$$M(m - 1, n) \geq M(m, n). \quad (1.2)$$ 

Ji and Zang also give the following monotonicity property of $M(m, n)$. 

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Theorem 1.3 (Ji and Zang). For any $n \geq 14$ and $0 \leq m \leq n - 2$, we have
\[ M(m, n) \geq M(m, n - 1). \]  

Our interest in this paper is to consider an analogue of Theorem 1.2 and Theorem 1.3 for overpartitions. Specifically, we will investigate the unimodal property for the first residue crank and the second residue crank. Recall that Corteel and Lovejoy defined an overpartition of $n$ as a partition of $n$ in which the first occurrence of a part may be overlined. For example, there are 14 overpartitions of 4:
\[
\begin{align*}
(4) & \quad (4) & \quad (3, 1) & \quad (3, 1) & \quad (3, 1) & \quad (2, 2) \\
(2, 2) & \quad (2, 1, 1) & \quad (2, 1, 1) & \quad (2, 1, 1) & \quad (1, 1, 1, 1) & \quad (1, 1, 1, 1)
\end{align*}
\]

Analogous to the crank of an ordinary partition, Bringmann, Lovejoy and Osburn defined the first and second residual crank of an overpartition. The first residual crank of an overpartition is defined as the crank of the subpartition consisting of non-overlined parts. The second residual crank is defined as the crank of the subpartition consisting of all of the even non-overlined parts divided by two.

For example, for $\lambda = (9, 9, 7, 6, 5, 4, 3, 1, 1)$, the partition consisting of non-overlined parts of $\lambda$ is $(9, 7, 5, 4, 3, 1, 1)$. The first residual crank of $\lambda$ is 4. The partition formed by even non-overlined parts of $\lambda$ divided by two is $(2)$. So the second residual crank of $\lambda$ is 2.

Since then, the researches on overpartitions have been extensively studied (see, for example, Corteel, Corteel and Lovejoy). In addition, there have been a great wealth of further results regarding the relations between the rank and crank of overpartitions (see, for example, Andrews, Chan, Kim and Osburn, Jennings-Shaffer, Lovejoy and Osburn, Jennings-Shaffer, Jennings-Shaffer, and the relations between spt-function and overpartitions (see, for example, Garvan and Jennings-Shaffer, Jennings-Shaffer). Let $\overline{M}(m, n)$ (resp. $\overline{M}2(m, n)$) denote the number of overpartitions of $n$ with first (resp. second) residual crank equal to $m$. Here we make the appropriate modifications based on the fact that for ordinary partitions we have $M(0, 1) = -1$ and $M(-1, 1) = M(1, 1) = 1$. For example, the overpartition $(7, 5, 2, 1)$ contributes a $-1$ to the count of $\overline{M}(0, 15)$ and a $+1$ to both $\overline{M}(-1, 15)$ and $\overline{M}(1, 15)$. For another example, $(10, 9, 9, 7, 6, 5, 3, 3, 2)$ contributes a $-1$ to the count of $\overline{M}2(0, 57)$ and a $+1$ to both $\overline{M}2(-1, 57)$ and $\overline{M}2(1, 57)$.

Bringmann, Lovejoy and Osburn derived the following generating function of $\overline{M}(m, n)$ and $\overline{M}2(m, n)$:
\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{M}(m, n)z^m q^n = \frac{(-q; q)_\infty(q; q)_\infty}{(zq; q)_\infty(q/z; q)_\infty}. \]  

(1.4)
\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M_2(m,n)z^m q^n = \frac{(-q;q)_\infty (q;q)_\infty}{(q;q^2)_\infty (zq;q)_\infty (q/z;q)_\infty}. \tag{1.5}
\]

From (1.1), (1.4) and (1.5), for fixed \(m\), the generating function of \(M(m,n)\) is equal to
\[
\sum_{n=0}^{\infty} M(m,n)q^n = (-q;q)_\infty \sum_{n=0}^{\infty} M(m,n)q^n \tag{1.6}
\]
and
\[
\sum_{n=0}^{\infty} M_2(m,n)q^n = \frac{(-q;q)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} M(m,n)q^{2n}. \tag{1.7}
\]

The main result of this paper is an analogue of Theorem 1.2 for overpartitions as stated below.

**Theorem 1.4.** For any \(m \geq 1\) and \(n \geq 0\),
\[\overline{M}(m-1,n) \geq \overline{M}(m,n),\] except for \((m,n) = (1,1)\) or \((m,n) = (1,2)\).

**Theorem 1.5.** For any \(m \geq 1\) and \(n \geq 0\),
\[\overline{M}_2(m-1,n) \geq \overline{M}_2(m,n).\] (1.9)

The proofs of Theorem 1.4 and Theorem 1.5 are based on the following Lemma. For the remainder of this paper, let \(\{b_n\}_{n=0}^{\infty}\) and \(\{c_n\}_{n=0}^{\infty}\) be any sequence of nonnegative integers but not necessarily the same sequence in different equations.

**Lemma 1.6.** The generating function of \(M(m-1,n) - M(m,n)\) can be decomposed as follows. For \(m = 1\),
\[
\sum_{n=0}^{\infty} (M(0,n) - M(1,n))q^n = (1 - q)^2 + q^2(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)
\]
\[+ (1 - q) \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n. \tag{1.10}\]

For \(m = 2\),
\[
\sum_{n=0}^{\infty} (M(1,n) - M(2,n))q^n = q(1 - q)(1 - q^3) + (1 - q) \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n. \tag{1.11}\]

For any \(m \geq 3\),
\[
\sum_{n=0}^{\infty} (M(m-1,n) - M(m,n))q^n = (1 - q) \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n. \tag{1.12}\]
In light of Lemma 1.6 and the relations (1.6) and (1.7), we give a proof of Theorem 1.4 and Theorem 1.5. Moreover, we also obtain the following monotonicity property for $M(m,n)$ and $M_2(m,n)$, which is an overpartition analogue of Theorem 1.3.

**Theorem 1.7.** For any $m \geq 0$ and $n \geq 1$,
\[
M(m,n) \geq M(m, n-1)
\]
and
\[
M_2(m,n) \geq M_2(m, n-1).
\]

As an application, we will use Theorem 1.4 to prove the unimodality of $k$-crank which was conjectured by Fu and Tang [17]. Recall that the $k$-colored partition is a $k$-tuple of partitions $\lambda = (\lambda(1), \lambda(2), \ldots, \lambda(k))$. For $k \geq 2$, Fu and Tang [17] defined the $k$-crank of $k$-colored partition as follows:

\[
k\text{-crank}(\lambda) = \ell(\lambda(1)) - \ell(\lambda(2)),
\]
where $\ell(\pi^{(i)})$ denotes the number of parts in $\pi^{(i)}$.

Let $M_k(m,n)$ denote the number of $k$-colored partitions of $n$ with $k$-crank $m$. The generating function of $M_k(m,n)$ can be derived by Bringmann and Dousse [14]:

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m,n) z^n q^m = \frac{(q;q)_{\infty}^{2-k}}{(zq;q)_{\infty} (z^{-1}q;q)_{\infty}}.
\]

Fu and Tang [17, Conjecture 4.1] raised the following conjecture on $\{M_k(m,n)\}_{m=-n}^{n}$:

**Conjecture 1.8** (Fu and Tang, 2018). For $n \geq 0$ and $k \geq 2$, the sequence of $\{M_k(m,n)\}_{m=-n}^{n}$ is unimodal except for $n = 1$, $k = 2$.

Fu and Tang also pointed out that using the asymptotic formula in [15] due to Bringmann and Manschot [15], one may give an asymptotic proof of Conjecture 1.8.

In this paper, we confirm Conjecture 1.8 with the aid of Theorem 1.4.

This paper is organized as follows: In Section 2, we give a proof of Lemma 1.6 with the aid of Theorem 1.2. Section 3 is devoted to prove Theorem 1.4. In Section 4, we demonstrate that Theorem 1.6 can be deduced by Theorem 1.4. In Section 5, we prove Theorem 1.7. Finally, in Section 6, Conjecture 1.8 will be confirmed with the aid of Theorem 1.4.

**2 The transformation of the generating function of $M(m-1,n) - M(m,n)$**

In this section, we give a proof of Lemma 1.6 in light of Theorem 1.2.
Proof of Lemma 1.6. From Theorem 1.2, it is routine to check that

$$\sum_{n=0}^{\infty} (M(0, n) - M(1, n))q^n = 1 - 2q + q^3 + q^4 - q^7 - q^9 + q^{10} - q^{11} + 2q^{12} - q^{13} + 2q^{14} - q^{15} + 2q^{16} - 2q^{17} + 3q^{18} - 3q^{19} + 3q^{20} - 2q^{21} + 3q^{22} - 3q^{23} + 6q^{24} - 4q^{25} + 6q^{26} - 2q^{27} + 7q^{28} - 4q^{29} + 11q^{30} - 5q^{31} + 12q^{32} - 3q^{33} + 13q^{34} - 4q^{35} + 20q^{36} - 6q^{37} + 22q^{38} - q^{39} + 27q^{40} - 3q^{41} + 37q^{42} - q^{43} + \sum_{n=44}^{\infty} b_n q^n.$$  \hspace{1cm} (2.1)

Set

$$f(q) = q^{10} + q^{14} + 2q^{16} + 3q^{18} + 2q^{20} + 3q^{22} + 4q^{24} + 2q^{26} + 4q^{28} + 5q^{30} + 3q^{32} + 4q^{34} + 6q^{36} + q^{38} + 3q^{40} + q^{42}$$

and

$$h(q) = q^{14} + q^{20} + 2q^{24} + 4q^{26} + 3q^{28} + 6q^{30} + 9q^{32} + 9q^{34} + 14q^{36} + 21q^{38} + 24q^{40} + 36q^{42}.$$  

It is trivial to check that

$$\sum_{n=0}^{\infty} (M(0, n) - M(1, n))q^n = (1 - q)^2 + q^2(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5) + (1 - q)f(q) + h(q) + \sum_{n=44}^{\infty} b_n q^n.$$  \hspace{1cm} (2.2)

Clearly $f(q)$ and $h(q)$ have nonnegative coefficients. This yields (1.10).

We next assume that $m = 2$. Similar as above, by Theorem 1.2, it can be checked that

$$\sum_{n=0}^{\infty} (M(1, n) - M(2, n))q^n = q - q^2 - q^4 + q^5 + q^7 + q^9 - q^{10} + q^{11} - q^{12} + q^{13} - q^{14} + 2q^{15} - q^{16} + 3q^{17} - q^{18} + 4q^{19} - q^{20} + 5q^{21} - q^{22} + 6q^{23} - q^{24} + 8q^{25} - q^{26} + \sum_{n=27}^{\infty} b_n q^n.$$  \hspace{1cm} (2.3)
Set
\[ f(q) = q^9 + q^{11} + q^{13} + q^{15} + q^{17} + q^{19} + q^{21} + q^{23} + q^{25} \]
and
\[ h(q) = q^7 + q^{15} + 2q^{17} + 3q^{19} + 4q^{21} + 5q^{23} + 7q^{25}. \]
Together with (2.3), we see that
\[ \sum_{n=0}^{\infty} (M(1, n) - M(2, n))q^n = q(1 - q)(1 - q^3) + (1 - q)f(q) + h(q) + \sum_{n=27}^{\infty} b_n q^n. \]
This yields (1.11).

For 3 \leq m \leq 7, it is routine to check that (1.12) is valid. We next assume that
\[ m \geq 8. \]
From Theorem 1.2, we see that for \( m \geq 43 \) and \( n \geq m + 1 \), we see that
\[ M(m - 1, n) - M(m, n) \geq 0. \]
It can be checked that when \( 8 \leq m \leq 42 \) and \( 44 \geq n \geq m + 1 \),
\[ M(m - 1, n) - M(m, n) \geq 0 \] also holds. When \( n = m \) or \( m - 1 \), we have
\[ M(m - 1, m - 1) - M(m, m - 1) = 1 \]
and
\[ M(m - 1, m) - M(m, m) = -1. \]
Note that for \( n \leq m - 2 \), \( M(m - 1, n) = M(m, n) = 0. \) Thus
\[ \sum_{n=0}^{\infty} (M(m - 1, n) - M(m, n))q^n = q^{m-1}(1 - q) + \sum_{n=m+1}^{\infty} b_n q^n. \] (2.4)
This completes the proof.

3 The proof of Theorem 1.4

This section is devoted to give a proof of Theorem 1.4. The main difficulty of the proof is to deal with the case \( m = 1 \). To solve this difficulty, we need two more lemmas.

Lemma 3.1. The coefficients of \( q^n \) in
\[ (1 - q)^2(-q; q)_\infty \]
is nonnegative for all \( n \geq 0 \) except for \( n = 1 \) or 4.

Proof. It is clear that
\[ (1 - q)^2(-q; q)_\infty = (1 - q)(-q; q)_\infty - q(1 - q)(-q; q)_\infty. \] (3.1)
We further expand the \((1 - q)(-q; q)_\infty\) and \(q(1 - q)(-q; q)_\infty\) as follows.
We first consider \((1 - q)(-q; q)_{\infty}\). Let \(d(n)\) denote the number of distinct partitions of \(n\). Clearly
\[
\sum_{n=0}^{\infty} d(n)q^n = (-q; q)_{\infty}. \tag{3.2}
\]
Moreover, we may classify the set of distinct partitions based on the largest part. To be specific, for \(j \geq 1\), let \(d_j(n)\) denote the number of distinct partitions of \(n\) with the largest part is equal to \(j\). Then it is clear that
\[
\sum_{n=0}^{\infty} d_j(n)q^n = q^j(-q; q)_{j-1}. \tag{3.3}
\]
Thus from (3.2) and (3.3) we see that
\[
(-q; q)_{\infty} = \sum_{n=0}^{\infty} d(n)q^n = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} d_j(n)q^n = 1 + \sum_{j=1}^{\infty} q^j(-q; q)_{j-1}. \tag{3.4}
\]
Thus we have
\[
(1 - q)(-q; q)_{\infty} = 1 - q + \sum_{j=1}^{\infty} q^j(-q; q)_{j-1} - \sum_{j=1}^{\infty} q^{j+1}(-q; q)_{j-1}
\]
\[
= 1 - q + \sum_{j=2}^{\infty} q^j(-q; q)_{j-1} - \sum_{j=2}^{\infty} q^j(-q; q)_{j-2}
\]
\[
= 1 + \sum_{j=2}^{\infty} q^{2j-1}(-q; q)_{j-2}
\]
\[
= 1 + q^3 + q^5(1 + q) + \sum_{j=4}^{\infty} q^{2j-1}(1 + q)(1 + q^2)(-q^3; q)_{j-4}. \tag{3.5}
\]
Moreover,
\[
\sum_{j=4}^{\infty} q^{2j-1}(1 + q)(1 + q^2)(-q^3; q)_{j-4}
\]
\[
= \sum_{j=4}^{\infty} q^{2j}(-q^2; q)_{j-3} + \sum_{j=4}^{\infty} q^{2j+1}(-q^3; q)_{j-4} + \sum_{j=4}^{\infty} q^{2j-1}(-q^3; q)_{j-4}. \tag{3.6}
\]
Substituting (3.6) into (3.5), we derive that
\[
(1 - q)(-q; q)_{\infty}
\]
\begin{align*}
&= 1 + q^3 + q^5(1 + q) + \sum_{j=4}^{\infty} q^{2j}(-q^2; q)_{j-3} + \sum_{j=4}^{\infty} q^{2j+1}(-q^3; q)_{j-4} + \sum_{j=4}^{\infty} q^{2j-1}(-q^3; q)_{j-4}.
\end{align*}

(3.7)

On the other hand, from (3.5), we see that

\[q(1 - q)(-q; q)_{\infty} = q + q^4 + q^6(1 + q) + \sum_{j=4}^{\infty} q^{2j}(1 + q)(1 + q^2)(-q^3; q)_{j-4}.
\]

(3.8)

We deduce that

\[\sum_{j=4}^{\infty} q^{2j}(1 + q)(1 + q^2)(-q^3; q)_{j-4}
\]

\[= \sum_{j=4}^{\infty} q^{2j}(-q^2; q)_{j-3} + \sum_{j=4}^{\infty} q^{2j+1}(-q^3; q)_{j-4} + \sum_{j=4}^{\infty} q^{2j+3}(-q^3; q)_{j-4}.
\]

(3.9)

Substituting (3.9) into (3.8), we have

\[q(1 - q)(-q; q)_{\infty}
\]

\[= q + q^4 + q^6(1 + q) + \sum_{j=4}^{\infty} q^{2j}(-q^2; q)_{j-3} + \sum_{j=4}^{\infty} q^{2j+1}(-q^3; q)_{j-4} + \sum_{j=4}^{\infty} q^{2j+3}(-q^3; q)_{j-4}.
\]

(3.10)

Substituting (3.7) and (3.10) into (3.1), we deduce that

\[(1 - q)^2(-q; q)_{\infty}
\]

\[= 1 - q + q^3 - q^4 + q^5 - q^7 + \sum_{j=4}^{\infty} q^{2j-1}(-q^3; q)_{j-4} - \sum_{j=4}^{\infty} q^{2j+3}(-q^3; q)_{j-4}
\]

\[= 1 - q + q^3 - q^4 + q^5 + q^7 + q^{12} + \sum_{j=6}^{\infty} q^{2j-1}(-q^3; q)_{j-6}(q^j + q^{j-2} + q^{2j-5}).
\]

(3.11)

Clearly (3.11) has nonnegative coefficients in \(q^n\) for all \(n \geq 0\) except for \(n = 1\) or \(n = 4\). This completes the proof.

Lemma 3.2. The coefficient of \(q^n\) in

\[(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)(-q; q)_{\infty}
\]

is nonnegative for all \(n \geq 1\).
Proof. From the Euler identity [1],

\[-q; q)_\infty = \frac{1}{(q; q^2)_\infty}, \tag{3.12}\]

we deduce that

\[
(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)(-q; q)_\infty \\
= \frac{(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)}{(q; q^2)_\infty} \\
= \frac{(-1 + q^2 + q^3 + q^4 - q^5)}{(1 - q^3)(q^7; q^2)_\infty} \\
= \frac{q^4}{(1 - q^3)(q^7; q^2)_\infty} - \frac{(1 - q^2)}{(q^7; q^2)_\infty}. \tag{3.13}\]

Let \( p_7(n) \) denote the number of partitions of \( n \) with each part odd and not less than 7. It is clear that

\[
\sum_{n=0}^{\infty} p_7(n)q^n = \frac{1}{(q^7; q^2)_\infty}. \]

Moreover, we may divide the set of partitions counted by \( p_7(n) \) into nonintersecting subsets based on the largest part. To be specific, for odd \( j \geq 7 \), let \( p_7,j(n) \) denote the number of partitions counted by \( p_7(n) \) such that the largest part is equal to \( j \). It is easy to see that

\[
\sum_{n=0}^{\infty} p_7,j(n)q^n = \frac{q^j}{(q^7; q^2)(j-5)/2}. \]

Thus we have

\[
\frac{1}{(q^7; q^2)_\infty} = 1 + \sum_{j \geq 7} \sum_{n=0}^{\infty} p_7,j(n)q^n = 1 + \sum_{j \geq 7} \sum_{j \text{ odd}} \frac{q^j}{(q^7; q^2)(j-5)/2}. \tag{3.14}\]

Using \((3.14)\), we may transform the second term in \((3.13)\) as follows.

\[
\frac{1 - q^2}{(q^7; q^2)_\infty} = 1 + \sum_{j \geq 7} \sum_{j \text{ odd}} \frac{q^j}{(q^7; q^2)(j-5)/2} - q^2 - \sum_{j \geq 7} \sum_{j \text{ odd}} \frac{q^{j+2}}{(q^7; q^2)(j-5)/2} \\
= 1 - q^2 + \frac{q^7}{1 - q^7} + \sum_{j \geq 9} \frac{q^j}{(q^7; q^2)(j-5)/2} - \sum_{j \geq 9} \frac{q^{j}}{(q^7; q^2)(j-7)/2} \\
= 1 - q^2 + \frac{q^7}{1 - q^7} + \sum_{j \geq 9} \frac{q^j}{(q^7; q^2)(j-7)/2} \left( \frac{1}{1 - q^j} - 1 \right)
\]
= 1 - q^2 + \frac{q^7}{1 - q^3} + \sum_{j \geq 9 \atop j \text{ odd}} \frac{q^{2j}}{(q^7; q^2)^{(j-5)/2}} \\
= 1 - q^2 + \frac{q^7}{1 - q^3} + \frac{q^{18}}{(1 - q^7)(1 - q^9)} + \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{2j}}{(q^7; q^2)^{(j-5)/2}}. 

(3.15)

We next transform the first term in (3.13) in light of (3.14) as follows.

\[
\frac{q^4}{(1 - q^3)(q^7; q^2)_\infty} = \frac{q^4}{1 - q^3} \left( 1 + \frac{q^7}{1 - q^3} + \frac{q^9}{(1 - q^7)(1 - q^9)} + \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-5)/2}} \right)
\]

\[
= \frac{q^4}{1 - q^3} + \frac{q^{11}}{(1 - q^3)(1 - q^7)(1 - q^9)} + \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-5)/2}}
\]

\[
= q^4 + q^7 + \frac{q^{10}}{1 - q^3} + \frac{q^{11}}{1 - q^7} + \frac{q^{14}}{1 - q^7} + \frac{q^{17}}{(1 - q^3)(1 - q^7)} + \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-5)/2}}. 

(3.16)

Notice that

\[
\frac{q^{13}}{(1 - q^3)(1 - q^7)(1 - q^9)} = \frac{q^{13}}{(1 - q^7)(1 - q^9)} + \frac{q^{16}}{(1 - q^3)(1 - q^7)(1 - q^9)}. 

(3.17)

Moreover,

\[
\sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-5)/2}} = \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-11)/2}(q^{j-2}; q^2)_2} \left( 1 + \frac{q^{j-4}}{1 - q^{j-4}} \right)
\]

\[
= \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)^{(j-11)/2}(q^{j-2}; q^2)} + \frac{q^{2j}}{(1 - q^3)(q^7; q^2)^{(j-5)/2}}. 

(3.18)\]
Substituting (3.17) and (3.18) into (3.16), we see that

\[
\frac{q^4}{(1 - q^3)(q^3; q^2)_\infty} = q^4 + \frac{q^{10}}{1 - q^3} + \frac{q^{11}}{1 - q^4} + \frac{q^{17}}{(1 - q^3)(1 - q^7)} \\
+ \frac{q^{13}}{(1 - q^7)(1 - q^9)} + \frac{q^{16}}{(1 - q^3)(1 - q^7)(1 - q^9)} \\
+ \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)_{(j-1)/2}(q^{j-2}; q^2)_2} \\
+ \sum_{j \geq 11 \atop j \text{ odd}} \frac{q^{2j}}{(1 - q^3)(q^7; q^2)_{(j-5)/2}}.
\]

(3.19)

By (3.15) and (3.19), we see that (3.13) can be rewritten as follows:

\[
(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)(-q; q)_\infty \\
= -1 + q^2 + q^4 + \frac{q^{10}}{1 - q^3} + \frac{q^{17}}{(1 - q^3)(1 - q^7)} \\
+ \sum_{j = 11 \atop j \text{ odd}}^{\infty} \frac{q^{j+4}}{(1 - q^3)(q^7; q^2)_{(j-1)/2}(q^{j-2}; q^2)_2} + \frac{q^{16}}{(1 - q^3)(1 - q^7)(1 - q^9)} \\
+ \left( \frac{q^{11}}{1 - q^7} + \frac{q^{13}}{(1 - q^7)(1 - q^9)} - \frac{q^{18}}{(1 - q^7)(1 - q^9)} \right) \\
+ \left( \sum_{j = 11 \atop j \text{ odd}}^{\infty} \frac{q^{2j}}{(1 - q^3)(q^7; q^2)_{(j-5)/2}} - \sum_{j = 11 \atop j \text{ odd}}^{\infty} \frac{q^{2j}}{(q^7; q^2)_{(j-5)/2}} \right). 
\]

(3.20)

It is clear that

\[
\frac{q^{18}}{(1 - q^7)(1 - q^9)} = \frac{q^{18}}{1 - q^7} \left( 1 + \frac{q^9}{1 - q^9} \right) = \frac{q^{18}}{1 - q^7} + \frac{q^{27}}{(1 - q^7)(1 - q^9)}. 
\]

(3.21)

Thus we have

\[
\frac{q^{11}}{1 - q^7} + \frac{q^{13}}{(1 - q^7)(1 - q^9)} - \frac{q^{18}}{(1 - q^7)(1 - q^9)} \\
= \frac{q^{11}}{1 - q^7} - \frac{q^{18}}{1 - q^7} + \frac{q^{13}}{(1 - q^7)(1 - q^9)} - \frac{q^{27}}{(1 - q^7)(1 - q^9)} \\
= q^{11} + \frac{q^{13}(1 + q^7)}{1 - q^9}. 
\]

(3.22)
Moreover, it is clear that

\[
\sum_{j \geq 11 \atop j \text{ odd}} q^{2j} \frac{(1 - q^3)(q^7; q^2)_{(j-5)/2}}{1 - q^{2j}} - \sum_{j \geq 11 \atop j \text{ odd}} q^{2j} = \sum_{j \geq 11 \atop j \text{ odd}} q^{2j+3} \frac{(1 - q^3)(q^7; q^2)_{(j-5)/2}}{1 - q^{2j+3}}. \tag{3.23}
\]

Substituting (3.22) and (3.23) into (3.20), we deduce that

\[
(1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)(-q; q)_{\infty}
\]

\[
= -1 + q^2 + q^4 + \frac{q^{10}}{1 - q^3} + \frac{q^{17}}{(1 - q^3)(1 - q^7)} + \frac{q^{16}}{(1 - q^3)(1 - q^7)(1 - q^9)}
\]

\[
+ \sum_{j \geq 11 \atop j \text{ odd}} q^{j+4} \frac{(1 - q^3)(q^7; q^2)_{(j-11)/2}(q^{j-2}; q^2)_2}{1 - q^{j+4}} + q^{11} + \frac{q^{13}(1 + q^7)}{1 - q^9}
\]

\[
+ \sum_{j \geq 11 \atop j \text{ odd}} q^{2j+3} \frac{(1 - q^3)(q^7; q^2)_{(j-5)/2}}{1 - q^{2j+3}}. \tag{3.24}
\]

Clearly, (3.24) has nonnegative coefficients in $q^n$ for all $n \geq 1$. This completes the proof.

We are now in a position to prove Theorem 1.4.

**Proof of Theorem 1.4.** From (1.6), it is clear to see that for fixed integer $m$,

\[
\sum_{n=0}^{\infty} (\tilde{M}(m-1, n) - \tilde{M}(m, n)) q^n = (-q; q)_{\infty} \sum_{n=0}^{\infty} (M(m-1, n) - M(m, n)) q^n. \tag{3.25}
\]

From (3.25) and Lemma 1.6, we see that for $m \geq 3$,

\[
\sum_{n=0}^{\infty} (\tilde{M}(m-1, n) - \tilde{M}(m, n)) q^n = (-q, q)_{\infty} \left( 1 - q \right) \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n. \tag{3.26}
\]

From (3.12), we deduce that

\[
\sum_{n=0}^{\infty} (\tilde{M}(m-1, n) - \tilde{M}(m, n)) q^n = \frac{(1 - q) \sum_{n=0}^{\infty} b_n q^n}{(q; q^2)_{\infty}} + \frac{\sum_{n=0}^{\infty} c_n q^n}{(q; q^2)_{\infty}}
\]

\[
= \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n. \tag{3.27}
\]

Clearly, (3.27) has nonnegative coefficients. This yields

\[
\tilde{M}(m-1, n) \geq \tilde{M}(m, n)
\]

for $m \geq 3$. 

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Similar as above, for $m = 2$, using Lemma 1.6, (3.12) and (3.25), we see that
\[
\sum_{n=0}^{\infty} (\overline{M}(1, n) - \overline{M}(2, n)) q^n = (-q, q)_{\infty} \left( q(1 - q)(1 - q^3) + (1 - q) \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n \right)
\]
\[
= \frac{q}{(q^2; q^2)_{\infty}} + \sum_{n=0}^{\infty} b_n q^n + \sum_{n=0}^{\infty} c_n q^n.
\]
(3.28)
Clearly (3.28) has nonnegative coefficients. This yields Theorem 1.4 holds for $m = 2$.

Finally, for $m = 1$, by Lemma 1.6, (3.12) and (3.25), we deduce that
\[
\sum_{n=0}^{\infty} (\overline{M}(0, n) - \overline{M}(1, n)) q^n
\]
\[
= q^2(-q; q)_{\infty} (1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)
\]
\[
+ (1 - q)^2(-q; q)_{\infty} + \frac{1 - q}{(q^3; q^2)_{\infty}} \sum_{n=0}^{\infty} b_n q^n + (-q; q)_{\infty} \sum_{n=0}^{\infty} c_n q^n
\]
\[
= q^2(-q; q)_{\infty} (1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5)
\]
\[
+ (1 - q)^2(-q; q)_{\infty} + \frac{1}{(q^3; q^2)_{\infty}} \sum_{n=0}^{\infty} b_n q^n + (-q; q)_{\infty} \sum_{n=0}^{\infty} c_n q^n.
\]
(3.29)
From Lemma 3.1 and Lemma 3.2 we see that
\[
q^2(-q; q)_{\infty} (1 - q)(1 - q^5)(-1 + q^2 + q^3 + q^4 - q^5) + (1 - q)^2(-q; q)_{\infty}
\]
has nonnegative coefficients in $q^n$ except for $n = 1, 2, 4$. Thus for $n \neq 1, 2, 4$, the coefficient of $q^n$ in (3.29) is nonnegative. It is trivial to check that $\overline{M}(0, 4) = \overline{M}(1, 4) = 2$. This completes the proof.

4 The proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5 with the aid of Theorem 1.4.

Proof of Theorem 1.5. By (1.7), it is clear that for any fixed integer $m$,
\[
\sum_{n=0}^{\infty} (\overline{M}(m - 1, n) - \overline{M}(m, n)) q^n
\]
\[
= (-q; q)_{\infty} \sum_{n=0}^{\infty} (M(m - 1, n) - M(m, n)) q^{2n}
\]
\[
= (-q; q^2)_{\infty} \left( (-q^2; q^2)_{\infty} \sum_{n=0}^{\infty} (M(m - 1, n) - M(m, n)) q^{2n} \right).
\]
(4.1)
From (3.25), we see that
\[
\sum_{n=0}^{\infty} (M(m-1,n) - M(m,n)) q^n = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty} \left( \sum_{n=0}^{\infty} (M(m-1,n) - M(m,n)) q^{2n} \right).
\] (4.2)
Hence by Theorem 1.4, for \( m \geq 2 \),
\[
\sum_{n=0}^{\infty} (M(m-1,n) - M(m,n)) q^n
\]
has nonnegative coefficients. This yields Theorem 1.5 holds for \( m \geq 2 \).

We next consider the case \( m = 1 \). In this case, by Theorem 1.4, it is easy to check
\[
\sum_{n=0}^{\infty} (M(m-1,n) - M(m,n)) q^{2n} = 1 - q^2 - q^4 + q^6 + \sum_{n=8}^{\infty} b_n q^n.
\] (4.3)
Thus it suffices to show that the coefficients of \( q^n \) in
\[
\frac{(-q; q^2)_\infty}{(q; q^2)_\infty} (1 - q^2 - q^4 + q^6)
\]
is nonnegative for all \( n \geq 0 \). Since
\[
\frac{(-q; q^2)_\infty}{(q; q^2)_\infty} (1 - q^2 - q^4 + q^6) = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty} (1 - q^2)(1 - q^4)
= (-q; q^2)_\infty (1 - q^4) \cdot \frac{1 + q}{(q^3; q^2)_\infty}.
\] (4.4)
We claim that
\[
(-q; q^2)_\infty (1 - q^4)
\]
has nonnegative coefficients in \( q^n \) for all \( n \geq 0 \).

Let \( sc(n) \) denote the number of distinct odd partitions of \( n \). It is clear that
\[
(-q; q^2)_\infty = \sum_{n=0}^{\infty} sc(n) q^n.
\]
We next divide the set of partitions counted by \( sc(n) \) into nonintersecting subsets based on the largest part. For \( j \geq 1 \) odd, let \( sc_j(n) \) denote the number of distinct odd partitions of \( n \) with largest part equal to \( j \). Then
\[
\sum_{n=0}^{\infty} sc_j(n) q^n = q^j (-q; q^2)_{(j-1)/2}.
\] (4.5)
So we see that

\[-q; q^2)_\infty = 1 + \sum_{j \geq 1 \atop j \text{ odd}} q^j (-q; q^2)_{(j-1)/2}. \tag{4.6}\]

Hence we deduce that

\[(1 - q^4)(-q; q^2) = 1 - q^4 + \sum_{j \geq 1 \atop j \text{ odd}} q^j (-q; q^2)_{(j-1)/2} - \sum_{j \geq 5 \atop j \text{ odd}} q^j (-q; q^2)_{(j-5)/2} \]
\[= 1 - q^4 + q + q^3 (1 + q) + \sum_{j \geq 5 \atop j \text{ odd}} q^j (-q; q^2)_{(j-1)/2} - \sum_{j \geq 5 \atop j \text{ odd}} q^j (-q; q^2)_{(j-5)/2} \]
\[= 1 + q + q^3 + \sum_{j \geq 5 \atop j \text{ odd}} q^j (-q; q^2)_{(j-5)/2}(q^{j-4} + q^{j-2} + q^{2j-6}). \tag{4.7}\]

Clearly, (4.7) has nonnegative coefficients, and our claim follows.

From the above claim, it is clear to see that (4.4) has nonnegative coefficients of $q^n$ for all $n \geq 0$. This completes the proof.

5 The monotonicity properties of $\overline{M}(m, n)$ and $\overline{M}2(m, n)$

In this section, we give a proof of Theorem 1.7.

Proof of Theorem 1.7: For fixed integer $m$,

\[\sum_{n=0}^{\infty} (\overline{M}(m, n + 1) - \overline{M}(m, n)) q^n = (1 - q) \sum_{n=0}^{\infty} \overline{M}(m, n) q^n. \tag{5.1}\]

From (1.6),

\[\sum_{n=1}^{\infty} (\overline{M}(m, n) - \overline{M}(m, n - 1)) q^n = (1 - q)(-q; q)_\infty \sum_{n=0}^{\infty} M(m, n) q^n. \tag{5.2}\]

Thus by (3.12), we have

\[\sum_{n=1}^{\infty} (\overline{M}(m, n) - \overline{M}(m, n - 1)) q^n = \frac{(1 - q)}{(q; q^2)_\infty} \sum_{n=0}^{\infty} M(m, n) q^n \]
\[= \frac{1}{(q^3; q^2)_\infty} \sum_{n=0}^{\infty} M(m, n) q^n, \tag{5.3}\]

which clearly has nonnegative coefficients. This yields (1.13). Using the same argument, we deduce that

\[\sum_{n=1}^{\infty} (\overline{M}2(m, n) - \overline{M}2(m, n - 1)) q^n = \frac{(-q; q)_\infty}{(q^3; q^2)_\infty} \sum_{n=0}^{\infty} M(m, n) q^{2n}, \tag{5.4}\]

which implies (1.14). This completes the proof.
6 The proof of Conjecture 1.8

In this section, we prove Conjecture 1.8 with the aid of Theorem 1.4.

**Proof of Conjecture 1.8**  From (1.4) and (1.16), it is clear that when \( k \geq 2 \)

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M_k(m,n)z^m q^n = \frac{(q;q)^{2-k}_\infty}{(q^2;q^2)_\infty(q/z;q)_\infty} \cdot \frac{1}{(q^2;q^2)_\infty(q;q)^{k-2}_\infty} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n)z^m q^n. \quad (6.1)
\]

Equating the coefficient of \( z^m \) on both sides, we find that for fixed \( m \),

\[
\sum_{n=0}^{\infty} M_k(m,n)q^n = \frac{1}{(q^2;q^2)_\infty(q;q)^{k-2}_\infty} \sum_{n=0}^{\infty} \overline{M}(m,n)q^n. \quad (6.2)
\]

Thus

\[
\sum_{n=0}^{\infty} (M_k(m-1,n) - M_k(m,n))q^n = \frac{1}{(q^2;q^2)_\infty(q;q)^{k-2}_\infty} \sum_{n=0}^{\infty} (\overline{M}(m-1,n) - \overline{M}(m,n))q^n. \quad (6.3)
\]

When \( m \geq 2 \), by Theorem 1.4 we find that \( \overline{M}(m-1,n) - \overline{M}(m,n) \geq 0 \). Thus by (6.3), we see that \( M_k(m-1,n) \geq M_k(m,n) \), as desired.

We now assume that \( m = 1 \). From Theorem 1.4 and simple calculation, we have

\[
\sum_{n=0}^{\infty} (\overline{M}(0,n) - \overline{M}(1,n))q^n = 1 - q - q^2 + q^3 + q^5 + \sum_{n=6}^{\infty} b_n q^n. \quad (6.4)
\]

Substituting (6.4) into (6.3),

\[
\sum_{n=0}^{\infty} (M_k(0,n) - M_k(1,n))q^n = \frac{1 - q - q^2 + q^3 + q^5}{(q^2;q^2)_\infty(q;q)^{k-2}_\infty} - \frac{1}{(q^2;q^2)_\infty(q;q)^{k-2}_\infty} \sum_{n=0}^{\infty} b_n q^n \]

\[
\sum_{n=6}^{\infty} b_n q^n. \quad (6.5)
\]

We claim that for \( n \geq 2 \), the coefficients of \( q^n \) in

\[
\frac{-q + q^3 + q^5}{(q^2;q^2)_\infty}
\]
is nonnegative. Recall that Andrews and Merca [6] proved the following inequality holds for \( n > 0 \):

\[
p(n) - p(n-1) - p(n-2) + p(n-5) \leq 0. \tag{6.6}
\]

Thus for \( n > 0 \),

\[
p(n) - p(n-1) - p(n-2) \leq 0. \tag{6.7}
\]

Therefore

\[
\sum_{n=0}^{\infty} (p(n-1) + p(n-2) - p(n))q^n = \frac{-1 + q + q^2}{(q; q)_\infty} = -1 + \sum_{n=1}^{\infty} b_n q^n. \tag{6.8}
\]

Setting \( q = q^2 \) in (6.8), we derive that

\[
\frac{-1 + q^2 + q^4}{(q^2; q^2)_\infty} = -1 + \sum_{n=1}^{\infty} b_n q^{2n}. \tag{6.9}
\]

Thus

\[
\frac{-q + q^3 + q^5}{(q^2; q^2)_\infty} = -q + \sum_{n=1}^{\infty} b_n q^{2n+1}. \tag{6.10}
\]

This yields our claim.

Substituting (6.10) into (6.5), and notice that the constant term in \( 1/(q^4; q^2)_\infty \) is equal to 1,

\[
\sum_{n=0}^{\infty} (M_k(0, n) - M_k(1, n))q^n = \frac{1 - q}{(q; q)_{k-2}} + \sum_{n=0}^{\infty} b_n q^n. \tag{6.11}
\]

Thus when \( k = 2 \), the coefficients of \( q^n \) in (6.11) is nonnegative for \( n \geq 2 \). When \( k \geq 3 \),

\[
\sum_{n=0}^{\infty} (M_k(0, n) - M_k(1, n))q^n = \frac{1}{(q; q)_{k-3}(q^2; q)_\infty} + \sum_{n=0}^{\infty} b_n q^n, \tag{6.12}
\]

which has nonnegative coefficients. This completes the proof. \( \blacksquare \)

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