Non-Commutative Gebauer-Möller Criteria

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Abstract
For an efficient implementation of Buchberger’s Algorithm, it is essential to avoid the treatment of as many unnecessary critical pairs or obstructions as possible. In the case of the commutative polynomial ring, this is achieved by the Gebauer-Möller criteria. Here we present an adaptation of the Gebauer-Möller criteria for non-commutative polynomial rings, i.e. for free associative algebras over fields. The essential idea is to detect unnecessary obstructions using other obstructions with or without overlap. Experiments show that the new criteria are able to detect almost all unnecessary obstructions during the execution of Buchberger’s procedure.

Keywords: Gröbner basis, free associative algebra, obstruction, Buchberger procedure

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1 Introduction

Ever since B. Buchberger’s thesis [3], Gröbner bases have become a fundamental tool for computations in commutative algebra and algebraic geometry. The most time-consuming part in Buchberger’s Algorithm is the computation of the normal remainder of an S-polynomial corresponding to a critical pair. Therefore a significant amount of energy has been spent on reducing the number of critical pairs which have to be treated. After the discovery of various criteria for discarding critical pairs ahead of time by B. Buchberger and H.M. Möller (see [4], [5] and [11]), this subject found an initial resolution via the Gebauer-Möller installation presented in [8] which offers a good compromise between efficiency and the success rate for detecting unnecessary critical pairs.

A very different picture presents itself for Gröbner basis computations for two-sided ideals in non-commutative polynomial rings. The basic Gröbner basis theory in this case was described by G.H. Bergman (see [2]), T. Mora (see [12] and [13]) and others, and obstructions, the non-commutative analogue of critical pairs, were studied in [13]. However, since only a few authors endeavoured to

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implement efficient versions of Buchberger’s Procedure for the non-commutative polynomial ring (i.e. the free associative algebra), the subject of minimizing the number of obstructions which have to be treated has received comparatively little attention, and merely a few rules were developed. For instance, the package *Plural* of the computer algebra system *Singular* implements a version of the product and the chain criterion, but not the multiply criterion or the leading word criterion. On the other hand, the system *Magma* appears to be based on a variant of the F4 Algorithm which does not use criteria for unnecessary obstructions. For an overview on rules which have been developed see for instance [7].

In this paper, we present generalizations of the Gebauer-Möller criteria for non-commutative polynomials. They cover not only the known cases of useless obstructions discussed in [13], Lemma 5.11 and [7], but form a complete analogue of the results in the commutative case. One of the key ingredients we use for this purpose is the consideration of obstructions without overlaps. We detect useless obstructions, i.e. obstructions that can be represented by other obstructions, using not only obstructions with overlaps but using also those without overlaps. We show that the consideration of obstructions without overlaps does not increase unnecessary computations, since a Gröbner representation is inherent in the S-polynomial of every obstruction without overlaps. Consequently, we reduce the number of obstructions efficiently and obtain a non-commutative version of the Gebauer-Möller criteria.

This paper is organised as follows. In Section 2 we recall the basic theory of Gröbner bases for two-sided ideals in non-commutative polynomial rings. In particular, we introduce and study obstructions (see Definitions 2.4 and 2.9 and Lemmas 2.6 and 2.8), present the Buchberger Criterion (see Proposition 2.10), and formulate the Buchberger Procedure (see Theorem 2.11). The non-commutative analogues of the Gebauer-Möller criteria are developed in Section 3. They are based on a careful study of the set of newly constructed obstructions which are produced during the execution of Buchberger’s Procedure. As a result, we are able to formulate the Non-Commutative Multiply Criterion (see Proposition 3.5), the Non-Commutative Leading Word Criterion (see Proposition 3.6) and the Non-Commutative Backward Criterion (see Proposition 3.12). When we combine these criteria, the result is a new Improved Buchberger Procedure 3.14.

The second author has implemented a version of the Buchberger Procedure for non-commutative polynomial rings in a package for the computer algebra system *ApCoCoA* which includes the non-commutative Gebauer-Möller criteria developed here (see [1]). In the last section, we present experimental results about the efficiency of the criteria for some cases of moderately difficult Gröbner basis computations.

Unless mentioned otherwise, we adhere to the definitions and terminology given in [10] and [11].
2 Gröbner Bases in \( K\langle X \rangle \)

In the following we let \( X = \{x_1, \ldots, x_n\} \) be a finite set of indeterminates (or a finite alphabet), and \( \langle X \rangle \) the monoid of all \textit{words} (or \textit{terms}) \( x_{i_1} \cdots x_{i_l} \) where the multiplication is concatenation of words. The empty word will be denoted by \( \lambda \). Furthermore, let \( K \) be a field, and let

\[
K\langle X \rangle = \{c_1 w_1 + \cdots + c_s w_s \mid s \in \mathbb{N}, c_i \in K \setminus \{0\}, w_i \in \langle X \rangle\}
\]

be the non-commutative polynomial ring generated by \( X \) over \( K \) (or the free associative \( K \)-algebra generated by \( X \)). We introduce basic notions of Gröbner basis theory in this setting.

\textbf{Definition 2.1.} A \textit{word ordering} on \( \langle X \rangle \) is a well-ordering \( \sigma \) which is compatible with multiplication, i.e. \( w_1 \geq_{\sigma} w_2 \) implies \( w_3 w_1 w_4 \geq_{\sigma} w_3 w_2 w_4 \) for all words \( w_1, w_2, w_3, w_4 \in \langle X \rangle \).

In the commutative case, a word ordering is usually called a \textit{term ordering} or \textit{monomial ordering}. For instance, the \textit{length-lexicographic ordering} \( \text{LLex} \) is a word ordering. It first compares the length of two words and then breaks ties using the non-commutative lexicographic ordering with respect to \( x_1 >_{\text{LLex}} \cdots >_{\text{LLex}} x_n \). Note that the non-commutative lexicographic ordering by itself is not a word ordering, since it is neither a well-ordering nor compatible with multiplication.

\textbf{Definition 2.2.} Let \( \sigma \) be a word ordering on \( \langle X \rangle \).

(a) Given a polynomial \( f \in K\langle X \rangle \setminus \{0\} \), there exists a unique representation \( f = c_1 w_1 + \cdots + c_s w_s \) with \( c_1, \ldots, c_s \in K \setminus \{0\} \) and \( w_1, \ldots, w_s \in \langle X \rangle \) such that \( w_1 >_{\sigma} \cdots >_{\sigma} w_s \). The word \( \text{Lw}_\sigma(f) = w_1 \) is called the \textit{leading word} of \( f \) with respect to \( \sigma \). The element \( \text{Lc}_\sigma(f) = c_1 \) is called the \textit{leading coefficient}. We let \( \text{Lm}_\sigma(f) = c_1 w_1 \) and call it the \textit{leading monomial} of \( f \).

(b) Let \( I \subseteq K\langle X \rangle \) be a two-sided ideal. The set \( \text{Lw}_\sigma(I) = \{\text{Lw}_\sigma(f) \mid f \in I \setminus \{0\}\} \subseteq \langle X \rangle \) is called the \textit{leading word set} of \( I \). The two-sided ideal \( \text{Lw}_\sigma(I) = \langle \text{Lw}_\sigma(f) \mid f \in I \setminus \{0\} \rangle \subseteq K\langle X \rangle \) is called the \textit{leading word ideal} of \( I \).

(c) A subset \( G \) of a two-sided ideal \( I \subseteq K\langle X \rangle \) is called a \textit{\( \sigma \)-Gröbner basis} of \( I \) if the set of the leading words \( \text{Lw}_\sigma(G) = \{\text{Lw}_\sigma(f) \mid f \in G \setminus \{0\}\} \) generates the leading word ideal \( \text{Lw}_\sigma(I) \).

In the following we focus on computations of Gröbner bases for two-sided ideals in \( K\langle X \rangle \). For readers who want to know further properties and applications of non-commutative Gröbner bases, we refer to [13] and [16]. Throughout this paper we assume that \( \sigma \) is a word ordering on \( \langle X \rangle \). The next algorithm is a central part of all Gröbner basis computations.
Theorem 2.3. (The Division Algorithm)

Let \( f \in K(X) \), \( s \geq 1 \), and \( G = \{ g_1, \ldots, g_s \} \subseteq K(X) \setminus \{0\} \). Consider the following sequence of instructions.

\[
\begin{align*}
\text{(D1)} & \quad \text{Let } k_1 = \cdots = k_s = 0, p = 0, \text{ and } v = f. \\
\text{(D2)} & \quad \text{Find the smallest index } i \in \{1, \ldots, s\} \text{ such that } \text{Lw}_\sigma(v) = w \text{Lw}_\sigma(g_i)w' \text{ for some words } w, w' \in \langle X \rangle. \text{ If such an } i \text{ exists, increase } k_i \text{ by } 1, \text{ set } c_{ik_i} = \frac{\text{Lw}_\sigma(g_k)}{\text{Lw}_\sigma(g_i)} w_{ik_i} = w, w'_{ik_i} = w', \text{ and replace } v \text{ by } v - c_{ik_i}w_{ik_i}g_iw'_{ik_i}. \\
\text{(D3)} & \quad \text{Repeat step (D2) until there is no more } i \in \{1, \ldots, s\} \text{ such that } \text{Lw}_\sigma(v) \text{ is a multiple of } \text{Lw}_\sigma(g_i). \text{ If now } v \neq 0, \text{ then replace } p \text{ by } p + \text{Lm}_\sigma(v) \text{ and } v \text{ by } v - \text{Lm}_\sigma(v), \text{ continue with step (D2)}. \\
\text{(D4)} & \quad \text{Return the tuples } (c_{11}, w_{11}, w'_{11}), \ldots, (c_{sk}, w_{sk}, w'_{sk}) \text{ and } p.
\end{align*}
\]

This is an algorithm which returns tuples \((c_{11}, w_{11}, w'_{11}), \ldots, (c_{sk}, w_{sk}, w'_{sk})\) and a polynomial \( p \in K(X) \) such that the following conditions are satisfied.

\[
\begin{align*}
\text{(a)} & \quad \text{We have } f = \sum_{i=1}^{s} \sum_{j=1}^{k_i} c_{ij}w_{ij}g_iw'_{ij} + p. \\
\text{(b)} & \quad \text{No element of } \text{Supp}(p) \text{ is contained in } \langle \text{Lw}_\sigma(g_1), \ldots, \text{Lw}_\sigma(g_s) \rangle. \\
\text{(c)} & \quad \text{For all } i \in \{1, \ldots, s\} \text{ and all } j \in \{1, \ldots, k_i\}, \text{ we have } \text{Lw}_\sigma(w_{ij}g_iw'_{ij}) \leq \sigma \text{Lw}_\sigma(f). \text{ If } p \neq 0, \text{ we have } \text{Lw}_\sigma(p) \leq \sigma \text{Lw}_\sigma(f). \\
\text{(d)} & \quad \text{For all } i \in \{1, \ldots, s\} \text{ and all } j \in \{1, \ldots, k_i\}, \text{ we have } \text{Lw}_\sigma(w_{ij}g_iw'_{ij}) \notin \langle \text{Lw}_\sigma(g_1), \ldots, \text{Lw}_\sigma(g_{i-1}) \rangle.
\end{align*}
\]

Note that the resulting tuples \((c_{11}, w_{11}, w'_{11}), \ldots, (c_{sk}, w_{sk}, w'_{sk})\) and polynomial \( p \) satisfying conditions (a)-(d) are not unique. This is due to the fact that in step (D2) of the Division Algorithm there might exist more that one pair \((w, w')\) satisfying \( \text{Lw}_\sigma(v) = w \text{Lw}_\sigma(g_i)w' \) (see [10], Example 3.2.2). A polynomial \( p \in K(X) \) obtained in Theorem 2.3 is called a normal remainder of \( f \) with respect to \( G \) and is denoted by \( \text{NR}_\sigma(G)(f) \).

For \( s \geq 1 \), we let \( F_s = (K(X) \otimes_K K(X))^s \) be the free two-sided \( K(X) \)-module of rank \( s \) with the canonical basis \( \{e_1, \ldots, e_s\} \), where \( e_i = (0, \ldots, 0, 1 \otimes 1, 0, \ldots, 0) \) with \( 1 \otimes 1 \) occurring in the \( i \)-th position for \( i = 1, \ldots, s \), and we let \( T(F_s) \) be the set of terms in \( F_s \), i.e. \( T(F_s) = \{we_1w' \mid i \in \{1, \ldots, s\}, w, w' \in \langle X \rangle \} \).

Definition 2.4. Let \( G = \{ g_1, \ldots, g_s \} \subseteq K(X) \setminus \{0\} \) with \( s \geq 1 \), and let \( i, j \in \{1, \ldots, s\} \) such that \( i \leq j \).

\( \text{(a)} \) If there exist some words \( w_i, w'_i, w_j, w'_j \in \langle X \rangle \) such that \( w_i \text{Lw}_\sigma(g_i)w'_i = w_j \text{Lw}_\sigma(g_j)w'_j \), then we call the element

\[
\alpha_{i,j}(w_i, w'_i; w_j, w'_j) = \frac{1}{\text{Lc}_\sigma(g_i)}w_i e_i w'_i - \frac{1}{\text{Lc}_\sigma(g_j)}w_j e_j w'_j \in F_s \setminus \{0\}
\]

an obstruction of \( g_i \) and \( g_j \). If \( i = j \), it is called a self obstruction of \( g_i \). We will denote the set of all obstructions of \( g_i \) and \( g_j \) by \( \text{Obs}(i, j) \).
(b) Let \( \alpha_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i,j) \) be an obstruction of \( g_i \) and \( g_j \). The polynomial

\[
S_{i,j}(w_i, w'_i; w_j, w'_j) = \frac{1}{\text{Le}(g_i)} w_i g_i w'_i - \frac{1}{\text{Le}(g_j)} w_j g_j w'_j \in K(X)
\]

is called the S-polynomial of \( \alpha_{i,j}(w_i, w'_i; w_j, w'_j) \).

Using these definitions, we can characterize Gröbner bases in the following way.

**Proposition 2.5.** Let \( G = \{g_1, \ldots, g_s\} \subseteq K(X) \setminus \{0\} \) be a set of polynomials which generate a two-sided ideal \( I = \langle G \rangle \subseteq K(X) \). Then the following conditions are equivalent.

(a) The set \( G \) is a \( \sigma \)-Gröbner basis of \( I \).

(b) For every obstruction \( \alpha_{i,j}(w_i, w'_i; w_j, w'_j) \) in the set \( \bigcup_{1 \leq i \leq j \leq s} \text{Obs}(i,j) \), its S-polynomial \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) has a representation

\[
S_{i,j}(w_i, w'_i; w_j, w'_j) = \sum_{k=1}^{\mu} c_k w_k g_{i_k} w'_k
\]

with \( c_k \in K, w_k, w'_k \in (X) \), and \( g_{i_k} \in G \) for all \( k \in \{1, \ldots, \mu\} \) such that \( Lw_\sigma(g_j w'_j) >_\sigma Lw_\sigma(w_k g_{i_k} w'_k) \) if \( c_k \neq 0 \) for some \( k \in \{1, \ldots, \mu\} \).

**Proof.** See [10], Proposition 4.1.2. □

A presentation of \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) as in Proposition 2.5b is called a (weak) Gröbner representation of \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) in terms of \( G \).

Observe that there are infinitely many obstructions in each set \( \text{Obs}(i,j) \), due to the following two types of trivial obstructions.

(T1) If \( \alpha_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i,j) \), then, for all \( w, w' \in (X) \), we have \( \alpha_{i,j}(ww_i, w'_i w'; ww_j, w'_j w') \in \text{Obs}(i,j) \).

(T2) For all \( w \in (X) \), we have \( \alpha_{i,j}(Lw_\sigma(g_j) w, 1; 1, w Lw_\sigma(g_i)), \alpha_{i,j}(1, w Lw_\sigma(g_i); Lw_\sigma(g_j) w, 1) \in \text{Obs}(i,j) \).

Before going on, let us get rid of these two types of trivial obstructions. The following lemma handles trivial obstructions of type (T1).

**Lemma 2.6.** If the S-polynomial of \( \alpha_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i,j) \) has a Gröbner representation in terms of \( G \), then, for all \( w, w' \in (X) \), the S-polynomial of \( \alpha_{i,j}(ww_i, w'_i w'; ww_j, w'_j w') \) also has a Gröbner representation in terms of \( G \).

**Proof.** Without loss of generality, we assume that \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) is non-zero. We write \( S_{i,j}(w_i, w'_i; w_j, w'_j) = \sum_{k=1}^{\mu} c_k w_k g_{i_k} w'_k \), where \( c_k \in K \setminus \{0\} \), \( w_k, w'_k \in (X) \), and \( g_{i_k} \in G \) such that \( Lw_\sigma(g_j w'_j) >_\sigma Lw_\sigma(w_k g_{i_k} w'_k) \) for all
Let $G = \{g_1, \ldots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$.

(a) Let $w_1, w_2 \in \langle X \rangle$ be two words. If there exist some words $w, w', w'' \in \langle X \rangle$ and $w \neq 1$ such that $w_1 = w'w$ and $w_2 = w''w$, or $w_1 = w''w'$ and $w_2 = w'w''$, or $w_1 = w'w''w'$ and $w_2 = w$, then we say $w_1$ and $w_2$ have an overlap at $w$. Otherwise, we say that $w_1$ and $w_2$ have no overlap.

(b) Let $o_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ be an obstruction. If $Lw_\sigma(g_i)$ and $Lw_\sigma(g_j)$ have an overlap at $w \in \langle X \rangle \setminus \{1\}$ and if $w$ is a subword of $w_1 Lw_\sigma(g_i) w'_i$, then we say that $o_{i,j}(w_i, w'_i; w_j, w'_j)$ has an overlap at $w$. Otherwise, we say that $o_{i,j}(w_i, w'_i; w_j, w'_j)$ has no overlap.

Thus, as shown in (T2), there are infinitely many obstructions without overlaps in each $\text{Obs}(i, j)$. The following lemma gets rid of these trivial obstructions.

Lemma 2.8. If $o_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ has no overlap, then $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a Gröbner representation in terms of $G$.

Proof. See [13], Lemma 5.4.

Observe that Lemma 2.8 is indeed a non-commutative version of the product criterion (or criterion 2) of Buchberger (cf. [5]).

Definition 2.9. Let $G = \{g_1, \ldots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$.

(a) Let $i, j \in \{1, \ldots, s\}$ and $i < j$. An obstruction in $\text{Obs}(i, j)$ is called non-trivial if it has an overlap and is of the form $o_{i,j}(w_i, 1; w'_j)$, or $o_{i,j}(1, w'_i; w_j, 1)$, or $o_{i,j}(w_i, w'_i; 1, 1)$, or $o_{i,j}(1, 1; w_j, w'_j)$ with $w_i, w'_i, w_j, w'_j \in \langle X \rangle$.

(b) Let $i \in \{1, \ldots, s\}$. A self obstruction in $\text{Obs}(i, i)$ is called non-trivial if it has an overlap and is of the form $o_{i,i}(1, w'_i; w_i, 1)$ with $w_i, w'_i \in \langle X \rangle \setminus \{1\}$.

(c) Let $i, j \in \{1, \ldots, s\}$ and $i \leq j$. The set of all non-trivial obstructions of $g_i$ and $g_j$ will be denoted by $\text{NTObs}(i, j)$.

In the literature, a non-trivial obstruction of the form $o_{i,j}(w_i, 1; 1, w'_j)$ is called a left obstruction, a non-trivial obstruction of the form $o_{i,j}(1, w'_i; w_j, 1)$ is called a right obstruction, and a non-trivial obstruction of the form $o_{i,j}(w_i, w'_i; w_j, w'_j)$. 

$k \in \{1, \ldots, \mu\}$. For all $w, w' \in \langle X \rangle$, it is clear that $S_{i,j}(ww_i, w'_i; ww_j, w'_j) = \sum_{k=1}^{\mu} c_k w_k g_k i, w'_k w'$. Since the word ordering $\sigma$ is compatible with multiplication, we have $w Lw_\sigma(w_j g_j w'_j) w' \gg w Lw_\sigma(w_i g_i w'_i) w'$ for all $k \in \{1, \ldots, \mu\}$. Hence we have $Lw_\sigma(w w_j g_j w'_j) w' \gg Lw_\sigma(w w_i g_i w'_i) w'$ for all $k \in \{1, \ldots, \mu\}$ and $S_{i,j}(ww_i, w'_i; ww_j, w'_j) = \sum_{k=1}^{\mu} c_k w w_k g_k w_k w'$ is a Gröbner representation in terms of $G$. 

To deal with trivial obstructions of type (T2), we introduce some terminology as follows.

Definition 2.7. Let $G = \{g_1, \ldots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$.

(a) Let $o_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ be an obstruction. If $Lw_\sigma(g_i)$ and $Lw_\sigma(g_j)$ have an overlap at $w \in \langle X \rangle \setminus \{1\}$ and if $w$ is a subword of $w_1 Lw_\sigma(g_i) w'_i$, then we say that $o_{i,j}(w_i, w'_i; w_j, w'_j)$ has an overlap at $w$. Otherwise, we say that $o_{i,j}(w_i, w'_i; w_j, w'_j)$ has no overlap.
1, 1) or $o_{i,j}(1, 1; w_j, w'_j)$ is called a center obstruction. We picture four types of obstructions as follows.

| $w_i$ | $Lw_\sigma(g_i)$ |
|-------|------------------|
| $Lw_\sigma(g_i)$ | $w'_i$ |
| $w_j$ | $Lw_\sigma(g_j)$ |
| $Lw_\sigma(g_j)$ | $w'_j$ |

At this point we can refine the characterization of Gröbner bases given in Proposition 2.5 in the following way.

**Proposition 2.10. (Buchberger Criterion)**

Let $G = \{g_1, \ldots, g_s\} \subseteq K\langle X \rangle$ be a set of non-zero polynomials which generate a two-sided ideal $I = \langle G \rangle \subseteq K\langle X \rangle$. Then the set $G$ is a $\sigma$-Gröbner basis of $I$ if and only if, for each non-trivial obstruction $o_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j)$, its S-polynomial $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a Gröbner representation in terms of $G$.

**Proof.** This follows directly from Proposition 2.5 and Lemmas 2.6 and 2.8. In view of Lemma 2.8 it suffices to consider each obstruction with overlap, which is either a non-trivial obstruction or a multiple of a non-trivial obstruction. Further, Lemma 2.6 treats a multiple of a non-trivial obstruction via the corresponding non-trivial obstruction. Therefore, it is sufficient to consider only non-trivial obstructions. 

The Buchberger Criterion enables us to formulate the following procedure for computing Gröbner bases of two-sided ideals. Note that, in the procedure, by a fair strategy we mean a selection strategy which ensures that every obstruction is selected eventually. Since these Gröbner bases need not be finite, we have to content ourselves with an enumerating procedure.

**Theorem 2.11. (The Buchberger Procedure)**

Let $s \geq 1$, and let $G = \{g_1, \ldots, g_s\} \subseteq K\langle X \rangle$ be a set of non-zero polynomials which generate a two-sided ideal $I = \langle G \rangle \subseteq K\langle X \rangle$. Consider the following sequence of instructions.

1. **(B1)** Let $B = \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j)$.

2. **(B2)** If $B = \emptyset$, return the result $G$. Otherwise, select an obstruction $o_{i,j}(w_i, w'_i; w_j, w'_j) \in B$ using a fair strategy and delete it from $B$.

3. **(B3)** Compute the S-polynomial $S = S_{i,j}(w_i, w'_i; w_j, w'_j)$ and its normal remainder $S' = \text{NR}_{\sigma,G}(S)$. If $S' = 0$, continue with step (B2).

4. **(B4)** Increase $s$ by one, append $g_s = S'$ to the set $G$, and append the set of obstructions $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ to the set $B$. Then continue with step (B2).
This is a procedure that enumerates a $\sigma$-Gr"{o}bner basis $G$ of $I$. If $I$ has a finite $\sigma$-Gr"{o}bner basis, the procedure stops after finitely many steps and the resulting set $G$ is a finite $\sigma$-Gr"{o}bner basis of $I$.

Proof. Note that this is a straightforward generalization of the commutative version of Buchberger’s algorithm to the non-commutative case. We refer to [13] for the original form of this procedure and to [16], Theorem 4.1.14 for a detailed proof.

3 Non-Commutative Gebauer-M"{o}ller Criteria

In this section we present non-commutative Gebauer-M"{o}ller criteria. They check whether an obstruction can be represented by “smaller” obstructions. If so, we declare such obstructions to be unnecessary. Before going into details, we define a certain well-ordering $\tau$ on $T(F_s) = \{we_iw' \mid i \in \{1,\ldots,s\}, w, w' \in \langle X \rangle\}$ and use it to order obstructions. In the following, let $s \geq 1$, and let $G = \{g_1,\ldots,g_s\} \subseteq K \langle X \rangle \setminus \{0\}$ be a set of non-commutative polynomials.

Definition 3.1. Let us define a relation $\tau$ on $T(F_s)$ as follows. For two terms $w_1e_iw'_1, w_2e_jw'_2 \in T(F_s)$, we let $w_1e_iw'_1 \geq_\tau w_2e_jw'_2$ if

(a) $w_1 Lw_\sigma(g_i)w'_1 >_{\sigma} w_2 Lw_\sigma(g_j)w'_2$, or

(b) $w_1 Lw_\sigma(g_i)w'_1 = w_2 Lw_\sigma(g_j)w'_2$ and $i > j$, or

(c) $w_1 Lw_\sigma(g_i)w'_1 = w_2 Lw_\sigma(g_j)w'_2$ and $i = j$ and $w_1 \geq_\sigma w_2$.

One can check that $\tau$ is a well-ordering and is compatible with scalar multiplication. The relation $\tau$ is called the module term ordering induced by $(\sigma,G)$ on $T(F_s)$.

By definition, for every obstruction $o_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s} Ob(i, j)$, we have $w_ie_iw'_i \leq_\tau w_je_jw'_j$. We extend the ordering $\tau$ to the set of obstructions $\bigcup_{1 \leq i \leq j \leq s} Ob(i, j)$ by committing the following slight abuse of notation.

Definition 3.2. Let $\tau$ be the module term ordering induced by $(\sigma,G)$ on $T(F_s)$. Let $o_{i,j}(w_i, w'_i; w_j, w'_j), o_{k,l}(w_k, w'_k; w_l, w'_l)$ be two obstructions in the set $\bigcup_{1 \leq i \leq j \leq s} Ob(i, j)$. If we have $w_je_iw'_j >_{\tau} w_je_iw'_j$, or if we have $w_je_iw'_j = w_je_iw'_j$ and $w_kw'_k \geq_{\tau} w_kw'_k$, then we let $o_{i,j}(w_i, w'_i; w_j, w'_j) \geq_{\tau} o_{k,l}(w_k, w'_k; w_l, w'_l)$. The ordering $\tau$ is called the ordering induced by $(\sigma,G)$ on the set of obstructions.

One can verify that $\tau$ is also a well-ordering on $\bigcup_{1 \leq i \leq s} Ob(i, j)$ and compatible with scalar multiplication.

Now we are ready to generalize the commutative Gebauer-M"{o}ller criteria (see [6] and [8]) to the non-commutative case. Recall that, in step (B4) of the Buchberger Procedure, when a new generator $g_s$ is added, we immediately construct new obstructions $\bigcup_{1 \leq i \leq s} NTObs(i, s)$. We want to detect unnecessary
obstructions in the set $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ of newly constructed obstructions as well as in the set $\bigcup_{1 \leq i \leq j \leq s-1} \text{NTObs}(i, j)$ of previously constructed obstructions. We achieve this goal via the following three steps. Firstly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ with the aid of other obstructions also in this set. This step is called a head reduction step in $\mathcal{H}$. Secondly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ with the aid of obstructions in the set $\bigcup_{1 \leq i \leq j \leq s-1} \text{NTObs}(i, j)$. This step is called a tail reduction step in $\mathcal{H}$. Thirdly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq j \leq s-1} \text{NTObs}(i, j)$ with the aid of the new generator $g_s$. Indeed, the first step corresponds to the commutative Gebauer-Möller criteria $M$ and $F$, and the last step corresponds to criterion $B_k$ (c.f. $\mathcal{H}$, Subsection 3.4).

The following lemma helps us to implement the first step, that is, to detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ of newly constructed obstructions via other obstructions in this set.

**Lemma 3.3.** Let $o_{i,s}(w_i, w_i'; u_s, u_s')$ and $o_{j,s}(w_j, w_j'; v_s, v_s')$ be two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ with two words $w, w' \in \{X\}$ satisfying $u_s = wv_s$ and $u_s' = v_s'w'$.

(a) If $i < j$ and $ww' \neq 1$, then we have

$$o_{i,s}(w_i, w_i'; u_s, u_s') = w o_{j,s}(w_j, w_j'; v_s, v_s') w' + o_{i,j}(w_i, w_i'; ww_j, w_j'w')$$

with $o_{i,s}(w_i, w_i'; u_s, u_s') > \tau o_{j,s}(w_j, w_j'; v_s, v_s')$ and $o_{i,j}(w_i, w_i'; u_s, u_s') > \tau o_{i,j}(w_i, w_i'; u_s, u_s')$. Further, if the $S$-polynomials $S_{j,s}(w_j, w_j'; v_s, v_s')$ and $S_{i,j}(w_i, w_i'; vw_j, w_j'w')$ have Gröbner representations in terms of $G$, then so does $S_{i,s}(w_i, w_i'; u_s, u_s')$.

(b) If $i > j$, then we have

$$o_{i,s}(w_i, w_i'; u_s, u_s') = w o_{j,s}(w_j, w_j'; v_s, v_s') w' - o_{j,i}(wv_j, w_j'w'; w_i, w_i')$$

with $o_{i,s}(w_i, w_i'; u_s, u_s') > \tau o_{j,s}(w_j, w_j'; v_s, v_s')$ and $o_{j,i}(wv_j, w_j'w'; w_i, w_i') > \tau o_{j,i}(wv_j, w_j'w'; w_i, w_i')$. Further, if the $S$-polynomials $S_{j,s}(w_j, w_j'; v_s, v_s')$ and $S_{j,i}(wv_j, w_j'w'; w_i, w_i')$ have Gröbner representations in terms of $G$, then so does $S_{i,s}(w_i, w_i'; u_s, u_s')$.

(c) If $i = j$ and $ww' \neq 1$ or if $i = j$ and $ww' = 1$ and $w_i > \tau w_j$, then we have

$$o_{i,s}(w_i, w_i'; u_s, u_s') = w o_{j,s}(w_j, w_j'; v_s, v_s') w' + o_{i,j}(w_i, w_i'; ww_j, w_j'w')$$

with $o_{i,s}(w_i, w_i'; u_s, u_s') > \tau o_{j,s}(w_j, w_j'; v_s, v_s')$ and $o_{i,j}(w_i, w_i'; u_s, u_s') > \tau o_{i,j}(w_i, w_i'; u_s, u_s')$. Further, if the $S$-polynomials $S_{j,s}(w_j, w_j'; v_s, v_s')$ and $S_{i,j}(w_i, w_i'; vw_j, w_j'w')$ have Gröbner representations in terms of $G$, then so does $S_{i,s}(w_i, w_i'; u_s, u_s')$.

**Proof.** We prove case (a). Cases (b) and (c) can be proved similarly. The equation in case (a) follows from Definition 2.4a and from the conditions $u_s =
\[ \text{with } w_{v, u'_{i,s}} = v'_i w' \text{ and } i < j. \] Because of \( w w' > 1 \), we have \( u_s Lw(g_s)u'_s = w_{v, u'_{i,s}} Lw(g_s)v'_i w' \). Consequently, we get \( u_s e_s u'_s \succ_v v'_i e_s v'_{l,s} \) and \( \alpha_i, s(w_i, w'_j; u_s, u'_s) \succ_{o_i,j} \alpha_j, s(w_j, w'_j; v_s, v'_s). \) From \( u_s Lw(g_s)u'_s = w_{v, u'_{i,s}} Lw(g_s)v'_i w' \) and \( s > j \), we get the inequalities \( u_s e_s u'_s \succ_{w w_j e_j w'_j w'} \) and \( \alpha_i, s(w_i, w'_j; u_s, u'_s) \succ_{o_i,j} \alpha_j, s(w_i, w'_j; w w_j, w'_j w'). \)

Next we show that, if \( S_{i,s}(w_i, w'_j; v_s, v'_s) \) and \( S_{i,j}(w_i, w'_j; w w_j, w'_j w') \) have Gröbner representations in terms of \( G \), then so does \( S_{i,s}(w_i, w'_j; u_s, u'_s) \). Clearly we have

\[ S_{i,s}(w_i, w'_j; u_s, u'_s) = w S_{j,s}(w_j, w'_j; v_s, v'_s) w' \]

Without loss of generality, we assume that \( S_{i,s}(w_i, w'_j; u_s, u'_s), S_{j,s}(w_j, w'_j; v_s, v'_s) \) and \( S_{i,j}(w_i, w'_j; w w_j, w'_j w') \) are non-zero. Since there is a Gröbner representation for \( S_{j,s}(w_j, w'_j; v_s, v'_s) \), we have

\[ S_{j,s}(w_j, w'_j; v_s, v'_s) = \sum_{k=1}^{\mu} a_k w_k g_i w'_k \]

with \( a_k \in K \setminus \{0\}, w_k, w'_k \in \langle X \rangle, g_i \in G \) for all \( k \in \{1, \ldots, \mu\} \), such that \( Lw_{\sigma}(v_s g_s v'_s) \succ_{\sigma} Lw_{\sigma}(a_k w_k g_i w'_k) \). Similarly, for \( S_{i,j}(w_i, w'_j; w w_j, w'_j w') \) we have

\[ S_{i,j}(w_i, w'_j; w w_j, w'_j w') = \sum_{l=1}^{\nu} b_l w_l g_i w'_l \]

with \( b_l \in K \setminus \{0\}, w_l, w'_l \in \langle X \rangle, g_i \in G \) for all \( l \in \{1, \ldots, \nu\} \), such that \( Lw_{\sigma}(w w_j g_j w' l w') \succ_{\sigma} Lw_{\sigma}(b_l w_l g_i w'_l) \). Therefore we have

\[ S_{i,s}(w_i, w'_j; u_s, u'_s) = w(\sum_{k=1}^{\mu} a_k w_k g_i w'_k) w' + \sum_{l=1}^{\nu} b_l w_l g_i w'_l \]

\[ = \sum_{k=1}^{\mu} a_k w w_k g_i w'_k w' + \sum_{l=1}^{\nu} b_l w_l g_i w'_l. \]

As \( u_s Lw_{\sigma}(g_s) u'_s = w v, Lw_{\sigma}(g_s) v'_s \), we have \( Lw_{\sigma}(u_s g_s u'_s) = Lw_{\sigma}(w v, g_s v'_s) \) \( \succ_{\sigma} Lw_{\sigma}(w w_k g_i w'_k w') \) for all \( k \in \{1, \ldots, \mu\} \). By Definition 2.4, we have \( Lw_{\sigma}(u_s g_s u'_s) = Lw_{\sigma}(w_i g_i u'_s) = Lw_{\sigma}(w w_j g_j w' l w') \succ_{\sigma} Lw_{\sigma}(b_l w_l g_i w'_l) \) for all \( l \in \{1, \ldots, \nu\} \). Therefore

\[ S_{i,s}(w_i, w'_j; u_s, u'_s) = \sum_{k=1}^{\mu} a_k w w_k g_i w'_k w' + \sum_{l=1}^{\nu} b_l w_l g_i w'_l \]

is a Gröbner representation of \( S_{i,s}(w_i, w'_j; u_s, u'_s) \). \( \square \)

The following example shows that the obstruction \( \alpha_{i,j}(w_i, w'_j; w w_j, w'_j w') \) in case (a) of Lemma 3.3 can be a non-trivial obstruction, i.e. a multiple of a non-trivial obstruction or an obstruction without overlap. Similar phenomena occur in cases (b) and (c) of Lemma 3.3 as well as in Lemmas 3.7 and 3.10.
Example 3.4. Consider polynomials $G = \{g_1, g_2, g_3\}$ in the non-commutative polynomial ring $K\langle x, y \rangle$.

(a) Assume that $\text{Lm}_\sigma(g_1) = y^3$, $\text{Lm}_\sigma(g_2) = x^2y^2$ and $\text{Lm}_\sigma(g_3) = xy^2x$. Then we have $o_{1,3}(x y x^2; 1, 1, y^2), o_{2,3}(x y, 1, y) \in \bigcup_{1 \leq i \leq 3} \text{NTObs}(i, 3)$, and

$$o_{1,3}(x y x^2; 1, 1, y^2) = o_{2,3}(x y; 1, 1, y) + o_{1,2}(x y x x; 1, x y).$$

Observe that $o_{1,2}(x y x x; 1, x y) = x y o_{1,2}(x y x^2; 1, 1, y)$ is a multiple of the non-trivial obstruction $o_{1,2}(x^2; 1, 1, y)$.

(b) Now assume that $\text{Lm}_\sigma(g_1) = (x y)^2$, $\text{Lm}_\sigma(g_2) = y$ and $\text{Lm}_\sigma(g_3) = x y x^2 y$. Then we have $o_{1,3}(x y x x; 1, 1, y), o_{2,3}(x, x^2 y; 1, 1) \in \bigcup_{1 \leq i \leq 3} \text{NTObs}(i, 3)$, and

$$o_{1,3}(x y x x; 1, 1, y) = o_{2,3}(x, x^2 y; 1, 1) x y + o_{1,2}(x y x x; 1, x x y x).$$

One can check that $o_{1,2}(x y x x; 1, x x y x)$ is an obstruction without overlap.

In the following, we present the non-commutative multiply criterion and the leading word criterion. They are non-commutative analogues of the Gebauer-Möller criteria M and F, respectively.

**Proposition 3.5. (Non-Commutative Multiply Criterion)**

Suppose that $o_{i,s}(w_i, u_i^s; v_s, u_s^s)$ and $o_{j,s}(w_j, u_j^s; v_s, u_s^s)$ are two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ such that there exist two words $w, w'$ in $\langle X \rangle$ satisfying $u_s = w v_s$ and $u'_s = v'_s w'$. Then we can remove the obstruction $o_{i,s}(w_i, u_i^s; v_s, u_s^s)$ from $\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$ in the execution of the Buchberger Procedure if $w w' \neq 1$.

**Proof.** By the previous lemma, the obstruction $o_{i,s}(w_i, u_i^s; v_s, u_s^s)$ can be represented as

$$o_{i,s}(w_i, u_i^s; v_s, u_s^s) = w o_{j,s}(w_j, u_j^s; v_s, u_s^s) w' + a o_{k,l}(w_k, u_k^l; w_l, u_l^l)$$

with $a \in \{1, -1\}$ and $k = \min\{i, j\}$, $l = \max\{i, j\}$. To prove that $o_{i,s}(w_i, u_i^s; v_s, u_s^s)$ is strictly larger than $o_{j,s}(w_j, u_j^s; v_s, u_s^s)$ and $o_{k,l}(w_k, u_k^l; w_l, u_l^l)$, we consider two cases. If $i > j$, then the result follows from Lemma 3.3b; if $i \leq j$, then the result follows from Lemma 3.3a and 3.3c and the condition $w w' \neq 1$. Moreover, the S-polynomial $S_{i,s}(w_i, u_i^s; v_s, u_s^s)$ has a Gröbner representation in terms of $G$ if $S_{j,s}(w_j, u_j^s; v_s, u_s^s)$ and $S_{k,l}(w_k, u_k^l; w_l, u_l^l)$ have Gröbner representations in terms of $G$. Theorem 2.11 ensures that $S_{j,s}(w_j, u_j^s; v_s, u_s^s)$ has a Gröbner representation in terms of $G$. Note that the obstruction $o_{k,l}(w_k, u_k^l; w_l, u_l^l)$ can be either a multiple of a non-trivial obstruction or an obstruction without overlap (for instance, see Example 3.4). If $o_{k,l}(w_k, u_k^l; w_l, u_l^l)$ is a multiple of a non-trivial obstruction, then Lemma 2.6 and Theorem 2.11 guarantee that $S_{k,l}(w_k, u_k^l; w_l, u_l^l)$ has a Gröbner representation in terms of $G$. If
Proposition 3.6. (Non-Commutative Leading Word Criterion)
Suppose that \( o_{i,s}(w_i, w'_i; u_s, w'_s) \) and \( o_{j,s}(w_j, w'_j; v_s, v'_s) \) are two distinct non-trivial obstructions in \( \bigcup_{1 \leq s \leq s} \text{NTObs}(i, s) \) such that there exist two words \( w, w' \) in \( \langle X \rangle \) satisfying \( u_s = w v_s \) and \( u'_s = v'_s w' \). Then \( o_{i,s}(w_i, w'_i; u_s, u'_s) \) can be removed from \( \bigcup_{1 \leq s \leq s} \text{NTObs}(i, s) \) in the execution of the Buchberger Procedure if one of the following conditions is satisfied.

(a) \( i > j \).

(b) \( i = j \) and \( w w' = 1 \) and \( w_i >_{>_{\tau}} w_j \).

Proof. Observe that condition (a) corresponds to Lemma 2.3.b, while condition (b) corresponds to Lemma 3.3.c. We represent \( o_{i,s}(w_i, w'_i; u_s, u'_s) \) as

\[
o_{i,s}(w_i, w'_i; u_s, u'_s) = w o_{j,s}(w_j, w'_j; v_s, v'_s) w' - o_{j,i}(w w_j, w'_j w'; w_i, w'_i).
\]

By Lemma 3.3.b and 3.3.c, we have \( o_{i,s}(w_i, w'_i; u_s, u'_s) \) is strictly larger than \( o_{j,s}(w_j, w'_j; v_s, v'_s) \) and \( o_{j,i}(w w_j, w'_j w'; w_i, w'_i) \). Moreover, if the S-polynomials \( S_{j,i}(w_j, w'_j; v_s, v'_s) \) and \( S_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) have Gröbner representations in terms of \( G \), then so does \( S_{i,s}(w_i, w'_i; u_s, u'_s) \). Theorem 2.11 ensures \( S_{j,s}(w_j, w'_j; v_s, v'_s) \) and \( S_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) have a Gröbner representation in terms of \( G \). Note that the obstruction \( o_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) can be either a multiple of a non-trivial obstruction or an obstruction without overlap (for instance, see Example 3.4). If \( o_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) is a multiple of a non-trivial obstruction, then Lemma 2.6 and Theorem 2.11 guarantee that \( S_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) has a Gröbner representation in terms of \( G \). If \( o_{j,i}(w w_j, w'_j w'; w_i, w'_i) \) is an obstruction without overlap, then, by Lemma 2.8, its S-polynomial has a Gröbner representation in terms of \( G \). Now the conclusion follows from Proposition 2.10 and Theorem 2.11.

Next we work on detecting unnecessary obstructions in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \) via obstructions in the set \( \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j) \) of previously constructed obstructions.

Lemma 3.7. Let \( o_{j,s}(u_j, u'_j; w_s, w'_s) \) and \( o_{i,j}(w_i, w'_i; v_j, v'_j) \) be non-trivial obstructions in \( \bigcup_{1 \leq s \leq s} \text{NTObs}(i, s) \) and \( \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j) \), respectively. If there exist two words \( w, w' \in \langle X \rangle \) such that \( u_j = w v_j \) and \( u'_j = v'_j w' \), then we have

\[
o_{j,s}(u_j, u'_j; w_s, w'_s) = -w o_{i,j}(w_i, w'_i; v_j, v'_j) w' + o_{i,s}(w w_i, w'_i w'; w_s, w'_s)
\]

where the inequalities \( o_{j,s}(u_j, u'_j; w_s, w'_s) \) and \( o_{i,j}(w_i, w'_i; v_j, v'_j) \) hold. Further, if the S-polynomials \( S_{i,s}(w_i, w'_i; v_j, v'_j) \) and \( S_{j,s}(w w_i, w'_i w'; w_s, w'_s) \) have Gröbner representations in terms of \( G \), then so does \( S_{j,s}(u_j, u'_j; w_s, w'_s) \).
Proof. The claimed equality follows from Definition 2.4.a and from the conditions $u_j = wv_j$ and $u_j' = v_j' w$. We have $o_{j,s}(u_j, u_j'; w_s, w_s') >_\tau o_{i,j}(w_i, w_i'; v_j, v_j')$ for $w_s e_s w_s' > \tau u_j e_j u_j' = w v_j e_j v_j'$ $w \ge \tau v_j e_j v_j'$. From the inequality $u_j e_j u_j' = w v_j e_j v_j'$ $w > \tau w w e_j w w'$, it follows that $o_{j,s}(u_j, u_j'; w_s, w_s') >_\tau o_{i,s}(w w_i, w_i' w'; w_s, w_s')$. Again, we can prove the second part by following the same argument as in the proof of Lemma 3.8.a.

Note that the obstruction $o_{i,s}(w w_i, w_i' w'; w_s, w_s')$ in Lemma 3.7 can be either a multiple of a non-trivial obstruction or an obstruction without overlap. However, it suffices for us to consider only the latter case, since the former case has been considered in Proposition 3.5 and, more precisely, in Lemma 3.8.b.

Proposition 3.8. (Non-Commutative Tail Reduction)

Suppose that $o_{j,s}(u_j, u_j'; w_s, w_s')$ and $o_{i,j}(w_i, w_i'; v_j, v_j')$ are non-trivial obstructions in $\bigcup_{1 \leq i \leq s} NTObs(i, s)$ and $\bigcup_{1 \leq i \leq j \leq s-1} NTObs(i, j)$, respectively, such that there exist two words $w, w' \in (X)$ satisfying $u_j = w v_j$ and $u_j' = v_j' w'$. If the word $w w_i$ is a multiple of $w_i Lw_s(g_s)$, or if the word $w_i' w'$ is a multiple of $Lw_s(g_s)w_i'$, then $o_{j,s}(u_j, u_j'; w_s, w_s')$ can be removed from $\bigcup_{1 \leq i \leq s} NTObs(i, s)$ in the execution of the Buchberger Procedure.

Proof. By Lemma 3.7, the obstruction $o_{j,s}(u_j, u_j'; w_s, w_s')$ can be represented as

$$o_{j,s}(u_j, u_j'; w_s, w_s') = -w o_{i,j}(w_i, w_i'; v_j, v_j') w' + o_{i,s}(w w_i, w_i' w'; w_s, w_s')$$

where the inequalities $o_{j,s}(u_j, u_j'; w_s, w_s') >_\tau o_{i,j}(w_i, w_i'; v_j, v_j')$ and $o_{j,s}(u_j, u_j'; w_s, w_s') > o_{i,s}(w w_i, w_i' w'; w_s, w_s')$ hold. Further, if the $S$-polynomials $S_{i,j}(w_i, w_i'; v_j, v_j')$ and $S_{i,s}(w w_i, w_i' w'; w_s, w_s')$ have Gröbner representations in terms of $G$, then so does $S_{j,s}(u_j, u_j'; w_s, w_s')$. Theorem 2.11 ensures that $S_{i,j}(w_i, w_i'; v_j, v_j')$ has a Gröbner representation in terms of $G$. Note that $w w_i$ is a multiple of $w_i Lw_s(g_s)$ or $w_i' w'$ is a multiple of $Lw_s(g_s)w_i'$. This implies that $o_{i,s}(w w_i, w_i' w'; w_s, w_s')$ has no overlap. By Lemma 2.8, $S_{i,s}(w w_i, w_i' w'; w_s, w_s')$ has a Gröbner representation in terms of $G$. Now the conclusion follows from Proposition 2.10 and Theorem 2.11.

Remark 3.9. Our experiments in the final section show that, after applying the previous two criteria, the Non-Commutative Tail Reduction is unlikely to apply in the Buchberger Procedure. This may be due to the fact that frequently the Non-Commutative Multiply Criterion and the Non-Commutative Leading Word Criterion have already detected all unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s} NTObs(i, s)$ of newly constructed obstructions.

So far we have detected unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s} O(i, s)$ of newly constructed obstructions. Intuitively, we are also able to detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq j \leq s-1} Obs(i, j)$ of previously constructed obstructions. Thus, in the last step, we detect unnecessary obstructions in this set by using the new generator $g_s$. 

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Lemma 3.10. Let \( o_{i,j}(w_i, w_i'; w_j, w_j') \in \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j) \) be a non-trivial obstruction. If there are two words \( w, w' \in \langle X \rangle \) satisfying \( w_j \text{Lw}_\sigma(g_j)w_j' = w \text{Lw}_\sigma(g_s)w' \), then we can represent \( o_{i,j}(w_i, w_i'; w_j, w_j') \) as

\[
o_{i,j}(w_i, w_i'; w_j, w_j') = o_{i,s}(w_i, w_i'; w, w') - o_{j,s}(w_j, w_j'; w, w').
\]

Moreover, if \( S_{i,s}(w_i, w_i'; w, w') \) and \( S_{j,s}(w_j, w_j'; w, w') \) have Gröbner representations in terms of \( G \), then so does \( S_{i,j}(w_i, w_i'; w_j, w_j') \).

Proof. The claimed equality follows from Definition 2.4.a and the condition of a non-trivial obstruction, say \( o_1 \). If there are two words \( w, w' \) such that \( w_j \text{Lw}_\sigma(g_j)w_j' = w \text{Lw}_\sigma(g_s)w' \), then we can represent \( o_{i,j}(w_i, w_i'; w_j, w_j') \) as

\[
o_{i,j}(w_i, w_i'; w_j, w_j') = o_{i,s}(w_i, w_i'; w, w') - o_{j,s}(w_j, w_j'; w, w').
\]

Example 3.11. Consider polynomials \( G = \{g_1, g_2, g_3\} \) in the non-commutative polynomial ring \( K\langle x, y \rangle \) with \( \text{Lm}_\sigma(g_1) = x^3yx, \text{Lm}_\sigma(g_2) = x^2 \) and \( \text{Lm}_\sigma(g_3) = x \). We have \( o_{1,2}(1, 1; x, yx) \in \bigcup_{1 \leq i \leq j \leq 2} \text{NTObs}(i, j) \) and \( x \text{Lw}_\sigma(g_2)yx = x^3yx = x^3y \text{Lw}_\sigma(g_3) \) and

\[
o_{1,2}(1, 1; x, yx) = o_{1,3}(1, 1; x^3y, 1) - o_{2,3}(x, yx; x^3y, 1).
\]

One can check that \( o_{1,3}(1, 1; x^3y, 1) \) is a non-trivial obstruction in \( \text{NTObs}(1, 3) \) and \( o_{1,2}(1, 1; x, yx) < o_{1,3}(1, 1; x^3y, 1) \). Moreover, \( o_{2,3}(x, yx; x^3y, 1) \) is an obstruction without overlap.

The following is a non-commutative analogue of the Gebauer-Möller criterion \( B_k \), which is also known as the chain criterion (or criterion 1) of Buchberger (cf. [5]).

Proposition 3.12. (Non-Commutative Backward Criterion)
Suppose that \( o_{i,j}(w_i, w_i'; w_j, w_j') \in \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j) \) is a non-trivial obstruction. Then in the execution of the Buchberger Procedure \( o_{i,j}(w_i, w_i'; w_j, w_j') \) can be removed from \( \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j) \) if the following three conditions are satisfied.

(a) There are two words \( w, w' \in \langle X \rangle \) such that \( w_j \text{Lw}_\sigma(g_j)w_j' = w \text{Lw}_\sigma(g_s)w' \).

(b) The obstruction \( o_{i,s}(w_i, w_i'; w, w') \) is either an obstruction without overlap or a multiple of a non-trivial obstruction in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \).

(c) The obstruction \( o_{j,s}(w_j, w_j'; w, w') \) is either an obstruction without overlap or a multiple of a non-trivial obstruction in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \).
Proof. By Lemma 3.10 we can represent \( o_{i,j}(w_i, w'_i; w_j, w'_j) \) as
\[
o_{i,j}(w_i, w'_i; w_j, w'_j) = o_{i,s}(w_i, w'_i; w, w') - o_{j,s}(w_j, w'_j; w, w')
\]
Moreover, \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) has a Gröbner representations in terms of \( G \) if \( S_{i,s}(w_i, w'_i; w, w') \) and \( S_{j,s}(w_j, w'_j; w, w') \) have Gröbner representations in terms of \( G \). If \( o_{i,s}(w_i, w'_i; w, w') \) is an obstruction without overlap, then, by Lemma 2.8 its S-polynomial has a Gröbner representations in terms of \( G \). If it is a multiple of a non-trivial obstruction in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \), then Lemma 2.8 and Theorem 2.11 ensure that \( S_{i,s}(w_i, w'_i; w, w') \) has a Gröbner representations in terms of \( G \). By the same argument, one can show that \( S_{j,s}(w_j, w'_j; w, w') \) has a Gröbner representations in terms of \( G \). Now the conclusion follows from Proposition 2.10 and Theorem 2.11.

We would like to mention that the Non-Commutative Backward Criterion given in Proposition 3.12 covers in particular all useless obstructions presented by T. Mora in [13], Lemma 5.11.

Remark 3.13. In order to apply Propositions 3.5, 3.6, 3.8 and 3.12 to remove unnecessary obstructions during the execution of the Buchberger Procedure, it is crucial to make sure that the S-polynomials of those removed obstructions have Gröbner representations.

(a) Propositions 3.5, 3.6 and 3.8 remove unnecessary non-trivial obstructions, say \( o_{i,s}(w_i, w'_i; w_s, w'_s) \), from the set \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \) of newly constructed obstructions. The Gröbner representation of the S-polynomial \( S_{i,s}(w_i, w'_i; w_s, w'_s) \) depends on the Gröbner representations of the S-polynomials of two smaller obstructions in the set \( \bigcup_{1 \leq i \leq j < s-1} \text{Obs}(i, j) \) and the set \( \bigcup_{1 \leq i \leq s} \text{Obs}(i, s) \).

(b) Proposition 3.12 removes unnecessary obstructions, say \( o_{i,j}(w_i, w'_i; w_j, w'_j) \), from the set \( \bigcup_{1 \leq i \leq j < s-1} \text{Obs}(i, j) \) of previously constructed obstructions. The Gröbner representation of \( S_{i,j}(w_i, w'_i; w_j, w'_j) \) depends on the Gröbner representations of the S-polynomials of two obstructions, say \( o_{k,s}(w_k, w'_k; u_s, u'_s) \) and \( o_{l,s}(w_l, w'_l; v_s, v'_s) \), in \( \bigcup_{1 \leq i \leq s} \text{Obs}(i, s) \), which are not necessarily smaller than \( o_{i,j}(w_i, w'_i; w_j, w'_j) \). If \( o_{k,s}(w_k, w'_k; u_s, u'_s) \) is a multiple of a non-trivial obstruction, say \( o_{k,s}(w_k, w'_k; u_s, u'_s) \), in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \), then, before removing \( o_{i,j}(w_i, w'_i; w_j, w'_j) \), it is important to ensure that \( o_{k,s}(w_k, w'_k; u_s, u'_s) \) is in \( \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s) \). The same check should be applied to \( o_{l,s}(w_l, w'_l; v_s, v'_s) \).

Observe that Propositions 3.5, 3.6 and 3.12 are actually generalizations of the well-known Gebauer-Möller criteria (see [6] and [8]) in commutative polynomial rings. More precisely, Propositions 3.5, 3.6 and 3.12 correspond to criterion \( M \), criterion \( F \) and criterion \( B_k \), respectively (c.f. [8], Subsection 3.4).

Using the Gebauer-Möller criteria, we can improve the Buchberger Procedure as follows.
Theorem 3.14. (Improved Buchberger Procedure)
In the setting of Theorem 2.11 we replace step (B4) by the following sequence of instructions.

(4a) Increase $s$ by one. Append $g_s = S'$ to the set $G$, and form the set of non-trivial obstructions $\text{NTObs}(s) = \bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)$.

(4b) Remove from $\text{NTObs}(s)$ all obstructions $\alpha_{i,s}(w_i, w'_i; u_s, u'_s)$ such that there exists an obstruction $\alpha_{j,s}(w_j, w'_j; v_s, v'_s) \in \text{NTObs}(s)$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $u_s = wv_s$, $u'_s = v'_s w'$ and $ww' \neq 1$.

(4c) Remove from $\text{NTObs}(s)$ all obstructions $\alpha_{i,s}(w_i, w'_i; u_s, u'_s)$ such that there exists an obstruction $\alpha_{j,s}(w_j, w'_j; v_s, v'_s) \in \text{NTObs}(s)$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $u_s = wv_s$, $u'_s = v'_s w'$, and such that $i > j$, or $i = j$ and $ww' = 1$ and $w_i >_\sigma w_j$.

(4d) Remove from $\text{NTObs}(s)$ all obstructions $\alpha_{i,j}(w_i, w'_i; w_s, w'_s)$ such that there exists an obstruction $\alpha_{j,s}(w_j, w'_j; v_s, v'_s) \in \text{B}$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $u_j = wv_j$, $u'_j = v'_j w'$, and such that $o_{i,s}(ww_i, w'_i w'; w_s, w'_s)$ has no overlap.

(4e) Remove from $\text{B}$ all obstructions $\alpha_{i,j}(w_i, w'_i; w_j, w'_j)$ such that there exist two words $w, w' \in \langle X \rangle$ satisfying $w \text{Lw}_\sigma(g_s) w' = w_j \text{Lw}_\sigma(g_j) w'_j$, and such that the following conditions are satisfied:

(i) $o_{i,s}(w_i, w'_i; w, w')$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\text{NTObs}(s)$.

(ii) $o_{j,s}(w_j, w'_j; w, w')$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\text{NTObs}(s)$.

(4f) Replace $\text{B}$ by $\text{B} \cup \text{NTObs}(s)$ and continue with step (B2).

Then the resulting set of instructions is a procedure that enumerates a $\sigma$-Gröbner basis $G$ of $I$. If $I$ has a finite $\sigma$-Gröbner basis, it stops after finitely many steps and the resulting set $G$ is a finite $\sigma$-Gröbner basis of $I$.

Proof. This follows from Theorem 2.11 and Propositions 3.5, 3.6, 3.8 and 3.12. □

4 Experiments and Conclusions

In this section we want to present some experimental data which illustrate the performance of the Gebauer-Möller criteria presented in Propositions 3.5, 3.8 and 3.12. The computations are based on an implementation (using C++) in an experimental version of the ApCoCoA library (see [1]) by the second author.
Example 4.1. Consider the non-commutative polynomial ring $\mathbb{Q}\langle a, b \rangle$ equipped with the word ordering $\text{LLex}$ on $\langle a, b \rangle$ such that $a >_{\text{LLex}} b$. We take the list of finite generalized triangle groups from [14], Theorem 2.12 and construct a list of ideals in $\mathbb{Q}\langle a, b \rangle$. For $k = 1, \ldots, 13$ let $I_k = \langle G_k \rangle \subseteq \mathbb{Q}\langle a, b \rangle$ be the ideal generated by the following set of polynomials $G_k \subseteq \mathbb{Q}\langle a, b \rangle$. 

\begin{align*}
G_1 &= \{a^2 - 1, b^3 - 1, (abab^2ab^2)^2 - 1\}, \\
G_2 &= \{a^2 - 1, b^3 - 1, (abab^2)^3 - 1\}, \\
G_3 &= \{a^2 - 1, b^3 - 1, (ab)^2 - 1\}, \\
G_4 &= \{a^2 - 1, b^3 - 1, (abab^2)^2 - 1\}, \\
G_5 &= \{a^2 - 1, b^5 - 1, (abab^2)^3 - 1\}, \\
G_6 &= \{a^2 - 1, b^5 - 1, (abab^2)^2 - 1\}, \\
G_7 &= \{a^2 - 1, b^5 - 1, (abab^2)^3 - 1\}, \\
G_8 &= \{a^2 - 1, b^4 - 1, (abab^3)^2 - 1\}, \\
G_9 &= \{a^2 - 1, b^3 - 1, (abab^3)^2 - 1\}, \\
G_{10} &= \{a^2 - 1, b^3 - 1, (abab^2)^2 - 1\}, \\
G_{11} &= \{a^2 - 1, b^3 - 1, (abab^2)^3 - 1\}, \\
G_{12} &= \{a^2 - 1, b^3 - 1, (abab^2ab^2)^2 - 1\}, \\
G_{13} &= \{a^2 - 1, b^3 - 1, (ababab^2)^2 - 1\}.
\end{align*}

The following table lists some numbers of polynomials and obstructions treated by the Improved Buchberger Procedure given in Theorem 3.14.

| $k$ | #(Gb) | #(RGb) | #(Tot) | #(Sel) | #(M) | #(F) | #(Bk) | $\rho$ |
|-----|-------|--------|--------|--------|------|------|-------|------|
| 1   | 62    | 35     | 7032   | 248    | 6512 | 48   | 224   | 0.0353 |
| 2   | 133   | 96     | 31700  | 533    | 30571| 70   | 526   | 0.0168 |
| 3   | 50    | 40     | 2828   | 197    | 2489 | 11   | 131   | 0.0697 |
| 4   | 64    | 28     | 4702   | 253    | 4185 | 46   | 218   | 0.0538 |
| 5   | 35    | 21     | 1580   | 115    | 1348 | 24   | 93    | 0.0728 |
| 6   | 199   | 164    | 51175  | 882    | 49126| 26   | 1141  | 0.0172 |
| 7   | 200   | 164    | 51864  | 886    | 49818| 17   | 1143  | 0.0170 |
| 8   | 53    | 37     | 3756   | 192    | 3357 | 19   | 188   | 0.0511 |
| 9   | 11    | 5      | 150    | 31     | 98   | 8    | 13    | 0.2067 |
| 10  | 22    | 15     | 741    | 74     | 605  | 18   | 44    | 0.0999 |
| 11  | 30    | 21     | 1573   | 116    | 1324 | 50   | 83    | 0.0737 |
| 12  | 97    | 70     | 16841  | 365    | 15989| 97   | 390   | 0.0217 |
| 13  | 220   | 194    | 87673  | 1021   | 85136| 153  | 1363  | 0.0116 |

Here we used the following abbreviations.

- #(Gb) is the number of elements of the Gröbner basis returned by the procedure.
- #(RGb) is the cardinality of the reduced Gröbner basis of the corresponding ideal.
• \#(Tot) is the total number of non-trivial obstructions constructed during the Buchberger Procedure.

• \#(Sel) is the number of actually selected and analysed non-trivial obstructions.

• \#(M) is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Multiply Criterion given in Proposition 3.5.

• \#(F) is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Leading Word Criterion given in Proposition 3.6.

• \#(Bk) is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Backward Criterion given in Proposition 3.12.

• \(\rho = \#(Sel)/\#(Tot)\).

Note that \#(RGb) is an invariant of the ideal which only depends on chosen word ordering. Other numbers in the table rely also on the selection strategy. In our experiments we used the normal strategy which first chooses the obstruction whose S-polynomial has the lowest degree and then breaks ties by choosing the obstruction whose S-polynomial has the smallest leading word with respect to the word ordering. In these experiments, the Non-Commutative Tail Reduction given in Proposition 3.8 detected no of unnecessary non-trivial obstruction (see Remark 3.9). Apparently, the Non-Commutative Multiply Criterion and the Non-Commutative Leading Word Criterion had already detected all unnecessary obstructions in the set \(\bigcup_{1 \leq i \leq s} \text{NTObs}(i, s)\) of newly constructed obstructions. The low ratios \(\rho\) in the table indicate that the non-commutative Gebauer-Möller criteria we obtained can detect most unnecessary obstructions during the procedure.

**Example 4.2.** The following ideals \(\text{braid3}\) and \(\text{braid4}\) in the non-commutative polynomial ring \(\mathbb{Q}\langle x_1, x_2, x_3 \rangle\) are taken from [15], Section 5. More precisely, \(\text{braid3}\) is the ideal generated by the set \{-\(x_2x_3x_1 + x_3x_1x_3, x_2x_1x_2 - x_3x_2x_3, x_1x_2x_1 - x_3x_1x_2, x_1^2 + x_1x_2x_3 + x_3^2 + x_2^3\}\}, and \(\text{braid4}\) is the ideal generated by the set \{-\(x_2x_3x_1 + x_3x_1x_3, x_2x_1x_2 - x_3x_2x_3, x_1x_2x_1 - x_3x_1x_2, x_1^2 + x_1x_2x_3 + x_3^2 + x_2^3\}\}. These ideals are generated by sets of homogeneous generators. The following table lists the results of the computations of Gröbner bases truncated at degree 11 with respect to \(\text{LLex}\) on \(\langle x_1, x_2, x_3 \rangle\) such that \(x_1 >_\text{LLex} x_2 >_\text{LLex} x_3\), via the Improved Buchberger Procedure.

|        | \#(Gb) | \#(Tot) | \#(Sel) | \(\rho\) |
|--------|--------|---------|---------|----------|
| braid3-11 | 726    | 289642  | 1663    | 0.0057   |
| braid4-11 | 416    | 93252   | 1150    | 0.0123   |

The meaning of the symbols is the same as in Example 4.1. In this experiment we also used the normal strategy. Moreover, since we compute truncated Gröbner bases, we discard those obstructions whose S-polynomial have degrees
larger than the degree of truncation. Thus the ratios \( \rho \) in the table are lower than the ratios in the table of Example 4.1. Again, the non-commutative Gebauer-Möller criteria detect most unnecessary obstructions during the procedure.

The experimental data in Examples 4.1 and 4.2 show that the generalizations of the Gebauer-Möller criteria presented in Propositions 3.5, 3.6 and 3.12 can successfully detect a large number of unnecessary obstructions. In fact, they apparently detect almost all unnecessary obstructions during the Buchberger Procedure.

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