CANONICAL BASES AND QUIVER VARIETIES

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Abstract. We prove the existence of canonical bases in the $K$-theory of quiver varieties. This existence was conjectured by Lusztig.

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1. Introduction

Lusztig proposed in [16] to construct a signed basis of the equivariant $K$-theory of a quiver variety. As in [14], this signed basis should be characterized by an involution and a metric. He suggested a formula for the involution and the metric and he conjectured the existence of the signed basis. This signed basis should also satisfy some positivity property, related, hopefully, to the positivity of the structural constants of the product and the coproduct of the modified quantum algebra in the canonical basis, for all simply laced types. The main purpose of this paper is to give a precise definition of this signed basis and to prove its existence. It was conjectured in [24] that the $K$-theory of the quiver variety, with the action of the quantized enveloping algebra of affine type defined in [20] (see also [23] for the type $A$ case), is isomorphic to the 'maximal integrable module' introduced by Kashiwara in [8]. This module has a canonical basis, see loc. cit. The conjectures in [9, §13] suggest that Kashiwara’s canonical basis and the geometric one are related, see Remark 7.2.2.

We thank the referee for useful suggestions.

Both authors are partially supported by EU grant # ERB FMRX-CT97-0100.

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2. The algebra $U$

2.1. Let $\mathfrak{g}$ be a simple, simply laced, complex Lie algebra. Let $(a_{ij})_{i,j \in I}$ be the Cartan matrix. The quantum loop algebra associated to $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra $U'$ generated by $x_{ij}^\pm_k$, $k_i^\pm = k_{ii}^\pm$ ($i \in I$, $r \in \mathbb{Z}$, $s \in \pm \mathbb{N}$) modulo the following defining relations

\[ k_i k_i^{-1} = 1 = k_i^{-1} k_i, \quad [k_i, x_{ij}^+, k_j^+ z] = 0, \]

\[ k_i x_{jr}^\pm k_i^{-1} = q^{\pm a_{ij}} x_{jr}^\pm, \]

\[ (w - q^{\pm a_{ij}} z) k_j^\pm w = (q^{\pm a_{ij}} w - z) x_j^\pm (w), \]

\[ (z - q^{\pm a_{ij}} w) x_j^\pm (z) x_j^\pm (w) = (q^{\pm a_{ij}} z - w) x_j^\pm (w) x_j^\pm (z), \]

\[
\sum_{m} \sum_{p=0}^{m} (-1)^p \binom{m}{p} x_{ir(w(1))}^+ \cdots x_{ir(w(p))}^+ x_{jr(w(p+1))}^+ \cdots x_{ir(w(m))}^+ = 0,
\]

where $i \neq j$, $m = 1 - a_{ij}$, $r_1, \ldots, r_m \in \mathbb{Z}$, and $w \in S_m$. We have set $[n] = q^{1-n} + q^{3-n} + \ldots + q^{n-1}$ if $n \geq 0$, $[n]! = [n][n-1] \ldots [2]$, and

\[
\binom{m}{p} = \frac{[m]!}{[p]![m-p]!}.
\]

We have also set $\varepsilon = +$ or $-$, and

\[ k_i^\varepsilon (z) = \sum_{r \geq 0} k_{i, \pm r} z^{\mp r}, \quad x_i^\varepsilon (z) = \sum_{r \in \mathbb{Z}} x_{ir}^\varepsilon z^{\mp r}. \]

2.2. Put $A = \mathbb{Z}[q, q^{-1}]$. Consider the $A$-subalgebra $U \subset U'$ generated by the quantum divided powers $(x_{ij}^\pm)^{(n)} = (x_{ij}^\pm)^n/[n]!$, the Cartan elements $k_i^{\pm 1}$, and the coefficients of the series

\[
\sum_{s \geq 0} p_{i, \pm s} z^s = \exp \left( \sum_{s \geq 1} \frac{h_{i, \pm s}}{s} z^s \right),
\]

where the elements $h_{i, s}$ are such that

\[ k_i^\varepsilon (z) = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{s \geq 1} h_{i, \pm s} z^{\mp s} \right). \]

Observe that $U$ coincides with the $A$-subalgebra generated by the elements $(e_i)^n/[n]!$, $(f_i)^n/[n]!$, and $k_i^{\pm 1}$, $i \in I \cup \{0\}$, where $e_i, f_i, k_i^{\pm 1}$ are the Kac-Moody generators, see [3, Proposition 2.2 and 2.6].
2.3. Let $\Delta$ be the coproduct of $U'$ defined in terms of the Kac-Moody generators as follows

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i.$$ 

Let $\tau, \psi, S$ be the anti-automorphisms of $U'$ such that

$$\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(k_i) = k_i^{-1}, \quad \tau(q) = q^{-1},$$

$$\psi(e_i) = qk_i f_i, \quad \psi(f_i) = qk_i^{-1} e_i, \quad \psi(k_i) = k_i, \quad \psi(q) = q,$$

$$S(e_i) = -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i, \quad S(k_i) = k_i^{-1}, \quad S(q) = q.$$ 

The map $S$ is the antipode. Let $x \mapsto \bar{x}$ be the algebra automorphism of $U'$ such that

$$\bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{k}_i = k_i^{-1}, \quad \bar{q} = q^{-1}.$$ 

2.4. Let $\hat{U}'$ be the modified algebra of $U'$, and let $\hat{U}$ be the corresponding $A$-form. Let $\eta_\lambda \in \hat{U}$ be the idempotent denoted by $1_\lambda$ in [13, §23.1].

3. THE BRAID GROUP

3.1. Let $P, Q$ be the integral weight lattice, and the root lattice of $\mathfrak{g}$. Let $\omega_i, \alpha_i, i \in I$, be the fundamental weights and the simple roots. Let $Q^+ \subset Q$, $P^+ \subset P$ be the subsemigroups generated by the simple roots and the fundamental weights. We set $\rho = \sum_{i \in I} \omega_i$. Let $a_i, i \in I$, be the positive integers such that the element $\theta = \sum_{i \in I} c_i \alpha_i \in Q^+$ is the highest root. The integer $c = 1 + \sum_i c_i$ is the Coxeter number of $\mathfrak{g}$.

Let $\delta$ be the smallest positive imaginary root of the corresponding affine root system. Recall that the affine root $\alpha_0$ is $\delta - \theta$. We set $\hat{P} = P \oplus \mathbb{Z}\delta$.

Let $W$ be the Weyl group of $\mathfrak{g}$. Let $w_0 \in W$ be the longest element. The extended affine Weyl group is the semi-direct product $\hat{W} = W \ltimes \hat{P}$. For any element $w \in \hat{W}$ let $l(w)$ be the length of $w$. Let $s_i \in \hat{W}, i \in I \cup \{0\}$, be the affine simple reflexions. The affine Weyl group is the normal subgroup $\hat{W} \subset W$ generated by the elements $s_i, i \in I \cup \{0\}$. Let $\Gamma$ be the quotient group $\hat{W}/W$. It is identified with a group of diagram automorphisms of the extended Dynkin diagram of $\mathfrak{g}$. In particular $\Gamma$ acts on $U, W$ in the obvious way.

Let $B_W, B_{\hat{W}}$ be the braid groups of $W, \hat{W}$. The group $B_{\hat{W}}$ is generated by elements $T_w, w \in \hat{W}$, with the relation $T_w T_{w'} = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$. The group $B_W$ is the subgroup generated by the elements $T_w, w \in W$. For simplicity we set $T_i = T_{si}$ for any $i \in I \cup \{0\}$, and $\theta_i = T_{\omega_i}$ for any $i \in I$. The group $B_{\hat{W}}$ acts on $U$ by algebra automorphisms. Let $T_i$ be the operator denoted by $T'_{i,1}$ in [13, §37.1.3]. If $i \neq j$ we have

$$T_i(e_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q^{-s} e_i (-a_{ij} - s) e_i^{(s)}, \quad T_i(e_i) = -f_i k_i,$$

$$T_i(f_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q^{s} f_j^{(s)} f_i f_i^{-a_{ij} - s}, \quad T_i(f_i) = -k_i^{-1} e_i.$$
We have also $T_i(k_j) = k_j k_i^{-a_{ij}}$ for all $i, j$.

For a future use, we introduce the following notations:

- let $\sigma$ be the automorphism of $\hat{B}_W$ such that $\sigma(T_w) = T_w^{-1}$ for all $w \in \hat{W}$;
- for any $\alpha \in \hat{Q}$, let $U_\alpha \subset U$ be the subset of the elements $x$ such that $k_i x k_i^{-1} = q^{(\alpha, \alpha)} x$ for all $i$;
- for any $i \in I$ let $i \in I$ be the unique element such that $w_0(\alpha_i) = -\alpha_i$;
- let $(, ): \hat{P} \times \hat{P} \to \hat{Q}$ be the pairing such that $(\omega_i, \alpha_j) = \delta_{ij}$;
- for any $\alpha \in \hat{Q}$ we set $|\alpha|^2 = \sum_i (\omega_i, \alpha)^2$.

3.2. Let $\gamma s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression for the element $\omega_i \in \hat{W}$. Set

$$\gamma_i = \sum_{\ell=1}^k \gamma s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell}) \in \hat{P}.$$ 

**Lemma.** We have $(\gamma_i, \alpha_i) = -c$.

**Proof.** Let $\Delta_\pm \subset \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the sets of positive and negative roots. Let $\hat{\Delta}_\pm \subset \Delta + \mathbb{Z}\delta$ be the sets of positive and negative affine roots. We put $\Delta = \Delta_+ \cup \Delta_-$, $\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_-$ and $\hat{\Delta}(\omega_i) = \hat{\Delta}_+ \cap \omega_i(\Delta_-)$. Then,

$$\gamma_i = \sum_{\beta \in \hat{\Delta}(\omega_i)} \beta.$$ 

Recall that

$$\hat{\Delta}_+ = \Delta_+ \cup \bigcup_{n \geq 1} (n\delta + \Delta), \quad \hat{\Delta}_- = \Delta_- \cup \bigcup_{n \geq 1} (-n\delta + \Delta),$$

and that $\omega_i(\alpha) = \alpha - (\omega_i, \alpha)\delta$ for all affine root $\alpha$. Thus,

$$\hat{\Delta}(\omega_i) = \{ \alpha - (n - a_i)\delta \mid \alpha \in \Delta_-, a_i > n \geq 0 \},$$

where we set $a_i = - (\omega_i, \alpha)$. Thus,

$$\gamma_i = \sum_{\alpha \in \Delta_-} a_i (\alpha + \frac{1 + a_i}{2}\delta).$$

Let $\kappa$ be the Killing form. We get

$$(\gamma_i, \alpha_i) = - \sum_{\alpha \in \Delta_-} (\omega_i, \alpha) \cdot (\alpha_i, \alpha)$$

$$= -\kappa(\omega_i, \alpha_i)/2$$

$$= -c,$$

see [6, Exercice 6.2].

We fix the Drinfeld generators of $U$ in such a way that

$$(3.2.1) \quad x_{ir}^- = c_i^r \theta_i^r(f_i), \quad x_{ir}^+ = c_i^r \theta_i^{-r}(e_i),$$

where $c_i = \pm 1$ and $c_i + c_j = 0$ if $a_{ij} < 0$, see [2, Definition 4.6]. Note that there is exactly two choices for the map $i \mapsto c_i$. A case-by-case computation shows that the integer $a_i a_j$ does not depend on $i$: it is equal to $(-1)^e$. 

\[\square\]
Proposition.

1. There are unique $\mathbb{A}$-algebra automorphisms $A, B : U \to U$ such that

\[
A(x^\pm_{ir}) = -q^{\mp 1}x^\pm_{ir}, \quad B(x^\pm_{ir}) = -x^\pm_{ir}k_i, \quad B(x^-_{ir}) = -k^-_{ir}x^-_{ir}.
\]

2. We have $\tau(x^\pm_{ir}) = x^\pm_{i-r}, \tau(k^\pm_{ir}) = k^\pm_{i-r}$.

3. We have $\psi(x^\pm_{ir}) = q^{-re}T_{\omega_0}A(x^\pm_{i-r}), x^\pm_{ir} = q^{re}T_{\omega_0}B(x^-_{ir})$.

Proof. Claim 2 is known, see [2]. Claim 1 is a consequence of the identities 3. Let us prove 3. Let $\overline{T}_i$, $\overline{T}'_i$ be the automorphisms of the algebra $U$ such that $\psi(T_i(x)) = \overline{T}_i(x)$, $\overline{T}'_i(x) = \overline{T}_i(x)$ for all $x \in U$. By [13, §37] we have $\overline{T}'_i = T''_{i-1}$. A case-by-case computation gives also $\overline{T}'_i = T''_{i-1}$. If $x \in U_\alpha$, $\alpha \in Q$, we have

\[
T''_{i-1}(x) = (-q)^{-(\alpha,\alpha)}T^{-1}_i(x)
\]

for all $i$, see [13, §37]. Thus,

\[
\overline{T}'_i(x) = \overline{T}_i(x) = (-q)^{-(\alpha,\alpha)}\sigma(\theta_i)(x)
\]

where $\beta_i = \alpha_{ik} + s_{ik}(\alpha_{ik}) + \cdots + s_{ik} \cdots s_{ij}(\alpha_{i})$. Note that $\alpha, \gamma_i = -\alpha, \beta_i$ since $\gamma_i = -\omega_i(\beta_i)$. The weight $\omega_i$ being dominant we have $T_{\omega_0}, T_{\omega_i} = T_\omega T_{\omega_0}$, i.e. $T_{\omega_0}\sigma(\theta_i) = T^{-1}_i$. Recall that

\[
T_{\omega_0}(e_i) = -f_i k_i, \quad T_{\omega_0}(f_i) = -k^-_i e_i, \quad T_{\omega_0}(k_i) = k^-_i, \quad \forall i \neq 0.
\]

Note that $\theta_i(k_i) = k_i$, see [2]. Using (3.2.1) we get

\[
(x^\pm_{ir}) = o_i(-q)^{re}\sigma(\theta_i)^{-\tau}(e_i) = -o_i(-q)^{re}T_{\omega_0}\sigma(\theta_i)^{-\tau}(e_i) = -q^{-re}T_{\omega_0}k^-_i x^-_{ir}.
\]

Similarly we have

\[
\psi(x^\pm_{ir}) = o_i(-q)^{re}\sigma(\theta_i)^{-\tau}(qk_i f_i) = -o_i(-q)^{re}T_{\omega_0}\sigma(\theta_i)^{-\tau}(q^{-1}e_i) = -q^{-1}-reT_{\omega_0}(x^\pm_{ir}).
\]

The case of $x^-_{ir}$ is identical.

\[\square\]

4. Reminder on quiver varieties

4.1. Let the couple $(J, H)$ denote the quiver such that $J$ is the set of vertices, $H$ is the set of arrows. If $h \in H$ let $h', h'' \in J$ be the incoming and the outcoming vertex of $h$. Let $\overline{h}$ denote the arrow opposite to $h$. We will consider the following cases:

- $\Pi = (I, H)$ where $I$ is as in 2.1 and $H$ is such that there are $2\delta_{ij} - a_{ij}$ arrows from $i$ to $j$ for all $i, j$. Then, let $\Omega \subset H$ be any set such that $H = \Omega \cup \overline{\Omega}$. Let $n_{ij}$ (resp. $\bar{n}_{ij}$) be the number of arrows in $\Omega$ (resp. $\overline{\Omega}$) from $i$ to $j$. Note that $n_{ij} = \bar{n}_{ij}$.

- Fix a set $I^1$ with a bijection $I \to I^1$, $i \to i^1$. The quiver $\Pi' = (I^1, H')$ is such that $I^1 = I \cup I^1$, $H' = H \cup \{i \to i^1, i^1 \to i | i \in I\}$.  

4.2. Fix $V = \bigoplus_{i \in I} \mathbb{C}^{a_i}$, $W = \bigoplus_{i \in I} \mathbb{C}^{b_i}$.

Convention. Fix $(m_i) \in \mathbb{Z}^I$. Hereafter let $\mu, \lambda, \alpha$ denote elements in $P$, $P^+$, $Q^+$ respectively such that $\mu = \sum_i m_i \omega_i$, $\lambda = \sum_i \ell_i \omega_i$, $\alpha = \sum_i a_i \alpha_i$. The dimension of the graded vector space $V$ is identified with the root $\alpha$ while the dimension of $W$ is identified with the weight $\lambda$.

The space

$$M_{\lambda \alpha} = \bigoplus_{h \in H} M_{a_i, a_{i'}^\vee} (\mathbb{C}) \bigoplus \bigoplus_{i \in I} (M_{a_i, \ell_i} (\mathbb{C}) \bigoplus M_{\ell_i, a_i} (\mathbb{C}))$$

is identified with the set of representations of the quiver $\Pi^e$ on $V \oplus W$. For any $(B, p, q) \in M_{\lambda \alpha}$ let $B_h$ be the component of the element $B \in \text{Hom}(V_h^\vee, V_h)$ and set

$$m_{\lambda \alpha} (B, p, q) = \sum_{h \in H} \varepsilon(h) B_h B_h^\vee + pq \in \bigoplus_{i} \text{Hom} (V_i, V_i),$$

where $\varepsilon$ is a function $\varepsilon : H \to \mathbb{C}^\times$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$. Put $G_{\lambda} = \prod_i \text{GL}_{\ell_i}$, $G_{\alpha} = \prod_i \text{GL}_{a_i}$. The group $\mathbb{C}^\times \times G_{\lambda} \times G_{\alpha}$ acts on $M_{\lambda \alpha}$ by

$$(z, g_{\lambda}, g_{\alpha}) \cdot (B, p, q) = (z g_{\lambda} B g_{\lambda}^{-1}, z g_{\lambda} p g_{\lambda}^{-1}, z g_{\lambda} q g_{\lambda}^{-1}).$$

Following [19], we consider the varieties

$$Q^{(\mu)}_{\lambda \alpha} = \text{Proj} \left( \bigoplus_{n \geq 0} A_n^{(\mu)} \right) \quad \text{and} \quad N_{\lambda \alpha} = m_{\lambda \alpha}^{-1}(0) / G_{\alpha},$$

where $/ \! /$ is the categorical quotient,

$$A_n^{(\mu)} = \left\{ f \in \mathbb{C}[m_{\lambda \alpha}^{-1}(0)] \mid f \left( g_{\alpha} \cdot (B, p, q) \right) = \chi_{\mu} (g_{\alpha})^{-n} f(B, p, q) \right\},$$

and $\chi_{\mu} (g_{\alpha}) = \prod_i \text{Det} (g_{\alpha_i})^{m_i}$. The obvious projection $\pi_{\lambda \alpha} : Q^{(\mu)}_{\lambda \alpha} \to N_{\lambda \alpha}$ is a projective map. If $\mu, \mu'$ are such that $m_i, m_i' > 0$ for all $i$, or $m_i, m_i' < 0$ for all $i$, then the varieties $Q^{(\mu)}_{\lambda \alpha}, Q^{(\mu')}_{\lambda \alpha}$ are canonically isomorphic. There is an open subset $m_{\lambda \alpha}^{-1}(0)^{\mu} \subset m_{\lambda \alpha}^{-1}(0)$ whose points are called $\mu$-semistable, such that there is a good quotient of $m_{\lambda \alpha}^{-1}(0)^{\mu}$ by the group $G_{\alpha}$ and we have

$$m_{\lambda \alpha}^{-1}(0)^{\mu} / G_{\alpha} = Q^{(\mu)}_{\lambda \alpha},$$

see [18, §1.7] for instance. Moreover, if $\mu$ is a regular weight then (4.2.1) is a geometric quotient and the variety $Q^{(\mu)}_{\lambda \alpha}$ is smooth, see [21, Proposition 2.6]. If $\mu$ is regular dominant, i.e. if $m_i > 0$ for all $i$, we set $Q_{\lambda \alpha} = Q^{(\mu)}_{\lambda \alpha}$.

Convention. Hereafter, we assume that $(\mu, \alpha) \neq 0$ for any root $\alpha$.

4.3. Put $d_{\lambda \alpha} = \text{dim} Q_{\lambda \alpha}$. It is known that $d_{\lambda \alpha} = (\alpha, 2\lambda - \alpha)$. If $\alpha \geq \beta$ the extension by zero of representations of the quiver gives a closed embedding $N_{\lambda \beta} \hookrightarrow N_{\lambda \alpha}$. For any $\alpha, \alpha'$, we consider the fiber product

$$Z_{\lambda \alpha \alpha'} = Q_{\lambda \alpha} \times_{\pi} Q_{\lambda \alpha'}. $$
If $\alpha' = \alpha + n\alpha$, $n > 0$, let $X_{\lambda\alpha\alpha'} \subset Z_{\lambda\alpha\alpha'}$ be the set of pairs $(x, x')$ which are the $G_{\alpha'}$-orbits of $V^*\text{-modules } y, y'$ in $M_{\lambda\alpha}, M_{\lambda\alpha'}$ with $y$ a subrepresentation of $y'$. If $\alpha' = \alpha - n\alpha$, put $X_{\lambda\alpha\alpha'} = \phi(X_{\lambda\alpha'}) \subset Z_{\lambda\alpha\alpha'}$, where $\phi$ is the automorphism of $Q_{\lambda} \times Q_{\lambda}$ taking an element $(x, y)$ to $(y, x)$. The variety $X_{\lambda\alpha\alpha'}$ is smooth, see [20, §5.3]. Consider the following varieties

$$N_{\lambda} = \bigcup_{\alpha} N_{\lambda\alpha}, \quad Q_{\lambda} = \bigcup_{\alpha} Q_{\lambda\alpha}, \quad Z_{\lambda} = \bigcup_{\alpha, \alpha'} Z_{\lambda\alpha\alpha'}, \quad X_{\lambda} = \bigcup_{\alpha, \alpha'} X_{\lambda\alpha\alpha'}, \quad F_{\lambda} = \bigcup_{\alpha} F_{\lambda\alpha},$$

where $\alpha, \alpha'$ take all the possible values in $Q^+$ and $F_{\lambda\alpha} = \pi_{\lambda\alpha}^{-1}(0)$.

**4.4.** For any complex algebraic linear group $G$, and any quasi-projective $G$-variety $X$ let $K^G(X)$ be the Grothendieck group of $G$-equivariant coherent sheaves on $X$. We put $R^G = K^G(\text{point})$. Let $X^G \subset R^G$ be the set of the simple modules. If the $G$-equivariant sheaf $\mathcal{E}$ is locally free, let $\wedge^i \mathcal{E}$ is its $i$-th wedge power, and $\wedge_{\mathcal{E}}$ be its maximal wedge power. Note that $\wedge_{\mathcal{E}}$ is still defined, in the obvious way, whenever $\mathcal{E}$ is a $G$-equivariant complex on $X$.

**Convention.** Hereafter, let $f_*, f^*, \otimes$, denote the derived functors $Rf_*, Lf^*, \otimes^L$ when they exist. Here $\otimes$ is the tensor product of coherent sheaves. We use the same notation for a sheaf and its class in the Grothendieck group.

**4.5.** Set $\tilde{G}_{\lambda} = G_{\lambda} \times \mathbb{C}^\times$. Let $q$ denote also the character of the group $\mathbb{C}^\times$ such that $z \mapsto z$. The canonical bundle of the variety $Q_{\lambda\alpha}$ is

$$(4.5.1) \quad \Omega_{Q_{\lambda\alpha}} = q^{-d_{\lambda\alpha}},$$

see [24, §6.4] for instance. Let $V_i, W_i$ be the vectorial representations of the groups $GL_{\lambda_i}, \text{GL}_{\epsilon_i}$. Consider the following elements in $R^{\tilde{G}_{\lambda} \times G_{\alpha}}$

$$F_i^+ = q^{-1}V_i - q^{-2}V_i + q^{-1} \sum_{a_i = -1} V_j, \quad F_i^- = -V_i, \quad F_i = F_i^+ + F_i^-.$$

The group $\tilde{G}_{\lambda}$ acts on the variety $Q_{\lambda}^{(\mu)}$. If $E$ is a $\tilde{G}_{\lambda} \times G_{\alpha}$-module, let $E^{(\mu)} = m^{-1}_{\lambda\alpha}(0)^{\mu} \times_{G_{\alpha}} E$ be the induced $\tilde{G}_{\lambda}$-bundle on $Q_{\lambda}^{(\mu)}$. There is a unique ring homomorphism

$$R^{\tilde{G}_{\lambda} \times G_{\alpha}} \rightarrow K^{\tilde{G}_{\lambda}}(Q_{\lambda}^{(\mu)})$$

such that $E \mapsto E^{(\mu)}$ for all $E$. If $\mu$ is dominant we set $V_i = V_i^{(\mu)}$ and similarly for $W_i, F_{i}^{\pm}, F_i$. We set also $\mathcal{V} = \bigoplus_{i} V_i, \mathcal{W} = \bigoplus_{i} W_i$.

**Convention.** The restriction to $Q_{\lambda\alpha}^{(\mu)}$ of a sheaf $\mathcal{E}$ on $Q_{\lambda}^{(\mu)}$ is denoted by $\mathcal{E}_{\alpha}$. For simplicity we set $F_{i; \alpha} = (F_i)_{\alpha}, \text{etc.}$

**4.6.** Consider the map

$$\dagger : M_{\lambda\alpha} \rightarrow M_{\lambda\alpha}, \quad (B, p, q) \mapsto (B, p, q)^\dagger = (p^t, q, t^q),$$

where the upperscript $t$ stands for the transpose map. Note that $\dagger$ does not commute to the action of the group $\tilde{G}_{\lambda} \times G_{\alpha}$. Let $\dagger$ be the group automorphism of $\tilde{G}_{\lambda} \times G_{\alpha}$ such that

$$\dagger : (z, g_{\lambda}, g_{\alpha}) \mapsto (z, g_{\lambda}, g_{\alpha})^\dagger = (z, t g_{\lambda}^{-1}, t g_{\alpha}^{-1}).$$
Then \((g \cdot x)^\dagger = g^\dagger \cdot x^\dagger\) for all \(g \in \tilde{G}_\lambda \times G_\alpha\), \(x \in M_\lambda\). The induced map \(\dagger : Q^{(\mu)}_{\lambda_\alpha} \rightarrow Q^{(-\mu)}_{\lambda_\alpha}\) is an isomorphism of algebraic varieties. Let

\[
\dagger : R^{\tilde{G}_\lambda \times G_\alpha} \rightarrow R^{\tilde{G}_\lambda \times G_\alpha}, \ E \mapsto E^\dagger
\]

be the ring automorphism induced by the group automorphism \(\dagger\). For any element \(E \in R^{\tilde{G}_\lambda \times G_\alpha}\), let \((E^{(\mu)})^\dagger \in K^{\tilde{G}_\lambda}(Q^{(\mu)}_{\lambda_\alpha})\) be the pull-back of \(E^{(-\mu)} \in K^{\tilde{G}_\lambda}(Q^{(-\mu)}_{\lambda_\alpha})\) by the automorphism \(\dagger\). We have

\[
(E^{(\mu)})^\dagger = (E^{\dagger})^{(\mu)}.
\]

For any \(w \in W\) we set \(w \ast \alpha = \lambda - w(\lambda) + w(\alpha)\). The element \(w \ast \alpha\) depends on the weight \(\lambda\). However, since \(\lambda\) is fixed in the whole paper the notation \(w \ast \alpha\) should not make any confusion. There is \(\tilde{G}_\lambda\)-equivariant isomorphism of algebraic varieties \(S_w : Q^{(\mu)}_{\lambda_\alpha} \rightarrow Q^{(\lambda,w \ast \alpha)}_{\lambda_\alpha}\) for each \(w\), such that

\[
S_i^2 = 1\quad \text{and}\quad S_{ww'} = S_w S_{w'}\quad \text{if} \quad l(ww') = l(w) + l(w'),
\]

see [15], [18], [21] (for simplicity we set \(S_i = S_{s_i}\), where \(s_i\) is the simple reflexion with respect to the root \(\alpha_i\)). The precise definition of \(S_w\) is given in the proof of Lemma 4.6. Consider the composed map \(\omega = S_{w_0} \dagger\). This choice is motivated by [16] and [21, Theorem 11.7]. The map \(\omega\) is an isomorphism of algebraic varieties \(Q_{\lambda_\alpha} \simto Q_{\lambda_{w_0 \ast \alpha}}\).

**Lemma.**

1. We have \(\omega^*(F_i) = -q^c F_1^\dagger\), \(\omega^*(W_i) = W_1^\dagger\) and

\[
\sum_j [a_{ij}][\omega^*(V_j) + q^c V_j^\dagger] = W_1^\dagger + q^c W_1^\dagger.
\]

2. We have \(\omega^2 = Id\).

3. We have \(\omega(F_{\lambda_\alpha}) = F_{\lambda_{w_0 \ast \alpha}}\).

**Proof.** We use the construction of the operator \(S_w\) given in [18], see also [15]. Let us recall it briefly. Set \(\alpha' = s_i \ast \alpha\), \(\mu' = s_i(\mu)\), \(\mu' = \sum_i m'_i \omega_i\). Let first assume that \(m_i < 0\). Then \(m'_i > 0\). Following [15, §3.2], let

\[
Z_i^\mu \subset m_{\lambda_\alpha}^{-1}(0)^{(\mu)} \times m_{\lambda_\alpha'}^{-1}(0)^{(-\mu')}
\]

be the set of pairs \((x, x')\), where \(x = (B, p, q), x' = (B', p', q')\) are such that

- the sequence of \(\tilde{G}_\lambda \times G_\alpha\)-modules

\[
0 \rightarrow q^{-2}V_i^{a(x')} \rightarrow q^{-1}W_i \oplus q^{-1} \bigoplus_{a_{ij} = -1} V_j^{b(x)} V_i \rightarrow 0
\]

such that \(a(x') = (q', B')\), \(b(x) = p_i + \varepsilon h B_h\) is exact,

- we have \(a(x)b(x) - a(x')b(x') = 0\),

- we have \(B_h = B'_h\) if \(h, h' \neq i\), and \(p_j = p'_j\), \(q_j = q'_j\) if \(j \neq i\).
Note that \(a(x')\) is injective and \(b(x)\) is surjective, see [18, Lemma 38]. Thus \(Z_i^\mu\) is a closed subset of \(m_{i\alpha}^{-1}(0)^\mu \times m_{i\alpha'}^{-1}(0)^{\mu'}\). Consider the group \(G_{\alpha\alpha'} = GL_{a_i} \times GL_{a_i'} \times \prod_{j \neq i} GL_{a_j}\). The categorical quotient

\[
Q_i^\mu = Z_i^\mu \sslash G_{\alpha\alpha'},
\]

is a smooth variety. Moreover, the obvious projections are isomorphisms of algebraic varieties

\[
P_{1,\alpha}^{(\mu)} : Q_i^\mu \xrightarrow{\sim} Q_{\alpha\alpha}^{(\mu)}, \quad P_{2,\alpha'}^{(\mu')} : Q_i^\mu \xrightarrow{\sim} Q_{\alpha'\alpha'}^{(\mu')},
\]

see [18, Proposition 40]. The group \(\tilde{G}_\Lambda\) acts in the obvious way on \(Q_i^\mu\), making the maps \(P_{1,\alpha}^{(\mu)}, P_{2,\alpha'}^{(\mu')}\) equivariant. By construction, for any \(i \neq j\) we have

\[
\begin{align*}
(p_{1,\alpha}^{(\mu)})^* (F_i^{(\mu)} + q^{-2} V_i^{(\mu)}) &= (p_{2,\alpha'}^{(\mu')})^* (q^{-2} V_i^{(\mu')}), \\
(p_{1,\alpha}^{(\mu)})^* (V_j^{(\mu')}) &= (p_{2,\alpha'}^{(\mu')})^* (V_j^{(\mu')}).
\end{align*}
\]

We set (recall that \(m_i < 0\))

\[
S_i = P_{2,\alpha'}^{(\mu')} (P_{1,\alpha}^{(\mu)})^{-1} : Q_{\alpha\alpha}^{(\mu)} \to Q_{\alpha'\alpha'}^{(\mu')}.
\]

If \(m_i > 0\) we set

\[
S_i = P_{1,\alpha}^{(\mu')} (P_{2,\alpha'}^{(\mu)})^{-1} : Q_{\alpha\alpha}^{(\mu)} \to Q_{\alpha'\alpha'}^{(\mu')}.
\]

Using (4.6.2) we get, if \(m_i < 0\) and \(i \neq j\),

\[
S_i^* (V_i^{(\mu')}) = q^2 F_i^{(\mu)} + V_i^{(\mu)}, \quad S_i^* (V_j^{(\mu')}) = V_j^{(\mu)}.
\]

Note that the map \(S_i\) commutes to the action of the group \(\tilde{G}_\Lambda\). Thus,

\[
S_i^* (W_j^{(\mu)}) = W_j^{(\mu')}
\]

for all \(j\). Set \(\varepsilon_i = +1\) if \(m_i > 0\), \(\varepsilon_i = -1\) if \(m_i < 0\). A case-by-case analysis gives the following equalities in \(K^{\tilde{G}_\Lambda}(Q_\Lambda^{(\mu')})\)

\[
S_i^* (F_j^{(\mu)}) = \begin{cases} 
-q^{2\varepsilon_i} F_j^{(\mu')} & \text{if } i = j \\
F_j^{(\mu')} & \text{if } a_{ij} = 0 \\
F_j^{(\mu')} + q^{\varepsilon_i} F_i^{(\mu')} & \text{if } a_{ij} = -1.
\end{cases}
\]

The general formula is

\[
(4.6.3) \quad S_i^* (F_j^{(\mu)}) = F_j^{(\mu')} - q^{\varepsilon_i} [a_{ij}] F_i^{(\mu')}.
\]

We now assume that the weight \(\mu\) is dominant. Thus, \(F_j^{(\mu)} = F_j\). Fix an element \(w\) in the Weyl group. Let us prove that

\[
(4.6.4) \quad w(\alpha_j) = \alpha_j \Rightarrow S_w^* (F_j) = q^{a(w, i)} F_i^{(w^{-1}(\mu))},
\]
where
\[ a(w, i) = \frac{1}{2} \sum_{\alpha \in \Delta_+ \cap w^{-1} \Delta_-} (\alpha_i, \alpha)^2. \]

We may assume that \( l(w) > 0 \) and that (4.6.4) holds for any \( x \) with \( l(x) < l(w) \). Fix \( k \in I \) such that \( w(\alpha_k) \in -Q^+ \). Let \( \langle s_i, s_k \rangle \) be the subgroup generated by \( s_i, s_k \). Let \( x \) be the element of minimal length in the set \( w(\langle s_i, s_k \rangle) \). Then, \( x(\alpha_i), x(\alpha_k) \in Q^+ \).

One of the following two cases holds, see [12, Proof of Proposition 1.8].

- Either \( a_{ik} = 0, w = xs_k, l(w) = l(x) + 1 \). Then, \( x(\alpha_i) = \alpha_j \). Using (4.6.4) for \( x \), and (4.6.3), we get
  \[ S_w^*(F_{\alpha_i}) = q^{a(x, i)} S_k^*(F_{\alpha_j}^{(x^{-1}(\mu))}) = q^{a(x, i)} F_{\alpha_j}^{(\mu)}. \]

Using the identity
\[ \Delta_+ \cap w^{-1} \Delta_- = s_k(\Delta_+ \cap x^{-1} \Delta_-) \cup \{ \alpha_k \} \]
we get also \( a(w, i) = a(x, i) \). Thus (4.6.4) holds.

- Either \( a_{ik} = -1, w = x s_i s_k, l(w) = l(x) + 2 \). Then, \( x(\alpha_k) = \alpha_j \). Using (4.6.4) for \( x \) we get
  \[ S_w^*(F_{\alpha_j}) = q^{a(x, k)} S_k^* S_i^*(F_{\alpha_j}^{(x^{-1}(\mu))}). \]

We are reduced to the \( A_2 \) case. Set \( \nu = x^{-1}(\mu) \). We have \( w^{-1}(\mu) = s_k s_i(\nu) < s_i(\nu) < \nu \). A direct computation using (4.6.3) gives
\[ S_k^* S_i^*(F_{\nu}^{(\mu)}) = q F_{\nu}^{(s_k s_i(\nu))}. \]

Using the identity
\[ \Delta_+ \cap w^{-1} \Delta_- = s_k s_i(\Delta_+ \cap x^{-1} \Delta_-) \cup \{ \alpha_k, \alpha_i + \alpha_k \} \]
we get also \( a(w, i) = a(x, k) + 1 \). Thus (4.6.4) holds.

Setting \( w, i, j \rightarrow w_0 s_i s_j \) in (4.6.4) and using the formula for \( S_{w_0}^* \), we get \( S_{w_0}^*(F_i) = -q^{a(w_0 s_i s_j \mu) + 2} F_{\mu}^{(w_0(\mu))}. \) Thus
\[ \omega^*(F_i) = -q^{a(w_0 s_i s_j \mu)} F_{\mu}^{(w_0(\mu))}. \]  

(4.6.5)

see (4.6.1). Moreover we have, see 3.2,
\[ a(w_0 s_i s_j) = \frac{1}{2} \sum_{\alpha \in \Delta_+ \setminus \langle \alpha_j \rangle} (\alpha_i, \alpha)^2 = \frac{1}{2} \kappa(\alpha_i, \alpha_j) - 2 = c - 2. \]

By definition we have \( q F_i = W_i - \sum_j [a_{ij}] V_j \). Using (4.6.1), (4.6.5) and the equality \( a_{ij} = a_{ij}^\dagger \) we get the identity
\[ \sum_j [a_{ij}] \omega^*(V_j) = W_i^\dagger + q^c W_i^\dagger - q^c \sum_j [a_{ij}] V_j^\dagger. \]
Claim 1 is proved.

From the definition of the operator $S_i$ we get $\dagger S_i = S_i \dagger$ for all $i$. Claim 2 follows immediately.

We now prove Claim 3. Using Claim 2 it is sufficient to prove that $\omega(F_{\lambda\alpha}) \subseteq F_{\lambda_wu\alpha}$. Assume that $\alpha' = s_i \alpha$, $\mu' = s_i(\mu)$ as above. It is sufficient to prove that $S_i(F_{\lambda\alpha}^{(\mu)}) \subseteq F_{\lambda\alpha}^{(\mu')}$. By [10, §1.3] the ring of $G_\lambda$-invariant polynomials on $m_{\lambda\alpha}^{-1}(0)$ is generated by the following two types of functions:

(i) $\text{tr}_V(B_{h_1} B_{h_2} \cdots B_{h_n})$ for any sequence $h_1, h_2, \ldots, h_n \in H$ such that $j = h'_1$, $h''_1 = h'_{n-1} = h'_n, h''_n = j$.

(ii) $\varphi(q_j B_{h_1} B_{h_2} \cdots B_{h_n} p_k)$ for any sequence $h_1, h_2, \ldots, h_n \in H$ such that $j = h'_1, h''_1 = h'_2, h''_2 = h'_3, h''_3 = \cdots = h'_n, h''_n = k$, and any linear form $\varphi$ on $\text{Hom}(W_k, W_j)$.

We may assume that $m_i < 0$. Fix an element $(x, x')$ in $Z_i^{\mu}$. Set $x = (B_h, p_j, q_j)$, $x' = (B'_h, p'_j, q'_j)$. In particular we have

$$B_h = B'_h \text{ if } h', h'' \neq i; \quad B_h B_{h_k} = B'_h B'_k \text{ if } h''_1 = h'_2 = i.$$ 

Thus any function of type (i) coincide on $x$ and $x'$. We have also

$$q_j = q'_j, \quad p_j = p'_j \text{ if } j \neq i; \quad q_i p_i = q'_i p'_i;$$

$$q_i B_h = q_i' B'_h \text{ if } h' = i; \quad B_h p_i = B'_h p'_i \text{ if } h'' = i.$$ 

Thus any function of type (ii) coincide on $x$ and $x'$. In particular $x \in F_{\lambda\alpha}^{(\mu)}$ iff $x' \in F_{\lambda\alpha}^{(\mu')}$. We are done.

Remark. The dual of the $\tilde{G}_\lambda$-bundle $E^{(\mu)}$ on $Q^{(\mu)}$ is $(E^*)^{(\mu)}$, where $E^*$ is the dual module, obtained by composing the $\tilde{G}_\lambda$-action by the group automorphism $(z, g_\lambda, g_\alpha) \mapsto (z^{-1}, g_\lambda^{-1}, g_\alpha^{-1})$. Note that, in the particular case where $E = V_i, W_i$ we have $\mathcal{V}_i^* = \mathcal{V}_i, \mathcal{W}_i^* = \mathcal{W}_i$.

Convention. Put $1_{\lambda\alpha} = \mathcal{O}_{F_{\lambda\alpha}}, 1'_{\lambda\alpha} = \mathcal{O}_{Q_{\lambda\alpha}}$. It is convenient to set also $1_{\lambda} = 1_{\lambda0}, 1'_{\lambda} = 1'_{\lambda0}$. To simplify the notations we put $\nu = w_0 \circ$. 

5. The involution on the convolution algebra

5.1. Given smooth quasi-projective $G$-varieties $X_1, X_2, X_3$, consider the projection $p_{ab} : X_1 \times X_2 \times X_3 \to X_a \times X_b$ for all $1 \leq a, b \leq 3$, $a \neq b$. Fix closed subvarieties $Z_{ab} \subseteq X_a \times X_b$ such that the restriction of $p_{13}$ to $p_{12}^{-1} Z_{12} \cap p_{23}^{-1} Z_{23}$ is proper and maps to $Z_{13}$. The convolution product is the map

$\ast : \mathbf{K}^G(Z_{12}) \times \mathbf{K}^G(Z_{23}) \to \mathbf{K}^G(Z_{13}), \quad (E, \mathcal{F}) \mapsto p_{13\ast}((p_{12}^* E) \otimes (p_{23}^* \mathcal{F})).$

If $Z_{12} = Z_{23} = Z_{13} = Z$, the map $\ast$ endows $\mathbf{K}^G(Z)$ with the structure of an algebra. See [4] for more details.
5.2. Let $D_{X_a}$ be the Serre-Grothendieck duality operator on $K^G(X_a)$. Assume that $X_a$ is connected. Let $\Omega_{X_a}$ be the canonical bundle of $X_a$, and let $\mathcal{O}_{X_a}$ be the structural sheaf. We have

$$D_{X_a}(\mathcal{E}) = (-1)^{\dim X_a} \mathcal{E}^* \otimes \Omega_{X_a}$$

for any $G$-equivariant locally free sheaf $\mathcal{E}$ on $X_a$. Assume that there is a character $q$ of the group $G$ such that $\Omega_{X_a} = q^{-\dim X_a}$ for all $a$. Consider the operator $D_{Z_{ab}} = q^{d_{ab}} D_{Z_{ab}}$, where $d_{ab} = (\dim X_a + \dim X_b)/2$. Recall that the automorphism $\phi : X_a \times X_b \to X_b \times X_a$ is the flip.

**Lemma.** Fix $x \in K^G(Z_{12})$, $y \in K^G(Z_{23})$.

1. $\phi^*(x \star y) = \phi^*(y) \star \phi^*(x)$, $D_{Z_{12}}(x) \star D_{Z_{23}}(y) = D_{Z_{12}}(x \star y)$, $\phi^* D_{Z_{ab}} = D_{Z_{ab}} \phi^*$.
2. If $Z_{12} = Z_{23} = Z_\lambda$ then $(\omega \times \omega)^*(x) \star (\omega \times \omega)^*(y) = (\omega \times \omega)^*(x \star y)$.
3. If $Z_{12} = Z_\lambda$, $Z_{23} = Q_\lambda$ or $F_\lambda$ then $(\omega \times \omega)^*(x) \star \omega^*(y) = \omega^*(x \star y)$.

See [14] for more details.

5.3. We consider the maps $\gamma_\lambda$, $\gamma'_\lambda$, $\Gamma_\lambda$, $\zeta_\lambda$ on $K^{G_\lambda}(F_\lambda)$, $K^{G_\lambda}(Q_\lambda)$, $K^{G_\lambda}(Z_\lambda)$ such that

$$\gamma_\lambda = \bigoplus q^{d_{ab}/2} \omega^* \mathcal{D}_{F_{ab}}, \quad \gamma'_\lambda = \bigoplus q^{d_{ab}/2} \omega^* \mathcal{D}_{Q_{ab}},$$

$$\Gamma_\lambda = \bigoplus (\omega \times \omega)^* D_{Z_{ab}}, \quad \zeta_\lambda = (\omega \times \omega)^* \phi^*$$

(see Lemma 4.6.3 for $\gamma_\lambda$). Let

$$\gamma : R^{\tilde{G}_\lambda} \to R^{\tilde{G}_\lambda}, \quad V \mapsto \tilde{V}$$

be the ring automorphism induced by the group automorphism $\tilde{G}_\lambda \to \tilde{G}_\lambda$, $(z, g_\lambda) \mapsto (z^{-1}, g_\lambda)$. By Lemma 4.6.1 the operators $\omega^*$, $\zeta_\lambda$ are $\dagger$-semilinear automorphisms of $R^{\tilde{G}_\lambda}$-modules, and $\gamma_\lambda$, $\gamma'_\lambda$, $\Gamma_\lambda$ are $\dagger$-semilinear. Let $\kappa : F_\lambda \mapsto Q_\lambda$ be the closed embedding.

**Lemma.** The following identities hold:

1. $\omega^* \mathcal{D}_{F_\lambda} = \mathcal{D}_{F_\lambda} \omega^*$, $\omega^* \mathcal{D}_{Q_\lambda} = \mathcal{D}_{Q_\lambda} \omega^*$, $\omega^* D_{Z_\lambda} = D_{Z_\lambda} \omega^*$, $(\omega \times \omega)^* \phi^* = (\omega \times \omega)^* \phi^*$.
2. $\kappa_* \omega^* = \omega^* \kappa_*$, $(\omega \times \omega)^* \phi^* = (\omega \times \omega)^* \phi^*$.
3. $\gamma_\lambda(u \star x) = \Gamma_\lambda(u) \star \gamma_\lambda(x)$, for any $x \in K^{G_\lambda}(F_\lambda)$, $u \in K^{G_\lambda}(Z_\lambda)$.
4. $\gamma'_\lambda(u \star x) = q^{d_{ab} - d_{ab}} \Gamma_\lambda(u) \star \gamma'_\lambda(x)$, for any $x \in K^{G_\lambda}(Q_{ab'})$, $u \in K^{G_\lambda}(Z_{ab'})$.

5.4. Let $A_{\lambda a''}$ be the quotient of the $R^{\tilde{G}_\lambda}$-module $K^{G_\lambda}(Z_{\lambda a''})$ by its torsion submodule. We set $A_\lambda = \bigoplus_{\alpha, a'} A_{\lambda a''}$. Setting $Z_{12} = Z_{23} = Z_\lambda$ in 5.1, we get an associative product on the space $A_\lambda$. The rings $R^{G_\lambda}$, $A$ are identified as in 4.5.

An $A$-algebra homomorphism $\Phi_\lambda : U \to A_\lambda$ is given in [20]. In this subsection we fix a particular normalization for $\Phi_\lambda$. Let $\delta : Q_\lambda \mapsto Q_\lambda \times Q_\lambda$ be the diagonal embedding, and let $p, p' : Q_\lambda \times Q_\lambda \to Q_\lambda$ be the first and the second projection. Let $f_{i\alpha}, f_{i\alpha}^\pm, v_{i\alpha}$ be the ranks of $F_{i\alpha}$, $F_{i\alpha}^\pm, V_{i\alpha}$. We have

$$f_{i\alpha} = (\alpha_i, \lambda - \alpha), \quad f_{i\alpha}^- = -v_{i\alpha} = -(\omega_i, \alpha).$$

Set $t_\alpha = (\alpha, 2\lambda - \alpha)/2 + |\alpha|^2$, and

$$r_{i\alpha}^+ = (\lambda, \alpha_i) - (\omega_i - \sum_j n_{ij} \omega_j, \alpha), \quad r_{i\alpha}^- = - (\omega_i - \sum_j n_{ij} \omega_j, \alpha).$$
Let $1_{\lambda\alpha\alpha'} \in A_\lambda$ be the class of the structural sheaf of $X_{\lambda\alpha\alpha'}$. For any $r \in \mathbb{Z}$ we put
\[
x_{ir}^+ = q^{(1-e)r} \sum_{\alpha' = \alpha + \alpha_i} (-1)^{i_{\alpha'}}(q^{-1} \Lambda_{iv}^{-1} \otimes \Lambda_{iv}^{-1})^{r+f_{i\alpha'}} \otimes p'^* \Lambda_{iv}^{-1} \otimes \Lambda_{iw}^{-t_{\alpha}} \otimes 1_{\lambda\alpha\alpha'},
\]
\[
x_{ir}^- = q^{(1-e)r} \sum_{\alpha' = \alpha - \alpha_i} (-1)^{i_{\alpha'}}(q^{-1} \Lambda_{iv}^{-1} \otimes \Lambda_{iv}^{-1})^{r+f_{i\alpha'}} \otimes p'^* \Lambda_{iv}^{-1} \otimes \Lambda_{iw}^{-t_{\alpha}} \otimes 1_{\lambda\alpha\alpha'}.
\]
Let also $k_{ir}^\pm(x)$ be the expansion at $z = \infty$ or 0 of
\[
\delta_\lambda \sum_{\alpha} q^{f_{i\alpha}} \left( \sum_{r \geq 0} (-q^{-e}/z)^r \otimes \Lambda_{iv}^{-1} \otimes \Lambda_{iw}^{-t_{\alpha}} \otimes 1_{\lambda\alpha\alpha'} \right)^{-1}.
\]
The map $\Phi_\lambda$ takes $x_{ir}^\pm$ to $x_{ir}^\pm$, and $k_{ir}^\pm$ to $k_{ir}^\pm$. For a future use, let us mention the following.

**Lemma.** For any $n > 0$ we have
\[
(x_{ir}^\pm)^{(n)} = \pm \sum_{\alpha' = \alpha + \alpha_i} (q^{-n} \Lambda_{iv}^{-1} \otimes \Lambda_{iv}^{-1})^{f_{i\alpha'}} \otimes p'^* \Lambda_{iv}^{-n} \otimes \Lambda_{iw}^{-t_{\alpha}} \otimes 1_{\lambda\alpha\alpha'},
\]
and similarly for $(x_{ir}^-)^{(n)}$.

**Proof.** By the same argument as in [20, §11.1-3] it is enough to check this relation for type $A_1$. In this case, using the faithful representation introduced in [22] the formula follows from a direct computation: the formula for the action of the Weyl group of $\mathbb{W}$ identifies $H_\lambda$-modules with $\mathbb{W}$-modules.

**Convention.** Hereafter we may omit the maps $\delta_\lambda, p^*, p'^*, \otimes$, hoping that it makes no confusion.

5.5. Let $H_\lambda \subset G_\lambda$ be any closed subgroup. Set $H_\lambda = H_\lambda \times C^\times$. For simplicity we set $\mathbb{W}_{H_\lambda, \alpha} = K^H_\lambda(F_{\alpha})$, $\mathbb{W}_{H_\lambda, \alpha} = K^H_\lambda(Q_{\alpha})$, $\mathbb{W}_{H_\lambda} = \oplus_{\alpha} \mathbb{W}_{H_\lambda, \alpha}$, $\mathbb{W}_{H_\lambda} = \oplus_{\alpha} \mathbb{W}_{H_\lambda, \alpha}$. Taking $Z_{12} = Z_\lambda$, $X_3 = \{\text{point}\}$, and $Z_{13} = Z_{23} = Q_\lambda$ or $F_\lambda$ in 5.1 we get a left $U$-action on the $R^{H_\lambda}$-modules $\mathbb{W}_{H_\lambda}$, $\mathbb{W}_{H_\lambda}$ such that
\[
(u, x) \mapsto u \cdot x = \Phi_\lambda(u) \ast x.
\]
Taking $Z_{23} = Z_\lambda$, $X_1 = \{\text{point}\}$, and $Z_{12} = Z_{13} = Q_\lambda$ or $F_\lambda$ we get a right $U$-action on $\mathbb{W}_{H_\lambda}$, $\mathbb{W}_{H_\lambda}$ such that $x \cdot u = x \ast \Phi_\lambda(u) = \delta^* \Phi_\lambda(u) \ast x$, see Lemma 5.2. We fix a maximal torus $T_\lambda \subset G_\lambda$.

**Lemma.**
(1) The $R^{H_\lambda}$-modules $\mathbb{W}_{H_\lambda}$, $\mathbb{W}_{H_\lambda}'$ are free of finite type, and we have $\mathbb{W}_{H_\lambda} = \mathbb{W}_{G_\lambda} \otimes_{R_\lambda} R^{H_\lambda}$, $\mathbb{W}_{H_\lambda}' = \mathbb{W}_{G_\lambda}' \otimes_{R_\lambda} R^{H_\lambda}$. Moreover, there is a canonical action of the Weyl group of $G_\lambda$ on $\mathbb{W}_{T_\lambda}$, $\mathbb{W}_{T_\lambda}'$ such that the forgetful map identifies $\mathbb{W}_{G_\lambda}$, $\mathbb{W}_{G_\lambda}'$ with the subspaces of invariant elements in $\mathbb{W}_{T_\lambda}$, $\mathbb{W}_{T_\lambda}'$.

(2) We have $\mathbb{W}_{H_\lambda} = U \cdot (R^{H_\lambda} \otimes 1_\lambda)$.

(3) We have $\mathbb{W}_{G_\lambda} = U \cdot 1_\lambda$. 

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Proof. Claim 1 is proved in [20, Theorem 7.3.5]. See also [4, Chapter 5]. Claim 2 is proved as in [20, Proposition 12.3.2]. Let us prove that, if \( H_\lambda = G_\lambda \), then \( R^{G_\lambda} \otimes 1_\lambda \in U \cdot 1_\lambda \). By construction \( k^\pm_r(z) \cdot 1_\lambda \) is the expansion of

\[
q^{k^0_r} \left( \sum_{r \geq 0} (-1/qz)^r \wedge^r W_i \right) \otimes \left( \sum_{r \geq 0} (-1/qz)^r \wedge^r W_i \right)^{-1}
\]

in \( R^{G_\lambda}[[z^{-1}]] \). Thus, the elements \( p_{i,s} \in R^{GL_i} \) are the elementary symmetric polynomials or zero. Claim 3 follows. \( \square \)

Convention. Although most of our constructions are meaningful for any closed subgroup \( H_\lambda \subseteq G_\lambda \), hereafter \( H_\lambda \) will be either \( G_\lambda \) or \( T_\lambda \). For simplicity, if \( H_\lambda = G_\lambda \) we put \( W_\lambda = W_{G_\lambda} \), \( W'_\lambda = W'_{G_\lambda} \), \( R_\lambda = R^{G_\lambda} \), \( X_\lambda = X^{G_\lambda} \).

5.6. The same construction as in 5.5 yields also a left and a right action of \( A_\lambda \) on \( W_{H_\lambda} \), \( W'_{H_\lambda} \). Since \( W_{H_\lambda} \), \( W'_{H_\lambda} \) are integrable left \( U \)-modules, they admit a left \( \hat{R}^\lambda \)-linear action of the group \( B_W \). We normalize this action in such a way that the element \( T_\ell \in B_W \) acts as the operator \( T_v^n \) in [13, §5.2.1]. Similarly let \( T_w \) be the left action of the element \( T_w \in B_W \) associated to the right \( U \)-action.

Proposition.

1. There is a unique action of the group \( B_W \) on \( A_\lambda \) by \( \hat{R}^\lambda \)-algebra automorphisms such that

\[
T_w(x \star y) = T_w(x) \star T_w(y), \quad \forall x \in A_\lambda, y \in W_{H_\lambda} \text{ or } W'_{H_\lambda}, \quad w \in W.
\]

2. We have \( T_w \Phi_\lambda = \Phi_\lambda T_w \), and \( \check{T}_w(y \star x) = \check{T}_w(y) \star \sigma(T_w)(x) \) for any \( x, y \) as above.

3. There is an invertible element \( r_\lambda \in \hat{R}^\lambda \) such that \( T_{w_\lambda}(1_{\lambda,w}) = r_\lambda \otimes 1_{\lambda,w} \),

\[
T_{w_\lambda}(1_{\lambda,w}) = r_\lambda \otimes 1_{\lambda,w}.
\]

4. There is an invertible element \( s_\lambda \in \hat{R}^\lambda \) such that \( T_{w_\lambda}(1_{\lambda,w}) = s_\lambda \otimes 1_{\lambda,w} \),

\[
T_{w_\lambda}(1_{\lambda,w}) = s_\lambda \otimes 1_{\lambda,w}.
\]

5. There is an invertible element \( \vartheta \in A \) such that \( r_\lambda \otimes 1_{\lambda,w} = \vartheta \otimes A_{w} \otimes \bigotimes_i \Lambda_{1_{\lambda,w}}^{x_i} \). Moreover \( r_\lambda s_\lambda = (-q)^{q/2} \otimes \vartheta \) and \( \vartheta = \pm q^{1/2} \otimes (\lambda,\lambda)^{1/2} \).

Proof. Let us prove Claim 1. We fix elements \( x \in I \), \( \lambda \in P^+ \). Let \( U'_i \subset U' \) be the subalgebra generated by \( e_i, f_i, k_i^{\pm 1} \). The proof uses an element \( t_i \) introduced in [17]. The element \( t_i \) belongs to a ring completion of \( U'_i \), is invertible, and satisfies the following identity

\[
t_i \cdot x \cdot t_i^{-1} = T_{i,1}(x), \quad \forall x \in U.
\]

Let us recall the construction of \( t_i \), following [5].

For any \( Q(q) \)-vector space \( V \) we set \( V^* = \text{Hom}_{Q(q)}(V, Q(q)) \). Let \( R'_i \subset U'^*_i \) be the \( Q(q) \)-space spanned by the matrix elements of the finite dimensional \( U'_i \)-modules. It is a Hopf algebra. Let \( x_{0} x_{1} \) denote the image of the element \( x \in R'_i \) by the coproduct. The space \( R'^*_i \) is a ring such that \( (f \cdot g)(x) = \sum f(x_0) g(x_1) \) for all \( x \in R'_i \). The canonical map \( U'_i \rightarrow R'^*_i \) is a ring homomorphism. An integrable \( U' \)-module \( V \) restricts to an integrable \( U'^*_i \)-module via the canonical embedding \( U'_i \subset U' \). It is also a \( R'^*_i \)-comodule for the co-action \( V \rightarrow R'_i \otimes V \), \( v \rightarrow \sum v_1 \otimes v_2 \).
such that \( x \cdot v = \sum v_1(x) v_2 \) for all \( x \in U'_t \). Therefore the ring \( R^*_t \) acts on \( V \) by \( f \cdot v = \sum f(v_1) v_2 \). This action restricts to the original \( U'_t \)-action via the map \( U'_t \to R^*_t \). Let \( \tau_i \in R^*_t \) be the element denoted by \( i \) in [5, §1.6]. It is invertible. Let \( t_i \) be the operator on \( V \) taking \( v \) to \( \tau_i \cdot v \). It is invertible and the inverse takes \( v \) to \( \tau_i^{-1} \cdot v \). We have

\[
t_i(v) = T''_{i,-1}(v), \quad t_i x t_i^{-1}(v) = T'''_{i,-1}(x) \cdot v,
\]

for all \( x \in U' \) and \( v \in V \), by [5, §2]. Therefore

\[
\tau_i \cdot x \cdot \tau_i^{-1} = T''_{i,-1}(x), \quad \forall x \in U',
\]

where we take the product in the ring \( R^*_t \) in the left hand side.

For any \( n \in \mathbb{N} \) let \( \Lambda_i(n) \) be the simple \( U'_i \)-module with highest weight \( n\omega_i \). Let \( R^*_i \subset R^*_t \) be the subspace spanned by matrix elements of the module \( \bigoplus_{n \leq n'} \Lambda_i(n') \). It is a subcoalgebra. Let \( I'_n \subset U'_t \) be the annihilator of \( \bigoplus_{n \leq n'} \Lambda_i(n') \). It is a two-sided ideal. The canonical map \( U'_i \to R^*_i \) factorizes through an isomorphism \( U'_i/I'_n \cong (R^*_i)^{\ast} \). Since \( R^*_i = \lim_{\leftarrow n} R'_i \), we get a \( \mathbb{Q}(q) \)-algebra isomorphism

\[
\lim_{\leftarrow n} (U'_i/I'_n) \cong R^*_i.
\]

For each \( n \) we choose \( \tau_{in} \in U'_t \) such that

\[
\tau_{in} - \tau_i \in \lim_{\leftarrow n' \geq n} (I'_n/I'_n')
\]

Since \( U'_i \) embeds in \( U' \), the space \( A_\Lambda \otimes A \mathbb{Q}(q) \) is a \( U'_i \otimes \mathbb{Q} H_\Lambda \)-bimodule of finite type over \( R^H_\Lambda \otimes A \mathbb{Q}(q) \). In particular there is an integer \( n \) such that the ideal \( I'_n \) acts trivially on \( A_\Lambda \otimes \mathbb{Q} H_\Lambda \). Fix such an integer \( n \). Then the operator \( T''_{i,-1} \) acts on \( A_\Lambda \otimes \mathbb{Q} H_\Lambda \) via the conjugation by the element \( \Phi_\Lambda(\tau_{in}) \in A_\Lambda \otimes \mathbb{Q} H_\Lambda \). Moreover, the formulas in [5, §2] imply that the left and right product by \( \Phi_\Lambda(\tau_{in}) \), \( \Phi_\Lambda(\tau_{in}^{-1}) \) preserves the subspace \( A_\Lambda \subset A_\Lambda \otimes \mathbb{Q} H_\Lambda \).

Recall that \( T_i \) acts as Lusztig's operator \( T''_{i,1} \). The element \( \tau'_i \) yielding the action of \( T''_{i,1} \) on \( A_\Lambda \) can be constructed as \( \tau_i \), using the identity

\[
T''_{i,1}(x) = (-q)_{(\alpha,\alpha_i)}(T'''_{i,-1})^{-1}(x)
\]

for any \( x \in U_\alpha \), see [13, §37.2.4].

Recall that \( N_\Lambda \) is a cone over the point \( 0 \) equal to the class of the trivial representation, and that the fixed points subset \( (N_\Lambda)^{C_\ast} \) is reduced to \( \{0\} \). Hence \( A_\Lambda \otimes \mathbb{Q} H_\Lambda \) coincides with the tensor product \( (W_G_{\Lambda} \otimes R^H_\Lambda W_G_{\Lambda}) \otimes \mathbb{Q} H_\Lambda \) by the Kunneth isomorphism, see [4, 5.6] and [20, §7]. Since \( A_\Lambda \) is torsion free over \( A \), it embeds in \( A_\Lambda \otimes \mathbb{Q} H_\Lambda \). Then, a standard argument implies that \( W_G_{\Lambda} \) is a faithful \( A_\Lambda \)-module, see [4, §5] for more details. For the same reason \( W_{H_\Lambda} \) is a faithful \( A_\Lambda \)-module. The unicity in Claim 1 follows.

Claim 2 is obvious from the previous construction. Claims 3,4 are obvious either since \( T'''_{i,0} \) is an invertible \( R^H_{\Lambda} \)-linear homomorphism from \( W_{H_\Lambda,0} \) to \( W_{H_\Lambda,\nu} \), and both \( R^H_{\Lambda} \)-modules are free of rank one, generated by \( 1'_\Lambda, 1'_{\Lambda,\nu} \) respectively.
Let us prove Claim 5. Part two of the claim follows from [11, 5.4.(a) and Corollary 5.9] (note that the formula for $s'$ in [11, 5.4.(a)] should be replaced by $s' = \sum_{i,j} a_{ij}t_it_j/2 - \sum_i t_i(1 + d_i)$). Let us prove Part one. Consider an element $w \in W$ such that $l(w) > 0$. Fix $i \in I$ such that $l(s_iw) = l(w) - 1$. Put $w' = s_iw$. Set $\alpha' = w' * \nu$, $\alpha = w * \nu$, $n = (ww_0(\lambda), \alpha_i)$. Thus $n > 0$ and $\alpha = \alpha' - n\alpha_i$. We have

$$T_i(1'_{\lambda\alpha'}) = e_i^{(a)}(1'_{\lambda\alpha'})$$

see [11, Lemma 5.6]. Let $r_w \in R_{H^\alpha}$ be the unique element such that $T_w(1'_{\lambda\alpha'}) = r_w1_{\lambda\alpha}$. The varieties $Q_{\lambda\alpha}$, $Q_{\lambda\alpha'}$ are reduced to a point. Thus, using Lemma 5.4 we get

$$T_w(1'_{\lambda\alpha'}) = \pm(q^{-n}\Lambda_{\nu}^{-1}\Lambda_{V_{\nu}}) f_{\nu}^{n} \Lambda_{F_{\nu}}^{-1}\Lambda_{W_{\nu}}^{t_{\nu} - t_{\nu}}(r_{w'}1'_{\lambda\alpha'}).$$

The classes of the $H_{\alpha'}$-equivariant sheaves $V_{\nu_{\alpha}}, V_{\nu_{\alpha'}}, F_{\nu_{\alpha}}^+$ are identified with elements of $R_{H^\alpha}$ in the obvious way. Let first assume that we have

$$\Lambda_{F_{\nu_{\alpha}}}^+ = \Lambda_{V_{\nu_{\alpha}}} \in R_{H^\alpha}.$$ 

Then

$$r_w = \pm q^{|s|} \Lambda_{W_{\nu_{\alpha}}}^{-t_{\mu}} \Lambda_{V_{\nu_{\alpha}}}^{t_{\mu} - t_{\alpha}} r_{w'},$$

where $s = (\alpha' - \alpha, \bar{\nu})(\alpha', \bar{\nu})$.

By induction on $l(w)$ we get

$$r_w = \pm q^{|s|} \Lambda_{W_{\nu_{\alpha}}}^{t_{\mu} - t_{\alpha}} \prod_j \Lambda_{V_{\nu_{\alpha}}}^{-t_{\mu}} \Lambda_{V_{\nu_{\alpha}}}^{t_{\mu} - t_{\alpha}}.$$ 

Setting $w = w_0$ in (5.6.2) we get $r_{\lambda} = \vartheta \Lambda_{W_{\nu_{\alpha}}}^{t_{\mu} - t_{\alpha}} \prod_j \Lambda_{V_{\nu_{\alpha}}}^{-t_{\mu}} \Lambda_{V_{\nu_{\alpha}}}^{t_{\mu} - t_{\alpha}}$, with $\vartheta \in \pm q^{|s|}$. A direct computation gives $\vartheta \in \pm q^{|s|}$, where

$$t = |n|^2/2 + \sum_{\alpha \in \Delta_{+}} (\lambda, \alpha)^2/2$$

$$= |n|^2/2 + c(\lambda, \lambda)/2,$$

see [6, Exercice 6.2].

Finally, we prove (5.6.1). We have an isomorphism of $G_{\lambda}$-varieties $Q_{\lambda\alpha}^{(\mu)} \rightarrow Q_{\lambda\alpha'}^{(\mu')}$, see the proof of Lemma 4.6 and the notations therein. This isomorphism takes $V_i^{(\mu)}$ to $(F_i^+)^{(\mu')}$. Assume that $\mu$ is regular dominant, so that $Q_{\lambda\alpha}^{(\mu)} = Q_{\lambda\alpha}$ and $V_i^{(\mu)} = V_i^{(\mu)}$. Since the $G_{\lambda}$-variety $Q_{\lambda\alpha}$ is reduced to a point, it is canonically isomorphic to $Q_{\lambda\alpha}^{(\mu)}$, and the isomorphism takes $F_{\nu_{\alpha}}$ to $(F_i^+)^{(\mu')}$. □

5.7. For each $i, \alpha$ we consider the elements

$$g_{i,\alpha} = -1 + (c - 1)f_{\nu_{\alpha}}^+ + f_{\nu_{\alpha}}^+ - c f_{\nu_{\alpha}}^{-}\in \mathbb{Z},$$

$$h_{i,\alpha} = r_{\nu_{\alpha}}^+ + d_{\nu_{\alpha}} - a_{\nu_{\alpha}} + r_{\nu_{\alpha}}^{-}\in \mathbb{Z}/2\mathbb{Z}.$$ 

Convention. The elements $r_{\nu_{\alpha}}^+$ depend on the choice of the orientation $\Omega$. Hereafter we assume that $n_{ij} = \bar{n}_{ij}$ for all $i, j$ if $c$ is even, and $n_{ij} = \bar{n}_{ij}$ for all $i, j$ if
c is odd (i.e. if g is of type $A_{2n}$). The existence of such an orientation is checked case-by-case.

Using the convention above, a direct computation gives

$$g_{i;\alpha + \alpha_j} + g_{j;\alpha} = g_{j;\alpha + \alpha_i} + g_{i;\alpha}, \quad h_{i;\alpha + \alpha_j} + h_{j;\alpha} = h_{j;\alpha + \alpha_i} + h_{i;\alpha}$$

for all $i, j$. Thus there are unique quadratic maps $x : Q \to \mathbb{Z}$, $y : Q \to \mathbb{Z}/2\mathbb{Z}$ such that

$$x(\nu) = (c - 1)|\nu|^2 - c(\lambda, \lambda), \quad x(\alpha + \alpha_i) - x(\alpha) = cv_{i,\nu} - g_{i;\alpha},$$

(5.7.1) \quad $$y(\nu) = 0, \quad y(\alpha + \alpha_i) - y(\alpha) = h_{i;\alpha},$$

for all $\alpha, i$. Put $\xi(\alpha) = q^{\alpha(x)}$, $g(\alpha) = (-1)^{y(\alpha)}$, $t_{\alpha\alpha'} = t_{w_0*\alpha} - t_{\alpha'} - t_{\alpha_0*\alpha} + t_{\alpha}$. Consider the element

$$c_{\lambda\alpha} = g(\alpha)\xi(\alpha)q^{-\alpha(\alpha)}\Lambda_{W_i}^{\lambda_0} \otimes \bigotimes_i \left(\Lambda_{V_i,\alpha}^{2\nu_i,\nu_i} \otimes \Lambda_{V_i,\alpha}^{\nu_i,\nu_i} \otimes V_i^{\alpha,\alpha'}\right) \otimes r_{\lambda} \in W_{H_{\lambda,\alpha}}^\alpha.$$

Put $c_{\lambda} = \sum_{\alpha} c_{\lambda\alpha}$.

**Lemma.**

1. If $\alpha' = \alpha - \alpha_j$ then the restriction of $c^{-1}_\lambda \otimes c_{\lambda}$ to $X_{\lambda\alpha\alpha'}$ is

$$(-1)^{h_{j,\alpha'}}q^{g_{j,\alpha'}}(\Lambda_{V_i} \otimes \Lambda_{V_i}^{-1})^{-f_{j,\alpha'}}^{-f_{j,\alpha'}}\Lambda_{W_i} \otimes \Lambda_{V_i}^{\alpha,\alpha'} \otimes p^{\nu_i}_{\alpha,\alpha'} \otimes V_i^{\alpha,\alpha'}.$$

2. We have $\omega^*(c_{\lambda\alpha}) = c_{\lambda,\alpha_0*\alpha}$, $c_{\lambda\nu} = r_{\lambda}^{-1} \otimes 1_{\nu}^\alpha$.

**Proof.** Fix $\alpha, \alpha'$ such that $\alpha' = \alpha - \alpha_j$. For any $i$ we consider the following elements in $W_{H_{\lambda}}$:

$$U_i = q^\nu \omega^*(V_i^\alpha) + V_i, \quad c_{\lambda} = \bigotimes_i \Lambda_{V_i}^\nu \otimes V_i^\nu.$$

The rank of $U_{i;\alpha}$ is $v_{i;\alpha} + v_{i;w_0*\alpha} = v_{i;\nu}$. From Lemma 4.6.1 we have

$$\sum_j [a_{ij}]U_{j;\alpha} = q^\nu W\alpha_i + W_{i;\alpha}.$$

The quantum Cartan matrix (i.e. the $I \times I$-matrix whose $(i, j)$-th entry is $[a_{ij}]$) is invertible over $\mathbb{Q}(q)$. Thus, for any $\alpha, \alpha', i$ we get

(5.7.2) \quad $$(1_{\lambda\alpha} \otimes U_{i;\alpha'})|_{X_{\lambda\alpha\alpha'}} = (U_{i;\alpha} \otimes 1_{\alpha\alpha'})|_{X_{\lambda\alpha\alpha'}} \in K_{H\lambda}(X_{\lambda\alpha\alpha'}).$$

We have $c_{\lambda} = \bigotimes_i \Lambda_{V_i}^\nu \otimes U_i$ and $F_i^\nu = -V_i$. Thus, using (5.7.2) we get

$$(c_{\lambda,\alpha}^{-1} \otimes c_{\lambda}^\alpha)|_{X_{\lambda\alpha\alpha'}} = (\Lambda_{V_i} \otimes \Lambda_{V_i}^{-1})^{\nu_i,\nu_i}|_{X_{\lambda\alpha\alpha'}} \otimes p^{\nu_i}_{\alpha,\alpha'} \otimes \Lambda_{F_i}^{\alpha,\alpha'}.$$

Note that

$$c_{\lambda\alpha} = \xi(\alpha)\phi(\alpha)r_{\lambda} \otimes c_{\lambda\alpha}^\nu \bigotimes_i \Lambda_{V_i}^{2\nu_i,\nu_i} \otimes \Lambda_{V_i}^{\nu_i,\nu_i}.$$
Thus Claim 1 follows, using (5.7.1) and the identity \( v_{j,\nu} = f^{+}_{j,\alpha'} + f^{+}_{j,\mu_{0} * \alpha'} \). Let us prove Claim 2. By definition of \( c'_{\lambda} \) we have \( \omega^* (c'_{\lambda}) = c'_{\alpha, w_0 * \alpha} \) and \( c'_{\lambda 0} = 1'_{\lambda} \). Thus, using Proposition 5.6.4 we get

\[
c_{\lambda \nu} = \vartheta^{-2} r_{\lambda} \otimes \Lambda_{W}^{-2t_{\nu}} \otimes \bigotimes_{i} \Lambda_{V_{i,\nu}}^{2t_{\nu}} = r_{\lambda}^{-1} \otimes 1'_{\lambda \nu}.
\]

Since \( Q_{\lambda \nu} \) is a point, we identify the equivariant sheaf \( \Lambda_{V_{i,\nu}} \) with an element in \( R^\lambda \) in the obvious way. Using (5.7.2) we get

\[
\bigotimes_{i} \Lambda_{q^{-\epsilon/2}\omega^* (V_{i,\nu}) + q^{\epsilon/2} V_{i,\nu}^{\ast} \otimes_{w_0 * \alpha}}^{2t_{\nu}} = \left( \bigotimes_{i} \Lambda_{q^{-\epsilon/2}\omega^* (V_{i,\nu}) + q^{\epsilon/2} V_{i,\nu}^{\ast}}^{2t_{\nu}} \right) \otimes 1'_{\lambda, w_0 * \alpha} = \vartheta^{-2} q^{-|\nu|^2 e_{r_{\lambda}}^{-1}} \otimes \Lambda_{W}^{-2t_{\nu}} \otimes 1'_{\lambda, w_0 * \alpha}.
\]

Thus, \( \omega^* (r_{\lambda}) = q^2 q^{-|\nu|^2 e_{r_{\lambda}}^{-1}} \) and

\[
\omega^* \left( r_{\lambda} \otimes \Lambda_{W}^{t_{\nu}} \otimes \bigotimes_{i} \Lambda_{q^{-\epsilon/2} V_{i,\nu}}^{2t_{\nu}} \right) = q^2 q^{-|\nu|^2 e_{r_{\lambda}}^{-1}} \otimes \Lambda_{W}^{t_{\nu}} \otimes \bigotimes_{i} \Lambda_{q^{-\epsilon/2} V_{i,\nu}}^{2t_{\nu}}.
\]

A direct computation (see the Appendix) shows that \( \xi (\alpha) = \xi (w_0 * \alpha) \), \( q (\alpha) = 1' (w_0 * \alpha) \) for all \( \alpha \). We are done. \( \square \)

Let \( C_{\lambda} \) be the \( R_{\lambda} \)-linear automorphism of \( A_{\lambda} \) such that

\[
C_{\lambda} (x) = x \otimes (c_{\lambda} \boxtimes c_{\lambda}^{-1}).
\]

**Proposition.**

1. The map \( C_{\lambda} \Gamma_{\lambda} \) is an algebra involution of \( A_{\lambda} \) such that \( C_{\lambda} \Gamma_{\lambda} (q) = q^{-1} \), \( C_{\lambda} \Gamma_{\lambda} (x_{\lambda}^{\pm}) = q^{c} x_{\lambda}^{\mp} \).

2. The map \( C_{\lambda} \zeta_{\lambda} \) is an algebra anti-involution of \( A_{\lambda} \) such that \( C_{\lambda} \zeta_{\lambda} (q) = q \), \( C_{\lambda} \zeta_{\lambda} (x_{\lambda}^{\pm}) = q^{-c} x_{\lambda}^{\mp} \).

**Proof:** The variety \( X_{\lambda \alpha'} \) is smooth of dimension \( d_{\lambda \alpha} := (d_{\lambda} + d_{\lambda \alpha'}) / 2 \). Let \( \Omega_{\lambda \alpha'} \) be its canonical bundle. If \( \alpha' = \alpha + \alpha_i \), using (4.5.1), Lemma 4.6.1 and Remark 4.6 we get

\[
(5.7.3) \quad \Omega_{\lambda \alpha'} = q^{f_{i,\alpha'} - d_{\lambda \alpha'} (q^{-1} \Lambda_{V_{i,\alpha}} \boxtimes \Lambda_{V_{i,\alpha}'}) f_{i,\alpha'} \otimes p^{\ast} \Lambda_{X_{i,\alpha}}^{-1};
\]

\[
(5.7.4) \quad d_{\lambda \alpha'} - d_{\lambda \alpha'} + f_{i,\alpha'} = -1;
\]

\[
(5.7.5) \quad (\omega \times \omega)^\ast (\Lambda_{V_{i,\alpha}} \boxtimes \Lambda_{V_{i,\alpha}'}^{-1}) = q^{-1} \Lambda_{W_{\mu_0 * \alpha}} \boxtimes \Lambda_{V_{i,\alpha}'}^{-1};
\]

Using (5.7.3-4) we get

\[
D_{X_{i,\alpha}} (x_{\lambda}^{\pm}) = \sum_{\alpha' = \alpha + \alpha_i} (-1)^{i_{i,\alpha'}^{\ast} + d_{\lambda \alpha'} (q^{-1} \Lambda_{V_{i,\alpha}} \boxtimes \Lambda_{V_{i,\alpha}'})^{-r} f_{i,\alpha'} \otimes p^{\ast} \Lambda_{X_{i,\alpha}}^{-1} \otimes \Lambda_{W_{\mu_0 * \alpha}} \otimes \Omega_{\lambda \alpha'};
\]

\[
= \sum_{\alpha' = \alpha + \alpha_i} (-1)^{i_{i,\alpha'}^{\ast} + d_{\lambda \alpha'} (q^{-1} \Lambda_{V_{i,\alpha}} \boxtimes \Lambda_{V_{i,\alpha}'})^{-r} f_{i,\alpha'}^{\ast} \otimes p^{\ast} \Lambda_{X_{i,\alpha}^{-1}} \otimes \Lambda_{W_{\mu_0 * \alpha}} \otimes 1_{\lambda \alpha'}.
\]
Thus, using (5.7.5) we get
\[ \Gamma_\lambda(x_i^+) = \sum_{\alpha''=\alpha-\alpha'} (-1)^{\beta''} q^{-1-r-1+e\ell_{i;w_{\alpha}}}(q\Lambda_{\alpha'} \otimes \Lambda_{\alpha''}) \otimes \Lambda_{\alpha-\alpha''} \otimes 1_{\lambda_\alpha''} \]
\[ = x_{i-} \sum_{\beta''} (-1)^{\beta''} q^{e_{\beta''-1+(e-2)\ell_{i;w_{\alpha}}}} (q^{-1}\Lambda_{\alpha'} \otimes \Lambda_{\alpha''}) \otimes \Lambda_{\alpha-\alpha''} \otimes 1_{\lambda_\alpha''} \]
\[ = q^e x_{i-} \sum_{\substack{\beta'' \\ \alpha, \alpha''}} (-1)^{\beta''} q^{e_{\beta''-1+(e-2)\ell_{i;w_{\alpha}}}} (q^{-1}\Lambda_{\alpha'} \otimes \Lambda_{\alpha''}) \otimes \Lambda_{\alpha-\alpha''} \otimes 1_{\lambda_\alpha''} \]
where \( e_{\beta''} = \ell_{i;w_{\alpha}} + d_{i;w_{\alpha}} \), and \( g_{i;\alpha}, h_{i;\alpha} \) are as at the beginning of 5.7.

Using Lemma 5.7.1 we get
\[ C_\lambda \Gamma_\lambda(x_i^+) = q^e x_{i-} \]

Using Lemma 5.2.1 and Lemma 5.3.1-2 we get
\[ \zeta_\lambda = \Gamma_\lambda \phi^* D_{Z_\lambda} = \phi^* D_{Z_\lambda} \Gamma_\lambda, \quad C_\lambda \phi^* D_{Z_\lambda} = \phi^* D_{Z_\lambda} C_\lambda. \]

Thus, \((C_\lambda \Gamma_\lambda)^2 = (C_\lambda \zeta_\lambda)^2\). Using Lemma 5.7.2 we get
\[ (C_\lambda \Gamma_\lambda)^2 = (C_\lambda \zeta_\lambda)^2 = \text{Id}. \]

Thus, \( C_\lambda \Gamma_\lambda(x_i^+) = q^e x_{i-}^+ \) either. Recall that \( \phi^* D_{Z_\lambda} \phi_\lambda = \phi_\lambda \tau \), see [24, Lemma 6.5]. Then Claim 2 follows from Proposition 3.2.2 and (5.7.6).

5.8. Let \( A_\lambda, B_\lambda : A_\lambda \to A_\lambda \) be the \( \mathbf{R}_\lambda \)-algebra automorphisms such that
\[ A_\lambda(x) = (-q)^{(\rho,\alpha-\alpha')} x, \quad B_\lambda(x) = (-q)^{(\rho,\alpha-\alpha')} q^{(\alpha'-\alpha,2\lambda-\alpha'-\alpha)/2} x, \]
for any element \( x \in A_{\lambda_\alpha''} \). Then
\[ \Phi_\lambda A = A_\lambda \Phi_\lambda, \quad \Phi_\lambda B = B_\lambda \Phi_\lambda. \]

We consider the automorphisms \( \beta_{Z_\lambda}, \psi_{Z_\lambda} \) of the ring \( A_\lambda \) such that
\[ \beta_{Z_\lambda} = T_{w_0} B_\lambda C_\lambda \Gamma_\lambda, \quad \psi_{Z_\lambda} = T_{w_0} A_\lambda C_\lambda \zeta_\lambda. \]

Corollary.

1. The map \( \beta_{Z_\lambda} \) is \( \dagger \)-semilinear, the map \( \psi_{Z_\lambda} \) is \( \dagger \)-semilinear. Moreover, we have \( \beta_{Z_\lambda}^2 = \psi_{Z_\lambda}^2 = \text{Id} \).

2. For any \( u \in U \) we have \( \Phi_\lambda (\bar{u}) = \beta_{Z_\lambda} \Phi_\lambda (u), \Phi_\lambda \psi(u) = \psi_{Z_\lambda} \Phi_\lambda (u) \).

Proof. From Proposition 5.7.2 the map \( C_\lambda \zeta_\lambda \) is an antihomomorphism of \( A_\lambda \) such that \( q \mapsto q, \; x_i^+ \mapsto x_i^+ \) for all \( i \). Thus, using [13, §37] we get
\[ A_\lambda C_\lambda \zeta_\lambda = C_\lambda \zeta_\lambda A_\lambda, \quad T_{w_0} A_\lambda^{-1} = A_\lambda T_{w_0}, \quad T_{w_0} C_\lambda \zeta_\lambda = C_\lambda \zeta_\lambda T_{w_0}^{-1}. \]

Thus \( \psi_{Z_\lambda} \) is an idempotent. From Proposition 5.7.1 and [13, §37] we get also
\[ B_\lambda C_\lambda \Gamma_\lambda T_{w_0} = C_\lambda \Gamma_\lambda T_{w_0} B_\lambda^{-1}, \quad T_{w_0} C_\lambda \Gamma_\lambda = C_\lambda \Gamma_\lambda T_{w_0}^{-1}. \]
Thus \( \beta_{Z_\lambda} \) is an idempotent. Claim 2 is immediate.
6. The metric and the involution on standard modules

6.1. Set \( a_{\lambda \alpha} = (-q)^{(\rho, \alpha)}, b_{\lambda \alpha} = (-q)^{(\rho, -\lambda)}q^{-d_{\lambda \alpha}/2} \). Let \( a_\lambda, b_\lambda \) be the automorphisms of the \( R^{\hat{H}_\lambda} \)-module \( W_{H_\lambda} \) (resp. \( W'_{H_\lambda} \)) such that \( a_\lambda(x) = a_{\lambda \alpha}x, b_\lambda(x) = b_{\lambda \alpha}x \) for any element \( x \in W_{\lambda \alpha} \) (resp. \( x \in W'_{\lambda \alpha} \)). Using (5.8.1) we get

\[
  b_\lambda(u \cdot x) = B(u) \cdot b_\lambda(x), \quad a_\lambda(u \cdot x) = A(u) \cdot a_\lambda(x).
\]

Let \( \beta_\lambda, \beta'_\lambda \) be the automorphisms of \( W_{H_\lambda}, W'_{H_\lambda} \) respectively such that

\[
  \beta_\lambda = T_{u_0}b_\lambda c_\lambda \gamma_\lambda, \quad \beta'_\lambda = T_{u_0}b_\lambda c_\lambda \gamma'_\lambda.
\]

**Proposition.**

1. We have \( \beta_\lambda(u \cdot x) = \beta_{Z_\lambda}(u) \beta_\lambda(x) \) for any \( u \in A_\lambda, x \in W_{H_\lambda} \).
2. We have \( \beta'_\lambda(u \cdot x) = q^{d_{\lambda, \alpha}} \beta_{Z_\lambda}(u) \beta'_\lambda(x) \), for any \( u \in K^{\hat{H}_\lambda}(Z_{\lambda \alpha} \alpha') \), 
   \( x \in W'_{H_\lambda} \).
3. The maps \( \beta_\lambda, \beta'_\lambda \) are \( \sim \)-semilinear. Moreover we have \( \beta_\lambda^2 = \text{Id}, \beta'_\lambda^2 = \text{Id} \).
4. We have \( \beta_\lambda(1_\lambda) = 1_\lambda, \beta'_\lambda(1'_\lambda) = 1'_\lambda \).

**Proof.** Claim 1 follows from Lemma 5.3.3. Claim 2 follows from Lemma 5.3.4. Claim 3 follows from Corollary 5.8.1 and the equality \( F_{\lambda \alpha} = Z_{\lambda \alpha 0} \). Using Lemma 5.7.2 we get

\[
  \gamma_\lambda(1_\lambda) = 1_{\lambda'}, \quad \gamma'_\lambda(1'_\lambda) = 1'_{\lambda'}, \quad b_{\lambda \nu} = 1, \quad c_{\lambda \nu} = r_{\lambda}^{-1}1'_{\lambda \nu}.
\]

Thus Claim 4 follows from Proposition 5.6.3. \( \square \)

**Remark.** For any closed subgroup \( H'_{\lambda'} \subset H_\lambda \), the forgetful maps \( W_{H_\lambda} \to W'_{H'_{\lambda'}}, W'_{H_\lambda} \to W'_{H'_{\lambda'}} \) commute with the involutions \( \beta_\lambda, \beta'_\lambda \).

6.2. For any \( \kappa \)-module \( M \), let \( \hat{M} \) be the set of formal series in \( q^{-1} \) with coefficients in \( M \). We get (see 5.5)

\[
  W_{H_\lambda} = W_{H_\lambda} \otimes_{R^{\hat{H}_\lambda}} \hat{R}, \quad W'_{H_\lambda} = W'_{H_\lambda} \otimes_{R^{\hat{H}_\lambda}} \hat{R}.
\]

Recall that if \( \lambda = \lambda_1 + \lambda_2 \) in \( P^+ \) then the direct sum of representations of the quiver \( \Pi^\kappa \) gives an embedding \( \varpi : Q_{\lambda_1} \times Q_{\lambda_2} \hookrightarrow Q_\lambda \). Fix a pair of ring isomorphisms

\[
  R^{T_{\lambda_1}} \simeq Z[x_{1}^{\pm 1}, \ldots, x_{\ell_1}^{\pm 1}], \quad R^{T_{\lambda_2}} \simeq Z[y_{1}^{\pm 1}, \ldots, y_{\ell_2}^{\pm 1}],
\]

We have \( R^{T_{\lambda}} \simeq R^{T_{\lambda_1}} \otimes R^{T_{\lambda_2}} \). Set

\[
  R_{\lambda_1/\lambda_2} = Z[[q^{-1}, y_{i}/x_{j}; i, j]] \otimes Z[q^{-1}, y_{i}/x_{j}; i, j] R^{T_\lambda},
\]

where \( 1 \leq i \leq \ell_1 \) and \( 1 \leq j \leq \ell_2 \). Recall that \( \kappa \) is the closed embedding \( F_\lambda \hookrightarrow Q_\lambda \).

**Lemma.**

1. The direct image map \( \kappa_* \) is an isomorphism \( \hat{W}_{H_{\lambda}} \simeq \hat{W}'_{H_{\lambda}} \).
2. Assume that \( \lambda = \lambda_1 + \lambda_2 \) in \( P^+ \). Then, there is a unique isomorphism of \( \hat{R}_{\lambda_1/\lambda_2} \otimes U \)-modules

\[
  \varpi_{\lambda_1/\lambda_2} : \hat{R}_{\lambda_1/\lambda_2} \otimes_{\hat{R}} (W_{T_{\lambda_1}} \otimes_{\hat{R}} W_{T_{\lambda_2}}) \simeq \hat{R}_{\lambda_1/\lambda_2} \otimes_{\hat{R}} W_{T_{\lambda}}
\]

such that \( 1_{\lambda_1} \otimes 1_{\lambda_2} \mapsto 1_\lambda \).
Proof. Let first prove Claim 1. Assume that $H_{\lambda} = T_{\lambda}$. We set $\ell = \sum \ell_i$. There is an isomorphism of rings $R^T_{\lambda} \cong \mathbb{Z}[z_1^{\pm 1}, \ldots, z_{\ell}^{\pm 1}]$. Fix $R^T_{\lambda}$-bases in $W_{\lambda}$, $W'_{\lambda}$. By Thomason’s concentration theorem in equivariant $K$-theory and by [20, Proposition 4.2.2], the determinant of the map $\kappa_{*}$ in those bases belongs to the set

$$(R^T_{\lambda})^\times \cdot \prod_{k} (1 - q^{n_k} z_{i_k} / z_{j_k})$$

for some $i_k, j_k \in [1, \ell], n_k \in \mathbb{Z} \setminus \{0\}$. We can assume that $n_k < 0$ for all $k$. Thus this determinant is invertible in the ring $R^T_{\lambda}$. The case of a general group $H_{\lambda}$ follows from Lemma 5.5. Let us prove Claim 2. In [24, Proposition 7.10.(v)] we define an embedding of $R^T_{\lambda} \otimes U$-modules

$$\Delta_W : W_{T_{\lambda}} \to W_{T_{\lambda_1}} \otimes_A W_{T_{\lambda_2}}.$$ 

By [24, Theorem 7.12] the map $\Delta_W$ is an isomorphism whenever $q, x_j, y_i$ are specialized to non-zero complex numbers such that $y_i / x_j \notin q^{1+\mathbb{N}}$ for all $i, j$. Hence it yields an isomorphism of $R_{\lambda_1/\lambda_2} \otimes U$-modules

$$\hat{R}_{\lambda_1/\lambda_2} \otimes_{R^T_{\lambda}} (W_{T_{\lambda_1}} \otimes_A W_{T_{\lambda_2}}) \cong \hat{R}_{\lambda_1/\lambda_2} \otimes_{R^T_{\lambda}} W_{T_{\lambda}}.$$ 

The unicity follows from Lemma 5.5. \hfill $\Box$

6.3. Let $a$ be the map from $Q_{\lambda}$ to the point. We consider the pairing of $R^{\hat{H}_{\lambda}}$-modules

$$( : ) : W_{H_{\lambda}} \times W'_{H_{\lambda}} \to R^{\hat{H}_{\lambda}}$$

given by $(x : y) = a_{*}(x \otimes y)$, where $\otimes$ is the tor-product relative to the smooth variety $Q_{\lambda}$. The pairing $( : )$ is perfect, see [20]. Note that, $W'_{H_{\lambda}}$ being a free $A$-module, there is an embedding $W'_{H_{\lambda}} \subset W_{H_{\lambda}}$. Let us consider the pairings

$$(||) : W_{H_{\lambda}} \times W'_{H_{\lambda}} \to R^{\hat{H}_{\lambda}},$$

$$(|) : W_{H_{\lambda}} \times W_{H_{\lambda}} \to R^{\hat{H}_{\lambda}}, \quad (|') : W'_{H_{\lambda}} \times W'_{H_{\lambda}} \to R^{\hat{H}_{\lambda}}$$

such that

$$(x||y) = (c^{-1}_{\lambda} a_{*} x : \omega T^{\pm 1}_v(y)), \quad (x|y) = (x||\kappa_{*}(y)), \quad (x|y)' = (\kappa_{*}^{-1}(x)||y),$$

see Lemma 6.2.1. Let $\partial : R^{\hat{H}_{\lambda}} \to A$ be the group homomorphism such that $\partial(q) = q$, and $\partial(V) = 0$ if $V$ is a non trivial simple $H_{\lambda}$-module.

Proposition.

1. We have $(x|y) = (y|x)^\dagger$.
2. We have $(x \gamma_1 y \gamma_1) = x y^\dagger$, for all $x, y \in R^{\hat{H}_{\lambda}}$.
3. We have $(u \cdot x|y) = (x|\psi(u) \cdot y)$.
4. The pairing $(|)$ is uniquely determined by conditions 2 and 3.
5. We have $(\beta_{\lambda}(x)||y) = (x||\beta_{\lambda}(y))$.
6. The pairing of $A$-modules $\partial(||)$ is perfect.
7. Claims 1 and 2 hold for the pairing $(|')$ also.
Proof. First, note that \(a_{\lambda 0} = 1\). By Lemma 5.7.2 we have \(c_{\lambda 0} = (r_{\lambda}^1)^{-1}1_{\lambda}'\). By Proposition 5.6.3 we have \(\omega^*T_{w_0}^{-1}(1_\lambda) = (r_{\lambda}^1)^{-1}1_{\lambda}\). Thus

\[
(x1_\lambda | y1_\lambda) = (c_{\lambda 0}^{-1}a_{\lambda 0}x1_\lambda : \omega^*T_{w_0}^{-1}(y1_\lambda)) = (x1_\lambda : y1_\lambda),
\]

Thus we get (6.7.6) and (5.8.3) we get

\[
\omega
\]

Claim 2 is proved. Fix \(u \in U, x \in W_{H_{\lambda,\alpha}}, y \in W'_{H_{\lambda,\alpha}}\). For all \(w \in W\) let \(1_{\lambda, w=0} \otimes x \in K^{H_{\lambda}}(Z_{\lambda, w=0, \alpha})\) be the obvious element. We have

\[
(1_{\lambda, w=0} \otimes x) \ast y = (x : y)1_{\lambda, w=0}.
\]

Thus, the associativity of \(\ast\) gives

\[
(x \ast u : y)1_\lambda = (1_\lambda \otimes x) \ast \Phi_\lambda(u) \ast y = (x : u \cdot y)1_\lambda.
\]

Thus we get

(6.3.0) \[
(u \ast x : y) = (x : \Phi_\lambda(u) \ast y).
\]

Assume now that \(x \in W_{H_{\lambda,\alpha}}, y \in W'_{H_{\lambda,\alpha}}\). Using (6.3.0), Lemma 5.2.3, Proposition 5.6.1, (5.6.7) and (5.8.3) we get

\[
(u \ast x : y) = (c_{\lambda 0}^{-1}a_{\lambda 0}x : \omega^*T_{w_0}^{-1}1_{\lambda}'(y)) = (c_{\lambda 0}^{-1}a_{\lambda 0}x : \omega^*T_{w_0}^{-1}1_{\lambda}') = (xT_{w_0}\Phi_\lambda(u) \ast y) = (xT_{w_0}\Phi_\lambda(u) \ast y) = (x|\psi_{Z_{\lambda}}\Phi_\lambda(u) \ast y).
\]

Then, apply Corollary 5.8.2. Claim 3 is proved. Claim 4 follows from Lemma 5.5.2. Assume now that \(x \in W_{H_{\lambda,\alpha}}, y \in W'_{H_{\lambda,\alpha}}\) as above. For all \(w\) we have

\[
1_{\lambda, w=0} \ast (1_{\lambda, w=0} \otimes x) = x.
\]

Fix \(u_\lambda\) such that \(\tilde{T}_{w_0}(1_{\lambda'}) = u_\lambda1_\lambda\). Using Proposition 5.6 we get

\[
u_\lambda 1_\lambda \ast T_{w_0}^{-1}(1_{\lambda'} \otimes x) = \tilde{T}_{w_0}(x), \quad T_{w_0}^{-1}(1_{\lambda'} \otimes x) \ast T_{w_0}^{-1}(y) = s_{\lambda}^{-1}(x : y)1_\lambda,
\]

i.e.

(6.3.1) \[
T_{w_0}^{-1}(1_{\lambda'} \otimes x) = u_\lambda^{-1}1_\lambda \otimes \tilde{T}_{w_0}(x),
\]

(1_{\lambda} \otimes \tilde{T}_{w_0}(x)) \ast T_{w_0}^{-1}(y) = u_\lambda s_{\lambda}^{-1}(x : y)1_\lambda.

This yields

(6.3.2) \[
(\tilde{T}_{w_0}(x) : T_{w_0}^{-1}(y)) = u_\lambda s_{\lambda}^{-1}(x : y).
\]

Claim 5 is analogous to [14, Lemma 12.15]. First, using (4.5.1) one gets

(6.3.3) \[
x : q_{\lambda,\alpha}(\tilde{D}_{\lambda,\alpha}(x)) = q_{\lambda,\alpha}(\tilde{D}_{\lambda,\alpha}(x)).
\]
Note that

\[(6.3.4)\quad b_{\lambda,w_0^*x}\alpha = q^{-d_{\lambda\alpha}/2}.\]

Using (6.3.0), Lemma 5.7.2, (6.3.4) and (6.3.3) we get

\[
(x||\beta(y)) = q^{-3d_{\lambda\alpha}/2}(c_{\alpha\lambda}^{-1}a_{\lambda\alpha}x : c_{\alpha\lambda}b_{\lambda,w_0^*x}\alpha D_{Q_{\lambda\alpha}}(y)) \\
= q^{-d_{\lambda\alpha}}(x : D_{Q_{\lambda\alpha}}(y)) \\
= (D_{F_{\lambda\alpha}}(x) : y)^\dagger.
\]

Using Lemma 5.7.2 and (6.3.1) we get

\[T_{w_0}^{-1}C_\lambda\zeta_\lambda(x \boxtimes 1_\lambda) = T_{w_0}^{-1}C_\lambda(1_{\lambda^\circ} \boxtimes \omega^*x) = u_\lambda^{-1}r_\lambda^{-1}1_\lambda \boxtimes \tilde{T}_{w_0}c_\lambda^{-1}\omega^*x.\]

The same argument as for (6.3.1) gives

\[T_{w_0}(x \boxtimes 1_\lambda) = s_\lambda^{-1}T_{w_0}(x) \boxtimes 1_{\lambda^\circ}.\]

Recall that \(\omega\) is \(\dagger\)-linear, \((r_\lambda s_\lambda)_\lambda = r_\lambda s_\lambda\), and \(c_{\lambda 0} = (r_\lambda)_\lambda^{-1}1_\lambda^\dagger\). Thus we get

\[C_\lambda\zeta_\lambda T_{w_0}(x \boxtimes 1_\lambda) = r_\lambda^{-1}s_\lambda^{-1}(1_\lambda \boxtimes c_\lambda^{-1}\omega^*T_{w_0}x).\]

Thus, (5.8.3) gives \(\tilde{T}_{w_0}c_\lambda^{-1}\omega^*(x) = u_\lambda s_\lambda^{-1}c_\lambda^{-1}\omega^*T_{w_0}(x)\), i.e.

\[(6.3.5)\quad T_{w_0}c_\lambda\omega^*(x) = c_\lambda\omega^*\tilde{T}_{w_0}(u_\lambda^{-1}s_\lambda x)\]

for all \(x \in W_{H_\lambda}\). Using (6.3.4), (6.3.5), (6.3.2) we get

\[
(\beta(x)||y) = q^{d_{\lambda\alpha}/2}(c_{\alpha\lambda}^{-1}a_{\lambda\alpha}T_{w_0}b_{\lambda,w_0^*x}\alpha \omega^*D_{F_{\lambda\alpha}}(x) : \omega^*T_{w_0}^{-1}(y)) \\
= (T_{w_0}c_{\lambda,w_0^*x}\alpha \omega^*D_{F_{\lambda\alpha}}(x) : c_{\alpha\lambda}^{-1}\omega^*T_{w_0}^{-1}(y)) \\
= (u_\lambda^{-1}s_\lambda T_{w_0}D_{F_{\lambda\alpha}}(x) : T_{w_0}^{-1}(y))^\dagger \\
= (D_{F_{\lambda\alpha}}(x) : y)^\dagger.
\]

Claim 5 is proved. Claim 1 follows from (6.3.2), (6.3.5) and Lemma 5.7.2. Indeed

\[
(x|y) = (c_{\alpha\lambda}^{-1}a_{\lambda\alpha}x : \omega^*T_{w_0}^{-1}\kappa_\lambda(y)) \\
= (u_\lambda s_\lambda^{-1}T_{w_0}^{-1}\omega^*c_{\lambda\alpha}^{-1}a_{\lambda\alpha}(x) : y)^\dagger \\
= (\omega^*c_{\alpha\lambda}^{-1}T_{w_0}^{-1}a_{\lambda\alpha}(x) : y)^\dagger \\
= (y|x)^\dagger.
\]

Claim 6 follows from the Schur Lemma and the fact that \((::)\) is a perfect pairing of \(R^{H_\lambda}\)-modules. 

\[\square\]

Remarks.

1. The pairings \((::)\), \((::)^\dagger\) are obviously compatible with the forgetful maps, see Remark 6.1.

2. The \(U\)-module \(W_{H_\lambda}^*\) has the following algebraic interpretation: let \(W_{H_\lambda}^*\) be \(W_{H_\lambda}^*\) with the new action of \(U\), denoted by \(\circ\), such that \(u \circ x = \phi^*\Phi_\lambda S(u^*x)\), where \(S\) is the antipode; then, the \(U\)-module \(W_{H_\lambda}^*\) is the right dual of \(W_{H_\lambda}\).
7. Construction of the signed basis

7.1. Following Lusztig we consider the sets

\[ B'_{H,\lambda} = \{ b \in W'_{H,\lambda} \mid \beta'_\lambda(b) = b, \partial(b)b' \in 1 + q^{-1}\mathbb{Z}[[q^{-1}]] \}, \]

\[ B_{H,\lambda} = \{ b \in W_{H,\lambda} \mid \beta_\lambda(b) = b, \partial(b)b \in 1 + q^{-1}\mathbb{Z}[q^{-1}] \}. \]

We also set \( B'_\lambda = B'_{G,\lambda}, B_\lambda = B_{G,\lambda} \).

Proposition.

1. If the subset \( B \subset B_{H,\lambda} \) satisfies:
   - \( B \) is a basis of the \( \mathbb{A} \)-module \( W_{H,\lambda} \),
   - for any elements \( b, b' \in B \) we have \( \partial(b)b' \in \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}] \).

Then \( B_{H,\lambda} = \pm B \). The similar statement holds for \( B'_{H,\lambda} \).

2. We have \( x_{\lambda}x_{\lambda}1_{\lambda\nu} \in B_{H,\lambda} \), and \( x_{\lambda}'x_{\lambda}'1_{\lambda\nu} \in B'_{H,\lambda} \), for any \( x \in X^{H,\lambda} \).

Proof. Claim 1 is standard, see [14, §12.20] for instance. We reproduce a proof here for the convenience of the reader. Fix an element \( b \in B_{H,\lambda} \). Set \( b = \sum_i p_i b_i \), where \( b_i \in B \) and \( p_i \in \mathbb{A} \). Fix \( n \in \mathbb{Z} \) such that \( p_i \in q^n\mathbb{Z}[q^{-1}] \) for all \( i \) and \( p_i \notin q^n\mathbb{Z}[q^{-1}] \) for some \( i \). For all \( i \) let \( p_{in} \in \mathbb{Z} \) be such that \( p_i = p_{in}q^n + q^{-n}\mathbb{Z}[q^{-1}] \). Then, \( \sum_i p_{in}^2 > 0 \). Thus,

\[ \partial(b)b \in q^{2n}\sum_i p_{in}^2 + q^{2n-1}\mathbb{Z}[q^{-1}] \]

On the other hand, we have \( \partial(b)b \in 1 + q^{-1}\mathbb{Z}[q^{-1}] \). It follows that \( n = 0 \) and \( \sum_i p_{in}^2 = 1 \). Since \( \beta_\lambda(b) = b \) and \( \beta_\lambda(b_i) = b_i \) for all \( i \), we must have \( p_i = p_i \) for all \( i \). Hence \( p_i \in \mathbb{Z} \) for all \( i \). Then \( \sum_i p_i^2 = 1 \). Thus \( b \in \pm B \). Let us prove Claim 2. By Proposition 6.1.4 and 6.3.2 we have \( x_{\lambda}x_{\lambda}1_{\lambda\nu} \in B_{H,\lambda} \), \( x_{\lambda}'x_{\lambda}'1_{\lambda\nu} \in B'_{H,\lambda} \). Hence, using [11] we get \( T_{w_0}(x_{\lambda}) \in B_{H,\lambda}, \ T_{w_0}'(x_{\lambda}') \in B'_{H,\lambda} \). Finally, Proposition 5.6.3 gives \( T_{w_0}^{-1}(x_{\lambda}) = x_{\lambda}1_{\lambda\nu} \). We are done. \( \square \)

Remark. In general \( 1_{\lambda\nu} \notin B_{H,\lambda} \).

7.2. For any \( \lambda \in P^+ \) let \( V(\lambda) \) be Kashiwara’s maximal integrable module. By definition, \( V(\lambda) \) is the free \( \mathbb{A} \)-module with the action of the algebra \( U \) such that there is a weight vector \( v_\lambda \) of weight \( \lambda \) which generates \( V(\lambda) \) and satisfies the following defining relations:

\[ U_\alpha(v_\lambda) = 0 \text{ for any } \alpha \in Q \setminus \{0\} \text{ s.t. } (\alpha, \lambda) \geq 0, \]

\[ f_i^{1+\ell}(v_\lambda) = 0 \text{ if } i \neq 0, \]

\[ e_0^{1+\beta}(v_\lambda) = 0, \]

see [9, §5.1]. It is proved in [8] that the module \( V(\lambda) \) admits a global basis. Let \( B(\lambda) \) be this basis. The element \( v_\lambda \) belongs to \( B(\lambda) \). Let \( ^{-}: V(\lambda) \rightarrow V(\lambda) \) be the unique \( \mathbb{A} \)-antilinear map such that \( b = b \) for all elements \( b \in B(\lambda) \). It is conjectured in [24, Remark 7.19] that there is an isomorphism of \( \mathbb{U} \)-modules \( V(\lambda) \rightarrow W(\lambda) \) such that \( \nu_{\lambda} \rightarrow 1_{\lambda} \). Let us first consider the case \( \lambda = \omega_i \). Let \( W(\omega_i') \) be the fundamental simple finite dimensional \( \mathbb{U}' \)-module associated to the weight \( \omega_i \), see [9, (5.7)], [1, §1.3]. Let \( W(\omega_i) \subset W(\omega_i') \) be the corresponding \( \mathbb{A} \)-form. For any \( \mathbb{U} \)-module \( M \) and any formal variable \( z \), let \( M_z \) be the representation of \( U \) on the space \( M[z^{\pm}] \) such that \( (x_{jr}^{\pm})^{(n)} \rightarrow (x_{jr}^{\pm})^{(n)} \otimes z^r, k_{jr}^{\pm} \rightarrow k_{jr}^{\pm} \otimes z^r \). Fix a weight
vector \( w_{\omega_i} \in W(\omega_i) \) of weight \( \omega_i \). By [9, Theorem 5.15.(viii)] there is a unique isomorphism of \( U \)-modules

\[
(7.2.2) \quad V(\omega_i) \xrightarrow{\sim} W(\omega_i)_z
\]
such that \( v_{\omega_i} \mapsto w_{\omega_i} \). The product by \( z \) is an automorphism of \( U \)-modules. It preserves the basis \( B(\omega_i) \). There is a unique basis \( B^0(\omega_i) \) of \( W(\omega_i) \) such that the map (7.2.2) takes \( B(\omega_i) \) to \( \bigsqcup_{n \in \mathbb{Z}} z^n B^0(\omega_i) \), see [9, Theorem 5.15.(iii)].

The group \( G_{\omega_i} \), being isomorphic to \( \mathbb{C}^\times \), we identify \( R^{G_{\omega_i}} \), with \( \mathbb{Z}[z_i^{\pm 1}] \) in the usual way.

**Theorem A.**

1. There is a unique element \( a_i \in \mathbb{Q}(q)^x \) and a unique isomorphism of \( U \)-modules \( \phi : V(\omega_i) \xrightarrow{\sim} W_{\omega_i} \) such that \( v_{\omega_i} \mapsto 1_{\omega_i} \) and the multiplication by \( z \) is mapped to the multiplication by \( a_i z_i^2 \). Moreover \( a_i = \pm 1 \).

2. Assume that \( (\cdot) : V(\omega_i) \times V(\omega_i) \rightarrow \mathbb{A} \) is a symmetric perfect pairing of \( \mathbb{A} \)-modules such that \( (z^m v_{\omega_i} | z^n v_{\omega_i}) = \delta_{n,m} \) and \( \langle u \cdot x | y \rangle = \langle x | \psi(u) \cdot y \rangle \). Then

\[
\pm B(\omega_i) = \{ b \in V(\omega_i) | b = b, (b|b') \in 1 + q^{-1} \mathbb{Z}[q^{-1}] \}.
\]

Moreover, \( (b|b') \in q^{-1} \mathbb{Z}[q^{-1}] \) if \( b, b' \in B(\omega_i) \) and \( b \neq b' \).

3. \( B_{G_{\omega_i}} = \pm \phi(\mathbb{B}(\omega_i)) \). It is a signed basis of \( W_{\omega_i} \).

**Proof.** Let us prove Claim 1. We identify \( K^{C^\times}(F_{\omega_i}) \) with the specialization of the \( U \)-module \( W_{\omega_i} \) at the maximal ideal of \( R^{G_{\omega_i}} \) associated to \( 1 \in G_{\omega_i} \). There is a unique element \( a_i \in \mathbb{Q}(q)^x \) and a unique isomorphism of \( U' \)-modules \( \mathbb{Q}(q) \otimes_{\mathbb{A}} K^{C^\times}(F_{\omega_i}) \xrightarrow{\sim} W(\omega_i)_{a_i} \) such that \( 1_{\omega_i} \mapsto w_{\omega_i} \), since both \( U' \)-modules are simple, see [20]. The \( U \)-modules \( K^{C^\times}(F_{\omega_i}), W(\omega_i)_{a_i} \), being cyclic generated by \( 1_{\omega_i}, w_{\omega_i} \), we get an isomorphism \( K^{C^\times}(F_{\omega_i}) \xrightarrow{\sim} W(\omega_i)_{a_i} \) such that \( 1_{\omega_i} \mapsto w_{\omega_i} \). The identification of the group \( G_{\omega_i} \) with \( C^\times \) is such that for any \( (B, p, q) \in M_{\omega_i} \) and any \( g_{\omega_i} \in G_{\omega_i} \) we have

\[
(1, g_{\omega_i}, 1) \cdot (B, p, q) = (B, g_{\omega_i}^{-1} p, g_{\omega_i} q).
\]

Since \( p_j = 0, q_j = 0 \) if \( j \neq i \) we have

\[
(1, g_{\omega_i}, 1) \cdot (B, p, q) = (1, g_{\omega_i}) \cdot (B, p, q)
\]

for \( g_{\omega_i} = (g_{\omega_i})_j \) with \( g_{\omega_i} = g_{\omega_i}^{-1} \text{Id}_{W_{\omega_i}} \). Then the group \( G_{\omega_i} \) acts trivially on \( F_{\omega_i} \) and the natural isomorphism of \( \mathbb{A}[z_i^{\pm 1}] \)-modules

\[
W_{\omega_i} = K^{G_{\omega_i} \times C^\times}(F_{\omega_i}) \xrightarrow{\sim} K^{C^\times}(F_{\omega_i})[z_i^{\pm 1}]
\]
takes \( \mathcal{V}_i \) to \( \mathcal{V}_i \otimes z_i \), and \( \mathcal{W}_j \) to \( \mathcal{W}_j \otimes z_i \). In particular we have

\[
\bigwedge_{\mathcal{V}_i} \mapsto \bigwedge_{\mathcal{V}_i} \otimes z_i^{\langle \rho, \alpha \rangle}, \quad \bigwedge_{\mathcal{F}^+_j} \mapsto \bigwedge_{\mathcal{F}^+_j} \otimes z_i^{f^{+}_{j, \alpha}}, \quad \bigwedge_{\mathcal{F}^-_j} \mapsto \bigwedge_{\mathcal{F}^-_j} \otimes z_i^{f^{-}_{j, \alpha}}
\]

\(^1\) H. Nakajima remarked that Theorem A.1 was not stated correctly in a previous version of the paper. He also mentioned to us that the isomorphism of \( U \)-modules \( V(\omega_i) \xrightarrow{\sim} W_{\omega_i} \) was known to him.
and \( x^\pm_{jr} \mapsto x^\pm_{jr} \otimes z^i_\lambda, \ k^\pm_{jr} \mapsto k^\pm_{jr} \otimes z^i_\lambda \), because \( t_{\alpha + \omega} - t_\alpha = f^+_{j;\alpha} - f^-_{j;\alpha} \). Hence \( \mathbf{W}_\omega \cong (W(\omega))_{i_0} \), where \( z \). 

The map \( \phi \) takes the involution \( v \mapsto \bar{v} \) on \( V(\omega) \) to the involution \( \beta_\omega \) on \( \mathbf{W}_\omega \) since both \( \mathbf{U} \)-modules are cyclic and the involutions are compatible with \( u \mapsto \bar{u} \) on \( \mathbf{U} \). The map \( \beta_\omega \) is \( z \)-linear by Corollary 5.8.1, the map \( v \mapsto \bar{v} \) on \( V(\omega) \) is \( z \)-linear since both are cyclic by \( z \) preserves \( \mathbf{B}(\omega) \). Hence \( \bar{a}_i = a_i \). The \( j \)-th Drinfeld polynomial of \( K^{\otimes} F(\omega) \) is \( P_j(t) = (t - q^{-e})^{\delta_i} \). Hence the \( j \)-th Drinfeld polynomial of \( W(\omega) \) is \( P_j(t) = (t - a_i^{-1} q^{-e})^{\delta_i} \). An easy computation shows that the elements \( h_{\pm 1} \) act on the vector \( w_\omega \in V(\omega) \) as follows \( h_{\pm 1}(w_\omega) = a_i^{\pm 1} q^{\pm e} w_\omega \). Since \( h_{\pm 1} \in \mathbf{U} \) the element \( a_i q^{-e} \) belongs to \( \mathbf{A} \) and is invertible. Thus \( a_i = \pm 1 \) because \( a_i \in \mathbb{Z} \) and \( a_i = \bar{a}_i \).

Let us prove Claim 2. We first recall some well-known fact. Let \( \bar{B} \) be the canonical basis of \( \mathbf{U} \), see \([13, 25.2]\). By \([8, 58]\) there is a subset \( \mathbf{I}(\lambda) \subset \bar{B} \) such that the space \( \mathbf{I}(\lambda) = \bigoplus_{b \in \mathbf{B}(\lambda)} M_b \subset \mathbf{U}_\eta \) is a left \( \mathbf{U} \)-submodule, and such that there is a unique isomorphism of \( \mathbf{U} \)-modules \( \mathbf{U}_\eta / \mathbf{I}(\lambda) \rightarrow \mathbf{V}(\lambda) \) which takes \( \eta \) to \( v_\lambda \). The \( \mathbf{U} \)-module \( \mathbf{V}(\lambda) \) being integrable, Kashiwara’s modified operators \( \bar{e}_j, \bar{f}_j \), \( j \in I \cup \{ 0 \} \), act on \( \mathbf{V}(\lambda) \). Let \( \mathbf{L}(\lambda) \subset \mathbf{V}(\lambda) \) be the \( \mathbb{Z}[q^{-1}] \)-lattice linearly spanned by \( \mathbf{B}(\lambda) \). It is stable by the operators \( \bar{e}_j, \bar{f}_j \), see \([8, \text{Proposition 9.1}] \), and contains the element \( v_\lambda \). The induced operators on the quotient \( \mathbf{V}(\lambda)/\mathbf{L}(\lambda) \) are still denoted by \( \bar{e}_j, \bar{f}_j \). Let \( \pi : \mathbf{L}(\lambda) \rightarrow \mathbf{L}(\lambda)/\mathbf{L}(\lambda) \) be the projection. We set \( \mathbf{B}(\lambda) = \pi(\mathbf{B}(\lambda)) \). It is known that \( \bar{e}_j, \bar{f}_j \) take \( \mathbf{B}(\lambda) \) to \( \mathbf{B}(\lambda) \cup \{ 0 \} \).

We now assume that \( \lambda = \omega_i \). Then the operators \( \bar{e}_j, \bar{f}_j \) are \( z \)-linear. Let \( \mathbf{L}(\omega_i) \) be the \( \mathbb{Z}[q^{-1}] \)-module spanned by \( \mathbf{B}(\omega_i) \). Let \( \mathbf{L}(\omega_i) \) be the projection of \( \mathbf{B}(\omega_i) \) in \( \mathbf{L}(\omega_i)/\mathbf{L}(\omega_i) \). There is an isomorphism of crystals

\[
(B(\omega_i), \mathbf{L}(\omega_i)) \cong (B(\omega_i), \mathbf{L}(\omega_i))/\mathbf{L}(\omega_i).
\]

Any element in \( B(\omega_i) \) can be reached at \( w_\omega \) after applying a monomial in the operators \( \bar{e}_j, j \in I \cup \{ 0 \} \), see \([1, \text{Lemma 1.5.(1) and (2)}] \) and \([9, \text{Proposition 5.4.(i)}] \). Thus any element in \( B(\omega_i) \) can be reached at \( \{ z^m v_\omega : m \in \mathbb{Z} \} \) after applying a monomial in the operators \( \bar{e}_j, j \in I \cup \{ 0 \} \).

Set \( \mathbf{L}(\omega_i) = \bigcup_{k \geq 0} \mathbf{L}(\omega_i)^k \), where

\[
\mathbf{L}(\omega_i)^k = \bigoplus_{\ell \leq k} \bigoplus_{j_1, \ldots, j_\ell} \mathbb{Z}[q^{-1}, z^{\pm 1}] \bar{f}_{j_1} \cdots \bar{f}_{j_\ell} (v_\omega).
\]

We claim that

\[
(7.2.3) \quad \mathbf{L}(\omega_i) = \mathbf{L}(\omega_i)^\infty + q^{-1}\mathbf{L}(\omega_i),
\]

\[
(7.2.4) \quad \langle \mathbf{L}(\omega_i) \mid \mathbf{L}(\omega_i) \rangle \subseteq \mathbb{Z}[q^{-1}], \quad \langle \bar{f}_j(x) y \rangle = \langle x \bar{e}_j(y) \rangle + q^{-1} \mathbb{Z}[q^{-1}],
\]

\forall j \in I \cup \{ 0 \}, \forall x, y \in \mathbf{L}(\omega_i). \] Claim (7.2.3) is obvious. To prove (7.2.4) we use the following Lemma, whose proof is given after the proof of the proposition.

**Lemma.** Fix \( j \in I \cup \{ 0 \} \). For any \( x \in V(\omega_i) \) fix elements \( x_r \in V(\omega_i), r \in [0, t] \), such that \( x = \sum_{r=0}^t \bar{f}_j^{(r)}(x_r) \) and \( \bar{e}_j(x_r) = 0 \).

1. If \( x \in \mathbf{L}(\omega_i) \) then \( x_r \in \mathbf{L}(\omega_i) \).
The corresponding weight subspaces. Then (7.2.6) for $k > 0$ reduces to
\[ ⟨z^n v_ω, z^n v_ω⟩ = δ_{mn}. \]
Similarly, (7.2.7) for $k = 0$ reduces to
\[ ⟨f_j(z^n v_ω) | y⟩ ∈ ⟨z^n v_ω | e_j(y)⟩ + q^{-1}Z[q^{-1}]. \]
This is obvious if $j = 0$ because $⟨f_j(z^n v_ω) | y⟩ ≠ 0$ or $⟨z^n v_ω | e_j(y)⟩ ≠ 0$ implies that $y ∈ \bigoplus_n V(ω_i)_{ω_i + θ + nδ}$, and $V(ω_i)_{ω_i + θ + nδ} = \{0\}$ for all $n$, see [9, Proposition 5.14.(i)] for instance. If $j ≠ 0$, (7.2.8) is proved as follows. Since $e_j(z^n v_ω) = 0$, we have $f_j(z^n v_ω) = f_j(z^n v_ω)$. Fix elements $y_s ∈ V(ω_i)$, $s ∈ [0, u]$, as in Lemma 7.2.3. Then, $y_s ∈ L(ω_i)$ by Lemma 7.2.1. We must show that
\[ ⟨f_j(z^n v_ω) | ∑_{s≥0} f_j(s)(y_s)⟩ ∈ ⟨z^n v_ω | ∑_{s≥1} f_j(s−1)(y_s)⟩ + q^{-1}Z[q^{-1}]. \]
The computation in [13, Proposition 19.1.3] gives the result, since $⟨z^n v_ω, y_s⟩ ∈ Z[q^{-1}]$ for all $s$, by (7.2.6) for $k = 0$. We may therefore assume that (7.2.6), (7.2.7) are already known for $k − 1$ with $k > 0$. Using (7.2.5) we see that (7.2.6) for $k$ follows from (7.2.6), (7.2.7) for $k − 1$. Finally, (7.2.7) for $k$ is proved as in [13, Proposition 19.1.3] using (7.2.6) for $k$, and Lemma 7.2.3.

From (7.2.6) we get
\[ ⟨L(ω_i) | L(ω_i)⟩ ⊆ Z[q^{-1}]. \]
On the other hand there is a positive integer $a$ such that
\[ ⟨L(ω_i) | L(ω_i)⟩ ⊆ q^aZ[q^{-1}]. \]
because $B^0(\omega_i)$ is finite. Using (7.2.3) it yields
\[ \langle L(\omega_i) | L(\omega_i) \rangle \subseteq \mathbb{Z}[q^{-1}] . \]
Similarly, given $x, y \in L(\omega_i)$ we fix $x^\infty \in L(\omega_i)^\infty$ such that $x - x^\infty \in q^{-1}L(\omega_i)$, see (7.2.3). Then (7.2.7) yields
\[ \langle \hat{f}_j(x^\infty) | y \rangle \in \langle x^\infty | \hat{c}_j(y) \rangle + q^{-1}\mathbb{Z}[q^{-1}] . \]
Thus
\[ \langle \hat{f}_j(x) | y \rangle = \langle \hat{f}_j(x^\infty) | y \rangle + q^{-1}\mathbb{Z}[q^{-1}] = (x^\infty | \hat{c}_j(y) \rangle + q^{-1}\mathbb{Z}[q^{-1}] \]
\[ = (x | \hat{c}_j(y) \rangle + q^{-1}\mathbb{Z}[q^{-1}] . \]
We have proved (7.2.4).

Then, Claim 2 is proved as in [13, Lemma 19.1.4], using (7.2.3), (7.2.4) and Proposition 7.1.1. More precisely for any element $b \in B(\omega_i)$, let $\ell(b)$ be the smallest $k \geq 0$ such that $b \in L(\omega_i)^k + q^{-1}L(\omega_i)$, see (7.2.3). For any $b, b' \in B(\omega_i)$ we prove by induction on $\ell(b)$ that
\[ (b | b') \in q^{-1}\mathbb{Z}[q^{-1}] \text{ if } b \neq b', \]
\[ (b | b) \in 1 + q^{-1}\mathbb{Z}[q^{-1}] . \]
If $\ell(b) = 0$ then $b = z^m\nu_{\omega_i}$ for some $n \in \mathbb{Z}$. Thus, if $(b | b') \neq 0$ then $b' = z^m\nu_{\omega_i}$ for some $m$ and both statements are obvious. Fix $k > 0$. Assume that (7.2.9), (7.2.10) hold for any $b, b'$ such that $\ell(b) < k$. Fix $b, b'$ such that $\ell(b) = k$. By (7.2.3) there is an integer $j \in I \cup \{0\}$ and an element $b_1 \in B(\omega_i)$ such that $\hat{f}_j(b_1) \in b + q^{-1}L(\omega_i)$ and $\ell(b_1) = k - 1$. Using (7.2.4) we get
\[ \langle b | b' \rangle \in \langle \hat{f}_j(b_1) | b' \rangle + q^{-1}\mathbb{Z}[q^{-1}] = \langle b_1 | \hat{e}_j(b') \rangle + q^{-1}\mathbb{Z}[q^{-1}] . \]
We have $\hat{e}_j(b) \in b_1 + q^{-1}L(\omega_i)$. Thus,
\[ (b | b) \in \langle b_1 | b_1 \rangle + q^{-1}\mathbb{Z}[q^{-1}] . \]
Hence (7.2.10) for $k$ follows from (7.2.10) for $k - 1$. If $b \neq b'$, either $\hat{e}_j(b') \in q^{-1}L(\omega_i)$, and then
\[ (b | b') \in q^{-1}\mathbb{Z}[q^{-1}] \]
by (7.2.4), or there is an element $b'_1 \in B(\omega_i)$ such that $\hat{e}_j(b') = b'_1 + q^{-1}L(\omega_i)$. In the last case $b_1 \neq b'_1$ (else applying $\hat{f}_j$ we would get $b \in b' + q^{-1}L(\omega_i)$, and thus $b = b'$). Hence (7.2.9) for $k$ follows from (7.2.9) for $k - 1$. Finally, Claim 2 follows from (7.2.9), (7.2.10) and Proposition 7.1.1.

Claim 3 is obvious from Claim 1 and Claim 2.

\textbf{Proof of Lemma 7.2.} Claims 1,2 generalize [13, Lemma 18.2.2] to the non-highest weight module case. The proof follows [13, Lemma 18.2.2]. Note that we only use Claim 1; Claim 2 is given for the sake of completeness.
We prove Claim 1 by induction on $t$. It is obvious if $t = 0$. Since
\[
\tilde{e}_j(x) = \sum_{r=0}^{t-1} f_j^{(r)}(x_{r+1}), \quad \tilde{e}_j(L(\omega_i)) \subseteq L(\omega_i),
\]
we get $\sum_{r=0}^{t-1} f_j^{(r)}(x_{r+1}) \in L(\omega_i)$. The induction hypothesis for $t - 1$ gives $x_{r+1} \in L(\omega_i)$ for all $r \in [0, t - 1]$. Since $e_j(x_{r+1}) = 0$ we have $f_j^{(r+1)}(x_{r+1}) = f_j^{r+1}(x_{r+1})$. Since $\tilde{f}_j(L(\omega_i)) \subseteq L(\omega_i)$ we get $f_j^{(r+1)}(x_{r+1}) \in L(\omega_i)$. Using $x \in L(\omega_i)$, $f_j^{(r)}(x_r) \in L(\omega_i)$ for all $r \in [1, t]$ we get $x_0 \in L(\omega_i)$.

We prove Claim 2 by induction on $t$. It is obvious if $t = 0$. Since $x \in B(\omega_i)$ we have $\tilde{e}_j(x) \in B(\omega_i) + q^{-1}L(\omega_i)$ or $\tilde{e}_j(x) \in q^{-1}L(\omega_i)$. In the second case we get, using Claim 1, $x_r \in q^{-1}L(\omega_i)$ for all $r \in [1, t]$. Thus $x_0 \in x + q^{-1}L(\omega_i)$ and we are done. Consider now the first case. Using the induction hypothesis for $t - 1$ we get an integer $r_0 \in [0, t - 1]$ such that $x_{r_0+1} \in B(\omega_i) + q^{-1}L(\omega_i)$ and $x_{r+1} \in q^{-1}L(\omega_i)$ for all $r \in [0, t - 1] \setminus \{r_0\}$. Thus $\tilde{e}_j(x) \in \tilde{f}_j^{r_0}(x_{r_0+1}) + q^{-1}L(\omega_i)$. Hence
\[
x \in \tilde{f}_j \tilde{e}_j(x) + q^{-1}L(\omega_i) = \tilde{f}_j^{r_0+1}(x_{r_0+1}) + q^{-1}L(\omega_i).
\]
Hence, necessarily $x_0 \in q^{-1}L(\omega_i)$. We are done.

We prove Claim 3. To simplify we set $x_r = y_s = 0$ if $r > t$, $s > u$. For any $a, b \geq 0$ we have, see [13, Proposition 19.1.3],
\[
(f_j^{(a)}(x_r) | f_j^{(b)}(y_s)) = \delta_{a,b} \delta_{r+1,s} C_{a,s}(x_r | y_s),
\]
where
\[
C_{a,s} = \left[ q^{a^2-a(\omega+j),\omega}) \right].
\]
From Claim 1 we have $y_s \in L(\omega_i)_{\mu+\omega j}$ for all $s$. Since $e_j(y_s) = 0$ we have $f_j^{(s-1)}(y_s) = f_j^{s-1}(y_s)$. In particular, $f_j^{(s-1)}(y_r) \in L(\omega_i)_{\mu+\omega j}$. Thus $\langle x | f_j^{(s-1)}(y_s) \rangle \in \mathbb{Z}[q^{-1}]$. On the other hand,
\[
\langle x | f_j^{(s-1)}(y_s) \rangle = \sum_{r=0}^{t} \langle f_j^{(r)}(x_r) | f_j^{(s-1)}(y_s) \rangle = C_{s-1,s} \langle x_{s-1} | y_s \rangle.
\]
Now
\[
C_{s-1,s} = \left[ q^{(s-1)2-(s-1)(\omega+j),\omega}) \right] \in 1 + q^{-1}\mathbb{Z}[q^{-1}].
\]
Thus $\langle x_{s-1} | y_s \rangle \in \mathbb{Z}[q^{-1}]$ for all $s \geq 1$. If $s \neq r + 1$ then $\langle x_r | y_s \rangle = 0$ since $x_r, y_s$ have different weights. \hfill \Box

Here is the main result of the paper.

**Theorem B.**

1. The sets $B_{T_X}, B'_{T_X}$ are signed basis of $W_{T_X}, W'_{T_X}$. Moreover, for any $b, b' \in B_{T_X}$ (resp. $b, b' \in B'_{T_X}$) we have
   \[
   \partial(b|b') = \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}] \quad \text{(resp. $\partial(b|b') = \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}]$)}
   \]
2. The signed basis $B_{T_X}, B'_{T_X}$ are dual with respect to the pairing $\partial(\|وسط) in the following sense: for all $b \in B_{T_X}$ there is a unique element $b' \in B'_{T_X}$ such that $\partial(b||b') = 1$, and we have $\partial(b||b'') = 0$ whenever $b'' \neq \pm b'$.
admits a unique involution \( c^{\text{nor}} \) such that
\[
e^{\text{nor}}(v_1, \ldots, i_\ell) = v_1, \ldots, i_\ell, \quad e^{\text{nor}}(u \cdot v_1, \ldots, i_\ell) = \tilde{u} \cdot v_1, \ldots, i_\ell, \quad \forall u \in U.
\]
Set \( R^{T_{i_\ell}} = \mathbb{Z}[z_k^{\pm 1}] \) for all \( k \). Set also
\[
\tilde{R}_{i_1/\ldots/i_\ell} = \mathbb{Z}[q^{-1}, z_{k+1}/z_k; k] \otimes_{\mathbb{Z}[q^{-1}, z_{k+1}/z_k]} R^{T_k},
\]
where \( k \) takes all possible values in \([1, \ell]\). The tensor product
\[
\mathbb{R}_k W_{T_{i_\ell}} := \tilde{R}_{i_1/\ldots/i_\ell} \otimes_{R^{T_k}} \tilde{W}_{T_{i_\ell}}
\]
is endowed with the unique pairing of \( \tilde{R}_{i_1/\ldots/i_\ell} \)-modules such that
\[
(\mathbb{R}_k x_k | \mathbb{R}_k y_k)_{i_1/\ldots/i_\ell} = \mathbb{R}_k (x_k | y_k), \quad \forall x_k, y_k \in \tilde{W}_{T_{i_\ell}},
\]
and the pairing \((\cdot)\) on each factor is as in 6.3. As in 7.1 let \( \partial(\cdot)_{i_1/\ldots/i_\ell} \) be the corresponding pairing
\[
\mathbb{R}_k W_{T_{i_\ell}} \times \mathbb{R}_k W_{T_{i_\ell}} \rightarrow \mathbb{Z}((q^{-1})).
\]
We have an isomorphism of \( U \)-modules \( W_{T_{i_\ell}} \simeq V(\omega_i) \) such that \( 1_{\omega_i} \mapsto v_{\omega_i} \), see Theorem 7.2.A.1. By Theorem 7.2.A.2 we have
\[
\pm \mathbb{R}_k B(\omega_{i_k}) + \mathbb{R}_k q^{-1} L(\omega_{i_k}) = \{ b \in \mathbb{R}_k W_{T_{i_\ell}} | \partial(b)_{i_1/\ldots/i_\ell} = 1 + q^{-1} \mathbb{Z}[q^{-1}] \}.
\]
Thus, by [9, Theorem 8.5 and Proposition 8.6] the set
\[
\{ b \in N | e^{\text{nor}}(b) = b, \partial(b)_{i_1/\ldots/i_\ell} = 1 + q^{-1} \mathbb{Z}[q^{-1}] \}
\]
is a signed basis of \( N \). Iterating \((\ell - 1)\)-times the map in Lemma 6.2.2 we get an isomorphism of \( U \)-modules
\[
\varpi_{i_1/\ldots/i_\ell} : \mathbb{R}_k W_{T_{i_\ell}} \simeq R_{i_1/\ldots/i_\ell} \otimes_{R^{T_k}} W_{T_k}, \quad v_{\omega_1} \otimes \cdots \otimes v_{\omega_\ell} \mapsto 1_{\lambda}.
\]
The \( U \)-module \( W_{T_{i_\ell}} \) is generated by \( R^{T_k} \otimes 1_{\lambda} \) by Lemma 5.5.2. Thus \( \varpi_{i_1/\ldots/i_\ell}(N) = W_{T_{i_\ell}} \). By [7] the pairing \((\cdot)_{i_1/\ldots/i_\ell} \) still satisfies Proposition 6.3.3 with \( 1_{\lambda} \) instead of \( v_{i_1, \ldots, i_\ell} \). It is easy to see that Proposition 6.3.2 also holds in this setting. Thus, by Proposition 6.3.4 the pairings \((\cdot)_{i_1/\ldots/i_\ell} \) and \((\cdot)\) on \( N \) and \( W_{T_{i_\ell}} \) coincide. Moreover, Proposition 6.1 and Corollary 5.8.2 give \( \varpi_{i_1/\ldots/i_\ell} e^{\text{nor}} = \beta_{\lambda} \varpi_{i_1/\ldots/i_\ell} \). Thus,
\[
\varpi_{i_1/\ldots/i_\ell}(\{ b \in N | e^{\text{nor}}(b) = b, \partial(b)_{i_1/\ldots/i_\ell} = 1 + q^{-1} \mathbb{Z}[q^{-1}] \}) =
\]
\[
= \{ b \in W_{T_{i_\ell}} | \beta_{\lambda}(b) = b, \partial(b)_{i_1/\ldots/i_\ell} \in 1 + q^{-1} \mathbb{Z}[q^{-1}] \}.
\]

Proof. Fix a decomposition \( \lambda = \sum_{k=1}^{\ell} \omega_{i_k} \). We set \( v_{i_1, \ldots, i_\ell} = v_{\omega_1} \otimes \cdots \otimes v_{\omega_\ell} \). By [9, §8] the \( U \)-submodule
\[
N := U \cdot (R^{T_k} \otimes v_{i_1, \ldots, i_\ell}) \subset \mathbb{R}_k V(\omega_{i_k})
\]

is endowed with the unique pairing of \( \mathbb{R}_k \)

\[
e^{\text{nor}}(v_1, \ldots, i_\ell) = v_1, \ldots, i_\ell, \quad e^{\text{nor}}(u \cdot v_1, \ldots, i_\ell) = \tilde{u} \cdot v_1, \ldots, i_\ell, \quad \forall u \in U.
\]
In particular, $\mathcal{B}_{T_\lambda}$ is a signed basis of $\mathbf{W}_{T_\lambda}$ such that

\[(7.2.11) \quad \partial(b|b') \in \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}]\]

for all $b,b' \in \mathcal{B}_{T_\lambda}$. By Proposition 6.3.6 there is a signed basis $\mathcal{B}_{T_\lambda}^* \subset \mathbf{W}_{T_\lambda}^*$ dual to $\mathcal{B}_{T_\lambda}$ with respect to the pairing $\partial(\| )$. Let $b^* \in \mathcal{B}_{T_\lambda}^*$ be the element dual to $b \in \mathcal{B}_{T_\lambda}$. Let $E : \mathbf{W}_{T_\lambda} \rightarrow \mathbf{W}_{T_\lambda}'$ be the unique $\mathbb{A}$-modules isomorphism such that $E(b) = b^*$ for all $b \in \mathcal{B}_{T_\lambda}$. Using (7.2.11) we get

$$(\kappa_* - E)(\bigoplus_{b \in \mathcal{B}_{T_\lambda}} \mathbb{Z}[q^{-1}]b) \subset \bigoplus_{b \in \mathcal{B}_{T_\lambda}} q^{-1}\mathbb{Z}[q^{-1}]b^*.$$

Thus, the map $\kappa_* : \mathbf{W}_{T_\lambda} \rightarrow \mathbf{W}_{T_\lambda}'$ is invertible (see also Lemma 6.2.1) and we have

$$(\kappa_*)^{-1} = \sum_{n \geq 0} (-1)^n E^{-1}((\kappa_* - E)E^{-1})^n.$$

In particular we get

$$\partial((\kappa_*)^{-1}(b^*)|b'') \in \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}],$$

i.e.

$$\partial(b^*|b'')' \in \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}],$$

for all $b,b' \in \mathcal{B}_{T_\lambda}$. Using Proposition 6.3.5 we get also $\beta'_\lambda(b^*) = b^*$ for all $b \in \mathcal{B}_{T_\lambda}$. Thus $\mathcal{B}_{T_\lambda}^* \subset \mathcal{B}_{T_\lambda}'$. Then, apply Proposition 7.1.1. We are done. \qed

Remarks.

1. It is easy to see that $\mathbf{X}_{T_\lambda}$ is a subgroup of the multiplicative group of $\mathbb{R}\mathbf{F}_\lambda$ such that $\mathbf{X}_{T_\lambda} \mathbf{B}_{T_\lambda} = \mathbf{B}_{T_\lambda}$. Thus there is a subset $\mathbf{B}_{T_\lambda}^0 \subset \mathbf{B}_{T_\lambda}$ which is an $\mathbb{R}\mathbf{F}_\lambda$-basis of $\mathbf{W}_{T_\lambda}$. In particular for any maximal ideal $I \subset \mathbb{R}\mathbf{F}_\lambda$ the set $\mathbf{B}_{T_\lambda}^0 \otimes 1$ is a $\mathbb{Z}$-basis of $\mathbf{W}_{T_\lambda} \otimes \mathbb{R}\mathbf{F}_\lambda (\mathbb{R}\mathbf{F}_\lambda/I)$. If $\hat{\mathcal{I}}(I) = I$ then the involution $\beta_{T_\lambda}$ and the metric $\partial(\| )$ descend to $\mathbf{W}_{T_\lambda} \otimes \mathbb{R}\mathbf{F}_\lambda (\mathbb{R}\mathbf{F}_\lambda/I)$. It is not clear if $\pm \mathbf{B}_{T_\lambda}^0 \otimes 1$ admits a similar characterization as $\mathbf{B}_{T_\lambda}$ in 7.1.

2. Probably the sets $\mathbf{B}_{\lambda}, \mathbf{E}_{\lambda}'$ are signed bases of $\mathbf{W}_{\lambda}, \mathbf{W}_{\lambda}'$. The conjectures in [9, \S 13] and the previous theorem suggest that Kashiwara's canonical basis of $V(\lambda)$ coincide with $\mathbf{B}_{\lambda}, \mathbf{E}_{\lambda}'$, up to signs.

7.3. We do not assume any more that $\mathfrak{g}$ is simply laced. Fix $i \in I$. The fundamental module $W(\omega_i)$ is as in [9]. Let $W(\omega_i)[z^{\pm}]$ be the affinized module, see [9, \S 4.2]. Set $d_i = \max(1, (\alpha_i, \alpha_i)/2)$. Then $V(\omega_i)$ is isomorphic to the $U$-submodule $W(\omega_i)[z^{\pm d_i}] \subset W(\omega_i)[z^{\pm}]$, see [9, Theorem 5.15.(viii)]. Set $z_i = z^{d_i}Id : V(\omega_i) \rightarrow V(\omega_i)$. Let us mention the following fact, which is not used in the paper.

Proposition. For any $\mathfrak{g}$ (not necessarily simply laced) and any $i \in I$ there is a unique pairing of $\mathbb{A}$-modules $(\| ) : V(\omega_i) \times V(\omega_i) \rightarrow \mathbb{A}$ such that

$$\langle z_i^n v_{\omega_i} | z_i^m v_{\omega_i} \rangle = \delta_{n,m}, \quad \langle u \cdot x | y \rangle = \langle x | \psi(u) \cdot y \rangle.$$

This pairing is perfect and symmetric.
Proof. The proof is similar to the proof of [13, Proposition 19.1.2]. By [9, Proposition 5.14.(iii)] the space \( V(\omega_1)_{\hat{\mu}} \) is finite-dimensional for any \( \hat{\mu} \in \hat{P} \). Set
\[
V(\omega_1)^* = \bigoplus_{\hat{\mu} \in \hat{P}} \text{Hom}(V(\omega_1)_{\hat{\mu}}, \mathbb{A}).
\]

Since \( \psi \) is an anti-automorphism, there is a unique \( U \)-module structure on \( V(\omega_1)^* \) such that
\[
(u \cdot f)(x) = f(\psi(u) \cdot x), \quad \forall u \in U, \forall x \in V(\omega_1).
\]
The \( U \)-module \( V(\omega_1)^* \) is endowed with the \( \hat{P} \)-grading such that
\[
V(\omega_1)^{\mu+n\delta} = \text{Hom}(V(\omega_1)_{\mu+n\delta}, \mathbb{A}).
\]
Recall that \( V(\omega_1)_{\omega_1} = \mathbb{A} \cdot v_\omega_1 \). Let \( f_{\omega_1} \in V(\omega_1)^* \) be the unique linear form such that \( f_{\omega_1}(v_\omega_1) = 1 \), and \( f_{\omega_1}(v) = 0 \) for all \( v \in V(\omega_1)_{\hat{\mu}} \) with \( \hat{\mu} \neq \omega_1 \). Hence, \( f_{\omega_1} \in V(\omega_1)_{\omega_1}^* \).

We must prove that there is a unique morphism of \( U \)-modules \( V(\omega_i) \to V(\omega_j)^* \) which takes \( v_{\omega_i} \) to \( f_{\omega_1} \), and that it is invertible. The spaces \( V(\omega_i)_{\hat{\mu}} \) have the same dimension for all \( \mu \). Thus the set of the weights \( \mu \in P \) such that \( V(\omega_i)_{\mu+n\delta} \neq \{0\} \) for some \( n \in \mathbb{Z} \) is contained in \( \omega_i - \sum_{j \in I} N \alpha_j \), since this is true for \( V(\omega_1) \).

Hence \( f_{\omega_i} \) is an extremal vector of weight \( \omega_i \), see [9, Theorem 5.3]. By the universal property of \( V(\omega_1) \), there is a unique morphism of \( U \)-modules \( \phi : V(\omega_i) \to U \cdot f_{\omega_i} \subset V(\omega_1)^* \) which takes \( v_{\omega_i} \) to \( f_{\omega_1} \). Moreover, we have \( \phi(V(\omega_i)_{\hat{\mu}}) \subset V(\omega_i)^{\mu+n\delta} \) for all \( \hat{\mu} \in \hat{P} \).

Since \( V(\omega_i)_{\hat{\mu}} \) is finite dimensional it is sufficient to prove that the map \( \phi \) is injective. Let the operator \( z_i \) acts on \( V(\omega_i)^* \) by
\[
(z_i \cdot f)(x) = f(z_i^{-1} \cdot x), \quad \forall x \in V(\omega_i), \forall f \in V(\omega_i)^*.
\]
Then \( \phi \) commutes to \( z_i \). Since \( W(\omega_i) \cong V(\omega_i)/(z_i-1)V(\omega_i) \), the map \( \phi \) induces a non-zero morphism of \( U \)-modules \( W(\omega_i) \to W(\omega_i) \). It is injective since \( W(\omega_i) \) is simple. Thus \( \phi \) is injective.

The pairing is symmetric because it is unique and \( \psi^2 = \text{Id} \).

\[
\square
\]

8. Example

We assume that \( \Pi = \{\{1\}, \emptyset\} \). We set \( \lambda = \ell \omega_1, \alpha = a \omega_1 \). To simplify we omit the subscripts 1 and we set \( Q_{\ell a} = Q_{\ell a}, d_{\ell a} = d_{\lambda a}, \) etc. Set
\[
\tilde{x}_+^a = \sum_{a'=a+1} (\ell-\ell-a') q^{-2r-a'} (\Lambda_{V}^{-r-\ell+a'} \Lambda_{W}^{r+\ell-a} \otimes \Lambda_{W}^{r+\ell-a} \otimes 1_{fa a'}),
\]
\[
\tilde{x}_-^a = \sum_{a'=a-1} (\ell-a') q^{-2r+a'} (\Lambda_{V}^{-a-a'} \Lambda_{W}^{-r+a} \otimes 1_{fa a'}).
\]
We have \( Q_{\ell a} \neq \emptyset \) if and only if \( 0 \leq a \leq \ell \). More precisely \( F_{\ell a} \) is smooth and isomorphic the Grassmanian of \( a \)-dimensional subspaces in \( C^\ell \), and \( Q_{\ell a} = T^* F_{\ell a} \). An element in \( Q_{\ell a} \) may be viewed as a couple \((V, u)\), where \( V \subset C^\ell \) is a \( a \)-dimensional subspace, and \( u \in \text{End}(C^\ell) \) is a nilpotent map such that \( \text{Im} u \subset V \subset \text{Ker} u \). The element \( z \in C^* \) acts on \( T^* F_{\ell a} \) by multiplication by the scalar \( z^2 \) along the fibers.
The automorphism \( \omega : Q_{t\alpha} \to Q_{t\ell,t\alpha} \) takes the couple \((V, \rho)\) to \((V^\perp, \tau\rho)\), where \(V^\perp \subseteq \mathbb{C}^\ell\) is the subspace orthogonal to \(V\), with respect to the canonical scalar product on \(\mathbb{C}^\ell\). Let \(E_a\) be the tautological rank \(a\) vector bundle on \(T^*F_{t\alpha}\). Let \(E_a\) be the restriction of \(E'_a\) to \(F_{t\alpha}\). Set \(Q_a = W_a/E'_a\), \(Q_a = W_a/E_a\). Set \(E = \bigoplus_a E_a\), \(E' = \bigoplus_a E'_a\), etc.

A direct computation gives

\[ V = qE', \quad \tilde{x}_r^+ = \wedge_{\ell} x^+ \wedge_{\mathbb{Z}^\ell} \wedge_{\mathbb{Z}^\ell}, \quad \tilde{x}_r^- = \wedge_{\ell} x^- \wedge_{\mathbb{Z}^\ell} \wedge_{\mathbb{Z}^\ell}. \]

Hereafter we omit the operators \(\kappa_a\) and \(\otimes\). The following lemma is immediate, see for instance \([22]\).

**Lemma.**

1. We have \(\tilde{x}_0^+(1_{t\alpha}) = [(\ell - a + 1)1\ell_a - 1, \tilde{x}_0^-(1_{t\alpha}) = [a + 1]1\ell_a + 1\).
2. We have \(1_{t\alpha} = \sum_{a'=0}^{a(\ell-a)}(-1)^{q-2i} \wedge^i (Q_a E'_a)\).

A direct computation gives

**Proposition.**

1. We have \(B'_1 = B_1 = \{x_{110}, x_{11}^{-1}; x \in X_1\}\).
2. We have \(B'_2 = \{x_{120}, x_{12}^{-1}; x \in X_2\}\).

Appendix

Let us check that \(x(w_0 + a) = x(a)\) and \(y(w_0 + a) = y(a)\), for all \(a \in Q\), where \(x : Q \to \mathbb{Z}\) and \(y : Q \to \mathbb{Z}/2\mathbb{Z}\) are the quadratic maps which satisfy \((5.7.1)\). We give the proof for the map \(x\), the case of the map \(y\) is left to the reader. Set

\[ \alpha = \sum_j a_j \alpha_j, \quad \lambda = \sum_j \ell_j \omega_j, \quad \nu = \lambda - w_0(\lambda) = \sum_j k_j \alpha_j. \]

Write \(x(\alpha) = Q(\alpha) + L(\alpha) + a\), where \(Q\) is a quadratic form, \(L\) is a linear form and \(a \in \mathbb{Z}\). Put

\[ Q(\alpha) = \sum_{i,j} q_{ij} a_i a_j, \quad L(\alpha) = \sum_j b_j a_j. \]

A direct computation gives \(g_{i;\alpha} = c_i + \sum_j c_{ij} a_j\) with

\[ c_i = -1 + (2 - c)\ell_i + (2c - 1)k_i, \quad c_{ij} = (c - 2)a_{ij} + \delta_{ij}(2 - 2c). \]

Using the relation \(x(\alpha + a_i) - x(\alpha) = c k_i - g_{i;\alpha}\), we get

\[ b_i = c k_i - c_i + \frac{1}{2} c_{ii} = (c - 2)\ell_i + (1 - c)k_i, \]

\[ q_{ij} = -\frac{1}{2} c_{ij} = (1 - \frac{c}{2}) a_{ij} + \delta_{ij}(c - 1), \]

that is

\[ Q(\alpha) = (c - 1)|\alpha|^2 + (1 - \frac{c}{2})(\alpha, \alpha), \quad L(\alpha) = (c - 2)(\lambda, \alpha) + (1 - c) \sum_j k_j a_j. \]
The identity $Q(\nu) + L(\nu) + a = x(\nu) = (c - 1)|\nu|^2 - c(\lambda, \lambda)$, gives

$$a = -c(\lambda, \lambda) + \left(\frac{c}{2} - 1\right)(\nu, \nu) - (c - 2)(\lambda, \nu) + (c - 1)|\nu|^2 = (c - 1)|\nu|^2 - c(\lambda, \lambda),$$

since $2(\lambda, \nu) = (\nu, \nu)$. In particular

$$a = x(0) = x(\nu).$$

We want now to check that $x(w_0 \ast \alpha) = x(\alpha)$. Since $x(\nu) = x(0)$, we have $Q(\nu) + L(\nu) = 0$. Moreover (2) gives $Q(\alpha) = Q(w_0(\alpha))$. Thus it is enough to prove that

$$L(w_0(\alpha)) - L(\alpha) = 2 \sum_{i,j} k_i q_{ij} a_j.$$ 

By (2), the left hand side is equal to

$$(2 - c)(\nu, \alpha) + (c - 1) \sum_j k_j (a_j + a_{\frac{\lambda}{2}}).$$

By (1), the right hand side is equal to

$$(2 - c)(\nu, \alpha) + 2(c - 1) \sum_j k_j a_{\frac{\lambda}{2}}.$$

Since $w_0(\nu) = -\nu$ we have $k_j = k_{\frac{\lambda}{2}}$. We are done.

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