Homogeneous linear matrix difference equations of higher order: Singular case

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Abstract: In this article, we study the singular case of an homogeneous generalized discrete time system with given initial conditions. We consider the matrix pencil singular and provide necessary and sufficient conditions for existence and uniqueness of solutions of the initial value problem.

Keywords: linear difference equations, matrix, singular, system.

1 Introduction

Many authors have studied generalized discrete & continuous time systems, see [1-28], and their applications, see [29-35]. Many of these results have already been extended to systems of differential & difference equations with fractional operators, see [36-45]. In this article, our purpose is to study the solutions of a generalized initial value problem of linear matrix difference equations into the mainstream of matrix pencil theory. Thus, we consider

\[ A_n X_{k+n} + A_{n-1}X_{k+n-1} + ... + A_1 X_{k+1} + A_0 X_k = 0 \]

with known initial conditions

\[ X_0, X_1, ..., X_{0+n-1}, \]

where \( A_i, i = 0, 1, ..., n \in \mathcal{M}(m_1 \times r_1; \mathcal{F}) \), (i.e. the algebra of square matrices with elements in the field \( \mathcal{F} \)) with \( X_k \in \mathcal{M}(m_1 \times 1; \mathcal{F}) \) and \( \det A_n = 0 \) if \( A_n \) is square. In the sequel we adopt the following notations

\[
\begin{align*}
Y_{k,1} &= X_k, \\
Y_{k,2} &= X_{k+1}, \\
\ldots \\
Y_{k,n-1} &= X_{k+n-2}, \\
Y_{k,n} &= X_{k+n-1}.
\end{align*}
\]

and

\[
\begin{align*}
Y_{k+1,1} &= X_{k+1} = Y_{k,2}, \\
Y_{k+1,2} &= X_{k+2} = Y_{k,3}, \\
\ldots \\
Y_{k+1,n-1} &= X_{k+n-1} = Y_{k,n}, \\
A_n Y_{k+1,n} &= A_n X_{k+n} = -A_{n-1}Y_{k,n} - ... - A_1 Y_{k,2} - A_0 Y_{k,1}.
\end{align*}
\]
Let \( m_1 n = r \) and \( m_1 n + r_1 - m_1 = m \). Then the above system can be written in Matrix form in the following way

\[ FY_{k+1} = GY_k, \quad (1) \]

with known initial conditions

\[ Y_{k_0}. \quad (2) \]

Where

\[
F = 
\begin{bmatrix}
I_{m_1} & 0_{m_1,m_1} & \cdots & 0_{m_1,m_1} & 0_{m_1,m_1} \\
0_{m_1,m_1} & I_{m_1} & \cdots & 0_{m_1,m_1} & 0_{m_1,m_1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{m_1,m_1} & 0_{m_1,m_1} & \cdots & I_{m_1} & 0_{m_1,m_1} \\
0_{m_1,r_1} & 0_{m_1,r_1} & \cdots & 0_{m_1,r_1} & A_n
\end{bmatrix},
\]

\[
G = 
\begin{bmatrix}
0_{m_1,m_1} & I_{m_1} & \cdots & 0_{m_1,m_1} \\
0_{m_1,m_1} & 0_{m_1,m_1} & \cdots & 0_{m_1,m_1} \\
\vdots & \vdots & \ddots & \vdots \\
0_{m_1,m_1} & 0_{m_1,m_1} & \cdots & I_{m_1} \\
-A_0 & -A_1 & \cdots & -A_{n-1}
\end{bmatrix},
\]

and

\[
Y_k = \begin{bmatrix} Y_{k,1}^T & Y_{k,2}^T & \cdots & Y_{k,m_1}^T \end{bmatrix}^T.
\]

With dimensions \( F, G \in \mathcal{M}(r \times m; \mathcal{F}) \), (i.e. the algebra of matrices with elements in the field \( \mathcal{F} \)) with \( Y_k \in \mathcal{M}(m \times 1; \mathcal{F}) \). For the sake of simplicity we set \( \mathcal{M}_m = \mathcal{M}(m \times m; \mathcal{F}) \) and \( \mathcal{M}_{rm} = \mathcal{M}(r \times m; \mathcal{F}) \). The matrices \( F \) and \( G \) can be non-square (when \( r \neq m \)) or square (when \( r = m \)) and \( F \) singular (\( \det F = 0 \)).

## 2 Singular matrix pencils: Mathematical background and notation

In this section we will give the mathematical background and the notation that is used throughout the paper.

**Definition 2.1** Given \( F, G \in \mathcal{M}_{rm} \) and an indeterminate \( s \in \mathcal{F} \), the matrix pencil \( sF - G \) is called regular when \( r = n \) and \( \det(sF - G) \neq 0 \). In any other case, the pencil will be called singular.

**Definition 2.2** The pencil \( sF - G \) is said to be *strictly equivalent* to the pencil \( s\tilde{F} - \tilde{G} \) if and only if there exist nonsingular \( P \in \mathcal{M}_m \) and \( Q \in \mathcal{M}_m \) such as

\[
P(sF - G)Q = s\tilde{F} - \tilde{G}.
\]

In this article, we consider the case that the pencil is *singular*. Unlike the case of the regular pencils, the characterization of a singular matrix pencil, apart from the set of the
determinantal divisors requires the definition of additional sets of invariants, the minimal indices. Let \( \mathcal{N}_r, \mathcal{N}_l \) be right, left null space of a matrix respectively. Then the equations

\[
(sF - G)U(s) = 0_{m, 1}
\]

and

\[
V^T(s)(sF - G) = 0_{1, m}
\]

have solutions in \( U(s), V(s) \), which are vectors in the rational vector spaces \( \mathcal{N}_r(sF - G) \) and \( \mathcal{N}_l(sF - G) \) respectively. The binary vectors \( X(s) \) and \( Y^T(s) \) express dependence relationships among the columns or rows of \( sF - G \) respectively. \( U(s), V(s) \) are polynomial vectors. Let \( d = \dim \mathcal{N}_r(sF - G) \) and \( t = \dim \mathcal{N}_l(sF - G) \). It is known \([46-53]\) that \( \mathcal{N}_r(sF - G), \mathcal{N}_l(sF - G) \), as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees

\[
\epsilon_1 = \epsilon_2 = \ldots = \epsilon_g = 0 < \epsilon_{g+1} \leq \ldots \leq \epsilon_d
\]

and

\[
\zeta_1 = \zeta_2 = \ldots = \zeta_h = 0 < \zeta_{h+1} \leq \ldots \leq \zeta_t
\]

respectively. The set of minimal indices \( \epsilon_i \) and \( \zeta_j \) are known \([46-53]\) as column minimal indices (c.m.i.) and row minimal indices (r.m.i) of \( sF - G \) respectively. To sum up in the case of a singular matrix pencil, we have invariants, a set of elementary divisors (e.d.) and minimal indices, of the following type:

- e.d. of the type \((s - a)^{p_j}, \) finite elementary divisors (nz. f.e.d.)
- e.d. of the type \( s^q = \frac{1}{s^p}, \) infinite elementary divisors (i.e.d.).
- m.c.i. of the type \( \epsilon_1 = \epsilon_2 = \ldots = \epsilon_g = 0 < \epsilon_{g+1} \leq \ldots \leq \epsilon_d, \) minimal column indices
- m.r.i. of the type \( \zeta_1 = \zeta_2 = \ldots = \zeta_h = 0 < \zeta_{h+1} \leq \ldots \leq \zeta_t, \) minimal row indices

**Definition 2.3.** Let \( B_1, B_2, \ldots, B_n \) be elements of \( \mathcal{M}_n. \) The direct sum of them denoted by \( B_1 \oplus B_2 \oplus \ldots \oplus B_n \) is the blockdiag \([ B_1 \ B_2 \ \ldots \ B_n \ ].\)

The existence of a complete set of invariants for singular pencils implies the existence of canonical form, known as Kronecker canonical form \([46-53]\) defined by

\[
sF_K - Q_K := sI_p - J_p \oplus sH_q - I_q \oplus sF_e - G_e \oplus sF_\zeta - G_\zeta \oplus 0_{h,g}
\]

where \( sI_p - J_p \) is uniquely defined by the set of f.e.d.

\[
(s - a_1)^{p_1}, \ldots, (s - a_\nu)^{p_\nu}, \quad \sum_{j=1}^\nu p_j = p
\]

of \( sF - G \) and has the form

\[
sI_p - J_p := sI_{p_1} - J_p(a_1) \oplus \ldots \oplus sI_{p_\nu} - J_p(a_\nu)
\]
The $q$ blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$s^{q_1}, \ldots, s^{q_\sigma}, \sum_{j=1}^{\sigma} q_j = q$$

of $sF - G$ and has the form

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \cdots \oplus sH_{q_\sigma} - I_{q_\sigma}$$

Thus, $H_q$ is a nilpotent element of $M_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_q^\tilde{q} = 0_{q,q},$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in M_{p_j},$$

$$J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & \cdots & 0 & 0 \\ 0 & a_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_j & 1 \\ 0 & 0 & \cdots & 0 & a_j \end{bmatrix} \in M_{p_j},$$

$$H_{q_j} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_{q_j}.$$
3 Main results

Following the given analysis in section 2, there exist non-singular matrices $P, Q$ such that

$$PFQ = F_K, \quad PGQ = G_K.$$  \hfill (3)

Let

$$Q = \begin{bmatrix} Q_p & Q_q & Q_\epsilon & Q_\zeta & Q_g \end{bmatrix}$$  \hfill (4)

where $Q_p \in \mathcal{M}_{rp}, Q_q \in \mathcal{M}_{rq}, Q_\epsilon \in \mathcal{M}_{re}, Q_\zeta \in \mathcal{M}_{r\zeta}$ and $Q_g \in \mathcal{M}_{rg}$

**Lemma 3.1.** System (1) is divided into five subsystems:

$$Z_{k+1}^p = J_p Z_k^p$$  \hfill (5)

the subsystem

$$H_q Z_{k+1}^q = Z_k^q$$  \hfill (6)

the subsystem

$$F_\epsilon Z_{k+1}^\epsilon = G_\epsilon Z_k^\epsilon$$  \hfill (7)

the subsystem

$$F_\zeta Z_{k+1}^\zeta = G_\zeta Z_k^\zeta$$  \hfill (8)

and the subsystem

$$0_{h,g} \cdot Z_{k+1}^g = 0_{h,g} \cdot Z_k^g$$  \hfill (9)

**Proof.** Consider the transformation

$$Y_k = Q Z_k$$

By substituting this transformation into (1) we obtain

$$FQZ_{k+1} = GQZ_k.$$  

Whereby, multiplying by $P$ and using (3), we arrive at

$$F_K Z_{k+1} = G_K Z_k + PV_k.$$  

Moreover, we can write $Z_k$ as

$$Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \\ Z_k^\epsilon \\ Z_k^\zeta \\ Z_k^g \end{bmatrix}$$

where $Z_k^p \in \mathcal{M}_{p1}, Z_q^k \in \mathcal{M}_{q1}, Z_\epsilon^k \in \mathcal{M}_{\epsilon 1}, Z_\zeta^k \in \mathcal{M}_{\zeta 1}$ and $Z_g^k \in \mathcal{M}_{h1}$. Taking into account the above expressions, we arrive easily at the subsystems (5), (6), (7), (8), and (9).

Solving the system (1) is equivalent to solving subsystems (5), (6), (7), (8), and (9).
Remark 3.1. The subsystem (5) is a regular type system and its solution is given from, see [1-28],

$$Z_k^p = j_k^{k-ko} Z_k^p, \quad (10)$$

Remark 3.2. The subsystem (6) is a singular type system but its solution is very easy to compute, see [8-18],

$$Z_k^q = 0_{q,1}. \quad (11)$$

Proposition 3.1. The subsystem (7) has infinite solutions and can be taken arbitrary

$$Z_k^i = C_{k,1}. \quad (12)$$

Proof. If we set

$$Z_k = \begin{bmatrix} Z_k^{i+1} \\ Z_k^{i+2} \\ \vdots \\ Z_k^d \end{bmatrix},$$

by using the analysis in section 2, system (22) can be written as:

$$\text{blockdiag} \{ L_{\epsilon, g+1}, \ldots, L_{\epsilon, d} \} \begin{bmatrix} Z_k^{i+1} \\ Z_k^{i+2} \\ \vdots \\ Z_k^d \end{bmatrix} = \text{blockdiag} \{ L_{\epsilon, g+1}, \ldots, L_{\epsilon, d} \} \begin{bmatrix} Z_k^{i+1} \\ Z_k^{i+2} \\ \vdots \\ Z_k^d \end{bmatrix}.$$

Then for the non-zero blocks a typical equation can be written as

$$L_{\epsilon, i} Z_{k+1}^{\epsilon_i} = L_{\epsilon, i} Z_k^{\epsilon_i}, \quad i = g + 1, g + 2, \ldots, d,$$

or

$$\begin{bmatrix} I_{\epsilon, i} & 0_{\epsilon, 1} \end{bmatrix} Z_{k+1}^{\epsilon_i} = \begin{bmatrix} 0_{\epsilon, 1} & I_{\epsilon, i} \end{bmatrix} Z_k^{\epsilon_i},$$

or

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{i+1} \\ \epsilon_{i+2} \\ \vdots \\ \epsilon_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{i+1} \\ \epsilon_{i+2} \\ \vdots \\ \epsilon_{k+1} \end{bmatrix},$$

or

$$\begin{align*}
\epsilon_{i+1} &= \epsilon_{i+2} \\
\epsilon_{i+2} &= \epsilon_{i+3} \\
\vdots \\
\epsilon_{k+1} &= \epsilon_{k+2}
\end{align*}$$

This is of a regular type system of difference equations with $\epsilon_i$ equations and $\epsilon_i + 1$ unknowns. It is clear from the above analysis that in every one of the $d - g$ subsystems
one of the coordinates of the solution has to be arbitrary by assigned total. The solution of the system can be assigned arbitrary

\[ Z^i_k = C_{k,1} \]

**Proposition 3.2.** The subsystem (8) has the unique solution

\[ Z^\zeta_k = 0_{\zeta,1}. \quad (13) \]

**Proof.** If we set

\[ Z^\zeta_k = \begin{bmatrix} Z^{\zeta_{h+1}}_k \\ Z^{\zeta_{h+2}}_k \\ \vdots \\ Z^{\zeta_t}_k \end{bmatrix}, \]

then the subsystem (8) can be written as:

\[
\text{blockdiag} \{ L_{\zeta_{h+1}}, \ldots, L_{\zeta_t} \} \begin{bmatrix} Z^{\zeta_{h+1}}_k \\ Z^{\zeta_{h+2}}_k \\ \vdots \\ Z^{\zeta_t}_k \end{bmatrix} = \text{blockdiag} \{ \tilde{L}_{\zeta_{h+1}}, \ldots, \tilde{L}_{\zeta_t} \} \begin{bmatrix} Z^{\zeta_{h+1}}_k \\ Z^{\zeta_{h+2}}_k \\ \vdots \\ Z^{\zeta_t}_k \end{bmatrix}.
\]

Then for the non-zero blocks, a typical equation can be written as

\[
L_{\zeta_j} Z^{\zeta_j}_{k+1} = \tilde{L}_{\zeta_j} Z^{\zeta_j}_k, \quad j = h + 1, h + 2, \ldots, t,
\]

or

\[
\begin{bmatrix} I_{\zeta_j} \\ 0_{1,\zeta_j} \end{bmatrix} Z^{\zeta_j}_{k+1} = \begin{bmatrix} 0_{1,\zeta_j} \\ I_{\zeta_j} \end{bmatrix} Z^{\zeta_j}_k,
\]

or

\[
\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} z_{k+1}^{\zeta_{1,1}} \\ z_{k+1}^{\zeta_{1,2}} \\ \vdots \\ z_{k+1}^{\zeta_j} \\ z_{k+1}^{\zeta_{j,j}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} z_{k}^{\zeta_{1,1}} \\ z_{k}^{\zeta_{1,2}} \\ \vdots \\ z_{k}^{\zeta_j} \end{bmatrix},
\]

or

\[
\begin{align*}
\zeta_{k+1}^{\zeta_{1,1}} &= 0 \\
\zeta_{k+1}^{\zeta_{1,2}} &= \zeta_{k}^{\zeta_{1,1}} \\
& \quad \vdots \\
\zeta_{k+1}^{\zeta_{j,j}} &= \zeta_{k}^{\zeta_{j,j-1}} \\
0 &= \zeta_{k}^{\zeta_{j,j}}
\end{align*}
\]

7
We have a system of \( \zeta_j + 1 \) difference equations and \( \zeta_j \) unknowns. Starting from the last equation we get the solutions

\[
\begin{align*}
\zeta_j \zeta_j &= 0 \\
\zeta_j \zeta_j^{-1} &= 0 \\
\zeta_j \zeta_j^{-2} &= 0 \\
&\quad \vdots \\
\zeta_j \zeta_j^{-1} &= 0
\end{align*}
\]

Hence, the system (8) has the following unique solution

\[
Z_k^0 = 0_{\zeta,1}
\]

**Remark 3.3.** The subsystem (9) has an infinite number of solutions that can be taken arbitrary

\[
Z_k^p = C_{k,2}
\]

We can state the following Theorem

**Theorem 3.1.** Consider the system (1), with known initial conditions (2) and a singular matrix pencil \( sF - G \). Then its solution is unique if and only if the c.m.i. are zero

\[
dim N_r(sF - G) = 0
\]

and

\[
Y_{k_0} \in \text{colspan} Q_p
\]

The unique solution is then given from the formula

\[
Y_k = Q_p Z_{k_0}^{k-k_0} Z_k^p
\]

where \( Z_k^p \) is the unique solution of the algebraic system \( Y_{k_0} = Q_p Z_{k_0}^p \). In any other case the system has infinite solutions.

**Proof.** First we consider that the system has non zero c.m.i and non zero r.m.i. By using transformation

\[
Y_k = Q Z_k
\]

then from (10), (11), (12), (13) and (14) the solutions of the subsystems (5), (6), (7), (8) and (9) respectively are

\[
Z_k = \begin{bmatrix}
Z_k^p \\
Z_k^q \\
Z_k^q \\
Z_k^q \\
Z_k^q
\end{bmatrix} = \begin{bmatrix}
J_{p-k_0}^k Z_{k_0}^p \\
0_{q,1} \\
C_{k,1} \\
0_{l-h,1} \\
C_{k,2}
\end{bmatrix}.
\]

Then by using (4)

\[
Y_k = Q Z_k = \begin{bmatrix}
Q_p & Q_q & Q_r & Q_\zeta & Q_s
\end{bmatrix} \begin{bmatrix}
J_{p-k_0}^k Z_{k_0}^p \\
0_{q,1} \\
C_{k,1} \\
0_{l-h,1} \\
C_{k,2}
\end{bmatrix}
\]
and

\[ Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p + Q_s C_{k,1} + Q_g C_{k,2} \]

Since \( C_{k,1} \) and \( C_{k,2} \) can be taken arbitrary, it is clear that the general singular discrete time system for every suitable defined initial condition has an infinite number of solutions. It is clear that the existence of c.m.i. is the reason that the systems (7) and consequently (9) exist. These systems have always infinite solutions. Thus a necessary condition for the system to have unique solution is not to have any c.m.i. which is equal to

\[ \text{dim} \mathcal{N}(sF - G) = 0. \]

In this case the Kronecker canonical form of the pencil \( sF - G \) has the following form

\[ sF_K - Q_K := sI_p - J_p \oplus sH_q - I_q \oplus sF_\zeta - G_\zeta \]

and then the system (1) is divided into the three subsystems (5), (6), (8) with solutions (10), (11), (13) respectively. Thus

\[ Y_k = QZ_k = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix} \begin{bmatrix} J_p^{k-k_0} Z_{k_0}^p \\ 0_{q,1} \\ 0_{1-t_1,1} \end{bmatrix} \]

and

\[ Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p. \]

The solution that exists if and only if

\[ Y_{k_0} = Q_p Z_{k_0}^p, \]

or

\[ Y_{k_0} \in \text{colspan} Q_p. \]

In this case the system has the unique solution

\[ Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p. \]

\section*{References}

[1] T. M. Apostol; Explicit formulas for solutions of the second order matrix differential equation \( Y'' = AY \), Amer. Math. Monthly 82 (1975), pp. 159-162.

[2] R. Ben Taher and M. Rachidi; Linear matrix differential equations of higher-order and applications, E. J. of Differential Eq., Vol. 2008 (2008), No. 95, pp. 1-12.

[3] S. L. Campbell, C.D. Meyer and N.J. Rose; Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, SIAM J. Appl. Math. 31(3) 411-425, (1976).

[4] S. L. Campbell; Comments on 2-D descriptor systems, Automatica (Journal of IFAC), v.27 n.1, p.189-192. (1991).
[5] S. L. Campbell; *Singular systems of differential equations*, Pitman, San Francisco, Vol. 1, 1980; Vol. 2, 1982.

[6] S. L. Campbell; *Singular systems of differential equations*, Pitman, San Francisco, Vol. 1, 1980; Vol. 2, 1982.

[7] L. Dai, *Impulsive modes and causality in singular systems*, International Journal of Control, Vol 50, number 4 (1989).

[8] I.K. Dassios, *Homogeneous linear matrix difference equations of higher order: regular case*, Bull. Greek Math. Soc. 56, 57-64 (2009).

[9] I. Dassios, *On a boundary value problem of a class of generalized linear discrete time systems*, Advances in Difference Equations, Springer, 2011:51 (2011).

[10] I.K. Dassios, *On non-homogeneous linear generalized linear discrete time systems*, Circuits systems and signal processing, Volume 31, Number 5, 1699-1712 (2012).

[11] I. Dassios, *On solutions and algebraic duality of generalized linear discrete time systems*, Discrete Mathematics and Applications, Volume 22, No. 5-6, 665–682 (2012).

[12] I. Dassios, *On stability and state feedback stabilization of singular linear matrix difference equations*, Advances in difference equations, 2012:75 (2012).

[13] I. Dassios, *On robust stability of autonomous singular linear matrix difference equations*, Applied Mathematics and Computation, Volume 218, Issue 12, 6912–6920 (2012).

[14] I.K. Dassios, G. Kalogeropoulos, *On a non-homogeneous singular linear discrete time system with a singular matrix pencil*, Circuits systems and signal processing (2013).

[15] I. Dassios, G. Kalogeropoulos, *On the relation between consistent and non consistent initial conditions of singular discrete time systems*, Dynamics of continuous, discrete and impulsive systems Series A: Mathematical Analysis, Volume 20, Number 4a, pp. 447–458 (2013).

[16] Dassios I., *On a Boundary Value Problem of a Singular Discrete Time System with a Singular Pencil*, Dynamics of continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 22(3): 211-231 (2015).

[17] E. Grispos, S. Giotopoulos, G. Kalogeropoulos; *On generalised linear discrete-time regular delay systems.*, J. Inst. Math. Comput. Sci., Math. Ser. 13, No.2, 179-187, (2000).

[18] E. Grispos, *Singular generalised autonomous linear differential systems.*, Bull. Greek Math. Soc. 34, 25-43 (1992).

[19] Kaczorek, T.; *General response formula for two-dimensional linear systems with variable coefficients*. IEEE Trans. Aurom. Control Ac-31, 278-283, (1986).

[20] Kaczorek, T.; *Equivalence of singular 2-D linear models*. Bull. Polish Academy Sci., Electr. Electrotechnics, 37, (1989).
[21] Kalogeropoulos, Grigoris, and Charalambos Kontzalis. Solutions of Higher Order Homogeneous Linear Matrix Differential Equations: Singular Case. arXiv preprint arXiv:1501.05667 (2015).

[22] J. Klamka, J. Wyrwa, Controllability of second-order infinite-dimensional systems. Syst. Control Lett. 57, No. 5, 386–391 (2008).

[23] J. Klamka, Controllability of dynamical systems, Matematyka Stosowana, 50, no.9, pp.57-75, (2008).

[24] C. Kontzalis, G. Kalogeropoulos. Controllability and reachability of singular linear discrete time systems. arXiv preprint arXiv:1406.1489 (2014).

[25] F. L. Lewis; A survey of linear singular systems, Circuits Syst. Signal Process. 5, 3-36, (1986).

[26] F.L. Lewis; Recent work in singular systems, Proc. Int. Symp. Singular systems, pp. 20-24, Atlanta, GA, (1987).

[27] F. L. Lewis; A review of 2D implicit systems, Automatica (Journal of IFAC), v.28 n.2, p.345-354, (1992).

[28] L. Verde-Star; Operator identities and the solution of linear matrix difference and differential equations, Studies in Applied Mathematics 91 (1994), pp. 153-177.

[29] I. Dassios, A. Zimbidis, The classical Samuelson’s model in a multi-country context under a delayed framework with interaction, Dynamics of continuous, discrete and impulsive systems Series B: Applications & Algorithms, Volume 21, Number 4-5b pp. 261–274 (2014).

[30] I. Dassios, A. Zimbidis, C. Kontzalis. The Delay Effect in a Stochastic Multiplier-Accelerator Model. Journal of Economic Structures 2014, 3:7.

[31] I. Dassios, G. Kalogeropoulos, On the stability of equilibrium for a reformulated foreign trade model of three countries. Journal of Industrial Engineering International, Springer, Volume 10, Issue 3, pp. 1-9 (2014). 10.71 DOI 10.1007/s40092-014-0071-9.

[32] Ogata, K: Discrete Time Control Systems. Prentice Hall, (1987)

[33] W.J. Rugh; Linear system theory, Prentice Hall International (Uk), London (1996).

[34] J.T. Sandefur; Discrete Dynamical Systems, Academic Press, (1990).

[35] A. P. Schinnar, The Leontief dynamic generalized inverse. The Quarterly Journal of Economics 92.4 pp. 641-652 (1978).

[36] F. M. Atici, P. W. Eloe, Initial value problems in discrete fractional calculus, Proceedings of the American Mathematical Society, vol. 137, no. 3, pp. 981–989 (2009).

[37] D. Baleanu, K. Diethelm, E. Scalas, Fractional Calculus: Models and Numerical Methods, World Scientific (2012).

[38] I.K. Dassios, Optimal solutions for non-consistent singular linear systems of fractional nabla difference equations, Circuits, Systems and Signal Processing, Springer, Volume 34, Issue 6, pp. 1769-1797 (2015). DOI 10.1007/s00034-014-9930-2
[39] I.K. Dassios, D. Baleanu, *On a singular system of fractional nabla difference equations with boundary conditions*, Boundary Value Problems, 2013:148 (2013).

[40] I.K. Dassios, D.I. Baleanu. *Duality of singular linear systems of fractional nabla difference equations*. Applied Mathematical Modeling, Elsevier, Volume 39, Issue 14, pp. 4180-4195 (2015). DOI 10.1016/j.apm.2014.12.039

[41] I. Dassios, D. Baleanu, G. Kalogeropoulos, *On non-homogeneous singular systems of fractional nabla difference equations*, Applied Mathematics and Computation, Volume 227, 112–131 (2014).

[42] T. Kaczorek, *Selected problems of fractional systems theory*. Vol. 411. Springer (2011).

[43] T. Kaczorek, *Application of the Drazin inverse to the analysis of descriptor fractional discrete-time linear systems with regular pencils*. Int. J. Appl. Math. Comput. Sci 23.1, 2013: 29–33 (2014).

[44] Kontzalis, Charalambos P., and Grigoris Kalogeropoulos. *A note on the relation between a singular linear discrete time system and a singular linear system of fractional nabla difference equations*. arXiv preprint arXiv:1412.2380 (2014).

[45] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, p. xxiv+340. Academic Press, San Diego, Calif, USA (1999).

[46] H.-W. Cheng and S. S.-T. Yau; *More explicit formulas for the matrix exponential*, Linear Algebra Appl. 262 (1997), pp. 131-163.

[47] B.N. Datta; *Numerical Linear Algebra and Applications*, Cole Publishing Company, 1995.

[48] L. Dai, *Singular Control Systems*, Lecture Notes in Control and information Sciences Edited by M.Thoma and A.Wyner (1988).

[49] R. F. Gantmacher; *The theory of matrices I, II*, Chelsea, New York, (1959).

[50] G. I. Kalogeropoulos; *Matrix pencils and linear systems*, Ph.D Thesis, City University, London, (1985).

[51] Kontzalis, Charalambos P., and Panayiotis Vlamos. *Solutions of Generalized Linear Matrix Differential Equations which Satisfy Boundary Conditions at Two Points*. Applied Mathematical Sciences 9.10 (2015): 493-505.

[52] I. E. Leonard; *The matrix exponential*, SIAM Review Vol. 38, No. 3 (1996), pp. 507-512.

[53] G.W. Steward and J.G. Sun; *Matrix Perturbation Theory*, Oxford University Press, (1990).