We study an off-policy contextual pricing problem where the seller has access to samples of prices which customers were previously offered, whether they purchased at that price, and auxiliary features describing the customer and/or item being sold. This is in contrast to the well-studied setting in which samples of the customer’s valuation (willingness to pay) are observed. In our setting, the observed data is influenced by the historic pricing policy, and we do not know how customers would have responded to alternative prices. We introduce suitable loss functions for this pricing setting which can be directly optimized to find an effective pricing policy with expected revenue guarantees without the need for estimation of an intermediate demand function. We focus on convex loss functions. This is particularly relevant when linear pricing policies are desired for interpretability reasons, resulting in a tractable convex revenue optimization problem. We further propose generalized hinge and quantile pricing loss functions, which price at a multiplicative factor of the conditional expected value or a particular quantile of the valuation distribution when optimized, despite the valuation data not being observed. We prove expected revenue bounds for these pricing policies respectively when the valuation distribution is log-concave, and provide generalization bounds for the finite sample case. Finally, we conduct simulations on both synthetic and real-world data to demonstrate that this approach is competitive with, and in some settings outperforms, state-of-the-art methods in contextual pricing.

Key words: Machine learning, Pricing, Revenue Management, Off-Policy Learning

1. Introduction

There is an increasing amount of data being collected and stored on customers and sales histories. This has led to the development of more targeted pricing algorithms to try and increase revenues, whereby contextual prices are offered to customers depending on the attributes of the customer and/or product. There has been interest in utilizing abundant historical, posted-price data on whether or not each customer purchased a particular item at the price they were offered. As an example studied further in Section 9.2, we explore offering personalized prices to customers for grocery items, based only on previous transaction data on whether each customer purchased an item based on the price they were offered on that day, and demographic information from their participation in a loyalty program. Using such observational data is less costly than running randomized trials (Dubé and Misra 2017), or online algorithms which balance exploration and
exploitation over time (Den Boer 2015), and that may lead to a significant loss of revenue in the short term. Our observational posted-price setting has received less attention than the setting where the seller has access to customer valuation (willingness to pay) samples or distribution information (Mohri and Medina 2014, Dhangwatnotai et al. 2015, Devanur et al. 2016, Medina and Vassilvitskii 2017, Huang et al. 2018, Babaioff et al. 2018, Daskalakis and Zampetakis 2020, Allouah et al. 2021, Beyhaghi et al. 2021). A challenge in the posted-price setting is that we do not observe counterfactual data on whether customers would have purchased if offered different prices, an instance of the fundamental problem of causal inference (Holland 1986). Furthermore, the observed data is influenced by the historic pricing policy, which can make it difficult to estimate how customers will respond to prices that are uncommon (Shalit et al. 2017).

A popular approach in practice is a predict then optimize, or direct method approach, whereby an intermediate contextual demand function is estimated to predict the probability a customer purchases at a given price, and then optimized to maximize revenue (Chen et al. 2015, Ferreira et al. 2016, Dubé and Misra 2017, Alley et al. 2019, Baardman et al. 2020, Biggs et al. 2021b). In practice, often sellers do not know the functional form of demand, leading to a model selection problem. Although advances in machine learning have introduced new models that can capture the complexities of contextual demand more accurately, these models can result in complex revenue maximization problems. Note, this is a consequence of the choice of the demand model used, rather than the inherent properties of the true demand. For example, although tree ensemble or neural network models can result in accurate demand estimation (Ferreira et al. 2016, Mišić 2017, Chen and Mišić 2021, Feldman et al. 2021), both are highly non-linear functions of their inputs, resulting in a non-linear estimated revenue surface when price is optimized. This holds even when considering simple pricing policies, such as linear functions. Furthermore, it is unknown if there exist bounds on expected revenue from optimizing such complex demand functions.

An alternative, more direct approach, is to formulate a loss function for the pricing setting and optimize such loss directly through empirical risk minimization without estimating demand first. At a high level, a loss function provides a way to measure how well a policy performs directly from data. Unlike in typical supervised learning settings, we do not observe the ideal price to charge each customer (their valuation or willingness to pay) which could potentially be considered the labels we are trying to learn. As such, it is not clear how to use the distance from this label as our criteria, as is often done in regression. In the classification setting, surrogate loss functions are often justified through desirable properties of the policy which minimizes them, such as Bayes consistency (predict 1 if $P[Y|X] > 0.5$, 0 otherwise, which is the optimal classification policy (Bartlett et al. 2006)). It is less clear how to define and determine desirable properties of a loss function in our posted price setting. We propose convex pricing loss functions where the pricing policy obtained from optimizing
the loss function has attractive expected revenue bounds. However, unlike the classification setting, we show that there is no convex loss function which can always find the optimal pricing policy, so there is always some gap in expected revenue.

2. Contributions
We propose loss functions for contextual pricing with observational posted price data, where by customers are offered a price as a function of customer and/or feature attributes, which address the aforementioned issues.

Specifically, we propose loss functions that are *convex*, so they can be optimized in a computationally efficient manner. This is particularly relevant when implementing linear pricing policies, which are desirable for interpretability and generalizability, as this results in a convex revenue optimization problem. The importance of interpretable pricing is documented in Amram et al. (2020) and Biggs et al. (2021b). Transparency allows sellers to understand how the algorithm is pricing items, verify this matches their intuition, and ensure the algorithm satisfies any regulatory requirements.

In particular, we propose two loss functions with *bounds on the expected revenue* relative to the optimal revenue achievable from the optimal contextual pricing policy which has access to each customers valuation. The first is a *hinge pricing loss function*, which has similarities with the classification hinge loss function but is adapted to the posted-price setting. This function captures the intuition that if the item sells at the posted price, a higher price should be charged, while if it doesn’t sell, the price needs to be lowered. The second is a *quantile pricing loss function*, which intuitively prescribes prices at a given quantile of prices at which similar customers purchased. We derive optimality conditions for these loss functions, and show that despite not observing valuation data, the resulting pricing policies price at the scaled conditional expected value and at a quantile of the valuation distribution, respectively. Assuming the distribution of customer valuations is log-concave, we prove revenue bounds of 0.566 for the hinge pricing policy and 0.749 for the quantile pricing policy with robustly chosen parameters. To the best of our knowledge, these are the first bounds for the contextual posted-price setting. We also show generalization bounds for performance from a finite sample of observations.

Finally, we provide simulations on both synthetic and real-world data to demonstrate that this approach is competitive with, and in some settings outperforms, state-of-the-art methods in contextual pricing, which although known to work well in practice, do not have known expected revenue guarantees.
3. Other Related Literature

A significant body of literature focuses on the online pricing setting, where the seller chooses prices to balance learning and earning over time (Kleinberg and Leighton 2003, Gallego et al. 2006, Feng 2010, Broder and Rusmevichientong 2012, Harrison et al. 2012, Cheung et al. 2017, Besbes et al. 2018, den Boer and Keskin 2020, Calmon et al. 2021, Keskin et al. 2021). Within this area there has also been a focus on incorporating contextual data to offer more targeted prices (Javanmard and Nazerzadeh 2016, Qiang and Bayati 2016, Bertsimas and Vayanos 2017, Cohen et al. 2018, Nambiar et al. 2019, Cohen et al. 2020, Ban and Keskin 2020, Zhang et al. 2021, Chen and Gallego 2021, Liu et al. 2021). A comprehensive review of earlier work can be found in Den Boer (2015).

As experimentation can be costly, we focus on a setting where we want to maximize revenue in the short term, using only existing observational data.

An important differentiating feature in the pricing literature is how much information the seller has access to. There is a substantial body of work that focuses on a seller with limited samples of valuation data (Dhangwatnotai et al. 2015, Huang et al. 2018, Babaioff et al. 2018, Daskalakis and Zampetakis 2020, Derakhshan et al. 2020, Allouah et al. 2021), including contextual side information (Mohri and Medina 2014, Devanur et al. 2016, Medina and Vassilvitskii 2017). In the setting with a single valuation sample and no contextual data, assuming the valuation distribution is regular, Dhangwatnotai et al. (2015) prove expected revenue bounds of 0.5 by setting the next customer's price equal to the valuation of the first customer. Huang et al. (2018) improve the revenue bound to 0.589 by pricing at a 0.85 multiplicative factor of the valuation observed, provided the valuation distribution is log-concave distribution. Daskalakis and Zampetakis (2020) study the case with two samples and prove bounds of 0.558 for regular distributions, while Allouah et al. (2021) improve to 0.615 and also provide upper and lower bounds for any number of samples in the regular and log-concave valuation distribution setting using a dynamic programming approach. Huang et al. (2018) also studied a data-rich regime, showing that in order to find a $1 - \epsilon$ optimal price, the sample complexity scales polynomially in $1/\epsilon$.

Algorithms also exist under alternative knowledge about the valuation distribution. Cohen et al. (2015) provide bounds when only the support of the valuation distribution is known, Azar et al. (2013) and Chen et al. (2019) use the mean and variance of the valuation distribution, Elmachtaoub et al. (2020) uses the coefficient of deviation of the valuation distribution, while Bergemann and Schlag (2011) use a neighbourhood containing the true valuation distribution. In contrast to all this work, we have posted-price samples on whether the item sells or not at the price they were given, rather than samples or other knowledge of the valuation distribution. Hence, the observed posted-price sales samples are affected by the historic pricing policy (which the valuation samples are not), which needs to be accounted for in any pricing algorithm. Some of this work also focuses
on the ‘value of price discrimination’ rather than practical algorithms for pricing that can be solved tractably.

Closest to our work are efforts to formulate pricing loss functions which can be optimized directly to find pricing policies. [Mohri and Medina 2014] propose a loss function when setting reserve prices for a second price auction, while assuming valuation samples (the maximum each customer is willing to pay) are available. The surrogate functions they propose are non-convex, hence challenging to optimize. Similar to our setting, [Biggs et al. 2021a] propose pricing loss functions for the observational posted-price setting when prices are restricted to a discrete price ladder. Here we focus on continuous prices. In the same setting we study, [Ye et al. 2018] proposes a customized \( \epsilon \)-insensitive loss, used for contextual pricing at Airbnb and is described in more detail in Appendix B. This captures the intuition that if the item sold at the price it was offered, it is desirable to offer a higher price, while if it didn’t sell, a lower price should likely be offered. While this approach is shown to perform well in simulations, there are no guarantees on what the revenue will be from following a pricing policy obtained by optimizing this loss function. Here we provide revenue bounds.

A potential approach to pricing using observational data is to use methods from the off-policy learning literature such as inverse propensity scoring (IPS) [Rosenbaum and Rubin 1983, Beygelzimer and Langford 2009, Li et al. 2011]. Although IPS methods are typically used when the treatment is binary, there has been extensions to continuous treatments, which better captures pricing (i.e., Austin and Stuart 2015, Kallus and Zhou 2018). In Kallus and Zhou (2018) the reward is estimated using a weighted average according to how far the given treatment in the historic data is from the proposed policy as evaluated by a kernel. Kallus and Zhou (2018), however, leads to non-convex optimization problems for most practical choices of the kernel. Our approach borrows the use of inverse propensity weights from this literature.

4. Model
We study a fundamental contextual pricing problem of a monopolist with no inventory constraints which wants to set prices based on historical data to maximize sales in the short term. Each customer is described by features \( X \in \mathcal{X} \) and has an unobserved valuation \( V \in \mathbb{R}_+ \), which is the maximum amount they are willing to pay, both drawn from some distribution. The customer is offered the realization \( P \in \mathbb{R}_+ \) of a stochastic price from a historical pricing policy follows a known conditional distribution with density \( \phi(P|X) \). This assumption supposes that the practitioners know or have recorded the pricing policy used in the past. We observe whether the customer purchases \( Y \in \{0,1\} \), depending on whether the price is above or below their valuation. Specifically,
We do not observe the counterfactual outcomes associated with the customer being given a different price from what was assigned by the historic pricing policy, nor do we observe the valuation of each customer. As such, we have access to an i.i.d dataset of samples $S_n = \{(Y_i, P_i, X_i)\}_{i=1}^n$. For identifiability \cite{Swaminathan2015}, we assume overlap and ignorability.

**Definition 1.** (Overlap) $\phi(p|X) > 0$, $\forall p \in \mathbb{R}_+$

**Definition 2.** (Ignorability) $Y(p) \perp P|X$, $\forall p \in \mathbb{R}_+$

The overlap assumption requires that each price must have a non-zero probability of being offered to each customer. The known impossibility result of counterfactual evaluation also applies when it is not satisfied \cite{Langford2008}. The ignorability assumption requires there are no hidden confounding variables influencing the pricing decision and the customers purchasing decision. It is commonly satisfied as long as the factors which drove historical pricing decisions are available in the observed data \cite{Bertsimas2020}. For clarity of exposition, we also assume that all customers have a positive valuation, i.e. all customers will purchase if given the product for free, $P(V > 0|X) = \bar{F}_V(0) = 1$ and that the complementary CDF (survival function) $\bar{F}_V(v) = \mathbb{P}(V > v|X)$, is log-concave. Recall a function is log-concave if its domain is a convex set and it satisfies

$$f(\theta x + (1-\theta)y) \geq f(x)\theta f(y)^{1-\theta} \quad \forall x, y \in \text{dom } f, \ 0 < \theta < 1$$

The log-concavity assumption encompasses a broad range of valuation distributions including commonly used distributions such as normal, exponential, and uniform \cite{Bagnoli2005}. Furthermore, it is known that if the density function is log-concave, then so is the complementary CDF \cite{Bagnoli2005}. Log-concavity also implies that the hazard rate is monotone, a common assumption in pricing \cite{Cole2015, Huang2018, Allouah2021}. Without log-concavity there are simple examples showing that the revenue gap may be unbounded \cite{Cole2015}.

Our goal is to find a pricing policy, $\pi \in \Pi : X \rightarrow \mathbb{R}_+$, which prescribes a price for each customer. The expected revenue obtained from a policy conditioned on $X$ is:

$$\mathcal{R}(\pi(X)) = \mathbb{E}_V[\pi(X)1\{\pi(X) \leq V\}|X]$$

(2)

That is, if the price offered $\pi(X)$ is less than the customer’s valuation $V$, the item sells and the revenue obtained is the price offered, while the revenue is zero if the price is above the customer’s valuation. Unfortunately, it is not possible to directly optimize $\mathcal{R}(\pi(X))$ because the distribution of
customer valuations is neither known, nor are samples observed. Furthermore, optimizing $R(\pi(X))$, which is non-convex and discontinuous, is challenging (Mohri and Medina 2014). Instead, we need to find a loss function that provides good prices when optimized using the samples we observe $Y, P$. In particular, we aim to find loss functions that are convex, so they can be optimized efficiently, yet still have provable guarantees on expected revenue, relative to the optimal contextual pricing policy, $\arg\max_{\pi(X)} R(\pi(X))$, which has access to the valuation distribution. We start by showing an impossibility result, which states that there is no expected loss function with only access to posted price data able to recover the optimal price, for all valuation distributions. This Proposition is a minor extension from Theorem 2 of Mohri and Medina (2014) which proves that there is no convex surrogate for which the global minimum price is obtained in the case with access to valuation data. We denote a loss function using observational posted-price data as $L(\pi(X), Y, P) : \mathbb{R}_+ \times \{0, 1\} \times \mathbb{R}_+ \to \mathbb{R}$ belonging to the class of functions which are convex in their first argument $L$.

**Proposition 1.** There is no non-constant function $E_{Y, P}[L(\cdot, Y, P) | V = v]$, left continuous in $v$ and with $L(\cdot, Y, P)$ convex with respect to its first argument, such that for any distribution $(Y, P, V) \sim D$ on $\{0, 1\} \times \mathbb{R}_+ \times \mathbb{R}_+$ satisfying Equation [1], there exists a non-negative optimal price $p^* \in \arg\max_p R(p)$, satisfying $\min_p E_{Y, P}[L(p, Y, P)] = E_{Y, P}[L(p^*, Y, P)]$.

The requirement of left continuity of the expected loss is a minor technical requirement that follows from a left continuity assumption in Mohri and Medina (2014). This contrasts with the well-studied classification setting, where there are convex surrogates for the 0/1 loss which can recover the optimal classification policy, a property known as Bayes consistency or classification calibration (Bartlett et al. 2006). For convenience, we provide a table of commonly used notation in Table 1.

### Table 1  A summary of frequently used notation

| Notation | Description |
|----------|-------------|
| $V$      | Valuation of the customer (*unobserved*) |
| $Y$      | Whether the customer purchased the item |
| $P$      | Price posted in the past |
| $X$      | Contextual features describing the customer and or product |
| $\phi(P|X)$ | Conditional historic probability density of a price being offered to a customer |
| $\pi(X)$ | Contextual pricing policy (decision) |
| $R(\cdot)$ | Expected conditional revenue |
| $p^*$    | Optimal price |
| $c$      | Parameter the seller can set for hinge pricing loss |
| $\tau$   | Parameter the seller can set for quantile pricing loss |
| $p_h$    | Price set by the hinge pricing loss function |
| $p_q$    | Price set by the quantile pricing loss function |
5. Pricing Hinge Loss Function

Since it is not possible to find a convex loss function that recovers the optimal pricing policy, we focus on loss functions that are capable of achieving a high proportion of the optimal revenue. We propose the hinge pricing loss function, which bears similarities to the hinge loss function used for classification tasks, as defined in Definition 3 and visualized in Figure 1.

**Definition 3.** The hinge pricing loss function is given by

\[
L_h^c(\pi(X), Y, P) = \frac{1}{\phi(P|X)}[cY(P - \pi(X))^+ + (1 - cY)(\pi(X) - P)^+]
\]

Where \( c \) is a parameter chosen by the seller. This loss function penalizes prices that are below the listed price when the item was sold, and penalizes prices that are above the listed price when the item wasn’t sold. This is reasonable since if an item sold, the customer’s valuation is likely higher than the listed price, so pricing above the listed price is more attractive. Conversely, if an item did not sell, then the customer’s valuation is likely below the listed price, so prescribing below this price should be encouraged. Each item is given by a weight inversely proportional to the probability of receiving the price, \( \phi(P|X) \), to counteract the imbalance due to the historical pricing policy.

When we minimize this loss function, we end up pricing at the expected valuation for each customer, scaled by a parameter \( c \). This is proved in Lemma 1. In some sense \( c \) controls how aggressive the pricing policy is, with a \( c < 1 \) resulting in prices less than the expected valuation, while \( c > 1 \) results in prices above the expected valuation. We focus on \( c < 1 \) as this results in stronger revenue bounds. We provide guidance on how to choose \( c \) following our analysis. If we condition on \( X \) and for notational simplicity denote \( p_h = \arg\min_{\pi(X)} \mathbb{E}_{Y,P}[L_h^c(\pi(X), Y, P)|X] \), then:

**Lemma 1.** (Hinge pricing loss optimality condition): Let \( p_h \) be the minimizer of the hinge pricing loss function, then \( p_h = c\mathbb{E}[V|X] \).
Proof of Lemma 2:

\[ E_{Y,P}[L^h_c(p_h, Y, P)|X] = \int_{\theta}^{p_h} \left( \frac{c\mathbb{P}(Y = 1|p, X)}{\phi(p|X)}(p - p_h)^+ + \frac{1 - c\mathbb{P}(Y = 1|p, X)}{\phi(p|X)}(p_h - p)^+ \right) \phi(p|X)dp \]

Applying Leibniz rule to find the optimality conditions:

\[ \frac{\partial}{\partial p_h} E_{Y,P}[L^h_c(p_h, Y, P)|X] = \int_{p_h}^{\infty} -c\mathbb{P}(Y = 1|p, X)dp + \int_{0}^{p_h} 1 - c\mathbb{P}(Y = 1|p, X)dp \]

\[ = \int_{0}^{p_h} 1dp - c\int_{0}^{\infty} \bar{F}_V(p)dp = p_h - cE[V|X]dp = 0 \]

Therefore, \( p_h = cE[V|X] \) at optimality. \( \square \)

When we optimize the hinge pricing loss function, we can find bounds on the optimal revenue, relative to the best contextual pricing policy which has access to the valuation distribution. We first define the conditional optimal price as \( p^* = \arg \max_p \mathcal{R}(p) \).

**Theorem 1.** For \( 0 < c \leq 1 \):

\[ \frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min \left\{ e^{-c}, \min_{0 < f < 1} \frac{c(f - 1)e^{c(f-1)}}{f \ln(f)} \right\} \]

To prove this bound we need to examine two cases, where the prescribed price is below the optimal price \( p_h < p^* \), and where it is above \( p_h > p^* \). First, we prove two Lemmas which give revenue bounds in each case. These results rely on the valuation distribution being log-concave to restrict how quickly the valuation distribution can diminish. We start with the case where \( p_h < p^* \).

**Lemma 2.** Consider a pricing policy which prescribes a price \( p_h \) such that \( p_h \geq c\int_{0}^{\infty} \bar{F}_V(p)dp \) and \( p_h < p^* \), then:

\[ \frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \min_{0 < f < 1} \frac{c(f - 1)e^{c(f-1)}}{f \ln(f)}. \]

**Proof of Lemma 2:**

\[ \int_{0}^{\infty} \bar{F}_V(p)dp \geq \int_{0}^{p^*} \bar{F}_V(p)dp \geq \int_{0}^{p^*} \bar{F}_V(p^*)\frac{p^*}{\bar{F}_V(p^*)}dp = \left[ \frac{p^*\bar{F}_V(p^*)p^*}{\ln(\bar{F}_V(p^*))} \right]_{0}^{p^*} = \frac{p^*(\bar{F}_V(p^*) - 1)}{\ln(\bar{F}_V(p^*))} \] (3)

Where here the second inequality follows due to log-concavity, \( \bar{F}_V(p) \geq \bar{F}_V(0)\theta \bar{F}_V(p^*)^{1-\theta} \) for \( 0 < \theta < 1 \), and since \( \bar{F}_V(0) = 1 \). Furthermore, by substituting [3] into condition \( p_h \geq c\int_{0}^{\infty} \bar{F}_V(p)dp \):

\[ \frac{p_h}{p^*} \geq \frac{c}{p^*} \int_{0}^{\infty} \bar{F}_V(p)dp \geq \frac{c(\bar{F}_V(p^*) - 1)}{\ln(\bar{F}_V(p^*))} \] (4)
Therefore:
\[
\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h \bar{F}_V(p_h)}{p^* \bar{F}_V(p^*)} \geq \frac{p_h \bar{F}_V(p^*)^{p_h}}{p^* \bar{F}_V(p^*)} \tag{5}
\]

Where the inequality follows from log-concavity. Furthermore, since \(\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \leq 1\), \(\frac{p_h \bar{F}_V(p^*)^{p_h}}{p^* \bar{F}_V(p^*)} \leq 1\). We also have that \(0 \leq \bar{F}_V(p^*) \leq 1\) and \(0 \leq \frac{p_h}{p^*} < 1\). Therefore, according to Lemma 8, which can be found in the Appendix, \(\frac{p_h \bar{F}_V(p^*)^{p_h}}{p^* \bar{F}_V(p^*)}\) is a monotone increasing function within this range, which validates the substitution of (4) into (5):

\[
\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq \frac{c (\bar{F}_V(p^*) - 1)}{\ln(\bar{F}_V(p^*))} \geq \frac{c (\bar{F}_V(p^*) - 1)}{\bar{F}_V(p^*) \ln(\bar{F}_V(p^*))} e^{c(\bar{F}_V(p^*) - 1)} \tag{5} \]

We note that the condition in the above Lemma, \(p_h \geq c \int_0^\infty \bar{F}_V(p) dp\), is clearly satisfied by the optimality conditions for the hinge pricing loss function. We prove an expected revenue bound for the case where the prescribed price is greater than the optimal price \(p_h \geq p^*\) in Lemma 3.

**Lemma 3.** If there exists a pricing rule \(p_h\) satisfying optimality conditions from Lemma 1 and \(p_h > p^*\), then:

\[
\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} \geq e^{-c}
\]

**Proof of Lemma 3:** The function \(\ln(\bar{F}_V(p))\) is concave, so it lies below its tangent at \(p_h\):

\[
\ln(\bar{F}_V(p)) \leq \ln(\bar{F}_V(p_h)) + s(p - p_h) \implies \bar{F}_V(p) \leq \bar{F}_V(p_h)e^{s(p-p_h)}
\]

Where the gradient is \(s = \frac{\partial \ln(\bar{F}_V(p))}{\partial p} \bigg|_{p=p_h} = \frac{\bar{F}_V'(p_h)}{\bar{F}_V(p_h)} < 0\). Furthermore, because \(\bar{F}_V(p) \geq \bar{F}_V(p_h)\frac{p_h}{p^*}\) for \(0 \leq p \leq p_h\) and \(\bar{F}_V(p) \leq \bar{F}_V(p_h)\frac{p_h}{p}\) for \(p_h < p \leq \infty\), it follows that \(\bar{F}_V'(p_h) \leq \frac{\partial \bar{F}_V(p_h)}{\partial p} \bigg|_{p=p_h} = \bar{F}_V'(p_h)\frac{\ln(\bar{F}_V(p_h))}{p_h}\). Furthermore, since \(\bar{F}_V(p) \leq 1\), a refined upper bound is

\[
\bar{F}_V(p) \leq \begin{cases} 
1 & \text{if } p \leq p_h - \frac{\ln(\bar{F}_V(p_h))}{s} \\
\bar{F}_V(p_h)e^{s(p-p_h)} & \text{if } p \geq p_h - \frac{\ln(\bar{F}_V(p_h))}{s}
\end{cases}
\]

Where \(p_h - \frac{\ln(\bar{F}_V(p_h))}{s}\) is where the upper bounds intersect. From the optimality condition in Lemma 1,

\[
p_h = c \int_0^\infty \bar{F}_V(p) dp \leq c \left( \int_0^{p_h - \frac{\ln(\bar{F}_V(p_h))}{s}} 1 dp + \int_{p_h - \frac{\ln(\bar{F}_V(p_h))}{s}}^\infty \bar{F}_V(p) e^{s(p-p_h)} dp \right) = c \left( p_h - \frac{\ln(\bar{F}_V(p_h))}{s} + \frac{\bar{F}_V(p_h)}{s} e^{-\ln(\bar{F}_V(p_h))} \right)
\]

With some rearranging, this implies \(sp_h(\frac{1}{c} - 1) + 1 \geq -\ln(\bar{F}_V(p_h))\). Since \(s \leq \frac{\ln(\bar{F}_V(p_h))}{p_h}\), for \(0 < c \leq 1\):

\[
\ln(\bar{F}_V(p_h))(\frac{1}{c} - 1) + 1 \geq -\ln(\bar{F}_V(p_h)) \implies \bar{F}_V(p_h) \geq e^{-c}
\]

Since \(p_h > p^*\) and \(\bar{F}_V(p^*) \leq 1\), \(\frac{\mathcal{R}(p_h)}{\mathcal{R}(p^*)} = \frac{p_h \bar{F}_V(p_h)}{p^* \bar{F}_V(p^*)} \geq e^{-c}\). □
Lemma 3 is a generalization of Lemma 5.4 in Lovász and Vempala (2007), who proved a similar result for the case when $c = 1$. We can combine these results to prove Theorem 1.

Proof of Theorem 1 follows from applying the optimality conditions in Lemma 1 and combining the bounds in Lemma 2 and 3.

The bound from Theorem 1 is difficult to simplify further due to the need to solve transcendental equations which do not admit closed form solutions. However, since Theorem 1 only has parameters $f$ and $c$, it is possible to find the best revenue bound across all valuation distributions with simulation:

**Corollary 1.** $\max_{0 \leq c \leq 1} \frac{R(p_h)}{R(p^*)} \geq 0.566$, which occurs at $c^* = 0.5685$.

To the best of our knowledge, these are the first bounds for the contextual posted-price setting we study. The fact that the optimal parameter $c$ is less than 1 captures the intuition that it is better to price slightly below the customer’s expected valuation than above it. Indeed, if the price is above the customer’s valuation there will not be a sale and no revenue is gained, whereas if the price is below the customer’s valuation, there will still be a sale and revenue gained even if it is less than could optimally be achieved.

We also note that the seller can use their intuition or knowledge of the problem to guide the choice of $c$. For example, in the case where the selling outcomes are deterministic, i.e. when the valuation is a scalar (or a deterministic function of $X$), then pricing slightly less than the expected valuation is close to optimal. This can be achieved by setting $c$ slightly less to 1, due to the asymmetry of revenue noted above. More variation in the valuation distribution requires a more conservative choice of $c$.

The hinge pricing loss can also be used to adapt the results from Medina and Vassilvitskii (2017) to this setting. Medina and Vassilvitskii (2017) provide pricing algorithms based on access to an estimate of the valuation, such as that obtained by regressing on valuation samples. Lemma 1 shows that using the hinge pricing loss function, for example using $c = 1$, will provide an estimate of the expected value of the valuation distribution which can be used in their algorithm. However, their approach is based on forming clusters of customers with similar valuations and setting a price for each, so it does not result in the interpretable linear policies we desire.

The hinge pricing loss function and analysis can also be used to provide some insight to the pricing loss function proposed in Ye et al. (2018), and described in more detail in Appendix B. In particular, when the parameters in Ye et al. (2018) are set to $c_1 = \infty$ and $c_2 = -\infty$, and the historic pricing policy is uniform, then the loss proposed in Ye et al. (2018) is the same as the hinge pricing loss function. In this case, our bounds apply to Ye et al. (2018).
6. Quantile Pricing Loss Function

While the hinge pricing loss function is intuitive and works well in many practical settings, we also provide an alternative convex loss function which has better expected revenue guarantees:

**Definition 4.** The quantile pricing loss function is given by

\[
L_{q}^{\tau}(\pi(X), Y, P) = \frac{Y}{\phi(P|X)} \left[ (1 - \tau)(P - \pi(X))^{-} + \tau(\pi(X) - P)^{-} \right] 
\]

This is visualized in Figure 2. This can be considered as a weighted quantile loss function for price, only using data for which the product sold. If the product doesn’t sell, there is no contribution to the loss. The weight is inversely proportional to the probability of the product receiving each price \(\phi(P|X)\), to counteract the imbalance due to the historical pricing policy.

Lemma 4 shows that when this loss function is optimized, we price at the \((1 - \tau)^{th}\) percentile of the customers’ valuations. This is despite not having access to valuation data. Let \(p_q = \arg\min E_{Y,P}[L_{q}^{\tau}(\pi(X), Y, P)|X]\).

**Lemma 4.** (Quantile optimality condition): Let \(p_q\) be the minimizer of the quantile pricing loss function, then \(\tau \int_{p_q}^{p} \tilde{F}_{V}(p)dp = (1 - \tau) \int_{p_q}^{\infty} \tilde{F}_{V}(p)dp\).

**Proof of Lemma 4:**

\[
E_{Y,P}[L_{q}^{\tau}(p_q, Y, P)|X] = \int_{p} E[Y|p, X] \frac{\phi(p|X)}{\phi(P|X)} \left[ (1 - \tau)(p - p_q) + \tau(p_q - p) \right]dp 
\]

\[
= \int_{p_q}^{\infty} (1 - \tau)(p - p_q) \tilde{F}_{V}(p)dp + \int_{0}^{p_q} \tau(p_q - p) \tilde{F}_{V}(p)dp 
\]

Where the second equality follows since \(E[Y|p, X] = P(V > p|X)\) from Equation 1. Applying the first order optimality condition, and the Leibniz integral rule:

\[
\frac{\partial}{\partial p_q} E_{Y,P}[L_{q}^{\tau}(p_q, Y, P)|X] = \int_{0}^{p_q} \tau \tilde{F}_{V}(p)dp - \int_{p_q}^{\infty} (1 - \tau) \tilde{F}_{V}(p)dp = 0 
\]

It follows that \(\tau \int_{0}^{p_q} \tilde{F}_{V}(p)dp = (1 - \tau) \int_{p_q}^{\infty} \tilde{F}_{V}(p)dp\). \(\square\)
Using this lemma it is straightforward to show this corresponds to pricing at the \((1 - \tau)^{th}\) percentile of the customers’ valuations. While to the best of our knowledge this loss function is not currently used in practice, it does bear resemblance to some commonly used \textit{ad-hoc} contextual pricing strategies, whereby the price is set to be close to similar products which have recently sold. In the setting with a limited number of valuation samples and no contextual data, Allouah et al. (2021) propose pricing at scaled fractions of order statistics of the valuation sample, which is a similar idea albeit in a different setting. If we condition on \(X\), then Theorem 2 provides a bound on the revenue by optimizing this loss function relative to the optimal revenue.

**Theorem 2.**

\[
\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min \left\{ \frac{\min_{z \geq \tau} z \tau (\ln(z) + 1) - z^2}{\tau - z}, \min_{0 < f < 1} \frac{(1 - \tau) e^{(1 - \tau)(f - 1)}}{f \ln(f)} \right\}
\]

The proof for Theorem 2 requires exploring two cases, where the price chosen by minimizing the quantile pricing loss is above the optimal price and when it is below. For the case where \(p_q < p^*\), the situation is very similar to the hinge pricing loss and Lemma 2 can be used. Next, we bound the case where the price obtained by minimizing the quantile pricing loss function is greater than the optimal price.

**Lemma 5.** If there exists a pricing rule \(p_q\) satisfying optimality conditions from Lemma 2 and \(p_q > p^*\), then:

\[
\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \frac{z \tau (\ln(z) + 1) - z^2}{\tau - z}.
\]

**Proof of Lemma 5:**

\[
\int_{0}^{p_q} \tilde{F}_V(p)dp + \int_{p_q}^{p^*} \tilde{F}_V(p)dp \geq \int_{p_q}^{p^*} \tilde{F}_V(p)dp + \int_{p_q}^{p^*} \tilde{F}_V(p^*) \frac{p_q - p}{p_q - p^*} \tilde{F}_V(p_q) \frac{p - p^*}{p_q - p^*} dp
\]

On the other hand,

\[
\int_{p_q}^{\infty} \tilde{F}_V(p)dp \leq \int_{p_q}^{\infty} \tilde{F}_V(p^*) \frac{p_q - p}{p_q - p^*} \tilde{F}_V(p_q) \frac{p - p^*}{p_q - p^*} dp = \frac{-\tilde{F}_V(p_q)(p_q - p^*)}{\ln(\tilde{F}_V(p_q)) - \ln(\tilde{F}_V(p^*))}
\]

Therefore, from Lemma 2 we have that:

\[
\begin{align*}
\int_{0}^{p_q} \tilde{F}_V(p)dp &= \frac{(1 - \tau)}{\tau} \int_{p_q}^{\infty} \tilde{F}_V(p)dp \\
\Rightarrow p^* \tilde{F}_V(p^*) + \frac{(\tilde{F}_V(p_q) - \tilde{F}_V(p^*)) (p_q - p^*)}{\ln(\tilde{F}_V(p_q)) - \ln(\tilde{F}_V(p^*))} &\leq \frac{(1 - \tau)}{\tau} \frac{-\tilde{F}_V(p_q)(p_q - p^*)}{\ln(\tilde{F}_V(p_q)) - \ln(\tilde{F}_V(p^*))} \\
\Rightarrow \frac{\tilde{F}_V(p^*) \left( \ln \left( \frac{\tilde{F}_V(p_q)}{\tilde{F}_V(p^*)} \right) + 1 \right) - \frac{1}{\tau} \tilde{F}_V(p_q)}{(\tilde{F}_V(p^*) - \frac{1}{\tau} \tilde{F}_V(p_q))} &\leq \frac{p_q}{p^*} \\
\Rightarrow \frac{\tilde{F}_V(p_q) \left( \tilde{F}_V(p^*) \left( \ln \left( \frac{\tilde{F}_V(p_q)}{\tilde{F}_V(p^*)} \right) + 1 \right) - \frac{1}{\tau} \tilde{F}_V(p_q) \right)}{\tilde{F}_V(p^*) \left( \tilde{F}_V(p^*) - \frac{1}{\tau} \tilde{F}_V(p_q) \right)} &\leq \frac{p_q \tilde{F}_V(p_q)}{p^* \tilde{F}_V(p^*)} = \frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)}
\end{align*}
\]
Where (6) follows from $\ln(\bar{F}_V(p_q)) - \ln(\bar{F}_V(p^*)) < 0$ since $p_q > p^*$, and requires $\bar{F}_V(p_q) \geq \tau$, proven in Lemma 9 which can be found in the Appendix. Substituting $z = \frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}$,

$$\frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)(\bar{F}_V(p^*) - \frac{1}{z} \bar{F}_V(p_q))} (\bar{F}_V(p^*) (\ln(z) + 1) - \frac{1}{z} \bar{F}_V(p^*) z) \leq \frac{\bar{F}_V(p_q)}{\bar{F}_V(p^*)}$$

Where the requirement $z \geq \tau$ follows from $\bar{F}_V(p^*) \geq \bar{F}_V(p_q)$, since $p_q > p^*$.

Using these results, we are able to prove Theorem 2.

**Proof of Theorem 2.** From optimality condition in Lemma 1,

$$\tau \int_0^{p_q} \bar{F}_V(p) dp = (1 - \tau) \int_{p_q}^{\infty} \bar{F}_V(p) dp \implies \int_0^{p_q} \bar{F}_V(p) dp = (1 - \tau) \int_0^{\infty} \bar{F}_V(p) dp$$

When $p_q < p^*$, since $p_q \geq \int_0^{p_q} \bar{F}_V(p) dp$, then the conditions for Lemma 2 are satisfied. When $p_q > p^*$, we can apply Lemma 5. Therefore, $\frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq \min \left\{ \min_{z \geq \tau} \frac{z \tau (\ln(z) + 1)}{\tau - z}, \min_{0 < f < 1} \frac{(1 - \tau)(f - 1)e^{(1 - \tau)(f - 1)}}{f \ln(f)} \right\}$.

Similar to the bounds for the quantile pricing function, these bounds are difficult to evaluate analytically. However, the bound can be evaluated empirically using simulation.

**Corollary 2.** $\max_{0 \leq \tau \leq 1} \frac{\mathcal{R}(p_q)}{\mathcal{R}(p^*)} \geq 0.749$, which occurs at $\tau^* = 0.209$.

This corresponds to setting the price to approximately the 79th percentile, out of the items that sold. In the proof for Theorem 2 there is a balance between setting the quantile too low, in which case an adversary can choose a distribution which has a high probability of selling at high prices, versus setting the quantile too high in which case the adversary can make the selling probability low at prices higher than the optimal. However, how quickly the selling probability can decrease with price is limited by the log-concave distribution. The result from Corollary 2 provides practical guidance to the practitioner on how to manage this balance in a robust manner. Similar to the hinge pricing loss function, if the valuation is deterministic (a point mass), a suitable choice of parameter for $\tau$ is close to zero, corresponding to pricing at close to the 100th percentile of tickets sold. We note that although the quantile pricing loss function has a better worst case bound than the hinge pricing loss function, as we will see in Section 9, this doesn’t always yield the highest revenue.
7. Cross-validation for Parameter Selection
The parameters $c$ and $\tau$ can be chosen in such a way to robustly maximize revenue across all valuation distributions, as per Corollary 1 or 2. However, is often too conservative in practice. We can utilize pricing practitioner’s partial knowledge of the valuation distribution or demand to set these parameters. We present a simple heuristic for choosing $c$ and $\tau$ for the specific valuation distribution encountered based on ideas from cross-validation. To evaluate the effectiveness of the model parameters, we propose using a demand model estimated from the available data. Specifically, for each parameter value $\tau_k$ in a discretized set, we find an optimal policy $\hat{\pi}_{\tau_k}$ by optimizing the corresponding hinge or quantile loss function. We then estimate the revenue obtained from this pricing policy using the estimated demand model, and pick the parameter which corresponds to the highest estimated revenue. This approach allows us to utilize an accurate non-parametric demand model (such as a tree-ensemble or neural network), while not needing to be concerned about the difficulty of optimizing such a non-convex model.

Algorithm 1: Cross-validation for contextual pricing

Estimate demand model $\hat{f}(X,P)$ from data ;

for $k \in \{1, \ldots, N\}$ do

Find optimal policy using parameter $\tau_k$, $\hat{\pi}_k = \arg \min_{\pi \in \Pi} \frac{1}{n} \sum_{i=1}^{n} L_{\tau_k}(\pi(X_i),Y_i,P_i)$;

Evaluate estimated reward $\hat{R}_k = \frac{1}{n} \sum_{i=1}^{n} \hat{\pi}_k(X_i)\hat{f}(X_i,\hat{\pi}_k(X_i))$;

end

Find maximum reward $k^* = \arg \max_k \hat{R}_k$, choose $\pi_{k^*}$ as policy;

8. Linear Policies and Generalization Bounds
An important class of pricing policies is the class of linear functions, where $\pi_\theta(X) = \langle \theta, X \rangle$. This class of functions is valued for its simplicity, interpretability and ease of optimization and is well studied in the pricing literature (i.e. [Besbes and Zeevi (2015)]). Furthermore, because of its simplicity, it is less likely to overfit the data, allowing generalization bounds to be established.

The bounds which have been introduced in previous sections apply when optimizing the loss function over the distribution in expectation, which is not known in practice. It is of interest to study how the solution will generalize when optimized using a finite sample of data to form generalization bounds. To achieve these bounds we use regularized loss minimization, in which we jointly minimize the empirical risk and a regularization function which penalizes large values of $\theta$.

$$\hat{\theta}_\lambda = \arg \min_{||\theta|| \leq B} \frac{1}{n} \sum_{i=1}^{n} L(\langle \theta, X_i \rangle, Y_i, P_i) + \lambda ||\theta||^2$$
Generalization bounds are well studied for regularized convex loss functions which are $\rho$-Lipschitz (i.e. Shalev-Shwartz and Ben-David (2014)). In Lemma 6 we show that the quantile and hinge pricing loss functions are $\rho$-Lipschitz. For these results, we will assume that the price range is bounded, $p \in [a, b]$, so that the historical probability of assigning a price $\phi(p|X)$ can be bounded away from 0 by a constant.

**Lemma 6.** Assume there exists $d \leq \phi(p|X)$ for all $X \in X$, $p \in [a, b]$. Then $L^q_\tau(\cdot, Y, P)$ and $L^h_\tau(\cdot, Y, P)$ are $\rho$-Lipschitz, with $\rho = \max\{\tau, 1 - \tau\}/d$.

This simple result follows from taking the maximum absolute gradient of each loss function. This allows us to apply known generalization bounds to this setting, such as Corollary 13.9 from Shalev-Shwartz and Ben-David (2014). First, denote $\theta^* = \arg\min_{\theta} E[L(\langle \theta, X \rangle, Y, P)]$.

**Lemma 7.** For either $L = L^h_\tau(\cdot, Y, P)$ or $L = L^q_\tau(\cdot, Y, P)$, let $\lambda = \sqrt{\frac{2\max\{\tau, 1-\tau\}^2}{d^2B^2n}}$. For any training set of size $n$

$$
E[L(\langle \hat{\theta}_\lambda, X \rangle, Y, P)] \leq E[L(\theta^T X, Y, P)] + \frac{B \max\{\tau, 1 - \tau\} \sqrt{8}}{d\lambda^2} \sqrt{\frac{8}{n}}.
$$

In particular, if $\epsilon > 0$, and $n \geq \frac{8(\max\{\tau, 1-\tau\})^2B^2}{d^2\epsilon^2}$, then $E[L(\langle \hat{\theta}_\lambda, X \rangle, Y, P)] \leq E[L(\langle \theta^*, X \rangle, Y, P)] + \epsilon$.

An advantage of this bound is that it doesn’t explicitly depend on the dimension of the covariate space $X$, although there is a dependence on $B$ since we require $||\theta_\lambda|| \leq B$. Therefore, it is still possible to get good bounds when the covariate space is high-dimensional yet sparse. This bound suggests a large sample size $n$ is needed for small $d$, which occurs when there are some prices which have a very low probability of being assigned $\phi(p|x)$. This happens when the pricing policy is unbalanced. This unbalanced data issue is well known in the off-policy learning community and affects inverse propensity weighting methods, because to account for observations at uncommon prices, large weights must be used, leading to high variance estimates. Approaches including normalization via re-weighting (Lunceford and Davidian 2004, Austin and Stuart 2015), and trimming of the weights to reduce the variance of the estimates (Elliot 2008, Ionides 2008) have been shown to help mitigate this problem. It is also possible to apply generalization bounds to non-linear function classes, although this may result in non-convex revenue optimization which may be challenging to solve.

### 9. Numerical Experiments

We test the proposed loss functions using synthetic and real-world datasets. We benchmark against commonly used direct method approaches that first estimate demand using logistic regression ($dm\_log$, i.e. Chen et al. 2015) and a tree ensemble model ($dm\_lgbm$, i.e. Misić 2017), then optimize estimated revenue to find a pricing policy. In particular we adopt the logistic regression
model implemented in sci-kit learn (Pedregosa et al. 2011), and the lightgbm boosted tree package (Ke et al. 2017) with default parameters in each case. We also benchmark against an Inverse Propensity Weighting (IPW) approach (kern_ipw) adapted to the continuous action setting (Kallus and Zhou 2018), whereby revenue is estimated using a weighted average according to how far the historic price is from the the proposed policy as evaluated by a kernel. We use a Gaussian kernel and set the bandwidth parameter as 0.2, a value with worked well in our setting. The hinge and quantile pricing loss functions (denoted hinge and quant respectively) are solved using the cross-validation technique from Section 7 to find the parameters ($\tau$ and $c$), with 20 rounds of cross-validation. We use the lightgbm model to evaluate the reward in this procedure. In all experiments, we restrict the pricing policy to be a linear function of the covariates, with no intercept term. In all algorithms, the optimal policy is found using the popular BFGS non-linear optimization algorithm (Nocedal and Wright 2006), implemented in scipy (Virtanen et al. 2020), with the maximum number of iterations set to 100. This can handle non-differentiable demand functions such as tree based ensemble methods.

9.1. Synthetic Data

Synthetic experiments are important in this setting due to the lack of publicly available pricing data with counterfactual outcomes on whether a customer would have purchased if a different price been offered. As a result, estimating the revenue of different pricing policies from historical data is challenging. However, with synthetic data, the underlying probability distributions which govern customer behavior are known so pricing policies can be evaluated.

In our synthetic experiments, we propose a number of different data generating situations to test the algorithms under varied demand. We run experiments where valuation distribution is uniformly distributed, with an offset that depends on $X$, such that $V \sim \text{Uniform}(g(X), g(X) + 3)$. We set $X \sim \text{Uniform}(1,2)^m$, and $P \sim \text{Uniform}(1,3)$. As described in Section 4 we do not observe $V$, but only $Y$ as generated by Equation 1. We also generate valuations according an shifted exponential distribution $V \sim \text{Exp}(g(X), \text{loc} = 5), X \sim \text{Uniform}(1,5)^m, P \sim \text{Uniform}(0,15)$. For each distribution, we study a number of different dependencies on $X$, both linear and non-linear, including a linear function $g(X) = \frac{1}{m} \sum_{i=1}^{m} X_{ij}$, a step function, $g(X) = \frac{1}{m} \sum_{i=1}^{m} 1\{X_{ij} \geq \bar{X}_i\}$, where $\bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$ is the average value for each dimension, and an (inverted) absolute value function $g(X) = \frac{1}{m} (X_{i}^{(1)} - \sum_{i=1}^{m} |X_{ij} - \bar{X}_i|)$, where $X_{i}^{(1)} = \max_{j} X_{ij}$ corresponds to the largest value for each dimension. To initialize the BFGS optimizer, we use an initial solution $\theta_{0i} = \frac{1}{m}$, for $i \in [m]$.

We compare the policies generated to the optimal pricing policy in each situation, found by minimizing the valuation loss function (Equation 2), although this valuation data isn’t available to the other algorithms. We report the distance from the proposed to the optimal
solution, $\frac{1}{m} \sum_{i=1}^{m} |\pi^*(X_i) - \hat{\pi}(X_i)|$. We vary the dimension $m \in \{1, 2, 4\}$ and dataset size $n \in \{300, 3000, 30000, 300000\}$. We repeat each simulation 20 times, and report the average with plus/minus one standard error shown shaded in Figures 3, 4 and 5. These figures correspond to the linear, step and absolute function covariate dependence of the customer valuation, respectively.

### 9.1.1. Discussion:
We observe that the hinge and quantile pricing loss functions are competitive with the three benchmarks, and while the best performing method depends on the distribution and the dataset size, hinge and quantile pricing loss tend to perform the best in settings with larger datasets and with valuations which have an exponential distribution. Furthermore, we emphasize that the hinge and quantile pricing loss algorithms have guarantees on expected revenue, whereas the algorithms we benchmark against do not. In nearly all instances, we observe that the direct method using a logistic regression demand model performs well for very small data sets. This is likely due to the bias-variance trade-off inherent in demand models with varying complexity. The logistic regression model is biased, even in the linear setting, since the logistic function is not
able to exactly model uniform or exponential complementary CDF. Despite this bias, for small datasets, it has a much lower variance than the boosted tree or kernel models, which results in better pricing policies for small datasets \((n = 300, 3000)\). While not convex, the estimated logistic regression revenue function is quasi-convex, and empirically the BFGS optimizer is often able to find the minimum, which is not true for the boosted tree or kernel models. We note that none of the approaches we benchmark against result in a convex policy optimization problem, so it is possible that any approach could get stuck, an issue that the hinge and quantile pricing loss functions do not have.

Conversely, for larger datasets \((n = 30000, 300000)\), we observe that the kernel IPW method improves significantly, while the logistic regression does not. This is likely due to the relatively larger bias of the logistic regression approach, which doesn’t decrease with the number of samples. Furthermore, as the number of samples increases, we empirically observe that the demand of the
Figure 5  Distance to optimality, absolute value function

We observe that the hinge and quantile pricing models tend to perform well for larger datasets \((n = 30000, 300000)\), and are often the best models in this range. We hypothesize this is because the boosted tree evaluator used in the cross-validation procedure has a high variance and is not accurate for small samples, but can more accurately estimate revenue for larger samples and pick a suitable parameter via cross-validation. However, we also observe that the hinge and quantile pricing models tend to choose better policies than the boosted tree model used in the cross-validation step. This is because optimizing the boosted tree is very challenging since it is non-continuous with many local minima, whereas for each parameter instance in the cross-validation the hinge and quantile regressions are convex. The lack of convergence to a global optimum solution for kernel and boosted tree methods is also likely why the hinge and quantile regression models have a lower variance than...
these methods, as shown by smaller error bounds. This suggests non-parametric demand models are a risky choice for pricing practitioners.

We observe that the hinge and quantile pricing models do comparatively better for the exponential valuation distribution than the uniform distribution relative to the logistic model. This is likely because there is less bias in the logistic regression model for the uniform distribution, as the logistic function can better approximate the uniform CDF than the exponential CDF. Likewise, the logistic regression models seem to perform worse when there is a non-linear dependence in $X$.

Interestingly, the policies generated by the boosted tree ensemble was inferior to the logistic regression in nearly all simulations, despite the boosted tree having a significantly higher predictive accuracy for larger datasets, empirically suggesting that the predictive performance on its own is not a good criterion for choosing a demand model to optimize.

9.2. Case Study: Personalized Prices for Groceries

We also test the proposed approaches on a real-world dataset, which tracks individual customers and whether they purchased strawberries from a chain of grocery stores at the posted price. In addition, demographic data is collected on the individual customers based on enrollment in a loyalty program, specifically, income, age, gender, marital status, homeownership, size of household, and whether the customer has children. Using this data, Amram et al. (2020) and Biggs et al. (2021b) have studied the potential of offering personalized prices to customers, based on these features to maximize revenue, using interpretable tree-based models. This data was originally collected by the analytics firm Dunnhumby, and we use the cleaned and processed version of the data from Amram et al. (2020), who provide a detailed description of the data. The data size is 97295 rows, corresponding to unique trips to the supermarket, with 3.49% of trips resulting in a sale of strawberries. Once the features described above have been one-hot encoded, it results in a dataset with dimension $m = 34$. Unlike the setting studied in this paper, the historical pricing policy is not known. However, based on the available data, a distribution of historical prices $\phi(p)$ offered is estimated using a lognormal distribution. This pricing policy has no dependence on customer features $X$ since the grocery is not currently using personalized pricing.

To evaluate the pricing policies, we use the approach used in Biggs et al. (2021b). Specifically, since the data doesn’t show the counterfactual sales outcomes associated with different pricing policies, we used a model-based approach to estimate the contextual selling probabilities. These probabilities are estimated using a model trained on a held-out dataset, to avoid bias due to a finite data sample. To achieve this, the data is initially split into a prescription dataset and an evaluation dataset. All models used to prescribe prices (including estimating demand if necessary) are trained on the prescription set. A predictive model is trained on the evaluation dataset, which
can estimate the revenue of any given pricing policy. Following Biggs et al. (2021b), we use a \textit{lightgbm} boosted tree model as the evaluator model as it had the highest out-of-sample accuracy of the methods we tested (81% AUC, compared 71% for logistic regression). In Figure 6, we show the estimated revenue of a linear policy, as evaluated by the out of sample \textit{lightgbm} model, where the linear policy is optimized using the approaches previously discussed. We show results for when the prescription dataset, used to find the pricing policy, increases in size $n$. The remaining data is used as the evaluation set. The simulations are repeated 10 times for each dataset size, and the standard error of the average reward is shown in the shaded area. We omitted 2 trails where the \textit{dm_lgbm} did not converge to a solution.

We observe that the hinge pricing loss function performs the best across all dataset sizes, with the quantile pricing loss function improving to comparable performance for large datasets. Interestingly, we observe that the hinge and quantile pricing loss functions outperform the \textit{dm_lgbm} method, even though the \textit{lightgbm} is the model used to evaluate the estimated revenue. As in the synthetic experiments, this is likely due to the difficulty of optimizing the highly non-linear estimated revenue curve produced by the boosted tree and is also perhaps why there is no significant improvement as the data size increases. The kernel IPW function has a very high variance and is unreliable, while the logistic regression model does not perform well, possibly due to non-linear relationships between covariates in the data which it is not able to capture.

10. Conclusion
We have proposed methods to address the problem of contextual pricing using observational posted-price data rather than the well-studied setting where willingness to pay (valuation) data is available.
We have focused on pricing algorithms which can achieve bounds on expected revenue and can also be computed tractably. In particular, we have proposed two loss functions, the quantile and hinge pricing loss functions, which are convex and can be easily optimized. We have also shown how to choose the relevant parameters to optimize bounds on expected revenue according to an adversarially chosen valuation distribution, and how to heuristically choose parameter values when the seller has access to an estimated demand model which is accurate but challenging to optimize, such as neural networks or tree ensembles. Using both real-world and synthetic data, we have shown that the proposed loss functions are competitive against commonly used contextual pricing approaches, which are known to work well in practice, but do not have theoretical expected revenue bounds and may be unpredictable due to non-convexity.

In terms of limitations and future work, it would be interesting to extend the setting to include inventory and multiple products, which are common in practice. We have focused on presenting expected revenue bounds in the case where the valuation distribution follows a log-concave distribution, which encompasses most commonly used distributions in pricing. A possible avenue for future research is to consider regular valuation distributions (a relaxation of log-convexity).

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**Appendix A: Proofs**

Proof of Lemma 1: If $L(p, (Y,P))$ is convex with respect to its first argument, then so is $E_{Y,P}[L(p, (Y,P)) | V]$ (Boyd et al. (2004), Section 3.2.1). Therefore $E_{Y,P}[L(p, (Y,P)) | V]$ satisfies the conditions for $L_c(p,V)$ from Mohri and Medina (2014).

**Lemma 8.** In the domain $0 \leq x \leq 1$, $0 \leq b \leq 1$ and $xb^{x-1} \leq 1$, $xb^{x-1}$ is monotone increasing.

**Proof of Lemma 8**

$$\frac{\partial}{\partial x} xb^{x-1} = \frac{\partial}{\partial x} x e^{x \ln(b)} = \frac{e^{x \ln(b)}}{b} + \frac{x \ln(b)}{b} e^{x \ln(b)} = \frac{e^{x \ln(b)}}{b} [1 + x \ln(b)]$$

$\frac{e^{x \ln(b)}}{b} \geq 0$ for $b \geq 0$. Therefore the derivative is positive if $1 + x \ln(b) > 0$, or equivalently, $x < \frac{-1}{\ln(b)}$, and negative for $x > \frac{-1}{\ln(b)}$, with the maximum occurring at $x = \frac{-1}{\ln(b)}$. Therefore, the maximum value $xb^{x-1}$ can take is $\frac{e^{-1}}{\ln(b)}$.

It can be verified that $\frac{e^{-1}}{\ln(b)} \geq 1$, for example by taking the derivative, the maximum occurs when $b = e^{-1}$. Finally, $xb^{x-1} = 1$ at $x = 1$. Therefore, $xb^{x-1}$ is monotone increasing in the range $0 \leq x < \frac{1}{\ln(b)}$, and for $\frac{-1}{\ln(b)} \leq x \leq 1$, $xb^{x-1} \geq 1$.

**Lemma 9.** If $p^* \leq p_q$, then $F_V(p_q) \geq \tau$. 

Proof of Lemma \ref{lem:quantile} Due to log-concavity:
\[
\bar{F}_V(p) \geq \bar{F}_V(p_q)^{\frac{p_q}{p}} \quad \forall p \leq p_q \quad \text{and} \quad \bar{F}_V(p_q) \geq \bar{F}_V(p)^{\frac{p_q}{p}} \quad \forall p \geq p_q
\]
Therefore,
\[
\int_0^{p_q} \bar{F}_V(p)dp \geq \int_0^{p_q} \bar{F}_V(p_q)^{\frac{p_q}{p}} = \left[ p_q \bar{F}_V(p_q)^{\frac{p_q}{p_q}} \frac{p_q}{\ln(F_V(p_q))} \right]_0^{p_q} = \frac{p_q(\bar{F}_V(p_q) - 1)}{\ln(F_V(p_q))}
\]
\[
\int_{p_q}^{\infty} \bar{F}_V(p)dp \leq \int_{p_q}^{\infty} \bar{F}_V(p_q)^{\frac{p_q}{p}} = \left[ p_q \bar{F}_V(p_q)^{\frac{p_q}{p_q}} \frac{p_q}{\ln(F_V(p_q))} \right]_{p_q}^{\infty} = -\frac{p_q \bar{F}_V(p_q)}{\ln(F_V(p_q))}
\]
From optimality condition in Lemma \ref{lem:optimality}
\[
\int_0^{p_q} \bar{F}_V(p)dp = \frac{1-\tau}{\tau} \int_{p_q}^{\infty} \bar{F}_V(p)dp \implies p_q(\bar{F}_V(p_q) - 1) \frac{p_q}{\ln(F_V(p_q))} \leq -\frac{1-\tau}{\tau} p_q \bar{F}_V(p_q)
\]
\[
\implies p_q(\frac{1}{\tau} \bar{F}_V(p_q) - 1) \frac{p_q}{\ln(F_V(p_q))} \leq 0
\]
Therefore, $\bar{F}_V(p_q) \geq \tau$, since $\ln(F_V(p_q)) < 0$.

Appendix B: \textit{$\epsilon$-insensitive pricing loss function}

Ye et al. (2018) proposes the a customized $\epsilon$-insensitive loss used for contextual pricing at Airbnb. Using notation from Section \ref{sec:notation}, the proposed loss is:
\[
L(\pi(X), Y, P) = Y[(P - \pi(X))^+ + (\pi(X) - c_1 P)^+] + (1-Y)[(\pi(X) - P)^+ + (c_2 P - \pi(X))^+]
\]
Where $c_1 > 0$ and $c_2 < 0$ are parameters set by the user. This loss function shown in Figure \ref{fig:loss_function}. This loss function captures the intuition that if the item sold ($Y = 1$), then the customer's valuation is higher than the price the customer was given $P$. Therefore, pricing below the price the customer was given ($\pi(X) \leq P$) results in an opportunity cost as a higher price could have been charged with the customer still purchasing. To increase revenue, the customer should be given a price higher than the price they were offered, although it is not clear how much higher, since eventually the customer will not purchase if the price gets too high. This is reflected in in there being no loss between $P$ and $c_1 P$, but an increasing loss being incurred above $c_1 P$. Similarly, if the item doesn’t sell for a given customer, then it is likely the price should be lowered, but again, it is not clear by how much. One drawback of this approach is that it is not clear how to choose
constants $c_1$ and $c_2$. An advantage, of this approach relative to the predict then optimize approaches, is that
the loss function is convex so it is easier to optimize. While this is desirable, it is of little use if the policy
it converges to is not justified. While this approach is shown to perform well in simulations and is intuitive,
there are no guarantees on what the revenue will be from following a pricing policy obtained by optimizing
this loss function. Furthermore, there isn’t any analysis of how this approach performs for the pricing setting
in particular.