PLATONIC AND ALTERNATING 2-GROUPS

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ABSTRACT. We recall Schur’s work on universal central extensions and develop the analogous theory for categorical extensions of groups. We prove that the String 2-groups are universal in this sense and study in detail their restrictions to the finite subgroups of the Spin groups. Of particular interest are subgroups of the 3-sphere Spin(3), as well as the spin double covers of the alternating groups, whose categorical extensions turn out to be governed by the stable 3-stem $\pi_3(S^3)$.

1. Introduction

By a categorical group or 2-group, we mean a small monoidal groupoid $(\mathcal{G}, \bullet, 1)$ with weakly invertible objects. We will think of such a $\mathcal{G}$ as a categorical extension

$$\begin{array}{ccc}
1/\mathbb{A} & \longrightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
1/ \mathbb{U}(1) & \longrightarrow & String(3) \longrightarrow S^3,
\end{array}$$

where $\mathcal{G} = \pi_0\mathcal{G}$ is the group of isomorphism classes of objects in $\mathcal{G}$ and the abelian group

$$\mathbb{A} = \pi_1\mathcal{G} = \text{aut}_{\mathcal{G}}(1)$$

is the center of $\mathcal{G}$. The purpose of this note is to study two families of finite categorical groups, sitting inside the Lie 2-groups $String(n)$. First, we discuss the platonic 2-groups, which are categorical extensions of the finite subgroups of the three sphere and have as center a cyclic group of order $|G|$. The platonic 2-groups are of interest, because the finite subgroups of

$$S^3 = SU(2) = Spin(3)$$

are the protagonists of the McKay correspondence. Their list consists of the cyclic and the binary dihedral groups, plus the three exceptional cases: the binary tetrahedral group $2T \cong \tilde{A}_4$, the binary octahedral group $2O \cong \tilde{S}_4$, and the binary icosahedral group $2I \cong \tilde{A}_5$. The fact that there are canonical categorical extensions of all these groups suggests a categorical aspect of McKay correspondence that seems worth exploring.

The second family of examples consists of the alternating 2-groups $\mathbb{A}_n$, which are related to the stable homotopy groups of spheres by the tower

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Here

\[ O = \text{colim}O(n) \]

is the infinite orthogonal group, and the homotopy groups turning up are

\[ \pi_1(S^0) \cong \mu_2, \quad \pi_2(S^0) \cong \mu_2, \quad \pi_3(S^0) \cong \mu_{24}, \quad \text{and} \quad B\pi_3(O) = U(1). \]

The homomorphism \( \tilde{\varrho}_n \) is the permutation representation, and

\[ e: \pi_3(S^0) \longrightarrow U(1) \]

is the Adams e-invariant. In the philosophy of [Kap15], the stable 1-stem yields the sign governing super-symmetry, while the stable 2-stem provides the sign governing categorified supersymmetry. It was Kapranov’s question about a conceptual description of the stable 3-stem in this context that motivated our work. For \( n \) sufficiently large, the alternating 2-groups turn out to be universal in an appropriate sense. A consequence of this is the following result.

**Theorem 1.1.** The restriction of \( \text{String}(n) \) to \( \tilde{A}_n \) has exact order 24 for all \( n \geq 4 \). The restriction of \( \text{String}(3) \) to a finite subgroup \( G \subseteq S^3 \) has exact order \( |G| \).

It would be interesting to have a direct proof of Theorem 1.1 using any of the known constructions of the String 2-groups. Further, one can think of \( \pi_3(S^0) \) as framed bordism group, generated by the three sphere in its invariant framing and use a \( K3 \)-surface with little holes cut out as a null-bordism of 24\( S^3 \), suggesting a potential connection with the categorical groups turning up in Mathieu Moonshine.

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**2. Extensions and group cohomolgy**

Let \( G \) be a group, and write

\[ H_*(G) = H_*^{gp}(G, \mathbb{Z}) \]

for the group homology of \( G \) with coefficients in the trivial \( G \)-module \( \mathbb{Z} \). Then

\[ H_1(G) \cong G^{ab} \]
is the abelianization of $G$, and $H_2(G)$ is the Schur multiplier of $G$. We will refer to $H_3(G)$ as the categorical Schur multiplier of $G$. Recall that a group $G$ is called perfect if its abelianization is trivial and that a perfect group is called superperfect if its Schur multiplier also vanishes. The smallest non-trivial example of a superperfect group is the binary icosahedral group, whose categorical Schur multiplier is

$$H_3(2I) \cong \mu_{120},$$

see [Hau78]. A list of the categorical Schur multipliers of some superperfect groups exists as HAP library.

**Definition 2.1.** A central extension

$$A_{uni} \rightarrow \tilde{G}_{uni} \rightarrow G$$

of finite dimensional Lie groups is called a Schur cover of $G$, if it is universal in the following sense: for any finite dimensional central extension

$$A \rightarrow \tilde{G} \rightarrow G$$

of $G$ there exists a unique map of central extensions

$$A_{uni} \rightarrow \tilde{G}_{uni} \rightarrow G$$

$$A \rightarrow \tilde{G} \rightarrow G.$$

If it exists, the Schur cover of $G$ is unique up to unique isomorphism. We recall two classical results about Schur covers.

**Theorem 2.2** (Schur 1904). Let $G$ be a perfect discrete group. Then $G$ possesses a Schur cover, whose central subgroup is the Schur multiplier

$$A_{uni} = H_2(G).$$

**Lemma 2.3** (Second Whitehead Lemma). Let $G$ be a semisimple, compact and connected Lie group. Then the universal covering group of $G$ is a Schur cover. Its central subgroup is the fundamental group

$$A_{uni} = \pi_1(G).$$

To emphasize the analogy between these two statements, let $BG$ be the classifying space of $G$. Then we have

$$\pi_i(BG) = \pi_{i-1}(G).$$

So, the Lie group $G$ is connected if and only if $\pi_1(BG)$ is trivial. In this case, we have

$$H_1(BG; \mathbb{Z}) = 0$$

and

$$H_2(BG; \mathbb{Z}) = \pi_2(BG) \cong \pi_1(G).$$

The goal of this section is to develop the theory of Schur covers in the context of categorical central extensions. Let $G$ be a finite dimensional Lie group, and write $\text{Ext}(G)$ for the bicategory of finite dimensional central Lie 2-group extensions.
as in [SP11]. Central in this context means that the conjugation action of $G$ on $A$ is trivial, Lie means that $\mathcal{G}$ is a finite dimensional Lie groupoid and that the additional data (tensor multiplication, associator, etc.) are required to be locally continuous and smooth in an appropriate sense.

**Definition 2.4.** A categorical Schur cover of $G$ is an initial object in $\mathcal{E}xt(G)$.

Explicitly,

\[
\begin{array}{ccc}
1/\!/ A & \longrightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
1/\!/ A & \longrightarrow & G
\end{array}
\]

is a categorical Schur cover of $G$ if for any other finite dimensional central Lie 2-group extension as above there exists a 1-morphism

\[
\begin{array}{ccc}
1/\!/ A & \longrightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
1/\!/ A & \longrightarrow & G
\end{array}
\]

in $\mathcal{E}xt(G)$, which is unique up to unique 2-isomorphism. If it exists, the categorical Schur cover of $G$ is unique up to equivalence, which in turn is unique up to unique isomorphism. The goal of this section is to prove the following result.

**Theorem 2.5.**

(1) Let $G$ be a superperfect discrete group. Then $G$ possesses a categorical Schur cover, whose center is

\[ A_{uni} = H_3(G). \]

(2) Let $G$ be a simply connected compact Lie group, and let $s$ be the number of simple factors of $G$. Then $G$ possesses a categorical Schur cover, whose center is

\[ A_{uni} = U(1)^s. \]

Note that simply connected compact Lie groups are automatically semi-simple [MT91, Thm.5.29], so that the statement in (2) makes sense. Let

\[
\begin{array}{cccccc}
C^0 & \overset{d}{\longrightarrow} & C^1 & \overset{d}{\longrightarrow} & \cdots & \overset{d}{\longrightarrow} & C^{n-1} & \overset{d}{\longrightarrow} & Z^n
\end{array}
\]

be a cochain complex of abelian groups. Recall that the Dold-Kan $n$-groupoid of $C^\bullet$ is the (strictly symmetric monoidal) strict $n$-groupoid with objects $Z^n$ and arrows

\[
\begin{array}{ll}
1\text{Hom}(\alpha, \beta) & = \{ \gamma \in C^{n-1} \mid d\gamma = \beta - \alpha \} \\
2\text{Hom}(\gamma, \delta) & = \{ \epsilon \in C^{n-2} \mid d\epsilon = \delta - \gamma \} \\
\ldots
\end{array}
\]

Composition of arrows is given by addition, and so is the monoidal structure. The following theorem summarizes results of Schur (1911), Singh (1976), Schommer-Pries [SP11, Thm.99], Wagemann and Wockel [WW15], Schreiber [Sch13].
**Theorem 2.6.** Let $G$ and $A$ be finite dimensional Lie groups with $A$ abelian. Let $C_\text{gp}^\bullet(G; A)$ be the cochain complex of locally continuous group cocycles on $G$ with values in the trivial $G$-module $A$ as in [WW15, Def.I.1]. Then the Dold-Kan groupoid of

$$C^1_{\text{gp}}(G; A) \xrightarrow{d} Z^2_{\text{gp}}(G; A)$$

is equivalent to the symmetric monoidal category of central extensions of the form

$$A \longrightarrow \tilde{G} \longrightarrow G.$$ 

The Dold-Kan 2-groupoid of

$$C^1_{\text{gp}}(G; A) \xrightarrow{d} C^2_{\text{gp}}(G; A) \xrightarrow{d} Z^3_{\text{gp}}(G; A)$$

is equivalent to the symmetric monoidal bicategory of categorical central extensions of the form

$$1//A \longrightarrow \mathcal{G} \longrightarrow G.$$

To be specific, let $\alpha$ be an $A$-valued 3-cocycle on $G$. Then we have the (skeletal) groupoid

$$\mathcal{G}_\alpha = \begin{pmatrix} G \times A \\ \downarrow \\ G \end{pmatrix}$$

The group multiplications give a monoidal structure on $\mathcal{G}_\alpha$ with the associator encoded in $\alpha$, and the unit maps trivial. This is the categorical group associated to $\alpha$ by the equivalence in the theorem.

**Corollary 2.7 ([Bro94], [SP11]).** Assume we are given finite dimensional Lie groups $G$ and $A$ with $A$ abelian. Then the following hold.

1. Classes in $H^2_{\text{gp}}(G; A)$ are in one to one correspondence with isomorphism classes of central extensions of $G$ by $A$.
2. If we have

$$H^1_{\text{gp}}(G; A) \cong 0,$$

then a degree 2 class as in (1) determines the corresponding central extension uniquely up to unique isomorphism.
3. Classes in $H^3_{\text{gp}}(G; A)$ are in one to one correspondence with equivalence classes of categorical central extensions of $G$ with center $A$.
4. If we have

$$H^i_{\text{gp}}(G; A) \cong 0$$

for $i = 1$ and $i = 2$, then a degree 3 class as in (3) determines the corresponding categorical central extension uniquely up to equivalence, which in turn is unique up to unique isomorphism.
Let $G$ be a discrete group. Write $BG$ for the classifying space (geometric realization of the nerve) of $G$. Then we have isomorphisms
\[ H^i_{gp}(G) \cong H_i(BG), \]
\[ H^i_{gp}(G; A) \cong H^i(BG; A), \]
where on the right-hand side we have singular (co)homology. The universal coefficient theorem gives the short exact sequence
\[ 0 \rightarrow Ext(H_{i-1}(G), A) \rightarrow H^i(G; A) \rightarrow Hom(H_i(G), A) \rightarrow 0, \]
natural in $A$.

**Proof of Theorems 2.2 and 2.5 (1).** Assume that $G$ is perfect, and let $A$ be an abelian group, viewed as trivial $G$-module. Then the universal coefficient theorem with $i = 1$ implies
\[ 0 = H^1(G; A) = Z^1(G, A). \]
By Theorem 2.6 it follows that the groupoid of central extensions of $G$ by $A$ is discrete in the sense that there are no non-identity automorphisms. Using the universal coefficient theorem with $i = 2$, it follows that the isomorphism classes of said groupoid are parametrised by $Hom(H_2(G), A)$. Now allow $A$ to vary. Then we obtain an equivalence from the category of central extensions of $G$ to the under category $H_2(G) \downarrow Ab$, sending a central extension to the homomorphism classifying it. In particular, there is a universal central extension, which is characterised uniquely, up to unique isomorphism, by the fact that it is classified by $id_{H_2(G)}$. The proof for Theorem 2.5 (1) is analogous.

We also have the following corollary of the universal coefficient theorem (using injectivity of the circle group).

**Corollary 2.8.** If $G$ is finite, we have a non-canonical isomorphism
\[ H^i(G; U(1)) \cong H^i(G) \cong H_i(G). \]
In particular, the (categorical) Schur multiplier of a finite group $G$ classifies (categorical) central extensions of $G$ by the circle group.

These categorical central extensions by the circle group are of interest for the theory of projective 2-representations \cite{GU14}, just like central group extensions by the circle group are of interest for the theory of projective representations.

**Definition 2.9.** Let $G$ be a categorical group with center $A$. Let $\phi: A \rightarrow B$ be a homomorphism to another abelian group. Then the categorical group with center $B$ associated to $G$ (via $\phi$) is the groupoid $G[\phi]$ with objects identical to those of $G$ and arrows the pairs $(f, b)$ with $f$ an arrow of $G$ and $b \in B$, modulo the equivalence relation
\[ (f \bullet a, b) \sim (f, \phi(a) + b), \quad a \in A. \]
The multiplication data are inherited from $G$.

Given an abelian group $H$, we will write $H \downarrow Ab$ for the category of abelian groups under $H$.

**Theorem 2.10.** (1) For a perfect group $G$ with Schur cover $\tilde{G}_{uni}$, the functor
sending the homomorphism

\[ \phi : H_2(G) \to A \]

to the balanced product

\[ A \times_{H_2(G)} \tilde{G}_{\text{uni}} \]

is an equivalence of categories.

(2) For a superperfect group \( G \) with categorical Schur cover \( \tilde{G}_{\text{uni}} \), the functor

\[ \tilde{G}_{\text{uni}}[-] : H_3(G) \downarrow \text{Ab} \to \text{Ext}(G) \]

is an equivalence of (bi)categories.

**Proof.** The first part is classical, we prove (2). The universal coefficient theorem implies that the bicategory \( \text{Ext}(G) \) has only identity 2-morphisms and that we have an abstract equivalence

\[ H_3(G) \downarrow \text{Ab} \sim \text{Ext}(G). \]

The identity map of \( H_3(G) \) is an initial object of \( H_3(G) \downarrow \text{Ab} \). Under the isomorphism of the universal coefficient theorem, this corresponds to the class of a 3-cocycle \( \alpha_{\text{uni}} \) with values in \( H_3(G) \), and for arbitrary \( A \), the universal coefficient isomorphism is

\[ H^3(G; A) \cong \text{Hom}(H_3(G), A) \]

\[ [\phi, \alpha_{\text{uni}}] \mapsto \phi, \]

by naturality. If

\[ \tilde{G}_{\text{uni}} = \tilde{G}_{\alpha_{\text{uni}}}, \]

then, by construction,

\[ \tilde{G}_{\phi, \alpha_{\text{uni}}} \simeq \tilde{G}_{\text{uni}}[\phi]. \]

An inverse of the functor \( \tilde{G}_{\text{uni}}[-] \) restricts the unique 1-morphism \( \tilde{G}_{\text{uni}} \to G \) to centers. \( \square \)

**Definition 2.11.** Let \( G \) be a discrete group, not necessarily perfect. Assume that the Schur multiplier \( H_2(G) \) vanishes. Then we still have the cohomology class \( [\alpha_{\text{uni}}] \) and \( \tilde{G}_{\text{uni}} \) as in the above proof. We will refer to any Lie 2-group extension equivalent to \( \tilde{G}_{\text{uni}} \) as a **weak categorical Schur cover** of \( G \).

In the situation of the definition,

\[ \tilde{G}_{\text{uni}}[-] : H_3(G) \downarrow \text{Ab} \to \text{Ext}(G) \]

is still essentially bijective, but may no longer be an equivalence of bicategories. Let now \( G \) be a simply connected compact Lie group, and let \( T \) be a finite dimensional abelian Lie group with cocharacter lattice

\[ \tilde{T} := \text{Hom}(U(1), T). \]
Theorem 2.12. Let $s$ be the number of simple factors of $G$. Then
\[ H^1_{gp}(G; T) = H^2_{gp}(G; T) = 0, \]
and we have an isomorphism
\[ H^3_{gp}(G; T) \cong \tilde{T}^s, \]
which is natural in $G$.

Proof. We have
\[ \pi_i(BG) = \pi_{i-1}(G) = \begin{cases} 0 & i \leq 3 \\ \mathbb{Z}^s & i = 4, \end{cases} \]
[MT91 Thm. 4.17]. Using Hurewitz and the universal coefficient theorem, this implies
\[ H^i_{gp}(G; A) = H^i(BG; A) = \begin{cases} 0 & 1 \leq i \leq 3 \\ A^s & i = 4 \end{cases} \]
for discrete coefficients $A$. If $T_0$ is the connected component of 0, then the short exact sequence
\[ T_0 \longrightarrow T \longrightarrow T/T_0, \]
gives isomorphisms
\[ H^i_{gp}(G; T_0) \cong H^i_{gp}(G; T) \]
for $1 \leq i \leq 3$. Let $S \subseteq T_0$ be a maximal compact subgroup. Then $T_0$ is the product of $S$ with $\mathbb{R}^m$ for some $m$. By [Hu52 Thm. 2.8], the cohomology of a compact Lie group with coefficients in $\mathbb{R}^m$ vanishes for $i \geq 1$. Hence the short exact sequence
\[ S \longrightarrow T_0 \longrightarrow T_0/S \]
gives isomorphisms
\[ H^i_{gp}(G; S) \cong H^i_{gp}(G; T_0). \]
At the same time, we have
\[ \tilde{S} = \tilde{T}. \]
We may therefore assume, without loss of generality, that $T$ is a compact torus. Finally, the long exact cohomology sequence for
\[ \tilde{T} \hookrightarrow t \longrightarrow T, \]
with $t$ the Lie algebra of $T$, gives isomorphisms
\[ H^i_{gp}(G; T) \cong H^{i+1}(BG; \tilde{T}). \]
\[ \square \]

The statements of Lemma 2.3 and Theorem 2.5 (2) can be derived from this result:

Proof of the Second Whitehead Lemma. By Weyl’s theorem, the simply connected group $\tilde{G}$ is again compact, and by Theorem 2.12 it has no non-trivial central extension with finite dimensional center. \[ \square \]
Proof of Theorem 2.5 (2). We have functorial isomorphisms
\[
\tilde{T}^s = \text{Hom}(\mathbb{Z}^s, \text{Hom}(U(1), T)) \\
\cong \text{Hom}(U(1) \otimes \mathbb{Z}^s, T) \\
= \text{Hom}(U(1)^s, T).
\]
So, Theorem 2.12 implies that the bicategory \( \mathcal{E}xt(G) \) is equivalent to the category of finite dimensional abelian Lie groups under \( U(1)^s \).

Example 2.13 ([SP11, Thm.100]). For \( n = 3 \) and \( n \geq 5 \), the string extension
\[
\frac{1}{U(1)} \longrightarrow \text{String}(n) \longrightarrow \text{Spin}(n)
\]
is the universal central Lie 2-group extension of the simple and simply connected Lie group \( \text{Spin}(n) \).

3. The Cyclic Groups

As a warm-up to the platonic and alternating case, we study the categorical extensions of the finite subgroups of the circle group. The finite cyclic groups have integral homology
\[
H_i(\mu_n) = \begin{cases} 
\mathbb{Z} & i = 0, \\
\mu_n & i \text{ odd, and} \\
0 & \text{else.}
\end{cases}
\]
This implies that \( \mu_n \) possesses a weak categorical Schur cover \( C_n \), whose centre is \( \mu_n \). Let \( \mathcal{U}(1)^- \) be the categorical extension of the circle group classified by the standard generator of \( H^3_{\text{gp}}(U(1); U(1)) \cong H^4(BU(1); \mathbb{Z}) \cong \mathbb{Z} \).

We will see that there is a 1-morphism of categorical central extensions
\[
\begin{array}{c}
\frac{1}{\mu_n} \longrightarrow C_n \longrightarrow \mu_n \\
\downarrow \downarrow \\
\frac{1}{U(1)} \longrightarrow \mathcal{U}(1)^- \longrightarrow U(1),
\end{array}
\]
identifying \( C_n \) with a sub-categorical group of \( \mathcal{U}(1)^- \). Let \( \mathbb{R} \) act on \( \mathbb{Z} \times U(1) \) by
\[
x \cdot (m, z) := (m, z \cdot e^{-2\pi imx}),
\]
and recall that
\[
\mathcal{U}(1)^- = \left( \mathbb{R} \ltimes (\mathbb{Z} \times U(1)) \right)
\]
is constructed as the strict categorical group corresponding to the crossed module

\footnote{In [Gan14], we make the convention that the basic categorical extension of the circle group is the 2-group \( \mathcal{U}(1) \) classified by the other generator. These two 2-groups differ by a sign in the action.}
\( \mathbf{v} : \mathbb{Z} \times U(1) \to \mathbb{R} \)
\[
(m, z) \mapsto m,
\]

see [Gan14]. In other words, \( U(1)^- \) has as objects \( \mathbb{R} \) and as arrows
\[
\left\{ x \xrightarrow{z} x + m \mid x \in \mathbb{R}, m \in \mathbb{Z}, \text{ and } z \in U(1) \right\},
\]
composing two arrows means multiplying their labels, and the strict monoidal structure is
given by the respective group structures of objects and arrows.

**Lemma 3.1.** The weak Schur cover \( C_n \) of \( \mu_n \) can be constructed as the strict categorical group corresponding to the sub-crossed module
\[
\begin{array}{ccc}
\mathbb{Z} \times \mu_n & \to & \mathbb{Z} \times U(1) \\
\kappa \downarrow & & \downarrow \mathbf{v} \\
\frac{1}{n} \mathbb{Z} & \to & \mathbb{R}.
\end{array}
\]

**Proof.** The circle group \( U(1) \) acts by multiplication on the spheres \( S^{2k-1} \subset \mathbb{C}^k \), and on
\[
S^\infty = \text{colim}_k S^{2k-1}.
\]
We have
\[
BU(1) \simeq S^\infty / U(1) = \mathbb{C}P^\infty
\]
\[
B\mu_n \simeq S^\infty / \mu_n = L^\infty_n
\]
(infinite dimensional lens space). Let \( i: \mu_n \hookrightarrow U(1) \) be the inclusion map. Then
\[
Bi: B\mu_n \to BU(1)
\]
is identified with the quotient map
\[
L^\infty_n \to \mathbb{C}P^\infty.
\]
This is a fibration with fiber \( S^1 \cong U(1) / \mu_n \). Its homology Leray-Serre spectral sequence has \( E^2 \)-term

For cohomology with coefficients in \( A \), we obtain an \( E_2 \)-term of the form
with
\[ d_2(a) = a + \cdots + a. \]

These spectral sequences collapse to give the familiar minimal resolutions
\[ \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{n \cdot}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{n \cdot}{\rightarrow} \mathbb{Z} \overset{\cdots}{\rightarrow} \]
and
\[ A \overset{0}{\rightarrow} A \overset{n \cdot}{\rightarrow} A \overset{0}{\rightarrow} A \overset{n \cdot}{\rightarrow} A \overset{\cdots}{\rightarrow} \]

We claim that we have a commuting diagram
\[
\begin{array}{c}
\mathbb{Z} \overset{q}{\rightarrow} H^4(CP^n; \mathbb{Z}) \overset{\sim}{\rightarrow} H^3_{gp}(U(1); U(1)) \\
\mathbb{Z}/n\mathbb{Z} \overset{\approx}{\rightarrow} H^4(L_n^\infty; \mathbb{Z}) \overset{\sim}{\rightarrow} H^3(\mu_n; U(1)) \overset{\approx}{\rightarrow} H^3(\mu_n; \mu_n) \\
\mu_n \overset{\sim}{\rightarrow} \text{Hom}(\mu_n, U(1)) \overset{\approx}{\rightarrow} \text{Hom}(\mu_n, \mu_n),
\end{array}
\]
where \( q \) is the quotient map, and the equal signs refer to the standard identifications. The commutativity of the top left square follows from the fact that \( Bi^* \) is the edge homomorphism in our cohomology spectral sequence. The commutativity of the bottom left square is a diagram chase, involving the minimal resolutions for cohomology with coefficients in \( \mathbb{Z}, \mathbb{R} \) and \( U(1) \). It follows that the restriction of \( U(1)^- \) to \( \mu_n \) is a choice of \( C_n[i] \). This categorical group \( C_n[i] \) with center \( U(1) \) associated to \( C_n \) determines \( C_n \) up to equivalence. It follows that the categorical group associated to \( \kappa \) is a choice of \( C_n \). \( \square \)

There is an alternative description of the categorical group \( C_n \). For \( a \in \frac{1}{n}\mathbb{Z} \), we write
\[ a = [a] + a', \]
where the Gauß bracket \([a]\) denotes the largest integer less than or equal to \( a \).

**Definition 3.2 ([HLY14], [JS93, Sec.3, p.49]).** Let \( C_n' \) be the skelettal 2-group constructed from the \( \mu_n \)-valued 3-cocycle
\[ \alpha(a|b|c) := \exp([a' + b']c') = \exp([a' + b']c) \]
Lemma 3.3. We have an equivalence of 2-groups between $C'_n$ and $C_n$.

Proof. We define a monoidal equivalence from $C'_n$ to $C_n$. On objects, we let $F$ be the map

$$F: \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n}\mathbb{Z}$$

$$[a] \longmapsto a',$$

and on arrows, we let $F$ be the map

$$F: \left(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right) \times \mu_n \longrightarrow \frac{1}{n}\mathbb{Z} \times (\mathbb{Z} \times \mu_n)$$

$$([a], z) \longmapsto (a', 0, z).$$

We then define the natural transformation

$$\phi: F([a]) + F([b]) \longrightarrow F([a] + [b])$$

given by the arrows

$$(a' + b', -[a' + b'], 1)$$
in $C_n$. It is elementary to check that $(F, \phi)$ is indeed a monoidal equivalence. □

4. Platonic 2-groups

The previous section will serve as blueprint for our discussion of the Platonic 2-groups. Let $G$ be a finite subgroup of the three sphere. It is well known\footnote{Periodicity is a theorem by Artin and Tate \cite{AT68}, the full statement is a combination of \cite[XII.2(4),XII.11.1, XVI.9 Application 4]{CE99}. See also \cite[Cor. 3.1]{FHHP04} for a direct proof (following Schur) that the Schur multiplier vanishes and \cite{TZ08} for an explicit resolution and a description of the product structure in cohomology.} that the (co)homology of $G$ is periodic with period 4, with the reduced integral homology concentrated in odd degrees,

$$H_i(G) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0, \\
G^{ab} & \text{if } i \equiv 1 \mod 4, \\
\mu_{|G|} & \text{if } i \equiv 3 \mod 4, \text{ and} \\
0 & \text{if } i > 0 \text{ is even},
\end{cases}$$

and the integral cohomology concentrated in even degrees,

$$H^i(G) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0, \\
G^{ab} & \text{if } i \equiv 2 \mod 4, \\
\mu_{|G|} & \text{if } i \text{ is a positive multiple of 4, and} \\
0 & \text{else.}
\end{cases}$$

In particular, $G$ possesses a weak categorical Schur cover with center $\mu_{|G|}$. The following proposition shows that $G_{uni}$ can be realized as a sub-categorical group of the third String 2-group.
**Proposition 4.1.** The restriction of $\text{String}(3)$ to $G$ is equivalent to the categorical group with center $U(1)$ associated to $\mathcal{G}_{\text{uni}}$ via the canonical inclusion $i: \mu_{|G|} \hookrightarrow U(1)$,

$$\text{String}(3)|_G \simeq \mathcal{G}_{\text{uni}}[i].$$

**Proof.** We follow the argument in the proof of Lemma 3.1, with the difference that the circle group of complex units, $U(1) \subset \mathbb{C}$, is replaced by the three sphere of unit quaternions, $S^3 \subset \mathbb{H}$. Viewing $S^\infty = \text{colim} S^{4n-1}$ as the colimit of the spheres in $\mathbb{H}^n$, we have

$$BS^3 \simeq S^\infty / S^3 = \mathbb{HP}^\infty,$$

and

$$BG \simeq S^\infty / G.$$ 

If $j$ is the inclusion of $G$ in $S^3$, then $Bj$ becomes the fibration

$$S^3/G \xleftarrow{j} BG \xrightarrow{Bj} \mathbb{HP}^\infty.$$ 

The fibre is the spherical three manifold $S^3/G$ and, in particular, connected and oriented. We get the following picture of the Leray-Serre spectral sequence for integral homology.

The only non-trivial differential $d_4$ is multiplication by $n = |G|$. The remainder of the proof is identical to that of Lemma 3.1. □

**Remark 4.2.** The spherical three manifolds turning up as fibres in the above proof have been the object of intense study. For instance, if $G$ is the binary icosahedral group, then the space $S^3/G$ is the exotic homology 3-sphere of Poincaré.

**Remark 4.3.** In the abelian case, where $G$ is a finite cyclic subgroup of $S^3$, the inclusion $j$ factors through a maximal torus,

$$G \xleftarrow{j} S^3 \xrightarrow{i} S^1 \xleftarrow{k} S^3,$$
and the commuting diagram

\[
\begin{array}{c}
H^3_{gp}(\mathbb{S}^3; U(1)) \xrightarrow{k^*} H^3_{gp}(\mathbb{S}^1; U(1)) \\
\downarrow \quad \downarrow \\
H^*(\mathbb{H}P^\infty; \mathbb{Z}) \xrightarrow{(Bk)^*} H^*(\mathbb{C}P^\infty; \mathbb{Z}) \\
\end{array}
\]

|v| = 4 \quad \mathbb{Z}[v] \quad \mathbb{Z}[x] \quad |x| = 2

\[
v \quad \longrightarrow \quad x^2
\]

identifies the restriction of String(3) to \(\mathbb{S}^1\) with the categorical group \(U(1)^-\) of the previous section.

5. The string covers of the alternating groups

Let \(S_n\) be the symmetric group on \(n\) elements, and let \(\varrho_n\) be its permutation representation. The alternating group \(A_n \subset S_n\) is the subgroup of even permutations. We will write \(\tilde{A}_n\) for its spin double cover. In this section, we will introduce a family of categorical groups \(\mathscr{A}_n\), fitting into commuting diagrams

\[
\begin{array}{c}
1/\pi_3(\mathbb{S}^3) \xrightarrow{e} 1/U(1) \\
\downarrow \quad \downarrow \\
\mathscr{A}_n \xrightarrow{\varrho_n} String(n) \\
\downarrow \quad \downarrow \kappa_n \\
\tilde{A}_n \xrightarrow{\bar{\varrho}_n} Spin(n) \\
\downarrow \quad \downarrow \ \\
A_n \xrightarrow{\varrho_n} SO(n) \\
\downarrow \quad \downarrow \epsilon_n \\
S_n \xrightarrow{\varrho_n} O(n).
\end{array}
\]

Here \(e\) is the Adams \(e\)-invariant, the arrows with Greek names are the canonical maps, and in each tower, the top two vertical arrows describe a categorical central extension. These \(\mathscr{A}_n\) are characterized, up to equivalence, by

\[
\mathscr{A}_n[e] \cong String(n)|\tilde{A}_n.
\]

**Definition 5.1.** We will refer to categorical groups \(\mathscr{A}_n\) as above as the string covers of the alternating groups or simply as the alternating 2-groups.

5.1. The Whitehead tower of the plus construction. The content of this section is folklore, see for instance the Mathoverflow discussion Plus construction considerations. Let
Let \( X \) be a connected CW-complex with basepoint, whose fundamental group has perfect commutator subgroup
\[
P = [\pi_1(X), \pi_1(X)].
\]
Let
\[
p: X \longrightarrow X^+
\]
be a homology isomorphism such that
\[
p_*(P) = 0 \subseteq \pi_1(X^+).
\]
These conditions are satisfied if and only if the map \( p \) is universal, in the homotopy category, with respect to the property (1). This universal property of the plus construction determines \( X^+ \) up to unique isomorphism in the homotopy category. We use the notation \( X^+ \) whenever the above conditions are satisfied, even when we are working in the strict category. Given \( p \) as above, we may pull back the Whitehead tower of \( X^+ \) to a tower of fibrations over \( X \),
\[
X = W_1 \leftarrow \xi_1 \leftarrow X_2 \leftarrow \xi_2 \leftarrow X_3 \leftarrow \cdots
\]
\[
p \downarrow \quad p_1 \downarrow \quad p_2 \downarrow \quad p_3
\]
\[
X^+ = W_1 \leftarrow W_2 \leftarrow W_3 \leftarrow \cdots
\]
Using the Lerray-Serre spectral sequence, one shows inductively that the \( p_i \) are homology isomorphisms. So, the fundamental group of \( X_i \) is perfect for \( i > 1 \), and
\[
p_i: X_i \longrightarrow W_i
\]
satisfies the universal property for its plus construction. In particular, \( W_i \) is a choice for \( X^+_i \), and the homology of the tower \( X_* \) encodes the homotopy groups of the plus construction of \( X \). More precisely,
\[
\tilde{H}_i(X_j) = \begin{cases} 0 & \text{if } i < j, \\ \pi_i(X^+) & \text{if } i = j. \end{cases}
\]
It is possible to construct the tower \( X_* \) directly from \( X \). For this, we let \( X_1 = X \) and then proceed inductively, as follows: once \( X_i \) has been constructed, let \( K(H_i(X_i), i) \) be the \( i \)th Eilenberg-MacLane space for the group \( H_i(X_i) \), and define \( \xi_i \) as the fibration classified by the map
\[
f_i: X_i \longrightarrow K(H_i(X_i), i)
\]
corresponding to \( id_{H_i(X_i)} \) under the universal coefficient theorem. We will refer to the resulting tower as the homology tower of \( X \). For instance, \( X_2 = \tilde{X}/P \) is the quotient of the universal cover of \( X \) by the perfect group \( P \).

**Theorem 5.2.** We have a diagram of pull-back squares,
Here $Q\Sigma^0 = \text{colim } \Omega^n S^n$ is the infinite loop space of the sphere spectrum, and $(Q\Sigma^0)_0$ the connected component of its basepoint, while $\eta: S^0 \to KO$ is the unit map. The composition of the solid horizontal arrows gives the maps induced, respectively, by the representation $\varrho_n$ and its lifts $\bar{\varrho}_n$ and $\tilde{\varrho}_n$.

**Proof of Theorem 5.2** It is well known that $\eta$ induces isomorphisms on

$$
\pi_0 = \mathbb{Z} \quad \text{and} \quad \pi_1 = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_2 = \mathbb{Z}/2\mathbb{Z}.
$$

So, the map $Q\eta$ pulls back the first three steps of the Whitehead tower of $Z \times BO$ to the first three steps of the Whitehead tower of $Q\Sigma^0$. The Barratt-Quillen-Priddy theorem yields a homology isomorphism

$$
p: BS_\infty \to (Q\Sigma^0)_0,
$$

satisfying

$$
p_*(A_\infty) = 0 \quad \text{and} \quad (Q\eta) \circ p = B\varrho_\infty.
$$

So, the Whitehead tower of $Q\Sigma^0$ is identified with the plus construction of the homology tower of $BS_\infty$. It remains to identify this homology tower in the relevant degrees. The first step is the pull-back of $B\varepsilon_\infty$ along $B\varrho_\infty$. This is the non-trivial double cover $B\bar{\varrho}_\infty$, classified by the map

$$
B(sgn): BS_\infty \to B\bar{\varrho}_\infty \to BO \to B\text{det} \to B\{\pm 1\}.
$$

Indeed,

$$
X_2 = \tilde{X}/P = ES_\infty/A_\infty.
$$

Next, the double cover of $B\bar{\varrho}_\infty$ is $B\bar{\bar{\varrho}}_\infty$ and pulls back $B\kappa_\infty$ to the fibration $\xi_2 = B\sigma_\infty$. The classifying map $f_2$ represents the class

$$
[f_2] \in H^2(BA_\infty; H_2(BA_\infty))
$$

classifying the Schur cover of $A_\infty$. Finally, the lift of $B\bar{\bar{\varrho}}_\infty$ to $B\tilde{A}_\infty$ is $B\bar{\bar{\varrho}}_\infty$. \qed
As an immediate consequence of the theorem, we obtain half of the tower promised in the introduction as restrictions of the short exact sequence

\[ A_\infty \longrightarrow S_\infty \longrightarrow \pi_1(S^0), \]

the Schur cover of \( \tilde{A}_\infty \)

\[ \pi_2(S^0) \longrightarrow \tilde{A}_\infty \longrightarrow A_\infty, \]

and the categorical Schur cover of the superperfect group \( \tilde{A}_\infty \)

\[ 1/\pi_3(S^0) \longrightarrow \mathcal{A}_\infty \longrightarrow \tilde{A}_\infty. \]

In other words, we can now construct the \( n \)th alternating 2-group as

\[ \mathcal{A}_n := \mathcal{A}_\infty|\tilde{A}_n. \]

Consider the homomorphisms

\[
\begin{align*}
 b_1: H_1(S_n) &\longrightarrow H_1(S_\infty) \cong \pi_1(S^0) \cong \mu_2 \\
 b_2: H_2(A_n) &\longrightarrow H_2(A_\infty) \cong \pi_2(S^0) \cong \mu_2 \\
 b_3: H_3(\tilde{A}_n) &\longrightarrow H_3(\tilde{A}_\infty) \cong \pi_3(S^0) \cong \mu_{24}.
\end{align*}
\]

where the middle isomorphisms are induced by the Barratt-Priddy-Quillen map.

**Lemma 5.3** ([Hau78 7.2.3]). The map \( b_1 \) is an isomorphism for \( n \geq 2 \), the map \( b_2 \) is an isomorphism for \( n = 4, 5 \) or \( n \geq 8 \), and the map \( b_3 \) is an isomorphism for \( n = 4, n = 8 \) or \( n \geq 11 \).

**Proof.** It is well known that the abelianization of \( S_n \) is \( \mathbb{Z}/2\mathbb{Z} \) for \( n \geq 2 \). The values where \( b_2 \) is an isomorphism are also well known. This goes back to work of Schur. For \( 5 \leq n \leq \infty \), the group \( A_n \) is perfect. In this range, we have a compatible system of isomorphisms

\[ H_2(A_n) \cong \pi_2(BA_n^+). \]

Similarly, we have compatible isomorphisms

\[ H_3(\tilde{A}_n) \cong \pi_3(B\tilde{A}_n^+), \]

for \( n = 5 \) and \( 8 \leq n \leq \infty \). Further, when \( A_n \) is perfect, the fibration

\[ B\{\pm 1\} \longrightarrow B\tilde{A}_n^+ \longrightarrow BA_n^+. \]

[Hau78] Prop.7.1.3] yields an isomorphism

\[ \pi_3(B\tilde{A}_n^+) \cong \pi_3(BA_n^+). \]

Apart from the case \( n = 4 \), which we will treat in Lemma [5.9], the statement of the Lemma can now be read off from the proof of Proposition A in [Hau78]. \( \square \)

In low degrees, we still have:

**Corollary 5.4.** When \( A_n \) is perfect, then its spin extension \( \tilde{A}_n \) is classified by the homomorphism \( b_2 \). When \( \tilde{A}_n \) is superperfect, then its string extension \( \mathcal{A}_n \) is classified by the homomorphism \( b_3 \).
5.2. The Adams $e$-invariant. Given a ring spectrum $E$ with unit map $\eta = \eta_E$, we may form the exact triangle

$$E[-1] \longrightarrow \overline{E} \longrightarrow \mathbb{S}^0 \overset{\eta}{\longrightarrow} E$$

in the stable homotopy category. In the case $E = KO$, we have

$$\pi_{4k-1}(KO) = 0 \quad \text{and} \quad \pi_{4k}(KO) = \mathbb{Z}.$$ 

In positive degrees, the stable homotopy groups of spheres are finite. It follows that for $k \geq 1$, the map $\pi_{4k}(\eta)$ is zero, so that we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_{4k-1}(KO) \longrightarrow \pi_{4k-1}(\mathbb{S}^0) \longrightarrow 0.$$ 

For a finite abelian group $\pi$, we further have the isomorphism

$$(2) \quad Ext(\pi, \mathbb{Z}) \cong Hom(\pi, \mathbb{Q}/\mathbb{Z})$$

resulting from the injective resolution

$$Z \longrightarrow Q \longrightarrow \mathbb{Q}/\mathbb{Z}.$$ 

**Definition 5.5.** For $k \geq 1$, we let

$$e: \pi_{4k-1}(\mathbb{S}^0) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

be the homomorphism classifying the extension $[\pi_{4k-1}(KO)]$ above.

**Lemma 5.6.** Our definition of $e$ agrees with the definition of the Adams $e$-invariant in \[AS74\] (1.1) and \[APS75\] (4.11).

**Proof.** Following the discussion of the complex $e$-invariant in \[CF66\] III,16], Atiyah and Smith identify the real $e$-invariant of a framed manifold $M$ of dimension $4k - 1$ as

$$e(M) = \begin{cases} \hat{A}(B) & k \text{ even,} \\ \frac{1}{2}\hat{A}(B) & k \text{ odd,} \end{cases}$$

where $B$ is any spin manifold with boundary $\partial B = M$. They argue that this is a well-defined element of $\mathbb{Q}/\mathbb{Z}$ by the integrality result \[AH59\] Cor.2(ii). To understand this formulation, consider the maps of exact triangles

$$\begin{array}{cccccc}
\mathbb{S}^0 & \longrightarrow & \mathbb{S}^0 & \longrightarrow & \mathbb{S}^0 & \longrightarrow \\
\downarrow \eta & \quad & \downarrow \eta & \quad & \downarrow & \\
MSpin & \longrightarrow & \hat{A} & \longrightarrow & KO & \longrightarrow \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \\
MSpin[1] & \longrightarrow & KO[1] & \longrightarrow & KO_Q[1] & \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \\
\mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & \mathbb{S}_Q^1,
\end{array}$$

---

3 This is the first step in the construction of the $E$-based Adams-Novikov spectral sequence.
where \( \hat{A} \) is the Atiyah-Bott-Shapiro orientation \(^{[ABS64]} \). Following \(^{[LM89]} (7.9), (7.13), (7.17) \), this yields a diagram with exact columns

\[
\begin{array}{cccccc}
\Omega_{4k}^{Spin} & \xrightarrow{\text{ind}} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{4k}^{Spin, fr} & \xrightarrow{\pi_{4k-1}(KO)} & \pi_{4k-1}(KO)_\mathbb{Q} & \xrightarrow{1} & 0, \\
\downarrow & & \downarrow & & \\
\Omega_{4k-1}^{fr} & & \pi_{4k-1}(S^0) & & \\
\end{array}
\]

where \( \text{ind} \) is the Atiyah-Milnor-Singer invariant,

\[
\text{ind}(X) = \begin{cases} 
\hat{A}(X) & \text{k even}, \\
\frac{1}{2} \hat{A}(X) & \text{k odd}.
\end{cases}
\]

We claim that, for even \( k \), the composite of the red arrows sends a spin manifold with framed boundary to the integral over its \( \hat{A} \)-class. Indeed, this relative \( \hat{A} \)-genus is a homomorphism from \( \Omega_{4k}^{Spin, fr} \) to \( \mathbb{Q} \), which for closed manifolds agrees with the \( \hat{A} \)-genus. Since the inclusion

\[
\Omega_{4k}^{Spin} \xrightarrow{\sim} \Omega_{4k}^{Spin, fr}
\]

becomes an isomorphism after tensoring with \( \mathbb{Q} \), this property determines the relative \( \hat{A} \)-genus uniquely. By the identical argument, the red arrows compose to half the relative \( \hat{A} \)-genus for \( k \) odd. We may now reformulate the definition \(^{[AS74] (1.1)} \) as follows: Given an element \( x \) of \( \pi_{4k-1}(S^0) \), choose a pre-image \( \overline{x} \) of \( x \) in \( \pi_{4k-1}(KO) \) and take \( e(x) \) to be the image of \( \overline{x} \) in

\[
\pi_{4k-1}(KO)_\mathbb{Q} \xrightarrow{\sim} \pi_{4k}(KO)_\mathbb{Q} \xrightarrow{\sim} \mathbb{Q}
\]

modulo

\[
\pi_{4k}(KO) = \mathbb{Z}.
\]

This description of \( e \) coincides with the classifying map of the extension \([\pi_{4k-1}(KO)]\). \( \square \)

**Lemma 5.7.** The composite

\[
H^4(BSpin; \mathbb{Z}) \xrightarrow{(B\tilde{g}_\infty)^*} H^4(B\tilde{A}_\infty; \mathbb{Z}) \sim \ Ext(H_3(B\tilde{A}_\infty), \mathbb{Z})
\]

sends the preferred generator of \( H^4(BSpin; \mathbb{Z}) \) to the extension \([\pi_3KO]\) of

\[
H_3(B\tilde{A}_\infty) \cong \pi_3(S^0),
\]

used in Definition \( 5.3 \).

**Proof.** The naturality of the universal coefficient theorem (the isomorphism in the lemma) allows us to replace \( B\tilde{A}_\infty \) with \( B\tilde{A}^+_\infty \) and \( B\tilde{g}_\infty \) with \( B\tilde{g}^+_\infty \). Let

\[
\xi: BSpin \longrightarrow K(\mathbb{Z}, 4)
\]
represent the preferred generator. Then $(B\tilde{\varrho}_\infty)^+([\xi])$ is represented by the composite

\[ \xi' = \xi \circ B\tilde{\varrho}_\infty^+. \]

We have a homotopy commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & H_3(hofib(\xi')) & \rightarrow & H_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(K(\mathbb{Z}, 3)) & \rightarrow & \pi_3(hofib(\xi')) & \rightarrow & \pi_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(Spin) & \rightarrow & \pi_3(hofib(B\tilde{\varrho}_\infty^+)) & \rightarrow & \pi_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(O) & \rightarrow & \pi_3(QKO) & \rightarrow & \pi_3(QS^0) & \rightarrow & 0.
\end{array}
\]

whose rows are homotopy fiber sequences. Using the long exact sequence of (unstable) homotopy groups, we find that all the spaces in the top two rows are 2-connected. In fact, the second row forms the 2-connected cover of the third row. Using Hurwicz and the fact that there are no non-trivial homomorphisms from a finite group to $\mathbb{Z}$, we arrive at the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & H_3(hofib(\xi')) & \rightarrow & H_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(K(\mathbb{Z}, 3)) & \rightarrow & \pi_3(hofib(\xi')) & \rightarrow & \pi_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(Spin) & \rightarrow & \pi_3(hofib(B\tilde{\varrho}_\infty^+)) & \rightarrow & \pi_3(B\tilde{A}_\infty^+) & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \rightarrow & \pi_3(O) & \rightarrow & \pi_3(QKO) & \rightarrow & \pi_3(QS^0) & \rightarrow & 0.
\end{array}
\]

The universal coefficient theorem identifies the class $[\xi']$ with the extension on the top row, while the bottom row is the extension in Definition 5.5. \qed

**Corollary 5.8.** The restriction of $\text{String}(n)$ to $\tilde{A}_n$ is equivalent, in a manner unique up to unique isomorphism, to the categorical group with center $U(1)$ associated to $\mathcal{A}_n$ via the Adams $e$-invariant,

\[ \mathcal{A}_n[e] \simeq \text{String}(n)|_{\tilde{A}_n}. \]

**Proof.** This follows from the commutativity of the diagram.
where the horizontal isomorphisms on the right are given by the universal coefficient theorem, and the right-most vertical isomorphism is $[2]$. The left two vertical isomorphisms come from the long exact sequence associated to the short exact sequence of coefficients $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$.

One interpretation of the isomorphism $[2]$ uses the fact that the circle group $U(1)$ is a classifying space for $\mathbb{Z}$. So, the central extensions of $\pi$ by $\mathbb{Z}$ are classified by homotopy classes of group homomorphisms from $\pi$ to $U(1)$, and we have

$$K(\mathbb{Z}, 4) = B^3U(1).$$

Applying the construction $B(-)^+$ to the categorical central extensions of this section, we obtain the map of exact triangles

\[
\begin{array}{cccc}
B^2\pi_3(S^0) & \xrightarrow{B^2e} & B^2U(1) \\
\downarrow & & \downarrow \\
B\mathcal{A}^+ & \xrightarrow{B\tilde{\varrho}^+} & B\text{Spin} \\
\downarrow & & \downarrow \\
B\tilde{A}^+ & \xrightarrow{B\tilde{\varrho}^+} & B\text{String} \\
\end{array}
\]

whose middle square adds another floor to the map of Whitehead towers in Theorem 5.2. In particular,

$$H_4(B\mathcal{A}; \mathbb{Z}) \cong \pi_4(S^0) = 0.$$

**Lemma 5.9.** The canonical inclusion of $\tilde{\mathcal{A}}_4$ in $\tilde{\mathcal{A}}_\infty$ induces an isomorphism in degree three homology, sending the fundamental class of the tetrahedral spherical 3-form to the second Hopf map $\nu: S^7 \rightarrow S^4$,

\[
H_3(S^\infty/\tilde{\mathcal{A}}_4) = H_3(B\tilde{\mathcal{A}}_4) \xrightarrow{\cong} H_3(B\tilde{\mathcal{A}}_\infty) \cong \pi_3(S^0)
\]

$$[S^3/\tilde{\mathcal{A}}_4] \xrightarrow{\nu} [\nu].$$
Proof. Applying the plus construction (with respect to $\tilde{\mathcal{A}}_\infty$) to the fibration

\[
\begin{array}{c}
\text{Spin}/\tilde{\mathcal{A}}_\infty \\
\downarrow \\
B\tilde{\mathcal{A}}_\infty
\end{array}
\longrightarrow
\begin{array}{c}
B\tilde{\mathcal{A}}_\infty \\
\downarrow \\
B\tilde{\varrho}_\infty \\
\downarrow \\
B\text{Spin},
\end{array}
\]

we obtain the identification

\[
\left(\text{Spin}/\tilde{\mathcal{A}}_\infty\right)^+ = hofib\left(B\tilde{\varrho}_\infty^+\right),
\]

see [Far96, 3.D.3(2)]. From the proof of Lemma 5.7, we therefore have the short exact sequence

\[
\begin{array}{c}
0 \\
\parallel \\
0
\end{array}
\longrightarrow
\begin{array}{c}
H_3(\text{Spin}) \\
\parallel \\
Z
\end{array}
\longrightarrow
\begin{array}{c}
H_3(\text{Spin}/\tilde{\mathcal{A}}_\infty) \\
\parallel \\
24\cdot
\end{array}
\longrightarrow
\begin{array}{c}
H_3(B\tilde{\mathcal{A}}_\infty) \\
\parallel \\
Z
\end{array}
\longrightarrow
\begin{array}{c}
Z/24Z
\end{array}
\longrightarrow
0,
\]

whose first map can be identified with the differential

\[
d_4 : H_4(B\text{Spin}) \longrightarrow H_3(\text{Spin}/\tilde{\mathcal{A}}_\infty)
\]

in the Leray-Serre spectral sequence for $B\tilde{\varrho}_\infty$. This can be compared to the scenario for the platonic 2-groups. In particular, we have the commuting diagram

\[
\begin{array}{c}
0 \\
\parallel \\
0
\end{array}
\longrightarrow
\begin{array}{c}
Z
\end{array}
\longrightarrow
\begin{array}{c}
24\cdot
\end{array}
\longrightarrow
\begin{array}{c}
Z
\end{array}
\longrightarrow
\begin{array}{c}
Z/24Z
\end{array}
\longrightarrow
0
\]

\[
\begin{array}{c}
0 \\
\parallel \\
0
\end{array}
\longrightarrow
\begin{array}{c}
H_4(B\text{Spin}(3))
\end{array}
\longrightarrow
\begin{array}{c}
d_4 \\
\parallel \\
\cong \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow
\end{array}
\begin{array}{c}
H_3(\text{Spin}(3)/\tilde{\mathcal{A}}_4)
\end{array}
\longrightarrow
\begin{array}{c}
H_5(B\tilde{\mathcal{A}}_4)
\end{array}
\longrightarrow
0
\]

\[
\begin{array}{c}
0 \\
\parallel \\
0
\end{array}
\longrightarrow
\begin{array}{c}
H_4(B\text{Spin})
\end{array}
\longrightarrow
\begin{array}{c}
d_4 \\
\parallel \\
\cong \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow \\
\parallel \\
\downarrow
\end{array}
\begin{array}{c}
H_3(\text{Spin}/\tilde{\mathcal{A}}_\infty)
\end{array}
\longrightarrow
\begin{array}{c}
H_3(B\tilde{\mathcal{A}}_\infty)
\end{array}
\longrightarrow
0.
\]

Here we are using a non-standard inclusion of $\text{Spin}(3)$ inside $\text{Spin}(4) \subset \text{Spin}$, covering the orthogonal complement of the trivial summand of the permutation representation. This map still gives an isomorphism in $H_3$, implying that all the vertical arrows are isomorphisms. It follows that the generator of

\[
H_3(B\tilde{\mathcal{A}}_\infty) \cong \pi_3(S^0)
\]

with $c$-invariant $\frac{1}{24}$ is the image of the fundamental class of $\text{Spin}(3)/\tilde{\mathcal{A}}_4$ under its inclusion in $B\tilde{\mathcal{A}}_\infty$. \hfill \Box

**Corollary 5.10.** The fourth alternating 2-group, $\mathcal{A}_4$, is the weak categorical Schur cover of the binary tetrahedral group, while

\[
\mathcal{A}_3 \cong \mathcal{C}_6
\]

is the weak categorical Schur cover of the cyclic group on six elements.
6. Explicit constructions

This section recalls the construction of the String 2-groups given in [Woc11] and [WW15]. We only discuss the restriction to our finite subgroups. Following [BtD95], we identify the maximal torus of $\text{Spin}(n)$ with

$$T = \mathbb{R}|_{\frac{n}{2}}/\mathbb{Z}|_{\frac{n}{2}},$$

where

$$t = \mathbb{R}|_{\frac{n}{2}}$$

is the Lie algebra and

$$Λ^\vee = \mathbb{Z}|_{\frac{n}{2}} = \{m \in \mathbb{Z}|_{\frac{n}{2}} | \langle m, m \rangle\}$$

is the coweight lattice. The basic bilinear form $\langle -, - \rangle$ on $\text{spin}(n)$ is then the multiple of the Killing form that restricts to the standard scalar product on $\mathbb{R}|_{\frac{n}{2}}$. The Cartan three form is the invariant three form $\nu$ on $\text{Spin}(n)$ with

$$\nu_1(\xi, \zeta, \eta) = \langle [\xi, \zeta], \eta \rangle.$$

Restricted to $S^3 = \text{Spin}(3)$, we have

$$\nu_1(\xi, \zeta, \eta) = \langle \xi \times \zeta, \eta \rangle = \det(\xi, \zeta, \eta).$$

So, $\nu$ is the volume form.

Let now $G \subset \text{Spin}(n)$ be a finite subgroup, and let

$$\mathbb{Z} \leftarrow \text{Bar}_\bullet G$$

be the bar resolution,

$$\text{Bar}_k G = \mathbb{Z}[G][G]^k.$$

Let $C_\bullet(\text{Spin}(n))$ be the singular chain complex of $\text{Spin}(n)$. Since $\text{Spin}(n)$ is 2-connected, and $\text{Bar}_\bullet G$ is free, we may choose maps

$$f_i: \text{Bar}_i G \rightarrow C_i(\text{Spin}(n)),$$

for $0 \leq i \leq 3$, such that $f_0$ maps $g()$ to the 0-simplex $g$ in $\text{Spin}(n)$, and the $f_i$ fit together to form a map of truncated chain complexes of $\mathbb{Z}[G]$-modules. Here $G$ acts on the simples in $\text{Spin}(n)$ by left translation.

Explicitly, a choice of $f$ amounts to, for each $g \in G$, a path $\gamma_g$ from 1 to $g$, for each pair $(g|h)$ of elements of $G$, a 2-simplex $\Delta_{g,h}$ bounding

$$\gamma_g - \gamma gh + g\gamma_h,$$

and for each triple $(g|h|k)$, a 3-simplex $W_{g,h,k}$ bounding

$$-\Delta_{g,h} + \Delta_{g,hk} - \Delta_{gh,k} + g\Delta_{h,k}.$$

**Definition 6.1.** For a fixed choice of $f_\bullet$, let

$$\alpha: G^3 \rightarrow \mathbb{R}/\mathbb{Z}$$

be the 3-cocycle

$$\alpha(g|h|k) = \frac{1}{2\pi^2} \int_{W_{g,h,k}} \nu \mod \mathbb{Z}.$$

**Lemma 6.2.** Different choices of $f_\bullet$ yield cohomologous choices of $\alpha.$
Proof. Let $f'$ be a second choice for $f$, and let $\alpha'$ be the resulting 3-cocycle. Employing again the 2-connectedness of $\text{Spin}(n)$, we obtain a chain homotopy

$$
\begin{array}{cccc}
\text{Bar}_0G & \overset{\delta}{\rightarrow} & \text{Bar}_1G & \overset{\delta}{\rightarrow} & \text{Bar}_2G & \overset{\delta}{\rightarrow} & \text{Bar}_3G \\
0 & \downarrow & 0 & \downarrow & H_1 & \downarrow & H_2 & \downarrow & f_3-f'_3 \\
C_0(\text{Spin}(n)) & \overset{\delta}{\rightarrow} & C_1(\text{Spin}(n)) & \overset{\delta}{\rightarrow} & C_2(\text{Spin}(n)) & \overset{\delta}{\rightarrow} & C_3(\text{Spin}(n))
\end{array}
$$

relating $f$ and $f'$ up to degree 2 and such that

$$f_3-f'_3 - H_2 \circ \delta$$

takes values in the 3-cycles $Z_3(\text{Spin}(n))$. Letting $\beta$ be the 2-cocycle

$$\beta(g|h) = \frac{1}{2\pi^2} \int_{H_2(g|h)} \nu \mod \mathbb{Z},$$

it follows that

$$\alpha - \alpha' = \delta^* \beta.$$

\[\Box\]

Remark 6.3. In [FGMNS], Femina, Galves, Neto and Sreafico describe the fundamental domain of the action of $2T$ on the three sphere as an octahedron (the join of two geodesic segments). This yields a specific description of the fundamental class of $S^3/2T$. It would be interesting to identify this class with an explicit group cocycle or to give a more direct relationship with the second Hopf map.

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