Effect of piezoelectric patch on natural frequencies of Timoshenko beam made of functionally graded material

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Abstract
The present paper addresses developing the Dynamic Stiffness Method (DSM) for natural frequency analysis of functionally graded beam with piezoelectric patch based on the Timoshenko beam theory and power law of material grading. Governing equations and general solution of free vibration are conducted for the beam element with piezoelectric layer that is modelled as a homogeneous Timoshenko beam. The obtained solution allows establishing dynamic stiffness matrix for modal analysis of FGM beam with bonded piezoelectric distributed sensors/actuators. Effect of thickness and position of the smart sensors/actuators and material parameters on natural frequencies is studied with the aim for dynamic testing and health monitoring of FGM structures. The theoretical developments are validated and illustrated by numerical examples.

1. Introduction

Due to increasingly employed functionally graded materials in high-tech industries, studying behavior of structural components such as beams or plates made of that material under various loadings becomes vitally essential. The most important achievements in modelling and analysis of the material and structures were reported in the surveys given by Birman and Byrd [1] and Gupta and Talha [2]. Various problems in dynamic analysis of functionally graded beams were studied in the widespread literature, for instance, in the works [3–7]. A large number of works is devoted also to study vibrations of the beams with localized damages such as cracks [8–13]. Recently, some procedures were proposed by Yu and Chu [14]; Banerjee et al [15] and Khiem and Huyen [16] to detect cracks in functionally graded beams with natural frequencies measured by the traditional technique of modal testing. As well known, the traditional modal testing is restricted to use a limited number of discrete sensors and actuators that are usually unable to gather sufficient amount of data for solving the problem of damage detection in structural health monitoring. Therefore, using distributed sensors and actuators for modal testing would be surely promising to enhance solution of the damage detection problem.

Tzou and Tseng [17] demonstrated the necessity in using distributed piezoelectric sensor/actuator for dynamic measurement/control of distributed parameter systems such as flexible structures. The authors developed also the so-called piezoelectric finite element approach to free vibration of a plate with distributed piezoelectric actuator on the top and sensor at the bottom. Lee and moon [18] developed a theory of distributed sensor/actuator that can be adopted to measure/excite specific modes of plates and beams. Rao and Sunar [19] accomplished a comprehensive survey on the use of piezoelectric materials for disturbance sensing and control of flexible structures. Lee and Jiang [20] studied the electromechanical properties of a piezoelectric laminae that can be used as distributed sensor/actuator for measurement/control of the distributed parameter systems. Recent progress in structural health monitoring by the use of distributed piezoelectric transducers was reported in [21, 22]. Particularly, Wang and Quek [23] showed that the buckling and flutter capacities of an elastic column could be enhanced by using piezoelectric patches bonded to both sides of the column as actuators with an
applied voltage. Wang et al [24] revealed an effect of a piezoelectric patch bonded to a beam on natural frequency of the beam and demonstrated an interesting fact that piezoelectric patch used as an actuator could restore the healthy condition of a cracked beam. Mateescu et al [25] presented a method for crack detection in beam and plate using piezoelectric sensors bonded on both sides of the structures.

Using piezoelectric material for sensing and controlling a structure behavior is essentially leading to analysis of the structures with piezoelectric components [26, 27] such as beams or plates with layers or patches. Namely, Yang and Lee [28] used the stepped beam model for modal analysis of Timoshenko beam with piezoelectric patches symmetrically bonded onto both the top and bottom and demonstrated that stiffness and inertia of the piezoelectric material, as well as shear deformation and rotary inertia of the base beam may make change in natural frequencies of the coupled beam. The model of multistep beam was employed also by Maurini et al [29] for modal analysis of classical beam with numerous pairs of piezoelectric patches using different techniques including the so-called assumed modes method proposed by themselves. Wang and Quek [30] used the sandwich beam model for modal analysis of a Euler–Bernoulli beam embedded with piezoelectric layers and they found that natural frequency of the sandwich beam is function of stiffness and thickness of the piezoelectric layers. Lee and Kim [31] first proposed to apply the spectral element method (SEM) for vibration analysis of also Euler–Bernoulli beam bonded with a piezoelectric layer (two-layer beam model) and declared that the SEM may provide reliable dynamic characteristics of the elastic–piezoelectric two-layer beams. Then, the SEM have been developed for modelling and analysis of homogeneous [32] and composite [33] Timoshenko beams with piezoelectric layers.

While the most of the aforementioned studies are concerned with the homogeneous beams, there is found in the literature very few works devoted to functionally graded beams with piezoelectric patches except some mentioned below. The stability of an FGM Timoshenko beam embedded by the top and bottom piezoelectric layers/actuators has been investigated in [34] and it is found a significant effect of both the piezoelectric actuators and FGM parameters on the critical buckling loads. Li et al [35] even proposed a model of functionally graded piezoelectric beam for its vibration analysis and revealed the increase of natural frequency and decrease of electric potential with increasing gradient index of the material. Recently, Bendine et al [36] studied the problem for active vibration control of functionally graded beams with upper and lower surface-bonded piezoelectric layers using the finite element method.

The present paper addresses developing the Dynamic Stiffness Method (DSM) for natural frequency analysis of functionally graded Timoshenko beam with a bonded piezoelectric patch. The DSM is preferred to develop herein because of the following reasons. First, DSM allows one to obtain response of a structure in arbitrarily high-frequency range that is typical for piezoelectric material. Second, when a number of piezoelectric patches are bonded to a uniform beam it becomes nonuniform one of stepwise varying cross section, for analysis of which the DSM is most efficient. Thus, the subject of this study is to examine the effect of thickness and location of the piezoelectric patch mutually with gradient index of the material on the beam’s natural frequencies in different cases of boundary conditions. Theoretical development is validated and illustrated by numerical results.

2. Governing equations

Consider an FGM beam of length $L$, cross sectional area $A_s = b \times h_b$ (figure 1). It is assumed that the material properties of the beam vary along the thickness direction by the power law distribution as follows

$$
\Omega(z) = \Omega_b + (\Omega_t - \Omega_b)(z/h + 0.5)^n;
-\frac{h_b}{2} \leq z \leq \frac{h_b}{2},
$$

where $\Omega$ stands for Young’s, shear modulus and material density $E$, $G$, $\rho$; subscripts $t$ and $b$ denote the top and bottom material respectively; $n$ is power law exponent; $z$ is ordinate of point along the beam height from the mid plane. Assuming small deformation in the framework of Timoshenko beam theory, the constituting equations

![Figure 1. FGM beam element with piezoelectric layer.](image)
for the beam at section \( x \) are
\[
\begin{align*}
\sigma_x &= E(z)\varepsilon_x; \quad \tau_{xz} = G(z)\gamma_{xz}, \\
\varepsilon_x &= \partial u_0 / \partial x - (z - h_0)\partial / \partial x; \quad \gamma_{xz} = \partial w_0 / \partial x - \theta, \\
\end{align*}
\]
(2)
where \( u(x, z, t), w(x, z, t) \) are axial and transverse displacements in cross-section at \( x \); \( u_0(x, t), w_0(x, t) \) are the displacements on the neutral plane and \( \theta \) is rotation of the cross-section; \( \varepsilon_x, \gamma_{xz}, \sigma_x, \tau \) are deformation and strain components; \( \kappa \) is geometry correction factor; \( h_0 \) is acknowledged as exact position of neutral plane measured from the beam midplane. Based on the condition for neutral plane of the FGM beam, the actual position of neutral axis is calculated as
\[
h_0 = n(r_e - 1)h/2(n + 2)(n + r_e); \quad r_e = E_i/E_b. 
\]
(3)
Using the constitutive equation (2) strain energy of the beam is calculated as
\[
\Pi_b = (1/2) \int \int \int (\sigma_x \varepsilon_x + \tau_{xz} \gamma_{xz}) \, dV_b
\]
\[
= (1/2) \int \int \int [E(z)\varepsilon_x^2 + \kappa G(z)\gamma_{xz}^2] \, dV_b
\]
\[
= \int_0^L \left( A_{11} u_0'^2 - 2A_{12} u_0' \theta' + A_{33}(w_0' - \theta)^2 \right) \, dx,
\]
(4)
where comma denotes derivative with respect to spatial variable \( x \) and
\[
A_{11} = \int_A E(z) \, dA = bhE_b \varphi_1(r_e, n);
\]
\[
A_{12} = \int_A E(z)(z - h_0) \, dA = bh^2E_b \varphi_2(r_e, n);
\]
\[
A_{22} = \int_A E(z)(z - h_0)^2 \, dA = bh^3E_b \varphi_3(r_e, n);
\]
\[
A_{33} = \kappa \int_A G(z) \, dA = bh\kappa G_b \varphi_1(r_e, n);
\]
\[
\varphi_1(x, n) = (x + n) / (1 + n); \quad \varphi_2(x, n) = (2x + n) / 2(2 + n) - \alpha(x + n) / (1 + n);
\]
\[
r_e = \rho_i / \rho_b; \quad r_e = E_i/E_b; \quad \alpha = 1/2 + h_0 / h_b
\]
(5)
On the other hand, kinetic energy of the beam is
\[
T_b = (1/2) \int \int \int (\dot{u}^2 + \dot{w}^2) \, dV
\]
\[
= (1/2) \int_0^L \left( I_{11} \dot{u}_0^2 - 2I_{12} \dot{u}_0 \dot{\theta} + I_{22} \dot{\theta}^2 + I_{14} \dot{w}_0^2 \right) \, dx,
\]
(6)
with
\[
I_{11} = \int_A \rho(z) \, dA = bh\rho_b \varphi_1(r_e, n);
\]
\[
I_{12} = \int_A \rho(z)(z - h_0) \, dA = bh^2\rho_b \varphi_2(r_e, n);
\]
\[
I_{22} = \int_A \rho(z)(z - h_0)^2 \, dA = bh^3\rho_b \varphi_3(r_e, n).
\]
(7)
Let’s now consider the piezoelectric layer as a homogeneous Timoshenko beam element, so that constitutive equations can be expressed as
\[
\begin{align*}
\varepsilon_{px} &= u'_p - \theta'_p, \quad \gamma_p = w'_p - \theta_p, \\
\sigma_{px} &= C_{11}^{p} \varepsilon_{px} - h_{13} D; \quad \tau_p = C_{33}^{p} \gamma_p; \quad \varepsilon = -h_{33} \varepsilon_{px} + \beta_3^{p} D,
\end{align*}
\]
(8)
where \( C_{11}^{p}, h_{13}, \beta_3^{p} \) are elastic modulus, piezoelectric and dielectric constants respectively. \( \varepsilon \) and \( D \) are electric field and displacement of the piezoelectric layer.
Perfect bonding of the base beam with the piezoelectric layer is represented by the conditions

\[
\begin{align*}
\begin{cases}
    u(x, \frac{h_0}{2}, t) = u_p(x, \frac{-h_p}{2}, t), \\
    w(x, \frac{h_0}{2}, t)
\end{cases}
\end{align*}
\]

\[= w_p(x, \frac{-h_p}{2}, t), \tag{9}\]

that yield

\[u_{p0} = u_0 - \theta h/2, \quad h = h_0 + h_p, \quad w_{p0} = w_0, \quad \theta = \theta_p. \tag{10}\]

Therefore

\[
\begin{align*}
\varepsilon_{px} &= u'_0 - (z + h/2)\theta', \\
\gamma_p &= w'_0 - \theta;
\end{align*}
\]

and

\[
\begin{align*}
\Pi_p &= (1/2) \int \int \int \left( \sigma_{px} \varepsilon_{px} + \tau_p \gamma_p + \varepsilon D \right) dV_p \\
&= (1/2) \int \int \left[ C_{11} \varepsilon_{px}^2 - 2h_{13} D \varepsilon_{px} + C_{35}^p \gamma_p^2 + \beta_{35}^p D^2 \right] dV_p \\
&= (1/2) \int_0^L \left\{ C_{11}^p A_p u_0'^2 - C_{11}^p A_p h u_0 \theta' + \\
&\quad - (2h_{13} D(u'_0 - h\theta/2) - \beta_{35}^p A_p D^2) \right\} dx
\end{align*}
\]

\[
T_p = (1/2) \int \int \int \rho_p (\ddot{u}_p^2 + \ddot{w}_p^2) dV
\]

\[= (1/2) \int_0^L \left\{ \rho_p A_p \ddot{u}_0^2 - \rho_p A_p h \ddot{u}_0 \theta \\
+ (\rho_p A_p + \rho_p A_p h^2/4) \ddot{\theta} + \rho_p A_p \ddot{w}_0 \right\} dx, \tag{12}\]

where \(A_p = bh_p; \quad I_p = bh_p^3/12\). Therefore, total strain and kinetic energies of the system are

\[
\begin{align*}
\Pi &= \Pi_b + \Pi_p \\
&= (1/2) \int_0^L \left\{ A_{11} u_0'^2 - 2 A_{12} u_0' \theta' + A_{22} \theta'^2 | \\
&\quad - 2h_{13} A_p D(u'_0 - h\theta/2) + \beta_{35}^p A_p D^2 \right\} dx, \\
T &= T_p + T_p = (1/2) \int_0^L \left\{ I_{11} \dot{\theta}^2 - 2I_{12} \dot{u}_0 \theta + \\
&\quad + I_{22} \ddot{\theta}^2 + I_{33} \ddot{w}_0 \right\} dx, \tag{13}\]
\]

where

\[
\begin{align*}
A_{11}^* &= A_{11} + C_{11}^p A_p; \quad A_{12}^* = A_{12} + C_{11}^p A_p h/2; \\
A_{22}^* &= A_{22} + C_{11}^p (I_p + A_p h^2/4); \quad A_{33}^* = \kappa A_{33} + C_{35}^p A_p; \\
I_{11}^* &= I_{11} + \rho_p A_p; \quad I_{12}^* = I_{12} + \rho_p A_p h/2; \tag{14a} \\
I_{22}^* &= I_{22} + \rho_p I_p + \rho_p A_p h^2/4. \tag{14b}
\end{align*}
\]

Figure 2. Nodal displacements and forces of a beam element.
Putting expressions (13) to the Hamilton's principle

\[ \int_{t_0}^{t_f} \delta (T - H) dt = 0, \]

one gets

\[
\begin{align*}
(I^*_1 u_0^t - A^*_1 u_0^t) - (I^*_2 \ddot{\theta} - A^*_1 \dot{\theta}) + h_{13} A_P D' = 0; \\
(I^*_1 \dot{w}_0^t - A^*_3 (w_0^t - \theta)) = 0; \\
(I^*_2 u_0^t - A^*_2 u_0^t) - (I^*_3 \ddot{\theta} - A^*_2 \dot{\theta}) + (A^*_3 (w_0^t - \theta) + h_{13} A_P h D')/2 = 0; \\
h_{13} A_P (u_0^t - h \dot{\theta}'/2) - \beta^*_3 A_P D = 0
\end{align*}
\]

(15a)

and

\[
\begin{align*}
[(A^*_1 u_0^t - A^*_2 u_0^t) + h_{13} A_P D)] \delta u_0^t|^{t_f}_{t_0} = 0; \\
[(A^*_2 u_0^t - A^*_2 u_0^t) + h_{13} A_P h D')/2)] \delta \dot{\theta}|^{t_f}_{t_0} = 0; \\
[(A^*_3 (w_0^t - \theta)) \delta w_0^t|^{t_f}_{t_0} = 0.
\end{align*}
\]

(15b)

The last equation in (15a) allows one to find \( D = h_{13}(u_0^t - h \dot{\theta}'/2) / \beta^*_3 \), so that remaining equations in (15) can be written as

\[
\begin{align*}
(I^*_1 u_0^t - B^*_1 u_0^t) - (I^*_2 \ddot{\theta} - B^*_1 \dot{\theta}) = 0; \\
(I^*_2 \dot{w}_0^t - B^*_2 (w_0^t - \theta)) = 0; \\
(I^*_3 u_0^t - B^*_3 u_0^t) - (I^*_3 \ddot{\theta} - B^*_3 \dot{\theta}) + A^*_3 (w_0^t - \theta) = 0;
\end{align*}
\]

(16)

and

\[
\begin{align*}
[(B^*_1 u_0^t - B^*_1 u_0^t)] \delta u_0^t|^{t_f}_{t_0} = 0; \\
[(B^*_2 \dot{w}_0^t - B^*_2 \dot{\theta}) \delta \dot{\theta}|^{t_f}_{t_0} = 0; \\
[(A^*_3 (w_0^t - \theta)) \delta w_0^t|^{t_f}_{t_0} = 0,
\end{align*}
\]

where the constants \( B_1^*, B_2^*, B_3^* \) are

\[
\begin{align*}
B_1^* &= A^*_1 - A_P h^2/\beta^*_3 = A_{11} + E_P A_P, \\
B_2^* &= A^*_2 - A_P h^2/2\beta^*_3 = A_{12} + E_P A_P h/2, \\
B_3^* &= A^*_3 - A_P h^2/4\beta^*_3 = A_{22} + C_{11} A_P + E_P A_P h^2/4; \\
E_P &= C_{11} = h^2/\beta^*_3.
\end{align*}
\]

Transferring equation (16) into the frequency domain, one gets

\[
[A] \{ Z''(x, \omega) \} + [B] \{ Z'(x, \omega) \} + [C] \{ Z(x, \omega) \} = 0,
\]

(17)

where vectors

\[
Z = \{ U(x, \omega), \Theta(x, \omega), W(x, \omega) \},
\]

\[
Z' = \frac{dZ}{dx}, Z'' = \frac{d^2Z}{dx^2},
\]

Figure 3. Model of FGM beam with piezoelectric patch.
If the piezoelectric layer is employed as a distributed sensor, charge output of which can be calculated as

\[
Q = \int D dA = b \int_0^L D dx = (b h_{14}/\beta_{33}^p) (u_0 - h \theta/2) L
\]

(18)
That can be easily solved to give three roots $\eta_1, \eta_2, \eta_3$, so that one obtains

$$\lambda_{1,4} = \pm ki; \quad \lambda_{1,5} = \pm k; \quad \lambda_{3,6} = \pm k; \quad k_j = \sqrt{\eta_j}, \quad j = 1, 2, 3.$$

As a consequence, general solution of equation (17) is represented as

$$[z_0] = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{16} \\ d_{21} & d_{22} & \ldots & d_{26} \\ d_{31} & d_{32} & \ldots & d_{36} \\ \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \begin{bmatrix} e^{\lambda_1 x} \\ \vdots \\ e^{\lambda_6 x} \end{bmatrix}$$

3. Dynamic stiffness matrix for FGM beam element with piezoelectric layer

3.1. General solution of vibration in FGM beam with piezoelectric layer

Seeking solutions of equation (17) in the form: $Z_0 = d e^{\lambda x}$ leads to characteristic equation

$$\det[\lambda^2 A + \lambda B + C] = 0$$

This is in fact a cubic algebraic equation with respect to $\eta = \lambda^2$ that can be easily solved to give three roots $\eta_1, \eta_2, \eta_3$, so that one obtains

$$\lambda_{1,4} = \pm ki; \quad \lambda_{1,5} = \pm k; \quad \lambda_{3,6} = \pm k; \quad k_j = \sqrt{\eta_j}, \quad j = 1, 2, 3.$$
The latter expressions show that
\[
\alpha_4 = \alpha_1; \quad \alpha_5 = \alpha_2; \quad \alpha_6 = \alpha_3; \quad \beta_4 = -\beta_1; \quad \beta_5 = -\beta_3; \quad \beta_6 = -\beta_2.
\]
Therefore, general solution of (17) can be rewritten in the form
\[
\{z_0(x, \omega)\} = \{G_0(x, \omega)\} \{C\}
\]  
(20)
where \(\{C\} = (G_1, \ldots, G_6)^T\) and \(\{G_0(x, \omega)\}\) is the matrix
\[
\begin{bmatrix}
G_{1x} e^{k_1 x} & G_{2x} e^{k_2 x} & G_{3x} e^{k_3 x} & G_{4x} e^{k_4 x} & G_{5x} e^{k_5 x} & G_{6x} e^{k_6 x}
\end{bmatrix}.
\]  
(21)

3.2. Dynamic stiffness matrix formulation

Considering a beam element as shown in figure 2, where the following nodal displacement and force vectors are introduced
Figure 4. Natural frequencies of SS-beam in dependence on normalized thickness of piezoelectric patch bonded onto different positions on beam and gradient index of functionally graded material: (a) first; (b) second and (c) third frequency.

\[
\{ U(\omega) \} = \{ U_1, \Theta_1, W_1, U_2, \Theta_2, W_2 \}^T ; \\
\{ P(\omega) \} = \{ N_1, M_1, Q_1, N_2, M_2, Q_2 \}^T
\]  

With

\[
U_1 = U(0, \omega); \quad \Theta_1 = \Theta(0, \omega); \quad W_1 = W(0, \omega); \\
U_2 = U(L, \omega); \quad \Theta_2 = \Theta(L, \omega); \quad W_2 = W(L, \omega); \\
N_1 = (B_{12}^* \partial_x \Theta - B_{13}^* \partial_x W)_{x=0}; \\
M_1 = (B_{12}^* \partial_x U - B_{13}^* \partial_x \Theta)_{x=0}; \\
Q_1 = A_{333}(\Theta - \partial_x W)_{x=0}; \\
N_2 = (B_{14}^* \partial_x U - B_{15}^* \partial_x \Theta)_{x=L};
\]
Rewrite the latter equations in matrix form

\[
\begin{align*}
M_2 &= (B_{23}^x \partial_\Theta \Theta - B_{24}^x \partial_\Theta U)_{x=L}; \\
Q_2 &= A_{33}(\partial_W \Theta - \Theta)_{x=L}.
\end{align*}
\]

where

\[
\begin{align*}
\{U_1, \Theta_1, W_1, U_2, \Theta_2, W_2\}^T &= [Z(\omega)] \{G\}; \\
\{N_1, M_1, Q_1, N_2, M_2, Q_2\}^T &= [\tilde{Q}(\omega)] \{G\},
\end{align*}
\]

(23)

(24)
and $\mathbb{R}$ is differential operator

$$
[\mathbb{R}] = \begin{bmatrix}
B_{11}^a \partial_{x_k} & -B_{12}^a \partial_{x_k} & 0 \\
-B_{12}^a \partial_{x_k} & B_{22}^a \partial_{x_k} & 0 \\
0 & -A_{33}^a & A_{33}^a \partial_{x_k}
\end{bmatrix}.
$$

Eliminating vector $C$ from equation (23) leads to

$$
[Q_c] = [D_c(\omega)] [U_l],
$$

where matrix

$$
[D_c(\omega)] = [Q(\omega)] \cdot [Z(\omega)]^{-1}
$$

is called hereby dynamic stiffness matrix for the FGM Timoshenko beam element.

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**Figure 6.** Natural frequencies of CF-beam in dependence on normalized thickness of piezoelectric patch bonded onto different positions on beam and gradient index of functionally graded material: (a) first; (b) second and (c) third frequency.
In general case, when a given structure consists of a number of beam elements, the total dynamic stiffness matrix for the structure is assembled accordingly to that as accomplished in the finite element method. Namely, the dynamic stiffness matrix is assembled by

$$[D(\omega)] = \sum_{n=1}^{p-1} [T_n]^{-1} [D_n(\omega)] \cdot [T_c].$$

4. Numerical results for illustration and validation

Consider an FGM beam bonded by a piezoelectric patch as shown in figure 3. Recalling the notations for size (thickness and length) of the piezoelectric patch and host beam, we can note that the single beam segment without piezoelectric patch is a particular case of the two-layer one when \(h_p = 0\). So that the FGM beam bonded with a piezoelectric patch is now considered as a structure consisting of three beam elements, one of which is the beam element with piezoelectric layer considered above the other two elements are simple ones without piezoelectric layer. Position of piezoelectric patch bonded on beam is defined by the distance measured from the left end of beam to the left end of piezoelectric patch. The particular case considered below is that shown in

Figure 7. Variation of normalized first (left) and second (right) frequencies versus relative thickness of piezoelectric patch bonded onto (a) the ends; (b) the middle and (c) full-length of SS-beam for different gradient index \((n = 0.2 - 0.5 - 1.0 - 2.0 - 5.0 - 10)\) of the material.
Figure 3 where length of three elements is the same and position of piezoelectric patch is acknowledged as the left, right (boundary) and intermediate (middle) one.

Suppose that constants of FGM and piezoelectric material are

\[
\begin{align*}
E_t &= 390 \text{ GPa}, \quad \rho_t = 3960 \text{ kg m}^{-3}, \quad \mu_t = 0.25; \\
E_b &= 210 \text{ GPa}, \\
\rho_b &= 7800 \text{ kg m}^{-3}, \quad \mu_b = 0.31; \\
C_{11}^p &= 69.0084 \text{ GPa}, \\
C_{66}^p &= 21.0526 \text{ GPa}, \\
\rho_p &= 7750 \text{ kg m}^{-3}, \\
h_{13} &= -7.70394 \times 10^8 \text{ V m}^{-1}, \\
\beta_{33}^p &= 7.3885 \times 10^7 \text{ m F}^{-1}.
\end{align*}
\]

So, natural frequency parameter \( \lambda = \omega (L^2/\theta) \sqrt{\rho_b / E_b} \) are computed as function of thickness ratio \( h_p / h_b \) (called below normalized thickness of piezoelectric patch) for various gradient index \( n \) of the material. Five lowest frequency parameters computed for three traditional cases of boundary conditions such as simply supported (SS), clamped–clamped (CC) and clamped-free (CF) end beam are given in tables 1–3. For comparison, there are provided also in the tables the frequency parameters computed for the FGM beam without piezoelectric patch by DSM proposed in Su and Banerjee, 2015 that is shortly noticed in the Tables as S&B. Agreement between the frequency parameters computed in the present study for the case if \( h_p = 0 \) and those obtained in the Reference mentioned above is a fact that validates reliability of the proposed theoretical development. Also, three lowest natural frequencies of the beams in dependence of the normalized thickness of
the piezoelectric patch bonded on various positions on beam and gradient index of the functionally graded material are illustrated in figures 1–3.

For illustrating a more typical effect of the piezoelectric patch on variation of natural frequencies, the ratio of natural frequencies of beam with piezoelectric patch to those of beam without the patch is introduced and acknowledged herein as normalized natural frequencies. The ratio computed for two modes of FGM beam with

Figure 9. Variation of normalized first (left) and second (right) frequency versus thickness of piezoelectric patch bonded onto (a) the clamped end; (b) the free end; (c) the middle and (d) full-length of cantilever beam in various gradient index \( n = 0.2 \)–0.5–1.0–2.0–5.0–10 of the material.
piezoelectric patch bonded on three different positions (two at boundaries and one at the middle) and the beam with full-length bonded piezoelectric layer is presented in the subsequent figures 4–6. The latter figures are provided in two columns representing two the frequencies and four rows corresponding to the locations where the piezoelectric patch is bonded onto. Since the boundary conditions of SS- and CC-beams are symmetrical, natural frequencies are independent upon which end the piezoelectric patch is bonded on. So that in the figures 4–5 corresponding to SS- and CC-beams there are only three rows, while in figure 6 representing the normalized frequencies of CF-beam we have got four rows.

Observing the data given in tables 1–3 and figures 4–6 one can make a discussion as follows: comparing the natural frequencies computed in this study for FGM beam with piezoelectric patch of zero thickness with those computed for FGM beam without piezoelectric patch given in Su and Banerjee, 2015 shows very good agreement between the results. This can be acknowledged as a validation of the theory developed above for FGM beam with piezoelectric patch. Natural frequencies of an FGM beam bonded with a piezoelectric patch remain to decrease with increasing gradient index \( n \) of the material as those of the beam without piezoelectric patch. Natural frequencies of FGM beam bonded with piezoelectric patch as function of the patch thickness are dependent on boundary conditions and where the piezoelectric patch is bonded on beam. Namely, for the beam of symmetric boundary conditions such as SS- and CC-beams, first two natural frequencies are both monotonically decreasing with increasing thickness of piezoelectric patch bonded onto the beam middle. In the case when the piezoelectric patch is attached to the clamped ends of CC-beam, first frequency of the beam is monotonically increasing while second and third frequencies are monotonically decreasing. In the latter case, all three natural frequencies of SS-beam are varying not monotonically, but first slightly decreasing then increasing with the thickness. More attractive behavior is observed for natural frequencies of CF-beam that are all monotonically reducing with growing of the patch thickness when the piezoelectric patch is bonded to the beam free end. While the first and second frequencies are both monotonically growing when the patch is attached to the clamped end. This was well known fact in studying cantilever beam with attached mass.

A more fruitful insight to the variation of natural frequencies versus thickness of piezoelectric patch and gradient index of material can be provided by examining the ratio of natural frequencies of beam with piezoelectric patch to those of the base beam alone. The ratios called above normalized frequencies are computed in dependence upon thickness of the piezoelectric patch for various material gradient index and different patch locations. Graphs of the normalized natural frequencies given in figures 7–9 corresponding to the cases of conventional boundary conditions allow one to make the following notices. First, it is observed that the ratios are increasing with material gradient index \( n \) what is opposite to the variation of the natural frequencies themselves. Next, the normalized natural frequencies of beam covered with full-length piezoelectric layer vary in the same (parabolic) mode for all the three cases (SS/CC/CF) of boundary conditions. A similar (to the latter) mode of variation appears also for second frequency of SS- and CC-beam with piezoelectric patch bonded onto the beam ends. The normalized first frequency of CC- and CF- beams with piezoelectric patch bonded onto clamped end is monotonically increasing with the patch thickness. Both two normalized frequencies of CC-beam with intermediately bonded piezoelectric patch and CF-beam with the patch bonded onto free end are monotonically decreasing when the thickness is growing. Finally, the normalized frequencies of SS-beam with piezoelectric patch are varying in non-monotonous mode, but it can be noticed herein that variation of the normalized first frequency along thickness of piezoelectric patch attached to the beam end is similar to variation of normalized second frequency when the patch is immediately bonded on the beam.

All the above made notices demonstrate the fact that effect of a bonded piezoelectric patch on the dynamic characteristics of an FGM beam is strongly coupled with the effect of functionally graded material properties. This exhibits an interaction between the electro-elasticity of piezoelectric and functionally graded materials.

5. Conclusions

Natural frequencies are examined for an FGM beam bonded with a piezoelectric patch that includes also the case of beam covered with full-length piezoelectric layer using the dynamic stiffness method. First, a model of FGM beam element with a piezoelectric layer is proposed using the power law of material property grading and Timoshenko beam theory. The piezoelectric layer has the same width as the host beam and is modeled as a homogeneous Timoshenko beam element. The dynamic stiffness model of the beam element is first developed and then used for modal analysis of the FGM beam with piezoelectric patch as a multistep beam structure.

Numerical analysis has been carried out to study dependence of the natural frequencies on the piezoelectric patch thickness, location and gradient index of the functionally graded material. It was demonstrated that piezoelectric patch bonded to a beam does not change the basic properties of the beam material, but it can make change (either increase or decrease) in natural frequencies of the beam. Namely, piezoelectric patch makes the
natural frequencies increased/decreased when it is bonded closely to the clamped/free end of beam. In general, thin/thick piezoelectric patch or layer reduces/increases the stiffness of the beam.

The sensor and actuator problem of the piezoelectric patch has not been investigated in the present work; it would be a subject for further study of the authors.

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