THE HASTINGS-MCLEOD SOLUTION TO THE GENERALIZED SECOND PAINLEVÉ EQUATION

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Abstract. The generalized second Painlevé equation \( \Delta y - x_1 y - 2y^3 = 0 \) in \((x_1, x_2) \in \mathbb{R}^2\), plays an important role in the theory of light-matter interactions in nematic liquid crystals [6]. Hastings-McLeod [16] showed the existence of a unique, positive, entire solution of the ODE \( y'' - xy - 2y^3 = 0 \). In this paper we show the existence of the corresponding solution of the PDE on the plane. It has a form of a quadruple connection between the Airy function \( \text{Ai}(x) \), two one dimensional Hastings-McLeod solutions \( \pm h(x) \) and the heteroclinic solution \( \tanh(x/\sqrt{2}) \) of the one dimensional Allen-Cahn equation.

1. Introduction

The second Painlevé equation
\[
y'' - xy - 2y^3 = 0, \quad x \in \mathbb{R},
\]
is known to play an important role in the theory of integrable systems [1], random matrices [13, 14, 9], Bose-Einstein condensates [2, 3, 18, 22] and other problems [4, 17, 19]. Recently [10] it has been shown that when the right hand side of (1.1) is allowed to be a constant \( \alpha \in \mathbb{R} \) then it describes local profiles of the so-called shadow kink in the theory of light-matter interaction of nematic liquid crystals. In [6, 11, 12] further relation between other types of non topological defects (shadow vortices, shadow domain walls) and the generalized Painlevé equation
\[
\Delta y - x_1 y - 2y^3 = 0, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,
\]
was established showing that their local structure is determined by special solutions of (1.2). One of the characteristics of these solutions is that they should be entire, another is that they should be minimal. To explain what this means, let \( A \in \mathbb{R}^2 \) be a bounded subset of \( \mathbb{R}^2 \) and
\[
E_{\text{P}II}(u, A) = \int_A \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} x_1 u^2 + \frac{1}{2} u^4 \right],
\]
be the functional associated to the generalized second Painlevé equation. By definition a solution of (1.2) is minimal if
\[
E_{\text{P}II}(y, \text{supp } \phi) \leq E_{\text{P}II}(y + \phi, \text{supp } \phi)
\]
for all \( \phi \in C_0^\infty(\mathbb{R}^2) \). This notion of minimality is standard for many problems in which the energy of a localized solution is actually infinite due to non compactness of the domain. The study of minimal solutions of (1.1) has been recently initiated in [10] where we have showed that the Hastings-McLeod solution, denoted in this paper by \( h \), is, up to the sign change, the only minimal solution which is bounded at \( +\infty \). We recall (cf. [16]) that \( h : \mathbb{R} \to \mathbb{R} \) is positive, strictly decreasing \( (h' < 0) \) and such that
\[
h(x) \sim \text{Ai}(x), \quad x \to \infty,
\]
\[
h(x) \sim \sqrt{|x|}/2, \quad x \to -\infty.
\]
The asymptotic behaviour of \( h \) is determined by the location of the global minima of the potential \( H(x, y) = \frac{1}{2} xy^2 + \frac{1}{2} y^4 \) associated to the equation (1.1) that can alternatively be written \( y'' - H_y(x, y) = 0 \). Indeed, for

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for all $\phi$ such that
\begin{equation}
\lim_{t \to -\infty} y(t, x_1) = O(1), \text{ as } x_1 \to \infty \text{ (uniformly in } x_2).}
\end{equation}

There exists a solution to
\begin{equation}
\Delta y - x_1 y - 2y^3 = 0, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,
\end{equation}
such that
\begin{enumerate}
\item $y$ is positive in the upper-half plane and odd with respect to $x_2$ i.e. $y(x_1, x_2) = -y(x_1, -x_2)$.
\item $y$ and its derivatives are bounded in the half-planes $[s_0, \infty) \times \mathbb{R}, \forall s_0 \in \mathbb{R}$.
\item $y$ is minimal with respect to perturbations $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\phi(x_1, x_2) = -\phi(x_1, -x_2)$.
\item $\frac{\mu(x_1,x_2)}{AC(x_1)} = O(1)$, as $x_1 \to \infty$ (uniformly in $x_2$).
\item For every $x_2 \in \mathbb{R}$ fixed, let $\tilde{y}(t_1, t_2) := \frac{\sqrt{2}}{(-\frac{3}{2}t_1)^\frac{3}{2}} y\left(-\left(-\frac{3}{2}t_1\right)^\frac{3}{2}, x_2 + t_2\left(-\frac{3}{2}t_1\right)^{-\frac{1}{2}}\right)$. Then
\begin{equation}
\lim_{t \to -\infty} \tilde{y}(t_1 + l, t_2) = \begin{cases} 
\tanh(t_2/\sqrt{2}) & \text{when } x_2 > 0, \\
-1 & \text{when } x_2 < 0,
\end{cases}
\end{equation}
for the $C^1_{\text{loc}}(\mathbb{R}^2)$ convergence.
\item $y_{x_1}(x_1, x_2) < 0, \forall x_1 \in \mathbb{R}, \forall x_2 > 0$.
\item $y_{x_2}(x_1, x_2) > 0, \forall x_1, x_2 \in \mathbb{R}$, and $\lim_{l \to \pm \infty} y(x_1, x_2 + l) = \pm h(x_1)$ in $C^2_{\text{loc}}(\mathbb{R}^2)$, where $h$ is the Hastings-McLeod solution of (1.1).
\end{enumerate}

Comparing (iv) with (1.4) we see that as $x_1 \to \infty$ the function $y(x_1, x_2)$ behaves similarly as the Hastings-McLeod solution $h(x_1)$. At the same time, as $x_2 \to \pm \infty$ we have $y(x_1, x_2) \to \pm h(x_1)$, $x_2 \to \pm \infty$. Perhaps the most interesting aspect of the above solution $y$ is stated in property (v), since after rescaling we obtain as $x_1 \to -\infty$, the convergence to the heteroclinic orbit $\eta(x) = \tanh(x/\sqrt{2})$ ($\eta: \mathbb{R} \to (-1,1)$) of the Allen-Cahn ODE $\eta'' = \eta^3 - \eta$. We recall that this orbit connecting the two minima $\pm 1$ of the corresponding potential $W(u) = \frac{1}{4}(1 - u^2)^2$, plays a crucial role in the study of minimal solutions of the Allen-Cahn equation
\begin{equation}
\Delta u = u^3 - u, \quad u: \mathbb{R}^n \to \mathbb{R}.
\end{equation}

Again, we say that $u$ is a minimal solution of (1.7) if
\begin{equation}
E_{\text{AC}}(u, \supp \phi) \leq E_{\text{AC}}(u + \phi, \supp \phi),
\end{equation}
for all $\phi \in C_0^\infty(\mathbb{R}^2)$, where
\begin{equation}
E_{\text{AC}}(u, \Omega) := \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{4}(1 - u^2)^2
\end{equation}
is the Allen-Cahn energy associated to (1.7). It is known [21] that in dimension $n \leq 7$, any minimal solution $u$ of (1.7) is either trivial i.e. $u \equiv \pm 1$ or one dimensional i.e. $u(x) = \eta((x - x_0) \cdot \nu)$, for some $x_0 \in \mathbb{R}^n$, and some unit vector $\nu \in \mathbb{R}^n$. In the proof of Theorem 1.1 it is shown that a minimal solution of (1.5) rescaled as in (v), converges as $x_1 \to -\infty$ to a minimal solution of (1.7). This deep connection of the structure of the Painlevé equation with the Allen-Cahn PDE, suggests that several properties of the Allen-Cahn equation should be transferred to the Painlevé equation. Although by construction the solution $y$ is only minimal for odd perturbations, we expect that $y$ is actually minimal for general perturbations, and plays a similar role that the heteroclinic orbit for the Allen-Cahn equation. What’s more the two global minimizers $\pm 1$ of the functional $E_{\text{AC}}$ have their counterparts in the two minimal solutions $\pm h$ of the Painlevé equation. Indeed, property (vii) establishes that $y$ connects monotonically along the vertical direction $x_2$, the two minimal solutions $\pm h(x_1)$, in the same way that $\eta$ connects monotonically the two global minimizers $\pm 1$. This analogy between the Painlevé and the Allen-Cahn equation is natural if seen from the point of view of the potential.
$H(x_1, y) = \frac{1}{2}x_1y^2 + \frac{1}{2}y^4$ (cf. the expression of $E_{\rho_1}$) compared with $W(u) = \frac{1}{4}(1 - u^2)^2$. For the latter the phase transition connects two minima ±1 while for the former the phase transition connects the two branches $\pm \sqrt{(-x_1)^{3/2}}$ of minima of the potential $H$ parametrized by $x_1$.

We believe that in higher dimension $y : \mathbb{R}^{n+1} \to \mathbb{R}$, $(n \geq 1)$ the structure of solutions of (1.5) exactly mirrors that of (1.7), and going further, one may ask: is it true that in dimension $n \leq 7$, any minimal solution $Y : \mathbb{R}^{n+1} \to \mathbb{R}$ of (1.5) is either $Y(x_1, x_2, \ldots, x_{n+1}) = \pm h(x_1)$ or $Y(x_1, x_2, \ldots, x_{n+1}) = y(x_1, (x_2, \ldots, x_{n+1}) \cdot n + b)$, for some constant $b \in \mathbb{R}$, and some unit vector $n \in \mathbb{S}^{n-1}$?

2. Odd minimizers of the Ginzburg-Landau type energy

We consider the energy functional

$$E(u) = \int_{\mathbb{R}^2} \frac{\epsilon}{2} |\nabla u|^2 - \frac{1}{2\epsilon} \mu(x) u^2 + \frac{1}{4\epsilon} u^4,$$

where $u \in H^1(\mathbb{R}^2)$ and $\epsilon > 0$ is a small parameter. We suppose that $\mu \in C^\infty(\mathbb{R}^2)$ is radial i.e. $\mu(x) = \mu_{rad}(|x|)$, with $\mu_{rad} \in C^\infty(\mathbb{R})$ an even function. In addition we assume that

$$\mu \in L^\infty(\mathbb{R}^2), \mu'_{rad} < 0 \text{ in } (0, \infty), \text{ and } \mu_{rad}(\rho) = 0 \text{ for a unique } \rho > 0,$$

In the physical context described in [6] the function $\mu$ is specific

$$\mu(x) = e^{-|x|^2} - \chi, \quad \text{with some } \chi \in (0, 1), \quad f(x) = -\frac{1}{2} \nabla \mu(x),$$

but this particular form is irrelevant here. The Euler-Lagrange equation of $E$ is

$$(2.3) \quad \epsilon^2 \Delta u + \mu(x) u - u^3 = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

and we also write its weak formulation:

$$(2.4) \quad \int_{\mathbb{R}^2} -\epsilon^2 \nabla u \cdot \nabla \psi + \mu u \psi - u^3 \psi = 0, \quad \forall \psi \in H^1(\mathbb{R}^2),$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$. Note that due to the radial symmetry of $\mu$ the energy (2.1) and equation (2.3) are invariant under orthogonal transformations in the domain, and sign change in the range. Our strategy to construct the solution of (1.5) enjoying the properties described in Theorem 1.1 is to find first an odd with respect to $x_2$ minimizer $u_\epsilon$ of $E$ and then scaling and passing to the limit $\epsilon \searrow 0$ recover $y$ - this gives us existence. Second, in section 3 we show all the properties of $y$ stated in Theorem 1.1.

We explain, formally at the moment, the relation between (1.5) and the energy $E$. Looking at the energy density of $E$ it is evident that as $\epsilon \to 0$ the modulus of the global or odd minimizer $u_\epsilon$ should approach a nonnegative root of the polynomial

$$-\mu(x) z + z^3 = 0,$$

or in other words, $|u_\epsilon| \to \sqrt{\mu_{\tau}}$ as $\epsilon \to 0$ in some, perhaps weak, sense. This function, called the Thomas-Fermi limit of the minimizer is not in $H^1(\mathbb{R}^2)$ and therefore the transition near the set $\mu(x) = 0$ has to be mediated somehow. To see this let us consider a point $\xi$ such that $\mu(\xi) = 0$. By (2.2) $\xi = \rho e^{i\theta}$. At $\xi$ introduce local orthogonal frame $(e^{i\theta}, ie^{i\theta})$ and coordinates $s = (s_1, s_2)$ associated with it. Let $u_\epsilon$ be any solution of (2.3) and

$$z(s) = e^{-1/3} u(\xi + \epsilon^{2/3} s).$$

Noting that $\mu(\xi + \epsilon^{2/3} s) = \epsilon^{2/3} s_1 \mu_1 + \ldots$ with $\mu_1 < 0$ we get that $z$ satisfies

$$\Delta_s z + s_1 \mu_1 z - z^3 = o(1), \quad \text{as } \epsilon \searrow 0.$$

The equation on the left becomes the second Painlevé equation after passing to the limit and suitable scaling. Now, suppose that $u_\epsilon$ is the odd minimizer of $E$, i.e. $u_\epsilon(x_1, x_2) = -u_\epsilon(x_1, -x_2)$. Except for the points $\vec{x} = (\pm \rho, 0)$ the limiting function $z$ could be one of the Hastings-McLeod one dimensional solutions. However, at $(\pm \rho, 0)$ we should have $z(s_1, s_2) = -z(s_1, -s_2)$, which means that $z$ genuinely depends on both variables. This is the idea behind the proof of the existence part in Theorem 1.1. Showing properties of the solution is a different story and depends on rather tricky application of the moving plane method.
Our first purpose in this paper is to study qualitative properties of the global minimizers of $E$ as $\epsilon \searrow 0$.

In our previous work [10] we studied the following energy

$$E(u, \mathbb{R}) = \int_{\mathbb{R}} \left( \frac{\epsilon}{2} |u_x|^2 - \frac{1}{4\epsilon} \mu(x) u^2 + \frac{1}{4\epsilon} |u|^4 - a f(x) u \right), \quad u : \mathbb{R} \to \mathbb{R},$$

where $a \geq 0$ is a parameter and $f = -\frac{1}{2} \mu'$, and in [11] we studied its analog for maps $u : \mathbb{R}^2 \to \mathbb{R}^2$.

By proceeding as in [11], one can see that under the above assumptions there exists a global minimizer $v$ of $E$ in $H^1(\mathbb{R}^2)$, namely that $E(v) = \min_{H^1(\mathbb{R}^2)} E$. In addition, we show that $v$ is a classical solution of (2.3), and $v$ is radial. Similarly, in the class $H^1_{\text{odd}}(\mathbb{R}^2) := \{ u \in H^1(\mathbb{R}^2) : u(x_1, x_2) = -u(x_1, -x_2) \}$ of odd functions with respect to $x_2$, there exists an odd minimizer $u$ which also solves (2.3) and satisfies $u(x_1, x_2) = u(-x_1, x_2)$. Although in the sequel we will focus on the odd minimizer for completeness we chose to present our next result in a slightly more general framework.

**Theorem 2.1.** For $\epsilon \ll 1$ let $u_\epsilon$ be a solution of (2.3) converging to 0 as $|x| \to \infty$ (which may be the odd or global minimizer). Let $\rho > 0$ be the zero of $\mu_{\text{rad}}$ and let $\mu_1 := \mu_{\text{rad}}(\rho) < 0$. For every $\xi = \rho e^{i\theta}$, we consider the local coordinates $s = (s_1, s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$, and the rescaled functions:

$$(2.5) \quad w_\epsilon(s) = 2^{-1/2}(-\mu_1 \epsilon)^{-1/3} u_\epsilon \left( \frac{\xi + \epsilon^{2/3} s}{(-\mu_1)^{1/3}} \right).$$

As $\epsilon \to 0$, the function $w_\epsilon$ converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ up to subsequence, to a function $y$ bounded in the half-planes $[s_0, \infty) \times \mathbb{R}$, for every $s_0 \in \mathbb{R}$, which is a solution of

$$(2.6) \quad \Delta y(s) - s_1 y(s) - 2y^3(s) = 0, \quad \forall s = (s_1, s_2) \in \mathbb{R}^2.$$

In particular, if we take $u_\epsilon$ to be the odd minimizer of $E$ and $\xi = (\pm \rho, 0)$, then the solution $y$ satisfies $y(s_1, s_2) = -y(s_1, -s_2)$, and is minimal with respect to perturbations $\phi \in C^0_{\text{loc}}(\mathbb{R}^2)$, $\phi(s_1, s_2) = -\phi(s_1, -s_2)$. On the other hand, if we take $u_\epsilon$ to be the odd minimizer then $y(s_1, s_2) = h(s_1)$ or $y(s_1, s_2) = -h(s_1)$.

We observe that as a corollary of [12, Theorem 1.1.] it can be proven that $|v_\epsilon| \to \sqrt{\mu_{\text{rad}}} in C^0_{\text{loc}}(D(0; \rho))$. Because of the analogy between the functional $E$ and the Gross-Pitaevskii functional in the theory of Bose-Einstein condensates we will call $\sqrt{\mu_{\text{rad}}}$ the Thomas-Fermi limit of $v_\epsilon$. Theorem 2.1 gives account on how non-smoothness of the limit of $v_\epsilon$ is mediated near the circumference $|x| = \rho$, where $\mu$ changes sign, through the solution of (2.6). We should mention here that detailed description of the minimizers for yet more general setting of the energy can be found in [11, 12].

Before proving the theorem we gather general results for minimizers and solutions that are valid for any values of the parameters $\epsilon > 0$. For the rest of this paper $v$ or $v_\epsilon$ will be the global minimizer and $u$ or $u_\epsilon$ will be the odd minimizer or a critical point of $E$. We first prove the existence of global and odd minimizers.

**Lemma 2.2.** For every $\epsilon > 0$ there exists $v \in H^1(\mathbb{R}^2)$ such that $E(v) = \min_{H^1(\mathbb{R}^2)} E$. As a consequence, $v$ is a $C^\infty$ classical solution of (2.3). Moreover, for $\epsilon \ll 1$ the global minimizer $v$ is unique up to change of $v$ by $-v$, and it can be written as $v(x) = v_{\text{rad}}(|x|)$, with $v_{\text{rad}} \in C^\infty(\mathbb{R})$, positive, even, and such that $\lim_{|x| \to \infty} v_{\text{rad}} = 0$.

**Proof.** We proceed as in [11, Lemma 2.1] to establish that the global minimizer exists and is a smooth solution of (2.3) converging to 0 as $|x| \to \infty$. Next, we notice that $v \equiv 0$ for $\epsilon \ll 1$. Indeed, by choosing a test function $\psi \neq 0$ supported in $D(0; \rho) \cap \{ x_2 > 0 \}$, and such that $\psi^2 < 2\mu$, one can see that

$$E(\psi) = \frac{\epsilon}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 + rac{1}{4\epsilon} \int_{\mathbb{R}^2} \psi^2(\psi^2 - 2\mu) < 0, \quad \epsilon \ll 1.$$

Let $x_0 \in \mathbb{R}^2$ be such that $v(x_0) \neq 0$. Without loss of generality we may assume that $v(x_0) > 0$. Next, consider $\tilde{v} = |v|$ which is another global minimizer and thus another solution. Clearly, in a neighborhood of $x_0$ we have $v = |v|$, and as a consequence of the unique continuation principle (cf. [20]) we deduce that $v \equiv \tilde{v} > 0$ on $\mathbb{R}^2$. Furthermore, the maximum principle implies that $v > 0$, since $v \neq 0$. To prove that $v$ is radial we consider the reflection with respect to the line $x_1 = 0$. We can check that $E(v, \{ x_1 > 0 \}) = E(v, \{ x_1 < 0 \})$, since otherwise by even reflection we can construct a map in $H^1$ with energy smaller than $v$. Thus, the map $\hat{v}(x) = v(|x_1|, x_2)$ is also a minimizer, and since $\hat{v} = v$ on $\{ x_1 > 0 \}$, it follows by unique continuation that $\hat{v} \equiv v$ on $\mathbb{R}^2$. Repeating the same argument for any line of reflection, we deduce that $v$ is radial. To complete
clearly $u$ is a solution of (2.4), we find for any solution $u \in H^1(\mathbb{R}^2)$ of (2.3) the following alternative expression of the energy:

\begin{equation}
E(u) = -\int_{\mathbb{R}^2} \frac{u^4}{4e}.
\end{equation}

Formula (2.7) implies that $v$ and $\tilde{v}$ intersect for $|x| = r > 0$. However, setting

$$w(x) = \begin{cases} v(x) & \text{for } |x| \leq r \\ \tilde{v}(x) & \text{for } |x| \geq r, \end{cases}$$

we can see that $w$ is another global minimizer, and again by the unique continuation principle we have $w \equiv v \equiv \tilde{v}$. This completes the proof of Lemma 2.2. \hfill \Box

On the other hand, in the class $H^1_{\text{odd}}(\mathbb{R}^2) := \{ u \in H^1(\mathbb{R}^2) : u(x_1, x_2) = -u(x_1, -x_2) \}$ of odd functions with respect to $x_2$, there exists an odd minimizer with the following properties:

**Lemma 2.3.** For every $\epsilon > 0$ there exists $u \in H^1_{\text{odd}}(\mathbb{R}^2)$ such that $E(u) = \min_{H^1_{\text{odd}}(\mathbb{R}^2)} E$. As a consequence, $u$ is a $C^\infty$ classical solution of (2.3). Moreover

(i) $u(x) \to 0$ as $|x| \to \infty$,
(ii) $u(x_1, x_2) = u(-x_1, x_2)$,
(iii) up to transformation $u \mapsto -u$ we have $u(x_1, x_2) > 0$, $\forall (x_1, x_2) \in \mathbb{R} \times (0, \infty)$, provided that $\epsilon \ll 1$.

**Proof.** The existence of $u \in H^1_{\text{odd}}(\mathbb{R}^2)$ such that $E(u) = \min_{H^1_{\text{odd}}(\mathbb{R}^2)} E$, follows as in [11, Lemma 2.1], and clearly $u$ is a critical point of $E$ in the subspace $H^1_{\text{odd}}(\mathbb{R}^2)$. In view of the radial symmetry of $\mu$ it is easy to see that the Euler-Lagrange equation (2.4) holds also for every $\phi \in H^1(\mathbb{R}^2)$, such that $\phi(x_1, x_2) = \phi(x_1, -x_2)$.

As a consequence, $u$ is a $C^\infty$ classical solution of (2.3).

For the proof of (i) we refer to [11, Lemma 2.1]. To show that $u(x_1, x_2) = u(-x_1, x_2)$, we first note that $E(u, [0, \infty) \times \mathbb{R}) = E(u, (-\infty, 0] \times \mathbb{R})$. Indeed, if we assume without loss of generality that $E(u, [0, \infty) \times \mathbb{R}) < E(u, (-\infty, 0] \times \mathbb{R})$, the function

\begin{equation}
\tilde{u}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{when } x_1 \geq 0 \\ u(-x_1, x_2) & \text{when } x_1 \leq 0, \end{cases}
\end{equation}

has strictly less energy than $u$, which is a contradiction. Thus, $E(u, [0, \infty) \times \mathbb{R}) = E(u, (-\infty, 0] \times \mathbb{R})$, and as a consequence the function $\tilde{u}$ is also an odd minimizer and a solution. It follows by unique continuation [20] that $\tilde{u} \equiv u$, that is, $u(x_1, x_2) = u(-x_1, x_2)$.

Now, it remains to establish the uniqueness of the odd minimizer $u$, when $\epsilon \ll 1$. Proceeding as in Lemma 2.2, we can see that $u \neq 0$ for $\epsilon \ll 1$, and that either $u > 0$ or $u < 0$ on $\mathbb{R} \times (0, \infty)$. Assume that $u_1$ and $u_2$ are two minimizers of $E$ in $H^1_{\text{odd}}(\mathbb{R}^2)$ such that $u_1 > 0$ and $u_2 > 0$ on $\mathbb{R} \times (0, \infty)$. Next, define the maps

\begin{equation}
u_+(x_1, x_2) = \begin{cases} \min(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \geq 0 \\ \max(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \leq 0, \end{cases}
\end{equation}

\begin{equation}
u_-(x_1, x_2) = \begin{cases} \max(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \geq 0 \\ \min(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \leq 0, \end{cases}
\end{equation}

and the set $A := \{(x_1, x_2) \in \mathbb{R} \times (0, \infty) : u_1(x_1, x_2) < u_2(x_1, x_2)\}$. We can see that $E(u_1, A) = E(u_2, A)$ since otherwise we have either $E(u_+) < E(u_2)$ or $E(u_-) < E(u_1)$, which contradicts the minimality of $u_1$ and $u_2$. As a consequence, $E(u_+) = E(u_2) = E(u_1) = E(u_-)$, and it follows that $u_+$ and $u_-$ are also minimizers and solutions. Next, by unique continuation [20], we obtain that either $u_1 \equiv u_+$ or $u_1 \equiv u_-$, i.e. we have either $0 \leq u_1 \leq u_2$ or $u_1 \geq u_2 \geq 0$ on $\mathbb{R} \times [0, \infty)$. Finally, applying (2.7) to $E(u_1) = E(u_2)$, we conclude in view of the ordering of $u_1$ and $u_2$ that $u_1 \equiv u_2$. This completes the proof. \hfill \Box

To study the limit of solutions as $\epsilon \to 0$, we need uniform bounds. Modifying slightly the arguments in [11, Section 2], we obtain:
Lemma 2.4. For every $\epsilon > 0$ let $u_\epsilon$ be a solution of (2.3) converging to 0 as $|x| \to \infty$. Then, $u_\epsilon$ are uniformly bounded.

Proof. We drop the index and write $u := u_\epsilon$. Since $\mu$ is bounded, the roots of the cubic equation $u^3 - \mu(x)u = 0$ belong to a bounded interval, for all values of $x$. If $u$ takes positive values, then it attains its maximum $0 \leq \max_{R^2} u = u(x_0)$, at a point $x_0 \in \mathbb{R}^2$. In view of (2.3):

$$0 \geq \epsilon^2 \Delta u(x_0) = u^3(x_0) - \mu(x_0)u(x_0),$$

thus it follows that $u(x_0)$ is uniformly bounded above. In the same way, we prove the uniform lower bound for $u$. \hfill \Box

Lemma 2.5. For $\epsilon \ll 1$ let $u_\epsilon$ be a solution of (2.3) converging to 0 as $|x| \to \infty$. Then, there exist a constant $K > 0$ such that

$$|u_\epsilon(x)| \leq K(\sqrt{\max(\mu(x),0)} + \epsilon^{1/2}), \quad \forall x \in \mathbb{R}^2.$$  

As a consequence, if for every $\xi = \rho e^{i\theta}$ we consider the local coordinates $s = (s_1, s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$, then the rescaled functions $w_\epsilon(s)$ defined in (2.5) are uniformly bounded on the half-planes $[s_0, \infty) \times \mathbb{R}$, $\forall s_0 \in \mathbb{R}$.

Proof. As above we write $u := u_\epsilon$. Let us define the following constants

- $M > 0$ is the uniform bound of $|u_\epsilon|$ (cf. Lemma 2.4),
- $\lambda > 0$ is such that $3\mu_{rad}(\rho - h) \leq 2\lambda h$, $\forall h \in [0,\rho]$,
- $\kappa > 0$ is such that $\kappa^4 \geq 6\lambda$.

Next, we construct the following comparison function

$$\chi(x) = \begin{cases} 
\lambda(\rho - |x| + \epsilon^{2/3}) & \text{for } |x| \leq \rho, \\
\frac{\lambda}{\sqrt{2}}(|x| - \rho - \epsilon^{2/3})^2 & \text{for } \rho \leq |x| \leq \rho + \epsilon^{2/3}, \\
0 & \text{for } |x| \geq \rho + \epsilon^{2/3}.
\end{cases}$$

One can check that $\chi \in C^1(\mathbb{R}^2 \setminus \{0\}) \cap H^1(\mathbb{R}^2)$ satisfies $|\Delta \chi| \leq \frac{2\lambda}{\epsilon^{2/3}}$ in $H^1(\mathbb{R}^2)$. Finally, we define the function $\psi := \frac{|u|^2}{2} - \chi - \kappa^2\epsilon^{2/3}$, and compute:

$$c^2 \Delta \psi = c^2(|\nabla u|^2 + u \Delta u - \Delta \chi) \\
\geq -\mu|u|^2 + |u|^4 - c^2 \Delta \chi \\
\geq -\mu|u|^2 + |u|^4 - 2\epsilon^{4/3}\lambda.$$  

(2.13)

Now, one can see that when $x \in \omega := \{x \in \mathbb{R}^2 : \psi(x) > 0\}$, we have $\frac{|u|^4}{3} - \mu|u|^2 \geq 0$, since

$$x \in \omega \cap D(0;\rho) \Rightarrow \frac{|u|^4}{3} \geq \frac{2\lambda}{3} \left(\rho - |x| + \frac{\epsilon^{2/3}}{2}\right)|u|^2 \geq \mu|u|^2.$$  

In the open set $\omega$ we also have: $|\frac{|u|^4}{3} \geq \frac{2\lambda}{3} \epsilon^{4/3} \geq 2\epsilon^{4/3}\lambda$, thus $\Delta \psi \geq 0 \in \omega$ in the $H^1$ sense. To conclude, we apply Kato’s inequality that gives: $\Delta \psi^+ \geq 0$ on $R^2$ in the $H^1$ sense. Since $\psi^+$ is subharmonic with compact support, we obtain by the maximum principle that $\psi^+ \equiv 0$ or equivalently $\psi \leq 0$ in $\mathbb{R}^2$. The statement of the lemma follows by adjusting the constant $K$. \hfill \Box

After this preparation we are ready to prove the main result of this section.

Proof Theorem 2.1. For every $\xi = \rho e^{i\theta}$ we consider the local coordinates $s = (s_1, s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$, and we rescale the solutions by setting $\tilde{u}(s) = \frac{u(\xi + s e^{2/3})}{\epsilon^{2/3}}$. Clearly $\Delta \tilde{u}(s) = \epsilon \Delta u(\xi + s e^{2/3})$, thus

$$\Delta \tilde{u}(s) + \frac{\mu(\xi + s e^{2/3})}{\epsilon^{4/3}} \tilde{u}(s) - \tilde{u}^3(s) = 0, \quad \forall s \in \mathbb{R}^2.$$  

Writing $\mu(\xi + \eta) = \mu_1 h_1 + h \cdot A(\eta)$, with $\mu_1 := \mu'_{rad}(\rho) < 0$, $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and $A(0) = 0$, we obtain

$$\Delta \tilde{u}(s) + (\mu_1 s_1 + A(s e^{2/3}) \cdot \xi) \tilde{u}(s) - \tilde{u}^3(s) = 0, \quad \forall s \in \mathbb{R}^2.$$  

(2.14)
Next, we define the rescaled energy by
\begin{equation}
(2.15) \quad \tilde{E}(\tilde{u}) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla \tilde{u}(s)|^2 - \frac{\mu (\xi + s e^{2/3})}{2e^{2/3}} \tilde{u}^2(s) + \frac{1}{4} \tilde{u}^4(s) \right) \, ds.
\end{equation}
With this definition \( \tilde{E}(\tilde{u}) = \frac{1}{e^{2/3}} E(u) \). From Lemma 2.5 and (2.14), it follows that \( \Delta \tilde{u} \), and also \( \nabla \tilde{u} \), are uniformly bounded on compact sets. Moreover, by differentiating (2.14) we also obtain the boundedness of the second derivatives of \( \tilde{u} \). Thanks to these uniform bounds, we can apply the theorem of Ascoli via a diagonal argument to obtain the convergence of \( \tilde{u} \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \) (up to a subsequence) to a solution \( \tilde{z} \) of
\begin{equation}
(2.16) \quad \Delta \tilde{z}(s) + \mu_1 s_1 \tilde{z}(s) - \tilde{z}^3(s) = 0, \quad \forall s \in \mathbb{R}^2,
\end{equation}
which is associated to the functional
\begin{equation}
(2.17) \quad \tilde{E}_0(\phi, J) = \int_{\mathbb{R}} \left( \frac{1}{2} |\nabla \phi(s)|^2 - \frac{\mu_1}{2} s_1 \phi^2(s) + \frac{1}{4} \phi^4(s) \right) \, ds.
\end{equation}

Given \( \tilde{\psi}(s) \) a test function supported in the compact set \( K \), let \( \psi(x) := e^{1/3} \tilde{\psi}(\frac{x - \xi}{e^{1/3}}) \) \( \Leftrightarrow \) \( \tilde{\psi}(s) = \psi(\tilde{\xi} + e^{2/3} s) \). In the case where we take \( u \) to be the global minimizer \( v \), since \( E(v + \psi, supp \psi) \geq E(v, supp \psi) \), we have \( \tilde{E}(\tilde{v} + \psi, K) \geq \tilde{E}(\tilde{v}, K) \), and at the limit \( \tilde{E}_0(\tilde{z} + \psi, K) \geq \tilde{E}_0(\tilde{z}, K) \). Thus, \( \tilde{z} \) is a minimal solution of (2.16). In addition, the radial symmetry of \( u \), implies that \( \tilde{z} \) depends only on the variable \( s_1 \). Indeed, noticing that \( \lim_{e \to 0} \frac{[\tilde{\xi} + e^{2/3}(s_1, s_2)] - \rho}{e^{2/3}} = s_1 \), it follows that \( \tilde{v}_e(s_1, s_2) = \tilde{v}_1(s_1 + o(1), 0) \), and \( \tilde{z}(s_1, s_2) = \tilde{z}(s_1, 0) \). Similarly, in the case where we take \( u \) to be the odd minimizer \( \xi = (\pm \rho, 0) \), we can see that \( \tilde{z} \) is a minimal solution of (2.16) for perturbations such that \( \tilde{\psi}(s_1, s_2) = -\tilde{\psi}(s_1, -s_2) \). Finally, setting \( y(s) := \frac{1}{\sqrt{2(\rho^2/3^2)}} \tilde{z}(s) - (\rho s_1, \rho s_2) \), (2.16) reduces to (2.6), that is, \( y \) solves (2.6). In the case where we take \( u \) to be the global minimizer \( v \), we can see that either \( y(s_1, s_2) = h(s_1) \) or \( y(s_1, s_2) = -h(s_1) \), since \( \pm h \) are the only minimal solutions of (1.1) (cf. [10, Theorem 1.3]). On the other hand, in the case where we take \( u \) to be the odd minimizer and \( \xi = (\pm \rho, 0) \), it is clear that \( y \) is odd with respect to \( s_2 \), and minimal for perturbations such that \( \tilde{\psi}(s_1, s_2) = -\tilde{\psi}(s_1, -s_2) \).

3. PROOF OF THEOREM 1.1

We will proceed in few steps. The proof of (i), (ii) and (iii) follows from Theorem 2.1, Lemma 2.5, and the fact that a minimal solution of 1.5 cannot be identically zero. To establish (v) we proceed as in Theorem 2.1. After rescaling appropriately \( y \) as \( x_1 \to -\infty \), we compute uniform bounds of the rescaled functions. Then, by the theorem of Ascoli, we obtain at the limit a minimal solution of the Allen-Cahn equation (1.7). The proof of (vi) and (vii) is based on the moving plane method applied in a sector contained in the upper half-plane. The main difficulty is due to the unboundedness of the domain and to the availability of boundary conditions only on the \( x_1 \) axis where \( y(x_1, 0) = 0 \). We also utilize the asymptotic behaviour of \( y \), as \( x_1 \to \pm \infty \), provided respectively by (v) and Lemma 3.2. Our main tool is a version of the maximum principle in unbounded domains (cf. Lemma 3.1), that allows us to compute bounds for \( y(x_1, y) \) when \( x_1 \) is large enough and \( x_2 > 0 \) (cf. Lemmas 3.3 and 3.4). Next, these bounds are extended to the whole half-plane \( x_2 > 0 \) by applying the sliding method (cf. Lemma 3.5).

Proof of (i), (ii) and (iii). By applying Theorem 2.1 in a neighborhood of the point \( \xi = (\rho, 0) \) to the odd minimizer \( u \), such that \( u > 0 \) on \( \mathbb{R} \times (0, \infty) \), it is clear that we obtain a solution \( y \) of (2.6) which is odd with respect to the second variable \( s_2 \), and such that \( y \geq 0 \), on \( \mathbb{R} \times (0, \infty) \). For the sake of convenience in what follows we substitute the variables \( (s_1, s_2) \) by \( (x_1, x_2) \). The properties (ii) and (iii) are also straightforward by Theorem 2.1 and Lemma 2.5. Thus, it remains to show that \( y(x_1, x_2) > 0 \), \( \forall x_1 \in \mathbb{R} \times (0, \infty) \). Assume by contradiction that \( y(x_1, x_2) = 0 \), for some \( x \in \mathbb{R} \times (0, \infty) \), then it follows from the maximum principle that \( y \equiv 0 \). To conclude we are going to show that a solution \( y \) of (1.5) which is minimal for odd perturbations, cannot be identically zero. Indeed, the minimality of \( y \) implies that the second variation of the energy \( E_{P_{11}} \) is nonnegative:
\begin{equation}
(3.1) \quad \int_{\mathbb{R}^2} (|\nabla \phi(x)|^2 + 6y^2(x) + x_1 \phi^2(x)) \, dx \geq 0, \quad \forall \phi \in C_0^1(\mathbb{R}^2), \text{ such that } \phi(x_1, x_2) = -\phi(x_1, -x_2).
\end{equation}
Clearly (3.1) does not hold when $y \equiv 0$, if we take $\phi(x) = \phi_0(x_1 + l, x_2)$, with $l \to \infty$, and $\phi_0 \in C_0^1(\mathbb{R}^2)$ fixed, such that $\phi_0(x_1, x_2) = -\phi_0(x_1, -x_2)$, and $\phi_0 \not\equiv 0$.

Next we recall a useful version of the maximum principle in unbounded domains [7, Lemma 2.1].

**Lemma 3.1.** Let $D$ be a domain (open connected set) in $\mathbb{R}^n$, possibly unbounded. Assume that $\overline{D}$ is disjoint from the closure of an infinite open connected cone $\Sigma$. Suppose there is a function $z$ in $C(\overline{D})$ that is bounded above and satisfies for some continuous function $c(x)$

$$\Delta z - c(x)z \geq 0 \text{ in } D \text{ with } c(x) \geq 0$$

$$z \leq 0 \text{ on } \partial D.$$

Then $z \leq 0$ in $D$.

As a first application of Lemma 3.1 we prove the exponential convergence of $y$ to 0, as $x_1 \to \infty$.

**Lemma 3.2.** $|y(x_1, x_2)| = O(e^{-\frac{3}{2}x_1^2/2})$, as $x_1 \to \infty$ (uniformly in $x_2$).

**Proof.** We define $\psi(x_1, x_2) := Me^{-\frac{3}{2}x_1^2/2}$, in the domain $D := \{(x_1, x_2) : x_1 > 1, x_2 > 0\}$, where $M \geq e^{\frac{3}{2}} \sup_{x_2 > 0} \psi(1, x_2)$ is a constant. It is easy to see that $\Delta \psi \leq x_1 \psi$ in $D$, and $\Delta (y - \psi) \geq 1(y - \psi)$ in $D$. Since $y - \psi \leq 0$ on $\partial D$, it follows from Lemma 3.1 that $y \leq \psi$ in $D$.

**Proof of (v).** We set $(x_1, r) := (-\frac{2}{3}x_1^2, (x_1)^{1/2}r)$, where $x_1 \leq -1$ and $r \in \mathbb{R}$. Equivalently we have $(x_1, r) = (-\frac{3}{2}t_1^2, t_2(-\frac{3}{2}t_1)^{-\frac{1}{2}})$. Next we define $\tilde{y}(t_1, t_2) := \frac{\sqrt{r}}{(-\frac{3}{2}t_1)^{1/2}} y(x_1, r + x_2)$, for every $x_2 \in \mathbb{R}$ fixed, or equivalently

$$(2.2)\quad y(x_1, r + x_2) = \frac{(-x_1)^{1/2}}{\sqrt{2}} \tilde{y}(t_1, t_2).$$

We are going to show that $\tilde{y}(t_1, t_2)$ is uniformly bounded up to the second derivatives, when $t_2$ belongs to a compact interval and $t_1 \to -\infty$. By differentiating (2.2) with respect to $s_1$ and $r$ we obtain

$$(3.3a)\quad \sqrt{2}y_{x_1}(x_1, r + x_2) = (-x_1)\tilde{y}_{t_2}(t_1, t_2),$$

$$(3.3b)\quad \sqrt{2}y_{x_2}(x_1, r + x_2) = (-x_1)^{1/2} \tilde{y}_{t_2}(t_1, t_2),$$

$$(3.3c)\quad \sqrt{2}y_{x_1}(x_1, r + x_2) = \frac{1}{2}(-x_1)^{-1/2} \tilde{y}_{t_2}(t_1, t_2) + (-x_1)\tilde{y}_{t_2}(t_1, t_2) - \frac{r}{2} \tilde{y}_{t_2}(t_1, t_2),$$

$$(3.3d)\quad \sqrt{2}y_{x_2}(x_1, r + x_2) = -\tilde{y}_{t_2} + (-x_1)^{1/2} \tilde{y}_{t_2} - \frac{r}{2}(-x_1)^{-1/2} \tilde{y}_{t_2},$$

Since by construction (cf. (2.11) in Lemma 2.5) $y$ satisfies $|y(x_1, x_2)| = O(|x_1|^{1/2})$ as $x_1 \to -\infty$ (i.e. $\tilde{y}$ is bounded), we obtain by (1.5) and standard elliptic estimates [15, §3.4 p. 37] that

$$(3.4)\quad |\nabla y(x_1, x_2)| = O(|x_1|^{1/2}) \text{ and } |D^2 y(x_1, x_2)| = O(|x_1|^{1/2}), \text{ as } x_1 \to -\infty.$$

From (3.4) and (3.3) it follows that

$$(3.5)\quad |\nabla \tilde{y}(t_1, t_2)| = O(|x_1|^{1/2}) \text{ and } |D^2 \tilde{y}(t_1, t_2)| = O(|x_1|), \text{ as } x_1 \to -\infty,$$

provided that $(t_1, t_2) \in \Sigma_{t_0, t_0} := \{(t_1, t_2) : t_1 \leq t_0, |t_2| \leq r_0(-\frac{3}{2}t_1)^{1/2}\}$, where $t_0 < 0$ and $r_0 > 0$ are arbitrary constants. In particular, we have $\sqrt{2} \Delta \tilde{y}(x_1, x_2) = (-x_1)^{1/2} \Delta \tilde{y}_{t_2}(t_1, t_2) + O(|x_1|^{1/2})$, for $(t_1, t_2) \in \Sigma_{t_0, r_0}$. On the other hand it is clear by (1.5) that $\sqrt{2} \Delta \tilde{y}(x_1, x_2) = (-x_1)^{1/2} (\tilde{y}_{t_2}(t_1, t_2) - \tilde{y}(t_1, t_2))$, thus

$$(3.6)\quad |\Delta \tilde{y}(t_1, t_2)| \text{ and } |\nabla \tilde{y}(t_1, t_2)| \text{ are bounded, } \forall (t_1, t_2) \in \Sigma_{t_0, r_0}.$$

Similarly, by differentiating once more equations (3.3) with respect to $x_1$ and $r$, one can show that

$$(3.7)\quad |D^2 \tilde{y}(t_1, t_2)| \text{ is bounded, } \forall (t_1, t_2) \in \Sigma_{t_0, r_0}.$$
Next, we apply the theorem of Ascoli to the sequence $\tilde{y}(t_1 + l, t_2)$ as $l \to -\infty$. Up to a subsequence $l_n \to -\infty$, we obtain via a diagonal argument, the convergence in $C^1_{loc}(\mathbb{R}^2)$ of $\tilde{y}_n(t_1, t_2) := \tilde{y}(t_1 + l_n, t_2)$ to a bounded function $\tilde{z}(t_1, t_2)$ that we are going to determine. Our claim is that the limit $\tilde{z}$ is a minimal solution of the Allen-Cahn equation (1.7), which is independent of the subsequence $l_n$. The proof of this property is based on the following energy considerations. Let $(e_1, e_2)$ be the canonical basis of $\mathbb{R}^2$. The energy functional

$$E_{P_{tt}}(y, A) = \int_{A-x_2e_2} \left[ \frac{1}{2} |\nabla y(x_1, r + x_2)|^2 + \frac{1}{2} x_1 y^2(x_1, r + x_2) + \frac{1}{2} y^4(x_1, r + x_2) \right] dx_1 dr,$$

associated to (1.5), becomes after changing variables as in (3.2)

$$E_{P_{tt}}(y, A) = \hat{E}_{P_{tt}}(\tilde{y}, \tilde{A}) = \hat{F}(\tilde{y}, \tilde{A}) + \hat{R}(\tilde{y}, \tilde{A}),$$

where

$$\tilde{A} := \left\{ (t_1(x_1), t_2(x_1, r)) : (x_1, r) \in A - x_2e_2 \right\},$$

and

$$\hat{F}(\tilde{y}, \tilde{A}) := \int_{A} \left\{ \left( -\frac{3}{2} \right) \tilde{y}^2 \right\} \hat{I} \left[ \frac{1}{2} \nabla \tilde{y}(t_1, t_2)^2 - \frac{\tilde{y}^2(t_1, t_2)}{2} + \frac{\tilde{y}^4(t_1, t_2)}{4} \right] dt_1 dt_2,$$

and

$$\hat{R}(\tilde{y}, \tilde{A}) := \int_{A} \left[ \frac{(\tilde{y} + t_2 \tilde{y}_r)^2}{16(\frac{-3}{2} t_1)^{\frac{3}{2}}} + \frac{(\tilde{y} + t_2 \tilde{y}_r) \tilde{y}_{tt}}{4(\frac{-3}{2} t_1)^{\frac{3}{2}}} \right] dt_1 dt_2.$$

Let $\hat{\phi}(t_1, t_2) \in C_0^\infty(\mathbb{R}^2)$ be a test function such that $\hat{B} := \text{supp} \hat{\phi} \subset \left\{ (t_1, t_2) : c - d \leq t_1 \leq c \right\}$, for some constants $c \in \mathbb{R}$ and $d > 0$. Given $l \in \mathbb{R}$, we consider the translated functions $\hat{\phi}^{-l}(t_1, t_2) := \hat{\phi}(t_1 - l, t_2)$, and $\tilde{y}^{-l}(t_1, t_2) := \tilde{y}(t_1 + l, t_2)$. Note that $\hat{B}^l := \text{supp} \hat{\phi}^{-l} = \hat{B} + le_1$, and $\text{supp} \hat{\phi}^{-l} \subset \left\{ (t_1, t_2) : t_1 < -1 \right\}$ when $l < 1 - c$. Thus, for $l < 1 - c$, we can define $\phi^{-l} \in C_0^\infty(\mathbb{R}^2)$ by $\phi^{-l}(x_1, r + x_2) = \left(\frac{-x_1^2}{2} \frac{-3}{2} t_1 \right)^{\frac{3}{2}} \hat{\phi}^{-l}(t_1, t_2)$ as in (3.2).

Let $B^l := \left\{ (x_1(t_1), r(t_1, t_2) + x_2) : (t_1, t_2) \in \hat{B}^l \right\}$.

We first examine the case where $x_2 = 0$, and assume that $\hat{\phi}(t_1, t_2) = -\hat{\phi}(t_1, -t_2)$. In view of the minimality of $y$ and (3.9), we have

$$\hat{E}_{P_{tt}}(\tilde{y} + \hat{\phi}^{-l}, \hat{B}^l) = \hat{E}_{P_{tt}}(y + \phi^{-l}, B^l) \geq E_{P_{tt}}(y, B^l) = \hat{E}_{P_{tt}}(\tilde{y}, \hat{B}^l).$$

On the one hand, it is clear that the boundedness of $\tilde{y}$ and (3.6) imply that $\lim_{l \to -\infty} \hat{R}(\tilde{y} + \hat{\phi}^{-l}, \hat{B}^l) = 0$ and $\lim_{l \to -\infty} \hat{E}_{P_{tt}}(\tilde{y}, \hat{B}^l) = 0$. Next, setting $t_0 := c + l$, we have

$$\left( -\frac{3}{2} t_1 \right)^{\frac{3}{2}} \leq \left( -\frac{3}{2} t_0 \right)^{\frac{3}{2}} + d \left( -\frac{3}{2} t_0 \right)^{\frac{3}{2}}, \forall t_1 \in [t_0 - d, t_0].$$

Thus, we obtain

$$\hat{F}(\tilde{y}, \hat{B}^l) = \frac{1}{2} \left( -\frac{3}{2} \right) \tilde{y}^2 \hat{G}(\tilde{y}^l, \hat{B}^l) + O(|t_0|^{-\frac{3}{2}}) = \frac{1}{2} \left( -\frac{3}{2} \right) \tilde{y}^2 \hat{G}(\tilde{y}^l, \hat{B}^l) + O(|t_0|^{-\frac{3}{2}}),$$

and

$$\hat{F}(\tilde{y} + \hat{\phi}^{-l}, \hat{B}^l) = \frac{1}{2} \left( -\frac{3}{2} \right) \tilde{y}^2 \hat{G}(\tilde{y}^l + \hat{\phi}^{-l}, \hat{B}^l) + O(|t_0|^{-\frac{3}{2}}) = \left( -\frac{3}{2} \right) \tilde{y}^2 \hat{G}(\tilde{y}^l + \hat{\phi}, \hat{B}) + O(|t_0|^{-\frac{3}{2}}),$$

where we have set $\hat{G}(\tilde{z}, \hat{B}) := \int_{\hat{B}} \left( \frac{1}{2} |\nabla \tilde{z}|^2 - \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^4}{4} \right) dt$. Finally, since $\tilde{y}^l - (t_1, t_2) \to \tilde{z}(t_1, t_2)$ in $C^1_{loc}(\mathbb{R}^2)$, as $n \to \infty$, we conclude that

$$\hat{G}(\tilde{z} + \hat{\phi}, \hat{B}) = \lim_{n \to -\infty} \left( -\frac{3}{2} \right) \hat{E}_{P_{tt}}(\tilde{y}^l + \hat{\phi}, \hat{B}) \geq \lim_{n \to -\infty} \left( -\frac{3}{2} \right) \hat{E}_{P_{tt}}(\tilde{y}^l + \hat{\phi}, \hat{B}) = \hat{G}(\tilde{z}, \hat{B}),$$

or equivalently $E_{AC}(\tilde{z} + \hat{\phi}, \hat{B}) \geq E_{AC}(\tilde{z}, \hat{B})$. This means that $\tilde{z}$ is a minimal solution of the Allen-Cahn equation (1.7) for odd perturbations $\hat{\phi}$. In addition, since $\tilde{y}(t_1, 0) = 0$, we also have that $\tilde{z}(t_1, 0) = 0$. Thus, from [8, Theorem 1.5], it follows that $\tilde{z}$ is a function of only $t_2$. Finally, from the minimality of $\tilde{z}$ and the fact that $t_2 \geq 0 \Rightarrow \tilde{z} \geq 0$, it is immediate that $\tilde{z}(t_1, t_2) = \eta(t_2) = \text{tanh}(t_2/\sqrt{2})$ is the heteroclinic connection. The uniqueness of $\tilde{z}$ also implies that the convergence $\tilde{y}^l(t_1, t_2) \to \tilde{z}(t_1, t_2)$ holds as $l \to -\infty$. 

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It remains to examine the case where $x_2 \neq 0$. Without loss of generality we assume that $x_2 > 0$. Now (3.13) holds for arbitrary test functions $\phi(t_1,t_2) \in C^\infty_c(\mathbb{R}^2)$, since $B^l \subset \{(x_1,x_2) : x_2 > 0\}$ as $l \to -\infty$. Repeating the previous arguments we find that $\tilde{z}$ is a nonnegative minimal solution of (1.7). Applying [5, Corollary 5.2], we deduce that $\tilde{z} \equiv 1$. This completes the proof of (v).

**Proof of (vi) and (vii).** The proofs of (vi) and (vii) which are based on the moving plane method, follow from the next lemmas.

**Lemma 3.3.** We have $y_{x_1}(x_1,x_2) < 0$, $\forall x_1 \geq 0$, $\forall x_2 > 0$. In addition, for every $d > 0$, there holds $\sup_{x_2 \geq d} y_{x_1}(1,x_2) < 0$, and $\inf_{x_2 \geq d} \psi(1,x_2) > 0$.

**Proof.** Given $\lambda \geq 0$, we define the function $\psi(x_1,x_2) := y(x_1,x_2) - y(-x_1 + 2\lambda, x_2)$ for $(x_1,x_2) \in D_\lambda := \{(x_1,x_2) : x_1 > \lambda, x_2 > 0\}$. One can check that $\psi_\lambda = 0$ on $\partial D_\lambda$, and

$$\Delta \psi - c(x_1,x_2)\psi_\lambda = 2(x_1 - \lambda)\psi(-x_1 + 2\lambda, x_2) \geq 0,$$

with $c(x_1,x_2) = x_1 + 2(y^2(x_1,x_2) + y(x_1,x_2)(-x_1 + 2\lambda, x_2) + y^2(-x_1 + 2\lambda, x_2)) \geq 0$. Furthermore, $\psi_\lambda$ is bounded above by Theorem 1.1 (ii), and not identically zero by Theorem 1.1 (v). As a consequence of Lemma 3.1, it follows that $\psi_\lambda(x_1,x_2) < 0$, $\forall x_1 > \lambda$, $\forall x_2 > 0$, and thus by Hopf’s Lemma $\frac{\partial \psi_\lambda}{\partial n}(\lambda,x_2) = 2y_{x_1}(\lambda,x_2) < 0$, $\forall x_2 > 0$. To establish that $\sup_{x_2 \geq d} y_{x_1}(1,x_2) < 0$, we proceed by contradiction and assume the existence of a sequence $\{\tilde{x}_n\}$ such that $\lim_{n \to \infty} \tilde{x}_n = \infty$ and $\lim_{n \to \infty} y_{x_1}(1,\tilde{x}_n) = 0$. Next, we set $\tilde{y}_n(x_1,x_2) = y(x_1,x_2 + \tilde{x}_n)$. In view of the bounds provided in Theorem 1.1 (ii), we obtain by the theorem of Ascoli that (up to subsequence) $\tilde{y}_n$ converges in $C^1_{\text{loc}}$ to a nonnegative minimal solution $\tilde{y}$ of (1.5). Since $\tilde{y}_{x_1}(1,0) = \lim_{n \to \infty} y_{x_1}(1,\tilde{x}_n) = 0$, and $\tilde{y}_{x_1}(x_1,x_2) \leq 0$, $\forall x_1 \geq 0$, $\forall x_2 \in \mathbb{R}$, the maximum principle applied to (3.18)

$$\Delta \tilde{y}_{x_1} = \tilde{y} + (x_1 + 6\tilde{y}^2)\tilde{y}_{x_1} \geq (x_1 + 6\tilde{y}^2)\tilde{y}_{x_1},$$

implies that $\tilde{y}_{x_1}(x_1,x_2) = 0$, $\forall x_1 \geq 0$, $\forall x_2 \in \mathbb{R}$. Then, since $\lim_{x_1 \to \infty} \tilde{y}(x_1,x_2) = 0$, $\forall x_2 \in \mathbb{R}$, it follows that $\tilde{y} \equiv 0$ in the half-plane $x_1 \geq 0$. Finally, we deduce by unique continuation that $\tilde{y} \equiv 0$ in $\mathbb{R}^2$, which is a contradiction since $\tilde{y}$ is minimal. Thus we have established that $\sup_{x_2 \geq d} y_{x_1}(1,x_2) < 0$. The proof that $\inf_{x_2 \geq d} \psi(1,x_2) > 0$ is similar.

**Lemma 3.4.** For every vector $n = e^{i(\theta + \frac{\pi}{7})} \in \mathbb{C} \sim \mathbb{R}^2$, with $\theta \in (0, \frac{\pi}{7})$, there exists $s_n > 0$ such that $\nabla y(x_1,x_2) \cdot n > 0$, $\forall x_1 > s_n$, $\forall x_2 > 0$.

**Proof.** Our first claim is that there is a constant $k_1 > 0$, such that $k_1 y_{x_1}(x_1,x_2) \leq -\sqrt{x_1} y(x_1,x_2)$, $\forall x_1 \geq 1$, $\forall x_2 \geq 0$. Indeed, let $\psi(x_1,x_2) = k_1 y_{x_1}(x_1,x_2) + \sqrt{x_1} y(x_1,x_2)$ for $(x_1,x_2) \in D := \{x_1 > 1, x_2 > 0\}$, where the constant $k_1$ will be adjusted later. It is clear that $\psi(x_1,0) = 0$, $\forall x_1 \geq 1$. We also note that $y_{x_1}(x_1,0) < 0$, since the function $y_{x_1}$ vanishes at $(1,0)$, is negative in $\{x_1 > 0, x_2 > 0\}$, and satisfies (3.18). This and $\sup_{x_2 \geq d} y_{x_1}(x_1,2) < 0$, $\forall d > 0$, imply that when $k_1$ is large enough, we have $\psi(x_1,2) \leq 0$, $\forall x_2 \geq 0$. Next, we compute

$$\Delta \psi = \left( x_1 + 6y^2 + \frac{1}{k_1 \sqrt{x_1}} \right) k_1 y_{x_1} + \left( x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2} \right) \sqrt{x_1} y = \left( x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2} \right) \psi + \left( 4y^2 + \frac{k_1}{k_1 \sqrt{x_1}} - \frac{1}{4x_1^2} \right) k_1 y_{x_1}.$$

By choosing $k_1$ large enough we can ensure that $(x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2}) \geq 0$ and $(4y^2 + \frac{k_1}{k_1 \sqrt{x_1}} - \frac{k_1}{4x_1^2} + \frac{1}{4x_1^2}) \leq 0$, when $x_1 \geq 1$ and $x_2 \geq 0$. Thus, by applying Lemma 3.1, our claim follows.

Similarly, we are going to establish that there is a constant $k_2 > 0$, such that $y_{x_2}(x_1,x_2) \geq -k_2 y(x_1,x_2)$, $\forall x_1 \geq 1$, $\forall x_2 \geq 0$. To do this we let $\psi(x_1,x_2) = -k_2 y(x_1,x_2)$ for $(x_1,x_2) \in D$, where the constant $k_2$ will again be adjusted later. We first note that $y_{x_2}(x_1,0) > 0$, $\forall x_1 \in \mathbb{R}$, since the function $y$ vanishes at $(x_1,0)$, is positive in $\{x_2 > 0\}$, and satisfies (1.5). This and $\inf_{x_2 \geq d} y(x_1,2) > 0$, $\forall d > 0$, imply that when $k_2$ is large enough, we have $\psi(x_1,2) \leq 0$, $\forall x_2 \geq 0$. On the other hand, it is clear that $\psi(x_1,0) < 0$, $\forall x_1 \geq 1$. Next, we compute

$$\Delta \psi = (x_1 + 6y^2)(-y_{x_2}) + (x_1 + 2y^2)(-k_2 y) \geq (x_1 + 6y^2) \psi.$$
Finally, setting $\frac{\nabla y}{|\nabla y|} = e^{i\phi}$ when $(x_1, x_2) \in D$ (with $\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})$), we find that

$$\tan \phi = \frac{y_{x_2}}{y_{x_1}} \leq \frac{k_1 k_2}{\sqrt{x_1}} \implies \phi \leq \pi - \arctan \left( \frac{k_1 k_2}{\sqrt{x_1}} \right).$$

As a consequence, we have $\nabla y(x_1, x_2) \cdot n > 0$ if $\theta \in (\arctan \left( \frac{k_1 k_2}{\sqrt{x_1}} \right), \frac{\pi}{2})$, that is, if $x_1 > s_n := \left( \frac{k_1 k_2}{\tan \frac{\pi}{2}} \right)^2$.

\[\square\]

**Lemma 3.5.** Let $\theta \in (0, \frac{\pi}{2})$ be fixed, and consider for every $\lambda \in \mathbb{R}$ the reflection $\sigma_\lambda$ with respect to the line $\Gamma_\lambda := \{(x_1, x_2) : x_2 = \tan \theta (x_1 - \lambda)\}$, and the domain $D_\lambda := \{(x_1, x_2) : 0 < x_2 < \tan \theta (x_1 - \lambda)\}$. Then, the function $\psi_\lambda(x_1, x_2) := y(x_1, x_2) - y(\sigma_\lambda(x_1, x_2))$ is negative in $D_\lambda$, for every $\lambda \in \mathbb{R}$.

**Proof.** We set $n = e^{i(\theta + \frac{\pi}{2})}$ as in Lemma 3.4, and denote by $(p', q')$ the image by $\sigma_\lambda$ of a point $(p, q) \in D_\lambda$, and by $D'_\lambda$ the set $\sigma_\lambda(D_\lambda)$. It is obvious that $\psi_\lambda(x_1, 0) < 0$, for every $\lambda$, and that $\psi_\lambda(x_1, x_2) = 0$, for $(x_1, x_2) \in \Gamma_\lambda$. Moreover, $\psi_\lambda$ satisfies

$$\Delta \psi_\lambda(p, q) - c(p, q) \psi_\lambda = (p - p') y(p', q') \geq 0, \quad \forall (p, q) \in D_\lambda,$$

with $c(p, q) = p + 2(y^2(p, q) + y(p, q)y(p', q') + y^2(p', q'))$. For each $\lambda \in \mathbb{R}$ we consider the statement

$$\psi_\lambda(p, q) < 0, \quad \forall (p, q) \in D_\lambda.$$  

We shall first establish Lemma 3.5 in the case where $\theta \in (0, \frac{\pi}{2})$. According to Lemma 3.4, (3.19) is valid for each $\lambda \geq s_n$. Set $\lambda_0 = \inf\{\lambda \in \mathbb{R} : \psi_\lambda < 0 \text{ holds in } D_{\lambda_0}, \text{ for every } \mu \geq \lambda\}$. We will prove $\lambda_0 = -\infty$. Assume instead $\lambda_0 \in \mathbb{R}$. Then, there exist a sequence $\lambda_n \downarrow \lambda_0$ such that $\lim_{\lambda \to \infty} \lambda_n = \lambda_0$, and a sequence $(p_n, q_n) \in D_{\lambda_n}$, such that $y(p_n, q_n) \geq y(p'_n, q'_n)$. According to Lemma 3.4, we have $p'_n \leq s_n$, thus the sequence $(p_n, q_n)$ is bounded. Up to subsequences we may assume that $\lim_{\lambda \to \infty} (p_n, q_n) = (p_0, q_0) \in D_{\lambda_0}$, with $p'_0 \leq s_n$. By definition of $\lambda_0$, we have $\psi_{\lambda_0} \leq 0$ in $D_{\lambda_0}$, and $\psi_{\lambda_0}(p_0, q_0) = 0$ i.e. $y(p_0, q_0) = y(p'_0, q'_0)$. Now we distinguish the following cases. If $(p_0, q_0) \in D_{\lambda_0}$, the maximum principle implies that $\psi_{\lambda_0} \equiv 0$ in $D_{\lambda_0}$. Clearly, this situation is excluded, since $y$ is positive in the half-plane $\{x_2 > 0\}$. On the other hand, the maximum principle also implies that $\frac{\partial \psi_{\lambda_0}}{\partial n}(p, q) = 2 \frac{\partial \psi_{\lambda_0}}{\partial y}(p, q) > 0$, provided that $(p, q) \in \Gamma_0$ and $q > 0$. Furthermore, the previous inequality still holds at the vertex $(p, q) = (\lambda_0, 0)$, since $y_{x_2}(x_1, 0) > 0$ and $y_{x_1}(x_1, 0) = 0$, for every $x_1 \in \mathbb{R}$ (cf. the proof of Lemma 3.4). As a consequence, in a neighborhood of the line segment $\{(x_1, x_2) : x_2 = \tan \theta (x_1 - \lambda), 0 \leq x_1 \leq s_n\}$, we have that $\frac{\partial \psi_{\lambda_0}}{\partial n} > 0$, and it follows that $(p_0, q_0)$ cannot belong to $\Gamma_0$. Finally, since the case where $p_0 > \lambda_0$ and $q_0 = 0$ is ruled out (because $y$ is positive in the half-plane $\{x_2 > 0\}$), we have reached a contradiction.

Next, we establish Lemma 3.5 in the case where $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, which is a little bit more involved. When $\theta = \frac{\pi}{2}$, it is clear that (3.19) is valid for each $\lambda \geq s_n$. Otherwise, when $\theta = \frac{\pi}{4}$, let $A'_{\lambda_0} := \{(p', q') \in D'_{\lambda_0} : p' \leq s_n\}$, and let $A_\lambda = \sigma_\lambda(A'_{\lambda_0})$. Our first claim is that $m := \inf A'_{\lambda_0} > 0$. Indeed, proceeding as in the proof of Theorem 1.1 (v), one can see that $$(x_1, x_2) \in A'_{\lambda_0}, x_1 \to -\infty \frac{\sqrt{2}}{\sqrt{-x_1}} y(x_1, x_2) = 1.$$ In addition, proceeding as in the proof of Lemma 3.4, we obtain that $\inf y(x_1, x_2) : (x_1, x_2) \in A'_{\lambda_0}, s_n - l \leq x_1 \leq s_n, (x_1, x_2) \in A_\lambda = 0$, since $\lim_{\lambda_0 \to \infty} \inf \{x_1 \in A_{\lambda_0} : (x_1, x_2) \in A_{\lambda_0} \} = 0$ (cf. Lemma 3.2). As a consequence when $\lambda \geq s_n + 1$ is large enough, we have $y(p', q') \geq m > y(p, q), \forall (p, q) \in A_\lambda$, and also $y(p', q') > y(p, q), \forall (p, q) \in D_{\lambda_0} \setminus A_\lambda$, by definition of $s_n$. This establishes that (3.19) holds for $\lambda$ large enough. Then, defining $\lambda_0$ as previously, we assume by contradiction that $\lambda_0 \in \mathbb{R}$, and deduce in a similar way the existence of the sequences $\lambda_n$ and $(p_n, q_n) \in D_{\lambda_n}$.

We need to show that $(p_n, q_n)$ is bounded. For $\nu > \lambda$, let $M_\nu := (\nu, \tan(\frac{\nu - \lambda}{2})) \in \Gamma_\lambda$, and let $M_\nu := \{(x_1, x_2) : x_2 = \tan(\theta + \frac{\pi}{2})(x_1 - \nu) + \tan(\nu - \lambda)\}$ be the line parallel to $n$ and passing through $M_\nu$. Let also $B'_{\nu} := \{(p', q') \in A'_{\lambda_0} : q' \geq \tan(\theta + \frac{\pi}{2})(p' - \nu) + \tan(\nu - \lambda)\}$. Proceeding as previously, we can see that $\forall \nu > \lambda_0 + 2, \forall \nu > \lambda_0 - 1$, we have $\inf_{B'_{\nu}} y > m$ for some constant $m > 0$, while $\lim_{\lambda_0 \to \infty} \{x_1 \in A_{\lambda_0} : (x_1, x_2) \in B_{\lambda_0} \} = 0$. As a consequence, for $\nu$ large enough and $\lambda > \lambda_0 - 1$, we have $y(p', q') \geq m > y(p, q), \forall (p, q) \in B_{\lambda_0}$, and thus
To complete the proof we utilize the same arguments detailed in the case where \( \theta \)

\[ x_n \]

where \( n \) and \( y \) (resp. Theorem 1.1. Moreover, in the half-plane \( x_2 > 0 \), \( y_{x_1} \) and \( y_{x_2} \) satisfy respectively \( \Delta y_{x_1} \geq (x_1 + 6y^2)y_{x_1} \) and \( \Delta y_{x_2} = (x_1 + 6y^2)y_{x_2} \), thus \( y_{x_1} \) (resp. \( y_{x_2} \)) cannot vanish in the open half-plane \( x_2 > 0 \), since otherwise we would obtain by the maximum principle \( y_{x_1} \equiv 0 \) (resp. \( y_{x_2} \equiv 0 \)). These situations are excluded by the fact that \( y > 0 \) in the open half-plane \( x_2 > 0 \), and \( y_{x_2}(x_1,0) > 0 \), \( \forall x_1 \in \mathbb{R} \). Therefore we have proved that \( y_{x_1}(x_1,x_2) < 0 \), \( \forall x_1 \in \mathbb{R} \), \( \forall x_2 > 0 \), and \( y_{x_2}(x_1,x_2) > 0 \), \( \forall x_1,x_2 \in \mathbb{R} \). Finally, setting \( \bar{y}_i(x_1,x_2) = y(x_1, x_2 + l) \), we obtain by the Theorem of Ascoli, that up to a subsequence \( l \to \infty \), \( \bar{y}_i \) converges in \( C^2 \) to a nonnegative minimal solution \( \bar{y}_\infty \) of (1.5). Furthermore, the monotonicity of \( y \) along the \( x_2 \) direction implies that \( \bar{y}_\infty \) is independent of \( x_2 \). Thus, since \( h \) is the only nonnegative minimal solution of (1.5) (cf. [10, Theorem 1.3]), we deduce that \( \bar{y}_\infty(x_1,x_2) = h(x_1) \), and that \( \lim_{l \to \infty} y(x_1, x_2 + l) = h(x_1) \) is independent of the subsequence \( l_k \). We also note that \( \|y(x_1,x_2)\| < h(x_1) \), \( \forall (x_1,x_2) \in \mathbb{R}^2 \), from which Theorem 1.1 (iv) follows. This completes the proof of Theorem 1.1.

Lemma 3.5 implies that \( \forall \theta \in (0, \frac{\pi}{2}) \), \( \forall \lambda \in \mathbb{R} \), and \( (p,q) \in \Gamma \lambda \) with \( q > 0 \), we have \( \frac{\partial y}{\partial n}(p,q) = 2 \frac{\partial y}{\partial n}(p,q) > 0 \), where \( n = e^{i(\theta + \frac{\pi}{2})} \). It follows that \( y_{x_1}(x_1,x_2) \leq 0 \), and \( y_{x_2}(x_1,x_2) \geq 0 \), \( \forall x_1 \in \mathbb{R} \), \( \forall x_2 \geq 0 \). Moreover, the monotonicity of \( \Gamma \lambda \), \( \Delta_{\lambda,s} \), and \( \Delta_{\lambda,v} \), where \( \theta > s_n \) and \( \lambda < s_n \).

\[ \begin{align*}
&\text{Figure 1. The sets } A_\lambda, A'_\lambda, B_{\lambda,v}, B'_{\lambda,v}, \text{ and the lines } \Gamma_\lambda, \Delta_v, \text{ in the case where } \lambda > s_n \text{ and } \\
&\lambda < s_n. 
\end{align*} \]

\((p_k, q_k) \notin B_{\lambda,v} \). Furthermore, since \( p'_k \leq s_n \) by Lemma 3.4, we have established the boundedness of \((p_k, q_k)\).

To complete the proof we utilize the same arguments detailed in the case where \( \theta \in (0, \frac{\pi}{4}) \).

\( \square \)

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