CONTACT HANDLES, DUALITY, AND SUTURED FLOER HOMOLOGY

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ABSTRACT. We give an explicit construction of the Honda–Kazez–Matić gluing maps in terms of contact handles. We use this to prove a duality result for turning a sutured manifold cobordism around, and to compute the trace in the sutured Floer TQFT. We also show that the decorated link cobordism maps on the hat version of link Floer homology defined by the first author via sutured manifold cobordisms and by the second author via elementary cobordisms agree.

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1. INTRODUCTION

The purpose of this paper is to provide an explicit construction of the Honda–Kazez–Matić gluing map [HKM08] in terms of contact handles, and use this to prove several results about the sutured Floer TQFT defined by the first author [Juh16]. We finally show that the decorated link cobordism maps on the hat version of link Floer homology defined via sutured manifold cobordisms by the first author [Juh16], and the maps defined using elementary link cobordisms by the second author [Zem16] agree.

1.1. The contact gluing map. Sutured manifolds were introduced by Gabai [Gab83] to construct taut foliations on 3-manifolds, and are also ubiquitous in contact topology. In this paper, a sutured manifold is a pair \((M, \gamma)\), where \(M\) is a compact oriented 3-manifold with boundary, and the set of sutures \(\gamma \subseteq \partial M\) is an oriented 1-manifold that divides \(\partial M\) into subsurfaces \(R_+ (\gamma)\) and \(R_- (\gamma)\) that meet along \(\gamma\). For example, if \(M\) carries a contact structure such that \(\partial M\) is convex with dividing set \(\gamma\), then \((M, \gamma)\) is a sutured manifold.

We say that \((M, \gamma)\) is balanced if \(M\) has no closed components, each component of \(M\) contains a suture, and \(\chi (R_+ (\gamma)) = \chi (R_- (\gamma))\). Sutured Floer homology, defined by the first author [Juh06], assigns an \(F_2\)-vector space \(SFH (M, \gamma)\) to a balanced sutured manifold \((M, \gamma)\). It is a common extension of the hat version of Heegaard Floer homology of closed 3-manifolds and link Floer homology, both due to Ozsváth and Szabó [OS04c, OS04b, OS08], to 3-manifolds with boundary.

Let \((M, \gamma)\) and \((M', \gamma')\) be sutured manifolds such that \(M \subseteq \text{int} (M')\). Given a contact structure \(\xi\) on \(M' \cap \text{int} (M)\) such that \(\partial M \cup \partial M'\) is convex with dividing set \(\gamma \cup \gamma'\), Honda–Kazez–Matić [HKM08] define a gluing map

\[
\Phi_\xi : SFH (-M, -\gamma) \to SFH (-M', -\gamma')
\]

using partial open book decompositions that satisfy a “contact compatibility” condition near the boundary. However, the contact compatibility condition makes working with and computing the gluing map impractical. In the first part of this paper, we give a new definition of the contact gluing map based on contact handle attachments, prove invariance via contact cell decompositions, and show that our map agrees with the Honda–Kazez–Matić gluing map. In particular, this allows us to give a simple diagrammatic description of the gluing map for a single contact handle attachment; for a precise statement about the gluing map associated to a contact handle attachment, see Proposition 5.6. Contact handles were introduced by Giroux [Gir91]; see Definition 5.5.

We now describe the map \(C_{h_i}\) that we assign to attaching a contact \(i\)-handle \(h_i\) for \(i \in \{0, 1, 2, 3\}\). Let \((\Sigma, \alpha, \beta)\) be a diagram of \((M, \gamma)\); then \((\Sigma, \alpha, \beta)\) is a diagram of \((-M, -\gamma)\). Attaching a contact 0-handle corresponds to taking the disjoint union of \(\Sigma\) with a disk. A contact 1-handle corresponds to attaching a 1-handle to \(\partial \Sigma\); see Figure 1.1. Adding a disk or a 1-handle to \(\partial \Sigma\) does not change the sutured Floer complex, and we define \(C_{h_i}\) to be the tautological map on intersection points. A contact 2-handle is attached to \(\partial M\) along a curve \(l\) that intersects \(\gamma\) in two points. Let \(\lambda_\pm \subseteq \Sigma\) be the properly embedded arc corresponding to \(l \cap R_\pm (\gamma)\). As in Figure 1.1, we glue a 1-handle \(H\) to \(\Sigma\) along \(\partial \Sigma\), and add a curve \(\alpha\) to \(\alpha\) and a curve \(\beta\) to \(\beta\) that intersect in \(H\) in a single point \(c\), and such
that \( \alpha \cap \Sigma = \lambda_- \) and \( \beta \cap \Sigma = \lambda_+ \). Then, given a generator \( x \in T_\alpha \cap T_\beta \), we let \( C_{h^2}(x) = x \times \{e\} \). Finally, suppose that we attach a contact 3-handle \( h^3 \) along an \( S^2 \) component \( S \) of \( \partial M \) containing the suture \( \gamma_S = \gamma \cap S \), giving rise to the sutured manifold \((M', \gamma')\). We then choose a diagram where \( \gamma_S \) is encircled by a curve \( \alpha \in \alpha \) and a curve \( \beta \in \beta \) such that \( \alpha \cap \beta = \{x, y\} \), and such that there are no other \( \alpha \) or \( \beta \) curves between \( \alpha \) and \( \gamma_S \) or \( \beta \) and \( \gamma_S \). Let \( \Sigma \) be the result of gluing a disk to \( \Sigma \) along \( \gamma_S \). Then \((\Sigma', \alpha \setminus \{\alpha\}, \beta \setminus \{\beta\})\) is a diagram of \((M', \gamma')\); see Figure 1.1. We let \( C_{h^3}(x \times \{x\}) = 0 \) and \( C_{h^3}(x \times \{y\}) = x \), where \( x \) has the larger Maslov grading in \((\bar{\Sigma}, \alpha, \beta)\).

Note that Zarev [Zar10] has also defined a type of gluing map in sutured Floer homology, corresponding to a convex decomposition. Combining this with the \( EH \) invariant [HKM09], one can define a map for gluing a contact structure to a sutured manifold. Zarev conjectured that this map agrees with the Honda–Kazez–Matić gluing map, though we will not address Zarev’s construction in this paper.

1.2. The sutured Floer TQFT. The second author [Juh16] defined the category of balanced sutured manifolds and sutured manifold cobordisms, and extended \( SFH \) to a functor on this category. A sutured manifold cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) is a triple \( \mathcal{W} = (W, Z, [\xi]) \), where \( W \) is a 4-manifold with boundary and corners, \( Z \) is a codimension-0 compact submanifold of \( \partial W \) such that \( \partial W \setminus \text{int}(Z) = -M_0 \cup M_1 \), and \([\xi]\) is a certain equivalence class of a contact structure on \( Z \) such that \( \partial Z \) is convex with dividing set \( \gamma_0 \cup \gamma_1 \). The sutured cobordism map

\[ F_{\mathcal{W}}: SFH(M_0, \gamma_0) \rightarrow SFH(M_1, \gamma_1) \]

is the composition of the contact gluing map for \( -\xi \) and 4-dimensional handle maps.
Let $\xi_{I \times \partial M}$ denote the $I$-invariant contact structure on $-I \times \partial M$ that induces the dividing set $\gamma$ on $\partial M$. Consider the trace cobordism

$$\Lambda_{(M, \gamma)} = (I \times M, \xi_{I \times \partial M})$$

from $(M, \gamma) \sqcup (-M, \gamma)$ to $\emptyset$, and the cotrace cobordism

$$V_{(M, \gamma)} = (I \times M, \xi_{I \times \partial M})$$

from $\emptyset \to (M, \gamma) \sqcup (-M, \gamma)$. In Theorem 8.1, we positively answer [Juh16, Conjecture 11.13]:

**Theorem 1.1.** The trace cobordism $\Lambda_{(M, \gamma)}$ induces the canonical trace map

$$\text{tr}: SFH(M, \gamma) \otimes SFH(-M, \gamma) \to \mathbb{F}_2,$$

obtained by evaluating cohomology on homology. The cotrace cobordism $V_{(M, \gamma)}$ induces the canonical cotrace map

$$\text{cotr}: \mathbb{F}_2 \to SFH(M, \gamma) \otimes SFH(-M, \gamma).$$

The proof relies on the following deep technical result, which is Theorem 7.1. Before stating it, recall that, given a sutured triple diagram $(\Sigma, \alpha, \beta, \gamma)$, we can associate to it a sutured manifold cobordism $W_{\alpha, \beta, \gamma}$ from $(M, \beta, \gamma, \alpha, \beta) \sqcup (M, \beta, \gamma, \alpha, \beta)$ to $(M, \gamma, \alpha, \gamma, \beta)$.}

**Theorem 1.2.** Let $T = (\Sigma, \alpha, \beta, \gamma)$ be an admissible balanced sutured triple diagram. Then the cobordism map

$$F_{W_{\alpha, \beta, \gamma}}: CF(\Sigma, \alpha, \beta) \otimes CF(\Sigma, \beta, \gamma) \to CF(\Sigma, \alpha, \gamma)$$

is chain homotopic to the map $F_{\alpha, \beta, \gamma}$ defined in [Juh16, Definition 5.13] that counts holomorphic triangles on the triple diagram $T$.

One can obtain from Theorem 1.1 a positive answer to [Juh16, Question 11.9]:

**Theorem 1.3.** If $W: (M, \gamma) \to (M', \gamma')$ is a balanced cobordism of sutured manifolds, and $W^\lor$ is the cobordism obtained by turning around $W$, then

$$F_{W^\lor} = (F_W)^\lor,$$

with respect to the trace pairing.

We also give a self-contained proof of this result, without invoking Theorem 8.1. As a special case, we obtain that the decorated link cobordism maps $F_{X}^Z$ of the first author satisfy an analogous duality property when we turn a decorated link cobordism $X$ around; see [JM17, Section 5.7]. Indeed, these maps are defined by assigning a sutured manifold cobordism to a decorated link cobordism, and applying the $SFH$ functor.

The second author [Zem16] later gave a different construction of link Floer cobordism maps $F_{X}^Z$ by composing maps defined for elementary link cobordisms, and showing independence of the decomposition. Note that this construction makes sense for all versions of link Floer homology, not just the hat version. In the last section, we prove that the two maps agree:

**Theorem 1.4.** Given a decorated link cobordism $X$, we have $F_{X}^I = \widehat{F}_{X}^Z$. A key technical lemma that we use throughout the paper gives a simple formula for the naturality map for a compound stabilization operation on a sutured diagram (called a $(k,0)$- or $(0,l)$-stabilization in [JT12]), which consists of a simple stabilization, followed by handlesliding some $\alpha$-curves over the new $\alpha$-curve, or some $\beta$-curves over the new $\beta$-curve; see Proposition 2.2.
1.3. Notation and conventions. Throughout this paper, if $A$ and $B$ are smooth manifolds, then we write $A \cong B$ if $A$ and $B$ are diffeomorphic. If $A$ and $B$ are submanifolds of $C$, then $A \approx B$ denotes that $A$ and $B$ are $C^0$ close. Given a submanifold $A$ of $C$, we write $N(A)$ for a regular neighborhood of $A$ in $C$. We denote Heegaard diagrams by $\mathcal{H}$, and handle decompositions by $\mathcal{H}$. If $M$ is an oriented $n$-manifold, then we will denote the same manifold with its orientation reversed by $\overline{M}$ when $n$ is even, and by $-M$ when $n$ is odd. The closure of a set $X$ is $cl(X)$. If $\xi$ is a co-oriented 2-plane field on $M$, we will write $-\xi$ for the co-orientated 2-plane field obtained by reversing the co-orientation of $\xi$.

We orient the boundary of a manifold using the “outward normal first” convention. To be consistent with this convention, all our cobordisms go from left to right.

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2. 1-handle and 3-handle maps and triangle maps: compound stabilizations

In this section, we describe several results about the interactions between holomorphic triangles and 1-handle, 3-handle, and stabilization maps. These will be used in later sections.

2.1. 1-handle and 3-handle maps. Let $(\Sigma, \alpha, \beta)$ be an admissible sutured triple diagram, and let $p_1, p_2 \in \Sigma$ be a pair of points that are both in components of $\Sigma \setminus (\alpha \cup \beta)$ that intersect $\partial \Sigma$. We construct the admissible sutured diagram $(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$ by removing disks centered at $p_1$ and $p_2$, and adding an annulus $A$ connecting the boundaries of the disks. Furthermore, $\alpha_0$ and $\beta_0$ are homologically nontrivial curves in $A$ that intersect transversely at two points $\theta^+_{\alpha_0, \beta_0}$ and $\theta^-_{\alpha_0, \beta_0}$, such that $\theta^+_{\alpha_0, \beta_0}$ has the larger relative Maslov grading. The first author [Juh16, Section 7] defined the 1-handle map

$$F_1^{\alpha_0, \beta_0}: CF(\Sigma, \alpha, \beta) \to CF(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}),$$

and the 3-handle map

$$F_3^{\alpha_0, \beta_0}: CF(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}) \to CF(\Sigma, \alpha, \beta),$$

$$x \mapsto x \times \theta^+_{\alpha_0, \beta_0} \quad \text{and} \quad x \times \theta^-_{\alpha_0, \beta_0} \mapsto x.$$

If $T = (\Sigma, \alpha, \beta, \gamma)$ is an admissible sutured triple, then it induces a holomorphic triangle map

$$F_T: CF(\Sigma, \alpha, \beta) \otimes CF(\Sigma, \beta, \gamma) \to CF(\Sigma, \alpha, \gamma).$$

If $p_1, p_2 \in \Sigma \setminus (\alpha \cup \beta \cup \gamma)$ are distinct points such that they are in components of $\Sigma \setminus (\alpha \cup \beta \cup \gamma)$ that intersect $\partial \Sigma$, then we can similarly form the admissible Heegaard triple

$$T' := (\Sigma', \alpha' = \alpha \cup \{\alpha_0\}, \beta' = \beta \cup \{\beta_0\}, \gamma' = \gamma \cup \{\gamma_0\}),$$

where $\Sigma'$ is obtained by adding a 1-handle $A$ with feet at $p_1$ and $p_2$, and three new curves, $\alpha_0$, $\beta_0$, and $\gamma_0$, that are homologically nontrivial in $A$ and pairwise intersect in two points; see Figure 2.1. If $x$, $y$ are sets of attaching curves on some Heegaard surface $S$, then we denote the diagram $(S, x, y)$ by $H_{x,y}$. 


Proof. Consider the first diagram. The assumption that the points θ⁺₀β₀, θ⁺₀γ₀, and θ⁺₀α₀ are marked by solid circles, and θ⁻₀α₀, θ⁻₀β₀, and θ⁻₀γ₀ by empty circles. On the left, the only index 0 triangle connecting θ⁺₀α₀, θ⁺₀β₀, θ⁺₀γ₀, and θ⁻₀α₀ is shaded. On the right, the only index 0 triangle connecting θ⁺₀α₀, θ⁻₀β₀, θ⁻₀γ₀, and θ⁻₀α₀ is shaded.

**Proposition 2.1.** With the above notation, the following diagrams are commutative:

![Diagram with commuting arrows](image)

\[
\begin{array}{c}
\text{CF}(\mathcal{H}_\alpha, \beta) \otimes \text{CF}(\mathcal{H}_\beta, \gamma) \\
\downarrow F'_1 \otimes F''_1 \otimes F''_3 \otimes F''_3 \\
\text{CF}(\mathcal{H}_\alpha, \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\end{array}
\]

\[
\begin{array}{c}
\text{CF}(\mathcal{H}_\alpha, \beta) \otimes \text{CF}(\mathcal{H}_\beta, \gamma) \\
\downarrow F'_1 \otimes F''_1 \otimes F''_3 \otimes F''_3 \\
\text{CF}(\mathcal{H}_\alpha, \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\end{array}
\]

\[
\begin{array}{c}
\text{CF}(\mathcal{H}_\alpha', \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\downarrow F'_1 \otimes F''_1 \otimes F''_3 \otimes F''_3 \\
\text{CF}(\mathcal{H}_\alpha', \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\end{array}
\]

\[
\begin{array}{c}
\text{CF}(\mathcal{H}_\alpha, \beta) \otimes \text{CF}(\mathcal{H}_\beta, \gamma) \\
\downarrow F'_1 \otimes F''_1 \otimes F''_3 \otimes F''_3 \\
\text{CF}(\mathcal{H}_\alpha, \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\end{array}
\]

\[
\begin{array}{c}
\text{CF}(\mathcal{H}_\alpha', \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\downarrow F'_1 \otimes F''_1 \otimes F''_3 \otimes F''_3 \\
\text{CF}(\mathcal{H}_\alpha', \beta') \otimes \text{CF}(\mathcal{H}_\beta', \gamma') \\
\end{array}
\]

**Proof.** Consider the first diagram. The assumption that the points p₁ and p₂ are in components of \(\Sigma \setminus (\alpha \cup \beta \cup \gamma)\) that intersect \(\partial \Sigma\) allows one to reduce the claim to the model computation

(1) \[ F_{\alpha_0, \beta_0, \gamma_0}(\theta^+_{\alpha_0, \beta_0} \otimes \theta^+_{\beta_0, \gamma_0}) = \theta^+_{\alpha_0, \gamma_0} \]

in the annulus \(A\), which was established in the proof of [Juh16, Theorem 7.6]; see the left-hand side of Figure 2.1.

We now show commutativity of the second diagram. Let \(x \in T_\alpha \cap T_\beta\) and \(y \in T_\beta \cap T_\gamma\). Then

\[ F_T \circ (\text{id}_{\text{CF}(\mathcal{H}_\alpha, \beta)} \otimes F''_3 \otimes F''_3)(x \otimes (y \otimes \theta^-_{\beta_0, \gamma_0})) = F_T(x \otimes y). \]

On the other hand,

\[
F''_3 \circ F'_1 \circ (\text{id}_{\text{CF}(\mathcal{H}_\alpha, \beta')} \otimes F''_3 \otimes F''_3)
\]

\[
\left(F''_3 \circ F'_1 \circ (\text{id}_{\text{CF}(\mathcal{H}_\alpha, \beta')} \otimes F''_3 \otimes F''_3)\right)(x \otimes (y \otimes \theta^-_{\beta_0, \gamma_0})) = F''_3 \circ F'_1 \circ \left(F''_3 \circ F'_1 \circ (\text{id}_{\text{CF}(\mathcal{H}_\alpha, \beta')} \otimes F''_3 \otimes F''_3)\right)(x \otimes y).
\]

Figure 2.1. The annulus \(A\) is bounded by the two dashed circles. The intersection points \(\theta^+_{\alpha_0, \beta_0}, \theta^+_{\beta_0, \gamma_0},\) and \(\theta^+_{\gamma_0, \alpha_0}\) are marked by solid circles, and \(\theta^-_{\alpha_0, \beta_0}, \theta^-_{\beta_0, \gamma_0},\) and \(\theta^-_{\gamma_0, \alpha_0}\) by empty circles. On the left, the only index 0 triangle connecting \(\theta^+_{\alpha_0, \beta_0},\) \(\theta^+_{\beta_0, \gamma_0},\) and \(\theta^+_{\gamma_0, \alpha_0}\) is shaded. On the right, the only index 0 triangle connecting \(\theta^+_{\alpha_0, \beta_0},\) \(\theta^-_{\beta_0, \gamma_0},\) and \(\theta^-_{\gamma_0, \alpha_0}\) is shaded.
Hence, commutativity for the generator \( x \otimes (y \times \theta_{\beta_0, \gamma_0}^-) \) follows from
\[
F_{\alpha_0, \beta_0, \gamma_0}^+ (\theta_{\alpha_0, \beta_0}^+ \otimes \theta_{\beta_0, \gamma_0}^-) = \theta_{\alpha_0, \gamma_0}^-,
\]
which can be shown similarly to equation (1); see the right-hand side of Figure 2.1. Note that there is a unique index 0 pseudo-holomorphic triangle in \( A \) connecting \( \theta_{\alpha_0, \beta_0}^+ \), \( \theta_{\beta_0, \gamma_0}^- \), and \( \theta_{\alpha_0, \gamma_0}^- \), and there is none connecting \( \theta_{\alpha_0, \beta_0}^+ \), \( \theta_{\beta_0, \gamma_0}^- \), and \( \theta_{\alpha_0, \gamma_0}^- \).

On a generator of the form \( x \otimes (y \times \theta_{\beta_0, \gamma_0}^-) \), we have
\[
F_T \circ (\text{id}_{CF(H_{\alpha, \beta})} \otimes F_3^{\beta_0, \gamma_0}) (x \otimes (y \times \theta_{\beta_0, \gamma_0}^+)) = 0,
\]
and, using equation (1),
\[
F_{\alpha_0, \gamma_0}^{\alpha_0, \gamma_0} \circ F_T \circ \left( F_{1, \beta_0}^{\alpha_0, \beta_0} \otimes \text{id}_{CF(H_{\theta, \gamma})} \right) (x \otimes (y \times \theta_{\beta_0, \gamma_0}^+)) =
\]
\[
F_{\alpha_0, \gamma_0}^{\alpha_0, \gamma_0} (F_T (x, y) \times F_{\alpha_0, \beta_0, \gamma_0} (\theta_{\alpha_0, \beta_0}^+ \otimes \theta_{\gamma_0, \beta_0}^+)) = F_{\alpha_0, \gamma_0}^{\alpha_0, \gamma_0} (F_T (x, y) \times \theta_{\alpha_0, \gamma_0}^-) = 0.
\]

This establishes commutativity of the second diagram. Commutativity of the third diagram is analogous.

\[\square\]

2.2. Compound stabilization. In this section, we describe an elaboration of the usual stabilization operation on Heegaard diagrams. Suppose that \( H = (\Sigma, \alpha, \beta) \) is an admissible sutured diagram, and that \( \lambda \) is an embedded path on \( \Sigma \) between two distinct points on \( \partial \Sigma \) that avoids the \( \alpha \) curves. We define the compound stabilization of \( H \) along \( \lambda \), as follows.

First, construct a surface \( \Sigma' \) by pushing \( \lambda \) into the sutured compression body \( U_\lambda \), and add a tube that is the boundary of a regular neighborhood of \( \lambda \). Let \( \alpha_0 \) be a longitude of the tube, concatenate a portion of the curve \( \lambda \) on \( \Sigma \). Furthermore, let \( \beta_0 \) be a meridian of the tube. The curve \( \alpha_0 \) may intersect other \( \beta \) curves; however, \( \beta_0 \) intersects only \( \alpha_0 \). The construction is shown in Figure 2.2. Let us denote by \( H' \) the Heegaard diagram \( (\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}) \). This is an instance of a \((k, 0)\)-stabilization using the terminology of [JT12, Definition 6.26], where \( k = |\alpha_0 \cap \beta| \). If \( \lambda \) avoids \( \beta \), then we can perform an analogous operation, with the roles of \( \alpha \) and \( \beta \) swapped, which is an instance of a \((0, l)\)-stabilization. We also call this a compound stabilization. In the opposite direction, we say that \( H \) is obtained from \( H' \) by a compound destabilization.

We denote the unique intersection point of \( \alpha_0 \) and \( \beta_0 \) by \( c_{\alpha_0, \beta_0} \). There is a map
\[
\sigma_{\alpha_0, \beta_0} : SFH(H) \rightarrow SFH(H'),
\]
defined by
\[
\sigma_{\alpha_0, \beta_0} (x) = x \times c_{\alpha_0, \beta_0},
\]
which is a chain isomorphism since the tube is added near \( \partial \Sigma \).

On the other hand, there is also a naturality map \( \Phi_{H \rightarrow H'} \). One would expect these to be equal. Indeed, we prove the following (compare [HKM08, Proposition 3.7]):

**Proposition 2.2.** The compound stabilization map \( \sigma_{\alpha_0, \beta_0} \) is chain homotopic to the naturality map \( \Phi_{H \rightarrow H'} \).

One strategy to prove the above theorem would be to handleslide all the \( \beta \) curves that intersect \( \alpha_0 \) across \( \beta_0 \). The map from naturality induced by these handleslides can be computed by counting holomorphic triangles. To do this, one could analyze how holomorphic triangles degenerate as one stretches two necks (one on each end of the tube we are adding). While this can be done, we will give a somewhat indirect argument that avoids performing a neck-stretching argument.

Suppose that \( T = (\Sigma, \alpha, \beta, \beta') \) is an admissible sutured triple with a path \( \lambda \) from \( \partial \Sigma \) to itself that does not intersect any \( \alpha \) curves. Then we can perform the compound stabilization procedure on \( (\Sigma, \alpha, \beta, \beta') \) to obtain a Heegaard triple
\[
T' = (\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}, \beta' \cup \{\beta_0'\}),
\]
where \( (\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}) \) is the compound stabilization of \( (\Sigma, \alpha, \beta) \) along \( \lambda \). Furthermore, the curve \( \beta_0' \) is isotopic to \( \beta_0 \) and \( |\beta_0 \cap \beta_0'| = 2 \), while \( |\alpha_0 \cap \beta_0| = |\alpha_0 \cap \beta_0'| = 1 \). An example is shown
in Figure 2.3. Let \( \theta^+_{\beta_0, \beta'_0} \) be the point of \( \beta_0 \cap \beta'_0 \) with the higher relative Maslov grading, and write \( \alpha_0 \cap \beta_0 = \{c_{\alpha_0, \beta_0}\} \) and \( \alpha_0 \cap \beta'_0 = \{c_{\alpha_0, \beta'_0}\} \).

**Lemma 2.3.** If \( T = (\Sigma, \alpha, \beta, \beta') \) is a sutured triple and

\[
\mathcal{T}' = (\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}, \beta' \cup \{\beta'_0\})
\]

is a compound stabilization of \( T \), as described in the previous paragraph, then

\[
F_{\mathcal{T}'}(x \times c_{\alpha_0, \beta_0}, y \times \theta^+_{\beta_0, \beta'_0}) = F_{\mathcal{T}}(x, y) \times c_{\alpha_0, \beta'_0}.
\]

**Proof.** Since the tube is added near \( \partial\Sigma \), the result is obtained by a model computation inside the tube. This is shown in Figure 2.3. \(\square\)
Remark 2.4. Despite the notation, the triangle map computation of Lemma 2.3 does not assume that the curves $\beta$ and $\beta'$ appearing in the triple $T$ are related by a sequence of handleslides or isotopies. However, we will only need the result for examples where that is the case.

Analogously, we need to consider moves of the $\alpha_0$ curve appearing in a compound stabilization. To this end, suppose that $T = (\Sigma, \alpha, \beta)$ is a sutured triple with two paths, $\lambda_1$ and $\lambda_2$, from $\partial \Sigma$ to itself, such that $\lambda_1$ and $\lambda_2$ have the same endpoints and disjoint interiors. Furthermore, suppose that $\lambda_1$ avoids $\alpha'$ and $\lambda_2$ avoids $\alpha$. We can construct a compound stabilization of the triple $(\Sigma, \alpha', \beta, \beta)$ to obtain

$$T' = (\Sigma', \alpha' \cup \{\alpha'_0\}, \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}),$$

where $(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$ is the compound stabilization of $(\Sigma, \alpha, \beta)$ along $\lambda_1$. Furthermore, the curve $\alpha'_0$ is a concatenation of a portion of the curve $\lambda_2$ on $\Sigma$ with a longitude of the tube $\Sigma' \setminus \Sigma$, such that $|\alpha_0 \cap \alpha'_0| = 2$ and $|\alpha_0 \cap \beta_0| = |\alpha'_0 \cap \beta_0| = 1$. In the tube, $\alpha_0$, $\alpha'_0$, and $\beta_0$ are configured as in Figure 2.4. Let $\theta^{+}_{\alpha'_0, \alpha_0}$ be the point of $\alpha'_0 \cap \alpha_0$ with the larger relative Maslov grading, and write $\alpha_0 \cap \beta_0 = \{c_{\alpha_0, \beta_0}\}$ and $\alpha'_0 \cap \beta_0 = \{c_{\alpha'_0, \beta_0}\}$.

Lemma 2.5. If $T = (\Sigma, \alpha', \beta, \beta)$ is an admissible sutured triple and

$$T' = (\Sigma', \alpha' \cup \{\alpha'_0\}, \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$$

is a compound stabilization, as described in the previous paragraph, then

$$F_{T'}(x \times \theta^{+}_{\alpha'_0, \alpha_0}, y \times c_{\alpha_0, \beta_0}) = F_T(x, y) \times c_{\alpha'_0, \beta_0}.$$

Proof. As before, since the ends of the tube $\Sigma' \setminus \Sigma$ are near $\partial \Sigma$, we obtain constraints on the multiplicities of any homology class of triangles which has holomorphic representatives. An easy model computation shows that triangles with representatives have homology class $\psi \cup \psi_0$, where $\psi$ is a homology class on $(\Sigma, \alpha', \beta, \beta)$ and $\psi_0$ is a homology class supported entirely on the tube. The appropriate model computation is shown in Figure 2.4.

![Figure 2.4](image)

**Figure 2.4.** The model computation of Lemma 2.5. The $\alpha'$ curves are shown as dashed red, the $\alpha$ curves are shown as solid red, and the $\beta$ curves are shown as solid blue.

Using the above two lemmas, we now prove Proposition 2.2.

Proof of Proposition 2.2. Let $H' = (\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$ denote a compound stabilization of $(\Sigma, \alpha, \beta)$ using a path $\lambda$ with ends on $\partial \Sigma$, and let $B$ be the tube attached. Let $H = (\Sigma, \alpha \cup \{\alpha_1\}, \beta \cup \{\beta_1\})$ be another compound stabilization of $H$, along a path that is parallel to $\lambda$. Write $B$ for the attached tube. Let $H' = (\Sigma', \alpha \cup \{\alpha_0, \alpha_1\}, \beta \cup \{\beta_0, \beta_1\})$ denote the two-fold compound stabilization of $\Sigma$ along both paths. Write $\{c_0\} = \alpha_0 \cap \beta_0$ and $\{c_1\} = \alpha_1 \cap \beta_1$.

We claim that

$$(2) \quad \Phi_{H' \to \tilde{H}}(x \times c_0) = \Phi_{H' \to \tilde{H}}(x) \times c_0.$$ 

To see this, we note that a sequence of diagrams from $H'$ to $H'$ can be constructed by starting with $H'$, performing a simple stabilization near the boundary, and then moving one foot of the new
tube along $\Sigma$, parallel to $\alpha_0$. At various points, we will have to handleslide a $\beta$ curve across $\beta_1$. It is not obvious what the holomorphic triangle count will be for each handleslide. However, by the holomorphic triangle count from Lemma 2.3, it is unchanged by the presence of the compound stabilization along $\alpha_0$. In particular, the triangles counted by going from $\mathcal{H}'$ to $\mathcal{H}''$ are the same as the ones counted in the analogous sequence of diagrams from $\mathcal{H}$ to $\mathcal{H}'$, so equation (2) follows.

We now consider the path of Heegaard diagrams from $\mathcal{H}'$ to $\mathcal{H}$ shown in Figure 2.5. The diagram $\mathcal{H}'''$ is obtained by handlesliding $\beta_1$ over $\beta_0$. We let $\beta'_1$ denote the curve resulting from this handleslide. The diagram $\mathcal{H}'''$ is obtained by isotopying the Heegaard surface by sliding the foot of the tube marked $D$ inside $\beta'_1$ over the tube marked $B$, carrying $\alpha_0$ and $\beta_0$ along, and then handlesliding $\alpha_0$ over $\alpha_1$, giving rise to a new curve $\alpha'_0$. Note that $\mathcal{H}'''$ is a simple stabilization of $\mathcal{H}$, and hence there is a destabilization map from $\mathcal{H}'''$ to $\mathcal{H}$.

Using the presence of the boundary $\partial \Sigma$ to simply the computation, one can see that the only holomorphic triangles contributing to the change of diagrams map $\Phi_{\mathcal{H}' \to \mathcal{H}''\prime}$ have homology class $\psi \cup \psi_0$, where $\psi$ is a holomorphic triangle on an unstabilized Heegaard triple $(\Sigma, \alpha, \beta, \beta')$, where $\beta'$ is a small isotopy of $\beta$, and $\psi_0$ is the homology class shown in Figure 2.6. Using an additional triangle map to move the $\beta'$ back to $\beta$ (and only isotoping $\beta_0$ and $\beta'_1$ a small amount) that can be analyzed similarly, we have that

$$\Phi_{\mathcal{H}' \to \mathcal{H}''}(x \times c_0 \times c_1) = x \times c_0 \times c'_1,$$

where $c'_1 = \alpha_1 \cap \beta'_1$.

By Lemma 2.5, we have that

$$\Phi_{\mathcal{H}' \to \mathcal{H}''}(x \times c_0 \times c'_1) = x \times c' \times c'_1,$$

where $c_0 = \alpha'_0 \cap \beta_0$. Equations (3) and (4) imply that

$$\Phi_{\mathcal{H}' \to \mathcal{H}''}(x \times c_0 \times c_1) = x \times c'_0 \times c'_1.$$

The diagrams $\mathcal{H}'''$ and $\mathcal{H}$ are related by a destabilization of the curves $\alpha'_0$ and $\beta_0$. Hence

$$\Phi_{\mathcal{H}''' \to \mathcal{H}}(x \times c'_0 \times c'_1) = x \times c_1.$$

In $\mathcal{H}$, the only $\alpha$-curve that intersects $\beta_1$ is $\alpha_1$, and $\alpha_1 \cap \beta_1 = \{c_1\}$, hence every generator in $\mathcal{H}$ is of the form $y \times c_1$ for some $y \in T_\alpha \cap T_\beta$. So, there are constants $c_{x,y} \in \mathbb{F}_2$ such that

$$\Phi_{\mathcal{H} \to \mathcal{H}}(x) = \sum_{y \in T_\alpha \cap T_\beta} c_{x,y} (y \times c_1).$$

If we substitute this into equation (2), we get that

$$\Phi_{\mathcal{H}' \to \mathcal{H}''}(x \times c_0) = \sum_{y \in T_\alpha \cap T_\beta} c_{x,y} (y \times c_0 \times c_1).$$

Together with equations (5) and (6), we arrive at the equality

$$\Phi_{\mathcal{H}' \to \mathcal{H}}(x \times c_0) = (\Phi_{\mathcal{H}''' \to \mathcal{H}} \circ \Phi_{\mathcal{H}' \to \mathcal{H}''} \circ \Phi_{\mathcal{H} \to \mathcal{H}})(x \times c_0) = \sum_{y \in T_\alpha \cap T_\beta} c_{x,y} (y \times c_1) = \Phi_{\mathcal{H} \to \mathcal{H}}(x).$$

Hence, we obtain that

$$\Phi_{\mathcal{H}' \to \mathcal{H}}(x \times c_0) = (\Phi_{\mathcal{H} \to \mathcal{H}} \circ \Phi_{\mathcal{H}' \to \mathcal{H}})(x \times c_0) = (\Phi_{\mathcal{H} \to \mathcal{H}} \circ \Phi_{\mathcal{H} \to \mathcal{H}})(x) = x.$$

We note that the last equality follows from naturality. Hence $\Phi_{\mathcal{H}' \to \mathcal{H}}(x \times c_0) = x$, completing the proof of Proposition 2.2. \hfill $\square$
In this section, we give a definition of the contact gluing map using contact cell decompositions, and prove invariance. The construction is similar to the one due to Honda, Kazez, and Matić [HKM08].
On a formal level, the gluing map is described as follows. Suppose that \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\) and that \(\xi\) is a cooriented contact structure on \(M' \setminus \text{int}(M)\) such that \(\partial M\) is a convex surface with dividing set \(\gamma\) and \(\partial M'\) is a convex surface with dividing set \(\gamma'\). Note that this implies that 

\[ R_+(\xi) \cap \partial M = R_-(M, \gamma) \quad \text{and} \quad R_-(\xi) \cap \partial M = R_+(M, \gamma). \]

In this situation, there is an induced map

\[ \Phi_\xi : SFH(-M, -\gamma) \to SFH(-M', -\gamma'), \]

called the gluing map.

3.1. Contact cell decompositions. In this section, we describe some background on contact cell decompositions. The technical content is due to Honda, Kazez, and Matić [HKM09, Section 1.1] and Giroux [Gir02]. Before we discuss cell decompositions of contact 3-manifolds, we need the following notion of cell decomposition of a surface with divides:

**Definition 3.1.** Let \(F\) be a closed, orientable surface, and \(\gamma \subseteq F\) a dividing set. A **sutured cell decomposition** of \((F, \gamma)\) consists of

- a collection of pairwise disjoint disks \(B_1, \ldots, B_n \subseteq F\) (that we think of as fattened 0-cells) such that each \(B_i \cap \gamma\) is an arc and each component of \(\gamma\) intersects some \(B_i\), and
- a collection of pairwise disjoint properly embedded arcs \(\lambda_1, \ldots, \lambda_m \subseteq F \setminus \bigcup_{i=1}^{n} \text{int}(B_i)\) disjoint from \(\gamma\) such that the closure of each component of \(F \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m)\) is a disk with piecewise smooth boundary.

For an illustration of a sutured cell decomposition, see the left-hand side of Figure 3.1.

**Remark 3.2.** Let \(D\) be a component of \(F \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m)\). Since each component of \(\gamma\) intersects some \(B_i\), the dividing set \(\gamma \cap D\) contains no closed curves. Hence, according to Giroux’s criterion [Hon00a, Theorem 3.5], if \(D\) is embedded in a contact structure such that it is convex with dividing set \(\gamma \cap D\) and \(\partial D\) is Legendrian, then it has a tight neighborhood.

**Definition 3.3.** Suppose that \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\) and \(\xi\) is a contact structure on \(Z := M' \setminus \text{int}(M)\) such that \(\partial Z\) is convex with dividing set \(\gamma \cup \gamma'\). A **contact cell decomposition** of \((Z, \xi)\) consists of the following data:

1. A non-vanishing contact vector field \(v\) defined on a neighborhood of \(\partial Z\) in \(Z\) and transverse to \(\partial Z\) such that it induces the dividing set \(\gamma \cup \gamma'\). The flow of \(v\) induces a diffeomorphism of \(\partial Z \times I\) with a collar neighborhood of \(\partial Z\) in \(Z\). Under this diffeomorphism, \(v\) corresponds to \(\partial /\partial t\), the boundary \(\partial M\) is identified with \(\partial M \times \{0\}\), and \(\partial M'\) is identified with \(\partial M' \times \{1\}\).

We let \(\nu = v|_{\partial M \times I}\) and \(\nu' = v|_{\partial M' \times I}\).
(2) “Barrier” surfaces $S \subseteq \partial M \times (0,1)$ and $S' \subseteq \partial M' \times (0,1)$ in $Z$ that are isotopic to $\partial M$ and $\partial M'$, respectively, and are transverse to $v$. Write $N$ and $N'$ for the collar neighborhoods of $\partial M$ and $\partial M'$ in $Z$ that are bounded by $S$ and $S'$, respectively, and set $Z' = Z \setminus \text{int}(N \cup N')$.
(3) A Legendrian graph $\Gamma \subseteq \mathcal{Z}$ that intersects $\partial Z'$ transversely in a finite collection of points along the dividing set of $\partial Z'$ with respect to the vector field $v$. Furthermore, $\Gamma$ is tangent to the vector field $v$ near $\partial Z'$.
(4) A choice of regular neighborhood $N(\Gamma)$ of $\Gamma$ such that $\xi|_{N(\Gamma)}$ is tight and $\partial N(\Gamma) \setminus \partial Z'$ is a convex surface. Furthermore, $N(\Gamma) \cap \partial Z'$ is a collection of disks $D$ with Legendrian boundary such that $tb(D) = -1$. We assume that the edge rounding procedure of Honda [Hon00a, Section 3.2.2] has been performed such that $N(\Gamma)$ meets $\partial Z'$ tangentially along the Legendrian unknots forming $\partial N(\Gamma)$.
(5) A collection of convex 2-cells $D_1, \ldots, D_n$ inside $Z' \setminus \text{int}(N(\Gamma))$ with Legendrian boundary on $\partial Z' \cup \partial N(\Gamma)$ and $tb(\partial D_i) = -1$.

Furthermore, the following hold:

(a) The complement of $N(\Gamma) \cup D_1 \cup \cdots \cup D_n$ in $Z'$ is a finite collection of topological 3-balls, and $\xi$ is tight on each.
(b) The disks in $N(\Gamma) \cap \partial Z'$ and the Legendrian arcs $\partial D_i \cap \partial Z'$ induce a sutured cell decomposition of $\partial Z'$, with the dividing set induced by $v$ (Definition 3.1).

Remark 3.4. Given surfaces $S$ and $S'$ in $Z$ and a transverse contact vector field $v$ defined in a neighborhood of $\partial Z$ that satisfy (1) and (2), it is not always possible to construct a contact cell decomposition of $Z$ with $S$ and $S'$ as barrier surfaces. For example, the characteristic foliations on $S$ and $S'$ may obstruct the existence of sutured cell decompositions (Definition 3.1) such that the arcs $\lambda_i$ and the curves $\partial B_i$ are Legendrian.

We now describe an important example of contact cell decompositions:

Example 3.5. Suppose that $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$ and $\xi$ is a compatible contact structure on $Z = M' \setminus \text{int}(M)$. Furthermore, suppose that $(Z, \xi)$ is contactomorphic to $F \times I$ with an $I$-invariant contact structure for a surface $F \cong \partial M \cong \partial M'$. Suppose that the disks $B_1, \ldots, B_n \subseteq F$ and the (not necessarily Legendrian) properly embedded arcs $\lambda_1, \ldots, \lambda_m \subseteq F \setminus \text{int}(B_1 \cup \cdots B_n)$ give a sutured cell decomposition of $F \times \{0\}$, as in Definition 3.1.

There is an induced contact cell decomposition of $Z$, called the product contact cell decomposition, that we describe presently. For an illustration, see Figure 3.1. Let $G \subseteq F$ be the graph consisting of $G = \partial B_1 \cup \cdots \cup \partial B_n \cup \lambda_1 \cup \cdots \cup \lambda_m$; this is not necessarily Legendrian. However, using Giroux’s flexibility theorem, we can find a surface $S \subseteq F \times I$ that is the image of a $C^0$ small isotopy of $F \times \{\frac{1}{2}\}$ that is transverse to $\partial/\partial t$ and such that the image of $G \times \{\frac{1}{2}\}$ is Legendrian. We let $S'$ be the translation of $S$ by $\frac{1}{2}$ in the $\partial/\partial t$ direction. We use $S$ and $S'$ for the barrier surfaces of our contact cell decomposition. We will also write $B_1, \ldots, B_n$ and $\lambda_1, \ldots, \lambda_m$ for the images on $S$ of the corresponding cells on $F \times \{\frac{1}{2}\}$, under the chosen $C^0$ small isotopy. Let $N$ and $N'$ be the collars of $\partial M$ and $\partial M'$ bounded by $S$ and $S'$, respectively, and write $Z' = Z \setminus \text{int}(N \cup N')$.

For every $i \in \{1, \ldots, n\}$, choose a point $p_i \in \text{int}(\gamma \cap B_i)$. We define the Legendrian graph $\Gamma$ as $\bigcup_{i=1}^n (\{p_i\} \times I) \cap Z'$, and set $N(\Gamma) = \bigcup_{i=1}^n (B_i \times I) \cap Z'$. The contact 2-cells of the decomposition are defined to be $D_i := (\lambda_i \times I) \cap Z'$. Using Giroux’s flexibility theorem on $\partial N(\Gamma)$, we can assume that each $D_i$ has Legendrian boundary. The disks $D_i$ have $tb(\partial D_i) = -1$ since their boundaries intersect the dividing set exactly twice. If $C$ is a component of $S \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m)$, then $(C \times I) \cap Z'$ is a tight contact ball in the complement of $N(\Gamma) \cup D_1 \cup \cdots \cup D_n$ in $Z'$.

To show invariance of the gluing map, we need to describe how two contact cell decompositions are related. We have the following; cf. [HKM09, Theorem 1.2]:
Figure 3.1. The left shows a sutured cell decomposition of a surface $F$ with dividing set $\gamma$. The right shows the induced product contact cell decomposition of an $I$-invariant contact structure on $F \times I$ with dividing set $\gamma$ from Example 3.5. The barrier surfaces are $S \approx F \times \{\frac{1}{2}\}$ and $S' \approx F \times \{\frac{2}{3}\}$. The fattened 0-cell $B_1$ is shown on the left in gray. The dividing set $\gamma$ is shown in red. The Legendrian arcs $\lambda_j$ and $\lambda_k$, as well as the corresponding convex 2-cells $D_j$ and $D_k$, are shown in green.

**Proposition 3.6.** Suppose that $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$, and $\xi$ is a contact structure on $Z = M' \setminus \text{int}(M)$ such that $\partial Z$ is convex with dividing set $\gamma \cup \gamma'$. If $C_1$ and $C_2$ are contact cell decompositions of $Z$, then there is a sequence $C_{(1)}, \ldots, C_{(\ell)}$ of contact cell decompositions with $C_1 = C_{(1)}$ and $C_2 = C_{(\ell)}$, such that $C_{(i+1)}$ is obtained from $C_{(i)}$ by one of the following moves, or its inverse:

1. (Isotopy) Replacing a contact cell decomposition $C$ with a contact cell decomposition of the form $\phi(C)$, where $\phi \in \text{Diff}(Z)$ (not necessarily a contactomorphism) fixes $\partial Z$ pointwise and is isotopic to the identity relative to $\partial Z$.

2. (Index 0/1 cancellation) Subdividing a Legendrian edge of $\Gamma$, or adding a Legendrian edge that has one endpoint on $\Gamma$, but which is otherwise disjoint from $\Gamma$ and the 2-cells of $C$.

3. (Index 1/2 cancellation) Adding a Legendrian edge $\lambda$ to $\Gamma$, and adding a convex 2-cell $D$ with $\text{tb}(\partial D) = -1$, such that the interiors of $\lambda$ and $D$ are disjoint from all the other cells, and $\partial D = c' \cup c''$, where $c'$ is a Legendrian arc on a neighborhood of $\lambda$, and $c'' \subseteq \partial Z' \cup \partial N(\Gamma)$ is a Legendrian arc that is disjoint from the dividing set on $\partial Z' \cup \partial N(\Gamma)$.

4. (Index 2/3 cancellation) Adding a convex disk $D$ with $\partial D \subseteq \partial Z' \cup \partial N(\Gamma)$ and $\text{tb}(\partial D) = -1$ disjoint from the other cells.

**Proof.** The result follows from an adaptation of the subdivision techniques due to Giroux [Gir02] and Honda, Kazez, and Matić [HKM09, Theorem 1.2].

**Step 1:** Suppose $C_1$ and $C_2$ are contact cell decompositions with the same barrier surfaces $S$ and $S'$, the same contact vector fields $\nu$ and $\nu'$, such that $C_1$ and $C_2$ induce the same sutured cell decompositions $D$ and $D'$ of $S$ and $S'$, and such that the Legendrian 1-skeletons of $C_1$ and $C_2$ intersect $S$ and $S'$ at the same points. Then $C_1$ and $C_2$ can be connected by Moves (1)–(4).

The proof follows from the strategy of [HKM09, Theorem 1.2] that we summarize using our present notation. Firstly, we can construct two contact automorphisms $\psi_1$ and $\psi_2$ of $(Z, \xi)$ that are isotopic to the identity relative to $\partial Z$, such that $\psi_1(C_1)$ and $\psi_2(C_2)$ are cell decompositions with Legendrian graphs $\psi_1(\Gamma_1)$ and $\psi_2(\Gamma_2)$, respectively, that intersect $\partial M \times I$ and $\partial M' \times I$ along arcs of the form $\{p\} \times [a, 1]$ or $\{p'\} \times [0, a]$ for various $p \in \gamma$, $p' \in \gamma'$, and $a > 0$. Furthermore, by applying Move (1), we may assume that the intersection of each 2-cell with $\partial Z \times I$ is $C^0$ close to a subset of the form $\lambda \times I$ for some arc $\lambda$ in $\partial Z$. Next, we take sutured cell decompositions of $\partial M \times \{1\}$ and $\partial M' \times \{0\}$ with Legendrian 1-cells, and with 2-cells $D$ with Legendrian boundary and $\text{tb}(\partial D) = -1$. We add the Legendrian 1-cells and convex 2-cells to $C_1$ and $C_2$ using Moves (2)–(4). Then we apply the subdivision procedure of Giroux [Gir02] away from $\partial Z \times I$. 
**Step 2:** Let $C$ be a contact cell decomposition with barrier surface $S$ and induced sutured cell decomposition $D$ of $S$. Then any isotopy of $S$ through surfaces that are transverse to $\nu = \partial / \partial t$ that fixes the cells of $D$ can be achieved using Move (1). The same holds for $S'$, $\nu'$, and $D'$.

Without loss of generality, we may assume that $C$ looks like a product decomposition on $\partial M \times I$, as in Example 3.5. In particular, near $S$, the graph $\Gamma$ is a union of Legendrians of the form $\{ p \} \times [a, 1]$ for $p \in \gamma$ and some $a > 1$, and the 2-cells are small perturbations of subsets of $\lambda \times I$ for arcs $\lambda \subseteq \partial M$. Suppose that $f : S \to \partial M \times I$ is an embedding that is the identity on $D$, and whose image is transverse to $\partial / \partial t$. We can extend $f$ to an automorphism of $Z$ that is the identity outside $\partial M \times (0, 1)$. It is straightforward to see that $f(C)$ is a contact cell decomposition, and that $C$ and $f(C)$ are related by Move (1).

**Step 3:** If $C_1$ and $C_2$ are contact cell decompositions with the same vector field $\nu$, but with different barrier surfaces and induced sutured cell decompositions, then $C_1$ with $C_2$ can be connected using Moves (1)–(4).

Let $D_i$ and $D'_i$ be the sutured cell decompositions induced on the barrier surfaces $S_i$ and $S'_i$ of the contact cell decomposition $C_i$ for $i \in \{1, 2\}$. Write $\pi : \partial M \times I \to \partial M$ and $\pi' : \partial M' \times I \to \partial M'$ for the projections. Note that $\pi(D_1)$ and $\pi(D_2)$ are sutured cell decompositions of $\partial M$, and $\pi'(D'_1)$ and $\pi'(D'_2)$ are sutured cell decompositions of $\partial M'$. It is straightforward to see that $\pi(D_1)$ and $\pi(D_2)$ can be connected by a sequence of the following moves and their inverses:

1. **(D-1)** Adding an edge $\lambda \subseteq \partial M$ that has both ends on boundaries of fattened 0-cells in $\pi(D_1)$, but which is otherwise disjoint from all the 0- and 1-cells of $\pi(D_1)$, as well as $\gamma$.
2. **(D-2)** Adding a new fattened 0-cell $B$ and a new arc $\lambda \subseteq \partial M$ to $\pi(D_1)$ that connects $B$ to another 0-cell in $\pi(D_1)$, but is otherwise disjoint from the 0- and 1-cells of $\pi(D_1)$.

Moves (D-1) and (D-2) are illustrated in Figure 3.2.

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**Figure 3.2.** Moves (D-1) and (D-2) between sutured cell decompositions of $\partial M$. The grey disks are the fattened 0-cells, and the green arcs marked $\lambda$ are the Legendrians. The red arcs marked $\gamma$ form the dividing set.

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We first describe how to connect $C_1$ to a cell decomposition $C'_2$ that has barrier surfaces and sutured cell decompositions that are $C^0$ close to the barrier surfaces and sutured cell decompositions of $C_2$. Let us focus on the barrier surfaces $S_1$ and $S_2$ and sutured cell decompositions $D_1$ and $D_2$ near $\partial M$. As in Step (1), we can reduce to the case when $Z$ is a product as in Example 3.5, by adding a cellular decomposition of $\partial M \times \{1\}$ to $C_1$ and $C_2$. 

We claim that a sequence of moves (D-1) and (D-2) relating $\pi(D_1)$ and $\pi(D_2)$ can be $C^0$ approximated by applying moves (1)–(4) to $C_1$. Using Step (2), we can isotope portions of $S_1$ that are not near $D_1$. To realize Move (D-1) and add a new arc $\lambda$ to $\pi(D_1)$, we apply Step (2), so that the image of $\lambda$ on $S$ is $C^0$ approximated by a Legendrian $\lambda'$, and then we apply Move (4) to $C_1$ and a new convex 2-cell. As we have reduced to product cell decompositions, we can assume that $\Gamma$ consists of arcs of the form $\{p\} \times [a,b]$ with $p \in \gamma$ and $0 < a < b < 1$. The new 2-cell $D$ is the intersection of $\pi(\lambda') \times I$ with $Z \setminus (\text{int}(N) \cup \text{int}(N') \cup N(\Gamma))$, where $N$ and $N'$ are the collars bounded by $S$ and $S'$.

Move (D-2) can be achieved, up to $C^0$ approximation, by applying Move (3) that adds a Legendrian arc to $\Gamma$ of the form $\{p\} \times [a,b]$ with $p \in \gamma$ and a convex 2-cell with Legendrian boundary, similar to the 2-cell added to realize move (D-1). Note that, as in Step (1), we will have to perform a $C^0$ small isotopy of $S_1$ to make sure that the new 2-cell has Legendrian boundary.

The procedure described above yields a decomposition $C_2'$ that has sutured cell decompositions whose projections to $\partial M$ and $\partial M'$ are $C^0$ close to the projections of $D_2$ and $D_2'$. It is not hard to see that, by refining this procedure and using Step (2), we can obtain barrier surfaces of $C_2'$ that are $C^0$ close to the barrier surfaces for $C_2$.

Finally, since $C_2'$ and $C_2$ are $C^0$ close and are both product decompositions, we can construct an automorphism $\phi$ of $Z$ such that $\phi(C_2') = C_2$. In the case of non-product contact cell decompositions, we use Step (1) to connect $\phi(C_2')$ to $C_2$ using Moves (1)–(4).

**Step 4:** If $C_1$ and $C_2$ are arbitrary contact cell decompositions (with potentially different choices of contact vector fields), then $C_1$ and $C_2$ can be connected by Moves (1)–(4).

We will focus on changing the vector field $\nu_1$ of $C_1$, defined on a neighborhood of $\partial M$, to the contact vector field $\nu_2$ of $C_2$. The idea is that the space of germs of contact vector fields defined on open neighborhoods of $\partial M$ that induce the dividing set $\gamma$ is convex and hence contractible. Suppose $\nu_1$ and $\nu_2$ are two choices of contact vector fields, and $C_1$ and $C_2$ are two contact cell decompositions that use the vector fields $\nu_1$ and $\nu_2$ and have barrier surfaces $S_1$ and $S_2$, respectively. It is easy to see that there is an open neighborhood of $\partial M$ where the convex combinations $\nu_t = (2-t)\nu_1 + (t-1)\nu_2$ are non-vanishing for $t \in [1,2]$. Clearly, $\nu_t$ is also transverse to $\partial M$ for $t \in [1,2]$. We can define an isotopy $\psi_t$ of a neighborhood of $\partial M$, by flowing a point $p \in M$ backward along $\nu_1$, until it hits $\partial M$ (say, at a time $-T(p)$) and then composing with the flow along $\nu_t$ for time $T(p)$. This yields a 1-parameter family of contactomorphisms $\psi_t$ of all of $M$ for $t \in [1,2]$ by writing $\psi_t$ as the integral of a time dependent contact vector field $X_t$ (i.e., $X_t$ is a contact vector field for each $t$) defined over $\partial M \times I$, and then extending $X_t$ to all of $Z$ using time dependent contact Hamiltonians that vanish outside a neighborhood of $\partial M$. Using the previous steps, we can move $S_1$ into this neighborhood of $\partial M$, and then simply push $C_1$ forward under $\psi_2$, then connect $\psi_2(C_1)$ with $C_2$ using the previous steps.

**Remark 3.7.** Giroux’s techniques [Gir02] for finding a common subdivision of two contact cell decompositions involve the following move: Replace a convex 2-cell $D$ with a pair of convex 2-cells $D_1$ and $D_2$ with $\text{tb}(\partial D) = -1$ that meet along a Legendrian arc $\lambda$ such that $D_1 \cup D_2 = D$ and $\lambda \subseteq D$ is an arc that intersects the dividing set of $D$ transversely at a single point. We note that this move can be achieved as a composition of the moves listed in Proposition 3.6. To see this, let $D$ be a convex 2-cell, and let $D_1$ and $D_2$ be the convex 2-cells in a subdivision, meeting along the Legendrian arc $\lambda$. Let $D'$ be a parallel copy of $D$, intersecting $D$ only along $\partial D$. Let $D_1' \subseteq D'$, $D_2' \subseteq D'$, and $\lambda' \subseteq D'$ be the images of $D_1$, $D_2$, and $\lambda$, respectively. We can add $D_1'$ and $\lambda'$ to the decomposition using Move (3). Then we can add $D_2'$ to the decomposition using Move (4). Then we remove $D$ from the decomposition using the inverse of Move (4). Finally, we use Move (1) to move $\lambda'$ into the position of $D_1$, $D_2$, and $\lambda$, while preserving all the other cells.

It is possible to directly show invariance of the contact gluing map under subdividing a 2-cell into two contact 2-cells that meet along a Legendrian arc. Indeed, the topological manipulation described above gives a recipe for doing so. Nonetheless, the model computations required to show invariance of the gluing map under Moves (1)–(4) are simpler.
3.2. Contact handle maps. We first recall the definition of contact handles in dimension 3 due to Giroux [Gir91]; see also Özbagci [Özb11].

**Definition 3.8.** For \( k \in \{0, 1, 2, 3\} \), a \( 3 \)-dimensional contact handle of index \( k \) attached to a sutured manifold \( (M, \gamma) \) is a standard contact ball \( (B_0, \xi_0) \) (possibly with corners) that is attached via a map \( \phi: S \to \partial M \) for some subset \( S \subseteq \partial M \). Furthermore, the dividing set of \( \xi_0 \) on \( S \) is mapped into \( \gamma \) under \( \phi \), and we have the following requirements, depending on the index:

- **(Index 0):** \( S = \emptyset \) and \( B = D^3 \) has no corners. The dividing set on \( \partial B \) is a single circle.
- **(Index 1):** As a manifold with corners, \( B \) is \( I \times D^2 \), and \( S = \partial I \times D^2 \). The dividing set on \( \partial I \times D^2 \) consists of one arc on each component. The dividing set on \( I \times \partial D^2 \) consists of two arcs, each connecting the two components of \( \partial I \times \partial D^2 \).
- **(Index 2):** As a manifold with corners, \( B \) is \( D^2 \times I \) and \( S = \partial D^2 \times I \). The dividing set is the same as on a contact 1-handle.
- **(Index 3):** \( S = \partial D^3 \) and \( B = D^3 \), with no corners. The dividing set on \( \partial B \) is a single circle.

In Figure 3.3, we have drawn the dividing sets on contact handles.

**Figure 3.3.** Contact handles. On the left is a picture of a contact 0-handle or 3-handle. On the right is a contact 1-handle or 2-handle.

3.2.1. **Contact 0-handle map.** Adding a contact 0-handle \( h^0 \) amounts to adding a copy of the product sutured manifold \( (D^2 \times [-1, 1], S^1 \times \{0\}) \) to \( (M, \gamma) \). Noting that

\[
SFH(-D^2 \times [-1, 1], -S^1 \times \{0\}) \cong \mathbb{F}_2,
\]

the contact 0-handle map is simply the tautological map

\[
C_{h^0}: SFH(-M, -\gamma) \xrightarrow{\cong} SFH(-M, -\gamma) \otimes \mathbb{F}_2 \xrightarrow{\cong} SFH(-M, -\gamma) \otimes SFH(-D^2 \times [-1, 1], -S^1 \times \{0\}) \xrightarrow{\cong} SFH((-M, -\gamma) \cup (-D^2 \times [-1, 1], -S^1 \times \{0\})).
\]

On the chain level, it is given as follows. Choose a diagram \( \mathcal{H} = (\Sigma, \alpha, \beta) \) of \( (-M, -\gamma) \). As \( \mathcal{H}_0 = (\bar{D}^2, \emptyset, \emptyset) \) is a diagram of \( (-D^2 \times [-1, 1], -S^1 \times \{0\}) \), the disjoint union \( \mathcal{H} \cup \mathcal{H}_0 \) is a diagram of \( (-M, -\gamma) \cup (-D^2 \times [-1, 1], -S^1 \times \{0\}) \). Let \( i: \Sigma \to \Sigma \cup \bar{D}^2 \) be the inclusion. Then, for \( x = (x_1, \ldots, x_d) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \), we set \( C_{h^0}(x) = (i(x_1), \ldots, i(x_d)) \). This clearly induces \( C_{h^0} \) on homology.

3.2.2. **Contact 1-handle map.** A contact 1-handle \( h^1 \) determines two points along the sutures. If \( (\Sigma, \alpha, \beta) \) is a Heegaard surface for \( (M, \gamma) \), glue a strip to \( \partial \Sigma \) where the feet of the 1-handle are attached; see Figure 3.4. Add no new \( \alpha \) or \( \beta \) curves. The map on complexes is the tautological map induced by the inclusion of Heegaard surfaces. Let us call this map \( C_{h^1} \).

**Lemma 3.9.** The map \( C_{h^1} \) is natural; i.e., it commutes with change of diagrams maps.

**Proof.** The proof is straightforward, since the change of diagrams maps are either tautological (stabilization, isotopy) or involve counting holomorphic curves that do not intersect \( \partial \Sigma \times \text{Sym}^{k-1}(\Sigma) \subseteq \text{Sym}^k(\Sigma) \) (continuation maps for changes of the almost complex structure, triangle maps for changes of the \( \alpha \) and \( \beta \) curves).
3.2.3. Contact 2-handle map. We now define a map for contact 2-handles. Let \((\Sigma, \alpha, \beta)\) be an admissible diagram of \((M, \gamma)\). Suppose that \(l\) is the curve on \(\partial M\) along which we add a 2-handle \(h^2\), and \(l\) intersects \(\gamma\) (the sutures) exactly twice. Let \(p_1, p_2 \in \gamma\) be the two points of intersection and \(l_\pm = l \cap R_{\pm}(\gamma)\). We denote the result of the 2-handle attachment by \((M', \gamma')\).

We now construct a diagram of \((M', \gamma')\). Choose a sutured Morse function \(f\) and gradient-like vector field \(v\) on \((M, \gamma)\) that induce \((\Sigma, \alpha, \beta)\), and follow the flow of \(v\) from \(l_+\) and \(l_-\) onto \(\Sigma\). Writing \(\lambda_+\) and \(\lambda_-\) for the resulting arcs on \(\Sigma\), we note that \(\lambda_+\) avoids the \(\beta\) curves and \(\lambda_-\) avoids the \(\alpha\) curves. Hence, we can form a diagram \((\Sigma', \alpha \cup \{\alpha'\}, \beta \cup \{\beta'\})\), as in Figure 3.5. The surface \(\Sigma'\) is obtained by adding a band \(B\) to the boundary of \(\Sigma\) at \(p_1\) and \(p_2\). The curve \(\alpha'\) is obtained by concatenating the curve \(\lambda_-\) with an arc in the band. The curve \(\beta'\) is obtained by concatenating the curve \(\lambda_+\) with an arc in the band. We assume that \(\alpha'\) and \(\beta'\) intersect in a single point in the band (and possibly other places outside the band). Furthermore, we assume that the intersection point of \(\alpha'\) and \(\beta'\) has the configuration shown in Figure 3.5; i.e., locally, it looks like there is a holomorphic disk on \((\bar{\Sigma}, \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})\) going towards the intersection point that does not intersect \(\partial \Sigma'\).

To see that \((\Sigma', \alpha \cup \{\alpha'\}, \beta \cup \{\beta'\})\) is indeed a diagram of \((M', \gamma')\), let

\[
J = [-1, 1] \times \{0\} \subseteq D^2 \subseteq \mathbb{R}^2.
\]

Consider the arc \(a = J \times \{1/2\}\) in the 2-handle \(D^2 \times I \subseteq M'\) connecting \(p_1\) and \(p_2\). Furthermore, consider the neighborhood

\[
N(a) = ([[-1, 1] \times (-\varepsilon, \varepsilon)) \cap D^2 \times I
\]

of \(a\), where \(\varepsilon \in (0, 1)\). Then \(N(a) \cap R_{\pm}(\gamma')\) is a regular neighborhood of \(\gamma' \cap \Gamma = J \times \{0, 1\}\), and \(N(a)\) is a product 1-handle. On the sutured diagram level, attaching \(N(a)\) corresponds to adding the band \(B = J \times I\). Then \((D^2 \times I) \setminus N(a)\) is the disjoint union of two 3-dimensional 2-handles, attached to \((M \cup N(a), \gamma')\) along \(l_\pm \cup s_\pm\), where \(s_\pm\) is a properly embedded arc in \(N(a) \cap R_{\pm}(\gamma')\). These attaching curves correspond to \(\alpha'\) and \(\beta'\) on the Heegaard surface \(\Sigma'\). The diagram \((\Sigma', \alpha \cup \{\alpha'\}, \beta \cup \{\beta'\})\) is also admissible.

The contact 2-handle map

\[
C_{h^2} : SFH(\Sigma, \alpha, \beta) \to SFH(\Sigma', \alpha \cup \{\alpha'\}, \beta \cup \{\beta'\})
\]

is defined by the formula

\[
C_{h^2}(x) := x \times c,
\]

where \(c \in \alpha' \cap \beta'\) is the intersection point in the band.
Lemma 3.10. The following hold:

1. The map $C_{h^2}$ is a chain map.
2. The map $C_{h^2}$ is independent of the choices made in the construction; i.e., the choice of diagram $(\Sigma, \alpha, \beta)$, and the choice of arcs $\lambda_+$ and $\lambda_-$ obtained by projecting the two arcs $l_+$ and $l_-$ onto $\Sigma$.

Proof. The claim that $C_{h^2}$ is a chain map is straightforward. We wish to show that $\partial \circ C_{h^2} = C_{h^2} \circ \partial$, and hence we need to check that $\partial(x \times c) = \partial x \times c$. To this end, we note that any disk counted by $\partial(x \times c)$ must have zero multiplicity around $c$, since the boundary $\partial \Sigma$ is nearby. Hence, the domain is equal to a disk on $(\Sigma, \alpha, \beta)$ with a constant disk at $c$ added.

We now show that $C_{h^2}$ is independent of the choices made in the construction (i.e., commutes with the change of diagrams maps, appropriately). We note that there are two sources of ambiguity: the curves $\lambda_+$ and $\lambda_-$ in $\Sigma$, and the diagram $(\Sigma, \alpha, \beta)$. Let us address the ambiguity of $\lambda_+$ and $\lambda_-$. Since they are gotten by flowing curves in $\partial M$ under the gradient-like vector field of a Morse function, any two choices of $\lambda_+$ are related by handleslides over $\beta$ curves (as well as isotopies of $\lambda_+$, relative $\partial \Sigma$, such that it never intersects any $\beta$ curves). Similarly, any two choices of $\lambda_-$ are related by handleslides over the $\alpha$ curves and isotopies within $\Sigma \setminus \alpha$. To see that the change of diagrams map commutes with $C_{h^2}$, one realizes the handleslide and isotopy maps as triangle maps, and uses the local computation at the end of the proof of [HKM09, Lemma 3.5], which is illustrated...
by [HKM09, Figure 8]. For the reader’s convenience, we have reproduced the relevant picture in Figure 3.6, since we will use the local computation later, as well.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_6.png}
\caption{The local computation required to show that the contact 2-handle map is independent of handleslides of the $\alpha'$ curve used in the definition of the contact 2-handle map. A local computation using the vertex multiplicities (identical to the one in [HKM09, Lemma 3.5]) shows that any holomorphic triangle with vertices at $c$ and $\theta^+_{\alpha_1,\alpha_2}$ also has a vertex at $c'$, and has domain equal to the shaded region. The orientation of $\Sigma$ is shown.}
\end{figure}

3.2.4. Contact 3-handle maps. We now define the contact 3-handle map. Let $(M',\gamma')$ be the result of gluing a contact 3-handle $h^3$ to the balanced sutured manifold $(M,\gamma)$, and suppose that $(M',\gamma')$ is also balanced. The 3-handle is attached to a 2-sphere component $S \subseteq \partial M$ such that $\gamma_S = \gamma \cap S$ is a single closed curve. One defines the contact 3-handle map $C_{h^3}$ to be equal to the composition of the 4-dimensional 3-handle map obtained by surgering $(-M, -\gamma)$ along the 2-sphere obtained by pushing $S$ into $\text{int}(M)$, followed by the inverse of the contact 0-handle map corresponding to removing the resulting copy of $(-D^2 \times [-1,1], -S^1 \times \{0\})$.

On the diagram level, we choose a diagram $(\Sigma, \alpha, \beta)$ of $(M,\gamma)$ such that there are curves $\alpha \in \alpha$ and $\beta \in \beta$ that are parallel to $\gamma_S \subseteq \partial \Sigma$, intersect transversely in two points $x$ and $y$, and there are no other $\alpha$ or $\beta$ curves between $\alpha$ and $\gamma_S$ or $\beta$ and $\gamma_S$. Let $\Sigma'$ be the result of gluing a disk to $\Sigma$ along $\gamma_S$. Then $(\Sigma', \alpha \setminus \{\alpha\}, \beta \setminus \{\beta\})$ is a diagram of $(M',\gamma')$. Suppose that $x$ has larger relative grading than $y$ in $(\bar{\Sigma}, \alpha, \beta)$. For $x \in T_\alpha \cap T_\beta$, we set $C_{h^3}(x \times \{x\}) = 0$ and $C_{h^3}(x \times \{y\}) = x$.

3.3. Definition of the contact gluing map. We now define the gluing map, in terms of a contact cell decomposition (Definition 3.3). Suppose that $(M,\gamma)$ and $(M',\gamma')$ are balanced sutured manifolds, $(M,\gamma)$ is a sutured submanifold of $(M',\gamma')$, and $\xi$ is contact structure on $Z = M' \setminus \text{int}(M)$ that has dividing set $\gamma \cup \gamma'$. Let $C$ be a contact cell decomposition of $Z$. Let $\nu$ and $\nu'$ be the chosen contact vector fields, and $S$ and $S'$ the chosen barrier surfaces near $\partial M$ and $\partial M'$, respectively. Recall that we write $N$ and $N'$ for the collar neighborhoods of $\partial M$ and $\partial M'$ bounded by $S$ and $S'$, respectively. Let $\Gamma \subseteq Z \setminus \text{int}(N \cup N')$ denote the Legendrian 1-skeleton and $D_1, \ldots, D_n$ the convex 2-cells.

A neighborhood of each vertex of $\Gamma$ that is not contained in $S$ or $S'$ determines a contact 0-handle. A neighborhood of each edge of $\Gamma$ is a contact 1-handle. A neighborhood of each convex 2-cell $D_i$ is a contact 2-handle. Finally, after removing neighborhoods of the graph $\Gamma$ and the disks $D_i$, we are left with a collection of tight contact 3-balls that we view as a collection of contact 3-handles. Write $h_1, \ldots, h_n$ for these handles, ordered such that their indices are nondecreasing.

Let $\gamma_0$ be the dividing set on $S$ with respect to $\nu$ and $\xi$, and write $\gamma'_0$ for the dividing set on $S'$ with respect to $\nu'$ and $\xi$. The flow of $\nu$ induces a diffeomorphism 

$$\psi^\nu : (M,\gamma) \to (M \cup N, \gamma_0)$$
that is well-defined up to isotopy. Note that \((N', \gamma'_0 \cup \gamma')\) is a balanced sutured manifold, and \(\xi\) induces the dividing set \(\gamma'_0 \cup \gamma'\) on \(\partial N'.\) There is a canonical isomorphism
\[
SFH(-M \cup -N', -\gamma \cup -\gamma'_0 \cup -\gamma') \cong SFH(-M, -\gamma) \otimes SFH(-N', -\gamma'_0 \cup -\gamma').
\]
Hence, we can define a map
\[
\Phi_{\cup N'}: SFH(-M, -\gamma) \to SFH(-M \cup -N', -\gamma \cup -\gamma'_0 \cup -\gamma')
\]
via the formula
\[
\Phi_{\cup N'}(x) = x \otimes EH(N', \gamma'_0 \cup \gamma', \xi_{|N'}),
\]
where \(EH(N', \gamma'_0 \cup \gamma', \xi_{|N'}) \in SFH(-N', -\gamma'_0 \cup -\gamma')\) is the contact invariant of \(\xi_{|N'},\) constructed using a partial open book decomposition, as defined by Honda, Kazez, and Matić [HKM09]. We describe the invariant \(EH(N', \gamma'_0 \cup \gamma', \xi_{|N'})\) in more detail in Section 4, when we prove some properties of the gluing map.

We now define the contact gluing map as the composition
\[
(7)
\Phi_{\xi, C} := C_{h_\xi} \circ \cdots \circ C_{h_1} \circ (\psi_\ast \otimes \id) \circ \Phi_{\cup N'},
\]
where \(C_{h_i}\) is the map induced by the contact handle \(h_i,\) as in Section 3.2.

### 3.4. Invariance of the contact gluing map

In this section, we prove the following:

**Theorem 3.11.** The contact gluing map \(\Phi_{\xi, C}\) is independent of the contact cell decomposition \(C.\)

As a first step, the following lemma is helpful:

**Lemma 3.12.** If \(h_1\) and \(h_2\) are two contact handles (of arbitrary index) that are disjoint and are attached to \((M, \gamma),\) then
\[
C_{h_1} \circ C_{h_2} = C_{h_2} \circ C_{h_1}.
\]

**Proof.** The proof follows by analyzing the formulas for the two maps. If one of \(h_1\) and \(h_2\) is a 0-handle or a 1-handle, then the statement is obvious. Let us consider the case when \(h_1\) and \(h_2\) are disjoint 2-handles. Let us recall how the maps \(C_{h_i}\) are defined, for \(i \in \{1, 2\}.\) We first attach a band \(B_i\) to the boundary of a sutured diagram for \((-M, -\gamma).\) The band \(B_i\) is attached where the attaching circle of \(h_i\) intersects \(\gamma \subseteq \partial M.\) We pick curves \(\alpha_i\) and \(\beta_i,\) according to the attaching circle of \(h_i,\) that intersect at a single point \(c_i\) in \(B_i.\) The map \(C_{h_i}\) is defined by \(C_{h_i}(x) = x \times c_i.\) Since the handles \(h_1\) and \(h_2\) are disjoint, the bands \(B_1\) and \(B_2\) can be chosen to be disjoint, and the curves \(\alpha_i\) and \(\beta_i\) can be assumed to not intersect the band \(B_j\) when \(i \neq j.\) Hence, it follows that we can use the curves \(\alpha_1, \beta_1, \alpha_2,\) and \(\beta_2\) to compute both compositions \(C_{h_2} \circ C_{h_1}\) and \(C_{h_2} \circ C_{h_1}\), and since the formulas for the two compositions clearly agree, we conclude that \(C_{h_2} \circ C_{h_1} = C_{h_1} \circ C_{h_2}.\) In a similar manner, one can show that the same formula holds if at least one of \(h_1\) and \(h_2\) is a 3-handle. \(\square\)

**Proof of Theorem 3.11.** Independence of the relative ordering of cells of the same index follows from Lemma 3.12, so it is sufficient to check invariance under the moves in Proposition 3.6.

Invariance under Move (1) (isotopy). If \(C\) and \(C'\) are two contact cell decompositions that differ by Move (1), then the maps \(\Phi_{\xi, C}\) and \(\Phi_{\xi, C'}\) differ by post-composition with a map \(\phi_*: SFH(M', \gamma') \to SFH(M', \gamma')\) for a diffeomorphism \(\phi: M' \to M'\) that is isotopic to the identity relative to \(\partial M'.\) By naturality of sutured Floer homology [JT12, Theorem 1.9], the map \(\phi_*\) is the identity.

We now consider Move (2) (index 0/1 cell cancellation). We need to check that subdividing an edge of \(\Gamma,\) or adding a Legendrian edge \(\lambda\) to \(\Gamma\) that meets \(\Gamma\) at a single vertex and intersects none of the other cells does not change the map. Let us first consider subdivision. The contact 0-handle map adds a disk to the Heegaard surface with no new \(\alpha\) or \(\beta\) curves, and the contact 1-handle map attaches a band to the boundary of the Heegaard surface with no new \(\alpha\) or \(\beta\) curves. When we subdivide a Legendrian edge into two Legendrian edges that meet at a single vertex, the contact handle map changes by first adding a disk to \(S\) (the 0-handle map), followed by two bands, each of which has one foot on the new disk, and another foot at a foot of the original band. Clearly, the induced maps agree. Invariance under adding a Legendrian edge \(\lambda\) that intersects \(\Gamma\) at a single vertex follows similarly.
Invariance under Move (3) (index 1/2 contact cell cancellation) can be verified by a model computation, as we now describe. The computation is summarized in Figure 3.7. Recall that, by definition of an index 1/2 contact cell cancellation, the original Legendrian skeleton $\Gamma$ is modified by adding an additional Legendrian edge $\lambda$, and the new 2-cell intersects $S \cup \partial N(\Gamma) \cup S'$ (where $S$ and $S'$ denote the barrier surfaces of $C$) along an arc that does not intersect the dividing set. For the sake of demonstration, let us assume that the new 2-cell intersects $S \cup \partial N(\Gamma) \cup S'$ along $R_+$. If $(\Sigma, \alpha, \beta)$ is a sutured Heegaard diagram for $M \cup N \cup N'(\Gamma) \cup N'$, the effect of the 1-cell corresponding to $\lambda$ is to attach a band $B$ to the boundary of $\bar{\Sigma}$. The attaching cycle of the 2-cell intersects the dividing set of $\partial N(\lambda)$ at two points. On the level of diagrams, the effect is to attach a band $B'$ to $B$, and add new curves $\alpha_0$ and $\beta_0$, as in Figure 3.7. Note that the effect of adding $B$ and $B'$ to $\bar{\Sigma}$ is to attach a tube to two points along the interior of $\bar{\Sigma}$ near $\partial \Sigma$, and the curve $\beta_0$ is a meridian of the tube, while the curve $\alpha_0$ is the concatenation of a longitude of the tube with a path on $\Sigma$ between the two ends of the tube. The curves $\alpha_0$ and $\beta_0$ intersect at a single point $c$, and the composition of the contact 1-handle and 2-handle maps is

$$(C_{h2} \circ C_{h1})(x) = x \times c.$$ 

This is the compound stabilization map from Section 2.2. By Proposition 2.2, this is equal to the transition map from the naturality of sutured Floer homology.

Finally, we consider invariance under Move (4) (index 2/3 contact cell cancellation). This is a similar model computation to index 1/2 cell cancellation. In the case of a contact 2/3 cell cancellation, we add a convex 2-cell $D$ to the decomposition that has Legendrian boundary and $tb(\partial D) = -1$, and such that $\partial D$ intersects the dividing set on $S \cup \partial N(\Gamma) \cup S'$ exactly twice. By definition of a contact cell decomposition, the disk $D$ cuts one of the contact 3-cells in our decomposition into two contact 3-cells. Hence, the affect on the contact gluing map from Equation (7) is to insert a contact 2-handle map $C_{h2}$, followed by a contact 3-handle map $C_{h3}$. The composition is easily seen to be a diffeomorphism map, as demonstrated in Figure 3.8.
Indeed, for notational simplicity, assume that $h^2$ is attached to $(M, \gamma)$. Let $(M', \gamma')$ be the result of attaching $h^2$ to $(M, \gamma)$, and $(M'', \gamma'')$ the result of attaching $h^3$ to $(M', \gamma')$. Given a diagram $(\Sigma, \alpha, \beta)$ of $(M, \gamma)$, we get a diagram of $(M', \gamma')$ by attaching a band $B$ to a component of $\partial \Sigma$, and add curves $\alpha'$ to $\alpha$ and $\beta'$ to $\beta$ parallel to the suture $\gamma_{S}$ along which $h^3$ is attached, and such that $\alpha' \cap \beta' \cap B = \{c\}$. Furthermore, $|\alpha' \cap \beta'| = 2$. For $x \in T_{\alpha} \cap T_{\beta}$, we have $C_{h^2}(x) = x \times c$. By construction, in $(\bar{\Sigma}, \alpha \cup \{\alpha'\}, \beta \cup \{\beta'\})$, the bigons between the two points of $\alpha' \cap \beta'$ go to $c$ from the other intersection point, hence $c$ has the smaller relative grading. It follows that $(C_{h^3} \circ C_{h^2})(x) = C_{h^3}(x \times c) = x$. □

Figure 3.8. The composition of a contact 2-handle map followed by a contact 3-handle map, for a canceling pair of contact 2- and 3-cells

4. CONTACT HANDLES AND PARTIAL OPEN BOOK DECOMPOSITIONS

4.1. Partial open book decompositions and sutured Heegaard diagrams. In order to prove basic properties of the gluing map, and to compare our construction to the one due to Honda, Kazez, and Matić [HKM08], we need the following definition from [HKM09, Section 1.2]:

**Definition 4.1.** A *partial open book decomposition* $(S, P, h)$ is a triple consisting of a compact, connected, oriented surface $S$ with non-empty boundary, a proper compact subsurface $P \subseteq S$ (the *page*), such that $S$ is obtained from $\text{cl}(S \setminus P)$ by successively attaching 1-handles, and a smooth embedding $h : P \to S$ (the *monodromy*) such that $h|_{\partial S \cap P} = \text{id}$.

An abstract (non-embedded) partial open book decomposition $(S, P, h)$ defines a sutured manifold $(M, \gamma)$ via the formula

$$M = (S \times [0, \frac{1}{2}] \cup P \times [\frac{1}{2}, 1]) / \sim_h,$$

where $\sim_h$ is the equivalence relation defined as

- $(x, t) \sim_h (x, t')$ if $x \in \partial S$ and $t, t' \in [0, \frac{1}{2}]$,
- $(x, t) \sim_h (x, t')$ if $x \in \partial P \cap \partial S$ and $t, t' \in [\frac{1}{2}, 1]$, and
- $(x, 1) \sim_h (h(x), 0)$ if $x \in P$.

The manifold $M$ contains the properly embedded surface

$$\Sigma := (S \times \{\frac{1}{2}\}) \cup (P \times \{\frac{3}{4}\}).$$

The curves $\gamma = \partial M \cap \Sigma$ divide $\partial M$ into two subsurfaces that meet along $\gamma$. Furthermore, $(M, \gamma)$ is a balanced sutured manifold and $\Sigma$ is a sutured Heegaard surface for $(M, \gamma)$. Using our orientation
collections of pairwise disjoint, properly embedded arcs \( \alpha \) such that

\[
\alpha = (a_1 \cup \cdots \cup a_k)
\]

with ends on \( P \cap \partial S \) such that \( P \setminus (a_1 \cup \cdots \cup a_k) \) deformation retracts onto \( \partial P \setminus \partial S \) (or equivalently, \( S \setminus (a_1 \cup \cdots \cup a_k) \) deformation retracts onto \( c(S \setminus P) \)).

**Definition 4.2.** If \( (S, P, h) \) is a partial open book decomposition, a basis of arcs for \( P \) in \( S \) is a collection of pairwise disjoint, properly embedded arcs \( a = \{a_1, \ldots, a_k\} \) on \( P \) with ends on \( P \cap \partial S \) such that \( P \setminus (a_1 \cup \cdots \cup a_k) \) deformation retracts onto \( \partial P \setminus \partial S \).

Given a basis of arcs for \( P \) in \( S \), we can construct attaching curves \( \alpha \) and \( \beta \) on \( \Sigma \) that make \( (\Sigma, \alpha, \beta) \) a Heegaard diagram for \( (M, \gamma) \). Let \( b_i \) be an isotopic copy of \( a_i \) obtained by moving the ends of \( a_i \) along \( \partial \Sigma \) in the positive direction such that \( a_i \cap b_j = \delta_{ij} \). Then set

\[
\alpha_i = (a_i \times \{1\}) \cup (a_i \times \{2\}) \quad \text{and} \quad \beta_i = (b_i \times \{1\}) \cup (h(b_i) \times \{1\}).
\]

A partial open book decomposition \( (S, P, h) \) determines a unique contact structure \( \xi \) on \( (M, \gamma) \), up to equivalence, as follows; see [OE09, Proposition 1.2]. Let

\[
U_1 := \left( S \times [0, \frac{1}{2}] \right)/\sim_1 \quad \text{and} \quad U_2 := \left( P \times \left[ \frac{1}{2}, 1 \right] \right)/\sim_2,
\]

where \( \sim_1 \) and \( \sim_2 \) are the relations defined by

\[ (x, t) \sim_1 (x, t') \quad \text{if} \quad x \in \partial S, \quad \text{and} \quad (x, t) \sim_2 (x, t') \quad \text{if} \quad x \in \partial P \cap \partial S. \]

Then \( \xi|_{U_1} \) is the unique tight contact structure on \( U_1 \) with dividing set \( \partial S \times \{\frac{1}{2}\} \), and \( \xi|_{U_2} \) is the unique tight contact structure on \( U_2 \) with dividing set \( \partial P \times \{\frac{1}{2}\} \). Hence, we say that \( (S, P, h) \) is a partial open book decomposition of the contact 3-manifold \( (M, \gamma, \xi) \) if we are given a contactomorphism between the contact 3-manifold defined by \( (S, P, h) \) and \( (M, \gamma, \xi) \).

Given a partial open book decomposition \( (S, P, h) \) of \( (M, \gamma, \xi) \) and a basis of arcs \( a = \{a_1, \ldots, a_k\} \), let \( x_i = a_i \cap b_i \) for \( i \in \{1, \ldots, k\} \) and \( x_\xi = x_1 \times \cdots \times x_k \). Then Honda, Kazez, and Matić [HKM09] showed that \( x_\xi \) is a cycle whose homology class \( EH(M, \gamma, \xi) \in SFH(\gamma) \) is independent of the choice of partial open book and basis of arcs, and is hence an invariant of \( \xi \).

### 4.2. Partial open books and contact handles

Contact handle decompositions were defined by Giroux [Gir91]. In this section, we describe some useful relations between contact handle decompositions and the Honda–Kazez–Matić definition of a partial open book; compare [OE09].

Given a partial open book decomposition \( (S, P, h) \) for the contact sutured manifold \( (M, \gamma, \xi) \), we can naturally construct a contact handle decomposition of \( (M, \gamma, \xi) \) with no 3-handles, as follows. Recall that the manifold \( M \) is obtained by gluing the contact handlebodies \( U_1 \) and \( U_2 \) together, as described in Section 4.1, along a portion of their boundaries, using the map \( h \).

We start by constructing a contact handle decomposition of \( U_1 \) consisting of only 0-handles and 1-handles. Such a description is obtained by giving the surface \( S \) a decomposition into 2-dimensional 0-handles and 1-handles. Next, we extend this decomposition to a contact handle decomposition of all of \( M \). To do this, pick a basis of arcs \( a \) for \( P \) in \( S \). The closed curves

\[ l_i := (a_i \times \{1\}) \cup (h(a_i) \times \{0\}) \]

bound disks in \( U_2 \) that intersect the dividing set \( \partial S \times \{0\} \) of \( \partial U_1 \) exactly twice. By perturbing \( U_1 \) and \( U_2 \) slightly inside \( M \), we can assume that the curves \( l_i \) are Legendrian. As the curve \( l_i \) intersects the dividing set of \( \partial U_1 \) exactly twice, it follows that they bound convex disks with \( \text{tb} = -1 \), and that neighborhoods of these convex disks are contact 2-handles. Furthermore, after attaching these contact 2-handles, we obtain the sutured manifold \( M \).

In the opposite direction, given a contact handle decomposition \( H \) of \( (M, \gamma, \xi) \) with handles ordered with nondecreasing index and no 3-handles, viewed as a cobordism from \( \emptyset \) to \( \partial M \), we can construct a partial open book decomposition \( (S, P, h) \), as follows. The handlebody \( U_1 \) is the union of the 0- and 1-handles. If \( \gamma_1 \) is the dividing set of \( \xi \) on \( \partial U_1 \), then \( (U_1, \gamma_1) \) is diffeomorphic to the product sutured manifold \( (S \times [0, \frac{1}{2}])//\sim \times \partial S \times \{\frac{1}{2}\}) \), where \( S := R_+ (\gamma_1) \subseteq \partial U_1 \). Let \( U_2 \) be the union of the 2-handles, and let \( \gamma_2 \) be the dividing set of \( \xi \) on \( \partial U_2 \). Then \( (U_2, \gamma_2) \) is a product sutured manifold of the form \( (P \times [\frac{1}{2}, 1])//\sim \times \partial P \times \{\frac{1}{2}\}) \) for \( P := U_2 \cap S \). We finally set \( h \) to be \( \pi_1 \circ \pi_2 : P \to S \), where \( \pi_1 : S \times [0, \frac{1}{2}] \to S \) and \( \pi_2 : P \times [\frac{1}{2}, 1] \to P \times \{1\} \subseteq R_-(\gamma_1) \) are the projections.
Lemma 4.3. Let $h_1, \ldots, h_n$ be the handles of a contact handle decomposition $H$ of $(M, \gamma, \xi)$, ordered such that their indices are nonincreasing. If there are no 3-handles, then

$$EH(M, \gamma, \xi) = (C_{h_1} \circ \cdots \circ C_{h_n})(1) \in SFH(-M, -\gamma),$$

where $1 \in F_2 \cong SFH(0)$.

Proof. Let $(S, P, h)$ be the partial open book corresponding to $H$, as above. Suppose that $h_1, \ldots, h_k$ are the 2-handles. On the level of diagrams, $C_{h_{k+1}} \circ \cdots \circ C_{h_n}$ corresponds to adding a disk for each 0-handle, and a band for each 1-handle. The union of these is $S \times \{ \frac{1}{2} \}$. Adding a 2-handle $h_i$ for $i \in \{1, \ldots, k\}$ corresponds to attaching a band $B_i$ to $S$, and adding curves $\alpha_i$ and $\beta_i$. Then $B_1 \cup \cdots \cup B_k = P \times \{ \frac{1}{2} \}$, and $\alpha_i$ and $\beta_i$ are obtained from the basis of arcs $\{a_1, \ldots, a_k\}$ as described in Section 4.1, where $a_i$ is the core of $B_i$. Let $x_i = \alpha_i \cap \beta_i \cap B_i$ for $i \in \{1, \ldots, k\}$. Then the element $(C_{h_1} \circ \cdots \circ C_{h_n})(1) = x_1 \times \cdots \times x_k$ tautologically agrees with the cycle $x_k$ representing $EH(M, \gamma, \xi)$, as defined by Honda, Kazez, and Matić [HKM09] for the partial open book decomposition $(S, P, h)$ and the basis of arcs $\{a_1, \ldots, a_k\}$.

We now describe the effect of attaching a single contact handle on the level of partial open books. We begin with the effect of attaching a contact 1-handle:

Lemma 4.4. Suppose that we obtain the contact sutured manifold $(M', \gamma', \xi')$ from $(M, \gamma, \xi)$ by attaching a contact 1-handle $h^1$. Let $(S, P, h)$ be a partial open book decomposition for the contact structure $\xi$ on $(M, \gamma)$. A partial open book decomposition $(S', P', h')$ for the contact structure $\xi'$ on $(M', \gamma')$ can be obtained by setting

1. $S' = S \cup B$, where $B$ is a band attached to $\partial S$,
2. $P' = P$, and
3. $h' = \iota_{S} \circ h$.

Proof. First, we isotope the partial open book $(S, P, h)$ in $M$ such that the attaching sphere of $h^1$ lies in $\partial S \setminus P$. We set $S' = S \cup B$, where the band $B \subseteq h^1$ has boundary equal to the dividing set on $h^1$. Then

$$(S' \times [0, \frac{1}{2}] \cup P \times [\frac{1}{2}, 1]) / \sim_{h'} = (S \times [0, \frac{1}{2}] \cup P \times [\frac{1}{2}, 1]) / \sim_{h'} \cup (B \times [0, \frac{1}{2}]) / \sim_{h'}.$$

Furthermore,

$$(S \times [0, \frac{1}{2}] \cup P \times [\frac{1}{2}, 1]) / \sim_{h'} = (S \times [0, \frac{1}{2}] \cup P \times [\frac{1}{2}, 1]) / \sim h = M$$

and $(B \times [0, \frac{1}{2}]) / \sim_{h'} = h^1$.

We now consider the effect of attaching a contact 2-handle $h^2$ to $(M, \gamma)$. Let $\xi'$ denote the contact structure on $M \cup h^2$ with dividing set $\gamma'$, obtained by gluing the tight contact structure $\xi_2$ on $h^2$ to $\xi$. Let $p_1, p_2 \in \gamma$ denote the two points of intersection of the attaching circle of $h^2$ with $\gamma$. Furthermore, the attaching circle of $h^2$ consists of a path $\lambda_+$ in $R_+(\gamma)$ from $p_1$ to $p_2$, concatenated with the reverse of a path $\lambda_-$ in $R_-(\gamma)$ from $p_1$ to $p_2$. We can isotope the partial open book decomposition $(S, P, h)$ such that $p_1$ and $p_2$ lie in $\partial S \setminus P$. Since we have identifications

$S \setminus P \cong R_+(M)$ and $S \setminus h(P) \cong R_-(M),$

we can view $\lambda_+$ as a path in $S \setminus P$, and $\lambda_-$ as a path in $S \setminus h(P)$. Using the above notation, we are now prepared to describe a partial open book decomposition for the contact structure $\xi'$ on $(M \cup h^2, \gamma')$.

Lemma 4.5. Suppose that $h^2$ is a contact 2-handle attached to $(M, \gamma, \xi)$, and let $\xi'$ be the resulting contact structure on $(M \cup h^2, \gamma')$. Given a partial open book decomposition $(S, P, h)$ for $\xi$ on $(M, \gamma)$, a partial open book decomposition for $\xi'$ on $(M \cup h^2, \gamma')$ is given by $(S', P', h')$, where

1. $S' = S$,
2. $P' = P \cup N(\lambda_+)$,
3. $h'|_{P} = h$, and $h'$ maps $N(\lambda_+)$ to $N(\lambda_-)$. 

Proof. Using equation (8), we write $M$ as the union of the subsets $U_1 = (S \times [0, \frac{1}{2}]) / \sim_1$ and $U_2 = (P \times [\frac{1}{2}, 1]) / \sim_2$. We note that

$$(P' \times [\frac{1}{2}, 1]) / \sim_2 = (P \times [\frac{1}{2}, 1]) / \sim_2 \cup (N(\lambda_+) \times [\frac{1}{2}, 1]) / \sim_2.$$  

Let $U'_2 = (P' \times [\frac{1}{2}, 1]) / \sim_2$ and $h^2 = (N(\lambda_+) \times [\frac{1}{2}, 1]) / \sim_2$. Then $(U_1 \cup U'_2) / \sim_h$ is obtained from $(U_1 \cup U_2) / \sim_h$ by attaching $h^2$ along a neighborhood of the circle $(\lambda_+ \times [\frac{1}{2}, 1]) \cup (\lambda_- \times \{0\})$. By definition of a partial open book decomposition, it follows that $(S', P', h')$ is a partial open book for $\xi'$ on $(M \cup h^2, \gamma')$.

\[\square\]

4.3. Positive stabilizations and contact handle cancellations. Honda, Kazez, and Matić [HKM09] extended the notion of positive stabilizations to partial open book decompositions, adapting Giroux’s construction [Gir02] for open books of closed manifolds. In this section, as an instructive example, we show how to interpret their construction in terms of canceling pairs of contact handles.

**Definition 4.6.** Suppose that $(S, P, h)$ is a partial open book decomposition of the contact 3-manifold $(M, \gamma, \xi)$, and suppose that $c$ is a properly embedded arc on $S$. Let $\delta c$ be a properly embedded arc on $S$. We claim that $(S, P, h)$ can be obtained from $(S', P', h')$ by inserting a pair of canceling index 1 and 2 contact handles. The new partial monodromy map $h' : P' \rightarrow S'$ is defined as

$$h' := R_\tau \circ (h \cup \text{id}_B),$$

where $R_\tau$ is a right-handed Dehn twist along $\tau$, with respect to the orientation of $S$.

We now wish to relate positive stabilizations to canceling pairs of contact handles. As described in Section 4.2, the partial open book decomposition $(S, P, h)$, together with a choice of handle decomposition of the surface $S$ into 0-handles and 1-handles, as well as a basis of arcs $a$ for $P$, determine a contact handle decomposition of $(M, \gamma)$. We now show that the contact handle decomposition arising from a positive stabilization $(S', P', h')$ of $(S, P, h)$ can be obtained from a contact handle decomposition arising from $(S, P, h)$ by inserting a pair of canceling index 1 and 2 contact handles.

**Lemma 4.7.** Suppose $(S, P, h)$ is a partial open book decomposition of $(M, \gamma, \xi)$, and that $(S', P', h')$ is a positive stabilization of $(S, P, h)$ along a properly embedded arc $c \subseteq S$. Let $U_1$ and $U_2$ be the tight contact handlebodies defined in equation (8) associated to $(S, P, h)$, whose union is $M$. Consider a contact handle decomposition $H$ of $(M, \gamma, \xi)$ arising from the partial open book $(S, P, h)$ as in Section 4.2. Then a handle decomposition $H'$ arising from the positive stabilization $(S', P', h')$ can be obtained from $H$ by adding a pair of canceling index 1 and 2 contact handles between $U_1$ and $U_2$.

**Proof.** By concatenating the arc $c \times \{0\} \subseteq \partial U_1$ with $h^{-1}(c) \times \{\frac{3}{4}\} \subseteq U_2$, obtained by pushing $c \times \{0\}$ into $U_2$, we get the closed curve

$$q = (c \times \{0\}) \cup (h^{-1}(c) \times \{\frac{3}{4}\}).$$

Note that there is a convex disk $D \subseteq U_2$ with $\partial D = q$ and $tb(\partial D) = -1$, obtained by perturbing $h^{-1}(c) \times [\frac{1}{2}, 1]$. A neighborhood $h^2$ of $D$ is a 2-handle according to Remark 3.2. A neighborhood of the Legendrian arc $h^{-1}(c) \times \{\frac{3}{4}\}$ is a contact 1-handle $h^1$. The contact handles $h^1$ and $h^2$ cancel, and hence determine a new contact handle decomposition of $M$. We now claim that this handle decomposition is compatible with the partial open book decomposition $(S', P', h')$.

Define $U''_1 = U_1 \cup h^1$, and let $U''_2$ denote the closure of $M \setminus U''_1$. Note that, after attaching the contact 2-handle $h^2$ to $U''_1$, which cancels the 1-handle $h^1$, we are left with a tight contact handlebody that is isotopic to $U_1$ in $M$.

As in Section 4.2, after we insert the canceling handles $h^1$ and $h^2$ between $U_1$ and $U_2$, the new collection of contact 0-handles and 1-handles in our handle decomposition determines a new surface $S''$, and the new 2-handles and their attaching circles determine a page $P'' \subseteq S''$ and a monodromy map $h''$, which we describe in more detail below. We claim that $(S'', P'', h'')$ is the positive stabilization $(S', P', h')$ of $(S, P, h)$ along the arc $c \subseteq S$. 

The surface $S''$ is obtained by attaching the characteristic surface of the handle $h^1$ to $S$. The effect of this is to attach a band $B$ to $S$, and hence $S'' = S'$. We now describe how to obtain the page $P''$ from our new handle decomposition, and demonstrate that $P'' = P'$. Write $U''_t = (S' \times [0, \frac{1}{2}]) / \sim_1$, where $(x, t) \sim (x, t')$ whenever $x \in \partial S'$. The 2-handles of our new decomposition are attached along $\partial U''_t = S' \times \{0, \frac{1}{2}\}$. The way the attaching circles of the original 2-handles intersect $S \times \{\frac{1}{2}\} \subseteq S' \times \{\frac{1}{2}\}$ has not changed, since we attached the canceling contact 1- and 2-handle to a neighborhood of $S \times \{0\}$. The attaching circle of the new contact 2-handle $h^2$ intersects $S' \times \{\frac{1}{2}\}$ along the co-core of the band $B$. The page $P''$ is obtained by taking a neighborhood of the intersection of $S' \times \{\frac{1}{2}\}$ with the attaching circles of the 2-handles, and hence $P'' = P \cup B$, which coincides with $P'$.

Finally, we consider the monodromy $h'' : P' \to S'$ induced by our new contact handle decomposition. Let $l$ be the attaching circle of a contact 2-handle. The map $h''$ is determined, up to isotopy, by sending $l \cap (S' \times \{\frac{1}{2}\})$ to $l \cap (S' \times \{0\})$. First note that the intersection of the attaching circle of $h^2$ with $S' \times \{0\}$ is the co-core of $B$, composed with a right-handed Dehn twist along the curve $\tau$, as shown in Figure 4.1.

We now describe the attaching circles of the remaining contact 2-handles. Note that, since the new contact 2-handle $h^2$ is attached to the union of $S \times \{0\}$ and the boundary of the contact 1-handle $h^1$, the images of the attaching circles of the original 2-handles on $S' \times \{0\}$ will not coincide with their original attaching circles on $S \times \{0\}$. Suppose that $l$ is the attaching circle of a contact 2-handle from our original contact handle decomposition. Let $a \subseteq S$ be such that $a \times \{\frac{1}{2}\} = l \cap (S \times \{\frac{1}{2}\})$, and let $b \subseteq S$ be such that $b \times \{0\} = l \cap (S \times \{0\})$. If $b \cap c = \emptyset$, the corresponding contact 2-handle in our new decomposition will still be attached along $(a \times \{\frac{1}{2}\}) \cup (b \times \{0\}) \subseteq \partial U''$. However, if $b \cap c \neq \emptyset$, then the contact 2-handle corresponding to $l$ will be attached along $(a \times \{\frac{1}{2}\}) \cup (b' \times \{0\})$ for some other curve $b' \subseteq S' \times \{0\}$. This is shown in Figure 4.1. It is evident that $b'$ is the result of a right-handed Dehn twist around the curve $\tau$ on $S'$, obtained by concatenating $c$ with the core of $B$. As $h''$ is determined, up to isotopy, by the images of the attaching curves of the contact 2-handles on $S' \times \{\frac{1}{2}\}$, it follows that $h'' = R_\tau \circ h$, which is $h'$.

\section{Properties of the gluing map}  

\subsection{The gluing map for $I$-invariant contact structures} In this section, we prove that gluing on a copy of $\partial M \times I$ induces the identity map, in a sense that we now describe. Suppose that $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$ and $\xi$ is a contact structure on $M' \setminus \text{int}(M)$. Furthermore, suppose that there is a Morse function $f$ on $M \setminus \text{int}(M)$ such that $f|_{\partial M} \equiv 0$, $f|_{\partial M'} \equiv 1$, $f$ has no critical points, and there is a contact vector field $\nu$ such that $\nu(f) > 0$. Furthermore, suppose that the dividing set of $\xi$ on $\partial M \cup \partial M'$, with respect to $\nu$, is equal to $\gamma \cup \gamma'$. The vector field $\nu$ induces a diffeomorphism

$$\phi^\nu : (-M, -\gamma) \to (-M', -\gamma'),$$

which is well-defined up to isotopy, relative to $\partial M$. The induced diffeomorphism map

$$\phi^\nu_* : \text{SFH}(-M, -\gamma) \to \text{SFH}(-M', -\gamma')$$

has a simple description. If $((\tilde{\Sigma}, \alpha, \beta))$ is a sutured Heegaard diagram for $(-M, -\gamma)$, then we can construct a Heegaard diagram for $(-M', -\gamma')$ as

$$(\tilde{\Sigma} \cup \tilde{A}, \alpha, \beta),$$

where $A$ is the characteristic surface of $\xi$, with respect to $\nu$; i.e.,

$$A := \{ p \in M' \setminus \text{int}(M) : \nu_p \in \xi_p \}.$$

The surface $A$ is a collection of annuli. With respect to these two diagrams, the diffeomorphism map takes the form

$$\phi^\nu_*(x) = x.$$

Our gluing map satisfies the following analogue of [HKM08, Theorem 6.1].
Figure 4.1. Inducing a positive stabilization of a partial open book decomposition with a canceling pair of contact 1- and 2-handles. On the top-left is the surface $S$ and the arc $c$. On the top-right is the stabilized surface $S'$. Middle-left is the tight contact handlebody $U_1$. Middle-right is the contact handle body $U''_1$ obtained by adding a contact 1-handle $h^1$ to $U_1$ and, in green, a canceling contact 2-handle $h^2$. In the bottom row, we show how the partial monodromy map $h$ changes. On the bottom-left, the curve $b$ is the intersection of the attaching circle of a contact 2-handle with $S \times \{0\}$, before the canceling 1- and 2-handles are attached. On the bottom-right, we show the image $b'$ of the attaching curve of the same contact 2-handle on $S'$ after inserting a canceling 1-handle and 2-handle. Note that $b'$ is obtained from $b$ by applying a right-handed Dehn twist about the curve $\tau \subseteq S'$. This looks like a left-handed Dehn twist in our picture; however, the orientation of $S$ is clockwise with respect to the page, because we are attaching the contact 1-handle and 2-handle to $S \times \{0\}$ (i.e., the picture is of $(S \times [0,1/2]) / \sim_1$ turned “upside down”).

**Proposition 5.1.** Suppose that $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$ and $\xi$ is a contact structure on $M' \setminus \text{int}(M)$. Furthermore, suppose that there is a Morse function $f$ on $M' \setminus \text{int}(M)$ such that $f|_{\partial M} \equiv 0$, $f|_{\partial M'} \equiv 1$, and there is contact vector field $\nu$ such that $\nu(f) > 0$. Under the above assumptions, the contact gluing map $\Phi_\xi$ satisfies

$$\Phi_\xi = \phi^\nu,$$

where $\phi^\nu : (-M, -\gamma) \to (-M', -\gamma')$ is the diffeomorphism described above.

Before we begin with the proof, we need the following definition regarding sutured cell decompositions of surfaces:

**Definition 5.2.** We say that the sutured cell decompositions $D = (B_1, \ldots, B_n, \lambda_1, \ldots, \lambda_m)$ and $D^* = (B'_1, \ldots, B'_n, \lambda'_1, \ldots, \lambda'_m)$ of the surface with divides $(F, \gamma)$ are dual if the following hold:

1. $B_i \cap B'_j = \emptyset$ for all $i$ and $j$.
2. Each component of $F \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m)$ intersects $\gamma$ in a single arc, and contains a single fattened 0-cell $B'_k$. The same statement holds with the roles of $D$ and $D^*$ reversed.
3. $|\lambda_i \cap \lambda'_j| = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta.

A sutured cell decomposition $D$ that admits a dual is called dualizable.

**Remark 5.3.** Not all sutured cell decompositions $D$ admit dual cell decompositions. For example, if the intersection of a component of $F \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m)$ with $\gamma$ is disconnected, then it is not dualizable.
Remark 5.4. Every surface with divides $(F, \gamma)$ such that $\pi_0(\gamma) \to \pi_0(F)$ is surjective admits a dualizable cell decomposition. To construct one, we pick sets of arcs $A_0$ and $A_1$ along $\gamma$, such that the arcs in $A_0 \cup A_1$ are pairwise disjoint, and $A_0$ and $A_1$ each contain exactly one arc in every component of $\gamma$. We can then view $R_+(\gamma)$ as a cobordism with product boundary from $A_0$ to $A_1$. Hence, we can cut $R_+(-\gamma)$ along a collection of arcs $\lambda_1, \ldots, \lambda_k$ with boundary on $A_0$ such that we are left with a set of disks. We pick another collection of arcs $\lambda_{k+1}, \ldots, \lambda_m$ in $R_-(\gamma)$ with boundary on $A_0$ that cut $R_-(-\gamma)$ into disks. We pick a mutated cell decomposition with a fattened 0-cell $B_i$ along each arc of $A_0$, and we use the collection of arcs $\lambda_1, \ldots, \lambda_m$. The 0-cells $B_1, \ldots, B_n$ and the arcs $\lambda_1, \ldots, \lambda_m$ determine a handle decomposition of $F$, and the dual mutated cell decomposition is obtained by using the dual handle decomposition.

Proof of Proposition 5.1. Let $D$ and $D^*$ be dual mutated cell decompositions of $(\partial M, \gamma)$. Let $C$ be the product contact cell decomposition of $M' \setminus \text{int}(M)$ constructed from $D$ as in Example 3.5, with barrier surfaces $S$ and $S'$. Write $N$ and $N'$ for the collar neighborhoods of $\partial M$ and $\partial M'$ in $M' \setminus \text{int}(M)$ that are bounded by $S$ and $S'$, respectively. We denote the dividing set on $S'$ by $\gamma_{0'}^\ast$.

For each fattened 0-cell $B_i$ of $D$, there is a contact 1-handle $h_{B_i}$ of $C$. For each arc $\lambda$ of $D$, there is a contact 2-handle $h_{\lambda}$ of $C$. For each 2-cell $c$ of $D$, there is a contact 3-handle $h_c$ of $C$. Let $h_1, \ldots, h_n$ be an enumeration of these handles with nondecreasing index. Using equation (7), the gluing map is defined as

$$\Phi(x) := (C_{h_n} \circ \cdots \circ C_{h_1}) (\psi''(x) \otimes EH(N', \gamma' \sqcup \gamma_{0'}, \xi_{N'})) \quad (9)$$

The element $EH(N', \gamma' \sqcup \gamma_{0'}, \xi_{N'})$ is defined using a partial open book decomposition of $(N', \gamma' \sqcup \gamma_{0'}, \xi_{N'})$. In Section 4.2, we described how a contact handle decomposition of $N'$ with no 3-handles, viewed as a cobordism from 0 to $\partial N'$, can be used to construct a partial open book. By turning around our construction of a contact cell decomposition $C$ from $D$, we can also construct a contact handle decomposition of $N'$ from a sutured cell decomposition of $(\partial M, \gamma)$; however, the indices of the corresponding handles will be different. For our argument to work, we will actually consider the handle decomposition $H'$ of $N'$ induced by the dual sutured cell decomposition $D^*$. For each fattened 0-cell $B_i$ of $D^*$, there is a contact 2-handle $h_{B_i}$ of $H'$. For each arc $\lambda'$ of $D^*$, there is a contact 1-handle $h_{\lambda'}$ of $H'$.

By the above construction, there is a correspondence between the $k$-cells of $D^*$ and the $(2-k)$-handles of $H'$. By construction, there is a correspondence between the $k$-cells of $D$ and the $(k+1)$-handles of $C$. By definition, there is a correspondence between the $k$-cells of $D$ and the $(2-k)$-cells of $D^*$. Combining these, we get a correspondence between the $k$-handles of $H'$ and the $(k+1)$-handles of $C$. Let $h_1', \ldots, h_n'$ be an enumeration of the contact handles of $H'$ such that $h_i'$ corresponds to $h_i$ under the above correspondence. By Lemma 4.3,

$$EH(N', \gamma' \sqcup \gamma_{0'}, \xi) = (C_{h_n'} \circ \cdots \circ C_{h_1'}) (1) \quad (10)$$

where 1 is the generator of $SFH(\emptyset) \cong \mathbb{F}_2$. Using Lemma 3.12, we can commute contact handle maps for disjoint contact handles, so we can rearrange equation (9) as

$$\Phi(x) \equiv \left((C_{h_n} \circ C_{h_{n-1}}) \circ \cdots \circ (C_{h_1} \circ C_{h_1'})\right) (\psi''(x)) \quad (1)$$

The contact handles $h_i$ and $h_i'$ do not always form a canceling pair in the sense of Proposition 3.6. However, they are close enough to a canceling pair to allow us to reduce the above composition to the diffeomorphism map $\psi''$ by performing a sequence of handle cancellations and isotopies, as we now describe.

Let us first consider the case when $h_i$ and $h_i'$ correspond to a 0-cell $B$ of $D$. Under the previously described correspondence, there is a 2-cell $B^*$ of $D^*$ that $B$ corresponds to. Using the previous notation, we have $h_i = h_B$ and $h_i' = h_{B^*}$. Then $h_B$ and $h_{B^*}$ form a pair of canceling index 0 and 1 contact handles; see Figure 5.1. Hence $C_{h_B} \circ C_{h_{B^*}}$ is a diffeomorphism map by the computation in the proof of Theorem 3.11. We now consider the case when $h_i$ and $h_i'$ correspond to a 1-cell $\lambda$ of $D$. Let $\lambda^*$ denote the corresponding dual arc of $D^*$. Let $B_1$ and $B_2$ be the fattened 0-cells of $D$ at $\partial \lambda$ (note that we do not exclude the possibility that $B_1 = B_2$). Let $h_{B_1}$ and $h_{B_2}$ denote the corresponding 1-handles
of $C$. The arc $\lambda$ corresponds to a 2-handle $h_\lambda$ of $C$. Let $B_1^*$ and $B_2^*$ denote the 2-cells of $D^*$ that correspond to $B_1$ and $B_2$, respectively. The 2-cells $B_1^*$ and $B_2^*$ correspond to 0-handles $h_{B_1}$ and $h_{B_2}$ of $H'$. The dual arc $\lambda^*$ of $D^*$ induces a contact 1-handle of $H'$. As described above, the handles $h_{B_1}$ and $h_{B_2}$ form a canceling pair of index 1 and 2 contact handles. After canceling these two handles, the handles $h_\lambda$ and $h_{\lambda^*}$ do not quite form a canceling pair of index 1 and 2 handles in the sense of Proposition 3.6, because the attaching circle of the 2-handle $h_\lambda$ does not intersect the dividing set along $h_{\lambda^*}$. Instead, it intersects the dividing set near the feet of the 1-handle $h_{\lambda^*}$. This is shown in Figure 5.2. After performing an isotopy of $h_\lambda$, the handles $h_\lambda$ and $h_{\lambda^*}$ form a canceling pair of index 1 and 2 contact handles. Hence the composition $C_{h_i} \circ C_{h_i^*}$ induces a diffeomorphism map, by the computation in the proof Theorem 3.11.

Finally, we consider the case when $h_i$ and $h_i^*$ correspond to a 2-cell $c$ of $D$. Corresponding to $c$, there is a fattened 0-cell $c^*$ of $D^*$. The 2-cell $c$ induces a 3-handle $h_c = h_i$ of $C$, as well as a 2-handle $h_{c^*} = h_i^*$ of $H'$. The handles $h_c$ and $h_{c^*}$ form a canceling pair of index 2 and 3 contact handles. Hence, the composition $C_{h_i} \circ C_{h_i^*}$ is equal to a diffeomorphism map, by the same computation as in the proof of Theorem 3.11.

We have shown that $C_{h_i} \circ C_{h_i^*}$ is equal to a diffeomorphism map for every $i \in \{1, \ldots, n\}$, induced by cancelling the handles $h_i$ and $h_i^*$ that are stacked horizontally in the $\nu$-direction. It follows from equation (10) that $\Phi_{\xi}$ is equal to the diffeomorphism map $\phi_{\nu}^\ast$. \hfill $\Box$

### 5.2. Morse-type contact handles.

In Section 3.2, we defined maps for gluing a contact handle $h$ onto the boundary of a sutured manifold $(M, \gamma)$. We note that $M$ is not a sutured submanifold of $M \cup h$, so the Honda–Kazez–Matić framework does not assign a gluing map to the inclusion $M \subseteq M \cup h$. Nonetheless, there is a natural notion of contact handle that fits into the Honda–Kazez–Matić TQFT framework:

**Definition 5.5.** Suppose that $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$, and $\xi$ is a contact structure on $Z = M' \setminus \text{int}(M)$ with dividing set $\gamma \cup \gamma'$. We say that $(Z, \xi)$ is a **Morse-type contact handle of index $k$** if there is a contact vector field $\nu$ on $Z$ that points into $Z$ on $\partial M$ and out of $Z$ on $\partial M'$, as well as a decomposition $Z = Z_0 \cup h$, such that

1. $Z_0$ is diffeomorphic to $\partial M \times I$,
2. $\nu$ is non-vanishing on $Z_0$, points into $Z_0$ on $\partial M \times \{0\}$ and out of $Z_0$ on $\partial M \times \{1\}$, and each flowline of $\nu$ is an arc from $\partial M \times \{0\}$ to $\partial M \times \{1\}$,
3. $h$ is a topological 3-ball with piecewise smooth boundary, and $\xi$ is tight on $h$.

Furthermore, $h$ is a contact $k$-handle attached to $M \cup Z_0$, as in Definition 3.8, with corners smoothed.
Figure 5.2. A canceling pair of index 1 and 2 contact handles induced by a 1-cell $\lambda$ of $D$, and the dual arc $\lambda^*$ of $D^*$. The first picture shows the sutured cell decomposition $D$ near $\lambda$. The second shows the contact handles associated to the cells $B_1$, $B_2$, $B_1^*$, $B_2^*$, $\lambda$, and $\lambda^*$. The third picture is obtained by canceling $h_{B_i}$ against $h_{B_i^*}$ for $i \in \{1, 2\}$. After this cancellation and a small isotopy of $h_{\lambda}$, the handles $h_{\lambda}$ and $h_{\lambda^*}$ form a canceling pair of contact handles.

If $(M, \gamma)$ is a sutured submanifold of $(M', \gamma')$, and $(Z, \xi) = (M' \setminus \text{int}(M), \xi)$ is a Morse-type contact handle of index $k$ attached to $(M, \gamma)$, then we call a choice of contact vector field $\nu$ and decomposition $Z = Z_0 \cup h$ a parametrization of $(Z, \xi)$. Given a parametrization of $(Z, \xi)$, there is a natural candidate for the contact gluing map $\Phi_\xi$, namely

$$C_h \circ \phi_\nu^{\mid Z_0} : SFH(-M, -\gamma) \to SFH(-M', -\gamma').$$

In the above equation,

$$\phi_\nu^{\mid Z_0} : SFH(-M, -\gamma) \to SFH(-M \cup -Z_0, -\gamma_0)$$

is the diffeomorphism map induced by the vector field $\nu$, as discussed in Section 5.1, where $\gamma_0$ is the dividing set of $\xi$ on $\partial(M \cup Z_0)$. Furthermore,

$$C_h : SFH(-M \cup -Z_0, -\gamma_0) \to SFH(-M', -\gamma')$$

is the contact handle map, as defined in Section 3.2. Indeed, we will prove the following:
Proposition 5.6. Suppose \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\), and \((Z, \xi) = (M' \setminus \text{int}(M), \xi)\) is a Morse-type contact handle, with a parametrizing contact vector field \(\nu\) and decomposition \(Z = Z_0 \cup h\). Then the contact gluing map \(\Phi_\xi\) is equal to the composition \(C_h \circ \phi_*^{[z_0]}\).

Proof. The proof is essentially the same for all handle indices, so for definiteness we will focus on 2-handles. By assumption, the contact vector field \(\nu\) is non-vanishing on \(Z_0\), and on a collar neighborhood of \(\partial M'\). Let \(D \subseteq Z\) be a core of \(h\). Pick an incoming barrier surface \(S \subseteq Z_0\). Extend \(D\) down into \(Z_0\) such that \(\partial D \subseteq S\) is Legendrian with \(tb(\partial D) = -1\). Then we can perturb \(D\) while fixing \(\partial D\) such that it becomes convex. Let \(N\) denote the collar of \(\partial M\) bounded by \(S\), and let

\[ Z' := \text{cl}(Z \setminus (N \cup N(D))). \]

We can pick \(N(D)\) such that \(\nu\) is non-vanishing on \(Z'\). Using the flow of \(\nu\), one can construct a Morse function \(f\) on \(Z'\) that is 0 on \(\partial Z' \setminus \partial M'\) and 1 on \(\partial M'\), and such that \(\nu(f) > 0\).

The image of \(\partial N(D) \setminus S\) in \(\partial M'\) under the flow of \(\nu\) consists of two disks, \(D_1\) and \(D_2\). Pick a dualizable sutured cell decomposition \(\mathcal{D}\) of \((\partial M', \gamma')\) with no 0-cells or 1-cells that intersect \(D_1\) or \(D_2\). Let \(B_1, \ldots, B_m\) be the 0-cells of \(\mathcal{D}\), and let \(\lambda_1, \ldots, \lambda_n\) be the 1-cells. Write \(c_1, \ldots, c_k\) for the 2-cells. Adapting Example 3.5, after performing a \(C^0\) small isotopy of \(S\), and picking a barrier surface \(S'\) that bounds a collar neighborhood \(N'\) of \(\partial M'\), we can construct a contact cell decomposition \(\mathcal{C}\) of \(Z\) that has barrier surfaces \(S\) and \(S'\). The contact cell decomposition \(\mathcal{C}\) has no 0-cells, one 1-cell for each 0-cell \(B_1, \ldots, B_m\) of \(\mathcal{D}\), one 2-cell for each 1-cell \(\lambda_1, \ldots, \lambda_n\) of \(\mathcal{D}\), one 3-cell for each 2-cell \(c_1, \ldots, c_k\) of \(\mathcal{D}\), and also the 2-cell \(D\). Let us write \(h_1, \ldots, h_n\) for the handles induced by \(\mathcal{D}\), and write \(h_D\) for \(N(D)\), viewed as a contact 2-handle. By definition

\[ \Phi_\xi(x) = (C_{h_n} \circ \cdots \circ C_{h_1} \circ C_{h_0}) \left( \phi_*^{[z_0]}(x) \otimes EH(N', \xi|_{N'}) \right). \]

A dual sutured cell decomposition \(\mathcal{D}'\) of \((\partial M', \gamma')\) gives rise to a contact handle decomposition \(\mathcal{H}'\) of \((N', \xi|_{N'})\) starting at the empty sutured manifold. We can compute \(EH(N', \xi|_{N'})\) by applying Lemma 4.3 to \(\mathcal{H}'\). Exactly as in the proof of Proposition 5.1, the handles of \(\mathcal{H}'\) cancel \(h_1, \ldots, h_n\) pairwise, and we can reduce equation (11) to

\[ \left( \phi_*^{[z_0]} \circ C_{h_D} \circ \phi_*^{[z_0]} \right)(x). \]

Given the description of the diffeomorphism maps \(\phi_*^{[z_0]}\) and \(\phi_*^{[z_0]}\) from Section 5.1, the above expression is clearly equal to \(C_h \circ \phi_*^{[z_0]}\). \(\Box\)

5.3. Functoriality of the gluing map. We now show that the gluing map defined in this paper satisfies the functoriality property of the Honda–Kazez–Matić construction [HKM08, Proposition 6.2]. This property will be useful when we prove the equivalence of our construction with the Honda–Kazez–Matić construction.

Proposition 5.7. Suppose that we have a chain of contact submanifolds

\[ (M, \gamma) \subseteq (M', \gamma') \subseteq (M'', \gamma''), \]

as well as a contact structure \(\xi\) on \(M'' \setminus \text{int}(M)\) such that \(\partial M, \partial M', \text{ and } \partial M''\) are convex with dividing sets \(\gamma, \gamma', \text{ and } \gamma''\), respectively. Writing \(\epsilon'\) for \(\xi|_{M' \setminus \text{int}(M)}\) and \(\xi''\) for \(\xi|_{M'' \setminus \text{int}(M)}\), we have

\[ \Phi_\xi = \Phi_{\xi'} \circ \Phi_{\xi''}. \]

Proof. Define \(Z' := M' \setminus \text{int}(M)\) and \(Z'' := M'' \setminus \text{int}(M')\). Let \(\mathcal{C}'\) and \(\mathcal{C}''\) be contact cell decompositions of \((Z', \xi')\) and \((Z'', \xi'')\). Let us write \(h_1', \ldots, h_m'\) for the contact handles of \(\mathcal{C}'\), and \(h_1'', \ldots, h_m''\) for the contact handles of \(\mathcal{C}''\). Let \(\nu'\) and \(\nu''\) denote the contact vector fields chosen on the incoming ends of \(Z'\) and \(Z''\), let \(N_1\) and \(N_2\) denote the incoming layers of \(Z'\) and \(Z''\), and let \(N_1'\) and \(N_2'\) denote the outgoing layers, respectively, as described in Definition 3.3. By definition, the composition \(\Phi_{\xi'} \circ \Phi_{\xi''}\) is equal to

\[ (C_{h_m'} \circ \cdots \circ C_{h_1'}) \circ \Phi_{\text{int}(N_2)} \circ \phi_*^{[z_0]} \circ (C_{h_m'} \circ \cdots \circ C_{h_1'}) \circ \Phi_{\text{int}(N_1')} \circ \phi_*^{[z_0]} \circ (C_{h_m'} \circ \cdots \circ C_{h_1'}). \]
The map \( \Phi_{\cup N_2} \) is given by tensoring with \( EH(N'_i, \xi|N'_i) \), for \( i \in \{1, 2\} \). As in Lemma 4.3, the element \( \Phi_{\cup N_2} \) can be written as a composition of contact handle maps \( C_{h_1} \circ \cdots \circ C_{h_t} \), for a sequence of contact 0-, 1-, and 2-handles \( h_1, \ldots, h_t \). Hence, the composition in equation (12) can be written as

\[
(C_{h'_n} \circ \cdots \circ C_{h'_1}) \circ \Phi_{\cup N_2'} \circ \phi^{\nu'|N_2} \circ (C_{h'_n} \circ \cdots \circ C_{h'_1}) \circ (C_{h_0} \circ \cdots \circ C_{h_1}) \circ \phi^{\nu|N_1}.
\]

By picking \( \nu'' \) and \( N_2 \) appropriately, we can assume that the diffeomorphism \( \phi^{\nu''|N_2} : M' \to M' \cup N_2 \) is a contactomorphism on all of \( Z' \), and is the identity on \( \partial M \). Write \( h_k = \phi^{\nu''|N_2}(h_k) \) and \( h'_k = \phi^{\nu'|N_2}(h'_k) \). Using the diffeomorphism invariance of the contact handle maps, we can rewrite equation (13) as

\[
(C_{h'_n} \circ \cdots \circ C_{h'_1}) \circ \Phi_{\cup N_2'} \circ (C_{h'_n} \circ \cdots \circ C_{h'_1}) \circ (C_{h_0} \circ \cdots \circ C_{h_1}) \circ \phi^{\nu'|N_1}.
\]

The map \( \Phi_{\cup N_2} \) can be commuted with all the contact handle maps to the right of it, by Lemma 3.12. After possibly isotoping some of the remaining contact handles, we can apply Lemma 3.12 and reorder the handles such that they are attached with nondecreasing index. Furthermore, after isotoping some of the handles, we can assume that the handles in the above composition are induced by a contact cell decomposition (i.e., the 0-handles and 1-handles are induced by a Legendrian graph, and the 2-handles are induced by a sequence of convex disks with \( tb = -1 \) attached to a neighborhood of the graph and \( \partial((Z' \cup Z'') \setminus \text{int}(N_1 \cup N'_2)) \)). It follows that equation (14) is equal to \( \Phi_{\xi, C} \) for some contact cell decomposition \( C \) of \( (Z, \xi) \), completing the proof.

5.4. Equivalence with the Honda–Kazez–Matić construction. In this section, we prove that our construction of the gluing map from Section 3.3 is equivalent to the original construction due to Honda, Kazez, and Matić [HKM08]. We will write \( \Phi^{HKM}_\xi \) for the map defined using their construction.

Theorem 5.8. Suppose \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\) with no isolated components, and that \( \xi \) is a contact structure on \( M' \setminus \text{int}(M) \) with convex boundary and dividing set \( \gamma \cup \gamma' \). Then the Honda–Kazez–Matić gluing map \( \Phi^{HKM}_\xi \) is equal to the gluing map \( \Phi_\xi \) we defined in Section 3.3.

Proof. Using the composition law for both constructions of the gluing map (Proposition 5.7 and [HKM08, Proposition 6.2]), it is sufficient to show the claim when \( M' \setminus \text{int}(M) \) consists of a single Morse-type contact handle of index 0, 1, or 2.

For a Morse-type index 0 handle, the claim is straightforward. Write \( M' \setminus \text{int}(M) \) as \( Z_0 \cup h^0 \) where \( Z_0 \cong (I \times \partial M) \), and \( h^0 \) is a 3-ball. The contact structure \( \xi \) is the union of the \( I \)-invariant contact structure on \( Z_0 \) and the unique tight contact structure on \( h^0 \). Write \( \gamma_0 \) for the suture on \( M \cup Z_0 \), and suppose that \( \nu \) is a parametrizing contact vector field on \( Z_0 \).

Under the identification

\[
SFH(M \cup Z_0 \cup h^0, \gamma'_0 \cup \gamma_0) \cong SFH(M \cup Z_0, \gamma'_0) \otimes SFH(h^0, \gamma_0),
\]

where \( \gamma_0 \) consists of a single suture on \( h^0 \), both gluing maps

\[
\Phi_\xi, \Phi^{HKM}_\xi : SFH(M, \gamma) \to SFH(M \cup Z_0, \gamma'_0) \otimes SFH(h^0, \gamma_0)
\]

take the form

\[
x \mapsto \phi^\nu(x) \otimes EH(h^0, \gamma_0, \xi_0) = \phi^\nu(x) \otimes 1,
\]

where 1 denotes the generator of \( SFH(h^0, \gamma_0) \cong \mathbb{F}_2 \). This follows from [HKM08, Proposition 6.1] for \( \Phi^{HKM}_\xi \), and from Proposition 5.6 for our map \( \Phi_\xi \).

Before we consider index 1 and 2 Morse-type contact handles, we must first give a more detailed description of the construction Honda, Kazez, and Matić [HKM08]. The definition of the map \( \Phi^{HKM}_\xi \) uses the description of partial open books from [HKM09]. Given a contact sutured manifold \((M, \gamma, \xi)\), one picks a properly embedded Legendrian graph \( K \subseteq M \) that intersects \( \partial M \) along a collection of univalent vertices in \( \gamma \), such that \( M \setminus \text{int}(N(K)) \) is product disk decomposable. Here \( N(K) \) denotes a standard contact neighborhood, which is also product disk decomposable. It follows that \( M \setminus \text{int}(N(K)) \) is contactomorphic to \((S \times I)/\sim_1\) for a compact surface \( S \) with boundary, and \( N(K) \) is contactomorphic to \((P \times I)/\sim_2\) for a compact surface with boundary \( P \). Here \( P \) has piecewise smooth
boundary whose edges can naturally be divided into two types: those that intersect \( \partial N(K) \setminus \partial M \), and those that intersect \( \partial M \). By \((P \times I)/\sim_2 \), we mean the space obtained by quotienting out the \( I \) direction along the edges that are contained in \( \partial N(K) \). Since \((P \times I)/\sim_1 \) meets \((S \times I)/\sim_1 \) along \( \partial N(K) \setminus \partial M \), the surfaces \( P \times \{0\} \) and \( P \times \{1\} \) give two embeddings of \( P \) into \( S \). Using the projection of \( S \times I \) onto \( S \), we identify \( P \times \{0\} \subseteq S \times \{1\} \) with a subset of \( S \), for which we also write \( P \). The surface \( P \times \{1\} \) then gives another smooth embedding \( h: P \to S \), which is the monodromy map.

The Honda–Kazez–Matić map is easiest to define if one picks a contact structure \( \zeta \) on \((M, \gamma)\), such that \( \partial M \) is convex with dividing set \( \gamma \). To define the map \( \Phi^{HKM}_\xi \), one picks Legendrian graphs \( K \subseteq M \) and \( K' \subseteq M' \setminus \text{int}(M) \), whose complements are product disk decomposable. After modifying \( K' \) in a neighborhood of \( \partial M \), one extends \( K \) to a Legendrian graph on all of \( M' \), whose complement is product disk decomposable. Furthermore, outside a small neighborhood of \( \partial M \), the extension agrees with \( K \) and \( K' \). Let us write \( \tilde{K} \) for this Legendrian graph. The graph \( K \) is also required to satisfy a contact compatibility condition near \( \partial M \), described in [HKM08], though the specific form is not important for our present argument. The graph \( K \subseteq M \) induces a partial open book \((S, P, h)\) for \((M, \gamma, \zeta)\), which induces a diagram \((\Sigma, \alpha, \beta)\) for \((M, \gamma)\). The graph \( \tilde{K} \) induces a partial open book \((S', P', h')\) for \((M', \gamma')\), which gives rise to the diagram \((\Sigma', \alpha', \beta')\). Furthermore \( \Sigma' = \Sigma \cup \Sigma'' \), \( \alpha' = \alpha \cup \alpha'' \), and \( \beta = \beta \cup \beta'' \), for a surface \( \Sigma'' \) and a collection of curves \( \alpha'' \) and \( \beta'' \) on \( \Sigma' \). There is a canonical intersection point \( x_\xi \in T_{\alpha''} \cap T_{\beta''} \), and the map \( \Phi^{HKM}_\xi \) is defined by the formula

\[
\Phi^{HKM}_\xi(x) = x \times x_\xi.
\]

We now consider a Morse-type contact 1-handle addition. In this case,

\[
(M' \setminus \text{int}(M), \xi) \cong (Z_0 \cup h_1, \xi_0 \cup \xi_1),
\]

where \((Z_0, \xi_0)\) is an \( I \)-invariant contact structure on \( I \times \partial M \), and \((h_1, \xi_1)\) is a contact 1-handle. Let \( \tilde{K} \subseteq M \cup Z_0 \) denote an extension that can be used to compute the gluing map. We note that if \( K' \subseteq Z_0 \) is a graph whose complement is product disk decomposable, then the complement of \( K' \) in \( Z_0 \cup h_1 \) is also product disk decomposable. It follows that we can use the same graph

\[
\tilde{K} \subseteq M \cup Z_0 \subseteq M \cup Z_0 \cup h_1
\]
to compute the gluing map for \( Z_0 \cup h_1 \). Write \((S, P, h)\) for the partial open book of \((M, \gamma)\) induced by \( K \). Write \((S'_0, P'_0, h'_0)\) for the partial open book of \((M \cup Z_0, \gamma'_0)\) induced by \( \tilde{K} \subseteq M \cup Z_0 \), and write \((S', P', h')\) for the partial open book of \( M \cup Z_0 \cup h_1 \) induced by \( \tilde{K} \).

Since \((M \cup Z_0 \cup h_1) \setminus \partial(N(K))\) is obtained by attaching a contact 1-handle to \((M \cup Z_0) \setminus \partial(N(K))\), we can apply Lemma 4.4 to see that the partial open book \((S', P', h')\) is obtained from \((S'_0, P'_0, h'_0)\) by attaching a 1-handle to \( S' \setminus \partial P' \), and setting \( P' = P'_0 \) and \( h' = h'_0 \). The same basis of arcs \( a' \) for \( P' \) in \( S' \) can be used for \( P'_0 \) in \( S'_0 \), which we assume extends a basis of arcs \( a \) for \( P \) in \( S \). Let \((\Sigma, \alpha, \beta), (\Sigma'_0, \alpha'_0, \beta'_0)\), and \((\Sigma', \alpha', \beta')\) be the diagrams induced by \((S, P, h), (S'_0, P'_0, h'_0)\), and \((S', P', h')\) with the bases \( a, a' \) and \( a' \), respectively.

The Heegaard surface \( \Sigma' \) is thus obtained by attaching a 1-handle along the boundary of \( \Sigma'_0 \). Notice that since \( \alpha'_0 = \alpha' \) and \( \beta'_0 = \beta' \), and \( \Sigma' \) is obtained from \( \Sigma'_0 \) by attaching a band along \( \partial \Sigma'_0 \), the groups \( SFH(\Sigma'_0, \alpha'_0, \beta'_0) \) and \( SFH(\Sigma', \alpha', \beta') \) are naturally isomorphic, and the isomorphism is given by the contact 1-handle map, defined in this paper. Furthermore, we observe that

\[
\Phi^{HKM}_{\xi_0 \cup \xi_1}(x) = C_{h_1} \circ \Phi^{HKM}_{\xi_0}.\tag{15}
\]

By [HKM08, Proposition 6.1], the above expression is equal to \( C_{h_1} \circ \phi^{\gamma'}_* \), where

\[
\phi^{\gamma'}_*: SFH(\Sigma, \alpha, \beta) \to SFH(\Sigma'_0, \alpha'_0, \beta'_0)
\]
is the composition of the tautological map induced by a diffeomorphism, and the transition maps induced by naturality. By Proposition 5.1, we see that the equation in equation (15) agrees with the definition of \( \Phi_{\xi_0 \cup \xi_1} \), in this paper.

The argument when

\[
(M' \setminus \text{int}(M), \xi) = (Z_0 \cup h^2, \xi_0 \cup \xi_2)
\]
is a Morse-type contact 2-handle is similar. Suppose that $K \subseteq M$ is a Legendrian graph such that $M \setminus \text{int}(N(K))$ is product disk decomposable, and induces a partial open book that satisfies the contact compatibility condition near $\partial M$. We then let $\hat{K}_0$ denote a Legendrian extension to $M \cup Z_0^\prime$, whose complement is product disk decomposable, and which can be used to compute the map $\Phi_{\hat{K}_0}^{\text{HKM}}$ for gluing $(Z_0, \xi_0)$ to $M$. We can define an extension $\hat{K}$ of $K$ into all of $M \cup Z_0 \cup h^2$ by setting

$$\hat{K} := \hat{K}_0 \cup c,$$

where $c$ is a Legendrian co-core of the 2-handle $h^2$. We note that $(M \cup Z_0 \cup h^2) \setminus \text{int}(N(\hat{K}))$ is product disk decomposable. Let $(S, P, h), (S_0^\prime, P_0^\prime, h_0^\prime)$, and $(S^\prime, P^\prime, h^\prime)$ denote the partial open books induced by $K \subseteq (M, \gamma)$, $\hat{K}_0 \subseteq (M \cup Z_0, \gamma_0^\prime)$, and $\hat{K} \subseteq (M \cup Z_0 \cup h^2, \gamma^\prime)$, respectively. Let $\pi: S \times I \to S$ be the projection. Writing the 2-handle $h^2$ as $(B \times I)/\sim_2$, where $B$ is a square, we observe that $P^\prime$ is obtained from $P_0^\prime$ by adding $\pi(B \times \{0\})$ to $P_0^\prime$. The monodromy is extended to $P^\prime$ by mapping $\pi(B \times \{1\}) \subseteq S$ to $\pi(B \times \{1\}) \subseteq S$.

We start with a basis of arcs $a$ for $P \subseteq S$, and extend $a$ to a basis $a_0'$ for $P_0' \subseteq S_0'$. A basis of arcs $a'$ for $P^\prime \subseteq S^\prime$ can then be obtained from $a_0'$ by adding a new arc $a'$, which is a core of the band $\pi(B \times \{0\})$. Write $(\Sigma, (\alpha, \beta), (\Sigma_0^\prime, (\alpha_0', \beta_0'))$, and $(\Sigma', (\alpha', \beta'))$ for the diagrams obtained from the partial open books $(S, P, h), (S_0^\prime, P_0^\prime, h_0^\prime)$, and $(S^\prime, P^\prime, h^\prime)$, with bases $a, a_0'$, and $a'$, respectively. Also, let us write $\alpha_0' = \alpha \cup \alpha_0'$ and $\beta_0' = \beta \cup \beta_0'$, where $\alpha_0'$ and $\beta_0'$ are the curves induced by the arcs in $a_0' \setminus a$, and $x_{\xi_0'}$ for the canonical intersection point in $T_{\alpha_0'} \cap T_{\beta_0'}$. Finally, write $a'$ and $\beta'$ for the curves induced by the new basis arc $a'$, and write $c'$ for the canonical intersection point of $a' \cap \beta'$.

The map $\Phi_{\xi_0', \xi_2}^{\text{HKM}}$ is defined by the formula

$$\Phi_{\xi_0', \xi_2}^{\text{HKM}}(x) = x \times x_{\xi_0'} \times c'.$$

Noting that $\Phi_{\xi_0'}(x) = x \times x_{\xi_0}$, we see that

$$\Phi_{\xi_0', \xi_2}^{\text{HKM}} (x) = (C_{h_2} \circ \Phi_{\xi_0'}^{\text{HKM}})(x).$$

By the same argument as for contact 1-handles, this is equal to $(C_{h_2} \circ \phi_{\xi_0'}^\gamma)(x) = \Phi_{\xi_0', \xi_2}(x)$, completing the proof.

6. Turning around cobordisms of sutured manifolds and duality

In this section, we compute the effect of turning around a cobordism of sutured manifolds, proving Theorem 1.3.

6.1. The canonical trace pairing. As described in [FJR11, Proposition 2.14] and [Juh16, Section 11.2], there is duality between $SFH(M, \gamma)$ and $SFH(-M, \gamma)$. If $(\Sigma, (\alpha, \beta))$ is a diagram for $(M, \gamma)$, then $(\Sigma, (\beta, \alpha))$ is a diagram for $(-M, \gamma)$. Since $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is equal to $\mathbb{T}_\beta \cap \mathbb{T}_\alpha$, we can define a map

$$\text{tr}: CF((\Sigma, (\alpha, \beta)) \otimes CF((\Sigma, (\beta, \alpha)) \to \mathbb{F}_2$$

by the formula

$$\text{tr}(x \otimes y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that tr is a chain map, since $J$-holomorphic discs on $(\Sigma, (\alpha, \beta))$ from $x$ to $y$ are in bijection with $J$-holomorphic discs on $(\Sigma, (\beta, \alpha))$ from $y$ to $x$. In particular, tr is the usual pairing between homology and cohomology. Note that the trace pairing gives a natural isomorphism

$$CF((\Sigma, (\beta, \alpha)) \cong CF((\Sigma, (\alpha, \beta))^\vee := \text{Hom}_{\mathbb{F}_2}(CF((\Sigma, (\alpha, \beta)), \mathbb{F}_2).$$

In [Juh16], the first author defined a pairing

$$(\cdot, \cdot): SFH(M, \gamma) \otimes SFH(-M, -\gamma) \to \mathbb{F}_2.$$

This agrees with tr under the canonical isomorphism $SFH(-M, \gamma) \cong SFH(-M, -\gamma)$.

In the opposite direction, there is the cotrace map,

$$\text{cotr}: \mathbb{F}_2 \to CF((\Sigma, (\alpha, \beta)) \otimes CF((\Sigma, (\beta, \alpha)).$$
The cotrace map is defined by the formula
\[ \text{cotr}(1) = \sum_{x \in \gamma_{a}, \gamma_{b}} x \otimes x. \]

Finally, we note that if \( V \) is a finite dimensional vector space over \( \mathbb{F}_2 \), then there are canonical isomorphisms
\[
\text{Hom}_{\mathbb{F}_2}(V, V) \cong \text{Hom}_{\mathbb{F}_2}(V \otimes V^\vee, \mathbb{F}_2) \cong \text{Hom}_{\mathbb{F}_2}(V^\vee \otimes V).
\]
Under these isomorphisms, the \( \text{tr} \) and \( \text{cotr} \) maps are identified with \( \text{id}_V \in \text{Hom}_{\mathbb{F}_2}(V, V) \).

6.2. Sutured manifold cobordisms and the induced maps. In this section, we review the definition of sutured manifold cobordisms, special cobordisms, boundary cobordisms, and the construction of the sutured cobordism maps. We finally give a simpler definition of the cobordism maps using our gluing map from Section 3.3. The following is [Juh16, Definition 2.3].

**Definition 6.1.** We say that the contact structures \( \xi_0 \) and \( \xi_1 \) on the sutured manifold \( (M, \gamma) \) are equivalent if they can be connected by a 1-parameter family \( \{ \xi_t : t \in I \} \) of contact structures on \( (M, \gamma) \), such that \( \partial M \) is convex with dividing set \( \gamma \) for each \( \xi_t \). In this case, we write \( \xi_1 \sim \xi_2 \), and denote the equivalence class of \( \xi \) by \( [\xi] \).

Sutured manifold cobordisms were defined in [Juh16, Definition 2.4].

**Definition 6.2.** Let \( (M_0, \gamma_0) \) and \( (M_1, \gamma_1) \) be sutured manifolds. A cobordism from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \) is a triple \( W = (W, Z, [\xi]) \) such that
- \( W \) is a compact, oriented 4-manifold with boundary and corners,
- \( Z \) is a codimension-0 submanifold of \( \partial W \), and \( \partial W \setminus \text{int}(Z) = M_0 \sqcup M_1 \),
- \( \xi \) is a positive contact structure on \( (Z, \gamma_0 \cup \gamma_1) \).

Note that equivalent contact structures can have different characteristic foliations on \( \partial M \), which gives us enough flexibility to compose cobordisms. The sutured manifold cobordism \( W \) is balanced if \( (M_0, \gamma_0) \) and \( (M_1, \gamma_1) \) are balanced sutured manifolds. Furthermore, we say that \( Z_0 \) is an isolated component of \( Z \) if \( Z_0 \cap M_1 = \emptyset \). The following is [Juh16, Definition 5.1]

**Definition 6.3.** We say that the cobordism \( W = (W, Z, [\xi]) \) from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \) is special if
- \( W \) is balanced,
- \( \partial M_0 = \partial M_1 \), and \( Z = -I \times \partial M_0 \) is the trivial cobordism between them,
- \( \xi \) is an \( I \)-invariant contact structure on \( Z \) such that each \( \{ t \} \times \partial M_0 \) is convex with dividing set \( \{ t \} \times \gamma_0 \) for every \( t \in I \), with respect to the contact vector field \( \partial/\partial t \).

Given a special cobordism \( W \) from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \), we define the map
\[
F_W : SFH(M_0, \gamma_0) \to SFH(M_1, \gamma_1)
\]
by composing maps associated to 4-dimensional handle attachments along the interior of \( M_0 \); see [Juh16, Section 8]. The following is equivalent to [Juh16, Definition 10.4].

**Definition 6.4.** A sutured cobordism \( (W, Z, [\xi]) \) from \( (M, \gamma) \) to \( (M', \gamma') \) is called a boundary cobordism if \( M \subseteq \text{Int}(M') \), \( W = I \times M' / \sim \), where \( (t, x) \sim (t', x) \) for every \( x \in \partial M' \) and \( t \in I \), and \( \xi \) is a contact structure on \( Z = -\{ 0 \} \times (M' \setminus \text{int}(M)) \) inducing the sutures \( \{ 0 \} \times \gamma \) on \( \{ 0 \} \times \partial M \) and \( \{ 0 \} \times \gamma' \) on \( \{ 0 \} \times \partial M' \).

If \( W = (W, Z, [\xi]) \) is a boundary cobordism from \( (M, \gamma) \) to \( (M', \gamma') \), then we can view \((-M, -\gamma)\) as a sutured submanifold of \((-M', -\gamma')\), and \(-\xi\) is a positive contact structure on \( Z = -M' \setminus \text{int}(-M) \) with dividing set \(-\gamma_0 \cup -\gamma_1\). If \( Z \) has no isolated components, then the map \( F_W : SFH(M, \gamma) \to SFH(M', \gamma') \) is defined as the Honda–Kazez– Matić gluing map \( \Phi_{-\xi} \).

Every balanced cobordism \( W = (W, Z, [\xi]) \) from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \) can uniquely be written as a composition \( W^a \circ W^b \), where
\[
W^b = (I \times (M_0 \cup -Z) / \sim, \{ 0 \} \times Z, [\xi])
\]
is a boundary cobordism from \((M_0, \gamma_0)\) to \((M_0 \cup -Z, \gamma_1)\). Furthermore,

\[ \mathcal{W}^s = (W, -I \times \partial M_1, [\eta]) \]

is a special cobordism from \((M_0 \cup -Z, \gamma_1)\) to \((M_1, \gamma_1)\), where \(-I \times \partial M_1\) is a collar of \(\partial M_1\) in \(Z\), and \(\eta\) is an \(I\)-invariant contact structure with dividing set \(\{t\} \times \gamma_1\) on \(\{t\} \times \partial M_1\) for every \(t \in I\), with respect to \(\partial / \partial t\). We call \(\mathcal{W}^s\) the special part, and \(\mathcal{W}^b\) the boundary part of \(\mathcal{W}\). If \(Z\) has no isolated components, then the cobordism map \(F_\mathcal{W}\) is defined as \(F_{\mathcal{W}^s} \circ F_{\mathcal{W}^b}\).

According to [Juh16, Definition 10.1], in the general case, we choose a standard contact ball \(B_0 \subseteq \text{int}(Z_0)\) with convex boundary and dividing set \(\delta_0\) in each isolated component \(Z_0\) of \(Z\). We write \((B, \delta)\) for the union of the balls \((B_0, \delta_0)\), and consider the cobordism \(\mathcal{W}' = (W, Z', [\xi'])\) from \((M_0, \gamma_0)\) to \((M_1, \gamma_1) \sqcup (B, \delta)\), where \(Z' = Z \setminus \text{int}(B)\) and \(\xi' = \xi|_{Z'}\). Since \(Z'\) has no isolated components and

\[ SFH((M_1, \gamma_1) \sqcup (B, \delta)) \cong SFH(M_1, \gamma_1), \]

we can define \(F_{\mathcal{W}} := F_{\mathcal{W}'}\). This is independent of the choice of \(B\).

The gluing map that we defined in Section 3.3 also assigns maps to contact 3-handles, and hence \(\Phi_{-\xi}\) makes sense even if \(Z\) has isolated components, giving rise to an alternative definition of \(F_{\mathcal{W}^b}\). We now show that the two constructions agree.

**Proposition 6.5.** Let \(\mathcal{W} = (W, Z, \xi)\) be a sutured manifold cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\), possibly with \(Z\) having isolated components. Then

\[ F_{\mathcal{W}} = F_{\mathcal{W}^s} \circ \Phi_{-\xi}, \]

where \(F_{\mathcal{W}}\) is the cobordism map defined in [Juh16, Definition 10.1] using the Honda–Kazez–Matić gluing map, and \(\Phi_{-\xi}\) is the gluing map from Section 3.3.

**Proof.** By definition, \(F_{\mathcal{W}} = i \circ F_{\mathcal{W}'} = i \circ F_{\mathcal{W}^s} \circ F_{\mathcal{W}^b}\), where

\[ \mathcal{W}' = (W, Z', [\xi']): (M, \gamma) \to (M', \gamma') \sqcup (B, \delta) \]

is the cobordism defined above, and

\[ i: SFH((M_1, \gamma_1) \sqcup (B, \delta)) \xrightarrow{\sim} SFH(M_1, \gamma_1) \]

is the canonical isomorphism. As \(Z'\) has no isolated components, \(F_{\mathcal{W}'}\) is the Honda–Kazez–Matić gluing map \(\Phi_{-\xi'}\), which agrees with our gluing map from Section 3.3 by Theorem 5.8.

On the other hand, \(\Phi_{-\xi} = \Phi_{-\xi_B} \circ \Phi_{-\xi'}\), where \(\xi_B = \xi|_B\). Hence, it suffices to show that

\[ i \circ F_{\mathcal{W}'} = F_{\mathcal{W}^s} \circ \Phi_{-\xi_B}. \]

Let \(W_B\) be the special cobordism from \((M_0 \cup -Z', \gamma_1 \cup \delta)\) to \((M_0 \cup -Z, \gamma_1) \sqcup (B, \delta)\) obtained by pushing \(\partial B\) slightly into \(Z'\), and attaching a 4-dimensional 3-handle along each component. Then

\[ (W')^s = (W^s \sqcup \text{id}_{(B, \delta)}) \circ W_B. \]

By definition, the contact 3-handle map \(\Phi_{-\xi_B} = j \circ F_{W_B}\), where

\[ j: SFH((M_0 \cup -Z, \gamma_1) \sqcup (B, \delta)) \xrightarrow{\sim} SFH(M_0 \cup -Z, \gamma_1) \]

is the canonical isomorphism; see Section 3.2.4. Hence, it suffices to show that

\[ i \circ F_{\mathcal{W}^s \circ \text{id}_{SFH(B, \delta)}} = F_{\mathcal{W}^s} \circ j. \]

This holds since \(F_{\mathcal{W}^s \circ \text{id}_{SFH(B, \delta)}} = F_{\mathcal{W}^s} \otimes \text{id}_{SFH(B, \delta)}\) and \(SFH(B, \delta) \cong \mathbb{F}_2. \) \(\square\)
6.3. **Turning around sutured cobordisms.** In this section, we use our contact handle maps to prove a first result about duality in sutured Floer homology. If

\[ W = (W, Z, [\xi]): (M, \gamma) \to (M', \gamma') \]

is a cobordism of sutured manifolds, then we can form the cobordism

\[ W^\vee := (W, Z, [\xi]): (-M', \gamma') \to (-M, \gamma) \]

by reversing which ends of \( W \) are viewed as incoming or outgoing. The main result of this section is the following:

**Theorem 6.6.** If \( W: (M, \gamma) \to (M', \gamma') \) is a balanced cobordism of sutured manifolds, and \( W^\vee \) is the cobordism obtained by turning around \( W \), then

\[ F_{W^\vee} = (F_W)^\vee, \]

with respect to the trace pairing from Section 6.1.

Suppose that \((M, \gamma)\) is a sutured submanifold of \((M', \gamma')\), and let \(\xi\) be a contact structure on \(Z = -M' \setminus \text{int}(M)\) with dividing set \(\gamma \cup \gamma'\) on the convex surface \(\partial Z\). Consider the boundary cobordism

\[ W := (I \times M'/\sim, \{0\} \times Z, [\xi]) \]

from \((M, \gamma)\) to \((M', \gamma')\). Then \(W^\vee\) is a sutured cobordism from \((-M', \gamma')\) to \((-M, \gamma)\). In general, \(W^\vee\) will be neither a special cobordism nor a boundary cobordism. It is the product of a boundary cobordism

\[ (W^\vee)^b: (-M', \gamma') \to (-M' \cup -Z, \gamma), \]

and a special cobordism

\[ (W^\vee)^s: (-M' \cup -Z, \gamma) \to (-M, \gamma). \]

The 4-manifold underlying \((W^\vee)^s\) is also \(M' \times I/\sim\). We need the following topological description of the special cobordism \((W^\vee)^s\):

**Lemma 6.7.** Suppose that

\[ W^\vee := (I \times M'/\sim, \{0\} \times Z, [\xi]): (-M', \gamma') \to (-M, \gamma) \]

is the sutured cobordism described above, and suppose that \((Z, \xi)\) has a contact handle decomposition relative to \(\partial M\) with an associated Morse function \(f: Z \to I\). Then the special part \((W^\vee)^s\) of \(W^\vee\) has a Morse function \(F\) whose critical points are in bijective correspondence with the critical points of \(f\). Furthermore, if \(p\) is a critical point of \(f\) and \(p'\) is the associated critical point of \(F\), then

\[ \text{ind}_p(F) = 4 - \text{ind}_p(f). \]

The intersection of the descending manifold of a critical point of \(F\) with

\[-M' \cup_{\partial M} -Z = -M \cup_{\partial M} Z \cup_{\partial M'} \partial Z\]

is equal to the union of the ascending flow lines of the corresponding critical point of \(f\) in \(Z\), together with their images in \(-Z\).

**Proof.** We first define an auxiliary function

\[ G: (I \times [-1, 2])/(I \times \{2\}) \to I. \]

We require \(G\) to satisfy the following:

- \(G(t, s) = t\) for \(s = -1\).
- \(\nabla G \neq 0\) for all \((t, s)\).
- If \((t, s) \in I \times I\), then \((\partial G/\partial t)(t, s) = 0\) if and only if \(t = \frac{1}{2}\).
- If \((t, s) \in \{\frac{1}{2}\} \times I\), then \((\partial G/\partial s) < 0\).
- \(G|_{\{0\} \times [-1, 2]} \equiv G|_{I \times \{2\}} \equiv G|_{\{1\} \times [0, 2]} \equiv 0\).
The graph of such a function $G(t, s)$ is shown in Figure 6.1.

Let us view $M'$ as

$$M' \cong M \cup (\partial M \times [-1, 0]) \cup -Z \cup (\partial M' \times [1, 2]),$$

and extend $f$ over all of $M'$ such that

- $f|_{\partial M} \equiv -1$,
- $f|_{\partial M \times [-1, 0]}(x, s) = s$, and
- $f|_{\partial M' \times [1, 2]}(x, s) = s$.

We then consider the function $F: M' \times I/\sim \to I$ given by

$$F(t, x) := G(t, f(x)).$$

It is then straightforward to verify that $F$ has the stated properties. \hfill \Box

**Remark 6.8.** For an index 1 critical point of $f$, the attaching sphere of the corresponding critical point of $F$ will be a knot $K$. The framing of $K$ depends on some auxiliary choices, such as a choice of Riemannian metric, and the precise choice of $G$. However, up to isotopy, the framing is determined uniquely by the property that the framing of $K \cap Z$ is the mirror of the framing of $K \cap (-Z)$.

**Proof of Theorem 6.6.** The claim was shown for special cobordisms in [Juh16, Theorem 11.8]. Hence, it only remains to verify it for boundary cobordisms. By the composition law for the gluing map, it is sufficient to prove the claim when the boundary cobordism $W = (W, Z, [\xi])$ is formed by adding a single contact $k$-handle for $k \in \{0, 1, 2, 3\}$.

The cobordism map

$$F_W: SFH(M, \gamma) \to SFH(M', \gamma')$$

is the gluing map $\Phi_{-\xi}$, corresponding to the sutured submanifold $(-M, -\gamma)$ of $(-M', -\gamma')$.

For a contact 0-handle, the map $\Phi_{-\xi}$ is the tautological one from $SFH(M, \gamma)$ to $SFH(M', \gamma')$.

On the other hand, consider $W^\vee$ from $(-M', \gamma')$ to $(-M, \gamma)$. Here $Z$ has an isolated component $Z_0$ corresponding to the contact 0-handle; i.e., $Z_0 \cap M = \emptyset$. Hence, by [Juh16, Definition 10.1], the map $F_{W^\vee}$ is defined by removing a standard contact ball $B$ with connected dividing set $\delta$ on $\partial B$ from the interior of $Z_0$, and adding $(B, \delta)$ to $(-M, \gamma)$. The resulting cobordism is a product from $(-M', \gamma') = (-M, \gamma) \cup (B, \delta)$ to $(-M', \gamma')$. Hence $F_{W^\vee}$ is $\text{id}_{SFH(-M', \gamma')}$, followed by the canonical identification of $SFH(-M', \gamma')$ with $SFH(-M, \gamma)$, which is $\Phi^{\vee}_{-\xi}$.

Now suppose that $W$ is formed by adding a contact 1-handle to $(M, \gamma)$. In this case, $F_W = \Phi_{-\xi}$, which is obtained by adding a strip to the boundary of a Heegaard diagram for $(-M, -\gamma)$. By Lemma 6.7, the cobordism $W^\vee$ is obtained by gluing $(Z, -\xi)$ along $\partial M'$ to $M = M \cup_{\partial M} -Z$, then attaching a 4-dimensional 3-handle. Note that we attach $Z$ along $\partial M'$, not $\partial M$, so it becomes...
a contact 2-handle. The 4-dimensional 3-handle is attached to a 2-sphere in $Z \cup_{\partial M'} -Z$. As in Lemma 6.7, the 2-sphere is the union of the ascending flowlines of $f$ in $Z$ (i.e., the co-core of $Z$, viewed as a contact 1-handle attached to $-M$) together with its image in the copy of $-Z$ that we glue onto $-M'$. Diagrammatically, this is shown in Figure 6.2. An easy model computation shows that this is equal to the dual of the contact 1-handle map.

![Figure 6.2. Computing the cobordism map for a turned-around contact 1-handle.](image)

Orientations on the Heegaard surface are shown.

We now consider a sutured cobordism $W$ formed by a contact 2-handle attachment to the sutured manifold $(M, \gamma)$. In this case, the dual cobordism $W^\vee$ is formed by gluing $(Z, -\xi)$ to $M' = M \cup -Z$ along $\partial M'$, and then attaching a 4-dimensional 2-handle. Gluing $(Z, -\xi)$ to $M'$ is now a contact 1-handle attachment. As described in Lemma 6.7, the knot that we attach a 2-handle along is given as the union of the co-core of $Z$ (viewed as a contact 2-handle) as well as its image in $-Z$. Let $\mathcal{H}'$ be an admissible diagram of $(-M', \gamma')$. After adding the contact 1-handle and the 4-dimensional 2-handle, we get a diagram that is a compound stabilization of $\mathcal{H}'$. After performing a compound destabilization, we get back to $\mathcal{H}'$. An easy triangle map computation for the 2-handle map shows that the composition of the triangle map and the compound destabilization map is dual to the contact 2-handle map. That the compound stabilization map agrees with the map from naturality is shown in Proposition 2.2. A schematic for the turned-around contact 2-handle cobordism is shown in Figure 6.3. The triangle map is shown in more detail in Figure 6.4.

Finally, we consider the case when $W$ is formed by adding a contact 3-handle. In this case, $Z$ has an isolated component, and the cobordism map $F_W$ is obtained by removing a standard contact ball from the 3-ball we are adding. Then the map is computed as a trivial gluing map, followed by a 3-handle map. The 3-handle map is for an embedded 2-sphere that is a push-off of the boundary 2-sphere on which we are attaching the contact 3-handle. The dual cobordism $W^\vee$ is formed by adding a contact 0-handle, followed by a 4-dimensional 1-handle. Hence, the claim that $F_W^\vee = F_W$ follows from the fact that the 4-dimensional 1-handle and 3-handle maps are dual to each other, as in [OS06, Theorem 3.5].

### 7. Triangle cobordisms

If $T = (\Sigma, \alpha, \beta, \gamma)$ is a balanced sutured triple diagram, then we can form the sutured cobordism $W_{\alpha, \beta, \gamma} = (W_{\alpha, \beta, \gamma}, Z_{\alpha, \beta, \gamma}, \xi_{\alpha, \beta, \gamma})$. 

□
Figure 6.3. Computing the cobordism map for a turned-around contact 2-handle.

Figure 6.4. The triangle map computation for the 4-dimensional 2-handle map in the cobordism map for a turned-around contact 2-handle. The orientation of the surface is shown. The two dashed lines with arrows on the left and right are identified. The only homology classes of triangles that have a vertex at $c$ have multiplicity 1 in the shaded region, and zero in the other regions shown.
as in [Juh16, Section 5]. We note, however, that the construction of the contact structure $\xi_{\alpha,\beta,\gamma}$ in [Juh16, Section 5] was incorrect, as it involved gluing contact structures along annuli whose boundaries did not intersect the dividing set. In this section, we will provide a different description, which we will take as the definition.

Before describing the 4-manifold $W_{\alpha,\beta,\gamma}$, we establish some notation. For $\tau \in \{\alpha,\beta,\gamma\}$, let $U_\tau$ be the sutured compression body obtained from $\Sigma \times I$ by attaching 3-dimensional 2-handles along $\tau \times \{0\} \subseteq \Sigma \times \{0\}$. We view $\Sigma$ as being embedded into $\partial U_\tau$ as $\Sigma \times \{1\}$. Similarly, we denote by $R_\tau \subseteq \partial U_\tau$ the result of surgering $\Sigma \times \{0\}$ along $\tau \times \{0\}$. Using this notation,

$$\partial U_\tau = \Sigma \cup (\partial \Sigma \times I) \cup R_\tau.$$

For $\tau, \tau' \in \{\alpha, \beta, \gamma\}$, consider the 3-manifold

$$M_{\tau,\tau'} := U_\tau \cup \Sigma - U_{\tau'},$$

and let $R_{\tau,\tau'}$ denote the surface

$$R_{\tau} \cup_{\partial \Sigma} R_{\tau'}.$$

Here, we write $-U_{\tau'}$ for the 3-manifold $U_{\tau'}$ with the opposite orientation, and $R_{\tau'}$ for the surface $R_{\tau}$ with the opposite orientation. Note that, using this orientation convention, we have

$$\partial M_{\tau,\tau'} = R_{\tau,\tau'}.$$

The oriented 1-manifold $\partial \Sigma$ has a natural embedding into $\partial M_{\tau,\tau'}$ that we denote $\gamma_{\tau,\tau'}$, and $(M_{\tau,\tau'},\gamma_{\tau,\tau'})$ is a sutured manifold with diagram $(\Sigma,\tau,\tau')$. Let $\Delta$ be a regular triangle in $\mathbb{C}$ with edges $e_\alpha, e_\beta$, and $e_\gamma$ appearing clockwise, and give $\Delta$ the complex orientation; see Figure 7.1.

![Figure 7.1. A triangle $\Delta$, oriented as in the construction of $W_{\alpha,\beta,\gamma}$. The boundary orientations of the edges $e_\alpha, e_\beta$ and $e_\gamma$ are also shown.](image)

We define the 4-manifold

$$W_{\alpha,\beta,\gamma} := (\Delta \times \Sigma) \cup (e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma)/\sim,$$

where $\sim$ denotes gluing $\Delta \times \Sigma$ to $e_\tau \times U_\tau$ along $e_\tau \times \Sigma$ for $\tau \in \{\alpha, \beta, \gamma\}$. Furthermore, let

$$Z_{\alpha,\beta,\gamma} := (\Delta \times \partial \Sigma) \cup (e_\alpha \times R_\alpha) \cup (e_\beta \times R_\beta) \cup (e_\gamma \times R_\gamma) \subseteq \partial W_{\alpha,\beta,\gamma},$$

which we orient as the boundary of $W_{\alpha,\beta,\gamma}$. We then have the decomposition

$$\partial W_{\alpha,\beta,\gamma} = -M_{\alpha,\beta} \cup -M_{\beta,\gamma} \cup M_{\alpha,\gamma} \cup Z_{\alpha,\beta,\gamma}.$$

Using our orientation conventions,

$$\partial Z_{\alpha,\beta,\gamma} = R_{\alpha,\beta} \cup R_{\beta,\gamma} \cup \bar{R}_{\alpha,\gamma}.$$

There is a natural collection of sutures $\gamma_{\alpha,\beta,\gamma}$ on $\partial Z_{\alpha,\beta,\gamma}$, defined as

$$\gamma_{\alpha,\beta,\gamma} := \{v_{\alpha,\beta}, v_{\beta,\gamma}, v_{\alpha,\gamma}\} \times \partial \Sigma,$$

where $v_{\sigma,\tau}$ is the vertex of $\Delta$ between $e_\sigma$ and $e_\tau$. There is a natural contact structure $\xi_{\alpha,\beta,\gamma}$ on $Z_{\alpha,\beta,\gamma}$ with dividing set $\gamma_{\alpha,\beta,\gamma}$, which is positive for the orientation of $Z_{\alpha,\beta,\gamma}$ we have described. We delay our description of $\xi_{\alpha,\beta,\gamma}$ until Section 7.1. Then $W_{\alpha,\beta,\gamma} = (W_{\alpha,\beta,\gamma}, Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$ is a sutured manifold cobordism from $(M_{\alpha,\beta}, \gamma_{\alpha,\beta}) \cup (M_{\beta,\gamma}, \gamma_{\beta,\gamma})$ to $(M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})$. The main goal of this section is to prove the following:
Theorem 7.1. Let \((\Sigma, \alpha, \beta, \gamma)\) be an admissible balanced sutured triple diagram. Then the cobordism map
\[
F_{W_{\alpha, \beta, \gamma}} : CF(\Sigma, \alpha, \beta) \otimes CF(\Sigma, \beta, \gamma) \to CF(\Sigma, \alpha, \gamma)
\]
is chain homotopic to the map \(F_{\alpha, \beta, \gamma}\) defined in [Juh16, Definition 5.13] that counts holomorphic triangles on the triple diagram \((\Sigma, \alpha, \beta, \gamma)\).

The proof occupies the remainder of this section. The first step is to define the contact structure \(\xi_{\alpha, \beta, \gamma}\) in detail, and describe some useful properties. Next, we introduce some special diagrams called “doubled diagrams” and “weakly conjugated diagrams” that appear when computing the cobordism map for \(W_{\alpha, \beta, \gamma}\). We then compute convenient formulas for the contact gluing map for \((Z_{\alpha, \beta, \gamma}, \xi_{\alpha, \beta, \gamma})\), and for the 4-dimensional handle attachments of the cobordism \(W_{\alpha, \beta, \gamma}\). Finally, we put all the pieces together, using associativity of the triangle maps and some other relations to show that the cobordism map \(F_{W_{\alpha, \beta, \gamma}}\) is chain homotopic to the triangle map \(F_{\alpha, \beta, \gamma}\).

7.1. The contact structure \(\xi_{\alpha, \beta, \gamma}\) on \(Z_{\alpha, \beta, \gamma}\). The sutured manifold \((Z_{\alpha, \beta, \gamma}, \gamma_{\alpha, \beta, \gamma})\) has a natural contact structure \(\xi_{\alpha, \beta, \gamma}\) that we define by decomposing \(Z_{\alpha, \beta, \gamma}\) along convex surfaces. The construction generalizes to the case of sutured multi-diagrams \((\Sigma, \eta_1, \ldots, \eta_n)\) for arbitrary \(n \geq 1\); though, in this paper, we will only need it for \(n \in \{1, 2, 3\}\).

To define the contact structure \(\xi_{\alpha, \beta, \gamma}\), it will be more convenient to view \(\Delta\) as a 2-disk with smooth boundary, and \(Z_{\alpha, \beta, \gamma}\) as
\[
(\partial \Sigma \times \Delta) \cup (R_\alpha \times e_\alpha) \cup (R_\beta \times e_\beta) \cup (R_\gamma \times e_\gamma).
\]
Let us write \(Z_0\) for \(\partial \Sigma \times \Delta\), which is a union of \(\partial \Sigma\) solid tori. Let \(\gamma_0\) consist of two parallel longitudinal sutures on each component of \(\partial Z_0\), such that their projections to \(\Delta\) wind positively around \(\partial \Delta\) with respect to the orientation of \(\Delta\) (i.e., they wind counterclockwise around \(\partial \Sigma \subseteq \Sigma\)). Since \((Z_0, \gamma_0)\) is product disc decomposable, it admits a unique tight contact structure \(\xi_0\), up to equivalence, which has \(\partial Z_0\) as a convex surface with dividing set \(\gamma_0\).

Write \((Z_\tau, \gamma_\tau)\) for the product sutured manifold \((R_\tau \times e_\tau, \partial R_\tau \times \{m_\tau\})\), where \(m_\tau\) is a midpoint of \(e_\tau\). The sutured manifold \((Z_\tau, \gamma_\tau)\) is product disc decomposable, and hence admits a unique tight contact structure \(\xi_\tau\) with dividing set \(\gamma_\tau\), up to equivalence. Let \(s_\alpha\), \(s_\beta\), and \(s_\gamma\) denote small translations of the sutures \(\gamma_\alpha\), \(\gamma_\beta\), and \(\gamma_\gamma\), respectively, such that each component of \(s_\tau\) intersects the corresponding component of \(\gamma_\tau\) transversely at exactly two points, for each \(\tau \in \{\alpha, \beta, \gamma\}\). Let \(N(s_\tau) \subseteq \partial Z_\tau\) denote a small regular neighborhood of \(s_\tau\) that intersects \(\gamma_\tau\) along two arcs. Using Giroux’s Legendrian realization principle, we may assume that each \(\partial N(s_\tau)\) is Legendrian.

We now describe how the subsurfaces \(N(s_\tau) \subseteq Z_\tau\) are glued to \(Z_0\). By picking \(\gamma_0\) appropriately, we may assume that each component of \(\partial \Sigma \times \{m_\tau\}\) intersects \(\gamma_0\) transversely at two points. For each \(\tau \in \{\alpha, \beta, \gamma\}\), we pick a small neighborhood \(N(\partial \Sigma \times \{m_\tau\}) \subseteq \partial Z_0\), such that each component of \(N(\partial \Sigma \times \{m_\tau\})\) intersects \(\gamma_0\) along two arcs. Using Legendrian realization, we may assume that \(\partial N(\partial \Sigma \times \{m_\tau\})\) is Legendrian.

We glue \((Z_\alpha, \xi_\alpha)\), \((Z_\beta, \xi_\beta)\), and \((Z_\gamma, \xi_\gamma)\) to \((Z_0, \xi_0)\) by identifying \(N(s_\tau)\) and \(N(\partial \Sigma \times \{m_\tau\})\) for \(\tau \in \{\alpha, \beta, \gamma\}\). We let \(\xi_{\alpha, \beta, \gamma}\) be the resulting contact structure. After rounding the Legendrian corners, the contact structure \(\xi_{\alpha, \beta, \gamma}\) has dividing set isotopic to \(\gamma_{\alpha, \beta, \gamma}\). This is shown in Figure 7.2. As \(N(s_\tau)\) is unique up to isotopy, \(\xi_{\alpha, \beta, \gamma}\) is well-defined up to equivalence.

If \((\Sigma, \alpha, \beta)\) is a sutured Heegaard diagram, the above construction adapts easily to yield a sutured manifold \((Z_{\alpha, \beta, \gamma}, \gamma_{\alpha, \beta})\) with a contact structure \(\xi_{\alpha, \beta}\). The following will be useful throughout the paper:

Lemma 7.2. If \((\Sigma, \alpha, \beta)\) is a sutured Heegaard diagram, then the sutured manifold \((Z_{\alpha, \beta, \gamma}, \gamma_{\alpha, \beta})\) is diffeomorphic to \((R_{\alpha, \beta} \times I, \partial \Sigma \times \{0, 1\})\), and the contact structure \(\xi_{\alpha, \beta}\) is isotopic to the \(I\)-invariant contact structure on \(R_{\alpha, \beta} \times I\) with dividing set \(\partial \Sigma \times \{t\}\) on \(R_{\alpha, \beta} \times \{t\}\) for every \(t \in I\).

Proof. The claim that \((Z_{\alpha, \beta, \gamma}, \gamma_{\alpha, \beta})\) and \((R_{\alpha, \beta} \times I, \partial \Sigma \times \{0, 1\})\) are diffeomorphic is obvious. To see that the contact structure \(\xi_{\alpha, \beta}\) is isotopic to the \(I\)-invariant contact structure in the statement, we will construct a decomposition of the latter along convex annuli that cut \(R_{\alpha, \beta} \times I\) into the disjoint union of the three contact manifolds \((Z_{\alpha}, \xi_{\alpha})\), \((Z_{\beta}, \xi_{\beta})\), and \((Z_0, \xi_0)\). Let \(s_1, s_2 \subseteq R_{\alpha, \beta}\) be two
disjoint curves that are both small translates of the dividing set $\gamma_{\alpha,\beta} = \partial \Sigma$. We assume that $s_1$ and $s_2$ each intersects $\gamma_{\alpha,\beta}$ transversely at two points. Using Legendrian realization, we can assume that both $s_1$ and $s_2$ are Legendrian.

We will cut $R_{\alpha,\beta} \times I$ along the two annuli $s_1 \times I$ and $s_2 \times I$. We note that since $s_1$ and $s_2$ are Legendrians, the characteristic foliations on $s_1 \times I$ and $s_2 \times I$ are simple to describe. They consist of the horizontal leaves $s_i \times \{t\}$ for all $t \in I$, as well as two vertical lines of singularities along $\{p\} \times I$ for $p \in s_i \cap \gamma_{\alpha,\beta}$. This is shown in Figure 7.3. We note that the characteristic foliation satisfies the Poincaré-Bendixson property (the limit set of a flow-line consists of a singular point, a periodic orbit, or a union of singular points and connecting orbits), and has no closed orbits or retrograde saddle connections. So, by the work of Giroux [Gir00], the surfaces $s_i \times I$ are convex. Furthermore, the dividing set on $s_i \times I$ consists of two curves of the form $\{q\} \times I$, where $q \in s_i \setminus \gamma_{\alpha,\beta}$.

After rounding the Legendrian corners that appear when we cut along $s_1 \times I$, we obtain the disjoint union of the sutured manifolds $(Z_{\alpha}, \xi_{\alpha})$, $(Z_{\beta}, \xi_{\beta})$, and $(Z_0, \gamma_0)$. Furthermore, the contact structures obtained on the three pieces are isotopic to $\xi_{\alpha}$, $\xi_{\beta}$, and $\xi_0$, since they are tight by Giroux’s criterion [Hon00a, Theorem 3.5], and $\xi_{\alpha}$, $\xi_{\beta}$, and $\xi_0$ are the unique tight contact structures, by definition. A picture of the convex decomposition of $R_{\alpha,\beta} \times I$ along $s_1 \times I$ and $s_2 \times I$ is shown in Figure 7.4.
As in Lemma 7.2, we work backwards, and provide a convex decomposition of \( (Z, \partial Z) \), described above. To see this, let \( \alpha, \beta, \gamma \) be a Legendrian arc on \( \partial Z \) that perturb the dividing sets \( \alpha, \beta \), and similarly for \( \beta \). Let \( \bar{R}_{\beta} \subseteq R_{\beta} \times \{0\} \), oriented as the boundary of \( R_{\beta} \times I \), denote the image of \( R_{\beta} \times \{0\} \subseteq R_{\beta} \times \{0\} \), and let \( R_{\beta} \times \{0\} \) denote the image of \( R_{\beta} \times \{0\} \subseteq R_{\beta} \times \{0\} \). Using the Legendrian realization principle, we may assume that \( s_{\alpha, \beta} \) and \( s_{\beta, \gamma} \) are Legendrian. The following description of \( (Z_{\alpha, \beta, \gamma}, \xi_{\alpha, \beta, \gamma}) \) will be useful for our purposes:

**Lemma 7.3.** The contact structure \( (Z_{\alpha, \beta, \gamma}, \xi_{\alpha, \beta, \gamma}) \) is equivalent to the one described above along the surfaces \( I, \xi_{\alpha, \beta, \gamma} \) and \( (Z_{\beta, \gamma}, \xi_{\beta, \gamma}) \) together along the surfaces \( R_{\beta} \times I, \xi_{\alpha, \beta} \) and \( R_{\beta} \times I, \xi_{\beta, \gamma} \) (which are convex with Legendrian boundary), described above.

**Proof.** As in Lemma 7.2, we work backwards, and provide a convex decomposition of \( (Z_{\alpha, \beta, \gamma}, \xi_{\alpha, \beta, \gamma}) \) into the disjoint union of \( (R_{\alpha, \beta} \times I, \xi_{\alpha, \beta}) \) and \( (R_{\beta, \gamma} \times I, \xi_{\beta, \gamma}) \). By definition, \( Z_{\alpha, \beta, \gamma} \) is the union of \( Z_{\alpha}, Z_{\beta}, Z_{\gamma} \), and \( Z_0 \). Let us view \( Z_{\beta} \) as \( R_{\beta} \times I \) with \( I \)-invariant contact structure, and Legendrian corners along \( \partial R_{\beta} \times \{0, 1\} \), as on the left side of the middle level of Figure 7.4. We start with the surface \( S_{\beta} := R_{\beta} \times \{ \frac{1}{2} \} \), which is convex with Legendrian boundary. We can view \( \partial S_{\beta} \) as a Legendrian arc on \( \partial Z_0 \) that intersects the dividing set on \( \partial Z_0 \) at two points. There is an annulus \( S_{\alpha} \subseteq Z_0 \) with one boundary component on \( \partial S_{\beta} \), and another boundary component on \( \partial Z_0 \) between \( Z_{\gamma} \) and \( Z_{\alpha} \). Furthermore, after perturbing the surface \( S_{\alpha} \) to be convex, it cuts \( (Z_0, \xi_0) \) into two copies of \( (Z_0, \xi_0) \). To see this, we consider \( (Z_0, \xi_0) \) as the \( I \)-invariant contact manifold (with Legendrian corners) in the center of the middle of Figure 7.4. In this picture, an example of an annulus cutting

![Figure 7.4](image-url)
Figure 7.5. Gluing \((R_{\alpha,\beta} \times I, \xi_{\alpha,\beta})\) and \((R_{\beta,\gamma} \times I, \xi_{\beta,\gamma})\) together to obtain \((Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})\). We glue along the green shaded subsurfaces labeled \(R'_{\beta}\) and \(\bar{R}'_{\beta}\). The dividing sets \(\gamma_{\alpha,\beta}\) and \(\gamma_{\beta,\gamma}\) are shown in red. The curves \(s_{\alpha,\beta}\) and \(s_{\beta,\gamma}\) are small perturbations of the dividing sets, and are Legendrian.

\[ \gamma_{\alpha,\beta}, \bar{R}'_{\beta}, R'_{\alpha} \]

\[ \gamma_{\beta,\gamma}, \bar{R}'_{\beta}, R'_{\alpha} \]

\[ \text{glue} \]

\[ \text{Figure 7.5.} \]

\( (Z_0, \xi_0) \) into two copies of \((Z_0, \xi_0)\) would be the intersection of \(Z_0\) with a slice \(R_{\alpha,\beta} \times \{t\}\). We let our decomposing surface \(S\) be the union \(S_{\beta} \cup S_0\). This is shown schematically in Figure 7.6. When we cut along \(S\), we get two components, one of which is \(Z_\alpha \cup Z_0 \cup Z_\beta\), and one of which is \(Z_\beta \cup Z_0 \cup Z_\gamma\).

Lemma 7.2 identifies these latter two contact manifolds with \(R_{\alpha,\beta} \times I\) and \(R_{\beta,\gamma} \times I\) with \(I\)-invariant contact structures.

\[ \square \]

\[ (R_{\beta,\gamma} \times I, \xi_{\beta,\gamma}) \]

\[ (R_{\alpha,\beta} \times I, \xi_{\alpha,\beta}) \]

\[ \text{Figure 7.6.} \]

7.2. A handle decomposition of \((W_{\alpha,\beta,\gamma})^s\). Recall that \(W_{\alpha,\beta,\gamma}\) is a sutured manifold cobordism from \((M_{\alpha,\gamma} \cup M_{\beta,\gamma}, \gamma_{\alpha,\beta})\) to \((M_{\alpha,\gamma} \cup M_{\beta,\gamma}, \gamma_{\alpha,\beta})\). The boundary cobordism \((W_{\alpha,\beta,\gamma})^b\) corresponds to gluing \((Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})\) to \((-M_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})\) \((M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})\) to \((M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})\). In light of Lemma 7.3, we can topologically view this as gluing \(\bar{R}_{\beta} \subseteq -\partial M_{\alpha,\beta}\) to \(R_{\beta} \subseteq -\partial M_{\beta,\gamma}\).

Hence, the special cobordism \((W_{\alpha,\beta,\gamma})^s\) goes from \((M_{\alpha,\beta} \cup R_{\beta}, M_{\beta,\gamma}, \gamma_{\alpha,\gamma})\) to \((M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})\). In this section, we give a topological description of the special cobordism \((W_{\alpha,\beta,\gamma})^s\) in terms of 4-dimensional handle attachments.

A handle decomposition of \((W_{\alpha,\beta,\gamma})^s\) can be constructed from a sutured Morse function on \(U_{\beta}\), as we now describe. Let \(f_{\beta}: U_{\beta} \to I\) be a Morse function induced by the diagram \((\Sigma, \beta)\); i.e, we view
$U_\beta$ as a collection of 2-handles glued to $\Sigma \times I$ along $\beta \times \{0\} \subseteq \Sigma \times \{0\}$. Furthermore, we pick $f_\beta$ such that $f_\beta^{-1}(1) = \Sigma \times \{1\}$, $f_\beta^{-1}(0) = R_\beta$, and $f_\beta(y, t) = t$ for $y \in \partial \Sigma$. We also assume that $f_\beta$ has $|\beta|$ index 1 critical points, whose ascending manifolds intersect $\Sigma \times \{1\}$ along the $\beta$ curves, and that $f_\beta$ has no other critical points. For a curve $\beta_i \in \beta$, let $\lambda_i \subseteq U_\beta$ denote the descending manifold of the critical point of $f_\beta$ corresponding to $\beta_i$. We have the following topological description of $W_\alpha, \beta, \gamma$:

**Lemma 7.4.** The special cobordism $(W_\alpha, \beta, \gamma)^*$ from $(M_\alpha, \beta \cup \gamma_\alpha, \gamma)$ to $(M_\alpha, \gamma, \gamma_\alpha)$ consists of $|\beta|$ 2-handle attachments. The 2-handles are attached along the link formed by concatenating the arcs $\lambda_i \subseteq U_\beta$ (that have boundary on $\bar{R}_\beta$) with their reflections in $-U_\beta$. The framing on this link is determined by picking an arbitrary framing on $\lambda_i$ and concatenating it with the mirrored framing on the image of $\lambda_i$ in $-U_\beta$.

**Proof.** This can be proven similarly to Lemma 6.7, by using a Morse function built from $f_\beta$ as well as another auxiliary function. We leave the details to the reader. □

### 7.3. Doubled diagrams

Suppose that $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a sutured Heegaard diagram for $(M, \gamma)$. There is a special way of constructing a new diagram from $\mathcal{H}$, which will be important for computing the cobordism map $F_{W_\alpha, \beta, \gamma}$. Let us first pick two collections of subintervals $I_0$ and $I_1$ of $\partial \Sigma$ such that each component of $\partial \Sigma$ contains exactly one subinterval from $I_0$ and one subinterval from $I_1$ that are disjoint.

We now define the *Heegaard surface that is doubled along $R_\beta$* to be

$$D_\beta(\Sigma) := \Sigma \natural I_0 \bar{\Sigma} \natural I_1 R_\beta,$$

where $\natural$ denotes the boundary connected sum operation. Here $\bar{\Sigma}$ denotes a pushoff of $\Sigma$ into $U_\beta$, with the opposite orientation.

Before we define the attaching curves, let us first describe how $D_\beta(\Sigma)$ is embedded in $M_{\alpha, \beta}$. A schematic is shown in Figure 7.7. Strictly speaking, we have changed the sutures. An isotopy supported in a neighborhood of the original sutures moves the new sutures to the original sutures.

![Figure 7.7](image)

**Figure 7.7.** The doubled Heegaard surface $D_\beta(\Sigma)$. A neighborhood of a portion of the sutures $\gamma$ in $M$ is shown. The sutures are drawn in red. Strictly speaking, the new sutures $\partial D_\beta(\Sigma)$ are different from $\gamma$; however, an isotopy supported in a neighborhood of $\gamma$ moves $\partial D_\beta(\Sigma)$ back to $\gamma$. 
We now describe compressing curves on \( D_\beta(\Sigma) \). First, pick a collection of pairwise disjoint arcs on \( \Sigma \) with boundary on \( \partial_0 \) that form a basis of \( H_1(\Sigma, \partial_0) \). One then doubles these across \( \partial_0 \), and obtains a collection of curves \( \Delta_\Sigma \subseteq \Sigma \amalg_1 \Sigma \subseteq D_\beta(\Sigma) \). Similarly, one picks a collection of pairwise disjoint arcs on \( R_\beta \) (or equivalently, a collection of arcs on \( \Sigma \) that avoid the \( \beta \) curves) with boundary on \( \partial_1 \) that form a basis of \( H_1(R_\beta, \partial_1) \). Doubling these curves across \( \partial_1 \) yields closed curves \( \delta_\beta \subseteq \Sigma \amalg_1 \Sigma \subseteq D_\beta(\Sigma) \).

Write \( \bar{\beta} \) for the images of the \( \beta \) curves on \( \Sigma \). Note that the \( \alpha, \beta, \) and \( \delta_\beta \) curves are all disjoint. An easy computation shows that
\[
|\alpha| + |\beta| + |\delta_\beta| = |\Delta_\Sigma|.
\]

The doubled Heegaard diagram is now defined as
\[
D_\beta(\mathcal{H}) = (D_\beta(\Sigma), D_\beta(\alpha), \Delta_\Sigma) = (\Sigma \amalg_0 \Sigma, \Sigma \amalg_1 \Sigma, R_\beta, \alpha \cup \beta \cup \delta_\beta, \Delta_\Sigma).
\]

**Remark 7.5.** Note that there is some asymmetry in the above construction, since we took \( \Sigma \) and connected it to a surface \( \Sigma \amalg_1 \Sigma \) that was in the \( U_\beta \) handlebody. We could instead connect \( \Sigma \) along \( \partial_0 \) to the surface \( R_\beta \) that was in the \( U_\alpha \) handlebody, and construct analogous attaching curves \( \Delta_\Sigma \) and \( \delta_\alpha \cup \bar{\alpha} \cup \beta \) for the Heegaard surface \( D_\alpha(\Sigma) := R_\alpha \amalg_1 \Sigma \amalg_0 \Sigma \). We will write
\[
D_\alpha(\mathcal{H}) = (D_\alpha(\Sigma), \Delta_\Sigma, \delta_\alpha \cup \bar{\alpha} \cup \beta)
\]
for this Heegaard diagram. If there is any ambiguity, we will call the Heegaard diagram \( D_\beta(\mathcal{H}) \) the \( \beta \)-double, and \( D_\alpha(\mathcal{H}) \) the \( \alpha \)-double.

**7.4. Weakly conjugated diagrams.** Given a Heegaard diagram \( (\Sigma, \alpha, \beta, w) \) of a based 3-manifold, we can consider the conjugate diagram \( (\bar{\Sigma}, \bar{\alpha}, \bar{\beta}, w) \) that represents the same based 3-manifold. This was described by Ozsváth and Szabó [OS04c], and was explored further by Hendricks and Manolescu [HM17]. Given a sutured diagram \( (\Sigma, \alpha, \beta) \) for \( (M, \gamma) \), one can consider the sutured diagram \( (\Sigma, \beta, \alpha) \); however, this is now a diagram for \( (M, -\gamma) \). So, unlike in the case of closed 3-manifolds, this operation does not induce a conjugation action on \( SFH(M, \gamma) \).

Nonetheless, a similar diagrammatic manipulation appears when we compute the cobordism map for \( \mathcal{W}_{\alpha, \beta, \gamma} \). In analogy to the terminology for the conjugation action on Heegaard diagrams for closed 3-manifolds, we will say that the diagrams that appear are *weakly conjugated*. We describe the construction of weakly conjugated diagrams in this section.

As with the doubled diagrams, we pick collections of subintervals \( I_0 \) and \( I_1 \) in \( \partial \Sigma \) such that each component of \( \partial \Sigma \) contains exactly one subinterval from \( I_0 \) and from \( I_1 \) that are disjoint. We can then form the weakly conjugated Heegaard surface
\[
C(\Sigma) := R_\alpha \amalg_1 \Sigma \amalg_1 R_\beta.
\]
This is shown in Figure 7.8. As described, \( \partial C(\Sigma) \) is different from \( \partial \Sigma \), but an isotopy supported in a neighborhood of \( \partial \Sigma \) moves \( \partial C(\Sigma) \) to \( \partial \Sigma \).

We now describe compressing curves on \( C(\Sigma) \). Note that \( \alpha \) and \( \beta \) still bound compressing disks on \( \Sigma \). As curves on \( \Sigma \), we denote them by \( \bar{\alpha} \) and \( \bar{\beta} \). However, these are not complete collections of compressing curves, as we have increased the genus of the Heegaard surface by attaching \( R_\alpha \) and \( R_\beta \). Hence, we pick a collection of pairwise disjoint arcs on \( R_\alpha \) that form a basis of \( H_1(R_\alpha, \partial_0) \), and double them across \( \partial_0 \), to get a collection of curves \( \delta_\alpha \) on \( R_\alpha \amalg_1 \Sigma \). Similarly, we pick a collection of pairwise disjoint arcs on \( R_\beta \) that form a basis of \( H_1(R_\beta, \partial_1) \), and double them across \( \partial_1 \) to get a collection of curves \( \delta_\beta \) on \( \Sigma \amalg_1 R_\beta \). We define the weakly conjugated Heegaard diagram of \( \mathcal{H} \) to be
\[
C(\mathcal{H}) = (C(\Sigma), C(\beta), C(\alpha)) := (R_\alpha \amalg_1 \Sigma \amalg_1 R_\beta, \bar{\beta} \cup \delta_\beta, \bar{\alpha} \cup \delta_\alpha).
\]

**7.5. The change of diagrams map from \( D_\beta(\mathcal{H}) \) to \( \mathcal{H} \).** In this section, we prove a relatively simple formula for the change of diagrams map from the \( \beta \)-double of a diagram \( D_\beta(\mathcal{H}) \) to the original diagram \( \mathcal{H} = (\Sigma, \alpha, \beta) \). Recall that
\[
D_\beta(\mathcal{H}) = (\Sigma \amalg_1 \Sigma, R_\beta, \alpha \cup \bar{\beta} \cup \delta_\beta, \Delta_\Sigma).
\]
framing of is not hard to see that the triple $$(D, \delta, \beta)$$ pushing a curve $$\beta$$ disks in surgery on. Let $$U$$ (here, the second copy of $$D$$).

**Proof.** The idea is simple: The map to the change of diagrams map, by the well-definedness of the sutured cobordism maps [Juh16].

We will write $$\beta' := \beta \cup \delta \beta$$. Note that $$(D_\beta(\Sigma), \Delta, \beta \cup \beta')$$ is the $$\alpha$$-double of the diagram $$(\Sigma, \beta, \beta)$$ (see Remark 7.5), which represents

$$
\left((S^1 \times S^2)^{[\beta]}(\partial \Sigma)\#(R_\beta \times I), \partial R_\beta \times I\right),
$$

where $$(S^1 \times S^2)^{[\beta]}(\partial \Sigma)$$ is obtained by removing $$|\partial \Sigma|$$ 3-balls from $$(S^1 \times S^2)^{[\beta]}$$, and adding a connected suture to each boundary component. Hence, its sutured Floer homology has a top-graded generator that we will denote by $$\Theta_{\Delta, \beta, \beta'}$$. So, there is a holomorphic triangle map $$F_2: SFH(D_\beta(\Sigma), \alpha \cup \beta', \Delta(\Sigma)) \to SFH(D_\beta(\Sigma), \alpha \cup \beta', \beta \cup \beta')$$ defined by the formula

$$F_2 := F_{\alpha \cup \beta', \Delta, \beta, \beta'}(-, \Theta_{\Delta, \beta, \beta'}).$$

Notice that $$\delta \beta$$ intersects none of the collections $$\alpha$$, $$\beta \subseteq \Sigma$$, nor $$\beta \subseteq \bar{\Sigma}$$. We can therefore define a 3-handle map

$$F_3 := F_{3\beta', \beta'}: SFH(D_\beta(\Sigma), \alpha \cup \beta', \beta \cup \beta') \to SFH(\Sigma, \alpha \cup \beta')$$

(here, the second copy of $$\beta'$$ is a small Hamiltonian translate of $$\beta'$$, though we omit this from the notation).

**Lemma 7.6.** The composition $$F_3 \circ F_2$$ is chain homotopic to the change of diagrams map from $$CF(D_\beta(\mathcal{H}))$$ to $$CF(\mathcal{H})$$.

**Proof.** The idea is simple: The map $$F_2$$ can be interpreted as a composition of 2-handle maps, and the map $$F_3$$ can be interpreted as a composition of 3-handle maps, for a collection of 4-dimensional 2-handles and 3-handles that topologically cancel. Thus, the induced composition is chain homotopic to the change of diagrams map, by the well-definedness of the sutured cobordism maps [Juh16].

We now explain the technical details. Let us first describe the framed link that $$F_2$$ represents surgery on. Let $$U'$$ denote the $$\beta$$-handlebody of the diagram $$D_\beta(\mathcal{H})$$ (so that $$\Delta$$ bound compressing disks in $$U'$$). For each curve in $$\beta$$ and each curve in $$\delta \beta$$, we will construct a component of $$L$$. To a curve $$\beta_i \in \beta$$, there is a knot $$K_{\beta_i} \subseteq U'$$, obtained by pushing $$\beta_i$$ into $$U'$$ slightly. We choose the framing of $$K_{\beta_i}$$ to be parallel to $$\Sigma$$. Similarly, given $$\delta_k \in \delta \beta$$, we can construct a framed knot $$K_{\delta_k}$$ by pushing $$\delta_k$$ into $$U'$$, and take the framing induced by the tangent space of the surface $$D_\beta(\Sigma)$$. The construction is illustrated in Figure 7.9. We define the framed link $$L$$ to be the union

$$L := \bigcup_{\tau \in \beta \cup \delta \beta} K_{\tau}.$$

Note that $$(D_\beta(\Sigma), \alpha \cup \beta', \beta \cup \beta')$$ is a diagram for the surgered manifold $$M(L)$$. Furthermore, it is not hard to see that the triple $$(D_\beta(\Sigma), \alpha \cup \beta', \Delta, \beta \cup \beta')$$ can be connected via handleslides and isotopies to a triple which is subordinate to a bouquet for $$L$$. Therefore the map $$F_2$$ corresponds to
the map induced by surgery. The map $F_3$ is by definition the 3-handle map for a collection of $|\beta'|$ 2-spheres in $M(\mathbb{L})$ that topologically cancel the link $\mathbb{L}$. It follows that the composition $F_3 \circ F_2$ is chain homotopic to the change of diagrams map from naturality.

7.6. **The change of diagrams map from $C(\mathcal{H})$ to $D_\beta(\mathcal{H})$.** Analogously to the change of diagrams maps from the previous section, it is convenient to describe the change of diagrams map from $C(\mathcal{H})$ to $D_\beta(\mathcal{H})$ in concrete terms.

Pick a collection of pairwise disjoint arcs on $R_\alpha$ with endpoints in $\mathcal{I}_0$ that form a basis of $H_1(R_\alpha, \mathcal{I}_0)$. These induce arcs on $\Sigma$ disjoint from $\alpha$, and let $\Delta_\alpha \subseteq \Sigma \cup \mathcal{I}_0, \Sigma$ be the curves formed by doubling them across $\mathcal{I}_0$. By construction, the curves in $\Delta_\alpha$ are disjoint from $\alpha$ and $\bar{\alpha}$. Also, let $\delta_\alpha$ denote the images of the curves $\Delta_\alpha$ on $R_\alpha, \Sigma$. Let $\alpha' := \bar{\alpha} \cup \Delta_\alpha$ and $\beta' := \beta \cup \delta_\beta$.

Recall that the surface $D_\beta(\Sigma)$ is $\Sigma \cup \bar{\alpha}, \Sigma \cup \mathcal{I}_0, \Sigma \cup R_\beta$, and $C(\Sigma)$ is defined as $R_\alpha, \Sigma \cup \mathcal{I}_0, \Sigma \cup R_\beta$. Since surgering the surface $\Sigma$ on the $\alpha$ curves yields $R_\alpha$, we see that there is a 1-handle map

$$G_1 := F_1^{\alpha, \alpha'}: CF(C(\mathcal{H})) = CF(C(\Sigma), \beta', \bar{\alpha} \cup \delta_\alpha) \to CF(D_\beta(\Sigma), \alpha \cup \beta', \bar{\alpha} \cup \alpha').$$

We note that $(D_\beta(\Sigma), \alpha \cup \alpha', \Delta_\Sigma)$ is a diagram for

$$\left( (S^1 \times S^2)^{|\alpha|}(\bar{\Sigma}), \partial R_\beta \times I, \partial R_\beta \times I \right).$$

Indeed, on this diagram, all the attaching curves are disjoint from $R_\beta$. If we cut $R_\beta$ off of $D_\beta(\Sigma)$, we are left with the diagram $(\Sigma \cup \mathcal{I}_0, \Sigma, \alpha \cup \alpha', \Delta_\Sigma)$, which is the double of a diagram for the sutured manifold $(S^1 \times S^2)^{|\alpha|}(\bar{\Sigma})$ obtained by removing $\partial \Sigma$ 3-balls from $(S^1 \times S^2)^{|\alpha|}$, and adding a connected suture to each boundary component. In particular, there is a top-graded generator

$$\Theta_{\alpha \cup \alpha', \Delta_\Sigma} \in SFH(D_\beta(\Sigma), \alpha \cup \alpha', \Delta_\Sigma),$$

and hence we can also define a triangle map

$$G_2 := F_{\alpha \cup \beta', \alpha \cup \alpha', \Delta_\Sigma}(-, \Theta_{\alpha \cup \alpha', \Delta_\Sigma}),$$

where $\Theta_{\alpha \cup \alpha', \Delta_\Sigma}$ is the top-graded generator of $SFH(D_\beta(\Sigma), \alpha \cup \alpha', \Delta_\Sigma)$.

**Lemma 7.7.** The composition $G_2 \circ G_1$ is chain homotopic to the change of diagrams map from $C(\mathcal{H})$ to $D_\beta(\mathcal{H})$.

**Proof.** As in the proof of Lemma 7.6, the composition $G_2 \circ G_1$ can be interpreted as the cobordism map for a canceling collection of 4-dimensional 1-handles and 2-handles. □
7.7. A diagram for \( M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma} \) and a formula for the special cobordism map \((W_{\alpha,\beta,\gamma})^*\).

Let \( T = (\Sigma, \alpha, \beta, \gamma) \) be an admissible sutured triple diagram. We now describe a diagram \( A(T) \) for the sutured manifold \((M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma}, \gamma_{\alpha,\gamma})\), which has sutures along \( \partial R_\beta \), where the two manifolds are glued together.

Analogous to the doubled diagram, we let \( I_0, I_1 \subseteq \partial \Sigma \) be disjoint collections of subintervals, such that each component of \( \partial \Sigma \) contains exactly one subinterval from \( I_0 \) and \( I_1 \). We form the diagram

\[
A(T) = (D_\gamma(\Sigma), D_\gamma(\alpha), A(\beta)) = (\Sigma \sharp I_0 \Sigma \sharp I_1 R_\gamma, \alpha \cup \gamma \cup \delta_\gamma, \beta \cup \bar{\beta} \cup \Delta_\beta),
\]

where the component \( \Sigma \) of \( D_\gamma(\Sigma) \) is embedded in \( M_{\alpha,\beta} \) and the component \( \bar{\Sigma} \) in \( M_{\beta,\gamma} \). We call \( A(T) \) the amalgamation of \( T \) along \( R_\beta \). Here \( \delta_\gamma \subseteq \Sigma \sharp I_1 R_\gamma \) is obtained by doubling a collection of arcs forming a basis of \( H_1(R_\gamma, I_1) \). Similarly, \( \Delta_\beta \subseteq \Sigma \sharp I_0 \bar{\Sigma} \) is obtained by doubling a basis of arcs in \( H_1(R_\beta, I_0) \) across \( I_0 \).

Note that the doubling construction is an instance of amalgamation, as

\[
D_\beta(\Sigma, \alpha, \beta, \gamma) = A(\Sigma, \alpha, \emptyset, \beta).
\]

Here we interpret \( R_\emptyset \) as \( \Sigma \), so that \( \Delta_\beta \) is the collection \( \Delta_\Sigma \) defined in the construction of a doubled diagram.

**Lemma 7.8.** The diagram \( A(T) \) is a diagram for \((M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma}, \gamma_{\alpha,\gamma})\).

**Proof.** The diagram \( A(T) \) is obtained by first replacing \( R_\beta \) with \( \Sigma \) in

\[
C(H_{\beta,\gamma}) = (R_\beta \sharp I_0 \Sigma \sharp I_1 R_\gamma, \gamma \cup \delta_\gamma, \beta \cup \bar{\beta}),
\]

after which \( \delta_\beta \) becomes \( \Delta_\beta \). As compressing \( \Sigma \) along \( \beta \) gives \( R_\beta \), if we add \( \beta \), the resulting diagram

\[
(\Sigma \sharp I_0 \Sigma \sharp I_1 R_\gamma, \gamma \cup \delta_\gamma, \beta \cup \bar{\beta} \cup \delta_\beta)
\]

represents \((-U_\beta \cup_{R_\beta} U_\beta) \cup \Sigma - U_\gamma, \partial \Sigma \). Finally, we add \( \alpha \), which amounts to gluing \( U_\alpha \) to \(-U_\beta \cup_{R_\beta} U_\beta \cup \Sigma - U_\gamma \) along \( \Sigma \subseteq \partial(-U_\beta) \).

By Lemma 7.4, the special cobordism

\[
(W_{\alpha,\beta,\gamma})^*: (M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma}, \gamma_{\alpha,\gamma}) \to (M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})
\]

is a 2-handle cobordism, for surgery on a framed link \( L \subseteq M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma} \). We recall the description of the framed link \( L \). One takes a Morse function \( f_\beta \) on \( U_\beta \) that is 1 on \( \Sigma \) and 0 on \( \bar{R}_\beta \), and has \( \beta \) as the intersection of the ascending manifolds of the critical points of \( f_\beta \) with \( \Sigma \). Then the descending manifolds of the critical points of \( f_\beta \) determine a collection of \( |\beta| \) properly embedded arcs \( \lambda_i \subseteq U_\beta \) that have both ends on \( R_\beta \). The link \( L \) is obtained by taking the union of the arcs \( \lambda_i \subseteq -U_\beta \subseteq M_{\alpha,\beta} \), together with their images in \( U_\beta \subseteq M_{\beta,\gamma} \).

Let \( \Delta_\Sigma \subseteq \Sigma \sharp I_0 \Sigma \) be a collection of curves obtained by picking arcs forming a basis of \( H_1(\Sigma, I_0) \), and then doubling them across \( I_0 \). One can assume that the chosen basis of \( H_1(\Sigma, I_0) \) extends the basis of \( H_1(R_\beta, I_0) \) we chose in the construction of the \( \delta_\beta \) curves, so that \( \delta_\beta \subseteq \Delta_\Sigma \), though this is not essential.

We note that

\[
SFH(D_\gamma(\Sigma), A(\beta), \Delta_\Sigma) = SFH(\Sigma \sharp I_0 \Sigma \sharp I_1 R_\gamma, \beta \cup \bar{\beta} \cup \Delta_\beta, \Delta_\Sigma)
\]

admits a top-graded generator \( \Theta_{A(\beta), \Delta_\Sigma} \). Indeed, there are no curves in this diagram on \( R_\gamma \). Upon cutting \( R_\gamma \) out, we are left with \( (\Sigma \sharp I_0 \Sigma \cup \beta \cup \bar{\beta} \cup \Delta_\beta, \Delta_\Sigma) \), which is a doubled diagram for \((S^1 \times S^2)|_{\beta}\) with \( |\partial \Sigma| \) 3-balls removed, and a single suture added to each boundary \( S^2 \). Hence \( (D_\gamma(\Sigma), A(\beta), \Delta_\Sigma) \) is a diagram for \((S^1 \times S^2)|_{\beta}\) \(|_{\partial \Sigma}| \#(R_\gamma \times I, \partial R_\gamma \times I) \).

Note that \((D_\gamma(\Sigma), D_\gamma(\alpha), \Delta_\Sigma) \) is a doubled diagram for \((M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})\). We have the following:

**Lemma 7.9.** The special cobordism map

\[
F_{(W_{\alpha,\beta,\gamma})^*}: CF(M_{\alpha,\beta} \cup_{R_\beta} M_{\beta,\gamma}, \gamma_{\alpha,\gamma}) \to CF(M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})
\]

is chain homotopic to the triangle map

\[
F_{D_\gamma(\alpha), A(\beta), \Delta_\Sigma}(-, \Theta_{A(\beta), \Delta_\Sigma}).
\]
Proof. By Lemma 7.4, the special cobordism \((W_{\alpha,\beta,\gamma})^s\) is a 2-handle cobordism, for a framed link \(\mathbb{L} \subseteq -U_\beta \cup \mathcal{R}_\beta \cup U_\beta\) described above. It is not hard to see that the triple 
\[(D_\gamma(S), D_\gamma(A), A(\beta), \Delta)\]
can be related by a sequence of handleslides and isotopies to a triple that is subordinate to a bouquet for \(\mathbb{L}\). By definition of the cobordism maps, it follows that \(F_{(W_{\alpha,\beta,\gamma})^s}\) is chain homotopic to the triangle map above. \(\square\)

7.8. A holomorphic triangle description of the gluing map. Let \(\mathcal{T} = (\Sigma, \alpha, \beta, \gamma)\) be a sutured Heegaard triple. In this section, we present a natural candidate for the map for gluing a holomorphic triangle description of the gluing map.

Let 
\[\mathcal{C}(\mathcal{H}_{\beta,\gamma}) = (R_\beta \cup \Sigma \cup R_\gamma, \gamma \cup \delta_\beta, \beta \cup \delta_\beta)\]
be a weak conjugate of \(\mathcal{H}_{\beta,\gamma} = (\Sigma, \beta, \gamma)\), as described in Section 7.4. We now construct our candidate

\[\Psi : CF(\mathcal{H}_{\alpha,\beta}) \otimes CF(C(\mathcal{H}_{\beta,\gamma})) \to CF(A(T))\]

for the gluing map. Notice that the domain of the map \(\Psi\) is \(CF(M_{\alpha,\beta} \cup R_\beta, M_{\beta,\gamma} \gamma, \alpha, \gamma)\), and its range is \(CF(M_{\alpha,\beta} \cup R_\beta, M_{\beta,\gamma} \gamma, \alpha, \gamma)\).

The definition of the map \(\Psi\) is formally similar to the map for connected sums due to Ozsváth and Szabó [OS04b]. We will call \(\Psi\) the \textit{amalgamation map}, since its image is in the Floer homology of the amalgamated diagram from the previous section. We also remark that the Ozsváth–Szabó maps that inspire the construction of \(\Psi\) were called the \textit{intertwining maps} in [Zem18], where they played a similar role as in our present context.

Note that there is a 1-handle map
\[\Phi^{\gamma,\delta_\gamma,\delta_\gamma,\delta_\gamma}_1 : CF(\mathcal{H}_{\alpha,\beta}) \to CF(D_\gamma(S), D_\gamma(A), D_\gamma(A)),\]
where \(D_\gamma(A) = \alpha \cup \gamma \cup \delta_\gamma\) and \(D_\gamma(A) = \beta \cup \gamma \cup \delta_\beta\). Similarly, there is also a 1-handle map
\[\Phi^{\beta,\beta}_1 : CF(C(\mathcal{H}_{\beta,\gamma})) \to CF(D_\gamma(S), D_\gamma(A), A(\beta)).\]

The 1-handle map \(\Phi^{\beta,\beta}_1\) is obtained by adding the \(\beta\) curves to \(R_\beta \subseteq C(\mathcal{H}_{\beta,\gamma})\), as 1-handles. Note that, when we apply the 1-handle maps to add in the \(\beta\) curves to \(R_\beta\), we obtain \(\Sigma\), and the curves \(\delta_\beta\) become \(\Delta_\beta\). Finally, we define

\[\Psi := \Phi^{\gamma,\delta_\gamma,\delta_\gamma,\delta_\gamma}_1(\Phi^{\beta,\beta}_1) \otimes \left(\Phi^{\gamma,\delta_\gamma,\delta_\gamma,\delta_\gamma}_1 \otimes \Phi^{\beta,\beta}_1\right).\]

A key ingredient in our analysis of the sutured cobordism \(W_{\alpha,\beta,\gamma}\) is the following:

**Proposition 7.10.** The amalgamation map
\[\Psi : SFH(M_{\alpha,\beta} \cup R_\beta, M_{\beta,\gamma} \gamma, \alpha, \gamma) \to SFH(M_{\alpha,\beta} \cup R_\beta, M_{\beta,\gamma} \gamma, \alpha, \gamma)\]
defined above is chain homotopic to the contact gluing map for gluing \((Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})\) to \((-M_{\alpha,\beta} \cup -\gamma, -M_{\beta,\gamma} \cup -\gamma, \gamma)\).

**Proof.** First, we describe a contact handle decomposition of \((Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})\), relative to \(R_{\alpha,\beta} \cup R_{\beta,\gamma}\). We pick a collection of subintervals \(I_0 \subseteq \partial R_\beta\) such that each component of \(\partial R_\beta\) contains exactly one subinterval. Now we pick a collection of pairwise disjoint arcs \(\lambda_1, \ldots, \lambda_n\) on \(R_\beta\) with boundary on \(I_0\) that form a basis of \(H_1(R_\beta, I_0)\). Note that similar collections of arcs were used to construct doubled diagrams and weakly conjugated diagrams.

We claim that a contact handle decomposition of \((Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})\), relative to \(R_{\alpha,\beta} \cup R_{\beta,\gamma}\), can be constructed as follows:

1. the contact 1-handles are the components of \(N(I_0 \times I)\) (hence, we have one 1-handle for each component of \(\partial R_\beta\));
2. the contact 2-handles are \(N(\lambda_i \times I)\) for \(i \in \{1, \ldots, n\}\).
The 1-handles are simply added with feet along the corresponding subintervals of $I_0 \subseteq R_\beta \subseteq \partial M_{\alpha,\beta}$ and $\tilde{I}_0 \subseteq \tilde{R}_\beta \subseteq \partial M_{\beta,\gamma}$. The 2-handles are attached along curves obtained by concatenating an arc $\lambda_i$ on $R_\beta$ with its mirror on $\tilde{R}_\beta$.

To show that the above description determines a contact handle decomposition, we use Lemma 7.3. This allows us to write $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$ as $(R_{\alpha,\beta} \times I, -\xi_{\alpha,\beta})$ and $(R_{\beta,\gamma} \times I, -\xi_{\beta,\gamma})$ glued together along two subsurfaces $\tilde{R}_\beta \subseteq R_{\alpha,\beta} \times \{0\}$ and $R_\beta \subseteq R_{\beta,\gamma} \times \{0\}$ that are small perturbations of $\tilde{R}_\beta \subseteq \tilde{R}_{\alpha,\beta} \times \{0\}$ and $R_\beta \subseteq R_{\beta,\gamma} \times \{0\}$, respectively. We pick $R_\beta'$ and $\tilde{R}_\beta'$ such that $\tilde{I}_0 \subseteq R_\beta'$ and $\tilde{I}_0 \subseteq \tilde{R}_\beta'$. We identify $R_\beta'$ and $\tilde{R}_\beta'$ with their images on $\partial M_{\alpha,\beta}$ and $\partial M_{\beta,\gamma}$, respectively. The above description implies that the cores of the 1-handles in our contact handle decomposition are Legendrian, and the attaching curves of the 2-handles cross the dividing set exactly twice. Using Legendrian realization after attaching the contact 1-handles, we can assume the attaching curves of the 2-handles cross the dividing set exactly twice. Using the above description determines a contact handle decomposition of $Z_{\alpha,\beta,\gamma}$, starting at $R_{\alpha,\beta} \sqcup R_{\beta,\gamma}$ and ending at $R_{\alpha,\gamma}$.

Using the above contact handle decomposition of $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$, we can now give a description of the gluing map associated to $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$ on the level of Heegaard diagrams.

The contact 1-handle maps have a simple description in terms of diagrams; one simply adds a band with ends at the feet of the 1-handle. There are two dividing arcs on the band, and they are distinguished: One intersects $R_\beta' \times I \subseteq Z_{\alpha,\beta,\gamma}$ nontrivially, while one is disjoint from $R_\beta' \times I$; see Figure 7.10. Let us call the edge that intersects $R_\beta' \times I$ the “inner” dividing arc of the 1-handle. The other edge we call the “outer” dividing arc. We will write $e_o$ for the outer arcs, and $e_i$ for the inner arcs.

![Figure 7.10](image)

Figure 7.10. A contact 1-handle added to $-M_{\alpha,\beta} \sqcup -M_{\beta,\gamma}$. On the 1-handle, the inner edge $e_i$ of the dividing set is contained in $R_\beta' \times I$, and the outer edge $e_o$ is disjoint from $R_\beta' \times I$.

Note that the attaching circles of the 2-handles only intersect the dividing set along the contact 1-handles, and only along the inner arcs $e_i$. Hence, on the level of diagrams, the contact 2-handle map involves adding a band between two points on inner edges of the bands for the contact 1-handles we added, and also adding an $\alpha$-curve and a $\beta$-curve.

Recall that the contact 2-handles were obtained by picking a basis of arcs for $H_1(R_\beta, I_0)$. A collection of pairwise disjoint arcs on $R_\beta$ that form a basis of $H_1(R_\beta, I_0)$ is the same as a handle decomposition of $R_\beta$ into 1-handles that are attached to a neighborhood of $I_0$ in $R_\beta$. Hence, when we add all the 2-handle bands to the 1-handle bands, we get a copy of $R_\beta$ (the orientation of the new copy of $R_\beta$ becomes apparent if one analyzes the orientation of the band added by a contact 2-handle in Section 3.2.3).

We call the collection of $\beta$-curves that we add with the contact 2-handles $\delta_\beta$, and we call the $\alpha$-curves that we add $\epsilon$. By assumption, we start at the disjoint union of $H_{\alpha,\beta} = (\Sigma, \alpha, \beta)$ and a
weak conjugate
\[ C(H_{\beta, \gamma}) = (R_\beta \# \Sigma_2 R_\gamma, \gamma \cup \delta_\gamma, \beta \cup \delta_\beta) \]
of \( H_{\beta, \gamma} = (\Sigma, \beta, \gamma) \). After adding all the 2-handles, we obtain the Heegaard diagram
\[ G(T) = (\Sigma(T), \alpha(T), \beta(T)) := (\Sigma_2 R_\beta \# \Sigma_2 R_\gamma, \alpha \cup \epsilon \cup \tilde{\gamma} \cup \delta_\gamma, \beta \cup \delta_\beta \cup \tilde{\beta}) \]
for \( M_{\alpha, \beta} \cup M_{\beta, \gamma} \). Note that the diagram \( G(T) \) is similar to, but not equal to, the Heegaard diagram \( A(T) \) of \( M_{\alpha, \beta} \cup R_\delta M_{\beta, \gamma} \) that appears in the target of the map in Proposition 7.10.

Let us now describe the \( \delta_\beta \), \( \epsilon \), and \( \tilde{\delta}_\beta \) curves more explicitly. Let us view \( R_\beta \) as being built from a single 0-handle for each contact 1-handle (equivalently, for each component of \( \partial \)) such that \( \tilde{\beta} \). An example is shown in Figure 7.12.

Similarly, we can view the band added by each contact 1-handle as having four edges lying in \( \tilde{\lambda}_0 \), \( e_\alpha \), \( \lambda_0 \), and \( e_\epsilon \). Each contact 2-handle corresponds to adding a band along \( e_\epsilon \), and adding a curve in \( \epsilon \) and a curve in \( \tilde{\beta} \). Let \( c_i \) be the core of the band of the contact 2-handle associated to the arc \( \lambda_i \) on \( R_\beta \). We extend the curves \( c_i \) across the bands of the 1-handles so that they have both ends on \( e_\epsilon \); see the top left and top right of Figure 7.11. If we isotope each \( c_i \) near \( e_\epsilon \), such that its ends are in \( \tilde{\lambda}_0 \), and then double it across \( \tilde{\lambda}_0 \) onto \( R_\beta \), then we get the \( \epsilon \) curves. If we isotope each \( c_i \) near \( e_\epsilon \) in the opposite direction until its ends are in \( \tilde{\lambda}_0 \), and then we double it across \( \tilde{\lambda}_0 \) onto \( \Sigma \), then we the \( \delta_\beta \) curves. Examples of the curves \( \delta_\beta \) and \( \epsilon \) on \( R_\beta \) are shown in Figure 7.11.

Let \( \tilde{\lambda}_1 \) denote the collection of subarcs of \( \partial R_\beta \) that are used to connect \( R_\beta \) to \( \Sigma \) in the weakly conjugated diagram \( C(H_{\beta, \gamma}) \). Note that we can pick an orientation reversing diffeomorphism between the new copy of \( R_\beta \) and \( \tilde{R}_\beta \) (viewed as a subset of the weakly conjugated Heegaard surface \( R_\beta \# \Sigma_2 R_\gamma \)) such that \( \tilde{\lambda}_0 \subseteq \tilde{R}_\beta \) is mapped to \( \tilde{\lambda}_0 \subseteq R_\beta \), and \( \tilde{\lambda}_0 \subseteq \tilde{R}_\beta \) is mapped to \( \tilde{\lambda}_1 \subseteq R_\beta \). Furthermore, we can assume that \( \epsilon \cap \tilde{R}_\beta \) is mapped to \( \epsilon \cap R_\beta \), and that \( \tilde{\delta}_\beta \cap \tilde{R}_\beta \) is mapped to \( \tilde{\delta}_\beta \cap R_\beta \).

The next step to understanding the gluing map is to destabilize the region \( R_\beta \# \tilde{R}_\beta \). Unfortunately, the curves \( \tilde{\delta}_\beta \) and \( \epsilon \) are not suitable for this (even if we use the compound destabilization operation from Section 2.2). In order to present the curves in a reasonable manner, we need to do some handleslides amongst the \( \delta_\beta \), \( \tilde{\delta}_\beta \), and \( \epsilon \) curves, as we now describe.

We perform two moves. First, we modify the \( \epsilon \) curves, as follows. Recall that we obtained \( \tilde{\delta}_\beta \) and \( \epsilon \) by isotoping the cores of the 2-dimensional 1-handles near the boundary of \( \tilde{R}_\beta \). Let us write \( c_i^\epsilon \) for the co-core of the handle with core \( c_i \). By construction, \( |c_i^\epsilon \cap c_j| = \delta_{ij} \).

We now isotope each \( c_i^\epsilon \) in a neighborhood of \( c_i \cup \lambda_0 \) such that its ends lie in \( \tilde{\lambda}_0 \). Let \( \epsilon' \) denote the closed curves obtained by doubling the resulting curves across \( \tilde{\lambda}_0 \), onto \( R_\beta \). Note that we perform this isotopy after \( \tilde{\lambda}_1 \) the 2-handle bands have been added (not after just the corresponding 2-handle has been attached). An example is shown in Figure 7.12.

Since
\[ |(\delta_\beta); \cap \epsilon'_j| = |(\tilde{\delta}_\beta); \cap \epsilon'_j| = \delta_{ij}, \]
we can handleslide \( \delta_\beta \) over \( \delta_\beta \) along \( \epsilon' \) in such a way that the resulting curves \( \tilde{\delta}_\beta \) do not intersect the \( \epsilon' \) curves. With this configuration, we note that \( \epsilon' \) intersects only \( \tilde{\delta}_\beta \), and, furthermore, the two sets of curves come in pairs. A sequence of destabilizations can then be used to surger out the \( \epsilon' \) curves, while removing the \( \tilde{\delta}_\beta \) curves. Once we do this, we are left with the diagram \( A(T) \) of \( M_{\alpha, \beta} \cup R_\delta M_{\beta, \gamma} \) described in Section 7.7.

We now describe the effect of these Heegaard moves on the sutured Floer complexes, and relate it to the desired triangle map formula.

As a first step, by the same triangle map computation that shows that the gluing map is invariant under the choice of diagrams (shown in Figure 3.6), the gluing map satisfies
\[ \Phi_{-\xi} = F_{\alpha(T), \beta \cup \gamma', \beta(T) \circ (\bar{F}_{1}^{\gamma'} \gamma') \circ (\bar{F}_{1}^{\beta \beta} \circ C_{\epsilon' \tilde{\gamma}} \circ C_{\epsilon \tilde{\gamma}})}, \]
where \( \gamma' := \epsilon \cup \bar{\gamma} \cup \tilde{\gamma} \).
In the above expression, the contact handle map $C_{\epsilon, \bar{\delta}}$ is defined by adding the surface $R_{\beta}^\circ \Sigma R_{\gamma}$ to $R_{\beta \Sigma} \Sigma R_{\gamma}$, and then adding in the curves $\epsilon$ and $\bar{\delta}_{\beta}$. On the level of complexes, it is defined by the formula $x \mapsto x \times c$, where $c$ is the distinguished intersection point of $T_{\epsilon} \cap T_{\bar{\delta}_{\beta}}$. The copy of $R_{\beta}$ that is added should be thought of as $\Sigma$ surgered along the $\beta$ curves. Note that $C_{\epsilon, \bar{\delta}_{\beta}}$ can be described by attaching contact 0-handles and 1-handles to add $R_{\beta}$, and then attaching contact 2-handles to add $\bar{R}_{\beta}$ and the curves $\epsilon$ and $\bar{\delta}_{\beta}$.

We now change $\epsilon$ to $\epsilon'$. Define

$$\gamma'':=\epsilon' \cup \gamma \cup \delta_{\gamma}.$$ 

Using naturality and associativity,

$$\Phi^{\beta, \gamma'}_{\alpha(T) \rightarrow \alpha \cup \gamma''} \circ \Phi_{-\epsilon} \simeq F_{\alpha \cup \gamma, \beta \cup \gamma', \beta(T)} \circ \left( \left( \Phi^{\beta, \gamma'}_{\alpha(T) \rightarrow \alpha \cup \gamma''} \circ F_{\gamma', \gamma} \right) \otimes \left( \Phi^{\beta, \gamma'}_{\beta, \beta \cup \gamma'} \circ F_{1}^{\beta, \beta} \circ \Phi^{\beta, \gamma'}_{\beta} \right) \right).$$

We claim that

$$\Phi^{\beta, \gamma'}_{\alpha(T) \rightarrow \alpha \cup \gamma''} \circ F_{\gamma', \gamma} \simeq F_{1}^{\gamma'', \gamma''}.$$
Figure 7.12. The portion of the Heegaard surface identified with $\bar{R}_\beta \natural R_\beta$. In this case, $R_\beta$ is a genus 1 surface with one boundary component. The curves $\epsilon$ are shown in the second row. The curves $\epsilon'$ are shown in the third. The curves $\delta_\beta$ and $\bar{\delta}_\beta$ are shown in the last row.

To see this, first note that the domain of both maps is $SFH(\Sigma, \alpha, \beta)$. The 1-handle maps $F_1^{\gamma', \gamma''}$ are defined by taking the boundary connected sum with $\Sigma' := \bar{R}_\beta \natural R_\beta \natural \bar{\Sigma}$. The boundary connected sum operation yields $|\partial \Sigma|$ arcs on $\Sigma(T) = \Sigma \natural \Sigma'$ that separate $\Sigma$ from $\Sigma'$, and intersect none of the attaching curves in $\alpha$, $\beta$, $\gamma'$, or $\gamma''$. Hence, the change of diagrams map appearing on the left-hand side of equation (16) involves only counting holomorphic triangles with image on the disjoint union of $\Sigma$ and $\Sigma'$. Since there is a unique top-graded generator of $SFH(\Sigma', \gamma', \gamma'')$, and $\gamma''$ is obtained from $\gamma'$ by a sequence of handleslides, that generator will be preserved by the change of diagrams map from $SFH(\Sigma', \gamma', \gamma'')$ to $SFH(\Sigma', \gamma'', \gamma'')$. Hence, the 1-handle map will be preserved. Equation (16) now follows.

It follows that the gluing map is chain homotopic to

$$F_{\alpha \cup \gamma'', \beta \cup \gamma', \beta(T)} \circ \left( F_1^{\gamma'', \gamma'} \otimes \left( \phi_\beta(T)_{\beta \cup \gamma'' \rightarrow \beta \cup \gamma'} \circ F_1^{\beta, \beta} \circ C_{\epsilon, \delta_\beta} \right) \right).$$

Note that, on the diagram $(\Sigma(T), \beta \cup \gamma'', \beta(T))$, the $\delta_\beta$ curves each have only one intersection point, which occurs with an $\epsilon'$ curve. The $\epsilon'$ curves still intersect both $\delta_\beta$ and $\bar{\delta}_\beta$. Further, each $\epsilon'$ curve intersects exactly one $\delta_\beta$ curve. Hence, we can consider the compound stabilization map $\sigma_{\epsilon', \delta_\beta}$ defined in Section 2.2. It agrees with the map from naturality by Proposition 2.2.

We claim that

$$\phi_\beta(T)_{\beta \cup \gamma'' \rightarrow \beta \cup \gamma'} \circ F_1^{\beta, \beta} \circ C_{\epsilon', \delta_\beta} \simeq F_1^{\beta, \beta} \circ \sigma_{\epsilon', \delta_\beta} : SFH(C(H_\beta, \gamma)) \rightarrow SFH(\Sigma(T), \beta \cup \gamma'', \beta(T)).$$
By Proposition 2.1, it suffices to show the claim with the β curves surgered out, and with no 1-handle maps (note that it is easy to show that on Σ, there is a path from each β curve to ∂Σ that avoids δβ ∩ Σ, so the hypotheses of the previously mentioned proposition are satisfied). Therefore, it suffices to show that
\[ σ′.δβ \simeq \Phi^{-1}_{Tγ → γ′} \circ Cε.δβ, \]
or, equivalently, that
\[ id \simeq (σ′.δβ)^{-1} \circ \Phi^{-1}_{Tγ → γ′} \circ Cε.δβ. \]
However, this holds because of the functoriality of the gluing map; i.e., because the right-hand side represents the map induced by gluing a trivial a copy of Rβ × I to −Mβ.γ.

Hence, we conclude that the gluing map Φ−ε is chain homotopic to
\[ Fα∪γ′′,β,γ(Σ) \circ \left( F_{1}^{β,γ} \circ (F_{1}^{β,γ'} \circ σ′.δβ) \right). \]
We still need to handleslide the δβ over δβ to become δβ, and then compound destabilize. That is, the gluing map is chain homotopic to
\[ (σ′.δβ)^{-1} \circ \Phi^{-1}_{Tγ → γ′} \circ Fα∪γ′′,β,γ(Σ) \circ \left( F_{1}^{β,γ} \circ (F_{1}^{β,γ'} \circ σ′.δβ) \right). \]
where β(T)' = β ∪ δβ ∪ δβ ∪ δβ. By associativity, this is chain homotopic to
\[ (σ′.δβ)^{-1} \circ \left( Fα∪γ′′,β,γ(Σ) \circ \left( F_{1}^{β,γ'} \circ \left( F_{1}^{β,γ'} \circ σ′.δβ \right) \right) \right). \]
By an analog of Lemma 2.3, concerning triangle maps and compound stabilizations, this now becomes
\[ F_{Dγ(α),Dγ(β),Aβ}(Σ) \circ \left( F_{1}^{C(γ),C(γ)} \circ (σ′.δβ)^{-1} \circ \Phi_{Tγ → γ′} \circ F_{1}^{β,γ} \circ σ′.δβ \right) \]
where C(γ) = γ ∪ δγ. Applying the triangle counts from Proposition 2.1 to the map F1β,γ for 1-handles added near the boundary, and also commuting the map F1β,γ with the destabilization map, this becomes
\[ F_{Dγ(α),Dγ(β),Aβ}(Σ) \circ \left( F_{1}^{C(γ),C(γ)} \circ (σ′.δβ)^{-1} \circ \Phi_{Tγ → γ′} \circ F_{1}^{β,γ} \circ σ′.δβ \right) \]
We claim that
\[ (σ′.δβ)^{-1} \circ \Phi_{Tγ → γ′} \circ F_{1}^{β,γ} \circ σ′.δβ \simeq id, \]
which follows simply from naturality, as it is a loop in the space of Heegaard diagrams. We have now arrived at our desired formula, concluding the proof of Proposition 7.10.

### 7.9. Computation of the triangle map

We now prove that the triangle cobordism map for Wα,β,γ is chain homotopic to the map that counts holomorphic triangles on a single Heegaard triple, by using the formula from the previous section for the contact gluing map.

**Theorem 7.11.** If (Σ, α, β, γ) is a sutured Heegaard triple, then the sutured cobordism map
\[ F_{Wα,β,γ}: CF(Σ, α, β) ⊗ CF(Σ, β, γ) → CF(Σ, α, γ) \]
is chain homotopic to the map that counts holomorphic triangles on the triple (Σ, α, β, γ).

**Proof.** We first compose with the change of diagrams map id ⊗ Φ_{Hβ,γ → C(β,γ)}. The next step is to use the gluing map to glue the two copies of Rβ together. Then we performs surgery on a |β|-component framed link. See Section 7.7 for a description of the triangle diagram
\[ (Dγ(Σ), Dγ(α), Aβ, ΔΣ) \]
used for computing the 2-handle cobordism map. This yields (omitting writing the first change of diagrams map)
\[ F_{Dγ(α),Aβ,ΔΣ} \circ \left( F_{Dγ(α),Dγ(β),Aβ} \circ \left( F_{1}^{γ∪δγ,γ′∪δγ} \circ F_{1}^{β,β'} \right) \right) \circ Θ_{Aβ,ΔΣ} \].
We now use associativity to see that this is chain homotopic to
\[ F_{D_\gamma(\alpha),D_\gamma(\beta),\Delta_E} \circ \left( F_1^{\alpha \cup \beta, \gamma} \otimes \left( F_{D_\gamma(\beta),A(\beta),\Delta_E} \circ \left( F_1^{\beta, \Omega A(\beta),\Delta_E} \right) \right) \right) \]

Note that
\[ \Phi := F_{D_\gamma(\beta),A(\beta),\Delta_E} \circ \left( F_1^{\beta, \Omega A(\beta),\Delta_E} \right) \]
is the change of diagrams map \( \Phi_{C(H_{\beta, \gamma})} \) from a weakly conjugated diagram to a doubled diagram, by Lemma 7.7. Thus, the cobordism map is
\[ (17) \]
\[ F_{D_\gamma(\alpha),D_\gamma(\beta),\Delta_E} \circ \left( F_1^{\gamma_1 \cup \delta, \gamma_2} \otimes \Phi \right). \]

The range of this map is not \( SFH(H_{\beta, \gamma}) \), but instead a double of this diagram along \( R_\gamma \), so we must compose with the change of diagrams map from \( D_\gamma(H_{\beta, \gamma}) \) to \( H_{\beta, \gamma} \), which is
\[ F_3^{\gamma_1 \cup \delta_1, \gamma_2 \cup \delta_2} \circ \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( F_1^{\gamma \cup \delta} \otimes \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( \Phi \otimes \Omega \Delta_E,D_\gamma(\gamma) \right) \right) \right) \right) \]

by Lemma 7.7. We now post compose equation (17) with this expression, and use associativity, and we see that the composition is chain homotopic to
\[ F_{D_\gamma(\alpha),D_\gamma(\beta),\Delta_E} \circ \left( F_1^{\gamma_1 \cup \delta_1, \gamma_2 \cup \delta_2} \circ \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( F_1^{\gamma, \gamma_2 \cup \delta_2} \otimes \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( \Phi \otimes \Omega \Delta_E,D_\gamma(\gamma) \right) \right) \right) \right) \right). \]

Using the 3-handle and triangle map computation from Proposition 2.1 (note that it is an easy exercise to verify that the hypotheses of that proposition are satisfied), this is equal to
\[ F_{\alpha,\beta,\gamma} \circ \left( - \otimes F_3^{\gamma_1 \cup \delta_1, \gamma_2 \cup \delta_2} \circ \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( \Phi \otimes \Omega \Delta_E,D_\gamma(\gamma) \right) \right) \right). \]

We note that
\[ F_3^{\gamma_1 \cup \delta_1, \gamma_2 \cup \delta_2} \circ \left( F_{D_\gamma(\beta),\Delta_E,D_\gamma(\gamma)} \circ \left( \Phi \otimes \Omega \Delta_E,D_\gamma(\gamma) \right) \right) \]
is just a change of diagrams map. Writing \( \Psi \) for the compositions of all the three change of diagrams maps (the initial one from \( H_{\beta, \gamma} \) to a weakly conjugated diagram \( C(C(H_{\beta, \gamma})) \), which we have been omitting writing, and then the last two, going back to \( H_{\beta, \gamma} \) through a doubled diagram), the composition is thus equal to
\[ F_{\alpha,\beta,\gamma} \circ \left( - \otimes \Psi \right) \simeq F_{\alpha,\beta,\gamma} \circ \left( - \otimes \Phi \right). \]

since \( \Phi \simeq \Omega \Delta_E,D_\gamma(\gamma) \) by naturality. \( \square \)

8. Trace and Cotrace Cobordisms

Let \( \xi_{I \times \partial M} \) denote the \( I \)-invariant contact structure on \( -I \times \partial M \) that induces the dividing set \( \gamma \) on \( \partial M \). In this section, we consider the trace cobordism
\[ \Lambda_{(M, \gamma)} = (I \times M, -I \times \partial M, [\xi_{I \times \partial M}]) \]
from \( (M, \gamma) \sqcup (-M, \gamma) \) to \( \emptyset \), and the cotrace cobordism
\[ V_{(M, \gamma)} = (I \times M, -I \times \partial M, [\xi_{I \times \partial M}]) \]
from \( \emptyset \) to \( (M, \gamma) \sqcup (-M, \gamma) \). The main result of this section is the following:

**Theorem 8.1.** The trace cobordism \( \Lambda_{(M, \gamma)} : (M, \gamma) \sqcup (-M, \gamma) \to \emptyset \) induces the canonical trace map
tr: \( SFH(M, \gamma) \otimes SFH(-M, \gamma) \to F_2 \).

The cotrace cobordism \( V_{(M, \gamma)} : \emptyset \to (M, \gamma) \sqcup (-M, \gamma) \) induces the canonical cotrace map
cotr: \( F_2 \to SFH(M, \gamma) \otimes SFH(-M, \gamma) \).

To prove Theorem 8.1, we first need a convenient topological description of the trace cobordism. Suppose that \( (\Sigma, \alpha, \beta) \) is a diagram for \( (M, \gamma) \). We will write \( (M_{\alpha, \beta}, \gamma_{\alpha, \beta}) \) for the sutured manifold constructed from the diagram \( (\Sigma, \alpha, \alpha) \), and we note that \( (M_{\alpha, \beta}, \gamma_{\alpha, \beta}) \) can be obtained by surgering \( (R_\alpha \times I, \partial R_\alpha \times I) \) along \( k \) 0-spheres. We will consider the triangular sutured manifold cobordism
\[ W_{\alpha, \beta, \gamma} = (W_{\alpha, \beta, \alpha}, Z_{\alpha, \beta, \alpha}, [\xi_{\alpha, \beta}], \gamma_{\alpha, \beta}) \]
from \((M, \gamma) \sqcup (-M, \gamma)\) to \((M_{\alpha, \beta}, \gamma_{\alpha, \beta})\), defined in Section 7.
There is also a cobordism $W_\alpha = (W_\alpha, Z_\alpha, (\xi_\alpha))$ from $(M_{\alpha,\alpha}, \gamma_{\alpha,\alpha})$ to $\emptyset$. The 4-manifold $W_\alpha$ is defined as

$$W_\alpha = (P_1 \times \Sigma) \cup_{e_\alpha \times \Sigma} (e_\alpha \times U_\alpha).$$

Here $P_1$ denotes a monogon, viewed as having a single boundary edge $e_\alpha$.

**Lemma 8.2.** The cobordism $\Lambda_{(M,\gamma)}$ is equivalent to the composition $W_\alpha \circ W_{\alpha,\beta,\gamma}$.

**Proof.** It follows from [Juh16, Proposition 6.6] that $W_\alpha \cup W_{\alpha,\beta,\gamma}$ is diffeomorphic to $I \times M_{\alpha,\beta}$, and furthermore, that $Z_{\alpha,\beta,\gamma} \cup Z_{\alpha}$ is diffeomorphic to $-I \times \partial M$. The fact that $\xi_\alpha$ and $\xi_{\alpha,\beta,\gamma}$ glue up too give $\xi_{I \times \partial M}$ can be proven by adapting Lemma 7.3. Schematically, the decomposition of $\Lambda_{(M,\gamma)}$ into $W_{\alpha,\beta,\gamma}$ and $W_\alpha$ is shown in Figure 8.1.

![Figure 8.1](image)

**Figure 8.1.** The decomposition $I \times M \cong W_{\alpha,\beta,\gamma} \cup M_{\alpha,\alpha} \cup W_{\alpha}$. It is convenient to view the polygons $P_1$, $P_2$, and $P_3$ as having fattened vertices.

Let $\alpha'$ denote be a small Hamiltonian translate of $\alpha$. Note that, of course, $(M_{\alpha,\alpha}, \gamma_{\alpha,\alpha})$ and $(M_{\alpha,\alpha'}, \gamma_{\alpha,\alpha'})$ are homeomorphic. It follows from Theorem 7.1 and the composition law that the map

$$F_{\Lambda_{(M,\gamma)}} : CF(\Sigma, \alpha, \beta) \otimes CF(\Sigma, \beta, \alpha) \to \mathbb{F}_2$$

is equal to the composition

$$F_{W_\alpha} \circ F_{\alpha,\beta,\alpha'} \circ (id_{CF(\Sigma, \alpha, \beta)} \otimes \Phi(\Sigma, \beta, \alpha) \to (\Sigma, \beta, \alpha')),$$

where $F_{\alpha,\beta,\alpha'}$ is the map that counts holomorphic triangles on $(\Sigma, \alpha, \beta, \alpha')$.

It remains to compute the cobordism map for $W_\alpha$. Note that, on $(\Sigma, \alpha, \alpha')$, there is a canonical bottom-graded intersection point $\Theta_{\alpha,\alpha'}$.

**Proposition 8.3.** The cobordism $W_\alpha$ from $(M_{\alpha,\alpha'}, \gamma_{\alpha,\alpha'})$ to $\emptyset$ induces the map

$$x \mapsto \begin{cases} 1 & \text{if } x = \Theta_{\alpha,\alpha'} \\ 0 & \text{otherwise.} \end{cases}$$

Before we prove the above result, we need to find a convenient Morse function for the cobordism $W_\alpha$. As a first step, we prove the following:

**Lemma 8.4.** Let $U_\alpha$ be the sutured compression body formed by attaching 3-dimensional 2-handles to $\Sigma \times I$ along the curves $\alpha \times \{0\}$. The space $U_\alpha$ is a manifold with corners, but, after rounding corners, we can view $U_\alpha$ as a (non-sutured) handlebody of genus $|\alpha| - \chi(R_\alpha) + 1$ and boundary

$$\partial U_\alpha = (\Sigma \times \{1\}) \cup \tilde{R}_\alpha,$$

where $R_\alpha$ is the surface obtained by surgering $\Sigma$ along the $\alpha$ curves. Furthermore, a (possibly overcomplete) set of compressing disks for $U_\alpha$ can be obtained by taking $|\alpha|$ compressing disks $D_\alpha$. 

with boundary \( \alpha \times \{1\} \) for \( \alpha \in \mathfrak{A} \), as well as disks of the form \( D_{c^*_i}^\bullet := c^*_i \times I \), for pairwise disjoint, embedded arcs \( c^*_1, \ldots, c^*_b(\alpha) \) in \( \Sigma \) that avoid the \( \alpha \) curves, and form a basis of \( H_1(R_\alpha, \partial R_\alpha) \). These cut \( U_\alpha \) into \( b_0(\alpha) \) 3-balls.

**Remark 8.5.** A basis of arcs \( c^*_1, \ldots, c^*_b(\alpha) \) can be obtained by picking a Morse function \( g: R_\alpha \to I \) (viewing \( R_\alpha \) as a cobordism from \( \emptyset \) to \( \partial R_\alpha \)) that has \( b_0(\alpha) \) local minima, and \( b_1(\alpha) \) index 1 critical points. The Morse function \( g \) determines a handle decomposition of \( R_\alpha \) with \( b_0(\alpha) \) 0-handles (i.e., disks) and \( b_1(\alpha) \) 1-handles. The arcs \( c^*_1, \ldots, c^*_b(\alpha) \) can be taken as the co-cores of the 1-handles in this decomposition.

**Proof of Lemma 8.4.** View \( U_\alpha \) as a cobordism from \( \partial U_\alpha = (\Sigma \times \{1\}) \cup R_\alpha \) to the empty set. The \( \alpha \) curves determine 3-dimensional 2-handles in \( U_\alpha \). After attaching these 2-handles, the remaining cobordism is homeomorphic to \( R_\alpha \times I \) (with corners smoothed), viewed as a cobordism from \( R_\alpha \cup \partial R_\alpha \) to \( \emptyset \). In such a way, we reduce the argument to the case when there are no \( \alpha \) curves, where it is straightforward. \( \square \)

We need an additional Morse theory argument:

**Lemma 8.6.** Suppose that \( U_\alpha \) is the sutured compression body induced by the sutured monodiagram \((\Sigma, \alpha)\). Then \( I \times U_\alpha \) can be viewed, after smoothing corners, as a (non-sutured) cobordism from the closed manifold \( U_\alpha \cup \partial U_\alpha \) to \( \emptyset \). Furthermore, there is a Morse function \( F: I \times U_\alpha \to I \) such that

- \( F^{-1}(0) = U_\alpha \cup \partial U_\alpha \) to \( I \times \{0\} \),
- \( F \) has no index 0, 1, or 2 critical points,
- \( F \) has \( |\alpha| + b_1(\alpha) \) index 3 critical points, and
- \( F \) has \( b_0(\alpha) \) index 4 critical points.

The attaching spheres of the 3-handles are obtained taking the union of the disks \( D_\alpha \) and \( D_{c^*_i} \) (defined in Lemma 8.4) in \( U_\alpha \), together with their images in \( -U_\alpha \).

**Proof.** A model for the 4-manifold obtained by rounding the corners of \( I \times U_\alpha \) can be taken to be \( X := I \times U_\alpha / \sim \), where \( (t, x) \sim (t', x) \) if \( x \in \partial U_\alpha \). Using Lemma 8.4, we can construct a Morse function \( f: U_\alpha \to I \) that has \( f^{-1}(0) = \partial U_\alpha \), such that \( f \) has no index 0 or 1 critical points, \( |\alpha| + b_1(\alpha) \) index 2 critical points, and \( b_0(\alpha) \) index 3 critical points. Furthermore, the descending manifolds of the index 2 critical points in \( U_\alpha \) are the disks \( D_\alpha \) and \( D_{c^*_i} \).

To construct a Morse function on \( X \) with the stated critical points, the argument is now a modification of Lemma 6.7. More precisely, we will construct an auxiliary function

\[
G: (I \times I)/(I \times \{0\}).
\]

We view \((I \times I)/(I \times \{0\})\) as having the same smooth structure at the point \( I \times \{0\} \) as the upper half-plane; see Figure 8.6. Furthermore, we assume that

- \( G|_{I \times \{0\}} \equiv G|_{\{0\} \times I} \equiv G|_{\{1\} \times I} \equiv 0 \),
- \( \partial G/\partial s > 0 \) in \( (0,1) \times I \),
- \( (\partial G/\partial t)(t,s) = 0 \) if and only if \( (t,s) \in \{1/2\} \times I \).

An example of such a function \( G \) is shown in Figure 8.6. We then consider the function \( F: X \to I \) defined by

\[
F(t,y) = G(t, f(y)).
\]

It is straightforward to verify that \( F \) on \( I \times U_\alpha \) with its corners rounded is Morse, and has critical points with attaching spheres as stated. \( \square \)

We are now ready to prove Proposition 8.3:

**Proof of Proposition 8.3.** The sutured cobordism \( W_\alpha = (W_\alpha, Z_\alpha, [\xi_\alpha]) \) from \((M_\alpha, \gamma_\alpha, \alpha')\) to \( \emptyset \) is the composition of the boundary cobordism \( W_\alpha^b \) obtained by gluing \( Z_\alpha := R_\alpha \times I \) to \( \partial M_\alpha, \alpha' \cong R_\alpha \cup \partial R_\alpha = \bar{R}_\alpha \), and the special cobordism \( W_\alpha^s \) (between closed 3-manifolds) diffeomorphic to \( U_\alpha \times I \).
from $U_{\alpha} \cup -U_{\alpha}$ to $\emptyset$. We use Lemma 8.6 to give a handle decomposition of the cobordism $W_{\alpha}$ consisting of $|\alpha| - \chi(R_{\alpha}) + 1$ 3-handles and $b_0(R_{\alpha})$ 4-handles.

Note that this description of the cobordism $W_{\alpha}$ does not quite allow us to compute the cobordism map $F_{W_{\alpha}}$, because when we glue $Z_{\alpha} = R_{\alpha} \times I$ to $M_{\alpha'}$, we obtain a closed (i.e., non-sutured manifold). The necessary modification is to instead remove $b_0(R_{\alpha})$ 3-balls from $Z_{\alpha}$, to obtain a sutured manifold cobordism from $(M_{\alpha, \alpha'}, \gamma_0, \alpha)$ to $\bigcup_{i=1}^{b_0(R_{\alpha})} (B^3, \gamma_0)$ (where $\gamma_0 \subseteq \partial B^3$ is a simple closed curve) and then compose with the natural isomorphism

$$\bigotimes_{i=1}^{b_0(R_{\alpha})} SFH(B^3, \gamma_0) \cong F_2.$$ 

Let us write $Z'_{\alpha}$ for $Z_{\alpha}$ with $b_0(R_{\alpha})$ 3-balls removed, and $W'$ for the induced special cobordism from $(M_{\alpha, \alpha'} \cup Z'_{\alpha}, \bigcup_{i=1}^{b_0(R_{\alpha})} \gamma_0)$ to $\bigcup_{i=1}^{b_0(R_{\alpha})} (B^3, \gamma_0)$. Note that $W'$ can be given a handle decomposition that is the same as the handle decomposition for $W_{\alpha}'$ with the 4-handles removed.

We write $S_{\alpha} \subseteq M_{\alpha, \alpha'} \cup Z'_{\alpha}$ for the 2-spheres obtained by doubling the compressing disk $D_{\alpha}$ for $\alpha \in \alpha$, and $S_{\alpha}' \subseteq M_{\alpha, \alpha'} \cup Z'_{\alpha}$ for the 2-spheres obtained by doubling the compressing disk $D_{\alpha}' \subseteq U_{\alpha}$. We recall that the curves $c_\alpha'$ were obtained by picking a Morse function on $R_{\alpha}$ that had $b_0(R_{\alpha})$ index 0 critical points and $b_1(R_{\alpha})$ index 1 critical points. The index 1 critical points each determine a 2-dimensional 1-handle, added to the handles of index 0. The curves $c_\alpha'$ are then the co-cores of these handles.

Note that, by the composition law for sutured cobordisms, we can commute the 3-handle maps for the spheres $S_{\alpha}$ with the contact glueing map for gluing in $Z'_{\alpha}$. As the composition of the 3-handle maps for the spheres $S_{\alpha}$ has the same formula as the one in the statement of the proposition, under the identification of

$$SFH(R_{\alpha} \times I, \partial R_{\alpha} \times I) \cong F_2,$$

we thus reduce to the case when there are no $\alpha$ curves; i.e., when $(\Sigma, \alpha)$ is the diagram $(R_{\alpha}, \emptyset)$.

To see the claim when there are no $\alpha$ curves, we note that the cobordism map is obtained by first gluing $Z'_{\alpha}$ to the product sutured manifold $(R_{\alpha} \times I, \partial R_{\alpha} \times I)$, and then attaching $b_1(R_{\alpha})$ 4-dimensional 3-handles. The contact manifold $Z'_{\alpha}$ is obtained by attaching $b_1(R_{\alpha})$ contact 2-handles. If $c_i^\alpha$ denotes the arc on $R_{\alpha}$ from Lemma 8.4, obtained as the co-core of a handle decomposition of $R_{\alpha}$, as above, then the attaching 1-spheres $s_i$ for the contact 2-handles forming $Z'_{\alpha}$ are given by

$$s_i = (c_i \times \{0, 1\}) \cup (\partial c_i \times I).$$

Note that $s_i$ bounds the disk $D_{c_i}$, described in Lemma 8.4, which is the descending manifold of the Morse function $f$, constructed on $U_{\alpha}$ in the proof of Lemma 8.6. The 2-sphere $S_{c_i}$ along which we attach the $b_1(R_{\alpha})$ 3-handles are then equal to the union of $D_{c_i}$, together with the core of the corresponding contact 2-handle. Now an easy model computation shows that the composition of
these $b_1(R_\alpha)$ contact 2-handle maps, and the $b_1(R_\alpha)$ 3-handle maps, sends $1 \in SFH(R_\alpha \times I, \partial R_\alpha \times I)$ to $1 \in \bigotimes_{i=1}^{b_1(R_\alpha)} SFH(B^3, \gamma_0)$. This model computation is shown in Figure 8.3.

Using the composition law for sutured cobordisms, the formula for the cobordism map $F_{W_\alpha}$ now follows.

\[ F_{W_\alpha}(\theta^-) \to \theta^- \]

**Figure 8.3.** $R_\alpha$ and $R_\alpha \times I$. On the left, a 1-handle $h_i$ from a handle decomposition of $R_\alpha$ is shown, as well as the core $c_i$ and the co-core $c_i^\ast$. On the right is the product manifold $(R_\alpha \times I, \partial R_\alpha \times I)$, as well as the closed curve $s_i$, along which we attach a contact 2-handle. The red lines on the right indicate the sutures of $R_\alpha \times I$.

**Figure 8.4.** The model computation from Proposition 8.3. The contact 2-handle map takes the form $1 \mapsto \theta^-$, and the 3-handle map sends $\theta^-$ to 1.

We can now prove the main theorem of this section:

**Proof of Theorem 8.1.** Let us consider the trace cobordism map. Recall from equation (18) that $\Lambda_{(M, \gamma)}$ can be written as the composition of $W_\alpha$ with $W_{\alpha, \beta, \gamma}$. Noting that $W_{\alpha, \beta, \gamma}$ is equivalent
to \( W_{\alpha, \beta, \alpha'} \) (where \( \alpha' \) is a small Hamiltonian translate of \( \alpha \)), and using the formula for \( F_{W_{\alpha}} \) from Proposition 8.3, we know that

\[
F_{\lambda(M, \gamma)} : CF(\Sigma, \alpha, \beta) \otimes CF(\Sigma, \beta, \alpha) \to \mathbb{F}_2
\]

takes the form

\[
F_{\lambda(M, \gamma)}(x \otimes y) = \left\langle F_{\alpha, \beta, \alpha'}(x, \Phi_{\beta}^{\alpha} \circ \Phi_{\alpha}^{\beta}(y)), \Theta_{\alpha, \alpha'}^{-} \right\rangle,
\]

where

\[
\langle z, z' \rangle := \begin{cases} 
1 & \text{if } z = z', \\
0 & \text{otherwise}.
\end{cases}
\]

Note, however, that the triangle map \( F_{\alpha, \beta, \alpha'} \) counts the same triangles as the triangle map \( F_{\alpha', \alpha, \beta} \), and that \( \Theta_{\alpha, \alpha'}^{-} \in CF(\Sigma, \alpha, \alpha') \) is the same intersection point as \( \Theta_{\alpha', \alpha}^{+} \in CF(\Sigma, \alpha', \alpha) \). Hence

\[
\left\langle F_{\alpha, \beta, \alpha'}(x, \Phi_{\beta}^{\alpha} \circ \Phi_{\alpha}^{\beta}(y)), \Theta_{\alpha, \alpha'}^{-} \right\rangle = \left\langle F_{\alpha', \alpha, \beta}(\Theta_{\alpha', \alpha}^{+}, x), \Phi_{\beta}^{\alpha} \circ \Phi_{\alpha}^{\beta}(y) \right\rangle.
\]

However, \( F_{\alpha', \alpha, \beta}(\Theta_{\alpha', \alpha}^{+}, \cdot) \) is the transition map \( \Phi_{\alpha, \alpha'}^{\beta}(\cdot) \), from naturality. Hence the cobordism map becomes simply the composition

\[
\left\langle \Phi_{\alpha}^{\beta} \circ \Phi_{\alpha}^{\beta}(x), \Phi_{\alpha}^{\beta} \circ \Phi_{\alpha}^{\beta}(y) \right\rangle,
\]

which is easily seen to be \( \langle \Phi_{\alpha}^{\beta} \circ \Phi_{\alpha}^{\beta}(x), y \rangle = \langle x, y \rangle \).

The formula for the cotrace cobordism map \( F_{\lambda(M, \gamma)} \) follows from the above formula for the trace cobordism map \( F_{\lambda(M, \gamma)} \), combined with Theorem 6.6. \( \square \)

9. Equivalence of Two Link Cobordism Map Constructions

In this section, we describe an application of the techniques of this paper to link Floer homology.

9.1. Background on link Floer homology. Knot Floer homology is an invariant of knots embedded in 3-manifolds constructed by Ozsváth and Szabó [OS04a], and independently by Rasmussen [Ras03]. Link Floer homology is a generalization to links, constructed by Ozsváth and Szabó [OS08].

Definition 9.1. A multi-based link \( L = (L, w, z) \) in a 3-manifold \( Y \) is an oriented link \( L \subseteq Y \), together with two disjoint collections of basepoints \( w, z \subseteq L \) such that

1. each component of \( L \) has at least two basepoints,
2. the basepoints along a link component of \( L \) alternate between \( w \) and \( z \), as one traverses the link.

To a multi-based link \( L \) in \( Y \), link Floer homology associates an \( \mathbb{F}_2 \)-module

\[
\widehat{HFL}(Y, L).
\]

To construct the modules, one picks a Heegaard diagram for the pair \( (Y, L) \), in the following sense:

Definition 9.2. A Heegaard diagram \( (\Sigma, \alpha, \beta, w, z) \) for an oriented multi-based link \( (Y, (L, w, z)) \) consists of the following:

1. A Heegaard diagram \( (\Sigma, \alpha, \beta) \) for \( Y \), such that \( Y \setminus \Sigma \) is the union of two handlebodies \( U_\alpha \) and \( U_\beta \) that meet along \( \Sigma \).
2. \( \Sigma \cap L = w \cup z \).
3. Each component of \( \Sigma \setminus (\alpha \cup \beta) \) contains exactly one \( w \) basepoint and one \( z \) basepoint.
4. The arcs of \( L \cap U_\alpha \) are isotopic in \( U_\alpha \), relative their endpoints, to arcs that lie in \( \Sigma \setminus \alpha \). The arcs of \( L \cap U_\beta \) are isotopic in \( U_\beta \), relative their endpoints, to arcs that lie in \( \Sigma \setminus \beta \).
5. The link \( L \) intersects \( \Sigma \) positively at the \( z \) basepoints, and negatively at the \( w \) basepoints.
A somewhat more concise way of defining a Heegaard diagram for a multi-based links is as a diagram for the sutured manifold $Y(\mathbb{L})$, obtained by removing a neighborhood of the link $L$ from $Y$, and adding sutures to $\partial(Y \setminus N(L))$ that are positively oriented meridians of $L$ over the $w$ basepoints and negatively oriented meridians of $L$ over the $z$ basepoints. Link Floer homology, as defined by Ozsváth and Szabó [OS08], is easily seen to satisfy the isomorphism

$$\widehat{HFL}(Y, L) \cong SFH(Y(\mathbb{L})).$$

If $(\Sigma, \alpha, \beta, w, z)$ is a diagram for $(Y, \mathbb{L})$, then the link Floer complex $\widehat{CFL}(\Sigma, \alpha, \beta, w, z)$ is generated over $\mathbb{F}_2$ by intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and the differential counts Maslov index 1 pseudo-holomorphic discs that go over none of the $w$ or $z$ basepoints.

For links in $S^3$ or null-homologous links in $(S^1 \times S^2)^{\# n}$, there is some additional structure on $\widehat{HFL}(S^3, \mathbb{L})$. If $(\Sigma, \alpha, \beta, w, z)$ is a diagram for a link in $S^3$, one can define three relative gradings, $gr_w$, $gr_z$, and $A$, on $\widehat{CFL}(\Sigma, \alpha, \beta, w, z)$. If $x$ and $y$ are two intersection points, then the gradings $gr_w$ and $gr_z$ are defined by picking a homology class $\phi \in \pi_2(x, y)$ and setting

$$gr_w(x, y) := \mu(\phi) - 2n_w(\phi) \quad \text{and} \quad gr_z(x, y) := \mu(\phi) - 2n_z(\phi),$$

where $n_w(\phi)$ and $n_z(\phi)$ denote the sum of the multiplicities of $\phi$ over the $w$ or $z$ basepoints, respectively. It is easy to see that the formulas for $gr_w$ and $gr_z$ are independent of the choice of homology class $\phi$. An absolute lift of the relative grading $gr_w$ can be fixed by requiring that $\widehat{HF}(\Sigma, \alpha, \beta, w)$, which is isomorphic as a relatively graded group to $\bigotimes_{w} |w|^{-1} \left( (\mathbb{F}_2)_-^1 \oplus (\mathbb{F}_2)_+^1 \right)$, have top-graded generator in grading $|w|/2$. An absolute lift of the grading $gr_z$ can be specified similarly. Finally, the Alexander grading $A$ can be defined as

$$A := \frac{1}{2} (gr_w - gr_z).$$

### 9.2. The link Floer homology TQFT

In [Juh16], the first author provided a construction of cobordism maps for decorated link cobordisms. The construction used the following notion of cobordism between multi-based links:

**Definition 9.3.** Let $Y_1$ and $Y_2$ be 3-manifolds containing multi-based links $\mathbb{L}_1 = (L_1, w_1, z_1)$ and $\mathbb{L}_2 = (L_2, w_2, z_2)$, respectively. A **decorated link cobordism** from $(Y_1, \mathbb{L}_1)$ to $(Y_2, \mathbb{L}_2)$ is a triple $(X, S, A)$, where

1. $X$ is an oriented cobordism from $Y_1$ to $Y_2$,
2. $S$ is a properly embedded oriented surface in $X$ with $\partial S = -L_1 \cup L_2$, and
3. $A$ is a properly embedded 1-manifold in $S$ that divides $S$ into two subsurfaces $S_w$ and $S_z$ that meet along $A$, such that $w_1, w_2 \subseteq S_w$ and $z_1, z_2 \subseteq S_z$.

Note that the above definition is slightly different from [Juh16, Definition 4.5], and follows [Zem16]. The equivalence of the two definitions is explained in [JM17, Section 2.3].

If $X = (X, S, A)$ from $(Y_1, \mathbb{L}_1)$ to $(Y_2, \mathbb{L}_2)$ is a decorated link cobordism, then there is a well-defined cobordism $W(X) = (W, Z, [\xi])$ of sutured manifolds from $Y(\mathbb{L}_1)$ to $Y(\mathbb{L}_2)$, as we now describe. The 4-manifold $W$ is defined by the formula

$$W := X \setminus N(S),$$

where $N(S)$ denotes a regular neighborhood of $S$, viewed as the unit normal disk bundle of $S$. The set $Z$ is defined as the unit normal circle bundle of $S$, oriented as a submanifold of $\partial W$. Since $S$ is an oriented surface, $Z$ is a principal $S^1$-bundle over $S$. According to Lutz [Lut77] and Honda [Hon00b], the dividing set $A$ uniquely determines an $S^1$-invariant contact structure on $Z$ with dividing set $A$ on $S$, up to isotopy. The contact structure $[\xi]$ is defined to be this $S^1$-invariant contact structure. The link cobordism map

$$F^L_{X} : \widehat{HFL}(Y_1, \mathbb{L}_1) \to \widehat{HFL}(Y_2, \mathbb{L}_2)$$

is defined to be the sutured cobordism map

$$F_{W(X)} : SFH(Y_1(\mathbb{L}_1)) = \widehat{HFL}(Y_1, \mathbb{L}_1) \to SFH(Y_2(\mathbb{L}_2)) = \widehat{HFL}(Y_2, \mathbb{L}_2).$$
The second author [Zem16] constructed another link cobordism map \( F^{Z, s}_{X} \), where \( s \in \text{Spin}^c(X) \) is a Spin\(^c\) structure on \( X \), that did not use the Honda–Kazez–Matić gluing map. It instead was defined by writing a link cobordism as a composition of elementary link cobordisms. The map \( F^{Z, s}_{X} \) is defined on a more general version of link Floer homology than \( F^{J, s}_{X} \), though it induces a map \( \hat{F}^{Z, s}_{X} \) on the hat version. Let

\[
\hat{F}^{Z}_{X} := \sum_{s \in \text{Spin}^c(X)} \hat{F}^{Z, s}_{X}.
\]

It is not obvious that the maps \( F^{J, s}_{X} \) and \( \hat{F}^{Z, s}_{X} \) agree.

The first author and Marengon [JM17] made some steps towards computing the maps \( F^{J, s}_{X} \) when \( X \) is an elementary link cobordism, though most of the computational results from [JM17] are still in terms of the Honda–Kazez–Matić gluing map, and hence it is challenging to directly compare the maps \( F^{J, s}_{X} \) and \( \hat{F}^{Z, s}_{X} \). Nonetheless, combining several results from [JM17] with the results of this paper, we are able to prove the following:

**Theorem 9.4.** Given a decorated link cobordism \( \mathcal{X} \), we have \( F^{J, s}_{X} = \hat{F}^{Z, s}_{X} \).

### 9.3. Elementary link cobordisms.

In this section, we provide the following definition:

**Definition 9.5.** We say a decorated link cobordism \( \mathcal{X} = (X, S, A) : (Y_1, L_1) \to (Y_2, L_2) \) is an elementary link cobordism if one of the following is satisfied:

1. **(Identity cobordism)** \( (Y_1, L_1) = (Y_2, L_2) = (Y, L) \) and \( (X, S, A) = (I \times Y, I \times L, I \times p) \), where \( p \subseteq L \) consists of exactly one point in each component of \( L \setminus (w \cup z) \).
2. **(1-, 2-, or 3-handle attachment)** The cobordism \( (X, S, A) \) is obtained by attaching a 4-dimensional 1-, 2-, or 3-handle, with framed attaching sphere disjoint from \( L_1 \).
3. **(0- or 4-handle attachment)** The cobordism \( X \) is obtained by attaching a 4-dimensional 0-handle or 4-handle, viewed as a smooth 4-ball, that contains a standard disk intersecting the boundary of the 4-ball in an unknot, with dividing set consisting of a single arc on the disk.
4. **(Saddle cobordism)** The cobordism \( \mathcal{X} \) has underlying 4-manifold \( X = I \times Y \), and a surface \( S \) such that projection to \( I \) induces a Morse function that has a single index 1 critical point that occurs in a \( w \)-region or a \( z \)-region, and such that the dividing arcs all travel from \( L_1 \) to \( L_2 \).
5. **(Stabilization cobordism)** The cobordism \( \mathcal{X} \) has underlying 4-manifold \( X = I \times Y \) and surface \( S = I \times L \). Furthermore, exactly one arc of \( A \) goes from \( L_1 \) to \( L_1 \) or from \( L_2 \) to \( L_2 \). All other arcs are of the form \( I \times \{p\} \) for various \( p \in L \). A stabilization cobordism is **positive** if it adds two basepoints, and is **negative** if it removes two basepoints.

A schematic of a saddle cobordism can be found in Figure 9.7. Examples of stabilization cobordisms are shown in Figure 9.1.

**Figure 9.1.** Two examples of stabilization cobordisms. The two surfaces are each of the form \( I \times L \), and sit inside of \( I \times Y \). The shaded regions are the \( w \) regions, and the unshaded regions are the \( z \) regions.

It is straightforward to see that an arbitrary link cobordism \( \mathcal{X} \), such that \( \pi_0(S) \to \pi_0(X) \) is a surjection, can be decomposed into a sequence of link cobordisms that are each diffeomorphic to one
of the elementary link cobordisms in the above list. We remark that the above list is over-complete, in the following sense:

Remark 9.6. An elementary positive stabilization cobordism can be written as a composition of a 0-handle cobordism (adding an unknot with two basepoints) followed by a 1-handle, followed by a saddle cobordism. Similarly an elementary negative stabilization cobordism can be written as a composition of a saddle cobordism, followed by a 3-handle and a 4-handle.

9.4. Link triple diagrams and contact structures.

Definition 9.7. A link triple diagram

$$(\Sigma, \alpha, \beta, \gamma, w, z)$$

is a Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$ with $2(n-g(\Sigma)+1)$ basepoints $w \cup z \subseteq \Sigma$, where $n = |\alpha| = |\beta| = |\gamma|$, such that each component of $\Sigma \setminus \tau$ for $\tau \in \{\alpha, \beta, \gamma\}$ is planar and contains exactly one $w$ basepoint and exactly one $z$ basepoint.

Given a link triple diagram $T = (\Sigma, \alpha, \beta, \gamma, w, z)$, we can construct a decorated link cobordism

$$X_T = (X_T, S_T, A_T),$$

as follows. The 4-manifold $X_T$ is constructed as in [OS06], by the formula

$$X_T := (\Delta \times \Sigma) \cup (e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma) / \sim.$$

Here, the handlebody $U_\tau$ for $\tau \in \{\alpha, \beta, \gamma\}$ is obtained by attaching 3-dimensional 2-handles to $\Sigma \times I$ along $\tau \times \{0\}$, and filling in the resulting sphere boundary components with 3-dimensional 0-handles.

We obtain the surface $S_T$ as follows. Pick Morse functions $f_\alpha$, $f_\beta$, and $f_\gamma$ on $U_\alpha$, $U_\beta$, and $U_\gamma$ that induce the attaching curves $\alpha$, $\beta$, and $\gamma$, respectively. By concatenating the ascending flow-lines passing through the basepoints in $w$ and $z$, we get a collection of $|w|$ arcs $K_\alpha$, $K_\beta$, and $K_\gamma$ in each of $U_\alpha$, $U_\beta$, and $U_\gamma$, respectively. Each arc has exactly one endpoint in $w$ and one endpoint in $z$, and we orient it from $w$ to $z$. Then the surface $S_T$ is defined as

$$S_T := (\Delta \times (\Sigma \setminus (w \cup z))) \cup (e_\alpha \times K_\alpha) \cup (e_\beta \times K_\beta) \cup (e_\gamma \times K_\gamma).$$

Finally, we describe the dividing set $A_T$ on $S_T$. Let $p_\alpha \subseteq K_\alpha$ denote a collection of points obtained by picking a single point in each arc of $K_\alpha$. Define $p_\beta$ and $p_\gamma$ similarly. Then

$$A_T := (e_\alpha \times p_\alpha) \cup (e_\beta \times p_\beta) \cup (e_\gamma \times p_\gamma),$$

is a dividing set on $S_T$.

Given a link triple diagram $T = (\Sigma, \alpha, \beta, \gamma, w, z)$, we naturally obtain a sutured Heegaard triple $T_0 = (\Sigma_0, \alpha, \beta, \gamma)$, where $\Sigma_0 = \Sigma \setminus N(w \cup z)$. Let $W_{T_0} = (W_{T_0}, Z_{T_0}, [\xi_{T_0}])$ denote the associated sutured cobordism.

Proposition 9.8. Let $T = (\Sigma, \alpha, \beta, \gamma, w, z)$ be a link triple diagram, and $T_0 = (\Sigma_0, \alpha, \beta, \gamma)$ the corresponding sutured triple diagram. Then $Z_{T_0}$ is the unit normal circle bundle of $S_T$, and there is a projection map $\pi: Z_{T_0} \rightarrow S_T$ with fiber $S^1$. The contact structure $\xi_{T_0}$ on $Z_{T_0}$ described in Section 7 is equivalent to the $S^1$-invariant contact structure on $Z_{T_0}$ with respect to the dividing set $A_T$ and the projection map $\pi$.

Proof. Let us write $\xi_{S^1}$ for the $S^1$-invariant contact structure on $Z_{T_0}$. The proof of the proposition will be to describe a convex decomposition of $(Z_{T_0}, \xi_{S^1})$ into the disjoint union of the contact manifolds $(Z_0, \xi_0)$, $(Z_\alpha, \xi_\alpha)$, $(Z_\beta, \xi_\beta)$, and $(Z_\gamma, \xi_\gamma)$. Note that $S_T$ has no closed components, so $Z_{T_0}$ is diffeomorphic to $S_T \times S^1$, with the map $\pi$ given by projection onto the first factor.

We will decompose $Z_{T_0}$ along $3|\partial \Sigma|$ convex annuli. To construct the annuli, it is convenient to view $\Delta$ as a smooth 2-disc, and view the edges $e_\alpha$, $e_\beta$, and $e_\gamma$ as being closed, disjoint subintervals of $\partial \Delta$. We let $A_0$ denote the set of annuli of the form $e_\alpha \times \{w\} \times S^1$ and $e_\gamma \times \{z\} \times S^1$ inside $S_T \times S^1$. Note that we cannot decompose along the annuli in $A_0$, since their boundaries are disjoint from the dividing set, and hence we cannot use Legendrian realization to ensure that $\partial A_0$ is Legendrian and...
\(A_0\) is convex. Instead, we perform a finger move along each boundary component of each annulus in \(A_0\) until it intersects the dividing set. Let \(A\) denote the resulting collection of annuli. The configuration of the annuli \(A_0\) and \(A\) are shown in Figure 9.2.

![Figure 9.2](image)

**Figure 9.2.** The annuli \(A_0\) in \(S_T \times S^1\) (left), and the annuli \(A\) obtained by performing finger moves along the boundaries toward the dividing set on \(\partial S_T \times S^1\) (right). We prove that the dividing set on each annulus in \(A\) is as shown in the bottom. The hexagonal region in the middle corresponds to a component of \(\Delta \times \{x\}\), for a basepoint \(x \in w \cup z\). The orientation of \(\Delta \times \{x\}\) is shown, and we take the product orientation on \(S_T \times S^1\) in the picture.

We can perturb the annuli in \(A\) so that they are convex with Legendrian boundary (note this requires using Legendrian realization along \(\partial S_T \times S^1\), and hence involves replacing \(\xi_S\) with an equivalent contact structure that is no longer \(S^1\)-invariant near the boundary).

We now claim that the dividing sets on the annuli in \(A\) are as in the bottom of Figure 9.2, consisting of two arcs that go from one boundary component of the annulus to the other, and which do not wind around the annulus. To see this, we first claim that, for an appropriately chosen \(S^1\)-invariant contact 1-form, we can embed a neighborhood of each annulus inside an \(e_T\)-invariant contact structure on \(e_T \times I \times S^1\) that has dividing set on \((\partial e_T) \times I \times S^1\) equal to \((\partial e_T) \times \{\frac{1}{2}\} \times S^1\). To do this, we recall that an \(S^1\)-invariant contact 1-form on \(S_T \times S^1\) can be defined as \(\beta + f \cdot d\theta\), where \(\beta\) is a 1-form on \(S_T\) and \(f : S_T \to \mathbb{R}\) is a function that is zero exactly on \(A_T\). It is straightforward to write down conditions for such a 1-form to be a contact form on \(S_T \times S^1\). Since the contact form \(\beta + f \cdot d\theta\) is essentially determined by the characteristic foliation \(\ker \beta\) on \(S_T \times \{p\}\), we will focus on constructing a singular foliation with the appropriate dividing set that is invariant under translation along a non-vanishing vector field (thought of as \(\partial/\partial e_T\)) in a neighborhood of each annulus in \(A\). It is an elementary, though somewhat tedious, exercise to explicitly construct an appropriate contact 1-form by picking \(\beta\) and \(f\) appropriately, so we will leave that step to the reader. An example of an appropriately chosen characteristic foliation on \(S_T \times \{p\}\) is shown in Figure 9.3.

We can choose a neighborhood of an annulus in \(A\) that embeds into an \(e_T\)-invariant contact structure on \(e_T \times I \times S^1\) as in Figure 9.4. Note that, inside \(e_T \times I \times S^1\), the annulus \(A\) pictured on the right of Figure 9.4 is isotopic, relative to its boundary, to a surface of the form \(e_T \times s\), for a Legendrian \(s\) in \((\partial e_T) \times I \times S^1\). The characteristic foliation and dividing set on \(e_T \times S\) are the same as on the annulus shown in Figure 7.3. Namely, the characteristic foliation consists of horizontal leaves lying in \(t \times s\), as well as two vertical singular sets of the form \(e_T \times \{p\}\), for two points \(p\) in \(s\). The dividing set on \(e_T \times s\) consists of two vertical arcs, as well.
Figure 9.3. The characteristic foliation of $S_T \times \{p\}$ of an $S^1$-invariant contact 1-form on $S_T \times S^1$ with dividing set $A_T$.

Figure 9.4. A neighborhood of an annulus $A \in \mathbf{A}$ in $S_T \times S^1$, which embeds into an $e_\tau$-invariant contact structure on $e_\tau \times I \times S^1$.

Note that, if we could show that $A$ and $e_\tau \times s$ were isotopic through convex surfaces, we would be done, since the dividing set on $A$ would be isotopic to the one on $A$, which is what we are trying to show. This is somewhat geometrically hard to prove, so we argue as follows. Perturb $e_\tau \times s$ slightly, such that there is a Legendrian loop $\ell$, intersecting one of the dividing arcs twice. This does not change the isotopy type of the dividing set. Let $D_0 \subseteq e_\tau \times s$ be the disk bounded by $\ell$. Now take a convex disk $D$ in $e_\tau \times I \times S^1$, with $D \cap (e_\tau \times s) = \ell$. The dividing set on $D$ consists of a single arc, since we can assume $D$ lies in a tight contact ball. We can replace $e_\tau \times s$ by $((e_\tau \times s) \setminus D_0) \cup D$, and then rounding along the Legendrian corner $\ell$. When we do this, we do not change the isotopy type of the dividing set, and we can move $e_\tau \times s$ to $A$ via a sequence of such moves. The move is shown in Figure 9.5. This establishes that the annuli $A$ have a dividing set isotopic to the one shown in Figure 9.2.

Having determined the dividing sets along the convex annuli in $\mathbf{A}$, we can cut along the annuli in $\mathbf{A}$, then round the Legendrian corners. The dividing set on the sutured manifold corresponding to $\Delta \times \partial \Sigma$ is shown in Figure 9.6. As a sutured manifold, this is the same as $(Z_0, \gamma_0)$. Similarly, it is easy to see that rounding the Legendrian corners on the other 3 pieces yields the sutured manifolds $(Z_\alpha, \gamma_\alpha)$, $(Z_\beta, \gamma_\beta)$, and $(Z_\gamma, \gamma_\gamma)$. On the other hand, we note that $\xi_{S^1}$ is tight, since the dividing set on $S_T$ contains no contractible components. It follows that $\xi_{S^1}$ restricts to tight contact structures on $(Z_0, \gamma_0)$, $(Z_\alpha, \gamma_\alpha)$, $(Z_\beta, \gamma_\beta)$, and $(Z_\gamma, \gamma_\gamma)$. However, each of these four sutured manifolds are product disk decomposable, so, up to equivalence, there is a unique tight contact structure on each one. Hence, the restrictions of $\xi_{S^1}$ are equivalent to $\xi_0$, $\xi_\alpha$, $\xi_\beta$, and $\xi_\gamma$, respectively. Since $\xi_{\alpha, \beta, \gamma}$
Figure 9.5. Attaching a convex disk to move the annulus $e \times s \subseteq e \times I \times S^1$ to the position of an annulus $A \in A$. On the left is (a $C^0$-small perturbation of) the annulus $e \times s$. In the middle is a convex disk with Legendrian boundary $\ell$ with $tb(\ell) = -1$. On the right is a convex annulus that we can take to be $A$, obtained by edge rounding along $\ell$.

is constructed by gluing together $\xi_0$, $\xi_\alpha$, $\xi_\beta$, and $\xi_\gamma$, it follows that $\xi_{S^1}$ and $\xi_{\alpha,\beta,\gamma}$ are equivalent on $Z_T$. □

Figure 9.6. Rounding corners after cutting $S_T \times S^1$ along the annuli in $A$. Shown is the dividing set on the boundary of a solid torus component of $(S_T \times S^1) \setminus A$ corresponding to $\Delta \times \{x\} \times S^1$, for a basepoint $x \in w \cup z$. On the top left, we show the dividing before rounding the Legendrian corners. On the top right, we show the result of rounding corners. On the bottom, we show the result of isotoping the dividing set. We view $Z_0$ as being “below” the surface shown.

9.5. Saddle cobordisms and link triple diagrams. In this section, we review the construction of saddle cobordism maps [Zem16, Section 6]. As an important step towards proving Theorem 9.4, we show that the maps from [Zem16] and [Juh16] agree for such cobordisms.

Definition 9.9. Suppose that $Y$ is a 3-manifold containing an oriented multi-based link $L = (L, w, z)$. We say that an oriented square $B \subseteq Y$ is a $\beta$-band for a multi-based link $L$ in $Y$ if $B$ is smoothly embedded in $Y$, it is identified with $[-1,1] \times [-1,1]$, and

1. $B \cap L = [-1,1] \times \{1,-1\}$,
2. the boundary orientation of $B$ agrees with the orientation of $-L$,
3. $B \cap (w \cup z) = \emptyset$, and both ends of $B$ are in regions of $L \setminus (w \cup z)$ that go from $z$ to $w$.

Note that if $B$ is a $\beta$-band for the link $L$ in $Y$, then there is a well defined multi-based link

$L(B) = (L(B), w, z)$

obtained by band surgery on $B$. The following is [Zem16, Definition 6.4]
Definition 9.10. We say the link triple diagram
\[(\Sigma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}, \beta_n, w, z),\]
is subordinate to the \(\beta\)-band \(B\) if the following hold:
\begin{enumerate}
\item \((\Sigma_0, \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1})\) is a diagram for the sutured manifold \(Y(\Sigma) \setminus N(B)\), where \(\Sigma_0 = \Sigma \setminus (w \cup z)\),
\item \((\Sigma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, w, z)\) is a diagram for \((Y, \Sigma)\),
\item the curves \(\beta_1', \ldots, \beta_{n-1}'\) are small Hamiltonian translates of the curves \(\beta_1, \ldots, \beta_{n-1}\),
\item the curve \(\beta_n'\) is induced by the band \(B\), and \((\Sigma, \alpha_1, \ldots, \alpha_n, \beta_1', \ldots, \beta_{n-1}', w, z)\) is a diagram for \((Y, L(B))\).
\end{enumerate}

Notice that, if we ignore the basepoints \(w\) and \(z\), then the curve \(\beta_n'\) is related to \(\beta_n\) by a sequence of handleslides and isotopies. It follows that the 4-manifold \(X_{\alpha, \beta, \beta'}\) induced by a Heegaard triple subordinate to a \(\beta\)-band is diffeomorphic to \(I \times Y\) with a neighborhood of \(\{\frac{1}{2}\} \times U_{\beta}\) removed.

The band \(B\) induces a saddle cobordism \(S(B) \subseteq I \times Y\) from \(L\) to \(L(B)\). The surface \(S(B)\) is obtained by rounding the corners of the surface
\[S_0(B) := ([0, \frac{1}{2}] \times L) \cup (\{\frac{1}{2}\} \times B) \cup ([\frac{1}{2}, 1] \times L(B)).\]

We note that \(S(B)\) has two natural choices of dividing sets. To construct them, pick points \(p \subseteq L \setminus (w \cup z)\) and \(q \subseteq L(B) \setminus (w \cup z)\), such that the following hold:
\begin{enumerate}
\item Each component of \(L \setminus (w \cup z)\) contains exactly one point of \(p\), and each component of \(L(B) \setminus (w \cup z)\) contains exactly one point of \(q\).
\item If \(C \subseteq L \setminus (w \cup z)\) is a component disjoint from \(B\), then \(p \cap C = q \cap C\).
\item If \(C\) is a component of \(L \setminus (w \cup z)\) that intersects \(B\), and \(\{p\} = C \cap p\), then \(p \in B\). Similarly, if \(C'\) is a component of \(L(B) \setminus (w \cup z)\) that intersects \(B\), and \(\{q\} = C' \cap q\), then \(q \in B\).
\end{enumerate}

A dividing \(A_0\) on \(S(B) \setminus (\{\frac{1}{2}\} \times B)\) is then specified by the equation
\[A_0 := ([0, \frac{1}{2}] \times p) \cup ([\frac{1}{2}, 1] \times q).\]

There are two natural ways to extend the dividing set \(A_0\) to \(B\). We write \(A_w\) and \(A_z\) for the two possible extensions, as shown in Figure 9.7.

Lemma 9.11. If \((\Sigma, \alpha, \beta, \beta', w, z)\) is a subdiagram of a \(\beta\)-band \(B\), then \((\Sigma, \beta, \beta', w, z)\) represents an unlink \(U\) in \((S^1 \times S^2)^{\#g(\Sigma)}\), where all components of \(U\) have two basepoints, except for one component that has four basepoints. With respect to each of the Maslov gradings \(\text{gr}_w\) and \(\text{gr}_z\), there is a top-graded generator of \(\text{HFL}(\Sigma, \beta, \beta', w, z)\), for which we write \(\Theta_w\) and \(\Theta_z\).

Proof. For \(i \in \{1, \ldots, n-1\}\), the curves \(\beta_i\) and \(\beta'_i\) are Hamiltonian translates of each other, and hence determine a 2-sphere in the manifold \(U_{\beta_i} \cup U_{\beta'_i}\). After surgering all of these out, we are left with \(|w| - 2\) copies of \(S^3\) with a doubly based unknotted in them, and one copy of \(S^3\) with an unknotted containing four basepoints. After surgering all of these spheres out, it is easy to see \(\text{HFL}\) is generated by two elements, one of which is in \((\text{gr}_w, \text{gr}_z)\) grading \((\frac{1}{2}, -\frac{1}{2})\), and one of which is in grading \((\frac{1}{2}, +\frac{1}{2})\). The effect of undoing the surgeries we did on the 2-spheres corresponds to adding back in a collection of 1-handles, which clearly preserves the property of having a top \(\text{gr}_w\) graded element, and a top \(\text{gr}_z\) graded element. \(\square\)

Lemma 9.12. Consider the link cobordism \(X = (X, S, A)\) from the empty set to an unlink in \((S^1 \times S^2)^{\#k}\), constructed by setting \(X\) to be a 4-dimension genus \(k\) handlebody, \(S\) to be a collection of \(n\) standardly embedded disks in \(X\), intersecting \(\partial X \cong (S^1 \times S^2)^{\#k}\) in \(n\) disjoint unknotted, and letting \(A\) consist of a single arc on each component of \(S\), except for one component of \(S\), where \(A\) consists of two arcs. Then
\[F_{X}^1(1) = \hat{F}_{X}^2(1) = \begin{cases} \Theta_w & \text{if } \chi(S_w) = \chi(S_w) + 1, \\ \Theta_z & \text{if } \chi(S_w) = \chi(S_w) - 1, \end{cases}\]
where \(1\) denotes the generator of \(\text{HFL}(\emptyset, \emptyset) \cong \mathbb{F}_2\).
Figure 9.7. A portion of the surface $S(B) \subseteq I \times Y$, as well as the two dividing sets $A_w$ and $A_z$ on $S(B)$. The $w$ basepoints are shown as solid dots, while the $z$ basepoints are open dots. The $w$ regions are shown as shaded, the $z$ regions are unshaded.

**Proof.** We decompose the cobordism $\mathcal{X}$ into a composition of elementary link cobordisms (Definition 9.5). Write $\mathcal{X} = X_4 \circ X_3 \circ X_2 \circ X_1$ where

- $X_1$ is a 0-handle cobordism, which adds a doubly based unknot in $S^3$;
- $X_2$ is a stabilization cobordism, which adds two basepoints to the unknot in $S^3$;
- $X_3$ consists of $(n - 1)$ 0-handles, each adding a doubly based unknot;
- $X_4$ consists of $(k + n - 1)$ 1-handles.

Note that if $U \subseteq S^3$ is a doubly based unknot, then $\widehat{HFL}(S^3, U) \cong \mathbb{F}_2$. Noting that the 0-handle maps are nonzero, since they can be canceled topologically, using multiplicativity of link Floer homology under disjoint unions, it follows that

$$F_{X_i} = \widehat{F}_{X_i}$$

for $i = 1, 3$.

Now if $U'$ is an unknot in $S^3$ with four basepoints, then, by Lemma 9.11, we have $\widehat{HFL}(S^3, U') \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. Furthermore, $\widehat{HFL}(S^3, U')$ is generated by two elements, $\Theta^w$ and $\Theta^z$, which are distinguished by grading. The element $\Theta^w$ has $(\text{gr}_{w, z}^{w}, \text{gr}_{w, z}^{z})$ bigrading $(+\frac{1}{2}, -\frac{1}{2})$ while $\Theta^z$ has bigrading $(-\frac{1}{2}, +\frac{1}{2})$.

It follows immediately from the definition of the maps in [Zem16] that

$$\widehat{F}_{X_2}(1) = \begin{cases} 
\Theta^w & \text{if } \chi(S_w) = \chi(S_z) + 1, \\
\Theta^z & \text{if } \chi(S_w) = \chi(S_z) - 1,
\end{cases}$$
since the cobordism map for a stabilization cobordism is defined using the quasi-stabilization map [Zem16, Section 3.2]. On the other hand, a straightforward functoriality argument shows that \( F^J_{\hat{X}} \) must be nonzero. By [JM17, Theorem 5.18], it follows that \( F^J_{\hat{X}}(1) \) has the same \( \text{gr}_w \) and \( \text{gr}_z \) grading as \( \hat{F}^Z_{\hat{X}}(1) \). Since \( \overline{HF}(S^3, \emptyset) \) is a 2-dimensional vector space over \( \mathbb{F}_2 \), it follows that \( F^J_{\hat{X}} = \hat{F}^Z_{\hat{X}} \).

Finally, the two cobordism map constructions use the same 4-dimensional handle attachment maps, so
\[
F^J_{\hat{X}} = \hat{F}^Z_{\hat{X}}.
\]

Furthermore, as in Lemma 9.11, the 1-handle maps preserve the top graded elements \( \Theta^w \) and \( \Theta^z \). Composing all the maps, the claim now follows. \( \square \)

In [Zem16], the link cobordism maps for \( X_w = (I \times Y, S(B), A_w) \) and \( X_z = (I \times Y, S(B), A_z) \) are defined to be
\[
\hat{F}^Z_{X_w}(-) := F_{\alpha,\beta}\beta'(- \otimes \Theta^z_{\beta,\beta'}) \quad \text{and} \quad \hat{F}^Z_{X_z}(-) := F_{\alpha,\beta}\beta'(\beta \otimes \Theta^w_{\beta,\beta'}). \tag{19}
\]

**Lemma 9.13.** For the decorated saddle cobordisms \( X_w \) and \( X_z \), defined above, we have \( F^J_{X_w} = \hat{F}^Z_{X_w} \) and \( F^J_{X_z} = \hat{F}^Z_{X_z} \).

**Proof.** The key observation is that if \((\Sigma, \alpha, \beta', w, z)\) is a Heegaard triple subordinate to a \( \beta \)-band, then the 4-manifold \( X_{\alpha,\beta,\beta'} \) is equal to \( I \times Y \) with a neighborhood of \( \{\frac{1}{2}\} \times U_\beta \) removed, and the surface with divides
\[
(S_{\alpha,\beta,\beta'}, A_{\alpha,\beta,\beta'}) = (S(B) \setminus \left( \{\frac{1}{2}\} \times B \right), A_0).
\]

There is a cobordism \( X_3 : \emptyset \to Y_{\beta,\beta'} \) consisting of 0-handles and 1-handles. Inside \( X_\beta \), there is a surface \( S_0 \) that consists of \( |w| - 1 \) disks. We note that \( |w| - 2 \) of the disks have boundary equal to a doubly based unknot in \( Y_{\beta',\beta'} \), but one disk has boundary equal to an unknot with four basepoints. It is clear that if we fill in the \( Y_{\beta',\beta'} \) boundary of the link cobordism \((X_{\alpha,\beta,\beta'}, S_{\alpha,\beta,\beta'}) \) with \((X_\beta, S_0)\), then we obtain the (undecorated) link cobordism \((I \times Y, S(B))\). On the other hand, as described above, there are two natural dividing sets on \( S_0 \), shown in Figure 9.7. Define
\[
A_{w,0} = S_0 \cap A_w \quad \text{and} \quad A_{z,0} = S_0 \cap A_z.
\]

Using Lemma 9.12, we have that
\[
F^J_{X_{\beta,S_0},A_{w,0}}(1) = \Theta^z_{\beta,\beta'} \quad \text{and} \quad F^J_{X_{\beta,S_0},A_{z,0}}(1) = \Theta^w_{\beta,\beta'},
\]
as maps from \( \overline{HF}(\emptyset, \emptyset) \cong \mathbb{F}_2 \) to \( \overline{HF}(\Sigma, \beta, \beta', w, z) \).

Using Theorem 7.1 and Proposition 9.8, we know that the sutured link cobordism map \( F^J_{X_{\alpha,\beta,\beta'}} \) for the decorated link cobordism
\[
X_{\alpha,\beta,\beta'} = (X_{\alpha,\beta,\beta'}, S_{\alpha,\beta,\beta'}, A_{\alpha,\beta,\beta'})
\]
is the map \( F_{\alpha,\beta,\beta'} \) that counts holomorphic triangles on the Heegaard triple \((\Sigma, \alpha, \beta, \beta')\). It follows from the composition law that
\[
F^J_{X}(\alpha,\beta,\beta') = F_{\alpha,\beta,\beta'}(- \otimes F_{X_{\beta,S_0},A_{z,0}}(1)) = F_{\alpha,\beta,\beta'}(- \otimes \Theta^w_{\beta,\beta'}) = \hat{F}^Z_{X_{\beta}}(-).
\]
The result for \( F^J_{X_{\hat{X}}} \) follows similarly. \( \square \)

**9.6. Proof of Theorem 9.4.** We can now prove that the link cobordism map constructions from [Juh16] and [Zem16] agree:

**Proof of Theorem 9.4.** Given an arbitrary decorated link cobordism \( \mathcal{X} = (X, S, \mathcal{A}) \), it is easy to see that one can decompose \( \mathcal{X} \) into a sequence of link cobordisms which are each diffeomorphic to an elementary link cobordism (Definition 9.5). Using the composition law, it remains to verify the claim for each type of elementary link cobordism. The maps obviously agree for identity cobordisms, and 0-, 1-, 2-, 3- or 4-handle cobordisms. By Lemma 9.13, they agree for decorated saddle cobordisms. Finally, as in Remark 9.6, a stabilization cobordism can be decomposed into a composition of the other elementary cobordisms, and hence having established the claim for the other types of elementary cobordisms, it follows as well for stabilization cobordisms. \( \square \)
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