THE ARNOUX–YOCCHZ MAPPING CLASSES VIA PENNER’S CONSTRUCTION

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Abstract. We give a new description of the Arnoux–Yoccoz mapping classes as a product of two Dehn twists and a finite order element. The construction is analogous to Penner’s construction of mapping classes with small stretch factors.

1. Introduction

In 1981, Arnoux and Yoccoz [AY81] constructed a pseudo-Anosov mapping class \( \tilde{h}_g \) on the closed orientable surface \( S_g \) for each \( g \geq 3 \). These Arnoux–Yoccoz mapping classes were the first examples to illustrate various interesting phenomena, which we will discuss shortly. The original construction of these mapping classes used interval exchange transformations, and to this day no alternative descriptions of these mapping classes were known.

The goal of the paper is to present a new description as product of two Dehn twists and finite order element. One motivation for such a construction is that in most computer programs mapping classes are specified as a products of such “simple” mapping classes. This new description might also help construct families of mapping classes in new ways that are analogous to the Arnoux–Yoccoz examples and hence might also serve as interesting examples. Finally, our construction shows that powers of the Arnoux–Yoccoz mapping classes arise from Penner’s construction [Pen88], which allows one to describe the invariant train track and the stable and unstable foliations in new ways.

Our main theorem is the following.

Theorem 1. The Arnoux–Yoccoz mapping class \( \tilde{h}_g \) on the surface \( S_g \) is conjugate to \( \tilde{f}_g = r \circ T_a \circ T_b^{-1} \), where \( T_a \) and \( T_b^{-1} \) are positive and negative Dehn twists about the curves \( a \) and \( b \) pictured on Figure 1, and \( r \) is a rotation of the surface by one click in either direction.

![Figure 1](image.png)

Figure 1. The surface \( S_g \) with a rotational symmetry of order \( g \). This figure shows the case \( g = 6 \).

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For the proof, we use the fact, shown by the second author in Section 5 of [Str17], that the mapping classes $\tilde{h}_g$ arise as lifts of mapping classes on nonorientable surfaces. More precisely, there is a pseudo-Anosov mapping class $h_g$ on the closed nonorientable surface $N_{g+1}$ of genus $g+1$ for each $g \geq 3$ such that $\tilde{h}_g$ is the lift of $h_g$ by the double cover $S_g \to N_{g+1}$. We will deduce Theorem 1 from the following.

**Theorem 2.** The nonorientable Arnoux–Yoccoz mapping class $h_g$ on the surface $N_{g+1}$ is conjugate to $f_g = r \circ T_c$, where $T_c$ is a Dehn twist about the two-sided curve $c$ pictured on Figure 2 and $r$ is a rotation of the surface by one click in either direction.

![Figure 2](image)

**Figure 2.** The circle with an X inside it indicates a crosscap: the inside of the circle is not part of the surface and antipodal points of the circle are identified. A disk with one crosscap is therefore a Möbius band. So this figure shows a nonorientable surface obtained by attaching $g$ twisted bands to the boundary of a Möbius band. The surface has one boundary component. By gluing a disk to the boundary component, we obtain the closed surface $N_{g+1}$.

On a nonorientable surface, there is no notion of positive or negative twisting, so we specify the direction of the Dehn twist $T_c$ by the coloring of the curve $c$ on Figure 2 as follows. By cutting out the crosscap in the middle and cutting the twisted bands, we obtain an orientable surface with an embedding in $\mathbb{R}^2$ coming from the figure. Our cut-up surface inherits the orientation of $\mathbb{R}^2$. The blue and red parts of our curve indicate the parts where the twisting behaves like a positive and negative twist, respectively, with respect to the orientation of the cut-up surface. See Figure 5B later for an alternative description of the twisting direction.

**Penner’s construction.** In both the orientable and nonorientable cases, the $g$th power of the mapping class arises from Penner’s construction, analogously to the way Penner constructed pseudo-Anosov mapping classes with small stretch factors in [Pen91].

To see this, note that

$$f_g^g = T_{-(g-1)c} \circ \cdots \circ T_c$$
and
\[ \tilde{f}_g^3 = T_{r^{-2}}(c) \circ T_{r^{-1}}(c) \circ T_c \circ T_b^{-1}. \]

In the nonorientable case, it is possible to orient regular neighborhoods of the curves \( c, \ldots, r^{g-1}(c) \) in a way that at all the intersections, the orientations of the two neighborhoods disagree (Figure 3A). Furthermore, any pair of curves intersects exactly once and hence minimally, and the complement of the union of the curves consists of discs. In the orientable case, \( A = \{a, \ldots, r^{g-1}(a)\} \) and \( B = \{b, \ldots, r^{g-1}(b)\} \) are filling multicurves, and we twist only positively along curves in \( A \) and negatively along curves in \( B \), as required in Penner’s construction [Pen88].

**History and motivation.** One reason that might explain why no alternative construction were known before is that these mapping classes cannot be constructed using Thurston’s construction [Thu88]. This was shown by Hubert and Lanneau [HL06], using the fact that the extension fields \( \mathbb{Q}(\lambda_g + \lambda_g^{-1}) \) are not totally real, where \( \lambda_g \) denotes the stretch factor of \( \tilde{h}_g \). The Arnoux–Yoccoz examples were the first known examples with this property.

Some other interesting properties include:

- Unlike the generic case, the stretch factor and its inverse are not Galois conjugates.
- The SAF invariant of the flat surfaces corresponding to the Arnoux–Yoccoz examples vanishes. The Arnoux–Yoccoz examples were among the first known to have this property. Pseudo-Anosov maps with vanishing SAF invariant have been studied in [AS09, CS13, DS16, Str17].
- It is not known whether the Veech group of the flat surface of a pseudo-Anosov map can be cyclic [HMSZ06, Problem 6]. In the \( g = 3 \) case, Hubert, Lanneau and Möller [HLM09] showed that the Veech group of the Arnoux–Yoccoz surface is not cyclic, but computer experiments suggest that the Arnoux–Yoccoz surfaces are good candidates for having cyclic Veech groups when \( g > 3 \).

For other work on the Arnoux–Yoccoz mapping classes and their flat surfaces, see [Arn88, Bow10, Bow13, McM15, HW18].

**Generalizations.** The Arnoux–Yoccoz example in the \( g = 3 \) case has been generalized by Arnoux and Rauzy in Section 3 of [AR91], see also Section 4.2 of [PLV08]. On the surface \( \mathcal{N}_4 \), the first example in the Arnoux–Rauzy family is the third power of the Arnoux–Yoccoz mapping class \( h_3 \), which is conjugate to \( T_{r^{-2}}(c) \circ T_{r^{-1}}(c) \circ T_c \) by our Theorem 2. We believe that the members of Arnoux–Rauzy family are the mapping classes \( T_{r^{-2}}(c) \circ T_{r^{-1}}(c) \circ T_k \) where \( k \geq 1 \), but we will not give a proof of this in this paper.

In a follow-up paper [LS18], we will generalize the twist-and-rotation construction in Theorem 2 in a different way in order to construct minimal pseudo-Anosov stretch factors on various different nonorientable surfaces. In particular, we will show that the Arnoux–Yoccoz example \( h_3 \) has minimal stretch factor on \( \mathcal{N}_4 \) among pseudo-Anosov mapping classes with an orientable invariant foliation.
Order and directions. On Figure 1, the rotation by 180 degrees about the axis intersecting $a$ and $b$ symmetrically commutes with both $T_a$ and $T_b^{-1}$, but conjugates $r$ to $r^{-1}$. Hence, the direction in which we rotate in Theorem 1 does not matter up to conjugation. The twists $T_a$ and $T_b$ commute, so the order of the mapping classes $T_a$, $T_b^{-1}$ also does not matter. Conjugating $r \circ T_a \circ T_b^{-1}$ by $r$ yields $T_a \circ T_b^{-1} \circ r$ so whether we twist or rotate first also does not matter up to conjugation.

On the other hand, it is clear that if we change the direction of both Dehn twists, then we get a mapping class which is conjugate to the inverse of the Arnoux–Yoccoz mapping class. It is less obvious that if we change the direction in which $b$ winds around the hole in order to avoid $a$, we also get the inverse. We will make a remark about this at the end of the paper.

It can be shown by studying their flat surfaces that $\tilde{h}_g$ is conjugate to $\tilde{h}_g^{-1}$ when $g = 3$, but not when $g > 3$. Therefore one indeed needs to be careful about some of the choices.

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2. Proofs

In this section, we give the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1. By the following claim, the proof of Theorem 1 reduces to the study of the unstable foliation of $f_g$ and the image of the core curve $\gamma$ of the crosscap under $f_g$.

Claim. The pseudo-Anosov mapping class $h_g$ in [Str17, Section 5] is uniquely determined by its unstable foliation, a one-sided curve $\gamma$ transverse to this foliation and its image $\gamma' = h_g(\gamma)$.

Proof of the claim. The unstable foliation induces an interval exchange map on the boundary of the regular neighborhood of $\gamma$. The same is true for $\gamma'$. These interval exchange maps are conjugate, apart from the fact that the intervals for $\gamma'$ are $\lambda^{-1}$ times shorter than the intervals for $\gamma$. These interval exchange maps do not have any symmetries, so there is a unique way to map $\gamma$ to $\gamma'$ that conjugates the interval exchanges to each other.

The complements of $\gamma$ and of $\gamma'$ are unions of foliated rectangles, and the homeomorphism that maps foliated rectangles in the complement of $\gamma$ to foliated rectangles in the complement of $\gamma'$ by mapping leaves to leaves and scaling the measure of the foliation by $\lambda^{-1}$ is unique up to isotopy along the leaves of the foliation. Therefore the mapping class is indeed uniquely determined by the foliation and the two curves. This proves the claim.

To show that the mapping class $f_g$ is conjugate to $h_g$, we will describe the unstable foliation of $f_g$ and show that it is isometric to the unstable foliation of $h_g$. Then we check that the image of the curve $\gamma$ is the same under $f_g$ as under $h_g$. By the claim above, this will imply that $f_g$ and $h_g$ are conjugate.

We now describe the unstable foliation of $f_g$. As described by Penner in [Pen88], an invariant bigon track for $f_g$ can be obtained by smoothing out
the intersections of the \( g \) curves on Figure 3A. This bigon track is shown on Figure 3B.

\[ \begin{align*}
\lambda_6 & \quad \lambda_5 \\
\lambda_4 & \quad \lambda_3 \\
\lambda_2 & \quad \lambda_1 \\
1 & \quad 1 \\
1 & \quad 1 \\
1 & \quad 1 \\
1 & \quad 1 \\
1 & \quad 1 \\
1 & \quad 1 \\
\end{align*} \]

FIGURE 3

For \( i = 0, \ldots, g - 1 \), let \( \mu_i \) be the characteristic measure of the curve \( r^{-i}(c) \), that is, the measure that takes the value 1 on the branches traversed by \( r^{-i}(c) \) and value 0 on all other branches. The cone generated by the characteristic measures \( \mu_i \) is invariant under both the Dehn twist \( T_c \) and the rotation \( r \). This cone therefore contains the measure corresponding to the unstable foliation of \( f_g \). To find this measure, we consider the action of \( f_g \) on the cone. The mapping class \( T_c \) acts by the matrix which has 1s on the diagonal and in the first row, and 0s otherwise, and \( r \) acts by a permutation matrix. When \( g = 7 \), the action of \( f_g = r \circ T_c \) is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

the companion matrix of the polynomial \( x^7 - x^6 - \cdots - x - 1 \). In general, the characteristic polynomial is \( x^g - x^{g-1} - \cdots - x - 1 \). The stretch factor \( \lambda \) of \( f_g \) is the Perron-Frobenius eigenvalue of this matrix, that is, the largest real root of its characteristic polynomial. The corresponding eigenvector is \((1, \lambda, \lambda^2, \ldots, \lambda^{g-1})\). Therefore, the unstable foliation of \( f_g \) is given by the measure \( \sum_{i=0}^{g-1} \lambda^i \mu_i \) on our bigon track.

Now observe that Figure 3B can be redrawn as shown on Figure 4. One way to see this is that a large regular neighborhood of \( \gamma \) on Figure 3B is the disk with the crosscap in the middle (hence a Möbius band), but without the twisted bands attached. The effect of attaching the twisted bands is that intervals on the boundary of the Möbius band get identified. The identifications do not preserve the orientation near the boundary of the Möbius band.
band, therefore the intervals are identified by translations on Figure 4. The widths of the twisted bands according to the unstable measured foliation are $1, \lambda, \ldots, \lambda^{g-1}$, the measures on the branches of the bigon track inside the twisted bands, therefore these are the lengths of the intervals on Figure 4.

![Figure 4](image)

**Figure 4.** The left and the right edges are identified by a flip, and each pair of two intervals labeled by $\lambda^i$ is identified by a translation. The boundary of Figure 3B is contracted to one point. The vertical foliation equals the foliation defined by the measure $\sum_{i=0}^{g-1} \lambda^i \mu_i$ on the bigon track in Figure 3B, which is the unstable foliation of $f_g$.

The pattern in which the rectangles are glued together is the same as on Figure 5.1 in [Str17], and the foliations are identical up to a constant factor ($\lambda^{g-1}$).

![Figure 5](image)

**Figure 5.** The curves $\gamma$, $T_c(\gamma)$, and $\gamma' = r(T_c(\gamma))$. To go from the second figure to the third, we isotope a small piece of $T_c(\gamma)$ through the crosscap. This is possible, because the antipodal points of the circle containing the X are identified.

The curve $\gamma' = h_g(\gamma)$ on Figure 5.1 in [Str17] corresponds to the curve $\gamma'$ on Figures 3B and 4. It remains to show that $\gamma' = f_g(\gamma)$. After applying the twist $T_c$ on the curve $\gamma$ on Figure 5A, we obtain the curve shown on Figure 5B. After rotation by one click, this curve indeed maps to $\gamma'$.

**Proof of Theorem**. Consider the orientable double cover of the nonorientable surface in Figure 2. One way to construct this covering surface is to cut along the twisted bands on Figure 3B, remove the central crosscap, and glue together two copies of the resulting surface. We can think about
the two copies as the upper and lower half of the cylinder pictured on Figure 6. The upper and lower boundaries of this cylinder are subdivided into $2g$ intervals, which correspond to the $2g$ intervals obtained by cutting the twisted bands. The orientation-reversing involution of this cylinder that identifies the upper and lower half is the reflection about the center of the picture in the ambient 3-dimensional space. When the intervals along the boundaries are identified in the manner shown, the quotient of the surface is our nonorientable surface with the twisted bands, with the boundary collapsed to one point.

Note that the rotation $r$ downstairs lifts to the rotation of the cylinder by two intervals.

By flattening out the cylinder, we obtain the representation on Figure 7A. The lift of the curve $c$ along which we twist in the definition of $f_g$ has two components, shown on Figure 7A. A Dehn twist about the curve $c$ downstairs lifts to the product of a positive twist along one of the lifts and a negative twist about the other lift.

To find out which twist is positive and which twist is negative, recall that $T_c(\gamma)$ is a curve that runs in a small neighborhood of one of the twisted bands (Figures 5A and 5B). The green curve on Figure 7A is the lift of $\gamma$, so its image under the two twists should run in a small neighborhood of the boundary of the annulus. The way that happens is when the twist is positive along the blue curve and negative along the red curve (see Figure 7B).

After changing Figure 7A by rotating the inner boundary by 180 degrees, we obtain the representation shown on Figure 7C. By subdividing this surface along the arcs shown and their rotated copies, we can see that this surface can be represented in $\mathbb{R}^3$ as the surface on Figure 7D chopped up in the analogous way. The two twisting curves correspond to the curves shown on Figure 1, and we indeed twist positively along the curve $a$ and negatively along the curve $b$.

In the introduction we made the remark that if the curve $b$ loops around the hole in the other direction, then we get a mapping class which is conjugate to the inverse of the Arnoux–Yoccoz mapping class. This can be seen by rotating the inner boundary of the annulus on Figure 7A in the opposite way and drawing the subdivision on Figure 7D slightly differently. The result is still conjugate to the Arnoux–Yoccoz mapping class. However, when compared to the statement of Theorem 1, both directions of the Dehn twists,
as well as the way in which b loops around the hole, have changed. Since changing the direction of both Dehn twists gives the inverse mapping class, changing only the way in which b loops around the hole yields a mapping class which is conjugate to the inverse of the Arnoux–Yoccoz example.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Figure 7}
\end{figure}

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