The Relativistic Generalization of the Gravitational Force for Arbitrary Spacetimes

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Abstract

It has been suggested that re-expressing relativity in terms of forces could provide fresh insights. The formalism developed for this purpose only applied to static, or conformally static, space-times. Here we extend it to arbitrary space-times. It is hoped that this formalism may lead to a workable definition of mass and energy in relativity.

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1 Introduction

Despite the elegance of Einstein’s geometrodynamics [1] the concept of ”force” is still used extensively. This is probably due largely to the inertia involved in the radical change of the concepts required. Accepting the need to maintain the contact with the earlier concepts, it was argued [2,3] that it would be worthwhile to re-introduce the force concept into General Relativity (GR). At the very least it should provide a fresh way of looking at the consequences of the theory and new insights into its working. Hopefully it could lead to new tests of GR which had not been suggested by the purely geometrical formulation. In fact this re-expression has provided a further understanding of the dynamics of a neutral test particle in the field of a charged, rotating,
massive point source [4,5]. It has also been used to suggest possible relativistic explanations of some hitherto unexplained astrophysical phenomena [6].

There is a more compelling reason to consider forces in relativity. As has been argued earlier [7], it may provide a way to avoid the necessity of going to higher (than four) dimensions in an attempt to unify the forces of nature. A geometrical unification must necessarily enlarge space-time so that the special role of the usual four-metric only appears in a projective theory but it (the four-metric) occurs on a par with other fields in the full theory. Alternatively gravity must be treated as just another field which happens to be of spin two and happens to have an Einstein-Hilbert Lagrangian. In terms of the relativistic equations of motion, gravity appears on the left-hand side of the equation (in the Christoffel symbol) while all the other forces appear on the right-hand side. Either one enlarges the space to include the other fields in the metric (and the Christoffel symbols) or one can remove gravity from the left-hand side and display it explicitly on the right-hand side to see how it can be related to the other forces. Of course this is not an attempt at a final solution of the problem, but rather an attempt to find some ”signposts” for an alternative way to tackle the problem.

There is another hope for this approach. The force will be defined operationally. It will take a simple form in some particular frame of reference. A sequence of space-like hypersurfaces defined by these frames of reference would provide a physical basis for a 3+1 split of space-time. The hope is that this split may provide a suitable basis for a canonical quantization attempt.

The gravitational force is extracted from the left-hand side of the equation of motion by determining the ”force” that gives the same ”bent path” in a ”flattened out” background space-time as would be given by the curvature of space-time. The relativistic analogue of the gravitational force has been called the pseudo-Newtonian ($\psi N$) force. It was originally calculated for the Schwarzschild and Reissner-Nordstrom geometries and an attempt was made to deal with the Kerr-Newmann metric [2]. The former already gives ”electro-gravitic unification” in that it predicts a repulsive force of a charged source on neutral matter [4]. Though very different from the current paradigm of unification, this effect is no less a physical manifestation of unification, at least in principle, than the ”mixing of photons and weak neutral bosons” in the electro-weak theory of Glashow-Salam-Weinberg [8]. The extension to other static metrics was achieved later [3]. The formalism yielded a single ”gravitational potential”, in a special frame of reference, which turned out
to be the conjectured potential mentioned by Hawking and Ellis [9].

The restriction to static space-times is easily understood. Classically, force is related to energy, and energy is a conserved quantity only if there exists a timelike isometry. This restriction does not allow any time dependence of the \( \psi N \) force. Thus there remains a problem of definition of energy for arbitrary space-times. Of course there can be no hope of constructing a realistic field theory based on static space-times. In particular, there would then be no canonically conjugate "momentum" field. Attempts to extend to conformally static space-times, while successful [10], were not very fruitful as, in effect, the time dependence was eliminated there.

To pursue such a programme, it is necessary to obtain an expression for the extremal tidal force. This extremal tidal force will lead to an expression for the extended \( \psi N(e\psi N) \) force four-vector. The zero component of the force four-vector will represent the rate of change of energy of the particle due to a change in the gravitational field. Usually we can write the force as the gradient of a scalar potential. Here we obtain two scalar quantities corresponding to this \( e\psi N \) force. The quantity which corresponds to the time component of the force will give the potential energy of the test particle which contributes to its time variation.

The plan of the paper is as follows. In the next section we briefly review the relevant aspects of the \( \psi N \)-formalism for the purposes of application. In sect. 3 we discuss the extrema of the general tidal force. In the next section we calculate the general \( \psi N \) force. In sect. 5 we obtain the extension of the \( \psi N \) potential to arbitrary space-times. In sect. 6 we discuss two cosmological examples and finally, in the last section, we summarize and discuss our results.

2 The Pseudo-Newtonian Formalism

Though the gravitational force is not detectable in a freely falling frame \( (\text{FFF}) \) that is so only at a point, it is detectable over a finite spatial extent as the tidal force. For example, it could be measured by an "accelerometer" consisting [2] of two masses connected by a spring ending in a needle which moves on one of the masses marked off as a dial. Stretching or squashing the spring causes the needle to move one side of the zero position or the other. The usual tidal force would cause the spring to stretch. However, a repulsion would result in squashing. Thus this accelerometer could measure
and identify attraction and repulsion.

Mathematically, the tidal acceleration is given by

\[ A^\mu = R^\mu_{\nu \rho \pi} t^\nu l^\rho t^\pi, \quad (\mu, \nu, ... = 0, ..., 3), \]  

(1)

where \( R \) is the Riemann tensor, \( t \) a timelike vector and \( l \) is the spacelike "separation" vector representing the accelerometer. In its rest frame the tidal force on a test particle of mass \( m \) was taken to be

\[ F^\mu_T = m R^\mu_{000} l^\nu. \]  

(2)

The external values of the tidal force can be obtained by requiring that \( l \) be an eigenvector of \( R \) in eq.(2). Since \( l \) has no time component in the chosen frame, neither does \( F^*_T \), the extremal value of the tidal force. Physically the extremal value can be obtained by turning the accelerometer about, till it gives the maximal reading, and noting the direction given by it.

For the Schwarzschild metric the relevant principal direction is the radial direction. Modulo a local Lorentz factor, \( f \), eq. (2) gives the usual Newtonian tidal force for \( F^*_T \). The effect of introducing a charge is to reduce the tidal force by \( mQ^2l/r^3 \), in gravitational units. If we also include rotation the principal direction is no longer radial in general, but lies in the radial polar (\( \phi = \text{constant} \)) plane. Its angle of inclination to the radial direction depends on the radial and polar coordinates of the test particle. Generally eq.(2) yields a cubic equation for the eigenvalue (which is essentially \( |F^*_T| \)). This will always have at least one real root.

If the metric is of Carter's "circular" form \([11]\), or can be otherwise broken into blocks so that there are isometries in the block containing the time component \([3]\), eq.(2) can be reduced to the form of a directional derivative along \( l \), by using Riemann normal coordinates (RNCs) \([1]\),

\[ F^i_T = ml.\nabla \Gamma^i_{00}, \quad (i, j, ... = 1, 2, 3). \]  

(3)

Thus, up to an "integration constant" \( m\Gamma^i_{00} \) must give the \( \psi N \) force, \( F^i \). This term is fixed by requiring that there be no \( \psi N \) force in a Minkowski space. The \( \psi N \) force is now

\[ F^i = m\Gamma^i_{00}. \]  

(4)

We can then write this force as the gradient of a scalar \( \psi N \) potential, \( V \). Requiring that the potential also be zero in a Minkowski space gives

\[ F_i = -V_{,i} = \frac{1}{2} m(1 - g_{00})_{,i}. \]  

(5)
That this should be the value of \( V \) is the conjecture mentioned earlier \[9\].

For the Schwarzschild metric \( F_i \) is simply the Newtonian gravitational force for a point mass and the \( \psi_N \) potential the usual Newtonian gravitational potential. The inclusion of a charge introduces a repulsive component of the gravitational force \[2\]. The entire structure of the \( \psi_N \) force for the Kerr-Newmann metric may be seen in embryonic form in the \( \psi_N \) potential.

\[
V = -m(2Mr - Q^2)/2(r^2 + \alpha^2 \cos^2 \vartheta)
\]  

(6)

where \( Q \) is the charge and \( \alpha \) the angular momentum per unit mass of the gravitational source. Clearly there will be a polar component of the \( \psi_N \) force since \( V \) is \( \vartheta \) dependent.

3 The Extermal Tidal Force

In developing the \( \psi_N \)-formalism \( t \) was identified with the timelike Killing vector so as to provide an easily integrable expression for the tidal force. Despite the necessity of staticity for energy conservation, and hence the usual force concept, the accelerometer would still show a deflection in a nonstatic situation. As such there must be a "force" embedded in the geometry. For example, in a Friedmann-model universe the accelerometer needle must show this deflection over sufficiently long periods of time and the model must, therefore, have a "force" in it. This will be so regardless of the fact that there is no apparent gravitational source in the model. The deflection, which the needle of the accelerometer indicates, would be the same for all orientations of the accelerometer. This deflection would, therefore, be attributed to an expansion of the universe as a whole. It comes from the gravitational field and not from any gravitational source. The requirement is to determine the extremal value of the tidal force for an arbitrary space-time without any time isometry.

To study the consequences of time variation in terms of forces it is necessary to give up any timelike symmetry and squarely face the time variability. The timelike Killing vector used earlier must be replaced by the unit tangent vector to the world line of the observer, \( t \). In this rest frame the four-vector force representing the accelerometer, \( l^\nu \), will again have no zero component and the tangent vector can be written as

\[
t^\mu = f(x)\delta_0^\mu,
\]  

(7)
where \( f(x) = (g_{00})^{-1/2} \). On account of this change of value of \( t^\mu \) the tidal-force expression becomes

\[
F_T^\mu = m f^2(x) R_{0j0}^\mu l^j,
\]

which differs from eq.(2) due to the extra \( f^2(x) \) factor. Since the Killing vector was not a unit vector it introduced a scaling which appeared as a local Lorentz factor. This local Lorentz factor was removed there by hand but here it is adjusted automatically. Thus, while a re-scaled time appeared there, we are using proper time here.

We need to take coordinates that are essentially synchronous coordinates [12] but without the restriction \( g_{00} = 1 \). Thus \( g_{0i} = 0 \). Now both ends of the accelerometer are spatially free, i.e. both move and do not stay attached to some spatial point. However, there is a "memory" of the initial time built into the accelerometer in that the zero position is fixed then. Any change is registered that way. Thus "time" behaves very differently from "space". We must, therefore, use RNCs only for the spatial and not for the temporal direction.

Thus using the block-diagonalised form of the metric with spatial RNCs, eq.(8) can be written explicitly as

\[
F_T^\mu = m f^2(\Gamma_{00,j}^\nu - \Gamma_{0j,0}^\nu + \Gamma_{0j}^\nu \Gamma_{00}^\nu - \Gamma_{0k}^\nu \Gamma_{0j}^k) l^i.
\]

This force, when extremised, can have no time component as \( I \) must be proportional to \( F_T \) in that case.

Consider an observer in an FFF, equipped with an accelerometer by which he can detect the tidal forces experienced by him. He can then swivel the accelerometer till he gets a maximum reading on the dial. Regarding eq.(9) as an eigenvalue equation with \( \mu \) replaced by \( i \), the eigenvalue problem becomes

\[
m f^2(\Gamma_{00,j}^i - \Gamma_{0j,i}^0 + \Gamma_{0j}^i \Gamma_{00}^0 - \Gamma_{0k}^i \Gamma_{0l}^k) l^j = \lambda l^i,
\]

where \( \lambda \) is the eigenvalue and \( l^i \) is the corresponding eigenvector. Writing eq.(10) in matrix form we see that it has a nontrivial solution only if \( \lambda \) satisfies the cubic equation

\[
\lambda^3 + 3a_1 \lambda^2 + 3a_2 \lambda + a_3 = 0,
\]

\[
\begin{align*}
a_1 &= -(A + E + K)/3, \\
a_2 &= (AE + AK - BD - CG + EK - FH)/3, \\
a_3 &= -AEK + AFH + BDK - BFG + CEG - CDH;
\end{align*}
\]
\[
\begin{align*}
A &= m f^2 \left[ \Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^1_{10,0} - (\Gamma^1_{01})^2 \right], \\
B &= m f^2 \Gamma^1_{00,2}, \quad C = m f^2 \Gamma^1_{00,3}, \quad D = m f^2 \Gamma^2_{00,1}, \\
E &= m f^2 \left[ \Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^2_{01,0} - (\Gamma^2_{02})^2 \right], \\
F &= m f^2 \Gamma^2_{00,3}, \quad G = m f^2 \Gamma^3_{00,1}, \quad H = m f^2 \Gamma^3_{00,2}, \\
K &= m f^2 \left[ \Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^3_{03} \Gamma^3_{00} - (\Gamma^3_{03})^2 \right].
\end{align*}
\] (13)

Equation (11) can be solved to yield three roots and provide the corresponding separation vector [13]. The general solution does not provide much wisdom. As such we shall only make some observations regarding the solution here. In the generic case there will be three distinct eigenvalues. One of them, at least, will always be real. If the other two are complex, the real value will give the required \(F_T^*\) and the corresponding eigenvector will give its direction. If all three are real the maximum magnitude eigenvalue gives the required tidal force, \(F_T^*\), and the corresponding eigenvector gives its direction. Further, if all three eigenvalues are real and equal we have isotropy. If, due to extra symmetries, one of the eigenvalues is zero we get a quadratic equation. In this case the roots are always real. It is also possible that, due to further symmetries, two roots are zero and we get a linear equation. Again \(F_T^*\) and the corresponding eigenvector are easily obtained.

4 The Extended \(\psi N\) Force

Recall our definition of the relativistic analogue of the Newtonian gravitational force. It is that quantity whose directional derivative along the accelerometer, placed along the principal direction, gives the extremised tidal force and which is zero in a Minkowski space. Thus the \(e\psi N\) force, \(F^\mu\), satisfies the equation

\[ F_T^* = l^\nu F_T^\nu. \] (14)

The fact that the zero component of the left side is zero does not guarantee that the zero component of \(F^\mu\) is zero. With the appropriate gauge choice and using RNCs spatially, eq.(14) can be written in a space and time break up as

\[ l^i(F^0_i + \Gamma^0_{ij}F^j) = 0, \] (15)

\[ l^j(F^i_j + \Gamma^i_{0j}F^0) = F_T^{*i}. \] (16)

Notice that the expression in the brackets is essentially \(F_T^* \delta^i_j\) up to a scaling by the length of the accelerometer.
A simultaneous solution of the above equations can be obtained by taking
the ansatz \[13\]
\[
F^0 = m\left[(\ln A)_0 - \Gamma^0_{00} + \Gamma^i_{0j}/A\right]f^2,
\]
\[
F^i = m\Gamma^i_{00}f^2,
\]
where
\[
A = (\ln \sqrt{-g})_0, \quad g = \text{det}(g_{ij}),
\]
Now for these metrics
\[
\Gamma^0_{00} = \frac{1}{2}g^{00}g_{00,0}, \quad \Gamma^i_{00} = -\frac{1}{2}g^{ij}g_{00,j}, \quad \Gamma^i_{0j} = \frac{1}{2}g^{ik}g_{jk,0}.
\]
Thus the covariant form of the $e\psi N$ force, with the appropriate choice of
frame, is given by
\[
F_0 = m\left[(\ln A)_0 + g^{ik}g_{jk,0}g^{it}g_{it,0}/4A\right],
\]
\[
F_j = m(\ln f)_i.
\]
The new feature of the $e\psi N$ force is its zero component. In special rela-
tivistic terms, which are relevant for discussing forces in a Minkowski space,
the zero component of the four-vector force corresponds to a proper rate of
change of energy of the test particle. Further, we know that in general an
accelerated particle either radiates or absorbs energy according as
$dE/dt$ is
greater or less than zero. Thus $F_0$, here, should also correspond to energy
absorption or emission by the background space-time. In fact we could have
separately anticipated that there should be energy non-conservation as there
is no timelike isometry. In that sense $F_0$ gives a measure of the extent to
which the space-time lacks time isometry.

Another way of interpreting $F_0$ is that it gives a measure of the change
of the ”gravitational potential energy” in the space-time. In classical terms,
neglecting this component of the $e\psi N$ force would lead to erroneous conclu-
sions regarding the ”energy content” of the gravitational field. Contrariwise,
including it enables us to revert to classical concepts while dealing with a
general relativistically valid treatment. It can be hoped that this way of
looking at energy in relativity might provide a pointer to the solution of the
problem of definition of mass and energy in GR.
5 The $e\psi N$ Potentials

It is clear that the gauge freedom could not be used to reduce the gravitational potential to a single quantity for an arbitrary space-time (as had been attempted earlier). Equations (17), (18) and (19) provide five such quantities. The $e\psi N$ force, for arbitrary space-times, can be expressed in the form of the derivatives of two quantities. Now for our block-diagonalised metrics

\[ g^{ik}g_{ik,0} = -g_{00}g_{ik}. \]  

(23)

Thus the last term in eq.(21) can be reduced to

\[ g^{ik}g_{jk,0}g^{jl}g_{il,0} = -g_{00}g_{ij,0}. \]  

(24)

Hence we can write

\[ F_0 = -U_0, \quad F_i = -V_{ij}, \]  

(25)

with

\[ U = m[\ln(Af/B) - \int (g_{00}g_{ij,0}/4A)dt], \]  

(26)

\[ V = -m \ln f, \]  

(27)

where $B$ is a constant with units of time inverse, so as to make $A/B$ dimensionless.

It is clear that $V$ is the generalization of the classical gravitational potential and, for small variations from a Minkowski space,

\[ V \sim \frac{1}{2}m(g_0 - 1), \]  

(28)

which is the $\psi N$ potential. In fact the $e\psi N$ force for a static spacetime is simply the $\psi N$ force with the Lorentz factor adjusted. (Notice that it is reminiscent of the Kahler potential.) It is in this sense that the $e\psi N$ potential is more natural to use than the $e\psi N$ potential.

The quantity $U$ clearly represents a potential energy of the test particle that contributes to its time variation. It is important that the entire metric tensor (all ten components) is contained in it. However, only the time-varying art of these components is relevant. It is on account of this fact that the static metric has only $g_{00}$ as the relevant potential. This is not the case for a gravitational wave [13,14], for example. If $U$ is neglected, the ”Newtonian” institution will mislead us.
6 Application to Some Geometries

To better comprehend the significance of the $e\psi N$ force and potential we consider two concrete examples. (In a separate paper we discuss the application of this formalism to the problem of the energy in gravitational waves [14]). The two examples are:

a) the De Sitter metrics,
b) the Friedmann metrics.

a) The De Sitter metrics

The De Sitter metrics, with the observation point at the origin of the polar coordinates, is defined by

$$ds^2 = (I - r^2/D^2)dt^2 - (1 - r^2/D^2)^{-1}dr^2 - r^2d\Omega^2,$$

(29)

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\phi^2$ is the solid angle element and $D = \sqrt{3/\Lambda}$ is the "radial distance to the event horizon", where $\Lambda$ is the cosmological constant. Then the tidal force is

$$F^0_T = F^2_T = F^3_T = 0, \quad F^1_T = -ml^1/D^2$$

(30)

and hence the maximum tidal force is simply $-ml/D^2$. This gives the $e\psi N$ force

$$F^0 = F^2 = F^3 = 0, \quad F^1 = -mr/D^2$$

(31)

which is the usual force of "cosmical repulsion". The corresponding $e\psi N$ potentials are then

$$U = 0, \quad V = m\ln\sqrt{1 - r^2/D^2}.$$

(32)

Written in its exponentially expanding form the De Sitter metric is

$$ds^2 = d\tau^2 - \exp[2\tau/D](dr^2 + r^2d\Omega^2).$$

(33)

In this form the tidal force is isotropic, but it has the same maximal value as in the previous form. Here the $e\psi N$ force becomes

$$F^0 = F_0 = -m/D, \quad F^i = 0,$$

(34)
and the corresponding $e\psi N$ potential is

$$U = -mr/D, \quad V = 0$$  \hspace{1cm} (35)$$

The repulsive force has been replaced by a time-dependent "potential energy", which again provides the red-shift. Here the universal expansion has been built in, thus obviating the necessity for a "force".

The $e\psi N$ formalism requires the use of the latter metric form, eq.(33), instead of the former, eq.(29). The reason is that the Lie derivative of the separation vector, $l$, is not zero along the unit time-like vector, $t$ given by

$$t^\mu = 1 - \frac{r^2}{D^2} - \frac{1}{2}\delta_0^\mu$$  \hspace{1cm} (36)$$

required for the former metric form. In physical terms, our accelerometer is very small but non-negligible on the cosmological scale. To the extent that it is non-negligible the time parameter is re-scaled from one end of it to the other. In other words, in effect the separation vector chosen does not lie in the purely spatial direction, but has a temporal component. This is not so in the latter case. Clearly, for cosmological purposes, we need to use a synchronous coordinate system and must take $g_{00} = 1$.

b) The Friedmann metrics

In the Friedmann cosmological models, due to the conservation of mass energy, the energy density decreases with time as the universe expands. We shall discuss only matter-dominated Friedmann models

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \alpha^2(\chi)d\Omega^2\right],$$  \hspace{1cm} (37)$$

where $\chi$ is the hyperspherical angle, $\sigma(\chi)$ is $\sin h\chi$, $\chi$ or $\sin \chi$ according as the model is open ($k = -1$), flat ($k = 0$) or closed ($k = 1$) and $a(t)$ is the corresponding scale factor. The tidal force in this case is

$$F_T^0 = 0, \quad F_T^i = -m\ddot{a}^i/a,$$  \hspace{1cm} (38)$$

where dot "." denotes differentiation with respect to coordinate time $t$, and not $s$. The maximal value is clearly $-m\ddot{a}/a$.

For the flat Friedmann model, the extremal tidal force is

$$F_T^i = 2ml/9t^2.$$  \hspace{1cm} (39)$$
Thus $F_T^* \sim t^{-2}$ and hence $F_{T0}^* \to \infty$ as $t \to 0$ and $F_{T0}^* \to 0$ as $t \to \infty$

To discuss the general behaviour of the Friedmann models, let us, first consider the extremal tidal force at the very early stages of their evolution. For the early stages of the open (or closed) Friedmann model, the scale factor will be $a_0^{1/3} t^{2/3+\epsilon}$, where $\epsilon$ is a small positive (or negative) quantity, and the extremal tidal force takes the form

$$F_T^* \sim (1 - 3\epsilon/2)F_{T0}^*.$$  \hfill (40)

Thus the extremal tidal force will be less (or greater) than for the flat case.

For the open Friedmann model at later times, the extremal value of the tidal force is

$$F_T^* = 4ml/a_0^2(cosh\eta - 1)^3.$$  \hfill (41)

Thus $F_T^* \sim t^{-2}$ for large $t$, as in the case of $F_{T0}^*$.

For the closed model, the extremal tidal force is

$$F_T^* = 4ml/a_0^2(1 - cos \eta)^3.$$  \hfill (42)

Thus $F_T^*$ reaches a minimum value of

$$F_T^* = ml/2a_0^2,$$  \hfill (43)

at $\eta = \pi$ (i.e. at $t = a_0\pi/2$) and again becomes infinite at $\eta = 2\pi$ (or $t = a_0\pi$). The $e\psi N$ force, for the Friedmann models, is simply

$$F^0 = F_0 = -m\ddot{a}/\dot{a}, \quad F_i = 0,$$  \hfill (44)

and the corresponding $e\psi N$ potentials

$$U = m\ln(\dot{a}/b), \quad V = 0,$$  \hfill (45)

where $b$ is an arbitrary constant with unit of $\dot{a}$. For a flat Friedmann model, eq.(4) yields

$$F^0 F_{00} = m/3t, \quad F(i) = 0.$$  \hfill (46)

The corresponding $e\psi N$ potentials are

$$U_0 = \frac{1}{3}m\ln(t/T), \quad V = 0$$  \hfill (47)

for an appropriate choice of $b$. Thus $F_0$ is proportional to $t^{-1}$ and hence $F_0$ goes to infinity as $t$ approaches zero and it tends to zero when $t$ tends to

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infinity. Since $F_0$ is positive, it corresponds to the energy absorption [13] by the background space-time. The comparison of the behaviour of the early stages of the open and closed models with the flat, for the $e\psi N$ force and potential, follows the tidal-force pattern exactly.

For the open Friedmann model at arbitrary times, the time component of the $e\psi N$ force turns out to be

$$F_0^- = 2m/a_0 \sinh(\cosh \eta - 1),$$

and the corresponding $e\psi N$ potential is

$$U_- = m \ln[\sinh(\cosh \eta - 1)].$$

Hence $F_0^-$ goes as $t^{-1}$ for large $t$ as in the case of $F_{00}$. Notice that $U_- \to 0$ as $t \to \infty$ whereas $U_0 \to -\infty$ as $t \to \infty$. This odd feature of the flat Friedmann model may indicate a problem with our ansatz solution.

For the closed Friedmann universe there is a problem. The Christoffel symbol appearing in eq.(16) is zero, for this case, when $F^*_T$ reaches a minimum value. According to the ansatz used this gives an infinite $e\psi N$ force at that instant. This is clearly absurd. It was verified that obtaining the general solution to eqs.(15) and (16) does not resolve this problem. However, there is an arbitrariness in what we choose to call the "zero" of the accelerometer. There is no a priori reason to set it at any particular value. We can then choose to set it at zero at the phase of maximum expansion, $\eta = \pi$, so as to avoid the infinity in the $e\psi N$ force. Using this resetting, the $e\psi N$ force becomes

$$F^0 = m\frac{(4 + 3 \sin^2 \eta + 3 \cos \eta + \cos^3 \eta)}{4a_0(1 - \cos \eta) \sin \eta}, \quad F^i = 0$$

This gives $F^\mu = 0$ at the phase of maximum expansion of the universe.

The $e\psi N$ potentials, here, are

$$U = m\left[\frac{1}{1 - \cos \eta} + \frac{1}{4} \ln(1 - \cos \eta)\right]/a_0, \quad V = 0.$$  

Hence, at the phase of maximum expansion the gravitational potential energy is

$$U = m(1 + \ln \sqrt{2})/2a_0, \quad V = 0.$$  

Generally, as $t \to 0$, the $e\psi N$ force and potential for the Friedmann metrics tend to infinity. In the closed model they again become infinite at $\eta = 2\pi$, i.e. $t = \pi a_0$.  

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7 Conclusion

It has been shown that the $e\psi N$-formalism, which had been useful in providing some insights into the consequences of relativity for a static space-time, can be extended to arbitrary space-times. The $e\psi N$ potential, in this case, approximates the $\psi N$ potential for small variations from the Minkowski space-time. The difference arises for nonstatic space-times. The fundamentally new feature of the $e\psi N$ force is its zero component. This does not contribute directly to the tidal force, since the accelerometer has no zero component in its own rest frame, but it enters through the Christoffel symbols $\Gamma^0_{ij}$ and $\Gamma^i_{0j}$. We find that this force can be described in terms of two $e\psi N$ potentials in general.

We would like to reiterate, here, the importance of the choice of frame. In doing so, to avoid possible confusion, we make it clear that we refer not to the coordinate system but to the frame. All too often coordinates are regarded as merely labelling some point in a manifold. While mathematically correct, sight is lost of the physical point that in GR it includes the choice of the reference frame. All physics is done with resect to some frame and for any given experiment there will always be a preferred frame. This argument does not militate against the principle of general covariance but, rather, complements it. For any particular process there must be a frame in which it can be most simply described. The point of general covariance is that physical laws, generally, have no preferred frame which can be universally used for all purposes and in all circumstances. To discuss the centrifugal force it is necessary to go into a corotating frame [15]. Similarly, for the purpose of providing a $3 + 1$ split of the space-time metric, we need to enter the freely falling rest frame, which gives the $e\psi N$ force and potentials.

In view of the fact that an apparently internally consistent expression for energy seems to be forthcoming from the $e\psi N$ formalism, it may be hoped that it will provide a handle to tackle the problem of definition of mass and energy in GR. It could even be hoped that such a definition of energy could ultimately lead to a viable canonical quantization programme. However, there are still some problems to be resolved. For one thing there is an arbitrariness in setting the zero of the extremal tidal force, which may be nontrivial. This zero of the extremal tidal force measured by the accelerometer can be fixed according to the observer's choice. We are using the RNCs spatially. This means that we have chosen the three arbitrary constants, one in each spatial direction, to be zero. Thus we have an arbitrariness in the zero setting of
the extremal tidal force which we can set by fixing the suitable arbitrary constants.

There is another problem. The solution of eqs.(15) and (16) is not unique but has been obtained by taking an ansatz. In fact this solution is the ”particular integral” part of the general solution. There is another, ”complementary”, part to be obtained for the general solution of the set of equations. It would be of interest to solve the set of coupled, linear, inhomogenous, partial differential equations given by eqs.(15) and (16). This solution would provide further understanding of the definition of energy in GR given by the $\psi N$ approach.

It is worthwhile to point out that the $e\psi N$ approach is not an alternative approach to GR. In fact it is very much a part of GR. It is an attempt to understand the implications of this theory better. The purpose is to recast the consequences in Newtonian—”force” terms as an aid to our intuition. Other similar attempts tend to give a ”weak-field” or ”linearized” effect. We, on the other hand, use a method which allows us to consider strong-field, non-linearized, effects. This would be vital in any attempt to ”quantize relativity”. (Here we would draw a distinction between ”quantizing gravity” and ”relativity”. The latter is a theory of motion [2-4, 15].)

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