Abstract. In this paper we introduce \( n\mathbb{Z} \)-abelian and \( n\mathbb{Z} \)-exact categories by axiomatising properties of \( n\mathbb{Z} \)-cluster tilting subcategories. We study this categories and show that every \( n\mathbb{Z} \)-cluster tilting subcategory of an abelian (resp., exact) category has a natural structure of an \( n\mathbb{Z} \)-abelian (resp., \( n\mathbb{Z} \)-exact) category. Also we show that every small \( n\mathbb{Z} \)-abelian category arise in this way, and discuss the problem for \( n\mathbb{Z} \)-exact categories.

1. Introduction

Higher Auslander-Reiten theory was introduced by Iyama in [9, 12]. It deals with \( n \)-cluster tilting subcategories of abelian and exact categories. Jasso in [14] introduced \( n \)-abelian and \( n \)-exact categories as a higher-dimensional analogue of abelian and exact categories, that are axiomatisation of \( n \)-cluster tilting subcategories. Jasso proved that each \( n \)-cluster tilting subcategory of an abelian (resp., exact) category is \( n \)-abelian (resp., \( n \)-exact). Also every \( n \)-abelian category has been shown to arise in this way [5, 17].

Special kind of \( n \)-cluster tilting subcategories, called \( n\mathbb{Z} \)-cluster tilting subcategories have nicer properties than general \( n \)-cluster tilting subcategories. An \( n \)-cluster tilting subcategory is said to be an \( n\mathbb{Z} \)-cluster tilting subcategory if satisfies the following additional condition.

- If \( \text{Ext}^k(M, M) \neq 0 \) then \( k \in n\mathbb{Z} \).

In this paper we give the following characterisation of \( n\mathbb{Z} \)-cluster tilting subcategories of exact categories. We refer the reader to Section 2 for the definitions of \( n \)-cluster tilting subcategory and \( n \)-exact sequence.

Theorem 1.1. Let \( M \) be an \( n \)-cluster tilting subcategory of an exact category \( \mathcal{E} \). The following conditions are equivalent.

1. \( M \) is an \( n\mathbb{Z} \)-cluster tilting subcategory.
2. For every \( X \in M \) and every \( n \)-exact sequence \( Y : Y^0 \to Y^1 \to \cdots Y^n \to Y^{n+1} \) the following induced sequence of abelian groups is exact.

\[
0 \to \text{Hom}_\mathcal{E}(X, Y^0) \to \text{Hom}_\mathcal{E}(X, Y^1) \to \cdots \to \text{Hom}_\mathcal{E}(X, Y^n) \to \text{Hom}_\mathcal{E}(X, Y^{n+1}) \\
\to \text{Ext}^n_\mathcal{E}(X, Y^0) \to \text{Ext}^n_\mathcal{E}(X, Y^1) \to \cdots \to \text{Ext}^n_\mathcal{E}(X, Y^n) \to \text{Ext}^n_\mathcal{E}(X, Y^{n+1}) \\
\to \text{Ext}^{2n}_\mathcal{E}(X, Y^0) \to \text{Ext}^{2n}_\mathcal{E}(X, Y^1) \to \cdots \to \text{Ext}^{2n}_\mathcal{E}(X, Y^n) \to \text{Ext}^{2n}_\mathcal{E}(X, Y^{n+1}) \\
\to \cdots.
\]
For every $Y \in \mathcal{M}$ and every $n$-exact sequence $X : X^0 \to X^1 \to \cdots X^n \to X^{n+1}$ the following induced sequence of abelian groups is exact.

$$0 \to \text{Hom}_\mathcal{E}(X^{n+1}, Y) \to \text{Hom}_\mathcal{E}(X^n, Y) \to \cdots \to \text{Hom}_\mathcal{E}(X^1, Y) \to \text{Hom}_\mathcal{E}(X^0, Y)$$

$$\to \text{Ext}^n_\mathcal{E}(X^{n+1}, Y) \to \text{Ext}^n_\mathcal{E}(X^n, Y) \to \cdots \to \text{Ext}^n_\mathcal{E}(X^1, Y) \to \text{Ext}^n_\mathcal{E}(X^0, Y)$$

$$\to \text{Ext}^{2n}_\mathcal{E}(X^{n+1}, Y) \to \text{Ext}^{2n}_\mathcal{E}(X^n, Y) \to \cdots \to \text{Ext}^{2n}_\mathcal{E}(X^1, Y) \to \text{Ext}^{2n}_\mathcal{E}(X^0, Y)$$

$$\to \cdots$$

Iyama and Jasso in [13, Definition-Proposition 2.15] proved the above theorem for an $n$-cluster tilting subcategory $\mathcal{M} \subseteq \text{mod}(\mathcal{A})$, where $\mathcal{A}$ is a dualizing $R$-variety. They also showed that $\mathcal{M} \subseteq \text{mod}(\mathcal{A})$ is an $n\mathbb{Z}$-cluster tilting subcategory if and only if $\mathcal{M}$ closed under $n$th syzygies if and only if $\mathcal{M}$ closed under $n$th cosyzygies. The proof of Iyama and Jasso of this theorem in [13], heavily biased on the enough projectives and enough injectives properties of $\mathcal{A}$. But in Theorem [11] we don’t have such assumptions.

Iyama in [12] Appendix A showed that for an $n$-cluster tilting subcategory $\mathcal{M} \subseteq \text{mod} \Lambda$, where $\Lambda$ is an Artin algebra and for every two objects $M, N \in \mathcal{M}$, every element in $\text{Ext}^n_\mathcal{A}(M, N)$ is Yoneda equivalent to a unique (up to homotopy) $n$-fold extension of $M$ by $N$ with terms in $\mathcal{M}$. In the following theorem we give the following more general version of this result for any $n$-cluster tilting subcategory of an exact category $\mathcal{E}$.

**Theorem 1.2.** Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$ and

$$\xi : 0 \to X^0 \to E^1 \to E^2 \to \cdots \to E^n \to X^{n+1} \to 0,$$

with $X^0, X^{n+1} \in \mathcal{M}$ be an acyclic sequence in $\mathcal{E}$. Then there is a unique (up to homotopy) $n$-exact sequence

$$0 \to X^0 \to X^1 \to X^2 \to \cdots \to X^n \to X^{n+1} \to 0,$$

Yoneda equivalent to $\xi$.

If moreover $\mathcal{M}$ be an $n\mathbb{Z}$-cluster tilting subcategory and

$$\xi : 0 \to X^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{kn-2}} E^{kn-1} \xrightarrow{f^{kn-1}} E^k \xrightarrow{f^k} X^{kn+1} \to 0,$$

be a $kn$-fold extension with $X^0, X^{kn+1} \in \mathcal{M}$, then $\xi$ is Yoneda equivalent to splicing of $k, n$-exact sequences.

Let $\mathcal{M}$ be an $n$-exact category. For every two object $X, Y \in \mathcal{M}$ we can define $\text{nExt}^1(X, Y)$ as Yoneda equivalence classes of $n$-exact sequences starting from $Y$ and ending with $X$. Also for a positive integer $k$, $\text{nExt}^k(X, Y)$ is defined as Yoneda equivalence classes of $k$-fold $n$-extensions. We refer the reader to Section 2 for the definition of $n$-exact category and to Section 5 for the definitions of $\text{nExt}^1(X, Y)$ and $\text{nExt}^k(X, Y)$. Motivated by Theorems [11] and [12] we define $n\mathbb{Z}$-exact (resp. $n\mathbb{Z}$-abelian) categories as axiomatisation of $n\mathbb{Z}$-cluster tilting subcategories of exact (resp. abelian) categories (See Definition [5.2]). Let $\mathcal{M}$ be an $n$-exact (resp. $n$-abelian) category. We say that $\mathcal{M}$ is an $n\mathbb{Z}$-exact (resp. $n\mathbb{Z}$-abelian) category if for every object $X \in \mathcal{M}$, $\text{nExt}^*(X, -)$ and $\text{nExt}^*(-, X)$ induce long exact sequences for all $n$-exact sequences.

We show that an $n$-cluster tilting subcategory of an abelian (resp. exact) category is an $n\mathbb{Z}$-exact (resp. $n\mathbb{Z}$-abelian) category if and only if it is an $n\mathbb{Z}$-cluster tilting subcategory
of an abelian (resp. exact) category. This shows that \(n\mathbb{Z}\)-exact (resp. \(n\mathbb{Z}\)-abelian) categories are good axiomatisation of \(n\mathbb{Z}\)-cluster tilting subcategories of exact (resp. abelian) categories.

The paper is organized as follows. In section 2 we recall the definitions of \(n\)-cluster tilting subcategories of abelian and exact categories, \(n\)-exact and \(n\)-abelian categories and some of their basic properties. In section 3 we recall the Gabriel-Quillen embedding for \(n\)-exact categories and prove that it has expected properties as the classical Gabriel-Quillen embedding. In section 4 we study \(n\mathbb{Z}\)-cluster tilting subcategories and prove Theorem 1.1 and Theorem 1.2. In section 5, motivated by the results of previous sections we introduce \(n\mathbb{Z}\)-abelian and \(n\mathbb{Z}\)-exact categories, and we prove that every small \(n\mathbb{Z}\)-abelian category is equivalent to an \(n\mathbb{Z}\)-cluster tilting subcategories of an abelian category. For small \(n\mathbb{Z}\)-exact categories we prove a similar result using the Gabriel-Quillen embedding.

1.1. **Notation.** Throughout this paper, unless otherwise stated, \(n\) always denotes a fixed positive integer. All categories we consider are assumed to be additive and by subcategory we mean full subcategory which is closed under isomorphisms.

2. **PRELIMINARIES**

In this section we recall the definitions of \(n\)-exact category, \(n\)-abelian category and \(n\)-cluster tilting subcategory. Also we recall some basic results that we need in the rest of the paper. For further information the readers are referred to [9, 10, 12, 14].

2.1. **\(n\)-exact categories.** Let \(\mathcal{M}\) be an additive category and \(f : A \to B\) a morphism in \(\mathcal{M}\). A weak cokernel of \(f\) is a morphism \(g : B \to C\) such that for all \(C' \in \mathcal{M}\) the sequence of abelian groups

\[
\text{Hom}(C, C') \xrightarrow{(g \circ C')} \text{Hom}(B, C') \xrightarrow{(f \circ C')} \text{Hom}(A, C')
\]

is exact. The concept of weak kernel is defined dually. Let \(d^0 : X^0 \to X^1\) be a morphism in \(\mathcal{M}\). An \(n\)-cokernel of \(d^0\) is a sequence

\[(d^1, \ldots, d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}\]

of objects and morphisms in \(\mathcal{M}\) such that for each \(Y \in \mathcal{M}\) the induced sequence of abelian groups

\[0 \to \text{Hom}(X^{n+1}, Y) \to \text{Hom}(X^n, Y) \to \cdots \to \text{Hom}(X^1, Y) \to \text{Hom}(X^0, Y)\]

is exact. Equivalently, the sequence \((d^1, \ldots, d^n)\) is an \(n\)-cokernel of \(d^0\) if for all \(1 \leq k \leq n - 1\) the morphism \(d^k\) is a weak cokernel of \(d^{k-1}\), and \(d^n\) is moreover a cokernel of \(d^{n-1}\) [14, Definition 2.2]. The concept of \(n\)-kernel of a morphism is defined dually.

**Definition 2.1.** [18, Definition 2.4] Let \(\mathcal{M}\) be an additive category. A left \(n\)-exact sequence in \(\mathcal{M}\) is a complex

\[X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}\]

such that \((d^0, \ldots, d^{n-1})\) is an \(n\)-kernel of \(d^n\). The concept of right \(n\)-exact sequence is defined dually. An \(n\)-exact sequence is a sequence which is both a right \(n\)-exact sequence and a left \(n\)-exact sequence.
Definition 2.2. ([14, Definition 3.1]) An \textit{n-abelian} category is an additive category \( \mathcal{M} \) which satisfies the following axioms.

(i) The category \( \mathcal{M} \) is idempotent complete.
(ii) Every morphism in \( \mathcal{M} \) has an \( n \)-kernel and an \( n \)-cokernel.
(iii) For every monomorphism \( d^0 : X^0 \to X^1 \) in \( \mathcal{M} \) and for every \( n \)-cokernel \((d^1, \ldots, d^n)\) of \( d^0 \), the following sequence is \( n \)-exact:
\[
X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}.
\]
(iv) For every epimorphism \( d^n : X^n \to X^{n+1} \) in \( \mathcal{M} \) and for every \( n \)-kernel \((d^0, \ldots, d^{n-1})\) of \( d^n \), the following sequence is \( n \)-exact:
\[
X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}.
\]

Let \( X \) and \( Y \) be two \( n \)-exact sequences. We remained that a morphism \( f : X \to Y \) of \( n \)-exact sequences is a morphism of complexes. A morphism \( f : X \to Y \) of \( n \)-exact sequences is called \textit{a weak isomorphism} if \( f^k \) and \( f^{k+1} \) are isomorphisms for some \( k \in \{0, 1, \cdots, n+1\} \) with \( n + 2 := 0 \) [14, Definition 4.1].

Let
\[
\begin{array}{ccccccc}
X & \xrightarrow{f^0} & X^0 & \xrightarrow{d^0_X} & X^1 & \xrightarrow{d^1_X} & \cdots & \xrightarrow{d^{n-2}_X} & X^{n-1} & \xrightarrow{d^{n-1}_X} & X^n \\
\downarrow{f} & & \downarrow{f^0} & & \downarrow{f^1} & & \cdots & & \downarrow{f^{n-1}} & & \downarrow{f^n} \\
Y & \xrightarrow{d^0_Y} & Y^0 & \xrightarrow{d^1_Y} & \cdots & \xrightarrow{d^{n-2}_Y} & Y^{n-1} & \xrightarrow{d^{n-1}_Y} & Y^n \\
\end{array}
\]

be a morphism of complexes in an additive category. Recall that the \textit{mapping cone} \( C = C(f) \) of \( f \) is the complex

\[
X^0 \xrightarrow{d^0_C} X^1 \oplus Y^0 \xrightarrow{d^1_C} \cdots \xrightarrow{d^{n-2}_C} X^n \oplus Y^{n-1} \xrightarrow{d^{n-1}_C} Y^n,
\]

where
\[
d^k_C := \begin{pmatrix}
-d^{k+1}_X & 0 \\
 f^{k+1} & d^k_Y
\end{pmatrix} : X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}
\]

for each \( k \in \{-1, 0, \ldots, n - 1\} \). In particular \( d^{-1}_C = \begin{pmatrix}
-d^n_X \\
 f^0_Y
\end{pmatrix} \) and \( d^{-1}_C = \begin{pmatrix}
f^n & d^{n-1}_Y
\end{pmatrix} \).

- The above diagram is called an \textit{n-pullback} of \( Y \) along \( f^n \) if the complex (2.1) is a \emph{left} \( n \)-exact sequence.
- The above diagram is an \textit{n-pushout} of \( X \) along \( f^0 \) if the complex (2.1) is a \emph{right} \( n \)-exact sequence [14, Definition 2.11].

Definition 2.3. ([14, Definition 4.2]) Let \( \mathcal{M} \) be an additive category. An \textit{n-exact structure} on \( \mathcal{M} \) is a class \( \mathcal{X} \) of \( n \)-exact sequences in \( \mathcal{M} \), closed under weak isomorphisms of \( n \)-exact sequences, and which satisfies the following axioms:

(E0) The sequence \( 0 \to 0 \to \cdots \to 0 \to 0 \) is an \( \mathcal{X} \)-admissible \( n \)-exact sequence.
(E1) The class of $\mathcal{X}$-admissible monomorphisms is closed under composition.

(E1') The class of $\mathcal{X}$-admissible epimorphisms is closed under composition.

(E2) For each $\mathcal{X}$-admissible $n$-exact sequence $X$ and each morphism $f : X^0 \to Y^0$, there exists an $n$-pushout diagram of $(d_0^X, \cdots, d_{n-1}^X)$ along $f$ such that $d_0^X$ is an $\mathcal{X}$-admissible monomorphism. The situation is illustrated in the following commutative diagram:

\[
\begin{array}{cccccc}
X^0 & \xrightarrow{d_0^X} & X^1 & \xrightarrow{d_1^X} & \cdots & \xrightarrow{d_{n-1}^X} & X^n & \xrightarrow{d_n^X} & X^{n+1} \\
\downarrow{f} & & \downarrow{d_1^X} & & & & \downarrow{d_{n-1}^X} & & \\
Y^0 & \xrightarrow{d_0^Y} & Y^1 & \xrightarrow{d_1^Y} & \cdots & \xrightarrow{d_{n-1}^Y} & Y^n & \xrightarrow{d_n^Y} & Y^{n+1}
\end{array}
\]

(E2') For each $\mathcal{X}$-admissible $n$-exact sequence $Y$ and each morphism $g : X^{n+1} \to Y^{n+1}$, there exists an $n$-pullback diagram of $(d_1^Y, \cdots, d_n^Y)$ along $g$ such that $d_n^Y$ is an $\mathcal{X}$-admissible epimorphism. The situation is illustrated in the following commutative diagram:

\[
\begin{array}{cccccc}
X^1 & \xrightarrow{d_1^X} & \cdots & \xrightarrow{d_{n-1}^X} & \xrightarrow{d_n^X} & X^{n+1} \\
\downarrow{d_0^Y} & & & & & \downarrow{d_n^Y} \\
Y^0 & \xrightarrow{d_0^Y} & Y^1 & \xrightarrow{d_1^Y} & \cdots & \xrightarrow{d_{n-1}^Y} & Y^n & \xrightarrow{d_n^Y} & Y^{n+1}
\end{array}
\]

An $n$-exact category is a pair $(\mathcal{M}, \mathcal{X})$ where $\mathcal{M}$ is an additive category and $\mathcal{X}$ is an $n$-exact structure on $\mathcal{M}$. If the class $\mathcal{X}$ is clear from the context, we identify $\mathcal{M}$ with the pair $(\mathcal{M}, \mathcal{X})$. The members of $\mathcal{X}$ are called $\mathcal{X}$-admissible $n$-exact sequences, or simply admissible $n$-exact sequences when $\mathcal{X}$ is clear from the context. Furthermore, if $X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$ is an admissible $n$-exact sequence, $d^0$ is called admissible monomorphism and $d^n$ is called admissible epimorphism.

Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. $\mathcal{B}$ is called covariantly finite in $\mathcal{A}$ if for every $A \in \mathcal{A}$ there exists an object $B \in \mathcal{B}$ and a morphism $f : A \to B$ such that, for all $B' \in \mathcal{B}$, the sequence of abelian groups $\text{Hom}_\mathcal{A}(B, B') \to \text{Hom}_\mathcal{A}(A, B') \to 0$ is exact. Such a morphism $f$ is called a left $\mathcal{B}$-approximation of $A$. The notions of contravariantly finite subcategory of $\mathcal{A}$ and right $\mathcal{B}$-approximation are defined dually. A functorially finite subcategory of $\mathcal{A}$ is a subcategory which is both covariantly and contravariantly finite in $\mathcal{A}$ [I, Page 113].

**Definition 2.4.** ([I, Definition 4.13]) Let $(\mathcal{E}, \mathcal{X})$ be an exact category and $\mathcal{M}$ a subcategory of $\mathcal{E}$. $\mathcal{M}$ is called an $n$-cluster tilting subcategory of $(\mathcal{E}, \mathcal{X})$ if the following conditions are satisfied.

(i) Every object $E \in \mathcal{E}$ has a left $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible monomorphism $E \cong M$. 

\[
\begin{array}{cccccc}
X^0 & \xrightarrow{d^0} & X^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} \\
\downarrow{f} & & \downarrow{g} & & & & \downarrow{h} & & \\
Y^0 & \xrightarrow{d^0} & Y^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & Y^n & \xrightarrow{d^n} & Y^{n+1}
\end{array}
\]
(ii) Every object $E \in \mathcal{E}$ has a right $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible epimorphism $M' \to E$.

(iii) We have

$$\mathcal{M} = \{ E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i_\mathcal{E}(E, \mathcal{M}) = 0 \}$$

$$= \{ E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i_\mathcal{E}(\mathcal{M}, E) = 0 \}.$$

We call $\mathcal{M}$ an $n\mathbb{Z}$-cluster tilting subcategory of $(\mathcal{E}, \mathcal{X})$ if the following additional condition is satisfied:

(iv) If $\text{Ext}^k_\mathcal{E}(\mathcal{M}, \mathcal{M}) \neq 0$ then $k \in n\mathbb{Z}$.

An $n$-cluster tilting subcategory of abelian category $\mathcal{E}$ is $n$-cluster tilting subcategory of the exact category $\mathcal{E}$ with the exact structure of all short exact sequences.

Note that $\mathcal{E}$ itself is the unique 1-cluster tilting subcategory of $\mathcal{E}$.

A full subcategory $\mathcal{M}$ of an exact or abelian category $\mathcal{E}$ is called $n$-rigid, if for every two objects $M, N \in \mathcal{M}$ and for every $k \in \{1, \cdots, n-1\}$, we have $\text{Ext}^k_\mathcal{E}(\mathcal{M}, \mathcal{M}) = 0$. Any $n$-cluster tilting subcategory $\mathcal{M}$ of an exact category $\mathcal{E}$ is $n$-rigid.

The following theorem gives the main source of $n$-exact categories.

**Theorem 2.5.** ([14, Theorem 4.14]) Every $n$-cluster tilting subcategory of an abelian (resp. exact) category inherit natural structure of an $n$-abelian (resp. $n$-exact) category.

**Theorem 2.6.** ([5, 17]) Every small $n$-abelian category is equivalent to an $n$-cluster tilting subcategory $\mathcal{M}$ of an abelian category.

The first author in [3] showed that there are $n$-exact categories which are not $n$-cluster tilting subcategories. However there is a nice embedding of small $n$-exact categories in abelian categories that we recall it in the next section.

3. **Gabriel-Quillen embedding for $n$-exact categories**

Let $\mathcal{M}$ be a small $n$-exact category. In this section we first recall the Gabriel-Quillen embedding theorem [3, Theorem]. Then we show that it detects $n$-exact sequences and its essential image is closed under $n$-extensions, up to Yoneda equivalence. This allows us to compute the group of $n$-extensions introduced in [19] in a Grothendieck category.

Let $\mathcal{M}$ be a small $n$-exact category. Recall that Mod($\mathcal{M}$) is the category of all additive contravariant functors from $\mathcal{M}$ to the category of all abelian groups. It is an abelian category with all limits and colimits, which are defined point-wise. Also by Yoneda’s lemma, representable functors are projective and the direct sum of all representable functors $\Sigma_{X \in \mathcal{M}} \text{Hom}(-, X)$, is a generator for Mod($\mathcal{M}$). Thus Mod($\mathcal{M}$) is a Grothendieck category [6, Proposition 5.21].

A functor $F \in \text{Mod}(\mathcal{M})$ is called weakly effaceable, if for each object $X \in \mathcal{M}$ and $x \in F(X)$ there exists an admissible epimorphism $f : Y \to X$ such that $F(f)(x) = 0$. We denote by Eff($\mathcal{M}$) the full subcategory of all weakly effaceable functors. For each $k \in \{1, \cdots, n\}$ we denote by $L_k(\mathcal{M})$ the full subcategory of Mod($\mathcal{M}$) consist of all functors like $F$ such that for every $n$-exact sequence

$$X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$$
the sequence of abelian groups
\[ 0 \to F(X^{n+1}) \to F(X^n) \to \cdots \to F(X^{n-k}) \]
is exact. Also for a Serre subcategory $C$ of an abelian category $A$ we set $C^{\perp k} = \{ A \in A | \text{Ext}_{A}^{k}(C, A) = 0 \}$. In particular $C^{\perp 1} = C^\perp$ is the subcategory of $C$-closed objects [16, Page 39]. In the following proposition we collect the basic properties of the Gabriel-Quillen embedding theorem.

**Proposition 3.1.** ([3 Section 3])

(i) $\text{Eff}(\mathcal{M})$ is a localising subcategory of $\text{Mod}(\mathcal{M})$.

(ii) $\text{Eff}(\mathcal{M})^{\perp} = \mathcal{L}_1(\mathcal{M})$.

(iii) For every $k \in \{1, \cdots, n\}$, $\text{Eff}(\mathcal{M})^{\perp k} = \mathcal{L}_k(\mathcal{M})$.

Denote by $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ the composition of the Yoneda functor $\mathcal{M} \to \text{Mod}(\mathcal{M})$ with the localization functor $\text{Mod}(\mathcal{M}) \to \text{Mod}(\mathcal{M})_{\text{Eff}(\mathcal{M})} \simeq \text{Eff}(\mathcal{M})^{\perp} = \mathcal{L}_1(\mathcal{M})$. Thus $H(X) = (-, X) : \mathcal{M}^{\text{op}} \to \text{Ab}$. For simplicity we denote $(-, X)$ by $H_X$. Then

(i) For every $n$-exact sequence $X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ in $\mathcal{M}$, 
\[ 0 \to H_{X^0} \to H_{X^1} \to \cdots \to H_{X^n} \to H_{X^{n+1}} \to 0 \]
is exact in $\mathcal{L}_1(\mathcal{M})$.

(ii) The essential image of $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ is $n$-rigid.

In the following proposition we prove that the canonical functor $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ detects $n$-exact sequences.

**Proposition 3.2.** Let $Y : Y^0 \to Y^1 \to \cdots \to Y^n \to Y^{n+1}$ be a complex of objects in $\mathcal{M}$ such that
\[ 0 \to H_{Y^0} \to H_{Y^1} \to \cdots \to H_{Y^n} \to H_{Y^{n+1}} \to 0 \]
is exact in $\mathcal{L}_1(\mathcal{M})$. Then $Y$ is an admissible $n$-exact sequence in $\mathcal{M}$.

**Proof.** Because the essential image of $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ is $n$-rigid, by the similar argument as in the proof of [15 Proposition 2.2] for each object $Z \in \mathcal{M}$ we have the following exact sequence of abelian groups.
\[ 0 \to \text{Hom}(H_Z, H_{Y^0}) \to \text{Hom}(H_Z, H_{Y^1}) \to \cdots \to \text{Hom}(H_Z, H_{Y^n}) \to \text{Hom}(H_Z, H_{Y^{n+1}}). \]

Thus by Yoneda’s lemma $Y$ is a left $n$-exact sequence. Dually it is a right $n$-exact sequence. We need to show that $Y$ is an admissible $n$-exact sequence. The cokernel of $H_{Y^n} \to H_{Y^{n+1}}$, denoted by $C$, is weakly effaceable. In particular, there exist $X^n \in \mathcal{M}$ and an admissible epimorphism $X^n \to Y^{n+1}$ in $\mathcal{M}$, such that $C(Y^{n+1}) \to C(X^n)$ carries the image of $1_{Y^{n+1}}$ to $0$. This means that there is a commutative diagram with exact rows in $\mathcal{L}_1(\mathcal{M})$ of the following form for an admissible $n$-exact sequence $X : X^0 \to X^1 \to \cdots \to X^n \to Y^{n+1}$ in $\mathcal{M}$.
\[
\begin{array}{cccccccc}
0 & \to & H_{X^0} & \to & H_{X^1} & \to & \cdots & \to & H_{X^n} & \to & H_{Y^{n+1}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \ddots & & \ddots & & \downarrow & & \downarrow \\
0 & \to & H_{Y^0} & \to & H_{Y^1} & \to & \cdots & \to & H_{Y^n} & \to & H_{Y^{n+1}} & \to & 0,
\end{array}
\]
where the dotted arrows are induced by the factorization property of $n$-kernel. Because the top row is induced by an admissible $n$-exact sequence, by the dual of obscure axiom ([14, Proposition 4.11]) and Yoneda’s lemma the bottom row is also induced by an admissible $n$-exact sequence. □

In the classical case, the Gabriel-Quillen embedding $\mathcal{E} \hookrightarrow \text{Lex}(\mathcal{E})$ identifies $\mathcal{E}$ with an extension closed subcategory of the abelian category $\text{Lex}(\mathcal{E})$. We prove a similar result for $n$-exact categories. First let recall some facts about Yoneda extension groups in exact categories.

Let $\mathcal{E}$ be an exact category, $k$ a positive integer and $A, C \in \mathcal{E}$. An exact sequence

$$A \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_0 \rightarrow C$$

in $\mathcal{E}$ is called a $k$-fold extension of $C$ by $A$. Two $k$-fold extensions of $C$ by $A$, $\xi : A \rightarrow B_{k-1} \rightarrow \cdots \rightarrow B_0 \rightarrow C$ and $\xi' : A \rightarrow B'_{k-1} \rightarrow \cdots \rightarrow B'_0 \rightarrow C$

are said to be Yoneda equivalent if there is a chain of $k$-fold extensions of $C$ by $A$

$$\xi = \xi_0, \xi_1, \ldots, \xi_{l-1}, \xi_l = \xi'$$

such that for every $i \in \{0, \cdots, l-1\}$, we have either a chain map $\xi_i \rightarrow \xi_{i+1}$ starting with $1_A$ and ending with $1_C$, or a chain map $\xi_{i+1} \rightarrow \xi_i$ starting with $1_A$ and ending with $1_C$. $\text{Ext}^k_\mathcal{E}(C, A)$ is defined as the set of Yoneda equivalence classes of $k$-fold extensions of $C$ by $A$ [20, 21].

**Remark 3.3.** Let $\mathcal{E}$ be an exact category and $X \rightarrow Y \rightarrow Z$ be a conflation in $\mathcal{E}$. The for every object $A \in \mathcal{E}$, in the long exact sequence

$$0 \rightarrow \text{Hom}_\mathcal{E}(A, X) \rightarrow \text{Hom}_\mathcal{E}(A, Y) \rightarrow \text{Hom}_\mathcal{E}(A, Z) \rightarrow \text{Ext}^1_\mathcal{E}(A, X) \rightarrow \text{Ext}^1_\mathcal{E}(A, Y) \rightarrow \text{Ext}^1_\mathcal{E}(A, Z) \rightarrow \cdots$$

for every $i \geq 1$, the connecting morphism $\text{Ext}^i_\mathcal{E}(A, Z) \rightarrow \text{Ext}^{i+1}_\mathcal{E}(A, X)$ is given by splicing an $i$-fold extension

$$Z \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \rightarrow E^i \rightarrow A$$

with the short exact sequence $X \rightarrow Y \rightarrow Z$ and obtaining the following $(i+1)$-fold extension.

$$X \rightarrow Y \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \rightarrow E^i \rightarrow A.$$

**Remark 3.4.** Let $K \rightarrow B_0 \rightarrow C$ and $A \rightarrow B'_1 \rightarrow K'$ be two conflation in an exact category $\mathcal{E}$ and $\sigma : K \rightarrow K'$ be a morphism. Then by taking pullback and pushout along $\sigma$, we have the following commutative diagram with exact rows.
These two rows are Yoneda equivalent in $\text{Ext}^2_{\mathcal{E}}(C, A)$. Now consider the following commutative diagram with exact rows

\[
\begin{array}{c}
A \rightarrow W_1 \rightarrow B_0 \rightarrow C \\
| \downarrow K | \\
A \rightarrow B'_1 \rightarrow W_0 \rightarrow C \\
| \downarrow K' |
\end{array}
\]

By taking a pullback

\[
\begin{array}{c}
Y^1 \rightarrow \text{Ker}(f^2) \\
\downarrow \\
Y^1 \rightarrow \text{Ker}(g^2)
\end{array}
\]

we obtain the following commutative diagram with exact rows, where the bottom row is Yoneda equivalent to $\xi$.

\[
\begin{array}{c}
X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \\
\downarrow \\
Y^0 \rightarrow Y^1 \rightarrow X^2 \rightarrow X^3
\end{array}
\]

In the proof of the following proposition we use this argument several times.

**Theorem 3.5.** Let $\mathcal{M}$ be a small $n$-exact category. The embedding $\mathcal{M} \hookrightarrow \mathcal{L}_1(\mathcal{M})$ is closed under $n$-extensions, up to Yoneda equivalence.

**Proof.** Let $k \in \{1, \ldots, n-1\}$, $E^0, E^{n+1} \in \mathcal{L}_1(\mathcal{M})$ and

\[
\xi : 0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n \rightarrow E^{n+1} \rightarrow 0
\]

be a $k$-fold extension of $E^{n+1}$ by $E^0$ in $\mathcal{L}_1(\mathcal{M})$. There exist $Y, X^{n+1} \in \mathcal{M}$ such that $H(Y) = E^0$ and $H(X^{n+1}) = E^{n+1}$. Then the cokernel of $E^n \rightarrow H(X^{n+1})$, denoted by $C$, is effaceable. In particular, there exist $X^n \in \mathcal{M}$ and an admissible epimorphism $X^n \rightarrow X^{n+1}$ in $\mathcal{M}$, such that $C(X^{n+1}) \rightarrow C(X^n)$ carries the image of $1_{X^{n+1}}$ to $0$. This means that there is a commutative diagram with exact rows in $\mathcal{L}_1(\mathcal{M})$ of the following form for an admissible $n$-exact sequence $X : X^0 \hookrightarrow X^1 \rightarrow \cdots \rightarrow X^n \rightarrow X^{n+1}$ in $\mathcal{M}$. 
By repeating this argument we obtain the following commutative diagram with exact rows, such that the bottom row is Yoneda equivalent to $\xi$.

$$0 \rightarrow H(X^0) \xrightarrow{f^0} H(X^1) \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} H(X^n) \xrightarrow{f^n} H(X^{n+1}) \rightarrow 0$$

$$0 \rightarrow H(Y) \xrightarrow{g^0} E^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{n-1}} E^n \xrightarrow{g^n} H(X^{n+1}) \rightarrow 0.$$

By taking pullback along $\operatorname{Ker}(f^n) \rightarrow \operatorname{Ker}(g^n)$ and by Remark 3.4, we obtain the following commutative diagram with exact rows, such that the bottom row is Yoneda equivalent to $\xi$.

$$0 \rightarrow H(X^0) \rightarrow H(X^1) \rightarrow \cdots \rightarrow H(X^{n-2}) \rightarrow H(X^{n-1}) \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0$$

$$0 \rightarrow H(Y) \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-2} \rightarrow E_{pb}^{n-1} \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0.$$

Now for $j \in \{1, \ldots, n-1\}$ we set $C^j := \operatorname{Im}(H(X^j) \rightarrow H(X^{j+1}))$ in $L_1(M)$. By applying the functor $\operatorname{Hom}_{L_1(M)}(-, H(Y))$ to the short exact sequence

$$\eta_{n-1} : 0 \rightarrow C^{n-2} \rightarrow H(X^{n-1}) \rightarrow C^{n-1} \rightarrow 0$$

we obtain the following exact sequence.

$$\operatorname{Ext}^{n-2}_{L_1(M)}(H(X^{n-1}), H(Y)) \rightarrow \operatorname{Ext}^{n-2}_{L_1(M)}(C^{n-2}, H(Y)) \xrightarrow{\alpha} \operatorname{Ext}^{n-1}_{L_1(M)}(C^{n-1}, H(Y))$$

$$\rightarrow \operatorname{Ext}^{n-1}_{L_1(M)}(H(X^{n-1}), H(Y)) = 0.$$

Therefore $\alpha$ is a surjective map, and so by Remark 3.3 there is an exact sequence

$$\epsilon : 0 \rightarrow H(Y) \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots \rightarrow F^{n-2} \rightarrow C^{n-2} \rightarrow 0$$

such that $\epsilon \eta_{n-1}$ is Yoneda equivalent to

$$0 \rightarrow H(Y) \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \rightarrow E^{n-2} \rightarrow E_{pb}^{n-1} \rightarrow C^{n-1} \rightarrow 0.$$

Thus we have the following diagram, where the bottom row is still Yoneda equivalent to $\xi$.

$$0 \rightarrow H(X^0) \rightarrow H(X^1) \rightarrow \cdots \rightarrow H(X^{n-2}) \rightarrow H(X^{n-1}) \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0$$

$$0 \rightarrow H(Y) \rightarrow F^1 \rightarrow \cdots \rightarrow F^{n-2} \rightarrow H(X^{n-1}) \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0.$$

By repeating this argument we obtain the following commutative diagram where the bottom row is Yoneda equivalent to $\xi$.

$$0 \rightarrow H(X^0) \rightarrow H(X^1) \rightarrow H(X^2) \rightarrow \cdots \rightarrow H(X^{n-1}) \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0$$

$$0 \rightarrow H(Y) \rightarrow G^1 \rightarrow H(X^2) \rightarrow \cdots \rightarrow H(X^{n-1}) \rightarrow H(X^n) \rightarrow H(X^{n+1}) \rightarrow 0.$$
Now by applying $\text{Hom}_{\mathcal{L}_1(\mathcal{M})}(H(X^1), -)$ to the short exact sequence $0 \to H(Y) \to G^1 \to C^1 \to 0$ we have the following exact sequence of abelian groups.

$$\text{Hom}_{\mathcal{L}_1(\mathcal{M})}(H(X^1), G^1) \to \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(H(X^1), C^1) \to \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^1(H(X^1), H(Y)) = 0.$$ 

So there is a morphism $h^1 : H(X^1) \to G^1$ that make the following diagram commutative. By the universal property of kernel there is also a morphism $h^0 : H(X^0) \to H(Y)$ that make the following diagram commutative.

$$
\begin{array}{c}
0 \to H(X^0) \to H(X^1) \to H(X^2) \to \cdots \to H(X^{n-1}) \to H(X^n) \to H(X^{n+1}) \to 0 \\
\downarrow h^0 \quad \downarrow h^1 \\
0 \to H(Y) \to G^1 \to H(X^2) \to \cdots \to H(X^{n-1}) \to H(X^n) \to H(X^{n+1}) \to 0.
\end{array}
$$

Let

$$
X^0 \to X^1 \to X^2 \to \cdots \to X^{n-1} \to X^n \to X^{n+1} \\
\downarrow h \\
Y \to Y^1 \to Y^2 \to \cdots \to Y^{n-1} \to Y^n \to X^{n+1}
$$

be the $n$–pushout diagram along $h$, where $H(h) = h^0$. Then applying the functor $H$, by \cite[Proposition 5.1]{20} we see that $\xi$ is Yoneda equivalent to

$$
0 \to H(Y) \to H(Y^1) \to \cdots \to H(Y^n) \to H(Y^{n+1}) \to 0.
$$

\hfill \box

**Proposition 3.6.** Let $\mathcal{M}$ be a small $n$-exact category. If two $n$-fold extension

$$\xi : 0 \to H_{X^0} \to H_{X^1} \to \cdots \to H_{X^n} \to H_{X^{n+1}} \to 0$$

and

$$\eta : 0 \to H_{X^0} \to H_{Y^1} \to \cdots \to H_{Y^n} \to H_{X^{n+1}} \to 0$$

be Yoneda equivalent in $\mathcal{L}_1(\mathcal{M})$, then $X : X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ and $Y : X^0 \to Y^1 \to \cdots \to Y^n \to X^{n+1}$ are homotopy equivalent $n$-exact sequences in $\mathcal{M}$.

**Proof.** By Proposition \ref{3.2} $X$ and $Y$ are $n$-exact sequences. The rest of proof is similar to the proof of \cite[Proposition A.1]{12}. Since $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ is full and faithful, we identify $\mathcal{M}$ with the essential image of $H$. Because $\mathcal{M}$ is an $n$-rigid subcategory of $\mathcal{L}_1(\mathcal{M})$ by \cite[Proposition 2.2]{15} we have the exact sequences

$$
0 \to (-, X^0) \to (-, X^1) \to \cdots \to (-, X^n) \to (-, X^{n+1}) \overset{\alpha}{\to} \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(-, X^0)
$$

and

$$
0 \to (-, X^0) \to (-, Y^1) \to \cdots \to (-, Y^n) \to (-, X^{n+1}) \overset{\beta}{\to} \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(-, X^0).
$$

Since $X$ and $Y$ are Yoneda equivalent, the image of two functor $\alpha$ and $\beta$ are the same. So we have the following commutative diagram, where dotted arrows are induced by the factorization property of $n$-kernel.
Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$. Using the result of previous section we show that every $n$-fold extension between two object in $\mathcal{M}$ is Yoneda equivalent to a unique (up to homotopy) $n$-exact sequence in $\mathcal{M}$. This is a generalization of a result by Iyama ([12, Proposition A.1]). We also show a similar result for $n\mathbb{Z}$-cluster tilting subcategories.

**Proposition 4.1.** ([4, Proposition 3.9]) Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$. The restriction functor $\mathbb{R} : \text{Mod}(\mathcal{E}) \to \text{Mod}(\mathcal{M})$ induces an equivalence $\dfrac{\text{Mod}(\mathcal{E})}{\text{Eff}(\mathcal{E})} \cong \dfrac{\text{Mod}(\mathcal{M})}{\text{Eff}(\mathcal{M})}$ making the following diagram commutative.
Theorem 4.2. Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$ and
\[ \xi : 0 \rightarrow X^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \rightarrow E^n \rightarrow X^{n+1} \rightarrow 0 \]
be an acyclic sequence in $\mathcal{E}$ with $X^0, X^{n+1} \in \mathcal{M}$. Then there is a unique (up to homotopy) $n$-exact sequence
\[ 0 \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \rightarrow X^n \rightarrow X^{n+1} \rightarrow 0 \]
that is Yoneda equivalent to $\xi$.

Proof. By the above proposition we have the following commutative diagram of functors.

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{E}) & \xrightarrow{\mathbb{R}} & \text{Mod}(\mathcal{M}) \\
\downarrow \text{Q}_\mathcal{E} & & \downarrow \text{Q}_\mathcal{M} \\
\text{Eff}(\mathcal{E}) & \xrightarrow{\mathbb{R}} & \text{Eff}(\mathcal{M})
\end{array}
\]

where $i$ is an inclusion functor. Then the result follows from Theorem 3.5 and Proposition 3.6.

In the following theorem we show that for $n\mathbb{Z}$-cluster tilting subcategories, every $kn$-fold extension is Yoneda equivalence to splicing of $k$, $n$-exact sequences.

Theorem 4.3. Let $\mathcal{M}$ be an $n\mathbb{Z}$-cluster tilting subcategory of an exact category $\mathcal{E}$ and
\[ \xi : 0 \rightarrow X^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{kn-2}} E^{kn-1} \xrightarrow{f^{kn-1}} E^{kn} \xrightarrow{f^k} X^{kn+1} \rightarrow 0 \]
be a $kn$-fold extension with $X^0, X^{kn+1} \in \mathcal{M}$. Then $\xi$ is Yoneda equivalent to splicing of $k$, $n$-exact sequences.

Proof. We use the induction on $k$. The case $k = 1$ was proved in Theorem 4.2. Now let $k \geq 2$ and assume that the result follows for any $m$, $m < k$. Let $X^{kn} \rightarrow E^{kn}$ be a deflation with $X^{kn} \in \mathcal{M}$. So the composition $X^{kn} \rightarrow E^{kn} \rightarrow X^{kn+1}$ is a deflation. Thus we have the following commutative diagram where the top row is an $n$-exact sequence.

\[
\begin{array}{cccccccc}
0 & \rightarrow & X^{(k-1)n} & \rightarrow & \cdots & \rightarrow & X^{kn} & \rightarrow & X^{kn+1} & \rightarrow & 0 \\
& & \downarrow & & & & \downarrow & & & & \\
0 & \rightarrow & E^0 & \rightarrow & E^1 & \rightarrow & \cdots & \rightarrow & E^{(k-1)n} & \rightarrow & \cdots & \rightarrow & X^{kn} & \rightarrow & X^{kn+1} & \rightarrow & 0
\end{array}
\]
The proof of Theorem 3.5 carries over to show that $\xi$ is Yoneda equivalent to a $kn$-fold extension of the following form.

$$0 \rightarrow \tilde{X}^0 \rightarrow \tilde{E}^1 \rightarrow \cdots \rightarrow \tilde{E}^{(k-1)n} \rightarrow X^{(k-1)n+1} \rightarrow \cdots \rightarrow X^{kn} \rightarrow X^{kn+1} \rightarrow 0$$

Then the result follows from induction hypothesis.

\begin{proof}

The result follows from induction hypothesis.
\end{proof}

**Remark 4.4.** Let $\mathcal{M}$ be a small $n$-exact category. Viewing $\mathcal{M}$ as an $n$-rigid subcategory of $\mathcal{L}_1(\mathcal{M})$, by a similar argument as in the proof of Theorem 4.3 we can see that the following conditions are equivalent.

1. For every $i \in \{1, \cdots, kn - 1\} \setminus n\mathbb{Z}$ we have $\text{Ext}^i_{\mathcal{L}_1(\mathcal{M})}(\mathcal{M}, \mathcal{M}) = 0$.
2. Every $kn$-fold extension of two objects in $\mathcal{M}$ is Yoneda equivalence to splicing of $k$, $n$-exact sequences.

For the proof of Theorem 1.1 we need the following lemma.

**Lemma 4.5.** Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$ and $Y : 0 \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^n \rightarrow Y^{n+1} \rightarrow 0$ be an $n$-exact sequence in $\mathcal{M}$. For $j \in \{1, \cdots, n-1\}$ set $C^j := \text{Im}(X^j \rightarrow X^{j+1})$ in $\mathcal{E}$. Indeed, split $Y$ into short exact sequences as follows.

$$0 \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^{n-1} \rightarrow Y^n \rightarrow Y^{n+1} \rightarrow 0$$

**Diagram 4.1**

1. Let $k \in \{1, \cdots, n-1\}$ such that $\text{Ext}^k_{\mathcal{E}}(\mathcal{M}, C^j) \neq 0$, then $k = n - j$.
2. Assume that $\text{Ext}^i_{\mathcal{E}}(\mathcal{M}, \mathcal{M}) = 0$ for every $i \in \{n + 1, \cdots, 2n - 1\}$. Let $k \in \{n + 1, \cdots, 2n - 1\}$ such that $\text{Ext}^k_{\mathcal{E}}(\mathcal{M}, C^j) \neq 0$, then $k = 2n - j$.
3. More generally let $\mathcal{M}$ be an $n\mathbb{Z}$-cluster tilting subcategory. If $\text{Ext}^k_{\mathcal{E}}(\mathcal{M}, C^j) \neq 0$, then $k \in n\mathbb{Z}$ or $k \in n\mathbb{Z} \setminus \{j\}$.

**Proof.** For $j \in \{1, \cdots, n-2\}$ by applying the functor $\text{Hom}_{\mathcal{E}}(X, -)$ to the short exact sequence

$$0 \rightarrow C^j \rightarrow Y^{j+1} \rightarrow C^{j+1} \rightarrow 0,$$

we have the exact sequence

$$\text{Hom}_{\mathcal{E}}(X, Y^{j+1}) \rightarrow \text{Hom}_{\mathcal{E}}(X, C^{j+1}) \rightarrow \text{Ext}^1_{\mathcal{E}}(X, C^j) \rightarrow \text{Ext}^k_{\mathcal{E}}(X, Y^{j+1}) = 0.$$  

Since $Y^{j+1} \rightarrow C^{j+1}$ is a right $\mathcal{M}$-approximation, the first map is epimorphism. Thus $\text{Ext}^1_{\mathcal{E}}(X, C^j) = 0$ for $j \in \{1, \cdots, n-2\}$. The rest of proof is straightforward and we leave it to the reader.

\end{proof}

Now we are ready to prove Theorem 1.1, which gives a characterization of $n\mathbb{Z}$-cluster tilting subcategories.
Proof of Theorem 1.1. We first show that (1) $\Rightarrow$ (2). By [15, Proposition 2.2] we have the following exact sequence

$$0 \rightarrow \text{Hom}_E(X, Y^0) \rightarrow \text{Hom}_E(X, Y^1) \rightarrow \cdots \rightarrow \text{Hom}_E(X, Y^n) \rightarrow \text{Hom}_E(X, Y^{n+1})$$

$$\rightarrow \text{Ext}_E^n(X, Y^0) \rightarrow \text{Ext}_E^n(X, Y^1).$$

Also, using the argument in the proof of [15, Proposition 2.2], it is not hard to see that for every positive integer $k$ we have the following exact sequence.

$$\text{Ext}^{(k-1)n}_E(X, Y^n) \rightarrow \text{Ext}^{(k-1)n}_E(X, Y^{n+1}) \rightarrow \text{Ext}^k_E(X, Y^0) \rightarrow \text{Ext}^k_E(X, Y^1).$$

Now by looking at Diagram 4.1 and using Lemma 4.5 we have the following exact sequences

$$\text{Ext}^n_E(X, Y^0) \rightarrow \text{Ext}^n_E(X, Y^1) \rightarrow \text{Ext}^n_E(X, C^1) \rightarrow \text{Ext}^{n+1}_E(X, Y^0) = 0,$n -exact sequence, we obtain the following commutative diagram.

Using the dual of Remark 3.4, we can obtain the following diagram where the top row is Yoneda equivalent to $\xi$.

Note that in this diagram $E_2$ is differ from $E_2$ in the previous diagram, but for simplicity we use the same notation. By the minimality of $t$ it is easy to see that the sequence

$$0 \rightarrow \text{Hom}_E(X, Y^0) \rightarrow \text{Hom}_E(X, Y^1) \rightarrow \cdots \rightarrow \text{Hom}_E(X, Y^n) \rightarrow \text{Hom}_E(X, Y^{n+1})$$

$$\rightarrow \cdots$$

$$\rightarrow \text{Ext}^k_E(X, Y^0) \rightarrow \cdots \rightarrow \text{Ext}^k_E(X, Y^{i-1}) \rightarrow \text{Ext}^k_E(X, Y^i),$$
is exact. Now consider the Diagram 4.1 for \( Y \). By using the minimality of \( t \) we have
\[
\text{Ext}^r_E(X, C^1) = 0, \quad r \in \{kn + 1, \ldots, kn + i - 2\}
\]
\[
\text{Ext}^r_E(X, C^2) = 0, \quad r \in \{kn + 1, \ldots, kn + i - 3\}
\]
\[
\vdots
\]
\[
\text{Ext}^r_E(X, C^{i-2}) = 0, \quad r \in \{kn + 1\}.
\]
Using this equations and by the dual argument of the proof of Theorem 3.5 we obtain the following commutative diagram with exact rows.

\[
\begin{array}{c}
0 \to Y \to \cdots \to Y^{i-1} \to Y^i \to E^{i+1} \to \cdots \to E^{n+1} \to \cdots \to E^t \to X \to 0 \\
0 \to Y \to \cdots \to Y^{i-1} \to Y^i \to Y^{i+1} \to \cdots \to Y^{n+1} \to 0
\end{array}
\]

For simplicity we set \( f^i : Y^i \to Y^{i+1} \) as composition \( g^i : Y^i \to C^i \) with \( h^i : C^i \to Y^{i+1} \). We claim that the induced sequence
\[
\text{Ext}^{kn}_E(X, Y^{i-1}) \to \text{Ext}^{kn}_E(X, Y^i) \to \text{Ext}^{kn}_E(X, Y^{i+1}),
\]
is not exact. First by applying \( \text{Hom}_E(X, -) \) to the short exact sequence \( C^{i-1} \to Y^i \to C^i \) we have the following exact sequence of abelian groups.
\[
\text{Ext}^{kn}_E(X, Y^i) \to \text{Ext}^{kn}_E(X, C^i) \to \text{Ext}^{kn+1}_E(X, C^{i-1}) \to \text{Ext}^{kn+1}_E(X, Y^{i-1}).
\]
Now consider the nonzero element
\[
\xi : 0 \to C^i \to E^{i+1} \to \cdots \to E^t \to X \to 0,
\]
in \( \text{Ext}^{kn}_E(X, C^i) \). By Remark 3.3
\[
\eta := \theta(\xi) = [0 \to C^{i-1} \to Y^i \to C^i \to 0] \xi.
\]
Obviously \( \text{Ext}^{kn+1}_E(X, h^{i-1})(\eta) = 0 \). Thus \( \text{Ext}^{kn}_E(X, g^i) \) is not an epimorphism. But by applying \( \text{Hom}_E(X, -) \) to the short exact sequence \( C^i \to Y^{i+1} \to C^{i+1} \) we can see that
\[
\text{Ext}^{kn}_E(X, C^i) = \ker \left( \text{Ext}^{kn}_E(X, g^{i+1}) \right) \subseteq \ker \left( \text{Ext}^{kn}_E(X, f^{i+1}) \right).
\]
So we have
\[
\text{Im} \left( \text{Ext}^{kn}_E(X, f^i) \right) \neq \ker \left( \text{Ext}^{kn}_E(X, f^{i+1}) \right),
\]
which gives a contradiction and the result follows. \( \square \)

**Remark 4.6.** Let \( \mathcal{M} \) be an \( n \)-rigid subcategory of an exact category \( \mathcal{E} \). By the proof of Theorem 3.4 the following conditions are equivalent.

1. For every \( j \in \{1, \ldots, kn+i-1\} \setminus \mathbb{Z} \) where \( 1 \leq i \leq n-1 \) we have \( \text{Ext}^j_E(\mathcal{M}, \mathcal{M}) = 0 \).
(2) For every $X \in \mathcal{M}$ and every $n$-exact sequence $Y : Y^0 \to Y^1 \to \cdots Y^n \to Y^{n+1}$ the following induced sequence of abelian groups is exact.

$$0 \to \text{Hom}_\mathcal{E}(X, Y^0) \to \text{Hom}_\mathcal{E}(X, Y^1) \to \cdots \to \text{Hom}_\mathcal{E}(X, Y^n) \to \text{Hom}_\mathcal{E}(X, Y^{n+1})$$

$$\to \cdots$$

$$\to \text{Ext}^k_{\text{E}}(X, Y^0) \to \cdots \to \text{Ext}^k_{\text{E}}(X, Y^{i-1}) \to \text{Ext}^k_{\text{E}}(X, Y^i).$$

5. $n\mathbb{Z}$-ABELIAN AND $n\mathbb{Z}$-EXACT CATEGORIES

In this section, inspired by the results of the previous section, we define $n\mathbb{Z}$-abelian and $n\mathbb{Z}$-exact categories and show that they are axiomatisation of $n\mathbb{Z}$-cluster tilting subcategories of abelian and exact categories, respectively.

Let $M$ be an $n$-exact category. For an object $M \in \mathcal{M}$ we denote the diagonal and codiagonal map by

$$\Delta_M = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) : M \to M \oplus M$$

and

$$\nabla_M = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) : M \oplus M \to M,$$

respectively. We write $\Delta$ and $\nabla$ when $M$ is clear from the context. Now let $L, N \in \mathcal{M}$. An $n$-exact sequence

$$\xi : 0 \to L \to M^1 \to M^2 \to \cdots \to M^n \to N \to 0$$

is called an $n$-extension of $N$ by $L$. Let $f : L \to L'$ be an arbitrary morphism. By taking $n$-pushout along $f$ we obtain the following morphism between $n$-exact sequences.

$$0 \to L \to M^1 \to \cdots \to M^n \to N \to 0$$

$$\downarrow f \quad \downarrow \quad \downarrow \quad \downarrow$$

$$0 \to L' \to W^1 \to \cdots \to W^n \to N \to 0.$$

The bottom row is denoted by $f\xi$. For a morphism $g : N' \to N$, $\xi g$ is defined dually by taking $n$-pullback along $g$.

For another $n$-extension of $N$ by $L$ like

$$\xi' : 0 \to L \to M'^1 \to M'^2 \to \cdots \to M'^n \to N \to 0$$

their Baer sum is defined as $\nabla(\xi \oplus \xi')\Delta$. Two $n$-extensions of $N$ by $L$ are said to be Yoneda equivalent if there is a morphism of $n$-exact sequences with identity end terms, from one to another. By [14, Proposition 4.10] this is an equivalence relation and we denote by $n\text{Ext}^1_{\mathcal{M}}(M, L)$ the set of Yoneda equivalence classes of $n$-extension of $N$ by $L$. This is an abelian group (ignoring set theoretical difficulties) by the Baer sum operation. Also for a positive integer $k$, $k$-fold $n$-extension is defined similarly. Indeed a $k$-fold $n$-extension is splicing of $k$, $n$-extensions. The Yoneda equivalence classes of $k$-fold $n$-extension of $N$ by $L$ is denoted by $n\text{Ext}^k_{\mathcal{M}}(M, L)$ and it is an abelian group with Baer sum operation (for more details see [19, Section 5.2]).

Now let $\mathcal{M}$ be an $n$-exact category and

$$(5.1) \quad 0 \to L \to M^1 \to M^2 \to \cdots \to M^n \to N \to 0$$
be an $n$-exact sequence in $\mathcal{M}$. For every object $X \in \mathcal{M}$ there is the following induced sequence of abelian groups.

$$ 0 \to \text{Hom}_\mathcal{M}(X, L) \to \text{Hom}_\mathcal{M}(X, M^1) \to \cdots \to \text{Hom}_\mathcal{M}(X, M^n) \to \text{Hom}_\mathcal{M}(X, N) $$
$$ \to \text{nExt}^1_\mathcal{M}(X, L) \to \text{nExt}^1_\mathcal{M}(X, M^1) \to \cdots \to \text{nExt}^1_\mathcal{M}(X, M^n) \to \text{nExt}^1_\mathcal{M}(X, N) $$
$$ \to \text{nExt}^2_\mathcal{M}(X, L) \to \text{nExt}^2_\mathcal{M}(X, M^1) \to \cdots \to \text{nExt}^2_\mathcal{M}(X, M^n) \to \text{nExt}^2_\mathcal{M}(X, N) $$
$$ \to \cdots $$

(5.2)

Dually, we have the following induced sequence of abelian groups.

$$ 0 \to \text{Hom}_\mathcal{M}(N, X) \to \text{Hom}_\mathcal{M}(M^n, X) \to \cdots \to \text{Hom}_\mathcal{M}(M^1, X) \to \text{Hom}_\mathcal{M}(L, X) $$
$$ \to \text{nExt}^1_\mathcal{M}(N, X) \to \text{nExt}^1_\mathcal{M}(M^n, X) \to \cdots \to \text{nExt}^1_\mathcal{M}(M^1, X) \to \text{nExt}^1_\mathcal{M}(L, X) $$
$$ \to \text{nExt}^2_\mathcal{M}(N, X) \to \text{nExt}^2_\mathcal{M}(M^n, X) \to \cdots \to \text{nExt}^2_\mathcal{M}(M^1, X) \to \text{nExt}^2_\mathcal{M}(L, X) $$

(5.3)

It is natural to ask about the exactness of these sequences. The following theorem gives the answer.

**Theorem 5.1.** Let $\mathcal{M}$ be an $n$-exact category realized as an $n$-cluster tilting subcategory of an exact category $\mathcal{E}$. Then the following conditions are equivalent.

1. $\mathcal{M}$ is an $n\mathbb{Z}$-cluster tilting subcategory of $\mathcal{E}$.
2. For every $n$-exact sequence like (5.1) and every $X \in \mathcal{M}$ the induced sequence of abelian groups (5.2) is exact.
3. For every $n$-exact sequence like (5.1) and every $X \in \mathcal{M}$ the induced sequence of abelian groups (5.3) is exact.

**Proof.** We only prove that (1) and (2) are equivalent. The equivalence of (1) and (3) is dual.

Because $\mathcal{M}$ is an $n\mathbb{Z}$-cluster tilting subcategory, by Theorem 4.3 we have $\text{Ext}^k_X(Y, X) \cong \text{Ext}^k_\mathcal{M}(X, Y)$. Thus (1) $\Rightarrow$ (2) follows from Theorem 4.1.

(2) $\Rightarrow$ (1). By Theorem 4.2 $\text{Ext}^k_Y(X, Y) \cong \text{Ext}^k_\mathcal{M}(X, Y)$ and hence we have the following exact sequence of abelian groups.

$$ 0 \to \text{Hom}_\mathcal{E}(X, L) \to \text{Hom}_\mathcal{E}(X, M^1) \to \cdots \to \text{Hom}_\mathcal{E}(X, M^n) \to \text{Hom}_\mathcal{E}(X, M^{n+1}) $$
$$ \to \text{Ext}^k_\mathcal{E}(X, L) \to \text{Ext}^k_\mathcal{E}(X, M^1) \to \cdots \to \text{Ext}^k_\mathcal{E}(X, M^n) \to \text{Ext}^k_\mathcal{E}(X, M^{n+1}) $$

Therefor by Remark 4.6 for every $k \in \{n+1, \ldots , 2n-1\}$ we have $\text{Ext}^k_\mathcal{E}(\mathcal{M}, \mathcal{M}) = 0$. But this implies that $\text{Ext}^k_\mathcal{E}(X, Y) \cong \text{nExt}^k_\mathcal{M}(X, Y)$ by Remark 4.4. So we have the following exact sequence

$$ 0 \to \text{Hom}_\mathcal{E}(X, L) \to \text{Hom}_\mathcal{E}(X, M^1) \to \cdots \to \text{Hom}_\mathcal{E}(X, M^n) \to \text{Hom}_\mathcal{E}(X, M^{n+1}) $$
$$ \to \text{Ext}^k_\mathcal{E}(X, L) \to \text{Ext}^k_\mathcal{E}(X, M^1) \to \cdots \to \text{Ext}^k_\mathcal{E}(X, M^n) \to \text{Ext}^k_\mathcal{E}(X, M^{n+1}) $$
$$ \to \text{Ext}^k_\mathcal{E}(X, L) \to \text{Ext}^k_\mathcal{E}(X, M^1) \to \cdots \to \text{Ext}^k_\mathcal{E}(X, M^n) \to \text{Ext}^k_\mathcal{E}(X, M^{n+1}) $$

The result follows by repeating this argument. \( \square \)

Motivated by Theorem 5.1 we give the following definition as axiomatisation of $n\mathbb{Z}$-cluster tilting subcategories.
Definition 5.2. Let $\mathcal{M}$ be an $n$-exact (resp. $n$-abelian) category. We say that $\mathcal{M}$ is an $n\mathbb{Z}$-exact (resp. $n\mathbb{Z}$-abelian) category if it satisfies the following two conditions.

1. For every $n$-exact sequence like (5.1) and every $X \in \mathcal{M}$ the induced sequence of abelian groups (5.2) is exact.
2. For every $n$-exact sequence like (5.1) and every $X \in \mathcal{M}$ the induced sequence of abelian groups (5.3) is exact.

Let $\mathcal{M}$ be a small $n$-abelian category. A functor $F \in \text{Mod}(\mathcal{M})$ is called finitely presented or coherent, if there exists an exact sequence of the form

\[ \text{Hom}_{\mathcal{M}}(-, X) \xrightarrow{\text{Hom}_{\mathcal{M}}(-, f)} \text{Hom}_{\mathcal{M}}(-, Y) \rightarrow F \rightarrow 0. \]

We denote by $\text{mod}(\mathcal{M})$ the full subcategory of $\text{Mod}(\mathcal{M})$ consist of all finitely presented functors. Since every morphism in $\mathcal{M}$ has a weak kernel, $\text{mod}(\mathcal{M})$ is an abelian category \[7\], Theorem 1.4).

A functor $F \in \text{mod}(\mathcal{M})$ is called effaceable, if there is an exact sequence

\[ \text{Hom}_{\mathcal{M}}(-, X) \xrightarrow{\text{Hom}_{\mathcal{M}}(-, f)} \text{Hom}_{\mathcal{M}}(-, Y) \rightarrow F \rightarrow 0, \]

for some epimorphism $f : X \rightarrow Y$. The full subcategory of effaceable functors is denoted by $\text{eff}(\mathcal{M})$.

Corollary 5.3. Let $\mathcal{M}$ be a small $n$-abelian category. Then the following statements are equivalent.

1. $\mathcal{M}$ is equivalent to an $n\mathbb{Z}$-cluster tilting subcategory of $\frac{\text{mod}(\mathcal{M})}{\text{eff}(\mathcal{M})}$.
2. $\mathcal{M}$ is an $n\mathbb{Z}$-abelian category.

Proof. By \[4\] \[17\] $\mathcal{M}$ is equivalent to an $n$-cluster tilting subcategory of $\frac{\text{mod}(\mathcal{M})}{\text{eff}(\mathcal{M})}$. Thus the result follows from Theorem 5.1. \qed

By Theorem \[1\] for $n$-cluster tilting subcategories the conditions (1) and (2) of Definition 5.2 are equivalent. So it is natural to ask about the equivalence of these conditions for all $n$-exact categories. By the following proposition for any small $n$-exact categories these conditions are equivalent.

Proposition 5.4. Let $\mathcal{M}$ be a small $n$-exact category. If we take $\mathcal{M}$ as a subcategory of $\mathcal{L}_1(\mathcal{M})$ (by Gabriel-Quillen embedding), then the following statements are equivalent.

1. $\mathcal{M}$ is an $n\mathbb{Z}$-exact category.
2. For every $X \in \mathcal{M}$ and every exact sequence $Y^0 \rightarrow Y^1 \rightarrow \cdots \rightarrow Y^n \rightarrow Y^{n+1}$ in $\mathcal{L}_1(\mathcal{M})$ with terms in $\mathcal{M}$ the the following induced sequence of abelian groups is exact.

\[ 0 \rightarrow \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(X, Y^0) \rightarrow \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(X, Y^1) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(X, Y^n) \rightarrow \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(X, Y^{n+1}) \]
\[ \rightarrow \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(X, Y^0) \rightarrow \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(X, Y^1) \rightarrow \cdots \rightarrow \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(X, Y^n) \rightarrow \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^n(X, Y^{n+1}) \]
\[ \rightarrow \cdots \]
(3) For every $X \in \mathcal{M}$ and every exact sequence $Y^0 \to Y^1 \to \cdots Y^n \to Y^{n+1}$ in $\mathcal{L}_1(\mathcal{M})$ with terms in $\mathcal{M}$ the following induced sequence of abelian groups is exact.

$$0 \to \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(Y^{n+1}, X) \to \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(Y^n, X) \to \cdots \to \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(Y^1, X) \to \text{Hom}_{\mathcal{L}_1(\mathcal{M})}(Y^0, X)$$

$$\to \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^1(Y^{n+1}, X) \to \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^1(Y^n, X) \to \cdots \to \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^1(Y^1, X) \to \text{Ext}_{\mathcal{L}_1(\mathcal{M})}^1(Y^0, X)$$

$$\to \cdots$$

Proof. The proof is similar to the proof of Theorem 5.1 and is left to the reader. □

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