STRUCTURE CONSTANTS FOR PREMODULAR CATEGORIES

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Abstract

In this paper we study conjugacy classes for pivotal fusion categories. In particular we prove a Burnside type formula for the structure constants concerning the product of two conjugacy class sums of a such fusion category. For a braided weakly integral fusion category $C$ we show that these structure constants multiplied by dim($C$) are non-negative integers, extending some results obtained by Zhou and Zhu (see [ZZ19]) in the settings of semisimple quasitriangular Hopf algebras.

1. Introduction

Conjugacy classes for finite groups are a very important tool in the study of representations of finite groups. Their associated class sums form a basis for the center of the group algebra and the structure constants obtained by multiplying two such class sums are integers. A precise formula for these structure constants, in terms of characters, was given by Burnside, see e.g. [Isa76, pp. 45].

In the spirit of Zhu’s work, [Zhu97], Cohen and Westreich in [CW00] have defined a notion of a conjugacy class of a semisimple Hopf algebra. Using this new concept they were able to extend some results from finite group representations to semisimple Hopf algebra representations.

More recently, Shimizu introduced in [Shi17], the notion of conjugacy classes for fusion categories, extending the previous notion of Cohen and Westreich mentioned above. In the same paper Shimizu associated to each conjugacy class a central element called also conjugacy class sum. These class sums play the role of the sum of group elements of a conjugacy class in group theory.

In [Shi17] Shimizu asked what results from [CW00, CW14] can be extended from semisimple Hopf algebras to fusion categories.

In this paper we give an analogue of Burnside formula for the structure constants of any pivotal fusion category extending the results obtained in [CW00] for semisimple Hopf algebras.

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Theorem 1.1. Let $\mathcal{C}$ be a pivotal fusion category with a commutative Grothendieck ring. With the above notations one has

$$c^k_{ij} = \sum_{s=0}^{m} \frac{\chi_s(C_i) \chi_s(C_j) \chi_s^*(C_k)}{\dim(C) \dim(C^k) d_s}.$$  

To prove the above theorem, we use the theory of probability groups, introduced by Harrison in 1979 in [Har79], as a tool to study the character ring of a pivotal fusion category. Extending the results from [ZZ19], for a pivotal fusion category $\mathcal{C}$, we prove that the algebra generated by the dual of the probability group $\text{CF}(\mathcal{C})$ is isomorphic to the center $\text{CE}(\mathcal{C})$ as introduced by Shimizu in [Shi17], see also Section 3.

Cohen and Westreich [CW00, Theorem 2.6] proved that the product of two class sums of $H$ is an integral combination up to a factor of $\dim(H)^{-2}$ of the class sums of $H$, i.e.

$$C_i C_j = \frac{1}{\dim(H)^2} \left( \sum_{k=0}^{m} \tilde{c}_{ij}^k C_k \right)$$

where $\{C_i \mid 0 \leq i \leq m\}$ is the set of class sums of $H$ and each $\tilde{c}_{ij}^k$ is a non-negative integer. Recently in [ZZ19, Theorem 5.6] the authors have shown that this factor can be replaced by $\dim(H)$. Next theorem generalizes the result obtained by Zhu and Zhou in [ZZ19] from quasi-triangular Hopf algebras to integral premodular categories.

**Theorem 1.3.** Let $\mathcal{C}$ be a premodular category. Then the structure constants $c_{j_1,j_2}^{i}$ multiplied by $\dim(\mathcal{C})$ are algebraic integers.

Moreover, if $\mathcal{C}$ is weakly integral, then the same numbers multiplied by $\dim(\mathcal{C})$ are non-negative integers.

The proof of theorem has as a key step a result of Ostrik from [Ost15, Theorem 2.13], concerning the image by the forgetful functor of the central primitive idempotents of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ from $\text{CF}(\mathcal{Z}(\mathcal{C}))$ to $\text{CF}(\mathcal{C})$.

We also show that the character ring $\text{CF}(\mathcal{C})$ of a modular fusion category $\mathcal{C}$ is self dual, generalizing the result obtained by Zhu and Zhou in [ZZ19] for the Drinfeld double $D(H)$ of a semisimple Hopf algebra $H$.

This paper is organized as follows. In Section 2 we recall the basic facts on probability groups and prove a formula for the (hypergroup) algebra structure of the dual of a probability group. In Section 3 we recall the basics on fusion categories and the describe the dual probability group structure of the character ring as the space of central elements. Theorem 1.1 is proven in Subsection 3.1. Section 4 contains the proof of Theorem 1.3.
We work over the ground field \( \mathbb{C} \) of complex numbers. All algebras and fusion categories under consideration are \( \mathbb{C} \)-linear.

### 2. Probability groups

First we recall the definition of a probability group as in \cite{Har79}. A probability group is a set \( A \) together with a function

\[
p : A^3 \to \mathbb{R}_{\geq 0}, \ (a, b, c) \mapsto p(a.b = c)
\]

satisfying the following conditions:

1. For all \( a, b \in A \), the values \( p(a.b = c) = 0 \) for all but finitely many \( c \in A \), and

\[
p(a.b = c) \geq 0, \quad \sum_{c \in A} p(a.b = c) = 1
\]

2. For all \( a, b, c, d \in A \) one has

\[
\sum_{x \in A} p(ab = x)p(xc = d) = \sum_{y \in A} p(ay = d)p(bc = y)
\]

3. There is an identity element \( 1 \in A \) such that for any \( a \in A \)

\[
p(1.a = a) = p(a.1 = a) = 1
\]

4. For any \( b \in A \) there is a unique \( b^* \in A \) such that

\[
p(b.b^* = 1) > 0
\]

5. For any \( a, b, c \in A \) one has that

\[
p(a.b = c) = p(b^*.a^* = c^*)
\]

6. For all \( a \in A \) it follows

\[
p(a.a^* = 1) = p(a^*.a = 1)
\]

We also use the notation \( p_c(a, b) \) for \( p(a.b = c) \). If \( (A, p) \) is a probability group, one easily checks that the identity element \( 1 \) is unique, \( 1^* = 1 \) and \( (a^*)^* = a \) for all \( a \in A \).

Given a set \( A \) we denote by \( \mathbb{C}[A] \) the complex vector space with linear basis \( a \in A \). Then \( (A, p) \) is a probability group if and only if \( \mathbb{C}[A] \) is a \( \mathbb{k} \)-algebra with multiplication given by

\[
a.b = \sum_{c \in A} p(ab = c)c
\]

for all \( a, b \in A \). Note that the associativity of the product as defined above is equivalent to Equation (2.2).
Based on the above properties of \( p \) it is easy to see that

\[
(2.7) \quad u_A := \frac{1}{n(A)} \sum_{a \in A} h_{aa}
\]

is an idempotent element of \( A \) and \( au_A = u_A = u_Aa \) for any \( a \in A \).

We define a bilinear map \( m : \mathbb{C}[A] \times \mathbb{C}[A] \to \mathbb{C} \) where \( m(a, x) \) is the coefficient of \( a \) in writing \( x \) as linear combination of the standard basis \( A \), for any \( a \in A \) and \( x \in \mathbb{C}[A] \). Therefore \( m(a, bc) = p_a(b, c) \) for all \( a, b, c \in A \). Extending this by linearity, in general one has that

\[
(2.8) \quad m_A(\sum_{a \in A} \alpha_a a, \sum_{a \in A} \beta_a a) = \sum_{a \in A} \alpha_a \beta_a a.
\]

Recall that a probability group is called \emph{abelian} if \( p_c(a, b) = p_c(b, a) \) for all \( a, b, c \in A \). This equivalent to \( \mathbb{C}[A] \) being a commutative algebra.

For the rest of this section we let \( A \) be a finite abelian probability group with the cardinal \( |A| = m + 1 \).

Define \( \hat{A} := \{ \mu_j : \mathbb{C}[A] \to \mathbb{C} \} \) the set of all algebra morphisms of \( \mathbb{C}[A] \). Since \( A \) is abelian and \( \mathbb{C}[A] \) is a semisimple commutative algebra (see \cite{Har79}) it follows that \( \hat{A} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}[A]^\ast \). Therefore one has \( |A| = |\hat{A}| \). Moreover, one can define a multiplication on \( \hat{A} \) by declaring

\[
(2.9) \quad [\mu_i \ast \mu_j](a) := \mu_i(a) \mu_j(a), \text{ for all } a \in A.
\]

Extending linearly \( \mu_i \ast \mu_j \) on the entire \( \mathbb{C}[A] \) one obtains an algebra structure on \( \mathbb{C}[A]^\ast = \mathbb{C}[\hat{A}] \). It follows that there are some non-zero scalars \( \hat{p}_k(i, j) \in \mathbb{C} \) such that

\[
(2.10) \quad \mu_i \ast \mu_j = \sum_{k=0}^{m} \hat{p}_k(i, j) \mu_k.
\]

2.1. Recall \cite{W193} that a \emph{generalized hypergroup} is a set \( A \) such that \( A := \mathbb{C}[A] \) is a *-algebra with unit \( a_0 \) over \( \mathbb{C} \) and \( A = \{ a_0, a_1, \ldots, a_n \} \) is a basis of \( A \) with \( A^\ast = A \) for which the structure constants \( n_{ij}^k \) defined by

\[
c_i c_j = \sum_{k} n_{ij}^k c_k
\]

satisfy the following two conditions:

\[
c_i^\ast = c_j \iff n_{ij}^0 > 0, \text{ and } c_i^\ast \neq c_j \iff n_{ij}^0 = 0.
\]

A generalized hypergroup \( A \) is called \emph{abelian} if \( c_i c_j = c_j c_i \) for all \( i, j \); it is called \emph{real} if \( n_{ij}^k \in \mathbb{R} \) for all \( i, j, k \); \emph{positive} if \( n_{ij}^k \geq 0 \) for all \( i, j, k \).
and normalized if $\sum_k n_{ij}^k = 1$ for all $i, j$. Therefore a probability group is a real normalized generalized hypergroup with $n_{ii}^0 = n_{ii}^{0*}$ for all $i$.

Let $A$ be an abelian probability group. Define the linear functional $\tau : \mathbb{C}[A] \to \mathbb{C}$, $a \mapsto \delta_{a,1}$. Note that $\tau(x) = m_A(1, x)$ for all $x \in \mathbb{C}[A]$. Since $\tau : \mathbb{C}[A] \to \mathbb{C}$ is a non-degenerate trace there are some non-zero complex numbers $n_j$ such that $\tau(F_j) = \frac{1}{n_j}$. It follows that

$$\tau = \sum_{j=0}^{m} \frac{\mu_j n_j}{n_j},$$

and both $\{a, h_a a^*\}$ and $\{F_j, \frac{1}{n_j} F_j\}$ are dual bases for $\tau$. Therefore

$$\sum_{j=0}^{m} n_j F_j \otimes F_j = \sum_{a \in A} a \otimes h_a a^*$$

Note that $\text{aug} : \mathbb{C}[A] \to \mathbb{C}$, $a \mapsto 1$ is a morphism of $\mathbb{C}$-algebras. Without loss of generality we may suppose that $\mu_0 = \text{aug}$. It follows that $\mu_0 \ast \mu_j = \mu_j \ast \mu_0 = \mu_j$ for all $j$. Moreover, the central primitive idempotent $F_0$ associated to $\mu_0$ coincides to the element $u_A$ from Equation (2.7). Moreover, note that $n_0 = n(A)$ since

$$\frac{1}{n_0} = \tau(F_0) = \frac{1}{n(A)} \left( \sum_{a \in A} h_a \tau(a) \right) = \frac{h_1}{n(A)} = \frac{1}{n(A)}.$$

Let $A$ be an abelian probability group as above and $F_j$ a primitive central idempotent corresponding to a character $\mu_j : \mathbb{C}[A] \to \mathbb{C}$. Applying $\text{id} \otimes \mu_j$ to Equation (2.12) one obtains:

$$F_j = \frac{1}{n_j} \left( \sum_{a \in A} h_a \mu_j(a^*) a \right).$$

Given an algebra homomorphism $\mu_j : \mathbb{C}[A] \to \mathbb{C}$ as above, it is easy to see that $\mu_j^* : \mathbb{C}[A] \to \mathbb{C}$, $a \mapsto \mu_j(a)$ is also an algebra homomorphism. We let $j^*$ be the index for which $\mu_j = \mu_{j^*}$. With the above notations by [Har79, Proposition 2.10] it follows that $p_0(j, j^*) > 0$, for all $0 \leq j \leq m$. We let $\hat{h}_j := \frac{1}{p_0(j, j^*)}$ and $n(\hat{A}) := \sum_{j=0}^{m} \hat{h}_j$.

Following the same [Har79, Proposition 2.10] for any finite abelian probability group $A$ one has that

1. $\hat{h}_j$ is a real positive number.
2. $n(\hat{A}) = n(A)$.
3. For any $j_1, j_2$ the first orthogonality relation is written as:

$$\sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a^*) = \delta_{j_1, j_2} n(\hat{A})$$
For any \( a, b \in A \) the second orthogonality relation is written as:

\[
\sum_{j=0}^{m} \hat{h}_j h_a \mu_j(a) \mu_j(b^*) = n(A) \delta_{a,b}.
\]  

Recall that an abelian probability group is called dualizable if \( \hat{p}_k(i,j) \geq 0 \) for any \( i, j, k \). In this case it can be proven that \( \hat{A} \) is also a probability group in which \( \mu_j^* \) is defined by \( \mu_j^*(a) = \mu_j^*(a) = \mu_j(a) \), see [Har79].

**Proposition 2.16.** Let \( A \) be an abelian probability group. With the above notations it follows that

\[
\hat{p}_k(j_1, j_2) = \frac{1}{n_k} \left( \sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a) \mu_k(a^*) \right)
\]

for all \( 0 \leq j_1, j_2, k \leq m \).

**Proof.** Evaluating both sides of Equation (2.10) at \( F_k \) and expanding the formula for \( F_k \) from Equation (2.13) one obtains:

\[
\hat{p}_k(j_1, j_2) = \left( \sum_l \hat{p}_l(j_1, j_2) \mu_l \right)(F_k) = [\mu_{j_1} \ast \mu_{j_2}](F_k)[\mu_{j_1} \ast \mu_{j_2}](F_k)
\]

\[
= \frac{1}{n_k} \left( \sum_{a \in A} h_a [\mu_{j_1} \ast \mu_{j_2}](a) \mu_k(a^*) \right)
\]

\[
= \frac{1}{n_k} \left( \sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a) \mu_k(a^*) \right)
\]

which gives the formula from Equation (2.17). \( \square \)

**Corollary 2.18.** Let \( \hat{A} \) be a finite probability group. With the above notations:

\[
\hat{p}_0(j_1, j_2) = \frac{n_{j_1}}{n(A)} \delta_{j_1, j_2}.
\]
Proof. For $k = 0$, in Equation (2.17) one has

$$ \hat{p}_0(j_1, j_2) = \frac{1}{n_0} \left( \sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a) \mu_0(a^*) \right) $$

$$ = \frac{1}{n(A)} \left( \sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a) \right) $$

$$ = \frac{1}{n(A)} \left( \sum_{a \in A} h_a \mu_{j_1}(a) \mu_{j_2}(a^*) \right) $$

$$ = \frac{1}{n(A)} \mu_{j_1} \left( \sum_{a \in A} h_a \mu_{j_2}(a^*) a \right) $$

2.13  $$ \frac{1}{n(A)} \mu_{j_1} (n_{j_2} F_{j_2^*}) $$

$$ = \frac{n_{j_2^*}}{n(A)} \delta_{j_1, j_2^*} $$

$$ = \frac{n_{j_1}}{n(A)} \delta_{j_1, j_2^*}. $$

\[ \square \]

2.2. Quotient probability group. Let $S$ be a subring of a probability ring $A$. One can define a quotient ring by the following equivalence relation: $a \sim_S b$ if and only if there are $s_1, s_2 \in S, x \in A$ such that $m_A(1, as_1 x^*) > 0$ and $m_A(1, xb^* s_2) > 0$.

Define $A//S$ as the set of equivalence classes of $\sim_S$. For an element $a \in A$ one can see that $[a]_S = [b]_S$ if and only if $u_s a u_s = u_s b u_s$. Therefore there is a set bijection

$$ \mathbb{C}[A//S] \xrightarrow{\phi} u_S \mathbb{C}[A] u_S, \ [a]_S \mapsto u_s a u_s. $$

Then $A//S$ becomes a probability group with the multiplication inherited from $u_s a u_s$ via the above isomorphism. We denote by $\overline{p}_{[c]}([a], [b])$ the probability structure of $A//S$. Therefore

$$ [a][b] = \sum_{[c] \in A//S} \overline{p}_{[c]}([a], [b])[c]. $$

We write shortly $[a]$ instead of $[a]_S$ when no confusion arises. Following [Har79] in the case $A$ is abelian one can show that

$$ (2.20) \quad \overline{p}_{[c]}([a], [b]) = \sum_{v \in [c]} p_v(u, s). $$

with $u \in [a]_S$ and $s \in [b]_S$ arbitrarily chosen.
It was proven in [Har79, Proposition 2.11] that if $A$ is an abelian dualizable probability group then
\[(2.21) \quad S^\perp \overset{\alpha}{\to} A/\hat{S}, \chi \mapsto \alpha(\chi), \text{ with } \alpha(\chi)([a]_S) = \chi(a).\]
is an isomorphism of probability groups.

2.3. Let $A$ be a ring which is free as $\mathbb{Z}$-module. As in [EGNO15, Chapter 3] we define a $\mathbb{R}_{\geq 0}$-ring as a ring $A$ which is free as $\mathbb{Z}$-module with a special basis $B = \{b_i\}_{i \in I}$ such that $b.ib_j = \sum_{k \in I} N_{ij}^{k}b_k$ with $N_{ij}^{k} \in \mathbb{R}_{\geq 0}$. Moreover, a unital $\mathbb{R}_{\geq 0}$-ring is a $\mathbb{R}_{\geq 0}$-ring whose unit $1 \in B$. For any unital $\mathbb{R}_{\geq 0}$-ring $(R, B)$ define a functional $\tau : R \to \mathbb{C}$ which on the basis $B$ is given by $\tau(b) := \delta_{b,1}$.

Then a $\mathbb{R}_{\geq 0}$-based ring is a unital $\mathbb{R}_{\geq 0}$-ring together with an involution $i \mapsto i^*$ on $I$ such that the map
\[\sum_{i \in I} \alpha_i b_i \mapsto \sum_{i \in I} \alpha_i b_i^*\]
is a multiplicative anti-involution on $A$ and
\[(2.22) \quad \tau(b_i b_j) = \delta_{i,j}^*,\]
for all $i, j \in I$.

For the basis $B$ of a $\mathbb{R}_{\geq 0}$-based ring $(R, B)$ we can define a bilinear form $m : R \times R \to \mathbb{C}$ given by $m(\sum_{i \in I} \alpha_i b_i, \sum_{i \in I} \beta_i b_i) = \sum_{i \in I} \alpha_i \beta_i$.

By [EGNO15, Proposition 3.1.6] the constants $N_{ij}^{k*}$ are invariant under cyclic permutations of $\{i, j, k\}$ since $\tau$ is a trace on $R$, i.e $\tau(xy) = \tau(yx)$ for all $x, y \in R$. Therefore one can write that
\[N_{ij}^{k*} = \tau(b_i b_j b_k) = \tau(b_j b_k b_i) = N_{jk}^{i*} = \tau(b_k b_i b_j) = N_{ki}^{j*}\]
for all $i, j, k \in I$. Therefore any $\mathbb{R}_{\geq 0}$-based ring $(R, B)$ has the property that
\[(2.23) \quad m(z, xy) = m(x^*, yz^*) = m(y, z^*x).\]
for all $x, y, z \in R$. Moreover since $\tau(x) = \tau(x^*)$ for all $x \in R$ it follows that
\[m(x, yz) = m(x^*, z^*y^*).\]

Recall that by [EGNO15, Proposition 3.3.9] one has $\text{FPdim}(a) = \text{FPdim}(a^*)$ for all $a \in R$.

**Example 2.24.** Given a based ring $(R, B)$ one can consider the basis $B' = \{b'\}_{b \in B}$ where $b' := \frac{b}{\text{FPdim}(b)}$ and one obtains that $(B', p)$ with
\[(2.25) \quad p(a'b' = c') = Na_{ab} \left[ \frac{\text{FPdim}(c)}{\text{FPdim}(a)\text{FPdim}(b)} \right].\]
is a probability group. Clearly $C[B] = R \otimes_Z C$.

Note that in this case one has $s(a') = FPdim(a)FPdim(a^*)$ and $n(B') := \sum_{a \in B} FPdim(a)FPdim(a^*) = FPdim(R)$.

3. Fusion categories and their Grothendieck groups

Recall that a fusion category is a semisimple finite tensor category. We refer to [EGNO15] for the basic theory of tensor categories.

Throughout this paper $C$ denotes a fusion category and $1$ the unit object of a $C$. Given a monoidal category $C$ one can construct a braided fusion category $Z(C)$ called the monoidal centre (or Drinfeld centre) of $C$, see e.g., [Kas95, XIII.3] for details. The objects of $Z(C)$ consist of pairs $(V, \sigma_V)$ of an object $V \in C$ and a natural isomorphism $\sigma_{V,X} : V \otimes X \to X \otimes V$ for all $X \in O(C)$, satisfying a part of the hexagon axiom. A morphism $f : (V, \sigma_V) \to (W, \sigma_W)$ in $Z(C)$ is a morphism in $C$ such that $(id_X \otimes f) \circ \sigma_{V,X} = \sigma_{W,X} \circ (f \otimes id_X)$ for all objects $X$ of $C$. The composition of morphisms in $CF(Z(C))$ is defined via the usual composition of morphisms in $C$.

Let $C$ be a finite tensor category and $F : Z(C) \to C$ be the forgetful functor. It is well known that $F$ admits a right adjoint functor $R : C \to Z(C)$ such that $Z := FR : C \to C$ is a Hopf comonad. Following [Shi17, Subsection 2.6] one also has that

$$Z(V) \simeq \int_{X \in C} X \otimes V \otimes X^*.$$  

If $\pi_{V;X} : Z(V) \to X \otimes V \otimes X^*$ are the universal dinatural transformation associated to the above end $Z(V)$ then the counit $\epsilon : Z \to id_C$ is given by $\epsilon_V := \pi_{V;1}$. Moreover, using Fubini’s theorem for ends, the comultiplication $\delta : Z \to Z^2$ of $Z$ is also described in terms of the dinatural transformation $\pi$, see [Shi17, Subsection 3.2].

The object $A := Z(1)$ has the structure of a central commutative algebra in $Z(C)$. It is called the adjoint algebra of $A$.

Its multiplication $m_A : A \otimes A \to A$ and its unit $u_A : 1 \to A$ are uniquely determined by by the universal property of the end $Z$ as:

$$\pi_{1;X} \circ u_A = coev_X,$$

$$\pi_{1;X} \circ m_A = (id_X \otimes ev_X \otimes id_X) \circ (\pi_{1;X} \otimes \pi_{1;X})$$

Moreover $\epsilon_1 : A \to 1$ is a morphism of algebras, see [Shi17].

Recall that a pivotal structure $j$ on a tensor category $C$ is a tensor isomorphism $j : id_C \to (\cdot)^{**}$. Given a pivotal structure one can construct for any object $X$ of $C$ a right evaluation $\tilde{ev}_X : X \otimes X^* \xrightarrow{j \otimes id} X^{**} \otimes$
There is a right action of $CE(C)$ on $\mathcal{C}$ by pairing for all $f \in M$ we may suppose that categorical dimensions of the simple objects. Without loss of generality it is also a elements. \cite[Theorem 3.10]{Shimizu2017} for a pivotal fusion category $C$. Definition 3.3].

For brevity, we denote $\chi_\rho \in f, g$ (3.4):

$$
\delta \in f \leftarrow b = f \circ m \circ (b \otimes id_A) \quad \text{for all } f \in CF(C) \text{ and } b \in CE(C).
$$

Then the usual right pivotal trace of an endomorphism $f : X \to X$ is obtained as a particular case for $A = B = 1$. In particular, the right dimension of $X$ with respect to $j$ is defined as the right trace of the identity of $X$. A pivotal structure (or the underlying fusion category) is called spherical if $\dim(X) = \dim(X^*)$ for all objects $X$ of $C$, see \cite[Section 2.2]{ENO2005}.

Given an object $X$ of $\mathcal{C}$ the internal character $\text{ch}(X)$ is defined as the partial pivotal trace

$$
\text{ch}(X) := tr_{A,1}^X(\rho_X) : A \to 1.
$$

where $\rho_X : A \otimes X \to X$ is the canonical action of $A$ on $X$, see \cite[Definition 3.3]{Shimizu2017}.

The space $CF(\mathcal{C}) := \text{Hom}_\mathcal{C}(A, 1)$ is called the space of class functions of $\mathcal{C}$ and it is a $\mathbb{C}$-algebra where the multiplication of two class functions $f, g \in CF(\mathcal{C})$ is defined via $f \ast g := f \circ Z(g) \circ \delta_1$. Here the map $\delta : Z \to Z^2$ is the comultiplication structure of $Z$ recalled above. By \cite[Theorem 3.10]{Shimizu2017} for a pivotal fusion category $\mathcal{C}$ one has that $\text{ch}(X \otimes Y) = \text{ch}(X) \text{ch}(Y)$ for any two objects $X$ and $Y$ of $\mathcal{C}$ and $Gr_\mathcal{C}(\mathcal{C}) \to CF(\mathcal{C}), [X] \mapsto \text{ch}(X)$ is an isomorphism of algebras.

The space $CE(\mathcal{C}) := \text{Hom}_\mathcal{C}(1, A)$ is called the space of central elements. It is also a $\mathbb{C}$-algebra where the multiplication on $CE(\mathcal{C})$ is given by $a \cdot b := m \circ (a \otimes b)$ for any $a, b \in CE(\mathcal{C})$. There is a non-degenerate pairing $\langle , \rangle : CF(\mathcal{C}) \times CE(\mathcal{C}) \to \mathbb{C}$, given by $\langle f, a \rangle \text{id}_1 = f \circ a$, for all $f \in CF(\mathcal{C})$ and $a \in CE(\mathcal{C})$. We also denote $f(a) := \langle f, a \rangle$. There is a right action of $CE(\mathcal{C})$ on $CF(\mathcal{C})$ denoted by $\leftarrow$ given by $f \leftarrow b = f \circ m \circ (b \otimes id_A)$ for all $f \in CF(\mathcal{C})$ and $b \in CE(\mathcal{C})$.

Given a fusion category $\mathcal{C}$ we let $M_0, M_1, \ldots, M_m$ be a complete set of representatives for the isomorphism classes of simple objects of $\mathcal{C}$. For brevity, we denote $\chi_i := \text{ch}(M_i) \in CF(\mathcal{C})$ and $d_i := \chi_i(1)$ the categorical dimensions of the simple objects. Without loss of generality we may suppose that $M_0 = 1$ and therefore $d_0 = 1$. Moreover we denote by $i^*$ the unique index for which $M_i^* \simeq M^i$.

Recall that also from Equation (2.12) one has

$$
\sum_{j=0}^m \frac{\dim(C)}{\dim(C_j)} F_j \otimes F_j = \sum_{i=0}^m \chi_i \otimes \chi_i^*.
$$

(3.4)
Let \( \mathcal{C} \) be a pivotal fusion category. Recall that the global dimension of \( \mathcal{C} \) is defined as 
\[
\dim(\mathcal{C}) := \sum_{i=0}^{m} d_{i}d_{i}^{*} \in \mathbb{C}.
\]
It is well-known that in this case \( \dim(\mathcal{C}) \neq 0 \). The cointegral \( \lambda \) of \( \mathcal{C} \) is defined as 
\[
\lambda = \frac{1}{\dim(\mathcal{C})} \left( \sum_{i=0}^{m} d_{i} \chi_{i} \right) \in \text{CF}(\mathcal{C}).
\]
Up to a scalar \( \lambda \) is the unique element of \( \text{CF}(\mathcal{C}) \) with the property that 
\[
\chi_{\lambda} = \chi(1) \lambda \text{ for all } \chi \in \text{CF}(\mathcal{C}).
\]
Moreover, \( \lambda(u_{A}) = 1 \) and \( \lambda^{2} = \lambda \). The Fourier transform of \( \mathcal{C} \) associated to \( \lambda \) is the linear map 
\[
F_{\lambda} : \text{CE}(\mathcal{C}) \to \text{CF}(\mathcal{C})\] 
where 
\[
S : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C})\] 
on \( \text{CF}(\mathcal{C}) \) is the antipodal map of \( \text{CE}(\mathcal{C}) \), see [Shi17, Definition 3.6].

Let \( \mathcal{C} \) be a pivotal fusion category with commutative Grothendieck ring. Recall that 
\( R : \mathcal{C} \to \mathbb{Z}(\mathcal{C}) \) is the right adjoint of the forgetful functor \( F : \mathbb{Z}(\mathcal{C}) \to \mathcal{C} \). Note that in this case by [Shi17, Theorem 6.6] the object \( R(1) \in \text{O}(\mathbb{Z}(\mathcal{C})) \) is multiplicity-free.

A conjugacy class of \( \mathcal{C} \) is defined as a simple subobject of \( R(1) \) in \( \mathbb{Z}(\mathcal{C}) \). Since the monoidal center \( \mathbb{Z}(\mathcal{C}) \) is also a fusion category we can write 
\( R(1) = \bigoplus_{j=0}^{m} \mathcal{C}_{j} \) as a direct sum of simple objects in \( \mathbb{Z}(\mathcal{C}) \). Thus \( \mathcal{C}_{0}, \ldots, \mathcal{C}_{m} \) are the conjugacy classes of \( \mathcal{C} \). Since the unit object \( 1_{\mathbb{Z}(\mathcal{C})} \) is always a subobject of \( R(1) \), we may assume \( \mathcal{C}_{0} = 1_{\mathbb{Z}(\mathcal{C})} \).

Let also \( F_{0}, F_{1}, \ldots, F_{m} \) be the central primitive idempotents of \( \text{CF}(\mathcal{C}) \). We define 
\[
\mathcal{C}_{j} := F_{\lambda}^{-1}(F_{j}) \in \text{CE}(\mathcal{C})
\]
for the conjugacy class sums corresponding to the conjugacy class \( \mathcal{C}_{j} \).

For a fusion category \( \mathcal{C} \) we denote by \( \overline{\text{Irr}(\mathcal{C})} \) the set of elements 
\[
\{ \frac{\mathcal{C}_{j}}{\dim(\mathcal{C}_{j})} \}_{j=0}^{m} \in \text{CF}(\mathcal{C}).
\]
Since \( \text{CF}(\mathcal{C}) \simeq \text{Gr}_{C}(\mathcal{C}) \) is a based ring, by Example (2.24) it follows that \( \overline{\text{Irr}(\mathcal{C})} \) is a probability group with \( \mathbb{C}[\overline{\text{Irr}(\mathcal{C})}] = \text{CF}(\mathcal{C}) \). By [Bur19, Equation 4.8] one has that
\[
\mu_{j}(\chi) = \chi(\frac{C_{j}}{\dim(\mathcal{C}_{j})}).
\]
where \( \lambda \in \text{CE}(\mathcal{C}) \) is the idempotent integral of \( \mathcal{C} \).

We also denote by \( \overline{\text{Cls}(\mathcal{C})} \) the set of central elements 
\[
\{ \frac{C_{j}}{\dim(\mathcal{C})} \}_{j=0}^{m} \in \text{CE}(\mathcal{C}).
\]
We denote by \( \overline{\text{CF}(\mathcal{C})} \) the dual complex algebra \( \mathbb{C}[\overline{\text{Irr}(\mathcal{C})}] \).

By the proof of [Shi17, Theorem 6.12] for all \( \chi \in \text{CF}(\mathcal{C}) \) one has that
\[
\mu_{j}(\chi) = \chi(\frac{C_{j}}{\dim(\mathcal{C}_{j})}).
\]

Since \( \overline{\text{Irr}(\mathcal{C})} \) is an abelian probability group orthogonality relations form Equations (2.14) and (2.15) can be written as
\[
\sum_{k=0}^{m} \frac{\chi_{i}(C_{k})\chi_{j}^{*}(C_{k})}{\dim(\mathcal{C}_{k})} = \delta_{i,j} \dim(\mathcal{C})
\]
and respectively

\[ \sum_{i=0}^{m} \chi_i(C_i) \chi_i^*(C_k) = \delta_{l,k} \dim(C^k) \dim(C) \]

**Corollary 3.9.** Let \( \mathcal{C} \) be a fusion category with a commutative Grothendieck ring.

\[ \frac{\dim(C)}{\dim(C')} \in A \]

**Proof.** Since as above \( \mu_j(\chi_i) = \frac{\chi_i(C_j)}{\dim(C_j)} \) (see the proof of [Shi17, Theorem 6.12]) the first orthogonality relation (3.7) can be written as \( \sum_{i=0}^{m} \mu_k(\chi_i) \mu_k(\chi_i^*) = \frac{\dim(C)}{\dim(C')} \). On the other hand it is known that \( \mu_{l \overline{Z}(M)}(\chi_i) \) are cyclotomic integers for all \( 0 \leq l \leq m \). \( \square \)

**Lemma 3.10.** For any \( \chi \in \text{CF}(\mathcal{C}) \) and \( z, z' \in \text{CE}(\mathcal{C}) \) one has

\[ d(\chi)(zz') = \chi(z)\chi(z') \]

**Proof.** If \( z = \sum_{i=0}^{n} z_i E_i \) and \( z' = \sum_{i=0}^{n} z'_i E_i \) then \( zz' = \sum_{i=0}^{n} z_i z'_i E_i \). Then \( \chi_i(z)\chi_i(z') = z_i z'_i d_i^2 \) and \( d(\chi_i)\chi_i(zz') = d_i^2 z_i z'_i = \chi_i(z)\chi_i(z') \). \( \square \)

Next theorem extends the result obtained in [ZZ19] for semisimple Hopf algebras.

**Theorem 3.12.** Let \( \mathcal{C} \) be a fusion category. Then \( \overline{\text{Cls}(\mathcal{C})} \) is an abelian generalized hypergroup and \( \overline{\text{Cls}(\mathcal{C})} \cong \overline{\text{Irr}(\mathcal{C})} \) as generalized hypergroups. More precisely, the map

\[ \overline{\text{Irr}(\mathcal{C})} \to \overline{\text{Cls}(\mathcal{C})}, \mu_j \mapsto \frac{C_j}{\dim(C_j)} \]

is bijective and induces an isomorphism of \( \mathbb{C} \)-algebras \( \overline{\text{CF}(\mathcal{C})} \cong \overline{\text{CE}(\mathcal{C})} \).

**Proof.** Clearly this map is bijective, it remains to show that it induces an isomorphism of \( \mathbb{C} \)-algebras \( \alpha : \overline{\text{CF}(\mathcal{C})} \to \overline{\text{CE}(\mathcal{C})}, \mu_j \mapsto \frac{C_j}{\dim(C_j)} \).

One has to prove that \( \alpha(\mu_i \mu_j) = \alpha(\mu_i) \alpha(\mu_j) \) for all \( i, j \). Suppose as above that \( \mu_i \mu_j = \sum_k \tilde{p}_k(i,j) \mu_k \). Evaluating at \( \frac{X_s}{d_s} \) this identity it follows that

\[ [\mu_i \mu_j](\frac{X_s}{d_s}) = \sum_k \tilde{p}_k(i,j) \mu_k(\frac{X_s}{d_s}) = \frac{X_s}{d_s} \sum_k \tilde{p}_k(i,j) \frac{C_k}{\dim(C_k)} \]
On the other hand
\[ [\mu_i \mu_j] \left( \frac{\chi_s}{d_s} \right) = \mu_i \left( \frac{\chi_s}{d_s} \right) \mu_j \left( \frac{\chi_s}{d_s} \right) = \frac{\chi_s}{d_s} \left( \frac{C_i}{\dim(C_i)} \right) \frac{\chi_s}{d_s} \left( \frac{C_j}{\dim(C_j)} \right). \]

Thus
\[ \sum_k \hat{p}_k(i, j) \frac{C_k}{\dim(C_k)} = \frac{C_i}{\dim(C_i)} \frac{C_j}{\dim(C_j)} \]
which shows that \( \alpha(\mu_i \mu_j) = \alpha(\mu_i) \alpha(\mu_j) \). \( \Box \)

3.1. **Proof of Theorem 1.1** In this section we give an analogue of Burnside formula for the structure constants. Since \( \{C_j\}_j \) form a \( \mathbb{C} \)-linear basis for \( \text{CE}(\mathcal{C}) \) one has that

\[ C_{j_1} C_{j_2} = \sum_{l=0}^m c_{j_1, j_2}^l C_l \]

form some scalars \( c_{j_1, j_2}^l \in \mathbb{C} \). These scalars are called the structure constants of \( \mathcal{C} \). The last equation from the proof above shows that

\[ c_{i j}^k = \frac{\dim(C_i) \dim(C_j)}{\dim(C_k)} \hat{p}_k(i, j). \]

Combined with Equation (2.17) one obtains the result of Theorem 1.1.

4. **Proof of Theorem 1.3**

A braided category is called premodular if it has a spherical structure. Equivalently, this is a ribbon fusion category, that is, a fusion category equipped with a braiding and a twist (also called a balanced structure), see [ENO05a].

Then \( \mathcal{C} \) is called pseudo-unitary if \( \text{FPdim}(\mathcal{C}) = \dim(\mathcal{C}) \). If such is the case, then by [ENO05a, Proposition 8.23], \( \mathcal{C} \) admits a unique spherical structure with respect to which the categorical dimensions of simple objects are all positive. It is called the canonical spherical structure. For this structure, the categorical dimension of an object coincides with its Frobenius-Perron dimension, i.e. \( \text{FPdim}(X) = \dim(X) \) for any object \( X \) of \( \mathcal{C} \).

Recall [ENO05b] that a fusion category \( \mathcal{C} \) is called weakly integral if its Frobenius-Perron dimension is an integer. If \( \mathcal{C} \) is such a fusion category then \( \mathcal{C} \) is pseudo-unitary by [ENO05a, Proposition 8.24].

Moreover, if \( \mathcal{C} \) is weakly integral then by [ENO05a, Proposition 8.27] the dimensions of simple objects in \( \mathcal{C}_{ad} \) are integers. Since the conjugacy
classes \( C^j \) are sum of simple object of the adjoint subcategory it follows that \( \dim(C^j) \) are integers.

Recall that premodular fusion category \( C \) is called modular if its \( S \)-matrix is non-degenerate.

**Theorem 4.1.** Let \( C \) be a modular fusion category. Then \( \text{Irr}(C) \) is a dualizable probability group and
\[
\text{Irr}(C) \simeq \hat{\text{Irr}}(C)
\]
as probability groups.

**Proof.** By [Shi17, Example 6.14] the map \( f_Q : \text{CF}(C) \to \text{CE}(C), \chi_i \mapsto \sum_{j=0}^{m} s_{ij} e_j \) is an isomorphism of algebras. Moreover from Proposition [Bur19, Theorem 6.5] one has that \( f_Q(\chi_i d_i) = g_i = \frac{C_i}{\dim(C)} \) which shows that \( f_Q \) sends bijectively \( \text{Irr}(C) \) in \( \text{Cls}(C) \). This shows that \( \text{Cls}(C) \) is also a probability group and \( f_Q : \text{Irr}(C) \to \text{Cls}(C) \) is an isomorphism of probability groups. On the other hand by Theorem 3.12 one has that in this situation \( \text{Cls}(C) \simeq \hat{\text{Irr}}(C) \) as probability groups. \( \square \)

Let \( C \) be a pivotal fusion category with a commutative Grothendieck ring \( \text{CF}(C) \simeq \text{Gr}_k(C) \). Let also \( F_0, F_1, \ldots, F_m \) be the primitive central idempotents of \( \text{CF}(C) \) and \( \mu_0, \mu_1, \ldots \mu_m \) be their corresponding characters on \( \text{CF}(C) \). Therefore
\[
\mu_i : \text{CF}(C) \to \mathbb{C}, \quad \mu_i(F_j) = \delta_{i,j}.
\]
We also denote by \( C^0, C^1, \ldots C^m \) the conjugacy classes of \( C \) corresponding in this order to the primitive idempotents \( F_0, F_1, \ldots F_m \).

Let also \( M_0, M_1, \ldots M_m \) be a complete set of representatives for the isomorphism classes of simple objects of \( C \). As above, without loss of generality we may assume that \( M_0 = 1 \) is the unit object of \( C \).

As above we denote by \( V_0, V_1, \ldots V_r \) a complete set of simple objects of \( \mathcal{Z}(C) \) and by \( \hat{E}_0, \hat{E}_1, \ldots \hat{E}_r \) their associated primitive idempotents in \( \text{CE}(\mathcal{Z}(C)) \). We denote also by \( \hat{\chi}_0, \hat{\chi}_1, \ldots \hat{\chi}_r \), the characters associated to \( V_0, V_1, \ldots V_r \) and let \( \hat{d}_0, \hat{d}_1, \ldots \hat{d}_r \) be their quantum dimensions. Therefore \( \hat{d}_s = \hat{\chi}_s(1) \) for all \( s \). We may also assume that \( V_0 = 1 \) is the unit object of \( \mathcal{Z}(C) \). Without loss of generality we may also suppose that \( V_i = C^i \) for any \( 0 \leq i \leq m \). Since the Drinfeld map \( F_Q : \text{CF}(\mathcal{Z}(C)) \to \text{CE}(\mathcal{Z}(C)) \) is bijective it follows that \( \hat{F}_j := F_Q^{-1}(\hat{E}_j) \) is a complete set of primitive orthogonal idempotents of \( \text{CF}(\mathcal{Z}(C)) \).

Let also \( F : \mathcal{Z}(C) \to C \) be the forgetful functor. It is well known, see [ENO05a, Lemma 8.49], that the induced map \( \text{Res} = F_* : \text{CF}(\mathcal{Z}(C)) \to \text{CF}(C) \) is surjective.
Moreover, Ostrik showed in [Ost15, Theorem 2.13] that for any primitive idempotent $F_j \in \text{CF}(C)$ of a spherical category $C$ there is a unique primitive idempotent $\hat{F}_{\sigma(j)} \in \text{CF}(Z(C))$ whose restriction to $\text{CF}(C)$ is $F_j$ and moreover $V_{\sigma(j)} = C_{\sigma(j)}$ is a conjugacy class of $C$ with
\[
\dim(V_{\sigma(j)}) = \dim(C_j), \text{ i.e } \hat{d}_{\sigma(j)} = \hat{d}_j.
\]
Thus $F_*(\hat{F}_{\sigma(j)}) = F_j$, for all $0 \leq j \leq m$. Note also that $F_*(\hat{F}_s) = 0$ for $s \neq \sigma(j)$ for some $j$. We denote by $\hat{s}_{ij}$ the $S$-matrix of the braided category $Z(C)$.

4.1. For the rest of this section we suppose that $C$ is a braided spherical fusion category, i.e a premodular category. Since in this case $Z(C)$ is a modular tensor category it follows by [BK01, Theorem 3.1.12] that the irreducible characters of $\text{CF}(Z(C))$ are indexed by the simple object of $Z(C)$. More precisely, if $V_i$ is a simple object of $Z(C)$ then $\hat{\mu}_i : \text{CF}(Z(C)) \to \mathbb{C}$, $\hat{\mu}_i([V_j]) = \frac{\hat{s}_{ij}}{d_i}$ is an algebra homomorphism.

By [DGNO10, Section 2.10] there is also a braided tensor functor
\[
\iota : C \hookrightarrow Z(C), X \mapsto (X, c_X, -).
\]
that is fully faithful. It follows that $\iota(M_i) \simeq V_i$ for some $0 \leq i \leq r$. Note that $\{0, \tilde{1}, \ldots, \tilde{m}\} \cap \{0, 1, \ldots, m\} = \{0\}$. Indeed, for $i > 1$ $\iota(M_i)$ cannot be a conjugacy class since $M_i = F(\iota(M_i))$ does not contain the unit object $1$ of $C$. Since $\text{Fev} = \text{id}$, note that $F_*(\hat{\chi}_i) = \chi_t$, for all $0 \leq t \leq m$.

Consider the inclusion $\text{CF}(C) \subseteq \text{CF}(Z(C))$ induced by the inclusion functor of (4.2). For any $0 \leq j \leq m$ we define a class of characters
\[
\mathcal{A}_j := \{\hat{\mu}_i \in \text{CF}(Z(C)) \mid \hat{\mu}_i|_{\text{CF}(C)} = \mu_j\}.
\]

**Proposition 4.3.** Let $C$ be a premodular category. With the above notations one has $\sigma(j) \in \mathcal{A}_j$ for any $0 \leq j \leq m$.

**Proof.** Following [Shi17, Example 6.14], inside $\text{CF}(Z(C))$ one can write that
\[
\hat{\chi}_i = \sum_{l=0}^{r} \frac{\hat{s}_{il}}{d_i} \hat{F}_l,
\]
for all $0 \leq i \leq m$. Applying the morphism $F_* : \text{CF}(Z(C)) \to \text{CF}(C)$ induced by the forgetful functor $F$ one obtains that
\[
\chi_i = \sum_{j=0}^{r} \frac{\hat{s}_{ij}}{d_j} F_*(\hat{F}_j) = \sum_{j=0}^{m} \frac{\hat{s}_{i\sigma(j)}}{d_{\sigma(j)}} F_j.
\]
This implies that $\mu_j(\chi_t) = \frac{\hat{s}_{\sigma(j)}(t)}{d_{\sigma(j)}} = \hat{\mu}_{\sigma(j)}(\hat{\chi}_t)$, thus $\hat{\mu}_{\sigma(j)} \in A_j$. \hfill \Box

4.2. **Proof of the main Theorem 1.3.** We denote by $P_v(s, u)$ the probability structure of $\text{Irr}(Z(\mathcal{C}))$ and by $\hat{P}_v(i, j)$ the probability structure of the dual abelian probability group $\text{Irr}(Z(\mathcal{C}))$. By Example (2.24) we have

$$\hat{P}_v(s, u) = P_v(s, u) = \frac{\hat{N}_{s, u} d_v}{d_s d_u},$$

*Proof.* Since $\mathcal{C}$ is braided we have an inclusion of braided categories

$$\mathcal{C} \hookrightarrow Z(\mathcal{C}), X \mapsto (X, c_{X, -}).$$

This shows that $\text{CF}(\mathcal{C}) \subseteq \text{CF}(Z(\mathcal{C}))$ and therefore $\text{Irr}(\mathcal{C}) \leq \text{Irr}(Z(\mathcal{C}))$ is a probability subgroup. By [Har79, Proposition 2.11] it follows that $\text{Irr}(\mathcal{C})$ is a dualizable probability group and

$$[\text{Irr}(\mathcal{C})] = \text{CF}(\mathcal{C}) \simeq \text{CF}(Z(\mathcal{C}))/\text{CF}(\mathcal{C})^\perp$$

where $\text{CF}(\mathcal{C})^\perp = \{ \hat{\mu}_j \in \text{CF}(Z(\mathcal{C})) \mid \hat{\mu}_j(\hat{\chi}_s) = \text{FPdim}(\chi_s), \text{ for all } 0 \leq s \leq m \}$. Moreover, under the isomorphism (2.21) one has

$$[\hat{\mu}_u]_{\text{CF}(\mathcal{C})^\perp} = [\hat{\mu}_v]_{\text{CF}(\mathcal{C})^\perp} \iff u, v \in A_j,$n

for some $j$.

By Equation (2.20) one has

$$\hat{p}_k(i, j) = \sum_{v \in A_k} \hat{P}_v(s, u) = \sum_{v \in A_k} P_v(s, u) = \sum_{v \in A_k} \frac{\hat{N}_{s, u} d_v}{d_s d_u}.$$

Since $C_i = \dim(C^i) g_i$ it follows that

$$C_i C_j = \sum_{k=1}^m \left( \sum_{v \in A_k} \frac{\hat{N}_{s, u} d_v \dim(C^i) \dim(C^j)}{d_s d_u \dim(C^k)} \right) C_k$$

By Proposition 1.3 one has $\sigma(i) \in A_i$. Therefore if $s = \sigma(i)$ and $u = \sigma(j)$, then

$$C_i C_j = \sum_{k=1}^m \left( \sum_{v \in A_k} \frac{\hat{N}_{s, u} \dim(C^i) \dim(C^j)}{\dim(C^k)} \right) C_k$$

since $\hat{d}_{\sigma(i)} = \dim(C^i)$ and $\hat{d}_{\sigma(j)} = \dim(C^j)$.

By Corollary 3.9 one has $\dim(C^k) \mid \dim(C)$ and therefore

$$\dim(C) C_{ij}^k = \frac{\dim(C)}{\dim(C^k)} \left( \sum_{v \in A_k} \hat{N}_{s, u} \hat{d}_v \right)$$
is an algebraic integer. Moreover, if $\mathcal{C}$ is weakly integral then by the above formula, the same numbers are non-negative integers. Indeed, by [GN08, Theorem 3.10] one has $\hat{d}_v$ is integer for any $v \in \mathcal{A}_k$ since $\mathcal{C}_{\sigma(i)}$ and $\mathcal{C}_{\sigma(j)}$ have integer dimensions. Moreover $\frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^i)}$ is a rational number which is also an algebraic integer, therefore an integer. □

Remark 4.6. Taking $j = i^*$ in the proof of the above theorem and $u = s^*$ it follows that
\[
\frac{1}{\dim(\mathcal{C}^i)} = \hat{p}_1(i, i^*) = \sum_{v \in \mathcal{A}_i} \frac{\hat{N}_v}{d_v^2} \hat{d}_v^2
\]
i.e. $\dim(\mathcal{C}^i)|\hat{d}_s^2$ for all $s \in \mathcal{A}_i$.

In particular, if $\hat{d}_s = 1$ and $s \in \mathcal{A}_j$ it follows that $\dim(\mathcal{C}^j) = 1$.

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