What attracts to attractors?

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Whether, how, and to what extent solutions of Bjorken-expanding systems become insensitive to aspects of their initial conditions is of great importance to the phenomenology of heavy-ion collisions. Here we study 1+1D and phenomenologically relevant 3+1D systems in which initial conditions decay to a universal attractor solution. We show that this early-time decay is governed by a power-law, in marked contrast to the exponential decay to attractor solutions at late times. Therefore, the physical mechanisms of hydrodynamization operational at late times do not drive the approach to the attractor at early times, and the early-time attractor is reached prior to hydrodynamization. Hydrodynamization is subsequently achieved as the attractor approaches its fluid-dynamic approximation at late times.

In a dynamical system, an attractor is the particular solution to which arbitrary initial conditions within the basin of attraction relax at sufficiently late times. In general, the attractor is characterized by the competition between the expansion rate that drives the system towards local anisotropy, and the isotropizing interaction rate \([1]\). Attractors are easily found empirically by evolving a set of different initial conditions (see Fig. 1 for an example). Recently, such attractor solutions have received attention in the context of ultra-relativistic heavy-ion collisions. Their form is of interest for understanding the onset of fluid-dynamic behaviour \([2,21]\) and the origin of the non-thermal fixed-point behaviour in far-from-equilibrium dynamics \([1,21,25]\). For the phenomenology of heavy-ion collisions, these studies are needed to clarify to what extent different observables inform us either about the details of the initial conditions or about the material properties of the system.

Whether an attractor solution exists at arbitrarily early times depends on the dynamics that drives the initial conditions to the attractor. Obviously, if the mechanisms of relaxation to the attractor take a finite time, a unique attractor solution does not exist prior to that time. Here we point out that different models undergoing Bjorken expansion do exhibit attractor behaviour at arbitrarily early times. The existence of the early-time attractor is a consequence of the singular geometry of the longitudinal expansion which renders heavy-ion phenomenology insensitive to the unknown details of the longitudinal structure of the initial state.

Israel-Stewart theory: For instance, in Bjorken-expanding Israel-Stewart (IS) theory \([20]\) with transverse translational symmetry, an attractor solution exists for the ratio of longitudinal pressure \(p_L\) over energy density \(\varepsilon\),

\[
\partial_\tau \varepsilon + \frac{1}{\tau} [\varepsilon + p_L] = 0, \tag{1}
\]

\[
\partial_\tau \phi + \frac{4}{3} \frac{\phi}{\tau} = -\frac{1}{\tau_R} \left[ \phi - \frac{4}{3} \frac{\eta}{\tau} \right]. \tag{2}
\]

Here \(\tau\) is the proper time and \(\phi \equiv \frac{1}{3} \varepsilon - p_L\). The time governing the relaxation to fluid-dynamic constitutive equations is \(\tau_R = \frac{5}{a} \frac{\eta}{\varepsilon - p_L}\), where \(a\) is a free parameter, conventionally fixed to \(a = 1\). This is the timescale on which linearized non-hydrodynamic excitations decay.

We work for a conformal equation of state \(\varepsilon = 3P\) and...
constant $\eta/s \propto \eta/\varepsilon^{3/4}$. The equation of motion for the ratio $x \equiv p_\perp/\varepsilon$ (written for convenience in the rescaled time $t = \tau/\tau_R$) reads

$$\left(\frac{3}{4} - \frac{x}{4}\right) \frac{dx}{dt} = \frac{45x^2 - 30x + 5 + 15(t - 3x) - 16a}{45t} \left.\right|_{t \equiv 0}. \quad (3)$$

The limit $a \to 0$ at finite $\tau_R$ is equivalent to an ideal IS theory with $\eta = 0$. In this simplest case, the attractor is the equilibrium $x_A = 1/3$, and how the attractor is approached is given by rewriting (3) in terms of the deviation $\delta = x - x_A$

$$\left(\frac{3}{4} - \frac{1 + 3\delta}{12}\right) \frac{d\delta}{dt} = \frac{\delta^2}{t} - \delta. \quad (4)$$

Depending on whether $t > \delta$ or $t < \delta$, the approach to the attractor is governed by the expansion rate $t^{-1}$ or by the interaction rate $t^0$, respectively. At all times, sufficiently small deviations from the attractor decay exponentially $\delta \sim \varepsilon^{3/2}$, which is characteristic for linearized non-hydrodynamic perturbations around thermal equilibrium [6]. The factor $3/2$ arises from the non-trivial time-evolution of the background.

For finite $a$, eq. (3) corresponds to the first-order IS theory, which has two solutions that remain regular for $t_0 \to 0$ with limits $\lim_{t_0 \to 0} x_{\pm}(t_0) = \frac{1}{15} (5 \pm 4\sqrt{5}a)$, respectively. The solution $x_-(t)$ is the attractor solution $x_A(t)$, while $x_+(t)$ limits the basin of attraction from above. While we do not have an analytic solution $x_-(t)$, the attractor can be expanded at late and early times. It is well-known that the late time (fluid-dynamic) expansion in powers of $1/t$,

$$x_-(t) = 1 + \frac{1}{3} \sum_{i=1}^{\infty} \frac{h_i}{t^i}, \quad (5)$$

$$h_1 = -\frac{1}{3}(16a), \quad h_2 = -\frac{176a}{27}, \quad \ldots \quad (6)$$

is an asymptotic, non-convergent but Borel-resummable series [6]. The asymptotic nature of this series is seen in Fig. 2, where inclusion of successive orders worsens eventually the agreement with the exact numerical solution at any finite $t$. Here, we contrast this hydrodynamic expansion with the corresponding early-time (or single-hit [27])

$$\partial_\tau f + \vec{v}_\perp \cdot \partial_{\vec{x}_\perp} f - \frac{p_\perp}{\tau} \partial_{p_\perp} f = -\frac{(-u_\perp u^{\mu})}{\tau_R} [f - f_{eq}]. \quad (10)$$

We emphasize that the timescale of this decay becomes increasingly rapid and ultimately instantaneous with decreasing $t_0$ (see Fig. 1). This is the hallmark of a decay governed by the expansion rate. It is qualitatively different from what one expects from the decay of non-hydrodynamic modes, and it forces the decay to the attractor prior to hydrodynamization.

Kinetic theory: Features similar to the above can also be seen in Bjorken-expanding massless kinetic theory in the relaxation time approximation (RTA)

Here, the distribution function $f(\tau, \vec{x}_\perp; \vec{p}_\perp, p_z)$ depends on $p^\mu = (p, \vec{p}_\perp, p_z), p = \sqrt{p_\perp^2 + p_z^2}$, and on the proper time $\gamma u_\perp$ denotes the rest frame of the energy density and $\vec{v}_\perp = \vec{p}_\perp/p, v_z = p_z/p$ are transverse and longitudinal velocities, respectively. We work with a conformal relaxation time $\tau_R^{-1} = \gamma \varepsilon^{1/4}$. 

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1 We note that within IS theory, $\alpha$ is a free parameter. The choice $\alpha = 1$ amounts to equating $\tau_R$ to the second order hydrodynamic coefficient $\tau_\eta$ of RTA kinetic theory. The choice $\alpha = 5/16$ would instead insure that the early-time attractor of IS theory coincides with that of kinetic theory.
For systems with transverse translational symmetry, this Boltzmann equation can be reduced to a tower of moment equations

\[ \partial_t p_l + \frac{1}{\tau} [(2l + 1)p_l - (2l - 1)p_{l+1}] = -\frac{1}{\tau_\text{R}} \left[ p_l - \frac{1}{2l + 1} \frac{\varepsilon}{\tau} \right], \tag{11} \]

where \( p_l \equiv \int_0^1 dx \int \frac{4\pi d^2 p}{(2\pi)^3} f \nu^2 \frac{d^2 v}{d^2 p} \); this definition implies \( 0 \leq \cdots \leq p_2 \leq p_1 \leq p_0 \). Energy density and longitudinal pressure correspond to the first two moments, \( \varepsilon = p_0 \), \( p_L = p_1 \). The first two equations in the hierarchy \((11)\) result in

\[ \left( \frac{3}{4} - \frac{x}{4} \right) \frac{dx}{dt} = \frac{3x^2 - 6x + 3\gamma + t(1 - 3x)}{3t}, \tag{12} \]

where \( y \equiv p_2/\varepsilon \). The attractor is found amongst the regular solutions. Solutions that remain regular for \( t_0 \to 0 \) satisfy \( \lim_{t_0 \to 0} x_+(t_0) = 1 \pm \sqrt{1 - y(0)} \). All physical values \( x \leq 1 \) lie within the basin of attraction since \( x_+ > 1 \). Since \( 0 < y < x \), the attractor solution at early times is \( x_A(0) = 0 \), and therefore also all higher moments \( p_l(0) \) vanish. At late times, it follows trivially from \((12)\) that the attractor approaches equilibrium, \( \lim_{t \to \infty} x_A(t) = \frac{1}{3} \).

The late-time fluid-dynamic expansion of this transport theory has been computed to high orders \( [9] \). In complete analogy to IS theory, it is a non-convergent asymptotic and Borel-resummmable series. The early-time, single-hit expansion results in a convergent series

\[ x(t) = \left(1 - \frac{3h - 5}{8} \right) t + \left(15h - 5 - 8(h - 2) \right) t^2 + O(t^3), \tag{13} \]

\[ h_n = \frac{4}{3} F_2 \left( \frac{1}{3}, \frac{1}{2}; \frac{1}{2}; \frac{3}{2}, \frac{3}{2} + 2; \frac{1}{3} \right) \frac{4}{n + 8} + \frac{4}{n + 12}, \tag{14} \]

that we have evaluated to high order. In complete analogy to IS theory, it can be analytically continued to any \( t \) beyond its radius of convergence, see Fig. 2.

Also the transient dynamics according to which generic initial conditions approach the attractor shares the main qualitative features of the IS theory discussed above. At late initializations, \( t_0 > 1 \), eqs. \((12)\) and \((4)\) govern identical exponential decays of linearized non-hydrodynamic modes. For early times, \( t_0 \ll 1 \), the decay of \( \delta = x - x_A \) to the attractor depends on \( y \), and through \( y \) on the initial conditions of all higher moments. Because \( 6x > 3y \) for any system, an approximate solution of the approach to the attractor can be obtained for a generic initial condition by neglecting \( y \) in \((12)\) which leads to the power law decay \( \delta \sim t^{-8/3} \).

Similar reasoning suggests that \( y \) would approach its attractor \( \sim t^{-16/3} \) thus justifying the above approximation; in fact, the same reasoning gives for all higher orders \( p_l(t)/\varepsilon(t) \propto x(t)^l \). These power laws are easily seen in numerical solutions of eq. \((10)\) (see supplemental material).

**Attractors in 3+1D:** So far we have discussed the attractor solutions for an idealized geometry with transverse translational symmetry. It is, however, necessary to go beyond this simplified setup \([13, 14, 29]\) to eventually make contact with phenomenology \([29]\). Because the early-time approach to the attractor is governed by the longitudinal expansion rate, breaking the translational symmetry in the transverse directions can change the above picture only to the extent to which transverse gradients are not negligible compared to the longitudinal one. Therefore, at sufficiently early initialization, independent of the transverse geometry and for all transverse positions \( r \), arbitrary initial conditions in 3+1D evolve towards the 1+1D attractor. However, while the late-time evolution of the attractor does not depend on the higher moments of the distribution function (which decay early towards their attractor solutions), it does depend on the transverse profile of energy and transverse momentum.

These features are realized in 3+1D solutions of eq. \((10)\), initialized with a Gaussian transverse energy profile with central energy density \( \varepsilon_0 \) and r.m.s. radius \( R \), see Fig. 3. For early initialization time \( t_0 \) keeping \( \varepsilon_0 t_0 \) fixed, eq. \((10)\) can be rescaled such that the evolution depends on only one dimensionless combination of model parameters, the opacity \( \hat{\gamma} = \gamma R^{1/4} (\varepsilon_0 t_0)^{1/4} = \)}
The opacity of a system increases with coupling strength ($\gamma$), transverse system size ($R$) and initial central energy density ($\epsilon_0$); physical collision systems were estimated to correspond to a range of opacities, $\hat{\gamma} \lesssim 2$ for proton-nucleus collision, $2 \lesssim \hat{\gamma} \lesssim 4$ for semi-peripheral PbPb collisions and somewhat higher values in central PbPb collisions [29].

The physical time in Fig. 3 is rescaled by a position- and time-dependent relaxation time $\tau_R^{-1}(r,v) = \gamma \epsilon(r,v)^{1/4}$. Therefore, for a system in which energy density decreases faster than $\propto \tau^{-4}$ due to transverse expansion, the relation between physical and rescaled time is not monotonous; this is the reason $t$ decreases for sufficiently late $\tau$ in the finite-$\hat{\gamma}$ curves of Fig. 3. Moreover, because of this rescaling, the deviation of the $r = 0$ attractor solution from the 1+1D one, and the deviation of the attractor solutions at finite $r$ from the one at $r = 0$ arise solely from the radial expansion. For fixed $\hat{\gamma}$, the $r$-dependence is remarkably mild. Low orders in the single-hit expansion are seen to be sufficient to describe systems characterized by values of $\hat{\gamma}$ that are within experimental reach. What Fig. 3 makes abundantly clear is that what remains universal across collision geometries is not the late-time attractor but the early-time attractor. That is, what remains universal is what follows from few-hit dynamics and not what follows from hydrodynamization.

In summary, in Bjorken expanding systems the attractor is reached prior to hydrodynamization. We have identified a power-law approach to the attractor that is operational far from equilibrium. This mechanism is not based on the decay of linearized non-hydrodynamic modes but instead it is driven by the longitudinal expansion. It is thus dominant if the initialization occurs at sufficiently early times. In particular, the time it takes to reach the attractor is arbitrarily fast as the system is initialized arbitrarily early. We note the qualitative difference to the decay of linearized non-hydrodynamic modes that would not lead to a unique early-time attractor at $t \lesssim 1$. This is because, while the decay time $\tau_R(\tau_0)$ of non-hydrodynamic modes does go to zero in the $\tau_0 \to 0$ limit, the decay time $\tau_R(\tau)$ grows so fast due to the expansion that the attractor is reached only at $\tau \sim \tau_R(\tau)$. That is, non-hydrodynamic modes could not decay at times $t \lesssim 1$ while the systems we study show such decays.

The above arguments about an expansion-driven early-time approach to the attractor carry over to 3+1D systems to the extent that the longitudinal gradients dominate over the transverse ones. Therefore, that the attractor is reached quickly and without requiring hydrodynamization may be at the basis of understanding why hallmarks of collectivity exhibit universal features down to the smallest hadronic collision systems.
Radius of convergence of the single-hit expansion in the IS theory

Here, we provide further evidence that the early-time single-hit expansion is a convergent series. The coefficients of the series of eq. \( (7) \) are shown in the upper panel of Fig. 4. The high-order coefficients determine the convergence properties of the series. That the high-order coefficients seem to saturate to \( \sim 0.24^n \) suggests that the radius of convergence is that of a geometric series \( \sum (0.24)^n \), that is \( |t| < 4 \). Consistent with that, the solution continued to complex \( t \) shows non-analytic structures away from the real axis at \( |t| \sim 4 \).

Early-time power law decay to the RTA attractor

In the RTA kinetic theory \((10)\), evolution of \( x = p_L/\varepsilon \) is coupled to higher moments. To characterize the early-time power law decay to the attractor, we rewrite the
hierarchy of RTA moment equations \([11]\) for \(x_l \equiv p_l/\varepsilon\),
\[
\left(\frac{3}{4} - \frac{x}{4}\right) \frac{dx_l}{dt} = -\frac{\left(2l - x\right)x_l - \left(2l - 1\right)x_{l+1}}{t} + \left(\frac{1}{2l+1} - x_l\right). \tag{15}
\]

Here, \(x = x_1\); eq. \([12]\) is the first \((l = 1)\) of these moment equations. Fig. 5 shows that the early-time power law decay of \(x(t)\) towards the attractor becomes \(\delta \sim t^{-8/3}\), and it is independent of the initial condition for higher moments (see Fig. 5). Similarly, the early-time decay of \(y(t)\) is consistent with \(\sim t^{-16/3}\).

![FIG. 5: The early-time decay of \(x(t)\) to the RTA attractor (thick blue line same as in Fig. 1) follows a power-law almost independent of the initial condition for the higher moment \(y(t)\) (dashed and straight red line). This power law decay seizes when the interaction rate becomes comparable to the expansion rate (see grey line and discussion in text).](image)

According to the right hand side of \([12]\), we expect the interaction rate \((\propto t^0)\) to balance the expansion rate \((\propto t^{-1})\) as soon as \(x(t) \approx \frac{1}{6} t (1 - 3\epsilon(t))\). As seen from Fig. 5, it is at this time that the power law decay to the attractor seizes to persist. This further illustrates that expansion forces the decay to the attractor prior to any contribution from the interaction rate.

**A rapidly converging approximation to the IS attractor solution**

In the course of the present study, we stumbled upon a rapidly converging, analytic approximation. While this solution is not needed for any step of the present paper, we document it in this supplemental material.

The approximation is obtained by inserting into \([3]\) the Taylor series

\[
x_-(t) = \sum_{l=0}^{l_{\text{max}}} c_l(t_*) (t - t_*)^l. \tag{16}
\]

Here, expansion is around an arbitrary time \(t_*\). Collecting powers of \(t\) in \([3]\), one can express all coefficients \(c_l\), \(l \geq 1\), as rational functions of \(c_0\) and \(t_*\). Since the derivative \(dx/dt\) of the attractor solution must not diverge in the limit \(t \to 0\), one requires

\[
x_-(t = 0) = \frac{1}{15} \left(5 - 4\sqrt{5}\right) = \sum_{l=0}^{l_{\text{max}}} c_l(t_*) (-t_*)^l, \tag{17}
\]

where the right-hand side is now an explicitly known rational function of \(c_0(t_*)\) and \(t_*\). Thus, \([17]\) defines \(c_0(t_*)\) implicitly in terms of \(t_*\). By construction, see eq. \([16]\), \(c_0(t) = x(t)\) is the attractor solution, and \([17]\) therefore provides an implicit expression for this solution. As seen in Fig. 6, even for a truncation at the lowest order \(l_{\text{max}} = 1\), the accuracy of this procedure is comparable to that of the slow-roll approximation \([8]\), and it improves rapidly upon including higher orders in the truncated ansatz \([16]\).

![FIG. 6: Upper panel: The IS attractor solution, compared to its slow roll expansion, and compared to the approximation \(x(t) = c_0(t)\) where \(c_0\) is determined from expanding \([16]\) to first, second and third order respectively. Lower panel: the difference between the full IS attractor solution and the result obtained in the various approximations.](image)