HIGHER RANK THIN MONODROMY IN $O(5)$

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ABSTRACT. In this article, we study 19 examples of higher rank orthogonal hypergeometric groups of degree five. These examples appeared in [3, Table 6]. We establish the thinness of 12 of these hypergeometric groups. Some of these examples are associated with Calabi-Yau 4-folds.

This produces the first examples in the cyclotomic family of Zariski dense non-arithmetic orthogonal hypergeometric monodromy groups of real rank two.

1. INTRODUCTION

Let $\theta = z\frac{dz}{z}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$. Then, the hypergeometric differential equation of order $n$ is defined by

$$z(\theta + \alpha_1) \cdots (\theta + \alpha_n) - (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) u(z) = 0. \tag{1.1}$$

It is defined on the thrice-punctured Riemann sphere $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, and it has $n$ linearly independent solutions. Thus, the fundamental group $\pi_1$ of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ acts on the (local) solution space of (1.1): calling $V$ the solution space in the neighbourhood of a point of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, we get the monodromy representation $\rho : \pi_1 \to \text{GL}(V)$. The subgroup $\rho(\pi_1)$ of $\text{GL}(V)$ is called the monodromy group of the hypergeometric differential equation (1.1). We also call it the hypergeometric group associated to $\alpha, \beta \in \mathbb{C}^n$.

Levelt (cf. [7, Thm. 3.5]) showed that if $\alpha_j - \beta_k \notin \mathbb{Z}$ for all $1 \leq j, k \leq n$, then there exists a basis of the solution space of (1.1) with respect to which the associated hypergeometric group is the subgroup of $\text{GL}_n(\mathbb{C})$ generated by the companion matrices $A$ and $B$ of the polynomials

$$f(x) = \prod_{j=1}^n (x - e^{2\pi i \alpha_j}), \quad g(x) = \prod_{j=1}^n (x - e^{2\pi i \beta_j})$$

respectively. More precisely, if

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \quad g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0,$$

then, in an appropriate basis, the monodromy representation is given by

$$g_{\infty} \mapsto A = \begin{pmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & \cdots & -a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{n-1} \end{pmatrix}, \quad g_0 \mapsto B^{-1} = \begin{pmatrix} 0 & 0 & \cdots & -b_0 \\ 1 & 0 & \cdots & -b_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -b_{n-1} \end{pmatrix}^{-1}, \quad g_1 \mapsto A^{-1}B,$$

\vspace{1cm}

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where \( g_0, g_1, g_\infty \) are the loops around 0, 1, \( \infty \) respectively, and the hypergeometric group is \( \Gamma(f, g) := \langle A, B \rangle \subseteq \text{GL}_n(\mathbb{C}) \). The condition \( \alpha_j - \beta_k \notin \mathbb{Z} \) for all \( 1 \leq j, k \leq n \) ensures that \( f \) and \( g \) do not have any common root.

**Definition 1.** A hypergeometric group \( \Gamma(f, g) \) is called arithmetic if it is of finite index in \( G(\mathbb{Z}) \), and thin if it has infinite index in \( G(\mathbb{Z}) \), where \( G \) is the Zariski closure of \( \Gamma(f, g) \) inside \( \text{GL}_n(\mathbb{C}) \).

Sarnak’s question [16] about classifying the pairs of polynomials \( f, g \) for which \( \Gamma(f, g) \) is arithmetic or thin has witnessed many interesting developments. For a detailed account on recent progress see the introduction of [2].

In this article we consider the case \( n = 5 \), and we assume that \( \alpha, \beta \in \mathbb{Q}^5 \) and \( f, g \in \mathbb{Q}[x] \). This implies that \( f, g \) are products of cyclotomic polynomials and that in fact they sit in \( \mathbb{Z}[x] \). Up to taking a conjugate of the original group, we may assume \( \alpha_j, \beta_k \in [0, 1) \), which leads to a list of finitely many cases. Out of these, we only consider the cases that satisfy the following conditions:

1. \( f \) and \( g \) have no common roots, such that Levelt’s result applies;
2. the Zariski closure of \( \Gamma(f, g) \) is \( O_Q \) for some non-degenerate quadratic form \( Q \);
3. \( \Gamma(f, g) \subseteq O_Q \) is primitive (see [7, Def. 5.1]).

It is sufficient to study the problem for pairs \( f, g \) up to scalar shift (see [7, Def. 5.5]); in other words, studying the group \( \Gamma(f, g) \) is equivalent to studying \( \Gamma(f', g') \), with \( f'(x) = -f(-x) \) and \( g'(x) = -g(-x) \) ([19, Rem. 1.2] also applies here). From the results of [7] it follows that, up to scalar shift, there are exactly 77 cases that satisfy the above conditions. These are discussed in [3]. Four of these cases correspond to finite monodromy groups and 17 cases correspond to monodromy groups for which the Zariski closure is of type \( O(4, 1) \), hence has real rank one. For the remaining 56 pairs the Zariski closure is \( O(3, 2) \), with real rank two, see [3]. Out of these 56 cases, 37 have already been proven to be arithmetic: 11 cases [3, Table 2] by the results of Venkataramana [21], 2 cases [3, Table 3] by the results of Singh [18], 23 cases [3, Table 4] by Bajpai–Singh [3], and 1 case by Bajpai–Singh–Singh [4]. In the end, we are left with 19 cases of type \( O(3, 2) \) whose arithmeticity or thinness was still undetermined. These cases are listed in Table 1 and Table 2 below. We will show that the 12 cases of Table 1 are thin.

It is to be noted that Venkataramana [21] has constructed 11 infinite families of higher rank arithmetic orthogonal hypergeometric groups, and Fuchs–Meiri–Sarnak [12] have constructed 7 infinite families of hyperbolic thin orthogonal hypergeometric groups. In dimension \( n = 5 \), it follows from the latter work that 7 out of the 17 mentioned cases with monodromy groups of real rank one [3, Table 1] are thin. The remaining 10 real rank one cases [3, Table 10] are still open.

Of particular relevance among the 56 examples of type \( O(3, 2) \) are the 14 pairs with **maximally unipotent monodromy**, that is, those hypergeometric groups \( \Gamma(f, g) \) where the polynomial \( f \) is associated to \( \alpha = (0, 0, 0, 0, 0) \).

It is well known that in dimension \( n = 4 \), the 14 symplectic hypergeometric groups with maximally unipotent monodromy emerge as images of monodromy representations arising from Calabi-Yau 3-folds, see [11]. It was shown [17, 20, 9] that exactly half of these groups are arithmetic and half are thin. Similarly, for \( n = 6 \) it is expected that many of the 40
symplectic hypergeometric groups with maximally unipotent monodromy will arise from Calabi-Yau 5-folds; see the introduction of [1]. Out of these groups 20 are known to be arithmetic and 17 are known to be thin, with the status of the remaining 3 groups still unknown [1, 2].

Now, for \( n = 5 \), it is known that at least some of the 14 orthogonal hypergeometric groups with maximally unipotent monodromy arise from Calabi-Yau 4-folds; see [8, Sec. 3.9.3] for a detailed account. One of the purposes of this work is to investigate the dichotomy between arithmetic and thin monodromy among the 14 groups with maximally unipotent monodromy. Out of these cases, 2 have been shown to be arithmetic by Singh [18], and in this article we prove that 9 of them are thin. The other 3 cases remain open. But our result means that in dimension \( n = 5 \) more than half of the hypergeometric groups with maximally unipotent monodromy are thin, meaning that the dichotomy observed in dimension \( n = 4 \) and expected for dimension \( n = 6 \) breaks down for \( n = 5 \).

One interesting example is example 5 in Table 1. This is the sextic case, where \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}) \). We refer to [14, Sec. 6] for an account on this particular case.

Table 1. List of the 12 monodromy groups for which the associated quadratic form Q has signature (3,2) and thinness is shown in this article.

| No. | \( \alpha \) | \( \beta \) | No. | \( \alpha \) | \( \beta \) |
|-----|-------------|-------------|-----|-------------|-------------|
| 1   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) | 2   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) |
| 3   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) | 4   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) |
| 5   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) | 6   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \) |
| 7   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9} \) | 8   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9} \) |
| 9   | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9} \) | 10  | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9} \) |
| 11  | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{5}{7} \) | 12  | (0, 0, 0, 0, 0) | \( \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{5}{7} \) |

1.1. Results. With a ping pong argument very similar to that in [9] and [1] we obtain the following result.

**Theorem 2.** Let \( G \) be any one of the groups in Table 1, let \( A, B \) be the generators from Equation 1.2 and let \( T = BA^{-1} \). Then

- \( G = \langle \pm T \rangle \ast \langle B \rangle \) if \( B^k = -I \) for some \( k \in \mathbb{N} \), and
- \( G = \langle T \rangle \ast \langle B \rangle \) otherwise.

**Corollary 3.** The groups in Table 1 are thin.

**Proof.** The groups \( \langle T \rangle, \langle \pm T \rangle \) and \( \langle B \rangle \) are abelian and hence have the Haagerup property [13, Ex. 2.4(2)]. Then \( G \) is also Haagerup [13, Prop. 2.5(3)]. On the other hand, \( \text{SO}(3, 2, \mathbb{R}) \), as a simple Lie group of real rank 2, satisfies property \( (T) \) [6, Thm. 1.6.1], as
Table 2. List of the 7 monodromy groups for which the associated quadratic form $Q$ has signature $(3,2)$ and arithmeticity or thinness is unknown.

| No. | $\alpha$ | $\beta$ | No. | $\alpha$ | $\beta$ |
|-----|----------|--------|-----|----------|--------|
| 1   | $(0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ | 2   | $(0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ |
| 3   | $(0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ | 4   | $(0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ |
| 5   | $(0,0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ | 6   | $(0,0,0,0,0,0)$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ |
| 7   | $(0, \frac{3}{10}, \frac{3}{10}, \frac{7}{10}, \frac{7}{10})$ | $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$ |

does the finite extension $O(3,2,\mathbb{R})$ [6, Prop. 1.7.6]. If $G$ were a finite-index subgroup of $O(3,2,\mathbb{Z})$, then it would be a lattice in $O(3,2,\mathbb{R})$ and hence have property (T). This is a contradiction, since non-compact groups cannot both be Haagerup and have property (T) [10, Sect. 1.2.1]. Therefore, $G$ must be thin.

Out of the hypergeometric groups of orthogonal type with real rank two in dimension $n = 5$, we are now left with 7 cases whose arithmeticity or thinness is still unknown, listed in Table 2. For more details and an initial account on this problem, we refer the interested reader to the articles [3] and [5].

Out of the hypergeometric groups of type $O(3,2)$ with maximally unipotent monodromy, 2 cases were shown to be arithmetic in [18]. Combining this result with our Corollary 3, we get the following.

**Corollary 4.** Out of the 14 degree five orthogonal hypergeometric groups with a maximally unipotent monodromy, listed in Tables 2 & 6 in [3, Section 2], at least 2 are arithmetic and at least 9 are thin.

2. **Ping-pong**

Our proof is very similar to that in [1], which, in turn, is an adaptation of the methods of Brav and Thomas [9]. Using the notation of Theorem 2, our goal is to apply the following version of the ping pong lemma to the case where $G_1 = \langle T \rangle$, $G_2 = \langle B \rangle$, $H = \{I\}$, respectively to $G_1 = \langle \pm T \rangle$, $G_2 = \langle B \rangle$, $H = \{\pm I\}$.

**Theorem 5** (see [15], Prop. III.12.4). Let $G$ be a group generated by two subgroups $G_1$ and $G_2$, whose intersection $H$ has index $> 2$ in $G_1$ or $G_2$. Suppose that $G$ acts on a set $W$, and suppose that there are disjoint non-empty subsets $X,Y \subseteq W$, such that $(G_1 \setminus H)Y \subseteq X$ and $(G_2 \setminus H)X \subseteq Y$, with $HY \subseteq Y$ and $HX \subseteq X$. Then $G = G_1 \ast_H G_2$.

We apply the ping pong lemma to the canonical action of $G$ on $\mathbb{R}^5$. The two halves $X$, $Y$ of the ping pong table will both decompose into a union of open cones (with the origin removed) that is invariant under multiplication with $-I$. The table halves are constructed from a single non-empty convex cone $F$ that we will explicitly provide for each case. The exact construction depends on the order of $B$. Note, that $T$ has always order 2.
If $B$ has finite order, we set
\[ X = F \cup -F, \quad Y = \bigcup_{B^i \neq \pm I} B^i F \cup -B^i F. \]

To verify that we get a valid ping pong, we then have to check that $X$ and $Y$ are disjoint and that $T$ maps $Y$ into $X$. The other conditions on the ping pong table are automatically satisfied.

If $B$ has infinite order, we let
\[ X = F_0 \cup -F_0, \quad F_0 = F \cap -EF, \]
for a certain matrix $E$ that satisfies $E^2 = EBE^{-1}B = ETE^{-1}T = I$. We also let $\eta = \min \{i > 0 \mid (B^i - I)^5 = 0\}$ and let $Y$ be the finite union
\[ Y = Y^+ \cup Y^-, \quad Y^+ = \bigcup_{1 \leq i \leq \eta} B^i F \cup -B^i F, \quad Y^- = EY^+. \]

To verify that we get a valid ping pong, we again have to check that $X$ and $Y$ are disjoint and that $T$ maps $Y$ into $X$. The condition that $B^i \neq 0$ maps $X$ into $Y$ is no longer automatic. To verify this in a finite number of steps, we check that $B$ maps both $X$ and $Y^+$ into $Y^+$, and that $B^{-1}$ maps both $X$ and $Y^-$ into $Y^-$, see the right diagram in Figure 1.

**Figure 1.** The ping pong table when $B$ has finite (left) and infinite order (right)

3. **SageMath code**

Here we present the SageMath code that we used to verify the conditions of the ping pong lemma. The cones are implemented by the library class `ConvexRationalPolyhedralCone`, which does all calculations with exact arithmetic in rational numbers. The sets $X, Y, Y^+, Y^-$ that form our ping pong table are represented by lists of such cones.

In contrast to our proof, the library class works with closed convex cones, which we think of as the closure of our cones. To check that a cone is non-empty, we ask that its closure has the same dimension as the ambient vector space. To check that two open cones (without origin) are disjoint, we ask that the intersection of their closures has less than full dimension.
from itertools import count

def companion_matrix(polynomial):
    return block_matrix([[block_matrix([[matrix(1,4), [matrix.identity(4)]]),
                             -matrix(polynomial.list()[:-1]).T]])

# check if the two sets of open cones are disjoint
def are_disjoint(CC, DD):
    return all(C.intersection(D).dim() < 5 for C in CC for D in DD)

# check if the first set of cones is contained in the second
def contained_in(CC, DD):
    return all(any(is_subcone(C, D) for C in CC) for D in DD)

# check if the first cone is contained in the second
def is_subcone(C, D):
    return Cone(block_matrix([[D.rays().column_matrix(),
                               C.rays().column_matrix()]]).T).is_equivalent(D)

# apply the linear transformation L to the set of cones
def transform_set(CC, L):
    return [Cone((L*C.rays().column_matrix()).T) for C in CC]

def verify_finite_order(B, T, F):
    if not T^2 == 1:
        return False # we assume that T has order 2
    eta = next(i for i in count(1) if (B^i-1)^5 == 0)
    if not B^eta == 1:
        return False # B not of finite order
    if not F.dim() == 5:
        return False # F has empty interior
    X = [F] + transform_set([F], -E)
    Y = [C for i in [1..eta-1] for C in transform_set(X, B^i) if not B^i == -1]
    if not are_disjoint(X, Y):
        return False # ping pong table halves not disjoint
    if not contained_in(transform_set(Y, T), X):
        return False # T does not map Y into X
    return True # remaining conditions automatic, ping pong works

def verify_infinite_order(B, T, F):
    if not T^2 == 1:
        return False # we assume that T has order 2
    if not F.dim() == 5:
        return False # F has empty interior
    E = B*Permutation([5,4,3,2,1]).to_matrix()
    F0 = F.intersection(transform_set([F], -E)[0])
    X = [F0] + transform_set([F0], -1)
    eta = next(i for i in count(1) if (B^i-1)^5 == 0)
Yplus = [C for i in [1..eta] for sgn in [1,-1] 
        for C in transform_set([F], sgn*B^i)]
Yminus = transform_set(Yplus, E)
if not are_disjoint(X, Yplus+Yminus):
    return False # ping pong table halves not disjoint
if not contained_in(transform_set(Yplus+Yminus, T), X):
    return False # T does not map Y into X
if not contained_in(transform_set(X+Yplus, B), Yplus):
    return False # B does not map both X and Y^+ into Y^+
if not contained_in(transform_set(X+Yminus, B.inverse()), Yminus):
    return False # B^-1 does not map both X and Y^- into Y^- 
    return True # remaining conditions automatic, ping pong works

To use the above code, one has to call the method verify_finite_order, respectively 
verify_infinite_order, with the matrices B, T and the cone F as arguments. Below is 
an example how to verify thinness for a single case.

A=companion_matrix(cyclotomic_polynomial(1)^5)
B=companion_matrix(cyclotomic_polynomial(2)*cyclotomic_polynomial(8))
M=matrix([[ -8, -8, -1, -1, 0, 0, -1, -3, -3, -1],
           [ 5, 35, 0, 4, 1, 5, 14, 4, 2],
           [-45, -59, -4, -6, -3, -9, -25, -21, -8],
           [ 59, 45, 6, 4, 3, 8, 21, 25, 9],
           [-43, -13, -5, -1, -1, -3, -7, -17, -6]])
print(verify_finite_order(B, B*A.inverse(), Cone(M.T)))

4. Thinness

We now provide specific cones F that make the ping pong in Section 2 work. For each 
case in Table 1 we write whether the order of the companion matrix B is finite or infinite, 
and we give the cone F in terms of a matrix M, whose column vectors span the cone.

Case 1. \( \alpha = (0, 0, 0, 0, 0) \), \( \beta = (\frac{1}{7}, \frac{1}{2}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}) \), and B has infinite order.

\[
M = \begin{pmatrix}
   3 & -1 & -31 & -245 & -5 & -5 & -1 & -7 & 0 & -1 & -8 \\
  -22 & -4 & 158 & 1194 & 23 & 24 & -4 & -24 & 4 & -6 & 32 & 1 & -5 & 39 \\
  -84 & -6 & -332 & -2292 & -41 & -44 & -14 & -44 & -6 & -10 & -58 & -3 & -9 & -85 \\
  -90 & -4 & 226 & 2118 & 33 & 40 & -4 & -40 & 4 & -10 & -48 & 3 & -9 & 69 \\
 -31 & -1 & -245 & -999 & -10 & -15 & -9 & -15 & -1 & -5 & -15 & -1 & -4 & -47
\end{pmatrix}
\]

Case 2. \( \alpha = (0, 0, 0, 0, 0) \), \( \beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}) \), and B has infinite order.

\[
M = \begin{pmatrix}
   -33 & -33 & -31 & -31 & -1 & -1 & -1 & -1 & -1 & -21 & -2 & -11 & -11 & -2 & 0 \\
  -37 & 158 & -63 & 142 & -3 & -3 & -2 & 4 & 5 & -73 & 9 & 53 & -6 & 3 & 1 \\
 -489 & -288 & -419 & -252 & -4 & -11 & -6 & -6 & -3 & -104 & -15 & -101 & -172 & -31 & -3 \\
  57 & 258 & 35 & 202 & -3 & -1 & 4 & 1 & -83 & 17 & 95 & 24 & 1 & 3 \\
-290 & -95 & -266 & -61 & -1 & -8 & -9 & -1 & -2 & -31 & -11 & -38 & -97 & -17 & -1
\end{pmatrix}
\]
Case 3. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}), \) and \( B \) has infinite order.

\[
M = \begin{pmatrix}
-1 & -13 & -11 & -1 & -1 & -1 & -1 & -2 & -3 & -12 & 0 \\
-2 & -4 & 62 & -12 & 50 & -2 & -1 & 4 & 5 & 32 & 9 \\
-2 & -150 & -112 & -114 & -88 & -8 & -101 & -101 & -32 & -32 & -327 & -431 & -139 & -25 & -3 \\
-2 & 60 & 98 & 44 & 70 & 2 & 1 & 4 & 1 & -32 & 17 & 95 & 279 & 175 & 57 & 7 & 3 \\
-1 & -101 & -35 & -83 & -21 & -7 & -8 & -1 & -2 & -21 & -11 & -38 & -95 & -293 & -86 & -15 & -1
\end{pmatrix}
\]

Case 4. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}), \) and \( B \) has infinite order.

\[
M = \begin{pmatrix}
-1 & -13 & -11 & -1 & -1 & -2 & -2 & -11 & -38 & -38 & -11 & -2 & -3 & -3 & 0 \\
-1 & -3 & 58 & -1 & 0 & 4 & 7 & -3 & 9 & 53 & 179 & 17 & 16 & 7 & 13 & 8 & 1 \\
0 & -91 & -100 & -5 & 0 & -6 & -3 & 0 & -13 & -101 & -327 & -317 & -108 & -19 & -33 & -24 & -3 \\
-1 & 87 & 78 & 5 & 2 & 4 & -1 & -2 & 17 & 97 & 279 & 289 & 90 & 11 & 21 & 30 & 3 \\
-1 & -84 & -23 & -6 & -9 & -1 & -2 & -3 & -11 & -38 & -93 & -255 & -75 & -13 & -14 & -19 & -1
\end{pmatrix}
\]

Case 5. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6}, \frac{5}{6}), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-20 & -20 & -2 & -2 & -1 & 0 & 0 & -1 & -3 & -3 & -1 \\
15 & 89 & 0 & 1 & 0 & 4 & 1 & 1 & 5 & 14 & 4 & 2 \\
-139 & -153 & -3 & -1 & -5 & -6 & -3 & -2 & -9 & -25 & -24 & -9 \\
133 & 119 & -1 & 1 & 5 & 4 & 3 & 2 & 8 & 21 & 22 & 8 \\
-109 & -35 & -3 & -2 & -5 & -1 & -1 & -3 & -7 & -17 & -6
\end{pmatrix}
\]

Case 6. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{3}{5}), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-6 & -6 & -4 & -1 & -1 & -2 & -11 & -38 & -38 & -11 & -2 & -1 & 0 \\
-1 & 27 & 2 & 19 & -1 & 4 & 0 & 7 & 16 & 19 & 179 & 53 & 9 & 5 & 1 \\
-49 & -47 & -37 & -34 & -6 & -6 & -1 & -21 & -117 & -355 & -327 & -101 & -15 & -3 & -3 \\
35 & 37 & 26 & 29 & 4 & 4 & 1 & 11 & 79 & 251 & 279 & 95 & 17 & -1 & 3 \\
-39 & -11 & -27 & -10 & -6 & -1 & -7 & -13 & -75 & -255 & -95 & -38 & -11 & -2 & -1
\end{pmatrix}
\]

Case 7. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{5}{6}, \frac{7}{8}), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-8 & -8 & -1 & -1 & 0 & 0 & -1 & -3 & -3 & -1 \\
5 & 35 & 0 & 4 & 1 & 1 & 5 & 14 & 4 & 2 \\
-45 & -59 & -4 & -6 & -3 & -2 & -9 & -25 & -21 & -8 \\
59 & 45 & 6 & 4 & 3 & 2 & 8 & 21 & 25 & 9 \\
-43 & -13 & -5 & -1 & -1 & -3 & -7 & -17 & -6
\end{pmatrix}
\]

Case 8. \( \alpha = (0, 0, 0, 0, 0), \beta = (\frac{1}{2}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-10 & -10 & -1 & 0 & 0 & -1 & -3 & -7 & -7 & -3 & -1 \\
15 & 43 & 1 & 4 & 1 & 1 & 5 & 14 & 32 & 14 & 7 & 3 \\
-53 & -71 & -4 & -6 & -3 & -2 & -9 & -25 & -56 & -45 & -21 & -8 \\
71 & 53 & 6 & 4 & 3 & 2 & 8 & 21 & 45 & 56 & 25 & 9 \\
-43 & -15 & -4 & -1 & -1 & -3 & -7 & -14 & -32 & -14 & -5
\end{pmatrix}
\]
Case 9. \( \alpha = (0, 0, 0, 0, 0), \beta = \left( \frac{1}{2}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \right), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-12 & -12 & -1 & -1 & -3 & -7 & -7 & -3 & -1 \\
5 & 51 & 0 & 4 & 1 & 1 & 5 & 14 & 32 & 7 & 4 & 2 \\
-49 & -83 & -3 & -6 & -3 & -2 & -9 & -25 & -56 & -38 & -18 & -7 \\
95 & 67 & 7 & 4 & 3 & 2 & 8 & 21 & 45 & 63 & 28 & 10 \\
-63 & -17 & -5 & -1 & -1 & -3 & -7 & -14 & -39 & -17 & -6
\end{pmatrix}
\]

Case 10. \( \alpha = (0, 0, 0, \frac{1}{4}, \frac{3}{4}), \beta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \) and \( B \) has infinite order.

\[
M = \begin{pmatrix}
-1 & -1 & -1 & -5 & -1 & -40 & -8944 & -8944 & -40 & -1 & -23 & -54 & -97 & -97 & -54 & -23 \\
-3 & -3 & 2 & -17 & 19 & 119 & 20877 & 14600 & -60 & 36 & 68 & 139 & 237 & -257 & -119 & -38 \\
-4 & -9 & -2 & -24 & -22 & -141 & -56171 & -68806 & -418 & -27 & -56 & -148 & -249 & -919 & -538 & -226 \\
-3 & -5 & 2 & -19 & 20 & 138 & 6198 & 64737 & -139 & 15 & 65 & 160 & 240 & -130 & -230 & -105 \\
-1 & -6 & -1 & -7 & -40 & -100 & -50376 & -56653 & -279 & -23 & -54 & -97 & -131 & -625 & -355 & -160
\end{pmatrix}
\]

Case 11. \( \alpha = (0, 0, 0, \frac{1}{2}, \frac{1}{2}), \beta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \) and \( B \) has infinite order.

\[
M = \begin{pmatrix}
-1 & 0 & -3 & -61 & -223 & -465 & -715 & -715 & -465 & -223 & -61 & -3 & -35 & -1 & -1 \\
-3 & 11 & 241 & 831 & 1637 & 2395 & -1935 & -1145 & -427 & -21 & 49 & -121 & -3 & 3 \\
-4 & -2 & 0 & -416 & -1320 & -2424 & -3368 & -7586 & -5190 & -2706 & -854 & -42 & -172 & -10 & -4 \\
-3 & 2 & 21 & 427 & 1145 & 1935 & 2581 & -1637 & -831 & -241 & -11 & -21 & -137 & -3 & 3 \\
-1 & -1 & -61 & -223 & -465 & -715 & -925 & -5255 & -3497 & -1723 & -485 & -23 & -51 & -7 & -1
\end{pmatrix}
\]

Case 12. \( \alpha = (0, 0, 0, \frac{1}{6}, \frac{5}{6}), \beta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \) and \( B \) has finite order.

\[
M = \begin{pmatrix}
-1 & -1 & -27 & -64 & -99 & -121 & -132 & -121 & -99 & -64 & -27 & -18 & -18 & -1 & -1 \\
-1 & 3 & 16 & 107 & 229 & 332 & 385 & 407 & -119 & -110 & -77 & -29 & 10 & 6 & 71 & 25 \\
-5 & -4 & -32 & -173 & -341 & -464 & -515 & -539 & -673 & -625 & -550 & -403 & -211 & -116 & -101 & -46 \\
2 & 3 & 44 & 157 & 275 & 352 & 383 & 409 & 275 & 273 & 266 & 213 & 119 & 65 & 80 & 30 \\
-5 & -1 & -27 & -64 & -99 & -121 & -132 & -145 & -671 & -627 & -530 & -357 & -161 & -107 & -42 & -18
\end{pmatrix}
\]

5. Open cases

For the remaining cases listed in Table 2, our ping pong approach did not succeed. As in [1] we checked that this particular ping pong setup must fail for any choice of a cone \( F \).

Indeed, the closure of our table half \( X \) must always contain certain limit directions. The requirement that \( X \) is a union of a convex cone and its negative and that it is closed under ping pong will then force \( X \) to become the whole vector space in each of the above cases.

It might still be that a more complicated ping pong, perhaps even one based on convex cones, could work for these cases. But we suspect that many, if not all, of the remaining cases are, in fact, arithmetic.

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