Riemann-Hilbert problem of the three-component coupled Sasa-Satsuma equation and its multi-soliton solutions

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Abstract

In this work, the inverse scattering transform of the three-component coupled Sasa-Satsuma equation is investigated via the Riemann-Hilbert method. Firstly we consider a Lax pair associated with a $7 \times 7$ matrix spectral problem for the equation. Then we present the spectral analysis of the Lax pair, from which a kind of Riemann-Hilbert problem is formulated. Moreover, $N$-soliton solutions to the equation are constructed through a particular Riemann-Hilbert problem with vanishing scattering coefficients. Finally, the dynamics of the soliton solutions are discussed with some graphics.

Key words: The three-component coupled Sasa-Satsuma equation; Riemann-Hilbert problem; Multi-soliton solutions.

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1 Introduction

It is well known that much physical phenomena can be proved or obtained by the soliton solutions of nonlinear evolutions equations (NLEEs). As a consequence, the investigation of soliton solutions for NLEEs equations has become more and more attractive. A variety of approaches have been proposed over these years for seeking soliton solutions [1]-[5]. Among those methods, the Riemann-Hilbert (RH) approach is one of the most effective methods to seek soliton solutions for integrable systems. The main process of this method is to seek a corresponding Riemann-Hilbert problem (RHP) on the spectral analysis of integrable systems. In 1980s, Beals and his co-workers [6] investigated direct and inverse...
scattering for Ablowitz-Kaup-Newell-Segur systems on the line with integrable matrix-valued potentials and, derived partial characterization of the scattering data. These works are very familiar with the various boundary conditions for the Jost function, under the condition of the location of the spectral parameter. More importantly, it is very helpful for us to construct the RHP of integrable systems. Recently, the RH approach is also extended to consider initial boundary value problems and asymptotics of integrable equations \cite{7,15}.

The standard nonlinear Schrödinger (NLS) equation has been paid much attention due to its widespread applications in optics, Bose-Einstein condensates, hydrodynamics, plasma physics, molecular biology and even finance. To well describe other important types of nonlinear physical phenomena in a similar way, it is necessary to go beyond the standard NLS description. One prime research is to add higher-order terms and/or dissipative terms to the NLS equation to accurately model extreme wave events in some nonlinear wave systems such as micro-structured optical fibres and fibre lasers. Another important development focuses on the investigation of coupled-wave systems, as many physical systems comprise interacting wave components of distinct modes, frequencies or polarizations. Recently, the coupled NLS equations have become a topic of intense research, since the components are usually more than one practically for many physical phenomena. Therefore, the chief aim of this work is to construct the RHP and soliton solutions for the following three-coupled higher-order NLS equation, whose form reads \cite{29}

\[
\begin{align*}
iq_1 + \frac{1}{2}q_{1XX} + q_1 \sum_{N=1}^{3} |q_N|^2 + i \left[ q_{1XXX} + 6q_{1X} \sum_{N=1}^{3} |q_N|^2 \right] = 0, \\
+3q_1 \left( \sum_{N=1}^{3} |q_N|^2 \right) = 0,
\end{align*}
\]

\[
\begin{align*}
+3q_2 \left( \sum_{N=1}^{3} |q_N|^2 \right) = 0,
\end{align*}
\]

\[
\begin{align*}
+3q_3 \left( \sum_{N=1}^{3} |q_N|^2 \right) = 0,
\end{align*}
\]

where \( q_1 = q_1(X,T), \) \( q_2 = q_2(X,T) \) and \( q_3 = q_3(X,T) \) are three complex functions of \((X,T)\). As mentioned in Ref.\cite{29}, in order to investigate Eq.\( \text{(1.1)} \) conveniently, it is very necessary to rewrite \( \text{(1.1)} \) in the three-component coupled Sasa-Satsuma equation

\[
\begin{align*}
\begin{cases}
 u_{1,t} + u_{1,xxx} + 6 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right) u_{1,x} + 3u_1 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right)_x = 0, \\
u_{2,t} + u_{2,xxx} + 6 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right) u_{2,x} + 3u_2 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right)_x = 0, \\
u_{3,t} + u_{3,xxx} + 6 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right) u_{3,x} + 3u_3 \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right)_x = 0,
\end{cases}
\end{align*}
\]

(1.2)

2
by making use of the following three transformations (i.e., gauge, Galilean and scale transformations)

\[
\begin{align*}
\begin{aligned}
    u_1(x, t) &= q_1(X, T) \exp \left\{ -\frac{i}{6} \left( X - \frac{T}{18} \right) \right\}, \\
    u_2(x, t) &= q_2(X, T) \exp \left\{ -\frac{i}{6} \left( X - \frac{T}{18} \right) \right\}, \\
    u_3(x, t) &= q_3(X, T) \exp \left\{ -\frac{i}{6} \left( X - \frac{T}{18} \right) \right\},
\end{aligned}
\end{align*}
\]

(1.3)

As we all know that the RH approach is an effective way to construct soliton solutions [16]-[28]. Nonetheless, since the Eq. (1.2) involves a 7 × 7 matrix spectral problem, the RHP for the Eq. (1.2) is rather hard to deal with. The research in this direction, to our best knowledge, has not been considered so far. The chief purpose of the present article is to discuss the RHP and soliton solutions of the Eq. (1.2) by utilizing the RH approach.

The layout of this paper is arranged as follows. In section II, we present the spectral analysis of the Eq. (1.2), from which a kind of RHP is established. In section III, by considering a specific RHP, we obtain multi-soliton solutions of the Eq. (1.2). Besides, some figures are presented to understand the dynamics of these solutions. The last section summarizes the results of this work.

2 Riemann-Hilbert problem

The Eq. (1.2) admits the following Lax pair [29]

\[
\begin{align*}
\begin{aligned}
    \Phi_x &= U \Psi, \\
    \Phi_t &= V \Psi,
\end{aligned}
\end{align*}
\]

(2.1)

with

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & u_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{u}_1 \\
0 & 0 & 0 & 0 & 0 & 0 & u_2 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{u}_2 \\
0 & 0 & 0 & 0 & 0 & 0 & u_3 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{u}_3 \\
-\bar{u}_1 & -u_1 & -\bar{u}_2 & -u_2 & -\bar{u}_3 & -u_3 & 0
\end{pmatrix},
\]

(2.2)

where \(\sigma_3 = \text{diag}(1, 1, 1, 1, 1, 1, -1)\), and \(\Phi = \Phi(x, t, \lambda)\) is a column vector function of the spectral parameter \(\lambda\). Additionally, the compatibility condition of the Lax pair (2.1) yields the Eq. (1.2).

For convenience, we consider a new matrix spectral function \(\Psi = \Psi(x, t, \lambda)\) given by

\[
\Phi(x, t, \lambda) = \Psi(x, t, \lambda) e^{i(\lambda x + 4\lambda^3 t)} \sigma_3.
\]

(2.3)
Then the spectral problem (2.1) can be rewritten as
\[
\begin{cases}
\Psi_x + i\lambda[\Psi, \sigma_3] = Q\Psi, \\
\Psi_t + 4i\lambda^3[\Psi, \sigma_3] = \widetilde{Q}\Psi,
\end{cases}
\] (2.4)
with
\[
\widetilde{Q} = 4\lambda^2Q + 2i\lambda (Q^2 + Q_x) \sigma_3 + Q_xQ - QQ_x - Q_{xx} + 2Q^3.
\] (2.5)

Throughout this article, \( \mathbb{C}^+ = \{ z \in \mathbb{C} | \Im(z) > 0 \} \) and \( \mathbb{C}^- = \{ z \in \mathbb{C} | \Im(z) < 0 \} \) represent the upper half-plane and the lower half-plane, respectively. Let us next consider the two matrix functions
\[
\Psi_{\pm} = [(\Psi_{\pm})_1, (\Psi_{\pm})_2, (\Psi_{\pm})_3, (\Psi_{\pm})_4, (\Psi_{\pm})_5, (\Psi_{\pm})_6, (\Psi_{\pm})_7],
\] (2.6)
under the following asymptotic conditions
\[
\Psi_+ \rightarrow \mathbb{I}, \quad x \rightarrow +\infty \quad \text{and} \quad \Psi_- \rightarrow \mathbb{I}, \quad x \rightarrow -\infty.
\] (2.7)

Here each \((\Psi_{\pm})_m\) represents the mth column of \(\Psi_{\pm}\), respectively. The matrix \(\mathbb{I}\) is the 7 \(\times\) 7 unit matrix, and the subscripts of \(\Psi\) refer to which end of the x-axis the boundary conditions are required. Utilizing the above boundary condition (2.7), the two matrix solutions \(\Psi_{\pm}\) can be given by the following two Volterra integral equations
\[
\begin{cases}
\Psi_+(x, \lambda) = \mathbb{I} + \int_{-\infty}^{x} e^{i\lambda\sigma_3(x-y)}Q(y)\Psi_+(y, \lambda)e^{i\lambda\sigma_3(y-x)}dy, \\
\Psi_-(x, \lambda) = \mathbb{I} + \int_{-\infty}^{x} e^{i\lambda\sigma_3(x-y)}Q(y)\Psi_-(y, \lambda)e^{i\lambda\sigma_3(y-x)}dy,
\end{cases}
\] (2.8)
respectively. It is not hard to check that
\[
(\Psi_+)_1, (\Psi_+)_2, (\Psi_+)_3, (\Psi_+)_4, (\Psi_+)_5, (\Psi_+)_6, (\Psi_+)_7,
\] (2.9)
can be analytically extendible to \(\mathbb{C}^+\). Besides,
\[
(\Psi_-)_1, (\Psi_-)_2, (\Psi_-)_3, (\Psi_-)_4, (\Psi_-)_5, (\Psi_-)_6, (\Psi_-)_7,
\] (2.10)
can be analytically extendible to \(\mathbb{C}^-\).

In the following, we study the analytic properties of \(\Psi_{\pm}\). Actually, noticing \(\text{tr}(Q) = \text{tr}(\widetilde{Q}) = 0\), we have \(\text{det}(\Psi_{\pm}) = 1, \quad \lambda \in \mathbb{R}\). Besides, \(\Psi_{\pm}\) can be linearly related by
\[
\Psi_-e^{i\lambda\sigma_3x} = \Psi_+e^{i\lambda\sigma_3x}\Omega(\lambda), \quad \lambda \in \mathbb{R},
\] (2.11)
where \(\Omega(\lambda) = (\Omega_{ij})_{7 \times 7}, \lambda \in \mathbb{R}\). Noticing that \(\text{det}(\Omega(\lambda)) = 1\) since \(\text{det}(\Psi_{\pm}) = 1\). From the analytic property of \(\Psi_-\), we know that \(\Omega_{77}\) can be analytically extended to \(\mathbb{C}^+\), otherwise \(\Omega_{kj}(1 \leq k, j \leq 6)\) can be analytically extended to \(\mathbb{C}^-\). Generally, \(\Omega_{k7}, \Omega_{7j}\) are not extended off the real \(\lambda\)-axis.

By considering the analytic properties of \(\Psi_{\pm}\), we can obtain
\[
P_1 = [(\Psi_+)_1, (\Psi_+)_2, (\Psi_+)_3, (\Psi_+)_4, (\Psi_+)_5, (\Psi_+)_6, (\Psi_-)_7],
\] (2.12)
which is analytic in $\mu \in \mathbb{C}^+$. In addition, we have the following asymptotic behavior of $P_1$ (i.e., $P_1(\lambda) \to \mathbb{I}$, $\lambda \to \infty$.) In order to further derive RHP for the Eq. (1.12), we must construct a analytic matrix $P_2$ in $\mathbb{C}^-$. For this purpose, we first introduce the adjoint equation of the first expression in (2.1)

$$K_x + i\lambda [K, \sigma_3] = QK.$$  \hfill (2.13)

It is not hard to know that $\Psi^\pm_-$ meet the above expression (2.13). Then let us introduce

$$\Psi_+^{-1} = \begin{pmatrix} (\Psi_+^{-1})^1 \\ (\Psi_+^{-1})^2 \\ (\Psi_+^{-1})^3 \\ (\Psi_+^{-1})^4 \\ (\Psi_+^{-1})^5 \\ (\Psi_+^{-1})^6 \\ (\Psi_+^{-1})^7 \end{pmatrix}, \quad \Psi_-^{-1} = \begin{pmatrix} (\Psi_-^{-1})^1 \\ (\Psi_-^{-1})^2 \\ (\Psi_-^{-1})^3 \\ (\Psi_-^{-1})^4 \\ (\Psi_-^{-1})^5 \\ (\Psi_-^{-1})^6 \\ (\Psi_-^{-1})^7 \end{pmatrix}. \hfill (2.14)$$

Here each $(\Psi_\pm)^m$ represents the mth row of $\Psi_\pm^-$, respectively. Hence we can find that

$$\begin{cases} [(\Psi_+^{-1})^1, (\Psi_+^{-1})^2, (\Psi_+^{-1})^3, (\Psi_+^{-1})^4, (\Psi_+^{-1})^5, (\Psi_+^{-1})^6, (\Psi_+^{-1})^7], \\ [(\Psi_-^{-1})^1, (\Psi_-^{-1})^2, (\Psi_-^{-1})^3, (\Psi_-^{-1})^4, (\Psi_-^{-1})^5, (\Psi_-^{-1})^6, (\Psi_-^{-1})^7], \end{cases} \hfill (2.15)$$

can be analytically extended to $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. As a result, we can give a matrix function

$$\begin{pmatrix} (\Psi_+^{-1})^1 \\ (\Psi_+^{-1})^2 \\ (\Psi_+^{-1})^3 \\ (\Psi_+^{-1})^4 \\ (\Psi_+^{-1})^5 \\ (\Psi_+^{-1})^6 \\ (\Psi_+^{-1})^7 \end{pmatrix}, \hfill (2.16)$$

which is analytic in $\lambda \in \mathbb{C}^-$, and the asymptotic behavior of $P_2$ yields $P_2(\lambda) \to \mathbb{I}$, $\lambda \to \infty$. Moreover, we can easily find that $e^{-i\lambda \sigma_3 x} \Psi_+^{-1}$ can be linearly related by a scattering matrix $\Theta(\lambda) = \Omega^{-1}(\lambda)$

$$e^{-i\lambda \sigma_3 x} \Psi_+^{-1} = \Theta(\lambda) e^{-i\lambda \sigma_3 x} \Psi_+^{-1}, \quad \lambda \in \mathbb{R}. \hfill (2.17)$$

where

$$\Theta(\lambda) = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} \\ \Theta_{21} & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25} & \Theta_{26} & \Theta_{27} \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} & \Theta_{37} \\ \Theta_{41} & \Theta_{42} & \Theta_{43} & \Theta_{44} & \Theta_{45} & \Theta_{46} & \Theta_{47} \\ \Theta_{51} & \Theta_{52} & \Theta_{53} & \Theta_{54} & \Theta_{55} & \Theta_{56} & \Theta_{57} \\ \Theta_{61} & \Theta_{62} & \Theta_{63} & \Theta_{64} & \Theta_{65} & \Theta_{66} & \Theta_{67} \\ \Theta_{71} & \Theta_{72} & \Theta_{73} & \Theta_{74} & \Theta_{75} & \Theta_{76} & \Theta_{77} \end{pmatrix}, \hfill (2.18)$$

Similar to the above scattering coefficients $\Omega_{kj}$, we can know that $\Theta_{77}$ allows an analytical extension to $\mathbb{C}^-$. Otherwise $\Theta_{kj}(1 \leq k, j \leq 6)$ are analytically extendible to $\mathbb{C}^+$. Besides, $\Theta_{k7}, \tau_{7j}(1 \leq k, j \leq 6)$ can be defined on the real $\lambda$-axis.
Summarizing the above analysis, we have derived two matrix functions $P_1$ and $P_2$, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Next we notice that the limit of $P_2$ from the right-hand side of the real $\lambda$-line as $P^-$, and the limit of $P_1$ from the left-hand side the real $\lambda$-line as $P^+$, Hence we can get an RHP for the Eq. (2.19)

$$G(x, \lambda) = P^-(x, \lambda)P^+(x, \lambda), \quad \lambda \in \mathbb{R},$$

(2.19)

where

$$G = \begin{pmatrix} \mathbf{I}_{6\times6} & \Omega_{7j} e^{2i\lambda x} \\ \Theta_{7j} e^{2i\lambda x} & 1 \end{pmatrix}, \quad j = 1, 2, \ldots, 6,$$

(2.20)

and the canonical normalization condition for above RHP (2.19) yields

$$\begin{cases} P_1(x, \lambda) \to \mathbb{I}, & \lambda \in \mathbb{C}^+ \to \infty, \\ P_2(x, \lambda) \to \mathbb{I}, & \lambda \in \mathbb{C}^- \to \infty. \end{cases}$$

(2.21)

Next we suppose that the RHP (2.19) is irregular. Here it indicates both $\det(P_1)$ and $\det(P_2)$ have certain zero in the analytic domains. From the definitions of $P_1$ and $P_2$, we have

$$\det(P_1(\lambda)) = \Omega_{77}(\lambda) \quad \text{and} \quad \det(P_2(\lambda)) = \Theta_{77}(\lambda).$$

(2.22)

To explain these zero well, we introduce a symmetry relation $Q^\dagger = -Q$, where "$^\dagger$" means the Hermitian of a matrix. Hence from the relation (2.13), we have

$$\Psi_{\pm}^\dagger(\bar{\lambda}) = \Psi_{\mp}^{-1}(\lambda).$$

(2.23)

The it follows from (2.22) that

$$\Omega^\dagger(\bar{\lambda}) = \Omega^{-1}(-\lambda) \rightarrow \begin{cases} \Omega_{17}(\lambda) = \Theta_{71}(\lambda), & \Omega_{27}(\lambda) = \Theta_{72}(\lambda), \quad \lambda \in \mathbb{R}, \\ \Omega_{37}(\lambda) = \Theta_{73}(\lambda), & \Omega_{47}(\lambda) = \Theta_{74}(\lambda), \quad \lambda \in \mathbb{R}, \\ \Omega_{57}(\lambda) = \Theta_{75}(\lambda), & \Omega_{67}(\lambda) = \Theta_{76}(\lambda), \quad \lambda \in \mathbb{R}, \\ \Omega_{77}(\lambda) = \tilde{\Theta}_{55}(\bar{\lambda}), \quad \lambda \in \mathbb{C}^+. \end{cases}$$

(2.24)

In addition, the following expression also holds

$$P_{1}^\dagger(\bar{\lambda}) = P_2(\lambda), \quad \lambda \in \mathbb{C}^-.$$

(2.25)

To further settle the RHP (2.19), we must introduce another symmetry relation $Q = \sigma Q \sigma$, where

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

(2.26)

It follows from (2.26) that

$$\sigma \Psi_{\pm}(\bar{\lambda}) \sigma = \Psi_{\pm}(\lambda), \rightarrow \sigma \Omega(\bar{\lambda}) \sigma = \Omega(\lambda).$$

(2.27)
Apparently, the above relation means that
\[
\begin{align*}
\Omega_{17}(\lambda) &= \tilde{\Omega}_{27}(-\lambda), \quad \Omega_{37}(\lambda) = \tilde{\Omega}_{47}(-\lambda), \quad \Omega_{57}(\lambda) = \tilde{\Omega}_{67}(-\lambda), \quad \lambda \in \mathbb{R}, \\
\Omega_{77}(\lambda) &= \tilde{\Omega}_{77}(-\lambda), \quad \lambda \in \mathbb{C}^+.
\end{align*}
\tag{2.28}
\]
In addition, from the relation (2.27), we have
\[
\sigma \tilde{P}_1(-\lambda) \sigma = P_1(\lambda), \quad \lambda \in \mathbb{C}^+.
\tag{2.29}
\]
Thus, we know that if \(\lambda_j\) is a zero of \(\det(P_1)\), then \(\tilde{\lambda}_j = \lambda_j\) is a zero of \(\det(P_2)\). Furthermore, in terms of (2.28), we find that \(-\lambda_j\) is also a zero of \(\det(P_1)\). Next we consider the zeros of \(\det(P_1)\) in the following two cases.

**Case (I):** we suppose that \(\det(P_1)\) admits a total number of \(2N\) zeros \(\lambda_j(1 \leq j \leq 2N)\) satisfying \(\lambda_{N+j} = -\tilde{\lambda}_j(1 \leq j \leq N)\), which are all in \(\mathbb{C}^+\). Likewise, \(\det(P_2)\) admits \(2N\) zeros \(\lambda_j(1 \leq j \leq 2N)\) satisfy \(\lambda_j = -\tilde{\lambda}_j\), which all lie in \(\mathbb{C}^-\).

**Case (II):** \(\det(P_1)\) admits also \(N\) simple zeros \(\lambda_j(1 \leq j \leq N)\) in \(\mathbb{C}^+\), where all \(\lambda_j\) are pure imaginary. \(\det(P_2)\) admits \(N\) zeros \(\tilde{\lambda}_j\) in \(\mathbb{C}^-\), where \(\lambda_j = \tilde{\lambda}_j\). Under these assumptions, \(\ker(P_1(\lambda_j)\) and \(P_2(\lambda_j)\) are one-dimensional, and they are spanned by
\[
P_1(\lambda_j)v_j = 0, \quad \hat{v}_j P_2(\tilde{\lambda}_j) = 0, \quad 1 \leq j \leq 2N,
\tag{2.30}
\]
respectively. For the first type of zeros, from the relations (2.25) and (2.30), we have \(\hat{v}_j = v_j^\dagger, 1 \leq j \leq 2N\). Similarly, form (2.29), we can obtain the following relation \(v_j = \sigma \tilde{v}_j, N + 1 \leq j \leq 2N\). Next we should construct the vectors \(v_j(1 \leq j \leq N)\). To this end, we set the x-derivative of \(P_1(\lambda_j)v_j = 0\). The using the first expression in (2.4), we can obtain \(v_j = e^{i\lambda_j \sigma_3x}v_j, 1 \leq j \leq N\), in which \(v_{j0}\) is independent of \(x\). Thus utilizing the above results, the two vectors \(\hat{v}_j\) and \(v_j\) can be obtained explicitly. To construct the multi-soliton solutions for the Eq. (1.2), we should choose \(G = 1\). Therefore, the solution for the particular RHP reads
\[
\begin{align*}
P_1(\lambda) &= I - \sum_{k=1}^{2N} \sum_{j=1}^{2N} \frac{v_k v_j (M^{-1})_{kj}}{\lambda - \lambda_j}, \\
P_2(\lambda) &= I + \sum_{k=1}^{2N} \sum_{j=1}^{2N} \frac{v_k v_j (M^{-1})_{kj}}{\lambda - \lambda_k},
\end{align*}
\tag{2.31}
\]
where \(M = (M_{kj})_{2N \times 2N}\) is a matrix whose entries yield \(M_{kj} = \frac{v_k v_j}{\lambda_j - \lambda_k}\). For second type of zeros, the vectors \(v_j, \hat{v}_j(1 \leq j \leq N)\) are defined by
\[
\hat{v}_j = v_j^\dagger, \quad v_j = e^{i\lambda_j \sigma_3x}v_j, 0.
\tag{2.32}
\]
Utilizing these vectors, the RHP (2.19) in this case can be also solved
\[
\begin{align*}
P_1(\lambda) &= I - \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{v_k v_j (M^{-1})_{kj}}{\lambda - \lambda_j}, \\
P_2(\lambda) &= I + \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{v_k v_j (M^{-1})_{kj}}{\lambda - \lambda_k},
\end{align*}
\tag{2.33}
\]
where $M = (M_{kj})_{N \times N}$ is a matrix whose entries yield $M_{kj} = \frac{i k_j}{\gamma_j - \lambda_k}$. In terms of $P_1$ in (2.31), we can recover the potentials $u_1, u_2, u_3$. Actually, we can expand $P_1(\lambda)$ as

$$P_1(\lambda) = I + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + \left( \frac{1}{\lambda^3} \right), \quad \lambda \to \infty. \quad (2.34)$$

Then inserting (2.33) into the first expression in (2.4), and collecting $O(1)$ terms yields

$$Q = i \begin{bmatrix} P_1^{(1)}, \sigma_3 \end{bmatrix},$$

which means that $u_1, u_2, u_3$ can be expressed as

$$u_1 = -2i \left( P_1^{(1)} \right)_{17}, \quad u_2 = -2i \left( P_1^{(1)} \right)_{37}, \quad u_3 = -2i \left( P_1^{(1)} \right)_{57}, \quad (2.35)$$

where $(P_1^{(1)})_{kj}$ is the $(k,j)$-entry of the function $P_1^{(1)}$. Here from (2.31) and (2.33), the matrix functions can be rewritten as

$$\begin{align*}
P_1^{(1)} &= -\sum_{k=1}^{2N} \sum_{j=1}^{2N} u_k v_j (M^{-1})_{kj}, \\
P_1^{(2)} &= -\sum_{k=1}^{N} \sum_{j=1}^{N} u_k v_j (M^{-1})_{kj}.
\end{align*} \quad (2.36)$$

### 3 Multi-soliton solutions

To obtain soliton solutions for the Eq. (1.2), we must consider the $t$-evolutions of the scattering data. From the second expression in (2.4) and (2.11), we get $\Omega_t = 4i\lambda^3[\sigma_3, \Omega]$, which gives the following results

$$\Omega_{17,t} = 8i\lambda^3 \Omega_{15}, \quad \Omega_{37,t} = 8i\lambda^3 \Omega_{37}, \quad \Omega_{57,t} = 8i\lambda^3 \Omega_{57}. \quad (3.1)$$

Additionally, utilizing the second expression in (2.4), we can obtain that $v_{j,t} = 4i\lambda^3 j \sigma_3 v_j$. Thus, for the first type of zeros, we have

$$v_j = \begin{cases} 
\theta_3^j \sigma_3 v_{j,0}, & 1 \leq j \leq N, \\
\sigma_3 \theta_3^{j-N} \sigma_3 v_{j-N,0}, & N + 1 \leq j \leq 2N,
\end{cases} \quad (3.2)$$

and

$$\hat{v}_j = \begin{cases} 
\eta_3^{j,0} e^{\theta_3^j}, & 1 \leq j \leq N, \\
\eta_3^{j-N,0} e^{\theta_3^{j-N}}, & N + 1 \leq j \leq 2N,
\end{cases} \quad (3.3)$$

where $\theta_j = i\lambda_j x + 4i\lambda_3^3 t$, and $v_{j,0}$ are constant vectors. Furthermore, for the second type of zeros, we have

$$v_j = e^{\theta_3^j \sigma_3} v_{j,0}, \quad \hat{v}_j = \eta_3^{j,0} e^{\theta_3^j \sigma_3}, \quad 1 \leq j \leq N, \quad (3.4)$$

where $\theta_j = i\lambda_j x + 4i\lambda_3^3 t$ with $\lambda_j$ being imaginary, and $v_{j,0}$ are constant vectors. For the fist type of zeros of $\det(P_1)$, we choose $v_{j,0} = (\alpha_j, \beta_j, \gamma_j, \mu_j, \rho_j, \delta_j, 1)^T$ to be complex constant
vectors. Then using Eqs. (2.35), (2.36), (3.2) and (3.3), the $N$-soliton solutions of the Eq. (1.2) can be obtained as follows

$$
\begin{align*}
\mathbf{u}_1 &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=1}^{N} \sum_{j=N+1}^{2N} \alpha_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj} \\
&\quad + 2i \sum_{k=N+1}^{2N} \sum_{j=1}^{N} \beta_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=N+1}^{2N} \sum_{j=N+1}^{2N} \beta_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj}, \\
\mathbf{u}_2 &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \gamma_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=1}^{N} \sum_{j=N+1}^{2N} \gamma_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj} \\
&\quad + 2i \sum_{k=N+1}^{2N} \sum_{j=1}^{N} \mu_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=N+1}^{2N} \sum_{j=N+1}^{2N} \mu_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj}, \\
\mathbf{u}_3 &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \rho_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=1}^{N} \sum_{j=N+1}^{2N} \rho_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj} \\
&\quad + 2i \sum_{k=N+1}^{2N} \sum_{j=1}^{N} \delta_k e^{\theta_k - \theta_j} (M^{-1})_{kj} + 2i \sum_{k=N+1}^{2N} \sum_{j=N+1}^{2N} \delta_k e^{\theta_k - \theta_j - N} (M^{-1})_{kj},
\end{align*}
$$

where $M = (M_{kj})_{2N \times 2N}$ is defined by

$$
M_{kj} = \begin{cases}
\Delta_1 e^{\theta_k + \theta_j} + e^{\theta_k + \theta_j} & 1 \leq k, j \leq N, \\
\Delta_2 e^{\theta_k + \theta_j - N} + e^{-\theta_k - \theta_j - N} & 1 \leq k \leq N, \quad N+1 \leq j \leq 2N, \\
\Delta_3 e^{\theta_j + \theta_k - N} + e^{-\theta_j - \theta_k - N} & 1 \leq j \leq N, \quad N+1 \leq k \leq 2N, \\
\Delta_4 e^{\theta_j - \theta_k - N} + e^{-\theta_j + \theta_k - N} & N+1 \leq k, j \leq 2N,
\end{cases}
$$

with

$$
\begin{align*}
\Delta_1 &= \bar{\alpha}_k \alpha_j + \bar{\beta}_k \beta_j + \bar{\gamma}_k \gamma_j + \bar{\mu}_k \mu_j + \bar{\rho}_k \rho_j + \bar{\delta}_k \delta_j, \\
\Delta_2 &= \alpha_k \bar{\beta}_j - N + \bar{\beta}_k \alpha_j - N + \bar{\gamma}_k \gamma_j - N + \bar{\mu}_k \mu_j - N + \bar{\rho}_k \rho_j - N + \bar{\delta}_k \delta_j, \\
\Delta_3 &= \alpha_j \bar{\beta}_k - N + \bar{\beta}_j \alpha_k - N + \bar{\gamma}_j \gamma_k - N + \bar{\mu}_j \mu_k - N + \bar{\rho}_j \rho_k - N + \bar{\delta}_j \delta_k - N, \\
\Delta_4 &= \alpha_j - N \alpha_k - N + \bar{\beta}_j - N \beta_k - N + \bar{\gamma}_j - N \gamma_k - N + \bar{\mu}_j - N \mu_k - N + \bar{\rho}_j - N \rho_k - N + \bar{\delta}_j - N \delta_k.
\end{align*}
$$

To show the one-soliton solution explicitly, we should choose the appropriate parameters in (3.5)-(3.7). In the following, we choose $\bar{\alpha}_1 = \beta_1$, $\gamma_1 = \mu_1$ and $\delta_1 = \rho_1$ in (3.5)-(3.7).
Then by taking \( \lambda_1 = \xi_1 + i\eta_1 \), a new breather solution for the Eq. (1.2) can be constructed

\[
\begin{align*}
    u_1 &= -\frac{2\sqrt{2}\alpha_1\xi_1\eta_1}{\sqrt{|\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2}} \xi_1 \cosh(X_1) \cos(Y_1) + \eta_1 \sinh(X_1) \sin(Y_1), \\
    u_2 &= -\frac{2\sqrt{2}\gamma_1\xi_1\eta_1}{\sqrt{|\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2}} \xi_1 \cosh(X_1) \cos(Y_1) + \eta_1 \sinh(X_1) \sin(Y_1), \\
    u_3 &= -\frac{2\sqrt{2}\rho_1\xi_1\eta_1}{\sqrt{|\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2}} \xi_1 \cosh(X_1) \cos(Y_1) + \eta_1 \sinh(X_1) \sin(Y_1),
\end{align*}
\]

(3.8)

where

\[
\begin{align*}
    X_1 &= -2\eta_1 \left( x + 4 \left( 3\xi_1^2 - \eta_1^2 \right) t \right) + \ln \sqrt{2 \left( |\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2 \right)}, \\
    Y_1 &= 2\xi_1 \left( x + 4 \left( \xi_1^2 - 3\eta_1^2 \right) t \right).
\end{align*}
\]

(3.9)

Fig.1 displays the dynamics of periodic breather-type solution (3.8).

We next discuss the case for \( N = 2 \) in Eqs. (3.5)-(3.7). As seen in Fig.2, the two solitons pass through each other, and their polarizations do not change. After collision when this right soliton passes to the left, its power has diminished dramatically. Particularly, Figs.2(b) and 2(c) also display that the components of \( u_2, u_3 \) even vanishes after collision. The another collision is displayed in Figs.2(d)-(2(f)). This collision is a little similar to that in Fig.2(a) and is another example of soliton interactions. The shapes of the solitons can be changed after collision. All the phenomena indicate that there is a lot of energy transfer has taken place between these two solitons during the collision. Due to the limit of length, the other corresponding figures are omitted here.

In the following, we assume that \( \det(P_1) \) admits \( N \) simple zeros \( \lambda_j \) in \( \mathbb{C}^+ \). Taking \( v_{j,0} = (\alpha_j, \bar{\alpha}_j, \gamma_j, \bar{\gamma}_j, \rho_j, \bar{\rho}_j, 1) \) to be complex vectors. Then by utilizing (2.35), (2.35), and (3.3), we can obtain another kind of \( N \)-soliton solutions as follows

\[
\begin{align*}
    u_1(x,t) &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_k e^{\theta_k - \theta_j} (M^{-1})_{kj}, \\
    u_2(x,t) &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \gamma_k e^{\theta_k - \theta_j} (M^{-1})_{kj}, \\
    u_3(x,t) &= 2i \sum_{k=1}^{N} \sum_{j=1}^{N} \rho_k e^{\theta_k - \theta_j} (M^{-1})_{kj},
\end{align*}
\]

(3.10)

where \( M = (M_{kj})_{N \times N} (1 \leq k, j \leq N) \) is defined by

\[
M_{kj} = \frac{(\alpha_k \bar{\alpha}_j + \bar{\alpha}_k \alpha_j + \gamma_k \bar{\gamma}_j + \bar{\gamma}_k \gamma_j + \rho_k \rho_j + \bar{\rho}_k \bar{\rho}_j) e^{\theta_k + \theta_j} + e^{-\theta_k - \theta_j}}{\lambda_j - \lambda_k}. \tag{3.11}
\]

Taking \( N = 1 \), the \( N \)-soliton solutions (3.10) yields a single soliton solution for the Eq.
and θ with N seeking the suitable parameters, which is useful for understanding the dynamical behaviors of the soliton solutions.

Next, taking $N = 2$, Eq. (3.10) can be reduced to a two-bell soliton solution of the Eq. (1.2) given by

$$
\begin{align*}
\begin{cases}
u_1(x, t) = 2i\alpha_1 \exp (\theta_1 - \bar{\theta}_1) (M^{-1})_{11} + 2i\alpha_1 \exp (\theta_1 - \bar{\theta}_2) (M^{-1})_{12} \\
+ 2i\alpha_2 \exp (\theta_2 - \bar{\theta}_1) (M^{-1})_{21} + 2i\alpha_2 \exp (\theta_2 - \bar{\theta}_2) (M^{-1})_{22}, \\
u_2(x, t) = 2i\gamma_1 \exp (\theta_1 - \bar{\theta}_1) (M^{-1})_{11} + 2i\gamma_1 \exp (\theta_1 - \bar{\theta}_2) (M^{-1})_{12} \\
+ 2i\gamma_2 \exp (\theta_2 - \bar{\theta}_1) (M^{-1})_{21} + 2i\gamma_2 \exp (\theta_2 - \bar{\theta}_2) (M^{-1})_{22}, \\
u_3(x, t) = 2i\rho_1 \exp (\theta_1 - \bar{\theta}_1) (M^{-1})_{11} + 2i\rho_1 \exp (\theta_1 - \bar{\theta}_2) (M^{-1})_{12} \\
+ 2i\rho_2 \exp (\theta_2 - \bar{\theta}_1) (M^{-1})_{21} + 2i\rho_2 \exp (\theta_2 - \bar{\theta}_2) (M^{-1})_{22},
\end{cases}
\end{align*}
$$

(3.13)

where $M = (T_{jm})_{2\times2}$ with

$$
\begin{align*}
T_{11} &= \frac{2 (|\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2) \exp (\theta_1 + \bar{\theta}_1) + \exp (-\theta_1 - \bar{\theta}_1)}{\lambda_1 - \lambda_1}, \\
T_{12} &= \nabla_1 \exp (\theta_1 + \bar{\theta}_2) + \exp (-\theta_1 - \bar{\theta}_2), \\
T_{21} &= \nabla_2 \exp (\theta_1 + \bar{\theta}_2) + \exp (-\theta_1 - \bar{\theta}_2), \\
T_{22} &= \frac{2 (|\alpha_1|^2 + |\gamma_1|^2 + |\rho_1|^2) \exp (\theta_2 + \bar{\theta}_2) + \exp (-\theta_2 - \bar{\theta}_2)}{\lambda_2 - \lambda_2}, \\
\nabla_1 &= (\bar{\alpha}_1\alpha_2 + \alpha_1\bar{\alpha}_2 + \bar{\gamma}_1\gamma_2 + \gamma_1\bar{\gamma}_2 + \bar{\rho}_1\rho_2 + \rho_1\bar{\rho}_2), \\
\nabla_2 &= (\alpha_1\alpha_2 + \alpha_1\bar{\alpha}_2 + \bar{\gamma}_1\gamma_2 + \gamma_1\bar{\gamma}_2 + \bar{\rho}_1\rho_2 + \rho_1\bar{\rho}_2),
\end{align*}
$$

(3.14)

and $\theta_m = i\lambda_m x + 4i\lambda_m^3 t (m = 1, 2)$. The two-bell soliton interactions given by Eq. (3.10) with $N = 2$ are shown in Fig. 4.
Figure 1. (Color online) Breather wave via solutions (3.8) \(|u_1|, |u_2|, |u_3|\) with parameters: 
\[
\alpha_1 = i/\sqrt{3}, \gamma_1 = \sqrt{2}i/\sqrt{3}, \rho_1 = \sqrt{2}i/\sqrt{3}, \lambda_1 = 0.5 + 0.5i.
\]

Figure 2. (Color online) Two-soliton solutions with parameters: (a,b,c): \(\lambda_1 = 0.4 + 0.5i, \lambda_2 = 0.7 + 0.8i, \alpha_1 = \gamma_1 = \mu_1 = \rho_1 = \beta_2 = \gamma_2 = \mu_2 = \rho_2 = 0, \beta_1 = \alpha_2 = \gamma_2 = 1\); (d,e,f): \(\lambda_1 = 0.5 + 0.5i, \lambda_2 = 0.4 + 0.6i, \alpha_1 = \beta_1 = \gamma_1 = \mu_1 = \rho_1 = \alpha_2 = 1, \gamma_2 = 2, \beta_2 = \mu_2 = \rho_2 = 0\).

Figure 3. (Color online) Bright-dark soliton wave via solutions (3.12) with parameters: \(\alpha_1 = 1, \gamma_1 = 2, \rho_1 = 3\). (a,b,c): \(\eta_1 = 1\). (d,e,f): \(\eta_1 = -1\).
This work is dedicated to the associated multi-soliton solutions of the three-component coupled Sasa-Satsuma equation (1.2). By using the Riemann-Hilbert method, we have obtained a class of multi-soliton solutions for the Eq. (1.2). Additionally, based on the soliton solution formulas, we find some interesting soliton solutions which contain breather-type solution, single-soliton solution etc. In order to help readers understand those soliton solutions better, the propagation behaviors have been shown by graphical simulations (i.e., Figs. 1-4). Particularly, the Eq. (1.2) we investigated in this work are fairly more general as they involve a $7 \times 7$ Lax pair. The celebrated Sasa-Satsuma equation, a crucial model in fiber optics, is its particular case. Another important reduction of the Eq. is the coupled Sasa-Satsuma equation. Consequently, the $N$-soliton solutions of the celebrated Sasa-Satsuma equation and the coupled Sasa-Satsuma equation [25] can be respectively obtained by reducing the $N$-soliton solutions of (1.2). More importantly, in certain physical situations, two or more wave packets of different carrier frequencies appear simultaneously, and their interactions can be governed by the coupled NLS equations. Examples include nonlinear light propagation in a birefringent optical fiber or a wavelength-division-multiplexed system [30, 31, 32], spinor Bose-Einstein condensates (BECs) [33, 34], the interaction of Bloch-wave packets in a periodic system [35], the evolution of two surface wave packets in deep water [36], etc. Since the coupled NLS equations arise in a wide variety of physical subjects such as nonlinear optics, water waves, BECs, etc, these results should prove useful to the investigations of those physical problems.

Finally, we remark that there are several methods to get exact solutions for nonlinear
evolution equations (NLEEs), such as the Darboux transformation [1, 37], the RH problem approach [3, 4], the Hirota method [5], the dressing method [38], the Wronskian technique [39], etc. Thus, it is very necessary to discuss whether the Eq. (1.2) can be solved by using these approaches? These will be left for future discussions. Recently, the Darboux transform method is used to derive the $N$-soliton solutions of the celebrated Sasa-Satsuma equation. Comparing with the soliton solution formulae obtained in this work and those constructed by Darboux transformation in Ref. [40], it is very clear that Eq. (3.10) is much simpler. In Ref. [11], the bilinear method is used to derive the soliton solutions of the integrable Sasa-Satsuma equation. Unfortunately, the construction of multisoliton solutions to this equation presents difficulties due to its complicated bilinearization. So the authors discuss briefly some previous attempts and then present the correct bilinearization based on the interpretation of the SS equation as a reduction of the three-component KP hierarchy. As a result, we find that the RH method can provide an effective and powerful mathematical tool to derive exact solutions of NLEEs, which should be suitable to analyze other models in mathematical physics and engineering.

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References

[1] Matveev VB, Salle MA. Darboux Transformation and Solitons. Springer Berlin (1991).

[2] Bluman GM, Kumei S. Symmetries and differential equations. Graduate Texts in Math, 81, Springer-Verlag, New York (1989).

[3] Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. Theory of Solitons: The Inverse Scattering Method. Consultants Bureau New York (1984).

[4] Ablowitz MJ, Clarkson PA, Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press Cambridge (1991).

[5] Hirota R. Direct Methods in Soliton Theory. Berlin: Springer (2004).

[6] Beals R, Deift P, and Tomei C. Direct and Inverse Scattering on the Line. Math. Surv. Mono. 28, AMS, Providence, RI (1988).

[7] Tian SF. Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method. J Differ Equ 2017; 262: 506-558.

[8] Tian SF. Initial-boundary value problems for the coupled modified Korteweg-de Vries equation on the interval. Commun Pure Appl Anal2018; 17: 923-957.
[9] Tian SF. The mixed coupled nonlinear Schrödinger equation on the half-line via the Fokas method. Proc R Soc Lond A 2016; 72: 20160588.

[10] Fokas AS, Lenells J. The unified method: I Nonlinearizable problems on the half-line. J Phys A 2012; 45: 195201.

[11] Yan ZY. An initial-boundary value problem for the integrable spin-1 Gross- Pitaevskii equations with a $4 \times 4$ Lax pair on the half-line. Chaos 2017; 27: 053117.

[12] Wang DS, Wang XL. Long-time asymptotics and the bright N-soliton solutions of the Kundu-Eckhaus equation via the Riemann-Hilbert approach. Nonlinear Anal 2018; 41: 334-361.

[13] Xu J, Fan EG. The unified transform method for the Sasa-Satsuma equation on the half-line. Proc R Soc A 2013; 469: 20130068.

[14] Xu J, Fan EG. Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: without solitons. J Differ Equ 2015; 259: 1098-1148.

[15] Lenells J, Fokas AS. The unified method: II. NLS on the half-line t-periodic boundary conditions. J Phys A: Math Theor 2012; 45: 195202.

[16] Ma WX. Riemann-Hilbert problems and N-soliton solutions for a coupled mKdV system. J Geom Phys 2018; 132: 45-54.

[17] Ma WX. Riemann-Hilbert problems of a six-component mKdV System and its soliton solutions. Act Math Sci 2019; 39: 509-523.

[18] Ma WX. The inverse scattering transform and soliton solutions of a combined modified Korteweg-de Vries equation. J Math Anal Appl 2019; 471: 796-811.

[19] Peng WQ, Tian SF, Wang XB, Zhang TT, Fang Y. Riemann-Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations. J Geom Phys 2019; 146: 103508.

[20] Wang DS, Zhang DJ, Yang J. Integrable properties of the general coupled nonlinear Schrödinger equations. J Math Phys 2010; 51: 023510.

[21] Geng X and Wu J. Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation. Wave Motion 2016; 60: 62-72.

[22] Ablowitz MJ and Fokas AS. Complex Variables: Introduction and Applications. Cambridge University Press, Cambridge (2003).

[23] Zhang YS, Cheng Y and He JS. Riemann-Hilbert method and N-soliton for two-component Gerdjikov-Ivanov equation. J Nonlinear Math Phys 2017; 24: 210-223.

[24] Wang XB, Han B. The pair-transition-coupled nonlinear Schrödinger equation: The Riemann-Hilbert problem and N-soliton solutions. Eur Phys J Plus 2019; 134: 78.

[25] Wu J, Geng X. Inverse scattering transform of the coupled Sasa-Satsuma equation by Riemann-Hilbert approach. Commun Theor Phys 2017; 67: 527-534.
[26] Guo B and Ling L. Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation. J Math Phys 2012; 53: 073506.

[27] Yang JK. Nonlinear Waves in Integrable and Nonintegrable Systems. SIAM, Philadelphia (2010).

[28] Shchesnovich VB and Yang JK. General soliton matrices in the Riemann-Hilbert problem for integrable nonlinear equations. J Math Phys 2003; 44: 4604.

[29] Nakkeeran K, Porsezian K, Sundaram P, Shanmugha and Mahalingam A. Optical Solitons in N-Coupled Higher Order Nonlinear Schrödinger Equations. Phys Rev Lett 1988; 80: 1425.

[30] Agrawal GP. Nonlinear Fiber Optics. Academic, San Diego (1989).

[31] Hasegawa A and Kodama Y. Solitons in Optical Communications. Clarendon, Oxford (1995).

[32] Menyuk CR. Nonlinear pulse propagation in birefringent optical fibers. IEEE J Quantum Electron 1987; 23: 174.

[33] Dalfovo F, Giorgini S, Pitaevskii LP and Stringari S. Theory of Bose-Einstein condensation in trapped gases. Rev Mod Phys 1999; 71: 463.

[34] Ieda J, Miyakawa T and Wadati M. Matter-Wave Solitons in an F=1 Spinor Bose-Einstein Condensate. J Phys Soc Jpn 2004; 73: 2996.

[35] Roskes GJ. Some nonlinear multiphase interactions. Stud Appl Math 1967; 55: 231.

[36] Shi Z and Yang J. Solitary waves bifurcated from Bloch-band edges in two-dimensional periodic media. Phys Rev E 2007; 75: 056602.

[37] Ma WX. Darboux transformations for a Lax integrable system in 2n dimensions. Lett Math Phys 1997; 39: 33-49.

[38] Lenells J. Dressing for a novel integrable generalization of the nonlinear Schrödinger equation. J Nonliner Sci 2010; 20: 709-722.

[39] Li CX, Ma WX, Liu XJ, Zeng YB. Wronskian solution of the Boussinesq equation-solitons, negatons, positons and complexitons. Inver Probl 2007; 23: 5279-5296.

[40] Xu T, Wang D, Li M, Liang H. Soliton and breather solutions of the Sasa-Satsuma equation via the Darboux transformation. Phys Scr 2014; 89: 075207.

[41] Gilson C, Hietarinta J, Nimmo J and Ohta Y. Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions. Phys Rev E 2003; 68: 016614.