Extrinsic and intrinsic acoustical cross-sections for a viscous fluid particle near a planar rigid boundary: circular cylinder example

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Abstract
The objective of this investigation is to derive closed-form partial-wave series expansions for the extrinsic and intrinsic acoustical scattering, extinction and absorption cross-sections of a fluid viscous cylindrical particle of arbitrary geometrical cross-section, located nearby a flat rigid boundary. Plane progressive waves with arbitrary incidence angle are considered in a non-viscous fluid. The multiple scattering between the particle and the boundary is described using the multipole expansion in cylindrical coordinates, the method of images and the translational addition theorem. The extrinsic cross-sections are determined by integrating the associated power flow densities over a virtual surface of large radius enclosing the object and its image, stemming from far-field expressions of the incident and scattered pressure fields. The expressions for the intrinsic cross-sections are established after defining an effective acoustic field incident on the particle, which includes the primary incident field, the reflected waves from the boundary and the scattered field from the image object. Subsequently, the incident effective field is used with the scattered field from the object to derive closed-form analytical expressions for the intrinsic cross-sections, based on a far-field scattering approach, which does not introduce any approximation in the evaluation of the expressions. The obtained expressions involve the angle of incidence, the expansion coefficients of the scatterer and its image, and the distance from the center of mass of the particle to the boundary. Numerical examples for a viscous fluid circular cylindrical cross-section are considered, and computations illustrate the analysis with particular emphasis on varying the size of the particle, the angle of incidence and the particle-wall distance. The results find potential applications in the quantitative predictions of the acoustical cross-sections from fluid/soft/compressible objects nearby a boundary, such as contrast agents in biomedicine, oceanography, sonochemistry, marine science, and (borehole) geophysical applications to name a few examples.

1. Introduction
Cross-sections are important physical observables used to characterize quantitatively an object (or multiple objects) in a wavefield from the standpoint of its scattering, absorption and extinction properties in a medium of wave propagation. For a passive material (i.e., with no emitting sources present in its core), the scattering and absorption cross-sections are defined as the ratios of the scattered and absorbed powers, respectively, to a characteristic intensity factor. For an active material, those are defined as radiation and amplification cross-sections [1–3]. The extinction cross-section is the sum of the two (p 13 in [4]), as defined by energy conservation law.

In the field of acoustics, the cross-sections can be used to predict the response of a bubble depending on a variety of suitable scattering models [5] for biomedical imaging applications with contrast agents. Moreover, they are directly connected with radiation force [6, 7] and torque [8], and can be used to validate the emergence of exotic effects such as the generation of a negative radiation force [9] and torque [10, 11], in agreement with energy conservation.
Several investigations have considered extended formalisms for the acoustical cross-sections of a scatterer immersed in a fluid [8, 12, 13] or encased in an elastic [14–20] or a viscoelastic [21–28] medium, although recent works considered the multiple scattering effects between two particles [29] in a non-viscous fluid medium. Extrinsic [29] (i.e., global) and intrinsic [30] (i.e., local) cross-sections have been defined, which provide adequate means to characterize the scattering, absorption and extinction properties of the scatterers.

In applications where a submerged particle is close to a boundary (for example, an object nearby the seabed in oceanography applications, a vessel/muscle wall in biomedicine, or a chamber edge in acoustofluidics), the numerical predictions for the acoustical cross-sections relying on the existing analytical models developed for an unbounded fluid medium of wave propagation become inaccurate, as multiple reflections/interactions with the boundary occur, and alter significantly the scattering, absorption and extinction properties of the scatterer. Therefore, it is of some importance to devise an extended formalism for the acoustical extrinsic and intrinsic cross-sections, taking the multiple reflections/scattering effects between the wall and the particle into account.

The aim of this work is directed toward the development of a novel rigorous analytical formalism, which introduces the acoustical extrinsic and intrinsic cross-sections for a particle made of a passive material of arbitrary geometrical cross-section (in 2D) valid for any scattering regime (i.e., Rayleigh, Mie or geometrical/ray acoustics). Particularly, it accounts for the multiple reflections/scattering effects occurring between the particle and the rigid boundary by means of the rigorous multipole expansion method in cylindrical coordinates [31], the method of images [32, 33] as well as the translational addition theorem for cylindrical wave functions [34].
The choice and the correlation between the multipole expansion method, the method of images, and the translational addition theorem of cylindrical wave functions used here provide a powerful mathematical tool to obtain exact closed-form series expressions for the extrinsic and intrinsic cross-sections without any approximations.

The methodology to determine the extrinsic acoustical cross-sections consists on integrating the time-averaged power flow densities associated with the acoustic wave motion, stemming from an analysis of the incident and scattered fields in the far-field region. Moreover, an effective acoustic pressure field incident on the particle is defined, which includes the primary incident field, the reflected waves from the boundary as well as the scattered field from the image object, then utilized with the scattered field from the object to derive the intrinsic acoustical cross-sections. The analysis is substantiated by numerical computations for a viscous fluid cylindrical particle of circular cross-section illuminated by plane progressive waves with arbitrary incidence. Particular emphases are given on the distance from the boundary, the dimensionless size parameter of the particle and the angle of incidence of the incoming waves. The present analytical formalism and its related results would be relevant for applications involving the numerical predictions of the acoustical cross-sections for a particle nearby a boundary using exact partial-wave series expansions (PWSEs). The results can serve to validate purely numerical approaches based on the finite-element method, the boundary element method, or other tools.

Figure 3. Panels (a)–(c) display the plots for the extrinsic scattering, extinction and absorption cross-sections, respectively, versus $kd$ and $\alpha$ for a small (subwavelength) non-viscous circular cylinder located near a boundary at $ka = 0.1$. 

The choice and the correlation between the multipole expansion method, the method of images, and the translational addition theorem of cylindrical wave functions used here provide a powerful mathematical tool to obtain exact closed-form series expressions for the extrinsic and intrinsic cross-sections without any approximations.
Moreover, the 2D cylindrical particle geometry serves as a benchmark solution where the anticipated effects in 2D can be expected to occur for a 3D particle nearby a boundary, and this analysis should assist along that direction of research.

In section 2, the two-dimensional scattering of acoustical plane progressive waves by a fluid particle near a rigid planar boundary is presented. Subsequently, the procedures to derive the closed-form expressions for the extrinsic and intrinsic cross-sections are developed in sections 3 and 4, respectively. In section 5, the case of a fluid viscous cylinder with circular geometrical cross-section is considered with the appropriate boundary conditions in order to determine the unknown scattering coefficients and compute the extrinsic and intrinsic non-dimensional cross-section factors. Numerical results are presented and discussed in sections 6 and 7 provides the conclusion of this work.

2. Two-dimensional scattering of acoustical plane progressive waves by a fluid particle of arbitrary geometrical cross-section near a rigid flat boundary

Consider a harmonic plane progressive continuous-wave field propagating in a nonviscous fluid, and incident upon a particle with arbitrary geometrical cross-section, with an angle of incidence $\alpha$ as shown in figure 1. The incident pressure field is expressed as
\[ p_{\text{inc}} = p_0 e^{i(kr - \omega t)} , \]

where \( p_0 \) is the pressure amplitude, \( k \) is the wave-vector, \( r \) is the vector position, and \( \omega \) is the angular frequency.

\[ p_{\text{inc}}(r, \theta, t) = p_0 e^{-i\omega t} e^{ikr \cos(\theta - \alpha)} = p_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n e^{-in\alpha} f_0(kr) e^{i\theta}, \]

where \( f_0(\cdot) \) is the cylindrical Bessel function of the first kind, and \( k = \omega/c \) is the wavenumber, where \( c \) is the speed of sound in the medium of wave propagation.

The plane wave pressure field reflected from the rigid/fixed boundary, and incident on the particle from the boundary side are also represented by a PWSE, such that

\[ p_R(r, \theta, t) = p_0 e^{-i\omega t} e^{ikr \cos(\theta - \pi + \alpha)} e^{2ikd \cos \alpha} \]

\[ = p_0 e^{-i\omega t} e^{2ikd \cos \alpha} \sum_{n=-\infty}^{\infty} i^n e^{-in(\pi - \alpha)} f_0(kr) e^{i\theta}, \]

Figure 5. Panels (a)–(c) display the plots for the extrinsic scattering, extinction and absorption cross-sections, respectively, versus \( kd \) and \( \alpha \) for a viscous (sound absorptive) circular cylinder located near a boundary at \( ka = 5 \).
where \( d \) is the distance from the center of mass of the particle to the boundary along the direction \( \theta = 0 \) (figure 1).

The scattered pressure field resulting from the interaction of the primary incident plane progressive wave field with the object is expressed as,

\[
\rho_{\text{sca}}(r, \theta, t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(kr) e^{in\theta},
\]

where \( p_0 \) is the initial pressure, \( H_n^{(1)}(\cdot) \) is the cylindrical Hankel function of the first kind of order \( n \), and \( C_n \) is the expansion coefficient that will be determined using adequate boundary conditions.

Since the object is made of a fluid material, the internal pressure field inside its core is expressed as

\[
\rho_{\text{int}}(r, \theta, t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} C_n^{\text{int}} h_n(\kappa_l r) e^{in\theta},
\]

where \( \kappa_l \) is the wavenumber corresponding to the longitudinal/compressional waves propagating inside the core material. For a viscous fluid medium, \( \kappa_l \) is a complex number, accounting for sound absorption. \( C_n^{\text{int}} \) is the expansion coefficient for the internal waves.

By applying the method of images, the scattering from the planar rigid boundary is substituted by the scattering from the image object mirrored by it. Thus, the scattered pressure field contributed by the image

Figure 6. The same as in figure 5 but the plots in panels (a)–(c) correspond to the intrinsic cross-sections.
object is expressed as
\[ p_{\text{image}}^\text{int}(r', \theta', t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} D_n \frac{H_n^{(1)}(kr')}{r'} e^{i\nu'}, \]
where \( D_n \) is the expansion coefficient of the image object.

In addition, the internal pressure field associated with the image object is expressed as
\[ p_{\text{int}}^\text{image}(r', \theta', t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} D_n^\text{int} H_n(\kappa L r') e^{i\nu'}, \]
where \( D_n^\text{int} \) is the expansion coefficient for the internal waves of the image object.

Notice that equations (4)–(7) are valid to express the internal and external fields for an object of noncircular geometrical cross-section and its image (in 2D). Similar PWSEs using the mathematical basis of cylindrical wave functions have been used previously in elastic wave [35–37] and acoustic scattering [38–42] based on the T-matrix formalism [43] for objects such as squares, ellipses and other complex shapes. (See also the related discussions in [11, 29, 30, 41, 42, 44, 45].)

Before applying the appropriate boundary conditions to determine the expansion coefficients, the expressions for the pressure fields have to be rewritten, all referred to a common system of coordinates. The adequate boundary conditions of continuity must be satisfied, either at the planar rigid boundary surface (applied previously in the context of the multiple acoustic scattering [46, 47] from a rigid boundary [48, 49]), or at the surface of the particle and its corresponding image in both systems of coordinates [50]. Both methods are equivalent, and lead to the same result. Using the latter approach, the translational addition theorem of cylindrical wave functions is used to express the corresponding PWSE in a particular coordinate system as an infinite sum of wave functions in the other system of coordinates.

The product of the cylindrical Hankel function with the complex exponential function is expressed as
\[ H_n^{(1)}(kr) e^{i\nu} = \begin{cases} \sum_{m=-\infty}^{+\infty} J_{m-n}(2kd) H_n^{(1)}(kr) e^{i\nu}, & r > 2d \\ \sum_{m=-\infty}^{+\infty} J_m(kr) H_n^{(1)}(2kd) e^{i\nu}, & r < 2d \end{cases}, \]
whereas,
\[ H_n^{(1)}(kr) e^{i\nu} = \begin{cases} \sum_{m=-\infty}^{+\infty} J_{m-n}(2kd) H_n^{(1)}(kr') e^{i\nu'}, & r' > 2d \\ \sum_{m=-\infty}^{+\infty} J_m(kr') H_n^{(1)}(2kd) e^{i\nu'}, & r' < 2d \end{cases}. \]

The total pressure field in the host fluid medium (outside the object) can be expressed in either systems of coordinates after applying the orthogonality condition for the complex exponential function as,
\[ p_{\text{tot}}(r, \theta, t)|_{r<2d} = p_{\text{inc}}(r, \theta, t) + p_{\text{obj}}(r, \theta, t) + p_{\text{sc}}^\text{image}(r, \theta, t)|_{r<2d}, \]
where,
\[ p_{\text{sc}}^\text{image}(r, \theta, t)|_{r<2d} = p_0 e^{-i\omega t} \left( \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} D_n H_n^{(1)}(2kd) \right) J_n(kr) e^{i\nu}, \]
\[ p_{\text{tot}}(r, \theta, t)|_{r>2d} = p_{\text{inc}}(r, \theta, t) + p_{\text{obj}}(r, \theta, t) + p_{\text{sc}}^\text{image}(r, \theta, t)|_{r>2d}, \]
where,
\[ p_{\text{sc}}^\text{image}(r, \theta, t)|_{r>2d} = p_0 e^{-i\omega t} \left( \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} D_n H_n^{(1)}(2kd) \right) H_n^{(1)}(kr) e^{i\nu}. \]

Similarly, the expression for the total pressure field in the primed system of coordinates is given by,
\[ p_{\text{tot}}(r', \theta', t)|_{r'<2d} = p_{\text{inc}}(r', \theta', t) + p_{\text{obj}}(r', \theta', t) + p_{\text{sc}}^\text{image}(r', \theta', t)|_{r'<2d} + p_{\text{sc}}^\text{image}(r', \theta', t), \]
where,
\[ p_{\text{inc}}(r', \theta', t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} i^n e^{-i\nu(n-\lambda)} J_n(kr') e^{in\theta'}, \tag{15} \]
\[ p_{\text{R}}(r', \theta', t) = p_0 e^{-i\omega t} e^{2\lambda d} \cos \alpha \sum_{n=-\infty}^{+\infty} i^n e^{-i\nu(n-\lambda)} J_n(kr') e^{in\theta'}, \tag{16} \]
and,
\[ p_{\text{object}}^{\text{object}}(r', \theta', t) = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} \left( \sum_{m=-\infty}^{+\infty} C_m H_n^{(1)}(2kd) \right) J_n(kr') e^{in\theta'}. \tag{17} \]

When \( r' > 2d \), the expression for the total pressure field becomes,
\[ p_{\text{tot}}(r', \theta', t) |_{r'>2d} = p_{\text{inc}}(r', \theta', t) + p_{\text{R}}(r', \theta', t) + p_{\text{object}}^{\text{object}}(r', \theta', t) |_{r'>2d} + p_{\text{image}}^{\text{image}}(r', \theta', t), \tag{18} \]
where,
\[ p_{\text{object}}^{\text{object}}(r', \theta', t) |_{r'>2d} = p_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} \left( \sum_{m=-\infty}^{+\infty} C_m H_n^{(1)}(2kd) \right) H_n^{(1)}(kr') e^{in\theta'}. \tag{19} \]

Since the particle is a sound penetrable fluid, the following conditions of continuity should be imposed in both systems of coordinates \((r, \theta)\) and \((r', \theta')\), such that,
\[ p_{\text{tot}}(r, \theta, t) |_{r=A_0} = p_{\text{int}}(r, \theta, t) |_{r=A_0}, \tag{20} \]
\[ \frac{1}{\rho} \nabla \Phi_{\text{tot}}(r, \theta, t) \cdot \mathbf{n} |_{r=A_0} = \frac{1}{\rho_f} \nabla \Phi_{\text{int}}^{\text{object}}(r, \theta, t) \cdot \mathbf{n} |_{r=A_0}, \tag{21} \]
\[ p_{\text{tot}}^{\text{object}}(r', \theta', t) |_{r'=A_0} = p_{\text{int}}^{\text{image}}(r', \theta', t) |_{r'=A_0}, \tag{22} \]
\[ \frac{1}{\rho} \nabla \Phi_{\text{tot}}^{\text{object}}(r', \theta', t) \cdot \mathbf{n} |_{r'=A_0} = \frac{1}{\rho_f} \nabla \Phi_{\text{int}}^{\text{image}}(r', \theta', t) \cdot \mathbf{n} |_{r'=A_0}, \tag{23} \]
where \( p_{\text{tot}} = i \omega \rho \Phi_{\text{tot}}, p_{\text{int}}^{\text{object}}, p_{\text{int}}^{\text{image}} \) are the total and internal velocity potentials, respectively, and \( \rho \) and \( \rho_f \) are the mass densities of the host fluid medium and fluid particle, respectively. The parameters \( \mathbf{n} \) and \( A_0 \) are the normal vector and the surface shape function of the particle of arbitrary geometry \([41, 42, 44, 45]\), respectively.

### 3. Extrinsic acoustical cross-sections

The mathematical expressions for the extrinsic scattering, absorption and extinction cross-sections involve the PWSEs for the incident and scattered pressure fields in the far-field in the system of coordinates \((r, \theta)\). Denoting the characteristic acoustic intensity by \( I_0 = |p_0|^2/(2\rho c) \), the extrinsic scattering cross-section is expressed as \([12]\),
\[
\sigma_{\text{sca}}^s = \frac{1}{I_0} \int_S \frac{\mathbf{p}_{\text{tot}} \cdot \mathbf{n}}{\mathbf{v}_{\text{tot}}} \cdot d\mathbf{S}, \tag{24}
\]
where \( d\mathbf{S} = \mathbf{n} \, dS \) is the differential vector surface element and \( \mathbf{n} \) the outward normal, \( p_{\text{tot}} = p_{\text{object}}^{\text{object}} + p_{\text{image}}^{\text{image}} \), \( \mathbf{p}_{\text{tot}} \) and \( \mathbf{p}_{\text{int}}^{\text{object}}, \mathbf{p}_{\text{int}}^{\text{image}} \) are the total and internal velocity potentials, respectively, and \( n \) the over-bar denotes time-averaging over the period of the wave. The integration in equation (24) is executed over a fixed cylindrical surface of large radius \( r \).

The absorption cross-section is also expressed as \([12]\),
\[
\sigma_{\text{abs}}^s = -\frac{1}{I_0} \int_S \frac{\mathbf{p}_{\text{tot}}(\mathbf{v}_{\text{tot}} + \mathbf{v}_i) \cdot d\mathbf{S}}, \tag{25}
\]
where \( \mathbf{v}_{\text{tot}}(kr, kr') \to \infty = \nabla ((p_{\text{inc}} + p_{\text{R}})/(i\omega \rho)). \)

The expression for the extinction cross-section is deduced from the sum of equations (24) with (25) such that,
\[
\sigma_{\text{ext}}^e = -\frac{1}{I_0} \iint_S \left( (p_{\text{inc}} + p_R) \mathbf{v}_l + \overline{p_R \mathbf{v}_l} \right) \cdot \mathbf{dS},
\]

where \( \iint_S (p_{\text{inc}} + p_R) \mathbf{v}_l \cdot \mathbf{dS} = 0 \) in a non-viscous fluid.

In the region far from the object and boundary \((kr \to \infty)\), the far-field expressions of the cylindrical Bessel and Hankel cylindrical functions are used by the following functions, respectively, such that

\[
J_n(kr) \approx \sqrt{\frac{2}{\pi kr}} \cos \left( kr - \frac{\pi n}{2} - \frac{\pi}{4} \right)
\]

and

\[
H_n^{(1)}(kr) \approx \sqrt{\frac{2}{\pi kr}} e^{ikr - \frac{\pi n}{2} - \frac{\pi}{4}}.
\]

In the system of coordinates \((r, \theta)\), the incident, reflected and scattered pressure fields (included in equation (12) for \( r > 2d \)) are expressed, respectively, as,

\[
p_i(r, \theta, t) = \sum_{r>2d} P_0 e^{-i\omega t} \frac{2}{\pi kr} \sum_{n=-\infty}^{+\infty} i^n e^{-i\omega n} \cos \left( kr - \frac{\pi n}{2} - \frac{\pi}{4} \right) e^{i\omega t},
\]

Figure 7. Panels (a)–(c) display the plots for the extrinsic scattering, extinction and absorption cross-sections, respectively, versus \( ka \) and \( \alpha \) for a viscous (sound absorptive) circular cylinder located near a boundary at \( kd = 5.5 \).
\[ p_{\text{eff}}(r, \theta, t) = p_{\text{inc}}(r, \theta, t)|_{r<2d} + p_{\text{sc}}(r, \theta, t) \]

\[ p_{\text{object}}(r, \theta, t) = p_{\text{inc}}(r, \theta, t) + p_{\text{sc}}(r, \theta, t) \]

\[ p_{\text{image}}(r, \theta, t) = p_{\text{inc}}(r, \theta, t) - p_{\text{sc}}(r, \theta, t) \]

The intrinsic cross-sections are now defined as follows. Similarly to the results in section 2, the intrinsic scattering cross-section is expressed as,

\[ \sigma_{\text{sc}}^s = \frac{1}{I_0} \int_S p_{\text{sc}} \cdot \mathbf{v}_{\text{sc}} \cdot dS, \]

where the superscript ‘i’ denotes the intrinsic nature of the cross-section, and the scalar product \((n \cdot \mathbf{v}_{\text{sc}})|_{r>kr} \approx p_{\text{sc}} \cdot \langle \mathbf{u} \rangle).$

The intrinsic absorption cross-section is defined here as,

\[ \sigma_{\text{abs}}^s = -\frac{1}{I_0} \int_S \left( p_{\text{inc}} + p_{\text{sc}} \right) \cdot \left( \mathbf{v}_{\text{inc}} + \mathbf{v}_{\text{sc}} \right) \cdot dS, \]

where \( \mathbf{v}_{\text{inc}}|_{r>kr} \approx \nabla p_{\text{inc}}/(i\omega \rho) \) is the effective vector field. Notice that if the particle is non-absorptive, \( \sigma_{\text{abs}}^s = 0. \)
The expression for the intrinsic extinction cross-section $\sigma_{\text{ext}}^i$ is deduced from the sum of equations (35) and (36) such that,

$$\sigma_{\text{ext}}^i = -\frac{1}{k^2} \int_S (p_{\text{inc}}^\text{object} \cdot \chi_{\text{inc}}^\text{ext} + p_{\text{sca}}^\text{object} \chi_{\text{sca}}^\text{ext}) \cdot \mathrm{d}S.$$  \hspace{1cm} (37)

After performing the adequate integration using the far-field expressions for the Bessel and Hankel functions in the expressions for the fields in equations (35)–(37), the intrinsic scattering, absorption and extinction cross-sections (per unit-length) are obtained as,

$$\sigma_{\text{sca}}^i = \frac{4}{k} \sum_{n=-\infty}^{\infty} |C_n|^2,$$  \hspace{1cm} (38)

$$\sigma_{\text{abs}}^i = -\frac{4}{k} \left\{ \sum_{n=-\infty}^{\infty} C_n^4 \left[ i^n (e^{-i\alpha} + e^{i\alpha} + 2kd \cos \alpha) + \sum_{m=-\infty}^{\infty} D_m H_n^{(1)}(2kd) \right] + |C_0|^2 \right\},$$  \hspace{1cm} (39)

Figure 8. The same as in figure 7 but the plots in panels (a)–(c) correspond to the intrinsic cross-sections.
5. Circular fluid cylinder example

Consider now the special case of a fluid cylinder of circular cross-section of radius $a$, as shown in figure 2. Applications of the boundary conditions given by equations (20) and (21) at $r = a$ using equations (5) and (10) as well as the orthogonality of the complex exponential functions $e^{in\theta}$, lead to the following coupled system of linear equations,

$$
\sigma_{\text{ext}}^i = -\frac{4}{k} \Re \left\{ \sum_{n=-\infty}^{\infty} C_n \left[ i^n (e^{-in\alpha} + e^{-in(\pi - \alpha) + 2i kd \cos \alpha}) + \sum_{m=-\infty}^{\infty} D_m H_{n-m}^{(1)}(2kd) \right] \right\}.
$$

Figure 9. Panels (a)–(c) display the plots for the extrinsic scattering, extinction and absorption cross-sections, respectively, versus $ka$ and $\alpha$ for a viscous (sound absorptive) circular cylinder located near a boundary at $kd = 15$. 

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where the primes in equation (42) (and equation (44), next page) denote the derivatives of the wave functions, and $c_f$ is the speed of sound of the compressional waves in the fluid cylinder.

Similarly, the application of the boundary conditions at the image cylinder surface using equations (7) and (14) leads to the following coupled system of linear equations,
The coupled system of equations (41)–(44) can be solved by performing adequate matrix inversion, so that the expansion coefficients $C_n$ and $D_n$ can be determined, which allows computing the extrinsic and intrinsic cross-sections. Adequate truncation is imposed, which ensures convergence of the series. A maximum truncation limit defined as $N_{\text{max}} = \lfloor \max(ka, kd) \rfloor + 35$ has been found suitable to produce accurate results with negligible truncation error in the order of $\sim 10^{-6}$.
6. Numerical results and discussions

A MATLAB numerical code is developed to solve numerically the coupled system of equations (41)–(44) for a fluid viscous circular cylinder located nearby a rigid flat boundary, and compute the acoustic extrinsic and intrinsic cross-sections (per-length). For the circular cylinder case, the cross-sectional surface is \( S_c = 2a \).

The physical properties of the fluid circular cylinder chosen here as an example to illustrate the analysis approximate those of a red blood cell fluid material, relevant to bio-acoustofluidics applications. The mass density of the fluid circular cylinder particles is \( \rho_f = 1099 \text{ kg m}^{-3} \), and the speed of sound for the compressional waves in its core is \( c_f = 1631 \text{ m s}^{-1} \). The host fluid medium is water (\( \rho = 1000 \text{ kg m}^{-3} \) and speed of sound \( c = 1500 \text{ m s}^{-1} \)). Inside the core cylinder material, the wavenumber for the compressional waves is \( k_c = \omega/c (1 + i\gamma) \), where \( \gamma \) is a dimensionless absorption coefficient.

Systematic examples are now considered to illustrate the analysis and compute the dimensionless cross-sections (per-length).

Initially, the angle of incidence of the plane progressive wave field and the dimensionless distance are varied, respectively, in the chosen ranges \(-90^\circ \leq \alpha \leq 90^\circ \) and \( ka < kd \leq 15 \), for fixed values of the size parameter \( ka \) of the circular cylinder. The example of a Rayleigh circular fluid (non-viscous, i.e., \( \gamma = 0 \)) cylinder near a boundary having \( ka = 0.1 \) is chosen first. Panels (a)–(c) of figure 3 display the results for the normalized
(i.e. dimensionless) extrinsic scattering, extinction and absorption cross-sections (per length), respectively. Panel (a) shows that the plot for $k\sigma_{sca}^e$ exhibits periodic oscillations versus $kd$ as the angle of incidence varies in the range $\sim -60^\circ \leq \alpha \leq 60^\circ$, while outside this bandwidth, the oscillatory behavior is less pronounced. Such oscillations are the results of multiple reflections effects occurring between the particle and wall. At angle values $\alpha > 60^\circ$, the multiple reflections become less pronounced due to geometrical consideration as the waves reflected from the boundary are minimally intercepted by the particle. Panel (b) shows the exact same plot shown in panel (a), which is expected since the circular cylindrical particle is a non-viscous fluid. This leads to a zero extrinsic absorption cross-section as shown in panel (c).

Computations for the normalized intrinsic scattering, extinction and absorption cross-sections (per length) are displayed in panels (a)–(c) of figure 4 for the Rayleigh cylinder having $ka = 0.1$ immersed in a nonviscous fluid. Panel (a) shows the plot for $k\sigma_{sca}^i$ where periodic oscillations are exhibited, with symmetry versus the axis $\alpha = 0^\circ$. The amplitude of $k\sigma_{sca}^i$ is about four times smaller than its extrinsic counterpart, as shown in panel (a) of figure 3. Moreover, panel (b) of figure 4 for the normalized intrinsic extinction cross-section per-length $k\sigma_{ext}^i$ displays a similar plot to panel (a), while $k\sigma_{abs}^i$ vanishes as shown in panel (c) for a non-absorptive cylinder.

The effect of increasing the size parameter to $ka = 5$ is considered now for an absorptive cylinder with $\gamma = 10^{-3}$, chosen for convenience to illustrate the analysis. The corresponding plots for the normalized extrinsic
scattering, extinction and absorption cross-sections (per length) are displayed in panels (a)--(c) of figure 5. As $ka$ increased, $k\sigma^c_{\text{sca}}$ is about $10^4$ times larger than its counterpart for the Rayleigh cylinder shown in panel (a) of figure 3. This also applied to the plot of $k\sigma^e_{\text{ext}}$ displayed in panel (b), which is larger than $k\sigma^c_{\text{sca}}$, leading to a non-zero $k\sigma^e_{\text{abs}} > 0$, shown in panel (c), as required by energy conservation. As $kd$ increases, more undulations are manifested in the plots of $k\sigma^c_{\text{sca}}$, $k\sigma^e_{\text{ext}}$ and $k\sigma^e_{\text{abs}}$. Those oscillations are induced by the multiple interferences between the waves scattered by the particle and those reflected from the boundary.

Similar properties are also observed for the normalized intrinsic scattering, extinction and absorption cross-sections (per length), displayed in panels (a)--(c) of figure 6. Nonetheless, their amplitudes compared to those of the normalized extrinsic cross-sections are reduced by about 50%.

Another case of interest is to vary the angle of incidence of the plane progressive wave field and the dimensionless size parameter, respectively, in the chosen ranges $-90^\circ \leq \alpha \leq 90^\circ$ and $ka = 1, 2, 3, 4$ and 5, for fixed values of the non-dimensional particle-boundary distance $kd$. Panels (a)--(c) of figure 7 display the plots for the normalized extrinsic cross-sections while those of figure 8 correspond to the normalized intrinsic cross-sections at $kd = 5.5$. In all the plots, the acoustic normalized extrinsic and intrinsic cross-sections increase as $ka$ increases, which is expected since at higher frequencies, the scattering and absorption increase. Moreover, the amplitudes of the normalized intrinsic cross-sections in figure 8 are reduced by about 50%, compared to those of the normalized extrinsic cross-sections displayed in figure 7.

Figure 14. The same as in figure 13 but the plots in panels (a)--(c) correspond to the intrinsic cross-sections.
Additional computations for a larger dimensionless distance $kd = 15$ are also performed, and the results for the normalized extrinsic and intrinsic cross-sections are displayed in figures 9 and 10, respectively. Similarly to the case where $kd = 5.5$, the amplitudes of the normalized intrinsic cross-sections in figure 10 are reduced by about 50%, compared to those of the normalized extrinsic cross-sections displayed in figure 9. Nonetheless, as $ka$ increases, undulations in the plots become more pronounced as the angle of incidence varies. At a larger particle-boundary dimensionless distance $kd$, the established (quas)standing wave field generated between the object and the flat wall at a fixed $ka$ value possesses more nodes and antinodes, which affects the amplitudes of the normalized extrinsic and intrinsic cross-sections.

Finally, the effect of varying the size parameter and the dimensionless distance, respectively, in the chosen ranges $0 < ka \leq 5$ and $ka < kd \leq 15$ is considered for fixed values of the angle of incidence $\alpha$. Because of practical considerations, the constraint $kd > ka$ should be always satisfied so as to lead physical results for realistic practical cases. After imposing this condition, the normalized extrinsic and intrinsic cross-sections per length are computed for $\alpha = 0^\circ$, and the corresponding results are displayed, respectively, in figures 11 and 12. All the panels in those figures display sinusoidal oscillations as $kd$ increases. Moreover, as $ka$ increases, the amplitude of the undulations become less pronounced since the scattering, absorption and consequently the extinction increase with frequency.

Increasing the angle of incidence to $\alpha = 45^\circ$ is also considered, and the corresponding computations for the normalized extrinsic and intrinsic cross-sections per length are displayed, respectively, in figures 13 and 14. As $\alpha$ deviates from zero, the normalized extrinsic cross-sections display smaller amplitudes as shown in figure 13, while the opposite phenomenon occurs for the normalized intrinsic cross-sections displayed in figure 14, where their amplitudes are larger. This effect indicates that the global (i.e., extrinsic) and local (i.e., intrinsic) properties behave differently depending on the choice of the values of $\alpha$, $ka$ and $kd$.

7. Conclusion

In summary, this work presented a complete comprehensive analysis to derive closed-form expansions for the acoustical extrinsic and intrinsic scattering, absorption and extinction cross-sections for a cylindrical particle located nearby a flat boundary. The present investigation is also substantiated with numerical computations for a circular viscous fluid cylinder immersed in a non-viscous fluid, and subjected to incident plane waves of arbitrary incidence. The coupling effects of the reflected waves, and those scattered by the particle and its image are taken into account in deriving the mathematical expressions for the extrinsic and intrinsic acoustical cross-sections. The multipole expansion method in cylindrical coordinates, the method of images, as well as the translational addition theorem of cylindrical wave functions are used. Methodical numerical examples are considered where a series of oscillatory behaviors are manifested in the plots of the normalized extrinsic and intrinsic cross-sections. Depending on the particle-boundary dimensionless distance $kd$, the incidence angle $\alpha$ as well as the dimensionless size parameter $ka$, situations have been predicted where the particle yields larger or weaker scattering, absorption and extinction of the incident waves, depending on the choice of the parameters $\alpha$, $ka$ and $kd$. The presented formalism corresponds to an extended version of the classical ‘optical theorem’ in acoustics for a 2D particle of arbitrary shape, located nearby a boundary, and illuminated by plane progressive waves. The results find useful applications involving multiple wave interacting systems for a particle near a chamber wall, particularly, in underwater/ocean acoustics, sonochemistry, acoustofluidics, fluid dynamics, reconfigurable metafluids and liquid crystals to name a few examples. The expressions for the extrinsic and intrinsic cross-sections obtained here can serve to predict similar behaviors for a 3D particle of arbitrary shape (or spherical form) nearby a boundary, and this analysis should assist in developing appropriate analytical modeling for the cross-sections using the multipole expansion method, the method of images and the addition theorem in spherical coordinates.

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