Coorbit Spaces with Voice in a Fréchet Space

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Outline

1. Reproducing representations
2. Classical coorbit theory
3. Generalized coorbit theory based on target spaces
4. Application to non integrable representations
5. Work in progress and open problems
Voice transform

- $\mathcal{H}$ Hilbert space of signals (e.g. $\mathcal{H} = L^2(\mathbb{R}^d)$);
- $G$ locally compact group with Haar measure $dx$;
- $\pi$ unitary representation of $G$ on $\mathcal{H}$;
- for a fixed vector $u \in \mathcal{H}$, the voice transform is defined by
  \[
  V_u : \mathcal{H} \longrightarrow L^\infty(G) \cap C(G) \quad V_u v(x) := \langle v, \pi(x)u \rangle_{\mathcal{H}};
  \]
- if $V_u$ defines an isometry $\mathcal{H} \hookrightarrow L^2(G)$, $\pi$ is called a reproducing representation and $u$ an admissible vector;
- the synthesis operator is given by the dual map
  \[
  V_u' : L^2(G) \longrightarrow \mathcal{H} \quad V_u'f = \int_G f(x)\pi(x)u \, dx;
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- thus one has the weak integral reconstruction
  \[
  v = V_u'V_u v = \int_G \langle v, \pi(x)u \rangle \pi(x)u \, dx.
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Reproducing kernels

- $V_u$ intertwines $\pi$ with the left regular representation of $G$;
- on the $L^2(G)$-side, we have the reproducing formula
  \[
  V_u v = V_u v * V_u u \quad \forall v \in \mathcal{H}
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  where the convolution is given by $f * g(x) = \int f(y)g(y^{-1}x) \, dy$;
- $K := V_u u$ is called a kernel for the representation;
- $\mathcal{H}$ is isometrically embedded into $L^2(G)$ via $V_u$ as the reproducing kernel Hilbert space
  \[
  \mathcal{M}^2 := \{ f \in L^2(G) \mid f = f * K \}.
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Example: the wavelets

- $H = \mathcal{F}^{-1}L^2(\mathbb{R}^+) < L^2(\mathbb{R})$;
- $G = \mathbb{R} \times \mathbb{R}^+ = ax + b \quad (b', a')(b, a) = (b' + a'b, a'a)$;
- $dx = a^{-2} db da$;
- $\pi(b, a)\nu(x) := a^{-1/2} \nu((x - b)/a) \quad \nu \in H$.

Calderón’s condition

$\pi$ is reproducing, and a vector $u \in H$ is admissible if and only if

$$\int_{\mathbb{R}^+} |\hat{u}(\xi)|^2 \frac{d\xi}{\xi} = 1.$$
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Motivation for coorbits

- Some reproducing systems are better than others, as long as they have additional good properties, e.g. a fast coefficient decay;
- the good properties are not detected by the Hilbert space structure, whereas they are typically measured by norms;
- nice reproducing kernels can be extended to reproduce a certain family of Banach spaces.

Coorbit space theory

Every sufficiently good reproducing representation generates an associated family of smoothness Banach spaces, the coorbit spaces.
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Test functions

Classical setup (Feichtinger–Gröchenig, 1986).

- \( \pi \) reproducing irreducible representation, \( u \) admissible vector;
- suppose \( K (= V_u u) \in L^1(G) \);
- \( \pi \) is called an integrable representation;
- define the space of test functions

\[
S := \{ \nu \in \mathcal{H} \mid V_u \nu \in L^1(G) \}
\]

with coorbit norm \( \| \nu \| := \| V_u \nu \|_1 \).

(i) \( S \subset \mathcal{H} \) is a dense \( \pi \)-invariant Banach space, independent of \( u \);
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\mathcal{M}^1 := \{ f \in L^1(G) \mid f = f * K \}.
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Take $S'$, the dual of $S$, as the space of distributions;

- we have dense embeddings $S \hookrightarrow \mathcal{H} \hookrightarrow S'$;
- extend the voice transform to $S'$ by

$$V_u^e T(x) := S' \langle T, \pi(x)u \rangle_S;$$

The extended voice transform

$$V_u^e : S' \longrightarrow L^\infty(G)$$

is bounded and injective.
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Coorbit spaces

- For any left-invariant Banach space of functions $Y$ (e.g. $Y = L^p(G)$) define the coorbit space

$$Co(Y) := \{ T \in S' \mid V^e_T \in Y \}$$

with norm $\|T\| := \|V^e_T\|_Y$.

(i) Every $Co(Y)$ is a $\pi$-invariant Banach space;
(ii) $V^e_u$ defines an isometry from $Co(Y)$ onto the reproducing kernel space

$$\mathcal{M}^Y := \{ f \in Y \mid f = f * K \}.$$

- In particular: $Co(L^\infty(G)) = S'$, $Co(L^2(G)) = \mathcal{H}$, $Co(L^1(G)) = S$.

For the wavelet representation: $Co(L^p_w(G)) = B^p,q_s(\mathbb{R})$ (Besov spaces).
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Limits of the classical theory

Classical coorbit theory works under two basic assumptions on the representation:

- $\pi$ has to be irreducible ...
- ... and integrable.

Many interesting reproducing representations, which naturally arise as restrictions of the metaplectic representation on triangular subgroups of the symplectic group, are neither irreducible nor integrable!
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A non integrable representation

- $\mathcal{H} = \mathcal{F}^{-1} L^2(\hat{\mathbb{R}}_+) \otimes L^2(S^1) < L^2(\mathbb{R} \times S^1)$;
- $G = (\mathbb{R} \times \mathbb{R}_+) \times S^1$;
- $dx = a^{-2} \, db \, da \, d\varphi/2\pi$;
- $\pi(b, a, \varphi) \nu(x, \vartheta) = a^{-1/2} \nu((x - b)/a, \vartheta - \varphi) \quad \nu \in \mathcal{H}$;

**Proposition**

(i) $\pi$ is reproducing;

(ii) there exist reproducing kernels $K$ such that
- $K \in L^p(G)$ for all $p > 1$;
- $K \notin L^1(G)$. 
Generalized coorbit theory

A non integrable representation

- $\mathcal{H} = \mathcal{F}^{-1}L^2(\hat{\mathbb{R}}_+) \otimes L^2(S^1) < L^2(\mathbb{R} \times S^1)$;
- $G = (\mathbb{R} \times \mathbb{R}_+) \times S^1$;
- $d\chi = a^{-2} db \, da \, d\varphi/2\pi$;
- $\pi(b, a, \varphi)\chi(x, \vartheta) = a^{-1/2} \chi((x - b)/a, \vartheta - \varphi)$ $\quad \chi \in \mathcal{H}$;

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Christensen–Ólafsson, 2011: \textit{Coorbit spaces for dual pairs} (axiomatic approach).

Which (non integrable) kernels are (still) good enough to reproduce coorbit spaces?
Christensen–Ólafsson, 2011: *Coorbit spaces for dual pairs* (axiomatic approach).

**Generalized coorbit space theory based on target spaces**

Which (non integrable) kernels are (still) good enough to reproduce coorbit spaces?
A coorbit space is the inverse image of some target space under the extended voice transform:

$$ Co(Y) \xrightarrow{V^e_u} Y; $$

the coorbit space $Co(Y)$ inherits its structure by isomorphism with the model space:

$$ \mathcal{M}^Y = \{ f \in Y \mid f = f \ast K \} < Y; $$

the inverse isomorphism is given by the weak integral

$$ \mathcal{M}^Y \ni f \mapsto \int_\mathcal{G} f(x)\pi(x)u \, dx \in Co(Y); $$

the space of test functions is itself a coorbit space, which is contained in $\mathcal{H}$ and dense in it.
A coorbit space is the inverse image of some target space under the extended voice transform:

$$\text{Co}(Y) \xrightarrow{V_e^u} Y;$$

the coorbit space $\text{Co}(Y)$ inherits its structure by isomorphism with the model space:

$$\mathcal{M}^Y = \{ f \in Y \mid f = f \ast K \} < Y;$$

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A notion is then focusing . . .

**Target space**

For a good target space it should make sense:

- the right convolution with the kernel: $f \ast K$, $f \in Y$;
- the weak integration: $\int f(x)\pi(x)u \, dx$, $f \in Y$.

**Basic target space**

The target space for the test functions should fulfill some additional properties: its weak integrals have further to

- converge in $\mathcal{H}$;
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Target spaces (2)

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- converge in $\mathcal{H}$;
- be dense in $\mathcal{H}$. 
For the generalized coorbit theory, we need to handle Fréchet spaces.

- Let $F$ be a conjugate left-invariant Fréchet space of measurable functions on $G$:
  - $F \subseteq L^0(G)$;
  - $\overline{f} \in \mathcal{T}$ for all $f \in F$;
  - $\ell(y)f \in \mathcal{T}$ for all $f \in F$ and $y \in G$, where $\ell(y)f(x) := f(y^{-1}x)$.

- The Köthe dual of $F$ is the space

$$F^\# := \{g \in L^0(G) \mid fg \in L^1(G) \ \forall f \in F\}.$$ 

- e.g. $(L^p)^\# = L^{p'}$ for $p \in [1, \infty]$, $(L^\infty)^\# = L^1$. 

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Stefano Vigogna (Università di Genova)
Extended voice

\(\pi : G \to U(\mathcal{H})\) reproducing (not necessarily irreducible) representation, \(u \in \mathcal{H}\) fixed admissible vector.

Take a conjugate left-invariant Fréchet space of functions \(\mathcal{T}\).

1. \(K \in \mathcal{T}^\#\);
2. Def \(\mathcal{M}^T := \{ f \in \mathcal{T} \mid f \ast K = f \};\)
3. Prop \(\mathcal{M}^T < \mathcal{T}\) is a \(\pi\)-invariant reproducing kernel Fréchet space.
4. \(V_u \mathcal{H} < (\mathcal{M}^T)^\#;\)
5. Def \(S := \{ v \in \mathcal{H} \mid V_u v \in \mathcal{T} \};\)
6. Prop \(S \simeq \mathcal{M}^T\) via \(V_u\).
7. \(u \in S\) (i.e. \(K \in \mathcal{T}\)) and be cyclic for \(\pi^S;\)
8. Prop \(S \hookrightarrow \mathcal{H} \hookrightarrow S'\) dense embeddings;
9. Def \(V_u^o : S' \to C(G), V_u^o T(x) := \langle T, \pi(x)u \rangle;\)
10. Prop \(V_u^o\) is continuous respect to the compact convergence, and injective.
Extended voice

\( \pi : G \to U(H) \) reproducing (not necessarily irreducible) representation, \( u \in H \) fixed admissible vector.

Take a conjugate left-invariant Fréchet space of functions \( T \).

H1 \( K \in T^\# \);
Def \( M^T := \{ f \in T \mid f \ast K = f \} \);
Prop \( M^T \subset T \) is a \( \pi \)-invariant reproducing kernel Fréchet space.

H2 \( V_uH \subset (M^T)^\# \);
Def \( S := \{ \nu \in H \mid V_u \nu \in T \} \);
Prop \( S \simeq M^T \) via \( V_u \).

H3 \( u \in S \) (i.e. \( K \in T \)) and be cyclic for \( \pi^S \);
Prop \( S \hookrightarrow H \hookrightarrow S' \) dense embeddings;
Def \( V^e_u : S' \to C(G), \ V^e_u T(x) := \langle T, \pi(x)u \rangle \);
Prop \( V^e_u \) is continuous respect to the compact convergence, and injective.
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Take a conjugate left-invariant Fréchet space of functions $T$.

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Def $M^T := \{ f \in T \mid f \ast K = f \}$;

Prop $M^T < T$ is a $\pi$-invariant reproducing kernel Fréchet space.

H2 $V_u \mathcal{H} < (M^T)^\#$;

Def $S := \{ \nu \in \mathcal{H} \mid V_u \nu \in T \}$;

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H2 \( V_u \mathcal{H} \subset (\mathcal{M}^{\mathcal{T}})^{\#} \);

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\begin{enumerate}
  \item \( K \in \mathcal{T}^\# \);
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  \item \text{Prop } \mathcal{M}^\mathcal{T} \subset \mathcal{T} \text{ is a } \pi \text{-invariant reproducing kernel Fréchet space.}
  \item \( V_u \mathcal{H} < (\mathcal{M}^\mathcal{T})^\# \);
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Coorbit spaces

Take a conjugate left-invariant Fréchet (Banach) space of functions $Y$.

**H4** $K \in Y^\#$;

**Def** $M^Y := \{ f \in Y \mid f \ast K = f \}$;

**Prop** $M^Y < Y$ is a $\pi$-invariant reproducing kernel Fréchet (Banach) space.

**H5** $V_\nu S < (M^Y)^\#$;

**Def** $Co(Y) := \{ T \in S' \mid V_\nu^e T \in Y \}$.

**H6** $V_\nu^e Co(Y) < M^Y$;

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Generalized coorbit theory

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Prop $\text{Co}(Y) \simeq \mathcal{M}^Y$ via $V^e_u$. 
The hypothesis H6 looks rather difficult to check. Some stronger conditions are:

**Proposition 1**

If $T^\pi = T'$ and $|K| \ast |K|$ exists in $T$, then hypothesis H6 is verified.

**Proposition 2**

If $S$ is reflexive and $K \in (V_u^e S')^\pi$, then hypothesis H6 is verified.
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If $\pi$ is not irreducible, then the construction is no longer independent on the choice of the analysing vector (counterexample by Fuhr, 2012);

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- Suppose $T \ast \tilde{T} < T$ and $f \mapsto f \ast \tilde{g}$ to be continuous for all $f, g \in T$;
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Admissible target spaces

- Classical coorbit theory falls into the generalized framework for $\mathcal{T} = L^1(G)$;
- one can obtain the classical theory also starting from a reflexive target space, replacing $L^1(G)$ with $L^1(G) \cap L^2(G)$.
- All the assumptions hold true taking the projective target space

$$\mathcal{T} = \bigcap_{p \in (1,\infty)} L^p(G), \quad \mathcal{T}^\# = \mathcal{T}' = \text{span} \bigcup_{p \in (1,\infty)} L^p(G),$$

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Thanks for your attention!