Finite-time scaling of dynamic quantum criticality

Shuai Yin, Xizhou Qin, Chaohong Lee, and Fan Zhong
State Key Laboratory of Optoelectronic Materials and Technologies, School of Physics and Engineering, Sun Yat-sen University, Guangzhou 510275, People’s Republic of China

We develop a theory of finite-time scaling for dynamic quantum criticality by considering the competition among an external time scale, an intrinsic reaction time scale and an imaginary time scale arising respectively from an external driving field, the fluctuations of the competing orders and thermal fluctuations. Through a successful application in determining the critical properties at zero temperature and the solution of real-time Lindblad master equation near a quantum critical point at nonzero temperatures, we show that finite-time scaling offers not only an amenable and systematic approach to detect the dynamic critical properties, but also a unified framework to understand and explore nonequilibrium dynamics of quantum criticality, which shows specificities for open systems.

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Detecting quantum phase transitions (QPTs) and understanding their real-time dynamics are of great importance [1–5]. Recent experimental breakthrough in ultracold atoms [6] promises new tools to study the quantum critical dynamics [7]. In the nonequilibrium critical dynamics of QPTs at zero temperature in which a controlling parameter is changed with time through a critical point [1, 5], the Kibble-Zurek mechanism (KZM), which was first introduced in cosmology by Kibble [8] and then in condensed matter physics by Zurek [9], has been found to describe the dynamics of QPTs well [4, 5, 10]. In this adiabatic–impulse–adiabatic approximation of the KZM, the system considered is assumed to cease evolving in the impulse regime within which adiabaticity breaks down due to critical slowing down [11]. Yet, dynamical scaling has been reported just within this regime [12, 13] and confirmed in both integrable and nonintegrable systems [13, 14]. In classical critical dynamics, an explanation based on coarsening has been developed [15], however in quantum phase transitions, a systematic understanding of the full scaling behavior is still lacking.

On the other hand, natural systems and their measurements exist inevitably in nonzero temperatures, though probably only initial thermal states need considering in ultracold atoms [10]. Thermal effects on a quantum critical state can give rise to a variety of exotic behavior in the famous quantum critical regime (QCR) [17] as exhibits in a wide range of strongly correlated systems [1–3, 18]. Yet, as both phases exhibit complex long-range quantum entanglement near the quantum critical point and are violently excited thermally, it is a great challenge to describe quantum critical dynamics at finite temperatures, let alone nonequilibrium real-time effects [12, 20]. Indeed, none of the analytic, semiclassical, or numerical methods of condensed-matter physics yields accurate results for dynamics in the QCR except for some special systems in 1D [2]. Even in isolated situations it is difficult to study the time evolution of nonequilibrium systems with many degrees of freedom [1–5, 22, 23]. Therefore, systematic approaches have to be invoked.

Time plays a fundamental role in quantum criticality owing to the interplay of static and dynamic behaviors. Specifically, by varying the distance to the critical point $g$ at a time rate $R$, a continuous QPT at a finite temperature $T$ is characterized mainly by three time scales. The first one is a reaction time $\tau$, that arises from the fluctuations of the competing orders and blows up as $\tau \sim |g|^{-\nu z}$ with the standard critical exponents $\nu$ and $z$ as $g$ vanishes [1]. The second one is an “imaginary” time scale $\tau_T = 1/T$ (the Plank and the Boltzmann constants have been set to 1) due to the finite $T$, since the real time is its analytical continuation to imaginary numbers through a Feynman path integral representation [1]. The third one is an externally imposed driving time scale $\tau_d$ that results from the driving and grows as $\tau_d \sim R^{-z/r}$ with a rate exponent $r$ that is related to $z$ and other static critical exponents [25]. It is the competition among $\tau$, $\tau_T$, and $\tau_d$ that lead to a diversity of equilibrium and dynamic universal phenomena near a quantum critical point.

FIG. 1: (color online) Schematic phase diagram under a sweep of $g$ near its critical value 0. Two equilibrium phases (light grey/green) dominated by the reaction time $\tau$, are separated by two crossover domains fanning out from the quantum critical point $O$. One domain is the QCR (dark/blue) controlled by the imaginary time scale $\tau_T$. The other is the new FTS (grey/red) regime governed by the driving time scale $\tau_d$. 
Here we study systematically the competition among the three characteristic time scales according to the theory of finite-time scaling (FTS) [25]. As seen in Fig. 1 besides the usual equilibrium regimes and the QCR which are respectively dominated by $\tau_s$ and $\tau_f$, our most important result is that a new nonequilibrium FTS domain is created. In this domain, $\tau_d$ is the shortest time among the trio and thus dominates, just as the well-known regime of finite-size scaling in which the characteristic size $L$ of the system is shorter than its correlation length. At $T = 0$, this indicates the FTS domain overlaps just the impulse regime of the KZM for sweeping $g$. As a consequence, although the system falls out of equilibrium, the state does not cease evolving; rather, it evolves according to the imposed time scale $\tau_d$ instead of $\tau_s$ with nonadiabatic excitations obeying FTS. Therefore, FTS improves the understanding of KZM on its dark impulse regime and produces naturally scaling forms suggested in [12–14]. In addition, FTS enables us to the study within the same framework other driving dynamics than the KZ protocols [20], which focus on changing non-symmetry breaking terms like $g$. We shall show that these provides a convenient method to determine the critical points and breaking terms like $\gamma$ that enables one to probe the zero-temperature scaling at $\tau_s$.

As another important result, we shall show that in nonequilibrium quantum critical dynamics of open systems one must include an additional variable such as the coupling to a heat bath to the intrinsic quantum dynamics [21]. This is an important difference from the classical case and must be considered when extending nonequilibrium quantum critical dynamics to finite temperatures [19]. We shall see that the master equation in the Lindblad form just offers such a variable and is thus an appropriate platform to study real-time nonequilibrium quantum criticality.

We start with an open many-body quantum system interacting with a heat bath [19] to study the interplay of quantum and thermal fluctuations. The state of such a system can be described by a density matrix operator $\rho$ according to quantum statistical physics. For weak system-environment couplings, after assuming Markovian and tracing over the bath variables, one obtains the master equation for $\rho$ in the Lindblad form [20–28],

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + cL\rho,$$

(1)

where $L\rho = -\sum_{i=1}^{N_E} \beta_i (V^\dagger_{i\rightarrow j} V^{\dagger}_{j\rightarrow i})\rho + \rho V^\dagger_{i\rightarrow j} V^{\dagger}_{j\rightarrow i}$ $- 2V_{i\rightarrow j} V^{\dagger}_{j\rightarrow i})/2$, $c$ is the dissipation rate and measures the coupling strength between the system and the bath, $N_E$ is the total number of energy levels, $\beta_i = \exp(-E_i/T)/\text{Tr} \exp(-H/T)$ with $E_i$ being the $i$th eigenvalue of $H$, and $V_{i\rightarrow j}$ is the thermal jump matrix whose element at the $j$th row and $i$th column is one or zero in the energy representation. $V_{i\rightarrow j}$ fulfills $\beta_i \rho E V_{i\rightarrow j} = \beta_j V_{i\rightarrow j} \rho E$ with the equilibrium density matrix operator $\rho_E \equiv \exp(-H/T)/\text{Tr} \exp(-H/T)$ whose eigenvalues are $\beta_i$. This can be regarded as a detailed balance condition in equilibrium. The Lindblad equation (1) is a real-time dynamical equations which integrates both the quantum and the thermal contributions. It has been widely used in quantum optics [29] and relaxation processes in open quantum systems [27, 30]. Although for large couplings, Eq. (1) may be inapplicable [31], this equation gives a reasonable description in the weak coupling limit, for instance, for the time-independent $H$, the steady solution of Eq. (1) is $\rho_E$ independent of $c$ [26–28], this is consistent with the foundation of statistical mechanics [32]. In the following, we focus on the weak coupling limit and consider the scaling properties of the Lindblad equation.

The theory of FTS [25] takes explicitly into account the rate $R$, which plays a role similar to $L^{-1}$ since it imposes on a system an additional time scale that manipulates its evolution. In classical critical dynamics, the nonequilibrium dynamic scaling can be generalized directly from the equilibrium ones as confirmed by the renormalization-group theory [23]. However, in the nonequilibrium quantum criticality, as pointed out, a coupling strength must be considered as an independent scaling variable. In the weak coupling limit, this strength can be reduced to the dissipation rate $c$. Accordingly, for a length rescaling of factor $b$, an order parameter $M$ transforms as

$$M(t, g, h_z, T, L, c, R) = b^{-\beta/\nu} M(t b^{-\delta/\nu}, g b^{1/\nu}, h_z b^\beta, c b^{\delta/\nu}, R b^\gamma),$$

(2)

where the two critical exponents $\beta$ and $\delta$ are defined as usual in classical critical phenomena by $M \propto g^\beta$ in the absence of an external probe field $h_z$ conjugate to $M$ and $M \propto h_z^\delta$ at $g = 0$, respectively. In the weak coupling limit, $c$ is small thus one can expect its scaling behavior is controlled by the the fixed point corresponding to the critical point at $c = 0$, thus the dimension of $c$ is identical with $t^{-1}$ as can be inspected from Eq. (1). This is checked latter by the numerical solution of Eq. (1).
With Eq. (2), one can describe in a unified framework different kinds of driven dynamics via changing \( g, h_z \) or \( T \) and readily define different regimes and their crossovers. Taking \( g = R t \) for instance, neglecting \( h_z \), suppressing one independent variable, and choosing \( b \) such that \( R b \) becomes a constant, one finds an FTS scaling form

\[
M(g, T, L, c, R) = R^{\beta/\nu_r} f_1(g R^{-1/\nu_r}, T R^{\zeta_R}, L^{-1} R^{-1/\nu_r}, c R^{-\zeta_R}),
\]

where \( \zeta_R = z + 1/\nu \) obtained from \( g = R t \) and its rescaling \[22\] and the function \( f_i \) with an integer \( i \) denotes a scaling function. FTS dominates when \( |g| R^{-1/\nu_r} \ll 1 \), \( T R^{\zeta_R} \ll 1 \), \( L^{-1} R^{-1/\nu_r} \ll 1 \), and \( c R^{-\zeta_R} \ll 1 \). The first gives \( \tau_d \sim R^{-\zeta_R} \ll |g|^{\nu_z} \sim \tau_s \), the second \( \tau_d \ll 1/T \tau_T \) as they ought to be. Crossovers to other regimes occur near \( |g| \sim R^{1/\nu_r} \) and \( T \sim R^{\zeta_R} \) as depicted in Fig. 4 and similar ones for \( L \) and \( c \). The first gives \( |t| \sim R^{-\nu_z/(1+\nu z)} \) because \( g = R t \). This is just the scaling of the KZM upon identifying \( \hat{t} \) with the freeze-out time instant \[3, 2, 10\] for a closed system \( c = 0 \) in the thermodynamic limit \( L \to \infty \) and at \( T = 0 \).

Several remarks are in order here. (a) Equation (4) is different from the similar scaling form for finite temperatures in \[20\] because \( c \) must be included to introduce the thermal fluctuation in the nonequilibrium situation. (b) To return to the equilibrium scaling form at finite-temperatures \[4\], the scaling function \( f_i \) must satisfy a constraint of \( \partial f_i / \partial c = 0 \) for \( R = 0 \). (c) Besides recovering the full scaling forms of finite-size for closed system in \[13, 14\] by fixing \( c = 0 \) and \( T = 0 \), the nonequilibrium dissipation scaling for spontaneous emissions in zero-temperature open quantum systems can also be studied by fixing \( T = 0 \) in Eq. (3). (d) Note that \( c \) should be small in the weak coupling limit and thus the regime dominated by \( c \) may be inaccessable.

Instead of sweeping \( g \), when \( h_z = R_z t \), one obtains similarly the order parameter

\[
M_h = R_z^{\beta/\nu_r} f_2(g R_z^{-1/\nu_r}, h_z R_z^{-\beta/\nu_r}, T R_z^{\zeta_R}, L^{-1} R_z^{-1/\nu_r}, c R^{-\zeta_R}),
\]

with \( r_z = z + \beta \delta / \nu \). Different regimes and their crossovers can be also readily defined. Different from sweeping \( g \) through the critical point as the ordinary KZM protocols \[12, 14, 20\], here we fix \( g \) and change the symmetry breaking field \( h_z \). This provides a method to determine the critical point from distinct critical behaviors for \( g = 0 \) and \( g \neq 0 \), a method which we shall utilize below and may also be realizable experimentally. Note that in this protocol, the form of \( \tau_d \) remains remarkably if \( R \) and \( r \) are replaced with their counterparts. However, in addition to the fixed \( \tau_s \) for the fixed \( g \), there exists another reaction time diverging with \( |h_z|^{-\nu_z/\beta \delta} \). These result in new competitions but act only as corrections in the FTS regime, showing an advantage of FTS.

Now we show that FTS can provide methods to detect quantum critical properties such as the critical point and critical exponents. For simplicity, we consider \( T = 0 \) and \( c = 0 \) in the thermodynamic limit \( L \to \infty \). According to Eq. (4), at \( h_z = 0 \), \( M_h \) reduces to

\[
M_0(g, R_z) = R_z^{\beta/\nu_r} f_3(g R_z^{-1/\nu_r}),
\]

while the field at \( M_h = 0 \), denoted by \( h_{z,0} \), scales as

\[
h_{z,0}(g, R_z) = R_z^{\beta/\nu_r} f_4(g R_z^{-1/\nu_r}).
\]

Differentiating \( M_h \) with respect to \( h_z \) in Eq. (4), one obtains the susceptibility at zero field,

\[
\chi(g, R_z) = R_z^{\beta(1-\delta)/\nu_r} f_5(g R_z^{-1/\nu_r}).
\]

To fix the critical point, we can define a cumulant

\[
C(g, R_z) = M_0(h_{z,0}) = f_6(g R_z^{-1/\nu_r})
\]

similar to the Binder cumulant in finite-size scaling \[11\]. As \( C \) is a function of only one independent variable, its curves for different \( R_z \) intersect at the critical point \( g = 0 \) at which \( C \) becomes a constant \( f_6(0) \) independent on \( R_z \). This gives the critical point with which all the critical exponents can then be estimated. For example, \( \beta/\nu_r \) and \( \beta \delta / \nu_r \) can be estimated respectively from Eqs. (4) and (6) by fitting \( M_0 \) and \( h_{z,0} \) for a series of \( R_z \) at \( g = 0 \). Similarly, from Eq. (3) at \( c = 0 \), \( T = 0 \), and \( L \to \infty \), \( \beta / \nu_r \) can be estimated by fitting \( M \) for a series of \( R \) at \( g = 0 \). From these three exponent ratios and the scaling law \[4\] \( \beta(1+\delta) = (d+\nu) \nu \) with the space dimensionality \( d \), one can determine all the critical exponents.

As an example of the FTS method to determine critical properties, we consider the one-dimensional (1D) transverse-field Ising model whose Hamiltonian is \[1\]

\[
H = -h_x \sum_{n=1}^{N} \sigma_n^x - \sum_{n=1}^{N-1} \sigma_n^x \sigma_{n+1}^z,
\]

and has been realized in CoNb\(_2\)O\(_6\) experimentally \[33\], where \( \sigma_n^x \) and \( \sigma_n^z \) are the Pauli matrices, \( h_x \) is the transverse field, and the Ising coupling has been set to unity as our energy unit. The model exhibits a continuous QPT from a ferromagnetic phase to a quantum paramagnetic phase at a critical point \( h_{xc} \) (and so \( g = h_z - h_{xc} \)) at \( T = 0 \) \[3\]. The order parameter is the magnetization \( M = \sum_{n=1}^{N} (\sigma_n^z) / N \) for the \( N \) spins with the angle brackets denoting the quantum and/or thermal average. As a method to probe the transition, we add to \( H \) a symmetry-breaking term \(-h_z \sum_{n=1}^{N} \sigma_n^z\).

We illustrate our approach at \( T = 0 \) and \( c = 0 \) at which Eq. (1) is same to Schrödinger’s equation and some exact results are available for comparison. We solve the model using the time-evolving block-decimation algorithm \[34\], which is capable of treating large system sizes. We determine the critical point in Fig. 2 and apply it purposely
FIG. 3: (color online) Estimation of critical exponents. Our agreement of the results collected in Table I shows the intersections is \( h_{xc} = 0.999(2) \), a good estimate of the exact value \( h_{xc} = 1 \). We choose a lattice size of \( L = 2000 \), which has been checked to produce a negligible size effect.

FIG. 2: (color online) Estimation of quantum critical point. Curves of the cumulant \( C \) for different \( R_z \) intersect at the critical point \( h_{qc} \) or \( q = 0 \). Owing to possible errors from the truncation of the singular values in the Schmidt decomposition \( [34] \), however, the intersections are slightly scattered as shown in the inset. Nevertheless, the average of all the intersections is \( h_{qc}^N = 0.999(2) \), a good estimate of the exact value \( h_{qc} = 1 \). We choose a lattice size of \( L = 2000 \), which has been checked to produce a negligible size effect.

FIG. 4: (color online) Nonequilibrium scaling at nonzero temperatures. (a) Data of \( M_h \) versus \( T \) plotted in the inset for the three different sets of \( R_z, c \), and \( L \) so choosing as to fix the value of \( L^{-1} R_z^{z/\nu} \) and \( c R_z^{z/\nu} \) collapse as expected on a single curve for the fixed \( LR_z^{z/\nu} = 1.166 \) and \( c R_z^{z/\nu} = 3.603 \) according to the FTS (4) at \( g = 0 \) \((h_\alpha = 0.999) \), \( h_\alpha = 0 \). (b) If, instead of \( c R_z^{z/\nu} \), we fix all \( c = 0.7 \), the value for \( L = 6 \), and keep others, the rescaled curves then do not collapse.

| \( h_{xc} \) | \( \beta \) | \( \delta \) | \( \nu \) | \( z \) |
|---|---|---|---|---|
| Numerical | 0.999(2) | 0.125(11) | 14.9(6) | 0.98(4) | 1.01(3) |
| Exact \([1]\) | 1 | 0.125 | 15 | 1 | 1 |

TABLE I: Critical point and exponents for the 1D transverse-field Ising model

In conclusion, FTS not only provides a unified understanding of the driving dynamics in general and lights up the dark impulse regime of KZM at zero temperature in particular, but also sheds light on the QCR at nonzero temperatures by establishing its own regime. It offers a powerful unified approach amenable to both numerics and experiments to study equilibrium and nonequilibrium dynamics of quantum criticality. We have shown that in the latter in open systems one must include the dissipation rate as an independent scaling variable and the Lindblad equation can be a valuable framework for such studies. Although we have studied a simple model for illustration, our approach should be applicable to more complex systems as well. In addition, our results indicate that the classical theory of FTS with proper modifications can well describe quantum criticality, new physics may be in action \([2]\) if it is violated.

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*Electronic address: zsuyinshuai@163.com
†Electronic address: lichaoh2@mail.sysu.edu.cn
‡Electronic address: stszf@mail.sysu.edu.cn