Lorentz Beams

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Abstract
A new kind of tridimensional scalar optical beams is introduced. These beams are called Lorentz beams because the form of their transverse pattern in the source plane is the product of two independent Lorentz functions. Closed-form expression of free-space propagation under paraxial limit is derived and pseudo non-diffracting features pointed out. Moreover, as the slowly varying part of these fields fulfills the scalar paraxial wave equation, it follows that there exist also Lorentz-Gaussian beams, i.e. beams obtained by multiplying the original Lorentz beam to a Gaussian apodization function. Although the existence of Lorentz-Gaussian beams can be shown by using two different and independent ways obtained recently from Kiselev [Opt. Spectr. 96, 4 (2004)] and Gutierrez-Vega et al. [JOSA A 22, 289-298, (2005)], here we have followed a third different approach, which makes use of Lie’s group theory, and which possesses the merit to put into evidence the symmetries present in paraxial Optics.

1 Introduction
Optical beams are electromagnetic fields with a well distinguishable mean direction of propagation (that, from now on, we indicate as z axis), in the nearness of which the most part of field’s energy is contained during propagation. After the publication of the fundamental work due to Durnin et al. [1], issued in 1987 about non-diffracting beams in free-space, there was a certain number of scientific investigations on optical fields with the structure of beams and which possess, or approximate at least, the diffraction features of the aforesaid beams [2]-[3]. Indeed, as an ideal non-diffracting beam is physically unrealizable, because on each plane z = constant it carries an infinitive amount of energy, it is possible to obtain diffraction-free like fields only with approximation: these fields are also known as pseudo non-diffracting beams. Examples of that are the well-known Gaussian beams, Bessel-Gauss beams [4] or other optical fields recently introduced by casting the propagation problem in coordinate systems different of rectangular and circular ones [5] that share the properties of maintaining a non-diffracting behaviour only inside a limited spatial range named...
Rayleigh distance. In the present work, we wish to introduce another class of pseudo non-diffracting realizable beams that we will call Lorentz beams (LB for short) as well as their Gaussian apodized version, that, to authors’s knowledge, were never been study before today. If, from a theoretical point of view, the research of new kinds of optical beams is interesting, it is particularly stimulating in the present case because of the physical realizability of the proposed field. This realizability is not due to a Gaussian term, as usually happen for almost all other known optical beams, but from a practical point of view it was shown \[7\]-\[8\] that certain laser sources produce fields that shows fundamental variations with respect to the canonical Gaussian beam. As well-known Gaussian beam is a minimum uncertainty field i.e. it possesses the minimum achievable angular spreading once the spatial extension is fixed; for certain laser sources, e.g. double heterojunction (DH) Ga\(_{1-x}\)Al\(_x\)As lasers, which produce highly diverging fields, a Gaussian description for the transverse fields fails: in this case it was shown that a Lorentzian distribution is a better approximation, as it take into account of the higher angular spreading, being equal the spatial extension \[8\]. The paper is organized as follows: in next section we introduce the LB and we study their propagation under Fresnel or paraxial approximation. In particular we give the closed-form expression for these kind of fields on a generical (\(x,y\)) plane. Furthermore, we will make use of a theory group approach to introduce another class of optical beams obtained by multiplying a Lorentz beams with a two-dimensional Gaussian envelope.

2 Lorentz Beams and paraxial propagation

Let us suppose we have on a source plane, that we decide to be (\(x,y,z=0\)) plane, the following scalar field distribution

\[ V_0(x,y) = \frac{A}{w_x w_y} \left[ \frac{1}{1 + (x/w_x)^2} \right] \left[ \frac{1}{1 + (y/w_y)^2} \right] \]

where \(A\) is a constant value and \(w_x\) and \(w_y\) are parameters related to the beam width, with \(A, w_x\) and \(w_y\) \(\in \mathbb{R}\). This kind of field is the product of two functions of \(x\) and \(y\) variables which have the form of a Lorentzian function of parameter \(w_x\) and \(w_y\). The Lorentzian is a well known bell-shaped curve used principally to describe the spectral lines of simple dinamical systems usually present in physics. Starting from the field in (1) we wish to derive the form that such a field acquires during free propagation. To do this, we pass to the Fourier domain, calculating the plane waves spectrum on \(z=0\). We have

\[ A_0(p,q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{A}{w_x w_y} \left[ \frac{1}{1 + (x/w_x)^2} \right] \left[ \frac{1}{1 + (y/w_y)^2} \right] \exp(-2\pi px) \times \right. \]

\[ \left. \exp(-2\pi qy) dxdy \right] = \frac{A}{w_x w_y} \int_{-\infty}^{\infty} \left[ \frac{1}{1 + (x/w_x)^2} \right] \exp(-2\pi px) dx \times \]

\[ \int_{-\infty}^{\infty} \left[ \frac{1}{1 + (y/w_y)^2} \right] \exp(-2\pi qy) dy = A\pi^2 \exp(-2\pi |p| w_x) \exp(-2\pi |q| w_y) \]
where $A_0(p,q)$ is the complex spectrum, $p$ and $q$ are the conjugated variables to $x$ and $y$ variables respectively. Once we know the spectrum of plane waves on $z = 0$ we can calculate it on a generical $(x,y)$-plane as follows

$$A_z(p, q) = A_0(p, q) \exp(2i\pi mz) = \pi^2 \exp(-2\pi|p|w_x) \exp(-2\pi|q|w_y) \exp(2i\pi mz)$$

because it is well known in which fashion a plane wave propagates in free-space.

The parameter $m$ is a complex variable that must fulfil the relationship

$$m = \sqrt{(1/\lambda^2 - p^2 - q^2)}$$

and it is imaginary for evanescent waves, in which $p^2 + q^2 \geq 1/\lambda^2$, and real for homogeneous waves where $p^2 + q^2 \leq 1/\lambda^2$, where $\lambda$ is the wavelength. We know that a field is said to be homogeneous when the spectrum $A_z(p,q)$ is different from zero only inside the circle $p^2 + q^2 \leq 1/\lambda^2$.

2.1 Fresnel or paraxial limit

Observing the (2) we deduce that if the values of $w_x$ and $w_y$ are enoughly greater than wavelength $\lambda$, the majority contribution to the field arises from homogeneous waves having the amplitude $A_z(p,q)$ corresponding to points, in Fourier plane $(p, q)$, near to the origin. In this case one is authorized to do the following expansion

$$m = \sqrt{(1/\lambda^2 - p^2 - q^2)} \approx 1 + (p^2 + q^2)\lambda$$

by retaining only the first two terms in the series, so that the plane wave spectrum in (3) becomes

$$A_{z_{parax}}(p, q) = \pi^2 \exp(-2\pi|p| w_x - i\pi\lambda z p^2) \exp(-2\pi|q| w_y - i\pi\lambda z q^2) \exp(ikz)$$

in which $k = 2\pi/\lambda$ is the wave number. Now, as we know the plane wave spectrum on $z$ we can also derive the full form of the field if we are able to inverse the two-dimensional Fourier transform

$$V(x, y, z) = \int_\infty^\infty \int_\infty^\infty [\pi^2 \exp(ikz) \exp(-2\pi|p| w_x - i\pi\lambda z p^2) \times \exp(-2\pi|q| w_y - i\pi\lambda z q^2)] dp dq$$

$$= \pi^2 \exp(ikz) \int_\infty^\infty \exp(-2\pi|p| w_x - i\pi\lambda z p^2) \exp(i2\pi p) dp \times \int_\infty^\infty \exp(-2\pi|q| w_y - i\pi\lambda z q^2) \exp(i2\pi q) dq$$

It is important to note that the role of paraxial approximation was to give a plane wave spectrum $A_{z_{parax}}(p,q)$ factorized in two terms, each one depending only from a single Fourier variable, $p$ or $q$; this is a properties that was not fulfilled by the exact spectrum in (5) for the presence of the term $\exp(i2\pi mz)$. As a consequence of that also the complex field $V(x, y, z)$ is in a similar factorized form. To obtain a solution of (4) let us focus on the integral

$$I = \int_\infty^\infty \exp(-2\pi|p| w_x - i\pi\lambda z p^2) \exp(i2\pi p) dp$$

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It can be written as

\begin{align*}
I &= \int_0^\infty \exp(-2\pi pw_x - i\pi \lambda z p^2) \exp(i2\pi px) dp + \\
&+ \int_{-\infty}^0 \exp(2\pi pw_x - i\pi \lambda z p^2) \exp(i2\pi px) dp
\end{align*}

For the first of two integral in right-hand side we have

\begin{align*}
I &= \int_0^\infty \exp(-2\pi pw_x - i\pi \lambda z p^2) \exp(i2\pi px) dp = \\
&= \exp(c) \frac{\exp(c)}{a} \int_0^\infty \exp(-s^2) ds
\end{align*}

where we have defined the following auxiliaries variables

\begin{align*}
a^2 &= i\pi \lambda z \\
b &= \pi (w_x - ix)/a \\
c &= b^2 \\
s &= ap + b
\end{align*}

and \textit{erf}(x) is the usual error function \textit{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-s^2) ds [6]. The other integral appearing in (11) can be easily calculated by observing that it is equal to that just derived in (13) after having substituted the variable \(x\) with \(-x\). On utilizing this result we finally obtain the full form of the field

\[ V(x, y, z) = \frac{A\pi^2 \exp(i kz)}{4i\lambda z} [V_x^+(x, z) + V_x^-(x, z)][V_y^+(y, z) + V_y^-(y, z)] \]

where

\[ V_x^\pm(r, z) = \frac{\exp[\pi(w_x \pm ir)^2/i\lambda z]}{\sqrt{i\pi \lambda z}} \{1 - \text{erf}[\pi(w_x \pm ir)/\sqrt{i\pi \lambda z}]\} \]

and \(r = x, y\). Equation (18) is the principal result of the present work and in next sections we analyse more in detail the propagation features of these beams.

### 2.2 Propagation and diffraction-free range

We expect that the field in (18) changes its shape during propagation as a consequence of diffraction. It is well known, however, that it is possible to define a diffraction-free range, i.e. a linear distance on \(z\)-axis, under which the beam remains essentially unchanged. To do this we write the beam to values near to the source plane \((z = 0)\) by taking advantage of the expansion of error function for high values of its argument

\[ \text{erf}(s) \approx 1 + \frac{s}{\pi} \exp(-s^2) \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k - 1/2)}{s^{2k}} \]

\( (20) \)
On utilizing this expansion it is easy to verify that \(18\) reduces to \(1\). Indeed one obtains, by keeping only the first term

\[
V(x, y, 0) = \frac{A\pi^2}{4} \lim_{z \to 0} \frac{1}{i\lambda z} [V^+_x(x, z) + V^-_x(x, z)]\left[V^+_y(y, z) + V^-_y(y, z)\right] =
\]

where we used the identity \(\Gamma(1/2) = \sqrt{\pi}\). The shape of the field will be practically unchanged as far as the second term in expansion will be negligible respect to the first. This conducts us to the following condition,

\[
\frac{\Gamma(1/2)}{\pi(w_r \pm i\lambda z)} \gg \frac{\Gamma(3/2)}{\pi(w_r \pm i\lambda z)} \quad \text{for } r = x, y.
\]

Equation \(22\) leads to

\[
\Gamma(1/2) \gg \Gamma(3/2) \quad \frac{\lambda}{\pi w^2 z}
\]

and finally (recall that \(\Gamma(3/2) = \sqrt{\pi}/2\)) we arrive to the evaluation of the diffraction-free range (or Rayleigh distance) for this kind of beams

\[
z_R = \frac{2\pi w^2}{\lambda}
\]

In \(24\) we have let \(w_x = w_y = w\) to simplify the analysis. If one does not make this assumption there will exist two different diffraction-free ranges, one to \(x\)-axis and another to \(y\)-axis. In figure \(1\) we report the modulus of field’s amplitude evaluated to different distances from the source. In particular we observe that, for distances sufficiently shorter than \(z_R\) the effect of diffraction is negligible, as expected, while when \(z \approx z_R\) the diffraction affects the field shape.

### 3 Lorentz-Gauss beams

So far we have evaluated the paraxial propagation of the beam essentially by using an integral approach. In fact, once we knew the field on the source plane \(V_0(x, y)\) we were able to obtain it on every plane as dictated by Fresnel theory, namely we had

\[
V(x, y, z) = -\frac{i\exp(ikz)}{\lambda z} \int \int \infty V_0(\xi, \eta) \exp\left[i\frac{k}{2z}(x - \xi)^2 + (y - \eta)^2\right] d\xi d\eta
\]
Actually we used (25) in Fourier space to obtain the plane wave spectrum on $z$ and from it, by mean an inverse Fourier transformation, we pointed out the field $V(x, y, z)$. If we let

$$V(x, y, z) = \exp(ikz)f(x, y, z)$$

(26)

where $f(x, y, z)$ is the slowly varying part of $V(x, y, z)$, we know that, if $V(x, y, z)$ fulfills the integral (24), then $f(x, y, z)$ fulfills a differential equation, known as paraxial wave equation, that in tridimensional space holds

$$\nabla^2_T f(x, y, z) + 2ikf_z(x, y, z) = 0$$

(27)

where $\nabla^2_T$ is the two-dimensional Laplace operator in the transverse plane, i.e. $\nabla^2_T f = f_{xx} + f_{yy}$ and $f_z$ is the partial derivative respect to $z$ variable. This equation has very special properties, and in particular we here are interested to its symmetry features. In fact there exists a beautiful theory, due to the mathematician Sophus Lie, that allows to perform an analysis on the symmetry groups associated to a particular differential equation (or, more in general, to a system of differential equations). Essentially, the theory says that there exist some differential transformations which act like operators on system’s solutions and that lead to others solutions of the same equation, when applied to an existing and known starting solution. The theory bases itself upon a certain number of theorems in the context of differential geometry, and we will not enter in details; however, in Appendix we put a proof of the derivation of the symmetry group we utilize in the following (see, for example, in [10] for the theory foundations). This theory was used, for example, by Winche [11] to show that Hermite-Gauss and Laguerre-Gauss beams (with complex argument) can be generated from the fundamental Gaussian beam simply by applying to it the powers of certain differential Lie operators. Among all the symmetry transformations associated to (27) we concentrate upon only one which states that if $f(x, y, z)$ is a solution of the aforesaid equation also will be the function $f^*(x, y, z)$ where

$$f^*(x, y, z) = \frac{1}{1 + iz/L} \exp[-\frac{x^2 + y^2}{w_0^2(1 + iz/L)}]f\left(\frac{x}{1 + iz/L}, \frac{y}{1 + iz/L}, \frac{z}{1 + iz/L}\right)$$

(28)

with $w_0$ and $L$ two real parameter. The property in (28) was recently proved, utilizing two different approaches, by Kiselev [9] that has utilized a separation variable method and by Gutierrez-Vega et al. [5], in which the authors obtained the same result by inserting a well-constructed trial function into paraxial wave equation. It is interesting to note that the way we indicated here is a third different method, a method which possesses the merit to underline which is the foundation of this result, i.e. an hidden symmetry [11] and which shows that all beams with a Gaussian envelope are intimately connected to the paraxial wave equation. As a consequence of (28) we immediately conclude that also exist the Lorentz-Gauss beams, namely

$$V^*(x, y, z) = \frac{A\pi^2}{4} \frac{\exp(ikz)}{i\lambda z} \left\{ V^+_{x}\left(\frac{x}{1 + iz/L}, \frac{z}{1 + iz/L}\right) + V^-_{x}\left(\frac{x}{1 + iz/L}, \frac{z}{1 + iz/L}\right) \right\} \times$$

$$\times \left\{ V^+_{y}\left(\frac{y}{1 + iz/L}, \frac{z}{1 + iz/L}\right) + V^-_{y}\left(\frac{y}{1 + iz/L}, \frac{z}{1 + iz/L}\right) \right\} \times$$

$$\times \exp[-\frac{x^2 + y^2}{w_0^2(1 + iz/L)}]$$

(29)
The shape of this kind of field depends by the choices of the parameters $w_x, w_y, w_0$ and $L$. In particular if we choose $w_0 < w_x, w_y$, we obtain a beams that behaves like a Lorentz beam near to the $z$ axis and like a Gaussian beam far from it. If we put $L = z_R$, Lorentz-Gauss beams [28] for $w_0 = w_x(w_y)$ shows a field shape more defined around $z$ axis with respect to the Lorentz beams [15], i.e. on the $x−y$ plane the LB has a wider bell-shaped field function with respect to [29].

4 Conclusion

A new kind of tridimensional, rectangularly-symmetric, realizable scalar optical beams has been introduced. On the source plane these beams are the product of two independent Lorentz function and the exact analytical expression for the field on a general plane, under paraxial regime, has been derived. In particular it is interesting to note that it deals of a rare case in which one is in the presence of physically realizable fields, the propagation of which is known in closed-form, that does not possess a Gaussian envelope term. This kind of field can represent a valid candidate to modelize the shape of fields generated by certain laser sources, as double heterojunction (DH) $Ga_{1-x}Al_xAs$ lasers. Using a Lie group approach we introduced also the solution obtained by modulating the aforesaid beams with a Gaussian envelope.

5 References

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A Symmetry groups of scalar paraxial wave equation

Consider the 3-D scalar paraxial wave equation

$$\nabla^2 u(x, y, z) + 2iku_z(x, y, z) = 0 \quad (30)$$

We wish to show that this equation admits a class of solutions that are modulated by a two-dimensional Gaussian envelope. First of all, to this equation it is possible to associate a manifold $M \in X \times U$, where $X = \mathbb{R}^p$, with $p = 3$ meaning number of independent variables and $U = \mathbb{R}^q$, with $q = 1$ meaning the number of dependent variables. On such a manifold it is also possible to define a tangent vector field in the following form

$$v = \xi(x, y, z, u) \partial_x + \eta(x, y, z, u) \partial_y + \tau(x, y, z, u) \partial_z + \phi(x, y, z, u) \partial_u \quad (31)$$

In the context of Lie group theory [10] tangent vector fields are the generators of symmetry transformations through the following relationship

$$\psi(\epsilon, v) = \exp(\epsilon v) \quad (32)$$

where $\psi$ is the transformation induced by the vector $v$ and $\epsilon$ is a real parameter characterizing the group. A symmetry transformation is a map that allows to pass from a starting point $(x, y, z, u)$ on the manifold $M$ to another point $(x', y', z', u')$ on the same manifold by mean the relation

$$(x', y', z', u') = \psi(\epsilon, v)(x, y, z, u) = \exp(\epsilon v)(x, y, z, u) \quad (33)$$

To find the explicit expression of the vector field in (31), one can utilize the following procedure. It is necessary to extend the space $X \times U$ in order that it also contains the second order derivatives, an operation said prolongation. By consequence the prolonged tangent vector field becomes

$$v = \xi \partial_x + \eta \partial_y + \tau \partial_z + \phi \partial_u + \phi^x \partial_{u_x} + \phi^y \partial_{u_y} + \phi^z \partial_{u_z} + \phi^{xx} \partial_{u_{xx}} + \phi^{yy} \partial_{u_{yy}} + \phi^{zz} \partial_{u_{zz}} + \phi^{xy} \partial_{u_{xy}} + \phi^{xz} \partial_{u_{xz}} + \phi^{yz} \partial_{u_{yz}} \quad (34)$$

where we have dropped the dependence from the variables $x, y, z, u$. We can rewrite it as

$$v = \sum_{i=1}^{p} \xi_i \partial_{x_i} + \phi \partial_u \sum_{J} \phi^J \partial_{u_J} \quad (35)$$

by defining $J = (j_1, j_2, ... j_l), 1 \leq j_l \leq p, 1 \leq l \leq n$ where $n$ is the equation order. All the coefficients in (31) are expressible in terms of $\xi, \eta, \tau$ and $\phi$ and their derivatives through the formula

$$\phi^J = D_J(\phi - \sum_{i=1}^{p} \xi_i u_{J,i}) + \sum_{i=1}^{p} \xi_i u_{J,i} \quad (36)$$

where $u_i = \partial u/\partial x_i$ and $u_{J,i} = \partial u_{J}/\partial x_i$ with $x_i$ generical variable and $D_J$ representing the total derivative. Among all such 13 coefficients only $\phi^{xx}, \phi^{yy}, \phi^{zz}$ are important for our purpose. Indeed, under certain hypotheses, it is possible
to obtain all the symmetry group of transformation of scalar paraxial wave equation through the condition

\[ v[uxx + uyy + 2iku_z] = 0 \]  \hspace{1cm} (37)

that, by taking into account relation (34) implies

\[ \phi^{xx} + \phi^{yy} + 2iku^5 = 0 \]  \hspace{1cm} (38)

On performing the calculations as in (36) we find

\[ \phi^{xx} = \phi_{xx} + u_z(2\phi_{xx} - \xi_{xx}) + uy(-\eta_{xx}) + uz(-\tau_{xx}) + u_x^2(\phi_{uu} - 2\xi_{uu}) + \\
+ uxuy(-2\eta_{ru}) + u_xuz(-2\tau_{ru}) + uxxux(-3\xi_u) + uxxuy(-\eta_u) + \\
+ u_{xx}uz(-\tau_u) + u_{xy}ux(-2\eta_x) + u_{xx}(\phi_u - 2\xi_x) + u_x^3(-3\xi_{uu}) + \\
+ u_{yy}(-2\eta_y) + u_{xx}(-2\tau_x) + uzux(-2\tau_u) + u_x^2uy(-\eta_u) + \\
+ u_x^2u_z(-\tau_{uu}) \]  \hspace{1cm} (39)

\[ \phi^{yy} = \phi_{yy} + uy(2\phi_{yy} - \eta_{yy}) + ux(-\xi_{yy}) + uz(-\tau_{yy}) + u_y^2(\phi_{uu} - 2\eta_{uu}) + \\
+ uxuy(-2\xi_{yy}) + uyuz(-2\tau_{yy}) + uyyuy(-3\eta_y) + u_{yy}ux(-\xi_u) + \\
+ u_{yy}uz(-\tau_u) + u_{xy}uy(-2\xi_y) + u_{yy}(\phi_u - 2\eta_y) + u_y^3(-3\eta_{uu}) + \\
+ u_{yy}(-2\xi_y) + u_{yy}(-2\tau_u) + uyuy(-2\tau_u) + u_y^2ux(-\xi_{uu}) + \\
+ u_y^2u_z(-\tau_{uu}) \]  \hspace{1cm} (40)

\[ \phi^5 = \phi_z + u_z(2\phi_u - \tau_z) + ux(-\xi_z) + uzux(-\xi_u) + uy(-\eta_z) + \\
+ uzuy(-\eta_u) + u_z^2(-\tau_u) \]  \hspace{1cm} (41)

Now we equate the right and left-hand side homologous terms appearing in (38) and finally we obtain

\[ \phi_{xx} + \phi_{yy} = \phi_t \]  \hspace{1cm} (42)

\[ 2(\phi_{xx} - \xi_{xx}) - \xi_{yy} = -\xi_t \]  \hspace{1cm} (43)

\[ 2(\phi_{yy} - \xi_{yy}) - \xi_{xx} = -\xi_y \]  \hspace{1cm} (44)

\[ 2\xi_x = \tau_t \]  \hspace{1cm} (45)

\[ 2\eta_y = \tau_t \]  \hspace{1cm} (46)

\[ \tau_x = \tau_y = \tau_u = 0 \]  \hspace{1cm} (47)

\[ \eta_u = \xi_u = 0 \]  \hspace{1cm} (48)

\[ \phi_{uu} = 0 \]  \hspace{1cm} (49)

where we have defined an auxiliary variable \( t = z/(2ik) \). Solving this system of equations is not difficult but we report here the result only, that can be verified by substitution,

\[ \tau = c_1 + 2c_2t + 4c_3t^2 \]  \hspace{1cm} (50)

\[ \xi = c_2x + 4c_3tx - 2c_4t + c_5 \]  \hspace{1cm} (51)

\[ \eta = c_2y + 4c_3ty - 2c_4t + c_6 \]  \hspace{1cm} (52)
\[ \phi = [-c_3(x^2 + y^2) + c_4(x + y) - 4c_3t + c_7]u + \alpha(x, y, t) \] (53)

where \( \alpha \) is a generical function and \( c_i(i = 1, 2, ... 7) \) are integration constants. Each of these constants is related to a particular generator of symmetry \( v_i \) which can be obtained by letting tidily all the constants to zero except the \( i \)th. Among all such a transformations there is the following one

\[ v = 4tx\partial_x + 4ty\partial_y + 4t^2\partial_t + (-x^2 - y^2 - 4t)u\partial_u \] (54)

From (33)-(54) follows that, if we indicate as \( u(x, y, s) \) a solution then also the following one represents a valid one

\[ u^*(x, y, s) = \frac{1}{1 + 4es} \exp[-\epsilon \left(\frac{x^2 + y^2}{1 + 4es}\right)]u\left(\frac{x}{1 + 4es}, \frac{y}{1 + 4es}, \frac{z}{1 + 4es}\right) \] (55)

On coming back to the old coordinate \( z \) and letting \( \epsilon/2k = 1/L \) and \( w^2 = 1/\epsilon \), it holds

\[ u^*(x, y, z) = \frac{1}{1 + iz/L} \exp[-\frac{x^2 + y^2}{w^2(1 + iz/L)}]u\left(\frac{x}{1 + iz/L}, \frac{y}{1 + iz/L}, \frac{z}{1 + iz/L}\right) \] (56)

that represents the result we were looking for.
Figure 1: Modulus of field’s amplitude of a Lorentz beam on: a) source plane, i.e. $z = 0$ b) when $z = 0.1z_R$ c) when $z = 0.5z_R$ d) when $z = z_R$. The parameters are chosen as follows: $\lambda = 0.6328\mu m$, $w_x = w_y = w = 10^3\lambda$. $z_R = 2\pi w^2/\lambda = 3.976m$