High-speed railway vehicles operate much faster than traditional railway vehicles. After a four-axle high-speed railcar is modeled, an analytical solution is employed in this paper to solve dynamic equations. According to this analytical solution, the coupling of four-axle high-speed railcar equations depends strictly on the adhesion coefficient. A novel parallel control strategy is then formulated to prevent wheels from slipping and track the desired velocity profile. The proposed control strategy includes feedback linearization and sliding mode controllers to achieve the desired performance. Finally, the simulation results indicated the effectiveness of the proposed control system in the high-speed railcar such that the tracking error is less than 12%.
presents a numerical solution for the validation of analytical results. In Section 4, a control system is designed to control the slip and track the desired velocity profile. Feedback linearization and sliding mode methods are employed in Sections 4.1 and 4.2 to design a slip controller and a velocity profile tracker. The closed-loop results are reported in Section 4.3, whereas validation, stability, and robustness of the proposed controller are evaluated in Sections 4.4, 4.5, and 4.6, respectively.

2. Modeling the Four-Axle Railcar

A high-speed multiple unit is a high-speed train on which the drive power is distributed underneath the floors of several railcars. The distribution of power over multiple railcars would allow for higher acceleration than a single locomotive. Different drive power distribution methods can be implemented through high-speed multiple units. Figure 1 demonstrates various types of high-speed multiple units.

According to Figure 1, each railcar is self-propelled in different types of high-speed multiple units. Therefore, the coupling force between the two railcars can be neglected [9]. For the longitudinal dynamic modeling of a four-axle railcar, a free-body diagram of only one railcar can be regarded in Figure 2. The car body is assumed to have no longitudinal movement in proportion to the wheels. Each wheelset is also being rotated independently; hence, the system has five degrees of freedom.

According to Figure 2, motion equations in the longitudinal direction are obtained as follows:

\[ M \ddot{x}_{\text{car}} = F_{a1} + F_{a2} + F_{a3} + F_{a4} - F_{\text{loss}}, \]  

\[ J_i \ddot{\omega}_i = T_{mi} - r_i F_{ai}, \quad i = 1, 2, 3, 4, \]

where \( F_{\text{loss}} = \frac{1}{2} C_d A_p \rho V_{\text{car}}^2, \) (3)

where \( T_{mi} \) refers to the traction motor torque, and \( F_{\text{loss}} \) denotes the drag force. According to equations (1) and (2), the dynamic equations of railcars are completely coupled due to the adhesion forces among wheels and rails. Calculated through the following equations, these forces are the major tractive forces in the acceleration and braking systems of a railcar [10]:

\[ F_{ai} = \mu_i N_i, \quad i = 1, 2, 3, 4, \]

\[ \mu = c_e (-a \nu_i) - d_e (-b \nu_i), \]

\[ \nu_i = r_i \omega - V_{\text{car}}, \]

where \( N_i \) indicates the normal force, and \( \mu_i \) shows the adhesion coefficient, whereas \( \nu_i \) represents the slip velocity in km/h. Moreover, parameters \( a, b, c, \) and \( d \) in equation (5) are constant values depending on the conditions of the contact surface between wheels and rails [10]. Figure 3 draws a comparison between equation (5) with experimental results by considering 8% uncertainty for parameters \( a \) and \( b \) [11].

According to Figure 3, equation (5) fits the experimental results in the peak area. However, after reaching its maximum value, the curve in equation (5) plunges more quickly than the experimental results curve. Hence, equation (5) presents a good accuracy in the peak area that can be used in the acceleration mode.

The equations of four-axle railcar can be written in the state-space form as follows:

\[ \begin{align*}
\dot{z}_1 &= \frac{1}{M} \left( T_{m1} - r_1 \left( c_e \left( -3.6a (r_1 z_1 - z_1) \right) - d_e \left( -3.6b (r_1 z_1 - z_1) \right) N_1 \right) \right), \\
\dot{z}_2 &= \frac{1}{J_2} \left( T_{m2} - r_2 \left( c_e \left( -3.6a (r_2 z_2 - z_1) \right) - d_e \left( -3.6b (r_2 z_2 - z_1) \right) N_2 \right) \right), \\
\dot{z}_3 &= \frac{1}{J_3} \left( T_{m3} - r_3 \left( c_e \left( -3.6a (r_3 z_1 - z_1) \right) - d_e \left( -3.6b (r_3 z_1 - z_1) \right) N_3 \right) \right), \\
\dot{z}_4 &= \frac{1}{J_4} \left( T_{m4} - r_4 \left( c_e \left( -3.6a (r_4 z_1 - z_1) \right) - d_e \left( -3.6b (r_4 z_1 - z_1) \right) N_4 \right) \right), \\
\dot{z}_5 &= \frac{1}{M} \left( \sum_{i=1}^{4} \left( c_e \left( -3.6a (r_i z_1 - z_1) \right) - d_e \left( -3.6b (r_i z_1 - z_1) \right) N_i \right) - \frac{1}{2} C_d A_p \rho V_{\text{car}}^2 \right), \\
z_1 &= \omega_1, \\
z_2 &= \omega_2, \\
z_3 &= \omega_3, \\
z_4 &= \omega_4, \\
z_5 &= V_{\text{car}}.
\end{align*} \]
3. Analytical Solution to Dynamic Equations

The Adomian decomposition method (ADM) is adopted in this section to offer an analytical solution for the state-space model of the system in response to the constant torque. The railcar model is considered a quarter to avoid the calculation complexity of Adomian polynomials for the whole model. By replacing the constant coefficients from Table 1, the state-space model is presented as follows:

\[
\begin{align*}
\dot{z}_1 &= -782e^{-0.89424z_1+1.944z_5} + 782e^{-1.9872z_1+4.32z_5} + 0.017T_m, \\
\dot{z}_5 &= 10e^{-0.89424z_1+1.944z_5} - 10e^{-1.9872z_1+4.32z_5} - 0.0000953382353z_5^2.
\end{align*}
\] (8)

The ADM is obtained from equation (8):

\[
\begin{align*}
\sum_{n=0}^{\infty} z_{1n} &= \bar{z}_{10} + L^{-1}\left(0.017T_m\right) + L^{-1}\left(\sum_{n=0}^{\infty} A_{1n}\right), \\
\sum_{n=0}^{\infty} z_{5n} &= \bar{z}_{50} + L^{-1}\left(\sum_{n=0}^{\infty} A_{5n}\right).
\end{align*}
\] (9)

After equation (9) is solved through the novel algorithm to determine Adomian polynomials [12], the following equations are obtained:

\[
\begin{align*}
z_{10} &= z_{11} + z_{12}, \\
z_{50} &= z_{51} + z_{52},
\end{align*}
\] (10)

\(z_{10}\) to \(z_{52}\) are shown in appendix A. According to equation (A.1), the longitudinal velocity of railcar \(z_5\) and the rotational velocity of wheel \(z_1\) depend strictly on the traction motor torque. Thus, the results are reported in two cases to reach an in-depth perception of the traction motor torque effect. The first case is the situation in which the traction motor torque is very high (in order of \(10^4\)). The other case is a state where the amount of traction motor torque is low (in order of \(10^2\)).

3.1. Case I. \(T_m = 20000\text{ N.m}\). Figure 4 reports the results of Case I.

According to Figure 4, the railcar reaches a very low velocity of 0.0416 km/h during 0.008 s (green curve) due to no appropriate adhesion coefficients. The appropriate adhesion coefficient exists only at the beginning of motion, and its amount reaches zero after nearly 0.008 s. This can also be proven by equation (A.2) which is a simplified version of equation (A.1). According to equation (A.2), coefficients of terms in which \(T_m\) exists in the equations belonging to \(z_{12}\) are in order of \(10^9\), whereas they are in order of \(10^7\) for terms of \(z_{52}\). Hence, the effect of the input torque on the wheel rotational speed is greater than the railcar velocity. As a result, the adhesion coefficient becomes negligible after 0.008 s.

3.2. Case II. \(T_m = 200\text{ N.m}\). The ADM solution is not valid when the input torque is 200 N.m due to an increase of adhesion in the contact area between wheels and rails. In other words, once \(T_m = 20000\) (see Figure 4), the adhesion coefficient has a considerable value only at the beginning of motion, whereas it is negligible after nearly 0.008 s. In this case, the coupling of equation (8) decreases, and the presented ADM method is feasible. However, when \(T_m = 200\), the adhesion coefficient has a remarkable value. Consequently, equation (8) is completely coupled.

Hence, it is better to extend the terms carrying power orders in the railcar-state equations until equations exit the power form. The Taylor expansion of order four around the point of \(v_1 = 1\text{ km/h}\) is employed to extend the power terms of adhesion coefficient relation.

\[
\begin{align*}
\mu_{\text{exp}} &= (e^{-a}\cdot c - d\cdot e^{-b}) + (be^{-b}d - ae^{-a}) (v_1 - 1) \\
&\quad + \left(-b^2e^{-b}d + a^2e^{-a}\right) (v_1 - 1)^2 \\
&\quad + \left(b^3e^{-b}d - a^3e^{-a}\right) (v_1 - 1)^3 \\
&\quad + \left(-b^4e^{-b}d + a^4e^{-a}\right) (v_1 - 1)^4.
\end{align*}
\] (11)

Ozis and Yıldırım [13] indicated that the ADM method was equivalent to the Homotopy perturbation method (HPM) for equations with exponential nonlinearity in which exponential terms are expanded as the Taylor series [14]. Furthermore, Li [15] illustrated that ADM and HPM were theoretically equivalent in solving nonlinear equations.
Table 1: Characteristics of the ICE3 high-speed railcar.

| Symbol | Definition (Unit) | Value |
|--------|-------------------|-------|
| $M$    | Total mass (tone) | 68    |
| $r$    | Wheel radius (m)  | 0.46  |
| $J$    | Axle inertia (kgm$^2$) | 100   |
| $C_d$  | Coefficient of drag force | 0.25  |
| $A$    | Front surface area of railcar (m$^2$) | 10.58 |

Figure 4: Comparison between equation (5) and experimental results. (a) Experimental results [11]. (b) Equation (5).

Figure 3: Results of the ADM solution for velocity and adhesion coefficient (Case I).
Therefore, HPM is employed to solve equation (7) when \( T_m = 200 \text{ N.m} \).

HPM form of equation (7) can be written as follows:

\[
(1 - p) \frac{dz_1}{dt} + p \left( \frac{dz_1}{dt} - \frac{1}{J_w} (T_{m1} - r_1 \mu \exp \alpha N_1) \right) = 0,
\]

\[
(1 - p) \frac{dz_2}{dt} + p \left( \frac{dz_2}{dt} - \frac{1}{J_w} (T_{m2} - r_2 \mu \exp \alpha N_2) \right) = 0,
\]

\[
(1 - p) \frac{dz_3}{dt} + p \left( \frac{dz_3}{dt} - \frac{1}{J_w} (T_{m3} - r_3 \mu \exp \alpha N_3) \right) = 0,
\]

\[
(1 - p) \frac{dz_4}{dt} + p \left( \frac{dz_4}{dt} - \frac{1}{J_w} (T_{m4} - r_4 \mu \exp \alpha N_4) \right) = 0,
\]

\[
(1 - p) \frac{dz_5}{dt} + p \left( \frac{dz_5}{dt} - \frac{1}{J_w} (\mu \exp \alpha N_1 + \mu \exp \beta N_2 + \mu \exp \gamma N_3) + \frac{\mu \exp \beta N_4}{2} C_d A p a z_5^2 \right) = 0.
\]

Variables \( z_1 \) to \( z_5 \) of equation (12) are defined as follows:

\[
z_1 = z_{10} + p z_{11},
\]

\[
z_2 = z_{20} + p z_{21},
\]

\[
z_3 = z_{30} + p z_{31},
\]

\[
z_4 = z_{40} + p z_{41},
\]

\[
z_5 = z_{50} + p z_{51}.
\]

4.1. Slip Control through Feedback Linearization. By differentiating equation (5) and equating it to zero, the slip velocity allows the highest coefficient of adhesion to be obtained nearly 1.21 km/h. Therefore, slip control in acceleration aims to reach this slip velocity and consequently reaches the highest coefficient of adhesion. The linearized equations of four-axle railcar movement can be achieved as follows by using the feedback linearization method (see appendix B for details):

\[
\psi_1 = \chi_1,
\]

\[
\psi_2 = \chi_2,
\]

\[
\psi_3 = \chi_3,
\]

\[
\psi_4 = \chi_4,
\]

\[
\psi_1 = \frac{1}{2M} \begin{pmatrix}
-A C_d \psi_1^2 + 2 N_f e^{-3.6 \psi_1} - 2 N_g e^{-3.6 \psi_1} + 2 N_f e^{-3.6 \psi_1} & \\
+2 N_f e^{-3.6 \psi_1} - 2 N_g e^{-3.6 \psi_1} + 2 N_f e^{-3.6 \psi_1} & 0
\end{pmatrix},
\]

where
\( \chi_1 = -k_1 e_1, \)
\( \chi_2 = -k_2 e_2, \)
\( \chi_3 = -k_3 e_3, \)
\( \chi_4 = -k_4 e_4. \) \hfill (16)

where \( e_1-e_4 \) indicate the error between the desired and existent values of slip velocity, and \( k_1-k_4 \) depict positive coefficients assuring that this error approaches zero.

4.2. Tracking the Desired Velocity Profile. The safe and efficient operation of a railway system depends mainly on the performance of its automatic train control (ATC) system [16, 17]. In fact, an ATC system consists of three subsystems, i.e., automatic train operation (ATO), automatic train protection (ATP), and automatic train supervision (ATS) [18], out of which the ATO plays a key role in the urban rail transit system. Generating the desired velocity profile, the ATO tries to track the created velocity profile. Therefore, the ATO can be adopted to help trains perform the operation plan accurately, improve transportation efficiency, enhance the comfort level of passengers, and save energy [19]. Numerous studies have analyzed the ATO design for trains [18–21] and high-speed trains [16].
In this section, regarding the velocity profile of a four-axle ICE3 railcar, a controller is designed to track the desired velocity profile. According to [22], Figure 9 shows the ICE3 profile in acceleration. Evidently, ICE3 reaches 300 km/h in 370 s. To pursue this velocity profile, it should first be estimated by using a function. For this purpose, a seventh-order Gaussian function is used as follows with coefficients shown in Table 2:

\[
V_d = a_1 \exp\left(-\frac{(b_1 - t)^2}{c_1^2}\right) + a_2 \exp\left(-\frac{(b_2 - t)^2}{c_2^2}\right) + a_3 \exp\left(-\frac{(b_3 - t)^2}{c_3^2}\right)
\]

\[
+ a_4 \exp\left(-\frac{(b_4 - t)^2}{c_4^2}\right) + a_5 \exp\left(-\frac{(b_5 - t)^2}{c_5^2}\right)
\]

\[
+ a_6 \exp\left(-\frac{(b_6 - t)^2}{c_6^2}\right) + a_7 \exp\left(-\frac{(b_7 - t)^2}{c_7^2}\right).
\]
According to Figure 9, this function fits the ICE3 velocity profile properly.

The sliding mode method is employed to track the velocity profile. This method has been implemented successfully in various engineering fields [23–25]. In this study, the control variable is the railcar velocity, the desired value of which is the velocity profile defined in equation (17). The control input can be obtained from \( \dot{s} = \dot{\hat{e}} + \lambda \dot{e} = 0 \) in the following way for the railcar equations:

\[
\begin{align*}
T_m &= u - \kappa \tanh(\lambda e), \\
0 &= \left( -0.555556 J_z \dot{\omega} \sum_i \left( a \left( \alpha (1 - \gamma)^{\gamma/4} \alpha (1 - \gamma)^{-\gamma/4} \alpha^{\gamma/4} \right) \right) \right) \right) - (a + \kappa \dot{e}) N_f \left( \frac{M_0 + M_r}{2} e^{-\lambda \dot{e}} \right) - \frac{0.27778N \left( M_0 - AC_{\text{gap}} \right)}{\left( 0.5N \dot{J}_w \rho_0 d \lambda \right)} - (0.5N J_w e^{\lambda \dot{e}}) e^{-\lambda \dot{e}} - (0.138889 \rho_0 d \lambda \dot{e}^2) - 0.138889 \rho_0 d \lambda \dot{e}^2
\end{align*}
\]

(18)

### Table 2: Coefficients of equation (17) for estimation of the ICE3 velocity profile.

| Coefficient | Value |
|-------------|-------|
| \(a_1\)     | 103.3 |
| \(a_2\)     | -22.2 |
| \(a_3\)     | 56.85 |
| \(a_7\)     | 42.82 |
| \(b_2\)     | 0.3331e5 |
| \(b_3\)     | 471.8 |
| \(b_6\)     | 1837 |
| \(c_1\)     | 1856 |
| \(c_3\)     | 292.5 |
| \(c_5\)     | 443.2 |
| \(c_7\)     | 603.3 |
| \(d_3\)     | 69.36 |
| \(d_4\)     | -33.33 |
| \(d_6\)     | 52.61 |
| \(d_1\)     | 3603 |
| \(d_2\)     | 378 |
| \(d_3\)     | 611.2 |
| \(c_2\)     | 1131 |
| \(c_4\)     | 0.7388e7 |
| \(c_6\)     | 294.9 |
| \(\kappa\)  | 965.7 |

#### 4.3. Simulation and Results.

The simulation process was performed within 3600 s. In the first 100 s requiring appropriate adhesion between wheels and rails for acceleration, \(\alpha = T_m / \alpha T_v + (1 - \alpha) T_v\) was set at 0.99. After 100 s and the creation of proper adhesion between wheels and rails, \(\alpha\) was set equal to 0.01 so that the speed control system would track the favorable velocity profile. The limitation for \(T_m\) was not to exceed 1980 N.m, for the traction motor torque of ICE3 was 1980 N.m. Figures 10 and 11 report the results.

According to Figure 10, the ICE3 velocity profile was properly tracked, and the maximum tracking error was reported nearly 12%. Figure 11 shows that the adhesion rate was kept within the 0.0246–0.0251 range due to having a significant portion on the creation of the input torque by the slip control system during the first 500 s. Then, there is a transition zone for the adhesion coefficient so that its value changes from 0.016 to 0.025 due to switching from the slip controller to the velocity profile tracker. Finally, this quantity decreased to nearly 0.0165 by reducing the effect of the slip control system and having no requirement for high adhesion coefficient.

#### 4.4. Validation.

An approximate solution can be adopted to verify the results achieved in Subsection 4.3. According to Figure 11, the adhesion coefficient is nearly constant at the beginning and end of motion. Therefore, the following solution can be achieved by using equations (1) to (4) and assuming that the coefficient of adhesion is constant:
Figure 12 reports the results obtained by placing constant values from Table 1 into equation (19) (the values of \( \mu_1, \mu_2, \mu_3, \mu_4 \) within the range of 0 to 500 s and 1200 to 3600 s are considered 0.0249 and 0.0165, respectively).

In addition, the numerical results are compared with the experimental data of the Chinese high-speed train (CRH3) [26] as well. As it can be seen from Figure 13, the experimental data verify the designed controller performance such...
that graphs for numerical methods and experimental data are the same on the transient traction area which is for acceleration. However, the maximum velocity for CRH3 is more which makes the dash-line graph go higher than the solid-line graph in the stable area.

4.5. Stability. The stability of the slip controller should be evaluated to ensure the stability of the proposed control structure. As discussed in Subsection 4.1, the feedback linearization method was employed to design the slip controller. Thus, for the evaluation of stability, it is important to ensure that the internal dynamics of the system are not affected by the input under any circumstances. For this purpose, the zero dynamic stability of the system should first be evaluated. Therefore, in the last equation of equation (B.2) related to \( \dot{\psi}_1 \), all quantities for \( v_{s1}, v_{s2}, v_{s3}, \) and \( v_{s4} \) are equal to zero to obtain zero dynamics of the system.

\[
\dot{\psi}_1 = \frac{(N_1 + N_2 + N_3 + N_4)c - (N_1 + N_2 + N_3 + N_4)d - (1/2)\rho AC_d\psi_1^2}{M}
\]  

(20)
The following parameter is obtained by solving equation (20) and using the constant value from Table 1:

\[ \psi_1(t) = \frac{41957.82621}{t}. \quad (21) \]

Equation (21) is evidently stable; hence, zero dynamics of the system are stable. Consequently, the internal dynamics of the system are also stable.

4.6. Robustness. In control problem literature, for the evaluation of robustness in the control structure, some unavoidable hardware limits, such as actuator dead zones, are usually studied [27, 28]. However, for railway vehicle control problems, some specific scenarios for the uncertainty of constant parameters must be considered.

4.6.1. Scenario I. In the first scenario, a wheelset of the four-axle high-speed railcar is abandoned and replaced with a newer wheelset due to a technical problem. Thus, the new wheelset radius differs from those of the other wheelsets. After the simulation was performed, the following results were achieved by setting the radii of wheelsets at

\[ r_1 = 0.46 + 0.02, \quad r_2 = 0.46, \quad r_3 = 0.46, \quad r_4 = 0.46. \]

According to Figure 14, the control structure can track the desired velocity profile properly even when there are uncertainties in the system parameters.

4.6.2. Scenario II. Although the track was assumed straight with no irregularities, it was possible to consider some uncertainties. Therefore, the second scenario included

\[ N_1 = 170000 \pm 1500, \quad N_2 = 170000 \pm 1500, \quad N_3 = 170000 \pm 1500, \quad N_4 = 170000 \pm 1500. \]

According to Figure 15, the control structure is robust against uncertainties and can track the desired speed accurately.

5. Conclusion

This paper proposed an analytical solution to the motion equations of a four-axle railcar. According to the results, the railcar acceleration time depended on the adhesion coefficient. Therefore, the slip control system is expected to adjust the input torque so that slip remains within the desired range. Consisting of feedback linearization and sliding mode methods, a parallel control structure was designed to control the railcar velocity and track the desired velocity profile. Finally, the results showed that the designed parallel control structure operated properly and that the railcar tracked the velocity profile with less than 12% tracking error.

In reality, traction motors of high-speed trains are mainly the three-phase asynchronous induction motors, the productive torque of which is a function of flux linkage and current in the rotor and the stator. Therefore, as a future work, a complete model of the dynamic motion of a railcar can be developed to account for the dynamic equations of the traction motor as well. In order to control slip in this complete model, a controller is needed to control the voltage applied to the traction motor.

Appendix

A. Variables in ADM solution

\[ z_{10} = 0.001 + \frac{1}{100}T_m t, \]

\[ z_{11} = \frac{1}{T_m} \begin{pmatrix} -48096.70581 + 87448.55968e^{-0.0001944t - 0.00894247T_m t} \\ -39351.85186e^{-0.000432t - 0.0198727T_m t} \end{pmatrix}, \]
By eliminating smaller terms versus larger ones, equation (A.1) can be simplified as follows:

\[
(z_{10} = 0.001 + \frac{1}{100} T_{m} t)
\]

\[
z_{11} = \frac{1}{T_{m}} (-48096 + 87448 e^{-0.0089424 T_{m} t} - 39351 e^{-0.019872 T_{m} t}),
\]

\[
z_{12} = \frac{1}{T_{m}} \left( (3.8 \times 10^{9}) e^{-0.0089424 T_{m} t} + (-3.8 \times 10^{9}) e^{-0.019872 T_{m} t} - 3.5 \times 10^{9} e^{-0.017884 T_{m} t} \right) + 5.82 \times 10^{9} e^{-0.0288144 T_{m} t} - 1.58 \times 10^{9} e^{-0.039744 T_{m} t} - 7.3 \times 10^{8}
\]

\[
z_{50} = 0.00036,
\]

\[
z_{51} = \frac{1}{T_{m}} \left( 9.0 \times 10^{9} e^{-0.0089424 T_{m} t} + 10.0 \times 10^{9} e^{-0.019872 T_{m} t} + 11.0 \times 10^{9} e^{-0.017884 T_{m} t} + 12.0 \times 10^{9} e^{-0.039744 T_{m} t} 
\]

\[
z_{52} = \frac{1}{T_{m}} \left( -4.9 \times 10^{7} e^{-0.0089424 T_{m} t} + (4.9 \times 10^{7}) e^{-0.019872 T_{m} t} + 4.4 \times 10^{7} e^{-0.017884 T_{m} t} - 7.4 \times 10^{7} e^{-0.0288144 T_{m} t} + 2 \times 10^{7} e^{-0.039744 T_{m} t} + 9.3 \times 10^{6} \right)
\]

B. Feedback Linearization Method

It is possible to formulate state-space equations of the four-axle railcar in a normal form by adopting the following variables:

\[
\begin{align*}
\psi_1 &= r_1 z_1 - z_5, \\
\psi_2 &= r_2 z_2 - z_5, \\
\psi_3 &= r_3 z_3 - z_5, \\
\psi_4 &= r_4 z_4 - z_5, \\
\psi_5 &= z_5.
\end{align*}
\]
With the placement of equations (B.1) into (7), the state-space equations for the railcar are created in the normal shape:

\[
\begin{align*}
\dot{\psi}_1 &= \frac{1}{2J_{w1}M} \left( -2cN_1\left(Mr_1^2 + J_{w1}\right)e^{-3.6av_{w1}} + 2dN_1\left(Mr_1^2 + J_{w1}\right)e^{-3.6bv_{w1}} + AC_{d}J_{w1}\psi_1^2 - 2J_{w1}N_1ce^{-3.6av_{w1}} + 2J_{w1}N_3de^{-3.6bv_{w1}} ight) \\
\dot{\psi}_2 &= \frac{1}{2J_{w2}M} \left( -2cN_2\left(Mr_2^2 + J_{w2}\right)e^{-3.6av_{w2}} + 2dN_2\left(Mr_2^2 + J_{w2}\right)e^{-3.6bv_{w2}} + AC_{d}J_{w2}\psi_1^2 - 2J_{w2}N_1ce^{-3.6av_{w2}} + 2J_{w2}N_3de^{-3.6bv_{w2}} - 2J_{w2}N_4ce^{-3.6av_{w2}} - 2J_{w2}N_4de^{-3.6bv_{w2}} + 2Mr_1T_{s1} \\
\dot{\psi}_3 &= \frac{1}{2J_{w3}M} \left( -2cN_3\left(Mr_3^2 + J_{w3}\right)e^{-3.6av_{w3}} + 2dN_3\left(Mr_3^2 + J_{w3}\right)e^{-3.6bv_{w3}} + AC_{d}J_{w3}\psi_1^2 - 2J_{w3}N_1ce^{-3.6av_{w3}} + 2J_{w3}N_3de^{-3.6bv_{w3}} - 2J_{w3}N_4ce^{-3.6av_{w3}} - 2J_{w3}N_4de^{-3.6bv_{w3}} + 2Mr_2T_{s2} \\
\dot{\psi}_4 &= \frac{1}{2J_{w4}M} \left( -2cN_4\left(Mr_4^2 + J_{w4}\right)e^{-3.6av_{w4}} + 2dN_4\left(Mr_4^2 + J_{w4}\right)e^{-3.6bv_{w4}} + AC_{d}J_{w4}\psi_1^2 - 2J_{w4}N_1ce^{-3.6av_{w4}} + 2J_{w4}N_3de^{-3.6bv_{w4}} - 2J_{w4}N_4ce^{-3.6av_{w4}} - 2J_{w4}N_4de^{-3.6bv_{w4}} + 2Mr_3T_{s3} \\
\psi_1 &= \frac{1}{2M} \left( -AC_{d}\psi_1^2 + 2N_1ce^{-3.6av_{w1}} - 2N_1de^{-3.6bv_{w1}} + 2N_2ce^{-3.6av_{w2}} - 2N_2de^{-3.6bv_{w2}} + 2N_3ce^{-3.6av_{w3}} - 2N_3de^{-3.6bv_{w3}} + 2N_4ce^{-3.6av_{w4}} - 2N_4de^{-3.6bv_{w4}} \right)
\end{align*}
\]

where $\psi_1, \psi_2, \psi_3,$ and $\psi_4$ are related to the dynamics associated with the linearized part of the system, whereas $\psi_1$ is associated with internal dynamics.

It is now possible to make the system linear by imposing the input torque as follows:

\[
\begin{align*}
T_{s1} &= \frac{1}{Mr_1} \left( -\frac{1}{2} AC_{d}J_{w1}\psi_1^2 + cN_1\left(Mr_1^2 + J_{w1}\right)e^{-3.6av_{w1}} - dN_1\left(Mr_1^2 + J_{w1}\right)e^{-3.6bv_{w1}} - J_{w1}N_3ce^{-3.6av_{w1}} - J_{w1}N_2de^{-3.6bv_{w1}} + J_{w1}N_3de^{-3.6bv_{w1}} - J_{w1}N_4ce^{-3.6av_{w1}} + J_{w1}N_4de^{-3.6bv_{w1}} + J_{w1}M \\
T_{s2} &= \frac{1}{Mr_2} \left( -\frac{1}{2} AC_{d}J_{w2}\psi_1^2 + cN_2\left(Mr_2^2 + J_{w2}\right)e^{-3.6av_{w2}} - dN_2\left(Mr_2^2 + J_{w2}\right)e^{-3.6bv_{w2}} - J_{w2}N_3ce^{-3.6av_{w2}} - J_{w2}N_2de^{-3.6bv_{w2}} + J_{w2}N_3de^{-3.6bv_{w2}} - J_{w2}N_4ce^{-3.6av_{w2}} + J_{w2}N_4de^{-3.6bv_{w2}} + J_{w2}M \\
\end{align*}
\]
Data Availability

The output data obtained to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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