POINCARÉ INEQUALITIES IN BILATERAL GRAND LEBESGUE SPACES

E. Ostrovsky
  e-mail: galo@list.ru

L. Sirota
  e-mail: sirota@zahav.net.il

E. Rogover
  e-mail: rogoove@gmail.com

Abstract.
In this paper we obtain the non-asymptotic estimations of Poincare type between function and its gradient in the so-called Bilateral Grand Lebesgue Spaces. We also give some examples to show the sharpness of these inequalities.

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary 34A34, 65M20, 42B25.

Key words and phrases: norm, Grand and ordinary Lebesgue Spaces, integral and other singular operator, Poincaré domain and inequalities, exact estimations, Young theorem, Hölder inequality.

1. Introduction

"The term Poincaré type inequality is used, somewhat loosely, to describe a class of inequalities that generalize the classical Poincaré inequality"

\[ \int_D |f(x)|^p \, dx \leq A(p, D) \int_D | | \text{grad} f(x) |^p | \, dx, \]

(0)

see [1], chapter 8, p.215, and the source work of Poincaré [23].

We will call "the Poincaré inequality" some improved, or modified inequality

\[ |T_\delta f(\cdot)|_p \leq C(D) \cdot \frac{p}{|p-d|} \cdot | | \text{grad} f(\cdot) | \, |_p, \]

(1)

or more generally the inequality of a view:

\[ |T_\delta f(\cdot)(/ (\delta(x))^{\alpha})|_p \leq C(\alpha)(D) \cdot \frac{p}{|p-d(1+\alpha)|} \cdot | | \text{grad} f(x) | / (\delta(x)^\alpha | \, |_p. \]

(2)

and we will call the domains \{D\}, \( D \subset R^d, d = 2, 3, \ldots \) which satisfied the inequalities (1) or (2) for any functions \( f \) in the Sobolev class \( W(1, p) \) as a Poincaré domains.

There are many publications about these inequalities and its applications, see, for instance, [5], [6], [17], [24], [27] and the classical monographs [2], [10]; see also reference therein.

Here

\[ \delta = \delta(x) = \delta_D(x) = \inf_{y \in \partial D} |x - y| \]
is the distance between the point \( x, \ x \in D \) and the boundary \( \partial D \) of the set \( D \), \( p \in [1, \infty), \ p \neq d(1+\alpha) \) in the case when the domain \( D \) is bounded and \( \delta(x) = |x| \) otherwise;

\[
\alpha = \text{const} \in (-1, \infty); \ x \in \mathbb{R}^d \Rightarrow |x| = (x, x)^{1/2};
\]

following, for the real value \( x \) \(|x| \) denotes usually absolute value of \( x \);

\[
T_\delta f(x) = \frac{f^0(x)}{\delta(x)}; \ T_{\alpha,\delta} f(x) = \frac{f^0(x)}{(\delta(x))^{1+\alpha}} = T_\delta f(x) / [(\delta(x))^{\alpha}],
\]

where

\[
f^0(x) = f(x) - \int_D f(x) dx / |D|, \ |D| = \text{meas}(D) \in (0, \infty)
\]

in the case of bounded domain \( D \) (a "centering" of a function \( f, \) ) and \( f^0(x) = f(x) \) otherwise.

Note that in the case of bounded domain \( D \) the inequalities (1) and (2) may be rewritten as follows:

\[
\inf_{c \in \mathbb{R}} |(f(x) - c)/(\delta(x))^{1+\alpha}|_p \leq C_\alpha(D) \cdot \frac{1}{|p - d(1+\alpha)|} \cdot |\text{grad} f(x)|/\delta(x)^\alpha |_p.
\]

We will called the operators \( T_\delta \) and \( T_{\alpha,\delta} \) as a Poincaré operators.

It is known that if the domain \( D \) is open, bounded, contain the origin and has a Lipschitz or at last Hölder boundary, or consists on the finite union of these domains, that it is Poincaré domain.

**We will assume that the considered set \( D \) is Poincaré domain, at last in some parameter \( \alpha \).**

**We will distinguish a two cases: the first, or bounded case is when the domain \( D \) is bounded and satisfied the Poincaré condition, and the second case of unbounded domain also with Poincaré property.**

In the first case the value \( p \) might belongs to the semi-closed interval \( 1 \leq p < d \) or in more general case when \( \alpha \neq 0 \) we suppose \( d(1+\alpha) > 1 \) and in the second case or correspondingly \( d(1+\alpha) < p < \infty \).

Note that in the case of the bounded domain \( D \) the Poincaré inequality (1) may be rewritten as follows:

\[
|T_\delta f^0(x)/\delta(x)^\alpha|_p \leq C_\alpha(D) \cdot \frac{1}{|p - d(1+\alpha)|} \cdot |\text{grad} f(x)|/\delta(x)^\alpha |_p,
\]

We denote as usually the classical \( L_p \) Lebesgue norm

\[
|f|_p = |f|_{p,D} = \left( \int_D |f(x)|^p dx \right)^{1/p}; \ f \in L_p \Leftrightarrow |f|_p < \infty,
\]

and denote for arbitrary measurable subset \( D_1 \) of the set \( D \) : \( D_1 \subset D \)

\[
|f|_{p,D_1} = \left( \int_{D_1} |f(x)|^p dx \right)^{1/p};
\]

and denote also \( L(a, b) = \cap_{p \in (a, b)} L_p \).

**Lemma 1.** If \( D_1, D_2 \subset D, \ D_1 \cap D_2 = \emptyset, \) then
The proof follows immediately from the triangle inequality for the $L_p$ norm.

**Lemma 2.** Let $D$ be a measurable set with non-trivial finite Lebesgue measure: $0 < \mu(D) < \infty$. Let also $f(x), f : D \to R$ be positive a.e. bounded function. We assert that for all the values $p \in [1, \infty)$

$$0 < C_1 \leq |f|_p \leq C_2 < \infty.$$ 

*Proof.*

1. **Upper bound.** Let us denote $M = \sup_{x \in D} f(x)$. We have:

$$|f|^p_p \leq M^p \cdot \mu(D),$$

following

$$|f|^p_p \leq M \cdot [\mu(D)]^{1/p} \leq M \max(1, \mu(D)) \overset{def}{=} C_2.$$ 

2. **Low bound.** Let $D_1$ be some subset of the set $D$ with positive measure and $m$ be some positive number such that

$$\forall p \in D_1 f(x) \geq m.$$

We have:

$$|f|^p_p \geq m^p \cdot \mu(D_1),$$

following

$$|f|^p_p \geq m \cdot [\mu(D_1)]^{1/p} \geq m \cdot \min(1, \mu(D_1)) \overset{def}{=} C_1.$$ 

Our aim is a generalization of the estimation (2), (3) on the so-called Bilateral Grand Lebesgue Spaces $BGL = BGL(\psi) = G(\psi)$, i.e. when $f(\cdot) \in G(\psi)$ and to show the precision of obtained estimations by means of the constructions of suitable examples.

We recall briefly the definition and needed properties of these spaces. More details see in the works [8], [9], [11], [12], [20], [21], [15], [13], [14] etc. More about rearrangement invariant spaces see in the monographs [3], [16].

For $a$ and $b$ constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p), p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a + 0)$ and $\psi(b - 0)$, with conditions $\inf_{p \in (a, b)} > 0$ and $\min\{\psi(a + 0), \psi(b - 0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b)$.

The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : R^d \to R$ endowed with the norm

$$\|f\|G(\psi) \overset{def}{=} \sup_{p \in (a, b)} \left[ \frac{|f|^p_p}{\psi(p)} \right],$$

if it is finite.

In the article [21] there are many examples of these spaces. For instance, in the case when $1 \leq a < b < \infty, \beta, \gamma \geq 0$ and

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta}(b - p)^{-\gamma};$$
we will denote the correspondent $G(\psi)$ space by $G(a, b; \beta, \gamma)$; it is not trivial, non-reflexive, non-separable etc. In the case $b = \infty$ we need to take $\gamma < 0$ and define

$$
\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta}, p \in (a, h);
$$

$$
\psi(p) = \psi(a, b; \beta, \gamma; p) = p^{-\gamma} = p^{-|\gamma|}, p \geq h,
$$

where the value $h$ is the unique solution of a continuity equation

$$(h - a)^{-\beta} = h^{-\gamma}$$

in the set $h \in (a, \infty)$.

The $G(\psi)$ spaces over some measurable space $(X, F, \mu)$ with condition $\mu(X) = 1$ (probabilistic case) appeared in [15].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [4], [13].

It was proved also that in this case each $G(\psi)$ space coincides with the so-called exponential Orlicz space, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b; \delta))$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

**Remark 1.** If we introduce the discontinuous function

$$
\psi_r(p) = 1, \ p = r; \psi_r(p) = \infty, \ p \neq r, \ p, r \in (a, b)
$$

and define formally $C/\infty = 0, \ C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the $L_r$ norm:

$$
||f||G(\psi_r) = |f|_r.
$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces $L_r$.

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [8], [11], theory of probability in Banach spaces [13], [15], [20], in the modern non-parametrical statistics, for example, in the so-called regression problem [20].

The article is organized as follows. In the next section we obtain the main result: upper bounds for Poincaré operators in the Bilateral Grand Lebesgue spaces. In the third section we construct some examples in order to illustrate the precision of upper estimations.

The last section contains some slight generalizations of obtained results.

We use symbols $C(X, Y), C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot), \ p \in (A, B)$, where $g = g(p), \ h = h(p), \ g, h : (A, B) \rightarrow R_+$, denotes as usually

$$
0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.
$$

The symbol $\sim$ will denote usual equivalence in the limit sense.
We will denote as ordinary the indicator function
\[ I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A; \]
here \( A \) is a measurable set.

All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

2. **Main result: upper estimations for Poincaré operator**

Let \( \psi(\cdot) \in \Psi(a, b) \), where \( 1 = a < b = d \) in the case of bounded domain \( D \) and \( d = a < b = \infty \) otherwise.

Define for the arbitrary function \( \psi \in \Psi(a, b) \) the auxiliary function of the variable \( p \)
\[
\psi_{\alpha,d}(p) = \frac{p}{|d - p(1 + \alpha)|} \cdot \psi(p),
\]
\[
\psi_d(p) = \frac{p}{|d - p|} \cdot \psi(p) = \psi_{0,d}(p).
\]

Notice that only the values \( p = \infty \) and \( p = d(1 + \alpha) \) are critical points; another points are not interest.

**Theorem 1.** Let \( f \in G(\psi) \), \( \psi \in \Psi(a, b) \) and let the domain \( D \) be the Poincaré domain. Then

\[
\|T_{\alpha,\delta} f\|_{G(\psi_{\alpha,d})} \leq C(D, \alpha, d) \| | \text{grad} f | \|_{G(\psi)}.
\]  

**Example 1.** When the domain \( D \) is bounded (the first case), \( a = 1, b = d, \beta, \gamma > 0 \), and \( f \in G(1, d; \beta, \gamma), \ f \neq 0 \), then
\[
T_\delta f(\cdot) \in G(1, d; \beta, \gamma + 1).
\]

**Example 2.** When the domain \( D \) is Poincaré and unbounded (the second case), \( a = d, b = \infty, \beta > 0, \gamma < 0 \), and \( f \in G(d, \infty; \beta, \gamma), \ f \neq 0 \), then
\[
T_\delta f(\cdot) \in G(\infty; \beta + 1, \gamma + 1).
\]

**Proof** of the theorem 1 is very simple. Denote for the simplicity \( u = T_{\alpha,\delta} f^0; u : D \to R \). We suppose \( f(\cdot) \in G(\psi) \); otherwise is nothing to prove.

We can assume without loss of generality that \( \| | \text{grad} f | \|_{G(\psi)} = 1 \); this means that

\[
\forall p \in (a, b) \Rightarrow | | \text{grad} f | |_p \leq \psi(p).
\]

Using the inequality (1) we obtain:

\[
|u|_p \leq C(D) \frac{p}{|d - p(1 + \alpha)|} \cdot \psi(p) = C(D) \psi_{\alpha,d}(p) = C(D) \psi_{\alpha,d}(p) \| | \text{grad} f | \|_{G(\psi)}.
\]

The assertion of theorem 1 follows after the dividing over the \( \psi_{\alpha,d}(p) \), tacking the supremum over \( p \), \( p \in (a, b) \) and on the basis of the definition of the \( G(\psi) \) spaces. \( \square \)
3. Low bounds for Poincaré inequality.

In this section we build some examples in order to illustrate the exactness of upper estimations. We consider both the cases \( a = 1, \ b = d \) and \( a = d, \ b = \infty \).

A. Bounded domain.

Note that in the bounded case only the value \( p = p_0 \) is the critical value. It is presumed in this (sub)section that \( d(1 + \alpha) > 1 \); otherwise is nothing to prove.

Let us denote for the mentioned values \( p \in (1, d(1 + \alpha)) \) and for the function \( f \in \cap_{p \in (1, d(1 + \alpha))} L^p, f \neq 0 \) the quantity

\[
V_{\alpha,d}(f,p) = V(f,p) = |T_{\alpha,d}f|_p \cdot \left[(d - p(1 + \alpha))/p\right],
\]

From the inequality (2) follows that for some non-trivial (positive and finite) number \( C(1) \)

\[
\sup_{f \in L(1, d(1 + \alpha)), f \neq 0} \lim_{p \to d(1 + \alpha) - 0} V(f,p) \leq C(1).
\]

We intend to prove an inverse inequality at the critical point \( p \to d(1 + \alpha) - 0 \).

**Theorem 2.a.** For all the values \( \alpha \in (-1, \infty) \), such that \( d(1 + \alpha) > 1 \) there exists a constants \( C_1 = C_1(\alpha, d) \) such that \( \sup_{f \in L(1, d(1 + \alpha)), f \neq 0} \lim_{p \to d(1 + \alpha) - 0} V_{\alpha,d}(f,p) \geq C_1. \)

**Proof.** Let us consider a function (more exactly, a family of the functions) of a view

\[
u(x) = u_\Delta(x) = | \log |x||^\Delta \cdot I(|x| < 1/e), \quad \Delta = \text{const} > 1.\]

Here the domain \( D \) is the unit ball of the space \( R^d : D = \{x : |x| \leq 1\} \).

Note that the average value \( |D|^{-1} \int_D u(x) \, dx \) is bounded.

We have using the multidimensional spherical coordinates as \( p \to d(1 + \alpha) - 0 : \)

\[
|\delta(x)|^{-\alpha} \text{grad} u_p \asymp \int_0^{1/e} r^{d-1-p(1+\alpha)} | \log r|^{p(\Delta-1)} \, dr \\
\asymp \int_0^\infty \exp(-y(d-p(1+\alpha))) \, y^{p(\Delta-1)} \, dy \\
= [d - p(1 + \alpha)]^{-p(\Delta-1)-1} \Gamma(p(\Delta - 1)) \\
\asymp [d - p(1 + \alpha)]^{-p(\Delta-1)-1}, \quad (10)
\]

since we can suppose that the value \( p \) is bounded. Here the \( \Gamma(\cdot) \) denotes usually Gamma function.

It follows from the equality (10) that

\[
|\delta(x)|^{-\alpha} \text{grad} u_p \asymp [d - p(1 + \alpha)]^{-\Delta-1-(1+\alpha)/d}. \quad (11)
\]
Further, we find analogously:

\[
|\delta^{-1-\alpha}u^0_p|_p \asymp \int_1^\infty e^{-y(d-p(1+\alpha))} y^p \Delta \; dy \asymp [d-p(1+\alpha)]^{-p\Delta-1},
\]

therefore

\[
|\delta^{-1-\alpha}u^0_p|_p \asymp [d-p(1+\alpha)]^{-\Delta-(1+\alpha)/d}. \tag{12}
\]

We conclude substituting the estimations (11) and (12) into the expression for the value \(V(f,p)\) that it is bounded from below as \(p \to d(1+\alpha) - 0\).

But the function \(u = u_\Delta(x)\) is discontinuous. Let us redefine the function \(u = u(x)\) as following. Let \(\phi = \phi(r)\) be infinitely differentiable function with support on the set \(r \in [1/e, 2/e]\) :

\[
\phi(r) > 0 \iff r \in (1/e, 2/e)
\]

and such that

\[
\phi(1/e) = 1, \quad \phi'(1/e) = e \cdot \Delta, \quad \phi(2/e) = \phi'(2/e) = 0.
\]

Then the function

\[
\tilde{u}_\Delta(x) = |\log |x| |^\Delta \cdot I(|x| \in (1, 1/e) + \phi(|x|) \cdot I(|x| \in [1/e, 2/e]))
\]

def \(= u_\Delta(x) + w(x) = u_\Delta(x) I(|x| \in (0, 1/e) + w(x) I(|x| \in (1/e, 2/e))\).

gives us the example for the theorem 2.a. Indeed, since the supports for the functions \(u_\Delta\) and \(w(x)\) are adjoint, we have using the lemma 1 for the nominator for the expression of a function \(V\) (8) the following simple estimation from below:

\[
|\tilde{u}_\Delta|_p \geq |u_\Delta|_p.
\]

We need further to obtain the upper estimate for the denominator for the expression for the function \(V\). We have using the assertions of lemma 1 and lemma 2:

\[
|\text{grad } \tilde{u}_\Delta|_p \leq |\text{grad } u_\Delta|_p + |\text{grad } w|_p
\]

\[
\leq C_5 \left[ \frac{p}{p(1+\alpha) - d} \right]^{(\Delta-1)-(1+\alpha)/d} + C_6 \]

\[
\leq C_7 \left[ \frac{p}{p(1+\alpha) - d} \right]^{(\Delta-1)-(1+\alpha)/d}.
\]

This completes the proof of theorem 2.a.

**B. Unbounded domain.**

**Theorem 2.b.** For all the values \(\alpha \in (-1, \infty)\), such that \(d(1+\alpha) > 1\) there exists a constants \(C_2 = C_2(\alpha, d) \in (0, \infty)\) and the unbounded Poincaré domain \(D\) for which

\[
\sup_{f \in L(1,d(1+\alpha)), f \neq 0} \lim_{p \to d(1+\alpha) + 0} V_{\alpha,d}(f, p) \geq C_2.
\]
and
\[ \sup_{f \in L(1,d(1+\alpha)), f \neq 0} \lim_{p \to \infty} V_{\alpha,d}(f,p) \geq C_2. \]  

**Proof** is at the same as in the proof of theorem 2.a. It is sufficient to consider the domain \( D = \{ x : |x| \geq 1 \} \) and to choose the single function
\[ v(x) = v_\Delta(x) = |(\log |x|) |^\Delta I(|x| > e) + I(0.5e \leq |x| \leq e)\phi(|x|), \]
where \( \phi = \phi(r) \) is infinitely differentiable function with the support \( r \in (0.5e, e) \) such that
\[ \phi(e/2) = \phi'(e/2) = 0, \phi(e) = 1, \phi'(e) = \Delta/e; \]
\( \Delta = \text{const} > 1. \)

We have for the values \( p \in (d(1+\alpha) + 0, \infty) \) repeating the considerations for the proof of theorem 2.a used the lemmas 1.2:
\[ | | \text{grad} v_\Delta | |_{x}^{-\alpha} |_{p} \approx C_9 \Gamma\left(\frac{p\Delta - 1}{p(1 + \alpha) - d}\right) + C_{10}, \]
\[ \approx C_{11} \Gamma\left(\frac{p\Delta - 1}{p(1 + \alpha) - d}\right) + C_{10}, \]
\[ | | v_\Delta | |_{x}^{1+\alpha} |_{p} \approx C_{12} \Gamma\left(\frac{p\Delta + 1}{p(1 + \alpha) - d}\right) + C_{13}, \]
\[ \approx C_{14} \Gamma\left(\frac{p\Delta + 1}{p(1 + \alpha) - d}\right) + C_{13}. \]

therefore
\[ [v_\Delta | x |^{1+\alpha}]_{p} : [ | | \text{grad} v_\Delta | |_{x}^{-\alpha} |_{p}] \approx p/[d - p(1 + \alpha)]. \]

This completes the proof of theorem 2.b. We omitted the simple calculations used the Stirling’s formula as \( p \to \infty \) etc.

**Remark 2.** It follows from theorems 2.a and 2.b that the estimation of theorem 1 is exact in the \( G(\psi) \) spaces. Namely, for the function \( \psi_\Delta(p) \) of a view
\[ \psi_\Delta(p) = |u_\Delta(\cdot)|_{p} \]
or correspondingly
\[ \psi_\Delta(p) = |v_\Delta(\cdot)|_{p} \]
in the assertion of theorem 1 (5) take place the inverse asymptotical inequality, up to multiplicative constant.

\[ \square \]

4. Concluding remarks

We consider in this section some generalizations of inequalities (1), (2).

A. Let us denote

\[
\log^+ z = \max(1, |\log z|), \quad z > 0; \quad z_+ = \max(z, 0); \quad z \in (-\infty, \infty).
\]

We obtain using the works [5], [6], [17], [24], [27] etc. the following inequality

\[
\left| \frac{u^0 \cdot (\log^+ \delta)^{B_1}}{\delta^{1+\alpha}} \right|_p \leq C(d, \alpha, D, B_1, B_2) \times \\
\left[ \frac{p}{d - p(1 + \alpha)} \right]^{(1-B_2+B_1)} \cdot \left| \frac{|\grad u| \cdot (\log^+ \delta)^{B_2}}{\delta^\alpha} \right|_p.
\]

(15)

Here in addition \( B_1, B_2 = \text{const}, \ \min(B_1, B_2) > 0. \)

We assert analogously to the theorems 2.a,2.b and using at the same examples that the inequality (15) is asymptotically exact as \( p \to d(1 + \alpha) \) and as \( p \to \infty. \)

At the same result is true for more general operator:

\[
\left| \frac{u^0 \cdot (\log^+ \delta)^{B_1} \cdot S(\log^+ \delta)}{\delta^{1+\alpha}} \right|_p \leq C(d, \alpha, D, B_1, B_2) \times \\
\left[ \frac{p}{d - p(1 + \alpha)} \right]^{(1-B_2+B_1)} \cdot \left| \frac{|\grad u| \cdot (\log^+ \delta)^{B_2} \cdot S(\log^+ \delta)}{\delta^\alpha} \right|_p,
\]

(16)

where \( S(z), \ z \in (1, \infty) \) is positive continuous slow varying as \( z \to \infty \) function.

B. Variable exponent weight estimations.

We consider in this section the case of bounded domain \( D. \) Let \( w_1(x), \ w_2(x) \) be two positive measurable local integrable functions (weights).

We denote the weight \( L_{p,w} \) average

\[
u_w = u_{w,D} = \frac{\int_D u(x) \ w(x) \ dx}{\int_D w(x) \ dx}
\]

and correspondence \( L_{p,w} \) norm

\[
|u|_{p,w} = \left[ \int_D |u(x)|^p \ w(x) \ dx \right]^{1/p}, \ p \geq 1.
\]

It is obtained in many publications, under some natural conditions, see, for example, [24], the following variable exponent, or the so-called \( p,q;\ p \leq q \) estimations:

\[
|u - u_{w_1,D}|_{q,w_1} \leq K(p,q;\ w_1, w_2, D) \ |\grad u|_{p,w_2}.
\]

(17)

For example, \( q > p(1 + \alpha) \) \( \Rightarrow \)
\[
\left\| \frac{u - u_{w_1,D}}{\delta^{1+\alpha}} \right\|_{q} \leq C_{\alpha,D} \left| q - p(1 + \alpha) \right|^{-1+1/(p(1+\alpha))-1/q} \cdot \left\| \frac{\| \text{grad} u \|}{\delta^\alpha} \right\|_{p(1+\alpha)},
\]

and the last inequality (18) is sharp.

As a consequence: if
\[
\| \text{grad} u \| / \delta^\alpha \in G(\psi)
\]
and if we denote
\[
\nu(q) = \nu_{\alpha,D}(q) = \inf_{p \in [1, q/(1+\alpha))] \left\{ \left| q - p(1 + \alpha) \right|^{-1+1/(p(1+\alpha))-1/q} \cdot \psi(p) \right\},
\]
then
\[
\left\| \frac{u - u_{w_1,D}}{\delta^{1+\alpha}} \right\|_G(\nu) \leq C(\alpha, D) \cdot \| \text{grad} u \| / \delta^\alpha \cdot G(\psi).
\]

\[
(19)
\]

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