Purity of Zip Strata with Applications to Ekedahl-Oort
Strata of Shimura Varieties of Hodge Type

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Abstract. We show that the zip stratification given by an arbitrary \( \hat{G} \)-zip over a
scheme is pure. We deduce purity of the level-\( m \)-stratification for truncated Barsotti-
Tate groups and purity of the Ekedahl-Oort stratification for special fibers of good
models of Shimura varieties of Hodge type. We also show that all Ekedahl-Oort strata
are quasi-affine schemes and generalize several properties of the Bruhat stratification
from the PEL case to the case of Shimura varieties of Hodge type.

Introduction

Let us call a subscheme \( Z \) of a scheme \( X \) pure if the inclusion \( Z \hookrightarrow X \) is an affine
morphism. If \( X \) is locally noetherian, this implies that every irreducible component of
\( Z \) has codimension 1 in \( Z \).

Such notions of purity results play an important role in Algebraic Arithmetic Geom-
etry. Examples are the purity of the (complement of the) branch locus for quasi-finite
normal schemes over regular schemes or the purity of Newton strata proved by de Jong
and Oort ([dJOo]) and Vasiu ([Va2]).

Let \( \hat{G} \) be a linear algebraic group over a finite field \( \mathbb{F}_q \) and assume that its identity
component is reductive. In [PWZ2] the notion of a \( \hat{G} \)-zip over a scheme \( S \) was defined
and for each such \( \hat{G} \)-zip \( I \) the associated zip stratification was introduced. Details are
recalled in the beginning of Section 3. This stratification is a decomposition of \( S \) into
locally closed subschemes \( S^{[\hat{w}]} \). We refer to [PWZ2], [MW], [Lie], and to [Wd1] for ex-
amples how \( \hat{G} \)-zips arise. The first main result is the following theorem (Theorem 3.2).

Theorem 1. The subschemes \( S^{[\hat{w}]} \) of \( S \) are pure.

We prove the theorem by showing a purity result for the classifying stack of \( \hat{G} \)-zips,
in fact even more general for orbitally finite algebraic zip data (in the sense of [PWZ1]),
see Theorem 2.2.

We derive several applications from Theorem 1. Let \( X_m \) be a truncated Barsotti-
Tate group of level \( m \) \( (1 \leq m \leq \infty) \) over a scheme \( S \) of characteristic \( p > 0 \) in [NVW]
there is defined a level \( m \) stratification of \( S \). For \( m = 1 \) this is a variant of a (special
case of a) zip stratification and Theorem 1 can be used to show that all these strata
are pure for \( m = 1 \). We deduce the main result of [NVW] (Theorem 3.5) but with a
much shorter proof and without any restrictions on the characteristic.
Theorem 2. All level-$m$-strata of $S$ associated to $X_m$ are pure in $S$.

A further very interesting case of a zip stratification is the Ekedahl-Oort stratification of the special fiber $S$ of the canonical model of a Shimura variety of Hodge type at a place of good reduction defined by Zhang [Zh]. Hence we obtain as a direct application (Theorem 4.1):

Theorem 3. Each Ekedahl-Oort stratum of $S$ is pure in $S$.

Moreover we prove the following absolute result about Ekdehdahl-Oort strata (Theorem 4.2).

Theorem 4. Each Ekedahl-Oort stratum of $S$ is a quasi-affine scheme.

This generalizes analogue results for moduli spaces of abelian varieties by Oort ([Oo1]) and for Shimura varieties of PEL-Type by Viehmann and first author ([ViWd]). Finally, we show some general facts about the Bruhat stratification (introduced in [Wd2]) for the special fiber $S$ of a Shimura variety of Hodge type (Theorem 4.3). This generalizes the main result of [Wd3] where the case of Shimura varieties of PEL type has been considered (with a more complicated proof).

Notation

If $X$ is a scheme over some ring $A$ and $A \to B$ a homomorphism of rings, then $X_B$ denotes the base change $X \otimes_A B$. If $q$ is a prime power, the base change of objects via the absolute $q$-Frobenius is denoted by $(\ )^{(q)}$.

By a linear algebraic group over a field $k$ we mean a smooth affine group scheme over $k$. If $\hat{G}$ is a not necessarily connected linear algebraic group, we denote by $G$ its identity component and by $\pi_0(G) := \hat{G}/G$ the finite étale group scheme of connected components; similarly for other letters of the alphabet and for homomorphisms of linear algebraic groups. We call a homomorphism of connected linear algebraic groups $\varphi: G \to H$ an isogeny, if $\varphi$ is surjective and has finite kernel. For a connected linear algebraic group $P$ we denote by $R_u P$ its unipotent radical (if it exists over $k$).

1 Preliminaries

1.1 A version of the Lang-Steinberg theorem

Consider the following situation. Let $k$ be a field, let $G$ be a group scheme over $k$, and let $F: G \to G$ be an endomorphism of $k$-group schemes. Then $G$ acts on itself by $F$-conjugation, i.e. by $g \cdot h := ghF(g)^{-1}$. Recall the following theorem due to R. Steinberg.

Theorem 1.1. Assume that $G$ is connected and of finite type over $k$ and assume that $F$ is an isogeny (i.e., $F$ is surjective and hence has finite kernel). Then the following assertions are equivalent.

(i) The action of $G$ on itself by $F$-conjugation is transitive.
(ii) The action of $G$ on itself by $F$-conjugation has finitely many orbits.

(iii) The subgroup scheme $G^F := \{ g \in G : F(g) = g \}$ of $G$ is a finite $k$-scheme.

(iv) For every field extension $K$ of $k$ and for all $x_0 \in G(K)$ the generalized Lang morphism

\[ L_{x_0} : G_K \to G_K, \quad g \mapsto gx_0F(g_0)^{-1}, \]

is a finite morphism.

**Proof.** Passing to sufficiently big algebraically closed extension of $k$ we may assume that $k$ is algebraically closed and that $k = K$ in (iii). Passing to the underlying reduced group scheme, we may assume that $G$ is smooth over $k$. Then the result is due to Steinberg ([St]).

**Remark 1.2.** (1) In the situation of Theorem 1.1 the equivalent conditions hold if $\text{Lie}(F) : \text{Lie}(G) \to \text{Lie}(G)$ is a nilpotent map.

(2) Assume that $G$ is smooth over $k$. If the equivalent conditions of Theorem 1.1 hold, then $L_{x_0}$ is flat. If $\text{Lie}(F) = 0$, then $L_{x_0}$ is even étale and $G^F$ is a finite étale group scheme over $k$.

Both remarks apply in particular if $G$ is defined over a finite field and if $F$ is the Frobenius.

### 1.2 Affineness criteria

Let $k$ be a field, let $\hat{G}$ be an affine group scheme of finite type over $k$. We start with an easy lemma.

**Lemma 1.3.** Let $\hat{X}$ be a scheme of finite type over $k$ and let $\hat{G}$ be a linear algebraic group acting on $\hat{X}$. Let $\hat{Z} \subseteq \hat{X}$ be a $\hat{G}$-orbit. Then the following assertions are equivalent.

(i) $\hat{Z}$ is affine.

(ii) There exists a an affine $G$-orbit contained in $\hat{Z}$.

(iii) All $G$-orbits contained in $\hat{Z}$ are affine.

**Proof.** This follows from the easy fact that the $G$-orbits in $\hat{Z}$ are the connected components of $\hat{Z}$ and that all these connected components are isomorphic to each other because $\hat{G}$ acts transitively on them.

We denote by $\hat{G}_{\text{red}}$ the underlying reduced subgroup scheme – if this exists (which is automatically the case if $k$ is perfect). We recall the following results on affine quotients of $\hat{G}$ from [CPS].

**Proposition 1.4.** Let $\hat{J} \subseteq \hat{H}$ be $k$-subgroup schemes of $\hat{G}$ and let $K$ be an algebraically closed extension of $k$.

(1) $\hat{G}/\hat{H}$ is affine if and only if $\hat{G}_K/\hat{H}_K$ is affine.

(2) $\hat{G}/\hat{H}$ is affine if and only if $(\hat{G}_K)_\text{red}/(\hat{H}_K)_\text{red}$ is affine.

(3) $\hat{G}/\hat{H}$ is affine if and only if $G/H$ is affine.

(4) If $\hat{G}/\hat{H}$ and $\hat{H}/\hat{J}$ are affine, then $\hat{G}/\hat{J}$ is affine.

(5) If $G$ is unipotent, then $G/H$ is always affine.
(6) If $G$ is reductive, then $\hat{G}/\hat{H}$ is affine if and only if $(H_K)_\text{red}$ is reductive.

(7) Assume that $H_K \subseteq R_uG_K$, then $\hat{G}/\hat{H}$ is affine.

**Proof.** As the formation of quotients commutes with base change and as the property “affine” is stable under fpqc descent, we may assume that $k = K$ is algebraically closed. Then (1) is trivial and (2) is implied by Chevalley’s theorem that for a finite surjective morphism $f : X \to Y$ of noetherian schemes the scheme $X$ is affine if and only if $Y$ is affine. Assertion (3) follows from Lemma 1.3.

Using (2) and (3) we may assume in the proof of (4) – (7) that all groups are connected and smooth over $k$. Then using [CPS] Theorem 4.3, (4) follows from loc. cit. Prop. 2.2, and (5) follows from loc. cit. Corollary 4.5. Finally, (7) follows from (5) and (4).

We further collect some further results on affine morphisms which are well known but for which we do not know of a good reference.

**Lemma 1.5.** Let $f : X \to Y$ be a morphism on schemes. Let $X_\text{red}$ be the underlying reduced scheme of $X$. Then $f$ is affine if and only if the composition $g : X_\text{red} \hookrightarrow X \to Y$ is affine.

**Proof.** The condition is clearly necessary. Assume that $g$ is affine. To show that $f$ is affine, we may assume that $Y$ is affine. Then $X_\text{red}$ is affine and hence $X$ is affine by the general version of Chevalley’s theorem in the appendix of [Co].

**Proposition 1.6.** Let $X$ be a locally noetherian scheme and let $i : Z \hookrightarrow X$ be an affine immersion. Then the schematic closure $\bar{Z}$ of $Z$ in $X$ exists and for every irreducible component $T$ of $\bar{Z} \setminus Z$ one has $\text{codim}(T, \bar{Z}) = 1$.

**Proof.** The schematic closure $\bar{Z}$ exists because $i$ is quasi-compact ([GW] Remark 10.31). We may assume $X = \bar{Z}$. Clearly one has $\text{codim}(T, X) \geq 1$.

Assume that $\text{codim}(T, X) \geq 2$. Replacing $X$ by $\text{Spec}(\mathcal{O}_{X,T})$ we may assume that $X = \text{Spec} A$ for a locally noetherian ring $A$ and that $Z$ is the complement of the closed point $x$ of $\text{Spec} A$. Passing to the completion we even may assume that $A$ is complete and in particular excellent. Replacing $A$ by $A/p$, where $p$ is a minimal prime ideal we may assume that $A$ is an excellent integral domain. As $A$ is excellent, its normalization is finite and we may assume in addition that $A$ is normal.

Eventually we obtain an affine immersion $Z = \text{Spec} A \setminus \{x\} \hookrightarrow \text{Spec} A$ of normal noetherian schemes. As $\dim A \geq 2$ the algebraic analogue of Hartogs’ theorem ([GW] Theorem 6.45) implies $A = \Gamma(Z, \mathcal{O}_Z)$. This is a contradiction because $Z$ is affine.

## 2 Purity for algebraic zip data

### 2.1 Algebraic zip data

We recall the general definition of an algebraic zip datum from [PWZ1]. Let $k$ be a field.
Definition 2.1. An algebraic zip datum over $k$ is a tuple $\hat{Z} = (\hat{G}, \hat{P}, \hat{Q}, \hat{\varphi})$, where $\hat{G}$ is a linear algebraic group over $k$ such that $G$ is reductive, where $\hat{P}$ and $\hat{Q}$ are linear subgroups of $\hat{G}$ such that $P$ and $Q$ are parabolic subgroups of $G$, and where $\hat{\varphi}: \hat{P}/R_uP \to \hat{Q}/R_uQ$ is a homomorphism such that $\varphi: P/R_uP \to Q/R_uQ$ is an isogeny.

The zip group attached to $\hat{Z}$ is the linear algebraic group $E := E\hat{Z} := \{(p, q) \in \hat{P} \times \hat{Q} ; \varphi(\hat{p}) = \hat{q}\}$, where $\hat{p}$ (resp. $\hat{q}$) denotes the image of $p$ in $\hat{P}/R_uP$ (resp. of $q$ in $\hat{Q}/R_uQ$). It acts on $\hat{G}$ by restriction of the left action $$(\hat{P} \times \hat{Q}) \times G \to G, \quad ((p, q), g) \mapsto pgq^{-1}.$$ If $\hat{Z} = (\hat{G}, \hat{P}, \hat{Q}, \hat{\varphi})$ is an algebraic zip datum, then $Z := (G, P, Q, \varphi)$ is again an algebraic zip datum, called the connected algebraic zip datum attached to $\hat{Z}$.

There is the obvious notion of a base change $\hat{Z}_K$ to a field extension $K$ of $k$. An algebraic zip datum $\hat{Z}$ is called orbitally finite if $E\hat{Z}_K$ acts with finitely many orbits on $G_K$ for some (equivalently, for all) algebraically closed extension $K$ of $k$.

2.2 Description of the orbits

Assume that $k$ is algebraically closed. In this case we will confuse reduced schemes of finite type over $k$ and their $k$-valued points. For an orbitally finite connected algebraic zip datum $Z = (G, P, Q, \varphi)$ we have the following description of the $E$-orbits in $G$ (by [PWZ1] Theorem 5.14 and Prop. 7.1). We first choose a frame $(B, T, g)$ of $Z$ in the sense of [PWZ1] Definition 3.6. It determines Levi subgroups $^gT \subseteq L \subseteq P$ and $T \subseteq M \subseteq Q$.

Let $(W, I)$ be the Weyl group of $G$ with its set of simple reflections attached to $(T, B)$. For $w \in W$ we denote by $\ell(w)$ its length. For each $w \in W$ we choose a representative $\hat{w} \in N_G(T)$ such that $(w_1w_2) = \hat{w}_1\hat{w}_2$ for all $w_1, w_2 \in W$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

Let $J \subseteq I$ be the type of $P$ and let $K \subseteq I$ be the type of $Q$. Then
$$JW \leftrightarrow \{E\text{-orbits in } G\}, \quad w \mapsto G^w := E \cdot g\hat{w}$$ yields a parametrization of the $E$-orbits in $G$ independent of the choice of $\hat{w}$ and of the frame ([PWZ1] Prop. 5.8).

2.3 Affineness of orbits for zip data

Let $\hat{Z} = (\hat{G}, \hat{P}, \hat{Q}, \hat{\varphi})$ be an algebraic zip datum over a field $k$. The following is the main technical result.

**Theorem 2.2.** Assume that the algebraic zip datum $\hat{Z}$ is orbitally finite. Then the $E\hat{Z}$-orbits on $\hat{G}$ are affine.

We prove this theorem in several steps.
2.3.1 Reduction to an algebraically closed base field and to the connected case

Clearly, we may assume that $k$ is algebraically closed. As the identity component of $E_\mathbb{Z}$ is $E_\mathbb{Z}$, we may assume that $\hat{\mathbb{Z}} = \mathbb{Z} = (G, P, Q, \varphi)$ is a connected algebraic zip datum because of Lemma 1.3.

2.3.2 Passing to smaller zip data

For the proof we will use the following procedure of passing to zip data of smaller dimension introduced in [PWZ1] Section 4. We choose a frame $(T, B, g)$ and use the notation established in Subsection 2.1. The Bruhat decomposition yields a disjoint decomposition into locally closed subvarieties

\[(2.3.1) \quad G = \bigcup_{x \in W_K} Pg\hat{x}Q.\]

For $x \in J W^K$ we set

\[P_x := M \cap \hat{x}^{-1}g^{-1}P, \quad Q_x := \varphi( L \cap g\hat{x}Q).\]

Then $\varphi \circ \text{int}(g\hat{x})$ induces an isogeny $\varphi_\hat{x} : P_x/R_uP_x \to Q_x/R_uQ_x$ and

\[Z_\hat{x} := (M, P_x, Q_x, \varphi_\hat{x})\]

is a connected algebraic zip datum. We obtain a bijection

\[(2.3.2) \quad \{E\text{-orbits } X \text{ in } Pg\hat{x}Q\} \leftrightarrow \{E_{\mathbb{Z}_\hat{x}}\text{-orbits } Y \text{ in } M\}\]

defined by $Y = M \cap \hat{x}^{-1}g^{-1}X$ and $X = E \cdot (g\hat{x}Y)$ ([PWZ1] Proposition 4.7).

We set

\[U := R_uP, \quad V := R_uQ.\]

Now [PWZ1] Lemma 4.5 and its proof shows that there is a well defined homomorphism

\[(2.3.3) \quad e_\hat{x} : P \cap g\hat{x}Q \to E_{\mathbb{Z}_\hat{x}}, \quad p \mapsto (m, \varphi(\ell))\]

where $p = u\ell$ for $u \in U$, $\ell \in L$, and $\hat{x}^{-1}g^{-1}p = vm$ for $v \in V$, $m \in M$. It has a section

\[(2.3.4) \quad f_\hat{x} : E_{\mathbb{Z}_\hat{x}} \to P \cap g\hat{x}Q, \quad (p', q') \mapsto g\hat{x}p'.\]

2.3.3 Passing to $G/V$

Consider the quasi-projective quotient $G/V$ with the action of $P$ on it given by

\[(p, hV) \mapsto ph\varphi(\ell)^{-1}V,\]

where $p = u\ell$ for $u \in U$, $\ell \in L$. Restricting the projection $P \times Q \to P$ to $E$ yields a surjective homomorphism $E \to P$. Via this homomorphism we obtain an action of $E$ on $G/V$ making the quotient morphism $\pi : G \to G/V$ equivariant with respect to the $E$-action.
Lemma 2.3. The maps $X \mapsto \pi(X)$ and $Y \mapsto \pi^{-1}(Y)$ yield mutually inverse bijections between the set of $E$-orbits $X$ in $G$ and the set of $P$-orbits $Y$ in $G/V$. If $Y$ is affine, then $\pi^{-1}(Y)$ is affine.

Proof. As $\pi$ and $E \to P$ are surjective, $\pi(X)$ is a $P$-orbit in $G/V$ for every $E$-orbit in $G$. Conversely, if $Y = P \cdot hV \subseteq G/V$ is a $P$-orbit in $G/V$, then $\pi^{-1}(Y) = P \cdot gV \subseteq G$ is an $E$-orbit by the definition of $E$ and its action on $G$. Finally, as $G$ is affine and $G/V$ is separated, $\pi$ is affine. This shows the last assertion. \( \square \)

Lemma 2.4. Let $t \in M$. The homomorphism $e_{\bar{\xi}}$ restricts to a surjective homomorphism

$$\bar{\xi}_t : \text{Stab}_{P^g \hat{Q}}(g \bar{t}V) \to \text{Stab}_{E_{\bar{\xi}}}(t)$$

and the section $f_{\bar{\xi}}$ restricts to a section $\bar{f}_{\bar{\xi}}$ of $\bar{\xi}_t$. Moreover $\text{Ker}(\bar{\xi}_t)_{\text{red}} = U \cap g^2V$.

Proof. Write $p = u\ell$, $\ell^{-1}g^{-1}p = vm$ for $p \in P \cap g^2Q$ as in the definition of $e_{\bar{\xi}}$. Then

$$(*) \quad pg\bar{t}\varphi(\ell)^{-1} \in g\bar{t}V \iff vmt\varphi(\ell)^{-1} \in V \iff mt\varphi(\ell)^{-1} = 1.$$ 

It remains to show the last assertion. The definition of $e_{\bar{\xi}}$ and $(*)$ show that $U \cap g^2V \subseteq \text{Ker}(\bar{\xi}_t) \subseteq (\text{Ker} (\varphi)U) \cap g^2V$. As $\varphi : L \to M$ is an isogeny between connected algebraic groups, $\text{Ker}(\varphi)_{\text{red}}$ lies in the center of $L$ and hence is of multiplicative type. In particular $\text{Ker}(\varphi)_{\text{red}} \cap g^2V = 1$. As $U \cap g^2V$ is smooth, this shows the last assertion. \( \square \)

2.3.4 Proof of Theorem 2.2

Looking at dimensions one sees that $P = G$ if and only if $Q = G$. In this case $E$ acts transitively on $G$ by the Lang-Steinberg Theorem [1.1] and the affineness of the single orbit is clear. Hence we can assume that $P$ and $Q$ are proper parabolic subgroups of $G$.

Let $w \in J^W$ and $x \in J^W K$ the unique element of minimal length in $W_J w W_K$. Then $\hat{w} = \hat{v} \bar{t}$ for some $\hat{v} \in M$ representing an element $v \in W_K$. Let $X := E \cdot gw \subseteq P g\bar{t}Q$ be the corresponding $E$-orbit. By Lemma 2.3 it suffices to show that the image of $X$ in $G/V$ is affine. As this is a $P$-orbit it suffices to show that the quotient $P/\text{Stab}_{P}(gw\bar{t}V)$ is affine. Hence we are done by Proposition 1.4 if we have shown the following inclusion

$$(2.3.5) \quad S_{\bar{\xi}} := \text{Stab}_{P}(gw\bar{t}V)_{\text{red}}^0 \subseteq U = R_u P.$$ 

Proceeding by induction on $\dim G$ we may assume that $(2.3.5)$ holds for the connected algebraic zip datum $Z_{\bar{\xi}}$. In particular we have

$$\text{Stab}_{P_{\bar{\xi}}}(R_u Q_{\bar{\xi}})_{\text{red}}^0 \subseteq R_u P_{\bar{\xi}} = \hat{z}^{-1}g^{-1}U \cap M.$$ 

Hence we have $f_{\bar{\xi}}(R_u P_{\bar{\xi}}) \subseteq U$ and $S_{\bar{\xi}}$ is the product of two closed subgroups of $U$ by Lemma 2.4 which proves $(2.3.5)$.}

7
3 Applications

In this section we fix a finite field $\mathbb{F}_q$ with $q$ elements and a finite extension $k$ of $\mathbb{F}_q$. We recall the definition of $\hat{G}$-zips from [PWZ2] Subsection 3.2. Here $\hat{G}$ is a linear algebraic group over $\mathbb{F}_q$ such that $G$ is reductive. Let $\chi : \mathbb{G}_{m,k} \to G_k$ be a cocharacter and let $P_\pm = U_\pm \rtimes L$ be the attached pair of opposite parabolic subgroups of $G_k$ with common Levi subgroup $L = \text{Cent}_{G_k}(\chi)$. Fix a subgroup scheme $\Theta$ of the finite étale scheme $\pi_0(\text{Cent}_{\hat{G}_k}(\chi))$ and let $\hat{L}$ be the inverse image of $\Theta$ in $\text{Cent}_{\hat{G}_k}(\chi)$. Then $\hat{P}_\pm := U_\pm \rtimes \hat{L}$ is an algebraic subgroup of $\hat{G}_k$ with identity component $P_\pm$.

The algebraic zip datum associated to $\hat{G}$ and $(\chi, \Theta)$ is $\hat{Z} := (\hat{G}_k, \hat{P}_+, \hat{P}_-, \hat{\phi})$, where $\hat{\phi}$ is the Frobenius isogeny $\hat{P}_+/U_+ \cong \hat{L} \to \hat{L}(q) \cong \hat{P}_-(q) / U_-(q)$. This algebraic zip datum is orbitally finite by [PWZ1] Proposition 7.3.

**Definition 3.1.** Let $S$ be a scheme over $k$. A $\hat{G}$-zip of type $(\chi, \Theta)$ is a tuple $(I, I_+, I_-, \iota)$ consisting of a $\hat{G}_k$-torsor $I$ over $S$, a $\hat{P}_+$-toros $I_+ \subseteq I$, a $\hat{P}_-$-torse $I_- \subseteq I$ and an isomorphism of $\hat{L}(q)$-torse $\iota : I_+^{(q)} / U_+^{(q)} \sim I_- / U_-^{(q)}$.

We fix a $\hat{G}$-zip $\mathcal{I}$ of type $(\chi, \Theta)$ over a $k$-scheme $S$. As explained in [PWZ2] Section 3 the moduli stack of $\hat{G}$-zips of type $(\chi, \Theta)$ is an algebraic stack isomorphic to the quotient stack $[E_\hat{G}/\hat{G}_k]$ over $k$. Hence $\mathcal{I}$ corresponds to a morphism $\zeta : S \to [E_\hat{G}/\hat{G}_k]$ and the description of the underlying topological space of $[E_\hat{G}/\hat{G}_k]$ yields a decomposition of $S$ into locally closed subschemes

\[(3.0.6) \quad S = \bigsqcup_{[\hat{w}] \in \Gamma \backslash \Xi^{\chi,\Theta}} S^{[\hat{w}]},\]

called the zip stratification of $S$ corresponding to the $\hat{G}$-zip $\mathcal{I}$. Here $\Xi^{\chi,\Theta}$ is defined in [PWZ2] (3.19) and $\Gamma$ is the absolute Galois group of $k$. If $G = G$ is connected, then $\Xi^{\chi,\Theta} = JW$ with the notation introduced in Subsection 2.2.

In particular we obtain for any algebraically closed extension $K$ of $k$ a bijection

\[(3.0.7) \quad \Xi^{\chi,\Theta} \leftrightarrow \left\{ \text{isomorphism classes of } \hat{G} \text{-zips of type } (\chi, \Theta) \text{ over } K \right\}\]

which we denote by $\Xi^{\chi,\Theta} \ni \hat{w} \mapsto L_\hat{w}$.

Moreover, if $\Gamma \backslash \Xi^{\chi,\Theta}$ is endowed with the partial order $\preceq$ defined in [PWZ2] (3.16), then the corresponding topological space is homeomorphic to the underlying topological space of $[E_\hat{G}/\hat{G}_k]$ ([PWZ2] Theorem 3.20). In particular we obtain

\[(3.0.8) \quad S^{[\hat{w}]} \subseteq \bigsqcup_{[\hat{w}'] \in \Gamma \backslash \Xi^{\chi,\Theta}} S^{[\hat{w}']}\]

with equality if $\zeta$ is generizing (e.g., if $\zeta$ is open, or flat, or formally smooth).

As an application of Theorem 2.2 we obtain the following result.

**Theorem 3.2.** The inclusion of each zip stratum $S^{[\hat{w}]} \to S$ is an affine morphism.
Proof. Let $\hat{G}^{[\hat{w}]}$ be the reduced $E_2$-orbit on $\hat{G}$ corresponding to $[\hat{w}] \in \Gamma \setminus \Xi^{X, \Theta}$. By [PWZ2] Proposition 2.2 we may consider $[E_2 \setminus \hat{G}^{[\hat{w}]})$ as a smooth locally closed substack of $[E_2 \setminus \hat{G}_k]$. The inclusion $[E_2 \setminus \hat{G}^{[\hat{w}]}] \hookrightarrow [E_2 \setminus \hat{G}_k]$ is affine, because it becomes the inclusion $\hat{G}^{[\hat{w}]} \hookrightarrow \hat{G}_k$, which is affine by Theorem 2.2 after the fppf base change $\hat{G}_k \to [E_2 \setminus \hat{G}_k]$. By definition, $S^{[\hat{w}]} = \zeta^{-1}([E_2 \setminus \hat{G}^{[\hat{w}]}])$ which implies the claim because being affine is stable under base change.

Using Proposition 1.6 we obtain as a corollary [PWZ2] Proposition 3.33 which was announced there without proof.

**Corollary 3.3.** Let $S$ be a locally noetherian scheme over $k$, and let $Z$ be a closed subscheme of codimension $\geq 2$. Assume that $Z$ contains no embedded component of $S$ (which is automatic if $S$ is reduced). Let $L$ be a $\hat{G}$-zip over $S$ whose restriction to $S \setminus Z$ is fppf-locally constant. Then $L$ is fppf-locally constant.

Recall that a syntomic morphism $f: X \to S$ of schemes is a flat morphism locally of finite presentation which can be factorized locally on $X$ into a regular immersion followed by a smooth morphism. Then in the statement of the corollary we may replace “fppf-locally constant” by “constant after a finite syntomic base change” by the following strengthening of [PWZ2] Proposition 3.31.

**Proposition 3.4.** Assume that $G$ splits over $k$ and that $\pi_0(\hat{G}_k)$ is a constant scheme. Let $L$ be as before and let $S = \bigsqcup_{\hat{w} \in \Xi^{X, \Theta}} S^{\hat{w}}$ be the corresponding zip stratification. For $\hat{w} \in \Xi^{X, \Theta}$ fix a finite extension $k'$ of $k$ such that the corresponding $\hat{G}$-zip $L_{\hat{w}}$ (3.0.7) is defined over $k'$.

Then a morphism of schemes $f: T \to S$ factors through $S^{\hat{w}}$ if and only if there exists a finite syntomic surjective morphism $g: T' \to T$ where $T'$ is a $k'$-scheme such that $g^*(f^*L)$ is isomorphic to the constant $\hat{G}$-zip $L_{\hat{w}} \times_{k'} T'$.

Proof. The splitting assumptions on $G$ and on $\pi_0(\hat{G}_k)$ imply that the $E_2$-orbit $\hat{G}^{\hat{w}}$ in $\hat{G}$ corresponding to $\hat{w}$ is defined over $k$. The orbit morphism $a: E_2 \to \hat{G}^{\hat{w}}, e \mapsto e \cdot \hat{w}$, (2.2.1) yields an isomorphism $E_2/H_\hat{w} \sim \hat{G}^{\hat{w}}$, where $H_\hat{w}$ is the scheme theoretic stabilizer of $\hat{w}$ in $E_2$. Proceeding as in the proof of [PWZ2] Proposition 3.31 it suffices to show that there exists a finite surjective syntomic morphism $Z \to \hat{G}^{\hat{w}}$ such that $a \times \text{id}_Z: E_2 \times_{\hat{G}^{\hat{w}}} Z \to Z$ has a section.

Now [PWZ1] Theorem 8.1 shows that the reduced identity component $(H_\hat{w})^0_{\text{red}}$ of $H_\hat{w}$ is a connected unipotent algebraic group (over a perfect field) and hence the projection $E_2 \to E_2/(H_\hat{w})^0_{\text{red}}$ has a scheme-theoretic section. Moreover, $H_\hat{w}/(H_\hat{w})^0_{\text{red}}$ is a finite group scheme and hence finite syntomic ([SGA3] Exp. VIIB, Corollaire 5.4). Hence we can take $Z = E_2/(H_\hat{w})^0_{\text{red}}$.

These results can in particular be applied to the examples of $\hat{G}$-zips given in [PWZ2] Section 9 or in [Wd2] Subsection 3.3 (see also [Lie]):

1. $\text{GL}_d$-zips that arise from the De Rham cohomology of smooth proper Deligne-Mumford stacks $\mathcal{X}$ over $S$ such that the Hodge spectral sequence degenerates and such that the higher derived images of the differential sheaves $\Omega^k_{\mathcal{X}/S}$ are locally free.
\(O_S\)-modules. Here \(d\) is the rank of \(H^p_{DR}(\mathcal{X}/S)\). In middle degree \(n = \dim(\mathcal{X}/S)\), the De Rham cohomology carries via the cup product an extra structure yielding the structure of a \(\text{GSp}_d\)-zip or of a \(\text{GO}_d\)-zip, depending on the parity of \(n\).

(2) \(\text{GL}_h\)-zips that arise from at level 1 truncated Barsotti-Tate groups of height \(h\):

Recall that in \([\text{PWZ2}]\) Subsection 9.3 there is described a functor from the category of \(\text{BT}_1\) of height \(h\) and dimension \(d\) over an \(\mathbb{F}_p\)-scheme \(S\) to the category of \(\text{GL}_h\)-zips over \(S\) of type \(\chi\), where \(\chi\) is the cocharacter \(t \mapsto \text{diag}(t, \ldots, t, 1, \ldots, 1)\) where \(t\) is repeated \(h - d\) times. By Dieudonné theory this is an equivalence of categories if \(S\) is the spectrum of a perfect ring.

This allows us to deduce the main result of \([\text{NVW}]\) without any of the restrictions on the characteristic made there: Let \(k\) be a field of characteristic \(p > 0\), let \(D_m\) be an \(m\)-truncated Barsotti-Tate group (a \(\text{BT}_m\)) over \(k\) for \(1 \leq m < \infty\). For a \(k\)-scheme \(S\) and for a \(\text{BT}_m\) \(X_m\) over \(S\) let \(S_{D_m}(X_m)\) be the unique locally closed subscheme of \(S\) that satisfies the following property. A morphism \(f: T \to S\) factors through \(S_{D_m}(X_m)\) if and only if \(f^*(X_m)\) and \(D_m \times_k T\) are locally for the fpf-topology isomorphic.

**Theorem 3.5.** The inclusion \(S_{D_m}(X_m) \hookrightarrow S\) is affine. In particular, if \(S\) is locally noetherian, then every irreducible component of \(S_{D_m}(X_m) \setminus S_{D_m}(X_m)\) has pure codimension 1 in \(S_{D_m}(X_m)\).

As in \([\text{NVW}]\) Corollary 1.5 one obtains an analogous purity result for Barsotti-Tate groups.

**Proof.** We may assume that \(k\) is algebraically closed. Moreover we may assume that \(X_m\) has constant height, constant dimension, and constant codimension on \(S\) equal, respectively, to the height, dimension, and codimension of \(D_m\) because all these numerical invariants are locally constant on \(S\).

Consider first the case \(m = 1\). Let \(h\) be the height of \(D_1\). As recalled above, the isomorphism class of \(D_1\) corresponds to the isomorphism class of some \(\text{GL}_h\)-zip \(\mathcal{L}\) by Dieudonné theory and hence to some element \(w \in \mathcal{W} \supseteq S_h\) \((\text{PWZ2} 9.19)\), see also Moonen’s paper \([\text{Mo}]\), where this has been shown first). Let \(\mathcal{L}\) be the \(\text{GL}_h\)-zip attached to \(X_1\). Then \(S^w\) has the universal property that a morphism \(T \to S\) factors through \(S^w\) if and only if \(f^*\mathcal{L}\) is isomorphic to \(\mathcal{L}_T\) \((\text{Proposition 3.3} \text{ or } \text{PWZ2} \text{ Prop. 3.31})\). Hence if \(T = \text{Spec}(K)\) for a perfect field \(K\), then \(f\) factors through \(S^w\) if and only if it factors through \(S_{D_1}(X_1)\). Therefore the underlying topological spaces of \(S^w\) and of \(S_{D_1}(X_1)\) are equal. As \(S^w \to S\) is affine by \(\text{Theorem 5.2} S_{D_1}(X_1) \to S\) is affine \((\text{Lemma 1.5})\).

The case for \(1 < m < \infty\) can now be deduced from the case \(m = 1\) by \([\text{NVW}]\) Lemma 5.2 whose proof does not use the restrictions on \(p\) made for the proof in the case \(m = 1\) in loc. cit. \(\square\)

## 4 Ekedahl-Oort strata for Shimura varieties of Hodge type

Let \((G, X)\) be a Shimura datum of Hodge type, i.e., \((G, X)\) is a Shimura datum that can be embedded into the Siegel Shimura datum. We choose such an embedding. Hence \(G\) is a reductive group over \(\mathbb{Q}\) and \(X\) is a \(G(\mathbb{R})\)-conjugacy class of homomorphisms
$h: \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}} \to G_\mathbb{R}$ satisfying Deligne’s conditions. Let $p$ be a prime number such that $G(\mathbb{Q}_p)$ has a hyperspecial subgroup $K_p$. Choose a place $v$ of the reflex field $E$ of $(G, X)$ over $p$. Let $K = K_p K^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. If $K^p$ is sufficiently small, Kisin ([Ki]) and Vasiu ([Va1]) have shown the existence of integral canonical models for the Shimura variety attached to $(G, X)$ (with some restrictions for $p = 2$) over $O_{E,v}$. We denote by $S := S_K(G, X)$ the special fiber. It is a smooth quasi-projective over $k := \kappa(v)$ the residue field of the place $v$.

The hyperspecial group $K_p$ is the group of $\mathbb{Z}_p$-valued points of a reductive group scheme over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$. We denote by $G$ its special fiber. Hence $G$ is a reductive group over $\mathbb{F}_p$. To $X$ one may attach a conjugacy class $[\chi]$ of cocharacters of $G_k$ as explained in [Wo] Subsection 5.3. Choose $\chi \in [\chi]$. As explained in Subsection 3.1, $(G, \chi)$ yields a connected algebraic zip datum $Z := Z_{G, \chi} = (G_k, P, Q, \varphi)$. Zhang has constructed in [Zh] a $G$-zip of type $\chi$ over $S$ and he has shown in loc. cit. that the corresponding classifying morphism $S \to [E_Z \backslash G_k]$ is smooth. Here we use the (slightly different) Construction 5.13 of a $G$-zip $Z$ of type $\chi$ given in [Wo] and obtain a smooth morphism

$$\zeta : S \to [E_Z \backslash G_k]$$

yielding a zip stratification

$$S = \bigsqcup_{w \in J_W} S^w$$

of the special fiber of the Shimura variety of Hodge type. Here $J$ is the type of $P$. This stratification is the Ekedahl-Oort stratification of $S$. Note that as we follow here Wortmann’s construction, the type $\chi$ of the $G$-zip is the inverse of the type considered in [ViWd] in the special case of Shimura varieties of PEL type. This is because Wortmann uses systematically De Rham cohomology and contravariant Dieudonné theory (following Kisin) and in [ViWd] De Rham homology and covariant Dieudonné theory is used.

The smoothness of $\zeta$ immediately implies that all Ekedahl-Oort strata $S^w$ are smooth ([PWZ2] Proposition 3.30). Applying Theorem 3.2 and Proposition 1.6 to this special case we obtain the following result.

**Theorem 4.1.** The inclusion of an Ekedahl-Oort stratum $S^w \hookrightarrow S$ into the special fiber of the integral model of a Shimura variety of Hodge type at a place of good reduction is affine. In particular, every irreducible component of $\overline{S^w} \setminus S^w$ has codimension 1 in $\overline{S^w} = \bigsqcup_{w' \leq w} S^{w'}$.

The results above allow also to show the following theorem.

**Theorem 4.2.** Every Ekedahl-Oort stratum $S^w$ is quasi-affine.

This generalizes the analogue fact for Shimura varieties of PEL type from [ViWd].

**Proof.** Let $f : \mathcal{A} \to S$ be the universal abelian scheme (more precisely the restriction of the universal abelian scheme over the Siegel space by the chosen inclusion of $S$ into the Siegel space). Then $\omega := \det(f_* \Omega^1_{\mathcal{A}/S})$ is an ample line bundle as the restriction of an ample line bundle from the Siegel space. On the other hand Proposition 3.4
implies that we find a finite syntomic surjective morphism $T \to S^w$ such that $g^*L$ is constant. In particular the $P_+$-torsor of $g^*L$, which corresponds to the pullback of the Hodge filtration $f_*\Omega^1_{A/S} \subseteq H^1_{\text{DR}}(A/S)$ under $g^*$ is trivial. Therefore the ample line bundle $g^*\omega$ is trivial and hence $T$ is quasi-affine. This implies that $S^w$ is quasi-affine by [ViWd] Lemma 10.5.

We conclude with an application of the smoothness of $\zeta$ not related to purity. In [Wd2] it was shown that for every $G$-zip $L$ of some type $(\chi, \Theta)$ over a scheme $T$ the corresponding zip stratification is the refinement of a coarser stratification, the Bruhat stratification. Let $\mathcal{Z}$ be the algebraic zip datum corresponding to $(G, \chi, \Theta)$. The Bruhat stratification is obtained by composing the classifying morphism $T \to \mathcal{E}_G \backslash \mathcal{G}$ corresponding to $L$ with the canonical morphism

\[(4.0.9) \quad \beta: \mathcal{E}_G \backslash \mathcal{G} \to \mathcal{B}\]

constructed in [Wd2] Subsection 2.3. Here $\mathcal{B}$ is the Bruhat stack attached to the $(\mathcal{G}, \chi, \Theta)$ in loc. cit.

For the special fiber $S$ of the integral model of a Shimura variety of Hodge type the Bruhat stratification generalizes the stratification by the $a$-number from the Siegel case (Wd3) studied by Oort (Oo2). It is a decomposition $S = \bigsqcup_{[x] \in \Gamma \backslash JW} [x]S$, where $W$, $J$, and $K$ are defined as in Subsection 2.2 and where $\Gamma$ is the absolute Galois group of the finite field $k$.

The results above now allow to generalize many of the results of [Wd3] to Shimura varieties of Hodge type.

**Theorem 4.3.** All Bruhat strata $[x]S$ are smooth and either empty or of dimension $\ell(x^{J,K})$, where $x^{J,K}$ is the element of maximal length in $JW \cap WJxWK$. The closure of $[x]S$ is given by

$\overline{[x]S} = \bigsqcup_{[x'] \leq [x]} [x']S$,

where $\leq$ denotes the partial order on $\Gamma \backslash JW$ induced by the Bruhat order on $W$.

**Proof.** The canonical morphism $\beta$ (1.1.9) is smooth ([Wd2] Proposition 2.9). Therefore the morphism $a: S \to \mathcal{B}$ defining the Bruhat stratification is smooth as a composition of smooth morphisms. Hence all of the assertions follow from the analogous assertions for $\mathcal{B}$ (Wd2 Section 1).

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