Nonlinear Schrödinger equations with radially symmetric data of critical regularity

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Abstract

This paper is concerned with the global existence of small solutions to pure-power nonlinear Schrödinger equations subject to radially symmetric data with critical regularity. Under radial symmetry we focus our attention on the case where the power of nonlinearity is somewhat smaller than the pseudoconformal power and the initial data belong to the scale-invariant homogeneous Sobolev space. In spite of the negative-order differentiability of initial data the nonlinear Schrödinger equation has global in time solutions provided that the initial data have the small norm. The key ingredient in the proof of this result is an effective use of global weighted smoothing estimates specific to radially symmetric solutions.

Key Words and Phrases. Nonlinear Schrödinger equation, radial solutions, critical regularity.

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1 Introduction and result

In this paper we study the Cauchy problem for the nonlinear Schrödinger equation

\begin{equation}
    i\partial_t u + \Delta u = \lambda |u|^{p-1}u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n
\end{equation}

\(p > 1, \ \lambda \in \mathbb{C}\) subject to the initial data \(u(0, x) = \varphi(x)\). Our main concern is to investigate the problem of global existence of small \(H^s\)-solutions. Assuming

\begin{equation}
    p - 1 = \frac{4}{n-2s}, \quad 0 \leq s < \frac{n}{2}
\end{equation}

(or equivalently \(s = \frac{n}{2} - \frac{2}{p-1}, \quad 1 + \frac{4}{n} \leq p < \infty\))
and in addition \([s] < p - 1\) ([s] denotes the greatest integer not greater than \(s \geq 0\)) if \(p - 1\) is not an even integer, Cazenave and Weissler proved that, for any \(\varphi \in H^s(\mathbb{R}^n)\) with \(||D_x|^s \varphi||_{L^2(\mathbb{R}^n)}\) small, the associated integral equation has a unique global solution (Theorems 1.1 and 1.2 in [2]). From the point of view of the \(H^s\)-theory, their result concerns the critical case. Though they left it open to show the global existence of small \(H^s\)-solutions in the case where subcritical nonlinear terms occur in the equation together with the critical nonlinear term (see page 81 of [2]), this issue was resolved by Kato to some extent (Theorem 6.1 in [9]). Note that, for the proof of small data global existence, Kato assumed much less than Cazenave and Weissler did, but still he assumed the restrictive condition (F.3) on page 299 of [9] which keeps any \(L^2\)-subcritical term \(\lambda|u|^{p-1}u\) with \(p < 1 + 4/n\) from occurring in the equation. We also refer to the fine works of Ginibre, Ozawa and Velo [4], Nakamura and Ozawa [12], and Pecher [18]. In these papers they also devised how to relax the assumption on the nonlinear term Cazenave and Weissler made in [2].

Allowing the \(L^2\)-critical and/or the \(L^2\)-supercritical terms in the equation, they also established theorems on the global existence of small solutions in the \(H^s\)-framework for \(0 \leq s < n/2\). For the \(H^s\)-theory with \(s \geq n/2\) we refer the reader to Nakamura and Ozawa [13], [15].

In this paper we focus our attention on radially symmetric solutions to the pure-power nonlinear Schrödinger equation, and we intend to explore the subject concerning the global existence of small solutions when \(p\) is somewhat smaller than the \(L^2\)-critical power \(1 + 4/n\) and the radially symmetric initial data \(\varphi\) is in the scale-invariant homogeneous Sobolev space. Let \(U(t) = e^{it\Delta}\), and let \(\dot{H}^s_2(\mathbb{R}^n)\) denote the homogeneous Sobolev space \(\{v \in S'(\mathbb{R}^n) \mid ||D_x|^s v \in L^2(\mathbb{R}^n)\}\) for \(-n/2 < s < n/2\), where \(|D_x|^s v := \mathcal{F}^{-1}|\xi|^s \mathcal{F} v\). We shall prove

**Theorem 1.1.** Suppose that \(n \geq 3,\ 4/(n + 1) < p - 1 < 4/n,\ \lambda \in \mathbb{C}\). Set \(s_0 = -n/2 + 2/(p - 1)\). There exists a positive constant \(\delta\) depending on \(n, p, \lambda\) such that if radially symmetric data \(\varphi \in \dot{H}^{-s_0}_2(\mathbb{R}^n)\) is small so that \(||D_x|^{-s_0} \varphi||_{L^2(\mathbb{R}^n)} \leq \delta\), then the integral equation

\[
(1.3) \quad u(t) = U(t)\varphi - i\lambda \int_0^t U(t - \tau)|u(\tau)|^{p-1}u(\tau)d\tau
\]

has a unique radially symmetric solution \(u \in C(\mathbb{R}; \dot{H}^{-s_0}_2(\mathbb{R}^n))\) satisfying

\[
(1.4) \quad \sup_{t \in \mathbb{R}} ||D_x|^{-s_0} u(t, \cdot)||_{L^2(\mathbb{R}^n)} + ||x|^{-\alpha_0} u||_{L^{\infty}(\mathbb{R} \times \mathbb{R}^n)} \leq C\delta
\]
for a suitable constant $C > 0$. Here $q_0 > 2$ and $\alpha_0 \in \mathbb{R}$ are the exponents defined in (3.5), (3.6) below.

Note that $0 < s_0 < 1/2$ for $4/(n + 1) < p - 1 < 4/n$. Hence the key to showing this theorem is getting over the difficulty caused by the negative-order differentiability of initial data. The Strichartz estimates manifest the gain of integrability exponents for the free solutions with initial data in $L^2(\mathbb{R}^n)$ [22]. They have played an essential role in the development of the local and global (in time) $H^s$-theory for $s \geq 0$, together with space-time estimates of solutions to inhomogeneous equations as well as elaborate estimates of products of functions in fractional-order Sobolev or Besov spaces. For the purpose of carrying out the contraction-mapping argument in the present setting we therefore start by showing the Strichartz-type estimates with derivative gain for the free Schrödinger equation subject to initial data in $L^2(\mathbb{R}^n)$ (see Section 2). In order to obtain such estimates we adapt a technique of showing extended Strichartz estimates for the free wave equation with radially symmetric data (see, e.g., Sogge [20] on page 126), and in this way we shall prove weighted space-time $L^q(\mathbb{R}^{1+n})$ estimates with derivative gain for solutions to the free Schrödinger equation with radially symmetric data in $L^2(\mathbb{R}^n)$. In application to nonlinear problems it is also necessary to establish space-time estimates for the inhomogeneous Schrödinger equation. By virtue of the Christ-Kiselev lemma [3] along with the $TT^*$ argument, it is possible to derive some weighted estimates of radially symmetric solutions to the inhomogeneous equation directly from the weighted estimates for the free Schrödinger equation. Making use of these weighted estimates, we can carry out the contraction-mapping argument to show the main result.

In connection with Theorem 1.1 the referee has advised the author to make reference to the results of Y. Tsutsumi [25], Ginibre, Ozawa and Velo [4], and Nakanishi and Ozawa [16] who also studied the global existence for the nonlinear Schrödinger equation with the $L^2$-subcritical term $\lambda|u|^{p-1}u$ $(0 < p - 1 < 4/n, \lambda \in \mathbb{C})$. For $\lambda \in \mathbb{R}$ it was proved that the nonlinear Schrödinger equation admits a unique global solution for any data $\varphi \in L^2(\mathbb{R}^n)$ [25]. When $\lambda \in \mathbb{C}$ and

\[
\frac{2}{n} < p - 1 < \frac{4}{n} \quad (n \leq 3), \quad \frac{\sqrt{n^2 + 4n + 36} - n - 2}{4} < p - 1 < \frac{4}{n} \quad (n \geq 4),
\]

it follows from Theorem 2.1 of [16], which refines an earlier result of [4], that the nonlinear Schrödinger equation admits a unique global solution for any data $\varphi \in L^2(\mathbb{R}^n) \cap \mathcal{F}\dot{H}^{s_0}$ having its small $\mathcal{F}\dot{H}^{s_0}$-norm. Here $s_0$ is the same as in Theorem
1.1. In view of the dual Hardy inequality

\[ \| D_x \|^{-s_0} v \|_{L^2(\mathbb{R}^n)} \leq C \| |x|^{s_0} v \|_{L^2(\mathbb{R}^n)} \]

we find that under the spherical symmetry assumption Theorem 1.1 improves on the result of Nakanishi and Ozawa because it uses the weaker norm to measure the size of initial data. The author thanks the referee for drawing attention to the relation between Theorem 1.1 and Theorem 2.1 of [16].

Finally we mention a similar progress on the \( H^s \)-theory for the nonlinear wave equation

\[ \partial_t^2 u - \Delta u = \lambda |u|^{p-1} u. \]  

Global existence of small \( H^s \)-solutions has been studied in Lindblad and Sogge [11], Nakamura and Ozawa [14]. (See also [11], [8], [17], [14] on the local in time \( H^s \)-theory.) Some improvements have been made in the radial case thanks to extended Strichartz-type estimates specific to radially symmetric solutions. See Theorem 6.6.1 of Sogge [20] and Hidano [6], [7] for this matter.

This paper is organized as follows. Section 2 is devoted to the proof of weighted space-time \( L^q \) estimates for the free Schrödinger equation and the inhomogeneous equation. In Section 3 we prove Theorem 1.1 by using these key estimates.

## 2 Weighted space-time \( L^q \) estimates

In this section we first prove weighted \( L^q \) estimates for the free Schrödinger equation with radially symmetric data by adapting the method of showing improved radial Strichartz estimates for the free wave equation (see, e.g., Sogge [20] on page 126). Subsequently, using the celebrated lemma of Christ and Kiselev [3] (see also Smith and Sogge [19], Tao [24]) along with the \( TT^* \) argument, we derive some weighted \( \tilde{L}^{q'} - L^q \) estimates for the inhomogeneous equation from the weighted estimate for the free Schrödinger equation with radially symmetric data.

Let us start with showing the following theorem.

**Theorem 2.1.** Suppose \( n \geq 2 \). There exists a constant \( C > 0 \) depending on \( n, q, \alpha \), and the estimate

\[ \| |x|^{-\alpha} D_x |^s U(t) \|_{L^n(\mathbb{R} \times \mathbb{R}^n)} \leq C \| \varphi \|_{L^2(\mathbb{R}^n)} \]

holds for radially symmetric data \( \varphi \in L^2(\mathbb{R}^n) \) provided that

\[ -\alpha - s + \frac{n+2}{q} = \frac{n}{2}, \quad 2 \leq q < \infty, \quad \frac{n}{q} - \frac{n-1}{2} < \alpha < \frac{n}{q}. \]
Proof. We follow the proof of Strichartz-type estimates for the free wave equation with radially symmetric data. (See page 126 of [20]. See also [6], [7].) Let $d\sigma$ denote the Lebesgue measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and let $\hat{d\sigma}$ denote the Fourier transform of $d\sigma$, that is,

\begin{equation}
\hat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-i\omega \cdot \xi} d\sigma = d\sigma(\omega).
\end{equation}

Since $\varphi$ is radially symmetric and so is its Fourier transform $\hat{\varphi}$, we may write $\hat{\varphi}(\xi)$ as $\psi(|\xi|)$. Making the change of variables $\eta = \rho^2$, we have

\begin{equation}
|D_x|^s(U(-t)\varphi)(x)
= (2\pi)^{-n} \int_0^\infty e^{it\rho^2} \psi(\rho) \rho^{s+n-1} \hat{d\sigma}(\rho x) d\rho
= (2\pi)^{-n} \int_0^\infty e^{it\eta} \psi(\sqrt{\eta}) \eta^{(s+n-1)/2} \hat{d\sigma}(\sqrt{\eta} x) \frac{1}{2\sqrt{\eta}} d\eta
= (2\pi)^{-n+1} \left( (2\pi)^{-1} \int_{\mathbb{R}} e^{it\eta} H(\eta) \psi(\sqrt{\eta}) |\eta|^{(s+n-1)/2} \hat{d\sigma}(\sqrt{\eta} x) \frac{1}{2\sqrt{\eta}} d\eta \right).
\end{equation}

Here $H(\eta)$ denotes the Heaviside function. We therefore get by the Sobolev embedding theorem and the Plancherel theorem

\begin{equation}
\| |D_x|^s(U(\cdot)\varphi)(x)\|_{L^q(\mathbb{R}_+)}
\leq C \| |D_x|^{1/2-1/q} \left( (2\pi)^{-1} \int_{\mathbb{R}} e^{it\eta} H(\eta) \psi(\sqrt{\eta}) |\eta|^{(s+n-2)/2} \hat{d\sigma}(\sqrt{\eta} x) d\eta \right) \|_{L^2(\mathbb{R}_+)}
= C \| \eta^{1/2-1/q+(s+n-2)/2} \psi(\sqrt{\eta}) \hat{d\sigma}(\sqrt{\eta} x) \|_{L^2(\mathbb{R}_+)}
\end{equation}

($\mathbb{R}_+ = (0, \infty)$) for any fixed $x \in \mathbb{R}^n$. Using the Minkowski inequality and making the change of variables $y := \sqrt{\eta} x$, we obtain from (2.5)

\begin{equation}
\| |x|^{-\alpha} |D_x|^s(U(t)\varphi)(x)\|_{L^q(\mathbb{R}^n \times \mathbb{R}_+^2)}
\leq C \| \eta^{1/2-1/q+(s+n-2)/2} \psi(\sqrt{\eta}) \|_{L^q(\mathbb{R}_+)} \| |x|^{-\alpha} \hat{d\sigma}(\sqrt{\eta} x) \|_{L^2(\mathbb{R}_+)}
= C \| \eta^{1/2-1/q+(s+n-2)/2} \psi(\sqrt{\eta}) \eta^{\alpha/2-n/(2q)} \|_{L^q(\mathbb{R}_+)} \| |y|^{-\alpha} \hat{d\sigma}(y) \|_{L^q(\mathbb{R}^n \times \mathbb{R}_+^2)}
\leq C \| \eta^{1/2-1/q+(s+n-2)/2+\alpha/2-n/(2q)} \psi(\sqrt{\eta}) \|_{L^2(\mathbb{R}_+)}.
\end{equation}

Here we have handled the $L^q(\mathbb{R}^n)$-norm of $|y|^{-\alpha} \hat{d\sigma}(y)$ as

\begin{equation}
\| |y|^{-\alpha} \hat{d\sigma}(y)\|_{L^q(\{y \in \mathbb{R}^n : |y| < 1\})} + \| |y|^{-\alpha} \hat{d\sigma}(y)\|_{L^q(\{y \in \mathbb{R}^n : |y| > 1\})} \leq C,
\end{equation}

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using the assumption \((n/q) - (n-1)/2 < \alpha < n/q\) and the fact that \(\tilde{\sigma}\) is a smooth function on \(\mathbb{R}^n\) satisfying \(\tilde{\sigma}(y) = O(|y|^{-(n-1)/2})\) as \(|y| \to +\infty\) (See, e.g., Stein [21] on page 348.). To finish the proof, we make the change of variables \(\lambda = \sqrt{\eta}\), and we continue the estimate of (2.6) as

\[
\ldots = C\| \lambda^{1-(2/q)+s + n - 2 + \alpha} \psi(\lambda) \lambda^{1/2} \|_{L^2(\mathbb{R}^n_+)} = C\| \lambda^{(n-1)/2} \psi(\lambda) \|_{L^2(\mathbb{R}^n_+)} = C\| \varphi \|_{L^2(\mathbb{R}^n)}.
\]

We have finished the proof of Theorem 2.1. \hfill \Box

Remark. The referee has kindly pointed out that Theorem 2.1 is true also for \(q = \infty\). Indeed, using the weighted Sobolev inequality for radially symmetric functions (see, e.g., Appendix of [5]), we get for all \((t, x) \in \mathbb{R} \times \mathbb{R}^n\)

\[
|x|^{(n/2) - \bar{s}} |(U \varphi)(t, x)| \leq C\| D_x^s (U \varphi)(t, \cdot) \|_{L^2(\mathbb{R}^n)} = C\| D_x^s \varphi \|_{L^2(\mathbb{R}^n)}, \quad \frac{1}{2} < s < \frac{n}{2}.
\]

This immediately leads to the estimate (2.1) for \(q = \infty\). The referee has also pointed out that it is hence possible to derive the estimate (2.1) for \(2 < q < \infty\) via the complex interpolation between the above case \(q = \infty\) and the well-known estimate for \(q = 2\) obtained in [10], [1], [23], and [26].

Using the Christ-Kiselev lemma [3] (see also Smith and Sogge [19], Tao [24]), we can derive the following estimates for the inhomogeneous equation from the estimate (2.1) for the free solution \(U(t) \varphi\).

**Theorem 2.2.** Suppose \(n \geq 2, 2 \leq q < \infty, 2 \leq \tilde{q} < \infty, n/q - (n-1)/2 < \alpha < n/q, n/\tilde{q} - (n-1)/2 < \tilde{\alpha} < n/\tilde{q}\). Set

\[
s := -\alpha + \frac{n + 2}{q} - \frac{n}{2}, \quad \bar{s} := -\tilde{\alpha} + \frac{n + 2}{\tilde{q}} - \frac{n}{2}.
\]

The estimate

\[
\|x|^{-\alpha} |D_x|^s \int_0^t U(t - \tau) F(\tau) d\tau \|_{L^s(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\|x|^{\bar{s}} |D_x|^{-\bar{s}} F\|_{L^{2(\mathbb{R}^n \times \mathbb{R}^n)}}
\]

holds for radially symmetric (in \(x\)) \(F\).

**Theorem 2.3.** Suppose \(n \geq 2, 2 \leq \tilde{q} < \infty, n/\tilde{q} - (n-1)/2 < \tilde{\alpha} < n/\tilde{q}\). Set \(\bar{s} := -\tilde{\alpha} + (n + 2)/\tilde{q} - n/2\). The estimate

\[
\|D_x|^\bar{s} \int_0^t U(t - \tau) F(\tau) d\tau \|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \leq C\|x|^{\bar{s}} F\|_{L^{2}(\mathbb{R}^n \times \mathbb{R}^n)}
\]
holds for radially symmetric (in $x$) $F$.

**Corollary 2.4.** Suppose $n \geq 2$, $2 \leq q < \infty$, $2 \leq \tilde{q} < \infty$, $n/q - (n-1)/2 < \alpha < n/q$, $n/\tilde{q} - (n-1)/2 < \tilde{\alpha} < n/\tilde{q}$, and

$$-\alpha + \frac{n+2}{q} - \frac{n}{2} - \tilde{\alpha} + \frac{n+2}{\tilde{q}} - \frac{n}{2} = 0.$$  

The estimate

$$\left\| D_x^{-\tilde{\alpha} + (n+2)/\tilde{q} - n/2} \int_0^t U(t - \tau) F(\tau) d\tau \right\|_{L^\infty(R; L^2(R^n))}$$

$$+ \left\| x^{-\alpha} \int_0^t U(t - \tau) F(\tau) d\tau \right\|_{L^q(R \times R^n)} \leq C \left\| x^{\tilde{\alpha}} F \right\|_{L^{\tilde{q}'}(R \times R^n)}$$

holds for radially symmetric (in $x$) $F$.

**Proof of Theorem 2.2.** By Theorem 2.1 we know

$$\left\| U^* F \right\|_{L^2(R^n)} \leq C \left\| x^{\tilde{\alpha}} |D_x|^{-\tilde{s}} F \right\|_{L^{\tilde{q}'}(R \times R^n)}$$

and hence

$$\left\| x^{-\alpha} |D_x|^s U^* F \right\|_{L^q(R \times R^n)} \leq C \left\| x^{\tilde{\alpha}} |D_x|^{-\tilde{s}} F \right\|_{L^{\tilde{q}'}(R \times R^n)}.$$  

Here

$$(U^* F)(t, x) = U(t) \int_R U(-\tau) F(\tau) d\tau = \int_R U(t - \tau) F(\tau) d\tau.$$

If $\tilde{q}' < q$, then the estimate (2.9) is an immediate consequence of the Christ-Kiselev lemma [3] (see also Smith and Sogge [19], Tao [24]). Though the Christ-Kiselev lemma does not apply to the case $\tilde{q}' = q = 2$, Sugimoto [23] and Vilela [26] have already generalized the works of Kato and Yajima [10] and Ben-Artzi and Klainerman [1], and they have independently shown the estimate (2.9) for $\tilde{q}' = q = 2$ without assuming radial symmetry. The proof of Theorem 2.2 has been finished.

**Proof of Theorem 2.3.** By the $L^2$ conservation and Theorem 2.1 we have

$$\left\| U^* F \right\|_{L^\infty(R; L^2(R^n))} = \left\| U^* F \right\|_{L^2(R^n)} \leq C \left\| x^{\tilde{\alpha}} |D_x|^{-\tilde{s}} F \right\|_{L^{\tilde{q}'}(R \times R^n)}.$$  

The estimate (2.10) is an immediate consequence of the Christ-Kiselev lemma [3] and (2.15) as before. The proof has been finished.
3 Proof of Theorem 1.1

Our proof starts with the following elementary result.

**Proposition 3.1.** The inequality

\[
\max \left( \frac{1}{p}, \frac{2}{p-1} - \frac{n-1}{2} \right) < \frac{2}{p-1} - \frac{n+1}{2p}
\]

holds for all \( n \geq 3 \) and \( 1 + 4/(n+1) < p < 1 + 4/n \).

**Proof.** Though this result is not valid for \( n = 2 \), we shall keep \( n \) general until the end of our argument for clarity. By \( p_0(n) \) we write the larger root of the quadratic equation \((n-1)p^2 - (n+1)p - 2 = 0\), namely

\[ p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)}. \]

By direct computation we can find that \( 1 + 4/(n+1) < p_0(n) \) for all \( n \geq 2 \), and that \( 1 + 4/n < p_0(n) \) for \( n = 2, 3 \), \( 1 + 4/n = 2 = p_0(n) \) for \( n = 4 \), and \( p_0(n) < 1 + 4/n \) for \( n \geq 5 \). Because of the equivalence

\[ \frac{1}{p} < \frac{2}{p-1} - \frac{n-1}{2} \iff p < p_0(n), \]

the proof of Proposition 3.1 is reduced to showing

\[
\frac{2}{p-1} - \frac{n-1}{2} < \frac{2}{p-1} - \frac{n+1}{2p}, \quad n = 2, 3, 4, \quad 1 + \frac{4}{n+1} < p < 1 + \frac{4}{n},
\]

\[
\frac{2}{p-1} - \frac{n-1}{2} < \frac{2}{p-1} - \frac{n+1}{2p}, \quad n \geq 5, \quad 1 + \frac{4}{n+1} < p \leq p_0(n),
\]

\[
\frac{1}{p} < \frac{2}{p-1} - \frac{n+1}{2p}, \quad n \geq 5, \quad p_0(n) < p < 1 + \frac{4}{n}.
\]

The inequality (3.2) is equivalent to \( p > (n+1)/(n-1) \), which does not consist with \( p < 1 + 4/n \) for \( n = 2 \). This is the reason why the case \( n = 2 \) is ruled out in the proposition. On the other hand, by virtue of \((n+1)/(n-1) \leq 1 + 4/(n+1) \) for \( n \geq 3 \), the inequality \( p > (n+1)/(n-1) \) is automatically satisfied if \( n \geq 3 \). For the same reason the statement of (3.3) is true. Finally, we see that the inequality (3.4) is equivalent to \( p < 1 + 4/(n-1) \), which is certainly satisfied due to the assumption \( p < 1 + 4/n \). The proof of Proposition 3.1 has been finished. \( \square \)
In what follows we assume \( n \geq 3 \). Note that the assumption \( p > 1 + 4/(n+1) \) is equivalent to \( 2/(p-1) - (n-1)/2 < 1 \). Therefore, in view of (3.1) we can choose \( q_0 > 2 \) so that

\[
\max \left( \frac{1}{p}, \frac{2}{p-1} - \frac{n-1}{2} \right) < \frac{2}{q_0} < \frac{2}{p-1} - \frac{n+1}{2p},
\]

and set

\[
\alpha_0 := \frac{n+2}{q_0} - \frac{2}{p-1}, \quad s_0 := -\alpha_0 + \frac{n+2}{q_0} - \frac{n}{2}.
\]

Note that \( s_0 \) defined above is equal to \(-n/2 + 2/(p-1)\), so that \( s_0 > 0 \) for \( p < 1 + 4/n \). For the purpose of proving Theorem 1.1 we define the space \( X \) of tempered distributions on \( \mathbb{R}^{1+n} \) as follows:

\[
X := \{ v = v(t,x) \mid v \in C(\mathbb{R}; \dot{H}^{-s_0}_2(\mathbb{R}^n)) \text{ is radially symmetric in } x, \sup_{t \in \mathbb{R}} \| |D_x|^{-s_0} v(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \| |x|^{-\alpha_0} v\|_{L^q_0(\mathbb{R} \times \mathbb{R}^n)} < \infty \}.
\]

We carry out the contraction-mapping argument, using the estimate (2.11) with \( \tilde{q}, \tilde{\alpha} \) satisfying

\[
\frac{p\tilde{q}'}{q_0} = q_0, \quad -\tilde{\alpha}/p = \alpha_0.
\]

We rewrite \( \tilde{q}, \tilde{\alpha} \) of (3.8) as \( q_1, \alpha_1 \), respectively. To proceed, we must verify that these satisfy all the conditions of Corollary 2.4. It is easy to see that the condition

\[
-\alpha_0 + \frac{n+2}{q_0} - \frac{n}{2} - \alpha_1 + \frac{n+2}{q_1} - \frac{n}{2} = 0,
\]

which is reduced to \( \alpha_0 = (n+2)/q_0 - 2/(p-1) \), is surely satisfied. The condition \( n/q_0 - (n-1)/2 < \alpha_0 < n/q_0 \), which is reduced to

\[
\frac{2}{p-1} - \frac{n-1}{2} < \frac{2}{q_0} < \frac{2}{p-1}
\]

for \( \alpha_0 = (n+2)/q_0 - 2/(p-1) \), is satisfied by virtue of (3.5). The condition \( 1/q_1 \leq 1/2 \), which is equivalent to \( 1/p \leq 2/q_0 \), is assumed in (3.5). Since the definition of \( q_1 \) (see (3.8)) implies

\[
p \left( \frac{1}{p} - \frac{1}{q_0} \right) = \frac{1}{q_1},
\]

the condition \( 1/q_1 > 0 \) is equivalent to \( 1/q_0 < 1/p \). Because of

\[
\frac{2}{p-1} - \frac{n+1}{2p} < \frac{2}{p}
\]
for all $p > 1 + 4/(n + 1)$ we surely see that the inequality $1/q_0 < 1/p$ is true (see (3.5)). The condition $n/q_1 - (n - 1)/2 < \alpha_1$, which is equivalent to $2/q_0 < 2/(p - 1) - (n - 1)/(2p)$, is also assumed in (3.5). Finally, the condition $\alpha_1 < n/q_1$, which is equivalent to $2/(p - 1) - n/p < 2/q_0$, is automatically satisfied because $q_0$ has been chosen so that $2/(p - 1) - (n - 1)/2 < 2/q_0$, and we easily see $2/(p - 1) - n/p < 2/(p - 1) - (n - 1)/2$ for $p < 1 + 4/n$.

Since we have finished the verification of the fact that the exponents $q_j$ and $\alpha_j$ ($j = 0, 1$) satisfy all the conditions in Corollary 2.4, let us solve the associated integral equation

(3.9) \[ u(t) = U(t)\varphi - i \int_0^t U(t - \tau)f(u(\tau))d\tau \]

($f(u(\tau)) := \lambda |u(\tau)|^{p-1}u(\tau)$) by applying the key estimates (2.1) and (2.11). An application of (2.1) yields

**Proposition 3.2.** Suppose $n \geq 3$. If $\varphi \in \dot{H}^{-s_0}_2(\mathbb{R}^n)$ is radially symmetric, then $U(t)\varphi \in X$ and the estimate

(3.10) \[ \sup_{t \in \mathbb{R}} \| D_x |^{-s_0} U(t)\varphi \|_{L^2(\mathbb{R}^n)} + \| |x|^{-\alpha_0} U(t)\varphi \|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \| D_x |^{-s_0} \varphi \|_{L^2(\mathbb{R}^n)} \]

holds.

In view of (3.8) we also obtain the following by the application of (2.11).

**Proposition 3.3.** Suppose $n \geq 3$. If $v \in X$, then

(3.11) \[ \int_0^t U(t - \tau)f(v(\tau))d\tau \in X. \]

Moreover, the estimate

(3.12) \[ \sup_{t \in \mathbb{R}} \left\| D_x |^{-s_0} \int_0^t U(t - \tau)(f(v_1(\tau)) - f(v_2(\tau)))d\tau \right\|_{L^2(\mathbb{R}^n)} + \left\| |x|^{-\alpha_0} \int_0^t U(t - \tau)(f(v_1(\tau)) - f(v_2(\tau)))d\tau \right\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \]

\[ \leq C \left( \| |x|^{-\alpha_0} v_1 \|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} + \| |x|^{-\alpha_0} v_2 \|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \right)^{p-1} \]

\[ \times \| |x|^{-\alpha_0} (v_1 - v_2) \|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)}. \]

holds for $v_1, v_2 \in X$. 

10
Proof of Proposition 3.3. Recall that

\begin{equation}
-\alpha_1 + \frac{n+2}{q_1} - \frac{n}{2} = \alpha_0 - \frac{n+2}{q_0} + \frac{n}{2} = -s_0.
\end{equation}

To start with, we must note that it is actually possible to show

\begin{equation}
|D_x|^{-s_0} \int_0^t U(t-\tau)F(\tau)d\tau \in (C \cap L^\infty)(\mathbb{R}; L^2(\mathbb{R}^n))
\end{equation}

for radially symmetric (in \(x\)) \(F\) satisfying \(|x|^{\alpha_1}F \in L^{q_1}(\mathbb{R} \times \mathbb{R}^n)\). To verify the continuity in \(t\), by the density of \(C_0^\infty(\mathbb{R} \times \mathbb{R}^n)\) in \(L^{q_1}(\mathbb{R} \times \mathbb{R}^n)\) we can choose a sequence \(\{G_j\}_{j \in \mathbb{N}} \subset C_0^\infty(\mathbb{R} \times \mathbb{R}^n)\) such that \(G_j\) is radially symmetric in \(x\) and \(G_j \to |x|^{\alpha_1}F\) in \(L^{q_1}(\mathbb{R} \times \mathbb{R}^n)\) as \(j \to \infty\). Set \(F_j := |x|^{-\alpha_1}G_j\). We claim that \(|D_x|^{-s_0}F_j \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))\). For the verification of this claim it suffices to show \(|x|^{-\alpha_1} \in L^q(B_1)\) \((n/q = n/2 + s_0, 0 < s_0 < 1/2, B_1 := \{x \in \mathbb{R}^n : |x| < 1\})\) by the Sobolev embedding theorem \(L^q(\mathbb{R}^n) \subset \dot{H}^{-s_0}_2(\mathbb{R}^n)\). We easily see

\(|x|^{-\alpha_1} \in L^q(B_1) \iff \alpha_1 < \frac{n}{q} \left(= \frac{n}{2} + s_0\right)\)

\(\iff \alpha_1 - s_0 < \frac{n}{2} \iff \frac{n+2}{q_1} - \frac{n}{2} < \frac{n}{2} \quad \text{(by (3.13))}\)

\(\iff 1 - \frac{p}{q_0} < \frac{n}{n+2} \quad \text{(Recall } \frac{1}{q_1} = 1 - \frac{p}{q_0} \text{ by (3.8)}\)\)

\(\iff \frac{2}{q_0} > \frac{4}{p(n+2)}.\)

The last inequality obviously holds because of the assumptions \(2/q_0 > 1/p\) (see (3.5)) and \(n \geq 3\). Since we have shown \(|D_x|^{-s_0}F_j \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))\), we find by the standard argument

\begin{equation}
|D_x|^{-s_0} \int_0^t U(t-\tau)F_j(\tau)d\tau = \int_0^t U(t-\tau)|D_x|^{-s_0}F_j(\tau)d\tau \in C(\mathbb{R}; L^2(\mathbb{R}^n))
\end{equation}

\((j = 1, 2, \ldots)\) and hence we conclude by the estimate (2.11) that this is a Cauchy sequence in \((C \cap L^\infty)(\mathbb{R}; L^2(\mathbb{R}^n))\). Its limit exists in \((C \cap L^\infty)(\mathbb{R}; L^2(\mathbb{R}^n))\), which proves (3.14).

We are in a position to complete the proof of Proposition 3.3. Since \(pq_1' = q_0\) and \(-\alpha_1/p = \alpha_0\) (see (3.8)), the property (3.11) is a direct consequence of (2.11),
In view of the basic inequality $|f(v_1) - f(v_2)| \leq C(|v_1| + |v_2|)^{p-1}|v_1 - v_2|$, the estimate (3.12) follows in the same way. The proof of Proposition 3.3 has been finished. \hfill \Box

We prove Theorem 1.1. For $M > 0$ let us define

$$X_M := \{ v \in X | \sup_{t \in \mathbb{R}} \| D_x |^{-s_0} v(t, \cdot) \|_{L^2(\mathbb{R}^n)} + \| |x|^{-\alpha_0} v \|_{L^{q_0}(\mathbb{R} \times \mathbb{R}^n)} \leq M \}.$$ 

Endowed with the metric

$$d(v, w) := \sup_{t \in \mathbb{R}} \| D_x |^{-s_0} (v(t, \cdot) - w(t, \cdot)) \|_{L^2(\mathbb{R}^n)} + \| |x|^{-\alpha_0} (v - w) \|_{L^{q_0}(\mathbb{R} \times \mathbb{R}^n)},$$

the set $X_M$ is a complete metric space. Defining the operator

$$N(v) := U(t) \varphi - i \int_{0}^{t} U(t - \tau)f(v(\tau))d\tau \quad (v \in X),$$

we find by (3.10)–(3.12) that there exists a constant $\delta > 0$ depending on $n$, $p$, and $\lambda$ such that if $\| D_x |^{-s_0} \varphi \|_{L^2(\mathbb{R}^n)} \leq \delta$, then the operator $N$ has a unique fixed point in $X_{C\delta}$. Here $C$ is a suitable positive constant, and the fixed point is a solution to the integral equation (3.9) which is unique in $X_{C\delta}$. We have finished the proof of Theorem 1.1.

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