Next-to-Next-to-Leading Logarithmic Threshold Resummation for Deep-Inelastic Scattering and the Drell-Yan Process

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Abstract

The soft-gluon resummation exponents $G^N$ in moment space are investigated for the quark coefficient functions in deep-inelastic structure functions and the quark-antiquark contribution to the Drell-Yan cross section $d\sigma/dM$. Employing results from two- and three-loop calculations we obtain the next-to-next-to-leading logarithmic terms $\alpha_s(\alpha_s \ln N)^n$ of $G^N$ to all orders in the strong coupling constant $\alpha_s$. These new contributions facilitate a reliable assessment of the numerical effect and the stability of the large-$N$ expansion.
Deep-inelastic lepton-hadron scattering (DIS) and Drell-Yan (DY) lepton pair production in hadronic collisions are among the processes best suited for probing the short-distance structure of hadrons. The subprocess cross sections (coefficient functions) for these processes in perturbative QCD receive large logarithmic corrections, originating from soft-gluon radiation, at large values of the scaling variables $x$ corresponding to large Mellin moments $N$. These corrections have been resummed [1, 2] to all orders in the strong coupling constant $\alpha_s$ up to next-to-leading logarithmic accuracy (technically defined in eq. (10) below). As the next-to-leading contributions are often hardly suppressed against the leading terms, it is important to extend the resummation to the next-to-next-to-leading logarithms. In this paper we present corresponding results for the DIS structure functions $F_{1,2,3}(x, Q^2)$ (where $Q^2$ represents the resolution scale) and for the Drell-Yan cross section $d\sigma/dQ^2$ (where $Q$ stands for the invariant mass $M$ of the lepton pair).

The soft-gluon resummation of the DIS ($P \equiv 1$) and DY ($P \equiv 2$) quark coefficient functions $C^N_P(Q^2)$ in $N$-space is given by [1, 2]

$$C^N_P(Q^2) = g_{P,0}(Q^2) \cdot \exp[G^N_P(Q^2)] + O(N^{-1}\ln^N N).$$

The functions $C^N$, $g_0$ and $G^N$ also depend on the factorization scale $\mu_f$ and the renormalization scale $\mu_r$, a dependence which we will often suppress for brevity. For the quantities under consideration the contributions $g_0$ collecting the $N$-independent terms are known up to second order in $\alpha_s$ from the two-loop calculations performed in refs. [3]. In the standard MS renormalization and factorization scheme employed throughout this paper the exponents $G^N_P$ in eq. (1) can be written as

$$G^N_1(Q^2) = \ln \Delta_q(Q^2, \mu_f^2) + \ln J_q(Q^2) + \ln \Delta^\text{int}_1(Q^2),$$

$$G^N_2(Q^2) = 2 \ln \Delta_q(Q^2, \mu_f^2) + \ln \Delta^\text{int}_2(Q^2).$$

Closely following the notations of ref. [1], the components entering eq. (2) are

$$\ln \Delta_q(Q^2, \mu_f^2) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \int_{\mu_f^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A(a_s(q^2))$$

collecting of effects of soft-gluon radiation collinear to initial-state partons,

$$\ln J_q(Q^2) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \left[ \int_{(1-z)Q^2}^{(1-z)^2Q^2} \frac{dq^2}{q^2} A(a_s(q^2)) + B(a_s([1-z]Q^2)) \right]$$

taking into account collinear final-state radiation, and the process-dependent piece

$$\ln \Delta^\text{int}_P(Q^2) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} D_P(a_s([1-z]Q^2))$$

attributed to large-angle soft-gluon emissions. The integrands in eqs. (3)–(5) are given by

$$F(a_s) = \sum_{l=1}^{\infty} F_l a_s^l, \quad F = A, B, D_P,$$
where we normalize the expansion parameter as \( a_s = \alpha_s/(4\pi) \).

The constants \( A_l \) in eq. (6) are the coefficients of the \( 1/[1-x]_l \) terms of \( l \)-loop quark-quark splitting functions \( P_{qq}^{(l-1)}(x) \) — this actually completes the \( \overline{\text{MS}} \) definition of the coefficients \( B_l \) and \( D_{P,l} \) \(^2\) (see also ref. \(^3\)). Thus \( A_1 \) and \( A_2 \) are well-known, reading

\[
A_1 = 4C_F, \quad A_2 = 8C_F \left[ \left( \frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_f \right].
\]  

(7)

Here \( N_f \) denotes the number of effectively massless quark flavours, and the colour factors are \( C_F = 4/3 \) and \( C_A = 3 \) in QCD. The exact expression for \( P_{qq}^{(2)}(x) \) has not been completed yet. However, recently rather accurate approximations have been derived \(^4\) from the available partial results, most notably the lowest integer-\( N_f \) moments \(^7\), \(^8\). Together with the exact \( N_f^2 \) term determined in ref. \(^9\) the results of ref. \(^6\) yield

\[
A_3 = (1178.8 \pm 11.5) - (183.95 \pm 0.85)N_f - \frac{16}{27} C_f N_f^2. \quad (8)
\]

For \( N_f = 3 \ldots 5 \) the (independent) errors in eq. (8) can be combined to an overall uncertainty of \( \pm 12 \). For \( \alpha_s < 0.3 \) this uncertainty amounts to less than 0.1\% of the total three-loop value of \( A(a_s) \). Finally the constants \( B_1 \) and \( D_{P,1} \) in eq. (6) are given by \(^2\)

\[
B_1 = -P_{q,\delta}^{(0)} = -3C_F, \quad D_{P,1} = 0, \quad (9)
\]

where \( P_{q,\delta}^{(0)} \) is the coefficient of \( \delta(1-x) \) in the one-loop quark-quark splitting function.

The second-order terms \( B_2 \) and \( D_{P,2} \) will be discussed below.

After the integrations in eqs. (3)–(5) are performed, the functions \( G_P^N(Q^2) \) in eq. (2) take the form

\[
G_P^N(Q^2) = L g_1(\lambda) + g_2(\lambda) + a_s g_3(\lambda) + \ldots \quad (10)
\]

with \( L = \ln N, \lambda = \beta_0 a_s L \) and

\[
g_i(\lambda) = \sum_{k=1}^{\infty} g_{ik}(a_s L)^k. \quad (11)
\]

The first term in eq. (11), which depend on \( A_1 \) only (see eq. (10)) below), collects the leading logarithmic (LL) large-\( N \) contributions \( L(a_s L)^n \). The coefficients \( A_2 \) and \( B_1 \) determine the functions \( g_2 \) resumming the next-to-leading logarithmic (NLL) terms \( (a_s L)^n \). These functions have been determined in refs. \(^7\), \(^8\). The next-to-next-to-leading logarithmic (NNLL) approximation includes \( g_3 \) which is correspondingly fixed by \( A_3, B_2 \) and \( D_{P,2} \).

In order to calculate \( g_3 \) it is convenient to introduce

\[
X = 1 + a_s(\mu_r^2/\mu^2) \beta_0 \ln \frac{q^2}{\mu_r^2}. \quad (12)
\]

\(^1\)The error estimate (8) has been derived in ref. \(^6\) under an assumption (no \( \ln^3(1-x) \) terms) on the form of \( P_{qq}^{(2)}(x) \). Abandoning this constraint does not lead to a significant modification of eq. (8).
and to write the next-to-next-to-leading order coupling constant as

\[ a_s(q^2) = a_s(\mu_r^2) \frac{1}{X} - a_s^2(\mu_r^2) \frac{\beta_1}{\beta_0} \ln X \]

\[ + a_s^3(\mu_r^2) \left( \frac{\beta^2_2}{\beta^2_0} \ln^2 X - \ln X - 1 + \frac{\beta_2}{\beta_0} X \right) \]

\[ + O\left( \left( a_s(\mu_r^2) \ln(q^2/\mu_r^2) \right)^n \right) . \] (13)

Here \( \beta_0, \beta_1 \) and \( \beta_2 \) are the coefficients of the \( \beta \)-function of QCD up to three loops \[10\]. After inserting eq. (13) into eqs. (3) and (4), the inner integrations over \( A(\mu_r) \) can be readily carried out. A straightforward method to perform the \( z \)-integral is to write the integrand as an infinite series in \( \ln(1-x) \), to use

\[ \int_0^1 \frac{dz \ z^{N-1} - 1}{1-z} \ln^k(1-x) = \frac{(-1)^{k+1}}{k+1} \left\{ S_1^{k+1}(N) + \frac{1}{2} k(k+1) S_2^{k+1}(N) S_2(N) \right\} \]

\[ + O(S_1^{k-2}) \] (14)

(cf. ref. [11]) with \( S_1(N) = \sum_{j=1}^N 1/j^0 \) and

\[ S_1(N) = \ln N + \gamma_e + O(1/N) , \quad S_2(N) = \zeta_2 + O(1/N) , \] (15)

and to re-assemble the expansions in the end. In this way we arrive at

\[ g_1^{\text{DIS}}(\lambda) = \frac{A_1}{\beta_0} \left[ \lambda + (1-\lambda) \ln(1-\lambda) \right] \] (16)

\[ g_2^{\text{DIS}}(\lambda) = -\frac{A_1 \gamma_e - B_1}{\beta_0} \ln(1-\lambda) + \frac{A_1 \beta_1}{\beta_0^2} \left[ \lambda + \ln(1-\lambda) + \frac{1}{2} \ln^2(1-\lambda) \right] \]

\[ - \frac{A_2}{\beta_0^2} \left[ \lambda + \ln(1-\lambda) \right] + \ln \left( \frac{Q^2}{\mu_r^2} \right) \frac{A_1}{\beta_0} \ln(1-\lambda) + \ln \left( \frac{\mu_r^2}{\mu_r^2} \right) \frac{A_1}{\beta_0} \lambda \] (17)

and our new result

\[ g_3^{\text{DIS}}(\lambda) = + A_1 \left\{ \frac{1}{2} (\gamma_e + \zeta_2) \frac{\lambda}{1-\lambda} + \frac{\beta^2_2}{\beta_0^2} \frac{1}{1-\lambda} \left[ \frac{1}{2} \ln^2(1-\lambda) + \lambda \ln(1-\lambda) + \frac{1}{2} \lambda^2 \right] \right. \]

\[ - \frac{\beta_1 \gamma_e}{\beta_0^2} \frac{1}{1-\lambda} \left[ \lambda + \ln(1-\lambda) \right] + \frac{\beta_2^2}{\beta_0^2} \left[ \frac{1}{2} \ln^2(1-\lambda) + \ln(1-\lambda) + \lambda \right] \left\} \right. \]

\[ + A_2 \left\{ \frac{\gamma_e}{\beta_0} \frac{\lambda}{1-\lambda} - \frac{\beta_1}{\beta_0^2} \frac{1}{1-\lambda} \left[ \ln(1-\lambda) + \lambda + \frac{1}{2} \lambda^2 \right] \right\} \frac{A_3}{2 \beta_0} \frac{\lambda^2}{1-\lambda} \]

\[ - B_1 \left\{ \frac{\gamma_e}{\beta_0} \frac{\lambda}{1-\lambda} - \frac{\beta_1}{\beta_0^2} \frac{1}{1-\lambda} \left[ \ln(1-\lambda) + \lambda + \frac{1}{2} \lambda^2 \right] \right\} \frac{B_2}{\beta_0} \frac{\lambda}{1-\lambda} - \frac{D_{1,2}}{\beta_0} \frac{\lambda}{1-2\lambda} \]

\[ + \ln \left( \frac{Q^2}{\mu_r^2} \right) \left\{ \frac{A_1 \beta_1}{\beta_0^2} \frac{1}{1-\lambda} \left[ \lambda + \ln(1-\lambda) \right] + \left( B_1 - A_1 \gamma_e - \frac{A_2}{\beta_0} \right) \frac{\lambda}{1-\lambda} \right\} \]

\[ + \ln^2 \left( \frac{Q^2}{\mu_r^2} \right) \frac{A_1}{2} \frac{\lambda}{1-\lambda} + \ln \left( \frac{\mu_r^2}{\mu_r^2} \right) \frac{A_2}{\beta_0} \lambda - \ln^2 \left( \frac{\mu_r^2}{\mu_r^2} \right) \frac{A_1}{2} \lambda . \] (18)
The functions \( g_{2,3}^{DY} \) are obtained from eqs. (17) and (18) by substituting \( \lambda \to 2\lambda, \gamma_e \to 2\gamma_e \) and \( \zeta_2 \to 4\zeta_2 \) in the terms with \( A_i \), removing the terms with \( B_l \) (recall eq. (2)), and replacing \( D_{1,1} \) by \( D_{2,1} \). The result corresponding to eq. (14) reads \( g_{1}^{DY}(\lambda) = 2 g_{1}^{DIS}(2\lambda) \). The generalization of eqs. (16)–(18) to other processes involving eqs. (3)–(5) is obvious.

Now we are ready to address the second-order coefficients \( B_2 \) and \( D_{P,2} \). The first-order expansion coefficients \( g_{31}^{DIS} \) and \( g_{31}^{DY} \), as defined in eq. (11), are given by

\[
g_{31}^{DY} = 1/2 P^3 A_1(\gamma_e^2 + \zeta_2)\bar{\beta}_0 + P^2 A_2\gamma_e - \delta P_1(B_2 - B_1\gamma_\beta_0) - D_{P,2}
\]

with \( \delta_{kj} = 1 \) for \( k = j \) and \( \delta_{kj} = 0 \) else. On the other hand \( g_{31}^{(P)} \) can be determined by expanding eqs. (1) and (10) to order \( \alpha_s^2 \) and comparing to the two-loop results of refs. [3, 12], as done for the DIS case in ref. [13]. The result for the Drell-Yan coefficient function reads

\[
g_{31}^{DY} = C_F C_A \left( \frac{1616}{27} - 56\zeta_3 - 32\zeta_2\gamma_e + \frac{176}{3}\gamma_e^2 + \frac{1072}{9}\gamma_e \right) - C_F N_f \left( \frac{224}{27} + \frac{32}{3}\gamma_e^2 + \frac{160}{9}\gamma_e \right),
\]

yielding

\[
D_{2}^{DY} = C_F C_A \left( -\frac{1616}{27} + 56\zeta_3 + \frac{176}{3}\zeta_2 \right) + C_F N_f \left( \frac{224}{27} - \frac{32}{3}\zeta_2 \right).
\]

This term has already been derived in ref. [4], albeit without explicitly attributing it to the \( \alpha_s((1 - z)^2Q^2) \) contribution. Note that, as it has to be for a \( \Delta^{\text{int}} \) contribution [1], eq. (21) does not contain an Abelian (\( C_F^2 \)) term. We consider this as a first concrete check of the correctness of the setup (3)-(5) at the NNLL level. Using eq. (12) of ref. [13], the corresponding constraint for the DIS case reads

\[
B_2 + D_2^{DIS} = C_F \left( -\frac{3}{2} - 24\zeta_3 + 12\zeta_2 \right) + C_F C_A \left( -\frac{3155}{54} + 40\zeta_3 + \frac{44}{3}\zeta_2 \right) + C_F N_f \left( \frac{247}{27} - \frac{8}{3}\zeta_2 \right) = -P_{q,\delta}^{(1)} + \frac{1}{2} D_2^{DY} - 7\beta_0 C_F,
\]

where \( P_{q,\delta}^{(1)} \) is the coefficient of \( \delta(1 - x) \) in the two-loop quark-quark splitting function. Unlike the DY case (21) two new constants can occur at the NNLL level in DIS, hence the consistency of the resummed and the two-loop coefficient functions does not completely specify \( g_{32}^{DIS} \). For that either an extension of the calculations of refs. [1, 2] to the next order, or the \( \ln^2 \) \( N \) term of the three-loop coefficient function (fixing \( g_{32} \) which involves the combination \( B_2 + 2D_{1,2} \)) is required. An approximate result has been derived for the latter [15] from the constraints of refs. [7, 8, 13]. Within errors that result is consistent with \( D_2^{DIS} = 0 \), but, for instance, also with \( B_2 = -P_{q,\delta}^{(1)} + \xi \beta_0 C_F \) for \( \xi \simeq 8 \ldots 13 \).

Note that for \( g_3 \) the convention [14] differs from that in ref. [13]: there the second index \( i \) in \( g_{3i} \) refers to the overall power of \( \alpha_s \) in the expansion \( (10) \) of \( G^N \). Hence \( g_{31} \) is called \( g_{32} \) in ref. [13] etc.
The integrals in eqs. (3)–(8) are not well-defined, as they involve the running coupling at arbitrarily low scales. This feature leads to the poles at \( (2) \alpha_s \beta_0 \ln N = 1 \) in eqs. (10)–(15) and their DY counterparts, a problem which is usually dealt with by the ‘minimal-prescription’ contour for the Mellin inversion \([16]\). Another option, followed in ref. \([13]\) for the DIS case, is to re-expand eq. (1) in powers of the prescription contour for the Mellin inversion \([16]\). A\nother option, followed in ref. \([13]\) for the DIS case, is to re-expand eq. (1) in powers of \( \alpha_s \) and to keep only those terms \( \alpha_s^n \ln^{2n+1-k} N, k = 1, \ldots l \) (i.e., the first \( l \) ‘towers’ of logarithms) which are completely fixed by the known terms in eq. (1) and in \( g_0 \) of eq. (3). In connection with a two-loop result for \( g_0 \) the NLL and NNLL resummations lead to \( l = 4 \) and \( l = 5 \), respectively. General expressions for the first four towers can be found in eq. (14) of ref. \([13]\).\n
An improvement on the tower expansion for DIS in ref. \([13]\) will only be possible once \( B_2^{\text{DIS}} \) in eq. (22) is exactly determined. However, the present results for \( g_3^{\text{DIS}} \) are directly relevant to the ‘physical’ evolution kernels \( K^N \) for the scaling violations of the non-singlet structure functions investigated beyond order \( \alpha_s^2 \) in refs. \([13, 17]\). Identifying all scales for

Note that the third term of eq. (15) has been misprinted in the original preprint as well as the journal version of ref. \([13]\). \( 16/27 C_F N_f \) has to be replaced by \( 16/27 C_F N_f^2 \), and \( 70/9 C_F N_f^2 \) by \( 280/9 C_F^2 N_f \).
brevity these kernels can be written as

\[
\frac{dF^{N}_{a,NS}}{d\ln Q^2} = \left( P^{N}(a_s) + \frac{d\ln C^{N}_{a}(a_s)}{da_s} \beta(a_s) \right) F^{N}_{a,NS}(Q^2) \equiv K^{N}(a_s) F^{N}_{a,NS}(Q^2),
\]

where \( P^{N}(a_s) \) and \( C^{N}_{a}(a_s) \) \((a = 1, 2, 3)\) are the moments of the non-singlet splitting functions and coefficient functions, respectively. At large-\( x \)/large-\( N \) eq. (24) holds for the full structure functions up to corrections of order \( 1/N \). Inserting eqs. (11) and (10) into this equation one arrives at the NNLL resummation of this kernel,

\[
K^{N}_{a,\text{res}}(a_s) = - \ln N (A_1 a_s + A_2 a_s^2 + A_3 a_s^3) - \left( 1 + \frac{\beta_1}{\beta_0} a_s + \frac{\beta_2}{\beta_0} a_s^2 \right) \lambda^2 \frac{dg_1}{d\lambda} - \left( a_s \beta_0 + a_s^2 \beta_1 \right) \lambda \frac{dg_2}{d\lambda} - a_s^2 \beta_0 \frac{dg_3}{d\lambda} (\lambda g_3(\lambda)) + \mathcal{O}(a_s^3 (a_s \ln N)^n)
\]

with \( \lambda = a_s \beta_0 \ln N, A_i \) of eqs. (7) and (8), and \( g_i \) from eqs. (16)–(18) with \( \mu_f = \mu_r = Q \). Note that the leading large-\( N \) term of the quark-quark splitting function is given by \( \ln N \) at all orders in \( a_s \) [18].

Finally we briefly illustrate the numerical impact of the NNLL corrections. For these illustrations we choose \( \mu_r^2 = \mu_f^2 = Q^2, N_f = 4 \) and \( \alpha_s = 0.2 \). Depending on the precise value of \( \alpha_s(M_Z^2) \), the latter number corresponds to scales between about 25 and 50 GeV \(^2\), a range typical for fixed-target experiments both on DIS and the Drell-Yan process. The corresponding results for the functions \( G^{N}_{\text{DIS}} \) and \( G^{N}_{\text{DY}} \) are presented in fig. 1. Their convolutions with a schematic, but typical input are shown in fig. 2, where the minimal-prescription Mellin inversion [16] has been employed. The remaining ambiguity in the DIS case (discussed below eq. (22)) is illustrated by the results for \( D_2^{\text{DIS}} = 0 \) and for \( B_2 = -P^{(1)}_{a,\delta} + 13 \beta_0 C_F \) (denoted by \( \xi = 13 \) in the figures), the latter curves indicating the maximal NNLL effects consistent with the available three-loop information [4, 8, 13, 13]. It is obvious from both figures that knowledge of the \( \ln N (\alpha_s \ln N)^n \) and \( (\alpha_s \ln N)^n \) terms [4, 8] alone is not sufficient for reliably determining the functions \( G^{N}_{P} \) and their impact after convolution even for rather moderate values of \( N \) and \( x \) (often denoted \( \tau \) in the DY case). On the other hand the NNLL corrections presented in this paper are rather small over a wide range, e.g., less than 10% and 20% at \( x \leq 0.85 \) and \( \sqrt{x} \leq 0.75 \) for the DIS and DY results of fig. 2, respectively. This stabilization indicates that the soft-gluon exponents \( G^{N}_{P} \) and their effects can now be reliably estimated for these processes.

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**Fig. 1:** The LL, NLL and NNLL approximations for the resummation exponents $G^N(Q^2)$ in eq. (10) at $\mu_r^2 = \mu_f^2 = Q^2$ for $\alpha_s(Q^2) = 0.2$ and four flavours. The two NNLL curves in the DIS case indicate the present uncertainty as discussed below eq. (22).

**Fig. 2:** The convolutions of the results shown in Fig. 1 with a typical input shape.