On Vertices and Facets of Combinatorial 2-Level Polytopes

Manuel Aprile¹, Alfonso Cevallos¹, and Yuri Faenza²

¹ DISOPT - École Polytechnique Fédérale de Lausanne, Switzerland
{manuel.aprile,alfonso.cevallosmanzano}@epfl.ch

² IEOR, Columbia University, USA
{yf2414}@columbia.edu

Abstract. 2-level polytopes naturally appear in several areas of pure and applied mathematics, including combinatorial optimization, polyhedral combinatorics, communication complexity, and statistics. In this paper, we present a polyhedral study of 2-level polytopes arising in combinatorial settings. For all the known (to the best of our knowledge) such polytopes $P$ we show that $f_0(P)f_{d-1}(P) \leq d2^{d+1}$. Here $f_0(P)$ (resp. $f_{d-1}(P)$) is the number of vertices (resp. facets) of $P$, and $d$ is its dimension. Whether this holds for all 2-level polytopes was asked in [3], where experimental results showed it true for $d \leq 6$. The key to most of our proofs is an understanding of the combinatorial structures underlying those polytopes. This leads to a number of results that we believe to be of independent interest: a trade-off formula for the number of cliques and stable sets in a graph; a description of the facets of the base polytope of the 2-sum of matroids; a linear-size description of the base polytope of matroids that are 2-level in terms of cuts of an associated tree. We also give a self-contained proof of the characterization of the last class, a result that originally appeared in [13].

Keywords: 2-level Polytopes; Matroids; Polyhedral Combinatorics.

1 Introduction

Let $P \subseteq \mathbb{R}^d$ be a polytope. We say that $P$ is 2-level if, for each facet $F$ of $P$, all the vertices of $P$ that are not vertices of $F$ lie in the same translate of the affine hull of $F$. Equivalently, $P$ is 2-level if and only if it has theta-rank 1 [11], or all its pulling triangulations are unimodular [27], or it has a slack matrix with entries in $\{0,1\}$ [3]. Those last three definitions appeared in papers from the semidefinite programming, statistics, and polyhedral combinatorics communities respectively, showing that 2-level polytopes naturally arise in many areas of mathematics.

2-level polytopes generalize Birkhoff polytopes [30], Hanner polytopes [15], Hansen polytopes [16], order polytopes [29], spanning tree polytopes of series-parallel graphs [13], some min up/down polytopes [18], and stable set polytopes of perfect graphs [6]. A fundamental result in polyhedral combinatorics shows that the linear extension complexity of polytopes from the last class is subexponential in the dimension [28]. Whether this upper bound can be pushed down to polynomial is a major open problem. For a generic 2-level polytope, the situation is even worse: no non-trivial bound is known for its linear extension complexity.
On the other hand, 2-level polytopes admit a “smallest possible” semidefinite extension, i.e. of size $d + 1$, with $d$ begin the dimension of the polytope [11] Hence, they are prominent candidates for showing the existence of a strong separation between the expressing power of exact semidefinite and linear extensions of polytopes. Interest in 2-level polytopes is also motivated by their connection to the prominent log-rank conjecture in communication complexity [19]. If this was true, then 2-level polytopes would admit a linear extension of subexponential size. We defer details on this to the end of the current section.

Fig. 1: The first three polytopes (the simplex, the cross-polytope and the cube) are 2-level. The fourth one is not 2-level, because of the highlighted facet.

Because of their relevance, a solid understanding of 2-level polytopes would be desirable. Unfortunately, despite an increasing number of recent studies [3,11,12,13], such an understanding is yet to be obtained. We do not have e.g. any decent bound on the number of $d$-dimensional 2-level polytopes, nor do we have a structural theory of their slack matrices, of the kind that has been developed for totally unimodular matrices (see e.g. [23]).

Still, those works have suggested promising directions of research, either by characterizing specific classes of 2-level polytopes, or by proving general properties. For instance, in [19], by building on Seymour’s decomposition theorem for 3-connected matroids and on the description of uniform matroids in terms of forbidden minors, a characterization of 2-level polytopes that are base polytopes of matroids is given. On the other hand, it is shown in [11] that each $d$-dimensional 2-level polytope is affinely isomorphic to a 0/1 polytope, hence it has at most $2^d$ vertices. Interestingly, the authors of [11] also showed that a $d$-dimensional 2-level polytope also has at most $2^d$ facets. This makes 2-level polytopes quite different from “random” 0/1 polytopes, that have $(d / \log d)^{\Theta(d)}$ facets [2]. Experimental results from [30] suggest that this separation could be even stronger: up to $d = 7$, the product of the number of facets $f_{d-1}(P)$ and the number of vertices $f_0(P)$ of a $d$-dimensional 2-level polytope $P$ does not exceed $d2^{d+1}$. In [3], it is asked whether this always hold. Here, we turn their question into a conjecture.

Although 2-level polytopes are not the only polytopes with this property, they are exactly the class of polytopes for which such an extension can be obtained via a certain hierarchy. See [11] for details.
Conjecture 1 (Vertex/facet trade-off). Let $P$ be a $d$-dimensional 2-level polytope. Then

$$f_0(P)f_{d-1}(P) \leq d^{2d+1}.$$ 

Moreover, equality is achieved if and only if $P$ is affinely isomorphic to the cross-polytope or the cube.

It is immediate to check that the cube and the cross-polytope (its polar) indeed verify $f_0(P)f_{d-1}(P) = d^{2d+1}$. Conjecture 1 has an interesting interpretation as an upper bound on the “size” of slack matrices of 2-level polytopes, since $f_0(P)$ (resp. $f_{d-1}(P)$) is the number of columns (resp. rows) of the (smallest) slack matrix of $P$. Many fundamental results on linear extensions of polytopes are based on properties of their slack matrices. We believe that advancements on Conjecture 1 may lead to precious insights on the structure of (the slack matrices of) 2-level polytopes, similarly to how progresses on e.g. the outstanding Hirsch [22] and $3^d$ conjectures for centrally symmetric polytopes [17] shed some light on our general understanding of polytopes.

Our contributions. The goal of this paper is to present a polyhedral study of 2-level polytopes arising from combinatorial settings. Our main results are the following:

- We give some evidence supporting Conjecture 1 by proving it for all known classes (to the best of our knowledge) of 2-level polytopes with an underlying combinatorial structure. These include all those cited in the paper so far, plus all matroid cycle polytopes that are 2-level. We moreover show examples of 0/1 polytopes with a simple structure (including spanning tree and forest polytopes) that are not 2-level and do not satisfy Conjecture 1. This suggests that, even though there are clearly polytopes that are not 2-level and satisfy Conjecture 1, 2-levelness seem to be the “correct” hypothesis to prove a general positive result.
- We prove a number of properties of combinatorial objects associated to 2-level polytopes, that we believe to be of interest. These results include: a trade-off formula for the number of stable sets and cliques in a graph; a non-redundant characterization of facet-defining inequalities for base polytopes of matroids under the 2-sum operation; and a non-redundant linear description of 2-level base polytopes of matroids in terms of cuts of some trees associated to those matroids. In particular, we show that the number of facet-defining inequalities for these last polytopes grows linearly with the dimension. At a more philosophical level, these examples suggest that being 2-level is a very attractive feature for a (combinatorial) polytope, since it seems to imply a well-behaved underlying structure.
- We give an alternative proof of the characterization of 2-level polytopes that are base polytopes of matroids. Unlike the one in [13], our proof is self-contained – except for some basic results on matroids – and in particular, it does not rely on any known decomposition theorem or forbidden minor characterization of matroids.

Organization of the paper. We introduce some basic definitions and techniques in Section 2; those are enough to show that Conjecture 1 holds for Birkhoff and Hanner polytopes. In Section 3 we first prove an upper bound on the product of the number of stable sets and cliques of a graph (see Theorem 6). We then reduce the proof of Conjecture 1 for stable set
polytopes of perfect graphs, order polytopes, Hansen polytopes, and min up/down polytopes to statements on stable sets and cliques of associated graphs, which are also proved in Section 3. Hence, we call all those graphical 2-level polytopes. In Section 4, we give a non-redundant description of facets of the base polytope of the 2-sum of matroids in terms of the facets of the base polytopes of the original matroids (see Theorem 24). This is used to obtain a compact description of 2-level base polytopes of matroids (see Theorem 30) and a proof of Conjecture for this class (see Theorem 33). Once those results are obtained, one can easily characterize base polytopes of matroids that are 2-level (see Section 3.3). In Section 5, we prove the conjecture for the cycle polytopes of certain binary matroids, which generalizes all cut polytopes that are 2-level. In Section 6, we give examples showing that Conjecture does not trivially hold for all “well-behaved” 0/1 polytopes.

Related work. We already mentioned the paper [3] that provides an algorithm based on the enumeration of closed sets to list all 2-level polytopes, as well as papers [11,13,27] where equivalent definitions and/or families of 2-level polytopes are given. In [9], the algorithm from [3] is extended and the geometry of 2-level polytopes with some prescribed facet is studied. Among other results, in [11] it is shown that the stable set polytope of a graph $G$ is 2-level if and only if $G$ is perfect. A characterization of all base polytopes of matroids that are 2-level is given in [13], building on the decomposition theorem for matroids that are not 3-connected (see e.g. [21]). A similar characterization of 2-level cycle polytopes of matroids (which generalize cut polytopes) is given in [10].

As already pointed out, 2-level polytopes play an important role in the theory of linear and semidefinite extensions. The (linear) extension complexity $xc(P)$ of a polytope $P$ has recently imposed itself as an important measure of the complexity of $P$. It is the minimum number of inequalities in a linear description of an extended formulation for $P$. In [28] it is shown that the stable set polytope of a perfect graph with $d$ nodes has extension complexity $dO(\log d)$. Whether this bound can be improved or extended to all 2-level polytopes is unknown. On the other hand, the semidefinite extension complexity of a $d$-dimensional 2-level polytope is $d + 1$ [11]. 2-level polytopes are therefore a good candidate to show an exponential separation between the power of exact semidefinite and linear formulations for 0/1 polytopes. The log-rank conjecture aims at understanding the amount of information that needs to be exchanged between two parties in order to compute an input 0/1 matrix. Exact definitions are not interesting for the scope of this paper, and we refer the interested reader to e.g. [20]. Here it is enough to note that, if true, this conjecture would imply that the extension complexity of a $d$-dimensional 2-level polytope is at most $2^{poly \log(d)}$, hence subexponential.

2 Basics

We let $\mathbb{R}_+$ be the set of non-negative real numbers. For a set $S$ and an element $e$, we denote by $A + e$ and $A - e$ the sets $A \cup \{e\}$ and $A \setminus \{e\}$, respectively. For a point $x \in \mathbb{R}^I$, where $I$ is an index set, and a subset $J \subseteq I$, we let $x(J) = \sum_{i \in J} x_i$.

For basic definitions about polytopes and graphs, we refer the reader to [20] and [7], respectively. The polar of a polytope $P \subseteq \mathbb{R}^d$ is the polyhedron $P^\circ = \{y \in \mathbb{R}^d : y \cdot x \leq 1\}$. It is well known that, if $P \subseteq \mathbb{R}^d$ is a $d$-dimensional polytope with the origin in its interior,

\[ It \ immediately \ follows \ from \ e.g. \ [30] \ Theorem \ 2.11. \]
then so is $P^\Delta$, and one can define a one-to-one mapping between vertices (resp. facets) of $P$ and facets (resp. vertices) of $P^\Delta$. Thus, a polytope as above and its polar will simultaneously satisfy or not satisfy Conjecture \[1\] The $d$-dimensional cube is $[-1, 1]^d$, and the $d$-dimensional cross-polytope is its polar. A $0/1$ polytope is the convex hull of a subset of the vertices of $\{0, 1\}^d$. The following facts will be used many times:

**Lemma 2.** \[7\] Let $P$ be a 2-level polytope of dimension $d$. Then

1. $f_0(P), f_{d-1}(P) \leq 2^d$.
2. Any face of $P$ is again a 2-level polytope.

One of the most common operation with polytopes is the Cartesian product. Given two polytopes $P_1 \subseteq \mathbb{R}^{d_1}$, $P_2 \subseteq \mathbb{R}^{d_2}$, their Cartesian product is $P_1 \times P_2 = \{(x, y) \in \mathbb{R}^{d_1+d_2} : x \in P_1, y \in P_2\}$. This operation will be useful to us as it preserves 2-levelness and the bound of Conjecture \[7\]

**Lemma 3.** Two polytopes $P_1, P_2$ are 2-level if and only if their Cartesian product $P_1 \times P_2$ is 2-level. Moreover, if two 2-level polytopes $P_1$ and $P_2$ satisfy Conjecture \[7\], then so does $P_1 \times P_2$.

**Proof.** The first part follows immediately from the fact that $P_1 = \{x : A^{(1)}x \leq b^{(1)}\}$, $P_2 = \{y : A^{(2)}y \leq b^{(2)}\}$, then $P_1 \times P_2 = \{(x, y) : A^{(1)}x \leq b^{(1)}; A^{(2)}y \leq b^{(2)}\}$, and that the vertices of $P_1 \times P_2$ are exactly the points $(x, y)$ such that $x$ is a vertex of $P_1$ and $y$ a vertex of $P_2$.

For the second part, let $P = P_1 \times P_2$, $d_1 = d(P_1)$, $d_2 = d(P_2)$. Then it is well known that $d(P) = d_1 + d_2$, $f_0(P) = f_0(P_1)f_0(P_2)$, and $f_{d-1}(P) = f_{d_1-1}(P_1) + f_{d_2-1}(P_2)$. We conclude

$$f_0(P)f_{d-1}(P) = f_0(P_1)f_{d_1-1}(P_1)f_0(P_2) + f_0(P_2)f_{d_2-1}(P_2)f_0(P_1)$$

$$\leq d_12^{d_1+d_2+1} + d_22^{d_1+d_2+1} = d(P)2^{d(P)+1},$$

where the inequality follows by induction and from Lemma \[2\] Suppose now that $P$ satisfies the bound with equality. Then, for $i = 1, 2$, $P_i$ also satisfies the bound with equality and $f_0(P_i) = 2^{d(P_i)}$, which means that $P_i$ is a $d_i$-dimensional cube. Then $P$ is a $d$-dimensional cube. \qed

### 2.1 Hanner and Birkhoff polytopes

We start off with two easy examples. Hanner polytopes \[15\] are defined as the smallest family that contains the $[-1, 1]$ segment of dimension 1, and is closed under taking polars and Cartesian products.

**Lemma 4.** Hanner polytopes satisfy Conjecture \[7\].

**Proof.** $[-1, 1] \subseteq \mathbb{R}^1$ satisfies the conjecture. Successively taking polars and Cartesian products generates full-dimensional polytopes containing the origin in their interior. The thesis then follows from Lemma \[3\] and from the discussion on polars earlier in Section \[2\] \qed
For an integer \( n \geq 2 \), a permutation matrix is an \( n \times n \) 0/1 matrix with exactly one 1 per row and column. If we identify any such matrix with a 0/1 vector in \( \mathbb{R}^{n^2} \), then we can define the Birkhoff polytope \( B_n \subset \mathbb{R}^{n^2} \) as the convex hull of all permutation matrices. For \( n = 2 \), the polytope \( B_2 \) is affinely isomorphic to the [-1,1] segment of dimension 1, which is a Hamner polytope and hence verifies Conjecture [1].

For \( n \geq 3 \), \( B_n \) is known [30] to have exactly \( n! \) vertices, \( n^2 \) facets, and dimension \((n-1)^2\), and it is known to be 2-level.\(^5\) It is easy to check that the conjectured inequality is observed, and is loose for all \( n \geq 3 \).

**Lemma 5.** Birkhoff polytopes satisfy Conjecture [1].

### 3 Graphical 2-Level Polytopes

We present a general result on the number of cliques and stable sets of a graph. Proofs of all theorems from the current section will be based on it.

**Theorem 6 (Stable set/clique trade-off).** Let \( G = (V,E) \) be a graph on \( n \) vertices, \( C \) its family of non-empty cliques, and \( S \) its family of non-empty stable sets. Then

\[
|C||S| \leq n(2^n - 1).
\]

Moreover, equality is achieved if and only if \( G \) or its complement is a clique.

**Proof.** Consider the function \( f : C \times S \to 2^V \), where \( f(C,S) = C \cup S \). For a set \( W \subset V \), we bound the size of its pre-image \( f^{-1}(W) \). If \( W \) is a singleton, the only pair in its pre-image is \((W,W)\). For \(|W| \geq 2\), we claim that \(|f^{-1}(W)| \leq 2|W|\).

There are at most \(|W|\) intersecting pairs \((C,S)\) in \( f^{-1}(W) \). This is because the intersection must be a single element, \( C \cap S = \{v\} \), and once it is fixed every element adjacent to \( v \) must be in \( C \), and every other element must be in \( S \).

There are also at most \(|W|\) disjoint pairs in \( f^{-1}(W) \), as we prove now. Fix one such disjoint pair \((C,S)\), and notice that both \( C \) and \( S \) are non-empty proper subsets of \( W \). All other disjoint pairs \((C',S')\) are of the form \( C' = C \setminus A \cup B \) and \( S' = S \setminus B \cup A \), where \( A \subseteq C \), \( B \subseteq S \), and \(|A|,|B| \leq 1\). Let \( X \) (resp. \( Y \)) denote the set formed by the vertices of \( C \) (resp. \( S \)) that are anticomplete to \( S \) (resp. complete to \( C \)). Clearly, either \( X \) or \( Y \) is empty. We settle the case \( Y = \emptyset \), the other being similar. In this case \( \emptyset \neq A \subseteq X \), so \( X \neq \emptyset \).

If \( X = \{v\} \), then \( A = \{v\} \) and we have \(|S| + 1\) choices for \( B \), with \( B = \emptyset \) possible only if \(|C| \geq 2\), because we cannot have \( C' = \emptyset \). This gives at most \( 1 + |S| + |C| - 1 \leq |W| \) disjoint pairs \((C',S')\) in \( f^{-1}(W) \). Otherwise, \(|X| \geq 2 \) forces \( B = \emptyset \), and the number of such pairs is at most \( 1 + |X| \leq 1 + |C| \leq |W| \).

We conclude that \(|f^{-1}(W)| \leq 2|W|\), or one less if \( W \) is a singleton. Thus

\[
|C \times S| \leq \sum_{k=0}^{n} 2k \binom{n}{k} - n = n2^n - n,
\]

\(^5\) To see this, notice that each facet-defining inequality corresponds to the non-negativity constraint of a coordinate, and thus each vertex satisfies the constraint with a slack of either 0 or 1.
where the (known) fact $\sum_{k=0}^{n} 2k\binom{n}{k} = n2^n$ holds since
\[
n2^n = \sum_{k=0}^{n} (k + (n-k)) \binom{n}{k} = \sum_{k=0}^{n} k \binom{n}{k} + (n-k) \binom{n}{n-k} = 2 \sum_{k=0}^{n} k \binom{n}{k}.
\]

The bound is clearly tight for $G = K_n$ and $G = \overline{K_n}$. For any other graph, there is a subset $W$ of 3 vertices that induces 1 or 2 edges. In both cases, $|f^{-1}(W)| = 5 < 2|W|$, hence the bound is loose. □

**Corollary 7.** Let $G$, $\mathcal{C}$ and $\mathcal{S}$ be as in Theorem 6 and $\mathcal{C}' = \mathcal{C} \cup \{\emptyset\}$ and $\mathcal{S}' = \mathcal{S} \cup \{\emptyset\}$ be the families of (possibly empty) cliques and stable sets of $G$, respectively. Then
\[
|\mathcal{C}'||\mathcal{S}'| \leq (n+1)2^n,
\]
and equality is achieved if and only if $G$ or its complement is a clique.

**Proof.** We apply the previous inequality to obtain
\[
|\mathcal{C}'||\mathcal{S}'| = (|\mathcal{C}| + 1)(|\mathcal{S}| + 1) = |\mathcal{C}||\mathcal{S}| + (|\mathcal{C}| + |\mathcal{S}'|)
\leq n(2^n - 1) + (|\mathcal{C} \cup \mathcal{S}'| + |\mathcal{C} \cap \mathcal{S}'|)
\leq n(2^n - 1) + (2^n + n) = (n+1)2^n.
\]
Clearly the inequality is tight whenever $G$ or its complement is a clique, and from Theorem 6 we know that it is loose otherwise. □

### 3.1 Stable set polytopes of perfect graphs

For a graph $G = (V,E)$, its stable set polytope $\text{STAB}(G)$ is the convex hull of the characteristic vectors of all stable sets in $G$. It is known that $\text{STAB}(G)$ is 2-level if and only if $G$ is a perfect graph [11], or equivalently [6] if and only if $\text{STAB}(G) = \{x \in \mathbb{R}^V_+ : x(C) \leq 1 \text{ for all maximal cliques } C \text{ of } G\}$.

**Lemma 8.** Stable set polytopes of perfect graphs satisfy Conjecture [7].

**Proof.** For a perfect graph $G = (V,E)$ on $d$ vertices, the polytope $\text{STAB}(G)$ is $d$-dimensional. If we define $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{S}'$ as in Corollary 7 then the number of vertices in $\text{STAB}(G)$ is at most $|\mathcal{S}'|$. There are at most $d$ non-negativity constraints, and at most $|\mathcal{C}| = |\mathcal{C}'| - 1$ clique constraints, so the number of facets in $\text{STAB}(G)$ is at most $|\mathcal{C}'| + d - 1$. Hence
\[
f_0(\text{STAB}(G))f_{d-1}(\text{STAB}(G)) \leq (|\mathcal{C}'| + d - 1)|\mathcal{S}'|
= |\mathcal{C}'||\mathcal{S}'| + (d - 1)|\mathcal{S}'|
\leq (d + 1)2^d + (d - 1)2^d = d2^{d+1},
\]
where we used Corollary 7 and the trivial inequality $|\mathcal{S}'| \leq 2^d$. We see that the conjectured inequality is satisfied, and is tight only in the trivial cases $d = 1$ or $|\mathcal{S}'| = 2^d$. In the latter case, $G$ has no edges and $\text{STAB}(G)$ is affinely isomorphic to the cube. □
3.2 Order polytopes

Given a poset $P$ on $[d]$, with order relation $<_P$, its order polytope $O(P)$ is:

$$O(P) = \{ x \in [0,1]^d : x_i \leq x_j \forall i <_P j \}.$$  

A subset $I \subseteq [d]$ is called an upset if $i \in I$ and $i <_P j$ imply $j \in I$. In [25], the following characterization of vertices of an order polytope is given.

Lemma 9. The vertices of $O(P)$ are the characteristic vectors of upsets of $P$. In particular, the number of vertices of $O(P)$ is the number of upsets of $P$.

From this result it is clear that $O(P)$ is a 2-level polytope. Indeed, if all vertices of a polytope have 0/1 coordinates and all facet-defining inequalities can be written as $0 \leq c^T x \leq 1$ for integral vectors $c$, then the polytope is 2-level.

Given a poset $P$, we say that $j$ covers $i$ in $P$ if $i <_P j$ and there is no $k$ in $P$ such that $i <_P k <_P j$. We say that $\{i, j\}$ is a covering pair if $j$ covers $i$ or $i$ covers $j$. $P$ can be described by a graph called Hasse Diagram $G_P([d], E)$, with $i, j \in E$ if and only if $\{i, j\}$ is a covering pair. This graphical representation and Theorem [7] are the main ingredients to prove the following.

Lemma 10. Order polytopes satisfy Conjecture [7].

Proof. Let $P$ be a poset on $[d]$ and $O(P)$ the corresponding order polytope. It is easy to see that $O(P)$ is full dimensional, hence it has dimension $d$. Its Hasse Diagram $G_P([d], E)$ is a graph with $d$ vertices and $e = |E|$ edges. We will apply our trade-off bound over this graph to derive the desired inequality.

Recall that a subset $A$ of $P$ is an antichain if no two elements $i, j \in A$ satisfy $i <_P j$ or $i <_P j$. There is a bijection between the upsets and the antichains of a poset: each upset is uniquely determined by the subset of its minimal elements, which is an antichain, and each antichain can be seen as the set of minimal elements of an upset. This shows that the number of upsets of $P$ is equal to the number of antichains. In particular, an antichain does not contain any covering pair, so every antichain is a stable set in $G_P$. Hence for $G_P$, Lemma [9] implies

$$f_0(O(P)) = |\text{upsets}| = |\text{antichains}| \leq |S'|.$$  

Now, as observed in [25], in the definition of the polytope $O(P)$ the inequalities of the form $x_i \leq x_j$ are redundant if $\{i, j\}$ is not a covering pair, and can be ignored. Thus, the facet-defining inequalities reduce to at most $2d + e$. Notice that a direct consequence of the definition of covering pairs is that $G_P$ is triangle-free. Therefore the only cliques in $G_P$ are the edges, the vertices, and the empty set, i.e., $|C'| = e + d + 1$. Hence

$$f_{d-1}(O(P)) \leq 2d + e = |C'| + d - 1.$$  

Hence $f_0(O(P))f_{d-1}(O(P)) \leq |S'|(|C'|+d-1)$. This is the same upper bound we obtained in Lemma [5], hence the proof continues in a similar fashion. The conjectured inequality is therefore satisfied, and is tight only if $d = 1$ or $|S'| = 2^d$. In the latter case, $P$ is the trivial poset, and $O(P)$ is a cube. \qed
3.3 Hansen polytopes

Given a \((d-1)\)-dimensional polytope \(P\), the twisted prism of \(P\) is the \(d\)-dimensional polytope defined as the convex hull of \(\{(x,1) : x \in P\} \text{ and } \{(-x,-1) : x \in P\}\). For a perfect graph \(G\) with \(d-1\) vertices, its Hansen polytope \(\text{Hans}(G)\), represented by \(\text{Hans}(G)\), is defined as the twisted prism of \(\text{STAB}(G)\). Hansen polytopes are 2-level and centrally symmetric, see e.g. [3].

Lemma 11. Hansen polytopes satisfy Conjecture [4]

Proof. Let \(G = (V,E)\) be a perfect graph on \(d-1\) vertices, and let \(C'\) and \(S'\) be as in Corollary 7. Then \(\text{Hans}(G)\) has \(2|S'|\) vertices (from the definition), and \(2|C'|\) facets (see e.g. [16]). Using again Corollary 7, we get

\[
f_0(\text{Hans}(G))f_{d-1}(\text{Hans}(G)) \leq 4|S'||C'| \leq 4d^2d^{-1} = d^{2d+1}.
\]

The inequality is tight only if \(G\) is either a clique or an anti-clique. The Hansen polytopes of these graphs are affinely equivalent to the cross-polytope and cube, respectively. \(\square\)

3.4 Min up/down polytopes

Fix two integers \(0 < l < d\). For a 0/1 vector \(x \in \{0,1\}^d\) and index \(1 \leq i \leq d-1\), we call \(i\) a switch index of \(x\) if \(x_i \neq x_{i+1}\). Vector \(x\) satisfies the min up/down constraint (with parameter \(l\)) if for any two switch indices \(i < j\) of \(x\), we have \(j-i \geq l\). In other words, when \(x\) is seen as a bit-string then it consists of blocks of 0’s and 1’s each of length at least \(\ell\) (except possibly for the first and last blocks). The min up/down polytope \(P_d(l)\) is defined as the convex hull of all 0/1 vectors in \(\mathbb{R}^d\) satisfying the min up/down constraint with parameter \(l\). Those polytopes have been introduced in [18] in the context of discrete planning problems with machines that have a physical constraint on the frequency of switches between the operating and not operating states.\(^6\) In [18, Theorem 4], the following characterization of the facet-defining inequalities of \(P_d(l)\) is given.

Lemma 12. Let \(I \subset [d]\) be an index subset with elements \(1 \leq i_1 < i_2 < \cdots < i_k \leq d\), such that a) \(k = |I|\) is odd and b) \(i_k - i_1 \leq l\). Then, the two inequalities \(0 \leq \sum_{j=1}^{k} (-1)^{j-1}x_{i_j} \leq 1\) are facet-defining for \(P_d(l)\). Moreover, each facet-defining inequality in \(P_d(l)\) can be obtained in this way.

As argued before, it is clear from this result that \(P_d(l)\) is a 2-level polytope, because all vertices have 0/1 coordinates, and all facet-defining inequalities are of the form \(0 \leq c^T x \leq 1\) for integral vectors \(c\).

Lemma 13. 2-level min up/down polytopes satisfy Conjecture [4]

\(^6\) The more general definition given in [18] considers two parameters \(\ell_1\) and \(\ell_2\), which respectively restrict the minimum lengths of the blocks of 0’s and 1’s in valid vertices. The resulting polytope is 2-level precisely when \(\ell_1 = \ell_2\), thus in this section we restrict our attention to this case. General (non-2-level) min up/down polytopes do not satisfy Conjecture [4] (see Example 5).
Proof. Consider the 2-level min up/down polytope $P_d(l)$, for integers $0 < l < d$. $P_d(l)$ is full dimensional, hence it has dimension $d$. Define the graph $G([d-1], E)$, where $\{i,j\} \in E$ whenever $|j-i| \leq l - 1$, and let $C'$ and $S'$ be as in Corollary 7. We delay for a moment the proof of the following facts: a) $f_0(P_d(l)) = 2|S'|$; and b) $f_{d-1}(P_d(l)) = 2|C'|$. We obtain:

$$f_0(P_d(l))f_{d-1}(P_d(l)) = 4|S'||C'|.$$  

This is the same inequality that appears in the proof of Lemma 11, hence in a similar fashion we conclude that the conjectured inequality is satisfied, and it is tight only if $G$ is either a clique or an anti-clique. These cases correspond to $l = d - 1$ and $l = 1$, respectively, and it can be checked that $P_d(l)$ is then affinely equivalent to the cross-polytope or the cube.

Proof of fact a). For a vector $x \in \{0,1\}^d$, let $I_x \subseteq [d-1]$ be its set of switch indices. Then $x$ is (a vertex) in $P_d(l)$ iff $I_x$ is a stable set in $G$. Moreover, if two vertices $x, y \in P_d(l)$ have exactly the same switch indices, then either $x = y$ or $x + y = 1$ (the all-ones vector). Hence, there is a mapping from the set of vertices of $P_d(l)$ to $S'$, where each pre-image contains 2 elements. This proves the claim.

Proof of fact b). Let $\mathcal{I} \subseteq 2^{[d]}$ be the collection of all index sets $I \subseteq [d]$ satisfying the properties of Lemma 12. The lemma asserts that $f_{d-1}(P_d(l)) = 2|\mathcal{I}|$. To complete the proof, we present a bijection from $\mathcal{I}$ to $C'$. For $I \subseteq [d]$ in $\mathcal{I}$, let $i$ be the lowest index in $I$, let $j = \min\{i+1,d\}$, and define $I' = I \setminus \{j\}$. $I'$ is a clique in $G$. We conclude the proof by showing that the mapping can be inverted, hence it is bijective. Recall that $G$ has nodes indexed from 1 to $d - 1$. For $I' \in C'$, if $|I'|$ is odd, let $I = I'$; if $I' = \emptyset$, let $I = \{d\}$; otherwise, let $i$ be the lowest index in $I$ and $j = \min\{i+1,d\}$, and define $I = I' \cup \{j\}$. Clearly, in all cases $I \in \mathcal{I}$, and the preimages of two even cliques or two odd cliques are distinct. Now pick an even clique $I'$. If $I' = \emptyset$, then $I = \{d\}$ is not the preimage of an odd clique. If $I' \neq \emptyset$ and $i + 1 < d$, then $I$ is not a clique of $G$, hence, in particular, it cannot be an odd clique. If $d \leq i + l$, then $d \in I$, and the latter never occurs for odd cliques. □

4 2-Level Matroid Base Polytopes

In this section we investigate the base polytope of the 2-sum of matroids, and 2-level matroid base polytopes. We start by giving a non-redundant description of the base polytopes of the 2-sum $M_1 \oplus M_2$ of matroids in terms of the facets of the base polytopes of $M_1$ and $M_2$ (Corollary 24). We then focus on matroids whose base polytopes are 2-level, which have been characterized in 13. We give an explicit description of the associated base polytopes (Theorem 30), and prove that they verify Conjecture 1 (Theorem 33). Finally, in Section 4.3 we give an alternative, self-contained proof of the characterization from 13. For basic definitions and facts about matroids not appearing in the current section we refer to [21].

4.1 The Base Polytope of the 2-Sum of Matroids

We identify a matroid $M$ by the couple $(E, \mathcal{B})$, where $E = E(M)$ is its ground set, and $\mathcal{B} = \mathcal{B}(M)$ is its base set. Whenever it is convenient, we describe a matroid in terms of its independent sets or its rank function. Given $M = (E, \mathcal{B})$ and a set $F \subseteq E$, the restriction $M|F$ is the matroid with ground set $F$ and independent sets $\mathcal{I}(M|F) = \{I \in \mathcal{I}(M) : I \subseteq F\}$;
and the contraction $M/F$ is the matroid with ground set $M \setminus F$ and rank function $r_{M/F}(A) = r_M(A \cup F) - r_M(F)$. For an element $e \in E$, the removal of $e$ is $M - e = M\setminus (E - e)$. A set $F \subseteq E$ is a circuit if it minimally dependent, i.e. $F$ is dependent but every proper subset of it is independent; and $F \subseteq E$ is a flat if it is maximal for its rank, i.e. $r(F) < r(F + x)$ for all $x \in E \setminus F$. An element $p \in E$ is called a loop (respectively coloop) of $M$ if it appears in none (all) of the bases of $M$.

Consider matroids $M_1 = (E_1, B_1)$ and $M_2 = (E_2, B_2)$, with non-empty base sets. If $E_1 \cap E_2 = \emptyset$, we can define the direct sum $M_1 \oplus M_2$ as the matroid with ground set $E_1 \cup E_2$ and base set $B_1 \times B_2$. If, instead, $E_1 \cap E_2 = \{p\}$, where $p$ is neither a loop nor a coloop in $M_1$ or $M_2$, we let the 2-sum $M_1 \triangleleft M_2$ be the matroid with ground set $E_1 \cup E_2 - p$, and base set $\{B_1 \cup B_2 - p : B_i \in B_i \text{ for } i = 1, 2 \text{ and } p \in B_1 \triangle B_2\}$. A matroid is connected (2-connected for some authors) if it cannot be written as the direct sum of two matroids, each with fewer elements; and a connected matroid $M$ is 3-connected if it cannot be written as a 2-sum of two matroids, both with strictly fewer elements than $M$.

**Example 1.** A matroid $M(E, B)$ is uniform if $B = \binom{E}{k}$, where $k$ is the rank of $M$. We denote the uniform matroid with $n$ elements and rank $k$ by $U_{n,k}$. Notice that, if $M_1$ and $M_2$ are uniform matroids with $|E(M_1) \cap E(M_2)| = 1$, then $M_1 \triangleleft M_2$ is unique up to isomorphism, for any possible common element. Figure 2 depicts $M = U_{5,2} \triangleleft U_{6,3}$, with common element $p = 5$. $M$ has ground set $\{1, 2, 3, 4, 6, 7, 8, 9, 10\}$ and rank 4, and two of its bases are $\{1, 2, 6, 7\}$ and $\{1, 6, 7, 8\}$.

![Fig. 2: A representation of $M = U_{5,2} \oplus U_{6,3}$.](image)

The proofs of the following three classical results can be found e.g. in [21].

**Lemma 14 (Characterization of contraction).** Consider a matroid $M = (E, B)$ and a set $F \subseteq E$. For any fixed basis $B_F$ of $M|F$, one has $B(M/F) = \{B \subseteq E \setminus F : B \cup B_F \in B(M)\}$.

**Observation 15.** Let $M = M_1 \oplus M_2$, with $E(M_1) \cap E(M_2) = \{p\}$.

1. $M_1 \triangleleft M_2$ is connected if and only if so are $M_1$ and $M_2$.
2. $B(M_1 \triangleleft M_2) = B(M_1 - p) \times B(M_2/p) \cup B(M_1/p) \times B(M_2 - p)$.
3. $|B(M_i)| = |B(M_i - p)| + |B(M_i/p)|$, for $i = 1, 2$.
4. If $M_2 = M_2' \oplus M_2''$, where $E(M_1) \cap E(M_2') = \{p\}$ and $E(M_1) \cup E(M_2'') \cap E(M_2'') = \emptyset$, then $M_1 \triangleleft M_2 = (M_1 \triangleleft M_2') \oplus M_2''$. 
Observation 16. Let $M = M_1 \oplus M_2$, with $E(M_1) \cap E(M_2) = \{p\}$. For a set $I \subseteq E(M_1) - p$: $M[I] = M_1[I]$, and $r_M(I) = r_{M_i}(I)$.

We now list a series of lemmas that will be needed for the proof of the main results of the section.

Lemma 17. If $p \in E$ is not a coloop of matroid $M = (E, B)$, any independent set not containing $p$ can be extended to a basis of $M$ not containing $p$.

Proof. Let $I \in \mathcal{I}(M)$, $p \notin I$. Extend $I$ to a basis $B \in \mathcal{B}$. If $p \notin B$ we are done, so assume $p \in B$. Since $p$ is not a coloop of $M$, there exists a basis $B' \in \mathcal{B}$ such that $p \notin B'$. Now apply the basis exchange property of matroids to find an element $e \in B'$ such that $B + e - p$ is a basis in $B$. The result follows.

Lemma 18. Let $M = M_1 \oplus M_2$ and $I_1, I_2$ be independent sets of $M_1, M_2$ respectively, such that $p \notin I_1 \cup I_2$ and $I_1 \cup I_2$ is dependent in $M$. Then $I_i + p$ is dependent in $M_i$ for $i = 1, 2$.

Proof. Suppose that $I_1 + p$ is independent, hence it is contained in a basis $B_1$ of $M_1$. Since $p$ is not a coloop, $I_2$ can be extended using Lemma 17 to a basis $B_2$ of $M_2$ that does not contain $p$. Hence $I_1 \cup I_2$ is independent in $M$, a contradiction. Similarly we can deduce that $I_2 + p$ is not independent.

Lemma 19. Let $M = M_1 \oplus M_2$ and $C \subseteq E$ such that $C \cap E_i \neq \emptyset$ for $i = 1, 2$. $C$ is a circuit of $M$ if and only if $C_i = (C \cap E_i) + p$ is a circuit of $M_i$ for $i = 1, 2$.

Proof. Let $C$ be a circuit of $M$, i.e. $C$ is dependent and $C - e$ is independent for all $e \in C$. For $i = 1, 2$, that $C_i$ is dependent in $M_i$ follows from Lemma 18. $C \cap E_1$ and $C \cap E_2$ are independent in $M$, hence so are they in $M_1$ (resp. $M_2$) using Observation 16 while $C$ is dependent in $M$. Hence, $C_i = (C \cap E_i) + p$ is dependent in $M_i$, for $i = 1, 2$. $C_i - p$ is clearly independent in $M_i$. We are left to show that, for any $e \in C_i$, $e \neq p$, $C_i - e$ is independent in $M_i$. We consider without loss of generality the case $i = 1$. We know that $C - e$ is independent in $M$, so $C - e \subseteq B_1 \cup B_2 - p$ for bases $B_1$ of $M_1$ and $B_2$ of $M_2$ with $p \in B_1 \cap B_2$. Moreover, $C - e = (C_1 \cup (C_2 - p)) - e$. Hence, $C_2 - p \subseteq B_2$ but since $C_2$ is dependent in $M_2$, $p \notin B_2$, hence $p \notin B_1$. Therefore $C_1 - e \subseteq B_1$, i.e. $C_1 - e$ is independent in $M_1$.

Now let $C_1, C_2$ be circuits in $M_1, M_2$. If $C$ is independent in $M$, then either $C_1$ is independent in $M_1$ or $C_2$ in $M_2$, a contradiction. Let $e \in C$, assume without loss of generality $e \in E_1$. Note that $C - e \subseteq (C_1 - e) \cup (C_2 - p)$. By definition, $C_1 - e$ is independent in $M_1$ and $C_2 - p$ is independent in $M_2$. Since $p$ is not a loop of $M_1$ (resp. a coloop of $M_2$), we extend $C_1 - e$ (resp. $C_2 - p$) to a basis $B_1$ of $M_1$ (resp. $B_2$ of $M_2$) containing $p$ (resp. not containing $p$, using Lemma 17). As $C - e \subseteq C_1 - e \cup C_2 - p \subseteq B_1 \cup B_2 \in B(M)$, we conclude that $C - e$ is independent in $M$.

Lemma 20. Let $M = M_1 \oplus M_2$, $F \subseteq E$ satisfy $F \cap E_i \neq \emptyset$ for $i = 1, 2$, and define $F_i = F \cap E_i + p$ for $i = 1, 2$. Assume that $p$ is not a loop or a coloop of $M_i | F_i$ for $i = 1, 2$.

Then one has

$$M|F = M_1|F_1 \oplus M_2|F_2 \quad \text{and} \quad M/F = M_1/F_1 \oplus M_2/F_2.$$
Proof. First statement: By definition of $F_1, F_2$, the two matroids have the same ground set. We will show that they also have the same independent sets. Let $M | F$ an independent set of $M | F$, then $I \subseteq F$ i.e. there exist $B_1, B_2 \in B_2$ s.t. $I \subseteq B_1 \cup B_2 - p, p \in B_1 \cap B_2$. Now, assume without loss of generality that $p \in B_1$. Let $I_{F_1} = (B_1 \cap F) + p, I_{F_2} = B_2 \cap F$ and note that they are independent sets of $M | F_1$ and $M | F_2$ respectively. $I_{F_1}$ can be completed to a basis $B_1$ of $M | F_1$ which contains $p$ and, using Lemma $17$, $I_{F_2}$ can be completed to a basis $B_2$ of $M | F_2$ that does not contain $p$. Since $I \subseteq I_{F_1} \cup I_{F_2} - p \subseteq B_1 \cup B_2 - p$, we conclude that $I$ is independent in $M | F_1 \oplus_2 M | F_2$.

For the other direction, let $I$ be an independent set of the 2-sum, hence $I \subseteq B_{F_1} \cup B_{F_2} - p$ with $B_{F_1}$ basis of $M | F_1$ and $p \in B_{F_1} \cap B_{F_2}$. Say, $p \in B_{F_1}$. Using again the fact that $p$ is not a coloop of any of $B_{F_1} \cup B_{F_2}$, we can use Lemma $14$ to conclude that there exists $\tilde{B}_i \in B(M_i)$ for $i = 1, 2$, with $B_{F_1} \subseteq \tilde{B}_1, B_{F_2} \subseteq \tilde{B}_2$, and $p \notin \tilde{B}_2$. Hence, 

$I \subseteq B_{F_1} \cup B_{F_2} - p \subseteq \tilde{B}_1 \cup \tilde{B}_2 - p \subseteq B(M)$,

so $I$ is independent in $M$. Moreover $I \subseteq F$, which implies that $I$ is independent in $M | F$.

Second statement: Clearly the two matroids have the same ground set. We show that they have the same bases. Let $B \in B(M | F)$, and fix $B_F \in B(M | F)$. Then from Lemma $14$ we know $B \cup B_F = \tilde{B} \in B(M)$, hence there exists $B_1, B_2 \subseteq B_2$ such that $\tilde{B} = B_1 \cup B_2 - p$ and $p \in B_1 \Delta B_2$. By the first statement, we also have that $B_F = B_{F_1} \cup B_{F_2} - p$ with $B_{F_i}$ basis of $M_i | F_i$ and $p \in B_{F_i} \cap B_{F_2}$. We now argue that $B_{F_i} = B_i \cap F_i$, for $i = 1, 2$. The statement is clearly true if $p \notin B_{F_i} \Delta B_i$. In turn, the latter holds if we show that $p \in B_i$ implies $p \notin B_{F_i}$ for $i = 1, 2$. So suppose without loss of generality that $p \notin B_1 \cap B_{F_1}$. Then $B_1 \cap F_1$ is an independent set of $M_1 | F_1$. Clearly $B_{F_1} \subseteq B_1 \cap F_1$, and the containment is strict as $p \notin B_{F_1}$, a contradiction to the fact that $B_{F_1}$ is a basis of $M_1 | F_1$.

Now for $i = 1, 2$ define $B'_i = B_i \setminus B_{F_i}$. As $B_{F_i} = B_i \cap F_i$, we deduce that $B'_i \subseteq B_i \setminus F_i$. Since $B_i$ is a basis of $M_i$ and $B_{F_i}$ is a basis of $M_i | F_i$, we can use Lemma $14$ and deduce that $B'_i$ is a basis of $M_i | F_i$. Then $B = B'_1 \cup B'_2$ is a basis of $M_1 | F_1 \oplus_2 M_2 | F_2$.

For the other direction, let $B \in B(M_1 | F_1 \oplus_2 M_2 | F_2)$, hence $B = B'_1 \cup B'_2$ with $B'_i \in B(M_i | F_i)$. Then choose bases $B_{F_i}$ of $M_i | F_i$ with $p \in B_{F_i} \cap B_{F_2}$ (which is possible since $p$ is not a loop or coloop of any of $M_i | F_i$ for $i = 1, 2$). Then $B_F = (B_{F_1} \cup B_{F_2}) - p$ is a basis of $M_i | F_i$ by the first statement. From Lemma $14$ we deduce that $B_i = B_{F_i} \cup B'_i$ is a basis of $M_i$ for $i = 1, 2$. Moreover, $p \in B_1 \Delta B_2$, and

$$B \cup B_F = (B'_1 \cup B'_2) \cup (B_{F_1} \cup B_{F_2}) - p = (B'_1 \cup B_{F_1}) \cup (B'_2 \cup B_{F_2}) - p = (B_1 \cup B_2) - p$$

is a basis of $M$, which implies that $B$ is a basis of $M | F$ using again Lemma $14$. □

**Lemma 21.** Let $M = M_1 \oplus_2 M_2$ and $F \subset E_1 - p$ (resp. $E_2 - p$). Then $M | F = M_1 | F \oplus_2 M_2$ (resp. $M_1 \oplus_2 M_2 | F$).

**Proof.** The two matroids have the same ground set. Recall Lemma $14$. Let $B \in B(M | F)$. Then for any $B_F \in B(M | F)$, we have $B \cup B_F \in B(M)$, hence $B \cup B_F = (B_1 \cup B_2) - p$ for some $B_i \in B(M_i)$ for $i = 1, 2, p \in B_1 \Delta B_2$. Then, by definition, $B_1 \setminus B_F$ is a basis of $M_1 | F$ and $B = B_1 \setminus B_F \cup B_2 - p$ is a basis of $M_1 | F \oplus_2 M_2$. The other direction is proved similarly. □
The base polytope $B(M) \subseteq \mathbb{R}^E$ of a matroid $M = (E, \mathcal{B})$ is given by the convex hull of the characteristic vectors of its bases. For a matroid $M$, the following is known to be a description of $B(M)$.

$$B(M) = \{ x \in [0,1]^E : x(F) \leq r(F) \text{ for } F \subseteq E; x(E) = r(E) \}.$$  

(1)

When $M$ is connected the following characterization of the facet-defining inequalities for $[1]$ is given in $[8]$. (We report the statement as it appears in $[13]$.)

**Theorem 22.** Let $M = (E, \mathcal{B})$ be a connected matroid. For every facet $F$ of $B(M)$ there is a unique $S \subseteq E$, $S \neq \emptyset$, such that $F = B(M) \cap \{ x \in \mathbb{R}^E : x(S) = r(S) \}$. Moreover, a non-empty subset $S$ gives rise to a facet of $B(M)$ if and only if one of these two conditions holds:

1. $S$ is a flat such that $M|S$, $M/S$ are connected;
2. $S = E - e$ for some $e \in E$ such that $M|S$, $M/S$ are connected.

The subsets $S$ in 1. are called facets, and they are in one-to-one correspondence with the facet-defining inequalities in $[1]$ of the form $x(S) \leq r(S)$, including $x_e \leq 1$ for $e \in E$.

Let us remark that one can ignore the condition of $S$ being flat. Indeed, if $S$ is not a flat, there exists $e \in E \setminus S$ such that $r(S + e) = r(S)$. Then, using the definition of contraction, $r_{M/S}(e) = r_M(S + e) - r_M(S) = 0$, i.e., $M/S$ is not connected since it has $e$ as a loop. For this reason, in the following we will consider a facet to be a set $S \subseteq E$ such that $M|S$, $M/S$ are connected.

For $S = E - e$ satisfying the conditions in 2., we refer to element $e$ as defining a non-negativity facet. Indeed it can be easily seen that it defines the same facet as $x_e \geq 0$.

Throughout the rest of the section, we assume that $M_1(E_1, B_1)$, $M_2(E_2, B_2)$ are connected matroids, with, $E_1 \cap E_2 = \{ p \}$, and we define $M = M_1 \oplus_2 M_2$. It is well known that under these assumptions $M$ is also connected. By the arguments above, characterizing $B(M)$ essentially boils down to characterizing facets of $M_1 \oplus_2 M_2$.

**Theorem 23.** Let $F$ be a facet of $M$. One of the following holds:

1. $F = E_i \cup F' - p$, where $F'$ is a facet of $M_j$ containing $p$, and $i \neq j \in \{1, 2\}$.
2. $F$ is a facet of $M_i$ not containing $p$ for some $i \in \{1, 2\}$.
3. $F = E_i - p$ for some $i \in \{1, 2\}$.

Conversely, let $F_1$ be a facet of $M_1$, $F_1 \neq \{p\}$. Then

1. If $p \in F_1$, $F = E_2 \cup F_1 - p$ is a facet of $M$.
2. If $p \notin F_1$, $F_1$ is a facet of $M$.
3. If $M_2/p$ and $M_1 - p$ are connected, then $E_1 - p$ is a facet of $M$.

**Proof.** First statement. Case 1: Let us first assume $F \cap E_i \neq \emptyset$ for $i = 1, 2$. Let $F_i = F' \cap E_i + p$. In order to prove the theorem, we need to show that we are in case 1. We first show that $F_1, F_2$ satisfy Lemma 20 i.e. that $p$ is not a loop or coloop of $M_1|F_1$ and $M_2|F_2$.

By symmetry, we only argue for $M_1|F_1$. Since $F$ is a facet of $M$, $M|F$ is connected, hence it has a circuit $C \subseteq F$ of $M|F$ whose intersection with both $E_1$ and $E_2$ is non-empty. $C$ is clearly a circuit of $M$ as well. Then, using Lemma 19 we deduce that $\{p\} \subseteq C \cap E_1 + p \subseteq F_1$.
where \( C \cap E_1 + p \) is a circuit of \( M_1 \), hence a circuit of \( M_1 | F_1 \). In particular, \( p \) is not a loop or coloop of \( M_1 | F_1 \). We can then use Lemma \[20\] to deduce that \( M/F = M_1/F_1 \oplus M_2/F_2 \), i.e. \( M/F \) is a direct sum. But if both summands are non-empty, this would imply that \( M/F \) is not connected, a contradiction to \( F \) being a flacet. Hence we can assume without loss of generality that \( F = E_2 \cup F_1 - p \). We need to show that \( F \) is a flacet of \( M_1 \), for which we use again Lemma \[20\]. We have \( M/F = M_1/F_1 \oplus M_2/F_2 = M_1/F_1 \oplus M_2/F_2 \), and since \( M/F \) is connected, so is \( M_1/F_1 \). Moreover, \( M_1/F_1 = M_1/F_1 \oplus M_2/F_2 = M/F \), which is connected. We conclude that \( F \) is a flacet of \( M_1 \).

Case 2: Assume \( F \subseteq E_1 - p \) for some \( i \), say, \( i = 1 \). To prove that we are in case 2 of the statement we need to show that \( F \) is a flacet of \( M_1 \). That \( M_1/F \) is connected follows from the fact that \( M/F \) is connected and Observation \[16\].

To show that \( M_1/F \) is connected it suffices to observe that \( M/F = M_1/F \oplus M_2 \) due to Lemma \[21\] and the statement follows using Observation \[15\] since \( M/F \) is connected by hypothesis.

Case 3: \( F = E_1 - p \) or \( F = E_2 - p \). This coincides with case 3 of the statement.

Second statement. Case 1: The claim follows from Lemma \[20\] exactly as in the Case 1 of the first statement. The hypotheses of the lemma \[20\] are clearly satisfied since \( M_2, M_1 | F_1 \) are connected, hence they have no loop or coloop. Hence we have \( M/F = M_1/F_1 \oplus M_2/E_2 = M_1/F_1 \oplus M_2, M/F = M_1/F_1 \oplus M_2/E_2 = M_1/F_1 \), from which we get that \( M/F \) and \( M/F \) are connected.

Case 2: We have that \( M/F_1 = M_1/F_1 \), hence it is connected, and thanks to Lemma \[21\] \( M/F_1 = M_1/F_1 \oplus M_2 \), which implies that \( M/F \) is connected. Hence \( F_1 \) is also a flacet of \( M \).

Case 3: We claim that \( M/(E_1 - p) = M_2/p \), hence by hypothesis it is connected. The two matroids have the same ground set \( E_2 - p \) and both have as bases the bases of \( M_2 \) that contain \( p \). Similarly one can observe that \( M/(E_1 - p) = M_1 - p \), hence it is connected, which concludes the proof. \( \square \)

We remark that a statement similar to the first half of Theorem \[23\] for a somehow similar definition of 2-sum and flacet appeared in \[5\]. However, we were not able to convince ourselves that the proof from \[5\] is complete, and some of its statements appear to be wrong.

**Corollary 24.** Let \( M = M_1 \oplus M_2 \), with \( M_1, M_2 \) connected. The following is a non-redundant description of \( B(M) \):

\[
B(M) = \{ x \in \mathbb{R}^E : \\
x_e \geq 0 \quad \text{if } e \in E_i - p : M_i - e \text{ connected, } i = 1, 2 \\
x(E_i \cup F - p) \leq r(E_i \cup F - p) \quad F \text{ flacet of } M_i : \{ p \} \not\subseteq F, i \neq j \in \{ 1, 2 \} \\
x(F) \leq r(F) \quad F \text{ flacet of } M_i : p \not\in F, i \in \{ 1, 2 \} \\
x(E_1 - p) \leq r(E_1 - p) \quad \text{if } M_i - p, M_j/p \text{ connected, } i \neq j \in \{ 1, 2 \} \\
x(E) = r(E) \}.
\]

**Proof.** That \( e \) defines a non-negativity facet of \( B(M) \) if and only if \( M_i - e \) is connected, where \( e \in E_i - p \), can be proved as follows: first, assume without loss of generality that \( e \in E_1 - p \); then one can see, similarly as in Lemma \[21\] that \( M - e = (M_1 - e) \oplus M_2 \).
Hence $M - e$ is connected if and only if $M_1 - e$ is (Observation 15), which completes the proof due to Theorem 22. All the rest is a consequence of Theorem 23.

□

Example 1 continued. We describe the base polytope of the matroid $M$ from Example 1. Note that

$$B(U_{n,k}) = \{ x \in [0,1]^n : x(E) = k \}.$$  

(See Observation 29 for a proof). Applying Theorem 23 we deduce:

$$B(M) = \{ x \in \mathbb{R}^9 : x_i \geq 0 \text{ for } i \in \{10\} \setminus \{6\}, \quad x_i \leq 1 \text{ for } i \in \{10\} \setminus \{6\}, \quad x_1 + x_2 + x_3 + x_4 \leq 2, \quad x_6 + x_7 + x_8 + x_9 + x_{10} \leq 3, \quad x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9 + x_{10} = 4 \}. $$

Note that the sets $\{1, 2, 3, 4\}, \{6, 7, 8, 9, 10\}$ correspond to case 2 of the theorem.

Corollary 25. Let us write $f(M) = f_{d-1}(B(M))$, and similarly for $M_1, M_2$. Then $f(M_1) + f(M_2) - 2 \leq f(M) \leq f(M_1) + f(M_2) + 2$.

Proof. The corollary is a consequence of Corollary 24. The slack of $+2$ in the right-hand side is due to the possibility of $M$ having $E_1 - p, E_2 - p$ as facets. The negative slack in the left-hand side could in principle be 4, due to the fact that $p$ might define a non-negativity facet in $B(M_1), B(M_2)$ and that $\{p\}$ could also define a facet in $M_1, M_2$. However a simple case analysis shows that the slack is always at most 2. For instance, if $\{p\}$ is a facet of $M_1$ and $M_2$, and also $p$ defines a non-negativity facet in $B(M_1)$, then we have $E_1 - p$ and $M_2/p$ connected, which implies that $E_1 - p$ is a facet of $M$ due to Theorem 23, hence the slack is at most 2. The other cases proceed similarly. □

4.2 Linear Description of 2-Level Matroid Base Polytopes

A matroid $M(E, B)$ is uniform if $B = \binom{E}{k}$, where $k$ is the rank of $M$. We denote the uniform matroid with $n$ elements and rank $k$ by $U_{n,k}$. Notice that, if $M_1$ and $M_2$ are uniform matroids with $|E(M_1) \cap E(M_2)| = 1$, then $M_1 \oplus M_2$ is unique up to isomorphism, for any possible common element. Let $\mathcal{M}$ be the class of matroids whose base polytope is 2-level. $\mathcal{M}$ has been characterized in [13].

Theorem 26. The base polytope of a matroid $M$ is 2-level if and only if $M$ can be obtained from uniform matroids through a sequence of direct sums and 2-sums.

The following lemma implies that we can, when looking at matroids in $\mathcal{M}$, decouple the operations of 2-sum and direct sum.

Lemma 27. Let $M$ be a matroid obtained by applying a sequence of direct sums and 2-sums from the matroids $M_1, \ldots, M_k$. Then $M = M'_1 \oplus M'_2 \oplus \ldots \oplus M'_k$, where each of the $M'_i$ is obtained by repeated 2-sums from some of the matroids $M_1, \ldots, M_k$. 

Proof. Immediately from repeated applications of Observation 15, part 2.

Since the base polytope of the direct sum of matroids is the Cartesian product of the base polytopes, to obtain a linear description of \( B(M) \) for \( M \in \mathcal{M} \), we can focus on base polytopes of connected matroids obtained from the 2-sums of uniform matroids. Any connected matroid can be seen as a sequence of 2-sums, which can be represented via a tree (see Figure 3); the following is a version of [21, Proposition 8.3.5] tailored to our needs. We include a proof for completeness.

**Theorem 28.** Let \( M \) be a connected matroid. Then there are 3-connected matroids \( M_1, \ldots, M_t \), and a \( t \)-vertex tree \( T = T(M) \) with edges labeled \( e_1, \ldots, e_{t-1} \) and vertices labeled \( M_1, \ldots, M_t \), such that

1. \( E(M) \cap \{e_1, \ldots, e_{t-1}\} = \emptyset \), and \( E(M_1) \cup E(M_2) \cup \cdots \cup E(M_t) = E(M) \setminus \{e_1, \ldots, e_{t-1}\} \);
2. if the edge \( e_i \) joins the vertices \( M_{j_i} \) and \( M_{j_2} \), then \( E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\} \);
3. if no edge joins the vertices \( M_{j_1} \) and \( M_{j_2} \), then \( E(M_{j_1}) \cap E(M_{j_2}) = \emptyset \).

Moreover, \( M \) is the matroid that labels the single vertex of the tree \( T/e_1, \ldots, e_{t-1} \) at the conclusion of the following process: contract the edges \( e_1, \ldots, e_{t-1} \) of \( T \) one by one in order; when \( e_i \) is contracted, its ends are identified and the vertex formed by this identification is labeled by the 2-sum of the matroids that previously labeled the ends of \( e_i \).

Proof. We proceed by induction on \( n = |E(M)| \). For \( n = 1 \), \( M \) is 3-connected, \( T \) consists of only one vertex and there is nothing to show. For \( n > 1 \): if \( M \) is 3-connected, again there is nothing to show. Otherwise, \( M = M' \oplus_2 M'' \) for some matroids \( M', M'' \), that are connected (due to Observation 15) and that satisfy \( |E(M')|, |E(M'')| < n \). Hence by induction hypothesis the thesis holds for \( M', M'' \). Let \( T', T'' \) be their corresponding trees, with vertices labeled by the 3-connected matroids \( M'_1, \ldots, M'_t \), and \( M''_1, \ldots, M''_t \) respectively, edges labeled \( e'_1, \ldots, e'_{t-1} \) and \( e''_1, \ldots, e''_{t-1} \) respectively, and let \( t = t_1 + t_2 \). By definition of 2-sum there is exactly one element, which we denote by \( e_{t-1} \), in \( E(M') \cap E(M'') \). By induction we have:

\[
E(M) = E(M') \cup E(M'') \setminus \{e_{t-1}\} = (E(M'_1) \cup \cdots \cup E(M'_t)) \setminus \{e'_1, \ldots, e'_{t-1}\}) \cup (E(M''_1) \cup \cdots \cup E(M''_t)) \setminus \{e''_1, \ldots, e''_{t-1}\}) \setminus \{e_{t-1}\}.
\]

We can assume without loss of generality that \( \{e'_1, \ldots, e'_{t-1}\} \cap E(M') = \emptyset \) by renaming the elements of \( E(M') \), and similarly we can assume \( \{e''_1, \ldots, e''_{t-1}\} \cap E(M'') = \emptyset \). Since \( M' \) satisfies properties 1-3, there is exactly one matroid \( M'_i \) such that \( e_{t-1} \in E(M'_i) \), and similarly there is exactly one matroid \( M''_j \) such that \( e_{t-1} \in E(M''_j) \). Let \( T \) be the tree obtained by joining \( T', T'' \) through the edge \( (M'_i, M''_j) \). Now, it is easy to check that the matroids labeling the vertices of \( T \) will satisfy properties 1-3 after an appropriate renaming of the matroids and relabeling of the edges \( (M'_i) \) will be renamed \( M_i \), \( M''_j \) \( M_{i+t} \), and similarly for the elements \( e'_i, e''_j \). The statement about the contraction \( T/e_1, \ldots, e_{t-1} \) follows by induction: one first contracts the edges in \( T' (e_1, \ldots, e_{t-1}) \), then the edges in \( T'' (e_{t_1}, \ldots, e_{t-2}) \), obtaining vertices labeled by \( M' \) and \( M'' \). Then, contracting the edge \( e_{t-1} \) joining \( M', M'' \) one gets \( M' \oplus_2 M'' = M \). □
Example 2. Consider the matroid $M$ whose associated tree structure is given in Figure 3. The ground set of $M$ is $\{1, 2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 15\}$ and its rank, which can be computed as the sum of the ranks of the nodes minus the number of edges, is 4. $\{1, 2, 11, 13\}$ is a basis and $\{8, 9, 10, 12\}$ a circuit of $M$.

Fig. 3: The matroid from Example 2

Observation 29. If $M \in \mathcal{M}$ is connected and non-uniform, we can assume without loss of generality that every node in its tree structure given by Theorem 28 is a uniform matroid with at least 3 elements. Each of those uniform matroids has no flacets besides its singletons.

Proof. No uniform matroid in a 2-sum can have ground set of size one, since the 2-sum is defined when the common element is not a loop or a coloop of either summand. The only 3-connected uniform matroid on two elements is $U_{2,1}$. However, it is easy to see that for any matroid $M$, $M \oplus_2 U_{2,1}$ is isomorphic to $M$: if the ground set of $U_{2,1}$ is $\{p, e\}$, with $p$ being the element common to $M$, the 2-sum has the only effect of replacing $p$ by $e$ in $M$. Finally, let $F$ be a flacet of a uniform matroid $U_{n,k}$. Since $F$ is a flat, $|F| < k$, but then the restriction to $F$ does not have any cycle. Hence, for the restriction to be connected we must have $|F| = 1$.

For a connected matroid $M(E, B) \in \mathcal{M}$, Theorem 28 reveals a tree structure $T(M)$, where every node represents a uniform matroid, and every edge represents a 2-sum operation. We now give a simple description of the associated base polytope. Let $a$ be an edge of $T(M)$. The removal of $a$ breaks $T$ into 2 connected components $C_a^1$ and $C_a^2$. Let $E_a^1$ (resp. $E_a^2$) be the set of elements from $E$ that belong to uniform matroids from $C_a^1$ (resp. $C_a^2$). The following theorem shows that all the inequalities needed to describe $B(M)$ are the “trivial” inequalities $0 \leq x \leq 1$, plus $x(F) \leq r(F)$, where $F = E_a^1$ or $E_a^2$ for some edge $a$ of $T(M)$. If $M$ is 2-sum of uniform matroids $U_1, \ldots, U_t$, then clearly $T$ will have $t - 1$ edges. From Observation 29 we know that $E(U_i) \geq 3$ for any $i$. Hence, if $|E| = n$, we have

$$n = \sum_{i=1}^{t} |E(U_i)| - 2(t - 1) \geq 3t - 2(t - 1) = t + 2,$$
hence \( t \leq n - 2 \). Thus, the total number of inequalities needed is linear in the number of elements.

Theorem 30. Let \( M = (E, B) \in \mathcal{M} \) be a connected matroid obtained as 2-sums of uniform matroids \( U_i = U_{n_i,k_i}, \ldots, U_1 = U_{n_1,k_1} \). Let \( T(N, A) \) be the tree structure of \( M \) according to Theorem 28. For each \( a \in A \), let \( C^1_a, C^2_a, E^1_a, E^2_a \) be defined as above. Then

\[
B(M) = \{ x \in \mathbb{R}^E : \quad \begin{align*}
    x &\geq 0 \\
x &\leq 1 \\
x(F) &\leq r(F) \quad \text{for } F = E^i_a \text{ for some } i \in \{1, 2\} \text{ and } a \in A, \\
x(E) &= r(E) \}. 
\]

Moreover, if \( F = E^i_a \) for \( i \in \{1, 2\} \) and some \( a \in A \), then \( r(F) = 1 - |C^i_a| + \sum_{j, u_j \in C^i_a} k_j \).

Proof. Let us call a subset \( C \subseteq N \) a valid component for \( T \) if \( C = C^i_a \) for some \( i \in \{1, 2\} \) and \( a \in A \), and denote the set of all valid components of \( T \) by \( \mathcal{F} \). Each connected subtree of \( T(N, A) \) represents a connected matroid obtained as 2-sums of uniform matroids. Thus, we can prove the theorem by induction on \( t \). The statement on the rank is immediate. For \( t = 1 \), \( \mathcal{F} \) is empty and thanks to Observation 29, the remaining inequalities are enough to describe \( B(M) \). Now let \( t > 1 \). Thanks to Theorem 22, to prove the thesis it is enough to show that, if \( F \) is a flacet of \( M \) with \( |F| \geq 2 \), then \( F \in \mathcal{F} \). First notice that we can write, without loss of generality, \( M = M' \oplus U_i \), where \( U_i \) corresponds to a leaf \( v_i \) of \( T \) and \( M' \) is obtained as 2-sums of \( U_1, \ldots, U_{t-1} \), hence it satisfies the inductive hypothesis. Note that the tree corresponding to \( M' \) is then \( T - v_i \). Let us denote by \( v_i \) the only neighbor of \( v_i \) in \( T \). Let \( E' + p, E(U_i) = E_i + p \) be the ground sets of \( M' \), \( U_i \) respectively, where \( E' = \bigcup_{i=1}^{t-1} E_i \), and \( E_i = E \cap E(U_i) \) for \( i = 1, \ldots, t \). Clearly \( p \in E(U_i) \). Now, since \( F \) is a flacet of \( M \), we can apply Theorem 23 to get three possible cases. If \( F \) has non-empty intersection with both \( E(M') \) and \( E_i \), then we are in case 1 and either \( F = E(U_i) \cup F' - p \) or \( F = E' \cup F_i - p \), where \( F', F_i \) are flacetcs of \( M' \), \( U_i \) respectively, containing \( p \). However, the latter case is not possible because of Observation 29, so the only possibility is that \( F = E_i \cup F' \). By induction, \( F' \) belongs to \( \mathcal{F}' \) defined for \( M' \) as in the statement of the theorem. Moreover, since \( F' \) contains \( p \), its corresponding component \( C \) in \( T - v_i \) contains \( v_i \) and then \( C + v_i \) is a valid component for \( T \). Moreover \( |F' \cap E_i| \in \{0, |E_i|\} \) for any \( i = 1, \ldots, t - 1 \), which implies \( F \in \mathcal{F} \). Suppose now we are in case 2, i.e., \( F \) is strictly contained in one of \( E', E_i \). Then \( F \) is a flacet of one of \( M' \), \( U_i \), the latter not being possible again due to Observation 29. So \( F \) is a flacet of \( M' \) and it does not contain \( p \), hence by induction hypothesis its corresponding component \( C \) does not contain \( v_i \). But then \( C \) is a valid component of \( T \) and again \( F \in \mathcal{F} \). Finally, if we are in case 3 then \( F = E_i \) or \( F = E \), and in both cases \( F \in \mathcal{F} \).

Example 2, continued. We apply Theorem 30 to deduce a complete description of \( B(M) \), where \( M \) is the matroid from Example 3. Let \( S' = \{5, 6, 7\} \) and \( S = [15] \setminus S' \). We obtain:
\[ B(M) = \{ x \in \mathbb{R}^{12} : \begin{array}{l}
\text{for } i \in S, \ x_i 
\geq 0
\end{array} \text{ and for } i \in S, \ x_i 
\leq 1
\begin{array}{l}
x_1 + x_2 + x_3 + x_4 
\leq 2
\end{array}
\begin{array}{l}
x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 3
\end{array}
\begin{array}{l}
x_{11} + x_{12} 
\leq 1
\end{array}
\begin{array}{l}
x_{13} + x_{14} + x_{15} \leq 1
\end{array}
\begin{array}{l}
x_1 + x_2 + x_3 + x_4 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 4
\end{array}\} \].

Note that we omitted the inequalities associated to the cuts with right-hand side 4, since they are redundant.

Before, proving Conjecture \[1\] for 2-level base polytopes, we need two more technical lemmas.

**Lemma 31.** Let \( M \in \mathcal{M} \) be a connected matroid, and let \( p \in E \). Then we have:

1. \( \{p\} \) is a facet of \( M \) if and only if \( M/p \) is connected.
2. \( p \) defines a non-negativity facet of \( B(M) \) if and only if \( M - p = M|(E - p) \) is connected.

**Proof.** Part 1 is trivial: if \( \{p\} \) is a facet of \( M \) then \( M/p \) is connected by definition. For the other direction, if \( M/p \) is connected, then since \( M|\{p\} \) is connected by definition we have that \( \{p\} \) is a facet of \( M \).

Part 2 is a direct consequence of Theorem \[22\] part 2, if we observe that, if \( S = E - e \), \( M/S \) has only one element and it is then connected. \( \square \)

**Lemma 32.** Let \( M \in \mathcal{M} \) be such that \( M = M_1 \oplus_2 U \) where \( U = U'_{n',k'} \) is a 3-connected uniform matroid with \( n' \geq 3 \). Then \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2(n' - 1) \); and if \( n' = 3 \) then \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2 \).

**Proof.** Let us denote by \( E' \) the ground set of \( U \), by \( E_1 \) the ground set of \( M_1 \) and let \( E_1 \cap E' = \{p\} \) as usual. Let \( fl(M) \) be the number of facets of \( M \), and \( nn(M) \) the number of elements of \( M \) which define a non-negativity facet of \( B(M) \). From Theorem \[22\] we know that \( f_{d-1}(B(M)) = fl(M) + nn(M) \), and an analogous statement holds for \( M_1, U \). Now, from Theorem \[23\] and Observation \[29\] we have that

\[ fl(M) \leq fl(M_1) + (n' - 1) + 2 = fl(M_1) + n' + 1, \quad (3) \]

where the -1 comes from the fact that the singleton \( \{p\} \) is a facet for \( U \) but not for \( M \). Moreover,

\[ nn(M) \leq nn(M_1) + n' - 1 \quad (4) \]

can be seen as well. This gives us \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2n' \). To prove the thesis for \( n' \geq 4 \), we have to show that the inequality holds even if we add a -2 to the right hand side. We now consider a number of cases:

1. \( \{p\} \) if a facet of \( M_1 \). This happens if and only if \( M_1/p \) is connected, thanks to Lemma \[31\].
2. \( p \) defines a non-negativity facet of \( B(M_1) \). This happens if and only if \( M_1 - p \) is connected, again due to Lemma 31.

3. \( E' - p \) is a facet of \( M \). This happens if and only if \( M/(E' - p) \) is connected and \( M/(E' - p) \) is connected.

4. \( E_1 - p \) is a facet of \( M \). This happens if and only if \( M/(E_1 - p) \) is connected and \( M/(E_1 - p) \) is connected.

If case 1 is true, we can subtract 1 from the right-hand side of \( \text{(3)} \) since \( \{p\} \) should not be counted as a possible facet of \( M \). Similarly if case 3 is false or case 4 is false, since these two cases correspond to the +2 term in \( \text{(3)} \).

Assume case 2 is true, we can subtract one from the right-hand side of \( \text{(4)} \). Now, if both cases 1 and 2 are true, we are done. Similarly if cases 3, 4 are both false. Suppose now at least one of cases 1, 2 are false, and at least one of cases 3,4 is true. We claim that if case 1 is false, then case 3 is false, and that if case 2 is false, then case 4 is false. This implies that we can always safely subtract 2 from the right hand side of \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2n' \) and get the thesis for \( n' \geq 4 \). Assume case 1 is false, hence \( M_1/p \) is not connected. But it can be verified easily using the definition of 2-sum and contraction that \( M_1/p = M/(E' - p) \), hence case 3) is false. Assume now case 2 is false: then \( M_1 - p \) is not connected, but \( M_1 - p = M_1/(E_1 - p) \) hence case 4 is false.

Now consider the case \( n' = 3 \), which gives the two sub-cases \( U = U_{3,1} \) and \( U = U_{3,2} \). The inequality that we already proved gives \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 4 \) for \( n' = 3 \). To complete the proof we need to gain an additional \( -2 \) in the right hand side. If \( U = U_{3,1} \), one can notice that no singleton of \( U \) is a facet, since it is not a flat; hence we gain a \( -2 \) considering the two elements of \( U - p \) which are counted in inequality \( \text{(3)} \) as facets. On the other hand, if \( U = U_{3,2} \), no element of \( U \) defines a non-negativity facet, hence we gain a \( -2 \) in inequality \( \text{(4)} \).

\[ \square \]

**Theorem 33.** 2-level matroid base polytopes satisfy Conjecture 1.

**Proof.** We will use the fact that, for any \( n \geq 3 \) and any \( k \in \{0, \ldots, n\} \), \( \binom{n}{k} \leq \frac{2}{3} 2^{n-1} \). This can be easily proved by induction. We prove the conjecture on the polytope \( B(M) \), for each matroid \( M = (E, B) \in \mathcal{M} \), and we prove it by induction on the number of elements \( n = |E| \).

The base cases \( n \leq 3 \) can be easily verified.

If \( M \) is not connected, then \( M = M_1 \oplus M_2 \) for two matroids \( M_1, M_2 \in \mathcal{M} \), each with fewer elements than \( M \), so by induction hypothesis the conjecture holds for them. The base polytope \( B(M) \) is simply the Cartesian product of \( B(M_1) \) and \( B(M_2) \), so by Lemma 3 the conjecture also holds for \( B(M) \), and is tight only if \( B(M) \) is a cube.

Assume now on that \( M \) is connected. In [13], it is proven that the smallest affine subspace containing the base polytope of a connected matroid on \( n \) elements is of dimension \( d = n - 1 \). If \( M \) is uniform, \( M = U_{n,k} \), the number of vertices in \( B(M) \) is \( f_0 = |B| = \binom{n}{k} \leq \frac{2}{3} 2^{n-1} \), where we assumed \( n \geq 3 \). And in view of Observation 29 the constraints of the form \( 0 \leq x \leq 1 \) are sufficient to define \( B(M) \), hence the number of facets is \( f_{d-1} \leq 2n \). Therefore, \( f_0 f_{d-1} \leq \frac{3}{4} n 2^n \leq (n-1)2^n = d 2^{d+1} \), where the last inequality is loose for \( n \geq 5 \). The only examples with \( n \leq 4 \) for which the conjecture is tight correspond to cubes, and the 3-dimensional cross-polytope coming from \( U_{4,2} \).

Finally, assume that \( M \) is connected but is not uniform, so it is not 3-connected. Then \( M = M_1 \oplus_2 M_2 \), with matroids \( M_1, M_2 \in \mathcal{M} \) each with fewer elements than \( M \), so by...
induction hypothesis the conjecture holds for both of them. Let \( E(M_1) \cap E(M_2) = \{ p \} \). Both
\( M_1 \) and \( M_2 \) are connected, by Observation 15. We can assume without loss of generality that
\( E(M_1) = n_1 \geq n_2 = E(M_2) \), and that \( M_2 \) is uniform, \( M_2 = U_{n_2,k_2} \), with \( n_2 \geq 3 \) (by Observation 29). We consider two cases for the value of \( n_2 \).

Case \( n_2 \geq 4 \): first notice that the family \( \mathcal{M} \) is closed under removing or contracting an

element. This is because if \( e \in M \in \mathcal{M} \), the base polytopes \( B(M - e) \) and \( B(M/e) \) are
affinely isomorphic to the faces of \( B(M) \) that intersect the hyperplanes \( x_e = 0 \) and \( x_e = 1 \),
respectively, and by Lemma 2 these faces are also 2-level. Hence, we know from Observation 15 that

\[
\begin{align*}
    f_0 &= |\mathcal{B}(M)| = |\mathcal{B}_{M_1-p}| \cdot |\mathcal{B}_{U_{n_2,k_2}/p}| + |\mathcal{B}_{M_1/p}| \cdot |\mathcal{B}_{U_{n_2,k_2}-p}| \\
    &= \binom{n_2-1}{k_2-1} |\mathcal{B}_{M_1-p}| + \binom{n_2-1}{k_2} |\mathcal{B}_{M_1/p}| \\
    &\leq \frac{3}{4} 2^{n_2-2} (|\mathcal{B}_{M_1-p}| + |\mathcal{B}_{M_1/p}|) = \frac{3}{4} 2^{d_2-1} |\mathcal{B}(M_1)|.
\end{align*}
\]

From Lemma 32 the number of facets in \( B(M) \) is
\( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2(n_2 - 1) = f_{d-1}(B(M_1)) + 2d_2 \). We use the induction hypothesis in \( M_1 \), and the trivial bound
\( |\mathcal{B}(M_1)| \leq 2^{d_1} \) to obtain:

\[
\begin{align*}
    f_0 f_{d-1}(B(M)) &< \frac{3}{4} 2^{d_2-1} |\mathcal{B}(M_1)| (f_{d-1}(B(M_1)) + 2d_2) \\
    &\leq \frac{3}{4} 2^{d_2-1} (d_2 2^{d_1+1} + 2d_2 ) \\
    &= \frac{3}{4} (d_1 + d_2) 2^{d_1+d_2} < (d_1 + d_2 - 1) 2^{d_1+d_2} = d 2^{d+1}.
\end{align*}
\]

Where in the last inequality we used the fact that \( n_1 \geq n_2 \geq 4 \), so \( d_1 \geq d_2 \geq 3 \).

Case \( n_2 = 3 \): We can prove in a similar manner as before that

\[
    f_0 = |\mathcal{B}(M)| < \left( \frac{2}{1} \right) (|\mathcal{B}(M_1 - p)| + |\mathcal{B}(M_1/p)|) = 2|\mathcal{B}(M_1)|.
\]

And from Lemma 32 \( f_{d-1}(B(M)) \leq f_{d-1}(B(M_1)) + 2 \). Thus,

\[
    f_0 f_{d-1}(B(M)) < 2|\mathcal{B}(M_1)| (f_{d-1}(B(M_1)) + 2) \leq 2 (d_1 2^{d_1+1} + 2^{d_1} \cdot 2) = d 2^{d+1}.
\]

We conclude by remarking that, since the inequalities above hold strictly, the only 2-level base polytopes satisfying the bound of Conjecture 1 are cubes and cross-polytopes. \( \square \)

As the forest matroid of a graph \( G \) is in \( \mathcal{M} \) if and only if \( G \) is series-parallel 13, we deduce the following.

**Corollary 34.** Conjecture 7 is true for the spanning tree polytope of series-parallel graphs.

### 4.3 An alternative proof of the characterization of 2-level matroid base polytopes

We conclude the section by showing that our results, in particular Theorem 30 and 23 provide an alternative polyhedral proof of Theorem 26 the characterization of 2-level base polytopes given by 13. We restate it below for convenience.
**Theorem (26)** restated. Let $M$ be a matroid and $B(M)$ its base polytope. Then $B(M)$ is 2-level if and only if $M$ can be constructed from uniform matroids by taking direct sums or 2-sums.

**Proof.** Let us first remark that $B(M)$ is 2-level if and only if for any flacet $F$ of $M$, the quantity $|B \cap F|$, where $B$ is a base of $M$, takes exactly two values. In particular, it is easy to see that the two values are $r(F)$ and $r(F) - 1$, using the fact that two vertices $\chi^{B_1}, \chi^{B_2}$ of $B(M)$ are adjacent if and only if $B_1 = B_2 - e + f$ for some elements $e, f$ (see for instance [24]). The if direction is a direct consequence of Theorem 30. Indeed, Theorem 30 implies that any connected matroid which is the 2-sum of uniform matroids has flats of the form $F = E_i^p$ for some $i \in \{1, 2\}$ and $a$ an edge of the decomposition tree (note that in the proof we do not use in any way that $B(M)$ is 2-level). Now, fix on flacet $F = E_1^a$ without loss of generality, $M$ can be written as 2-sum of matroids $M_1, M_2$ with ground sets $E_1^a + p, E_2^a + p$ respectively, for some $p$. Then for any basis $B$, $B = B_1 \cup B_2 - p$ where as usual $B_i$ is a basis of $M_i$ for $i = 1, 2$ and $p \in B_1 \Delta B_2$. The quantity $|B \cap F| = |(B_1 - p) \cap F|$ is equal to $r(F)$ if $p \notin B_1$, and to $r(F) - 1$ otherwise. This implies that $B(M)$ is 2-level, and by the first part of Lemma [3] this extends to the case of $B(M)$ not connected (recall Lemma 27 and the subsequent remark about direct sum of matroids and Cartesian product of their corresponding base polytopes).

We now prove the only if direction. We first assume that $M$ is connected. We will use the fact that, since $B(M)$ is 2-level, $|B \cap F| \geq r(F) - 1$ for any basis $B$ and flacet $F$.

We will use the tree decomposition introduced in Theorem [28] a connected matroid can be decomposed as 2-sum of 3-connected matroids $M_1, \ldots, M_t$ and we associate a tree with $M_i$’s as nodes to this decomposition. We want to prove that all the $M_i$’s are uniform. We argue by induction on $t$.

The case $t = 1$ coincides with $M$ being 3-connected. We need to show that $M$ is uniform (equivalently, that $B(M)$ is a hypersimplex). From the definition of 3-connectedness (See [21] p. 273) we have that for any partition $(X, Y)$ of the ground set $E$ of $M$ satisfying $\min(|X|, |Y|) \geq 2, r(X) + r(Y) \geq r(E) + 2$ (i.e. $M$ does not have any 2-separation). We now apply this inequality with $X = F$ any flacet of $M$. Assuming $\min(|F|, |E - F|) \geq 2$, one has $r(F) + r(E - F) \geq r(E) + 2$. Let $B$ a basis of $M$ such that $|B \cap (E - F)| = r(E - F)$.

One has
\[
r(F) = |B \cap F| + |B \cap (E - F)| = |B \cap F| + r(E - F)
\]
but this is in contradiction with the $B(M)$ being 2-level. Hence, it must be that either $|F| \leq 1$, or $|E - F| \leq 1$. Notice that in both cases we can assume equality holds, since $F$ is a proper non-empty subset of $E$. Suppose that $|E - F| = 1$. Then, since $M$ is connected, $r(E - F) = 1$ and $1 + r(F) = r(E - F) + r(F) \geq r(E) + 1$ hence $r(F) = r(E)$, a contradiction to the fact that $F$ is a flat. Hence we must have that for any flacet $F$, $|F| = 1$. This means that $B(M) = \{x \in \mathbb{R}^E : 0 \leq x \leq 1, x(E) = r(E)\}$, i.e. it is a hypersimplex. But then $M$ is uniform.

For $t > 1$, let $M = M_1 \oplus M_2$, with $E_1 \cap E_2 = \{p\}$. If we show that $B(M_1), B(M_2)$ are 2-level, then we will be done by induction, since their respective decomposition trees have less than $t$ nodes. So assume that $B(M_1)$ is not 2-level. Then $M_1$ has a flacet $F_1$, and a basis
apply the second part of Theorem 23. We consider a number of cases:

1. $p \in F \cap B$. Then $F = E_2 \cup F_1 - p$ is a facet of $M$. By Lemma 36, $M|F = M_1|F_1 \oplus M_2$, hence $r(F) = r(F_1) + r(E_2) - 1$. Consider a basis $B$ of $M$ such that $B_1 - p \subseteq B$, so $B = B_1 \cup B_2 - p$, $B_2$ a basis of $M_2$ not containing $p$. Then

$$|B \cap F| = |B \cap E_2 - p| + |B \cap F_1| = |B_2 \cap E_2| + |B_1 \cap F_1 - p| = r(E_2) + r(F_1) - 3 = r(F) - 2,$$

which is a contradiction because $B(M)$ is 2-level.

2. $p \in F_1, p \notin B$. We proceed analogously as the previous case, but $|B \cap F_1| \leq r(F_1) - 2$ (instead of $r(F_1) - 3$), and $|B_2 \cap E_2 - p| \leq r(E_2) - 1$ since $p \in B_2$.

3. $p \notin F_1$. Then $F_1$ is a facet of $M$ as well. Completing $B_1$ to a basis $B$ of $M$, we get

$$|B \cap F_1| = |B_1 \cap F_1| \leq r(F_1) - 2,$$

again a contradiction.

The same argument shows that $M_2$ is 2-level, and we are done.

Finally, if $M$ is not connected, it can be written as a direct sum of connected matroids $M_1, \ldots, M_k$. We need to show that $B(M_i)$ is 2-level for every $i$. But $B(M)$ is the Cartesian product $B(M_1) \times \cdots \times B(M_k)$ and it is 2-level, hence we are done by Lemma 3.

5 Cut Polytope and Matroid Cycle Polytope

Given a graph $G$ with edge set $E$, its cut polytope $\text{CUT}(G) \subseteq \mathbb{R}^E$ is the convex hull of the characteristic vectors of the cuts of $G$. For general graphs, a linear description of $\text{CUT}(G)$ is not known. However, for graphs without $K_5$ as a minor, $\text{CUT}(G)$ is described by:

$$\text{CUT}(G) = \{x \in [0,1]^E : x(F) - x(C \setminus F) \leq |F| - 1 \forall F \in \mathcal{F}\},$$

where $\mathcal{F} = \{F \subseteq V(G) : F \subseteq C, C \text{ induced cycle of } G, |F| \text{ odd}\}$.

For a matroid $M = (E,B)$, a set $C \subseteq E$ is a cycle if $C = \emptyset$ or $C$ is a disjoint union of circuits. The cycle polytope $C(M)$ of $M$ is the convex hull of the characteristic vectors of its cycles $\mathbb{H}$. Cycle polytopes can be seen as a generalization of cut polytopes. Indeed, it can be shown that if $M$ is cographic, i.e. it is the dual of the forest matroid of some graph $G$, then the cycles of $M$ correspond to the cuts of $G$, hence $C(M) = \text{CUT}(G)$. The cycle polytope $C(M)$ is given by the convex hull of the characteristic vectors of its cycles, and it is a generalization of the cut polytope $\text{CUT}(G)$ for a graph $G$ $\mathbb{H}$.

A matroid is called binary if it can be represented over the finite field $GF_2$. Given a matroid $M$, we denote by $M^*$ its dual matroid. $M$ is binary if and only if $M^*$ is binary. An element $e$ of a matroid is a chord of a circuit $C$ if $C$ is the symmetric difference of two circuits whose intersection is $e$. A chordless circuit is a circuit with no chords and the same definition can be applied to cocircuits, that are circuits in the dual matroid. $F_2^*$ denotes the dual of the Fano matroid; $R_{10}$ is a binary matroid associated with the $5 \times 10$ matrix whose columns are the $10$ 0/1 vectors with 3 ones and 2 zeros; $M_{K_5}^*$ is the dual of the forest matroid of $K_5$.

In this section we prove Conjecture $\mathbb{H}$ for the cycle polytope $C(M)$ of the binary matroids $M$ that have no minor isomorphic to $F_2^*$, $R_{10}$, $M_{K_5}^*$ and are 2-level. When those minors are forbidden, a complete linear description of the associated polytope is known (see $\mathbb{H}$). This class includes all cut polytopes that are 2-level, and has been characterized in $\mathbb{H}$:
Theorem 35. Let $M$ be a binary matroid with no minor isomorphic to $F_7^*$, $R_{10}$, $M_{K_5}^*$. Then $C(M)$ is 2-level if and only if $M$ has no chordless cocircuit of length at least 5.

Corollary 36. The polytope $\text{CUT}(G)$ is 2-level if and only if $G$ has no minor isomorphic to $K_5$ and no induced cycle of length at least 5.

Recall that the cycle space of graph $G$ is the set of its Eulerian subgraphs (subgraphs where all vertices have even degree), and it is known (see for instance [14]) to have a vector space structure over the field $\mathbb{Z}_2$. This statement and one of its proofs easily generalizes to the cycle space (the set of all cycles) of binary matroids. We report the proof for completeness.

Lemma 37. Let $M$ be a binary matroid with $d$ elements and rank $r$. Then the cycles of $M$ form a vector space $C$ over $\mathbb{Z}_2$ with the operation of symmetric difference as sum. Moreover, $C$ has dimension $d - r$.

Proof. That $C$ is a vector space can be easily verified using the fact that $C$ is closed under taking symmetric difference. This immediately derives from a characterization of binary matroids that can be found in [21]. Theorem 9.1.2: $M$ is binary if and only if the symmetric difference of any set of circuits is a disjoint union of circuits. We will now give a basis for $C$ of size $d - r$. The construction is analogous to the construction of a fundamental cycle basis in the cycle space of a graph. Consider a basis $B$ of $M$. For any $e \in E \setminus B$, let $C_e$ denote the unique circuit contained in $B + e$ (note that $e \in C_e$). Since $|B| = r$, we have a family $B_C = \{C_{e_1}, \ldots, C_{e_{d-r}}\}$ of the desired size. Note that the $C_e$'s are all linearly independent: indeed, $C_e$ cannot be expressed as symmetric difference of other members of $B_C$ since it is the only one containing $e$. We are left to show that $B_C$ generates $C$. Let $C \in C$, $C \neq \emptyset$, and let $\{e_1, \ldots, e_{d-r}\} \cap C = e_{i_1}, \ldots, e_{i_k}$ for some $k \geq 1$ (indeed, $C \notin B$). Consider now $D = C \Delta C_{e_{i_1}} \Delta \ldots \Delta C_{e_{i_k}}$. $D$ is a cycle, however one can see that it is contained in $B$: for each $e \in E \setminus B$, if $e \in C$ then $e$ appears exactly twice in the expression of $D$, hence $e \notin D$; if $e \notin C$, $e$ does not appear in the expression at all. This implies that $D = \emptyset$, which is equivalent to $C = C_{e_{i_1}} \Delta \ldots \Delta C_{e_{i_k}}$. 

Corollary 38. Let $M$ be a binary matroid with $d$ elements and rank $r$. Then $M$ has exactly $2^{d-r}$ cycles.

The only missing ingredient is a description of the facets of the cycle polytope for the class of our interest, which extends the description of the cut polytope given in [5].

Theorem 39. [11] Let $M$ be a binary matroid, and let $\mathcal{C}$ be its family of chordless cocircuits. Then $M$ has no minor isomorphic to $F_7^*$, $R_{10}$, $M_{K_5}^*$ if and only if

$$C(M) = \{ x \in [0, 1]^E : x(F) - x(C \setminus F) \leq |F| - 1 \text{ for } C \in \mathcal{C}, F \subseteq C, |F| \text{ odd} \}.$$ 

Theorem 40. Let $M$ be a binary matroid with no minor isomorphic to $F_7^*$, $R_{10}$, $M_{K_5}^*$ such that $C(M)$ is 2-level. Then $C(M)$ satisfies Conjecture [7].

Proof. As remarked in [11] and [10], the following equations are valid for $C(M)$: a) $x_e = 0$, for $e$ coloop of $M$; and b) $x_e - x_f = 0$, for $\{e, f\}$ cocircuit of $M$.

The first equation is due to the fact that a coloop cannot be contained in a cycle, and the second to the fact that circuits and cocircuits have even intersection in binary matroids.
A consequence of this is that we can delete all coloops and contract e for any cocircuit \( \{c, f\} \) without changing the cycle polytope: for simplicity we will just assume that \( M \) has no coloops and no cocircuit of length 2. In this case \( C(M) \) has full dimension \( d = |E| \). Let \( r \) be the rank of \( M \). Corollary 38 implies that \( C(M) \) has \( 2^{d-r} \) vertices. Let now \( T \) be the number of cotriangles (i.e., cocircuits of length 3) in \( M \), and \( S \) the number of cocircuits of length 4 in \( M \). Thanks to Theorem 39 and to the fact that \( M \) has no chordless cocircuit of length at least 5, we have that \( C(M) \) has at most \( 2d + 4T + 8S \) facets. Hence the bound we need to show is:

\[
2^{d-r}(2d + 4T + 8S) \leq 2d^{d+1},
\]

which is equivalent to \( 2T + 4S \leq d(2^r - 1) \).

Since the cocircuits of \( M \) are circuits in the binary matroid \( M^* \), whose rank is \( d-r \), we can apply Corollary 38 to get \( T + S \leq 2^r - 1 \), where the \(-1\) comes from the fact that we do not count the empty set. Hence, if \( d \geq 4 \),

\[
2T + 4S \leq 4(T + S) \leq d(2^r - 1).
\]

The bound is loose for \( d \geq 5 \). The cases with \( d \leq 4 \) can be easily verified, the only tight examples being affinely isomorphic to cubes and cross-polytopes.

Corollary 41. 2-level cut polytopes satisfy Conjecture 7.

6 Conclusions

In this paper, we provided a thorough analysis of 2-level polytopes coming from combinatorial settings. We hope that the reader shares with us the opinion that those polytope are relevant for the optimization community, and the 2-levelness property seem to be strong enough to leave hope for deep theorems on their structure. While we proved Conjecture 1 for all 2-level polytopes we could characterize, it is open for the general case. Whether some techniques and ideas introduced in this paper can be extended to attack it also remains open. Here, we would like to discuss a different issue stemming from Conjecture 1: is 2-levelness the “right” assumption for proving \( f_{d-1}(P)f_0(P) \leq d2^{d+1} \), or is this bound valid for a much more general class of 0/1 polytopes? We now provide some examples of “well-behaved” 0/1 polytopes that do not verify Conjecture 1.

Example 3 (Forest polytope of \( K_{2,n} \)). Let \( P \) be the forest polytope of \( K_{2,n} \). Note that \( P \) has dimension \( d = 2n \). Conjecture 1 implies an upper bound of \( n2^{2(n+1)} = O(4^\varepsilon n) \) for \( f_0(P)f_{d-1}(P) \), for any \( \varepsilon > 0 \). Each subgraph of \( K_{2,n} \) that takes, for each node \( v \) of degree 2, at most one edge incident to \( v \), is a forest. Those graphs are \( 3^\text{a} \). Moreover, each induced subgraph of \( K_{2,n} \) that takes the nodes of degree \( n \) plus at least 2 other nodes is 2-connected, hence it induces a (distinct) facet of \( P \). Those are \( 2^n - (n + 1) \). In total \( f_0(P)f_{d-1}(P) = \Omega(6^n) \).

Example 4 (Spanning tree polytope of the skeleton of the 4-dimensional cube). Let \( G \) be the skeleton of the 4-dimensional cube, and \( P \) the associated spanning tree polytope. Through extensive computation\(^7\), we verified that \( f_0(P)f_{d-1}(P) \geq 1.603 \cdot 10^{11} \), while the upper bound from Conjecture 1 is \( \approx 1.331 \cdot 10^{11} \).

\(^7\) We computed the number of spanning trees of \( G \) using the well known Kirchhoff’s matrix tree theorem \( \text{[4]} \). The facets of a 2-connected graph are the 2-connected, induced subgraphs
Example 5 (3-level min up/down polytopes). Fix $d \geq 3$. A 0/1 vector $x \in \{0, 1\}^d$ is “bad” if there are indices $0 < i < j < d$ such that $x_i = x_{j+1} = 1$ and $x_{i+1} = x_j = 0$. In other words, when seen as a bit-string, $x$ is bad if it contains two or more separate blocks of 1’s. Let $P \subset \mathbb{R}^d$ be the convex hull of all 0/1 vectors that are not bad: this is a min up/down polytope, as defined in [18], with parameters $\ell_1 = 1$ and $\ell_2 = d - 1$.

Each non-zero vertex $x$ in $P$ contains exactly one block of 1’s, thus it is uniquely described by two indices $0 \leq i < j \leq d$, such that $x_k = 1$ if $i < k < j$, and $x_k = 0$ otherwise. Therefore (counting also the zero vector), $P$ contains $\binom{d+1}{2} + 1$ vertices. On the other hand, from the facet characterization presented in [18] we know that
\[
P = \left\{ x \in \mathbb{R}_+^d : \sum_{j=1}^{k} (-1)^{j-1} x_{i_j} \leq 1, \text{ for any indices } 1 \leq i_1 < \cdots < i_k \leq d \text{ s.t. } k \text{ is odd} \right\},
\]
where all inequalities above are facet-defining. Moreover, since the polytope is full-dimensional (it contains the $d$-dimensional standard simplex) and no inequality is a multiple of another, they all define distinct facets. This means that there are $d$ facets coming from non-negativity constraints, and $2^{d-1}$ facets that are in one-to-one correspondence with odd subsets of the index set $[d]$. Hence, the total number of facets is $2^{d-1} + d$. It is easy to check that for $d \geq 3$ we have
\[
f_0(P)f_{d-1}(P) = \left( \binom{d+1}{2} + 1 \right) \cdot [2^{d-1} + d] > d 2^{d+1},
\]
thus the polytope does not satisfy Conjecture I. Note that $P$ is a 3-level polytope: for each facet $F$ of $P$, there exist two translates of the affine hull of $F$ such that all the vertices of $P$ lie either in $F$ or in one of those two translates.

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whose contraction is 2-connected, and we compute them by exhaustive search. The Matlab code can be found at: [http://disopt.epfl.ch/files/content/sites/disopt/files/users/249959/flacets.zip](http://disopt.epfl.ch/files/content/sites/disopt/files/users/249959/flacets.zip)

Recall that the min up/down polytope is 2-level precisely when its parameters $\ell_1$ and $\ell_2$ are equal, and in that case the polytope satisfies Conjecture I see Lemma 13.
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