FREE ARAKI-WOODS FACTORS
AND CONNES’ BICENTRALIZER PROBLEM

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Abstract. We show that for any type III1 free Araki-Woods factor $M = \Gamma(H(\mathbf{R}), U)$, the bicentralizer of the free quasi-free state $\varphi_U$ is trivial. Using Haagerup’s Theorem, it follows that there always exists a faithful normal state $\psi$ on $M$ such that $(M^\psi)' \cap M = C$.

1. Introduction

Let $M$ be a separable type III1 factor and let $\varphi$ be a faithful normal (f.n.) state on $M$. For any $x, y \in M$, set $[x, y] = xy - yx$ and $[x, \varphi] = x\varphi - \varphi x$. The asymptotic centralizer of $\varphi$ is defined by

$$AC(\varphi) := \{ (x_n) \in \ell^\infty(\mathbb{N}, M) : \|[x_n, \varphi]\| \to 0 \}.$$ 

Note that $AC(\varphi)$ is a unital $C^*$-subalgebra of $\ell^\infty(\mathbb{N}, M)$. The bicentralizer of $\varphi$ is defined by

$$AB(\varphi) := \{ a \in M : [a, x_n] \to 0 \text{ ultrastrongly, } \forall (x_n) \in AC(\varphi) \}.$$ 

It is well-known that $AB(\varphi)$ is a von Neumann subalgebra of $M$, globally invariant under the modular group $(\sigma^\varphi_t)$. Moreover, $AB(\varphi) \subset (M^\varphi)' \cap M$. If $AB(\varphi) = C$, it follows from the Connes-Størmer Transitivity Theorem ([5]) that $AB(\psi) = C$ for any faithful normal state $\psi$ on $M$. We shall say in this case that $M$ has trivial bicentralizer. Connes conjectured that any separable type III1 factor should have trivial bicentralizer. If there exists a faithful normal state $\varphi$ on $M$ such that $(M^\varphi)' \cap M = C$, then $M$ has trivial bicentralizer. Haagerup proved in [7] that the converse holds true. Haagerup’s Theorem leads to the uniqueness of the amenable type III1 factor (see [2]). The following type III1 factors are known to have trivial bicentralizer:

1. The unique amenable III1 factor (Haagerup, [7]).
2. Full factors that have almost periodic states (Connes, [3]).
3. Free products $(M_1, \varphi_1) * (M_2, \varphi_2)$ such that the centralizers $M_1^\varphi_1$ have enough unitaries (Barnett, [1]).

In this paper, we show that the bicentralizer is trivial for a large class of type III1 factors, namely the free Araki-Woods factors of Shlyakhtenko ([14]). We briefly recall the construction here; see Section 2 for more details. To each real separable
Hilbert space $H_R$ together with an orthogonal representation $(U_t)$ of $R$ on $H_R$, one can associate a von Neumann algebra denoted by $\Gamma(H_R, U_t)''$, called the free Araki-Woods von Neumann algebra. This is the free analog of the factors coming from the CAR relations. The von Neumann algebra $\Gamma(H_R, U_t)''$ comes equipped with a unique free quasi-free state denoted by $\varphi_U$, which is always normal and faithful on $\Gamma(H_R, U_t)''$. If $\dim H_R \geq 2$, then $\Gamma(H_R, U_t)''$ is a full factor. It is of type $\text{III}_1$ when $(U_t)$ is non-periodic and non-trivial. If the representation $(U_t)$ is almost periodic, then $\varphi_U$ is an almost periodic state and it follows from [14] that the relative commutant of the centralizer of the free quasi-free state is trivial, i.e. if $\mathcal{M} := \Gamma(H_R, U_t)'', then $(\mathcal{M}''_\varphi)' \cap \mathcal{M} = C$. In the almost periodic case, results in [6] yield $\mathcal{M}''_\varphi \approx L(F_\infty)$.

When the representation $(U_t)$ has no eigenvectors (e.g. $U_t = \lambda_t$, the left regular representation of $R$ on $L^2(R, R)$), then the centralizer $\mathcal{M}''_\varphi$ is trivial. It was unknown in general whether or not $\Gamma(H_R, U_t)''$ has trivial bicentralizer. Even though the centralizer of the free quasi-free state $\varphi_U$ may be trivial, we will show that the bicentralizer of $\varphi_U$ is always trivial. The main result of this paper is the following:

**Theorem.** Let $\mathcal{M} := \Gamma(H_R, U_t)''$ be a free Araki-Woods factor of type $\text{III}_1$. Denote by $\varphi_U$ the free quasi-free state. Then $AB(\varphi_U) = C$. Consequently, there always exists a faithful normal state $\psi$ on $\mathcal{M}$ such that $(\mathcal{M}''_\psi)' \cap \mathcal{M} = C$.

2. Preliminaries

2.1. Preliminaries on spectral analysis. We shall need a few definitions and results from the spectral theory of abelian automorphism groups. Let $(\alpha_t)$ be an ultraweakly continuous one-parameter automorphism group on a von Neumann algebra $\mathcal{M}$. For $f \in L^1(\mathcal{M})$ and $x \in \mathcal{M}$, set

$$\alpha_f(x) = \int_{-\infty}^{+\infty} f(t)\alpha_t(x) dt.$$  

The $\alpha$-spectrum $\text{Sp}_\alpha(x)$ of $x \in \mathcal{M}$ is defined as the set of characters $\gamma \in \hat{\mathcal{R}}$ for which $\hat{f}(\gamma) = 0$, for all $f \in L^1(\mathcal{M})$ satisfying $\alpha_f(x) = 0$. We shall identify $\hat{\mathcal{R}}$ with $\mathcal{R}$ in the usual way such that

$$\hat{f}(\gamma) = \int_{-\infty}^{+\infty} e^{\gamma t} f(t) dt, \quad \forall \gamma \in \mathcal{R}, \forall f \in L^1(\mathcal{R}).$$

For $z \in \mathcal{C}$, denote by $\Im(z)$ its imaginary part.

**Lemma 2.1 ([7]).** Let $\mathcal{M}$ and $(\alpha_t)$ be as above. Let $x \in \mathcal{M}$ and $\delta > 0$. If the function $t \mapsto \alpha_t(x)$ can be extended to an entire (analytic) $\mathcal{M}$-valued function such that

$$\|\alpha_z(x)\| \leq Ce^{\delta|\Im(z)|}, \quad \forall z \in \mathcal{C},$$

for some constant $C > 0$, then $\text{Sp}_\alpha(x) \subset [-\delta, \delta]$.

Let $\varphi$ be a f.n. state on a von Neumann algebra $\mathcal{M}$. Denote by $(\sigma_x^\varphi)$ the modular group on $\mathcal{M}$ of the state $\varphi$. Denote by $L^2(\mathcal{M}, \varphi)$ the $L^2$-space associated with $\varphi$ and by $\xi_\varphi$ the canonical cyclic separating vector. We shall write $\|x\|_\varphi = \varphi(x^* x)^{1/2}$, for any $x \in \mathcal{M}$. On bounded subsets of $\mathcal{M}$, the topology given by the norm $\| \cdot \|_\varphi$ coincides with the strong operator topology. Recall that $S_\varphi : x\xi_\varphi \mapsto x^* \xi_\varphi$ is a closable (densely defined) operator on $L^2(\mathcal{M}, \varphi)$. Denote by $\mathcal{S}_\varphi$ its closure and
Thus, by approximating \( S_\phi = J_\phi \Delta^{1/2}_\phi \) for its polar decomposition. Note that \( L^2(\mathcal{M}, \phi) \) is naturally endowed with an \( \mathcal{M} \)-\( \mathcal{M} \) bimodule structure defined as follows:

\[
x \cdot \xi := x\xi,
\xi \cdot x := J_\phi x^* J_\phi \xi, \quad \forall x \in \mathcal{M}, \forall \xi \in L^2(\mathcal{M}, \phi).
\]

We shall denote \( x \cdot \xi \) and \( \xi \cdot x \) simply by \( x\xi \) and \( \xi x \). The next lemma is well-known, but we give a proof for the reader’s convenience.

**Lemma 2.2.** Let \( \mathcal{M} \) and \( \phi \) be as above. Let \( x \in \mathcal{M} \) and \( 0 < \delta < 1 \). Assume that \( \text{Sp}_{\sigma_\phi}(x) \subset [-\delta, \delta] \). Then \( \|x\xi_\phi - \xi_\phi x\| \leq \delta\|x\|_\phi \).

**Proof.** Let \( x \in \mathcal{M} \) and \( 0 < \delta < 1 \) be such that \( \text{Sp}_{\sigma_\phi}(x) \subset [-\delta, \delta] \). Let \( f \in L^1(\mathbb{R}) \) be such that the Fourier transform \( \hat{f} \) vanishes on \([-\delta, \delta]\). Since \( \text{Sp}_{\sigma_\phi}(\sigma_\phi^f(x)) \subset \text{Sp}_{\sigma_\phi}(x) \cap \text{support}(\hat{f}) = \emptyset \) (see [2]), it follows that \( \sigma_\phi^f(x) = 0 \). We have

\[
\hat{f}(\log \Delta_\phi)x \xi_\phi = \int_{-\infty}^{t} f(t) \Delta^{1/2}_\phi x \xi_\phi \; dt
= \int_{-\infty}^{t} f(t) \sigma_\phi^{*} x \xi_\phi \; dt
= \sigma_\phi^{*} x \xi_\phi
= 0.
\]

Thus, by approximating \( 1_{\mathbb{R}\setminus[-\delta, \delta]} \) by such functions \( \hat{f} \), we get

\[1_{\mathbb{R}\setminus[-\delta, \delta]}(\log \Delta_\phi)x \xi_\phi = 0;\]
i.e. \( x \xi_\phi \) is in the spectral subspace of \( \log \Delta_\phi \) corresponding to the interval \([-\delta, \delta]\). Notice that

\[\xi_\phi x = J_\phi x^* J_\phi \xi_\phi = J_\phi x^* \xi_\phi = J_\phi S_\phi x \xi_\phi = \Delta^{1/2}_\phi x \xi_\phi.\]

Clearly, \( \sup\{|e^{t/2} - 1| : t \in [-\delta, \delta]\} = e^{\delta/2} - 1 \). Moreover, one can see that the operator \((1 - \Delta^{1/2}_\phi)1_{[-\delta, \delta]}(\log \Delta_\phi)\) is bounded and, to be precise,

\[\|(1 - \Delta^{1/2}_\phi)1_{[-\delta, \delta]}(\log \Delta_\phi)\| \leq e^{\delta/2} - 1 \leq \delta,\]
since \( 0 < \delta < 1 \). Thus, we get

\[\|x \xi_\phi - \xi_\phi x\| = \|(1 - \Delta^{1/2}_\phi) x \xi_\phi\|
= \|(1 - \Delta^{1/2}_\phi)1_{[-\delta, \delta]}(\log \Delta_\phi)x \xi_\phi\|
\leq \|(1 - \Delta^{1/2}_\phi)1_{[-\delta, \delta]}(\log \Delta_\phi)\| \|x \xi_\phi\|
\leq \delta\|x\|_\phi.\]

\[\square\]

**Lemma 2.3 ([2]).** Let \( \mathcal{M} \) and \( \phi \) be as above. Let \( (x_n) \in \ell^\infty(\mathbb{N}, \mathcal{M}) \). Then

\[\lim_n \|x_n \xi_\phi - \xi_\phi x_n\| = 0 \iff \lim_n \|x_n \phi - \phi x_n\| = 0.\]
2.2. Preliminaries on Shlyakhtenko’s free Araki-Woods factors. Recall now the construction of the free Araki-Woods factors due to Shlyakhtenko ([14]). Let $H_{\mathbb{R}}$ be a real separable Hilbert space and let $(U_t)$ be an orthogonal representation of $\mathbb{R}$ on $H_{\mathbb{R}}$. Let $H = H_{\mathbb{R}} \otimes _{\mathbb{R}} \mathbb{C}$ be the complexified Hilbert space. Let $J$ be the canonical anti-unitary involution on $H$ defined by

$$J(\xi + i\eta) = \xi - i\eta, \quad \forall \xi, \eta \in H_{\mathbb{R}}.$$ If $A$ is the infinitesimal generator of $(U_t)$ on $H$, we recall that $j : H_{\mathbb{R}} \to H$ defined by $j(\zeta) = (2\pi i)^{1/2}\zeta$ is an isometric embedding of $H_{\mathbb{R}}$ into $H$. Moreover $JAJ = A^{-1}$ and $JA^t = A^tJ$, for every $t \in \mathbb{R}$. Let $K_{\mathbb{R}} = j(H_{\mathbb{R}})$. It is easy to see that $K_{\mathbb{R}} \cap iK_{\mathbb{R}} = \{0\}$ and that $K_{\mathbb{R}} + iK_{\mathbb{R}}$ is dense in $H$. Write $T = JA^{-1/2}$. Then $T$ is an anti-linear closed invertible operator on $H$ satisfying $T = T^{-1}$. Such an operator is called an involution on $H$. Moreover, $K_{\mathbb{R}} = \{\xi \in \text{dom}(T) : T\xi = \xi\}$.

We introduce the full Fock space of $H$:

$$\mathcal{F}(H) = C\Omega \oplus \bigoplus _{n=1}^{\infty} H_{\mathbb{R}}^{\otimes n}.$$ The unit vector $\Omega$ is called the vacuum vector. For any $\xi \in H$, define the left creation operator

$$\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) : \left\{ \right. \begin{array}{l} \ell(\xi)\Omega = \xi, \\
\ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{array}$$

We have $\|\ell(\xi)\| = \|\xi\|$ and that $\ell(\xi)$ is an isometry if $\|\xi\| = 1$. For any $\xi \in H$, we denote by $s(\xi)$ the real part of $\ell(\xi)$ given by

$$s(\xi) = \frac{\ell(\xi) + \ell(\xi)^*}{2}.$$ A crucial result of Voiculescu [16] claims that the distribution of the operator $s(\xi)$ with respect to the vacuum vector state $\varphi(x) = \langle x\Omega, \Omega \rangle$ is the semicircular law of Wigner supported on the interval $[-\|\xi\|, \|\xi\|]$.

**Definition 2.4** (Shlyakhtenko, [14]). Let $(U_t)$ be an orthogonal representation of $\mathbb{R}$ on the real Hilbert space $H_{\mathbb{R}}$. The free Araki-Woods von Neumann algebra associated with $(H_{\mathbb{R}}, U_t)$, denoted by $\Gamma(H_{\mathbb{R}}, U_t)^{\prime\prime}$, is defined by

$$\Gamma(H_{\mathbb{R}}, U_t)^{\prime\prime} := \{s(\xi) : \xi \in K_{\mathbb{R}}\}^{\prime\prime}.$$ The vector state $\varphi_U(x) = \langle x\Omega, \Omega \rangle$ is called the free quasi-free state and is faithful on $\Gamma(H_{\mathbb{R}}, U_t)^{\prime\prime}$. Let $\xi, \eta \in K_{\mathbb{R}}$ and write $\zeta = \xi + i\eta$. We have

$$2s(\xi) + 2is(\eta) = \ell(\zeta) + \ell(T\zeta)^*.$$ Thus, $\Gamma(H_{\mathbb{R}}, U_t)^{\prime\prime}$ is generated as a von Neumann algebra by the operators of the form $\ell(\zeta) + \ell(T\zeta)^*$ where $\zeta \in \text{dom}(T)$. Note that the modular group $(\sigma^U_t)$ of the free quasi-free state $\varphi_U$ is given by $\sigma^U_t = \text{Ad}(\mathcal{F}(U_t))$, where $\mathcal{F}(U_t) = \text{id} \oplus \bigoplus _{n=1}^{\infty} U_{\mathbb{R}}^{\otimes n}$. In particular, it satisfies

$$\sigma^U_t(\ell(\zeta) + \ell(T\zeta)^*) = \ell(U_t\zeta) + \ell(TU_t\zeta)^*, \quad \forall \zeta \in \text{dom}(T), \forall t \in \mathbb{R}.$$ The free Araki-Woods factors provided many new examples of full factors of type III [1] [2] [11]. We can summarize the general properties of the free Araki-Woods factors in the following theorem (see also [15]):
Theorem 2.5 (Shlyakhtenko, [11, 12, 13, 14]). Let \((U_i)\) be an orthogonal representation of \(\mathbb{R}\) on the real Hilbert space \(H_\mathbb{R}\) with \(\dim H_\mathbb{R} \geq 2\). Write \(\mathcal{M} := \Gamma(H_\mathbb{R}, U_i)''\).

1. \(\mathcal{M}\) is a full factor and Connes’ invariant \(\tau(\mathcal{M})\) is the weakest topology on \(\mathbb{R}\) that makes the map \(t \mapsto U_t\) strongly continuous.
2. \(\mathcal{M}\) is of type \(\text{II}_1\) iff \(U_t = \text{id}\), for every \(t \in \mathbb{R}\).
3. \(\mathcal{M}\) is of type \(\text{III}_\lambda\) (\(0 < \lambda < 1\)) iff \((U_t)\) is periodic of period \(\frac{2\pi}{|\log \lambda|}\).
4. \(\mathcal{M}\) is of type \(\text{III}_1\) in the other cases.
5. The factor \(\mathcal{M}\) has almost periodic states iff \((U_t)\) is almost periodic.

Let \(H_\mathbb{R} = \mathbb{R}^2\) and \(0 < \lambda < 1\). Let

\[
U_t^\lambda = \begin{pmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{pmatrix}.
\]

Notation 2.6 ([14]). Write \((T_\lambda, \varphi_\lambda) := (\Gamma(H_\mathbb{R}, U_t)'', \varphi_U)\) where \(H_\mathbb{R} = \mathbb{R}^2\) and \((U_t)\) is given by equation (1).

Using a powerful tool called the matricial model, Shlyakhtenko was able to prove the following isomorphism:

\[
(T_\lambda, \varphi_\lambda) \cong (\mathcal{B}(\ell^2(\mathbb{N})), \psi_\lambda) \ast (L^\infty[-1, 1], \mu),
\]

where \(\psi_\lambda(e_{ij}) = \delta_{ij} \lambda^{|1 - \lambda|}\), \(i, j \in \mathbb{N}\), and \(\mu\) is a non-atomic measure on \([-1, 1]\). The notation \(\cong\) means a state-preserving isomorphism. He also proved that \((T_\lambda, \varphi_\lambda)\) has the free absorption property, namely, that

\[
(T_\lambda, \varphi_\lambda) \ast L(F_\infty) \cong (T_\lambda, \varphi_\lambda).
\]

3. The main result

3.1. Technical lemmas. As we said before, the centralizer of the free quasi-free state may be trivial; this is the case for instance when the orthogonal representation \((U_t)\) on \(H_\mathbb{R}\) has no eigenvectors. Nevertheless, the following lemma shows that for any free Araki-Woods von Neumann algebra, there exists a non-trivial sequence of unitaries \((u_n)\) in the asymptotic centralizer of the free quasi-free state \(\varphi_U\).

Lemma 3.1 (Vaes, [15]). Let \(\mathcal{M} := \Gamma(H_\mathbb{R}, U_t)''\) be a free Araki-Woods von Neumann algebra. Denote by \(\varphi\) the free quasi-free state and by \((\sigma_t)\) the modular group of the state \(\varphi\). Then there exists a sequence of unitaries \((u_n)\) in \(\mathcal{M}\), entire (analytic) w.r.t. \((\sigma_t)\), such that

1. \(\|\sigma_z(u_n) - u_n\| \to 0\) uniformly on compact sets of \(\mathbb{C}\),
2. \(\varphi(u_n) \to 0\),
3. \((u_n) \in AC(\varphi)\).

Proof. This lemma, with the exception of item (3), is Vaes’ result (see Lemma 4.3 in [15]). Item (3) was not observed by Vaes but is immediate from the construction using Lemmas 2.1, 2.2 and 2.3. \(\square\)

The following lemma is a generalization of Barnett’s lemma (see [1]), which was itself a generalization of Murray and Neumann’s 14ε lemma.

Lemma 3.2 (Vaes, [15]). For \(i = 1, 2\), let \((\mathcal{M}_i, \varphi_i)\) be a von Neumann algebra endowed with an f.n. state. Denote by \((\mathcal{M}, \varphi) = (\mathcal{M}_1, \varphi_1) \ast (\mathcal{M}_2, \varphi_2)\) the free...
product. Let \( a \in \mathcal{M}_1 \) and \( b, c \in \mathcal{M}_2 \). Assume that \( a, b, c \) belong to the domain of \( \sigma_{i/2}^\varphi \). Then, for every \( x \in \mathcal{M} \),

\[
\|x - \varphi(x)1\|_\varphi \leq \mathcal{E}(a, b, c) \max \{ \|x, a\|_\varphi, \|x, b\|_\varphi, \|x, c\|_\varphi \} + \mathcal{F}(a, b, c)\|x\|_\varphi
\]

where

\[
\begin{align*}
\mathcal{E}(a, b, c) &= 6\|a\|^3 + 4\|b\|^3 + 4\|c\|^3, \\
\mathcal{F}(a, b, c) &= 3\mathcal{C}(a) + 2\mathcal{C}(b) + 2\mathcal{C}(c) + 12\|\varphi(cb^*)\|\|cb^*\|, \\
\mathcal{C}(a) &= 2\|a\|^2\|\sigma_{i/2}^\varphi(a) - a\| + 2\|a\|^2\|a^*a - 1\| \\
&\quad + 3(1 + \|a\|^2)\|aa^* - 1\| + 6\|\varphi(a)\|\|a\|.
\end{align*}
\]

3.2. Proof of the theorem. Let \( \mathcal{M} := \Gamma(H_\mathbb{R}, U_\varphi)'' \) be a free Araki-Woods factor of type III\( \lambda \) and denote by \( \varphi \) the free quasi-free state. We recall that such a factor can always be written as the free product of three free Araki-Woods von Neumann algebras (see the proof of Theorem 2.7 in [15]):

\[
(\mathcal{M}, \varphi) \cong (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2) * (\mathcal{M}_3, \varphi_3).
\]

Notice that \( \sigma_{-i/2}^\varphi = \sigma_{i/2}^\varphi \sigma_{i/2}^\varphi \sigma_{i/2}^\varphi, \forall t \in \mathbb{R} \).

Thanks to Lemma 3.1, we may choose three sequences of unitaries \( (u_n^j) \) for \( j \in \{1, 2, 3\} \), such that \( u_n^j \in \mathcal{U}(\mathcal{M}_j) \) is analytic w.r.t. \( \sigma_{i/2}^\varphi \) and satisfies conditions (1) – (3) of Lemma 3.1 for all \( j \in \{1, 2, 3\} \). The way the sequence of unitaries \( (u_n^j) \) is constructed in Lemma 3.1 (see Lemma 4.3 in [15]) shows that conditions (1) – (3) are satisfied for the state \( \varphi \); i.e. the sequence of unitaries \( (u_n^j) \) in \( \mathcal{M}_j \) satisfies, for every \( j \in \{1, 2, 3\} \),

\[
\begin{align*}
(1) \quad \|\sigma_{i/2}^\varphi(u_n^j) - u_n^j\| &\to 0 \text{ uniformly on compact sets of } \mathbb{C}, \\
(2) \quad \varphi(u_n^j) &\to 0, \\
(3) \quad \|u_n^j, \varphi\| &\to 0.
\end{align*}
\]

Moreover, by freeness, \( \varphi(u_n^j(u_n^j)^*) = \varphi(u_n^j)^*\varphi(u_n^j) \to 0 \).

Assume that \( a \in AB(\varphi) \). Fix \( \varepsilon > 0 \). Since \( (u_n^j) \in AC(\varphi) \), it follows that \( [a, u_n^j] \to 0 \) ultrastrongly for any \( j \in \{1, 2, 3\} \), and thus we may choose \( n \in \mathbb{N} \) large enough such that

\[
\begin{align*}
\|a, u_n^j\|_\varphi &\leq \varepsilon/28, \quad \forall j \in \{1, 2, 3\}, \\
\mathcal{F}(u_n^1, u_n^2, u_n^3\|a\|_\varphi &\leq \varepsilon/2.
\end{align*}
\]

Thus, thanks to Lemma 3.2 we get \( \|a - \varphi(a)1\|_\varphi \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( a = \varphi(a)1 \). Thus \( AB(\varphi) = \mathbb{C} \), and we are done.

3.3. Final remark. Set \( \mathcal{M} := \Gamma(L^2(\mathbb{R}, \mathbb{R}), \lambda_\varphi)'' \), the free Araki-Woods factor associated with the left regular representation \( \lambda_\varphi \) of \( \mathbb{R} \) on the real Hilbert space \( L^2(\mathbb{R}, \mathbb{R}) \). Shlyakhtenko showed in [14] that the continuous core of \( \mathcal{M} \) is isomorphic to \( L(\mathcal{F}_\infty) \otimes \mathcal{B}(\ell^2) \) and that the dual action is precisely the one constructed by Rădulescu in [9]. As observed in [10], for any f.n. state \( \varphi \) on \( \mathcal{M} \), the centralizer \( \mathcal{M}^\varphi \) is amenable. Indeed, first we have

\[
\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \simeq L(\mathcal{F}_\infty) \otimes \mathcal{B}(\ell^2).
\]

Choose on the left-hand side of (2) a non-zero projection \( p \in L(\mathbb{R}) \) such that \( \text{Tr}(p) < +\infty \). We know that \( p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p \simeq L(\mathcal{F}_\infty) \) is solid by Ozawa’s result ([8]). Since \( L(\mathbb{R})p \) is diffuse in \( p(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p \), its relative commutant must be amenable.
In particular $\mathcal{M}^\varphi \otimes L(R)p$ is amenable. Thus, $\mathcal{M}^\varphi$ is amenable. Consequently, we obtain

**Corollary 3.3.** Let $\mathcal{M} := \Gamma(L^2(R,R),\lambda_t)^\prime\prime$. Then there exists an f.n. state $\psi$ on $\mathcal{M}$ such that $(\mathcal{M}^\psi)^\prime \cap \mathcal{M} = C$. Moreover, $\mathcal{M}^\psi$ is isomorphic to the unique hyperfinite $\text{II}_1$ factor.

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