On the metric dimension of corona product graphs

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Abstract

Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph $G$, the metric representation of a vertex $v$ of $G$ with respect to $S$ is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$, $i \in \{1, ..., k\}$ denotes the distance between $v$ and $v_i$. $S$ is a resolving set for $G$ if for every pair of vertices $u, v$ of $G$, $r(u|S) \neq r(v|S)$. The metric dimension of $G$, $\text{dim}(G)$, is the minimum cardinality of any resolving set for $G$. Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively. The corona product $G \odot H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. We give several results on the metric dimension of $G \odot^k H$. For instance, we show that given two connected graphs $G$ and $H$ of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively, if the diameter of $H$ is at most two, then $\text{dim}(G \odot^k H) = n_1(n_2 + 1)^{k-1}\text{dim}(H)$. Moreover, if $n_2 \geq 7$ and
the diameter of $H$ is greater than five or $H$ is a cycle graph, then 
\[ \dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}\dim(K_1 \odot H). \]

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## 1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [10] and Slater [19], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [3, 4, 5, 6, 7, 16, 18, 20]. Slater described the usefulness of these ideas into long range aids to navigation [19]. Also, these concepts have some applications in chemistry for representing chemical compounds [14, 15] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [17]. Other applications of this concept to navigation of robots in networks and other areas appear in [6, 12, 16]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [18], locating domination [11], resolving domination [1] and resolving partitions [5, 8, 9, 21]. In this article we study the metric dimension of corona product graphs.

We begin by giving some basic concepts and notations. Let $G = (V, E)$ be a simple graph of order $n = |V|$. Let $u, v \in V$ be two different vertices in $G$, the distance $d_G(u, v)$ between two vertices $u$ and $v$ of $G$ is the length of a shortest path between $u$ and $v$. If there is no ambiguity, we will use the notation $d(u, v)$ instead of $d_G(u, v)$. The diameter of $G$ is defined as 
\[ D(G) = \max_{u, v \in V}\{d(u, v)\}. \] 
Given $u, v \in V$, $u \sim v$ means that $u$ and $v$ are adjacent vertices. Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph $G$, the **metric representation** of a vertex $v \in V$ with respect to $S$ is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. We say that $S$ is a **resolving set** for $G$ if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The **metric dimension** of $G$ is the minimum cardinality of any resolving set for $G$, and it is denoted by $\dim(G)$.

Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively. The corona product $G \odot H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the
$i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of $G$ and by $H_i = (V_i, E_i)$ the copy of $H$ such that $v_i \sim v$ for every $v \in V_i$. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. We also note that the order of $G \odot^k H$ is $n_1(n_2 + 1)^k$.

2 Metric dimension of corona product graphs

We begin by presenting the following useful facts.

**Lemma 1.** Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let $H$ be a graph of order at least two. Let $H_i = (V_i, E_i)$ be the subgraph of $G \odot H$ corresponding to the $i^{th}$-copy of $H$.

(i) If $u, v \in V_i$, then $d_{G \odot H}(u, x) = d_{G \odot H}(v, x)$ for every vertex $x$ of $G \odot H$ not belonging to $V_i$.

(ii) If $S$ is a resolving set for $G \odot H$, then $V_i \cap S \neq \emptyset$ for every $i \in \{1, ..., n\}$.

(iii) If $S$ is a resolving set for $G \odot H$ of minimum cardinality, then $V \cap S = \emptyset$.

(iv) If $H$ is a connected graph and $S$ is a resolving set for $G \odot H$, then for every $i \in \{1, .., n\}$, $S \cap V_i$ is a resolving set for $H_i$.

**Proof.** (i) Let $y = v_i \in V$. The result directly follows from the fact that $d_{G \odot H}(u, x) = d_{G \odot H}(u, y) + d_{G \odot H}(y, x) = d_{G \odot H}(v, x) = d_{G \odot H}(v, x)$.

(ii) We suppose $V_i \cap S = \emptyset$ for some $i \in \{1, ..., n\}$. Let $x, y \in V_i$. By (i) we have $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$ for every vertex $u \in S$, which is a contradiction.

(iii) We will show that $S' = S - V$ is a resolving set for $G \odot H$. Now let $x, y$ be two different vertices of $G \odot H$. We have the following cases.

Case 1: $x, y \in V_i$. By (i) we conclude that there exist $v \in V_i \cap S'$ such that $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$.

Case 2: $x \in V_i$ and $y \in V_j, i \neq j$. Let $v \in V_i \cap S'$. Then we have $d_{G \odot H}(x, v) \leq 2 < 3 \leq d_{G \odot H}(y, v)$.

Case 3: $x, y \in V$. Let $x = v_i$ and let $v \in V_i \cap S'$. Then we have $d_{G \odot H}(x, v) = 1 < 1 + d_{G \odot H}(y, x) = d_{G \odot H}(y, v)$. 

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Case 4: \( x \in V_i \) and \( y \in V \). If \( x \sim y \), then \( y = v_i \). Let \( v_j \in V \), \( j \neq i \), and let \( v \in V_j \cap S' \). Then we have \( d_{G \circ H}(x, v) = 1 + d_{G \circ H}(y, v) > d_{G \circ H}(y, v) \). For \( x \not\sim y = v_i \) we take \( v \in V_i \cap S' \) and we obtain \( d_{G \circ H}(x, v) = d_{G \circ H}(x, y) + d_{G \circ H}(y, v) > d_{G \circ H}(y, v) \).

Therefore, \( S' \) is a resolving set for \( G \circ H \).

(iv) Let \( S_i = S \cap V_i \). For \( x \in S_i \) or \( y \in S_i \) the result is straightforward. We suppose \( x, y \in V_i - S_i \). Since \( S \) is a resolving set for \( G \circ H \), we have \( r(x|S) \neq r(y|S) \). By (i), \( d_{G \circ H}(x, u) = d_{G \circ H}(y, u) \) for every vertex \( u \) of \( G \circ H \) not belonging to \( V_i \). So, there exists \( v \in S_i \) such that \( d_{G \circ H}(x, v) \neq d_{G \circ H}(y, v) \). Thus, either \( (v \sim x \text{ and } v \not\sim y) \) or \( (v \not\sim x \text{ and } v \sim y) \). In the first case we have \( d_{G \circ H}(x, v) = d_{H_i}(x, v) = 1 \) and \( d_{G \circ H}(y, v) = 2 \leq d_{H_i}(y, v) \). The case \( v \not\sim x \) and \( v \sim y \) is analogous. Therefore, \( S_i \) is a resolving set for \( H_i \).

**Theorem 2.** Let \( G \) and \( H \) be two connected graphs of order \( n_1 \geq 2 \) and \( n_2 \geq 2 \), respectively. Then,

\[
\dim(G \circ^k H) \geq n_1(n_2 + 1)^{k-1}\dim(H).
\]

**Proof.** Let \( S \) be a resolving set of minimum cardinality in \( G \circ H \). From Lemma 1 (iii) we have that \( S \cap V = \emptyset \). Moreover, by Lemma 1 (ii) we have that for every \( i \in \{1, \ldots, n_1\} \) there exist a nonempty set \( S_i \subset V_i \) such that \( S = \bigcup_{i=1}^{n_1} S_i \). Now, by using Lemma 1 (iv) we have that \( S_i \) is a resolving set for \( H_i \). Hence, \( \dim(G \circ H) = |S| = \sum_{i=1}^{n_1} |S_i| \geq \sum_{i=1}^{n_1} \dim(H) = n_1 \dim(H) \). As a result, the lower bound follows. \( \square \)

**Theorem 3.** Let \( G \) be a connected graph of order \( n_1 \geq 2 \) and let \( H \) be a graph of order \( n_2 \geq 2 \). If \( D(H) \leq 2 \), then

\[
\dim(G \circ^k H) = n_1(n_2 + 1)^{k-1}\dim(H).
\]

**Proof.** Let \( S_i \subset V_i \) be a resolving set for \( H_i \) and let \( S = \bigcup_{i=1}^{n_1} S_i \). We will show that \( S \) is a resolving set for \( G \circ H \). Let us consider two different vertices \( x, y \) of \( G \circ H \). We have the following cases.

Case 1: \( x, y \in V_i \). Since \( D(H_i) \leq 2 \), we have that \( r(x|S_i) \neq r(y|S_i) \) leads to \( r(x|S) \neq r(y|S) \).

Case 2: \( x \in V_i \) and \( y \in V_j \), \( i \neq j \). Let \( v \in S_i \). Hence we have \( d(x, v) \leq 2 < 3 \leq d(y, v) \).

Case 3: \( x, y \in V \). Let \( x = v_i \). Then for every vertex \( v \in S_i \) we have \( d(x, v) = 1 < d(y, x) + 1 = d(y, v) \).
Case 4: \( x \in V_i \) and \( y \in V \). If \( x \sim y \), then let \( v \in S_j \), for some \( j \neq i \). So we have \( d(x, v) = 1 + d(y, v) > d(y, v) \). Moreover, if \( x \not\sim y = v_j \), for \( v \in S_j \) we have \( d(x, v) = d(x, y) + d(y, v) > d(y, v) \).

Thus, for every different vertices \( x, y \) of \( G \odot H \), we have \( r(x|S) \neq r(y|S) \), as a consequence, \( \text{dim}(G \odot H) \leq n_1 \text{dim}(H) \). Therefore, we have \( \text{dim}(G \odot H) \leq n_1(n_2 + 1)^{k-1}\text{dim}(H) \). By Theorem 2 we conclude the proof. \( \square \)

In order to show a consequence of the above theorem we present the following well known result, where \( K_t \) denotes a complete graph of order \( t \), \( K_{s,t} \) denotes a complete bipartite graph of order \( s + t \) and \( N_t \) denotes an empty graph of order \( t \).

**Lemma 4.** [6] Let \( G \) be a connected graph of order \( n \geq 4 \). Then \( \text{dim}(G) = n - 2 \) if and only if \( G = K_{s,t}, \ (s, t \geq 1) \), \( G = K_s + N_t, \ (s \geq 1, t \geq 2) \), or \( G = K_s + (K_1 \cup K_t), \ (s, t \geq 1) \).

**Corollary 5.** Let \( G \) be a connected graph of order \( n_1 \geq 2 \) and let \( H \) be a graph of order \( n_2 \geq 4 \) and diameter \( D(H) \leq 2 \). Then

\[ \text{dim}(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 2) \]

if and only if \( H = K_{s,t}, \ (s, t \geq 1) \); \( H = K_s + N_t, \ (s \geq 1, t \geq 2) \), or \( H = K_s + (K_1 \cup K_t), \ (s, t \geq 1) \).

We recall that the wheel graph of order \( n+1 \) is defined as \( W_{1,n} = K_1 \odot C_n \), where \( K_1 \) is the singleton graph and \( C_n \) is the cycle graph of order \( n \). The metric dimension of the wheel \( W_{1,n} \) was obtained by Buczkowski et. al. in [2].

**Remark 6.** [2] Let \( W_{1,n} \) be a wheel graph. Then

\[
\text{dim}(W_{1,n}) = \begin{cases} 
3 & \text{for } n = 3, 6, \\
2 & \text{for } n = 4, 5, \\
\left\lfloor \frac{2n+2}{5} \right\rfloor & \text{otherwise.}
\end{cases}
\]

The fan graph \( F_{n_1,n_2} \) is defined as the graph join \( N_{n_1} + P_{n_2} \), where \( N_{n_1} \) is the empty graph of order \( n_1 \) and \( P_{n_2} \) is the path graph of order \( n_2 \). The case \( n_1 = 1 \) corresponds to the usual fan graphs. Notice that, for the metric dimension of fan graphs, it is possible to find an equivalent result to Remark 6 which was obtained by Caceres et. al. in [4].
Remark 7. [4] Let $F_{1,n}$ be a fan graph. Then

$$\dim(F_{1,n}) = \begin{cases} 1 & \text{for } n = 1, \\ 2 & \text{for } n = 2, 3, \\ 3 & \text{for } n = 6, \\ \left\lceil \frac{2n^2+2}{5} \right\rceil & \text{otherwise.} \end{cases}$$

As a particular case of the Theorem 3 we obtain the following results.

Corollary 8. Let $G$ be a connected graph of order $n_1 \geq 2$. If $H$ is a wheel graph or a fan graph of order $n_2 \geq 8$, then

$$\dim(G \circ^k H) = n_1(n_2 + 1)^{k-1} \left\lfloor \frac{2n_2}{5} \right\rfloor.$$ 

Theorem 9. Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 2$. Let $\alpha$ be the number of connected components of $H$ of order greater than one and let $\beta$ be the number of isolated vertices of $H$. Then

$$\dim(G \circ^k H) \leq \begin{cases} n_1(n_2 + 1)^{k-1}(n_2 - \alpha - 1) & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\ n_1(n_2 + 1)^{k-1}(n_2 - \alpha) & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\ n_1(n_2 + 1)^{k-1}(n_2 - 1) & \text{for } \alpha = 0. \end{cases}$$

Proof. We suppose $\alpha \geq 1$ and $\beta \geq 1$. Let $A_i$ be the set of vertices of $G \circ H$ formed by all but one of the vertices per each of the $\alpha$ connected components of $H_i$. If $\beta \geq 2$ we define $B_i$ to be the set of vertices of $G \circ H$ formed by all but one of the isolated vertices of $H_i$. If $\beta = 1$ we assume $B_i = \emptyset$. Let us show that $S = \bigcup_{i=1}^{n_1}(A_j \cup B_j)$ is a resolving set for $G \circ H$. Let $x, y$ be two different vertices of $G \circ H$. We suppose $x, y \notin S$. We have the following cases.

Case 1. $x = v_i \in V$ and $y \in V_i$. For every vertex $u \in V_j \cap S, j \neq i$, we obtain $d(y, u) = d(y, x) + d(x, u) > d(x, u)$.

Case 2. $x = v_i \in V$ and $y \notin V_i$. For every $v \in S \cap V_i$ we have $d(x, v) = 1 < d(y, v)$.

Case 3. $x \in V_i$ and $y \in V_j, j \neq i$. For every $u \in V_i \cap S$ we have $d(x, u) \leq 2 < 3 \leq d(y, u)$. 

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Case 4. $x, y \in V_i$. We consider, without loss of generality, that $x$ is not an isolated vertex in $H_i$. Then there exists $v \in V_i \cap S$ such that $v \sim x$, so $d(x, v) = 1 < 2 = d(y, v)$.

Thus, for every two different vertices $x, y$ of $G \odot H$, we obtain $r(x|S) \neq r(y|S)$ and, as a consequence, $\dim(G \odot H) \leq n_1(n_2 - \alpha - 1)$.

As above, if $\beta = 0$ then we take $S = \bigcup_{j=1}^{n_2} A_j$ and we obtain $\dim(G \odot H) \leq n_1(n_2 - \alpha)$ and if $\alpha = 0$, then we take $S = \bigcup_{j=1}^{n_1} B_j$ and we obtain $\dim(G \odot H) \leq n_1(n_2 - 1)$. Note that if $\alpha = 0$, then it is not necessary to consider Case 4. Thus, the result follows.

\[ \blacksquare \]

**Corollary 10.** Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be an unconnected graph of order $n_2 \geq 2$. Then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong N_{n_2}$.

**Proof.** In [13] the authors showed that $\dim(G \odot N_{n_2}) = n_1(n_2 - 1)$. Hence, $\dim(G \odot^k N_{n_2}) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$. Moreover, by the above theorem, if $H$ is unconnected and $H \not\cong N_{n_2}$, then $\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}(n_2 - 2)$.

\[ \blacksquare \]

**Theorem 11.** Let $G$ and $H$ be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 3$, respectively. Then

$$\dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong K_{n_2}$. Moreover, if $H \not\cong K_{n_2}$, then

$$\dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}(n_2 - 2).$$

**Proof.** Since $\dim(K_{n_2}) = n_2 - 1$, by Theorem 3 we conclude $\dim(G \odot^k K_{n_2}) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$. On the contrary, we suppose $H \not\cong K_{n_2}$. Given a set $X$ of vertices of $H$ and a vertex $v$ of $H$, $N_X(v)$ denotes the set of neighbors that $v$ has in $X$: $N_X(v) = \{u \in X : u \sim v\}$. Given two vertices $a, b$ of $H$, let $X_{a,b}$ be the set formed by all vertices of $H$ different from $a$ and $b$. Since $H$ is a connected graph and $H \not\cong K_{n_2}$, there exist at least two vertices $a, b$ of $H$ such that $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$. Let $a_i, b_i$ be the vertices corresponding to $a, b$, respectively, in the $i^{th}$-copy $H_i = (V_i, E_i)$ of $H$. Let $S = \bigcup_{i=1}^{n_2}(V_i - \{a_i, b_i\})$. We will show that $S$ is a resolving set for $G \odot H$. Let $x, y$ be two different vertices of $G \odot H$ such that $x, y \notin S$. We have the following cases.
Case 1. $x = a_i$ and $y = b_i$. Since $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$ we have $r(x|S) \neq r(y|S)$.

Case 2. $x = v_i \in V$ and $y \in V_i$. For every $v \in V_j - \{a_j, b_j\}$, $j \neq i$, we have $d(y, v) = d(y, x) + d(x, v) > d(x, v)$. If $x \in V_i$ and $y \in V_j$, $j \neq i$, then for every $v \in V_i - \{a_i, b_i\}$ we have $d(x, v) \leq 2 < 3 \leq d(y, v)$.

Case 3. $x, y \in V$. Say $x = v_i$. Then for every $v \in V_i - \{a_i, b_i\}$ we have $d(x, v) = 1 < d(y, v)$.

Hence, for every two different vertices $x, y$ of $G \circ H$, we obtain $r(x|S) \neq r(y|S)$. Thus, $\dim(G \circ H) \leq n_1(n_2 - 2)$. Therefore, the result follows. □

As we have shown in Corollary 5, the above bound is tight.

**Theorem 12.** Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 2$. Then

$$\dim(G \circ^k H) \leq n_1(n_2 + 1)^{k-1} \dim(K_1 \circ H).$$

**Proof.** We denote by $K_1 \circ H_i$ the subgraph of $G \circ H$, obtained by joining the vertex $v_i \in V$ with all vertices of $H_i$. For every $v_i \in V$, let $B_i$ be a resolving set of minimum cardinality of $K_1 \circ H_i$ and let $B = \bigcup_{i=1}^{n_1} B_i$. By Lemma 1 (iii) we have that $v_i$ does not belong to any resolving set of minimum cardinality for $K_1 \circ H_i$. So, $B$ does not contain any vertex from $G$. We will show that $B$ is a resolving set for $G \circ H$. Let $x, y$ be two different vertices in $G \circ H$. We consider the following cases.

Case 1: $x, y \in V_i$. There exists $u \in B_i$ such that $d_{K_1 \circ H_i}(x, u) \neq d_{K_1 \circ H_i}(y, u)$, which leads to $d_{G \circ H}(x, u) \neq d_{G \circ H}(y, u)$.

Case 2: $x \in V_i$ and $y \in V_j$, $i \neq j$. Let $v \in B_i$. We have $d_{G \circ H}(x, v) \leq 2 < 3 \leq d_{G \circ H}(y, v)$.

Case 3: $x, y \in V$. Suppose now that $x$ is adjacent to the vertices of $H_i$. Hence, for every vertex $v \in B_i$ we have $d_{G \circ H}(x, v) = 1 < d_{G \circ H}(y, x) + 1 = d_{G \circ H}(y, v)$.

Case 4: $x \in V_i$ and $y \in V$. If $x \sim y$, then for every vertex $v \in B_j$, with $j \neq i$, we have $d_{G \circ H}(x, v) = 1 + d_{G \circ H}(y, v) > d_{G \circ H}(y, v)$. Now, let us assume that $x \not\sim y$. Hence, there exists $v \in B_j$ adjacent to $y$, with $j \neq i$. So, we have $d_{G \circ H}(x, v) = d_{G \circ H}(x, y) + 1 = d_{G \circ H}(x, y) + d_{G \circ H}(y, v) > d_{G \circ H}(y, v)$.

Thus, for every two different vertices $x, y$ of $G \circ H$, we have $r(x|S) \neq r(y|S)$ and, as a consequence, $\dim(G \circ H) \leq n_1 \dim(K_1 \circ H)$. Therefore, the result follows. □
Theorem 13. Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 7$. If $D(H) \geq 6$ or $H$ is a cycle graph, then

$$dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}dim(K_1 \odot H).$$

Proof. Let $S$ be a resolving set of minimum cardinality in $G \odot H$. By Lemma 1 (iii) we have $S \cap V = \emptyset$, as a consequence, $S = \bigcup_{i=1}^{n_1}S_i$, where $S_i \subset V_i$. Notice that, by Lemma 1 (ii), $S_i \neq \emptyset$ for every $i \in \{1, ..., n_1\}$. Now we differentiate two cases in order to show that $r(x|S_i) \neq (1, ..., 1)$ for every $x \in V_i - S_i$.

Case 1. $H$ is a cycle graph of order $n_2 \geq 7$. If $r(a|S_i) = (1, 1)$ for some $a \in V_i - S_i$, then, since $n_2 \geq 7$, there exist two vertices $x, y \in V_i - S_i$ such that $d_{H_i}(x, y) > 1$ and $d_{H_i}(y, v) > 1$, for every $v \in S_i$. Hence, $d_{G \odot H}(x, v) = d_{G \odot H}(y, v) = 2$ for every $v \in S_i$, which is a contradiction because, by Lemma 1 (i), $d_{G \odot H}(x, v) = d_{G \odot H}(y, v)$ for every vertex $u$ of $S$ not belonging to $S_i$.

Case 2. $D(H) \geq 6$. Let $x, y \in V_i - S_i$. Since $S$ is a resolving set for $G \odot H$, we have $r(x|S) \neq r(y|S)$. As we have noted before, by Lemma 1 (i) we have that $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$ for every vertex $u$ of $G \odot H$ not belonging to $V_i$. So, there exists $v \in S_i$ such that $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$ and, as a consequence, either ($v \sim x$ and $v \neq y$) or ($v \neq x$ and $v \sim y$). Now we suppose that there exists a vertex $a \in V_i - S_i$ such that $r(a|S_i) = (1, 1, ..., 1)$. If there exists a vertex $b \in V_i - S_i$ such that $d_{H_i}(b, u) > 1$, for every $u \in S_i$, then for every $w \in V_i - (S_i \cup \{a, b\})$, there exists $v \in S_i$ such that $w \sim v$. Then $D(H_i) \leq 5$. Moreover, if for every $b \in V_i - S_i$ there exists $v_b \in S_i$ such that $v_b \sim b$, then $D(H) \leq 4$. Therefore, if $D(H) \geq 6$, then $r(a|S_i) \neq (1, 1, ..., 1)$ for every $a \in V_i - S_i$.

Now, we denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all vertices of the $i$th-copy of $H$. In both the above cases we have $r(v_i|S_i) = (1, 1, ..., 1) \neq r(x|S_i)$ for every $x \in V_i - S_i$, so $S_i$ is a resolving set for $K_1 \odot H_i$. Hence, $dim(K_1 \odot H_i) \leq |S_i|$, for every $i \in \{1, ..., n_1\}$. Thus, $dim(G \odot H) \geq n_1dim(K_1 \odot H_i)$ and, as a consequence, $dim(G \odot^k H) \geq n_1(n_2 + 1)^{k-1}dim(K_1 \odot H)$. We conclude the proof by Theorem 12. \hfill $\square$

Corollary 14. Let $G$ be a connected graph of order $n_1 \geq 2$.

(i) If $n_2 \geq 7$, then $dim(G \odot^k C_{n_2}) = n_1(n_2 + 1)^{k-1}\left\lfloor \frac{2n_2 + 2}{5} \right\rfloor$.

(ii) If $n_2 \geq 7$, then $dim(G \odot^k P_{n_2}) = n_1(n_2 + 1)^{k-1}\left\lfloor \frac{2n_2 + 2}{5} \right\rfloor$. 

9
All our previous results concern to $G \odot H$ for $H$ of order at least two. Now we consider the case $H \cong K_1$. We obtain a general bound for $\dim(G \odot^k K_1)$ and, when $G$ is a tree, we give the exact value for this parameter.

**Claim 15.** Let $G$ be a simple graph. If $v$ is a vertex of degree greater than one in $G$, then for every vertex $u$ adjacent to $v$ there exists a vertex $x \neq u, v$ of $G$, such that $d(v, x) \neq d(u, x) + 1$.

The following lemma obtained in [2] is useful to obtain the next result.

**Lemma 16.** [2] If $G_1$ is a graph obtained by adding a pendant edge to a nontrivial connected graph $G$, then $\dim(G) \leq \dim(G_1) \leq \dim(G) + 1$.

**Theorem 17.** For every connected graph $G$ of order $n \geq 2$,

$$\dim(G \odot^k K_1) \leq 2^{k-1}n - 1.$$  

*Proof.* If $G \cong K_2$, then $\dim(K_2 \odot K_1) = \dim(P_1) = 1$. So, let us suppose $G \not\cong K_2$. Let us suppose, without loss of generality, that $v_n$ is a vertex of degree greater than one in $G$ and let $S = V - \{v_n\}$. For every $i \in \{1, ..., n\}$, let $u_i$ be the pendant vertex of $v_i$ in $G \odot K_1$. We will show that $S$ is a resolving set for $G \odot K_1$. Let $x, y$ be two different vertices of $G \odot K_1$. If $x = u_i$ and $y = u_j$, $i \neq j$, then we have either $i \neq n$ or $j \neq n$. Let us suppose for instance $i \neq n$. So, we obtain that $d(x, v_i) = 1 \neq d(y, v_i)$. On the other hand, if $x = v_n$ and $y = u_i$, then let us suppose $d(x, v_i) = 1$. Since $v_n$ is a vertex of degree greater than one in $G$, by Claim 15, there exists a vertex $v_j \in S$ such that $d(x, v_j) \neq d(v_i, v_j) + 1$. So, we have $d(x, v_j) \neq d(v_i, v_j) + 1 = d(v_i, v_j) + d(u_i, v_i) = d(y, v_i) + d(v_i, v_j) = d(y, v_j)$. Therefore, for every different vertices $x, y$ of $G \odot K_1$ we have $r(x|S) \neq r(y|S)$ and, as a consequence, $\dim(G \odot K_1) \leq n-1$. Therefore, $\dim(G \odot^k K_1) \leq 2^{k-1}n - 1$.  

By Lemma 16 we have $\dim(K_n \odot K_1) \geq \dim(K_n) = n - 1$. Thus, for $k = 1$ the above bound is achieved for the graph $G = K_n$.

To present the next result, we need additional definitions. A vertex of degree at least 3 in a graph $G$ will be called a major vertex of $G$. Any vertex $u$ of degree one is said to be a terminal vertex of a major vertex $v$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $G$. The terminal degree of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ is an exterior major vertex if it has positive terminal degree. Given a graph $G$, $n_1(G)$ denotes the number of vertices of degree one and $ex(G)$ denotes the number of exterior major vertices of $G$. 
Lemma 18. [6, 10, 19] If $T$ is a tree that is not a path, then $\dim(T) = n_1(T) - \text{ex}(T)$.

Theorem 19. For any tree $T$ of order $n \geq 3$, $\dim(T \odot K_1^k) = \begin{cases} n_1(T) & \text{for } k = 1, \\ 2^{k-2}n & \text{for } k \geq 2. \end{cases}$

Proof. If $T$ is a path of order $n \geq 3$, then we have $\dim(T \odot K_1) = 2 = n_1(T)$. Now, if $T$ is not a path, then by using Lemma 18, since $T \odot K_1$ is a tree, $n_1(T \odot K_1) = n$ and $\text{ex}(T \odot K_1) = n - n_1(T)$, we obtain the result for $k = 1$. Since for every tree $T$ of order $n$ we have $n_1(T \odot K_1) = n$, we obtain the result for $k \geq 2$. \hfill \Box

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