Excitation Spectrum for Bose Gases beyond the Gross–Pitaevskii Regime

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Abstract

We consider Bose gases of \(N\) particles in a box of volume one, interacting through a repulsive potential with scattering length of order \(N^{-1+\kappa}\), for \(\kappa > 0\). Such regimes interpolate between the Gross-Pitaevskii and thermodynamic limits. Assuming that \(\kappa\) is sufficiently small, we determine the ground state energy and the low-energy excitation spectrum, up to errors vanishing in the limit \(N \to \infty\).

1 Introduction and main result

We consider systems of \(N \in \mathbb{N}\) bosons in the three-dimensional box \(\Lambda = [-1/2, 1/2]^3\), with periodic boundary conditions, interacting through a repulsive potential with effective range of the order \(N^{-1+\kappa}\), for \(\kappa \geq 0\) small enough. The Hamilton operator of the system has the form

\[
H_N = \sum_{i=1}^{N} -\Delta x_i + \sum_{1 \leq i < j \leq N} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j)) \tag{1.1}
\]

and it is densely defined on \(L^2(\Lambda^N)\), the subspace of \(L^2(\Lambda^N)\) consisting of permutation-invariant functions on \(\Lambda^N\). Throughout the paper we will assume that the potential \(V \in L^3(\mathbb{R})\) is non-negative, radial and compactly supported. We recall the zero-energy scattering equation

\[
\left[-\Delta + \frac{1}{2} V(x)\right] f(x) = 0,
\]

with the boundary condition \(f(x) \to 1\) as \(|x| \to \infty\). It is easy to see that, outside of the range of the potential \(V\), the solution must have the form \(f = 1 - a_0/|x|\) for some
unique positive real number \(a_0\) called the scattering length of \(V\). By scaling,

\[
-\Delta + \frac{1}{2} N^{2-2\kappa} V(N^{1-\kappa} x) f(N^{1-\kappa} x) = 0,
\]

so that the scattering length of the rescaled potential is given by \(a_N = a_0 / N^{1-\kappa}\). For \(\kappa < 2/3\) the Hamiltonian \((1.1)\) represents a “dilute limit”, in the sense that the rescaled density \(\rho a_N^2 = N^{-2+3\kappa}\) tends to zero, as \(N \to \infty\).

For \(\kappa = 0\), \((1.1)\) describes a Bose gas in the so-called Gross-Pitaevskii regime, for which several rigorous results have been obtained, including convergence of the many-body dynamics towards the time-dependent Gross-Pitaevskii equation \([13, 26, 3, 10]\), complete Bose-Einstein condensation \([20, 21, 25, 1, 7, 24, 17, 11]\), bounds on the ground state energy to leading order \([22, 25]\) and, recently, up to errors vanishing in the limit of large \(N\) \([6]\), as well as estimates on the low-energy excitation spectrum \([6]\). For \(\kappa = 2/3\), the Hamilton operator \((1.1)\) describes instead (after appropriate rescaling) a gas in the thermodynamic limit, at fixed density. In this regime, only the ground state energy has been determined rigorously, at leading order \([23]\) and, recently, including next-order corrections \([15]\). In this regime, the existence of Bose-Einstein condensation and the form of the excitation spectrum are still open and in fact they represent very challenging goals in mathematical physics (a review of a long-term project based on a renormalization group approach to these questions can be found in \([2]\)).

In this paper, we choose \(\kappa \in (0; 2/3)\) small enough, interpolating between the Gross-Pitaevskii and the thermodynamic limit. Our main theorem extends precise estimates on the ground state energy and on low-lying excitations from the Gross-Pitaevskii limit to regimes with \(\kappa > 0\).

**Theorem 1.1.** Let \(V \in L^3(\mathbb{R})\) be pointwise nonnegative, spherically symmetric and compactly supported and denote by \(a_0\) its scattering length, and let \(E_N\) be the ground state of the Hamiltonian \(H_N\) defined in \((1.1)\). For \(\kappa \geq 0\) small enough, there exists \(\varepsilon > 0\) such that, in the limit \(N \to \infty\),

\[
E_N = 4\pi a_0 N^{\kappa} (N - 1) + c_\Lambda(a_0 N^{\kappa})^2 + \frac{1}{2} \sum_{p \in \Lambda^*_+} \left[ \left( \sqrt{|p|^4 + 16\pi a_0 p^2 N^{\kappa}} - p^2 - 8\pi a_0 N^{\kappa} + \frac{(8\pi a_0 N^{\kappa})^2}{2p^2} \right) + O(N^{-\varepsilon}) \right],
\]

(1.2)

with the notation \(\Lambda^*_+ = 2\pi \mathbb{Z}^3 \setminus \{0\}\) and

\[
c_\Lambda = 2 - \lim_{M \to \infty} \sum_{p \in \mathbb{Z}^3 \setminus \{0\} : |p_1|, |p_2|, |p_3| \leq M} \frac{4 \cos(|p|)}{|p|^2},
\]

(1.3)

where, in particular, the limit exists. Moreover, for if \(\kappa, \mu > 0\) are small enough, there exists \(\varepsilon > 0\) such that the spectrum of \(H_N - E_N\) below the threshold \(N^{\kappa/2 + \mu}\) consists of eigenvalues given, in the limit \(N \to \infty\), by

\[
\sum_{p \in \Lambda^*_+} n_p \sqrt{|p|^4 + 16\pi a_0 N^{\kappa} p^2} + O(N^{-\varepsilon}),
\]

(1.4)
where \( n_p \in \mathbb{N} \) for all \( p \in \Lambda^*_+ \) and \( n_p \neq 0 \) only for a finite number of \( p \in \Lambda^*_+ \).

Remark: Approximating the sum on the r.h.s. of (1.2) with an integral, we find

\[
E_N = 4\pi a_0 N^{1+\kappa} + \frac{N^{5\kappa/2}}{2(2\pi)^3} \int \left[ \sqrt{|p|^4 + 16\pi a_0 p^2} - p^2 - 8\pi a_0 + \frac{(8\pi a_0)^2}{2p^2} \right] dp + \mathcal{O}(N^{2\kappa})
\]

\[
= 4\pi a_0 N^{1+\kappa} + 4\pi \cdot \frac{128}{15\sqrt{\pi}} a_0^{5/2} N^{5\kappa/2} + \mathcal{O}(N^{2\kappa})
\]

which is consistent with the Lee-Huang-Yang formula, derived in [15] for \( \kappa = 2/3 \) (after appropriate rescaling). The expansion (1.2) is more precise, since it identifies all contributions to the ground state energy that do not vanish as \( N \to \infty \) (but, of course, it only holds for \( \kappa > 0 \) small enough).

Remark: Eq. (1.4) implies that low-lying excited eigenvalues of the Hamiltonian \( H_N \) have energy of order \( N^{\kappa/2} \) (for this reason, the interval \([0; N^{\kappa/2} + \mu] \) considered in (1.4) contains several eigenvalues of the operator \( H_N \), for any \( \mu > 0 \), in the limit of large \( N \)). In fact, (1.4) also shows that, to leading order, the dispersion of the excitations is linear in \( |p| \). By Landau’s criterion, this gives a heuristic explanation of the emergence of superfluidity [18].

Remark: The bounds (1.2), (1.4) establish the validity of the predictions of Bogoliubov theory [8] for \( \kappa > 0 \) small enough, extending the results obtained in [6] for \( \kappa = 0 \) (and previous results obtained in [28, 16, 19, 12, 27, 9] for mean-field bosons and in [5] for regimes interpolating between the mean-field and the Gross-Pitaevskii limit).

The proof of Theorem 1.1 follows the strategy developed in [7] for the Gross-Pitaevskii regime with \( \kappa = 0 \). It makes use, crucially, of recent estimates obtained in [1], for sufficiently small \( \kappa > 0 \), for the expectation of the number and the energy of excitations of the condensate (with a rate that becomes optimal as \( \kappa \to 0 \)).

First of all, we factor out the Bose-Einstein condensate, focusing on its orthogonal excitations. To this end, we introduce a unitary operator \( U_N \), mapping the Hilbert space \( L^2_s(\Lambda^N) \) into the truncated Fock space \( \mathcal{F}^\leq N \), constructed on the orthogonal complement of the condensate wave function. This leads us to an excitation Hamiltonian \( \mathcal{L}_N = U_N H_N U_N^* \), acting on \( \mathcal{F}^\leq N \).

As a second step, we conjugate \( \mathcal{L}_N \) with a (generalized) Bogoliubov transformation, defining the renormalized excitation Hamiltonian \( \mathcal{G}_N = e^{-B} \mathcal{L}_N e^B \). The antisymmetric operator \( B \) is quadratic in (modified) creation and annihilation operators. Conjugation with \( e^B \) acts on high momenta; it creates short-scale correlations among particles. Even after conjugation with \( e^B \), there are still some important contributions to the energy hidden in parts of \( \mathcal{G}_N \) that are cubic and quartic in creation and annihilation operators. For this reason, we need to conjugate \( \mathcal{G}_N \) with another unitary operator of the form \( e^A \), this time with \( A \) cubic, rather than quadratic, in creation and annihilation operators. We obtain the twice renormalized Hamiltonian \( \mathcal{J}_N = e^{-A} \mathcal{G}_N e^A \). Up to error terms that can be estimated with the a-priori bounds from [1], \( \mathcal{J}_N \) is the sum of a constant term, a quadratic and a quartic contribution. The quartic part is positive and can be
neglected, when proving lower bounds on the ground state energy and on the excited eigenvalues. To show matching upper bounds, we only have to control the quartic potential on appropriate trial states. To conclude the proof of Theorem 1.1, we still need to diagonalize the quadratic part of \( J_N \). To this end, we use a last (generalized) Bogoliubov transformation to define the final excitation Hamiltonian \( \mathcal{M}_N = e^{-T} J_N e^T \), with a diagonal quadratic part.

While the previous steps of the analysis are quite similar to what was done in [6] for \( \kappa = 0 \), this last step differs substantially. The reason is that, for \( \kappa > 0 \), the (generalized) Bogoliubov transformation \( e^T \) diagonalizing the quadratic component of \( J_N \) creates a large number of excitations (the number of excitations of the Bose-Einstein condensate is of the order \( N^{3\kappa/2} \), and thus diverges, as \( N \to \infty \), for any \( \kappa > 0 \)). For this reason, it is more difficult to determine the action of \( e^T \) (being \( e^T \) a generalised Bogoliubov transformation, its action is not explicit; it has to be computed through an expansion, whose convergence depends on the size of \( T \) and to control the resulting growth of error terms. This part of the analysis, which is carried out in Section 4 and leads to the proof of Theorem 1.1 in Section 5, is the main novelty of our work. Together with the a-priori bounds on the number and the energy of excitations of the Bose-Einstein condensate from [1], this is also the main reason why we have to restrict our analysis to very small values of the parameter \( \kappa > 0 \). While we did not try to optimize the choice of \( \kappa \), it is clear that to make it substantially larger and to approach the thermodynamic limit at \( \kappa = 2/3 \), genuinely new ideas are needed. In this direction, let us mention the recent result obtained in [14], where the techniques of [15] have been applied to show the existence of Bose-Einstein condensation up to \( \kappa < 2/5 \); compared with the bounds of [1], this approach only controls the expectation of the number of particles and therefore it cannot be directly applied to show Theorem 1.1. It would in fact be possible to combine the estimate in [14] with the bounds in [1, Proposition 3.3] to get a better control of energy and high powers of number of excitations. Since, however, we would still have to restrict to very small values of \( \kappa \), we decided to use directly the bounds in [1], to keep our analysis as short as possible.

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2 Fock Space and the Excitation Hamiltonian

We describe excitations of the Bose-Einstein condensate on the truncated Fock space

\[ \mathcal{F}_+^{\leq N} = \bigoplus_{k=0}^{N} L^2_+ (\Lambda)^\otimes_k \]
built over the orthogonal complement $L_2^s(\Lambda)$ of the zero-momentum mode $\varphi_0 \equiv 1$. We map the original $N$-particle Hilbert space $L_2^s(\Lambda^N)$ into $\mathcal{F}_{\leq N}^\perp$, through the unitary operator $U_N : L_2^s(\Lambda^N) \to \mathcal{F}_{\leq N}^\perp$, defined by $U_N\psi_N = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N\}$, with $\alpha_j \in L_2^s(\Lambda)^{\otimes j}$, if

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes (N-1)} + \cdots + \alpha_N$$

where $\otimes_s$ denotes the symmetrized tensor product. The map $U_N$ factors out the Bose-Einstein condensate and allows us to focus on its orthogonal excitations. With $U_N$, we can define the excitation Hamiltonian $\mathcal{L}_N = U_N H_N U_N^*$ as a self-adjoint operator on a dense subspace of $\mathcal{F}_{\leq N}^\perp$. Proceeding as in [1, Sect. 2], we find

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)}$$

with

$$\mathcal{L}_N^{(0)} = \frac{N - 1}{2N} N^\kappa \hat{V}(0)(N - N_+) + \frac{N^\kappa \hat{V}(0)}{2N} N_+(N - N_+)$$

$$\mathcal{L}_N^{(2)} = \sum_{p \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[b_p^* b_p - \frac{1}{N} a_p^* a_p\right]$$

$$\mathcal{L}_N^{(3)} = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda^*_+, p \neq q} \hat{V}(p/N^{1-\kappa}) \left[a_q^* a_p a_p a_q + a_q^* a_p a_q a_p\right]$$

$$\mathcal{L}_N^{(4)} = \frac{N^\kappa}{2N} \sum_{p,q \in \Lambda^*_+, r \in \Lambda^*, r \neq p, -q} \hat{V}(r/N^{1-\kappa}) a_p^* a_q^* a_p a_q a_p a_q$$

where $\Lambda^*_+ = 2\pi \mathbb{Z}^3 \setminus \{0\}$ is the set of possible momenta of excitations of the condensate and where, for any $p \in \Lambda^*_+$, $a_p^*$, $a_p$ are the usual creation and annihilation operators. Moreover, for $p \in \Lambda^*_+$, we introduced generalized creation and annihilation operators

$$b_p^* = U_N a_p^* \frac{a_0}{\sqrt{N}} U_N^* = a_p^* \sqrt{\frac{N - N_+}{N}}$$

$$b_p = U_N a_0 \frac{a_p}{\sqrt{N}} a_0 U_N^* = \sqrt{\frac{N - N_+}{N}} a_p$$

satisfying the approximate CCR

$$[b_p, b_q] = [b_p^*, b_q^*] = 0, \quad [b_p, a_q^*] = \delta_{p,q} \left(1 - \frac{N_+}{N}\right) - \frac{1}{N} a_q^* a_p, \quad (2.2)$$

together with the useful commutation relations

$$[b_p, a_q^* a_r] = \delta_{p,q} b_r, \quad [b_p^*, a_q^* a_r] = -\delta_{p,r} b_q^*.$$
Observe that, while \( a_p, a_p^* \) do not preserve the truncation on the number of particles, the operators \( b_p, b_p^* \) (and also products of the form \( a_p^* a_q \)) are well-defined on \( \mathcal{F}_c^N \).

In the following, we will also use the notation

\[
\mathcal{K} = \sum_{p \in \Lambda_1^*} p^2 a_p^* a_p, \quad \mathcal{V}_N = \mathcal{L}_N^{(4)} = \frac{N^\kappa}{2N} \sum_{\substack{p \neq q \in \Lambda, r \in \Lambda^*}} \hat{V}(r/N^{1-\kappa}) a_p^* a_q a_p a_{q+r} \tag{2.3}
\]

for the kinetic and the potential energy operators. Moreover, we set \( \mathcal{H}_N = \mathcal{K} + \mathcal{V}_N \).

It will sometimes also be useful to switch to position space, introducing, for a fixed \( \ell \in (0; 1/2) \), the ground state \( f_\ell \) of the Neumann problem

\[
\left[ -\Delta + \frac{1}{2} V \right] f_\ell(x) = \lambda_\ell f_\ell \tag{3.1}
\]

on \( B_{\ell N^{1-\kappa}} \) with boundary condition \( f_\ell(x) = 1 \) for \( |x| = \ell N^{1-\kappa} \). By scaling, \( f_\ell(N^{1-\kappa}) \) satisfies the equation

\[
\left[ -\Delta + N^2 - 2\kappa \frac{1}{2} V(N^{1-\kappa}) \right] f_\ell(N^{1-\kappa}) = N^{2-2\kappa} \lambda_\ell f_\ell(N^{1-\kappa})
\]

on \( B_\ell \). We now define \( f_N \) to be the extension of \( f_\ell(N^{1-\kappa}) \) to \( \Lambda \), obtained by setting \( f_N(x) = 1 \) if \( x \in \Lambda \setminus B_\ell \). With \( \chi_\ell \) denoting the characteristic function of \( B_\ell \) we have

\[
\left[ -\Delta + N^2 - 2\kappa \frac{1}{2} V(N^{1-\kappa}) \right] f_N = N^{2-2\kappa} \lambda_\ell \chi_\ell f_N \tag{3.2}
\]

We further define \( w_\ell = 1 - f_\ell \) on \( B_{\ell N^{1-\kappa}} \) and its rescaled version \( w_N = w_\ell(N^{1-\kappa}) = 1 - f_N \) on \( \Lambda \), with Fourier coefficients

\[
\hat{w}_N(p) = \int_{\Lambda} e^{-ip \cdot x} w_N(x) dx = \frac{1}{N^{3-2\kappa}} \hat{w}_\ell(p/N^{1-\kappa}) = \delta_{p,0} - \hat{f}_N(p).
\]

Some important properties of these functions are collected in the next lemma, whose proof is a straightforward adaptation of \([6\text{, Lemma 3.1}])\.
Lemma 3.1. Fix $\ell \in (0, 1/2)$ and denote by $f_\ell$ the solution of (3.1). For $N \in \mathbb{N}$ large enough the following properties hold true:

i) We have
$$
\lambda_\ell = \frac{3a_0}{(\ell N^{1-\kappa})^3} \left( 1 + \frac{9}{5} \frac{a_0}{\ell N^{1-\kappa}} + \mathcal{O} \left( \frac{a_0^2}{(\ell N^{1-\kappa})^2} \right) \right) .
$$

(3.3)

ii) We have $0 \leq f_\ell, w_\ell \leq 1$, and there exists a constant $C > 0$ such that
$$
\left| \int V(x)f_\ell(x)dx - 8\pi a_0 \left( 1 + \frac{3}{2} \frac{a_0}{\ell N^{1-\kappa}} \right) \right| \leq \frac{C a_0^3}{(\ell N^{1-\kappa})^2} .
$$

(3.4)

iii) There exists a constant $C > 0$ such that
$$
0 \leq w_\ell \leq C \left( 1 + \frac{1}{|x|} \right) \quad \text{and} \quad |\nabla w_\ell| \leq \frac{C}{1 + |x|^2} ,
$$

(3.5)

and precisely
$$
\frac{1}{(\ell N^{1-\kappa})^2} \int_{\mathbb{R}^3} w_\ell(x)dx - \frac{2}{5} \pi a_0 \leq \frac{C a_0}{\ell N^{1-\kappa}} .
$$

(3.6)

iv) There exists a constant $C > 0$ such that
$$
\hat{w}_N(p) \leq \frac{C N^\kappa}{N^2|p|^2} ,
$$

(3.7)

for all $p \in \mathbb{R}^3$ and $N \in \mathbb{N}$ large enough (such that $N^\kappa/N \leq 1/\ell$).

We define now $\eta : \Lambda_+^* \to \mathbb{R}$ through
$$
\eta_p = -N \hat{w}_N(p) = -\frac{N^\kappa}{N^2-2\kappa} \hat{w}_\ell(p/N^{1-\kappa}) ,
$$

In particular, $\hat{f}_N(p) = \delta_{p,0} + \eta_p/N$. In position space, we have
$$
\hat{\eta}(x) = -N w_\ell(N^{1-\kappa}x)
$$

for $x \in \Lambda$. From Lemma 3.1 and in particular Eq. (3.5), (3.7), we obtain
$$
\eta_p \leq \frac{C N^\kappa}{|p|^2} , \quad ||\eta||_2 \leq C N^\kappa , \quad ||\hat{\eta}||_\infty \leq CN .
$$

(3.8)

Moreover, from (3.3) we deduce
$$
||\eta||_{H^1}^2 = N^{4-2\kappa} \left| \nabla w_\ell(N^{1-\kappa}:x) \right|_2^2 \leq \int_{B_\ell} dx \frac{CN^{4-2\kappa}}{(1 + N^{2-2\kappa} |x|^2)^2} \leq C N^{1+\kappa} .
$$

The scattering equation (3.2) translates to Fourier space into
$$
p^2 \hat{f}_N(p) + \frac{N^\kappa}{2N} (\hat{V}(-N^{1-\kappa}) * \hat{f}_N)_p = N^{2-2\kappa} \lambda_\ell (\hat{\chi}_\ell * \hat{f}_N)_p ,
$$

(3.9)
for \( p \in \Lambda^\alpha \), or, equivalently,
\[
p^2 \eta_p + \frac{1}{2} N^\kappa \hat{V}(p/N^{1-\kappa}) + \frac{1}{2N} \sum_{q \in \Lambda^\alpha} N^\kappa \hat{V}((p-q)/N^{1-\kappa}) \eta_q = N^3 - 2\kappa \lambda_0 \hat{\chi}_e(p) + N^{2-2\kappa} \lambda_d \sum_{q \in \Lambda^\alpha} \hat{\chi}_e(p-q) \eta_q. \tag{3.10}
\]

In order to make the \( \ell^2 \)-norm of \( \eta \) small (which is important to control the action of the corresponding generalized Bogoliubov transformation), we introduce an infrared cutoff, restricting \( \eta \) to high momenta. For an \( \alpha > 0 \) to be specified below, we define \( P_H := \{ p \in \Lambda^\alpha_+ : |p| \geq N^\alpha \} \) and the coefficients
\[
\eta_H(p) = \eta_p \chi_{P_H}(p).
\]
Then, we have
\[
\|\eta_H\|_2 \leq C N^{\kappa - \alpha/2}, \quad \|\eta_H\|_H \leq C N^{(1+\kappa)/2}, \quad \|\eta_H\|_\infty \leq C N^{\kappa - \alpha},
\]
and moreover, from Lemma 3.1 and (3.8),
\[
|\tilde{\eta}_H(x)| = \left| \sum_{|p| \geq N^{\alpha}} e^{ip \cdot x} \eta_p \right| \leq |\hat{\eta}(x)| + \sum_{|p| < N^\alpha} |\eta_p| \leq C(N + N^{\kappa + \alpha}).
\]

We assume throughout the following analysis that \( \alpha > 2\kappa, \kappa + \alpha < 1 \) so that
\[
\|\eta_H\|_2 \to 0, \quad \|\eta_H\|_\infty \to 0 \quad \text{as} \quad N \to \infty, \quad \|\tilde{\eta}_H\|_\infty \leq CN. \tag{3.11}
\]

With the coefficients \( \eta_H \), we define the antisymmetric operator
\[
B = \frac{1}{2} \sum_{p \in \Lambda^\alpha_+} \eta_H(p) [b^*_p b^*_{-p} - b_p b_{-p}] = \frac{1}{2} \sum_{p \in \Lambda^\alpha_+ : |p| > N^\alpha} \eta_p [b^*_p b^*_{-p} - b_p b_{-p}] \tag{3.12}
\]
and we consider the corresponding generalized Bogoliubov transformation \( e^B \). With (3.11), we can control the action of \( e^B \) on powers of the number of particles operator. The proof of the following lemma can be found in [10] Lemma 3.1 (see [3] for the analogue in the translation invariant setting).

**Lemma 3.2.** Assume \( \alpha > 2\kappa \). For every \( j \in \mathbb{N} \) there exists a constant \( C > 0 \) such that
\[
e^{-B}(\mathcal{N}_+ + 1)^j e^B \leq C(\mathcal{N}_+ + 1)^j.
\]

On states with few excitations, the generalized Bogoliubov transformation \( e^B \) acts approximately like a standard Bogoliubov transformation. To make this statement more precise, for \( p \in \Lambda^\alpha_+ \) we set \( \gamma_p = \cosh(\eta_H(p)) \), \( \sigma_p = \sinh(\eta_H(p)) \) and we define operators \( d_p, d^*_p \) through the identities
\[
e^{-B} b_p e^B = \gamma_p b_p + \sigma_p b^*_{-p} + d_p, \quad e^{-B} b^*_p e^B = \gamma_p b^*_p + \sigma_p b_{-p} + d^*_p. \tag{3.13}
\]
Observe that $\gamma_p = 1$ and $\sigma_p = 0$ for $p \in P_H$. For $p \in P_H$, on the other hand, we can use (3.8) and (3.11) to bound

$$
|\sigma_p| \leq \frac{CN^\kappa}{|p|^\alpha}, \quad |\sigma_p - \eta_p| \leq \frac{CN^{3\kappa}}{|p|^\alpha}, \quad |\gamma_p| \leq C, \quad |\gamma_p - 1| \leq \frac{CN^{2\kappa}}{|p|^\alpha}.
$$

(3.14)

In position space, we have

$$
\|\tilde{\sigma}\|_2 = \|\sigma\|_2 \leq CN^{\kappa - \alpha/2} \to 0, \quad \|\tilde{\sigma}\|_\infty \leq CN, \quad \|\tilde{\sigma} \ast \tilde{\gamma}\|_\infty \leq CN.
$$

(3.15)

In the next lemma, taken from [7, Lemma 2.3], we establish bounds for the remainder operators $d_p, d^*_p$.

**Lemma 3.3.** For $p \in \Lambda^*_N$, let $d_p, d^*_p$ be defined as in (3.13), $\alpha > 2\kappa$ and $N$ large enough. Then

$$
\| (\mathcal{N} + 1)^{n/2} d_p \| \leq C \left[ \| \eta_H(p) \| (\mathcal{N} + 1)^{(n+3)/2} \xi \| + \| \eta_H \|_2 \| b_p (\mathcal{N} + 1)^{(n+2)/2} \xi \| \right]
$$

$$
\| (\mathcal{N} + 1)^{n/2} d^*_p \| \leq C \left[ \| (\mathcal{N} + 1)^{(n+3)/2} \xi \| \right].
$$

(3.16)

Moreover, defining $\tilde{d}_x, \tilde{d}^*_x$ similarly as in (3.13) but in position space, we have

$$
\| (\mathcal{N} + 1)^{n/2} \tilde{d}_x \| \leq \frac{C}{N} \left[ \| (\mathcal{N} + 1)^{(n+3)/2} \xi \| + \| \tilde{b}_x (\mathcal{N} + 1)^{(n+2)/2} \xi \| \right]
$$

$$
\| (\mathcal{N} + 1)^{n/2} \tilde{d}_y \| \leq \frac{C}{N} \left[ \| \tilde{a}_x (\mathcal{N} + 1)^{(n+1)/2} \xi \| + (1 + |\tilde{\eta}(x - y)|) \| (\mathcal{N} + 1)^{(n+2)/2} \xi \| + \| \tilde{a}_y (\mathcal{N} + 1)^{(n+3)/2} \xi \| \right]
$$

$$
\| (\mathcal{N} + 1)^{n/2} \tilde{d}_x \| \leq \frac{C}{N^2} \left[ \| (\mathcal{N} + 1)^{(n+6)/2} \xi \| + |\tilde{\eta}(x - y)| \| (\mathcal{N} + 1)^{(n+4)/2} \xi \| + \| \tilde{a}_x (\mathcal{N} + 1)^{(n+5)/2} \xi \| + \| \tilde{a}_y (\mathcal{N} + 1)^{(n+5)/2} \xi \| \right].
$$

(3.17)

We can now define the renormalized excitation Hamiltonian

$$
\mathcal{G}_N = e^{-B} \mathcal{L}_N e^B.
$$

(3.18)

**Proposition 3.4.** Assume that $2\kappa < \alpha < 1/2$. Then we have that

$$
\mathcal{G}_N = C\mathcal{G}_N + Q\mathcal{G}_N + \mathcal{H}_N + \mathcal{C}_N + \mathcal{E}_N,
$$

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where
\[ C_{G_N} = \frac{N-1}{2} N^\kappa \hat{V}(0) + \sum_{p \in \Lambda^*_+} \left[ p^2 \sigma_p^2 + N^\kappa \hat{V}(p/N^{1-\kappa})(\sigma_p \gamma_p + \sigma_p^2) \right] \]
\[ + \frac{1}{N} \sum_{p \in P_H} \eta_p \left[ p^2 \eta_p + \frac{N^\kappa}{2N} \left( \hat{V}(\cdot/N^{1-\kappa}) \ast \eta \right)_p \right] \]
\[ + \frac{1}{2N} \sum_{p,q \in \Lambda^*_+} N^\kappa \hat{V}((p-q)/N^{1-\kappa}) \sigma_p \gamma_p \sigma_q \gamma_q \]
\[ - \frac{1}{N} \sum_{u \in \Lambda^*_+} \sigma_u \sum_{p \in \Lambda^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_p , \]

Moreover, we have that
\[ Q_{G_N} = \sum_{p \in \Lambda^*_+} \left[ \Phi_p b_p^* b_p + \frac{1}{2} \Gamma_p \left( b_p^* b_{-p} + b_p b_{-p} \right) \right] \] (3.19)

with (recall the convention that \( \gamma_p = 1 \) and \( \sigma_p = 0 \) for \( p \in P_H^c \))
\[ \Phi_p = 2p^2 \sigma_p^2 + N^\kappa \hat{V}(p/N^{1-\kappa})(\gamma_p + \sigma_p)^2 + \frac{2N^\kappa \gamma_p \sigma_p}{N} \left( \hat{V}(\cdot/N^{1-\kappa}) \ast \eta \right)_p \]
\[- \frac{N^\kappa (\gamma_p^2 + \sigma_p^2)}{N} \left( \hat{V}(\cdot/N^{1-\kappa}) \ast \eta \right)_0 \]
as well as
\[ \Gamma_p = 2p^2 \sigma_p \gamma_p + N^\kappa (\gamma_p + \sigma_p)^2 \hat{V}(p/N^{1-\kappa}) + (\gamma_p^2 + \sigma_p^2) \frac{N^\kappa}{N} \left( \hat{V}(\cdot/N^{1-\kappa}) \ast \eta \right)_p \]
\[- 2\gamma_p \sigma_p \frac{N^\kappa}{N} \left( \hat{V}(\cdot/N^{1-\kappa}) \ast \eta \right)_0 \]
and furthermore that
\[ C_N = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda^*_+} \hat{V}(p/N^{1-\kappa}) \left[ b_{p+q}^* b_p^* \left( \gamma_q b_q + \sigma_q b_{-q}^* \right) + h.c. \right] . \] (3.21)

Finally, the self-adjoint error term \( E_{G_N} \) satisfies the operator inequality
\[ \pm E_{G_N} \leq C N^{-1/2+3\alpha/2+\alpha} (H_N + N_+^2 + 1)(N_+ + 1) . \] (3.22)

The proof of Proposition 3.4 is similar to the proof of \[7, Prop. 3.2, part b)]]. For completeness, we sketch the proof in Appendix A. The proposition describes the main contributions to \( \mathcal{G}_N \), up to an error that can be estimated as in \( (3.22) \). The fact that this error is small on low-energy states is a consequence of the following theorem, which is the main result of \[8].
Theorem 3.5. Let $G_N$ be as in (3.18) and assume that $6\kappa < \alpha < 1/2 - 3\kappa/2$ and $\kappa \in [0; 1/44)$. Let $E_N$ be the ground state energy of $H_N$, as defined in (1.1) and $\psi_N \in L^2_s(\Lambda^N)$ so that $\psi_N = \chi((H_N - E_N \leq \zeta)\psi_N \in L^2_s(\Lambda^N)$ for a $\zeta > 0$. Moreover, let $\xi_N = e^{-B}U_N\psi_N \in \mathcal{F}_+^{\leq N}$ be the excitation vector associated to $\psi_N$. Then, for every $j \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $C > 0$ such that

$$\langle \xi_N, (H_N + 1)(N_+ + 1)^j \xi_N \rangle \leq C N^{20\kappa + \varepsilon \zeta^2 + N^{44\kappa + 2\varepsilon}}.$$  

While the vacuum expectation of the renormalized excitation Hamiltonian (3.18) captures the correct ground state energy, to leading order (the main contribution to (3.19) is exactly $4\pi a_0 N^{1+\kappa}$), there are still non-negligible next-order corrections hidden in cubic and quartic parts of $G_N$. To extract them, we conjugate $G_N$ with another unitary operator of the form $e^A$, where $A$ is now an antisymmetric phase, cubic in generalized creation and annihilation operators. More precisely, we define the low-momentum set $P_L = \{ p \in \Lambda^*: |p| < N^{\beta} \}$, depending on the parameter $\beta < \alpha$, and

$$A = \frac{1}{\sqrt{N}} \sum_{r \in P_L, v \in P_L} \eta_r [b^*_r v b^*_v b_v - \text{h.c.}]. \quad (3.23)$$  

The next lemma will be used to control the action of the unitary operator $e^A$ on powers of the number of particles operator $N_+$ and on the product $H_N N_+$. It can be proven similarly as [7, Prop. 4.2, Prop. 4.4] (the second estimate requires bounds on the commutator $[H_N, A]$ that are shown below, in Lemma B.5).

Lemma 3.6. Let $A$ be defined as in (3.23), and let $\kappa < \beta < \alpha$ satisfy

$$2\kappa < \beta < \alpha < 1/2 - 2\kappa, \quad 4\kappa < \alpha, \quad 3\beta + 4\kappa - 1 < \alpha. \quad (3.24)$$  

For any $j \in \mathbb{N}$, there exists $C > 0$ such that

$$e^{-A}(N_+ + 1)^j e^A \leq C (N_+ + 1)^j$$  

and

$$e^{-A}(N_+ + 1)(H_N + 1)e^A \leq C (H_N + N_+^2 + N^* N_+)(N_+ + 1).$$  

With $A$, we define a second renormalized excitation Hamiltonian

$$J_N = e^{-A}G_N e^A.$$  

Some important properties of $J_N$ are collected in the next proposition.

Proposition 3.7. Let $\kappa < \beta < \alpha$ satisfy

$$2\kappa < \beta < \alpha < 1/2 - 2\kappa, \quad 4\kappa < \alpha, \quad 3\beta + 4\kappa - 1 < \alpha. \quad (3.24)$$  

Then we have that

$$J_N = C_{J_N} + Q_{J_N} + V_N + E_{J_N},$$  

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where
\[ C_{\mathcal{J}_N} = C_{\tilde{\mathcal{J}}_N}, \quad Q_{\mathcal{J}_N} = \sum_{p \in \Lambda^*_+} \left[ F_p b^*_p b_p + \frac{1}{2} G_p (b^*_p b^*_{-p} + b_p b_{-p}) \right], \]  
for (recall that \( \gamma_p = 1 \) and \( \sigma_p = 0 \) for \( p \in P^c_H \))
\[ F_p = (\gamma_p^2 + \sigma_p^2) p^2 + (\gamma_p + \sigma_p)^2 N^\kappa (\hat{\omega} (\cdot / N^{1-\kappa}) \ast \hat{f}_N)_p \]  
and
\[ G_p = 2p^2 \sigma_p \gamma_p + (\gamma_p + \sigma_p)^2 N^\kappa (\hat{\omega} (\cdot / N^{1-\kappa}) \ast \hat{f}_N)_p \]
The self-adjoint error term \( \mathcal{E}_{\mathcal{J}_N} \) satisfies the bound
\[ \pm \mathcal{E}_{\mathcal{J}_N} \leq CN^{\frac{1}{2}+\frac{\kappa}{2}+\alpha} (\mathcal{H}_N + N^2_+ + 1)(N_+ + 1) \]
\[ + CN^{\frac{\kappa}{2}-\kappa} (\kappa + 1)(N_+ + 1). \]  
Moreover, for any \( \delta > 0 \), \( \mathcal{E}_{\mathcal{J}_N} \) also satisfies the estimate
\[ \pm \mathcal{E}_{\mathcal{J}_N} \leq CN^{\frac{1}{2}+\frac{\kappa}{2}+\alpha} (\mathcal{H}_N + N^2_+ + 1)(N_+ + 1) \]
\[ + C(N^{\frac{\kappa}{2}-\kappa} + N^{2\kappa-\frac{3}{2}})(N + 1) + CN^{\kappa-2\beta}(\kappa + 1) + \delta \mathcal{V}_N + C\delta^{-1}N^\kappa(N_+ + 1). \]  
Remark: the second bound (3.29) on the error term \( \mathcal{E}_{\mathcal{J}_N} \) is useful to establish upper bounds on the eigenvalues of \( \mathcal{J}_N \) (because, on trial states, we will be able to show that the contribution of \( \mathcal{V}_N \) is small, as \( N \to \infty \)).

The proof of Prop. 3.7 is similar to the proof of [7, Prop. 3.3]. For completeness, we describe it in Appendix B below.

4 Diagonalization of the Quadratic Hamiltonian

In this section we apply a final (generalized) Bogoliubov transformation to the renormalized Hamiltonian \( \mathcal{J}_N \) in order to diagonalize its quadratic part. First, we prove some bounds on the coefficients \( F_p, G_p \) defined in (3.26), (3.27).

Lemma 4.1. Assume (3.24). There exist constants \( C > c > 0 \) independent of \( N \), such that for \( N \) large enough, we have:

i) \( c(p^2 + N^\kappa) \leq F_p \leq C(p^2 + N^\kappa) \) for all \( p \in \Lambda^*_+ \),

ii) For all \( p \in \Lambda^*_+ \),
\[ c \leq 1 + \frac{G_p}{F_p} \leq 2, \quad c \cdot \min \left[ 1, \frac{p^2}{N^{2\kappa}} \right] \leq 1 - \frac{G_p}{F_p} \leq C. \]  

In particular, this implies that \( |G_p|/F_p \leq 1 - cN^{-\kappa} \), for all \( p \in \Lambda^*_+ \).
iii) \(|G_p| \leq CN^\kappa \) for \( p \in P_H^c \).

iv) \(|G_p| \leq CN^{2\kappa} |p|^{-2} \) for \( p \in P_H \).

**Proof.** We start by proving i). For \( p \in P_H^c \), we write

\[
F_p = p^2 + N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 + N^\kappa \left( (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 \right)
\]

and we observe that, by \((3.4)\), \((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 \geq C > 0\), uniformly in \( N \). With

\[
|((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0| \leq C |p|/N^{1-\kappa} \leq CN^{\alpha+\kappa-1},
\]

for all \( p \in P_H^c \), we conclude that

\[
\tilde{c}(p^2 + N^\kappa) - CN^{2\kappa+\alpha-1} \leq F_p \leq \tilde{C}(p^2 + N^\kappa) + CN^{2\kappa+\alpha-1}.
\]

This gives i) for \( N \) sufficiently large, thanks to \( 2\kappa + \alpha - 1 < 0 \). For \( p \in P_H \) we proceed similarly, using the fact that \(|((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p| \leq C\), uniformly in \( N \). As for ii), we compute \( F_p - G_p = (\gamma_p - \sigma_p)^2p^2 \) (which in particular implies that \( G_p \leq F_p \)). With i), distinguishing the cases \(|p| \leq N^{\kappa/2}\) and \(|p| > N^{\kappa/2}\) (and recalling that \( \gamma_p = 1, \sigma_p = 0 \) for \( p \in P_H^c \)), we obtain \((4.1)\). Part iii) follows immediately from \(|(\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p| \leq C\). To prove iv), we can use the scattering equation. Indeed, for \( p \in P_H \), we have by \((3.4)\) that

\[
2p^2|\sigma_p \gamma_p - \eta_p| \leq CN^{3\kappa} |p|^{-4} \leq CN^{2\kappa} |p|^{-2},
\]

\[
N^\kappa((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p |(\gamma_p + \sigma_p)^2 - 1| \leq CN^\kappa((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p |\sigma_p \gamma_p + \sigma_p^2|
\leq CN^{2\kappa} |p|^{-2}.
\]

Thus, it is enough to prove that \( p^2 \eta_p + N^\kappa((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p \) satisfies the desired bound. By the scattering equation \((3.9)\) and the bound \((4.3)\), we find

\[
p^2 \eta_p + N^\kappa((\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p = N^{3-2\kappa} \chi_\ell(\hat{\chi}_\ell \ast \hat{f}_N)_p \leq CN^\kappa |p|^{-2}.
\]

To show the last inequality, we remark that

\[
\tilde{\chi}_\ell(p) = \int_{|x| \leq \ell} e^{-ipx} = \frac{4\pi}{|p|^2} \left( \frac{\sin(\ell |p|)}{|p|} - \ell \cos(\ell |p|) \right),
\]

which in particular implies that \(|\tilde{\chi}_\ell(p)| \leq C/|p|^2\) (recall that \( \ell < 1/2 \)). Moreover, recalling that \( \hat{\eta} \) is supported in \( B_\ell \), we find, with \((3.5)\), \(|(\tilde{\chi}_\ell \ast \eta)_q| = |(\tilde{\chi}_\ell \hat{\eta})(q)| = |\eta_q| \leq CN^\kappa/q^2\) for all \( q \in \Lambda^\_+ \). Since \( \kappa < 1\), we conclude that

\[
|(\tilde{\chi}_\ell \ast \hat{f}_N)_p| \leq \frac{C}{|p|^2}.
\]

which implies \((4.2)\). This concludes the proof of the lemma. \( \Box \)
Part \( ii) \) of Lemma 4.1 allows us to define \( \tau \in \ell^1(\Lambda_*; \mathbb{R}) \subset \ell^2(\Lambda_*; \mathbb{R}) \) through

\[
\tanh(2\tau_p) = -\frac{G_p}{F_p}, \tag{4.4}
\]
or, equivalently, through

\[
\tau_p = \frac{1}{4} \left[ \log \left(1 - \frac{G_p}{F_p}\right) - \log \left(1 + \frac{G_p}{F_p}\right) \right], \tag{4.5}
\]

These coefficients will be used to define the generalized Bogoliubov transformation which is going to diagonalize the quadratic part of the Hamiltonian \( J_N \). In the next Lemma we collect some useful properties of \( \tau_p \).

**Lemma 4.2.** Assume (3.24) and let \( \tau \) be defined through (4.4). Then there exists a constant \( C > 0 \) such that

\[
\|\tau\|_\infty \leq C + \log N^{\kappa/4}, \quad \|\tau\|_2^2 \leq CN^{3\kappa/2}, \quad \|\tau\|_1 \leq CN^{\alpha + \kappa}, \quad \|\tau\|_{H^1} \leq CN^{\alpha + 2\kappa}, \tag{4.6}
\]

for every \( N \) large enough.

**Proof.** We observe that, for \( |p| < N^{\kappa/2} \), part \( ii) \) of Lemma 4.1 and (4.5) imply that

\[
|\tau_p| \leq \frac{1}{4} \log(N^\kappa/|p|^2) + C \tag{4.7}
\]

For \( N^{\kappa/2} \leq |p| < N^\alpha \), we use instead part \( i) \) and part \( iii) \) of Lemma 4.1 and we apply the mean value theorem to (4.5) to show that

\[
|\tau_p| \leq C|G_p|/F_p \leq CN^\kappa/p^2 \tag{4.8}
\]

Similarly, for \( |p| \geq N^\alpha \), part \( i) \) and part \( iv) \) of Lemma 4.1 lead us to

\[
|\tau_p| \leq C|G_p|/F_p \leq CN^{2\kappa}/p^4 \tag{4.9}
\]

The three estimates (4.7), (4.8), (4.9) immediately imply the bound for \( \|\tau\|_\infty \). To control the other norms, let us denote by \( \tau^{(1)} \), \( \tau^{(2)} \), \( \tau^{(3)} \) the restriction of \( \tau \) to the domains \( |p| < N^{\kappa/2} \), \( N^{\kappa/2} \leq |p| < N^\alpha \) and, respectively, \( |p| \geq N^\alpha \). With (4.8), we easily find that \( \|\tau^{(2)}\|_2^2 \leq CN^{3\kappa/2}, \|\tau^{(2)}\|_1 \leq CN^{\alpha + \kappa}, \|\tau^{(2)}\|_{H^1} \leq CN^{\alpha + 2\kappa} \). Similarly, with (4.9), we obtain \( \|\tau^{(3)}\|_2^2 \leq CN^{4\kappa - 3\alpha}, \|\tau^{(3)}\|_1 \leq CN^{2\kappa - \alpha}, \|\tau^{(3)}\|_{H^1} \leq CN^{4\kappa - 3\alpha} \). To compute norms of \( \tau^{(1)} \), we need to be a bit more careful. With (4.7) and with a dyadic decomposition, we find

\[
\|\tau^{(1)}\|_2^2 \leq CN^{3\kappa/2} + C \sum_{|p| < N^{\kappa/2}} \log^2(N^{\kappa/2}/|p|) \]

\[
\leq CN^{3\kappa/2} + C \sum_{j=0}^{\infty} \sum_{p} \chi(N^{\kappa/2}2^{-(j+1)} \leq |p| < N^{\kappa/2}2^{-j}) \log^2(N^{\kappa/2}/|p|) \]

\[
\leq CN^{3\kappa/2} + CN^{3\kappa/2} \sum_{j=0}^{\infty} 2^{-3j}(j+1)^2 \leq CN^{3\kappa/2}
\]

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Lemma 4.3. Assume energy \( K \) bounds in Lemma 4.2, we control the action of \( \tau \) in \( \Lambda^* \) \( \| \sum_{p \in \Lambda^*} \) where we used the fact that the ball \( |p| < N^{\kappa/2} 2^{-j} \) contains at most \( C N^{3\kappa/2} 2^{-3j} \) points in \( \Lambda^* \) (the sum could be truncated at \( j = C \kappa \log N \), for sufficiently large \( C \)). Similarly, we find \( \| \tau^{(1)} \|_1 \leq C N^{3\kappa} \) and \( \| \tau^{(1)} \|_{H^1} \leq C N^{5\kappa/2} \). With (3.24), we obtain (4.6). \( \square \)

With the coefficients \( \tau_p \) introduced in (4.4) we can define the antisymmetric operator
\[
T = \frac{1}{2} \sum_{p \in \Lambda^*} \tau_p (b^*_p b^*_r - b_p b_p)
\]
and we consider the corresponding generalized Bogoliubov transformation \( e^T \). Using the bounds in Lemma (4.2), we control the action of \( e^T \) on powers of \( \mathcal{N}_+ \) and on the kinetic energy \( \mathcal{K} \), with the following extension of Lemma 3.1 of [10].

Lemma 4.4. Assume \( (3.24) \) and let \( \tau \) be defined as in (4.4).

i) For every \( j \in \mathbb{N} \), there exists a constant \( C > 0 \) such that
\[
e^{-sT} \mathcal{N}_+^j e^{sT} \leq C N^{j(j+1)^\kappa/4} (N_+ + N^{3\kappa/2})^j
\]
for all \( s \in [0; 1] \).

ii) For every \( j \in \mathbb{N} \), there exists a constant \( C > 0 \) such that
\[
e^{-sT} \mathcal{K} \mathcal{N}_+^j e^{sT} \leq C N^{(j+1)^\kappa/2} (\mathcal{K} + N^{\alpha+2\kappa}) (N_+ + N^{3\kappa/2})^j
\]
for all \( s \in [0; 1] \).

iii) Let \( \mathcal{E}_N = -e^{-T} \mathcal{V}_N e^T - \mathcal{V}_N \). Then, we have
\[
\pm \mathcal{E}_N, \pm e^T \mathcal{E}_N e^{-T} \leq C N^{-\frac{1}{2} \alpha+\frac{3\kappa}{2} (\mathcal{K} + N^{\alpha+2\kappa}) (N_+ + N^{3\kappa/2})}.
\]

To prove Lemma 4.3 (in particular, part iii)), it is useful to bound the potential energy operator in terms of \( \mathcal{K} \) and \( \mathcal{N}_+ \).

Lemma 4.4. There exists a constant \( C > 0 \) such that on \( \mathcal{F}_+^{\leq N} \) we have
\[
\mathcal{V}_N \leq C \mathcal{K} \mathcal{N}_+.
\]

Proof. For \( \xi \in \mathcal{F}_+^{\leq N} \), we bound
\[
\langle \xi, \mathcal{V}_N \xi \rangle \leq \frac{1}{N^{1-\kappa}} \sum_{p,q,r \in \Lambda^*_+ \atop r \neq -p-q} |\mathcal{V}(\xi r / N^{1-\kappa})| ||a_{r+p+r} a_{q}|| ||a_{q+r} a_{p}||
\]
\[
\leq \frac{1}{N^{1-\kappa}} \sum_{r \in \Lambda^*_+} |\mathcal{V}(\xi r / N^{1-\kappa})| \sum_{p,q \in \Lambda^*_+ \atop r \neq -p-q} \frac{(p+r)^2 (q+r)^2} ||a_{r+p} a_{q}||^2
\]
\[
\leq \left[ \sup_{q \in \Lambda^*_+} \frac{1}{N^{1-\kappa}} \sum_{r \in \Lambda^*_+ \atop r \neq -q} |\mathcal{V}(\xi r / N^{1-\kappa})| (q+r)^2 \right] \left[ \mathcal{K}^{1/2} \mathcal{N}_+^{1/2} \xi \right]^2 \leq \frac{C}{N} \left[ \mathcal{K}^{1/2} \mathcal{N}_+^{1/2} \xi \right]^2.
\]
\( \square \)
We can now proceed with the proof of Lemma 4.3.

Proof of Lemma 4.3. To prove i) we will use induction over \( j \). For \( j = 0 \), the claim is trivial. So we assume (4.10) to hold with \( j \) replaced by \( j - 1 \), for some \( j \geq 1 \). To prove it holds also for \( j \), we compute the commutator

\[
[N_j^\pm, T] = \frac{1}{2} \sum_{p \in \Lambda^\pm_j} \tau_p b_p^* b_{-p}^* [(N_+ + 2)^j - N_j^\pm] + \text{h.c.} \tag{4.14}
\]

Since \((N_+ + 2)^j - N_j^\pm \leq 2jN_+^{j-1} + C(N_+ + 1)^{j-2}\), for a constant \( C > 0 \) depending on \( j \), we obtain

\[
|\langle \xi | [N_j^\pm, T] | \xi \rangle | \leq \sum_{p \in \Lambda^\pm_j} |\tau_p| \| b_p [N_j^\pm - (N_+ - 2)^j]^{1/2} \xi \| \| b_{-p}^* [(N_+ + 2)^j - N_j^\pm]^{1/2} \xi \|
\]

\[
\leq \sum_{p \in \Lambda^\pm_j} |\tau_p| \| b_p [T_j^\pm - (N_+ - 2)^j]^{1/2} \xi \|
\]

\[
\times \left( \| b_{-p} [(N_+ + 2)^j - N_j^\pm]^{1/2} \xi \| + \| (N_+ + 2)^j - N_j^\pm \|^{1/2} \xi \| \right)
\]

\[
\leq (2j|\tau|_\infty + C)\|N_j^\pm/2\| \|T\|^2 + C|\tau|_2^2 \left\| (N_+ + 1)^{j-1} \xi \right\|^2.
\]

For \( s \in [0; 1] \), we set \( \xi_s = e^{sT} \xi \) and \( \varphi(s) = \langle \xi_s | (N_+)^j | \xi_s \rangle \). With (4.14), Lemma 4.2 and the inductive assumption, we obtain, for \( N \) large enough,

\[
|\partial_s \varphi(s)| \leq (\log(N^{j\kappa/2}) + C)\varphi(s) + CN^{3\kappa / 2} \langle \xi_s, (N_+ + 1)^{(j-1)} \xi_s \rangle
\]

\[
\leq (\log(N^{j\kappa/2}) + C)\varphi(s) + CN^{j(j-1)\kappa/4} \langle \xi, (N_+ + N^{3\kappa/2}) \rangle \xi
\]

for every \( s \in [0; 1] \). Gronwall’s Lemma implies (4.10).

We proceed analogously to prove ii). For \( j = 0 \) we have

\[
[K, T] = \sum_p p^2 \tau_p (b_p^* b_{-p} + b_{-p} b_p),
\]

hence

\[
\langle \xi | [K, T] | \xi \rangle \leq 2 \sum_{p \in \Lambda^\pm_j} p^2 |\tau_p| \| b_p \xi \| (\| b_{-p} \xi \| + \| \xi \|)
\]

\[
\leq (2|\tau|_\infty + C)\|K^{1/2} \xi \|^2 + C|\tau|_2^2 \| \xi \|^2,
\]

and applying Gronwall’s Lemma as above yields the claim. For \( j \geq 1 \) we have

\[
[KN_j^\pm, T] = K[N_j^\pm, T] + [K, T]N_j^\pm. \tag{4.15}
\]
To bound the first term in the right-hand side of (4.15) we compute

\[
\mathcal{K}[N_+^j, T] = \sum_{p,q \in \Lambda_+^j} q^2 \tau_p a_p^* a_q b_p^* b_{-p} (N_+ + 2)^j - N_+^j \\
+ \sum_{p,q \in \Lambda_+^j} q^2 \tau_p a_p^* a_q b_p b_{-p} (N_+ - (N_+ - 2)^j \\
= \sum_{p,q \in \Lambda_+^j} q^2 \tau_p b_p^* b_p^* a_p^* a_q ((N_+ + 2)^j - N_+^j) \\
+ 2 \sum_{p \in \Lambda_+^j} p^2 \tau_p b_p^* b_{-p} (N_+ + 2)^j - N_+^j \\
+ \sum_{p,q \in \Lambda_+^j} q^2 \tau_p a_p^* b_p b_{-p} (N_+ - (N_+ - 2)^j) =: I + II + III,
\]

We have

\[
|\langle \xi | I | \xi \rangle| \leq \sum_{p,q \in \Lambda_+^j} q^2 |\tau_p| \|b_p b_{-p} (N_+^j - (N_+ - 2)^j)|^{1/2} \| \\
\times \left( \|a_{-p} a_q ((N_+ + 2)^j - N_+^j)|^{1/2} \| + \|a_q ((N_+ + 2)^j - N_+^j)|^{1/2} \| \right) \\
\leq (2j \|\tau\|_\infty + C) \|N_+^j/2 K_+^1/2 \xi\|^2 + C \|\tau\|_2 \|N_+ + 1\|^2 \|\xi\|^2,
\]

and similarly

\[
|\langle \xi | III | \xi \rangle| \leq (2j \|\tau\|_\infty + C) \|N_+^j/2 K_+^1/2 \xi\|^2 + C \|\tau\|_2 \|N_+ + 1\|^2 \|\xi\|^2.
\]

As for II, we find

\[
|\langle \xi | II | \xi \rangle| \leq C \|\tau\|_\infty \|N_+^{(j-1)/2} K_+^1/2 \xi\|^2 + C \|\tau\|_2 \|N_+ + 1\|^2 \|\xi\|^2.
\]

The second term on the right-hand side of (4.15) is given by

\[
[\mathcal{K}, T]N_+^j = \sum_{p \in \Lambda_+^j} p^2 \tau_p (b_p^* b_{-p} + b_p b_{-p}) N_+^j
\]

which can be bounded as above and satisfies the estimate

\[
\langle \xi, [\mathcal{K}, T]N_+^j \xi \rangle \leq (2 \|\tau\|_\infty + C) \|N_+^j/2 K_+^1/2 \xi\|^2 + C \|\tau\|_2 \|N_+^j/2 \xi\|^2.
\]

Putting things together we find

\[
\langle \xi, [\mathcal{K}N_+^j, T] \xi \rangle \leq [(4j + 2) \|\tau\|_\infty + C] \|N_+^j/2 K_+^1/2 \xi\|^2 + C \|\tau\|_2 \|N_+^j/2 \xi\|^2 \\
+ C \|\tau\|_2 \|N_+ + 1\|^2 \|N_+^j/2 \xi\|^2.
\]

Using Gronwall’s Lemma again as above, the inductive assumption, Lemma 4.2 and the bound (4.10) we obtain (4.11).
Finally, we turn to the proof of part \( iii \). We write
\[
\mathcal{E}_N = - \int_0^1 e^{-sT} [\mathcal{V}_N, T] e^{sT} ds,
\]
and using the expression (2.1) in position space, we find
\[
[\mathcal{V}_N, T] = \frac{1}{2} \int dxdy N^2 - 2\kappa V(N^{1-\kappa}(x-y)) \hat{\tau}(x-y)(\hat{\bb}_x \bb_y + \hat{\bb}_y \bb_x) + \int dxdy N^2 - 2\kappa V(N^{1-\kappa}(x-y))[\hat{\bb}_x \bb_y a^*(\hat{\tau}_y) \hat{a}_x + \text{h.c.}].
\]
With \( \|\hat{\tau}\|_\infty \leq \|\tau\|_1, \|\hat{\tau}_y\|_2 = \|\hat{\tau}(\cdot - y)\|_2 = \|\tau\|_2 \) and Lemma 4.2, we bound
\[
\|\langle \xi, [\mathcal{V}_N, T]\xi \rangle \| \leq \int dxdy N^2 - 2\kappa V(N^{1-\kappa}(x-y)) \|\mathcal{V}_N\|_1/2\|\xi\|_1 + \int dxdy N^2 - 2\kappa V(N^{1-\kappa}(x-y)) \|\mathcal{V}_N\|_1/2\|\xi\|_1\|\mathcal{N}\| \leq CN^{(\kappa-1)/4}\|\tau\|_1 \|\mathcal{V}_N\|_1/2\|\xi\|_1\|\mathcal{N}\|.
\]
Using (4.11) and Lemma 4.2 we get
\[
\pm e^{-sT} [\mathcal{V}_N, T] e^{sT} \leq CN^{\frac{1}{4} + \alpha + \frac{\alpha}{2}}(\mathcal{N}_+ + N^{3\kappa/2}),
\]
for every \( s \in [0, 1] \). Together with (4.16), this concludes the proof of \( iii \). \( \square \)

Notice that by Lemma 4.2, \( \|\tau\|_2 \) is not uniformly bounded in \( N \); this prevents us from applying Lemma 3.3 to understand the action of \( e^T \). Nevertheless, following [7], we obtain an expansion similar to (5.13), using the fact that \( \tau \in \ell^1 \) and that \( \|\tau\|_\infty \) only grows logarithmically in \( N \). Using the commutation relations (2.2) we write
\[
e^{-T} \bb_p e^T = \bb_p + \int_0^1 e^{-sB(\tau)} [\bb_p, T] e^{sB(\tau)} ds
\]
\[
= \bb_p + \int_0^1 e^{-sT} \left[ \tau_p^* b_{-p}^* - \left( \frac{\mathcal{N}_+}{N} \tau_p b_{-p}^* + \frac{1}{N} \sum_{q \in \Lambda^*_+} \tau_q b_q a_q a_{-q}^* b_{-p}^* \right) \right] e^{sT} ds
\]
\[
= \bb_p + \tau_p b_{-p}^* - \int_0^1 e^{-sT} \left[ \frac{\mathcal{N}_+}{N} \tau_p b_{-p}^* + \frac{1}{N} \sum_{q \in \Lambda^*_+} \tau_q b_q a_q^* a_{-q}^* b_{-p}^* \right] e^{sT} ds
\]
\[
+ \int_0^1 \int_0^1 e^{-s_2 T} [\tau_p b_{-p}^*, B(\tau)] e^{s_2 T} ds_2 ds_1
\]
Iterating we find, for every \( k \in \mathbb{N} \), the truncated expansion
\[
e^{-T} \bb_p e^T = \sum_{n=0}^k \frac{\tau_p^{2n}}{(2n)!} b_p + \sum_{n=1}^k \frac{\tau_p^{2n-1}}{(2n-1)!} b_{-p}^* + D_p^{(k)},
\]
(4.17)
where we defined the coefficients \( \tilde{\gamma}_p = \cosh \tau_p \), \( \tilde{\sigma}_p = \sinh \tau_p \) and where the remainder operators satisfy, for any \( n \in \mathbb{Z} \), the estimate

\[
\| (N_+ + 1)^{n/2} D_p \xi \| \leq C e^{\| \tau \|_\infty} \int_0^1 \left( |\tau_p| \left\| (N_+ + 1)^{(n+3)/2} e^{sT} \xi \right\| + |\tau|_1 \left\| a_p (N_+ + 1)^{(n+2)/2} e^{sT} \xi \right\| \right) ds
\]

\[
\leq C N^{-1+\kappa/4} \int_0^1 \left( |\tau_p| \left\| (N_+ + 1)^{(n+3)/2} e^{sT} \xi \right\| + N^{\alpha+\kappa} \left\| a_p (N_+ + 1)^{(n+2)/2} e^{sT} \xi \right\| \right) ds,
\]

(4.19)

Observe that, from Lemma 3.2 and from the bounds (4.17), (4.18), (4.19),

\[
|\tilde{\gamma}_p|, |\tilde{\sigma}_p| \leq C \frac{N^{\kappa/4}}{|p|^{1/2}} \quad \text{for } |p| \leq N^{\kappa/2},
\]

(4.20)

whereas \( |\tilde{\gamma}_p| \leq C \) if \( |p| > N^{\kappa/2} \) and

\[
|\tilde{\sigma}_p| \leq C |G_p|/F_p \leq C \left\{ \begin{array}{ll} N^{\kappa/p^2} & \text{if } N^{\kappa/2} \leq |p| \leq N^\alpha \\ N^{2\kappa/|p|} & \text{if } |p| \geq N^\alpha \end{array} \right.
\]

(4.21)

We can now study the action of \( e^T \) on the quadratic operator \( Q_{J_N} \), defined in (3.24).

Lemma 4.5. Assume (3.24). Then we have

\[
e^{-T} Q_{J_N} e^T = -\frac{1}{2} \sum_{p \in \Lambda_+^*} \left[ F_p - \sqrt{F_p^2 - G_p^2} \right] + \sum_{p \in \Lambda_+^*} \sqrt{F_p^2 - G_p^2} a_p^* a_p + \mathcal{E}_{Q_{J_N}},
\]

(4.22)

with the bound

\[
\pm \mathcal{E}_{Q_{J_N}} \leq C N^{-1+2\alpha+7\kappa/2} (K + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2})
\]

(4.23)
Proof. Applying (4.18) to (3.25), we find
\[
e^{-T}Q_J e^T = \sum_{p \in \Lambda^+_N} F_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) (\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^*_p) + \frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ G_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) (\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^*_p) + \text{h.c.} \right] + \mathcal{E}_1,
\]
with
\[
\mathcal{E}_1 = \sum_{p \in \Lambda^+_N} F_p D^*_p e^{-T} b_p e^T + \sum_{p \in \Lambda^+_N} F_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) D_p + \frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ G_p D^*_p e^{-T} b^*_p e^T + \text{h.c.} \right] + \frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ G_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) D^*_p + \text{h.c.} \right] \tag{4.24}
\]
\[
=: I + II + III + IV.
\]
The coefficients \( \tau_p \), defined in (4.4), are exactly chosen to approximately diagonalize
\[
\sum_{p \in \Lambda^+_N} F_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) (\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^*_p) + \frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ G_p (\tilde{\gamma}_p b^*_p + \tilde{\sigma}_p b_{-p}) (\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^*_p) + \text{h.c.} \right]
\]
\[
= -\frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ F_p - \sqrt{F^2_p - G^2_p} \right] + \sum_{p \in \Lambda^+_N} \sqrt{F^2_p - G^2_p} a^*_p a_p + \mathcal{E}_2,
\]
where, from the commutation relations (2.2),
\[
\mathcal{E}_2 = \sum_{p \in \Lambda^+_N} \sqrt{F^2_p - G^2_p} (b^*_p b_p - a^*_p a_p) - \frac{1}{2} \sum_{p \in \Lambda^+_N} \left[ F_p - \sqrt{F^2_p - G^2_p} \right] \left( \frac{N_+}{N} + \frac{a^*_p a_{-p}}{N} \right).
\]
Observe that, by Lemma 4.1
\[
\pm \sum_{p \in \Lambda^+_N} \sqrt{F^2_p - G^2_p} (b^*_p b_p - a^*_p a_p) \leq C \sum_{p \in \Lambda^+_N} F_p a^*_p a_p \frac{N_+}{N} a_p \leq CN^{-1} \sum_{p \in \Lambda^+_N} (p^2 + N^\kappa) a^*_p N_+ a_p \leq CN^{-1}(K + N^\kappa)N_+.
\]
With
\[
0 \leq F_p - \sqrt{F^2_p - G^2_p} \leq G^2_p / F_p
\]
and from the bound in Lemma 4.1, we conclude that
\[
\pm \mathcal{E}_2 \leq CN^{-1}(K + N^\kappa)N_+ + CN^{\alpha + 2\kappa - 1}N_+.
\]
We still have to show that the four terms on the r.h.s. of (4.24) satisfy (4.22). Using Lemma 4.1, Lemma 4.3 and the bound (4.19) with $n = -1$, we can bound

$$|\langle \xi | I | \xi \rangle| \leq C \sum_{p \in \Lambda_+^\times} (p^2 + N^\kappa)\|(N_+ + 1)^{1/2} e^{-T_p} b_p e^T \xi \||\|(N_+ + 1)^{-1/2} D_p \xi\|
$$

$$\leq CN^{-1+\kappa/2} \sum_{p \in \Lambda_+^\times} (p^2 + N^\kappa)\|a_p(N_+ + N^{3\kappa/2})^{1/2} e^T \xi\|
$$

$$\times \int_0^1 \left[|\tau_p|\|(N_+ + 1)e^{sT} \xi\| + N^{\alpha+\kappa}\|a_p(N_+ + 1)^{1/2} e^{sT} \xi\|\right] ds
$$

$$\leq CN^{-1+\alpha+7\kappa/2}\|(K + N^{\alpha+2\kappa})^{1/2}(N_+ + N^{3\kappa/2})^{1/2} \xi\|^2$$

To estimate the second term on the r.h.s. of (4.24) we split $I = I_1 + I_2$, where $I_1$ contains the sum over $|p| \leq N^{\kappa/2}$ and $I_2$ the sum over $|p| > N^{\kappa/2}$. With (4.20) and (4.21) we estimate

$$|\langle \xi | I_2 | \xi \rangle| \leq CN^{-1+\alpha+9\kappa/2}\|(N_+ + N^{3\kappa/2}) \xi\|^2$$

and

$$|\langle \xi | I_2 | \xi \rangle| \leq CN^{-1+\kappa/4} \sum_{|p| > N^{\kappa/2}} p^2 \left[\|a_p(N_+ + 1)^{1/2} \xi\| + |\tau_p|\|(N_+ + 1) \xi\|\right]
$$

$$\times \int_0^1 \left[|\tau_p|\|(N_+ + 1)e^{sT} \xi\| + N^{\alpha+\kappa}\|a_p(N_+ + 1)^{1/2} e^{sT} \xi\|\right] ds
$$

$$\leq CN^{-1+3\kappa/2+13\kappa/4}\|(K + N^{\alpha+2\kappa})^{1/2}(N_+ + N^{3\kappa/2})^{1/2} \xi\|^2$$

Noting that, by Lemma 4.1, \(\|G\|_2 \leq CN^{\kappa+3\alpha/2}\), \(\|G/|\|_2 \leq CN^{\kappa+\alpha/2}\), we further get

$$|\langle \xi, I_2 | \xi \rangle| \leq CN^{-1+5\kappa/4} \sum_{p \in \Lambda_+^\times} |G_p| \|(N_+ + N^{3\kappa/2}) \xi\|
$$

$$\times \int_0^1 \left[|\tau_p|\|(N_+ + 1)e^{sT} \xi\| + N^{\alpha+\kappa}\|a_p(N_+ + 1)^{1/2} e^{sT} \xi\|\right] ds
$$

$$\leq CN^{-1+3\kappa/2+17\kappa/4}(\xi, (K + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2}) \xi)$$

Finally, applying (4.19) with $n = -2$, we find

$$|\langle \xi | IV | \xi \rangle|
$$

$$\leq C\|(N_+ + 1) \xi\| \sum_{p \in \Lambda_+^\times} |G_p|\|(N_+ + 1)^{-1} D_p(\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^c_p) \xi\|
$$

$$\leq N^{-1+\alpha+5\kappa/4}\|(N_+ + 1) \xi\| \sum_{p \in \Lambda_+^\times} |G_p|\|\int_0^1 ds\|(N_+ + 1)^{1/2} e^{sT}(\tilde{\gamma}_p b_p + \tilde{\sigma}_p b^c_p) \xi\|
$$

$$\leq N^{-1+\alpha+3\kappa/2}\|(N_+ + 1) \xi\| \sum_{p \in \Lambda_+^\times} |G_p|\|\|(N_+ + N^{3\kappa/2})^{1/2} \xi\|$$

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Dividing the sum into the domains $|p| < N^{\kappa/2}$ and $|p| > N^{\kappa/2}$, and using the bounds (4.20), $\|G/|\|_2 \leq CN^{\kappa+\alpha/2}$ and Lemma 4.2, we find

$$\langle \xi, IV \xi \rangle \leq CN^{-1+2\alpha+7\kappa/2}||K^{-1/2}(N_++N^{3\kappa/2})^{1/2}||^2$$

Combined with the previous bounds, this implies (4.23).

We define now our final excitation Hamiltonian

$$\mathcal{M}_N = e^{-T} J_N e^T,$$

and we introduce the notation

$$E_{\text{error}} = \frac{1}{2} \sum_{p \in \Lambda^*_+} \left[ \sqrt{|p|^2 + 16\pi a_0 N^\kappa p^2} - p^2 - 8\pi a_0 N^\kappa + \frac{(8\pi a_0 N^\kappa)^2}{2p^2} \right]$$

Proposition 4.6. Assume $\kappa, \alpha, \beta$ satisfy the conditions (3.21). Then

$$\mathcal{M}_N = 4\pi a_0 N^\kappa (N-1) + e_\Lambda(a_0 N^\kappa)^2 + E_{\text{Bog}} + \sum_{p \in \Lambda^*_+} \sqrt{|p|^2 + 16\pi a_0 N^\kappa |p|^2} a_p^* a_p + \mathcal{E}_N + \mathcal{E}_{\mathcal{M}_N},$$

with $e_\Lambda$ as defined in (3.5) and where the error term is such that

$$\pm e^T \mathcal{E}_{\mathcal{M}_N} e^{-T} \leq C(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}) + C(N^{-1/2+\alpha+7\kappa/2} + N^{-1+2\alpha+11\kappa/2})(N_++N^{3\kappa/2}) + CN^{-1/2+7\kappa+\alpha}(H_N + N_+^2 + 1)(N_+ + 1) + CN^{\frac{5}{2}-\frac{\beta}{2}} (K+1)(N_+ + 1).$$

Moreover, for every $\delta > 0$, we find

$$\pm \mathcal{E}_{\mathcal{M}_N} \leq C(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}) + C N^{-\frac{1}{2}+7\kappa+\alpha}(K + N^{\alpha+2\kappa}) (N_++N^{3\kappa/2})^2 + C(N^{\kappa-\frac{\beta}{2}} + N^{\frac{7\kappa}{2}})(N_++N^{3\kappa/2}) + CN^{\frac{5}{2}-\frac{\beta}{2}} (K + N^{\alpha+2\kappa}) + \delta \mathcal{V}_N + C \delta N^{-1/2+\alpha+7\kappa/2} (K + N^{\alpha+2\kappa})(N_++N^{3\kappa/2}) + C \delta^{-1} N^{\frac{3}{2}}(N_++N^{3\kappa/2}).$$

Remark: Similarly as remarked after Prop. 3.7, the second estimate (4.27) for the error term $\mathcal{E}_{\mathcal{M}_N}$ will be useful to prove upper bounds on the eigenvalues of $\mathcal{M}_N$.

Proof. From Prop. 3.7 and Lemma 4.5 we obtain that

$$\mathcal{M}_N = C J_N - \frac{1}{2} \sum_{p \in \Lambda^*_+} \left[ F_p - \sqrt{F_p^2 - G_p^2} \right] + \sum_{p \in \Lambda^*_+} \sqrt{F_p^2 - G_p^2} a_p^* a_p + \mathcal{V}_N + \mathcal{E}_{\mathcal{M}_N} + e^{-T} \mathcal{E}_{\mathcal{M}_N} e^T$$

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where $\mathcal{E}_{J_N}$, $\mathcal{E}_{Q_{J_N}}$ satisfy the bounds (3.28), (4.23) while $\mathcal{E}_{V_N}$ and $e^T \mathcal{E}_{V_N} e^{-T}$ satisfy (4.12). Using Lemma 4.3 to control the action of $e^T$ on $\mathcal{E}_{Q_{J_N}}$, we find that

$$
\pm e^T \left[ \mathcal{E}_{V_N} + \mathcal{E}_{Q_{J_N}} + e^{-T} \mathcal{E}_{J_N} e^T \right] e^{-T} \\
\leq C(N^{-1/2+\alpha+7\kappa/2} + N^{-1+2\alpha+11\kappa/2})(\mathcal{K} + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2}) \\
+ C(N^{-1/2+\delta\kappa/2} + N^{\delta\kappa/2} + N^{\delta\kappa/2}) (\mathcal{K} + N^{\alpha+2\kappa}) + \mathcal{E}_{V_N} \\
+ C\delta N^{-1/2+\alpha+7\kappa/2}(\mathcal{K} + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2}) + C\delta^{-1}N^{\delta\kappa/2}(N_+ + N^{3\kappa/2}).
$$

(4.28)

Using (3.29), instead of (3.28), to control $\mathcal{E}_{J_N}$, we also find

$$
\pm e^T \left[ \mathcal{E}_{V_N} + \mathcal{E}_{Q_{J_N}} + e^{-T} \mathcal{E}_{J_N} e^T \right] \\
\leq C N^{-1/2+\alpha+7\kappa/2}(\mathcal{K} + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2})^2 \\
+ C(N^{-1/2+\alpha+7\kappa/2} + N^{\alpha+2\kappa})(\mathcal{K} + N^{\alpha+2\kappa}) + \mathcal{E}_{V_N} \\
+ C\delta N^{-1/2+\alpha+7\kappa/2}(\mathcal{K} + N^{\alpha+2\kappa})(N_+ + N^{3\kappa/2}) + C\delta^{-1}N^{\alpha+2\kappa/2}(N_+ + N^{3\kappa/2}).
$$

(4.29)

From (3.26), (3.27), we obtain

$$
\sqrt{F_p^2 - G_p^2} = \sqrt{|p|^4 + 2p^2 N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p},
$$

and with the bound $| (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 | \leq C |p|/N^{1-\kappa}$ and the approximation (4.34), we find

$$
|N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - 8\pi \alpha_0 N^\kappa| \leq C N^{2\kappa-1} |p|.
$$

(4.30)

Thus,

$$
\sum_{p \in \Lambda_0^+} \sqrt{F_p^2 - G_p^2} a_p^* a_p = \sum_{p \in \Lambda_0^+} \sqrt{|p|^4 + 16\pi \alpha_0 N^\kappa p^2 a_p^* a_p} + \mathcal{E}_1
$$

(4.31)

with $\pm \mathcal{E}_1 \leq C N^{-1+2\kappa} \mathcal{K}$ and, from Lemma 4.3

$$
\pm e^T \mathcal{E}_1 e^{-T} \leq C N^{-1+5\kappa/2}(\mathcal{K} + N^{\alpha+2\kappa}).
$$

Let us now turn our attention to the constant term

$$
C_{MN} := C_{J_N} - \frac{1}{2} \sum_{p \in \Lambda_0^+} [F_p - \sqrt{F_p^2 - G_p^2}].
$$

With the notation introduced in (1.3) and in (4.20), we will show that

$$
C_{MN} = 4\pi \alpha_0 N^\kappa (N - 1) + E_{Bog} + e_\Delta (\alpha_0 N^\kappa)^2 + O(N^{-1+3\kappa/2+\alpha} + N^{3\kappa/2}).
$$

(4.32)
Combined with (4.28), with (4.29) and with the bounds for the error $\delta_1$ introduced in (4.31), (4.32) shows (4.26) and (4.27) and completes therefore the proof of Prop. 4.6.

To show (4.32), we first observe that, from (3.19), (3.25), (3.26) and (3.27),

$$C_{MN} = \frac{N}{2}N^\kappa \hat{V}(0) + \sum_{p \in P_H} \left[ p^2 \sigma_p^2 + N^\kappa \hat{V}(p/N^{1-\kappa}) (\sigma_p \gamma_p + \sigma_p^2) \right]$$

$$+ \sum_{p \in P_H} \frac{N^\kappa}{2N} \left[ \hat{V}((p-q)/N^{1-\kappa}) \gamma_q \sigma_q \gamma_p \sigma_p - \frac{1}{N} \sum_{u \in \Lambda^*_+} \sigma_u^2 \sum_{p \in \Lambda^*} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_p \right]$$

$$+ \frac{1}{2} \sum_{p \in \Lambda^*_+} \left[ \sqrt{|p|^4 + 2p^2 N^\kappa \hat{V}((./N^{1-\kappa}) \hat{f}_N)_p} - (\gamma_p^2 + \sigma_p^2)p^2 \right]$$

$$- (\gamma_p + \sigma_p^2)N^\kappa \hat{V}((./N^{1-\kappa}) \hat{f}_N)_p \right].$$

Recalling that $\gamma_p = 1$ and $\sigma_p = 0$ for $p \in P_H$, we find that $p^2 \sigma_p^2 - (1/2)(\gamma_p^2 + \sigma_p^2)p^2 = -p^2/2$, for all $p \in \Lambda^*_+$. Using the bound

$$\sum_{p \in \Lambda^*_+} \hat{V}((p-q)/N^{1-\kappa})||\eta_p| \leq CN,$$

we find

$$\frac{1}{N} \sum_{u \in \Lambda^*_+} \sigma_u^2 \sum_{p \in \Lambda^*} N^\kappa \left| \hat{V}(p/N^{1-\kappa}) \eta_p \right| \leq CN^{3\kappa - \alpha}$$

as well as

$$\sum_{p \in P_H} N^\kappa \hat{V}(p/N^{1-\kappa}) (\sigma_p \gamma_p + \sigma_p^2) - \frac{1}{2} \sum_{p \in \Lambda^*_+} (\gamma_p + \sigma_p^2)N^\kappa \hat{V}((./N^{1-\kappa}) \hat{f}_N)_p$$

$$= -\frac{1}{2} \sum_{p \in \Lambda^*_+} N^\kappa \hat{V}((./N^{1-\kappa}) \hat{f}_N)_p - \frac{N^\kappa}{N} \sum_{p \in P_H} (\gamma_p \sigma_p + \sigma_p^2) \hat{f}_N)_p$$

$$= -\frac{1}{2} \sum_{p \in \Lambda^*_+} N^\kappa \hat{V}((./N^{1-\kappa}) \hat{f}_N)_p - \frac{N^\kappa}{N} \sum_{p \in P_H} \gamma_p \sigma_p \hat{f}_N)_p + \mathcal{O}(N^{3\kappa - \alpha}).$$
Thus, we have
\[ C_{MN} = \frac{N - 1}{2} N^\kappa \hat{V}(0) - \frac{N^\kappa}{N} \sum_{p \in P_H} \gamma_p \sigma_p (\hat{V}(\cdot/N^{1-\kappa}) * \eta)_p \]
\[ + \frac{1}{N} \sum_{p \in P_H} p^2 \eta_p^2 + \frac{N^\kappa}{2N} (\hat{V}(\cdot/N^{1-\kappa}) * \eta)_p \eta_p \]
\[ - \frac{N^2}{2N} \sum_{p \in \Lambda^*_H} \left[ \hat{V}((p-q)/N^{1-\kappa}) \gamma_q \sigma_q \gamma_p \sigma_p + E_{Bog,N} + O(N^{3\kappa - \alpha}) \right], \]
\[ (4.33) \]

where we defined
\[ E_{Bog,N} = \frac{1}{2} \sum_{p \in \Lambda^*_H} \left[ \sqrt{|p|^4 + 2p^2 N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p - p^2} \right. \]
\[ - N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p + \frac{N^{2\kappa} (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p^2}{2p^2} \]}
\[ = \frac{1}{2} \sum_{p \in \Lambda^*_H} e_{p,N}. \]

To compare \( E_{Bog,N} \) with its limiting value \( E_{Bog} \), we first compare the summands \( e_{p,N} \) in \( E_{Bog,N} \) with the corresponding summands
\[ e_p = \sqrt{|p|^4 + 16\pi a_0 N^\kappa |p|^2 - |p|^2 - 8\pi a_0 N^\kappa + \left( \frac{8\pi a_0 N^\kappa}{2 |p|^2} \right)^2} \]
in \( E_{Bog} \). On one hand, Taylor expanding the square root we see that
\[ |e_{p,N}|, |e_p| \leq CN^6 \kappa |p|^{-4}, \]
which yields
\[ \frac{1}{2} \sum_{|p| > N} |e_{p,N} - e_p| \leq CN^{6\kappa - 1}, \]
\[ (4.34) \]
for a constant \( C \) independent of \( p, N \). On the other hand, \( (4.30) \) implies, expanding once again the square roots in \( e_p, e_{p,N} \),
\[ |e_p - e_{p,N}| \leq \frac{CN^{6\kappa}}{|p|^4} \left| (8\pi a_0)^3 - (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p^3 \right| \leq \frac{CN^{8\kappa - 1}}{|p|^3}. \]
\[ (4.35) \]
Combining \( (4.34) \) and \( (4.35) \) we get, for \( N \) large enough,
\[ |E_{Bog} - E_{Bog,N}| \leq CN^{8\kappa - 1} \log N \leq CN^{-1+10\kappa}. \]
\[ (4.36) \]
We now analyze the remaining terms on the right-hand side of (4.33). Using the scattering equation (3.10), the bound (4.3) and the approximation (3.11), we find

\[- \frac{N^\kappa}{2} \hat{V}(0) + \frac{1}{N} \sum_{p \in P_H} \left[ p^2 \eta_p^2 + \frac{N^\kappa}{2N} (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_p \right] \]

\[= - \frac{N^\kappa}{2} \hat{V}(0) - \frac{N^\kappa}{2N} \sum_{p \in L^*} \hat{V}(p/N^{1-\kappa}) \eta_p + \mathcal{O}(N^{2\kappa+\alpha-1}) \]  \hspace{1cm} (4.37)

\[= - \frac{1}{2} N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N) \]  \hspace{1cm} + \mathcal{O}(N^{2\kappa+\alpha-1})

\[= - 4\pi N^\kappa a_0 + \mathcal{O}(N^{2\kappa+\alpha-1}), \]

with the remainder \(\mathcal{O}(N^{2\kappa+\alpha-1})\) arising from the missing low momenta in the sum on the first line. Next, we combine the second and the fifth term on the r.h.s. of (4.33). To this end, we write

\[- \frac{N^\kappa}{N} \sum_{p \in P_H} \gamma_p \sigma_p (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_p + \frac{N^\kappa}{2N} \sum_{p,q \in P_H} \hat{V}((p-q)/N^{1-\kappa}) \gamma_q \sigma_q \gamma_p \sigma_p \]

\[= \frac{N^\kappa}{N} \sum_{p \in P_H} \sum_{q \in P_H} \hat{V}((p-q)/N^{1-\kappa}) \left[ \frac{1}{2} \gamma_p \sigma_p \gamma_q \sigma_q - \gamma_p \sigma_p \eta_q \right] \]  \hspace{1cm} (4.38)

\[- \frac{N^\kappa}{N} \sum_{p \in P_H} \sum_{q \in P_H} \hat{V}((p-q)/N^{1-\kappa}) \gamma_p \sigma_p \eta_q. \]

To deal with the first term, we write

\[\frac{1}{2} \gamma_p \sigma_p \gamma_q \sigma_q - \gamma_p \sigma_p \eta_q \]

\[= \frac{1}{2} (\gamma_p \sigma_p - \eta_p + \eta_p)(\gamma_q \sigma_q - \eta_q + \eta_q) - \left( \gamma_p \sigma_p - \eta_p \right) \eta_q + \eta_p \eta_q \]

\[= \frac{1}{2} (\gamma_p \sigma_p - \eta_p)(\gamma_q \sigma_q - \eta_q) + \frac{1}{2} (\gamma_q \sigma_q - \eta_q) \eta_p - (\gamma_p \sigma_p - \eta_p) \eta_q - \frac{1}{2} \eta_p \eta_q. \]

With the bound \( |\gamma_p \sigma_p - \eta_p| \leq C |p|^{-5} \) (and noticing, exchanging \( p \) and \( q \), that the contribution of the terms in square brackets vanishes), we obtain

\[\frac{N^\kappa}{N} \sum_{p \in P_H} \sum_{q \in P_H} \hat{V}((p-q)/N^{1-\kappa}) \left[ \frac{1}{2} \gamma_p \sigma_p \gamma_q \sigma_q - \gamma_p \sigma_p \eta_q \right] \]

\[= - \frac{N^\kappa}{2N} \sum_{p \in P_H} \sum_{q \in P_H} \hat{V}((p-q)/N^{1-\kappa}) \eta_p \eta_q + \mathcal{O}(N^{-1+\kappa}). \]

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Inserting this into (4.38) we find (recall \( f_N = 1 + N^{-1} \eta \))

\[
- \frac{N^\kappa}{N} \sum_{p \in P_H} \gamma_p \sigma_p (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_p + \frac{N^\kappa}{2N} \sum_{p,q \in P_H} \hat{V}((p - q)/N^{1-\kappa}) \gamma_q \sigma_q \gamma_p \sigma_p
\]

\[
= - \frac{N^\kappa}{2} \sum_{p \in P_H} \left[ (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - \hat{V}(p/N^{1-\kappa}) \right] \eta_p
\]

\[
- \frac{N^\kappa}{N} \sum_{p \in P_H} \sum_{q \in P_H^0} \hat{V}((p - q)/N^{1-\kappa}) \left[ (\gamma_p \sigma_p - \eta_p) \eta_q + \frac{1}{2} \eta_p \eta_q \right] + O(N^{-1+\kappa}),
\]

In the second term, the contribution proportional to \((\gamma_p \sigma_p - \eta_p) \eta_q\) is small, of order \(N^{-1+\kappa}\); the other contribution leads, adding terms with \(p \in P_H^0\) (producing an error of order \(N^{-1+3\kappa+2\alpha}\)) to:

\[
- \frac{N^\kappa}{N} \sum_{p \in P_H} \gamma_p \sigma_p (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_p + \frac{N^\kappa}{2N} \sum_{p,q \in P_H} \hat{V}((p - q)/N^{1-\kappa}) \gamma_q \sigma_q \gamma_p \sigma_p
\]

\[
= - \frac{N^\kappa}{2} \sum_{p \in P_H} \left[ (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p - \hat{V}(p/N^{1-\kappa}) \right] \eta_p
\]

\[
- \frac{N^\kappa}{2N} \sum_{q \in P_H^0} (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_q \eta_q + O(N^{-1+3\kappa+2\alpha}).
\]

The second term in the square bracket can be combined with the leading term in (4.38). In fact, we find

\[
\frac{N^\kappa}{2} \left[ \hat{V}(0) + \frac{1}{N} \sum_{p \in P_H} \hat{V}(p/N^{1-\kappa}) \eta_p \right]
\]

\[
= \frac{N^{1+\kappa}}{2} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 - \frac{N^\kappa}{2} \sum_{p \in P_H^0} \hat{V}(p/N^{1-\kappa}) \eta_p
\]

\[
= 4\pi a_0 N^{1+\kappa} + \frac{N^{1+\kappa}}{2} \left[ (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 - 8\pi a_0 \right] - \frac{N^\kappa}{2} \sum_{p \in P_H^0} \hat{V}(p/N^{1-\kappa}) \eta_p.
\]

The last term can be combined with the other terms on the r.h.s. of (4.39). We find

\[
- \frac{N^\kappa}{2} \sum_{p \in P_H} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p \eta_p - \frac{N^\kappa}{2N} \sum_{p \in P_H^0} (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta)_p \eta_p - \frac{N^\kappa}{2} \sum_{p \in P_H^0} \hat{V}(p/N^{1-\kappa}) \eta_p
\]

\[
= - \frac{N^\kappa}{2} \sum_{p \in \Lambda_+^*} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_p \eta_p - \frac{N^\kappa}{2} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N)_0 \eta_0.
\]
Using again the scattering equation (3.9) and the approximation (3.4), we get
\[ C_{MN} = 4\pi a_0 N^\kappa (N - 1) + E_{Bog} \]
\[ - \frac{1}{2} \sum_{p \in \Lambda_+^*} N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p \eta_p - \sum_{p \in \Lambda_+^*} \frac{N^{2\kappa} (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p^2}{4p^2} \]
\[ - \frac{1}{2} N^\kappa (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_0 \eta_0 + \frac{N^{1+\kappa}}{2} [\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_0 - 8\pi a_0] \]
\[ + O(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}). \]

Using again the scattering equation (3.9) and the approximation (3.4), we get
\[ C_{MN} = -4\pi a_0 N^\kappa (N - 1) + E_{Bog} - \sum_{p \in \Lambda_+^*} \frac{N^{3-\kappa} \lambda_\ell (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p (\hat{\chi}_\ell * \hat{f}_N)_p}{2p^2} \]
\[ + \frac{N^{1+\kappa}}{2} [\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_0 - 8\pi a_0] - 4\pi a_0 N^\kappa \eta_0 + O(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}). \]

By Lemma 3.1, the definition of \( \eta_0 \) and (3.6) we have
\[ \frac{N^{1+\kappa}}{2} [\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_0 - 8\pi a_0] = \frac{6\pi a_0^2 N^{2\kappa}}{\ell} + O(N^{-1+3\kappa}), \]
\[ -4\pi a_0 N^\kappa \eta_0 = \frac{8(\pi a_0 N^\kappa)^2}{5} \ell^2 + O(N^{-1+3\kappa}), \]
and using (4.30) and the bound \(|(\hat{\chi}_\ell * \eta)_p| \leq C\eta^\kappa / p^2 \) (see argument before (4.3)) we get
\[ - \sum_{p \in \Lambda_+^*} \frac{N^{3-\kappa} (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p \lambda_\ell (\hat{\chi}_\ell * \hat{f}_N)_p}{2p^2} \]
\[ = \frac{3a_0}{\ell^3} \sum_{p \in \Lambda_+^*} \frac{N^{2\kappa} (\hat{V}(\cdot/N^{1-\kappa}) * \hat{f}_N)_p \lambda_\ell (\hat{\chi}_\ell * \hat{f}_N)_p + O(N^{-1+3\kappa})}{2p^2} \]
\[ = -\frac{12\pi a_0^2 N^{2\kappa}}{\ell^3} \sum_{p \in \Lambda_+^*} \frac{(\hat{\chi}_\ell * \hat{f}_N)_p}{p^2} + O(N^{-1+3\kappa}) \]
\[ = -\frac{12\pi a_0^2 N^{2\kappa}}{\ell^3} \sum_{p \in \Lambda_+^*} \frac{\hat{\chi}_\ell(p)}{p^2} + O(N^{-1+3\kappa}). \]

In conclusion, we have
\[ C_{MN} = 4\pi a_0 N^\kappa (N - 1) + E_{Bog} \]
\[ + 6\pi a_0^2 N^{2\kappa} \left[ \frac{1}{\ell} + \frac{4\pi}{15} \ell^2 - 2 \sum_{p \in \Lambda_+^*} \frac{\hat{\chi}_\ell(p)}{p^2} \right] + O(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}) \]
\[ = 4\pi a_0 N^\kappa (N - 1) + E_{Bog} + 6\pi a_0^2 N^{2\kappa} I_\ell + O(N^{-1+3\kappa+2\alpha} + N^{3\kappa-\alpha}). \]
It is possible to show that
\[ I_\ell = \frac{4\pi\ell^2}{3} - \frac{8\pi}{3} \lim_{M \to \infty} \sum_{p \in \Lambda^+ | \|p\|_1, |p_2|, |p_3| \leq M} \frac{\cos(\ell |p|)}{|p|^2} \]
and, in particular, that the limit on the r.h.s. exists for every \( \ell \in (0, 1/2) \). Furthermore, this quantity is in fact independent of the particular choice of \( \ell \in (0; 1/2) \). This is proved in [7, Lemma 5.4] and, choosing for instance \( \ell = 1/(2\pi) \), it implies (4.32) (recall the definition (1.3) of \( e_\Lambda \)).

5 Proof of Theorem 1.1

In this section we prove our main result. We assume here that the parameters \( \kappa, \alpha, \beta > 0 \) satisfy (3.24) and also the conditions \( 6\kappa < \alpha < 1/2 - 3\kappa/2 \) and \( \kappa \in [0; 1/44) \), so that Theorem 3.5 holds true.

We set
\[ E_{M_N} = 4\pi a_0 N^\kappa (N-1) + E_{\text{Bog}} + e_\Lambda (a_0 N^\kappa)^2 \]
and we consider the diagonal operator
\[ \mathcal{D} = \sum_{p \in \Lambda^+} \varepsilon_p a_p^* a_p, \]
with \( \varepsilon_p = \sqrt{|p|^4 + 16\pi a_0 N^\kappa |p|^2} \). With this notation, Proposition 4.6 reads
\[ \mathcal{M}_N - E_{M_N} = \mathcal{D} + \mathcal{V}_N + \mathcal{E}_{M_N} \quad (5.1) \]
where the error term \( \mathcal{E}_{M_N} \) satisfies the bounds (4.26) and (4.27).

To prove a lower bound on the ground state energy of \( M_N \) (and later on its excited eigenvalues), we need a-priori estimates on the number and energy of excitations in low-energy states. Suppose that \( \xi_N \in \mathcal{F}_{+}^\infty \), with \( \|\xi_N\| = 1 \) and
\[ \xi_N = \chi (\mathcal{M}_N - E_N \leq N^{\kappa/2+\mu}) \xi_N \quad (5.2) \]
for some \( \mu > 0 \). Recalling that \( \mathcal{M}_N = e^{-T} e^{-A} e^{-B} U H_N U^* e^B e^A e^T \) and defining \( \psi_N = U N e^B e^A e^T \xi_N \in L_2^2(\Lambda^N) \), we find \( \|\psi_N\| = 1 \) and
\[ \psi_N = \chi (H_N - E_N \leq N^{\kappa/2+\mu}) \psi_N \]
From Theorem 3.5 we conclude therefore that
\[ \langle e^{-B} U_N \psi_N, (H_N + 1)(N_+ + 1) e^{-B} U_N \psi_N \rangle \leq C \left[ N^{21\kappa+\epsilon} \zeta^2 + N^{44\kappa+2\epsilon} \right]^2 \]
\[ \langle e^{-B} U_N \psi_N, (N_+ + 1)^3 e^{-B} U_N \psi_N \rangle \leq C \left[ N^{21\kappa+2\mu+\epsilon} + N^{44\kappa+2\epsilon} \right]^3 \]
With $e^{-B} U_N \psi_N = e^A e^T \xi_N$ and with Lemma 3.6 we arrive at
\[
\langle e^T \xi_N, (H_N + 1)(N_+ + 1)e^T \xi_N \rangle \leq C \left( N^{21\kappa+2\mu+\varepsilon} + N^{44\kappa+2} \right)^3
\]
and
\[
\langle e^T \xi_N, (N_+ + 1)^3 e^T \xi_N \rangle \leq C \left( N^{21\kappa+2\mu+\varepsilon} + N^{44\kappa+2} \right)^3
\]
If $\kappa, \mu > 0$ are small enough, we can use this and (4.26) with an appropriate choice of $\alpha, \beta > 0$ (making sure, in particular, that (4.24) holds true) to show that there exists $\varepsilon > 0$ with
\[
\langle \xi_N, E_{M_N} \xi_N \rangle \leq C N^{-\varepsilon}
\]
for all $\xi_N \in F_+^{\leq N}$ satisfying (5.2).

If $\xi_N$ denotes now the ground state of the operator $M_N$, we have $M_N \xi_N = E_N \xi_N$ and (5.2) is certainly satisfied. With (5.1) and (5.3), we obtain
\[
E_N \geq E_{M_N} - C N^{-\varepsilon}
\]
if $\kappa, \varepsilon > 0$ are small enough. Testing (5.1) on the vacuum and using the bound (4.27) to bound the vacuum expectation of $E_{M_N}$ (choosing again $\alpha, \beta > 0$ appropriately), we also find the upper bound
\[
E_N \leq E_{M_N} + C N^{-\varepsilon}
\]
if $\kappa, \varepsilon > 0$ are small enough. We conclude that $|E_N - E_{M_N}| \leq C N^{-\varepsilon}$.

Let us now study excitations. We denote by $\lambda_j$ the eigenvalues of $M_N - E_N$ and by $\nu_j$ those of $D$, indexed in increasing order. Assume $\lambda_j \leq N^{\kappa/2+\mu}$. Then, with the notation $P_\zeta = \chi(M_N - E_N \leq N^{\kappa/2+\mu}) F_+^{\leq N}$ for the spectral subspace of $M_N$, we have (applying the min-max principle for the eigenvalues of $M_N - E_N$)
\[
\lambda_j = \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, (M_N - E_N) \xi_N \rangle
\]
\[
\quad = \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, (M_N - E_N) \xi_N \rangle
\]
\[
\quad \geq \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, (M_N - E_{M_N}) \xi_N \rangle - C N^{-\varepsilon}
\]
if $\kappa, \varepsilon > 0$ are small enough. Here we used the upper bound (5.3) for $E_N$. With (5.1) and using the positivity of $V_N$, we obtain
\[
\lambda_j \geq \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, (D + E_{M_N}) \xi_N \rangle - C N^{-\varepsilon}
\]
From (5.3), we conclude that
\[
\lambda_j \geq \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, D \xi_N \rangle - C N^{-\varepsilon}
\]
\[
\geq \inf_{Y \subset F_+^{\leq N}} \sup_{\xi_N \in Y \subset F_+^{\leq N}} \langle \xi_N, D \xi_N \rangle - C N^{-\varepsilon} = \nu_j - C N^{-\varepsilon}
\]
(5.6)
if $\kappa, \varepsilon > 0$ are small enough.

Finally, we need to establish upper bounds for the eigenvalues $\lambda_j$. To this end, we are going to use eigenvectors of $\mathcal{D}$ as trial states. Notice that the eigenvalues of $\mathcal{D}$ have the form

$$\nu_j = \sum_{p \in \Lambda^*_+} n_p(j) \varepsilon_p,$$

(5.7)

for a sequence $\{n_p(j)\}_{p \in \Lambda^*_+}$ with $n_p(j) \in \mathbb{N}$. An eigenvector of $\mathcal{D}$ associated with the eigenvalue $\nu_j$ has the form

$$\vartheta_j = C_j \prod_{p \in \Lambda^*_+} (a_p^*)^{n_p(j)} \Omega$$

(5.8)

for appropriate normalization constants $C_j$ ($\Omega$ denotes as usual the vacuum in Fock space). If $\nu_j$ is degenerate, the choice of $\vartheta_j$ is not unique, but in the following we work exclusively with eigenvectors of the form (5.8). We are going to need the following lemma, which controls the expectation of $\mathcal{V}_N$ on spaces spanned by the eigenvectors $\vartheta_j$.

**Lemma 5.1.** Let $\vartheta_1, \ldots, \vartheta_m$ be normalized eigenvectors for $\mathcal{D}$ as in (5.8), corresponding to its first $m$ eigenvalues $\nu_1 < \ldots \leq \nu_m < N^{\kappa/2+\mu}$. Then there exists a constant $C > 0$ such that

$$\langle \xi_N, \mathcal{V}_N \xi_N \rangle \leq C N^{-1+9\kappa/4+5\mu/2}$$

for all $\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_m)$ with $\|\xi_N\| = 1$.

**Proof.** The proof is very similar to the proof of [7, Lemma 6.1]. Since $\varepsilon_p > p^2$, we have $a_p \vartheta_j = 0$ if $|p| \geq N^{\kappa/4+\mu/2}$ and $j \in \{1, \ldots, m\}$, hence also for all $\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_m)$. This implies with $N_+ \leq CD$ and $[N_+, \mathcal{D}] = 0$ that

$$\langle \xi_N, \mathcal{V}_N \xi_N \rangle \leq \frac{N^\kappa}{2N} \sum_{|p,q,r| \leq N_+} \hat{V}(r/N^{1-\kappa}) |a_p a_q \xi_N||a_{q+r} a_p \xi_N||$$

$$\leq \frac{CN^\kappa}{N} \sum_{|r| \leq N^{\kappa/4+\mu/2}} \| (N_+ + 1) \xi_N \|^2$$

$$\leq C N^{-1+7\kappa/4+3\mu/2} \| (\mathcal{D} + 1) \xi_N \|^2 \leq C N^{-1+9\kappa/4+5\mu/2}$$

$\square$

Suppose now that $\lambda_j \leq N^{\kappa/2+\mu}$. Notice that, from the lower bound (5.6), this also implies that $\nu_j \leq C N^{\kappa/2+\mu}$. By the min-max principle, we have

$$\lambda_j = \inf_{Y \subset \mathcal{F}^2_{\geq N}} \sup_{\xi_N \in Y} \langle \xi_N, (\mathcal{M}_N - E_N) \xi_N \rangle \leq \sup_{\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_j)} \langle \xi_N, (\mathcal{M}_N - E_N) \xi_N \rangle$$

$$\leq \sup_{\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_j)} \langle \xi_N, (\mathcal{M}_N - E_{\mathcal{M}_N}) \xi_N \rangle + N^{-\varepsilon}$$

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by (5.4). With (5.1) and Lemma 5.1, we obtain
\[
\lambda_j \leq \nu_j + \sup_{\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_j)} \langle \xi_N, \mathcal{N}_N \xi_N \rangle + \langle \xi_N, \mathcal{E}_M \xi_N \rangle
\]
\[
\leq \nu_j + CN^{-\varepsilon} + \sup_{\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_j)} \langle \xi_N, \mathcal{E}_M \xi_N \rangle.
\]
if \(\kappa, \mu, \varepsilon > 0\) are small enough. Using now (4.27) (with an appropriate choice of \(\alpha, \beta > 0\)), estimating, for \(n = 0, 1, 2, \ldots\),
\[
\mathcal{N}_N^{n+1} \leq C\mathcal{N}_N^{n} \leq C\mathcal{D}_N^{n+1}
\]
and observing that \(\mathcal{D}_N^{n+1} \leq \nu_j^{n+1} \leq CN^{(\kappa/2+\mu)(n+1)}\) on \(\text{Span}(\vartheta_1, \ldots, \vartheta_j)\), since \(\nu_j \leq CN^{\kappa/2+\mu}\), we obtain that \(\langle \xi_N, \mathcal{E}_M \xi_N \rangle \leq CN^{-\varepsilon}\) for all normalized \(\xi_N \in \text{Span}(\vartheta_1, \ldots, \vartheta_j)\).

A Analysis of \(\mathcal{G}_N\)

In this section, we sketch the proof of Proposition 3.4. The proof goes along the same lines as the one of [7, Prop. 3.2], taking into account the different scaling. We write
\[
\mathcal{G}_N = \mathcal{G}_N^{(0)} + \mathcal{G}_N^{(2)} + \mathcal{G}_N^{(3)} + \mathcal{G}_N^{(4)},
\]
with \(\mathcal{G}_N^{(i)} = e^{-B} \mathcal{L}_N^{(i)} e^B\) for \(i = 0, 2, 3, 4\) and with \(B\) as defined in (3.12); we analyze these terms individually in the next subsections. First, we need some rough bound to control the growth of operators of the form \((H_N + 1)(\mathcal{N}_N + 1)^j\) under the action of \(B\). In the next lemma, as well as in the rest of the section, we assume \(\alpha > 2\kappa\).

**Lemma A.1.** For every \(j \in \mathbb{N}\) there exists a constant \(C\) such that:
\[
e^{-B} \mathcal{N}_N^{j} e^B \leq C \mathcal{N}_N^{j} + CN^{1+\kappa} \mathcal{N}_N^{j+1}
\]
\[
e^{-B} \mathcal{V}_N \mathcal{N}_N^{j} e^B \leq C \mathcal{V}_N \mathcal{N}_N^{j} + CN^{1+\kappa} \mathcal{N}_N^{j+1}.
\]

The proof is analogous to that of [7, Lemma 7.1] and we omit it. We have to take into account the different scaling, producing the growth \(\|\eta\|^2_{\mathcal{H}} \leq CN^{1+\kappa}\) instead of \(N\).

A.1 Analysis of \(\mathcal{G}_N^{(0)}\)

From (2.1) and Lemma 3.2 we immediately obtain
\[
\mathcal{G}_N^{(0)} = \frac{N - 1}{2} \bar{N}^{\kappa} \mathcal{V}(0) + \mathcal{E}_N^{(0)}\]
with
\[
\pm \mathcal{E}_N^{(0)} \leq CN^{-1+\kappa} \mathcal{N}_N^{1+1}.
\]
A.2 Analysis of $G^{(2)}_N$

First we write $\mathcal{L}^{(2)}_N = K + \mathcal{L}^{(2,V)}_N$, with

$$
\mathcal{L}^{(2,V)}_N = \sum_{p \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \left[ b_p^* b_p - \frac{1}{N} a_p^* a_p \right] + \frac{N^\kappa}{2} \sum_{p \in \Lambda^*_+} \hat{V}(p/N^{1-\kappa}) [b_p^* b_{-p}^* + b_p b_{-p}] ,
$$

and we proceed to analyze the terms individually. We start with the kinetic energy.

Lemma A.2. We have

$$
e^{-B} K e^B = K + \sum_{p \in \Lambda^*_+} \left[ p^2 \sigma_p^2 \left( 1 - \frac{N_+}{N} + \frac{1}{N} \right) + 2p^2 \gamma_p \sigma_p (b_p^* b_{-p}^* + h.c.) \right] + \sum_{p \in \Lambda^*_+} \frac{1}{N} p^2 \sigma_p^2 \sum_{q \in \Lambda^*_+} \left[ (\gamma^2_q + \sigma_q^2) b_q^* b_q + \gamma_q \sigma_q (b_q^* b_{-q}^* + h.c.) + \sigma_q^2 \right] + \sum_{p \in \Lambda^*_+} [p^2 \sigma_p b_{-p}^* d_p + h.c.] + \mathcal{E}^{(K)}_{\tilde{N}} ,
$$

(A.4)

with

$$\pm \mathcal{E}^{(K)}_{\tilde{N}} \leq N^{-\frac{1}{2}+\kappa}(K + \mathcal{N}^2_+ + 1)(\mathcal{N}_+ + 1) .$$

(A.5)

Proof. We first write $K$ in terms of the operators $b_p, b_p^*$ in order to apply the expansion (3.13). We find, with $b_p^* b_p = a_p^*(1 - \mathcal{N}_+ / N) a_p$,

$$
e^{-B} K e^B = \sum_{p \in \Lambda^*_+} p^2 e^{-B} b_p^* b_p e^B + \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 e^{-B} b_p^* b_q^* b_q b_p e^B + \tilde{E}_1 ,$$

(A.6)

where, with Lemma A.1

$$\pm \tilde{E}_1 \leq C N^{-2} e^{-B} K (\mathcal{N}_+ + 1)^2 e^B \leq C N^{-1+\kappa} [K (\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^2] .$$

Using (3.13) we compute

$$
\sum_{p \in \Lambda^*_+} p^2 e^{-B} b_p^* b_p e^B = \sum_{p \in \Lambda^*_+} p^2 \left[ (\gamma^2_p + \sigma^2_p) b_p^* b_p + \gamma_p \sigma_p (b_p^* b_{-p}^* + b_p b_{-p}) + \sigma^2_p \left( 1 - \frac{N_+}{N} \right) \right] + \sum_{p \in \Lambda^*_+} \left( p^2 \sigma_p^2 b_{-p}^* d_p + h.c. \right) + \tilde{E}_2 ,
$$

with

$$\tilde{E}_2 = \sum_{p \in \Lambda^*_+} p^2 \left[ \gamma_p (b_p^* d_p + h.c.) + d_p^* d_p - \frac{1}{N} \sigma^2_p a_p^* a_p \right] .$$
With (3.14), (3.16), we obtain

$$\pm \tilde{\mathcal{E}}_2 \leq N^{(-1+\kappa)/2} \left[ (K + 1)(N_+ + 1) + (N_+ + 1)^3 \right].$$

Next we consider the second term on the r.h.s. of (A.6). We first consider the operator

$$D = \sum_{q \in \Lambda^*} e^{-B} b_q^* b_q e^B = \sum_{q \in \Lambda^*_+} \left[ (\gamma_q^2 + \sigma_q^2) b_q^* b_q + \gamma_q \sigma_q (b_q^* b_{-q} + b_q^* b_{-q}) + \sigma_q^2 \right] + \tilde{\mathcal{E}}_3,$$

with $$\pm \tilde{\mathcal{E}}_3 \leq CN^{-1}(N_+ + 1)^2$$. From Lemma 3.2 we also have $$\pm D \leq C(N_+ + 1)$$. From these bounds we find

$$\frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 e^{-B} b_p^* b_q b_p e^B = \frac{1}{N} \sum_{p \in \Lambda^*_+} p^2 (\gamma_p b_p^* + \sigma_p b_{-p} + d_p^*) (\gamma_p b_p + \sigma_p b_{-p} + d_p)$$

$$= \frac{1}{N} \sum_{p \in \Lambda^*_+} p^2 \sigma_p^2 b_p D b_p^* + \tilde{\mathcal{E}}_4$$

$$= \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 (\gamma_q^2 + \sigma_q^2) b_p^* b_q b_p^* + \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 \sigma_q^2 b_p^* b_p$$

$$+ \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 \gamma_q \sigma_q b_p (b_{-q}^* + b_q) b_p^* + \tilde{\mathcal{E}}_5,$$

where

$$\pm \tilde{\mathcal{E}}_5 \leq CN^{-\frac{1}{2}+\kappa}(K + 2^2)(N_+ + 1).$$

Bringing the expression to normal order and bounding some additional errors, we find

$$\frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 e^{-B} b_p^* b_q b_p e^B = \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 (\gamma_q^2 + \sigma_q^2) b_q^* b_q + \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 \sigma_q^2$$

$$+ \frac{1}{N} \sum_{p,q \in \Lambda^*_+} p^2 \sigma_p^2 \gamma_q \sigma_q (b_{-q}^* + b_q) + \text{h.c.} + \frac{1}{N} \sum_{p \in \Lambda^*_+} p^2 \sigma_p^2 + \tilde{\mathcal{E}}_6,$$

with

$$\pm \tilde{\mathcal{E}}_6 \leq CN^{-\frac{1}{2}+\kappa}(K + 2^2 + 1)(N_+ + 1).$$

This concludes the proof of the lemma. \(\square\)

We proceed with the potential term $$\mathcal{G}_N^{(2,V)} = e^{-B} \mathcal{G}_N^{(2,V)} e^B$$. 

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Lemma A.3. We have
\[ G_{N}^{(2,V)} = N^{\kappa} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) \sigma_{p}^{2} + N^{\kappa} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) \gamma_{p} \sigma_{p}(1 - N_{+}/N) \]
\[ + \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})(\gamma_{p} + \sigma_{p})^{2}(2b_{p}^{*}b_{p}^{*} + b_{p}^{*}b_{p}^{*} + b_{p}b_{p}) \]
\[ + \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})[(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})d_{p}^{*} + d_{p}^{*}(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})] + \text{h.c.} \]  
(A.7)
with
\[ \pm \hat{E}_{\kappa}^{(V)} \leq CN^{-1/2}(N_{+} + 1)^{3}. \]  
(A.8)

Proof. From the definition of \( L_{N}^{(2,V)} \), using Lemma 3.2 and the bounds (3.16), we have
\[ G_{N}^{(2,V)} = \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) e^{-B}[2b_{p}^{*}b_{p}^{*} + b_{p}^{*}b_{p}^{*} + b_{p}b_{p}]e^{B} + \hat{E}_{\tau}, \]
with
\[ \pm \hat{E}_{\tau} \leq N^{-1}(N_{+} + 1). \]

Using again the bounds (3.16) we find
\[ \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) e^{-B}[2b_{p}^{*}b_{p}^{*} + b_{p}^{*}b_{p}^{*} + b_{p}b_{p}]e^{B} \]
\[ = N^{\kappa} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})[\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*}][\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*}] \]
\[ + \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})[(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})d_{p}^{*} + d_{p}^{*}(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})] + \text{h.c.} \]  
(A.9)
with \( \pm \hat{E}_{\kappa} \leq N^{-1/2}(N_{+} + 1)^{3} \) (for \( \kappa < 1/2 \)). Bringing (A.9) into normal order, we get
\[ G_{N}^{(2,V)} = N^{\kappa} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) \sigma_{p}^{2} + N^{\kappa} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa}) \gamma_{p} \sigma_{p}(1 - N_{+}/N) \]
\[ + \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})(\gamma_{p} + \sigma_{p})^{2}(2b_{p}^{*}b_{p}^{*} + b_{p}^{*}b_{p}^{*} + b_{p}b_{p}) \]
\[ + \frac{N^{\kappa}}{2} \sum_{p \in \Lambda_{+}^{3}} \hat{V}(p/N^{1-\kappa})[(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})d_{p}^{*} + d_{p}^{*}(\gamma_{p}b_{p}^{*} + \sigma_{p}b_{p}^{*})] + \text{h.c.} \]
\[ + \hat{E}_{\tau} + \hat{E}_{\kappa} + \hat{E}_{\kappa}, \]
with the remainder term

$$\tilde{\mathcal{E}}_0 = -\frac{N^\kappa}{2} \sum_{p \in \Lambda_+^*} \hat{V}(p/N-1+\kappa) \left[ \sigma_p \gamma_p (2\alpha_p a_p + \alpha_p^* a_{-p}^* + a_p a_{-p}) + 2\alpha_p^2 (N_+ + a_p^* a_p) \right]$$

that satisfies $\pm \tilde{\mathcal{E}}_0 \leq CN^{-1+\kappa}(N_+ + 1)$. This concludes the proof.

A.3 Analysis of $\mathcal{G}_N^{(3)}$

We have

$$\mathcal{G}_N^{(3)} = e^{-B} \mathcal{L}_N^{(3)} e^B = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N-1+\kappa) e^{-B} b_{p+q} b_{-p}^* a_q e^B + \text{h.c.}.$$

Lemma A.4. We have

$$\mathcal{G}_N^{(3)} = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N-1+\kappa) b_{p+q}^* b_{-p} b_q e^B + \text{h.c.} + \mathcal{E}_{\tilde{\mathcal{G}}_N}^{(3)}, \quad \text{(A.10)}$$

with the estimate

$$\pm \mathcal{E}_{\tilde{\mathcal{G}}_N}^{(3)} \leq CN^{-1/2+2\kappa}(\mathcal{V}_N + N_+ + 1)(N_+ + 1). \quad \text{(A.11)}$$

Proof. First we write everything in terms of generalized creation and annihilation operators. Using again $a_{-p}^* a_q = b_{-p}^* b_q + a_{-p}^* (N_/N) a_q$, we find

$$\mathcal{G}_N^{(3)} = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N-1+\kappa) \left[ e^{-B} b_{p+q}^* b_{-p}^* b_q e^B + \text{h.c.} \right] + \mathcal{E}_{\tilde{\mathcal{G}}_N}, \quad \text{(A.12)}$$

and switching to position space and using (A.1) we find

$$\pm \mathcal{E}_{\tilde{\mathcal{G}}_N} \leq N^{-1/2}(\mathcal{V}_N + N_+ + 1)(N_+ + 1).$$

We can now write the first term of (A.12) as

$$\frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N-1+\kappa) \left[ e^{-B} b_{p+q}^* b_{-p}^* b_q e^B \right] = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N-1+\kappa) e^{-B} b_{p+q}^* e^B e^{-B} b_{-p} e^B e^{-B} b_q e^B.$$

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Using the expansion (3.13), representing the potential in position space and applying the bounds (3.11), (3.14), (3.15) one finds

\[
G_N^{(3)} = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* \atop p+q \neq 0} \hat{V}(p/N^{1-\kappa}) \gamma_p \gamma_{p+q} b_p^* b_{p+q} \left[ (\gamma_q b_q + \sigma_q b_{-q}) + \text{h.c.} \right] + \tilde{E}_2, \quad (A.13)
\]

with

\[
\pm \tilde{E}_2 \leq CN^{(-1+\kappa)/2}(V_N + N_+ + 1)(N_+ + 1).
\]

We conclude by observing that, thanks to (3.14), we can replace the factors \(\gamma_{p+q}\) and \(\gamma_p\) in (A.13) by one, producing an error that satisfies the desired bound (A.11). \(\square\)

### A.4 Analysis of \(G_N^{(4)}\)

We now analyse the term

\[
G_N^{(4)} = e^{-B} \mathcal{L}_N^{(4)} e^B = \frac{N^\kappa}{\sqrt{N}} \sum_{p,q,r \in \Lambda_+^* \atop r \neq -p-q} \hat{V}(p/N^{1-\kappa}) e^{-B} a_p^* a_q^* a_p a_{q+r} e^B.
\]

**Lemma A.5.** We have

\[
G_N^{(4)} = V_N + \frac{N^\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) \sigma_q \sigma_p \gamma_p (1 + 1/N - 2N+/N)
\]

\[
+ \frac{N^\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) (\eta_H q) \left[ \gamma_p^2 b_p^* b_{-p}^* b_p + 2\gamma_p \sigma_p b_p^* b_p + 2\sigma_p b_p b_{-p}
\]

\[
+ d_p (\gamma_p b_{-p} + \sigma_p b_{-p}^*) + (\gamma_p b_p + \sigma_p b_{-p}^*) d_{-p} + \text{h.c.} \right] (A.14)
\]

\[
+ \frac{N^\kappa}{N^2} \sum_{p,q,u \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) (\eta_H q) (\eta_H u) \left[ \gamma_u^2 + \sigma_u^2 \right] b_u^* b_{-u} + \gamma_u \sigma_u b_u^* b_{-u}
\]

\[
+ \gamma_u \sigma_u b_u b_{-u} + \sigma_u^2 \right] + \mathcal{E}_N^{(4)},
\]

with the estimate

\[
\pm \mathcal{E}_N^{(4)} \leq CN^{-1/2+3\kappa}(V_N + N_+ + 1)(N_+ + 1). \quad (A.15)
\]

**Proof.** We write

\[
a_p^* a_q^* a_p a_{q+r} = b_p^* b_q^* b_p b_{q+r} \left( 1 - \frac{3}{N} + \frac{2N_+}{N} \right) + a_p^* a_q^* a_p a_{q+r} \Theta_{N_+},
\]

with

\[
\Theta_{N_+} = \left[ \frac{(N_+ + 2)(N_+ - 1) + (N_+ - 2)}{N^2} \right] - \left[ \frac{N_+^2 - 3N_+ - 2}{N^2} \right] \left[ \frac{(N_+ + 2)(N_+ - 1)}{N^2} \right]
\]

\[
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\]
satisfying \( \pm \Theta_{N+} \leq C(N_+ + 1)^2/N^2 \). From this and Lemma A.1, it follows that
\[
G_N^{(4)} = \frac{N^\kappa (N + 1)}{N^2} \sum_{p,q,r \in \Lambda_+^*} \sum_{r \neq p,-q} \hat{V}(p/N^{1-\kappa}) e^{-B} b_{p+r}^* b_q b_p b_{q+r} e^B
\]
\[+ \frac{N^\kappa}{N^2} \sum_{p,q,u \in \Lambda_+^*} \sum_{r \neq -p,-q} \hat{V}(p/N^{1-\kappa}) e^{-B} b_{p+r}^* b_q^* b_u b_p b_{q+r} e^B + \tilde{\epsilon}_1 \quad (A.16)
\]
with an error that satisfies
\[
\pm \tilde{\epsilon}_1 \leq CN^{-1+\kappa}(\mathcal{V}_N + N_+ + 1)(N_+ + 1).
\]
The first term on the right hand side of (A.16) can be expanded using (3.13). Bringing the \( b \) and \( b^* \) operators into normal order and using again Lemma (3.3) to bound all remainder terms, it is possible to show that
\[
\frac{N^\kappa (N + 1)}{N^2} \sum_{p,q,r \in \Lambda_+^*} \sum_{r \neq p,-q} \hat{V}(p/N^{1-\kappa}) e^{-B} b_{p+r}^* b_q b_p b_{q+r} e^B
\]
\[= \mathcal{V}_N + \frac{N^\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) \sigma_q \gamma_q \sigma_p \gamma_p (1 + 1/N - 2N_+/N)
\]
\[+ \frac{N^\kappa}{2N} \sum_{p,q \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) \sigma_q \gamma_q \left[ \gamma_p^2 b_p^* b_{-p}^* + 2 \gamma_p \sigma_p b_p^* b_p + \sigma_p^2 b_p b_{-p} + (d_p(\gamma_p b_{-p} + \gamma_p b_{u,-p} + \gamma_p b_{u,-p} d_{-p} + h.c.) \right] + \tilde{\epsilon}_2
\]
with error term bounded by
\[
\pm \tilde{\epsilon}_2 \leq CN^{-(1-\kappa)/2}(\mathcal{V}_N + N_+ + 1)(N_+ + 1).
\]
As for the remaining term in (A.16), using the fact that \( \sum_{u \in \Lambda_+^*} e^{-B} b_u^* b_u e^B = e^{-B} N_+ (1 - N_+/N) e^B \), applying Lemma 3.2 and expanding the remaining \( b \) and \( b^* \) operators, one can see that
\[
\frac{N^\kappa}{N^2} \sum_{p,q,r \in \Lambda_+^*} \sum_{r \neq p,-q} \hat{V}(r/N^{1-\kappa}) e^{-B} b_{p+r}^* b_q b_u b_p b_{q+r} e^B
\]
\[= \frac{N^\kappa}{N^2} \sum_{p,q,u \in \Lambda_+^*} \hat{V}((p-q)/N^{1-\kappa}) \sigma_q \gamma_q \sigma_u \gamma_u \left[ (\gamma_u^2 + \sigma_u^2) b_u b_u + \gamma_u \sigma_u b_u b_u b_{-u} + \gamma_u \sigma_u b_u b_{-u}^2 + \sigma_u^2 \right] + \tilde{\epsilon}_3.
\]
Here, the error $\tilde{E}_3$ satisfies the same bound as $\tilde{E}_2$. Bringing everything together, we find

$$G_N^{(4)} = V_N + \frac{N\kappa}{2N} \sum_{p,q \in \Lambda_*^+} \hat{V}((p - q)/N^{1-\kappa})\sigma_q\gamma_p\gamma_p\gamma_p(1 + 1/N - 2N_+/N)$$

$$+ \frac{N\kappa}{2N} \sum_{p,q \in \Lambda_*^+} \hat{V}((p - q)/N^{1-\kappa})\sigma_q\gamma_q\left[\gamma_p^2b_p^*b_p^* + 2\gamma_p\sigma_p\gamma_p + \gamma_p^2\sigma_p\gamma_p + d_p(\gamma_p\gamma_p + \sigma_p\gamma_p)\gamma_p + \gamma_p\gamma_p\right]$$

$$+ \frac{N\kappa}{N^2} \sum_{p,q,u \in \Lambda_*^+} \hat{V}((p - q)/N^{1-\kappa})\gamma_q\gamma_p\gamma_p\left[\gamma_u^2\sigma_u^2 + \gamma_u\gamma_u\right]$$

(A.17)

We conclude the proof by noticing that, with (3.14), we can replace $\gamma_q\sigma_q$ in the second line and $\sigma_q\gamma_p\gamma_p$ in the last line of (A.17) by resp. $(\eta_H)_p$, $(\eta_H)_p(\eta_H)_q$, producing an error that satisfies the bound (A.15).

A.5 Proof of Proposition 3.4

We now collect the results from the previous sections to prove Proposition 3.4. By (A.2), (A.4), (A.7), (A.10) and (A.14), we find

$$G_N = \tilde{C}_{G_N} + \tilde{C}_{G_N} + C_N + \mathcal{H}_N + \mathcal{D}_N + \tilde{E},$$

where the error term satisfies

$$\pm \tilde{E} \leq N^{-1/2+3\kappa}(\mathcal{H}_N + N^2_+ + 1)(N_+ + 1)$$

by (A.3), (A.5), (A.8), (A.11), (A.15), and where we defined

$$\tilde{C}_{G_N} = \frac{N - 1}{2} N^{\kappa}\hat{V}(0) + \sum_{p \in \Lambda_*^+} p^2\sigma_p^2 \left[1 + \frac{1}{N} + \frac{1}{N} \sum_{u \in \Lambda_*^+} \sigma_u^2\right]$$

$$+ \sum_{p \in \Lambda_*^+} N^{\kappa}\hat{V}(p/N^{1-\kappa})\sigma_p^2 + \sum_{p \in \Lambda_*^+} N^{\kappa}\hat{V}(p/N^{1-\kappa})\gamma_p\sigma_p$$

$$+ \frac{N^{\kappa}}{2N} \sum_{p,q \in \Lambda_*^+} \hat{V}((p - q)/N^{1-\kappa})\sigma_q\gamma_q\sigma_p\gamma_p(1 + 1/N)$$

(A.18)

$$+ \frac{N^{\kappa}}{N^2} \sum_{p \in \mathcal{P}_{H}, u \in \Lambda_*^+} (\hat{V}(/N^{1-\kappa}) \ast \eta_H)\eta_p\sigma_u^2,$$
\[ \tilde{Q}_{\nu N} = \sum_{p \in \Lambda^+_N} b^*_p b_p \left[ 2\sigma_p^2 p^2 + N^\kappa \hat{V}(p/N^{1-\kappa})(\gamma_p + \sigma_p)^2 + \frac{2N^\kappa}{N} \gamma_p \sigma_p (\hat{V}(/N^{1-\kappa}) \ast \eta_H)_p \right] \\
+ \sum_{p \in \Lambda^+_N} (b^*_p b^*_{-p} + b_p b_{-p}) \times \left[ p^2 \gamma_p \sigma_p + \frac{N^\kappa}{2} \hat{V}(p/N^{1-\kappa})(\gamma_p + \sigma_p)^2 + \frac{N^\kappa}{2N} \hat{V}(\cdot /N^{1-\kappa}) \ast \eta_H)_p (\gamma_p^2 + \sigma_p^2) \right] \\
- \frac{N^\kappa}{N} \sum_{p \in \Lambda^+_N} \left[ p^2 \sigma_p^2 + N^\kappa \hat{V}(p/N^{1-\kappa}) \gamma_p \sigma_p + \frac{N^\kappa}{N} \sum_{q \in \Lambda^+_N} \hat{V}((p - q)/N^{1-\kappa}) \gamma_p \sigma_p \gamma_q \sigma_q \right] \\
+ \frac{1}{N} \sum_{u \in \Lambda^-_N} \left[ (\gamma_u^2 + \sigma_u^2) b^*_u b_u + \gamma_u \sigma_u (b^*_u b^-_{-u} + b_u b^-_{-u}) \right] \\
\times \sum_{p \in \mathcal{P}_H} \left[ p^2 \sigma_p^2 + \frac{N^\kappa}{N} \hat{V}(\cdot /N^{1-\kappa}) \ast \eta_H)_p \right] \right], \tag{A.19} \]

and

\[ D_N = \sum_{p \in \Lambda^+_N} p^2 \sigma_p d^*_p d^-_{-p} + \text{h.c.} \]
\[ + \frac{1}{2} \sum_{p \in \Lambda^+_N} N^\kappa \left[ \hat{V}(p/N^{1-\kappa}) + \frac{1}{N} \sum_{q \in \Lambda^+_N} V((p - q)/N^{1-\kappa}) (\eta_H)_q \right] \\
\times \left[ (\gamma_p b^*_p + \sigma_p b^-_{-p}) d^*_p + d^*_p (\gamma_p b^*_{-p} + \sigma_p b_p) \right] + \text{h.c.} \]
\[ = \sum_{p \in \Lambda^+_N} \left[ p^2 \sigma_p + \frac{1}{2} N^\kappa \hat{V}(p/N^{1-\kappa}) + \frac{1}{2N} N^\kappa (\hat{V}(\cdot /N^{1-\kappa}) \ast \eta_H) \right] d^*_p b^-_{-p} + \text{h.c.} \]
\[ + \sum_{p \in \Lambda^+_N} \frac{1}{2} N^\kappa \left[ \hat{V}(p/N^{1-\kappa}) + \frac{1}{N} (\hat{V}(\cdot /N^{1-\kappa}) \ast \eta_H) \right] [(\gamma_p - 1) d^*_p b^-_{-p} + \sigma_p d^*_p b_p + \text{h.c.}] \]
\[ + \sum_{p \in \Lambda^+_N} \frac{1}{2} N^\kappa \left[ \hat{V}(p/N^{1-\kappa}) + \frac{1}{N} (\hat{V}(\cdot /N^{1-\kappa}) \ast \eta_H) \right] [\sigma_p d^*_p b^-_{-p} + \gamma_p d^*_p b^-_{-p} + \text{h.c.}] \right]. \tag{A.20} \]

\( D_N \) collects all terms arising from the conjugation of \( \mathcal{L}_N \) containing \( d, d^* \) operators that are not already included in \( \tilde{E} \). We want to extract relevant contributions from these terms. First observe that \( \eta_H \) can be replaced by \( \eta \) in the convolutions appearing on the r.h.s. of \( (A.20) \), producing an error smaller than \( N^{-1/2+3\kappa/2+\alpha/2}(N^\kappa + 1)^2 \) in the operator sense. Taking for example the term on the first line, using \( (3.16) \) and \( (3.8) \) we
can bound
\[ \pm \langle \xi, \frac{N^\kappa}{2N} \sum_{p \in \Lambda_A} \left[ (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta) - (\hat{V}(\cdot/N^{1-\kappa}) \ast \eta_H) \right] d_p^* b_{-p} \xi \rangle \leq CN^{-2+\kappa} \sum_{p \in \Lambda_+, |q| \leq N^\alpha} |\hat{V}((p-q)/N^{1-\kappa})\eta_q| \times \left[ |\eta_H(p)||N_+ + 1|\xi|| + ||\eta_H|| \right] \|b_p(N_+ + 1)^{1/2}\xi\| \|N_+\xi\| \leq CN^{-2+\kappa} \left\| \hat{V}(\cdot/N^{1-\kappa}) \right\|_2 \|\eta_H\|_2 \|N_+ + 1\| \xi\| \sum_{|q| \leq N^\alpha} |\eta_q| \leq CN^{-\frac{1}{2}+\frac{3\kappa+\alpha}{2}} \|N_+ + 1\| \xi\|^2. \]

The other errors can be bounded in the same way. We now denote by \( D_i \) the term on the \( i \)-th line of the right hand side of (A.20) after this replacement. By (3.10), (3.16) and (3.13), we conclude that \( \pm D_1, \pm D_2 \leq CN^{-1+\kappa}(N_+ + 1)^2. \) As for \( D_3, \) switching to position space, one can bound the term proportional to \( \gamma_p d_p^* b_{-p} \) by \( CN^{-1+\kappa/2}(N_+ + N_+ + 1)(N_+ + 1) \) and easily replace \( \sigma_p \) by \( \eta_H(p) \) in the other term. This produces an additional error of order \( N^{-1+\kappa}(N_+ + 1)^2. \) We are left with
\[ \tilde{D}_3 = \frac{1}{2} \sum_{p \in P_H} N^\kappa \langle \hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N \rangle \eta_p d_p^* b_{-p} + \text{h.c.}. \]

Here we don’t have enough decay in \( p \) to conclude by (3.16): this term contains contributions that are not negligible in the limit of large \( N \), at our desired level of precision. To isolate them, we write, for \( p \in P_H, \)
\[
d_p = (\gamma_p - 1)b_p - \sigma_p b_{-p}^* + e^{-B} b_p e^B - b_p = - \int_0^1 (\sigma_p(s)b_p + \gamma_p(s)b_{-p}^*) ds - \int_0^1 e^{-sB}[B, b_p]e^sB ds \]
\[
= \eta_p \int_0^1 d_p^{(s)} ds - \frac{\eta_p}{N} \int_0^1 e^{-sB} N_+ b_{-p} e^B ds - \frac{1}{N} \int_0^1 \sum_{q \in P_H} \eta_q e^{-sB} b_q a_{-q}^* a_p e^B ds,
\]

where for \( s \in [0, 1] \) we used the notation \( \sigma_p^{(s)}, \gamma_p^{(s)}, d_p^{(s)} \) to denote coefficients and operators built from \( s \eta \) instead of \( \eta \). The additional \( \eta_p \) factor in the first two terms lets us bound them as above. We can thus write
\[ \tilde{D}_3 = -\frac{N^\kappa}{2N} \int_0^1 \sum_{p, q \in P_H} \langle \hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N \rangle \eta_p \eta_q [e^{-sB} b_q^* a_{-q} a_p e^B b_p^* + \text{h.c.}] ds + \tilde{E}_1, \]

with \( \pm \tilde{E}_1 \leq CN^{-1+\kappa}(N_+ + 1)^2. \)

We now further expand
\[
e^{-sB} a_{-q}^* a_p e^B = a_{-q}^* a_p + \int_0^s e^{-tB} (\eta_p b_{-q}^* b_{-p} + \eta_q b_p b_q) e^{tB} dt
\]
and plug this into $\tilde{D}_3$. The term with the additional factor $\eta_p$ can also be bounded as above. In the term proportional to $e^{-sB}b_q^*e^{sB}a_{-q}^*a_p^*b_p^*$ we commute $b_p^*$ to the left, while in the last term that is left we expand $e^{-tB}b_qe^{tB} = \gamma_q(t)b_q + \sigma_q(t)b_{-q} + d_q(t)$ and we commute the $b_p^*$ to the left of $\gamma_q(t)b_q$. One can see that the quartic terms (in creation and annihilation operators) can now be bounded as above by Cauchy-Schwarz. We are left with the quadratic terms arising from the commutators:

$$
\tilde{D}_3 = -\frac{N^\kappa}{2N} \int_0^1 \sum_{p,q \in P_H} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N) \eta_p \eta_q e^{-sB}b_q^*e^{sB} dqs
$$

$$
-\frac{N^\kappa}{2N} \int_0^1 \int_0^s \sum_{p,q \in P_H} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N) \eta_p \eta_q^2 e^{-sB}b_q^*e^{sB} e^{-tB}b_q e^{tB} dtds
$$

+ h.c. + \tilde{\mathcal{E}}_2

with

$$
\pm \tilde{\mathcal{E}}_2 \leq CN^{-1+\kappa}(\mathcal{N}+1)^2.
$$

Expanding once more with (3.13) and collecting the terms with $d_q^{(s)*}, d_q^{(t)}$ in the error, and finally integrating in $t$ and $s$ we arrive at

$$
\mathcal{D}_N = -\frac{N^\kappa}{2N} \sum_{q \in \Lambda_+, p \in P_H} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N) \eta_p \gamma_q b_q^*b_{-q} + b_q b_{-q}) + (\sigma_q^2 + \gamma_q^2)b_q^*b_q + \sigma_q^2
$$

$$
+ \frac{N^\kappa}{2N} \sum_{q \in \Lambda_+, p \in P_H} (\hat{V}(\cdot/N^{1-\kappa}) \ast \hat{f}_N) \eta_p b_q^*b_q + \tilde{\mathcal{E}}_3.
$$

(A.21)

We denote the constant term on the r.h.s. of (A.21) by $\mathcal{D}_N^{(0)}$ and collect the quadratic ones in $\mathcal{D}_N^{(2)}$, so that $\mathcal{D}_N = \mathcal{D}_N^{(0)} + \mathcal{D}_N^{(2)} + \tilde{\mathcal{E}}_3$ with the error bound

$$
\pm \tilde{\mathcal{E}}_3 \leq CN^{-1+\frac{1}{2}}(\mathcal{V}_N + \mathcal{N}_+ + 1)(\mathcal{N}_+ + 1).
$$

Let us now go back to the term $\tilde{C}_{\eta_H}$ defined in (A.18). Using (3.8) and (3.14) we see that we can replace the $\eta_H$ in the convolution on the last line by $\eta$, producing an error smaller than $N^{-1}$. Adding the constant contribution coming from (A.21) and
Indeed, using \( \hat{\eta} \) remark that also in these terms, replacing bound (4.3) we find

\[
\tilde{C}_{GN} + D_N^{(0)} = \frac{N-1}{2} N^\kappa \hat{\nu}(0) + \sum_{p \in \Lambda_+^*} \left[ p^2 \sigma_p^2 + N^\kappa \hat{\nu}(p/N^{1-\kappa})(\sigma_p \gamma_p + \sigma_p^2) \right] \]

\[
+ \frac{1}{2N} \sum_{p,q \in \Lambda_+^*} N^\kappa \hat{\nu}((p-q)/N^{1-\kappa}) \sigma_p \gamma_p \sigma_q \gamma_q \]

\[
+ \frac{1}{N} \sum_{p \in P_H} \eta_p \left[ p^2 \eta_p + \frac{1}{2N} N^\kappa (\hat{\nu}(\cdot/N^{1-\kappa}) \star \eta)_p \right] \]

\[
+ \frac{1}{N} \sum_{u \in \Lambda_+^*} \sum_{p \in P_H} \eta_p \left[ p^2 \eta_p - \frac{1}{2N} N^\kappa (p/N^{1-\kappa}) \nonumber \right] \]

\[
+ \frac{1}{2N} N^\kappa (\hat{\nu}(\cdot/N^{1-\kappa}) \star \eta)_p \]

\[
+ \mathcal{O}(N^{-1+2\kappa+\alpha}), \]

where the error \( N^{-1+2\kappa+\alpha} \) arises from substituting the factors \( \sigma_q, \sigma_q \gamma_q \sigma_p \gamma_p \) with \( (\eta_H)_q \), \( (\eta_H)_q \sigma(\eta_H)_p \), and then \( \eta_p \) with \( \eta \) in the resulting convolution on the third line. This is, by the assumptions on \( \alpha \), smaller than \( N^{-1/2+2\kappa} \). Using now equation (3.10) and the bound (4.3) we find

\[
\tilde{C}_{GN} + D_N^{(0)} = \frac{N-1}{2} N^\kappa \hat{\nu}(0) + \sum_{p \in \Lambda_+^*} \left[ p^2 \sigma_p^2 + N^\kappa \hat{\nu}(p/N^{1-\kappa})(\sigma_p \gamma_p + \sigma_p^2) \right] \]

\[
+ \frac{1}{2N} \sum_{p,q \in \Lambda_+^*} N^\kappa \hat{\nu}((p-q)/N^{1-\kappa}) \sigma_p \gamma_p \sigma_q \gamma_q \]

\[
+ \frac{1}{N} \sum_{p \in P_H} \eta_p \left[ p^2 \eta_p + \frac{1}{2N} N^\kappa (\hat{\nu}(\cdot/N^{1-\kappa}) \star \eta)_p \right] \]

\[
- \frac{1}{N} \sum_{u \in \Lambda_+^*} \sum_{p \in P_H} N^\kappa (p/N^{1-\kappa}) \eta_p + \mathcal{O}(N^{-1/2+2\kappa}), \]

and finally in the last line we replace the sum over \( P_H \) with one over the whole \( \Lambda_+^* \) to get (3.19). This produces a negligible error, of order at most \( N^{-1+2\kappa+\alpha+2} \leq N^{-1/2+2\kappa} \). Indeed, using \( \hat{\nu} \in L^\infty(\Lambda^*) \) and (3.3) we see that

\[
\pm \frac{1}{N} \sum_{u \in \Lambda_+^*} \sum_{p \in P_H} N^\kappa (p/N^{1-\kappa}) \eta_p \leq CN^{-1+2\kappa+\alpha+2}. \]

Similarly, we combine \( \tilde{Q}_N \) defined in (A.19) with the quadratic terms in (A.21). Using again the scattering equation (3.10) and the bound (4.3), we get

\[
\tilde{Q}_N + D_N^{(2)} = Q_N + \tilde{E}_4 \]

with the bound \( \pm \tilde{E}_4 \leq N^{-1/2+3\kappa+2\alpha}(K+\mathcal{N}+1)(\mathcal{N}+1) \). We omit the details, but we remark that also in these terms, replacing \( \eta_H \) with \( \eta \) produces terms of the leading order in the error. This concludes the proof of the proposition.
B Analysis of $J_n$

In this section, we sketch the proof of Proposition 3.7. The proof is similar to the proof of [7, Prop. 3.3], taking into account the different scaling. We recall the antisymmetric operator $A$, defined in (3.23). Throughout this section, we assume that the parameters $\kappa, \alpha, \beta$ satisfy (3.24).

B.1 Analysis of $e^{-A}Q_{\mathcal{G}_N}e^A$

To control the action of $A$ on the quadratic operator $Q_{\mathcal{G}_N}$ introduced in (3.20), we will make use of the following lemma.

Lemma B.1. Let $\Gamma_p, \Phi_p$ be sequences satisfying

$$|\Gamma_p| \leq C N^\alpha \left( \chi_{\{|p| \leq N^\alpha\}} + 1/|p|^2 \right), \quad |\Phi_p| \leq C N^\kappa.$$

Then we have

$$\pm \sum_{p \in \Lambda^*_+} \Phi_p [b_p b_p^*, A] \leq C N^{-\frac{1}{2}} (N_+ + 1)^2 \quad (\text{B.1})$$

$$\pm \sum_{p \in \Lambda^*_+} \Gamma_p [(b_p b_p^*-p+b_p b_p), A] \leq C N^{-\frac{1}{2}} (K + 1)(N_+ + 1). \quad (\text{B.2})$$

Thanks to the scattering equation (3.1) and to the bounds (4.3), the coefficients $\Gamma_p, \Phi_p$ appearing in (3.20) satisfy the assumptions of Lemma B.1.

Lemma B.2. We have

$$e^{-A}Q_{\mathcal{G}_N}e^A = Q_{\mathcal{G}_N} + \mathcal{E}_{\mathcal{Q}_N},$$

with

$$\pm \mathcal{E}_{\mathcal{Q}_N} \leq C N^{-\frac{1}{2}+\kappa} (H + N_+^2 + 1)(N_+ + 1).$$

Proof of Lemma B.2. We expand

$$e^{-A}Q_{\mathcal{G}_N}e^A = Q_{\mathcal{G}_N} + \int_0^1 e^{-sA}[Q_{\mathcal{G}_N}, A]e^{sA}ds$$

and the claim follows from Lemmas B.1 and 3.6.

Proof of Lemma B.1. The proof is similar to that of [7, Lemma 8.2]. Using the commutation rules (2.2), we find

$$[b_p^*, A] = N^{-\frac{1}{2}} \sum_{r \in \mathcal{F}_H} \sum_{v \in \mathcal{F}_L} \eta_v \left[ b_r^* b_r \left( 1 - \frac{N_+}{N} \right) \delta_{p+r+v} + b_v^* b_{r+v} \left( 1 - \frac{N_+ - 1}{N} \right) \delta_{p-r} 
- b_r^* b_r^* \left( 1 - \frac{N_+}{N} \right) \delta_{p,v} \right]$$

$$- N^\frac{1}{2} \sum_{r \in \mathcal{F}_H} \sum_{v \in \mathcal{F}_L} \eta_r \left[ b_v^* (b_{r+v} a_r^* a_{r+v} + a_r^* a_{r+v} b_r + b_r b_r^* a_v^* a_v) - b_{r+v}^* b_{r+v}^* a_r^* a_v \right].$$

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Since $A^* = -A$, we have $[b_p^*, A]^* = [A^*, b_p] = -[A, b_p] = [b_p, A]$ and we can also compute

$$[b_p^* b_p, A] = \sum_{j=1}^{6} \Delta_j + \text{h.c.},$$

with

$$\Delta_1 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Phi_{r+v} \eta_r \left( 1 - \frac{N_+ - 1}{N} \right) b_{r+v}^* b_{r+v}^* b_v$$

$$\Delta_2 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Phi_r \eta_r \left( 1 - \frac{N_+ - 2}{N} \right) b_{r+v}^* b_{r+v}^* b_v$$

$$\Delta_3 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Phi_v \eta_r \left( 1 - \frac{N_+ - 2}{N} \right) b_{r+v}^* b_{r+v}^* b_v$$

$$\Delta_4 = N^{-3/2} \sum_{p \in \Lambda^*_r} \sum_{r \in P_H, v \in P_L} \Phi_p \eta_r \left( 1 - \frac{N_+ - 2}{N} \right) b_{r+v}^* b_{r+v}^* a_p b_v$$

$$\Delta_5 = N^{-3/2} \sum_{p \in \Lambda^*_r} \sum_{r \in P_H, v \in P_L} \Phi_p \eta_r \left( 1 - \frac{N_+ - 2}{N} \right) b_{r+v}^* b_{r+v}^* a_p b_v$$

$$\Delta_6 = N^{-3/2} \sum_{p \in \Lambda^*_r} \sum_{r \in P_H, v \in P_L} \Phi_p \eta_r \left( 1 - \frac{N_+ - 2}{N} \right) b_{r+v}^* b_{r+v}^* b_v.$$

We point out here that we get less terms than the corresponding ones in [7, Lemma 8.2]. The infrared cutoff that we introduced in sequence $\eta$ leads to a slightly different form of the operator $A$ than the one used in [7], which in turn causes several terms appearing there to vanish in our setting. This remark also applies to other similar expansions in the rest of this section. We now proceed to bound the individual terms $\Delta_i$. With $|\Phi_p| \leq CN^{\kappa}$, $||\eta_H||_2 \leq CN^{\kappa-\alpha/2}$ and $\alpha > 4\kappa$, Cauchy-Schwarz implies that

$$\pm \Delta_{1,2,3} \leq CN^{-1/2+2\kappa-\alpha/2} (N_+ + 1)^2 \leq CN^{-1/2} (N_+ + 1)^2,$$

Rearranging $\Delta_4$ in normal order, we obtain a commutator term, cubic in creation and annihilation operators, that can be estimated like $\Delta_1, \Delta_2, \Delta_3$. The remaining terms are easily seen to satisfy

$$\pm \Delta_4, \pm \Delta_5, \pm \Delta_6 \leq CN^{-1+2\kappa-\alpha/2} (N_+ + 1)^2,$$

concluding the proof of (B.1).

As for the off-diagonal terms, we have that

$$\sum_{p \in \Lambda^*_r} \Gamma_p ([b_p^* b_{p^*} + b_p b_{p^*}, A] = \sum_{j=1}^{7} \Upsilon_j + \text{h.c.},$$

where
\( \Upsilon_1 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Gamma_{r+v} \eta_r \left( 1 - \frac{N_+ + 1}{N} \right) \left( b_{r-v} b_{r-v} - \frac{1}{N} a_{r-v} a_{r-v} \right) b_v \)

\( \Upsilon_2 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Gamma_{r+v} \eta_r \left( 1 - \frac{N_+ + 1}{N} \right) b_{r-v} b_{r-v} \)

\( \Upsilon_3 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Gamma_r \eta_r \left( 1 - \frac{N_+ + 1}{N} \right) \left( b_{r+v} b_r - \frac{1}{N} a_{r+v} a_r \right) b_v \)

\( \Upsilon_4 = N^{-1/2} \sum_{r \in P_H, v \in P_L} \Gamma_r \eta_r \left( 1 - \frac{N_+ - 1}{N} \right) b_{r+v} b_r \)

\( \Upsilon_5 = -N^{-1/2} \sum_{r \in P_H, v \in P_L} \Gamma_r \eta_r \left[ \left( 1 - \frac{N_+}{N} \right) + \left( 1 - \frac{N_+ + 1}{N} \right) \right] b_{r+v} b_{r-v} \)

\( \Upsilon_6 = -N^{-3/2} \sum_{p \in \Lambda^*_+, r \in P_H, v \in P_L} \Gamma_p \eta_r \left[ b_p (a_{r+v} a_{r-p} b_{r-v} + b_{r+v} a_{r-p} a_r) b_v - b_p a_p a_{r-p} b_{r+v} \right] \)

\( \Upsilon_7 = -N^{-3/2} \sum_{p \in \Lambda^*_+, r \in P_H, v \in P_L} \Gamma_p \eta_r \left[ (a_{r+v} a_{r-p} b_{r-v} + b_{r+v} a_{r-p} a_r) b_v b_p - a_p a_{r-p} b_{r+v} b_p \right] . \)

We start by showing how to bound \( \Upsilon_2 \), the two terms in \( \Upsilon_1 \) are bounded in the same way. Using Cauchy-Schwarz we get

\[
|\langle \xi, \Upsilon_2 \xi \rangle| \leq N^{-\frac{1}{2}+\kappa-2\alpha} \left( \sum_{r \in P_H, v \in P_L} |\Gamma_{r+v}|^2 \| b_{r-v} \xi \|^2 \right)^{1/2} \left( \sum_{r \in P_H, v \in P_L} \| b_{r-v} b_v \xi \|^2 \right)^{1/2} \leq C N^{-\frac{1}{2}+\kappa+\frac{3}{2}+\beta-2\alpha} \| \Lambda_+^{1/2} \xi \| \| (\Lambda_+ + 1) \xi \| ,
\]

where we used the fact that \( |\eta_r| \leq C N^{\kappa-2\alpha} \) and \( \sum_{v \in P_L} |\Gamma_{r+v}|^2 \leq C N^\kappa |P_L|^{1/2} \leq C N^{\kappa+3\beta/2} \) uniformly in \( r \in P_H \). \( \Upsilon_3 \) and \( \Upsilon_4 \) are bounded similarly, and thanks to the assumptions on \( \alpha, \beta, \kappa \) we conclude

\[ \pm \Upsilon_{1,2,3,4} \leq N^{-\frac{1}{2}} (\Lambda_+ + 1)^2 . \]

As for \( \Upsilon_5 \), we use again Cauchy-Schwarz to bound

\[
|\langle \xi, \Upsilon_5 \xi \rangle| \leq N^{-\frac{1}{2}} \left( \sum_{r \in P_H, v \in P_L} |\Gamma_v|^2 |\eta_r|^2 |r|^{-2} \| \Lambda_+^{1/2} \xi \|^2 \right)^{1/2} \left( \sum_{r \in P_H, v \in P_L} |r|^2 \| b_{r-v} \xi \|^2 \right)^{1/2} \leq C N^{-\frac{1}{2}+2\kappa+\frac{3}{2}+\beta-2\alpha} \| \Lambda_+^{1/2} \xi \| \| (\Lambda_+ + 1)^{1/2} (\Lambda_+ + 1)^{1/2} \xi \| ,
\]

which together with the assumptions on \( \alpha, \beta, \kappa \) shows \( \pm \Upsilon_5 \leq N^{-1/2}(\kappa + 1)(\Lambda_+ + 1) \). We now turn our attention to \( \Upsilon_6, \Upsilon_7 \). Here we commute each of the terms in square
bracket to (partial) normal order and again bound it using Cauchy-Schwarz. We show how this is done for the first term in $\Upsilon_6$. We have

$$ N^{-3/2} \sum_{p \in \Lambda^*_1} \sum_{r \in P_H, v \in P_L} \Gamma_p \eta_r b_p a^*_{r+v} a_{-p} b^*_v b_v $$

$$ = N^{-3/2} \sum_{p \in \Lambda^*_1} \sum_{r \in P_H, v \in P_L} \Gamma_p \eta_r b_p b^*_r a^*_{r+v} a_{-p} b_v + N^{-3/2} \sum_{r \in P_H, v \in P_L} \Gamma_r \eta_r b_r b^*_r b_v, $$

and using $\|\Gamma\|_2 \leq CN^{\alpha+3\alpha/2}$ and $\|\chi_{PH} \eta\|_2 \leq CN^{\alpha-\alpha/2}$ we bound

$$ \pm \langle \xi, N^{-3/2} \sum_{p \in \Lambda^*_1} \sum_{r \in P_H, v \in P_L} \Gamma_p \eta_r b_p b^*_r a^*_{r+v} a_{-p} b_v \xi \rangle $$

$$ \leq N^{-3/2} \left( \sum_{p \in \Lambda^*_1} \sum_{r \in P_H, v \in P_L} |\Gamma_p|^2 \left| b^*_r a_{r+v} a_{-p} b_v \right|^2 \right)^{1/2} \left( \sum_{p \in \Lambda^*_1} \sum_{r \in P_H, v \in P_L} |\eta_r|^2 \left| a_{-p} b_v \xi \right|^2 \right)^{1/2} $$

$$ \leq CN^{-1+2\alpha+\alpha} \| (N_+ + 1) \xi \|^2, $$

and analogously for the cubic term. The other terms in $\Upsilon_6$ and $\Upsilon_7$ are bounded in the same way, and using the assumptions (3.24) we get

$$ \pm \Upsilon_6, \pm \Upsilon_7 \leq CN^{-1/2} (N_+ + 1)^2, $$

which concludes the proof of (B.2) and the lemma.

**B.2 Analysis of $e^{-A}C_N e^A$**

To control the action of $A$ on the cubic term $C_N$, defined in (3.21), we will make use of the following lemma.

**Lemma B.3.** We have

$$ [C_N, A] = \Xi_0 + \delta C_N $$

where

$$ \Xi_0 = 2N^{-1} \sum_{r \in P_H, v \in P_L} N^\kappa \left( \hat{V}(r/N^{1-\kappa}) + \hat{V}((r + v)/N^{1-\kappa}) \right) \eta_r b^*_r b_v $$

and

$$ \pm \delta C_N \leq CN^{-1/2} \left[ (N_+ + 1)(K + 1) + (N_+ + 1)^3 \right], $$

Moreover,

$$ \pm [\Xi_0, A] \leq CN^{-1/2} (N_+ + 1)^2. $$
Proof. Proceeding as in [7, Lemma 8.4], we find

$$[\mathcal{C}_N, A] = \Xi_0 + \sum_{j=1}^{8} \left[ \Xi_j + \text{h.c.} \right],$$

with $\Xi_0$ as in (B.3) and

$$\Xi_1 = N^{-1} \sum_{r \in P_H, v \in P_L} N^\kappa \hat{V}(r + v) \left[ (1 - \frac{N_+}{N})^2 - 1 \right] b_v$$

$$+ N^{-1} \sum_{r \in P_H} N^\kappa (r/N^{1-\kappa}) \eta_v \left[ (1 - \frac{N_+ + 1}{N}) \left( 1 - \frac{N_+}{N} \right) - 1 \right] b_v$$

$$\Xi_2 = N^{-1} \sum_{r \in P_H} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_v \left[ (1 - \frac{N_+ + 1}{N}) \left( b^{*}_{p} b_{r-v} - \frac{1}{N} a^{*}_{p} a_{r-v} \right) \right] \times \left( \gamma_{r+v-p} b_{r-v-p} + \sigma_{r+v-p} b^{*}_{p-r-v} \right)$$

$$\Xi_3 = N^{-1} \sum_{r \in P_H} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_v \left[ \frac{N_+ - 2}{N} b^{*}_{r+v} b_{r-v} b^{*}_{p} \right] \times \left( \gamma_{p+v} b_{v-p} + \sigma_{p-v} b^{*}_{p-v} \right)$$

$$\Xi_4 = N^{-2} \sum_{r \in P_H} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_v \left[ \frac{N_+ - 2}{N} b^{*}_{r+v} b_{r-v} b^{*}_{p} \right] \times \left( \gamma_{p+v} b_{v-p} + \sigma_{p-v} b^{*}_{p-v} \right)$$

$$\Xi_5 = N^{-2} \sum_{r \in P_H, p,q \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_v \left[ a^{*}_{p+q} a_{r-v} b_{r-v} + b_{r+v} a^{*}_{p+q} a_{-r} b^{*}_{p} \left( \gamma_{q} b_{q} + \sigma_{q} b^{*}_{-q} \right) \right]$$

$$\Xi_6 = N^{-2} \sum_{r \in P_H, p,q \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) \eta_v \left[ b^{*}_{p} b_{r+v} a^{*}_{p+q} a_{-r} b_{-p} \left( \gamma_{q} b_{q} + \sigma_{q} b^{*}_{-q} \right) \right]$$

$$\Xi_7 = N^{-1/2} \sum_{p,q \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) b^{*}_{p+q} \left[ b^{*}_{p}, A \right] \left( \gamma_{q} b_{q} + \sigma_{q} b^{*}_{-q} \right),$$

$$\Xi_8 = N^{-1/2} \sum_{p,q \in \Lambda^*_+} N^\kappa \hat{V}(p/N^{1-\kappa}) b^{*}_{p+q} \left[ b^{*}_{p}, A \right] \left( \gamma_{q} b_{q} + \sigma_{q} b^{*}_{-q} \right),$$

To get (B.3) we set $\delta_{j,v} = \sum_{j=1}^{8} \Xi_j + \text{h.c.}$ and we proceed to bound the individual terms $\Xi_i$. With $\sum_{r} \left| \hat{V}(r/N^{1-\kappa}) \eta_r \right| \leq CN$ we find $\pm \Xi_1 \leq CN^{-1}(N_+ + 1)^2$. As for $\Xi_2$, we use $\|\eta_r\|_2, ||\sigma||_2 \leq CN^{\kappa-\alpha/2}$ and $|P_L| \leq CN^{3\beta}$ to conclude that

$$\pm \Xi_2 \leq N^{-1+2\kappa+3\beta/2-\alpha/2}(N_+ + 1)^2 \leq CN^{-1/2}(N_+ + 1)^2.$$
Note that here we need the assumption $\alpha - \beta > 2\beta + 4\kappa - 1$ and the bound
\[
\sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} |\sigma_{r+v-p}|^2 \|b_{r-v} b^*_p\| \leq C |\mathcal{P}_L| \|\sigma\|_2 \|\mathcal{N}_+ + 1\|^2 \xi,
\]
The terms $\Xi_3, \Xi_4$ are bounded similarly to $\Xi_2$. As for $\Xi_5$, we keep the factor $(\gamma_q b_{-q} + \sigma_q b^*_{-q})$ as it is, rearranging the other operators in normal order. Then, we apply Cauchy-Schwarz’s inequality and the same bounds as above. We proceed analogously for $\Xi_6$ (here we only have to commute $a_v$ and $b^*_p$) and we find
\[
\pm \Xi_5, \pm \Xi_6 \leq C N^{-1/2} (\mathcal{N}_+ + 1)^3.
\]
Expanding $\Xi_7$ with (B.3) we find terms analogous to $\Xi_{3,4,6}$ that can be bounded as above. Expanding $\Xi_8$, however, we find some terms that need to be estimated with the kinetic energy $K$. To explain this point, let us write $\Xi_8 = \Xi_8^{(a)} + \Xi_8^{(b)}$ respectively for the terms proportional to $\gamma_q, \sigma_q$ in the commutator with $A$. We compute $\Xi_8^{(a)} = \sum_{j=1}^5 \Xi_8^{(a,j)}$, with
\[
\Xi_8^{(a,1)} = N^{-1+\kappa} \sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \tilde{V}(p/\mathcal{N}_+^{1-\kappa}) \eta_r \gamma_{r+v} b^*_p b^*_{p+v} b_{r-v} b_{r+v} \left(1 - \frac{\mathcal{N}_+}{N}\right) b_{r+v} b_v
\]
\[
\Xi_8^{(a,2)} = N^{-1+\kappa} \sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \tilde{V}(p/\mathcal{N}_+^{1-\kappa}) \eta_r \gamma_{r-v} b^*_p b^*_{p-v} b_{r+v} b_{r+v} \left(1 - \frac{\mathcal{N}_+}{N}\right) b_{r+v} b_v
\]
\[
\Xi_8^{(a,3)} = - N^{-1+\kappa} \sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \tilde{V}(p/\mathcal{N}_+^{1-\kappa}) \eta_r \gamma_{r-v} b^*_p b_{p+v} b^*_{p-v} b_{r+v} \left(1 - \frac{\mathcal{N}_+}{N}\right) b_{r+v} b_v
\]
\[
\Xi_8^{(a,4)} = - N^{-2+\kappa} \sum_{r \in \mathcal{P}_H} \sum_{p,q \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \tilde{V}(p/\mathcal{N}_+^{1-\kappa}) \eta_r \gamma_{q} b^*_p b_{p+q} b^*_{p+q} b_{r+v} \left(\gamma_{r+v} a^*_{q} b^*_{r}- b^*_{r+v} a^*_{q} b_{r}\right) b_v
\]
\[
\Xi_8^{(a,5)} = - N^{-2+\kappa} \sum_{r \in \mathcal{P}_H} \sum_{p,q \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \tilde{V}(p/\mathcal{N}_+^{1-\kappa}) \eta_r \gamma_{q} b^*_p b_{p+q} b^*_{p+q} a^*_{q} b_{r+v} b_{r+v} b_{r+v}.
\]
By Cauchy-Schwarz we find
\[
\langle \xi, \Xi_8^{(a,1)} \xi \rangle \leq N^{-1-\kappa} \left( \sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} |p|^2 \|b_{r-v} b^*_p\| \langle \mathcal{N}_+ + 1 \rangle^{-1/2} \xi \|^{2} \right)^{1/2} \times \left( \sum_{r \in \mathcal{P}_H} \sum_{p \in \mathcal{A}_+^*} \sum_{v \in \mathcal{P}_L} \frac{\tilde{V}(p/\mathcal{N}_+^{1-\kappa})}{|p|^2} \eta_r \|b_v\| \langle \mathcal{N}_+ + 1 \rangle^{1/2} \xi \|^{2} \right)^{1/2} \leq C N^{-1/2} \left[ \left\langle (\mathcal{K} + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \xi \|^{2} + \langle \mathcal{N}_+ + 1 \rangle^{1/2} \xi \|^{2} \right] \right].
\]
The other quartic terms in Ξ^{(a)}_8 are bounded in the same way, while the sestic terms are handled like Ξ^5. The terms arising from Ξ^{(b)}_8 are analogous and can be bounded similarly (we omit the details), and in the end we obtain

\[ \pm \Xi_8 \leq CN^{-1/2}(K + N_+^2 + 1)(N_+ + 1), \]

which concludes the proof of (B.6). Finally, the bound (B.7) is a consequence of Lemma B.1 and the fact that for \( v \in P_L \) we have the uniform bound

\[ \sum_{r \in P_H} N^{-1+\kappa}(\hat{V}(r/N^{1-\kappa}) + \hat{V}((r + v)/N^{1-\kappa}))\eta_r \leq CN^\kappa. \]

\[ \square \]

**Lemma B.4.** We have

\[ e^{-A} C_N e^A = C_N + 2N^{-1} \sum_{r \in P_H \atop v \in P_L} N^\kappa \left( \hat{V}(r/N^{1-\kappa}) + \hat{V}((v + r)/N^{1-\kappa}) \right) \eta_r b^*_v b_v + \mathcal{E}_{C_N}, \]

with

\[ \pm \mathcal{E}_{C_N} \leq CN^{-1/2+\kappa}(\mathcal{H}_N + N_+^2 + 1)(N_+ + 1). \]

**Proof.** With Lemma B.3 we find

\[ e^{-A} C_N e^A = C_N + \int_0^1 e^{-sA}[C_N, A]e^{sA} ds \]

\[ = C_N + \int_0^1 e^{-sA} \Xi_0 e^{sA} ds + \int_0^1 e^{-sA} \delta_{C_N} e^{sA} ds \]

\[ = C_N + \Xi_0 + \int_0^1 \int_0^{s_1} e^{-s_2 A}[\Xi_0, A]e^{s_2 A} ds_2 ds_1 + \int_0^1 e^{-sA} \delta_{C_N} e^{sA} ds, \]

and the lemma follows from (B.6) and (B.7) and from Lemma 3.6 \[ \square \]

**B.3 Analysis of** \( e^{-A}\mathcal{H}_N e^A \)

Finally, we have to control the action of A on the Hamilton operator \( \mathcal{H}_N = K + \mathcal{V}_N \), introduced in (2.3). To this end, we will make use of the following two lemmas.

**Lemma B.5.** We have

\[ |\langle \xi_1, [\mathcal{H}_N, A]\xi_2 \rangle| \leq C \langle \xi_1, \mathcal{H}_N + (N_+ + 1)^2 \xi_1 \rangle + C \langle \xi_2, \mathcal{H}_N + (N_+ + 1)^2 \xi_2 \rangle \]

\[ + CN^\kappa \langle \xi_1, (N_+ + 1)\xi_1 \rangle + CN^\kappa \langle \xi_2, (N_+ + 1)\xi_2 \rangle \]

**Moreover, we can decompose**

\[ [\mathcal{H}_N, A] = \Theta_0 + \delta_{\mathcal{H}_N} \]

(B.8)
where
\[
\Theta_0 = -N^{-1/2} \sum_{r \in P_H, v \in P_L} N^\kappa \tilde{V}(r/N^{1-\kappa}) b^*_r v b^*_v + \text{h.c.} \quad (B.10)
\]
and
\[
\pm \delta_{\mathcal{H}_N} \leq CN^{-\frac{1}{2} + \kappa} \frac{2}{\sqrt{2}} (\kappa + \mathcal{N}^2_+) + 1) (\mathcal{N}_+ + 1). \quad (B.11)
\]

**Proof.** Proceeding as in [7, Lemma 4.3], we find
\[
[H_N, A] = \sum_{j=0}^7 \Theta_j + \text{h.c.},
\]
with \( \Theta_0 \) as defined in (B.10) and
\[
\begin{align*}
\Theta_1 &= 2N^{-1/2} \sum_{r \in P_H, v \in P_L} N^\kappa \eta_r (r \cdot v) b^*_r + v b^*_v, \\
\Theta_2 &= N^{-3/2} \sum_{r \in P_H, q \in \Lambda^*_+, u \in \Lambda^*, v \in P_L} N^\kappa \tilde{V}(u/N^{1-\kappa}) \eta_r b^*_r + v b^*_u a_q v b_v, \\
\Theta_3 &= N^{-3/2} \sum_{r \in P_H, q \in \Lambda^*_+, u \in \Lambda^*, v \in P_L} N^\kappa \tilde{V}(u/N^{1-\kappa}) \eta_r b^*_r + v b^*_u a_q v b_v, \\
\Theta_4 &= -N^{-3/2} \sum_{r \in P_H, q \in \Lambda^*_+, u \in \Lambda^*} N^\kappa \tilde{V}((p - r)/N^{1-\kappa}) \eta_r b^*_r + v b^*_u a_q v b_v, \\
\Theta_5 &= -N^{-3/2} \sum_{p \in P_H, r \in P_L} N^\kappa \tilde{V}((p - r)/N^{1-\kappa}) \eta_r b^*_r + v b^*_u a_q v b_v, \\
\Theta_6 &= N^{-3/2} \sum_{r \in P_H, p \in P_L} N^\kappa \tilde{V}((p - r)/N^{1-\kappa}) \eta_r b^*_r + v b^*_u a_q v b_v, \\
\Theta_7 &= 2N^2 \sqrt{N} \lambda_l \sum_{r \in P_H, v \in P_L} (\hat{\chi}_1 * f_N)_r \eta_r b^*_r + v b^*_v.
\end{align*}
\]

This is (B.9), with \( \delta_{\mathcal{H}_N} = \sum_{j=1}^7 \Theta_j \). Note that to isolate \( \Theta_0 \) we need to use the scattering equation (B.10), which also produces the last error term \( \Theta_7 \). Switching to position space, it is simple to show that
\[
\pm(\Theta_0 + \text{h.c.}) = \pm \left( \int dx dy \ N^{5/2 - 2\kappa} V(N^{1-\kappa}(x - y)) \tilde{b}^*_x \tilde{b}^*_y \left( \sum_{v \in P_L} e^{i y v} b_v \right) + \text{h.c.} \right) \leq CN^\kappa (\mathcal{N}_+ + 1).
\]

For all other terms \( \Theta_1, \ldots, \Theta_7 \), we show (B.8) and (B.11). The contribution \( \Theta_1 \) can be bounded by Cauchy-Schwarz, yielding
\[
|\langle \xi_1 | \Theta_1 | \xi_2 \rangle| \leq CN^{-1/2} \| (\kappa + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \xi_1 \| \| (\mathcal{N}_+ + 1)^{1/2} \xi_2 \|,
\]

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which implies (B.8) and (B.11). To bound the quintic terms \( \Theta_2, \Theta_3, \Theta_4 \), we switch to position space. We find that

\[
\Theta_2 = N^{-1/2} \int_{\Lambda^2} N^{2-2\kappa} V(N^{1-\kappa}(x - y)) \hat{b}_x^* \hat{b}_y^* (\hat{\eta}_{H,x} \hat{\eta}_{H,y} \tilde{a}_x^* \tilde{a}_y^* \tilde{b}(\hat{\chi}_{L,x}) dxdy ,
\]

where \( \hat{\eta}, \hat{\chi}_{L,x} \) are defined by \( \hat{\eta}_{H,x}(y) = \hat{\eta}(y - x), \hat{\chi}_{L,x}(y) = \chi_{P_L}(y - x) \). Using the bounds \( \|\hat{\eta}_{H,x}\|_2 = \|\eta_H\|_2 \leq CN^{\kappa-\alpha/2}, \|\hat{\chi}_{L,x}\|_2 = \|\chi_{P_L}\|_2 \leq CN^{3\beta/2} \) we obtain

\[
\langle \xi_1 | \Theta_2 | \xi_2 \rangle \leq CN^{-1/2+\kappa+(3\beta-\alpha)/2} \int_{\Lambda^2} N^{2-2\kappa} V(N^{1-\kappa}(x - y)) \|\tilde{a}_x \tilde{a}_y \xi_1 \| \|\tilde{b}(\hat{\chi}_{L,x}) \xi_2 \|
\leq CN^{-1+(3\kappa+3\beta-\alpha)/2} \|V_N \xi_1 \| \|(N_+ + 1)^{3/2} \xi_2 \|
\leq CN^{-1/2} \|V_N \xi_1 \| \|(N_+ + 1)^{3/2} \xi_2 \|,
\]

where we used the assumption \( \alpha - \beta > 2\beta + 4\kappa - 1 \) in the last step. This bound implies both (B.8) and (B.11) for \( \Theta_2 \). The terms \( \Theta_3, \Theta_4 \) can be handled analogously. As for \( \Theta_5, \Theta_6 \), we use again the estimate

\[
\sum_{p \in P_L} N^\kappa \hat{V}((p-r)/N^{1-\kappa})^2 \leq CN
\]

which holds uniformly in \( r \in P_L \). With Cauchy-Schwarz, we find that

\[
|\langle \xi_1 | \Theta_5 | \xi_2 \rangle| \leq CN^{-1/2} \|(K + 1)^{1/2} (N_+ + 1)^{1/2} \xi_1 \| \|(N_+ + 1)^{1/2} \xi_2 \| ,
\]

and, similarly,

\[
|\langle \xi_1 | \Theta_6 | \xi_2 \rangle| \leq N^{-3/2} \left( \sum_{r \in P_L} \sum_{p \in P_L} N^\kappa \hat{V}((p-r)/N^{1-\kappa}) |\eta_r| |b_p \xi_2| \right)^{1/2} \times \left( \sum_{r \in P_L} \sum_{p \in P_L} N^\kappa \hat{V}((p-r)/N^{1-\kappa}) |\eta_r| |p| \|b_{p+u} - b_{p-\xi} \xi_1| \right)^{1/2} \leq CN^{-1+\beta+2\kappa/2} \|(K + 1)^{1/2} (N_+ + 1)^{1/2} \xi_1 \| \|(N_+ + 1)^{1/2} \xi_2 \| .
\]

Finally, it follows from (B.3), (B.13), and \( \alpha > 4\kappa \) that

\[
|\langle \xi_1 | \Theta_7 | \xi_2 \rangle| \leq N^{-1/2} \|(N_+ + 1)^{1/2} \xi_1 \| \|(N_+ + 1)^{1/2} \xi_2 \|. 
\]

This concludes the proof of the lemma. \( \square \)

**Lemma B.6.** Let \( \Theta_0 \) be defined as in (B.10). Then, we have

\[
[\Theta_0, A] = \Pi_0 + \delta_{\Theta_0} \tag{B.12}
\]
where

\[ \Pi_0 = -2N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \left( \hat{V}(r/N^{1-\kappa}) + \hat{V}((r + v)/N^{1-\kappa}) \right) \eta_v b_v^* b_v, \]  

(B.13)

and

\[ \pm \delta \Theta_0 \leq CN^{-1/2}(K + N_0^2 + 1)(N_+ + 1). \]  

(B.14)

Moreover,

\[ \pm [\Pi_0, A] \leq CN^{-1/2}(N_+ + 1)^2. \]  

(B.15)

Proof. Proceeding as in [7, Lemma 8.6], we obtain

\[ [\Theta_0, A] = \Pi_0 + \sum_{j=1}^{8} \left[ \Pi_j + \text{h.c.} \right] + \delta \]

with \( \Pi_0 \) defined as in (B.13),

\[ \delta = -2N^{-1+\kappa} \sum_{r \in P_H} \sum_{v \in P_L} \chi_{P_H}(r+v) \hat{V}((r+v)/N^{1-\kappa}) \eta_v b_v^* b_v \]

and

\[ \Pi_1 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}((r+v)/N^{1-\kappa}) \eta_v b_v^* \left[ \left( 1 - \frac{N_+}{N} \right)^2 - 1 \right] b_v, \]

\[ \Pi_2 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(r/N^{1-\kappa}) \eta_v b_v^* \left[ \left( 1 - \frac{N_+ + 1}{N} \right) \left( 1 - \frac{N_+}{N} \right) - 1 \right] b_v, \]

\[ \Pi_3 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(v/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+ + 1}{N} \right), \]

\[ \times \left( b_{w-r-v} b_r - \frac{1}{N} a_{w-r-v} a_r \right) b_w, \]

\[ \Pi_4 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(v/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+}{N} \right), \]

\[ \times \left( b_{w+v} b_{r+v} - \frac{1}{N} a_{w+v} a_{r+v} \right) b_w, \]

\[ \Pi_5 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(r/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+}{N} \right) b_{w-v} b_w, \]

\[ \Pi_6 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(v/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+}{N} \right) b_{w-v} b_w, \]

\[ \Pi_7 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(r/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+}{N} \right) b_{w-v} b_w, \]

\[ \Pi_8 = -N^{-1} \sum_{r \in P_H} \sum_{v \in P_L} N^{\kappa} \hat{V}(v/N^{1-\kappa}) \eta_v b_v^* \left( 1 - \frac{N_+}{N} \right) b_{w-v} b_w, \]

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\[ \Pi_5 = N^{-2} \sum_{r \in P_H} \sum_{s \in P_H} N^\kappa \hat{V}(s/N^{1-\kappa}) \eta_r b^*_s (b_{r+w} a^*_{r+w} a_{r+v} + a^*_s a_{r+w} a_{r+v}) b^*_{r+s} b_w , \]

\[ \Pi_6 = -N^{-2} \sum_{r \in P_H} \sum_{s \in P_H} N^\kappa \hat{V}(s/N^{1-\kappa}) \eta_r b^*_s b^*_{r+v} a^*_{s+w} a_v b^*_{s} b_w , \]

\[ \Pi_7 = -N^{-1/2} \sum_{r \in P_H} \sum_{v \in P_L} N^\kappa \hat{V}(r/N^{1-\kappa}) b^*_{r+v} [b^*_{-r}, A] b_v , \]

\[ \Pi_8 = -N^{-1/2} \sum_{r \in P_H} \sum_{v \in P_L} N^\kappa \hat{V}(r/N^{1-\kappa}) b^*_{r+v} b^*_{-r} [b_v, A] . \]

This is (B.12), with \( \delta_{\Theta_0} = \delta + \sum_{j=1}^8 \Pi_j + \text{h.c.} \). We proceed to bound the terms in the error to show (B.14). The characteristic function in \( \delta \) vanishes for any \( v \), if \( |r| > N^\alpha + N^\beta \):

\[ \langle \xi, \delta \xi \rangle \leq CN^{-1+\kappa} \langle \xi, N \xi \rangle \left( \sum_{|r| \leq 2N^\alpha} |\eta_r| \right) \leq CN^{-1+2\kappa+\alpha} \langle \xi, N \xi \rangle . \]

The term \( \Pi_1 \) is easily bounded by Cauchy-Schwarz’s inequality and \( \sum_{r \in A^*_L} |\hat{V}(r/N^{1-\kappa}) \eta_r| \leq CN \): we find \( \pm \Pi_1 \leq CN^{-1+\kappa}(N^{1/2} \leq CN^{-1/2}(N^+ + 1)^2 \). As for \( \Pi_2 \) we estimate

\[
\begin{aligned}
|\langle \xi, \Pi_2 \xi \rangle | & \leq CN^{-1+\kappa} \left( \sum_{r \in P_H} \sum_{w \in P_L} |\eta_r|^2 \|b_{r+w} b_v \xi \|^2 \right)^{1/2} \\
& \times \left( \sum_{r \in P_H} \sum_{w \in P_L} \|b_{r+w} \xi \|^2 \right)^{1/2} \\
& \leq CN^{-1+\kappa} ||\eta_H||_2 P_L^{1/2} \| (N^+ + 1)^2 \xi \|^2 \\
& \leq CN^{-1/2} \| (N^+ + 1)^2 \xi \|^2 ,
\end{aligned}
\]

where we used \( \alpha - \beta > 2\beta + 4\kappa - 1 \) in the last step. The terms \( \Pi_3, \Pi_4 \) can be bounded similarly. The terms \( \Pi_5, \Pi_6 \) are first rearranged in normal order, producing additional terms from commutators, and then they are bounded as we did for \( \Xi_5, \Xi_6 \) in the proof of Lemma B.3. We find \( \Pi_5, \Pi_6 \leq CN^{-1/2}(N^+ + 1)^3 \). Expanding the commutator in \( \Pi_7 \) we find terms similar to \( \Pi_1, ..., \Pi_6 \), which are bounded in the same way. Finally, the term \( \Pi_8 \) is analogous to \( \Xi_8 \) in Lemma B.3 and can be bounded similarly, using the kinetic energy operator. This yields \( \Pi_8 \leq CN^{-1/2}(K + N^+_L + 1)(N^+ + 1) \). Collecting all bounds, we get (B.14). Finally, the bound (B.15) follows from the fact that \( \Pi_0 = -\Xi_0 \) and from (B.7).

Applying the last two lemmas, we can now control the action of \( A \) on \( \mathcal{H}_N \).
Lemma B.7. We have
\[ e^{-A_H} N e^A = H_N - N^{-1/2} \sum_{r \in P_H, v \in P_L} N^\kappa \hat{V}(r/N^{1-\kappa}) \left[ b^*_r v b^*_v + \text{h.c.} \right] \]
\[ - N^{-1} \sum_{r \in P_H, v \in P_L} N^\kappa \left( \hat{V}(r/N^{1-\kappa}) + \hat{V}((r + v)/N^{1-\kappa}) \right) \eta_r b^*_v b_v + \mathcal{E}_{H_N}, \]
with
\[ \pm \mathcal{E}_{H_N} \leq C N^{-1/2+\beta/2+\gamma} (K + N^2_+ + 1)(N_+ + 1). \]

Proof. With Lemma B.5 and Lemma B.6, we find
\[ e^{-A_H} N e^A = H_N + \int_0^1 e^{-sA} \Theta_0 e^{sA} ds + \int_0^1 e^{-sA} \delta_{H_N} e^{sA} ds \]
\[ = H_N + \Theta_0 + \int_0^1 \int_0^1 e^{-s_2 A} \Pi_0 e^{s_2 A} ds_2 ds_1 + \int_0^1 e^{-sA} \delta_{H_N} e^{sA} ds \]
\[ = H_N + \Theta_0 + \frac{1}{2} \Pi_0 + \int_0^1 \int_0^1 \int_0^1 e^{-s_3 A} \Pi_0 A [\Pi_0, A] e^{s_3 A} ds_3 ds_2 ds_1 + \int_0^1 e^{-sA} \delta_{H_N} e^{sA} ds \]
\[ = H_N + \Theta_0 + \frac{1}{2} \Pi_0 + \mathcal{E}_{H_N}, \]
and the claim follows by (B.10) (B.13), the bounds (B.11), (B.14) and Lemma 3.6. \(\square\)

B.4 Proof of Proposition 3.7

Bringing together the results of Lemma B.2, Lemma B.4 and Lemma B.7, we find that
\[ e^{-A} G_N e^A = C_{G_N} + Q_{G_N} + C_N + H_N \]
\[ + N^{-1} \sum_{r \in P_H, v \in P_L} N^\kappa \left( \hat{V}(r/N^{1-\kappa}) + \hat{V}((r + v)/N^{1-\kappa}) \right) \eta_r b^*_v b_v \]
\[ - N^{-1/2} \sum_{r \in P_H, v \in P_L} N^\kappa \hat{V}(r/N^{1-\kappa}) \left[ b^*_r v b^*_v + \text{h.c.} \right] + \tilde{\mathcal{E}}_{J_N} + e^{-A} \mathcal{E}_{G_N} e^A, \]
\[ (B.16) \]
with
\[ \pm \tilde{\mathcal{E}}_{J_N} \leq C N^{-1/2+\gamma} (K + N^2_+ + 1)(N_+ + 1). \]

We also notice that, by Equation (3.22) and Lemma 3.6, we have
\[ \pm e^{-A} \mathcal{E}_{G_N} e^A \leq C N^{-1/2+5\kappa/2+\alpha} (H_N + N^2_+ + 1)(N_+ + 1). \]

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We consider
\[ C_N - N^{-1/2} \sum_{\substack{r \in P^c_h \setminus v \in P_r^c \setminus v \neq -r}} N^\kappa \hat{V}(r/N^{1-\kappa}) b_{r+v}^* b_{r-v} b_v + \text{h.c.} \]
\[ = N^{-1/2} \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} N^\kappa \hat{V}(r/N^{1-\kappa}) b_{r+v}^* b_{r-v} (\gamma_v b_v + \sigma_v b_{-v}) \]
\[ + N^{-1/2} \sum_{r \in P^c_h \setminus v \in P_r^c, v \neq -r} N^\kappa \hat{V}(r/N^{1-\kappa}) b_{r+v}^* b_{r-v} b_v + \text{h.c.} \]
\[ = Z_1 + Z_2 + \text{h.c.} \]

It is simple to check that
\[ \pm(Z_2 + \text{h.c.}) \leq CN^{-\frac{1}{2} + \frac{\delta}{2}} (K + 1)(\mathcal{N}_+ + 1). \]

The term \( Z_1 \), on the other hand, can be bounded in two ways, leading to the estimates (3.28) and, respectively, (3.29). In the first case, we have
\[ |\langle \xi, Z_1 \xi \rangle| \leq N^{-1/2} \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} N^\kappa \frac{\hat{V}(r/N^{1-\kappa})}{r} r \|b_{r+v} b_{r-v}\| \left( \|b_v\| + \|\sigma_v\| (\mathcal{N}_+ + 1)^{1/2} \|\xi\| \right) \]
\[ \leq N^{-1/2 - \beta + \kappa} \left( \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} r^2 \|b_{r+v} b_{r-v}\|^2 \right)^{1/2} \]
\[ \times \left( \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} \frac{\hat{V}(r/N^{1-\kappa})^2}{r^2} v^2 \|b_v\|^2 \right)^{1/2} \]
\[ + N^{-1/2 + \kappa} \left( \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} r^2 \|b_{r+v} b_{r-v}\|^2 \right)^{1/2} \]
\[ \times \left( \sum_{r \in \Lambda_1^c, v \in P_r^c, v \neq -r} \frac{\hat{V}(r/N^{1-\kappa})^2}{r^2} \sigma_v^2 (\mathcal{N}_+ + 1)^{1/2} \|\xi\|^2 \right)^{1/2} \]
\[ \leq C N^{\frac{\delta}{2} - \beta}(K + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \|\xi\|^2 \]
\[ + C N^{\frac{\delta}{2} - \frac{\beta}{2}} (K + 1)^{1/2} (\mathcal{N}_+ + 1)^{1/2} \|\xi\|^2 (\mathcal{N}_+ + 1)^{1/2} \|\xi\|, \]

In the second case, we switch to position space and find for any \( \delta > 0 \) that
\[ |\langle \xi | Z_1 | \xi \rangle| \leq \int dx dy N^{5/2 - 2\kappa} V(N^{1-\kappa}(x - y)) |\langle \xi, b_{x}^* b_{y} \sum_{v \in P_r^c} e^{ixy}(\gamma_v b_v + \sigma_v b_{-v})|\xi\rangle| \]
\[ \leq \delta(\xi | V_N | \xi) + C N^{\kappa} \delta^{-1} (\langle \xi | (\mathcal{N}_+ + 1)|\xi\rangle). \]
Finally, we combine $Q_{\mathcal{G}N}$, as defined in (3.20), with the quadratic terms on the r.h.s. of (3.16). Comparing with the definition of $Q_{\mathcal{J}N}$ in (3.25), we obtain

$$Q_{\mathcal{G}N} + K + N^{-1} \sum_{r \in P_H, v \in P_L} N^\kappa \left( \hat{V}(r/N^1 - \kappa) + \hat{V}((r + v)/N^1 - \kappa) \right) \eta_p b^*_v b_v = Q_{\mathcal{J}N} + \tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2 + \tilde{\mathcal{E}}_3$$

where

$$\tilde{\mathcal{E}}_1 = \sum_{p \in \Lambda^+_*} p^2 (a^*_p a_p - b^*_p b_p) = N^{-1} \sum_{p \in \Lambda^+_*} p^2 a^*_p N_p a_p$$

is easily bounded by $\pm \tilde{\mathcal{E}}_1 \leq N^{-1}(K + 1)(N_+ + 1)$, and

$$\tilde{\mathcal{E}}_2 = -N^{-1} \sum_{r \in P_H, v \in P_L^c} N^\kappa \left( \hat{V}(r/N^1 - \kappa) + \hat{V}((r + v)/N^1 - \kappa) \right) \eta_p b^*_v b_v$$

$$-N^{-1} \sum_{r \in P^c_H, v \in \Lambda^+_*} N^\kappa \left( \hat{V}(r/N^1 - \kappa) + \hat{V}((r + v)/N^1 - \kappa) \right) \eta_p b^*_v b_v$$

$$-2N^{-1} \sum_{p \in \Pi_H} \sigma^2_p N^\kappa \left( (\hat{V}(\cdot/N^1 - \kappa) \ast \eta)_p + (\hat{V}(\cdot/N^1 - \kappa) \ast \eta)_0 \right) b^*_p b_p$$

can be bounded by $\pm \tilde{\mathcal{E}}_2 \leq C N^{\kappa - 2\beta}(K + 1)$ (note that the most dangerous term is the one on the first line). Finally, the off-diagonal part of the error

$$\tilde{\mathcal{E}}_3 = \sum_{p \in \Lambda^+_*} (G_p - \Gamma_p)[b^*_p b^*_\pi - b_p b_\pi]$$

can be estimated by

$$|\langle \xi, \tilde{\mathcal{E}}_3 \xi \rangle| = |\langle \xi, \sum_{p \in \Lambda^+_*} (G_p - \Gamma_p)[b^*_p b^*_\pi - b_p b_\pi] \xi \rangle|$$

$$\leq C N^{-1 + \kappa} \sum_{p \in \Pi_H} \left( \sum_{q \in \Pi_H^c} \left| \hat{V}((p - q)/N^1 - \kappa) \right| |\eta_q| + \sum_{q \in \Lambda^+_*} |\sigma_p| \left| \hat{V}(q/N^1 - \kappa) \right| |\eta_q| \right) \|b^*_p \xi\| \|b_\pi \xi\|$$

$$\leq C(N^{-\frac{1}{2} + \frac{2}{\alpha} + \alpha} + N^{2\kappa - 3\alpha/2}) \|(K + 1)^{1/2} \xi\| \|(N_+ + 1)^{1/2} \xi\|,$$

to get (3.28), or, without using the kinetic energy for the second term, by

$$|\langle \xi, \tilde{\mathcal{E}}_3 \xi \rangle| \leq C N^{-\frac{1}{2} + \frac{2}{\alpha} + \alpha} \|(K + 1)^{1/2} \xi\| \|(N_+ + 1)^{1/2} \xi\| + C N^{2\kappa - \alpha/2} \|(N_+ + 1)^{1/2} \xi\|^2$$

to get (3.29). Collecting the estimates from above and taking into account the assumptions on $\kappa, \alpha, \beta$, we conclude the proof of Proposition 3.7.
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