A classical model for the Maxwell equations coupled with matter

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To the memory of my friend Antonio Ambrosetti

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Abstract

We present a simple model of interaction of the Maxwell equations with a matter field defined by the Klein-Gordon equation. A simple linear interaction and a nonlinear perturbation produce solutions of the equations containing hylomorphic solitons, namely stable, solitary waves whose existence is related to the ratio energy/charge. These solitons, at low energy, behave as pointwise charged particles in an electromagnetic field.

Key words: Maxwell equations, Nonlinear Klein-Gordon equation, solitons, Q-balls, variational methods.

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1
1 Introduction

In classical mechanics the coupling of the electromagnetic field is given by the following equation:

$$\frac{d}{dt} \left( m\dot{\xi} \right) = q \left( E + \dot{\xi} \times H \right)$$  \hspace{1cm} (1)

where \( m \) is the mass of a particle, \( \xi = \xi(t) \) is its position in space and \( \dot{\xi} \) is the time derivative of \( \xi \). Unfortunately this equation is not consistent with the Maxwell equations. One of the main reasons of this inconsistency comes from the fact that the Maxwell equations are relativistic invariant and hence the inertial mass/energy of a charged material point is infinite. If the material point is replaced by a sort of ball other problems are present such as the selfinteraction of the field produced by the particle and the difficulty of a relativistic description of a solid body. As far as I know there is not a satisfactory description of the dynamics of a microscopic charged "ball" in an e.m. field. With the advent of quantum mechanics, this problem has lost its relevance and quantum models have been sought to describe this interaction.

Here we recall the formally simplest of them, since it has some relevance for this paper. It is given by the interaction of the Klein-Gordon equation (which describes a spinless boson field) with the e.m. field. In this case the action functional is given by

$$\mathcal{A}_W := \frac{1}{2} \int \int \left( |(\partial_t + iq\varphi) \psi|^2 - |(\nabla - iqA) \psi|^2 + m^2 |\psi|^2 \right) dx \ dt.$$  \hspace{1cm} (2)

where \( \psi \) is the wave-function of the boson field, \((\varphi, A)\) is the gauge potential, \( m \) and \( q \) is the mass and the electric charge of a particle.

Despite the fact that quantum electrodynamics (QED) is a well-established theory, we think that the study of the possibility of a consistent classical electrodynamics (CED) is still a relevant issue that might shed new light also on the unsolved problems in QED.

The model proposed here is based on the idea that charged particles can be described by solitons which can be seen as bumps of a "matter field". The idea is not new; in the last half century, since the pioneering work of Rosen in 68 [15], a lot of papers have been written. The original point of this paper is the introduction of a very simple interaction between the "matter field" and the e.m. field which produces identical particles which obey the known laws of CED.
2 The model

2.1 The basic equations

The action of the electromagnetic field is defined by the Lagrangian density

\[ L_{\text{F}}[\varphi, A] = \frac{1}{2} \left( |\partial_t \varphi + \nabla \varphi|^2 - |\nabla \times A|^2 \right) \]  

(3)

Now, we need to choose an equation to describe the matter field. The simplest semilinear equation invariant for the Poincaré group is the following

\[ \Box \psi + W'(|\psi|) \frac{\psi}{|\psi|} = 0 \]  

(4)

where

\[ \Box := \partial^2_t - \Delta; \]

\( \psi \) takes values in \( \mathbb{C} \) and

\[ W(s) = \frac{1}{2} s^2 + N(s); \]  

(5)

\[ N \in C^2(\mathbb{R}^3), \quad N(0) = N'(0) = 0. \]

The use of the complex variable is important since it gives to the field \( \psi \) an internal degree of freedom represented by a phase shift given by

\[ \psi(t, x) \mapsto e^{i\theta} \psi(t, x) \]  

(6)

Usually, people refer to equation (4) as to the nonlinear Klein-Gordon equation since its linearization gives the Klein-Gordon equation:

\[ \Box \psi + \psi = 0 \]  

(KG)

Equation (4) has a variational structure and its Lagrangian density can be written as follows:

\[ L_{\text{m}}[u, S] = \frac{1}{2} \left( |\partial_t u|^2 - |\nabla u|^2 \right) - W(|u|) \]

\[ = \frac{1}{2} \left[ |\partial_t u|^2 - |\nabla u|^2 + (\partial_t S)^2 u^2 - |\nabla S|^2 u^2 \right] - W(u) \]  

(7)

where we have set

\[ \psi(t, x) = u(t, x) e^{iS(t, x)}; \quad u \geq 0. \]  

(8)

We want to couple the matter field \( \psi \) with the electromagnetic field in the most simple and natural way, taking care that the Lorentz invariance of the equations be satisfied. Since the Lagrangian of the electromagnetic field depend on the 4-vector \( (\varphi, A) \), it must be coupled with a 4-vector determined by \( \psi \). There are two possible candidates which are linear in \( u \) and invariant for the transformation (8):

\[ (\partial_t u, \nabla u) \]
\((\partial_t S, \nabla S) u\)

Notice that these vectors are controvariant. They lead to the following interaction Lagrangian densities:

\[
\mathcal{L}_0 [u, \varphi, A] = \beta (\varphi \partial_t u + A \cdot \nabla u)
\]

\[
\mathcal{L}_1 [u, S, \varphi, A] = \beta (\partial_t S \varphi + A \cdot \nabla S) u
\]

(9)

where \(\beta\) is the interaction constant which, in order to fix the ideas, we assume positive; on the contrary, the sign "+" in the above definitions is necessary since \((\partial_t u, \nabla u)\) and \((\partial_t S, \nabla S)\) are covariant with respect to the time-coordinate; a "−" would violate the time-reversal property and the equations would lose the invariance for the Poincaré group. Since \(\mathcal{L}_1\) is not locally gauge invariant, we assume the Lorentz condition

\[
\partial_t \varphi + \nabla \cdot A = 0.
\]

(10)

In this paper we will examine the case \(\mathcal{L}_1\) (with \(\beta > 0\)) which provides a very rich model.

So, we will study the equations relative to the following Lagrangian density:

\[
\mathcal{L} = \mathcal{L}_m + \mathcal{L}_i + \mathcal{L}_f
\]

\[
= \frac{1}{2} \left[ |\partial_t u|^2 - |\nabla u|^2 \right] dt - W(u) + \frac{1}{2} \left[ (\partial_t S)^2 - |\nabla S|^2 \right] u^2
\]

\[
+ \beta (A \cdot \nabla S + \varphi \partial_t S) u
\]

\[
+ \frac{1}{2} \left( |\partial_t A + \nabla \varphi|^2 - |\nabla \times A|^2 \right)
\]

Making the variation of the action functional

\[
\mathcal{A} = \int \int \mathcal{L} \, dx \, dt
\]

with respect to \(u, S, \varphi\) and \(A\) we get the following system of equations:

\[
\Box u + W'(u) - \left[ (\partial_t S)^2 - |\nabla S|^2 \right] u = \beta (A \cdot \nabla S + \varphi \partial_t S)
\]

(11)

\[
\partial_t \left( \partial_t S \, u^2 - \beta \varphi u \right) - \nabla \cdot (\nabla S \, u^2 + \beta A u) = 0
\]

(12)

\[
\nabla \cdot (\partial_t A + \nabla \varphi) = \beta \partial_t S u
\]

(13)

\[
\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \varphi) = \beta \nabla S u.
\]

(14)

We can express these equations with new variables in order to make the equations independent of \(\beta\) and to get the Maxwell equations:

\[
\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi
\]

(15)

\[
\mathbf{H} = \nabla \times \mathbf{A}
\]

(16)
\begin{align*}
\rho &= -\beta \partial_t S u \\
\mathbf{j} &= \beta \nabla S u 
\end{align*}

So we get:
\begin{align*}
\nabla \cdot \mathbf{E} &= \rho \quad \text{(GAUSS)} \\
\nabla \times \mathbf{H} - \partial_t \mathbf{E} &= \mathbf{j} \quad \text{(AMPERE)}
\end{align*}

and (15) and (16) give rise to the first couple of the Maxwell equations:
\begin{align*}
\nabla \times \mathbf{E} &= 0 \quad \text{(FARADAY)} \\
\nabla \cdot \mathbf{H} &= 0 \quad \text{(GAUSS FOR MAGNETISM)}
\end{align*}

The equations of the matter field become:
\begin{align*}
\Box u + W'(u) + \frac{j^2 - \rho^2 + \varphi \rho + \mathbf{A} \cdot \mathbf{j}}{u} &= 0 \quad (19) \\
\partial_t (\rho u - \varphi u) + \nabla \cdot (\mathbf{j}u + \mathbf{A}u) &= 0. \quad (20)
\end{align*}

Notice that also the equations (19) and (20) depend only from the gauge-independent variables \((u, \rho, \mathbf{j}, \mathbf{E}, \mathbf{H})\) since the dependence from \(\varphi\) and \(\mathbf{A}\) in these equations can be eliminated via the Gauss equations which gives \(\varphi\) and \(\mathbf{A}\) by the appropriate Green functions. The action can be rewritten as follows:
\begin{align*}
A &= \frac{1}{2} \iint \left( |\partial_t u|^2 - |\nabla u|^2 \right) dx dt - \iint W(u) dx dt \\
&\quad + \frac{1}{2} \iint (\rho^2 - j^2) dx dt \\
&\quad + \iint (\mathbf{A} \cdot \mathbf{j} - \varphi \rho) dx dt + \frac{1}{2} \iint (\mathbf{E}^2 - \mathbf{H}^2) dx dt. \quad (21)
\end{align*}

**Remark 1.** As we have already remarked, the Lagrangian (9) is not invariant for the local action of the gauge group of the electromagnetic field; however, it is invariant for the global action of a gauge transformation namely invariant for the 4-parameters group
\begin{align*}
(u, S, \varphi, \mathbf{A}) \mapsto (u, S - at - b \cdot x, \varphi + a, \mathbf{A} + b); \quad (a, b) \in \mathbb{R}^4 \quad (22)
\end{align*}

Moreover, we have the invariance (6) which can be rewritten as follows:
\begin{align*}
T_\theta S &= S + \theta; \quad \theta \in \mathbb{R} / 2\pi \mathbb{Z} \quad (23)
\end{align*}

and it is typical of the (4) equation. Notice that in this model the invariance (23) is independent of the gauge invariance (22) and hence it leads to a different conservation law (see Section (23)). Finally, we remark that the position (17) and (18) are appropriate if we put ourselves in a gauge where \((u, S, \varphi, \mathbf{A})\) vanish at infinity, or to be more precise, if \(u, S \in H^1(\mathbb{R}^3), \varphi \in D^{1,2}(\mathbb{R}^3)\) and \(\mathbf{A} \in D^{1,2}(\mathbb{R}^3, \mathbb{R}^3)\) (see (27)).
2.2 Conservation laws

Let us examine the main integral of motion that will be used in the following. We assume that \((u, S, \varphi, A)\) is a solution of (11),..., (14) and that all the quantities 
\(|\partial_t u|^2, |\nabla u|^2, (\partial_t S)^2, \) etc. be integrable; under these assumptions we will compute some integral of motion relevant for this paper.

**Conservation of Energy.** Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian. We have the following result:

**Proposition 2.** The energy takes the following form

\[
E = E_m + E_f + E_i
\]

where

\[
E_m[u, S] = \frac{1}{2} \int \left( |\partial_t u|^2 + |\nabla u|^2 \right) dx + \frac{1}{2} \int \left( (\partial_t S)^2 + |\nabla S|^2 \right) u^2 dx + \int W(u) dx
\]

\[
= \frac{1}{2} \int \left( |\partial_t u|^2 + |\nabla u|^2 \right) dx + \frac{1}{2} \int \left( \frac{\rho^2 + j^2}{\beta^2} \right) dx + \int W(u) dx
\]

is the matter field energy.

\[
E_f[\varphi, A] = \frac{1}{2} \int \left( |\partial_t A + \nabla \varphi|^2 + |\nabla \times A|^2 \right) dx
\]

\[
= \frac{1}{2} \int (E^2 + H^2) dx.
\]

is the e.m. field energy, and

\[
E_i[u, S, \varphi, A] = -\beta \int (\varphi \partial_t S + A \cdot \nabla S) u dx
\]

\[
= \int (\varphi \rho - A \cdot j) dx.
\]

is the interaction energy.

The names matter field energy, e.m. field energy, and interaction energy are motivated by the fact that \(E_m\) depend only on the matter variables \((u, S)\), \(E_f\) depend on the e.m. field variables \((\varphi, A)\) and only \(E_i\) depend on all the four variables.

**Proof:** By Noether’s Theorem, we have that the energy density \(E_m\) relative to the Lagrangian density \(\mathcal{L}_m\) is given by
\[
E_M = \frac{\partial L_m}{\partial (\partial_t u)} \cdot \partial_t u + \frac{\partial L_m}{\partial (\partial_t S)} \cdot \partial_t S - L_m
\]

\[
= (\partial_t u)^2 + (\partial_t S)^2 u^2 - \frac{1}{2} \left[ |\partial_t u|^2 - |\nabla u|^2 - W(u) + (\partial_t S)^2 u^2 - |\nabla S|^2 u^2 \right]
\]

\[
= \frac{1}{2} \left[ |\partial_t u|^2 + |\nabla u|^2 + (\partial_t S)^2 u^2 + |\nabla S|^2 u^2 \right] + W(u)
\]

The computation of energy density relative to the Lagrangian density \( L_f \) is the usual one and we report it for completeness:

\[
\frac{\partial L_f}{\partial (\partial_t A)} - L_f = \left( \partial_t A + \nabla \phi \right) \cdot \partial_t A - \frac{1}{2} \left( \partial_t A + \nabla \phi \right)^2 + \frac{1}{2} (\nabla \times A)^2
\]

\[
= -E \cdot (-E + \nabla \phi) - \frac{1}{2} E^2 + \frac{1}{2} H^2
\]

\[
= \frac{1}{2} E^2 + \frac{1}{2} H^2 - E \cdot \nabla \phi = \mathcal{E}_f - E \cdot \nabla \phi
\]

The energy density relative to the Lagrangian density \( L_i \) is given by

\[
\frac{\partial L_i}{\partial (\partial_t S)} \cdot \partial_t S - L_i = \beta \varphi u \partial_t S - \beta (A \cdot \nabla S + \partial_t S \varphi) u
\]

\[
= -\beta A \cdot \nabla S u = -A \cdot j
\]

If we set

\[
\mathcal{E}_i = -E \cdot \nabla \varphi - A \cdot j
\]

we have that

\[
E_i = \int \mathcal{E}_i \, dx = \int (-E \cdot \nabla \varphi - A \cdot j) \, dx
\]

\[
= \int (\nabla \cdot E \varphi - A \cdot j) \, dx = \int (\rho \varphi - A \cdot j) \, dx
\]

\[
= -\beta \int (\varphi \partial_t S + A \cdot \nabla S) \, u \, dx
\]

Then

\[
E = E_m + E_f + E_i.
\]

\[
\square
\]

**Conservation of Momentum.** Momentum, by definition, is the quantity which is preserved by virtue of the space invariance of the Lagrangian. Here we will compute only the matter-field moment since it is the only part needed in the rest of this paper.
Proposition 3. The momentum takes the following form
\[ P = P_m + P_f + P_i \] (27)
where
\[ P_m = \int (\partial_t u \nabla u + \partial_t S \nabla u^2) \, dx \] (28)
\[ = \int (\partial_t u \nabla u + \rho j) \, dx \] (29)
is the matter field momentum.

Proof: By Noether’s Theorem, we have that the momentum densities \( P_m \) relative to the Lagrangian densities \( L_m \) is given by
\[ P_m = \frac{\partial L_m}{\partial (\partial_t u)} \nabla u + \frac{\partial L_m}{\partial (\partial_t S)} \nabla S = \partial_t u \nabla u + \partial_t S \nabla u^2 \]
\[ \square \]

Conservation of electric charge: Even if our equations are not invariant for the whole gauge group, nevertheless the electric charge is preserved as it is a consequence of (22). In fact by (GAUSS) and (AMPERE), we get the continuity equation:
\[ \partial_t \rho = \nabla \cdot (\partial_t E) = \nabla \cdot (\nabla \times H - j) = - \nabla \cdot j \]
So, the total electric charge
\[ Q[u, S] = \int \rho \, dx = -\beta \int \partial_t S \, dx \] (30)
is preserved.

Conservation of hylenic charge: Following [3] and [7] the hylenic charge, by definition, is the quantity which is preserved by the invariance for the transformation (6). It is defined as follows:
\[ H[u, S, \varphi] = \int \left( \frac{\partial_t S u^2}{\beta} - \varphi u \right) \, dx = \int (\rho - \varphi) u \, dx \] (31)
By equation (12), we see directly that the hylenic charge is preserved.

2.3 The Cauchy problem
In order to study the Cauchy problem, it is more convenient to use the variable \( \psi \) rather that \( (u, S) \). To this end, we introduce the following operators:
\[ \mathfrak{D}_t (\psi) = \text{Im} \left( \frac{\partial_t \psi}{\psi} \right) = \text{Im} \left( \frac{\partial_t (ue^{iS})}{ue^{iS}} \right) \]
\[ = \text{Im} \left( \frac{\partial_t u e^{iS} + iu \partial_t S e^{iS}}{ue^{iS}} \right) = \text{Im} \left( \partial_t u + i \partial_t S \right) = \partial_t S \]}
\[ D_x (\psi) = \text{Im} \left( \frac{\nabla \psi}{\psi} \right) = \nabla S \]

Since we have assumed the Lorentz condition the equations (11),..., (14), using (5), can be rewritten as follows:

\[ \square \psi + \psi = -N'(|\psi|) \frac{\psi}{|\psi|} + A \cdot D_x (\psi) - \varphi D_t (\psi) \] (32)

\[ \square \varphi = D_t (\psi) |\psi| \] (33)

\[ \square A = D_x (\psi) |\psi| . \] (34)

We make the following (redundant) assumptions on \( N \):

\[ N, N' \text{ and } N'' \text{ are bounded}; \] (35)

\[ N(s) \geq -\frac{1}{2} (1 - \delta) s^2; \quad 0 < \delta < 1. \] (36)

**Theorem 4.** If (35), (36) hold and \( \beta \) is sufficiently small, the Cauchy problem relative to equations (32),..., (34) has a unique weak solution.

**Proof:** The proof of this theorem follows standard arguments and we will just give a sketch. The function space where to work is \( H^1 \times (D^{1,2})^4 \) where

\[ H^1 = \left\{ \psi \in L^2(\mathbb{R}^3, \mathbb{C}) \mid \int (|\nabla \psi|^2 + |\psi|^2) \, dx < +\infty \right\} \]

\[ D^{1,2} = \left\{ f \in L^6(\mathbb{R}^3) \mid \int |\nabla f|^2 \, dx < +\infty \right\} . \] (37)

We set

\[ U(t, x) = (\psi(t, x), \varphi(t, x), A(t, x)); \]

where \( \psi \in H^1(\mathbb{R}^3, \mathbb{C}), \varphi \in D^{1,2}(\mathbb{R}^3, \mathbb{R}), \) and \( A \in D^{1,2}(\mathbb{R}^3, \mathbb{R}^3) = [D^{1,2}(\mathbb{R}^3, \mathbb{R}^1)]^3 \).

So, we end with the Cauchy problem

\[ \square U + P_1 U = F(U) \] (38)

where \( P_1 \) is the projection of \( U \) on the first component i.e. \( P_1 U = \psi \).

We equip the phase space \( X := \left[ H^1 \times (D^{1,2})^4 \right] \times (L^2)^6 \), with its natural norm given by

\[ \| U \|^2 = \| \partial_t \psi \|^2_{L^2} + \| \partial_t \psi \|^2_{H^1} + \| \partial_t \varphi \|^2_{L^2} + \| \nabla \varphi \|^2_{L^2} + \| \partial_t A \|^2_{L^2} + \| \nabla A \|^2_{L^2} \]

It is well known that a sufficient condition for the Cauchy problem to have a unique solution for the initial data in \( X \) is:
• the energy inequality holds: there exists two positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \|U\|^2 \leq E[U] \leq c_2 \|U\|^2
\]
This inequality can be proved if \( \beta \) is sufficiently small and if (35) holds;
• \( F : X \to X' \) is locally compact; this fact holds since the embedding
\[
U_{loc} \to (L_{loc}^6)^6
\]
is compact;

Under these conditions the proof goes as follows:

1. we take a sequence of approximate solutions; for example we can use the Faedo-Galerkin procedure;
2. we take the weak limit of the approximated solutions which exists thank to the second energy inequality;
3. we pass to the limit in the weak formulation of the equations; we can take the limit in the nonlinear part \( F \) since it is locally compact;
4. we can prove the uniqueness thanks to the first energy inequality and the Gronwall’s inequality.

\[ \Box \]

**Remark 5.** The optimal conditions for the existence of solutions and the study of the their regularity is not in the aim of this paper and it is a question that, for the moment, is left open.

## 3 \( q \)-solitons

Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A soliton is a solitary wave which exhibits some form of stability so that it has a particle-like behavior (see e.g. [12], [14], [11], [7]).

It is well known that equation (4) presents solitons under suitable assumptions on \( W \). It has been largely studied during the 70’s and the 80’s of the last century. The first rigorous result about finite energy solution was due to Strauss [13] and later Berestycki and Lions [8] gave sufficient and “almost necessary” condition for the existence. In [7] there is a detailed analysis of the case in which \( W \geq 0 \). If we couple (4) with the Maxwell equation via the interaction (2) the solitons usually are called \( Q \)-balls (Coleman [9]). The first rigorous result about the existence of \( Q \)-balls has been establish in 2002, [5]. Afterwards, their stability has been proved in [6]. A detailed analysis on \( Q \)-balls and the references to the large literature can be found in [7]; in all these paper the interaction between the solitons and the e.m. field is established by the Lagrangian (2).
In this section we analyze the existence and the properties of solitons when the interaction with the Maxwell equation is simply given by the Lagrangian \( L \) and not by \( S \). They will be called \( q \)-solitons. The main difference between \( q \)-solitons and the \( Q \)-balls is that the former behave like single particles while the latter behave like a swarm of particles (see [7], Sections 4.1.2 and 5.1.5).

### 3.1 Existence of stationary waves

Let us prove the existence of some particular solution of equations (11),..., (14); first, we look for stationary solutions, namely solutions where \( \psi \) is a stationary wave, i.e.

\[
\psi(t, x) = u(x) e^{-i\omega t}; \quad u \geq 0
\]

We make the following ansatz:

\[
u = u(x); \quad S = -\omega t; \quad \varphi = \varphi(x); \quad A = 0.
\]

Replacing these variables in (11),..., (14), we have that eq. (12) and (14) are identically satisfied while eq. (11) and (13) become

\[
-\Delta u + W'(u) - \omega^2 u + \beta \omega \varphi = 0 \quad (41)
\]

\[
-\Delta \varphi = \beta \omega u \quad (42)
\]

These two equations have nontrivial solutions provided that suitable conditions on \( W \in C^2 \) be satisfied: we write \( W \) as follows,

\[
W(s) = \frac{1}{2} s^2 + N(s),
\]

In the model of our interest, \( N \) must be considered as a small perturbation of the parabola \( 1/2s^2 \). However, in order to get an existence result, it is sufficient to make the following assumptions on \( N \):

- (N-1) \( N(0) = N'(0) = N''(0) = 0 \);
- (N-2) \( \inf_{s \in \mathbb{R}} N(s) := N_{\inf} < 0 \), \( N_{\inf} \) is allowed also to be \(-\infty\);
- (N-3) there exist \( C > 0 \) and \( 2 < p < 6 \) such that \( |N'(s)| \leq C(1 + s^{p-1}) \).

We will show that, at least for \( \beta \) small, the above assumptions guarantee the existence of nontrivial solutions of eqs. (11), (12). In most of the literature relative to \( L \) usually we have the following choice of \( N \):

\[
N(s) = \frac{1}{p} |s|^p, \quad 2 < p < 6.
\]

This assumption implies the existence of nontrivial solutions also for eqs. (11), (12) for every \( \beta > 0 \). However, in our model, it is more interesting (see Th. 8) to choose a "bump-like" \( N \) such as

\[
N(s) = -\varepsilon^2 s^3 \exp \left( -\frac{|s - 1|}{\varepsilon} \right).
\]
or a "bell" function such as

\[ N(s) = \begin{cases} -\left[(s-1)^2 - \varepsilon^2\right]^2 & \text{if } |s-1| < \varepsilon \\ 0 & \text{if } |s-1| \geq \varepsilon \end{cases} \]  

(46)

where \( \varepsilon \) is a small parameter which makes \( W(s) \geq 0 \). Its relevance will be discussed in Th. 8.

We define the following bilinear form:

\[ a_\omega(u, u) = \frac{1}{2} \int \left[ |\nabla u|^2 + (1 - \omega^2) u^2 \right] dx + \beta^2 \omega^2 \int \int \frac{u(x)u(y)}{|x-y|^2} dx dy \]  

(47)

Notice that

\[ \int \int \frac{u(x)u(y)}{|x-y|^2} dx dy = \int (G * u) u dx \]

where

\[ G(x) = \frac{1}{4\pi |x|^2} \]

is the Green function relative to the Poisson equation

\[ -\Delta \phi = u \]  

(48)

namely \((G * u)(x) = (\Delta)^{-1} : \mathcal{D}^{1,2} \to \mathcal{D}^{1,2}\).

Now, let us introduce a number \( \omega_{\text{inf}} \) which is very relevant in this study of solitons:

\[ \omega_{\text{inf}} := \inf \left\{ \omega > 0 \mid \exists u \in H^1_c, \ a_\omega(u, u) + \int N(u) \ dx < 0 \right\} ; \]  

(49)

\( \omega_{\text{inf}} \) depends on \( \beta \) and the shape of \( N \). For example if \( N(u) \) is given by (44), then it is immediate to check that \( \omega_{\text{inf}} = 0 \) for every \( \beta > 0 \). If \( N(u) \) is given by (45) \( \omega_{\text{inf}} \) depends on \( \beta \) (see Cor. 7).

We have the following theorem.

**Theorem 6.** If (N-1) and (N-3) hold and if \( \omega_{\text{inf}} < 1 \), then for every \( \omega \in (\omega_{\text{inf}}, 1) \), eqs. (41), (42) have nontrivial solutions in \( H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \).

**Proof:** The couple of equations (41) and (42) can be easily solved by standard variational methods; we will give here a sketch of the proof avoiding standard estimates which are well known among people working in nonlinear analysis.

Set

\[ H^1_{\text{rad}}(\mathbb{R}^3) = \{ u \in H^1(\mathbb{R}^3) \mid u = u(|x|) \} ; \]

and

\[ V := \{ u \in H^1_{\text{rad}}(\mathbb{R}^3) \mid a_\omega(u, u) < +\infty \} \]
Since $\omega_{\text{sup}} < 1$, $V$ is a Hilbert space equipped with scalar product $a_\omega(u, v)$ and norm $\|u\|_V := \sqrt{a_\omega(u, v)}$. Moreover, by the definition of $\omega_{\text{sup}}$, $\exists \delta > 0$,

$$\|u\|^2_V \geq \delta \|u\|^2_{H^1}$$

Then, using the Gagliardo-Nirenberg-Sobolev estimate we can see that

$$V = H^1_{\text{rad}} + \left[D^1,2_{\text{rad}}(\mathbb{R}^3)\right] \subset H^1_{\text{rad}} + L^{6/5}$$

Now, we define on $V$ the following functional:

$$J[u] = \|u\|^2_V + \int N(u) dx \tag{51}$$

By (N-3), (50) and standard arguments, $J$ is a differentiable functional in $V$. So we have to prove two facts: (1) the critical points of $J$ solve eqs. (41) and (42) and (2) if $\omega \in (\omega_{\text{inf}}, 1)$, $J$ has at least a nontrivial critical point.

(1) We have that $\forall v \in V$,

$$dJ[u](v) = \int [\nabla u \nabla v + (1 - \omega^2) uv] \ dx + \omega^2 \beta^2 \int [(G * u) u + N(u)] v dx$$

and by well known arguments, we have that

$$-\Delta u + \omega^2 (G * u) + N'(u) = 0.$$

Taking account of (43)

$$-\Delta u + W'(u) + \omega^2 \beta^2 (G * u) - \omega^2 u = 0;$$

finally, setting $\varphi = \omega \beta (G * u)$, we get equation (41) while equation (42) follows from the definition of $\varphi$.

(2) The simplest way to prove the existence of critical points of $J$ is the use of the Mountain Pass theorem of Ambrosetti and Rabinowitz [1]. Following standard arguments, it is easy to prove that $J$ satisfies the Palais-Smale condition (for a very similar result see [5], Lemma 4.3). The interesting fact is to check the conditions which guarantee the geometry of the Mountain Pass theorem namely that

$$\exists r > 0, \|u\|^2_V = r \Rightarrow J[u] \geq b > 0 \tag{52}$$

and

$$\exists \bar{u}, \|\bar{u}\|^2_{H^1} > r, \ J[\bar{u}] \leq 0 \tag{53}$$

By the definition of $\|\cdot\|_V$,

$$\|u\|^2_V \geq \delta \|u\|^2_{H^1}, \ \delta > 0.$$
If $r > 0$, is sufficiently small, by (43), (N-1), (N-3) and standard computations, $\exists C, \eta > 0$ such that

$$\int |N(u)| \, dx \leq C \|u\|_H^{2+\eta} \leq \frac{C}{\delta} \|u\|_H^{2+\eta}$$

then, if $\|u\|_V = r$

$$J[u] = \frac{1}{2} \|u\|_V^2 + \int |N(u)| \, dx \geq \delta \|u\|_V^2 - \frac{C}{\delta} \|u\|_V^{2+\eta}$$

$$= \left[ \delta - \frac{Cr^\eta}{\delta} \right] r^2$$

Then if $r$ is sufficiently small $J[u] \geq b > 0$ and (52) is proved. (53) holds by the definition (49) of $\omega_{inf}$.

□

**Corollary 7.** If (N-1), (N-2) and (N-3) hold, then there exists $\beta_0 > 0$, such that for every $\beta \in (0, \beta_0)$ eqs. (41), (42) have nontrivial solutions in $H^1(\mathbb{R}^N)$.

**Proof:** By Th. 6 it is sufficient to prove that

$$\omega_{inf} < 1.$$

By (N-2), we can choose a point $s_1$ such that

$$N(s_1) = -h^2.$$

We set

$$u_r = \begin{cases} 
  s_1 & \text{if } |x| < r \\
  0 & \text{if } |x| > r + 1 \\
  \frac{|x|}{s_1} - [(r + 1)|x| - 1] s_1 & \text{if } r < |x| < r + 1
\end{cases}, \quad (54)$$

and

$$\bar{\omega} = \sqrt{1 - \frac{h^2}{s_1^2}}.$$

and

$$F[u] := \frac{1}{2} \int |\nabla u|^2 \, dx + \int \left( W(u) - \frac{1}{2} \bar{\omega}^2 u^2 \right) \, dx$$

Let us compute $F[u_r]$:

$$F[u_r] = \frac{1}{2} \int_{B_{r+1} \setminus B_r} |\nabla u_r|^2 \, dx + \int_{B_{r+1} \setminus B_r} \left( W(u) - \frac{1}{2} \bar{\omega}^2 u_r^2 \right) \, dx$$

$$+ \int_{B_r} \left( W(u_r) - \frac{1}{2} \bar{\omega}^2 u_r^2 \right) \, dx$$

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The first part can be estimated as follows:

\[
\frac{1}{2} \int_{B_{r+1} \setminus B_r} |\nabla u_r|^2 \, dx + \int_{B_{r+1} \setminus B_r} \left( W(u) - \frac{1}{2} \tilde{\omega}^2 u^2 \right) \, dx \leq C \cdot \text{meas} \, (B_{r+1} \setminus B_r) \\
\leq C_1 r^2
\]

For the second part we have that

\[
\int_{B_r} \left( W(u_r) - \frac{1}{2} \tilde{\omega}^2 u_r^2 \right) \, dx = \int_{B_r} \left[ \frac{1}{2} u_r^2 - \frac{1}{2} \tilde{\omega}^2 u_r^2 + N(u_r) \right] \, dx \\
= \int_{B_r} \left[ \frac{1}{2} \left( 1 - \tilde{\omega}^2 \right) s_1^2 + N(u_r (s_1)) \right] \, dx \\
\leq \int_{B_r} \left[ \frac{1}{2} \left( 1 - \left( 1 - \frac{h^2}{s_1^2} \right) \right) s_1^2 - h^2 \right] \, dx \\
= \int_{B_r} \left[ \frac{1}{2} h^2 - h^2 \right] \, dx = \frac{4}{3} \pi r^3 h^2 = \frac{2}{3} \pi r^3 h^2
\]

Then, we have that

\[
F[u_r] \leq C_1 r^2 - \frac{2}{3} \pi r^3 h^2.
\]

So, we can choose \( \tilde{r} \) so large that

\[
F[u_{\tilde{r}}] < -\tilde{r}^3 h^2 < -1
\]

and \( \beta \) so small that

\[
\beta^2 \tilde{\omega}^2 \int (G \ast u_{\tilde{r}}) u_{\tilde{r}} \, dx \leq 1.
\]

Then

\[
a_{\tilde{\omega}}(u_{\tilde{r}}, u_{\tilde{r}}) + \int N(u_{\tilde{r}}) \, dx = F[u_{\tilde{r}}] + \frac{1}{2} \beta^2 \tilde{\omega}^2 \int (G \ast u_{\tilde{r}}) u_{\tilde{r}} \, dx \leq -\frac{1}{2}
\]

and hence

\[
\omega_{\text{inf}} < \bar{\omega} = \sqrt{1 - \frac{h^2}{s_1^2}} < 1.
\]

\[\square\]

### 3.2 Stationary \( q \)-solitons

Using the equivariant Mountain Pass theorem and exploiting the fact that the functional \((61)\) is even, it is possible to prove that eqs. \((11), \ldots, (14)\), have infinitely many radially symmetric solutions of the form \((39), (40)\), namely solitary waves. We call **ground state solution**, the radially symmetric solution \(u_0 > 0\) which minimizes the following quantity:

\[
\Lambda [u, \omega] = \frac{E[u, \omega]}{\|H[u, \omega]\|} = \frac{\int \left( \frac{1}{2} |\nabla u| - \frac{1}{2} \omega^2 u^2 + W(u) + \beta^2 \omega^2 (G \ast u) \right) \, dx}{\| \int (-\beta^{-2} \omega u^2 - \omega (G \ast u) \) \, dx \|}
\]
in $H^1 \times \mathbb{R}^+$. Clearly, at least for a generic $W$, this solution is unique and it corresponds to the critical value determined by the Mountain Pass Theorem, having chosen $\omega$ which minimizes $\Lambda[u, \omega]$. Notice that $\Lambda[u, \omega]$ is the ratio of the matter energy (25) and the hylenic charge (31). If $W \geq 0$, the ground state solution, is a soliton in the sense that it is orbitally stable (see e.g. [6] or [7]).

From now on, $\sigma_0$ we will denote the ground state solution of equations (41), (42), namely the configuration

$$\sigma_0(x) = \begin{bmatrix} (u_0(x), 0, 0, 0) \\ (0, -\omega_0 + \theta, 0, 0) \end{bmatrix}$$

where $\theta$ is a possible phase shift which is not relevant and from now on, it will be neglected. Such a function will be called a $q$-soliton. We have chosen this name to emphasize the comparison with the $Q$-balls that are stable configurations of the equations determined by the action $\mathcal{A}_w + \mathcal{A}_f$ (see (2) and (3) and the discussion at the beginning of Sec. 3). Roughly speaking a $Q$-ball behaves like a swarm of charged particles kept close to each other by the gluing force determined by $N(s)$ (see [3] or [7] Sec. 5.1.5). Instead, as we will see in this and the next sections, a $q$-soliton behaves like a single particle of "matter" condensed by the gluing force determined by $N(s)$.

The $q$-soliton, has a positive electric charge $\rho_0 = \omega u_0$ (and hence, by (42), $\varphi_0(x) > 0$). However, the equations (11),...,(14), have also a solution with negative charge given by

$$\sigma_0^-(x) = \begin{bmatrix} (u_0(x), 0, 0, 0) \\ (0, \omega_0, 0, 0) \end{bmatrix}$$

Then (11),...,(14), have at least two orbitally stable solutions determined by a $q$-soliton $\sigma_0(x)$ and a $q$-antisoliton $\sigma_0^-(x)$:

$$U(t, x) = \begin{bmatrix} (u_0(x), 0, \varphi_0(x), 0) \\ (0, -\omega_0 t, 0, 0) \end{bmatrix} ; \quad U^-(t, x) = \begin{bmatrix} (u_0(x), 0, -\varphi_0(x), 0) \\ (0, \omega_0 t, 0, 0) \end{bmatrix}$$

Generally, they are unique up to space-time translations and phase shift. Rotations do not produce new solutions since $u_0$ is radially symmetric.

The "shape" of a soliton is determined by the nonlinear $N(s)$. In [2] there is a detailed analysis of this topic in the case $\beta = 0$. Clearly this analysis can be extended to the $q$-soliton when $\beta$ is small. The next theorem examines some properties of the $q$-solitons in the case in which $N(s)$ is a small bump such as (46):

**Theorem 8.** For every $\varepsilon > 0$, we can choose $N$ such that

- $W \geq 0$;
- $1 - \varepsilon < \omega_{\text{inf}} < 1$. 


• if $u_0$ is the Mountain Pass solution of eqs. (41) and (42), then

$$1 - \varepsilon \leq \|u_0\|_{L^\infty} \leq 1 + \varepsilon$$

**Proof.** We choose $N$ to be a bell function such as (46) so that

$$\min N = N(1) \geq -\varepsilon^2$$

and

$$\text{supp}(N) = [1 - \varepsilon, 1 + \varepsilon]$$

Then, the first inequality is trivially verified. In order to prove the second inequality, we see that

$$N(u) \geq -\frac{1}{2} \varepsilon^2$$

and so, if we put $\tilde{\omega}^2 = 1 - \varepsilon^2$, we have that

$$a_{\omega}(u, u) + \int N(u) \, dx \geq \frac{1}{2} \int \left[ |\nabla u|^2 + (1 - \tilde{\omega}^2) u^2 \right] \, dx + \int N(u) \, dx$$

$$= \frac{1}{2} \int \left[ |\nabla u|^2 + \varepsilon^2 u^2 \right] \, dx - \frac{1}{2} \varepsilon^2 \int u^2 \, dx \geq 0$$

So by the definition of $\omega_{\text{inf}}$ (49), we have that $\omega_{\text{inf}}^2 > \tilde{\omega}^2$ and hence $\omega_{\text{inf}} > \sqrt{1 - \varepsilon^2} > 1 - \varepsilon$.

The third inequality follows applying to eq. 41 the maximum principle. The details of the proof can be found in [2].

□

**Remark 9.** The picture which comes out from Cor. 7 and Th. 8 is the following: given the free electromagnetic field and the free matter field relative to KG, we get $q$-solitons provided that

- the interaction between them is given by a Lagrangian of type (2) with $\beta$ very small;
- KG is perturbed by a nonlinear term $N(s)$ (negative in some point) small with respect to 1 and large with respect to $\beta$.

If we want to analyze the properties of a $q$-soliton considered as a model for physical particles, it is useful to rewrite equation (11) with dimensional constants. We get the following equation, which is satisfied by $\sigma_0$ if suitably rescaled):

$$\partial_t^2 u - e^2 \Delta u + \alpha^2 u + \frac{e^2}{c^2} N'(u) - \left[ (\partial_t S)^2 - c^2 |\nabla S|^2 \right] u = \eta \beta (A \cdot \nabla S - \varphi \partial_t S)$$

In this equation,
• $c$ is the speed of light which makes the equation invariant for the Lorentz transformations with the parameter $c$;

• $u$ has the dimension of 
\[
\frac{\{\text{mass}\}^{\frac{1}{2}}}{\{\text{space}\}}
\]
this fact can be deduced e.g. by the fact that, by Th. [2]
\[
\frac{1}{2} \int \left[ |\partial_t u|^2 + c^2 |\nabla u|^2 \right] dx
\]
has the dimension of energy;

• $\alpha$ has the dimension of a frequency; if we linearize eq. (56) with $\beta = 0$, we get KG
\[
\partial_t^2 u - c^2 \partial_x^2 u + \alpha^2 u = 0.
\]
which has the following dispersion relations:
\[
\omega_{KG} = \alpha \sqrt{1 + \frac{c^2}{\alpha^2} k_{KG}^2}
\]
where $\omega_{KG}$ and $k_{KG}$ are the frequency and the wave number of the small perturbations of the matter field. Since $\omega_0 < \alpha < \omega_{KG}$ the oscillations of the $q$-soliton, having frequency $\omega_0$, do not excite dispersive waves in the surrounding matter field. This fact partially explains the stability of the soliton;

• if we give to $N'$ the same dimension of $u$, $\ell$ has the dimension of a length and it is of the order of the radius of the soliton in the sense that
\[
u(x), \nabla v(x) \equiv 0 \text{ for every } x \geq r_0 := k\ell
\]
where $\equiv$ means that the quantity is exponentially small and $k$ is a dimensionless variable which depends on $N$.

• here $S$ is supposed to be dimensionless;

• $\eta\beta$ represents the strength of the interaction of the matter field with the electromagnetic field; by (17)
\[
\dim \beta = \left\{\text{electric charge}\right\} \cdot \left\{\text{time}\right\} \cdot \left\{\text{space}\right\} \frac{1}{\left\{\text{mass}\right\}^{\frac{1}{2}}}
\]
and if we give to $\varphi$ the dimension of an electric field i.e.
\[
\left\{\text{electric charge}\right\} / \left\{\text{space}\right\},
\]
then
\[
\dim \eta = \left\{\text{mass}\right\} \left\{\text{electric charge}\right\}^2 \cdot \left\{\text{space}\right\}^2 \cdot \left\{\text{time}\right\}^2;
\]
using these variables, then $a_\omega(u, u)$ defined by (47) becomes

$$a_\omega(u, u) = \int \left[ c^2 |\nabla u|^2 + (\alpha^2 - \omega^2) u^2 \right] dx + \omega^2 \eta^2 \beta^2 \int \frac{u(x)u(y) \, dx \, dy}{|x - y|^2}$$

hence, if $\eta \beta$ is too large with respect to the other constants, then, by (49), if $\omega_{\text{inf}} \geq \alpha$ and there are no solitons. Actually there is a competition between the gluing force which increases with $c\ell^{-1}N$ and the electric force which increases with $\eta \beta$. The gluing force tends to concentrate the matter field while the electric force tends to spread it.

By this discussion, it results that

$$W(u) = \frac{1}{2} \alpha^2 u + \frac{c^2}{\ell^2} N(u)$$

represents the potential of the "nuclear force" which is repelling when $u$ is small and attractive when the values of $u$ are in range where $N(s)$ is negative. $N(s)$ is responsible of the nonlinear behavior of the matter field and hence of the existence of $q$-solitons. By these considerations, Th. 5 and Remark 6, a $q$-soliton is a good model for physical particles if, in the dimensionless equation

$$\beta^2 \ll \max N(s) < 1$$

Also the condition

$$W(u) \geq 0$$

is suitable for a physical model.

We denote by

$$\{\sigma_0, \varphi_0\} = \left[ \begin{array}{c} (u_0(x), 0, \varphi_0, 0) \\ (0, -\omega_0, 0, 0) \end{array} \right]$$

the equilibrium configuration containing a $q$-soliton. The energy of this configuration is given by:

$$E[\{\sigma_0, \varphi_0\}] = E_m[\{\sigma_0, \varphi_0\}] + E_i[\{\sigma_0, \varphi_0\}] + E_v[\{\sigma_0, \varphi_0\}]$$

$$= E[\sigma_0] + E_i[\{\sigma_0, \varphi_0\}] + E_v[\varphi_0]$$

where

$$E[\sigma_0] = \int \left[ \frac{1}{2} |\nabla u_0|^2 - \frac{1}{2} \omega_0^2 u_0^2 \right] dx + \int W(u_0) dx;$$

$$E_i[\{\sigma_0, \varphi_0\}] = \omega_0 \beta \int \varphi_0 u_0 dx;$$

$$E_v[\varphi_0] = \frac{1}{2} \int |\nabla \varphi_0|^2 dx.$$
All these terms are positive; \( E[\sigma_0] \) is positive by (25) and (59); \( E_i[\{\sigma_0, \varphi_0\}] \) is positive since, by eq. (12), \( \omega_0 \) and \( \varphi_0(x) \) have same sign. Thus this term is positive also for anti-solitons. The energy \( E[\sigma_0] + E_i[\{\sigma_0, \varphi_0\}] \) is concentrated around 0 in a region of radius \( r_0 \). In fact, since \( u \) decays exponentially, from the physical point of view, it can be considered null for \( |x| \) larger that a suitable \( r_0 \).

Finally, notice that the em. field energy of a soliton does not diverge as the energy of a pointwise particle would do.

### 3.3 Travelling \( q \)-solitons

The action functional is invariant for the group of the Lorentz boosts:

\[
t' = \frac{t - vx_1}{\sqrt{1 - v^2}}; \quad x' = \left( \frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3 \right).
\]

Hence if \( u(t, x), S(t, x), \varphi(t, x), A(t, x) \) is a solution of (11),..,(14), also \( u(t', x'), S(t', x'), \varphi'(t', x'), A'(t, x) \) is a solution.

Since \((\varphi, A)\) is a 4-vector it transforms as follows:

\[
\varphi'(t, x) = \frac{\varphi(t', x') - vA_1(t', x')}{\sqrt{1 - v^2}}
\]

\[
A'(t, x) = \left( \frac{A_1(t', x') - v\varphi(t', x')}{\sqrt{1 - v^2}}, A_2(t', x'), A_3(t', x') \right)
\]

As usual we set

\[
\gamma = \frac{1}{\sqrt{1 - v^2}}
\]

If \( \sigma_0 \) denotes the stationary \( q \)-soliton defined by (55) and if \( \mathbf{v} = (v, 0, 0) \), we get the following family of solutions:

\[
u_\mathbf{v}(t, x) := u_0(\gamma(x_1 - vt), x_2, x_3) = u_0(x')
\]

\[
S_\mathbf{v}(t, x) := -\omega_0 \gamma (t - vx_1) = -\omega_0 t' = k \cdot x - \omega_0 t
\]

where

\[
k = (k, 0, 0) = (\gamma \omega_0 v, 0, 0);
\]

\[
\omega_\mathbf{v} = \gamma \omega_0;
\]

\[
\varphi_\mathbf{v}(t, x) = \gamma \varphi_0(x')
\]

\[
A_\mathbf{v}(t, x) = -\gamma (v\varphi_0(x'), 0, 0)
\]

If \( \mathbf{v} = (v, 0, 0), |v| < 1 \), we define a moving solitons as follows:

\[
\sigma_\mathbf{v}(x) = \left[\begin{array}{c}
(u_\mathbf{v}(0, x), S_\mathbf{v}(0, x), 0, 0) \\
(\partial_t u_\mathbf{v}(0, x), \partial_t S_\mathbf{v}(0, x), 0, 0)
\end{array}\right]_{t=0}.
\]
The configuration
\[
\sigma_v(x) + \begin{bmatrix}
(0,0,\gamma \varphi_0(x'), -\gamma (v \varphi_0(x')) \\
(0,0,\gamma \partial_t \varphi_0(x'), -\gamma v \partial_t \varphi_0(x'))
\end{bmatrix}
\]
is initial condition of the solution (63), (64), (67), (68) of eqs. (11),...,(14).

If \(v\) is any vector with \(|v| < 1\), \(R \in O(3)\) is a rotation such that \(Rv = (|v|, 0, 0)\), we set,
\[
\sigma_v(x) = \sigma_{R^{-1}v}(x)
\]

Definition 10. A moving \(q\)-soliton with velocity \(v \in \mathbb{R}^3\) in the point \(\bar{x} \in \mathbb{R}^3\) is a function of the form
\[
\sigma_v(x - \bar{x}).
\]

The evolution of a free moving soliton is given by
\[
\sigma_v(x - vt - \bar{x}) = \sigma_{R^{-1}(x - R(vt - \bar{x}))}.
\]

3.4 Mechanical properties of \(q\)-solitons

First, we will investigate the intrinsic quantities of a moving \(q\)-soliton. Since these properties are independent of \(R\) and \(\bar{x}\) we will just consider \(\sigma_v\) with \(v = (v, 0, 0)\) and \(\bar{x} = 0\).

The simplest quantity to describe of a \(q\)-soliton is the electric charge. It is defined by (30) and in this case is
\[
q[\sigma_0] = \omega_0 \beta \int u_0 dx
\]
It depends only on the soliton and not on the configuration of the surrounding field. Moreover it has the following property:

Proposition 11. The electric charge of a moving soliton is independent of the motion:
\[
q[\sigma] := q[\sigma_0]
\]

Proof: By (30) and (66), making a change of variable \(x_1 = 1/\gamma x' + vt\), we have that
\[
q[\sigma_v] = -\beta \int \partial_t S_v \, dx = \beta \omega \int u_v(0, x) \, dx
\]
\[
= \gamma \omega_0 \beta \int u_v(x') \, dx' = \gamma \omega_0 \beta \int u_v(x') \frac{1}{\gamma} \, dx'
\]
\[
= \omega_0 \beta \int u_0 \, dx.
\]

From now on \(q\) will denote the charge of a \(q\)-soliton. Next, let us consider the mass:
Definition 12. The mass of a moving $q$-soliton is defined by

$$m[\sigma_v] := \frac{P[\sigma_v]}{v}$$

where $P[\sigma_v]$ is the momentum of the matter field (see Prop. 3), namely:

$$P[\sigma_v] = P_m[(\sigma_v, \varphi_v)] = \int (\partial_t u_v \nabla u_v + \partial_i S \nabla S u_v^2) \, dx$$

Remark 13. Notice that this definition of mass is intrinsic to the equations (11),..., (14) and it is independent of any physical interpretation; it can be interpreted as a "physical" mass whenever $x$ and $t$ are interpreted as variables of the physical space-time.

Let us compute explicitly the momentum:

Theorem 14. The momentum of a $q$-soliton takes the following form:

$$P_m[\sigma_v] = \gamma v \left[ \frac{1}{3} \int |\nabla u_0|^2 \, dx + \omega_0^2 \int u_0^2 \, dx \right]$$

(71)

Proof: By Prop. 3

$$P[\sigma_v] = \int (\partial_t u_v \nabla u_v + \partial_i S \nabla S u_v^2) \, dx;$$

assuming $v = (1, 0, 0)$, by (63),..., (66)

$$P_1[\sigma_v] = \int (\partial_t u_v \partial_x u_v + \partial_i S \nabla S u_v^2) \, dx$$

$$= \int \partial_t u_0(x') \partial_x u_0(x') \, dx + k \omega_v \int u_0^2(x') \, dx$$

$$= \int [\partial_x u_0(x') \partial x'] \left[ \partial_x u_0(x') \partial x \right] \, dx + v \gamma^2 \omega_0^2 \int u_0^2(x') \, dx$$

$$= v \gamma^2 \int [\partial_x u_0(x')]^2 \, dx + v \gamma^2 \omega_0^2 \int u_0^2(x') \, dx$$

Making a change of variable $x_1 = 1/\gamma x_1 + vt$, we get

$$P_1[\sigma_v] = v \gamma^2 \int \left[ \partial_x u_0(x') \right]^2 \frac{1}{\gamma} \, dx' + v \gamma \omega_0^2 \int u_0^2(x') \frac{1}{\gamma} \, dx'$$

$$= v \gamma \left[ \int \left[ \partial_x u_0(x) \right]^2 \, dx + \omega_0^2 \int u_0^2(x) \, dx \right]$$

Since $u_0$ is radially symmetric,

$$\int \partial_x u_0^2 \, dx = \frac{1}{3} \int |\nabla u_0|^2 \, dx$$

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then
\[ P_1 [\sigma_v] = v \gamma \left[ \frac{1}{3} \int |\nabla u_0|^2 \, dx + \omega_0^2 \int u_0^2(x) \, dx \right] \]

It is immediate to see that \( P_2 [\sigma_v] = P_3 [\sigma_v] = 0 \) and hence we get the conclusion.

So we have obtained the following result:

**Corollary 15.** The mass of a \( q \)-soliton takes the following value:

\[ m [\sigma_v] = \gamma m [\sigma_0] = \gamma \left[ \frac{1}{3} \int |\nabla u_0|^2 \, dx + \omega_0^2 \int u_0^2 \, dx \right] \tag{72} \]

From now on \( m \) will denote the rest mass of a \( q \)-soliton.

We define the energy of a moving soliton as follows:

\[ E [\sigma_v] = E_m [\{\sigma_v, \varphi_v\}] \]

The next proposition describes how the energy transforms in a moving soliton:

**Theorem 16.** The energy of a \( q \)-soliton is given by

\[ E [\sigma_v] := \gamma m + \frac{1}{\gamma} \left( \frac{5}{3} \omega_0^2 \int \varphi_0 u_0 \, dx \right) = \gamma m - \frac{1}{\gamma} \left( \frac{5}{3} \int \varphi_0 \rho_0 \, dx \right) \]

where \( \rho_0(x) = \omega_0^2 \beta u_0(x) \) (see (37)).

**Remark 17.** In a theory with \( \beta = 0 \), the energy of a soliton coincides with its mass \( \gamma m \) and hence it transforms as the time-component of a time-like vector. If \( \beta \neq 0 \) part of the energy transforms differently. This fact is not so surprising since the energy of a \( q \)-soliton includes the energy of the selfinteraction of the soliton with the e.m. field generated by itself. The energy-momentum of the e.m. field does not transform as the energy of a space-like vector since it is a light-like vector. Hence there is a term which is small of the order \( \beta \) which transforms differently. Since this terms is related to the interaction of the matter field with the e.m. field it might be related to a sort of classical counterpart of the fine-structure constant; however this point needs a further investigation.

In order to prove Th. 16 we need the following lemma which is a variant of the Pohozaev-Derrik theorem (11), (13):

**Lemma 18.** If \( u \) is any solution of eqs. (41), (42), then

\[ \int W(u) \, dx = \frac{1}{2} \omega^2 \int u^2 \, dx - \frac{1}{6} \int |\nabla u|^2 \, dx - \frac{5}{3} \omega^2 \beta^2 \int (G * u) \, u \, dx \]

**Proof:** Let

\[ J[u] = \int \left[ \frac{1}{2} |\nabla u|^2 + W(u) - \frac{1}{2} \omega^2 u^2 \right] \, dx + \omega^2 \beta^2 \int (G * u) \, u \, dx \tag{73} \]
be the functional $J$ defined by (51). Then, if $u$ is a solution of eqs. (41), (42), we have that $dJ[u] = 0$. Now, let us consider the "curve" $\lambda \mapsto u_\lambda$ in $V = H^1 + D^{1,2}$ defined by

$$u_\lambda = u \left( \frac{x}{\lambda} \right)$$

Then,

$$\left( \frac{d}{d\lambda} J[u_\lambda] \right)_{\lambda = 1} = 0.$$

Making the change of variable $x \mapsto x\lambda^{-1}$, we get that

$$J[u_\lambda] = \frac{\lambda}{2} \int |\nabla u|^2 \, dx + \lambda^3 \int W(u) \, dx - \frac{1}{2} \lambda^3 \int \omega^2 u^2 \, dx + \frac{1}{2} \omega^2 (G \ast u) \, u \, dx$$

Then,

$$0 = \frac{d}{d\lambda} J[u_\lambda]_{\lambda = 1}$$

$$= \left[ \frac{1}{2} \int |\nabla u|^2 \, dx + 3\lambda^3 \int W(u) \, dx - \frac{3}{2} \lambda^2 \int \omega^2 u^2 \, dx + \frac{1}{2} \omega^2 (G \ast u) \, u \, dx \right]_{\lambda = 1}$$

$$= \frac{1}{2} \int |\nabla u|^2 \, dx + 3 \int W(u) \, dx - \frac{3}{2} \int \omega^2 u^2 \, dx + \frac{1}{2} \int \omega^2 (G \ast u) \, u \, dx$$

Hence

$$\int W(u) \, dx = \frac{1}{2} \int \omega^2 u^2 \, dx - \frac{1}{6} \int |\nabla u|^2 \, dx - \frac{5}{3} \omega^2 \beta^2 \int (G \ast u) \, u \, dx.$$

□

**Corollary 19.** Given a stationary $q$-soliton $\sigma_0$, we have that

$$E[\sigma_0] = \frac{1}{3} \int |\nabla \sigma_0|^2 \, dx + \omega_0^2 \int u_0^2 \, dx - \frac{5}{3} \beta^2 \int (G \ast u_0) \, u_0 \, dx$$

**Proof:** Replacing $W$ in (61), and using Lemma 18, we get:

$$E[\sigma_0] = \frac{1}{2} \int \left[ |\nabla \sigma_0|^2 + \omega_0^2 u_0^2 \right] \, dx + \frac{1}{2} \int W(u_0) \, dx$$

$$= \frac{1}{2} \int \left[ |\nabla \sigma_0|^2 + \omega_0^2 u_0^2 \right] \, dx + \frac{1}{2} \int \omega^2 u^2 \, dx$$

$$- \frac{1}{6} \int |\nabla u|^2 \, dx - \frac{5}{3} \omega_0^2 \beta^2 \int (G \ast u_\lambda) \, u_\lambda \, dx$$

$$= \left( \frac{1}{2} - \frac{1}{6} \right) \int |\nabla \sigma_0|^2 \, dx + \left( \frac{1}{2} + \frac{1}{2} \right) \omega_0^2 \int u_0^2 \, dx - \frac{5}{3} \beta^2 \int (G \ast u_0) \, u_0 \, dx$$

□
Proof of Th.16: By Prop. 2 and (63),...,(68) we have that

$$E[\sigma_v] = \frac{1}{2} \int |\partial_t u_0(x')|^2 dx + \frac{1}{2} \int |\nabla u_0(x')|^2 dx$$

$$+ \frac{1}{2} (k^2 + \omega_v^2) \int u_0(x')^2 dx + \int W(u_0(x')) dx$$

making the change of the integration variable $x_1 = 1/\gamma x'_1 + vt$, we get

$$E[\sigma_v] = \frac{1}{2 \gamma} \int |\partial_t u_0(x')|^2 dx' + \frac{1}{2 \gamma} \int |\nabla u_0(x')|^2 dx'$$

$$+ \frac{1}{2 \gamma} (k^2 + \omega_v^2) \int u_0(x')^2 dx' + \frac{1}{\gamma} \int W(u_0(x')) dx'$$

Let us compute each piece individually:

$$A = \frac{1}{2 \gamma} \int |\partial_t u_0(x')|^2 dx' = \frac{1}{2} \int |\partial_{x'_1} u_0(x') \partial_{x_1}|^2 dx'$$

$$= \frac{1}{2} \int |\partial_{x'_1} u_0(x') \gamma^2 v^2|^2 dx' = \frac{v^2}{2} \int |\partial_{x'_1} u_0(x')|^2 dx'$$

$$= \frac{v^2}{2} \int |\partial_{x_1} u_0(x)|^2 dx$$

Since $v$ is radially symmetric,

$$\int |\partial_{x_1} u_0(x)|^2 dx = \frac{1}{3} \int |\nabla u_0|^2 dx$$

(74)
Then,

$$A = \frac{v^2}{6} \int |\nabla x u_0|^2 dx$$

Let us compute the second piece using (74) again:

$$B = \frac{1}{2 \gamma} \int |\nabla u_0(x')|^2 dx'$$

$$= \frac{1}{2 \gamma} \int \left[ |\partial_{x'_1} u_0(x') \partial_{x'_1} x'_1|^2 \right] dx' + \frac{1}{2 \gamma} \int \left[ |\partial_{x'_2} u_0(x')|^2 + |\partial_{x'_3} u_0(x')|^2 \right] dx'$$

$$= \frac{\gamma}{6} \int |\partial_{x_1} u_0(x)|^2 dx + \frac{1}{2 \gamma} \int \left[ |\partial_{x_2} u_0(x)|^2 + |\partial_{x_3} u_0(x)|^2 \right] dx'$$

$$= \frac{\gamma}{6} \int |\nabla u_0|^2 dx + \frac{1}{3 \gamma} \int |\nabla u_0|^2 dx = \left( \frac{\gamma}{6} + \frac{1}{3 \gamma} \right) \int |\nabla u_0|^2 dx$$

In order to compute the third piece, we need (65) and (66):

$$C = \frac{1}{2 \gamma} (k^2 + \omega_v^2) \int u_0(x)^2 dx = \frac{1}{2 \gamma} \left[ (\gamma \omega_0 v)^2 + (\gamma \omega_0)^2 \right] \int u_0(x)^2 dx$$

$$= \frac{1}{2 \omega_0^2 \gamma} (v^2 + 1) \int u_0(x)^2 dx$$
The computation of the fourth piece uses Lemma 18:

\[
\frac{1}{\gamma} \int W(u_0(x')) dx' = \frac{1}{\gamma} \int W(u_0(x)) dx
\]

\[
= \frac{1}{6\gamma} \int |\nabla u_0|^2 dx + \frac{1}{2\gamma} \omega_0^2 \int u_0^2 dx - \frac{5}{3} \omega_0^2 \beta^2 \int (G * u_0) u_0 dx
\]

\[
= E + F + G
\]

Then,

\[
E[\sigma_v] = A + B + C + E + F = (A + B + E) + (C + F) + G
\]

We have that

\[
\gamma^2 v^2 + \gamma^2 + 1 = \frac{v^2 + 1}{1 - v^2} + 1 = \frac{2}{1 - v^2} = 2\gamma^2
\]

then,

\[
A + B + E = \left( \frac{\gamma^2 v^2 + \gamma^2 + 1}{6} - \frac{1}{6\gamma} \right) \int |\nabla u_0|^2 dx
\]

\[
= \frac{1}{6\gamma} (\gamma^2 v^2 + \gamma^2 + 1) \int |\nabla u_0|^2 dx
\]

\[
= \frac{\gamma}{3} \int |\nabla u_0|^2 dx
\]

and

\[
C + F = \left[ \frac{1}{2} \omega_0^2 \gamma (v^2 + 1) + \frac{1}{2\gamma} \omega_0^2 \right] \int u_0(x)^2 dx
\]

\[
= \frac{\omega_0^2}{2} \left[ \gamma (v^2 + 1) + 1 \right] \int u_0(x)^2 dx
\]

\[
= \frac{\omega_0^2}{2} \gamma^2 \int u_0(x)^2 dx = \gamma \omega_0^2 \int u_0(x)^2 dx
\]

Concluding, using Cor. 19, we have that

\[
E[\sigma_v] = \frac{\gamma}{3} \int |\nabla u_0|^2 dx + \gamma \omega_0^2 \int u_0(x)^2 dx - \frac{5}{3} \omega_0^2 \beta^2 \int (G * u_0) u_0 dx
\]

\[
\square
\]

Placing a stationary \( q \)-soliton in an generic electromagnetic field with gauge potential \((\varphi, A)\) and using the notation \((55)\), we get the following configuration,

\[
\sigma_0 + \begin{bmatrix}
(0, 0, \varphi, A) \\
(0, 0, \partial_t \varphi, \partial_t A)
\end{bmatrix} = \begin{bmatrix}
(u_0, 0, \varphi, A) \\
(0, -\omega_0, \partial_t \varphi, \partial_t A)
\end{bmatrix}
\]

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By Prop. 2, the energy of this configuration is

\[ E[{\sigma_0, \varphi, A, \partial_t A}] = E_m[{\sigma_0, \varphi}] + E_0[{\sigma_0, \varphi, A, \partial_t A}] + E_r[{\varphi, A, \partial_t A}] \]

\[ = E[\sigma_0] + \omega_0 \beta \int \varphi u_0 dx + \frac{1}{2} \int (E^2 + H^2) dx \]

If the soliton is small with respect to \( \nabla \varphi \), (namely if \( r_0 = k\ell \) is small), then, by (70),

\[ \omega_0 \int \varphi(x) u_0(x) \, dx \approx \omega_0 \varphi(0) \beta \int u_0(x) \, dx = q \varphi(0) \]

where "\( \approx \)" means that the accuracy of this approximation is good if the quantities involved are large with respect to \( \beta \) (and to the radius of the soliton). In fact, the field \( \varphi_0(x) \) produced by the q-soliton, is of the order of \( \beta \ll 1 \), and hence, if \( \varphi \approx 1 \), we have that \( \varphi = \varphi_0 \approx \varphi \) and \( \nabla (\varphi - \varphi_0) \approx \nabla \varphi \). Then

\[ E[{\sigma_0, \varphi, A}] \approx E[\sigma_0] + q \varphi(0) + \frac{1}{2} \int (E^2 + H^2) dx \]

Therefore, thanks to Prop. 2 and our analysis if a soliton is placed in a e.m. field we can distinguish the **soliton energy** \( E[\sigma_0] \), the **potential energy** \( q \varphi(0) \) and the **e.m. field energy** \( \frac{1}{2} \int (E^2 + H^2) dx \). This distinction is crucial for the study of the dynamics of the soliton (see section 3.5). Finally, we remark that, the potential energy \( q \varphi(0) \) is localized within the radius of the soliton. This fact eliminates one of the difficulties posed by the dualism particle-field where the localization of the potential energy of a particle is a meaningless problem.

If the q-soliton is moving, extending the above arguments, we have the following result:

**Proposition 20.** If the q-soliton is small with respect to \( \nabla \varphi \) and \( \nabla A \) and \( \beta \ll 1 \), then

\[ E[{\sigma_v, \varphi, A}] = E_m[{\sigma_v, \varphi_v}] + E_r[{\sigma_v, \varphi_v, A}] \]

\[ \approx \gamma m + q[\varphi(0) + \mathbf{v} \cdot \mathbf{A}(0)] \]  \hspace{1cm} (75)

**Proof:** By Th. 16 \( E_m[{\sigma_v, \varphi_v}] = \gamma E[\sigma_0] \approx \gamma m \). Then by Prop. 2 and [65], we get

\[ E[{\sigma_v, \varphi, A}] = \gamma m - \int (\varphi \partial_t S_v + \mathbf{A} \cdot \nabla S_v) u_\varphi \, dx \]

\[ = \gamma m + \omega_0 \int \varphi(x) u_\varphi(x) \, \gamma dx + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(x) u_\varphi(x) \gamma \, dx \]

and, using [62], [63], [67], [68] and making a change of variables, we have that

\[ E[{\sigma_v, \varphi, A}] = \gamma m + \omega_0 \int \varphi(x) u_\varphi(x') \, dx' + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(x) u_\varphi(x') \, dx' \]

\[ = \gamma m + \omega_0 \int \varphi(L^{-1} x) u_\varphi(x) \, dx + \omega_0 \mathbf{v} \cdot \int \mathbf{A}(L^{-1} x) u_\varphi(x) \, dx \]
where \( L \) denotes the Lorentz boost defined by (62), namely \( L x = x' \). If the soliton is small with respect to \( \nabla \varphi \) and \( \nabla A \), then, using the definition (70) of \( q \),

\[
\omega_0 \int \varphi(L^{-1}x) u_0(x) \, dx \cong \varphi(L^{-1}0) \omega_0 \int u_0(x) \, dx = q \varphi(0)
\]

and similarly

\[
\omega_0 v \cdot \int A(L^{-1}x) u_0(x) \, dx \cong v \cdot A(L^{-1}0) \omega_0 \int u_0(x) \, dx = q \varphi v \cdot A(0)
\]

□

Notice that (75) is the energy of the soliton, namely the matter field energy plus the interaction energy contained in the radius of the soliton; the total energy of a configuration which contains a soliton, depends also on \( \partial_t A \) and, by Prop. 2 and Prop 20, it takes the following form:

\[
E_{\text{tot}} \left[ \{ \sigma, \varphi, A, \partial_t A \} \right] \cong \gamma m + q \left[ \varphi(0) + v \cdot A(0) \right] + \frac{1}{2} \int (E^2 + H^2) \, dx.
\]

Now let us examine a configuration containing several solitons

\[
\sigma_{\nu_k, \bar{x}_k} := \sigma_{\nu}(-\bar{x}_k), \quad k = 1, \ldots, N
\]

where \( \sigma_{\nu_k}(-\bar{x}_k) \) has been defined by Def. 10. We assume that

\[
|\bar{x}_k - \bar{x}_h| \geq 2r_0, \quad k \neq h
\]

(76)

where \( r_0 \) denote the radius of the solitons. We remember that \( u \) decays exponentially, so the matter field is essentially null out of a neighborhood of each soliton and hence

\[
E \left[ \sum_{k=1}^{N} \sigma_{\nu_k, \bar{x}_k} \right] \cong \sum_{k=1}^{N} E [\sigma_{\nu_k, \bar{x}_k}] \cong m \sum_{k=1}^{N} \gamma_k
\]

(77)

where

\[
\gamma_k = 1 \sqrt{1 - |v_k|^2}
\]

Notice that, in the configuration (77), also the \( q \)-antisolitons can be included. They have the same mass of solitons, but opposite electric charge.

If we embed this configuration in an external e.m. field, the total energy takes the following form:

\[
E_{\text{tot}} \left[ \left\{ \sum_{k=1}^{N} \sigma_{\nu_k, \bar{x}_k}, \varphi, A, \partial_t A \right\} \right] \cong m \sum_{k=1}^{N} \gamma_k + q \sum_{k=1}^{N} \left[ \varphi(\bar{x}_k) + v_k \cdot A(\bar{x}_k) \right] + \frac{1}{2} \int (E^2 + H^2) \, dx.
\]

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3.5 Dynamics of $q$-solitons

Now let examine the dynamics of a solitons in the presence of an "external" electromagnetic field. More exactly, we want to examine the behavior of the solution of the Cauchy problem with the following initial conditions:

$$U_0 = \sum_{k=1}^{N} \sigma v_k, \bar{x}_k + \begin{bmatrix} (0, 0, \varphi_0, A_0) \\ (0, 0, \varphi_1, A_1) \end{bmatrix}$$

(78)

where $v_k \in \mathbb{R}^3$ is such that $|v_k| < 1$.

It is well known that, thanks to the invariance of the hylenic ratio, the soliton is orbitally stable (see e.g. [7]). This means that if the perturbation field generated by $(\varphi_0(x), A_0(x)), (\varphi_1(x), A_1(x))$ is small (with respect to $\beta^{-1}$) around the soliton, then the solution of the Cauchy problem has the following form:

$$U_0(t, x) = \sum_{k=1}^{N} \sigma v_k(t), \bar{x}_k(t) + \begin{bmatrix} (u_p(t, x), S_p(t, x), \varphi(t, x), A(t, x)) \\ (\partial_t u_p(t, x), \partial_t S_p(t, x), \partial_t \varphi(t, x), \partial_t A(t, x)) \end{bmatrix}$$

(79)

where

- $u_p(t, x), S_p(t, x)$ are essentially null thanks to the orbital stability of the soliton and they will be neglected;
- $\sum_{k=1}^{N} \sigma v_k(t), \bar{x}_k(t)$ is the configuration of the $q$-solitons and its structure is determined by a $N$ function $\xi_k : \mathbb{R} \rightarrow \mathbb{R}^3$ such that $\xi_k(t) = \bar{x}_k(t); \dot{\xi}_k(t) = v_k(t)$;

Our aim to investigate the dynamics of the $q$-solitons under the following assumptions:

- **(A-1)** $\beta \ll 1$; as we have seen this condition implies that the Cauchy problem is well posed and that the energy of a $q$-solitons equals its mass (Th. 16);
- **(A-2)** the solitons are far from each other (i.e. (76) holds) during the time interval considered; this happens if
  - (i) this assumption is satisfied by the initial condition (78);
  - (ii) all the $q$-solitons have the same charge (namely there are not $q$-antisolitons), so that, during the evolution, the $q$-solitons repel each other;
  - (iii) the e.m. field is not locally too strong, so that the $q$-solitons cannot collide;
- **(A-3)** $|\dot{\xi}_k(t)| \ll 1$; this fact avoids the $q$-soliton to produce a strong radiation and, from the technical point of view, it simplify the computations. Clearly this happens if the e.m. field is not too strong.
We will show, that under these assumptions the $q$-solitons behave as classical particles. To this aim, we analyze the action functional relative to the configuration (79):

\[ A = \int \int (L_m + L_i + L_F) \, dx \, dt \]

(80)

Since we have assumed (A-2), then

\[ \int L_m dx \cong \sum_{k=1}^{N} \int L_i \left[ \sigma_{\xi_k(t),\xi_k(t)}; \varphi, A \right] dx \]

and

\[ \int L_i dx \cong \sum_{k=1}^{N} \int L_i \left[ \left\{ \sigma_{\xi_k(t),\xi_k(t)}; \varphi, A \right\} \right] dx \]

Let us compute each piece of the action separately:

**Lemma 21.** Under the assumptions (A-1), (A-2), (A-3), we have that

\[ \int L_m \left[ \sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx \cong -m \sqrt{1 - \left| \dot{\xi}_k(t) \right|^2}. \]

**Proof:** Since the Lagrangian $L_m$ does not depend explicitly on $t$ and $x$, we can choose a reference frame where, for a fixed $t$,

\[ \xi_k(t) = 0 \quad \text{and} \quad \dot{\xi}_k(t) = (v_k, 0, 0) \]

so that

\[ \int L_m \left[ \sigma_{\dot{\xi}_k(t),\xi_k(t)} \right] dx = \int L_m \left[ \sigma_{v_k} \right] dx \]

We recall that by (63) and (64),

\[ \sigma_{v_k}(x) = \left[ \begin{array}{c} u_0(x'), -\gamma_k \omega_0, 0, 0 \\ 0, \gamma_k \omega_0 v_k, 0, 0 \end{array} \right] \]

where we have set

\[ \gamma_k = \frac{1}{\sqrt{1 - \left| \dot{\xi}_k(t) \right|^2}}. \]
Then, by (7), and (63)....(66)

\[ \int \mathcal{L}_m \left[ \sigma_{\xi_k(t), \xi_k(t)} \right] dx = \int \mathcal{L}_m [\sigma_{v_k}] dx \]

\[ = \frac{1}{2} \int |\partial_t u_0(x')|^2 dx - \int |\nabla u_0(x')|^2 dx \]

\[ + \frac{1}{2} (k^2_k - \omega_{v_k}^2) \int u_0(x')^2 dx - \int W(u_0(x')) dx dt \]

If we assume that \( \ddot{\xi}_k \) is not too large (i.e. (A-2)),

\[ \partial_t x'_1 = \partial_t \frac{x' - \dot{\xi}_k(t) t}{\sqrt{1 - |\dot{\xi}_k(t)|^2}} \approx v_k \gamma_k; \]

then, arguing as in the proof of Th. 16 and using similar notations for each \( k \),

\[ \int \mathcal{L}_m \left[ \sigma_{\xi_k(t), \xi_k(t)} \right] dx = A_k - B_k + C_k^a - C_k^b - E_k - F_k - G_k \]

where

\[ C_k^a = \frac{1}{2} \omega_0^2 \gamma_k v_k^2 \int u_0(x)^2 dx; \quad C_k^b = \frac{1}{2} \omega_0^2 \gamma_k \int u_0(x)^2 dx. \]

Going on with our computation,

\[ A_k - B_k - E_k = \left[ \frac{\gamma_k v_k^2}{6} - \frac{\gamma_k}{6} + \frac{1}{6} \frac{1}{\omega_0^2 \gamma_k} \right] \int |\nabla u_0|^2 dx \]

\[ = \frac{1}{6 \gamma_k} \left[ v_k^2 \gamma_k - \gamma_k^2 - 1 \right] \int |\nabla u_0|^2 dx \]

\[ = \frac{1}{6 \gamma_k} \left[ \frac{v_k^2 - 1}{1 - v_k^2} - 1 \right] \int |\nabla u_0|^2 dx = \frac{1}{3 \gamma_k} \int |\nabla u_0|^2 dx \]

\[ C_k^a - C_k^b - F_k = \left[ \frac{1}{2} \omega_0^2 \gamma_k (v_k^2 - 1) - \frac{1}{2} \omega_0^2 \gamma_k \right] \int u_0^2 dx \]

\[ = \left[ - \frac{1}{2} \omega_0^2 \gamma_k - \frac{1}{2 \gamma_k} \right] \int u_0(x)^2 dx \]

\[ = - \frac{\omega_0^2}{\gamma_k} \int u_0(x)^2 dx \]

The term \( G_k \) will be ignored since we have assumed \( \beta \ll 1 \) (i.e. (A-2)). Then, by (72),

\[ \int \mathcal{L}_m \left[ \sigma_{\xi_k(t), \xi_k(t)} \right] dx = - \frac{1}{3 \gamma_k} \int |\nabla u_0|^2 dx - \frac{\omega_0^2}{\gamma_k} \int u_0(x)^2 dx \]

\[ = - \frac{1}{\gamma_k} m = - m \sqrt{1 - |\dot{\xi}_k(t)|^2} \]

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Now let us compute $\int \mathcal{L}_i dx$.

**Lemma 22.** If (A-1) and (A-2) hold, then

$$\int \mathcal{L}_1 \left[ \sigma_{\xi_k(t),\xi_{\epsilon_k}(t)} ; \varphi, A \right] dx \cong q \left[ \varphi(t, \xi_k(t)) - A(t, \xi_k(t)) \cdot \dot{\xi}_k(t) \right] dt.$$

**Proof:** As in the previous lemma, we can choose a reference frame where, for a fixed $t$, $\dot{\xi}_k(t) = \mathbf{v}$ and $\xi_k(t) = 0$. Then following the same arguments used in the proof of Prop. 20 we have that

$$\int \mathcal{L}_1 \left[ \sigma_{\mathbf{v}, \mathbf{x}} ; \varphi, A \right] dx \cong q (\mathbf{v} \cdot A(\mathbf{x}) - \varphi(\mathbf{x})).$$

□

The above lemmas give the following result:

**Theorem 23.** Let $U_0(t, x)$ be the solution of the Cauchy problem relative to equation (11)...(14) with the initial condition (78). Then If (A-1), (A-2) and (A-3) hold, we have that

$$\frac{d}{dt} \left( \frac{m \dot{\xi}_k}{\sqrt{1 - \left| \dot{\xi}_k \right|^2}} \right) \cong q \left( \mathbf{E} + \dot{\xi}_k \times \mathbf{H} \right)$$

(81)

$$\nabla \cdot \mathbf{E} = \sum_{k=1}^N \rho_0(x - \xi_k)$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{E} = \sum_{k=1}^N j_0(x - \xi_k)$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

**Proof:** The action (80) becomes

$$\mathcal{A} = \mathcal{A}_M + \mathcal{A}_A + \mathcal{A}_V$$

$$= P \sum_{k=1}^N \int \left[ \sqrt{1 - \left| \dot{\xi}_k(t) \right|^2} + \frac{q}{\epsilon} \left[ \mathbf{A}(t, \xi_k(t)) \cdot \dot{\xi}_k(t) - \varphi(t, \xi_k(t)) \right] \right] dt$$

$$+ \int \int \mathcal{L}_V [\varphi, A] dx dt$$

Making the variation of $\mathcal{A}_M + \mathcal{A}_A$ with respect to $\xi_k$, we get the Lorentz equation (81); making the variation of $\mathcal{A}$ given by (21) we get the Maxwell equations.

□

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Remark 24. Theorem 23 states that equations (11),..., (14) provide a model for material particles which, at low energies, agree with the well known physics. It is interesting to investigate the predictions of this model when the assumptions (A-2), (A-3) are violated. If (A-2)-(ii) is violated there are antisolitons which attract solitons since they have opposite charges; then also (A-2)-(i) will be eventually violated and the two particles will annihilate. Since our equation is invariant for time-reversal, also the creation of a couple particle-antiparticle might occur; of course this can happen only if there is sufficient energy, namely if (A-2),(iii) does not hold. If (A-3) does not hold, a numerical computation of the radiation when $\xi$ is large gives a spectrum which probably can be compared with the experimental data.

4 Conclusive remarks

More than 50 years ago, De Broglie wrote:

Des considérations sur lesquelles je reviendrai me conduisent aujourd’hui a penser que le corpuscule doit etre assimilé non pas à un véritable point singulier de $u$, mais à un très petite région singulière de l’espace où $u$ prendrait une très grande valeur et obéirait à une équation non linéaire dont l’équation linéaire de la Mécanique onduleaire ne serait qu’une forme approximative valable en dehors de la région singulière. L’idée que l’équation de propagation de $u$, contrairement à l’équation classique du $\Psi$, est en principe non linéaire m’apparait même maintenant comme tout à fait essentielle. ([10], Chap. IX, 1, p. 95.)

The development of the nonlinear analysis of the last half century allows to construct models of particles in line with the ideas of De Broglie. The model presented here is strongly based on Classical Mechanics and "a priori" has nothing to do with Quantum Mechanics (QM), in contrast with the ideas of De Broglie. Nevertheless, it is interesting to notice that it presents some feature which are considered peculiar of QM.

The first thing to remark is the fact that particles-like solutions of nonlinear equations with positive energy, in dimension 3, seems possible only if they have at least one internal degree of freedom, namely $\psi$ takes values in $\mathbb{C}$ and not in $\mathbb{R}$ (see e.g Derrik theorem [11]). This fact implies that

$$\psi(t,x) = u(t,x)e^{i(k \cdot x - \omega t)}$$

presents an undulatory aspect as desired by De Broglie. Furthermore, since the energy/momentum ($E, p$) of the particle and the wave number ($\omega, k$) are 4-vectors they must be proportional and hence

$$E = \hbar \omega \quad \text{and} \quad p = \hbar k$$
where $\hbar$ is a constant depending on the parameters of the problem. So we can say that eqs (11),...,(14) present one kind of intrinsic Plank constant. However, this similarity does not imply the De Broglie pilot wave theory or the Bohmian mechanics since the "interference" or the "entanglement" phenomena cannot be reproduced by this model.

The second remarkable fact is the existence of anti-particles, and the fact that an antiparticle is produced by time-reversion (or by charge inversion).

Another peculiarity is that the $q$-solitons are equal to each other and two of them cannot be in the same position. This fact implies that they are forced to follow the Bose-Einstein statistics which also is considered a quantum phenomenon.

However, we do not think that the $q$-solitons could be considered as a model for elementary particles. They are just an example (and probably the simplest one) that shows the possibility of a classical theory of electrodynamics and the fact that some quantum phenomena are consequences of a consistent field theory independently of the quantization.

Nevertheless it is possible to implement the ideas presented here to build a "classical" model of elementary particles. It is necessary to take $\psi$-functions with spinor values and a Lagrangian with a suitable symmetry. For example in [4] a $U(1) \times SU(2)$ symmetry is considered.

The final conclusion is the following: if the Maxwell equations are weakly coupled in the simplest way with a linear equation, invariant for the Poincaré group, then a small nonlinear perturbation (see Th. 3) is sufficient to produce not only a consistent electrodynamics theory, but also solitons which share some characteristic with quantum particles.

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