Global Classical Solutions to the Compressible Navier-Stokes Equations with Slip Boundary Conditions in 3D Exterior Domains

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Abstract

We are concerned with the global existence of classical solutions to the barotropic compressible Navier-Stokes equations with slip boundary condition in a three-dimensional (3D) exterior domain. We demonstrate that the classical solutions exist globally in time provided that the initial total energy is suitably small. It is worth noting that the initial density is allowed to have large oscillations and contain vacuum states. For our purpose, some new techniques and methods are adopted to obtain necessary a priori estimates, especially the estimates on the boundary. Moreover, we also give the large-time behavior of the classical solutions what we have gotten.

Keywords: compressible Navier-Stokes equations; global existence; slip boundary condition; exterior domain; vacuum; large-time behavior.

1 Introduction

The viscous barotropic compressible Navier-Stokes for isentropic flows reveal the principles of conservation of mass and momentum in the absence of exterior forces:

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + \nabla P(\rho) = 0,
\end{cases}$$

where \((x, t) \in \Omega \times (0, T]\), \(\Omega\) is a domain in \(\mathbb{R}^3\), \(t \geq 0\) is time, \(x\) is the spatial coordinate. \(\rho \geq 0, u = (u^1, u^2, u^3)\) and \(P(\rho) = a\rho^\gamma (a > 0, \gamma > 1)\) are the unknown fluid density,
velocity and pressure, respectively. The constants $\mu$ and $\lambda$ are the shear viscosity and bulk coefficients respectively satisfying the following physical restrictions:

$$\mu > 0, \quad 2\mu + N\lambda \geq 0. \quad (1.2)$$

This paper can be regarded as a continuation of our previous work [6]. Unlike the previous article, we study the global existence of classical solutions of (1.1) in the exterior of a simply connected bounded domain in $\mathbb{R}^3$ rather than in the inner bounded domain. More precisely, three-dimensional problem of the system (1.1) will be investigated under the assumptions: The domain $\Omega$ is the exterior of a simply connected bounded domain $D$ in $\mathbb{R}^3$, i.e. $\Omega = \mathbb{R}^3 - \bar{D}$ and its boundary $\partial \Omega$ is smooth. In addition, the system (1.1) is discussed subject to the given initial data

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = \rho_0 u_0(x), \quad x \in \Omega, \quad (1.3)$$

and slip boundary condition

$$u \cdot n = 0, \quad \text{curl} u \times n = 0 \text{ on } \partial \Omega, \quad (1.4)$$

with the far field behavior

$$u(x, t) \to 0, \quad \rho(x, t) \to \rho_\infty \geq 0, \text{ as } |x| \to \infty, \quad (1.5)$$

where $n = (n^1, n^2, n^3)$ is the unit normal vector to the boundary $\partial \Omega$ pointing outside $\Omega$, $\rho_\infty$ is a given constant. Our purpose is to show that such global classical solutions indeed exists provided that the initial energy is sufficiently small. It is worth noting that in this paper the initial density is allowed to have possibly large oscillations with constant state as far field which could be either vacuum or non-vacuum. Since $\Omega$ is no longer bounded, there are some distinguish differences from the previous work [6].

There are a considerable number of literatures on the large time existence and behavior of solutions to (1.1). The one-dimensional problem has been studied extensively, see [18, 27, 39, 40] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions are known in [35, 41] in the absence of vacuum and recently, for strong solutions also, in [7, 9, 28, 37] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura-Nishida [33] for initial data close to a non-vacuum equilibrium in $H^3$. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from vacuum. Later, Hoff [19, 21] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [31] (see also Feireisl [14, 16]), where the global existence of weak solutions when the exponent $\gamma$ is suitably large are achieved. The main restriction on initial data is that the initial total energy is finite, so that the density vanishes at far fields, or even has compact support. However, little is known on the structure of such weak solutions, particularly, the regularity and the uniqueness of such weak solutions remain open. Here we refer the reader to the books by Lions [31], Feireisl [14], Novotný & Straškraba [36] and Giga & Novotný [47] for further details.

The choice of the boundary conditions is more complex when one discusses the system (1.1) in a region with boundaries. One of the frequent choices is the homogeneous Dirichlet boundary condition $u = 0$ on $\partial \Omega$ in the case when $\partial \Omega$ represents a fixed
wall. This condition was formulated by G. Stokes in 1845, which is also called no-slip boundary condition expressing the physical fact that a real fluid adheres to \( \partial \Omega \). However, in the case where the obstacles have a rough boundary, the no-slip boundary condition is no longer valid (see for instance [42]). An alternative was suggested by H. Navier even before (in 1824), who proposed the conditions as follows:

\[
    u \cdot n = 0, \quad (2D(u)n + \partial u)_{tan} = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( D(u) = (\nabla u + (\nabla u)^T)/2 \) is the shear stress, \( \vartheta \) is a scalar friction function, and the symbol \( v_{tan} \) represents the projection of tangent plane of the vector \( v \) on \( \partial \Omega \). We also call it Navier-type slip condition in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip and the slip velocity is proportional to the shear stress. Such boundary conditions can be induced by effects of free capillary boundaries (see e.g., [3]), or a rough boundary as in [12], or a perforated boundary, which is then called Beavers-Joseph’s law, see [5,38], or an exterior electric field as in [10]. The Navier-type slip condition has been employed in a wide range of problem, including free surface problems (see e.g., [13]), turbulence modeling (see e.g., [17]) and inviscid limits (see e.g., [13,46]). It is a remarkable fact that when \( \partial \Omega \) is of constant curvature, (1.4) is equivalent to (1.6) with \( \vartheta = \kappa \) (see [6]), where \( \kappa \) is the corresponding principal curvature of \( \partial \Omega \). One can see [13,21,34,43,44,46] for the studies of Navier-Stokes equations with Navier-type slip boundary condition. However, as far as we know, there are few research works on the compressible Navier-Stokes equations with slip boundary condition in an unbounded domain, except for the results of [21] and [15], which give the global existence of weak solutions in a half-space and the weak-strong uniqueness property in the class of finite energy weak solutions in an unbounded domain respectively.

The outline of this paper is as follows. First, we will give our main results at the end of this section. In Section 2, some notations, known facts and elementary inequalities needed in later analysis are prepared for our discussion. Section 3 and Section 4 are devoted to deriving the necessary a priori estimates on classical solutions which can guarantee the local classical solution to be a global classical one. Finally, we will prove the main results, Theorems 1.1 and 1.2 in Section 5.

Before stating the main results, we explain some functional spaces, notations and conventions used throughout this paper. For a positive integer \( k \) and \( 1 \leq q < +\infty \), the standard homogeneous Sobolev spaces are denoted as follows:

\[
    D^{k,q}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \left| \left\| \nabla^k u \right\|_{L^q(\Omega)} < +\infty \right\} \right., \quad \| \nabla u \|_{D^{k,q}(\Omega)} \triangleq \| \nabla^k u \|_{L^q(\Omega)}; \\
    W^{k,q}(\Omega) = L^q(\Omega) \cap D^{k,q}(\Omega), \quad \text{with the norm} \quad \| u \|_{W^{k,q}(\Omega)} \triangleq \left( \sum_{|m| \leq k} \| \nabla^m u \|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}; \\
    D^k(\Omega) = D^{k,2}(\Omega), \quad H^k(\Omega) = W^{k,2}(\Omega).
\]

For simplicity, we denote \( L^q(\Omega), D^{k,q}(\Omega), D^k(\Omega), W^{k,q}(\Omega) \) and \( H^k(\Omega) \) by \( L^q, D^{k,q}, D^k, W^{k,q} \) and \( H^k \) respectively, and set

\[
    B_R \triangleq \{ x \in \mathbb{R}^3 | |x| < R \}, \quad \int f \, dx \triangleq \int \int f \, dx.
\]

For two \( 3 \times 3 \) matrices \( A = \{ a_{ij} \}, \) \( B = \{ b_{ij} \}, \) the trace of \( AB \) is presented by \( A: B, \) that is,

\[
    A: B \triangleq \text{tr}(AB) = \sum_{i,j=1}^3 a_{ij} b_{ji}.
\]
Finally, we denote $\nabla_i v = (\partial_i v^1, \partial_i v^2, \partial_i v^3)$, $i = 1, 2, 3$, and the material derivative of $v$ by $\dot{v} \triangleq v_t + u \cdot \nabla v$, when $v = (v^1, v^2, v^3)$. Furthermore, $v \cdot \nabla u \triangleq (v \cdot \nabla u^1, v \cdot \nabla u^2, v \cdot \nabla u^3)$ and $\nabla u \cdot v \triangleq (\nabla_1 u \cdot v, \nabla_2 u \cdot v, \nabla_3 u \cdot v)$.

Let $(\rho_0, u_0)$ and $\rho_{\infty}$ be given by (1.3) and (1.5) respectively. The initial total energy of (1.1) is defined by

$$C_0 = \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) \, dx,$$

where

$$G(\rho) \triangleq \rho \int_{\rho_{\infty}}^\rho \frac{P(s) - P(\rho_{\infty})}{s^2} \, ds.$$

Now we give our first result, which indicates that the existence and large-time behavior of global classical solutions to the problem (1.1)-(1.5).

**Theorem 1.1** Let $\Omega$ be the exterior of a simply connected bounded domain $D$ in $\mathbb{R}^3$ and its boundary $\partial \Omega$ is smooth. For some $q \in (3, 6)$ and two given constants $M$, $\bar{\rho} \geq \rho_{\infty} + 1$, assume that the initial data $(\rho_0, u_0)$ is given as follows:

$$u_0 \in \left\{ f \in D^1 \cap D^2 : f \cdot n = 0, \text{ curl} f \times n = 0 \text{ on } \partial \Omega \right\},$$

$$(\rho_0 - \rho_{\infty}, P(\rho_0) - P(\rho_{\infty})) \in H^2 \cap W^{2,q},$$

$$0 \leq \rho_0 \leq \bar{\rho}, \| \nabla u_0 \|_{L^2} \leq M,$$

$$\rho_0 \in L^{3/2} \text{ if } \rho_{\infty} = 0,$$

and the compatibility condition

$$- \mu \Delta u_0 - (\mu + \lambda) \text{div} u_0 + \nabla P(\rho_0) = \rho_0^\dagger g,$$

for some $g \in L^2$. Then there exists a positive constant $\varepsilon$ depending only on $\mu, \lambda, \gamma, a, \bar{\rho}, \Omega$ and $M$ such that if $C_0 \leq \varepsilon$, then the system (1.1)-(1.5) has a unique global classical solution $(\rho, u)$ in $\Omega \times (0, \infty)$ satisfying

$$0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad (x, t) \in \Omega \times (0, \infty),$$

$$\left\{ \begin{array}{l}
(\rho - \rho_{\infty}, P - P(\rho_{\infty})) \in C([0, \infty); H^2 \cap W^{2,q}), \\
\nabla u \in C([0, \infty); H^1) \cap L^{\infty}_{\text{loc}}(0, \infty; W^{2,q}), \\
u_t \in L^{\infty}_{\text{loc}}(0, \infty; D^1 \cap D^2) \cap H^1_{\text{loc}}(0, \infty; D^1), \\
\sqrt{\rho}u_t \in L^{\infty}(0, \infty; L^2).
\end{array} \right. \quad (1.12)$$

In addition, the following large-time behavior

$$\lim_{t \to \infty} \left( \| \rho - \rho_{\infty} \|_{L^r} + \| \rho^\dagger u \|_{L^4} + \| \nabla u \|_{L^2} \right) = 0,$$

holds for all $r \in (2, \infty)$ if $\rho_{\infty} > 0$ and $r \in (\gamma, \infty)$ if $\rho_{\infty} = 0$.

The following theorem shows the large-time behavior of the gradient of the density when vacuum states appear initially.
Theorem 1.2 Under the conditions of Theorem 1.1, assume further that \( \rho_\infty > 0 \) and there exists some point \( x_0 \in \Omega \) such that \( \rho_0(x_0) = 0 \). Then the unique global classical solution \((\rho, u)\) to the problem (1.1)-(1.5) obtained in Theorem 1.1 satisfies that for any \( r > 3 \),
\[
\lim_{t \to \infty} \| \nabla \rho(\cdot, t) \|_{L^r} = \infty.
\]  
(1.14)

Remark 1.1 Since \( q > 3 \), it follows from Sobolev’s inequality and (1.12) that 
\[
\rho - \rho_\infty, \nabla \rho \in C(\bar{\Omega} \times [0, T]).
\]
Moreover, it also follows from (1.12) and (1.12) that  
\[
u, \nabla u, \nabla^2 u, u_t \in C(\bar{\Omega} \times [\tau, T]),
\]
(1.16)
due to the following simple fact that  
\[
L^2(\tau, T; H^1(\tau, T; H^{-1})) \hookrightarrow C([\tau, T]; L^2).
\]
Finally, by (1.11), we have  
\[
\rho_t = -\nabla u \cdot \nabla \rho - \rho \text{div} u \in C(\bar{\Omega} \times [\tau, T]),
\]
which together with (1.15) and (1.16) shows that the solution obtained by Theorem 1.1 is a classical one.

Remark 1.2 One also can prove the same conclusions of Theorems 1.1 and 1.2 under the following more wide boundary condition:  
\[
u \cdot n = 0, \ \text{curl} u \times n = -Au \text{ on } \partial \Omega,
\]
(1.17)
where \( A = A(x) \) is \( 3 \times 3 \) symmetric matrix which has compact support in \( \mathbb{R}^3 \) and \( A \in W^{2,6} \) satisfying the conditions of Remark 1.3 in [6]. Since the treatments of some additional priori estimates are parallel to those of [6], we omit detailed proofs.

Due to the equivalence of norms \( \| \nabla v \|_{L^2} \simeq \| D(v) \|_{L^2} \) for any \( v \in D^1 \) with \( \nu \cdot n = 0 \) and \( v \to 0 \) as \( |x| \to \infty \) (see [12]), similar to the proof of [6, Theorem 1.4], we immediately have the following conclusion.

Theorem 1.3 Set \( A = B - 2D(n) \) in (1.17), where \( B \in W^{2,6}(\Omega) \) is a positive semi-definite \( 3 \times 3 \) symmetric matrix. Under the same assumptions of Theorem 1.1 expect that the first condition of (1.8) is replaced by  
\[
u_0 \in \{ f \in D^1 \cap D^2 : f \cdot n = 0, \ \text{curl} f \times n = -Au \text{ on } \partial \Omega \},
\]
the conclusions of Theorems 1.1, 1.2 with respect to the slip boundary condition (1.17) (instead of (1.4)) remain valid provided that \( 2\mu + 3\lambda > 0 \).

As a direct result of Theorem 1.3 for compressible Navier-Stokes equations (1.1) with Navier-type slip boundary condition (1.6), we give the following conclusion on the global existence and large-time behavior of classical solutions.
Corollary 1.4 Assume that \( \vartheta \in W^{2,6} \), the shear viscosity coefficient \( \mu \) and the bulk one \( \lambda \) satisfy one of the following two conditions:

(1) \( A = \vartheta I - 2D(n) \) satisfies the assumption given by Remark 1.2.
(2) \( 2\mu + 3\lambda > 0 \), \( \vartheta \geq 0 \).

Then, for the system (1.1)-(1.3) with Navier-type slip boundary condition (1.6) and the far field behavior (1.5), the conclusions of Theorems 1.1-1.3 hold.

We now comment on the analysis of this paper. Indeed, compared with the previous results (\[6\]) where we treated the problem in a simply connected bounded domain, since the domain is unbounded, there are two difficulties that we have to overcome: one of them is how to estimate \( \nabla u \) by means of \( \text{div} u \) and \( \text{curl} u \), the other one is how to control the boundary integrals, especially (see (3.34)),

\[
\int_{\partial \Omega} \sigma^m(\nabla F \cdot u)(\dot{u} \cdot n)ds.
\]

For the former, we establish some necessary inequalities in Section 2 related to \( \nabla u \), \( \text{div} u \) and \( \text{curl} u \), see Lemmas 2.5-2.9. For the latter, we introduce a smooth ‘cut-off’ function defined on a ball containing \( \mathbb{R}^3 - \Omega \), which not only enables us to eliminate the first derivative in the boundary integral above by using the method in [6] based on the divergence theorem and the fact \( u = u_\perp \times n \) (see (3.27)), but also reduces the boundary integral to the integral on the ball. The reader can see (3.34) for details. All these treatments are the key to achieve our purposes in this paper.

2 Preliminaries

2.1 Some known inequalities and facts

In this subsection, some facts and elementary inequalities, which will be used frequently later, are collected.

First, similar to the proof of [22, Theorem 1.4], we have the local existence of strong and classical solutions.

Lemma 2.1 Let \( \Omega \) be as in Theorem 1.1, assume that \( (\rho_0, u_0) \) satisfies (1.8) and (1.10). Then there exist a small time \( T > 0 \) and a unique strong solution \( (\rho, u) \) to the problem (1.1)-(1.4) on \( \Omega \times (0, T) \) satisfying for any \( \tau \in (0, T) \),

\[
\begin{align*}
(\rho - \rho_\infty, P - P(\rho_\infty)) &\in C([0, \infty); H^2 \cap W^{2,q}), \\
u &\in C([0, \infty); D^1 \cap D^2), \quad \nabla u \in L^2(0, T; H^2) \cap L^{p_0}(0, T; W^{2,q}), \\
\nabla u &\in L^\infty(\tau, T; H^2 \cap W^{2,q}) \\
u_t &\in L^\infty(\tau, T; D^1 \cap D^2) \cap H^1(\tau, T; D^1), \\
\sqrt{\rho} u_t &\in L^\infty(0, T; L^2),
\end{align*}
\]

where \( q \in (3, 6) \) and \( p_0 = \frac{9q-6}{10q-12} \in (1, \frac{3}{2}) \).

The following Lemma can be found in [11].

Lemma 2.2 (Gagliardo-Nirenberg) Assume that \( \Omega \) is the exterior of a simply connected domain \( D \) in \( \mathbb{R}^3 \). For \( p \in [2, 6], q \in (1, \infty) \) and \( r \in (3, \infty) \), there exists some
generic constant $C > 0$ which may depend on $p, q$ and $r$ such that for any $f \in H^1(\Omega)$ and $g \in L^q(\Omega) \cap D^{1,r}(\Omega)$,

$$
\|f\|_{L^p(\Omega)} \leq C \|f\|_{L^p}^{6-p} \|\nabla f\|_{L^2}^{2p-6},
$$
\[ (2.1) \]

$$
\|g\|_{C(\overline{\Omega})} \leq C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}.
$$
\[ (2.2) \]

Generally, (2.1) and (2.2) are called Gagliardo-Nirenberg’s inequalities.

Next, in order to arrive at the uniform (in time) upper bound of the density $\rho$, we need the following Zlotnik inequality \[48\].

**Lemma 2.3** Assume that $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$, and the function $y$ satisfy

$$
y'(t) = g(y) + b'(t) \text{ on } [0, T], \quad y(0) = y^0,
$$

If $g(\infty) = -\infty$ and

$$
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
$$
\[ (2.3) \]

for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then

$$
y(t) \leq \max \{y^0, \zeta\} + N_0 < \infty \text{ on } [0, T],
$$

where $\zeta$ is a constant such that

$$
g(\zeta) \leq -N_1 \quad \text{for} \quad \zeta \geq \zeta.
$$
\[ (2.4) \]

Consider the Neumann boundary value problem

$$
\begin{cases}
-\Delta v = \text{div} f & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = -f \cdot n & \text{on } \partial \Omega, \\
\nabla v \to 0, \text{ as } |x| \to \infty,
\end{cases}
$$
\[ (2.5) \]

where $v = (v^1, v^2, v^3)$ and $f = (f^1, f^2, f^3)$. Indeed, the problem is equivalent to

$$
\int \nabla v \cdot \nabla \eta dx = \int f \cdot \nabla \eta dx, \quad \forall \eta \in C_0(\mathbb{R}^3).
$$

Thanks to \[36\ Lemma 5.6], we have the following conclusion.

**Lemma 2.4** For the system (2.5), one has

(1) If $f \in L^q$ for some $q \in (1, \infty)$, then there exists a unique (modulo constants) solution $v \in D^{1,q}$ such that

$$
\|\nabla v\|_{L^q} \leq C(q, \Omega)\|f\|_{L^q}.
$$

(2) If $f \in W^{k,q}$ for some $q \in (1, \infty)$, $k \geq 1$, then $\nabla v \in W^{k,q}$ and

$$
\|\nabla v\|_{W^{k,q}} \leq C\|f\|_{W^{k,q}}.
$$

**Definition 2.1** Let $\Omega$ be a domain in $\mathbb{R}^3$. If the first Betti number of $\Omega$ vanishes, namely, any simple closed curve in $\Omega$ can be contracted to a point, we say that $\Omega$ is simply connected. If the second Betti number of $\Omega$ is zero, we say that $\Omega$ has no holes.
The following two lemmas are given in [2][5], more precisely, Theorem 3.2 in [5] and Propositions 2.6-2.9 in [2].

**Lemma 2.5** Let \( k \geq 0 \) be a integer, \( 1 < q < +\infty \), and assume that \( D \) is a simply connected bounded domain in \( \mathbb{R}^3 \) with \( C^{k+1,1} \) boundary \( \partial D \). Then for \( v \in W^{k+1,q}(D) \) with \( v \cdot n = 0 \) on \( \partial D \), there exists a constant \( C = C(q,k,D) \) such that

\[
\|v\|_{W^{k+1,q}(D)} \leq C(\|\text{div}v\|_{W^{k,q}(D)} + \|\text{curl}v\|_{W^{k,q}(D)}). \tag{2.6}
\]

In particular, for \( k = 0 \), we have

\[
\|\nabla v\|_{L^q(D)} \leq C(\|\text{div}v\|_{L^q(D)} + \|\text{curl}v\|_{L^q(D)}). \tag{2.7}
\]

**Lemma 2.6** Suppose that \( D \) is a bounded domain in \( \mathbb{R}^3 \) and its \( C^{k+1,1} \) boundary \( \partial D \) only has a finite number of \( 2 \)-dimensional connected components. Then for an integer \( k \geq 0 \) and \( 1 < q < +\infty \), there exists a constant \( C = C(q,k,D) \) such that every \( v \in W^{k+1,q}(D) \) with \( v \times n = 0 \) on \( \partial D \) satisfies

\[
\|v\|_{W^{k+1,q}(D)} \leq C(\|\text{div}v\|_{W^{k,q}(D)} + \|\text{curl}v\|_{W^{k,q}(D)} + \|v\|_{L^q(D)}).
\]

In particular, if \( D \) has no holes, then

\[
\|v\|_{W^{k+1,q}(D)} \leq C(\|\text{div}v\|_{W^{k,q}(D)} + \|\text{curl}v\|_{W^{k,q}(D)}).
\]

The following conclusion is shown in [5] Theorem 3.2.

**Lemma 2.7** Let \( D \) be a simply connected domain in \( \mathbb{R}^3 \) with \( C^{1,1} \) boundary, and \( \Omega \) is the exterior of \( D \). For \( v \in D^{1,q}(\Omega) \) with \( v \cdot n = 0 \) on \( \partial \Omega \), it holds that

\[
\|\nabla v\|_{L^q(\Omega)} \leq C(\|\text{div}v\|_{L^q(\Omega)} + \|\text{curl}v\|_{L^q(\Omega)}) \quad \text{for any } 1 < q < 3, \tag{2.8}
\]

and

\[
\|\nabla v\|_{L^q(\Omega)} \leq C(\|\text{div}v\|_{L^q(\Omega)} + \|\text{curl}v\|_{L^q(\Omega)} + \|\nabla v\|_{L^2(\Omega)}) \quad \text{for any } 3 \leq q < +\infty.
\]

Due to [32] Theorem 5.1] (choosing \( \alpha = 0 \)), we obtain

**Lemma 2.8** Let \( \Omega \) be given in Lemma 2.7, for any \( v \in W^{1,q}(\Omega) \) (\( 1 < q < +\infty \)) with \( v \times n = 0 \) on \( \partial \Omega \), it holds that

\[
\|\nabla v\|_{L^q(\Omega)} \leq C(\|v\|_{L^q(\Omega)} + \|\text{div}v\|_{L^q(\Omega)} + \|\text{curl}v\|_{L^q(\Omega)}).
\]

Using Lemmas 2.5, 2.8, we further have the following result.

**Lemma 2.9** Let \( D \) be a simply connected domain in \( \mathbb{R}^3 \) with smooth boundary, and \( \Omega \) is the exterior of \( D \). For any \( p \in [2,6] \) and integer \( k \geq 0 \), there exists some positive constant \( C \) depending only on \( p, k \) and \( D \) such that every \( v \in \{D^{k+1,p}(\Omega) \cap D^{1,2}(\Omega)|v(x,t) \to 0 \text{ as } |x| \to \infty \} \) with \( v \cdot n_{|\partial \Omega} = 0 \) or \( v \times n_{|\partial \Omega} = 0 \) satisfies

\[
\|\nabla v\|_{W^{k,p}(\Omega)} \leq C(\|\text{div}v\|_{W^{k,p}(\Omega)} + \|\text{curl}v\|_{W^{k,p}(\Omega)} + \|\nabla v\|_{L^2(\Omega)}). \tag{2.9}
\]
Proof. First, letting \( B_R \triangleq \{ x \in \mathbb{R}^d | |x| < R \} \) be a ball whose center is at the origin such that \( \bar{D} \subset B_R \), it follows from \((2.1)\) that there exists some \( C \) depending only on \( D \) such that every \( v \in \{ D^{1,2}(\Omega)|v(x, t) \to 0 \text{ as } |x| \to \infty \} \) satisfies for any \( p \in [2, \infty) \)

\[
\|v\|_{L^p(\bar{D} \cap \Omega)} \leq C\|v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)},
\]

which together with the standard Sobolev’s inequality gives

\[
\|v\|_{L^4(\partial\Omega)} \leq C\|v\|_{L^2(\bar{D} \cap \Omega)} + C\|\nabla v\|_{L^2(\bar{D} \cap \Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}.
\]

Then, it suffices to prove \((2.9)\) in the case \( v \cdot n = 0 \) since the case \( v \times n = 0 \) can be handled similarly, where we utilize Lemma \((2.6)\) instead of Lemma \((2.5)\). We introduce a cut-off function \( \eta(x) \in C_c^\infty(\mathbb{R}^d) \) satisfying \( \eta(x) = 1 \) for \( |x| \leq R, \eta(x) = 0 \) for \( |x| \geq 2R, \) \( 0 < \eta(x) < 1 \) for \( R < |x| < 2R, \) and \( |\partial^\alpha \eta(x)| < C(R, \alpha) \) for any \( 0 \leq |\alpha| \leq k + 1. \) Notice that \( B_{2R} \cap \Omega \) is a simply connected domain and \( \eta \cdot \vec{n} = 0 \) on \( \partial B_{2R} \cup \partial \Omega, \) consequently,

\[
\|\nabla (\eta v)\|_{L^p(\Omega)} = \|\nabla (\eta v)\|_{L^p(B_{2R} \cap \Omega)} \\
\leq C(\|\text{div}(\eta v)\|_{L^p(B_{2R} \cap \Omega)} + \|\text{curl}(\eta v)\|_{L^p(B_{2R} \cap \Omega)}) \\
\leq C(\|\text{div}v\|_{L^p(\Omega)} + \|\text{curl}v\|_{L^p(\Omega)} + \|v\|_{L^p(B_{2R} \cap \Omega)}).
\]

On the other hand, the standard \( L^p \) estimate implies that

\[
\|\nabla((1 - \eta)v)\|_{L^p(\mathbb{R}^d)} \leq C(\|\text{div}((1 - \eta)v)\|_{L^p(\mathbb{R}^d)} + \|\text{curl}((1 - \eta)v)\|_{L^p(\mathbb{R}^d)}) \\
\leq C(\|\text{div}v\|_{L^p(\Omega)} + \|\text{curl}v\|_{L^p(\Omega)} + \|v\|_{L^p(B_{2R} \cap \Omega)}),
\]

which together with \((2.12)\) leads to

\[
\|\nabla v\|_{L^p(\Omega)} \leq C(\|\text{div}v\|_{L^p(\Omega)} + \|\text{curl}v\|_{L^p(\Omega)} + \|v\|_{L^p(B_{2R} \cap \Omega)}).
\]

Hence, for \( k = 0 \) and \( p \in [2, \infty], \) by Holder’s and Gagliardo-Nirenberg’s inequalities,

\[
\|\nabla v\|_{L^p(\Omega)} \leq C(\|\text{div}v\|_{L^p(\Omega)} + \|\text{curl}v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}) \\
\leq C(\|\text{div}v\|_{L^p(\Omega)} + \|\text{curl}v\|_{L^p(\Omega)} + \|\nabla v\|_{L^2(\Omega)}),
\]

where in the second inequality we have used \((2.10)\).

Finally, choosing \( k = 1 \) in \((2.13)\) gives

\[
\|\nabla v\|_{L^1(\Omega)} \leq C(\|\text{div}v\|_{L^1(\Omega)} + \|\text{curl}v\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)}) \\
\leq C(\|\text{div}v\|_{L^1(\Omega)} + \|\text{curl}v\|_{L^1(\Omega)} + \|\nabla v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}) \\
\leq C(\|\text{div}v\|_{L^1(\Omega)} + \|\text{curl}v\|_{L^1(\Omega)} + \|\nabla v\|_{L^2(\Omega)}),
\]

where in the last inequality we have utilized \((2.14)\). Combining this, \((2.13)\), Gagliardo-Nirenberg’s inequality, and an inductive derivation leads to \((2.9)\) and finishes the proof of Lemma \((2.9)\).

Finally, we state the Beale-Kato-Majda type inequality which was first proved in \([4, 26]\) when \( \text{div} u \equiv 0 \) (see also \([23]\) for the case that \( \text{div} u \neq 0 \)).

**Lemma 2.10** For \( 3 < q < \infty, \) assume that \( u \cdot n = 0 \) and \( \text{curl} u \times n = 0, \) \( \nabla u \in W^{1,q}, \) there is a constant \( C = C(q) \) such that the following estimate holds

\[
\|\nabla u\|_{L^\infty} \leq C(\|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^2}) + C\|\nabla u\|_{L^2} + C.
\]

**Proof.** The proof is similar to that of \([3, \text{Lemma 2.7}]\), and we omit it. \(\Box\)
2.2 Estimates for \( F, \text{curl}u \) and \( \nabla u \)

Denote
\[
\text{curl} u \triangleq \nabla \times u, \quad F \triangleq (2\mu + \lambda)\text{div}u - (P - P(\rho_\infty)),
\]
where \( F \) is called the effective viscous flux and plays an important role in our analysis. Now we give some a priori estimates for \( F, \text{curl}u \) and \( \nabla u \), which will be used frequently later.

**Lemma 2.11** Assume \( \Omega \) is an exterior domain of some simply connected bounded domain in \( \mathbb{R}^3 \) and its boundary \( \partial \Omega \) is smooth. Let \((\rho, u)\) be a smooth solution of (1.1) in \( \Omega \) with slip condition. Then for any \( p \in [2, 6] \) and \( q \in (1, \infty) \), there exists a positive constant \( C \) which may depend only on \( p, q, \mu, \lambda \) and \( \Omega \) such that

\[
\|\nabla F\|_{L^q} \leq C\|\rho\dot{u}\|_{L^q},
\]

(2.16)

\[
\|\nabla \text{curl} u\|_{L^p} \leq C(\|\rho\dot{u}\|_{L^p} + \|\rho\ddot{u}\|_{L^2} + \|\nabla u\|_{L^2}),
\]

(2.17)

\[
\|F\|_{L^p} \leq C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}(\|\nabla u\|_{L^2}^2 + \|P - P(\rho_\infty)\|_{L^2}^2)^{(6-p)/(2p)},
\]

(2.18)

\[
\|\text{curl}u\|_{L^p} \leq C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}\|\nabla u\|_{L^2}^{(6-p)/(2p)} + C\|\nabla u\|_{L^2},
\]

(2.19)

\[
\|\nabla u\|_{L^p} \leq C\|\nabla u\|_{L^2}^{(6-p)/(2p)}(\|\rho\ddot{u}\|_{L^2} + \|P - P(\rho_\infty)\|_{L^6}^2)^{(3p-6)/(2p)} + C\|\nabla u\|_{L^2}.
\]

(2.20)

**Proof.** By (1.1)\(_2\), it is easy to find that \( F \) satisfies
\[
\int \nabla F \cdot \nabla \eta dx = \int \rho\ddot{u} \cdot \nabla \eta dx, \quad \forall \eta \in C_0^\infty(\mathbb{R}^3).
\]

Consequently, by Lemma 2.4
\[
\|\nabla F\|_{L^q} \leq C\|\rho\dot{u}\|_{L^q},
\]

(2.21)

and for any integer \( k \geq 1 \),
\[
\|\nabla F\|_{W^{k,q}} \leq C\|\rho\dot{u}\|_{W^{k,q}}.
\]

(2.22)

On the other hand, (1.1)\(_2\) can be rewritten as \( \mu \nabla \times \text{curl}u = \nabla F - \rho\dot{u} \). Notice that \( \text{curl} u \times n = 0 \) on \( \partial \Omega \) and \( \text{div}(\nabla \times \text{curl}u) = 0 \), we deduce from Lemmas 2.8, 2.9 and (2.21) that
\[
\|\nabla \text{curl} u\|_{L^q} \leq C(\|\nabla \times \text{curl}u\|_{L^q} + \|\text{curl}u\|_{L^q}) \leq C(\|\rho\dot{u}\|_{L^q} + \|\text{curl}u\|_{L^q}).
\]

(2.23)

and for any integer \( k \geq 1 \),
\[
\|\nabla \text{curl} u\|_{W^{k,p}} \leq C(\|\nabla \times \text{curl}u\|_{W^{k,p}} + \|\nabla \text{curl} u\|_{L^2}) \leq C(\|\rho\dot{u}\|_{W^{k,p}} + \|\rho\ddot{u}\|_{L^2} + \|\nabla u\|_{L^2}),
\]

(2.24)

Therefore, by Gagliardo-Nirenberg’s inequality and (2.23), for \( p \in [2, 6] \),
\[
\|\nabla \text{curl} u\|_{L^p} \leq C(\|\rho\dot{u}\|_{L^p} + \|\text{curl}u\|_{L^p}) \leq C(\|\rho\dot{u}\|_{L^p} + \|\nabla \text{curl} u\|_{L^2} + \|\text{curl}u\|_{L^2}) \leq C(\|\rho\ddot{u}\|_{L^p} + \|\rho\ddot{u}\|_{L^2} + \|\nabla u\|_{L^2}).
\]
which gives (2.17).

Furthermore, it follows from Gagliardo-Nirenberg’s inequality, (2.21) and (2.16) that for $p \in [2, 6]$,

\[
\|F\|_{L^p} \leq C \|F\|_{L^2}^{(6-p)/2p} \|\nabla F\|_{L^2}^{(3p-6)/2p} \\
\leq C \|\rho \dot{u}\|_{L^2}^{(3p-6)/2p} \left( \|\nabla u\|_{L^2} + \|P - P(\rho_\infty)\|_{L^2} \right)^{(6-p)/2p}. 
\tag{2.25}
\]

Similarly,

\[
\|\text{curl} u\|_{L^p} \leq C \|\text{curl} u\|_{L^2}^{(6-p)/(2p)} \|\nabla u\|_{L^2}^{(3p-6)/(2p)} \\
\leq C \left( \|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} \right)^{(3p-6)/(2p)} \|\nabla u\|_{L^2}^{(6-p)/(2p)} + C \|\nabla u\|_{L^2}. 
\tag{2.26}
\]

A combination of (2.25) and (2.26) yields (2.18).

Finally, by virtue of Lemma 2.7, (2.1), (2.16) and (2.19), it indicates that

\[
\|\nabla u\|_{L^p} \leq C \|\nabla u\|_{L^2}^{(6-p)/(2p)} \|\nabla u\|_{L^6}^{(3p-6)/(2p)} \\
\leq C \|\nabla u\|_{L^2}^{(6-p)/(2p)} \left( \|\rho \dot{u}\|_{L^2} + \|P - P(\rho_\infty)\|_{L^6} \right)^{(3p-6)/(2p)} + C \|\nabla u\|_{L^2},
\]

where in the second inequality we have used

\[
\|\nabla u\|_{L^6} \leq C \left( \|\text{div} u\|_{L^6} + \|\text{curl} u\|_{L^6} + \|\nabla u\|_{L^2} \right) \\
\leq C \left( \|F\|_{L^6} + \|\text{curl} u\|_{L^6} + \|P - P(\rho_\infty)\|_{L^6} + \|\nabla u\|_{L^2} \right) \\
\leq C \left( \|\rho \dot{u}\|_{L^2} + \|P - P(\rho_\infty)\|_{L^6} + \|\nabla u\|_{L^2} \right),
\]

due to both (2.25) and (2.26) with $p = 6$. The proof of Lemma 2.11 is finished. \hfill \Box

3 A priori estimates(I): lower order estimates

In this section, Assume $\Omega$ is always the exterior of a simply connected bounded domain in $\mathbb{R}^3$. Choosing a positive real number $R$ such that $\bar{D} \subset B_R$, one can extend the unit outer normal $n$ to $\Omega$ such that

\[
n \in C^3(\bar{\Omega}), \quad n \equiv 0 \text{ on } \mathbb{R}^3 \setminus B_{2R}. \tag{3.1}
\]

Let $T > 0$ be a fixed time and $(\rho, u)$ be the smooth solution to (1.1)–(1.5) on $\Omega \times (0, T]$ with smooth initial data $(\rho_0, u_0)$ satisfying satisfying (1.8) and (1.9). We are going to establish some necessary a priori bounds for smooth solutions of the problem (1.1)–(1.5) which can extend the local classical solution guaranteed by Lemma 2.1 to be a global one.

In what follows, we will use the convention that $C$ may denote a generic positive constant depending on $\mu, \lambda, \gamma, a, \bar{\rho}, \rho_\infty, \Omega$ and $M$, and use $C(\delta)$ to emphasize that $C$ depends on $\delta$. We have the following standard energy estimate for $(\rho, u)$ and preliminary $L^2$ bounds for $\nabla u$ and $\rho \dot{u} u$.

**Lemma 3.1** Let $(\rho, u)$ be a smooth solution of (1.1)–(1.5) on $\Omega \times (0, T]$. Then there is a positive constant $C$ depending only on $\mu, \lambda, \gamma$ and $\Omega$ such that

\[
\sup_{0 \leq t \leq T} \left( \|\rho \dot{u} u\|_{L^2}^2 + \|G(\rho)\|_{L^1} \right) + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C C_0. \tag{3.2}
\]
Proof. Multiplying (1.1) by $G'(\rho)$ and taking advantage of slip boundary condition (1.4) show that

$$\left(\int G(\rho)dx\right)_t + \int (P - P(\rho_\infty))\text{div}udx = 0. \quad (3.3)$$

Note that $-\Delta u = -\nabla\text{div}u + \nabla \times \text{curl}u$, we rewrite (1.1) as

$$\rho\dot{u} - (\lambda + 2\mu)\nabla\text{div}u + \mu\nabla \times \text{curl}u + \nabla (P - P(\rho_\infty)) = 0. \quad (3.4)$$

Multiplying (3.4) by $u$, along with (3.3) and (1.4), we have

$$\left(\frac{1}{2} \int \rho|u|^2dx + \int G(\rho)dx\right)_t + (\lambda + 2\mu) \int (\text{div}u)^2dx + \mu \int |\text{curl}u|^2dx = 0, \quad (3.5)$$

which together with (2.8) gives (3.2) and finishes the proof of Lemma 3.1. \(\square\)

For $\sigma(t) \triangleq \min\{1,t\}$, we define

$$A_1(T) \triangleq \sup_{0 \leq t \leq T} (\sigma\|\nabla u\|_2^2) + \int_0^T \sigma \int_0^T \rho|\dot{u}|^2dxdt, \quad (3.6)$$

$$A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^3 \int_0^T \rho|\dot{u}|^2dx + \int_0^T \sigma^3 \int_0^T \sigma^3 |\nabla u|^2dxdt, \quad (3.7)$$

and

$$A_3(T) \triangleq \sup_{0 \leq t \leq T} \int_0^T \rho|u|^3dx. \quad (3.8)$$

The rest of section is devoted to proving the following proposition, which guarantees the existence of a global classical solution of (1.1)–(1.5).

**Proposition 3.2** Under the conditions of Theorem 1.1, there exists a positive constant $\varepsilon$ depending on $\mu$, $\lambda$, $a$, $\gamma$, $\bar{\rho}$, $\rho_\infty$, $\Omega$ and $M$ such that if $(\rho, u)$ is a smooth solution of (1.1)–(1.5) on $\Omega \times (0,T]$ satisfying

$$\sup_{\Omega \times [0,T]} \rho \leq 2\bar{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{1/2}, \quad A_3(\sigma(T)) \leq 2C_0^{1/4}, \quad (3.9)$$

then

$$\sup_{\Omega \times [0,T]} \rho \leq \frac{7\bar{\rho}}{4}, \quad A_1(T) + A_2(T) \leq C_0^{1/2}, \quad A_3(\sigma(T)) \leq C_0^{1/4}, \quad (3.10)$$

provided $C_0 \leq \varepsilon$.

**Proof.** Proposition 3.2 is a direct consequence of Lemmas 3.5, 3.7 and 3.8 below. \(\square\)

The following Lemmas 3.3, 3.8 will be proven under the assumption (3.9).

**Lemma 3.3** Suppose $(\rho, u)$ is a smooth solution of (1.1)–(1.5) on $\Omega \times (0,T]$ satisfying (3.9). Then there exists a positive constant $C$ depending only on $\mu$, $\lambda$, $a$, $\gamma$, $\bar{\rho}$, $\rho_\infty$ and $\Omega$ such that

$$A_1(T) \leq CC_0 + C \int_0^T \int \sigma|\nabla u|^3dxdt, \quad (3.11)$$

and

$$A_2(T) \leq CC_0 + CA_1(T) + C \int_0^T \int \sigma^3|\nabla u|^4dxdt. \quad (3.12)$$
Proof. The proof is motivated by [6,19,24]. First, the following two identities will be used more than once:

\[
\text{div}(u \cdot \nabla u) = u \cdot \nabla (\text{div} u) + \nabla : u \nabla,
\]
(3.13)

\[
\text{curl}(u \cdot \nabla u) = u \cdot \nabla (\text{curl} u) + \nabla u \times \nabla u.
\]
(3.14)

Next, for \( m \geq 0 \), multiplying (1.1) by \( \sigma m \cdot \dot{u} \) yields

\[
\int \sigma m |\dot{u}|^2 dx = - \int \sigma m \cdot \nabla P dx + (\lambda + 2\mu) \int \sigma m \nabla \text{div} u \cdot \dot{u} dx
\]
\[- \mu \int \sigma m \nabla \times \text{curl} u \cdot \dot{u} dx
\]
\[= \sum_{i=1}^{3} I_i.
\]
(3.15)

We will estimate the three terms in the last equality. It follows from the slip boundary condition \( u \cdot n|_{\partial \Omega} = 0 \) that

\[
u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u \quad \text{on} \partial \Omega.
\]
(3.16)

For \( I_1 \), we check that

\[
I_1 = - \int \sigma m \cdot \nabla Pdx
\]
\[- \int \sigma m u_t \cdot \nabla (P - P(\rho_\infty)) dx - \int \sigma m u \cdot \nabla u \cdot \nabla P dx
\]
\[= \left( \int \sigma m (P - P(\rho_\infty)) \text{div} u dx \right) t - m \sigma^m \sigma' \int (P - P(\rho_\infty)) \text{div} u dx
\]
\[+ \int \sigma m P \nabla u : \nabla u dx + (\gamma - 1) \int \sigma m P \text{div} u^2 dx + \int_{\partial \Omega} \sigma m P u \cdot \nabla n \cdot u ds,
\]
(3.17)

where we have used (3.16), (3.13) and the relation

\[
P_t + \text{div}(Pu) + (\gamma - 1)P \text{div} u = 0.
\]

For the boundary term in the last of (3.17), one has by (2.11)

\[
\int_{\partial \Omega} \sigma m P u \cdot \nabla n \cdot u ds \leq C(\bar{\rho}) \sigma^m \|u\|_L^2(\partial \Omega)
\]
\[\leq C(\bar{\rho}) \sigma^m \|\nabla u\|_L^2.
\]

As a result, together with (3.12),

\[
I_1 \leq \left( \int \sigma m (P - P(\rho_\infty)) \text{div} u dx \right) t + C(\bar{\rho}) \|\nabla u\|_L^2 + C(\bar{\rho}) m \sigma^m \sigma' C_0.
\]
(3.18)
Similarly, by (3.13),

\[ I_2 = (\lambda + 2\mu) \int \sigma^m \nabla \div u \cdot \dot{u} \, dx \]

\[ = (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \div (\dot{u} \cdot n) \, ds - (\lambda + 2\mu) \int \sigma^m \div \dot{u} \, dx \]

\[ = (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \div (u \cdot \nabla u \cdot n) \, ds - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\div u)^2 \, dx \right)_t \]

\[ - (\lambda + 2\mu) \int \sigma^m \div \div u \, dx + \frac{m(\lambda + 2\mu)}{2} \sigma^{m-1} \sigma' \int (\div u)^2 \, dx \]  

(3.19)

\[ = (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \div (u \cdot \nabla u \cdot n) \, ds - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\div u)^2 \, dx \right)_t \]

\[ + \frac{\lambda + 2\mu}{2} \int \sigma^m (\div u)^3 \, dx - (\lambda + 2\mu) \int \sigma^m \div \nabla u : \nabla \dot{u} \, dx \]

\[ + \frac{m(\lambda + 2\mu)}{2} \sigma^{m-1} \sigma' \int (\div u)^2 \, dx. \]

On the other hand, it follows from Hölder’s inequality, (2.11), and (2.16) that

\[ \left| \int_{\partial \Omega} \div (u \cdot \nabla u \cdot n) \, ds \right| \]

\[ \leq \frac{1}{\lambda + 2\mu} \left( \int_{\partial \Omega} |F \cdot \nabla n \cdot u| \, ds \right) + \frac{1}{\lambda + 2\mu} \left( \int_{\partial \Omega} |P - P(\rho_{\infty})| u \cdot \nabla n \cdot u| \, ds \right) \]

\[ \leq C \left( \|F\|_{L^4(\partial \Omega)} \|u\|_{L^4(\partial \Omega)}^2 + \|u\|_{L^2(\partial \Omega)}^2 \right) \]

\[ \leq C \left( \|\nabla F\|_{L^2} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \]

\[ \leq \frac{1}{2} \|\rho \frac{1}{2} \dot{u}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4). \]

Hence,

\[ I_2 \leq -\frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\div u)^2 \, dx \right)_t + C \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 + C \int \sigma^m |\nabla u|^{3} \, dx \]  

\[ + \frac{1}{2} \sigma^m \|\rho \frac{1}{2} \dot{u}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^4 + C \|\nabla u\|_{L^2}^2. \]  

(3.20)
For $I_3$, a straightforward computation along with (3.11) gives that

$$I_3 = -\mu \int \sigma^m \nabla \times \text{curl} \cdot \dot{u} \, dx$$

$$= -\mu \int \sigma^m \text{div} (\text{curl} \times \dot{u}) \, dx \quad \mu \int \sigma^m \text{curl} \cdot \text{curl} \dot{u} \, dx$$

$$= -\frac{\mu}{2} \left( \int \sigma^m |\text{curl}u|^2 dx \right)_t + \frac{\mu m}{2} \sigma^{m-1} \sigma' \int |\text{curl}u|^2 dx$$

$$- \mu \int \sigma^m \text{curl} \cdot \text{curl} (u \cdot \nabla u) \, dx$$

$$= -\frac{\mu}{2} \left( \int \sigma^m |\text{curl}u|^2 dx \right)_t + \frac{\mu m}{2} \sigma^{m-1} \sigma' \int |\text{curl}u|^2 dx$$

$$= - \mu \int \sigma^m \text{curl} \cdot \text{curl} (u \cdot \nabla u) \, dx$$

$$+ \frac{\mu}{2} \int \sigma^m |\text{curl}u|^2 \text{div} u \, dx$$

$$\leq -\frac{\mu}{2} \left( \int \sigma^m |\text{curl}u|^2 dx \right)_t + C \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^3}^3. \quad (3.21)$$

Together with the estimates of $I_1, I_2$ and $I_3$ above, it follows from (3.15) that

$$\left( (\lambda + 2\mu) \int \sigma^m (\text{div} u)^2 \, dx + \mu \int \sigma^m |\text{curl}u|^2 \, dx \right)_t$$

$$- 2 \left( \int \sigma^m (P - P(\rho_\infty)) \, dx \right)_t + \int \sigma^m \rho |\dot{u}|^2 \, dx$$

$$\leq C(\bar{\rho}) m \sigma^{m-1} \sigma' C_0 + C \sigma^m \|\nabla u\|_{L^2}^4 + C(\bar{\rho}) \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^3}^3, \quad (3.22)$$

which together with Young’s inequality, Lemma 2.7, (3.2) and (3.9) gives (3.11).

Now it remains to prove (3.12). Rewrite (1.12) as

$$\dot{\rho} \dot{u} = \nabla F - \mu \nabla \times \text{curl} u. \quad (3.23)$$

Imposing $\sigma^m \dot{u}^j [\partial / \partial t + \text{div} (u \cdot \cdot)]$ on (3.23)$^j$, summing with respect to $j$, and integrating over $\Omega$, we obtain

$$\left( \frac{\sigma^m}{2} \int \rho |\dot{u}|^2 \, dx \right)_t - \frac{m}{2} \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 \, dx$$

$$= - \int \sigma^m \dot{u} \text{div} (\rho \dot{u} \cdot u) \, dx - \frac{1}{2} \int \sigma^m \rho_1 |\dot{u}|^2 \, dx$$

$$+ \int \sigma^m (\dot{u} \cdot \nabla F_t + \dot{u}^j \text{div} (u \partial_j F)) \, dx$$

$$- \mu \int \sigma^m (\dot{u} \cdot \nabla \times \text{curl} u_t + \dot{u}^j \text{div} (u (\nabla \times \text{curl} u)^j)) \, dx$$

$$= \int \sigma^m (\dot{u} \cdot \nabla F_t + \dot{u}^j \text{div} (u \partial_j F)) \, dx$$

$$+ \mu \int \sigma^m (-\dot{u} \cdot \nabla \times \text{curl} u_t - \dot{u}^j \text{div} (u (\nabla \times \text{curl} u)^j)) \, dx$$

$$= J_1 + J_2, \quad (3.24)$$
where we have utilized \((1.1)_1\) and the boundary condition \(u \cdot n = 0\).

Let’s estimate \(J_1\) and \(J_2\). For \(J_1\), by Gagliardo-Nirenberg’s, Young’s and Hölder’s inequalities, we deduce from \((3.13), (1.1)_1\) and \((2.16)\) that

\[
J_1 = \int \sigma^m \hat{u} \cdot \nabla F_t dx + \int \sigma^m \hat{u} \cdot \nabla (u \hat{\tau}_j F) dx = \int \sigma^m F_t \hat{u} dx + \int \sigma^m \hat{u} \cdot \nabla (u \hat{\tau}_j F) dx
\]

For the first boundary term \(\hat{J}_1\), denoting \(h \triangleq u \cdot (\nabla n + \nabla n^T)\), we have by \((3.16)\),
Lemma 2.11 and Gagliardo-Nirenberg’s, Young’s, and Hölder’s inequalities,

\[ J_1 = \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t \]

\[ = - \int_{\partial \Omega} \sigma^m F_t (u \cdot \nabla n \cdot u) ds + \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t \]

\[ = \int_{\partial \Omega} \sigma^m F h \cdot u ds + m\sigma^{m-1} \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \]

\[ = \int_{\partial \Omega} \sigma^m F h \cdot \dot{u} ds - \int_{\partial \Omega} \sigma^m F h \cdot ((u^\perp \times n) \cdot \nabla u) ds \]

\[ + m\sigma^{m-1} \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \]

\[ = \int_{\partial \Omega} \sigma^m F h \cdot \dot{u} ds - \int_{\partial \Omega} \sigma^m \nabla u^\perp \cdot \nabla (F h^i) dx \]

\[ + \int \sigma^m F h^i \nabla \times u^\perp \cdot \nabla u^i dx + m\sigma^{m-1} \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds, \]

where in the last inequality we have used the following key observation due to [6]

\[ - \int_{\partial \Omega} F h \cdot ((u^\perp \times n) \cdot \nabla u) ds \]

\[ = - \int \text{div}(F h^i \nabla u^i \times u^\perp) dx \]

\[ = - \int \nabla u^i \times u^\perp \cdot \nabla (F h^i) dx + \int F h^i \nabla \times u^\perp \cdot \nabla u^i dx. \]

It follows from (2.11) and (2.16) that

\[ \int_{\partial \Omega} |F h \cdot \dot{u}| ds \leq C ||F||_{L^4(\partial \Omega)} ||u||_{L^4(\partial \Omega)} ||\dot{u}||_{L^2(\partial \Omega)} \]

\[ \leq C ||\nabla F||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)} ||\nabla \dot{u}||_{L^2(\Omega)} \]

\[ \leq \delta \frac{C}{8} \||\nabla u||_{L^2(\Omega)}^2 + C(\delta) ||\rho^{1/2} \dot{u}||_{L^2(\Omega)}^2 \||\nabla u||_{L^2(\Omega)}^2 \]

\[ \leq \int \nabla u^i \times u^\perp \cdot \nabla (F h^i) dx \leq \int F h^i \nabla \times u^\perp \cdot \nabla u^i dx \]

\[ \leq C ||\nabla F||_{L^6} ||\nabla u||_{L^2} ||u||_{L^6}^2 + C ||F||_{L^6} ||\nabla u||_{L^2} ||u||_{L^6}^2 \]

\[ + C ||F||_{L^6} ||\nabla u||_{L^3} ||u||_{L^6} \]

\[ \leq \delta \frac{C}{8} \||\nabla u||_{L^2}^2 + C(\delta) ||\rho^{1/2} \dot{u}||_{L^2}^2 ||\nabla u||_{L^2}^2 + C(\delta) ||\nabla u||_{L^2}^2 + C(\delta) ||\nabla u||_{L^3}^4. \]

\[ \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \leq C ||F||_{L^4(\partial \Omega)} ||u||_{L^4(\partial \Omega)}^2 \]

\[ \leq C ||\nabla F||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)}^2 \]

\[ \leq \delta \frac{C}{8} \||\nabla u||_{L^2}^2 + C(\delta) ||\nabla u||_{L^2}^4. \]
\[
\tilde{J}_1^1 \leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right) + \frac{\delta}{8} \sigma^m \| \nabla \dot{u} \|_{L^2}^2 \\
+ C m \sigma^{m-1} \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 + C(\delta) \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 \| \nabla u \|_{L^2}^2 \\
+ C(\delta) \sigma^m (\| \nabla u \|_{L^4}^4 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^6}^6).
\] (3.33)

For the second boundary term \( \tilde{J}_1^2 \),

\[
\tilde{J}_1^2 = - \int_{\partial \Omega} (\nabla F \cdot u)(u \cdot \nabla n \cdot u) ds \\
= - \int_{\partial \Omega} (u \cdot \nabla n \cdot u) \nabla F \cdot (u^\perp \times n) ds \\
= - \int_{\partial \Omega} (u \cdot \nabla n \cdot u)(\nabla F \times u^\perp) \cdot n ds \\
= - \int_{\partial \Omega} \text{div}[(u \cdot \nabla n \cdot u)(\nabla F \times u^\perp)] dx \\
\leq \frac{1}{\delta} \| \nabla u \|_{L^2}^2 C \sigma^m \| u \|_{L^5}^3 \| \nabla F \|_{L^2} + C \sigma^m \| u \|_{L^6}^3 \| \nabla F \|_{L^2} + C \sigma^m \| u \|_{L^6}^3 \| \nabla u \|_{L^2}^2 \\
+ C(\delta) \| \nabla u \|_{L^2}^2 \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^6}^6). 
\] (3.34)

Similarly, for the last boundary term \( \tilde{J}_1^3 \),

\[
\tilde{J}_1^3 = - \frac{1}{\lambda + 2\mu} \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F^2 ds \\
\leq C \sigma^m \| u \|_{L^4(\partial \Omega)}^4 \| F \|_{L^3(\partial \Omega)}^2 \\
\leq C \sigma^m \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 \| \nabla u \|_{L^2}^2. 
\] (3.35)

Together with these estimates of the three boundary terms above, and by (3.25), we conclude that

\[
J_1 \leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right) - (\lambda + 2\mu) \int \sigma^m (\text{div} \dot{u})^2 dx \\
+ \frac{\delta}{2} \sigma^m \| \nabla \dot{u} \|_{L^2}^2 \\
+ C(\delta) \sigma^m \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 (\| \nabla u \|_{L^2}^4 + 1) \\
+ C(\delta) \sigma^m (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^4 + \| \nabla u \|_{L^4}^4). 
\] (3.36)
By (3.14), a direct computation shows that

\[ J_2 = -\mu \int \sigma^m \dot{\mathbf{u}} \cdot (\nabla \times \text{curl} u) d\mathbf{x} - \mu \int \sigma^m \mathbf{u} \cdot (\nabla \times \text{curl} u) \text{div} u d\mathbf{x} \]

\[ = -\mu \int \sigma^m |\text{curl} \mathbf{u}|^2 d\mathbf{x} - \mu \int \sigma^m \text{curl} \mathbf{u} \cdot \text{curl}(u \cdot \nabla u) d\mathbf{x} \]

\[ + \mu \int \sigma^m (\nabla \times \dot{\mathbf{u}}) \cdot \nabla \text{div} u d\mathbf{x} - \mu \int \sigma^m \text{div} u \text{curl} u \cdot \text{curl} \dot{u} d\mathbf{x} \]

\[ = -\mu \int \sigma^m |\text{curl} \mathbf{u}|^2 d\mathbf{x} - \mu \int \sigma^m \text{curl} \mathbf{u} \cdot (\nabla u \times \nabla \dot{u}) d\mathbf{x} \]

\[ + \mu \int \sigma^m (\nabla \times \dot{\mathbf{u}}) \cdot \nabla \text{div} u d\mathbf{x} - \mu \int \sigma^m \text{div} u \text{curl} u \cdot \text{curl} \dot{u} d\mathbf{x} \]

\[ = -\mu \int \sigma^m |\text{curl} \mathbf{u}|^2 d\mathbf{x} + \mu \int \sigma^m (\text{curl} \mathbf{u} \times \nabla u) \cdot \nabla \dot{u} d\mathbf{x} \]

\[ - \mu \int \sigma^m \text{curl} \mathbf{u} \cdot (\nabla u \times \nabla \dot{u}) d\mathbf{x} - \mu \int \sigma^m \text{div} u \text{curl} u \cdot \text{curl} \dot{u} d\mathbf{x} \]

\[ \leq -\mu \int \sigma^m |\text{curl} \mathbf{u}|^2 d\mathbf{x} + C \sigma^m \|\text{div} u\|_{L^2} \|\nabla u\|_{L^4}^2 \]

\[ \leq -\mu \int \sigma^m |\text{curl} \mathbf{u}|^2 d\mathbf{x} + \frac{\delta}{2} \sigma^m \|\text{div} u\|_{L^2}^2 + C(\delta) \sigma^m \|\nabla u\|_{L^4}^4. \]

Therefore, we deduce from (3.24), (3.36) and (3.37) that

\[ \frac{1}{2} \left( \sigma^m \|\rho^{\frac{1}{2}} \dot{\mathbf{u}}\|_{L^2}^2 + (\lambda + 2\mu) \sigma^m \|\text{div} \mathbf{u}\|_{L^2}^2 + \mu \sigma^m \|\text{curl} \mathbf{u}\|_{L^2}^2 \right) \]

\[ \leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F d\mathbf{s} \right) + \delta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 \]

\[ + C(\delta) \sigma^m \|\rho^{\frac{1}{2}} \dot{\mathbf{u}}\|_{L^2}^2 (\|\nabla u\|_{L^2}^4 + 1) \]

\[ + C(\delta) \sigma^m (\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^4}^4). \]

Next, combining (1.4), (3.27), and (3.16) gives

\[ (\dot{u} - (u \cdot \nabla n) \times u^\perp) \cdot n|_{\partial \Omega} = 0, \]

which together with (2.8) yields

\[ \|\nabla \dot{u}\|_{L^2} \leq C(\|\nabla \mathbf{u}\|_{L^2} + \|\text{curl} \mathbf{u}\|_{L^2} + \|\nabla[(u \cdot \nabla n) \times u^\perp]\|_{L^2}) \]

\[ \leq C(\|\nabla \mathbf{u}\|_{L^2} + \|\text{curl} \mathbf{u}\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^2), \]

where in the second inequality we have used

\[ \|\nabla[(u \cdot \nabla n) \times u^\perp]\|_{L^2(\Omega)} = \|\nabla[(u \cdot \nabla n) \times u^\perp]\|_{L^2(B_{2R})} \]

\[ \leq C(R)(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \]

\[ \leq C(R)(\|\nabla u\|_{L^2(B_{2R})}^2 + \|u\|_{L^2(B_{2R})}^2) \]

\[ \leq C(R)(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) \]

\[ \leq C(R)(\|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^4}^2), \]
due to \((3.1), (2.1)\) and Hölder’s inequality.

Now choose \(\delta\) small enough, it follows from \((3.39)\) and \((3.38)\) that

\[
\left( \sigma^m \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 \right) + (\lambda + 2\mu) \sigma^m \| \text{div} \dot{u} \|_{L^2}^2 + \mu \sigma^m \| \text{curl} \dot{u} \|_{L^2}^2 \\
\leq - \left( 2 \int_{\partial \Omega} \sigma^m (u \cdot \nabla \cdot u) F ds \right) + C \sigma^m \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 (\| \nabla u \|_{L^2}^4 + 1) \\
+ C \sigma^m (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^6 + \| \nabla u \|_{L^4}^4) \tag{3.40}
\]

Taking \(m = 3\) in \((3.40)\), and using \((3.39), (3.32), (3.9)\) and Lemma 2.11, we establish \((3.12)\) and complete the proof of Lemma 3.3 \(\Box\)

**Lemma 3.4** If \((\rho, u)\) be a smooth solution of \((1.1), (1.5)\) on \(\Omega \times (0, T)\) satisfying \((3.9)\) and \(\| \nabla u_0 \|_{L^2} \leq M\), then there exist two positive constants \(C = C(\bar{\rho}, M)\) and \(\varepsilon_1\) depending only on \(\mu, \gamma, a, \rho_\infty, \bar{\rho}, \Omega\) and \(M\), such that

\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla u \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx dt \leq C(\bar{\rho}, M), \tag{3.41}
\]

\[
\sup_{0 \leq t \leq \sigma(T)} t \int \rho |\dot{u}|^2 dx + \int_0^{\sigma(T)} t \int |\nabla \dot{u}|^2 dx dt \leq C(\bar{\rho}, M). \tag{3.42}
\]

**Proof.** Multiplying \((1.1.2)\) by \(u_t\) and integrating over \(\Omega\), a direct calculation gives

\[
\left( \frac{\lambda + 2\mu}{2} \int (\text{div} u)^2 dx + \frac{\mu}{2} \int |\text{curl} u|^2 dx - \int (P - P(\rho_\infty)) \text{div} u dx \right) + \int \rho |\dot{u}|^2 dx \\
= \int \rho \dot{u} \cdot (u \cdot \nabla u) dx - \int P_t \text{div} u dx \\
= \int \rho \dot{u} \cdot (u \cdot \nabla u) dx - \frac{1}{\lambda + 2\mu} \int (P - P(\rho_\infty)) F \text{div} u dx \\
- \frac{1}{\lambda + 2\mu} \int (P - P(\rho_\infty)) \nabla F \cdot u dx - \frac{1}{2(\lambda + 2\mu)} \int (P - P(\rho_\infty))^2 \text{div} u dx \\
+ \gamma \int P \text{div} u^2 dx \\
\leq C(\bar{\rho}) (\| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 \| \rho \frac{1}{2} u \|_{L^3} \| \nabla u \|_{L^6} + \| P - P(\rho_\infty) \|_{L^2} \| \nabla F \|_{L^2} \| u \|_{L^6}) \\
+ C(\bar{\rho}) (\| \nabla u \|_{L^2} \| F \|_{L^2} + \| P - P(\rho_\infty) \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla u \|_{L^4}^2) \\
\leq (C(\bar{\rho}) C_0^{1/2} + 1/4) \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 + C(\bar{\rho}) (\| \nabla u \|_{L^2}^2 + \| P - P(\rho_\infty) \|_{L^2}^2) \\
+ C(\bar{\rho}) \| P - P(\rho_\infty) \|_{L^2}^2, \tag{3.43}
\]

where we have taken advantage of \((3.9), (2.16), (2.18), (2.20)\), Hölder’s, Poincaré’s and Young’s inequalities.

Hence,

\[
\left( \lambda + 2\mu \right) \| \text{div} u \|_{L^2}^2 + \mu \| \text{curl} u \|_{L^2}^2 - 2 \int (P - \bar{P}) \text{div} u dx \right) + \int \rho |\dot{u}|^2 dx \\
\leq C(\bar{\rho}) \left( \| \nabla u \|_{L^2}^2 + \| P - P(\rho_\infty) \|_{L^2}^2 + \| P - P(\rho_\infty) \|_{L^6}^2 \right), \tag{3.44}
\]

provide that \(C_0 < \varepsilon_1 \triangleq (4C(\bar{\rho}))^{-12}\).
Therefore, it follows from Gronwall’s inequality, Lemmas 2.7 and 3.1 that (3.41) holds.

Finally, we will claim (3.42). Taking \( m = 1 \) in (3.40), and integrating over \( (0, \sigma(T)) \], we deduce from (3.41) that

\[
\sup_{0 \leq t \leq \sigma(T)} t \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 + \int_0^{\sigma(T)} t \| \nabla \dot{u} \|_{L^2}^2 dt
\]

\[
\leq C \int_0^{\sigma(T)} \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 dt + C \int_0^{\sigma(T)} t \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 (\| u \|_{L^2}^4 + 1) dt
\]

\[
+ C \int_0^{\sigma(T)} t (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^6) dt + C \int_0^{\sigma(T)} t \| \nabla u \|_{L^2}^4 dt
\]

\[
+ C \int_0^{\sigma(T)} \| \nabla u \|_{L^2}^4 dt + C t (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^4)
\]

\[
\leq C \int_0^{\sigma(T)} t \| \nabla u \|_{L^2}^4 dt + C(\bar{\rho}, M).
\]

(3.45)

On the other hand, by (2.20) and (3.41),

\[
\int_0^{\sigma(T)} t \| \nabla u \|_{L^2}^4 dt
\]

\[
\leq C \int_0^{\sigma(T)} t (\| u \|_{L^2}^3 \| \nabla u \|_{L^2} + \| P - P(\rho_{\infty}) \|_{L^6}^3 \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2}^4) dt
\]

\[
\leq C \int_0^{\sigma(T)} t^{\frac{1}{2}} (\| \nabla u \|_{L^2}^2 t^{\frac{1}{2}} (t \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2) \frac{1}{2} (\| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2) dt + C(\bar{\rho})
\]

\[
\leq C(\bar{\rho}, M) \left( \sup_{0 \leq t \leq \sigma(T)} t \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 \right)^{\frac{1}{2}} + C(\bar{\rho}).
\]

(3.46)

Therefore, together with (3.45) and (3.46), we get (3.42).

\[ \square \]

Lemma 3.5 Suppose \((\rho, u)\) is a smooth solution of (1.1) - (1.5) on \( \Omega \times (0, T) \) satisfying (3.9) and the assumption \( \| \nabla u_0 \|_{L^2} \leq M \), then there exists a positive constant \( \varepsilon_2 \) \((\leq \varepsilon_1)\) depending only on \( \mu, \lambda, \gamma, a, \bar{\rho}, \rho_{\infty}, \Omega \) and \( M \) such that

\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^T \rho |u|^3 dx \leq C_0^{\frac{1}{2}},
\]

(3.47)

provided \( C_0 < \varepsilon_2 \).

Proof. Multiplying (1.1) by 3|u|u, integrating over \( \Omega \), by Lemma 3.1 (2.20) and
Multiplying (3.51) by 3, we check that
\[
\left(\int \rho |u|^3 \, dx\right)_t = -3(\lambda + 2\mu) \int \text{div} \, \text{div} (|u|u) \, dx - 3\mu \int \text{curl} u \cdot \text{curl} (|u|u) \, dx
\]
\[+ 3 \int (P - P(\rho_\infty)) \text{div} (|u|u) \, dx\]
\[\leq C \int |u| |\nabla u|^2 \, dx + C \int |P - P(\rho_\infty)||u| |\nabla u| \, dx\]
\[\leq C \|u\|_{L^6} \|\nabla u\|_{L^6}^2 + C \|P - P(\rho_\infty)\|_{L^3} \|u\|_{L^6} \|\nabla u\|_{L^2}\]
\[\leq C \|\nabla u\|_{L^2}^2 (\|\rho \dot{u}\|_{L^2} + \|P - P(\rho_\infty)\|_{L^6} + \|\nabla u\|_{L^2})^\frac{1}{2} + C(\tilde{\rho})C_0^{\frac{1}{2}} \|\nabla u\|_{L^2}^\frac{1}{2}\]
\[\leq C(\tilde{\rho})\left(\|\nabla u\|_{L^2}^2\right)^{3/4} (\|\rho \dot{u}\|_{L^2}^2)^{\frac{1}{4}} \|\nabla u\|_{L^2} + C C_0^{\frac{1}{2}} (\|\nabla u\|_{L^2}^2)^{\frac{1}{2}} \|\nabla u\|_{L^2}\]
\[+ C(\|\nabla u\|_{L^2}) \|\nabla u\|_{L^2} + C(\tilde{\rho})C_0^{\frac{1}{2}} \|\nabla u\|_{L^2},\]
which along with (3.2) and (3.11) yields
\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 \, dx \leq C(\tilde{\rho}, M) \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 \, dt\right)^\frac{1}{2} + \int \rho_0 |u_0|^3 \, dx + CC_0 + CC_0^{\frac{1}{2}} \tag{3.48}
\]
\[\leq C(\tilde{\rho}, M)C_0^{\frac{1}{2}},\]
provided $C_0 < \varepsilon$, where in the last inequality we have used the simple fact
\[
\int \rho_0 |u_0|^3 \, dx \leq C(\tilde{\rho})\|\rho_0 \dot{u}_0\|_{L^2}^{\frac{3}{2}} \|\nabla u_0\|_{L^2}^{\frac{3}{2}} \leq C(\tilde{\rho}, M)C_0^{1/2}. \tag{3.49}
\]

Now let $\varepsilon_2 \overset{\Delta}{=} \min\{\varepsilon, (C(\tilde{\rho}, M))^{-1}\}$, and we establish (3.47). \qed

**Lemma 3.6** If $(\rho, u)$ is a smooth solution of (1.1) - (1.5) on $\Omega \times (0, T]$ satisfying (3.3), then
\[
\int_0^T \sigma^3 \|P - P(\rho_\infty)\|_{L^4}^4 \, dt \leq C(\tilde{\rho})C_0. \tag{3.50}
\]

**Proof.** It follows from (1.1) that $P - P(\rho_\infty)$ satisfies
\[
(P - P(\rho_\infty))_t + u \cdot \nabla (P - P(\rho_\infty)) + \gamma (P - P(\rho_\infty)) \text{div} u + \gamma P(\rho_\infty) \text{div} u = 0. \tag{3.51}
\]
Multiplying (3.51) by $3(P - P(\rho_\infty))^2$ and integrating over $\Omega$, after using $\text{div} u = \frac{1}{2p+\lambda} (F + P - P(\rho_\infty))$ and (2.18), we get
\[
3\gamma - \frac{1}{2\mu + \lambda} \|P - P(\rho_\infty)\|_{L^4}^4
\]
\[= - \left(\int (P - P(\rho_\infty))^3 \, dx\right)_t - \frac{3\gamma - 1}{2\mu + \lambda} \int (P - P(\rho_\infty))^3 \, F \, dx
\]
\[- 3\gamma P(\rho_\infty) \int (P - P(\rho_\infty))^2 \, \text{div} u \, dx\]
\[\leq - \left(\|P - P(\rho_\infty)\|_{L^3}^3 \right)_t + \delta \|P - P(\rho_\infty)\|_{L^4}^4 + C(\delta) \|F\|_{L^4}^4 + C(\delta) \|\nabla u\|_{L^2}^2
\]
\[\leq - \left(\|P - P(\rho_\infty)\|_{L^3}^3 \right)_t + \delta \|P - P(\rho_\infty)\|_{L^4}^4 + C(\delta) \|\rho \dot{u}\|_{L^2}^3 \|\nabla u\|_{L^2}
\]
\[+ C(\delta) \|\rho \dot{u}\|_{L^2}^4 \|P - P(\rho_\infty)\|_{L^2} + C(\delta) \|\nabla u\|_{L^2}^2.\]
Multiplying (3.52) by \( \sigma^3 \), then integrating over \((0,T)\), and choosing \(\delta\) suitably small, by (3.9), we obtain
\[
\int_0^T \sigma^3 \|P - P(\rho)\|_{L^4} dt \\
\leq C \sup_{0 \leq t \leq T} \|P - P(\rho)\|_{L^3}^3 + C \int_0^T \|P - P(\rho)\|_{L^3}^3 dt \\
+ C(\bar{\rho}) \int_0^T \sigma^3 \|\rho \bar{u}\|_{L^2}^3 \|\nabla u\|_{L^2} ds + C \int_0^T \sigma^3 \|\rho \bar{u}\|_{L^2}^3 \|P - P(\rho)\|_{L^2} ds + C(\bar{\rho})C_0 \\
\leq C(\bar{\rho}) \sup_{t \in (0,T)} \left( \sigma^{3/2} \|ho \bar{u}\|_{L^2} \right) \left( \sigma^{1/2} \|\nabla u\|_{L^2} \right) \int_0^T \sigma \|\rho \bar{u}\|_{L^2}^2 ds \\
+ C \sup_{t \in (0,T)} \left( \sigma^3 \|\rho \bar{u}\|_{L^2} \right) \int_0^T \frac{1}{2} \sigma \|\rho \bar{u}\|_{L^2}^2 ds + C(\bar{\rho})C_0 \\
\leq C(\bar{\rho})C_0.
\]
This completes the proof. 

\[ \square \]

**Lemma 3.7** There exists a positive constant \( \varepsilon_3 (\leq \varepsilon_2) \) depending only on \( \mu, \lambda, \gamma, a, \bar{\rho}, \rho_{\infty}, \Omega \) and \( M \) such that, if \((\rho, u)\) is a smooth solution of (1.1) - (1.5) on \( \Omega \times (0,T) \) satisfying (3.9) and \( \|\nabla u_0\|_{L^2} \leq M \), then
\[
A_1(T) + A_2(T) \leq C_0^{1/2},
\]  
provided \( C_0 \leq \varepsilon_3. \)

**Proof.** By Lemma 3.3 it indicates that
\[
A_1(T) + A_2(T) \leq C(\bar{\rho})C_0 + C(\bar{\rho}) \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 ds + C(\bar{\rho}) \int_0^T \sigma \|\nabla u\|_{L^3}^3 ds. \tag{3.54}
\]
Due to (2.20), (2.18), (2.19) and Lemma 3.6
\[
\int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 ds \\
\leq C \int_0^T \sigma^3 \|\nabla u\|_{L^2}^4 \|\rho \bar{u}\|_{L^2}^3 ds + C \int_0^T \sigma^3 \|\nabla u\|_{L^2}^2 ds \\
+ C \int_0^T \sigma^3 \|P - P(\rho_{\infty})\|_{L^6}^6 ds + C \int_0^T \sigma^3 \|\nabla u\|_{L^2}^4 ds \\
\leq C(\bar{\rho}) \sup_{t \in (0,T)} \left( \sigma^{3/2} \|\rho \bar{u}\|_{L^2} \right) \left( \sigma^{1/2} \|\nabla u\|_{L^2} \right) \int_0^T \int \sigma \|\rho \bar{u}\|^2 dx ds \\
+ C(\bar{\rho}) \int_0^T \sigma^3 \|P - P(\rho_{\infty})\|_{L^4}^4 ds + C \int_0^T (\sigma \|\nabla u\|_{L^2}^2) \sigma^2 \|\nabla u\|_{L^2}^2 ds \\
\leq C(\bar{\rho}) A_2^{1/2}(T) A_1^{1/2}(T) A_1(T) + CC_0 \\
\leq C(\bar{\rho})C_0.
\]  
(3.55)

Finally, we shall give the estimate of the last term on the right hand side of (3.54). By (3.55), it is easy to check that
\[
\int_{\sigma(T)}^T \int \sigma \|\nabla u\|_{L^2}^3 dx ds \leq \int_{\sigma(T)}^T \int (|\nabla u|^4 + |\nabla u|^2) dx ds \leq CC_0. \tag{3.56}
\]
Next, we deduce from (2.20), (3.41) and (3.9) that if \( C_0 \leq \varepsilon \), then
\[
\int_0^{\sigma(T)} \sigma \| \nabla u \|_{L^3}^3 dt \\
\leq C(\bar{\rho}) \int_0^{\sigma(T)} t \| \nabla u \|_{L^2}^{3/2} \left( \| \rho \dot{u} \|_{L^2}^{3/2} + C_0^{1/4} \right) dt + C \int_0^{\sigma(T)} \sigma \| \nabla u \|_{L^2}^3 dt \\
\leq C(\bar{\rho}) \int_0^{\sigma(T)} \| \nabla u \|_{L^2} \| \nabla u \|_{L^2}^{1/2} \left( t \int \rho |\dot{u}|^2 dx \right)^{3/4} dt + C(\bar{\rho})C_0 \\
\leq C(\bar{\rho}) \sup_{t \in (0, \sigma(T)]} \| \nabla u \|_{L^2} \int_0^{\sigma(T)} \| \nabla u \|_{L^2}^{1/2} \left( t \int \rho |\dot{u}|^2 dx \right)^{3/4} dt + C(\bar{\rho})C_0 \\
\leq C(\bar{\rho}, M)A_1^{3/4}C_0^{1/4} + C(\bar{\rho})C_0 \\
\leq C(\bar{\rho}, M)C_0^{5/8}.
\] (3.57)

Taking
\[
\varepsilon_3 \equiv \min \left\{ \varepsilon_2, (C(\bar{\rho}, M))^{-8} \right\},
\]

By (3.54) and (3.55)-(3.57), we immediately obtain (3.53) in the case \( C_0 < \varepsilon_3 \). \( \square \)

It is time to derive a uniform (in time) upper bound for the density, which plays a crucial role in deriving all the higher order estimates and thus extending the local classical solution to be a global one. The method we are going to employ can be found in [29] [30].

**Lemma 3.8** Let \((\rho, u)\) be a smooth solution of (1.1)–(1.5) on \( \Omega \times (0, T] \) satisfying (3.9) and \( \| \nabla u_0 \|_{L^2} \leq M \), then there exists a positive constant \( \varepsilon \) as described in Theorem 1.1 depending only on \( \mu, \lambda, \gamma, \alpha, \bar{\rho}, \rho_\infty, \Omega \) and \( M \) such that,
\[
\sup_{0 \leq t \leq T} \| \rho(t) \|_{L^\infty} \leq \frac{7\bar{\rho}}{4},
\]

provided \( C_0 \leq \varepsilon \).

**Proof.** Denote
\[
D_t\rho \equiv \rho_t + u \cdot \nabla \rho, \quad g(\rho) \equiv -\frac{\rho(P - P(\rho_\infty))}{2\mu + \lambda}, \quad b(t) \equiv -\frac{1}{2\mu + \lambda} \int_0^t \rho F dt,
\]
then (1.1) can be rewritten as
\[
D_t\rho = g(\rho) + b'(t).
\] (3.58)

In order to finish the proof, by Lemma 2.3 it is sufficient to check that the function \( b(t) \) must verify (2.3) with some suitable constants \( N_0, N_1 \).

Let \( \varepsilon_3 \) be given in Lemma 3.7. One deduces from Gagliardo-Nirenberg’s inequality,
Lemma 2.3 yields that
\[
|b(t_2) - b(t_1)| \leq C \int_0^{\sigma(T)} \| F(\cdot, t) \|_{L^\infty} dt
\]
\[
\leq C(\bar{\rho}) \int_0^{\sigma(T)} \| F(\cdot, t) \|^{1/2}_{L^6} \| \nabla F(\cdot, t) \|^{1/2}_{L^6} dt
\]
\[
\leq C(\bar{\rho}) \int_0^{\sigma(T)} \| \tilde{\rho} \|^{1/2}_{L^2} \| \nabla \tilde{u} \|^{1/2}_{L^2} dt
\]
\[
\leq C(\bar{\rho}) \int_0^{\sigma(T)} t^{-\frac{\alpha}{2\mu}} \left( t \| \rho \|_{L^2}^2 \right)^{\frac{1}{2}} \left( t \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{4}} dt
\]
\[
\leq C(\bar{\rho}, M) \left( \int_0^{\sigma(T)} t^{-\frac{\alpha}{2\mu}} \left( t \| \rho \|_{L^2}^2 \right)^{\frac{1}{2}} dt \right)^{3/4}
\]
\[
\leq C(\bar{\rho}, M) (A_1(\sigma(T)))^{\frac{3}{8}}
\]
\[
\leq C(\bar{\rho}, M) C_0^{\frac{3}{8}}.
\]
Therefore, for \( t \in [0, \sigma(T)] \), one can choose \( N_0 \) and \( N_1 \) in (2.3) as follows:
\[
N_1 = 0, \quad N_0 = C(\bar{\rho}, M) C_0^{\frac{3}{8}},
\]
and \( \bar{\zeta} = \rho_\infty \) in (2.4). It is easy to check that
\[
g(\zeta) = -\frac{a_{\zeta}}{2\mu + \lambda} (\zeta^2 - \rho_\infty) \leq -N_1 = 0, \quad \text{for all} \quad \zeta \geq \bar{\zeta} = \rho_\infty.
\]
So Lemma 2.3 yields that
\[
\sup_{t \in [0, \sigma(T)]} \| \rho \|_{L^\infty} \leq \max\{ \bar{\rho}, \rho_\infty \} + N_0 \leq \bar{\rho} + C(\bar{\rho}, M) C_0^{\frac{3}{8}} \leq \frac{3\bar{\rho}}{2}, \quad (3.59)
\]
provided
\[
C_0 \leq \min\{ \varepsilon_3, \varepsilon_4 \}, \quad \text{for} \quad \varepsilon_4 \triangleq \left( \frac{\bar{\rho}}{2C(\bar{\rho}, M)} \right)^{\frac{3}{8}}.
\]
On the other hand, by Gagliardo-Nirenberg’s inequality, (3.9) and (2.10), for all \( \sigma(T) \leq t_1 \leq t_2 \leq T \),
\[
|b(t_2) - b(t_1)| \leq C(\bar{\rho}) \int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^\infty} dt
\]
\[
\leq \frac{a}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \int_{\sigma(T)}^{T} \| F(\cdot, t) \|_{L^\infty}^4 dt
\]
\[
\leq \frac{a}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \int_{\sigma(T)}^{T} \| F(\cdot, t) \|_{L^6}^2 \| \nabla F(\cdot, t) \|_{L^6}^2 dt
\]
\[
\leq \frac{a}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \int_{\sigma(T)}^{T} \| \rho \|_{L^2}^2 \| \nabla u(\cdot, t) \|_{L^2}^2 dt
\]
\[
\leq \frac{a}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) C_0,
\]
Therefore, one can choose \( N_1 \) and \( N_0 \) in (2.3) as:

\[
N_1 = \frac{a}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho})C_0.
\]

Note that

\[
g(\zeta) = -\frac{a\zeta}{2\mu + \lambda}(\zeta^\gamma - \rho_\infty^\gamma) \leq -N_1 = -\frac{a}{2\mu + \lambda}, \quad \text{for all} \quad \zeta \geq \rho_\infty + 1.
\]

So one can set \( \bar{\zeta} = \rho_\infty + 1 \) in (2.4). By Lemma 2.3 and (3.59), we get

\[
\sup_{t \in [\sigma(T), T]} \|\rho\|_{L^\infty} \leq \max \left\{ \frac{3\bar{\rho}}{2}, \rho_\infty + 1 \right\} + N_0 \leq \frac{3\bar{\rho}}{2} + C(\bar{\rho})C_0 \leq \frac{7\bar{\rho}}{4},
\]

(3.60)

provided

\[
C_0 \leq \varepsilon \triangleq \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}, \quad \text{for} \quad \varepsilon_5 \triangleq \frac{\bar{\rho}}{4C(\bar{\rho})}. \tag{3.61}
\]

Combining (3.59) and (3.60), we finish the proof of Lemma 3.8.

4 A priori estimates (II): higher order estimates

From now on, we assume that the initial energy \( C_0 \) always satisfies (3.61) and the positive constant \( C \) may depend on \( T, \|g\|_{L^2}, \|\nabla u_0\|_{H^1}, \|\rho_0 - \rho_\infty\|_{H^2 \cap W^{2,q}}, \|P(\rho_0) - P(\rho_\infty)\|_{H^2 \cap W^{2,q}} \)

besides \( \mu, \lambda, \rho_\infty, a, \gamma, \bar{\rho}, \Omega \) and \( M \), where \( g \in L^2(\Omega) \) is given by the compatibility condition (1.10). This section is devoted to some necessary higher order estimates of the smooth solution \((\rho, u)\) which will make sure that the local existence of classical solution can be extended globally in time. The methods used are mainly from [30]. For completeness, we still write it down in detail.

**Lemma 4.1** There exists a positive constant \( C \), such that

\[
\sup_{0 \leq t \leq T} \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2} + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \leq C, \tag{4.1}
\]

\[
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2 \cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty} dt \leq C. \tag{4.2}
\]

**Proof.** First, by (3.41) and (3.53), we obtain

\[
\sup_{t \in [0, T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 dxdt \leq C. \tag{4.3}
\]

Choosing \( m = 0 \) in (3.40), we deduce from (2.20), (3.2) and (4.3) that

\[
\left( \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 \right)_t + \|\nabla \dot{u}\|_{L^2}^2 \leq C(\|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + 1) \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + C - \left( \int_{\partial \Omega} (u \cdot \nabla u \cdot u) F ds \right)_t. \tag{4.4}
\]
As a result, by Gronwall’s inequality, together with the compatibility condition (1.10), (4.3), (4.4) and (3.32), we give (4.11).

Observe that for $2 \leq p \leq 6$, $\|\nabla \rho \|^p$ satisfies

$$
\begin{aligned}
&\int (|\nabla \rho|^p \rangle + \div(|\nabla \rho|^p u) + (p - 1)|\nabla \rho|^p \div \nu u
\end{aligned}
$$

$$
+ p|\nabla \rho|^{p-2}(\nabla \rho)^* \nabla u(\nabla \rho) + p|\nabla \rho|^{p-2} \nabla \rho \cdot \div \nu u = 0.
$$

Thus, due to (2.16),

$$
(\|\nabla \rho\|_{L^p})^t \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C\|\nabla F\|_{L^p}
\leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C\|\rho \dot{u}\|_{L^p}. \tag{4.5}
$$

On the other hand, by Gagliardo-Nirenberg’s inequality, we deduce from (3.2), (4.3), (4.10), (4.12) and (4.7),

$$
\|\nabla^2 u\|_{L^p} \leq C(\|\nabla u\|_{L^\infty} + \|\curl u\|_{L^\infty})
\leq C(\|\nabla \rho \|^p + \|\rho \dot{u}\|_{L^2} + \|P - P(\rho\infty)\|_{L^\infty})
\leq C(\|\nabla \rho \|^p + \|\rho \dot{u}\|_{L^2} + \|P - P(\rho\infty)\|_{L^\infty})
\leq C(\rho)(\|\nabla \rho\|_{L^p}^2 + 1).
$$

Moreover, by Lemma 2.9 and (2.21)-(2.24), we have for $p \in [2,6],$

$$
\|\nabla^2 u\|_{L^p} \leq C(\|\nabla u\|_{L^\infty} + \|\curl u\|_{L^\infty})
\leq C(\|\rho \dot{u}\|_{L^p} + \|\rho \dot{u}\|_{L^2} + \|P - P(\rho\infty)\|_{W^{1,p}})
\leq C(\|\nabla \rho\|_{L^p} + \|\rho \dot{u}\|_{L^2} + \|P - P(\rho\infty)\|_{L^\infty})
\leq C(\rho)(\|\nabla \rho\|_{L^p}^2 + 1), \tag{4.6}
$$

which, along with Lemma 2.10 (4.1), and (4.6) yields

$$
\|\nabla u\|_{L^\infty} \leq C(\|\nabla \rho\|_{L^p} + \|\nabla u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^6}) + C\|\nabla u\|_{L^2} + C
\leq C(1 + \|\nabla u\|_{L^2}) \log(e + \|\nabla^2 u\|_{L^6}) + C\|\nabla \rho\|_{L^6}
\leq C(1 + \|\nabla \rho\|_{L^6}^2) + C(1 + \|\nabla u\|_{L^2}) \log(e + \|\nabla \rho\|_{L^6}). \tag{4.8}
$$

Hence,

$$
(\log(e + \|\nabla \rho\|_{L^6}))^t \leq C(\rho)(1 + \|\nabla \rho\|_{L^6}^2) + C(\rho)(1 + \|\nabla \rho\|_{L^6}) \log(e + \|\nabla \rho\|_{L^6}). \tag{4.9}
$$

And then by Gronwall’s inequality and (4.11), we find that

$$
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} \leq C. \tag{4.10}
$$

Moreover, by (4.8),

$$
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C. \tag{4.11}
$$

Taking $p = 2$ in (4.5), by Gronwall’s inequality, together with (4.3) and (4.11), we check that

$$
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2} \leq C. \tag{4.12}
$$

Furthermore, by (4.11), (4.3), (4.10), (4.12) and (4.7),

$$
\int_0^T \|\nabla^2 u\|_{L^6} dt \leq C, \sup_{0 \leq t \leq T} \|\nabla^2 u\|_{L^2} \leq C. \tag{4.13}
$$

So we finish the proof of Lemma 4.1. \qed
Lemma 4.2 There exists a constant $C$ such that

$$\sup_{0 \leq t \leq T} \| \rho^{\frac{1}{2}} u_t \|_{L^2}^2 + \int_0^T \int |\nabla u_t|^2 dx dt \leq C, \quad (4.14)$$

$$\sup_{0 \leq t \leq T} (\| \rho - \rho_\infty \|_{H^2} + \| P - P(\rho_\infty) \|_{H^2}) \leq C, \quad (4.15)$$

Proof. By Lemma 4.1, we have

$$\| \rho^{\frac{1}{2}} u_t \|_{L^2}^2 \leq \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 + \| \rho^{\frac{1}{2}} u \cdot \nabla u \|_{L^2}^2 \leq C + C \| \rho^{\frac{1}{2}} u \|_{L^2} \| u \|_{L^6} \| \nabla u \|_{L^6}^2 \quad (4.16)$$

and

$$\int_0^T \| \nabla u_t \|_{L^2}^2 dt \leq \int_0^T \| \nabla \dot{u} \|_{L^2}^2 dt + \int_0^T \| \nabla (u \cdot \nabla u) \|_{L^2}^2 dt \leq C + \int_0^T \| \nabla u \|_{L^4}^2 + \| u \|_{L_\infty}^2 \| \nabla^2 u \|_{L^2}^2 dt \quad (4.17)$$

which give (4.14).

Notice that $P$ satisfies

$$P_t + u \cdot \nabla P + \gamma P \text{div} u = 0, \quad (4.18)$$

and by (1.1), Lemma 4.1, (1.7) and (4.20), one has

$$\frac{d}{dt} (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) \leq C (1 + \| \nabla u \|_{L^\infty}^2) (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) + C \| \nabla \dot{u} \|_{L^2}^2 + C, \quad (4.19)$$

where we have used the following simple fact that for $p \in [2, 6]$,

$$\| \nabla^2 u \|_{L^p} \leq C (\| \text{div} u \|_{W^{2,p}} + \| \text{curl} u \|_{W^{2,p}}) \leq C (\| \rho \dot{u} \|_{W^{1,p}} + \| P - P(\rho_\infty) \|_{W^{2,p}} + \| \rho \dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| P - P(\rho_\infty) \|_{L^2}), \quad (4.20)$$

due to Lemma 2.3 and (2.21)-(2.24). Then, by Gronwall’s inequality, it follows from (4.19), (1.11) and Lemma 4.1 that

$$\sup_{0 \leq t \leq T} (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) \leq C.$$ 

And the proof is completed. \qed

Lemma 4.3 There exists a positive constant $C$ such that

$$\sup_{0 \leq t \leq T} (\| \rho_t \|_{H^1} + \| \rho_\infty \|_{H^1}) + \int_0^T (\| \rho \dot{u} \|_{L^2}^2 + \| P_t \|_{L^2}^2) dt \leq C, \quad (4.21)$$

$$\sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{L^2}^2 + \int_0^T \sigma \| \rho^{\frac{1}{2}} u_{tt} \|_{L^2}^2 dt \leq C. \quad (4.22)$$
\textbf{Proof.} By (4.18) and Lemma 4.1
\[ \|P_t\|_{L^2} \leq C\|u\|_{L^\infty} \|\nabla P\|_{L^2} + C\|\nabla u\|_{L^2} \leq C. \] (4.23)

Differentiating (4.18) leads to
\[ \nabla P_t + u \cdot \nabla \nabla P + \nabla u \cdot \nabla P + \gamma \nabla P \text{div} u + \gamma P \nabla \text{div} u = 0. \]

Hence, by Lemmas 4.1 and 4.2
\[ \|\nabla P_t\|_{L^2} \leq C\|u\|_{L^\infty} \|\nabla^2 P\|_{L^2} + C\|\nabla u\|_{L^3} \|\nabla P\|_{L^6} + C\|\nabla^2 u\|_{L^2} \leq C, \] (4.24)

which together with (4.23) implies
\[ \text{sup}_{0 \leq t \leq T} \|P_t\|_{H^1} \leq C. \] (4.25)

By (4.18) again, it is easy to check that \( P_{tt} \) satisfies
\[ P_{tt} + \gamma P_t \text{div} u + \gamma P \text{div} u_t + u_t \cdot \nabla P + u \cdot \nabla P_t = 0. \] (4.26)

Multiplying (4.26) by \( P_{tt} \) and integrating over \( \Omega \times [0, T] \), we deduce from (4.25), Lemmas 4.1 and 4.2 that
\[ \int_0^T \|P_{tt}\|^2_{L^2} dt \]
\[ = -\int_0^T \int \gamma P_{tt} P_t \text{div} u dx dt - \int_0^T \int \gamma P_{tt} P \text{div} u_t dx dt \]
\[ - \int_0^T \int P_{tt} u_t \cdot \nabla P dx dt - \int_0^T \int P_{tt} u \cdot \nabla P_t dx dt \]
\[ \leq C \int_0^T \|P_t\|_{L^2}(\|P_t\|_{L^3}\|\nabla u\|_{L^6} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}\|\nabla P\|_{L^3} + \|u\|_{L^\infty}\|\nabla P_t\|_{L^2}) dt \]
\[ \leq C \int_0^T \|P_t\|_{L^2}(1 + \|\nabla u_t\|_{L^2}) dt \]
\[ \leq \frac{1}{2} \int_0^T \|P_{tt}\|^2_{L^2} dt + C \int_0^T \|\nabla u_t\|^2_{L^2} dt + C \]
\[ \leq \frac{1}{2} \int_0^T \|P_{tt}\|^2_{L^2} dt + C, \]

which gives \( \int_0^T \|P_{tt}\|^2_{L^2} dt \leq C. \)

One can deal with \( \rho_t \) and \( \rho_{tt} \) similarly. Thus (4.21) is proved.

It remains to prove (4.22). First, introducing the function
\[ H(t) = (\lambda + 2\mu) \int (\text{div} u_t)^2 dx + \mu \int |\text{curl} u_t|^2 dx. \]

Since \( u_t \cdot n = 0 \) on \( \partial \Omega \), by Lemma 2.7, we have
\[ \|\nabla u_t\|^2_{L^2} \leq C(\Omega)H(t). \] (4.27)
Differentiating (1.1) with respect to $t$, then multiplying by $u_{tt}$, yield that
\[
\frac{d}{dt} \left( (\lambda + 2\mu) \int (\text{div} u_t)^2 dx + \mu \int |\text{curl} u_t|^2 dx \right) + 2 \int \rho |u_{tt}|^2 dx
= \frac{d}{dt} \left( -\int \rho_t |u_t|^2 dx - 2 \int \rho_t \cdot \nabla u \cdot u_t dx + 2 \int P \text{div} u_t dx \right) \\
+ \int \rho_{tt} |u_t|^2 dx + 2 \int (\rho_t \cdot \nabla u)_t \cdot u_t dx - 2 \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \\
- 2 \int \rho u \cdot \nabla u_t \cdot u_{tt} dx - 2 \int P_{tt} \text{div} u_t dx
\]
\[
(4.28)
\]
Now we have to estimate $I_i$, $i = 0, 1, \cdots, 5$.

It follows from (1.11), (1.12), (4.28), (4.33), (1.14) and (1.21) that
\[
|I_0| = \left| -\int \rho_t |u_t|^2 dx - 2 \int \rho_t \cdot \nabla u \cdot u_t dx + 2 \int P \text{div} u_t dx \right| \\
\leq \left| \int \text{div}(\rho u_t) |u_t|^2 dx \right| + C \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6} + C \|\nabla u_t\|_{L^2}
\leq C \int \rho |u_t| |\nabla u_t| dx + C \|\nabla u_t\|_{L^2}
(4.29)
\]
\[
\leq C \|u\|_{L^6} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}
\leq C \|\nabla u\|_{L^2} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{3/2} + C \|\nabla u_t\|_{L^2}
\leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta),
\]
\[
|I_1| = \left| \int \rho_{tt} |u_t|^2 dx \right| = \left| \int \text{div}(\rho u_t) |u_t|^2 dx \right|
= 2 \left| \int (\rho_t u + \rho u_t) \cdot \nabla u_t \cdot u_t dx \right|
\leq C \left( \|\rho_t\|_{H^1} \|u\|_{H^2} + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2} \right) \|\nabla u_t\|_{L^2}^2
(4.30)
\]
\[
\leq C \|\nabla u_t\|_{L^2}^4 + C
\leq C \|\nabla u_t\|_{L^2}^4 H(t) + C
\]
\[
|I_2| = 2 \left| \int (\rho u \cdot \nabla u)_t \cdot u_t dx \right|
= 2 \left| \int \rho_{tt} u \cdot \nabla u \cdot u_t + \rho u_{tt} \cdot \nabla u \cdot u_t + \rho u_t \cdot \nabla u_t \cdot u_t dx \right|
\leq \|\rho_{tt}\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6} + \|\rho_{tt}\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \|u_t\|_{L^6}
+ \|\rho u_{tt}\|_{L^2} \|u\|_{L^\infty} \|u_t\|_{L^2} \|u_t\|_{L^6}
\leq C \|\rho_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2,
(4.31)
\]
\[
|I_3| + |I_4| = 2 \left| \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx \right| + 2 \left| \int \rho u \cdot \nabla u_t \cdot u_{tt} dx \right|
\leq C \|\rho^{1/2} u_{tt}\|_{L^2} (\|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2})
\leq \delta \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^2,
(4.32)
\]
and
\[ |I_5| = 2 \left| \int P_t \operatorname{div} u_t \, dx \right| \]
\[ \leq C \| P_t \|_{L^2} \| \operatorname{div} u_t \|_{L^2} \]
\[ \leq C(\| P_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2). \tag{4.33} \]

Therefore, together with these estimates above, we have

\[ \frac{d}{dt} (\sigma H(t) - \sigma I_0) + \sigma \int \rho |u_t|^2 \, dx \]
\[ \leq C(1 + \| \nabla u_t \|_{L^2}^2) \sigma H(t) + C(1 + \| \nabla u_t \|_{L^2}^2 + \| \rho u_t \|_{L^2}^2 + \| P_t \|_{L^2}^2), \]

By Gronwall’s inequality, together with (4.14), (4.21) and (4.29), and choose \( \delta > 0 \), such that \( C(\Omega) \delta < \frac{1}{4} \), where \( C(\Omega) \) is given by (4.27), we get

\[ \sup_{0 \leq t \leq T} \sigma H(t) + \int_0^T \sigma \| \rho^{\frac{1}{2}} u_t \|_{L^2}^2 \, dt \leq C. \]

Hence, by (4.27),

\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{L^2}^2 + \int_0^T \sigma \| \rho^{\frac{1}{2}} u_t \|_{L^2}^2 \, dt \leq C. \]  \[ \blacksquare \]

**Lemma 4.4** It holds that for any \( q \in (3, 6) \), there exists a positive constant \( C \) such that

\[ \sup_{t \in [0, T]} (\| \rho - \rho_\infty \|_{W^{2,q}} + \| P - P(\rho_\infty) \|_{W^{2,q}}) \leq C, \tag{4.34} \]

\[ \sup_{t \in [0, T]} \sigma \| \nabla u \|_{H^2}^2 + \int_0^T (\| \nabla u \|_{H^2}^2 + \| \nabla^2 u \|_{W^{1,q}}^2 + \sigma \| \nabla u_t \|_{H^1}^2) \, dt \leq C, \tag{4.35} \]

where \( p_0 = \frac{9q-6}{6q-12} \in (1, \frac{7}{2}) \).

**Proof.** By (4.15) and Lemma 4.1,

\[ \| \nabla^2 u \|_{H^1} \leq C(\| \rho \hat{u} \|_{H^1} + \| P - \bar{P} \|_{H^2} + \| u \|_{L^2}) \]
\[ \leq C + C \| \nabla u_t \|_{L^2}, \tag{4.36} \]

where we have use the fact that

\[ \| \nabla (\rho \hat{u}) \|_{L^2} \leq \| | \rho | \| u_t \|_{L^2} + \| \rho \nabla u_t \|_{L^2} + \| \nabla | \rho | \| u \| \| \nabla u \|_{L^2} \]
\[ + \| \rho \|_{L^2} \| \nabla u \|_{L^2}^2 + \| \rho \|_{L^2} \| \nabla^2 u \|_{L^2}^2 \]
\[ \leq \| \nabla \rho \|_{L^3} \| u_t \|_{L^6} + C \| \nabla u_t \|_{L^2} + C \| \nabla \rho \|_{L^5} \| u \|_{L^\infty} \| \nabla u \|_{L^6} \]
\[ + C \| \nabla u \|_{L^3} \| \nabla u \|_{L^6} + C \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2} \]
\[ \leq C + C \| \nabla u_t \|_{L^2}. \tag{4.37} \]

So we deduce from (4.36), (4.2), (4.14), and (4.22) that

\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u \|_{H^2}^2 + \int_0^T \| \nabla u \|_{H^2}^2 \, dt \leq C. \tag{4.38} \]
By Lemma \eqref{4.4} and \eqref{4.5}, we have
\begin{align*}
\|\nabla^2 u_t\|_{L^2} &\leq C(\|\nabla (\rho \dot{u})\|_{L^2} + \|\nabla P_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&= C(\|\rho u_t + \rho_t u_t + \rho_t u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2} + \|\nabla u_t\|_{L^2}) + C \\
&\leq C(\|\rho u_t\|_{L^2} + \|\rho_t\|_{L^3}\|u_t\|_{L^6} + \|\rho_t\|_{L^3}||u_t||_{L^6}) \\
&\quad + C(\|u_t||_{L^6}\|\nabla u_t\|_{L^2} + ||u_t||_{L^\infty}||\nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&\leq C(\|\rho^2 u_t\|_{L^2} + C\|\nabla u_t\|_{L^2} + C),
\end{align*}
where in the first inequality, we have utilized a priori estimate similar to \eqref{4.8} since
\begin{align}
\left\{\begin{array}{ll}
\mu \Delta u_t + (\lambda + \mu) \nabla \text{div} u_t = (\rho \dot{u})_t + \nabla P_t & \text{in } \Omega, \\
u_t \cdot n = 0 \text{ and curl } u_t \times n = 0 & \text{on } \partial \Omega.
\end{array}\right.
\end{align}

Hence, due to \eqref{4.22},
\begin{align}
\int_0^T \sigma \|\nabla u_t\|_{H^1}^2 dt \leq C.
\end{align}

By Gagliardo-Nirenberg’s inequality, \eqref{4.2}, \eqref{4.15} and \eqref{4.22}, we conclude that for any $q \in (3, 6)$,
\begin{align}
\|\nabla (\rho \dot{u})\|_{L^q} &\leq C\|\nabla \rho\|_{L^\infty}(\|\nabla \dot{u}\|_{L^q} + \|\nabla u_t\|_{L^2}) + C\|\nabla \dot{u}\|_{L^q} \\
&\leq C\|\nabla u_t\|_{L^2} + C(\|\nabla u_t\|_{L^q} + \|\nabla (u \cdot \nabla u)\|_{L^q}) \\
&\leq C\|\nabla u_t\|_{L^2} + C + C\|\nabla u_t\|_{L^2}^{\frac{6-q}{L^2}}\|\nabla u_t\|_{L^6}^{\frac{3(q-2)}{L^6}} \\
&\quad + C(\|u\|_{L^\infty}\|\nabla^2 u\|_{L^q} + \|\nabla u\|_{H^1}\|\nabla u\|_{H^2}) \\
&\leq C\sigma^{-\frac{1}{2}} + C\|\nabla u\|_{H^2} + C\sigma^{-\frac{1}{2}}(\sigma\|\nabla u_t\|_{H^1}^2)^{\frac{3(q-2)}{4q} + C}.
\end{align}

Integrating this inequality over $[0, T]$, by \eqref{4.1} and \eqref{4.41}, we obtain
\begin{align}
\int_0^T \|\nabla (\rho \dot{u})\|_{L^q}^{p_0} dt \leq C.
\end{align}

On the other hand, by \eqref{4.12}, \eqref{4.20}, \eqref{4.11} and \eqref{4.15},
\begin{align}
\|\nabla^2 u\|_{W^{1,q}} &\leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla (\rho \dot{u})\|_{L^q} + \|\nabla^2 P\|_{L^q} + \|\nabla P\|_{L^q} \\
&\quad + \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|P - \bar{P}\|_{L^q}) \\
&\leq C(1 + \|\nabla u\|_{L^q} + \|\nabla (\rho \dot{u})\|_{L^q} + \|\nabla^2 P\|_{L^q}),
\end{align}

Together with \eqref{4.16} and \eqref{4.15}, which yields that
\begin{align}
\left(\|\nabla^2 P\|_{L^q}\right)_t &\leq C\|\nabla u\|_{L^\infty}\|\nabla^2 P\|_{L^q} + C\|\nabla^2 u\|_{W^{1,q}} \\
&\leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla^2 P\|_{L^q} + C(1 + \|\nabla u_t\|_{L^2}) \\
&\quad + C\|\nabla (\rho \dot{u})\|_{L^q},
\end{align}

Hence, by Gronwall’s inequality, it follows from \eqref{4.2}, \eqref{4.14} and \eqref{4.43} that
\begin{align}
\sup_{t \in [0,T]} \|\nabla^2 P\|_{L^q} \leq C,
\end{align}

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which along with (4.14), (4.15), (4.44) and (4.43) also gives

\[
\sup_{t \in [0,T]} \|P - \tilde{P}\|_{W^{2,q}} + \int_0^T \|\nabla^2 u\|_{W^{1,q}}^2 \, dt \leq C. \tag{4.47}
\]

Similarly, we have

\[
\sup_{0 \leq t \leq T} \|\rho - \tilde{\rho}\|_{W^{2,q}} \leq C,
\]

which together with (4.47) leads to (4.34). Thus we finish the proof of Lemma 4.4.

\[\square\]

**Lemma 4.5** There exists a constant $C$ such that

\[
\sup_{0 \leq t \leq T} \|\nabla u_t\|_{H^1} + \|\nabla u\|_{W^{2,q}} \tag{4.48}
\]

for any $q \in (3,6)$.

**Proof.** Differentiating (1.12) with respect to $t$ twice gives

\[
\rho u_{tt} + \rho u \cdot \nabla u_t - (\lambda + 2\mu) \nabla \text{div} u_t + \mu \nabla \times \text{curl} u_t
\]

\[
= 2 \text{div}(\rho u) u_t + \text{div}(\rho u) u_t - 2(\rho u_t) - \nabla u_t - (\rho u_t + 2p_t u_t) \cdot \nabla u \tag{4.49}
\]

\[- \rho u_{tt} \cdot \nabla u - \nabla P_{tt}.
\]

Then, multiplying (4.49) by $2u_t$ and integrating over $\Omega$, it is easy to check that

\[
\frac{d}{dt} \int \rho |u_t|^2 \,dx + 2(\lambda + 2\mu) \int (\text{div} u_t)^2 \, dx + 2\mu \int |\text{curl} u_t|^2 \, dx
\]

\[
= -8 \int \rho u_t u \cdot \nabla u_t^2 \, dx - 2 \int \rho u_t \cdot (\nabla (u_t \cdot u_t) + 2\nabla u_t \cdot u_t) \, dx
\]

\[- 2 \int (\rho u_t + 2p_t u_t) \cdot \nabla u \cdot u_t \, dx - 2 \int \rho u_{tt} \cdot \nabla u \cdot u_t \, dx
\]

\[+ 2 \int P_{tt} \text{div} u_{tt} \, dx \triangleq \sum_{i=1}^5 J_i.
\]

Let us estimate $J_i (i = 1, \cdots, 5)$ one by one. For $J_1$, by Hölder’s and Young’s inequalities,

\[
|J_1| \leq C \|\rho^{1/2} u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|u\|_{L^\infty}
\]

\[\leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \|\rho^{1/2} u_t\|_{L^2}^2. \tag{4.51}
\]

It follows from (1.14), (2.21), (2.22) and (4.11) that

\[
|J_2| \leq C \left( \|\rho u_t\|_{L^3} + \|\rho_t u_t\|_{L^3} \right) \left( \|u_t\|_{L^6} \|\nabla u_t\|_{L^2} + \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \right)
\]

\[\leq C \left( \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} + \|\rho_t \|_{L^6} \|u_t\|_{L^6} \right) \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\]

\[\leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \sigma^{-3/2}. \tag{4.52}
\]

\[
|J_3| \leq C \left( \|\rho_t \|_{L^2} \|u_t\|_{L^\infty} \|\nabla u_t\|_{L^3} + \|\rho_t \|_{L^6} \|u_t\|_{L^6} \|\nabla u_t\|_{L^2} \right) \|u_t\|_{L^6}
\]

\[\leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \|\rho_t\|_{L^2}^2 + C(\delta) \sigma^{-1}, \tag{4.53}
\]

\[33\]
and
\[ |J_4| + |J_5| \leq C\|\rho u_{tt}\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6} + C\|P_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2} \]
\[ \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta)\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C(\delta)\|P_{tt}\|_{L^2}^2. \]  
(4.54)

Substituting these estimates of \( J_i \) \((i = 1, \cdots, 5)\) into (4.50), using the fact that
\[ \|\nabla u_{tt}\|_{L^2} \leq C(\|\text{div}u_{tt}\|_{L^2} + \|\text{curl}u_{tt}\|_{L^2}), \]  
(4.55)
since \( u_{tt} \cdot n = 0 \) on \( \partial\Omega \), and then choosing \( \delta \) small enough, we get
\[ \frac{d}{dt}\|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 \]
\[ \leq C\big(\|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2\big) + C\sigma^{-3/2}, \]  
(4.56)
which, together with (4.21), (4.22) and by Gronwall’s inequality shows that
\[ \sup_{0 \leq t \leq T} \sigma\|\rho_{tt}\|_{L^2}^2 + \int_0^T \sigma^2\|\nabla u_{tt}\|_{L^2}^2dt \leq C. \]  
(4.57)
Furthermore, it follows from (4.39) and (4.22) that
\[ \sup_{0 \leq t \leq T} \sigma\|\nabla u_t\|_{H^1}^2 \leq C. \]  
(4.58)

Finally, by (4.41), (4.42), (4.22), (4.34), (4.35), (4.57) and (4.58), we find that
\[ \sigma\|\nabla^2 u\|_{W^{1,q}} \leq C \left( \sigma + \sigma\|\nabla u_t\|_{L^2} + \sigma\|\nabla(\rho u_t)\|_{L^2} + \sigma\|\nabla^2 P\|_{L^2} \right) \]
\[ \leq C(\sigma + \sigma^{1/2} + \sigma\|\nabla u_t\|_{H^2} + \sigma^{1/2}(\sigma\|\nabla u_t\|_{H^1}^{2(2q-2)})) \]
\[ \leq C\sigma^{1/2} + C\sigma^{1/2}(\sigma^{-1})^{3q-2} \]
\[ \leq C, \]  
(4.59)

which together with (4.57) and (4.58) yields (4.48) and completes the proof. \( \square \)

5 Proofs of Theorems 1.1 and 1.2

With all the a priori estimates in Section 3 and Section 4 at hand, we will prove the main results of this paper in this section.

**Proof of Theorem 1.1.** By Lemma 2.1, the system (1.1)–(1.5) has a unique classical solution \((\rho, u)\) on \( \Omega \times (0, T_*) \) for some \( T_* \). Now we will extend the classical solution \((\rho, u)\) globally in time.

First, by the definition of \( A_1(T), A_2(T) \) (see (3.6), (3.7)), the assumption of the initial data (1.9) and (3.49), we have
\[ A_1(0) + A_2(0) = 0, \quad 0 \leq \rho_0 \leq \bar{\rho}, \quad A_3(0) \leq C_0^{1/2}. \]
Hence, there exists a \( T_1 \in (0, T_*] \) such that
\[ 0 \leq \rho_0 \leq 2\bar{\rho}, \quad A_1(T_1) + A_2(T_1) \leq 2C_0^{1/2}, \quad A_3(\sigma(T_1)) \leq 2C_0^{1/2}. \]  
(5.1)
Next, set
\[ T^* = \sup \{ T \mid (5.1) \text{ holds} \} \tag{5.2} \]

Naturally, \( T^* \geq T_1 > 0 \). On the other hand, for any \( 0 < \tau < T \leq T^* \), one deduces from Lemmas 4.3-3.6 that
\[
\begin{cases}
\rho - \bar{\rho} \in C([0, T]; W^{2,q}), \\
\nabla u_t \in C([\tau, T]; L^q), \quad \nabla u, \nabla^2 u \in C([\tau, T]; C(\bar{\Omega})),
\end{cases}
\tag{5.3}
\]

where one has taken advantage of the standard embedding
\[ L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q), \quad \text{for any } q \in [2, 6). \]

By (4.14), (4.22), (4.48) and (1.1), one gets
\[
\int_{\tau}^{T} \left( \int_{\tau}^{T} |\rho| |u_t|^2 \, dx \right) \, dt \\
\leq \int_{\tau}^{T} (\|\rho\|_{L^1}^2 + 2\|\rho u_t \cdot u_t\|_{L^1}) \, dt \\
\leq C \int_{\tau}^{T} \left( \|\rho\|_{L^1}^2 \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^1}^2 \right) \, dt \\
\leq C \int_{\tau}^{T} \left( \|\rho\|_{L^1}^2 \|\nabla u\|_{L^\infty}^2 + \|\nabla u\|_{L^6}^2 \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u_t\|_{L^2} \right) \, dt \\
\leq C,
\]

which together with (5.3) indicates that
\[ \rho^{1/2} u_t, \quad \rho^{1/2} \dot{u} \in C([\tau, T]; L^2). \tag{5.4} \]

Finally, we will claim that
\[ T^* = \infty. \tag{5.5} \]

Otherwise, \( T^* < \infty \). Now by Proposition [5.2], it holds that
\[ 0 \leq \rho \leq \frac{T}{4} \bar{\rho}, \quad A_1(T^*) + A_2(T^*) \leq C_0^2, \quad A_3(\sigma(T^*)) \leq C_0^2. \tag{5.6} \]

we deduce from Lemmas 4.4-3.6 and (5.1) that \( (\rho(x, T^*), u(x, T^*)) \) satisfies the initial data condition (1.8)-(1.10), where \( g(x) \triangleq \rho^{1/2}\dot{u}(x, T^*), \quad x \in \Omega \). Thus, Lemma 2.1 asserts that there is a \( T^{**} > T^* \) such that (5.1) holds for \( T = T^{**} \), which contradicts the definition of \( T^* \). Hence, \( T^* = \infty \).

By Lemmas (2.1) and 4.3-3.6, \( (\rho, u) \) is really the unique classical solution defined on \( \Omega \times (0, T] \) for any \( 0 < T < T^* = \infty \).

It remains to prove (1.13). Multiplying (3.51) by \( 4(P - P(\rho_\infty))^3 \), we have
\[
\frac{d}{dt} \| P - P(\rho_\infty) \|^4_{L^1} \\
= - (4\gamma - 1) \int (P - P(\rho_\infty))^4 \, du \, dx - \gamma \int P(\rho_\infty)(P - P(\rho_\infty))^3 \, du \, dx \\
\leq C\| P - P(\rho_\infty) \|^4_{L^1} + C\| \nabla u \|^2_{L^2},
\]
by (3.2) and (3.50), yields
\[
\int_1^{\infty} \left| \frac{d}{dt} \| P - P(\rho) \|_{L^4} \right| dt \leq C
\]
together with (3.50), we have
\[
\lim_{t \to \infty} \| P - P(\rho_\infty) \|_{L^4} = 0.
\]
Hence, for all \( r \in (2, \infty) \) if \( \rho_\infty > 0 \) and \( r \in (\gamma, \infty) \) if \( \rho_\infty = 0 \), we get
\[
\lim_{t \to \infty} \| \rho - \rho_\infty \|_{L^r} = 0. \tag{5.8}
\]
Set
\[
\phi(t) \triangleq (\lambda + 2\mu) \| \text{div} u \|_{L^2}^2 + \mu \| \text{curl} u \|_{L^2}^2.
\]
By (3.2) and (3.53),
\[
\int_1^{\infty} |\phi(t)|^2 dt \leq C.
\]
Reviewing our derivation of (3.22), one can find that the first term on the left side can be signed with absolute value. Taking \( m = 0 \), we get
\[
|\phi'(t)| \leq C(\| \rho^\frac{1}{2} \hat{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \| \nabla u \|_{L^4}^2 + \| \nabla \hat{u} \|_{L^2}^2), \tag{5.9}
\]
which, along with (3.2), (3.53) and (3.55), gives
\[
\int_1^{\infty} |\phi'(t)|^2 dt \leq C.
\]
As a result,
\[
\lim_{t \to \infty} \| \nabla u \|_{L^2} = 0. \tag{5.10}
\]
Finally, due to
\[
\int \rho^\frac{1}{2} |u|^4 dx \leq C(\| \rho^\frac{1}{2} u \|_{L^2} \| u \|_{L^6}^3 \leq C \| \nabla u \|_{L^2}^3,
\]
and (5.10), we obtain
\[
\lim_{t \to \infty} \| \rho^\frac{1}{2} u \|_{L^4} = 0.
\]
The proof of Theorem 1.1 is completed.

**Proof of Theorem 1.2.** For \( T > 0 \), the Lagrangian coordinates of the system are given by
\[
\begin{align*}
\frac{\partial}{\partial \tau} X(\tau; t, x) &= u(X(\tau; t, x), \tau), \quad 0 \leq \tau \leq T \\
X(t; t, x) &= x, \quad 0 \leq t \leq T, \ x \in \bar{\Omega}.
\end{align*} \tag{5.11}
\]
By (1.12), the transformation (5.11) is well-defined. In addition, by (1.11), we find that
\[
\rho(x, t) = \rho_0(X(0; t, x)) \exp \left\{ - \int_0^t \text{div} u(X(\tau; t, x), \tau) d\tau \right\}. \tag{5.12}
\]
If there exists some point \( x_0 \in \Omega \) such that \( \rho_0(x_0) = 0 \), then there is a point \( x_0(t) \in \bar{\Omega} \) such that \( X(0; t, x_0(t)) = x_0 \). Hence, by (5.12), \( \rho(x_0(t), t) \equiv 0 \) for any \( t \geq 0 \).
Now we will prove Theorem 1.2 by contradiction. Suppose there exist some positive constant $C_1$ and a subsequence $t_{n_j}$, $t_{n_j} \to \infty$ as $j \to \infty$ such that $\|\nabla \rho(\cdot, t_{n_j})\|_{L^r} < C_1$. Consequently, by Gagliardo-Nirenberg’s inequality, we get that for $r \in (3, \infty)$ and

\[
\theta_1 = \frac{3}{2r} - \frac{3}{2r},
\]

\[
\rho_\infty \leq \|\rho(\cdot, t_{n_j}) - \rho_\infty\|_{C^1(\mathbb{R})} \leq C\|\rho(\cdot, t_{n_j}) - \rho_\infty\|_{L^3}^{\theta_1} \|\nabla \rho(\cdot, t_{n_j})\|^{1-\theta_1}_{L^r},
\]

(5.13)

which is in contradiction with [5.8]. So we complete the proof.

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References

[1] Achdou, Y.; Pironneau, O.; Valentin, F. Effective boundary conditions for laminar flows over periodic rough boundaries. J. Comput. Phys. 147 (1998), no. 1, 187-218.

[2] Aramaki, J., $L^p$ theory of the div-curl system, Int. J. Math. Anal. 8 (6)(2014), 259-271.

[3] Bänsch, E. Finite element discretization of the Navier-Stokes equations with a free capillary surface. Numer. Math. 88 (2001), no. 2, 203-235.

[4] Beale, J. T.; Kato, T.; Majda. A. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys. 94 (1984), 61-66.

[5] Beavers, G. S.; Joseph, D. D. Boundary conditions at a naturally permeable wall. J. Fluid Mech. 30 (1967), 197-207.

[6] Cai, G.C.; Li, J. Existence and exponential growth of global classical solutions to the compressible Navier-Stokes equations with slip boundary conditions in 3D bounded domains. arXiv:2102.06348

[7] Cho, Y.; Choe, H. J.; Kim, H. Unique solvability of the initial boundary value problems for compressible viscous fluid. J. Math. Pures Appl. 83 (2004), 243-275.

[8] Cho, Y.; Kim, H. On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. Manuscript Math. 120 (2006), 91-129.

[9] Choe, H. J.; Kim, H. Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. J. Differ. Eqs. 190 (2003), 504-523.

[10] Cioranescu, D.; Donato, P.; Ene, H. I. Homogenization of the Stokes problem with nonhomogeneous slip boundary conditions. Math. Methods Appl. Sci. 19 (1996), no. 11, 857-881.
[11] Crispo, F.; Maremonti, P. An interpolation inequality in exterior domains. Rend. Sem. Mat.Univ. Padova, 112, 2004.

[12] Dhifaoui, A.; Meslameni, M.; Razafison, U. Weighted Hilbert spaces for the stationary exterior Stokes problem with Navier slip boundary conditions, J. Math. Anal. Appl., 472 (2), 2019, 1846-1871.

[13] da Veiga, H. B. A challenging open problem: the inviscid limit under slip-type boundary conditions. Discrete Continuous Dynamical Systems-S, 3(2), 231-236 (2010).

[14] Feireisl, E. Dynamics of viscous compressible fluids. Oxford University Press, New York, 2004.

[15] Feireisl, E.; Jin, B.J.; Novotný, A. Relative Entropies, Suitable Weak Solutions, and Weak-Strong Uniqueness for the Compressible Navier-Stokes System. J. Math. Fluid Mech. 14, 717-730 (2012).

[16] Feireisl, E.; Novotny, A.; Petzeltová, H. On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. 3 (2001), no. 4, 358-392.

[17] Galdi G. P, Layton W. J., Approximation of the larger eddies in fluid motions II: A model for space-filtered flow[J]. Mathematical Models and Methods in Applied Sciences, 2000, 10(03): 343-350.

[18] Hoff, D. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Amer. Math. Soc. 303 (1987), no. 1, 169-181.

[19] Hoff, D. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differ. Eqs. 120 (1995), no. 1, 215-254.

[20] Hoff, D. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Rational Mech. Anal. 132 (1995), 1-14.

[21] Hoff, D. Compressible flow in a half-space with Navier boundary conditions. J. Math. Fluid Mech. 7 (2005), no. 3, 315-338.

[22] Huang, X. D. On local strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum. Sci China Math. 63, (2020) https://doi.org/10.1007/s11425-019-9755-3

[23] Huang, X. D.; Li, J.; Xin Z. P. Serrin type criterion for the three-dimensional compressible flows. SIAM J. Math. Anal., 43 (2011), no. 4, 1872–1886.

[24] Huang, X. D.; Li, J.; Xin, Z. P. Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Comm. Pure Appl. Math. 65 (2012), 549-585.

[25] Jäger, W.; Mikelić, A. On the roughness-induced effective boundary conditions for an incompressible viscous flow. J. Differential Equations 170 (2001), no. 1, 96-122.
[26] Kato, T. Remarks on the Euler and Navier-Stokes equations in $R^2$. Proc. Symp. Pure Math. Vol. 45, Amer. Math. Soc., Providence, 1986, 1-7.

[27] Kazhikhov, A. V.; Shelukhin, V. V. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. Prikl. Mat. Meh. 41 (1977), 282-291.

[28] Li, J.; Liang, Z. On classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum. J. Math. Pures Appl. (9) 102 (2014), no. 4, 640-671.

[29] Li, J.; Xin, Z. Some uniform estimates and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows. J. Differ. Eqs. 221 (2006), no. 2, 275-308.

[30] Li, J.; Xin, Z. Global Existence of Regular Solutions with Large Oscillations and Vacuum. In: Giga Y., Novotny A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer 2016.

[31] Lions, P. L. Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford University Press, New York, 1998.

[32] Louati H, Meslameni M, Razafison U. Weighted $L^p$ theory for vector potential operators in three-dimensional exterior domains[J]. Mathematical Methods in the Applied Sciences, 2014, 39(8):1990-2010.

[33] Matsumura, A.; Nishida, T. The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20 (1980), no. 1, 67-104.

[34] Mucha, P. B. On Navier-Stokes equations with slip boundary conditions in an infinite pipe[J]. Acta Applicandae Mathematica, 2003, 76(1): 1-15.

[35] Nash, J. Le problème de Cauchy pour les équations différentielles d’un fluide général. Bull. Soc. Math. France. 90 (1962), 487-497.

[36] Novotny, A.; Straskraba, I. Introduction to the Mathematical Theory of Compressible Flow, Oxford Lecture Ser. Math. Appl., Oxford Univ. Press, Oxford, 2004.

[37] Salvi, R.; Straskraba, I. Global existence for viscous compressible fluids and their behavior as $t \to \infty$. J. Fac. Sci. Univ. Tokyo Sect. IA. Math. 40 (1993), 17-51.

[38] Schwarz, G. Hodge decomposition - a method for solving boundary value problems. Lecture Notes in Mathematics, 1607. Springer, Berlin, 1995.

[39] Serre, D. Solutions faibles globales des équations de Navier-Stokes pour unfluide compressible. C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), 639-642.

[40] Serre, D. Sur l’équation monodimensionnelle d’un fluide visqueux, compressible et conducteur de chaleur. C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), 703-706.

[41] Serrin, J. On the uniqueness of compressible fluid motion. Arch. Rational. Mech. Anal. 3 (1959), 271-288.
[42] Serrin, J. Mathematical principles of classical fluid mechanics. 1959 Handbuch der Physik, Bd. 8/1, Strömungsmechanik I pp. 125-263 Springer-Verlag, Berlin-Göttingen-Heidelberg

[43] Solonnikov V. A., On the Stokes equations in domains with non-smooth boundaries and on viscous incompressible flow with a free surface[C]. Nonlinear Partial Differential and their Applications, College de France Seminar, Paris. Pitman, 1982: 340-423.

[44] Vaigant, V. A., Kazhikhov, A.V.: On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid (Russian). Sibirsk. Mat. Zh. 36(6), 1283-1316 (1995); translation in Sib. Math. J. 36(6), 1108-1141 (1995)

[45] Von Wahl, W. Estimating \( \nabla u \) by \( \text{div} u \) and \( \text{curl} u \). Math. Methods in Applied Sciences, Vol.15, 123-143 (1992).

[46] Xiao, Y.; Xin, Z. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 60(7), 1027-1055(2007).

[47] Yoshikazu Giga, Antonín Novotný, Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer, 2018.

[48] Zlotnik, A. A. Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Diff. Eqs. 36 (2000), 701-716.