A matrix subadditivity inequality for $f(A + B)$ and $f(A) + f(B)$

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Abstract.

In 1999 Ando and Zhan proved a subadditivity inequality for operator concave functions. We extend it to all concave functions: Given positive semidefinite matrices $A$, $B$ and a non-negative concave function $f$ on $[0, \infty)$,

$$\|f(A + B)\| \leq \|f(A) + f(B)\|$$

for all symmetric norms (in particular for all Schatten $p$-norms). The case $f(t) = \sqrt{t}$ is connected to some block-matrix inequalities, for instance the operator norm inequality

$$\left\| \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \right\|_\infty \leq \max\{ \|A + \|X\|_\infty : \|B\| + \|X^*\|_\infty \}$$

for any partitioned Hermitian matrix.

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1. A subadditivity inequality

Capital letters $A$, $B$, . . . , $Z$ mean $n$-by-$n$ complex matrices, or operators on an $n$-dimensional Hilbert space $\mathcal{H}$. If $A$ is positive semidefinite, resp. positive definite, we write $A \geq 0$, resp. $A > 0$. Recall that a symmetric (or unitarily invariant) norm $\| \cdot \|$ satisfies $\|A\| = \|UAV\|$ for all $A$ and all unitaries $U$, $V$. We will prove:

**Theorem 1.1.** Let $A, B \geq 0$ and let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function. Then, for all symmetric norms,

$$\|f(A + B)\| \leq \|f(A) + f(B)\|.$$ 

For the trace norm Theorem 1.1 is a classical inequality. In case of the operator norm, Kosem [7] recently gave a three-line proof! But the general case is much more difficult. When $f$ is operator concave, Theorem 1.1 has been proved by Ando and Zhan [1]. Their proof is not elementary and makes use of integral representations of
operator concave functions. By a quite ingenious process, Kosem [7] derived from Ando-Zhan’s result a related superadditive inequality:

**Theorem 1.2.** Let $A, B \geq 0$ and let $g : [0, \infty) \to [0, \infty)$ be a convex function with $g(0) = 0$. Then, for all symmetric norms,

$$
\| g(A) + g(B) \| \leq \| g(A + B) \|.
$$

The special case $g(t) = t^m$, $m = 1, 2, \ldots$ is due to Bhatia-Kittaneh [4]. The general case has been conjectured by Aujla and Silva [3].

In this note we first give a simple proof of these two theorems. Our method is elementary: we only use a simple inequality for operator convex/concave functions and some basic facts about symmetric norms and majorization. For background we refer to [9] and references herein.

Next, we consider some inequalities for block-matrices inspired by the observation that Theorem 1.1 for $f(t) = \sqrt{t}$ can be written as $\| \sqrt{A^2 + B^2} \| \leq \| A + B \|$, or equivalently

$$
\left\| \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \right\| \leq \| A + B \|.
$$

We naturally asked if a similar result holds when the zeros are replaced by arbitrary positive matrices. We got proofs from T. Ando, E. Ricard and X. Zhan. We thank them for their collaboration.

## 2. Proof of Theorems 1.1-1.2 and related results

First we sketch the simple proof from [8] for Theorem 1.1 in the operator concave case (Ando-Zhan’s inequality). Let us recall some basic facts about operator convex/concave functions on an interval $[a, b]$. If $g$ is operator convex and $A$ is Hermitian, $a \geq A \geq b$, then for all subspaces $S \subset \mathcal{H}$, Davis’ Inequality holds for compressions onto $S$,

$$
g(AS) \leq g(A)S.
$$

Assuming $0 \in [a, b]$, $g(0) \leq 0$, one can derive Hansen’s Inequality: $Z$ being any contraction,

$$
g(Z^*AZ) \leq Z^* g(A)Z.
$$

Of course, for an operator concave function $f$ on $[a, b]$ with $f(0) \geq 0$, the reverse inequality holds. For such an $f$ on the positive half-line and $A, B > 0$ we then have

$$
f(A) \geq A^{1/2}(A + B)^{-1/2} f(A + B)(A + B)^{-1/2} A^{1/2}
$$

since $Z = (A + B)^{-1/2} A^{1/2}$ is a contraction and $A = Z^* (A + B) Z$. Similarly

$$
f(B) \geq B^{1/2}(A + B)^{-1/2} f(A + B)(A + B)^{-1/2} B^{1/2}.
$$
Consequently

\[ f(A) + f(B) \geq A^{1/2} \frac{f(A + B)}{(A + B)} A^{1/2} + B^{1/2} \frac{f(A + B)}{(A + B)} B^{1/2}. \]  \tag{2}

Next, the main observation of [8] can be stated as:

**Proposition 2.1.** Let \( A, B \geq 0 \) and let \( g : [0, \infty) \to [0, \infty) \). If \( g(t) \) decreases and \( tg(t) \) increases, then for all symmetric norms,

\[ \| (A + B)g(A + B) \| \leq \| A^{1/2} g(A + B) A^{1/2} + B^{1/2} g(A + B) B^{1/2} \|. \]

Combining (2) and Proposition 2.1 with \( g(t) = f(t)/t \) yields the Ando-Zhan Inequality [1]:

**Corollary 2.2.** Theorem 1.1 holds when \( f \) is operator concave.

This means that the eigenvalues of \( f(A + B) \) are weakly majorised by those of \( f(A) + f(B) \). Suppose now that \( f \) is onto, thus \( f(0) = 0, f(\infty) = \infty \) and its inverse function \( g \) is convex, increasing. Therefore the eigenvalues of \( g(f(A + B)) = A + B \) are weakly majorised by those of \( g(f(A) + f(B)) \). Replacing \( A \) and \( B \) by \( g(A) \) and \( g(B) \) respectively, we get the second Ando-Zhan Inequality [1]:

**Corollary 2.3.** Let \( A, B \geq 0 \) and let \( g : [0, \infty) \to [0, \infty) \) be a one to one function whose inverse function is operator concave. Then, for all symmetric norms,

\[ \| g(A) + g(B) \| \leq \| g(A + B) \|. \]

Now we turn to a quite simple proof of Theorem 1.2. It suffices to consider Ky Fan \( k \)-norms \( \| \cdot \|_k \). Suppose that \( f \) and \( g \) both satisfy Theorem 1.2. Using the triangle inequality and the fact that \( f \) and \( g \) are nondecreasing,

\[ \|(f + g)(A) + (f + g)(B)\|_k \leq \|f(A) + f(B)\|_k + \|g(A) + g(B)\|_k \]
\[ \leq \|f(A + B)\|_k + \|g(A + B)\|_k = \|(f + g)(A + B)\|_k, \]

hence the set of functions satisfying Theorem 1.2 is a cone. It is also closed for pointwise convergence. Since any positive convex function vanishing at 0 can be approached by a positive combination of angle functions at \( a > 0 \),

\[ \gamma(t) = \frac{1}{2} \{|t - a| + t - a\}, \]

it suffices to prove Theorem 1.2 for such a \( \gamma \). By Corollary 2.3 it suffices to approach \( \gamma \) by functions whose inverses are operator concave. We take (with \( r > 0 \))

\[ h_r(t) = \frac{1}{2} \{ \sqrt{t^2 - a^2} + t - \sqrt{a^2 + r} \}, \]
whose inverse
\[
t - \frac{r/2}{2t + \sqrt{a^2 + r - a}} + \frac{\sqrt{a^2 + r + a}}{2}
\]
is operator concave since $1/t$ is operator convex on the positive half-line (inequality (1) is then a basic fact of Linear Algebra). Clearly, as $r \to 0$, $h_r(t)$ converges uniformly to $\gamma$.

From Theorem 1.2 we can derive Theorem 1.1:

**Proof of Theorem 1.1.** It suffices to prove the theorem for the Ky Fan $k$-norms $\| \cdot \|_k$. This shows that we may assume $f(0) = 0$. Note that $f$ is necessarily non-decreasing. Hence, there exists a rank $k$ spectral projection $E$ for $A + B$, corresponding to the $k$-largest eigenvalues $\lambda_1(A + B), \ldots, \lambda_k(A + B)$ of $A + B$, such that
\[
\|f(A + B)\|_k = \sum_{j=1}^{k} \lambda_j(f(A + B)) = \text{Tr} Ef(A + B)E.
\]
Therefore, using a well-known property of Ky Fan norms, it suffices to show that
\[
\text{Tr} Ef(A + B)E \leq \text{Tr} E(f(A) + f(B))E.
\]
This is the same as requiring that
\[
\text{Tr} E(g(A) + g(B))E \leq \text{Tr} Eg(A + B)E
\]
for all non-positive convex functions $g$ on $[0, \infty)$ with $g(0) = 0$. Any such function can be approached by a combination of the type
\[
g(t) = \lambda t + h(t)
\]
for a scalar $\lambda < 0$ and some non-negative convex function $h$ vanishing at 0. Hence, it suffices to show that (3) holds for $h(t)$. We have
\[
\begin{align*}
\text{Tr} E(h(A) + h(B))E &= \sum_{j=1}^{k} \lambda_j(E(h(A) + h(B))E) \\
&\leq \sum_{j=1}^{k} \lambda_j(h(A) + h(B)) \\
&\leq \sum_{j=1}^{k} \lambda_j(h(A + B)) \quad \text{(by Theorem 1.2)} \\
&= \sum_{j=1}^{k} \lambda_j(Eh(A + B)E) \\
&= \text{Tr} Eh(A + B)E
\end{align*}
\]
where the second equality follows from the fact that $h$ is non-decreasing and hence $E$ is also a spectral projection of $h(A + B)$ corresponding to the $k$ largest eigenvalues. □
The above proof is inspired by a part of the proof of the following result [5]:

**Theorem 2.4.** Let \( f : [0, \infty) \rightarrow [0, \infty) \) be a concave function. Let \( A \geq 0 \) and let \( Z \) be expansive. Then, for all symmetric norms,
\[
\|f(Z^*AZ)\| \leq \|Z^*f(A)Z\|.
\]

Here \( Z \) expansive means \( Z^*Z \geq I \), the identity operator. Besides such symmetric norms inequalities, there also exist interesting inequalities involving unitary congruences. For instance [2]:

**Theorem 2.5.** Let \( A, B \geq 0 \) and let \( f : [0, \infty) \rightarrow [0, \infty) \) be a concave function. Then, there exist unitaries \( U, V \) such that
\[
f(A + B) \leq Uf(A)U^* + Vf(B)V^*.
\]

This implies that \( \lambda_{j+k+1}f(A + B) \leq \lambda_{j+1}f(A) + \lambda_{k+1}f(B) \) for all integers \( j, k \geq 0 \).

Combining Theorem 2.5 with Thompson’s triangle inequality we get:

**Corollary 2.6.** For any \( A, B \) and any non-negative concave function \( f \) on \([0, \infty)\),
\[
f(|A + B|) \leq Uf(|A|)U^* + Vf(|B|)V^*
\]
for some unitaries \( U, V \).

Therefore, we recapture a result from [8]: the map \( X \rightarrow \|f(X)\| \) is subaditive. For the trace-norm, this is Rotfel’d Theorem.

A remark may be added about Theorem 1.1: It can be stated for a family \( \{A_i\}_{i=1}^m \) of positive operators,
\[
\|f(A_1 + \cdots + A_m)\| \leq \|f(A_1) + \cdots + f(A_m)\|.
\]
Indeed, Proposition 2.1 can be stated for a suitable family \( A, B, \ldots \).

### 3. Inequalities for block-matrices

In Section 1 we noted an inequality involving a partitioned matrix. The following two theorems are generalizations due to T. Ando, E. Ricard and X. Zhan (private communications). The symbol \( \| \cdot \|_\infty \) means the operator norm.

**Theorem 3.1.** For all block-matrices whose entries are normal matrices of same size and for all symmetric norms,
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\| \leq \|A| + |B| + |C| + |D|\|.
\]
**Theorem 3.2.** For all block-matrices whose entries are normal matrices of same size,
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_\infty \leq \max \{ \|A| + |B|\|_\infty; \|C| + |D|\|_\infty; \|A| + |C|\|_\infty; \|B| + |D|\|_\infty \}
\]

**Proof.** Let \( A_1, A_2, B_1, B_2 \) be positive and let \( C_1, C_2 \) be contractions. Note that
\[
A_1C_1B_1 + A_2C_2B_2 = (A_1 A_2) \left( \begin{array}{cc} C_1 & 0 \\ 0 & C_2 \end{array} \right) \left( \begin{array}{c} B_1 \\ B_2 \end{array} \right).
\]
Applying the Cauchy-Shwarz inequality \( \|XY\| \leq \|X^*X\|^{1/2}\|YY^*\|^{1/2} \) and using \( \|ST\| \leq \|S\|_\infty \|T\| \) show
\[
\|A_1C_1B_1 + A_2C_2B_2\| \leq \|A_1^2 + A_2^2\|^{1/2}\|B_1^2 + B_2^2\|^{1/2}.
\]
Considering polar decompositions \( A = |A^*|^{1/2}U|A|^{1/2} \) and \( B = |B^*|^{1/2}V|B|^{1/2} \) then shows that
\[
\|A + B\| \leq \|A| + |B|\|^{1/2}\|A^*| + |B^*\|^{1/2}
\]
for all \( A, B \). Replacing \( A \) and \( B \) in (4) by
\[
\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}
\]
and using normality of \( A, B, C, D \) yield
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |A| + |C| & 0 \\ 0 & |B| + |D| \end{pmatrix} \right\|^{1/2}\left\| \begin{pmatrix} |A| + |B| & 0 \\ 0 & |C| + |D| \end{pmatrix} \right\|^{1/2}.
\]
This proves Theorem 3.2. This also proves Theorem 3.1 by using the fact that
\[
\left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\| \leq \|X + Y\|
\]
for all \( X, Y \geq 0 \). \( \Box \)

**Corollary 3.3.** For any partitioned Hermitian matrix,
\[
\left\| \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \right\|_\infty \leq \max \{ \|A| + \|X\|_\infty; \|B| + \|X^*\|_\infty \}.
\]

**Proof.** We consider the block matrix
\[
\begin{pmatrix} 0 & 0 & 0 & X \\ 0 & A & X^* & 0 \\ 0 & X & B & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix}
\]
with normal blocks
\[
\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \quad \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}
\]
and we apply Theorem 3.2. □

A special case of (4) is the following statement:

**Proposition 3.4.** Let $A$, $B$ be normal. Then, for all symmetric norms,

$$\|A + B\| \leq \| |A| + |B| \|.$$

This can be regarded as a triangle inequality for normal operators. In case of Hermitian operators, a stronger triangle inequality holds [6]:

**Proposition 3.5.** Let $S$, $T$ be Hermitian. Then, for some unitaries $U$, $V$,

$$|S + T| \leq \frac{1}{2} \{ U(|S| + |T|)U^* + V(|S| + |T|)V^* \}$$

Proposition 3.5 implies Proposition 3.4 by letting $S = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$.

Proposition 3.4 shows that, given $A, B \geq 0$ and any complex number $z$,

$$\|A + zB\| \leq \|A + |z|B\|$$

so that for all integers $m = 1, 2...$,

$$\|(A + zB)^m\| \leq \|(A + |z|B)^m\|. \quad (5)$$

This was observed by Bhatia and Kittaneh. They also noted the identity

$$A^m + B^m = \frac{1}{m} \sum_{j=0}^{m-1} (A + wB)^j \quad (6)$$

where $w$ is the primitive $m$-th root of the unit. Combining (5) and (6), Bhatia and Kittaneh obtained [4]: Given $A, B \geq 0$

$$\|A^m + B^m\| \leq \|(A + B)^m\|$$

for all $m = 1, 2...$ and all symmetric norms. This result was the starting point of superadditive or subadditive inequalities for symmetric norms.

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