Abstract. This is an overview about natural sample spaces for differential equations driven by various noises. Appropriate sample spaces are needed in order to facilitate a random dynamical systems approach for stochastic differential equations. The noise could be white or colored, Gaussian or non-Gaussian, Markov or non-Markov, and semimartingale or non-semimartingale. Typical noises are defined in terms of Brownian motion, Lévy motion and fractional Brownian motion. In each of these cases, a canonical sample space with an appropriate metric (or topology that gives convergence concept) is introduced. Basic properties of canonical sample spaces, such as separability and completeness, are then discussed.

Moreover, a flow defined by shifts, is introduced on these canonical sample spaces. This flow has an invariant measure which is the probability distribution for Brownian motion, Lévy motion or fractional Brownian motion. Thus canonical sample spaces are much richer in mathematical structures than the usual sample spaces in probability theory, as they have metric or topological structures, together with a shift flow (or driving flow) defined on it. This facilitates dynamical systems approaches for studying stochastic differential equations.

1. Random dynamical systems

Stochastic differential equations (SDEs) or stochastic partial differential equations (SPDEs) arise as mathematical models for complex systems under various random influences in engineering and science. Here we only consider random dynamical systems defined by SDEs. Such SDEs define random dynamical systems (RDS) with appropriate sample spaces, much as ordinary differential equations define deterministic dynamical systems.

Theory of random dynamical systems allows to discuss the qualitative behavior of stochastic systems that are not only driven by a white noise, Markov processes and semimartingales, but also driven by non-Markov processes or by non-semimartingales (e.g., fractional Brownian motion). To analyze these more general noise cases, appropriate sample spaces and ergodic theory play an important role. In this article, we discuss canonical or natural sample spaces for SDEs with various noises.

We recall the definition of a random dynamical system (RDS) in the state space $H = \mathbb{R}^n$, with the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and with time $t$ varying in $\mathbb{T} = \mathbb{R} = (-\infty, \infty)$ or $\mathbb{T} = \mathbb{R}^+ = [0, \infty)$, as in Arnold [1]. The state space $\mathbb{R}^n$ is equipped with the

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Euclidean norm (or length) \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \) and the usual scalar product \( <x,y> = x_1y_1 + \cdots + x_ny_n \).

Note that a deterministic dynamical system on the state space \( H \) is a mapping \( \psi : T \times H \to H \), \((t,x) \mapsto \psi(t,x)\), such that the flow property is satisfied:

\[
\psi(0,x) = x, \quad \psi(t+s,x) = \psi(t,\psi(s,x)),
\]

for all \( t, s \in T \) and \( x \in H \).

For a random dynamical system, we need an extra ingredient, namely, a model for the noise. Moreover, the flow property has a twist (thus called cocycle property) due to the effect of noise.

**Definition 1.1. (Random dynamical system)**

A random dynamical system (RDS), denoted by \( \varphi \), consists of two ingredients:

(i) **Model for the noise**: A driving flow on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), i.e., a flow \((\theta_t)_{t \in \mathbb{T}}\) on the sample space \( \Omega \), such that \( \mathbb{P} \) is invariant, namely \( \theta_t \mathbb{P} = \mathbb{P} \) for all \( t \in \mathbb{T} \), and \((t, \omega) \mapsto \theta_t \omega \) is measurable from \( \mathbb{T} \times \Omega \) to \( \Omega \).

(ii) **Model for the evolution**: A cocycle \( \varphi \) over \( \theta \), i.e. a measurable mapping \( \varphi : \mathbb{T} \times \Omega \times H \to H \), \((t, \omega, x) \mapsto \varphi(t, \omega, x)\), such that the family \( \varphi(t, \omega, \cdot) = \varphi(t, \omega) : H \to H \) of random mappings satisfies the cocycle property:

\[
\varphi(0, \omega) = \text{id}_H, \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \text{ for all } t, s \in \mathbb{T}, \omega \in \Omega.
\]

Here the driving flow \( \theta_t \) describes stationary dynamics of noise in an appropriately chosen sample space (see below). The mathematical model for noises in engineering and science is usually a stationary generalized stochastic process [22].

When \((t, x) \mapsto \varphi(t, \omega, x)\) is continuous for all \( \omega \in \Omega \), we say that \( \varphi \) is a continuous RDS. Since the continuity in space \( x \) is quite common for RDS generated by stochastic differential equations, we usually do not specifically mention this spatial continuity. We often call \( \varphi \) a continuous-time or discrete-time RDS when it is continuous or discrete in time \( t \).

It follows from [1] that \( \varphi(t, \omega), t \in \mathbb{R}, \) is a homeomorphism of \( H \) and

\[
\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega).
\]

2. **Dynamical systems driven by white noises**

2.1. **Brownian Motion.** The physical phenomenon *Brownian motion*\(^1\) is due to the incessant hitting of pollen by the much smaller molecules of the liquid. The hits occur a large number of times in any small time interval, independently of each other and the effect of a particular hit is small compared to the total effect [43]. The physical theory of this motion, set up by Albert Einstein in 1905, suggests that the motion is random, and has the following properties:

i) the motion is continuous;

ii) it has independent increments;

\(^1\)The phenomenon was first observed by Jan Ingenhouz in 1785, but was subsequently rediscovered by Brown in 1828, according to sources used by Eric Weisstein’s *World of Physics*, which can be found on the Internet at [http://scienceworld.wolfram.com/physics/BrownianMotion.html](http://scienceworld.wolfram.com/physics/BrownianMotion.html)
iii) the increments are stationary and Gaussian random variables.

Figure 1 shows a sample path of the Brownian motion. Intuitively speaking, property i) says that the sample path of the Brownian motion is continuous. Property ii) means that the displacements of a pollen particle over disjoint time intervals are independent random variables. Property iii) is natural considering the Central Limit Theorem.

We now describe the Brownian motion in the mathematical language, i.e., introduce the first definition of the Brownian motion\cite{22,2,6}.

**Definition A:** A stochastic process \( \{B_t(\omega) : t \geq 0\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\) is called a Brownian motion or a Wiener process if the following conditions hold:
1) \( B_0(\omega) = 0 \) a.s.;
2) the sample paths \( t \rightarrow B_t(\omega) \) are a.s. continuous;
3) \( B_t(\omega) \) has stationary independent increments;
4) the increments \( B_t(\omega) - B_s(\omega) \) has the normal distribution with mean 0 and variance \( t-s \), i.e. \( B_t(\omega) - B_s(\omega) \sim N(0, t-s) \) for any \( 0 \leq s < t \).

Since the stochastic process \( B_t(\omega) \) is a mapping from a probability space to a metric space and is governed by its law, i.e. the probability measure on the metric space induced by \( B_t(\omega) \), we want to find it in order to have a better understanding of the Brownian motion. In fact, we can find the finite dimensional distribution of the Brownian motion using condition 3) and 4) in Definition A, and this gives another definition of the Brownian motion as we shall see. We first write down the second definition\cite{6} and then prove it is equivalent to Definition
A.

**Definition B:** A stochastic process \( \{B_t(\omega) : t \geq 0\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\) is called a **Brownian motion** or a **Wiener process** if the following conditions hold:

1') \( B_0(\omega) = 0 \) a.s.;
2') the sample paths \( t \to B_t(\omega) \) are a.s. continuous;
3') for any finite sequence of times \( 0 = t_0 < t_1 < t_2 < \cdots < t_n \) and Borel sets \( B_1, \cdots, B_n \subset \mathbb{R} \)

\[
P\{B_{t_1}(\omega) \in B_1, \cdots, B_{t_n}(\omega) \in B_n\} = \int_{B_1} \cdots \int_{B_n} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2) \cdots \]
\[
\cdots p(t_n - t_{n-1}, x_{n-1}, x_n)dx_1 \cdots dx_n
\]

where

\[
p(t; x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}
\]
defined for any \( x, y \in \mathbb{R} \) and \( t > 0 \) is called the **transition density**.

We can see the conditions 1'),2') in Definition B are completely the same as 1),2) in Definition A, respectively, the only thing we need to do in the proof of the equivalence of the two is to show 3'),4) \( \iff 3' \).

**Proof 3'),4) \( \Rightarrow 3' \)**
Firstly, we show that Brownian motion \( B_t \) has Markov property, by proving that the conditional distribution of \( B_{t+s} \) given \( \mathcal{F}_t \) is the same as that given \( B_t \), in terms of moment generating function [10]. In fact,

\[
E(e^{uB_{t+s}}|\mathcal{F}_t) = E(e^{u([B_{t+s} - B_t] + B_t)}|\mathcal{F}_t)
\]
\[
= e^{uB_t} E(e^{u(B_{t+s} - B_t)}|\mathcal{F}_t)
\]
\[
= e^{uB_t} E(e^{u(B_{t+s} - B_t)})
\]
\[
= e^{uB_t} e^{u^2s/2}
\]
\[
= e^{uB_t} E(e^{u(B_{t+s} - B_t)}|B_t)
\]
\[
= E(e^{uB_{t+s}}|B_t).
\]
Secondly, we compute the joint distribution of the Brownian motion.

\[ P\{ B(t_{k+1}) \leq x_{k+1}, B(t_k) \leq x_k \} \]

\[ = P\{ B(t_k) \leq x_k, [B(t_{k+1}) - B(t_k)] + B(t_k) \leq x_{k+1} \} \]

\[ = P\{ B(t_k) \leq x_k, [B(t_{k+1}) - B(t_k)] \leq x_{k+1} - B(t_k) \} \]

\[ = P\{ B(t_{k+1}) - B(t_k) \leq x_{k+1} - B(t_k) | B(t_k) \leq x_k \} P\{ B(t_k) \leq x_k \} \text{ (Conditional probability)} \]

\[ = \int_{-\infty}^{x_k} \int_{-\infty}^{x_{k+1} - x} \frac{1}{\sqrt{2\pi(t_{k+1} - t_k)}} e^{-\frac{x^2}{2(t_{k+1} - t_k)}} \, dy \frac{1}{\sqrt{2\pi t_k}} e^{-\frac{x^2}{2t_k}} \, dx \]

\[ = \int_{-\infty}^{x_k} \frac{1}{\sqrt{2\pi t_k}} e^{\frac{x^2}{2t_k}} \, dx \int_{-\infty}^{x_{k+1}} \frac{1}{\sqrt{2\pi(t_{k+1} - t_k)}} e^{-\frac{(y-x)^2}{2(t_{k+1} - t_k)}} \, dy \]

\[ = \int_{-\infty}^{x_k} p(t_k; 0, x)dx \int_{-\infty}^{x_{k+1}} p(t_{k+1} - t_k; x, y)dy. \]

Finally, we have the finite dimensional distribution.

\[ P\{ B(t_1) \leq x_1, B(t_2) \leq x_2, \ldots, B(t_n) \leq x_n \} \]

\[ = P\{ B(t_1) \leq x_1 | B(t_0) = 0 \} P\{ B(t_2) \leq x_2 | B(t_1) \leq x_1 \} \]

\[ \cdots \]

\[ P\{ B(t_n) \leq x_n | B(t_{n-1}) \leq x_{n-1} \} \text{ (Markov property)} \]

\[ = \frac{P\{ B(t_1) \leq x_1 \} P\{ B(t_2) \leq x_2, B(t_1) \leq x_1 \} \cdots P\{ B(t_n) \leq x_n, B(t_{n-1}) \leq x_{n-1} \}}{P\{ B(t_1) \leq x_1 \} P\{ B(t_2) \leq x_2 \} \cdots P\{ B(t_{n-1}) \leq x_{n-1} \}} \]

\[ = \int_{-\infty}^{x_1} p(t_1; 0, y_1)dy_1 \int_{-\infty}^{x_2} p(t_2 - t_1; y_1, y_2)dy_2 \cdots \int_{-\infty}^{x_n} p(t_n - t_{n-1}; y_{n-1}, y_n)dy_n \]

which is obviously the same as 3’).

Conversely, we need to show 3’ \(\Rightarrow\) 3),4)

However, this part of work is completely done in [6](see page 153-155).

2.2. Wiener Measure. In Section 1, we treat the Brownian motion as a stochastic process, i.e., a collection of time-parameterized random variables. Since a stochastic process \(\xi\) is governed by its law, i.e., the probability measure \(\mu := P\xi^{-1}\) on the space it maps to, one may ask the question why the Brownian motion can be determined only by its finite dimensional distributions although we have proved the so-defined Brownian motion (Definition B) coincides with the definition describing the phenomenon (Definition A). Motivated by this question, we shall treat the Brownian motion as a “random variable”, which we call random function [3], and this point of view introduces the Wiener measure, the probability measure defined on an appropriate space which gives us the Brownian motion. We will sketch the ideas showing the existence and uniqueness of this special probability measure, starting from three different spaces, i.e, \(\mathbb{R}[0, \infty), C[0, \infty), C^0_0[0, \infty)\), the spaces of arbitrary functions,
continuous functions, and continuous functions passing zero at time 0, defined on \([0, \infty)\), respectively.

**First approach** This way of showing the existence and uniqueness of the Wiener measure is the one most often used to construct a Markov Process and is rather technical. The main idea is the following: First define a set function on the algebra generated by the cylinder sets in \(\mathbb{R}[0, \infty)\) according to the finite-dimensional distribution of Brownian motion. Note that it is a set function rather than a probability measure since it is just finitely additive (not countably additive); then by the celebrated *Kolmogorov extension theorem* \(^{22}\), this set function is uniquely extended to a probability measure on the \(\sigma\)--algebra generated by the algebra mentioned above.

More precisely, we now give the definitions and theorems.

- **\(\mathbb{R}[0, \infty)\)** denotes the set of all real-valued functions on \([0, \infty)\).

- **Cylinder set** \(^{22}\) A subset of \(\mathbb{R}[0, \infty)\) of the form
  \[
  A = \{ \omega \in \mathbb{R}[0, \infty) : (\omega(t_1), \ldots, \omega(t_n)) \in B_n \}
  \]
  where \(B_n\) is a Borel subset of \(\mathbb{R}^n\) is called a cylinder set.

  Fixing \(t_1, \ldots, t_n\) but varying \(B_n\) over the entire Borel subsets of \(\mathbb{R}^n\), the class of such all cylinder sets forms a \(\sigma\)--algebra \(\mathfrak{B}^{(t_1, \ldots, t_n)}\).

- **Set function** A set function \(\Phi_{t_1, \ldots, t_n}\) is defined on the measurable space \((\mathbb{R}[0, \infty), \mathfrak{B}^{(t_1, \ldots, t_n)})\), given by
  \[
  \Phi_{t_1, \ldots, t_n}(A) := P\{ (B_{t_1}, \ldots, B_{t_n}) \in B_n \}
  \]
  By varying the choice of the finite time points \(\{t_1, \ldots, t_n\} \subset [0, \infty)\), we get a class of such set functions \(\Phi := \{ \Phi_{t_1, \ldots, t_n} \}\), and this class is independent of the cylindrical expression of the functions in \(\mathbb{R}[0, \infty)\), i.e. if the cylinder set \(A\) of \(\text{[4]}\) has another expression, say
  \[
  A = \{ \omega \in \mathbb{R}[0, \infty) : (\omega(s_1), \ldots, \omega(s_m)) \in B_m \},
  \]
  then
  \[
  \Phi_{t_1, \ldots, t_n}(A) = \Phi_{s_1, \ldots, s_m}(A).
  \]

- **Algebra and \(\sigma\)-algebra**
  - \(\mathfrak{U}[0, \infty)\) denotes the algebra of subsets of \(\mathbb{R}[0, \infty)\) consisting all cylinder sets
  - \(\mathfrak{B}[0, \infty)\) denotes the smallest \(\sigma\)--algebra containing \(\mathfrak{U}[0, \infty)\)
  - \(\Phi\) is a finitely additive measure on \((\mathbb{R}[0, \infty), \mathfrak{U}[0, \infty))\) such that the restriction of \(\Phi\) to \(\mathfrak{B}^{(t_1, \ldots, t_n)}\) coincides with \(\Phi_{t_1, \ldots, t_n}\)

**Theorem 2.1 (Kolmogorov extension theorem).** The set function \(\Phi\) on \((\mathbb{R}[0, \infty), \mathfrak{U}[0, \infty))\) is uniquely extendable to a probability measure \(\tilde{\Phi}\) on \((\mathbb{R}[0, \infty), \mathfrak{B}[0, \infty))\).
It has been proved \cite{22,32} that there exist a unique probability measure $P$ on $(\mathbb{R}[0, \infty), \mathcal{B}[0, \infty))$, under which the coordinate mapping process

$$B_t(\omega) := \omega(t); \ \omega \in \mathbb{R}[0, \infty), \ t \geq 0$$

satisfies condition 3) and 4) of Definition A in section 2.1, so it does not introduce a “Brownian motion” as we defined. Fortunately, by another famous theorem of Kolmogorov\cite{41}, the continuity problem has been solved.

**Theorem 2.2 (Kolmogorov continuity theorem).** Suppose that the process $X = X\{t\}_{t \geq 0}$ satisfies the following condition: for all $T > 0$ there exist positive constants $\alpha, \beta, D$ such that

$$E[|X_t - X_s|^\alpha] \leq D \cdot |t - s|^{1+\beta}; \ 0 \leq s, t \leq T$$

then there exists a continuous modification of $X$.

**Modification**\cite{41} Suppose that $\{X_t\}, \{Y_t\}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$, then we say that $\{X_t\}$ is a modification of $\{Y_t\}$ if

$$P(\{\omega; X_t(\omega) = Y_t(\omega)\}) = 1 \ \forall t$$

Note that if $X_t$ is a modification of $Y_t$, then they have the same finite-dimensional distributions.

Now there is one problem needs to be solved: Why is $B_0(\omega) = \omega(0) = 0$ a.s.?

**Definition**\cite{11} *Wiener measure*, denoted by $\mu_W$, is a probability measure on a measurable space $(C, \mathcal{C})$ having the following two properties:

i) each $\omega(t) \in C$ is normally distributed under $\mu_w$ with mean 0 and variance $t$, i.e.,

$$\mu_w\{\omega(t) \leq x\} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{u^2}{2t}} du, \ x \in \mathbb{R}.$$ 

For $t = 0$ this is interpreted to mean that $\mu_w\{\omega(0) = 0\} = 1$.

ii) the stochastic process $\{\omega(t) : t \geq 0\}$ has independent increments under $\mu_w$, i.e., $\forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n, \ \omega(t_1) - \omega(t_0), \omega(t_2) - \omega(t_1), \cdots, \omega(t_n) - \omega(t_{n-1})$ are independent under $\mu_w$.

**Remark** It may not seem so obvious that this approach in fact proves the existence of the Wiener measure. Also, we cannot obtain the Wiener measure simply by assigning measure one to $C[0, \infty)$; see \cite{32}. One might hope to construct the measure directly on $C[0, \infty)$. Indeed, this is the main idea of the second approach we are going to present.

**Second approach** There are some advantages if we start from $C[0, \infty)$ since we can make it *Polish* (complete and separable) by assigning an appropriate *metric*. It has been proved that the Borel $\sigma$–algebra generated by the open sets in $C[0, \infty)$ is equal to the $\sigma$–algebra generated by all the cylinder sets \cite{32}. Thus there is a totally different way of constructing Wiener measure.
Metric on $C[0, \infty)$

(5) \[ \rho(\omega_a, \omega_b) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_a(t) - \omega_b(t)| \wedge 1) \]

This metric introduces the topology of uniform convergence on compact intervals. For convenience, we denote $C[0, \infty)$ by $C$ and its Borel $\sigma$-algebra by $\mathcal{C}$.

The main idea is to construct a sequence of probability measure $\{P_n\}$ on $(C, \mathcal{C})$ such that $P_n \Rightarrow \mu_w$. Since (2) gives the f.d.d of the Wiener measure $\mu_w$, we must let the f.d.d of $\{P_n\}$ weakly converges to those of $\mu_w$. Although weak convergence in $C$ need not follow the weak convergence of the f.d.d alone in general, it does if we add the condition that $\{P_n\}$ is tight \[32\] \[11\]. In fact, if a metric space is separable and complete, then each probability measure on the measurable space is tight (see Theorem 1.3 in \[11\]). It can be shown that equipped with the metric defined by (5), the canonical space is separable and complete \[11\]. For detail, see \[32\] \[11\].

Remark What is still not natural is that $B_0(\omega) = \omega(0) = 0$ a.s. So next we will talk about constructing Wiener measure on $C_0[0, \infty)$. It appears that the above two different approaches may be adapted to this case.

We can define the Brownian motion $B_t$ for $t \in \mathbb{R}$ as follows: Taking two independent Brownian motions $\hat{B}_t$ and $\tilde{B}_t$, we define

(6) \[ B_t = \begin{cases} \hat{B}_t, & \text{if } t \geq 0; \\ \tilde{B}_{-t}, & \text{if } t < 0. \end{cases} \]

We can work on the space $C_0(\mathbb{R}, \mathbb{R}^n)$ for two-sided Brownian motion. The Wiener measure defined above should be similar defined in this space \[1\].

2.3. Canonical sample space. We consider a SDE

(7) \[ dX_t = b(X_t)dt + \sigma(X_t)dB(t). \]

The canonical sample space is $\Omega := C_0(\mathbb{R}; \mathbb{R}^n)$, space of continuous functions that are zero at time zero, equipped with the compact open topology, the Borel $\sigma$–field $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}^+, \mathbb{R}^n))$, and Wiener (probability) measure $\mu_W$.

We introduce a driving flow $\theta_t$ on this canonical sample space $\Omega$ is given by the Wiener shifts

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega = C_0(\mathbb{R}, \mathbb{R}^d).$$

In this case the measure $\mu_W$ is invariant \[1\], i.e.,

$$\mu_W(\theta_t^{-1}(A)) = \mu_W(A)$$

for all $A \in \mathcal{F}$. Moreover, this invariant measure is actually ergodic with respect to the flow $\theta$ \[13\].
3. Dynamical systems driven by colored noises

Colored noise, or noise with non-zero correlation (‘memory’) in time, are common in the physical, biological and engineering sciences [20]. A good candidate for modeling colored noise is the fractional Brownian motion.

3.1. Fractional Brownian motion. A fractional Brownian motion (fBM) $B^H(t), t \in \mathbb{R}$, with $H \in (0,1)$ the Hurst parameter, is still a Gaussian process. But it is characterized by the stationarity of its increments and a memory property. The increments of the fractional Brownian motion are not independent, except in the standard Brownian motion case ($H = \frac{1}{2}$). Thus it is not a Markov process except when $H = \frac{1}{2}$. Specifically, $B^H(0) = 0$ a.s., mean $\mathbb{E}B^H(t) = 0$, covariance $\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, and variance $\text{Var}[B^H(t) - B^H(s)] = |t-s|^{2H}$. It also exhibits power scaling and path regularity properties with Hölder parameter $H$, which are very distinct from Brownian motion. The standard Brownian motion is a special fBM with $H = 1/2$. Figure 2 is a sample path of the fractional Brownian motion with $H = 0.25$.

![Figure 2. A sample path of fractional Brownian motion $B^H(t)$, with $H = 0.25$](image)

Note that fBM $B^H(t)$ is non-Markov and non-semimartingale. Thus the usual stochastic integration [45] is not applicable, and other integration concepts are needed [40] [39].

3.2. Canonical sample space. The stochastic calculus involving fBM is currently being developed; see e.g. [40] [54] and references therein. This will lead to more advances in the study of SDEs driven by colored fBM noise:

$$dX_t = b(X_t)dt + \sigma(X_t)dB^H(t).$$

Since the fBM $B^H(t)$ is not Markov, the solution process $X_t$ is not Markov either. Thus the usual techniques from Markov processes will not be applicable to the study of SDEs driven...
by fBms. However, the random dynamical systems approach, as described in § above, looks promising \[38, 19\]. The theory of RDS, developed by Arnold and coworkers \[1\], describes the qualitative behavior of systems of stochastic differential equations in terms of stability, Lyapunov exponents, invariant manifolds, and attractors. See Theorem 2.3 in \[38, 9\], the canonical sample space is $\Omega := C_0(\mathbb{R}, \mathbb{R}^d)$, the set of continuous functions that is zero at zero, but the probability measure $\mu_{fBM}$ is generated by $B^H_t$ under the compact-open topology as defined in Section 2. The Borel $\sigma$-field is $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d))$. We can introduce a flow $\theta_t$ on this canonical sample space $\Omega$ defined by the shifts $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega = C_0(\mathbb{R}, \mathbb{R}^d)$.

In this case the measure $\mu_{fBM}$ is invariant, i.e.

$$\mu_{fBM}(\theta_t^{-1}(A)) = \mu_{fBM}(A)$$

for all $A \in \mathcal{F}$.

4. Dynamical systems driven by non-Gaussian noises

In the last two sections, we considered dynamical systems driven by Gaussian noises (white or colored), in terms of Brownian motion or fractional Brownian motion. In this section, we discuss differential equation driven by non-Gaussian Lévy noises.

4.1. Lévy Motions. Gaussian processes like Brownian motion have been widely used to model fluctuations in engineering and science. For a particle in Brownian motion, its sample paths are continuous in time almost surely (i.e., no jumps), its mean square displacement increases linearly in time (i.e., normal diffusion), and its probability density function decays exponentially in space (i.e., light tail or exponential relaxation) \[41\]. But some complex phenomena involve non-Gaussian fluctuations, with peculiar properties such as anomalous diffusion (mean square displacement is a nonlinear power law of time) \[12\] and heavy tail (non-exponential relaxation) \[57\]. For instance, it has been argued that diffusion in a case of geophysical turbulence \[52\] is anomalous. Loosely speaking, the diffusion process consists of a series of “pauses”, when the particle is trapped by a coherent structure, and “flights” or “jumps” or other extreme events, when the particle moves in a jet flow. Moreover, anomalous electrical transport properties have been observed in some amorphous materials such as insulators, semiconductors and polymers, where transient current is asymptotically a power law function of time \[49, 21\]. Finally, some paleoclimatic data \[15\] indicates heavy tail distributions and some DNA data \[52\] shows long range power law decay for spatial correlation.

Lévy motions are thought to be appropriate models for non-Gaussian processes with jumps \[48\]. Let us recall that a Lévy motion $L(t)$, or $L_t$, is a non-Gaussian process with independent and stationary increments, i.e., increments $\Delta L(t, \Delta t) = L(t + \Delta t) - L(t)$ are stationary (therefore $\Delta L$ has no statistical dependence on $t$) and independent for any non overlapping time lags $\Delta t$. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L(t) - L(t_0)| \geq \delta) \rightarrow 0$ as $t \rightarrow t_0$ for any positive $\delta$. With a suitable modification \[4\], these paths may be taken as càdlàg, i.e., paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time.
This generalizes the Brownian motion $B(t)$ or $B_t$, as $B(t)$ satisfies all these three conditions. But *Additionally*, (i) Almost every sample path of the Brownian motion is continuous in time in the usual sense and (ii) Brownian motion’s increments are Gaussian distributed.

Dynamical systems driven by non-Gaussian Lévy noises have attracted much attention recently \[4\, 30\, 50\]. Under certain conditions, the SDEs driven by Lévy motion generate stochastic flows \[35\, 4\], and also generate random dynamical systems (or cocycles) in the sense of Arnold \[1\]. Recently, exit time estimates have been investigated by Imkeller and Pavlyukevich \[27\, 28\], and Yang and Duan \[56\] for SDEs driven by Lévy motion. This shows some qualitatively different dynamical behaviors between SDEs driven by Gaussian and non-Gaussian noises.

Lévy motions are named in honor of the French probabilist Paul Lévy, who first studied them in 1930s. From a mathematical point of view, there are so many reasons why they are so important \[4\], such as:

- There are many important examples, such as Brownian motion, the Poisson process, stable processes, and subordinators.
- They are generalizations of random walks to continuous time.
- They are the simplest class of processes whose paths consist of continuous motion interspersed with jump discontinuities of random size appearing at random times.

**Definition** \[4\] A stochastic process $L = (L(t), t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a Lévy motion if:

- **(L1)** $L(0) = 0$ a.s.;
- **(L2)** $L$ has independent and stationary increments;
- **(L3)** $L$ is *stochastically continuous*, i.e. for all $\epsilon > 0$ and for all $s$

  $$\lim_{t \to s} P(|X(t) - X(s)| > \epsilon) = 0$$

With a suitable modification \[4\], these paths may be taken as càdlàg, i.e., paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time.

Figure 3 is a sample path for a Lévy motion.

One way to understand the structure of the Lévy motions is to employ Fourier analysis. It may be shown that each $L(t)$ is *infinitely divisible*. The infinitely divisible random variables are characterized completely through their characteristic functions by a beautiful formula, established by Paul Lévy and A. Ya. Khintchine in the 1930s. The related definitions and theorems are as follows.

**Definition** \[4\] The *characteristic function* of a stochastic process $X(t)$ taking values in $\mathbb{R}^d$ is the mapping $\Phi_t : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\Phi_t(u) = \mathbb{E}(e^{iu \cdot X(t)}) := \int_{\mathbb{R}^d} e^{iu \cdot y} p_t(dy)$$

where $p_t$ is the distribution of $X(t)$. 
Definition 4. X(t) is infinitely divisible if for each \( n \in \mathbb{N} \), there exists a probability measure \( p_{t,n} \) on \( \mathbb{R}^d \) with characteristic function \( \Phi_{t,n}(u) \) such that \( \Phi_t(u) = (\Phi_{t,n}(u))^n \), for each \( u \in \mathbb{R}^d \).

Theorem 4.1 (The Lévy-Khintchine Formula [4]). If \( L = (L(t), t \geq 0) \) is a Lévy motion, then

\[
\Phi_t(u) = e^{t\eta(u)} \quad \text{for each } t \geq 0, u \in \mathbb{R}^d
\]

where

\[
\eta(u) = ib \cdot u - \frac{1}{2} u \cdot au + \int_{\mathbb{R}^d - \{0\}} [e^{iu \cdot y} - 1 - iu \cdot y I_{|y|<1}(y)] \nu(dy)
\] (9)

for some \( b \in \mathbb{R}^d \), a non-negative definite symmetric \( d \times d \) matrix \( a \) and a Borel measure \( \nu \), called Lévy jump measure, on \( \mathbb{R}^d - \{0\} \) for which \( \int_{\mathbb{R}^d - \{0\}} (|y|^2 + 1) \nu(dy) < \infty \). Here \( I_S \) is the
indicator function of the set \( S \).

Conversely, given a mapping of the form (9), we can always construct a Lévy motion for which \( \Phi_t(u) = e^{tu} \) and call it a Lévy motion with the characteristics or generating triple \((a, b, \nu)\).

**Remark:** The so-defined Borel measure \( \nu \) in (9) is called the Lévy jump measure, which should not be confused with the probability measure induced by the Lévy motion, to be defined below.

The different terms which appear in the Lévy-Khintchine formula have a probabilistic significance emphasized in [46]. Every Lévy motion is obtained as a sum of independent processes with three types of characteristics \((0, b, 0), (a, 0, 0)\) and \((0, 0, \nu)\). Thus, the Lévy measure accounts for the jumps of \( L \) and the knowledge of \( \nu \) permits to give a probabilistic construction of \( L \); see [46].

### 4.2. Canonical sample space.

Consider a SDE driven by non-Gaussian Lévy noise

\[
dX_t = b(X_t)dt + \sigma(X_t)dL(t).
\]

The canonical space has to be enlarged to include all the cadlag functions, i.e. functions that are right-continuous and have left limits, defined on \( \mathbb{R} \) and taking values in \( \mathbb{R}^d \). This space is denoted as \( D(\mathbb{R}, \mathbb{R}^n) \).

We adopt the same point of view as in Section 2 that a stochastic process is also a random variable, i.e.

\[
L_t(\omega) : \omega \to D(\mathbb{R}, \mathbb{R}^n) \quad \omega \to L(t), t \in \mathbb{R}.
\]

**Remark 4.2.** Here the compact-open metric defined in (5) cannot make \( D(\mathbb{R}^+, \mathbb{R}^n) \) separable [45]. For example, if \( f_\alpha(t) = 1_{[\alpha, \infty)}(t), f_\beta(t) = 1_{[\beta, \infty)}(t) \), then \( \rho(f_\alpha, f_\beta) = 1/2 \) for all \( \alpha, \beta \) with \( 0 \leq \alpha < \beta \leq 1 \), and since there are uncountably many such \( \alpha, \beta \), the space is not separable.

However, it can be made complete and separable when endowed with the Skorohod metric [11]. With this special metric, we call \( D(\mathbb{R}, \mathbb{R}^d) \) a Skorohod space. The Skorohod metric on \( D(\mathbb{R}, \mathbb{R}^d) \) is defined as

\[
d(x, y) := \sum_{m=1}^{\infty} \frac{1}{2^m} (1 \wedge d_m^o(x^m, y^m)) \quad \text{for all } x, y \in D
\]

where \( x^m(t) := g_m(t)x(t), y^m(t) := g_m(t)y(t) \) with

\[
g_m(t) := \begin{cases} 
1, & \text{if } |t| \leq m-1 \\
m-t, & \text{if } m-1 \leq |t| \leq m, \\
0, & \text{if } |t| \geq m
\end{cases}
\]

and

\[
d_m^o(x, y) := \inf_{\lambda \in \Lambda} \left\{ \sup_{-m \leq s < t \leq m} \left| \log \frac{\lambda(t) - \lambda(s)}{t-s} \right| \vee \sup_{-m \leq t \leq m} |x(t) - y(\lambda(t))| \right\},
\]
where $\Lambda$ denotes the set of strictly increasing, continuous functions from $\mathbb{R}$ to itself.

The Borel $\sigma$–field under this topology is denoted as $\mathcal{S}$. For studying the weak convergence and tightness in $D$, the same approach adopted in $C$ can be applied except that the fact the natural projections are not continuous need to be noticed [11].

**Definition** The probability measure, $\mu_L$, in $(D(\mathbb{R}, \mathbb{R}^n), \mathcal{S})$ that makes every element in $D(\mathbb{R}, \mathbb{R}^n)$ a sample Lévy path is called the Lévy probability measure. Note that this measure is not to be confused with the Lévy jump measure $\nu$ mentioned above.

We can also introduce a flow $\theta = (\theta_t, t \in \mathbb{R})$ on this canonical sample space $\Omega$ by the shifts

\[(\theta_t \omega)(s) := \omega(t + s) - \omega(t).\]

The (Lévy) probability measure $\mu_L$ is invariant under this flow, i.e.

\[\mu_L(\theta_t^{-1}(A)) = \mu_L(A)\]

for all $A \in \mathcal{F}$; see [4] (Page 325) and [37]. This flow is an ergodic dynamical system [1] with respect to the above probability measure $\mu_L$.

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(Jinqiao Duan) DEPARTMENT OF APPLIED MATHEMATICS, ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO, IL 60616, USA
E-mail address, J. Duan: duan@iit.edu

(Xingye Kan) DEPARTMENT OF APPLIED MATHEMATICS, ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO, IL 60616, USA
E-mail address, X. Kan: xkan@iit.edu

(Björn Schmalfuß) INSTITUT FÜR MATHEMATIK, FAKULTÄT EIM, WARBURGER STRASSE 100, 33098, PADERBORN, GERMANY
E-mail address, Björn Schmalfuß: schmalfuss@upb.de