Generalized Tensor Analysis Method Applied to Non-time-orthogonal Coordinate Frames

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A generalized covariant method of analysis applicable to coordinate frames for which time is not orthogonal to space, such as spacetime around a star possessing angular momentum or on a rotating disk, is presented. Important aspects of such an analysis are shown to include i) use of the physically relevant contravariant or covariant component form for a given vector/tensor, ii) conversion of physical (measured) components to generalized coordinate components prior to tensor analysis, iii) use of generalized covariant constitutive equations during tensor analysis, and iv) conversion of coordinate components back to physical components after tensor analysis. The method is applied to electrodynamics in a rotating frame, and shown to predict the measured results of the Wilson and Wilson and Roentgen/Eichenwald experiments.

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I. INTRODUCTION

The literature on electrodynamic analysis in rotating frames, though fairly extensive, rarely addresses the issue of relating generalized coordinate values for electric and magnetic field strengths to those physical values measured in experiment. When it is addressed, ambiguity seems to arise.

For example, Crater[1], advocates “apparent freedom of choice for which components of the electromagnetic second-rank $F$ tensor, (contravariant, covariant, or mixed) one uses...” and that on the rotating frame, “... one would have two devices for measuring voltages, capacitances, inductances, etc., one for those physicists who prefer the covariant convention and one for those who prefer the contravariant convention.” This begs the question, which Crater does not bring up: What if one simply took the usual volt meter used in the lab onto the rotating frame. Which of these various coordinate types, if either, would it read?

As another example, Ridgely[2] notes that contravariant electromagnetic field tensors have a vector basis and covariant tensors have a one-form basis, but does not suggest which of them corresponds to values read on instruments located in a rotating frame. Nor does he consider the closely related issue of determining physical (measured) components from coordinate (mathematical) components (be they contravariant or covariant). The present article answers these questions and presents a general analysis method suitable for, but not limited to, electrodynamics in rotating frames.

In Section II a generalized tensor analysis method for mechanics and/or electrodynamics is developed, that while applicable to any frame, finds specific utility for those coordinate frames in which time is not orthogonal to space. Such coordinate frames have off diagonal space-time terms in the metric, and are designated “non-time-orthogonal” (NTO). They include rotating frames, as well as spacetime around objects having significant angular momentum.

Section III provides a brief mathematical review of rotating frames. That, along with the method of Section II is then applied in Section IV to the Wilson and Wilson experiment, and in Section V to the Roentgen/Eichenwald experiment. It is shown that the NTO nature of the rotating frame introduces terms into the analysis that would not be present in a time orthogonal (TO) frame analysis, and that agreement between theory and experiment is directly attributable to those additional terms.

II. NTO TENSOR ANALYSIS: RULES OF THE GAME

A. Contravariant or covariant?

Generalized position coordinates $x^\mu$ are contravariant in nature. That is, the generalized displacement four-vector between two infinitesimally separated 4D points (events) is, to be precise, $dx^\mu$, not $dx_\mu$. In the (special) case where basis vectors $e_\mu$ are orthogonal (i.e., the coordinate grid lines are all orthogonal to one another), this distinction is not critical. (See Appendix A.) In the more general case (e.g., the NTO case), however, where all basis vectors $e_\mu$ are not orthogonal to one another, the distinction between $dx^\mu$ and $dx_\mu$ becomes important, and one must keep in mind that they do not represent the same entity in the physical world. In an NTO frame, $dx^0$ corresponds to displacement along the time axis, whereas $dx_0$ corresponds to displacement in both time and space. Hence, in the most general and accurate sense, $dx^0$ (and not $dx_\mu$) is the correct representation of coordinate displacement.
The need to distinguish between the physical significance of contravariant and covariant components carries over to other four vectors. For example, the four velocity

\[ u^\mu = \frac{dx^\mu}{d\tau} \]  

(1)

must be a contravariant vector since \( dx^\mu \) is contravariant. As before, the covariant vector components \( u_\mu = g_{\mu\nu}u^\nu \) may be considered an equivalent form of the four velocity in an orthogonal coordinate system (such as Minkowski coordinates in a Lorentz frame), but not in the more general NTO case.

Four momentum is defined, where \( L \) is the Lagrangian, as

\[ p_{\mu} = \frac{\partial L}{\partial u^{\mu}} \]  

(2)

and hence, it must be covariant since \( u^\mu \) is contravariant. For a free particle \( L = \frac{1}{2}mu^\mu u_\mu \) and four-momentum is \( mu_{\mu} \), not \( mu^\mu \). In an NTO frame these quantities are distinctly different, and in fact, as the author has shown, the correct value for energy in an NTO rotating frame is found from \( p_0 \) in the rotating frame, not \( p^0 \).

The four current density \( J^\mu \) obeys the physical law of conservation of charge

\[ J^{\mu}_{\cdot \cdot} = 0, \]  

(3)

where we have generalized to non-Minkowski coordinate systems via the covariant derivative symbolized by the semi-colon. Since the covariant derivative is with respect to (the contravariant) four vector displacement, the only possibility whereby \( \mathbf{8} \) can equal a 4D scalar invariant (zero) is if the physical world four current density is represented in contravariant form. Certainly \( J^\mu_{\cdot \cdot} = 0 \) holds, but in the most general case, the contravariant derivative \( \partial^\mu \) (raised \( \mu \) is with respect to \( dx_\mu \)) is not a derivative with respect solely to \( \partial \), but rather with respect to an amalgam of both space and time. Hence \( J^\mu_{\cdot \cdot} = 0 \) would not equal the time rate of change of charge density we would measure with physical instruments, and \( J_0 \) would not correspond to charge density, even in a generalized sense.

In general, we can use physical laws, which hold covariantly in a generalized sense, to determine what form (covariant or contravariant) any given vector or tensor must be represented by in order for it to correspond to a physical world entity. This is, in fact, what we did with four current and the charge conservation law above. As a second example, four-force is the derivative of four momentum with respect to proper time and so must be covariant.

In electrodynamics, one can use Maxwell’s equations and the Lorentz force law formulated in 4D to determine the appropriate form of field tensors. In a Minkowski coordinate system the 3D electric and magnetic fields \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \) and \( \mathbf{H} \) form the 4D tensors \( \mathbf{4} \).

\[
F^{\mu\nu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-1 & B_3 & -B_2 & 0 \\
-E_3 & B_2 & -B_1 & 0 \\
-E_3 & B_2 & -B_1 & 0 \\
\end{pmatrix}
\]

(4)

The generalized Maxwell source equations in 4D for Gaussian units are

\[
H^{\mu\nu} = \frac{4\pi}{c} J^\nu,
\]

(5)

where \( H^{\mu\nu} = F^{\mu\nu} \) in vacuum. Since the derivative is covariant (lowered index) in form and the current density is contravariant, the \( H \) tensor (and hence also the \( F \) tensor) dependent on (arising from) charge and current sources must be contravariant.

Note, however, that finding the four-force (covariant) vector acting on a charge or three-current density is determined via

\[
f_\mu = \frac{1}{c} F_{\mu\nu} J^\nu.
\]

(6)

Hence, since four-force is covariant and four-current is contravariant, the \( F \) tensor must, in this case, be covariant. We conclude that \( F^{\mu\nu} \) represents the physical fields \textit{found from} charges [see \( \mathbf{4} \)] and currents in a vacuum, but \( F_{\mu\nu} \) represents the physical fields as they \textit{act on} charges and currents [see \( \mathbf{4} \)] to produce forces. Again, this distinction is not critical in TO frames, though it is quite relevant in NTO frames when one wishes to know which electric and magnetic field components (covariant or contravariant) will be monitored by physical instruments. Since the covariant form of \( F \) represents the physical forces acting on charges and currents (which form the basis of field measuring devices such as voltmeters and Gauss meters), those covariant components \( F_{\mu\nu} \) correlate with what one would measure in an experiment. [However, one must ascertain that the principle of operation of the measuring device is force based (covariant \( f_\mu \)) rather than motion based (contravariant \( mu^\mu \) on the LHS of \( \mathbf{4} \)).]

B. Physical vs. Coordinate Components

Getting the correct contravariant or covariant components is not quite enough, however, in order to compare theoretical results with measured quantities. A generalized vector component (e.g., four velocity component \( u^1 \))

\[ 1 \] Care must be taken, however, if the four force is to be related to motion. That is, \( p^\mu = mu^\mu \) must be found from \( f^\mu \), not \( f_\mu \), and \( u^\mu \) integrated with respect to proper time to find coordinate displacement \( \Delta x^\mu \).
is a mathematical entity whose value generally does not equal the value one would measure with physical instruments (e.g., the velocity measured using standard rods and clocks in the $x^1$ axis direction.)

If a given basis vector does not have unit length, the magnitude of the associated component will not equal the physical quantity measured. For example, a vector with a single non-zero component value of 1 in a coordinate system where the corresponding basis vector for that component has length 3 does not have an absolute (physical) length equal to 1, but to three. In general, the physical component (measured) value is equal to the generalized coordinate component value only in the special case where the basis vector has unit length (is normalized.) One example of this is Lorentz frames with Minkowski coordinates, which have unit basis vectors and vector/tensor components equal to the physically measured values.

To determine physical component values, we need to calculate, given the generalized coordinate component $v^\alpha$, what the component value would be for a unit length basis vector. In Appendix B, for reference, we derive this well-known relation\[8\], i.e.,

$$v^3 = \sqrt{g_{00}v^0} \quad v^0 = \sqrt{-g^{00}v_0} \quad (7)$$

where underlining implies no summation, carets over indices indicate physical components, spatial coordinates are designated by Roman script, and the minus signs are needed as $g^{00}$ and $g_{00}$ are negative.

This result is readily extended to second order tensors, i.e.,

$$T^{\mu\nu} = \sqrt{-g^{00}g_{00}} T^{00} \quad T_{\mu\nu} = \sqrt{-g^{00}g_{00}} T_{00} \quad (8)$$

where, to be precise, multiplication of metric components by (-1) is required whenever they are negative.

We note that physical components are a special case of anholonomic components\[11\]. We caution that physical components do not transform according to the vector/tensor transformation laws and are not components of vectors/tensors\[8\]. Hence we may calculate physical components to determine what we would measure via experiment, but we must use coordinate components in carrying out tensor analysis.

We note further that physical components $v^\alpha$ and $v_\alpha$ are identical in orthogonal axes systems, but are generally different in non-orthogonal axis systems. This underscores the theme of Section II A and the need to find physical components for the physically relevant (contravariant or covariant) tensor form in order to compare theory with experiment in an NTO frame.

### C. Covariant Constitutive Equations

The 3D plus time constitutive relations

$$D = \varepsilon E \quad B = \mu H \quad (9)$$

were expressed by Minkowski in the 4D covariant form\[8\]

$$H^{\mu\nu}u_\nu = \varepsilon F^{\mu\nu}u_\nu \quad (10)$$

$$\varepsilon\sigma_{\lambda\mu\nu}F^{\lambda\mu}u_\nu = \mu\varepsilon\sigma_{\lambda\mu\nu}H^{\lambda\mu}u_\nu. \quad (11)$$

We will use (10) and (11) to relate $F^{\mu\nu}$ to $H^{\mu\nu}$ in NTO frames.

### D. General Method of Analysis

Analysis of the most general type of problem therefore comprises the following steps.

1. Determine whether known (measured) component values are contravariant or covariant in physical character. [See (7) through (8).]

2. Convert the appropriate contravariant or covariant physical components to the corresponding coordinate components via (7) and/or (8).

3. Apply tensor analysis as relevant to the particular problem, using generalized covariant constitutive equations.

4. Convert the coordinate component (contravariant or covariant as physically appropriate) answer to physical components to determine what would be measured in the physical world.

### III. ROTATING FRAME TRANSFORMATION

For the rotating frame analysis we employ cylindrical coordinates with $(cT,R,\Phi,Z)$ for the lab and $(ct,r,\phi,z)$ for the rotating frame. The transformation below, between the lab and a rotating frame having angular velocity $\omega$ in the $Z$ direction, is well known, widely used, and argued elsewhere by the author\[10\] to be appropriate for analyses such as that herein.

$$cT = ct \quad R = r \quad \Phi = \phi + \omega t \quad Z = z. \quad (12)$$

In matrix form this may be expressed as

$$\Lambda^\alpha_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\omega}{c} & 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Lambda_\beta^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\omega}{c} & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (13)$$

\[2\] Anholonomic, or non-coordinate, components are those components associated with non-coordinate basis vectors. For the special case where these non-coordinate basis vectors have unit length, anholonomic components equal physical components.
where \( A \) and \( B \) here are upper case Greek, \( \Lambda_{\alpha\beta} \) transforms a contravariant lab vector \( \{e.g., \, dX^B = (cdT, dR, d\Phi, dZ)^T\} \) to the corresponding contravariant rotating frame vector \( \{dx^\alpha = (cdt, dr, d\phi, dz)^T\} \), and \( \Lambda^A_{\beta} \) transforms the latter back from the rotating frame to the lab.

The following relations, which we will use in Sections IV and V, are derived in Klauber\[13\] from (12). The rotating frame coordinate metric \( g_{\alpha\beta} \) and its inverse \( g^{\alpha\beta} \) are

\[
\begin{align*}
g_{\alpha\beta} &= \begin{bmatrix}
-(1 - \frac{c^2}{r^2}) & 0 & \frac{c^2}{r^2} & 0 \\
0 & 1 & 0 & 0 \\
\frac{c^2}{r^2} & 0 & r^2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
g^{\alpha\beta} &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{r^2}
\end{bmatrix}. \quad (14)
\end{align*}
\]

The lab metric \( G_{AB} \) and its inverse \( G^{AB} \) are

\[
\begin{align*}
G_{AB} &= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R^2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad G^{AB} &= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{r^2}
\end{bmatrix}. \quad (15)
\end{align*}
\]

IV. WILSON AND WILSON RESULT

The Wilson and Wilson\[14\] experiment comprised a rotating cylinder of magnetic permeability \( \mu \) and dielectric constant \( \varepsilon \) and an axially directed uniform magnetic field \( B_0 \). They measured a radially directed electric field (consonant with a magnetic field internal to the cylinder) in the lab of

\[
E_{\text{lab, int measured}} = \frac{vB_0}{c} \left( \frac{1}{\varepsilon} - \mu \right), \quad (16)
\]

where \( v = \omega r \).

Recently, Pellegrini and Swift\[15\] (PS) seemed to show that, based on the global coordinate transformation \[12\] in Cartesian, rather than cylindrical, form \( \{e.g., \, dX^B = (cdT, dR, d\Phi, dZ)^T\} \) to a rotating frame, the theoretical prediction for the Wilsons experiment disagreed with the actual test results. Subsequent testing by Hertzberg et al.\[16\] confirmed the validity of the Wilson and Wilson result. As shown by PS, and in greater detail by Weber\[17\], the correct answer could be found by using local Lorentz frames as surrogates for the global rotating frame.

PS noted their global transformation to the rotating frame resulted in a metric with off diagonal space-time components \( \{e.g., \, dX^B = (cdT, dR, d\Phi, dZ)^T\} \). PS suggested that the resolution to the issue might lie in the form taken by the constitutive equations in the NTO rotating frame. Subsequently, covariant expressions for the constitutive equations were found and used by Burrows\[18\] and Ridgely\[19\] to yield the correct prediction for the Wilsons experiment.

However, Burrows and Ridgely carried out much of their analyses with Maxwell’s equations expressed in 3D form, and did not use a fully 4D generalized covariant tensor method throughout. Further, Ridgely did not employ the widely accepted transformation to the rotating frame used by PS, leaving unanswered the question of the ultimate validity of that transformation. Still further, PS showed that with certain assumptions they could derive the Wilsons result, yet with those same assumptions, they arrived at an incorrect prediction for a different experiment performed by both Roentgen\[20\] and Eichenwald\[21\] (RE). Neither Burrows nor Ridgely\[3\] addressed the RE experiment directly, leaving open the question as to what their analyses would predict for that test.

In the following sections we employ the steps of section \( \boxed{10} \) to derive \( \boxed{16} \). We also use those steps in Appendix C where the issue of modeling of the magnetization by surface currents as described in PS is resolved.

A. Overview of Analysis Procedure for Wilsons

Experiment

Steps 1 and 2: Convert the physical component \( B_0 \) measured in the lab in the air to the associated covariant component of the 4D \( F \) tensor. Obtain the corresponding contravariant form of the \( F \) tensor.

Step 3: Transform the \( F \) tensor to the rotating frame. Take the \( F \) tensor as equal to the \( H \) tensor in air. Apply boundary conditions obtained from the 4D Maxwell equations at the boundary of the cylinder to find the \( H \) tensor inside the rotating cylinder. Apply the constitutive equations \[10\] and \[11\] to the \( H \) tensor to obtain the \( F \) tensor inside the cylinder. Transform the \( F \) tensor back to the lab.

Step 4: Convert the appropriate component of \( F \) to obtain the physical component in the radial direction of the lab electric field inside the cylinder as in \( \boxed{10} \).

Schematically, the individual mathematical steps in the analysis are outlined in \[17\] to \[19\]. “Ext” means external to the rotating permeable dielectric cylinder, and

\[3\] In fact, in ref. \[18\], p. 119 just prior to equation (45a), Ridgely states that for an axially directed laboratory electric field, the magnetic induction field \( B^\parallel = 0 \) in the rotating frame. This appears to contradict the experimental result of Roentgen and Eichenwald, refs. \[20\] and \[21\], and stated in ref. \[15\], p. 700 equation \[27\] and following paragraph therein.
Then in the rotating frame one finds between the external and internal boundary conditions at the rotating material boundary source term and the 4D Gauss divergence law to apply

Then use the source Maxwell equations with zero "Int" means internal to the cylinder.

\[ H^{\mu\nu}_{\text{rot,ext}} = \frac{4\pi}{c} J^\nu = 0. \] (25)

Then applying the Gauss divergence theorem in 4D one gets the boundary condition

\[ n_\mu H^\mu_{\text{ext}} = n_\mu H^\mu_{\text{int}} \] (26)

where \( n_\mu \) is a 4-vector in the direction of interest. By taking various values for \( n_\mu \) [such as \( n_\mu = (0,1,0,0) \)] at the cylinder boundaries one readily finds that

\[ H^\mu_{\text{rot,ext}} = H^\mu_{\text{rot,int}} \]

(27)

where (27) is the 4D \( \mathbf{D,H} \) tensor internal to the cylinder in rotating coordinates. To find the associated 4D \( \mathbf{E,B} \) tensor, use the covariant constitutive relations and (27) with \( u^\mu \) and \( u_\mu \).

For a fixed location in the rotating frame, the four-velocity is

\[ u^\nu = \frac{dx^\nu}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{dx^\nu}{dt} = \gamma \left[ \begin{array}{c} c \\ 0 \\ 0 \\ 0 \end{array} \right], \] (28)

where \( \gamma \) has the usual meaning and \( v = \omega r \). The covariant four vector obtained from lowering the index of the contravariant 4-velocity \( u^\nu \) is

\[ u_\nu = g_{\nu\alpha} u^\alpha = \left[ \begin{array}{c} -c \sqrt{1 - v^2/c^2} \\ 0 \\ r \omega \sqrt{1 - v^2/c^2} \\ 0 \end{array} \right]. \] (29)

Using (28) with (27) having the index \( \sigma = 3 \), one finds, to first order,

\[ F_{\text{rot,int}}^{AB} = -F_{\text{rot,ext}}^{AB} \equiv \frac{\mu B_0}{r}. \] (30)

4 Note that the components of \( e_{\sigma \lambda \mu \nu} \) for a non-orthonormal basis are not all 0 or 1. (See ref. Exercise 8.3, p. 207.) However, all such components differ from one by a factor equal to the determinant of the transformation from a local orthonormal basis to the non-orthonormal basis, and that factor cancels on both sides of (30), making calculations herein using (30) simpler.
where the initial equality results from the anti-symmetry of the tensor, and from here on we freely interchange numeric and letter indices, i.e., $\mu = t, r, \phi, \text{or} \ \varepsilon$ is equivalent to $\mu = 0, 1, 2, 3$, respectively.

Using (29) with (10) having the index $\mu = 1$, one obtains

$$2(H_{\text{rot,int}}^{10}u_0 + H_{\text{rot,int}}^{12}u_2) = 2\varepsilon \left(F_{\text{rot,int}}^{10}u_0 + F_{\text{rot,int}}^{12}u_2\right)$$

or

$$\frac{B_0}{r}\frac{r^2\omega}{\sqrt{1-v^2/c^2}} = \varepsilon \left(F_{\text{rot,int}}^{12} \frac{r^2\omega}{\sqrt{1-v^2/c^2}} - F_{\text{rot,int}}^{10}\sqrt{1-v^2/c^2}\right).$$

(32)

Using (30) and dropping higher order terms, one finds

$$F_{\text{rot,int}}^{10} = -F_{\text{rot,int}}^{01} \cong -\left(\frac{1}{\varepsilon} - \mu\right) \frac{vB_0}{c}.$$

(33)

Evaluation of other components in similar fashion shows them to be all zero and results in

$$F_{\text{rot,int}}^{\mu\nu} \cong \begin{bmatrix} 0 & \frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & 0 & 0 \\ -\frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & 0 & \frac{\mu B_0}{c} & 0 \\ 0 & -\frac{\mu B_0}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

(34)

As an aside, if we wished to know what $E$ and $B$ field values would be measured within the rotating cylinder, we would lower the index in (33) and calculate physical component values.

Transforming back to the lab, we get, again to first order

$$F_{\text{AB}}^{\mu\nu} = \Lambda_{A}^{\mu} \Lambda_{B}^{\nu} F_{\text{rot,int}}^{\mu\nu}$$

\cong \begin{bmatrix} 0 & \frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & 0 & 0 \\ -\frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & 0 & \frac{\mu B_0}{c} & 0 \\ 0 & -\frac{\mu B_0}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

(35)

Lowering the index to get the physically relevant covariant form$^5$ yields

$$F_{\text{AB}}^{\mu} = G_{A}^{\gamma} G_{B}^{\Omega} F_{\text{lab,int}}^{\gamma\Omega}$$

\cong \begin{bmatrix} 0 & -\frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & \mu B_0 & 0 \\ \frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right) & 0 & 0 & 0 \\ 0 & -\mu B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

(36)

It is interesting to note that the physical component for the $A = 1, B = 2$ (i.e., $Z$ direction) component above is $\mu B_0$, as might be expected. More importantly, the electric field in the radial direction in coordinate components is the $A = 1, B = 0$ component above. Hence, the experimentally measured value for the radial electric field in the lab is the physical component value

$$E_{\text{R}}^{\mu_{\text{lab, int}}} = \frac{vB_0}{c} \left(\frac{1}{\varepsilon} - \mu\right).$$

(37)

This is the Wilson and Wilson result.

It is important to note how this result hinges on the covariant constitutive equations (10) and (11) as expressed in an NTO frame. In (31) to (33) we get the unexpected result that $F_{\text{rot,int}}^{10} \neq 0$, even though $H_{\text{rot,int}}^{10} = 0$. This is due to the NTO nature of the rotating frame. That is, the lowering operation performed by $g_{\alpha\beta}$ (with off diagonal terms) yields a $u_{\mu}$ having a non-zero $v=2$ (i.e., $\phi$ direction) component [see (29)], even though $v^\mu = \gamma(c,0,0,0)^T$. This non-zero term in $u_{\mu}$ manifests in (34) [i.e., in (35)] in just the right way to give us two extra terms in (36) and result in (37).

V. THE ROENTGEN AND EICHENWALD RESULT

With an external electric field in the lab of $E_0$ in the axial (i.e., $Z$) direction, and the same transformation steps and boundary condition equations used above for deriving the Wilson and Wilson result, one finds, internal to the cylinder in the rotating frame,

$$H_{\text{rot,int}}^{\mu\nu} = H_{\text{rot,ext}}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & E_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\omega}{c} E_0 \\ -E_0 & 0 & \frac{\omega}{c} E_0 & 0 \end{bmatrix}. $$

(38)

Using $\sigma = 1$ in covariant constitutive relation (11) yields

$$F_{\text{rot,int}}^{23} = \mu H_{\text{rot,int}}^{23} = -\frac{\omega}{c} E_0.$$ 

(39)

From (10) with $\mu = 3$, $u_{\mu}$ of (29), and (38) one then finds

$$F_{\text{rot,int}}^{30} \cong \mu H_{\text{rot,int}}^{30} = -\frac{E_0}{\varepsilon}.$$ 

(40)

where we have again dropped higher order terms. Other values for indices $\sigma$ and $\mu$ in (10) and (11) result in all other tensor components of zero, and hence

$$F_{\text{rot,int}}^{\mu\nu} \cong \begin{bmatrix} 0 & 0 & 0 & \frac{E_0}{\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\omega}{c} E_0 \\ -\frac{E_0}{\varepsilon} & 0 & \frac{\omega}{c} E_0 & 0 \end{bmatrix}. $$

(41)

\[5\] We would find, to first order, an axial magnetic field of physical magnitude $\mu B_0$ and a radial electric field of physical magnitude $vB_0/c$.

\[6\] We do this step here (and at the beginning of the section) merely to remain consistent with the steps laid out in the general methodology of sections II.D and II.A. As noted, covariant physical components in orthogonal coordinate systems, such as the lab, equal contravariant physical components.
Lowering the indices, one obtains the covariant form, to first order

\[
F_{\mu
u}_{\text{rot, int}} = g_{\mu\alpha}g_{\nu\beta}F_{\alpha\beta}^{\text{rot, int}}
\]

\[
\approx \begin{bmatrix}
0 & 0 & 0 & -\frac{E_0}{c} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{v}{c}r \left( \frac{1}{\varepsilon} - \mu \right) E_0 \\
\frac{E_0}{\varepsilon} & 0 & -\frac{v}{c}r \left( \frac{1}{\varepsilon} - \mu \right) E_0 & 0
\end{bmatrix}.
\] (42)

The physical component for the radial direction in the rotating frame is

\[
B_r = F_{23}^{\text{rot, int}} = \sqrt{g^{22}} \sqrt{g^{33}} F_{23}^{\text{rot, int}}
\]

\[
\approx \frac{1}{r} F_{23}^{\text{rot, int}} \approx \left( \frac{1}{\varepsilon} - \mu \right) \frac{v}{c} E_0,
\] (43)

which for permeability \(\mu = 1\) is the RE result. (See ref. 17, p. 700, paragraph after eq. (27).)

VI. SUMMARY AND CONCLUSIONS

A tensor analysis method with very general applicability to mechanics and electrodynamics in any non-orthonormal basis vector coordinate system has been presented. Application of that method focused on one type of non-time-orthogonal (NTO) coordinate frame, the rotating frame, and correct predictions were obtained for the Wilson and Wilson and Roentgen/Eichenwald experiments.

Essential aspects of the method comprise i) use of the physically relevant contravariant or covariant form of vectors/tensors, ii) conversion of physical (measured) component values to generalized coordinate component values prior to tensor analysis, iii) use of generalized covariant constitutive equations during tensor analysis, and iv) conversion after tensor analysis of coordinate component answers to physical components for comparison with experiment.

The author cautions that we have not shown that the fashionable co-moving local Lorentz frame analysis method and the generalized tensor method presented herein are equivalent for NTO frames. While they yield identical results in many NTO cases, including those treated herein, they do not do so in all. See the author’s prior work 14 for a summary of differences and for logic supporting the generalized tensor method as the preferable approach.

APPENDIX A: PHYSICAL CHARACTER OF VECTOR FORMS

Since

\[
dx_{\mu} = g_{\mu\nu}dx^\nu
\] (A1)

if \(g_{\mu\nu}\) has \(g_{01} \neq 0\), then the \(x^0\) (time) axis is not orthogonal to the \(x^1\) spatial axis, and (with \(g_{02} = g_{03} = 0\))

\[
dx_0 = g_{00}dx^0 + g_{01}dx^1.
\] (A2)

Hence, unlike \(dx^0\), the component \(dx_0\) represents not a displacement purely through time, but an amalgam of displacement through both time (\(dx^0\)) and space (\(dx^1\)). If \(g_{01}\) were to equal zero (time orthogonal to space), then \(dx_0\) would comprise a displacement through time only.

In terms of basis vectors, the \(e_0\) basis vector is aligned with the time axis, and in the time-orthogonal case shares the same line of action as its associated basis one-form \(\omega^0\). Hence \(dx^0e_0 = dx_0\omega^0\) and both \(dx^0\) and \(dx_0\) represent only time (and no space) displacements.

In the more general case where the \(e_0\) (time) basis vector is not orthogonal to the \(e_i\) (space) basis vectors, then \(\omega^0\) does not share the same line of action as \(e_0\) and, \(dx_0\omega^0 = dx_0^0e_0 + dx_0^i e_i\). Hence, in this (NTO) case \(dx_0\) represents displacement in both space and time (i.e., along \(\omega^0\), not \(e_0\)), whereas \(dx^0\) represents displacement only in time. Thus, \(dx_0\) and \(dx^0\) can not be considered equivalent in any conceptual or physical sense. They do not represent the same physical entity.

APPENDIX B: PHYSICAL COMPONENTS

Consider an arbitrary vector \(v\) in a 2D space

\[
v = v^1 e_1 + v^2 e_2 = v^1 \hat{e}_1 + v^2 \hat{e}_2,
\] (B1)

where \(e_i\) are coordinate basis vectors and \(\hat{e}_i\) are unit length (non-coordinate) basis vectors pointing in the same respective directions. That is,

\[
\hat{e}_i = \frac{e_i}{|e_i|} = \frac{e_i}{\sqrt{e_i^2}} = \frac{e_i}{\sqrt{g_{ii}}},
\] (B2)

where underlining implies no summation. Note that \(e_1\) and \(e_2\) here do not, in general, have to be orthogonal. Note also, that physical components are those associated with unit length basis vectors and hence are represented by indices with carets in (B1).

Substituting (B2) into (B1), one readily obtains

\[
v^i = \sqrt{g_{ii}} v^i.
\] (B3)

Relation (B3) between physical and coordinate components is valid locally in curved, as well as flat, spaces and can be extrapolated as summarized in Section II B to 4D general relativistic applications, to higher order tensors, and to covariant components.

APPENDIX C: SURFACE CURRENT MODEL

NTO ANALYSIS

In PS 17, Section II, p. 697, magnetization in the rotating frame is modeled by an equivalent surface 3D cur-
rent density [see PS eq (6)]
\[ \mathbf{J}_S = \mathbf{M} \times \hat{n} \]  \hspace{0.5cm} (C1)

or
\[ J_{S i} = \varepsilon_{ijk} M^j n^k = \begin{bmatrix} 0 & \frac{\mu - 1}{4\pi} B_0 & 0 \\ \frac{\mu - 1}{4\pi} B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{\text{rot}} \end{bmatrix}, \]  \hspace{0.5cm} (C2)

where \( M^i \) is magnetization in the axial direction, \( n^k \) is a unit vector pointing inward (in the radial direction) from the cylinder outer surface, \( J_{S \phi} = \sigma_{\text{rot}} \) is the current density on the inner surface in the circumferential (i.e., the \( \phi \)) direction, and \(- J_{S \phi} = -\sigma_{\text{rot}} \) is the current density on the outer surface.

Note that (C2) is a “3-vector” in terms of physical components. We must convert it to coordinate components on the outer surface.

From (C2) and the definition of physical components
\[ \frac{(\mu - 1)}{4\pi} B_0 = \text{phys comp } J_{S \phi} = \sqrt{g^{\phi \phi}} J_S \phi = \sqrt{1 - v^2/c^2} J_S \phi \]  \hspace{0.5cm} (C3)

where, as before, \( v = \omega r \). Solving the LHS and RHS of (C3) for the coordinate component \( J_{S \phi} \) allows us to write the covariant four-vector as
\[ J_S \mu = \begin{bmatrix} 0 \\ 0 \\ \frac{(\mu - 1)}{4\pi} \frac{v B_0}{\sqrt{1 - v^2/c^2}} \\ 0 \end{bmatrix}. \]  \hspace{0.5cm} (C4)

The contravariant surface current four-vector is then
\[ J^\mu_S = g^{\mu \nu} J_S \nu \]
\[ = \begin{bmatrix} -1 & 0 & \frac{\mu - 1}{4\pi} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\mu - 1}{4\pi} & 0 & -\frac{v B_0}{\sqrt{1 - v^2/c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{(\mu - 1)}{4\pi} \frac{v B_0}{\sqrt{1 - v^2/c^2}} \\ 0 \end{bmatrix} \]
\[ = \begin{bmatrix} \frac{(\mu - 1)}{4\pi} v & 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 \\ 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 & 0 \\ \frac{\mu - 1}{4\pi} B_0 & 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 \\ 0 & \frac{(\mu - 1)}{4\pi} B_0 & 0 & 0 \end{bmatrix} \]  \hspace{0.5cm} (C5)

Note how the off diagonal term in the NTO metric \( g^{\mu \nu} \) results in a \( \mu =0 \) term that is not zero, and effectively “creates” charge. Transforming (C5) to the lab, we get
\[ \mathbf{J}_S^{\Lambda \beta} = \Lambda_\beta^\alpha J_S^\alpha \]
\[ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\mu - 1}{4\pi} \frac{v B_0}{\sqrt{1 - v^2/c^2}} & 0 & \sqrt{1 - v^2/c^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{(\mu - 1)}{4\pi} v & 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 \\ 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 & 0 \\ \frac{(\mu - 1)}{4\pi} B_0 & 0 & 0 & \frac{(\mu - 1)}{4\pi} B_0 \\ 0 & \frac{(\mu - 1)}{4\pi} B_0 & 0 & 0 \end{bmatrix} \]  \hspace{0.5cm} (C6)

where higher order terms were dropped on the RHS, and where we note, as did PS, that the transformation (12) to the lab does not create charge.

Physical charge density in the lab to first order is then
\[ \sigma_{\text{lab}} = J_{\partial S}^0 = \sqrt{-G_{00}} J_{S}^0 = \frac{(\mu - 1)}{4\pi} \frac{v}{c} B_0, \]  \hspace{0.5cm} (C7)

which equals (7) in PS and, again, yields the Wilson and Wilson result.

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