Potential flows in a core-dipole-shell system: numerical results

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Numerical solutions for: the integral curves of the velocity field (streamlines), the density contours, and the accretion rate of a steady-state flow of an ideal fluid with \( p = Kn^\gamma \) equation of state orbiting in a core-dipole-shell system are presented. For \( \gamma \neq 2 \), we found that the non-linear contribution appearing in the partial differential equation for the velocity potential has little effect in the form of the streamlines and density contour lines, but can be noticed in the density values. The study of several cases indicates that this appears to be the general situation. The accretion rate was found to increase when the constant \( \gamma \) decreases.

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I. INTRODUCTION

In a recent work [1] we study the streamlines and density contours of a stationary fluid in the presence of either a rigid sphere or a black hole both with a distant shell of matter modeled by a dipolar field. To fix ideas see Fig. 1 where this situation is depicted. The substitution of a halo or shell of matter by a fixed axisymmetric field is an old and common method in astrophysics, for example to simulate galaxy halos [2]. Numerical experiments confirm that this is usually a good approximation [3]. This method can be used in other physical situations where a core and a distant shell of matter are present.

In [1] we used for the fluid the stiff equation of state, i.e., a polytropic, \( p = Kn^\gamma \) with \( \gamma = 2 \). This stiff equation of state over simplifies the partial differential equation for the velocity potential field \( \Phi \) (it becomes linear). In this case the sound velocity in the fluid is equal to the speed of light. Therefore the stiff equation of state represents a limit situation not easily encountered in usual physical situations. Abrahams and Shapiro [4] studied the more realistic situation, a fluid with polytropic equation of state with \( 1 < \gamma < 2 \) in the presence of either a rigid sphere or a black hole. The sound velocity in this case is less than the speed of light and the partial differential equation for \( \Phi \) turns to be nonlinear.

In the present work, following Abrahams and Shapiro [4], we study a more realistic situation than the one examined in [1]. Now we compute the streamlines and baryon density contours for a fluid with polytropic equation of state with \( 1 < \gamma < 2 \) in the presence of either a rigid sphere or a black hole both with dipolar halos. In other words, now we consider the same situation as before, but for a general polytropic fluid. We consider also the accretion rate of particles into the black hole at its dependence with \( \gamma \). Furthermore, we complete the work of Abrahams and Shapiro [4] by examining the black hole case with \( 1 < \gamma < 2 \) and asymptotic constant velocity (case not studied in the quoted work). In particular, we study streamlines, density contours and accretion rate.

In section II we present the basic equations that describe potential flows and the nonlinear equation for the velocity potential for a polytropic equation of state of the form \( p = Kn^\gamma \). Section III is divided in three sub-sections. In sub-section III-A we introduce the metric that represents a central core with a distant dipolar shell of matter. This field is the first approximation to represent distant matter, like halos and rings. In sub-section III-B we present the numerical method used to solve the nonlinear partial differential equation presented in section II. In sub-section III-C we show some results and study the behavior of the streamlines and density contours of the fluid in the presence of either a rigid sphere or a black hole both with a dipolar halo. Also we compute the accretion rate of particles falling into a black hole with and without dipolar halo. Finally, in section IV we summarize our results.

II. BASIC EQUATIONS

The solution for potential flow of the equation of motion derived from the energy-momentum tensor for an ideal fluid, \( T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu} \), is

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\[ hU_\alpha = \Phi_{,\alpha}, \quad (1) \]

where \( h = (\rho + p)/n \) denotes the enthalpy per baryon, \( n \) the baryon density, \( p \) the pressure of the fluid, \( \rho \) the rest mass energy density, \( \epsilon \) the internal energy density, \( U^\mu \) represents the fluid four-velocity, and \( \Phi \) the scalar potential. Our conventions are \( G = c = 1 \), metric with signature \(+2\). Partial and covariant derivatives are denoted by commas and semicolons, respectively.

We recall that, by definition, an ideal fluid is adiabatic \(^5\), i.e. the entropy is constant along the world line of each fluid element. If in an instant \( t \) the entropy is constant throughout the volume of the fluid (isentropic case) then, for all time and any subsequent motion of the fluid, it retains everywhere the same constant value. In that case we find the solution for potential flows \(^6\). Hence, only isentropic flows can have potential flows \(^6\).

From Eq. (1) and the baryon number density conservation equation,

\[ (nU^\mu)_{,\mu} = 0, \quad (2) \]

we obtain the differential equation for the velocity potential,

\[ \Box \Phi + [\ln (\frac{n}{h})]_{,\alpha} \Phi_{,\alpha} = 0, \quad (3) \]

where, \( \Box \Phi = [\sqrt{-g}g^{\mu\nu}\Phi_{,\mu},_{\nu}] / \sqrt{-g} \). This equation is in general nonlinear and depends on the fluid equation of state. If we consider a polytropic equation of state \( p = Kn^\gamma \) and use the first law of thermodynamics for isentropic flows, \( d(\rho/n) + pd(1/n) = Ts \), (\( s \) is the entropy per baryon and \( T \) the temperature), we find

\[ \rho = \rho_0 + \frac{Kn^\gamma}{\gamma - 1}. \quad (4) \]

Now, assuming that the flow is relativistic we can neglect the rest mass energy density \( \rho_0 \) with respect to the internal energy density \( Kn^\gamma / (\gamma - 1) \), i.e. the flow satisfies a barotropic equation of state, \( p = (\gamma - 1)\rho \). In this case, the baryon number density can be written in terms of the enthalpy as

\[ n = \kappa h^{1/(\gamma - 1)}, \quad (5) \]

where \( \kappa = \left(\frac{\gamma - 1}{\gamma K}\right)^{1/(\gamma - 1)} \). Then, equation (3) can be written in the form

\[ \Box \Phi + \frac{2\gamma - 1}{\gamma - 1} [\ln h(\Phi)]_{,\alpha} \Phi_{,\alpha} = 0, \quad (6) \]

The simplest case is found when \( \gamma = 2 \), Eq. (3) reduces to a linear equation. In this case the barotropic sound speed, defined as \( dP/d\rho \equiv c_s^2 \), is equal to the speed of light, i.e. we have a stiff equation of state. In the general case \( 1 < \gamma < 2 \), the differential equation is nonlinear and the barotropic sound speed is \( c_s^2 = (\gamma - 1) \) is less than the speed of light.

The normalization condition, \( U_\alpha U^\alpha = -1 \), gives us a relation between the enthalpy and the scalar field,

\[ h = \sqrt{-\Phi_{,\alpha} \Phi^{,\alpha}. \quad (7) \]

This relation will be also useful to determine the baryon number density.

In this section we have followed the work of Moncrief \(^7\) which is in accord with Tabensky and Taub \(^8\) that consider a constant barotropic sound speed, \( c_s^2 = (\gamma - 1) \). We note that Abrahams and Shapiro \(^4\) consider a variable polytropic sound speed.

### III. POTENTIAL FLOWS IN BLACK HOLES AND RIGID SPHERES WITH DIPOLAR HALOS

#### A. The metric

To incorporate a dipolar field in the Schwarzschild metric we consider the solution found in \(^9\) where authors model the intermediate vacuum between a core and a distant shell of matter. This core-shell system was found solving the vacuum Einstein equations for a general static axially symmetric metric (Weyl solution \(^10\)). One of these equations
can be solved in terms of Legendre polynomials increasing with the distance. These shell-like structures can be used to model many situations of interest in astrophysics, such as: supernovas, nebulae, and galaxy halos that have a core and an exterior shell of matter. As a first approximation we will consider only the first term in the expansion, the dipolar term that in Newtonian gravity corresponds to a constant force. By letting in the solution presented in \[7\] the quadrupole and octopole moments equal zero, we obtain

\[
ds^2 = -\left(\frac{u-1}{u+1}\right)e^{2Du} dt^2 + m^2(u+1)^2 e^{2Du(2-u)-D^2[u^2(1-v^2)+v^2]} \left[\frac{du^2}{u^2-1} + \frac{dv^2}{1-v^2}\right] + (u+1)^2(1-v^2)e^{-2Du} dv^2,
\]

with

\[
u = \frac{1}{2m} \left[ \sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2} \right],
\]

\[ = \frac{r}{m} - 1, \quad u \geq 1,
\]

\[
v = \frac{1}{2m} \left[ \sqrt{\rho^2 + (z+m)^2} - \sqrt{\rho^2 + (z-m)^2} \right],
\]

\[
\varphi = \varphi, -1 \leq v \leq 1,
\]

where \((r, \theta, \varphi), (u, v, \varphi)\) and \((\rho, z, \varphi)\) are spherical, prolate spherical, and cylindrical coordinates, respectively. The metric \([8]\) represents a monopolar core in the presence of an external dipolar field \((D)\) that is associated to a distant shell or halo of matter. As we said before, this field can be seen as the first approximation to model an external concentration of matter like halos and rings.

### B. Numerical method

To solve equation \([8]\) in the space-time with metric \([8]\) we assume: a) That fluid is stationary, i.e., the function \(\Phi\) depends on time only through the addition of \(-at\), where \(a\) is a constant related to the zeroth component of the velocity. b) That due to the axial symmetry of the metric the potential \(\Phi\) does not depend on the variable \(\varphi\), and c) That the fluid is a test fluid, i.e. the metric does not evolve and it is given \textit{a priori}. Also we put \(m = 1\).

First we notice that due to the acceleration of the fluid produced by the dipole, the stationary condition leads to a region of instability away from the core (black hole or rigid sphere). So we have a region of stability \(\Omega\) that is depicted in Fig. \([2]\). The size of this region is very sensitive to the values of the dipolar field \((D)\) \([1]\) that accelerates the fluid. The stationary criteria is checked using the fact that baryon density \((5)\) is a positive defined quantity \([1]\). With the above mentioned assumptions Eq. \([8]\) reduces to an elliptical differential equation with an inner boundary conditions near the black hole or rigid sphere and an exterior boundary condition (asymptotic condition).

To solve Eq. \([8]\) we use a computational code based on a marching method with a second order precision, five point, finite difference. The numerical grid is evenly spaced in \(r\) and in \(\theta\). The marching method is the \textit{Stabilized Error Vector Propagation} (SEVP) which is very efficient in solving separable and non-separable elliptic equations. The main ingredient of the method is a clever superposition of a particular solution of the elliptic equation with and a homogeneous solution. For a detailed discussion of the method, see for instance \([1]\) \([2]\).

In the case of a rigid sphere, for the inner boundary condition, the usual condition of zero normal velocity in the surface of the sphere is employed. In general, the fluid velocity must be equal to the corresponding component of the velocity of the surface. Since usual stars have gaseous surfaces (not hard), this condition does not describe a typical flow around a star. In special astrophysical situations, like relativistic flows around a neutron star, this condition can be valid. For a black hole we use the condition that the fluid number density particle must remain finite on the black hole horizon \([3]\). This leads to the numerical condition \([4]\) that near \(u = 1\) (the black hole horizon),

\[
\frac{\partial}{\partial r^*} \left[ \frac{\Phi_{r^*} - \Phi_{\tau}}{u-1} \right] = 0,
\]

where \(r^* = (u + 1) + 2 \ln \left(\frac{u+1}{u-1}\right)\) is the tortoise radial coordinate \([3]\). With the assumption that \(\Phi_{\tau} = -a < 0\) the condition \([4]\) can also be satisfied in our case and it will be taken as the inner boundary condition. For numerical applications of this condition see \([1]\) \([2]\).
For the outer boundary condition, with help of the geodesic equation for the metric \[8\], we can find a characteristic value for the scalar field potential \[1\],

$$\Phi = \int \left( \frac{u + 1}{u - 1} \right) e^{2Dv(u) - \frac{L^2}{2} [u^2 (1 - v^2) + v^2]} \sqrt{\frac{u - 1}{u + 1}} e^{2Duv + k^2} du,$$

where the integration is performed along the lines of constant \(v\), usually between the surface of the sphere (or black hole horizon) and 60 Schwarzschild radius. Later, with the results obtained for the scalar field \(\Phi\), we check the stationary condition to see the value of the radius in which the system becomes non-stationary. The constant \(k\) is the fluid particle energy function.

To solve the non-linear equation \[9\] we first put the non-linear term equal to zero and calculate the linear part of the equation to find \(\Phi\). This solution is used as an initial guess for the non-linear problem. Then, from \[9\] we compute the fluid enthalpy. Finally, with this information we compute the non-linear term that is introduced in the non-linear equation to find new values of \(\Phi\). The process is repeated until the sum of the fractional change in the enthalpy for all interior points of the grid in one iteration is less than a certain error \(\epsilon\). Usually less than 10 iterations were required to reach an error of \(\epsilon \leq 10^{-6}\). Obviously, for the linear case we need only one iteration. The number of iterations depends on the value of \(\gamma\). For \(\gamma \approx 1\) the nonlinear factor tends to infinity and the program diverge. When \(\gamma = 1\) the fluid is pressureless (dust) and the fluid flow is geodesic, i.e., no longer obeys \(6\).

The code was tested for the case \(\gamma = 2\) using the analytical results for the steady flow of a fluid in the presence of either a hard sphere \[10\] or a black hole \[11\]. In this case the numerical solution agreed with the exact solution within an error better than 1.5\% for the radial velocity and 1\% for the angular velocity. Since we are interested in qualitative aspects of the fluid behavior this accuracy is sufficient for our purposes.

Another aspect of the fluid dynamics is the accretion rate of matter into a black hole. This accretion rate can be computed from \(1\) and \(5\), we find

$$\dot{N} = - \int_S nU^i \sqrt{-g} ds_i = - \int_S \kappa h \sqrt{-g} \Phi s^u g^{uu} \sqrt{-g} du dv,$$

where the integration is performed in a two-surface sphere centered on the black hole. The exact form of \(12\), for \(\gamma = 2\), is known \[13\]. It is \(\dot{N} = 16\pi M^2 n_\infty U_\infty^0\), where \(n_\infty\) and \(U_\infty^0\) are the asymptotic density and zeroth component of the 4-velocity, respectively. \(M\) is the black hole mass. We test our code to compute the accretion rate with this exact solution. We found an error less than 1\%.

C. Numerical results

In Figs. 3 and 4 for the case of a rigid sphere with a dipolar halo, we show the numerical results for the streamlines and density contour lines of the baryon number density. The constants are: radius of the sphere 2.5 (Schwarzschild radius 2), dipolar strength \(D = 0.001\), \(\gamma = 1.75\) and \(k = a = 1.25\), the outer boundary condition was set to 60 Schwarzschild radius. We see that the streamlines do not differ much from the linear case \[1\] and that the contour lines have the same qualitative behavior as in the linear case, \(\gamma = 2\). The numerical values for the baryon density are not the same due to the exponent \(1/(\gamma - 1)\) presented in the relation between the enthalpy and the baryon density. To ensure a physical result, in solving \(9\), we require that enthalpy be positive in every iteration of our code. The baryon number density is a positive quantity. We can have a negative enthalpy solution of \(n = \kappa h^{1/(\gamma - 1)}\), but in this case we have a negative baryon density.

For the same values of the constants, in Figs. 2 and 5 we show the numerical results for the streamlines and density contours for the case of a black hole with a dipolar halo. Again we see little difference compared to the linear case. These results tell us that the contribution of the non-linear term does not affect in a significant way the qualitative behavior of the fluid.

For the case of a black hole without dipole \((D = 0)\) we also found that, for a constant asymptotic velocity, that the streamlines and density contours do not distinguish between the linear and non linear case. The scaled accretion rate for the black hole \((\dot{N}/\kappa)\) increases when \(\gamma\) decreases, e.g., for \(\gamma = 1.9, 1.8, 1.7\), the scaled accretion rate increases in 6\%, 13\%, 24\%, respectively, the value of the asymptotic velocity used in this case is \(v_\infty = 0.6\), the same as in \[13\].

For \(\gamma = 2\), the linear case, we find that the value of the scaled accretion rate with or without dipolar halo differs by 1\%. We also find in this case, for the same values of Fig. 3, that the scaled accretion rate increases when \(\gamma\) decreases, e.g., for \(\gamma = 1.9, 1.8, 1.7\), the scaled accretion rate increases in 8\%, 18\%, 33\%, respectively. In the dipole case the
growth of the scaled accretion rate when \( \gamma \) decreases is greater than in the case without dipole. In both cases the growth of the scaled accretion rate is due to the factor involving the enthalpy [cf. Eq. (12)].

As mentioned before, for an accelerated fluid it is difficult to have stationary flow. For the cases studied along the paper we found that for \( D \leq 0.01 \) we can have a region of reasonable size (a ball greater than 45 Schwarzschild radius) where we have laminar flow. For \( D = 0.03 \) the radius of this ball shrinks to 5, and for \( D = 0.05 \) stationary flow does not exist. In summary, we believe that our results are representative of a generic situation when \( D \leq 0.01 \).

Finally, we want to point out that the variation of \( \Phi \) in the systems black hole and rigid sphere with or without dipole for the linear and nonlinear cases is, in almost every interior point of the grid, less than 5%.

IV. CONCLUSIONS

We have solved numerically the streamlines and the density contours for the baryon number density of an ideal fluid in the presence of either a rigid sphere or a black hole both with dipolar halo. The fluid has barotropic equation of state, \( p = (\gamma - 1) \) with \( 1 < \gamma < 2 \), that represents a more realistic physical situation than the linear case studied in [1]. We see that the nonlinear term does not affect qualitative the form of the streamlines and density contours when compared to the linear case, this is due to the low variation of the scalar field \( \Phi \) between the linear and nonlinear cases. When compared with other situations studied in the literature this fact appears to be generic. Another interesting result is that the accretion rate of the cases studied increases when the constant \( \gamma \) decreases. In the presence of the dipolar field this growth is bigger compared with the case without dipole.

The cases of a rotating black hole and rigid sphere with halos modeled by multipole moments beyond the dipole is under active consideration by the authors.

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[17] We found an oversight in equation (3.2) of [4], in the second term (the angular term) the expression inside the square bracket must be \( (1 - x^2) \) instead of \( (1 - x^2)^{1/2} \). This oversight is repeated in equation (A3), (A8) and (A10) of the appendix concerning the finite difference schemes used in the calculation of the particle density and the partial differential equation. As we mentioned before the differences between the cases with \( \gamma = 2 \) (linear case) and \( 1 < \gamma < 2 \) (nonlinear case) are small.
FIG. 1. The dipole approximation and the stationary condition leads to a finite region $\Omega$ of stationary flow, in this region a constant “force field” $\mathbf{F}$ is present. Outside $\Omega$ the fluid is not stationary.

FIG. 2. Numerical results for the fluid streamlines with $\gamma = 1.75$ when an external dipolar field with $D = 0.001$ is present and a rigid sphere of radius $r = 2.5$ (Schwarzschild radius equal 2) is placed as an obstacle. We set the values of the constant $k = a = 1.25$. The axes are defined as $X = r \sin \theta$ and $Z = r \cos \theta$, with $r = u + 1$ and $\cos \theta = v$. 
FIG. 3. Numerical results for the density contours \((n/\kappa)\) of the baryon number density when an external dipolar field with \(D = 0.001\) is present and a rigid sphere of radius \(r = 2.5\) (Schwarzschild radius equal 2) is placed as an obstacle. The axes and the constants are defined as in Fig. 2.

FIG. 4. Numerical results for the streamlines when an external dipolar field of value \(D = 0.001\) is present and a black hole is placed as an obstacle. The black hole has radius \(r = 2\) (Schwarzschild radius equal 2). The constants and the meaning of axes are the same of Fig. 2.
FIG. 5. Numerical results for the density contours \((n/\kappa)\) of the baryon number density when an external dipolar field of value \(D = 0.001\) is present and a black hole is placed as an obstacle. The black hole has radius \(r = 2\) (Schwarzschild radius equal 2). The constants and the meaning of the axes are the same of Fig. [2].