WELL-QUASI-ORDER OF PLANE MINORS
AND AN APPLICATION TO LINK DIAGRAMS

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Abstract. A plane graph \( H \) is a plane minor of a plane graph \( G \) if there is a sequence of vertex and edge deletions, and edge contractions performed on the plane, that takes \( G \) to \( H \). Motivated by knot theory problems, it has been asked if the plane minor relation is a well-quasi-order. We settle this in the affirmative. We also prove an additional application to knot theory. If \( L \) is a link and \( D \) is a link diagram, write \( D \to L \) if there is a sequence of crossing exchanges and smoothings that takes \( D \) to a diagram of \( L \). We show that, for each fixed link \( L \), there is a polynomial-time algorithm that takes as input a link diagram \( D \) and answers whether or not \( D \to L \).

1. Introduction

A highlight of the seminal Graph Minors theory of Robertson and Seymour is the proof of Wagner’s Conjecture: graphs are well-quasi-ordered under the graph minor relation \[46\]. We recall that a graph \( H \) is a minor of a graph \( G \) if a graph isomorphic to \( H \) can be obtained from \( G \) by a sequence of edge deletions, vertex deletions, and edge contractions. We also recall that a relation \( \leq \) on a set \( X \) is a quasi-order if it is reflexive and transitive, and it is a well-quasi-order if, in addition, for any infinite sequence of elements \( x_1, x_2, \ldots \) in \( X \), there exist integers \( i < j \) such that \( x_i \leq x_j \).

1.1. Our main result. As we discuss below, due to its connection with problems in knot theory, it has been asked if the plane minor relation is a well-quasi-order. A plane graph is a planar graph with a given embedding on the plane. Two plane graphs are combinatorially equivalent if there is a self-homeomorphism of the plane that takes one to the other. A plane graph \( H \) is a plane minor of a plane graph \( G \) (we write \( H \preceq_{\mathbb{R}^2} G \)) if there is a sequence of edge deletions, vertex deletions, and edge contractions that takes \( G \) to a plane graph combinatorially equivalent to \( H \).

For the sake of formality, we make precise the notion of a plane edge contraction. Let \( G \) be a plane graph, and let \( e \) be an edge of \( G \). Let \( \Delta \) be a region in the plane homeomorphic to a closed disk, such that \( \Delta \cap G \) consists of \( e \) and its endvertices. The plane contraction of \( e \) is the operation that consists of contracting the whole region \( \Delta \) to a point.

It is readily seen that \( \preceq_{\mathbb{R}^2} \) is a quasi-order on the set of all plane graphs. We prove the following.

Theorem 1.1. The relation \( \preceq_{\mathbb{R}^2} \) is a well-quasi-order on the collection of all plane graphs.

Besides its independent interest, this result is relevant in view of its applications to knot theory. Plane minors are related to linking graphs associated to positive braids \[6\] (see also \[1,40\]), and to the surface minor relation on embedded surfaces in \( \mathbb{R}^3 \) \[5\].

It is natural to consider the analogous notion of plane minor for graphs embedded in an arbitrary surface \( \Sigma \). If \( H \) and \( G \) are graphs embedded in a surface \( \Sigma \), then \( H \) is a \( \Sigma \)-minor of \( G \) if there is a sequence of vertex deletions, edge deletions, and edge contractions performed on \( \Sigma \) that take \( G \)
to a graph combinatorially equivalent to $H$. For each surface $\Sigma$, the $\Sigma$-minor relation is clearly a quasi-order.

Also motivated by a knot theory problem that we will shortly discuss, our own original question was whether the sphere minor relation is a well-quasi-order. A sphere graph is a graph with a given embedding into the sphere $S^2$. If $H$ and $G$ are sphere graphs, we write $H \preceq_{S^2} G$ if $H$ is a sphere minor of $G$. It is straightforward to see that Theorem 1.1 implies the following.

**Corollary 1.2.** The relation $\preceq_{S^2}$ is a well-quasi-order on the collection of all sphere graphs.

As in the Robertson-Seymour theory, knowing that a relation $\leq$ on a set $X$ is a well-quasi-order is particularly useful when accompanied by the existence of a polynomial-time algorithm to test $\leq$. As we will see, the next statement follows from the proof of the main result in [3].

**Theorem 1.3.** For each fixed sphere graph $H$, there is an algorithm that takes as input a sphere graph $G$ on $n$ vertices and decides whether $H \preceq_{S^2} G$ in time $O(n^2 \log n)$.

1.2. **Strategy of the proof of Theorem 1.1.** To prove Theorem 1.1 we make essential use of parameters and results in Topological Graph Theory, in particular on the tree-width and branch-width of a graph. We also use results on immersions of 2-regular digraphs, that is, digraphs in which each vertex has in-degree and out-degree two.

Theorem 1.1 states that if $G_1, G_2, \ldots$ is an infinite sequence of plane graphs, then there exist $i < j$ such that $G_i \preceq_{S^2} G_j$. It follows from classical results by Robertson and Seymour that if in this sequence there are plane graphs with arbitrarily large tree-width, then we are done. Thus we are left with the case in which the graphs in the sequence have bounded tree-width.

In this latter case, we exploit the very close relationship between plane graphs and 2-regular digraphs. As we review below, if $G$ is a plane graph then its directed medial graph $DM(G)$ is a 2-regular digraph in which each face is bounded by a directed closed walk. Hannie [20] proved that 2-regular digraphs of bounded tree-width are well-quasi-ordered under the digraph immersion relation. A result by Johnson [36] implies that if $G_1, G_2, \ldots$ is a sequence of plane graphs with bounded tree-width, then the corresponding sequence of directed medial graphs has bounded tree-width. With these two results, and the relationship between $\preceq_{S^2}$ and the immersion relation on 2-regular digraphs, we settle the case in which the graphs in $G_1, G_2, \ldots$ have bounded tree-width. See Section 2 for details.

1.3. **A further application to knot theory.** As in [5], we became interested in the plane minor relation (in our case, in the sphere minor relation) coming from a problem in knot theory.

Throughout this paper, we work in the piecewise linear category. All links under consideration are nonsplit, unordered, unoriented and contained in the 3-sphere $S^3$. We remark that when we speak of a link $L$ we include the possibility that $L$ is a link with only one component, that is, a knot. All diagrams under consideration are regular diagrams in the 2-sphere $S^2 \subset S^3$.

In Figure 1 we illustrate the crossing exchange and crossing smoothing operations in link diagrams. Let $L$ be a link, and let $D$ be a diagram. If there is a sequence of crossing exchanges and smoothings that takes $D$ to a diagram of $L$, then we write $D \sim L$.

![Figure 1](image)

**Figure 1.** In (a) we illustrate a crossing exchange operation, and in (b) and (c) the two crossing smoothing operations.
Question. For a fixed link \( L \), how difficult is it to decide whether a given diagram \( D \) satisfies that \( D \sim L \)?

There is an extensive body of literature dealing with the question of when two links (or link diagrams or projections) are related under some set of local operations [1, 2, 8, 14, 15, 17–19, 22, 24–35, 37–39, 41, 44, 48–50]. We refer the reader to [42] for a detailed discussion on these references.

Back to the question above, in order to clarify the issue of “how difficult” we need to turn to the formalism of computational complexity, and pose this question in the standard form of a decision problem, as follows.

Problem: \( \sim L \) (where \( L \) is a fixed link)

Input: A link diagram \( D \).

Question: Is it true that \( D \sim L \)?

In [42] it was proved that if \( L \) is a torus link \( T_{2,m} \) or a twist knot, then there is a polynomial-time algorithm that solves \( \sim L \). That is, there is a polynomial \( p(n) \) and an algorithm \( A_L \) that takes as input any diagram \( D \) with \( n \) crossings, and answers whether or not \( D \sim L \) in at most \( p(n) \) steps. It was also pointed out in that paper that it follows from results in [14] that if \( L \) is any prime link with crossing number at most 5, then there is also a polynomial-time algorithm that solves \( \sim L \).

It was conjectured in [42] that for each fixed link \( L \), there is a polynomial-time algorithm that solves \( \sim L \). Using Theorems 1.1 and 1.3, we settle this conjecture in the affirmative:

Theorem 1.4. For each fixed link \( L \), there is a polynomial-time algorithm that solves \( \sim L \). More specifically, there is an algorithm that takes as input a diagram \( D \) on \( n \) crossings, and decides in time \( O(n^2 \log n) \) whether \( D \sim L \).

2. Proof of Theorem 1.1

To prove Theorem 1.1 we exploit the close relationship between plane graphs and a particular class of 2-regular plane digraphs. We recall that a digraph is 2-regular if every vertex has in-degree and out-degree two. If \( D \) is a connected 2-regular plane digraph in which the rotation around each vertex goes inward-outward-inward-outward (as in Figure 3(a) below), then we say that \( D \) is a good digraph. Note that this is equivalent to the condition that each face is bounded by a directed closed walk. We let \( D \) denote the collection of all good digraphs.

As we now recall, good digraphs are closely related to connected plane graphs, via the notion of a directed medial graph. This connection is also discussed in [20, Chapter 1.1] and in [36, Chapter 2].

2.1. Directed medial graphs, good digraphs, and the relation \( \preceq \). Given a connected plane graph \( G \), its medial graph \( M(G) \) has a vertex for each edge of \( G \), and an edge between two vertices for each face of \( G \) in which their corresponding edges occur consecutively. See the left hand side of Figure 2. The edges of \( M(G) \) are drawn in the corresponding faces, and then \( M(G) \) is considered as a plane graph. We turn \( M(G) \) into a digraph as follows. Checkerboard colour the faces of \( M(G) \) [21], so that each black face contains a vertex of \( G \), as in the middle illustration in Figure 2. Orienting the face boundary of each black face clockwise induces a direction of each edge of \( M(G) \), and as a result we obtain the directed medial graph \( DM(G) \) of \( G \). See the right hand side illustration in Figure 2.

Clearly, the directed medial graph \( DM(G) \) of a connected plane graph \( G \) is a good digraph. Conversely, if \( D \) is a good digraph, then there is a unique (up to combinatorial equivalence) connected plane graph \( G \) such that \( D \) is the directed medial graph of \( G \). Thus there is (again, up to combinatorial equivalence) a one-to-one correspondence between the collection \( \mathcal{D} \) of all connected plane graphs and the collection \( \mathcal{D} \) of all good digraphs.
Figure 2. On the left hand side we illustrate a plane graph $G$ (black vertices and thick edges) and its medial graph $M(G)$ (white vertices and thin edges). In the middle we show the checkerboard colouring of $M(G)$ in which each black face contains a vertex of $G$. On the right hand side we illustrate the directed medial graph $DM(G)$ of $G$.

We exploit this bijection between $\mathcal{P}$ and $\mathcal{D}$ by identifying a quasi-order $\preceq$ on $\mathcal{D}$, and observing that $\preceq_{\mathbb{R}^2}$ and $\preceq$ are “parallel” quasi-orders: if $G$ and $H$ are connected plane graphs, then $H \preceq_{\mathbb{R}^2} G$ if and only if $DM(H) \preceq DM(G)$. This parallelism between $\preceq_{\mathbb{R}^2}$ and $\preceq$ will allow us to establish key results on $\preceq_{\mathbb{R}^2}$ based on results about $\preceq$.

To define $\preceq$, we note that for each vertex $v$ of a good digraph $D$, there are two natural split operations, illustrated in Figure 3(b) and (c). If $D$ and $D'$ are good digraphs, then write $D \preceq D'$ if a digraph combinatorially equivalent to $D$ can be obtained from $D'$ by a sequence of split operations. Clearly, $\preceq$ is a quasi-order on $\mathcal{D}$.

It is worth to remark that a split operation may disconnect a good digraph (which, by definition, is connected). This possibility is not a concern in our setting. By definition, $D \preceq D'$ if a sequence of split operations takes $D'$ to a digraph combinatorially equivalent to $D$. Since both $D$ and $D'$ are connected, then no split operation in this sequence can result in a disconnected digraph.

Figure 3. The two split operations on a vertex in a good digraph.

It is easy to see, as noted in [14, Proposition 1.6], that an edge contraction or deletion in a plane graph $G$ corresponds to a split operation in its directed medial graph $DM(G)$. We also note that if $H$ and $G$ are connected plane graphs and $H \preceq_{\mathbb{R}^2} G$, then a plane graph combinatorially equivalent to $H$ can be obtained from $G$ by some sequence of edge deletions and contractions (without the need of eliminating isolated vertices). These observations imply the parallelism between $\preceq_{\mathbb{R}^2}$ and $\preceq$ under the mapping $G \mapsto DM(G)$:

Fact 2.1. If $G$ and $H$ are connected plane graphs, then $H \preceq_{\mathbb{R}^2} G$ if and only if $DM(H) \preceq DM(G)$.

2.2. Proof of Theorem 1.1 We start by stating two key facts behind the proof of Theorem 1.1. These statements involve the tree-width of a graph, an important parameter developed and investigated in the Robertson-Seymour theory. We refer the reader to standard references in the field [12,43].
Fact 2.2. Let $G_1, G_2, \ldots$ be a sequence of connected plane graphs with bounded tree-width. Then the sequence of good digraphs $DM(G_1), DM(G_2), \ldots$ has bounded tree-width.

Fact 2.3. If $D_1, D_2, \ldots$ is a sequence of good digraphs with bounded tree-width, then there exist $i < j$ such that $D_i \preceq D_j$.

Deferring the proofs of these statements for the moment, we give the proof of Theorem 1.1. In the proof we also use a well-known fact (see for instance [12]): if $G$ is a fixed plane graph, then every plane graph with sufficiently large tree-width contains $G$ as a plane minor.

Proof of Theorem 1.1. The goal is to show that if $G_1, G_2, \ldots$ is a sequence of plane graphs, then there exists $i < j$ such that $G_i \preceq_{R^2} G_j$. We start by noting that it suffices to prove this under the assumption that $G_1, G_2, \ldots$ are connected, as the general case follows immediately from Higman’s lemma [23]. Thus we assume that $G_1, G_2, \ldots$ is a sequence of connected plane graphs.

If the tree-width of the graphs in the sequence $G_1, G_2 \ldots$ is unbounded, then from the statement immediately before this proof it follows that there is a $j$ such that $G_1 \preceq_{R^2} G_j$, and so we are done. If the tree-width of the graphs $G_1, G_2, \ldots$ is bounded, then by Fact 2.2 the sequence $DM(G_1), DM(G_2), \ldots$ has bounded tree-width, and thus by Fact 2.3 there exist $i < j$ such that $DM(G_i) \preceq DM(G_j)$. Using Fact 2.1 it follows that $G_i \preceq_{R^2} G_j$. □

2.3. Proofs of Facts 2.2 and 2.3. As we shall see, Facts 2.2 and 2.3 follow from known results on digraph immersions. For a review of the notion of digraph immersion, we refer the reader to [10,11]. For our current purposes, it suffices to recall that the immersion relation $\preceq_{imm}$ is a quasi-order on the set of all Eulerian digraphs (a digraph is Eulerian if the in-degree of each vertex equals its out-degree). If $D$ and $D'$ are Eulerian digraphs, we use $D \preceq_{imm} D'$ to denote that $D$ immerses into $D'$.

We make essential use of the following fact, which states that $\preceq_{imm}$ and $\preceq$ are the same relation, if we restrict ourselves to good digraphs. This is noted in [10,11] (see also [20, Proposition 1.2]).

Proposition 2.4. Let $D$ and $D'$ be good digraphs. Then $D \preceq D'$ if and only if $D \preceq_{imm} D'$.

In view of this statement, if we are dealing with good digraphs, then we can rewrite any known statement that involves $\preceq_{imm}$ with the corresponding statement involving $\preceq$.

In the proof of Fact 2.2 we follow the custom to use $tw(G)$ to denote the tree-width of a graph $G$. We also recall that $P_k \Box P_k$ is the planar $(k \times k)$-grid.

Proof of Fact 2.2. Johnson [36, Theorem 2.5] proved that for each fixed positive integer $k$, every Eulerian digraph with sufficiently large tree-width satisfies $DM(P_k \Box P_k) \preceq_{imm} D$. Using Proposition 2.4 it follows (I) that for each fixed positive integer $k$, every Eulerian digraph with sufficiently large tree-width satisfies $DM(P_k \Box P_k) \preceq D$.

Let $G_1, G_2, \ldots$ be a sequence of connected plane graphs with bounded tree-width, and let $k$ be an integer such that $tw(G_i) < k$ for $i = 1, 2, \ldots$. By way of contradiction, suppose that the sequence of corresponding good digraphs $DM(G_1), DM(G_2), \ldots$ has unbounded tree-width. By (I), it follows that there is an integer $j$ such that $DM(P_k \Box P_k) \preceq DM(G_j)$. By Fact 2.1 then $P_k \Box P_k \preceq_{R^2} G_j$. Since $tw(P_k \Box P_k) = k$ and tree-width is monotone under taking minors (see [12]), it follows that $tw(G_j) \geq k$, a contradiction.

Proof of Fact 2.3. Hannie proved that if $D_1, D_2, \ldots$ is a sequence of 2-regular digraphs with bounded branch-width, then there exist $i < j$ such that $D_i \preceq_{imm} D_j$ [20, Theorem 3.2]. Good digraphs are a particular instance of 2-regular digraphs, and so, in view of Proposition 2.4 it follows that if $D_1, D_2, \ldots$ is a sequence of good digraphs with bounded branch-width, then there exist $i < j$ such that $D_i \preceq D_j$. Since the tree-width of any graph is greater than its branch-width [43], it follows that this last statement also holds if $D_1, D_2, \ldots$ is a sequence of good digraphs with bounded tree-width.
3. Proof of Theorem 1.3

Theorem 1.3 follows from a straightforward adaptation of the chain of arguments and results used by Adler, Dorn, Fomin, Sau, and Thilikos to prove [3, Theorem 1]. This is the main result in [3], and establishes the following. For each fixed planar graph \( H \), there is an algorithm that takes as input a planar graph \( G \) on \( n \) vertices and answers in time \( O(n^2 \log n) \) whether \( H \) is an (abstract) minor of \( G \).

We prove Theorem 1.3 by explaining which adjustments are needed to the arguments in the proof of [3, Theorem 1] in our current context of sphere, rather than planar, graphs.

We recall that if \( H \) is an (abstract) graph, then a model \( M \) of \( H \) is a graph whose edge set \( E(M) \) is partitioned into \( c \)-edges (contraction edges) and \( m \)-edges (minor edges) such that the graph resulting from contracting all \( c \)-edges is isomorphic to \( H \). If \( G \) is a graph that contains \( M \) as a subgraph, then \( M \) is a model of \( H \) in \( G \). An easy but key observation is that \( H \) is a minor of \( G \) if and only if there is a model \( M \) of \( H \) in \( G \).

There is an analogous notion for sphere graphs. If \( \overline{H} \) is a sphere graph, then a sphere model \( \overline{M} \) of \( \overline{H} \) is a sphere graph whose edge set \( E(\overline{M}) \) is partitioned into \( c \)-edges (contraction edges) and \( m \)-edges (minor edges) such that the sphere graph resulting from contracting all \( c \)-edges in the sphere is combinatorially equivalent to \( \overline{H} \). If \( \overline{G} \) is a sphere graph that contains \( \overline{M} \) as a subgraph, then \( \overline{M} \) is a sphere model of \( \overline{H} \) in \( \overline{G} \). Again, an easy but key observation is that \( \overline{H} \) is a sphere minor of \( \overline{G} \) if and only if there is a sphere model \( \overline{M} \) of \( \overline{H} \) in \( \overline{G} \).

3.1. Four key steps in [3], and their counterparts for sphere minors. As clearly laid out in [3], at a high level the proof of [3, Theorem 1] follows from four statements. It uses the notion of a sphere cut decomposition (or simply sc-decomposition), a concept introduced in [47]. For further details on this notion, in our current context of interest, we refer the reader to [3]. We adhere to the usual convention to let \( \text{bw}(G) \) denote the branch-width of a graph \( G \).

It is known that the branch-width \( \text{bw}(G) \) of a graph \( G \) is closely related to its tree-width \( \text{tw}(G) \): \( \text{bw}(G) - 1 \leq \text{tw}(G) \leq \left\lfloor \frac{3}{2} \text{bw}(G) \right\rfloor - 1 \). Branch-width has an advantage over tree-width, since the branch-width of a planar graph is computable in quadratic time [47], while no polynomial-time algorithm for computing the tree-width is known.

In the upcoming discussion, \( H \) is a planar graph with \( h \) vertices, and \( G \) is a planar graph with \( n \) vertices; \( \overline{H} \) is a fixed embedding of \( H \) into the sphere, and \( \overline{G} \) is a fixed embedding of \( G \) into the sphere. That is, \( \overline{H} \) and \( \overline{G} \) are sphere graphs.

We will state results (1), (2), (3), and (4) that are established in [3], that put together easily imply that one can decide in time \( O(n^2 \log n) \) whether \( H \) is a minor of \( G \). In parallel, we will state related results (1'), (2'), (3'), and (4') that apply for \( \overline{G} \) and \( \overline{H} \), justifying at each point why these statements follow by an easy adaptation of their respective counterparts for \( G \) and \( H \).

We remark that in [3] a plane graph is defined to be a graph with a given embedding on the sphere, which is our notion of a sphere graph.

(1) One can compute in time \( O(n^2 \log n) \) an sc-decomposition of a sphere embedding of \( G \) with branch-width \( \text{bw}(G) \).

In [3] it is argued that such an sc-decomposition can be found for an arbitrary sphere embedding of \( G \). Therefore the following can be achieved on \( \overline{G} \).

(1') One can compute in time \( O(n^2 \log n) \) an sc-decomposition of \( \overline{G} \) with branch-width \( \text{bw}(G) \).

(2) If \( \text{bw}(G) > 42h \), then \( H \) is a minor of \( G \).

This is proved in [3] using a standard argument from the Robertson-Seymour theory: (i) if \( \text{bw}(G) \) is large enough, then any sphere embedding of \( G \) (in particular \( \overline{G} \)) contains a large grid as a minor, and
hence as a sphere minor, since grids have a unique embedding on the sphere; and (ii) every sufficiently large grid on the sphere contains any sphere embedding of $H$ (in particular $\overline{H}$) as a sphere minor. Thus we have the following.

(2') If $\text{bw}(G) > 42h$, then $\overline{H}$ is a sphere minor of $\overline{G}$.

(3) If $\text{bw}(G) \leq 42h$, then one can compute in time $2^{O(h)}$ the collection $\mathcal{A}$ of all sphere graphs that have between $h$ and $\frac{3}{2} \cdot \text{bw}(G) + h$ vertices and are sphere models of some sphere embedding of $H$.

We adapt this step to $\overline{H}$ as follows. Let $\mathcal{A}$ be the collection of sphere graphs in (3), and let $\mathcal{B}$ be the subcollection of $\mathcal{A}$ that consists of those sphere graphs that are sphere models of the particular sphere embedding $\overline{H}$ of $H$. One can obviously compute $\mathcal{B}$ from $\mathcal{A}$ in an amount time that only depends on $h$. Thus we obtain the following.

(3') There is a computable function $f(h)$ such that the following holds. If $\text{bw}(G) \leq 42h$, then one can compute in time $f(h)$ the collection $\mathcal{B}$ of all sphere graphs that have between $h$ and $\frac{3}{2} \cdot \text{bw}(G) + h$ vertices and are sphere models of $\overline{H}$.

The next statement is at the heart of the proof of [3, Theorem 1]. It involves the technique partially embedded dynamic programming, developed in [3], which is a refinement of the embedded dynamic programming technique introduced by Dorn in [13].

(4) If $\text{bw}(G) \leq 42h$, one can run partially embedded dynamic programing on the collection $\mathcal{A}$ from (3), in time $2^{O(h)} \cdot n$, to check whether some sphere graph in $\mathcal{A}$ is a sphere subgraph of the sphere embedding of $G$ chosen in (1).

We can follow exactly the same sequence of steps and arguments as in [3], but instead we run partially embedded dynamic programming on the collection $\mathcal{B}$ from (3'), in time $2^{O(h)} \cdot n$, to check whether some sphere graph in $\mathcal{B}$ is a sphere subgraph of $G$.

Just as [3, Theorem 1] is a straightforward consequence of (1)--(4), Theorem 1.3 is a straightforward consequence of (1')--(4').

3.2. Proof of Theorem 1.3

As in the discussion in the previous subsection, let $\overline{H}$ be a sphere graph with $h$ vertices, and let $\overline{G}$ be a sphere graph with $n$ vertices. Perform (1'), which takes time $O(n^2 \log n)$. If $\text{bw}(G) > 42h$, then by (2') we are done. Otherwise, perform (3') and (4'). Clearly $\overline{H}$ is a sphere minor of $\overline{G}$ if and only if there is a sphere model of $\overline{H}$ in $\overline{G}$ that has between $h$ and $\frac{3}{2} \cdot \text{bw}(G) + h$ vertices. Performing (3') and (4') takes time $O(f(h) + g(h) \cdot n)$. Thus the whole procedure to check whether $\overline{H}$ is a sphere minor of $\overline{G}$ takes time $O(n^2 \log n)$.

4. Proof of Theorem 1.4

Recall that all links under consideration are non-split, and all diagrams under consideration are regular diagrams on the sphere. An observation that dates back to P.G. Tait in the 19th century is that to each link diagram $A$ we can naturally associate two connected sphere graphs, the Tait graphs...
of \( A \), essentially following the reverse process we used in Section 2 to obtain the directed medial graph \( DM(G) \) of a connected plane graph \( G \).

\[ \text{Proj}(\mathcal{A}) \]

\[ \mathcal{B}(\mathcal{A}) \]

\[ \mathcal{W}(\mathcal{A}) \]

Figure 4. The Tait graphs associated to a link diagram.

4.1. The Tait graphs of a link diagram. We refer the reader to Figure 4. We start with a diagram \( \mathcal{A} \), and obtain its projection \( \text{Proj}(\mathcal{A}) \), which is the sphere graph obtained by omitting the over/under information at the crossings in \( \mathcal{A} \). Checkerboard colour \( \text{Proj}(\mathcal{A}) \) in either of the two possible ways. Let \( \mathcal{B}(\mathcal{A}) \) be the black Tait graph of this checkerboard colouring, that is, the sphere graph whose medial graph is \( \text{Proj}(\mathcal{A}) \), where \( \mathcal{B}(\mathcal{A}) \) has one vertex inside each black face. Analogously, let \( \mathcal{W}(\mathcal{A}) \) be the white Tait graph of this checkerboard colouring, that is, the sphere graph whose medial graph is \( \text{Proj}(\mathcal{A}) \), where \( \mathcal{W}(\mathcal{A}) \) has one vertex inside each white face.

If we perform the same process using the other checkerboard colouring of \( \text{Proj}(\mathcal{A}) \), then we evidently obtain the same Tait graphs, only with the colours interchanged.

The Tait graphs of a diagram are duals to each other. We use \( \mathcal{T}(\mathcal{A}) \) to denote the set \( \{ \mathcal{B}(\mathcal{A}), \mathcal{W}(\mathcal{A}) \} \). Note that, by the observation in the previous paragraph, \( \mathcal{T}(\mathcal{A}) \) does not depend on which of the two checkerboard colourings of \( \text{Proj}(\mathcal{A}) \) we use to construct the Tait graphs.

We make the important remark that, formally, one cannot speak about the black Tait graph or the white Tait graph of a diagram, since these objects are not unique: to begin with, one may choose to place a vertex in the (say) black Tait graph in any point of a black face. On the other hand, both Tait graphs are unique up to combinatorial equivalence. To avoid this subtlety, throughout this section we regard two combinatorially equivalent sphere graphs as the same sphere graph.

We emphasize that there is no drawback on adopting this view of considering combinatorial equivalence classes of sphere graphs as opposed to individual sphere graphs. The previous results we use on sphere graphs are Theorem 1.1 which clearly holds under this setting, and Theorem 1.3. Regarding
this last theorem we note that the algorithms involved use sphere graphs as combinatorial entities, so it fits seamlessly into this setting.

Similarly, throughout this section we make no distinction between two diagrams that are isotopic.

We recall that a sphere isotopy operation on a diagram is a topological deformation in the sphere that preserves the structure of the crossings. Two diagrams that are related by a sphere isotopy are isotopic. As our main goal is to establish an algorithmic result on links and diagrams, and for this purpose diagrams must be regarded as combinatorial entities, it is valid to work with isotopy classes of diagrams.

4.2. Proof of Theorem 1.4. If $A$ and $B$ are diagrams, then we write $A \triangleleft B$ if $A$ can be obtained from $B$ by a sequence of crossing smoothings and exchanges. Clearly, $\triangleleft$ is a quasi-order on the set of all diagrams.

We now establish a key statement for the proof of Theorem 1.4.

Fact 4.1. The following statements hold.

1. Let $A$ and $B$ be diagrams. Then $A \triangleleft B$ if and only if $T(A) \leq_{S^2} T(B)$.
2. The relation $\triangleleft$ is a well-quasi-order on the collection of all diagrams.
3. For each fixed diagram $A$, there is an algorithm that takes as input a diagram $B$ with $n$ crossings, and decides in time $O(n^2 \log n)$ whether $A \triangleleft B$.

Proof. If $A$ and $B$ are diagrams such that $A$ can be obtained from $B$ by a sequence of crossing exchanges, then their respective projections $\text{Proj}(A)$ and $\text{Proj}(B)$ are combinatorially equivalent, and so $T(A) = T(B)$. We also recall, as noted in [14, Proposition 1.6], that if $A$ and $B$ are diagrams such that $A$ can be obtained from $B$ by a crossing smoothing, then one Tait graph of $A$ is obtained by deleting one edge of one Tait graph of $B$, and the other Tait graph of $A$ is obtained by contracting in the sphere one edge of the other Tait graph of $B$. Combining these observations, (1) follows.

Statement (2) is an immediate consequence of (1) and the fact that $\leq_{S^2}$ is a well-quasi-order on the set of all sphere graphs (Corollary 1.2).

To prove (3), let $A$ be a fixed diagram. Checkerboard the projection $\text{Proj}(A)$ of $A$ in any of the two possible ways, and obtain $\mathcal{B}(A)$. Let $\mathcal{A}$ be an algorithm that takes as input a sphere graph $G$ on $s$ crossings, and decides in time $O(s^2 \log s)$ whether $\mathcal{B}(A) \leq_{S^2} G$. The existence of $\mathcal{A}$ is guaranteed from Theorem 1.3.

Now let $\mathcal{A}'$ be the algorithm that takes as input a diagram $B$ with $n$ crossings, and works as follows. First, $\mathcal{A}'$ checkerboard colours the projection $\text{Proj}(B)$ in any of the two possible ways, and calculates $\mathcal{B}(B)$ and $\mathcal{W}(B)$. Since $B$ has $n$ crossings, then $\text{Proj}(B)$ is a 4-regular sphere graph with $n$ vertices, and so $\text{Proj}(B)$ has $2n$ edges. By Euler’s formula, it follows that $\text{Proj}(B)$ has $n + 2$ faces. From this it easily follows that each of $\mathcal{B}(B)$ and $\mathcal{W}(B)$ has at most $n$ vertices.

Next, $\mathcal{A}'$ invokes $\mathcal{A}$ twice, once to check whether (i) $\mathcal{B}(A) \leq_{S^2} \mathcal{B}(B)$ and once to check whether (ii) $\mathcal{B}(A) \leq_{S^2} \mathcal{W}(B)$. Since each of $\mathcal{B}(B)$ and $\mathcal{W}(B)$ has at most $n$ vertices, each of these two queries is performed in time $O(n^2 \log n)$, and so the whole step is performed in time $O(n^2 \log n)$. We conclude that $\mathcal{A}'$ verifies in time $O(n^2 \log n)$ whether (i) holds and whether (ii) holds.

Recall that one Tait graph of $A$ is a sphere minor of a Tait graph of $B$ if and only if the other Tait graph of $A$ is a sphere minor of the other Tait graph of $B$. Thus (i) is equivalent to (i)$' \mathcal{B}(A) \leq_{S^2} \mathcal{B}(B)$ and $\mathcal{W}(A) \leq_{S^2} \mathcal{W}(B)$. Analogously, (ii) is equivalent to (ii)$' \mathcal{B}(A) \leq_{S^2} \mathcal{W}(B)$ and $\mathcal{W}(A) \leq_{S^2} \mathcal{B}(B)$.

We finally note that $T(A) \leq_{S^2} T(B)$ if and only if one of (i)$'$ and (ii)$'$ holds. Thus $\mathcal{A}'$ decides whether $T(A) \leq_{S^2} T(B)$. Since by Fact 4.1 we have that $A \triangleleft B$ if and only if $T(A) \leq_{S^2} T(B)$, it follows that $\mathcal{A}'$ decides whether $A \triangleleft B$. □
4.3. **Proof of Theorem 1.4.** We recall that if $\leq$ is a quasi-order on a set $X$, then a subset $Y$ of $X$ is an upper ideal for $\leq$ if for all $x, y \in X$ such that $x \leq y$ and $x \in Y$, we have that $y \in Y$. The following well-known fact is an easy exercise on well-quasi-order theory.

**Fact 4.2.** Let $\leq$ be a well-quasi-order on a set $X$, and let $Y$ be an upper ideal in $X$. Then there is a finite set $\{y_1, \ldots, y_n\} \subseteq Y$ with the following property. Let $x \in X$. Then $x \in Y$ if and only if $y_i \leq x$ for some $i \in \{1, \ldots, n\}$.

**Proof of Theorem 1.4.** Let $L$ be a fixed link. If $A$ and $B$ are diagrams such that $A \leadsto L$ and $A \triangleleft B$, then clearly $B \leadsto L$. That is, the collection $\mathcal{D}_L$ of all diagrams $D$ such that $D \leadsto L$ forms an upper ideal under $\triangleleft$. Thus it follows from Fact 4.1(2) and Fact 4.2 that there is a finite collection $\{C_1, \ldots, C_m\}$ of diagrams such that a diagram $D$ satisfies $D \leadsto L$ if and only if $C_i \triangleleft D$ for some $i \in \{1, \ldots, m\}$.

For each $i = 1, \ldots, m$, let $\mathcal{A}_i$ be an algorithm that takes as input a diagram $D$ on $n$ crossings and decides in time $O(n^2 \log n)$ whether $C_i \triangleleft D$. The existence of $\mathcal{A}_i$ is guaranteed from Fact 4.1(3).

Now let $\mathcal{A}$ be the algorithm that takes as input a diagram $D$ on $n$ crossings, and works as follows. The algorithm $\mathcal{A}$ applies $\mathcal{A}_1, \ldots, \mathcal{A}_m$ to $D$. Note that $D \leadsto L$ if and only if for some $i \in \{1, \ldots, m\}$ the algorithm $\mathcal{A}_i$ returns that $C_i \triangleleft D$. Thus $\mathcal{A}$ decides whether $D \leadsto L$. Finally we note that since $m$ is fixed (as $L$ is fixed), then $\mathcal{A}$ runs in time $O(n^2 \log n)$. □

5. **Two open questions**

We proved that the plane minor relation, and thus the sphere minor relation, is a well-quasi-order. There remains the question of whether the $\Sigma$-minor relation is a well-quasi-order for each surface $\Sigma$.

In the final section of [42] the following open question was included, remarking that if it were shown that if for every fixed link $L$ there is a polynomial-time algorithm that solves $\leadsto L$, then this would be the next natural problem to tackle. Since in this paper we have precisely settled the issue for each fixed link $L$, it seems worth reiterating this open problem.

Consider the following decision problem, in which the link $L$ is not fixed, but it is part of the input.

**Problem:** $\leadsto$

**Input:** A link $L$, and a link diagram $D$.

**Question:** Is it true that $D \leadsto L$?

What is the computational complexity of $\leadsto$? We remark that, in the language of computational complexity, our Theorem 1.3 provides an FPT (Fixed Parameter Tractable) solution to Problem $\leadsto$.

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