Discrete Space Structure of the 3D Wigner Quantum Oscillator

R.C. King†
Faculty of Mathematical Studies, University of Southampton,
Southampton SO17 1BJ, U.K.;

T.D. Palev‡
Abdus Salam International Centre for Theoretical Physics,
PO Box 586, 34100 Trieste, Italy;

N.I. Stoilova§ and J. Van der Jeugt¶
Department of Applied Mathematics and Computer Science, University of Ghent,
Krijgslaan 281-S9, B-9000 Gent, Belgium.

Abstract

The properties of a noncanonical 3D Wigner quantum oscillator, whose position and momentum operators generate the Lie superalgebra $sl(1|3)$, are further investigated. Within each state space $W(p)$, $p = 1, 2, \ldots$, the energy $E_q$, $q = 0, 1, 2, 3$, takes no more than 4 different values. If the oscillator is in a stationary state $\psi_q \in W(p)$ then measurements of the non-commuting Cartesian coordinates of the particle are such that their allowed values are consistent with it being found at a finite number of sites, called “nests”. These lie on a sphere centered on the origin of fixed, finite radius $\rho_q$. The nests themselves are at the vertices of a rectangular parallelepiped. In the typical cases ($p > 2$) the number of nests is 8 for $q = 0$ and 3, and varies from 8 to 24, depending on the state, for $q = 1$ and 2. The number of nests is less in the atypical cases ($p = 1, 2$), but it is never less than two. In certain states in $W(2)$ (resp. in $W(1)$) the oscillator is “polarized” so that all the nests lie on a plane (resp. on a line). The particle cannot be localized in any one of the available nests alone since the coordinates do not commute. The probabilities of measuring particular values of the coordinates are discussed. The mean trajectories and the standard deviations of the coordinates and momenta are computed, and conclusions are drawn about uncertainty relations. The rotational invariance of the system is also discussed.

†E-mail: R.C.King@maths.soton.ac.uk
‡E-mail: tpalev@inrne.bas.bg. Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria.
§E-mail: Neli.Stoilova@rug.ac.be. Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria.
¶E-mail: Joris.VanderJeugt@rug.ac.be.
1 Introduction

In the present paper we continue the investigation of new quantum systems originating from
the representation theory of basic classical Lie superalgebras. In particular, we study fur-
ther the properties of a 3-dimensional (3D) Wigner quantum oscillator whose mathematical
background involves the Lie superalgebra $sl(1|3)$ \[1, 2\].

The idea itself behind these investigations stems from the 1950’s paper of Wigner Do the
equations of motion determine the quantum mechanical commutation relations? \[3\]. In this
paper Wigner has generalized a result of Ehrenfest \[4\]. The latter stated (up to ordering
details, which are irrelevant in our case) that in the Heisenberg picture of quantum mechan-
ics Hamilton’s (resp. the Heisenberg) equations are a unique consequence of the canonical
commutation relations (CCRs) and the Heisenberg (resp. Hamilton’s) equations. Wigner has
proved a stronger statement. He has shown through an example that Hamilton’s equations
can be identical to the Heisenberg equations even if the position and momentum operators
do not satisfy the CCRs.

The above considerations justify the following definition \[1, 2\]:

**Definition 1** A system with Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{n} \frac{\hat{p}_\alpha^2}{2m_\alpha} + V(\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n),$$  \[1.1\]

which depends on the $6n$ variables $\hat{r}_\alpha$ and $\hat{p}_\alpha$, with $\alpha = 1, 2, \ldots, n$, to be interpreted as
(Cartesian) coordinates and momenta, respectively, is said to be a Wigner quantum system
(WQS) if the following conditions hold:

**P1** The state space $W$ is a Hilbert space. To every physical observable $O$ there corresponds
a Hermitian (self-adjoint) operator $\hat{O}$ acting in $W$.

**P2** The observable $O$ can take on only those values which are eigenvalues of $\hat{O}$. The
expectation value of the observable $O$ in a state $\psi$ is given by $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)/\langle \psi, \psi \rangle$, where $(\psi, \phi)$ denotes the scalar product of $\psi, \phi \in W$.

**P3** Hamilton’s equations and the Heisenberg equations hold and are identical (as operator
equations) in $W$.

Postulates **P1** and **P2** are common to any quantum system. The difference with canonical
quantum mechanics comes from the last postulate **P3**. In the canonical case instead of **P3**
one postulates the validity of the Heisenberg equations and the CCRs, then as mentioned
above, Hamilton’s equations hold too.

In \[3\] Wigner has considered, as an example of such a system, a one-dimensional oscillator
with a Hamiltonian ($m = \omega = \hbar = 1$) $\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$. Abandoning the requirement $[\hat{q}, \hat{p}] = i$,
Wigner searched for all operators $\hat{q}$ and $\hat{p}$ such that Hamilton’s equations $\dot{q} = \hat{p}$ and $\dot{p} = -\hat{q}$
are identical with the Heisenberg equations $\dot{q} = i[\hat{H}, \hat{q}]$ and $\dot{p} = i[\hat{H}, \hat{p}]$. In addition to the
canonical solution he found infinitely many other solutions.
Different aspects of Wigner’s idea were studied by several authors. Among the earlier references we mention \([5, 6, 7, 8, 9, 10, 11]\), but the subject still remains of interest \([12, 13, 14, 15, 16, 17, 18, 19, 20]\).

It is perhaps worth mentioning that postulate \(P3\) can be weakened (so far only in the 1D case) in a manner consistent with Wigner’s ideas \([11]\) so that deformed quantum oscillators \([21, 22]\) and, more generally, Daskaloyannis oscillators \([11, 23]\) can be viewed as (generalized) Wigner oscillators.

Our approach to Wigner quantum oscillators is essentially based on two observations. The first one, due to Kamefuchi and Ohnuki \([24]\), is the proof that all solutions found by Wigner are different representations of just one pair of para-Bose (pB) creation and annihilation operators (CAOs) \(B^+ = (q \mp p)/\sqrt{2}\). More generally, we recall that the (representation independent) pB operators, which generalize Bose statistics, are defined by the relations \([25]\):

\[
\{B^\xi_i, B^\eta_j\} = (\epsilon - \xi)\delta_{ij} B^\eta_k + (\epsilon - \eta)\delta_{jk} B^\xi_i, \quad i, j, k = 1, 2, \ldots, N, \quad \xi, \eta, \epsilon = \pm \text{or} \pm 1. \quad (1.2)
\]

Here and throughout the paper \(\{x, y\} = xy + yx\) and \([x, y] = xy - yx\) for any \(x, y\).

The second relevant observation is that any \(N\) pairs of pB operators \(B^{\pm}_1, \ldots, B^{\pm}_N\) are odd elements, generating a Lie superalgebra \([26]\), isomorphic to the orthosymplectic Lie superalgebra \(osp(1|2N)\) \([27]\). The Fock spaces of any \(N\) pairs of parabosons and in particular of bosons are irreducible \(osp(1|2N)\) modules. In this terminology the oscillator of Wigner can be called \(osp(1|2)\) oscillator, its position and momentum operators are the odd generators of \(osp(1|2)\), the Hamiltonian is a simple polynomial of the position and momentum operators and the solutions found by Wigner are different irreducible representations of this Lie superalgebra.

The \(osp(1|2N)\) Lie superalgebra is a basic classical Lie superalgebra from class \(B\) in the classification of Kac \([28]\). In fact \((1.2)\) yields one possible definition of \(osp(1|2N)\): the associative superalgebra with unity, subject to the relations \((1.2)\) is the universal enveloping algebra of \(osp(1|2N)\).

The results of Wigner can be easily extended to any \(N\)-dimensional harmonic oscillator, turning it into a Wigner quantum oscillator (WQO). To this end one has first to express the Hamiltonian via Bose operators

\[
[b^+_i, b^-_j] = \delta_{ij}, \quad [b^+_i, b^+_j] = [b^-_i, b^-_j] = 0, \quad i, j = 1, \ldots, N, \quad (1.3)
\]

and their anticommutators and subsequently replace them with pB operators \((1.2)\). The corresponding solutions are now associated with infinite-dimensional irreducible representation of the Lie superalgebra \(osp(1|2N)\). Since this superalgebra is of class \(B\) we refer to the statistics of the canonical quantum oscillator (CQO) (and its pB generalization) as being \(B\)-superstatistics.

Having observed all this, it was natural to ask whether one can satisfy the postulates of Definition 1 with position and momentum operators which generate algebras from the other, different from \(B\), classes of basic Lie superalgebras. A positive answer to this question was given in \([1, 2]\) with operators \(A^\pm_i, \quad i = 1, \ldots, N\), which satisfy certain relations that we will specify in the next section. These ensure that they generate the Lie superalgebra \(sl(1|N)\). The corresponding solutions are this time associated with a Wigner quantum system that
takes the form of an $N$-dimensional non-canonical Wigner quantum oscillator. Since the special linear Lie superalgebra $sl(1|N)$ is of class $A$, we refer to the statistics of this WQO as being $A$-superstatistics.

In a similar approach Barut and Bracken [29] have described the internal dynamics (Zitterbewegung) of Dirac’s electron. Their creation and annihilation operators satisfy similar triple relations as in our case (Eqs. (2.11)), but instead of a Lie superalgebra they generate the Lie algebra $so(5)$.

In the present paper we study further the properties of the 3D WQO, related to the $sl(1|3)$ superalgebra and initiated in [1, 2]. The paper is organized as follows.

In Section 2 we outline the mathematical structure of the 3D noncanonical oscillator. The compatibility between the Heisenberg equations and the Hamilton’s equations is achieved with operators $A_1^\pm, A_2^\pm, A_3^\pm$ which satisfy triple relations similar to those for the para-Bose case (1.2), but this time they generate the Lie superalgebra $sl(1|3)$. The Fock spaces $W(p)$ of these operators are defined. The inequivalent representations are labeled by one positive integer $p$. For $p > 2$ all Fock spaces are 8-dimensional, whereas in the case $p = 1$ (resp. $p = 2$) $\dim W(1) = 4$ (resp. $\dim W(2) = 7$). In the terminology of Kac [28] the $p = 1, 2$ representations are called atypical representations. For this reason we refer to the Fock spaces $W(1)$ and $W(2)$ as atypical, to the corresponding oscillator as atypical etc. We shall see in the next sections that the properties of the atypical oscillators are very different from those with $p > 2$.

In Section 3 we recall the known [1, 2] physical properties of the $sl(1|3)$ WQOs. Firstly, the oscillator has finite space dimensions and the Hamiltonian has no more than 4 different eigenvalues. In the stationary states the distance of the particle to the origin is quantized so that the particle is constrained to move on one of 4 possible spheres. Secondly, the geometry of the oscillator is noncommutative. Neither coordinates nor the momenta commute with each other. Therefore the position of the particle on the corresponding sphere cannot be localized. In this respect the WQO belongs to the class of models of noncommutative quantum oscillators [30, 31, 32, 33] and, more generally, to theories with noncommutative geometry [34, 35]. The literature on the subject is vast. Moreover the subject is not any longer of purely theoretical interest. Most recently papers predicting (experimentally) measurable deviations from the commutativity of the coordinates have been published [36, 37, 38].

All results after Section 3 are new. In Section 4 the probabilistic distribution of the particle is analyzed. The basis consists of stationary states. The main result is the following: if the particle is in one of the basis states with $p > 2$, then measurements of its coordinates are consistent with it only being found at 8 particular points on a sphere which form the vertices of a rectangular parallelepiped (see Figure 1). Thus as in [39], the coordinates of the particle are observables with a quantized spectrum just like energy, angular momentum, etc. The number of the points, called “nests”, can be even less in the atypical cases. In certain states with $p = 2$ the oscillator is becoming a flat object with 4 vertices (Figure 2). There are three states in the $p = 1$ case when the oscillator is even one-dimensional (Figure 3). In Section 5 the mean trajectories and the standard deviations of the position and momenta operators for an arbitrary state are written down. It is shown (Conclusion 3) that there exists no nontrivial analogue of the Heisenberg uncertainty relations since one can always find a state $x$ for which either $(\Delta r_k)_x = 0$ or $(\Delta p_k)_x = 0$. 

4
In Section 6 we show that despite the fact that the \( sl(1|3) \) oscillator is very different from the 3D Bose oscillator, they still have some features in common. In particular we show that to each \( \rho = 1 \) mean trajectory of the \( sl(1|3) \) oscillator there corresponds exactly the same trajectory of the 3-dimensional canonical oscillator. Finally, in Section 7, we discuss the rotational invariance of the WQO.

## 2 Mathematical structure of the 3D WQO

Let \( \hat{H} \) be the Hamiltonian of a three-dimensional harmonic oscillator, that is

\[
\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2} \hat{R}^2. \tag{2.1}
\]

We proceed to view this oscillator as a Wigner quantum system and work throughout in the Heisenberg picture in which the operators are, in general, time dependent. According to postulate P3 the operators \( \hat{R} \) and \( \hat{P} \) have to be defined in such a way that Hamilton’s equations

\[
\dot{\hat{P}} = -m\omega^2 \hat{R}, \quad \dot{\hat{R}} = \frac{1}{m} \hat{P}. \tag{2.2}
\]

and the Heisenberg equations

\[
\dot{\hat{P}} = i\hbar [\hat{H}, \hat{P}], \quad \dot{\hat{R}} = i\hbar [\hat{H}, \hat{R}] \tag{2.3}
\]

are both valid, and are identical as operator equations. These equations are compatible only if

\[
[\hat{H}, \hat{P}] = i\hbar m\omega^2 \hat{R}, \quad [\hat{H}, \hat{R}] = -\frac{i\hbar}{m} \hat{P}. \tag{2.4}
\]

The most general solution of (2.2) and (2.3) is not known. Here we mention the canonical Bose solution. Expressed via boson creation and annihilation operators it reads:

\[
r_k(t) = \sqrt{\frac{\hbar}{2m\omega}} (b_k^+ e^{i\omega t} + b_k^- e^{-i\omega t}), \quad p_k(t) = i \sqrt{\frac{m\omega\hbar}{2}} (b_k^+ e^{i\omega t} - b_k^- e^{-i\omega t}). \tag{2.5}
\]

In this setting \( r_k \) and \( p_k \) are position and momentum operators, defined in a Bose Fock space \( \Phi \) with orthonormal basis states

\[
|n_1, n_2, n_3\rangle = \frac{(b_1^+)^{n_1}(b_2^+)^{n_2}(b_3^+)^{n_3}}{\sqrt{n_1! n_2! n_3!}} |0\rangle, \quad n_1, n_2, n_3 \in \mathbb{Z}_+ \tag{2.6}
\]

subject to the known transformation relations:

\[
b_k^+ |\ldots, n_k, \ldots\rangle = \sqrt{n_k + 1} |\ldots, n_k + 1, \ldots\rangle, \quad b_k^- |\ldots, n_k, \ldots\rangle = \sqrt{n_k} |\ldots, n_k - 1, \ldots\rangle. \tag{2.7}
\]

As mentioned already in the Introduction, this Bose solution belongs to the class of \( B \)-superstatistics.
In the present paper we deal with solutions of (2.2) and (2.3) for which the position and momentum operators generate a Lie superalgebra from the class $A$, more precisely $\mathfrak{sl}(1|3)$. To make the connection with $\mathfrak{sl}(1|3)$ we write the operators $\hat{P} \equiv (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ and $\hat{R} \equiv (\hat{R}_1, \hat{R}_2, \hat{R}_3)$ in terms of new operators:

$$A^\pm_k = \sqrt{\frac{m\omega}{2\hbar}} \hat{R}_k \pm i \sqrt{\frac{1}{2m\omega\hbar}} \hat{P}_k, \quad k = 1, 2, 3.$$  \hfill (2.8)

The Hamiltonian $\hat{H}$ of (2.1) and the compatibility conditions (2.4) then take the form:

$$\hat{H} = \frac{\omega\hbar}{2} \sum_{i=1}^3 \{A_i^+, A_i^-\},$$  \hfill (2.9)

$$\sum_{i=1}^3 [\{A_i^+, A_i^-\}, A_k^\pm] = \mp 2A_k^\pm, \quad i, k = 1, 2, 3.$$  \hfill (2.10)

As a solution to (2.10) we chose operators $A_i^\pm$ that satisfy the following triple relations:

$$[\{A_i^+, A_j^-\}, A_k^\pm] = \delta_{jk} A_i^+ - \delta_{ij} A_k^+, \quad (2.11a)$$

$$[\{A_i^+, A_j^-\}, A_k^-] = -\delta_{ik} A_j^- + \delta_{ij} A_k^-, \quad (2.11b)$$

$$\{A_i^+, A_j^+\} = \{A_i^-, A_j^-\} = 0.$$  \hfill (2.11c)

In our case $i, j, k = 1, 2, 3$. Equations (2.11) are defined however for $i, j, k = m, m+1, \ldots, n$, where $m$ and $n$ are any integers (including $m = -\infty$ and $n = \infty$).

**Proposition 1** The operators $A_i^\pm$, $i = 1, \ldots, n$, satisfying (2.11), are odd elements generating the Lie superalgebra $\mathfrak{sl}(1|n)$ \[41\].

The generators $A_i^\pm$, $i = 1, \ldots, n$ are said to be creation and annihilation operators of $\mathfrak{sl}(1|n)$. These CAOs are the analogue of the Jacobson generators for the Lie algebra $\mathfrak{sl}(n+1)$ \[11\] and could also be called Jacobson generators of $\mathfrak{sl}(1|n)$.

Coming back to the 3D oscillator, we emphasize again that all considerations here are in the Heisenberg picture. The position and momentum operators depend on time. Hence also the CAOs depend on time. Writing this time dependence explicitly, one has:

**Hamilton’s equations** $$\dot{A}_k^\pm(t) = \mp i\omega A_k^\pm(t),$$  \hfill (2.12)

**Heisenberg equations** $$\dot{A}_k^\pm(t) = \frac{i\omega}{2} \sum_{i=1}^3 [\{A_i^+(t), A_i^-(t)\}, A_k^\pm(t)].$$  \hfill (2.13)

The solution of (2.12) is evident,

$$A_k^\pm(t) = \exp(\mp i\omega t) A_k^\pm(0)$$  \hfill (2.14)

and therefore if the defining relations (2.11) hold at a certain time $t = 0$, i.e., for $A_k^\pm \equiv A_k^\pm(0)$, then they hold as equal time relations for any other time $t$. From (2.11) it follows also that
the equations (2.12) are identical with equations (2.13). For further use we write the time dependence also of \( \hat{R} = (\hat{R}_1, \hat{R}_2, \hat{R}_3) \) and \( \hat{P} = (\hat{P}_1, \hat{P}_2, \hat{P}_3) \) explicitly:

\[
\hat{R}_k(t) = \sqrt{\frac{\hbar}{2m\omega}} \left( A_k^+ e^{-i\omega t} + A_k^- e^{i\omega t} \right),
\]

(2.15a)

\[
\hat{P}_k(t) = -i \sqrt{\frac{m\omega \hbar}{2}} \left( A_k^+ e^{-i\omega t} - A_k^- e^{i\omega t} \right),
\]

(2.15b)

where \( k = 1, 2, 3 \).

Finally, the single particle angular momentum operators \( \hat{M}_j \) defined in [2] by

\[
\hat{M}_j = -\frac{1}{\hbar} \sum_{k,l=1}^{3} \epsilon_{jkl} \{ \hat{R}_k, \hat{P}_l \}, \quad j = 1, 2, 3,
\]

(2.16)

take the following form in terms of the CAOs (2.11)

\[
\hat{M}_j = -i \sum_{k,l=1}^{3} \epsilon_{jkl} \{ A_k^+, A_l^- \}, \quad j = 1, 2, 3.
\]

(2.17)

It is straightforward to verify that with respect to this choice of angular momentum operator \( \hat{M} \) the operators \( \hat{R}, \hat{P} \) and \( \hat{M} \) all transform as 3-vectors.

The state spaces which we consider here are those irreducible \( sl(1|3) \) modules that may be constructed by means of the usual Fock space technique precisely as in the parastatistics case [25]. To this end we require that the representation space, \( W(p) \), contains (up to a multiple) a unique cyclic vector \( |0\rangle \) such that

\[
A_i^- |0\rangle = 0, \quad A_i^- A_j^+ |0\rangle = p\delta_{ij} |0\rangle, \quad i, j = 1, 2, 3.
\]

(2.18)

The above relations are enough for the construction of the full representation space \( W(p) \). This space defines an indecomposable finite-dimensional representation of the CAOs (2.11) and hence of \( sl(1|3) \) for any value of \( p \). However we wish to impose the further physical requirements that:

(a) \( W(p) \) is a Hilbert space with respect to the natural Fock space inner product;

(b) the observables, in particular the position and momentum operators (2.13), are Hermitian operators.

Condition (b) reduces to the requirement that the Hermitian conjugate of \( A_i^+ \) should be \( A_i^- \), i.e.

\[
(A_i^\pm)\dagger = A_i^\mp.
\]

(2.19)

The condition (a) is then such that \( p \) is restricted to be a positive integer [40], in fact any positive integer.
Let $\Theta \equiv (\theta_1, \theta_2, \theta_3)$. The state space $W(p)$ of the system is spanned by the following orthonormal basis (called the $\Theta$-basis):

$$|p; \Theta\rangle \equiv |p; \theta_1, \theta_2, \theta_3\rangle = \sqrt{(p-q)!/p!} (A_1^+)^{\theta_1} (A_2^+)^{\theta_2} (A_3^+)^{\theta_3} |0\rangle,$$  \hspace{1cm} (2.20)$$

where

$$\theta_i \in \{0, 1\} \text{ for all } i = 1, 2, 3$$ \hspace{1cm} (2.21)

and

$$0 \leq q \equiv \theta_1 + \theta_2 + \theta_3 \leq \min(p, 3).$$ \hspace{1cm} (2.22)

The transformation of the basis states (2.20) under the action of the CAOs reads as follows:

$$A_1^+ |p; \ldots, \theta_i, \ldots\rangle = \theta_i (-1)^{\theta_1+\ldots+\theta_{i-1}} \sqrt{p-q+1} |p; \ldots, \theta_i - 1, \ldots\rangle,$$ \hspace{1cm} (2.23a)

$$A_1^- |p; \ldots, \theta_i, \ldots\rangle = (1 - \theta_i) (-1)^{\theta_1+\ldots+\theta_{i-1}} \sqrt{p-q} |p; \ldots, \theta_i + 1, \ldots\rangle.$$ \hspace{1cm} (2.23b)

The factors $\theta_i$ and $(1 - \theta_i)$ ensure that the only non-vanishing cases are those for which $|p; \ldots, \theta_i \pm 1, \ldots\rangle$ do indeed belong to the set of basis states defined by (2.20)-(2.22).

Note the first big difference between this non-canonical WQO and the case of a conventional CQO:

**Observation 1** *Contrary to the CQO with an infinite-dimensional state space, each state space $W(p)$ of the WQO is finite-dimensional.*

In fact $\dim W(p) = 8$ for $p > 2$, whereas $\dim W(1) = 4$ and $\dim W(2) = 7$.

### 3 Known properties of 3D WQOs

Here we recall the physical properties of the Wigner quantum oscillators as given in [1, 2].

The first thing we note is that the representation of $sl(1|3)$ was chosen such that, as in the case of a 3D CQO, the physical observables $\hat{H}$, $\hat{R}$, $\hat{P}$ and $\hat{M}$ are, in the case of the WQO, all Hermitian operators within every Hilbert space $W(p)$ for each $p = 0, 1, \ldots$ (in accordance with postulate **P1**).

Secondly, in the case of the WQO the Hamiltonian $\hat{H}$ is diagonal in the basis (2.20)-(2.22), i.e. the basis vectors $|p; \Theta\rangle$ are stationary states of the system. As in the 3D CQO the energy levels are equally spaced with the same spacing $\hbar\omega$. Contrary to the CQO each Hilbert space $W(p)$ has no more than four equally spaced energy levels, with spacing $\hbar\omega$. More precisely,

$$\hat{H} |p; \Theta\rangle = E_q |p; \Theta\rangle \text{ with } E_q = \frac{\hbar\omega}{2} (3p - 2q).$$ \hspace{1cm} (3.1)

So we can define stationary states $\psi_q$ as superpositions of states $|p; \Theta\rangle$ with the same $q$: 

\begin{align*}
\psi_0 &= |p; 0, 0, 0\rangle, \hspace{1cm} (3.2a) \\
\psi_1 &= \alpha(1, 0, 0) |p; 1, 0, 0\rangle + \alpha(0, 1, 0) |p; 0, 1, 0\rangle + \alpha(0, 0, 1) |p; 0, 0, 1\rangle, \hspace{1cm} (3.2b) \\
\psi_2 &= \alpha(1, 1, 0) |p; 1, 1, 0\rangle + \alpha(1, 0, 1) |p; 1, 0, 1\rangle + \alpha(0, 1, 1) |p; 0, 1, 1\rangle, \hspace{1cm} (3.2c) \\
\psi_3 &= |p; 1, 1, 1\rangle. \hspace{1cm} (3.2d)
\end{align*}
where $\alpha(\theta_1, \theta_2, \theta_3)$ are complex numbers. The stationary states satisfy $\hat{H}\psi_q = E_q\psi_q$. Only the states with $q \leq p$ belong to the space $W(p)$. Note that in the atypical cases ($p = 1, 2$) the lowest energy level is degenerate: there are three linearly independent states with the same ground state energy.

Perhaps the most striking difference between the WQO and the CQO is that the geometry of the Wigner oscillators is noncommutative: the position operators $\hat{R}_1, \hat{R}_2, \hat{R}_3$ of the oscillating particle do not commute with each other,

$$[\hat{R}_i, \hat{R}_j] \neq 0 \quad \text{for } i \neq j = 1, 2, 3.$$  \hfill (3.3)

Hence for the Wigner oscillators a coordinate representation (x-representation) does not exist. Similarly,

$$[\hat{P}_i, \hat{P}_j] \neq 0 \quad \text{for } i \neq j = 1, 2, 3$$  \hfill (3.4)

and therefore also a momentum representation (p-representation) cannot be defined.

On the other hand $\hat{R}^2$ and $\hat{P}^2$ commute with the Hamiltonian. More than that, they are proportional to $\hat{H}$:

$$\hat{\epsilon} \equiv \frac{2}{\omega h} \hat{H} = \frac{2m\omega}{h} \hat{R}^2 = \frac{2m\omega}{h} \hat{P}^2 = \sum_{i=1}^{3} \{A_i^+, A_i^-\}. \hfill (3.5)$$

And thus:

$$\hat{R}^2 \left| p; \Theta \right> = \frac{h}{2m\omega} (3p - 2q) \left| p; \Theta \right>, \hfill (3.6a)$$

$$\hat{P}^2 \left| p; \Theta \right> = \frac{m\omega h}{2} (3p - 2q) \left| p; \Theta \right>, \hfill (3.6b)$$

for $0 \leq q \equiv \theta_1 + \theta_2 + \theta_3 \leq \min(p, 3)$. Eq. (3.6a) indicates that if the oscillator is in a stationary state $\psi_q$ with energy $E_q = \frac{m\omega h}{2} (3p - 2q)$, then the distance $\varrho_q$ between the oscillating particle and the origin of the coordinate system is

$$\varrho_q = \sqrt{\frac{h}{2m\omega} (3p - 2\theta_1 - 2\theta_2 - 2\theta_3)} \hfill (3.7)$$

and this distance is an integral of motion, it is preserved in time. For further references we formulate the following observation.

**Conclusion 1** Each stationary state $\psi_q$, which is a superposition of states $|p; \Theta\rangle$ with one and the same $q = \theta_1 + \theta_2 + \theta_3$, corresponds to a configuration in which the particle is somewhere at a distance $\varrho_q$ from the centre of the coordinate system. However, the position of the particle on the sphere of radius $\varrho_q$ cannot be localized because the coordinates do not commute with one another.

The maximum distance of the particle from the centre is

$$\varrho_{\text{max}} \equiv \varrho_0 = \sqrt{\frac{3hp}{2m\omega}} \hfill (3.8)$$
and this corresponds to the state $|p; 0, 0, 0\rangle$, which carries also the maximal energy $E_{\text{max}} = \frac{3}{2}\hbar\omega p$. Thus the WQO occupies a finite volume. The oscillating particle is locked in a sphere with radius (3.8), which is another property very different from the CQO for which there is no finite upper bound on the radial distance.

Let us now consider the angular momentum of the stationary states $|p; \Theta\rangle$ of the WQO. One finds:

$$
\hat{M}_3^2|p; \Theta\rangle = (\theta_1 - \theta_2)^2 |p; \Theta\rangle, \quad (\text{3.9a})
$$
$$
\hat{M}_3^2|p; \Theta\rangle = \begin{cases} 
0 & \text{if } \theta_1 = \theta_2 = \theta_3; \\
2|p; \Theta\rangle & \text{otherwise.} 
\end{cases} \quad (\text{3.9b})
$$

Representing, as usual, the $\hat{M}_3^2$ eigenvalue as $M(M+1)$, we see that the $\Theta$-basis vectors have fixed total angular momentum 0 or 1. Although $\hat{M}_3^2$ and $\hat{M}_3^2$ are diagonal in the $\Theta$-basis, $\hat{M}_3$ is not. In fact, we have:

$$
\hat{M}_3|p; 0, 0, \theta_3\rangle = 0, \quad \hat{M}_3|p; 1, 1, \theta_3\rangle = 0, \quad (\text{3.10a})
$$
$$
\hat{M}_3\left(\frac{1}{\sqrt{2}}|p; 1, 0, \theta_3\rangle \pm \frac{i}{\sqrt{2}}|p; 0, 1, \theta_3\rangle\right) = \pm\left(\frac{1}{\sqrt{2}}|p; 1, 0, \theta_3\rangle \pm \frac{i}{\sqrt{2}}|p; 0, 1, \theta_3\rangle\right), \quad (\text{3.10b})
$$

so the orthonormed $\hat{M}_3$ eigenvectors are easy to construct in the $\Theta$-basis.

The properties of the states $|p; \Theta\rangle$ which were mentioned so far are summarized in Table 1. In the atypical case with $p = 2$ (resp. $p = 1$) one should skip the last row (resp. the last 4 rows) thus getting the correct dimensions for the state space.

| $p \geq q$ | $|p; \Theta\rangle$ | $q$ | $E_q/(\frac{3}{2}\hbar\omega)$ | $M$ | $M_3^2$ |
|---|---|---|---|---|---|
| $p \geq 0$ | $|p; 0, 0, 0\rangle$ | 0 | $3p$ | 0 | 0 |
| $p \geq 1$ | $|p; 1, 0, 0\rangle$ | 1 | $3p - 2$ | 1 | 1 |
| $p \geq 1$ | $|p; 0, 1, 0\rangle$ | 1 | $3p - 2$ | 1 | 1 |
| $p \geq 1$ | $|p; 0, 0, 1\rangle$ | 1 | $3p - 2$ | 1 | 0 |
| $p \geq 2$ | $|p; 1, 1, 0\rangle$ | 2 | $3p - 4$ | 1 | 0 |
| $p \geq 2$ | $|p; 1, 0, 1\rangle$ | 2 | $3p - 4$ | 1 | 1 |
| $p \geq 2$ | $|p; 0, 1, 1\rangle$ | 2 | $3p - 4$ | 1 | 1 |
| $p \geq 3$ | $|p; 1, 1, 1\rangle$ | 3 | $3p - 6$ | 0 | 0 |
4 On the position and momentum of the oscillating particle

The results in the previous section are not very precise about the position of the oscillating particle in one of its stationary states $\psi_q$ or $|p; \Theta\rangle$: the only conclusion is that the particle is localized on a sphere with radius $\varrho_q$.

We shall first investigate the probabilistic distribution of the particle on the sphere corresponding to the states $|p; \Theta\rangle$ or $\psi_q$. In particular we shall show, with respect to measurements of $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$, that in the stationary states $|p; \Theta\rangle$ the particle can be found at only 8 points on the sphere (we call them “nests”) with radius $\varrho_q$, see (3.5), and the number of such nests is even less in the atypical cases $p = 1$ and $p = 2$.

The main tool to obtain these results is based on the observation that the set of operators $\hat{H}, \hat{R}^2_1, \hat{R}^2_2, \hat{R}^2_3, \hat{P}^2_1, \hat{P}^2_2, \hat{P}^2_3$ mutually commute and therefore can be diagonalized simultaneously. Observe that $\hat{R}^2_k$ and $\hat{P}^2_k$ and more generally all even elements are independent of the time $t$, which is why we do not write $\hat{R}^2_k(t)$ and $\hat{P}^2_k(t)$.

The sequence (4.1) contains in fact only 3 independent integral of motions, for instance $\hat{R}^2_1, \hat{R}^2_2, \hat{R}^2_3$, since

$$\hat{P}^2_k = m^2 \omega^2 \hat{R}^2_k, \quad \hat{H} = m \omega^2 \hat{R}^2, \quad \hat{P}^2 = m^2 \omega^2 \hat{R}^2, \quad \text{and} \quad \hat{R}^2 = \hat{R}^2_1 + \hat{R}^2_2 + \hat{R}^2_3. \quad (4.2)$$

All these are Hermitian operators in $W(p)$. Hence we can choose a basis consisting of common eigenvectors to all of them. In this case, we are lucky in the sense that all these operators are already diagonal in the $\Theta$-basis.

At this point it is convenient to introduce dimensionless notation for the energy, the coordinates and the momenta:

$$\dot{\varepsilon} = \frac{2}{\omega \hbar} \hat{H}, \quad \dot{r}_i(t) = \sqrt{\frac{2m\omega}{\hbar}} \hat{R}_i(t), \quad \dot{p}_i(t) = \sqrt{\frac{2}{m\omega \hbar}} \hat{P}_i(t), \quad i = 1, 2, 3. \quad (4.3)$$

Then $\dot{r}_i^2 = \dot{p}_i^2$, $i = 1, 2, 3$ and

$$\dot{\varepsilon} = \dot{r}^2 = \dot{p}^2 = \sum_{i=1}^{3} \{A_i^+, A_i^-\}. \quad (4.4)$$

4.1 The basis vectors of $W(p)$ with $p > 2$ (typical case)

For $p > 2$ all state spaces $W(p)$ of the system are 8-dimensional. The following holds:

$$\dot{r}_k^2 |p; \Theta\rangle = \dot{p}_k^2 |p; \Theta\rangle = (p - q + \theta_k) |p; \Theta\rangle, \quad k = 1, 2, 3. \quad (4.5)$$

What are the conclusions, which we can draw from Eqs. (4.3)? Let us answer this question first for one particular state, e.g. $|p; 1, 1, 0\rangle$. If measurements of the observables...
corresponding to \( \hat{r}^1, \hat{r}^2, \hat{r}^2_1, \hat{r}^2_2, \hat{r}^2_3 \) are performed, then according to postulate \( \textbf{P2} \) they will give the eigenvalues of these operators, namely

\[
  r^2 = 3p - 4, \quad r^2_1 = r^2_2 = p - 1, \quad r^2_3 = p - 2.
\]

Moreover since the operators \( \hat{r}^2, \hat{r}^2_1, \hat{r}^2_2, \hat{r}^2_3 \) commute the results (4.6) can be measured simultaneously. The latter means that if several measurements of the coordinates are performed, then they will discover all of the time that the particle is accommodated in one of 8 nests with coordinates

\[
  r_1 = \pm \sqrt{p - 1}, \quad r_2 = \pm \sqrt{p - 1}, \quad r_3 = \pm \sqrt{p - 2},
\]

of a sphere with radius \( \rho = \sqrt{3p - 4} \).

Similarly, the measurements of the projections of the momenta will give (due to (4.5)):

\[
  p_1 = \pm \sqrt{p - q + \theta_1}, \quad p_2 = \pm \sqrt{p - q + \theta_2}, \quad p_3 = \pm \sqrt{p - q + \theta_3}.
\]

The generalization of this result to any \( \Theta \)-state is evident:

**Conclusion 2** If the system is in one of the \( \Theta \)-basis states \( |p; \Theta \rangle \) then measurements of \( r_1, r_2 \) and \( r_3 \) imply that the oscillating particle can be found in no more than 8 nests with coordinates

\[
  r_1 = \pm \sqrt{p - q + \theta_1}, \quad r_2 = \pm \sqrt{p - q + \theta_2}, \quad r_3 = \pm \sqrt{p - q + \theta_3},
\]

on a sphere with radius \( \rho_q = \sqrt{3p - 2q} \). The measured values of the momenta can take also only 8 different values,

\[
  p_1 = \pm \sqrt{p - q + \theta_1}, \quad p_2 = \pm \sqrt{p - q + \theta_2}, \quad p_3 = \pm \sqrt{p - q + \theta_3}.
\]

Conclusion 2 significantly enhances the properties of the WQO known so far, and collected in Conclusion 1. The particle is not just anywhere on the sphere. In every \( \Theta \)-state \( |p; \Theta \rangle \) the particle can be spotted in no more than 8 points of the sphere with radius \( \rho_q \). This is what we can say so far. What we cannot say yet is whether some of these nests are not forbidden for “visits” or what is the probability of finding the particle in any one of them.

In order to investigate this last question we shall need the eigenvectors and the eigenvalues of all the operators of the coordinates and of the momenta. Before that a short remark related to the properties of any WQS will be in order.

Let \( \hat{O} \) be an observable and let \( x_1, \ldots, x_n \) be an orthonormed basis of eigenvectors of \( \hat{O} \): \( \hat{O} x_i = O_i x_i \). Assume that the system is in a state \( \psi = \alpha_1 x_1 + \ldots + \alpha_n x_n \) normalized to 1. Postulate \( \textbf{P2} \) tells us that the expectation value \( \langle \hat{O} \rangle_\psi \) of the observable \( \hat{O} \) in the state \( \psi \) is

\[
  \langle \hat{O} \rangle_\psi = \langle \psi, \hat{O} \psi \rangle = |\alpha_1|^2 O_1 + \ldots + |\alpha_n|^2 O_n.
\]

It follows that \( |\alpha_i|^2 \) gives the probability of measuring the eigenvalue \( O_i \) of the operator \( \hat{O} \). This is just the superposition principle of QM. The conclusion is that this principle holds for any WQS.
Thus in order to examine the probability for the particle to be in one of the 8 nests, one has to introduce as a first step an \( \hat{r}_k \)-basis, namely an orthonormal basis of eigenvectors of \( \hat{r}_k \) for any \( k = 1, 2, 3 \). The second step is to express the \( \Theta \)-basis via the \( \hat{r}_k \)-basis for any \( k = 1, 2, 3 \), and to apply the superposition principle.

One has to proceed in a similar way in order to examine the probability for the particle to have each one of the possible values of momentum.

Let us first concentrate on the probabilities for the position of the particle, when the system is in the basis state \( |p; \Theta \rangle \). We shall see that the noncommutativity of the operators \( \hat{r}_k(t) \) plays an important role, in the sense that \( \hat{r}_1(t) \), \( \hat{r}_2(t) \) and \( \hat{r}_3(t) \) have no common eigenvectors. This will lead to a certain amount of uncertainty about the position probabilities for the 8 nests that can be occupied by the particle.

In this analysis, we need explicit expressions for the eigenvectors of \( \hat{r}_k(t) \) \( (k = 1, 2, 3) \). Let us define, for any \( k \in \{1, 2, 3\} \) and any \( \Theta \) satisfying (2.21), the following vectors in \( W(p) \):

\[
v_k(\Theta) = \frac{1}{\sqrt{2}} (|p; \Theta_{\theta_k=0} \rangle + (-1)^{\theta_1+\cdots+\theta_k} e^{-i\omega t}|p; \Theta_{\theta_k=1} \rangle). \tag{4.12}
\]

Herein, \( \Theta_{\theta_k=0} \) stands for the \( \Theta \)-value specified by the LHS of (1.12) in which \( \theta_k \) is replaced by 0 (and similarly for \( \Theta_{\theta_k=1} \)). Thus \( v_k(\Theta) \) depends on \( \theta_k \) only through the sign factor \((-1)^{\theta_1+\cdots+\theta_k}\). A careful computation shows that these (time-dependent) vectors \( v_k(\Theta) \) constitute an orthonormal basis of eigenvectors of \( \hat{r}_k(t) \) in \( W(p) \):

\[
\hat{r}_k(t) v_k(\Theta) = (-1)^{\theta_k} \sqrt{p - q + \theta_k} v_k(\Theta). \tag{4.13}
\]

The physical interpretation of each eigenvector \( v_k(\Theta) \) is clear (Postulate P2): if (at the time \( t \)) the oscillating particle is in a state \( v_k(\Theta) \) then its \( k \)-th coordinate is \((-1)^{\theta_k} \sqrt{p - q + \theta_k}\).

The inverse relations of (4.12) are also easy to write down:

\[
|p; \Theta \rangle = \frac{1}{\sqrt{2}} (-1)^{\theta_1+\cdots+\theta_{k-1}}\theta_k e^{i\omega t \theta_k} (v_k(\Theta_{\theta_k=0}) + (-1)^{\theta_k} v_k(\Theta_{\theta_k=1})). \tag{4.14}
\]

The main observation needed is that in the inverse transformations (1.14) only two different vectors \( v_k \) appear, each with a coefficient of which the square modulus is 1/2. In order to understand the importance of this observation, consider an example, say \( |p; 1, 1, 0 \rangle \). The expansion of this vector in the \( \hat{r}_k(t) \) eigenvectors (for \( k = 1, 2, 3 \)) reads:

\[
|p; 1, 1, 0 \rangle = \frac{e^{i\omega t}}{\sqrt{2}} (v_1(0, 1, 0) - v_1(1, 1, 0)) \tag{4.15a}
\]

\[
= -\frac{e^{i\omega t}}{\sqrt{2}} (v_2(1, 0, 0) - v_2(1, 1, 0)) \tag{4.15b}
\]

\[
= \frac{1}{\sqrt{2}} (v_3(1, 1, 0) + v_3(1, 1, 1)). \tag{4.15c}
\]

We see that the coefficients of \( v_1(0, 1, 0) \) and \( v_1(1, 1, 0) \) are equal up to a sign, and moreover their square modulus is 1/2. Therefore the superposition principle asserts that with equal probability 1/2 the first coordinate \( r_1 \) of the particle is either \( +\sqrt{p - 1} \) or \( -\sqrt{p - 1} \). In other
words, the probability of finding the particle somewhere in the four nests above the $r_2r_3$-plane is $1/2$; and the probability to find the particle somewhere in the four nests below the $r_2r_3$-plane is also $1/2$. Let us underline that this conclusion is time independent. The time dependent basis which we have used in order to derive it was playing only an intermediate role.

By means of the same arguments, using (4.15b), (4.15c) and (4.13), one concludes that also with probability $1/2$ the second coordinate $r_2$ and the third coordinate $r_3$ of the particle take values $\pm \sqrt{p-1}$ and $\pm \sqrt{p-2}$, respectively for the state $|p; 1, 1, 0\rangle$.

Taking the three results (about the probabilities for $r_1, r_2$ and $r_3$) together does however not lead to a unique solution for the probabilities to find the particle in a particular nest. Indeed, there are 8 probabilities to be determined (one for each nest). From (4.15a) we have deduced that the sum of four of them (above the $r_2r_3$-plane) is $1/2$, and the sum of the remaining four (below the $r_2r_3$-plane) is also $1/2$; so this yields 2 linear relations for the 8 unknown probabilities. Similarly, (4.15b) and (4.15c) each yield 2 linear relations. So in total there are 6 linear relations in 8 unknowns. A more detailed investigation even shows that only 4 of the 6 linear relations are independent. This leads to the conclusion that the probability of the particle being found in each nest cannot be determined by the present considerations: there remain certain degrees of freedom.

We have made this analysis for the example $|p; 1, 1, 0\rangle$, but from (4.14) it is clear that this conclusion generalizes to the case of all $\Theta$-states. This follows from the fact that in the inverse transformations (4.14) only two different vectors $v_k$ appear, each with a coefficient of which the square modulus is $1/2$.

We summarize the results in the next proposition.

**Proposition 2** If the system is in one of the $\Theta$-basis states $|p; \Theta\rangle$, then measurements of the position of the oscillating particle are such that it can only be observed to occupy one of the 8 nests with coordinates

$$r_{k\pm} = \pm \sqrt{p - q + \theta_k}, \quad k = 1, 2, 3,$$

on a sphere of dimensionless radius $\rho_q = \sqrt{3p - 2q}$. The probability $P(\pm \pm \pm)$ of finding the particle in the nest with coordinate $(r_{1\pm}, r_{2\pm}, r_{3\pm})$ cannot be determined. However, the probability of finding the particle somewhere in the four nests with first coordinate equal to $r_{1+}$ is $1/2$, and of finding it somewhere in the four nests with first coordinate equal to $r_{1-}$ is also $1/2$. The same holds for the second and third coordinates.

The measurement of the momentum of the particle can take one of the eight values

$$p_{k\pm} = \pm \sqrt{p - q + \theta_k}, \quad k = 1, 2, 3.$$

Again, the individual probabilities for each of the eight possible values of the momenta cannot be determined; but the probability of having a fixed component $p_{k+}$ is $1/2$, and that of a fixed component $p_{k-}$ is also $1/2$ ($k = 1, 2, 3$).

The proof of the second part of Proposition 2, related to the probabilities of the 8 possible values (4.17) for the measurement of the momentum of the particle, is essentially the same as
for the coordinates. This time however, one has to use the orthonormal basis of eigenvectors of $\hat{p}_k(t)$, given by:

$$\tilde{v}_k(\Theta) = \frac{1}{\sqrt{2}} (|p; \Theta_{\theta_k=0}⟩ - i(-1)^{\theta_1+\ldots+\theta_k} e^{-i\omega t} |p; \Theta_{\theta_k=1}⟩),$$

(4.18)

with

$$\hat{p}_k(t)\tilde{v}_k(\Theta) = (-1)^{\theta_k} \sqrt{p - q + \theta_k} \tilde{v}_k(\Theta).$$

(4.19)

The inverse relations of (4.18) are:

$$|p; \Theta⟩ = \frac{i^{\theta_k}}{\sqrt{2}} (-1)^{\theta_1+\ldots+\theta_{k-1}} e^{i\omega t} \tilde{v}_k(\Theta_{\theta_k=0}) + (-1)^{\theta_k} \tilde{v}_k(\Theta_{\theta_k=1}).$$

(4.20)

The properties deduced in this subsection are summarized in Figure 1. The eight pictures of Figure 1 correspond to the 8 stationary states $|p; \Theta⟩$ of $W(p)$. The top-most picture corresponds to $\Theta = (0, 0, 0)$, the state has highest energy $E_0$, and the particle is on a sphere with maximum radius $q_0 = q_{\text{max}}$, see (3.7) and (3.8). The eight small circles on the sphere indicate the 8 possible results of measuring the three coordinates of the particle (the nests). The next three pictures correspond to the stationary states with $q = 1$ and energy $E_1$; the 8 nests are now on a sphere with radius $q_1 < q_0$. Then follow the three pictures corresponding to the stationary states with $q = 2$ and energy $E_2$; the 8 nests are now on a sphere with radius $q_2 < q_1$. Finally, the last picture corresponds to the stationary state $|p; 1, 1, 1⟩$ with lowest energy $E_3$; again there are 8 nests on a sphere with smallest radius $q_3$. Note that the 8 points on the sphere are the corner points of a cube only for $\Theta = (0, 0, 0)$ and $\Theta = (1, 1, 1)$.

We shall see later in Section 7 that there is a certain rotational invariance of the WQO that would support the expectation that within each set of nests of our stationary states the probability that the measured values of the coordinates coincide with those of an individual nest are the same for all 8 nests. A similar conclusion could be reached by the following semi-classical argument.

Let $s = (\pm \pm \pm)$ be a sequence of signs specifying the sites of the 8 possible nests associated with a particular stationary state $|p; \Theta⟩$ by signifying the signs of the corresponding coordinates $(r_1, r_2, r_3)$. Let $P(s)$ be the probability of finding the particle in the nest specified by $s$. Then clearly

$$\sum_{s=(\pm \pm \pm)} P(s) = P(++++) + P(+-+-) + P(+-+-) + P(---+) + P(---+) + P(---+) + P(---+) = 1.$$  

(4.21)

Now consider any coordinate of this nest, say $z(s) = \sin \theta \cos \phi \hat{r}_1 + \sin \theta \sin \phi \hat{r}_2 + \cos \theta \hat{r}_3$, as measured in some arbitrary direction determined by $\theta$ and $\phi$. Then one might expect that for all $m$ the $m$th moment, $z(s)^m$, of $z(s)$ weighted by the probability $P(s)$ and summed over all nests, might be equal to the expectation value of the corresponding operator $\hat{z}(s) = \sin \theta \cos \phi \hat{r}_1 + \sin \theta \sin \phi \hat{r}_2 + \cos \theta \hat{r}_3$, that is:

$$\sum_{s=(\pm \pm \pm)} P(s) (z(s))^m = (|p; \Theta⟩, (\hat{z}(s))^m |p; \Theta⟩),$$

(4.22)
for all $\theta$ and $\phi$.

Applying this argument in the case $m = 1$, the right hand side of (4.22) is 0 for all stationary states $|p; \Theta\rangle$, and taking into account (4.21) we obtain

\[
\mathcal{P}(+ + +) + \mathcal{P}(+ + -) + \mathcal{P}(+ - +) + \mathcal{P}(+ - -) = \frac{1}{2},
\]
(4.23a)

\[
\mathcal{P}(+ + +) + \mathcal{P}(+ - +) + \mathcal{P}(- + +) + \mathcal{P}(- + -) = \frac{1}{2},
\]
(4.23b)

\[
\mathcal{P}(+ + +) + \mathcal{P}(+ - -) + \mathcal{P}(- + +) + \mathcal{P}(- - +) = \frac{1}{2}.
\]
(4.23c)

These are just our previous conditions given in Proposition 2.

If in addition we impose the condition (4.22) with $m = 2$ we obtain additional information, that is again independent of the particular state $|p; \Theta\rangle$ under consideration, namely:

\[
\mathcal{P}(+ + +) + \mathcal{P}(+ + -) + \mathcal{P}(- - +) + \mathcal{P}(+ - -) = \frac{1}{2},
\]
(4.24a)

\[
\mathcal{P}(+ + +) + \mathcal{P}(- + +) + \mathcal{P}(- + -) + \mathcal{P}(+ - -) = \frac{1}{2},
\]
(4.24b)

\[
\mathcal{P}(+ + +) + \mathcal{P}(- + +) + \mathcal{P}(+ - -) + \mathcal{P}(+ - -) = \frac{1}{2}.
\]
(4.24c)

Combining this with (4.23) we find

\[
\mathcal{P}(+ + +) = \mathcal{P}(+ - -) = \mathcal{P}(- + -) = \mathcal{P}(- - +) = x,
\]
(4.25a)

\[
\mathcal{P}(+ + -) = \mathcal{P}(+ + -) = \mathcal{P}(- + +) = \mathcal{P}(+ - -) = \frac{1}{4} - x,
\]
(4.25b)

for any $x$ such that $0 \leq x \leq \frac{1}{4}$.

Finally, turning to the case $m = 3$ in (4.22), for which the right hand side is again 0 for all stationary states $|p; \Theta\rangle$, we find by considering all possible values of $\theta$ and $\phi$ one additional condition:

\[
\mathcal{P}(+ + +) + \mathcal{P}(+ + -) + \mathcal{P}(+ - +) + \mathcal{P}(+ - -) = \frac{1}{2}.
\]
(4.26)

When combined with (4.23), we obtain the unique solution

\[
\mathcal{P}(+ + +) = \mathcal{P}(+ + -) = \mathcal{P}(+ - +) = \mathcal{P}(+ - -) = \mathcal{P}(+ - -) = \mathcal{P}(- + +) = \frac{1}{8}.
\]
(4.27)

This indicates that for all stationary states $|p; \Theta\rangle$, the probability of finding the particle in any one of the 8 available nests is the same. All nests are equally likely.

It should be pointed out however, that this argument is semi-classical in the sense that the left hand side of (4.22) involving the weighted moments, is evaluated on the assumption that $z(s)$ is just a linear combination, determined by $\theta$ and $\phi$, of commuting coordinates $r_1$, $r_2$ and $r_3$, whereas on the right hand side $\hat{z}(s)$ is the same linear combination of non-commuting operators $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$. This limitation shows itself if we apply (4.22) in the case $m = 4$. We find that it is not satisfied by $\mathcal{P}(s) = 1/8$ for all $s$ for any stationary state $|p; \Theta\rangle$. 
4.2 The basis vectors of \(W(p)\) for \(p \leq 2\) (atypical cases)

So far we have considered the properties of almost all state spaces \(W(p)\). There are only two more cases left, namely those with \(p = 1\) and \(p = 2\). We shall see in this section that some of their physical properties are very different from those of the typical cases, considered above.

4.2.1 The state space \(W(2)\)

For \(p = 2\), the state space \(W(2)\) is 7-dimensional, since \(|p; 1, 1, 1\rangle = 0\). Equations (4.5) remain valid for all admissible \(\Theta\)-values (that is, for all \(\Theta\) with \(\Theta \neq (1, 1, 1)\)). This implies that also Conclusion 2 (with equations (4.9) and (4.10)) remains valid for the admissible \(\Theta\)-values. In this case, it is interesting to note that for the states with \(q = p = 2\), one of the operators \(\hat{r}_k^2\) has zero eigenvalue. For example, for the state \(|2; 1, 1, 0\rangle\) one finds

\[
\begin{align*}
    r_1^2 &= r_2^2 = 1, \\
    r_3^2 &= 0.
\end{align*}
\]

Thus the third coordinate of the particle is zero. Also \(p_3\), the third component of the momentum, is zero. So the system becomes flat, and the particle is “polarized” so as to lie in the \(r_1r_2\)-plane. The oscillator behaves as a two-dimensional object. The coordinates of the possible nests for this state are \((r_1, r_2, r_3) = (\pm 1, \pm 1, 0)\). So there are 4 nests where the particle can be found; similarly, it can have only four different momenta.

These nests are depicted in Figure 2. The seven pictures of Figure 2 correspond to the 7 stationary states \(|2; \Theta\rangle\) of \(W(2)\). For \(q = 0\) or \(q = 1\), the situation is not much different from the typical case, and the coordinates of the particle corresponds to one of the eight nests on the sphere, indicated by circles. For \(q = 2\), there are three independent lowest energy states. For these states, there are only 4 possible nests on a sphere with radius \(\sqrt{2}\). These 4 nests are in the \(r_1r_2\)-plane for \(\Theta = (1, 1, 0)\), in the \(r_1r_3\)-plane for \(\Theta = (1, 0, 1)\), and in the \(r_2r_3\)-plane for \(\Theta = (0, 1, 1)\).

The conclusions about the probabilities, formulated in Proposition 2, remain valid, but should be modified appropriately for the lowest energy states with \(q = 2\). For example, in the stationary state \(|2; 1, 1, 0\rangle\), the four nests have coordinates

\[(r_{1\pm}, r_{2\pm}, 0) = (\pm 1, \pm 1, 0).\]

The four probabilities \(\mathcal{P}(\pm \pm 0)\) cannot be determined, although we know that

\[
\begin{align*}
    \mathcal{P}(++0) + \mathcal{P}(+0) &= 1/2, \\
    \mathcal{P}(-+0) + \mathcal{P}(--0) &= 1/2, \\
    \mathcal{P}(++0) + \mathcal{P}(--0) &= 1/2, \\
    \mathcal{P}(-+0) + \mathcal{P}(--0) &= 1/2.
\end{align*}
\]

The derivation of these results for the probabilities is the same as for the typical case, and is based on the explicit expansion of the \(\Theta\)-states in terms of the (normalized) eigenvectors of \(\hat{r}_k\) and \(\hat{p}_k\) \((k = 1, 2, 3)\).

To construct the eigenvectors of \(\hat{r}_k\), one can still use the earlier expressions (4.12). Simply replace \(|2; 1, 1, 1\rangle\) by 0 in each of the vectors (4.12); then the resulting 7 vectors \(v_k(\Theta)\) with...
\[ \theta_1 + \theta_2 + \theta_3 \leq 2 \] are the eigenvectors of \( \hat{r}_k \). Observe that some of these eigenvectors need to be renormalized because of the substitution \( |2; 1, 1, 1\rangle = 0 \). For example, for \( p > 2 \) one has \( v_1(0, 1, 1) = (|p; 0, 1, 1\rangle + e^{i\omega t}|p; 1, 1, 1\rangle) / \sqrt{2} \). For \( p = 2 \) this becomes \( v_1(0, 1, 1) = |2; 0, 1, 1\rangle / \sqrt{2} \), and the corresponding normalized \( \hat{r}_1 \) eigenvector is \( |2; 0, 1, 1\rangle \).

Further consideration by means of a semi-classical argument, analogous to the use of (4.22), leads to an expectation that

\[ P(+ + 0) = P(+ - 0) = P(- + 0) = P(- - 0) = 1/4. \]

### 4.2.2 The state space \( W(1) \)

The state space \( W(1) \) is 4-dimensional. The admissible \( \Theta \)-values have \( \theta_1 + \theta_2 + \theta_3 \leq 1 \). For these admissible \( \Theta \)-values, (4.5) and Conclusion 2 remain valid. In this case, the interesting states are those with lowest energy with \( q = p = 1 \). For these states, two of the operators \( \hat{r}_k^2 \) have zero eigenvalue. For example, for the state \( |1; 1, 0, 0\rangle \) one finds

\[ r_1^2 = 1, \quad r_2^2 = r_3^2 = 0. \] (4.29)

The coordinates of the two possible nests for this state are \( (r_1, r_2, r_3) = (\pm 1, 0, 0) \). Similarly, \( p_2 = p_3 = 0 \) for this state. So the oscillating system becomes one-dimensional, the particle is “polarized” along the \( r_1 \)-axis.

These nests are depicted in Figure 3. The four pictures of Figure 3 correspond to the 4 stationary states \( |1; \Theta\rangle \) of \( W(1) \). For \( q = 0 \) the coordinates of the particle corresponds to one of the eight nests on the sphere, indicated by circles. As before, the probabilities of finding the particle in the nests cannot be determined individually, and we can only draw the conclusion that the probability of finding the particle somewhere in the four nests above or below the \( r_1r_2 \)-plane (resp. \( r_1r_3 \)-plane and \( r_2r_3 \)-plane) is \( 1/2 \).

For \( q = 1 \), there are again three independent lowest energy states. For these states, there are only 2 possible nests. This time, the same considerations about probabilities leads to a unique solution for the probabilities of finding the particle in one of the two nests. For each of the states \( |1; \Theta\rangle \) with \( q = p = 1 \), the probability of finding the particle in one of the two nests is \( 1/2 \). These nests are at the opposite poles on a sphere with radius 1.

The analysis of the probabilities is completely similar to the previous cases. The essential ingredient is again the explicit expansion of the \( \Theta \)-states in terms of the (normalized) eigenvectors of \( \hat{r}_k \) and \( \hat{p}_k \) \( (k = 1, 2, 3) \).

In order to construct the eigenvectors of \( \hat{r}_k \) in \( W(1) \), one can once more use the earlier expressions (L.12), simply by replacing all non-admissible \( \Theta \)-states by 0 and renormalizing the vectors if necessary.

### 4.3 Arbitrary vectors of \( W(p) \)

So far we were studying mainly the properties of the \( \Theta \)-states. Here we proceed to consider some properties of the coordinates and momenta for an arbitrary state \( x \in W(p) \) and for any representation label \( p \).
An arbitrary vector \( x \) from the state space \( W(p) \) can be represented as

\[
x = \sum_{\theta_{123} \leq p} \alpha(\theta_1, \theta_2, \theta_3)|p; \theta_1, \theta_2, \theta_3\rangle,
\]

(4.30)

where \( \alpha(\theta_1, \theta_2, \theta_3) \) are any complex numbers, such that

\[
\sum_{\theta_{123} \leq p} |\alpha(\theta_1, \theta_2, \theta_3)|^2 = 1.
\]

(4.31)

Above and throughout

\[
\theta_{ijk} = \theta_i + \theta_j + \theta_k \quad (4.32)
\]

and \( \sum_{\theta_{123} \leq p} \) denotes a sum over all \( \theta_1, \theta_2, \theta_3 \in \{0, 1\} \) with the additional restriction \( \theta_1 + \theta_2 + \theta_3 \leq p \). We shall be using also the polar form of \( \alpha(\theta_1, \theta_2, \theta_3) \)

\[
\alpha(\Theta) = |\alpha(\Theta)| e^{i\varphi(\Theta)},
\]

(4.33)

where \( \alpha(\Theta) = \alpha(\theta_1, \theta_2, \theta_3) \) and \( \varphi(\Theta) = \varphi(\theta_1, \theta_2, \theta_3) \).

The possible coordinates (and momenta) of the oscillator, in an arbitrary state \( x \), follows from the previous discussions and the superposition principle. For clarity, let us formulate it for \( W(p) \) with \( p > 2 \). Then a measurement of the position of the particle in the state \( x \) will yield one of the 64 possible nests

\[
(r_1, r_2, r_3) = (\pm \sqrt{p-q+\theta_1}, \pm \sqrt{p-q+\theta_2}, \pm \sqrt{p-q+\theta_3}), \quad \text{with} \quad q = \theta_1 + \theta_2 + \theta_3.
\]

The probability of finding the particle somewhere in the eight nests associated with \( |p; \theta_1, \theta_2, \theta_3\rangle \) is given by \( |\alpha(\theta_1, \theta_2, \theta_3)|^2 \), but the probability for each nest separately cannot be determined. Similarly, an arbitrary state \( x \) of \( W(2) \) can be in 44 possible nests; an arbitrary state \( x \) of \( W(1) \) can be found in 14 possible nests.

In order to give more properties of the position probabilities, it is again necessary to expand the general state \( x \) in terms of the orthonormalized eigenstates of \( \hat{r}_k(t) \). Let us do it here explicitly for \( p > 2 \); (4.30) and (4.14) imply:

\[
x = \sum_{\Theta} \frac{1}{\sqrt{2}} \left( \alpha(\Theta_{\theta_k=0}) + (-1)^{\theta_1+\ldots+\theta_{k-1}} e^{i\omega t} \alpha(\Theta_{\theta_k=1}) \right) v_k(\Theta).
\]

(4.34)

Then the square modulus of the coefficient in front of \( v_k(\Theta) \) yields the probability of the particle in the state \( x \) being observed to have \( \hat{r}_k(t) \) eigenvalue \( (-1)^{\theta_k} \sqrt{p-q+\theta_k} \).

Many of our formulas to be presented later will look rather complicated in the general state \( x \), so sometimes we shall concentrate on a particular example of such a normalized state which carries all the main features of the general picture. We take as our standard example one of the simplest non-stationary states (we assume that \( p > 2 \), but most of the results, apart from the number of the nests hold for \( p = 1 \) and 2)

\[
z = \frac{1}{\sqrt{2}} |p; 0, 0, 0\rangle + \frac{1}{\sqrt{2}} |p; 0, 0, 1\rangle.
\]

(4.35)
The consideration of such an example will help to understand some of the peculiar features of the WQO in a general state. Let us explicitly deduce what can be said about the position of the particle when the system is in the state $z$. First of all, only two $\Theta$-states are involved in (4.35), each of these $\Theta$-states corresponding to 8 nests. All of these nests are different, so the particle can be detected in 16 possible nests. The probability to detect the particle somewhere in the 8 nests corresponding to $|p; 0, 0, 0 \rangle$ or to $|p; 0, 0, 1 \rangle$ is $1/2$; these probabilities are just the square moduli of the coefficients in (4.33).

The state (4.35) of the oscillator corresponds to a configuration in which (see Fig. 1) 8 nests have value of $r_3 = \sqrt{p}$ and the other 8 states have $r_3 = -\sqrt{p}$. We cannot determine the probabilities of the 16 nests separately, but we can draw conclusions about the probability $P(r_3 = \pm \sqrt{p})$ of detecting the particle in the nests with a given $r_3$-value. To this end consider the expansion of the state $z$ in terms of the eigenvectors of $\hat{r}_3(t)$:

$$z = \frac{1}{2} (1 + e^{i\omega t}) v_3(0, 0, 0) + \frac{1}{2} (1 - e^{i\omega t}) v_3(0, 0, 1).$$

(4.36)

Then the square moduli of the coefficients give the probabilities of finding the particle in the nests with a particular $r_3$-value. So we find:

$$r_3 = -\sqrt{p} \quad \text{with probability} \quad P(r_3 = -\sqrt{p}) = \frac{1 - \cos(\omega t)}{2},$$

(4.37a)

$$r_3 = \sqrt{p} \quad \text{with probability} \quad P(r_3 = \sqrt{p}) = \frac{1 + \cos(\omega t)}{2}.$$  

(4.37b)

Equations (4.37) describe an interesting new phenomenon, which does not show up whenever the oscillator is in one of the $\Theta$-basis states or more generally in any stationary state (3.2). As it should be $P(r_3 = -\sqrt{p}) + P(r_3 = \sqrt{p}) = 1$. But the probabilities are time dependent. There is an oscillation of the probabilities: the probabilities for the particle to be found in the nests either with $r_3 = \sqrt{p}$ or with $r_3 = -\sqrt{p}$ vary from zero to one.

Contrary to this the probabilities of finding the particle in the nests with a fixed $r_1$-value, or with a fixed $r_2$-value are time independent. This follows from the expansion of the $z$-state in terms of the eigenvectors of $\hat{r}_1(t)$ and $\hat{r}_2(t)$, which yields:

$$P(r_1 = \sqrt{p}) = P(r_1 = -\sqrt{p}) = P(r_1 = \sqrt{p - 1}) = P(r_1 = -\sqrt{p - 1}) = \frac{1}{4},$$

(4.38)

and

$$P(r_2 = \sqrt{p}) = P(r_2 = -\sqrt{p}) = P(r_2 = \sqrt{p - 1}) = P(r_2 = -\sqrt{p - 1}) = \frac{1}{4}.$$  

(4.39)

In the case of $p = 1$ (4.36) and (4.37) still hold, so the oscillations of the probabilities along the $r_3$-axis remain unaltered. In this case however two of the 10 nests, those associated with $|1; 0, 0, 1 \rangle$, are on the third axis, which yields:

$$P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2}, \quad P(r_1 = 0) = \frac{1}{4},$$

(4.40)

and

$$P(r_2 = 1) = P(r_2 = -1) = \frac{1}{4}, \quad P(r_2 = 0) = \frac{1}{2}.$$  

(4.41)
Based on (4.37)-(4.39) we can compute the average values of the coordinates in the state $z$:

$$
\langle \hat{r}_3(t) \rangle_z = -\sqrt{p} \frac{1 - \cos(\omega t)}{2} + \sqrt{p} \frac{1 + \cos(\omega t)}{2} = \sqrt{p} \cos(\omega t). \quad (4.42a)
$$
$$
\langle \hat{r}_1(t) \rangle_z = \langle \hat{r}_2(t) \rangle_z = 0. \quad (4.42b)
$$

Let us consider also the probability properties of the stationary states $\psi_q$, with $q = 1$ and $q = 2$ ($q = 0$ and $q = 3$ correspond to basis states $|p; 0, 0, 0\rangle$ and $|p; 1, 1, 1\rangle$ and the possible coordinates of these states have already been discussed).

In the state $\psi_1$, the particle is detected in one of the 24 nests on the same sphere, corresponding to the union of the nests for $|p; 1, 0, 0\rangle$, $|p; 0, 1, 0\rangle$ and $|p; 0, 0, 1\rangle$. The probability of finding the particle somewhere in the 8 nests of the basis state $|p; 1, 0, 0\rangle$ is given by $|\alpha(1, 0, 0)|^2$, and similarly for the other two basis states involved. The probabilities for each nest separately cannot be determined, but one can make definite statements about the probability of finding the particle in the nests with a fixed height with respect to the coordinate planes. An expansion of $\psi_1$ in terms of the $\hat{r}_1(t)$ eigenvectors $v_1(\Theta)$ by means of (4.34) yields: $r_1$ can take the values $\pm \sqrt{p}$, each with probability $|\alpha(1, 0, 0)|^2/2$; or $r_1$ can take the values $\pm \sqrt{p-1}$, each with probability $|\alpha(0, 1, 0)|^2/2 + |\alpha(0, 0, 1)|^2/2$. The probabilities related to $r_2$ and $r_3$ are similar.

The analysis for $\psi_2$ is identical. An expansion of $\psi_2$ in terms of the $\hat{r}_1(t)$ eigenvectors yields: $r_1$ can take the values $\pm \sqrt{p-1}$, each with probability $|\alpha(1, 1, 0)|^2/2 + |\alpha(1, 0, 1)|^2/2$; or $r_1$ can take the values $\pm \sqrt{p-2}$, each with probability $|\alpha(0, 1, 1)|^2/2$.

### 5 Mean trajectories and standard deviations of positions and momenta

Now that we have discussed some properties of the position operators in more general states, let us next compute the mean trajectories or time dependent expectation values of the coordinates and momenta and their standard deviations in a general state $x$. We shall then specify our results to the stationary states $\psi_q$. We shall indicate also (Conclusion 3) that for the WQS there exists no (nontrivial) Heisenberg uncertainty relations.

For the mean trajectory of the coordinates in an arbitrary state $x$ we obtain:

$$
\langle \hat{r}_k(t) \rangle_x = \langle x, \hat{r}_k(t)x \rangle = \sum_{\theta_{123} \leq p} (-1)^{\theta_1+\cdots+\theta_{k-1}} \sqrt{p - q + \theta_k} |\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})| \times \cos(\omega t - \varphi(\Theta_{\theta_k=0}) + \varphi(\Theta_{\theta_k=1})),
$$

(5.1)

where as in (1.12) $\Theta_{\theta_k=0}$ stands for the $\Theta$-value in which $\theta_k$ is replaced by 0 (and similarly for $\Theta_{\theta_k=1}$). Observe that in (5.1), the contributions come in equal pairs; e.g. for $k = 1$, the contribution coming from $\Theta = (0, \theta_2, \theta_3)$ is the same as that coming from $\Theta = (1, \theta_2, \theta_3)$, since $\sqrt{p - q + \theta_k}$ is independent of $\theta_k$. Similarly, one finds:

$$
\langle \hat{p}_k(t) \rangle_x = \langle x, \hat{p}_k(t)x \rangle = -\sum_{\theta_{123} \leq p} (-1)^{\theta_1+\cdots+\theta_{k-1}} \sqrt{p - q + \theta_k} |\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})| \times \sin(\omega t - \varphi(\Theta_{\theta_k=0}) + \varphi(\Theta_{\theta_k=1})).
$$

(5.2)
For instance, for our standard example (1.28), we find
\[
\langle \hat{r}_1(t) \rangle_z = 0, \quad \langle \hat{r}_2(t) \rangle_z = 0, \quad \langle \hat{r}_3(t) \rangle_z = \sqrt{p} \cos(\omega t),
\]
\[
\langle \hat{p}_1(t) \rangle_z = 0, \quad \langle \hat{p}_2(t) \rangle_z = 0, \quad \langle \hat{p}_3(t) \rangle_z = -\sqrt{p} \sin(\omega t).
\]
(5.3a) (5.3b)

Note that each term in the right hand sides of (5.1)-(5.2) contains multiples
\[\alpha(\theta_1, \theta_2, \theta_3)\alpha(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)\] such that \(\theta_1 + \theta_2 + \theta_3 \neq \tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3.\)

Therefore the right hand sides of (5.1)-(5.2) vanish if the system is in a stationary state \(\psi_q.\)

The latter stems from the observation that in the stationary states, see (3.2), \(x\) is a linear combination of basis states \(|p; \theta_1, \theta_2, \theta_3\rangle\) with fixed \(\theta_1 + \theta_2 + \theta_3,\) namely all nonzero coefficients \(\alpha(\theta_1, \theta_2, \theta_3)\) in (4.23) have one and the same \(q = \theta_1 + \theta_2 + \theta_3.\) Thus we have

**Conclusion 3** The mean trajectories of the coordinates and momenta vanish if the system is in a stationary state \(\psi_q.\)

In order to draw conclusions about the uncertainty of the coordinates and momenta, more precisely about their standard deviations, we also need the mean square deviations of \(\hat{r}_k\) and \(\hat{p}_k,\) \(k = 1, 2, 3.\) It follows from (4.3) that
\[
\langle \hat{r}_k(t)^2 \rangle_x = \langle \hat{p}_k(t)^2 \rangle_x = \sum_{\theta_{123} \leq p} (p - q + \theta_k)|\alpha(\Theta)|^2.
\]
(5.5)

Recall that the general definition of the standard deviation \(\Delta X\) of an observable \(X\) in a state \(x\) is given by
\[
\Delta X_x = \sqrt{\langle X^2 \rangle_x - \langle X \rangle_x^2}.
\]
(5.6)

So from the previous formulas one can write down the standard deviation of \(\hat{r}_k(t)\) and \(\hat{p}_k(t)\) in an arbitrary state \(x:\)
\[
\Delta \hat{r}_k(t)_x = \left[ \sum_{\theta_{123} \leq p} (p - q + \theta_k)|\alpha(\Theta)|^2 - \left( \sum_{\theta_{123} \leq p} (-1)^{\theta_1 + \cdots + \theta_{k-1}} \sqrt{p - q + \theta_k} \right)^2 \right]^{1/2},
\]
(5.7a)
\[
\Delta \hat{p}_k(t)_x = \left[ \sum_{\theta_{123} \leq p} (p - q + \theta_k)|\alpha(\Theta)|^2 - \left( \sum_{\theta_{123} \leq p} (-1)^{\theta_1 + \cdots + \theta_{k-1}} \sqrt{p - q + \theta_k} \right)^2 \right]^{1/2}.
\]
(5.7b)

Because of the double products in the expansion of the square, (5.7) cannot be simplified further for an arbitrary state vector \(x.\)

Let us just observe that in any one of the stationary states \(\psi_q\) the standard deviations become very simple and are time independent:
\[
\Delta \hat{r}_j(t)_{\psi_0} = \Delta \hat{p}_j(t)_{\psi_0} = \sqrt{p},
\]
(5.8a)
\[
\Delta \hat{r}_j(t)_{\psi_1} = \Delta \hat{p}_j(t)_{\psi_1} = \sqrt{p - 1 + |\alpha(\theta_j = 1, \theta_k = \theta_l = 0)|^2} \geq \sqrt{p - 1},
\]
(5.8b)
\[
\Delta \hat{r}_j(t)_{\psi_2} = \Delta \hat{p}_j(t)_{\psi_2} = \sqrt{p - 1 - |\alpha(\theta_j = 0, \theta_k = \theta_l = 1)|^2} \geq \sqrt{p - 2},
\]
(5.8c)
\[
\Delta \hat{r}_j(t)_{\psi_3} = \Delta \hat{p}_j(t)_{\psi_3} = \sqrt{p - 2},
\]
(5.8d)
where \( j \neq k \neq l \neq j \in \{1, 2, 3\} \).

Although formulas (5.7) look complicated, they are easy to apply. For instance, for our standard example (4.28), we find

\[
\begin{align*}
\Delta \hat{r}_1(t)_z &= \Delta \hat{p}_1(t)_z = \Delta \hat{r}_2(t)_z = \Delta \hat{p}_2(t)_z = \sqrt{\frac{2p - 1}{2}}, \\
\Delta \hat{r}_3(t)_z &= \sqrt{p} \sin(\omega t), \quad \Delta \hat{p}_3(t)_z = \sqrt{p} \cos(\omega t).
\end{align*}
\] (5.9)

Equations (5.7) can be used for independent verification of some of the properties of the WQOs. Consider for instance the state \([2; 1, 1, 0]\). We know, see (4.21), that in this state the particle is polarized in the \(r_1r_2\)-plane, both \(r_3 = 0\) and \(p_3 = 0\). Equation (5.7) confirms this:

\[
\Delta \hat{r}_3(t)_y = \Delta \hat{p}_3(t)_y = 0 \quad \text{in the state} \quad y = [2; 1, 1, 0].
\] (5.10)

Let us note more generally that for any \(p\) and \(k = 1, 2, 3\) there exists a state \(x_k\) and a time \(t_k\) such that \(\Delta \hat{r}_k(t)_x \neq 0\) or \(\Delta \hat{p}_k(t)_x \neq 0\). For instance

\[
\begin{align*}
\Delta \hat{r}_1(0)_{x_1} &= 0 \quad \text{for} \quad x_1 = \frac{1}{\sqrt{2}}(|p; 0, 1, 0) + |p; 1, 1, 0\rangle, \\
\Delta \hat{r}_2(0)_{x_2} &= 0 \quad \text{for} \quad x_2 = \frac{1}{\sqrt{2}}(|p; 0, 0, 1) + |p; 0, 1, 1\rangle, \\
\Delta \hat{r}_3(0)_{x_3} &= 0 \quad \text{for} \quad x_3 = \frac{1}{\sqrt{2}}(|p; 1, 0, 0) + |p; 1, 0, 1\rangle.
\end{align*}
\] (5.11)

As an immediate consequence we have

**Conclusion 4**  The position and momentum operators of a WQO do not satisfy an uncertainty-like relation of the form

\[
\Delta \hat{r}_k(t)_x \Delta \hat{p}_k(t)_x \geq \gamma
\] (5.12)

for any \(\gamma > 0\) holding simultaneously for all states \(x\) of the system at all times \(t\) (as is the case for a CQO and, more generally, for any canonical quantum system).

If however, \(x\) is any stationary state \(\psi_q\), then in the typical case with \(p > 2\), (5.8) yields

\[
\Delta \hat{r}_k(t)_{\psi_q} = \Delta \hat{p}_k(t)_{\psi_q} \geq \sqrt{p - 2}
\] (5.13)

for all \(k = 1, 2, 3\). Therefore for any stationary state and any time

\[
\begin{align*}
\Delta \hat{r}_i(t)_{\psi_q} \Delta \hat{r}_j(t)_{\psi_q} &\geq p - 2, \quad \Delta \hat{r}_i(t)_{\psi_q} \Delta \hat{p}_j(t)_{\psi_q} \geq p - 2, \quad \Delta \hat{p}_i(t)_{\psi_q} \Delta \hat{p}_j(t)_{\psi_q} \geq p - 2,
\end{align*}
\] (5.14)

with \(p - 2 > 0\).

Returning to the case of an arbitrary state \(x \in W(p)\), uncertainty-like relations of the type (5.12) certainly will exist, but the uncertainty parameter \(\gamma\) may be zero. They can be derived from the general uncertainty relation [42]

\[
\Delta \hat{F}(t)_x \Delta \hat{G}(t)_x \geq \frac{1}{2} \left| \langle [\hat{F}(t), \hat{G}(t)] \rangle_x \right|.
\] (5.15)
that applies to any two Hermitian operators $\hat{F}$ and $\hat{G}$ for any $x \in W(p)$.

Applying this in the case $\hat{F}(t) = \hat{r}_k(t)$ and $\hat{G}(t) = \hat{p}_k(t)$ and $x$ as defined in (4.23) gives

$$\Delta \hat{r}_k(t)x \Delta \hat{p}_k(t)x \geq \left| \sum_{\theta_1 \leq \theta \leq p} (-1)^{\theta} (p - q + \theta)|\alpha(\Theta)|^2 \right|. \quad (5.16)$$

It should be noted that the sign factors $(-1)^{\theta}$ are such that cancellations may occur and may yield zero on the right hand side, as is the case, for example, if $k = 3$ and $x$ is the state $x_3 = \frac{1}{\sqrt{2}}|p; 1, 0, 0\rangle + |p; 1, 0, 1\rangle$ introduced in (5.11a).

On the other hand if (5.16) is restricted to the stationary state $\psi_0$, for example, then it yields a relation very similar, when properly dimensionalized, to the Heisenberg uncertainty relation, namely:

$$\Delta \hat{R}_k(t)\psi_0 \Delta P_k(t)\psi_0 \geq \frac{\hbar p}{2}. \quad (5.17)$$

Formulae of the type (5.16) in the case of $\Delta \hat{r}_k(t)x \Delta \hat{r}_l(t)x$, $\Delta \hat{r}_k(t)x \Delta \hat{p}_l(t)x$ and $\Delta \hat{p}_k(t)x \Delta \hat{p}_l(t)x$ with $k \neq l$ are however much more involved and will not be analyzed here. They are to be found in Appendix A. It should be noted that as a consequence of the non-commutativity of the coordinates, we find for example,

$$\Delta \hat{r}_1(t)x \Delta \hat{r}_2(t)x \geq \left| 2\sqrt{p(p-1)}|\alpha(0,0,0)\alpha(1,1,0)| \sin(2\omega t - \phi(0,0,0) + \phi(1,1,0)) \\
+ 2\sqrt{(p-1)(p-2)}|\alpha(0,0,1)\alpha(1,1,1)| \sin(2\omega t - \phi(0,0,1) + \phi(1,1,1)) \\
+ (2p-1)|\alpha(1,0,0)\alpha(0,1,0)| \sin(\phi(1,0,0) - \phi(0,1,0)) \\
+ (2p-3)|\alpha(1,0,1)\alpha(0,1,1)| \sin(\phi(1,0,1) - \phi(0,1,1)) \right|. \quad (5.18)$$

6 Comparison with the canonical quantum oscillator

From the discussions so far it becomes clear that the WQOs are very different from the Bose canonical quantum oscillators (CQOs). Therefore it is somewhat of a surprise that one can establish a one-to-one correspondence between some mean trajectories of the 3D CQO and the $p = 1$ mean trajectories of the WQO. This will be the main topic of the present section.

In dimensionless units, see (4.3), the coordinates $\bar{r}_k$ and momenta $\bar{p}_k$, $k = 1, 2, 3$ of a 3D canonical oscillator (23) read:

$$\bar{r}_k(t) = b_k^+ e^{i\omega t} + b_k^- e^{-i\omega t}, \quad \bar{p}_k(t) = i (b_k^+ e^{i\omega t} - b_k^- e^{-i\omega t}). \quad (6.1)$$

Let us consider first a simple example. As a Bose analogue of our standard state $z$ we set

$$\bar{z} = \frac{1}{\sqrt{2}}|0,0,0\rangle + \frac{1}{\sqrt{2}}|0,0,1\rangle. \quad (6.2)$$

It is a simple computation to show that the mean trajectories of the coordinates and momenta in the state $\bar{z}$ of the Bose oscillator read:

$$\langle \bar{r}_1(t) \rangle_z = 0, \quad \langle \bar{r}_2(t) \rangle_z = 0, \quad \langle \bar{r}_3(t) \rangle_z = \cos(\omega t), \quad (6.3a)$$
$$\langle \bar{p}_1(t) \rangle_z = 0, \quad \langle \bar{p}_2(t) \rangle_z = 0, \quad \langle \bar{p}_3(t) \rangle_z = -\sin(\omega t). \quad (6.3b)$$
The above trajectories are the same as those of the WQO given in (1.3) provided in the latter that $p = 1$. This was the first indication that some of the trajectories of the WQO are the same as those of the canonical Bose oscillator. The question is how far does this similarity go. In the next proposition we summarize the results which we are able to establish.

**Proposition 3** To each $p = 1$ mean trajectory in the phase space of the Wigner quantum oscillator there corresponds an identical trajectory of the 3D Bose canonical quantum oscillator.

Let $p = 1$. By a straightforward computation one shows that the mean trajectory of the WQO in the state

$$x = \alpha(0, 0, 0)|1; 0, 0, 0\rangle + \alpha(1, 0, 0)|1; 1, 0, 0\rangle + \alpha(0, 1, 0)|1; 0, 1, 0\rangle + \alpha(0, 0, 1)|1; 0, 0, 1\rangle \quad (6.4)$$

is the same as the mean trajectory of the Bose oscillator in the state

$$x^* = \alpha(0, 0, 0)^*|0, 0, 0\rangle + \alpha(1, 0, 0)^*|1, 0, 0\rangle + \alpha(0, 1, 0)^*|0, 1, 0\rangle + \alpha(0, 0, 1)^*|0, 0, 1\rangle. \quad (6.5)$$

The $\ast$ in the right hand side of (6.3) denotes complex conjugation.

Explicitly the mean trajectories corresponding to (6.4) and (6.5) read:

$$\langle \hat{r}_k(t) \rangle_x = \langle \hat{r}_k(t) \rangle_{x^*} = 2|\alpha(0, 0, 0)\alpha(0, 0, 0)_{\theta_k=1}| \cos(\omega t + \varphi(0, 0, 0)_{\theta_k=1} - \varphi(0, 0, 0)), \quad (6.6a)$$

$$\langle \hat{p}_k(t) \rangle_x = \langle \hat{p}_k(t) \rangle_{x^*} = -2|\alpha(0, 0, 0)\alpha(0, 0, 0)_{\theta_k=1}| \sin(\omega t + \varphi(0, 0, 0)_{\theta_k=1} - \varphi(0, 0, 0)), \quad (6.6b)$$

where $\alpha(0, 0, 0)_{\theta_k=1}$ denotes $\alpha(1, 0, 0)$, $\alpha(0, 1, 0)$, $\alpha(0, 0, 1)$ according as $k = 1, 2, 3$, respectively.

However, although the standard deviations of the coordinates and momenta of the WQO in a state $x$ and of the CQO in a state $x^*$ also look somewhat similar, they are in fact different:

$$\text{WQO:} \quad \Delta \hat{r}_k(t)_{x} = \left[|\alpha(0, 0, 0)|^2 + |\alpha(0, 0, 0)_{\theta_k=1}|^2ight]^{1/2} \quad (6.7)$$

$$-4|\alpha(0, 0, 0)\alpha(0, 0, 0)_{\theta_k=1}|^2 \cos^2(\omega t - \varphi(0, 0, 0) + \varphi(0, 0, 0)_{\theta_k=1})), \quad k = 1, 2, 3.$$

$$\text{CQO:} \quad \Delta \hat{r}_k(t)_{x^*} = \left[1 + 2|\alpha(0, 0, 0)_{\theta_k=1}|^2ight]^{1/2} \quad (6.8)$$

$$-4|\alpha(0, 0, 0)\alpha(0, 0, 0)_{\theta_k=1}|^2 \cos^2(\omega t - \varphi(0, 0, 0) + \varphi(0, 0, 0)_{\theta_k=1})), \quad k = 1, 2, 3.$$

Our standard state $z$ provides a good illustration of the difference:

$$\text{WQO:} \quad \Delta \hat{r}_1(t)_{z} = \Delta \hat{r}_2(t)_{z} = \frac{1}{\sqrt{2}}, \quad \Delta \hat{r}_3(t)_{z} = |\sin(\omega t)|, \quad (6.9)$$

$$\text{CQO:} \quad \Delta \hat{r}_1(t)_{z^*} = \Delta \hat{r}_2(t)_{z^*} = 1, \quad \Delta \hat{r}_3(t)_{z^*} = (1 + \sin^2(\omega t))^{1/2}. \quad (6.10)$$
Let us go further and compare the standard deviations corresponding to the state $y = |1; 1, 0, 0\rangle$ and its Bose “partner” $y^* = |1, 0, 0\rangle$.

\[
\begin{align*}
\text{WQO: } & \quad \Delta \hat{r}_1(t)_y = \Delta \hat{p}_1(t)_y = 1, \quad \Delta \hat{r}_k(t)_y = \Delta \hat{p}_k(t)_y = 0, \quad k = 1, 2, \quad (6.11) \\
\text{CQO: } & \quad \Delta \bar{r}_1(t)_y = \Delta \bar{p}_1(t)_y = \sqrt{3}, \quad \Delta \bar{r}_k(t)_y = \Delta \bar{p}_k(t)_y = 1, \quad k = 1, 2. \quad (6.12)
\end{align*}
\]

Equation (6.11) confirms that the oscillator in the state $|1; 1, 0, 0\rangle$ “lives” on the $r_1$-axis. This is not the case for the CQO, either in the state $|1, 0, 0\rangle$ or in any other state, since the right hand side of (1.8) never vanishes, as is implied also by the Heisenberg uncertainty relations.

We have compared also the sizes of the WQO and CQO corresponding to the basis states. For the CQO

\[
\langle \bar{r}(t)^2 \rangle_{x^*} = 5 - 2|\alpha(0, 0, 0)|^2. \quad (6.14)
\]

Thus we have

\[
\begin{align*}
\text{WQO: } & \quad \langle \hat{r}(t)^2 \rangle_{|1,0,0,0\rangle} = 3, \quad \langle \hat{r}(t)^2 \rangle_{|1;1,0,0\rangle} = \langle \hat{r}(t)^2 \rangle_{|1,0,1,0\rangle} = \langle \hat{r}(t)^2 \rangle_{|1,0,0,1\rangle} = 1, \quad (6.15) \\
\text{CQO: } & \quad \langle \bar{r}(t)^2 \rangle_{|0,0,0\rangle} = 3, \quad \langle \bar{r}(t)^2 \rangle_{|0,0,0\rangle} = \langle \bar{r}(t)^2 \rangle_{|0,1,0\rangle} = \langle \bar{r}(t)^2 \rangle_{|0,0,1\rangle} = 5. \quad (6.16)
\end{align*}
\]

Thus only the states $|1; 0, 0, 0\rangle$ of the WQO and $|0, 0, 0\rangle$ of the CQO have one and the same space dimensions. This is perhaps not surprising since only these states have one and the same energy $\epsilon = 3$ (in units of $\omega\hbar/2$). The energy of the other WQO states is 1, whereas for the other CQO states it is 5.

### 7 Rotational invariance

In our analysis so far the description of the WQO has been carried out in terms of a fixed Cartesian coordinate system. However, it should be possible to analyse the WQO equally well with respect to a Cartesian coordinate system obtained from the original one by means of any rotation $g \in SO(3)$, the group of rotations in the underlying 3D space.

Clearly the picture of the WQO as developed so far will be selfconsistent if its properties remain unaltered under any simultaneous rotation of both the coordinate frame and the oscillator states. In that case the coordinates $(r'_1, r'_2, r'_3)$ of any rotated nest measured with respect to the rotated frame of reference should be the same as the coordinates $(r_1, r_2, r_3)$ of the original nest measured with respect to the original frame of reference. For each rotation $g \in SO(3)$ we have $g: \hat{r} \mapsto \hat{r}' = \hat{r}g$, so that for

\[
g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \quad (7.1)
\]
we have

\[ \begin{align*}
\hat{r}'_1 &= \hat{r}_1 g_{11} + \hat{r}_2 g_{21} + \hat{r}_3 g_{31}, \\
\hat{r}'_2 &= \hat{r}_1 g_{12} + \hat{r}_2 g_{22} + \hat{r}_3 g_{32}, \\
\hat{r}'_3 &= \hat{r}_1 g_{13} + \hat{r}_2 g_{23} + \hat{r}_3 g_{33}.
\end{align*} \]

(7.2a, 7.2b, 7.2c)

That the values of the new coordinates coincide with those of the old can be verified directly by finding the eigenvalues of \( \hat{r}'_1, \hat{r}'_2 \) and \( \hat{r}'_3 \). As expected for any \( g \) it turns out that in each case the eigenvalues are once again \( \pm \sqrt{p}, \pm \sqrt{p-1} \) and \( \pm \sqrt{p-2} \) with, as before, multiplicities 1, 2 and 1, respectively. More precisely, we can identify a new basis of \( W(p) \) consisting of common eigenvectors \(| p; \Theta \rangle_g \) of \( \hat{H}, (\hat{r}_1')^2, (\hat{r}_2')^2 \) and \( (\hat{r}_3')^2 \) as in (7.1) with eigenvalues \( p-q+\theta_k \) for \( k = 1, 2 \) and 3, respectively, where \( q = \theta_1 + \theta_2 + \theta_3 \). This leads to the conclusion that the measured values of the coordinates of the nests after both simultaneous rotations are given by

\[ \begin{align*}
\hat{r}_1' &= \pm \sqrt{p-q+\theta_1}, \\
\hat{r}_2' &= \pm \sqrt{p-q+\theta_2}, \\
\hat{r}_3' &= \pm \sqrt{p-q+\theta_3},
\end{align*} \]

(7.3)
in complete agreement with (4.9).

The explanation of this finding that all rotated coordinate systems are equivalent, lies in the observation that for each \( g \in SO(3) \) we can define new CAOs \( A_k^\pm (g) \) by

\[ A_k^\pm (g) = \sum_{i=1}^{3} A_i^\pm g_{ik} \]

(7.4)
such that

\[ g : \left| p; \Theta \right> = \sqrt{\frac{(p-q)!}{p!}} (A_1^+)^{\theta_1} (A_2^+)^{\theta_2} (A_3^+)^{\theta_3} \left| 0 \right> \]

\[ \mapsto \left| p; \Theta \right>_g = \sqrt{\frac{(p-q)!}{p!}} (A_1^+(g))^{\Theta_1} (A_2^+(g))^{\Theta_2} (A_3^+(g))^{\Theta_3} \left| 0 \right> \].

(7.5)

The crucial point is that under this transformation of the CAOs the new operators \( A_k^\pm (g) \) provide another solution to the triple relations (2.11). That is for all \( g \in SO(3) \) we have

\[ \begin{align*}
\{ A_i^+(g), A_j^+(g) \}, A_k^+(g) &= \delta_{ik} A_j^+(g) - \delta_{ij} A_k^+(g), \\
\{ A_i^+(g), A_j^-(g) \}, A_k^+(g) &= -\delta_{ik} A_j^-(g) + \delta_{ij} A_k^-(g), \\
\{ A_i^+(g), A_j^-(g) \} &= \{ A_i^-(g), A_j^+(g) \} = 0.
\end{align*} \]

(7.6a, 7.6b, 7.6c)

This implies that for all \( g \in SO(3) \) the operators \( A_k^\pm (g) \), for \( k = 1, 2, 3 \), generate the same Lie superalgebra \( sl(1|3n) \) whose irreducible representations specified by \( p \) define our Fock spaces \( W(p) \). In fact, for each \( g \in SO(3) \) the states \( | p; \Theta \rangle_g \) with \( \Theta_i \in \{0,1\} \) for all \( i = 1, 2, 3 \), subject to the familiar constraints \( 0 \leq q = \Theta_1 + \Theta_2 + \Theta_3 \leq \min(p,3) \), constitute an orthonormal basis for \( W(p) \). The transformation of these basis states is precisely as in (2.23), namely

\[ \begin{align*}
A_i^-(g) | p; \ldots, \Theta_i, \ldots \rangle_g &= \theta_i (-1)^{\Theta_1 \ldots \Theta_i-1} \sqrt{p-q+1} | p; \ldots, \Theta_i-1, \ldots \rangle_g, \\
A_i^+(g) | p; \ldots, \Theta_i, \ldots \rangle = (1-\theta_i) (-1)^{\Theta_1 \ldots \Theta_i-1} \sqrt{p-q} | p; \ldots, \Theta_i+1, \ldots \rangle_g.
\end{align*} \]

(7.7a, 7.7b)

As a consequence we have the following
Observation 2 Let \( \hat{O} \equiv \hat{O}(A_1^\pm, A_2^\pm, A_3^\pm) \) be any operator that takes the form of a multinomial in the CAO’s \( A_1^\pm, A_2^\pm \) and \( A_3^\pm \), and let \( v \equiv v(A_1^\pm, A_2^\pm, A_3^\pm) | 0 \rangle \) and \( u \equiv u(A_1^\pm, A_2^\pm, A_3^\pm) | 0 \rangle \) be any two vectors from \( W(p) \). Then for any \( g \in SO(3) \)
a) \[ (u, \hat{O}v) = (u_g, \hat{O}_g v_g), \]  
where \[ \hat{O}_g \equiv \hat{O}(A_1^\pm(g), A_2^\pm(g), A_3^\pm(g)); \]  
and \[ v_g \equiv v(A_1^\pm(g), A_2^\pm(g), A_3^\pm(g)) | 0 \rangle, \quad u_g \equiv u(A_1^\pm(g), A_2^\pm(g), A_3^\pm(g)) | 0 \rangle. \]
b) In particular if \( v \) is an eigenvector of \( \hat{O} \) with an eigenvalue \( \lambda \), so that \[ \hat{O}v = \lambda v, \quad \text{then} \quad \hat{O}_g v_g = \lambda v_g. \]

Returning to the case where \( |p; \Theta \rangle_g \), for various \( \Theta \), are a set of eight simultaneous eigenstates of \( \langle \hat{r}_1^1 \rangle^2 \), \( \langle \hat{r}_2^2 \rangle^2 \) and \( \langle \hat{r}_3^3 \rangle^2 \), with \( \hat{r}_1^1, \hat{r}_2^2, \hat{r}_3^3 \) defined in terms of \( \hat{r}_1, \hat{r}_2, \hat{r}_3 \) and the matrix elements of \( g \) by \( (7.2) \), we can find the transformation matrix \( T_g \) such that \( |p; \Theta \rangle_g = |p; \Theta \rangle T_g \) by explicitly expanding the formula \( (7.5) \) for \( |p; \Theta \rangle_g \). With respect to our usual ordering (cf. Table 1) of the states labelled by \( \Theta \), we find from \( (7.5) \) that
\[
T_g = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{11} & g_{12} & g_{13} & 0 & 0 & 0 & 0 \\
0 & g_{21} & g_{22} & g_{23} & 0 & 0 & 0 & 0 \\
0 & g_{31} & g_{32} & g_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{33} & g_{32} & g_{31} & 0 \\
0 & 0 & 0 & 0 & g_{23} & g_{22} & g_{21} & 0 \\
0 & 0 & 0 & 0 & g_{13} & g_{12} & g_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \det(g)
\end{pmatrix},
\]

where \( g_{ij} \) is the \((ij)\)th minor of \( g \), that is the determinant of the \(2 \times 2\) submatrix of \( g \) obtained by deleting the \(i\)th row and \(j\)th column, and of course \( \det(g) = 1 \) for all \( g \in SO(3) \). As can be seen from their block diagonal structure, the set of matrices \( T_g \) for \( g \in SO(3) \) constitute a reducible 8-dimensional representation of \( SO(3) \) which governs the transformation of the basis states of \( W(p) \) from \( |p; \Theta \rangle \) to \( |p; \Theta \rangle_g \) defined by \( (7.3) \).

8 Concluding remarks

It is clear that while alternative, non-canonical solutions to the compatibility equations \( (2.4) \) between Hamilton’s equations and the Heisenberg equations exist in our \( sl(1|3) \) WQO model, they are in several very important respects quite different from the canonical solutions.

Firstly, each state space \( W(p) \) of our one particle 3D WQO is finite-dimensional; 8-dimensional in the case of typical representations of \( sl(1|3) \), and either 7 or 4-dimensional in the case of atypical representations.
Secondly, both the energy and the angular momentum are quantized, with equally spaced energy levels, all positive and with separation $\frac{1}{2} \hbar \omega$, and with the angular momentum restricted to be 0 or 1. Since there are only a finite number of energy levels, the energy is bounded. The degeneracies are always either 3 or 1. The lowest energy state is non-degenerate in all the typical cases, but degenerate in each of the atypical cases.

Thirdly, the spectrum of coordinates is also quantized, to the extent that in any stationary state measurements of the coordinates $r_1$, $r_2$, and $r_3$ give values consistent only with the particle being found at a finite number of possible sites, namely the various nests that we have identified. In the typical, $p > 2$ case, the number of possible nests is 64, while in the atypical cases it is 44 if $p = 2$ and only 14 if $p = 1$. In all cases the distance of the particle from the origin is bounded and may take on only the values $\sqrt{(\hbar/2m\omega)(3p - 2q)}$ with $q \in \{0, 1, 2, 3\}$ such that $q \leq p$.

Fourthly, not only is the mean trajectory of the particle in any stationary state zero, but there exist both typical and atypical states for which the standard deviation of some coordinate $r_k$, or some component of linear momentum $p_k$, is also zero. This implies that for the WQO there exists no uncertainty relation involving a non-zero uncertainty parameter $\gamma$ that applies to all states at all times.

Fifthly, the atypical case is distinguished from the typical case in possessing stationary states of dimension lower than three, namely two-dimensional in the case $p = 2$ and one-dimensional in the case $p = 1$.

Many of these non-standard results are a consequence of the fact that the underlying geometry of this WQO model is non-commutative. This means that their interpretation must be undertaken carefully. In particular it should be stressed that it is not possible to specify precisely the position of the particle. Fortunately, the square operators $\hat{r}_1^2$, $\hat{r}_2^2$ and $\hat{r}_3^2$ not only mutually commute but also commute with the Hamiltonian. Their common eigenstates are the stationary states $|p; \Theta\rangle$, for which the eigenvalues of $\hat{r}_1^2$, $\hat{r}_2^2$ and $\hat{r}_3^2$ are simultaneously fixed to be either $p$, $p - 1$ or $p - 2$. Thus the spectrum of the measured values of each of the coordinates themselves, $r_1$, $r_2$ and $r_3$, is necessarily restricted to the set of values $\pm \sqrt{p}$, $\pm \sqrt{p - 1}$ and $\pm \sqrt{p - 2}$. Any measurement of a coordinate, $r_1$ say, results in one or other of the allowed positive or negative values of $r_1$ with, as we have shown, equal probability, leaving the signs of the other coordinates undetermined. Thus the particle in a stationary state $|p; \Theta\rangle$ has a certain probability of being within one or other of the relevant nests, but it is not to be thought of as localized in any particular nest. The peculiarities of the non-commutative geometry are such that these observations are independent of the choice of coordinates $r_1$, $r_2$ and $r_3$. The rotational invariance of the WQO is such that the same conclusion about the nests is reached for measurements with respect to any set of three coordinates obtained from $r_1$, $r_2$ and $r_3$ by means of a rotation. There is no preferred coordinate system for the description of the WQO.

In this article we have restricted ourselves to the relatively simple case of an $n = 1$, single particle 3-dimensional WQO with a relatively low, $2^3$-dimensional Fock space $W(p)$ associated with typical irreducible representations of $sl(1|3)$, and even lower dimensions for atypical representations. A very natural next step is to generalize the results to the case of an $n$-body 3-dimensional WQO as introduced in [2]. In such a case the dimension of the Fock space is $2^3n$-dimensional for typical representations, and again lower dimensions
for atypical representations. The relevant calculations are somewhat intricate and will be the subject of a separate article, in which the restriction from the Lie superalgebra \( \text{sl}(1|3n) \) to the simple Lie algebra \( \text{so}(3) \) of the rotation group plays a key role in the determination of the possible angular momentum states of our multiparticle system. It suffices to say at this stage that not only are the energy and angular momentum quantized and bounded but, in the corresponding stationary states of fixed energy and angular momentum, so also are the single particle coordinates and components of linear momentum. As is to be expected the corresponding nest structure is more complicated and we have to contend with the relevant class \( A \) statistics, and account for the numbers of particles whose coordinates, when measured, can coincide with those of the nests, as well as more complicated patterns of degeneracy.

Further generalizations also come to mind. In particular the class of irreducible representations considered here are those specified by the parameter \( p \). There exist other finite-dimensional irreducible representations of \( \text{sl}(1|3) \), and more generally of \( \text{sl}(1|3n) \), that are specified not just by a single positive integer \( p \), but by a partition or equivalently a sequence of Kac-Dynkin indices \([28]\). These can be expected to provide other interesting models of the Wigner quantum oscillator in the one-particle case and, more particularly, in multi-particle cases. At the same time it would be interesting to explore in the same way other non-oscillator Wigner quantum systems. For examples of this kind see \([7, 10, 13]\).

**Acknowledgements**

TDP is thankful to Prof. Randjbar-Daemi for the kind invitation to visit the High Energy Section of the Abdus Salam International Centre for Theoretical Physics and to Prof. D. Trifonov for the numerous discussions. NIS has been supported by a Marie Curie Individual Fellowship of the European Community Programme “Improving the Human Research Potential and the Socio-Economic Knowledge Base” under contract number HPMF-CT-2002-01571. This work was supported also by the Royal Society Joint Project Grant UK-Bulgaria H01R381 and by NATO (Collaborative Linkage Grant).

**Appendix A**

Here we gather together expressions for \((\Delta r_k)_x(\Delta l)_x\), \((\Delta p_k)_x(\Delta p_l)_x\) and \((\Delta r_k)_x(\Delta p_l)_x\). They have been calculated by means of the general uncertainty relation \((5.13)\), applied in the case of the arbitrary state \(x \in W(p)\) defined in \((4.23)\).

First of all in the case \(k = l\) the only non-vanishing commutator gives:

\[
\Delta \hat{r}_k(t)_x \Delta \hat{p}_k(t)_x \geq \left| \sum_{\theta_{23} \leq p} (-1)^{\theta_k} (p - q + \theta_k)|\alpha(\Theta)|^2 \right|.
\]

(A.1)

For \(k \neq l\) the evaluation of the relevant commutators yields:

\[
(\Delta \hat{r}_k)_x(\Delta \hat{p}_l)_x \geq -2\sigma_{kl} \sqrt{p(p - 1)} |\alpha(0, 0, 0)\alpha(\theta_k = \theta_l = 1, \theta_m = 0)|
\]
\[ \times \cos(2\omega t - \varphi(0, 0, 0) + \varphi(\theta_k = \theta_l = 1, \theta_m = 0)) \]
\[ -2\varepsilon_{klm}\sqrt{(p-1)(p-2)|\alpha(1, 1, 1)\alpha(\theta_k = \theta_l = 0, \theta_m = 1)| \]
\[ \times \cos(2\omega t - \varphi(\theta_k = \theta_l = 0, \theta_m = 1) + \varphi(1, 1, 1)) \]
\[ +(2p-1)|\alpha(\theta_k = 1, \theta_l = \theta_m = 0)\alpha(\theta_l = 1, \theta_k = \theta_m = 0)| \]
\[ \times \cos(\varphi(\theta_l = 1, \theta_k = \theta_m = 0) - \varphi(\theta_l = 1, \theta_k = \theta_m = 0)) \]
\[ +\sigma_{kl}\varepsilon_{klm}(2p-3)|\alpha(\theta_k = 0, \theta_l = \theta_m = 1)\alpha(\theta_l = 0, \theta_k = \theta_m = 1)| \]
\[ \times \cos(\varphi(\theta_l = 0, \theta_k = \theta_m = 1) - \varphi(\theta_l = 0, \theta_k = \theta_m = 1)) \]; \quad (A.2) \]

\[ (\Delta \hat{r})_x (\Delta \hat{r})_x \geq \left| -2\sigma_{kl}\sqrt{p(p-1)|\alpha(0, 0, 0)\alpha(\theta_k = \theta_l = 1, \theta_m = 0)|} \]
\[ \times \sin(2\omega t - \varphi(0, 0, 0) + \varphi(\theta_k = \theta_l = 1, \theta_m = 0)) \]
\[ -2\varepsilon_{klm}\sqrt{(p-1)(p-2)|\alpha(1, 1, 1)\alpha(\theta_k = \theta_l = 0, \theta_m = 1)|} \]
\[ \times \sin(2\omega t - \varphi(\theta_k = \theta_l = 0, \theta_m = 1) + \varphi(1, 1, 1)) \]
\[ +\sigma_{kl}(2p-1)|\alpha(\theta_k = 1, \theta_l = \theta_m = 0)\alpha(\theta_l = 1, \theta_k = \theta_m = 0)| \]
\[ \times \sin(\varphi(\theta_l = 1, \theta_k = \theta_m = 0) - \varphi(\theta_l = 1, \theta_k = \theta_m = 0)) \]
\[ -\varepsilon_{klm}(2p-3)|\alpha(\theta_k = 0, \theta_l = \theta_m = 1)\alpha(\theta_l = 0, \theta_k = \theta_m = 1)| \]
\[ \times \sin(\varphi(\theta_l = 0, \theta_k = \theta_m = 1) - \varphi(\theta_l = 0, \theta_k = \theta_m = 1)) \]; \quad (A.3) \]

\[ (\Delta \hat{p})_x (\Delta \hat{p})_x \geq \left| 2\sigma_{kl}\sqrt{p(p-1)|\alpha(0, 0, 0)\alpha(\theta_k = \theta_l = 1, \theta_m = 0)|} \]
\[ \times \sin(2\omega t - \varphi(0, 0, 0) + \varphi(\theta_k = \theta_l = 1, \theta_m = 0)) \]
\[ +2\varepsilon_{klm}\sqrt{(p-1)(p-2)|\alpha(1, 1, 1)\alpha(\theta_k = \theta_l = 0, \theta_m = 1)|} \]
\[ \times \sin(2\omega t - \varphi(\theta_k = \theta_l = 0, \theta_m = 1) + \varphi(1, 1, 1)) \]
\[ +\sigma_{kl}(2p-1)|\alpha(\theta_k = 1, \theta_l = \theta_m = 0)\alpha(\theta_l = 1, \theta_k = \theta_m = 0)| \]
\[ \times \sin(\varphi(\theta_l = 1, \theta_k = \theta_m = 0) - \varphi(\theta_l = 1, \theta_k = \theta_m = 0)) \]
\[ -\varepsilon_{klm}(2p-3)|\alpha(\theta_k = 0, \theta_l = \theta_m = 1)\alpha(\theta_l = 0, \theta_k = \theta_m = 1)| \]
\[ \times \sin(\varphi(\theta_l = 0, \theta_k = \theta_m = 1) - \varphi(\theta_l = 0, \theta_k = \theta_m = 1)) \] \quad \text{)}; \quad (A.4) \]

where \( k, l, m \in \{1, 2, 3\} \) are all different; \( \sigma_{kl} = +1 \) if \( k < l \) and \( -1 \) if \( k > l \); \( \varepsilon_{klm} = \pm 1 \) is the signature of the permutation \((k, l, m)\) of \((1, 2, 3)\).

**References**

[1] T.D. Palev, *J. Math. Phys.* **23**, 1778 (1982).
T.D. Palev, *Czech. J. Phys.* **B32**, 680 (1982).

[2] T.D. Palev and N.I. Stoilova, *J. Math. Phys.* **38**, 2506 (1997).

[3] E.P. Wigner, *Phys. Rev.* **77**, 711 (1950).
[4] P. Ehrenfest, Z. Phys. 4, 455 (1927).
[5] L.M. Yang, Phys. Rev. 84, 788 (1951).
[6] D.G. Boulvare and S. Deser, Nuovo Cim. 30, 230 (1963).
[7] L. O’Raifeartaigh and C. Ryan, Proc. Roy. Irish. Acad. 62 A, 83 (1963).
[8] S. Okubo, Phys. Rev. D 22, 919 (1980).
[9] N. Mukunda, E.C.G. Sudarshan, J.K. Sharma and C.L. Mehta, J. Math. Phys. 21, 2386 (1980).
[10] S. Okubo, Phys. Rev. A 23, 2776 (1980).
[11] K. Odaka, T. Kishi and S. Kamefuchi, J. Phys. A 24, L591 (1991).
[12] V.I. Man’ko, G. Marmo, E.C.G. Sudarshan, and F. Zaccaria, “Wigner problem and alternative commutation relations”, preprint quant-ph/9612007 (1996).
[13] R. López-Ptña, V.I. Man’ko and G. Marmo, Phys. Rev. A 56, 1126 (1997).
[14] M. Arik, N.M. Atakishiyev and K.B. Wolf, J. Phys. A 32, L371 (1999).
[15] P.C. Stichel, Lect. Notes Phys. 539 (2000).
[16] N.M. Atakishiyev, G.S. Pogosyan, L.I. Vicent and K.B. Wolf, J. Phys. A 34, 9381 (2001).
[17] R. de Lima Rodrigues, A.F. de lima, K. Araújo Ferreira and A.N. Vaidya, “Quantum oscillators in the canonical coherent states”, preprint hep-th/0205173 (2002).
[18] E. Kapuscik, Czech. J. Phys. 50, 1279 (2000).
[19] A. Horzela, Czech. J. Phys. 50, 1245 (2000).
[20] A. Horzela, Tr. J. Phys. 1, 1 (2000).
[21] L.C. Biedenharn, J. Phys. A 22, L873 (1989).
[22] A.J. Macfarlane, J. Phys. A 22, 4581 (1989).
[23] C. Daskaloyannis, J. Phys. A 24, L789 (1991).
[24] Y. Ohnuki and S. Kamefuchi, J. Math. Phys. 19, 67 (1978).
[25] H.S. Green, Phys. Rev. 90, 270 (1953).
[26] M. Omote, Y. Ohnuki and S. Kamefuchi, Prog. Theor. Phys. 56, 1948 (1976).
[27] A.Ch. Ganchev and T.D. Palev, J. Math. Phys. 21, 797 (1980); Preprint JINR P2-11941 (1978) (in Russian).
[28] V.G. Kac, *Lect. Notes Math.* **676**, 597 (1978).

[29] A.O. Barut and A.J. Bracken, *J. Math. Phys.* **26**, 2515 (1985).

[30] A. Smailagic and E. Spallucci, *Phys. Rev.* **D 65**: 107701 (2002); hep-th/0203260.

[31] A. Smailagic and E. Spallucci, *J. Phys. A** D 35**, L363 (2002); hep-th/0205242.

[32] B. Mathukumar and P. Mitra, *Phys. Rev. D** **66**, 027701 (2002); hep-th/0204143.

[33] A. Hatzinikitas and I. Smyrakis, *J. Math. Phys.* **43**, 113 (2002); hep-th/0103074.

[34] A. Connes, *Non-commutative geometry.* Academic Press, San Diego (1994).

[35] L. Castellini, *Class. Quant. Gravity* **17**, 3377 (2000); hep-th/0005210.

[36] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, *Phys. Rev. Lett.* **86**, 2716 (2001); hep-th/0010173.

[37] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Phys. Lett. B** **527**, 149 (2002); hep-th/0012173.

[38] H. Falomir, J. Gamboa, M. Loewe, F. Mendez and J.G. Rojas, *Phys. Rev. D** **66**, 045018 (2002); hep-th/0203260.

[39] R. Jagannathan, *Int. J. Theor. Phys.* **22**, 1105 (1983).

[40] T.D. Palev, *J. Math. Phys.* **21**, 1283 (1980).

[41] T.D. Palev and J. Van der Jeugt, *J. Math. Phys.* **43**, 3850 (2002); hep-th/0010107.

[42] See for instance J.L. Powell and B. Crasemann, *Quantum mechanics*, Addison-Wesley Pub Co. Reading Massachusetts, USA, 1961 or the original paper H.P. Robertson, *Phys. Rev.* **34**, 163 (1929).
Figure 1. Identification of the results of all possible measurements of the coordinates of the particle as 8 points (or nests) on a sphere for each of the stationary states $|p; \Theta\rangle$, with $p > 2$. $W(p > 2)$ is 8-dimensional. The lowest energy level corresponding to $|p; 1, 1, 1\rangle$ is non-degenerate. In this example, $p = 3$. |p; 0, 0, 0\rangle

|p; 1, 0, 0\rangle

|p; 1, 0, 0\rangle

|p; 1, 1, 0\rangle

|p; 1, 0, 1\rangle

|p; 0, 0, 1\rangle

|p; 1, 1, 0\rangle

|p; 0, 1, 0\rangle

|p; 0, 1, 1\rangle

|p; 1, 1, 1\rangle

|p; 1, 1, 1\rangle

|p; 1, 1, 1\rangle

|p; 1, 1, 1\rangle
Figure 2. Identification of the results of all possible measurements of the coordinates of the particle as points (or nests) on a sphere for each of the stationary states $|p; \Theta \rangle$, with $p = 2$. $W(p = 2)$ is only 7-dimensional. The lowest energy level is threefold degenerate.

$|2; 0, 0, 0 \rangle$

$|2; 1, 0, 0 \rangle$

$|2; 1, 1, 0 \rangle$

$|2; 1, 0, 1 \rangle$

$|2; 0, 1, 1 \rangle$

$|2; 0, 0, 1 \rangle$
Figure 3. Identification of the results of all possible measurements of the coordinates of the particle as points (or nests) on a sphere for each of the stationary states $|p; \Theta\rangle$, with $p = 1$. $W(p = 1)$ is 4-dimensional. The lowest energy level is threefold degenerate.

$|1; 0, 0, 0\rangle$

$|1; 1, 0, 0\rangle$

$|1; 0, 1, 0\rangle$

$|1; 0, 0, 1\rangle$