White dwarfs as test objects of Lorentz violations

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Abstract
In the present work, the thermodynamical properties of bosonic and fermionic gases are analysed under the condition that a modified dispersion relation is present. This last condition implies a breakdown of Lorentz symmetry. The implications for the condensation temperature will be studied, as well as for other thermodynamical variables such as specific heat, entropy, etc. Moreover, it will be argued that those cases entailing a violation of time reversal symmetry of the motion equations could lead to problems with the concept of entropy. Concerning the fermionic case, it will be shown that Fermi temperature suffers a modification due to the breakdown of Lorentz symmetry. The results will be applied to white dwarfs and the consequences upon the Chandrasekhar mass–radius relation will be shown. The possibility of resorting to white dwarfs for the testing of modified dispersion relations is also addressed. It will be shown that the comparison of the current observations against the predictions of our model allows us to discard some values of one of the parameters appearing in the modifications of the dispersion relation.

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1. Introduction
A long-standing puzzle in modern physics concerns the issue of a possible quantization of the gravitational field. Some of the current efforts in this direction entail, unavoidably, the breakdown of Lorentz symmetry [1–3]. In the bedrock of modern physics lies Lorentz symmetry, and in consequence, it has been subjected to some of the highest precision tests in physics [4–8]. Clearly, the theoretical analysis contained in [4] does not contemplate the study of the consequences of a modified dispersion relation, since at that time this idea had not yet appeared, they rather explore other possibilities. For instance, Mansouri and Sexl consider an ether model which maintains absolute simultaneity and elicits the physical conclusions of this premise.
It has to be mentioned that up to now there has been no experimental evidence purporting a possible violation of this symmetry. At this point we have to be more careful with our language because the phrase violation of Lorentz symmetry could mean many things, since this feature embodies several characteristics, for instance, local Lorentz invariance, or local position invariance [8]. In our case the meaning of the phrase Lorentz violation will take a very precise expression. Indeed, one of the possible ways in which Lorentz symmetry could be violated is related to the modification of the dispersion relation. This feature emerges in some models, for instance, non-critical string theory, non-commutative geometry, and canonical gravity [7] that try to quantize the gravitational field. In other words, we may find in the extant attempts to quantize gravity some models that predict new physics, since they involve a region in which one of the fundamental symmetries of modern physics becomes only an approximation. It is needless to say that this possibility has spurred lot of work in this direction, but as we will point out below, more work is required in order, either to understand better the theoretical background of this kind of effects or to propose experiments which could detect, or at least, put bounds upon the corresponding parameters emerging from these models.

On the other hand, the possibility of having Lorentz symmetry only as an approximation also appears in relation with other scenarios. Indeed, in the phenomenological realm it has been suggested an energy-dependent speed of light as a possible solution to the GZK paradox [9], i.e., the observation of ultra-high energy cosmic rays above the expected GZK threshold for interaction of such cosmic rays with the cosmic microwave background [9, 10]. In other words, there is evidence, stemming from sources with very diverse origins, that Lorentz symmetry could be only an approximate feature of nature, and therefore the analysis of the consequences of the breakdown of the aforementioned symmetry and of the possible ways in which its effects could be detected needs no further justification.

In the experimental quest for this kind of effect, interferometry has played a fundamental role [11–14], though it has to be underlined that it is not the unique scenario in which experimental proposals have been introduced. In this direction we may mention, as an interesting case, the modifications upon the standard model that a Lorentz invariance violation could have [15]. For instance, the fact that the maximum attainable velocity for particles is not the speed of light and the possible detection of this difference in speed by the Auger experiment [16]. Within this framework photons and neutrinos have different maximum attainable velocities, a fact that could be detected in the next generation of neutrino detectors as NUBE [17].

The idea in the present work is to introduce a deformed dispersion relation as a fundamental fact for the dynamics of massive bosons and fermions. Afterwards we analyse the effects of this assumption upon the thermodynamics of the corresponding gas. Here we must justify why the thermodynamics could suffer modifications due to the breakdown of Lorentz symmetry. The main reason is very simple. Statistical mechanics teaches us that if the relation between the momentum \( p \) and the energy \( \epsilon \) of a particle (for the case of \( l \)-spacelike dimensions) satisfies the relation \( \epsilon \sim ps \), then the relation between the pressure of the gas, \( P \), and the energy density, \( u \), reads \( P = u/\ell \) [18]. Clearly, a deformed dispersion relation modifies the usual functional dependence upon energy and momentum, and therefore, the thermodynamical properties do suffer changes as an unavoidable consequence of this kind of Lorentz violation.

Clearly, the tiny effects involved in the possible deformations of this relation entail, unfortunately, an almost insurmountable experimental difficulty in the case of terrestrial experimental proposals, and in consequence, we may wonder if this kind of analysis could shed some light in this respect. Though this last remark is true, it has also to be underlined that white dwarfs can be considered as an example of a fermionic gas in the highly degenerated regime [18]. The quantal behaviour of a fermionic gas is responsible for the emergence
of the Chandrasekhar mass–radius relationship, which embodies the equilibrium between the pressure and the gravitational interaction of the star. As will be shown, a deformed dispersion relation modifies the thermodynamical parameters, among them the pressure, and, in consequence, the relation between pressure and the mass of the star must change. In other words, white dwarfs are astrophysical objects that could be used as a system in the quest for this kind effects, i.e., the present proposal is then to look for deviations in the Chandrasekhar mass–radius relationship that could be explained by this kind of violation of Lorentz symmetry.

The possibility in this direction is supported by the observational data [19] where some white dwarfs have a radius smaller than that deduced from a normal electron-degenerate equation of state. Though there are theoretical efforts which try to explain this discrepancy in several ways, for instance, strange-quark matter within the white dwarf core, this is not the only possibility and, in consequence, this discrepancy could also involve in its explanation a change in the Chandrasekhar mass–radius relationship stemming from a violation to the Lorentz symmetry. Currently some authors consider eight candidates in which this discrepancy appears [19].

2. Deformed dispersion relations and quantal properties of gases

As mentioned above, several quantum-gravity models predict a modified dispersion relation [1–3], the one can be characterized, phenomenologically, through corrections hinging upon Planck’s length, $l_p$

$$E^2 = p^2[1 - \alpha(E/l_p)^n] + m^2. \tag{1}$$

Here $\alpha$ is a coefficient, whose precise value depends upon the considered quantum gravity model, while $n$, the lowest power in Planck’s length leading to a non-vanishing contribution, is also model dependent. Casting (1) in ordinary units we have ($E_p = \sqrt{c^5\hbar/G}$ denotes Planck’s energy)

$$E^2 = p^2c^2[1 - \alpha(E/E_p)^n] + (mc^2)^2. \tag{2}$$

Our present analysis will be restricted to massive particles; the case of massless particles has already been studied [20]. At this point we will divide our study into two parts, each one of them associated with a particular quantal statistics.

2.1. Bosonic statistics

2.1.1. General case. Let us consider massive bosons, according to (2) the relation between energy and momentum becomes now

$$p = \frac{1}{c} \sqrt{\frac{E^2 - m^2c^4}{1 - \alpha(E/E_p)^n}}. \tag{3}$$

The number of microstates is given by

$$\Sigma = \frac{s}{(2\pi\hbar)^3} \int \int d\vec{r} \, d\vec{p}. \tag{4}$$

In this last expression, $s$ is a weight factor arising from the internal structure of the particles, i.e., spin. If our gas is inside a container of volume $V$

$$\Sigma = \frac{4s\pi V}{(2\pi\hbar)^3} \int p^2 \, dp. \tag{5}$$
Casting the number of particles in terms of an integral of the energy
\[ \Sigma = \frac{4s \pi V}{(2\pi \hbar)^3} \int_0^\infty \frac{E^2 - m^2c^4}{c^3} \sqrt{1 - \alpha (E/E_p)^n} \left\{ \frac{E + \alpha \{(n-1)E^2 - nm^2c^4\}E_{\nu}^n}{[1 - \alpha (E/E_p)^n]^2} \right\} dE. \]  

(6)

The density of states per energy unit is easily calculated
\[ \Omega_1(E) = \frac{4s \pi V}{(2\pi \hbar)^3} \left( \frac{2c \pi}{\bar{\hbar}} \right)^3 \sqrt{E^2 - m^2c^4} \frac{1}{1 - \alpha (E/E_p)^n} \left\{ \frac{E + \alpha \{(n-1)E^2 - nm^2c^4\}E_{\nu}^n}{[1 - \alpha (E/E_p)^n]^2} \right\}. \]  

(7)

If in (7) we set \( \alpha = 0, s = 2 \) and \( m = 0 \), then we recover the density of states for photons
\[ \Omega_1(E) = \frac{8s \pi V}{(2\pi \hbar)^3} E^2. \]  

(8)

The average number of particles reads
\[ N = N_0 + N_e, \]  

(9)

\[ N_0 = s \frac{1}{1 - \lambda \exp(-mc^2/\kappa T)} \]  

(10)

\[ N_e = s \int_{mc^2/\kappa T}^{\infty} \Omega(E) \frac{1}{\lambda e^{E/\kappa T} - 1} dE. \]  

(11)

Here \( N_0 \) denotes the number of particles in the ground state, whereas \( N_e \) is the number of particles in the excited states. Additionally, \( \kappa \) and \( \lambda \) are Boltzmann’s constant and the fugacity, respectively. Performing the integration we obtain
\[ N_e = \frac{4s \pi V}{(2\pi \hbar)^3} (\kappa T)^3 \left\{ \Gamma(3)g_{(3)}(\lambda \exp(-mc^2/\kappa T)) + \alpha (n + 3/2)(\kappa T/E_p)^n \right\} \times \left[ \Gamma(3 + n)g_{(3+n)}(\lambda \exp(-mc^2/\kappa T)) \right. \]
\[ \left. + \frac{n + 1}{2} \Gamma(1 + n)(mc^2/\kappa T)^2 g_{(1+n)}(\lambda \exp(-mc^2/\kappa T)) \right\}. \]  

(12)

In (12) we have that \( g_{(n)}(z) \) are the Bose–Einstein functions [18] and \( \Gamma(n) \) the Gamma-functions [21]. Additionally, \( g_{(n)}(z) \) is a monotonically increasing function, and, if \( n > 1 \), then it is bounded \( 0 < z < 1 \). In other words, \( N_e \) has a maximum defined by the fulfillment of the condition
\[ \lambda \exp(-mc^2/\kappa T) = 1. \]  

(13)

This entails that the number of particles that can be located in the excited states has a maximum
\[ N_{e_{\text{max}}} = \frac{4s \pi V}{(2\pi \hbar)^3} (\kappa T)^3 \left\{ 2\xi(3) + \alpha (n + 3/2)(T/T_p)^n \right\} \times \left[ (n + 2)!\xi(3 + n) + \frac{(n + 1)!}{2} (mc^2/\kappa T)^2 \xi(n + 1) \right]. \]  

(14)

We have introduced the definition of Planck’s temperature \( T_p = E_p/\kappa \), and the Riemann Zeta function \( \xi(z) \) [22]. If we impose the condition \( \alpha = 0 \) then
\[ N_{e_{\text{max}}} = \frac{8s \pi V}{(2\pi \hbar)^3} (\kappa T)^3 \xi(3). \]  

(15)
White dwarfs as test objects of Lorentz violations

In other words, we recover a fact already known; see expression (13) in [23]. If \( \alpha > 0 \) then the maximum number of particles in the excited states grows, whereas, if \( \alpha < 0 \), this quantity diminishes. Hence, in the former case the number of particles in the ground state, \( N_0 \), diminishes, i.e., the Bose–Einstein condensation ‘slows down’, for the latter case the condensation ‘speeds up’. In the context of Bose–Einstein condensation the breakdown of Lorentz symmetry appears as a modification of the number of particles that can be located in the excited states. In order to evaluate this change note that

\[
\left\| N_e^{(\text{max} - n)} - N_e^{(\text{max})} \right\| / N_e^{(\text{max})} = |\alpha| \frac{n + 3/2}{\xi(3)} (T/T_p)^n [(n + 2)!\xi(3 + n) + (n + 1)! (mc^2/\kappa T)^2 \xi(n + 1)].
\]

Unfortunately the effect depends upon the ratio \( T/T_p \), hence

\[
\left\| N_e^{(\text{max} - n)} - N_e^{(\text{max})} \right\| / N_e^{(\text{max})} \sim 0.1 \rightarrow T \sim \left[ \frac{1}{|\alpha|(n + 3/2)} \right]^{1/n} T_p.
\]

The required temperature is a linear function of Planck’s temperature, a fact that experimentally is a serious drawback, since \( T_p \sim 10^{32} \text{ K} \), and in order to have a noticeable change we must achieve temperatures close enough to \( T_p \).

Let us now address the issue of the number of particles in the ground state. The condition for the existence of Bose–Einstein condensation reads

\[
N > N_e^{(\text{max} - n)}.
\]

As in the case in which Lorentz symmetry is present, here a critical temperature, \( T_c \), appears

\[
N = \frac{4\pi \lambda}{(2\pi \hbar)} \left( kT_c \right)^3 \left\{ 2\xi(3) + \alpha(n + 3/2)(T_c/T_p)^n [(n + 2)!\xi(3 + n) + \frac{(n + 1)!}{2} (mc^2/\kappa T_c)^2 \xi(n + 1)] \right\}.
\]

Since \( N = N_0 + N_e \), then \( N_0/N = 1 - N_e/N \), and, according to (19) and (14), we have that

\[
N_e/N = \left( \frac{T}{T_c} \right)^3 \frac{2\xi(3) + \alpha(n + 3/2)(T_c/T_p)^n [(n + 2)!\xi(3 + n) + \frac{(n + 1)!}{2} (mc^2/\kappa T_c)^2 \xi(n + 1)]}{2\xi(3) + \alpha(n + 3/2)(T_c/T_p)^n [(n + 2)!\xi(3 + n) + \frac{(n + 1)!}{2} (mc^2/\kappa T_c)^2 \xi(n + 1)]}.
\]

This last expression allows us to calculate the ratio \( N_0/N \). Indeed, if \( T > T_c \), then \( N_e/N = 1 \), whereas if \( T < T_c \) we must resort to (20) to evaluate \( N_e/N \). Setting \( \alpha = 0 \) we recover the usual expression [18, 23]. The breakdown of Lorentz symmetry does impinge upon the condensation temperature.

Let us now analyse some other thermodynamical parameters, for instance, the internal energy, \( U \).

For our case

\[
U = \frac{smc^2}{\lambda^{-1} \exp[mc^2/\kappa T] - 1} + \frac{4\pi V}{(2\pi \hbar)^3} \times \int_{mc^2}^{\infty} \left[ E^2 \sqrt{E^2 - m^2c^4} + \alpha \frac{n + 3/2}{E_p} E^{n+2} \sqrt{E^2 - m^2c^4} \right] \frac{dE}{\lambda^{-1} \exp[E/\kappa T] - 1}.
\]
We may cast the internal energy in the following form:

\[ U = \sum_{mc^2}^{2}\lambda \exp\left(-mc^2/\kappa T\right)N_0 + 4s\pi V(2\pi c\bar{\hbar})^3 \int_0^\infty \frac{E^2 \sqrt{E^2 + m^2c^4}}{\lambda - 1} \exp\left[\sqrt{E^2 + m^2c^4}/\kappa T\right] - 1 \]

\[ = \frac{4s\pi V}{(2\pi c\bar{\hbar})^3} \alpha(n + 3/2)(\kappa T)^4(T/T_p)^n \Sigma_{l=0}^{n+1} \frac{(n+1)!}{l!(n+1-l)!} \]

\[ \times \left(\frac{mc^2}{\kappa T}\right)^l \Gamma(n + 4 - l)g(n+4-l)(\lambda \exp(-mc^2/\kappa T)). \] (22)

This shows that the internal energy is modified if Lorentz symmetry is broken. The energy grows if \( \alpha > 0 \), and diminishes when \( \alpha < 0 \). Unfortunately the effect, as pointed out before, behaves as \( (T/T_p)^n \), remember that \( T_p = E_p/\kappa \) denotes Planck’s temperature.

Let us now analyse the pressure of our gas. With our assumptions we may obtain that the pressure is given by

\[ P = \frac{4s\pi V}{(2\pi c\bar{\hbar})^3} \left\{ \frac{1}{3} \int_0^\infty \frac{E^4 \sqrt{E^2 + m^2c^4}}{\lambda - 1} \exp\left[\sqrt{E^2 + m^2c^4}/\kappa T\right] - 1 \right\} \]

\[ + \frac{4s\pi V}{(2\pi c\bar{\hbar})^3} \alpha(n + 3/2)(\kappa T)^4(T/T_p)^n \Sigma_{l=0}^{n+1} \frac{(n+1)!}{l!(n+1-l)!} \]

\[ \times \left(\frac{mc^2}{\kappa T}\right)^l \Gamma(n + 4 - l)g(n+4-l)(\lambda \exp(-mc^2/\kappa T)). \] (23)

It is important to mention that if \( \alpha > 0 \), then the pressure grows, with respect to the case in which Lorentz symmetry is present.

This last remark allows us to interpret the breakdown of Lorentz symmetry for massive bosons as a repulsive interaction, if \( \alpha > 0 \). Indeed, the presence of a repulsive interaction (among the particles of a gas) entails the increase of the pressure, compared against the corresponding value for an ideal gas.

Let us explain this point deeper. A fleeting glance at the cluster expansion and its relation to the virial coefficients [18] clearly shows that the first correction to the ideal gas state equation expressed in terms of the virial state equation \( (PV)/(NKT) = \Sigma_{l=1}^\infty a_l(T)(N\lambda^3/V)^{-l} \), where \( N \) denotes the number of particles, corresponds to a virial coefficient that can be written as a function of the potential energy of interaction between the \( i \)th and the \( j \)th particle \( v_{ij} \)

\[ a_2 = -\frac{1}{\lambda} \int f_{12} d^3r_{12}. \] (24)

Where \( \exp(-v_{12}/K T) = 1 + f_{12}, \) and \( \lambda = \sqrt{2\pi h^2/mK T} \) is the thermal wavelength [18].

A repulsive interaction means that \( v_{12} > 0 \), and in consequence, \( f_{12} < 0 \), and therefore, \( a_2 > 0 \). If we introduce this condition into the virial expression we obtain a pressure larger than that corresponding to an ideal gas. In other words, the introduction of a repulsive interaction among the particles comprising the gas entails an increase of the pressure, compared to the pressure of an ideal gas.

It is in this sense that we say that the loss of the symmetry appears, at the bulk level, as the emergence of a repulsive interaction (if \( \alpha > 0 \) ), and in consequence, at least in principle, we could detect some effects stemming from loop quantum gravity, non-commutative geometry, etc. If \( \alpha \) turns out to be negative, then the breakdown is equivalent, for massive bosons, to an attractive interaction.

Another interesting point is related to the density of number of states; see (7). Clearly, if \( \alpha > 0 \), then the density grows, whereas, if \( \alpha < 0 \), this parameter decreases. This last remark
allows us to state that, since the entropy is a function of the number of microstates available to a macrostate, then a positive $\alpha$ must yield a larger entropy (compared to the case in which Lorentz symmetry is present), and if $\alpha$ is negative, then we must have a smaller entropy. This assertion can be checked recalling that the entropy $S$ satisfies the relation \[ S = \frac{U + PV}{N\kappa T} - \frac{\mu}{\kappa T}. \] (25)

In this last expression $\mu$ denotes the chemical potential. Our previous results allow us to write

\[
S = \frac{4s\pi V}{(2\pi c \hbar)^3} T \left\{ \frac{4}{3} \int_0^\infty \frac{E^4}{\sqrt{E^2 + m^2 c^2}} \lambda^{-1} \exp \left[ \sqrt{E^2 + m^2 c^2}/\kappa T \right] - 1 \right. \\
+ \alpha(n + 3/2)(\kappa T)^4(T/Tp)^n \left[ \sum_{l=0}^n \frac{(n)!}{l!(n-l)!} \left( \frac{m^2 c^2}{\kappa T} \right)^l \Gamma(n + 3 - l) g_{(n+3-l)} \right] \\
\times \left. (\lambda \exp(-mc^2/\kappa T) + \sum_{l=0}^{n+1} \frac{(n+1)!}{l!(n+1-l)!} \left( \frac{m^2 c^2}{\kappa T} \right)^l \Gamma(n + 4 - l) g_{(n+4-l)}) \right\} - N\kappa \ln \lambda + \lambda mc^2 \exp(-mc^2/\kappa T) N_0 / T. \] (26)

Indeed, if $\alpha > 0$, then we obtain an entropy larger than the corresponding value with Lorentz symmetry. The remaining possibility, $\alpha < 0$, embodies a smaller entropy. In the limit $T \to 0$ the entropy $S$ vanishes, as happens in the usual case [18].

2.1.2. Reversible adiabatic processes. Let us now resort to (26) and consider the case $n = 1$, the entropy becomes

\[
S_{(1)} = \frac{4s\pi V}{(2\pi c \hbar)^3} T \left\{ \frac{4}{3} \int_0^\infty \frac{E^4}{\sqrt{E^2 + m^2 c^2}} \lambda^{-1} \exp \left[ \sqrt{E^2 + m^2 c^2}/\kappa T \right] - 1 \right. \\
+ \alpha(5/2)(\kappa T)^4(T/Tp)^1 \left[ 3! g_{(5)}(\lambda \exp(-mc^2/\kappa T)) \right] \\
+ 2! \left( \frac{m^2 c^2}{\kappa T} \right) g_{(4)}(\lambda \exp(-mc^2/\kappa T)) + 4! g_{(5)}(\lambda \exp(-mc^2/\kappa T)) \right\} - N\kappa \ln \lambda + \lambda mc^2 \exp(-mc^2/\kappa T) N_0 / T. \] (27)

We know that the breakdown of Lorentz symmetry implies also the violation of the CPT theorem [24], since it assumes the presence of Lorentz symmetry. Therefore, we may wonder if this violation could have some consequences at macroscopic level. For instance, let us assume that the case $n = 1$ is related to the breakdown of time reversal invariance, i.e., it entails a theory in which the dynamics is not time reversal invariant. Then, the concept of reversible adiabatic process is lost, since it necessarily requires time reversal invariant laws of motion. An acceptable definition of entropy must satisfy eight criteria [25]; among them we may find the fact that the entropy must be invariant in all reversible adiabatic processes, and increase in any irreversible adiabatic process (the second criterion of [25]). Within this context consider the case in which our bosonic gas is inside a container whose walls are adiabatic, and now let us introduce the following condition:

\[ S_{(1)} = \text{const}. \] (28)
It is an adiabatic process in which the entropy does not change. Since we have assumed the violation of time reversal invariance, then (28) has no physical meaning, since it is related to reversible processes, which do not exist if time reversal invariance is lost. In other words, in those cases in which the breakdown of Lorentz symmetry involves a violation of time reversal invariance we will end up with a concept of entropy which will define curves without physical meaning. Then it seems that we face two possibilities: (i) either thermodynamics loses its validity within this context; (ii) or those cases implying the loss of time reversal invariance have to be discarded, since they violate thermodynamics.

2.2. Fermionic statistics

Let us now consider the case of a fermionic gas in which, once again, we have a deformed dispersion relation as that given by (2). The argument mentioned above (explaining the modifications of the state equation for a bosonic gas) is valid here, and, in consequence, we must expect changes in the thermodynamical variables of the gas, if Lorentz symmetry is broken. One of these parameters is the Fermi momentum, which defines, in the limit $T \to 0$, the border between single-particle states empty and occupied. Indeed, in the aforementioned limit the mean occupation numbers of single-particle state, $\langle n_E \rangle$, are equal to 0 if $E > \mu_0$, and 1 if $E < \mu_0$, where $\mu_0$ is the chemical potential at $T = 0$ [18].

For our case the zero-point energy of an electronic gas (with $N$ particles in a volume $V$) becomes

$$\epsilon_0 = 2 \Sigma_{p< P_F} \sqrt{p^2 c^2 [1 - \alpha (E/E_p)^n]} + (mc^2)^2. \quad (29)$$

The sum has to be done considering only those states with a momentum smaller than Fermi momentum, $P_F$, since those with a higher momentum are empty [18].

In the continuum limit we obtain that

$$\epsilon_0 = \frac{2V}{(2\pi \hbar)^3} \int_0^{P_F} 4\pi p^2 \sqrt{(pc)^2 + (mc^2)^2} \left\{ 1 - \frac{\alpha}{2} \frac{(pc)^2}{(pc)^2 + (mc^2)^2} \left( \frac{\sqrt{(pc)^2 + (mc^2)^2}}{E_p} \right)^{n} \right\} dp. \quad (30)$$

This last expression can be cast in the following form:

$$\epsilon_0 = \frac{8\pi V}{(2\pi \hbar)^3} (m^4 c^5) \left( \frac{mc^2}{E_p} \right)^n \left\{ f(x_F) - \frac{\alpha}{2} \frac{(mc^2)}{E_p} g(x_F) \right\}, \quad (31)$$

$$x_F = \frac{P_F}{mc}, \quad (32)$$

$$P_F = \hbar \left( \frac{3\pi^2 N}{V} \right)^{1/3}, \quad (33)$$

$$g(x_F) = \int_0^{x_F} x^4 (1 + x^2)^{(n-1)/2} dx, \quad (34)$$

$$f(x_F) = \frac{1}{16} \left\{ \frac{1}{4} \left[ (x_F + \sqrt{1 + x_F^2})^4 - (x_F + \sqrt{1 + x_F^2})^{-4} \right] - \frac{1}{2} \ln \left[ x_F + \sqrt{1 + x_F^2} \right] \right\}. \quad (35)$$

It is readily seen that the dependence upon $n$ of the breakdown of Lorentz symmetry is encoded in $g(x_F)$ and in the parameter $\left( \frac{mc^2}{E_p} \right)^n$. For the sake of simplicity let us consider the
case $n = 1$, then

$$
\epsilon_0 = \frac{8\pi V}{(2\pi \hbar)^3} (m^4 c^5 \bar{\hbar}) \left\{ f(x_F) - \frac{amc^2}{10E_p x_F^5} \right\}. \quad (36)
$$

Since the pressure of the zero-point energy is given by $P_0 = -\left(\frac{\partial \epsilon_0}{\partial V}\right)$ [18], we obtain for this case ($n = 1$)

$$
P_0 = \frac{8\pi}{(2\pi \hbar)^3} (m^4 c^5 \bar{\hbar}) \left\{ \frac{x_F}{3} \frac{df(x_F)}{dx_F} - f(x_F) - \frac{amc^2}{15E_p x_F^5} \right\}. \quad (37)
$$

If we consider $\alpha > 0$ then, the pressure decreases with respect to the case in which Lorentz symmetry is not broken, whereas, if $\alpha < 0$ the pressure grows. Once again, we say that the loss of the symmetry appears, at the bulk level, as an attractive interaction (if $\alpha > 0$). When $\alpha$ turns out to be negative, then the breakdown is equivalent, for massive fermions, to a repulsive interaction.

3. White dwarfs

At this point we may wonder if a fermionic system could provide us with some experimental proposal which could lead us to have a more optimistic scenario than that appearing in connection with bosonic gasses. Historically, Fermi statistics was first applied in astrophysics, namely, to the thermodynamic equilibrium of white dwarfs [26, 27]. An idealized white dwarf consists of helium in an almost complete state of ionization, and hence the microscopic constituents of the star may be taken as $N$ electrons and $N/2$ helium nuclei at temperature in which the dynamics of the electrons is in the relativistic limit (in other words, $x_F \gg 1$ [18]); additionally, the helium nuclei do not contribute significantly to the dynamics of the problem. In other words, in a first approximation we may neglect the presence of the nuclei in the system. The temperature of the star ($T \sim 10^7$ K) allows us to consider the electron gas in a state of almost complete degeneracy ($T_F = 10^{10}$ K).

The equilibrium configuration of a white dwarf is given by the fact that the pressure of the gas competes with the gravitational attraction. Clearly, (37) shows us that the pressure of the zero-point energy is modified as a consequence of the breakdown of Lorentz symmetry, and therefore, we must expect a change in the analysis of the equilibrium state of a white dwarf. In other words, Chandrasekhar shows us that the equilibrium of white dwarfs defines a relationship between the mass of the star and its radius [27], and in consequence taking into account our arguments we may seek for violations of Lorentz symmetry looking for changes in the Chandrasekhar mass–radius relationship.

Let us assume that the star has a spherical configuration (of radius $R$) and that the electron gas is uniformly distributed over the body of the star; this restriction is easily removed, but we consider it in our first-order approximation. If an adiabatic change in $V$ takes place, then the change in the zero-point energy is given by

$$
d\epsilon_0 = -4\pi R^2 P_0(R) dR. \quad (38)
$$

The change in the gravitational potential energy is given by

$$
dE_g = \frac{GM^2}{R^2} dR. \quad (39)
$$

In this last expression $M$ is the mass of the star and $G$ the gravitational constant [27]. If the system is in equilibrium, then the net change in its total energy, $(\epsilon_0 + E_g)$, for an infinitesimal change in its size, should be zero, thus, for equilibrium

$$
P_0(R) = \frac{GM^2}{4\pi R^2}. \quad (40)
$$
Note that (37) entails that the pressure depends upon the density (see (33)), and therefore

\[ P_F = \hbar \left( \frac{9\pi N}{4R^3} \right)^{1/3}, \]  

(41)

\[ x_F = \frac{\hbar}{mcR} \left( \frac{9\pi N}{4} \right)^{1/3}. \]  

(42)

Since \( M \approx 2Nm_p \), where \( m_p \) is the proton mass, then (\( m \) is the electron mass)

\[ x_F = \frac{\hbar}{mcR} \left( \frac{9\pi M}{8m_p} \right)^{1/3}. \]  

(43)

3.1. Case \( n=1 \)

The equilibrium condition reads

\[ \frac{GM^2}{4\pi R^4} = \frac{8\pi}{(2\pi \hbar)^3} \left( m^4 c^5 \right) \left\{ \frac{\lambda_F}{3} \frac{df(x_F)}{dx_F} - f(x_F) - \frac{a mc^2}{15E_p} \left( x_F \right)^2 \right\}. \]  

(44)

From this expression we may find the dependence of \( R \) upon \( M \), the one is given by

\[ R = \Gamma \sqrt{1 - \frac{12\beta \tau}{\Gamma^4}} \left\{ 1 - \frac{2a \omega}{5} \left[ 1 - \frac{12\beta \tau}{\Gamma^4} \right]^{-3/2} \right\}. \]  

(45)

\[ \Gamma = \frac{\hbar}{mc} \left( \frac{9\pi M}{8m_p} \right)^{1/3}, \]  

(46)

\[ \omega = \frac{mc^2}{E_p}, \]  

(47)

\[ \beta = \frac{GM^2}{4\pi}, \]  

(48)

\[ \tau = \frac{8\pi}{(2\pi \hbar)^3} \left( m^4 c^5 \right). \]  

(49)

The modification in the radius due to the breakdown of Lorentz symmetry appears in the term \( 6a \omega \left[ 1 - \frac{12\beta \tau}{\Gamma^4} \right]^{-3/2} \). If \( \alpha = 0 \), then we recover the usual prediction [27]. If we denote Chandrasekhar prediction by \( R_{\text{ch}} \), then

\[ R = R_{\text{ch}} \left\{ 1 - \frac{2}{5a \omega} \left[ 1 - \left( \frac{M}{M} \right)^{2/3} \right]^{3/2} \right\}. \]  

(50)

\[ (M)^{2/3} = \frac{\hbar}{3\pi G} \left( \frac{9\pi}{8m_p} \right)^{4/3}. \]  

(51)

If \( \alpha > 0 \), then the allowed radii are smaller than the corresponding value when Lorentz symmetry is present, whereas \( \alpha < 0 \) yields larger radii. In order to see the possibilities that our approach provides let us remember that [27]

\[ R_{\text{ch}} = \frac{(9\pi)^{1/3}}{2} \frac{\hbar}{mc} \left( \frac{M}{m_p} \right)^{1/3} \sqrt{1 - \left( \frac{M}{M} \right)^{2/3}}. \]  

(52)
This last expression entails that if $M \to \tilde{M}$ (from below), then $R_{\text{ch}} \to 0$. Let us now analyse the consequences of (50). If $\alpha < 0$, then in the limit $M \to \tilde{M}$ the breakdown of Lorentz symmetry predicts a non-vanishing radius for the white dwarf

$$R \to -\frac{2}{5} \alpha \omega \Gamma \left\{ 1 - \left( \frac{M}{M} \right)^{2/3} \right\}^{-1}.$$ (53)

In other words, we have found for the case $\alpha < 0$ a criterion that could allow us to test this kind of violations to Lorentz symmetry. Indeed, it predicts very large radii for white dwarfs with mass very close to $\tilde{M} \sim 1.44M_{\odot}$, where $M_{\odot}$ denotes the mass of the sun. Those cases in which $\alpha > 0$ yield a smaller radius than Chandrasekhar’s prediction.

3.2. Case $n = 2$

In this case, the pressure becomes

$$P_0 = \frac{8\pi}{(2\pi \hbar)^3} \left( m^4 c^5 \right) \left\{ \frac{x_F^4 - x_F^2}{12} - \frac{10}{15} \alpha \omega x_F^6 \right\}.$$ (54)

The equilibrium condition for the white dwarf reads

$$\frac{GM^2}{4\pi R^4} = \frac{8\pi}{(2\pi \hbar)^3} \left( m^4 c^5 \right) \left\{ \frac{x_F^4 - x_F^2}{12} - \frac{10}{15} \alpha \omega x_F^6 \right\}.$$ (55)

This last equation defines a dependence of the radius $R$ in terms of the mass of the white dwarf $M$.

$$R = R_{\text{ch}} \left\{ 1 - \frac{4}{7} \alpha \omega^2 \left[ 1 - \left( \frac{M}{M} \right)^{2/3} \right]^{-2} \right\}.$$ (56)

In this case expression $\Gamma$ and $R_{\text{ch}}$ are given by (46) and (52), respectively. Imposing the condition $\alpha = 0$ allows us to recover from (56) Chandrasekhar’s relationship. Once again, the present model predicts a non-vanishing radius for the case $\alpha < 0$ in the limit $M \to \tilde{M}$, though in this case

$$R \to -\frac{4}{7} \alpha \omega^2 \Gamma \left\{ 1 - \left( \frac{M}{M} \right)^{2/3} \right\}^{-3/2}.$$ (57)

If we divide (57) by (53) we find

$$\frac{R_2}{R_1} = \frac{10}{7} \omega \left\{ 1 - \left( \frac{M}{M} \right)^{2/3} \right\}^{1/2}.$$ (58)

This last expression means that the limit $M \to \tilde{M}$ (for the case $\alpha < 0$) diverges faster for $n = 2$ than for $n = 1$. In other words, the effects of the breakdown of Lorentz symmetry become more detectable as $n$ goes from 1 to 2.

3.3. General case

The general equilibrium condition for a white dwarf is given by the following expression:

$$\frac{GM^2}{4\pi R^4} = \frac{8\pi}{(2\pi \hbar)^3} \left( m^4 c^5 \right) \left\{ \frac{x_F^4 - x_F^2}{12} - \alpha \omega^2 \left\{ \frac{2}{15} x_F^5 (1 + x_F^2)^{(n-1)/2} + \frac{n - 1}{35} x_F^7 (1 + x_F^2)^{(n-3)/2} \right\} \right\}.$$ (59)
It is readily seen that the case $\alpha < 0$ entails a pressure larger than the corresponding value when Lorentz symmetry is present. In other words, the breakdown of Lorentz symmetry in the form of $\alpha < 0$ can always be interpreted as the emergence of a repulsive interaction among the particles of an electronic gas. The question concerning the behaviour of the radius in the limit $M \rightarrow \bar{M}$ for any value of $n$ requires a careful analysis, though we may conjecture that it could always embody a divergent radius, i.e., for any value of $n$ it seems that the radius becomes $(a$ and $l$ are positive constants depending upon the value of $n$)

$$R = R_{ch} \left\{ 1 - a\alpha\omega^n \left[ 1 - \left( \frac{M}{\bar{M}} \right)^{2/3} \right]^{-l} \right\}. \quad (60)$$

Hence in the limit $M \rightarrow \bar{M}$, once again, we obtain a divergent radius.

Finally, in the same line of reasoning, the case $\alpha > 0$ implies for a fermionic gas the emergence of a lower pressure, and therefore, the breakdown of Lorentz symmetry could always be interpreted for this kind of matter equivalent to the emergence of an attractive interaction among the particles of the gas.

4. Conclusions

In the present work, a deformed dispersion relation has been introduced as a fundamental fact for the dynamics of massive bosons and fermions. The effects of this assumption upon the thermodynamics of the corresponding gas have been analysed.

For the case of massive bosons, it has been proved that $\alpha > 0$ is tantamount to the emergence of a repulsive interaction among the particles, whereas, $\alpha < 0$ is related to the appearance of an attractive interaction. In other words, the breakdown of Lorentz symmetry does impinge upon the thermodynamic properties of a bosonic gas, entropy, state equation, specific heat, etc, though the possibility of detecting them is hindered by the fact that the extra terms related to the loss of the symmetry behave like $T/T_p$, where $T$ is the temperature of the system and $T_p \sim 10^{32}$ K is Planck’s temperature. It has also been argued that for those violations related to the breakdown of time reversal invariance we may find that the entropy defines curves without physical meaning, and that for these cases there are no reversible adiabatic processes. In connection with this last remark we confront two possibilities: (i) either thermodynamics loses its validity within this context; (ii) or thermodynamics is valid and those cases entailing the loss of time reversal invariance have to be discarded.

At this point it is noteworthy to comment that in the extant literature we may find some results claiming that the case $n = 1$ has to be discarded due to violations to black hole thermodynamics [28, 29]. The present results support the conclusions contained in [28, 29], since they coincide in the fact that the linear case has problems with the concept of entropy. The new point in our work consists in the fact that these problems with thermodynamics can be found also in the context of ordinary matter, for instance, a bosonic gas.

Additionally, the case of fermionic statistics has been analysed, and in the quest for a system that could provide a feasible experimental proposal for the detection of this kind of violations of Lorentz symmetry the modifications that the Chandrasekhar’s mass–radius relationship suffers have been considered. This allow us to introduce astrophysical objects and try to understand if they could shed some light upon this issue. For the case of massive fermions it has been proved that $\alpha < 0$ is equivalent to the emergence of a repulsive interaction among the particles, whereas, $\alpha > 0$ is related to the appearance of an attractive interaction. Additionally, it has been proved that the case $\alpha < 0$ embodies a behaviour (in the limit $M \rightarrow \bar{M}$) very different from the predictions of the usual model for white dwarfs. The current
data seem to discard negative values of \( \alpha \) [30]. In other words, our approach, together with the current data, could allow us to consider only positive values of \( \alpha \) as physically meaningful. At this point, it is noteworthy to comment that more observations are required to check the prediction of the model for the case \( \alpha < 0 \), since the closest value to our critical mass, \( \tilde{M} \sim 1.44M_\odot \) (for Sirius B, \( M/M_\odot = 1.0034 \pm 0.026 \) [30]) lies not close enough to the needed value.

It is interesting to mention that the observations contain a puzzling feature, namely, some stars do have radii which are significantly smaller than the theoretical predictions [19, 30]. There are several models which (without resorting to any kind of breakdown of Lorentz symmetry) try to solve this puzzle. Nevertheless, at this point we must mention that these observations are compatible with \( \alpha > 0 \), though do not necessarily provide a proof for the existence of non-vanishing values of \( \alpha \). Let us explore the possibilities that these observations could mean in this context. and denote the difference between observation and Chandrasekhar’s model in the radius by \( \Delta R \), for \( \alpha > 0 \). Then, for \( n = 1 \)

\[
\frac{\Delta R}{2 \omega_1} \left[ 1 - \left( \frac{M}{\tilde{M}} \right)^{2/3} \right] \geq \alpha. \tag{61}
\]

Let us now consider the following two white dwarfs with the same mass, i.e., G156-64 (a strange white dwarf) and Wolf 485 A (this white dwarf satisfies Chandrasekhar’s relationship); see table I in [19]. In this case \( \Delta R \approx 2.78 \times 10^8 \text{ cm} \), and therefore \( \alpha \leq 10^{22} \). Taking into account further physical aspects, for instance, Coulomb correction, lattice energy, or more realistic density distribution for the electronic gas, etc, shall provide a much lower bound for \( \alpha \).

Finally, the present approach could also be implemented in connection with other schemes, for instance, \( \kappa \)–Poincaré dispersion relation in order to obtain constraints upon the quantum \( \kappa \)–Poincaré algebra. This possibility would be, in some sense, a continuation of work already done [31].

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