1 Introduction. Basic definitions

In this paper, notions of compatible and almost compatible Riemannian and pseudo-Riemannian metrics, which are motivated by the theory of compatible (local and nonlocal) Poisson structures of hydrodynamic type and generalize the notion of flat pencil of metrics (this notion plays an important role in the theory of integrable systems of hydrodynamic type and the Dubrovin theory of Frobenius manifolds \cite{1}, see also \cite{2}–\cite{3}), are introduced and studied. Compatible metrics generate compatible Poisson structures of hydrodynamic type (these structures are local for flat metrics \cite{4} and they are nonlocal if the metrics are not flat \cite{5}–\cite{9}) and their investigation is necessary for the theory of integrable systems of hydrodynamic type. In “nonsingular” case, when eigenvalues of pair of metrics are distinct, in this paper the complete explicit description of compatible and almost compatible metrics is obtained. The “singular” case of coinciding eigenvalues of pair of metrics is considerably more complicated for the complete analysis and has still not been completely studied. Nevertheless, the problem on two-component compatible flat metrics is completely investigated. All such pairs, both “nonsingular” and “singular”, are classified by ours. In this paper we present the complete description of nonsingular pairs of two-component flat metrics. The problems of classification of compatible flat metrics and compatible metrics of constant Riemannian curvature are of particular interest, in particular, from the viewpoint of the theory of Hamiltonian systems of hydrodynamic type. More detailed classification results for these problems will be published in another paper. In the given paper we prove that the approach proposed by Ferapontov in \cite{4} for the study of flat pencils of metrics can be also applied (with the corresponding modifications and corrections) to pencils of metrics of constant Riemannian curvature and to the general compatible Riemannian and pseudo-Riemannian metrics. We also correct a mistake which is in \cite{4} in the criterion of compatibility of local nondegenerate Poisson structures of hydrodynamic type (or, in other words, compatibility of flat metrics).

We shall use both contravariant metrics $g^{ij}(u)$ with upper indices, where $u = (u^1, ..., u^N)$ are local coordinates, $1 \leq i, j \leq N$, and covariant metrics $g_{ij}(u)$ with lower indices, $g^{is}(u)g_{sj}(u) = \delta^i_j$. The indices of coefficients of the Levi–Civita connections $\Gamma^i_{jk}(u)$ (the Riemannian connections generated by the corresponding metrics) and tensors of Riemannian curvature $R^{ij}_{kl}(u)$ are raised and lowered by the metrics corresponding to them:

$$
\Gamma^i_{jk}(u) = g^{is}(u)\Gamma^s_{jk}(u), \quad \Gamma^i_{jk}(u) = \frac{1}{2}g^{is}(u)\left(\frac{\partial g_{sk}}{\partial u^j} + \frac{\partial g_{sj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^s}\right),
$$

$$
R^{ij}_{kl}(u) = g^{is}(u)R^{s}_{kl}(u), \quad R^{ij}_{kl}(u) = -\frac{\partial \Gamma^i_{jl}}{\partial u^k} + \frac{\partial \Gamma^i_{jk}}{\partial u^l} - \Gamma^i_{lk}(u)\Gamma^p_{jl}(u) + \Gamma^i_{pl}(u)\Gamma^p_{jk}(u).
$$

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\textbf{Definition 1.1} Two contravariant metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) of constant Riemannian curvature \( K_1 \) and \( K_2 \), respectively, are called compatible if for any linear combination of these metrics

\[
 g^{i^j}(u) = \lambda_1 g^{i^j}_1(u) + \lambda_2 g^{i^j}_2(u), \tag{1.1}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants such that \( \det(g^{i^j}(u)) \neq 0 \), is a metric of constant Riemannian curvature \( \lambda_1 K_1 + \lambda_2 K_2 \) and the coefficients of the corresponding Levi–Civita connections are related by the same linear formula:

\[
 \Gamma^{i^j}_k(u) = \lambda_1 \Gamma^{i^j}_{1,k}(u) + \lambda_2 \Gamma^{i^j}_{2,k}(u). \tag{1.2}
\]

We shall also say in this case that the metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) form a pencil of metrics of constant Riemannian curvature.

\textbf{Definition 1.2} Two Riemannian or pseudo-Riemannian contravariant metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) are called compatible if for any linear combination of these metrics

\[
 g^{i^j}(u) = \lambda_1 g^{i^j}_1(u) + \lambda_2 g^{i^j}_2(u), \tag{1.3}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants such that \( \det(g^{i^j}(u)) \neq 0 \), the coefficients of the corresponding Levi–Civita connections and the components of the corresponding tensors of Riemannian curvature are related by the same linear formula:

\[
 \Gamma^{i^j}_k(u) = \lambda_1 \Gamma^{i^j}_{1,k}(u) + \lambda_2 \Gamma^{i^j}_{2,k}(u), \tag{1.4}
\]

\[
 R^{i^j}_{kl}(u) = \lambda_1 R^{i^j}_{1,kl}(u) + \lambda_2 R^{i^j}_{2,kl}(u). \tag{1.5}
\]

We shall also say in this case that the metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) form a pencil of metrics.

\textbf{Definition 1.3} Two Riemannian or pseudo-Riemannian contravariant metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) are called almost compatible if for any linear combination of these metrics\( g^{i^j}(u) \) relation \( (1.3) \) is fulfilled.

\textbf{Definition 1.4} Two Riemannian or pseudo-Riemannian metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) are called nonsingular pair of metrics if the eigenvalues of this pair of metrics, that is, the roots of the equation

\[
 \det(g^{i^j}(u) - \lambda g^{i^j}_2(u)) = 0, \tag{1.6}
\]

are distinct.

These definitions are motivated by the theory of compatible Poisson brackets of hydrodynamic type. In the case if the metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) are flat, that is, \( R^{i^j}_{1,kl}(u) = R^{i^j}_{2,kl}(u) = 0 \), relation \( (1.5) \) is equivalent to the condition that an arbitrary linear combination of the flat metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) is also a flat metric and Definition \( 1.2 \) is equivalent to the well-known definition of a flat pencil of metrics or, in other words, a compatible pair of local nondegenerate Poisson structures of hydrodynamic type \( \mathbb{I} \) (see also \( \mathbb{II} \), \( \mathbb{III} \)). In the case if the metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) are metrics of constant Riemannian curvature \( K_1 \) and \( K_2 \), respectively, that is,

\[
 R^{i^j}_{1,kl}(u) = K_1 (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k), \quad R^{i^j}_{2,kl}(u) = K_2 (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k),
\]

relation \( (1.5) \) gives the condition that an arbitrary linear combination of the metrics \( g^{i^j}_1(u) \) and \( g^{i^j}_2(u) \) \( (1.3) \) is a metric of constant Riemannian curvature \( \lambda_1 K_1 + \lambda_2 K_2 \) and Definition \( 1.2 \) is
equivalent to Definition 1.1 of a pencil of metrics of constant Riemannian curvature or, in other words, a compatible pair of the corresponding nonlocal Poisson structures of hydrodynamic type which were introduced and studied by the author and Ferapontov in [10]. Compatible metrics of more general type correspond to compatible pairs of nonlocal Poisson structures of hydrodynamic type which were introduced and studied by Ferapontov in [11]. They arise, for example, if we shall use a recursion operator generated by a pair of compatible Poisson structures of hydrodynamic type and determining, as is well-known, an infinite sequence of corresponding Poisson structures.

As was earlier noted by the author in [6]–[8], condition (1.5) follows from condition (1.4) in the case of certain special reductions connected with the associativity equations (see also Theorem 3.2 below). Of course, it is not by chance. Under certain very natural and quite general assumptions on metrics (it is sufficient but not necessary, in particular, that the eigenvalues of the pair of the metrics under consideration are distinct), compatibility of the metrics follows from their almost compatibility but, generally speaking, in the general case, it is not true even for flat metrics (we shall present here below the corresponding counterexamples). Correspondingly, we would like to emphasize that condition (1.4) which is considerably more simple than condition (1.5) “almost” guarantees compatibility of metrics and deserves a separate investigation but, in the general case, it is necessary to require also the fulfillment of condition (1.5) for compatibility of the corresponding Poisson structures of hydrodynamic type. It is also interesting to find out, does condition (1.5) guarantee the fulfillment of condition (1.4) or not.

2 Compatible local Poisson structures of hydrodynamic type

The local homogeneous Poisson bracket of the first order, that is, the Poisson bracket of the form

\[ \{ u^i(x), u^j(y) \} = g^{ij}(u(x)) \delta_x(x-y) + b^{ij}_k(u(x)) u^k_x \delta(x-y), \]

where \( u^1, ..., u^N \) are local coordinates on a certain smooth \( N \)-dimensional manifold \( M \), is called a local Poisson structure of hydrodynamic type or Dubrovin–Novikov structure [9]. Here, \( u^i(x), 1 \leq i \leq N \), are functions (fields) of a single independent variable \( x \), and the coefficients \( g^{ij}(u) \) and \( b^{ij}_k(u) \) of bracket (2.1) are smooth functions on \( M \).

In other words, for arbitrary functionals \( I[u] \) and \( J[u] \) on the space of fields \( u^i(x), 1 \leq i \leq N \), a bracket of the form

\[ \{ I, J \} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x)) u^k_x \right) \frac{\delta J}{\delta u^j(x)} dx \]

is defined and it is required that this bracket is a Poisson bracket, that is, it is skew-symmetric:

\[ \{ I, J \} = -\{ J, I \}, \]

and satisfies the Jacobi identity

\[ \{ \{ I, J \}, K \} + \{ \{ J, K \}, I \} + \{ \{ K, I \}, J \} = 0 \]

for arbitrary functionals \( I[u], J[u] \) and \( K[u] \). The skew-symmetry (2.3) and the Jacobi identity (2.4) impose very strict conditions on the coefficients \( g^{ij}(u) \) and \( b^{ij}_k(u) \) of bracket (2.2) (these conditions will be considered below).

The local Poisson structures of hydrodynamic type (2.1) were introduced and studied by Dubrovin and Novikov in [9]. In this paper, they proposed a general Hamiltonian approach to
the so-called \textit{homogeneous systems of hydrodynamic type}, that is, to evolutionary quasilinear systems of first-order partial differential equations

\[
    u^i_t = V_j^i(u) u^j_x
\]  

that corresponds to structures (2.1).

This Hamiltonian approach was motivated by the study of the equations of Euler hydrodynamics and the Whitham averaging equations that describe the evolution of slowly modulated multiphase solutions of partial differential equations [15].

Local bracket (2.2) is called \textit{nondegenerate} if \( \det(g_{ij}(u)) \neq 0 \). For general nondegenerate brackets of form (2.2), Dubrovin and Novikov proved the following important theorem.

\textbf{Theorem 2.1 (Dubrovin, Novikov [9])} \textit{If} \( \det(g_{ij}(u)) \neq 0 \), \textit{then bracket (2.2) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity if and only if}

1. \( g_{ij}(u) \) is an arbitrary flat pseudo-Riemannian contravariant metric (a metric of zero Riemannian curvature),
2. \( b^i_{jk}(u) = -g^{is}(u) \Gamma_{jk}^s(u) \), where \( \Gamma_{jk}^s(u) \) is the Riemannian connection generated by the contravariant metric \( g^{ij}(u) \) (the Levi–Civita connection).

Consequently, for any local nondegenerate Poisson structure of hydrodynamic type, there always exist local coordinates \( v^1, \ldots, v^N \) (flat coordinates of the metric \( g^{ij}(u) \)) in which the coefficients of the brackets are constant:

\[
    g^{ij}(v) = \eta^{ij} = \text{const}, \quad \Gamma_{jk}^i(v) = 0, \quad b^i_{jk}(v) = 0, \quad \text{(2.6)}
\]

that is, the bracket has the constant form

\[
    \{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \delta J \frac{d}{dx} \eta^{ij} \frac{d}{dx} \delta v^j(x) dx,
\]

where \( (\eta^{ij}) \) is a nondegenerate symmetric constant matrix:

\[
    \eta^{ij} = \eta^{ji}, \quad \eta^{ij} = \text{const}, \quad \det(\eta^{ij}) \neq 0. \quad \text{(2.8)}
\]

On the other hand, as early as 1978, Magri proposed a bi-Hamiltonian approach to the integration of nonlinear systems [16]. This approach demonstrated that the integrability is closely related to the bi-Hamiltonian property, that is, to the property of a system to have two compatible Hamiltonian representations. As was shown by Magri in [16], compatible Poisson brackets generate integrable hierarchies of systems of differential equations. Therefore, the description of compatible Poisson structures is very urgent and important problem in the theory of integrable systems. In particular, for a system, the bi-Hamiltonian property generates recurrent relations for the conservation laws of this system.

Beginning from [16], quite extensive literature (see, for example, [17]–[21] and the necessary references therein) has been devoted to the bi-Hamiltonian approach and to the construction of compatible Poisson structures for many specific important equations of mathematical physics and field theory. As far as the problem of description of sufficiently wide classes of compatible Poisson structures of defined special types is concerned, apparently the first such statement was considered in [22], [23] (see also [24], [25]). In those papers, the present author posed and completely solved the problem of description of all compatible local scalar first-order and third-order Poisson brackets, that is, all Poisson brackets given by arbitrary scalar first-order and third-order ordinary differential operators. These brackets generalize the well-known compatible pair of the Gardner–Zakharov–Faddeev bracket [26], [27] (first-order bracket) and the Magri bracket [16] (third-order bracket) for the Korteweg–de Vries equation.
In the case of homogeneous systems of hydrodynamic type, many integrable systems possess compatible Poisson structures of hydrodynamic type. The problems of description of these structures for particular systems and numerous examples were considered in many papers (see, for example, [28], [33]). In particular, in [28] Nutku studied a special class of compatible two-component Poisson structures of hydrodynamic type and the related bi-Hamiltonian hydrodynamic systems. In [12] Ferapontov classified all two-component homogeneous systems of hydrodynamic type possessing three compatible local Poisson structures.

In the general form, the problem of description of flat pencil of metrics (or, in other words, compatible nondegenerate local Poisson structures of hydrodynamic type) was considered by Dubrovin in [1], [2] in connection with the construction of important examples of such flat pencils of metrics, generated by natural pairs of flat metrics on the spaces of orbits of Coxeter groups and on other Frobenius manifolds and associated with the corresponding quasi-homogeneous solutions of the associativity equations. In the theory of Frobenius manifolds introduced and studied by Dubrovin [1], [2] (they correspond to two-dimensional topological field theories), a key role is played by flat pencils of metrics, possessing a number of special additional (and very strict) properties (they satisfy the so-called quasi-homogeneity property). In addition, in [3] Dubrovin proved that the theory of Frobenius manifolds is equivalent to the theory quasi-homogeneous compatible nondegenerate Poisson structures of hydrodynamic type. The general problem of compatible nondegenerate local Poisson structures was also considered by Ferapontov in [1].

The author’s papers [5], [8] are devoted to the general problem of classification of local Poisson structures of hydrodynamic type, to integrable nonlinear systems which describe such compatible Poisson structures and to special reductions connected with the associativity equations.

**Definition 2.1 (Magri [16])** Two Poisson brackets \{ , \}_1 and \{ , \}_2 are called compatible if an arbitrary linear combination of these Poisson brackets

\[
\{ , \} = \lambda_1 \{ , \}_1 + \lambda_2 \{ , \}_2,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants, is also always a Poisson bracket. In this case, one can say also that the brackets \{ , \}_1 and \{ , \}_2 form a pencil of Poisson brackets.

Correspondingly, the problem of description of compatible nondegenerate local Poisson structures of hydrodynamic type is pure differential-geometric problem of description of flat pencils of metrics (see [1], [2]).

In [1], [2] Dubrovin presented all the tensor relations for the general flat pencils of metrics. First, we introduce the necessary notation. Let \( \nabla_1 \) and \( \nabla_2 \) be the operators of covariant differentiation given by the Levi–Civita connections \( \Gamma_{1,k}^i(u) \) and \( \Gamma_{2,k}^i(u) \), generated by the metrics \( g_{1ij}^s(u) \) and \( g_{2ij}^s(u) \), respectively. The indices of the covariant differentials are raised and lowered by the corresponding metrics: \( \nabla_1^i = g_1^{is}(u)\nabla_1^s, \nabla_2^i = g_2^{is}(u)\nabla_2^s \). Consider the tensor

\[
\Delta^{ijk}(u) = g_1^{is}(u)g_2^{jp}(u)\left(\Gamma_{2,ps}^k(u) - \Gamma_{1,ps}^k(u)\right),
\]

introduced by Dubrovin in [1], [2].

**Theorem 2.2 (Dubrovin [1], [2])** If metrics \( g_{1ij}^s(u) \) and \( g_{2ij}^s(u) \) form a flat pencil, then there exists a vector field \( f^i(u) \) such that the tensor \( \Delta^{ijk}(u) \) and the metric \( g_{1ij}^s(u) \) have the form

\[
\Delta^{ijk}(u) = \nabla_2^i\nabla_2^j f^k(u),
\]

\[
g_{1ij}^s(u) = \nabla_2^i f^j(u) + \nabla_2^j f^i(u) + cg_{2ij}^s(u),
\]

where \( c \) is a certain constant, and the vector field \( f^i(u) \) satisfies the equations

\[
\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
\]

\[
\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
\]

\[
\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
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\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
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\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
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\[
\Delta_{ij}^s(u)\Delta^{ik}(u) = \Delta_{ik}^s(u)\Delta_{ij}^k(u),
\]
Lemma 2.1 ([5]) (An explicit criterion of compatibility for Poisson structures of hydrodynamic type)

Lemma 2.1 ([5]) (An explicit criterion of compatibility for two Poisson structures of hydrodynamic type is formulated, that is, it is shown what explicit form is sufficient and necessary for the Poisson structures of hydrodynamic type to be compatible.

For the moment, we are able to formulate such explicit general criterion only namely in terms of Poisson structures but not in terms of metrics as in Theorem 2.2.

Theorem 2.3 ([4])

Theorem 2.3 is actually a criterion of almost compatibility of flat metrics that does not guarantee compatibility of the corresponding nondegenerate local Poisson structures of hydrodynamic type.

where

\[ \Delta^{ij}_{k}(u) = g_{2,k,s}(u)\Delta^{sij}(u) = \nabla_{2,k}\nabla_{2,f}f^{j}(u), \]  

(2.14)

and

\[ (g^{1}_{i}(u)g^{2}_{j}(u) - g^{1}_{2}(u)g^{2}_{1}(u))\nabla_{2,s}\nabla_{2,p}f^{k}(u) = 0. \]  

(2.15)

Conversely, for the flat metric \( g^{1}_{2}(u) \) and the vector field \( f^{1}(u) \) that is a solution of the system of equations (2.13) and (2.14), the metrics \( g^{2}_{2}(u) \) and (2.13) form a flat pencil.

The proof of this theorem immediately follows from the relations that are equivalent to the criterion of compatibility for two Poisson structures.

In my paper, an explicit and simple criterion of compatibility for two Poisson structures of hydrodynamic type is formulated, that is, it is shown what explicit form is sufficient and necessary for the Poisson structures of hydrodynamic type to be compatible.

For the moment, we are able to formulate such explicit general criterion only namely in terms of Poisson structures but not in terms of metrics as in Theorem 2.2.

Lemma 2.1 ([5]) (An explicit criterion of compatibility for Poisson structures of hydrodynamic type) Any local Poisson structure of hydrodynamic type \( \{I, J\} \) is compatible with the constant nondegenerate Poisson bracket (2.3) if and only if it has the form

\[ \{I, J\} = \int \frac{\delta I}{\delta v^{k}(x)} \left( \left( \eta^{is} \frac{\partial h^{j}}{\partial v^{s}} + \eta^{js} \frac{\partial h^{i}}{\partial v^{s}} \right) \frac{d}{dx} + \eta^{ks} \frac{\partial^{2} h^{j}}{\partial v^{s}\partial v^{k} \partial x} \right) \frac{\delta J}{\delta v^{j}(x)} dx, \]  

(2.16)

where \( h^{i}(v) \), \( 1 \leq i \leq N \), are smooth functions defined on a certain neighbourhood.

We do not require in Lemma 2.1 that the Poisson structure of hydrodynamic type \( \{I, J\} \) is nondegenerate. Besides, it is important to note that this statement is local.

In 1995, in the paper [4], Ferapontov proposed an approach to the problem on flat pencils of metrics, which is motivated by the theory of recursion operators, and formulated the following theorem as a criterion of compatibility of nondegenerate local Poisson structures of hydrodynamic type:

Theorem 2.3 ([4]) Two local nondegenerate Poisson structures of hydrodynamic type given by flat metrics \( g^{1}_{2}(u) \) and \( g^{2}_{2}(u) \) are compatible if and only if the Nijenhuis tensor of the affinor \( v^{j}_{i}(u) = g^{1}_{i}(u)g_{2,i,s}(u) \) vanishes, that is

\[ N^{i}_{k}(u) = v^{i}_{s}(u) \frac{\partial v^{k}_{s}}{\partial u^{s}} - v^{s}_{k}(u) \frac{\partial v^{i}_{s}}{\partial u^{s}} + v^{k}_{s}(u) \frac{\partial v^{i}_{s}}{\partial u^{s}} - v^{i}_{s}(u) \frac{\partial v^{k}_{s}}{\partial u^{s}} = 0. \]  

(2.17)

Besides, it is noted in the remark in [4] that if the spectrum of \( v^{j}_{i}(u) \) is simple, the vanishing of the Nijenhuis tensor implies the existence of coordinates \( R^{1}, ..., R^{N} \) for which all the objects \( v^{j}_{i}(u), g^{1}_{i}(u), g^{2}_{i}(u) \) become diagonal. Moreover, in these coordinates the \( i \)th eigenvalue of \( v^{j}_{i}(u) \) depends only on the coordinate \( R^{i} \). In the case when all the eigenvalues are nonconstant, they can be introduced as new coordinates. In these new coordinates \( v^{j}_{i}(R) = \text{diag} (R^{1}, ..., R^{N}) \), \( g^{1}_{i}(R) = \text{diag} (\text{diag}(R^{1}, ..., g^{N}(R)), g^{1}_{i}(R) = \text{diag} (R^{1}g^{1}(R), ..., R^{N}g^{N}(R)). \)

Unfortunately, in the general case the Theorem 2.3 is not true and, correspondingly, it is not a criterion of compatibility of flat metrics. Generally speaking, compatibility of flat metrics does not follow from the vanishing of the corresponding Nijenhuis tensor. In this paper we shall present the corresponding counterexamples. In the general case, as it will be shown here, the Theorem 2.3 is actually a criterion of almost compatibility of flat metrics that does not guarantee compatibility of the corresponding nondegenerate local Poisson structures of hydrodynamic type.
But if the spectrum of \( v^j_i(u) \) is simple, that is, all the eigenvalues are distinct, then we proves here that the Theorem 2.3 is not only true but also can be essentially generalized for the case of arbitrary compatible Riemannian or pseudo-Riemannian metrics, in particular, for the especially important cases in the theory of systems of hydrodynamic type, namely, the cases of metrics of constant Riemannian curvature or the metrics generating the general nonlocal Poisson structures of hydrodynamic type. Namely, the following theorem is true.

**Theorem 2.4** 1) If for any linear combination \( g^1(u) \) of two metrics \( g^1_i(u) \) and \( g^2_i(u) \) condition (1.4) is fulfilled, then the Nijenhuis tensor of the affinor

\[
v^j_i(u) = g^i_j(u)g_2_{;s}(u)
\]

vanishes. Thus, for compatible and almost compatible metrics, the corresponding Nijenhuis tensor always vanishes.

2) If a pair of metrics \( g^1_i(u) \) and \( g^2_i(u) \) is nonsingular, that is, the roots of the equation

\[
det(g^{ij}_1(u) - \lambda g^{ij}_2(u)) = 0,
\]

are distinct, then it follows from the vanishing of the Nijenhuis tensor of the affinor \( v^j_i(u) = g^i_j(u)g_2_{;s}(u) \) that the metrics \( g^1_i(u) \) and \( g^2_i(u) \) are compatible. Thus, a nonsingular pair of metrics is compatible if and only if the metrics are almost compatible.

### 3 Almost compatible metrics and Nijenhuis tensor

Let us consider two contravariant Riemannian or pseudo-Riemannian metrics \( g^1_i(u) \) and \( g^2_i(u) \), and also the corresponding coefficients of the Levi–Civita connections \( \Gamma^{ij}_1(u) \) and \( \Gamma^{ij}_2(u) \).

We introduce the tensor

\[
M^{ijk}(u) = g^{is}_1(u)\Gamma^{jk}_{2,s}(u) - g^{js}_2(u)\Gamma^{ik}_{1,s}(u) - g^{is}_1(u)\Gamma^{jk}_{2,s}(u) + g^{js}_2(u)\Gamma^{ik}_{1,s}(u).
\] (3.1)

It follows from the following representation that \( M^{ijk}(u) \) is actually a tensor:

\[
M^{ijk}(u) = g^{is}_1(u)g^{jp}_2(u)(\Gamma^{k}_{2;ps}(u) - \Gamma^{k}_{1;ps}(u)) - g^{js}_1(u)g^{ip}_2(u)(\Gamma^{k}_{2;ps}(u) - \Gamma^{k}_{1;ps}(u)) = \Delta^{ijk}(u) - \Delta^{ijk}(u).
\] (3.2)

**Lemma 3.1** The tensor \( M^{ijk}(u) \) vanishes if and only if the metrics \( g^1_i(u) \) and \( g^2_i(u) \) are almost compatible.

Let us introduce the affinor

\[
v^j_i(u) = g^i_j(u)g_2_{;s}(u)
\] (3.3)

and consider the Nijenhuis tensor of this affinor

\[
N^{kj}(u) = v^k_i(u)\frac{\partial v^k_j}{\partial u^s} - v^k_j(u)\frac{\partial v^k_i}{\partial u^s} + v^s_i(u)\frac{\partial v^s_j}{\partial u^k} - v^s_j(u)\frac{\partial v^s_i}{\partial u^k}.
\] (3.4)

following [4], where were similarly considered the affinor \( v^j_i(u) \) and its Nijenhuis tensor for two flat metrics.

**Theorem 3.1** The metrics \( g^1_i(u) \) and \( g^2_i(u) \) are almost compatible if and only if the corresponding Nijenhuis tensor \( N^{kj}(u) \) vanishes.
Lemma 3.2 The following identities are always fulfilled:

\[ g_{1,sp}(u)N_{pq}^{r}(u)g_{2}^{rj}(u)g_{2}^{jk}(u) = M^{kij}(u) + M^{jki}(u) + M^{jik}(u), \]  
(3.5)

\[ 2(M^{jki}(u) + M^{jik}(u)) = g_{1,sp}(u)N_{rq}^{p}(u)g_{2}^{ri}(u)g_{2}^{qj}(u)g_{2}^{sk}(u) + g_{1,sp}(u)N_{rq}^{p}(u)g_{2}^{rk}(u)g_{2}^{qj}(u)g_{2}^{si}(u), \]  
(3.6)

\[ 2M^{kij}(u) = g_{1,sp}(u)N_{rq}^{p}(u)g_{2}^{ri}(u)g_{2}^{qj}(u)g_{2}^{sk}(u) - g_{1,sp}(u)N_{rq}^{p}(u)\frac{g_{2}^{rk}(u)g_{2}^{qj}(u)g_{2}^{si}(u)}{2}. \]  
(3.7)

Corollary 3.1 The tensor \( M^{ijk}(u) \) vanishes if and only if the Nijenhuis tensor \( \Phi(u) \) vanishes.

In the papers \([3, 5] \), the present author studied reductions in the general problem on compatible flat metrics, connected with the associativity equations, namely, the following ansatz in formula (2.16):

\[ h^{i}(v) = \eta^{ij}_{q} \frac{\partial \Phi}{\partial v^{q}}, \]

where \( \Phi(v^{1}, ..., v^{N}) \) is a function of \( N \) variables.

Correspondingly, in this case the metrics have the form:

\[ g_{1}^{ij}(v) = \eta_{ij}, \quad g_{2}^{ij}(v) = \eta^{ij}_{p} \frac{\partial^{2} \Phi}{\partial v^{p} \partial v^{q}}. \]  
(3.8)

Theorem 3.2 If metrics \([3, 5] \) are almost compatible, then they are always compatible. Moreover, in this case, the metric \( g_{2}^{ij}(v) \) is necessarily also flat, that is, metrics \([3, 5] \) form a flat pencil of metrics. Condition of almost compatibility of metrics \([3, 5] \) has the form

\[ \eta^{sp} \frac{\partial^{2} \Phi}{\partial v^{p} \partial v^{q}} \frac{\partial^{3} \Phi}{\partial v^{r} \partial v^{s} \partial v^{t}} = \eta^{sp} \frac{\partial^{2} \Phi}{\partial v^{r} \partial v^{s} \partial v^{t}} \frac{\partial^{3} \Phi}{\partial v^{p} \partial v^{q} \partial v^{r}} \]  
(3.9)

and coincides with the condition of compatible deformation of two Frobenius algebras which was derived and studied by the author in \([3, 5] \).

In particular, in the author’s papers \([3, 5] \), it is proved that in the two-component case \( (N = 2) \), for \( \eta^{ij} = \varepsilon^{i} \delta^{ij}, \varepsilon^{i} = \pm 1 \), condition (3.9) is equivalent to the following linear second-order partial differential equation with constant coefficients:

\[ \alpha \left( \varepsilon^{1} \frac{\partial^{2} \Phi}{\partial (v^{1})^{2}} - \varepsilon^{2} \frac{\partial^{2} \Phi}{\partial (v^{2})^{2}} \right) = \beta \frac{\partial^{2} \Phi}{\partial v^{1} \partial v^{2}}, \]  
(3.10)

where \( \alpha \) and \( \beta \) are arbitrary constants.

4 Compatible metrics and Nijenhuis tensor

Let us prove the second part of Theorem 2.4. In the previous section it is proved, in particular, that it always follows from compatibility (and even almost compatibility) of metrics that the corresponding Nijenhuis tensor vanishes.

Assume that a pair of metrics \( g_{1}^{ij}(u) \) and \( g_{2}^{ij}(u) \) is nonsingular, that is, the eigenvalues of this pair of the metrics are distinct, and assume also that the corresponding Nijenhuis tensor vanishes. Let us prove that, in this case, the metrics \( g_{1}^{ij}(u) \) and \( g_{2}^{ij}(u) \) are compatible (their almost compatibility follows from the previous section).
Obviously, that the eigenvalues of the pair of the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ coincide with the eigenvalues of affinor $v^{ij}_1(u)$. But it is well-known that if all eigenvalues of an affinor are distinct, then it always follows from the vanishing of the Nijenhuis tensor of this affinor that there exist local coordinates such that in these coordinates the affinor reduces to a diagonal form in the corresponding neighbourhood (see also [35]).

So we can consider further that the affinor $v^{ij}_1(u)$ is diagonal in the local coordinates $u^1, ..., u^N$,

$$v^{ij}_1(u) = \lambda^i(u)\delta^j_i,$$  \hspace{1cm} (4.1)

where is no summation over the index $i$, and \(\lambda^i(u), i = 1, ..., N,\) are the eigenvalues of the pair of metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$, which are assumed distinct:

$$\lambda^i \neq \lambda^j, \text{ for } i \neq j.$$

\hspace{1cm} (4.2)

**Lemma 4.1** If the affinor (3.3) is diagonal in a certain local coordinates and all its eigenvalues are distinct, then, in these coordinates, the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ are also necessarily diagonal.

Actually, we have

$$g^{ij}_1(u) = \lambda^i(u)g^{ij}_2(u).$$

It follows from symmetry of the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ that for any $i, j$

$$(\lambda^i(u) - \lambda^j(u))g^{ij}_2(u) = 0,$$  \hspace{1cm} (4.3)

where is no summation over indices, that is,

$$g^{ij}_2(u) = g^{ij}_1(u) = 0 \text{ for } i \neq j.$$

**Lemma 4.2** Let an affinor $w^{ij}_1(u)$ be diagonal in a certain local coordinates $u = (u^1, ..., u^N)$, that is, $w^{ij}_1(u) = \mu^i(u)\delta^j_i$.

1) If all the eigenvalues of an diagonal affinor are distinct, that is, $\mu^i(u) \neq \mu^j(u)$ for $i \neq j$, then the Nijenhuis tensor of this affinor vanishes if and only if the $i$th eigenvalue $\mu^i(u)$ depend only on the coordinate $u^i$.

2) If all the eigenvalues coincide, then the Nijenhuis tensor vanishes.

3) In the general case of a diagonal affinor, the Nijenhuis tensor vanishes if and only if

$$\frac{\partial \mu^i}{\partial u^j} = 0$$  \hspace{1cm} (4.4)

for all $i, j$ such that $\mu^i(u) \neq \mu^j(u)$.

Actually, for any diagonal affinor $w^{ij}_1(u) = \mu^i(u)\delta^j_i$, the Nijenhuis tensor $N^{ij}_1(u)$ has the form:

$$N^{ij}_1(u) = (\mu^i - \mu^k)\frac{\partial \mu^j}{\partial u^k}\delta^{ki} - (\mu^j - \mu^k)\frac{\partial \mu^i}{\partial u^k}\delta^{ki}$$

(no summation over indices). Thus, the Nijenhuis tensor vanishes if and only if for all $i, j$

$$(\mu^i(u) - \mu^j(u))\frac{\partial \mu^i}{\partial u^j} = 0,$$
where is no summation over indices.

It follows immediately from Lemmas 4.1 and 4.2 that for any nonsingular pair of almost compatible metrics there always exist local coordinates in which the metrics have the form

\[ g_2^{ij}(u) = g^i(u)\delta^{ij}, \quad g_1^{ij}(u) = \lambda^i(u')g^i(u)\delta^{ij}, \quad \lambda^i = \lambda^i(u'), \quad i = 1, \ldots, N. \]

Moreover, we derive immediately that any diagonal metrics of the form \( g_2^{ij}(u) = g^i(u)\delta^{ij} \) and \( g_1^{ij}(u) = f^i(u')g^i(u)\delta^{ij} \) for any nonzero functions \( f^i(u') \), \( i = 1, \ldots, N \), (they can be here, for example, coinciding nonzero constants, that is, the pair of metrics may be “singular”) are always almost compatible. We shall prove now that they are always compatible. Then Theorem 2.4 will be completely proved.

Consider diagonal metrics \( g_2^{ij}(u) = g^i(u)\delta^{ij} \) and \( g_1^{ij}(u) = f^i(u')g^i(u)\delta^{ij} \), where \( f^i(u') \), \( i = 1, \ldots, N \), are arbitrary functions of a single variable, which are not equal to zero identically, and consider their arbitrary linear combination

\( g^{ij}(u) = (\lambda_2 + \lambda_1 f^i(u'))g^i(u)\delta^{ij}, \)

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants such that \( \det(g^{ij}(u)) \neq 0 \).

Let us prove that relation (1.4) is always fulfilled for the corresponding tensors of Riemannian curvature.

Recall that for any diagonal metric \( \Gamma_{ij}^{jk}(u) = 0 \) if all the indices \( i, j, k \) are distinct. Correspondingly, \( R_{ij}^{kl}(u) = 0 \) if all the indices \( i, j, k, l \) are distinct. Besides, as a result of the well-known symmetries of the tensor of Riemannian curvature we have:

\[ R_{kl}^{i}(u) = R_{lk}^{i}(u) = 0, \]

\[ R_{ij}^{kl}(u) = -R_{ij}^{kl}(u) = R_{ij}^{kl}(u) = -R_{ij}^{kl}(u). \]

Thus, it is sufficient to prove relation (1.3) only for the following components of the tensor of Riemannian curvature: \( R_{ij}^{kl}(u) \), where \( i \neq j, \quad i \neq l \).

For any diagonal metric \( g_2^{ij}(u) = g^i(u)\delta^{ij} \) we have

\[ \Gamma_{2,ik}^{i}(u) = \Gamma_{2,kj}^{i}(u) = -\frac{1}{2g^i(u)} \frac{\partial g^i}{\partial u^k}, \quad \text{for any } i, k; \]

\[ \Gamma_{2,jj}^{i}(u) = \frac{1}{2} \frac{g^i(u)}{(g^i(u))^2} \frac{\partial g^j}{\partial u^j}, \quad i \neq j. \]

\[ R_{2,il}^{ij}(u) = g^i(u)R_{2,il}^{ij}(u) = \]

\[ g^i(u) \left( -\frac{\partial \Gamma_{2,ii}^{i}}{\partial u^i} + \frac{\partial \Gamma_{2,ii}^{i}}{\partial u^j} - \sum_{s=1}^{N} \Gamma_{2,ii}^{s}(u)\Gamma_{2,il}^{s}(u) + \sum_{s=1}^{N} \Gamma_{2,ii}^{s}(u)\Gamma_{2,il}^{s}(u) \right). \quad (4.5) \]

It is necessary to consider separately two different cases.

1) \( j \neq l \).

\[ R_{2,il}^{ij}(u) = g^i(u) \left( \frac{\partial \Gamma_{2,ii}^{i}}{\partial u^i} - \Gamma_{2,ii}^{i}(u)\Gamma_{2,il}^{i}(u) + \Gamma_{2,jj}^{i}(u)\Gamma_{2,il}^{i}(u) + \Gamma_{2,il}^{i}(u)\Gamma_{2,ii}^{i}(u) \right) = \]

\[ \frac{1}{2} g^i(u) \frac{\partial (g^i(u))}{\partial u^i} \left( \frac{g^j(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} + \frac{1}{4} \frac{g^i(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} \frac{\partial g^i}{\partial u^j} - \right) \]

\[ - \frac{1}{4g^i(u)} \frac{\partial g^j}{\partial u^i} \frac{\partial g^j}{\partial u^j} + \frac{1}{4} g^i(u) \frac{\partial g^i}{\partial u^i} \frac{\partial g^i}{\partial u^i}. \quad (4.6) \]
are not equal identically to zero).

An arbitrary nonsingular pair of metrics is compatible if and only if there exist local coordinates \( u = (u^1, \ldots, u^N) \) such that \( g^{ij}(u) = g^i(u)\delta^{ij} \) and \( g^{ij}(u) = f^i(u^s)g^i(u)\delta^{ij}, \) where \( f^i(u^s), \ i = 1, \ldots, N, \) are arbitrary functions of single variables (of course, in the case of nonsingular pair of metrics, these functions are not equal to each other if they are constants and they are not equal identically to zero).

Respectively, for the metric

\[
g^{ij}(u) = (\lambda_2 + \lambda_1 f^i(u^s))g^i(u)\delta^{ij},
\]

we obtain (we use here that all the indices \( i, j, l \) are distinct):

\[
R^{ij}_{kl}(u) = (\lambda_2 + \lambda_1 f^i(u^s))\left[ \frac{1}{2} g^{ij}(u) \frac{\partial}{\partial u^k} \left( \frac{g^{ji}(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^l} \right) + \frac{1}{4} g^{ij}(u) \frac{\partial g^i}{\partial u^l} \frac{\partial g^i}{\partial u^l} \right] = \lambda_1 R^{ij}_{kl}(u) + \lambda_2 R^{ij}_{2l}(u).
\]  

(4.7)

\( 2) \ j = l. \)

\[
R^{ij}_{2l}(u) = g^i(u)\left( -\frac{\partial \Gamma^j_{2l}}{\partial u^i} + \frac{\partial \Gamma^j_{2l}}{\partial u^j} - \Gamma^j_{2l}(u)\Gamma^j_{2l}(u) - \Gamma^j_{2s}(u)\Gamma^j_{2s}(u) \right) =
\]

\[
\Gamma^j_{2s}(u)\Gamma^j_{2s}(u) + \sum_{s=1}^{N} \Gamma^j_{2s}(u)\Gamma^j_{2s}(u) =
\]

\[
\frac{1}{2} g^i(u) \frac{\partial}{\partial u^j} \left( \frac{1}{2} g^{ij}(u) \frac{\partial g^i}{\partial u^j} \right) + \frac{1}{2} g^i(u) \frac{\partial}{\partial u^j} \left( \frac{g^i(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^l} \right) + \frac{1}{4} g^i(u) \frac{\partial g^i}{\partial u^l} \frac{\partial g^i}{\partial u^l} - \sum_{s \neq l}^{N} \frac{1}{4} g^i(u) \frac{\partial g^i}{\partial u^l} \frac{\partial g^i}{\partial u^l}.
\]  

(4.8)

Respectively, for the metric

\[
g^{ij}(u) = (\lambda_2 + \lambda_1 f^i(u^s))g^i(u)\delta^{ij},
\]

we obtain (we use here that the indices \( i, j \) are distinct):

\[
R^{ij}_{ij}(u) = \frac{1}{2} (\lambda_2 + \lambda_1 f^i(u^s)) g^i(u) \frac{\partial}{\partial u^j} \left( \frac{1}{g^i(u)} \frac{\partial g^i}{\partial u^j} \right) + \frac{1}{2} g^i(u) \frac{\partial}{\partial u^j} \left( \frac{\lambda_2 + \lambda_1 f^i(u^s)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} \right) + \frac{1}{4} (\lambda_2 + \lambda_1 f^i(u^s)) \frac{g^i(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} \frac{\partial g^i}{\partial u^j} - \frac{1}{4} (\lambda_2 + \lambda_1 f^i(u^s)) g^i(u) \frac{\partial g^i}{\partial u^j} \frac{\partial g^i}{\partial u^j} - \sum_{s \neq i, s \neq j}^{N} \frac{1}{4} g^i(u) g^i(u) \frac{\partial g^i}{\partial u^j} \frac{\partial g^i}{\partial u^j} =
\]

\[
\lambda_1 R^{ij}_{1,i}(u) + \lambda_2 R^{ij}_{2,i}(u).
\]  

(4.9)

Theorem 4.1 is proved. Moreover, the complete explicit description of nonsingular pairs of compatible and almost compatible metrics is obtained and the following theorem is proved:

**Theorem 4.1** An arbitrary nonsingular pair of metrics is compatible if and only if there exist local coordinates \( u = (u^1, \ldots, u^N) \) such that \( g^{ij}(u) = g^i(u)\delta^{ij} \) and \( g^{ij}(u) = f^i(u^s)g^i(u)\delta^{ij}, \) where \( f^i(u^s), \ i = 1, \ldots, N, \) are arbitrary functions of single variables (of course, in the case of nonsingular pair of metrics, these functions are not equal to each other if they are constants and they are not equal identically to zero).
Let us consider here the problem on nonsingular pairs of compatible flat metrics. It follows from Theorem 4.3 that it is sufficient to classify flat metrics of the form $g^{ij}_2(u) = g^i(u)\delta^{ij}$ and $g^{ij}_1(u) = f^i(u')g^j(u)\delta^{ij}$, where $f^i(u')$, $i = 1, \ldots, N$, are arbitrary functions of single variables.

The problem of description of diagonal flat metrics, that is, flat metrics $g^{ij}_2(u) = g^i(u)\delta^{ij}$, is a classical problem of differential geometry. This problem is equivalent to the problem of description of curvilinear orthogonal coordinate systems in a pseudo-Euclidean space and it was studied in detail and mainly solved in the beginning of the 20th century (see [36]). Locally, such coordinate systems are determined by $n(n-1)/2$ arbitrary functions of two variables (see [37], [38]). Recently, Zakharov showed that the Lamé equations describing curvilinear orthogonal coordinate systems can be integrated by the inverse scattering method [39] (see also an algebraic-geometric approach in [40]). The condition that the metric $g^{ij}(u) = f^i(u')g^j(u)\delta^{ij}$ is flat gives exactly $n(n-1)/2$ equations which are linear with respect to the functions $f^i(u')$. Note that, in this case, components $\Gamma^{ij}_k$ of tensor of Riemannian curvature automatically vanish as a result of formula (4.7). And the vanishing of components $\Gamma^{ij}_k$ gives the corresponding $n(n-1)/2$ equations. In particular, in the case $N = 2$ this completely solves the problem of description of nonsingular pairs of compatible flat metrics. In the next section we give their complete description. But without a doubt this problem is integrable in general for any $N$. We are going to devote to this question another work. In particular, it is very interesting to classify all the $n$-orthogonal curvilinear coordinate systems in a pseudo-Euclidean space, for which the functions $f^i(u') = (u')^n$ define compatible flat metrics (respectively, separately for $n = 1; n = 1, 2; n = 1, 2, 3$, and so on).

5 Two-component compatible flat metrics

We present here the complete description of nonsingular pairs of two-component compatible flat metrics (see also [37], [38], where an integrable homogeneous system of hydrodynamic type, describing all the two-component compatible flat metrics, was derived and investigated).

It is proved above that for any nonsingular pair of two-component compatible metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ there always exist local coordinates $u^1, \ldots, u^N$ such that

$$(g^{ij}_2(u)) = \left( \begin{array}{cc} \frac{1}{(b^1(u))^2} & 0 \\ \frac{1}{(b^2(u))^2} & 0 \end{array} \right), \quad (g^{ij}_1(u)) = \left( \begin{array}{cc} \frac{\varepsilon^1 f^1(u)}{(b^1(u))^2} & 0 \\ \frac{\varepsilon^2 f^2(u)}{(b^2(u))^2} & 0 \end{array} \right),$$

(5.1)

where $\varepsilon^i = \pm 1$, $i = 1, 2$; $b^i(u)$ and $f^i(u')$, $i = 1, 2$, are arbitrary nonzero functions of the corresponding single variables.

Lemma 5.1 An arbitrary diagonal metric $g^{ij}_2(u)$ (5.1) is flat if and only if the functions $b^i(u)$, $i = 1, 2$, are solutions of the following linear system:

$$\frac{\partial b^2}{\partial u^1} = \varepsilon^1 \frac{\partial F}{\partial u^2} b^1(u), \quad \frac{\partial b^1}{\partial u^2} = -\varepsilon^2 \frac{\partial F}{\partial u^1} b^2(u),$$

(5.2)

where $F(u)$ is an arbitrary function.

Theorem 5.1 The metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ (5.1) form a flat pencil of metrics if and only if the functions $b^i(u)$, $i = 1, 2$, are solutions of the linear system (5.2), where the function $F(u)$ is a solution of the following linear equation:

$$2 \frac{\partial^2 F}{\partial u^1 \partial u^2} (f^1(u^1) - f^2(u^2)) + \frac{\partial F}{\partial u^2} \frac{df^1(u^1)}{du^1} - \frac{\partial F}{\partial u^1} \frac{df^2(u^2)}{du^1} = 0.$$  

(5.3)
In the case, if the eigenvalues of the pair of the metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) are not only distinct but also are not constants, we can always choose local coordinates such that \( f^1(u^1) = u^1 \), \( f^2(u^2) = u^2 \) (see also remark in \([4]\)). In this case, equation (5.3) has the form
\[
2 \frac{\partial F}{\partial u^1} \frac{\partial^2 F}{\partial u^2} (u^1 - u^2) + \frac{\partial F}{\partial u^2} - \frac{\partial F}{\partial u^1} = 0. \tag{5.4}
\]

Let us continue this recurrent procedure for the metrics \( G^{ij}_{n+1}(u) = v^{ij}_n G^{ij}_n(u) \) with the help of the affinor \( v^{ij}_n(u) = u^i \delta^j \).

**Theorem 5.2** Three metrics
\[
(G^{ij}_n(u)) = \begin{pmatrix}
\frac{\varepsilon^1(u^1)^n}{(b^1(u))^2} & 0 \\
0 & \frac{\varepsilon^2(u^2)^n}{(b^2(u))^2}
\end{pmatrix}, \quad n = 0, 1, 2, \tag{5.5}
\]
form a flat pencil of metrics (pairwise compatible) if and only if the functions \( b^i(u), i = 1, 2 \), are solutions of the linear system (5.2), where
\[
F(u) = c \ln(u^1 - u^2), \tag{5.6}
\]
c is an arbitrary constant. Already the metric \( G^{ij}_3(u) \) is flat only in the most trivial case, when \( c = 0 \), and, respectively, \( b^1 = b^1(u^1), b^2 = b^2(u^2) \).

The metric \( G^{ij}_3(u) \) is a metric of nonzero constant Riemannian curvature \( K \neq 0 \) if and only if
\[
(b^1(u))^2 = (b^2(u))^2 = \frac{\varepsilon^2}{4K}(u^1 - u^2), \quad \varepsilon^1 = -\varepsilon^2, \quad c = \pm \frac{1}{2}. \tag{5.7}
\]

6 Almost compatible metrics which are not compatible

**Lemma 6.1** Two-component diagonal conformally Euclidean metric
\[
g^{ij}(u) = \exp(a(u)) \delta^{ij}, \quad 1 \leq i, j \leq 2,
\]
is flat if and only if the function \( a(u) \) is harmonic, that is,
\[
\Delta a \equiv \frac{\partial^2 a}{\partial(u^1)^2} + \frac{\partial^2 a}{\partial(u^2)^2} = 0. \tag{6.1}
\]

In particular, the metric \( g^{ij}_1(u) = \exp(u^1 u^2) \delta^{ij}, \quad 1 \leq i, j \leq 2 \), is flat. Obviously, that the flat metrics \( g^{ij}_1(u) = \exp(u^1 u^2) \delta^{ij}, \quad 1 \leq i, j \leq 2 \), and \( g^{ij}_2(u) = \delta^{ij}, \quad 1 \leq i, j \leq 2 \), are almost compatible, for them the Nijenhuis tensor (3.4) vanishes. But it follows from Lemma 6.1 that these metrics are not compatible, their sum is not a flat metric.

Similarly it is possible to construct also other counterexamples to Theorem 2.3. Moreover, the following statement is true.

**Proposition 6.1** Any nonconstant real harmonic function \( a(u) \) defines a pair of almost compatible metrics \( g^{ij}_1(u) = \exp(a(u)) \delta^{ij}, \quad 1 \leq i, j \leq 2 \), and \( g^{ij}_2(u) = \delta^{ij}, \quad 1 \leq i, j \leq 2 \), which are not compatible. These metrics are compatible if and only if \( a = a(u^1 \pm iu^2) \).

Let us construct also almost compatible metrics of constant Riemannian curvature, which are not compatible.
Lemma 6.2 Two-component diagonal conformally Euclidean metric
\[ g^{ij}(u) = \exp(a(u))\delta^{ij}, \quad 1 \leq i, j \leq 2, \]
is a metric of constant Riemannian curvature \( K \) if and only if the function \( a(u) \) is a solution of the Liouville equation
\[ \Delta a \equiv \frac{\partial^2 a}{\partial(u_1)^2} + \frac{\partial^2 a}{\partial(u_2)^2} = 2Ke^{-a(u)}. \quad (6.2) \]

Proposition 6.2 For the metrics \( g_{ij}^1(u) = \exp(a(u))\delta_{ij}, \quad 1 \leq i, j \leq 2 \), and \( g_{ij}^2(u) = \delta_{ij}, \quad 1 \leq i, j \leq 2 \), the corresponding Nijenhuis tensor vanishes, that is, they are always almost compatible. But they are compatible metrics of constant Riemannian curvature \( K \neq 0 \) and \( 0 \), respectively, if and only if the function \( a(u) \) is constant.

Note, that all the one-component “metrics” are always compatible, and all the one-component local Poisson structures of hydrodynamic type are also always compatible. Let us construct for any \( N > 1 \) examples of almost compatible metrics which are not compatible.

Proposition 6.3 The metrics \( g_{ij}^1(u) = b(u)\delta_{ij}, \quad 1 \leq i, j \leq N \) and \( g_{ij}^2(u) = \delta_{ij}, \quad 1 \leq i, j \leq N \), where \( b(u) \) is an arbitrary function, are always almost compatible, the corresponding Nijenhuis tensor vanishes. But they are compatible real metrics only in the most trivial case, when the function \( b(u) \) is constant. Complex metrics are compatible if and only if either the function \( b(u) \) is constant, or \( N = 2 \) and \( b(u) = b(u^1 \pm iu^2) \).

7 Compatible metrics and nonlocal Poisson structures of hydrodynamic type

Nonlocal Poisson structures of hydrodynamic type were introduced and studied in the work of the present author and Ferapontov [10] (see also [11]–[14]). They have the following form:
\[ \{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^i_k(u(x)) u^k_x + Ku^i_x \right) \left( \frac{d}{dx} \right)^{-1} u^j_x \frac{\delta J}{\delta \omega^j(x)} dx, \quad (7.1) \]

where \( K \) is an arbitrary constant.

The bracket of form (7.1) is called nondegenerate if \( \det(g^{ij}(u)) \neq 0 \).

Theorem 7.1 ([10]). If \( \det(g^{ij}(u)) \neq 0 \), then bracket (7.1) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. \( g^{ij}(u) \) is an arbitrary pseudo-Riemannian contravariant metric of constant Riemannian curvature \( K \),
2. \( b^i_k(u) = -g^{is}(u)\Gamma^j_{sk}(u) \), where \( \Gamma^j_{sk}(u) \) is the Riemannian connection generated by the contravariant metric \( g^{ij}(u) \) (the Levi–Civita connection).

Proposition 7.1 Nonlocal nondegenerate Poisson brackets of form (7.1) are compatible if and only if their metrics are compatible.
In [11] Ferapontov introduced and studied more general nonlocal Poisson brackets of hydrodynamic type, namely, the brackets of the following form:

\[
\{I, J\} = \int \frac{\delta I}{\delta u(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^i_j(u(x)) u^k_x \right) + \sum_{\alpha=1}^L (w^\alpha)^i_k(u) u^k_x \left( \frac{d}{dx} \right)^{-1} (w^\alpha)^j_l(u) u^l_x \frac{\delta J}{\delta u^j(x)} dx, \quad \det(g^{ij}(u)) \neq 0. \tag{7.2}
\]

**Theorem 7.2** ([11]) Bracket (7.2) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. \( b^i_j(u) = -g^{i\alpha}(u) \Gamma^j_{sk}(u) \), where \( \Gamma^j_{sk}(u) \) is the Riemannian connection generated by the contravariant metric \( g^{ij}(u) \) (the Levi–Civita connection),

2. the pseudo-Riemannian metric \( g^{ij}(u) \) and the set of affinors \( (w^\alpha)^i_j(u) \) satisfy the relations:

\[
\begin{align*}
g_{ik}(u)(w^\alpha)^j_k(u) &= g_{jk}(u)(w^\alpha)^i_k(u), \quad \alpha = 1, \ldots, L, \tag{7.3} \\
\nabla_k(w^\alpha)^j_i(u) &= \nabla_j(w^\alpha)^i_k(u), \quad \alpha = 1, \ldots, L, \tag{7.4} \\
R_{kl}^{ij}(u) &= \sum_{\alpha=1}^L \left( (w^\alpha)^i_k(u)(w^\alpha)^j_l(u) - (w^\alpha)^j_k(u)(w^\alpha)^i_l(u) \right). \tag{7.5}
\end{align*}
\]

Moreover, the family of affinors \( w^\alpha(u) \) is commutative: \( [w^\alpha, w^\beta] = 0 \).

**Proposition 7.2** If nonlocal Poisson brackets of form (7.2) are compatible, then their metrics are also compatible.

Actually, if nonlocal Poisson brackets of form (7.2) are compatible, then it follows from the conditions of compatibility and from Theorem 7.2 that, first, relation (7.2) is fulfilled and, secondly, the curvature tensor for the metric \( g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u) \) has the form

\[
R_{kl}^{ij}(u) = \sum_{\alpha=1}^{L_1} \lambda_1 \left( (w^\alpha_1)^i_k(u)(w^\alpha_1)^j_l(u) - (w^\alpha_1)^j_k(u)(w^\alpha_1)^i_l(u) \right) + \sum_{\alpha=1}^{L_2} \lambda_2 \left( (w^\alpha_2)^i_k(u)(w^\alpha_2)^j_l(u) - (w^\alpha_2)^j_k(u)(w^\alpha_2)^i_l(u) \right) = \lambda_1 R_{1,kl}^{ij}(u) + \lambda_2 R_{2,kl}^{ij}(u).
\]

Apparently, the converse statement is also always true (this is only our conjecture which is not strictly proved in the most general case at this moment).

Relations (7.3)–(7.5) are nothing but the Gauss–Peterson–Codazzi equations for \( N \)-dimensional surfaces \( M \) with flat normal connections in a pseudo-Euclidean space \( E^{N+L} \). Here \( g_{ij}(u) \) is the first fundamental form of the surface \( M \), and \( w^\alpha(u) \) are the Weingarten operators [11].

We consider more in detail the case of compatible nonlocal Poisson structures of hydrodynamic type that correspond to surfaces with holonomic net of curvature lines (see [4]). This case is the most interesting for applications (here we do not give numerous important examples, see, for example, in [11], [12], [12] or in the author’s survey [41]).

**Proposition 7.3** Let two nonlocal Poisson brackets of form (7.2) correspond to surfaces with holonomic net of curvature lines and be given in coordinates of curvature lines. In this case, if the corresponding pair of metrics is nonsingular, then the nonlocal Poisson structures are compatible if and only if their metrics are compatible.
In this case the metrics \( g_{ij}^1(u) = g^1_i(u)\delta^{ij} \) and \( g_{ij}^2(u) = g^2_i(u)\delta^{ij} \), and also the Weingarten operators \( (w^\alpha_1)_i^j(u) = (w^\alpha_1)_i^j(u)\delta^j_i \) and \( (w^\alpha_2)_i^j(u) = (w^\alpha_2)_i^j(u)\delta^j_i \) are diagonal in the coordinates under consideration. For any such “diagonal” case, condition (7.3) is automatically fulfilled, all the Weingarten operators commute, conditions (7.4) and (7.5) have the following form, respectively:

\[
2g^i(u) \frac{\partial (w^\alpha)^i}{\partial u^k} = ((w^\alpha)^i - (w^\alpha)^k) \frac{\partial g^i}{\partial u^k} \text{ for all } i \neq k, \tag{7.6}
\]

\[
R_{ij}^{12}(u) = \sum_{\alpha=1}^L (w^\alpha)^i(u)(w^\alpha)^j(u). \tag{7.7}
\]

It follows from nonsingularity of the pair of the metrics and from compatibility of the metrics that the corresponding Nijenhuis tensor vanishes and there exist functions \( f^i(u^i), i = 1, \ldots, N, \) such that:

\[
g^1_i(u) = f^i(u^i)g^2_i(u).
\]

Using relations (7.6) and (7.7), it is easy to prove that in this case it follows from compatibility of the metrics that an arbitrary linear combination of nonlocal Poisson brackets under consideration is also a Poisson bracket.

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