How is a chordal graph like a supersolvable binary matroid?

Raul Cordovil, David Forge and Sulamita Klein

To the memory of Claude Berge

Abstract. Let $G$ be a finite simple graph. From the pioneering work of R. P. Stanley it is known that the cycle matroid of $G$ is supersolvable if $G$ is chordal (rigid): this is another way to read Dirac’s theorem on chordal graphs. Chordal binary matroids are not in general supersolvable. Nevertheless we prove that, for every supersolvable binary matroid $M$, a maximal chain of modular flats of $M$ canonically determines a chordal graph.

1. Introduction and notations

Throughout this note $M$ denotes a matroid of rank $r$ on the ground set $[n] := \{1, 2, \ldots, n\}$. We refer to [7, 9] as standard sources for matroid theory. We recall and fix some notation of matroid theory. The restriction of $M$ to a subset $X \subseteq [n]$ is denoted $M|_X$. A matroid $M$ is said to be simple if all circuits have at least three elements. A matroid $M$ is binary if the symmetric difference of any two different circuits of $M$ is a union of disjoint circuits. Graphic and cographic matroids are extremely important examples of binary matroids. The dual of $M$ is denoted $M^*$. Let $C = C(M)$ [resp. $C^* = C^*(M) = C(M^*)$] be the set of circuits [resp. cocircuits] of $M$. Let $C_\ell := \{C \in C : |C| \leq \ell\}$. In the following the singleton $\{x\}$ is denoted by $x$. We will denote by

$$cl(X) := X \cup \{x \in [n] : \exists C \in C, C \setminus X = x\},$$

the closure in $M$ of a subset $X \subseteq [n]$. We say that $X \subseteq [n]$ is a flat of $M$ if $X = cl(X)$. The set $F(M)$ of flats of $M$, ordered by inclusion, is a geometric lattice. The rank of a flat $F \in F$, denoted $r(F)$, is equal to $m$ if there are $m + 1$ flats in a maximal chain of flats from $\emptyset$ to $F$.

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The flats of rank 1, 2, 3 and \( r - 1 \) are called points, lines, planes, and hyperplanes respectively. A line \( L \) with two elements is called trivial and a line with three elements is called nontrivial (a binary matroid has no line with more than three points). Given a set \( X \subseteq [n] \), let \( r(X) := r(\text{cl}(X)) \). A pair \( F, F' \) of flats is called modular if
\[
 r(F) + r(F') = r(F \lor F') + r(F \land F').
\]
A flat \( F \in \mathcal{F} \) is modular if it forms a modular pair with every other flat \( F' \in \mathcal{F} \).

**Definition 1.1.** A matroid \( M \) on \([n]\) of rank \( r \) is supersolvable if there is a maximal chain of modular flats \( \mathcal{M} \)
\[
\mathcal{M} := F_0(= \emptyset) \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r(= [n]).
\]
We call \( \mathcal{M} \) an \( M \)-chain of \( M \). To the \( M \)-chain \( \mathcal{M} \) we associate the partition \( \mathcal{P} \) of \([n]\)
\[
\mathcal{P} := F_1 \uplus \cdots \uplus (F_i \setminus F_{i-1}) \uplus \cdots \uplus (F_r \setminus F_{r-1}).
\]
We call \( \mathcal{P} \) an \( M \)-partition of \( M \).

We recall that a graph \( G \) is said chordal (or rigid or triangulated) if every cycle of length at least four has a chord. Chordal graphs are treated extensively in Chapter 4 of [6]. The notion of a “chordal matroid” has also been recently explored in the literature, see [2].

**Definition 1.2** ([1] p. 53). Let \( M \) be an arbitrary matroid (not necessarily simple or binary). A circuit \( C \) of \( M \) has a chord \( i_0 \) if there are two circuits \( C_1 \) and \( C_2 \) such that \( C_1 \cap C_2 = i_0 \) and \( C = C_1 \Delta C_2 \). In this case, we say that the chord \( i_0 \) splits the circuit \( C \) into the circuits \( C_1 \) and \( C_2 \). We say that a matroid is \( \ell \)-chordal if every circuit with at least \( \ell \) elements has a chord. A simple matroid \( M \) is chordal if it is 4-chordal.

In this paper we always suppose that the edges of a graph \( G \) are labelled with the integers of \([n]\). If nothing else is indicated we suppose \( G \) is a connected graph. Let \( \mathcal{M}(G) \) be the cycle matroid of the graph \( G \): i.e., the elementary cycles of \( G \), as subsets of \([n]\), are the circuits of \( \mathcal{M}(G) \). In the same way, the minimal cutsets of a connected graph \( G \) (i.e., a set of edges that disconnect the graph) are the circuits of a matroid on \([n]\), called the cocycle matroid of \( G \). A matroid is graphic (resp. cographic) if it is the cycle (resp. cocycle) matroid of a graph. The cocycle matroid of \( G \) is dual to the cycle matroid of \( G \) and both are binary. The cocycle matroids of the complete graph \( K_5 \) and of the
complete bipartite graph $K_{3,3}$ are examples of binary but not graphic matroids; see Section 13.3 in [7] for details. The Fano matroid is an example of a supersolvable binary matroid that is neither graphic nor cographic. Finally, note that an elementary cycle $C$ of $G$ has a chord iff $C$ seen as a circuit of the matroid $M(G)$ has a chord.

Example 1.3. Consider the chordal graph $G_0 = G_0(V, [7])$ in Figure 1 and the corresponding cycle matroid $M(G_0)$. It is clear that

$$M := \emptyset \subsetneq \{1\} \subsetneq \{1, 2, 3\} \subsetneq \{1, 2, 3, 4, 5\} \subsetneq [7]$$

is an $M$-chain. The associated $M$-partition is

$$\mathcal{P} := \{1\} \cup \{2, 3\} \cup \{4, 5\} \cup \{6, 7\}.$$ 

The linear order of the vertices is such that for every $i$ in $\{2, 3, 4, 5\}$ the neighbors of the vertex $v_i$ contained in the set $\{v_1, \ldots, v_{i-1}\}$ form a clique; this is Dirac’s well known characterization of chordal graphs (see [5, 6]). This is also a characterization of graphic supersolvable matroids (see Proposition 2.8 in [8]). That is, a graphic matroid $M(G)$ is supersolvable iff the vertices of $G$ can be labeled as $v_1, v_2, \ldots, v_m$ such that, for every $i = 2, \ldots, m$, the neighbors of $v_i$ contained in the set $\{v_1, \ldots, v_{i-1}\}$ form a clique. We say that a linear order of the vertices of $G$ with the above properties is an $S$-label of the vertices of $G$.

Ziegler proved that every supersolvable binary matroid without a Fano submatroid is graphic (Theorem 2.7 in [10]). In the next section we answer the following natural question:

$\circ$ For a generic binary matroid, what are the relations between the notions of “chordal” and “supersolvable”?

2. CHORDAL AND SUPERSOLVABLE MATROIDS

Lemma 2.1. Let $M$ be a simple binary matroid. The following two conditions are equivalent for every circuit $C$ of $M$:

(2.1) $C \subsetneq \text{cl}(C)$,
(2.1.2) \( C \) has a chord.

For nonbinary matroids only the implication (2.1.2) \( \Rightarrow \) (2.1.1) holds.

**Proof.** If \( i \in \text{cl}(C) \setminus C \), then there is a circuit \( D \) such that \( i \in D \) and \( D \setminus i \not\subseteq C \). As \( M \) is binary \( D' = D \setminus C \) is also a circuit of \( M \). So \( i \) is a chord of \( C \). If \( i \) is a chord of \( C \), then clearly \( i \in \text{cl}(C) \). Finally, in the uniform rank-two nonbinary matroid \( U_{2,4} \), the set \( C = \{1, 2, 3\} \) is a circuit without a chord but \( C \not\subseteq \text{cl}(C) = [4] \). \( \square \)

**Theorem 2.2.** A binary supersolvable matroid \( M \) is chordal but the converse does not hold in general.

**Proof.** Let \( M := \emptyset \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = [n] \) be an \( M \)-chain of \( M \). Suppose by induction that the restriction of \( M \) to \( F_{r-1} \) is chordal. The result is clear in the case that \( C^* := [n] \setminus F_{r-1} \) is a singleton. Suppose that \( |C^*| > 1 \) and consider a circuit \( C \) of \( M \) not contained in the modular hyperplane \( F_{r-1} \). Then there are two elements \( i, j \in C \cap C^* \) and the line \( \text{cl}\{i, j\} \) meets \( F_{r-1} \). So \( C \not\subseteq \text{cl}(C) \) and we know from Lemma 2.1 that \( C \) has a chord.

A counterexample of the converse is \( M^*(K_{3,3}) \), the cocycle matroid of the complete bipartite graph \( K_{3,3} \). It is easy to see from its geometric representation that it is chordal but not supersolvable (see 10 and page 514 in 7 for its geometric representation). \( \square \)

**Definition 2.3** (4). Let \( M \) be an arbitrary matroid and consider an integer \( \ell \geq 2 \). The matroid \( M \) is \( \ell \)-closed if the following two conditions are equivalent for every subset \( X \subseteq [n] \):

(2.3.1) \( X \) is closed,

(2.3.2) for every subset \( Y \) of \( X \) with at most \( \ell \) elements we have \( \text{cl}(Y) \subseteq X \).

We note that Condition (2.3.2) is equivalent to:

(2.3.2') for every circuit \( C \) of \( M \) with at most \( \ell + 1 \) elements

\[ |C \cap X| \geq |C| - 1 \implies C \subseteq X. \]

**Definition 2.4.** Let \( C' \) be a subset of \( C \), the set of circuits of \( M \). Let \( \text{cl}_\Delta(C') \) denote the smallest subset of \( C \) such that:

(2.4.1) \( C' \subseteq \text{cl}_\Delta(C') \),

(2.4.2) whenever a circuit splits into two circuits \( C_1 \) and \( C_2 \) that are in \( \text{cl}_\Delta(C') \) then \( C \) is also in \( \text{cl}_\Delta(C') \).

**Theorem 2.5.** For every simple binary matroid \( M \) the following three conditions are equivalent:

(2.5.1) \( M \) is \( \ell \)-closed,
Proposition 2.6. Let $M$ be a supersolvable matroid and
\[ \mathcal{M} := F_0 \subseteq \cdots \subseteq F_{r-1} \subseteq F_r \]
an $M$-chain. Let $F$ be a flat of $M$. Then $M|F$, the restriction of $M$ to the flat $F$, is a supersolvable matroid and \{ $F_i \cap F : F_i \in \mathcal{M}$ \} is the set of (modular) flats of an $M|F$-chain. \hfill $\Box$

Definition 2.7. Let $\mathcal{P} = P_1 \uplus \cdots \uplus P_r$ be an $M$-partition of a supersolvable matroid $M$. We associate to $(M, \mathcal{P})$ a graph $G_\mathcal{P}$ such that:

- $V(G_\mathcal{P}) = \{ P_i : i = 1, 2, \ldots, r \}$ is the vertex set of $G_\mathcal{P}$,
- $\{ P_i, P_j \}$ is an edge of $G_\mathcal{P}$ iff there is a nontrivial line $L$ of $M$ meeting $P_i$ and $P_j$.

We call $G_\mathcal{P}$ the S-graph of the pair $(M, \mathcal{P})$.

Note that every nontrivial line $L$ of the binary supersolvable matroid $M$ meets exactly two $P_i$’s and if $L$ meets $P_i$ and $P_j$, with $i < j$, necessarily $|P_i \cap L| = 1$ and $|P_j \cap L| = 2$. Indeed $F_{j-1} = \bigcup_{i=1}^{j-1} P_i$ is a modular flat disjoint from $P_j$, so $|F_{j-1} \cap L| = 1$. This simple property
will be used extensively in the proof of Theorem 2.10. Given a chordal graph $G$ with a fixed S-labeling, we get an associated supersolvable matroid $M(G)$ and an associated $M$-partition $P$. We say that $G_P$, the S-graph determined by $(M(G), P)$, is the derived S-graph of $G$ for this S-labeling.

**Remark 2.8.** Note that the derived S-graph $G_P$ of a chordal graph $G$ is a subgraph of $G$. Indeed set $V(G_P) = \{P_1, \ldots, P_m\}$ and consider the map $P_\ell \to v_{\ell+1}$, $\ell = 1, \ldots, m$. Let $\{P_i, P_j\}$, $1 \leq i < j \leq m$, be an edge of $G_P$. From the definitions we see that $\{v_{i+1}, v_{j+1}\}$ is necessarily an edge of $G$.

**Example 2.9.** Consider the S-labeling of the graph $G_0$ given in Figure 1 and the associated $M$-partition $P$ (see Example 1.3). The derived S-graph $G_P$ is a path from $P_1$ to $P_4$. Consider now the $M$-partition of $M(G_0)$:

$$P' := \{4\} \cup \{3, 5\} \cup \{1, 2\} \cup \{6, 7\}.$$  

In this case the corresponding S-graph $G'_P$ is $K_{1,3}$ with $P_2$ being the degree-3 vertex. It is easy to prove that for any $M$-partition $P$ of the cycle matroid of the complete graph $K_\ell$, the S-graph $G_P$ is the complete graph $K_{\ell-1}$.

Our main result is:

**Theorem 2.10.** Let $M$ be a simple binary supersolvable matroid with an $M$-partition $P$. Then the S-graph $G_P$ is chordal.

**Proof.** Let $P = P_1 \cup \cdots \cup P_r$. We claim that $P_r$ is a simplicial vertex of $G_P$. Suppose that $\{P_i, P_j\}$ and $\{P_r, P_j\}$, $i < j$, are two different edges of $G_P$ and that there are two nontrivial lines $L_1 := \{x, y, z\}$ and $L_2 := \{x', y', z'\}$ where $x, y, x', y' \in P_r$ and $z \in P_i$, $z' \in P_j$. We will consider two possible cases:

- Suppose first that two of the elements $x, y, x', y'$ are equal; w.l.o.g., we can suppose $x = x'$. As $M$ is binary the elements $x, y, y'$ can’t be colinear, so $\text{cl}(\{x, y, y'\})$ is a plane. From modularity of $F_{r-1}$, we know that $\text{cl}(\{x, y, y'\}) \cap F_{r-1}$ is a line. So the line $\text{cl}(\{y, y'\})$ meets the modular hyperplane $F_{r-1}$ in a point $a$. Now the line $\{z, z', a\}$ is a nontrivial line which meets $P_i$ and $P_j$. Then by definition $\{P_i, P_j\}$ is an edge of $G_P$.

- Suppose now that the elements $x, y, x', y'$ are different. Then as $M$ is binary we have $r(\{x, y, x', y'\}) = 4$. From modularity of $F_{r-1}$, we know that $r(\text{cl}(\{x, y, x', y'\}) \cap F_{r-1}) = 3$. Then the six lines $\text{cl}(\{\alpha, \beta\})$, for $\alpha$ and $\beta$ in $\{x, y, x', y'\}$ meet $F_{r-1}$ in six coplanar
points; let these points be labelled as in Figure 2. Let $P_\ell$ be the set that contains $a$. We will consider three subcases.

- Suppose first that $i < j < \ell$. From the property given immediately after Definition 2.7, we have that $c$ is also in $P_\ell$. Consider the modular flat $F_{\ell-1} = \bigcup_{h=1}^{\ell-1} P_h$. We know that the plane $\text{cl} \{a, c, z, z'\}$ meets $F_{\ell-1}$ in a line, so $\text{cl} \{z, z'\}$ is a non-trivial line meeting $P_i$ and $P_j$ and so $\{P_i, P_j\}$ is an edge of $G_\mathcal{P}$.

- Suppose now that $\ell < i < j$. Then the nontrivial line $\{a, d, z\}$ meets $P_i$ and $P_\ell$ and we have $d \in P_i$. So the nontrivial line $\{c, d, z'\}$ meets $P_i$ and $P_j$ and $\{P_i, P_j\}$ is an edge of $G_\mathcal{P}$.

- Suppose finally that $i \leq \ell \leq j$. The nontrivial line $\{a, d, z\}$ meets $P_\ell$ and $P_i$ so $d \in P_\ell$. The nontrivial line $\{c, d, z'\}$ meets $P_\ell$ and $P_j$ and necessarily we have $c \in P_j$. We conclude that the nontrivial line $\{b, c, z\}$ meets $P_i$ and $P_j$ and $\{P_i, P_j\}$ is an edge of $G_\mathcal{P}$.

By induction we conclude that $G_\mathcal{P}$ is chordal.  

We say that two $M$-chains

$$\mathcal{M} := \emptyset \subset \cdots \subset F_{r-1} \subset F_r = [n]$$

and

$$\mathcal{M}' := \emptyset \subset \cdots \subset F'_{r-1} \subset F'_r = [n]$$

are
are related by an elementary deformation if they differ by at most one flat. We say that two $M$-chains are equivalent if they can be obtained from each other by elementary deformations.

**Proposition 2.11.** Every two $M$-chains of the same matroid $M$ are equivalent.

**Proof.** We prove it by induction on the rank. The result is clear for $r = 2$. Suppose it true for all matroids of rank at most $r - 1$. Consider two different $M$-chains $M := \emptyset \subset \cdots \subset F_{r-1} \subset F_r = [n]$ and $M' := \emptyset \subset \cdots \subset F'_{r-1} \subset F'_r = [n]$.

Let $F_\ell$ be the flat of highest rank of the $M$-chain $M$ contained in $F'_{r-1}$. We know that $F_j \cap F'_{r-1}$, $j = 0, 1, \ldots, r$, is a modular flat of the matroid $M$ and that $r(F_j \cap F'_{r-1}) = j - 1$, for $j = \ell + 2, \ldots, r - 1$.

Let $M_0 := M$ and for $i$ from 1 to $r - 1 - \ell$, let $M_i$ be the $M$-chain $\emptyset \subset \cdots \subset F_\ell \subset F_{\ell+2} \cap F'_{r-1} \subset \cdots \subset F_{\ell+i+1} \cap F'_{r-1} \subset F_{\ell+i+1} \cdots \subset [n]$.

We have clearly by definition that for $i$ from 0 to $r - 2 - \ell$, the $M$-chains $M_i$ and $M_{i+1}$ are equivalent. This sequence of equivalences shows that $M$ is equivalent to $M_{r-1-\ell}$. Finally by the induction hypothesis we have that $M'$ is equivalent to $M_{r-1-\ell}$ which concludes the proof. \qed

**Remark 2.12.** Proposition 2.11 can be used to obtain all the S-labels of a given chordal graph $G$ from a fixed one. If $G$ is doubly-connected the number of $M$-chains of $M(G)$ is equal to the half of the number of such labelings, see [8], Proposition 2.8.

It is natural to ask if, given a chordal graph $G$, there is a supersolvable matroid $M$ together with an $M$-partition $\mathcal{P}$ such that $G = G_\mathcal{P}$. Can the matroid $M$ be supposed graphic? The next proposition gives a positive answer to these questions:

**Proposition 2.13.** Let $G = (V, E)$ be a chordal graph with an $S$-labeling $v_1, \ldots, v_m$ of its vertices, and $\tilde{G}$ the extension of $G$ by a vertex $v_0$ adjacent to all the vertices, i.e.:

$$V_{\tilde{G}} = V_G \cup v_0 \quad \text{and} \quad E_{\tilde{G}} = E_G \cup \{v_i, v_0\}, \ i = 1, \ldots, m.$$  

Then $G_\mathcal{P}$, the derived $S$-graph of $\tilde{G}$ for the $S$-labeling $v_0, v_1, \ldots, v_m$ is isomorphic to $G$.  

Proof. As \( v_0 \) is adjacent to every vertex \( v_i, \ i = 1, \ldots, m \), it is clear that \( v_0, v_1, \ldots, v_m \) is an S-labeling of \( \tilde{G} \). Let \( \mathcal{P} \) and \( \tilde{\mathcal{P}} \) denote the corresponding \( M \)-partitions of the graphic matroids \( M(G) \) and \( M(\tilde{G}) \).

We have \( \mathcal{P} = P_1 \uplus \cdots \uplus P_{m-1} \) and \( \tilde{\mathcal{P}} = \tilde{P}_1 (=\{v_0,v_1\}) \uplus \tilde{P}_2 \uplus \cdots \uplus \tilde{P}_m \) with \( \tilde{P}_i = P_{i-1} \uplus \{v_0,v_i\} \), for \( i = 2, \ldots, m \). Now we can see that if \( \{v_i,v_j\}, \ 0 \leq i < j \leq m - 1 \), is an edge of \( G \) then \( \{\tilde{P}_i,\tilde{P}_j\} \) is an edge of \( G_{\tilde{\mathcal{P}}} \). From Remark 2.8 we get that reciprocally \( G_{\tilde{\mathcal{P}}} \) is a subgraph of \( G \). \( \square \)

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Departamento de Matemática,
INSTITUTO SUPERIOR TÉCNICO
AV. ROVISCO PAIS - 1049-001 LISBOA - PORTUGAL
E-mail address: cordovil@math.ist.utl.pt

LRI, UMR 8623, BATIMENT 490 UNIVERSITÉ PARIS-SUD
91405 ORSAY CEDEX, FRANCE
E-mail address: forge@lri.fr
Instituto de Matemática and COPPE-Sistemas, Universidade Federal do Rio de Janeiro, Caixa Postal 68511, 21945-970, Rio de Janeiro, RJ, Brasil
E-mail address: sula@cos.ufrj.br