A DEGENERATE ELLIPTIC PROBLEM FROM
SUBSONIC-SONIC FLOWS IN CONVERGENT NOZZLES

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ABSTRACT. This paper concerns continuous subsonic-sonic potential flows in a two dimensional convergent nozzle, which is governed by a free boundary problem of a quasilinear degenerate elliptic equation. It is shown that for a given nozzle which is a perturbation of an straight one, and a given mass flux, there exists uniquely a continuous subsonic-sonic flow whose velocity vector is along the normal direction at the inlet and the sonic curve. Furthermore, the sonic curve of this flow is a free boundary, where the flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up at the sonic state.

1. Introduction. In this paper we study subsonic-sonic flows in convergent nozzles. Such problems naturally arise in physical experiments and engineering designs, and there are many experiments and numerical simulations and rigorous theories involved in this field [2, 6]. Two-dimensional steady compressible fluids satisfy the Euler system:

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0, \\
\frac{\partial (P + \rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} = 0, \\
\frac{\partial (\rho uv)}{\partial x} + \frac{\partial (P + \rho v^2)}{\partial y} = 0,
\]

where $(u, v)$, $P$ and $\rho$ represent the velocity, pressure and density of the flow, respectively. The flow is assumed to be isentropic so that $P = P(\rho)$ is a smooth function. In particular, for a polytropic gas with adiabatic exponent $\gamma > 1$,

\[
P(\rho) = \frac{1}{\gamma \rho^\gamma}
\]

is the normalized pressure. Assume further that the flow is irrotational, i.e.

\[
\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}
\]
Then, the Bernoulli law ([2, 6]) yields
\[
\rho(q^2) = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{1/(\gamma - 1)}, \quad q = \sqrt{u^2 + v^2}, \quad 0 < q < \sqrt{2/(\gamma - 1)}. \tag{1.6}
\]
The sound speed \( c \) is defined as \( c^2 = P'(\rho) \). At the sonic state, the speed is \( c_s = \sqrt{2/(\gamma + 1)} \), which is critical in the sense that the flow is subsonic when \( q < c_s \), sonic when \( q = c_s \), and supersonic when \( q > c_s \). The Euler system (1.1)–(1.5) can be transformed into the full potential equation
\[
\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0, \tag{1.7}
\]
where \( \varphi \) is a velocity potential with \( \nabla \varphi = (u, v) \), \( \rho \) is the function given by (1.6). It is noted that (1.7) is elliptic in the subsonic region, degenerate at the sonic state, while hyperbolic in the supersonic region.

Subsonic and subsonic-sonic flows past a profile or in a nozzle have been studied for a long time. In the outstanding work [1], L. Bers showed that there exists a unique subsonic flow past a two-dimensional profile if the freestream Mach number is less than a critical value; furthermore, the maximum flow speed tends to the sound speed as the freestream Mach number tends to the critical value. The same results for multi-dimensional cases were obtained in [10, 7]. In these works, the flow with the critical freestream Mach number has not been considered. Based on the compensated compactness method, it was shown in [4] that the two-dimensional flows with sonic points past a profile may be realized as weak limits of sequences of strictly subsonic flows. However, their regularity and uniqueness are unknown yet. Similar situation occurs for subsonic and subsonic-sonic flows in an infinitely long nozzle. For a two-dimensional infinitely long nozzle, it was proved in [20] that there exists a critical value such that a strictly subsonic flow exists uniquely as long as the incoming mass flux is less than a critical value; furthermore, the maximum flow speed tends to the unique subsonic flow past a two-dimensional profile if the freestream Mach number is less than a critical value, and subsonic-sonic flows exist as weak limits of strictly subsonic flows. The multi-dimensional cases were dealt with in [9, 5]. There are also many studies on rotational subsonic flows and we refer to [3, 8, 21] and the references therein. Moreover, [18, 16, 17] studied smooth transonic flows of Meyer type in de Laval nozzles.

Continuous subsonic-sonic flow problems in convergent nozzles were studied recently. Assume that the inlet, the lower and upper walls of the nozzle are given by \( \Gamma_{in} : x = g(y) (0 \leq y \leq \mathcal{R} \sin \vartheta) \), \( \Gamma_{lw} : y = 0 (-\mathcal{R} \leq x \leq 0) \) and \( \Gamma_{upw} : y = f(x) (-\mathcal{R} \cos \vartheta \leq x \leq 0) \), respectively, where \( \mathcal{R} > 0 \), \( \vartheta \in (0, \pi/2) \), \( g(\mathcal{R} \sin \vartheta) = -\mathcal{R} \cos \vartheta \) and \( f(-\mathcal{R} \cos \vartheta) = \mathcal{R} \sin \vartheta \). The outlet is chosen to be the sonic curve of the flow where the velocity of the flow is along the normal direction, which is a free boundary from the upper wall to the lower wall, and is denoted by \( \Gamma_{out} : x = S(y) (0 \leq y \leq f(-\mathcal{R} \cos \vartheta)) \). The flow satisfies the slip conditions at the walls, and its mass flux is \( m \) and its incoming flow angle is along the normal direction at the inlet. Such a subsonic-sonic flow problem is to seek \( (\varphi, S, \zeta) \) solving
\[
\begin{align*}
\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) &= 0, & (x, y) &\in \Omega, \tag{1.8} \\
\varphi(g(y), y) &= 0, & 0 < y < \mathcal{R} \sin \vartheta, \tag{1.9} \\
\frac{\partial \varphi}{\partial y}(x, 0) &= 0, & -\mathcal{R} \leq x \leq S(0), \tag{1.10} \\
\frac{\partial \varphi}{\partial y}(x, f(x)) - f'(x) \frac{\partial \varphi}{\partial x}(x, f(x)) &= 0, & -\mathcal{R} \cos \vartheta \leq x \leq -\mathcal{R} \cos \vartheta, \tag{1.11} \\
|\nabla \varphi(S(y), y)| &= c_s, & \varphi(S(y), y) = \zeta, & 0 < y < f(-\mathcal{R} \cos \vartheta), \tag{1.12}
\end{align*}
\]
\[ |\nabla \varphi(x, y)| < c_*, \quad (x, y) \in \Omega, \quad (1.13) \]
\[
\int_0^{R \sin \vartheta} |\nabla \varphi(g(y), y)| \rho(|\nabla \varphi(g(y), y)|^2) (1 + (g'(y))^2)^{1/2} dy = m, \quad (1.14) \]

where \( \Omega \) is the domain bounded by \( \Gamma_{in}, \Gamma_{lw}, \Gamma_{upw} \) and \( \Gamma_{out} \).

If \( g \) and \( f \) take the forms of
\[ g(y) = -\sqrt{R^2 - y^2}, \quad 0 \leq y \leq R \sin \vartheta, \quad (1.15) \]
and
\[ f(x) = -x \tan \vartheta, \quad -R \cos \vartheta \leq x \leq 0, \quad (1.16) \]
respectively, then the problem (1.8)–(1.14) admits a radially symmetric subsonic-sonic flow, whose sonic curve is the arc centered at the origin with radius \( \hat{R} = m/(c_* \rho(c_2^*) \vartheta) \). For this radially symmetric subsonic-sonic flow, its speed is \( C^{1/2} \) Hölder continuous and its acceleration blows up at the sonic curve. A natural question is whether this radially symmetric subsonic-sonic flow is stable, which was first studied in [15]. It was shown in [15] that for \( c_- \rho(c_2^*) \vartheta < m < c_* \rho(c_2^*) \vartheta \), the radially symmetric subsonic-sonic flow is stable with respect to a perturbation of \( \Gamma_{in} \), where \( c_- \in (0, c_*) \) is a constant (see (2.12)). Furthermore, the sonic curve of the subsonic-sonic flow in [15] is a free boundary, where the flow is singular in the sense that the speed is only \( C^{1/2} \) Hölder continuous and the acceleration blows up at the sonic curve. Later in [12], the structural stability was proved for the case that both the inlet and the incoming flow angle are perturbed. Therefore, the remaining question is that whether this radially symmetric subsonic-sonic flow is stable with respect to a perturbation of the wall.

In this paper, we study the structural stability of the radially symmetric subsonic-sonic flow with respect to a perturbation of the upper wall. From the compatible condition, the sonic curve must intersect the upper wall at a point where the curvature is zero. Furthermore, the intersecting point between the sonic curve and the upper wall is free. So we perturb the upper wall away from the sonic state of the radially symmetric subsonic-sonic flow. More precisely, it is assumed that \( g \) takes the form of (1.15), and \( f \in C^{3,\alpha}([-R \cos \vartheta, 0]) \) satisfies
\[ f(-R \cos \vartheta) = R \sin \vartheta, \quad f'(-R \cos \vartheta) = -\tan \vartheta, \quad (1.17) \]
\[ f''(-R \cos \vartheta) = f'''(-R \cos \vartheta) = 0, \]
\[ f(x) = -x \tan \vartheta, \quad -\mathcal{P} \cos \vartheta \leq x \leq 0, \quad (1.19) \]
\[ \|f' + \tan \vartheta\|_{C^2([-R \cos \vartheta, 0])} \leq \delta, \quad (1.20) \]

where \( \alpha \in (0, 1) \) and \( \mathcal{P} \in (\bar{R}, R) \) are constants. For \( c_- \rho (c_0^2 - \mathcal{P}) \vartheta < m < c_+ \rho (c_0^2 - \mathcal{P}) \vartheta \), we solve the problem (1.8)–(1.14). There are also studies on the perturbation of the upper wall ([14, 13, 19]). But this is a different problem. Indeed, it is assumed in [14, 13, 19] that the sonic curve intersects the upper wall at a fixed point and the mass flux is free. In physical view, it is more reasonable to prescribe the mass flux instead of the assumption that the sonic curve intersects the upper wall at a fixed point.

The problem (1.8)–(1.14) is a free boundary problem of a quasilinear degenerate elliptic equation. Furthermore, the degeneracy occurs at the free boundary, and the degeneracy is characteristic ([22]). We solve such a free boundary problem in the potential plane as in [15, 14, 13, 12, 19]. The reason is that the shape of the sonic curve is unknown in the physical plane, while known in the potential plane; furthermore, it is more convenient to estimate the speed of the flow in the potential plane than in the physical plane since it is a solution in the potential plane, while the absolute value of the gradient of a solution in the physical plane. In the potential plane, the problem (1.8)–(1.14) is a quasilinear degenerate elliptic problem with free parameters and nonlocal boundary conditions. We use the Schauder fixed point theorem to prove the existence of subsonic-sonic flows: for the given speed at the inlet and the upper wall in a suitable space, we first formulate and solve a related free boundary problem with known boundary conditions, and then show that the fixed point theorem can be applied to seek a solution to the problem (1.8)–(1.14). It is noted that the speed at the inlet is a perturbation of a constant, while the speed at the upper wall is a perturbation of a function. Furthermore, the intersecting point between the sonic curve and the upper wall is free. Hence we need more elaborate estimates than ones in [15, 14, 13, 12, 19] to get the desired solution. As to the uniqueness of the subsonic-sonic flow, in [15, 14, 13, 12, 19] one can fix the free boundaries into fixed boundaries and transform the nonlocal boundary conditions into common boundary conditions by a suitable coordinates transformation. However, there is not such a coordinates transformation because the wall is not straight and its intersecting point with the sonic curve is free. For this reason, we do energy estimates in the potential plane directly. A key step is to estimate the potential at the sonic curve and the nonlocal term at the upper wall, which is much complicated. Summing up, it is shown in this paper that for \( c_- \rho (c_0^2 - \mathcal{P}) \vartheta < m < c_+ \rho (c_0^2 - \mathcal{P}) \vartheta \), there is a unique subsonic-sonic flow to the problem (1.8)–(1.14) if \( \delta \) is suitably small. Furthermore, the speed of the flow is only \( C^{1/2} \) Hölder continuous and the acceleration blows up at the sonic curve.

The paper is arranged as follows. In § 2, we formulate the problem (1.8)–(1.14) by the speed of the flow in the potential plane, and then state the main results (existence and uniqueness) of the paper. The existence and uniqueness results are proved in § 3 and § 4, respectively.

2. Formulation in the potential plane and main results. In this section, we formulate the subsonic-sonic flow problem in the potential plane and state the main results of the paper.
2.1. Problem in the potential plane. Define a velocity potential $\varphi$ and a stream function $\psi$, respectively, by
\[
\frac{\partial \varphi}{\partial x} = q \cos \theta, \quad \frac{\partial \varphi}{\partial y} = q \sin \theta, \quad \frac{\partial \psi}{\partial x} = -\rho q \sin \theta, \quad \frac{\partial \psi}{\partial y} = \rho q \cos \theta,
\]
where $\theta$ is the flow angle. The full potential equation (1.7) can be reduced to the following Chaplygin equations ([2]):
\[
\frac{\partial \theta}{\partial \psi} + \frac{\rho(q^2) + 2q^2\rho'(q^2)}{q\rho^2(q^2)} \frac{\partial q}{\partial \varphi} = 0, \quad \frac{1}{q} \frac{\partial q}{\partial \psi} - \frac{1}{\rho(q^2)} \frac{\partial \theta}{\partial \varphi} = 0
\]
in the potential-stream coordinates $(\varphi, \psi)$. And the coordinate transformations between the two coordinate systems are valid at least in the absence of stagnation points. Eliminating $\theta$ from (2.1) yields the following second-order quasilinear equation
\[
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (2.2)
\]
where
\[
A(q) = \int_{c_s}^{q} \frac{\rho(s^2) + 2s^2\rho'(s^2)}{s\rho^2(s^2)} ds, \quad B(q) = \int_{c_s}^{q} \frac{\rho(s^2)}{s} ds, \quad 0 < q < \sqrt{2/(\gamma - 1)}.
\]
Here, $B(\cdot)$ is strictly increasing in $(0, \sqrt{2/(\gamma - 1)})$, while $A(\cdot)$ is strictly increasing in $[0, c_s]$ and strictly decreasing in $[c_s, \sqrt{2/(\gamma - 1)})$. We use $A^{-1}(\cdot)$ to denote the inverse function of $A(\cdot)|_{[0, c_s]}$ in this paper.

Assume that $(-R, 0)$ in the physical plane is transformed into the origin in the potential plane. Then $\Omega$ is transformed into $(0, \zeta) \times (0, m)$. Since $g$ satisfies (1.15), we can rewrite $\Gamma_{in}$ as
\[
x(s) = -R \cos \frac{s}{R}, \quad y(s) = R \sin \frac{s}{R}, \quad s \in [0, R \vartheta].
\]
Denote the coordinate transformation from $\{0\} \times [0, m]$ to $\Gamma_{in}$ by $S_{in}(\psi)$. Then $S_{in}(0) = 0, S_{in}(m) = R \vartheta$, and
\[
S_{in}'(\psi) = 1/(q(0, \psi)\rho(q^2(0, \psi))), \quad \psi \in [0, m].
\]
Hence
\[
\int_{0}^{m} \frac{1}{q(0, \psi)\rho(q^2(0, \psi))} d\psi = R \vartheta.
\]
It follows from the first equation in (2.1) that
\[
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = -\frac{\partial \theta}{\partial \psi}(0, \psi) = \frac{1}{R} S_{in}'(\psi) = \frac{1}{R q(0, \psi)\rho(q^2(0, \psi))}, \quad \psi \in (0, m).
\]
Assume that the speed of the flow at the upper wall is denoted by
\[
q(x, f(x)) = \mathcal{Q}_{up}(x), \quad -R \cos \vartheta \leq x \leq -R \cos \vartheta.
\]
The potential function at the upper wall can be expressed as
\[
\varphi(x, f(x)) = \Phi_{up}(x) = \int_{-R \cos \vartheta}^{x} \mathcal{Q}_{up}(s) \sqrt{1 + (f'(s))^2} ds, \quad -R \cos \vartheta \leq x \leq -R \cos \vartheta, \quad (2.3)
\]
The inverse function of $\Phi_{up}$ is denoted by
\[
X_{up}(\varphi) = \Phi_{up}^{-1}(\varphi), \quad 0 \leq \varphi \leq \zeta, \quad (2.4)
\]
where \( \zeta = \Phi_{\text{up}}(\xi, \Upsilon) \). It follows from the second equation in (2.1) that
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = \frac{\partial \theta}{\partial \varphi}(\varphi, 0) = 0, \quad \psi \in (0, m),
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = \frac{\partial \theta}{\partial \varphi}(\varphi, m) = \left( \frac{f''(x)}{1 + (f'(x))^2} \right)^{3/2} \mathcal{D}_{\text{up}}(x) \bigg|_{x=X_{\text{up}}(\varphi)}, \quad \psi \in (0, m).
\]

Therefore, in the potential plane, the subsonic-sonic flow problem (1.8)–(1.14) can be formulated as follows
\[
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \quad (2.5)
\]
\[
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = \frac{1}{\mathcal{R}q(0, \psi)\rho(q^2(0, \psi))}, \quad \psi \in (0, m), \quad (2.6)
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta), \quad (2.7)
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = \left( \frac{f''(x)}{1 + (f'(x))^2} \right)^{3/2} \mathcal{D}_{\text{up}}(x) \bigg|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (0, \zeta), \quad (2.8)
\]
\[
q(\zeta, \psi) = c_*, \quad \psi \in (0, m), \quad (2.9)
\]
\[
\int_0^m \frac{1}{q(0, \psi)\rho(q^2(0, \psi))} d\psi = \mathcal{R} \vartheta, \quad (2.10)
\]
\[
q(\varphi, m) = \mathcal{D}_{\text{up}}(X_{\text{up}}(\varphi)), \quad \varphi \in [0, \zeta], \quad (2.11)
\]

where \((q, \zeta)\) is unknown.

**Remark 2.1.** From (1.19), one gets that
\[
\frac{f''(x)}{1 + (f'(x))^2} \mathcal{D}_{\text{up}}(x) \bigg|_{x=X_{\text{up}}(\varphi)} = 0, \quad \varphi \in (\xi, \zeta),
\]
where \( \xi = \Phi_{\text{up}}(-\mathcal{P} \cos \vartheta) \).

**Remark 2.2.** It is noted that (2.11) is equivalent to
\[
\mathcal{D}_{\text{up}}(x) = q(\Phi_{\text{up}}(x), m), \quad x \in [-\mathcal{R} \cos \vartheta, -R \cos \vartheta].
\]

Solutions to the problem (2.5)–(2.9) are defined as follows.

**Definition 2.1.** For \( \zeta > 0 \) and \( m > 0 \), a function \( q \in L^\infty((0, \zeta) \times (0, m)) \) is said to be a solution to the fixed boundary problem (2.5)–(2.9), if

\[
\int_0^\zeta \int_0^m \left( A(q(\varphi, \psi)) \frac{\partial^2 \Upsilon}{\partial \psi^2}(\varphi, \psi) + B(q(\varphi, \psi)) \frac{\partial^2 \Upsilon}{\partial \varphi^2}(\varphi, \psi) \right) d\psi d\varphi
\]

\[
- \int_0^m \frac{\Upsilon(0, \psi)}{\mathcal{R}q(0, \psi)\rho(q^2(0, \psi))} d\psi
\]

\[
+ \int_0^\zeta \frac{f''(x)}{(1 + (f'(x))^2)^{3/2} \mathcal{D}_{\text{up}}(x)} \bigg|_{x=X_{\text{up}}(\varphi)} \Upsilon(\varphi, m) d\varphi = 0
\]

holds for each \( \Upsilon \in C^2([0, \zeta] \times [0, m]) \) with
\[
\frac{\partial \Upsilon}{\partial \psi} (\cdot, 0) \bigg|_{(0, \zeta)} = 0, \quad \frac{\partial \Upsilon}{\partial \varphi} (0, \cdot) \bigg|_{(0, \zeta)} = \Upsilon(\cdot, 0) \bigg|_{(0, \zeta)} = 0.
\]
2.2. Radially symmetric continuous subsonic-sonic flows. If \( f \) takes the form of (1.16), then the nozzle is straight, and there is a radially symmetric continuous subsonic-sonic flow. For \( 0 < m < c_-\rho(c_2^2)\mathcal{R}\theta \), the radially symmetric subsonic-sonic flow can be described as
\[
\hat{\varphi}(x, y) = \hat{\phi}(-\sqrt{x^2 + y^2}), \quad \hat{R} \leq \sqrt{x^2 + y^2} \leq \mathcal{R}, \quad 0 \leq y \leq -x \tan \vartheta,
\]
where \( \hat{R} = m/(c_\ast \rho(c_2^2)\theta) \), the free boundary is given by
\[
\hat{\Gamma}_{\text{out}} : x = \hat{S}(y) = -\sqrt{\hat{R}^2 - y^2}, \quad 0 \leq y \leq \hat{R} \sin \vartheta,
\]
and \( \hat{\phi} \) solves
\[
\hat{r}\hat{\phi}'(r)\rho((\hat{\phi}'(r))^2) = -\frac{m}{\vartheta}, \quad -\mathcal{R} \leq r \leq -\hat{R},
\]
\[
\hat{\phi}'(-\hat{R}) = c_\ast, \quad \hat{\phi}(-\mathcal{R}) = 0.
\]
At the inlet, the speed is \( \hat{\phi}'(-\mathcal{R}) = q_0 \), where \( q_0 \in (0, c_\ast) \) solves
\[
q_0\rho(q_0^2) = \frac{m}{\mathcal{R}\vartheta}.
\]
In the potential plane, the symmetric flow \((\hat{q}, \hat{\zeta})\) is given by
\[
\hat{q}(\varphi, \psi) = A^{-1}(A(q_0) + \frac{\vartheta\varphi}{m}), \quad (\varphi, \psi) \in [0, \hat{\zeta}] \times [0, m],
\]
\[
\hat{\zeta} = \hat{\phi}(-\mathcal{R}) = -\frac{mA(q_0)}{\vartheta} = -\mathcal{R}q_0\rho(q_0^2)A(q_0).
\]
Set \( \hat{\zeta} = \hat{\phi}(-\mathcal{P}) \).

It was shown in [15] that for \( c_-\rho(c_2^2)\mathcal{R}\theta < m < c_\ast\rho(c_2^2)\mathcal{R}\theta \), the radially symmetric subsonic-sonic flow is stable with respect to a perturbation of \( \Gamma_{\text{in}} \), where \( c_- \in (0, c_\ast) \) solves
\[
\rho(c_-^2)|A(c_-)| = 1.
\]
(2.12)

It is noted that the lower bound of the mass flux is necessary in view of the mathematical analysis since the linearized problem of the radially symmetric continuous subsonic-sonic flow is unstable if the incoming mass flux is less than or equal to this lower bound ([15]). For the radially symmetric subsonic-sonic flow, \( c_-\rho(c_2^2)\mathcal{R}\theta < m < c_\ast\rho(c_2^2)\mathcal{R}\theta \) is equivalent to \( c_- < q_0 < c_\ast \).

2.3. Main results. The main results of the paper are the existence and uniqueness of the solution to the problem (2.5)–(2.11) for \( c_-\rho(c_2^2)\mathcal{R}\theta < m < c_\ast\rho(c_2^2)\mathcal{R}\theta \) and suitable small \( \delta > 0 \).

**Theorem 2.1.** Assume that \( f \in C^3,\alpha([-\mathcal{R}\cos \vartheta, 0]) \) satisfies (1.17)–(1.20), where \( \alpha \in (0, 1) \) and \( \mathcal{P} \in (\hat{R}, \mathcal{R}) \) are constants. For two constants \( m \) and \( \beta \) satisfying
\[
c_-\rho(c_2^2)\mathcal{R}\theta < m < c_\ast\rho(c_2^2)\mathcal{R}\theta, \quad 0 < \beta < 1,
\]
(2.13)

there exists a constant \( \delta_0 > 0 \) depending only on \( \gamma, \mathcal{R}, \vartheta, \mathcal{P}, m \) and \( \beta \), such that for \( 0 < \delta \leq \delta_0 \), the problem (2.5)–(2.11) admits a unique solution \((q, \zeta)\) with \( q \in C^{2,\alpha}([0, \zeta] \times [0, m]) \cap C^{1/2}([0, \zeta] \times [0, m]) \) satisfying
\[
|\zeta - \hat{\zeta}| \leq \kappa_0\delta^\beta,
\]
(2.14)
\[
|q(0, \psi) - q_0| \leq \delta^\beta, \quad \psi \in [0, m],
\]
(2.15)
\[
|q(\zeta\phi, \psi) - q(\hat{\zeta}\phi, \psi)| \leq M\delta^\beta(1 - \phi)^{1/2}, \quad (\phi, \psi) \in (0, 1) \times (0, m),
\]
(2.16)
where $\hat{q}$, $\hat{\zeta}$, $c_-$ and $q_0$ are given in § 2.2,

$$\kappa_0 = \frac{(1 - \rho(q_0^2)\rho)\rho(q_0^2) + 2q_0^2\rho'(q_0^2))\mathcal{R}}{2\rho(q_0^2)}.$$ 

$M > 0$ is a constant depending only on $\gamma$, $\mathcal{R}$, $\vartheta$, $\mathcal{P}$, $m$ and $\beta$.

**Remark 2.3.** In Theorem 2.1, $q \in C^{1/2}([0, \zeta] \times [0, m])$ and $1/2$ is the optimal Hölder exponent. That is to say, the subsonic-sonic flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up at the sonic curve.

In the physical plane, Theorem 2.1 can be stated as follows.

**Theorem 2.2.** Assume that $g$ is given by (1.15), and $f \in C^{3, \alpha}([-\mathcal{R}\cos \vartheta, 0])$ satisfies (1.17)–(1.20), where $\alpha \in (0, 1)$ and $\mathcal{P} \in (\hat{\mathcal{R}}, \mathcal{R})$ are constants. For two constants $m$ and $\beta$ satisfying (2.13), there exists a constant $\delta_0 > 0$ depending only on $\gamma$, $\mathcal{R}$, $\vartheta$, $\mathcal{P}$, $m$ and $\beta$, such that for $0 < \delta \leq \delta_0$, the problem (1.8)–(1.14) admits a unique solution $(\varphi, S, \zeta)$ which satisfy (2.14)–(2.17) in the potential plane.

3. Proof of the existence theorem. In this section, we prove the existence result in Theorem 2.1 by using the Schauder fixed point theorem.

3.1. Well-posedness of a fixed boundary problem. In this subsection, we investigate the existence and uniqueness of the solution to the following boundary value problem

\[
\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m),
\]

(3.1)

\[
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = h_{\text{in}}(\psi), \quad \psi \in (0, m),
\]

(3.2)

\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta),
\]

(3.3)

\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = h_{\text{up}}(\varphi), \quad \varphi \in (0, \zeta),
\]

(3.4)

\[
q(\zeta, \psi) = h_{\text{out}}(\psi), \quad \psi \in (0, m),
\]

(3.5)

where $\zeta > 0$, $m > 0$, $0 \leq h_{\text{in}} \in C([0, m])$, $h_{\text{up}} \in C([0, \zeta])$, and $h_{\text{out}} \in C([0, m])$ satisfies $0 < \inf_{(0, m)} h_{\text{out}} \leq \sup_{(0, m)} h_{\text{out}} \leq c_*$.

First we state the definition of solutions and a comparison principle for the problem (3.1)–(3.5).

**Definition 3.1.** A function $q \in L^{\infty}((0, \zeta) \times (0, m))$ is said to be a supersolution (subsolution) solution to the problem (3.1)–(3.5), if $0 < \inf_{(0, \zeta) \times (0, m)} q \leq \sup_{(0, \zeta) \times (0, m)} q \leq c_*$, and

\[
\int_0^\zeta \int_0^m \left( A(q(\varphi, \psi)) \frac{\partial^2 \Upsilon}{\partial \varphi^2}(\varphi, \psi) + B(q(\varphi, \psi)) \frac{\partial^2 \Upsilon}{\partial \psi^2}(\varphi, \psi) \right) d\varphi d\psi
\]

\[
- \int_0^m h_{\text{in}}(\psi) \Upsilon(0, \psi) d\psi + \int_0^\zeta h_{\text{up}}(\varphi) \Upsilon(\varphi, m) d\varphi
\]
Proposition 3.1. Only on \( \delta > (3.5) \) if \((\ref{3.1}), \text{Proposition 3.2})\)

Lemma 3.1

Proposition 3.1. Assume that \( l_1 \leq \xi \leq l_2 < l_3 \leq \zeta \leq l_4 \), and

\[ l_5 \leq h_{in}(\psi) \leq l_6, \quad h_{out}(\psi) = c_s, \quad 0 < \psi < m, \]

\[ |h_{up}(\varphi)| \leq l_7 \delta, \quad 0 < \varphi < \zeta, \]

where \( \delta > 0 \) and \( l_i > 0 \) \((1 \leq i \leq 7)\) are constants. There exists \( \delta_1 > 0 \) depending only on \( \gamma, m \) and \( l_i \) \((1 \leq i \leq 7)\), such that if \( 0 < \delta \leq \delta_1 \), then the problem \((3.1)-(3.5)\) admits uniquely a solution. Furthermore, the solution satisfies

(i) \( q \in C^\infty((0, \zeta) \times (0, m)) \cap C((0, \zeta) \times [0, m]) \) satisfies

\[ q_-(\varphi, \psi) \leq q(\varphi, \psi) \leq q_+(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \]

where for \( (\varphi, \psi) \in [0, \zeta] \times [0, m] \), \( q_+ \) and \( q_- \) are given by

\[ q_+(\varphi, \psi) = A^{-1}\left( \inf_{(0, m)} h_{in} - \mu_1 \psi^2 \right) (\varphi - \zeta) - \mu_2 \delta (\varphi^2 - \zeta^2), \]

\[ q_-(\varphi, \psi) = A^{-1}\left( \sup_{(0, m)} h_{in} + \mu_1 \psi^2 \right) (\varphi - \zeta) + \mu_2 \delta (\varphi^2 - \zeta^2), \]

respectively, while \( 0 < \mu_1 \leq \mu_2 \) depend only on \( \gamma, m \) and \( l_i \) \((1 \leq i \leq 7)\).

(ii) There exist two constants \( \lambda \in (0, 1/2] \) and \( \mu_3 > 0 \) depending only on \( \gamma, m \) and \( l_i \) \((1 \leq i \leq 7)\) such that \( q \in C^1((0, \zeta) \times [0, m]) \), and \( \|q\|_{C^\lambda((0, \zeta) \times [0, m])} \leq \mu_3 \).

Proof. The uniqueness follows directly from Lemma 3.1. We establish the existence result by using the method of elliptic regularization. For each positive integer \( n \), consider the following regularized problem

\[ \frac{\partial^2 A(q_n)}{\partial \psi^2} + \frac{\partial^2 B(q_n)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \quad (3.7) \]

\[ \frac{\partial A(q_n)}{\partial \psi}(0, \psi) = h_{in}(\psi), \quad \psi \in (0, m), \quad (3.8) \]

\[ \frac{\partial B(q_n)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta), \quad (3.9) \]

\[ \frac{\partial B(q_n)}{\partial \psi}(\varphi, m) = h_{up}(\varphi), \quad \varphi \in (0, \zeta), \quad (3.10) \]

\[ q_n(\zeta, \psi) = \frac{nc_s}{n+1}, \quad \psi \in (0, m). \quad (3.11) \]

Note that \((3.7)\) is a uniformly elliptic equation provided that \( 0 < \tilde{c} \leq q_n \leq \bar{c} < c_s \) for some constants \( \tilde{c} \) and \( \bar{c} \). Therefore, to get the existence of the solution to
the problem (3.7)–(3.11), it suffices to construct suitable upper and sub solutions. Denote

\[ P_1(q) = \frac{B'(q)}{A'(q)} = \left( 1 - \frac{\gamma + 1}{2} q^2 \right)^{-1} \left( 1 - \frac{\gamma - 1}{2} q^2 \right)^{(\gamma + 1)/(\gamma - 1)}, \]

\[ P_2(q) = \left( \frac{B'(q)}{A'(q)} \right)' \frac{1}{A'(q)} = (\gamma + 1) q^4 \left( 1 - \frac{\gamma + 1}{2} q^2 \right)^{-1} \left( 1 - \frac{\gamma - 1}{2} q^2 \right)^{(\gamma + 2)/(\gamma - 1)}, \]

where \( q \in (0, c_\ast) \). Set

\[ C_1 = \inf_{(\zeta, c_\ast)} P_1(q), \quad C_2 = \sup_{(\zeta, c_\ast)} P_1(q)(-A(q))^{1/2}, \quad C_3 = \sup_{(\zeta, c_\ast)} P_2(q)(-A(q))^{3/2}, \]

where

\[ c = A^{-1} \left( C_2 - 2 l_4 l_6 \right). \]

For \( (\varphi, \psi) \in [0, \zeta] \times [0, m] \), define

\[ q_{n,+}(\varphi, \psi) = A^{-1} \left( \inf_{(0,m)} h_{in} - \mu_1 \delta \psi^2 \right) (\varphi - \zeta) - \mu_2 \delta (\varphi^2 - \zeta^2) + A \left( \frac{nc_\ast}{n + 1} \right), \]

\[ q_{n,-}(\varphi, \psi) = A^{-1} \left( \sup_{(0,m)} h_{in} + \mu_1 \delta \psi^2 \right) (\varphi - \zeta) + \mu_2 \delta (\varphi^2 - \zeta^2) + A \left( \frac{nc_\ast}{n + 1} \right), \]

where

\[ \mu_1 = \frac{l_7}{2 C_1 m (l_5 - l_2)}, \]

\[ \mu_2 = \sqrt{2} C_2 \mu_1 l_4^{1/2} + \frac{4 \sqrt{2} C_3 \mu_1^2 m^2 l_4^{1/2}}{l_5^{3/2}}. \]

Choose

\[ \delta_1 = \min \left\{ 1, \frac{l_5}{2 (\mu_1 m^2 + 2 \mu_2 l_4)} \right\}. \]

For \( 0 < \delta \leq \delta_1 \), it follows from the definition \( \delta_1 \) that

\[ \mu_1 \delta \psi^2 (\zeta - \varphi) + \mu_2 \delta (\zeta^2 - \varphi^2) \leq \frac{l_5}{2} (\zeta - \varphi). \]

Hence

\[ A(\zeta) \leq A(q_{n,\pm}(\varphi, \psi)) \leq -\frac{l_5}{2} (\zeta - \varphi), \quad (\varphi, \psi) \in [0, \zeta] \times [0, m]. \] (3.12)

For \( 0 < \delta \leq \delta_1 \), one gets from (3.12) and the definition of \( \mu_1, \mu_2, \delta_1 \) that

\[ \frac{\partial B(q_{n,+})}{\partial \psi} (\varphi, m) = P_1(q_{n,+}(\varphi, m)) \frac{\partial A(q_{n,+})}{\partial \psi} (\varphi, m) = 2 \mu_1 m \delta (\zeta - \varphi) P_1(q_{n,+}(\varphi, m)) \]

\[ \geq 2 C_1 \mu_1 m \delta (\zeta - \varphi) \geq h_{up}(\varphi), \quad \varphi \in (0, \zeta), \]

and

\[ \frac{\partial^2 A(q_{n,+})}{\partial \psi^2} (\varphi, \psi) + \frac{\partial^2 B(q_{n,+})}{\partial \psi^2} (\varphi, \psi) \]

\[ = \frac{\partial^2 A(q_{n,+})}{\partial \psi^2} (\varphi, \psi) + P_1(q_{n,+}(\varphi, \psi)) \frac{\partial^2 A(q_{n,+})}{\partial \psi^2} (\varphi, \psi) \]

\[ + P_2(q_{n,+}(\varphi, \psi)) \left( \frac{\partial A(q_{n,+})}{\partial \psi} (\varphi, \psi) \right)^2 \]

\[ = -2 \mu_2 \delta + 2 \mu_1 \delta (\zeta - \varphi) P_1(q_{n,+}(\varphi, \psi)) + (2 \mu_1 \delta \psi (\zeta - \varphi))^2 P_2(q_{n,+}(\varphi, \psi)) \]
Additionally, it is clear that
\[ q_1 \leq q_2 \leq \cdots \leq q_{n-1} \leq q_n, \]
and
\[ \langle 0 \rangle \leq (\zeta - \varphi)^{1/2} \]
\[ \leq 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m). \]

Additionally, it is clear that
\[ \frac{\partial A(q_{n,+})}{\partial \varphi}(0, \psi) = \inf_{(0,m)} h_{in} - \mu_1 \delta \psi^2 \leq h_{in}(\psi), \quad \psi \in (0, m), \]
\[ \frac{\partial B(q_{n,+})}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta), \]
\[ q_{n,+}(\zeta, \psi) = \frac{nc_s}{n+1}, \quad \psi \in (0, m). \]

Hence, \( q_{n,+} \) is a supersolution to the problem (3.7)–(3.11). Turn to \( q_{n,-} \). For \( 0 < \delta \leq \delta_1 \), one gets from (3.12) and the definition of \( \mu_1, \mu_2, \delta_1 \) that
\[ \frac{\partial B(q_{n,-})}{\partial \psi}(\varphi, m) = -P_1(q_{n,-}(\varphi, m)) \frac{\partial A(q_{n,-})}{\partial \psi}(\varphi, m) = -2\mu_1 m \delta (\zeta - \varphi) P_1(q_{n,-}(\varphi, m)) \]
\[ \leq -2C_1 \mu_1 m \delta (\zeta - \varphi) \leq h_{up}(\varphi), \quad \varphi \in (0, \zeta), \]
and
\[ \frac{\partial^2 A(q_{n,-})}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(q_{n,-})}{\partial \psi^2}(\varphi, \psi) \]
\[ = \frac{\partial^2 A(q_{n,-})}{\partial \varphi^2}(\varphi, \psi) + P_1(q_{n,-}(\varphi, \psi)) \frac{\partial^2 A(q_{n,-})}{\partial \psi^2}(\varphi, \psi) \]
\[ + P_2(q_{n,-}(\varphi, \psi)) \left( \frac{\partial A(q_{n,-})}{\partial \psi}(\varphi, \psi) \right)^2 \]
\[ = 2\mu_2 \delta - 2\mu_1 \delta (\zeta - \varphi) P_1(q_{n,-}(\varphi, \psi)) - (2\mu_1 \delta \psi (\zeta - \varphi))^2 P_2(q_{n,-}(\varphi, \psi)) \]
\[ \geq \frac{\partial^2 A(q_{n,-})}{\partial \varphi^2}(\varphi, \psi) + P_1(q_{n,-}(\varphi, \psi)) \frac{\partial^2 A(q_{n,-})}{\partial \psi^2}(\varphi, \psi) \]
\[ \geq 2\mu_2 \delta - \frac{2C_2 \mu_1 \delta (\zeta - \varphi)^{1/2}}{l_5^{1/2}} - \frac{4C_3 \mu_1^2 \delta^2 \psi^2}{l_5^{3/2}} (\zeta - \varphi)^{1/2} \]
\[ \geq 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m). \]

Additionally, it is clear that
\[ \frac{\partial A(q_{n,-})}{\partial \varphi}(0, \psi) = \sup_{(0,m)} h_{in} + \mu_1 \delta \psi^2 \geq h_{in}(\psi), \quad \psi \in (0, m), \]
\[ \frac{\partial B(q_{n,-})}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta), \]
\[ q_{n,-}(\zeta, \psi) = \frac{nc_s}{n+1}, \quad \psi \in (0, m). \]

Hence, \( q_{n,-} \) is a subsolution to the problem (3.7)–(3.11).

Thanks to the above argument, one gets that for \( 0 < \delta \leq \delta_1 \), the problem (3.7)–(3.11) admits a solution \( q_n \in C^\infty((0, \zeta) \times (0, m)) \cap C([0, \zeta] \times [0, m]) \) satisfying
\[ 0 < q_{n,-}(\varphi, \psi) \leq q_n(\varphi, \psi) \leq q_{n,+}(\varphi, \psi) < c_s, \quad (\varphi, \psi) \in [0, \zeta] \times [0, m]. \]

(3.13)

For \( 1 \leq n_1 \leq n_2 \), it follows from Lemma 3.1 that
\[ q_{n_1}(\varphi, \psi) \leq q_{n_2}(\varphi, \psi), \quad (\varphi, \psi) \in [0, \zeta] \times [0, m]. \]
Set
\[ q(\varphi, \psi) = \lim_{n \to +\infty} q_n(\varphi, \psi), \quad (\varphi, \psi) \in [0, \zeta] \times [0, m]. \]

It is noted that
\[ q_{\pm}(\varphi, \psi) = \lim_{n \to +\infty} q_{n, \pm}(\varphi, \psi), \quad (\varphi, \psi) \in [0, \zeta] \times [0, m]. \]  

Thanks to (3.13), (3.14) and the classical theory on elliptic equations, one can get that
\[ q \in C^\infty((0, \zeta) \times (0, m)) \cap C([0, \zeta] \times [0, m]) \] is a solution to the problem (3.1)–(3.5), and satisfies (3.6). Similarly to [15], Corollary 3.1 and Proposition 3.3, one can get that
\[ \partial B(q_{\pm}) / \partial \psi \in L^\infty((\zeta/2, \zeta) \times (0, m)), \quad |q(\varphi_1, \psi_1) - q(\varphi_2, \psi_2)| \leq \mu \left( |\varphi_1 - \varphi_2|^{1/2} + |\psi_1 - \psi_2| \right), \]
where \( \mu \) is a positive constant depending only on \( \gamma, m \) and \( l_i \) (1 \( \leq i \leq 7 \)). Then (ii) follows from (3.6), the Harnack inequality in [11] and (3.15).

### 3.2. Existence result for the free boundary problem

We prove the existence theorem by the fixed point method in this subsection. Let \( Q \in C([0, m]) \) satisfy
\[ \|Q - q_0\|_{L^\infty(0, m)} \leq \delta^\beta, \]
\[ \int_0^m \frac{1}{Q(\psi) \rho(Q^2(\psi))} d\psi = R \vartheta. \]

Extend \( f \) such that
\[ \tilde{f}(x) = \begin{cases} f(x), & -R \cos \vartheta \leq x \leq 0, \\ -x \tan \vartheta, & x \geq 0. \end{cases} \]

For a given \( \mathcal{D} \in C([-R \cos \vartheta, +\infty)) \) satisfying
\[ \frac{q_0}{2} \leq \mathcal{D}(x) \leq c_*, \quad x \geq -R \cos \vartheta, \]
set
\[ \Phi(x) = \int_{-R \cos \vartheta}^x \mathcal{D}(s) \sqrt{1 + (\tilde{f}'(s))^2} ds, \quad x \geq -R \cos \vartheta, \]
whose inverse function is denoted by
\[ X(\varphi) = \Phi^{-1}(\varphi), \quad \varphi \geq 0. \]

Choose a function \( \eta \in C^2((0, +\infty)) \) such that
\[ \eta(\varphi) = \begin{cases} 1, & 0 \leq \varphi \leq \frac{3\xi + \hat{\zeta}}{4}, \\ \frac{3\xi + \hat{\zeta}}{4} < \varphi < \frac{\xi + 3\hat{\zeta}}{4}, \\ 0, & \varphi \geq \frac{\xi + 3\hat{\zeta}}{4}, \end{cases} \]
and
\[ \|\eta\|_{L^\infty(0, +\infty)} \leq \frac{4}{\zeta - \xi}, \quad \|\eta''\|_{L^\infty(0, +\infty)} \leq \frac{16}{(\zeta - \xi)^2}. \]

Consider the following free boundary problem
\[ \frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \]
\[
\frac{\partial A(q)}{\partial \varphi}(0, \psi) = \frac{1}{RQ(\psi)\rho(Q^2(\psi))}, \quad \psi \in (0, m), \tag{3.20}
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \xi), \tag{3.21}
\]
\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = \frac{\eta(\varphi)\bar{f}''(x)}{(1 + (\bar{f}'(x))^2)^{3/2}D(x)} \bigg|_{x = \chi(\varphi)}, \quad \varphi \in (0, \xi), \tag{3.22}
\]
\[
q(\zeta, \psi) = c_s, \quad \psi \in (0, m), \tag{3.23}
\]
\[
\int_0^m q(0, \psi)\rho(q^2(0, \psi)) \, d\psi = R\vartheta, \tag{3.24}
\]
where \((q, \zeta)\) is unknown.

Define
\[
\mathcal{S} = \left\{ (Q, \mathcal{D}) \in C([0, m]) \times C([-R \cos \vartheta, +\infty)) : Q \text{ satisfies (3.16) and } \mathcal{D} \text{ satisfies (3.18)} \right\}
\]
with the norm
\[
\|(Q, \mathcal{D})\|_{\mathcal{S}} = \max \left\{ \|Q\|_{L^\infty(0, m)}, \|\mathcal{D}\|_{L^\infty(-R \cos \vartheta, +\infty)} \right\}, \quad (Q, \mathcal{D}) \in \mathcal{S},
\]
and set
\[
\hat{\mathcal{S}} = \left\{ (Q, \mathcal{D}) \in \mathcal{S} : Q \text{ satisfies (3.17)} \right\}.
\]

**Proposition 3.2.** Assume that \(f \in C^{3,\alpha}([-R \cos \vartheta, 0])\) satisfies (1.17)–(1.20), where \(\alpha \in (0, 1)\) and \(P \in (\hat{R}, \mathcal{R})\) are constants. For two constants \(m\) and \(\beta\) satisfying (2.13), there exists a constant \(\delta_2 \in (0, 1)\) depending only on \(\gamma, \mathcal{R}, \vartheta, P, m\) and \(\beta\), such that for \((Q, \mathcal{D}) \in \hat{\mathcal{S}}\) with \(0 < \delta \leq \delta_2\), the free boundary problem (3.19)–(3.24) admits a unique solution \((q, \zeta)\) such that \(q \in C([0, \xi] \times [0, m])\) and
\[
|\zeta - \hat{\zeta}| \leq \kappa_0\delta^\beta, \tag{3.25}
\]
\[
|q(0, \psi) - \hat{q}_0| \leq \delta^\beta, \quad \psi \in [0, m]. \tag{3.26}
\]
Furthermore, \(q\) satisfies
\[
|q(\zeta, \psi) - \hat{q}^{\ast}(\zeta, \psi)| \leq M\delta^\beta(1 - \phi)^{1/2}, \quad (\phi, \psi) \in [0, 1] \times [0, m], \tag{3.27}
\]
\[
|q|_{L^\infty([0, m] \times [0, m])} \leq M, \tag{3.28}
\]
where \(M > 0\) and \(\lambda \in (0, 1/2)\) are constants depending only on \(\gamma, \mathcal{R}, \vartheta, P, m\) and \(\beta\).

**Proof.** It follows from (1.20) and the definition of \(\hat{f}\) that
\[
\left\| \frac{\hat{f}''}{(1 + (\hat{f}')^2)^{3/2}D} \right\|_{L^\infty(-\mathcal{R} \cos \vartheta, +\infty)} \leq \frac{2}{q_0^2}. \tag{3.29}
\]
Set
\[
\zeta_\pm = \hat{\zeta} \pm \kappa_0\delta^\beta, \quad \delta_1 = \min \left\{ 1, \left(\frac{q_0 - c_s}{2}\right)^{1/\beta}, \left(\frac{c_s - q_0}{2}\right)^{1/\beta}, \left(\frac{\hat{\zeta} - \hat{\xi}}{8\kappa_0}\right)^{1/\beta} \right\}.
\]
Thanks to Proposition 3.1 and the definition of \(\eta\), there exists \(\delta_1 \in (0, \delta_1)\) such that for \(\zeta \in [\zeta_-, \zeta_+]\), the problem (3.19)–(3.23) with \(\zeta = \hat{\zeta}\) admits a unique solution \(\hat{q}(\zeta) \in C^\infty([0, \xi] \times [0, m]) \cap C^\lambda([0, \hat{\zeta}] \times [0, m])\) satisfying
\[
\hat{q}(\zeta, -)(\varphi, \psi) \leq \hat{q}(\zeta)(\varphi, \psi) \leq \hat{q}(\zeta, +)(\varphi, \psi), \quad (\varphi, \psi) \in [0, \hat{\zeta}] \times [0, m], \tag{3.30}
\]
\[
\|\hat{q}(\zeta)\|_{C^1([0,\zeta] \times [0,m])} \leq \mu_3,
\]  
(3.31)

where for \((\varphi, \psi) \in [0,\zeta] \times [0,m],\)

\[
\hat{q}(\zeta) = A^{-1}\left(\frac{1}{R_0 + \delta^3} R_0 (q_0 + \delta^3)^2 + \mu_1 \delta \psi^2 \right)(\varphi - \zeta) + \mu_2 \delta (\varphi^2 - \zeta^2),
\]  
(3.32)

while \(\delta_1, \lambda, \mu_1, \mu_2\) and \(\mu_3\) are positive constants depending only on \(\gamma, R, \vartheta, P, m\) and \(\beta\). It follows from (3.32) that

\[
A(\hat{q}(\zeta) + (0,\psi)) = -\left(\frac{1}{R_0 + \delta^3} R_0 (q_0 + \delta^3)^2 + \mu_1 \delta \psi^2 \right)\tilde{\zeta} + \mu_2 \delta \tilde{\zeta}^2, \quad \psi \in (0,m).
\]

It is noted that \(\tilde{\zeta} = -R_0 \rho(q_0^2) A(q_0)\). Hence

\[
\hat{q}(\zeta) = q_0 + \frac{1 + \rho(q_0^2) A(q_0)}{2} \delta^3 + O(\delta^2) + O(\delta), \quad \psi \in (0,m), \quad (\delta \to 0^+),
\]

which, together with (3.30), leads to, when \(\delta \to 0^+\),

\[
\hat{q}(\zeta (\varphi , \psi )) \leq q_0 + \frac{1 + \rho(q_0^2) A(q_0)}{2} \delta^3 + O(\delta^2) + O(\delta), \quad \psi \in (0,m),
\]  
(3.33)

\[
\hat{q}(\zeta (\varphi , \psi )) \geq q_0 - \frac{1 + \rho(q_0^2) A(q_0)}{2} \delta^3 + O(\delta^2) + O(\delta), \quad \psi \in (0,m).
\]  
(3.34)

In the proof of this proposition, \(O(\cdot)\) depends only on \(\gamma, R, \vartheta, P, m\) and \(\beta\). It is noted that

\[
\frac{1}{2} < \frac{1 + \rho(q_0^2) A(q_0)}{2} < 1.
\]  
(3.35)

Thanks to (3.33)–(3.35), there exists a constant \(\delta_2 \in (0, \delta_1)\) such that for \(0 < \delta \leq \delta_2\), it holds that

\[
\hat{q}(\zeta (\varphi , \psi )) \leq q_0 + \delta^3, \quad \hat{q}(\zeta (\varphi , \psi )) \geq q_0 - \delta^3, \quad \psi \in (0,m).
\]  
(3.36)

For \(\tilde{\zeta} \in [\zeta_-, \zeta_+]\), it follows from Lemma 3.1 and (3.36) that

\[
q_0 - \delta^3 \leq \hat{q}(\zeta (\varphi , \psi )) \leq \hat{q}(\zeta (\varphi , \psi )) \leq q_0 + \delta^3, \quad \psi \in (0,m).
\]  
(3.37)

Define

\[
F(\tilde{\zeta}) = \int_0^m A(\hat{q}(\tilde{\zeta} (\varphi , \psi ))) \frac{1}{\hat{q}(\tilde{\zeta} (\varphi , \psi ))) d\psi, \quad \tilde{\zeta} \in [\zeta_-, \zeta_+].
\]

Similarly to [[15], Lemma 3.2 and Proposition 3.5], one can show that \(F\) is a strictly increasing and continuous function on \([\zeta_-, \zeta_+]\). Integrating (3.19) with respect to \(\psi\) over \((0,m)\), one gets from (3.21)–(3.23) with \(\zeta = \zeta_\pm\) and (2.29) that

\[
\frac{d^2}{d\varphi^2} \int_0^m A(\hat{q}(\zeta_\pm (\varphi , \psi ))) d\psi = O(\delta), \quad \varphi \in (0,\zeta_\pm),
\]

\[
\frac{d}{d\varphi} \int_0^m A(\hat{q}(\zeta_\pm (\varphi , \psi ))) d\psi = \vartheta, \quad \int_0^m A(\hat{q}(\zeta_\pm (\varphi , \psi ))) d\psi = 0,
\]

which lead to

\[
\int_0^m A(\hat{q}(\zeta_\pm (\varphi , \psi ))) d\psi = -\vartheta \zeta_\pm + O(\delta) = -\vartheta (\tilde{\zeta} \pm \kappa_0 \delta^3) + O(\delta).
\]  
(3.38)
It follows from (3.37) and (3.38) that
\[
F(\zeta) - R \vartheta = \int_0^m \left( \frac{1}{\tilde{q}_{\zeta}(0, \psi)} \rho(\tilde{q}_{\zeta}^2(0, \psi)) - \frac{1}{q_0 \rho(q_0^2)} \right) d\psi
= - \int_0^m \left( \frac{1}{q_0} + O(\delta^\beta) \right) \left( A(\tilde{q}_{\zeta}(0, \psi)) - A(q_0) \right) d\psi
\]
\[
= \frac{\delta(\zeta + \kappa_0 \delta^\beta) + mA(q_0)}{q_0} + O(\delta^\beta)
= \pm \frac{\kappa_0 \delta^\beta}{q_0} + O(\delta^\beta).
\]
Therefore, there exists a constant \(\delta_2 \in (0, \delta_1)\) such that for \(0 < \delta \leq \delta_2\), it holds that
\[
F(\zeta^-) < R \vartheta < F(\zeta^+).
\]
Hence there exists \(\zeta \in (\zeta_-, \zeta_+)\) such that \(F(\zeta) = R \vartheta\). That is to say, the free boundary problem (3.19)–(3.24) admits a unique solution \((q, \zeta)\) satisfying (3.25) and (3.26). Moreover, one can get (3.27) and (3.28) from (3.30)–(3.32).

We are ready to prove the existence result in Theorem 2.1, which may be stated as follows.

**Theorem 3.1.** Assume that \(f \in C^{3,\alpha}([-R \cos \vartheta, 0])\) satisfies (1.17)–(1.20), where \(\alpha \in (0, 1)\) and \(P \in (\hat{R}, R)\) are constants. For two constants \(m\) and \(\beta\) satisfying (2.13), there exists a constant \(\delta_0 > 0\) depending only on \(\gamma, R, \vartheta, P, m,\) and \(\beta\), such that for \(0 < \delta \leq \delta_0\), the problem (2.5)–(2.11) admits at least one solution \((q, \zeta)\) with \(q \in C^{2,\alpha}([0, \zeta] \times [0, m]) \cap C^{1/2}([0, \zeta] \times [0, m])\) satisfying (2.14)–(2.17).

**Proof.** For a given \((Q, \mathcal{Q}) \in \mathcal{S}\), define \(Q\) as follows:

(i) If \(\int_0^m \frac{1}{Q(\psi) \rho(Q^2(\psi))} d\psi < R \vartheta\), then there exists a unique \(t_- \in (0, 1)\) such that
\[
\int_0^m \frac{1}{(t_- Q(\psi) + (1 - t_-)((q_0 - \delta^\beta)\rho((t_- Q(\psi) + (1 - \lambda_-)(q_0 - \delta^\beta))^2))} d\psi = R \vartheta,
\]
and we define
\[
Q(\psi) = t_- Q(\psi) + (1 - t_-)(q_0 - \delta^\beta), \quad \psi \in [0, \psi];
\]

(ii) If \(\int_0^m \frac{1}{Q(\psi) \rho(Q^2(\psi))} d\psi = R \vartheta\), then we define
\[
Q(\psi) = Q(\psi), \quad \psi \in [0, \psi];
\]

(iii) If \(\int_0^m \frac{1}{Q(\psi) \rho(Q^2(\psi))} d\psi > R \vartheta\), then there exists a unique \(t_+ \in (0, 1)\) such that
\[
\int_0^m \frac{1}{(t_+ Q(\psi) + (1 - t_+)((q_0 + \delta^\beta)\rho((t_+ Q(\psi) + (1 - t_+)(q_0 + \delta^\beta))^2))} d\psi = R \vartheta,
\]
and we define
\[
Q(\psi) = t_+ Q(\psi) + (1 - t_+)(q_0 + \delta^\beta), \quad \psi \in [0, \psi].
\]
Hence \((Q, \mathcal{Q}) \in \mathcal{S}\). Thanks to Proposition 3.2, if \(0 < \delta \leq \delta_2\), then the free boundary
problem (3.19)–(3.24) admits a unique solution \((q, \zeta)\) such that \(q \in C([0, \zeta] \times [0, m])\) and
\[
|\zeta - \tilde{\zeta}| \leq \kappa_0 \delta^\beta, \quad (3.39)
\]
\[
|q(0, \psi) - q_0| \leq \delta^\beta, \quad \psi \in [0, m], \quad (3.40)
\]
\[
|q(\zeta \phi, \psi) - q(\tilde{\zeta} \phi, \psi)| \leq M \delta^\beta (1 - \phi)^{1/2}, \quad (\phi, \psi) \in [0, 1] \times [0, m], \quad (3.41)
\]
\[
|q|_{\lambda, (0, \zeta) \times (0, m)} \leq M, \quad (3.42)
\]
where \(\delta_0 \in (0, 1), M > 0\) and \(\lambda \in (0, 1/2)\) are constants depending only on \(\gamma, \mathcal{R}, \vartheta, \mathcal{P}, m\) and \(\beta\). Set
\[
\tilde{\mathcal{Q}}(\psi) = q(0, \psi), \quad \psi \in [0, m],
\]
and
\[
\tilde{\mathcal{Q}}(x) = \begin{cases} 
q(\Phi(x), m), & \text{if } \Phi(x) \leq \zeta, \\
c_*, & \text{if } \Phi(x) > \zeta,
\end{cases} \quad x \geq -\mathcal{R} \cos \vartheta.
\]
Due to (3.39) and (3.41), there exists a constant \(\delta_0 \in (0, \delta_2)\) depending only on \(\gamma, \mathcal{R}, \vartheta, \mathcal{P}, m\) and \(\beta\), such that if \(0 < \delta \leq \delta_0\), then
\[
\frac{q_0}{2} \leq \tilde{\mathcal{Q}}(x) \leq c_*, \quad x \geq -\mathcal{R} \cos \vartheta, \quad (3.43)
\]
\[
\int_{-\mathcal{R} \cos \vartheta}^{-\vartheta \cos \vartheta} \tilde{\mathcal{Q}}(s) \sqrt{1 + (\hat{f}'(s))^2} ds \leq \frac{3c + \tilde{\zeta}}{4}. \quad (3.44)
\]
Let \(0 < \delta \leq \delta_0\). It follows from \(q \in C([0, \zeta] \times [0, m]), (3.40)\) and (3.43) that \((\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}) \in \tilde{\mathcal{S}} \subset \mathcal{S}\). Hence one can define the mapping \(J : \mathcal{S} \rightarrow \tilde{\mathcal{S}} \subset \mathcal{S}\) by \(J(\mathcal{Q}, \mathcal{Q}) = (\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}})\). Using (3.42) and the similar argument to [15, Proposition 4.7 and Lemma 4.3], one can prove that \(J\) is compact and continuous. Therefore, it follows from the Schauder fixed point theorem that \(J\) admits a fixed point \((\mathcal{Q}_*, \mathcal{Q}_*) \in \tilde{\mathcal{S}}\), and denote \((q_*, \zeta_*)\) to be the solution to the free boundary problem (3.19)–(3.24) with \(Q = \mathcal{Q}_*\) and \(\mathcal{Q} = \mathcal{Q}_*\). It follows from (3.44), (1.19) and the definition of \(\eta\) and \(X\) that
\[
\eta(\varphi) \tilde{f}''(x) \left| \begin{array}{c}
\left(1 + (\tilde{f}'(x))^2\right)^{3/2} \mathcal{D}_*(x) \\
\end{array} \right|_{x = X_*(\varphi)} = \frac{\tilde{f}''(x)}{1 + (\tilde{f}'(x))^2} \left| \begin{array}{c}
\left(1 + (\tilde{f}'(x))^2\right)^{3/2} \mathcal{D}_*(x) \\
\end{array} \right|_{x = X_*(\varphi)}, \quad \varphi \in (0, \zeta_*),
\]
\[
q_*(\varphi, m) = \mathcal{D}_*(X_*(\varphi)), \quad \varphi \in [0, \zeta_*],
\]
where \(X_*\) is the corresponding coordinate transformation for \(\mathcal{D} = \mathcal{D}_*\). Therefore, \((q_*, \zeta_*)\) is a solution to the problem (2.5)–(2.11). It is noted that (2.14)–(2.16) follow from (3.39)–(3.41). Similarly to the proof of [15, Proposition 3.4 and Corollary 3.1], one can get (2.17), which, together with (2.16) and the classical theory on elliptic equations, leads to \(q_* \in C^{2,\alpha}([0, \zeta_*] \times [0, m]) \cap C^{1/2}([0, \zeta_*] \times [0, m])\).

4. Proof of the uniqueness theorem. In this section, we prove the uniqueness result in Theorem 2.1, which may be stated as follows.

**Theorem 4.1.** Assume that \(f \in C^3([-\mathcal{R} \cos \vartheta, 0])\) satisfies (1.17)–(1.20), where \(\mathcal{P} \in (\mathcal{R}, \mathcal{R})\) is a constant. For two constants \(m\) and \(\beta\) satisfying (2.13), there exists a constant \(\delta_0 > 0\) such that for \(0 < \delta \leq \delta_0\), there exists at most one solution \((q, \zeta)\) to the problem (2.5)–(2.11) such that \(q \in C^1([0, \zeta] \times [0, m]) \cap C([0, \zeta] \times [0, m]),
\]
\[
|\zeta - \tilde{\zeta}| \leq M \delta^\beta, \quad (4.1)
\]
\[
|q(\zeta \phi, \psi) - q(\tilde{\zeta} \phi, \psi)| \leq M \delta^\beta (1 - \phi)^{1/2}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (4.2)
\]
\[
\left| \frac{\partial A(q)}{\partial \varphi}(\varphi, \psi) \right| \leq M, \quad \left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq M \delta^3(\zeta - \varphi)^{1/2}, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m),
\]

where \( M > 0 \) is a constant, and \( \delta_0 \) depends only on \( \gamma, R, \theta, P, m, \beta \) and \( M \).

**Proof.** Let \((q_1, \zeta_1)\) and \((q_2, \zeta_2)\) be two solutions to the problem (2.5)–(2.11). Denote \( \Phi_{up,k} \) and \( X_{up,k} \) to be the coordinate transformations given by (2.3) and (2.4) corresponding to \( q_k (k = 1, 2) \). Set

\[
w_k(\phi, \psi) = A(q_k(\zeta_k, \phi, \psi)), \quad (\phi, \psi) \in [0, 1] \times [0, m], \quad k = 1, 2.
\]

Then, \( w_k \in C^1([0, 1] \times [0, m]) \cap C([0, 1] \times [0, m]) \) \((k = 1, 2)\) solves the problem

\[
\begin{align*}
1 & \frac{\partial^2 w_k}{\partial \phi^2} + \frac{\partial^2 w_k}{\partial \psi^2} = 0, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (4.4) \\
\frac{\partial w_k}{\partial \phi}(0, \psi) &= \Upsilon_{in,k}, \quad \psi \in (0, m), \quad (4.5) \\
\frac{\partial K(w_k)}{\partial \psi}(\phi, 0) &= 0, \quad \phi \in (0, 1), \quad (4.6) \\
\frac{\partial K(w_k)}{\partial \psi}(\phi, m) &= \Upsilon_{up,k}(\phi), \quad \phi \in (0, 1), \quad (4.7) \\
w_k(1, \psi) &= 0, \quad \psi \in (0, m), \quad (4.8) \\
\int_0^m \frac{1}{A^{-1}(w_k(0, \psi))\rho((A^{-1}(w_k(0, \psi)))^2)} d\psi &= R_\theta, \quad (4.9)
\end{align*}
\]

where \( K(s) = B(A^{-1}(s)) \) for \( s \leq 0 \), and

\[
\begin{align*}
\Upsilon_{in,k}(\psi) &= \frac{1}{R_\zeta_k A^{-1}(w_k(0, \psi))\rho((A^{-1}(w_k(0, \psi)))^2)}, \quad \psi \in (0, m), \\
\Upsilon_{up,k}(\phi) &= -\frac{f''(X_{up,k}(\zeta_k \phi))}{(1 + (f' (X_{up,k}(\zeta_k \phi)))^2)^{3/2}A^{-1}(w_k(\phi, m))}, \quad \phi \in (0, 1).
\end{align*}
\]

Set

\[ w(\phi, \psi) = w_1(\phi, \psi) - w_2(\phi, \psi), \quad (\phi, \psi) \in [0, 1] \times [0, m]. \]

For \( k = 1, 2 \), one gets from (4.4)–(4.8) that

\[
\begin{align*}
\int_0^1 \int_0^m \frac{1}{\zeta_k^2} \frac{\partial^2 w_k}{\partial \phi^2} (\phi, \psi) \frac{\partial w_k}{\partial \psi} (\phi, \psi) d\phi d\psi + \int_0^1 \int_0^m \frac{\partial K(w_k)}{\partial \psi} (\phi, \psi) \frac{\partial w_k}{\partial \psi} (\phi, \psi) d\phi d\psi \\
= -\int_0^m \Upsilon_{in,k}(\psi) w(0, \psi) d\psi + \int_0^1 \Upsilon_{up,k}(\phi) w(\phi, m) d\phi.
\end{align*}
\]

Hence

\[
\begin{align*}
\int_0^1 \int_0^m \left( \frac{1}{\zeta_k^2} \frac{\partial w_k}{\partial \phi} \right)^2 (\phi, \psi) + K'(w_1(\phi, \psi)) \left( \frac{\partial w_1}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi \\
= \int_0^1 \int_0^m \left( (\zeta_1 - \zeta_2) \frac{\partial w_2}{\partial \phi} \right) (\phi, \psi) \frac{\partial w_2}{\partial \phi} (\phi, \psi) d\phi d\psi \\
&- \int_0^1 \int_0^m \varrho(\phi, \psi) w(\phi, \psi) \frac{\partial w}{\partial \psi} (\phi, \psi) d\phi d\psi - \int_0^m (\Upsilon_{in,1}(\psi) - \Upsilon_{in,2}(\psi)) w(0, \psi) d\psi \\
&+ \int_0^1 (\Upsilon_{up,1}(\phi) - \Upsilon_{up,2}(\phi)) w(\phi, m) d\phi,
\end{align*}
\]
where
\[ g(\phi, \psi) = \frac{\partial w_2}{\partial \psi}(\phi, \psi) \int_0^1 K''(tw_1(\phi, \psi) + (1-t)w_2(\phi, \psi))dt, \]
\[ (\phi, \psi) \in (0, 1) \times (0, m). \]

It follows from (4.11)–(4.3) that
\[ M_1(1-\phi)^{-1/2} \leq K'(w_1(\phi, \psi)) \leq M_2(1-\phi)^{-1/2}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \]
\[ |g(\phi, \psi)| \leq M_2\delta(1-\phi)^{-1/2}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \]
\[ \left| \frac{\partial w_2}{\partial \phi}(\phi, \psi) \right| \leq M_2, \quad (\phi, \psi) \in (0, 1) \times (0, m), \]
\[ |\zeta_k - \hat{\zeta}_k| \leq M_2\delta, \quad |\zeta_k - \hat{\zeta}_k| \leq M_2\delta, \quad k = 1, 2, \]
\[ |q_k(\zeta_k, \psi) - \hat{q}_k(\zeta_k, \psi)| \leq M_2\delta(1-\phi)^{1/2}, (\phi, \psi) \in (0, 1) \times (0, m), \quad k = 1, 2, \]
where \(0 < M_1 < M_2\) are constants depending only on \(\gamma, R, \vartheta, P, m, \beta\) and \(M\), and
\[ \zeta_k = \Phi_{up,k}(-P \cos \theta) \text{ for } k = 1, 2. \]

We first estimate \(\zeta_1 - \zeta_2, \Upsilon_{in,1} - \Upsilon_{in,2}\) and \(\Upsilon_{up,1} - \Upsilon_{up,2}\). Thanks to (4.14), there exists \(\delta_0 \in (0, 1)\) depending only on \(\gamma, R, \vartheta, P, m, \beta\) and \(M\), such that
\[ \hat{\zeta}/2 < \zeta_k < (\hat{\zeta} + \hat{\zeta})/2 < \zeta_k < 2\hat{\zeta}, \quad k = 1, 2. \]

For \(k = 1, 2\), one gets from (2.5)–(2.9) that
\[ \frac{d^2}{d\varphi^2} \int_0^m A(q_k(\varphi, \psi))d\psi = - \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}} \bigg|_{x=X_{up,k}(\varphi)}, \quad \varphi \in (0, \zeta_k), \]
\[ \frac{d}{d\varphi} \int_0^m A(q_k(0, \psi))d\psi = \vartheta, \quad \int_0^m A(q_k(\zeta_k, \psi))d\psi = 0. \]

Hence
\[ \frac{d}{d\varphi} \int_0^m A(q_k(\varphi, \psi))d\psi = - \arctan f'(X_{up,k}(\varphi)), \quad \varphi \in (0, \zeta_k), \quad k = 1, 2, \]
and thus
\[ \int_0^m w_k(0, \psi)d\psi = \int_0^m A(q_k(0, \psi))d\psi = \int_0^{\zeta_k} \arctan f'(X_{up,k}(\varphi))d\varphi, \]
\[ \text{where } k = 1, 2. \]

Thanks to (4.17), (4.16) and (1.19), one gets that
\[ \zeta_1 - \zeta_2 = - \frac{1}{\vartheta} \int_0^m w(0, \psi)d\psi + \frac{1}{\vartheta} \int_0^{(\hat{\zeta} + \hat{\zeta})/2} e_0(\varphi)X(\varphi)d\varphi, \]
\[ \text{where } X(\varphi) = X_{up,1}(\varphi) - X_{up,2}(\varphi), \quad \varphi \in [0, (\hat{\zeta} + \hat{\zeta})/2], \]
\[ e_0(\varphi) = \int_0^1 \frac{f''(x)}{1 + (f'(x))^2} \bigg|_{x=tX_{up,1}(\varphi)+(1-t)X_{up,2}(\varphi)} dt, \quad \varphi \in [0, (\hat{\zeta} + \hat{\zeta})/2]. \]

It follows from (1.20) that
\[ e_0(\varphi) = O(\delta), \quad \varphi \in [0, (\hat{\zeta} + \hat{\zeta})/2], \]
which, together with (4.18), leads to
\[ |\zeta_1 - \zeta_2| \leq \frac{1}{\vartheta} \left| \int_0^m w(0, \psi)d\psi \right| + O(\delta) \int_0^{(\hat{\zeta} + \hat{\zeta})/2} |X(\varphi)|d\varphi. \]
In the proof of this theorem, \( O(\cdot) \) depends only on \( \gamma, R, \theta, \mathcal{P}, m, \beta \) and \( M \). Thanks to (2.4), one gets that \( X \in C^1([0, (\xi + \zeta)/2]) \) satisfies \( X(0) = 0 \), and

\[
X'(\varphi) = e_1(\varphi)X(\varphi) + e_2(\varphi)(w_1(\varphi/\zeta_1, m) - w_2(\varphi/\zeta_2, m))
\]

\[
= e_1(\varphi)X(\varphi) + e_2(\varphi)w(\varphi/\zeta_1, m) + e_2(\varphi)e_3(\varphi)(\zeta_1 - \zeta_2),
\]

\( \varphi \in (0, (\xi + \zeta)/2), \quad (4.20) \)

where

\[
e_1(\varphi) = \frac{1}{q_1(\varphi, m)} \int_0^1 E_1(tX_{up,1}(\varphi) + (1 - t)X_{up,2}(\varphi))dt, \quad \varphi \in [0, (\xi + \zeta)/2],
\]

\[
E_1(s) = -\frac{f'(s)f''(s)}{(1 + (f'(s))^2)^{3/2}}, \quad -R \cos \vartheta \leq s \leq 0,
\]

\[
e_2(\varphi) = \frac{1}{(1 + (f'(X_{up,2}(\varphi))^2)^{1/2}} \int_0^1 E_2(tw_1(\varphi/\zeta_1, m) + (1 - t)w_2(\varphi/\zeta_2, m))dt,
\]

\( \varphi \in [0, (\xi + \zeta)/2], \)

\[
E_2(s) = \left( \frac{1}{A^{-1}(s)} \right)^\prime, \quad s \leq 0,
\]

\[
e_3(\varphi) = -\frac{\varphi}{\zeta_1\zeta_2} \int_0^1 \frac{\partial w_2}{\partial \phi} \left( \frac{t\varphi}{\zeta_1} + \frac{(1 - t)\varphi}{\zeta_2}, m \right)dt, \quad \varphi \in [0, (\xi + \zeta)/2].
\]

It follows from (1.20), (4.13) and (4.14) that

\[
e_1(\varphi) = O(\delta), \quad e_2(\varphi) = O(1), \quad e_3(\varphi) = O(1), \quad \varphi \in [0, (\xi + \zeta)/2].
\]

Hence one can get from (4.20) and \( X(0) = 0 \) that

\[
|X(\varphi)| \leq O(1) \int_0^1 |w(\phi, m)|d\phi + O(1)|\zeta_1 - \zeta_2|, \quad \varphi \in [0, (\xi + \zeta)/2]. \quad (4.21)
\]

It follows from (4.9) and (4.15) that

\[
\int_0^m e_4(\psi)w(0, \psi)d\psi = 0,
\]

where

\[
e_4(\psi) = \int_0^1 \frac{1}{tq_1(0, \psi) + (1 - t)q_2(0, \psi)}dt = \frac{1}{q_0} + O(\delta^3), \quad \psi \in [0, m].
\]

Hence

\[
\left| \int_0^m w(0, \psi)d\psi \right| \leq O(\delta^3) \int_0^m |w(0, \psi)|d\psi,
\]

which, together with (4.19) and (4.21), yields

\[
|\zeta_1 - \zeta_2| \leq O(\delta^3) \int_0^m |w(0, \psi)|d\psi + O(\delta) \int_0^1 |w(\phi, m)|d\phi. \quad (4.22)
\]

It follows from (4.14) and (4.15) that

\[
Y_{in,1}(\psi) - Y_{in,2}(\psi) = e_5(\psi)w(0, \psi) + e_6(\psi)(\zeta_1 - \zeta_2), \quad \psi \in (0, m),
\]

where

\[
e_5(\psi) = -\frac{1}{\mathcal{R}_1} \int_0^1 \frac{1}{tq_1(0, \psi) + (1 - t)q_2(0, \psi)}dt = -\frac{1}{\mathcal{R}_1\mathcal{R}_0} + O(\delta^3), \quad \psi \in (0, m),
\]

\[
e_6(\psi) = \frac{1}{\mathcal{R}_2} \int_0^1 \frac{1}{tq_2(0, \psi) + (1 - t)q_1(0, \psi)}dt = \frac{1}{\mathcal{R}_2\mathcal{R}_0} + O(\delta^3), \quad \psi \in (0, m),
\]
\[
e_6(\psi) = - \frac{1}{R\zeta_1 q_2(0, \psi) \rho(q_2^2(0, \psi))} - \frac{1}{R\zeta_2 q_0(0, \psi)} + O(\delta^3), \quad \psi \in (0, m).
\]

Hence
\[
\Upsilon_{\text{in}, 1}(\psi) - \Upsilon_{\text{in}, 2}(\psi) = \left( - \frac{1}{R\zeta_0} + O(\delta^3) \right) w(0, \psi) + O(1)(\zeta_1 - \zeta_2), \psi \in (0, m). \tag{4.23}
\]

Similarly, one can get from (1.19), (1.20), (4.14) and (4.15) that
\[
\Upsilon_{\text{up}, 1}(\phi) - \Upsilon_{\text{up}, 2}(\phi) = e_7(\phi) w(\phi, m) + e_8(\phi)(\zeta_1 - \zeta_2) + e_9(\phi) X(\zeta_1 \phi), \phi \in (0, 1), \tag{4.24}
\]

where \(e_7, e_8, e_9\) are functions satisfying
\[
e_7(\phi) = O(\delta), \quad e_8(\phi) = O(\delta), \quad e_9(\phi) = O(\delta), \quad \phi \in [0, \tau], \tag{4.25}
e_7(\phi) = e_8(\phi) = e_9(\phi) = 0, \quad \phi \in [\tau, 1], \tag{4.26}
\]

and \(\tau = \hat{\xi}/(2\hat{\zeta}) + 1/2\).

Now we estimate each term in (4.10). It follows from (4.11) and (4.14) that
\[
\int_0^1 \int_0^m \left( \frac{1}{\zeta_1} \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) + K'(w_1(\phi, \psi)) \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) \right) d\phi d\psi
\geq \left( \frac{1}{\zeta_2} + O(\delta^3) \right) \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi
+ M_1 \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi. \tag{4.27}
\]

Using (4.12), (4.13), (4.14), (4.22) and the Hölder inequality, one gets that
\[
\left| \int_0^1 \int_0^m \left( \frac{\zeta_1 + \zeta_2}{\zeta_1 \zeta_2} \right)(\zeta_1 - \zeta_2) \frac{\partial w_2}{\partial \phi} (\phi, \psi) \frac{\partial w}{\partial \phi} (\phi, \psi) d\phi d\psi \right|
\leq O(1)|\zeta_1 - \zeta_2| \int_0^1 \int_0^m \left| \frac{\partial w}{\partial \phi} (\phi, \psi) \right| d\phi d\psi
\leq \left( O(\delta^3) \int_0^m |w(0, \psi)| d\psi + O(\delta) \int_0^1 |w(\phi, m)| d\phi \right) \int_0^1 \int_0^m \left| \frac{\partial w}{\partial \phi} (\phi, \psi) \right| d\phi d\psi
\leq O(\delta^3) \left( \int_0^m w^2(0, \psi) d\psi + \int_0^1 w^2(\phi, m) d\phi + \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi \right), \tag{4.28}
\]

and
\[
\int_0^1 \int_0^m \phi(\phi, \psi) w(\phi, \psi) \frac{\partial w}{\partial \psi} (\phi, \psi) d\phi d\psi
\leq O(\delta^3) \int_0^1 \int_0^m (1 - \phi)^{-1/2} w(\phi, \psi) \frac{\partial w}{\partial \psi} (\phi, \psi) d\phi d\psi
\leq O(\delta^3) \int_0^1 \int_0^m (1 - \phi)^{-1/2} w^2(\phi, \psi) d\phi d\psi
+ O(\delta^3) \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi. \tag{4.29}
\]

It follows from (4.23), (4.22) and the Hölder inequality that
\[
- \int_0^m (\Upsilon_{\text{in}, 1}(\psi) - \Upsilon_{\text{in}, 2}(\psi)) w(0, \psi) d\psi
\]
\[
\begin{aligned}
&= \int_0^m \left( \left( \frac{1}{R \zeta q_0} + O(\delta^2) \right) w^2(0, \psi) + O(1)(\zeta_1 - \zeta_2)w(0, \psi) \right) d\psi \\
&\leq \left( \frac{1}{R \zeta q_0} + O(\delta^2) \right) \int_0^m w^2(0, \psi) d\psi + O(\delta) \int_0^1 w^2(\phi, m) d\phi. \\
\end{aligned}
\]

Similarly, one can get from (4.21)–(4.26) and the Hölder inequality that
\[
\begin{aligned}
\int_0^1 (\Upsilon_{\text{up,1}}(\phi) - \Upsilon_{\text{up,2}}(\phi)) w(\phi, m) d\phi \\
= \int_0^r (e_7(\phi) w(\phi, m) + e_8(\phi)(\zeta_1 - \zeta_2) + e_9(\phi) X(\zeta_1 \phi)) w(\phi, m) d\phi \\
\leq O(\delta) \int_0^m w^2(0, \psi) d\psi + O(\delta) \int_0^1 w^2(\phi, m) d\phi.
\end{aligned}
\]

Substituting (4.27)–(4.31) into (4.10) leads to
\[
\begin{aligned}
&\left( \frac{1}{\tilde{\zeta}^2} + O(\delta^2) \right) \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi \\
&\quad + (M_1 + O(\delta^2)) \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi \\
&\leq O(\delta^2) \int_0^1 \int_0^m (1 - \phi)^{-1/2} w^2(\phi, \psi) d\phi d\psi + \left( \frac{1}{R \zeta q_0} + O(\delta^2) \right) \int_0^m w^2(0, \psi) d\psi \\
&\quad + O(\delta^2) \int_0^1 w^2(\phi, m) d\phi.
\end{aligned}
\]

It follows from (4.8) and the Hölder inequality that
\[
\begin{aligned}
w^2(\tilde{\phi}, \psi) &= \left( \int_\phi^1 \frac{\partial w}{\partial \phi}(\phi, \psi) d\phi \right)^2 \leq (1 - \tilde{\phi}) \int_\phi^1 \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi, (\tilde{\phi}, \psi) \in [0, 1] \times [0, m].
\end{aligned}
\]

Hence
\[
\int_0^m w^2(0, \psi) d\psi \leq \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi,
\]
and
\[
\begin{aligned}
\int_0^1 \int_0^m (1 - \tilde{\phi})^{-1/2} w^2(\tilde{\phi}, \psi) d\tilde{\phi} d\psi \\
&\leq \int_0^1 \int_0^m \int_\phi^1 (1 - \tilde{\phi})^{1/2} \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\tilde{\phi} d\psi \\
&\leq \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi \int_\phi^1 (1 - \tilde{\phi})^{1/2} d\tilde{\phi} \\
&= \frac{2}{3} \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi.
\end{aligned}
\]

Using (4.34) and the Hölder inequality, one can obtain
\[
\begin{aligned}
\int_0^1 w^2(\phi, m) d\phi \\
&\leq \int_0^1 \left( \frac{1}{m} \int_0^m |w(\phi, \psi)| d\psi + \int_0^m \left| \frac{\partial w}{\partial \psi}(\phi, \psi) \right| d\psi \right)^2 d\phi
\end{aligned}
\]
\[
\leq O(1) \int_0^1 \int_0^m w^2(\phi, \psi) d\phi d\psi + O(1) \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi
\]
\[
\leq O(1) \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 d\phi d\psi + O(1) \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi. \quad (4.35)
\]
Substitute (4.33)–(4.35) into (4.32) to get
\[
\left( \frac{1}{\xi^2} - \frac{1}{R\gamma_0} + O(\delta^\beta) \right) \int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi
\]
\[+ (M_1 + O(\delta^\beta)) \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi \leq 0. \quad (4.36)
\]
Thanks to (4.36) and \( \hat{\zeta} = -R\gamma_0 \rho(\gamma_0^2) A(q_0) < R\gamma_0 \), there exists 0 < \( \delta_0 \) depending only on \( \gamma, R, \theta, \mathcal{P}, m, \beta \) and \( M \), such that
\[
\int_0^1 \int_0^m \left( \frac{\partial w}{\partial \phi} \right)^2 (\phi, \psi) d\phi d\psi + \int_0^1 \int_0^m (1 - \phi)^{-1/2} \left( \frac{\partial w}{\partial \psi} \right)^2 (\phi, \psi) d\phi d\psi \leq 0
\]
provided that 0 < \( \delta < \delta_0 \). Hence for 0 < \( \delta < \delta_0 \), it holds that
\[
w(\phi, \psi) = 0, \quad (\phi, \psi) \in [0, 1] \times [0, m],
\]
which, together with (4.22), leads to \( \zeta_1 = \zeta_2 \) and thus \( q_1 = q_2 \). That is to say, \( (q_1, \zeta_1) \) and \( (q_2, \zeta_2) \) are the same solution to the problem (2.5)–(2.11) if 0 < \( \delta \leq \delta_0 \). \( \square \)

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