CHENG EQUATION: A REVISIT THROUGH SYMMETRY ANALYSIS

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Abstract. The symmetry analysis of the Cheng Equation is performed. The Cheng Equation is reduced to a first-order equation of either Abel’s Equations, the analytic solution of which is given in terms of special functions. Moreover, for a particular symmetry the system is reduced to the Riccati Equation or to the linear nonhomogeneous equation of Euler type. Henceforth, the general solution of the Cheng Equation with the use of the Lie theory is discussed, as also the application of Lie symmetries in a generalized Cheng equation.

1. Introduction

In 1984 Cheng⁵ discussed a pair of nonlinear partial differential equations with applications in the field of photosensitive molecules, namely

\begin{align*}
    u_x &= -auv \\
    v_t &= bu_x,
\end{align*}

(1.1)

where \( u(t, x) \) and \( v(t, x) \) represent the light intensity and density of the molecules, respectively. More specifically, the first equation says that the amount of light absorbed by molecules in a film irradiated by a light beam is proportional to the product of the light intensity and density of the molecules⁵, while the second equation reads that the density of the molecules in time is proportional to the light absorbed. The constants \( a \) and \( b \) describe the absorption and proportionality constants, respectively. The importance of the latter system is that it provides the first example of exact solitary waves solutions on a failing film, where the kind of waves are different from the solitons, observed experimentally by Kapitza & Kapitza in 1949ⁱ⁰. Some later studies on the subject are⁴,¹¹,¹⁸; for a review on the solitary waves we refer the reader in³. While a detailed review on solitary waves on failing film is given in⁴.

In the past work had been done to determine the solution of the Cheng equation through Hirota’s bilinearisation¹⁹ method and Painlevé analysis²⁰. The general
travelling wave solution of (1.1) had also been discussed in [20]. In this paper we revisit this equation using Lie’s method of symmetry analysis and discuss the possible reductions of the system and their solutions. The underlying Lie algebraic structure is also discussed. The Cheng Equation reduces to Abel’s equations of the First and Second kinds, the solution of which can be given in terms of special functions and to the Riccati Equation and to a linear nonhomogeneous equation of Euler type. To the authors’ knowledge the reduction of the Cheng equation to Abel’s and linear Euler-type equation have not been discussed in the literature and hence forms our main result. It is worthwhile mentioning that we found a new form of general solution which cannot follow from the general solutions as discussed before in the literature. Lie’s approach makes the analysis more complete. The paper also discusses the case where the parameters of the Cheng Equation are space dependent.

The paper is arranged as follows. In Section 2 the Lie symmetries of (1.1) are given. In subsequent sections the symmetry analysis corresponding to various cases of the arbitrary functions in the point symmetries is discussed along with the case for which the parameters are space dependent. The Conclusion and proper references are mentioned subsequently.

2. Symmetry Calculation of the Cheng Equation

For the convenience of the reader, we give a briefly discussion in the theory of Lie point symmetries. In particular, we present the basic definitions and main steps for the determination of Lie point symmetries for a give differential equation.

Consider $H^A (t,x,u^A,u^A_i)=0$, to be a set of differential equations, where $u^A = (u,v)$ and $u^A_i = \frac{\partial u^A}{\partial y^i}$ in which $y^i = (t,x)$. Then under the action of the infinitesimal one-parameter point transformation

\begin{align}
  t' &= t(t,x,u^A;\varepsilon) \\
  x' &= x(t,x,u^A;\varepsilon) \\
  u^A_i &= u^A_i(t,x,u^A;\varepsilon)
\end{align}

in which $\varepsilon$ is an infinitesimal parameter, the set of differential equations $H^A$ is invariant iff

\begin{equation}
  L_{\Gamma} (H^A) = 0.
\end{equation}

The later expression is the definition of the Lie derivative $L$ of $H^A$ along the direction

\begin{equation}
  \Gamma = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'}{\partial \varepsilon} \partial_x + \frac{\partial u^A}{\partial \varepsilon} \partial_{u^A}.
\end{equation}

Hence, we shall say that the vector field $\Gamma$ will be a Lie point symmetry for the set of differential equations $H^A$ if and only if the following condition is true

\begin{equation}
  \mathcal{L}_\Gamma (H^A) = 0.
\end{equation}
From (2.7) a set of linear differential equations is given for the functions \( \xi^t = \frac{\partial u^t}{\partial \varepsilon} \), \( \xi^x = \frac{\partial u^x}{\partial \varepsilon} \), and \( \eta^A \frac{\partial u^A}{\partial \varepsilon} \), whose solution determine the explicit form of the Lie point symmetries.

We omit the calculations and we give that the application of the Lie symmetry condition (2.7) for the system (1.1) provides with the Lie symmetries

\[
\begin{align*}
\Gamma_1 &= -v g'(x) \partial_v + g(x) \partial_x \\
\Gamma_2 &= h(t) \partial_t - u h'(t) \partial_u,
\end{align*}
\]

(2.8)

where \( h(t) \) and \( g(x) \) are arbitrary functions. The admitted Lie algebra of system (1.1) is \( 2A_1 \).

We consider various possibilities for the arbitrary functions and conduct the reductions.

- **a.** \( h(t) \) and \( g(x) \) are constant functions, which provide the translation symmetries.
- **b.** \( h(t) \) and \( g(x) \) are the identity function, which provide the scaling symmetries.
- **c.** The general case.

In the next three sections the reduction of (1.1) with respect to various cases is discussed.

### 3. CASE I: \( h(t) \) AND \( g(x) \) ARE CONSTANT FUNCTIONS

\[
\begin{align*}
\Gamma_{1A} &= \partial_x \\
\Gamma_{2A} &= \partial_t.
\end{align*}
\]

(3.1)

\( c \Gamma_{1A} + \Gamma_{2A} \) gives a travelling-wave solution.

The similarity variables are

\[
\begin{align*}
 f &= x - ct \\
u(x, t) &= w(f) \\
v(x, t) &= k(f),
\end{align*}
\]

(3.2)

where \( f \) is the new independent variable and \( w(f) \) and \( k(f) \) are the new dependent variables. This reduces system (1.1) to

\[
\begin{align*}
w'(f) &= -ak(f)w(f) \\
k'(f) &= -bw'(f).
\end{align*}
\]

(3.3)

System (3.3) can be written as a second-order equation with respect to \( w \), namely

\[
\frac{-cw'(f)^2}{aw(f)^2} + \frac{cw''(f)}{aw(f)} = bw'(f).
\]

(3.4)

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1. We use the Mubarakzyanov Classification Scheme [13, 14, 15, 16].
The symmetries of (3.4) are
\[ \Gamma_{1B} = \partial f, \quad \Gamma_{2B} = f \partial f - w \partial w. \]
(3.5)

We consider \( \Gamma_{1B} \) for reduction. The canonical coordinates are
\[ n = w(f) \quad \text{and} \quad m(n) = \frac{1}{w'(f)}. \]

This reduces (3.4) to
\[ m'(n) = -\frac{nm(n)^2ab}{c} \frac{m(n)}{n}. \]
(3.6)
The differential invariants,
\[ n = w(f) \quad \text{and} \quad m = w'(f), \]
reduce (3.4) to
\[ m'(n) = \frac{nab}{c} + \frac{m(n)}{n}. \]
(3.7)
Equation (3.8) is a Riccati equation and equation (3.7) is a linear equation of Euler type. Next we consider \( \Gamma_{2B} \) for reduction. The canonical coordinates are
\[ n = fw(f) \quad \text{and} \quad m(n) = \frac{1}{f(fw'(f) + w(f))}. \]
These reduce (3.4) to
\[ m'(n) = \frac{(n^3 ab + cn^2)m(n)^3}{cn} + \frac{(-n^2 ab - cn)m(n)^2}{cn} - \frac{m(n)}{n}. \]
(3.8)
The differential invariants,
\[ n = fw(f) \quad \text{and} \quad m(n) = f^2 w'(f), \]
reduce (3.4) to
\[ m'(n) = \frac{m(n)(n^2 ab + m(n) + 2cn)}{cn(m(n) + n)}. \]
(3.9)
Equations (3.8) and (3.9) are Abel’s equations of first and second kinds, respectively. The solution of (3.8) is
\[ m(n) = \frac{c}{(nab + cc_0)n}, \]
(3.10)
where \( C_0 \) is an arbitrary constant. The solutions of (3.8) and (3.9) are given in terms of Lambert W functions.

We use (3.10) to derive the solutions for system (1.1). They are
\[ u(x, t) = \frac{cc_0}{-e^{-cc_0(x-t)+C_1} + ab}, \]
\[ v(x, t) = \frac{cc_0}{a(-1 + abe^{cc_0(x-t)+C_1})}. \]
(3.11)
4. Case II: \( h(t) \) and \( g(x) \) are the identity function

The Lie point symmetries are now

\[
\Gamma_{1C} = x\partial_x - v\partial_v \\
\Gamma_{2C} = t\partial_t - u\partial_u.
\]

(4.1)

The similarity variables are

\[
\begin{align*}
  f &= \frac{t}{x}, \\
  u(x,t) &= \frac{w(f)}{t} \quad \text{and} \\
  v(x,t) &= \frac{k(f)}{x},
\end{align*}
\]

(4.2)

where \( f \) is the new independent variable and \( w(f) \) and \( k(f) \) are the new dependent variables. These reduce system \([2.1]\) to

\[
\begin{align*}
  fw'(f) &= ak(f)w(f) \\
  k'(f) &= -bw'(f).
\end{align*}
\]

(4.3)

Equations (4.3) can be written as a second-order equation with respect to \( w \). It is

\[
\frac{w'(f)}{aw(f)} + \frac{fw''(f)}{aw(f)} - \frac{fw'(f)^2}{aw(f)^2} + bw'(f) = 0.
\]

(4.4)

The symmetries of (4.3) are

\[
\begin{align*}
  \Gamma_{1D} &= \partial_f \\
  \Gamma_{2D} &= -f \log(f) \partial_f + w \partial_w.
\end{align*}
\]

(4.5)

The analysis is similar to the previous case and the equation reduces to Riccati, Linear Euler and Abel’s of the first and second kind, respectively. Next we perform the analysis by another set of similarity variables.

They are

\[
\begin{align*}
  f &= \frac{t}{x}, \\
  u(x,t) &= \frac{w(f)}{x} \quad \text{and} \\
  v(x,t) &= \frac{k(f)}{x},
\end{align*}
\]

(4.6)

where \( f \) is the new independent variable and \( w(f) \) and \( k(f) \) are the new dependent variables. These reduce system \([1.1]\) to

\[
\begin{align*}
  fw'(f) + w(f) &= ak(f)w(f) \\
  k'(f) &= -b(fw'(f) + w(f)).
\end{align*}
\]

(4.7)
The system (4.7) can be written as a second-order equation in $w$, namely

\[ fw(f)w''(f) + abw(f)^3 + w(f)w'(f) + abfw(f)^2w'(f) - fw'(f)^2 = 0. \]

The symmetries of (4.8) are

\[ \Gamma_{1E} = f \partial_f - w \partial_w \quad \text{and} \quad \Gamma_{2E} = f \log(f) \partial_f - (\log(f) + 1)w \partial_w. \]

(4.9)

The canonical coordinates with respect to $\Gamma_{1E}$ are

\[ r = w(f) \quad \text{and} \quad v(r) = \frac{1}{f(fw'(f) + w(f))}. \]

(4.10)

Equation (4.8) reduces to

\[ v'(r) = v(r)^2rab - \frac{v(r)}{r}. \]

The differential invariants with respect to $\Gamma_{1E}$ are

\[ r = fw(f) \quad \text{and} \quad v(r) = f^2w'(f). \]

(4.12)

Equation (4.8) reduces to

\[ v'(r) = -rab + \frac{v(r)}{r}. \]

Equation (4.11) is a Riccati equation and equation (4.13) is an equation of Euler type. The canonical coordinates with respect to $\Gamma_{2E}$ are

\[ r = w(f)f \log(f) \quad \text{and} \quad v(r) = \frac{1}{r}. \]

Equation (4.8) reduces to

\[ v'(r) = \frac{(abr^3 - r^2)v(r)^3}{r} - \frac{(-abr^2 + r)v(r)^2}{r} - \frac{v(r)}{r}. \]

The differential invariants with respect to $\Gamma_{2E}$ are

\[ r = w(f)f \log(f) \quad \text{and} \quad v(r) = (w(f) \log(f) + w(f) + f \log(f)w'(f))f \log(f). \]

(4.15)

Equation (4.8) reduces to

\[ v'(r) = \frac{v(r)}{r} - rab + 1 + \frac{abr^2 - r}{v(r)}. \]

Equations (4.14) and (4.15) are Abel's equation of First kind and Second kind, respectively. The solution of which can be given in terms of Lambert W function.
5. The general case

The general symmetry is considered. The similarity variables are

\[ f = h_a(t) - g_a(x), \]
\[ u(x, t) = h'_a(t)w(f) \quad \text{and} \]
\[ v(x, t) = g'_a(x)k(f), \]

(5.1)

where \( f \) is the new independent variable, \( w(f) \) and \( k(f) \) are the new dependent variables and \( h_a(t) \) and \( g_a(x) \) are given as

\[ h_a(t) = \int \frac{1}{h(t)} dt \quad \text{and} \]
\[ g_a(x) = \int \frac{1}{g(x)} dx. \]

(5.2)

These reduce (1.1) to

\[ w'(f) = aw(f)k(f) \quad \text{and} \]
\[ k'(f) = -bw'(f). \]

(5.3)

This is similar to the analysis of Case I. When the similarity variables are chosen as in (4.2), the equation reduces to Abel’s Equation of the First and Second kind, Riccati and nonhomogeneous equation of Euler type. The general solution is given as

\[ u(x, t) = \frac{C_0h'_a(t)}{e^{-C_0(x-t)+C_1} + ab} \quad \text{and} \]
\[ v(x, t) = -\frac{C_0g'_a(x)}{a(-1 + abe^{C_0(x-t)+C_1})}. \]

(5.4)

Similarly, when the similarity variables are chosen as in (4.6), i.e.,

\[ f = h_a(t) - g_a(x), \]
\[ u(x, t) = gb(x)w(f) \quad \text{and} \]
\[ v(x, t) = g'_a(x)k(f), \]

(5.5)

where \( h_a(t) \) and \( g_a(x) \) are as defined in (5.2) and \( gb(x) \) is given in terms of \( h(t) \) and \( g(x) \). The results follows as in Case II.

6. Parameters are space dependent

From the first equation in equation (1.1), we have

\[ \frac{-1}{a} \frac{u_x}{u} = v \]
so that
\[
v_t = -\frac{1}{a} \left( \frac{u_x}{u} \right)_t = -\frac{1}{a} \left( \frac{u_x u_t}{u^2} \right).
\]
Finally
\[
\left( \frac{u_{xt}}{u} - \frac{u_x u_t}{u^2} \right) + \frac{b}{a} u_x = 0.
\]
The symmetry vectors are
\[
\lambda(x) \partial_x, \quad \tau(t) \partial_t - \tau'(t) u \partial_u.
\]
If \(a = a(x)\) and \(b = b(x)\), then
\[
(6.1)
\left( \frac{u_{xt}}{u} - \frac{u_x u_t}{u^2} \right) + c(x) u_x = 0
\]
where \(c(x) = \frac{b}{a}\) and the symmetry vectors are
\[
c(x) \partial_x - c'(x) u \partial_u, \quad \tau(t) \partial_t - \tau'(t) u \partial_u.
\]
The similarity variable with respect to the first symmetry vector is
\[
(6.2)
u(x, t) = s(x) = c(x) x,
\]
where \(c(x) = \frac{b}{a}\) and the symmetry vectors are
\[
\lambda(x) \partial_x, \quad \tau(t) \partial_t - \tau'(t) u \partial_u.
\]
The similarity variable with respect to the first symmetry vector is
\[
(6.3)
\frac{c'(x)}{s'(x)} - \frac{c(x)}{s(x)} = 0.
\]
The linear combination of the two symmetry vectors is
\[
c(x) \partial_x - c'(x) u \partial_u + C_2 \left( \tau(t) \partial_t - \tau'(t) u \partial_u \right)
\]
where \(C_2\) is an arbitrary constant. The similarity variables are
\[
(6.4)
f_a = c_a(x) - \tau_a(t) \quad \text{and} \quad u(x, t) = w_a(f_a),
\]
where \(f_a\) is the new independent variable, \(w_a(f_a)\) is the new dependent variables and \(c_a(x)\) and \(\tau_a(t)\) are given as
\[
(6.5)
c_a(x) = \int \frac{1}{c(x)} dx \quad \text{and} \quad \tau_a(t) = \int \frac{1}{\tau(t)} dt.
\]
For \(c_a(x) = x\) and \(h_a(t) = t\), the equation \((6.1)\) reduces to
\[
(6.6)
\frac{w''_a(f_a)}{w_a(f_a)} = \frac{w'_a(f_a)}{w_a(f_a)^2} + w'_a(f_a).
\]
The Lie symmetries are

\[
\Gamma_a = \partial f_a, \\
\Gamma_b = f_a \partial f_a - w_a \partial w_a.
\]  

(6.7)

The equation (6.6) behaves similarly to (3.4). The symmetry \(\Gamma_a\) reduces (6.6) to the Riccati equation and \(\Gamma_b\) reduces (6.6) to Abel’s Equation.

7. CONCLUSION

In this work we applied the theory of invariant transformations to the study and the determination of analytical solutions for the Cheng equation. We found that the similarity solutions are expressed in terms of solutions of the Abel equations, that is, Cheng equation, when it is reduced from a second-order partial differential equation to a first-order ordinary differential equation with the use of point transformations, is equivalent with the Abel equation.

However, we found that the symmetry vectors depend upon arbitrary functions and we have performed our analysis for for various choices of similarity variable. The results of the various cases are similar, as for all the cases the system is reduced to either an Abel’s or a Riccati equation. The paper also made a note of the Lie algebra. Finally, the case when the parameters are space dependent is also discussed. The general solution of the later equation have been presented by the Lie symmetry approach.

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