An algebraic theory of non-relativistic spin

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Abstract

In this paper we present a new, elementary derivation of non-relativistic spin using exclusively real algebraic methods. To do this, we formulate a novel method to decompose the domain of a real endomorphism according to its algebraic properties. We reveal non-commutative multipole tensors as the primary physically meaningful observables of spin, and indicate that spin is fundamentally geometric in nature. In so doing, we demonstrate that neither dynamics nor complex numbers are essential to the fundamental description of spin.

1. Introduction

1.1. Current mathematical description of spin

The usual description of a spin-$s$ system is as a finite-dimensional, irreducible representation $(V^{(s)}, \rho^{(s)})$ of the real Lie algebra $\mathfrak{su}(2, \mathbb{C})$,

$$ \mathfrak{su}(2, \mathbb{C}) := \text{span}_\mathbb{R} \{ S_1, S_2, S_3 \} $$

(1.1)

of the Lie group $SU(2, \mathbb{C})$, with Lie product,

$$ S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c $$

(1.2)

where,

$$ \rho^{(s)} : \mathfrak{su}(2, \mathbb{C}) \rightarrow \text{End}(V^{(s)}) $$

(1.3)

and maps the Lie product to a commutator, $\forall a, b \in \mathfrak{su}(2, \mathbb{C})$,

$$ \rho^{(s)}(a \times b) = \rho^{(s)}(a) \circ \rho^{(s)}(b) - \rho^{(s)}(b) \circ \rho^{(s)}(a) $$

(1.4)

The traditional approach [1, 2] to find these representations considers instead the complexification $\mathfrak{su}(2, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, and uses the ladder operator basis,

$$ \rho^{(s)}(S_a) := -i(\rho^{(s)}(S_a) \pm i\rho^{(s)}(S_0)) $$

(1.5a)

$$ \rho^{(s)}(H) := -i(\rho^{(s)}(S_0)) $$

(1.5b)

to explore the representation’s root system. All finite-dimensional, irreducible representations are found this way, and are labelled in relation to the value the Casimir operator,

$$ S^2_{\rho^{(s)}} := \sum_{a=1}^{3} \rho^{(s)}(S_a) \circ \rho^{(s)}(S_a) = -s(s + 1) \text{id}, $$

(1.6)

takes in that representation.

The operators (1.5a) and (1.5b) have a physical interpretation through their action on the spinor space $V^{(s)}$: considering spin as a form of angular momentum, the (1.5a) increase/decrease the amount of alignment or anti-alignment the spin angular momentum has with the $z$-spatial-direction, which (1.5b) reveals. $S^2_{\rho^{(s)}}$ is then said to represent (the negative of) the total spin angular momentum of the representation.
1.2. The case for an elementary study of spin

Despite the success of the traditional methods of representation theory, they yield minimal insight into the fundamental physical properties of the systems the representations describe.

Firstly, it tells us very little about the meaningful observables of the representation. The ladder operators are not observable, so the only observables the formalism highlights are \( \rho^\alpha(S^\alpha) \) and \( \rho^\alpha(S^\beta) \). Since in this picture these two observables are sufficient to characterise each representation, one might consider these to be the only relevant ones. However, it is known that there are higher-order observables within the representation, such as the quadrupole and higher-order moments [3]. As these play no role in this formalism, it is not clear if their existence is significant.

Secondly, there are issues surrounding the standard interpretation of the operators (1.5a), (1.5b), and (1.6). It is certainly a valid one in the complexified theory, but cannot be considered so for the original real Lie algebra \( \mathfrak{su}(2, \mathbb{C}) \), since (1.5a) nor (1.5b) can be defined there. This raises questions regarding the role of representations in the interpretation of physics, and the meaning of the complex numbers brought into the formalism.

Finally, the physical significance of the group \( \text{SU}(2, \mathbb{C}) \) is also unclear. Some insight can be gained by recognising that \( \text{SU}(2, \mathbb{C}) \) is the double cover (and in this case also universal cover) of the homogeneous symmetry group of Euclidean three-space \( \text{SO}(3, \mathbb{R}) \). Unlike \( \text{SU}(2, \mathbb{C}), \text{SO}(3, \mathbb{R}) \) has a direct physical interpretation as the group of rotations. Furthermore, \( \text{su}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{R}) \) as Lie algebras. This connection between spin and geometric symmetry is not just mathematical, it is demonstrably non-trivial: fermionic systems require a \( 4\pi \) rotation to return to their original state.

These observations lead us to motivate a more physically grounded and mathematically elementary study of spin, working exclusively with the real structures associated with the rotation group \( \text{SO}(3, \mathbb{R}) \). As such, notions of dynamics shall be avoided. It is worth highlighting that the rotations of \( \text{SO}(3, \mathbb{R}) \) are not fundamentally dynamical, i.e. gradual transformations of a physical space over time; they are atemporal maps encoding a relationship between two configurations of the physical space. Since they preserve the Euclidean metric between any two elements of the space, they are fundamentally geometric in character. Therefore, studying spin through the rotation group alone will directly connect it to Euclidean geometry; we will expound this relationship in a future paper.

Our decision to avoid introducing complex numbers presents an immediate challenge: without their algebraic closure, we have no guarantee of eigenvalues for our operators. This makes the usual root system analysis inaccessible to us. The solution to this problem can be seen by noting that the map \( \rho^\alpha \) from the usual representation theory is mapping elements from an algebra in which only the Jacobi identity is satisfied, to elements from an associative algebra. Since, associativity is a stronger condition than satisfying the Jacobi identity, the map \( \rho^\alpha \) is effectively introducing new structure to turn the Lie algebra into an associative algebra.

Combining these observations with the presence of the identity (1.6), we are compelled to seek unital associative algebras \( A^\alpha \) which factor the map \( \rho^\alpha \),

\[
\rho^\alpha = \mu^\alpha \circ \alpha^\alpha
\]

where \( \mu \) is the product of \( A^\alpha \). This splits the problem of representation theory in half: first one finds all unital associative algebras which embed the structure of the Lie algebra, then, if desired, one finds associative algebra representations of these. The initial step in this splitting can be considered an algebraic approach to the representation theory of Lie algebras, and shall be the focus of this paper. We will show that such an approach yields more meaningful descriptions of all spins entirely in terms of their physical observables. From this point, the maps \( \alpha^\alpha \) we will be developing will not be written explicitly for the sake of readability.

1.3. Algebraic theories in physics

Before proceeding with this programme, let us consider the status of algebras and algebraic methods in physics more generally.

The pursuit of algebraic theories in physics was advocated by Einstein [4] as a means to more naturally describe quanta than a continuum theory. Though rarely used exclusively, many aspects of algebraic approaches have been behind several seminal results in physics. For example, Dirac’s standard bra/ket [5] enabled him to construct quantum theory almost entirely in terms of operators. Furthermore, his derivation of the Dirac equation from the Klein–Gordon equation necessitated the definition of an algebra between the \( \alpha \) and \( \beta \). The emergence of spin in that setting was not a result of relativity however, as Lévy-Leblond derived the Pauli equation from the Schrödinger equation [6, 7] by similar means. This, again, required the acceptance of an
algebraic structure inherent to the system he was describing. Though somewhat disguised by the analysis, Von Neumann defined an idempotent element with respect to the position-momentum algebra in his proof of the Stone-Von Neumann theorem [8].

Algebraic methods have wide application within physics: Gilmore et al [9] use algebraic methods to study the quantum defect; Iachello [10] presents a variety of algebraic techniques with a wide array of applications; Furey et al [11, 12] utilise non-associative algebraic structures to study the standard model of particle physics; and Hestenes [13], Doran and Lasenby [14], and Hiley and Callaghan [15, 16] have used Clifford algebras to study the Schrödinger, Pauli, and Dirac equations, indicating that more extensive algebraic study of even quantum theory is possible.

The study of spin in physics is traditionally approached analytically, and within the context of a larger theory: Weinberg’s study of relativistic spin [17] is performed using the language of quantum field theory; as is the work of Giraud et al [18] it inspired. Field theory also underpins the modern study of higher spins in higher-dimensional Minkowski space [19, 20]. Despite this, describing systems with spin using algebraic methods is not uncommon: Racah’s spherical tensor operator formalism [21] is perhaps best known; Zemach [22] presents methods which use spin-tensorial objects; and Giraud et al [18, 23] utilise similar, but distinct, objects to describe density operators. In all of these cases and more broadly, the theory under consideration almost always has symmetries and structures that are not intrinsic to the real symmetry that underpins spin. Such theories then cannot form the basis for the elementary analysis we have proposed.

Instead, let us consider some existing models with less additional structure that more closely relate geometry and spin. An important example mentioned earlier is the Clifford/Geometric Algebra [14] for Euclidean three-space, within which it is possible to model a spin $-\frac{1}{2}$ system using objects with definite geometric character [24, 25]. Subspaces of tensor products of this algebra [26] also underpin the definition of many classic higher-spin models [27–29]. This approach necessarily imparts a substructure to the resulting algebraic description due to the underlying vector spaces and Clifford products of the constituent algebras. This substructure is of indefinite physical character within a model for fundamental spin; as such, we shall pursue an alternative model. Beyond Clifford algebras, there are models which seek to avoid the ‘internal space’ common to models of spin: Savasta et al [30, 31] associate spin not with Lorentz symmetry but the $SO(4, \mathbb{R})$ symmetry present between three-velocities and rates of change of proper time in Minkowski space; Kaparulin et al [32] relate spin to geometric qualities of a particle worldline, also in Minkowski space; and Bühler [33] connects spin to polarisations of a wave in Euclidean three-space via the Lie group $SL(2, \mathbb{R})$. However, none of these models align with our stated goals of directly utilising only the $SO(3, \mathbb{R})$ symmetry of real Euclidean three-space. Finally, Colatto et al [34] present the closest model to our aims, connecting spin with the non-commutative geometry of position operators in Euclidean three-space; this possibility will be explored in relation to the present work in a future paper.

Besides these models, there are some algebraic theories from which we can draw inspiration. There exist real algebras, independent of quantum mechanics, which embed the structure of the real $\mathfrak{so}(3, \mathbb{R})$ and capture the properties of spin $-\frac{1}{2}$ and spin $-1$. These are the Clifford [14],

$$S_a S_b + S_b S_a + \frac{1}{2} \delta_{ab} = 0,$$

and Kemmer [35, 36],

$$S_a S_b S_c + S_c S_b S_a + \delta_{ab} S_c + \delta_{bc} S_a = 0,$$

algebras respectively. Together with (1.4) and (1.6) these identities completely specify the properties of spin $-\frac{1}{2}$ and spin $-1$, including the eigenspectrum of the generators $S_a$. These algebras demonstrate that such real algebraic descriptions for at least some spins are possible. However, it is unclear how to generalise them to arbitrary spin, and still does not address the question of meaningful observables. What they do indicate is that algebras of generators $S_a$ are robust enough to capture the essential physics of our systems. We shall therefore focus our attention on constructing such algebras.

The tools and procedures required to do this will be the focus of the remainder of this paper. In section 2, we will outline how to go about isolating the necessary structures to define real algebras to describe every spin. This will include decomposing a general algebra of spin generators into irreducible ‘multipole’ tensors, which form a complete set of physically distinct observables for spin systems. In section 3, we will present these algebras and examine their structure to justify the claim that they indeed describe all spins. In so doing, we will find that each spin is completely determined by its largest non-zero multipole tensor. In section 4, we will contrast these new algebras with many of the existing formalisms to describe arbitrary spins, and discuss the implications of this work on the interpretation and origin of spin in quantum mechanics.
2. Method

2.1. General algebras of generators
Let us now begin our analysis, starting from the Lie group of rotational symmetry \( \text{SO}(3, \mathbb{R}) \). It is well known that \( \text{SO}(3, \mathbb{R}) \) is generated by the Lie algebra \( \mathfrak{so}(3, \mathbb{R}) \).

**Definition 2.1 (\( \mathfrak{so}(3, \mathbb{R}) \)).** \( \mathfrak{so}(3, \mathbb{R}) \) is a real three-dimensional Lie algebra,
\[
\mathfrak{so}(3, \mathbb{R}) := \text{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}),
\]

with Lie product,
\[
S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c
\]

which is alternating, bilinear, and Jacobi.

The Lie product of \( \mathfrak{so}(3, \mathbb{R}) \) is isomorphic to the cross-product, so we adopt this notation to avoid confusion with commutators.

**Remark.** Care should be taken when using the results of this paper: all shall be given in terms of the basis (2.1) and product (2.2) above, which differs from the usual physics generators \( \hat{S}_a \) by,
\[
\hat{S}_a = iS_a.
\]

Since we wish to find unital associative algebras of the generators \( S_a \), consider the most general such algebra:

**Definition 2.2 (\( T(\mathfrak{so}(3, \mathbb{R})) \)).** The tensor algebra \([37]\) \( T(\mathfrak{so}(3, \mathbb{R})) \) of \( \mathfrak{so}(3, \mathbb{R}) \),
\[
T(\mathfrak{so}(3, \mathbb{R})) \cong \mathbb{R} \oplus \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R})^{\oplus 2} \oplus \mathfrak{so}(3, \mathbb{R})^{\oplus 3} \oplus ..., \]

is a unital, associative algebra with product \( \otimes \) and multiplicative identity 1.

The elements of this algebra are called ‘tensors’, and are all \( \mathbb{R} \)-linear combinations of ‘\( k \)-adic’ tensors,

**Definition 2.3 (\( k \)-adic tensors).** An element \( A \in T(\mathfrak{so}(3, \mathbb{R})) \) is \( k \)-adic for some \( k \in \mathbb{N} \) when,
\[
A = \otimes_{j=1}^{n} v_j = v_1 \otimes ... \otimes v_n, \ v_j \in \mathfrak{so}(3, \mathbb{R}) \quad k \in \mathbb{Z}^+.
\]

**Definition 2.4 (Tensor order).** We define the ‘tensor order’ of a \( k \)-adic to be \( k \). We extend tensor order to arbitrary linear combinations of \( k \)-adics by taking the largest tensor order amongst the terms.

\( T(\mathfrak{so}(3, \mathbb{R})) \) is infinite-dimensional and, by definition, (1.8) does not hold on it. We may impose the identity (1.8) by constructing the universal enveloping algebra \([38]\) \( U(\mathfrak{so}(3, \mathbb{R})) \) of \( \mathfrak{so}(3, \mathbb{R}) \).

**Definition 2.5 (\( U(\mathfrak{so}(3, \mathbb{R})) \)).** The universal enveloping algebra \( U(\mathfrak{so}(3, \mathbb{R})) \) is the quotient algebra,
\[
U(\mathfrak{so}(3, \mathbb{R})) \cong \frac{T(\mathfrak{so}(3, \mathbb{R}))}{I(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b)},
\]

where the \( I(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b) \) is the ideal generated by elements of the form in its argument.

The quotient in (2.5) embeds the Lie product (2.2) into the commutator of the algebra, as in (1.8). Like \( T(\mathfrak{so}(3, \mathbb{R})) \), \( U(\mathfrak{so}(3, \mathbb{R})) \) is infinite-dimensional, and its elements can be written as linear combinations of \( k \)-adic tensors.

**Remark.** Here we have abused notation by denoting both the products of \( T(\mathfrak{so}(3, \mathbb{R})) \) and \( U(\mathfrak{so}(3, \mathbb{R})) \) by \( \otimes \), but which product is meant at any given time will be clear from context.

2.2. Considerations for the decomposition of \( U(\mathfrak{so}(3, \mathbb{R})) \)
By construction, \( U(\mathfrak{so}(3, \mathbb{R})) \) is the most general unital associative algebra of the elements of \( \mathfrak{so}(3, \mathbb{R}) \) on which (1.8) holds. Thus, all other algebras sharing these properties must derive from it. All representations for particle spin are finite-dimensional, so it is clear that \( U(\mathfrak{so}(3, \mathbb{R})) \) is too general to correspond to any of them. If we consider the known spin representations as faithful associative algebra representations, we imply the existence of
finite-dimensional associative algebras with the same structure. But since these algebras must derive from $U(\mathfrak{so}(3, \mathbb{R}))$, the question is how to construct them?

**Definition 2.6.** $\forall k \in \mathbb{N},$

$$\mathfrak{so}(3, \mathbb{R})^{\otimes k} := \begin{cases} \mathfrak{so}(3, \mathbb{R}) \otimes \mathfrak{so}(3, \mathbb{R})^{\otimes k-1} & k \neq 0 \\ \mathbb{R} & k = 0 \end{cases}$$

In the case of $T(\mathfrak{so}(3, \mathbb{R}))$, we may find finite-dimensional algebras by quotienting all tensors above a certain tensor order $k[37]$, $T(\mathfrak{so}(3, \mathbb{R}))$

$$\frac{I(\mathfrak{so}(3, \mathbb{R}))}{I(\mathfrak{so}(3, \mathbb{R})^{\otimes(k+1)})}$$

where $I(\mathfrak{so}(3, \mathbb{R})^{\otimes(k+1)})$ is the ideal generated [37] by the order $-(k+1)$ tensors. However, we cannot expect this method to produce meaningful algebras from $U(\mathfrak{so}(3, \mathbb{R}))$, since in $U(\mathfrak{so}(3, \mathbb{R}))$ tensor order is not well-defined. This can be seen from the defining identity of the algebra, $\forall a, b \in \mathfrak{so}(3, \mathbb{R}),$

$$\mathfrak{so}(3, \mathbb{R})^{\otimes 2} \ni a \otimes b - b \otimes a = a \times b \in \mathfrak{so}(3, \mathbb{R}).$$

To progress, we must find a scheme for constructing ideals which depends on some well-defined property of the objects of $U(\mathfrak{so}(3, \mathbb{R})).$

Towards this, let us define the ‘adjoint action’ [1],

**Definition 2.7.** (Adjoint action). The adjoint action, $\text{ad} : U(\mathfrak{so}(3, \mathbb{R})) \to \text{End}(U(\mathfrak{so}(3, \mathbb{R}))),$

$$\text{ad}(u) = v \mapsto \begin{cases} uv & u \in \mathbb{R} \\ u \otimes v - v \otimes u & u \in \mathfrak{so}(3, \mathbb{R}) \\ \text{ad}(a) \circ \text{ad}(b)(v) & u = a \otimes b, \end{cases}$$

where $\text{End}(U(\mathfrak{so}(3, \mathbb{R})))$ is the space of all linear operators on $U(\mathfrak{so}(3, \mathbb{R})).$

It is easily verified that both $\text{ad}$ and $\text{ad}(A), \forall A \in U(\mathfrak{so}(3, \mathbb{R})),$ are well-defined on $U(\mathfrak{so}(3, \mathbb{R}))$, in that their outputs $\text{ad}(B)$ and $\text{ad}(A)(B)$ do not depend on our choice of representative for their inputs $B \in U(\mathfrak{so}(3, \mathbb{R})).$

**Lemma 2.8.** $\forall a, b \in \mathfrak{so}(3, \mathbb{R}), \text{ad}(a)(b) = a \times b.$

**Proof.** $\text{ad}(a)(b) = a \otimes b - b \otimes a = a \times b$ by the definition of $U(\mathfrak{so}(3, \mathbb{R})).$ \hfill \qed

**Lemma 2.9.** $\forall a \in \mathfrak{so}(3, \mathbb{R}), \text{ad}(a)$ is a derivation on $U(\mathfrak{so}(3, \mathbb{R})).$

**Proof.** Consider $A \otimes B \in U(\mathfrak{so}(3, \mathbb{R})).$ Then,

$$\text{ad}(a)(A \otimes B) = a \otimes A \otimes B - A \otimes B \otimes a$$

$$= (a \otimes A - A \otimes a) \otimes B + A \otimes (a \otimes B - B \otimes a)$$

$$= (\text{ad}(a)(A)) \otimes B + A \otimes (\text{ad}(a)(B)).$$ \hfill \qed

Taking lemmas 2.8 and 2.9, the action of $\text{ad}(A)$ is ultimately determined by the structure of the Lie product $\times$ of $\mathfrak{so}(3, \mathbb{R}).$ In view of these properties, the adjoint action will allow us to probe structures within $U(\mathfrak{so}(3, \mathbb{R}))$ which are inherently compatible with its algebraic structure.

To develop this further, let us define some terms for recurring structures in this paper,

**Definition 2.10 (Orthogonal).** Let $T, U \subset V$ be subspaces of a vector space $V$, and $A \in \text{End}(V).$ Then $T$ and $U$ are $A$-orthogonal iff $T \cap U = \{0\}$ and both $T$ and $U$ are closed under the action of $A.$ Additionally, $T$ and $U$ are $\{A_j\}$-orthogonal iff $\forall j$ they are $A_j$-orthogonal. When the operator or family of operators is clear from context, we will just refer to the subspaces as orthogonal.

**Definition 2.11 (Orthogonal Decomposition).** A family of mutually $A$-orthogonal vector spaces $\{B_j \subset C\}$, with $A \in \text{End}(C)$, is an $A$-orthogonal decomposition for $C$ iff

$$C \cong \bigoplus_{j=0}^{\infty} B_j,$$ (2.9)
This extends in the obvious way for mutually $\{A_j\}$-orthogonal spaces. Again, when the operator(s) are clear from context we will refer simply to orthogonal decompositions.

The utility of orthogonal decomposition is best seen in the context of an algebra. By construction, all representatives for each element of $C$ are contained in the same component $B_k$ for some $k$. This illustrates that quotienting $C$ by any of these components yields a new algebra which inherits both the structure of $C$, and additional structure based on the consequences of the quotient.

This observation offers us a route forward towards constructing algebraic descriptions for all spins: we must use the adjoint action to find an appropriate orthogonal decomposition of $U(\mathfrak{so}(3, \mathbb{R}))$. We will pursue one which is ad($U(\mathfrak{so}(3, \mathbb{R}))$)-orthogonal. We may simplify this requirement by noting that to be closed under $\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))$, is sufficient for a subspace to be closed under $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$. From definition (2.8), it is clear that $\forall k, \mathfrak{so}(3, \mathbb{R})^{\oplus k} \subseteq U(\mathfrak{so}(3, \mathbb{R}))$ is closed under $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$. Thus, we may focus our efforts on decomposing each $\mathfrak{so}(3, \mathbb{R})^{\oplus k}$.

If we were to use a matrix representation for $\mathfrak{so}(3, \mathbb{R})^{\oplus k}$ as an $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$-module [39], we could do this by changing to a basis where $\text{ad}(S_{\alpha})$ were in block diagonal form $\forall S_{\alpha}$. This would become exponentially difficult however as $\dim(\mathfrak{so}(3, \mathbb{R})^{\oplus k}) = 3^k$. What this does indicate however is that the action of $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$ contains all the information required to isolate these closed subspaces. If we had extended scalars to $\mathbb{C}$, we could make some progress simply finding each eigenspace of a given $\text{ad}(S_\alpha)$, as these must be orthogonal. The core of these ideas is using the properties of $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$ to understand the structure of the space on which we are acting, i.e. $\mathfrak{so}(3, \mathbb{C})^{\oplus 2}$; as we shall see, this information is in fact available to us independent of basis and without the need to complexify.

### 2.3. Natural decomposition of a vector space by a real operator

In this section, we shall develop a novel, basis-independent, algebraic method of decomposing a vector space according to the properties of one of its real operators. Let $V$ be a vector space over $\mathbb{F}$ and $A \in \text{End}(V)$.

**Definition 2.12.** (Annihilating Polynomial). An annihilating polynomial [40] of $A$ is a polynomial $n(x)$ with coefficients in $\mathbb{F}$ such that,

$$n(A) = 0,$$  \hspace{1cm} (2.10)

on $V$, where 0 is the zero map on $V$.

**Definition 2.13.** (Minimal Polynomial). The minimal polynomial of $A$ is the annihilating polynomial of least order.

**Lemma 2.14.** The minimal polynomial of $A$ unique up to scalar multiple and always exists when $\dim(V) \in \mathbb{N}$.  

**Proof.** See [40].

**Lemma 2.15.** All annihilating polynomials $n(x)$ are of the form $n(x) = 1(x)m(x)$, where $m(x)$ is the minimal polynomial.

**Proof.** See [40].

Let us factorise the unique monic minimal polynomial $m(x)$ into a product of non-zero powers of irreducible monic polynomials over $\mathbb{F}$,

$$m(x) = \prod_{j=1}^{n} f_j^{d_j}(x),$$  \hspace{1cm} (2.11)

where $\forall j_1 \neq j_2, \gcd(f_{j_1}(x), f_{j_2}(x)) = 1$. As we are working with a monic polynomial, none of these factors are constant. Let us choose one value of $j = k$ and write,

$$m(x) = p_k(x)q_k(x)$$  \hspace{1cm} (2.12a)

$$p_k(x) = f_k^{d_k}(x)$$  \hspace{1cm} (2.12b)

$$q_k(x) = \prod_{j=1, j \neq k} f_j^{d_j}(x).$$  \hspace{1cm} (2.12c)
Lemma 2.16. (Bézout’s Identity). Bézout’s Identity [41] states that for any two polynomials \( p(x) \) and \( q(x) \) there exist polynomials \( a(x) \) and \( b(x) \) such that,
\[
    a(x)p(x) + b(x)q(x) = \gcd(p(x), q(x)).
\]

Proof. See [42].

Applying Lemma 2.16 in our case, there exist polynomials \( a_k(x), b_k(x) \) such that,
\[
    a_k(x)p_k(x) + b_k(x)q_k(x) = \gcd(p_k(x), q_k(x)) = 1,
\]
(2.13)
where the final equality follows by construction.

Lemma 2.17. We may always find polynomials \( a_k(x) \) and \( b_k(x) \) in (2.13) which satisfy,
\[
\begin{align*}
    |a_k(x)| + |p_k(x)| &< |m(x)| \quad \text{(2.14a)} \\
    |b_k(x)| + |q_k(x)| &< |m(x)|. \quad \text{(2.14b)}
\end{align*}
\]

Proof. Let \((\alpha_j, \beta_j)\) denote the pair of coefficients of \( a_k(x)p_k(x) \) and \( b_k(x)q_k(x) \) at \( j \)-th order respectively. By equation (2.13), \( \forall j \geq 1 \), must have \((0, 0)\) or \((-\alpha_j, -\beta_j)\). Suppose \( t \geq |m(x)| \) is the largest order with a pair of non-zero coefficients \((\alpha_j, -\alpha_j)\). Since both \( p_k(x) \) and \( q_k(x) \) are monic, we must have,
\[
\begin{align*}
    a_k(x) &= \alpha_j x^{|m(x)|} + O(x^{|m(x)|-1}) \\
    b_k(x) &= -\alpha_j x^{|m(x)|} + O(x^{|m(x)|-1}).
\end{align*}
\]
Therefore, let us define new polynomials \( a'_k(x) \) and \( b'_k(x) \) by,
\[
\begin{align*}
    a'_k(x) &= a_k(x) - \alpha_j x^s q_k(x) \\
    b'_k(x) &= b_k(x) + \alpha_j x^s p_k(x),
\end{align*}
\]
where \( s = t - |m(x)| \). Clearly \( a'_k(x) \) and \( b'_k(x) \) have order \(|a'_k(x)| < |a_k(x)|\) and \(|b'_k(x)| < |b_k(x)|\), and together still satisfy (2.13). We may repeat this process for all orders \( j \geq |m(x)| \) until we find a pair \( a''_k(x) \) and \( b''_k(x) \) which satisfy (2.14a) and (2.14b) respectively.

\( a_k(x) \) and \( b_k(x) \) may in general be computed by, for example, the extended GCD algorithm.

We observe that the polynomial ring \( \mathbb{F}[A] \), where \( A \) has minimal polynomial \( m(x) \), is naturally ring isomorphic to,
\[
\mathbb{F}[A] \cong \frac{\mathbb{F}[x]}{I(m(x))},
\]
since their polynomials differ only by \( A \mapsto x \). This implies an identity in \( \mathbb{F}[A] \),
\[
    a_k(A) \circ p_k(A) + b_k(A) \circ q_k(A) = \text{id}_A,
\]
(2.15)
with \( \circ \) denoting composition. Note, if we choose \( a_k(x) \) and \( b_k(x) \) to satisfy (2.14a) and (2.14b) respectively, these polynomials are unaltered after the quotient.

Lemma 2.18. Suppose two idempotent maps \( A, B \in \text{End}(V) \) satisfy \( A \circ B = 0 \). Then, \( \text{Im}(A) \cap \text{Im}(B) = \{0\} \).

Proof. Consider \( \forall v, w \in V \) such that \( A(v) = B(w) = d \). Assume \( d \neq 0 \). Then,
\[
    A(v) = A \circ A(v) = A \circ B(w) = 0
\]
so \( A(v) = d = 0 \), contradicting our assumption.

Lemma 2.19. Suppose we have a family of maps \( \{A_j \in \text{End}(V)\} \) indexed over a set \( J \) such that,
\[
    \text{id}_V = \sum_{j \in J} A_j,
\]
and \( \forall \ p \neq q \in J \)
\[
    A_p \circ A_q = 0_V.
\]
Then, \( \forall j \in J \)
\[
    A_j \circ A_j = A_j.
\]

Proof. \( \forall k \in J \),
Lemma 2.20. \( V \) has an \( A \)-orthogonal decomposition,

\[
V \cong \text{Im}(I_k) \oplus \text{Im}(I_k),
\]

where,

\[
I_k := a_k(A) \circ p_k(A) \quad (2.17a)
\]
\[
\Pi_k := b_k(A) \circ q_k(A) \quad (2.17b)
\]

such that,

\[
id_V = I_k + \Pi_k. \quad (2.18)
\]

**Proof.** The existence of (2.17a) and (2.17b) follows immediately from (2.15), and (2.18) follows from (2.15). By equation (2.12a) we see that \( p_k(A) \circ q_k(A) = 0 \), and so \( I_k \) and \( \Pi_k \) satisfy,

\[
I_k \circ \Pi_k = \Pi_k \circ I_k = 0. \quad (2.19)
\]

Thus, by Lemma 2.19,

\[
\Pi_k \circ I_k = \Pi_k \quad (2.20a)
\]
\[
I_k \circ I_k = I_k. \quad (2.20b)
\]

Therefore, by Lemma 2.18, we have (2.16). To see this direct sum is \( A \)-orthogonal, note that,

\[
[A, \Pi_k] = [A, I_k] = 0,
\]

and so \( \text{Im}(I_k) \) and \( \text{Im}(I_k) \) are closed under \( A \).

**Lemma 2.21.** The polynomials \( q_k(x) \) and \( p_k(x) \) are minimal polynomials for \( A \) restricted to \( \text{Im}(I_k) \) and \( \text{Im}(\Pi_k) \) respectively.

**Proof.** Since \( m(A) = 0 \), we have,

\[
q_k(x) \circ I_k = 0 \quad (2.21a)
\]
\[
p_k(x) \circ \Pi_k = 0, \quad (2.21b)
\]

which shows that \( q_k(x) \) and \( p_k(x) \) are annihilating polynomials on \( \text{Im}(I_k) \) and \( \text{Im}(\Pi_k) \) respectively. Suppose \( p_j(x) \) is not minimal on \( \text{Im}(I_k) \), and denote the minimal polynomial \( r(x) \). Then by (2.18), \( r(A)q_k(A) = 0 \) on \( V \) which contradicts the minimality of \( m(A) = 0 \). Therefore, \( p_k(x) \) is minimal. The same logic shows this is also true for \( q_k(x) \).

**Theorem 2.22.** For each subset \( S \subseteq \{1, \ldots, n\} = N \), there is an \( A \)-orthogonal decomposition of \( V \),

\[
V \cong \text{Im}(I_S) \oplus \bigoplus_{j \in S} \text{Im}(I_j),
\]

where \( I_j \) is defined in (2.17b) and,

\[
I_S := a_S(A)p_S(A) \quad (2.23a)
\]
\[
p_S(x) = \prod_{j \in S} f^d_j(x), \quad (2.23b)
\]

for some polynomial \( a_S(x) \), such that,

\[
id_V = I_S + \sum_{j \in S} \Pi_j. \quad (2.24)
\]

**Proof.** To begin, let us further define,

\[
q_S(x) := \prod_{j \notin S} f^d_j(x) \quad (2.25a)
\]

for \( S \subseteq N \). For \( S = \emptyset \), \( p_S(x) = a_S(x) = 1 \), and thus \( I_S = \text{Id}_V \). We have already proven the case when \( S \) is a singleton in Lemma 2.20. Since each \( S \) has a finite number of elements, we may prove the remaining cases by induction. Suppose the Lemma is true for \( S \subseteq N \). Then, by the same logic used in the proof of Lemma 2.20, we find \( \forall j \neq k \in S \),

\[
I_j \circ \Pi_k = 0 \quad (2.26a)
\]
\[
\Pi_j \circ \Pi_k = 0 \quad (2.26b)
\]
By the same logic used in Lemma 2.21, \( q(x) \) is the minimal polynomial for \( A \) on \( \text{Im}(I) \). Now consider a set \( S' \supset S \) such that \( S'/S \) is a singleton. It is easy to see that,

\[
q_S(x) = p_{S'/S}(x)q_{S'}(x) \tag{2.27a}
\]

\[
p_{S'/S}(x)p_S(x) = p_{S'}(x) \tag{2.27b}
\]

\[
q_{S'}(x)p_S(x) = q_{S'/S}(x). \tag{2.27c}
\]

Using (2.27a) and Lemma 2.20 on \( \text{Im}(I) \) to find,

\[
I_S = a_{S'/S}(A)o_{S'/S}(A)I_S + b_{S'}(A)o_{S'}(A)I_S. \tag{2.28}
\]

We may use (2.27b) and (2.27c) to simplify the first and second terms respectively,

\[
a_{S'/S}(A)o_{S'/S}(A)I_S = a_{S'/S}(A)o_{S'}(A)o_{S'/S}(A) = I_{S'} \tag{2.29a}
\]

\[
b_{S'}(A)o_{S'}(A)I_S = b_{S'}(A)o_{S'}(A)o_{S'/S}(A) = \Pi_{S'/S}, \tag{2.29b}
\]

so that,

\[
\text{id}_V = I_S + \sum_{j \in S} \Pi_j = I_{S'} + \sum_{j \in S} \Pi_j = I_{S'} + \sum_{j \in S'} \Pi_j. \tag{2.30}
\]

Finally, observe that \( I_S \) and \( \{\Pi_j \mid j \in S'\} \), also satisfy (2.26a)–(2.26d).

**Corollary 2.23.** \( V \) has an \( A \)-orthogonal decomposition,

\[
V \cong \bigoplus_{j=1}^{n} \text{Im}(\Pi_j), \tag{2.31}
\]

where,

\[
\text{id}_V = \sum_{j=1}^{n} \Pi_j = \Pi_{S'/S}, \tag{2.32a}
\]

\[
p_j(A)\Pi_j = 0. \tag{2.32b}
\]

**Proof.** This follows from choosing \( S = N \) in Theorem 2.22, as \( I_N = a_N(A)m(A) = 0 \).

Thus, from the minimal polynomial of \( A \) we have arrived at a basis independent orthogonal decomposition of \( V \) through the projectors \( \Pi_j \). This has been achieved without the use of complex numbers, or reference to vectors in \( V \); it has been derived entirely from the algebraic properties of \( A \).

**Remark.** In the case where there is only one multiplicand in equation (2.11) with \( d_j > 1 \), we cannot find a resolution of the identity into more than one projector by this method alone. Furthermore, there may be subspaces closed under the action of \( A \) within each \( \text{Im}(\Pi_j) \) which we cannot differentiate using only the above. These limitations will not hamper our analysis of spin however, as we will within each, corresponding to the number of different irreducible polynomials the minimal polynomial contains. The equality of these non-zero idempotents between the two decompositions is assured by Lemma 2.17.

**Corollary 2.24 (A-Orthogonal Decompositions using Non-Minimal Polynomials).** Say we perform two \( A \)-orthogonal decompositions of \( V \) as in Corollary 2.23, one using the minimal polynomial of \( A \), and the other using a non-minimal annihilating polynomial of \( A \). Then, these two decompositions are isomorphic.

**Proof.** Since all annihilating polynomials are a polynomial multipole of the minimal polynomial by Lemma 2.15, performing the decomposition in Corollary 2.23 necessarily produces idempotents which contain the whole minimal polynomial as a factor. As such, these idempotents are identically zero. There can only be as many non-zero idempotents in the latter decomposition as there are in the former, corresponding to the number of different irreducible polynomials the minimal polynomial contains. The equality of these non-zero idempotents between the two decompositions is assured by Lemma 2.17.

It is instructive to relate what has just been developed to existing concepts. A traditional eigenspace is an \( \text{Im}(\Pi_k) \): \( (A - \lambda I_d)\Pi_k = 0 \), so \( |k| = 1 \) and \( |d_k| = 1 \). A generalised eigenspace [40] is an \( \text{Im}(\Pi_k) \): \( (A - \lambda I_d)\Pi_k = 0 \) where \( d \in \mathbb{Z}^+ \) is minimal, so \( |k| = 1 \) and \( |d_k| = d \). The case when \( |k| > 1 \) does not occur when considering operators on complex vector spaces, as \( \mathbb{C} \) is algebraic closed. A simple example of a
real operator with such a subspace is a planar rotation by an angle \( \theta \): the subspace \( \text{Im}(\Pi_k) \) in the plane of rotation satisfies \( (A^2 - 2 \cos \theta A + \text{id}) \circ \Pi_k = 0 \), which is irreducible over \( \mathbb{R} \).

**Corollary 2.25.** Given a finite collection of mutually commuting operators \( \{A_n\} \), there exists a unique \( \{A_n\} \)-orthogonal decomposition,

\[
\text{id}_V = \sum_{a_1,a_2,...} \Pi_{a_1}(A_1) \circ \Pi_{a_2}(A_2) \circ ... .
\]

**Proof.** Since the identity map is idempotent, compose together all \( A_n \)-orthogonal decompositions from Corollary 2.23. \( \square \)

**Remark.** Doing this, we cannot guarantee that the image of each combined projector is non-trivial; exactly which projectors have non-trivial image depends on the relationship between the operators in the collection. This is a basis-independent generalisation of simultaneous diagonalisation of operators to the real operator case, where even generalised eigenspaces may fail to exist. If the operators do not all mutually commute the situation is markedly more complex.

Of particular interest to our present problem is the case where all \( |f_k| = 1 \) and \( d_k = 1 \).

**Lemma 2.26.** When

\[
p_k(x) = x - \lambda_k,
\]

we find \( b_k(A) \) from \( \Pi_k \) in (2.17b),

\[
b_k(A) = \frac{1}{q_k(\lambda_k)}. \tag{2.35}
\]

**Proof.** By (2.14b) \( |b_k| = 0 \), i.e. constant. Now, we substitute (2.17b) into the idempotency condition (2.20a) and expand \( q_k(A) \) as a series in \( A - \lambda_k \text{id} \). Using (2.21b) completes the proof since \( \Pi_k \neq 0 \). \( \square \)

The method of algebraic orthogonal decomposition of a vector space developed here, in the particular case of \( |f_k| = 1 \) and \( d_k = 1 \), will be used extensively in our analysis.

### 2.4. Decomposition of \( U(\mathfrak{so}(3, \mathbb{R})) \)

#### 2.4.1. Method of decomposition

As established in section 2.2, we may achieve a complete orthogonal decomposition of \( U(\mathfrak{so}(3, \mathbb{R})) \) by orthogonally decomposing \( \mathfrak{so}(3, \mathbb{R})^\otimes k, \forall k \) into subspaces closed under the action of \( \text{ad}(\mathfrak{so}(3, \mathbb{R})) \). The necessary mathematics to do this for a single operator, or a set of commuting operators, on a real space was established in 2.3. However, it is not clear how we must apply this formalism to a set of non-commuting operators such as \( \text{ad}(S_j), \forall S_j \). Fortunately, we may avoid this complication entirely by instead working with the commuting set of operators \( \text{ad}(Z(U(\mathfrak{so}(3, \mathbb{R})))) \), where,

**Definition 2.27.** \((Z(U(\mathfrak{so}(3, \mathbb{R}))))\). The centre \( Z(U(\mathfrak{so}(3, \mathbb{R}))) \) of \( U(\mathfrak{so}(3, \mathbb{R})) \) is the subalgebra,

\[
Z(U(\mathfrak{so}(3, \mathbb{R}))) = \{ z \in U(\mathfrak{so}(3, \mathbb{R})) \mid z \otimes A = A \otimes z, \forall A \in U(\mathfrak{so}(3, \mathbb{R})) \}. \tag{2.36}
\]

**Lemma 2.28.** The \( \mathbb{R} \)-linear span of \( k \)-adic tensors \( \mathfrak{so}(3, \mathbb{R})^\otimes k \) is closed under \( \text{ad}(U(\mathfrak{so}(3, \mathbb{R}))) \), \( \forall k \in \mathbb{N} \).

**Proof.** It is sufficient to show \( \mathfrak{so}(3, \mathbb{R})^\otimes k \) is closed under \( \text{ad}(\mathfrak{so}(3, \mathbb{R})) \). On 0-adics, \( \text{ad}(\mathfrak{so}(3, \mathbb{R})) \) acts as 0. From Lemma 2.29, \( \forall a \in \mathfrak{so}(3, \mathbb{R}), \text{ad}(a) \) is a derivation on \( \mathfrak{so}(3, \mathbb{R})^\otimes k \). Furthermore, Lemma 2.28 shows that \( \text{ad}(a) \) on a 1-adic tensor gives a 1-adic tensor. Thus, \( \text{ad}(a) \) preserves tensor order for \( k \geq 1 \). \( \square \)

**Remark.** We will use lemmas 2.9 and 2.28 to allow the adjoint action to act on \( \mathfrak{so}(3, \mathbb{R})^\otimes k \subset T(\mathfrak{so}(3, \mathbb{R})) \). When we are using the adjoint action in this way will be clear from context.

**Lemma 2.29.** \( \forall z \in Z(U(\mathfrak{so}(3, \mathbb{R}))), k \in \mathbb{N}, \text{an ad}(z)\)-orthogonal decomposition of \( \mathfrak{so}(3, \mathbb{R})^\otimes k \) as defined in Corollary 2.23 is also an \( \text{ad}(U(\mathfrak{so}(3, \mathbb{R}))) \)-orthogonal decomposition.

**Proof.** \( \mathfrak{so}(3, \mathbb{R})^\otimes k \) is finite-dimensional, and so all operators on it have minimal polynomials by Lemma 2.14. Since \( \forall z \in Z(U(\mathfrak{so}(3, \mathbb{R}))), A \in U(\mathfrak{so}(3, \mathbb{R})), [z, A] = 0 \) we have that \( \text{ad}(z), \text{ad}(A) = \text{ad}(z, A) = 0 \). Now, fix an element \( w \in Z(U(\mathfrak{so}(3, \mathbb{R}))) \) and use Corollary 2.23 to \( \text{ad}(w) \)-orthogonally decompose \( \mathfrak{so}(3, \mathbb{R})^\otimes k \), resulting in,
\[ \text{id} = \sum_{j=1}^{n} \Pi_j(\text{ad}(w)) \]

Therefore, \([\Pi_j(\text{ad}(w)), \text{ad}(A)] = 0, \forall j, \] since by Lemma 2.28 \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\) is closed under \(\text{ad}(U(\mathfrak{so}(3, \mathbb{R}))\), this implies that \(\text{Im}(\Pi_j(\text{ad}(w)))\) is closed under the action of \(\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))\). As our choice of \(w \in Z(U(\mathfrak{so}(3, \mathbb{R}))\) was arbitrary, this completed the proof. \(\Box\)

Thus, using \(\text{ad}(Z(U(\mathfrak{so}(3, \mathbb{R}))))\), we may find \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\)-orthogonal subspaces of \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\) while avoiding the complication of non-commutativity amongst the \(\text{ad}(\mathfrak{so}(3, \mathbb{R}))\). It is also unnecessary to consider the whole of \(Z(U(\mathfrak{so}(3, \mathbb{R})))\): it can be generated by algebraic combinations of \(U\) and the Casimir element,

**Definition 2.30 (S^2).** The quadratic Casimir element \(S^2 \in Z(U(\mathfrak{so}(3, \mathbb{R})))\) of \(U(\mathfrak{so}(3, \mathbb{R}))\) is,

\[ S^2 := \sum_{a=1}^{3} S_a \otimes S_a. \] (2.37)

Therefore, only the action of \(\text{ad}(S^2)\) need be considered (since all \(\text{ad}(\mathbb{R})\) just act as scalings).

**Definition 2.31 (E and \( \varepsilon(k) \)).** To simplify the following mathematics, let us introduce the notation,

\[ E := \text{ad}(S^2) \] (2.38a)

\[ \varepsilon(k) := \text{ad}(S^2) + k(k+1) \text{id}_{U(\mathfrak{so}(3, \mathbb{R}))}. \] (2.38b)

We will defer explaining the \(k(k+1)\) term in \((2.38b)\) until slightly later.

Since there are infinitely many \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\) we cannot simply find the minimal polynomial of \(E\) on each of them. Instead, let us work iteratively and build up our decomposition order by order.

**Definition 2.32 (Left multiplication).** We define the left multiplication action \(L(A)\) on \(U(\mathfrak{so}(3, \mathbb{R}))\), is defined as, \(\forall A \in U(\mathfrak{so}(3, \mathbb{R}))\),

\[ L : U(\mathfrak{so}(3, \mathbb{R})) \rightarrow \text{End}(U(\mathfrak{so}(3, \mathbb{R}))) \]

\[ L(A) : B \rightarrow A \otimes B. \] (2.39)

We will also abuse notation slightly and use \(L(a), \forall a \in \mathfrak{so}(3, \mathbb{R})\), as a map \(\mathfrak{so}(3, \mathbb{R})^{\oplus k} \rightarrow \mathfrak{so}(3, \mathbb{R})^{\oplus k+1}\), \(\forall k \in \mathbb{N}\), or as a map \(T(\mathfrak{so}(3, \mathbb{R})) \rightarrow T(\mathfrak{so}(3, \mathbb{R}))\). When we are using the left multiplication in these ways will be clear from context.

**Theorem 2.33.** Given an \(E\)-orthogonal decomposition of \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\), we may not only promote this to an \(E\)-orthogonal decomposition of \(\mathfrak{so}(3, \mathbb{R})^{\oplus k+1}\), but may refine it to isolate more \(\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))\)-closed subspaces.

**Proof.** See Appendix A. \(\Box\)

Thus, Theorem 2.33 reveals that we may decompose all \(\mathfrak{so}(3, \mathbb{R})^{\oplus k}\) by starting with a scalar, then repeatedly applying \(L(\mathfrak{so}(3, \mathbb{R}))\) and decomposing the result according to the action of \(E\) at each step. Along the way, we will discover relationships between families of these orthogonal subspaces which will indicate a natural set of ideals, and therefore a natural set of finite-dimensional algebras, to construct from \(U(\mathfrak{so}(3, \mathbb{R}))\).

**2.4.2. Relationship between \(L(v)\) and \(E\)**

To successfully execute the scheme outlined in section 2.4.1, we must understand how \(L(\mathfrak{so}(3, \mathbb{R}))\) and \(E\) interact.

**Lemma 2.34.** \(\forall v \in \mathfrak{so}(3, \mathbb{R})\),

\[ [E, [E, [E, L(v)]]]] + 2[E^2, L(v)] = 0_{U(\mathfrak{so}(3, \mathbb{R}))} \] (2.40)

where \([\cdot, \cdot]\) is the commutator, and every implied product is composition. This identity holds on the whole of \(U(\mathfrak{so}(3, \mathbb{R}))\).

**Proof.** See Appendix B. \(\Box\)

**Definition 2.35.** Let us denote by \(\Pi_{f(A)} \in \text{End}(U(\mathfrak{so}(3, \mathbb{R})))\) an idempotent for which \(f(x)\) is the minimal polynomial for \(A\) on its image. Equivalently, \(f(x)\) is the polynomial of least order such that,

\[ f(A) \circ \Pi_{f(A)} = 0. \]

To understand the consequences of Lemma (2.34) it is instructive to consider its action on a subspace \(\text{Im}(\Pi_{E+1})\).
Lemma 2.36. On $\text{Im}(L(v)\circ \Pi_{E,t})$, with $t \geq -\frac{1}{4}$, $E$ has annihilating polynomial,

$$p(x) = (x + t)(x + (t + 1 + \sqrt{4t + 1}))(x + (t + 1 - \sqrt{4t + 1})).$$ (2.41)

Proof. Directly apply Lemma 2.34 to $\Pi_{E,t}$ and factorise.

Lemma 2.37. $\sqrt{4t + 1} \in \mathbb{N}$ iff $t = m(m + 1), m \in \mathbb{N}$.

Proof. The backwards direction is trivial. For the forwards direction, our assumption entails $n^2 = 4t + 1$ for some $n \in \mathbb{N}$. Therefore, $n^2$ is odd, so $n$ is odd, and we may write $n = 2m + 1, m \in \mathbb{N}$. Thus, $t = m(m + 1)$.

Lemma 2.38. When $t = m(m + 1), m \in \mathbb{N}$,

$$p(x) = (x + m(m + 1))(x + (m + 1)(m + 2))(x + (m - 1)m).$$ (2.42)

Proof. Direct substitution.

Remark. For $m = 0$ all three roots are consecutive naturals of the form $m(m + 1)$, and when $m = 0$ the roots are 0, 0, and 2.

To see the significance of this observation, let us begin our order-by-order decomposition.

Lemma 2.39. On $\mathbb{R}$ and $\mathfrak{so}(3, \mathbb{R})$ the action of $E$ has minimal polynomials,

$$m(x) = x$$ (2.43a)

$$m'(x) = x + 2;$$ (2.43b)

respectively.

Proof. Direct application of $E = \text{ad}(S^2)$ on $\alpha \in \mathbb{R}$ and $v \in \mathfrak{so}(3, \mathbb{R})$.

Remark. Since these contain a power of a single irreducible polynomial, no further decomposition can be made here.

Lemma 2.40. On all subspaces $S \subset U(\mathfrak{so}(3, \mathbb{R}))$ on which $E$ has a minimal polynomial, its minimal polynomial must be of the form,

$$m(x) = \prod_{n \in N} (x + n(n + 1)),$$ (2.44)

for some finite $N \subset \mathbb{N}$.

Proof. Noting that $0 = 0(0 + 1)$ and $2 = 1(1 + 1)$, we see that the minimal polynomials (2.43a) and (2.43b) are of this form. Therefore, our iterative process of $E$-orthogonal decomposition outlined in section 2.4.1 is initialised by subspaces annihilated by linear polynomials with natural number constant terms. Therefore, by Lemma 2.38, on every subspace we reach, $E$ has minimal polynomial of the form (2.44). By Theorem 2.33, $E$-orthogonal decompositions of every order of tensor in $U(\mathfrak{so}(3, \mathbb{R}))$ can be achieved by our process when starting from $\mathbb{R}$. Thus, since all elements of $U(\mathfrak{so}(3, \mathbb{R}))$ are linear combinations of $k$-adic tensors, on all subspaces generated by a single element $E$ has minimal polynomials of the form (2.44). Therefore, if a subspace has a minimal polynomial it must be of the form (2.44).

Remark. This accounts for the constant in (2.38b).

Remark. While (2.42) is always an annihilating polynomial on $\text{Im}(L(v)\circ \Pi_{E,t})$, we can see from (2.43a) and (2.43b) that it is not always minimal, and that its minimality depends on the particular subspace we are acting on. Regardless, we may use it to decompose $\text{Im}(L(v)\circ \Pi_{E,t})$ according to the methods of section 2.3. If indeed the polynomial is not minimal on a given subspace, we will produce idempotents that are $0_{U(\mathfrak{so}(3,\mathbb{R}))}$ along with the complete set of non-zero ones, as in Corollary 2.24; as such, this will not cause any issues.

2.4.3. Recursive decomposition of $k$-adic tensors

Since we have already found $\mathbb{R}$ and $\mathfrak{so}(3, \mathbb{R})$ to be indecomposable, we may focus on decomposing $k$-adic tensors for $k \geq 2$. It is convenient to capture each subspace we derive as the image of a map $\mathfrak{so}(3, \mathbb{R})^\otimes k \rightarrow U(\mathfrak{so}(3, \mathbb{R}))$, which will usually be expressed as,
\[ v_1 \otimes v_2 \otimes \ldots \otimes v_k \mapsto f(E) \circ L(v_1) \Pi_{\varepsilon(m)}(v_2 \otimes \ldots \otimes v_k), \]  
\quad \text{(2.45)}

for some \( m \in \mathbb{N} \) and some function \( f \) of \( E \). The order \( k \) will correspond to the total number of \( L(v_j) \) that we have applied in our recursive scheme. For these maps to make sense, their domains are \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \subset \mathcal{T}(\mathfrak{so}(3, \mathbb{R})) \), not \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \subset U(\mathfrak{so}(3, \mathbb{R})) \).

We may describe \( \text{Im}(L(v) \circ \Pi_{\varepsilon(m)}) \) conveniently by combining the left multiplication and decomposition by \( E \) steps into three new actions.

**Lemma 2.41.** Let \( \Pi_{\varepsilon(m)} \in \text{End}(U(\mathfrak{so}(3, \mathbb{R}))) \) be a map on whose image \( E \) has minimal polynomial \( \varepsilon(m), m \in \mathbb{N} \), then \( \forall v \in \mathfrak{so}(3, \mathbb{R}), B \in \mathfrak{so}(3, \mathbb{R})^{\otimes k-1}, \)

\[
L(v)\circ \Pi_{\varepsilon(m)}(B) = (L(v) + L^-(v) + L^+(v)) \circ \Pi_{\varepsilon(m)}(B)
\]

\quad \text{where,}

\[
L(v)\circ \Pi_{\varepsilon(m)}(B) = \begin{cases} 0 & m = 0 \vspace{1em} \\ \varepsilon(m)\varepsilon(m+1) & m \in \mathbb{Z}^+ \end{cases}
\]

\quad (2.46a)

\[
L^-(v)\circ \Pi_{\varepsilon(m)}(B) = \begin{cases} (E - 2 \text{ id})\varepsilon(1) & m = 0 \\ \varepsilon(m-1)\varepsilon(m+1) & m \in \mathbb{Z}^+ \end{cases}
\]

\quad (2.46b)

\[
L^+(v)\circ \Pi_{\varepsilon(m)}(B) = \begin{cases} \varepsilon(0)\varepsilon(0) & m = 0 \\ \varepsilon(m-1)\varepsilon(m) & m \in \mathbb{Z}^+ \end{cases}
\]

\quad (2.46c)

**Proof.** This follows directly from \( E \)-orthogonally decomposing \( \text{Im}(L(v) \circ \Pi_{\varepsilon(m)}) \) using Lemma 2.34. \( \square \)

**Definition 2.42.** We will call the \( L(v)/L^-(v)/L^+(v) \) the ‘step-down/step-level/step-up’ operators respectively.

**Corollary 2.43.** \( \forall m \in \mathbb{N}, \)

\[
\varepsilon(m-1)\circ L(v)\circ \Pi_{\varepsilon(m)}(B) = 0
\]

\quad (2.48a)

\[
\varepsilon(0)^2\circ L^-(v)\circ \Pi_{\varepsilon(0)}(B) = 0
\]

\quad (2.48b)

\[
\varepsilon(m)\circ L^+(v)\circ \Pi_{\varepsilon(m)}(B) = 0
\]

\quad (2.48c)

**Proof.** This follows by construction. \( \square \)

**Remark.** We shall soon see that, in fact, \( L^-(v)\circ \Pi_{\varepsilon(0)} = 0_{U(\mathfrak{so}(3, \mathbb{R}))} \) due to (2.42) not being minimal on \( \text{Im}(L(v) \circ \Pi_{\varepsilon(m)}) \).

Using the step operators, our scheme of decomposition for \( U(\mathfrak{so}(3, \mathbb{R})) \) is equivalent to applying all sequences of step operations by \( v_j \in \mathfrak{so}(3, \mathbb{R}) \) to 1 and keeping those that yield non-trivial subspaces. We are guaranteed to decompose \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \) fully in terms of the non-trivial subspaces reached after \( k \) steps due to (2.46) holding at every step. This process is summarised graphically in figure 1.

**2.4.4. Decomposition into multipoles**

Figure 1 shows a part of the \( E \)-orthogonal decomposition of \( U(\mathfrak{so}(3, \mathbb{R})) \), and combines results from the whole of this paper. This shown part, which may be verified through explicit computation, reveals that \( \forall k \in \{0, \ldots, 4\} \) there is exactly one subspace annihilated by \( \varepsilon(k) \) amongst all subspaces reached in \( k \) steps from 1. This subspace was reached by stepping-up from the unique subspace reached in \( k - 1 \) steps that is annihilated by \( \varepsilon(k - 1) \). Furthermore, there are no subspaces annihilated by \( \varepsilon(k) \) reachable in fewer than \( k \) steps. Let us extend these observations to \( \forall k \in \mathbb{N} \).

**Definition 2.44 (Multipoles).** Let us recursively define a family of maps,

\[ \{ M^{(k)} : \mathfrak{so}(3, \mathbb{R})^{\otimes k} \to U(\mathfrak{so}(3, \mathbb{R})) \mid k \in \mathbb{N} \}, \]
We refer both to \( M_k \) (and its image) as the ’multipole’ of order \( k \), and the family \( \{ M_k \} \) as the ’multipoles’.

Remark. We see by definition, \( \forall k \in \mathbb{N}, \epsilon(k) \circ M(k) = 0. \)

Lemma 2.45. \( \forall p, q \in \mathbb{N}, p \neq q, \quad \text{Im}(M(p)) \cap \text{Im}(M(q)) = \{0\}. \)

Proof. Suppose \( d \in \text{Im}(M(p)) \cap \text{Im}(M(q)). \) Then,

\[ \epsilon(p)(d) = \epsilon(q)(d) = 0, \]

and so,

\[ (p(p + 1) - q(q + 1))d = 0. \]

Since \( p \neq q, d = 0. \) \( \square \)
Lemma 2.46 (Properties of Multipoles). \( \forall k \in \mathbb{N} \), each multipole \( M^{(k)} \), is closed under the adjoint action of \( U(\mathfrak{so}(3, \mathbb{R})) \),
\[
\forall A \in U(\mathfrak{so}(3, \mathbb{R})), \quad \text{ad}(A) \circ M^{(k)} = M^{(k)} \circ \text{ad}(A),
\]
\[
(2.52a)
\]
is totally symmetric,
\[
\forall \tau \in S_k, \quad M^{(k)} \circ \tau = M^{(k)},
\]
\[
(2.52b)
\]
and for \( k \geq 2 \), is contractionless,
\[
\forall m \approx n \in \{1, \ldots, k\}, \quad \sum_{a_{m,a_n}=1}^{3} \delta_{a_{m,a_n}}M^{(k)}(\bigotimes_{j=1}^{k} S_{a_j}) = 0.
\]
\[
(2.52c)
\]
Proof. See Appendix C. \( \square \)

Lemma 2.47. Step-levels and step-downs from multipoles \( M^{(k)} \), \( \forall k \in \mathbb{N} \), can be written in terms of lower order multipoles,
\[
\begin{align*}
L^+(S_a) \circ M^{(k)} &= \left\{ \frac{L(4S^2 + (k-1)(k+1))}{4(4k^2 - 1)} \sum_{p=1}^{k} \left( (2k - 1)\delta_{a,b} M^{(k-1)} \left( \bigotimes_{j=p}^{k} S_{b_j} \right) - \sum_{q=1, q \neq p}^{k} \delta_{b,q} M^{(k-1)} \left( S_a \bigotimes \bigotimes_{j=p,q}^{k} S_{b_j} \right) \right) \right\} \quad k \geq 2 \\
&= \frac{1}{3} \delta_{a,b} \quad k = 1 \\
&= 0 \quad k = 0.
\end{align*}
\]
\[
(2.53a)
\]
\[
L^-(S_a) \circ M^{(k)} &= \left\{ \frac{1}{2} \sum_{p=1}^{k} \sum_{l=1}^{k} \delta_{a,b,c} M^{(k)} \left( S_l \bigotimes \bigotimes_{j=p}^{k} S_{b_j} \right) \right\} \quad k \geq 2 \\
&= \frac{1}{2} \sum_{c=1}^{3} \delta_{a,b,c} S_c \quad k = 1 \\
&= 0 \quad k = 0.
\]
\[
(2.53b)
\]
Proof. See Appendix D. \( \square \)

Lemma 2.48 (Minimal Polynomial for \( E \) on Multipoles). \( (2.42) \) is minimal on \( \text{Im}(L(\nu) \circ M^{(k)}) \) for \( k \in \mathbb{Z}^+ \), and \( (2.43b) \) is minimal on \( \text{Im}(L(\nu) \circ M^{(0)}) \).

Proof. See Appendix E. \( \square \)

Corollary 2.49. Lemma 2.48 implies,
\[
L^-(\nu) \circ M^{(0)} = 0 \quad (2.54a)
\]
\[
L^+(\nu) \circ M^{(0)} = L(\nu) \circ M^{(0)} \quad (2.54b)
\]
Proof. \((2.54a)\) follows from \( \varepsilon(2) \circ M^{(1)} = 0 \), and \((2.54b)\) from applying \((2.54a)\) to \((2.46)\). \( \square \)

Table 1 shows the image of \( M^{(4)} \) on a \( k \)-adic tensor for the first five values of \( k \). These objects look like non-commutative generalisations of the Cartesian multipole tensors, which supports our naming of the maps \( M^{(k)} \) ‘multipoles’. The multipoles agree with the forms implied by \([43]\), though their algebraic properties and interrelationships are much clearer from this method.

The significance of the multipoles to our decomposition can be seen by considering the following.

Theorem 2.50 (Multipole Decomposition of \( U(\mathfrak{so}(3, \mathbb{R})) \)).
\[
U(\mathfrak{so}(3, \mathbb{R})) \cong \bigoplus_{j=0}^{\infty} M^{(j)},
\]
\[
(2.55)
\]
where,
\[
M^{(j)} = \text{span}_{\mathbb{R}[S^2]}(\text{Im}(M^{(j)})),
\]
\[
(2.56)
\]
and \( \mathbb{R}[S^2] \) is the real polynomial ring over \( S^2 \).
Proof. Lemma (2.41) and Lemma 2.47 applied to a multipole $M^{(k)}$, $\forall k \in \mathbb{N}$, reveals that left multiplication of a multipole by $\mathfrak{so}(3, \mathbb{R})$ can be written as $\mathbb{R}[S^2]$-linear combinations of multipoles. Since we have defined $M^{(0)} = \mathbb{R}$, we may therefore write any $k$-adic tensors, and thus any finite $\mathbb{R}$-linear combination of $k$-adic tensors, using multipoles. Therefore, $U(\mathfrak{so}(3, \mathbb{R}))$ is a sum space of the $\{M^{(j)}\}$. That this is a direct sum follows from Lemma 2.45. \hfill \Box

Remark. Theorem 2.50 enables us to find the minimal polynomial of $E$ on any subspace of $U(\mathfrak{so}(3, \mathbb{R}))$ via Lemma 2.38.

3. Results

Unlike the tensor order decomposition (2.4) of $U(\mathfrak{so}(3, \mathbb{R}))$, each summand in the multipole decomposition (2.50) is trivially intersecting. Thus, we may derive a family of algebras from it.

Definition 3.1 (Spin-\(k\) Algebra).

\[
A^{(k)} := \frac{U(\mathfrak{so}(3, \mathbb{R}))}{I(\text{Im}(M^{(k+1)}))}.
\] (3.1)

Lemma 3.2. $\forall k \in \mathbb{N}$, $A^{(k)}$ is finitely generated.

Proof. The quotient by $I(M^{(k+1)})$ imposes the identity $M^{(k+1)} = 0$ on $A^{(k)}$. By (2.49), this implies $M^{(n)} = 0$, $\forall n \geq k + 1$. \hfill \Box

What is not obvious is that this process will yield a finite-dimensional real algebra, since each summand in (2.50) is a module over $Z(U(\mathfrak{so}(3, \mathbb{R})))$.

Lemma 3.3. In $A^{(k)}$, $\forall k \in \mathbb{N}$,

\[
L(S^2) = \frac{-k(k+2)}{4}.
\] (3.2)

Proof. See Appendix F. \hfill \Box

Remark. Since, (3.2) applies to the whole of $A^{(k)}$, this means that in $A^{(k)}$ the Casimir element $S^2$ becomes a real scalar. Reindexing $k = 2s$, we see that $S^2 = -s(s+1)$ in $A^{(s)}$, exactly what is expected for the spin-$s$ representation.

In fact, this connection is more extensive than this.

Lemma 3.4.

\[
\dim(\text{Im}(M^{(k)})) = 2k + 1
\] (3.3)

Proof. See Appendix G. \hfill \Box

Remark. Accordingly, $A^{(k)}$ has dimension $\sum_{j=0}^{k} 2j + 1 = (k + 1)^2 = (2s + 1)^2$. This is exactly the complex dimension of the operators in the usual complexified spin-$s$ representation.

Table 1. Images of the multipoles $k = 0, \ldots, 4$ on $k$-adic tensors.

| Multipole | Explicit Form |
|-----------|---------------|
| $M_0$     | 1             |
| $M_2$     | $\sum_{\sigma \in \text{Perm}(k)} \frac{1}{2} \delta_{\sigma(0)\sigma(0)} S_{\sigma(0)}^2 \circ S_{\sigma(0)} - \frac{1}{6} \delta_{\sigma(0)\sigma(0)} S_{\sigma(0)}^2$ |
| $M_{ab}$  | $\sum_{\sigma \in \text{Perm}(k)} \frac{1}{6} \delta_{\sigma(0)\sigma(0)} S_{\sigma(0)} \circ S_{\sigma(0)} - \frac{1}{30} \delta_{\sigma(0)\sigma(0)} (S_{\sigma(0)}^2 + 1) \circ S_{\sigma(0)}$ |
| $M_{abc}$ | $\sum_{\sigma \in \text{Perm}(k)} \frac{1}{24} \delta_{\sigma(0)\sigma(0)} S_{\sigma(0)} \circ S_{\sigma(0)} \circ S_{\sigma(0)} \circ S_{\sigma(0)}$ |
| $M_{abcd}$| $-\frac{1}{168} \delta_{\sigma(0)\sigma(0)} S_{\sigma(0)}^2 \circ S_{\sigma(0)} \circ S_{\sigma(0)} \circ S_{\sigma(0)}$ |


Using the table 1, we may calculate Evidence. Conjecture 3.5. Im\( (M^{(k+1)}) = \{0\} \) implies that if \( k \) is odd,

\[
\sum_{j=0}^{\frac{1}{2}(k-1)} S_a \otimes S_a + \left( j + \frac{1}{2} \right)^2 = 0,
\]

(3.4)

and if \( k \) is even,

\[
S_a \otimes \sum_{j=1}^{\frac{1}{2}k} (S_a \otimes S_a + j^2) = 0.
\]

(3.5)

Evidence. Using the table 1, we may calculate \( M^{(k+1)}(\times_j^{k+1} S_a) \) in \( A(\frac{k}{2}) \). This is in full agreement with the predicted spectra above. Further results may be easily found using Corollary C.9, from which we find the recurrence relation,

\[
M^{(k+1)} \left( \bigotimes_{j=1}^{k+1} S_a \right) = L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_a \right) - \frac{k^2}{4(4k^2 - 1)} L(4S^2 + (k - 1)(k + 1)) M^{(k-1)} \left( \bigotimes_{j=1}^{k-1} S_a \right),
\]

where \( S^2 \) will depend on the algebra we are working in.

Remark. This successfully predicts the complete eigenspectrum expected for our basis.

Due to this correspondence we are justifying in naming the algebra \( A(\frac{s}{2}) \) the ‘spin-s algebra’. More concretely, Conjecture 3.6. The spin-\( s \) representation is simply an associative algebra representation of \( A(\frac{s}{2}) \), and derives its bulk structure from it.

In deriving the algebras \( A(s) \), we have established that any spin may be specified entirely by its largest non-zero multipole, and completely described by: a finite collection of multipoles \( \{ M^{(n)} \} \) \( n \in \{0, \ldots, 2s\} \); the identity \( L(S^2) = L(-s(s+1)) \); and a multiplication table, the start of which is given in table 2. Such a table may be extended using the explicit forms of multipoles in table 1, and the step identities of section (2.4.4).

4. Discussion

4.1. Mathematical comparison with other non-relativistic models

The spin algebras \( A(\frac{s}{2}) \) with \( s = \frac{1}{2} \) and \( s = 1 \) are completely consistent with the Clifford and Kemmer algebras on \( \{ S_a \} \) defined by (1.4), (1.6), and equations (1.9) and (1.10) respectively. The identity (1.9) is directly implied by \( M^{(2)} = 0 \) in \( A(\frac{1}{2}) \), and the identity (1.10) is a consequence of \( M^{(3)} = 0 \) in \( A(1) \), but requires some manipulation of the decomposed identity map for 3-adic tensors to prove. Larger Clifford and Kemmer algebras which contain their \( \{ S_a \} \) counterparts are definable on the physical three-space \( [14, 35, 36] \) using modified identities (1.9) and (1.10), however it is unclear if any instructive comparisons can be made between them and the spin algebras.

The counterparts to the Clifford and Kemmer algebras for spins greater than 1 are mostly unknown. The traditional method for constructing real, algebraic theories for arbitrary spins is to form \( C_l(\mathbb{R}) \otimes^k \) from the Clifford algebra \( C_l(\mathbb{R}) \) [26], wherein we may find a subalgebra which describes spin \( -\frac{k}{2} \); this is similar to
forming a single system of arbitrary spin from multiple spin—\(\frac{1}{2}\) systems. However, this approach is wasteful in the sense that the majority of \(\text{Cl}_3(\mathbb{R})^\otimes\) does not describe our desired spin, and consistently isolating the correct subalgebra can be challenging a priori. This process also introduces an interpretational problem: if this is necessary to describe a higher spin system algebraically, are all such systems necessarily composite? This challenges the notion that, for example, spin-1 particles like the photon can be fundamental. If they can be, then does this mean the algebraic approach is inappropriate for such systems? The \(A^{(\omega)}\) provide a solution to these problems, as they: can be constructed systematically without needing an algebra for a lower spin; avoid the algebraic substructure the traditional method necessarily imparts.

Racah’s spherical tensor operator formalism [21] is also related to the spin algebras: if we complexify \(U(\mathfrak{so}(3, \mathbb{R}))\) and choose some preferred primary element of \(\mathfrak{so}(3, \mathbb{R})\), we can associate each of the \(2k + 1\) independent components of the multipole \(M^{(3)}\) with a unique component of the rank-\(k\) spherical tensor operator. The commutator between two spherical tensor operators is given by the adjoint action of one upon the other. As a particular choice of basis, the complexified spin algebras could be written in terms of the spherical tensor operators. However, since defining the spherical tensor operators requires complex numbers and a preferred element of \(\mathfrak{so}(3, \mathbb{R})\), the multipole spin algebras are more elementary and general.

The most important comparison yet to be made is with the standard matrix representations [2] of a spin-\(s\) system. As discussed in section 3, the matrix representations are associative algebra representations of the \(A^{(\omega)}\), and thus the underlying structures of both theories are identical. However, there are certain aspects of the standard formalism that are implied but not immediately accessible in the real algebraic theory. For example, the non-zero eigenvalues of \(S_p\) predicted by (3.4) and (3.5) are pure imaginary, so projectors into the corresponding eigenspaces cannot be constructed within the real \(A^{(\omega)}\). Since this is the case for \(M^{(2m+1)}\), \(m \in \mathbb{N}\), it suggests these multipoles are not observable, unlike the \(M^{(2m)}\), \(m \in \mathbb{N}\), which have real eigenvalues. In the usual matrix representations, the customary extension of scalars to \(\mathbb{C}\) enables the definition of a complete set of observable multipoles with constructible eigenstate projectors.

Though it may appear so, this is not a defect within the spin algebra formalism, nor is it an indication that the transition to matrix representations is essential to do physics. Instead, it indicates that observable multipoles and spin eigenstates are not fundamental in the description of spin; rather, they are an emergent phenomenon, predicted by a larger algebraic theory, specifically one with some real, central, algebraic element which squares to a negative real. Though this might sound contrived, quantum mechanics expressed as a real algebra contains such a structure; this implies that, in physics, observable spin multipole moments and eigenstates are an emergent, non-trivial prediction of quantum mechanics.

4.2. A comment on some relativistic models
Since we have developed the spin algebras in a very restricted, non-relativistic setting, any comparison of them with objects from relativistic or more physically complete domains will be qualitative at best. However, in the case of Weinberg [17] and later Giraud et al [18, 23] the comparison is more precise. They define tensorial objects which are totally symmetric, and contractionless out of spin operators extended to Minkowski space. These spin operators have the identity as their timelike element, and the resulting tensors are similar to our multipoles, but lack any particular relationship with \(E\). This makes the significance of these objects to the symmetries of Minkowski space much less clear than the relationship the multipoles have to rotational symmetry.

4.3. Interpretational comparison of spin with standard descriptions
In this work, we have established that \(A^{(\omega)}\) captures the essential structure of a spin-\(s\) system. By using real algebraic methods, we have shown that this structure can be derived without the use of dynamics, matrix representations, or complex numbers, amongst other things; they require only those structures naturally associated with the geometric symmetry group \(\text{SO}(3, \mathbb{R})\) of Euclidean three-space. As such, we must conclude that spin is fundamentally a geometric quality, not a dynamical one.

In most formulations of quantum mechanics, it is assumed that spin is an intrinsic form of angular momentum. Other more conservative descriptions, regard it as some fundamental property that \textit{transforms like} and phenomenologically \textit{behaves like} angular momentum. In section 4.1, we highlighted the differences between our description and the standard matrix theory these formulations refer to. Accordingly, the present work supports the latter view, insofar as it has revealed that there are actually two distinct, but equally relevant, notions of spin in physics.

The first notion of spin is the dynamical ‘phenomenological spin’, which is the angular momentum-like property familiar to all physicists. This spin is of primary interest to those wishing to make phenomenological predictions of a physical system from a model. To these efforts, each \(A^{(\omega)}\) offers a comprehensive account of all the meaningful, orthogonal observables for that theory, enabling us to keep track of all the possible interactions our theory can have.
For example, in spin-1 systems, spin quadrupole interactions are often included by squares of a single spin generator \([44, 45]\); examining \(A^{(1)}\) reveals that terms like \(M_{xy}\) are missing from both single- and two-site interactions, and that using \(S_i^2\) instead of \(M_{zz}\) in the Hamiltonian inadvertently introduces a constant energy shift due to the presence of a monopole \(M\) term. This empowers the model builder to more systematically and precisely explore the resulting physics.

The second notion of spin is the geometric ‘fundamental spin’, which is derived solely from rotational symmetry and determines the basic properties of the phenomenological spin. This spin is of primary interest to those interested in quantum foundations due to the insight it offers into our physical theories. Furthermore, the methods we developed to derive this spin can be used to tame other physical models with more extensive or exotic symmetries. The study of this spin, and the fundamental structures that underpin other properties of physical objects, reveal which aspects of a phenomenon are fundamental, and which are actually emergent, and in so doing enrich our understanding of physics.

5. Conclusion

In this paper we have constructed a completely algebraic theory of non-relativistic spin from spatial symmetry using only elementary arguments, without the use of quantisation, dynamics, calculus, matrix representations, or complex numbers. To do this we developed a formalism appropriate to the study of real operators, which can readily be applied to study other symmetries, such as those arising in field theory, amongst other mathematical and physical contexts. Through this formalism, we have shown that a spin-\(s\) system is a finite collection of non-commutative generalisations of Cartesian multipole tensors, and completely determined by specifying only the largest non-zero multipole. In working exclusively with structures naturally related to a geometric symmetry group, we have indicated that spin is fundamentally geometric in nature.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Proof of the promotion and refinement of an \(EE\)-orthogonal decomposition

**Definition A.1.** \((\mathcal{L})\) Let us abuse notation to define the linear maps, \(\forall k \in \mathbb{N},\) using the isomorphism \(\mathfrak{so}(3, \mathbb{R})^\otimes k+1 \cong \mathfrak{so}(3, \mathbb{R}) \otimes \mathfrak{so}(3, \mathbb{R})^\otimes k,\)

\[
\mathcal{L}: \text{End}(\mathfrak{so}(3, \mathbb{R})^\otimes k) \rightarrow \text{End}(\mathfrak{so}(3, \mathbb{R})^\otimes (k+1))
\]

\[
\mathcal{L}(A):= \text{id}_{\text{so}(3, \mathbb{R})} \otimes A = (v \otimes X) \mapsto v \otimes A(X).
\]  

(A.1)

**Lemma A.2.** For all \(k \in \mathbb{N},\)

\[
\text{id}_{\text{so}(3, \mathbb{R})} \otimes \text{id}_{\text{so}(3, \mathbb{R})} = \mathcal{L}(\text{id}_{\text{so}(3, \mathbb{R})^\otimes k}).
\]  

(A.2)

**Proof.** Clear from the definition of \(\mathcal{L}.\)

\(\square\)

As it is slightly less straightforward than for \(U(\mathfrak{so}(3, \mathbb{R})),\) let us be precise in our definition of the adjoint action \(\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))\) on \(\mathfrak{so}(3, \mathbb{R})^\otimes k,\)

**Definition A.3 (Adjoint action on \(\mathfrak{so}(3, \mathbb{R})^\otimes k).** For \(k = 0,\)

\[
\text{ad}^{(0)}: U(\mathfrak{so}(3, \mathbb{R})) \rightarrow \text{End}(\mathbb{R})
\]

\[
\text{ad}^{(0)}(u):= \alpha \mapsto \begin{cases} \mu u & u \in \mathbb{R} \\ 0 & u \not\in \mathbb{R} \end{cases}.
\]  

(A.3)
For all $k \geq 1$, we use the isomorphism $\mathfrak{so}(3, \mathbb{R})^\otimes k \cong \mathfrak{so}(3, \mathbb{R}) \otimes \mathfrak{so}(3, \mathbb{R})^{k-1}$ to define,

$$\text{ad}^{(k)}: U(\mathfrak{so}(3, \mathbb{R})) \to \text{End}(\mathfrak{so}(3, \mathbb{R})^\otimes k)$$

\begin{align*}
\text{ad}^{(k)}(u) &\colon v \otimes X \mapsto \begin{cases} 
(u(v \otimes X)) & u \in \mathbb{R} \\
(u \times v) \otimes X + v \otimes \text{ad}^{k-1}(u)(X) & u \in \mathfrak{so}(3, \mathbb{R}) \\
\text{ad}(a) \circ \text{ad}(b)(v \otimes X) & u = a \otimes b, 
\end{cases} 
\end{align*}

(A.4)

All definitions are extended by linearity in the first argument.

**Remark.** The properties of $\text{ad}^{(k)}$ are identical to $\text{ad}$ defined on $U(\mathfrak{so}(3, \mathbb{R}))$. We will abuse notation by dropping the superscript when the tensor order is clear.

**Lemma A.4.** Consider $A \in \text{End}(\mathfrak{so}(3, \mathbb{R})^\otimes k)$ such that $\text{Im}(A)$ is closed under the action of $\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))$. Then, $\text{Im}(\mathcal{L}(A))$ is closed under the action of $\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))$.

**Proof.** As in the main text, we need only show that $\text{Im}(\mathcal{L}(A))$ is closed under the action of $\text{ad}(\mathfrak{so}(3, \mathbb{R}))$. Using the isomorphism $\mathfrak{so}(3, \mathbb{R})^\otimes k \cong \mathfrak{so}(3, \mathbb{R}) \otimes \mathfrak{so}(3, \mathbb{R})^{k-1}$, $\forall u \in \mathfrak{so}(3, \mathbb{R})$,

$$\text{ad}(u)(v \otimes A(X)) = (u \times v) \otimes A(X) + v \otimes \text{ad}(u) \circ A(X)$$

$$= (u \times v) \otimes A(X) + v \otimes A(Y)$$

$$\in \text{Im}(\mathcal{L}(A)).$$

□

**Lemma A.5.** Suppose we have an $E$-orthogonal decomposition for $\mathfrak{so}(3, \mathbb{R})^\otimes k$, derived from Corollary 2.23, with resolution of the identity,

$$\text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} = \sum_{j=1}^{n} \Pi_j(E),$$

(A.5)

Then $\forall k \in \mathbb{N}$, we may extend this to an $E$-orthogonal decomposition on $\mathfrak{so}(3, \mathbb{R})^\otimes k+1$ with resolution of the identity,

$$\text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k+1} = \sum_{j=1}^{n} \mathcal{L}(\Pi_j(E)).$$

(A.6)

**Proof.** By Lemma A.2, (A.6) follows by applying $\mathcal{L}$ to (A.5). We must now verify that it constitutes an $E$-orthogonal decomposition. First, note that $\forall p \neq q$,

$$\mathcal{L}(\Pi_p(E)) \circ \mathcal{L}(\Pi_p(E)) = \text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes (\Pi_p(E) \circ \Pi_p(E)) = \mathcal{L}(\Pi_p(E))$$

$$\mathcal{L}(\Pi_p(E)) \circ \mathcal{L}(\Pi_q(E)) = \text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes (\Pi_p(E) \circ \Pi_q(E)) = 0,$n

and so by Lemma 2.18, $\forall p \neq q$, $\text{Im}(\mathcal{L}(\Pi_p(E))) \cap \text{Im}(\mathcal{L}(\Pi_q(E))) = \{0\}$. Thus,

$$\mathfrak{so}(3, \mathbb{R})^\otimes k+1 \cong \bigoplus_{j=1}^{n} \text{Im}(\mathcal{L}(\Pi_j(E))).$$

By Lemma A.4, each $\text{Im}(\mathcal{L}(\Pi_j(E)))$ is closed under $E$, so we are done. □

So we have shown that we may promote any $E$-orthogonal decomposition of $\mathfrak{so}(3, \mathbb{R})^\otimes k$ to one of $\mathfrak{so}(3, \mathbb{R})^\otimes k+1$. Let us now show this decomposition may be refined.

**Lemma A.6.** Consider $A \in \text{End}(\mathfrak{so}(3, \mathbb{R})^\otimes k)$. If $[\text{ad}^{(k)}(B), A] = 0$ for some $B \in U(\mathfrak{so}(3, \mathbb{R}))$, then $[\text{ad}^{(k+1)}(B), \mathcal{L}(A)] = 0$.

**Proof.** We need only show this is the case for $\text{ad}^{(k+1)}(v)$, $\forall v \in \mathfrak{so}(3, \mathbb{R})$. Thus,

$$\text{ad}^{(k+1)}(v) \circ \mathcal{L}(A) = \text{ad}^{(k+1)}(v) \circ (\text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes A)$$

$$= \text{ad}^{(k)}(v) \otimes A + \text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes (\text{ad}^{(k)}(v) \circ A)$$

$$= \text{ad}^{(k)}(v) \otimes A + \text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes (A \circ \text{ad}^{(k)}(v))$$

$$= (\text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes A) \circ (\text{ad}^{(k)}(v) \otimes \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} + \text{id}_{\mathfrak{so}(3, \mathbb{R})} \otimes \text{ad}^{(k)}(v))$$

$$= \mathcal{L}(A) \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k}$$

$$= \mathcal{L}(A) \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k}$$

$$= \mathcal{L}(A) \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k}$$

$$= \mathcal{L}(A) \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k} \circ \text{id}_{\mathfrak{so}(3, \mathbb{R})^\otimes k}$$

□
Theorem A.7. The $E$-orthogonal decomposition of $\mathfrak{so}(3, \mathbb{R})^{\otimes k+1}$ from Lemma A.5,
\[
\mathfrak{so}(3, \mathbb{R})^{\otimes k+1} \cong \bigoplus_{j=1}^{n} \text{Im}(\mathcal{L}(\Pi_j(E))),
\]
may be refined to an $E$-orthogonal decomposition,
\[
\mathfrak{so}(3, \mathbb{R})^{\otimes k+1} \cong \bigoplus_{j=1}^{m} \text{Im}(P_{j,l_j}(E)),
\]
where $m_j \in \mathbb{Z}^+$,
\[
\text{Im}(\mathcal{L}(\Pi_j(E))) \cong \bigoplus_{l_j=1}^{m_j} \text{Im}(P_{j,l_j}(E)),
\]
is an $E$-orthogonal decomposition for $\text{Im}(\mathcal{L}(\Pi_j(E)))$, and $\forall (p, r_p) \neq (q, s_q)$ the $\{P_{j,l_j}\}$ satisfy,
\[
\text{id}_{\mathfrak{so}(3, \mathbb{R})^{\otimes k+1}} = \sum_{j=1}^{m} \sum_{l_j=1}^{m_j} P_{j,l_j}(E) \quad \text{(A.7a)}
\]
\[
P_{(p,r_p)}(E) \circ P_{(p,r_p)}(E) = P_{(p,r_p)}(E) \quad \text{(A.7b)}
\]
\[
P_{(p,r_p)}(E) \circ P_{(q,s_q)}(E) = P_{(q,s_q)}(E) \circ P_{(p,r_p)}(E) = 0. \quad \text{(A.7c)}
\]

Proof. As $\text{Im}(\mathcal{L}(\Pi_j(E)))$ is finite-dimensional there must exist an $E$-orthogonal decomposition on it yielding a resolution of $\mathcal{L}(\Pi_j(E))$,
\[
\mathcal{L}(\Pi_j(E)) = \sum_{l_j=1}^{m_j} P_{j,l_j}(E).
\]
The positive integers $m_j$ and the forms of the $\{P_{j,l_j}\}$ depend on the form of the minimal polynomial for $E$ restricted to $\mathfrak{so}(3, \mathbb{R})^{\otimes k+1}$. Given such $E$-orthogonal decompositions $\{P_{j,l_j}\}$, it suffices for us to verify the properties (A.7a)–(A.7c). The identity resolution (A.7a) is obvious, and so by Lemma 2.19, (A.7b) follows from (A.7c).

To prove (A.7c) note that from Theorem 2.22,
\[
P_{j,l_j}(E) = f_{j,l_j}(E) \circ \mathcal{L}(\Pi_j(E)),
\]
for determined polynomial functions $f_{j,l_j}(E)$. When $p = q, \forall r_p \neq s_p$,
\[
P_{(p,r_p)}(E) \circ P_{(p,r_p)}(E) = 0,
\]
by definition. When $p \neq q$, we may use Lemma A.6, to show $\forall r_p, s_q$
\[
P_{(p,r_p)}(E) \circ P_{(q,s_q)}(E) = f_{(p,r_p)}(E) \circ \mathcal{L}(\Pi_p(E)) f_{(q,s_q)}(E) \mathcal{L}(\Pi_q(E)) = f_{(p,r_p)}(E) \circ f_{(q,s_q)}(E) \mathcal{L}(\Pi_p(E) \circ \Pi_q(E)) = 0.
\]

Appendix B. Proof of left action identity

To facilitate this and other proofs we must first discuss some identities. Let us define the right multiplication,
\[
R(\nu) = A \mapsto A \otimes \nu, \quad \text{(B.1)}
\]
where we note this is not a Lie algebra action on $U(\mathfrak{so}(3, \mathbb{R}))$, since,
\[
R(a) \circ R(b) - R(b) \circ R(a) = R(b \times a) - R(a \times b). \quad \text{(B.2)}
\]
This allows us to describe the adjoint action of $\nu \in \mathfrak{so}(3, \mathbb{R})$,
\[
\text{ad}(\nu) = L(\nu) = R(\nu). \quad \text{(B.3)}
\]
Noting $\forall A, B \in U(\mathfrak{so}(3, \mathbb{R}))$,
\[
[L(A), R(B)] = 0, \quad \text{(B.4)}
\]
we easily see the commutators,
\[
[\text{ad}(S_a), L(S_b)] = [L(S_a), L(S_b)] = L(S_a \times S_b). \quad \text{(B.5)}
\]
Next, let us examine $E$ more closely. For central elements $z \in Z(U(\mathfrak{so}(3, \mathbb{R}))),$

\[ [L(z), A] = 0 \]  \hspace{1cm} (B.7)

\[ L(z) = R(z). \]  \hspace{1cm} (B.8)

Then,

\[ E = \sum_{a=1}^{3} \text{ad}(S_a) \circ \text{ad}(S_a) = 2L(S^2) - 2 \sum_{a=1}^{3} L(S_a) \circ R(S_a). \]  \hspace{1cm} (B.9)

We may now proceed with the proof.

\[
\begin{align*}
[E, L(S_b)] &= \sum_{a=1}^{3} [\text{ad}(S_a) \circ \text{ad}(S_a), L(S_b)] \\
&= \sum_{a=1}^{3} \text{ad}(S_a) \circ [\text{ad}(S_a), L(S_b)] + [\text{ad}(S_a), L(S_b)] \circ \text{ad}(S_a) \\
&= \sum_{a=1}^{3} \text{ad}(S_a) \circ L(S_a \times S_b) + L(S_a \times S_b) \circ \text{ad}(S_a) \\
&= \sum_{a,c=1}^{3} \delta_{abc} (\text{ad}(S_a) \circ L(S_c) + L(S_c) \circ \text{ad}(S_a)) \\
&= \sum_{a,c=1}^{3} \delta_{abc} (L(S_a) - R(S_a)) \circ L(S_c) + L(S_c) \circ (L(S_a) - R(S_a))) \\
&= \sum_{a,c=1}^{3} \delta_{abc} L(S_a) \circ L(S_c) + L(S_c) \circ L(S_a) - 2 \sum_{a,c=1}^{3} \delta_{abc} L(S_c) \circ R(S_a).
\end{align*}
\]

Since total contraction of a symmetric and antisymmetric object yields zero we find,

\[ [E, L(S_b)] = -2 \sum_{c,a=1}^{3} \delta_{bca} L(S_c) \circ R(S_a) := -2F(S_b). \]  \hspace{1cm} (B.10)

If we instead had calculated $[E, R(S_b)]$ we would discover that,

\[ [E, L(S_a)] = [E, R(S_a)]. \]  \hspace{1cm} (B.11)

Next let us consider,

\[ [E, F(S_b)] = \sum_{c,a=1}^{3} \delta_{bca} [E, L(S_c) \circ R(S_a)]. \]

From (B.7) and (B.9) we see,

\[
\begin{align*}
[E, F(S_b)] &= -2 \sum_{d,c,a=1}^{3} \delta_{bca} [L(S_d) \circ R(S_a), L(S_c) \circ R(S_a)] \\
&= -2 \sum_{d,c,a=1}^{3} \delta_{bca} (L(S_d) \circ L(S_c) \circ [R(S_a), R(S_a)] + [L(S_d), L(S_c)] \circ R(S_a) \circ R(S_a)) \\
&= -2 \sum_{d,c,a=1}^{3} \delta_{bca} (L(S_d) \circ L(S_c) \circ R(S_a \times S_d) + L(S_d \times S_c) \circ R(S_a) \circ R(S_d)) \\
&= -2 \sum_{e,d,c,a=1}^{3} \delta_{bca} \delta_{ade} L(S_d) \circ L(S_c) \circ R(S_a) - 2 \sum_{e,d,c,a=1}^{3} \delta_{bca} \delta_{ace} L(S_d) \circ R(S_a) \circ R(S_d).
\end{align*}
\]

Utilising,

\[ \sum_{x=1}^{3} \delta_{xap} \delta_{xqs} = \delta_{ps} \delta_{qs} - \delta_{ps} \delta_{qs}, \]  \hspace{1cm} (B.12)
we find,

\[
[E, F(S_b)] = -2 \sum_{c,d=1}^3 (\delta_{bc} \delta_{ce} - \delta_{bd} \delta_{de}) L(S_b) \circ L(S_c) \circ R(S_d)
\]

\[
- 2 \sum_{c,d=1}^3 (\delta_{bd} \delta_{ac} - \delta_{bc} \delta_{ad}) L(S_b) \circ R(S_a) \circ R(S_d)
\]

\[
= -2 \sum_{c=1}^3 L(S_b) \circ L(S_c) \circ R(S_d) + 2 \sum_{c=1}^3 L(S_c) \circ L(S_b) \circ R(S_d)
\]

\[
-2 \sum_{d=1}^3 L(S_b) \circ R(S_a) \circ R(S_d) + 2 \sum_{d=1}^3 L(S_d) \circ R(S_a) \circ R(S_b)
\]

\[
= L(S_b) \circ (E - 2L(S^2)) + 2L(S^2) \circ R(S_b)
\]

\[
+ (E - 2L(S^2)) \circ R(S_b) + 2L(S_b) \circ L(S^2),
\]

and thus,

\[
[E, F(S_b)] = L(S_b) \circ E + E \circ R(S_b) = R(S_b) \circ E + E \circ L(S_b),
\]

with the final equality following from (B.11).

Hence, combining (B.10) and (B.13),

\[
[E, [E, [E, L(S_b)]]]] = -2[E, [E, F(S_b)]]
\]

\[
= -2[E, (L(S_b) \circ E + E \circ R(S_b))]
\]

\[
= -2[E, L(S_b)] \circ E - 2E \circ [E, R(S_b)]
\]

\[
= -2([E, L(S_b)] \circ E + E \circ [E, L(S_b)]),
\]

thus, finally,

\[
[E, [E, [E, L(S_b)]]]] = -2[E^2, L(S_b)].
\]

\[
\Box
\]

**Appendix C. Proof of the properties of the multipoles**

**C.1. Closed under \(\text{ad}(U(\mathfrak{so}(3, \mathbb{R})))\)**

**Lemma C.1.** \(\forall v \in \mathfrak{so}(3, \mathbb{R}), \forall A \in U(\mathfrak{so}(3, \mathbb{R})),\)

\[
[\text{ad}(v), L(A)] = L(\text{ad}(v)(A)).
\]

**Proof.** \(\forall W \in U(\mathfrak{so}(3, \mathbb{R})),\)

\[
[\text{ad}(v), L(A)](W) = \text{ad}(v)(A \otimes W) - A \otimes (\text{ad}(v)(W))
\]

\[
= ((\text{ad}(v)(A)) \otimes W + A \otimes (\text{ad}(v)(W))) - A \otimes (\text{ad}(v)(W))
\]

\[
= L(\text{ad}(v)(A))(W)
\]

where we have used the derivation property of \(\text{ad}(v)\) in the second line. \(\Box\)

**Theorem C.2.** \(\forall k \in \mathbb{N}, A \in U(\mathfrak{so}(3, \mathbb{R})),\)

\[
\forall A \in U(\mathfrak{so}(3, \mathbb{R})), \text{ad}(A) \circ M^{(k)} = M^{(k)} \circ \text{ad}(A).
\]

**Proof.** As ever we need only prove that (C.1) is true \(\forall v \in \mathfrak{so}(3, \mathbb{R}).\) When \(k = 0, \forall \alpha \in \mathbb{R},\)

\[
M^{(0)}(\alpha) = \alpha,
\]

and so (C.1) follows from,

\[
\text{ad}(v)(\alpha) = 0.
\]

When \(k \geq 1, \forall w \in \mathfrak{so}(3, \mathbb{R}),\)

\[
M^{(1)}(w) = w,
\]

and so (C.1) is clear. The, assuming the claim is true for the case \(k = m,\) by definition we have, \(\forall w_j \in \mathfrak{so}(3, \mathbb{R}),\)

\[
M^{(m+1)} \left( w_1 \otimes \bigotimes_{j=2}^{m+1} w_j \right) = \frac{\varepsilon(m-1) \circ \varepsilon(m)}{4(m+1)(2m+1)} \circ L(w_1) \circ M^{(m)} \left( \bigotimes_{j=2}^{m+1} w_j \right)
\]

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and so, using Lemma C.1,

\[
\text{ad}(v) \circ M^{(m+1)} \left( w_1 \otimes \bigotimes_{j=2}^{m+1} w_j \right) = \text{ad}(v) \circ \frac{\varepsilon(m - 1) \circ \varepsilon(m)}{4(m + 1)(2m + 1)} \circ L(w_i) \circ M^{(m)} \left( \bigotimes_{j=2}^{m+1} w_j \right)
\]

\[
= \frac{\varepsilon(m - 1) \circ \varepsilon(m)}{4(m + 1)(2m + 1)} \circ \text{ad}(v) \circ L(w_i) \circ M^{(m)} \left( \bigotimes_{j=2}^{m+1} w_j \right)
\]

\[
= \frac{\varepsilon(m - 1) \circ \varepsilon(m)}{4(m + 1)(2m + 1)} \circ \left( L(\text{ad}(v)(w_i)) \circ M^{(m)} \left( \bigotimes_{j=2}^{m+1} w_j \right) + L(w_i) \circ M^{(m)} \circ \text{ad}(v) \left( \bigotimes_{j=2}^{m+1} w_j \right) \right)
\]

\[
= M^{(m+1)} \left( \text{ad}(v)(w_i) \otimes \bigotimes_{j=2}^{m+1} w_j + w_i \otimes \text{ad}(v) \left( \bigotimes_{j=2}^{m+1} w_j \right) \right)
\]

\[
= M^{(m+1)} \circ \text{ad}(v) \left( w_i \otimes \bigotimes_{j=2}^{m+1} w_j \right),
\]

where the final line follows from the derivation property of \text{ad}(v).

C.2. Totally symmetric and contractionless

Lemma C.3. The quadruple \(M^{(2)}\) satisfies properties (2.52b) and (2.52c).

Proof. By explicit computation we find,

\[
M^{(2)}(S_a \otimes S_b) = \frac{1}{2}(S_a \otimes S_b + S_b \otimes S_a) - \frac{1}{3}\delta_{ab} S^2.
\]

Verifying the properties on this tensor is trivial. □

Definition C.4. For \(M^{(k)}\) satisfying (2.52b) and (2.52c),

\[
A := L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right)
\]

(2.2)

\[
B \circ L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right) := \sum_{p=1}^{k-3} \delta_{ab_1} L(S_a) \circ M^{(k)} \left( S_d \otimes \bigotimes_{j=1}^{k} S_b \right)
\]

(2.3)

\[
C \circ L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right) := \sum_{p=1}^{k} L(S_a) \circ M^{(k)} \left( S_a \otimes \bigotimes_{j=1}^{k} S_b \right)
\]

(2.4)

\[
D \circ L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right) := \sum_{p=1}^{k} \sum_{p=1}^{k} \sum_{p=1}^{k} \delta_{b_1b_2} L(S_a) \circ M^{(k)} \left( S_a \otimes S_d \otimes \bigotimes_{j=1}^{k} S_b \right)
\]

(2.5)

k \geq 2

k = 1.

Lemma C.5. For \(M^{(k)}\) satisfying (2.52b) and (2.52c),

\[
E \circ A = ((-2 - k(k + 1))id + 2B - 2C) \circ A
\]

(2.6)

\[
E \circ E \circ A = ((k + 1)^2(k^2 + 4)id - 4(k^2 + 2)B + 4(k + 1)^2C - 4D) \circ A.
\]

(2.7)

Proof. First, we see from (B.5),

\[
\text{ad}(S_a) \circ L(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right) = L(\text{ad}(S_a)(S_d)) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right)
\]

\[
+ L(S_a) \circ \text{ad}(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_b \right),
\]
and so,
\[ E_0L(S_0) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_j \right) \]
\[ = \sum_{c=1}^{3} \text{ad}(S_c) \circ \text{ad}(S_c) \circ L(S_c) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_j \right) \]
\[ = (-2 - k(k + 1)) L(S_0) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_j \right) + 2 \sum_{c=1}^{3} L(\text{ad}(S_c) \circ \text{ad}(S_c) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_j \right). \]

We may evaluate this second term with the help of (B.12),
\[ 2 \sum_{c=1}^{3} L(\text{ad}(S_c) \circ \text{ad}(S_c) \circ M^{(k)} \left( \bigotimes_{j=1}^{k} S_j \right) \]
\[ = 2 \sum_{p=1, d, e=1}^{k} \delta_{ad,b} L(S_d) \circ M^{(k)} \left( S_d \bigotimes_{j=1, j\neq p}^{k} S_j \right) - 2 \sum_{p=1}^{k} L(S_b) \circ M^{(k)} \left( \bigotimes_{j=1, j\neq p}^{k} S_j \right). \]

The expression for $E_0A$ then follows. Applying a second $E$, we may reuse the results so far to find,
\[ E_0B \circ A = -k(k - 1)B \circ A \]
\[ E_0C \circ A = (-2k + 2 + B - k(k + 3)C + 2D) \circ A. \]

The expression for $E_0A \circ A$ then follows. \[ \square \]

**Lemma C.6.** For $M^{(k)}$ satisfying (2.52b) and (2.52c),
\[ M^{(k+1)} \left( \bigotimes_{j=1}^{k+1} S_j \right) = \frac{1}{k + 1} \sum_{p=1}^{k+1} L(S_{a_p}) \circ M^{(k+1)} \left( \bigotimes_{j=1, j\neq p}^{k+1} S_j \right) \]
\[ - \frac{1}{(k + 1)(2k + 1)} \sum_{p=1}^{k+1} \sum_{q=1}^{k+1} \sum_{d=1}^{3} \delta_{a_p,a_q} L(S_d) \circ M^{(k+1)} \left( S_d \bigotimes_{j=1, j\neq p,q}^{k+1} S_j \right). \]

**Proof.** By Lemma C.5, we see that for $k \geq 1$,
\[ M^{(k+1)} \left( \bigotimes_{j=1}^{k+1} S_j \right) = \frac{\varepsilon(k - 1) \circ \varepsilon(k)}{4(k + 1)(2k + 1)} \circ L(S_{a_k}) \circ M^{(k+1)} \left( \bigotimes_{j=2}^{k+1} S_j \right) \]
\[ = \frac{1}{4(k + 1)(2k + 1)} \left( 4(2k + 1)id - 8B + 4(2k + 1)C - 4D \right) \]
\[ \circ L(S_{a_k}) \circ M^{(k+1)} \left( \bigotimes_{j=2}^{k+1} S_j \right). \]

Noting that,
\[ (2B + D) \circ L(S_{a_k}) \circ M^{(k)} \left( \bigotimes_{j=2}^{k+1} S_j \right) = \left( \sum_{p=1}^{k+1} \sum_{q=1}^{k+1} \sum_{d=1}^{3} \delta_{a_p,a_q} L(S_d) \circ M^{(k)} \left( S_d \bigotimes_{j=1, j\neq p,q}^{k+1} S_j \right) \right) \]
\[ = \sum_{p=1}^{k+1} \sum_{q=1}^{k+1} \sum_{d=1}^{3} \delta_{a_p,a_q} L(S_d) \circ M^{(k)} \left( S_d \bigotimes_{j=1, j\neq p,q}^{k+1} S_j \right), \]

we are done. \[ \square \]

**Lemma C.7.** For $k \in \mathbb{Z}_+$,
\[ \sum_{c=1}^{3} L(S_c) \circ M^{(k)} \left( S_c \bigotimes_{j=1}^{k-1} S_j \right) = \frac{k}{4(2k - 1)} L(4S^2 + (k - 1)(k + 1)) \circ M^{k-1} \left( \bigotimes_{j=1}^{k-1} S_j \right). \]
Proof. For \( k = 1 \), we see,
\[
\sum_{e=1}^{3} L(S_e) \circ M^{(1)}(S_e) = \sum_{e=1}^{3} L(S_e) \circ L(S_e) \circ M^{(0)}(1) = L(S^2) \circ M^{(0)}(1).
\]
For \( k > 1 \), consider,
\[
\sum_{e=1}^{3} L(S_e) \circ M^{(k)}(S_e) = k^{-1} j=1 L(S_e) \circ \varepsilon(k-2) \circ \varepsilon(k-1) L(S_e) \circ M^{(k-1)}(S_e) + 0
\]
and by (D.1), (B.10) and (B.9) we find,
\[
\sum_{e=1}^{3} L(S_e) (E_o, [E, L(S_e)]) - 2(k-1)[E, L(S_e)] M^{(k-1)}(S_e)
\]
This result is consistent with the \( k = 1 \) case.

Corollary C.8. \( \forall k \in \mathbb{Z}^+; \)
\[
B_o L(S_e) \circ M^{(k)}(\times_{j=1}^{k} S_j) := \frac{k}{4(2k-1)} L(4S^2 + (k-1)(k+1)) \circ \sum_{p=1}^{k} \delta_{0p} M^{(k-1)}(\times_{j=1, j \neq p}^{k} S_j)
\]
\[
D_o L(S_a) \circ M^{(k)}(\times_{j=1}^{k} S_j) := \begin{cases} \frac{k}{4(2k-1)} L(4S^2 + (k-1)(k+1)) \circ \sum_{p=1}^{k} \sum_{p=1}^{k} \delta_{pj} M^{(k-1)}(\times_{j=1, j \neq p}^{k} S_j) & k \geq 2 \\ 0 & k = 1. \end{cases}
\]
Proof. Direct substitution.

Corollary C.9. For \( M^{(k)} \) satisfying (2.52b) and (2.52c),
\[
M^{(k+1)}(\times_{j=1}^{k+1} S_j) = \frac{1}{k+1} \sum_{p=1}^{k+1} L(S_{ap}) \circ M^{(k)}(\times_{j=1, j \neq p}^{k+1} S_j)
\]
\[
= \frac{k}{4(k+1)(4k^2 - 1)} L(4S^2 + (k-1)(k+1)) \circ \sum_{p=1}^{k+1} \sum_{p=1}^{k+1} \delta_{ap} M^{(k-1)}(\times_{j=1, j \neq p}^{k+1} S_j).
\]
Proof. Direct substitution.

Theorem C.10. \( \forall k \in \mathbb{N}, k \geq 2, M^{(k)} \) satisfies (2.52b) and (2.52c).
Proof. Our proof is by induction. The base case \( k = 2 \) is proven in Lemma C.3. Assuming the theorem is true for \( M^{(m)} \), \( m \in \{2, \ldots, n\} \), for some \( n \in \mathbb{N} \), then \( M^{(m+1)} \) has form given in Corollary C.9. That \( M^{(m+1)}(\otimes_{j=1}^{k+1}S_{a_j}) \) is totally symmetric is clear. To establish contractionlessness, fix two indices \( r = s \in \{1, \ldots, k + 1\} \). If a contraction occurs between two indices within \( M^{(k)} \) or \( M^{(k-1)} \), then by assumption these give zero. Therefore, we need only check contractions occurring partially inside or completely outside \( M^{(k)} \) and \( M^{(k-1)} \). Let us first consider the first term in Corollary C.9,

\[
\frac{1}{k + 1} \sum_{a_1, a_2} \delta_{a_1, a_2} \sum_{p=1}^{k+1} L(S_{a_2}) \circ M^{(k)}(\otimes_{j=1,j \neq p}^{k+1} S_{a_j}) = -\frac{2}{k + 1} \sum_{d=1}^{k+1} L(S_d) \circ M^{(k)}(\otimes_{j=1,j \neq p,s}^{k+1} S_{a_j}),
\]

which by Lemma C.7 becomes,

\[
\ldots = \frac{k}{2(k+1)(2k-1)} L(4S^2 + (k-1)(k+1)) \circ M^{(k-1)}(\otimes_{j=1,j \neq p,s}^{k+1} S_{a_j}). \tag{C.10}
\]

Now consider the second term in Corollary C.9,

\[
= -\frac{k}{4(k+1)(4k^2 - 1)} \sum_{a_1, a_2} \delta_{a_1, a_2} L(4S^2 + (k-1)(k+1)) \circ \sum_{p=1}^{k+1} \sum_{q=1}^{k+1} \delta_{a_p, a_q} M^{(k-1)}(\otimes_{j=1,j \neq p,q}^{k+1} S_{a_j})
\]

\[
+ \sum_{q=1}^{k+1} \delta_{d_q} M^{(k-1)}(S_d \otimes \otimes_{j=1,j \neq q,r}^{k+1} S_{a_j}) + \sum_{p=1}^{k+1} \delta_{a_p} M^{(k-1)}(S_d \otimes \otimes_{j=1,j \neq r,q}^{k+1} S_{a_j})
\]

\[
= -\frac{k}{2(k+1)(2k-1)} L(4S^2 + (k-1)(k+1)) \circ M^{(k-1)}(\otimes_{j=1,j \neq p,s}^{k+1} S_{a_j}).
\]

Thus, we get exact cancellation between the two terms, and so \( \forall \ r = s \in \{1, \ldots, k + 1\} \),

\[
\sum_{a_1, a_2} \delta_{a_1, a_2} M^{(k+1)}(\otimes_{j=1}^{k+1} S_{a_j}) = 0.
\]

\[
\square
\]

Appendix D. Derivation of the images of multipoles under step-level and step-down

Here we will prove the results given in (2.53a) and (2.53b).

D.1. Step-level image

The step-level \( \nu \in \mathfrak{so}(3, \mathbb{R}) \) of a multipole \( M^{(k)}(\cdot) \) is given by,

\[
L^-(\nu) \circ M^{(k)}(\cdot) = \frac{\varepsilon(k-1) \circ (k+1)}{-4k(k+1)} \circ L(\nu) \circ M^{(k)}(\cdot).
\]

Commuting through the \( \varepsilon(\cdot) \) we find,

\[
\ldots = \frac{1}{-4k(k+1)} ([E, [E, L(\nu)]) + 2[E, L(\nu)] - 4k(k+1) L(\nu) \circ M^{(k)}(\cdot).
\]

From (B.13) we see that,

\[
[E, [E, L(\nu)]) = -2(R(\nu) \circ E + E \circ L(\nu))
\]

\[
= -2[E, L(\nu)] - 2(R(\nu) \circ E + L(\nu) \circ E), \tag{D.1}
\]

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which we combine with the previous equation and (B.3) to find,

$$
L^*(\nu) \circ M^{(k)} = \frac{1}{-4k(k + 1)}(-2(R(\nu) + L(\nu)) \circ E - 4k(k + 1)L(\nu)) \circ M^{(k)}
$$

$$
= \frac{-2k(k + 1)}{-4k(k + 1)}(L(\nu) - R(\nu)) \circ M^{(k)}
$$

$$
= \frac{1}{2}\text{ad}(\nu) \circ M^{(k)}
$$

$$
= \frac{1}{2}M^{(k)} \circ \text{ad}(\nu),
$$

from which (2.53b) follows.

D.2. Step-down image
The step-down by \( S_a \in \mathfrak{so}(3, \mathbb{R}) \) of a multipole \( M^{(k)} \) is given by,

$$
L^j(S_a) \circ M^{(k)} = \frac{\varepsilon(k) \circ \varepsilon(k + 1)}{4k(2k + 1)} \circ L(S_a) \circ M^{(k)}
$$

$$
= \frac{(E^2 + 2(k + 1)^2E + k(k + 1)^2(k + 2)\text{id}) \circ L(S_a) \circ M^{(k)}}{4k(2k + 1)}.
$$

Using Lemma C.5 we find,

$$
L^j(S_a) \circ M^{(k)} \left( \bigotimes_{j=1}^k S_{b_j} \right) = \frac{(2k - 1)B - D}{k(2k + 1)} \circ A.
$$

The identity (2.53a) follows by direct substitution using Lemma C.8.

D.3. Right multiplication images
The results of the previous subsections may be utilised to derive the form of a right multiplication of a multipole. This is essential to expand the table 2.

We observe that from the definition of \( \text{ad}(\nu) \) where \( \nu \in \mathfrak{so}(3, \mathbb{R}) \),

$$
0 = \text{ad}(\nu) \circ \varepsilon(k) \circ M^{(k)} = \varepsilon(k) \circ \text{ad}(\nu) \circ M^{(k)} = \varepsilon(k) \circ (L(\nu) - R(\nu)) \circ M^{(k)},
$$

and so,

$$
\varepsilon(k) \circ L(\nu) \circ M^{(k)} = \varepsilon(k) \circ R(\nu) \circ M^{(k)},
$$

which implies that,

$$
R^j(\nu) \circ M^{(k)} = L^j(\nu) \circ M^{(k)}
$$

$$
R^j(\nu) \circ M^{(k)} = L^j(\nu) \circ M^{(k)}.
$$

While,

$$
R^\dagger(\nu) \circ M^{(k)} = \frac{\varepsilon(k - 1) \circ \varepsilon(k + 1)}{-4k(k + 1)} \circ R(\nu) \circ M^{(k)}
$$

$$
= \frac{\varepsilon(k - 1) \circ \varepsilon(k + 1)}{-4k(k + 1)} \circ (L(\nu) - \text{ad}(\nu)) \circ M^{(k)}
$$

$$
= L^\dagger(\nu) \circ M^{(k)} - \text{ad}(\nu) \circ M^{(k)}
$$

$$
= -\frac{1}{2}\text{ad}(\nu) \circ M^{(k)},
$$

which from (D.2) gives,

$$
R^\dagger(\nu) \circ M^{(k)} = -L^\dagger(\nu) \circ M^{(k)}.
$$

Appendix E. Proof of the minimal polynomial of \( E \) left multiplied multipoles

Lemma E.1. \( M^{(0)} \) and \( M^{(1)} \) are non-zero.

Proof. By definition. □

Lemma E.2. If \( M^{(k)} \neq 0 \) is non-zero from some \( k \in \mathbb{N}, \forall n \leq k \ M^{(n)} \neq 0 \) is also non-zero.
Proof. If \( M^{(n)} = 0 \) is zero, then \( M^{(k)} = 0 \) by their relationship via step-ups. The claim is the contraposition of this fact.

**Lemma E.3.** If for some \( k \in \mathbb{N} \), \( \exists \ a \in \{1, 2, 3\} \), such that,

\[
M^{(k)} \left( \otimes_{j=1}^{k} S_{a_{j}} \right) = 0.
\]

Then,

\[
M^{(k+1)} \left( \otimes_{j=1}^{k+1} S_{a_{j}} \right) = 0.
\]

**Proof.** The case \( k = 0 \) is trivial, so consider \( M^{(k+1)} \) with \( k \geq 1 \) written as in Lemma C.6. Then we may write,

\[
M^{(k+1)} \left( \otimes_{j=1}^{k+1} S_{a_{j}} \right) = L(S_{a}) \circ M^{(k)} \left( \otimes_{j=1}^{k} S_{a_{j}} \right) - \frac{k}{(2k + 1)} \sum_{d=1}^{3} L(S_{d}) \circ M^{(k)} \left( S_{d} \otimes \otimes_{j=1}^{k} S_{a_{j}} \right)
\]

That each of these terms is linearly independent follows from the Poincaré-Birkhoff-Witt theorem [46]. Thus, since the first term is non-zero by assumption, we have established the claim.

**Lemma E.4.** (2.43b) is minimal on \( \text{Im}(L(v) \circ M^{(0)}) \).

**Proof.** This follows from Lemma 2.39.

**Theorem E.5.** (2.42) is minimal on \( \text{Im}(L(v) \circ M^{(k)}) \), \( \forall \ k \in \mathbb{Z}^{+} \).

**Proof.** Consider that the step operators contain pairs of factors from the annihilating polynomial \( \varepsilon(k-1) \circ \varepsilon(k) \varepsilon(k+1) \) of (2.38). Lemma E.3 established that \( M^{(k)} = 0 \), \( \forall \ k \in \mathbb{Z}^{+} \), and so,

\[
M^{(k+1)} \left( w_{1} \otimes \otimes_{j=2}^{k+1} w_{j} \right) = \frac{\varepsilon(k-1) \circ \varepsilon(k) \circ L(w_{1}) \circ M^{(k)}}{4(k + 1)(2k + 1)} \left( \otimes_{j=2}^{k+1} w_{j} \right) = 0
\]

Thus, \( \varepsilon(k-1) \circ \varepsilon(k) \) is not annihilating on \( \text{Im}(L(v) \circ M^{(k)}) \). Now consider the step-down from the multipole \( M^{(k)} \) of equation (2.53a) on the tensor \( \otimes_{j=1}^{k+1} S_{a_{j}} \),

\[
L^{-}(S_{a}) \circ M^{(k)} \left( \otimes_{j=1}^{k} S_{a_{j}} \right) = \left\{ \begin{array}{ll}
\frac{k^{2}}{4(4k^{2} - 1)} L(4S_{2}^{2} + (k-1)(k+1)) \circ M^{(k-1)} \left( \otimes_{j=1}^{k-1} S_{a_{j}} \right) & k \geq 2 \\
\frac{1}{3} S_{2}^{k} & k = 1
\end{array} \right.
\]

This is also non-zero, so,

\[
L^{-}(S_{w}) \circ M^{(k)} \left( \otimes_{j=1}^{k+1} S_{a_{j}} \right) = \frac{\varepsilon(k) \circ \varepsilon(k+1) \circ L(w_{1}) \circ M^{(k)}}{4k(2k+1)} \left( \otimes_{j=2}^{k+1} S_{a_{j}} \right) = 0.
\]

Thus, \( \varepsilon(k) \circ \varepsilon(k+1) \) is not annihilating on \( \text{Im}(L(v) \circ M^{(k)}) \). Finally, consider the form of the step-level from the multipole \( M^{(k)} \) of equation (D.2),

\[
L^{-}(S_{a}) \circ M^{(k)} = \frac{1}{2} \text{ad}(S_{a}) \circ M^{(k)}.
\]

If \( L^{-}(S_{a}) \circ M^{(k)} = 0 \), then \( \text{ad}(\mathfrak{so}(3, \mathbb{R})) \) must annihilate \( M^{(k)} \). This means that \( E \) must also annihilate \( M^{(k)} \). But by definition \( \varepsilon(k) \) annihilates \( M^{(k)} \), so,

\[
\varepsilon(k) \circ M^{(k)} = 0 = E \circ M^{(k)}
\]

i.e.

\[
k(k+1)M^{(k)} = 0.
\]
This implies $M^{(k)} = 0$, which contradicts Lemma F.3. So from,
\[
L^{-}(S_{w_i}) \sigma M^{(k)} \left( \bigotimes_{j=2}^{k+1} S_{w_j} \right) = \frac{1}{2}(k-1) \sigma (k + 1) - 4k(k + 1) \sigma L(v_i) \sigma M^{(k)} \left( \bigotimes_{j=2}^{k+1} S_{w_j} \right) = 0,
\]
$\varepsilon(k - 1) \sigma (k + 1)$ is not annihilating on $\text{Im}(L(v) \sigma M^{(k)})$. Therefore, no pair of factors from (2.38) is annihilating on $\text{Im}(L(v) \sigma M^{(k)})$. Therefore, there is no annihilating polynomial of lower order within (2.38).
Thus, from Lemma 2.15 (2.38) is the minimal polynomial. \hfill \square

**Appendix F. Proof of scalar multiple Casimir action**

Here we will prove that on the quotient algebra $A^\left( \frac{3}{2} \right)$,
\[
A^\left( \frac{3}{2} \right) = \frac{U(\mathfrak{so}(3, \mathbb{R}))}{I(\text{Im}(M^{(k+1)}))},
\]
of $U(\mathfrak{so}(3, \mathbb{R}))$ by the ideal generated by $\text{Im}(M^{(k+1)}), k \in \mathbb{N}$, that the Casimir element $S^2$ acts as a scalar,
\[
L(S^2) = L\left( \frac{-k(k + 2)}{4} \right).
\]
To do this, consider $\text{Im}(f)$ where,
\[
f := S_a \bigotimes_{j=1}^{k+1} S_{b_j} \mapsto L^1(S_a) \sigma L^1(S_{b_1}) \sigma M^{(k)} \left( \bigotimes_{j=2}^{k+1} S_{b_j} \right) \quad (F.1)
\]
From (2.49) we know,
\[
L^1(S_{b_1}) \sigma M^{(k)}(A) = M^{(k+1)}(S_{b_1} \otimes A) = 0, \quad (F.2)
\]
with the final equality following from $M^{(k+1)} = 0$ in $A^\left( \frac{3}{2} \right)$, and thus $f = 0$. However, we know from the main analysis of $U(\mathfrak{so}(3, \mathbb{R}))$ that $\text{Im}(f)$ can be written as a linear combination of central multiples of $M^{(k)}$. Since $M^{(k)}$ is non-zero in $A^\left( \frac{3}{2} \right)$, and $\text{Im}(f)$ is non-trivial in $U(\mathfrak{so}(3, \mathbb{R}))$, there must be some new identity amongst the central multiples causing $\text{Im}(f)$ to be trivial.

Equations (F.1) and (F.2) show that we are studying a step-down from the multipole $M^{(k+1)}$. When $k = 0$,
\[
0 = f(S_a \otimes S_b) = \frac{1}{3} S^2 \delta_{ab} = \frac{1}{3} L(S^2) \sigma M^{(0)}(1),
\]
from (2.53a). Since $A^{(0)}$ is spanned by $M^{(0)}$, we conclude that on the whole of $A^{(0)}$,
\[
L(S^2) = 0. \quad (F.4)
\]
If $k > 0$ we use (2.53a) to find,
\[
0 = \frac{L(4S^2 + k(k + 2))}{4(2k + 3)(2k + 1)} \sum_{p=1}^{k+1} \left( 2k + 1 \right) \delta_{ab}M^{(k)} \left( \bigotimes_{j=p}^{k+1} S_{b_j} \right) - \sum_{q=1, q \neq p}^{k+1} \delta_{b_qb_p}M^{(k)} \left( S_a \otimes \bigotimes_{j=p, q}^{k+1} S_{b_j} \right)
\]
\[
= \frac{L(4S^2 + k(k + 2))}{4(2k + 3)(2k + 1)} \sum_{p=1}^{k+1} \sum_{q=1, q \neq p}^{k+1} \left( \frac{2k + 1}{k} \delta_{ab}S_{b_q} - \delta_{b_qb_p}S_a \right) \otimes \bigotimes_{j=p, q}^{k+1} S_{b_j}.
\]
In $U(\mathfrak{so}(3, \mathbb{R}))$ the prefactor in the above has trivial kernel, since $U(\mathfrak{so}(3, \mathbb{R}))$ contains no zero divisors [47]. Furthermore, it is clear that there are enough arguments $\{C_j\}$ of the form in the argument of $M^{(k)}$ above such that $\{M^{(k)}(C_j)\}$ spans $\text{Im}(f)$, and therefore $\text{Im}(M^{(k)})$. Since $\text{Im}(M^{(k)})$ is non-trivial in $A^\left( \frac{3}{2} \right)$, and $k > 0$ we must have that,
\[
L(4S^2 + k(k + 2)) \sigma M^{(k)} = 0. \quad (F.5)
\]
As $4S^2 + k(k + 2)$ is central we may repeat the process by stepping-down (F.5), producing a family of identities $\forall n \in \{0, \ldots, k\}$,
\[
L \left( 4S^2 + j(j + 2)) \sigma M^{(n)} \right) = 0, \quad (F.6)
\]
where $\bigotimes_{j=n}^{k}$ denotes composition over the indexed maps.
A priori, any combination of these $L (4S^2 + j(j + 2))$ could be responsible for annihilating $\text{Im}(M^{(n)})$. To make progress, let us first consider a non-empty subset $I \subset \{0, \ldots, k - 1\}$ and suppose,
\[(\bigotimes_{j \in I} (4S^2 + j(j + 2))) \circ M^{(n)} = 0, \quad \text{(F.7)}\]

for some \(n \in \{0, \ldots, k\}\). Since \(M^{(k)}\) may be written as a series of step-ups from \(M^{(n)}\) by (2.49), we may use the fact that the composition in (F.7) commutes with step-ups to find,

\[(\bigotimes_{j \in I} (4S^2 + j(j + 2))) \circ M^{(k)} = 0. \quad \text{(F.8)}\]

Now, observe that for any \(p\) we may write,

\[L(4S^2 + k(k + 2)) = L(4S^2 + p(p + 2)) + L((k - p)(k + p + 2)), \quad \text{(F.9)}\]

which we may use to rewrite (F.8) as,

\[\left(\bigotimes_{j \in I} (4S^2 + k(k + 2)) - L((k - j)(k + j + 2)))\right) \circ M^{(k)} = 0. \quad \text{(F.10)}\]

We note that since \(k \notin I\) that there is a left multiplication of a non-zero scalar in (F.10). Thus, from (F.5) we find,

\[\prod_{j \in I} (k - j)(k + j + 2) M^{(k)} = 0, \quad \text{(F.11)}\]

which implies \(\text{Im}(M^{(k)})\) is trivial. This is in contradiction with our construction of \(A^{\left(\frac{k}{2}\right)}\) and thus (F.7) must be impossible \(\forall n \in \{0, \ldots, k\}\). This means any annihilating action of a composition of factors \(L(4S^2 + p(p + 2))\) must include the factor with \(p = k\).

With this in hand, let us consider the identity (F.6) on \(M^{(0)}\),

\[\bigotimes_{j = 0}^{k} (4S^2 + j(j + 2)) \circ M^{(0)} = 0, \quad \text{(F.12)}\]

and notice that it is an annihilating polynomial for \(L(S^2)\) on \(\text{Im}(M^{(0)})\). Thus, using the results of section 2.3 we may resolve the identity on \(\text{Im}(M^{(0)})\),

\[M^{(0)} = \sum_{j = 0}^{k} \bigotimes_{p = 0, p \neq j}^{k} L(4S^2 + p(p + 2)) - j(j + 2) + p(p + 2) \circ M^{(0)} = \sum_{j = 0}^{k} \Pi_j. \quad \text{(F.13)}\]

From our earlier argument, no annihilating composition like (F.7) can exist, and thus we must conclude that \(\text{Im}(M^{(0)}) \cap \text{Im}(\Pi_j) = \{0\}\), since \(\Pi_j\) contains no factor \(L(4S^2 + k(k + 2))\) by definition. However, \(\text{dim}(\text{Im}(M^{(0)})) = 1\), and since \(\Pi_j\) is linear, we must conclude that \(\text{Im}(M^{(0)}) \subset \text{Im}(\Pi_j)\) and \(\text{Im}(M^{(0)}) \cap \text{Im}(\Pi_j) = \{0\}\) for \(j \neq k\) by orthogonality.

Thus, for \(j \neq k\), \(\Pi_j = 0\), and so we have a family of annihilating polynomials for \(L(S^2)\) on \(\text{Im}(M^{(0)}), \forall j \in \{0, \ldots, k-1\}\),

\[\bigotimes_{p = 0, p \neq j}^{k} L(4S^2 + p(p + 2)) \circ M^{(0)} = 0. \quad \text{(F.14)}\]

But every annihilating polynomial must be a polynomial multiple of the minimal polynomial [40]. Since the family (F.14) have only one factor in common, we must conclude that,

\[L(4S^2 + k(k + 2)) \circ M^{(0)} = 0. \quad \text{(F.15)}\]

By the recursive relationship between the multipoles (2.49), all \(M^{(j)}\) begin from repeated stepping-up from \(M^{(0)}\). \(L(4S^2 + k(k + 2))\) is commutative with step-ups, thus we find \(\forall n \in \{0, \ldots, k\}\),

\[L(4S^2 + k(k + 2)) \circ M^{(n)} = 0. \quad \text{(F.16)}\]

Since the multipoles \(\{M^{(n)}\} n \in \{0, \ldots, k\}\) form a basis for \(A^{\left(\frac{k}{2}\right)}\), from (F.16) we must finally conclude that on the whole of \(A^{\left(\frac{k}{2}\right)}\),

\[L(4S^2 + k(k + 2)) = 0. \quad \text{(F.17)}\]

**Appendix G. Proof of the dimension of the multipoles**

**Lemma G.1.** Let \(f : \mathfrak{so}(3, \mathbb{R})^{\otimes k} \rightarrow \mathfrak{so}(3, \mathbb{R})^{\otimes k}\) be a projector into a subspace of \(k\)-th order tensors which satisfy the properties 2.46. Then, \(\text{Im}(f)\) is annihilated by \(\varepsilon(k)\).
Proof.

\[ Eof \left( \bigotimes_{j=1}^{k} S_{j} \right) = \sum_{i=1}^{3} \text{ad}(S_{i}) \circ \text{ad}(S_{i}) \circ \left( \bigotimes_{j=1}^{k} S_{j} \right) \]

\[ = \sum_{p=1}^{k} \sum_{d,e=1}^{3} \varepsilon_{cd,e} \text{ad}(S_{d}) \circ \left( S_{d} \otimes \left( \bigotimes_{j=1, j \neq p}^{k} S_{j} \right) \right) \]

\[ = \sum_{p=1}^{k} \sum_{d,e=1}^{3} \varepsilon_{cd,e} \text{ad}(S_{d}) \circ \left( S_{d} \otimes \left( \bigotimes_{j=1, j \neq p}^{k} S_{j} \right) \right) + \sum_{p=1}^{k} \sum_{q=1}^{k} \sum_{e=1}^{3} \varepsilon_{a_{d},q} \delta_{de} - \delta_{a_{d},e} \delta_{de} \circ \left( S_{d} \otimes S_{e} \otimes \left( \bigotimes_{j=1, j \neq p,q}^{k} S_{j} \right) \right) \]

\[ = -2k - k(k - 1) \circ \left( \bigotimes_{j=1}^{k} S_{j} \right) \]

\[ = -k(k + 1) \circ \left( \bigotimes_{j=1}^{k} S_{j} \right). \]

\[ \square \]

Lemma G.2. In \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \), the maximal subspace of totally symmetric contractionless tensors has dimension \( 2k + 1 \).

Proof. There are \( \binom{k + 2}{k} = \frac{(k + 2)(k + 1)}{2} \) symmetric tensors within \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \). Every possible contraction between these tensors reduces the number of linearly independent tensors by 1. There are \( \frac{k(k - 1)}{2} \) ways to contract a \( k \)-th order tensor, since contraction is a symmetric process. Therefore, there are,

\[ \frac{(k + 2)(k + 1)}{2} - \frac{k(k - 1)}{2} = 2k + 1, \]

linearly independent, totally symmetric, contractionless tensors.

\[ \square \]

Lemma G.3. \( \forall k \in \mathbb{N}, \text{Im}(M^{(k)}) \text{ is unique subspace annihilated by } \varepsilon(k) \text{ within the E-orthogonal decomposition of } \mathfrak{so}(3, \mathbb{R})^{\otimes k}. \) Furthermore, there are no non-trivial subspaces within \( \mathfrak{so}(3, \mathbb{R})^{\otimes k} \text{ annihilated by } \varepsilon(n) \text{ for } n \in \mathbb{N}, n > k. \)

Proof. The case \( k = 0 \) is trivial. We may proceed by induction. Suppose the statement is true for \( k = m \), then by Lemma 2.48 and Theorem 2.50 (applied to \( T(\mathfrak{so}(3, \mathbb{R})) \)), stepping-up from any subspace of \( \mathfrak{so}(3, \mathbb{R})^{\otimes m} \) other than \( \text{Im}(M^{(m)}) \) results in a subspace annihilated by \( \varepsilon(n) \) with \( n < k \). On the other hand, by assumption, there are no subspaces annihilated by \( \varepsilon(m + 1) \) or \( \varepsilon(m + 2) \) that we may step-level or step-down from respectively. Thus, the only subspace which steps-up to one annihilated by \( \varepsilon(m + 1) \) is \( \text{Im}(M^{(m)}) \). This subspace is by definition \( \text{Im}(M^{(m + 1)}) \), and from this argument is clearly unique and has the largest \( n \) amongst subspaces annihilated by \( \varepsilon(n) \).

\[ \square \]

Lemma G.4. No totally symmetric contractionless tensors are forced to zero during the quotient from \( T(\mathfrak{so}(3, \mathbb{R})) \) to \( U(\mathfrak{so}(3, \mathbb{R})) \).

Proof. This is a consequence of the Poincaré-Birkhoff-Witt theorem [46].

\[ \square \]

Corollary G.5. \( \text{Im}(M^{(k)}) \text{ contains all the totally symmetric, contractionless tensors of order } k, \text{ and therefore has dimension } 2k + 1. \)

Proof. This follows easily from the lemmas established in this appendix.

\[ \square \]

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