Essential on-Brane Equations for the Braneworld Gravity under the Schwarzschild Ansatz

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(Dated: May 2, 2014)

PACS numbers: 04.50.-h, 04.50.Kd, 11.10.Kk, 11.25.Mj

I. INTRODUCTION

Einstein gravity is successful in explaining (i) the Newton's law of universal gravitation for moderate distances, and (ii) the post-Newtonian evidences (light deflections and planetary perihelion precessions due to solar gravity, etc.). The explanations are achieved via the Schwarzschild solution of the Einstein equation. We show the three essential equations for the braneworld gravity, which are exactly solvable with two arbitrary functions. The arbitrariness may affect the predictive powers on the Newtonian and the post-Newtonian evidences.

It is argued that the braneworld gravity under the Schwarzschild ansatz should obey three essential equations on the brane, and they are solved exactly. We express the general solution of the fundamental equations of the braneworld in power series of the brane normal coordinate. The power series are derived by solving three independent components of the bulk Einstein equation, and the coefficients are recursively determined in terms of functions on the brane. The extension of the bulk Einstein equation are automatically satisfied, as far as the five on-brane functions obey three essential equations. They are the radial-extra and the extra-extra components of the bulk Einstein equation on the brane, and the equation of motion of the brane. Therefore the solution includes two arbitrary functions on the brane. We show that the essential equations are exactly solved by choosing an appropriate set of the two arbitrary functions. The arbitrariness may affect the predictive powers on the Newtonian and the post-Newtonian evidences.

II. FUNDAMENTAL EQUATIONS

In order to seek for the general solution, we begin with examining what are the equations to be solved. For definiteness, we consider the 3+1 dimensional Nambu-Goto type brane in 4+1 curved spacetime with Einstein type gravity. Let $X^{I}$ be the bulk coordinate, and $g_{IJ}(X^{K})$ be the bulk metric at the point $X^{K}$ [40]. Let the brane be located at $X^{I} = Y^{I}(x^{\mu})$ in the bulk, where $x^{\mu}$ ($\mu = 0, 1, 2, 3$) are parameters which serve as the brane coordinate. The dynamical variables of the system are $g_{IJ}(X^{K})$, $Y^{I}(x^{\mu})$, and matter fields. The $Y^{I}(x^{\mu})$ should
be taken as the collective modes of the brane formed by matter interactions. Note that it is inappropriate to take the induced metric $g_{ij} = Y_{i}^{p} Y_{j}^{q} g_{pq}(Y^{K})$ as a dynamical variable, since it alone cannot completely specify the state of the braneworld [40]. Then, the action integral is given by

$$\int \sqrt{-g}(\kappa^{-1}R - 2\lambda)d^{4}X - 2\lambda \int \sqrt{-\tilde{g}}d^{4}x + S_{m}, \quad (1)$$

where $S_{m}$ is the matter action, $g = \det g_{ij}$, $R = R_{ij}^{\mu}R_{j}^{\mu}$, $R_{ij}^{\mu}$ is the bulk curvature tensor written in terms of $g_{ij}$, $\tilde{g} = \det \tilde{g}_{\mu\nu}$, and $\lambda$, $\kappa$, and $\lambda$ are constants. We add no artificial fine-tuning terms. The equations of motion are derived by varying the action (1) with respect to the dynamical variables $g^{ij}$, $Y^{i}$ and the field variables:

$$\mathcal{E}_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} + \kappa(T_{ij} + \lambda g_{ij}) = 0, \quad (2)$$

$$\mathcal{N}^{i} \equiv (\tilde{g}^{\mu\nu} + \tilde{T}^{\mu\nu})Y_{\mu\nu}^{i} - 0, \quad (3)$$

and those for matters, where $T_{ij}$ and $\tilde{T}_{\mu\nu}$ are the energy momentum tensors with respect to $g_{ij}$ and $\tilde{g}_{\mu\nu}$, respectively [40]. Eq. (2) is the bulk Einstein equation, and (3) is the Nambu-Goto equation for the braneworld dynamics.

In accordance with the ansatz of staticity and sphericity, we introduce time, redial, polar and azimuth coordinates, $X^{0} = t$, $X^{1} = r$, $X^{2} = \theta$, and $X^{3} = \varphi$, respectively, and the normal geodesic coordinate $X^{4} = z$, such that $X^{\mu} = \tilde{x}^{\mu}$ ($\mu = 0, \cdots, 3$) and $z = 0$ on the brane. According to staticity and sphericity we can generally choose the metric tensor $g_{ij}$ of the form

$$ds^{2} = g_{ij}dX^{i}dX^{j} = f dt^{2} - h dr^{2} - k(d\theta^{2} + \sin^{2}\theta d\varphi^{2}) - dz^{2}, \quad (4)$$

where $f$, $h$ and $k$ are functions of $r$ and $z$ only, and we choose as $k|_{z=0} = r^{2}$ using defformism. Asymptotic flatness of the brane implies that

$$f, h \to 1 \quad \text{as} \quad r \to \infty \quad \text{at} \quad z = 0. \quad (5)$$

The independent non-vanishing components of the Ricci tensor $R_{ij}$ are

$$R_{00} = -f_{z}^{2}/2 + f_{z}^{2}/f_{r} + f_{z}2/h + f_{z}k_{z}/2/k, \quad (6)$$

$$R_{11} = h_{z}^{2}/2 - h_{z}^{2}/4h + h_{z}k_{z}/2/k + +f_{r}^{2}/f_{r}^{2} - f_{r}^{2}/4h^{2} - f_{r}h_{r}/2kh, \quad (7)$$

$$R_{22} = k_{z}^{2}/2 + f_{z}k_{z}/4h + h_{z}k_{z}/4h + +k_{r}^{2}/2h + f_{r}k_{r}/4h - h_{r}k_{r}/4h^{2} - h_{r}k_{r}/2kh, \quad (8)$$

$$R_{44} = -f_{z}^{2}/2f_{r} + h_{z}k_{z}/2h + k_{z}^{2}/f_{r}^{2} - h_{z}k_{z}/4h^{2} - h_{z}k_{z}/2kh, \quad (9)$$

$$R_{14} = f_{z}/2f_{r} + k_{r}/k_{z} - f_{z}k_{z}/4h^{2} - h_{z}k_{z}^{2}/2k^{2} - k_{z}k_{r}/2k^{2}, \quad (10)$$

where subscripts $r$ and $z$ indicate partial differentiations. The ansatz of staticity and sphericity also indicate that the only independent nonvanishing components of the energy momentum tensors are $T_{00}, T_{11}, T_{22}, T_{14}, T_{44}$, $\tilde{T}_{00}, \tilde{T}_{11}$ and $\tilde{T}_{22}$.

### III. Off-brane solution of the bulk Einstein equation

We first solve the bulk Einstein equation (2) alone without brane dynamics (3). We have five equations for three functions $f$, $h$ and $k$. The equations are not all independent because the conservation law holds:

$$g^{JK}\xi_{J,K} = 0. \quad (11)$$

We rewrite the equation (2) into the equivalent form

$$\mathcal{R}_{ij} \equiv R_{ij} + \kappa(T_{ij} - Tg_{ij}/3 - 2\lambda g_{ij}/3) = 0, \quad (12)$$

which is more convenient in deriving solutions below. In terms of $\mathcal{R}_{ij}$, we have $\mathcal{E}_{ij} = \mathcal{R}_{ij} - \mathcal{R}_{gij}/2$. For $\mathcal{R}_{ij}$ with (6)–(10), the conservation law (11) is written as

$$\mathcal{R}_{14,4} = -\frac{1}{2}\mathcal{R}_{00,1} - \frac{1}{2h}\mathcal{R}_{11,1} + \frac{1}{k}\mathcal{R}_{22,1} + \frac{1}{2}\mathcal{R}_{44,1} - \left(\frac{f_{r}^{2}}{f_{r}} - \frac{h_{r}^{2}}{4h} + \frac{k_{r}^{2}}{k}\right)\mathcal{R}_{14} - \left(\frac{f_{r}^{2}}{f_{r}} + \frac{h_{z}^{2}}{2h} + \frac{k_{z}^{2}}{k}\right)\mathcal{R}_{14}, \quad (13)$$

$$\mathcal{R}_{44,4} = -\frac{1}{2}\mathcal{R}_{00,4} + \frac{1}{h}\mathcal{R}_{11,4} + \frac{2}{k}\mathcal{R}_{22,1} - \frac{2}{h}\mathcal{R}_{14} - \left(\frac{f_{r}^{2}}{f_{r}} - \frac{h_{r}^{2}}{h} + \frac{2k_{r}^{2}}{k}\right)\mathcal{R}_{44}. \quad (14)$$

We expand quantities in terms of $z$. We denote by $F^{[n]}(r)$ the coefficient of the $z^{n}$ term in the expansion of any function $F(r,z)$:

$$F(r,z) = \sum_{n=0}^{\infty} F^{[n]}(r)z^{n}. \quad (15)$$

The operations of $^{[n]}$ obey the reduction rules

$$(F + G)^{[n]} = F^{[n]} + G^{[n]}, \quad (cF)^{[n]} = cF^{[n]}, \quad (16)$$

$$(FG)^{[n]} = \sum_{k=0}^{n} F^{[k]}G^{[n-k]}, \quad (17)$$

$$F^{-1}^{[n]} = -\sum_{k=0}^{n-1} F^{-1}[k]F^{[n-k]}F^{-1}[0], \quad (18)$$

where $F$ and $G$ are functions and $c$ is a constant. Let us assume $\mathcal{R}_{00} = \mathcal{R}_{11} = \mathcal{R}_{22} = 0$. Then, we have

$$\mathcal{R}_{14}^{[n]} = \frac{1}{n}\left[\frac{1}{2}\mathcal{R}_{14,4,1} - \left(\frac{f_{r}^{2}}{f_{r}} + \frac{h_{r}^{2}}{4h} + \frac{k_{r}^{2}}{k}\right)\mathcal{R}_{14}\right]^{[n-1]}, \quad (19)$$

$$\mathcal{R}_{44}^{[n]} = \frac{1}{n}\left[-\frac{2}{h}\mathcal{R}_{14,4,1} - \left(\frac{f_{r}^{2}}{f_{r}} + \frac{h_{r}^{2}}{h} + \frac{2k_{r}^{2}}{k}\right)\mathcal{R}_{14}\right]^{[n-1]} - \left(\frac{f_{r}^{2}}{f_{r}} + \frac{h_{r}^{2}}{h} + \frac{2k_{r}^{2}}{k}\right)\mathcal{R}_{44}. \quad (20)$$
for \( n \geq 1 \). According to (16)–(18), the right-hand side of eqs. (19)–(20) are linear combinations of \( R_{14}^{[j]} \) and \( R_{14}^{[1]} \) with \( 0 \leq j \leq n - 1 \) and their \( r \)-derivatives. Therefore, if we have \( R_{14}^{[0]} = R_{44}^{[0]} = 0 \), we can conclude that \( R_{14}^{[n]} = R_{14}^{[1]} = 0 \) for any \( n \geq 1 \). Thus, the independent equations to be solved are

\[
R_{00} = R_{11} = R_{22} = 0 \quad \text{in the bulk and} \quad (21)
\]

\[
R_{14} = R_{44} = 0 \quad \text{on the brane}. \quad (22)
\]

If we keep (21), we can replace (22) by

\[
E_{14} = E_{44} = 0 \quad \text{on the brane}. \quad (23)
\]

This is because

\[
E_{14} = R_{14}, \quad (24)
\]

\[
E_{44} = (R_{44} - R_{00} - R_{11} - 2R_{22})/2. \quad (25)
\]

When (6)–(10) are applied, the last equation of (22) includes second derivatives \( f_{zz}, h_{zz} \) and \( k_{zz} \), while that of (25) does not.

Here, we consider regions where \( T_{ff} = 0 \). Then the equations in (21) imply the recursion formulae

\[
f^{[n]} = \frac{1}{n(n-1)} \left[ \frac{f_z^2 - f_z h_z - f_z k_z}{2f} - \frac{f_{zz}}{h} \right]
\]

\[
- \frac{f_{rr}}{h} + \frac{f_z^2}{2h} + \frac{f_{zr} + f_z k_r}{kh} - 4 \frac{\kappa \lambda f}{3} \right]^{n-2}, \quad (26)
\]

\[
h^{[n]} = \frac{1}{n(n-1)} \left[ \frac{h_z^2 - f_z h_z - h_z k_z}{2h} + \frac{f_{zz}}{k} - \frac{f_{zr}}{2f^2}
\]

\[
+ \frac{f_z h_r}{2f^2} + \frac{2k_r}{k} + \frac{f_z k_r}{kh} - \frac{4 \kappa \lambda h}{3} \right]^{n-2}, \quad (27)
\]

\[
k^{[n]} = \frac{1}{n(n-1)} \left[ - \frac{f_z k_z}{2f} - \frac{h_z k_z}{h} + \frac{k_{rr}}{2f} + \frac{h_{zr}}{2h} + 2 \frac{4 \kappa \lambda k}{3} \right]^{n-2} \quad (28)
\]

for \( n \geq 2 \). According to (16)–(18), the right-hand side of eqs. (26)–(28) are written in terms of \( f^{[j]}, h^{[j]} \) and \( k^{[j]} \) for \( 0 \leq j \leq n - 1 \). For example, for \( n = 2 \),

\[
f^{[2]} = \frac{1}{2} \left[ \frac{f^{[2]} - f^{[1]} h^{[1]}}{2 f^{[0]}} - \frac{f^{[1]} k^{[1]}}{f^{[0]}} - \frac{f^{[2]} - f^{[1]} h^{[1]}}{4 \kappa \lambda f^{[0]}} \right]^{3}, \quad (29)
\]

\[
h^{[2]} = \frac{1}{2} \left( \frac{h^{[2]} - f^{[2]} h^{[1]}}{2 h^{[0]}} - \frac{h^{[1]} k^{[1]}}{r^2}
\]

\[
- \frac{f^{[0]} - f^{[0]} h^{[0]}}{2 f^{[0]} h^{[0]}} + \frac{f^{[0]} h^{[0]}}{2 h^{[0]} r^2} + \frac{2 f^{[0]} h^{[0]}}{2 h^{[0]} r^2} - \frac{4 \kappa \lambda h^{[0]}}{3} \right), \quad (30)
\]

\[
k^{[2]} = \frac{1}{2} \left( \frac{k^{[2]} - f^{[1]} k^{[1]}}{2 f^{[0]}} - \frac{h^{[1]} k^{[1]}}{h^{[0]}} - \frac{2 f^{[0]} r}{h^{[0]}} + \frac{h^{[0]} r}{2 h^{[0]}} + 2 \frac{4 \kappa \lambda r^2}{3} \right) \quad (31)
\]

where we used \( k^{[0]} = r^2 \) chosen using diffeomorphism. Similarly, we can inductively determine \( f^{[n]}, h^{[n]} \) and \( k^{[n]} \) in terms of \( f^{[0]}, h^{[0]}, f^{[1]}, h^{[1]} \) and \( k^{[1]} \) for any \( n > 2 \). Therefore, if \( f, h \) and \( k \) and their \( z \)-derivatives are given at the brane, \( f, h \) and \( k \) off the brane are given in the form (15) as far as it converges. We expect that there exist regions where it converges around the brane and off the core of the sphere. Continuations would be possible by expanding the functions again around other points. The regions where continuation fails can be taken to exhibit some physical structures there. Thus, what determine the braneworld solution are the five functions \( f^{[0]}, h^{[0]}, f^{[1]}, h^{[1]} \) and \( k^{[1]} \) of \( r \).

\section{IV. ON-BRANE SOLUTION OF THE BULK EINSTEIN EQUATION}

The five functions \( f^{[0]}, h^{[0]}, f^{[1]}, h^{[1]} \) and \( k^{[1]} \) should obey eq. (22) which is written as

\[
f^{[1]} + \frac{k^{[1]}}{r^2} = \frac{f^{[1]} f^{[0]}}{4 f^{[0]}} + \frac{h^{[1]} f^{[0]}}{4 h^{[0]}} + \frac{h^{[1]}}{r^2}, \quad (32)
\]

\[
f^{[2]} + \frac{k^{[2]}}{r^2} + 2 \frac{f^{[2]}}{2 f^{[0]}} + \frac{h^{[1]} k^{[1]}}{2 h^{[0]}} + \frac{k^{[1]}}{4 f^{[0]}} - 2 \frac{\kappa \lambda}{3} \quad (33)
\]

If we eliminate \( f^{[2]}, h^{[2]} \) and \( k^{[2]} \) from (33) with (29)–(31), we get

\[
f^{[1]} h^{[0]} + \frac{f^{[1]} k^{[1]}}{2 f^{[0]} r^2} + \frac{h^{[1]} k^{[1]}}{2 h^{[0]}} + \frac{k^{[1]}}{4 r^2} = \tilde{R} - \kappa \lambda, \quad (34)
\]

where

\[
\tilde{R} = - \frac{1}{h^{[0]}} \left[ \frac{f^{[0]}}{f^{[0]}} - \frac{f^{[0]} h^{[0]}}{2 f^{[0]} f^{[0]}} + \frac{2 f^{[0]} h^{[0]}}{h^{[0]}} - \frac{2 h^{[0]}}{r^2} + \frac{2}{r^2} \right] \quad (35)
\]

is the scalar curvature of the brane. Eq. (34) is equivalent to \( E_{44} = 0 \) and eq. (32) is the same as \( E_{14} = 0 \) since \( R_{14} = E_{14} \). We have two independent differential equations (32) and (34) for five functions \( f^{[0]}, h^{[0]}, f^{[1]}, h^{[1]} \) and \( k^{[1]} \). Hence, their solution includes three arbitrary on-brane functions. They are solvable linear differential equations as will be seen below. Once the on-brane solution is obtained, its off-brane form is always given by (15) with the coefficients inductively determined by (26)–(28).

We assume nothing about the bulk other than the ansatz (a)–(d), allowing even singularities. Therefore, the on-brane equations (32) and (34) are essential to determine the general solution.

Let us solve the essential equations (32) and (34). Let the functions

\[
u = - \frac{2 f^{[1]}}{f^{[0]}}, \quad v = - \frac{2 h^{[1]}}{h^{[0]}}, \quad w = - \frac{2 k^{[1]}}{k^{[0]}} \quad (36)
\]
be arbitrary. Then, (32) and (34) become
\[ u_r + 2w_r + (u - v)f_r^{[0]}/2f^{[0]} + 2(w - v)/r = 0, \]
\[ \tilde{R} = 2uv + 4uw + 4vw + 2w^2 + 2k\lambda. \]

If \( u \neq v \), eqs. (37) and (38) become the linear differential equations
\[ f^{[0]}_r - U f^{[0]} = 0, \]
\[ (1/h^{[0]}_r)_{,r} + P/h^{[0]} = Q \]
for \( f^{[0]} \) and \( 1/h^{[0]} \) with
\[ U = 2[-u_r - 2w_r + (2v - w)/r]/(u - v), \]
\[ P = [2U_r + (U + 2/r)^2]/(U + 4/r), \]
\[ Q = -(\kappa\lambda + 1/r^2 - vw - 2uw - 2w^2)/U + (4/r). \]

The solution of eqs. (39) and (40) is given by
\[ f^{[0]} = e^{-\int_r^\infty Udr}, \]
\[ h^{[0]} = e^{-\int_r^\infty Pdr} \left[ 1 - \int_r^\infty Qe^{-\int_r^\infty Pdrdr} \right]^{-1}. \]

If \( u = v \), the functions \( f^{[0]} \) becomes arbitrary, and eqs. (37) and (38) become linear differential equations for \( w \) and \( 1/h^{[0]} \). The solution is given by
\[ w = -r^{-1} \int_r^\infty (u - ru_r/2)dr \]
and \( h^{[0]} \) in (45) with \( P \) and \( Q \) obtained by the following replacements in (42) and (43). Replace \( U \) by \( f^{[0]}_r/f^{[0]} \) and \( w \) by the expression in (46). The ranges of integration in (44), (45) and (46) are chosen in accordance with the asymptotic flatness of the brane. The denominators in (42) and (43) are not identically vanishing because, if so, we have \( f^{[0]}_r = C/r^4 \) with a constant \( C \), and the brane cannot be asymptotically flat. Then, the solutions for \( f, h \) and \( k \) given by (15) and (26)–(28) with \( f^{[0]}, h^{[0]}, f^{[1]}, h^{[1]} \) and \( k^{[1]} \) by (36), (44) and (45) expire all the solutions of the bulk Einstein equation (2) (alone) under the ansatz \( (a) \)–(d) with \( T_{1,1} = 0 \).

V. ESSENTIAL EQUATIONS FOR BRANEWORLD GRAVITY

Now we turn to the solution for the “braneworld”. It is a thin physical object accompanied by matter distribution in the region \( |z| < \delta \), where \( \delta \) is the infinitesimal thickness of the brane. The expansions in \( z \) in the previous chapter is possible in the empty regions \( D^\pm \) with \( \pm z > \delta \). At \( |z| < \delta \), however, the bulk Einstein equation with concentrated energy distribution indicates that \( u, v \) and \( w \) should have a gap across the brane. We assume dominance of the collective mode \( Y^I \) in \( S_Y \) in (1) over the other terms from \( S_m \) in the energy-momentum tensor. In the present coordinate system, the collective mode is given by \( z = 0 \). Then, the bulk Einstein equation (2) at \( |z| < \delta \) implies for \( \alpha = u, v \) and \( w \)
\[ \alpha_{z=0} = (\alpha_{z=\delta} + \alpha_{z=-\delta})/2 \equiv \tilde{\alpha} \]
where the last equality in (47) defines \( \tilde{u}, \tilde{v} \) and \( \tilde{w} \). The Nambu-Goto equation (3) implies
\[ \tilde{u} + \tilde{v} + 2\tilde{w} = 0. \]
We apply the solution in the previous chapter in \( D^\pm \). At \( z = \pm\delta \), we have the similar equations as (37) and (38) with \( u(\pm), v(\pm) \) and \( w(\pm) \) in the places of \( u, v \) and \( w \), respectively. The sums of them at \( z = \pm\delta \) give
\[ \tilde{u}_r + 2\tilde{w}_r + (\tilde{u} - \tilde{v})f^{[0]}_r/f^{[0]} + 2(\tilde{w} - \tilde{v})/r = 0, \]
\[ \tilde{R} = 2\tilde{w} + 4\tilde{u} + 4\tilde{v} + 2\tilde{w}^2 + 2\kappa^2\lambda^2/3 + 2k\lambda, \]
while the differences trivially hold as far as we have (49). Now, we have three equations (49)–(51) for five functions \( f^{[0]} \) and \( h^{[0]} \), \( \tilde{u}, \tilde{v} \) and \( \tilde{w} \). Hence, their solution includes two arbitrary on-brane functions. They are solvable linear differential equations. If we choose \( \tilde{u} \) and \( \tilde{v} \) arbitrarily, we have the same solution as (44), (45) and (46) with \( u, v, w, \lambda \) replaced by \( \tilde{u}, \tilde{v}, -(\tilde{u} + \tilde{v})/2 \) and \( \lambda + \kappa\lambda^2/6 \), respectively. Once the on-brane solution is obtained, its off-brane form is always given by (15) with the coefficients inductively determined by (26)–(28). It expires all the solutions of our premised system. Therefore, we conclude that the on-brane equations (49)–(51) are essential to determine the general solution. They are the equations which determine how the brane is curved according to the dynamics, and, hence, they are the essential equations for the braneworld gravity.

The general solution involve large arbitrariness, which may affect the predictive powers of the theory on the Newtonian and the post-Newtonian evidences. It includes not only the the ordinary Schwarzschild solution on the brane, but also continuously deformed solutions which do not satisfy the brane Einstein equation. It implies arbitrarily deformed Newtonian potentials, and arbitrary amounts of light deflections and planetary perihelion precessions due to solar gravity. We need further physical prescriptions to make these predictions. For example, conditions for the bulk behavior may improve the situation. It is an urgent open problem of the braneworld theories. The idea of the brane induced gravity [5], [13], [21], [22], [24], [31], [32] may give a way out of this difficulty. The brane Einstein gravity emerges through the quantum effects of the brane [42]. The composite metric serves as the effective dynamical variable, just like the quantum-induced composite field which are well understood in some classes of models [43].

This work was supported by Grant-in-Aid for Scientific Research, No. 13640297, 17500601, and 22500819 from
Japanese Ministry of Education, Culture, Sports, Science and Technology.

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