Time and Energy in Gravity Theory

Ivanhoe B. Pestov

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia

Abstract

A new concept of internal time (viewed as a scalar temporal field) is introduced which allows one to solve the energy problem in General Relativity. The law of energy conservation means that the total energy density of the full system of interacting fields (including gravitational field) does not vary with time, thus being the first integral of the system. It is demonstrated that direct introduction of the temporal field permits to derive the general covariant four dimensional Maxwell equations for the electric and magnetic fields from the equations of electromagnetic fields considering in General Relativity. It means that the fundamental physical laws are in full correspondence with the essence of time. Theory of time presented here predicts the existence of matter outside the time.

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1 Introduction

In the theory of gravitational field the problems connected with the energy conservation exist in a literal sense since the time of its creation when Einstein set up the problem of including gravity into the framework of the Faraday concept of field. Thorough and deep analysis of the problem of gravity field given by him in the works [1], enables to formulate the key principles of gravity physics (General Relativity). However, till now in the framework of these principles there is no adequate solution to the energy conservation problem [2-5]. In presented paper we give the simple solution of the problem in question, which is based on the connection between the time and energy and necessarily follows from the first principles of General Relativity if one puts them into definite logical sequence. The energy conservation means that total energy density of gravity field and all other fields does not vary with time and hence represents the first integral of the system.

New understanding of time presented here may have implications for the problem of time in quantum gravity. In fact, a major conceptual problem in this field is the notion of time and how it should be treated in the formalism. The importance of this issue was recognized at the beginning of the history of quantum gravity [6], but the problem is still unresolved.
and has drawn recently an increasing attention (see, for example [7], [8] and [9] and further references therein).

The paper is organized as follows. In section 2 we formulate the fundamentals of general covariant theory of time with the guiding idea that the manifold is the main notion in physics and that time itself is a scalar field on the manifold which defines the evolution of the full system of fields being one of them. The system of interacting fields is considered to be full if the gravitational field is included in it.

All known dynamical laws of nature have the following form: the rate of change of certain quantity with time equals to the results of action of some operator on this quantity. So, a general covariant definition of rate of change of any field with time is one of the main results of the theory of time presented here. It is also a starting point and relevant condition for the consideration of the concept of evolution and the problem how to write the field equations in the general covariant evolution form.

In section 3 the connection of the temporal field with the Einstein gravitational potentials is established. This gives the possibility to derive general covariant law of energy conservation and produce the general covariant expression for the energy density of the gravity field.

To demonstrate a concrete application of general theory, in section 4 it is shown how evolution equations for the vectors of electric and magnetic fields in the four dimensional general covariant form can be derived from the equations for the bivector of electromagnetic field. Through this it is shown that the original Maxwell equations (which as a matter of fact express the fundamental physical law) are in full correspondence with fundamentals of the theory of time. In section 5 different representations for the energy density of the gravity field are derived.

2 Central role of time in gravity theory

In accordance with the principles of gravity physics, in this section the fundamentals of the theory of time are formulated.

According to Einstein, in presence of gravitational field all the systems of coordinates are on equal footing and in general, coordinates have neither physical nor geometrical meaning. Thus, in the gravity theory the coordinates play the same role as the Gauss coordinates in his internal geometry of surfaces from which all buildings of the modern geometry are grown. Hence, one needs to construct the internal theory of physical fields analogous to the Gauss internal geometry of surfaces. In other words, the problem is how the modern differential geometry transforms into the physical geometry.

The fields characterize the events and fill in the geometrical space. In view of what has been said above, this space is a four dimensional smooth manifold because this structure does not distinguish intrinsically between different coordinate systems (the principle of general covariance is naturally included into this notion). Hence it follows that the notion of smooth manifold is primary one not only in differential geometry but in the theoretical physics as well.
that deal with gravitational phenomena. This means that all other definitions, notions and laws should be introduced into the theory through the notion of smooth manifold. Indeed if some notion, definition or law is in agreement with the structure of smooth manifold, then they are general covariant, i.e., do not depend on the choice of coordinate system.

Definition of manifold is considered to be known and we only notice that all information on this topic can be found for example in reference [4] or [5]. For our purposes it is enough to keep in mind that all smooth manifolds can be realized as surfaces in the Euclidean space. In what follows we shall consider only four dimensional manifolds. That is evident from the physical point of view. However there is also a deep purely mathematical reason for this choice. Smooth manifold consists of topological manifold and differential structure defined on it. It is known [10], that a topological manifold always admits differential structure if and only if its dimension is not larger than four.

It is clear that, in general, manifold should be in some relation with its material content i.e., the fields. As it comes out, the relation between them can be viewed as a form and content. In view of this it is very important to know how material content designs manifold. If we take the point of view that manifold apriori is arena for the physical events, then it is natural to select simplest manifold, for example, manifold of special theory of relativity. However, Einstein argued (in the papers mentioned above) that nature of gravity field is not compatible with an apriori defined manifold.

It can be shown that manifold as a surface in the Euclidean space is designed by the covariant symmetrical tensor field \( g_{ij} \) on the manifold, for which adjoined quadratic differential form (Riemann metric)

\[
ds^2 = g_{ij} du^i du^j,
\]

is positive definite. Thus, covariant positive definite symmetrical tensor field \( g_{ij}(u) \) is the necessary element of any general covariant intrinsically self-consistent physical theory. Here we only give the defining system of differential equations

\[
g_{ij}(u^1, u^2, u^3, u^4) = \delta_{ab} \frac{\partial F^a}{\partial u^i} \frac{\partial F^b}{\partial u^j}, \quad a, b = 1, \cdots, 4 + k, \quad k \geq 0
\]

avoiding detailed consideration of this point. If the functions \( g_{ij}(u^1, u^2, u^3, u^4) \) are known in local system of coordinates \( u^1, u^2, u^3, u^4 \), then solving this system of equations we obtain the functions \( F^a(u^1, u^2, u^3, u^4) \) and hence the region of manifold, defined by the equations

\[
x^a = F^a(u^1, u^2, u^3, u^4),
\]

where \( x^a \) are the Cartesian coordinates of embedding Euclidean space.

An important conclusion that follows from this consideration is that there is one and only one way for the other fields to design manifold which can be explained as follows. Let \( u^1, u^2, u^3, u^4 \) be a local system of coordinates in the vicinity of some point of an abstract manifold. Let us determine in such a vicinity the system of differential equations that connect basis field \( g_{ij}(u) \) with other ones. Solving this system of equations we find field \( g_{ij}(u) \) and
by doing so, we design a local manifold of physical system in question. In what follows, a smooth manifold that corresponds to a physical system will be called a physical manifold.

For further consideration of the principles of internal field theory we simply note that there is a fundamental difference between physics and geometry. In geometry there is no motion that is tightly connected with the concept of time. Thus, to be logical, we need to introduce time into the theory using its first principles. Since, in general, coordinates have no physical sense, time can be presented as a set of functions of four independent variables (or in more strict manner as a geometrical object on the manifold). It is quite obvious from the logical point of view.

We put forward the idea that the time is a scalar field on the manifold. By this we get a simple answer to the question with long standing history "What is time ?" It should be emphasized, that temporal field (together with other fields) designs manifold as it was explained above but it has also another functions which will be considered below.

Temporal field with respect to the coordinate system $u^1, u^2, u^3, u^4$ in the region $U$ of smooth four dimensional manifold $M$ is denoted as

$$ f(u) = f(u^1, u^2, u^3, u^4). $$

If the temporal field is known, then to any two points $p$ and $q$ of manifold one can put in correspondence an interval of time

$$ t_{pq} = f(q) - f(p) = \int_{p}^{q} \partial_i f du^i. \quad (2) $$

Unlike time, space is not an independent entity. Instead of space we shall consider space cross-sections of the manifold $f^{-1}(t)$, which are defined by the temporal field. For the real number $t$, space cross-section is defined by the equation

$$ f(u^1, u^2, u^3, u^4) = t. \quad (3) $$

One can call the number $t$ ”the height” of the space cross-section of manifold. If a point $p$ belongs to the space cross-section $f^{-1}(t_1)$, and a point $q$ to the space cross-section $f^{-1}(t_2)$, then the time interval (2) is equal to the difference of the heights $t_{pq} = t_2 - t_1$. It is clear that $t_{pq} = 0$ if $p$ and $q$ belong to the same space cross-section. Thus, to the every segment of curve one can put in correspondence a time interval.

Given the general covariant definition of time, one should show that it is constructive in all respects. First of all we consider how the temporal field defines the form of physical laws. It is known that the general form of physical laws is very simple and is based on the following recipe: the rate of change of a certain quantity with time is equal to the result of action of some operator on this quantity. To be concrete, let us consider Maxwell equations. We know that the rate of change of electrical and magnetic fields enter the dynamical equations of electromagnetic field. Thus, we need to give a general covariant definition of the rate of
change of any field with time and in particular this definition should be applicable for the case of electromagnetic field.

This problem has fundamental meaning, since it is impossible to speak about physics when one has no mathematically rigorous definition of evolution. It is clear that correct general covariant definition should be conjugated with simple condition: if the rate of change of some field with time is equal to zero in one coordinate system, then in any other coordinate system the result will be the same.

On the manifold there is only one general covariant operation that can be considered as a basis for the definition of the rate of change with time of any field quantity. This general covariant operation is called derivative in given direction and is defined by the vector field on the manifold and the structure of the manifold itself. Thus, the problem is to connect the temporal field \( f(u^1, u^2, u^3, u^4) \) with some vector field \( t^i(u^1, u^2, u^3, u^4) \). Since a temporal field is a scalar one, the partial derivatives define covector field \( t^i = \frac{\partial f}{\partial u^i} \). Now, the following definition becomes self-evident: the gradient of temporal field (or the stream of time) is the vector field of the type

\[
 t^i = (\nabla f)^i = g^{ij} \frac{\partial f}{\partial u^j} = g^{ij} \partial_j f = g^{ij} t_j, \tag{4}
\]

where \( g^{ij} \) are the contravariant components of the Riemann metric (1), which is defined as usual, \( g^{ij} g_{kj} = \delta^i_k \). The gradient of the field of time defines the direction of the most rapid increase (decrease) of the field of time. We define now the rate of change of some quantity with time as the derivative in the direction of the gradient of the field of time and denote this operation by the symbol \( D_t \).

Let us find the expression for the rate of change with time of the temporal field itself. We have,

\[
 D_t f = t^i \partial_i f = g^{ij} \partial_i f \partial_j f.
\]

Since \( D_t f \) is a general covariant generalization of the evident relation \( \frac{d}{dt} t = 1 \), the temporal field should obey the fundamental equation

\[
 (\nabla f)^2 = g^{ij} \frac{\partial f}{\partial u^i} \frac{\partial f}{\partial u^j} = 1. \tag{5}
\]

Equation (5) means that the rate of change of the temporal field with time is a constant quantity, the most important constant of the theory. From the geometrical point of view the equation (5) simply shows that the gradient of the temporal field is unit vector field on the manifold with respect to the scalar product that is defined as usual by the metric (1),

\[
 (V, W) = g_{ij} V^i W^j = g^{ij} V_i W_j = V^i W_i = ||V|| ||W|| \cos \phi.
\]

It should be noted that the equation \((\nabla f)^2 = 1\) is the main equation of the geometrical optics. In view of this one can consider equation (5) as the equation of 4-optics. This analogy can be useful for consideration of some special problems in the theory of time.
The rate of change of the vector field and symmetrical tensor field with time is respectively given by the expressions

\[ D_t V^i = t^k \frac{\partial V^i}{\partial u^k} - V^k \frac{\partial t^i}{\partial u^k}, \]

\[ D_t g_{ij} = t^k \frac{\partial g_{ij}}{\partial u^k} + g_{kj} \frac{\partial t^i}{\partial u^j} + g_{ik} \frac{\partial t^j}{\partial u^i}. \]

(6)

(7)

Similar formulas can be presented for any other geometrical quantities. In mathematical literature the derivative with respect to the given direction is usually called Lie derivative. Thus, one can say that the rate of change of any field with time is the Lie derivative with respect to the direction of the stream of time.

Consider the notion of time reversal and the invariance with respect to this symmetry that is very important for what follows. It is almost evident that in general covariant form the time reversal invariance means that theory is invariant with respect to the transformations

\[ t^i \rightarrow -t^i. \]

(8)

It is clear that theory will be time reversal invariant if the gradient of temporal fields will appear in all formulae only as an even number of times, like \( t^i t^j \).

Within the scope of special relativity and quantum mechanics, time and energy are tightly connected. It is natural to suppose that in gravity physics the link between time and energy even more deep and energy conservation follows from the invariance of the Lagrangian theory with respect to the transformations

\[ f(u) \rightarrow f(u) + a, \]

(9)

where \( a \) is arbitrary constant. This invariance means that all the space sections of manifold of system in question are equivalent.

Einstein himself put in correspondence to the gravity field symmetrical tensor field \( \tilde{g}_{ij} \), which is characterized by the condition that adjoined quadratic differential form

\[ ds^2 = \tilde{g}_{ij} du^i du^j, \]

(10)

has the signature of the interval in special relativity. In accordance with the principle of gravity physics discussed above, it is natural to assume that Einstein’s interval (10) has a structure that is defined by the form-generating field \( g_{ij} \) (Riemann’s metric (1)) and temporal field. If disclosed, this structure will give a simple method to introduce temporal field into the equations of gravitational physics. It is quite evident that the metric (1) has an Euclidean signature and hence it has no structure. To be transparent in our consideration, let us give a simple mathematical construction. Let \( (V, V) = g_{ij} V^i V^j = |V|^2 \) be usual scalar product that is defined by the metric (1). One can consider tensor field \( S^i_j \) as linear transformation \( \tilde{V}^i = S^i_j V^j \) in the vector space in question. If the operator \( S \) is self-adjoint, that is \( (V, SW) = (SV, W) \), then it is always possible to introduce the scalar product associated with operator
S via the formula $<V,V> = (V,SV)$. It is clear that associated scalar product will be in general indefinite and with respect to the initial scalar product it has a structure. Thus, in general, the connection between the forms (1) and (10) is given by the relation

$$\tilde{g}_{ij} = g_{ik} S^k_j. \quad (11)$$

We shall give now simple expression for the operator $S^i_j$, which defines the Einstein interval. To this end we shall consider T-symmetry or time reversal invariance (8) as a fundamental physical principle. Further, we shall define T-symmetry in the space of the vector field in order to have transformation (8) as a particular case. We say that vector fields $\tilde{V}^i$ and $V^i$ are T-symmetrical, if the sum of this fields is orthogonal to the gradient of temporal field and their difference is collinear to it,

$$(\tilde{V}^i + V^i)t_i = 0, \quad \tilde{V}^i - V^i = \lambda t^i.$$  

From this we find,

$$\tilde{V}^i = V^i - 2n^i (V, n) = (\delta^i_j - 2n^i n_j)V^j,$$

where

$$n^i = \frac{t^i}{\sqrt{(t,t)}}, \quad (t,t) = g_{ij} t^i t^j.$$  

From this formula it follows that the fields $t^i$ and $-t^i$ are T-symmetrical and hence the definition of T-symmetry given in the space of the vector field is correct. It is not difficult to verify that

$$(\tilde{V}, W) = (V, \tilde{W}), \quad (\tilde{V}, \tilde{V}) = (V, V).$$  

From the above consideration it follows that operator $S$ is T-operator that is

$$S^i_j = \delta^i_j - 2n^i n_j$$

and in accordance with (11) for the Einstein’s potentials we obtain the following expression

$$\tilde{g}_{ij} = g_{ik}(\delta^k_j - 2n^k n_j) = g_{ij} - 2n^i n_j, \quad (12)$$

which is invariant under the transformations (8). To the tensor field $\tilde{g}_{ij}$ one can put in correspondence contravariant tensor field $\tilde{g}^{ij} = g^{ij} - 2n^i n^j$, which obeys the relation

$$\tilde{g}^{ik} \tilde{g}_{jk} = \delta^i_j.$$  

Let us give the physical meaning of the Einstein’s scalar product associated with T-symmetry. Since

$$<V,V> = (V, V) - 2(V, n)^2 = |V|^2 (1 - 2 \cos^2 \phi) = -|V|^2 \cos 2\phi,$$  

7
where $\phi$ is the angle between the vectors $V^i$ and $n^i$, then the Einstein’s scalar product is indefinite and can be positive, negative or equal to zero according to the value of the angle $\phi$. In particular, $<V, V> = 0$, if $\phi = \pi/4$. Thus, the Einstein’s $T$- symmetrical scalar product permits to classify all the vectors depending on which angle they form with the gradient of the temporal field. As we see, the temporal field and $T$-symmetry define the Einstein’s form (10) as the metric of the normal hyperbolic type. Hence, the gradient of the temporal field defines the causal structure on the physical manifold and can be identified with it. It is the physical meaning of the Einstein’s interval.

Let us show that the causal structure (the gradient of the temporal field) can be reduced to the canonical form $(0, 0, 0, 1)$ by the suitable coordinate transformation at once in all points of some (may be small) patch of any point on the physical manifold. Local coordinates with respect to which gradient of the temporal field has the form $(0, 0, 0, 1)$ will be called compatible with causal structure.

Geometrically the stream of time is defined as a congruence of lines (lines of time) on the manifold. We recall that the congruence of lines is a set of lines characterized by the fact that only one element of the set passes through each point of manifold (or its open region). According to the definition, lines belonging to the congruence do not intersect and fill either the whole manifold or a part of it. The simple non-trivial example is the congruence of rays coming from one point of the Euclidean space.

Analytically the lines of time are defined as the solutions of the autonomous system of differential equations

$$\frac{du^i}{dt} = g^{ij} \frac{\partial f}{\partial u^j} = g^{ij} \partial_j f = (\nabla f)^i, \quad (i = 1, 2, 3, 4). \quad (13)$$

It can be shown that if the functions $\varphi^i(t)$ are solutions to Eqs. (13), the functions $\psi^i(t) = \varphi^i(t + a)$, where $a$ is constant, will also be solutions to them. Since $ds^2 = g_{ij}du^i du^j$, from Eqs. (5) and (13) it follows that $ds/dt = \pm 1$. Thus, the length of the time line is a linear function of the parameter $t$, $s = \pm t + a$. After these preliminary remarks, it is time to give solution of the above set problem.

Let

$$u^i(t) = \varphi^i(u_0^1, u_0^2, u_0^3, u_0^4, t) = \varphi^i(u_0, t) \quad (14)$$

be the solution to equations (13) with initial data $\varphi^i(u_0, t_0) = u_0^i$ so that

$$\frac{\partial \varphi^i(u_0, t_0)}{\partial u_0^j} = \delta_j^i. \quad (15)$$

Substituting $u^i(t) = \varphi^i(u_0, t)$ into the function $f(u^1, u^2, u^3, u^4)$ we obtain $p(t) = f(\varphi(u_0, t))$. Differentiating this function with respect to $t$, by virtue of (5) and (13), one finds

$$\frac{dp(t)}{dt} = \frac{\partial f}{\partial u^i} \frac{du^i}{dt} = g^{ij} \frac{\partial f}{\partial u^i} \frac{\partial f}{\partial u^j} = 1.$$
It leads to
\[ f(\varphi(u_0, t)) = t - t_0 + f(u_0). \]  
(16)

Suppose that all initial data belong to the space section
\[ f(u^1_0, u^2_0, u^3_0, u^4_0) = t_0. \]  
(17)

Rewriting Eq. (17) in the parametric form
\[ u^i_0 = \psi^i(x^1, x^2, x^3), \]
Eqs. (14) can be written as the system of relations
\[ u^i = \phi^i(x^1, x^2, x^3, t). \]  
(18)

The right hand side of Eqs.(18) has continuous partial derivatives with respect to variable \( x^1, x^2, x^3, t \). From (14) and (15) it follows that the functional determinant of the system (18) is not equal to zero and hence it designs internal system of coordinate of the dynamical system in question. Now one can show that in such a system of coordinates the covariant and contravariant components of the gradient of the temporal field and some components of the field \( g_{ij} \) take a simple numerical value
\[ t^i = (0, 0, 0, 1) = t_i, \quad g_{44} = g^{44} = 1, \quad g_{\mu 4} = g^{\mu 4} = 0, (\mu = 1, 2, 3) \]
and hence this coordinate system is compatible with causal structure. Indeed, since in the system of coordinates \( x^1, x^2, x^3, t \) temporal field has a simple form
\[ f(x^1, x^2, x^3, t) = f(\phi(x, t)) = t, \]
then in this coordinate patch
\[ \frac{\partial f}{\partial x^1} = \frac{\partial f}{\partial x^2} = \frac{\partial f}{\partial x^3} = 0, \quad \frac{\partial f}{\partial t} = 1 \]
and hence \( t_i = (0, 0, 0, 1) \). Since coordinates \( x^1, x^2, x^3 \) do not vary along the lines of time, then
\[ \frac{dx^\mu}{dt} = g^{\mu 4} = 0, \quad (\mu = 1, 2, 3) \quad \frac{dt}{dt} = g^{44} = 1. \]
Thus, in the internal system of coordinates
\[ g^{\mu 4} = 0, \quad g^{44} = 1, \quad t^i = g^{ij} t_j = g^{i4} = (0, 0, 0, 1), \]
which is what we had to prove. Therefore, it follows that in the system of coordinates compatible with causal structure, metric (1) takes the form
\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu + (dt)^2, \quad \mu, \nu = 1, 2, 3, \]
(19)
since $t_i = g_{ij}t^j = g_{i4}$.

So, for any point of physical manifold we can indicate a local coordinate neighborhood with the coordinates compatible with causal structure. Covering manifold with such charts we get atlas on the manifold that is compatible with its causal structure. In this atlas the field equations have the most simple form in the sense that all components of the gradient of temporal field take numerical values. It should be noted that the system of coordinates compatible with causal structure is similar to the Darboux system of coordinate in the theory of symplectic manifolds which is a geometrical basis for the Hamilton mechanics.

From the above consideration it also follows that variable $t$, parametrizing the line of time can be considered as coordinate of time of the physical system in question. This name is justified particularly by the fact that in accordance with (6) and (7) the rate of change of any field with time is equal to the partial derivative with respect to $t$, i.e., $D_t = \partial/\partial t$ in the system of coordinate compatible with causal structure. This is due to the fact that $f(x^1, x^2, x^3, t) = f(\phi(x,t)) = t$.

Furthermore, if we reverse time putting $\tilde{t}^i = -t^i$, then the lines of time will be parametrized by new variable $\tilde{t}$. From the equations (13) it is not difficult to derive that there is one-to-one and mutually continuous correspondence between the parameters $t$ and $\tilde{t}$ given by the relation $\tilde{t} = -t$. From here it is clear that in the system of coordinate defined by the time reversal, the variable $-t$ will be the coordinate of time. Thus, the general covariant definition of time reversal given by (8) is adjusted with the familiar definition that is connected with the transformation of coordinates. Finally, we note one more important fact related to the time coordinate. Transformations (9) take the well-known form $t \rightarrow t + a$ in the system of coordinates compatible with the causal structure.

### 3 Equations of the gravity field

Consideration made above gives the evident method of constructing Lagrangians in gravity physics. It is clear that dynamics of gravity field is defined by the Lagrangian

$$L = \tilde{R},$$

where $\tilde{R}$ is a scalar that is constructed from the Einstein’s gravitational potentials by the standard way.

Since the Einstein interval is defined by the two fields connected by the equation (5), it is necessary to pay special attention when deriving the equations of gravitation field. A standard method is to incorporate the constraint (5) via a Lagrange multiplier $\varepsilon = \varepsilon(u)$, rewrite the action density for gravity field in the form

$$L_g = \tilde{R} + \varepsilon(g^{ij}t_i t_j - 1)$$

and treat the components of the fields $g_{ij}$ and $f$ as independent variables.
Let us consider the action

\[ A = \frac{1}{2} \int \tilde{R} \sqrt{g} \, d^4 u + \int L_F (\tilde{g}, q) \sqrt{g} \, d^4 u + \frac{1}{2} \int \varepsilon (g^{ij} t_i t_j - 1) \sqrt{g} \, d^4 u, \]  

(21)

where \( g = \text{Det}(g_{ij}) > 0 \) and \( L_F (\tilde{g}, q) \) is the Lagrangian density of the system of other fields \( q \) which incorporates the Einstein’s gravitational potentials in the conventional form. Such method of introduction of causal structure into the equations of material fields does not require special explanation. Nevertheless, it will be demonstrated on the example of the derivation of the Maxwell equations for the electric and magnetic fields in general covariant four-dimensional and evolutionary form since this problem remains unresolved up-to-date.

Since Einstein’s gravitational potentials are the functions of \( g_{ij} \) and \( f \), then it is necessary to use the rule of differentiation of the complex function. We have, \( \delta \tilde{R} = \tilde{g}^{ij} \delta \tilde{R}_{ij} + \tilde{R}_{ij} \delta \tilde{g}^{ij} \).

Further we denote the Christoffel symbols as \( \Gamma^i_{jk} \), \( \tilde{\Gamma}^i_{jk} \) belonging to the fields \( g_{ij} \), \( \tilde{g}_{ij} \), respectively. From (12), after some calculations, we obtain the formula

\[ \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + H^i_{jk}, \]  

(22)

where

\[ H^i_{jk} = n^i (\nabla_j n_k + \nabla_k n_j) + (2n^i n^l - g^{ij})(n_j \delta_k^m + n_k \delta_j^m)(\nabla_m n_l - \nabla_l n_m). \]  

(23)

In (23) \( \nabla_j \) is covariant derivative with respect to the connection \( \Gamma^i_{jk} \). It is easy to derive from (23) that \( H^{ji}_j = 2n^i \nabla_j n_i = 0 \), since \( n^i n_i = 1 \). Therefore, \( \tilde{\Gamma}^j_{jk} = \Gamma^j_{jk} \), and hence

\[ \tilde{\nabla}_l V^l = \nabla_l V^l = \frac{1}{\sqrt{g}} \partial_l (\sqrt{g} V^l), \]

where \( \tilde{\nabla}_l \) is the covariant derivative with respect to the connection \( \tilde{\Gamma}^j_{jk} \). From this it follows that \( \tilde{g}^{ij} \delta \tilde{R}_{ij} \) can be omitted as a perfect differential. Varying now \( \tilde{g}^{ij} \), we get

\[ \delta \tilde{g}^{ij} = \delta g^{ij} + P_{kl}^{ij} \delta g^{kl}, \]

where

\[ P_{kl}^{ij} = 2n^i n^j n_k n_l - n^i (n_k \delta_l^j + n_l \delta_k^j) - n^j (n_k \delta_l^i + n_l \delta_k^i) \]

is a tensor field symmetrical in covariant and contravariant indices. Thus,

\[ \delta (\tilde{R} \sqrt{g}) = (\tilde{R}_{ij} + \tilde{R}_{kl} P_{ij}^{kl} - \frac{1}{2} \tilde{R} g_{ij}) \delta \tilde{g}^{ij} \sqrt{g} \]

with neglect of a perfect differential. Let \( G_{ij} = \tilde{R}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{R} \) be the Einstein’s tensor. Observing that \( \tilde{g}_{ij} + \tilde{g}_{kl} P_{ij}^{kl} = g_{ij} \), it is easy to verify that variation \( \delta (\tilde{R} \sqrt{g}) \) can be presented in the form

\[ \delta (\tilde{R} \sqrt{g}) = (G_{ij} + G_{kl} P_{ij}^{kl}) \delta \tilde{g}^{ij} \sqrt{g}. \]
Further we put $\delta L = \frac{1}{2}M_{ij}\delta g^{ij}$, and introduce by the standard way the energy-momentum tensor $T_{ij} = M_{ij} - \tilde{g}_{ij}L_{F}$. Then $\delta(L_{F}\sqrt{g}) = \frac{1}{2}(M_{ij} + M_{kl}P^{kl}_{ij} - g_{ij}L_{F})\delta g^{ij}\sqrt{g}$ and total variation of the action can be presented in the following form

$$\delta A = \frac{1}{2}\int (G_{ij} + G_{kl}P_{ij}^{kl} + T_{ij} + T_{kl}P_{ij}^{kl} + \varepsilon t_{i}t_{j})\delta g^{ij}\sqrt{g}d^{4}u.$$  \hfill (24)

One can consider tensor $P_{ij}^{kl}$ as operator $P$ acting in the space of symmetrical tensor fields. The characteristic equation of this operator has the form $P^{2} + 2P = 0$, and hence $(P+1)^{2} = 1$. Thus, operator $P + 1$ is inverse to itself. Since $t_{i}t_{j} + t_{k}t_{l}P_{ij}^{kl} = -t_{i}t_{j}$, then it follows that Einstein equations have the form

$$G_{ij} + T_{ij} = \varepsilon \partial_{i}f \partial_{j}f, \quad g^{ij}\partial_{i}f \partial_{j}f = 1.$$  \hfill (25)

The equations (25) constitute the full system of equations of gravity field. As it is shown above, these equations emerge from the first principles of gravity physics formulated by Einstein. Soon it will be shown that the Lagrange multiplier $\varepsilon$ has a physical meaning of energy density of the system in question. From the Eqs. (25) it follows that

$$\varepsilon = G_{ij}t^{i}t^{j} + T_{ij}t^{i}t^{j}.$$  \hfill (26)

Varying the action with respect to $f$ and taking into account that $\delta f \tilde{g}^{ij} = Q^{kij}\delta t_{k}$, where

$$Q^{kij} = \frac{2}{\sqrt{(t,t)}}(2n^{k}n^{i}n^{j} - g^{k}n^{j} - g^{k}n^{i}),$$

we get the following equation

$$\frac{1}{2}\nabla_{k}((G_{ij} + T_{ij} - \varepsilon t_{i}t_{j})Q^{kij} + \nabla_{k}(\varepsilon t^{k}) = 0.$$  \hfill (27)

From Eqs. (25) and (27) we have

$$\nabla_{k}(\varepsilon t^{k}) = 0.$$  \hfill (28)

Equation (28) expresses the law of energy conservation in gravitational physics which, evidently, is general covariant. To make sure that we indeed deal with conservation of energy, it is sufficient to figure out that action (21) is invariant with respect to transformation $f(u) \rightarrow f(u) + a$, where $a$ is constant and $f(u)$ is a temporal field. Thus, the equation (28) results from the Noether’s theorem and the invariance of the action with respect to the one parametric group of transformations (9).

Consider the law of energy conservation from the various points of view. The so-called local energy conservation is written as follows

$$\tilde{\nabla}_{i}T^{ij} = 0,$$
where $T^{ij} = T_{kl} \tilde{g}^{ik} \tilde{g}^{jl}$. These equations are fulfilled on the equations of the fields $q$ that contribute to the energy-momentum tensor. Since

$$\tilde{\nabla}_i G^{ij} = 0$$

identically, then from the Einstein’s field equations (25) it follows that

$$\tilde{\nabla}_i T^{ij} = \varepsilon t^i (\tilde{\nabla}_t t^j) + t^j \tilde{\nabla}_i (\varepsilon t^i).$$

We now show that $t^i \tilde{\nabla}_i t^j = 0$. From (5) we have $n_i = t_i$, $n_j - n_i = \nabla_i t_j - \nabla_j t_i = 0$. With this and from (22), (23) we get

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2 t^j \nabla_i t_k. \quad (29)$$

Thus, $t^i \tilde{\nabla}_i t^j = t^i \nabla_i t^j + 2 t^j t^i t^k \nabla_i t_k = 0$. Since $\tilde{\nabla}_i (\varepsilon t^i) = \nabla_i (\varepsilon t^i)$, then finally we have

$$\tilde{\nabla}_i T^{ij} = t^j \nabla_i (\varepsilon t^i).$$

in view of this the energy conservation law can be treated as the condition of compatibility of the field equations. In this sense, the law of energy conservation is analogous to the law of charge conservation.

Show that rate of change of the energy density $D_t (\sqrt{g} \varepsilon)$ equal to zero and hence this quantity is a first integral of the system in question. We have $D_t (\sqrt{g} \varepsilon) = t^i \partial_i (\sqrt{g} \varepsilon) + \sqrt{g} \partial_i (\sqrt{g} t^i)$. Since $\sqrt{g} \nabla_k (\varepsilon t^k) = \partial_k (\sqrt{g} \varepsilon t^k)$, then from the law of energy conservation (28) it follows that energy density is the first integral of the system

$$D_t (\sqrt{g} \varepsilon) = 0. \quad (30)$$

In the system coordinates, compatible with causal structure, this equation has a more customary form

$$\frac{\partial}{\partial t} (\sqrt{g} \varepsilon) = 0.$$

Compare the law of energy conservation with the law of charge conservation which can be written in general covariant form as follows $\nabla_i J^i = 0$. Putting $J^i = t^i (t, J) + J^i - t^i (t, J) = \rho t^i + P^i$, we find that $D_t (\sqrt{g} \rho) + \partial_i (\sqrt{g} P^i) = 0$. Thus, we see that the charge density is not in general the first integral of the system while energy density always is.

4 Time in the theory of electric and magnetic fields

In this chapter the methods of gravitational physics and theory of time are applied for the derivation of the general covariant Maxwell equations for the electric and magnetic fields.

In connection with this we remind that the process of raising and lowering the indices is carried out with the help of the metric tensor $g_{ij}$ and its reciprocal $g^{ij}$. Symbol $\nabla_i$ denotes...
the covariant derivative with respect to the connection $\Gamma^i_{jk}$ of the metric $g_{ij}$, $\varepsilon_{ijkl}$ and $\varepsilon^{ijkl}$ are covariant and contravariant components of the Levi-Chivita tensor normalized as $\varepsilon_{1234} = \sqrt{g}$, and $\varepsilon^{1234} = 1/\sqrt{g}$, where $g$ is determinant of the metric tensor. Since metric is positive definite then $g > 0$.

Let $A_i$ be the vector potential of the electromagnetic field. Let us define the tensor of electromagnetic field as usual

$$F_{ij} = \partial_i A_j - \partial_j A_i.$$  \hfill (31)

According to the principle of gravity physics we introduce temporal field into the theory of electromagnetic field through the gauge invariant and general covariant Lagrangian

$$L_{em} = \frac{1}{4} F_{ij} F_{kl} \tilde{g}^{ik} \tilde{g}^{jl}. \hfill (32)$$

From (32) we derive the energy-momentum tensor

$$T_{ij} = F_{ik} F_{jl} \tilde{g}^{kl} - \tilde{g}_{ij} L_{em}. \hfill (33)$$

Let us show that the equations for the tensor of the electromagnetic field can be written in the form

$$\nabla_i \tilde{F}^{ij} = 0, \quad \nabla_i F^{ij} = 0,$$  \hfill (34)

where

$$\tilde{F}^{ij} = \frac{1}{2} \varepsilon^{ijkl} F_{kl}, \quad F^{ij} = F_{kl} \varepsilon^{ik} \tilde{g}^{jl}.$$  \hfill (35)

Indeed, from (31) we have

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$  

In this equation partial derivatives can be substituted by the covariant ones and after contraction with the Levi-Chivita tensor $\varepsilon^{ijkl}$ we get first of the equations (34). Further, from (29) it follows that

$$\tilde{\nabla}_i F^{ij} = \nabla_i F^{ij} + H^{ij}_i F^{ij} + H^{ij}_d F^{jd} = \nabla_i F^{ij},$$

Q.E.D.

Now we derive from the equations (34) the general covariant four dimensional Maxwell equations for the electric and magnetic fields which in fact express the fundamental physical laws. First of all we formulate the main relations of the vector algebra and vector analysis on the four dimensional physical manifold which is interesting by itself.

A scalar product of two vector fields $A^i$ and $B^i$ is defined as usual $(A, B) = g_{ij} A^i B^j = A^i B_i = A_i B^i = g^{ij} A_i B_j = |A||B| \cos \varphi$. A vector product of two vector fields $A^i$ and $B^i$ we shall construct as follows

$$C^i = [A B]^i = \varepsilon^{ijkl} t_j A_k B_l.$$  \hfill (35)

It is evident that $[A B] + [B A] = 0$. From geometrical point of view a vector product is a vector field that is tangent to any space section of the manifold in any point. By the direct
calculation it can be shown that $|[AB]| = |A||B| \sin \varphi$, \quad $[A[BC]] = B(A,C) - C(A,B)$. It should be noted that vector product is invariant with respect to transformations
\[
A^i \rightarrow A^i + \lambda t^i, \quad B^i \rightarrow B^i + \mu t^i,
\]
where $\lambda$ and $\mu$ are scalar fields.

Differential operators of the vector analysis on the physical manifold are defined as natural as algebraic ones. For the divergence and gradient we have respectively
\[
\text{div} A = \nabla_i A^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} A^i), \quad (36)
\]
\[
(\text{grad} \phi)^i = (g^{ij} - t^i t^j) \partial_j \phi = g^{ij} \partial_j \phi - t^i D_t \phi. \quad (37)
\]
From (36), (37) we derive
\[
\text{div} \text{grad} \phi = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \phi) - \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} t^i D_t \phi) = \nabla_i \nabla^i \phi - \nabla_i (t^i D_t \phi).
\]

A rotor of the vector field $A$ is defined as a vector product of $\partial$ and $A$
\[
(\text{rot} A)^i = \varepsilon^{ijkl} t_j \partial_k A_l = \frac{1}{2} \varepsilon^{ijkl} t_j (\partial_k A_l - \partial_l A_k). \quad (38)
\]
It is easy to verify that
\[
\text{rot} \text{grad} \phi = 0.
\]
Since the temporal field satisfies the equations $\partial_i t_j - \partial_j t_i = 0$, then rotor is invariant with respect to the transformations $A^i \rightarrow A^i + \lambda t^i$, where $\lambda$ is a scalar field. It is evident that all operations so defined are general covariant.

There is only one direct way to derive from the eqs. (34) the fundamental physical laws first formulated by Maxwell. Consider the rate of change with time of the potentials of electromagnetic field. By definition, we have
\[
D_t A_i = t^k \partial_k A_i + A_k \partial_i t^k = t^k (\partial_k A_i - \partial_i A_k) + \partial_i (t^k A_k) = t^k F_{ki} - \partial_i \phi,
\]
where $\phi = -t^k A_k$. Thus, the rate of change for the electromagnetic potentials can be presented as the difference of two covector fields with one of them having the form
\[
E_i = t^k F_{ik}. \quad (39)
\]
From this it follows that strength of the electric field is general covariant quantity that is defined by the equation
\[
E_i = -D_t A_i - \partial_i \phi. \quad (40)
\]
Now it is quite clear how one can give general covariant definition of the magnetic field strength. According to (38), the axial vector field

$$H^i = (\text{rot} A)^i = \varepsilon^{ijkl} t_j \partial_k A_l = \frac{1}{2} \varepsilon^{ijkl} t_j (\partial_k A_l - \partial_l A_k) \quad (41)$$

is the strength of the magnetic field. Out of (41) it follows that

$$H_i = t^k \tilde{F}_{ik}, \quad (42)$$

where

$$\tilde{F}_{ij} = g_{ik} g_{jl} \tilde{F}^{kl}. \quad (45)$$

Given electric and magnetic fields we see that the tensor of the electromagnetic field is simply a suitable notation. It is evident that vectors $E$ and $H$ are orthogonal to gradient of temporal field

$$t^i H_i = t^i E_i = 0. \quad (43)$$

Thus, vectors $E^i$ and $H^i$ belong to the three dimensional linear space of the vector fields orthogonal to the gradient of the temporal field.

Now it is not difficult to derive the Maxwell equations for the electric and magnetic fields staying in the framework of the vector analysis. However, for some reasons it has a certain interest to give derivation of the fundamental equations of physics precisely from the eqs. (34).

Resolving equations (39), (42) over $F_{ik}$, we obtain

$$F_{ij} = -t_i E_j + t_j E_i - \varepsilon_{ijkl} t_k H_l. \quad (44)$$

Thus, on the physical manifold there is general covariant one-to-one algebraic relation between the electric and magnetic fields and tensor of the electromagnetic field that is given by the equations (39), (42), (43), (44). Out of (44) we find

$$\tilde{F}^{ij} = -t^i H^j + t^j H^i - \varepsilon^{ijkl} t_k E_l, \quad (45)$$

$$F^{ij} = t^i E^j - t^j E^i - \varepsilon^{ijkl} t_k H_l. \quad (46)$$

Substituting (45), (46) into (34), we shall obtain the Maxwell equations for the strengths of the electric and magnetic fields in the following general covariant form

$$D_t H^i + H^i \nabla_k t^k = -\varepsilon^{ijkl} t_j \nabla_k E_l \quad (47)$$

$$\text{div} \ H = \nabla_i H^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} H^i) = 0 \quad (48)$$

$$D_t E^i + E^i \nabla_k t^k = \varepsilon^{ijkl} t_j \nabla_k H_l \quad (49)$$

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\[ \text{div} \, E = \nabla_i E^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} E^i) = 0. \quad (50) \]

Derive, for example, first set of the Maxwell equations. Substituting (45) into the first of Eqs. (34), we get

\[ -\nabla_i \tilde{F}^{ij} = \nabla_i (t^i H^j - t^j H^i + \varepsilon^{ijkl} t_k E_l) = t^i \nabla_i H^j - H^i \nabla_i t^j + H^j \nabla_i t^i - t^j \nabla_i H^i + \varepsilon^{ijkl} t_k \nabla_i E_l. \]

Since, according to (6),

\[ D_t H^j = t^i \partial_i H^j - H^i \partial_i t^j = t^i \nabla_i H^j - H^i \nabla_i t^j, \]

we have

\[ -\nabla_i \tilde{F}^{ij} = D_t H^j + \varepsilon^{ijkl} t_k \nabla_i E_l + H^j \nabla_i t^i - t^j \nabla_i H^i. \]

Taking into account that

\[ D_t t_i = D_t t^i = 0, \quad t_i H^i = 0, \]

we derive from the last equations the equation (48) and then equation (47).

Since \( \sqrt{g} \nabla_i t^i = \partial_i (\sqrt{g} t^i) \), \( D_t \sqrt{g} = t^i \partial_i \sqrt{g} + \sqrt{g} \partial_i t^i = \partial_t (\sqrt{g} t^i) \), then the equations (47) and (49) may be written in more symmetrical form

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} H^i) = -\varepsilon^{ijkl} t_j \nabla_k E_l, \]

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} E^i) = \varepsilon^{ijkl} t_j \nabla_k H_l \]

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} H) = -\text{rot} E \quad (51) \]

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} E) = \text{rot} H. \quad (52) \]

From Eqs. (51), (52) it follows that

\[ D_t (\partial_t (\sqrt{g} H^i)) = 0, \quad D_t (\partial_t (\sqrt{g} E^i)) = 0, \]

and therefore, Eqs. (48), (50) may be considered as the initial conditions. From Eqs. (51), (52) one can also derive that

\[ D_t (\sqrt{g} t_i E^i) = 0, \quad D_t (\sqrt{g} t_i H^i) = 0, \]

which shows that the orthogonality of electric and magnetic fields to the gradient of the temporal field may be considered as the initial conditions.
Thus, it is shown that the principles of the gravity physics are in full correspondence with the fundamental physical laws and hence they can be considered as a method to derive new fundamental equations. To complete this discussion with the Maxwell equations as the main topic, we should only note that the second set of the Maxwell equations can be derived from the principle of least action with the Lagrangian

$$L_{em} = \frac{1}{2}(|E|^2 - |H|^2).$$

We also write the expression for the components of the energy-momentum tensor in terms of electric and magnetic field strength

$$T_{ij} = \frac{1}{2}g_{ij}(|E|^2 + |H|^2) - E_i E_j - H_i H_j - t_i P_j - t_j P_i,$$

where $P_i$ are covariant components of the Pointing vector

$$P^i = \varepsilon^{ijkl} t_j E_k H_l, \quad P = [EH].$$

To conclude this section we mention the existence of the Lagrangian

$$L_{CS} = \frac{q}{2} t_i A_j \tilde{F}^{ij}$$

that is not invariant with respect to time reversal. The modification of electrodynamics due to the Lagrangian of this form was considered in [11], where an analogue of the gradient of the temporal field was introduced to $L_{CS}$ and interpreted as a mass for the photon. It would be very interesting to reconsider the results of the paper [11] from the new point of view presented here.

5 Energy

Let us consider the energy density

$$\varepsilon = \tilde{G}_{ij} t^i t^j + T_{ij} t^i t^j = \varepsilon_g + \varepsilon_m.$$  

(54)

of the full system of interacting fields. Here $\varepsilon_g$ and $\varepsilon_m$ are the energy density of the gravitational and material fields, respectively. Let us write $\varepsilon_m$ for the case of the electromagnetic field. From (53), (54) we find the energy density of the electromagnetic field

$$\varepsilon_m = \frac{1}{2}(E^2 + H^2)$$

(55)

which is a reasonable result.
For the completeness of the picture we shall formulate the energy conservation law starting from the general covariant Maxwell equations. Out of (47), (49) we immediately get

\[ E_t D_t E^l + H_t D_t H^l + (E^2 + H^2) \nabla_i t^i + \nabla_i P^i = 0. \]

Taking into account the relations

\[ \frac{1}{2} D_t (E_t E^l) = \frac{1}{2} D_t |E|^2 = E_t D_t E^l + \frac{1}{2} E^l E^k D_t g_{lk}, \]

\[ \nabla_i t^i = \frac{1}{2} g^{ij} (\nabla_i t_j + \nabla_j t_i) = \frac{1}{2} g^{ij} D_t g_{ij} = \frac{1}{\sqrt{g}} D_t \sqrt{g}, \]

this equation may be written in the form

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} \varepsilon_{em}) + \nabla_i P^i = -\frac{1}{2} T^{ij} D_t g_{ij}, \] (56)

where \( T^{ij} = \tilde{g}^{ik} \tilde{g}^{jl} T_{kl} \). Let us now assume that the electromagnetic field is considered on the background of the physical manifold of some full system of fields. Moreover, it is known that gravity field of this system is static, i.e., \( D_t g_{ij} = 0 \). In such approximation, when physical manifold is external with respect to the electromagnetic field, from (56) we obtain that the energy density of the electromagnetic field satisfies the equation

\[ \frac{1}{\sqrt{g}} D_t (\sqrt{g} \varepsilon_{em}) + \nabla_i P^i = 0. \]

It is exactly the energy conservation law of the electromagnetic field in the above mentioned approximation.

Let us now consider the energy density of gravity field itself in different representations. Since

\[ \varepsilon_g = \tilde{G}_{ij} t^i t^j = \frac{1}{2} \tilde{R}_{ij} g^{ij}, \]

then in the system of coordinate compatible with causal structure for the energy density of gravity field we have \( \varepsilon_g = \tilde{R}_{44} + \tilde{R}_{\mu \nu} g^{\mu \nu} \). From this it is not difficult to find that in this system of coordinate energy density of gravity field can be presented in the form

\[ \varepsilon_g = \frac{1}{8} ((Tr K)^2 - Tr (K^2)) + \frac{1}{2} P, \] (57)

where the matrix \( K \) is defined by the relation

\[ K^\mu_\nu = g^{\mu \tau} K_{\nu \tau} = g^{\mu \tau} \partial_t g_{\nu \tau}, \]

and \( P \) is a scalar curvature of the space section of the physical manifold of the system in question. In the found expression, it can be seen that there is no term that contains second
derivatives with respect to the time coordinate. First term in the right hand side of the equation (57) can be interpreted as the density of the kinetic energy and second one is the density of potential energy of the gravity field, \( \varepsilon_g = \varepsilon_k + \varepsilon_p \), where

\[
\varepsilon_k = \frac{1}{8}((TrK)^2 - Tr(K^2)), \quad \varepsilon_p = \frac{1}{2} P.
\]

To get general covariant expression for the density of kinetic and potential energy let us consider another representation for \( \varepsilon_g \). From (29) it follows that \( \tilde{R}_{ij} = R_{ij} + 2\nabla_i(t^t \nabla_t t_j) \). Thus,

\[
\varepsilon_g = \frac{1}{2} \tilde{R}_{ij} g^{ij} = \frac{1}{2} R + (\nabla_i t^i)^2 + D_t(\nabla_i t^i).
\]

Out of the Ricci identity \( \nabla_i \nabla_k t^i - \nabla_k \nabla_i t^i = R_{ikj} t^j \), we get \( t^i \nabla_i \nabla_k t^k = t^i \nabla_k \nabla_i t^k - R_{ij} t^j t^i \). Since \( \nabla_i t_k = \nabla_k t_i, \quad t^i t^j = 1 \), then \( t^i \nabla_k \nabla_i t^k = t^i \nabla_k \nabla_k t_i = -\nabla_k t^i \nabla_i t^k \). Taking into account the relation \( 2\nabla_k t_l = D_t g_{kl} \), for \( \varepsilon_g \) we finally get

\[
\varepsilon_g = \frac{1}{4} (g^{ik} g^{jl} - g^{ij} g^{kl}) D_t g_{ik} D_t g_{jl} + \frac{1}{2} (R - 2R_{ij} t^i t^j).
\]

If we now move to the system of coordinate compatible with causal structure and compare found expression with that established earlier, then one can conclude that density of the kinetic energy of the gravity field can be presented in the following general covariant form

\[
\varepsilon_k = \frac{1}{8} (g^{ik} g^{jl} - g^{ij} g^{kl}) D_t g_{ik} D_t g_{jl},
\]

whereas density of the potential energy is given as follows

\[
\varepsilon_p = \frac{1}{8} (g^{ik} g^{jl} - g^{ij} g^{kl}) D_t g_{ik} D_t g_{jl} + \frac{1}{2} (R - 2R_{ij} t^i t^j).
\]

Let us find the solution of the Einstein equations (25) for the case when energy-momentum tensor is equal to zero and physical manifold is flat, i.e., the Riemann tensor of the metric (1) equal to zero, \( R_{ijkl} = 0 \). In such case physical manifold can be considered as four-plane and hence \( g_{ij} = \delta_{ij} \). Under such conditions equation (5) has the following regular everywhere solution \( f(u) = a_i u^i \), where \( a_1, a_2, a_3, a_4 \) are constants constrained by the algebraic equation \( a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \). Thus, gradient of the temporal field is a constant unit vector with arbitrary direction on the four-plane. If one considers the manifold as a background, then any physical consequences should not depend on this arbitrariness. Thus, the described solution depends on three arbitrary parameters and it is not difficult to verify that for this solution \( \varepsilon_g = 0 \). Selecting one direction from the continuum we transmute considered physical manifold into the Minkowski space-time, which as is well-known, is a good background for the description of the physical phenomena. However for deeper understanding of the conceptual picture of the modern theoretical physics we need to develop internal field theory.
From the theory of time presented here it follows directly that there is **matter outside the time**. For example, the gravity field and the electromagnetic field exterior to time are described by the equations

\[
R_{ij} - \frac{1}{2}g_{ij}R = g_{ij}F^2 - F_{ik}F_{jl}g^{kl},
\]

\[
\nabla_i F^{ij} = 0, \quad F^{ij} = F_{kl}g^{ik}g^{jl}, \quad F_{ij} = \partial_i A_j - \partial_j A_i,
\]

where \( F^2 = \frac{1}{4}F_{ij}F^{ij} \). In this context it is very important to understand the nature of the emergence of time.

## 6 Conclusion

Let us now sum up the obtained results and focus on some of the problems. It is shown that gravity physics is an internal field theory by nature which does not contain apriori elements and can be characterized as follows. In the theory there is no internal reason that could objectively distinguish one arbitrary coordinate system from another. It is unrolled on the smooth four-manifold that does not exist apriori but is defined by the physical system itself. Manifold is a basic primary notion in physics and representation about space and time is given on this ground. All notions, definitions and laws are formulated in coordinate independent form, i.e., within the framework of the structure of smooth manifold. Einstein’s gravitational potentials are determined by the positive definite symmetrical tensor field (Riemann metric) and temporal field. Being a scalar field on the manifold the temporal field is introduced into the theory in the form that provide, in general, time reversal invariance. The system evolves in its proper time and it is important that we need not to compare intrinsic time with some other times. Central role of the time in the internal field theory is that time determines causal structure of the field theory and first of all it characterizes the internal nature of the gravity field as the simplest closed system. It is established that energy density is the first integral of the closed system of interacting fields. The general covariant definition of the electric and magnetic fields is given and the Maxwell equations for this fields are derived (of great interest is the problem of the evolution form of the Dirac equation on the physical manifold). It is shown how to reveal important internal properties of physical system when coordinates have no physical sense. This means that physical sense of general covariance is recognized and there exists a reparametrization-invariant description (see [9] for an excellent and detailed survey on this subject).
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