POSITIVITY IN FOLIATED MANIFOLDS AND GEOMETRIC APPLICATIONS

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ABSTRACT. We introduce the notion of positivity for a real basic (1, 1) class in basic Bott-Chern cohomology group on foliated manifolds, and study the relationship between this positivity and the negativity of transverse holomorphic sectional curvature and give some geometric applications.

1. Introduction

Since there is no general method to solve differential equations even in $\mathbb{R}$, mathematicians rather try to study the geometrical and topological properties of global manifolds and their asymptotic behaviors. The exact purpose of foliation theory is the qualitative study of differential equations which was initiated by the works of Poincaré and Bendixson, and developed later by Ehresmann, Reeb, Haefliger, and many others. Since then the subject has been a wide field in mathematical research and there are still many open questions in some directions in the theory of foliations (see, for example, [15] and references therein).

The notion of transverse structure plays a key role in the study and classification of foliations. This is a research object of global analysis, and we refer to [15] and references therein for a detailed survey. Since El Kacimi-Alaoui [14] proves the transverse Calabi-Yau theorem, there are many transverse counterparts on foliated manifolds (especially Sasakian manifolds) of the famous results on complex manifolds (especially the Kähler manifolds), such as the existence of canonical metrics on Sasakian manifolds [19], Sasaki-Einstein metrics and K-(semi-)stability on Sasakian manifolds [8,7], the Frankel conjecture on Sasakian manifolds [24,25], the Uhlenbeck-Yau theorem [50] about the existence of Hermitian-Einstein structure [4], foliated Hitchin-Kobayashi correspondence [2], the geometric pluripotential theory [23] on Sasakian manifolds, the transverse fully nonlinear equations [18] corresponding to [43,44,39], and the Higgs bundle on foliated manifolds [58].

In this paper, we introduce notions of the positivity of basic (1, 1) classes in the Bott-Chern cohomological group $H^{1,1}_{BC}(X/F, \mathbb{R})$ using the transverse invariant measure. Then we consider the relationship between this positivity and negativity of transverse holomorphic sectional curvature and also give some geometric applications.

A fundamental conjecture of Yau in 1970’s predicts that a compact Kähler manifold admitting a Kähler metric of negative holomorphic sectional curvature has an ample canonical line bundle, which has been confirmed by Wu-Yau [55] and Tosatti-Yang [47] (also see [13,50,59,62,63] for some further developments). Our first theorem is a transverse version of the works of Wu-Yau [55] and Tosatti-Yang [47].
Theorem 1.1. Let \((X, \mathcal{F})\) be a closed oriented, taut, transverse Kähler foliated manifold, where \(\mathcal{F}\) is the foliation with complex codimension \(n\) and \(\omega\) is a transverse Kähler metric with nonpositive transverse holomorphic sectional curvature. Then the normal canonical bundle \(K_{X/\mathcal{F}}\) is transverse nef. If furthermore \(\omega\) is a transverse Kähler metric with negative transverse holomorphic sectional curvature, then there exists a smooth basic function \(u \in C^\infty(X/\mathcal{F}, \mathbb{R})\) such that \(\omega_u := -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u\) is the transverse Kähler-Einstein metric with \(\text{Ric}(\omega_u) = -\omega_u\).

As an application of Theorem 1.1 and results of Touzet [48, Theorem 1.2], we are able to prove the following

Theorem 1.2. Suppose that \(X\) is a compact Kähler manifold and \(\mathcal{F}\) is (regular) holomorphic with complex codimension \(n\). Assume also that \(c_1(T\mathcal{F}) = 0\) and that there exists a transverse Kähler metric with negative transverse holomorphic sectional curvature. Then the lift of \(\mathcal{F}\), up to some finite covering of \(X\), is defined by a locally trivial fibration over a manifold whose first Chern class is quasi-negative; indeed, \(\mathcal{F}\) is defined by the Iitaka-Kodaira fibration of \(X\).

The outline of the paper is as follows. In Section 2, we collect preliminaries of global analytic and geometric aspects of foliated manifolds. In section 3, we study the geometric partial differential equations on foliated manifolds and as its geometric applications, we prove Theorem 1.1 and Theorem 1.2. In Appendix A, we collect preliminaries for distribution and current in order to deduce the transverse versions of the Poincaré lemma and the Dolbeault-Grothendieck lemma.

After the first version of this paper was posted on the arXiv, we learned that the Sasakian case of Theorem 1.1 was independently proved by Yong Chen, the Wu-Yau theorem on Sasakian manifolds, arXiv:2109.05414.

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2. Preliminaries

In this section, we collect some preliminaries of global analytic and geometric aspects of foliated manifolds (see [3, 15] for example). In what follows, Greek indices, Latin indices and capital Latin indices run from 1 to \(r\), 1 to \(n\) and 1 to \(k\) respectively, unless otherwise indicated.

2.1. Foliation.

Definition 2.1. Let \(X\) be a smooth manifold with \(\dim_{\mathbb{R}} X = k + n\). Then a foliation \(\mathcal{F}\) on \(X\) with real dimension \(k\) and real codimension \(n\) is defined by an atlas \(\mathcal{A}\) on \(X\) consisting of homeomorphism \(\kappa\) of open set \(U_\kappa \subset X\) to \(\tilde{V}_\kappa \times \tilde{U}_\kappa \subset \mathbb{R}^k \times \mathbb{R}^n\) such that

\[
\kappa \circ \kappa'^{-1} : \kappa'(U_\kappa \cap U_{\kappa'}) \to \kappa(U_\kappa \cap U_{\kappa'}), \quad (t, x) \mapsto (s(t, x), y(x)),
\]

with \((t, x), (s, y) \in \mathbb{R}^k \times \mathbb{R}^n\).

This kind of local coordinate patch \((U_\kappa, \kappa)\) is called to be distinguished for the foliation \(\mathcal{F}\).

We denote by \(T\mathcal{F}\) the tangent bundle to \(\mathcal{F}\), and by \(\nu\mathcal{F}\) the quotient \(T_X/T\mathcal{F}\), which is the normal bundle to \(\mathcal{F}\). Let \(\mathcal{X}(\mathcal{F})\) denote the space of all the smooth sections of \(T\mathcal{F}\). If \(g\) is a Riemannian
metric on $X$, then $(X, g, \mathcal{F})$ is called a Riemannian foliated manifold \[51\]. In the distinguished coordinate patch $(U; t, x)$, we set

\[ (2.1) \quad e_i := \frac{\partial}{\partial x^i} - \sum_{P=1}^{k} A_i^P \frac{\partial}{\partial t^P}, \quad 1 \leq i \leq n, \]

such that

\[ (2.2) \quad g \left( \frac{\partial}{\partial t^P}, e_i \right) = 0, \quad 1 \leq i \leq n, \quad 1 \leq P \leq k. \]

A direct calculation yields that

\[ (2.3) \quad A := \sum_{1 \leq i \leq n, 1 \leq P \leq k} A_i^P \frac{\partial}{\partial t^P} \otimes dx^i \]

is a well-defined tensor and that

\[ (2.4) \quad T_X := T_F \oplus T_F^\perp, \]

with respect to the Riemannian metric $g$, where

\[ (2.5) \quad T_F|U = \text{Span}_\mathbb{R} \left\{ \frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^k} \right\}, \quad T_F^\perp|U := \text{Span}_\mathbb{R} \{ e_1, \cdots, e_n \}. \]

Note that $\nu\mathcal{F}$ is smoothly isomorphic to $T_F^\perp$. Let

\[ (2.6) \quad \theta^P := dt^P + A_i^P dx^i, \quad 1 \leq P \leq k. \]

Then $\{ dx^i, \theta^P \}$ is the dual of $\{ e_i, \frac{\partial}{\partial t^P} \}$.

A differential form $u$ is called of the type $\{ p, q \}$ and order $s \in \mathbb{N} \cup \{ \infty \}$ if in the distinguished local coordinate patch $(U; t, x)$ it has the expression

\[ (2.7) \quad u = \frac{1}{p!q!} u_{i_1 \cdots i_p; j_1 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{j_q}, \]

where $u_{i_1 \cdots i_p; j_1 \cdots j_q} \in C^s(U, \mathbb{C})$ is skew-symmetric separately in the indices $i$ and the indices $P$.

Each differential $r$-form has unique decomposition as a sum of form of the type $\{ p, q \}$ with $p + q = r$ and the same order (see for example \[51\]). Let $\mathcal{D}_{\mathcal{F}}^p(X)$ (resp. $\mathcal{D}_{\mathcal{F}}^p(X)$) denote the set of the differential forms of type $\{ p, k \}$ and of order $s$ (resp. with compact support).

A differential $r$ form $\varphi \in \Omega^r(X)$ (resp. a $r$ current $T \in ^r\mathcal{G}(X)$) is said to be basic if it satisfies $i_\xi \varphi = \mathcal{L}_\xi \varphi = 0$ (resp. $i_\xi T = \mathcal{L}_\xi T = 0$) for each $\xi \in \mathfrak{X}(\mathcal{F})$. Let $\Omega^r(X/\mathcal{F})$ (resp. $^r\mathcal{G}(X/\mathcal{F})$) denote the space of basic forms (resp. basic current) of degree $r$ on the foliated manifold $(X, \mathcal{F})$. In particular, a function $f : X \to \mathbb{C}$ is called basic if $f(\varphi_\xi(t))$ is independent of $t$, where $\varphi_\xi(t)$ is the integral curve of $\xi$ for each $\xi \in \mathfrak{X}(\mathcal{F})$. Let $C^k(X/\mathcal{F}, \mathbb{C})$ denote the set of basic functions in $C^k(X, \mathbb{C})$ with $k \in \mathbb{N} \cup \{ \infty \}$.

**Lemma 2.1.** Let $X$ be a smooth manifold with $\dim_{\mathbb{R}} X = k + n$, and $\mathcal{F}$ a foliation on $X$ with real dimension $k$. Then any basic function defined on a distinguished chart $(U, \kappa)$ with compact support is zero.

**Proof.** Without loss of generality, we write

$$ \kappa : U \to \tilde{V} \times \tilde{U} \subset \mathbb{R}^k \times \mathbb{R}^n. $$

Then the conclusion follows from the fact that

$$ \partial(\tilde{V} \times \tilde{U}) = \left( (\partial \tilde{V}) \times \tilde{U} \right) \cup \left( \tilde{V} \times (\partial \tilde{U}) \right). $$
It follows from Lemma 2.1 that there does not exist a partition of unity subordinate to a given cover by basic functions (See for example [2]).

Note that a basic current \( T \in \mathcal{D}^{m-r} (X/F) \) is a continuous linear functional

\[
T : \mathcal{D}^p (X) \to \mathbb{C}
\]

with \( L_\xi T = 0 \) for each \( \xi \in \mathcal{X}(F) \) (see for example [31, 51]).

In the distinguished coordinate patch \((U; t^1, \ldots, t^k, x^1, \ldots, x^n)\), a basic \( p \) current \( T \) can be written as

\[
T = \sum_{i_1 < \cdots < i_p} T_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},
\]

where \( \frac{\partial T_{i_1, \ldots, i_p}}{\partial x^\alpha} = 0 \) for \( 1 \leq \alpha \leq k \). Hence it follows from Theorem A.14 that the basic cohomology group \( H^*(X/F) \) is given by

\[
H^p(X/F, \mathbb{R}) := \left\{ \frac{\varphi \in \Omega^p(X/F) : d\varphi = 0}{d\Omega^{p-1}(X/F)} \right\} \simeq \left\{ \frac{T \in \mathcal{D}^p(X/F) : dT = 0}{d\mathcal{D}^{p-1}(X/F)} \right\}.
\]

**Definition 2.2.** A codimension \( n \) foliation on \( X \) is defined by an open cover \( \mathcal{U} := \{U_i\}_{i \in I} \) with submersions \( \tau_i : U_i \to T \) over a transverse manifold \( T \) with \( \dim_\mathbb{R} T = n \), and for each nonempty intersection \( U_i \cap U_j \), a diffeomorphism

\[
\gamma_{ij} : \tau_i(U_i \cap U_j) \to \tau_j(U_i \cap U_j)
\]

satisfying \( \tau_j(x) = \gamma_{ij} \circ \tau_i(x) \) for all \( x \in U_i \cap U_j \). We say that \( \{U_i, \tau_i, T, \gamma_{ij}\} \) is a foliated cocycle defining \( F \).

The foliation \( F \) is called to be transversely orientable if \( T \) can be given an orientation preserved by all the local diffeomorphisms \( \{\gamma_{ij}\} \).

The foliation \( F \) is called to be Riemannian if there exists a Riemannian metric on \( T \) such that all the local diffeomorphisms \( \{\gamma_{ij}\} \) are isometries. Using the submersions \( f_i : U_i \to T \) one can construct on \( M \) a Riemannian metric which can be written in local coordinates as

\[
g = \sum_{\gamma=1}^k \theta^\gamma \otimes \theta^\gamma + \sum_{p,q=1}^n g_{pq}(x) dx^p \otimes dx^q.
\]

This metric is called bundle-like.

The foliation \( F \) is called to be transversely holomorphic (resp. Kähler) if \( T \) is a complex manifold (resp. a Kähler manifold) and the all the diffeomorphisms \( \{\gamma_{ij}\} \) are local biholomorphisms (resp. in addition preserving the Kähler form on \( T \)).

It follows from [16] that \( \dim_\mathbb{R} H^n(X/F, \mathbb{R}) = 0 \) or \( 1 \). If \( \dim_\mathbb{R} H^n(X/F, \mathbb{R}) = 1 \), then \( F \) is called homologically orientable, which is equivalent to the existence of a (real) volume form on the leaves \( \chi \) which is \( F \)-relatively closed, i.e., \( d\chi(\xi_1, \ldots, \xi_k, \cdot) = 0 \) for \( \xi_1, \ldots, \xi_k \in \mathcal{X}(F) \) (cf. [31] based on [37, 36, 22]). In this case, we can complete the transverse metric by a Riemannian metric along the leaves to obtain a Riemannian metric on the whole manifold for which the leaves are minimal and \( \chi \) is associated to this metric and we also say that \( F \) is taut. This hypothesis will enable one to define an inner product on \( \Omega^r(X/F) \) without using the basic manifold \( T \).
Lemma 2.2 (Basic Stokes’ theorem [2]). Let \((X, \mathcal{F})\) be a closed foliated manifold with a taut foliation \(\mathcal{F}\) of \(\dim \mathcal{F} = k\) and codimension \(n\). Then we have

\[
\int_X (\text{d}_B \varphi) \wedge \chi = 0, \quad \forall \varphi \in \Omega^{n-1}(X/\mathcal{F}),
\]

where \(\chi\) is the volume form of leaves which is \(\mathcal{F}\)-relatively closed.

Let \(\pi : E \to X\) be a complex vector bundle defined by a cocycle \(\{U_\kappa, g_{\kappa \kappa'}, G\}\), where \(\{U_\kappa\}\) is an open cover of \(X\) and \(g_{\kappa \kappa'} : U_\kappa \cap U_{\kappa'} \to G \subset GL(n, \mathbb{C})\) are the transition functions. Let \(\Gamma(E)\) (resp. \(\Gamma_c(E)\)) denote the set of all smooth sections (resp. with compact support) of \(E\).

A connection on the vector bundle \(E\) is a map

\[
\nabla : \mathcal{X}(X) \times \Gamma(E) \to \Gamma(E), \quad (V, \xi) \mapsto \nabla_V \xi
\]

such that for all \(f, h, u, v \in C^\infty(X, \mathbb{C})\) and \(\xi, \eta \in \Gamma(E)\), there holds

\[
\nabla_{fV + hv}(u\xi + v\eta) = V(u)f \xi + Y(v)h \xi + fu\nabla_V \xi + f v\nabla_V \eta + hu\nabla_Y \xi + hv\nabla_Y \eta.
\]

The curvature \(\mathcal{R}\) of \(\nabla\) is defined by

\[
\mathcal{R}(V, Y)Z := \nabla_V \nabla_Y Z - \nabla_Y \nabla_V Z - \nabla_{[V, Y]} Z, \quad \forall V, Y, Z \in \mathcal{X}(X).
\]

It follows from [27] that the vector bundle \(E\) is foliated if and only if it admits a linear connection such that its curvature \(\mathcal{R}\) satisfies

\[
\mathcal{R}(\xi_1, \xi_2) = 0, \quad \forall \xi_1, \xi_2 \in \mathcal{X}(\mathcal{F}),
\]

and that the vector bundle \(E\) is an \(\mathcal{F}\)-bundle if and only if it admits a linear connection (called basic connection) such that its curvature \(\mathcal{R}\) satisfies

\[
\mathcal{R}(\xi, \cdot) = 0, \quad \forall \xi \in \mathcal{X}(\mathcal{F}).
\]

In the Čech language, a vector bundle \(E\) is foliated if and only if it can be defined by a cocycle \(\{U_\kappa, g_{\kappa \kappa'}, G\}\) such that the components of \(g_{\kappa \kappa'}\) are basic functions, if and only if it admits a connection \(\nabla\) such that the connection form \(\omega\) satisfies \(i_\xi \omega = 0\) for each \(\xi \in \mathcal{X}(\mathcal{F})\). A vector bundle \(E\) is the \(\mathcal{F}\)-bundle if and only if it admits a connection \(\nabla\) such that the connection form \(\omega\) is a basic 1-form.

A section \(\eta \in \Gamma(E)\) is called basic if \(\nabla_\xi \eta = 0\) for each \(\xi \in \mathcal{X}(\mathcal{F})\), where \(\nabla\) is the basic connection on the \(\mathcal{F}\)-bundle. Let \(\Gamma(E/\mathcal{F})\) denote the set of all basic sections of \(E\).

If a vector bundle \(E\) is the \(\mathcal{F}\)-bundle, then the dual bundle \(E^*\) and all of its exterior and symmetric powers \(\wedge^* E\) and \(S^* E\) are the \(\mathcal{F}\)-bundle. In particular, \(E^* \otimes E^*\) is also the \(\mathcal{F}\)-bundle. If there exists \(h \in \Gamma(E^* \otimes E^*)\) such that \(h\) is positive and basic with respect to the basic connection, then \(E\) is called a Hermitian \(\mathcal{F}\)-bundle.

Let \(\mathcal{F}\) be a transverse holomorphic foliation on \(X\) with real dimension \(k\) and complex codimension \(n\), and let \(\nu\) be the complexified normal bundle \(\nu \mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}\) of \(\nu \mathcal{F}\). Let \(J\) be the automorphism of \(\nu\) associated to the complex structure; \(J\) satisfies \(J^2 = -\text{id}\) and then it has two eigenvalues, \(\sqrt{-1}\) and \(-\sqrt{-1}\), with associated eigensubbundles denoted \(\nu^{1,0}\) and \(\nu^{0,1}\), respectively. The holomorphic \(\mathcal{F}\)-line bundle \(K_{\mathcal{F}/\mathcal{F}} := \wedge^* (\nu^{1,0})^*\) is called the normal canonical line bundle. We have a splitting \(\nu^* = (\nu^{1,0})^* \oplus (\nu^{0,1})^*\) which gives rise to a decomposition

\[
\wedge^r \nu^* = \bigoplus_{p+q=r} \wedge^p \nu^q,
\]
where $\Lambda^{p,q} = \Lambda^p(\nu^{1,0})^* \otimes \Lambda^q(\nu^{0,1})^*$. Basic sections of $\Lambda^{p,q}$ consist of basic forms of type $(p,q)$. They form a vector space denoted by $\Omega^p_q(X/F)$. We have

$$\Omega^p_q(X/F) = \bigoplus_{p+q=r} \Omega^p_q(X/F).$$

The exterior differential $d$ decomposes into a sum of two operators

$$\partial : \Omega^{p,q}(X/F) \to \Omega^{p+1,q}(X/F), \quad \bar{\partial} : \Omega^{p,q}(X/F) \to \Omega^{p,q+1}(X/F).$$

A basic function $f \in C^1(X/F, \mathbb{C})$ is called basic holomorphic if $\bar{\partial}f = 0$.

For the basic currents, we also have

$$\mathcal{D}^p_q(X/F) = \bigoplus_{p+q=r} \mathcal{D}^p_q(X/F), \quad \mathcal{D}^p_q(X/F) = \bigoplus_{p+q=r} \mathcal{D}^p_q(X/F).$$

The space $\mathcal{D}^{p,q}(X/F)$ is called the space of basic currents of bidimension $(n-p,n-q)$ and bidegree $(p,q)$ on $X$, and it is also denoted by $\mathcal{D}^{n-p,n-q}(X/F)$. It follows from Lemma A.15 that the basic Dolbeault cohomology $H^{p,q}(X/F)$ of the foliation $F$ is given by

$$H^{p,q}(X/F, \mathbb{C}) := \{ \varphi \in \Omega^{p,q}(X/F) : \bar{\partial}\varphi = 0 \} \sim \{ T \in \mathcal{D}^{p,q}(X/F) : \bar{\partial}T = 0 \}.$$ 

We also introduce the basic Bott-Chern cohomology group defined by

$$H^{p,q}_{BC}(X/F, \mathbb{C}) := \{ \varphi \in \Omega^{p,q}(X/F) : \bar{\partial}\varphi = 0 \} \sim \{ T \in \mathcal{D}^{p,q}(X/F) : \bar{\partial}T = 0 \}.$$ 

It follows from [14] that both $\dim_{\mathbb{R}} H^p(X/F)$ and $\dim_{\mathbb{C}} H^{p,q}_{BC}(X/F)$ are finite (See [15] for more details). If $(X, F)$ is a transverse Kähler foliated manifold, then $H^{p,q}(X/F, \mathbb{C}) = H^{p,q}_{BC}(X/F, \mathbb{C})$ by the basic $\partial\bar{\partial}$-lemma [2].

Let $F$ be a transverse holomorphic foliation on $X$ with real dimension $k$ and complex codimension $n$. Then for a holomorphic Hermitian $F$-bundle $(E, h)$ with rank $r$, the adapted Chern connection $\nabla$ on $E$ is the unique basic connection which preserves $h$ and satisfies $\nabla^{(1,0)} = \bar{\partial}$. We denote by $c(E, h)$ the curvature of the connection $\nabla$, which is a basic $(1,1)$ form with values in End($E$).

By the Chern-Weil theory [4], the Chern form $c_i(E, h)$ of the holomorphic Hermitian $F$-bundle $(E, h)$ is defined by

$$\det \left( \text{Id}_E + \frac{\sqrt{-1}}{2\pi} c(E, h) \right) = 1 + \sum_{i\geq 1} c_i(E, h),$$

where $c_i(E, h)$ is a closed basic real $(i,i)$ form for $i \geq 1$. We also call $c_1(E, h)$ the Chern-Ricci form. We say that

$$c_i^{BC}(E/F) = [c_i(E, h)] \in H^{i,i}_{BC}(X/F, \mathbb{R}) := \frac{\{ \text{d-closed basic real } (i,i) \text{ forms} \}}{\sqrt{-1} \partial \bar{\partial} \{ \text{basic real } (i-1,i-1) \text{ forms} \}}$$

is the $i^{th}$ basic Chern class of $E$. In particular, the first basic Chern class $c_1(X/F)$ of $(X/F)$ is defined by $-c_1(K_{X/F})$.

Let $\{s_1, \cdots, s_r\}$ be a local basic basis of transverse holomorphic Hermitian $F$-bundle $E$ in the distinguished chart $(U; t^1, \cdots, t^k, z^1, \cdots, z^n)$. Then $h = (h_{\alpha\beta})$ with $h_{\alpha\beta} \in C^\infty(X/F, \mathbb{C})$ is the transverse Hermitian metric on $E$. Let

$$e_i = \partial_i + \sum_{P=1}^k A^P P \partial_i^P, \quad 1 \leq i \leq n$$

(2.19)
be a local basic basis of $\nu^{1,0}$ with dual $\{dz^1, \ldots, dz^n\}$. Then we use the notation
$$\nabla_i := \nabla_{e_i}, \quad \nabla_j := \nabla_{e_j}$$
and
$$\nabla_is_\alpha = \sum_\beta \Gamma^\beta_{i\alpha} s_\beta, \quad \text{with} \quad \Gamma^\beta_{i\alpha} = \sum_\gamma h^{i\beta\gamma} \partial_i h_{\alpha\gamma}, \quad \text{and} \quad \sum_\gamma h^{\alpha\beta\gamma} h_{\beta\gamma} = \delta^\beta_\beta.$$ Note that when $\nabla$ acts on basic sections, we have
$$\nabla_i = \nabla_{\partial_i}, \quad \nabla_j = \nabla_{\partial_j}, \quad \nabla_{[e_i, e_j]} = \nabla_{[\overline{e_i}, e_j]} = \nabla_{[\overline{e_i}, e_j]} = 0.$$ In what follows, we will always study basic sections. Hence we will use this fact directly without explanation.

The curvature is defined by
$$\nabla_i \nabla_j s_\alpha - \nabla_j \nabla_i s_\alpha - \nabla_{[e_i, e_j]} s_\alpha = R_{ij\alpha}^\beta s_\beta, \quad R_{ij\alpha}^\beta = -\partial_j \Gamma^\beta_{i\alpha}.$$ We use the notation $R_{ij\alpha\beta} := R_{ij\alpha}^\gamma h_{\gamma\beta}$, and have
$$R_{ij\alpha\beta} := -\partial_j \partial_i h_{\alpha\beta} + h^{\gamma} \partial_i h_{\alpha\gamma} \partial_j h_{\gamma\beta}.$$ The Chern-Ricci form $c_1(E, h)$ is given by
$$2\pi c_1(E, h) = -\overline{\partial} \overline{\partial} \log \det (h_{ij}).$$ For the transverse Hermitian metric $\omega = -\overline{\partial} \overline{\partial} \log \det (g_{ij})$ with $g_{ij} \in C^\infty(X/F, \mathbb{C})$, the curvature tensor $R = \{R_{ik\ell}\}$ of $\omega$ is given by (see [14])
$$R_{ik\ell} = -\partial_i \partial_k g_{\ell\ell} + g^{p\delta} (\partial_i g_{p\delta}) (\partial_k g_{\ell\ell}).$$
Given $x \in X$ and basic vector field $W \in \nu^{1,0}_x \setminus \{0\}$, the transverse holomorphic sectional curvature of $\omega$ at $x$ in the direction $W$ is
$$H_x(W) := \frac{\Re(W, \overline{W}, W, \overline{W})}{|W|^4}.$$ We say that $\omega$ has negative transverse holomorphic sectional curvature if
$$H_x(W) < 0, \quad \forall \, x \in X, \quad \forall \, W \in \nu^{1,0}_x \setminus \{0\}.$$ We set
$$H^\omega_x := \sup \{H^\omega_x(W) | W \in T^1_0 X \setminus \{0\} \}$$ and
$$\mu_\omega := \sup_{x \in X} H^\omega_x.$$ The Chern-Ricci form $\text{Ric}(\omega)$ is defined by
$$\text{Ric}(\omega) = -\overline{\partial} \overline{\partial} \log \det (g_{ij})$$ and there holds
$$2\pi c_1(X/F) = [\text{Ric}(\omega)] \in H^{1,1}_{BC}(X/F, \mathbb{R}).$$ We also introduce the transverse “torsion” tensor as in Hermitian geometry.
$$T^k_{ij} := g^{kj} (\partial_i g_{pq} - \partial_j g_{pq}), \quad T_{ik\ell} := T^k_{ij} g_{k\ell}.$$ For a basic $(1, 0)$ form $a = a_\ell dz^\ell$, define covariant derivative $\nabla \partial_\ell a_\ell$ by
$$\nabla_i a_\ell := \partial_\ell a_\ell = -\Gamma^p_{i\ell} a_p.$$ Then we can deduce
$$[\nabla_i, \nabla_j]a_\ell = -R_{ij\ell}^p a_p, \quad [\nabla_i, \nabla_j a_m] = R_{ijm}^p a_p.$$
where $R_{ij} = R_{ij}^p g^{pl} y_{lm}$. For each basic function $u \in C^2(X/F, \mathbb{R})$, one can infer
\begin{equation}
\nabla_i u = \partial_i u, \quad \nabla^k T = \partial_{tk} u, \quad \nabla^k \nabla_i u = \partial_{tk} \partial_i u, \quad [\nabla_i, \nabla_j] u = -T_{ij}^p \nabla_p u.
\end{equation}
A direct calculation yields that (cf. [46] for Hermitian manifold)
\begin{equation}
\nabla_i \nabla^k j \nabla_i u = \nabla_i \nabla^k j \nabla_k u + R_{kij}^p \nabla_j \nabla_p u - R_{ij}^p \nabla_k \nabla_p u
\end{equation}
and
\begin{equation}
\nabla_i \nabla^k j \nabla_i u = \nabla_i \nabla^k j \nabla_k u + R_{kij}^p \nabla_j \nabla_p u + T_{ij}^p \nabla_j \nabla_i u + T_{ij}^p \nabla_j \nabla_i u, \quad \forall u \in C^\infty(X/F, \mathbb{R}).
\end{equation}

**Lemma 2.3.** Let $F$ be a transverse holomorphic foliation on $X$ with real dimension $k$ and complex codimension $n$. If $f \in \mathcal{D}^{p,0}(X/F)$ satisfies that $\partial f \in \mathcal{E}^{p,1}(X/F)$, then $f \in \mathcal{E}^{p,0}(X/F)$.

**Proof.** The result is local, and hence we may assume that $X = V \times U \subset \mathbb{R}^k \times \mathbb{C}^n$ is a distinguish patch. Then one infers from Proposition 2.3 that $f = 1 \otimes \tilde{f}$ where $\tilde{f} \in \mathcal{D}^{p,0}(U)$ satisfies $\partial \tilde{f} \in \mathcal{E}^{p,1}(U)$, and hence it follows from [12] Corollary I-3.30 that $\tilde{f} \in \mathcal{E}^{p,0}(U)$, which means $f = 1 \otimes \tilde{f} \in \mathcal{E}^{p,0}((V \times U)/F|_{V \times U})$, as desired. \hfill \square

Let $\nu$ be a Hermitian $F$-bundle and $\omega$ denote the basic Hermitian metric on $\nu$. Then $\omega^n \wedge \chi$ is the canonical orientation on $X$ and $\omega^n$ is the canonical transverse orientation. In the distinguished chart $(U; t^1, \ldots, t^k, z^1, \ldots, z^n)$, each $u \in \Gamma(A^{p,q})$ can be written as
\begin{equation}
u = \frac{1}{p! q!} u_{i_1 \cdots i_p, j_1 \cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},\end{equation}
where $u_{i_1 \cdots i_p, j_1 \cdots j_q} = u_{i_1 \cdots i_p, j_1 \cdots j_q}(t, z)$.

However, given Lemma 2.1, we should introduce the notion of positivity similar to the one in [30] carefully (c.f. [12] Remark 1.15 of Section 1 of Chapter 3) and [52]).

A form $u \in \Gamma(A^{p,q})$ is called positive (resp. strictly positive) if on each distinguished chart $(U; t^1, \ldots, t^k, z^1, \ldots, z^n)$
\begin{equation}
u|_U \wedge \sqrt{-1} \alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge \sqrt{-1} \alpha_{n-1} \wedge \bar{\alpha}_{n-1} \wedge \chi \geq 0 \quad \text{resp.} \quad \nu|_U \wedge \sqrt{-1} \alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge \sqrt{-1} \alpha_{n-1} \wedge \bar{\alpha}_{n-1} \wedge \chi > 0
\end{equation}
on $U$ for each non-zero basic $\alpha_j \in \Gamma(\nu_{U}^{1,0})$ with constant coefficients under the coordinate $(z^1, \ldots, z^n)$ and $1 \leq j \leq n-p$.

In order to introduce the notion of positivity of basic $(p, p)$ current, let us recall the notion of invariant transverse measure. Let $F$ be a transverse holomorphic foliation on $X$ with real dimension $k$ and complex codimension $n$. Then an invariant transverse measure $\mu$ is a measure on the local leaf space in each chart which is preserved by transition functions. We can identify the invariant transverse measure $\mu$ with the basic $(n, n)$ current $C_\mu$ given by
\begin{equation}
C_\mu := (\sqrt{-1})^n \sum (1 \otimes \mu) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n
\end{equation}
in the distinguished chart $(t^1, \ldots, t^k, z^1, \ldots, z^n)$. In what follows, we will not distinguish $\mu$, $C_\mu$ and $1 \otimes \mu$.

A basic $(p, p)$ current $T$ is called positive if on each distinguished chart $(U; t^1, \ldots, t^k, z^1, \ldots, z^n)$
\begin{equation}
T|_U \wedge \sqrt{-1} \alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge \sqrt{-1} \alpha_{n-1} \wedge \bar{\alpha}_{n-1}
\end{equation}
is a positive invariant transverse measure on $U$ for each non-zero basic $\alpha_j \in \Gamma(\nu_{U}^{1,0})$ with constant coefficients in the given coordinates $(z^1, \ldots, z^n)$ and $1 \leq j \leq n - p$. The set of positive basic currents of bi-dimension $(n - p, n - p)$ will be denoted by

\[ \mathcal{D}^+_{n-p,n-p}(X/F) . \]

In the distinguished chart $(U; t^1, \ldots, t^k, z^1, \ldots, z^k)$, it follows from [12 Lemma III-1.4] that $\wedge^{p,p}$ admits a basis consisting of

\[ \alpha_s = \sqrt{-1} \alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \cdots \wedge \sqrt{-1} \alpha_{s,p} \wedge \bar{\alpha}_{s,p}, \quad 1 \leq s \leq \binom{n}{p}^2 \]

where $\alpha_s$ is of the type $dz^j \pm dz^k$ or $dz^j \pm \sqrt{-1} dz^k$, $1 \leq j, k \leq n$. This, together with the argument in [12 Proposition III-1.14], yields that each positive basic current

\[ T = \frac{(\sqrt{-1})^{p^2}}{p!p!} T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \bar{d}z^{\bar{j}_1} \wedge \cdots \wedge \bar{d}z^{\bar{j}_p} \in \mathcal{D}^+_{n-p,n-p}(X/F) \]

is real and of order 0, i.e., its coefficients $T_{I,J}$ are invariant transverse complex measures and satisfy

\[ T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p} = T_{\bar{j}_1,\ldots,\bar{j}_p,i_1,\ldots,i_p} . \]

Moreover $T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p}$ is positive invariant transverse measure, and the total variation $|T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p}|$ of the measures $T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p}$ satisfy the inequality

\[ \lambda_{i_1} \cdots \lambda_{i_p} \lambda_{\bar{j}_1} \cdots \lambda_{\bar{j}_p} |T_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_p}| \leq 2^p \sum_{m_1,\ldots,m_p} (\lambda_{m_1} \cdots \lambda_{m_p})^2 T_{m_1,\ldots,m_p,m_1,\ldots,m_p} , \]

where $\lambda_k \geq 0$ and

\[ \{i_1,\ldots,i_p\} \cap \{\bar{j}_1,\ldots,\bar{j}_p\} \subset \{m_1,\ldots,m_p\} \subset \{i_1,\ldots,i_p\} \cup \{\bar{j}_1,\ldots,\bar{j}_p\} . \]

We denote by $*\mathcal{D}^{p,q}(X,F)$ the set of $(k + p + q)$-forms given locally by

\[ \frac{1}{p!q!} \mathcal{C}_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q} \theta^1 \wedge \cdots \wedge \theta^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \bar{d}z^{\bar{j}_1} \wedge \cdots \wedge \bar{d}z^{\bar{j}_q} \]

with $\mathcal{C}_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q} = \mathcal{C}_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q}(t,z) \in C^\infty(U,\mathbb{R})$ and $\{\theta^1,\ldots,\theta^k\}$ defined similarly by (2.6).

The dual of $*\mathcal{D}^{p,q}(X,F)$ is denoted by $*\mathcal{D}^{n-p,n-q}(X,F)$ or $*\mathcal{D}'_{p,q}(X,F)$. With these notations, we get that $\mathcal{D}^+_{n-p,n-p}(X/F)$ is weakly-* closed cone in $0\mathcal{D}_{p,p}(X,F)$ (see [9,10]). Note that $0\mathcal{D}_{p,p}(X,F)$ is a separable topological vector space.

**Lemma 2.4.** Let $F$ be a transverse holomorphic foliation on $X$ with real dimension $k$ and complex codimension $n$, and let $\delta$ be a positive continuous function on $X$. Then the set given by

\[ \mathcal{F} := \left\{ T \in \mathcal{D}^+_{p,p}(X/F) : \int_X \delta T \wedge \omega^p \wedge \chi \leq 1 \right\} \]

is weakly-* compact and weakly-* sequentially compact.

**Proof.** This is a transverse version of [12 Proposition III-1.23]. Since

\[ V := \left\{ u \in 0\mathcal{D}^{p,p}(X,F) : -\delta \omega^p \wedge \chi < u < \delta \omega^p \wedge \chi \right\} \]

is an open neighborhood of 0 in the separable topological vector space $0\mathcal{D}^{p,p}(X,F)$ (actually we can check this directly by using the topology of $0\mathcal{D}^{p,p}(X,F)$), we get that

\[ \mathcal{F} := \left\{ T \in 0\mathcal{D}'_{p,p}(X,F) : \langle T, u \rangle \leq 1, \quad \forall \, u \in V \right\} \]
is weakly-* compact and weakly-* sequently compact from [29] Theorem 3.64, Theorem 3.67 & Corollary 3.68. Hence Lemma 2.4 follows from the fact that \( \mathcal{F} \) is a weakly-* closed subset contained in \( \tilde{\mathcal{F}} \).

2.2. Notions of positivity. Let \([\alpha] \in H^{1,1}_{BC}(X/\mathcal{F}, \mathbb{R})\), where \(\alpha\) is a smooth closed real \((1,1)\) basic form. We define the following positivity notions which are easily seen to be independent of the choice of \(\omega\) and hence without loss of generality we assume that \(\omega\) is the basic Gauduchon metric (i.e., \(\omega\) is a positive \((1,1)\) basic form such that \(\partial\overline{\partial}\omega = 0\)). It follows from [2] that there exists a unique basic Gauduchon metric up to scaling (when \(n \geq 2\)) in the conformal class of each Hermitian metric.

- \([\alpha]\) is transverse Kähler if it contains a representative which is a basic Kähler form, i.e., there is a smooth basic function \(\varphi\) such that \(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geq \varepsilon \omega\) on \(X\), for some \(\varepsilon > 0\).
- \([\alpha]\) is transverse nef if for every \(\varepsilon > 0\) there is a smooth basic function \(\varphi_{\varepsilon}\) such that \((2.27)\)

\[
\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_{\varepsilon} \geq -\varepsilon \omega
\]

holds on \(X\).
- \([\alpha]\) is transverse big if it contains a basic Kähler current \(T\), i.e., there exists a closed basic current \(T \in [\alpha]\) such that \(T \geq \varepsilon \omega\) holds weakly as currents on \(X\) for some \(\varepsilon > 0\).
- \([\alpha]\) is transverse pseudoeffective if it contains a closed positive basic current.

The set of all the transverse Kähler classes (resp. transverse pseudoeffective classes, transverse nef classes, transverse big classes) is denoted by \(\mathcal{K}_{X/\mathcal{F}}\) (resp. \(\mathcal{E}_{X/\mathcal{F}}, \mathcal{N}_{X/\mathcal{F}}, \mathcal{B}_{X/\mathcal{F}}\)).

Lemma 2.5. Let \(\mathcal{F}\) be a transverse holomorphic foliation on \(X\) with real dimension \(k\) and complex codimension \(n\). Then both \(\mathcal{E}_{X/\mathcal{F}}\) and \(\mathcal{N}_{X/\mathcal{F}}\) are closed cone. Moreover, there holds that \(\mathcal{N}_{X/\mathcal{F}} \subset \mathcal{E}_{X/\mathcal{F}}\) and \(\mathcal{E}_{X/\mathcal{F}} \cap (-\mathcal{E}_{X/\mathcal{F}}) = \{0\}\).

Proof. We just check the closeness by adapting the idea of [11] Proposition 6.1. Without out loss of generality, we assume that \(\omega\) is a transverse Gauduchon metric. For the closeness of \(\mathcal{E}_{X/\mathcal{F}}\), we assume that \(\{T_i\}_{i=1}^\infty\) is a sequence of closed positive basic currents such that

\[
[T_i] \to [\alpha], \quad \text{as} \quad i \to \infty.
\]

Then it follows from (2.10) that

\[
\int_M T_i \wedge \omega^{n-1} \wedge \chi = \int_M [T_i] \wedge \omega^{n-1} \wedge \chi \to \int_M [\alpha] \wedge \omega^{n-1} \wedge \chi, \quad \text{as} \quad i \to \infty.
\]

This, together with Lemma 2.3, yields that there exists a subsequence \(\{T_{i_j}\}_{j=1}^\infty\) converging to a closed positive basic current \(T\) weakly. Hence we can deduce \([T_{i_j}] \to [T]\) as \(j \to \infty\) and that \([T]\) by the uniqueness of weak limit is transverse pseudoeffective, as required.

For the closeness of \(\mathcal{N}_{X/\mathcal{F}}\), we assume that \(\{\alpha_i\}_{i=1}^\infty\) is a sequence of closed smooth real basic \((1,1)\) form such that

\[
[\alpha_i] \to [\alpha], \quad \text{as} \quad i \to \infty.
\]

We select a sequence of smooth representatives \(\gamma_j \in [\alpha] - [\alpha_j]\) which converges to 0 in \(C^\infty(X/\mathcal{F}, \mathbb{R})\). Fix \(\varepsilon > 0\). If \([\alpha_j]\) is transverse nef, then there exists a smooth representative \(\alpha_{j,\varepsilon}\) such that \(\alpha_{j,\varepsilon} \geq -\frac{\varepsilon}{2} \omega\). On the other hand, there exists a \(k_0 \in \mathbb{N}\) such that \(\gamma_j \geq -\frac{\varepsilon}{2} \omega\) with \(j \geq k_0\). Thus, one can infer that \([\alpha] = [\alpha_j] + [\gamma_j]\) contains a representative \(\alpha_{j,\varepsilon} + \gamma_j\) with

\[
\alpha_{j,\varepsilon} + \gamma_j \geq -\varepsilon \omega, \quad \forall j \geq k_0.
\]

This yields that \([\alpha]\) is transverse nef and hence that \(\mathcal{N}_{X/\mathcal{F}}\) is closed.
Let $[\alpha]$ is a transverse nef class. Then for each $\varepsilon > 0$, there exists a smooth basic function $\varphi_\varepsilon \in C^\infty(X/\mathcal{F}, \mathbb{R})$ such that
\[
\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon > -\varepsilon \omega.
\]
This yields that
\[
-\varepsilon \int_X \omega^n \wedge \chi \leq \int_X (\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon) \wedge \omega^{n-1} \wedge \chi = \int_X \alpha \wedge \omega^{n-1} \wedge \chi.
\]
This, together with Lemma 2.4, yields that we can assume that without loss of generality \(\lim_{\varepsilon \to 0}(\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon) = T\) in the current sense, where $T$ is a closed positive basic current. Hence $[\alpha] = [T]$ is transverse pseudoeffective.

If $T$ is a closed positive basic current on $X$ and the class $-[T]$ is also transverse pseudoeffective, then there is a closed positive basic current $\tilde{T} = -T + \sqrt{-1}\partial\bar{\partial}\varphi$ for some basic distribution $\varphi$, and so $\sqrt{-1}\partial\bar{\partial}\varphi = T + \tilde{T} > 0$ in the current sense. It follows from [21, Proposition 1.43] that we can see $\varphi$ as a basic plurisubharmonic function and hence $\varphi$ is a constant by the maximum principle. This yields that both $T$ and $\tilde{T} = -T$ are positive in the current sense and hence $T = 0$, as desired. $\square$

**Lemma 2.6.** Let $\mathcal{F}$ be a transverse Kähler foliation on $X$ with real dimension $k$ and complex codimension $n$. Then the the transverse Kähler cone $\mathcal{K}_{X/\mathcal{F}}$ is an open and convex cone inside $H^{1,1}(X/\mathcal{F}, \mathbb{R})$. Furthermore, there holds that $\mathcal{K}_{X/\mathcal{F}} \cap (-\mathcal{K}_{X/\mathcal{F}}) = \{0\}$.

**Proof.** This is a transverse version of [40] for the openness of Kähler cone and we use the idea adapted from [40]. By saying that $\mathcal{K}_{X/\mathcal{F}}$ is a cone we mean that if we are given $[\alpha] \in \mathcal{K}_{X/\mathcal{F}}$ and $\lambda \in \mathbb{R}_{>0}$, then $\lambda [\alpha] \in \mathcal{K}_{X/\mathcal{F}}$, which is obvious. The convexity of $\mathcal{K}_{X/\mathcal{F}}$ follows immediately from the fact that if $\omega_1$ and $\omega_2$ are transverse Kähler metrics on $(X, \mathcal{F})$ and $0 \leq \lambda \leq 1$, then $\lambda \omega_1 + (1-\lambda)\omega_2$ is also a transverse Kähler metric. For the openness of $\mathcal{K}_{X/\mathcal{F}}$, we fix closed real basic $(1,1)$ forms $\{\alpha_1, \ldots, \alpha_k\}$ on $(X, \mathcal{F})$ such that $\{[\alpha_1], \ldots, [\alpha_k]\}$ is a basis of $H^{1,1}(X/\mathcal{F}, \mathbb{R})$. Given a transverse Kähler class $[\alpha] \in \mathcal{K}_{X/\mathcal{F}}$ we can write $[\alpha] = \sum_{i=1}^k \lambda_i [\alpha_i]$, for some $\lambda_i \in \mathbb{R}$. Since $[\alpha] \in \mathcal{K}_{X/\mathcal{F}}$, there exists a basic function $\varphi \in C^\infty(X/\mathcal{F}, \mathbb{R})$ such that
\[
\sum_{i=1}^k \lambda_i \alpha_i + \sqrt{-1}\partial\bar{\partial}\varphi > 0.
\]
Since $X$ is compact, it follows that
\[
\sum_{i=1}^k \tilde{\lambda}_i \alpha_i + \sqrt{-1}\partial\bar{\partial}\varphi > 0,
\]
for all $\tilde{\lambda}_i$ sufficiently close to $\lambda_i$ ($1 \leq i \leq k$), and so all $(1,1)$ classes in a neighborhood of $[\alpha]$ contain a transverse Kähler metric.

If $\omega$ is a transverse Kähler metric on $X$ and the class $-[\omega]$ is also transverse Kähler, then there is a transverse Kähler metric $\tilde{\omega} = -\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some basic function $\varphi \in C^\infty(X, \mathcal{F})$, and so $\sqrt{-1}\partial\bar{\partial}\varphi = \omega + \tilde{\omega} > 0$ everywhere on $X$. This is impossible, since $\sqrt{-1}\partial\bar{\partial}\varphi \leq 0$ at the points where $\varphi$ attains its maximum. Hence we have $\mathcal{K}_{X/\mathcal{F}} \cap (-\mathcal{K}_{X/\mathcal{F}}) = \emptyset$. $\square$

**Lemma 2.7.** Let $\mathcal{F}$ be a transverse Kähler foliation on $X$ with real dimension $k$ and complex codimension $n$. Then there holds $\mathcal{E}_{X/\mathcal{F}} = \overline{\mathcal{K}}_{X/\mathcal{F}}$ and $\mathcal{E}^\circ_{X/\mathcal{F}} = \mathcal{B}_{X/\mathcal{F}}$. 


Lemma 3.1. Let $\alpha \in \mathcal{K}_{X/F}$ be a transverse Kähler metric.

Condition in definition of transverse nef (2.27) is equivalent to $\alpha + \varepsilon \omega \in \mathcal{K}_{X/F}$, for all $\varepsilon > 0$, which certainly implies that $[\alpha] \in \mathcal{K}_{X/F}$. Conversely, if $[\alpha] \in \mathcal{K}_{X/F}$, then there is a sequence $\{\beta_i\}$ of closed real $(1,1)$ forms such that $\alpha + \beta_i > 0$ for all $i$, and $[\beta_i] \to 0$ in $H^{1,1}(X/F, \mathbb{R})$. As before we fix closed real $(1,1)$ forms $\{\alpha_1, \ldots, \alpha_k\}$ on $X$ such that $\{[\alpha_1], \ldots, [\alpha_k]\}$ is a basis of $H^{1,1}(X/F, \mathbb{R})$, and for each $i$ we write

$$
[\beta_i] = \sum_{j=1}^{k} \lambda_{ij}[\alpha_j],
$$

with $\lambda_{ij} \in \mathbb{R}$. Since $[\beta_i] \to 0$, and $\{[\alpha_1], \ldots, [\alpha_k]\}$ is a basis, we conclude that $\lambda_{ij} \to \infty$ as $i \to \infty$, for each fixed $j$. If we let

$$
\tilde{\beta}_i = \sum_{j=1}^{k} \lambda_{ij}\alpha_j,
$$

then the forms $\tilde{\beta}_i$ converge smoothly to zero, as $i \to \infty$, and we can find smooth basic functions $\varphi_i \in C^\infty(X/F, \mathbb{R})$ such that $\beta_i = \tilde{\beta}_i + \sqrt{-1}\partial\bar{\partial}\varphi_i$. For every $\varepsilon > 0$ we choose $i$ sufficiently large so that $\tilde{\beta}_i < \varepsilon \omega$ on $X$, and so

$$
\alpha + \varepsilon \omega + \sqrt{-1}\partial\bar{\partial}\varphi_i > \alpha + \tilde{\beta}_i + \sqrt{-1}\partial\bar{\partial}\varphi_i = \alpha + \beta_i > 0,
$$

which proves (2.27).

The fact that transverse big class contains in $\mathcal{E}^0_{X/F}$ is obvious. On the other hand, for each $[\alpha] \in \mathcal{E}^0_{X/F}$, we have

$$
[\alpha] = \lim_{k \to \infty} \left( [\alpha] + \frac{1}{k}[\omega] \right),
$$

with $[\omega] \in \mathcal{K}_{X/F}$. Since $([\alpha] + \frac{1}{k}[\omega])$ is transverse big, it follows from the closeness of $\mathcal{E}^0_{X/F}$ that

$$
\mathcal{E}^0_{X/F} \subset \text{closure of transverse big cone} \subset \mathcal{E}^0_{X/F},
$$

which yields $\mathcal{E}^0_{X/F} = \mathcal{B}_{X/F}$. \hfill \square

Clearly every transverse Kähler class is transverse nef and transverse big, and every transverse big class is transverse pseudoeffective. Aslo, a transverse nef class is transverse pseudoeffective.

There are in general no other implications among these notions since we can see $H^{1,1}_{BC}(M, \mathbb{R})$ as $H^{1,1}_{BC}(X/F, \mathbb{R})$, where $M$ is a complex manifold and $F$ is defined by a holomorphic submersion $\pi : X \to M$ (cf. [11]).

3. Proof of the Main Theorem

Lemma 3.1. Let $(X, F)$ be a closed oriented, taut, transverse Hermitian foliated manifold, where $F$ is the foliation with complex codimension $n$, and $\omega$ denote a transverse Hermitian metric. Given a smooth basic function $h \in C^\infty(X/F, \mathbb{R})$, there exists a unique smooth basic function $u \in C^\infty(X/F, \mathbb{R})$ such that

$$
(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^{u+h}\omega^n, \quad \omega + \sqrt{-1}\partial\bar{\partial}u > 0.
$$

Proof. This is a transverse version of [11 61]. We will adapt the idea from [38 39 45] (cf. [18]) to prove it. We use the method of continuity and consider a family of equations

$$
\mathcal{H}(u, t) := \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n} - u - th = 0, \quad \omega + \sqrt{-1}\partial\bar{\partial}u > 0.
$$
We set
\[ \mathcal{T} := \left\{ t \in [0, 1] : (3.2) \text{ has a solution } u \in C^{2,\alpha}(X/F, \mathbb{R}) \text{ with } \alpha \in (0, 1) \text{ fixed} \right\}. \]

Note that \( 0 \in \mathcal{T} \) since \( \mathcal{H}(0, 0) = 0 \). For the openness of \( \mathcal{T} \), we fix \( t \in \mathcal{T} \). A direct calculation yields that
\[ (D_u \mathcal{H})_{(u, t)} \eta = \Delta \eta - \eta, \quad \forall \eta \in C^{2,\alpha}(X/F, \mathbb{R}), \]
where
\[ \Delta \eta := \frac{n\sqrt{-1} \partial \bar{\partial} \eta \wedge \bar{\omega}^{n-1}}{\bar{\omega}^n}, \quad \bar{\omega} := \omega + \sqrt{-1} \partial \bar{\partial} u_t > 0. \]

We claim that \( (D_u \mathcal{H})_{(u, t)} \) is bijective. Indeed, since \( \bar{\omega} \) is a positive basic \((1, 1)\) form, both \( \bar{\Delta} \) and \( (D_u \mathcal{H})_{(u, t)} \) are strictly transverse elliptic operators. For each \( \eta \in \ker(D_u \mathcal{H})_{(u, t)} \), at the point \( x_{\max} \) (resp. \( x_{\min} \)) where \( u \) attains its maximum (resp. its minimum), there holds
\[ -\eta(x_{\max}) \geq 0 \text{ (resp. } -\eta(x_{\min}) \leq 0), \]
which yields that \( \eta \equiv 0 \). Hence \( \ker(D_u \mathcal{H})_{(u, t)} = \{ 0 \} \), i.e., \( (D_u \mathcal{H})_{(u, t)} \) is injective.

Since the index of \( (D_u \mathcal{H})_{(u, t)} \) is the same as the index of the transverse Laplacian defined by \( \bar{\omega} \) which is zero by [2, Lemma 3.5], the injectivity of the operator directly implies that the operator \( (D_u \mathcal{H})_{(u, t)} \) is surjective.

Since \( (D_u \mathcal{H})_{(u, t)} \) is bijective, the implicit function theorem yields that \( \mathcal{T} \) is open at the point \( t \in \mathcal{T} \).

For the closeness of \( \mathcal{T} \), we need the a priori estimates for the solutions to \( (3.2) \).

**A uniform bound**
\[ (3.3) \quad \sup_X |u| \leq C \]
follows immediately from the standard maximum principle. Indeed, at the point \( x_{\max} \) where \( u \) attains its maximum, we have (see [14])
\[ (\sqrt{-1} \partial \bar{\partial} u) (x_{\max}) \leq 0, \]
and from \( (3.2) \) the upper bound of \( u \leq -t \inf_X h \) follows. The lower bound of \( u \) is similar. Here and henceforth, \( C \) will denote a uniform constant independent of \( t \) that may change from line to line.

We denote \( F := u + th \). We assume that \( \sup_X u \leq 0 \) by modifying \( \tilde{u} := u + t \inf_X h \) and \( \tilde{h} = h - \inf_X h \) such that \( F = \tilde{u} + \tilde{h} \).

**For the second order estimate**, we claim
\[ (3.4) \quad \sup_M |\partial \bar{\partial} u|_\omega \leq CK, \]
with \( K := 1 + \sup_M |\partial \bar{\partial} u|_\omega^2. \)

In the distinguished chart \((U; t^1, \cdots, t^k, z^1, \cdots, z^n)\), we write
\[ (3.5) \quad \bar{\omega} := \omega + \sqrt{-1} \partial \bar{\partial} u_t = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j, \quad \omega = \sqrt{-1} g_{ij} d\bar{z}^i \wedge d\bar{z}^j. \]

It follows from \( (3.2) \) and the inequality of arithmetic and geometric mean that
\[ (3.6) \quad \text{tr}_{\bar{\omega}} \omega \geq n \left( \frac{\inf_X \omega^n}{\bar{\omega}^n} \right)^{\frac{1}{n}} \geq \tau > 0, \]
where \( \tau \) is a constant independent of \( t \).
Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0 \) be the eigenvalues of \( \tilde{\omega} \) with respect to \( \omega \). We consider the quantity

\[
H(x) := \log \lambda_1(x) + \varphi(|\partial u|_g^2(x)) + \psi(u(x)), \quad \forall x \in M,
\]

where we define

\[
\varphi(t) = \frac{t}{K}, \quad t \geq 0, \quad \text{and} \quad \psi(t) = e^{-At}, \quad t \leq 0,
\]

with

\[
K = \sup_x |\partial u|_g^2 + 1,
\]

and \( A > 0 \) to be determined later. A direct calculation yields that

\[
-\psi' \geq A > 0, \quad \psi'' = -A\psi'.
\]

We assume that \( H \) attains its maximum at the point \( x_0 \in X \). In the following, we will calculate at the point \( x_0 \) under the distinguished coordinate \((t^1, \cdots, t^k, z^1, \cdots, z^n)\) for which \( \omega \) is the identity and \((\tilde{g}_{ij})\) is diagonal with entries \( \tilde{g}_{ii} = \lambda_i \) for \( 1 \leq i \leq n \), unless otherwise indicated.

Since \( \lambda_1 \) may not be smooth at \( x_0 \), we introduce a smooth function \( \phi \) on \( M \) by (cf. [5, Lemma 5] and [45, Proof of Theorem 3.1])

\[
H(x_0) \equiv \log f(x) + \varphi(|\partial u|_g^2(x)) + \psi(u(x)), \quad \forall x \in M.
\]

Note that \( f \) satisfies

\[
f(x) \geq \lambda_1(x) \quad \forall x \in X, \quad f(x_0) = \lambda_1(x_0).
\]

Note that

\[
\Delta_{\tilde{\omega}} u = \tilde{g}^{ij} \partial_i \partial_j u = n - \text{tr}\tilde{\omega} \omega.
\]

Applying the operator \( \Delta_{\tilde{\omega}} \) defined in (3.9) to (3.7), one infers

\[
0 = \frac{1}{\lambda_1} \Delta_{\tilde{\omega}} f - \frac{1}{\lambda_1^2} \tilde{g}^{ij} |\nabla_i f|^2 + \varphi' \Delta_{\tilde{\omega}}(|\partial u|_\omega^2) + \varphi'' \tilde{g}^{ij} |\partial_i |\partial u|_\omega| |^2 + \psi' \Delta_{\tilde{\omega}} u + \psi'' \tilde{g}^{ij} |\partial_i u|^2.
\]

Differentiating (3.7) one can deduce

\[
0 = \frac{\nabla_i f}{f} + \varphi' \nabla_i (|\partial u|_\omega^2) + \psi' (\nabla_i u).
\]

**Lemma 3.2.** Let \( \mu \) denote the multiplicity of the largest eigenvalue of \((\tilde{g}_{ij})\) with respect to \( (g_{ij}) \) at \( x_0 \), so that \( \lambda_1 = \cdots = \lambda_\mu > \lambda_{\mu+1} \geq \cdots \geq \lambda_n \). Then at \( x_0 \), for each \( i \) with \( 1 \leq i \leq n \), there hold

\[
\nabla_i \bar{g}_{k\ell} = (\nabla_i f) g_{k\ell}, \quad \text{for} \quad 1 \leq k, \ell \leq \mu,
\]

\[
\nabla_i \nabla_i f \geq \nabla_i \nabla_i \bar{g}_{11} + \sum_{q>\mu} \frac{|\nabla_i \bar{g}_{q1}|^2 + |\nabla_i \bar{g}_{1q}|^2}{\lambda_1 - \lambda_q}.
\]

**Proof.** See [45, Lemma 3.2]. \( \square \)

Since we work with \((z^1, \cdots, z^n)\) at the point \( x_0 \) under the distinguished coordinate \((t^1, \cdots, t^k, z^1, \cdots, z^n)\), the conclusion follows from the same argument of [45, Proof of Theorem 3.1] with vanishing gradient term by replacing \( u_{ij} - u_{ji} = T_p^u u_p \), the linearized operator \( L \) in [45, Formula (3.1)], [45, Formula (3.13)] and [45, Formula (3.14)], \( \Delta_{\tilde{\omega}} \), (2.21), (2.20), and (2.25) respectively. We also point out that here \( F = u + th \) contains the solution \( u \) which is harmless. Indeed, it follows from (3.6) that

\[
\nabla_i \nabla_i F = O(\lambda_1)
\]
will be absorbed into $\lambda_1 (\text{tr} \hat{\omega})$, and that, together with the definition of $\varphi$, we know that
\[
\varphi' (\partial_\rho F) (\partial_\rho u) = O(1)
\]
is harmless.

**The first order estimate**

\[(3.14) \quad \sup_M |\partial u|_\omega \leq C\]

follows from (3.4) and the blow-up argument (see the details in [18]).

Given (3.3), (3.4) and (3.14), $C^2, \alpha$-estimate for some $0 < \alpha < 1$ follows from the Evans-Krylov theory [17, 28, 49] (see also [42]).

Differentiating the equations and using the Schauder theory (see for example [20]), we then deduce uniform a priori $C^k$ estimates for all $k \geq 0$.

Finally, by routin arguments, the proof of Lemma 3.1 is completed. $\Box$

**Lemma 3.3.** Let $(X, F)$ be a closed oriented, taut, transverse Kähler foliated manifold, where $F$ is the foliation with complex codimension $n$, and let $\omega$ and $\hat{\omega}$ be transverse Kähler metrics such that $\omega$ has transverse holomorphic sectional curvature bounded above by a constant $-\kappa \leq 0$, and that $\hat{\omega}$ satisfies
\[(3.15) \quad \text{Ric}(\hat{\omega}) \geq -\lambda \hat{\omega} + \nu \omega,\]
for some constants $\lambda, \nu > 0$. Then one can infer
\[(3.16) \quad \Delta_{\hat{\omega}} \log \text{tr} \hat{\omega} \omega \geq \left( \frac{n+1}{2n} \kappa + \frac{\nu}{n} \right) \text{tr} \hat{\omega} \omega - \lambda.\]
In particular, there holds
\[(3.17) \quad \sup_X \text{tr} \hat{\omega} \omega \leq \frac{\lambda}{\frac{n+1}{2n} \kappa + \frac{\nu}{n}}.\]

Proof. This a transverse version of [55, Proposition 9] (cf. [57, 54]) which is an application of Yau’s Schwarz Lemma [60] and Royden’s trick [34]. We refer to [32, 53] for the simplified calculation.

In the distinguished chart $(U; t^1, \cdots, t^k, z^1, \cdots, z^n)$, we work with the qualities of $(z^1, \cdots, z^n)$. Hence (3.16) follows from the same calculation in the proof of [55, Proposition 9] and [17, Lemma 2.1]. At the point where $\log \text{tr} \hat{\omega} \omega$ attain its maximum, there holds
\[
\Delta_{\hat{\omega}} \log \text{tr} \hat{\omega} \omega \leq 0,
\]
which yields (3.17). $\Box$

**Lemma 3.4.** Let $(X, F)$ be a closed oriented, taut, transverse Kähler foliated manifold, where $F$ is the foliation with complex codimension $n$, and $\omega$ denote a transverse Kähler metric. If the normal canonical line bundle $K_{X/F}$ is transverse nef, then there hold
\[(1) \quad \text{For each } \varepsilon > 0, \text{ there exists a smooth basic function } u_\varepsilon \in C^\infty (X/F, \mathbb{R}) \text{ such that }
\]
\[
\omega_{\varepsilon}^n = e^{u_\varepsilon} \omega^n, \quad \text{with} \quad \omega_{\varepsilon} := -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u_\varepsilon + \varepsilon \omega > 0.
\]
Moreover, there holds
\[(3.18) \quad \text{Ric}(\omega_{\varepsilon}) = -\omega_{\varepsilon} + \varepsilon \omega \geq -\omega_{\varepsilon},\]
and

\[(3.19) \quad \sup_X u_\varepsilon \leq C,\]

where the constant \(C > 0\) depends only on \(\omega\) and \(n\).

(2) For each \(k = 1, \ldots, n\), there holds

\[(3.20) \quad \int_X (-c_1(X/F))^k \wedge \omega^{n-k} \wedge \chi \geq e^{(k-n)C/n} \int_X (-c_1(X/F))^n \wedge \chi \geq 0,
\]

where the constant \(C\) is the same as that in (3.19).

**Proof.** This is a transverse version of [55, Proposition 8]. We adapt the idea in [55] to prove it.

We show Item (1). For each \(\varepsilon > 0\), there exists a smooth basic function \(f_\varepsilon \in C^\infty(X/F, \mathbb{R})\) such that

\[(3.21) \quad \omega f_\varepsilon := -\text{Ric}(\omega) + \sqrt{-1} \bar{\partial} f_\varepsilon + \varepsilon \omega > 0\]

since \(K_{X/F}\) is transverse nef.

Fix \(\varepsilon > 0\). It follows from Lemma 3.1 that there exists a unique \(v_\varepsilon \in C^\infty(X/F, \mathbb{R})\) such that

\[(3.22) \quad (\omega f_\varepsilon + \sqrt{-1} \bar{\partial} v_\varepsilon)^n = e^{\varepsilon v_\varepsilon + f_\varepsilon} \omega^n, \quad \omega_\varepsilon := \omega f_\varepsilon + \sqrt{-1} \bar{\partial} v_\varepsilon > 0.\]

A direct calculation, together with (2.21), (3.21) and (3.22), yields that

\[\text{Ric}(\omega_\varepsilon) = \text{Ric}(\omega) - \sqrt{-1} \bar{\partial} f_\varepsilon - \sqrt{-1} \bar{\partial} v_\varepsilon = \varepsilon \omega - \omega f_\varepsilon - \sqrt{-1} \bar{\partial} v_\varepsilon = - \omega_\varepsilon + \varepsilon \omega.\]

We set \(u_\varepsilon := f_\varepsilon + v_\varepsilon\). It follows from the maximum principle, (3.21) and (3.22) that

\[\sup_X u_\varepsilon \leq C := \log \left( \frac{(\varepsilon_0 \omega - \text{Ric}(\omega))^n}{\omega^n} \right), \quad \forall \varepsilon < \varepsilon_0,\]

which yields (3.19).

We show Item (2). Since \(K_{X/F}\) is transverse nef, it follows from Lemma 2.2 and (3.22) that

\[\int_X \omega_\varepsilon^n \wedge \chi = \int_X (-\text{Ric}(\omega) + \varepsilon \omega)^n \wedge \chi > 0, \quad \forall \varepsilon > 0,\]

which yields that

\[(3.23) \quad \int_X (-c_1(X/F))^n \wedge \chi = \lim_{\varepsilon \to 0^+} \int_X \omega_\varepsilon^n \wedge \chi \geq 0.\]

We set

\[\sigma_k = \frac{\omega_\varepsilon^k \wedge \omega_\varepsilon^{n-k}}{\omega_\varepsilon^n}, \quad 0 \leq k \leq n.\]

It follows from Maclaurin’s inequality that

\[\sqrt[4]{\sigma_k} \geq \sqrt[4]{\sigma_n},\]

which, together with (3.19) and (3.22), yields that

\[(3.24) \quad \frac{\omega_\varepsilon^k \wedge \omega_\varepsilon^{n-k}}{\omega_\varepsilon^n} \geq \left( \frac{\omega^n}{\omega_\varepsilon^n} \right)^k \left( \frac{\omega^n}{\omega_\varepsilon^n} \right)^{n-k} = \left( e^{u_\varepsilon} \right)^{k-1} \geq e^{(k-n)C/n}.\]
It follows from Lemma 3.6 and (3.24) that
\begin{equation}
\int_X (-\text{Ric}(\omega) + \varepsilon \omega)^k \wedge \omega^{n-k} \wedge \chi \geq e^{(k-n)\nu} \int_X (-\text{Ric}(\omega) + \varepsilon \omega)^n \wedge \chi, \quad \forall \varepsilon > 0.
\end{equation}
Letting \( \varepsilon \to 0^+ \) in (3.25) shows (3.20). \( \square \)

**Lemma 3.5.** Let \((X, \mathcal{F})\) be a closed oriented, taut, transverse Kähler foliated manifold, where \(\mathcal{F}\) is the foliation with complex codimension \(n\) and \(\omega\) is a transverse Kähler metric with nonpositive transverse holomorphic sectional curvature. Then the normal canonical bundle \(K_{X/\mathcal{F}}\) is transverse nef.

**Proof.** This is a transverse version of [47, Theorem 1.1] and we adapt the idea from [47] to show it. If \(K_{X/\mathcal{F}}\) is not transverse nef, then there exists \(\varepsilon_0 > 0\) such that the class \(\varepsilon_0 [\omega] - 2 \pi c_1 (X/\mathcal{F})\) is transverse nef but not Kähler. As in the proof of Lemma 3.6, it follows from Item (1) of Lemma 3.4 that for each \(\varepsilon > 0\) there exists transverse Kähler metric \(\omega_\varepsilon := \varepsilon \omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u_\varepsilon\) satisfying \(\omega_\varepsilon^n = e^{u_\varepsilon} \omega^n\) and
\begin{equation}
\text{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + (\varepsilon + \varepsilon_0) \omega, \quad \max_X u_\varepsilon \leq C, \quad C^{-1} \omega \leq \omega_\varepsilon \leq C \omega
\end{equation}
where \(C = C(\omega, n, \varepsilon_0) > 0\) is a constant independent of \(\varepsilon\). Here we should use (3.17) with \(\kappa = 0, \lambda = 1, \nu = (\varepsilon + \varepsilon_0)\).

As in the proof of Lemma 3.6 there still holds
\begin{equation}
\|\omega_\varepsilon\|_{C^k(X, \omega)} \leq C_k,
\end{equation}
where \(C_k > 0\) is a constant independent of \(\varepsilon\).

It follows from (3.26), (3.27), the Ascoli-Arzelà theorem and a diagonal argument that there exists a sequence \(\{\varepsilon_i\}\) with \(\lim_{i \to \infty} \varepsilon_i = 0\) such that \(\omega_{\varepsilon_i}\) converge smoothly to a transverse Kähler metric \(\omega_0\) which satisfies
\[ [\omega_0] = \varepsilon_0 [\omega] - 2 \pi c_1 (X/\mathcal{F}), \]
which contradicts to the fact that \(\varepsilon_0 [\omega] - 2 \pi c_1 (X/\mathcal{F})\) is not transverse Kähler, as desired. \( \square \)

**Lemma 3.6.** Let \((X, \mathcal{F})\) be a closed oriented, taut, transverse Kähler foliated manifold, where \(\mathcal{F}\) is the foliation with complex codimension \(n\), and \(\omega\) denote the transverse Kähler metric with negative transverse holomorphic sectional curvature with upper bound \(-\kappa < 0\). Then there exists a smooth basic function \(u \in C^\infty(X/\mathcal{F}, \mathbb{R})\) such that \(\omega_u := -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u\) is the transverse Kähler-Einstein metric with \(\text{Ric}(\omega_u) = -\omega_u\).

**Proof.** This is a transverse version of [55, Theorem 9] and we adapt the idea from [55] to show it.

It follows from Lemma 3.3 and Lemma 3.5 that for each \(\varepsilon > 0\) there exists transverse Kähler metric \(\omega_\varepsilon := \varepsilon \omega - \text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u_\varepsilon\) satisfying \(\omega_\varepsilon^n = e^{u_\varepsilon} \omega^n\) and
\begin{equation}
\text{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + \varepsilon \omega, \quad \max_X u_\varepsilon \leq C,
\end{equation}
where \(C = C(\omega, n) > 0\) is a constant independent of \(\varepsilon\).

It follows from (3.17) with \(\lambda = 1, \nu = \varepsilon\) and (3.28) that
\begin{equation}
\text{tr}_{\omega_\varepsilon} \omega \leq \frac{2n}{(n+1)\kappa}.
\end{equation}
The uniform upper bound for \(u_\varepsilon \leq C\) yields that
\begin{equation}
\sup_X \omega_\varepsilon^n \omega_\varepsilon \leq C.
\end{equation}
Thanks to (3.29), (3.30) and the elementary inequality
\[ \text{tr}_\omega \omega_k \leq \frac{1}{(n-1)!} (\text{tr}_\omega \omega)^{n-1} \frac{\omega^n}{\omega^n}, \]
we can deduce that
\[ (3.31) \sup_X \text{tr}_\omega \omega \leq C, \]
where \( C = C(\omega, n, \kappa) > 0 \) is a constant independent of \( \varepsilon \).

It follows from (3.29) and (3.31) that
\[ (3.32) \quad C^{-1} \omega \leq \omega \leq C \omega, \]
which yields that
\[ (3.33) \quad \inf_X u \geq -C, \]
where \( C = C(\omega, n, \kappa) > 0 \) is a constant independent of \( \varepsilon \).

We claim
\[ (3.34) \quad \| \omega \|_{C^k(X, \omega)} \leq C_k, \]
where \( C_k > 0 \) is a constant independent of \( \varepsilon \). Indeed, in the distinguished chart
\[ (U; t^1, \ldots, t^k, z^1, \ldots, z^n), \]
we work with the qualities of \((z^1, \ldots, z^n)\). Hence (3.16) follows from the same argument in [47] following the work of Yau [61].

It follows from (3.33), (3.34), the Ascoli-Arzelà theorem and a diagonal argument that there exists a sequence \( \{u_{\varepsilon_i}\} \) with \( \lim_{i \to \infty} \varepsilon_i = 0 \) converge smoothly to a smooth basic function \( u \in C^\infty(X/\mathcal{F}, \mathbb{R}) \) which satisfies
\[ (-\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u)^n = e^n \omega^n, \quad -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u > 0. \]
This shows that \( \omega_u := -\text{Ric}(\omega) + \sqrt{-1} \partial \bar{\partial} u > 0 \) satisfies \( \text{Ric}(\omega_u) = -\omega_u \), as required. \( \square \)

Now we finish the proofs of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** The conclusion follows from Lemmas 3.5 and 3.6. \( \square \)

**Proof of Theorem 1.2.** It follows from Theorem 1.1 that there exists a transverse Kähler metric \( \tilde{\omega} \) such that \( 2\pi c_1(X/\mathcal{F}) = [\text{Ric}(\tilde{\omega})] = [-\tilde{\omega}] \). Since \( c_1(T\mathcal{F}) = 0 \), one infers from the adjunction formula that
\[ 2\pi c_1(X) = 2\pi c_1(X/\mathcal{F}) = [-\tilde{\omega}]. \]
Hence \( c_1(X) \) is represented by a semi-negative closed real \((1,1)\) form and \( c_1(X)^n \neq 0 \). Then Theorem 1.2 follows from [48, Theorem 1.2]. \( \square \)
APPENDIX A. PRELIMINARIES FOR DISTRIBUTION AND CURRENT

In this appendix, we collect preliminaries for distribution and current in order to deduce the transverse versions of the Poincaré Lemma and the Dolbeault-Grothendieck lemma.

Let \( X \) be a smooth oriented differential manifold with \( \dim_{\mathbb{R}} X = n \) and countable base. We first introduce a topology on the space of differential forms \( C^s(X, \bigwedge^p T^*_X) \). Let \( U \subset X \) be a coordinate open set and \( u \) a \( p \) form on \( X \), written \( u(x) = \sum_{i_1 < \cdots < i_p} u_{i_1, \cdots, i_p} dx^{i_1} \cdots dx^{i_p} \) on \( U \) with multi-indices of the type \( I = (i_1, \cdots, i_p) \). To every compact subset \( K \subset U \) and every integer \( s \in \mathbb{N} \), we associated a semi-norm

\[
\| u \|_K^s := \sup_{x \in K} \max_{|\alpha| \leq s} |\partial^\alpha u(x)|,
\]

where \( \alpha = (\alpha_1, \cdots, \alpha_n) \) runs over \( \mathbb{N}^n \) and \( \partial^\alpha = \partial^{|\alpha|}/\partial(x^1)^{\alpha_1} \cdots \partial(x^n)^{\alpha_n} \) is a derivation of order \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

**Definition A.1.** We introduce as follows spaces of \( p \) forms on manifolds.

1. We denote by \( \mathcal{E}^p(X) \) (resp. \( \mathcal{E}^s(X) \)) the space \( C^\infty(X, \bigwedge^p T^*_X) \) (resp. the space \( C^s(X, \bigwedge^p T^*_X) \)), equipped with the topology defined by all semi-norms \( \| u \|_K^s \) when \( s, L \) are integers and vary (resp. when \( L, U \) vary). We use the notation \( \mathcal{E}(X) := \mathcal{E}^0(X) \) and \( \mathcal{E}^s(X) := \mathcal{E}^s(X) \).

2. If \( K \subset X \) is a compact subset, \( \mathcal{D}^p(K) \) will denote the subspace of elements \( u \in \mathcal{E}^p(X) \) with support contained in \( K \), together with the induced topology; \( \mathcal{D}^p(X) \) will stand for the set of all elements with compact support, i.e., \( \mathcal{D}^p(X) := \bigcup_K \mathcal{D}^p(K) \). We use the notation \( \mathcal{D}(X) := \mathcal{D}^0(X) \) and \( \mathcal{D}^s(X) := \mathcal{D}^s(X) \).

3. The spaces of \( C^s \) forms \( \mathcal{D}^s_p(X) \) and \( \mathcal{D}^s_p(X) \) are defined similarly. We use the notation \( \mathcal{D}^s(X) := \mathcal{D}^s_0(X) \) and \( \mathcal{D}^s(X) := \mathcal{D}^s_0(X) \).

Since \( X \) is separable, the topology of \( \mathcal{E}^p(X) \) can be defined by means of a countable set of semi-norms \( \| u \|_K^s \), hence \( \mathcal{E}^p(X) \) (and likewise \( \mathcal{E}^s(X) \)) is a Fréchet space. The topology of \( \mathcal{D}^p(X) \) is induced by an arbitrary set of semi-norms \( \| u \|_K^s \) such that the compact sets \( K \) cover \( X \); hence \( \mathcal{D}^p(X) \) is a Banach space. It should be observed however that \( \mathcal{D}^p(X) \) is not a Fréchet space; in fact \( \mathcal{D}^p(X) \) is dense in \( \mathcal{E}^p(X) \) and thus non complete for the induced topology.

**Definition A.2.** Let \( X \) be a complex manifold with \( \dim_{\mathbb{C}} X = n \) and countable base. Then there are decompositions

\[
\mathcal{D}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{D}^p q(X, \mathbb{C}), \quad \mathcal{D}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{D}^p q(X, \mathbb{C}).
\]

The space \( \mathcal{D}^p q(X, \mathbb{C}) \) is called the space of currents of bidimension \( (p, q) \) and bidegree \( (n-p, n-q) \) on \( X \), and is also denoted \( \mathcal{D}^p_{n-p,q}(X, \mathbb{C}) \).

**A.1. Distribution.** A distribution on \( U \subset \mathbb{R}^n \) is a continuous linear form on \( \mathcal{D}(U) \).

**Definition A.3.** A distribution on \( U \subset \mathbb{R}^n \) is a linear form \( u \) on \( \mathcal{D}(U) \) such that for each compact set \( K \subset U \) there exist constants \( C \) and \( s \in \mathbb{N} \) such that

\[
|u(\varphi)| \leq C p^s_K(\varphi), \quad \forall \varphi \in \mathcal{D}(K).
\]

The set of all distributions in \( U \) is denoted by \( \mathcal{D}'(U) \).

We also use the notation \( \langle u, \varphi \rangle := u(\varphi), \quad \forall \varphi \in \mathcal{D}(U) \). The continuity condition in Definition A.3 is often stated as a sequential continuity.
**Theorem A.1.** Let $U \subset \mathbb{R}^n$ be an open set, and $u$ a complex valued linear form on $\mathcal{D}(U)$. Then $u$ is a distribution if and only $u(\varphi_j) \to 0$ when $j \to \infty$ for every sequence $(\varphi_j)_{j \in \mathbb{N}}$, converging to $0$ in the sense that $p_K(\varphi_j) \to 0$ as $j \to \infty$ for each fixed $s$ and $\text{supp}\varphi_j \subset K$ for all $j$ with $K \subset U$ compact; if and only if there exist continuous functions $\rho_\alpha$ for each $\alpha \in \mathbb{N}^n$ such that

$$|u(\varphi)| \leq \sum_{\alpha \in \mathbb{N}^n} |\rho_\alpha \partial^\alpha \varphi|, \ \forall \varphi \in \mathcal{D}(U),$$

and on each compact set $K \subset U$ all but finite number of the functions $\rho_\alpha$ vanish identically.

**Proof.** See [26, Theorem 2.1.4 & 2.1.5]. \(\square\)

A distribution is determined by the restriction to the sets in an open covering.

**Theorem A.2.** Let $U \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(U)$. Then we have

1. If for each point of $U$ there exists a neighborhood to which the restriction of $u$ is $0$, then $u = 0$.
2. Let $\{U_i\}_{i \in I}$ be an arbitrary open covering of $U$. Then if $u_i \in \mathcal{D}'(U_i)$ and $u_i = u_j$ on $U_i \cap U_j$ for all $i, j \in I$, then there exists one and only one $u \in \mathcal{D}'(U)$ such that $u_i$ is the restriction of $u$ to $U_i$ for each $i \in I$.

**Proof.** See [26, Theorem 2.2.1 & Theorem 2.2.4]. \(\square\)

**Theorem A.3.** Let $U \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(U)$. Then if $u(\varphi) \geq 0$ for each non-negative $\varphi \in \mathcal{D}(U)$, then $u$ is a positive regular Borel measure.

**Proof.** See [26, Theorem 2.1.7] and [35, Theorem 2.14 and Theorem 2.18]. \(\square\)

One can of course multiply a distribution $u$ in $\mathcal{D}'(U)$ by a smooth function $f \in C^\infty(U)$, and define partial derivatives $\partial^\alpha u$ of a distribution $u$ by the formulae

(A.2) \[ \langle fu, \varphi \rangle := \langle u, f \varphi \rangle, \]

(A.3) \[ \langle \partial^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle, \ \forall \alpha \in \mathbb{N}^n, \forall \varphi \in \mathcal{D}(U). \]

Indeed, these linear forms defined in (A.2) and (A.3) are continuous on $\mathcal{D}(U)$ and hence are well defined.

**Proposition A.4.** Let $U \subset \mathbb{R}^n$ be an open set, and $u \in \mathcal{D}'(U)$. Let also $\varphi \in C^\infty(U \times \mathbb{R}^q)$. If there exists a compact set $K \subset U$ such that $\text{supp}\varphi \subset K \times \mathbb{R}^q$, then the function

$$G : \mathbb{R}^q \to \mathbb{R}, \ \ y \mapsto \langle u, \varphi(\cdot, y) \rangle$$

is $C^\infty$, and there holds

$$\partial^\alpha G(y) = \langle u, \partial^\alpha_y \varphi(\cdot, y) \rangle, \ \forall \alpha \in \mathbb{N}^q.$$

**Proof.** See [33, Proposition 4.1.1] or [26, Theorem 2.1.3]. \(\square\)

**Remark A.1.**

1. We have written $\langle u, \varphi(\cdot, y) \rangle$ in place of $\langle u, \varphi_y \rangle$, where $\varphi_y \in C^\infty_0(\mathbb{R}^n)$ is the function given by $\varphi_y(x) = \varphi(x, y)$.

2. The assumption $\text{supp}\varphi \subset K \times \mathbb{R}^q$ means that for any $y \in \mathbb{R}^q$, the support of $\varphi_y$ is included in $K$. It holds in particular when $\varphi \in C^\infty_0(\Omega \times \mathbb{R}^q)$. 


(3) For a regular distribution \( u = u_f \) with \( f \in L^1_{\text{loc}}(\Omega) \), we have

\[
G(y) = \int f(x) \varphi(x, y) dx,
\]

so that, under the above assumptions, we get \( G \in C^\infty(\mathbb{R}^q) \) and

\[
\partial^\alpha G(y) = \int f(x) \partial^\alpha_y \varphi(x, y) dx, \quad \forall \alpha \in \mathbb{N}^q.
\]

**Proposition A.5.** Let \( U \subset \mathbb{R}^k \) be an open set, and \( u \in \mathcal{D}'(U) \). Let also \( \varphi \in C^\infty(U \times \mathbb{R}^q) \). Then

\[
\int_{\mathbb{R}^q} \langle u, \varphi(\cdot, y) \rangle dy = \langle u, \int_{\mathbb{R}^q} \varphi(\cdot, y) dy \rangle.
\]

**Proof.** See [33, Proposition 4.1.3]. \( \square \)

Let \( f : \mathbb{R}^p \to \mathbb{C} \) and \( g : \mathbb{R}^q \to \mathbb{C} \) be two functions. Then the function \( f \otimes g \) is defined on \( \mathbb{R}^{p+q} \) by

\[
(f \otimes g)(x_1, x_2) := f(x_1)g(x_2), \quad \forall x = (x_1, x_2) \in \mathbb{R}^{p+q} \text{ with } x_1 \in \mathbb{R}^p, \ x_2 \in \mathbb{R}^q.
\]

This function is called the tensor product of \( f \) and \( g \).

**Proposition A.6.** Let \( U \subset \mathbb{R}^n \) be an open set and \( u \in \mathcal{D}'(U) \). Then

\[
\partial_j u = 0, \quad \forall 1 \leq j \leq n \iff \exists C \in \mathbb{C} \text{ such that } \langle u, \varphi \rangle = C \int \varphi, \quad \forall \varphi \in \mathcal{D}(U).
\]

**Proof.** See [33, Proposition 4.2.3]. \( \square \)

The tensor product of distributions is defined by

**Proposition A.7.** Let \( (u_1, u_2) \in \mathcal{D}'(U_1) \times \mathcal{D}'(U_2) \). Then one has

1. for each \( \varphi \in \mathcal{D}(U_1 \times U_2) \), the function \( \psi : x \mapsto \langle u_2, \varphi(x, \cdot) \rangle \) belongs to \( \mathcal{D}(U_1) \);
2. the linear form \( u : \mathcal{D}(U_1 \times U_2) \ni \varphi \mapsto \langle u_1, \psi \rangle \) is continuous in \( \mathcal{D}(U_1 \times U_2) \), and it is called the tensor product, denoted by \( u = u_1 \otimes u_2 \), of \( u_1 \) and \( u_2 \). It is the only distribution in \( \mathcal{D}'(U_1 \times U_2) \) such that

\[
\langle u, \varphi_1 \otimes \varphi_2 \rangle = \langle u_1, \varphi_1 \rangle \langle u_2, \varphi_2 \rangle, \quad \forall (\varphi_1, \varphi_2) \in \mathcal{D}(U_1) \times \mathcal{D}(U_2).
\]

**Proof.** See [33, Proposition 4.2.4 & Proposition 4.2.5]. \( \square \)

**Proposition A.8.** Let \( (u_1, u_2) \in \mathcal{D}'(U_1) \times \mathcal{D}'(U_2) \), where \( U_1 \subset \mathbb{R}^k \) and \( U_2 \subset \mathbb{R}^n \) are open sets. Then for \( j \in \{1, \cdots, n+k\} \), one has

\[
\partial_j (u_1 \otimes u_2) = \begin{cases} 
(\partial_j u_1) \otimes u_2, & \text{for } 1 \leq j \leq k, \\
 u_1 \otimes (\partial_j u_2), & \text{for } k+1 \leq j \leq n+k.
\end{cases}
\]

**Proof.** See [33, Proposition 4.2.7]. \( \square \)

**Proposition A.9.** Let \( u \in \mathcal{D}'(U_1 \times U_2) \), where \( U_1 \subset \mathbb{R}^k \) and \( U_2 \subset \mathbb{R}^n \) are open sets. Then the following two assertions are equivalent:

1. \( \partial_j u = 0 \) for all \( j \in \{1, \cdots, k\} \);
2. There exists \( v \in \mathcal{D}'(U_2) \) such that \( u = 1 \otimes v \).
Proof. \((2) \Rightarrow (1)\) follows from Proposition A.8

\((1) \Rightarrow (2)\) follows from [24 Theorem 3.1.4’] for \(n = 1\), and for general \(n\), we first claim that for each \(\chi \in \mathcal{D}(U_1 \times U_2)\) there holds

\[
\chi \in \text{Im} (\partial_1 \circ \cdots \circ \partial_k) = \bigcap_{\ell=1}^{k} \text{Im} \partial_\ell
\]

if and only if

\[
(A.4) \quad \int_{\mathbb{R}^k} \chi(t^1, \ldots, t^k, x^{k+1}, \ldots, x^{k+n}) dt^1 \wedge \cdots \wedge dt^k = 0.
\]

Indeed, for ‘only if’ direction, we assume that \(\chi \in \text{Im} (\partial_1 \circ \cdots \circ \partial_k) \cap \mathcal{D}(U_1 \times U_2)\), i.e., \(\chi\) satisfies

\[
(A.5) \quad \frac{\partial^k \psi}{\partial x^1 \cdots \partial x^k} = \chi
\]

for \(\psi \in \mathcal{D}(U_1 \times U_2)\). Equality \((A.4)\) holds by a direct calculation.

For the ‘if’ direction, the function

\[
\psi(x) := \int_{-\infty}^{x^1} \cdots \int_{-\infty}^{x^k} \chi(t^1, \ldots, t^k, x^{k+1}, \ldots, x^{k+n}) dt^1 \wedge \cdots \wedge dt^k
\]

satisfies \((A.5)\). In addition, \(\psi \in \mathcal{D}(U_1 \times U_2)\) if \((A.4)\) holds.

We choose \(\psi_0 \in \mathcal{D}(U_1)\) with \(\int_{U_1} \psi_0 = 1\) and define

\[
\langle v, \chi \rangle := \langle u, \chi_0 \rangle, \quad \forall \ \chi \in \mathcal{D}(U_2)
\]

with

\[
\chi_0 := \chi(x^{k+1}, \ldots, x^{k+n}) \psi_0(x^1, \ldots, x^k).
\]

A direct check yields that \(v \in \mathcal{D}'(U_2)\). If \(\phi \in \mathcal{D}(U_1 \times U_2)\) and we set

\[
I(\phi) := \int_{\mathbb{R}^k} \phi(t^1, \ldots, t^k, x^{k+1}, \ldots, x^{k+n}) dt^1 \wedge \cdots \wedge dt^k,
\]

and have that

\[
\int_{\mathbb{R}^n} (\phi(t^1, \ldots, t^k, x^{k+1}, \ldots, x^{k+n}) - I(\phi) \psi_0) dt^1 \wedge \cdots \wedge dt^k = 0.
\]

Hence we have \(\phi - I(\phi) \psi_0 \in \text{Im} (\partial_1 \circ \cdots \circ \partial_k)\), which, together with the fact that \(\partial_j u = 0\) for \(j = 1, \cdots, k\), yields that

\[
\langle u, (\phi - I(\phi) \psi_0) \rangle = 0,
\]

i.e.,

\[
\langle u, \phi \rangle = \langle u, (I(\phi))_0 \rangle = \langle v, I(\phi) \rangle = \langle v, \int_{\mathbb{R}^k} \phi(x^1, \cdots, x^{k+n}) dx^1 \wedge \cdots \wedge dx^k \rangle = \int_{\mathbb{R}^k} \langle v, \phi \rangle dx^1 \wedge \cdots \wedge dx^k
\]

by Proposition A.5 as desired. \(\square\)
Let us recall the pullback of a distribution by some smooth map. Let $U_j \subset \mathbb{R}^{n_j}$ for $j = 1, 2$ be open sets, and $f : U_1 \to U_2$ a smooth map such that $f'(x)$ is surjective for each point $x \in \mathbb{R}^{n_1}$ (which yields that $n_1 \geq n_2$). Then for each $x_0 \in U_1$ fixed we choose a smooth map $g : U_1 \to \mathbb{R}^{n_1-n_2}$ (e.g., a linear map) such that the direct sum $f \oplus g$
\[ U_1 \ni x \mapsto (f(x), g(x)) \in \mathbb{R}^{n_1} = \mathbb{R}^{n_2} \times \mathbb{R}^{n_1-n_2} \]
has a bijective differential at $x_0$. By the inverse theorem there exists an open neighborhood $V_1 \subset U_1$ such that the restriction of $f \oplus g$ to $V_1$ is a diffeomorphism on an open neighborhood $V_2$ of $(f(x_0), g(x_0))$. We denote by $h$ its inverse. For any $\phi \in \mathcal{D}(V_1)$, we define
\[ (f^*u)(\phi) := (u \otimes 1_{\mathbb{R}^{n_1-n_2}})(\Phi), \quad \Phi(y) := \phi(h(y))|\det h'(y)|, \]
where $1_{\mathbb{R}^{n_1-n_2}}$ is the function 1 on $\mathbb{R}^{n_1-n_2}$. If in addition $u \in C^0(U_2)$, then we have $f^*u = u \circ f$.

**Theorem A.10.** Let $U_j \subset \mathbb{R}^{n_j}$ for $j = 1, 2$ be open sets, and $f : U_1 \to U_2$ a smooth map such that $f'(x)$ is surjective for each point $x \in \mathbb{R}^{n_1}$. Then there exists a unique continuous linear map $f^* : \mathcal{D}'(U_2) \to \mathcal{D}'(U_1)$ such that $f^*u = u \circ f$ when $u \in C^0(U_2)$. It maps $\mathcal{D}^k(U_2)$ into $\mathcal{D}^k(U_1)$ for each $k$. We call $f^*u$ the pullback of $u$ by $f$.

**Proof.** See [26, Theorem 6.1.2].

**Definition A.4.** Let $X$ be a smooth oriented manifold with dim$_\mathbb{R} X = n$ and countable basis and $\mathcal{A}$ an atlas on $X$ consisting of homeomorphism $\kappa$ of open set $U_\kappa \subset X$ to $\tilde{U}_\kappa \subset \mathbb{R}^n$. If for each atlas $\kappa$ one is given a distribution $u_{\kappa} \in \mathcal{D}'(\tilde{U}_\kappa)$ such that
\[ u_{\kappa'} := (\kappa \circ \kappa'^{-1})^*u_{\kappa} \quad \text{in} \quad \kappa'(U_\kappa \cap U_{\kappa'}), \]
then the system $u_{\kappa}$ is called a distribution in $X$. The set of all distributions in $X$ is denoted by $\mathcal{D}'(X)$.

The following theorem shows in particular that Definition A.4 coincides with our previous one if $M$ is an open subset of $\mathbb{R}^n$.

**Theorem A.11.** Let $X$ be a smooth oriented manifold with dim$_\mathbb{R} X = n$ and countable basis and $\mathcal{A}$ an atlas consisting of homeomorphism $\kappa$ of open set $U_\kappa \subset \mathbb{R}^n$ to $\tilde{U}_\kappa \subset \mathbb{R}^n$. If for each $\kappa \in \mathcal{A}$ we have a distribution $u_{\kappa} \in \mathcal{D}'(\tilde{U}_\kappa)$ and (A.7) is valid when $\kappa$ and $\kappa'$ belong to $\mathcal{A}'$, then there exists one and only one distribution $u \in \mathcal{D}'(X)$ such that $(\kappa'^{-1})^*u = u_{\kappa}$ for each $\kappa \in \mathcal{A}$.

**Proof.** See [26, Theorem 6.3.4].

A.2. **Current.** According to De Rham [10] currents are analogy with the usual definition of distributions.

**Definition A.5.** The space of currents of dimension $p$ (or degree $m-p$) on $X$ is the space $\mathcal{D}_p^m(X)$ of linear forms $T$ on $\mathcal{D}^p(X)$ such that the restriction of $T$ to all subspaces $\mathcal{D}^p(K)$, $K \subset X$, is continuous. The degree is indicated by raising the index, hence we set
\[ \mathcal{D}^{m-p}(X) = \mathcal{D}_p^m(X) := (\mathcal{D}^p(X))', \]
The space $s\mathcal{D}_p^m(X) = s\mathcal{D}^{m-p}(X) := (s\mathcal{D}^p(X))'$ is defined similarly and is called the space of currents of order $s$ on $X$. 

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Let $U$ be an open set in $\mathbb{R}^n$. Then a distribution $u$ can be seen as a current with degree 0 by
$$\langle u, \varphi dx^1 \wedge \cdots \wedge dx^n \rangle := \langle u, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(U).$$
For each $T \in \mathcal{D}'_p(U)$ can be written as
$$T := \frac{1}{p!} a_I dx^I$$
with $a_I \in \mathcal{D}'(U)$ defined by
$$\langle a_I, \varphi \rangle = \langle u, \varphi dx^1 \wedge \cdots \wedge dx^n \rangle = \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle T, \varphi dx_I^{f_p} \rangle,$$
where $\delta_{I_p, I_p'}$ is the multi-index Kronecker delta. A current is determined by the restriction to the sets in an open covering and Theorem A.2 holds for $p$ currents in $\mathcal{D}'_p(U)$. In addition, we have an analogy of Theorem A.11 as follows.

**Theorem A.12.** Let $X$ be a smooth oriented manifold with $\dim_{\mathbb{R}} X = n$ and countable basis and $\mathcal{A}$ an atlas on $X$ consisting of homeomorphism $\kappa$ of open set $U_\kappa \subset X$ to $\tilde{U}_\kappa \subset \mathbb{R}^n$. Then there exists a family of $p$ currents $(a_\kappa)_{\kappa \in \mathcal{A}}$ ($1 \leq p \leq n$) with $a_\kappa \in \mathcal{D}'_{n-p}(\tilde{U}_\kappa)$ given by

$$(A.8) \quad a_\kappa := \frac{1}{p!} a_{\kappa;i_1,\ldots,i_p} dx^{i_1}_\kappa \wedge \cdots \wedge dx^{i_p}_\kappa$$
such that

$$(A.9) \quad a_{\kappa'}^{i_1,\ldots,i_p} = ((\kappa \circ \kappa'^{-1})^* a_{\kappa;i_1,\ldots,i_p}) \frac{\partial x^{i_1}_\kappa}{\partial x^{j_1}_{\kappa'}} \cdots \frac{\partial x^{i_p}_{\kappa'}}{\partial x^{j_p}_{\kappa'}} \quad \text{in} \quad \kappa'(U_\kappa \cap U_{\kappa'}),$$

if and only if there exists unique $p$ current $T \in \mathcal{D}'_p(X)$ such that

$$(A.10) \quad \langle a_\kappa, \varphi \rangle = \langle T, \kappa^* \varphi \rangle, \quad \forall \varphi \in \mathcal{D}'(\tilde{U}_\kappa)$$
for each $\kappa \in \mathcal{A}$.

**Proof.** For ‘if’ direction, let $T \in \mathcal{D}'_p(X)$ and $\psi := \kappa \circ \kappa'^{-1}$ defined on $\kappa'(\tilde{U}_\kappa \cap \tilde{U}_{\kappa'})$. Then for each $f \in \mathcal{D}(\kappa(U_\kappa \cap U_{\kappa'}))$ and each multi-index $I_p = \{i_1, \ldots, i_p\}$ fixed, one infers

$$(A.11) \quad \langle a_\kappa; I_p, f dx_\kappa \rangle = \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle a_\kappa, f dx_I^{f_p}_\kappa \rangle$$
$$= \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle T, (f \circ \kappa) \kappa^* \left( dx_I^{f_p}_\kappa \right) \rangle$$
$$= \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle T, ((f \circ \psi) \circ \kappa') \kappa'^* \left( \psi^* dx_I^{f_p}_\kappa \right) \rangle$$
$$= \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle T, ((f \circ \psi) \frac{\partial x^{j+1}_{\kappa'}}{\partial x^{j}_{\kappa'}} \cdots \frac{\partial x^{j-p}_{\kappa'}}{\partial x^{j-p}_{\kappa'}}) \circ \kappa'^* \left( dx_I^{J_p}_{\kappa'} \right) \rangle$$
$$= \frac{1}{(n-p)!} \delta_{I_p, I_p'} \langle a_{\kappa'}, (f \circ \psi) \frac{\partial x^{j+1}_{\kappa'}}{\partial x^{j}_{\kappa'}} \cdots \frac{\partial x^{j-p}_{\kappa'}}{\partial x^{j-p}_{\kappa'}} dx_I^{J_p}_{\kappa'} \rangle$$
$$= \frac{1}{p! (n-p)!} \delta_{I_p, I_p'} \delta_{J_p, J_p'} \langle a_{\kappa'}; J_p, (f \circ \psi) \frac{\partial x^{j+1}_{\kappa'}}{\partial x^{j}_{\kappa'}} \cdots \frac{\partial x^{j-p}_{\kappa'}}{\partial x^{j-p}_{\kappa'}} dx_I^{J_p}_{\kappa'} \rangle,$$
where $I_p^c$ (resp. $J_p^c$) is the supplementary set of $I_p$ (resp. $J_p$) such that $I_p \cup I_p^c = \{1, \ldots, n\}$ (resp. $J_p \cup J_p^c = \{1, \ldots, n\}$).
A direct calculation, together with (A.6), yields that
\[(\psi^* a_{\kappa; I_p})(f \circ \psi) = (\det(\psi'))^{-1} \langle a_{\kappa; I_p}, f \circ \psi^{-1} \rangle = (\det(\psi'))^{-1} \langle a_{\kappa; I_p}, f \rangle.\]

It follows from (A.11) and (A.12) that
\[
(A.13) \quad \frac{1}{p!(n-p)!} \delta_{I_p,I_p'} \delta_{J_p,J_p'} a_{\kappa', I_p} \frac{\partial x_{i_1}^{j_1}}{\partial x_{k_{i_1}}^{j_{k_{i_1}}}} \cdots \frac{\partial x_{i_p}^{j_p}}{\partial x_{k_{i_p}}^{j_{k_{i_p}}}} = (\det((\kappa \circ \kappa'^{-1})) (\kappa \circ \kappa'^{-1}) a_{\kappa; I_p},
\]
which is equivalent to (A.9). Indeed, it follows from (A.13) that
\[
\langle \det((\kappa \circ \kappa'^{-1})) (\kappa \circ \kappa'^{-1}) a_{\kappa; I_p}, \frac{\partial x_{i_1}^{j_1}}{\partial x_{k_{i_1}}^{j_{k_{i_1}}}} \cdots \frac{\partial x_{i_p}^{j_p}}{\partial x_{k_{i_p}}^{j_{k_{i_p}}}} \rangle = \frac{1}{p!(n-p)!} \delta_{I_p,I_p'} \delta_{J_p,J_p'} a_{\kappa', I_p} \frac{\partial x_{i_1}^{j_1}}{\partial x_{k_{i_1}}^{j_{k_{i_1}}}} \cdots \frac{\partial x_{i_p}^{j_p}}{\partial x_{k_{i_p}}^{j_{k_{i_p}}}}
\]
\[
= \frac{1}{p!(n-p)!} \delta_{J_p,J_p'} a_{\kappa', I_p} (\det((\kappa \circ \kappa'^{-1})) (\kappa \circ \kappa'^{-1}) a_{\kappa; I_p},
\]
as desired.

For ‘only if’ direction, we choose a partition of unitary \(1 = \sum_j \chi_{\kappa_j} \) with \(\chi_{\kappa_j} \in \mathcal{D}(\hat{U}_{\kappa_j})\) for some \(\kappa_j \in \mathcal{A}\). A direct calculation yields that the linear form \(T\) defined by
\[
(A.14) \quad \langle T, \varphi \rangle := \sum_j \langle a_{\kappa_j}, (\kappa_j^{-1})^*(\chi_{\kappa_j} \varphi) \rangle, \quad \forall \varphi \in \mathcal{D}^{n-p}(X)
\]

is continuous in \(\mathcal{D}^{n-p}(X)\). Let \(T_{\kappa} \in \mathcal{D}_{n-p}(\hat{U}_{\kappa})\) defined by (A.10). Then for each \(\varphi \in \mathcal{D}^{n-p}(\hat{U}_{\kappa})\), we have
\[
\langle T_{\kappa}, \varphi \rangle = \langle T, \kappa^* \varphi \rangle
\]
\[
= \sum_j \langle a_{\kappa_j}, (\kappa_j^{-1})^*(\chi_{\kappa_j} \kappa^* \varphi) \rangle
\]
\[
= \sum_j \langle a_{\kappa_j}, (\chi_{\kappa_j} \circ \kappa_j^{-1}) (\kappa \circ \kappa_j^{-1})^* \varphi \rangle
\]
\[
= \sum_j \langle a_{\kappa_j}, ((\chi_{\kappa_j} \circ \kappa_j^{-1}) \varphi) \rangle
\]
\[
= \sum_j \langle a_{\kappa}, ((\chi_{\kappa_j} \circ \kappa_j^{-1}) \varphi) \rangle
\]
\[
= \langle a_{\kappa}, \varphi \rangle,
\]
where we use the fact that Equation (A.9) is equivalent to the fact that
\[
\langle a_{\kappa'}, (\kappa \circ \kappa'^{-1})^* \varphi \rangle = \langle a_{\kappa}, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}^{n-p}(\kappa(U_{\kappa} \cap U_{\kappa'})).
\]

It follows from Definition (A.4) and Theorem (A.12) that a distribution \(u \in \mathcal{D}'(X)\) on \(X\) can be viewed as a current with degree 0.

Many of the operations available for differential forms can be extended to currents by simple duality arguments. In general, if \(A : \bigoplus \mathcal{D}^p(X) \to \bigoplus \mathcal{D}^p(X)\) is a map of vector spaces which is continuous on \(\bigoplus \mathcal{D}^p(K)\) for each compact set \(K\), then it is possible to extend \(A\) to a mapping currents.
Lemma A.13 (Transverse Poincaré Lemma). Let $T \in {}^s\mathcal{D}^{p}(X) = {}^s\mathcal{D}^{p}_{m-p}(X)$. The exterior derivative

$$dT \in {}^{s+1}\mathcal{D}^{p+1}(X) = {}^{s+1}\mathcal{D}^{p}_{m-p-1}(X)$$

is defined by

$$\langle dT, \varphi \rangle = (-1)^{p+1}\langle T, d\varphi \rangle, \quad \forall \varphi \in {}^{s+1}\mathcal{D}^{m-p-1}(X).$$

The continuity of the linear form $dT$ on ${}^{s+1}\mathcal{D}^{m-p-1}(X)$ follows from the continuity of the map $d : {}^{s+1}\mathcal{D}^{m-p-1}(K) \to {}^{s}\mathcal{D}^{m-p}(K)$.

Let $T \in {}^{s}\mathcal{D}^{p+1}(X) = {}^{s}\mathcal{D}^{p}_{m-p-1}(X)$ and $\xi \in \mathfrak{X}(X)$ a vector field. The interior product of $\xi$ and $T$

$$i_{\xi}T \in {}^{s+1}\mathcal{D}^{p}(X) = {}^{s+1}\mathcal{D}^{p}_{m-p}(X)$$

is defined (see for example [15]) by

$$\langle i_{\xi}T, \varphi \rangle = (-1)^{p}\langle T, i_{\xi}\varphi \rangle, \quad \forall \varphi \in {}^{s+1}\mathcal{D}^{m-p}(X).$$

The continuity of the linear form $i_{\xi}T$ on ${}^{s+1}\mathcal{D}^{m-p}(X)$ follows from the continuity of the map $i_{\xi} : {}^{s+1}\mathcal{D}^{m-p}(K) \to {}^{s}\mathcal{D}^{m-p+1}(K)$.

Let $T \in {}^{s}\mathcal{D}^{p}(X) = {}^{s}\mathcal{D}^{p}_{m-p}(X)$. Then since the Lie derivative

$$L_{\xi}T \in {}^{s+1}\mathcal{D}^{p}(X) = {}^{s+1}\mathcal{D}^{p}_{m-p}(X)$$

by

$$\langle L_{\xi}T, \varphi \rangle = -\langle T, L_{\xi}\varphi \rangle, \quad \forall \varphi \in {}^{s+1}\mathcal{D}^{m-p}(X).$$

Proof. Since $i_{\partial/\partial t}T = L_{\partial/\partial t}T = 0$ with $1 \leq i \leq k$, we have

$$T = \sum_{1 \leq i_{1} < \cdots < i_{p} \leq n} T_{i_{1}, \ldots, i_{p}} dx^{i_{1}} \wedge \cdots \wedge dx^{i_{p}}$$

with $\partial T_{i_{1}, \ldots, i_{p}} / \partial x^{\ell} = 0$ for $1 \leq \ell \leq k$. It follows from (A.18) Proposition A.9 that

$$T_{i_{1}, \ldots, i_{p}} = 1 \otimes \tilde{T}_{i_{1}, \ldots, i_{p}},$$

where $\tilde{T}_{i_{1}, \ldots, i_{p}} \in \mathcal{D}^{p}(U_{2})$ and that

$$d_{\mathbb{R}^{n}}\tilde{T} = 0,$$

where

$$\tilde{T} = \sum_{1 \leq i_{1} < \cdots < i_{p} \leq n} \tilde{T}_{i_{1}, \ldots, i_{p}} dx^{i_{1}} \wedge \cdots \wedge dx^{i_{p}}.$$

It follows the standard Poincaré Lemma (see for example [12 Section 2.D.4]) and (A.20) that

$$\tilde{T} = \Theta + d_{\mathbb{R}^{n}}\tilde{S}$$

where $\tilde{S} \in {}^{s}\mathcal{D}^{p}_{m-p}(U_{2})$ given by

$$\tilde{S} = \sum_{1 \leq i_{1} < \cdots < i_{p-1} \leq n} \tilde{S}_{i_{1}, \ldots, i_{p-1}} dx^{i_{1}} \wedge \cdots \wedge dx^{i_{p-1}}$$
and $d_{\mathbb{R}^n}$ closed $p$ form $\Theta \in \mathcal{E}^p(U_2)$ given by

$$\Theta = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \Theta_{i_1, \cdots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with $\frac{\partial \Theta_{i_1, \cdots, i_p}}{\partial t^\ell} = 0$ for $1 \leq \ell \leq k$. Hence $\Theta$ and $S$ given by

$$S = \sum_{1 \leq i_1 < \cdots < i_p \leq n} (1 \otimes \hat{S}_{i_1, \cdots, i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p-1}.$$  

are the required currents from Proposition A.8. \hfill \Box

Now we have $d\Theta = 0$ and we deduce from applying the usual Poincaré lemma (see for example [12, (1.22)]) to $\Theta$ that

**Theorem A.14.** Let $T \in s\mathcal{D}_{n-p}^r(U_1 \times U_2)$, where $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^n$ are open sets and in addition $U_2$ is star-shaped open. We denote by $t = (t^1, \cdots, t^k)$ (resp. $x = (x^1, \cdots, x^n)$) the coordinates of $\mathbb{R}^k$ (resp. $\mathbb{R}^n$). If $dT = 0$ and $i_{\partial/\partial t^i} T = \mathcal{L}_{\partial/\partial t^i} T = 0$ with $1 \leq i \leq k$, then there exists a $S \in s\mathcal{D}_{n-p+1}^r(U_1 \times U_2)$ such that $T = dS$ and $i_{\partial/\partial t^i} S = \mathcal{L}_{\partial/\partial t^i} S = 0$ with $1 \leq i \leq k$.

Similar argument gives the transverse Dolbeault-Grothendieck lemma as follows.

**Lemma A.15** (Transverse Dolbeault-Grothendieck Lemma). Let $u \in s\mathcal{E}^{p,q}(U_1 \times U_2, \mathbb{C})$ (resp. $T \in s\mathcal{D}_{n-p,n-q}^r(U_1 \times U_2)$), where $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{C}^n$ containing 0 are open sets. We denote by $t = (t^1, \cdots, t^k)$ (resp. $z = (z^1, \cdots, z^n)$) the coordinates of $\mathbb{R}^k$ (resp. $\mathbb{C}^n$). If $\partial_{\mathbb{C}^n} u = 0$ and $i_{\partial/\partial t^i} u = \mathcal{L}_{\partial/\partial t^i} u = 0$ with $1 \leq i \leq k$, then we have

1. If $q = 0$, then $u = \sum_{1 \leq i_1 < \cdots < i_p \leq n} u_{i_1, \cdots, i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ where $u_{i_1, \cdots, i_p}$ is independent of $t$ and holomorphic for $z$.

2. If $q \geq 1$, then there exists a neighborhood $V_2 \subset U_2$ of 0 and a form $v \in s\mathcal{E}^{p,q-1}(U_1 \times V_2, \mathbb{C})$ (resp. $v \in s\mathcal{D}_{n-p,n-q+1}(U_1 \times V_2)$) such that $u = \partial_{\mathbb{C}} v$ on $U_1 \times V_2$ and $i_{\partial/\partial t^i} v = \mathcal{L}_{\partial/\partial t^i} v = 0$ with $1 \leq i \leq k$.

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