Constructing non-orthogonal split-split-plot designs using some resolvable block designs

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SUMMARY

We consider a new method of constructing non-orthogonal (incomplete) split-split-plot designs (SSPDs) for three (A, B, C) factor experiments. The final design is generated by some resolvable incomplete block design (for the factor A) and by square lattice designs for factors B and C using a modified Kronecker product of those designs (incidence matrices). Statistical properties of the constructed designs are investigated under a randomized-derived linear model. This model is strictly connected with a four-step randomization of units (blocks, whole plots, subplots, sub-subplots inside each block). The final SSPD has orthogonal block structure (OBS) and satisfies the general balance (GB) property. The statistical analysis of experiments performed in the SSPD is based on the analysis of variance often used for multistratum experiments. We characterize the SSPD with respect to the stratum efficiency factors for the basic estimable treatment contrasts. The structures of the vectors defining treatment contrasts are also given.

Key words: General balance property, Efficiency factors, Incomplete split-split-plot designs, Khatri–Rao product of matrices, Resolvable designs

1. Introduction

The main purpose of this paper is to provide a new method of constructing non-orthogonal (incomplete) split-split-plot designs (SSPDs) for experiments with three or more factors. To simplify the discussion, our considerations will be limited to three factors only. The traditional SSPD applied to three-factor experiments commonly used in agricultural research is an extension of a split-
plot design serving to accommodate the third factor. In practice, it is mostly applied to orthogonal (complete) experiments (by which we mean here that all treatment combinations occur once in each block); see e.g. Gomez and Gomez (1984). Giving up the requirement of completeness of the SSPD allows the use of larger numbers of levels of the factors (treatments). Generally, in constructions of incomplete designs the ordinary Kronecker product of matrices is often used. Then different experimental designs from the rich class of block designs (orthogonal, non-orthogonal balanced, partially balanced designs, etc.) are usually taken into account as seed designs to generate a final SSPD (e.g. Mejza, 1997a, 1997b; Ambroży and Mejza, 2011, 2012, 2013). Such designs indeed have many interesting statistical properties; they make it possible, for instance, to use control treatments with a different number of replicates than other treatments, but then the experiments are quite large.

In this paper we propose a new construction of an incomplete SSPD with a smaller number of experimental units, using the *Khatri–Rao product* (also called *semi-Kronecker product*) of matrices, which was defined by Khatri and Rao (1968) as a modification of the ordinary Kronecker product (see also Rao and Mitra, 1971; Gupta and Mukerjee, 1989).

The considerations are based on a linear model of observations, called a randomized-derived model (e.g. Mejza, 1997a; Ambroży and Mejza, 2011, 2013). We will limit our considerations to an incomplete SSPD with orthogonal block structure (OBS) and the general balance (GB) property (see e.g. Mejza, 1992). These statistical properties of the considered design allow one to apply Nelder’s approach to the analysis of variance for multistratum experiments (Nelder, 1965a, 1965b; Houtman and Speed, 1983). The analysis is presented in terms of basic contrasts, introduced by Pearce et al. (1974).

The paper can be treated as a generalization of some earlier ideas on the investigation of statistical properties and constructions of incomplete split-plot designs (cf. for example Mejza and Mejza, 1984; Mejza et al., 2001; Ozawa et al., 2004, 2018; Kuriki and Nakajima, 2007) for three-factor designs of split-plot type.
2. Linear model

Let us consider a three-factor experiment in which the first factor, say $A$, has $v_1$ levels (also called the whole plot treatments or $A$ treatments), the second factor, say $B$, has $v_2$ levels (called the subplot treatments or $B$ treatments), and the third factor, say $C$, has $v_3$ levels (called the sub-subplot treatments or $C$ treatments). Thus $v = v_1 v_2 v_3$ denotes the number of all treatment combinations in the experiment. It is assumed that the experimental material can be divided into $b$ blocks with $k_1$ ($< v_1$) whole plots. Then each whole plot is divided into $k_2$ ($< v_2$) subplots with $k_3$ ($< v_3$) sub-subplots. The blocks, the whole plots, the subplots and the sub-subplots are independently randomized before conducting the experiment. As a result of certain assumptions and four independent randomization processes in the experiment, the mixed linear model of the vector $y$ of $n (= bk_1k_2k_3)$ observations has the form:

$$y = D' \tau + \sum_{f=1}^{4} D'_f \eta_f + e,$$

(2.1)

with the following properties:

$$E(y) = D' \tau \quad \text{and} \quad \text{Cov}(y) = \sum_{f=1}^{4} D'_f V_f D_f + \sigma^2_e I_n,$$

(2.2)

where $D'$ is a known design matrix for $v$ treatment combinations, $\tau$ ($v \times 1$) is the vector of fixed treatment combination effects, and $D'_1, D'_2, D'_3, D'_4$ are respectively ($n \times b$), ($n \times bk_1$), ($n \times bk_1k_2$), ($n \times bk_1k_2k_3$) design matrices for blocks, the whole plots (within the blocks), the subplots (within the whole plots inside the blocks), and the sub-subplots (within the subplots inside the whole plots and blocks). They are expressed by:

$$D'_1 = I_b \otimes 1_{k_1} \otimes 1_{k_2} \otimes 1_{k_3}, \quad D'_2 = I_b \otimes I_{k_1} \otimes 1_{k_2} \otimes 1_{k_3}, \quad D'_3 = I_b \otimes I_{k_1} \otimes I_{k_2} \otimes 1_{k_3}, \quad D'_4 = I_b \otimes I_{k_1} \otimes I_{k_2} \otimes I_{k_3} = I_n,$$

where $I_x$ is the identity matrix of order $x$, $1_x$ is the $x$-dimensional vector of ones, and $\otimes$ denotes the Kronecker product of matrices. The $\eta_f$ ($f = 1, 2, 3, 4$) are, respectively, random effect vectors of the blocks, the whole plots, the subplots, the sub-subplots, with $E(\eta_f) = 0$, and $\text{Cov}(\eta_f) = V_f$, $\text{Cov}(\eta_f, \eta_{f'}) = 0$ for all $f \neq f'$. Also $e$ is an $n \times 1$ random vector of technical errors with $E(e) = 0$ and $\text{Cov}(e) = \sigma^2_e I_n$. According to the orthogonal block
structure of the considered designs, the covariance matrix $\text{Cov}(y)$ can be expressed by

$$
\text{Cov}(y) = \gamma_0 P_0 + \gamma_1 P_1 + \gamma_2 P_2 + \gamma_3 P_3 + \gamma_4 P_4,
$$

(2.3)

where $\gamma_f \geq 0$ and $P_f$ is a family of known pairwise orthogonal matrices summing to the identity matrix (cf. Mejza, 1997b). The range space of $P_f$ for $f = 0, 1, 2, 3, 4$ is termed the $f$-th stratum of the model, and $\gamma_f$ are unknown stratum variances (cf. Houtman and Speed, 1983).

In the incomplete SSPD the matrices $P_f$ represent, respectively, five strata: the total area stratum (zero stratum), the inter-block stratum (the first stratum, I), the inter-whole plot stratum (the second stratum, II), the inter-subplot stratum (the third stratum, III) and the inter-sub-subplot stratum (the fourth stratum, IV).

3. Construction of an incomplete SSPD

The incomplete SSPD is characterized by the following incidence matrices: the $v \times b$ incidence matrix $N_1 = \Delta D_1'$ (with respect to the blocks), the $v \times bk_1$ incidence matrix $N_2 = \Delta D_2'$ (with respect to the whole plots), and the $v \times bk_1k_2$ incidence matrix $N_3 = \Delta D_3'$ (with respect to the subplots). In the design of an experiment the incidence matrix $N_1$ plays the most important role. Its form and properties derive uniquely from the adopted method of construction. The remaining incidence matrices follow from the matrix $N_1$, but their general forms are not unique. However, the concurrence matrices $N_iN_i'$, $i = 1, 2, 3$, are unique.

In the method of constructing the incomplete SSPD we need resolvable designs (e.g. Mejza et al., 2012). A design with $v$ treatments is denoted by $D(v, r, k)$ if each treatment is replicated $r$ times and each block contains $k$ treatments. If the collection of blocks of a $D(v, r, k)$ can be grouped in such a way that every treatment occurs precisely once in every group, then the design is said to be resolvable and is denoted by $\text{RD}(v, r, k)$. Thus, each group is constituted of one replication of all treatments. An $\text{RD}(v, r, k)$ such that $v = s^2$, $r \leq s + 1$ and $k = s$ for a positive integer $s$ is called a square lattice design if any two blocks from different groups contain exactly one common treatment, and
it is then denoted by $\text{SLD}(s^2, r, s)$. If $r = s + 1$, it is called a \textit{balanced square lattice design}. It is well known that there exists a balanced square lattice design if $s$ is a prime or a prime power (see Raghavarao, 1971).

Let $N_A, N_B, N_C$ be incidence matrices of so-called generating subdesigns for the factors $A$, $B$ and $C$, respectively. In this construction

$$N_A = [N_{A1}, N_{A2}, \ldots, N_{Ar}]$$

is the incidence matrix of a resolvable design $\text{RD}(v', r, k_1)$, where the $v' \times m$ matrix $N_{Ai}$ corresponds to the $i$-th replicate, $v' (= v_1)$ denotes the number of $A$ treatments, $r$ is the number of their replicates, $k_1$ is the number of whole plots inside each block in the SSPD, and $m = \frac{v'}{k_1}$. Then let

$$N_B = [N_{B1}, N_{B2}, \ldots, N_{Br}] \text{ and } N_C = [N_{C1}, N_{C2}, \ldots, N_{Cr}]$$

be the incidence matrices of two square lattice designs $\text{SLD}(s_1^2, r, s_1)$ and $\text{SLD}(s_2^2, r, s_2)$ with the same number of replicates $r$, where $N_{Bi}$ and $N_{Ci}$ correspond to the $i$-th replicates with $s_1 (= k_2)$ and $s_2 (= k_3)$ plots per block in the SLDs (i.e. the subplots inside each whole plot and the sub-subplots inside each subplot in the SSPD, respectively), and $s_1^2 = v_2$ and $s_2^2 = v_3$ denote the numbers of $B$ treatments and $C$ treatments respectively.

For the resolvable design $\text{RD}(v', r, k_1)$,

$$N'_{Ai}N_{Ai} = k_1 I_m$$

holds for $i = 1, 2, \ldots, r$. From (3.3), $N_{Ai}N'_{Ai}$ has the eigenvalues $k_1$ and 0 with multiplicities $m$ and $v' - m$ for each $i = 1, 2, \ldots, r$. Let $P^{(m)} = \begin{pmatrix} p^{(m)}_0, p^{(m)}_1, \ldots, p^{(m)}_{m-1} \end{pmatrix}$ be an orthogonal matrix of order $m$ with $p^{(m)}_0 = (m^{-1/2})1_m$. The mutually orthonormal eigenvectors of $N_{Ai}N'_{Ai}$ corresponding to the eigenvalue $k_1$ are given by

$$x_{ij} = \frac{1}{\sqrt{k_1}} N_{Ai}P^{(m)}_j$$

for $j = 0, 1, \ldots, m - 1$. In particular $x_{i0} = (v')^{-\frac{1}{2}}1_{v'}$. The eigenvectors of $N_{Ai}N'_{Ai}$ corresponding to the eigenvalue 0 are denoted by $x^*_{ij}$ for $j = 1, 2, \ldots, (v' - m)$. Here $x^*_{i0}, x^*_{i1}, \ldots, x^*_{im-1}, x^*_{i1'}, \ldots, x^*_{iv', m}$ are also mutually

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orthonormal. The eigenvalues and the corresponding mutually orthonormal eigenvectors of \( N_A' N_A' \) are denoted by \( \theta_j \) and \( h_j \) for \( j = 0, 1, \ldots, v' - 1 \), with \( \theta_0 = r k_1 \) and \( h_0 = \frac{1}{\sqrt{v'}} 1_{v'} \).

For the square lattice designs SLD\( (s_i^2, r, s_1) \) and SLD\( (s_j^2, r, s_2) \),

\[
N_{Bi}' N_{Bi} = s_1 I_{s_1}, N_{Ci}' N_{Ci} = s_2 I_{s_2}, \quad N_{Bi}' N_{Bi} N_{Bj}' = J_{s_1}, \quad N_{Ci}' N_{Ci} N_{Cj}' = J_{s_2},
\]

(3.4)

(3.5)

(3.6)

(3.7)

hold for \( i, j = 1, 2, \ldots, r, \ i \neq j \). From (3.4), \( N_{Bi}' N_{Bi}' \) has the eigenvalues \( s_1 \) and \( 0 \) with multiplicities \( s_1 \) and \( s_1(s_1 - 1) \) for each \( i = 1, 2, \ldots, r \). From (3.6), \( N_{B1}' N_{B1}' , N_{B2}' N_{B2}' , \ldots , N_{Br}' N_{Br}' \) are mutually commutative, so these concurrence matrices have common eigenvectors. For each \( i \)-th replicate, from (3.4), the mutually orthonormal eigenvectors of \( N_{Bi}' N_{Bi}' \) corresponding to the eigenvalue \( s_1 \) are given by \( z^{(s_1)}_{ij} = \frac{1}{\sqrt{s_1}} N_{Bi}(p^{(s_1)}_j) \) for \( j = 0, 1, \ldots, s_1 - 1 \), where \( p^{(s_1)}_j \) is the \( j \)-th column vector of the orthogonal matrix \( P^{(s_1)} = (p^{(s_1)}_0, p^{(s_1)}_1, \ldots, p^{(s_1)}_{s_1 - 1}) \). In particular \( z^{(s_1)}_{i0} = \frac{1}{s_1} 1_{s_1} \). From (3.4), the eigenvectors \( z^{(s_1)}_{ij} \) for \( j = 0, 1, \ldots, s_1 - 1 \) are also the eigenvectors of the concurrence matrix \( N_{Bi}' N_{Bi}' (h \neq i) \) for the other replicate, and the eigenvalues of \( N_{Bi}' N_{Bi}' \) corresponding to \( z^{(s_1)}_{i0} \) and \( z^{(s_1)}_{ij} (j \neq 0) \) are \( s_1 \) and \( 0 \) respectively. The mutually orthonormal common eigenvectors of \( N_{B1}' N_{B1}' , N_{B2}' N_{B2}' , \ldots , N_{Br}' N_{Br}' \) corresponding to the eigenvalue \( 0 \) are denoted by \( z^{(0)}_{ij} \) for \( j = 1, 2, \ldots, s_1^2 - r(s_1 - 1) - 1 \). The facts given are summarized in Table 1.

Here the common eigenvectors in Table 1 are mutually orthogonal. Similarly, from (3.5) and (3.7) we have the eigenvalues and the common eigenvectors of the concurrence matrices \( N_{C1}' N_{C1}' , N_{C2}' N_{C2}' , \ldots , N_{Cr}' N_{Cr}' \) in SLD\( (s_2^2, r, s_2) \), which are presented in Table 2.

In the SSPD considered, we can express the incidence matrix \( N_1 \) as

\[
N_1 = N_A \bigotimes N_B \bigotimes N_C = [N_{A1} \otimes N_{B1} \otimes N_{C1} : N_{A2} \otimes N_{B2} \otimes N_{C2} : \ldots : N_{Ar} \otimes N_{Br} \otimes N_{Cr}],
\]

(3.8)
Table 1. Eigenvalues and common eigenvectors in $\text{SLD}(s_1^2, r, s_1)$

| Eigenvalues | Common eigenvectors |
|-------------|---------------------|
| $N_{B1}N'_{B1}$ | $s_1^{-1}1_{s_1^2}$ |
| $s_1$ | $s_1^{-1}1_{s_1^2}$ |
| $0$ | $z_{rj}^B (j = 1, 2, ..., s_1 - 1)$ |
| $\vdots$ | $\vdots$ |
| $0$ | $z_{rj}^B (j = 1, 2, ..., s_1 - 1)$ |

Table 2. Eigenvalues and common eigenvectors in $\text{SLD}(s_2^2, r, s_2)$

| Eigenvalues | Common eigenvectors |
|-------------|---------------------|
| $N_{C1}N'_{C1}$ | $s_2^{-1}1_{s_2^2}$ |
| $s_2$ | $s_2^{-1}1_{s_2^2}$ |
| $0$ | $z_{rj}^C (j = 1, 2, ..., s_2 - 1)$ |
| $\vdots$ | $\vdots$ |
| $0$ | $z_{rj}^C (j = 1, 2, ..., s_2 - 1)$ |

which is called the semi-Kronecker product of $N_A, N_B$, and $N_C$. From (3.8), the concurrence matrices have the forms:

$$N_1N'_1 = \sum_{i=1}^r ((N_A N'_A) \otimes (N_{B1}N'_{B1}) \otimes (N_{C1}N'_{C1})), $$

$$N_2N'_2 = \sum_{i=1}^r (I_{v'} \otimes (N_{B1}N'_{B1}) \otimes (N_{C1}N'_{C1})), $$

$$N_3N'_3 = \sum_{i=1}^r (I_{v'} \otimes (N_{C1}N'_{C1})).$$

(3.9)

In this method of construction the number of treatment combinations is equal to $v = v_1v_2v_3 = v' s_1^2 s_2^2$, and the number of all units is $n = bk_1k_2k_3 = bk_1s_1s_2$. 

This means that the incomplete SSPD has \(v\)' whole plot (A) treatments, \(s_1^2\) subplot (B) treatments and \(s_2^2\) sub-subplot (C) treatments.

4. Analysis

The statistical analysis of submodels related to the different strata (2.3) is based on algebraic properties of stratum information matrices for the treatment combinations, which are defined as

\[ A_f = \Delta P_f \Delta' \]  

for \(f = 0, 1, 2, 3, 4\). In the SSPD considered, from (4.1) we have

\[ A_0 = \frac{r}{v} J_0, \quad A_1 = \frac{1}{s_1 s_2} N_1 N'_1 - \frac{r}{v} J_0, \quad A_2 = \frac{1}{s_1 s_2} N_2 N'_2 - \frac{1}{k_1 s_1 s_2} N_1 N'_1, \]

\[ A_3 = \frac{1}{s_2} N_3 N'_3 - \frac{1}{s_1 s_2} N_2 N'_2, \quad A_4 = r I_v - \frac{1}{s_2} N_3 N'_3, \]  

(4.2)

where \(v = v' s_1^2 s_2^2\).

Our considerations concern an incomplete SSPD with the general balance (GB) property only, i.e. the matrices \(A_f\) \((f = 0, 1, 2, 3, 4)\) mutually commute. From this fact let us note that the following relations hold:

\[ N_1 N'_1 N_2 N'_2 = N_2 N'_2 N_1 N'_1, \]

\[ N_1 N'_1 N_3 N'_3 = N_3 N'_3 N_1 N'_1, \]

\[ N_2 N'_2 N_3 N'_3 = N_3 N'_3 N_2 N'_2. \]  

(4.3)

This means that the matrices (4.2), and therefore (4.3), have a common set of eigenvectors \(\{p_h\}\) corresponding to eigenvalues \(\{\epsilon_{fh}\}\) with respect to \(r\), where \(0 \leq \epsilon_{fh} \leq 1\) and \(f = 0, 1, 2, 3, 4; \ h = 1, 2, \ldots, v\). Since \(A_f 1_v = 0\) for \(f > 0\), for instance, the last eigenvector may be chosen as \(p_v = n^{-1/2} 1_v\). The remaining eigenvectors \(\{p_h\}\) for \(h < v\), where \(p'_h p_v = 0\), form a basis for all vectors generating some contrasts. Also, from the relation \(A_0 1_v \neq 0\), it follows that the zero stratum (for \(f = 0\)) is connected with estimation of the general mean only.

It can be easily shown that the mutually orthonormal common eigenvectors of \(N_1 N'_1, N_2 N'_2\) and \(N_3 N'_3\) are as follows:
(1) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes \frac{1}{s_1} 1_{s_1} \otimes \frac{1}{s_2} 1_{s_2}$.

(2) $h_j \otimes \frac{1}{s_1} 1_{s_1} \otimes \frac{1}{s_2} 1_{s_2}$, $(j = 1, 2, \ldots, v' - 1)$.

(3) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes \frac{1}{s_1} 1_{s_1} \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, s_2 - 1)$.

(4) $x_{ij} \otimes \frac{1}{s_1} 1_{s_1} \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, m - 1, j' = 1, 2, \ldots, s_2 - 1)$.

(5) $x_{ij}^m \otimes \frac{1}{s_1} 1_{s_1} \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, v' - m, j' = 1, 2, \ldots, s_2 - 1)$.

(6) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes \frac{1}{s_1} 1_{s_1} \otimes z_{ij}^c$, $(j = 1, 2, \ldots, v' - 1, j' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.

(7) $h_j \otimes \frac{1}{s_1} 1_{s_1} \otimes z_{ij}^c$, $(j = 1, 2, \ldots, v' - 1, j' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.

(8) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes z_{ij}^b \otimes \frac{1}{s_2} 1_{s_2}$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, s_2 - 1)$.

(9) $x_{ij} \otimes z_{ij}^b \otimes \frac{1}{s_2} 1_{s_2}$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, m - 1, j' = 1, 2, \ldots, s_2 - 1)$.

(10) $x_{ij}^m \otimes z_{ij}^b \otimes \frac{1}{s_2} 1_{s_2}$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, v' - m, j' = 1, 2, \ldots, s_2 - 1)$.

(11) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes z_{ij}^b \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, s_2 - 1, j' = 1, 2, \ldots, s_2 - 1)$.

(12) $x_{ij} \otimes z_{ij}^b \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, m - 1, j' = 1, 2, \ldots, s_2 - 1, j'' = 1, 2, \ldots, s_2 - 1)$.

(13) $x_{ij}^m \otimes z_{ij}^b \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, v' - m, j' = 1, 2, \ldots, s_2 - 1, j'' = 1, 2, \ldots, s_2 - 1)$.

(14) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes z_{ij}^b \otimes z_{ij}^c$, $(i \neq i', i, i' = 1, 2, \ldots, r, j = 1, 2, \ldots, s_2 - 1, j' = 1, 2, \ldots, s_2 - 1)$.

(15) $h_j \otimes z_{ij}^b \otimes z_{ij}^c$, $(j = 1, 2, \ldots, v' - 1, i \neq i', i, i' = 1, 2, \ldots, r, j' = 1, 2, \ldots, s_2 - 1, j'' = 1, 2, \ldots, s_2 - 1)$.

(16) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes z_{ij}^b \otimes z_{ij}^c$, $(i = 1, 2, \ldots, r, j = 1, 2, \ldots, s_2 - 1, j' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.

(17) $h_j \otimes z_{ij}^b \otimes z_{ij}^c$, $(j = 1, 2, \ldots, v' - 1, i = 1, 2, \ldots, r, j' = 1, 2, \ldots, s_2 - 1, j'' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.

(18) $\frac{1}{\sqrt{v'}} 1_{v'} \otimes z_{ij}^b \otimes \frac{1}{s_2} 1_{s_2}$, $(j = 1, 2, \ldots, v' - 1, j' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.

(19) $h_j \otimes z_{ij}^b \otimes \frac{1}{s_2} 1_{s_2}$, $(j = 1, 2, \ldots, v' - 1, j' = 1, 2, \ldots, s_2^2 - r(s_2 - 1) - 1)$.
In Table 3 we present eigenvalues and corresponding common eigenvectors of the concurrence matrices $\mathbf{N}_{i}$, $i = 1, 2, 3$.

| Eigenvalues | Common eigenvectors |
|-------------|---------------------|
| $\mathbf{N}_{1}$ | $\mathbf{N}_{1}'$ | $\mathbf{N}_{2}$ | $\mathbf{N}_{2}'$ | $\mathbf{N}_{3}$ | $\mathbf{N}_{3}'$ |
| $rk_{1}s_{1}s_{2}$ | $rs_{1}s_{2}$ | $rs_{2}$ | (1) |
| $s_{1}s_{2}^{j}$ | $j = 1, 2, ..., v' - 1$ | $rs_{1}s_{2}$ | (2) |
| $k_{1}s_{1}s_{2}$ | $s_{1}s_{2}$ | $s_{2}$ | (3), (4), (11), (12) |
| 0 | $s_{1}s_{2}$ | $s_{2}$ | (5), (13) |
| 0 | 0 | 0 | (6), (7), (16), (17), (22), (23) |
| $k_{1}s_{1}s_{2}$ | $s_{1}s_{2}$ | $rs_{2}$ | (8), (9) |
| 0 | $s_{1}s_{2}$ | $rs_{2}$ | (10) |
| 0 | 0 | $s_{2}$ | (14), (15), (20), (21) |
| 0 | 0 | $rs_{2}$ | (18), (19) |

We can note that any vector $\mathbf{c}_{h} = r\mathbf{p}_{h}$ such that the eigenvector $\mathbf{p}_{h}$ satisfies the condition

$$A_{f}\mathbf{p}_{h} = r\varepsilon_{fh}\mathbf{p}_{h}$$

(4.4)

for $f = 1, 2, 3, 4; h = 1, 2, ..., v - 1$ defines an orthogonal (basic) contrast $\mathbf{c}_{h}'\mathbf{t}$ (cf. Pearce et al., 1974). The $\varepsilon_{fh}$ is called the stratum efficiency factor for the
SSPD. The contrasts are connected with comparisons among main effects of the considered factors and interaction effects between them. They can be expressed through the eigenvectors given in Table 3. The vectors (1)–(23) are also the common eigenvectors of the stratum information matrices (4.2) with respect to $r$.

Finally we present potential stratum efficiency factors for the incomplete SSPD.

| Type of contrasts (No. of vectors) | Strata |
|-----------------------------------|--------|
|                                   | I      | II     | III    | IV     |
| $A$ (2)                           | 0/(rk) | 1 − 0/(rk) | –      | –      |
| $B$ (8)                           | 1/r    | –      | 1 − 1/r| –      |
|                                   | –      | –      | 1      | –      |
| $C$ (3)                           | 1/r    | –      | –      | 1 − 1/r|
|                                   | –      | –      | –      | 1      |
| $A \times B$ (9)                  | 1/r    | –      | 1 − 1/r| –      |
|                                   | –      | 1/r    | 1 − 1/r| –      |
|                                   | –      | –      | 1      | –      |
| $A \times C$ (4)                  | 1/r    | –      | –      | 1 − 1/r|
|                                   | –      | 1/r    | –      | 1 − 1/r|
|                                   | –      | –      | –      | 1      |
| $B \times C$ (11)                 | 1/r    | –      | –      | 1 − 1/r|
|                                   | –      | –      | 1/r    | 1 − 1/r|
|                                   | –      | –      | –      | 1      |
| $A \times B \times C$ (12)       | 1/r    | –      | –      | 1 − 1/r|
|                                   | –      | 1/r    | –      | 1 − 1/r|
|                                   | –      | –      | 1/r    | 1 − 1/r|
|                                   | –      | –      | –      | 1      |

Here $j = 1, 2, \ldots, v′ − 1$, and I, II, III and IV denote the inter-block stratum, the inter-whole plot stratum, the inter-subplot stratum and the inter-sub-sub-subplot stratum, respectively.
5. Example and discussion

Consider a $(6 \times 4 \times 9)$-experiment arranged in an incomplete SSPD. The $A$ treatments are allocated in a resolvable design with the incidence matrix $N_A = (N_{A1}; N_{A2}; N_{A3})$ (RD$(6, 3, 3)$) (cf. Mejza et al., 2012). The RD$(6, 3, 3)$ is an optimal $\alpha$-design generated by the CycDesigN software (see Whitaker et al., 2002). The parameters are as follows: $v' = 6, r = k_1 = 3$.

In turn, the $B$ treatments $(s_1^2 = 4)$ and $C$ treatments $(s_2^2 = 9)$ are allocated in different square lattice designs with the incidence matrices $N_B = (N_{B1}; N_{B2}; N_{B3})$ (SLD$(4, 3, 2)$) and $N_C = (N_{C1}; N_{C2}; N_{C3})$ (SLD$(9, 3, 3)$) respectively. Assume that these incidence matrices have the following forms (see Clatworthy, 1973): 

$$
N_A = \begin{bmatrix}
1 & 0 & : & 1 & 0 & : & 1 & 0 \\
1 & 0 & : & 1 & 0 & : & 0 & 1 \\
1 & 0 & : & 0 & 1 & : & 1 & 0 \\
0 & 1 & : & 0 & 1 & : & 0 & 1 \\
0 & 1 & : & 0 & 1 & : & 1 & 0 \\
0 & 1 & : & 1 & 0 & : & 0 & 1
\end{bmatrix},
$$

$$
N_B = \begin{bmatrix}
1 & 0 & : & 1 & 0 & : & 1 & 0 \\
1 & 0 & : & 0 & 1 & : & 0 & 1 \\
0 & 1 & : & 1 & 0 & : & 0 & 1 \\
0 & 1 & : & 0 & 1 & : & 1 & 0
\end{bmatrix},
$$

$$
N_C = \begin{bmatrix}
1 & 0 & : & 1 & 0 & : & 1 & 0 \\
1 & 0 & : & 0 & 1 & : & 0 & 1 \\
1 & 0 & : & 0 & 1 & : & 0 & 1 \\
0 & 1 & : & 1 & 0 & : & 0 & 1 \\
0 & 1 & : & 0 & 1 & : & 1 & 0 \\
0 & 0 & : & 1 & 0 & : & 0 & 1 \\
0 & 0 & : & 0 & 1 & : & 1 & 0 \\
0 & 0 & : & 0 & 1 & : & 1 & 0 \\
0 & 0 & : & 0 & 1 & : & 1 & 0 \\
0 & 0 & : & 0 & 1 & : & 1 & 0
\end{bmatrix}
$$

with 3 replicates.

From (3.3), $N_{Ai}N_{Ai}'$ has the eigenvalues $k_1 = 3$ and 0 with multiplicities $m$ ($= v' k_1^{-1}$) $= 2$ and $v' - m = 4$ for each $i = 1, 2, 3$. The eigenvectors of $N_{Ai}N_{Ai}'$ corresponding to the eigenvalue 0 are denoted by $x_{ij}'$ for $j = 1, 2, 3, 4$. Furthermore, the eigenvalues and the corresponding eigenvectors of $N_A N_A'$ are
denoted by $\theta_j$ and $h_j$ for $j = 0, 1, \ldots, 5$, with $\theta_0 = rk_1 = 9$ and $h_0 = \frac{1}{\sqrt{v'}} 1_v$, where $1_v = (1, 1, \ldots, 1)$. From Tables 1–2 we have that $N_{i} N'_{i}$ (i = 1, 2, 3) has the eigenvalues $s_1 = 2$ and $s_1(s_1 - 1) = 2$ respectively. In turn, $N_{ci} N'_{ci}$ (i = 1, 2, 3) has the eigenvalues $s_2 = 3$ and 0, with multiplicities $s_2 = 3$ and $s_2(s_2 - 1) = 6$.

Using the semi-Kronecker product $N_1 = N_A \boxtimes N_B \boxtimes N_C$ (see (3.8)) we obtain the incomplete SSPD with parameters: $v = v' s_1^2 s_2^2 = 216$, $b = rms_1 s_2 = 36$, $k = k_1 s_1 s_2 = 18$, $r = 3$ and $n = 648$, where $m = \frac{v'}{k_1} = 2$, $s_1 = 2$, $s_2 = 3$ and $k_1 = 3$.

**Table 5.** Eigenvalues and common eigenvectors of $N_1 N'_1$, $N_2 N'_2$ and $N_3 N'_3$

| Eigenvalues | Common eigenvectors |
|-------------|---------------------|
| $rk_1 s_1 s_2 = 54$ | $rs_1 s_2 = 18$ $rs_2 = 9$ (1) |
| $s_1 s_2 \theta_j = 6 \theta_j j = 1, \ldots, 5$ | $rs_1 s_2 = 18$ $rs_2 = 9$ (2) |
| $k_1 s_1 s_2 = 18$ | $s_1 s_2 = 6$ $s_2 = 3$ (3), (4), (11), (12) |
| $0$ | $s_1 s_2 = 6$ $s_2 = 3$ (5), (13) |
| $0$ | $0$ $0$ (6), (7), (16), (17), (22), (23) |
| $k_1 s_1 s_2 = 18$ | $s_1 s_2 = 6$ $rs_2 = 9$ (8), (9) |
| $0$ | $s_1 s_2 = 6$ $rs_2 = 9$ (10) |
| $0$ | $0$ $s_2 = 3$ (14), (15), (20), (21) |
| $0$ | $0$ $rs_2 = 9$ (18), (19) |

Let us note that the vectors (1)–(17) are also the common eigenvectors of the stratum information matrices (4.2) with respect to $r$. The vectors (18)–(23) do not occur in the example considered.

To present a sample layout of the considered SSPD, we introduce an abbreviation. Let $\{A_i, A_j, A_d | B_w, B_z | C_l, C_m, C_p\}$ denote a block such that $A_i, A_j, A_d$, where $i, j, d \in \{1, 2, 3, 4, 5, 6\}$, $i \neq j \neq d \neq i$, are the whole plot treatments inside the block, $B_w, B_z$, where $w, z \in \{1, 2, 3, 4\}$, $w \neq z$, are the subplot treatments in each of the whole plots, and $C_l, C_m, C_p$, where $l, m, p \in$
blocks} is then equal to 324, and the number of subplots and 3, we choose the levels of three factors in one block. The contents of the blocks can be expressed as follows (before randomization of units):

\[ \{A_1, A_2, A_3 \mid B_1, B_2 \mid C_1, C_2, C_3\}, \{A_1, A_2, A_3 \mid B_1, B_2 \mid C_4, C_5, C_6\}, \{A_1, A_2, A_3 \mid B_1, B_2 \mid C_7, C_8, C_9\}, \]

Below we show a sample layout (after three-step randomization). Let us consider the last of them, i.e. \( \{A_2, A_4, A_6 \mid B_2, B_3 \mid C_5, C_4, C_3\} \).

![Figure 1. Random assignment of the levels of three factors in one block of the SSPD](image)

We can notice that using the proposed construction method, the resulting incomplete SSPD has \( b = 36 \) blocks with size equal to \( k = 18 \) (three whole plots, 2 subplots and 3 sub-subplots). We also note that using the usual Kronecker product of matrices with \( \otimes \), the incomplete SSPD would be 9 times larger (the number of blocks is then equal to 324, and the number of experimental units \( n = 5832 \).
Table 6. Stratum efficiency factors of the SSPD in this example

| Type of contrast | df | I     | II    | III   | IV    |
|------------------|----|-------|-------|-------|-------|
|                  | (No. of vectors) |     |       |       |       |
| A                | (2) | 1     | 8/9   | –     | –     |
|                  | (2) | 2     | 4/9   | 5/9   | –     | –     |
|                  | (2) | 2     | –     | 1     | –     | –     |
| B                | (8) | 3     | 1/3   | –     | 2/3   | –     |
| A × B            | (9) | 3     | 1/3   | –     | 2/3   | –     |
|                  | (10)| 12    | –     | 1/3   | 2/3   | –     |
| C                | (3) | 6     | 1/3   | –     | –     | 2/3   |
|                  | (6) | 2     | –     | –     | –     | 1     |
| A × C            | (4) | 6     | 1/3   | –     | –     | 2/3   |
|                  | (5) | 24    | –     | 1/3   | –     | 2/3   |
|                  | (7) | 10    | –     | –     | –     | 1     |
| B × C            | (11)| 6     | 1/3   | –     | –     | 2/3   |
|                  | (14)| 12    | –     | –     | 1/3   | 2/3   |
|                  | (16)| 6     | –     | –     | –     | 1     |
| A × B × C        | (12)| 6     | 1/3   | –     | –     | 2/3   |
|                  | (13)| 24    | –     | 1/3   | –     | 2/3   |
|                  | (15)| 60    | –     | –     | 1/3   | 2/3   |
|                  | (17)| 30    | –     | –     | –     | 1     |

The eigenvalues (efficiency factors) and eigenvectors of the constructed SSPD depend on the efficiency factors of the component designs. Hence, the statistical properties of the component designs used in this construction lead to efficiency factors of the final design (Table 6) which are different from the efficiency factors in Table 4.

Because of some incompleteness of the SSPD, we have lost a little efficiency for some treatment contrasts, and we have a loss of estimability of some treatment contrasts in some strata. The general hypothesis concerning the effects of main sources of variability in the experiment can also be tested using the full set of orthogonal basic contrasts. It is only necessary to have a stratum (or strata) in which all basic treatment contrasts are estimable.

In particular, the general hypothesis concerning main effects of the factor A is testable in stratum II, while the hypotheses concerning the main effects of the factor B and the interaction effects of A × B are testable in stratum III. The other hypotheses are testable in stratum IV. To improve the statistical properties of
inference in the area of estimation, as well as in the area of hypothesis testing, some methods of combining information can be applied.

6. Conclusion

We have proposed a method of construction to obtain an incomplete split-split-plot design with very desirable statistical properties, especially the utilization of a small number of observations. As the main tools, resolvable block designs and the semi-Kronecker product of matrices are used. In particular, the generating incidence matrices for factors $A$, $B$, $C$, i.e. $N_A$, $N_B$ and $N_C$, must have the same number of partitions to apply the semi-Kronecker product. Additionally, for the factor $A$ we can use any resolvable block design, while for factors $B$ and $C$ square lattice designs are used. This is a slight disadvantage of the method, because square lattice designs exist only for some sets of design parameters. In this example, for the factor $A$ with $v' = 6$ levels we use a resolvable block design. For the remaining factors there are square lattice designs, where for the factor $B$ it is a balanced square $(r = s_1 + 1 = 2+1 = 3)$, and for the factor $C$ it is not a balanced square $(r = 3 < s_2 + 1 = 4)$.

The main advantage of the construction method proposed is that the size of the final experiment with desirable statistical properties is small. The requirement for properties such as general balance or orthogonal block structure entails a larger number of blocks in the experiment, but the number is always smaller than in the traditional method where the usual Kronecker product of generating matrices is used. It is an obvious fact that in experiments, complete designs, which are orthogonal designs, are the best in terms of statistical properties. They make it possible to estimate with full efficiencies (efficiency factors equal to 1) all comparisons of the treatment effects. However, there are situations where they cannot always be applied. For example, for various reasons, there may be a smaller block size involved in an experiment relative to the number of treatment combinations. One of the solutions to the problem is to use some incomplete balanced and/or partially balanced designs in the experiment, like the incomplete
SSPD proposed in this paper. Note that some treatment contrasts in the proposed design can be estimated with full efficiencies as in the complete (orthogonal) SSPD. Having the structure of the vectors (1)–(23) generating treatment contrasts, we can decide the level of the efficiency of estimation of particular treatment effects.

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