Solvability of an Operator Riccati Integral Equation in a Reflexive Banach Space

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If $X_{1,2}$ are Banach spaces, then by $\mathcal{L}(X_1, X_2)$ we will denote the space of bounded operators acting from $X_1$ to $X_2$. By $C_s(\mathcal{I}; \mathcal{L}(X_1, X_2))$ we denote the space of strongly continuous operator functions on interval $\mathcal{I} = [0, T]$ with the topology of strongly uniform convergence.

An operator function $\{U_{t,s}\}_{0 \leq s \leq t \leq T}$ on a Banach space is called forward (in time) evolution family if $U_{t,t} = I$ and $U_{t,s} = U_{t,r}U_{r,s}$ for all $0 \leq s \leq r \leq t \leq T$.

An operator function $\{V_{s,t}\}_{0 \leq s \leq t \leq T}$ on a Banach space is called backward (in time) evolution family if $V_{t,t} = I$ and $V_{s,t} = V_{s,r}V_{r,t}$ for all $0 \leq s \leq r \leq t \leq T$.

An evolution family is called strongly continuous if it is strongly continuous in $t$ (for fixed $s$) and in $s$ (for fixed $t$).

Let $X$ be a reflexive Banach space with duality pairing $\langle f, x \rangle$ ($x \in X, f \in X^*$). If $A_1 \in \mathcal{L}(X, X^*)$ then taking into account the canonical isomorphism between $X$ and $X^{**}$ one can consider that the adjoint operator $A_1^* \in \mathcal{L}(X, X^*)$. Operator $A_1 \in \mathcal{L}(X, X^*)$ is self-adjoint if $A_1 = A_1^*$. Self-adjoint operator $A_1 \in \mathcal{L}(X, X^*)$ is non-negative if $\langle A_1x, x \rangle \geq 0$ for all $x \in X$.

Analogously if $A_2 \in \mathcal{L}(X^*, X)$ then one can consider that the adjoint operator $A_2^* \in \mathcal{L}(X^*, X)$. Operator $A_2 \in \mathcal{L}(X^*, X)$ is self-adjoint if $A_2 = A_2^*$. Self-adjoint operator $A_2 \in \mathcal{L}(X^*, X)$ is non-negative if $\langle x, A_2x \rangle \geq 0$ for all $x \in X$.

Note that if $U_{t,s}$ is strongly continuous evolution family in reflexive space $X$ then $V_{s,t} = U_{t,s}^*$ is strongly continuous backward evolution family in $X^*$.

**Theorem.** Let $X$ be a reflexive Banach space and the following assumptions hold:

1. $\{U_{t,s}\}_{0 \leq s \leq t \leq T}$ is strongly continuous and uniformly bounded forward evolution family in $\mathcal{L}(X)$. Then $V_{s,t} = U_{t,s}^*$ is strongly continuous and uniformly bounded backward evolution family in $\mathcal{L}(X^*)$.
2. operator functions $C \in C_s(I; \mathcal{L}(X, X^*))$ and $B \in C_s(I; \mathcal{L}(X^*, X))$ 

3. $C(t) = C^*(t) \geq 0$ and $B(t) = B^*(t) \geq 0$ for all $t \in I$.

Then for all self-adjoint non-negative $G \in \mathcal{L}(X, X^*)$ the (backward) integral Riccati equation

$$P(t) = V_{t,T}G U_{T,t} + \int_t^T V_{t,s}\{C(s) - P(s)B(s)P(s)\}U_{s,t}ds$$

has a unique self-adjoint non-negative solution $P \in C_s(I; \mathcal{L}(X, X^*))$.

Some applications to the solvability of a system of forward-backward evolution linear equations

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t) \\ -C(t) & -A^*(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad x(0) = x_0, \quad y(T) = Gx(T) \quad t \in [0, T].$$

and mean-field game system of PDE will be given.

REFERENCES

1. N. Artamonov, Solvability of an Operator Riccati Integral Equation in a Reflexive Banach Space // Differential Equations, 2019, Vol. 55, No. 5, pp. 718–728