WEAKLY AMENABLE GROUPS

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ABSTRACT. We construct the first examples of finitely generated non–amenable groups whose left regular representations are not uniformly isolated from the trivial representation.

1. INTRODUCTION.

Recall that a locally compact group $G$ is called amenable if there exists a finitely additive measure $\mu$ on the set of all Borel subsets of $G$ which is invariant under the left action of the group $G$ on itself and satisfies $\mu(G) = 1$. The class of amenable groups, $AG$, has been introduced by von Neumann [16] in order to explain the Hausdorff–Banach–Tarski paradox and was investigated by a number of authors.

One of the most interesting characterizations of amenable groups was obtained by Hulaniski [11] in terms of $L^2$–representations.

Definition 1. One says that the left regular representation $L_G$ of a locally compact group $G$ on the Hilbert space $L^2(G)$ weakly contains the trivial representation, if for any $\varepsilon > 0$ and any compact subset $S \subseteq G$, there exists $v \in L^2(G)$ such that $\|v\| = 1$ and

\[ |\langle v, sv \rangle - 1| < \varepsilon \]

for any $s \in S$.

Theorem 2 (Hulaniski). A locally compact group $G$ is amenable if and only if the left regular representation of $G$ weakly contains the trivial representation.

Given a locally compact group $G$ and a compact subset $S \subseteq G$, we define $\alpha(G, S)$ as the supremum of all $\varepsilon \geq 0$ such that for any vector $v \in L^2(G)$ of norm $\|v\| = 1$, there exists an element $s \in S$ satisfying the inequality

\[ \|sv - v\| \geq \varepsilon. \]
In case the group \( G \) is discrete and finitely generated, the existence of a finite generating set \( S \) such that \( \alpha(G, S) > 0 \), implies the inequality \( \alpha(G, S') > 0 \) for any other generating set \( S' \) of \( G \). Thus it is natural to consider the quantity

\[
\alpha(G) = \inf_S \alpha(G, S),
\]

where \( S \) ranges over all finite generating sets of \( G \). The following definition can be found in [22]

**Definition 3.** The left regular representation of a finitely generated group \( G \) is said to be **uniformly isolated from the trivial representation** if \( \alpha(G) > 0 \).

Obviously one has

\[
(2) \quad \alpha(G) = 0
\]

for any finitely generated amenable group. Indeed, it is easy to check that (1) implies \( \|sv - v\| < \sqrt{2\varepsilon} \). Thus (2) follows from Theorem 1.2. On the other hand, it is not clear whether the equality (2) is equivalent to the amenability of the group \( G \). The following problem was suggested by Shalom in [22].

**Problem 4.** Is the left regular representation of any non–amenable finitely generated group uniformly isolated from the trivial representation?

In [22], the positive answer was obtained in the particular case of residually finite hyperbolic groups. However, the question remained open in general. The main purpose of the present note is to show that the answer is negative and can be obtained by using the methods developed in [20].

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2. **Main results**

The main results of the paper are gathered in this section. We call a finitely generated group \( G \) **weakly amenable** if it satisfies (2) and denote by \( WA \) the class of all weakly amenable groups.

Two families of non–amenable weakly amenable groups are constructed in the present paper. The idea of the first construction is similar to one from [5].

**Theorem 5.** Let \( A \) be a finitely generated abelian group. Suppose that there exist two monomorphisms \( \lambda, \mu : A \rightarrow A \) with the following properties.


1) \( \lambda \circ \mu \equiv \mu \circ \lambda \).
2) The subgroup generated by \( \lambda(A) \cup \mu(A) \) coincides with \( A \).
3) \( \lambda(A) \cup \mu(A) \neq A \).

Then the HNN–extension
\[
\langle A, t : t^{-1}\lambda(a)t = \mu(a), a \in A \rangle
\]
is a finitely generated weakly amenable non–amenable group.

Example 6. Suppose that \( A = \mathbb{Z} \) and \( \lambda, \mu \) are defined by \( \lambda(1) = m, \mu(1) = n \). If \( m, n \) are relatively prime, and \( |m| \neq 1, |n| \neq 1 \), one can easily verify the conditions of Theorem 2.1. Taking the HNN–extension, we obtain the Baumslag–Solitar group
\[
BS(m, n) = \langle a, t : t^{-1}a^mt = a^n \rangle.
\]
Using the Britton lemma on HNN–extensions [15, Ch. IV, Sec. 2], one can prove that the elements \( t \) and \( a^{-1}ta \) generates a free subgroup of rank 2. This shows that the class \( WA \) is not closed under the taking of subgroups.

In the last section of the present paper we give another way to construct a weakly amenable non–amenable group using limits of hyperbolic groups. The proof involves the tools of hyperbolic group theory developed in [18] and certain results from [20].

Recall that a locally compact group \( G \) is said to have property (T) of Kazhdan if the one–dimensional trivial representation is an isolated point of the set of all irreducible unitary representations of \( G \) endowed with the Fell topology (we refer to [12], [14] and [9] for more details). It follows easily from the definition and Hulaniski’s theorem that every discrete finitely generated amenable group having property (T) is finite. In contrast, we obtain the following unexpected result in the case of weakly amenable groups.

**Theorem 7.** There exists a 2–generated infinite periodic weakly amenable group \( Q \) having property (T) of Kazhdan. In particular, \( Q \) is non–amenable.

We also consider a variant of Day’s question which goes back to the papers [2], [16] and known as the so called ”von Neumann problem”. Let \( NF \) denote the class of all groups containing no non–abelian free subgroups, and \( AG \) denote the class of all amenable groups. Obviously \( AG \subseteq NF \) since any non–abelian free group is non–amenable and the class \( AG \) is closed under the taking of subgroups [16]. The question is whether \( NF = AG \).

Ol’shanskii [17] shown that certain groups constructed by him earlier (torsion groups with unbounded orders of elements in which all proper subgroups are cyclic) are non–amenable and thus the answer is negative. Further, in [11] Adian proved that the free Burnside groups \( B(m, n) \) of sufficiently large odd exponent \( n \) and rank
$m > 1$ are non–amenable. It is a natural stronger version of Day’s question, whether the inclusion

$$WA \cap NF \subset AG$$

is true. We note that all groups constructed in Theorem 2.1 contain non–abelian free subgroups (see Lemma 3.10 below). Furthermore, $B(m, n) \notin WA$ for any $m > 1$ and any $n$ odd and large enough, as follows from the main result of [21]. Thus these groups do not provide an answer. On the other hand the negative answer is an immediate consequence of Theorem 2.3.

**Corollary 8.** There exists a finitely generated weakly amenable non–amenable group which contains no non–abelian free subgroups.

In conclusion we note that our construction of the group $Q$ from Theorem 2.3 is closely related to the question whether any finitely generated group of exponential growth is of uniform exponential growth (see Section 4 for definitions). Originally, this problem was formulated in [8] and studied intensively during the last few years (we refer to [10] for survey). In [13], Koubi proved that the exponential growth rate $\omega(G)$ of every non–elementary hyperbolic group $G$ satisfies the inequality $\omega(G) > 1$. On the other hand, the following question is still open.

**Problem 9.** Is the set

$$\Omega_H = \{\omega(G) : G \text{ is non–elementary hyperbolic}\}$$

bounded away from the identity?

In Section 4, we observe that the negative answer would imply the existence of a finitely generated group having non–uniform exponential growth.

3. **Non–Hopfian weakly amenable groups**

Let $F_m$ be the free group of rank $m$, $X = \{x_1, x_2, \ldots, x_m\}$ a free generating set of $F_m$. We begin this section by describing the Grigorchuk’s construction of a topology on $G_m$, the set of all normal subgroups of $F_m$ (or, equivalently, on the set of all group presentations with the same generating set).

**Definition 10.** The Cayley graph $\Gamma = \Gamma(G, S)$ of a group $G$ generated by a set $S$ is an oriented labeled 1–complex with the vertex set $V(\Gamma) = G$ and the edge set $E(\Gamma) = G \times S$. An edge $e = (g, s) \in E(\Gamma)$ goes from the vertex $g$ to the vertex $gs$ and has the label $\phi(e) = s$. As usual, we denote the origin and the terminus of the edge $e$, i.e., the vertices $g$ and $gs$, by $\alpha(e)$ and $\omega(e)$ respectively. One can endow the group $G$ (and, therefore, the vertex set of $\Gamma$) with a length function by assuming $\|g\|_S$, the length of an element $g \in G$, to be equal to the length of a shortest word in the alphabet $S \cup S^{-1}$ representing $g$. 

Let $N \in \mathcal{G}_m$. To simplify our notation we will identify the set $X$ with the generating set of the quotient group $F_m/N$ naturally obtained from $X$. Now let $N_1, N_2$ be two normal subgroups of $F_m$ and $G_1 = F_m/N_1$, $G_2 = F_m/N_2$. By $B_i(r), i = 1, 2$, we denote the ball of radius $r$ around the identity in the Cayley graph $\Gamma_i = \Gamma(G_i, X)$, i.e., the oriented labeled subgraph with the vertex set
\[ V(B_i(r)) = \{ g \in G_i : \|g\|_{X_i} \leq r \} \]
and the edge set
\[ E(B_i(r)) = \{ e \in E(\Gamma_i) : \alpha(e) \in V(B_i(r)) \text{ and } \omega(e) \in V(B_i(r)) \} \].
One says that the groups $G_1$ and $G_2$ are locally $r$-isomorphic (being considered quotients of $F_m$) and writes $G_1 \sim_r G_2$ if there exists a graph isomorphism
\[ \iota : B_1(r) \to B_2(r) \]
that preserves labels and orientation.

**Definition 11.** For every $N \in \mathcal{G}_m$ and $r \in \mathbb{N}$, we consider the set
\[ W_r(N) = \{ L \in \mathcal{G}_m : F_m/N \sim_r F_m/L \} \].
One defines the topology on $\mathcal{G}_m$ by taking the collection of the sets $W_r(N)$ as the base of neighborhoods.

**Example 12.** Suppose that $\{N_i\}$ is a sequence of normal subgroups of $F_m$ such that $N_1 \geq N_2 \geq \ldots$. Then the limit of the sequence coincides with $\bigcap_{i=1}^{\infty} N_i$. Symmetrically if $N_1 \leq N_2 \leq \ldots$, then the limit is the union $\bigcup_{i=1}^{\infty} N_i$. The proof is straightforward and is left as an exercise to the reader.

We need the following result, which is proved in [20] (up to notation).

**Theorem 13.** Suppose that $\{N_i\}_{i \in \mathbb{N}}$ is a sequence of elements of $\mathcal{G}_m$ which converges to an element $N \in \mathcal{G}_m$. If the group $G = F_m/N$ is amenable, then
\[ \lim_{i \to \infty} \alpha(F_m/N_i, X) = 0. \]

**Remark 14.** Let $\mathcal{AG}_m$ denote the subset of all elements $N \in \mathcal{G}_m$ such that the quotient group $F_m/N$ is amenable. Essentially the theorem says that the map $\alpha : \mathcal{G}_m \to [0, +\infty)$ which takes each $N \in \mathcal{G}_m$ to $\alpha(F_m/N, X)$ is continuous at any point $N \in \mathcal{AG}_m$. It is not hard to see that $\alpha$ is not continuous at arbitrary point of $\mathcal{G}_m$. Indeed, consider the sequence of subgroups $N_1 \geq N_2 \geq \ldots$ of finite index in $F_m$ such that
\[ \bigcap_{i=1}^{\infty} N_i = \{1\} \]
(such a sequence exists since any free group is residually finite). One can easily check that \( s \) implies
\[
\lim_{{i \to \infty}} N_i = \{1\}
\]
(see Example 3.3). Since the group \( F_m \) is non–amenable whenever \( m > 1 \), we have \( \alpha(\{1\}) > 0 \). However, \( \alpha(F_m/N_i, X) = 0 \) for any \( i \), as the quotient groups \( F_m/N_i \) are finite (and, therefore, amenable).

Now suppose that \( G \) is the group defined by \( 3 \). The following four lemmas are proved under the assumptions of Theorem 2.1. Consider the homomorphism \( \phi: G \to G \) induced by \( \phi(t) = t \) and \( \phi(a) = \lambda(a) \) for every \( a \in A \).

**Lemma 15.** The homomorphism \( \phi \) is well–defined.

**Proof.** We have to check that for any relation \( R = 1 \) of the group \( G \) one has \( \phi(R) = 1 \) in \( G \). There are two possibilities for \( R \).

1) First suppose that \( R = 1 \) is a relation of the group \( A \). Since the restriction of \( \phi \) to \( A \) coincides with the monomorphism \( \lambda \), we have \( \phi(R) = \lambda(R) = 1 \).

2) Assume that \( R \) has the form \( (\lambda(a))^t(\mu(a))^{-1} \). Taking into account the first condition of Theorem 2.1, we obtain
\[
\phi((\lambda(a))^t(\mu(a))^{-1}) = (\lambda \circ \lambda(a))^t(\lambda \circ \mu(a))^{-1} = \mu \circ \lambda(a)(\mu \circ \lambda(a))^{-1} = 1.
\]
\( \square \)

**Lemma 16.** The map \( \phi \) is surjective.

**Proof.** Observe that \( G \) is generated by \( t \) and \( A \). As \( t \in \phi(G) \), it suffices to prove that \( A \leq \phi(G) \). Clearly we have \( \lambda(A) = \phi(A) \in \phi(G) \) and \( \mu(A) = (\lambda(A))^t \in \phi(G) \). It remains to refer to the second condition of Theorem 2.1. \( \square \)

Let us denote by \( \phi^i \) the \( i \)–th power of \( \phi \) and by \( N_i \) its kernel. Put \( N = \bigcup_{{i=1}}^{\infty} N_i \).

Obviously the group \( \overline{G} = G/N \) is generated by the images of \( a \) and \( t \) under the natural homomorphism \( G \to \overline{G} \). To simplify our notation we will denote these images by \( a \) and \( t \) as well.

**Lemma 17.** The group \( \overline{G} \) is an extension of an abelian group by a cyclic one.

**Proof.** We denote by \( B \) the kernel of the natural homomorphism \( \overline{G} \to \langle t \rangle \). Let us show that \( B \) is abelian. It is clear that \( B \) is generated by the set \( \{a^i : a \in A, i \in \mathbb{Z}\} \). Therefore, it is sufficient to show that \( [a^i, a^j] = 1 \) for any \( a \in A, i, j \in \mathbb{Z} \). Without
loss of generality we can assume that $i \geq j$. Moreover, conjugating by a suitable power of $t$, we can assume $j = 0$. In these settings, we have

\[ \phi^i([a^t, a]) = [(\lambda^i(a))^t, \lambda^i(a)] = [\mu^i(a), \lambda^i(a)] = 1 \]

as $A$ is abelian. Therefore, the element $[a^t, a]$ belongs to $N_i$ and thus its image in $\overline{G}$ is trivial.

We note that in certain particular cases (including, for example, non–Hopfian Baumslag–Solitar groups) Lemma 3.8 follows from a result of Hirshon [6]. As any abelian group is amenable and the class of amenable groups is closed under group extensions, Lemma 3.8 yields

**Corollary 18.** The group $\overline{G}$ is amenable.

**Lemma 19.** The group $G$ contains a non–abelian free subgroup.

**Proof.** According to the third condition of Theorem 2.1 there exists an element $a \in A \setminus (\lambda(A) \cup \mu(A))$. The elements $t$ and $a^{-1}ta$ generate the free group of rank 2 by the Britton lemma on HNN–extensions. $\square$

**Proof of Theorem 2.1.** Let us note that the sequence \( \{N_i\} \) converges to $N$. Applying Corollary 3.9 and Theorem 3.4, we obtain $\lim_{i \to \infty} \alpha(F/N_i, X) = 0$. On the other hand, $F/N_i \cong G$, this means that $\alpha(G) = 0$, i.e., $G$ is weakly amenable. Finally, $G$ is non–amenable according to Lemma 3.10. $\square$

4. **Common quotient groups of all non–elementary hyperbolic groups.**

Let us recall just one of a number of equivalent definitions of hyperbolicity. A group $G$ with a finite generating set $X$ is *hyperbolic* (in the sense of Gromov) if its Cayley graph $\Gamma = \Gamma(G, X)$ is a hyperbolic space with respect to the natural metric. This means that any geodesic triangle in $\Gamma$ is $\delta$–thin for a fixed constant $\delta$, i.e., each of its sides belongs to the closed $\delta$–neighborhood of the union of other two sides.

It has been mentioned by Gromov [7] (see also [9]), that an element $g$ of infinite order in a hyperbolic group $G$ is contained in a unique maximal elementary subgroup $E_G(g)$ (elementary closure of $g$). For a subgroup $H$ of a hyperbolic group $G$, its elementarizer $E_G(H)$ is defined as $\cap E_G(h)$, where $h$ ranges over all elements of infinite order in $H$. If the subgroup $H$ is non–elementary, $E_G(H)$ is the unique maximal finite subgroup of $G$ normalized by $H$ [18, Proposition 1]; notice also that $E_G(G)$ is the kernel of the action of $G$ on the hyperbolic boundary $\partial G$ induced by left multiplication on $G$. 


The following is the simplification of Theorem 2 from [18] (see also [19, Lemma 5.1]).

**Lemma 20.** Let $H_1, \ldots, H_k$ be non-elementary subgroups of a hyperbolic group $G$ such that $E_G(H_1) = \ldots = E_G(H_k) = 1$. Then there exists a non-elementary hyperbolic quotient $K$ of $G$ such that the image of each subgroup $H_1, \ldots, H_k$ under the natural epimorphism $G \to K$ coincides with $K$.

**Corollary 21.** Let $P_1, \ldots, P_k$ be non-elementary hyperbolic groups. Then there exists a non-elementary hyperbolic group $Q$ that is a homomorphic image of $P_i$ for every $i = 1, \ldots, k$.

**Proof.** The proof can be extracted from the one of Theorem 2 in [19]. Here we provide it for convenience of the reader. Let us set $H_i = P_i / E_{P_i}(P_i)$. Clearly $E_{H_i}(H_i) = 1$, as $E_{P_i}(P_i)$ is the maximal normal finite subgroup of $P_i$. Moreover, since any quotient of a non-elementary hyperbolic group modulo a finite normal subgroup is also a non-elementary hyperbolic group [4, Corollary 23(ii)], it follows that $H_i$ is non-elementary hyperbolic. Now we take the free product 

$$G = H_1 \ast \ldots \ast H_k.$$ 

It is easy to check that $E_G(H_i) = 1$ for every $i$ as there are no finite subgroups of $G$ normalized by $H_i$. It remains to apply Lemma 4.1. \hfill \Box

We need one more lemma (the proof can be found in [18]).

**Lemma 22.** Let $G$ be a non-elementary hyperbolic group, $g$ an element of $G$. Then there exists $N \in \mathbb{N}$ such that the quotient group of $G$ modulo the normal closure of $g^N$ is non-elementary and hyperbolic.

Now we are going to describe the main construction of the present section.

**Theorem 23.** There exists a 2-generated infinite periodic group $Q$ such that for every non-elementary hyperbolic group $H$, there is an epimorphism $\rho : H \to Q$.

**Proof.** Since any hyperbolic group is finitely presented, the set of all non-elementary hyperbolic groups is countable. Let us enumerate this set $G_1, G_2, \ldots$ and elements of the first group $G_1 = \{g_1, g_2, \ldots\}$. Consider the following diagram, which is constructed by induction.

$$\begin{array}{cccccccc}
G_1 & \to & G_2 & \to & \ldots & \to & G_k & \to & \ldots \\
\downarrow^{\pi_1} & & \downarrow^{\pi_2} & & \ldots & & \downarrow^{\pi_k} \\
Q_1 & \to & R_1 & \phi_2 & Q_2 & \psi_2 & \ldots & R_{k-1} & \phi_k & Q_k & \psi_k & R_k & \phi_{k+1} & \ldots \\
\end{array}$$

Suppose $G_1 = Q_1$ and let $\pi_1$ denote the corresponding natural isomorphism. Assume that we have already defined the groups $Q_i$, $R_{i-1}$ and homomorphisms $\phi_i : R_{i-1} \to$
$Q_i$, $\psi_{i-1} : Q_{i-1} \to R_{i-1}$ for all $i \leq k$. Denote by $\tau_k : G_1 \to Q_k$ the composition $\phi_k \psi_{k-1} \ldots \phi_2 \psi_1 \pi_1$ and by $\bar{g}_i$ the image of $g_i$ in $Q_k$ under $\tau_k$. According to Lemma 4.3, there exists $N_i \in \mathbb{N}$ such that the quotient $Q_k / \langle \bar{g}_i^{N_i} \rangle Q_k$ is a non-elementary hyperbolic group. We set

$$R_k = Q_k / \langle \bar{g}_i^{N_i} \rangle Q_k$$

and denote by $\psi_k$ the natural homomorphism from $Q_k$ to $R_k$. Further, by Corollary 4.2, there is a non-elementary hyperbolic group $Q_{k+1}$ such that there exist epimorphisms

$$\phi_{k+1} : R_k \to Q_{k+1} \quad \text{and} \quad \pi_{k+1} : G_{k+1} \to Q_{k+1}.$$ 

The inductive step is completed.

Let us denote by $U_k$ the kernel of $\tau_k$. Evidently we have $\{1\} = U_1 \leq U_2 \leq \ldots$. Set $U = \bigcup_{i=1}^{\infty} U_i$ and consider the quotient group $Q = G_1 / U$. Note that one can assume $G_1$ to be 2-generated without loss of generality. Further, $Q$ is a quotient group of $Q_i$ for all $i$, hence $Q$ is a quotient of $G_i$ for all $i$. The periodicity of $Q$ follows directly from our construction. It remains to show that $Q$ is infinite. To do this, let us suppose that $Q$ is finite. Then $Q$ is finitely presented. Therefore, $Q_i$ is a quotient group of $Q$ for all $i$ big enough. In particular, $Q_i$ is elementary whenever $i$ is sufficiently big and we get a contradiction. The theorem is proved.

Let us denote by $\mathcal{H}_m$ the subset of all $N \in G_m$ such that $F_m / N$ is non-elementary and hyperbolic. Recall also that $\text{AG}_m$ denotes the subset of all $N \in G_m$ such that $F_m / N$ is amenable. The following two observations from [20] plays the crucial role in the studying of the group $Q$.

**Theorem 24.** For every $m \geq 2$, the intersection of the closure of $\mathcal{H}_m$ (with respect to the Cayley topology on $G_m$) and $\text{AG}_m$ is non-empty.

**Lemma 25.** Suppose that $G$ is a finitely generated group and $\phi : G \to P$ is a surjective homomorphism onto a group $P$. Then $\alpha(G) \geq \alpha(P)$.

Now we want to show that the group $Q$ from Theorem 4.4 has all properties listed at Theorem 2.3.

**Proof of Theorem 2.3.** By Theorem 3.4 and Theorem 4.5, there is a sequence of elements $N_i \in \mathcal{H}_2$, $i \in \mathbb{N}$, such that

$$\lim_{i \to \infty} \alpha(F_2 / N_i) = 0.$$ 

Let us denote by $G_i$ the quotient group $F_2 / N_i$. According to Theorem 4.4, there exists an epimorphism $\rho_i : G_i \to Q$ for every $G_i$. Combining Lemma 4.6 and [20], we obtain $\alpha(Q) = 0$. As is well known, there are non-elementary hyperbolic groups having property (T) of Kazhdan (for instance, uniform lattices in $Sp(n, 1)$). Since
the class of Kazhdan groups is closed under the taking of quotients, the group $Q$ has the property $T$. Recall that any discrete amenable Kazhdan group is finite; taking into account the infiniteness of $Q$, we conclude that $Q$ is non–amenable. □

In conclusion we discuss certain relations with growth functions of hyperbolic groups. The growth function $\gamma^X_G : \mathbb{N} \rightarrow \mathbb{N}$ of a group $G$ generated by a finite set $X$ is defined by the formula

$$\gamma^X_G(n) = \text{card} \{ g \in G : \|g\|_X \leq n \},$$

where $\|g\|_X$ denotes the word length of $g$ relative to $X$. The exponential growth rate of $G$ with respect to $X$ is the number

$$\omega(G, X) = \lim_{n \to \infty} \sqrt[n]{\gamma^X_G(n)}.$$

The above limit exists by submultiplicativity of $\gamma^X_G$. The group $G$ is said to be of exponential growth (respectively of subexponential growth) if $\omega(G, X) > 1$ (respectively $\omega(G, X) = 1$) for some generating set $X$.

It is easy to see that above definitions are independent of the choice of a generating set in $G$. Let us consider the quantity

$$\omega(G) = \inf_X \omega(G, X),$$

where the infimum is taken over all finite generating sets of $G$. One says that $G$ has uniform exponential growth if

$$(7) \quad \omega(G) > 1.$$ 

It is an open question whether any group of exponential growth satisfies (7). We observe that Theorem 4.4 provides an approach to the solution of this problem.

Lemma 26. Let $G$ be a group generated by a finite set $X$ and $\phi : G \rightarrow P$ be an epimorphism. Then $\omega(G, X) \geq \omega(P, \phi(X))$.

Proof. This observation is well known and quite trivial. The proof follows easily from the inequality $\|g\|_X \geq \|\phi(g)\|_{\phi(X)}$. We leave details to the reader. □

Obviously Lemma 4.7 and Theorem 4.2 yield the following.

Corollary 27. Suppose that for every $\varepsilon > 0$, there exists a non–elementary hyperbolic group $H$ such that $\omega(H) < 1 + \varepsilon$. Then the group $Q$ from Theorem 4.1 has non–uniform exponential growth, i.e., $\omega(Q, X) > 1$ for any finite generating set $X$ of $Q$ but $\omega(Q) = 1$. 
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