Remarks on the global large solution to the three-dimensional incompressible Navier-Stokes equations

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Abstract: In this paper, we derive a new smallness hypothesis of initial data for the three-dimensional incompressible Navier-Stokes equations. That is, we prove that there exist two positive constants $c_0, C_0$ such that

$$
\|u_0^1 + u_0^2, u_0^3\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} \exp\{C_0(\|u_0^2\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,2}} + \|u_0\|_{\dot{B}^{-\frac{1}{2}}_{\infty,1}})\} \leq c_0,
$$

then (1.1) has a unique global solution. As an application we construct two family of smooth solutions to the Navier-Stokes equations whose $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ norm can be arbitrarily large.

Keywords: Incompressible Navier-Stokes equations; Large solution.

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1 Introduction

In this paper, we focus on the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$

$$
(\text{NS}) \quad \begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \\
\text{div} u = 0, \\
u(0, x) = u_0.
\end{cases}
$$

(1.1)

where $u = (u^1(t, x), u^2(t, x), u^3(t, x)) \in \mathbb{R}^3$ denotes the divergence free velocity field and $p \in \mathbb{R}$ is the scalar pressure.

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Firstly, it should be noticed that the incompressible Navier-Stokes system (1.1) is translation and scaling invariant: if \((u, \pi)\) is a solution of (1.1) on \([0, T] \times \mathbb{R}^3\), then, for any positive \(\lambda\), the scaled triplet \((u, \pi)\lambda\) defined by

\[
(u, \pi)\lambda(x, t) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 \pi(\lambda x, \lambda^2 t))
\]

is also a solution of (1.1) on \([0, \lambda^{-2} T] \times \mathbb{R}^3\). Thus, this leads to the notion of critical spaces for well-posedness, that is, an adapted space must be translation and scaling invariant in the following sense: \(||(u, \pi)\lambda||_X \approx ||(u, \pi)||_X\). The reader may check that the following spaces have the above invariance

\[
\dot{H}^{\frac{2}{d}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,1}^{\frac{3}{p} - 1}(\mathbb{R}^3)|_{p < \infty} \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3).
\]

Now, let us recall some important progress about the global existence results for small data. In his seminal work, Leray [14] proved in 1934 that if \(\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}\) is small enough, then there exists a global regular solution of (NS). Then Fujita–Kato [11] proved in 1964 that if \(\|u_0\|_{\dot{H}^{\frac{2}{d}}}\) is small enough, then there exists a unique global solution in the space \(C_b(\mathbb{R}^+; \dot{H}^{\frac{2}{d}}) \cap L^4(\mathbb{R}^+; \dot{H}^1)\). Indeed, the theorem of Fujita–Kato [11] allows to construct local in time unique solution to (1.1) with initial data in the Lebesgue space \(L^3(\mathbb{R}^3)\) [17]. Moreover, if the initial norm \(\|u_0\|_{L^3}\) is sufficiently small, then the strong solution to (1.1) exists globally in time. The above result was extended by Cannone–Meyer–Planchon [8] for initial data in Besov spaces with negative index. More precisely, they proved that if the initial data belongs to the Besov space, \(\dot{B}_{p,1}^{\frac{3}{p} - 1}(\mathbb{R}^3)\) for some \(p \in (3, \infty)\) and its norm is sufficiently small, then (1.1) has a unique global solution. The typical example of such kind of initial data reads

\[
u_0, \varepsilon(x) = \varepsilon^{-\alpha} \sin \left(\frac{x_3}{\varepsilon}\right) (\partial_2 \phi, \partial_1 \phi, 0)(x) \quad \text{with} \quad \alpha \in (0, 1), \quad \phi \in S(\mathbb{R}^3; \mathbb{R}).
\]

We remark that this type of initial data is not small in either \(\dot{H}^{\frac{2}{d}}(\mathbb{R}^3)\) or \(L^3(\mathbb{R}^3)\).

Koch–Tataru in 2001 [18] proved that given initial data in the derivatives of BMO space and its norm \(\text{BMO}^{-1}\) is sufficiently small, then (1.1) has a unique global solution. We point out that the largest space and the norm of which is scaling invariant under (1.2), is \(\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)\). Hmidi–Li [12] showed that smallness of \(\dot{B}_{\infty, \infty}^{-1}\) norm of solution to d-dimensional \((d \geq 3)\) incompressible Navier-Stokes prevents blowups. Moreover, Bourgain–Pavlović [2] proved that (1.1) is actually ill-posed with initial data in \(\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)\). Let us simply notice that the above norms have the following relation

\[
\|u_0\|_{\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)} \lesssim \|u_0\|_{\text{BMO}^{-1}(\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{B}_{\infty, 2}^{-1}(\mathbb{R}^3)}
\]

with

\[
\|u_0\|_{\dot{B}_{\infty, 1}^{-1}(\mathbb{R}^3)} = \sup_{t > 0} t^\frac{d}{2} \|e^{t \Delta} u_0\|_{L^\infty} \quad \text{and} \quad \|u_0\|_{\dot{B}_{\infty, 2}^{-1}(\mathbb{R}^3)} = \|e^{t \Delta} u_0\|_{L^2(\mathbb{R}^+, L^\infty(\mathbb{R}^3))}
\]

A remark due to Y. Meyer [19] is that the norm in such a space is always greater than the norm in the Besov space \(\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)\). This leads to the definition of a large initial data for the
incompressible Navier-Stokes equations: A divergence free vector field $u_0$ is a large initial data for the incompressible Navier-Stokes system if its $\dot{B}^{-1}_{\infty,\infty}$ norm is large. Chemin–Gallagher [3] proved that if a certain nonlinear function of the initial data is small enough, then there is a global solution to the Navier-Stokes equations (1.1) in a Koch-Tataru type space. Meanwhile, they provide an example of initial data satisfying that nonlinear smallness condition, but whose norm is arbitrarily large in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. For more results of large initial data which generate unique global solutions to (1.1), we refer the reader to see [4, 5, 6, 7, 13, 16] and the references therein. Recently, Li–Yu–Zhu–Yin [15] obtained the global large solution for a special initial data with the first two component of the initial velocity field being large in Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. One can mention that Paicu–Zhang [22] proved the global well-posedness of incompressible inhomogeneous Navier-Stokes equation with the third component of the initial velocity field being large. Motivated by this work [3, 22, 15], we continue to study the global behavior of solutions to the incompressible Navier-Stokes equations (1.1) for arbitrary large initial in the present paper.

The main result of the paper read as follows:

**Theorem 1.1** Let $3 < p < 6$. Then there exist two constants $\delta, C > 0$ such that for any $u_0 = (u_0^1, u_0^2, u_0^3) \in \dot{B}^{-3}_{p,1}$ satisfying the condition

$$
\|u_0^1 + u_0^2, u_0^3\|_{\dot{B}^{-1+3/p}_{p,1}} < \frac{C}{\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}}} \exp \{\frac{C}{\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}}} \}
$$

then (1.1) admits a unique global solution $u$. 

**Remark 1.1** Our obtained result in Theorem 1.1 improves considerably the corresponding result in [21] when the initial data belongs to Besov spaces $\dot{B}^{-1+3/p}_{p,1}, 3 < p < 6$. From Example 1.1, we can obtain the global solution of the initial data as (1.4). This implies that our obtained result in Theorem 1.1 partially covers the result in [3] when $\alpha \in (\frac{6}{7}, 1)$, and our proof is more brief than that in [3].

Let us present some examples of initial data the norm of which are big in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, yet they satisfy the smallness condition (1.3). The first case is that the third component of the initial velocity field is large, and the first two components of the initial velocity field are small.

**Example 1.1** Let $p \in [5, 6)$ and $\alpha \in (\frac{n}{p+2}, 1)$. According to [3], we take the following initial data

$$
u_{0,\varepsilon}(x) = \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}} \varepsilon^{-1} \cos \left(\frac{x_1}{\varepsilon}\right) \varepsilon^\alpha \partial_3 \phi(x_1, x_2, x_3).
$$

From Lemma 3.1 of [3], we have

$$
\|\nabla \phi(x_1, x_2, x_3)\|_{\dot{B}^{-1+3/p}_{p,1}} \lesssim \varepsilon^{\frac{1}{p}+\frac{3}{p}},
$$

$$
\|\nabla \phi(x_1, x_2, x_3)\|_{\dot{B}^{-1}_{\infty,\infty}} \gtrsim \varepsilon.
$$
Thus, we can get
\[
\|u_{0, \varepsilon}\|_{\dot{B}_{\infty, \infty}^{-1}} \approx \|u_{0, \varepsilon}\|_{\dot{B}_{1, 1}^{-1}} \approx \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p}},
\]
\[
\|u_{0, \varepsilon}^3\|_{\dot{B}_{p, 1}^{-1+\frac{3}{p}}} \lesssim \varepsilon^{-\frac{2}{p} + \frac{2}{p}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p}},
\]
\[
\|u_{0, \varepsilon}^1, u_{0, \varepsilon}^2\|_{\dot{B}_{p, 1}^{-1+\frac{3}{p}}} \lesssim \varepsilon^{\alpha - \frac{2}{p} + \frac{2}{p}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{p}}.
\]
Then, direct calculations show that the left side of (1.3) becomes
\[
C \left(\log \frac{1}{\varepsilon}\right)^{\frac{2}{5}} \exp \left(C \left(\log \frac{1}{\varepsilon}\right)^{\frac{2}{5}}\right) \varepsilon^{\alpha - \frac{2}{p} + \frac{2}{p}},
\]
which implies (1.1) have a global solution for \(\varepsilon\) sufficiently small.

The second case is that the first two components of the initial velocity field are large and the third component of the initial velocity field is small.

**Example 1.2** Let \(p \in (3, 6)\). According to [15], we take the initial data \(u_{0, \varepsilon} = (\partial_2 a_{0, \varepsilon}, -\partial_1 a_{0, \varepsilon}, 0)\) with
\[
a_{0, \varepsilon}(x_1, x_2, x_3) = \varepsilon^{-1} \left(\log \log \frac{1}{\varepsilon}\right)^{\frac{1}{p}} \chi(x_1, x_2) \phi(x_3),
\]
where the smooth functions \(\chi, \phi\) satisfying \(\hat{\chi}(-\xi_1, -\xi_2) = \hat{\chi}(\xi_1, \xi_2), \hat{\phi}(-\xi_3) = \hat{\phi}(\xi_3),\)
\[
\text{supp} \hat{\chi} \subset \tilde{C}, \quad \hat{\chi}(\xi_1, \xi_2) \in [0, 1]; \quad \hat{\chi}(\xi_1, \xi_2) = 1 \quad \text{for} \quad (\xi_1, \xi_2) \in \tilde{C}_1,
\]
and
\[
\hat{\phi}(\xi_3) = 0 \quad \text{for} \quad |\xi_3| \in \left[\frac{2\sqrt{3}}{3}, \frac{3\sqrt{2}}{4}\right], \quad \hat{\phi}(\xi_3) \in [0, 1]; \quad \hat{\phi}(\xi_3) = 1 \quad \text{for} \quad |\xi_3| \in \left[\frac{\sqrt{34}}{6}, \frac{\sqrt{17}}{4}\right],
\]
where
\[
\tilde{C} \triangleq \left\{ \xi \in \mathbb{R}^2 : |\xi_1 - \xi_2| \leq \varepsilon, \frac{8}{9} \leq \xi_1^2 + \xi_2^2 \leq \frac{9}{8} \right\},
\]
\[
\tilde{C}_1 \triangleq \left\{ \xi \in \mathbb{R}^2 : |\xi_1 - \xi_2| \leq \frac{\varepsilon}{2}, \frac{17}{18} \leq \xi_1^2 + \xi_2^2 \leq \frac{17}{16} \right\}.
\]
In fact, one has
\[
||\hat{a}_0||_{L^{\infty} \rightarrow \tau} \approx \varepsilon^{-\frac{1}{p}} \left(\log \log \frac{1}{\varepsilon}\right)^{\frac{1}{p}}.
\]
Then, direct calculations show that the left side of (1.3) becomes
\[
C \varepsilon^{1 - \frac{2}{p}} \left(\log \log \frac{1}{\varepsilon}\right) \exp \left(C \log \log \frac{1}{\varepsilon}\right),
\]
which implies (1.1) have a global solution for \(\varepsilon\) sufficiently small. From Remark 2.1 of [15], it also holds
\[
||u_0||_{\dot{B}_{\infty, \infty}^{-1}} \approx ||u_0||_{L^{\infty}} \gtrsim \left(\log \log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}.
\]
2 Littlewood-Paley Analysis

Throughout this paper, we will denote by $C$ any constant which may change from line to line and write $A \lesssim B$ if $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We also shall use the abbreviated notation $\|f_1, \ldots, f_n\|_X = \|f_1\|_X + \cdots + \|f_n\|_X$ for some Banach space $X$.

Next, we recall the Littlewood-Paley theory, the definition of homogeneous Besov spaces and some useful properties.

Let us start by introducing the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $C = \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that
\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \text{for all } \xi \neq 0.
\]

The frequency localization operator $\hat{\Delta}_j$ and $\hat{S}_j$ are defined by
\[
\hat{\Delta}_j f = \varphi(2^{-j} D) f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f) \quad \text{and} \quad \hat{S}_j f = \sum_{k \leq j-1} \hat{\Delta}_k f \quad \text{for } j \in \mathbb{Z}.
\]

With a suitable choice of $\varphi$, one can easily verify that
\[
\hat{\Delta}_j \hat{\Delta}_k f = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \hat{\Delta}_j (\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if } |j - k| \geq 5.
\]

Next we recall Bony’s decomposition from [1]:
\[
uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v),
\]
with
\[
\hat{T}_u v = \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{\Delta}_j v, \quad \hat{\Delta}_j v = \sum_{|j' - j| \leq 1} \hat{\Delta}_{j'} v.
\]

**Definition 2.1** We denote by $\mathcal{Z}'(\mathbb{R}^d)$ the dual space of $\mathcal{Z}(\mathbb{R}^d)$, where we set
\[
\mathcal{Z}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d) : D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \right\}.
\]
Then we have the formal homogenous Littlewood-Paley decomposition
\[
f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f, \quad \forall f \in \mathcal{Z}'(\mathbb{R}^d).
\]

The operators $\hat{\Delta}_j$ help us recall the definition of the homogenous Besov space (see [1]).

**Definition 2.2** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by
\[
\dot{B}^s_{p,r} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,r}} \triangleq \left\| (2^{ks}\|\hat{\Delta}_k f\|_{L^p})_{k \in \mathbb{Z}} \right\|_{\ell^r} < +\infty \right\}.
\]
It should be noted that a distribution \( f \in \tilde{B}_{p,r}^s \) if and only if there exist a constant \( C > 0 \) and a non-negative sequence \( \{d_k\}_{k \in \mathbb{Z}} \) such that
\[
\forall k \in \mathbb{Z}, \quad \|\dot{\Delta}_k f\|_{L^p} \leq C d_k 2^{-ks} \|f\|_{B^s_{p,r}} \quad \text{with} \quad \|d_k\|_{\ell^1} = 1.
\]

**Lemma 2.1** Let \( t > 0 \) and \( 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \) satisfying
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad t = t_1 + t_2.
\]
Then there exists a positive constant \( C \) such that
\[
\|[\dot{T}_j g]\|_{\tilde{B}^s_{p,r}} \leq C \|f\|_{L^{p_1}} \|g\|_{\tilde{B}^s_{p_2,r}}, \tag{2.5}
\]
\[
\|[\dot{T}_j g]\|_{\tilde{B}^{s-r}_p} \leq C \|f\|_{\tilde{B}_{p_1}^{s-t_1}} \|g\|_{\tilde{B}_{p_2}^{r_2}}, \tag{2.6}
\]
\[
\|[\dot{R}(f, g)]\|_{\tilde{B}^s_{p,r}} \leq C \|f\|_{\tilde{B}_{p_1}^{s_1}} \|g\|_{\tilde{B}_{p_2}^{s_2}}. \tag{2.7}
\]

**Proof of Lemma 2.1** The third conclusion (2.7) is the direct result of Theorem 2.52 in [1]. The other conclusions come essentially from [1], we give the proof here for completeness. Then, by using the properties of spectral localization of the Littlewood-Paley decomposition, one has
\[
\|\dot{\Delta}_j \dot{T}_u v\|_{L^p} \leq C \|\sum_{|k-j| \leq 4} \dot{\Delta}_j (\dot{S}_{k-1} u \dot{\Delta}_k v)\|_{L^p}
\]
\[
\leq C \sum_{|k-j| \leq 4} \|\dot{S}_{k-1} u\|_{L^{p_1}} \|\dot{\Delta}_k v\|_{L^{p_2}}
\]
\[
\leq C 2^{-j s} \|u\|_{L^{p_1}} \sum_{|k-j| \leq 4} 2^{(j-k)s} 2^{jk} \|\dot{\Delta}_k v\|_{L^{p_2}}
\]
\[
\leq C d_j 2^{-j s} \|u\|_{L^{p_1}} \|v\|_{\tilde{B}^s_{p_2,r}} \quad \text{with} \quad \|d_j\|_{\ell^1} = 1 \tag{2.8}
\]
and
\[
\|\dot{\Delta}_j \dot{T}_u v\|_{L^p} \leq C \|\sum_{|k-j| \leq 4} \dot{\Delta}_j (\dot{S}_{k-1} u \dot{\Delta}_k v)\|_{L^p}
\]
\[
\leq C \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} \|\dot{\Delta}_{k'} u\|_{L^{p_1}} \|\dot{\Delta}_k v\|_{L^{p_2}}
\]
\[
\leq C 2^{-j(s-t)} \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} 2^{(k-j)(-s+t)} 2^{k' t} \|\dot{\Delta}_{k'} u\|_{L^{p_1}} 2^{k s} \|\dot{\Delta}_k v\|_{L^{p_2}}
\]
\[
\leq C d_j 2^{-j(s-t)} \|u\|_{\tilde{B}_{p_1}^{s-t}} \|v\|_{\tilde{B}^s_{p_2,r}}. \tag{2.9}
\]
(2.8) and (2.9) result the desired (2.5) and (2.6), respectively. This ends the proof of Lemma 2.1.

Next, we present the following product estimate which will be used in the sequel.

**Lemma 2.2** [9] Let \( 2 \leq p \leq \infty, \; s_1 \leq \frac{d}{p} \) and \( s_2 \leq \frac{d}{p} \) with \( s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\} \). Then there holds
\[
\|[fg]\|_{\tilde{B}^{s_1+s_2}_{p_1}} \leq C \|f\|_{\tilde{B}^{s_1}_{p_1}} \|g\|_{\tilde{B}^{s_2}_{p_1}}.
\]
Finally, we recall the optimal regularity estimates for the heat equations.

**Lemma 2.3** [10] Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Assume that $u_0 \in \dot{B}^s_{p,r}$ and $G \in \dot{L}^{q_2}_T (\dot{B}^{s+\frac{2}{q_2}-2}_{p,r})$. Then the heat equations

$$
\begin{aligned}
\frac{\partial}{\partial t} u - \Delta u &= G, \\
u(0, x) &= u_0,
\end{aligned}
$$

has a unique solution $u \in \dot{L}^{q_2}_T (\dot{B}^{s+\frac{2}{q_2}}_{p,r})$ satisfying for all $T > 0$

$$
||u||_{\dot{L}^{q_2}_T (\dot{B}^{s+\frac{2}{q_2}}_{p,r})} \lesssim ||u_0||_{\dot{B}^s_{p,r}} + ||G||_{\dot{L}^{q_2}_T (\dot{B}^{s+\frac{2}{q_2}-2}_{p,r})}.
$$

## 3 Proof of the Main Result

**Proof of Theorem 1.1.** Let $U = e^{t\Delta} u_0$ be the solutions generated by the following heat equations

$$
\begin{aligned}
\frac{\partial}{\partial t} U - \Delta U &= 0, \\
\text{div} U &= 0, \\
U|_{t=0} &= u_0,
\end{aligned}
$$

(3.10)

Introducing the new quantity $v = u - U$, the system (1.1) can be reduced to

$$
\begin{aligned}
\frac{\partial}{\partial t} v + v \cdot \nabla v - \Delta v + \nabla p &= -v \cdot \nabla U - U \cdot \nabla v - U \cdot \nabla U, \\
\text{div} v &= 0, \\
v_0 &= 0.
\end{aligned}
$$

(3.11)

To element the pressure term, applying the Leray operator $P$ to the equation (1.1), one has

$$
\begin{aligned}
\frac{\partial}{\partial t} v - \Delta v &= -P(v \cdot \nabla v - \text{div}(v \otimes U) - \text{div}(U \otimes v) - U \cdot \nabla U), \\
\text{div} v &= 0, \\
v_0 &= (0, 0, 0).
\end{aligned}
$$

(3.12)

Invoking Lemma 2.3 to the above system (3.12) yields

$$
||v||_{\dot{L}^{\infty}_t (\dot{B}^{-1+\frac{2}{q}}_{p,1})} + ||v||_{\dot{L}^1_t (\dot{B}^{1+\frac{2}{q}}_{p,1})} \lesssim \int_0^t ||v \cdot \nabla v||_{\dot{B}^{\frac{2}{q}-1}_{p,1}} \, d\tau + \int_0^t ||U \cdot \nabla U||_{\dot{B}^{\frac{2}{q}-1}_{p,1}} \, d\tau
$$

$$
+ \int_0^t ||\text{div}(v \otimes U) + \text{div}(U \otimes v)||_{\dot{B}^{\frac{2}{q}-1}_{p,1}} \, d\tau,
$$

(3.13)

where we have used the fact that $P$ is a smooth homogeneous of degree 0 Fourier multipliers which maps $\dot{B}^{\frac{2}{q}-1}_{p,1}$ to itself.
For the term $I_1$, using the product estimate (see Lemma 2.2), we obtain
\[
I_1 \lesssim \int_0^t \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \, d\tau \lesssim \|v\|_{L^\infty(B^{1+\frac{3}{p}}_{p,1})} \|v\|_{L^1(B^{1+\frac{3}{p}}_{p,1})}, \tag{3.14}
\]
For the term $I_2$, notice that $\text{div}U=0$, we have
\[
U \cdot \nabla U^1 = (U^1 + U^2)\partial_1 U^1 + U^2 \partial_2 (U^1 + U^2) + U^2 \partial_3 U^3 + U^3 \partial_3 U^1, \\
U \cdot \nabla U^2 = (U^1 + U^2)\partial_2 U^2 + U^1 \partial_1 (U^1 + U^2) + U^1 \partial_3 U^3 + U^3 \partial_3 U^2, \\
U \cdot \nabla U^3 = U^1 \partial_1 U^3 + U^2 \partial_2 U^3 - U^3 (\partial_1 U^1 + \partial_2 U^2).
\]
Using the lemma 2.2 again gives
\[
I_2 = \int_0^t \|U \cdot \nabla U\|_{B^{1+\frac{3}{p}}_{p,1}} \, d\tau \lesssim \int_0^t \|U^1 + U^2, U^3\|_{B^{\frac{3}{p}}_{p,1}} \|U^1, U^2\|_{B^{\frac{3}{p}}_{p,1}} \, d\tau \tag{3.15}
\]
\[
\lesssim \|u_0^1 + u_0^2, u_0^3\|_{B^{\frac{3}{p}}_{p,1}} \|u_0^1, u_0^2\|_{B^{\frac{3}{p}}_{p,1}}. \tag{3.16}
\]
To deal with the term $I_3$, by Bony’s decomposition, one has
\[
v \cdot \nabla U^i = \sum_{j=1}^3 \partial_j (U^i v^j) = \sum_{j=1}^3 [\partial_j (\tilde{T}_U v^j) + \tilde{T}_v \partial_j U^i + \partial_j \hat{R}(U^i, v^j)]
\]
and
\[
U \cdot \nabla v^i = \sum_{j=1}^3 \partial_j (U^j v^i) = \sum_{j=1}^3 [\tilde{T}_U \partial_j v^i + \partial_j (\tilde{T}_v U^j) + \partial_j \hat{R}(U^j, v^i)].
\]
Using (2.5)–(2.7) from Lemma 2.1, respectively, we obtain
\[
\|\partial_j (\tilde{T}_U v^j), \tilde{T}_U \partial_j v^j\|_{B^{1+\frac{3}{p}}_{p,1}} \lesssim \|U\|_{L^\infty}\|v\|_{B^{\frac{3}{p}}_{p,1}}, \tag{3.17}
\]
\[
\|\tilde{T}_v \partial_j U^i, \partial_j (\tilde{T}_v U^j)\|_{B^{1+\frac{3}{p}}_{p,1}} \lesssim \|U\|_{B_{\infty,\infty}^1} \|v\|_{B^{1+\frac{3}{p}}_{p,1}}, \tag{3.18}
\]
\[
\|\hat{R}(U^i, v^j), \hat{R}(U^j, v^i)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|U\|_{B_{\infty,\infty}^1} \|v\|_{B^{1+\frac{3}{p}}_{p,1}}. \tag{3.19}
\]
Combining (3.17)–(3.19) implies
\[
\int_0^t \|v \cdot \nabla U + U \cdot \nabla v\|_{B^{1+\frac{3}{p}}_{p,1}} \, d\tau \lesssim \int_0^t \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \|U\|_{L^\infty} \, d\tau + \int_0^t \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \|U\|_{B_{\infty,\infty}^1} \, d\tau
\]
\[
\lesssim \int_0^t \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \|U\|_{B_{\infty,\infty}^1} \, d\tau + \int_0^t \|U\|_{L^\infty} \|v\|_{B^{\frac{3}{p}}_{p,1}} \|v\|_{B^{\frac{3}{p}}_{p,1}} \, d\tau
\]
\[
\lesssim \int_0^t \left( \|U\|_{L^\infty}^2 + \|U\|_{B_{\infty,\infty}^1} \right) \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \, d\tau + \frac{1}{2} \|v\|_{L^1(B^{1+\frac{3}{p}}_{p,1})}. \tag{3.20}
\]
Putting the estimates (3.14), (3.15) and (3.20) together with (3.13) yields
\[
\|v\|_{L^\infty(B^{1+\frac{3}{p}}_{p,1})} + \|v\|_{L^1(B^{1+\frac{3}{p}}_{p,1})} \lesssim \|v\|_{L^\infty(B^{1+\frac{3}{p}}_{p,1})} + \|v\|_{L^1(B^{1+\frac{3}{p}}_{p,1})} + \|u_0^1 + u_0^2, u_0^3\|_{B^{\frac{3}{p}}_{p,1}} \|u_0^1, u_0^2\|_{B^{\frac{3}{p}}_{p,1}}
\]
\[
+ \int_0^t \left( \|U\|_{L^\infty}^2 + \|U\|_{B_{\infty,\infty}^1} \right) \|v\|_{B^{1+\frac{3}{p}}_{p,1}} \, d\tau. \tag{3.21}
\]
Now, we define
\[ \Gamma \triangleq \sup \left\{ t \in [0, T^*) : \|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} \leq \eta \ll 1 \right\}, \]
where \( \eta \) is a small enough positive constant which will be determined later on.

Thus, for all \( t \in [0, \Gamma] \), choosing \( \eta \) small enough, we infer from (3.21)
\[
\|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} + \|v\|_{L_t^1(B_{p,1}^{\frac{3}{2} + \frac{1}{p}})} \lesssim \|u_0\|_{B_{p,1}^\frac{3}{2} - 1} + \|u_0\|_{B_{p,1}^\frac{3}{2} - 1} + \int_0^t \left( \|U\|_{L_t^\infty}^2 + \|U\|_{\dot{B}_{\infty,1}^1} \right) \|v\|_{B_{\infty,1}^{\frac{3}{2} - \frac{1}{p}}} d\tau. \tag{3.22}
\]
Notice that
\[
\int_0^t \left( \|U\|_{L_t^\infty}^2 + \|U\|_{\dot{B}_{\infty,1}^1} \right) d\tau \lesssim \|u_0\|_{B_{\infty,2}^{\frac{3}{2} - 1}} + \|u_0\|_{\dot{B}_{\infty,1}^{\frac{3}{2} - 1}},
\]
by Gronwall’s inequality, we have for all \( t \in [0, \Gamma] \)
\[
\|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} + \|v\|_{L_t^1(B_{p,1}^{\frac{3}{2} + \frac{1}{p}})} \lesssim \|u_0\|_{B_{p,1}^\frac{3}{2} - 1} + \|u_0\|_{B_{p,1}^\frac{3}{2} - 1} \exp \left\{ C \left( \|u_0\|_{B_{p,1}^\frac{3}{2} - 1} + \|u_0\|_{\dot{B}_{\infty,1}^{\frac{3}{2} - 1}} \right) \right\} \leq C \delta \tag{3.23}
\]
provided that the condition (1.3) holds.

Choosing \( \eta = 2C\delta \), thus we can get
\[
\|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} \leq \frac{\eta}{2} \ \text{for} \ \ t \leq \Gamma.
\]

So if \( \Gamma < T^* \), due to the continuity of the solutions, we can obtain that there exists \( 0 < \epsilon \ll 1 \) such that
\[
\|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} \leq \eta \ \text{for} \ \ t \leq \Gamma + \epsilon < T^* ,
\]
which is contradiction with the definition of \( \Gamma \).

Thus, we can conclude \( \Gamma = T^* \) and
\[
\|v\|_{L_t^\infty(B_{p,1}^{\frac{3}{2} - \frac{1}{p}})} \leq C < \infty \ \text{for all} \ \ t \in (0, T^*),
\]
which implies that \( T^* = +\infty \).

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