A SIMPLE CHARACTERISTIC-FREE PROOF OF THE
BRILL-NOETHER THEOREM

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Abstract. We describe how the use of a different degeneration from that considered by Eisenbud and Harris leads to a simple and characteristic-independent proof of the Brill-Noether theorem using limit linear series. As suggested by the degeneration, we prove an extended version of the theorem allowing for imposed ramification at up to two points. Although experts in the field have long been aware of the main ideas, we address some technical issues which arise in proving the full version of theorem.

1. Introduction

The Brill-Noether theorem addresses the fundamental question of the dimension and nonemptiness of spaces of linear series on smooth projective curves. The original proof was characteristic independent, although Griffiths and Harris [5] wrote their portion of the argument in the context of complex varieties. Gieseker’s proof of the Petri conjecture [4] was explicitly written as holding in all characteristics, and gave a new proof of the Griffiths-Harris result. The later degeneration proof by Eisenbud and Harris using limit linear series (Theorem 5.1 of [1]; see also [2]) in fact used the characteristic-0 hypotheses in nontrivial ways. This last proof is quite simple, so it is desirable to have a characteristic-independent version. At the same time, in Theorem 4.5 of [3], Eisenbud and Harris generalized the Brill-Noether theorem to consider linear series with imposed ramification. Not only does their argument use characteristic 0, but as stated the result fails in positive characteristic due to inseparability phenomena.

The purpose of the present note is to give a simple proof of the most general form of the Eisenbud-Harris result which holds in all characteristics: the case that ramification is imposed at at most two points. As is known to the experts in the field, the dependence on characteristic in the earlier Eisenbud-Harris proof of the classical Brill-Noether theorem can be largely avoided if one changes the degeneration considered, working instead with a degeneration to a chain of elliptic curves as considered by Welters in [10]. We describe this proof, and in the process observe that there is a nontrivial technical obstacle to this approach which can be circumvented via the alternative construction of limit linear series introduced in [8] (see Remark 3.13 for details). Our main result is the following:

Theorem 1.1. Let $C$ be a smooth, projective curve of genus $g$ over an algebraically closed field $k$, and $P_1, P_2$ distinct points on $C$. Given $r, d, \alpha$ sequences $0 \leq \alpha_k^1, \alpha_k^2 \leq$

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The characteristic 0 hypothesis is used implicitly in Lemma 8.
\[ \cdots \leq \alpha_i \leq d - r \text{ for } i = 1, 2, \text{ set} \]
\[ \rho := (r + 1)(d - r) - rg - \sum_{i=1}^{2} \sum_{j=0}^{r} \alpha_j^i. \]

Now, suppose that we have
\[ (1.1) \sum_{j: \alpha_j^1 + \alpha_j^2 + r \geq d + 1 - g} \alpha_j^1 + \alpha_j^2 + r - d + g \leq g. \]

Then the space \( G_d^r(C, P_1, P_2, \alpha^1, \alpha^2) \) parametrizing \( g_r^d \)'s on \( C \) with ramification sequence at least \( \alpha^i \) at \( P_i \) for \( i = 1, 2 \) is nonempty with every component of dimension at least \( \rho \).

Furthermore, if \((C, P_1, P_2)\) is a general 2-marked curve, then in fact \( G_d^r(C, P_1, P_2, \alpha^1, \alpha^2) \) is nonempty if and only if \((1.1)\) is satisfied, and in this case is pure of dimension \( \rho \).

Recall the ramification terminology: if \((L, V)\) is a \( g_r^d \) on \( C \), and \( P \in C \), the ramification sequence \( \alpha_0(P), \ldots, \alpha_r(P) \) is determined by \( \alpha_j(P) = a_j(P) - j \) for all \( j \), where \( a_0(P), \ldots, a_r(P) \) is the strictly increasing sequence of orders of vanishing at \( P \) of sections in \( V \).

Remarks 1.2. (i) One easily verifies that \((1.1)\) implies that \( \rho \geq 0 \). In addition, if the \( \alpha_j^i \) are all equal to 0, so that we are in the classical case without imposed ramification, and if \( \rho \geq 0 \), then \((1.1)\) is satisfied. Thus, we recover in particular the classical statement of the Brill-Noether theorem.

(ii) Note that the theorem fails in positive characteristic for ramification imposed at more than 2 points, even in the case of \( g_r^1 \)'s on \( \mathbb{P}^1 \), so the statement is sharp.

(iii) Although typically the dimensional upper bound for general curves is viewed as deeper than the nonemptiness statement, in the context of the type of degeneration argument we use, the opposite is true: one may prove the dimension statement without the nonemptiness statement, but not vice versa.

(iv) The restriction to algebraically closed base fields is a matter of convenience, insofar as the \( G_d^r \) spaces may be defined over an arbitrary base field (or scheme), and commute with base change. However, although one obtains a more general statement via this reduction, the notion of a general curve is less useful in the arithmetic setting, since there is no guarantee of having even one \( k \)-rational point inside a Zariski open subset of the moduli space.

Remark 1.3. We recall a standard reduction step for proving the Brill-Noether theorem. The dimensional lower bound is immediate from the construction of the space \( G_d^r(C, P_1, P_2, \alpha^1, \alpha^2) \), which realizes the space as an intersection of relative Schubert cycles inside a relative Grassmannian over \( \text{Pic}^d(C) \). Furthermore, this construction works for families of curves to produce proper \( G_d^r \) moduli spaces whose components always have relative dimension at least \( \rho \); it follows that to prove Theorem 1.1, it suffices to produce a single smooth curve \( C \) (over any extension of the base field) for which \( G_d^r(C, P_1, P_2, \alpha^1, \alpha^2) \) is as asserted in the theorem.

We mention that the same degeneration we use can also be used to give a simple characteristic-independent proof of the Gieseker-Petri theorem, and without the need to use the limit linear series construction of [8]. In a similar vein, while Lazarfeld’s proof [6] of the Gieseker-Petri theorem requires characteristic 0,
it seems likely that his argument yields the classical Brill-Noether theorem in arbitrary characteristic. Nonetheless, our proof of Theorem 1.1 has several desirable characteristics: it gives an all-in-one approach to both the dimensional statements and nonemptiness statements of the Brill-Noether theorem; it avoids the additional machinery of the Petri map; and, although restrictive, the generalization to ramification at two points ought to be of independent interest. For instance, we can conclude that spaces of Eisenbud-Harris limit linear series on chains of curves made up of general marked curves have the expected dimension in any characteristic; this is not true for arbitrary curves of compact type.

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2. The base case

Assuming the machinery of limit linear series, the most technical part of the proof of Theorem 1.1 is in fact the base case, which proceeds by direct case-by-case analysis as follows.

Lemma 2.1. Theorem 1.1 holds when \( g = 0 \) or 1, and moreover in this case the generality hypothesis can be described explicitly: for \( g = 0 \), no additional conditions are required, while for \( g = 1 \), we may take any \( C \), and it suffices to assume that \( P_1 \) does not differ from \( P_2 \) by \( m \)-torsion, with \( m \leq d \).

Proof. It will be convenient to set the following notation for vanishing sequences: \( a^i_j := a^i_j + j \) for all \( i, j \). We begin with the following observation, which follows from a dimension count: if \((\mathcal{L}, V)\) is a \( g^d_0 \) with vanishing sequence (at least) \( a^1_j \) at \( P_1 \) for \( i = 1, 2 \), then for any \( j = 0, \ldots, r \), then there exists \( s \in V \) vanishing to order at least \( a^1_j \) at \( P_1 \) and \( a^2_{r-j} \) at \( P_2 \). We then easily conclude the asserted emptiness statement under the generality hypothesis as follows: we immediately see that we must have \( a^1_j + a^2_{r-j} \leq d \) for all \( j \), which is precisely the desired statement for \( g = 0 \). On the other hand, if \( a^1_j = a \) and \( a^2_{r-j} = d-a \), we also conclude that we can only have a \( g^d_0 \) with this vanishing if the underlying line bundle is \( \mathcal{O}(aP_1 + (d-a)P_2) \). This can occur for two different values of \( a \) only if \( \mathcal{O}(aP_1 + (d-a)P_2) \cong \mathcal{O}(a'P_1 + (d-a')P_2) \), or \( \mathcal{O}((a-a')P_1 + (a'-a)P_2) \cong \mathcal{O} \). This implies that \( P_1 - P_2 \) is an \( |a-a'| \)-torsion point in the Jacobian of \( C \), so this in turn violates our generality hypothesis for the \( g = 1 \) case, and we conclude the asserted emptiness.

We next consider the dimension assertion, recalling that as discussed in Remark 1.3, the dimensional lower bound is an immediate consequence of the construction of \( G^d_0(C, P_1, P_2, \alpha^1, \alpha^2) \). We thus direct our attention to the upper bound, starting with the case \( g = 0 \). The only line bundle of degree \( d \) is \( \mathcal{O}(d) \), so the space of all \( g^d_0 \)'s is the Grassmannian \( G(d, H^0(\mathbb{P}^1, \mathcal{O}(d))) \), where we can identify \( H^0(\mathbb{P}^1, \mathcal{O}(d)) \) with the space of polynomials of degree \( d \). Directly from the definitions, we have that \( G^d_0(C, P_1, P_2, \alpha^1, \alpha^2) \) is described as an intersection of two Schubert cycles, associated to the flags of polynomials vanishing to different orders at \( P_1 \) and \( P_2 \), respectively. By unique factorization of polynomials, these flags are complementary in \( H^0(\mathbb{P}^1, \mathcal{O}(d)) \), so they are general, and thus by Kleiman's theorem, if the intersection of the Schubert cycles is nonempty, it has the expected dimension.

For the case \( g = 1 \), to prove the dimension statement we only need an upper bound, and we work over one point of \( \text{Pic}^d(C) \) at a time. According to the Riemann-Roch theorem, there are only two possibilities for the vanishing sequence
of a complete linear series of a line bundle $\mathcal{L}$ at a point $P$: if $\mathcal{L} \not\cong \mathcal{O}(dP)$, the vanishing sequence is $0$, while if $\mathcal{L} \cong \mathcal{O}(dP)$, the vanishing sequence is $0, 1, \ldots, d-1$. In particular, if $a^1_j = d$, then $G^d_j(C, P_1, P_2, \alpha^1, \alpha^2)$ is necessarily supported over the point corresponding to $\mathcal{O}(dP_1)$, and similarly for $a^2_j = d$. By the generality hypothesis, $\mathcal{O}(dP_1) \not\cong \mathcal{O}(dP_2)$, so we conclude that in order for the space to be nonempty, we must have at most one $a^i_j = d$.

There are two types of line bundles to consider: first, if $\mathcal{L} \not\cong \mathcal{O}(aP_1 + (d-a)P_2)$ for $0 \leq a \leq d$, we see that the flags obtained by imposing vanishing at $P_1$ and $P_2$ are transverse, hence general as before, so we conclude that for such an $\mathcal{L}$, if the corresponding fiber of $G^d_j(C, P_1, P_2, \alpha^1, \alpha^2)$ is nonempty, it has dimension

$$(r + 1)(d - 1 - r) - \sum_{i=1}^{2} \sum_{j=0^{r}} \alpha^i_j = \rho - 1.$$ 

Since the space of such line bundles $\mathcal{L}$ has dimension $1$, we conclude that $G^d_j(C, P_1, P_2, \alpha^1, \alpha^2)$ can have dimension at most $\rho$ over the open locus of line bundles of this type.

Next, suppose that $\mathcal{L} \cong \mathcal{O}(aP_1 + (d-a)P_2)$ for $a$ and $d$ are transverse, hence we may assume without loss of generality that $a = 0$ or $d$. In the case $a = 0$ or $d$, we see that the flags obtained in $H^0(C, \mathcal{L})$ from vanishing at $P_1$ or $P_2$ are still transverse, and thus the Schubert cycles intersect transversely, but the Schubert cycle indexing can be off by one from the vanishing sequence if $a = 0$ and $a^2_j = d$, or if $a = d$ and $a^1_j = d$. In either case, we have that the dimension of the corresponding fiber is at most

$$(r + 1)(d - 1 - r) - \sum_{i=1}^{2} \sum_{j=0^{r}} \alpha^i_j + 1 = \rho.$$ 

Finally, if $0 < a < d$, we find that the flag indexing corresponds to the Schubert cycle indexing, but the flags fail to be transverse at the spaces corresponding to vanishing of order $a$ at $P_1$ and $d - a$ at $P_2$. This only affects the dimension of intersection if for some $j$, we have $a^1_j = a$ and $a^2_{r-j} = d - a$. Since we have already proved the emptiness assertion if $a^1_j + a^2_{r-j} = d$ for more than one index $j$, we may assume that there is no $j' \neq j$ with $a^1_{j'} + a^2_{r-j'} = d$, and in particular if $j > 0$, we must have $a^1_{j-1} \leq a - 2$. Thus, regardless of whether or not $j > 0$, we can still obtain a valid vanishing sequence if we replace $a^1_j$ by $a - 1$. In this case, we have already seen that the Schubert cycles still intersect in the expected dimension, which is now

$$(r + 1)(d - 1 - r) - \sum_{i=1}^{2} \sum_{j=0^{r}} \alpha^i_j + 1 = \rho$$

because we changed $a^1_j$. This gives us the desired statement.

It remains to check the asserted nonemptiness statement, which does not require any generality hypothesis. In the case $g = 0$, if $a^1_j + a^2_{r-j} \leq d$ for all $r$, we can simply take $r$ sections of $\mathcal{O}(d)$, with the $j$th section vanishing to order exactly $a^1_j$ at $P_1$ and $a^2_{r-j}$ at $P_2$, and we obtain the desired $g^d_j$. Similarly, if $g = 1$ and we have $a_1, a_2$ with $a_1 + a_2 \leq d - 1$, then for any line bundle $\mathcal{L}$ on $C$ we have a section vanishing to order exactly $a_1$ at $P_1$ and $a_2$ at $P_2$ as long as we do not have $a_1 + a_2 = d - 1$ and $\mathcal{L} \cong (a_1P_1 + (d-a_1)P_2)$ or $\mathcal{L} \cong \mathcal{O}(d-a_2)P_1 + a_2P_2)$. Thus, if $a^1_j + a^2_{r-j} \leq d - 1$ for all $j$, we can choose any $\mathcal{L} \not\cong \mathcal{O}(aP_1 + (d-a)P_2)$ for $a = 0, \ldots, d$, and then
construct the desired $g_Y^n$ via a basis as above. Finally, if $a_j^1 + a_{r-j}^2 = d$ for some $j$, and $a_j^1 + a_{r-j}^2 \leq d - 1$ for $j' \neq j$, set $\mathcal{L}' = \mathcal{O}(a_j^1 \mathcal{P}_1 + a_{r-j}^2 \mathcal{P}_2)$, and since we cannot have $a_j^1 = a_j^1 - 1$ or $a_j^1 = a_{r-j}^2 - 1$ for any $j'$, we are still able to find a basis of sections with precisely the desired vanishing. This completes the proof of the lemma.

\[ \square \]

3. The degeneration

We begin by reviewing the machinery of limit linear series. For our argument, it is enough to consider degenerations to curves with two components. Specifically, we will consider the following situation:

**Situation 3.1.** Let $X_0$ be a proper nodal curve over $k$ obtained by gluing two smooth curves $Y$ and $Z$ to one another at a single node $Q$.

We recall the Eisenbud-Harris definition of a limit linear series from [3]:

**Definition 3.2.** An Eisenbud-Harris limit $g_d^n$ on $X_0$ is a pair $((\mathcal{L}_Y, V_Y), (\mathcal{L}_Z, V_Z))$ of $g_d^n$’s on $Y$ and $Z$ respectively, such that for $j = 0, \ldots, r$ we have

\begin{equation}
\alpha_j^Y + \alpha_j^Z \geq d - r,
\end{equation}

where $\alpha_j^Y, \ldots, \alpha_r^Y$ and $\alpha_j^Z, \ldots, \alpha_r^Z$ are the ramification sequences at $Q$ of $(\mathcal{L}_Y, V_Y)$ and $(\mathcal{L}_Z, V_Z)$, respectively.

**Notation 3.3.** The space of Eisenbud-Harris limit $g_d^n$’s on $X_0$ is denoted by $G_d^{\text{EH}}(X_0)$.

**Remark 3.4.** In fact, the Eisenbud-Harris terminology differs slightly in that our “limit series” are their “crude limit series.”

We also note that we can allow for imposed ramification of limit series as follows: given a smooth point $P$ of $X_0$, we say an Eisenbud-Harris limit $g_d^n$ given by the pair $((\mathcal{L}_Y, V_Y), (\mathcal{L}_Z, V_Z))$ has ramification at least $\alpha$ at $P$ if $(\mathcal{L}_Y, V_Y)$ or $(\mathcal{L}_Z, V_Z)$ does, depending on whether $P$ lies on $Y$ or $Z$.

**Notation 3.5.** Given nondecreasing integer sequences $\alpha^Y, \alpha^Z$ of length $r + 1$ in $[0, d - r]$ satisfying

\begin{equation}
\alpha_j^Y + \alpha_j^Z \geq d - r
\end{equation}

for $j = 0, \ldots, r$, let $G_d^{\text{EH}}(X_0; \alpha^Y, \alpha^Z)$ be the space of Eisenbud-Harris limit $g_d^n$’s $((\mathcal{L}_Y, V_Y), (\mathcal{L}_Z, V_Z))$ on $X_0$ with $(\mathcal{L}_Y, V_Y)$ having ramification sequence $\alpha^Y$ at $Q$, and $(\mathcal{L}_Z, V_Z)$ having ramification sequence $\alpha^Z$ at $Q$.

The power of the Eisenbud-Harris theory derives from its inductive structure. Specifically, we have the following basic observations, which are immediate consequences of the definition.

**Proposition 3.6.** The spaces $G_d^{\text{EH}}(X_0; \alpha^Y, \alpha^Z)$ give a stratification of $G_d^{\text{EH}}(X_0)$ by (locally closed) subschemes. Moreover, each space $G_d^{\text{EH}}(X_0; \alpha^Y, \alpha^Z)$ is isomorphic to an open subscheme of $G_d^p(Y, Q, \alpha^Y) \times G_d^p(Z, Q, \alpha^Z)$.

Alternatively, $G_d^{\text{EH}}(X_0)$ is the union over all sequences $\alpha^Y, \alpha^Z$ achieving equality in (3.2) for all $j$ of the spaces

$G_d^p(Y, Q, \alpha^Y) \times G_d^p(Z, Q, \alpha^Z)$.

The same descriptions hold if ramification is imposed at smooth points of $X_0$. 

In [8], a different definition of limit linear series is given. We will not need the
definition itself, but will need notation for the resulting space:

Notation 3.7. The space of limit \( g_r \)'s on \( X_0 \) as defined in [8] is denoted by \( G^r_d(X_0) \).

The two spaces of limit linear series are different, but we have the following result
comparing them:

**Theorem 3.8.** There is a map

\[
G^r_d(X_0) \to G^r_{d, EH}(X_0)
\]

which is surjective, and has fiber dimension bounded as follows: given an Eisenbud-
Harris limit \( g_d^r \) on \( X_0 \) as in Definition 3.2, the corresponding fiber of (3.3) has
dimension at most

\[
\sum_{j=0}^r \alpha_j Y + \alpha_{r-j} Z - d + r,
\]

This result combines Proposition 6.6 of [8] with Corollary 5.5 of [9]; see also
Theorem 2.3 of [7] for a simplified proof of the latter.

We can then define ramification conditions on \( G^r_d(X_0) \) by taking the preimage
under (3.3) of the corresponding locus of \( G^r_{d, EH}(X_0) \).

**Remark 3.9.** Technically, the map (3.3) is not known to exist in general on a scheme-
theoretic level. The precise statement is that \( G^r_{d, EH}(X_0) \) is a closed subscheme of
\( G^r_d(Y) \times G^r_d(Z) \); and (3.3) is in fact a morphism \( G^r_d(X_0) \to G^r_{d, EH}(Y) \times G^r_d(Z) \) which
factors set-theoretically through \( G^r_{d, EH}(X_0) \). However, since we are only interested
in questions of (non)emptiness and dimension, this technical distinction is irrelevant
to us.

We next recall the behavior of limit series in smoothing families.

**Situation 3.10.** Suppose \( X/B \) is a flat, proper morphism, with \( B \) the spectrum
of a DVR, and \( X \) a regular scheme having special fiber \( X_0 \) as above, and generic
fiber \( X_\eta \) a smooth proper curve.

**Theorem 3.11.** There exists a scheme \( G^r_d(X/B) \), proper over \( B \), with special fiber
\( G^r_d(X_0) \), generic fiber \( G^r_d(X_\eta) \), and such that every component of \( G^r_d(X/B) \) has
dimension at least \( \rho + \dim B = \rho + 1 \). The same holds with imposed ramification
along smooth sections of \( X/B \).

This is Theorem 5.3 of [8].

Using the dimension and properness assertions of the theorem, we conclude:

**Corollary 3.12.** If \( G^r_d(X_0) \) is empty, so is \( G^r_d(X_\eta) \). If \( G^r_d(X_0) \) is nonempty, every
component has dimension at least \( \rho \); if it has pure dimension \( \rho \), then \( G^r_d(X_\eta) \) is also
nonempty of pure dimension \( \rho \). The same holds with imposed ramification along
smooth sections of \( X/B \).

We can now easily prove Theorem 1.1. Rather than explicitly working with a
degeneration to a chain of elliptic curves, it is more convenient to work by induction
on genus, but in either case the spirit is the same.

**Proof of Theorem 1.1.** The base cases are \( g = 0, 1 \), and treated in Lemma 2.1.
Given \( g \geq 2 \), suppose Theorem 1.1 holds in all genera strictly less than \( g \). Given
Given sequences $\alpha^Y, \alpha^Z$ as in Notation 3.5, by the induction hypothesis together with Proposition 3.6, we conclude that if $G_{d,\alpha^Y,\alpha^Z}(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ is nonempty, it has pure dimension

$$\rho_{Y,\alpha^Y} + \rho_{Z,\alpha^Z} = \rho - \sum_{j=0}^{r} (\alpha^Y_j + \alpha^Z_{r-j} - d + r),$$

where

$$\rho_{Y,\alpha^Y} = (r + 1)(d - r) - r - \sum_{j=0}^{r} \alpha^Y_j - \sum_{j=0}^{r} \alpha^Z_j$$

and

$$\rho_{Z,\alpha^Z} = (r + 1)(d - r) - (g - 1)r - \sum_{j=0}^{r} \alpha^2_j - \sum_{j=0}^{r} \alpha^Z_j$$

are the expected dimensions of $g^r_d$'s on $Y$ and $Z$ with the required ramification at $P_1$, $P_2$ and $Q$. Working stratum by stratum, we thus conclude from Theorem 3.8 that $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ can have dimension at most $\rho$, and is nonempty if and only if $G^r_{d,\alpha^Y,\alpha^Z}(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ is nonempty. Since we know that every component of $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ always has dimension at least $\rho$, we conclude that $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ has pure dimension $\rho$ whenever it is nonempty. We thus conclude from Corollary 3.12 that if $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ is nonempty, it must have pure dimension $\rho$, and that nonemptiness of $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ is equivalent to nonemptiness of $G^r_{d,\alpha^Y,\alpha^Z}(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$.

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**Figure 3.1.** $Y$ is smooth of genus 1, and $Z$ is smooth of genus $g - 1$. 

$r, d, \alpha^1, \alpha^2$, let $Y$ be a smooth genus-1 curve with points $P_1$, $Q$ not differing by $m$-torsion for $m \leq d$, and let $Z$ be a general 2-marked smooth curve of genus $g - 1$, with marked points $Q$ and $P_2$. Let $X_0$ be the nodal curve obtained by gluing $Y$ to $Z$ at $Q$. Given this choice of $X_0$, let $X/B$ be as in Situation 3.10, and let $P_1, P_2$ be sections of $X/B$ extending the chosen points of $X_0$. That such a family always exists is well known; see for instance Theorem 3.4 of [8].
According to Remark 1.3, we have thus proved the dimension portion of the Brill-Noether theorem, and in particular the emptiness assertion in the case $\rho < 0$. It remains to analyze the sharp nonemptiness statement. Now, nonemptiness of $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ is equivalent to the existence of sequences $\alpha^Y, \alpha^Z$ achieving equality in (3.2) such that the spaces $G^r_d(Y; (P_1, \alpha^1), (Q, \alpha^Y))$ and $G^r_d(Z, (Q, \alpha^Z), (P_2, \alpha^2))$ are both nonempty. By the induction hypothesis, the latter nonemptiness conditions are determined precisely by the inequality (1.1).

Consider all possibilities for $\alpha^Y$ such that (1.1) is satisfied on $Y$. Then we have $\alpha^r_j + \alpha^r_{r-j} \leq d - r$ for all $j$, with equality holding for at most a single value of $j$. Since $\alpha^r_j = d - r - \alpha^r_{r-j}$, we then have $\alpha^r_j \geq \alpha^r_j$ for all $j$, with equality holding for at most a single value of $j$. It follows that as long as (1.1) holds on $Y$, then we have

$$
\alpha^r_j + \alpha^r_{r-j} + r - d + g - 1 \geq \sum_{j: \alpha^r_j + \alpha^r_{r-j} + r - d + g - 1} \alpha^r_j + \alpha^r_{r-j} + r - d + g - 1.
$$

It immediately follows that if (1.1) is violated for $\alpha^1, \alpha^2$, then it is impossible to find $\alpha^Y, \alpha^Z$ satisfying (1.1) on both $Y$ and $Z$. We thus conclude the desired emptiness statement for $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$, and hence for $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$.

Conversely, suppose first that (1.1) is satisfied with equality. It is enough to see that we can choose $\alpha^Y, \alpha^Z$ so that (3.4) is satisfied with equality. Since $g > 0$, we must have at least some indices $j$ such that $\alpha^r_j + \alpha^r_{r-j} + r \geq d + 1 - g$; let $j_0$ be the minimal such index. Then we set $\alpha^r_j = d - r - 1 - \alpha^r_{r-j}$ for all $j \neq j_0$, and $\alpha^r_{j_0} = d - r - \alpha^r_{r-j_0}$. Note that this gives a valid nondecreasing sequence because of the minimal of $j_0$. If we then set $\alpha^r_j = d - r - \alpha^r_{r-j}$ for all $j$, we will achieve equality in (3.4), as desired. Lastly, suppose that (1.1) is satisfied with strict inequality. Then we can set $\alpha^r_j = d - r - 1 - \alpha^r_{r-j}$ and $\alpha^r_j = d - r - \alpha^r_{r-j}$ for all $j$, and we will still have (1.1) satisfied on $Z$. We thus conclude that if (1.1) is satisfied, $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ – and hence $G^r_d(X_0; (P_1, \alpha^1), (P_2, \alpha^2))$ – is nonempty. Again according to Remark 1.3, we conclude the sharp nonemptiness assertion of Theorem 1.1, completing the proof of the theorem.

\[\square\]

Remark 3.13. The importance of the limit linear series construction of [8] arises in proving the dimension upper bound in the case that $\rho \geq 0$. In [3], Eisenbud and Harris do not construct a proper family of limit linear series, instead constructing a space whose special fiber consists of “refined limit series” – those which satisfy (3.1) with equality. In characteristic 0, this does not cause serious problems, as they show that a limit series arising from a linear series in a smoothing family is refined if and only if there is no ramification specializing to the node. They can then start with any linear series on the generic fiber and obtain a refined limit series on the special fiber after base change and blowup. However, this argument fails in positive characteristic, again due to the presence of inseparability. Thus, when working in positive characteristic, there is no obvious way to reduce to the refined case when comparing dimensions on the special and generic fibers.

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