Operator mixing in massless QCD-like theories and Poincarè–Dulac theorem

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Abstract
Recently, a differential-geometric approach to operator mixing in massless QCD-like theories – that involves canonical forms, obtained by means of gauge transformations, based on the Poincarè–Dulac theorem for the system of linear differential equations that defines the renormalized mixing matrix in the coordinate representation \( Z(x, \mu) \) – has been proposed in [1]. Specifically, it has been determined under which conditions a renormalization scheme exists where \( Z(x, \mu) \) may be set in a diagonal canonical form that is one-loop exact to all perturbative orders – the nonresonant diagonalizable \( \gamma_0 \beta_0 \) case (I) –. Moreover, the remaining cases, (II), (III) and (IV), of operator mixing, where such diagonalization is not possible, have also been classified in [1]. Accordingly, if \( \gamma_0 \beta_0 \), with \( \gamma(g) = \gamma_0 g^2 + \cdots \) the matrix of the anomalous dimensions and \( \beta(g) = -\beta_0 g^3 + \cdots \) the beta function, either is diagonalizable but a resonant condition for its eigenvalues and the system holds (II) or is nondiagonalizable and nonresonant (III), or is nondiagonalizable and resonant (IV), \( Z(x, \mu) \) is nondiagonalizable. In the cases (II), (III) and (IV), we demonstrate that its canonical form may be factorized into the exponential of a linear combination of upper triangular nilpotent constant matrices with coefficients that asymptotically in the UV are logs of the running coupling, i.e., asymptotically loglogs of the coordinates, and a diagonal matrix as in the nonresonant diagonalizable case (I). Hence, its ultraviolet asymptotics differs intrinsically from the case (I) and, for asymptotically free theories, this is the closest analog of logCFTs. We also work out actual physics realizations of the cases (I) and (II), while we argue that the cases (III) and (IV) are ruled out by a unitarity argument in the gauge-invariant sector.

1 Introduction and physics motivations

The aim of the present paper is to reconsider the operator mixing and the associated ultraviolet (UV) asymptotics of the renormalized mixing matrix \( Z(x, \mu) \) in the coordinate representation in asymptotically free Yang–Mills (YM) theories massless to all perturbative orders (massless QCD-like theories for short), in order to analyze further implications of the differential-geometric approach to operator mixing initiated in [1], where an essential role is played by the Poincarè–Dulac theorem [2] in the framework of canonical forms [3] for linear systems of differential equations.

In fact, \( Z(x, \mu) \) is a pivotal ingredient to work out the UV asymptotics of gauge-invariant correlators and OPE coefficients that will be considered in a forthcoming paper [4]. One problem addressed in [1], which is hardly discussed in the literature, has been to determine under which conditions the operator mixing may be essentially reduced to the multiplicatively renormalizable case. This is the case (I) – worked out extensively in [1] – of the classification based on the Poincarè–Dulac theorem introduced in [1].

The remaining cases, (II), (III) and (IV), of the aforementioned classification, where such a reduction is not actually possible, are studied in greater detail in the present paper. There are several physics motivations for doing so, since the UV asymptotics of operator mixing enters a number of applications of the renormalization group (RG), which range from the deep inelastic scattering [5] in QCD to the evaluation of the ratio \( \frac{\epsilon'}{\epsilon} \) [6–8] for the possible implications of new physics – if any – and to the constraints [9–14] to the eventual nonperturbative solution of the large-\( N \) limit [15–18] of massless QCD-like theories.

\textsuperscript{1} We only consider YM theories with a single gauge coupling. Our methods may extend to theories with multiple couplings, at the price of increasing mathematical complication.
A further motivation for working out such asymptotics, for the general case of operator mixing as well, occurs as a part of the program, christened the asymptotically free bootstrap in [11], of verifying whether a candidate [11] nonperturbative S matrix arises from the nonperturbative ’t Hooft large-N expansion [15] of a massless QCD-like theory, where the multiplicative renormalization of gauge-invariant operators has already played a key role [10].

In this respect, a fundamental physics application of [1] and the present paper, in order to work out the UV asymptotics of the nonperturbative glueball one-loop effective action [19], consists in showing that twist-2 operators in the program, christened the asymptotically free bootstrap in [11], of verifying whether a candidate [11] nonperturbative action [19], consists in showing that twist-2 operators in the present paper: They exploit the freedom of making (formal) holomorphic reparametrization of the gauge coupling.

Indeed, it has been known for some time that exploiting the freedom of changing renormalization scheme may lead to significant advantages. Perhaps, the most famous example is the ’t Hooft scheme [21], where all the coefficients of the beta function, $\beta(g) = -\beta_0 g^2 - \beta_1 g^3 + \cdots$, but the first two, $\beta_0$, $\beta_1$, may be set to 0 by a suitable (formal) holomorphic reparametrization of the gauge coupling.

In fact, the aforementioned coefficients, $\beta_2$, $\beta_3$, $\ldots$, may be set to an arbitrary value by a reparametrization of the coupling, and this freedom has been exploited in various contexts [22–27], including the supersymmetric one in relation to the exact NSVZ beta function [28].

As another example, all the coefficients but the first one, $\gamma_0$, of the anomalous dimension, $\gamma(g) = \gamma_0 g^2 + \cdots$, of a multiplicatively renormalizable operator may be set to 0 by a similar [29] – but in general different – reparametrization of the coupling.

These examples are well known, but are not relevant in the present paper: They exploit the freedom of making (formal) holomorphic diffeomorphisms in the space of the coupling, while the change of scheme that we refer to is actually the (formal) holomorphic nonabelian gauge freedom [1] in the choice of the basis of operators that mix under renormalization.

Of course, once the asymptotic estimates for $Z(x, \mu)$ have been obtained in the canonical scheme dictated by the Poincaré–Dulac theorem, they can be transferred to any other scheme by means of gauge transformations, in a way that we summarize as follows.

2 A summary of [1]

The key idea in [1] has been to employ the time-honored theory of canonical forms [3] – obtained by (formal) holomorphic gauge transformations – for linear systems of differential equations – specifically, the Poincaré–Dulac theorem [2] – in order to find a sufficient condition by which a renormalization scheme exists where the matrix $\frac{\gamma(g)}{\beta(g)}$ in Eq. (43) can be set in the canonical form:

$$\frac{\gamma(g)}{\beta(g)} = \frac{\gamma_0}{\beta_0}$$  \hspace{1cm} (1)

that is one-loop exact to all orders of perturbation theory, with:

$$\gamma(g) = -\frac{\partial Z}{\partial \log \mu} Z^{-1} = \gamma_0 g^2 + \gamma_1 g^4 + \gamma_2 g^6 + \cdots$$  \hspace{1cm} (2)

the matrix of the anomalous dimensions, and:

$$\frac{\partial g}{\partial \log \mu} = \beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 + \cdots$$  \hspace{1cm} (3)

the beta function, with $g = g(\mu)$ the renormalized coupling.

A sufficient condition [1] for a renormalization scheme to exist where $\frac{\gamma(g)}{\beta(g)}$ admits the canonical form in Eq. (1) is that the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the matrix $\frac{\gamma_0}{\beta_0}$, in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$, do not differ by a positive even integer:

$$\lambda_i - \lambda_j - 2k \neq 0$$  \hspace{1cm} (4)

for $i \leq j$ and $k$ a positive integer.

If such a renormalization scheme exists, the mixing has been dubbed nonresonant in [1]. Otherwise, it has been dubbed resonant. This terminology in [1] derives directly from the application of the Poincaré–Dulac theorem to the operator mixing, as we recall in the present paper.

Moreover, if in addition $\frac{\gamma_0}{\beta_0}$ is diagonalizable by a further change of the operator basis, the renormalized mixing matrix in the coordinate representation:

$$Z(x, \mu) = P \exp \left( \int_{g(\mu)}^{g(x)} \frac{\gamma(g)}{\beta(g)} dg \right)$$  \hspace{1cm} (5)

that enters the solution:

$$G(x) = Z(x, \mu) G(x, g(\mu), \mu) Z^T(x, \mu)$$  \hspace{1cm} (6)

of the Callan–Symanzik equation [1,4,30–33]:

$$\left( x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2D \right) G + \gamma(g) G + G \gamma^T(g) = 0$$
for 2-point correlators\(^4\) in Euclidean space-time:

\[
G_{ik}(x) = \langle O_i(x) O_k(0) \rangle
\]

of renormalized local gauge-invariant operators \(O_i(x)\):\(^5\)

\[
O_i = Z_{ik} O_{Bk}
\]

with \(O_{Bk}\) the bare operators that mix \(^6\) under renormalization and \(Z\) the bare mixing matrix, is diagonalizable as well, and its UV asymptotics reduces in the diagonal basis to the multiplicatively renormalizable case:

\[
Z_i(x, \mu) = \exp \left( \int \frac{\gamma_i}{g(x)} dg \right) = \left( \frac{g(\mu)}{g(x)} \right)^{\gamma_i/\beta_0} \tag{10}\]

with \(Z_i(x, \mu)\) and \(\gamma_i\) the eigenvalues of the corresponding matrices.

The key step in [1] to obtain the above result has been the differential-geometric interpretation [1] of a change of basis of renormalized operators, i.e., of a (finite) change of renormalization scheme:

\[
O_i'(x) = S_{ik}(x) O_k(x) \tag{11}\]

as a (formal) holomorphic invertible gauge transformation \(S(g)\) of \(A(g)\):

\[
A(g) = -\frac{\nu(g)}{\beta(g)} = \frac{\gamma_0}{\beta_0} + \cdots \tag{12}\]

as a (formal) meromorphic connection, with a Fuchsian singularity \(^7\) – i.e., a simple pole – at \(g = 0\), that transforms by the gauge transformation \(S(g)\) as:

\[
A'(g) = S(g) A(g) S^{-1}(g) + \frac{\partial S(g)}{\partial g} S^{-1}(g) \tag{13}\]

of \(\mathcal{D}\) as the corresponding covariant derivative:

\[
\mathcal{D} = \frac{\partial}{\partial g} - A(g) \tag{14}\]

that defines the linear system:

\[
\mathcal{D} X = \left( \frac{\partial}{\partial g} - A(g) \right) X = 0 \tag{15}\]

whose solution \(X(g)\), with a suitable initial condition, is \(Z(x, \mu)\), and finally of \(Z(x, \mu)\):

\[
Z(x, \mu) = P \exp \left( \int g(\mu) A(g) dg \right) = P \exp \left( -\int g(\mu) \frac{\gamma(g)}{\beta(g)} dg \right) \tag{16}\]

as a Wilson line that transforms as:

\[
Z'(x, \mu) = S(g(\mu)) Z(x, \mu) S^{-1}(g(x)) \tag{17}\]

for the gauge transformation \(S(g)\).

Following the interpretation above, the easiest way to compute the UV asymptotics of \(Z(x, \mu)\) consists in setting the meromorphic connection \(A(g)\) in a canonical form by a suitable holomorphic gauge transformation according to the Poincaré–Dulac theorem.

Consequently, the classification in [1] of operator mixing is as follows:

If a renormalization scheme exists where \(-\frac{\nu(g)}{\beta(g)}\) can be set in the canonical form of Eq. (1), we refer to the mixing as nonresonant that by Eq. (4) is the generic case. Otherwise, we refer to the mixing as resonant.

Besides, \(\frac{\nu_0}{\beta_0}\) may be either diagonalizable or nondiagonalizable.

Therefore, there are four cases of operator mixing:

- (I) Nonresonant diagonalizable \(\frac{\nu_0}{\beta_0}\).
- (II) Resonant diagonalizable \(\frac{\nu_0}{\beta_0}\).
- (III) Nonresonant nondiagonalizable \(\frac{\nu_0}{\beta_0}\).
- (IV) Resonant nondiagonalizable \(\frac{\nu_0}{\beta_0}\).

In the case (I), \(Z(x, \mu)\) is diagonalizable [1] to all orders of perturbation theory, since the mixing is nonresonant and \(\frac{\nu_0}{\beta_0}\) is diagonalizable.

The remaining cases, where \(Z(x, \mu)\) is not actually diagonalizable, are analyzed in greater detail in the present paper.

We believe, as already remarked in [1], that the geometric interpretation above and the employment of the Poincaré–Dulac theorem make the subject of operator mixing in the physics literature more transparent than in previous treatments \([5,6,37,38]\).
3 Plan of the paper

In Sect. 4, just as a preamble, we work out by elementary methods three examples of operator mixing that, in the special case of two operators, are paradigmatic of the general case.

In Sect. 5, which contains our main arguments and results, we analyze the four cases, (I), (II), (III) and (IV), in the classification above based on the Poincarè–Dulac theorem. Specifically, we work out the corresponding canonical forms for $-\frac{\gamma N}{\beta}$ and $Z(x, \mu)$ and its UV asymptotics. Moreover, we argue that the cases (III) and (IV) – where $\gamma_0$ is nondiagonalizable – are ruled out by unitarity of the free conformal limit at $g = 0$ in the gauge-invariant Hermitian sector of a massless QCD-like theory.

In Sect. 6, we revisit our elementary computation in Sect. 4 in the light of the Poincarè–Dulac theorem, obviously finding perfect agreement.

In Sect. 7, as an application of the general theory in the present paper, we provide physical realizations of the cases (I) and (II). Specifically, we demonstrate that the case (II) is actually realized for the mixing of four-quark operators in SU(N) massless QCD with $N_f = N$ flavors of quarks for every $N \geq 4$. Besides, we show that unitarity is implemented at $g = 0$ for the mixing of scalar dimension-8 operators in large-$N$ SU(N) YM theory, despite the aforementioned mixing would be potentially nonunitary, since $\frac{\gamma_0}{\beta_0}$ is potentially nondiagonalizable in this case, due to the degeneration of some of its eigenvalues.

In Appendix A we discuss asymptotic versus exact correlators.

4 A preamble: three examples for the mixing of two operators by elementary methods

In case $A(g) = -\frac{\gamma(g)}{\beta(g)}$ is upper triangular, we may compute $Z(x, \mu)$ from:

$$\frac{\partial Z}{\partial g} = A(g)Z$$

(18)

directly, avoiding the intricacies of the Poincarè–Dulac theorem.

Indeed, in this case $A(g)$ may be decomposed into the sum of the diagonal, $A_A(g)$, and nilpotent, $A_N(g)$, contributions:

$$A(g) = A_A(g) + A_N(g)$$

(19)

Then, $Z(x, \mu)$ may be computed in the form:

$$Z(x, \mu) = Z_A(x, \mu)Z_N(x, \mu)$$

(20)

provided that:

$$\frac{\partial Z_N}{\partial g} = Z_A^{-1}A_N(g)Z_A$$

and:

$$\frac{\partial Z_A}{\partial g} = A_A(g)Z_A$$

(22)

Therefore:

$$Z_A(x, \mu) = \exp \left( \int_{g(x)}^{g(\mu)} A_A(g)dg \right)$$

(23)

and:

$$Z_N(x, \mu) = P \exp \left( \int_{g(x)}^{g(\mu)} Z_A^{-1}A_N(g)Z_A dg \right)$$

(24)

Hence, since $Z_A^{-1}A_N(g)Z_A$ is nilpotent as well, the expansion of the path-ordered exponential for $Z_N(x, \mu)$ terminates at a finite order, and $Z(x, \mu)$ is computable in a closed form.

4.1 Nonresonant versus resonant mixing

In a massless QCD-like theory, according to Eq. (43):

$$A(g) = \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_{2n}g^{2n} \right)$$

(25)

with:

$$A_0 = \frac{\gamma_0}{\beta_0}$$

(26)

We work out for two operators three examples of mixing, which are paradigmatic of the general case.

Firstly, we set:

$$A_0 = A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(27)

diagonal, with $\lambda_1 \geq \lambda_2$, i.e., we display the eigenvalues of $A_0$ in nonincreasing order. Clearly, by Eq. (26), this is the case that $\gamma_0$ is diagonalizable.

Besides, we set:

$$A_{2k} = N_{2k} = \begin{pmatrix} 0 & v_{12} \\ 0 & 0 \end{pmatrix}$$

(28)

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8 These examples do not necessarily arise from a massless QCD-like theory. Physical examples are worked out in Sect. 7.
upper triangular, with \( \nu \) a real nonvanishing number, and \( \lambda_{2n} = 0 \) for \( n \neq k \). Hence, in our example:

\[
\frac{\partial Z}{\partial g} = \left( A g^{-1} + N_{2k} g^{2k-1} \right) Z
\]  

(29)

For \( A_0 \) diagonal, we consider the following two cases that we dub respectively, according to the terminology in Sect. 2, nonresonant diagonalizable \( \frac{Z_0}{\nu} \):

\[
\lambda_1 - \lambda_2 \neq 2k
\]  

(30)

and resonant diagonalizable \( \frac{Z_0}{\nu} \):

\[
\lambda_1 - \lambda_2 = 2k
\]  

(31)

with \( k \) a positive integer.

Secondly, we consider the case:

\[
\lambda_1 = \lambda_2 = \lambda
\]  

(32)

and:

\[
A_0 = A + N_0 = \begin{pmatrix} \lambda & \nu_{12} \\ 0 & \lambda \end{pmatrix}
\]  

(33)

which we dub, according to the terminology in Sect. 2, non-resonant nondiagonalizable \( \frac{Z_0}{\nu} \), as now \( A_0 \) is not diagonalizable, since its eigenvalues coincide and \( \nu_{12} \) is assumed to be nonzero.

We compute the corresponding \( Z(x, \mu) \) by the formulas above.

\textbf{4.1.1 Nonresonant diagonalizable \( \frac{Z_0}{\nu} \)}

\( Z_A \) is diagonal and computed by exploiting Eq. (23):

\[
Z_A(x, \mu) = \exp \int_{g(x)}^{g(\mu)} A d g = \begin{pmatrix} \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_1} & 0 \\ 0 & \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_2} \end{pmatrix}
\]  

(34)

Moreover, by direct evaluation:

\[
Z_A^{-1} A_N(g) Z_A = \begin{pmatrix} 1 & \nu_{12} g^{2k-1-\lambda_1+\lambda_2}(\mu) g^{\lambda_1-\lambda_2}(x) \\ 0 & 0 \end{pmatrix}
\]  

(35)

As a consequence:

\[
Z_N(x, \mu) = P \exp \left( \int_{g(x)}^{g(\mu)} Z_A^{-1} A_N(g) Z_A d g \right)
\]

Finally, combining Eq. (34) for \( Z_A \) and the above result for \( Z_N \), we get:

\[
Z(x, \mu) = \begin{pmatrix} \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_1} & 0 \\ 0 & \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_2} \end{pmatrix} \times \begin{pmatrix} 1 & \nu_{12} g^{2k-1-\lambda_1+\lambda_2}(\mu) g^{\lambda_1-\lambda_2}(x) \\ 0 & 0 \end{pmatrix}
\]

(36)

Hence, \( Z(x, \mu) \) is gauge equivalent to the diagonal \( Z_A(x, \mu) \).

\textbf{4.1.2 Resonant diagonalizable \( \frac{Z_0}{\nu} \)}

\( Z_A \) is given again by Eq. (34), but now:

\[
Z_A^{-1} A_N(g) Z_A = \begin{pmatrix} 1 & \nu_{12} g^{2k}(\mu) \\ 0 & 0 \end{pmatrix}
\]  

(39)

It follows from Eq. (24) that:

\[
Z_N(x, \mu) = P \exp \left( \int_{g(x)}^{g(\mu)} Z_A^{-1} A_N(g) Z_A d g \right)
\]
Finally, we specialize the case above to $k = 0$. We get for
\[ Z(x, \mu) = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & \lambda \\ \end{pmatrix} \begin{pmatrix} g(\mu) \\ g(x) \end{pmatrix} \]
and $Z_N(x, \mu)$, we obtain:
\[
Z(x, \mu) = \begin{pmatrix} (g(\mu)/g(x))^{\lambda_1} & 0 \\ 0 & (g(\mu)/g(x))^{\lambda_2} \end{pmatrix} \times \begin{pmatrix} I + \left( 0 \ v_{12}g^{2k}(x) \log \frac{g(\mu)}{g(x)} \right) \end{pmatrix} = \begin{pmatrix} (g(\mu)/g(x))^{\lambda_1} & 0 \\ 0 & (g(\mu)/g(x))^{\lambda_2} \end{pmatrix}
\]
Again, because of the occurrence of the term containing $g^{2k}(x)/g(x)$, $Z(x, \mu)$ is not diagonalizable by a holomorphic gauge transformation.

5 Main arguments and results: operator mixing by the Poincarè–Dulac theorem

5.1 Resonant canonical form of $-\frac{\gamma(g)}{\beta(g)}$ by the Poincarè–Dulac theorem

In a massless QCD-like theory the meromorphic connection $A(g)$ (Sect. 2) admits the (formal) expansion [1]:
\[
A(g) = -\frac{\gamma(g)}{\beta(g)} = \frac{1}{g} \left( A_0 + \sum_{k=1}^{\infty} A_{2k} g^{2k} \right)
\]
in odd powers of $g$, with the first few coefficients given by:
\[
A_0 = \frac{\gamma_0}{\beta_0} \quad A_2 = \frac{\beta_0 \gamma_1 - \beta_1 \gamma_0}{\beta_0^2} \quad A_4 = \frac{\beta_1^2 \gamma_0 - \beta_0 \beta_2 \gamma_0 - \beta_0 \beta_1 \gamma_1 + \beta_0^2 \gamma_2}{\beta_0^3}
\]
In general, by the Poincarè–Dulac theorem [2] (Sect. 5.5), $A(g)$ can be set by a (formal) holomorphic invertible gauge transformation in the canonical resonant form:
\[
A'(g) = \frac{1}{g} \left( A + N_0 + \sum_{k=1}^{\infty} N_{2k} g^{2k} \right)
\]
where:
\[
A_0 = A + N_0
\]
is upper triangular, with eigenvalues $\text{diag}(\lambda_1, \lambda_2, \ldots) = A$ in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$, and nilpotent part, $N_0$, in normal Jordan form. The upper triangular nilpotent matrices $N_{2k}$ satisfy:
\[
g^A N_{2k} g^{-A} = g^{2k} N_{2k}
\]
i.e., their only nonzero entries, $(N_{2k})_{ij}$, are such that:
\[ \lambda_i - \lambda_j = 2k \]
for $i < j$ and $k$ a positive integer. Besides:
\[ g^A N_0 g^{-A} = N_0 \quad (51) \]
as well, since $[A, N_0] = 0$ according to the Jordan normal form of $A_0$. Moreover, the sum in Eq. (47) contains only a finite number of terms, contrary to Eq. (43). Indeed, the number of differences of the eigenvalues is finite, and therefore, because of Eq. (49), so it is the number of terms in Eq. (47).

Equation (50) is the resonance condition for the eigenvalues of the linear system associated to $A(g)$:
\[ DX = \left( \frac{\partial}{\partial g} - A(g) \right) X = 0 \quad (52) \]
whose solution with a suitable initial condition (Sects. 5.6 and 5.7) is $Z(x, \mu)$.

In fact, from the proof (Sect. 5.5) of the Poincarè–Dulac theorem it follows that, once $A_0$ has been set in Jordan normal form by a global gauge transformation, precisely only the resonant terms in Eq. (47) may survive after the gauge transformation that sets Eq. (43) in the aforementioned canonical form. In this case, the linear system is resonant and consequently the associated operator mixing has been dubbed resonant in [1].

5.2 Nonresonant canonical form of $-\frac{\gamma(x)}{\beta(x)}$ by the Poincarè–Dulac theorem

Hence, a sufficient condition for all the resonant terms to be absent in Eq. (47) is that the eigenvalues of $A_0$ in nonincreasing order, $\lambda_i \geq \lambda_j$ for $i \leq j$, satisfy [1]:
\[ \lambda_i - \lambda_j \neq 2k \quad (53) \]
with $k$ a positive integer. The advantage of this sufficient condition is that it is easily verified a priori from the only knowledge of the eigenvalues of $A_0 = \frac{\gamma_0}{\beta_0}$ — a one-loop quantity — [1].

With more effort, we can refine the sufficient condition above into a necessary and sufficient condition: If we set $A_0$ in the Jordan normal form of Eq. (48), the necessary and sufficient condition for the linear system in Eq. (52), with $A(g)$ defined by Eq. (43), to admit by a holomorphic gauge transformation the nonresonant canonical form:
\[ A'(g) = \frac{\Lambda + N_0}{g} \quad (54) \]
is that all of the matrix elements $(N_{2k})_{ij}$ in Eq. (47), with $\lambda_i - \lambda_j = 2k$, vanish.

Of course, if $\lambda_i - \lambda_j = 2k$, this may only be verified by constructing iteratively (Sect. 5.5) the canonical form above. Otherwise, if $\lambda_i - \lambda_j \neq 2k$, no nonvanishing $N_k$ in Eq. (47) may occur.

In both the cases above the linear system is nonresonant, and consequently the associated operator mixing has been dubbed nonresonant in [1].

5.3 Classification of operator mixing by the Poincarè–Dulac theorem

Therefore, the Poincarè–Dulac theorem reduces the classification of operator mixing to the four cases [1] that we summarize below.

5.3.1 (I) Nonresonant diagonalizable $\frac{\gamma_0}{\beta_0}$

The linear system is nonresonant and $\frac{\gamma_0}{\beta_0}$ is diagonalizable.

For the system to be nonresonant, it is sufficient that the eigenvalues of $\frac{\gamma_0}{\beta_0}$ in nonincreasing order satisfy $[1]$:
\[ \lambda_i - \lambda_j \neq 2k \quad (55) \]
with $i \leq j$ and $k$ a positive integer.

For $\frac{\gamma_0}{\beta_0}$ to be diagonalizable, it is sufficient that its eigenvalues are all different.

As we have mentioned above, for the system to be nonresonant, the necessary and sufficient condition is that in the canonical form of Eq. (47) all the resonant terms vanish.

5.3.2 (II) Resonant diagonalizable $\frac{\gamma_0}{\beta_0}$

The linear system is resonant and $\frac{\gamma_0}{\beta_0}$ is diagonalizable.

For the system to be resonant, a necessary condition is that, for at least two eigenvalues in nonincreasing order, it holds:
\[ \lambda_i - \lambda_j = 2k \quad (56) \]
with $i < j$ and $k$ a positive integer.

In this case, a necessary and sufficient condition is that, correspondingly, at least one $N_{2k}$ in the canonical resonant form does not vanish.

The sufficient condition for $\frac{\gamma_0}{\beta_0}$ to be diagonalizable is as in the case (I).

5.3.3 (III) Nonresonant nondiagonalizable $\frac{\gamma_0}{\beta_0}$

The linear system is nonresonant and $\frac{\gamma_0}{\beta_0}$ is nondiagonalizable.

The nonresonant condition is as in the case (I).

The necessary condition for $\frac{\gamma_0}{\beta_0}$ to be nondiagonalizable is that at least two of its eigenvalues coincide.
The linear system is resonant and \( \frac{\partial x}{\partial t} \) is nondiagonalizable. The resonant condition is as in the case (II).

The necessary condition for \( \frac{\partial x}{\partial t} \) to be nondiagonalizable is as in the case (III).

5.4 A unitarity constraint

A massless QCD-like theory is conformal [39] invariant to the leading, \( O(g^0) \), and next-to-leading, \( O(g^2) \), perturbative order, since the beta function affects the solution of the Callan–Symanzik equation only starting from order \( g^4 \).

We argue that, if we assume that it is also unitary in its free conformal limit at \( g = 0 \) in the sector defined by gauge-invariant Hermitian operators in Minkowskian space-time, then the corresponding \( \frac{\partial x}{\partial t} \) should be diagonalizable. Thus, the aforementioned unitarity rules out the cases (III) and (IV).

The unitarity assumption above is satisfied in a massless QCD-like theory with a compact gauge group and matter fields satisfying the spin statistics theorem in the Hermitian gauge-invariant sector, as unitary gauges exist where the free limit is certainly unitary for the gluon and matter fields, and the gauge-fixing ghosts decouple in the correlators of gauge-invariant operators.

We demonstrate momentarily the aforementioned link between unitarity and diagonalizability of \( \frac{\partial x}{\partial t} \) for scalar operators in the conformal free limit. The analog argument for higher-spin operators will appear in a forthcoming paper [4].

Firstly, if \( \frac{\partial x}{\partial t} \) is nondiagonalizable, to order \( g^2 \) a logarithmic conformal field theory (logCFT) arises, which is known to be nonunitary [40,41].

Specifically, either in a CFT or logCFT, the 2-point correlators of Euclidean [12] scalar primary conformal operators corresponding [44] to Hermitian operators in Minkowskian space-time, \( G_{\text{conf}}(x) \), satisfy the Callan–Symanzik equation:

\[
x \cdot \frac{\partial}{\partial x} G_{\text{conf}}(x) + \Delta G_{\text{conf}}(x) + G_{\text{conf}}(x) \Delta T = 0 \tag{57}
\]

whose general solution is:

\[
G_{\text{conf}}(x) = \langle O(x) O(0) \rangle = e^{-\Delta \log \sqrt{x^2 \mu^2}} G e^{-\Delta T \log \sqrt{x^2 \mu^2}} \tag{58}
\]

in matrix notation, where \( \Delta \) is the matrix of the conformal dimensions, \( G \) is a real symmetric matrix [44] independent of space-time and the tensor product between repeated \( O \) is understood.

If \( \Delta \) is diagonalizable, a CFT occurs. Otherwise, if \( \Delta \) is nondiagonalizable, a logCFT [40,41] arises.

Moreover, both in a CFT and logCFT, for Euclidean primary conformal operators corresponding to Hermitian Minkowskian operators [44], the operators/states correspondence holds [41,44]:

\[
O(0)|0\rangle = |O_{in}\rangle
\]

\[
\langle O_{out} | O_{in} \rangle = \lim_{x \to \infty} \langle 0 | e^{2\Delta \log \sqrt{x^2 \mu^2}} O(x) \rangle = \lim_{x \to \infty} e^{2\Delta \log \sqrt{x^2 \mu^2}} e^{-\Delta T \log \sqrt{x^2 \mu^2}} \langle O_{out} | O_{in} \rangle = \lim_{x \to \infty} e^{\Delta \log \sqrt{x^2 \mu^2}} G e^{-\Delta T \log \sqrt{x^2 \mu^2}} \tag{59}
\]

As a consequence, the scalar product in matrix notation reads:

\[
\langle O_{out} | O_{in} \rangle = \lim_{x \to \infty} \langle 0 | e^{2\Delta \log \sqrt{x^2 \mu^2}} O(x) \rangle \tag{60}
\]

In order to be well defined, the scalar product in Eq. (60) must be independent of the variable \( \sqrt{x^2 \mu^2} \). Expanding the last line of Eq. (60) in powers of \( \log \sqrt{x^2 \mu^2} \), we get:

\[
\langle O_{out} | O_{in} \rangle = \left(I + \Delta G \log \sqrt{x^2 \mu^2} + \cdots \right) G \times \left(I - \Delta T \log \sqrt{x^2 \mu^2} + \cdots \right) = G + \left(\Delta G - \Delta T \log \sqrt{x^2 \mu^2} + \cdots \right) \tag{61}
\]

Then, the independence of the coordinates implies:

\[
\Delta G - \Delta T = 0 \tag{62}
\]

and:

\[
\langle O_{out} | O_{in} \rangle = G \tag{63}
\]

Besides, in a massless QCD-like theory, because of the existence of the perturbative expansion, it holds to order \( g^2 \):

\[
\Delta(g) = D I + g^2 \gamma_0 + \cdots \tag{64}
\]

\[
\mathcal{G}(g) = G_0 + g^2 G_1 + \cdots
\]

in the conformal renormalization scheme [39], with \( D \) the canonical dimension of the operators \( O \). Hence, expanding Eq. (62) to order \( g^2 \), we obtain:

\[
\gamma_0 G_0 - G_0 \gamma_0^T = 0 \tag{65}
\]

Most interestingly, Eq. (65) constrains \( G_0 \), which arises to order \( g^0 \), by means of \( \gamma_0 \), which arises to order \( g^2 \), the reason being the existence of the conformal structure to order \( g^2 \). The consequences of Eq. (65) follow:
If $\gamma_0$ is diagonalizable, by Eq. (65) $G_0$ commutes with $\gamma_0$ in the diagonal basis and thus in any basis. Besides, $G_0$, being a real symmetric matrix, is diagonalizable as well, and therefore $G_0$ and $\gamma_0$ are simultaneously diagonalizable.

This is the CFT case, where unitarity in the conformal free limit at $g = 0$ requires that $G_0$ has positive eigenvalues according to Eqs. (63) and (64) specialized to $g = 0$.

If instead $\gamma_0$ is nondiagonalizable, i.e., in the logCFT case, $G_0$ has necessarily negative eigenvalues, i.e., the theory is nonunitary in its free conformal limit at $g = 0$ in the gauge-invariant Hermitian sector.

Indeed, if $\gamma_0$ is nondiagonalizable, Eq. (62) nontrivially constrains the structure of $G_0$. Firstly, in this case:

$$G_0 = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_n \\ g_2 & g_3 & \ddots & \vdots \\ g_3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
g_n & \cdots & \cdots & 0 \end{pmatrix}$$

for some real $g_i$ in the basis where $\gamma_0$ has the canonical Jordan form:

$$\gamma_0 = \gamma_0 D I + N_0$$

(67)

with $\gamma_0D$ the eigenvalue of the Jordan block and $N_0$ nilpotent and upper diagonal with all the nonvanishing entries equal to 1. Then, Eq. (65) reads:

$$N_{ia}G_{0aj} = G_{0ia}N_{aj}^T$$

(68)

with:

$$N_{0kj} = \begin{cases} \delta_{j-1} & i = 1, \ldots, n; \ j = 2, \ldots, n \\ 0 & i = 1, \ldots, n; \ j = 1 \end{cases}$$

(69)

and:

$$N_{0kj}^T = \begin{cases} \delta_{i-1} & i = 2, \ldots, n; \ j = 1, \ldots, n \\ 0 & i = 1; \ j = 1, \ldots, n \end{cases}$$

(70)

Therefore, Eq. (68) implies:

$$G_{0i+1j} = G_{0ij+1} \quad i, \ j = 1, \ldots, n - 1$$

$$G_{0i+1n} = G_{0nj+1} = 0 \quad i, \ j = 1, \ldots, n - 1$$

(71)

that fixes the form of $G_0$ in Eq. (66).

Moreover, by a constant gauge transformation $S$ of the form:

$$S = \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ 0 & s_0 & \cdots & s_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_0 \end{pmatrix}$$

(72)

that commutes with $N_0$, $G_0$ transforms as [4]:

$$G'_0 = SG_0S^T$$

(73)

and may be set in the canonical form [41]:

$$G'_0 = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 \end{pmatrix}$$

(74)

It turns out that $G'_0$ has at least $\left\lceil r/2 \right\rceil$ positive eigenvalues and $\left\lfloor r/2 \right\rfloor$ negative eigenvalues, with $r$ the rank of $G_0'$ and $\left\lceil r/2 \right\rceil$ the integer part of $r/2$ [41]. To prove the preceding statement, we observe that:

$$G''_0 = I$$

(75)

Indeed, $G''_{0ij} = \delta_{i,n-j+1}$ and as a consequence:

$$G''_{0ij} = G'_{0ia}G'_{0aj} = \delta_{i,n-a+1}\delta_{a,n-j+1} = \delta_{ij}$$

(76)

Hence, the eigenvalues of $G'_0$ are $\pm 1$. Moreover, the trace of $G'_0$ is either 0 or 1, depending on whether $r$ is even or odd respectively.

As a consequence, since the trace of a matrix is the sum of its eigenvalues, if $r$ is even, $G'_0$ has $r/2$ positive eigenvalues and $r/2$ negative eigenvalues, otherwise, if $r$ is odd, $G'_0$ has $\left\lceil r/2 \right\rceil + 1$ positive eigenvalues and $\left\lfloor r/2 \right\rfloor$ negative eigenvalues.

By summarizing, the key point of the argument above is that the nondiagonalizability of $\gamma_0$ and the existence of the conformal structure to order $g^2$ determine the structure of $G_0$ that controls the scalar product in the free conformal limit, in such a way that the free conformal limit is nonunitary if $\gamma_0$ is nondiagonalizable.

Finally, we may extend the perturbative argument above about the existence of the scalar product to order $g^2$ to all orders of perturbation theory, by considering a massless QCD-like theory at its conformal Wilson-Fisher fixed point $g_*$, with $\beta(g_*, \epsilon) = -g_*\epsilon + \beta(g_*) = 0$, introduced.

---

$^{13}$ Not all $g_i$ may vanish, otherwise the correlator would vanish in the free conformal limit.
in [45, 46] to perform higher-loop computations in dimensional regularization – in \(d = 4 - 2\epsilon\) dimensions – of the anomalous-dimension matrices in massless QCD.

Indeed, the anomalous-dimension matrix \(\gamma(g_*)\) at the fixed point has the same coefficients [45, 46] – as a series in \(g_*\) – as the anomalous-dimension matrix \(\gamma(g)\) – as a series in \(g\) – and specifically the same \(\gamma_0\). Moreover, since the theory is conformal to all perturbative orders at the fixed point, the associated scalar product exists to all orders in \(g_*\).

Either way, the perturbative conformal symmetry to order \(g^2\) or the conformal symmetry to all orders at the aforementioned Wilson-Fisher fixed point, and the lowest-order unitarity, rule out the cases (III) and (IV) of operator mixing in the gauge-invariant Hermitian sector of a massless QCD-like theory. The statement above does not necessarily apply to operators outside the gauge-invariant sector, whose correlators may be affected by the mixing with the ghost sector, which does not need to be unitary.

5.5 A condensed proof of the Poincarè–Dulac theorem

We provide a condensed proof of (the linear version of) the Poincarè–Dulac theorem following [2].

**Poincarè–Dulac theorem:**

The most general linear system with a Fuchsian singularity at \(g = 0\), where the meromorphic connection \(A(g)\) admits the (formal) expansion:

\[
A(g) = \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right)
\]

may be set, by a (formal) holomorphic invertible gauge transformation, in the Poincarè–Dulac–Levelt normal form: 14

\[
A'(g) = \frac{1}{g} \left( \Lambda + N_0 + \sum_{k=1}^{\infty} N_k g^k \right)
\]

where \(\Lambda + N_0\) is the Jordan normal form of \(A_0\), its eigenvalues \(\text{diag}(\lambda_1, \lambda_2 \ldots) = \Lambda\) are in nonincreasing order \(\lambda_1 \geq \lambda_2 \geq \cdots\), \(N_0\) is nilpotent and upper triangular, and the nilpotent upper triangular matrices \(N_k\) satisfy:

\[
g^A N_k g^{-A} = g^k N_k
\]

for \(k = 1, 2, \ldots\), i.e., the only possibly nonvanishing matrix elements, \((N_k)_{ij}\), of the \(N_k\) are associated to the resonant eigenvalues:

\[
\lambda_i - \lambda_j = k
\]

with \(i < j\) and \(k\) a positive integer.

Incidentally, also \(g^A N_0 g^{-A} = N_0\), since \([N_0, \Lambda] = 0\) by the Jordan normal form of \(A_0\).

Of course, if either the eigenvalues are nonresonant or the resonant matrix coefficients \(N_k\) – associated to the resonant eigenvalues – vanish, the linear system collapses into the Euler form 15 [2]:

\[
A'(g) = \frac{1}{g} (A + N_0)
\]

We only report the key aspects of the proof, leaving more details to [2].

**Proof** The demonstration proceeds by induction on \(k = 1, 2, \ldots\) by proving that, once \(A_0\) and the first \(k - 1\) matrix coefficients, \(A_1, \ldots, A_{k-1}\), have been set in the Poincarè–Dulac–Levelt normal form above, a holomorphic gauge transformation exists that leaves them invariant and also puts the \(k\)-th coefficient, \(A_k\), in normal form.

The step 0 of the induction consists just in putting \(A_0\) in Jordan normal form – with eigenvalues in nonincreasing order and \(N_0\) upper triangular, as in the statement of the theorem – by a global (i.e., constant) gauge transformation.

At the \(k\)-th step, we choose the holomorphic gauge transformation of the form:

\[
S_k(g) = 1 + g^k H_k
\]

with \(H_k\) a matrix to be found momentarily. Its inverse is:

\[
S_k^{-1}(g) = (1 + g^k H_k)^{-1} = 1 - g^k H_k + \cdots
\]

where the dots represent terms of order higher than \(g^k\). The gauge action of \(S_k(g)\) on the connection \(A(g)\) furnishes:

\[
A'(g) = kg^{k-1} H_k (1 + g^k H_k)^{-1}
\]

\[
+ (1 + g^k H_k) A(g)(1 + g^k H_k)^{-1}
\]

\[
= kg^{k-1} H_k (1 + g^k H_k)^{-1}
\]

\[
+ (1 + g^k H_k) \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right) (1 + g^k H_k)^{-1}
\]

\[
= kg^{k-1} H_k (1 - \cdots)
\]

\[
+ (1 + g^k H_k) \frac{1}{g} \left( A_0 + \sum_{n=1}^{\infty} A_n g^n \right) (1 - g^k H_k + \cdots)
\]

\[
= kg^{k-1} H_k + \frac{1}{g} \left( A_0 + \sum_{n=1}^{k} A_n g^n \right)
\]

\[
+ g^{k-1}(H_k A_0 - A_0 H_k) + \cdots
\]

\[
= g^{k-1}(k H_k + A_0 A_0 - A_0 H_k) + A_{k-1}(g) + g^{k-1} A_k + \cdots
\]

where we have skipped in the dots all the terms that contribute to an order higher than \(g^{k-1}\), and we have put:

\[
A_{k-1}(g) = \frac{1}{g} \left( A_0 + \sum_{n=1}^{k-1} A_n g^n \right)
\]

14 In the present paper, we refer to it as the resonant canonical form.

15 In the present paper, we refer to it as the nonresonant canonical form.
that is the part of $A(g)$ that is not affected by the gauge transformation $S_k(g)$ and thus verifies the hypothesis of the induction.

Therefore, by Eq. (84) the $k$-th matrix coefficient, $A_k$, may be eliminated from the expansion of $A'(g)$ to order $g^{k-1}$ provided that an $H_k$ exists such that:

$$A_k + (kH_k + H_kA_0 - A_0H_k) = A_k + (k - adA_0)H_k = 0 \quad (86)$$

with $adA_0Y = [A_0, Y]$. If the inverse of $adA_0 - k$ exists, the unique solution for $H_k$ is:

$$H_k = (adA_0 - k)^{-1}A_k \quad (87)$$

Therefore, the only matrix coefficients that may not be removed from the expansion of $A'(g)$ at the $k$-th step of the induction belong to the subspace where $adA_0 - k$ is not invertible.

Hence, we should demonstrate that, for $k$ positive, the elements $Y_0$ of the aforementioned subspace satisfy the condition in Eq. (79) for $N_k$, according to the statement of the theorem.

To understand what is going on, it is convenient to suppose initially that $N_0 = 0$, i.e., that $A_0$ is diagonalizable [1].

In this case, $adA_0 - k = adA - k$, as a linear operator that acts on matrices, is diagonal with eigenvalues $\lambda_i - \lambda_j - k$ and the matrices $E_{ij}$, whose only nonvanishing entries are $(E_{ij})_{ij}$, as eigenvectors. Moreover, $adA - k$ is invertible if and only if its kernel contains only the zero matrix.

The eigenvectors $E_{ij}$, normalized in such a way that

$$E_{ij} |E'_{i'}j'| \delta_{i'i'} = (\delta_{ii'}\delta_{jj'}) \delta_{i'i'}$$

with $\langle A|B \rangle = Tr(\bar{A}B)$ and $\bar{A}$ the adjoint of the matrix $A$.

Thus, $E_{ij}$ belongs to the kernel of $adA - k$ if and only if $\lambda_i - \lambda_j - k = 0$ and $i < j$, as $k$ is a positive integer.

As a consequence, the $E_{ij}$ in the kernel satisfy Eq. (79), according to the statement of the theorem:

$$g^A E_{ij} g^{-A} = g^{\lambda_i - \lambda_j} E_{ij} = g^{k}E_{ij} \quad (88)$$

Now we suppose that $N_0$ does not vanish, i.e., $A_0$ is non-diagonalizable.

Hence, $A_0$ admits a canonical Jordan form as in the statement of the theorem.

The key point is that now $adA_0 - k$, as a linear operator that acts on matrices, is lower triangular for the following ordering of the matrix basis.

We may choose an increasing sequence, $\text{diag}(q_1, q_2, \ldots) = Q$, of rationally independent weights, $q_i$, [2] in such a way that the corresponding weight for $E_{ij}$ is $q_j - q_i - \delta_{ij}$, as computed via $g^{-Q} E_{ij} g^Q = g^{q_j - q_i - \delta_{ij}} E_{ij}$. Thus, we may order our basis in such a way that the sequence of basis vectors $E_l$ with $l = 1, 2, \ldots$ coincides with the following sequence of the $E_{ij}$ ordered with nondecreasing weights: The $E_{ij}$ for $i \neq j$ with strictly increasing weights, and the $E_{ii}$ – which have weight 0 – with $i$ increasing.

The action of $adA$ on the above basis leaves the weights of the $E_{ij}$ with $i \neq j$ invariant, and sends to zero the $E_{ii}$, in such a way that the action of $adA$ is diagonal on the entire basis.

Instead, the action of $adN_0$ on the entire basis produces a linear combination of terms with strictly increased weights, since $N_0$ is upper triangular and, therefore, it is the sum of terms with positive weights, and for each of these terms the commutator with any $E_{ij}$ strictly increases the weights.

Moreover:

$$(adA + N_0 - k)E_i = E_k (adA + N_0 - k)E_i$$

$$= E_k (adA + N_0 - k)_{hl} \quad (89)$$

where the sum on the index $h$ is understood. Hence, with this ordering of the basis, the matrix:

$$(adA + N_0 - k)_{hl} = (E_k)(adA + N_0 - k)E_i$$

is lower triangular and its eigenvalues coincide with the eigenvalues of $adA - k$ [2].

Now $adA + N_0 - k$ is not invertible if and only if at least one of its eigenvalues vanishes. But its eigenvalues coincide with the eigenvalues of $adA - k$. Therefore, $adA + N_0 - k$ is invertible on the orthogonal complement of the kernel of $adA - k$, as it is for $adA - k$.

Hence, every matrix coefficient $A_k$ orthogonal to the kernel of $adA - k$ may be removed from $A'(g)$, as in the diagonalizable case with $N_0 = 0$ above.

Obviously, the resonant matrix coefficients $N_k$ are finite in number, because there are only a finite number of differences of the eigenvalues.

As consequence, from a certain point on, all the remaining terms in the expansion of $A'(g)$ may be removed, because they belong to the orthogonal complement of the kernel of $adA_0 - k$, and the proof is complete.

5.6 Fundamental solution of the linear system

A fundamental solution of the linear system in Eq. (52) in the canonical resonant form of Eq. (78) is [2]:

$$X(g) = g^A g^N \quad (91)$$

with $N = N_0 + \sum_{k=1}^{r} N_k$, as we verify by direct computation [2]:

$$\frac{\partial X(g)}{\partial g} X^{-1}(g) = g^A \frac{A + N}{g} g^N g^{-N} g^{-A}$$

$$= g^A \frac{A + N}{g} g^{-A}$$

$$= \frac{A + g^A N g^{-A}}{g}$$
Moreover, \( X(g) \) may be computed in a closed form, since the expansion of \( g^N \) in powers of \( \log g \) terminates because \( N \) is nilpotent.

Correspondingly, the solution \( X(g)X^{-1}(g_0) \) of Eq. (52) in the canonical form of Eq. (78) that reduces to the identity at \( g = g_0 \) may be computed in a closed form as well:

\[
X(g)X^{-1}(g_0) = g^A g^N g_0^{-N} g_0^{-A} = g^A (g_0 g)^N g_0^{-A} = (g_0 g)^A g_0 e^{N \log \frac{g_0}{g}} g_0^{-A} = (g_0 g)^A e^{g_0^N N g_0^{-A} \log \frac{g_0}{g}} = (g_0 g)^A e^{\sum_{k=0}^N g_k^N N_k \log \frac{g_0}{g}}
\]

(93)

5.8 UV asymptotics of \( Z(x, \mu) \)

In the cases (II), (III) and (IV), as the canonical form of \( Z(x, \mu) \) is non-diagonal, its UV asymptotics is intrinsically different from the diagonal case (I).

Indeed, by Eq. (94) \( Z(x, \mu) \) may be factorized into the exponential of a linear combination of upper triangular nilpotent matrices with coefficients that asymptotically in the UV are logs of the running coupling, i.e., asymptotically logs of the coordinates, and a diagonal matrix as in the nonresonant diagonal case (I):

\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^A e^{\sum_{k=0} N g_k \log \frac{g(\mu)}{g(x)}} = \left( \frac{g(\mu)}{g(x)} \right)^A e^{\sum_{k=0}^N g_k^2 N g_k \log \frac{g(\mu)}{g(x)}}
\]

(96)

This is the closest analog of logCFTs that may occur in asymptotically free theories.

Moreover, some subtleties arise in computing the UV asymptotics of \( Z(x, \mu) \), since it follows from Eq. (96) that the factorization of \( Z(x, \mu) \) actually depends on the order of the factors, in such a way that:

\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^A Z(x, \mu) = e^{\sum_{k=0}^N g_k^2 N g_k \log \frac{g(\mu)}{g(x)}}
\]

(97)

but:

\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^A Z(x, \mu) = e^{\sum_{k=0}^N g_k^2 N g_k \log \frac{g(\mu)}{g(x)}} = e^{\sum_{k=0}^N g_k^2 N g_k \log \frac{g(\mu)}{g(x)}}
\]

(98)

Therefore, the limits for \( x \to 0 \) of Eqs. (97) and (98) in general do not coincide.

Specifically, in the case (II), i.e., for \( N_0 = 0 \), the limit is the identity \( I \) for Eq. (97), but it is not finite for Eq. (98).

6 Three examples for the mixing of two operators revisited by the Poincarè–Dulac theorem

For completeness, we verify that the elementary computation in Sect. 4 coincides with the solution of the linear system in canonical form according to the Poincarè–Dulac theorem.

6.1 Nonresonant diagonalizable \( \frac{\beta_0}{\beta_0} \)

\( N_0 = 0 \), because \( \frac{\beta_0}{\beta_0} \) is diagonal. Moreover:

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]

(99)
and:
\[ N_{2k} = 0 \]  
for \( k = 1, 2, \ldots \) because the system is nonresonant. Therefore, by Eq. (94) we obtain:
\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_1} \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]
that matches Eq. (37) up to a holomorphic gauge transformation.

6.2 Resonant diagonalizable \( \frac{N_0}{\beta_0} \)
\[ N_0 = 0 \], because \( \frac{N_0}{\beta_0} \) is diagonal. Moreover:
\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]
and:
\[
N = N_{2k} = \begin{pmatrix}
0 & \nu_{12} \\
0 & 0
\end{pmatrix}
\]
with \( \lambda_1 - \lambda_2 = 2k \), since the system is resonant. Therefore, by Eq. (94) we obtain:
\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_1} \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]
that matches Eq. (41).

6.3 Nonresonant nondiagonalizable \( \frac{N_0}{\beta_0} \)
\( \frac{N_0}{\beta_0} \) is not diagonalizable. Hence:
\[
\Lambda = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}
\]
and:
\[
N = N_0 = \begin{pmatrix}
0 & \nu_{12} \\
0 & 0
\end{pmatrix}
\]
Therefore, by Eq. (94) we obtain:
\[
Z(x, \mu) = \left( \frac{g(\mu)}{g(x)} \right)^{\lambda_1} \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]
that matches Eq. (42).

7 A physical realization of the cases (II) and (I)

7.1 Flavor-changing four-quark operators in SU(\( N \)) QCD
with \( N_f = N \) flavors of quarks

We work out a physical realization of the case (II): The operator mixing of flavor-changing four-quark operators computed in [7]. Specifically, we consider the two sets of operators [7]:
\[
Q_1^{\text{VLR}} = (\bar{s}^a \gamma_\mu P_L d^\beta)(\bar{u}^\beta \gamma^\mu P_R c^a)
\]
\[
Q_2^{\text{VLR}} = (\bar{s}^a \gamma_\mu P_L d^a)(\bar{u}^\beta \gamma^\mu P_R c^\beta)
\]
and:
\[
Q_1^{\text{SLR}} = (\bar{s}^a P_L d^\beta)(\bar{u}^\beta P_R c^a)
\]
\[
Q_2^{\text{SLR}} = (\bar{s}^a P_L d^a)(\bar{u}^\beta P_R c^\beta)
\]
where \( \bar{s}, d, \bar{u} \) and \( c \) are quark operators, and \( P_{L,R} = \frac{1}{2} (1 \mp \gamma_5) \). In [7] the anomalous-dimension matrices have been computed to order \( g^4 \):
\[
\gamma(g) = g^2 \gamma_0 + g^4 \gamma_1 + \cdots
\]
where $\gamma_0$ and $\gamma_1$ are:

$$
\gamma_0^{VLR} = \frac{1}{(4\pi)^2} \begin{pmatrix}
-6 + \frac{6}{N^2} & 0 \\
-6 & \frac{6}{N^2}
\end{pmatrix}
$$

$$
\gamma_1^{VLR} = \frac{1}{(4\pi)^2} \begin{pmatrix}
-203 / 6 + 479 / 6N^2 + 15 / 3N^3 + 10 / 3N_f - 22 / 3N_f \\
-100 / 3 + 5 / 3N^2 + 22 / 3N_f
\end{pmatrix} - \frac{71}{2} - \frac{18}{N^2} + \frac{4N_f}{N} \frac{137}{6N^2} + \frac{15}{3N^2} = \frac{22}{3N_f} \frac{N_f}{N}
$$

(111)

for the VLR operators, and:

$$
\gamma_0^{SLR} = \frac{1}{(4\pi)^2} \begin{pmatrix}
\frac{6}{N^2} & -\frac{6}{N^2} \\
0 & -6 + \frac{6}{N^2}
\end{pmatrix}
$$

$$
\gamma_1^{SLR} = \frac{1}{(4\pi)^4} \begin{pmatrix}
\frac{137}{6N^2} + \frac{15}{3N^2} - \frac{22}{3N_f}N_f \\
-\frac{71}{2} - \frac{18}{N^2} + \frac{4N_f}{N} \frac{137}{6N^2} + \frac{15}{3N^2} = \frac{22}{3N_f} \frac{N_f}{N}
\end{pmatrix}
$$

(112)

for the SLR operators, with $N$ and $N_f$ the number of colors and flavors respectively.

The eigenvalues of $\gamma_0^{VLR}$ in nonincreasing order are:

$$
\lambda_1^{VLR} = \frac{1}{(4\pi)^2} \frac{6}{N^2}
$$

$$
\lambda_2^{VLR} = \frac{1}{(4\pi)^2} \frac{6}{N^2} \left(-1 + \frac{1}{N^2}\right)
$$

(113)

that coincide with the eigenvalues of $\gamma_0^{SLR}$:

$$
\lambda_1^{SLR} = \frac{1}{(4\pi)^2} \frac{6}{N^2}
$$

$$
\lambda_2^{SLR} = \frac{1}{(4\pi)^2} \frac{6}{N^2} \left(-1 + \frac{1}{N^2}\right)
$$

(114)

We set $N_f = N$, in such a way that $\beta_0$ and $\beta_1$ read respectively:

$$
\beta_0 = \frac{1}{(4\pi)^2} \frac{11}{3} - \frac{2}{3} N_f = \frac{3}{(4\pi)^2}
$$

$$
\beta_1 = \frac{1}{(4\pi)^2} \left(\frac{34}{3} - \frac{13N_f}{3N} + \frac{N_f}{N^3}\right)
$$

$$
= \frac{1}{(4\pi)^4} \left(7 + \frac{1}{N^2}\right)
$$

(115)

Therefore, the differences of the eigenvalues satisfy the resonant condition in Eq. (31) with $k = 1$:

$$
\frac{\lambda_1^{VLR}}{\beta_0} - \frac{\lambda_2^{VLR}}{\beta_0} = 2
$$

(116)

$$
\frac{\lambda_1^{VLR}}{\beta_0} = \frac{\lambda_2^{SLR}}{\beta_0} = 2
$$

thus realizing in a physical theory the case (II).

We construct the holomorphic gauge transformations that bring the corresponding connections:

$$
- \frac{\gamma^{VLR}}{\beta(g)} = \frac{1}{g} \left( \gamma_{0D}^{VLR} + g^2 \beta_0 \gamma_{1D}^{VLR} \right) + \cdots
$$

and:

$$
- \frac{\gamma^{SLR}}{\beta(g)} = \frac{1}{g} \left( \gamma_{0D}^{SLR} + g^2 \beta_0 \gamma_{1D}^{SLR} \right) + \cdots
$$

(117)

(118)

(119)

in the Poincarè–Dulac–Levelt normal form of Eq. (78).

Firstly, we get the bases of operators where $\gamma_0^{VLR}$ and $\gamma_0^{SLR}$ are diagonal, by means of the global gauge transformations:

$$
S_0^{VLR} = \begin{pmatrix}
-\frac{1}{N} & 1 \\
1 & 0
\end{pmatrix}
$$

(120)

and:

$$
S_0^{SLR} = \begin{pmatrix}
1 & -\frac{1}{N} \\
0 & 1
\end{pmatrix}
$$

(121)

respectively. Correspondingly:

$$
\gamma_{0D}^{VLR} = \frac{1}{(4\pi)^2} \begin{pmatrix}
\frac{6}{N^2} & 0 \\
0 & 6 / N - 6
\end{pmatrix}
$$

$$
\gamma_{1D}^{VLR} = \frac{1}{(4\pi)^4} \begin{pmatrix}
\frac{51}{2N_f^3} + \frac{47}{N} \frac{18}{N_f^3} + \frac{9N_f}{2} - \frac{45}{2N} \\
-\frac{18}{N^3} - \frac{63}{N_f} \frac{41}{N} - \frac{21}{2N^3} - \frac{61N_f}{2}
\end{pmatrix}
$$

(122)

$$
\gamma_{0D}^{SLR} = \frac{1}{(4\pi)^2} \begin{pmatrix}
\frac{6}{N^2} & 0 \\
0 & 6 / N - 6
\end{pmatrix}
$$

$$
\gamma_{1D}^{SLR} = \frac{1}{(4\pi)^4} \begin{pmatrix}
\frac{51}{2N_f^3} + \frac{47}{N} \frac{18}{N_f^3} + \frac{9N_f}{2} - \frac{45}{2N} \\
-\frac{18}{N^3} - \frac{63}{N_f} \frac{41}{N} - \frac{21}{2N^3} - \frac{61N_f}{2}
\end{pmatrix}
$$

(123)

Secondly, we choose the gauge transformations in Eq. (82) for $k = 2$:

$$
S_2^{VLR}(g) = I + g^2 H_2^{VLR} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
+ g^2 \begin{pmatrix}
\alpha_{11}^{VLR} & 0 \\
\alpha_{21}^{VLR} & \alpha_{22}^{VLR}
\end{pmatrix}
$$

(124)
respectively, by requiring that the only terms that do not vanish in the gauge-transformed Eqs. (118) and (119) are the resonant ones:

\[
A_{2}^{VLR} + (2I - \text{ad}_{A_{0}^{VLR}})H_{2}^{VLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
0 & 6N + \frac{3N}{2} - \frac{15}{2\pi N}
0 & 0
\end{pmatrix}
\]

(126)

and:

\[
A_{2}^{SLR} + (2I - \text{ad}_{A_{0}^{SLR}})H_{2}^{SLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
0 & 6N + \frac{3N}{2} - \frac{15}{2\pi N}
0 & 0
\end{pmatrix}
\]

(127)

respectively, where:

\[
A_{0}^{VLR} = \frac{2}{\beta_{0}^{2}} \begin{pmatrix}
0 & 0 & 2
0 & 0 & -2
\end{pmatrix}
\]

\[
A_{2}^{VLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
-\frac{2}{3N^{2}} + \frac{17}{2N^{3}} - \frac{14}{3N^{2}} + \frac{47}{3N}
-\frac{6}{N} - \frac{21}{2N}
-\frac{14}{3N^{2}} - \frac{15}{2N^{2}} + \frac{4}{N^{2}} + \frac{41}{3N} + 14
\end{pmatrix}
\]

(128)

and:

\[
A_{0}^{SLR} = \frac{2}{\beta_{0}^{2}} \begin{pmatrix}
0 & 0 & 2
0 & 0 & -2
\end{pmatrix}
\]

\[
A_{2}^{SLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
-\frac{2}{3N^{2}} + \frac{17}{2N^{3}} - \frac{14}{3N^{2}} + \frac{47}{3N}
-\frac{6}{N} - \frac{21}{2N}
-\frac{14}{3N^{2}} - \frac{15}{2N^{2}} + \frac{4}{N^{2}} + \frac{41}{3N} + 14
\end{pmatrix}
\]

(129)

Therefore:

\[
A_{2}^{VLR} + (2I - \text{ad}_{A_{0}^{VLR}})H_{2}^{VLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
0 & 6N + \frac{3N}{2} - \frac{15}{2\pi N}
0 & 0
\end{pmatrix}
\]

(130)

with:

\[
c_{1}^{VLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
-\frac{2}{3N^{4}} + \frac{17}{2N^{3}} - \frac{14}{3N^{2}} + \frac{47}{3N}
-\frac{6}{N} - \frac{21}{2N}
\end{pmatrix}
\]

(131)

and:

\[
A_{2}^{SLR} + (2I - \text{ad}_{A_{0}^{SLR}})H_{2}^{SLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
0 & 6N + \frac{3N}{2} + \frac{3}{2}
0 & 0
\end{pmatrix}
\]

(132)

with:

\[
c_{1}^{SLR} = \frac{1}{(4\pi)^{2}} \begin{pmatrix}
-\frac{2}{3N^{4}} + \frac{17}{2N^{3}} - \frac{14}{3N^{2}} + \frac{47}{3N}
-\frac{6}{N} - \frac{21}{2N}
\end{pmatrix}
\]

(133)

The solutions are:

\[
H_{2}^{VLR} = \frac{1}{(3N^{2})^{2}} \begin{pmatrix}
\frac{1}{3N^{2}} - \frac{17}{4N^{3}} + \frac{7}{4N} - \frac{47}{6N}
0
\end{pmatrix}
\]

(134)

and:

\[
H_{2}^{SLR} = \frac{1}{(3N^{2})^{2}} \begin{pmatrix}
\frac{1}{3N^{2}} - \frac{17}{4N^{3}} + \frac{7}{4N} - \frac{47}{6N}
0
\end{pmatrix}
\]

(135)
The corresponding gauge-transformed connections $A'(g)$ in the\nPoincaré–Dulac–Levelt normal form read:

\[
-\left[ \gamma^{VLR}(g) \right]' - \frac{\gamma^{VLR}(g)}{\beta(g)} \left( S_2^{VLR}(g) \right)^{-1}
+ \frac{\partial S_2^{VLR}(g)}{\partial g} \left( S_2^{VLR}(g) \right)^{-1}
= \frac{1}{g} \left( \frac{\gamma_0^{VLR}}{\beta_0} + g^2 A_2^{VLR} \right)
= \frac{1}{g} \left[ \frac{2}{N^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 N^2 - 2 \\ 0 \end{pmatrix} + \frac{g^2}{(4\pi)^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 N^2 - 2 \\ 0 \end{pmatrix} \right]
\]  

(136)

and:

\[
-\left[ \gamma^{SLR}(g) \right]' = -S_2^{SLR}(g) \frac{\gamma^{SLR}(g)}{\beta(g)} \left( S_2^{SLR}(g) \right)^{-1}
+ \frac{\partial S_2^{SLR}(g)}{\partial g} \left( S_2^{SLR}(g) \right)^{-1}
= \frac{1}{g} \left( \frac{\gamma_0^{SLR}}{\beta_0} + g^2 A_2^{SLR} \right)
= \frac{1}{g} \left[ \frac{2}{N^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 N^2 - 2 \\ 0 \end{pmatrix} + \frac{g^2}{(4\pi)^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 N^2 - 2 \\ 0 \end{pmatrix} \right]
\]  

(137)

As a consequence, the corresponding $Z(x, \mu)$ can be read from Eq. (41) with $k = 1$.

7.2 Dimension-8 operators in large-$N$ YM theory

We demonstrate by explicit computation that both the case (I) and the resonant condition of the case (II) are realized in the large-$N$ YM theory, and that the unitarity constraint (Sect. 5.4) in the free conformal limit is satisfied as well.

We consider the dimension-8 gauge-invariant Hermitian scalar operators in SU$(N)$ YM theory [47,48]:

\[
O_{B841} = \frac{1}{N^4} d_{abcd} F_{\mu} F_{\nu} F_{\rho} F_{\sigma} F_{\rho} F_{\nu}
\]
\[
O_{B842} = \frac{1}{N^4} d_{abcd} F_{\mu} F_{\nu} F_{\rho} F_{\sigma} F_{\rho} F_{\nu}
\]
\[
O_{B843} = \frac{1}{N^4} d_{abcd} F_{\mu} F_{\nu} F_{\rho} F_{\sigma} F_{\rho} F_{\nu}
\]
\[
O_{B844} = \frac{1}{N^4} d_{abcd} F_{\mu} F_{\nu} F_{\rho} F_{\sigma} F_{\rho} F_{\nu}
\]
\[
O_{B845} = \frac{1}{N^4} d_{abcd} F_{\mu} F_{\nu} F_{\rho} F_{\sigma} F_{\rho} F_{\nu}
\]

(138)

where $F_{\mu\nu}$ is:

\[
F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c
\]

with:

\[
d_{abcd} = d_{ab} d_{cde}
\]

(139)

where $f^{abc}$, $d^{abc}$ are defined by:

\[
\begin{pmatrix} T^a, T^b \end{pmatrix} = i f^{abc} T^c
\]
\[
\begin{pmatrix} T^a, T^b \end{pmatrix} = \frac{1}{2} d_{abc} \begin{pmatrix} 1 & I \end{pmatrix}
\]

(140)

We refer to the operators $O_{B841} \cdots O_{B844}$ and $O_{B845} \cdots O_{B848}$ as to double-trace and single-trace operators respectively. They mix among themselves under renormalization [47,48]:

\[
O = Z O_B
\]

(141)

where $O$ is the column vector of renormalized operators, whose transpose, $O^T$, reads:

\[
O^T = (O_{B841}, O_{B842}, O_{B843}, O_{B844}, O_{B845}, O_{B846}, O_{B847}, O_{B848})
\]

(142)

with $O_B$ the vector of the bare operators.

The corresponding $\gamma_0$ reads [47]:

\[
\gamma_0 = \sum_{k=0}^{3} \frac{1}{N^k} \gamma_0^k
\]

(143)

For the matrix of 2-point correlators in the free conformal limit we get [20]:

\[
G^{(2)}(x) = \frac{1}{(x^2)^8} G_0
\]

(144)

with:

\[
G_0 = \sum_{k=0}^{6} \frac{1}{N^k} G_0^k
\]

(145)
We only report the leading-order terms, $\gamma_{00}$ and $G_{00}$, in the large-$N$ expansion [20]:

\[ \gamma_{00} = \frac{1}{(4\pi)^2} \begin{pmatrix}
0 & -\frac{11}{6} & 0 & 0 & 0 \\
-\frac{14}{3} & 0 & -\frac{10}{3} & 4 & \frac{1}{6} \\
-\frac{28}{3} & 8 & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & -\frac{22}{3} & 0 \\
0 & 0 & 0 & -\frac{19}{5} & -6 & -\frac{2}{3} & \frac{16}{3}
\end{pmatrix} \]

\[ G_{00} = \frac{1}{\pi^8} \begin{pmatrix}
576 & 384 & 768 & 1152 & 0 & 0 & 0 & 0 & 0 & 0 \\
384 & 768 & 1152 & 384 & 0 & 0 & 0 & 0 & 0 & 0 \\
768 & 1152 & 2688 & 768 & 0 & 0 & 0 & 0 & 0 & 0 \\
1152 & 384 & 768 & 4608 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5376 & 1920 & 3456 & 1536 & 0 & 0 \\
0 & 0 & 0 & 0 & 1920 & 1056 & 1728 & 960 & 0 & 0 \\
0 & 0 & 0 & 0 & 3456 & 1728 & 4416 & 1920 & 0 & 0 \\
0 & 0 & 0 & 0 & 1536 & 960 & 1920 & 1152 & 0 & 0
\end{pmatrix}
\]

(148)

Therefore:

\[ \gamma'_{00} = \frac{1}{(4\pi)^2} \begin{pmatrix}
-\frac{22}{3} & 0 & 0 & 0 \\
0 & -\frac{22}{3} & 0 & 0 \\
0 & 0 & 0 & -\frac{14}{3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{5}{2} \\
0 & 0 & 0 & -\frac{6}{5}
\end{pmatrix} \left(3\sqrt{41} + 5\right) \]

and:

\[ G'_{00} = \frac{1}{\pi^8} \begin{pmatrix}
1280 \left(18\sqrt{13} + 65\right) & 0 & 0 \\
0 & 1280 \left(65 - 18\sqrt{13}\right) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(150)

Hence, to the leading large-$N$ order, single- and double-trace operators only mix separately among themselves [20].

According to Eq. (65), $\gamma_{00}$ and $G_{00}$ are simultaneously diagonalizable by the global gauge transformation [20]:

\[ S = \begin{pmatrix}
0 & -\frac{1}{5} & \frac{5\sqrt{13}}{6} & 3 & 0 \\
0 & -\frac{2}{3} & \frac{3 - \sqrt{13}}{6} & 0 \\
-2 & \frac{7}{3} & \frac{1}{3} & 0 \\
2 & 0 & 0 & -\frac{1}{6}
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[ \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{656} \left(41 - 3\sqrt{41}\right) \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{656} \left(3\sqrt{41} + 41\right) \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{16\sqrt{697}} - \frac{1}{16} \\
0 \\
0 \\
0
\end{pmatrix}
\]

(149)

Interestingly, the system above satisfies the resonant condition for some eigenvalues in the double-trace sector [20]:

\[ \frac{\gamma'_{004}}{\beta_0} - \frac{\gamma'_{001}}{\beta_0} = 2 \]

\[ \frac{\gamma'_{004}}{\beta_0} - \frac{\gamma'_{002}}{\beta_0} = 2 \]

(152)

Moreover, despite $\gamma_{00}$ would be potentially nondiagonalizable because of the two coinciding eigenvalues in Eq. (150),
it is actually diagonalizable – and $G_{00}$ as well – according to the unitarity constraint (Sect. 5.4) in the free conformal limit. Moreover, we verify that the eigenvalues of $G_{00}$ are all positive numbers [20]:

\[
G'_{001} = \frac{1}{\pi^8} 1280 \left( 18\sqrt{3} + 65 \right) = 17.52 \ldots \\
G'_{002} = \frac{1}{\pi^8} 1280 \left( 65 - 18\sqrt{3} \right) = 0.013 \ldots \\
G'_{003} = \frac{1}{\pi^8} 1280 = 0.1349 \ldots \\
G'_{004} = \frac{1}{\pi^8} 1152 = 0.1214 \ldots \\
G'_{005} = \frac{1}{\pi^8} \left( 144 - \frac{816}{\sqrt{41}} \right) = 0.001 \ldots \\
G'_{006} = \frac{1}{\pi^8} 432 + \frac{7056}{\sqrt{697}} = 0.017 \ldots \\
G'_{007} = \frac{1}{\pi^8} \left( 144 + \frac{816}{\sqrt{41}} \right) = 0.028 \ldots \\
G'_{008} = \frac{1}{\pi^8} 432 - \frac{7056}{\sqrt{697}} = 0.017 \ldots \\
\tag{153}
\]

according to the aforementioned unitarity.

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**Appendix A: Asymptotic versus exact correlators**

We comment on the asymptotic versus exact form of the correlators in massless QCD-like theories.

The closed form of $Z(x, \mu)$ in Eq. (5) relies implicitly on the perturbative definition of $\gamma(g)$ and $\beta(g)$ that are believed to be formal series, at best asymptotic for $g \to 0$ thanks to the asymptotic freedom.

Correspondingly, the asymptotic solution of the Callan–Symanzik equation in Eq. (6), with $G(x, g(\mu), \mu) \sim \tilde{G}(x, \gamma(x))$, where $\gamma(x, g(x))$ relies on the RG-improvement of perturbation theory as well, is believed to be only asymptotic in the UV to the exact 2-point correlator thanks to the asymptotic freedom.

The above statement may be verified directly in the large-$N$ limit of confining massless QCD-like theories following [10], where it has been shown how the aforementioned asymptotics works in the multiplicatively renormalizable case.

Indeed, as remarked in [1], nonperturbatively, according to the RG, massless QCD-like theories develop a nontrivial dimensionful scale that labels the RG trajectory – the RG invariant – $A_{RGI}$:

\[
A_{RGI} \sim \mu e^{-\frac{1}{2\sqrt{\pi}} \frac{g}{\Lambda^2} c_0(1 + \sum_{n=1}^{\infty} c_n g^{2n})} \tag{A.1}
\]

– the only free parameter [12,13] in the nonperturbative S matrix of confining massless QCD-like theories – which any physical mass scale must be proportional to.

As a consequence, nonperturbatively in the large-$N$ limit of confining massless QCD-like theories [15–18], the leading contribution to the exact Euclidean 2-point correlators of gauge-invariant operators must be an infinite sum of free-field propagators [17,18], with every mass in the propagators proportional to $A_{RGI}$.

In the momentum representation, after the analytic continuation to Minkowskian space-time, the sum of free propagators is a sum of pure poles, while the analytic continuation of the RG-improved [10] perturbative solution of the Callan–Symanzik equation has only cuts, involving logs and loglogs of the momentum [10].

Therefore, the exact and all-order RG-improved 2-point Euclidean correlators cannot coincide, otherwise their analytic continuations would coincide as well, though we have just shown that they do not.

Hence, RG-improved perturbation theory may only be UV asymptotic in large-$N$ confining QCD-like theories, and in fact, as remarked above, it is believed to be such because of the asymptotic freedom.

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