EQUIPARTITION OF ENERGY FOR NONAUTONOMOUS WAVE EQUATIONS

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Abstract. Consider wave equations of the form

\[ u''(t) + A^2 u(t) = 0 \]

with \( A \) an injective selfadjoint operator on a complex Hilbert space \( H \). The kinetic, potential, and total energies of a solution \( u \) are

\[ K(t) = \|u'(t)\|^2, \quad P(t) = \|Au(t)\|^2, \quad E(t) = K(t) + P(t). \]

Finite energy solutions are those mild solutions for which \( E(t) \) is finite. For such solutions \( E(t) = E(0) \), that is, energy is conserved, and asymptotic equipartition of energy

\[ \lim_{t \to \pm \infty} K(t) = \lim_{t \to \pm \infty} P(t) = \frac{E(0)}{2} \]

holds for all finite energy mild solutions iff \( e^{itA} \to 0 \) in the weak operator topology. In this paper we present the first extension of this result to the case where \( A \) is time dependent.

1. Introduction. Let \( A = A^* \geq 0 \) be an injective selfadjoint operator on a complex Hilbert space \( H \). The corresponding initial value problem for the wave equation is

\[ u''(t) + A^2 u(t) = 0, \quad u(0) = f, \quad u'(0) = g; \]
and let \( u \in C^2(\mathbb{R}, \mathcal{H}) \) be a strong solution. Let

\[
U = \begin{pmatrix} Au \\ u' \end{pmatrix}.
\]

Then

\[
U'(t) = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} U(t) = M U(t)
\]

where

\[
M = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} = A \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -M^*.
\]

on \( \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H} \). The functional calculus for selfadjoint operators says (cf. [4])

\[
e^{tM} = \begin{pmatrix} \cos(tA) & \sin(tA) \\ -\sin(tA) & \cos(tA) \end{pmatrix}.
\]

This can be verified by differentiating both sides with respect to \( t \). It is easily seen that \( \{e^{tM} : t \in \mathbb{R}\} \) is a \((C_0)\) unitary group on \( \mathcal{H}^2 \), and thus the total energy

\[
E(t) = K(t) + P(t)
\]

\[
:= \|u'(t)\|^2 + \|Au(t)\|^2
\]

(2)

is conserved for all strong solutions \( u \in C^2(\mathbb{R}, \mathcal{H}) \) which correspond to \( f \in \mathcal{D}(A^2) \) and \( g \in \mathcal{D}(A) \) in [1]. The same is true for all mild solutions which correspond to \( \{e^{tM} \begin{pmatrix} Af \\ g \end{pmatrix} : t \in \mathbb{R}\} \) for \( f \in \mathcal{D}(A) \) and \( g \in \mathcal{H} \). Here \( K \) [resp. \( P \)] represents the kinetic [resp. potential] energy, and \( E(t) = E(0) \) is the total energy which is conserved.

Equi-partition of energy, i.e.

\[
\lim_{t \to \pm \infty} K(t) = \lim_{t \to \pm \infty} P(t) = \frac{E}{2}
\]

(with \( E = E(0) \)) for all finite energy mild solutions holds iff

\[
\langle e^{itA} h, k \rangle \to 0
\]

as \( t \to \pm \infty \) for all \( h, k \in \mathcal{H} \). A sufficient condition for this condition to hold is that \( A \) is spectrally absolutely continuous (SAC). To define this, write

\[
A = \int_{(0, \infty)} \lambda dE(\lambda)
\]

by the spectral theorem using a resolution of the identity. Then \( A \) is SAC means the bounded monotone function

\[
\lambda \to \|E(\lambda)f\|^2
\]

is absolutely continuous on \([0, \infty]\) (with limits 0 at 0 and \( \|f\|^2 \) at \( \infty \)) for all \( f \in \mathcal{H} \). This was proved in [2], 1969, [3], 1970.

Our goal in this paper is to extend this result to the context of replacing \( A \) by \( A(t) \), a family of time dependent commuting nonnegative injective selfadjoint operators on \( \mathcal{H} \), such that the new result reduces to the old result when \( A \) does not depend on \( t \) and is SAC. This will be the first asymptotic equipartition of energy result for a nonautonomous system generalizing [1].

The outline of the paper is as follows. In Section 2 we review the proof of asymptotic equipartition of energy for [1]. In Sections 3 and 4 we formulate the problem
and prove the wellposedness theorem. In Section 5 we establish the equipartition of energy results. Section 6 contains an example.

2. The autonomous wave equation. The unique mild solution to \([1]\) can be written as

\[ u(t) = e^{itA}F + e^{-itA}G \]  

where

\[
\begin{align*}
F, G &\in \mathcal{D}(A), f \in \mathcal{D}(A), g \in \mathcal{H} \\
AF &= \frac{1}{2}(Af - ig) \\
AG &= \frac{1}{2}(Af + ig) \\
f &= F + G \\
g &= i(AF - AG).
\end{align*}
\]

Then

\[
\begin{align*}
K(t) &= \|u'(t)\|^2 = \|e^{itA}AF - e^{-itA}AG\|^2, \\
P(t) &= \|Au(t)\|^2 = \|e^{itA}AF + e^{-itA}AG\|^2.
\end{align*}
\]

The law of cosines together with unitarity of \(e^{itA}\) implies

\[ E(t) = K(t) + P(t) = 2(\|AF\|^2 + \|AG\|^2) = E(0), \]

and

\[ K(t) - P(t) = -4Re \langle e^{itA}AF, AG \rangle. \]

Thus, energy is conserved, and energy is asymptotically equipartitioned iff

\[ K(t) - P(t) \rightarrow 0 \text{ as } t \rightarrow \pm\infty \]

for all \(F, G \in \mathcal{D}(A)\) iff

\[ 0 \leftarrow \langle e^{itA}h, h \rangle = \int_0^\infty e^{it\lambda}d(\|E(\lambda)h\|^2) \]

for all \(h \in \mathcal{H}\) as \(t \rightarrow \pm\infty\). In the SAC case, the Riemman-Lebesgue Lemma gives the conclusion.

The factorization

\[ 0 = \left( \frac{d^2}{dt^2} - A^2 \right) u = \left( \frac{d}{dt} - A \right) \left( \frac{d}{dt} + A \right) u \]

led to the d’Alembert formula

\[ u(t) = e^{itA}F + e^{-itA}G \]

since

\[ \mathcal{N}\left[ \left( \frac{d}{dt} - A \right) \left( \frac{d}{dt} + A \right) \right] = \mathcal{N}\left( \frac{d}{dt} - A \right) + \mathcal{N}\left( \frac{d}{dt} + A \right). \]

This uses the fact the space-time operators in parenthesis are injective and commute. Here \(\mathcal{N}\) denotes the null space. Thus, \(\{e^{\pm itA}h : t \in \mathbb{R}\}\) gives a general mild solution of

\[ u' = \pm iAu \]

with \(u(0) = h\), as \(h\) varies. Care must be used in finding a nonautonomous version of the preceding.
3. The nonautonomous framework. Let $B = B^*$ on $\mathcal{H}$. Then $u(t) = e^{itB}h$ is the unique mild solution of

$$u' - iBu = 0$$

$$u(0) = h$$

as $h \in \mathcal{H}$ varies. Moreover, by the Spectral Theorem,

$$B = U_o M_b U_o^{-1}$$

where $U_o : \mathcal{H} \rightarrow L^2(\Omega, \Sigma, \mu)$ is unitary from $\mathcal{H}$ to some concrete complex $L^2$ space, $b : \Omega \rightarrow \mathbb{R}$ is $\Sigma$-measurable and

$$(M_b g)(w) = b(w)g(w)$$

for $g \in M_b = \{b \in L^2 = L^2(\Omega, \Sigma, \mu) : gb \in L^2\}$. If $\{B(t) : t \in \mathbb{R}\}$ is a family of commuting selfadjoint operators on $\mathcal{H}$, then

$$B(t) = U_o M_{b(t)} U_o^{-1}$$

for $U_o, L^2$ as above and for $b(t) : \Omega \rightarrow \mathbb{R}, \Sigma$-measurable for all $t \in \mathbb{R}$. For simplicity, we will restrict to nonnegative time, $t \geq 0$.

Next we state our first assumption.

**HYP 1:** Let $\{A(t) : t \in \mathbb{R}^+ = [0, \infty)\}$ be nonnegative selfadjoint operators on $\mathcal{H}$ satisfying the condition

$$D_o = \mathcal{D}(A(t)), \quad D_1 = \mathcal{D}(A^2(t))$$

are both independent of $t$. Moreover, the commutator $[A(t), A(s)] = A(t)A(s) - A(s)A(t) = 0$ in the sense that the bounded operators $e^{itA(t)}$ and $e^{isA(s)}$ commute for all $t, s, \tau, \sigma \in \mathbb{R}^+$. Further, assume

$$A(\cdot)f \in C^1(\mathbb{R}^+, \mathcal{H})$$

for all $f \in D_o$, and there exists a bounded function $k_1 \in C([0, \infty), \mathbb{R})$ with $k_1 \geq \epsilon > 0$ such that

$$\|A(t)f\| \leq k_1(t)\|A(0)f\|$$

(5)

for all $f \in D_o$ and all $t \in \mathbb{R}^+$.

Note that (5) implies that $0 \notin \rho(A(t))$ holds either for all $t \in \mathbb{R}^+$ or for no $t \in \mathbb{R}^+$. In the former case

$$\|f\|_t := \|A(t)f\|$$

is equivalent to the graph norm of $A(t)$. Let

$$B(t) := \int_0^t A(s)ds.$$

(6)

If $A(t) = U_o M_{a(t)} U_o^{-1}$, then

$$B(t) = U_o M_{\int_0^t a(s)ds} U_o^{-1}.$$

(7)

We may view $a(t) : \Omega \rightarrow (0, \infty)$ as $a(t, \omega)$ with $a(\cdot, \cdot) : \mathbb{R}^+ \times \Omega \rightarrow (0, \infty)$. By (HYP1) and old theorem of J. L. Doob [1], without loss of generality we may assume $a(\cdot, \cdot)$ is jointly measurable on $\mathbb{R}^+ \times \Omega$ in the (Borel sets $\times \Sigma$) sense. Note that here each $a(t, \cdot)$ is defined on $\Omega \setminus N_t$ where $\mu(\Omega / N_t) = 0$ for each $t \in \mathbb{R}^+$. There are uncountably many null sets $N_t$, but Doob’s theorem says this is not a problem; they
can be chosen so that $a(t, \omega)$ is jointly measurable and certain integrals over $\mathbb{R}^+ \times \Omega$ will exist.

Another form of the Spectral Theorem says that, due to the commuting hypothesis in (HYP1), there is a function

$$ F : \mathbb{R}^+ \times (0, \infty) \rightarrow (0, \infty) $$

such that for each $t \in \mathbb{R}^+$,

$$ F(t, A(0)) = A(t); $$

moreover $F(t, x)$ is an $C^1$ function of $t \in \mathbb{R}^+$ for each fixed $x$. Later we shall assume more regularity on $F(t, x)$ as a function on $\mathbb{R}^+ \times (0, \infty)$.

Now using (6)-(7) we have

$$ B(t) = \int_0^t A(s)ds = U_0M_P(t)U_0^{-1} $$

where

$$ P(t) = \int_0^t a(s)ds. $$

Also, by (8) and (9), for $t \in \mathbb{R}^+$,

$$ B(t) = G(t, A(0)) $$

where

$$ G(t, x) = \int_0^t F(s, x)ds. $$

Now let $F_1, F_2 \in \mathcal{D}_1$ and $t \geq 0$. Define

$$ w(t) = e^{iB(t)}F_1 + e^{-iB(t)}F_2. $$

Suppressing the argument $t$, we get

$$ w' = e^{iB}iAF_1 - e^{-iB}iAF_2 $$

and

$$ w'' = e^{iB}(-A^2F_1 + iA'F_1) + e^{-iB}(-A^2F_2 - iA'F_2). $$

Let

$$ \tilde{w}(t) = e^{iB(t)}F_1 - e^{-iB(t)}F_2. $$

A similar calculation gives

$$ \tilde{w}'' = e^{iB}(-A^2F_1 + iA'F_1) - e^{-iB}(-A^2F_2 - iA'F_2). $$

Combining (13) and (14) yields

$$ \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}'' = -A^2 \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + iA' \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} -A^2 & iA' \\ iA' & -A^2 \end{pmatrix} \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}. $$

We can rewrite this as

$$ W'' = QW $$

where $W = \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}$ and

$$ Q(t) = -M^2 + iA'(t) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. $$
where \(-M^2\) is identified with \(-M(t)^2\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)\) for convenience.

Several conclusions follow. In the autonomous case, \(Q = -M^2\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)\), and each component \(Au, u'\) of \(U = \left(\begin{array}{c}Au \\ u'\end{array}\right)\) satisfies the same wave equation \(v''+M^2v = v''+A^2v = 0\). But in the nonautonomous case, the two components \(w, \tilde{w}\) of \(W\) satisfy different equations, namely
\[
\begin{align*}
  w'' + A^2w &= iA'\tilde{w} \\
  \tilde{w}'' + A^2\tilde{w} &= iA'w.
\end{align*}
\]
Still, this reduces to a single \(2 \times 2\) system as given by (15) for \(W = \left(\begin{array}{c}w \\ \tilde{w}\end{array}\right)\). In other words,
\[
z_{\pm}(t) = e^{\pm iB(t)}f_{\pm}
\]
for \(f_{\pm} \in \mathcal{D}_1\) satisfies
\[
z_{\pm}'' + A^2z_{\pm} = \pm iA'z_{\pm}
\]
and these are different second order single equations. Furthermore, \(Q(t)\) (see (15)) is normal but not selfadjoint because the imaginary part, \(iA'\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)\), is nonzero.

So the relevant second order equation here that is wellposed is a second order system in \(\mathcal{H}\) or, equivalently, a second order equation in \(\mathcal{H}^2\). For this system, we now establish wellposedness. Asymptotic equipartition of energy can be considered here, even though the equation does not conserve energy.

4. **The nonautonomous system.** We consider the system
\[
W''(t) = Q(t)W(t)
\]
in \(\mathcal{H}^2\) where
\[
Q(t) = -M^2 + iM'(t)
\]
\[
= -M^2 + iA'(t)\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)
\]
\[
= \left(\begin{array}{cc}-M^2 & iA' \\ iA' & -M^2\end{array}\right)(t)
\]
where as before \(-M^2\) is identified with \(-M(t)^2\left(\begin{array}{cc}I & 0 \\ 0 & I\end{array}\right)\) for convenience. As mentioned before, in the autonomous case, \(A' = 0\), and both \(w\) and \(\tilde{w}\) satisfy
\[
w'' + A^2w = 0.
\]
But when \(A\) depends in a nontrivial way on \(t\), neither \(w\) nor \(\tilde{w}\) satisfy a single second order equation. Still, the pair
\[
W = \left(\begin{array}{c}w \\ \tilde{w}\end{array}\right)
\]
does satisfy a second order equation governed by the evolution operator family 
\{Z(t, s) : t \geq s \geq 0\} which can be described as follows. Now \(W'' = QW\) implies

\[
\hat{W}'' = \left( \begin{array}{c} W \\ W' \end{array} \right) = \left( \begin{array}{cc} 0 & I \\ Q & 0 \end{array} \right) \left( \begin{array}{c} W \\ W' \end{array} \right) =: \hat{Q} \left( \begin{array}{c} W \\ W' \end{array} \right) = \hat{Q}\hat{W}
\]

Let

\[Z(t, s) = \int_s^t \hat{Q}(r)dr.\]

Then

\[\hat{W}(t) = Z(t, s)\hat{W}(s)\]

for \(t \geq s \geq 0\) and \(Z(t, s)\) is not unitary because \(\hat{Q}(t)\) is normal but not skewadjoint

for each \(t \in \mathbb{R}^+\).

**Theorem 4.1** (Wellposedness Theorem). Let (HYP1) hold. Then the problem

\[
\frac{d^2W}{dt^2}(t) = Q(t)W(t) \\
W(0) = H_1, \quad W'(0) = H_2
\]

with \(H_1, H_2 \in \mathcal{D}_1^2\) has a unique strong solution in \(C^2(\mathbb{R}^+, \mathcal{H}^2)\) of the form

\[W(t) = \left( \begin{array}{c} w(t) \\ \tilde{w}(t) \end{array} \right)\]

where

\[
w(t) = e^{iB(t)}F_1 + e^{-iB(t)}F_2, \quad (18)
\]

\[
\tilde{w}(t) = e^{iB(t)}F_1 - e^{-iB(t)}F_2, \quad (19)
\]

\[H_1 = \left( \begin{array}{c} F_1 + F_2 \\ F_1 - F_2 \end{array} \right), \quad H_2 = iA(0) \left( \begin{array}{c} F_1 - F_2 \\ F_1 + F_2 \end{array} \right)\]

and each component of \(\frac{1}{2}(A(0)-1)H_2 + H_1\) is \(F_1\), and each component of \(\frac{1}{2}(A(0)^{-1}H_2 - H_1)\) is \(F_2\).

**Proof.** The proof is a straightforward consequence of (HYP1), the previous calculations, and some elementary algebra showing the relationship between \((H_1, H_2)\) and \((F_1, F_2)\).

The point is that solutions are built from linear combinations of

\[e^{\pm iB(t)}F_\pm,\]

which is the spirit of a nonautonomous d’Alembert’s formula. This is a new feature of our first result.

\[\square\]

5. **Asymptotic equipartition of energy.** Let

\[W = \left( \begin{array}{c} w \\ \tilde{w} \end{array} \right) \in C^1(\mathbb{R}^+, \mathcal{H}^2)\]

be as in the preceding theorem. We want to obtain asymptotic equipartition of energy. Define the total kinetic energy of the system as

\[K(t) = \|W'(t)\|^2 \quad (20)\]

and the total potential energy as

\[P(t) = \|A(t)W(t)\|^2. \quad (21)\]
It is convenient to consider refinements of these, namely
\[
E_1(t) = \|w(t)\|^2, \quad E_2(t) = \|\dot{w}(t)\|^2, \\
E_3(t) = \|w'(t)\|^2, \quad E_4(t) = \|\dot{w}'(t)\|^2, \\
E_5(t) = \|A(t)w(t)\|^2, \quad E_6(t) = \|A(t)\dot{w}(t)\|^2.
\] (22)

Using (18), (19) and the Law of Cosines, we get
\[
E_1(t) = \|F_1\|^2 + \|F_2\|^2 + 2\text{Re} \left< e^{2iB(t)}F_1, F_2 \right>, \\
E_2(t) = \|F_1\|^2 + \|F_2\|^2 - 2\text{Re} \left< e^{2iB(t)}F_1, F_2 \right>, \\
E_3(t) = \|A(t)F_1\|^2 + \|A(t)F_2\|^2 - 2\text{Re} \left< e^{2iB(t)}A(t)F_1, A(t)F_2 \right>, \\
E_4(t) = \|A(t)F_1\|^2 + \|A(t)F_2\|^2 + 2\text{Re} \left< e^{2iB(t)}A(t)F_1, A(t)F_2 \right>, \\
E_5(t) = E_4(t), \quad E_6(t) = E_3(t).
\]

Moreover, we see that
\[
E_1(t) + E_2(t) = 2 \left( \|F_1\|^2 + \|F_2\|^2 \right), \\
E_3(t) + E_4(t) = E_5(t) + E_6(t) = 2 \left( \|A(t)F_1\|^2 + \|A(t)F_2\|^2 \right)
\]

Thus, \(E_3(t) + E_4(t)\) is conserved, while \(E_3(t) + E_4(t) = E_5(t) + E_6(t)\) are not, but will be “conserved at infinity” if \(A(t) \rightarrow A(\infty)\) as \(t \rightarrow \infty\) in a suitable sense.

We now make this notion more precise.

**HYP 2** Note that \(k_1\) of (HYP1) satisfies

\[
0 < \varepsilon_1 \leq k_1(t) \leq \frac{1}{\varepsilon_1}
\]

for some \(\varepsilon_1 > 0\) and all \(t \in \mathbb{R}^+\). Assume that for every \(f \in D_o\)

\[
A'()f \in L^1(\mathbb{R}^+, \mathcal{H}).
\]

It follows that

\[
A(t)f = \int_0^t A'(s)f \, ds + A(0)f \rightarrow \int_0^\infty A'(s)f \, ds + A(0)f =: A(\infty)f
\]

for all \(f \in D_o\), and we further assume

\[
\int_0^\infty \|A(t)f - A(\infty)f\|dt < \infty
\]

for all \(f \in D_o\), and that \(A(\infty)\) (which is symmetric) is selfadjoint and (HYP1) holds for \(\{A(t) : t \in [0, \infty]\}\) and \(k_1 \in C([0, \infty], (0, \infty))\).

Recall that if \(a(\tau) = b(\tau) = 1\) and \(t > \tau\), then

\[
a(t) - b(t) = \int_\tau^t \frac{d}{ds}(a(s)b(t + \tau - s)) \, ds = \int_\tau^t (a'(s)b(t + \tau - s) - a(s)b'(t + \tau - s)) \, ds
\]
whence for $f \in D_o, t > \tau$,  
\[ \|e^{\pm i(B(t) - B(\tau))} f - e^{\pm i(t - \tau) A(\infty)} f\| = \left\| \int_{\tau}^{t} e^{\pm i(B(s) - B(\tau))}(A(t + \tau - s) - A(\infty))e^{\pm isA(\infty)} f \, ds \right\| \]
\[ \leq \int_{\tau}^{t} \|A(s)f - A(\infty)f\| \, ds \]
by the contractivity and commutative hypotheses. Thus, using (HYP2), we have  
\[ \|e^{\pm i(B(t) - B(\tau))} f - e^{\pm i(t - \tau) A(\infty)} f\| \rightarrow 0 \]
as $\tau \rightarrow \infty$ with $t > \tau$ arbitrary.

Returning to (20), (21), we see that  
\[ K(t) = E_3(t) + E_4(t) = 2(\|A(t)F_1\|^2 + \|A(t)F_2\|^2) \]
\[ \rightarrow 2(\|A(\infty)F_1\|^2 + \|A(\infty)F_2\|^2) \]
as $t \rightarrow \infty$. Similarly,  
\[ P(t) = E_5(t) + E_6(t) = K(t), \]
so $P(t)$ converges to the expression in (24).

The more interesting “partial energies” are $E_j(t), j = 1, 2, 3, 4$ (and 5, 6 but $E_5 = E_4$ and $E_6 = E_3$). By [2] the expressions  
\[ \text{Re} \left< e^{2itB(t)} L_1, L_2 \right> \]
all tend to 0 as $t \rightarrow \infty$ for all $L_1, L_2$ iff (since $B(t) = t \left( \frac{1}{t} \right) B(t)$ and $\left( \frac{1}{t} \right) B(t)f \rightarrow A(\infty)f$, for every $f \in D_o$ by (HYP2))  
\[ \text{Re} \left< e^{itA(\infty)} L_1, L_2 \right> \rightarrow 0 \]
as $t \rightarrow \infty$ for all $L_1, L_2 \in \mathcal{H}$ iff  
\[ \text{Re} \left< e^{itA(\infty)} L, L \right> \rightarrow 0 \]
as $t \rightarrow \infty$ for all $L \in \mathcal{H}$. The Riemann Lebesgue condition holds provided $A(\infty)$ is spectrally absolutely continuous (SAC). More specifically, by yet another form of the Spectral Theorem,  
\[ A(\infty) = \int_{(0, \infty)} \lambda dE(\lambda) \]
and  
\[ \left< e^{itA(\infty)} L, L \right> = \int_{0}^{\infty} e^{it\lambda} d(\|E(\lambda)L\|^2) \]
\[ = \int_{0}^{\infty} e^{it\lambda} g_L(\lambda) d\lambda \rightarrow 0 \]
as $t \rightarrow \infty$ when $d(\|E(\lambda)L\|^2) = g_L(\lambda) d\lambda$ with $g_L \in L^1(0, \infty)$.

We require one more hypothesis before we state our main theorem.

**HYP3** Recall from (HYP1) and (HYP2) that  
\[ A(t) = F(t, A(0)) \]
for $0 \leq t \leq \infty$ for a function $F = F(t, x)$,

$$F : [0, \infty] \times (0, \infty) \rightarrow (0, \infty).$$

We assume $F \in C^1([0, \infty] \times (0, \infty))$.

We now state our main result.

**Theorem 5.1 (Asymptotic Equipartition of Energy).** Assume (HYP1), (HYP2) and (HYP3). Then $A(t)$ is spectrally absolutely continuous for $0 \leq t \leq \infty$ provided $A(\infty)$ is spectrally absolutely continuous, which we assume. Then for all finite energy solutions, as $t \rightarrow \infty$,

$$E_i(t) \rightarrow \|F_i\|^2 + \|F_2\|^2 \text{ for } i = 1, 2$$

and

$$E_j(t) \rightarrow \|A(\infty)F_1\|^2 + \|A(\infty)F_2\|^2 \text{ for } j = 3, 4, 5, 6.$$ 

In fact, $A(\infty)$ will automatically be SAC, provided that $F(\infty, x)$, as a function of $x$, is piecewise strictly monotone. This follows from Lemma 4.1 in [5].

6. **An example.** Let

$$A(t) = \sum_{j=1}^{n} a_j(t) \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} ib_j(t) \frac{\partial}{\partial x_j}$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, where $0 < a_j \in C^2(\mathbb{R}^\tau)$ with $a_j, a_j', a_j'' \in C([0, \infty])$, and $b_j \in C^2(\mathbb{R}^\tau)$ with $b_j, b_j', b_j'' \in C([0, \infty])$ for $j = 1, \ldots, n$. Notice that (HYP1), (HYP2) and (HYP3) all hold if in addition,

$$a_j', a_j'', b_j', b_j'' \in L^1(\mathbb{R}^\tau).$$

We observe that

$$A'(t) = \sum_{j=1}^{n} \left( a_j'(t) \frac{\partial^2}{\partial x_j^2} + ib_j'(t) \frac{\partial}{\partial x_j} \right)$$

**Acknowledgments.** Gisele and Jerry Goldstein thank Mustapha Mokhtar-Kharroubi for his wonderful hospitality at the University of Besancon, where this paper was completed. Fabiana would like to thank CAPES - grant BEX 12220-13-2 - for supporting her visit to The University of Memphis and thanks to the two first authors for their hospitality.

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Received March 2015; revised September 2015.

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