Regularity and uniqueness for a class of solutions to the hydrodynamic flow of nematic liquid crystals

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Abstract

In this paper, we establish an $\varepsilon$-regularity criterion for any weak solution $(u, d)$ to the nematic liquid crystal flow (1.1) such that $(u, \nabla d) \in L^p_t L^q_x$ for some $p \geq 2$ and $q \geq n$ satisfying the condition (1.2). As consequences, we prove the interior smoothness of any such a solution when $p > 2$ and $q > n$. We also show that uniqueness holds for the class of weak solutions $(u, d)$ the Cauchy problem of the nematic liquid crystal flow (1.1) that satisfy $(u, \nabla d) \in L^p_t L^q_x$ for some $p > 2$ and $q > n$ satisfying (1.2).

1 Introduction

For any $n \geq 3$, the hydrodynamic flow of nematic liquid crystals in $\mathbb{R}^n \times [0, T]$, for some $0 < T < +\infty$, is given by

$$
\begin{align*}
&u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_n), \\
&\nabla \cdot u = 0, \\
&d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \\
&(u, d) = (u_0, d_0),
\end{align*}
$$

(1.1)

where $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is the velocity field of underlying incompressible fluid, $d : \mathbb{R}^n \times [0, T] \to S^2$ is the director field of nematic liquid crystal molecules, $P : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is the pressure function, $\nabla \cdot$ denotes the divergence operator on $\mathbb{R}^n$, $\nabla d \otimes \nabla d = \left( \frac{\partial d}{\partial x_i} \cdot \frac{\partial d}{\partial x_j} \right)_{1 \leq i, j \leq n}$ is the stress tensor induced by the director field $d$, $I_n$ is the identity matrix of order $n$, $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the initial velocity field with $\nabla \cdot u_0 = 0$, and $d_0 : \mathbb{R}^n \to S^2$ is the initial director field.

The system (1.1) is a simplified version of the Ericksen-Leslie system modeling the hydrodynamics of liquid crystal materials, proposed by Ericksen [2] and Leslie [15] in 1960’s. It is a macroscopic continuum description of the time evolution of the material under the influence of both the flow field and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals. The interested readers can refer to [2], [15], [16], and [18] for more detail. Mathematically,
the system \((1.1)\) is strongly coupling the Naiver-Stokes equations and the (transported) heat flow of harmonic maps into \(S^2\).

For \(n = 2\), Lin-Lin-Wang \([17]\) have proved the existence of global Leray-Hopf type weak solutions to \((1.1)\) with initial and boundary conditions, which is smooth away from finitely many possible singular times (see Hong \([6]\) and Xu-Zhang \([23]\) for related works). Lin-Wang \([19]\) proved the uniqueness for such weak solutions. It remains a very challenge open problem to prove the global existence of Leray-Hopf type weak solutions and partial regularity of suitable weak solutions to \((1.1)\) in higher dimensions. A BKM type blow-up criterion was obtained for the local strong solution to \((1.1)\) for \(n = 3\) by \([9]\), i.e., if \(0 < T_* < +\infty\) is the maximum time interval of the strong solution to \((1.1)\), then

\[
\int_0^{T_*} (\|\nabla \times u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2) \, dt = +\infty.
\]

Recently, the local well-posedness of \((1.1)\) was obtained for initial data \((L_p, L_q)\) of small norm for \(n = 3\) by \([7]\), see also \([1]\) for \(n \geq 4\). Wang \([21]\) proved smoothness of weak solutions \(u\) to the heat flow of harmonic maps such that \(\nabla u \in L_p^p L_q^q(\mathbb{R}^n \times [0, T])\) with \(\frac{2}{p} + \frac{n}{q} = 1\) for \(n \geq 4\) (or \(q \geq 4\) for \(2 \leq n < 4\), see \([11]\) for the case \(2 < q < 4\) when \(2 \leq n < 4\)). In \([11]\), the uniqueness of Serrin’s solutions to the heat flow of harmonic maps is also established when \(p > 2\) and \(q > n\). This results motivate us to investigate the regularity and uniqueness of Serrin’s \((p, q)\)-solutions to the system \((1.1)\) of nematic liquid crystal flows.

Before stating our main theorems, we need to introduce some notations.

**Notations:** For \(1 \leq p, q \leq +\infty\), \(0 < T \leq \infty\), define the Sobolev space

\[
H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n) = \left\{ f \in L^2([0, T], H^1(\mathbb{R}^n, \mathbb{R}^n)) : \partial_t f \in L^2([0, T], L^2(\mathbb{R}^n, \mathbb{R}^n)) \right\},
\]

\[
\mathbb{E}^p(\mathbb{R}^n \times [0, T], \mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n \times [0, T], \mathbb{R}^n) : \nabla \cdot f = 0 \right\},
\]

and the Morrey space \(M^{p, \lambda}(U)\) for \(0 \leq \lambda \leq n + 2\) and \(U = U_1 \times [t_1, t_2] \subset \mathbb{R}^n \times \mathbb{R}\):

\[
M^{p, \lambda}(U) = \left\{ f \in L^p_{loc}(U) : \| f \|_{M^{p, \lambda}(U)} < +\infty \right\},
\]
where
\[ \left\| f \right\|_{M^p,\lambda(U)} = \left( \sup_{(x,t) \in U} \sup_{0 < r < \min \delta(x,t), \partial_p U} r^{\lambda-n-2} \int_{P_r(x,t)} |f|^p \right)^{\frac{1}{p}}, \]

\[ B_r(x) = \{ y \in \mathbb{R}^n : |y - x| \leq r \}, \quad P_r(x,t) = B_r(x) \times [t - r^2, t], \partial_p U = (\partial U_1 \times \{t_1, t_2\}) \cup (U_1 \times \{t_1\}), \]

and
\[ \delta((x,t), \partial_p U) = \inf_{(y,s) \in \partial_p U} \delta((x,t), (y,s)), \text{ and } \delta((x,t), (y,s)) = \min \left\{ |x - y|, \sqrt{|t - s|} \right\}. \]

Denote \( B_r \) (or \( P_r \)) for \( B_r(0) \) (or \( P_r(0) \) respectively). We also recall the weak Morrey space, \( M^p,\lambda(U) \), that is the set of functions \( f \) on \( U \) such that
\[ \left\| f \right\|_{M^p,\lambda(U)}^p = \sup_{r > 0, (x,t) \in U} \left\{ r^{\lambda-n+2} \left\| f \right\|_{L^p,\lambda(P_r(x,t) \cap U)} \right\} < +\infty, \]

where \( L^p,\lambda(P_r(x,t) \cap U) \) is the weak \( L^p \)-space, that is the collection of functions \( v \) on \( P_r(x,t) \cap U \) such that
\[ \left\| v \right\|_{L^p,\lambda(P_r(x,t) \cap U)}^p = \sup_{a > 0} \left\{ a^p \mid \{ z \in P_r(x,t) \cap U : |v(z)| > a \} \right\} < +\infty. \]

Recall that \((u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)\) is a weak solution to (1.1) if \((u, d)\) satisfies (1.1) in sense of distribution and (1.1) in sense of trace. A weak solution \((u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)\) of (1.1) if called a Serrin’s \((p, q)\)-solution, if \((u, \nabla d) \in L^p_x L^q_t(\mathbb{R}^n \times [0, T])\) for some \((p, q)\) satisfying (1.2). Our first result concerns an \( \epsilon_0 \)-regularity criterion for Serrin’s \((p, q)\)-solutions to (1.1).

**Theorem 1.1** There exists \( \epsilon_0 > 0 \) such that if a weak solution \((u, d) \in H^1(P_{r_0}(x_0, t_0), \mathbb{R}^n \times S^2)\) to (1.1) satisfies
\[ \|u\|_{L^p_x L^q_t(P_{\frac{1}{10}}(x_0, t_0))} + \|\nabla d\|_{L^p_x L^q_t(P_{\frac{1}{10}}(x_0, t_0))} \leq \epsilon_0, \]
where \( p \geq 2 \) and \( q \geq n \) satisfy (1.2), then \((u, d) \in C^\infty(P_{\frac{1}{10}}(x_0, t_0))\), and
\[ r\|u\|_{L^\infty(P_{\frac{1}{10}}(x_0, t_0))} + r\|\nabla d\|_{L^\infty(P_{\frac{1}{10}}(x_0, t_0))} \leq C \left( \|u\|_{L^p_x L^q_t(P_{\frac{1}{10}}(x_0, t_0))} + \|\nabla d\|_{L^p_x L^q_t(P_{\frac{1}{10}}(x_0, t_0))} \right). \]

A direct corollary of Theorem 1.1 is the following regularity theorem for Serrin’s \((p, q)\)-solutions to (1.1).

**Corollary 1.2** For some \( 0 < T < +\infty \), suppose that \((u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)\) is a weak solution to (1.1) with \((u, \nabla d) \in L^p_x L^q_t(\mathbb{R}^n \times [0, T])\), for some \( p > 2 \) and \( q > n \) satisfying (1.2). Then \((u, d) \in C^\infty(\mathbb{R}^n \times (0, T], \mathbb{R}^n \times S^2)\).

**Remark 1.3** (i) For the heat flow of harmonic maps and the Navier-Stokes equations, Corollary 1.2 is valid for the end point case \((p, q) = (+\infty, n)\). It is an interesting open question to investigate the regularity of Serrin’s solutions to (1.1) in this end point case.

(ii) If \((u_0, \nabla d_0) \in L^\gamma(\mathbb{R}^n)\) for some \( \gamma > n \), then the local existence of Serrin’s solutions in \( L^p_x L^q_t \) for some \( p > 2 \) and \( q > n \) can be obtained by the fixed point argument (see e.g., [3], §4).
As a corollary of Theorem 1.1 and Corollary 1.2, we can prove the uniqueness of Serrin’s \((p, q)\)-solutions to (1.1).

**Theorem 1.4** For \(n \geq 2\), \(0 < T < +\infty\), and \(i = 1, 2\), if \((u_i, d_i) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \times S^2\) are two weak solutions to (1.1) with the same initial data \((u_0, d_0) : \mathbb{R}^n \to \mathbb{R}^n \times S^2\). Suppose, in additions, there exists \(p > 2\) and \(q > n\) satisfying (1.3) such that \((u_1, \nabla d_1), (u_2, \nabla d_2) \in L^p_t L^q_x(\mathbb{R}^n \times [0, T])\). Then \((u_1, d_1) \equiv (u_2, d_2)\) on \(\mathbb{R}^n \times [0, T]\).

**Remark 1.5** For \(n = 2\), Lin-Wang [19] have proved the uniqueness of (1.1) for \(p = q = 4\). More precisely, if, for \(i = 1, 2\),

\[
\begin{cases}
u_i \in L^\infty([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T], H^1(\mathbb{R}^2)); \\
\nabla d_i \in L^\infty([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T], H^1(\mathbb{R}^2))
\end{cases}
\]

are weak solutions to (1.1) under the same condition, then \(u_i, d_i \equiv (u_2, d_2)\) on \(\mathbb{R}^2 \times [0, T]\). For \(n \geq 3\), Lin-Wang [19] proved the uniqueness, provided that \(u_i \in C([0, T], L^n(\mathbb{R}^n))\) and \(\nabla d_i \in C([0, T], L^n(\mathbb{R}^n))\) for \(i = 1, 2\).

### 2 Proof of Theorem 1.1 and Corollary 1.2

In this section, we will prove Theorem 1.1 and Corollary 1.2 for nematic liquid crystal flows (1.1). The crucial step is to establish an \(\epsilon_0\)-regularity criterion.

**Lemma 2.1** There exists \(\epsilon_0 > 0\) such that if \((u, \nabla d) \in L^p_t L^q_x(P_\frac{1}{10}(0, 1))\), for some \(p \geq 2\) and \(q \geq n\) satisfying (1.3), is a weak solution to (1.1) that satisfies

\[
\|u\|_{L^p_t L^q_x(P_\frac{1}{10}(0, 1))} + \|\nabla d\|_{L^p_t L^q_x(P_\frac{1}{10}(0, 1))} \leq \epsilon_0, \tag{2.1}
\]

then \((u, d) \in C^\infty(P_\frac{1}{16}(0, 1))\), and

\[
\|u\|_{L^\infty(P_\frac{1}{16}(0, 1))} + \|\nabla d\|_{L^\infty(P_\frac{1}{16}(0, 1))} \leq C\epsilon_0. \tag{2.2}
\]

Before proving this lemma, we need the following inequality, due to Serrin [20].

**Lemma 2.2** For any open set \(U \subset \mathbb{R}^n\) and any open interval \(I \subset \mathbb{R}\), let \(f, g, h \in L^2_t H^1_x(U \times I)\) and \(f \in L^p_t L^q_x(U \times I)\) with \(3 \leq n \leq q \leq +\infty\) and \(2 \leq p \leq +\infty\) satisfying (1.3). Then

\[
\int_{U \times I} |f| |g| |\nabla h| \leq C \|\nabla h\|_{L^2(U \times I)} \|g\|_{L^p_t H^1_x(U \times I)} \left\{ \int_{I} \|f\|_{L^q(\mathbb{R}^n)}^p \|g\|_{L^2(\mathbb{R}^n)}^2 \right\}^{\frac{1}{p}}, \tag{2.3}
\]

where \(C > 0\) depends only on \(n\).

**Proof of 2.1** For any \((x, t) \in P_\frac{1}{2}(0, 1)\) and \(0 < r < \frac{1}{\epsilon_0}\), we have, by (2.1),

\[
\|u\|_{L^p_t L^q_x(P_r(x, t))} + \|\nabla d\|_{L^p_t L^q_x(P_r(x, t))} \leq \epsilon_0. \tag{2.4}
\]
We will divide the proof into two claims.

Claim 1. \( \nabla d \in L^\gamma(P_2(0,1)) \) for any \( 1 < \gamma < \infty \), and

\[
\| \nabla d \|_{L^\gamma(P_2(0,1))} \leq C(\gamma) \| \nabla d \|_{L^p_1 L^2(P_1(0,1))}.
\] (2.5)

To show it, let \( d_1 : P_r(x,t) \to \mathbb{R}^3 \) solve

\[
\begin{cases}
\partial_t d_1 - \Delta d_1 = 0, & \text{in } P_r(x,t) \\
\int_{-r}^0 d_1 = d, & \text{on } \partial P_r(x,t).
\end{cases}
\] (2.6)

Set \( d_2 = d - d_1 \). Multiplying (1.13) and (2.6) by \( d_2 \), subtracting the resulting equations and integrating over \( P_r(x,t) \), we obtain

\[
\sup_{-r^2 \leq \tau \leq t} \int_{B_r(x)} |d_2|^2(\cdot, \tau) + 2 \int_{P_r(x,t)} |\nabla d|^2 
\leq C \int_{P_r(x,t)} (|u||d_2||\nabla d| + |\nabla d||d_2||\nabla d|) = J_1 + J_2.
\] (2.7)

By (2.3), the Poincaré inequality and the Young inequality, we have

\[
|J_1| \leq \begin{cases}
\| \nabla d \|^p_{L^p(P_r(x,t))} \| \nabla d_2 \|^{\frac{n}{2}}_{L^2(P_r(x,t))} \left\{ \int_{-r^2}^t \| u \|^p_{L^q(P_r(x,t))} \| d_2 \|^2_{L^2(B_r(x))} \, d\tau \right\}^{\frac{p}{2}}, & p < +\infty \\
\| \nabla d \|^p_{L^p(P_r(x,t))} \| \nabla d_2 \|^p_{L^p(P_r(x,t))} \| u \|^p_{L^q(P_r(x,t))}, & p = +\infty,
\end{cases}
\]

\[
\leq \begin{cases}
\frac{1}{2} \| \nabla d_2 \|^2_{L^2(P_r(x,t))} + C\epsilon_0 \| \nabla d \|^2_{L^2(P_r(x,t))} + C\epsilon_0^\frac{p}{2} \| d_2 \|^2_{L^p(P_r(x,t))}, & p < +\infty \\
\frac{1}{2} \| \nabla d_2 \|^2_{L^2(P_r(x,t))} + C\epsilon_0 \| \nabla d \|^2_{L^2(P_r(x,t))}, & p = +\infty.
\end{cases}
\]

Similarly, for \( J_2 \), we have

\[
|J_2| \leq \begin{cases}
\frac{1}{2} \| \nabla d_2 \|^2_{L^2(P_r(x,t))} + C\epsilon_0 \| \nabla d \|^2_{L^2(P_r(x,t))} + C\epsilon_0^\frac{p}{2} \| d_2 \|^2_{L^p(P_r(x,t))}, & p < +\infty \\
\frac{1}{2} \| \nabla d_2 \|^2_{L^2(P_r(x,t))} + C\epsilon_0 \| \nabla d \|^2_{L^2(P_r(x,t))}, & p = +\infty.
\end{cases}
\]

Putting these estimates into (2.7), applying (2.4), and choosing sufficiently small \( \epsilon_0 \), we have

\[
\int_{P_r(x,t)} \| \nabla d_2 \|^2 \leq C\epsilon_0 \| \nabla d \|^2_{L^2(P_r(x,t))}.
\] (2.8)

This, combined with the standard estimate on \( d_1 \), implies that for any \( \theta \in (0,1) \),

\[
(\theta r)^{-n} \int_{P_{\theta r}(x,t)} \| \nabla d \|^2 \leq C(\theta^2 + \theta^{-n}\epsilon_0) r^{-n} \int_{P_r(x,t)} \| \nabla d \|^2.
\] (2.9)

By iterations, we obtain for any \( (x,t) \in P_{\frac{1}{2}}(0,1) \), \( 0 < r \leq \frac{1}{2} \) and \( 0 < \alpha < 1 \),

\[
r^{-n} \int_{P_r(x,t)} \| \nabla d \|^2 \leq C r^{2\alpha} \int_{P_{\alpha}(0,1)} \| \nabla d \|^2.
\] (2.10)

Hence \( \nabla d \in M^{2,2-2\alpha}(P_{\frac{1}{2}}(0,1)) \) and

\[
\| \nabla d \|_{M^{2,2-2\alpha}(P_{\frac{1}{2}}(0,1))} \leq C \| \nabla d \|_{L^p_1 L^2(P_1(0,1))}.
\] (2.11)
Now Claim 1 follows by the same estimate of Riesz potentials between parabolic Morrey spaces as in \cite{10} (Theorem 1.5) and \cite{19} (Lemma 2.1).

**Claim 2.** $u \in L^\gamma(P_{\frac{1}{4}}(0,1))$ for any $1 < \gamma < \infty$, and

\[
\|u\|_{L^\gamma(P_{\frac{1}{4}}(0,1))} \leq C(\gamma) \|u\|_{L_t^\gamma L_x^2(P(0,1))}. \tag{2.12}
\]

Let $E^\gamma$ be the closure in $L^\gamma(\mathbb{R}^n, \mathbb{R}^n)$ of all divergence-free vector fields with compact supports. Let $\mathbb{P} : L^2(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^2$ be the Leray projection operator. It is well-known that $\mathbb{P}$ can be extended to a bounded linear operator from $L^\gamma(\mathbb{R}^n, \mathbb{R}^n)$ to $E^\gamma$ for all $1 < \gamma < +\infty$. Let $\mathbb{A} = \mathbb{P} \Delta$ denote the Stokes operator.

For any $(x, t) \in P_{\frac{1}{4}}(0,1)$ and $0 < r < \frac{1}{16}$, let $\eta \in C_0^\infty(P_{2r}(x, t))$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $P_r(x, t)$, $|\nabla \eta| \leq 4r^{-1}$, and $|\partial_t \eta| \leq 16r^{-2}$. Let $(v, P^1) : \mathbb{R}^n \times (0,1) \to \mathbb{R}^n \times \mathbb{R}$ solve

\[
\begin{aligned}
&\partial_t v - \Delta v + \nabla P^1 = -\nabla \cdot \left( \eta^2(u \otimes u + \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_n) \right) \quad &\text{in } \mathbb{R}^n \times (0,1) \\
&\nabla \cdot v = 0 \quad &\text{in } \mathbb{R}^n \times (0,1) \\
&v = 0 \quad &\text{on } \mathbb{R}^n \times \{0\}. \tag{2.13}
\end{aligned}
\]

Define $w : P_r(x, t) \to \mathbb{R}^n$ by $w = u - v$. Then $w$ solves the Stokes equation in $P_r(x, t)$:

\[
\begin{aligned}
&\partial_t w - \Delta w + \nabla Q^1 = 0 \quad &\text{in } P_r(x, t) \\
&\nabla \cdot w = 0 \quad &\text{in } P_r(x, t). \tag{2.14}
\end{aligned}
\]

By the standard theory of linear Stokes’ equations, we have that $w \in C^\infty(P_r(x, t))$ and, for any $\theta \in (0,1)$,

\[
\|w\|_{L_t^\gamma L_x^2(P_{r\theta}(x, t))} \leq C\theta \|w\|_{L_t^\gamma L_x^2(P_r(x, t))}. \tag{2.15}
\]

To estimate $v$, we apply $\mathbb{P}$ to both sides of the equation (2.13) to obtain

\[
\partial_t v - \mathbb{A} v = -\mathbb{P} \nabla \cdot \left( \eta^2(u \otimes u + \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_n) \right) \quad \text{in } \mathbb{R}^n \times (0,1); \ v = 0 \text{ on } \mathbb{R}^n \times \{0\}.
\]

By the Duhamel formula, we have

\[
v(t) = -\int_0^t e^{-(t-\tau)\mathbb{A}} \mathbb{P} \nabla \cdot \left( \eta^2(u \otimes u + \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_n) \right) \ d\tau, \ 0 < t \leq 1. \tag{2.16}
\]

Now we can apply Fabes-Jones-Riviere \cite{13} Theorem 3.1 (see also Kato \cite{12} page 474, (2.3')) to conclude that $v \in L_t^\gamma L_x^2(\mathbb{R}^n \times [0,1])$ and

\[
\|v\|_{L_t^\gamma L_x^2(\mathbb{R}^n \times [0,1])} \leq C(\|\eta u\|_{L_t^\gamma L_x^2(\mathbb{R}^n \times [0,1])} + \|\eta \nabla d\|_{L_t^\gamma L_x^2(\mathbb{R}^n \times [0,1])}) \\
\leq C\epsilon_0(\|u\|_{L_t^\gamma L_x^2(P_{2r}(x, t)))} + \|\nabla d\|_{L_t^\gamma L_x^2(P_{2r}(x, t)))}. \tag{2.17}
\]

Putting (2.15) and (2.17) together, we have that for any $\theta \in (0,1)$,

\[
\|u\|_{L_t^\gamma L_x^2(P_{\theta r}(x, t)))} \leq C(\theta + \epsilon_0)\|u\|_{L_t^\gamma L_x^2(P_{r}(x, t)))} + C\epsilon_0 \|\nabla d\|_{L_t^\gamma L_x^2(P_{2r}(x, t)))}. \tag{2.18}
\]
By Claim 1, we have that for any $\alpha \in (0, 1)$, there exists $\epsilon_0 > 0$ depending on $\alpha$ such that
\[
\|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{r}{4}}(x,t))} \leq C r^\alpha \|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))}.
\] (2.19)

Substituting (2.19) into (2.18) yields
\[
\|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{r}{4}}(x,t))} \leq C(\theta + \epsilon_0)\|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{r}{4}}(x,t))} + C r^\alpha \|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))}.
\] (2.20)

It is standard that by choosing $\theta = \theta_0(\alpha) > 0$ and iterating (2.20) finitely many times, we conclude that for any $(x,t) \in P_{\frac{r}{4}}$, $0 < r \leq \frac{1}{4}$ and $0 < \alpha < 1$,
\[
\|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{r}{4}}(x,t))} \leq C\left(\|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))} + \|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))}\right)^\alpha.
\] (2.21)

By Hölder’s inequality, (2.21) implies that $u \in M^{2,2-2\alpha}(P_{\frac{r}{4}}(0,1))$, and
\[
\|u\|_{M^{2,2-2\alpha}(P_{\frac{r}{4}}(0,1))} \leq C\left[\|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))} + \|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))}\right].
\] (2.22)

The higher integrability estimate of $u$ on $P_{\frac{r}{4}}(0,1)$ can be done by the parabolic Riesz potential estimate in parabolic Morrey spaces. Here we will sketch it. Let $\phi \in C_0^\infty(P_{\frac{r}{4}}(0,1))$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $P_{\frac{1}{4}}(0,1)$, and
\[
|\partial_t \phi| + |\nabla \phi| + |\nabla^2 \phi| \leq C.
\]
Define $\tilde{u} : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ by
\[
\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A}P \nabla \left( \phi^2 (u \otimes u + \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_n) \right) d\tau, \quad 0 < t \leq 1.
\] (2.23)

Then, as in the proof of Theorem 3.1 (i) of [11], we have that for any $(x,t) \in \mathbb{R}^n \times (0,1)$,
\[
|\tilde{u}(x,t)| \leq C \int_0^t \int_{\mathbb{R}^n} \frac{1}{\delta^{n+1}(x,t,(y,s))} (|\phi u|^2 + |\phi \nabla d|^2)(y,s) dy ds.
\] (2.24)

Recall the parabolic Riesz potential of order 1, $I_1(\cdot)$, is defined by
\[
I_1(f)(z) := \int_{\mathbb{R}^{n+1}} \frac{|f(w)|}{\delta^{n+1}(z,w)} dw, \quad f \in L^1(\mathbb{R}^{n+1}).
\]
Then we have
\[
|\tilde{u}(x,t)| \leq CI_1(F)(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,1),
\] (2.25)

where
\[
F = \phi^2(|u|^2 + |\nabla d|^2).
\]

By Hölder’s inequality, (2.21), and (2.22), we have that $F \in M^{1,2-2\alpha}(\mathbb{R}^{n+1})$ and
\[
\|F\|_{M^{1,2-2\alpha}(\mathbb{R}^{n+1})} \leq C\left(\|\nabla d\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))} + \|u\|_{L^p_t L^{\frac{n}{q}}_x(P_{\frac{1}{4}}(0,1))}^2\right).
\] (2.26)
Hence, by \cite{11} Theorem 3.1 (ii), we conclude that \( \tilde{u} \in M^*_{\alpha} (\mathbb{R}^n \times [0,1]) \), and
\[
\|\tilde{u}\|_{M^*_{\alpha} (\mathbb{R}^n \times [0,1])}^{\frac{2-2\alpha}{2-2\alpha} (\mathbb{R}^n \times [0,1])} \leq C \|F\|_{M^{1,2-2\alpha} (\mathbb{R}^{n+1})} 
\leq C \left( \|\nabla d\|_{L^2_\gamma L^3_\delta (P(0,1))}^2 + \|u\|_{L^2_\gamma L^3_\delta (P(0,1))}^2 \right). \tag{2.27}
\]
As \( \lim_{\alpha \uparrow 1} \frac{2-2\alpha}{2-2\alpha} = +\infty \), we have that \( \tilde{u} \in L^\gamma (P_{\frac{1}{16}} (0,1)) \) for any \( 1 < \gamma < +\infty \), and
\[
\|\tilde{u}\|_{L^\gamma (P_{\frac{1}{16}} (0,1))} \leq C(\gamma) \left( \|\nabla d\|_{L^2_\gamma L^3_\delta (P(0,1))}^2 + \|u\|_{L^2_\gamma L^3_\delta (P(0,1))}^2 \right). \tag{2.28}
\]
Set \( \tilde{w} = u - \tilde{u} \) on \( P_{\frac{1}{16}} (0,1) \). Then it follows from \cite{11} and (2.23) that
\[
\partial_t \tilde{w} - \Delta \tilde{w} + \nabla \bar{Q} = 0; \quad \nabla \cdot \tilde{w} = 0 \quad \text{in} \quad P_{\frac{1}{16}} (0,1).
\]
By the standard theory of linear Stokes’ equations, we have that \( \tilde{w} \in L^\infty (P_{\frac{1}{16}} (0,1)) \), and
\[
\|\tilde{w}\|_{L^\infty (P_{\frac{1}{16}} (0,1))} \leq C(\gamma) \left( \|\nabla d\|_{L^2_\gamma L^3_\delta (P(0,1))} + \|u\|_{L^2_\gamma L^3_\delta (P(0,1))} \right) \leq C \left( \|\nabla d\|_{L^2_\gamma L^3_\delta (P(0,1))} + \|u\|_{L^2_\gamma L^3_\delta (P(0,1))} \right). \tag{2.29}
\]
It is clear that (2.29) follows from (2.28) and (2.29). This completes the proof of Claim 2.

Finally, it is not hard to see that by the \( W^{1,2}_\gamma \)-theory for the heat equation and the linear Stokes equation, and the Sobolev embedding theorem, we have that \( (u, \nabla d) \in L^\infty (P_{\frac{1}{16}} (0,1)) \). Then the Schauder’s theory and the bootstrap argument can imply that \( (u, d) \in C^\infty (P_{\frac{1}{16}} (0,1)) \). Furthermore, the estimate (2.2) holds. This completes the proof. \hfill \Box

**Proof of Corollary 1.2** It is easy to see that when \( p > 2 \), \( q > n \), for any \( (x, t) \in \mathbb{R}^n \times (0,T) \), we can find \( R_0 > 0 \) such that
\[
\|u\|_{L^p_\gamma L^q_\delta (P_{R_0} (x,t))} + \|\nabla d\|_{L^p_\gamma L^q_\delta (P_{R_0} (x,t))} \leq \epsilon_0,
\]
where \( \epsilon_0 \) is given in Lemma 2.1. By Theorem 1.1, we conclude that \( (u, d) \in C^\infty (P_{\frac{R_0}{16}} (x,t)) \). This completes the proof of Theorem 1.2. \hfill \Box

### 3 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. To do this, we need the following estimate.

**Lemma 3.1** For \( T > 0 \), suppose that \( (u, d) \) is a weak solution to (1.1) in \( \mathbb{R}^n \times (0,T) \), which satisfies the assumption of Theorem 1.4. Then \( (u, d) \in C^\infty (\mathbb{R}^n \times (0,T], \mathbb{R}^n \times S^2) \), and there exists \( t_0 > 0 \) such that for \( 0 < t \leq t_0 \), it holds
\[
\sup_{0 < \tau \leq t} \sqrt{t} \left( \|u(\tau)\|_{L^\infty (\mathbb{R}^n)} + \|\nabla d(\tau)\|_{L^\infty (\mathbb{R}^n)} \right) \leq C \left( \|u\|_{L^p_\gamma L^q_\delta (\mathbb{R}^n \times [0,t])} + \|\nabla d\|_{L^p_\gamma L^q_\delta (\mathbb{R}^n \times [0,t])} \right). \tag{3.1}
\]
In particular, we have
\[
\lim_{t \downarrow 0^+} \sqrt{t} \left( \|u\|_{L^\infty (\mathbb{R}^n)} + \|\nabla d\|_{L^\infty (\mathbb{R}^n)} \right) = 0. \tag{3.2}
\]
Proof. Let $\epsilon_0$ be given by Lemma 2.1. Since $p > 2$ and $q > n$ satisfy (1.2), for any $0 < \epsilon \leq \epsilon_0$ we can find $t_0 > 0$ such that for any $0 < \tau \leq \sqrt{t_0}$

$$
\|u\| L^p_t L^q_x(\mathbb{R}^n \times [0, \tau^2]) + \|\nabla d\| L^p_t L^q_x(\mathbb{R}^n \times [0, \tau^2]) \leq \epsilon. \quad (3.3)
$$

For any $x_0 \in \mathbb{R}^n$, define

$$
\bar{u}(y, s) = \tau u(x_0 + y\tau, s\tau^2) \quad \bar{P}(y, s) = \tau^2 P(x_0 + y\tau, s\tau^2) \quad \bar{d}(y, s) = d(x_0 + y\tau, s\tau^2).
$$

Then $(\bar{u}, \bar{P}, \bar{d})$ is a weak solution to (1.1) on $P_1(0, 1)$, and by (3.3),

$$
\|\bar{u}\| L^p_t L^q_x(P_1(0, 1)) + \|\nabla \bar{d}\| L^p_t L^q_x(P_1(0, 1)) \leq \epsilon. \quad (3.4)
$$

By Lemma 2.1, we conclude that

$$
|\bar{u}(0, 1)| + |\nabla \bar{d}(0, 1)| \leq C \left( \|\bar{u}\| L^p_t L^q_x(P_1(0, 1)) + \|\nabla \bar{d}\| L^p_t L^q_x(P_1(0, 1)) \right). \quad (3.5)
$$

By rescaling, this implies

$$
\tau \left( |u(x_0, \tau^2)| + |\nabla d(x_0, \tau^2)| \right) \leq C \left( \|u\| L^p_t L^q_x(\mathbb{R}^n \times [0, \tau^2]) + \|\nabla d\| L^p_t L^q_x(\mathbb{R}^n \times [0, \tau^2]) \right) \leq C\epsilon. \quad (3.6)
$$

Taking supremum over all $x_0 \in \mathbb{R}^n$ completes the proof. \qed

Proof of Theorem 1.4. By (3.2), we have that for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that

$$
\mathcal{A}(t_0) = \sum_{i=1}^{2} \left[ \sup_{0 \leq t \leq t_0} \sqrt{t}(\|u_i(t)\| L^\infty(\mathbb{R}^n) + \|\nabla d_i(t)\| L^\infty(\mathbb{R}^n))
+ (\|u_i\| L^p_t L^q_x(\mathbb{R}^n \times [0, t_0])) + \|\nabla d_i\| L^p_t L^q_x(\mathbb{R}^n \times [0, t_0])) \right] \leq \epsilon. \quad (3.7)
$$

It suffices to show $(u_1, d_1) = (u_2, d_2)$ on $\mathbb{R}^n \times [0, t_0]$. To do so, let $u = u_1 - u_2$ and $d = d_1 - d_2$. Applying $\mathcal{P}$ to both (1.1) for $u_1$ and $u_2$ and taking the difference of resulting equations, we have that

$$
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} u - \Delta u = -\nabla \cdot (u \otimes u_1 + u_2 \otimes u + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d + (|\nabla d_1| + |\nabla d_2|)|\nabla d| \mathbb{I}_n), \\
\nabla \cdot u = 0, \\
\frac{\partial}{\partial t} d - \Delta d = (|\nabla d_1 + \nabla d_2| \cdot \nabla d_1 + |\nabla d_1|^2 d) - |u \cdot \nabla d_1 + u_2 \cdot \nabla d|,
\end{cases}
\end{align*}
$$

(3.8)

By the Duhamel formula, we have that for any $0 < t \leq t_0$,

$$
u(t) = -\int_0^t e^{-(t-\tau) \mathcal{A}} \nabla \cdot \left( u \otimes u_1 + u_2 \otimes u + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d + (|\nabla d_1| + |\nabla d_2|)|\nabla d| \mathbb{I}_n \right) d\tau,$$
\[ d(t) = \int_0^t e^{-(t-\tau)\Delta} \left( (\nabla d_1 + \nabla d_2) \cdot \nabla d + |\nabla d_1|^2 d - u \cdot \nabla d_1 - u_2 \cdot \nabla d \right) d\tau. \] (3.9)

For \(0 < t \leq t_0\), set

\[ \Phi(t) = \|u\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} + \|\nabla d\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} + \sup_{0 \leq \tau \leq t} \|d(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)}. \]

By (3.9) and the standard estimate on the heat kernel, we obtain that

\[
\left\| \nabla d(t) \right\|_{L^2(\mathbb{R}^n)} \leq C \left[ \sum_{i=1}^{2} \int_0^t (t-\tau)^{\frac{1}{p}-1} \|\nabla d_i\|_{L^p(\mathbb{R}^n)} \|\nabla d\|_{L^2(\mathbb{R}^n)} d\tau + \|d\|_{L^\infty(\mathbb{R}^n)} \int_0^t (t-\tau)^{\frac{1}{p}-1} \|\nabla d_1\|_{L^p(\mathbb{R}^n)} d\tau \\
+ \int_0^t (t-\tau)^{\frac{1}{p}-1} \|\nabla d_1\|_{L^p(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)} d\tau + \int_0^t (t-\tau)^{\frac{1}{p}-1} u_2 \|\nabla d\|_{L^p(\mathbb{R}^n)} d\tau \right].
\] (3.10)

By the standard Riesz potential estimate in \(L^p\)-spaces (see [4] Theorem 3.0), we see that \(\nabla d \in L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])\), and

\[
\left\| \nabla d \right\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \leq C \left[ \sum_{i=1}^{2} \|\nabla d_i\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \|\nabla d\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} + \|\nabla d\|_{L^\infty(\mathbb{R}^n \times [0,t_0])} \|\nabla d_1\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \right. \\
+ \left. \|\nabla d_1\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \|u\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} + \|u_2\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \|\nabla d\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \right] \\
\leq C \mathcal{A}(t_0) \Phi(t_0).
\] (3.11)

Similarly, by using the estimate of Theorem 3.1 (i) of [4], we have that \(u \in L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])\), and

\[
\|u\|_{L^p_t L^2_x(\mathbb{R}^n \times [0,t_0])} \leq C \mathcal{A}(t_0) \Phi(t_0).
\] (3.12)

Now we need to estimate \(\sup_{0 \leq \tau \leq t_0} \|d(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)}\). We claim

\[
\|d\|_{L^\infty(\mathbb{R}^n \times [0,t_0])} \leq C \mathcal{A}(t_0) \Phi(t_0).
\] (3.13)

To show (3.13), let \(H(x,t)\) be the heat kernel of \(\mathbb{R}^n\). By (3.9), we have

\[
|d(x,t)| = \left| \int_0^t \int_{\mathbb{R}^n} H(x-y,t-\tau) \left( (\nabla d_1 + \nabla d_2) \cdot \nabla d + |\nabla d_1|^2 d - u \cdot \nabla d_1 - u_2 \cdot \nabla d \right) (y, \tau) dyd\tau \\
- \int_0^t \int_{\mathbb{R}^n} H(x-y,t-\tau) (u \cdot \nabla d_1 + u_2 \cdot \nabla d) (y, \tau) dyd\tau \right| \\
\leq C \left[ \int_0^t \int_{\mathbb{R}^n} H(x-y,t-\tau) K(y, \tau) dyd\tau \\
+ \int_0^t \int_{\mathbb{R}^n} H(x-y,t-\tau) |\nabla d_1|^2(y, \tau) dyd\tau \sup_{0 \leq \tau \leq t} \|d(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} \right],
\] (3.14)
where

\[ K(y, \tau) := \sum_{i=1}^{2} (|u_i| + |\nabla d_i|)(|u| + |\nabla d|)(y, \tau). \]

By (3.7), we have that for any 0 < t ≤ t₀,

\[
\int_{0}^{t} \int_{\mathbb{R}^n} H(x - y, t - \tau)K(y, \tau) \, dy \, d\tau \\
\leq \mathcal{A}(t_0) \int_{0}^{t} (t - \tau)^{-\frac{\theta}{2}}\tau^{-\frac{1}{2}} \int_{\mathbb{R}^n} (|u| + |\nabla d|) \exp \left( -\frac{|x - y|^2}{4(t - \tau)} \right) \, dy \, d\tau \\
\leq \mathcal{A}(t_0) \left( \int_{0}^{t} (t - \tau)^{-\frac{\theta}{2}}\tau^{-\frac{1}{2}} \right)^{\frac{2}{p}} \left( \mathcal{L}^p_L(R^n \times [0, t]) \| |u| + |\nabla d| \|_{\mathcal{L}^p_{L^q}(\mathbb{R}^n \times [0, t])} \right) \\
\leq C\mathcal{A}(t_0) \Phi(t_0),
\]

where we have used Hölder inequality and

\[
\left\| (t - \tau)^{-\frac{\theta}{2}}\tau^{-\frac{1}{2}} \right\|_{\mathcal{L}^p_{L^q}(\mathbb{R}^n \times [0, t])} = t^{(\frac{1}{2} - \frac{\theta}{2q + 1})} \tau^{-\frac{\theta}{2q + 1}} \int_{0}^{1} (1 - \tau)^{-\frac{np}{2p - 1} - \frac{np}{2(p - 1)}} d\tau = \int_{0}^{1} (1 - \tau)^{-\frac{2p - 2}{2p - 1} - \frac{2p}{2(p - 1)}} d\tau < +\infty,
\]

as (i) \( \frac{np}{2q} + \frac{1}{p} = \frac{1}{2} \), and (ii) \( 2 < p < +\infty \) yields \( \frac{p}{2(p - 1)} < 1 \) and \( \frac{2p}{2(p - 1)} < 1 \).

Similarly, we can obtain that for 0 ≤ t ≤ t₀,

\[
\int_{0}^{t} \int_{\mathbb{R}^n} H(x - y, t - \tau)|\nabla d_1|^2(y, \tau) \, dy \, d\tau \leq C\mathcal{A}^2(t_0).
\]

Putting (3.15) and (3.16) into (3.14) and taking supremum over \( (x, t) \in \mathbb{R}^n \times [0, t_0] \), we have

\[
\sup_{0 \leq t \leq t_0} \| d \|_{L^\infty(\mathbb{R}^n)} \leq C\mathcal{A}(t_0) \Phi(t_0) + C\mathcal{A}^2(t_0) \sup_{0 \leq t \leq t_0} \| d \|_{L^\infty(\mathbb{R}^n)}. \tag{3.17}
\]

Therefore, if we choose \( \epsilon \leq \frac{1}{2C} \) so that \( C\mathcal{A}^2(t_0) \leq C\epsilon^2 \leq \frac{1}{2} \), then we obtain (3.13).

Putting (3.11), (3.12) and (3.13) together, and choosing \( \epsilon \leq \frac{1}{2C} \), we obtain

\[ \Phi(t_0) \leq C\mathcal{A}(t_0) \Phi(t_0) \leq \frac{1}{2} \Phi(t_0). \]

This implies that \( \Phi(t_0) = 0 \) and hence \( (u_1, d_1) \equiv (u_2, d_2) \) on \( \mathbb{R}^n \times [0, t_0] \). If \( t_0 < T \), then we can repeat the argument for \( t \in [t_0, T] \) and eventually show that \( (u_1, d_1) \equiv (u_2, d_2) \) on \( \mathbb{R}^n \times [0, T] \). This completes the proof. \( \square \)

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