Computational Methods for the Analysis of Molten Metal Electromagnetic Confinement Systems

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(Received on March 20, 1989; accepted in the final form on July 21, 1989)

Closed form solutions as well as Finite Element, Volume integral and Boundary Integral numerical methods are reviewed from the viewpoint of electromagnetic confinement system analysis. Particular attention is given to the use of Boundary Integral methods to obtain a self-consistent solution of the electromagnetic field equations when the conducting body is molten metal with a free surface. The application of Finite Element and Volume integral methods to confinement problems involving an assumed free surface contour is reviewed. Closed form solutions that conveniently yield upper bound estimates of confinement and stirring forces are discussed. A 2-D electromagnetic casting system is used to focus the discussion of the various methods. Numerical methods (e.g., Fast Fourier Transform, eigenvalue decomposition, Preconditioned Conjugate Gradients) that are useful in the analysis of confinement problems are discussed. Free surface profiles for several 2-D applications are presented. Suitable methods for 3-D confinement problems are briefly discussed. A substantial reference list is included in the paper.

KEY WORDS: electromagnetic confinement; electromagnetic casting; eddy current; numerical method; finite element; boundary integral.

1. Introduction

Electromagnetic confinement of molten metal is increasingly being used in a broad range of metallurgical applications. These range from relatively mature applications such as the electromagnetic mould for casting aluminum and bronze\(^{(3)}\) to emerging or proposed applications such as:

i) levitation and/or confinement in the entry region between the tundish and a horizontal casting mould,

ii) edge confinement\(^{(5)}\) within a strip or plate caster, and

iii) control of shape (e.g., thickness) to near final form specifications.\(^{(4)}\)

In virtually all such applications, an alternating electromagnetic field is used to induce current to flow within the molten metal so as to create the desired confinement forces. The confinement forces work against a static head that, in many applications, is primarily gravitational in origin.

A major problem in most confinement applications is to design a coil configuration such that the developed electromagnetic pressure is compatible with the desired free surface contour. From an electromagnetic viewpoint, analysis is difficult since the unknown free surface contour critically influences the magnetic field distribution, and thus the electromagnetic force\(^{(6)}\) and pressure distributions. Additional factors that complicate the electromagnetic field analysis include the following:

1) In order to achieve stable confinement, the penetration of an a.c. induction field into the conducting body will generally be quite shallow. This can lead to an excessively large solution grid if methods such as Finite Elements, Finite Differences, or Volume Integrals are used to undertake the electromagnetic field analysis.

2) In general, many of the confinement and/or stirring applications involve a geometry that is 3-dimensional. The solution of true 3-D electromagnetic field problems that describe induction processes is still a very active area of research. Unless these factors are most carefully considered, the electromagnetic fields and forces that are obtained from analysis can be quite meaningless.

This paper presents a selective review of computational methods that can be used, with confidence, to analyse the electromagnetic fields in confinement and stirring applications. The application of various solution methods to 2-D electromagnetic field problems is reviewed. Closed form solutions, Finite Elements, and Volume Integrals are considered, with special reference being given to the identification of the strengths and weaknesses of such formulations. It is shown that none of the above methods is particularly well suited to the analysis of a true electromagnetic confinement problem where the shape of the free surface is unknown a priori. By contrast, Boundary Element techniques appear to offer significant advantages for the analysis of electromagnetic confinement problems in both 2-D and 3-D. The use of such methods to very efficiently solve free surface problems is reviewed and results are presented for the crucible induction furnace, the electromagnetic mould and the CREM casting system.
2. Electromagnetic Fields and Forces

The general problem being considered in this paper is best illustrated by the now classic electromagnetic casting mould (Fig. 1). A timed harmonic current, flowing in an induction coil, creates a magnetic field of strength $H$. The latter, in turn, induces an electric field of strength $E$ in the molten metal to be contained and in any other nearby conducting bodies; e.g., the field shaping short circuit turn. The currents being driven by $E$ interact with the magnetic fields so as to produce a volume distribution of forces within the molten metal being cast. These electromagnetic forces result in two coupled phenomena that are characteristic of induction-based confinement applications:

(1) The nonconservative component of the force distribution stirs the molten metal. Depending on the supply frequency and system geometry, this stirring can be very pronounced.

(2) The free surface of the molten metal assumes a shape that is governed by the balance of electromagnetic pressure acting against pressures derived from gravity, molten metal mass transfer and surface tension.

The discussion in the present paper will focus primarily on electromagnetic confinement and on computational methods that can include the effect of the molten metal free surface. It is worthwhile emphasizing, however, that the free surface also has a major effect on the electromagnetic stirring forces.

2.1. Electromagnetic Fields

The magnetic field strength $H$ and the electric field strength $E$ can be obtained by solving Maxwell’s equations which, in the case of time harmonic sources of frequency $\omega$, can be written as:

$$\nabla \times H = Y E \quad \text{.........(1-a)}$$

$$\nabla \times E = Z H \quad \text{.........(2-a)}$$

$$Z = -j\omega \quad \text{.........(2-b)}$$

where, $j$: the conventional complex operator defined by $j^2 = -1$

$\varepsilon$: electrical permittivity

$\mu$: magnetic permeability

$\sigma$: electrical conductivity

$\omega$: the angular frequency of the electrical supply.

Knowing the electric field strength $E$, the conduction current density $J$ is simply:

$$J = aE \quad \text{.........(3)}$$

When the electromagnetic problem can be described in terms of a 2-D model, it may be convenient to describe $H$ in terms of a magnetic vector potential $\vec{A}$:

$$H = \frac{1}{\mu} \nabla \times \vec{A} \quad \text{.........(4)}$$

with $\vec{A}$ satisfying the following divergence constraint:

$$\nabla \cdot \vec{A} = 0 \quad \text{.........(5)}$$

It is a simple matter to show that in 2-D time harmonic problems, the electric field $E$ can be described in terms of the following two potentials:

$$E = -\nabla \phi - j\omega \vec{A} \quad \text{.........(6)}$$

In Eq. (6), the first term represents the electric field due to an impressed voltage sources while the second term represents the induced electric field. While the magnetic vector potential can be used to advantage in 2-D problems, the same advantages do not necessarily extend to the 3-D case. The difficulties lie with the choice of an appropriate gauge condition and with the potential used to represent the free space region.

2.2. Electromagnetic Forces and Pressure

At any point within a metallic conductor where the time harmonic current density is $J$ and the magnetic field strength is $H$, the time average electromagnetic volume force density is:

$$f = \frac{1}{2} Re(J \times H^*) \quad \text{.........(7)}$$

where, $Re$: the real part of a complex quantity.

These forces are confined to a surface layer having a thickness of approximately $2\delta$, where $\delta$ is the electromagnetic penetration depth:

$$\delta = \sqrt{\frac{2}{\mu \varepsilon \omega}} \quad \text{.........(8)}$$

In confinement applications, $\delta$ is generally chosen to be small relative to the dimensional of the molten metal region in order to maximize the sensitivity of the confinement field to perturbations in the geometry of the molten metal.
It is a relatively simple matter to expand Eq. (7) in order to obtain:

\[
J = -\frac{\mu}{4}(H \cdot H^*) + \frac{\mu}{2} Re\{[H \cdot \bar{H}]H^*\} \quad \text{(9)}
\]

where the superscript* is used to denote a complex conjugate quantity and all time harmonic fields are assumed to be specified in terms of peak values. The first term in Eq. (9) represents a conservative force that is derived from an electromagnetic pressure \( p_m \):

\[
p_m = \frac{\mu H^2}{4} \quad \text{…..(10)}
\]

This force can support a static head (e.g., as in a confinement application), but cannot do work (e.g., as in stirring the molten metal). The nonconservative contribution to the force field is given by the second term in Eq. (9). It is this latter force component that results in fluid motion in stirring applications. It is important to note, however, that in a cylindrical geometry, the second term in Eq. (9) also contributes to the total electromagnetic pressure when \( \delta \) is large relative to the conductor dimensions; i.e., when there is full penetration of the magnetic flux.

Regardless of the solution method that is used to determine the electromagnetic field distribution in a confinement application, the free surface contour of the molten metal must be such that the following pressure balance is satisfied at each point on the contour:

\[
\rho g h' = \frac{\rho}{2}(\partial \cdot \partial) + \frac{\rho}{4}(H \cdot H^*) + K^\prime \phi + C \quad \text{…..(11)}
\]

where, \( C \): an integration constant
\( g \): the gravitational constant
\( h' \): the height of the column of metal that is being supported at the contour point under consideration
\( H \): the magnetic field strength
\( K^\prime \): the curvature of the contour
\( \partial \): the fluid velocity
\( \rho \): the mass density
\( \phi \): the surface tension.

In most electromagnetic confinement applications, surface tension effects can be neglected. This is a reasonable approximation provided the molten metal free surface has a radius of curvature that is large.

In applications where surface tension and the momentum transfer from the moving metal can be neglected, the shape of the molten metal free surface is determined by a simple balance between gravitation and electromagnetic pressures. A simple iteration can therefore be implemented in which the shape of the free surface is adjusted so as to achieve the required balance. In many important applications, this balance can be achieved in relatively few iterations since the free surface assumes a simple shape. In the case of the crucible induction furnace, for example, the meniscus can be accurately described by a parabolic surface of revolution.

2.3. Solution Methods for Electromagnetic Field Problems

A broad range of closed form solutions is available for 1-D and quasi 2-D induction problems (7-10); a quasi 2-D problem is defined as being one in which the field variation in one direction exhibits a spatial harmonic variation. In 2-dimensions, time harmonic electromagnetic field problems have been solved by Finite Difference (11,12), Finite Element (13,14), Volume Integral (15,16), and Boundary Integral (17-21). Today, the method of choice in 2-D is probably the Finite Element method. Several excellent packages, running on machines ranging from mainframes to personal computers, are available for this class of problem.

The development of reliable computational methods for 3-D eddy current problems is today an area of very active research. While Finite Element methods have been proposed for 3-D eddy current problems, (22-24), there remain difficulties with the specification of the problem geometry and with the treatment of the very important free space region. These problems, together with advantages to be gained from parallel computation, (25) has resulted in considerable interest being shown in Boundary Element methods for 3-D problems (26-30).

As the shape of the free surface is not known a priori in most electromagnetic confinement problems, the e.m. field problem must be iteratively solved subject to the constraint provided by Eq. (11). Electromagnetic problems of this type have recently begun to receive attention in the literature (15,21-34). For the most part, (volume) Integral Equation (35) and (point) Finite Difference (36) methods have been used to compute the required electromagnetic fields in confinement applications. However, these two methods are awkward to use for this class of problem for several reasons. First, the entire solution grid for the conducting region must be modified at each step in the iteration for the free surface shape. Second, both Volume Integral and Finite Difference methods models do not easily model smooth surfaces. Finally, point methods (Finite Difference or Finite Element) must include the exterior free space region in the solution of any induction e.m. problem. This considerably increases the magnitude of the numerical problem. For these reasons, formulations that include only the interface boundaries offer significant advantages when modelling electromagnetic confinement problems. Such formulations are discussed in a subsequent section of this paper.

3. Closed Form Solutions

Closed form solutions cannot directly be used to solve electromagnetic confinement problems as their range of applicability is generally limited to 1-D and quasi 2-D problem geometries. However, such solutions can be used to significant advantage to obtain upper bound estimates for confinement pressures and stirring forces. Additionally, the dependence of the forces on such parameters as supply frequency, coil dimensions and melt electrical conductivity is easily
obtained. Two solution methods that prove to be particularly well suited to electromagnetic confinement and pumping applications are reviewed in this section.

3.1. The Transfer Matrix Solution Method

In a classic series of papers, Freeman developed the Transfer Matrix Method to very efficiently formulate and solve quasi 2-D multi-layer induction problems having either translational\(^{31}\) or cylindrical\(^{36,37}\) geometries. In the case of a cylindrical geometry, the excitation currents could be either circumferentially\(^{30}\) or axially\(^{31}\) directed. This solution method has been used to determine the lift and thrust forces developed by a linear induction pump\(^{30}\) and to analyse the electromagnetic forces developed by a rotary, in-mould stirring unit.\(^{39}\)

To illustrate the Transfer Matrix Method, consider the multi-layer cylindrical geometry shown in Fig. 2. When the excitation currents at the \(r\)-th interface are polyphase in the azimuthal direction, an axially directed travelling wave having magnitude \(J_r\) and wave number \(k\) is produced:

\[ J_r = J_0 \exp(j(\omega - kr)) \] (12)

One obvious application for a model of this type would be to determine the lifting and radially directed confinement forces in the so-called Lowry Levitation Casting apparatus.\(^{41}\) Such a system is essentially a vertically oriented cylindrical induction pump.

The basis of the Transfer Matrix formulation is to obtain the electromagnetic fields strictly in terms of interface components. In Fig. 2, for example, \(E_r\) and \(H_r\) denote the azimuthal and axial components of \(E\) and \(H\), respectively, at the \(r\)-th interface. Knowing \(E_r\), it is a simple matter to show that \(B_r\), the radially directed component of magnetic flux density at the \(r\)-th interface, is given by:

\[ B_r = -\frac{k}{\omega} E_r \] (13)

Freeman\(^{30}\) has shown that the input impedance at the current sheet excitation is given by:

\[ Z_{10} = \frac{Z_r Z_{r+1}}{(Z_r + Z_{r+1})} \] (14)

where \(Z_{r+1}\) and \(Z_r\) are obtained recursively as follows:

\[ Z_N = j \frac{\alpha \mu_0}{\gamma} K_1(\gamma r_{N-1}) \] (15-a)
\[ Z_{r-1} = (b_{r-1} - Z_r d_{r-1})/(c_{r-1} Z_N - a_{r-1}) \] (15-b)
\[ Z_{r+1} = (b_{r+1} - Z_r d_{r+1})/(c_{r+1} Z_r - a_{r+1}) \] (15-c)

\[ Z_1 = \frac{j \alpha \mu_0}{\gamma} I_1(\gamma r_1) \] (15-d)
\[ Z_2 = (b_2 - a_2 Z_1)/(c_2 Z_1 - d_2) \] (15-e)
\[ Z_r = (b_r - a_r Z_{r-1})(c_r Z_{r-1} - d_r) \] (15-f)

\[ a_n = r_n r_{n-1} I_1(a_n) K_0(a_n) + I_0(a_n) K_1(a_n) \] (16-a)
\[ b_n = j \mu_0 n \omega r_n I_0(a_n) K_0(a_n) - I_0(a_n) K_1(a_n) \] (16-b)

\[ c_n = j \frac{r_n^2 r_{n-1}}{n \omega} I_0(a_n) K_0(a_n) - I_0(a_n) K_1(a_n) \] (16-c)
\[ d_n = r_n r_{n-1} I_0(a_n) K_1(a_n) + I_0(a_n) K_0(a_n) \] (16-d)

\[ \gamma_n = [k^2 + j \mu_0 n \omega r_n]^{1/2} \] (16-e)
\[ a_n = \gamma_n r_n \] (16-f)
\[ a_{n+1} = \gamma_n r_{n+1} \] (16-g)

while \(I_0, I_1, K_0\), and \(K_1\) are modified Bessel functions. Knowing \(Z_r\) and \(Z_{r+1}\), \(E_r\) and \(H_r\) (at the excitation interface) can be obtained by solving the following two equations:

\[ Z_r = -\frac{E_r}{H_r} \] (17-a)
\[ Z_{r+1} = \frac{E_r}{(H_r - J_r)} \] (17-b)

At any other interface:

\[ \begin{bmatrix} E_n \\ H_n \end{bmatrix} = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix} \] (18)

Knowing the interface values of \(E\) and \(H\), it is convenient to use the Poynting vector in order to determine the real power \(P_n\) and the reactive power \(Q_n\) in the \(n\)-th layer:

\[ P_n + j Q_n = \pi [r_n E_n H_n^* - r_{n-1} E_{n-1} H_{n-1}^*] \] (19)

Note that in Eq. (19), \(P_n\) and \(Q_n\) are defined as being per unit of length along the cylindrical conductor. Similarly, the net force acting on the \(n\)-th layer is most conveniently determined by integrating the Maxwell stress tensor.\(^{39,40}\)

This formulation of the multi-layer quasi 2-D problem is extremely simple to implement for numerical calculation and is much more convenient to use than are other solutions to similar problems that are avail-
able in the literature.

3.2. The Green’s Function—Fourier Transform Solution

An induction model that is particularly useful for determining upper bound estimates of confinement forces consists of an infinitely long, possibly multi-layer, conductor in the field of a discrete source. One example, consisting of a long conducting cylinder coaxial with a filament current source, is shown in Fig. 3. In general, the source for this class of problem could be a short coil having finite length and cross-section. The electromagnetic field distribution for such problems is very conveniently determined in terms of a Green’s function closed form solution. Numerical values for the fields can be efficiently computed by using Fast Fourier Transform (FFT) routines that are part of most good scientific software packages. This solution method has been used to obtain upper bound estimates of the electromagnetic pressure, as a function of frequency, melt conductivity and coil coupling, developed by an electromagnetic mould.

To illustrate this class of solution, consider the simple filamentary current source of magnitude $I$ and radius $a$ shown in Fig. 3. The source is located at $z=0$ in a cylindrical coordinate frame of reference. At radius $r=b$, this source can be expressed as:

$$ f(z) = \delta(z) \quad \text{(20)} $$

where, $\delta(z)$: the Dirac delta function.

This distribution has the following Fourier Transform:

$$ K(k) = I \int_{-\infty}^{+\infty} \delta(z) e^{-ikz} dz = I \quad \text{(21)} $$

In other words, in the domain of the wave number $k$, the filamentary source can be represented in terms of an infinite number of travelling wave current sheets. For each $k$, the quasi 2-D model of Fig. 2 applies. Finally, taking the inverse Fourier Transform and noting the symmetry with respect to $z$, it follows that:

$$ f(z) = \frac{1}{\pi} \int_{0}^{\infty} K(k) \cos kz \, dk \quad \text{(22)} $$

Using the same concepts, it is a simple matter to show that when the source is a coil having radius $b$ and length $2L$, Eq. (22) still applies except that:

$$ K(k) = \frac{\sin kL}{kL} \quad \text{(23)} $$

The magnetic vector potential solution for the problem shown in Fig. 3 can therefore be constructed by considering the more fundamental problem of a conducting cylinder coaxial with unit magnitude current sheet of radius $r=b$ and wave number $k$. A separation of variables solution for this fundamental problem shows that within the conducting cylinder, the magnetic vector potential has the following $r$ variation:

$$ \vec{A}(r; k) = N(r; k) I_0(\gamma r) \quad \text{(24-a)} $$

where $N(r; k)$ is given by:

$$ N(r; k) = \frac{b}{a} \frac{\mu_1 I_1(kb)}{\gamma I_0(\gamma a) K_1(ka) + \mu_1 I_1(\gamma a) K_1(ka)} \quad \text{(24-b)} $$

where $\gamma$ is given by Eq. (16-c) and the subscript 1 is used to denote the conducting cylinder. Clearly, the Transfer Matrix approach described in the previous section could also have been used to determine $N(r; k)$.

It follows from Eq. (22) that at any point within the conducting cylinder, the magnetic vector potential due to a discrete source is given by:

$$ A_d(r; z) = \frac{1}{\pi} \int_{0}^{\infty} K(k) N(r; k) I_0(\gamma r) \cos kz \, dk \quad \text{(25)} $$

Knowing $A_d(r; z)$, the induced current density can be obtained directly from Eqs. (3) and (6). In a similar fashion, the components of magnetic flux density are given by:

$$ B_i(r; z) = \frac{1}{\pi} \int_{0}^{\infty} kK(k) N(r; k) I_0(\gamma r) \sin kz \, dk \quad \text{(26-a)} $$

$$ B_i(r; z) = \frac{1}{\pi} \int_{0}^{\infty} \gamma K(k) N(r; k) I_0(\gamma r) \sin kz \, dk \quad \text{(26-b)} $$

The most convenient method of computing the integrals in Eqs. (25) and (26) is to use sin and cos Fast Fourier Transform algorithms. In this fashion, for fixed $r$, an appropriately chosen array of samples in the $k$ domain will yield an axial distribution of field values. This approach has been used to obtain the magnetic field strength values shown in Fig. 4.

4. Volume Integral Solutions

The closed form solution methods that were detailed in the previous section, while providing useful design information when applied to electromagnetic...
confinement applications, nevertheless have the following severe limitations:

1. The multi-layer conducting body must be infinitely long. As end effects are not included in the formulations, true confinement applications cannot be considered.

2. The formulations are limited to a single body, albeit one having multiple layers. Therefore, the effects of field shaping conductors cannot be considered.

Both limitations can be removed when the electromagnetic confinement problem is formulated using one of several available approximate methods. By far the simplest approximate solution method for the confinement problem is based on a Volume Integral formulation.

To illustrate the essential aspects of the Volume Integral formulation, consider a 2-D axisymmetric array of conducting bodies. The conducting bodies are nonmagnetic with constant electrical conductivity and the excitation is time harmonic with angular frequency $\omega$. This geometry includes, as a special case, the electromagnetic mould shown in Fig. 1. By applying the vector Green's theorem and using the differential equation for the electromagnetic potential as derived from Eqs. (1-a), (4), and (5), it is possible to show that at an arbitrary point $\mathbf{r}$ within the conducting region:

$$A_{\phi}(f) = A_{\phi_{0}}(f) + \gamma_{0} \int g(f, f') J_{s}(f') df' \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \l
proposed by Silvester.\textsuperscript{15}) With this approach, Eq. (31) is first transformed by multiplying by \( \hat{R}^{-1} \) to obtain a matrix equation of the following form:

\[
[\hat{U} + j\mu_0\sigma \hat{M}] \hat{I} = \hat{R}^{-1} \hat{P} = \hat{b} \quad \text{...........(32)}
\]

where, \( \hat{U} \): the unity matrix.

Let \( \lambda_i \) be the \( i \)-th eigenvalue of \( \hat{M} \), with \( \hat{m}_i \) being the corresponding eigenvector. Assume further that the eigenvalues are orthonormal. It then follows that the solution for \( \hat{I} \) can be obtained as an expansion in terms of the eigenvectors \( \hat{m}_i \):

\[
\hat{I} = \sum_{i=1}^{N} a_i \hat{m}_i \quad \text{............(33-a)}
\]

where,

\[
a_i = \frac{\langle \hat{b}, \hat{m}_i \rangle}{1 + j\mu_0\sigma \lambda_i} \quad \text{............(33-b)}
\]

Note that the numerator in Eq. (33-b) represents the inner, or dot, product of two vectors. This solution method has the following significant advantages:

1. Only the dominant eigenvalues need be used in determining the constants \( a_i \). Therefore, \( N \) can be much smaller than the number of unknowns currents.

2. The eigenvalues and eigenvectors need only be computed once for a given grid but can be used for any \( \omega \alpha \) product that is consistent with that grid. Thus, the frequency dependence of the currents and forces can be obtained very efficiently without resolving the entire problem.

Knowing the current distribution, the forces acting on each element can be determined. One approach would be to compute the magnetic flux density at the elements of interest and thus determine the Lorentz force density. A preferable approach, however, would be to use the methods suggested by Lupi\textsuperscript{47} or Fawzi

\textbf{5. Finite Element CAD Methods}

Today, Finite Elements are almost universally used for the analysis of 2-D static and dynamic magnetic field problems in which the material boundaries are fixed. As the basic theory of the method\textsuperscript{49} and the application to 2-D electromagnetic field problems\textsuperscript{13,14,50} is well understood, details will not be presented in the section. Rather, attention will be focussed on those aspects of the method that are particularly relevant to electromagnetic confinement problems.

The Finite Element analysis of 2-D eddy current problems is undertaken in terms of a single component vector field which most often is the magnetic vector potential \( A \). The basic Finite Element formulation for such problems is derived, either by applying variational techniques to an appropriate functional, or by seeking a Galerkin (weighted residual) approximate solution to the governing differential equation. In either case, the approximate solution must be obtained on a grid that encompasses the full problem domain, including free space. The latter is in marked contrast to an Integral Equation formulation where a solution grid need only be applied to the conducting regions.

Typical electromagnetic confinement applications can be classed an being \textit{open boundary} problems, with...
the magnetic fields extending to infinity, in principle. A common Finite Element approximation is to impose a Dirichlet boundary condition at some appropriately chosen large distance from the region of interest. Otherwise, the number of nodal points can become excessively large. A preferable alternative would be to satisfy the far field constraints exactly on a special purpose boundary.\textsuperscript{51,52} Such a boundary must enclose all of the conducting regions in the problem, but otherwise, there are relatively few constraints as to how close the boundary can be placed to the conductors. To date, most (if not all) of the available Finite Element CAD packages for 2-D electromagnetics use the simple Dirichlet approximation for exterior field problems.

A typical Finite Element grid, representing the mould region in a CREM casting process,\textsuperscript{50} is shown in Fig. 7. Relative to the Volume Integral solution discussed in the previous section, the triangular elements used here have the significant advantage that curved surfaces can be more closely modeled. Further, a very convenient formulation results when triangular elements are coupled with linear interpolation basis functions.\textsuperscript{18,54} In particular, general high order elements can be obtained for a range of differential operators and the differentiation of the Finite Element solution can be accomplished as a simple matrix operation.\textsuperscript{54} These latter two features have very important implications when the Finite Element Method is used for analysis of electromagnetic confinement problems.

Unlike an Integral Equation formulation which, when cast in discrete form results in a fully populated system matrix, the Finite Element Method is based on a local approximation of the governing differential equation. Therefore, a field variable at any given node is influenced only by its connected neighbouring nodes. Consequently, the resulting system of linear equations is large and very sparse. While direct solutions methods such as Gaussian Elimination can be used with such systems, provided consideration is given to sparsity, the solution method that is most commonly used for 2-D Finite Element eddy current problems is based on the Preconditioned Conjugate Gradient algorithm.\textsuperscript{55} This iterative solution method provides rapid convergence and has been found to be very robust.

Progress in the development of Finite Element Methods for 2-D electromagnetic field problems has now reached the stage where full-featured CAD packages that run efficiently on personal computers are readily available. Many of the issues that relate to the development of such packages have been described by Lowther and Silvester\textsuperscript{50} and also can be found in the literature.\textsuperscript{57} Features that are important for the analysis of electromagnetic confinement applications include the following:

1. The package should include provision for higher order elements. These are particularly important when determining electromagnetic Lorentz forces since the basic vector potential solution can then be differentiated without significant loss of accuracy. By contrast, linear elements were used in the CREM geometry of Fig. 7. To obtain a reliable estimate of the electromagnetic forces, a much finer grid than would be estimated simply from penetration depth considerations had to be used near the meniscus. Even then, anomalous forces were obtained at certain locations.

2. The package should have a flexible Post-Pro-
cessor stage, with the graphic display of the computational results (Fig. 8) being convenient and intuitive. More important, however, the user should be able to customize the Post-Processor. For example, in the case of electromagnetic confinement, this would include the ability to develop used-defined commands to calculate the Lorentz force distribution at specific regions within the melt and to determine the electromagnetic pressure acting on the melt surface.

Relative to closed form and Volume Integral solution methods, a good Finite Element CAD package offers the following significant advantages when used for the analysis of 2-D electromagnetic confinement applications:

1. The geometry can be modelled rapidly and with good accuracy. A relatively smooth estimate can be obtained for curved surfaces; e.g., the meniscus region.
2. Depending on the package, modification of the problem geometry can be relatively simple.
3. Provided high order elements are available, the Lorentz force and pressure differentials can be obtained with good accuracy.
4. Large problems (e.g., up to 3,000 unknowns) can be solved on personal computer class machines. This is not the case with an integral equation formulation where major modelling compromises are often made order to minimize the size of (fully populated) system matrix.

The principal limitation of the available Finite Element CAD packages, from the viewpoint of electromagnetic confinement, is that the shape of the meniscus must be assumed. Further, modification of the meniscus profile, based on an intermediate calculation of the electromagnetic pressure distribution, is awkward. Therefore, an iteration to determine the meniscus profile is quite difficult to perform.

6. Boundary Integral and Boundary Element Methods

In recent years, considerable interest has been shown in the Boundary Element Method of formulating eddy current problems.17-21,26-30,38-41 By formulating such problems in terms of the surfaces bounding the conducting regions, the problem dimensionality can be reduced by one. Using Fig. 1 as an example, the solution grid need only match only the surfaces bounding the metal column and the shielding ring. It is not necessary to directly grid the conductor interior, or the exterior free space region. These regions are directly accounted for in the formulation. In addition, the induction coil can be treated in terms of an incident magnetic field of known strength \( \mathbf{H} \). This is the field that the coil would produce in the absence of any conducting regions. The ability to separate the incident and reaction fields and to deal only with conductor boundaries represent significant advantages when modelling electromagnetic confinement problems. By contrast, Finite Difference, Volume Integral and Finite Element solutions of 2-D electromagnetic field problems require that a full 2-D solution grid be used. In the FD and FE cases, the grid must include the free space region.

Full details on the formulation and implementation of Boundary Integral and Boundary Element methods for a broad range of eddy current problems can be found elsewhere.18-20,26-30,38-41 The purpose of the present section is simply to highlight selected aspects of the formulation.

To simplify matters, consider a problem having a single conducting region denoted by \( \Omega \) and a free space region denoted by \( \Omega^- \). Within \( \Omega \) and \( \Omega^- \), the governing Maxwell equations (1) and (2) can be combined to obtain vector Helmholtz equations for \( \mathbf{H} \) and \( \mathbf{E} \). These equations are of the form:

\[
\nabla \times \mathbf{H} - \beta^2 \mathbf{H} = 0 \quad \text{(34a)}
\]

\[
\nabla \times \mathbf{E} - \beta^2 \mathbf{E} = 0 \quad \text{(34b)}
\]

\[
\beta^2 = \nabla \times \nabla \times \mathbf{H} \quad \text{(34c)}
\]

where \( \nabla \times \mathbf{H} \) and \( \nabla \times \mathbf{E} \) are defined by Eqs. (1-b) and (2-b), respectively. Note that the magnitude of \( \beta^2 \) is different in each of the regions \( \Omega \) and \( \Omega^- \), thus defining separate Helmholtz equations for each region.

An integral equation representation of either \( \mathbf{H}(\xi) \) or \( \mathbf{E}(\xi) \), where \( \xi \) is a point of observation within either \( \Omega \) or \( \Omega^- \), can be obtained by applying the vector Green’s theorem:

\[
\int_{\Gamma} \left( \mathbf{G} \times \nabla \times \mathbf{G} - \mathbf{G} \cdot \nabla \times \mathbf{G} \right) d\mathbf{v} = \int_{\Omega^-} \left[ \mathbf{G} \times \nabla \times \mathbf{G} - \mathbf{G} \cdot \nabla \times \mathbf{G} \right] \cdot \mathbf{n} ds \quad \text{(35)}
\]

In Eq. (35), \( \gamma \) is the point of integration, the subscript \( \gamma \) denotes an operation taken with respect to the point of integration and \( \mathbf{G} \) represents the vector field of interest. It is convenient to define the function \( \mathbf{G} \) as follows:

**Fig. 8.** Magnetic field lines obtained from the Finite Element analysis of the CREM casting process.
where $\hat{e}$ is an arbitrary unit vector, $g(\tilde{\xi}, \tilde{\eta})$ is the fundamental solution of the vector Helmholtz equation and $\beta$ has a value defined by the material constants for regions $\Omega$ or $\Omega_-$, as appropriate. The detailed application of Eq. (35) to derive Integral Equations that are valid within $\Omega$ and $\Omega_-$ can be found in the literature.\textsuperscript{17,18,26,58-60} Ahmed,\textsuperscript{39} for example, has considered the 2-D axisymmetric case. Note that at this stage, $\hat{E}$ and $\hat{B}$ are defined for a points of observation $\tilde{\xi}$ that are within $\Omega$ and $\Omega_-$, but not on the boundary between these two regions.

By letting the points of observation $\tilde{\xi}$ within $\Omega$ and $\Omega_-$ move to the surface interface between the two regions and by taking proper account of the limiting behaviour of single and double layer potentials,\textsuperscript{17,18,26,58-60} Boundary Integral Equations for the tangential components of $\hat{E}$ and $\hat{B}$ on the surface interface can be derived. In the case of a 2-D axisymmetric geometry (e.g., crucible induction furnace, c.m. mould for cylindrical billets), the resulting Boundary Integral Equations are of the form\textsuperscript{59,61}:

$$1/2E_{i}(\tilde{\xi}) = E_{i}(\tilde{\xi}) + 1/2H_{1}(\tilde{\xi}) + 1/2\int_{\Omega_{+}} L_{\alpha}^{\beta} H_{\alpha}(\eta)$$

$$1/2H_{1}(\tilde{\xi}) = H_{1}(\tilde{\xi}) + 1/2E_{1}^{\alpha}(\tilde{\xi}) + 1/2F_{\beta}^{\alpha} H_{\alpha}(\eta)$$

A similar pair of equations can be derived by starting with a point of observation $\tilde{\xi}$ within the conducting region $\Omega$. In that above equations, the superscript $i$ denotes an incident field, $\tilde{\xi}$ is a position vector for the point of integration, $Y_{\alpha} = -j\omega \sigma$ and $Z_{\alpha} = j\omega \sigma \mu_{0}$. Similarly, the subscript $\hat{\phi}$ represents the azimuthal direction on the conductor surface while the subscript $t$ represents a surface tangential direction that is orthogonal to $\phi$.

In the above equations, the integral operators $L$\textsuperscript{(a)} and $L$\textsuperscript{(b)} have the following form:\textsuperscript{59,61}

$$L(\alpha) \mathcal{U}(\tilde{\xi}) = \int_{S} [(\hat{\alpha} \times \mathcal{U}(\tilde{\eta})) \times \mathcal{G}(\tilde{\xi}, \tilde{\eta})]dS_{\tilde{\eta}}$$

$$L(\alpha) \mathcal{U}(\tilde{\xi}) = \int_{S} [(\hat{\alpha} \times \mathcal{U}(\tilde{\eta})) \times \mathcal{G}(\tilde{\xi}, \tilde{\eta})]dS_{\tilde{\eta}}$$

where $\mathcal{G}$ is given by Eq. (36), with $\beta$ being defined by the free space material properties and the unit vector $\hat{e}$ being in either the $\phi$ or the $t$ direction, as appropriate.

In the case of most electromagnetic confinement problems, the electromagnetic penetration depth $\delta$ is small relative to the dimensions of the molten metal region. This constraint is necessary in order to obtain a relatively stiff confinement field. Fortunately, when $\delta$ is small, modelling this electromagnetic field problem is simplified considerably when a Boundary Integral formulation is used. In particular, at shallow penetration depth values, the tangential components of the electric and magnetic field can be related to one another, on the surface of the conductor, by using the Leontovich Impedance Boundary Condition (IBC)\textsuperscript{66}:

$$\hat{a} \times \hat{E} = Z_{S}[\hat{a} \times \hat{a} \times \hat{H}]$$

where $Z_{S}$ is the surface impedance defined as follows:

$$Z_{S} = 1/2(1-j)\omega \sigma \mu_{0}$$

Using Eqs. (39) and (40) to eliminate $H_{1}$ from Eq. (37-a), one obtains:

$$1/2E_{\alpha}(\tilde{\xi}) = E_{\alpha}(\tilde{\xi}) + [L_{\alpha}^{\beta} - 1/2Y_{\alpha}^{\beta} Z_{S}] E_{\alpha}(\tilde{\xi})$$

Similar formulations can be derived for problems involving multiple conductors,\textsuperscript{59,61,65} for cases where $\delta$ is not small relative to conductor dimensions,\textsuperscript{18-20,58-61} for geometries that have translational symmetry\textsuperscript{17,59,60,61,66} and for 3-D geometries.\textsuperscript{59-60}

In the IBC limit, rather than seeking a Boundary Integral formulation directly in terms of a vector field quantity, leading to Eq. (41) for example, an attractive alternative would be to use the scalar magnetic potential. This certainly would pose no difficulty for simply connected conductors and, indeed, has recently been used by Li and Evans\textsuperscript{65} to analyse a 3-D slab casting application. However, when the problem contains multiply connected conductors, such as the field shaping ring that is used in some electromagnetic casting geometries, the magnetic scalar potential is no longer uniquely defined. In such cases, the use of fictitious currents, as suggested by Mayergoyz,\textsuperscript{61} may resolve the difficulties.

A Boundary Integral formulation, and particularly the IBC formulation given by Eq. (41), has several significant advantages when applied to electromagnetic confinement problems:

1. Being a boundary formulation, a solution grid is only required on the conductor surface(s). Therefore, it is very simple to modify the solution grid as the iteration for the free surface shape proceeds. This is not the case when methods such as Finite Differences, Finite Elements, or Volume Integrals are used. This advantage it true for any boundary formulation, IBC, or otherwise.

2. The surface electric and magnetic fields are obtained with equivalent accuracy. With other methods (FD, FE, VI), one field is obtained as the derivative of the other, thus compromising the accuracy of the derived forces and pressures.

3. When the fields and/or forces are required within the interior of the molten metal, these must be separately computed at each desired point when a Boundary Integral/Element solution is used. However, this additional computational cost is offset by the fact that the resulting values are of higher accuracy than would be obtained by using FD, FE, or
VI solutions on a comparable grid.

(4) When the IBC can be used, the computational cost is reduced significantly since:

i) The number of integral equations is reduced by one half as is the number of field unknowns.

ii) The IBC integral equation given by Eq. (41) only involves the parameters of the free space region and thus is not computationally intensive to solve. Equations similar to Eqs. (37) and (41) have been used as the computational core in an iterative process to determine the molten metal free surface profile in several 2-D electromagnetic confinement problems. The iteration simply involves the repetitive solution of the electromagnetic problem, with the conductor geometry being modified at each step based on the computed electromagnetic pressures. The iteration was deemed to converge when the electromagnetic and gravitational pressures balanced. Contributions due to fluid flow and surface tension were neglected. As a boundary method was being used, modification of the conductor geometry was quite simple.

The simplest problem to solve involved a single conductor. The particular cases considered were a laboratory scale crucible induction furnace and the CREM caster with a nonconducting mould. Fig. 9 shows the predicted meniscus profiles for a small induction furnace as a function of frequency while a comparison with published data is given in Fig. 10 for the CREM process. In both cases, the agreement between the predicted and measured profiles is excellent.

When the meniscus was predicted for a high powered induction furnace using this iterative procedure, it was found that the agreement with published profiles was relatively poor. This was due primarily to the fact that shielding yokes were not included in the electromagnetic model. The Boundary Element formulation was therefore extended to include multiple bodies, some of which could be laminated steel. With the laminations included, the agreement between measured and calculated values of meniscus height became quite acceptable.

In order to treat an electromagnetic caster, including the effect of the field shaping ring, the Boundary Element formulation was extended to include multiple conducting bodies. Furthermore, the formulation was mixed in that each conducting body could be treated in the full Boundary Integral Equation sense, or if the penetration depth was shallow, in the IBC sense. This is an important consideration in electromagnetic casting since the IBC does not apply to the field shaping ring.

The mixed formulation was used to predict the meniscus shape in an electromagnetic caster as a function of coil radius, shield placement and supply frequency. Typical results are shown in Fig. 11 for a

![Fig. 10. Measured and predicted meniscus profiles for the CREM aluminum casting process.](image_url)

![Fig. 11. Meniscus profile as a function of coil radius in an electromagnetic caster operating at 10 kHz.](image_url)
10 cm diameter aluminum column and a 10 kHz supply frequency. The lower tip of the field shaping shield coincided with the quiescent surface of the melt. The corresponding electromagnetic pressure distributions are shown in Fig. 12. These latter results clearly illustrate the linear variation of pressure that is required in equilibrium.

7. Conclusions

Numerical methods that can be used for the analysis of electromagnetic confinement processes have been reviewed in this paper. Closed form solutions, based on the Transfer Matrix and the Green's function, provide a convenient means of determining upper bound estimates of confinement forces in many applications. However, numerical approximate methods must be used for those cases where conductor end effects are important. While Volume Integral methods are very convenient to use, the resulting staircase approximation to the molten metal free surface can introduce significant errors in the critical meniscus region. Finite Element methods eliminate this problem but still require that the molten metal free surface contour be assumed. Of the available approximate methods, it is felt that the Boundary Integral or Boundary Element methods are the most suitable for electromagnetic confinement applications. These methods have been used with considerable success to predict free surface contours in several confinement applications.

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Fig. 12. Axial variation of electromagnetic pressure for the meniscus profiles shown in Fig. 11.
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