A real CKM matrix?

P. Checchia\textsuperscript{1}, E. Piotto\textsuperscript{2} and F. Simonetto\textsuperscript{1}

\textsuperscript{1} I.N.F.N sezione di Padova and Dipartimento di Fisica
Università di Padova,
Padua, Italy

\textsuperscript{2} I.N.F.N sezione di Milano and Dipartimento di Fisica
Università di Milano,
Milan, Italy

Abstract

The hypothesis of a real Cabibbo-Kobayashi-Maskawa matrix has been considered and found to be disfavoured by present measurements even when neglecting results from CP violation in neutral Kaon decay. This result contradicts statements reported in [1].
1 Introduction

The Cabibbo-Kobayashi-Maskawa \cite{4, 5} $3 \times 3$ unitary matrix

\[
V_{CKM} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\]  

(1)

can be parameterized in terms of four parameters $\lambda$, $A$, $\rho$ and $\eta$ \cite{4}:

\[
V_{CKM} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \frac{\lambda}{2} & A\lambda^3(\rho - i\eta) \\
-\frac{\lambda}{2} & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4). 
\]  

(2)

In this parameterization (or in its extension to $\mathcal{O}(\lambda^6)$ \cite{4}) the complex phase of the matrix is given by the parameter $\eta$ related to the CP violation of the weak interactions. The $\lambda$ and $A$ parameters are known with a good accuracy ($\sim 1\%$ and $\sim 4\%$, respectively) while many contributions to extract $\rho$ and $\eta$ from the available measurements exist in the literature \cite{1, 2, 3}. To this purpose, the measurements of the CP violation parameter in neutral Kaon decay $|\epsilon_k|$, of the difference between the mass eigenstates in the $B^0_d - \bar{B}^0_d$ system $\Delta m_d$ and of the ratio $|\frac{V_{ub}}{V_{cb}}|$ and the lower limit on the difference between the mass eigenstates in the $B^0_s - \bar{B}^0_s$ system $\Delta m_s$ can be used. On the other hand, since the only direct experimental evidence for CP violation is given by the fact that $|\epsilon_k| \neq 0$ and the effect could be explained in terms of models proposed in alternative to the Standard Model (see, for instance, \cite{7, 8} and references therein), it is suggested to remove the constraint coming from the neutral Kaon system and to investigate the results on parameter $\eta$ or to test the hypothesis of a real $V_{CKM}$ matrix. In this letter the latter hypothesis is tested with a different statistical approach with respect to \cite{1} coming to a different conclusion. In addition, the inclusion of different data sets is discussed and the corresponding results are presented.

2 Measurements and constraints on $V_{CKM}$ parameters

The $\lambda$ parameter is the sine of the Cabibbo angle \cite{4}:

\[
\lambda = |V_{us}| = \sin \theta_c = 0.2196 \pm 0.0023. 
\]  

(3)

The $A$ parameter depends on the matrix element $|V_{cb}| = 0.0395 \pm 0.0017$ (obtained from semileptonic decays of B hadrons \cite{4}) and on $\lambda$:

\[
A = \frac{|V_{cb}|}{\lambda^2} = 0.819 \pm 0.035. 
\]  

(4)

In order to constrain the parameters $\rho$ and $\eta$ without considering CP violation in the neutral Kaon system, three experimental input are used:
2.1 \( \mathcal{B}_d^0 \) oscillations.

The mass difference \( \Delta m_d \) between the mass eigenstates in the \( \mathcal{B}_d^0 - \overline{\mathcal{B}_d^0} \) system has been measured with high precision [9, 10]. In the Standard Model it can be related to the CKM parameters in the following way:

\[
[ (1 - \rho)^2 + \eta^2 ] = \frac{\Delta m_d}{G_F^2 m_t^2 m_{\mathcal{B}_d^0}^2 \left( f_{\mathcal{B}_d^0} \sqrt{\mathcal{B}_{\mathcal{B}_d^0}} \right)^2} \eta_B F(z) A^2 \lambda^6 \tag{5}
\]

where \( m_t \) is the top pole mass scaled according to [13] and \( z = m_t^2/m_W^2 \). The function \( F(z) \) is given by:

\[
F(z) = \frac{1}{4} + \frac{9}{4(1 - z)} - \frac{3}{2(1 - z)^2} - \frac{3z^2 \ln z}{2(1 - z)^3}. \tag{6}
\]

The values of all the parameters are given in Table 1.

In the \( \rho - \eta \) plane, the measurement of \( \Delta m_d \) corresponds to a circumference centered in (1,0). By constraining \( \eta \) to zero, an evaluation of \( \rho \) can be obtained. Unfortunately the term \( f_{\mathcal{B}_d^0} \sqrt{\mathcal{B}_{\mathcal{B}_d^0}} \), given by lattice QCD calculations, is known with a 20% order uncertainty [11, 12] and therefore it gives the largest contribution to error on the \( \rho \) determination.

| parameter | value used in [1] | published data only | new input |
|-----------|-------------------|---------------------|-----------|
| \( G_F \) | \( 1.16639(1) \times 10^{-5} \text{GeV}^{-2} \) | 80.41 ± 0.10 GeV | |
| \( \lambda \) | 0.2196 ± 0.0023 | | |
| \( A \) | 0.819 ± 0.035 | | |
| \( m_t \) | 166.8 ± 5.3 GeV | | |
| \( m_W \) | 80.375 ± 0.064 GeV | | |
| \( m_{\mathcal{B}_d^0} \) | 5.2792 ± 0.0018 GeV | | |
| \( m_{\mathcal{B}_s^0} \) | 5.3692 ± 0.0020 GeV | | |
| \( \eta_B \) | 0.55 ± 0.01 | | |
| \( \Delta m_d \) | 0.471 ± 0.0016 ps\(^{-1} \) | 0.464 ± 0.0018 ps\(^{-1} \) | |
| \( \Delta m_s \) | >12.4 ps\(^{-1} \) (95% C.L.) | >9.1 ps\(^{-1} \) (95% C.L.) | |
| \( |V_{ub}|/|V_{cb}| \) | 0.093 ± 0.016 | 0.100 ± 0.013 | 0.215±0.040−0.030 GeV |
| \( f_{\mathcal{B}_d^0} \sqrt{\mathcal{B}_{\mathcal{B}_d^0}} \) | 0.201 ± 0.042 GeV | 1.14±0.07−0.06 |
| \( \xi \) | 1.14 ± 0.08 | | |

Table 1: Physical parameters used in the formulae (5),(6),(7) and (8). In the second column the values used in [1] and in the third column the values obtained from published data (or combinations quoted in [9]) are given. In the fourth column new values for \( f_{\mathcal{B}_d^0} \sqrt{\mathcal{B}_{\mathcal{B}_d^0}} \) and \( \xi \) [12] are given and the preliminary DELPHI \( V_{ub} \) measurement is included.

2.2 \( \mathcal{B}_s^0 \) oscillations.

The mass difference \( \Delta m_s \) between the mass eigenstates in the \( \mathcal{B}_s^0 - \overline{\mathcal{B}_s^0} \) system is expected to be much larger than \( \Delta m_d \) and in the Standard Model is related to the CKM parameters:

\[
[ (1 - \rho)^2 + \eta^2 ] = \frac{\Delta m_d}{\Delta m_s \lambda^2 m_{\mathcal{B}_d^0}^2} \xi^2 \tag{7}
\]
Since the ratio

\[ \xi = \frac{f_{B_s} \sqrt{B_{B_s}}}{f_{B_d} \sqrt{B_{B_d}}} \]  

(8)

is computed by lattice QCD with a better precision than the single terms, a measurement of \( \frac{\Delta m_s}{\Delta m_d} \) could provide a much stronger constraint on the \( \rho - \eta \) plane. However, given the very high frequency in the \( B_s^0 - \bar{B}_s^0 \) system oscillation, only a lower limit on \( \Delta m_s \) is available, as shown in Table 1, which corresponds to a circular bound in the two parameter space or, if the assumption \( \eta = 0 \) is made, to a lower limit for \( \rho \).

2.3 \( |V_{ub}| \) measurements from semileptonic b decay.

Charmless semileptonic b decays have been used to measure \( |V_{ub}| \) or the ratio \( |V_{ub}| / |V_{cb}| \). The CLEO collaboration determined that parameter both by measuring the rate of leptons produced in B semileptonic decays beyond the charm end-point [14] and from direct reconstruction of charmless B semileptonic decay [15]. The two results are consistent but both methods are limited by theoretical uncertainties. In [15] they are not combined and a value \( |V_{ub}| / |V_{cb}| = 0.08 \pm 0.02 \) obtained from the former result is given. At LEP, ALEPH [16], L3 [17] and more recently DELPHI [18] have measured the inclusive charmless semileptonic transitions \( b \to ul\nu \). The average value of the LEP measurements with the previous value is given in Table 1. The ratio of the two CKM matrix elements is related to the \( \rho \) and \( \eta \) parameters by:

\[ \frac{|V_{ub}|}{|V_{cb}|} = \lambda \sqrt{\rho^2 + \eta^2} \]  

(9)

and hence, in the \( \rho - \eta \) plane, the measurement of \( |V_{ub}| / |V_{cb}| \) corresponds to a circumference centered at the origin. If \( \eta \) is assumed to be zero, it would be proportional to the \( \rho \) absolute value.

3 Data compatibility with a real CKM matrix hypothesis.

The assumption of a real CKM matrix implies that all the constraints described in the previous section are reduced to values of (or limits on) \( \rho \). The compatibility of the obtained values can then be used to estimate the goodness of the assumption itself. In [19] and in the explicit reference to it in [8], it is written that the hypothesis of a real CKM matrix can fit the data. However it is unclear which was the statistical approach followed to come to that statement [19] and a completely different conclusion can be obtained with a standard method. In addition the claim of [19], although using a different input-data-set, contradicts what is reported in [7]. Assuming \( \eta = 0 \) and using exactly the same input parameters as [19] (Table 1 second column), the values

\[ \chi^2 = 6.7 \]

corresponds to a reasonable Confidence Level for a real CKM matrix hypothesis.

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1In the cited paper, a result obtained by an underconstrained fit to \( \rho \) and \( \eta \) excludes the hypothesis \( \eta = 0 \) at the 99% C.L. and another fit with \( \eta = 0 \) gives \( \chi^2 = 6.7 \). Since in the fits there are bounded parameters clearly correlated with \( \rho \) (i.e. \( f_{B_d} \sqrt{B_{B_d}} \)) it is not specified why the mentioned \( \chi^2 \) value corresponds to a reasonable Confidence Level for a real CKM matrix hypothesis.
\[ \rho^{\Delta m_d} = 0.01^{+0.18}_{-0.26} \]  
(10)

and

\[ \rho^{V_{ub}} = \pm(0.42 \pm 0.07) \]  
(11)

are obtained from eq. (5) and (9), respectively.

The limit on \( \Delta m_s \) has been obtained by means of the amplitude method \[19\] which allows to know the exclusion Confidence Level for any value of \( \Delta m_s \). Therefore, by a convolution with the dominant uncertainty from the ratio \( \xi \) in eq. (7) it is possible to obtain:

\[ \rho^{\Delta m_s} > -0.05 \]  
(12)

at the 95% Confidence Level.

In a naive approach on which the errors on \( \rho^{\Delta m_d} \) and \( \rho^{V_{ub}} \) are assumed to be uncorrelated it is evident that the two values are fairly incompatible. The negative \( \rho^{V_{ub}} \) solution is clearly excluded by the \( \rho^{\Delta m_s} \) limit and therefore it can be discarded. In order to include all the correlations due to common terms contributing to the errors, namely \( |V_{cb}| \) and \( \lambda \), a two dimensional error matrix \( M \) has been written and the Best Linear Unbiased Estimator \[20\] has been used:

\[ \rho^{\text{BLUE}} = \rho_i \sum_{j=1}^{2} \frac{\rho_i (M^{-1})_{ij} \rho_j}{\rho_i (M^{-1})_{jj}} \]  
(13)

with the variance

\[ \sigma^2_{\rho} = \frac{1}{\rho_i (M^{-1})_{jj}}. \]  
(14)

The error matrix \( M \) includes correlated and uncorrelated contributions:

\[ M_{ij} = \delta_{ij} \sigma^\text{uncorr} \sigma^\text{uncorr} + \sum_{\alpha=1}^{m} \Delta_{\alpha i} \Delta_{\alpha j} \]  
(15)

where the indexes \( i \) and \( j \) run over the two \( \rho \) measurements and \( \Delta_{\alpha i} \) is the change (with sign) on measurement \( i \) when the common systematic parameter \( \alpha \) is moved by its error. The \( \chi^2 \) is obtained by:

\[ \chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \left( \left[ \rho_i - \rho^{\text{BLUE}} \right] (M^{-1})_{ij} \left[ \rho_j - \rho^{\text{BLUE}} \right] \right). \]  
(16)

With the table\[3\] (second column) parameters and taking the positive error in eq. (10),

\[ \rho^{\text{BLUE}} = (3.58 \pm 0.66) \times 10^{-1} \text{ and } \chi^2 = 4.3 \text{ (1 Degree of freedom)} \]  
(17)

are obtained. The corresponding \( \chi^2 \) probability is 3.9% and this is clearly in contradiction with \[1\]. The correlation between the two measurements is small given the dominance of the uncertainty on \( f_{B_d} \sqrt{B_{B_d}} \) in \( \rho^{\Delta m_d} \). The effect of the limit on \( \Delta m_s \) is negligible even though the information contained in the amplitude is taken into account as suggested in \[3\]: since the value of \( \Delta m_s \) that one would obtain by inserting \( \rho = \rho^{\text{BLUE}} \) and \( \eta = 0 \) in eq. (7) is \( \Delta m_s = 31 \text{ ps}^{-1} \), the present experimental sensitivity (13.8 ps\(^{-1}\)) does not give any sizeable information on that \( \rho \) region.

Some of the parameters used to determine \( \rho \) and \( \chi^2 \) in eq. (16) have been obtained from preliminary results. It is then important to quantify the compatibility of the real
CKM hypothesis with published data. For such a test, the previous evaluation is repeated
with the parameters listed in the third column of table and

$$\rho^{\text{BLUE}} = (3.53 \pm 0.65) \times 10^{-1} \text{ and } \chi^2 = 4.0 \text{ (1 Degree of freedom)} \quad (18)$$

are obtained with a corresponding $\chi^2$ probability of 4.6%. Here, in order to take into
account terms of the order up to $O(\lambda^5)$ [21], the substitution $\rho \rightarrow \bar{\rho} = \rho(1 - \lambda^2/2)$ has
been done in eq. (5) with $\eta = 0$. This result, given the little change on parameters,
is similar to the previous one and therefore the hypothesis of a real CKM matrix is
disfavoured also by published data.

If a more recent value of $f_{B_d} \sqrt{B_{B_d}}$ is taken and the DELPHI $V_{ub}$ preliminary result is
included (see table), the incompatibility of the two measurements is still present:

$$\rho^{\text{BLUE}} = (3.91 \pm 0.53) \times 10^{-1} \text{ and } \chi^2 = 4.6 \text{ (1 Degree of freedom)} \quad (19)$$

corresponding to a $\chi^2$ probability of 3.1%.

Since the dominant error in $\rho^{\Delta m_d}$ is due to the lattice QCD computation, the hypoth-
thesis of a flat error with the same R.M.S. on $f_{B_d} \sqrt{B_{B_d}}$ has been studied with a simple
simulation. Several experimental results for $\rho^{\Delta m_d}$ and $\rho^{V_{ub}}$ have been generated with cen-
tral values equal to $\rho^{\text{BLUE}}$. For $\rho^{\Delta m_d}$ this is achieved by shifting the value of $f_{B_d} \sqrt{B_{B_d}}$
and allowing it to vary within a flat distribution with the R.M.S. corresponding to the
quoted error. All the other parameters of eq. (5) are allowed to vary with a gaussian
distribution corresponding to their error. In each experiment the combined value and
the $\chi^2$ are computed according to eq. (13) and (16), respectively. Looking at the $\chi^2$
distribution, it is possible to determine the fraction of simulated experiments with a $\chi^2$
higher than the value found in the evaluation with real data. This procedure has been
repeated for the three data sets of table and for none of them the $\chi^2$ has been found to be
higher than the experimental values of eq. (17), (18) and (19) in more than 5% of the
cases.

4 Conclusions

The hypothesis of a real CKM matrix is tested on the basis of the present published and
preliminary data and lattice QCD calculations. With all the input data used, included
those suggested in [1], that hypothesis is excluded at more than 95% Confidence Level.
This result contradicts statements in [1] and agrees with the result indicated in [1] with
an older input data set.

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References

[1] S. Mele, CERN-EP/98-133 submitted to Phys. Lett. B.
[2] N. Cabibbo, *Phys. Rev. Lett.* **10** (1963) 531.

[3] M. Kobayashi and T. Maskawa, *Prog. Theor. Phys.* **49** (1973) 652.

[4] L. Wolfenstein, *Phys. Rev. Lett.* **51** (1983) 1945.

[5] P. Paganini et al., *Phys. Scripta* **58** (1998) 556.

[6] F. Parodi et al. hep-ph/9802289.

[7] R. Barbieri et al., *Phys. Lett.* **B 425** (1998) 119.

[8] H. Georgi and S. Glashow, HUTP-98/A048.

[9] Particle Data Group, *Eur. Phys. J.* **C 3** (1998) 1.

[10] The LEP B Oscillation working group, LEPBOSC 98/3 contribution to the Vancouver 1998 conference.

[11] J.M. Flynn and C.T. Sachraida, hep-lat/9710057. Proceedings of: 4th International Workshop on Progress in Heavy Quark Physics Rostock, Germany.

[12] T. Draper, hep-lat/9810065. 14th International Symposium on Lattice Field Theory: *Lattice ’98* Boulder, CO, UK.

[13] A. J. Buras et al., *Nucl. Phys.* **B 347** (1990) 491.

[14] CLEO collab., J.Bartelt et al., *Phys. Rev. Lett.* **71** (1993) 4111.

[15] CLEO collab, J.P. Alexander et al., *Phys. Rev. Lett.* **77** (1996) 5000.

[16] ALEPH collab., R. Barate et al., CERN-EP/98-067 accepted by *Euro. Phys. J.C.*

[17] L3 collab., M. Acciarri et al., *Phys. Lett.* **B 436** (1998) 174.

[18] DELPHI collab., M. Battaglia et al. DELPHI 98-97 CONF 165.

[19] H.G. Moser and A. Roussarie, *Nucl. Instr. and Methods* **A 384** (1997) 491.

[20] See for example L.Lyons at al. *Nucl. Instr. and Methods* **A 270** (1988) 110 and references therein; details of the method are also described in COMBOS manual available at [http://www.cern.ch/LEPBOSC/combos](http://www.cern.ch/LEPBOSC/combos).

[21] A. J. Buras et al., *Phys. Rev.* **D 50** (1994) 3433.