Blessings and curse of smoothness and phase transitions in nonparametric regressions: a nonasymptotic perspective*

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Abstract

When the regression function belongs to the standard smooth classes consisting of univariate functions with derivatives up to the \((\gamma + 1)\)th order bounded in absolute values by a common constant everywhere or a.e., it is well known that the minimax optimal rate of convergence in mean squared error (MSE) is \(\left(\frac{\sigma^2}{n}\right)^{2\gamma+3}\) when \(\gamma\) is finite and the sample size \(n \to \infty\). From a nonasymptotic viewpoint that does not take \(n\) to infinity, this paper shows that: for the standard Hölder and Sobolev classes, the minimax optimal rate is \(\frac{\sigma^2(\gamma+1)}{n} \left(\frac{\sigma^2}{n}\right)^{2\gamma+3}\) when \(\frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2\gamma+3}\) and \(\left(\frac{\sigma^2}{n}\right)^{2\gamma+3} \left(\frac{\sigma^2(\gamma+1)}{n}\right)\) when \(\frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2\gamma+3}\). To establish these results, we derive upper and lower bounds on the covering and packing numbers for the generalized Hölder class where the absolute value of the \(k\)th \((k = 0, ..., \gamma)\) derivative is bounded by a parameter \(R_k\) and the \(\gamma\)th derivative is \(R_{\gamma+1}\)-Lipschitz (and also for the generalized ellipsoid class of smooth functions). Our bounds sharpen the classical metric entropy results for the standard classes, and give the general dependence on \(\gamma\) and \(R_k\). By deriving the minimax optimal MSE rates under \(R_k = 1, R_k \leq (k - 1)!\) and \(R_k = k!\) (with the latter two cases motivated in our introduction below) for the smooth classes with the help of our new entropy bounds, we show several interesting results that cannot be shown with the existing entropy bounds in the literature. We further consider the Hölder class of \(d\)-variate functions. Our result suggests that the classical asymptotic rate \(\left(\frac{\sigma^2}{n}\right)^{2\gamma+2+d}\) could be an underestimate of the MSE in finite samples.

1 Introduction

Estimation of an unknown smooth function \(f\) from the nonparametric regression model

\[ y_i = f(x_i) + \epsilon_i, \quad i = 1, ..., n \] (1)

has been a central object of study in statistics, numerical analysis and machine learning. The typical assumption about \(f\) is that it belongs to the standard Hölder class (or the standard Sobolev class) consisting of univariate functions with derivatives up to the \((\gamma + 1)\)th order bounded in absolute values by a common constant everywhere or a.e.. When \(\gamma\) is finite and the sample size \(n \to \infty\), it is well understood that the minimax optimal rate of convergence in mean squared error (MSE) is \(\left(\frac{\sigma^2}{n}\right)^{2\gamma+2}\), where \(n\) is the sample size and \(\sigma^2\) is the variance of the noise term \(\epsilon_i\).

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This classical result gives rise to the so called “blessing of smoothness” (since \( \frac{x^{2+\gamma^2}}{2^{\gamma+1} \gamma^2} \) decreases as \( \gamma \) increases) and empirical researchers are often advised to exploit higher degree smoothness assumptions if they are facing a small sample size. This suggestion is particularly common in economic applications where researchers need to perform subsample analyses and in these applications (e.g., studies on intergenerational mobility\(^1\)), \( n \) often ranges from tens to a couple of hundreds.

If \( n \) is small enough, exploiting higher degree smoothness assumptions could bring a curse. To see this, recall that any function \( f \) in the Hölder class, \( U_{r+1}[−1, 1] \), can be written as

\[
f(x) = f(0) + \sum_{k=1}^{\gamma} \frac{x^k}{k!} f^{(k)}(0) + x^\gamma f^{(\gamma)}(z) - \frac{x^\gamma}{\gamma!} f^{(\gamma)}(0)
\]  

where \( z \) is some intermediate value between \( x \) and 0. Therefore, we can decompose \( U_{r+1} \) into a polynomial subspace \( U_{r+1,1} \) associated with \( f(0) + \sum_{k=1}^{\gamma} \frac{x^k}{k!} f^{(k)}(0) \) and a Hölder subspace \( U_{r+1,2} \) with \( f^{(k)}(0) = 0 \) for all \( k = 0, ..., \gamma \). A similar decomposition also applies to the Sobolev class \( \mathcal{S}_{\gamma+1} \); in particular, \( \mathcal{S}_{\gamma+1} = U_{r+1,1} + S_{\gamma+1} := \{ f_1 + f_2 : f_1 \in U_{r+1,1}, f_2 \in S_{\gamma+1} \} \), where the Sobolev subspace \( S_{\gamma+1} \) is imposed with the restrictions that \( f^{(k)}(0) = 0 \) for all \( k \leq \gamma \) and \( f^{(\gamma+1)} \) belongs to the space \( z^{2-\gamma} \). \(^2\) Higher smoothness increases the difficulty of estimating the polynomial component while decreases the difficulty of estimating the component in \( U_{r+1,2} \) and \( S_{\gamma+1} \) (as long as the magnitude of the derivatives is not too large). Understanding this trade-off in finite sample settings can lead to useful guidance for empirical researchers.

In particular, when applying regression discontinuity designs (RDD) to perform causal inference, two conditional mean functions of a pretreatment variable are estimated from \( \mathbb{P} \). As discussed in \(^3\), researchers frequently use up to sixth order polynomials for three reasons: (i) any smooth function on a compact set can be approximated by high order polynomials arbitrarily well; (ii) the fit from high order polynomials is expected to be smooth when the relationship between the pretreatment variable and outcome is strong; (iii) many textbooks recommend including relevant predictors in causal inference to reduce bias and when the sample size is large, it is expected that the reduction in bias by including high order polynomials is larger than the increase in variance. Through various empirical studies, Gelman and Imbens \(^3\) discover that using global high order polynomials in RDD analysis results in noisy estimates and poor coverage of confidence intervals. The authors conjecture that high order polynomials of the pretreatment variable can incur both bias and variance, hence resulting in a poor coverage. Motivated by this conjecture and the fact that the sum of squared bias and variance is the MSE, this paper focuses on the optimal MSE convergence rates associated with \( \mathbb{P} \) in finite sample settings and answers the question of how fast the sample size needs to grow (as a function of \( \gamma \)) for higher order polynomials to become clearly more beneficial than lower order polynomials. The optimality question in this paper is addressed through an algorithm–independent sense; that is, no methods can yield a MSE rate that is smaller than our optimal rate.

Curse of smoothness can also arise in another form through the derivatives. In noisy recovery of solutions to ordinary differential equations (ODE), researchers often use polynomials and spline bases to approximate the solutions to overcome computational challenges (e.g., \(^6\)). As an example from studies of AIDs, Liang and Wu \(^6\) use local polynomial regressions to estimate the ODE solutions \( y \) and their first derivatives \( y' \) from noisy measurements of plasma viral load and

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\(^1\) The author is grateful to Professor Esfandiar Maasoumi at Emory University for pointing out this motivation.

\(^2\) See \(^15\) (Chapter 1) and \(^19\) (Examples 12.17 and 12.29). Other decompositions of the Sobolev class are also possible, see, e.g. \(^21\).
CD4+ T cell counts; then, the authors regress the estimates \( \hat{y}' \) on \( f(\hat{y}; \theta) \) to obtain estimates of the parameters \( \theta \) in the ODE model. Liang and Wu [3] mentioned that higher order local polynomials for approximating the solutions can also be used, and doing so would require boundedness on the higher order derivatives of the solutions. Motivated by these statistical procedures for recovering ODEs in the literature, Zhu and Mirzaei [22] study how the smoothness of ODEs affects the smoothness of the underlying solutions. To illustrate, let us consider the autonomous ODE \( y'(x) = f(y(x)) \). Like other areas in nonparametric estimation, it can be desirable to only assume smoothness structures on \( f \) for hedging against misspecification of the functional form for \( f \). Zhu and Mirzaei [22] show that: (i) If \( |f^{(k)}(x)| \leq c_0 \) for all \( x \) on the domain and \( k = 0, \ldots, \gamma + 1 \), then \( |\hat{y}^{(k+1)}(x)| \leq c_0^{k+1}k! \); (ii) the factorial bounds are attainable by the solutions to some ODE (e.g., \( y' = e^{-y-\frac{1}{2}} \)) and therefore tight. To this phenomenon, [22] gives the name “loss of smoothness”.

Motivated by the “loss of smoothness” phenomenon, this paper generalizes the standard Hölder class in the literature to one where the absolute value of the \( k \)th \((k = 0, \ldots, \gamma)\) derivative of any member is bounded by a parameter \( R_k \) and the \( \gamma \)th derivative is \( R_{\gamma+1} \)-Lipschitz (i.e., \( R_k \) is allowed to depend on \( k \)). We also generalize the standard ellipsoid class of smooth functions in a similar fashion by allowing its RKHS (reproducing kernel Hilbert space) radius to be bounded from above by \( R_{\gamma+1} \); that is,

\[
\mathcal{H}_{\gamma+1} = \left\{ f = \sum_{m=1}^{\infty} \theta_m \phi_m : \text{for } (\theta_m)_{m=1}^{\infty} \in \ell^2(\mathbb{N}) \text{ such that } \sum_{m=1}^{\infty} \frac{\theta_m^2}{\mu_m} \leq R_{\gamma+1}^2 \right\}
\]

where \( \ell^2(\mathbb{N}) := \{(\theta_m)_{m=1}^{\infty} | \sum_{m=1}^{\infty} \theta_m^2 < \infty \} \); \((\mu_m)_{m=1}^{\infty} \) and \((\phi_m)_{m=1}^{\infty} \) are the eigenvalues and eigenfunctions (that forms an orthonormal basis of \( L^2([0, 1]) \)), respectively, of an RKHS with \( \mu_m = (cm)^{-2(\gamma+1)} \) for some positive constant \( c \). The decay rate of the eigenvalues follows the standard assumption for \((\gamma+1)\)-degree smooth functions in the literature (see, e.g., [13, 19, 21]) and \( R_{\gamma+1} = 1 \) in [3] gives the standard ellipsoid class of smooth functions in [19]. Moreover, [3] is equipped with the inner product \( \langle h, g \rangle_{\mathcal{H}} = \sum_{m=1}^{\infty} \frac{\langle h, \phi_m \rangle \langle g, \phi_m \rangle}{\mu_m} \) where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2([0, 1]) \). The ellipsoid class is a generalization of the Sobolev class.

In our nonasymptotic framework, none of \( n, \gamma \) and \( \{R_k\}_{k=0}^{\gamma+1} \) are taken to infinity. The objectives of this paper are to examine the impacts of \( \gamma \) and \( \{R_k\}_{k=0}^{\gamma+1} \) on (i) the size of the generalized Hölder and ellipsoid classes of smooth functions, and on (ii) the optimal MSE convergence rates associated with [1] in finite sample settings. To accomplish the first objective, we establish upper and lower bounds on the covering and packing numbers of the generalized \( \mathcal{U}_{\gamma+1}, \mathcal{U}_{\gamma+1,2} \) and \( \mathcal{H}_{\gamma+1} \); see Table 1 for a summary of the results. With the help of our new entropy bounds, we then derive information theoretic lower bounds [3] and matching upper bounds for the MSE under various \( R_k \); see Tables 2–3 for a summary of the results.

### 1.1 Implications of our results

For the Hölder and ellipsoid classes, Tables 2–3 suggest that the “blessings of smoothness” arises in two ways: (i) from the nonasymptotic viewpoint, when \( \mathcal{U}_{\gamma+1} \) is imposed with the restrictions that \( f^{(k)}(0) = 0 \) for all \( k \leq \gamma \) (hence, \( \mathcal{U}_{\gamma+1} = \mathcal{U}_{\gamma+1,2} \)), \( (R^*)^{2\gamma+3} \times 1 \) for \( \mathcal{U}_{\gamma+1,2} \), and \( R_{\gamma+1}^{\gamma+3} \times 1 \) for \( \mathcal{H}_{\gamma+1} \), Table 2 shows that the minimax optimal rate for the MSE is \( \left( \frac{\sigma^2}{n} \right)^{2\gamma+2} \), which decreases

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Following the literature on minimax optimality, we consider the standard setup of [1] where \( \{x_i\}_{i=1}^{n} \) are independently and uniformly distributed on a bounded interval, and \( \{\varepsilon_i\}_{i=1}^{n} \) are independent \( N(0, \sigma^2) \) and independent of \( \{x_i\}_{i=1}^{n} \).
as $\gamma$ increases; (ii) when $\gamma$ is finite and $n \to \infty$, Tables 2-3 show that the minimax optimal rate for the MSE is $\left(\frac{a^2}{n}\right)^{\frac{2+2\gamma}{2+3\gamma}}$. Another way to think about (ii) in terms of Table 3 is, if the resolution $\delta$ is small enough (as a result of that $\frac{a^2}{\delta^2} \gtrsim (\gamma + 1)^{2\gamma+3}$), the size of $U_{\gamma+1,2}$ dominates the size of $U_{\gamma+1,1}$; hence, in deriving the bounds for the MSE, one may simply take $\log(\delta - \text{covering number}) \times \log(\delta - \text{packing number}) \times \delta^{-\gamma}$ under the standard assumption $R_k = 1$. This practice is common in the literature and leads to the classical rate $\left(\frac{a^2}{n}\right)^{\frac{2+2\gamma}{2+3\gamma}}$.

In general, however, when $n$ is small enough, exploiting higher degree smoothness assumptions can bring a curse. To illustrate, note from Table 3 that even when $R_k = 1$ (the standard assumption), if $\gamma > 1$ and $\frac{a^2}{\delta^2} \gtrsim (\gamma + 1)^{2\gamma+3}$, the minimax optimal rate is $\frac{a^2(\gamma+1)}{n}$, which is greater than $\left(\frac{a^2}{n}\right)^{\frac{2+2\gamma}{2+3\gamma}}$. Obviously, this result also holds true for the Sobolev class $S_{\gamma+1} = U_{\gamma+1,1} + S_{\gamma+1}$ mentioned at the beginning. In view of this result, exploiting higher degree smoothness assumptions on nonparametric regressions when $n$ is small may not be a good idea, unless the researchers believe that $U_{\gamma+1,1}$ has some sparsity or approximate sparsity structures (e.g., $f(k)(0) = 0$ for most of $k \in \{0, \ldots, \gamma\}$). In these cases, for instance, when estimating an unknown smooth function, researchers should consider regularizing the polynomial component in $U_{\gamma+1,1}$ (e.g., with the $l_1$-penalty) besides regularizing the component in $H_{\gamma+1}$ with the typical Sobolev penalty.

The finding that the minimax optimal rate increases as $\gamma$ increases when $\frac{n}{\delta^2} \gtrsim (\gamma + 1)^{2\gamma+3}$ and $\gamma > 1$ echoes the conjecture in [3]: higher order polynomials could induce higher MSE overall even in large samples. Gelman and Imbens [3] leave an open question about whether the poor performance of a high-order polynomial approach is attributed to the least square fit. This question can be analyzed from various angles and the answer would vary from one perspective to another. From the information theoretic viewpoint, our minimax lower bounds for the MSE suggest that, the impact of high order polynomials is intrinsic to the function classes $U_{\gamma+1}$ and $S_{\gamma+1}$; therefore, any algorithm (whether global polynomials or local polynomials or any other smooth methods) can suffer from including higher order polynomials when the sample size is not large enough.

The nonasymptotic analysis for the higher dimensional problem is rather involved because of the additional interplay between the smoothness parameter $\gamma$ and the dimension $d$. We are only able to provide some partial answers regarding $U_{\gamma+1}^d$. Yet, these results suggest that the classical asymptotic minimax rate $\left(\frac{a^2}{n}\right)^{\frac{2+2\gamma}{2+3\gamma}}$ could be an underestimate of the MSE in finite sample settings.

## 1.2 Theoretical contributions

Our main theoretical contributions lie in a set of new metric entropy bounds. In contrast to the classical entropy bounds in the literature, our new entropy bounds make the delicate analysis of minimax optimal rates in the regime of finite $n$ possible. In what follows, let us discuss the novelty of our results. The lower bound max $\{B_1(\delta), B_2(\delta)\}$ and the upper bound $B_1(\delta)$ in Table 1 are original. The (less original) bounds $B_2(\delta)$ and $R_{\gamma+1,1}^{\gamma+1}$ generalize the upper bounds associated with $U_{\gamma+1,1}$ and $U_{\gamma+1,2}$, respectively, in [4]. It is worth pointing out that $B_2(\delta)$ holds for all $\delta \in (0, 1)$ (not just $\delta$ such that $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)R_k}{10^2} < 0$) but is far from being tight when $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)R_k}{10^2} > 0$. Obviously $B_1(\delta) \lesssim B_2(\delta)$. When it comes to deriving the upper bounds for the MSE under large enough $R_k$ (such as $R_k = (k-1)!$ or $R_k = k!$), $B_1(\delta)$ will be very useful. In particular, [22] applies the counting argument in [4] to derive an upper bound for the

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4 To derive an upper bound for the $\delta$-covering number of $U_{\gamma+1} = U_{\gamma+1,1} + U_{\gamma+1,2}$ under the assumption $R_k \asymp 1$, the proof in Kolmogorov and Tikhomirov [4] considers a grid of points $(x_{-1}, ..., x_{-1}, x_{1}, ..., x_{\gamma})$ on $[-1, 1]$ where
covering number of $U_{\gamma+1,1}$ under $R_k = (k - 1)!$ and then derive an upper bound for the MSE. With the new bound $\overline{B}_1(\delta)$ developed in this paper, the upper bound in Table 3 for the MSE improves the one in [22] by a factor of $\gamma \log \gamma$.

Note that our lower bound for the generalized $U_{\gamma+1,1}$ in Table 1 takes the maximum of two terms. The part $B_2$ (a constant bound valid for all $\delta$ below a threshold detailed in Section 2) is useful for deriving the minimax lower bounds on the MSE if $R_k \leq (k - 1)!$ for all $k$, while $\overline{B}_1(\delta)$ will be useful if $R_k = k!$. Taking the maximum of the lower bounds for $U_{\gamma+1,1}$ and $U_{\gamma+1,2}$ gives a lower bound on the $\log(\delta$--packing number) of $U_{\gamma+1}$. The lower bound for $U_{\gamma+1}$ in [3] (derived under the assumption that $R_k \asymp 1$) is $\delta^{\frac{1}{\gamma+1}}$, which does not take into account the contribution from $U_{\gamma+1,1}$. Therefore, our lower bound sharpens the classical result for the standard Hölder class.

Compared to bounds $\overline{B}_1(\delta)$ and $\overline{B}_2(\delta)$ in Table 1, $\overline{B}_2$ is relatively straightforward. To establish $\overline{B}_1(\delta)$ and $\overline{B}_2(\delta)$, we discard the argument in [3] and instead, consider two classes (equivalent to $U_{\gamma+1,1}$), each in the form of a $\gamma$-dimensional polyhedron. The lower bound $\overline{B}_1(\delta)$ is the more delicate part. In particular, for any $f \in U_{\gamma+1,1} [-1, 1]$, we write $f(x) = \sum_{k=0}^{\gamma} \hat{\theta}_k \phi_k(x)$, where $\phi_k$ are the Legendre polynomials. The key step is to derive sharp nonasymptotic lower bounds for the magnitude of $\left(\hat{\theta}_k\right)^{\gamma}_{k=0}$ for the worst case.

Let us turn to the covering and packing numbers of the generalized ellipsoid class of smooth functions, $\mathcal{H}_{\gamma+1}$. Under the assumption $R_{\gamma+1} = 1$, the upper and lower bounds in Wainwright [19] (the last two inequalities on p.131) scale as $\left(\gamma + 1\right)\delta^{\frac{1}{\gamma+1}}$ and $\delta^{\frac{1}{\gamma+1}}$, respectively, while our upper and lower bounds in Table 1 have the same scaling $\delta^{\frac{1}{\gamma+1}}$. We close the gap in [19] by finding the optimal “pivotal” eigenvalue that best balances the “estimation error” and the “approximation error” from truncating for a given resolution $\delta$. More generally, for the case of $R_{\gamma+1} \gtrsim \gamma + 1$, we consider two different truncations, one giving the upper bound $\delta^{\frac{1}{\gamma+1}}$ and the other giving the lower bound $\left(R_{\gamma+1} \delta^{-1}\right)^{\frac{1}{\gamma+1}}$. Note that $\left(R_{\gamma+1} \delta^{-1}\right)^{\frac{1}{\gamma+1}} \asymp \delta^{\frac{1}{\gamma+1}}$ when $R_{\gamma+1} \asymp 1$.

A couple of interesting facts are revealed by Tables 2–3. First, when $R_k = (k - 1)!$, Table 2 shows that the optimal rates differ between $U_{\gamma+1,2}$ and $\mathcal{H}_{\gamma+1}$. Note that this difference cannot be revealed in the regime where $\gamma$ is fixed and $n \to \infty$, and by the classical entropy bounds. Second, Table 3 shows that, as the scaling of $R_k$ is increased from $(k - 1)!$ to $k!$, the minimax optimal rate is increased from $\frac{\sigma^2(\gamma+1)}{n}$ to $\frac{\sigma^2(\gamma+1) \log(\gamma \vee 2)}{n}$ when the component in $U_{\gamma+1,1}$ dominates. Once $\frac{\lambda}{\delta} \gtrsim (\gamma + 1)^{2\gamma+3}$ in the case of $R_k = (k - 1)!$, and $\frac{\lambda}{\sigma} \gtrsim ((\gamma + 1) \log(\gamma \vee 2))^{2\gamma+3}$ in the case of $R_k = k!$, the optimal rate becomes $\left(\frac{\sigma^2}{n}\right)^{2\gamma+2}$ as now the component in $U_{\gamma+1,2}$ dominates. The terms $(k!)^{\gamma}_{k=0}$ in [2] play a more important role on the size of $U_{\gamma+1,1}$ when $R_k$ becomes large enough, which is why $\overline{B}_1(\delta)$ and $\overline{B}_2(\delta)$ in Table 1 are very useful for deriving the minimax optimal rate for the MSE under large $R_k$.

1.3 Notation and definitions

**Notation.** Let $\lceil x \rceil$ denote the largest integer smaller than or equal to $x$. For two functions $f(n)$ and $g(n)$, let us write $f(n) \gtrsim g(n)$ if $f(n) \geq cg(n)$ for a universal constant $c \in (0, \infty)$; similarly, we write $f(n) \lesssim g(n)$ if $f(n) \leq c'g(n)$ for a universal constant $c' \in (0, \infty)$; and $s \asymp \delta^{\frac{1}{\gamma+1}}$; the bound $\left(\frac{s}{\sigma} + 1\right) \log \frac{1}{\delta} \left[\frac{\delta}{\sigma}\right] + 1 \log \left[\frac{\delta}{\sigma}\right] + 1 \log \left[\frac{\delta}{\sigma}\right]$ in [3] is obtained by counting the number of possible values of $\left(\frac{f(k)\epsilon(\eta)}{\delta_k}\right)^{\gamma}_{k=0}$ given the $\delta_k$--covering accuracy for the $k$th derivative (where $\delta_0 = \delta$).
Table 1: Upper and lower bounds on the log(δ − covering number) and log(δ − packing number) of the generalized $U_{r+1,1}$, $U_{r+1,2}$ and $H_{γ+1}$ in $L_q$-norm

| $U_{r+1,1}$ $(q \in \{2, ∞\})$ | $U_{r+1,2}$ $(q \in \{2, ∞\})$ | $H_{γ+1}$ $(q = 2)$ |
|--------------------------------|--------------------------------|---------------------|
| $\begin{cases} B_1(δ) & \text{if min}_{k \in [0,...,γ]} \frac{R_0(R+1)R_k}{k^2} \geq 0 \\ B_2(δ) & \text{otherwise} \end{cases}$ | $R^* \delta^\frac{1}{γ+1}$ if $R_0 ≥ 1$ | $R_1^{γ+1} \delta^\frac{1}{γ+1}$ if $R+1 ≥ γ+1$ |
| $\geq$ max $\{B_1(δ), B_2(δ)\}$ | $(R^* R_0)^{γ+1} \delta^\frac{1}{γ+1}$ if $R_0 ≤ 1$ | $R_1^{γ+1} \delta^\frac{1}{γ+1}$ if $R_0 ≤ 1$ |

where: $B_1(δ) = \sum_{k=0}^{γ} log R_0(R+1)R_k$; $B_2(δ) = \left(\frac{γ+1}{2}\right) log \left(\frac{2^γ}{δ}\right) + \sum_{k=0}^{γ} log R_k$; $B_1(δ) = \sum_{k=0}^{γ} log (R_{k+2m}^*γ_{m+1}) + \sum_{k=0}^{γ} log \left(\frac{γ_{1/2}R_{k+2m}^*γ_{m+1}}{2}\right)$ (with $R_{k+2m}^* = 0$ for $k + 2m > γ$); $B_2 = Cγ$ (valid for all $δ$ below a threshold detailed in Section 2); $R^* = \left(\sum_{k=0}^{γ} \frac{R_k}{R_0}\right)^{γ+1}$; $R_0$ is bounded away from zero under the column for $U_{r+1,2}$; $R_{r+1}$ is bounded away from zero under the column for $H_{γ+1}$.

Table 2: Minimax optimal MSE rates of the generalized $U_{r+1,2}$ and $H_{γ+1}$

| $U_{r+1,2}$ | $H_{γ+1}$ |
|-------------|-----------|
| $\frac{1}{(\frac{R^*}{2})^{2γ+1}} \left(\frac{2^γ}{δ}\right)^{2γ+1}$ | $\frac{1}{(R^* R_0)^{γ+1}} \left(\frac{2^γ}{δ}\right)^{2γ+1}$ |

where: $R^* = \left(\max_{k \in [1,...,γ+1]} \frac{R_k}{R_0}\right)\frac{R_0}{R_0}$ and $R_0$ is bounded away from zero under the column for $U_{r+1,2}$; $R_{r+1}$ is bounded away from zero under the column for $H_{γ+1}$.

$f(n) \asymp g(n)$ if $f(n) \gtrsim g(n)$ and $f(n) \lesssim g(n)$. Throughout this paper, we use various $c$ and $C$ letters to denote positive universal constants that are: $\gtrsim 1$ and independent of $n, γ$, $\{R_k\}_{k=0}^{γ+1}$ and $d$; these constants may vary from place to place. For a $d$-dimensional vector $θ$, the $l_q$-norm $|θ|_q := \left[\sum_{j=1}^{d} |θ_j|^q\right]^{1/q}$ if $1 ≤ q < ∞$ and $|θ|_q := \max_{j \in [1,...,d]} |θ_j|$ if $q = ∞$. Let $B^q_\delta(R) := \{θ \in \mathbb{R}^d : |θ|_q ≤ R\}$. For functions, the $L^2(\mathbb{R}_n)$-norm of the vector $f := \{f(x_i)\}_{i=1}^{n}$, denoted by $|f|_n$, is $\left[\frac{1}{n} \sum_{i=1}^{n} \left(f(x_i)^2\right)^{1/2}\right]^{1/2}$; the supremum norm $|f|_\infty := \sup_{x \in [a,b]} |f(x) - g(x)|$ and the $L^2(\mathbb{P})$ norm $|f-g|_2 = \sqrt{\frac{1}{b-a} \int_a^b |f(x) - g(x)|^2 dx}$.

Definition (covering and packing numbers). Given a set $Λ$, a set $\{η^1, η^2, ..., η^N\} \subset Λ$ is $δ$–cover of $Λ$ in the metric $ρ$ if for each $η ∈ Λ$, there exists some $i ∈ \{1, ..., N\}$ such that $ρ(η, η^i) ≤ δ$. The $δ$–covering number of $Λ$, denoted by $N_δ(δ, Λ)$, is the cardinality of the smallest $δ$–cover. A set $\{η^1, η^2, ..., η^M\} \subset Λ$ is a $δ$–packing of $Λ$ in the metric $ρ$ if for any distinct $i, j ∈ \{1, ..., M\}$, $ρ(η^i, η^j) > δ$. The $δ$–packing number of $Λ$, denoted by $M_δ(δ, Λ)$, is the cardinality of the largest $δ$–packing. Throughout this paper, we use $N_q(δ, F)$ and $M_q(δ, F)$ to denote the $δ$–covering number and the $δ$–packing number, respectively, of a function class $F$ with respect to the function norm $|·|_q$ where $q \in \{2, ∞\}$.

The following is a standard textbook result that summarizes the relationships between covering and packing numbers:

$M_2(2δ, Λ) ≤ N_2(δ, Λ) ≤ M_2(δ, Λ)$.

Given this sandwich result, a lower bound on the packing number gives a lower bound on the covering number, and vice versa; similarly, an upper bound on the covering number gives an upper
bound on the packing number, and vice versa.

## 2 Hölder classes

Let \( p = (p_j)_{j=1}^{d} \) and \( P = \sum_{j=1}^{d} p_j \) where \( p_j \)'s are non-negative integers; \( x = (x_j)_{j=1}^{d} \) and \( x^P = \prod_{j=1}^{d} x_j^{p_j} \). Write \( D^p f (x) = \partial^p f / \partial x_1^{p_1} \cdots \partial x_d^{p_d} \).

For a non-negative integer \( \gamma \), let \( \mathcal{U}_{\gamma+1} \left( (R_k)_{k=0}^{\gamma+1}, [-1, 1]^d \right) \) be the class of functions such that any function \( f \in \mathcal{U}_{\gamma+1} \left( (R_k)_{k=0}^{\gamma+1}, [-1, 1]^d \right) \) satisfies: (1) \( f \) is continuous on \([-1, 1]^d\), and all partial derivatives of \( f \) exist for all \( p \) with \( P := \sum_{k=1}^{d} p_k \leq \gamma \); (2) \( |D^p f (x)| \leq R_k \) for all \( p \) with \( P = k \) \((k = 0, ..., \gamma)\) and \( x \in [-1, 1]^d \), where \( D^0 f (x) = f (x) \); (3) \( \left| D^p f (x) - D^p f (x') \right| \leq R_{\gamma+1} \left| x - x' \right|_\infty \) for all \( p \) with \( P = \gamma \) and \( x, x' \in [-1, 1]^d \).

Our focus in this section is on \( d = 1 \), where we use the shortform \( \mathcal{U}_{\gamma+1} \). Section 4 consider a general \( d \), where we use the shortform \( \mathcal{U}_{\gamma+1}^d \). Remember that none of \( n, \gamma, \{R_k\}_{k=0}^{\gamma+1} \) and \( d \) are taken to infinity in our framework.

In view of (2), we have the following decomposition:

\[
\mathcal{U}_{\gamma+1} = \mathcal{U}_{\gamma+1,1} + \mathcal{U}_{\gamma+1,2} := \{ f_1 + f_2 : f_1 \in \mathcal{U}_{\gamma+1,1}, f_2 \in \mathcal{U}_{\gamma+1,2} \}
\]

where

\[
\mathcal{U}_{\gamma+1,1} = \left\{ f (x) = \sum_{k=0}^{\gamma} \theta_k x^k : (\theta_k)_{k=0}^{\gamma} \in \mathcal{P}_\gamma, x \in [-1, 1] \right\}
\]

with the \((\gamma + 1)\)-dimensional polyhedron

\[
\mathcal{P}_\gamma = \left\{ (\theta_k)_{k=0}^{\gamma+1} \in \mathbb{R}^{\gamma+1} : \theta_k \in \left[ -\frac{R_k}{k!}, \frac{R_k}{k!} \right] \right\}
\]

and

\[
\mathcal{U}_{\gamma+1,2} = \left\{ f \in \mathcal{U}_{\gamma+1} : f^{(k)} (0) = 0 \text{ for all } k \leq \gamma \right\}.
\]

### 2.1 Metric entropy bounds

**Lemma 2.1.** If \( \delta \) is small enough such that \( \min_{k \in \{0,...,\gamma\}} \log \frac{4(\gamma+1)R_k}{k! \delta} \geq 0 \), we have

\[
\log N_2 (\delta, \mathcal{U}_{\gamma+1,1}) \leq \log N_\infty (\delta, \mathcal{U}_{\gamma+1,1}) \leq \sum_{k=0}^{\gamma} \log \frac{4(\gamma+1)R_k}{k! \delta}; \quad (4)
\]

\[
\frac{B_1 (\delta)}{}
\]
if $\delta$ is large enough such that $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)R_k}{k, \delta} < 0$, we have

\[
\log N_2(\delta, U_{\gamma+1,1}) \leq \log N_\infty(\delta, U_{\gamma+1,1}) \leq \left(\frac{\gamma}{2} + 1\right) \log \frac{1}{\delta} + \sum_{k=0}^{\gamma} \log R_k.
\]

(5)

In terms of the lower bounds, we have

\[
\log M_\infty(\delta, U_{\gamma+1,1}) \geq \log M_2(\delta, U_{\gamma+1,1}) \geq B_1(\delta)
\]

where $B_1(\delta) = \sum_{k=0}^{\gamma} \log (9^{-\gamma} \gamma^{-\gamma}) + \sum_{k=0}^{\gamma} \log \frac{C \sum_{m=0}^{\gamma/2} R_{k+2m}}{\delta}$ (with $R_{k+2m} = 0$ for $k + 2m > \gamma$) for some positive universal constant $C$. Let $R^\dagger = \left(\sum_{k=0}^{\gamma} \frac{R_k}{k!}\right) \vee 1$ and $k \in \arg \max_{k \in \{0, \ldots, \gamma\}} \frac{R_k}{k!}$. If $\frac{cR_k}{k!R^\dagger \delta} \geq 2^\gamma$ and $3R^\dagger \delta \leq \frac{2R_k}{k!}$, we also have

\[
\log M_\infty(\delta, U_{\gamma+1,1}) \geq \log M_2(\delta, U_{\gamma+1,1}) \geq B_2 = C' \gamma
\]

for some positive universal constant $C'$.

Remark. When $R_k \asymp 1$ for $k = 0, \ldots, \gamma$, $R^\dagger \asymp 1$; when $R_0 \asymp 1$ and $R_k \asymp (k - 1)!$ for $k = 1, \ldots, \gamma$, $R^\dagger \asymp \log(\gamma \vee 2)$; when $R_k \asymp k!$ for all $k = 0, \ldots, \gamma$, $R^\dagger \asymp (\gamma \vee 1)$.

The proof for Lemma 2.1 is given in Section 5.1.

The upper bound $B_1(\delta)$ and the lower bound max $\{B_1(\delta), B_2\}$ as well as their proofs are novel. Bound $B_2(\delta)$ is the extension of the upper bound associated with $U_{\gamma+1,1}$ in [4] and allows for general $R_k$. Obviously $B_1(\delta) \asymp B_2(\delta)$. It is worth mentioning that [3] holds for all $\delta \in (0, 1)$ (not just $\delta$ such that $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{d(\gamma+1)R_k}{k, \delta} < 0$) but is far from being tight when $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{d(\gamma+1)R_k}{k, \delta} \geq 0$. When it comes to deriving the upper bounds for the convergence rates concerning $U_{\gamma+1,1}$ with large enough $R_k$, [4] will be very useful. In particular, if $R_k = (k - 1)!$ or $R_k = k!$ for $k = 1, \ldots, \gamma$, bound (4) gives much sharper scalings (in terms of $\gamma$) than (5).

In terms of the packing numbers, $B_2$ will be useful for deriving the minimax lower bounds for the convergence rates if $R_k \asymp (k - 1)!$, while $B_1(\delta)$ will be useful if $R_k = k!$.

Lemma 2.2. Let $R^\dagger = \left(\max_{k \in \{1, \ldots, \gamma+1\}} \frac{R_k}{(k-1)!}\right) \vee 1$. We have

\[
\log N_2(\delta, U_{\gamma+1,2}) \leq \log N_\infty(\delta, U_{\gamma+1,2}) \lesssim R^\dagger \frac{\delta^{-1}}{\gamma+1} \quad \text{if } R^\dagger \frac{\delta^{-1}}{\gamma+1} \in (0, 1), \text{ for } k = 0, \ldots, \gamma.
\]

We also have

\[
\log M_\infty(\delta, U_{\gamma+1,2}) \geq \log M_2(\delta, U_{\gamma+1,2}) \gtrsim R^\dagger \frac{\delta^{-1}}{\gamma+1}, \quad \text{if } R_0 \gtrsim 1, \delta \in (0, 1); \\
\log M_\infty(\delta, U_{\gamma+1,2}) \geq \log M_2(\delta, U_{\gamma+1,2}) \gtrsim (R^\dagger R_0)^{\frac{1}{\gamma+1}} \delta^{-\frac{1}{\gamma+1}}, \quad \text{if } R_0 \gtrsim 1, \delta \in (0, 1).
\]

The proof for Lemma 2.2 is given in Section 5.2.

Lemma 2.2 extends [4] to allow for general $R_k$. When $R_k \gtrsim k!$ for all $k = 1, \ldots, \gamma + 1$, $R^\dagger \frac{1}{\gamma+1} \asymp 1$ and the bounds in Lemma 2.2 coincide with those associated with $U_{\gamma+1,2}$ in [4]. If $R_k \gtrsim k!$ for
all \( k = 0, \ldots, \gamma + 1, R^* \frac{k+1}{\gamma+1} \gtrsim 1 \); for example, taking \( R_k \gtrsim (k!)^2 \) for all \( k = 0, \ldots, \gamma + 1 \) yields \( R^* \frac{\gamma+1}{\gamma+1} \gtrsim \frac{2}{e} \).

**Theorem 2.1.** Given Lemmas 2.1 and 2.2, we have

\[
\log N_2 (2\delta, U_{\gamma+1}) \leq \log N_\infty (2\delta, U_{\gamma+1}) \leq \begin{cases} B_1 (\delta) + R^* \frac{1}{\gamma+1} \delta^{-1} & \text{if } \min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)R_k}{k^0} \geq 0 \\ B_2 (\delta) + R^* \frac{1}{\gamma+1} \delta^{-1} & \text{if } \min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)R_k}{k^0} < 0 \end{cases}
\]

and

\[
\log M_\infty (\delta, U_{\gamma+1}) \geq \log M_2 (\delta, U_{\gamma+1}) \gtrsim \begin{cases} \max \left\{ B_1 (\delta), B_2, R^* \frac{1}{\gamma+1} \delta^{-1} \right\} & \text{if } R_0 \gtrsim 1 \\ \max \left\{ B_1 (\delta), B_2, (R^* R_0) \frac{1}{\gamma+1} \delta^{-1} \right\} & \text{if } R_0 \lesssim 1 \end{cases}
\]

where these bounds are subject to the conditions in Lemmas 2.1 and 2.2.

Theorem 2.1 follows easily from Lemmas 2.1 and 2.2; see the proof of Theorem 2.1 in Section 5.3.

### 2.2 Minimax optimality

Following the literature on minimax optimality, let us consider (11) where \( \{x_i\}_{i=1}^n \) are independently and uniformly distributed on \([-1, 1]\), and \( \{\varepsilon_i\}_{i=1}^n \) are independent \( \mathcal{N}(0, \sigma^2) \) and independent of \( \{x_i\}_{i=1}^n \), and for the achievability results, let us consider the least squares estimator

\[
\hat{f} \in \arg\min_{f \in \mathcal{F}} \frac{1}{2n} \sum_{i=1}^n (y_i - \hat{f}(x_i))^2 .
\]

(7)

To facilitate the presentations of our results and focus on the key points, we assume \( \sigma \gtrsim 1 \) throughout the rest of this paper. This assumption is not critical and can be relaxed.

**Theorem 2.2.** Let \( \mathcal{F} = U_{\gamma+1,2} \) in (7). Suppose \( \sigma \gtrsim 1 \) and \( R_0 \gtrsim 1 \). If \( \frac{n}{\sigma^2} \gtrsim (R^*)^{\frac{\gamma(2\gamma+3)+1}{\gamma+1}} \),

we have

\[
\sup_{f \in U_{\gamma+1,2}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r^2 + \exp \left\{ -cn\sigma^{-2}r^2 \right\},
\]

\[
\inf_{\hat{f}} \sup_{f \in U_{\gamma+1,2}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r^2,
\]

where

\[
r^2 = (R^*)^{\frac{\sigma^2}{\gamma+1}} \left( \frac{\gamma^2}{n} \right)^{\frac{2(\gamma+1)}{\gamma+1}}.
\]

The proof for Theorem 2.2 is given in Section 5.4.

**Theorem 2.3.** Let \( \mathcal{F} = U_{\gamma+1} \) in (7). Suppose \( \sigma \lesssim 1 \), \( R_0 \asymp 1 \) and \( R_k \leq (k-1)! \) for \( k = 1, \ldots, \gamma+1 \). If

\[
(\gamma + 1) \gtrsim \frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2\gamma+3},
\]

we have

\[
\sup_{f \in U_{\gamma+1}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r_1^2 + \exp \left\{ -cn\sigma^{-2}r_1^2 \right\}
\]

where \( r_1^2 = (R^*)^{\frac{\gamma(2\gamma+3)+1}{\gamma+1}} \left( \frac{\gamma^2}{n} \right)^{\frac{2(\gamma+1)}{\gamma+1}} \).
where \( r_1^2 = \frac{\sigma^2(\gamma+1)}{n} \), and \( r_1^2 \gtrsim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \) under \( \frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2\gamma+3} \). If \( 4^\gamma r_1^2 \gtrsim \frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2\gamma+3} \) for any \( \gamma > 1 \), we also have

\[
\inf \sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim \frac{\sigma^2(\gamma+1)}{n} \quad \text{for any } \gamma > 1.
\]

On the other hand, if \( n \gtrsim (\gamma + 1)^{2\gamma+3} \), then we have

\[
\inf \sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim \sigma^2(\gamma+1) \quad \text{for any } \gamma > 1.
\]

\[\text{Remark.} \quad \text{Consider } \gamma > 1. \quad \text{When } R_k = 1 \quad \text{for } k = 0, \ldots, \gamma, \text{ (9) becomes } \frac{n}{\sigma^2} \gtrsim 4^\gamma; \text{ when } R_0 = 1 \text{ and } R_k = (k-1)! \text{ for } k = 1, \ldots, \gamma, \text{ (9) is satisfied if } \frac{n}{\sigma^2} \gtrsim 4^\gamma (\log \gamma)^2. \]

The proof for Theorem 2.3 is given in Section 5.5.

**Theorem 2.4.** Let \( F = U_{\gamma+1} \) in [7]. Suppose \( \sigma \gtrsim 1 \) and \( R_k = k! \quad \text{for } k = 0, \ldots, \gamma + 1 \). If

\[
(\gamma + 1)^{2\gamma+1} \gtrsim \frac{n}{\sigma^2} \gtrsim (\gamma + 1) (\log (\gamma \vee 2))^{2\gamma+3},
\]

we have

\[
\sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r_2^2 + \exp \{ -c\sigma^{-2}r_2^2 \}
\]

where \( r_2^2 = \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \), and \( r_2^2 \gtrsim \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \) under \( \frac{n}{\sigma^2} \gtrsim ((\gamma + 1) (\log (\gamma \vee 2)))^{2\gamma+3} \). For any \( \gamma > 1 \), if

\[
4^\gamma \log \gamma \gtrsim \frac{n}{\sigma^2} \gtrsim (\gamma + 1) (\log (\gamma \vee 2))^{2\gamma+3},
\]

we also have

\[
\inf \sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim \frac{\sigma^2 (\gamma + 1) (\log (\gamma \vee 2))}{n} \quad \text{for any } \gamma > 1.
\]

On the other hand, if

\[
\frac{n}{\sigma^2} \gtrsim (\gamma + 1) (\log (\gamma \vee 2))^{2\gamma+3},
\]

then we have

\[
\sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r_2^2 + \exp \{ -c\sigma^{-2}r_2^2 \},
\]

\[
\inf \sup_{f} E \left( \left| \hat{f} - f \right|^2 \right) \gtrsim r_2^2.
\]

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where \( r_2^2 = \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \), and \( r_2^2 \gtrsim \frac{\sigma^2}{n} (\gamma+1) \log(\gamma+2) \) under (15).

The proof for Theorem 2.4 is given in Section 5.6.

3 Ellipsoid classes

In this section, we turn to the generalized ellipsoid class of smooth functions, \( \mathcal{H}_{\gamma+1} \). Remember that in our framework, none of \( R_{\gamma+1}, \gamma \), and \( n \) are taken to infinity.

3.1 Metric entropy bounds

**Theorem 3.1.** If \( R_{\gamma+1} \gtrsim \gamma + 1 \), we have

\[ \log N_2 (\delta, \mathcal{H}_{\gamma+1}) \asymp (R_{\gamma+1}\delta^{-1})^\frac{1}{\gamma+1}. \]

If \( R_{\gamma+1} \lesssim \gamma + 1 \), we have

\[ \log N_2 (\delta, \mathcal{H}_{\gamma+1}) \lesssim \delta^\frac{1}{\gamma+1}, \]

\[ \log N_2 (\delta, \mathcal{H}_{\gamma+1}) \gtrsim (R_{\gamma+1}\delta^{-1})^\frac{1}{\gamma+1}. \]

The proof for Theorem 3.1 is given in Section 5.7.

When \( R_{\gamma+1} = 1 \), Theorem 3.1 sharpens the upper bound for \( \log N_2 (\delta, \mathcal{H}_{\gamma+1}) \) in [19] from \((\gamma+1)\delta^{-\frac{1}{\gamma+1}}\) to \(\delta^{-\frac{1}{\gamma+1}}\). We discover the cause of the gap lies in that the “pivotal” eigenvalue (that balances the “estimation error” and the “approximation error” from truncating for a given resolution \( \delta \)) in [19] is not optimal. The truncation in [19] is commonly used in the existing literature and seems to originate from Theorem 3 in [8]. We close the gap by finding the optimal “pivotal” eigenvalue.

More generally, for the case of \( R_{\gamma+1} \lesssim \gamma + 1 \), we consider two different truncations, one giving the upper bound \( \delta^{-\frac{1}{\gamma+1}} \) and the other giving the lower bound \( (R_{\gamma+1}\delta^{-1})^\frac{1}{\gamma+1} \). Note that \((R_{\gamma+1}\delta^{-1})^\frac{1}{\gamma+1} \asymp \delta^{-\frac{1}{\gamma+1}}\) when \( R_{\gamma+1} \asymp 1 \). For the case of \( R_{\gamma+1} \gtrsim \gamma + 1 \), we use only one truncation to show that both the upper bound and the lower bound scale as \((R_{\gamma+1}\delta^{-1})^\frac{1}{\gamma+1}\).

3.2 Minimax optimality

Let us consider (1) where \( \{x_i\}_{i=1}^n \) are independently and uniformly distributed on \([0, 1]\), and \( \{\epsilon_i\}_{i=1}^n \) are independent \( \mathcal{N} (0, \sigma^2) \).

**Theorem 3.2.** Let \( \mathcal{F} = \mathcal{H}_{\gamma+1} \) in \([7]\). Suppose \( \sigma \gtrsim 1 \), \( R_{\gamma+1} \gtrsim 1 \) and the kernel function \( K(\cdot, \cdot) \) associated with \( \mathcal{H}_{\gamma+1} \) satisfies \( K(x, x') \lesssim 1 \) for all \( x, x' \in [0, 1] \). If \( R_{\gamma+1}^2 \left( \frac{a^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \gtrsim 1 \), we have

\[ \sup_{f \in \mathcal{H}_{\gamma+1}} \mathbb{E} \left( \left| \hat{f} - f \right|_2^2 \right) \lesssim r^2 + \exp \left\{ -cn\sigma^{-2}r^2 \right\}, \]

\[ \inf \sup_{f \in \mathcal{H}_{\gamma+1}} \mathbb{E} \left( \left| \hat{f} - f \right|_2^2 \right) \gtrsim r^2, \]
where \( r^2 = R_{\gamma+1}^2 \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \).

The proof for Theorem 3.2 is given in Section 5.

Let us introduce the Sobolev class with the restrictions at zero:

\[
S_{\gamma+1} := \{ f : [0, 1] \to \mathbb{R} | f \text{ is } \gamma + 1 \text{ times differentiable a.e., } f^{(k)}(0) = 0 \text{ for all } k \leq \gamma, \text{ and } f^{(\gamma)} \text{ is absolutely continuous with } \\
\int_0^1 \left[f^{(\gamma+1)}(t)\right]^2 dt \leq R_{\gamma+1}^2 \}.
\]

Like the Hölder classes in Section 2, the Sobolev classes considered above can be extended to those without the conditions \( f^{(k)}(0) = 0 \) for all \( k \leq \gamma \). When \( R_k = 1 \) for all \( k = 0, ..., \gamma + 1 \), based on Lemma 2.1 and Theorem 3.1, using arguments similar to those for Theorems 2.3 and 3.2, we would arrive at the same claim as in Theorem 2.3 for \( S_{\gamma+1} = U_{r+1,1} + S_{\gamma+1} \).

### 4 Conclusion and some insights about multivariate smooth functions

In the nonasymptotic framework, we have shown that exploiting higher degree smoothness assumptions can bring a curse unless the polynomial subspace has some sparsity or approximate sparsity structures. We want to emphasize that the finite sample implications of our results are applicable in numerous applications. Many semiparametric estimators involve nonparametric regressions as an intermediate step; in addition, estimations of generalized additive models, partially linear models, and single index models are all built upon nonparametric regressions.

The extension of our analysis to \( d \)-variate smooth functions is a lot more complex, because of an additional interplay between the smoothness parameter \( \gamma \) and the dimension \( d \). We conclude the paper by providing some partial results about the higher dimensional generalized Hölder class.

Given any function \( f \in U_{\gamma+1}^d \), we have

\[
f(x) = \sum_{k=0}^{\gamma} \sum_{p:P=k} \frac{x^p D^p f(0)}{k!} + \sum_{p:P=\gamma} \frac{x^p D^p f(z)}{\gamma!} - \sum_{p:P=\gamma} \frac{x^p D^p f(0)}{\gamma!}
\]

for some intermediate value \( z \). Similar to Section 2, we have the following decomposition:

\[
U_{\gamma+1}^d = U_{\gamma+1,1}^d + U_{\gamma+1,2}^d := \left\{ f_1 + f_2 : f_1 \in U_{r+1,1}^d, f_2 \in U_{\gamma+1,2}^d \right\}
\]

where

\[
U_{\gamma+1,1}^d = \left\{ f = \sum_{k=0}^{\gamma} \sum_{p:P=k} x^p \theta_{(p,k)} : \{\theta_{(p,k)}\}_{(p,k)} \in \mathcal{P}_\Gamma, x \in [-1, 1]^d \right\}
\]

with the \( \Gamma := \sum_{k=0}^{\gamma} \binom{d+k-1}{d-1} \) -dimensional polyhedron

\[
\mathcal{P}_\Gamma = \left\{ \{\theta_{(p,k)}\}_{(p,k)} \in \mathbb{R}^\Gamma : \text{for any given } k \in \{0, ..., \gamma\}, \theta_{(p,k)} \in \left[\frac{-R_k}{k!}, \frac{R_k}{k!}\right] \text{ for all } p \text{ with } P \leq k \right\}
\]
where $\theta = \{ \theta_{(p,k)} \}_{(p,k)}$ denotes the collection of $\theta_{(p,k)}$ over all $(p,k)$ configurations. And,

$$\mathcal{U}^d_{\gamma+1,2} = \left\{ f \in \mathcal{U}^d_{\gamma+1} : D^p f (0) = 0 \text{ for all } p \text{ with } P \leq k, k = 0, \ldots, \gamma \right\}.$$  

**Lemma 4.1.** Let $R^* = \left( \max_{k \in \{1, \ldots, \gamma+1\}} \frac{D^*_k R_k}{(k-1)!} \vee 1 \right)$. We have

$$\log N_2 \left( \delta, \mathcal{U}^d_{\gamma+1,2} \right) \leq \log N_\infty \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \lesssim d^d R^* \frac{d \delta \gamma^d}{\gamma+1}, \quad \text{if } R^* \frac{k}{\gamma+1} \delta^1 \gamma^d \in (0, 1) \text{ for } k = 0, \ldots, \gamma;$$

$$\log M_\infty \left( \delta, \mathcal{U}^d_{\gamma+1,2} \right) \lesssim \log M_2 \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \lesssim d^d R^* \frac{d \delta \gamma^d}{\gamma+1} \quad \text{if } \delta \in (0, 1).$$

**Remark.** With Lemma 4.1, we can easily establish the minimax optimal MSE rate for $\mathcal{U}^d_{\gamma+1,2}$, using arguments almost identical to those for Theorem 2.2.

The proof for Lemma 4.1 is given in Section 5.9.

**Lemma 4.2.** If $\delta$ is small enough such that $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)D^*_k R_k}{\delta k!} \geq 0$, we have

$$\log N_2 \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \leq \log N_\infty \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \leq \sum_{k=0}^{\gamma} D^*_k \log \frac{4(\gamma+1)D^*_k R_k}{\delta k!},$$

where $D^*_k = \left( \frac{d+k-1}{d-1} \right)$; if $\delta$ is large enough such that $\min_{k \in \{0, \ldots, \gamma\}} \log \frac{4(\gamma+1)D^*_k R_k}{\delta k!} < 0$, we have

$$\log N_2 \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \leq \log N_\infty \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right) \lesssim \left( \sum_{k=0}^{\gamma} D^*_k \right) \log \frac{1}{\delta} + \sum_{k=0}^{\gamma} D^*_k \log R_k.$$  

**Remark.** As in Lemma 2.1, (16) holds for all $\delta \in (0, 1)$ (not just $\delta$ such that $\min_k \log \frac{4(\gamma+1)D^*_k R_k}{\delta k!} < 0$) but is too loose when $\delta$ is small enough.

**Remark.** A simple upper bound on $\sum_{k=0}^{\gamma} D^*_k$ is $\sum_{k=1}^{\gamma} d^k \asymp d^\gamma$. Let us show a lower bound on $\sum_{k=0}^{\gamma} D^*_k$ for the case of $\gamma \geq 2d^2$ to illustrate how large $\frac{d^2}{m} \sum_{k=0}^{\gamma} D^*_k$ can be. We can write $D^*_k = \frac{(k+d-1)!}{(d-1)! k!} = \prod_{j=1}^{d-1} \frac{k+j}{j}$. Because $\gamma \geq 2d^2$, we have

$$\sum_{k=0}^{\gamma} D^*_k = \left( \sum_{k=0}^{\gamma} \prod_{j=1}^{d-1} \frac{k+j}{j} \right) \geq \left( \sum_{k=d^2}^{\gamma} \prod_{j=1}^{d-1} \frac{k+j}{j} \right) \geq \left( d^2 \left( \frac{d^2}{d} + 1 \right)^{d-1} \right) \geq d^{d+1}.$$  

The proof for Lemma 4.2 is given in Section 5.10. In theory, our arguments for $\mathcal{B}_1 (\delta)$ in Lemma 2.1 can be extended for analyzing the lower bound for $\log M_2 \left( \delta, \mathcal{U}^d_{\gamma+1,1} \right)$. However, this extension is very intensive. Arguments similar to those for $\mathcal{B}_2$ in Lemma 2.1 will not lead to a useful bound for $\mathcal{U}^d_{\gamma+1,1}$. Despite the lack of lower bounds, we can still gain some insights from Lemma 4.2, as it implies

$$\mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \lesssim r^2 + \exp \left\{ -c n \sigma^{-2} r^2 \right\}$$

13
where \( r^2 = \frac{a^2}{n} \sum_{k=0}^{\gamma} D_k^* \) and \( \hat{f} \) is the estimator in (1) with \( \mathcal{F} = \mathcal{U}_{\gamma+1,1}^d \). The quantity \( \sum_{k=0}^{\gamma} D_k^* \) is the higher dimensional analogue of \( \gamma + 1 \) and arises from the fact that a function in \( \mathcal{U}_{\gamma+1,1}^d \) has \( D_k^* \) distinct \( k \)th partial derivatives. Therefore, there is a good reason to think the rate \( r^2 \) is minimax optimal.

Suppose \( R_k = 1 \) for all \( k = 0, ..., \gamma + 1 \). If \( d \) is small relative to \( \gamma \) and \( n \), Lemma 4.1 implies that the minimax optimal rate concerning \( \mathcal{U}_{\gamma+1,1}^d \) is roughly \( \left( \frac{a^2}{2n} \right)^{\frac{2\gamma+2}{2\gamma+2+d}} \), the classical rate for \( \mathcal{U}_{\gamma+1}^d \) derived under the regime where \( \gamma \) and \( d \) are finite but \( n \to \infty \). Observe that 
\[
\frac{a^2}{n} \sum_{k=0}^{\gamma} D_k^* \geq \left( \frac{a^2}{n} \right)^{\frac{2\gamma+2}{2\gamma+2+d}}\text{whenever} \frac{2\gamma}{2\gamma+2} \geq \left( \sum_{k=0}^{\gamma} D_k^* \right)^{\frac{2\gamma+2+d}{d}}, \text{and} \frac{a^2}{n} \sum_{k=0}^{\gamma} D_k^* \geq \left( \frac{a^2}{n} \right)^{\frac{2\gamma+2}{2\gamma+2+d}}\text{whenever} \frac{2\gamma}{2\gamma+2} \geq \left( \sum_{k=0}^{\gamma} D_k^* \right)^{\frac{2\gamma+2+d}{d}}.
\]
Therefore, the classical asymptotic minimax rate \( \left( \frac{a^2}{n} \right)^{\frac{2\gamma+2}{2\gamma+2+d}} \) could be an underestimate of the MSE in finite sample settings where \( n \) is not large enough.

5 Proofs

5.1 Proof for Lemma 2.1

The upper bound. Recall the definition of \( \mathcal{U}_{\gamma+1,1}^d \):

\[
\mathcal{U}_{\gamma+1,1} = \left\{ f = \sum_{k=0}^{\gamma} \theta_k x_k : (\theta_k)_{k=0}^{\gamma} \in \mathcal{P}_\gamma, x \in [-1, 1] \right\}
\]

with the \((\gamma + 1)\)-dimensional polyhedron 
\[
\mathcal{P}_\gamma = \left\{ (\theta_k)_{k=0}^{\gamma} \in \mathbb{R}^{\gamma+1} : \theta_k \in \left[ \frac{-R_k}{k!}, \frac{R_k}{k!} \right] \right\}
\]

where \( R_k \) is allowed to depend on \( k \) in \( \{0, ..., \gamma\} \) only. We first derive an upper bound for \( N_{\infty}(\delta, \mathcal{U}_{\gamma+1,1}^d) \). Because the \( L^2(\mathbb{P}) \) norm is no greater than the sup norm and a smallest \( \delta \)-cover of \( \mathcal{U}_{\gamma+1,1}^d \) with respect to the \( \| \cdot \|_{\infty} \) norm also covers \( \mathcal{U}_{\gamma+1,1}^d \) with respect to the \( \| \cdot \|_2 \) norm, we have 
\[
N_2(\delta, \mathcal{U}_{\gamma+1,1}^d) \leq N_{\infty}(\delta, \mathcal{U}_{\gamma+1,1}^d).
\]

To bound \( \log N_{\infty}(\delta, \mathcal{U}_{\gamma+1,1}^d) \) from above, note that for \( f, f' \in \mathcal{U}_{\gamma+1,1}^d \), we have

\[
| f - f' |_{\infty} \leq \sum_{k=0}^{\gamma} | \theta_k - \theta'_k |
\]

where \( f' = \sum_{k=0}^{\gamma} \theta'_k x_k \) such that \( \theta' \in \mathcal{P}_\gamma \). Therefore, the problem is reduced to bounding \( N_1(\delta, \mathcal{P}_\gamma) \).

Consider \((a_k)_{k=0}^{\gamma}\) such that \( a_k > 0 \) for every \( k = 0, ..., \gamma \) and \( \sum_{k=0}^{\gamma} a_k = 1 \). To cover \( \mathcal{P}_\gamma \) within \( \delta \)-precision, we find a smallest \( a_k \delta \)-cover of \( \left[ \frac{-R_k}{k!}, \frac{R_k}{k!} \right] \) for each \( k = 0, ..., \gamma, \{ \theta_k, ..., \theta_k \} \), such that for any \( \theta \in \mathcal{P}_\gamma \), there exists some \( i_k \in \{1, ..., N_k\} \) with 
\[
\sum_{k=0}^{\gamma} | \theta_k - \theta_k^{i_k} | \leq \delta.
\]

As a consequence, we have

\[
\log N_1(\delta, \mathcal{P}_\gamma) \leq \sum_{k=0}^{\gamma} \log \frac{4R_k}{a_k k! \delta} = - \sum_{k=0}^{\gamma} \log a_k + \sum_{k=0}^{\gamma} \log \frac{4R_k}{k! \delta}. \tag{17}
\]
For \((a_k)_{k=0}^\gamma = 1\), the function
\[
h(a_0, \ldots, a_\gamma) := -\sum_{k=0}^{\gamma} \log a_k = -\log \left( \prod_{k=0}^{\gamma} a_k \right)
\]
is minimized at \(a_k = \frac{1}{\gamma+1}\). Consequently, the minimum of \(\sum_{k=0}^{\gamma} \log \frac{4R_k}{a_k k!\delta}\) equals \(\sum_{k=0}^{\gamma} \log \frac{4(\gamma+1)R_k}{k!\delta}\) and we have
\[
\log N_1(\delta, P_\gamma) \leq \sum_{k=0}^{\gamma} \log \frac{4(\gamma+1)R_k}{k!\delta}.
\]
Therefore,
\[
\log N_2(\delta, U_{\gamma+1,1}) \leq \log N_\infty(\delta, U_{\gamma+1,1}) \leq \sum_{k=0}^{\gamma} \log \frac{4(\gamma+1)R_k}{k!\delta}. \tag{18}
\]

If \(\delta\) is large enough such that \(\min_{k\in\{0, \ldots, \gamma\}} \frac{4(\gamma+1)R_k}{k!\delta} < 0\), we can evoke the counting argument in [4] and obtain
\[
\log N_2(\delta, U_{\gamma+1,1}) \leq \log N_\infty(\delta, U_{\gamma+1,1}) \leq \left(\frac{\gamma}{2} + 1\right) \log \frac{1}{\delta} + \sum_{k=0}^{\gamma} \log R_k. \tag{19}
\]

**The lower bound.** We first derive a lower bound for \(N_2(\delta, U_{\gamma+1,1})\). Because the \(L^2(\mathbb{P})\) norm is no greater than the sup norm and a largest \(\delta\)–packing of \(U_{\gamma+1,1}\) with respect to the \(|\cdot|_2\) norm also packs \(U_{\gamma+1,1}\) with respect to the \(|\cdot|_\infty\) norm, we have
\[
N_\infty(\delta, U_{\gamma+1,1}) \geq N_2(\delta, U_{\gamma+1,1}).
\]

Let \((\phi_k)_{k=0}^\gamma\) be the Legendre polynomials on \([-1, 1]\). For any function \(f \in U_{\gamma+1,1}\), we can write
\[
f(x) = \sum_{k=0}^{\gamma} \tilde{\theta}_k \phi_k(x) \tag{20}
\]
such that
\[
\tilde{\theta}_k = \frac{(2k+1)}{2} \int_{-1}^{1} f(x) \phi_k(x) dx. \tag{21}
\]
In Lemma A.1 of Section 5.11 we carefully modify the argument in [2] to show that
\[
\tilde{\theta}_k = \binom{k + \frac{1}{2}}{\frac{1}{2}} \sum_{m=0}^{\gamma/2} \frac{f^{(k+2m)}(0)}{2k+2m \cdot m! \cdot \left(\frac{1}{2}\right)^{k+m+1}}
\]
where \((a)_k = a(a+1) \cdots (a+k-1)\) is known as the Pochhammer symbol. Recall \(|f^{(k)}(0)| \leq R_k\) for \(k = 0, \ldots, \gamma\) and \(f^{(k)}(0) = 0\) for \(k > \gamma\). We can re-write
\[
U_{\gamma+1,1} = \left\{ f = \sum_{k=0}^{\gamma} \tilde{\theta}_k \phi_k(x) : (\theta_k)_{k=0}^\gamma \in P_\gamma^L, \ x \in [-1, 1] \right\}
\]
with the \((\gamma+1)\)–dimensional polyhedron
\[
P_\gamma^L = \left\{ (\frac{\gamma}{k})_{k=0}^\gamma \in \mathbb{R}^{\gamma+1} : \tilde{\theta}_k \in [-R_k, R_k] \right\}
\]
[15]
where $R_k := \sum_{m=0}^{\lfloor \gamma/2 \rfloor} b_{k,m} R_{k+2m}$ and $b_{k,m} = \frac{(k+\frac{1}{2})}{2^{k+2m}m!\left(\frac{1}{2}\right)_{k+m+1}}$. If we can bound $R_k$ from below by $\overline{R}_k$, then we have

$$P_L^L \supseteq P_L^L = \left\{ \left(\bar{\theta}_k\right)_k^\gamma_{k=0} \in \mathbb{R}^{\gamma+1} : \bar{\theta}_k \in [-R_k, R_k] \right\}. \tag{22}$$

Let us derive $R_k$. Because $f^{(l)}(0) = 0$ for $l > \gamma$,

$$\frac{f^{(k+2m)}(0)}{2^{k+2m}m!\left(\frac{1}{2}\right)_{k+m+1}} = 0 \quad \text{if } k + 2m > \gamma.$$

There are at most $\gamma + 1$ terms that are multiplied in the product $m!\left(\frac{1}{2}\right)_{k+m+1}$. Note that $m \leq \frac{\gamma}{2} \leq \frac{3\gamma}{2} + 1$ and

$$\left(\frac{1}{2}\right)_{k+m+1} = \frac{1 \cdot 2 \cdot \ldots \cdot (k+1)}{2 \cdot 2 \cdot \ldots \cdot 2} \leq \frac{2 \cdot 2 \cdot \ldots \cdot 2 \cdot (k+1)}{2 \cdot 2 \cdot \ldots \cdot 2} = (k + m + 1)!$$

where $k + m + 1 \leq \frac{3\gamma}{2} + 1$. Hence, we have

$$m!\left(\frac{1}{2}\right)_{k+m+1} \leq m!(k + m + 1)! \leq 1 \cdot \left(\frac{3\gamma}{2} + 1\right)^\gamma \leq (3\gamma)^\gamma.$$

As a result, we have

$$\overline{R}_k = \sum_{m=0}^{\lfloor \gamma/2 \rfloor} b_{k,m} R_{k+2m} = \sum_{m=0}^{\lfloor \gamma/2 \rfloor} \frac{(k + \frac{1}{2})}{2^{k+2m}m!\left(\frac{1}{2}\right)_{k+m+1}} R_{k+2m} \geq \left(\frac{k + 1}{2}\right) 2^{-\gamma} 3^{-\gamma} \gamma^{-\gamma} \sum_{m=0}^{\lfloor \gamma/2 \rfloor} R_{k+2m} \geq \frac{9^{-\gamma} \gamma^{-\gamma}}{2} \sum_{m=0}^{\lfloor \gamma/2 \rfloor} R_{k+2m} =: \overline{R}_k. \tag{23}$$

Note that for any $f, g \in U_{\gamma+1,1}$ where $f(x) = \sum_{k=0}^{\gamma} \bar{\theta}_k \phi_k(x)$ and $g(x) = \sum_{k=0}^{\gamma} \bar{\theta}_k \phi_k(x)$, we have

$$|f - g|_2^2 = \sum_{k=0}^{\gamma} \left[ \sqrt{\frac{2}{2k+1}} \left( \bar{\theta}_k - \bar{\theta}_k' \right) \right]^2. \tag{24}$$

In view of (22) and (24), to construct a packing set of $U_{\gamma+1,1}$ within $\delta$--separation, we find a largest $\sqrt{\frac{2k+1}{2}}\frac{\delta}{\sqrt{\gamma+1}}$--packing of $[-\overline{R}_k, \overline{R}_k]$ for each $k = 0, \ldots, \gamma$, \{ $\bar{\theta}_k^1, \ldots, \bar{\theta}_k^M_k$ \}, such that for any distinct $\bar{\theta}_k^i$ and $\bar{\theta}_k^j$ in the packing sets,

$$\sum_{k=0}^{\gamma} \left[ \sqrt{\frac{2}{2k+1}} \left( \bar{\theta}_k^i - \bar{\theta}_k^j \right) \right]^2 > \delta^2.$$
Therefore,

$$\log M_2(\delta, U_{\gamma+1,1}) \gtrsim \sum_{k=0}^{\gamma} \log \frac{\sqrt{2(\gamma+1)R_k}}{\sqrt{2k+1}\delta}. \quad (25)$$

Bounds (25) and (23) together give

$$\log M_2(\delta, U_{\gamma+1,1}) \geq \sum_{k=0}^{\gamma} \log (9^{-\gamma-\gamma}) + \sum_{k=0}^{\gamma} \log \frac{C\sum_{m=0}^{\lfloor\gamma/2\rfloor} R_{k+2m}}{\delta} \quad (26)$$

for some positive universal constant $C$. Because the $L^2(\mathbb{P})$ norm is no greater than the sup norm, we have

$$\log M_\infty(\delta, U_{\gamma+1,1}) \geq \log M_2(\delta, U_{\gamma+1,1})$$

$$\geq \sum_{k=0}^{\gamma} \log (9^{-\gamma-\gamma}) + \sum_{k=0}^{\gamma} \log \frac{C\sum_{m=0}^{\lfloor\gamma/2\rfloor} R_{k+2m}}{\delta} =: B_1(\delta).$$

The following argument gives another useful bound for $\log M_2(\delta, U_{\gamma+1,1})$. Let $R^\dagger := \left(\sum_{k=0}^{\gamma} \frac{R_k}{k!}\right)\vee 1$ and $\tilde{k} \in \arg\max_{k \in \{0,\ldots,\gamma\}} \frac{R_k}{k!}$. We consider a $3R^\dagger\delta$–grid of points on $[-\frac{R_k}{k!}, \frac{R_k}{k!}]$ (that is, each point is $3R^\dagger\delta$ apart) and denote the collection of these points by $\left(\frac{\theta_{\gamma_i}^j}{k}\right)_{i=1}^{M_0}$ where $M_0 = \frac{cR_k}{\delta k! R^\dagger}$. We choose $\delta$ such that $M_0 \geq 2^\gamma$ and $3R^\dagger\delta \leq \frac{2R_k}{k!}$. Let us fix $\theta_{\gamma}^* \in \left[-\frac{R_k}{k!}, \frac{R_k}{k!}\right]$ for $k \in \{0,\ldots,\gamma\}\setminus\tilde{k}$ and define

$$f_{\lambda_i}(x) = \theta_{\gamma}^* + \sum_{k \in \{0,\ldots,\gamma\}\setminus\tilde{k}} \lambda_{i,k}\theta_{\gamma}^* \mathbf{x}^k, \quad x \in [-1, 1]$$

where $(\lambda_{i,k})_{k \in \{0,\ldots,\gamma\}\setminus\tilde{k}} =: \lambda_i \in \{0, 1\}^\gamma$ for all $i = 1,\ldots,2^\gamma$. For any $\lambda_i$, $\lambda_j \in \{0, 1\}^\gamma$ such that $i \neq j$, we have

$$\left| f_{\lambda_i} - f_{\lambda_j} \right|_2 \leq \frac{1}{\sqrt{2}} \left[ \int_{-1}^{1} \left( \theta_{\gamma}^* + \sum_{k \in \{0,\ldots,\gamma\}\setminus\tilde{k}} \left(1 \{\lambda_{i,k} \neq \lambda_{j,k}\} \theta_{\gamma}^* \mathbf{x}^k\right) \right)^2 \mathbf{dx} \right]^\frac{1}{2}$$

$$\geq \frac{1}{\sqrt{2}} \left[ \int_{-1}^{1} \left( \theta_{\gamma}^* - \sum_{k \in \{0,\ldots,\gamma\}\setminus\tilde{k}} \left(1 \{\lambda_{j,k} \neq \lambda_{i,k}\} \theta_{\gamma}^* \mathbf{x}^k\right) \right)^2 \mathbf{dx} \right]$$

$$\geq \frac{1}{\sqrt{2}} \left[ \int_{-1}^{1} \left( \theta_{\gamma}^* - \sum_{k \in \{0,\ldots,\gamma\}\setminus\tilde{k}} \left(1 \{\lambda_{i,k} \neq \lambda_{j,k}\} \theta_{\gamma}^* \mathbf{x}^k\right) \right)^2 \mathbf{dx} \right]$$

$$\geq \frac{1}{\sqrt{2}} \left( 3R^\dagger\delta - \delta \sum_{k=0}^{\gamma} \frac{R_k}{k!} \right)$$

$$\geq \frac{1}{\sqrt{2}} \left( 2\delta \sum_{k=0}^{\gamma} \frac{R_k}{k!} \right) \vee (2\delta) \geq \delta$$

where the third line follows from the Jensen’s inequality and the concavity of $\sqrt{\cdot}$ on $(0, 1)$, and the fourth line follows from the triangle inequality. Hence, we have constructed a $\delta$–packing set. The cardinality of this packing set is $2^\gamma$. Consequently, we have

$$\log M_\infty(\delta, U_{\gamma+1,1}) \geq \gamma =: B_2.$$
Remark. Note that the lower bound \( \log M_2(\delta, U_{\gamma+1,1}) \geq \gamma \) holds for all \( \delta \) such that \( \frac{eR^\delta}{k!R^\delta} \geq 2^{\gamma} \) and \( 3R!\delta \leq \frac{2R^\delta}{k!} \).

5.2 Proof for Lemma 2.2

The upper bound. The following derivations generalize [4]. In particular, properly choosing the grid of points on \([-1, 1]\) is the key modification here. Any function \( f \in U_{\gamma+1,2} \) can be written as

\[
f(x + \Delta) = f(x) + \Delta f'(x) + \sum_{k=0}^\gamma \frac{\Delta^k}{k!} f^{(k)}(x) + REM_0(x + \Delta)
\]

where \( x, x + \Delta \in (-1, 1) \) and \( z \) is some intermediate value. Let \( REM_0(x + \Delta) := f(x + \Delta) - F_{\gamma-1}(x) - \frac{\Delta^\gamma}{\gamma!} f^{(\gamma)}(x) \) and note that

\[
|REM_0(x + \Delta)| = \left| \frac{\Delta^\gamma}{\gamma!} \right| |f^{(\gamma)}(z) - f^{(\gamma)}(x)|
\]

\[
\leq \left| \frac{\Delta^\gamma+1}{\gamma!} \right| R_{\gamma+1}.
\]

(27)

In other words,

\[
f(x + \Delta) = \sum_{k=0}^\gamma \frac{\Delta^k}{k!} f^{(k)}(x) + REM_0(x + \Delta)
\]

where \( |REM_0(x + \Delta)| \leq \frac{\Delta^\gamma+1}{\gamma!} R_{\gamma+1} \). Similarly, any \( f^{(i)} \in U_{\gamma+1-i,2} \) for \( 1 \leq i \leq \gamma \) can be written as

\[
f^{(i)}(x + \Delta) = \sum_{k=0}^{\gamma-i} \frac{\Delta^k}{k!} f^{(i+k)}(x) + REM_i(x + \Delta)
\]

(28)

where \( |REM_i(x + \Delta)| \leq \frac{\Delta^\gamma+1-i}{(\gamma-i)!} R_{\gamma+1-i} \).

For some \( \delta_0, \ldots, \delta_\gamma > 0 \), suppose that \( |f^{(k)}(x) - g^{(k)}(x)| \leq \delta_k \) for \( k = 0, \ldots, \gamma \), where \( f, g \in U_{\gamma+1,2} \). Then we have

\[
|f(x + \Delta) - g(x - \Delta)| \leq \sum_{k=0}^\gamma \frac{|\Delta^k\delta_k}{k!} + 2\left| \frac{\Delta^\gamma+1}{\gamma!} \right| R_{\gamma+1}.
\]

Let \( \left( \max_{k \in \{1, \ldots, \gamma+1\}} \frac{R^k}{k!} \right) \vee 1 =: R^* \). Consider \( |\Delta| \leq (R^{*-1}\delta)^{\gamma+1} \) and \( \delta_k = R^* \frac{k+\delta+1}{k+\delta+1} \) for \( k = 0, \ldots, \gamma \) and \( \delta \) such that \( \delta_k \in (0, 1) \). Then,

\[
|f(x + \Delta) - g(x - \Delta)| \leq \delta \sum_{k=0}^\gamma \left( R^* \frac{k+\delta+1}{k!} \right) + 2R^* |\Delta|^{\gamma+1}
\]

\[
\leq \delta \sum_{k=0}^\gamma \frac{1}{k!} + 2\delta \leq 5\delta.
\]

(29)

Let us consider the following \( (R^{*-1}\delta)^{\gamma+1} \) grid of points in \([-1, 1]\):

\[
x_{-s} < x_{-s+1} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_{s-1} < x_s,
\]

where \( \delta_k \in (0, 1) \) for all \( k = 0, \ldots, \gamma \).
with

\[ x_0 = 0 \text{ and } s \lesssim (R^{s-1}\delta)^{-\frac{1}{s+1}}. \]

It suffices to cover the \( k \)th derivatives of functions in \( \mathcal{U}_{\gamma+1,2} \) within \( \delta_k \)-precision at each grid point. Then by (29), we obtain a \( 5\delta \)-cover of \( \mathcal{U}_{\gamma+1,2} \). Following the arguments in [4], bounding \( N_\infty (\delta, \mathcal{U}_{\gamma+1,2}) \) can be reduced to bounding the cardinality of

\[ \Lambda = \left\{ \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma, \right. \quad -s \leq i \leq s, \quad 0 \leq k \leq \gamma \} : f \in \mathcal{U}_{\gamma+1,2} \]

with \( \lfloor x \rfloor \) denoting the largest integer smaller than or equal to \( x \). Starting with \( x_0 = 0 \), the number of possible values of the vector \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \) when \( f \) ranges over \( \mathcal{U}_{\gamma+1,2} \) is 1. For \( i = 1, \ldots, s \), given the value of \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \), let us count the number of possible values of \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \). The counting for \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \) is similar. For each \( 0 \leq k \leq \gamma \), let \( B_{k,i-1} := \left\lfloor \frac{f^{(k)}(x_i)}{\delta_k} \right\rfloor \). Observe that \( B_{k,i-1} \delta_k \leq f^{(k)}(x_0) < (B_{k,i-1} + 1) \delta_k \).

Taking (28) with \( x = x_{i-1} \) and \( \Delta = x_i - x_{i-1} \) gives

\[
\left| f^{(i)}(x_i) - \sum_{k=0}^{\gamma-i} \frac{\Delta^k}{k!} f^{(i+k)}(x_{i-1}) \right| \leq \frac{|\Delta|^{\gamma+1-i}}{(\gamma-i)!} R_{\gamma+1-i}.
\]

As a result,

\[
\left| f^{(i)}(x_i) - \sum_{k=0}^{\gamma-i} \frac{\Delta^k}{k!} B_{i+k,i-1} \right| \\
\leq \left| f^{(i)}(x_i) - \sum_{k=0}^{\gamma-i} \frac{\Delta^k}{k!} f^{(i+k)}(x_{i-1}) \right| + \left| \sum_{k=0}^{\gamma-i} \frac{\Delta^k}{k!} \left( f^{(i+k)}(x_{i-1}) - B_{i+k,i-1} \right) \right| \\
\leq \frac{|\Delta|^{\gamma+1-i}}{(\gamma-i)!} R_{\gamma+1-i} + \sum_{k=0}^{\gamma-i} \frac{|\Delta|^k}{k!} \delta_k \\
\leq (R^{s-1}\delta)^{-\frac{1}{s+1}} R_{\gamma+1-i} + \sum_{k=0}^{\gamma-i} \left[ \frac{1}{k!} (R^{s-1}\delta)^{k} R^{s+i+k} \delta^{1-\frac{i+k}{s+1}} \right] \\
\leq R^{s+i+1} \delta^{1-\frac{i}{s+1}} + R^{s+i+1} \delta^{1-\frac{i}{s+1}} \sum_{k=0}^{\gamma-i} \frac{1}{k!} \leq 4\delta_i.
\]

Hence, the number of possible values of \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \) is at most 4 given the value of \( \left( \left| \frac{f^{(k)}(x_i)}{\delta_k} \right| \right)^\gamma \). Consequently, we have

\[
\text{card}(\Lambda) \lesssim 4^{2s} \lesssim 16(R^{s-1}\delta)^{\frac{1}{s+1}}
\]

which implies

\[
\log N_2(\delta, \mathcal{U}_{\gamma+1,2}) \leq \log N_\infty(\delta, \mathcal{U}_{\gamma+1,2}) \lesssim R^{\frac{1}{s+1}} \delta^{\frac{1}{s+1}}.
\]

### The lower bound

In the derivation of the lower bound, [4] considers a \( \delta^{\frac{1}{s+1}} \)-grid of points

\[
\begin{align*}
\cdots < a_1 < x_1 < a_2 < x_2 < \cdots < a_{2s} < x_{2s}
\end{align*}
\]
where $\overline{a_i} - a_i = \delta^{-\frac{1}{q+1}}$ and $s \gtrsim \delta^{-\frac{1}{q+1}}$. Recall that we have previously considered a $(R^{q+1}\delta)^{\frac{1}{q+1}}$ grid of points in $[-1,1]$ in the derivation of the upper bound for $\log N_{\infty}(\delta, U_{\gamma+1,2})$. To obtain a lower bound for $\log M_{\infty}(\delta, U_{\gamma+1,2})$ with the same scaling as our upper bound, the key modification we need is to replace the $\overline{a_i} - a_i = (R^{q+1}\delta)^{\frac{1}{q+1}}$ and $s \gtrsim \delta^{-\frac{1}{q+1}}$ with $s \gtrsim R^{q+1}\delta^{-\frac{1}{q+1}}$. The rest of the arguments are similar to those in [4]. In particular, let us consider

$$f_\lambda(x) = R^s \sum_{i=1}^{2s} \lambda_i (\overline{a_i} - a_i)^{\frac{1}{q+1}} h_0 \left( \frac{x - a_i}{\overline{a_i} - a_i} \right)$$

where $\lambda_i \in \{0, 1\}$ and $\lambda \in \{0, 1\}^{2s}$, and $h_0$ is a function on $\mathbb{R}$ satisfying: (1) $h_0$ restricted to $[-1, 1]$ belongs to $U_{\gamma,+1,2}$; (2) $h_0(x) = 0$ for $x \notin (0, 1)$ and $h_0(x) > 0$ for $x \in (0, 1)$; (3) $h_0 \left( \frac{1}{2} \right) = \max_{x \in [0, 1]} h_0(x) = R_0$. As an example, we can take $h_0(x) = \begin{cases} 0 & x \notin (0, 1) \\ be^{-\frac{1}{(1-x^2)}} & x \in (0, 1) \end{cases}$ for some properly chosen constant $b$ that can only depend on $R_0$. Note that the functions $h(x) := R^s (\overline{a_i} - a_i)^{\frac{1}{q+1}} h_0 \left( \frac{x - a_i}{\overline{a_i} - a_i} \right)$ and also $f_\lambda(x)$ belong to $U_{\gamma+1,2}$ if $\delta \in (0, 1)$. For any distinct $\lambda, \lambda' \in \{0, 1\}^{2s}$, we have

$$|f_\lambda - f_{\lambda'}|_\infty \gtrsim R^s (\overline{a_i} - a_i)^{\frac{1}{q+1}} h_0 \left( \frac{1}{2} \right) = R_0 \delta.$$

If $R_0 \gtrsim 1$, then $R_0 \delta \gtrsim \delta$ and

$$\log M_{\infty}(\delta, U_{\gamma+1,2}) \gtrsim R^s \delta^{-\frac{1}{q+1}}.$$

If $R_0 \lesssim 1$, then we obtain

$$\log M_{\infty}(\delta, U_{\gamma+1,2}) \gtrsim R^s \delta^{-\frac{1}{q+1}}$$

which implies that

$$\log M_{\infty}(\delta, U_{\gamma+1,2}) \gtrsim R^s \delta^{-\frac{1}{q+1}} \left( \frac{\delta}{R_0} \right)^{\frac{1}{q+1}}.$$ 

Standard argument in the literature based on the Vasharmov-Gilbert Lemma further gives

$$\log M_2(\delta, U_{\gamma+1,2}) \gtrsim \begin{cases} R^s \delta^{-\frac{1}{q+1}} & \text{if } R_0 \gtrsim 1 \\ (R^s R_0)^{\frac{1}{q+1}} \delta^{-\frac{1}{q+1}} & \text{if } R_0 \lesssim 1. \end{cases} \tag{31}$$

### 5.3 Proof for Theorem 2.1

To cover $U_{\gamma+1}$ within $2\delta$-precision, we find a smallest $\delta$-cover of $U_{\gamma+1,1}, \{f_{1,1}, f_{1,2}, \ldots, f_{1,N_1}\}$, and a smallest $\delta$-cover of $U_{\gamma+1,2}, \{f_{2,1}, f_{2,2}, \ldots, f_{2,N_2}\}$. Given that any $f \in U_{\gamma+1}$ can be expressed by $f = f_1 + f_2$ for some $f_1 \in U_{\gamma+1,1}$ and $f_2 \in U_{\gamma+1,2}$, there exist some $f_{1,i}$ and $f_{2,i'}$ from the covering sets such that

$$|f_1 + f_2 - f_1,i - f_2,i|_q \leq |f_1 - f_1,i|_q + |f_2 - f_2,i|_q \leq 2\delta, \; q \in \{2, \infty\}.$$ 

Consequently, we obtain

$$\log N_q(2\delta, U_{\gamma+1}) \leq \log N_q(\delta, U_{\gamma+1,1}) + \log N_q(\delta, U_{\gamma+1,2}), \; q \in \{2, \infty\}.$$ 

In terms of $\log M_q(\delta, U_{\gamma+1})$, we have

$$\log M_q(\delta, U_{\gamma+1}) \geq \max \{\log M_q(\delta, U_{\gamma+1,1}), \log M_q(\delta, U_{\gamma+1,2})\}, \; q \in \{2, \infty\}. $$

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5.4 Proof for Theorem 2.2

The upper bound. In view of (7), the basic inequality gives

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{f}(x_i) - f(x_i) \right)^2 \leq \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i \left( \hat{f}(x_i) - f(x_i) \right).
\]

Let Ω(r; \bar{F}) = \{ f \in \bar{F} : |f|_n \leq r \} with \bar{F} := \{ g = g_1 - g_2 : g_1, g_2 \in F \}. We follow the standard recipe in the literature (see, e.g., [1, 5, 16, 19]) to bound the right-hand-side of (32). In particular, we solve for any \( r \in (0, \sigma] \) such that

\[
\frac{c}{\sqrt{n}} \int_{r^2}^{r} \sqrt{\log N_\infty(\delta, \Omega(r; \bar{F}))} d\delta \leq \frac{r^2}{\sigma}
\]

where \( N_\infty(\delta, \Omega(r; \bar{F})) \) is the \( \delta \)-covering number of the set \( \Omega(r; \bar{F}) \) in the \(|·|_n\) norm. Note that

\[
\frac{1}{\sqrt{n}} \int_{r^2}^{r} \sqrt{\log N_n(\delta, \Omega(r; \bar{F}))} d\delta
\]

\[
\leq \frac{1}{\sqrt{n}} \int_{0}^{r} \sqrt{\log N_\infty(\delta, \bar{F})} d\delta
\]

\[
\approx \frac{1}{\sqrt{n}} \left( R^* \right)^{\frac{1}{2}} \frac{r^{\frac{\gamma+1}{2}}}{T(r)}.
\]

Setting \( \sigma T(r) \approx r^2 \) yields

\[
r^2 \approx \left( R^* \right)^2 \frac{\sigma^2}{n} \left( \frac{2^{(\gamma+1)}}{(2^{(\gamma+1)})+1} \right).
\]

By Corollary 14.15 and Proposition 14.25 in [19], and integrating the tail probability, if \( \sigma \gtrsim 1 \), we have

\[
\sup_{f \in \mathcal{U}_{\gamma+1,2}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \approx r^2 + \exp \left\{ -cn\sigma^{-2}r^2 \right\}.
\]

Note that in deriving \( T(r) \), we have used the upper bound in Lemma 2.2, \( \log N_\infty(\delta, \mathcal{U}_{\gamma+1,2}) \lesssim R^* \delta^{-\frac{1}{\gamma+1}} \delta^{1-\frac{k}{\gamma+1}} \), which is valid if \( R^* \delta^{\frac{1}{\gamma+1}} \delta^{1-\frac{k}{\gamma+1}} \in (0, 1) \) for \( k = 0, ..., \gamma \). These conditions are satisfied if

\[
R^* \delta^{\frac{1}{\gamma+1}} \left[ \left( R^* \right)^{\frac{1}{2^{(\gamma+1)+1}}} \left( \frac{\sigma^2}{n} \right)^{\frac{2^{(\gamma+1)}}{(2^{(\gamma+1)})+1}} \right]^{\frac{1}{\gamma+1}} = \left( R^* \right)^{\frac{\gamma(2\gamma+3)+1}{2^{(\gamma+1)+1}}} \left( \frac{\sigma^2}{n} \right)^{\frac{1}{2^{(\gamma+1)+1}}} < 1,
\]

which is equivalent to

\[
\frac{n}{\sigma^2} > \left( R^* \right)^{\frac{\gamma(2\gamma+3)+1}{2^{(\gamma+1)+1}}}.
\]

Moreover, Corollary 14.15 in [19] requires

\[
\left( R^* \right)^{\frac{2}{2^{(\gamma+1)+1}}} \left( \frac{1}{n} \right)^{\frac{2^{(\gamma+1)}}{2^{(\gamma+1)+1}}} \gtrsim 1
\]

which is satisfied whenever \( \frac{n}{\sigma^2} \gtrsim \left( R^* \right)^{\frac{\gamma(2\gamma+3)+1}{\gamma+1}} \) and \( \sigma \gtrsim 1 \).

The lower bound. The Yang and Barron version of Fano’s inequality (see, e.g., [19, 20]) gives
\[
\inf \sup \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \geq \sup_{\eta, \epsilon} \frac{\delta^2}{4} \left( 1 - \frac{\log 2 + \log N_{KL}(\epsilon, Q_U) + \epsilon^2}{\log M_2(\delta, U_{\gamma+1,2})} \right) \tag{34}
\]

where \(N_{KL}(\epsilon, Q_U)\) denotes the \(\epsilon\)-covering number of \(U_{\gamma+1,2}\) with respect to the square root of the \(KL\)-divergence. We denote the product distribution of \(\{x_i\}_{i=1}^n\) by \(U\), and the distribution of \(y\) given \(\{x_i\}_{i=1}^n\) by \(P_j\) when the truth is \(f_j\). Observe that

\[
D_{KL}(P_j \times U \parallel P_k \times U) = \mathbb{E}_X \left[ D_{KL}(P_j \parallel P_k) \right] = \frac{n}{2\sigma^2} \left| \hat{f} - f \right|^2
\]

and consequently, under (33),

\[
\log N_{KL}(\epsilon, Q_U) = \log N_2 \left( \frac{\sqrt{2}}{n} \sigma \epsilon, U_{\gamma+1,2} \right) \leq \left( \frac{R^* \sqrt{n}}{\sigma \epsilon} \right)^{\frac{\gamma+1}{\gamma+1+\gamma}}.
\]

Setting \(\left( \frac{R^* \sqrt{n}}{\sigma \epsilon} \right)^{\frac{\gamma+1}{\gamma+1+\gamma}} \propto \epsilon^2\) yields \(\epsilon^2 \propto \left( \frac{nR^*}{\sigma^2} \right)^{\frac{1}{\gamma+1+\gamma}} =: \epsilon^2\). Observe that setting

\[
\delta \propto (R^*)^{\frac{1}{\gamma+1+\gamma}} \left( \frac{\sigma^2}{n} \right)^{\frac{\gamma+1}{2(\gamma+1+\gamma)}}
\]

ensures

\[
(R^* \delta^{-1})^{\frac{1}{\gamma+1}} \propto (R^*)^{\frac{2}{\gamma+3}} \left( \frac{n}{\sigma^2} \right)^{\frac{1}{2(\gamma+1+\gamma)}} \geq \epsilon^2.
\]

Consequently, we have

\[
1 - \frac{\log 2 + \log N_{KL}(\epsilon, Q_U) + \epsilon^2}{\log M_2(\delta^*, U_{\gamma+1,2})} \geq \frac{1}{2}
\]

and

\[
\inf \sup \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \geq (R^*)^{\frac{2}{2(\gamma+1+\gamma)}} \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1+\gamma)}}.
\]

### 5.5 Proof for Theorem 2.3

**The upper bound.** Taking \(R_0 \propto 1\) and \(R_k \leq (k-1)!\) for \(k = 1, \ldots, \gamma + 1\) in (6) yields

\[
\log N_2(\delta, U_{\gamma+1}) \propto (\gamma + 1) \log \left( \frac{1}{\delta} + \left( \frac{1}{\delta} \right)^{\frac{1}{\gamma+1}} \right), \tag{35}
\]

where we have used the fact that \(R^* = 1\). Note that

\[
\frac{1}{\sqrt{n}} \int_0^r \sqrt{\log N_n(\delta, \Omega(\delta; \bar{F}))} d\delta \leq \frac{1}{\sqrt{n}} \int_0^r \sqrt{\log N_\infty(\delta, \bar{F})} d\delta
\]

\[
\geq \frac{1}{n} \frac{r^{\gamma+1} + 1}{T(r)}^{\frac{2(\gamma+1)}{2(\gamma+1+2)}}
\]

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where the last line follows from (35). Note that when \( R_0 \approx 1 \) and \( R_k \leq (k-1)! \) for \( k = 1, ..., \gamma + 1 \), (33) is reduced to \( \frac{\sigma}{\sigma^2} \approx 1 \). Setting \( \sigma T(r) \approx r^2 \) yields

\[
r^2 \approx \max \left\{ \frac{\sigma^2 (\gamma + 1)}{n}, \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \right\}.
\]

By Corollary 14.15 and Proposition 14.25 in [19], and integrating the tail probability, if \( \sigma \gg 1 \), we have

\[
\sup_{f \in \mathcal{U}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \lesssim r^2 + \exp \left\{ -c n \sigma^{-2} r^2 \right\}.
\]

Note that Corollary 14.15 in [19] requires

\[
\sup_{f \in \mathcal{U}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \approx \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}}\left( \frac{\gamma + 1}{n} \right).
\]

which is satisfied whenever \( \frac{\sigma^2}{n} \approx 1 \). If \( \sigma \approx 1 \) and \( \gamma + 1 \approx \frac{\sigma^2}{n} \approx (\gamma + 1)^{2(\gamma+1)+1} \), we have \( r^2 \approx \frac{\sigma^2 (\gamma + 1)}{n} \). Otherwise, we have \( r^2 \approx \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \).

**The lower bound.** The lower bound \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \) is well known and hence we only show the lower bound \( \frac{\sigma^2 (\gamma + 1)}{n} \) under (33). If \( \gamma = 0 \) (or \( \gamma = 1 \)), then as long as \( \frac{\sigma^2}{n} \approx 1 \), we have \( r^2 \approx \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \). Otherwise, we have \( r^2 \approx \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \).

Again, we use

\[
\inf_{\hat{f}} \sup_{f \in \mathcal{U}} \mathbb{E} \left( \left| \hat{f} - f \right|^2 \right) \geq \sup_{\eta, \epsilon} \delta^2 \left( 1 - \frac{\log 2 + \log N_{KL} (\epsilon, \mathcal{U}) + \epsilon^2}{\log M_2 (\delta, \mathcal{U}_{\gamma+1})} \right)
\]

where \( N_{KL} (\epsilon, \mathcal{U}) \) denotes the \( \epsilon \)-covering number of \( \mathcal{U}_{\gamma+1} \) with respect to the square root of the KL-divergence. Similar to Section 5.4 under \( \frac{\sigma^2}{n} \approx 1 \), we have

\[
\log N_{KL} (\epsilon, \mathcal{U}) = \log N_2 \left( \sqrt{\frac{2}{\sigma}} \epsilon, \mathcal{U}_{\gamma+1} \right) \approx \gamma \log \frac{\sqrt{n} \gamma}{\sigma \epsilon} - \sum_{k=0}^{\gamma} \log k! + \left( \frac{\sqrt{n}}{\sigma \epsilon} \right)^{\frac{1}{\gamma+1}}
\]

for any \( \gamma \in \{0, ..., \beta\} \), where the last line follows by choosing a sufficiently small \( \epsilon \). Setting \( \left( \frac{\sqrt{n}}{\sigma \epsilon} \right)^{\frac{1}{\gamma+1}} \approx \epsilon^2 \) yields \( \epsilon^2 \approx \left( \frac{\sqrt{n}}{\sigma \epsilon} \right)^{\frac{1}{2(\gamma+1)+1}} \). Setting \( \epsilon^2 \approx \left( \frac{\sqrt{n}}{2^{\gamma+1}} \right)^{\frac{1}{2(\gamma+1)+1}} \) yields \( \epsilon \approx \epsilon^2 \).

Next, we turn to \( \log M_2 (\delta, \mathcal{U}_{\gamma+1}) \) in (33). For classes with \( R_0 \approx 1 \) and \( R_k \approx (k-1)! \) for \( k = 1, ..., \gamma + 1 \), it turns out the lower bound \( \log M_2 (\delta, \mathcal{U}_{\gamma+1}) \approx \gamma \) will suffice. Recalling the
definition of $\frac{R_k}{k!}$ in Section 5.1 we have $\frac{R_k}{k!} \asymp 1$ in the cases of $R_0 \asymp 1$ and $R_k \leq (k-1)!$. Also recall that the lower bound $\log M_2 (\delta, U_{\gamma+1,1}) \gtrsim \gamma$ in Section 5.1 holds for all $\delta$ such that $\frac{cR_k}{k!R^2 \delta} \geq 2^\gamma$ and $3R^2 \delta \leq \frac{2R_k}{k!}$. If

$$\gamma + 1 \gtrsim \epsilon^* + 2 \left( \frac{\sigma^2}{n} \right)^{\frac{1}{2(\gamma+1)+1}},$$

which is equivalent to

$$\frac{n}{\sigma^2} \gtrsim (\gamma + 1)^{2(\gamma+1)+1},$$

then we obtain

$$1 - \log 2 + \log N_{KL} (\epsilon^*, Q_U) + \epsilon^* \log M_2 (\delta, U_{\gamma+1}) \geq \frac{1}{2}$$

for all $\delta$ such that $\delta \lesssim \frac{2^{\gamma+1}}{R^2}$. Take $\delta = \frac{\sigma^2}{n}$ and observe that $\delta^* \gtrsim \frac{2^{\gamma+1}}{R^2}$ if

$$\frac{n}{\sigma^2} \gtrsim 2^\gamma R^2 \delta^2 \tag{39}$$

where $R^2 \gtrsim \log \gamma$ in the cases of $R_0 \asymp 1$ and $R_k \leq (k-1)!$.

Thus, we have shown that under (38) and (39),

$$\inf \sup_{f \in U_{\gamma+1}} \mathbb{E} \left( \left| \bar{f} - f \right|_2^2 \right) \gtrsim \frac{\sigma^2 (\gamma + 1)}{n}.$$

### 5.6 Proof for Theorem 2.4

In view of (18) and (30) (where $R_0 \asymp \frac{1}{n}$), we have

$$\log N_2 (\delta, U_{\gamma+1}) \gtrsim (\gamma + 1) \log \frac{\gamma + 1}{\delta} + \left( \frac{1}{\delta} \right)^{\frac{1}{\gamma+1}}. \tag{40}$$

In view of (26) and (31) (where also $R_0 \asymp 1$), for $\gamma > 1$, we have

$$\log M_2 (\delta, U_{\gamma+1}) \geq c \left[ (\gamma + 1) \log \frac{1}{\delta} - \gamma + \left( \frac{1}{\delta} \right)^{\frac{1}{\gamma+1}} \right] - (\gamma + 1) \log \gamma + (\gamma + 1) \log (\gamma - 1)!
\approx \gamma \log \frac{1}{\delta} \left( T_1(\delta) \right) - \gamma^2 \left( T_2(\delta) \right) \left( T_3(\delta) \right).$$

The upper bound. In terms of the upper bound, we obtain

$$r^2 \asymp \max \left\{ \frac{\sigma^2 (\gamma + 1) \log (\gamma + 2)}{n}, \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \right\}. \tag{41}$$

Note that when $R_k = k!$ for $k = 1, \ldots, \gamma + 1$, (33) is satisfied if $\frac{n}{\sigma^2} \geq (\gamma + 1)^{2\gamma+1}$. Like in Section 5.5 in order to apply Corollary 14.15 in [19], we require

$$\max \left\{ \frac{(\gamma + 1) \log (\gamma + 2)}{n}, \left( \frac{1}{n} \right)^{\frac{2(\gamma+1)}{2(\gamma+1)+1}} \right\} \gtrsim 1.$$
which is satisfied whenever \( n \gtrsim (\gamma \lor 1)^{2\gamma + 1} \).

If \( \sigma \gtrsim 1 \) and \((\gamma \lor 1)^{2\gamma + 1} \gtrsim \frac{\sigma^2}{\sigma^2} \gtrsim ((\gamma + 1) \log (\gamma \lor 2))^{2(\gamma + 1)^2}\), we have \( r^2 \asymp \frac{\sigma^2(\gamma + 1) \log (\gamma \lor 2)}{n} \). If \( \frac{n}{\sigma^2} \gtrsim ((\gamma + 1) \log (\gamma \lor 2))^{2(\gamma + 1)^2} \), we have \( r^2 \asymp \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma + 1)}{2(\gamma + 1)^2 + 1}} \). Note that if \( \gamma = 0 \) or \( \gamma = 1 \), as long as \( \frac{n}{\sigma^2} \gtrsim 1 \), we have \( r^2 \asymp \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma + 1)}{2(\gamma + 1)^2 + 1}} \).

The lower bound. Again, the lower bound \( \left( \frac{\sigma^2}{n} \right)^{\frac{2(\gamma + 1)}{2(\gamma + 1)^2 + 1}} \) is well known and hence we only show the lower bound \( \frac{\sigma^2(\gamma + 1) \log (\gamma \lor 2)}{n} \) under \([12]\). If \( \gamma = 0 \) (or \( \gamma = 1 \)), then as long as \( \frac{n}{\sigma^2} \gtrsim 1 \), we have \( r^2 \asymp \left( \frac{\sigma^2}{n} \right)^{\frac{2}{3}} \) (respectively, \( r^2 \asymp \left( \frac{\sigma^2}{n} \right)^{\frac{1}{3}} \)) in the derivation of the upper bound. The minimax lower bound is trivially \( \left( \frac{\sigma^2}{n} \right)^{\frac{2}{3}} \) (respectively, \( \left( \frac{\sigma^2}{n} \right)^{\frac{1}{3}} \)). Therefore, in what follows, we assume \( \gamma > 1 \).

If \( \frac{n}{\sigma^2} > (\gamma \lor 1)^{2\gamma + 1} \), we have \( \log N_{KL}(\epsilon, Q_U) \gtrsim \left( \frac{\sqrt{n}}{\sigma^2} \right)^{\frac{1}{2(\gamma + 1)}} \) for any \( \gamma \in \{0, \ldots, \beta\} \) by choosing a sufficiently small \( \epsilon \). Setting \( \left( \frac{\sqrt{n}}{\sigma^2} \right)^{\frac{1}{2(\gamma + 1)}} \asymp \varepsilon^2 \) yields \( \varepsilon^2 \asymp \left( \frac{n}{\sigma^2} \right)^{\frac{1}{2(\gamma + 1)^2 + 1}} := \varepsilon^2 \). Whenever
\[
\delta \lesssim 2^{-\gamma},
\]
we have
\[
T_1(\delta) - T_2 \gtrsim \gamma^2.
\]
Let us consider \( \delta^* \asymp \frac{\sigma^2 \gamma \log \gamma}{n} \). Observe that \( \delta^* \asymp 2^{-\gamma} \) if
\[
\frac{n}{\sigma^2} \gtrsim 2^{2\gamma - \gamma \log \gamma}
\]
and hence
\[
T_1(\delta^*) - T_2 \gtrsim \gamma \log \gamma \asymp (\gamma + 1) \log (\gamma \lor 2) \quad \text{since} \quad \gamma^2 \geq \gamma \log \gamma.
\]
In view of \([14]\), if
\[
(\gamma + 1) \log (\gamma \lor 2) \gtrsim \varepsilon^2 \asymp \left( \frac{n}{\sigma^2} \right)^{\frac{1}{2(\gamma + 1)^2 + 1}},
\]
which is equivalent to
\[
\frac{n}{\sigma^2} \gtrsim ((\gamma + 1) \log (\gamma \lor 2))^{2(\gamma + 1)^2},
\]
then we obtain
\[
1 - \frac{\log 2 + \log N_{KL}(\epsilon^*, Q_U) + \varepsilon^2}{\log M_2(\delta^*, U_{\gamma + 1})} \geq \frac{1}{2}.
\]
Therefore,
\[
\inf_{f} \sup_{f \in U_{\gamma + 1}} \mathbb{E} \left( \left| \tilde{f} - f \right|^2 \right) \gtrsim \frac{\sigma^2 (\gamma + 1) \log (\gamma \lor 2)}{n}. \tag{46}
\]

5.7 Proof for Theorem 3.1

In the special case of \( R_{\gamma + 1} = 1 \), the argument below sharpens the upper bound for \( \log N_2(\delta, H_{\gamma + 1}) \) in \([19]\) from \((\gamma \lor 1)^{-\frac{1}{1+\gamma}} \) to \((\gamma \lor 1)^{-\frac{1}{1+\gamma}} \). We find the cause of the gap lies in that the “pivotal” eigenvalue (that balances the “estimation error” and the “approximation error” from truncating for a given resolution \( \delta \)) in \([19]\) is not optimal. We close the gap by finding the optimal “pivotal” eigenvalue.
More generally, for the case of $R_{\gamma+1} \leq \gamma + 1$, we consider two different truncations, one giving the upper bound $\delta \to \frac{1}{\gamma + 1}$ and the other giving the lower bound $(R_{\gamma+1} \delta^{-1})^{\frac{1}{\gamma + 1}}$. Note that $(R_{\gamma+1} \delta^{-1})^{\frac{1}{\gamma + 1}} \propto \delta^{\frac{1}{\gamma + 1}}$ when $R_{\gamma+1} \propto 1$. For the case of $R_{\gamma+1} \geq \gamma + 1$, we use only one truncation to show that both the upper bound and the lower bound scale as $(R_{\gamma+1} \delta^{-1})^{\frac{1}{\gamma + 1}}$.

In view of (3), given $(\phi_m)_{m=1}^\infty$ and $(\mu_m)_{m=1}^\infty$, we compute $N_2(\delta, \mathcal{H}_{\gamma+1})$, it suffices to compute $N_2(\delta, \mathcal{E}_{\gamma+1})$ where

$$E_{\gamma+1} = \left\{ (\theta_m)_{m=1}^\infty : \sum_{m=1}^\infty \frac{\theta_m^2}{\mu_m} \leq R_{\gamma+1}^2, \mu_m = (cm)^{-(\gamma+1)} \right\}.$$ 

Let us introduce the $M$-dimensional ellipsoid

$$\mathcal{E}_{\gamma+1} = \left\{ (\theta_m)_{m=1}^M \text{ coincide with the first } M \text{ elements of } (\theta_m)_{m=1}^\infty \text{ in } E_{\gamma+1} \right\}$$

where $M (= M (\gamma + 1, \delta))$ is the smallest integer such that, for a given resolution $\delta > 0$ and weight $w_{\gamma+1}, w_{\gamma+1}^2 \delta^{2} \geq \mu_M$. In other words, $\mu_m \geq w_{\gamma+1}^2 \delta^{2}$ for all indices $m \leq M$. Consequently, we have:

(1) $B_2^M (w_{\gamma+1} R_{\gamma+1} \delta) \subseteq \mathcal{E}_{\gamma+1};$ \hspace{1cm} (47)

(2) $\mu_{M-1} = (c (M - 1))^{-2(\gamma+1)} > w_{\gamma+1}^2 \delta^2$ and $\mu_{M-1} = (c (M + 1))^{-2(\gamma+1)} < w_{\gamma+1}^2 \delta^2$, which yield

$$M \asymp \left(w_{\gamma+1} \delta\right)^{-\frac{1}{\gamma + 1}}.$$ \hspace{1cm} (48)

Note that (47), (48), and the fact $E_{\gamma+1} \supseteq \mathcal{E}_{\gamma+1}$ give

$$\log N_2(\delta, \mathcal{E}_{\gamma+1}) \geq \log N_2(\delta, \mathcal{E}_{\gamma+1}) \geq M \log \left(w_{\gamma+1} R_{\gamma+1}\right) \asymp \left(w_{\gamma+1} \delta\right)^{-\frac{1}{\gamma + 1}} \log \left(w_{\gamma+1} R_{\gamma+1}\right).$$ \hspace{1cm} (49)

In the following, let $A_1 + A_2 := \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$ for sets $A_1$ and $A_2$. For the upper bound, we have

$$N_2(\delta, \mathcal{E}_{\gamma+1}) \leq \frac{\text{vol} \left( \frac{2}{\delta} \mathcal{E}_{\gamma+1} \right)}{\text{vol} \left( B_2^M (1) \right)} \leq \left( \frac{2}{\delta} \right)^M \frac{\text{vol} \left( \mathcal{E}_{\gamma+1} \right)}{\text{vol} \left( B_2^M (1) \right)} \leq \left( \frac{2}{\delta} \right)^M \max \left\{ \frac{\text{vol} \left( 2 \mathcal{E}_{\gamma+1} \right)}{\text{vol} \left( B_2^M (1) \right)}, \frac{\text{vol} \left( 2 B_2^M \left( \frac{\delta}{2} \right) \right)}{\text{vol} \left( B_2^M (1) \right)} \right\} \leq \max \left\{ \left( \frac{4 R_{\gamma+1}}{\delta} \right)^M \prod_{m=1}^M \sqrt{\mu_m}, 2^M \right\}$$ \hspace{1cm} (50)

where the first inequality follows from the standard volumetric argument, and the last inequality follows from the standard result for the volume of ellipsoids. The fact $\mu_m = (cm)^{-(\gamma+1)}$ and the
elementary inequality \( \sum_{m=1}^{M} \log m \geq M \log M - M \) give

\[
\log \left( \frac{4R_{γ+1}}{δ} \right)^{M} \prod_{m=1}^{M} \sqrt{μ_m} \leq M \left( \log (4R_{γ+1}) + γ + 1 \right) + \\\nM \left( \log \frac{1}{δ} - (γ + 1) \log (cM) \right) = M \left( \log (4R_{γ+1}) + γ + 1 \right) + M \left( \log \frac{1}{δ} - (γ + 1) \log (cM) + \log \frac{1}{w_{γ+1}} - \log \frac{1}{w_{γ+1}} \right) \leq M \log (w_{γ+1} + 1) + M \log w_{γ+1} \\\n≤ M \log (w_{γ+1} + 1) + M \log (w_{γ+1} + 1) \leq M \log (w_{γ+1} + 1) + M \log (w_{γ+1} + 1) + M \log \log (4R_{γ+1}) + γ + 1 + \log (cM) \qquad (51)
\]

where we have used the fact \( μ_M = (cM)^{-2(γ+1)} \leq w_{γ+1}^2 δ^2 \) in the second inequality. Inequalities (48), (50) and (51) together yield

\[
\log N_2 (δ, E_{γ+1}) \overset{\theta_{γ+1}}{\geq} \log \left( \frac{1}{w_{γ+1}} \right)^{γ+1} \max \{ \log (w_{γ+1} + 1) \}, \log 2 \}.
\]

For any \( θ \in E_{γ+1} \), note that for a given \( δ \), we have

\[
\sum_{m=M+1}^{∞} θ_m^2 \leq μ_M \sum_{m=M+1}^{∞} \frac{θ_m^2}{μ_m} \leq w_{γ+1}^2 R_{γ+1}^2 δ^2.
\]

To cover \( E_{γ+1} \) within \( \left( 1 + w_{γ+1}^2 R_{γ+1}^2 \right)^{γ+1} δ \)-precision, we find a smallest \( δ \)-cover of \( E_{γ+1} \), \( \{ θ^1, ..., θ^N \} \), such that for any \( θ \in E_{γ+1} \), there exists some \( i \) from the covering set with

\[
|θ - θ|^2 \leq \sum_{m=1}^{M} \left( θ_m - θ_m^i \right)^2 + w_{γ+1}^2 R_{γ+1}^2 δ^2 \leq \left( 1 + w_{γ+1}^2 R_{γ+1}^2 \right)^{γ+1} \delta^2
\]

where we have used (52). Consequently, we have

\[
\log N_2 (δ, E_{γ+1}) \overset{θ_{γ+1}}{\geq} \log \left( \frac{1}{w_{γ+1}} \right)^{γ+1} \max \{ \log (w_{γ+1} + 1) \}, \log 2 \}.
\]

**Case 1:** \( R_{γ+1} \overset{γ+1}{\geq} \). Setting \( w_{γ+1} ≃ R_{γ+1}^{-1} \) in (49) and (53) solves

\[
\left( w_{γ+1} δ \left( 1 + w_{γ+1}^2 R_{γ+1}^2 \right)^{γ+1} \right)^{γ+1} \max \{ \log (w_{γ+1} + 1) \}, \log 2 \}
\]

where we have used (52). Consequently, we have

\[
\log N_2 (δ, E_{γ+1}) \overset{θ_{γ+1}}{\geq} \left( R_{γ+1} δ^{-1} \right)^{γ+1}.
\]

**Case 2:** \( R_{γ+1} \overset{γ+1}{\geq} \). Setting \( w_{γ+1} ≃ (γ + 1)^{-1} \) in (53) gives

\[
\log N_2 (δ, E_{γ+1}) \overset{θ_{γ+1}}{\geq} \delta^{γ+1}.
\]
Note that the lower bound obtained by setting $w_{\gamma+1} \asymp (\gamma + 1)^{-1}$ in (49) is not particularly useful. Instead, we consider a different truncation with $w_{\gamma+1} \asymp R_{\gamma+1}^{-1}$. Then (49) with $w_{\gamma+1} \asymp R_{\gamma+1}^{-1}$ gives

$$\log N_2 (\delta, \mathcal{E}_{\gamma+1}) \gtrsim R_{\gamma+1}^\frac{1}{\gamma+1} \delta^{-1}.$$

### 5.8 Proof for Theorem 3.2

#### The upper bound.

For the upper bound associated with $\mathcal{H}_{\gamma+1}$, we use the approach in [7] to bound the local complexity. Then the derivation boils down to solving for $\tilde{r}$ such that

$$\sqrt{\frac{1}{n}} \sum_{m=1}^\infty (\tilde{r}^2 \wedge \mu_m) \lesssim \frac{R_{\gamma+1} \tilde{r}^2}{\sigma} \text{ where } \mu_m = (cm)^{-2(\gamma+1)}.$$ As a result, we obtain

$$r^2 = R_{\gamma+1}^2 r^2 \asymp R_{\gamma+1}^\frac{1}{2(\gamma+1)+1} \left( \frac{\sigma^2}{n} \right) \left( \frac{\gamma+1}{2(\gamma+1)+1} \right)$$

and the claim in Theorem 3.2 follows from [19].

#### The lower bound.

In this derivation, we apply the results in Theorem 3.1. Setting

$$\left( \frac{R_{\gamma+1} \sqrt{n} \sigma \epsilon}{\epsilon^2} \right) \gtrsim (\gamma+1)^{-1}$$

yields $\epsilon^2 \asymp \left( \frac{n R_{\gamma+1}^2}{\sigma^2} \right)^{\frac{1}{2(\gamma+1)+1}} =: \epsilon^*$. Observe that setting

$$\delta \asymp R_{\gamma+1}^\frac{1}{2(\gamma+1)+1} \left( \frac{\sigma^2}{n} \right) \left( \frac{\gamma+1}{2(\gamma+1)+1} \right)$$

ensures

$$(R_{\gamma+1}^\frac{1}{\gamma+1}) \gtrsim R_{\gamma+1}^\frac{2}{\gamma+3} \left( \frac{n}{\sigma^2} \right)^\frac{1}{2(\gamma+1)+1} \asymp \epsilon^*.$$ Consequently, we have

$$1 - \frac{\log 2 + \log N_{KL}(\epsilon^*, \mathcal{Q}_H) + \epsilon^2}{\log M_2 (\delta^*, \mathcal{H}_{\gamma+1})} \gtrsim \frac{1}{2}$$

and

$$\inf \sup_{f \in \mathcal{H}_{\gamma+1}} \mathbb{E} \left( \left| \tilde{f} - f \right|^2 \right) \gtrsim R_{\gamma+1}^\frac{2}{\gamma+1+1} \left( \frac{\sigma^2}{n} \right) \left( \frac{\gamma+1}{2(\gamma+1)+1} \right).$$

### 5.9 Proof for Lemma 4.1

Like in Section [5.2], the proper choice of the grid of points on each dimension of $[-1, 1]^d$ is the key in this case. Any function $f \in \mathcal{U}_{\gamma+1,2}$ can be written as

$$f(x + \Delta) = \sum_{k=0}^\gamma \sum_{p; P=k} \frac{\Delta^p D^p f(x)}{k!} + \sum_{p; P'=\gamma} \left[ \frac{\Delta^p D^p f(z)}{\gamma!} - \frac{\Delta^p D^p f(x)}{\gamma!} \right] \quad := REM_0(x+\Delta)$$
where \( x, x + \Delta \in (-1, 1)^d \) and \( z \) is some intermediate value. For a given \( k \in \{0, ..., \gamma\} \), recall 
\[
\text{card}\left(\{p : P = k\}\right) = \binom{d + k - 1}{d - 1} = D_k^*.
\]
Therefore, we have
\[
|REM_0(x + \Delta)| \leq \frac{D_*^* R_{\gamma + 1} |\Delta|_{\gamma + 1}^{\gamma + 1}}{\gamma!}.
\]

In a similar way, writing
\[
D^\delta f(x + \Delta) = \sum_{k=0}^{\gamma - \tilde{P}} \sum_{p : P = k} \frac{\Delta^p D^p \tilde{p} f(x)}{k!} + \sum_{p : P = \gamma - \tilde{P}} \left[ \frac{\Delta^p D^p \tilde{p} f(\tilde{z})}{(\gamma - \tilde{P})!} - \frac{\Delta^p D^p \tilde{p} f(x)}{(\gamma - \tilde{P})!} \right]
\]
for \( 1 \leq \tilde{P} := \sum_{j=1}^d \tilde{p}_j \leq \gamma \) and \( \tilde{p} = (\tilde{p}_j)_{j=1}^d \), we have
\[
|REM_{\tilde{P}}(x + \Delta)| \leq \frac{D_*^* P_{\gamma - \tilde{P} + 1} |\Delta|_{\gamma + 1}^{\gamma + 1}}{(\gamma - \tilde{P})!}.
\]

For some \( \delta_0, \ldots, \delta_\gamma > 0 \), suppose that \( |D^p f(w) - D^p g(w)| \leq \delta_k \) for all \( p \) with \( P = k \in \{0, \ldots, \gamma\} \), where \( f, g \in U_{\gamma + 1, 2}^d \). Then we have
\[
|f(x + \Delta) - g(x + \Delta)| \\
\leq \sum_{k=0}^{\gamma} \sum_{p : P = k} \frac{\Delta^p}{k!} (D^p f(x) - D^p g(x)) + 2 \frac{D_*^* R_{\gamma + 1} |\Delta|_{\gamma + 1}^{\gamma + 1}}{\gamma!} \\
\leq \sum_{k=0}^{\gamma} \frac{D_*^* |\Delta|_{\gamma + 1} k! \delta_k}{k!} + 2 \frac{D_*^* R_{\gamma + 1} |\Delta|_{\gamma + 1}^{\gamma + 1}}{\gamma!}.
\]

Let \( \left(\max_{k \in \{1, \ldots, \gamma + 1\}} \frac{D_*^* R_k}{(k - 1)!} + 1\right) =: R^* \). Consider \( |\Delta|_{\gamma + 1} \leq d^{-1} (R^{* - 1} \delta)_{\gamma + 1}^{\gamma + 1} \) and \( \delta_k = R_*^* \gamma + 1 \delta^{1 - \gamma + 1} \) for \( k = 0, \ldots, \gamma \) and \( \delta \) such that \( \delta_k \in (0, 1) \). Then,
\[
|f(x + \Delta) - g(x - \Delta)| \leq \delta \sum_{k=0}^{\gamma} \left( R_*^* \gamma + 1 \delta \frac{1}{k!} \right) + 2 R^* |\Delta|_{\gamma + 1}^{\gamma + 1} \\
\leq \delta \sum_{k=0}^{\gamma} \frac{1}{k!} + 2 \delta \leq 5 \delta
\]
(58)
where we have used the fact that \( D_*^* \leq d^k \). On each dimension of \([-1, 1]^d\), we consider a \( d^{-1} (R^{* - 1} \delta)_{\gamma + 1}^{\gamma + 1} \)-grid of points. The rest of the arguments follow closely those in [3].
5.10 Proof for Lemma 4.2

For a given \( k \in \{0, ..., \gamma \} \), let \( \text{card} (\{p : P = k\}) = \binom{d + k - 1}{d - 1} = \binom{d + k - 1}{k} = D_k^* \). Recall the definition of \( \mathcal{U}_{\gamma+1,1}^d \):

\[
\mathcal{U}_{\gamma+1,1}^d = \left\{ f = \sum_{k=0}^{\gamma} \sum_{p : P = k} x^p \theta_{(p,k)} : \theta_{(p,k)} \in \mathcal{P}_\Gamma, x \in [-1, 1]^d \right\}
\]

with the \( \Gamma := \sum_{k=0}^{\gamma} D_k^* \)-dimensional polyhedron

\[
\mathcal{P}_\Gamma = \left\{ \{\theta_{(p,k)}\}_{(p,k)} \in \mathbb{R}^\Gamma : \text{for any given } k, \{\theta_{(p,k)}\}_p \in \left\{ \frac{-R_k}{k!}, \frac{R_k}{k!} \right\} \right\}
\]

where \( \theta = \{\theta_{(p,k)}\}_{(p,k)} \) denotes the collection of \( \theta_{(p,k)} \) over all \((p,k)\) configurations and \( \{\theta_{(p,k)}\}_p \) denotes the collection of \( \theta_{(p,k)} \) over all \( p \) configurations for a given \( k \in \{0, ..., \gamma\} \).

To bound \( \log N_\infty (\delta, \mathcal{U}_{\gamma+1,1}^d) \) from above, note that for \( f, f' \in \mathcal{U}_{\gamma+1,1}^d \), we have

\[
\left| f - f' \right|_\infty \leq \sum_{k=0}^{\gamma} \sum_{p : P = k} \left| \theta_{(p,k)} - \theta'_{(p,k)} \right|
\]

where \( f' = \sum_{k=0}^{\gamma} \sum_{p : P = k} x^p \theta'_{(p,k)} \) such that \( \theta' = \{\theta'_{(p,k)}\}_{(p,k)} \in \mathcal{P}_\Gamma \). Therefore, the problem is reduced to finding \( N_1 (\delta, \mathcal{P}_\Gamma) \).

To cover \( \mathcal{P}_\Gamma \) within \( \delta \)-precision, using arguments similar to those in Section 5.1, we find a smallest \( \frac{\delta}{\left(\gamma+1\right)D_k^*} \)-cover of \( \left\{ \frac{-R_k}{k!}, \frac{R_k}{k!} \right\} \) for each \( k = 0, ..., \gamma \), \( \left\{ \theta^1_k, ..., \theta^{N_k}_k \right\} \), such that for any \( \theta \in \mathcal{P}_\Gamma \), there exists some \( i_{(p,k)} \in \{1, ..., N_k\} \) with

\[
\sum_{k=0}^{\gamma} \sum_{p : P = k} \left| \theta_{(p,k)} - \theta^{i_{(p,k)}}_{k} \right| \leq \delta.
\]

As a consequence, we have

\[
\log N_1 (\delta, \mathcal{P}_\Gamma) \leq \sum_{k=0}^{\gamma} D_k^* \log \frac{4 \left( \gamma + 1 \right) D_k^* R_k}{\delta k!}
\]

and

\[
\log N_2 (\delta, \mathcal{U}_{\gamma+1,1}^d) \leq \log N_\infty (\delta, \mathcal{U}_{\gamma+1,1}^d) \leq \sum_{k=0}^{\gamma} D_k^* \log \frac{4 \left( \gamma + 1 \right) D_k^* R_k}{\delta k!}.
\]

If \( \delta \) is large enough such that \( \min_{k \in \{0, ..., \gamma\}} \log \frac{4 \left( \gamma + 1 \right) D_k^* R_k}{\delta k!} < 0 \), we use the counting argument in [4] to obtain

\[
\log N_2 (\delta, \mathcal{U}_{\gamma+1,1}^d) \leq \log N_\infty (\delta, \mathcal{U}_{\gamma+1,1}^d) \lesssim \left( \sum_{k=0}^{\gamma} D_k^* \right) \log \frac{1}{\delta} + \sum_{k=0}^{\gamma} D_k^* \log R_k.
\]

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5.11 Lemma A.1 and its proof

**Lemma A.1.** Let \( \{ \phi_k \}_{k=1}^{\infty} \) be the Legendre polynomials on \([-1, 1]\). For any \( f \in U_{\gamma+1,1} [-1, 1] \), we have \( f(x) = \sum_{k=0}^{\infty} \theta_k \phi_k(x) \) such that

\[
\tilde{\theta}_k = \left( k + \frac{1}{2} \right) \frac{\gamma}{2k+2m+1!} \frac{f(k+2m)(0)}{4^k} \frac{1}{(2k+2m+1)!} \frac{1}{2k+2m+1}
\]

where \((a)_k = a(a + 1) \cdots (a + k - 1)\) is known as the Pochhammer symbol.

**Proof.** To obtain the correct formula for finite sums, we carefully modify the derivations in [2] which concerns infinite sums. The Legendre expansion of \( x^k \) yields

\[
\frac{x^k}{k!} = \frac{1}{2k} \sum_{m=0}^{\gamma/2} \frac{k - 2m + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{k-m+1}} \phi_{k-2m}(x). \quad (59)
\]

First, let us consider the case where \( \gamma \) is odd. Applying (59) gives

\[
f(x) = \sum_{k=0}^{\gamma} \frac{f(k)(0)}{2^k} \sum_{m=0}^{\gamma/2} \frac{k - 2m + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{k-m+1}} \phi_{k-2m}(x)
\]

\[
= \sum_{k=0}^{\gamma/2} \frac{f(2k)(0)}{2^{2k}} \sum_{m=0}^{k} \frac{2k - 2m + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{2k-m+1}} \phi_{2k-2m}(x) \quad \text{(even } k) \]

\[
+ \sum_{k=0}^{\gamma/2} \frac{f(2k+1)(0)}{2^{2k+1}} \sum_{m=0}^{k} \frac{2k - 2m + \frac{3}{2}}{m! \left( \frac{1}{2} \right)_{2k-m+2}} \phi_{2k-2m+1}(x) \quad \text{(odd } k)
\]

\[
= \sum_{m=0}^{\gamma/2} \sum_{k=m}^{\gamma/2} \frac{2k - 2m + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{2k-m+1}} \frac{f(2k)(0)}{2^{2k}} \phi_{2k-2m}(x) \quad \text{(interchanging sums)}
\]

\[
+ \sum_{m=0}^{\gamma/2} \sum_{k=m}^{\gamma/2} \frac{2k - 2m + \frac{3}{2}}{m! \left( \frac{1}{2} \right)_{2k-m+2}} \frac{f(2k+1)(0)}{2^{2k+1}} \phi_{2k-2m+1}(x)
\]

\[
= \sum_{m=0}^{\gamma/2} \sum_{l=0}^{\gamma/2} \frac{2l + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{2l+m+1}} \frac{f(2l)(0)}{2^{2l+2m}} \phi_{2l}(x) \quad \text{(letting } l = k - m)
\]

\[
+ \sum_{m=0}^{\gamma/2} \sum_{l=0}^{\gamma/2} \frac{2l + \frac{3}{2}}{m! \left( \frac{1}{2} \right)_{2l+m+2}} \frac{f(2l+1)(0)}{2^{2l+2m+1}} \phi_{2l+1}(x)
\]

\[
= \sum_{l=0}^{\gamma/2} \sum_{m=0}^{\gamma/2} \frac{2l + \frac{1}{2}}{m! \left( \frac{1}{2} \right)_{2l+m+1}} \frac{f(2l+2m)(0)}{2^{2l+2m}} \phi_{2l}(x) \quad \text{(interchanging sums)}
\]

\[
+ \sum_{l=0}^{\gamma/2} \sum_{m=0}^{\gamma/2} \frac{2l + \frac{3}{2}}{m! \left( \frac{1}{2} \right)_{2l+m+2}} \frac{f(2l+2m+1)(0)}{2^{2l+2m+1}} \phi_{2l+1}(x)
\]

which gives the claim in Lemma A.1.
For the case of even $\gamma$, note that the term in (60) takes the form

$$\sum_{k=0}^{\lfloor \gamma/2 \rfloor} \frac{f^{(2k+1)}(0)}{2^{2k+1}} \sum_{m=0}^{k} \frac{2k - 2m + \frac{3}{2}}{m! \left( \frac{1}{2} \right)^{2k-m+2}} \phi_{2k-2m+1}(x)$$

and hence the previous derivations go through.

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