NEW GENERAL DECAY RESULT FOR A SYSTEM OF VISCOPROLASTIC WAVE EQUATIONS WITH PAST HISTORY

ADEL M. AL-MAHDI* AND MOHAMMAD M. AL-GHARABLI

The Preparatory Year Program, King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

SALIM A. MESSAoudI

Department of Mathematics, University of Sharjah
P. O. Box, 27272, Sharjah, UAE

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Abstract. This work is concerned with a coupled system of viscoelastic wave equations in the presence of infinite-memory terms. We show that the stability of the system holds for a much larger class of kernels. More precisely, we consider the kernels $g_i : [0, +\infty) \to (0, +\infty)$ satisfying
$$g'_i(t) \leq -\xi_i(t) H_i(g_i(t)), \quad \forall \, t \geq 0$$
for $i = 1, 2$, where $\xi_i$ and $H_i$ are functions satisfying some specific properties. Under this very general assumption on the behavior of $g_i$ at infinity, we establish a relation between the decay rate of the solutions and the growth of $g_i$ at infinity. This work generalizes and improves earlier results in the literature. Moreover, we drop the boundedness assumptions on the history data, usually made in the literature.

1. Introduction. In this work, we consider the following coupled system of viscoelastic wave equations:

\[
\begin{align*}
  u_{tt} - \Delta u + \int_0^{+\infty} g_1(s) \Delta u(\cdot, t-s) ds + f_1(u, v) &= 0, & \text{in } \Omega \times (0, +\infty), \\
  v_{tt} - \Delta v + \int_0^{+\infty} g_2(s) \Delta v(\cdot, t-s) ds + f_2(u, v) &= 0, & \text{in } \Omega \times (0, +\infty), \\
  u = v &= 0, & \text{on } \partial \Omega \times (0, +\infty), \\
  u(\cdot, -t) = u_0(t), u_t(\cdot, 0) = u_1, v(\cdot, -t) = v_0(t), v_t(\cdot, 0) = v_1, & \text{in } \Omega,
\end{align*}
\]

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$, $u_0, u_1, v_0, v_1$ are given initial data, $g_1$ and $g_2$ are the relaxation functions and $f_1$ and $f_2$ are nonlinear functions describing the interaction between the two waves. The unknowns $u$ and $v$ represent the displacements of waves. This system can be considered as a

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*Corresponding author.
generalization of the well-known Klein-Gordon system that appears in the quantum field theory [3, 29],
\[
\begin{align*}
    u_{tt} - \Delta u + m_1 u + k_1 u v^2 &= 0 \\
    v_{tt} - \Delta v + m_2 v + k_2 u^2 v &= 0.
\end{align*}
\]
Since the pioneer works of Dafermos [8, 9] in 1970, where the general decay was discussed, problems related to viscoelasticity have attracted a great deal of attention and many results of existence and long-time behavior have been established. The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastic industry.

**Viscoelastic problems with finite memory:** Dafermos [8, 9] discussed a one-dimensional viscoelastic equation and proved the well-posedness of the problem and showed that the solutions decay asymptotically to zero. However, no rate of decay has been specified. Hrusa [15] considered a one-dimensional nonlinear viscoelastic equation of the form
\[
\begin{align*}
    u_{tt} - c u_{xx} + \int_0^t g(t-s) \left( \psi(u_x(x,s)) \right)_x ds &= f(x,t), \quad \text{in } (0, 1) \times (0, \infty), \\
    u(0, t) = u(1, t) = 0, \quad t \geq 0, \\
    u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [0, 1],
\end{align*}
\] (1.1)
and proved several global existence results for large data and an exponential decay result for strong solutions when \( m(s) = e^{-s} \) and \( \psi \) satisfies certain conditions.

For multi-dimensional viscoelastic problems, we refer to [24, 23]. Up to the year 2008, most of the studies of viscoelastic problems were concerned with relaxation functions satisfying
\[
g'(t) \leq -\xi g^p(t), \quad \forall t \geq 0, \tag{1.2}
\]
where \( \xi > 0 \) is a constant and \( 1 \leq p < \frac{3}{2} \), which, in turn, yielded either uniform or polynomial decay. In 2008, Messaoudi [17] considered a viscoelastic wave equation and proved a general decay rate for a relaxation functions, \( g \), satisfying
\[
g'(t) \leq -\xi(t) g(t), \quad \forall t \geq 0, \tag{1.3}
\]
where \( \xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a non-increasing differentiable function. In fact the exponential and polynomial decay rates are only special cases. Using the idea of [17], a series of papers have appeared. For instance, Han and Wang [14], Liu [16] and references therein. Now, there is a natural question that raised in dealing with the general decay of viscoelastic-type systems:

**Q.** What about a more general class of relaxation functions satisfying
\[
g'(t) \leq -\xi(t) H(g(t)), \quad \forall t \geq 0? \tag{1.4}
\]
In 2016, Messaoudi and Al-Khulafi [20] proved a general and optimal decay rate of the solution of a density dependent viscoelastic problem for a class of relaxation functions, satisfying
\[
g'(t) \leq -\xi(t) g^p(t), \quad \forall t \geq 0, \quad 1 \leq p < \frac{3}{2}. \tag{1.4}
\]
This result answered Q with \( \xi \) being a nonincreasing differentiable function and \( H(s) = s^p \), for \( 1 \leq p < \frac{3}{2} \). In 2017, Mustafa [26] gave a complete answer to Q by assuming that \( H \) is either linear or strictly increasing and strictly convex
function on \((0, r]\), for \(r \leq g(0)\) and \(\xi\) is a positive nonincreasing differentiable function. Farida et al. [5] considered the following problem

\[
\begin{aligned}
& u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m-2}u_t = 0, & \text{in} \Omega \times (0, +\infty) \\
& u(x, t) = 0, & \text{on} \partial \Omega \times (0, +\infty) \\
& u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & \text{in} \Omega \times (0, +\infty),
\end{aligned}
\]  

(1.5)

and established the existence of the solution and proved an explicit and general decay rate results, using the multiplier method and some properties of the convex functions. For viscoelastic systems, Andrade and Mognon [3] treated the following decay rate results, using the multiplier method and some properties of the convex functions.

In [28], Santos considered (1.6) with

\[
\begin{aligned}
& g_i(s) > 0, g_i'' \in L^1(0, \infty) & \text{for} \ i = 1, 2
\end{aligned}
\]

and for some positive constants \(\alpha\) and \(\beta\)

\[
-\alpha g_i(t) \leq g_i'(t) \leq -\beta g_i(t).
\]

In [28], Santos considered (1.6) with

\[
\begin{aligned}
& f_1(u, v) = |u|^{p-2}u|v|^p & \text{and} & f_2(u, v) = |v|^{p-2}v(t)|u|^p,
\end{aligned}
\]

where \(p > 1\) if \(N = 1, 2\) and \(1 < p \leq 2\) if \(N = 3\). They proved the well posedness under the following assumptions on the relaxation functions

\[
1 - \int_0^\infty g_i(s)\,ds > 0, \quad g_i'' \in L^1(0, \infty) \quad \text{for} \ i = 1, 2
\]

and for some positive constants \(\alpha\) and \(\beta\)

\[
-\alpha g_i(t) \leq g_i'(t) \leq -\beta g_i(t).
\]

In [25], Mustafa investigated the well-posedness and the asymptotic behavior of the system (1.6) for relaxation functions satisfying

\[
\begin{aligned}
& g_i'(t) \leq -\xi_1(t)g_1(t), \quad g_2'(t) \leq -\xi_2(t)g_2(t),
\end{aligned}
\]

(1.7)

and for more general forms of \(f_1\) and \(f_2\) and established a general decay result depending on \(g_1\) and \(g_2\). Recently, Al-Gharabli and Kafini [1] established a general decay result for (1.6) with the relaxation functions \(g_i'(s)\) satisfying

\[
\begin{aligned}
& g_i'(t) \leq -H_i(g_i(t)), & \forall t \geq 0, & \quad i = 1, 2
\end{aligned}
\]

(1.8)

with \(H_i : [0, \infty) \rightarrow [0, \infty)\) with \(H_i(0) = 0\) and each \(H_i\) is linear or strictly increasing and strictly convex \(C^2\) function on \((0, r]\) for some \(r > 0\). This latter result allowed a larger class of relaxation functions and generalizes, in some cases, those in [22, 25, 28]. Very recently, Messaoudi and Hassan [21] considered (1.6) with relaxation functions satisfying

\[
\begin{aligned}
& g_i'(t) \leq -\xi(t)H_i(g_i(t)), & \forall t \geq 0
\end{aligned}
\]

(1.9)

and proved a new general decay result that improves most of the existing results in the literature related to the system of viscoelastic wave equations with finite memory.
Viscoelastic problems with infinite memory: Giorgi et al. [10] considered the following semilinear hyperbolic equation in a bounded domain \( \Omega \subset \mathbb{R}^3 \)

\[
  u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+
\]

with \( K(0), K(\infty) > 0 \) and \( K' \leq 0 \) and gave the existence of global attractors for the solutions. Conti and Pata [7] considered the following semilinear hyperbolic equation in a bounded domain \( \Omega \subset \mathbb{R}^n \),

\[
  u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.9}
\]

and proved the existence of a regular global attractor under some conditions on \( K \) and \( g \). In [11], Guesmia considered the following abstract equation

\[
u'' + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0,
\]

where \( A \) and \( B \) are two positive self-adjoint operator satisfying some conditions, and established a more general decay result for a the problem. Messaoudi and Al-Gharabli [19] considered the following nonlinear wave equation

\[|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} g(s)\Delta u(t-s)ds = 0, \quad \text{in } \Omega \times (0, +\infty)\]

and proved a general decay result of the solution energy using an approach different from that introduced by Guesmia [11]. The same authors [18] considered the following system

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds + |u_t|^{\alpha-1}u_t = f(u,v), \quad \text{in } \Omega, t \in \mathbb{R}^+,
  \\
  v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)ds + |v_t|^{\beta-1}v_t = k(u,v), \quad \text{in } \Omega, t \in \mathbb{R}^+,
\end{cases}
\tag{1.10}
\]

under relaxation functions satisfying (1.7) and established a general stability result. Recently, Al-Mahdi and Al-Gharabli [2] considered the following viscoelastic problem:

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + |u_t|^{\alpha-1}u_t = 0, \quad \text{in } \Omega \times (0, +\infty)\\
  u(x, t) = 0, \quad \text{on } \partial \Omega \times (0, +\infty)\\
  u(x, -t) = u_0(x, t), u_t(x, 0) = u_1(x), \quad \text{in } \Omega \times (0, +\infty),
\end{cases}
\tag{1.11}
\]

with a relaxation function \( g \), satisfying \( g'(t) \leq -\xi(t)g^p(t), \ t > 0 \ 1 \leq p < \frac{3}{2} \). They establish general decay rate results, using the multiplier method and some properties of non-homogonies ordinary differential inequalities. Very recently, Guesmia [12] considered two models of wave equations with infinite memory and established general relations between the decay rate of solutions and the growth of \( g \) at infinity. The aim of this work is to investigate problem (P) with the general class of relaxation functions \( g_1 \) and \( g_2 \) in the presence of infinite memory terms and use the idea developed by Mustafa in [26, 21, 12] taking into consideration the difficulty coming from replacing the finite memory with infinite one, to prove a new general decay result. Our result generalizes and improves all the existing results related to system of viscoelastic equations with infinite memories. This paper is organized as follows: in Section 2, we state some preliminary results and our main result. In Section 3, we state and prove some technical lemmas needed for the entire work. We give the proof of our main result and some comments in Section 4.
2. Preliminaries. In this section, we give our assumptions, state the existence theorem and present some useful lemmas. We use $c$ to denote a positive generic constant. **Assumptions:** We assume the following hypotheses:

(A.1) $g_i : [0, +\infty) \to (0, +\infty)$ (for $i = 1, 2$) are non-increasing differentiable functions such that

$$g_i(0) > 0, \quad 1 - \int_0^{+\infty} g_i(s)ds =: l_i > 0.$$  

(A.2) There exist non-increasing differentiable functions $\xi_i : [0, +\infty) \to (0, +\infty)$ and $C^1$ functions $H_i : [0, +\infty) \to [0, +\infty)$ which are linear or strictly increasing and strictly convex $C^2$ functions on $(0, r], r \leq g_i(0)$, with $H_i(0) = H'_i(0) = 0$ such that

$$g'_i(t) \leq -\xi_i(t)H_i(g_i(t)), \quad \forall \ t \geq 0 \quad \text{and for } i = 1, 2.$$  

(A.3) $f_i : R^2 \to R$ (for $i = 1, 2$) are $C^1$ functions with $f_i(0, 0) = 0$ and there exists a function $F$ such that

$$f_1(x, y) = \frac{\partial F}{\partial x}(x, y), \quad f_2(x, y) = \frac{\partial F}{\partial y}(x, y),$$

$$F \geq 0, \quad xf_1(x, y) + yf_2(x, y) - F(x, y) \geq 0,$$

and

$$\left| \frac{\partial f_i}{\partial x}(x, y) \right| + \left| \frac{\partial f_i}{\partial y}(x, y) \right| \leq d \left( 1 + |x|^{b_i - 1} + |y|^{b_i - 1} \right), \quad \forall (x, y) \in R^2,$$  

(2.1)  

Remark 1. (1) If $H$ is a strictly increasing and strictly convex $C^2$ function on $(0, r]$, with $H(0) = H'(0) = 0$, then it has an extension $\bar{H}$ which is a strictly increasing and strictly convex $C^2$—function on $(0, +\infty)$. For instance, we can define $\bar{H}$, for any $t > r$, by

$$\bar{H}(t) := \frac{H''(r)}{2}t^2 + (H'(r) - H''(r)r)t + \left( H(r) + \frac{H''(r)}{2}r^2 - H'(r)r \right).$$

(2) Inequality (2.1) yields, for some positive constant $k$, that

$$|f_i(x, y)| \leq k (|x| + |y| + |x|^{b_i} + |y|^{b_i})$$  

for all $(x, y) \in R^2$ and $i = 1, 2$.  

For completeness, we state, without proof, the global existence and regularity result whose proof can be found in [25].

**Theorem 2.1.** Let $(u_0(0), u_1), (v_0(0), v_1) \in H^1_0(\Omega) \times L^2(\Omega)$ be given. Assume that (A.1) and (A.3) hold. Then, problem (P) has a unique weak solution

$$(u, v) \in C([0, \infty); H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega)) \cap C^2([0, \infty); H^{-1}(\Omega)).$$

Moreover, if $(u_0(0), u_1), (v_0(0), v_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$, then problem (P) has a unique strong solution

$$(u, v) \in L^\infty([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{1, \infty}([0, \infty); H^1_0(\Omega)) \cap W^{2, \infty}([0, \infty); L^2(\Omega)).$$

Now, we introduce the energy functional

$$E(t) := \frac{1}{2} \left[ \|u_t\|^2_2 + \left( 1 - \int_0^{+\infty} g_i(s)ds \right) \|\nabla u\|^2_2 + (g_1 \circ \nabla u)(t) \right]$$

$$+ \frac{1}{2} \left[ \|v_t\|^2_2 + \left( 1 - \int_0^{+\infty} g_2(s)ds \right) \|\nabla v\|^2_2 + (g_2 \circ \nabla v)(t) \right] + \int_\Omega F(u, v)dx,$$  

(2.3)
where, for any \( w \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega)) \) and \( i = 1, 2, \)
\[
(g_i \circ w)(t) := \int_0^+ g_i(s)\|w(t) - w(t-s)\|_2^2 \, ds.
\]

**Lemma 2.2.** \([21]\) Let \((u, v)\) be the solution of \((P)\). Then,
\[
E'(t) = \frac{1}{2} (g_1' \circ \nabla u)(t) + \frac{1}{2} (g_2' \circ \nabla v)(t) \leq 0, \quad \forall t \geq 0. \tag{2.4}
\]

As in \([26]\), we set, for any \( 0 < a < 1 \) and \( i = 1, 2, \)
\[
C_{a,i} := \int_0^\infty \frac{g_i^2(s)}{ag_i(s) - g_i'(s)} \, ds \quad \text{and} \quad h_i(t) := ag_i(t) - g_i'(t).
\]

**Lemma 2.3** \([26]\). Assume that condition \( (A.1) \) holds. Then for any \( w \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega)) \), we have
\[
\int_\Omega \left( \int_0^+ g_i(s)(w(t) - w(t-s)) \, ds \right)^2 \, dx \leq C_{a,i}(h_i \circ w)(t), \quad \forall t \geq 0, \quad \text{for } i = 1, 2. \tag{2.5}
\]

**Lemma 2.4** (Jensen’s inequality). Let \( G : [a, b] \rightarrow R \) be a convex function. Assume that the functions \( f : \Omega \rightarrow [a, b] \) and \( h : \Omega \rightarrow R \) are integrable such that \( h(x) \geq 0, \) for any \( x \in \Omega \) and \( \int_\Omega h(x) \, dx = k > 0. \) Then,
\[
G \left( \frac{1}{k} \int_\Omega f(x)h(x) \, dx \right) \leq \frac{1}{k} \int_\Omega G(f(x))h(x) \, dx.
\]

3. **Technical Lemmas.** In this section, we state and prove some lemmas needed to establish our main result.

**Lemma 3.1.** As in \([12]\), there exist two positive constant \( M_1 \) and \( M_2 \) such that
\[
\int_t^\infty g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds \leq M_1 h_0(t),
\]
\[
\int_t^\infty g_2(s)\|\nabla v(t) - \nabla v(t-s)\|_2^2 \, ds \leq M_2 \overline{h}_0(t), \quad \tag{3.1}
\]

where
\[
h_0(t) = \int_0^t g_1(t + s) \left( 1 + \|\nabla u_0(s)\|^2 \right) \, ds
\]
and
\[
\overline{h}_0(t) = \int_0^t g_2(t + s) \left( 1 + \|\nabla v_0(s)\|^2 \right) \, ds.
\]

**Proof.**
\[
\int_t^\infty g_1(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds
\]
\[
\leq 2\|\nabla u(t)\|^2 \int_t^\infty g_1(s) \, ds + 2 \int_t^\infty g_1(s)\|\nabla u(t-s)\|^2 \, ds
\]
\[
\leq 2 \sup_{s \geq 0} \|\nabla u(s)\|^2 \int_0^\infty g_1(t + s) \, ds + 2 \int_0^\infty g_1(t + s)\|\nabla u(-s)\|^2 \, ds
\]
\[
\leq \left( \frac{4}{t_1} E(s) \right) \int_0^\infty g_1(t + s) \, ds + 2 \int_0^\infty g_1(t + s)\|\nabla u_0(s)\|^2 \, ds
\]
\[
\leq \left( \frac{4}{t_1} E(0) \right) \int_0^\infty g_1(t+s)ds + 2 \int_0^\infty g_1(t+s) \|
abla u_0(s)\|^2 ds \\
\leq M_1 \int_0^\infty g_1(t+s) \left( 1 + \|u_0(s)\|^2 \right) ds,
\]

(3.2)

where \( M_1 = \max\{2, (\frac{4}{t_1} E(0))\}. \)

Similarly, we can establish the second bound in the above lemma.

**Lemma 3.2.** [21] Assume that (A.1)–(A.3) hold. Then, the functional \( I \) defined by

\[
I(t) := \int_{\Omega} u_t dx + \int_{\Omega} v_t dx
\]

satisfies, along the solution of (P), the estimate

\[
I'(t) \leq \|u_t\|^2 - \frac{l_1}{2} \|
abla u\|^2 + \sigma_1(h_1 \circ \nabla u)(t) + \|v_t\|^2 \\
- \frac{l_2}{2} \|
abla v\|^2 + \sigma_1(h_2 \circ \nabla v)(t) - \int_{\Omega} F(u,v) dx.
\]

(3.3)

**Lemma 3.3.** [21] Assume that (A.1)–(A.3) hold. Then, the functional \( K \) defined by

\[
K(t) := K_1(t) + K_2(t), \quad K_1(t) := - \int_{\Omega} u_t \int_0^{+\infty} g_1(s)(u(t)-u(t-s))ds dx
\]

and

\[
K_2(t) := - \int_{\Omega} v_t \int_0^{+\infty} g_2(s)(v(t)-v(t-s))ds dx
\]

satisfies, along the solution of (P) and for any \( 0 < d_0 < 1 \), the estimate

\[
K'(t) \leq -(1 - \ell_1 - d_0) \|u_t\|^2 + \sigma \|
abla u\|^2 + \frac{c}{d_0} (C_{a_1} + 1)(h_1 \circ \nabla u)(t) \\
- (1 - \ell_2 - d_0) \|v_t\|^2 + \sigma \|
abla v\|^2 + \frac{c}{d_0} (C_{a_2} + 1)(h_2 \circ \nabla v)(t).
\]

(3.4)

**Lemma 3.4.** Assume that (A.1)–(A.3) hold. Then, the functionals \( J_1 \) and \( J_2 \) defined by

\[
J_1(t) := \int_{\Omega} \int_0^t \psi_1(t-s) \|
abla u(s)\|^2 ds dx
\]

and

\[
J_2(t) := \int_{\Omega} \int_0^t \psi_2(t-s) \|
abla v(s)\|^2 ds dx
\]

with \( \psi_i(t) := \int_t^\infty g_i(s)ds \) (for \( i = 1,2 \)) satisfy, along the solution of (P), the estimates

\[
J_1'(t) \leq 3(1 - l) \|
abla u\|^2 - \frac{1}{2}(g_1 \circ \nabla u)(t) \\
+ \frac{1}{2} \int_{\Omega} \int_0^{+\infty} g_1(s)(\nabla u(t) - \nabla u(t-s))ds dx
\]

(3.5)

and

\[
J_2'(t) \leq 3(1 - l) \|
abla v\|^2 - \frac{1}{2}(g_2 \circ \nabla v)(t)
\]
where \( l = \min\{l_1, l_2\} \).

**Proof.** It follows from the proof of Lemma 3.4 in [26] and Lemma 3.7 in [13]. \( \square \)

**Lemma 3.5.** The functional \( L \) defined by
\[
L(t) := NE(t) + N_1 I(t) + N_2 K(t)
\]
satisfies, for a suitable choice of \( N, N_1, N_2 \geq 1 \),
\[
L(t) \sim E(t)
\]
and the estimate
\[
L'(t) \leq -4(1 - l)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \left( \ell_0 N_2 - \frac{l}{4c} - N_1 \right) \left( \|u_t\|_2^2 + \|v_t\|_2^2 \right)
\]
\[
- c \int \Omega F(u, v) \, dx + \frac{1}{4} \left[ (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right], \quad \forall t \geq 0,
\]
where \( l = \min\{l_1, l_2\} \).

**Proof.** For the proof of (3.7), we refer to [6]. To prove (3.8), set
\[
\ell_0 = \min \{ 1 - \ell_2, 1 - \ell_2 \} > 0, \quad \delta = \frac{l}{4cN_2} \quad \text{and} \quad C_a = \max \{ C_{a, 1}, C_{a, 2} \}.
\]

Exploiting (3.3), (3.4) and recalling that \( g_i' = ag_i - h_i \), we obtain, for any \( t \geq 0 \),
\[
L'(t) \leq -\frac{l}{4} (2N_1 - 1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \left( \ell_0 N_2 - \frac{l}{4c} - N_1 \right) \left( \|u_t\|_2^2 + \|v_t\|_2^2 \right)
\]
\[
- \frac{1}{2} N_1 \int \Omega F(u, v) \, dx + \frac{a}{2} N \left( (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \right)
\]
\[
- \left[ \frac{1}{2} N - \frac{4c^2}{l} N_2^2 - C_a \left( \frac{4c^2}{l} N_2^2 + cN_1 \right) \right] \left[ (h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t) \right].
\]

We start by choosing \( N_1 \) large enough so that
\[
\frac{l}{4}(2N_1 - 1) > 4(1 - l),
\]
then we select \( N_2 \) so large that
\[
\ell_0 N_2 - \frac{l}{4c} - N_1 > 1.
\]

As \( \frac{ag_i^2(s)}{ag_i(s) - g_i'(s)} < g_i(s) \) for \( i = 1, 2 \), it follows from the Lebesgue Dominated Convergence Theorem that
\[
\lim_{a \to 0^+} ac_{a, i} = \lim_{a \to 0^+} \int_0^\infty \frac{ag_i^2(s)}{ag_i(s) - g_i'(s)} \, ds = 0 \quad \text{for} \quad i = 1, 2.
\]

This gives
\[
\lim_{a \to 0^+} ac_a = 0.
\]

Consequently, there exists \( a_0 \in (0, 1) \) such that if \( a < a_0 \), then
\[
aC_a < \frac{1}{8 \left[ \frac{4c^2}{l} N_2^2 + cN_1 \right]}.
\]
Now, we choose $N$ large enough so that
\[ N > \max\left\{ \frac{16c^2}{l}N_2^2, \frac{1}{2a_0} \right\} \]
and set
\[ a = \frac{1}{2N}. \]
Then
\[ \frac{1}{4}N - \frac{4c^2}{l}N_2^2 > 0 \quad \text{and} \quad a = \frac{1}{2N} < a_0. \]
These imply
\[ \frac{1}{2}N - \frac{4c^2}{l}N_2^2 - C_\alpha \left[ \frac{4c^2}{l}N_2^2 + cN_1 \right] > \frac{1}{2}N - \frac{4c^2}{l}N_2^2 - \frac{1}{8a} = \frac{1}{4}N - \frac{4c^2}{l}N_2^2 > 0. \]
Hence, we arrive at the required estimate.

4. Stability. In this section we state and prove our main result. We start with the following lemmas.

**Lemma 4.1.** Assume that (A.1) holds. Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$, the following estimate
\[ \int_0^t E(s)ds < \tilde{m}h_1(t), \quad (4.1) \]
where $h_1(t) = (1 + \int_0^t (h_0 + \tilde{h}_0)(s)ds)$ and $h_0$ and $\tilde{h}_0$ are defined in Lemma 3.1.

**Proof.** Let $F(t) = L(t) + J_1(t) + J_2(t)$, then using Lemmas 3.4 and 3.5 to conclude that, for any $t \geq 0$, and some $m > 0$
\[ F'(t) \leq - \frac{1}{2} \int_\Omega (\nabla u)(t) - \nabla u(t-s)ds^2 dx + \frac{1}{2} \int_\Omega \int_0^t g_1(s) (\nabla u(t-s)ds^2 dx) \]
\[ \leq - mE(t) + \frac{1}{2} \int_\Omega \int_0^t g_1(s) (\nabla u(t-s)ds^2 dx) \]
Therefore,
\[ m \int_0^t E(s)ds \leq F(0) - F(t) + \frac{M_1}{2} \int_0^t \int_0^{\infty} g_1(\tau + s) (1 + |\nabla u_0(s)|ds)^2 d\tau ds \]
\[ + \frac{M_2}{2} \int_0^t \int_0^{\infty} g_2(\tau + s) (1 + |\nabla v_0(s)|ds)^2 d\tau ds \]
\[ \leq F(0) + \left( \frac{M_1 + M_2}{2} \right) \int_0^t (h_0 + \tilde{h}_0)(s)ds. \quad (4.2) \]
Therefore, (4.1) is established with $\tilde{m} = \max\{ \frac{F(0)}{m}, \frac{M_1 + M_2}{2m} \}$. \qed
Corollary 1. Using (2.3) and (4.1), we have, for \( t = \min \{ t_1, t_2 \} \),
\[
\int_\Omega \int_0^t (\nabla u(t) - \nabla u(t-s))^2 \, ds \, dx + \int_\Omega \int_0^t (\nabla v(t) - \nabla v(t-s))^2 \, ds \, dx \\
\leq 2 \int_0^t \int_0^t (|\nabla u(t)|^2 + |\nabla u(t-s)|^2 + |\nabla v(t)|^2 + |\nabla v(t-s)|^2) \, ds \, dx
\]
\[
\leq \left( \frac{4}{1-t} \right) \int_0^t (E(t) + E(t-s)) \, ds \, dx
\]
\[
\leq \left( \frac{8}{1-t} \right) \int_0^t E(s) \, ds \, dx \leq \left( \frac{8}{1-t} \right) \tilde{m} h_1(t), \forall t \geq 0. \quad (4.3)
\]

Lemma 4.2. Under the assumptions (A1) and (A2), we have the following estimates
\[
\int_0^t g_1(s)||\nabla u(t) - \nabla u(t-s)||^2_2 ds \leq \frac{1}{q(t)} \bar{H}_{1}^{-1} \left( \frac{q(t)\mu_1(t)}{\xi_1(t)} \right), \quad \forall t \geq 0 \quad (4.4)
\]
and
\[
\int_0^t g_2(s)||\nabla v(t) - \nabla v(t-s)||^2_2 ds \leq \frac{1}{q(t)} \bar{H}_{2}^{-1} \left( \frac{q(t)\mu_2(t)}{\xi_2(t)} \right), \quad \forall t \geq 0, \quad (4.5)
\]
where \( H_i \) (for \( i = 1, 2 \)) are \( C^2 \) extension of \( H_i \) (\( i = 1, 2 \)),
\[
\mu_1(t) := - \int_0^t g'_1(s)||\nabla u(t) - \nabla u(t-s)||^2_2 ds \leq -cE'(t),
\]
\[
\mu_2(t) := - \int_0^t g'_2(s)||\nabla v(t) - \nabla v(t-s)||^2_2 ds \leq -cE'(t)
\]
\[
q(t) := \frac{q_0}{H_1(t)} < 1 \quad \text{and} \quad 0 < q_0 < \min \left\{ 1, \left( \frac{1-t}{8m} \right) \right\}. \quad (4.6)
\]

Proof. We start by defining the functionals \( \eta_i \) (for \( i = 1, 2 \)) by
\[
\eta_1(t) := q(t) \int_0^t ||\nabla u(t) - \nabla u(t-s)||^2_2 \, ds
\]
and
\[
\eta_2(t) := q(t) \int_0^t ||\nabla v(t) - \nabla v(t-s)||^2_2 \, ds,
\]
Thus, by (4.1), we have
\[
\eta_i(t) < 1, \quad \forall t \geq 0 \quad \text{and} \quad i = 1, 2. \quad (4.7)
\]
We further assume that \( \eta_i(t) > 0 \), for any \( t \geq 0 \). In addition, it follows from the strict convexity of \( H_i \) and the fact that \( H_i(0) = 0 \) that
\[
H_i(s\tau) \leq sH_i(\tau), \quad \text{for} \quad 0 \leq s \leq 1, \quad \tau \in (0, r) \quad \text{and} \quad i = 1, 2.
\]
These facts, hypothesis (A2), estimates (4.7) and Jensen’s inequality lead to
\[
\mu_1(t) = - \frac{1}{q(t)\eta_1(t)} \int_0^t q(t)\eta_1(t)g'_1(s)||\nabla u(t) - \nabla u(t-s)||^2_2 ds
\]
\[
\geq \frac{1}{q(t)\eta_1(t)} \int_0^t q(t)\eta_1(t)\xi_1(s)H_1(g_1(s))||\nabla u(t) - \nabla u(t-s)||^2_2 ds
\]
\[
\geq \frac{\xi_1(t)}{q(t)\eta_1(t)} \int_0^t q(t)H_1(\eta_1(t)g_1(s))||\nabla u(t) - \nabla u(t-s)||^2_2 ds
\]
\[
\geq \frac{\xi_1(t)}{q(t)} \bar{H}_1 \left( \frac{1}{\eta_1(t)} \int_0^t q(t) \eta_1(t) g_1(s) \| \nabla u(t) - \nabla u(t-s) \|^2 ds \right)
\]
\[
= \frac{\xi_1(t)}{q(t)} \bar{H}_1 \left( \frac{q(t)}{t} \int_0^t g_1(s) \| \nabla u(t) - \nabla u(t-s) \|^2 ds \right)
\]
\[
= \frac{\xi_1(t)}{q(t)} \bar{H}_1 \left( q(t) \int_0^t g_1(s) \| \nabla u(t) - \nabla u(t-s) \|^2 ds \right), \quad \forall t \geq 0,
\]

Using the fact that \( \bar{H}_1 \) is strictly increasing and strictly convex on \((0, \infty)\), (4.4) is established. By repeating the same steps, we can prove (4.5). \( \square \)

**Notation:** We define the functions \( G_2, G_3 \) and \( G_4 \) by
\[
G_2(t) = tG'(\varepsilon t), \quad G_3(t) = tG''^{-1}(t), \quad G_4(t) = \overline{G_3}(t), \tag{4.8}
\]
which are convex and increasing functions on \((0, r]\).

**Theorem 4.3.** Let \((u_0(0), u_1), (v_0(0), v_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\) be given. Suppose that assumptions \((A.1)-(A.3)\) hold. Then there exist two positive constants \(c \) and \(d\) such that the solution to problem \( (P) \) satisfies the estimate
\[
E(t) \leq \left( \frac{E(0)}{q(t)} \right) G_2^{-1} \left[ C + \int_0^t \xi(s)G_4 \left( \frac{q(s)h_1(s)}{\int_0^s \xi(s) ds} \right) ds \right], \tag{4.9}
\]
where \( \xi(t) = \min \{ \xi_1(t), \xi_2(t) \} \) and \( h_1(t), q(t) \) are defined in Lemma 3.1 and (4.6) respectively.

**Proof.** By combining (2.3), (3.8), (4.4) and (4.5), we arrive at
\[
F'(t) \leq -mE(t) + \frac{c}{q(t)} \bar{H}_1^{-1} \left( \frac{q(t)\mu_1(t)}{\xi_1(t)} \right) + \frac{c}{q(t)} \bar{H}_1^{-1} \left( \frac{q(t)\mu_2(t)}{\xi_2(t)} \right) + ch_1(t), \quad \forall t \geq 0. \tag{4.10}
\]
Set \( G' = \min \{ \bar{H}_1', \bar{H}_2' \} \). For a fixed \( 0 < \varepsilon < r \), define a functional \( F_1 \) by
\[
F_1(t) := G' \left( \frac{\varepsilon q(t)E(t)}{E(0)} \right) F(t) + E(t), \quad \forall t \geq 0.
\]
Then, using the fact that \( E' \leq 0, \bar{H}_1' > 0 \) and \( \bar{H}_2' > 0 \), we deduce that \( F_1 \sim E \) and (4.10) becomes
\[
F_1'(t) \leq -mE(t)G' \left( \frac{\varepsilon q(t)E(t)}{E(0)} \right) + \frac{c}{q(t)} G' \left( \frac{\varepsilon q(t)E(t)}{E(0)} \right) \bar{H}_1^{-1} \left( \frac{q(t)\mu_1(t)}{\xi_1(t)} \right) \tag{4.11}
\]
\[
+ \frac{c}{q(t)} G' \left( \frac{\varepsilon q(t)E(t)}{E(0)} \right) \bar{H}_1^{-1} \left( \frac{q(t)\mu_2(t)}{\xi_2(t)} \right) + ch_1(t)G' \left( \frac{q(t)E(t)}{E(0)} \right), \forall t \geq 0.
\]
Let \( \bar{H}'_i \) be the convex conjugate of \( \bar{H}_i \) in the sense of Young (see [4, pp. 61-64]), which has the form
\[
\bar{H}'_i(s) = s(\bar{H}'_i)^{-1}(s) - \bar{H}_i \left[ (\bar{H}'_i)^{-1}(s) \right], \quad \text{for} \quad i = 1, 2, \tag{4.12}
\]
and satisfies the following generalized Young’s inequality
\[
AB_i \leq \bar{H}'_i(A) + \bar{H}_i(B_i), \quad \text{for} \quad i = 1, 2. \tag{4.13}
\]
By taking $A = G'(\frac{q(t)E(t)}{E(0)})$, $B_i = \tilde{H}_i^{-1}(\frac{q(t)\mu_i(t)}{\xi_i(t)})$, for $i = 1, 2$, and combining (4.11)–(4.13), we obtain, $\forall t \geq 0$,

$$F'_i(t)$$

$$\leq - mE(t)G'\left(\frac{q(t)E(t)}{E(0)}\right) + \frac{c}{q(t)}H_1'\left[H'_1\left(\frac{q(t)E(t)}{E(0)}\right)\right] + c\mu_1(t)$$

$$+ \frac{c}{q(t)}H_2'\left[H'_2\left(\frac{q(t)E(t)}{E(0)}\right)\right] + c\mu_2(t) + ch_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$\leq - mE(t)G'\left(\frac{q(t)E(t)}{E(0)}\right) + \frac{c}{q(t)}G'(H'_1)\left(\frac{q(t)E(t)}{E(0)}\right) + c\mu_1(t)$$

$$+ \frac{c}{q(t)}G'(H'_2)\left(\frac{q(t)E(t)}{E(0)}\right) + c\mu_2(t) + ch_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$\leq - (mE(0) - c\epsilon)G'\left(\frac{q(t)E(t)}{E(0)}\right) + c\mu_1(t) + c\mu_2(t) + c\epsilon G'\left(\frac{q(t)E(t)}{E(0)}\right).$$

Multiplying this estimate by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\} > 0$ and using $\frac{q(t)E(t)}{E(0)} < r$,

$$G'\left(\frac{q(t)E(t)}{E(0)}\right) = \min \left\{H'_1\left(\frac{q(t)E(t)}{E(0)}\right), H'_2\left(\frac{q(t)E(t)}{E(0)}\right)\right\}$$

and the fact that $\mu_1(t) + \mu_2(t) \leq -cE'(t)$, we obtain

$$\xi(t)F'_i(t) \leq - (mE(0) - c\epsilon)\xi(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$+ c(\mu_1(t) + \mu_2(t)) + c\epsilon h_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right).$$

Therefore, for all $t \geq 0$, we have

$$\xi(t)F'_i(t)$$

$$\leq - (mE(0) - c\epsilon)\xi(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$- cE'(t) + c\epsilon h_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right).$$

Take $\epsilon$ smaller, if needed, to get, for some $k_0 > 0$,

$$\xi(t)F'_i(t)$$

$$\leq - k_0 \xi(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$- cE'(t) + c\epsilon h_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right), \quad \forall t \geq 0.$$

Consequently, by setting $F_2 = \xi F_1 + cE$, we obtain, for some $a_1, a_2 > 0$

$$a_1 F_2(t) \leq E(t) \leq a_2 F_2(t), \quad \forall t \geq 0$$

(4.14)

and

$$F'_2(t) \leq - k_0 \xi(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$+ c\epsilon h_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right)$$

$$- k_0 \xi(t)G'\left(\frac{q(t)E(t)}{E(0)}\right) + c\epsilon h_1(t)G'\left(\frac{q(t)E(t)}{E(0)}\right), \quad \forall t \geq 0.$$
where $G_2(\tau) = \tau G'\left(\varepsilon \tau\right)$ on $[0, 1]$. Since $G'_2(t) = G''(\varepsilon t) + \varepsilon t G'''(\varepsilon t)$, then, using the strict convexity of $H_3$ on $(0, \varepsilon)$, we find that $G'_2(t), G_2(t) > 0$ on $(0, 1]$. Using the generalized Young inequality (4.13) on the last term in (4.15) with $A = \frac{E'(E(t)q(t))}{E(0)}$, $B = \frac{1}{\varepsilon t} h_1(t)$, we get

$$c h_1(t) G'\left(\varepsilon \frac{E(t)q(t)}{E(0)}\right) = \frac{d}{q(t)} \frac{c}{d} q(t) h_1(t) \left[G'\left(\varepsilon \frac{E(t)q(t)}{E(0)}\right)\right]$$

$$\leq \frac{d}{q(t)} G_3 \left[G'\left(\varepsilon \frac{E(t)q(t)}{E(0)}\right)\right] + \frac{d}{q(t)} G_3' \left[\frac{c}{d} q(t) h_1(t)\right]$$

$$\leq \frac{d}{q(t)} \left(\varepsilon \frac{E(t)q(t)}{E(0)}\right) G_4 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t) h_1(t)\right]$$

$$\leq \frac{d\varepsilon}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)} G_4 \left[\frac{c}{d} q(t) h_1(t)\right],$$

Now, combining (4.15) and (4.16) and choosing $d$ small enough, we arrive at

$$F'_2(t) \leq -k_1 \frac{\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)} G_4 \left[\frac{c}{d} q(t) h_1(t)\right]$$

$$\leq -k_1 \frac{\xi(t)}{q(t)} G_2 \left(\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)} G_4 \left[\frac{c}{d} q(t) h_1(t)\right].$$

(4.17)

where $k_1 = k_0 - \varepsilon d > 0$. Since $E' < 0$ and $q' < 0$, then $G_2(\frac{E(t)q(t)}{E(0)})$ is decreasing. Hence, for $0 \leq t \leq T$, we have

$$G_2 \left(\frac{E(T)q(T)}{E(0)}\right) \leq G_2 \left(\frac{E(t)q(t)}{E(0)}\right).$$

(4.18)

Combining (4.17) with (4.18) and multiplying by $q(t)$, we get

$$q(t) F'_2(t) + k_1 \xi(t) G_2 \left(\frac{E(T)q(T)}{E(0)}\right) \leq d\xi(t) G_4 \left[\frac{c}{d} q(t) h_1(t)\right].$$

(4.19)

Since $q' < 0$, then for all $0 \leq t \leq T$,

$$(q F'_2) t + k_1 \xi(t) G_2 \left(\frac{E(T)q(T)}{E(0)}\right) \leq d\xi(t) G_4 \left[\frac{c}{d} q(t) h_1(t)\right].$$

(4.20)

Integrating (4.20) over $[0, T]$ and using the fact $q(0) = 1$, we have

$$G_2 \left(\frac{E(T)q(T)}{E(0)}\right) \int_0^T \xi(t) dt \leq \frac{F'_2(0)}{k_1} + \int_0^T \xi(t) G_4 \left[\frac{c}{d} q(t) h_1(t)\right] dt.$$

(4.21)

Hence,

$$G_2 \left(\frac{E(T)q(T)}{E(0)}\right) \leq \left[\frac{F'_2(0)}{c} + \int_0^T \xi(t) G_4 \left(\frac{c}{d} q(t) h_1(t)\right) dt\right] G_2^{-1}. $$

(4.22)

Thus

$$\left(\frac{E(T)q(T)}{E(0)}\right) \leq \left[\frac{F'_2(0)}{c} + \int_0^T \xi(t) G_4 \left(\frac{c}{d} q(t) h_1(t)\right) dt\right]^{-1} G_2^{-1}.$$

(4.23)

which yields

$$E(T) \leq \left(\frac{E(0)}{q(T)}\right) G_2^{-1} \left[C + \int_0^T \xi(t) G_4 \left(\frac{c}{d} q(t) h_1(t)\right) dt\right].$$

(4.24)

where $C = \max\{1, \frac{F'_2(0)}{c}\}$. This finishes the proof of Theorem 5.4.
Our result is obtained under very general assumption on the relaxation functions and without imposing any boundedness condition on the history function $\nu$. We select $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and $G' = \min\{H_1', H_2\}$, then $G'(t) = a_0 t^4$.

We will discuss two cases:

Case 1: if $m_0 \leq 1 + \|\nabla u_0(c, t)\|^2 \leq m_1$. Then we have the following:

$$G_4(t) = a_1 t^{\nu + 1}, \quad G_2(t) = a_2 t^{\nu + 1},$$

$$a_3(1 + t)^{-\nu + 1} \leq h_1(t) \leq a_4(1 + t)^{-\nu + 1},$$

$$\int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h_1(t) \right) dt < +\infty,$$

$$G_2 \left[ \frac{C + \int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h_1(t) \right) dt}{\int_0^T \xi(t)dt} \right] \leq a_5 T^{-\left(\frac{\nu}{\nu + 1}\right)},$$

$$q(T) \leq a_6 \begin{cases} 1 + \ln(1 + T), & \nu = 2; \\ 2, & \nu > 2; \\ (1 + T)^{-\nu + 2}, & 1 < \nu < 2. \end{cases}$$

Then

$$E(T) \leq a_7 \begin{cases} \left( 1 + \ln(1 + T) \right) t^{-\left(\frac{\nu}{\nu + 1}\right)}, & \nu = 2; \\ T^{-\left(\frac{\nu}{\nu + 1}\right)}, & \nu > 2; \\ (1 + T)^{-\nu + 2}, & 1 < \nu < 2. \end{cases}$$

Thus for $\nu \geq 2$ or $\sqrt{2} < \nu < 2$ we have $\lim_{T \rightarrow +\infty} E(T) = 0$.

Case 2: if $m_0(1 + t)^r \leq 1 + \|\nabla u_0(c, t)\|^2 \leq m_1(1 + t)^r$, where $0 < r < \nu - 1$, then we have the following:

$$a_3(1 + t)^{-\nu + 1 + r} \leq h_1(t) \leq a_4(1 + t)^{-\nu + 1 + r},$$

$$\int_0^T \xi(t)G_4 \left( \frac{c}{d} q(t)h_1(t) \right) dt < +\infty,$$

$$q(T) \leq a_6 \begin{cases} 1 + \ln(1 + T), & \nu - r = 2; \\ 2, & \nu - r > 2; \\ (1 + T)^{-\nu + 2}, & 1 < \nu - r < 2. \end{cases}$$

Then

$$E(T) \leq a_7 \begin{cases} \left( 1 + \ln(1 + T) \right) t^{-\left(\frac{\nu}{\nu + 1}\right)}, & \nu - r = 2; \\ T^{-\left(\frac{\nu}{\nu + 1}\right)}, & \nu - r > 2; \\ (1 + T)^{-\left(\frac{\nu - 2}{\nu + 1}\right)}, & 1 < \nu - r < 2. \end{cases}$$

Thus for $\nu - r \geq 2$ or $\frac{1}{2} \left( r + \sqrt{r^2 + 4r + 8} \right) < \nu < r + 2$ we have $\lim_{T \rightarrow +\infty} E(T) = 0$.

**Remark 2.** Our result is obtained under very general assumption on the relaxation functions and without imposing any boundedness condition on the history function $u_0(c, t)$. Our results generalizes and improves many results in the literature such as [21, 18, 2, 27].
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E-mail address: almahdi@kfupm.edu.sa
E-mail address: mahfouz@kfupm.edu.sa
E-mail address: smessaoudi@sharjah.ac.ae