WKB approximation under Killingbeck potential and its application to double heavy meson mass spectrum

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As we all known that non-relativistic or semi-relativistic constituent quark models can describe a large number of the meson sand baryon properties with surprising accuracy. In this work, we studied Killingbeck potential by using WKB approximation in the constituent quark model. The implicit approximate solution of the energy eigenvalue of the Schrödinger equation is obtained by using appropriate quantization conditions. Under the Killingbeck potential with the WKB approximation, we studied the spectra of double heavy mesons and predict the masses of high excited states in $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$ families. Finally, we discuss the application scope of double heavy mesons in Killingbeck potential.

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I. INTRODUCTION

In the past decades, many potential energy functions that play important roles in describing physical phenomena have been proposed. However, the solution of the wave equation with some potentials are exactly solvable, whereas other potentials are insoluble. In this case, numerical methods and approximate analytical techniques are required to obtain the solution of quantum system [1–3]. Wenzel-Kramers-Brillouin (WKB) approximation is a semiclassical approximation method, which can be used to analyze Schrödinger equation and successfully deals with some significant and vital problems [4–7], for example, the theory of electromagnetic waves [8]. The Bohr-Sommerfeld quantization rule, and the WKB approximation base on the Schrödinger equation, are both expected to give good results of energy eigenvalues in limit of large quantum numbers, in accordance with the Correspondence Principle [9]. In general, the above two methods give significantly different results for low-lying states, with that from WKB is usually more accurate.

The WKB approximation was developed by J. S. Kang [1, 10] and C. Quigg [11]et al. in 1970, and had been applied to particle energy spectra in high energy physics. In 2005, Z. Q. Ma et al. proposed an accurate quantization condition for one-dimensional quantum system and three-dimensional spherically symmetric quantum system [12, 13]. F. Brau reported the relationship among energy, radial quantum number and orbital angular momentum [2]. H. F. Lalus et al. used the WKB method to analyze the physical conditions that the system must satisfy in the process of tunneling [14]. E. Omugbe et al, obtained the non-relativistic ro-vibrational energy spectra, expectation values and the thermodynamic properties of the Schrödinger potential function within the framework of the WKB approximation method. Moreover, the authors in Refs. [15–22] have used WKB method to study mass spectra under various potential energies. The bound state solution to the wave equation under the quark-antiquark interaction potential functions such as the Killingbeck or the Cornell potential have attracted much research interest in high energy physics [18]. However, there are few researches on the study of the mass spectrum via the WKB approximation directly, as well as investigating the WKB approximation under Killingbeck potential. This study set out to examine this questions.

The Killingbeck potential plus an inversely quadratic potential (KPIQP) comprises the sum of Cornell potential plus the Harmonic oscillator potentials. The main focus of this paper is to obtain the solution of Killingbeck potential by using WKB approximation in the constituent quark model, fit the mass spectra of the two double heavy mesons in Killingbeck potential, such as charmonium($c\bar{c}$), bottomonium($b\bar{b}$) and bottom-charmed($b\bar{c}$). Besides, we predict the high excited states of the two double heavy meson systems.

This paper is organized as follows. In Sec. II, the WKB approximation and the WKB solution of KPIQP potential are introduced. In Sec. III, we analyze the mass spectra of double heavy mesons. Furthermore, we compare the WKB mass spectra under the Killingbeck potential with those obtained by other analytical methods and available experimental data. The paper is ended with a conclusion in section IV

II. WKB APPROXIMATION

F. Brau who has calculated the energy spectrum under Cornell potential in the quark model, reported the relationship among energy, radial quantum number and orbital angular momentum [2]. Base on Ref. [2], we solve the mass spectra of double mesons under KPIQP potential.

The basic quantities in the Bohr-Sommerfeld quantization
The radial motion takes place between two turning points, \( r_+ \) and \( r_- \). The four roots, \( r_j (k = 1, 2, 3, 4) \), of \( p_r \) are:

\[
\begin{align*}
 r_1 &= -y + \frac{D + \sqrt{D^2 + 4Z}}{2} + \frac{\sqrt{2D - (D^2 + 4Z)^{1/2}} \sqrt{Z}}{4x}, \\
 r_2 &= -y + \frac{D + \sqrt{D^2 + 4Z}}{2} - \frac{\sqrt{2D - (D^2 + 4Z)^{1/2}} \sqrt{Z}}{4x}, \\
 r_3 &= -y - \frac{D + \sqrt{D^2 + 4Z}}{2} + \frac{\sqrt{2D + (D^2 + 4Z)^{1/2}} \sqrt{Z}}{4x}, \\
 r_4 &= -y - \frac{D + \sqrt{D^2 + 4Z}}{2} - \frac{\sqrt{2D + (D^2 + 4Z)^{1/2}} \sqrt{Z}}{4x},
\end{align*}
\]

where

\[
\begin{align*}
 x &= -2\mu b, \\
 y &= -2\mu a, \\
 D &= 4\mu^2 (3a^2 + 8b E'), \\
 A &= 2^8 b^2 \mu^3 (6bL^2 + \mu (E'' + 3ak)), \\
 B &= 2^{11} b^2 \mu^5 (3a^3 \mu + 14abE' \mu + a^2 (-3bL^2 + E'' \mu) - 2b(4bL^2 E' - 2E'' \mu + 9b^2 \mu)), \\
 F &= 2^8 \mu^6 (-3(3a^2 + 8b E') a^3 + 4abE' - 8b^2 \kappa^2 \mu^2 + (-2^3 b^3 L^2 + 3a^2 \mu + 2^4 a^2 b E' \mu + 2^4 b^2 \mu (E'^2 - ak))^2), \\
 Z_1 &= AD + 3\left(-B + \frac{\sqrt{B^2 - 4AF}}{2}\right), \\
 Z_2 &= AD + 3\left(-B - \frac{\sqrt{B^2 - 4AF}}{2}\right),
\end{align*}
\]

and

\[
Z = D^2 - D(\sqrt{Z_1} + \sqrt{Z_2}) + (\sqrt{Z_1} - \sqrt{Z_2})^2 - 3A.
\]

The turning points are \( r_+ = r_2 \) and \( r_- = r_3 \).

Quantization of \( J_\phi \) trivially gives \( L = l + c_\phi \), \( l \) is orbital angular momentum. It can be obtained by substituting \( p_r \) into (2.1):

\[
\begin{align*}
 \alpha_1 K(\eta) + \alpha_2 \Pi(\gamma, \frac{\pi}{2}, \eta) + \alpha_3 \Pi\left(\frac{r_1}{r_2}, \frac{\pi}{2}, \eta\right) + \alpha_4 E(\eta) \\
 - 2\alpha_4 (n + c_\phi) \psi r_4 - r_- \sqrt{r_+ - r_1} = 0,
\end{align*}
\]

where

\[
\begin{align*}
 \eta &= \frac{r_4 - r_1}{r_+ - r_-}, \\
 \gamma &= \frac{r_+ - r_-}{r_+ - r_1}, \\
 \alpha_1 &= \frac{a}{2} (r_+ - r_1)(r_- - r_1)(E' + a(r_4 - r_1)), \\
 \alpha_2 &= -2(r_+ - r_1)(a\kappa + 2m^2), \\
 \alpha_3 &= -2a^2 r_+ r_4 (r_- - r_1), \\
 \alpha_4 &= \frac{aE'}{2} (r_+ - r_-)(r_+ - r_1).
\end{align*}
\]

In Eq. (2.19), \( K(\eta) \), \( E(\eta) \) and \( \Pi(\gamma, \frac{\pi}{2}, \eta) \) are the first, second and third complete elliptic integrals, respectively. The symbol \( n \) is radial quantum number, this appears to be a rather complicated equation since it cannot be solved explicitly for the energy. However, it leads to very accurate results if we choose the Langer prescription \( c_\phi = c_\phi = \frac{1}{2} \).
III. THE MASS SPECTRA OF DOUBLE HEAVY MESONS UNDER KPIQP POTENTIAL

In this section, we compute the mass spectra of two double heavy mesons such as charmonium(c\bar{c}), bottomonium(b\bar{b}) and bottom-charmed(b\bar{c}). The mass spectra obtained by WKB approximation and finite difference method was compared with the experimental value and the results obtained by other methods.

To test the WKB approximate solution under the KPIQP potential, we fit the 13 experimental values of $B_c$, c\bar{c} and b\bar{b} mesons. We define:

$$\chi^2 = \sum_i \left( \frac{T_i^{th} - E_i^{exp}}{E_i} \right)^2,$$

where $T_i^{th}$, $E_i^{exp}$ and $E_i$ are the theoretical value, experimental value and error, respectively. The error is at 1$\sigma$ of the experimental value of the mass. The fitted parameters are shown in Table II.

| TABLE I: Experimental values of $B_c$, c\bar{c} and b\bar{b} meson masses. The unit of the mass is MeV. |
|---|---|---|---|---|
| Meson | State | Experimental value | This work | $E_r$ |
|---|---|---|---|---|
| $B_c$ | $1^S_0$ | 6273.7 ± 0.3 | 6271 | 6.3 |
| | $2^S_0$ | 6872.1 ± 1.3 ± 0.1 ± 0.8 | 6882 | 6.9 |
| | $2^S_1$ | 6841.2 ± 0.6 ± 0.1 ± 0.8 | 6904 | 6.8 |
| c\bar{c} | $1^P_1$ | 2983.9 ± 0.5 | 3020 | 3.0 |
| | $2^S_0$ | 3637.6 ± 1.2 | 3635 | 3.6 |
| | $1^S_1$ | 3096.9 ± 0.006 | 3081 | 3.6 |
| | $2^S_1$ | 3686.09 ± 0.01 | 3696 | 3.7 |
| b\bar{b} | $2^S_1$ | 3525.38 ± 0.1 | 3483 | 3.5 |
| $1^P_1$ | 3525.38 ± 0.1 | 3483 | 3.5 |
| $1^S_0$ | 9399 ± 2.3 | 9354 | 9.4 |
| $2^S_0$ | 9999 ± 3.5 | 10003 | 10.0 |
| $1^S_1$ | 9460.3 ± 0.26 | 9362 | 9.5 |
| $2^S_1$ | 10023.2 ± 0.31 | 10001 | 10.0 |
| $1^P_1$ | 9899.3 ± 0.8 | 9899 | 9.9 |

| TABLE II: The parameters fitted in this work. |
|---|---|
| Parameter | Value |
| $m_c$(GeV) | 1.846 |
| $m_b$(GeV) | 5.228 |
| $a$(GeV$^2$) | 0.2598 |
| $b$(GeV$^3$) | -0.01062 |
| $c$(GeV$^3$) | 0.2076 |
| $\kappa$ | -0.9712 |

### A. Mass spectrum of $B_c$ mesons under KPIQP potential

There are three experimental values of $B_c$ mesons shown in the Table III. The experimental masses of 1$^1S_0$ and 2$^1S_0$ states are 6273 MeV and 6872 MeV [26], respectively. The experimental mass of 1$^3S_1$ state is 6841 MeV from LHCb Collaboration [23]. Compared the results obtained by WKB approximation of $B_c$ mesons with the three experimental values, one sees that the mass of 1$^1S_0$ state and 2$^1S_0$ state agree well with the experimental values. The result of 2$^3S_1$ state is higher than the experimental value. The masses calculated via WKB approximation are smaller than those in GI model and...
the results in Ref. [25]. With the increase of radial quantum number, our approximate solution are closer to the masses in GI model and in Ref. [25]. Besides, the masses obtained by WKB approximation are in good agreement with that computed by finite difference method (numerical solution). For high excited states, the results from WKB approximation are higher than those obtained via finite difference method.

In addition, we also predict the masses of highly excited states such as $G, H$ and $I$ waves. The predicted masses of highly excited states $4^1G_1, 1^1H_5$ and $2^1H_4$ are 7702 MeV, 7566 MeV and 7672 MeV, respectively. More details are shown in Table III.

### B. Mass spectrum of heavy flavor meson $c\bar{c}$ under KPIQP potential

With the proposal of the quark model, a large number of mesons have been reported by experiments, such as $\Psi$ and $\Upsilon$ mesons. In this part, we obtain the mass spectrum of the heavy meson $c\bar{c}$ and compare our results with those in Refs. [27, 28] and from MGI model [29], which take screening potential into GI model. More information of MGI model can be seen in Refs. [28–31].

The masses from experiment, from MGI model, calculated by numerical solution like Refs. [27, 28] and obtained via WKB approximation and finite difference method by us are shown in Table IV. The masses we computed by WKB approximation are in excellent agreement with the experimental values for $2^1S_0$ and $2^3S_1$ states. For $1^3S_1$, $3^3S_1$ and $1^1P_1$ states, our results with WKB approximation is lower than those of experimental values. When it comes to the masses calculated by approximate solution (WKB approximation) and numerical solution, there is an important point deserved to be pointed out here, that is, our results calculated via approximate solution are very close to those obtained by numerical solution for ground and low excited states.

When compared with the masses obtained in MGI model and Refs. [27, 28], our results of approximate solution are lower than those except for a few states. That means, our method can depress the mass spectrum of high excited states.

We also predict the masses of $L = 5$ and 6 bound states. The masses of $3^1G_4, 4^1G_4$ and $1^1H_5$ states are 4325 MeV, 4333 MeV and 4285 MeV, respectively. More information of the prediction can be seen in Table IV.

### C. Mass spectrum of heavy flavor meson $b\bar{b}$ under KPIQP potential

In recent ten years, with the improvement of collision energy and detector accuracy, a large number of new $b\bar{b}$ meson states have been observed, and the quality of more $b\bar{b}$ states has been determined, which further improves the experimental information of $b\bar{b}$ meson spectroscopy. Combining a large amount of experimental information and the current theoretical framework, we systematically studied the $b\bar{b}$ meson spectroscopy. In addition, we give theoretical predictions for those mesons not found in the experiments, and hope that this information can be helpful for the experiments. The specific mass spectra of $b\bar{b}$ meson family is shown in Table V.

This system has more experimental information and theoretical values from relevant literatures [30, 34, 35] than $B_c$ and $c\bar{c}$ mesons. The authors of Refs. [29, 30] take the same experimental values with those we use, which facilitates the work of the comparison and verification. Compared our results of $b\bar{b}$ meson with the experimental values, we can see that the results of most states are in good agreement with the experimental values, except for a few states. It can be concluded

| State | Exp. [32] | Ref. [28] | Ref. [27] | MGI [29] | WKB | FD |
|-------|-----------|-----------|-----------|-----------|------|----|
| $1^3S_0$ | 2983 | 2982 | 2990 | 2981 | 3015 | 3020 |
| $2^1S_0$ | 3637 | 3630 | 3646 | 3642 | 3635 | 3635 |
| $3^1S_0$ | 4043 | 4072 | 4200 | 4260 | 4221 | 4209 |
| $4^1S_0$ | 4443 | 4433 | 4313 | 4313 | 4313 | 4313 |
| $6^1S_0$ | 4443 | 4433 | 4313 | 4313 | 4313 | 4313 |
| $7^1S_0$ | 4443 | 4433 | 4313 | 4313 | 4313 | 4313 |
| $1^1P_1$ | 3525 | 3516 | 3515 | 3538 | 3483 | 3483 |
| $2^1P_1$ | 3934 | 3945 | 3933 | 3898 | 3898 | 3898 |
| $3^1P_1$ | 4739 | 4334 | 4200 | 4170 | 4165 | 4165 |
| $4^1P_1$ | 4439 | 4389 | 4258 | 4242 | 4242 | 4242 |
| $5^1P_1$ | 4449 | 4316 | 4316 | 4410 | 4410 | 4410 |
| $6^1P_1$ | 4458 | 4458 | 4458 | 4458 | 4458 | 4458 |
| $1^3D_2$ | 3799 | 3807 | 3848 | 3771 | 3768 | 3768 |
| $2^3D_2$ | 4158 | 4174 | 4137 | 4088 | 4085 | 4085 |
| $3^3D_2$ | 4560 | 4343 | 4221 | 4237 | 4237 | 4237 |
| $4^3D_2$ | 4940 | 4316 | 4316 | 4410 | 4410 | 4410 |
| $5^3D_2$ | 4610 | 4610 | 4610 | 4610 | 4610 | 4610 |
| $1^3F_3$ | 4206 | 4206 | 4206 | 4206 | 4206 | 4206 |
| $2^3F_3$ | 4356 | 4372 | 4296 | 4221 | 4224 | 4224 |
| $3^3F_3$ | 4746 | 4746 | 4746 | 4746 | 4746 | 4746 |
| $4^3F_3$ | 4396 | 4396 | 4396 | 4396 | 4396 | 4396 |
| $5^3F_3$ | 4289 | 4289 | 4289 | 4289 | 4289 | 4289 |
| $1^3G_4$ | 4225 | 4247 | 4250 | 4172 | 4169 | 4169 |
| $2^3G_4$ | 4465 | 4465 | 4465 | 4465 | 4465 | 4465 |
| $3^3G_4$ | 4339 | 4339 | 4339 | 4339 | 4339 | 4339 |
| $4^3G_4$ | 4333 | 4333 | 4333 | 4333 | 4333 | 4333 |
| $1^1H_5$ | 4285 | 4285 | 4285 | 4285 | 4285 | 4285 |
| $2^1H_5$ | 4369 | 4369 | 4369 | 4369 | 4369 | 4369 |
| $3^1H_5$ | 4383 | 4383 | 4383 | 4383 | 4383 | 4383 |
| $1^1I_6$ | 4364 | 4364 | 4364 | 4364 | 4364 | 4364 |
| $2^1I_6$ | 4453 | 4453 | 4453 | 4453 | 4453 | 4453 |
TABLE V: Mass spectrum of $b\bar{b}$ mesons at KPIQP potential. FD is the abbreviation of the masses calculated via finite difference method. The unit of the mass is MeV.

| State | Exp [33] | Ref. [34] | MGI [30] | GI [35] | WKB | FD |
|-------|----------|-----------|----------|---------|------|----|
| $1^1S_0$ | 9399 | 9409 | 9398 | 9394 | 9379 | 9354 |
| $2^1S_0$ | 9999 | 9996 | 9989 | 9975 | 10000 | 10003 |
| $3^1S_0$ | - | 10334 | 10336 | 10333 | 10356 | 10359 |
| $4^1S_0$ | - | 10612 | 10597 | 10616 | 10616 | 10617 |
| $5^1S_0$ | - | 10865 | 10810 | 10806 | 10815 | 10816 |
| $6^1S_0$ | - | - | 10991 | 11079 | 10965 | 10960 |
| $7^1S_0$ | - | - | 11149 | 11281 | 11025 | 10967 |
| $8^1S_0$ | - | - | 11289 | 11470 | 11104 | 11056 |
| $1^3S_1$ | 9460 | 9458 | 9463 | 9459 | 9487 | 9362 |
| $2^3S_1$ | 10023 | 10012 | 10017 | 10004 | 10008 | 10011 |
| $3^3S_1$ | 10355 | 10345 | 10356 | 10354 | 10364 | 10366 |
| $4^3S_1$ | 10579 | 10623 | 10612 | 10633 | 10624 | 10625 |
| $5^3S_1$ | 10881 | 10870 | 10822 | 10875 | 10823 | 10824 |
| $6^3S_1$ | 11003 | - | 11001 | 11092 | 10973 | 10968 |
| $7^3S_1$ | - | - | 11157 | 11294 | 11031 | 10974 |
| $1^1P_1$ | 9899 | 9911 | 9894 | 9881 | 9899 | 9899 |
| $2^1P_1$ | 10260 | 10254 | 10259 | 10250 | 10275 | 10275 |
| $3^1P_1$ | - | 10531 | 10530 | 10530 | 10549 | 10548 |
| $4^1P_1$ | - | 10814 | 10751 | 10790 | 10761 | 10760 |
| $5^1P_1$ | - | - | 10938 | 11013 | 10925 | 10914 |
| $6^1P_1$ | - | - | 11101 | 11218 | 11024 | 10963 |
| $7^1P_1$ | - | - | - | - | 11051 | 11035 |
| $1^3P_2$ | - | 10154 | 10163 | 10148 | 10172 | 10172 |
| $2^3P_2$ | - | 10439 | 10450 | 10450 | 10465 | 10464 |
| $3^3P_2$ | - | - | 10681 | 10706 | 10693 | 10692 |
| $4^3P_2$ | - | - | 10876 | 10934 | 10872 | 10871 |
| $5^3P_2$ | - | - | 11046 | 11143 | 11005 | 10966 |
| $6^3P_2$ | - | - | - | - | 11024 | 11004 |
| $1^3F_3$ | - | 10339 | 10366 | 10354 | 10373 | 10373 |
| $2^3F_3$ | - | 10598 | 10609 | 10619 | 10618 | 10618 |
| $3^3F_3$ | - | - | 10812 | 10853 | 10813 | 10812 |
| $4^3F_3$ | - | - | 10988 | 11066 | 10964 | 10962 |
| $5^3F_3$ | - | - | - | - | 11024 | 10972 |
| $1^1G_4$ | - | 10534 | 10530 | 10530 | 10539 | 10539 |
| $2^1G_4$ | - | 10747 | 10770 | 10749 | 10749 | 10749 |
| $3^1G_4$ | - | - | 10929 | 10988 | 10916 | 10916 |
| $4^1G_4$ | - | - | - | - | 11024 | 10977 |
| $1^1H_5$ | - | - | - | - | 10682 | 10681 |
| $2^1H_5$ | - | - | - | - | 10864 | 10863 |
| $3^1H_5$ | - | - | - | - | 11031 | 10985 |
| $1^1I_6$ | - | - | - | - | 10807 | 10807 |
| $2^1I_6$ | - | - | - | - | 10964 | 10963 |

that the results of the approximate solution are in consonance with the results of the numerical solution.

In addition, our results are lower than those in GI and MGI model, which verified this method can also reduce the mass of highly excited states. We also predict that the masses of highly excited states $4^1G_4$, $1^1H_5$ are $11024$ MeV, $10682$ MeV, respectively. More details can be found in Table V.

IV. DISCUSSION AND CONCLUSION

In this section, we compare the masses obtained by WKB approximation and the numerical solution calculated via finite difference method. We can see the results are highly consistent for ground states as well as low excited states. For high excited states, the masses obtained by WKB approximation are higher than those obtained via finite difference method. Next we will discuss the potential in Eq. (2.3). Ref. [36] pointed out that the potential of a meson system meet

$$V(r) > 0, V(r)^2 < 0, V(r)^3 > 0,$$ (4.1)

in this work, the first derivative and the third derivative are positive while the second derivative is negative. When the second derivative is less than zero, there exists a maximum of the potential energy, which is $r = 12.4$ GeV$^{-1}$. That means, when the value of $r$ is less than that of maximum, the mass spectrum of the heavy double meson increases gradually, whereas decreases gradually when the value of $r$ exceeds the maximum value. Therefore, the mass spectra in this work are decrease when arrive at a quantitative number. This property does not hold for large power potentials, such as $V = r^3$. In addition, the Harmonic oscillator potential $br^2$ fitted in this paper is less than zero, which is mainly responsible for suppressing the mass spectrum of the highly excited state in the total potential energy, and its role is the same as $\mu$ that in the screening potential. Next, let’s talk about the screening potential.

The screening potential has a great influence on the masses of the high excited states and little influence on the ground states. The main reason for screening potential effect is the creation of quark-antiquark pairs from vacuum [37–39]. Taking GI model as an example, the linear potential $ar$ in GI model tends to be infinite with the increase of the distance between quark and antiquark. When the distance between quark and antiquark arrive at a number, a pair of quark-antiquark will be created from vacuum, and this pair of newly created quark-antiquark will play a certain screening effect on the color charge. This causes the fact that linear potential $ar$ cannot fully reflect the interaction between quarks, because of which it is necessary to modify it. In the MGI model, the linear confinement term $ar$ is replaced as follows [29–31, 37, 38]:

$$ar \rightarrow V_{scr}(r) = \frac{a(1 - e^{-\alpha r})}{\mu},$$ (4.2)

where $V_{scr}(r)$ is screening potential that is expressed as $ar$ in the near distance and tends to be a constant $\frac{a}{\mu}$ in the long distance. We can adjust the parameter $\mu$ to change the screening potential.

Compared the Taylor expanded Eq. (4.2) with the potential function, we can obtained $\alpha = 0.077$ GeV. Our results are similar to those in literature [29–31], and the parameters affecting the screening potential effect in literature [29–31] are all expressed as $\mu$. In Ref. [29], $\mu = 0.15$ GeV, the authors used this parameter to calculate the mass spectrum of double heavy $c\bar{c}$ meson and the results are in good agreement with the experimental values. In Ref. [30] and Ref. [31], the values of...
$\mu$ are 0.074 GeV and 0.1 GeV, respectively. The mass spectrums of $b\bar{b}$ and $K$ meson are calculated with their respective parameters, and the results are in good agreement with the experimental values too. Numerically, the higher the value of $\mu$ are taken, the lower of the high excited states’ mass are. The value of $\alpha$ in this paper is higher than the value of $\mu$ in literature [30], which leads to the fact that the high excited state energy calculated in this paper is lower than that in literature [30].

In addition, it can be seen from Fig. 1 that coulomb potential plays a major role in low excitation state, while linear potential and harmonic oscillator potential play an increasing role in high excitation states and coulomb potential can be ignored. Since the coefficient of the harmonic oscillator potential is negative and the linear potential is positive, the harmonic oscillator potential decreases as $r^2$ while the linear potential increase as $r$, so the harmonic oscillator potential decreases faster than the linear potential. This brings about the total potential energy will reaching a max value and it is un-stable beyond this peak, which is contrary to the physical law. It can be seen that the KPIQP potential maybe not applicable in high excited states, nonetheless this phenomenon could predicts that there is an upper limit corresponding to a high excited states for the mass spectrums of mesons.

Finally, we give a summary: we use WKB approximation method mainly to calculate the mass spectrums of double heavy meson $B_c$, $c\bar{c}$ and $b\bar{b}$ families under KPIQP potential with fitted parameters, and predict the mass spectra of highly excited states of three meson families. Besides, the results obtained by WKB approximation method are in good agreement with those calculated via finite difference method for low excited states. As for high excited states, the approximate solutions (by WKB approximation method) are higher than numerical solution (via finite difference method). Moreover, we also predict the masses of highly excited states of three mesons system. The masses of $3^1H_b$, $1^1I_b$ and $2^1I_b$ states in $B_c$ family is 7734 MeV, 7678 MeV and 7705 MeV, respectively. For $c\bar{c}$ meson, the masses of $3^1H_c$ and $1^1I_c$ states are 4369 MeV, 4382 MeV as well as the $1^1I_b$ and $2^1I_b$ are 10807 MeV, 10964 MeV, respectively. More mass information can be seen in Table III, IV and V.

Compared with the screening potential, the fitted parameter that affects the screening potential effect is $\alpha = 0.077$ GeV, which is not significantly different from the others in the relevant literature.

We expect that our work will contribute to the understanding of the role of the quark interaction potential in highly excited states and hope that it is helpful to study and establish the mass spectrum of the double heavy meson family.

![Potential function diagram](https://via.placeholder.com/150)

**FIG. 1: Potential function diagram.**

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