HARMONIC-SUPERSPACE METHOD OF SOLVING $N=3$ SUPER-YANG-MILLS EQUATIONS

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Abstract

We analyze the superfield constraints of the $D=4$, $N=3$ SYM-theory using light-cone gauge conditions. The $SU(3)/U(1) \times U(1)$ harmonic variables are interpreted as auxiliary spectral parameters, and the transform to the harmonic-superspace representation is considered. The harmonic superfield equations of motion are drastically simplified in our gauge, in particular, the basic matrix of the harmonic transform and the corresponding harmonic analytic gauge connections become nilpotent on-shell. It is shown that these harmonic SYM-equations are equivalent to the finite set of solvable linear iterative equations.

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1 Introduction

It is well known that the geometric superfield constraints in the $D=4$, $N=3$ super-Yang-Mills theory are equivalent to the equations of motion of the corresponding component fields [1]. The special projections of these constraints can be interpreted as conditions for zero curvatures associated with the Grassmann covariant derivatives [2, 3, 4].

The possible connection of these zero-curvature conditions with integrability or solvability of the super-Yang-Mills (SYM) theories with 16 or 12 supercharges has been discussed more than 20 years in the framework of different superfield approaches (see, e.g. [2]-[8]). Here we shall use the harmonic-superspace method [3] to analyze the solutions of the $N=3$ SYM-equations. (The preliminary short version of this work appears as Ref.[13].)

The harmonic-superspace (HSS) method has been introduced first to solve the $D=4$, $N=2$ off-shell superfield constraints [4]. Harmonic variables are analogous to twistor or auxiliary spectral variables used in integrable models. Harmonic and twistor methods give the explicit constructions of the general solutions to zero-curvature conditions in terms of independent functions on special analytic (super)spaces satisfying the generalized Cauchy-Riemann analyticity conditions. For instance, the off-shell $N=2$ superfield constraints have been solved via the conditions of the Grassmann (G-) analyticity [4].

In the standard harmonic formulation of the $D=4$, $N=2$ SYM-theory, the basic harmonic connection is G-analytic [4], and the 2-nd one (via the harmonic zero-curvature...
condition) appears to be a nonlinear function of the basic connection. The $N = 2$ equation of motion is linearly dependent on the 2-nd harmonic connection, but it is the nonlinear equation for the basic connection. It has been shown in Ref. [10] that one can alternatively choose the 2-nd harmonic connection as a dual superfield variable, so that the dynamical G-analyticity condition (or the Grassmann-harmonic zero-curvature condition) for the first connection becomes a new equation of motion. We shall use below the similar change of basic HSS variables for the $N = 3$ SYM-theory.

In the HSS-approach to the $D = 4$, $N = 3$ SYM-theory [3], the $SU(3)/U(1) \times U(1)$ harmonics have been used for a covariant reduction of the spinor coordinates and derivatives and for the off-shell description of the theory in terms of corresponding G-analytic superfields. Moreover, it was shown that the $N = 3$ SYM-constraints in the ordinary superspace [1] can been transformed to the zero-curvature equations for the analytic harmonic gauge connections. It has been found in Ref.[4] that there exist some set of analytic connections on HSS to which no solutions of equations in the ordinary superspace correspond. However, nobody has succeeded to find interesting solutions of equations for the harmonic connections.

For completeness, we shall remark two things. First, the alternative $SL(2, C)$-harmonic interpretation of the $N = 3$ SYM-equations and the corresponding harmonic zero-curvature equation for two G-analytic connections was considered in Ref.[8], however, the problem of solving the $SL(2, C)$-harmonic equations will not be discussed in this paper. Second, the very useful light-cone gauge conditions in the ordinary superspace [11] have been considered for solutions of the 4D self-dual SYM-equations [11] and for the 10D SYM-equations [12]. We shall discuss the analogous light-cone gauge conditions in the harmonic approach.

It should be underlined that the $N = 3$ harmonic equations of motion in the original formulation contain three independent G-analytic connections, so they are more complicated than the $SU(2)$-harmonic equations using the single analytic connection. However, if we use the non-analytic superfield matrix $v$ (bridge) as a basic superfield variable of the $N = 3$ SYM-equations then the zero-curvature equations for the harmonic connections are solved automatically provided the dynamical G-analyticity conditions should be imposed. The equivalent derivation of the same dynamical bridge equations can be also considered in the SYM-representation with the flat harmonic derivatives.

It will be shown that the most simple gauge condition for the matrix $v$ simplifies drastically all basic equations, although the Lorenz invariance is broken down to the $SO(1, 1)$ subgroup by this condition. A crucial feature of this gauge is the nilpotency of the bridge matrix, namely that $v^3 = 0$.

In Sec.2, we discuss the superfield constraints of the $D = 4$, $N = 3$ SYM-theory and the light-cone gauge conditions for gauge connections which simplify the SYM-equations.

Sec. 3 is devoted to the analysis of the SYM-constraints in the $SU(3)/U(1) \times U(1)$ harmonic superspace using the light-cone nilpotent gauge for the bridge $v$. It will be shown that a partial Grassmann decomposition of the non-analytic superfield matrix $v$ is determined by three G-analytic matrices denoted by $b_1$, $b_3$ and $d_3$. The basic equations for the Grassmann connections in the bridge representation are equivalent to the set of G-analytic nonlinear harmonic differential equations for these matrices. The nonlinear terms in our G-analytic SYM-equations are proportional to Grassmann coordinates and therefore nilpotent. Using this nilpotency one can obtain linear 2-nd order differential
constraints for all three basic matrices.

In Sec. 4 we analyze the equations for the harmonic connections and superfield strengths in a manifestly analytic representation. The analytic harmonic connections are nilpotent and contain the fermionic matrices $b^1$ and $\bar{b}_3$ only. We construct also the non-analytic harmonic connections in this representation and a family of geometric objects depending on the single coefficient $b^1$ of the $v$-decomposition.

In the Appendices the basic formulas of the $SU(3)/U(1) \times U(1)$ harmonic superspace are summarized and the iterative procedure for solving the SYM harmonic differential equations is considered. The resulting finite number of the linear iterative harmonic equations can be, in principle, directly solved. Since the harmonic expansions of the SYM-superfields become very short on-shell the infinite number of auxiliary fields vanishes. The harmonic equations in our approach admit a dimensional reduction, so the further simplifications appear.

Thus, we find the simplest gauge conditions for the $N = 3$ SYM-equations in the harmonic superspace and show that these equations can be transformed to the set of linear solvable matrix differential equations. It is evident that the supersymmetry is crucial in our approach, and we expect that the supersymmetric (Grassmann) version of integrability relations yields new classes of the regular and stable SYM-solutions. It will be interesting to analyze the formal reduction of the scalar and fermion degrees of freedom in the exact constructions of these SYM-solutions in order to study solutions of the non-supersymmetric Yang-Mills equations.

2 \quad D = 4, \quad N = 3 \quad \text{SYM constraints in reduced-symmetry representation}

The coordinates of the $D = 4, \quad N = 3$ superspace are

$$z^M = (x^{\alpha \dot{\alpha}}, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}}_i) ,$$

where $\alpha, \dot{\alpha}$ are the $SL(2, C)$ indices and $i = 1, 2, 3$ are indices of the fundamental representations of the group $SU(3)$.

We shall study solutions of the SYM-equations using the non-covariant notation for these coordinates

$$x^+ \equiv x^{11} = t + x^3, \quad x^- \equiv x^{22} = t - x^3, \quad y \equiv x^{12} = x^1 + i x^2, \quad \bar{y} \equiv x^{21} = x^1 - i x^2, \quad (\theta^+_i, \theta^-_i) \equiv \theta^a_i, \quad (\bar{\theta}^{\dot{+}}_i, \bar{\theta}^{\dot{-}}_i) \equiv \bar{\theta}^{\dot{a}}_i .$$

(2.2)

suitable when the Lorenz symmetry is reduced to $SO(1, 1)$. The general $N = 3$ superspace has the odd dimension (6,6) in this notation.

The coordinates have the following $SO(1, 1)$ weights (helicities)

$$w(x^+) = 2, \quad w(x^-) = -2, \quad w(y) = w(\bar{y}) = 0, \quad w(\theta^+_i) = w(\bar{\theta}^{\dot{+}}_i) = \pm 1 .$$

(2.3)

and the simple conjugation properties

$$(x^\dagger)^\dagger = x^+, \quad (x^\dagger)^\dagger = x^-, \quad y^\dagger = \bar{y}, \quad (\theta^+_i)^\dagger = \bar{\theta}^{\dot{+}}_i, \quad (\theta^-_i)^\dagger = \bar{\theta}^{\dot{-}}_i$$

(2.4)
For products of arbitrary differential operators $X,Y$ or superfields $f$ it is convenient to use the following rules of conjugations:

\[
(XY)^\dagger = Y^\dagger X^\dagger, \quad Xf = -(1)^{p(X)p(f)}\bar{X}\bar{f},
\]

where $p(X)$ and $p(f)$ are the $Z_2$-parities.

The algebra of $D = 4$, $N = 3$ spinor derivatives in the reduced-symmetry representation can be written in the form

\[
\begin{align*}
\{D^k_+, D^j_+\} &= 0, \quad \{D^k_+, D_{i+}\} = 0, \quad \{D^k_+, D_{l+}\} = 2i\delta^k_l\partial_+, \\
\{D^k_-, D^j_-\} &= 0, \quad \{D^k_-, D_{i-}\} = 0, \quad \{D^k_-, D_{l-}\} = 2i\delta^k_l\partial_-, \\
\{D^k_+, D_{l-}\} &= 2i\delta^k_l\partial_y, \quad \{D^k_-, D_{l+}\} = 2i\delta^k_l\bar{\partial}_y, \quad \{D^k_+, D^-\} = \{D^k_+, D_-\} = 0.
\end{align*}
\]

Recall that the last two relations can contain six central charges, however, in what follows we shall consider the basic superspace without central charges.

Let us define the gauge connections $A(z)$ and the corresponding covariant derivatives $\nabla$ in the $(4|6,6)$-dimensional superspace

\[
\begin{align*}
\nabla_i^+ &= D^i_+ + A^i_+, \quad \bar{\nabla}_{i+} = \bar{D}_{i+} + \bar{A}_{i+}, \\
\nabla_{i+} &= \partial_{i+} + A_{i+}, \quad \nabla_- = \partial_- + A_-, \quad \nabla_y = \partial_y + A_y, \quad \bar{\nabla}_y = \bar{\partial}_y + \bar{A}_y,
\end{align*}
\]

then the $D = 4$, $N = 3$ SYM-constraints have the following reduced-symmetry form:

\[
\begin{align*}
\{\nabla^k_+, \nabla^l_+\} &= 0, \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l+}\} = 0, \quad \{\nabla^k_+, \bar{\nabla}_{l+}\} = 2i\delta^k_l\nabla_+, \\
\{\nabla^k_+, \nabla^-\} &= \bar{W}^{kl}, \quad \{\nabla^k_+, \bar{\nabla}_{l-}\} = 2i\delta^k_l\nabla_y, \\
\{\nabla^k_-, \bar{\nabla}_{l+}\} &= 2i\delta^k_l\bar{\nabla}_y, \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l-}\} = W^{kl}, \\
\{\nabla^k_-, \nabla^-\} &= 0, \quad \{\nabla_{k-}, \nabla_{l-}\} = 0, \quad \{\nabla^k_-, \nabla_{l-}\} = 2i\delta^k_l\nabla_-, \\
\end{align*}
\]

where $W^{kl}$ and $\bar{W}^{kl}$ are the gauge-covariant superfield strengths constructed from the gauge connections. In particular, this reduced form of the 4D constraints is convenient for the study of dimensional reduction.

The equations of motion for the superfield strengths follow from the Bianchi identities

\[
\begin{align*}
\nabla_i^+ \bar{W}^{kl} + \nabla^k_+ \bar{W}^{il} &= 0, \\
\nabla_{i+} \bar{W}^{kl} &= \frac{1}{2}(\delta^k_j \nabla_{j+} \bar{W}^{il} - \delta^l_j \nabla_{j+} \bar{W}^{jk}).
\end{align*}
\]

Superfields $W^{kl}$ satisfy the conjugated equations.

Let us analyze first Eqs.\,(2.8) together with the relations

\[
\begin{align*}
[\nabla^k_+, \nabla^\dagger_+] = [\bar{\nabla}_{k+}, \nabla^\dagger_-] = 0.
\end{align*}
\]

These equations for the positive-helicity connections have only the following pure gauge solutions

\[
(\nabla^k_+, \bar{\nabla}_{k+} , \nabla^\dagger_-) = g^{-1}(D^k_+, \bar{D}_{k+}, \partial_-)g.
\]

Thus, when $g = 1$ the simplest light-cone gauge conditions can be taken in the form

\[
A^k_+ = 0, \quad \bar{A}_{k+} = 0, \quad A_+ = 0.
\]
Note that we do not discuss here the off-shell light-cone gauge superfields [14].

The analogous gauge conditions were considered in Ref. [11] for the self-dual 4D SYM-theory and in Ref. [12] for the 10D SYM equations. It should be underlined that these superfield gauge conditions break the Lorenz group, but does not break the residual invariance with respect to translations $P_\pm, P_z, P_y, \bar{P}_y$ and supersymmetry generators $Q^k_\pm, \bar{Q}_{k\pm}$. Of course, one can choose additional non-supersymmetric gauge conditions for the remaining connections. The group parameters of the residual local gauge invariance satisfy the conditions

\[(D^k_+, \bar{D}_{k+}, \partial_{\pm})\tau_r(z) = 0 . \quad (2.16)\]

Analogously to Refs. [11, 12], we can parametrize all $N=3$ connections with $w = -1$ by the following matrix superpotential $f_-:

A^k_+ = D^k_+ f_-, \quad \bar{A}_- = \bar{D}_{k+} f_-

The equation

\[\{\nabla^k_+, \bar{\nabla}^-_i\} - \frac{1}{3} \delta^k_i \{\nabla^i_+, \bar{\nabla}^-_i\} = 0 \quad (2.18)\]

is equivalent to the linear constraint

\[(D^k_+ \bar{D}_{i+} - \frac{1}{3} \delta^k_i D^i_+ \bar{D}_{i+}) f_- = 0 . \quad (2.19)\]

Equations (2.11) give the nonlinear relations for superpotential $f_-$

\[F^l_i - \frac{1}{3} \delta^l_i F^i_l = 0 , \quad F^l_i \equiv (\bar{D}_{k-} D^i_+ + D^i_- \bar{D}_{k+}) f_- = \{D^i_- f_-, \bar{D}_{k+} f_-\}, \quad (2.20)\]

\[(D^k_- D^i_+ + D^i_- D^k_+ ) f_- + \{D^i_- f_-, D^i_- f_-\} = 0 , \quad (2.21)\]

\[(\bar{D}_{k-} \bar{D}_{i+} + \bar{D}_{i-} \bar{D}_{k+} ) f_- + \{\bar{D}_{k+} f_-, \bar{D}_{i+} f_-\} = 0 . \quad (2.22)\]

Underline that this system of the second-order equations is equivalent to the following set of the first-order equations:

\[D^k_- f_- = D^k_- \Omega + \frac{1}{2} [f_-, D^k_- f_-] , \quad (2.23)\]

\[\bar{D}_{k-} f_- = \bar{D}_{k-} \Omega + \frac{1}{2} [f_-, \bar{D}_{k-} f_-] , \quad (2.23)\]

with a suitable auxiliary superfield matrix function $\Omega$.

The equations for superpotential $f_-$ can be analyzed directly, however, we prefer to use a harmonic-superspace method instead.

\section{3 Harmonic-superspace equations for the nilpotent bridge matrix}

The Lorenz-covariant $SU(3)/U(1) \times U(1)$ harmonic superspace was introduced in Ref. [3] for the off-shell description of the $N = 3$ SYM-theory. The dynamical SYM-equations in
this approach were transformed into the set of pure harmonic equations for G-analytic superfield prepotentials.

Now we shall study the $SU(3)/U(1) \times U(1)$ harmonic superspace in another (reduced-symmetry) representation which allows us to consider the non-covariant gauges and the dimensional reduction.

We use the $SU(3)$-matrix harmonic variables $u^I_+,$ $u^I_-$ and the analytic coordinates $\zeta = (X^+, X^-, Y, \bar{Y}, \theta^\pm_1, \theta^\pm_2, \bar{\theta}^{\pm 1}, \bar{\theta}^{\pm 2})$ which describes the analytic superspace $H(4, 6|4, 4)$ (see Appendix A.1).

It is crucial that we start from the specific gauge conditions (2.15) for the $N = 3$ SYM-connections which break $SL(2, C)$, but preserve $SU(3)$. Consider the harmonic transform of the covariant Grassmann derivatives via the projections on the $SU(3)$-harmonics. As result we get so called harmonized Grassmann covariant derivatives

\[
\nabla^I_+ \equiv u^I_+ D^I_+ = D^I_+ , \quad \nabla^I_+ \equiv u^I_+ \bar{D}^I_+ = \bar{D}^I_+ , \quad \{D^I_+, \bar{D}^K_+\} = 2i \delta^I_K \partial_+ , \quad (3.1)
\]

\[
\nabla^I_- \equiv u^I_- \nabla^I_- = D^I_- + A^I_- , \quad \nabla^I_- \equiv u^I_- \nabla^I_- = \bar{D}^I_- + \bar{A}^I_- , \quad (3.2)
\]

with the harmonized Grassmann connections defined by

\[
A^I_- = u^I_- A^I_- , \quad \bar{A}^I_- = u^I_- \bar{A}^I_- . \quad (3.3)
\]

The $SU(3)$-harmonic projections of superfield constraints (2.9, 2.11) can be derived from the basic set of the $N = 3$ zero-curvature (or G-integrability) conditions for two harmonized Grassmann connections:

\[
D^I_+ A^I_- = \bar{D}^I_+ A^I_- = D^I_+ \bar{A}^I_- = \bar{D}^I_+ \bar{A}^I_- = 0 , \quad (3.4)
\]

\[
D^I_+ A^I_- + (A^I_-)^2 = 0 , \quad \bar{D}^I_- A^I_- + (A^I_-)^2 = 0 , \quad (3.5)
\]

All projections of the SYM-equations can be obtained acting by the harmonic $SU(3)$ derivatives $D^I_K$ on this basic set of conditions.

The G-integrability equations have a very simple general solution, namely

\[
A^I_-(v) = e^{-v} D^I_- e^v , \quad \bar{A}^I_-(v) = e^{-v} \bar{D}^I_- e^v , \quad (3.6)
\]

where the bridge $v$ is a superfield matrix satisfying the light-cone analyticity condition

\[
(D^I_+, \bar{D}^I_+)(v) = 0 , \quad (3.7)
\]

which is compatible with on-shell representation (3.1). Thus, $v$ does not depend on $\theta^+_1$ and $\bar{\theta}^{+ 1}$ in analytic coordinates (A.8).

Consider the gauge transformations of bridge $v$

\[
e^v \Rightarrow e^\lambda v^v e^{\tau v} , \quad (3.8)
\]

where $\lambda \in H(4, 6|4, 4)$ is a G-analytic matrix parameter, and parameter $\tau_v$ (2.18) does not depend on harmonics. Matrix $e^v$ describes a map of gauge superfields $A^I_\pm, \bar{A}^I_\pm$ defined in the central basis (CB) to the those in the analytic basis (AB). By definition, the
parameters of gauge transformations in CB are independent of harmonics, while the gauge parameters $\lambda$ are G-analytic. It is important that the off-shell $N = 3$ HSS formalism uses the same G-analytic gauge parameters.

The dynamical SYM-equations for $v$ due to Eqs. (3.6) are reduced to the following harmonic differential conditions for the basic Grassmann connections:

\[
(D^1_2, D^2_3, D^3_3) \left( A^1_2(v), \tilde{A}_3(v) \right) = 0 .
\] (3.9)

Note that these $H$-analyticity relations are trivial for Grassmann connections in the CB-superfield representation (3.3), but they become the nontrivial differential equations for the connections in the bridge representation. Equations for $v$ (3.3) are completely equivalent to the G-integrability equations (3.3), and they can be treated as a new representation of the SYM-equations.

It is not difficult to built all Grassmann CB-connections in terms of basic ones, for example,

\[
A^2_-(v) = D^2_1 A^1_2(v) , \quad A^3_-(v) = D^3_2 A^1_2(v) ,
\]
\[
\tilde{A}_1_-(v) = -D^1_3 \tilde{A}_3_-(v) , \quad \tilde{A}_2_-(v) = -D^2_3 \tilde{A}_3_-(v) .
\] (3.10)

The harmonic projections of the non-Abelian CB-superfield strengthes can also be constructed in the similar way

\[
u^i u^k W_{ik} \equiv \mathcal{W}_{23} = D_{2+} \tilde{A}_{3-} = -D_{3+} \tilde{A}_{2-} ,
\]
\[
u^i u^k \tilde{W}^{ik} \equiv \tilde{\mathcal{W}}^{12} = -D_{+}^2 A_+^1 = D_{+}^1 A_+^2 .
\] (3.11)

By construction, these CB-superfield strengthes satisfy the following non-Abelian G-analyticity relations:

\[
(D^1_2, \tilde{D}_{2+}, \tilde{D}_{3+}) \mathcal{W}_{23} = 0 , \quad (\bar{\nabla}^1_-, \bar{\nabla}_{2-}, \bar{\nabla}_{3-}) \mathcal{W}_{23} = 0 ,
\]
\[
(D^1_+, D^2_+, \tilde{D}_{3+}) \tilde{\mathcal{W}}^{12} = 0 , \quad (\bar{\nabla}^1_-, \bar{\nabla}_{2-}, \bar{\nabla}_{3-}) \tilde{\mathcal{W}}^{12} = 0
\] (3.12) (3.13)

and the simple H-analyticity conditions

\[
(D^1_2, D^2_3, D^3_3)(\mathcal{W}_{23}, \tilde{\mathcal{W}}^{12}) = 0 .
\] (3.14)

The G- and H-analyticity conditions for non-Abelian superfields $\mathcal{W}_{23}$ and $\tilde{\mathcal{W}}^{12}$ generate the corresponding component SYM-equations of motion.\footnote{The analogous harmonic projections of the $D = 4, N = 3$ superfield strengthes have been considered in Ref.~[16] for the Abelian case. The Abelian harmonic superfields $\mathcal{W}_{23}$ and $\tilde{\mathcal{W}}^{12}$ describe the ultrashort on-shell representations of the $N = 3$ superconformal group.}

Now we shall determine the explicit form of bridge $v$. Using the off-shell (4, 4)-analytic $\lambda$-transformations

\[
\delta v = \lambda + \frac{1}{2} [\lambda, v] + \ldots
\] (3.15)

one can choose the following non-superantisymmetric nilpotent gauge condition for $v$

\[
v = \theta_1 b^{1} + \bar{\theta}^{3} \bar{b}_3 + \theta_1 \bar{\theta}^{3} - d^{3}_3 , \quad v^2 = \theta_1 \bar{\theta}^{3} [\bar{b}_3, b^{1}] , \quad v^3 = 0 ,
\] (3.16)

\[
e^{-v} = I - v + \frac{1}{2} v^2 = I - \theta_1 b^{1} - \bar{\theta}^{3} \bar{b}_3 + \theta_1 \bar{\theta}^{3} \left( \frac{1}{2} [\bar{b}_3, b^{1}] - d^{3}_3 \right) ,
\] (3.17)
where fermionic matrices $b^1, \bar{b}_3$ and bosonic matrix $d_3$ are analytic functions of coordinates $\zeta$ introduced in (A.3).

Note that nilpotent gauges for harmonic bridges are possible in the harmonic formalisms with off-shell analytic gauge groups only [3, 9]. Our gauge for $v$ combines the action of the analytic gauge group and the light-cone analyticity condition (3.7).

Next we shall use the harmonic tilde-conjugation (A.9, A.11) to describe the reality conditions for the gauge superfields in HSS. For instance, the Hermitian conjugation $\dagger$ of the superfield matrices includes transposition and this conjugation. The conditions for bridge $v$ in the gauge group $SU(n)$ are

$$\text{Tr } v = 0, \quad v^\dagger = -v,$$

so that matrices $b^1, \bar{b}_3$ and $d_3$ have the following properties in the group $SU(n)$:

$$\text{Tr } b^1 = 0, \quad \text{Tr } \bar{b}_3 = 0, \quad \text{Tr } d_3 = 0,$$

$$\left( b^1 \right)^\dagger = \bar{b}_3, \quad \left( d_3 \right)^\dagger = -d_3.$$

The fermionic matrices have specific properties of traces

$$\text{Tr } (b^1)^{2k} = 0, \quad \text{Tr } (b^1 \bar{b}_3) = -\text{Tr } (\bar{b}_3 b^1).$$

It is useful to consider the explicit parametrization of Grassmann connection $A_1^+(v)$ and $\bar{A}_{3-}(v)$ in terms of basic analytic matrices (3.10)

$$A_1^+(v) \equiv e^{-v} D_1 e^v = b^1 - \theta_1^- (b^1)^2 + \theta_3^- f_3^1 + \theta_1^- \theta_3^- [b^1, f_3^1],$$

$$\bar{A}_{3-} \equiv e^{-v} \bar{D}_{3-} e^v = \bar{b}_3 + \theta_1^- \bar{f}_3^1 - \theta_3^- (\bar{b}_3)^2 + \theta_1^- \theta_3^- [\bar{f}_3^1, \bar{b}_3],$$

where the following auxiliary superfields are introduced:

$$f_3^1 = d_3 - \frac{1}{2} \{ b^1, \bar{b}_3 \}, \quad \bar{f}_3^1 = -d_3^\dagger - \frac{1}{2} \{ b^1, \bar{b}_3 \}.$$

Equations $(D_2^1, D_3^1) A_1^+(v) = 0$ generate the following independent relations for the (4,4)-analytic matrices:

$$D_2^1 b^1 = -\theta_2^- (b^1)^2,$$

$$D_3^2 b^1 = -\theta_2^- f_3^1,$$

$$D_2^1 f_3^1 = \theta_2^- [f_3^1, b^1], \quad D_3^2 f_3^1 = 0.$$

Equations $(D_2^1, D_3^1) \bar{A}_{3-}(v) = 0$ are equivalent to the relations

$$D_2^1 \bar{b}_3 = \theta_2^- \bar{f}_3^1, \quad D_3^2 \bar{b}_3 = \theta_2^- (\bar{b}_3)^2,$$

$$D_2^1 \bar{f}_3^1 = 0, \quad D_3^2 \bar{f}_3^1 = \theta_2^- [\bar{b}_3, \bar{f}_3^1].$$

In the case of the gauge group $SU(n)$, the last equations are not independent, but can be obtained by conjugation from the equations for $b^1$ and $f_3^1$. 

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It is useful to derive the following relations for matrices $b^1$ and $\bar{b}_3$ which do not contain auxiliary matrices $d^1_3, f^1_3$ or $\bar{f}^1_3$:

\[
\begin{align*}
\theta_2^- D^1_2(b^1, \bar{b}_3) &= 0, \\
\bar{\theta}^2^- D^2_3(b^1, \bar{b}_3) &= 0, \\
\theta_2^- D^2_3b^1 + \bar{\theta}^2^- D^1_2\bar{b}_3 &= \theta_2^- \bar{\theta}^2^- \{b^1, \bar{b}_3\}.
\end{align*}
\] (3.30) (3.31)

Solutions of the linear equations for matrices $f^1_3$ and $\bar{f}^1_3$ satisfy the subsidiary condition

\[
f^1_3 + \bar{f}^1_3 = -\{b^1, \bar{b}_3\}.
\] (3.32)

The equation for independent matrix $d^1_3$ is more complicated than Eq. (3.27)

\[
D^1_2d^1_3 = \frac{1}{2}\theta_2^- \left( [d^1_3, b^1] + \frac{1}{2}((b^1)^2, \bar{b}_3) \right).
\] (3.33)

The nilpotency of the nonlinear parts in these equations yields the subsidiary linear conditions for the coefficient functions

\[
D^1_2D^1_2(b^1, \bar{b}_3, d^1_3) = 0, \quad D^2_3D^2_3(b^1, \bar{b}_3, d^1_3) = 0.
\] (3.34)

Note that one can also find the following additional relations:

\[
\begin{align*}
(D^1_2b^1)^2 &= (D^2_3)^2(b^1)^2 = 0, \\
(D^1_2b^1)^2 &= D^1_2b^1D^2_3\bar{b}_3 = D^1_2b^1D^1_2d^1_3 = 0.
\end{align*}
\] (3.35) (3.36)

The harmonic linear equations for analytic superfields $b^1, \bar{b}_3, d^1_3$ have simple (short) solutions, i.e. the solutions with the finite number of the harmonic on-shell field components. This shortness is an important property of the SYM-solutions in the harmonic approach.

Now we shall study the action of the non-analytic harmonic derivatives on the basic matrices. First, let us consider the following relation:

\[
D^1_2(D^1_2)^2A^1_1(v) = (D^1_2)^2D^1_2A^1_1(v) = 0.
\] (3.37)

Equation $(D^1_2)^2A^1_1(v) = 0$ produces the non-analytic equations for the analytic matrices

\[
\begin{align*}
(D^1_2)^2b^1 &= \theta_1^- (D^1_2)^2(b^1)^2 = 2\theta_1^- (D^1_2b^1)^2, \\
(D^1_2)^2f^1_3 &= \theta_1^- (D^1_2)^2[b^1, f^1_3].
\end{align*}
\] (3.38) (3.39)

Using relation $D^2_3A^1_{3-}(v) = 0$ one can obtain the following equations:

\[
D^2_3\bar{b}_3 = \theta_1^- \left( D^1_2d^1_3 + \frac{1}{2}\{\bar{b}_3, D^1_2b^1\} \right), \quad \theta_1^- D^2_3\bar{b}_3 = 0.
\] (3.40)

It is important that all differential harmonic equations (3.29)-(3.33) contain nilpotent elements $\theta_2^-$ or $\bar{\theta}^2$ in the nonlinear parts, so the simplest iteration procedure for finding their solutions can be obtained via a partial Grassmann decomposition. In Appendix A.2, we consider the iterative procedure of solving the basic non-Abelian harmonic differential equations for the (4,4) analytic matrices $b^1$ and $\bar{b}_3$ (3.30)-(3.31) using the partial
decomposition in the Grassmann variables $\theta^-, \bar{\theta}^\pm, \bar{\theta}^{1-}, \bar{\theta}^{2-}$. The matrix $(4,0)$ coefficients of this decomposition have dimensions $-1/2 \geq l \geq -5/2$. The first iterative $(4,0)$ equations ($l = -1/2$) are linear and homogeneous. The next harmonic iterative equations for the $(4,0)$ components with $l \leq -1/2$ are resolved in terms of functions of the highest dimensions or their harmonic derivatives and contain also the nonlinear sources constructed from the solutions of the previous iterative equations. Note that some $(4,0)$ iterative equations are pure algebraic relations which reduce the number of independent functions. The harmonic differential equations for the independent $(4,0)$ functions can be, in principle, explicitly solved using the corresponding superfield Green functions.

Thus, the $SU(3)/U(1) \times U(1)$ harmonic method transforms the $N = 3$ superfield SYM-constraints together with the simple gauge conditions (2.15) to the non-Abelian harmonic SYM-equations (3.9) which are equivalent to the finite set of the iterative solvable linear equations.

Let us consider now the inverse harmonic transform which determines the on-shell gauge superfields in the ordinary superspace

$$A_1^i(v) \Rightarrow A_1^i(v) = (u_1^i + u_2^i D_1^2 + u_3^i D_3^2) e^{-v} D_1^1 e^v,$$

$$A_3^i(v) \Rightarrow A_3^i(v) = (u_1^i - u_2^i D_1^2 - u_3^i D_3^2) e^{-v} D_3^1 e^v,$$

(3.41)

where definitions (3.10) and relations (A.3) are used. By construction, these superfields satisfy the $D = 4, N = 3$ CB-constraints (2.9-2.11) and the harmonic differential conditions

$$D_K^I \left( A_I^i(v), \bar{A}_i^j(v) \right) = 0 ,$$

(3.42)

if equations (3.9) or the equivalent equations for bridge $v$ are fulfilled.

## 4 Analytic representation of solutions

In order to understand more deeply the geometric structure of our harmonic-superspace solutions it is useful to represent them in the analytic basis. Remember that the following covariant Grassmann derivatives are flat in the analytic representation of the gauge group before the gauge fixing:

$$e^v \nabla_\pm^1 e^{-v} \equiv \hat{\nabla}_\pm^1 = D_\pm^1, \quad e^v \nabla_{3\pm}^1 e^{-v} \equiv \hat{\nabla}_{3\pm}^1 = \hat{D}_{3\pm} .$$

(4.1)

The harmonic transform of the covariant derivatives via matrix $e^v$ (3.8) determines in AB the on-shell harmonic connections as a function of $v$

$$\nabla_K^I \equiv e^v D_K^I e^{-v} = D_K^I + V_K^I(v) ,$$

$$V_K^I(v) = e^v (D_K^I e^{-v}) .$$

(4.2)

Note that the harmonic connections in the bridge representations satisfy automatically the harmonic zero-curvature equations, for instance,

$$D_2^1 V_3^1 - D_3^2 V_2^1 + [V_2^1, V_3^1] - V_3^1 = 0 ,$$

$$D_2^1 V_3^1 - D_3^2 V_2^1 + [V_2^1, V_3^1] = 0 .$$

(4.3)
In the off-shell $N = 3$ formalism [3], the connections $V_2^1, V_3^2$ and $V_3^3$ are G-analytic by construction, so the harmonic-zero curvature equations are interpreted as the basic equations of motion. We prefer the bridge representation (4.2), since it is directly connected with the classical SYM-solutions.

It is evident that basic equations (3.9) are equivalent to the following set of the dynamic G-analyticity relations for the composed harmonic connections:

$$ (D_1^1, \bar{D}_{3-}) \left( V_2^1(v), V_3^2(v), V_3^3(v) \right) = 0 , $$

(4.4)

The positive-helicity analyticity conditions

$$ (D_1^+, \bar{D}_{3+})(V_2^1, V_3^2, V_3^3)(v) = 0 $$

(4.5)

are satisfied automatically for bridge $v$ in gauge (3.7).

The analytic SYM-equations (3.25-3.29) are equivalent to the following relations for the nilpotent on-shell analytic connections:

$$ e_v D_2^1 e^{-v} = V_2^1 , \quad (V_2^1)^2 = 0 , $$

(4.6)

$$ e_v D_3^2 e^{-v} = -\bar{\theta}^2 \bar{b}_3 \equiv V_3^2 , \quad (V_3^2)^2 = 0 , $$

(4.7)

which can be also rewritten as the harmonic differential equations

$$ (D_1^1, D_3^2)e^v = -(V_2^1, V_3^2)e^v . $$

(4.8)

The Grassmann connections $a_I^I$ and $\bar{a}_{K\pm}$ in AB can be calculated analogously to Ref.[10]. We get

$$ \hat{\nabla}^I_{\pm} = [\nabla^I_{\pm}, D^I_{\pm}] = D^I_{\pm} + a_I^I , \quad I = 2, 3 $$

(4.9)

$$ a_2^I = -D_2^I V_2^1 , \quad a_3^I = -D_3^I V_3^1 , $$

(4.10)

$$ \hat{\nabla}_{K\pm} = -[\nabla^3_K, D_{3\pm}] = D_{K\pm} + \bar{a}_{K\pm} , \quad K = 1, 2 $$

(4.11)

$$ \bar{a}_{2\pm} = \bar{D}_{3\pm} V_2^3 , \quad \bar{a}_{1\pm} = \bar{D}_{3\pm} V_1^3 $$

(4.12)

where $V_2^1, V_3^1$ and $V_3^3$ are the non-analytic harmonic connections defined in (4.2).

The nilpotent on-shell analytic connections satisfy the subsidiary conditions

$$ D_2^1 V_2^1 = \theta_2^- D_2^1 b^1 = 0 , \quad D_3^2 V_3^2 = -\bar{\theta}^2 \bar{D}_3^2 \bar{b}_3 = 0 . $$

(4.13)

By using Eqs.(3.40) harmonic connection $V_1^2$ can be written in terms of superfield $b^1$ alone

$$ e^v D_1^2 e^{-v} = V_1^2 = -\theta_1^- D_1^2 b^1 . $$

(4.14)

Connection $V_2^3$ can be obtained by conjugation of $V_1^2$

$$ V_2^3 = -(V_1^2)^\dagger = -\bar{\theta}^3^- \bar{D}_2^3 \bar{b}_3 . $$

(4.15)

Both connections satisfy the partial G-analyticity conditions

$$ \bar{D}_{3\pm} V_2^2 = 0 , \quad D_{1\pm} V_3^2 = 0 . $$

(4.16)
The 3-rd harmonic analytic connection can be readily calculated
\[ V_3^1 = D_2^1 V_3^2 - D_3^2 V_2^1 + [V_2^1, V_3^2] = \theta_3^{-1} b^1 - \bar{\theta}_3^{-1} \bar{b}_3 , \] (4.17)
where Eq.(3.31) is used. One can check straightforwardly the relations
\[ (V_3^1)^3 = 0 , \quad D_3^1 V_3^1 = \theta_3^{-1} \bar{\theta}_3^{-1} \{ b^1, \bar{b}_3 \} , \quad (D_3^1)^2 V_3^1 = 0 . \] (4.18)

The harmonic equations
\[
\begin{align*}
D_3^1 b^1 &= -\theta_3^{-1} (b^1)^2 - \bar{\theta}_3 f_3^1 , \\
D_3^1 \bar{b}_3 &= \bar{\theta}_3^{-1} (\bar{b}_3)^2 + \theta_3 f_3^1 , \\
\theta_2^* D_3^1 b^1 + \bar{\theta}_2^* D_2^1 \bar{b}_3 &= -\theta_2^* \theta_3 (b^1)^2 + \theta_2^* \bar{\theta}_3 (\bar{b}_3)^2 + \theta_2^* \bar{\theta}_3 \{ b^1, \bar{b}_3 \} , \\
\theta_3^* D_3^1 b^1 + \bar{\theta}_3^* D_3^1 \bar{b}_3 &= \theta_3^* \bar{\theta}_3 \{ b^1, \bar{b}_3 \} , \\
D_3^1 f_3^1 &= \theta_3^* [f_3^1, b^1] .
\end{align*}
\]
(4.19)-(4.23)
can be also derived directly from Eqs.(3.25-3.29).

Finally, for the last non-analytic harmonic connection we get
\[ V_3^1 = D_2^3 V_2^1 - D_1^2 V_2^3 + [V_2^1, V_2^3] \equiv e^v D_1^3 e^{-v} . \] (4.24)

It is convenient to calculate the superfield strength in AB
\[ \bar{w}^{12} = -D_1^1 D_1^3 V_1^2 = -D_2^2 b^1 . \] (4.25)
where Eq.(4.14) is used.

Stress that the single coefficient matrix $b^1$ generates the family of the AB-geometric objects: $V_2^1, V_2^2, a_{1\pm}$ and $\bar{w}^{12}$. The conjugated $\bar{b}_3$-family of superfields contains $V_3^2, V_3^3, \bar{a}_{2\pm}$ and
\[ w_{23} = -\bar{D}_{3+} D_2 V_3^2 = \bar{D}_{2+} \bar{b}_3 . \] (4.26)

The AB-superfield strengths are directly connected with the gauge CB-superfield strengths (3.11), e.g. $w_{23} = e^v \mathcal{W}_{23} e^{-v}$. These superfields satisfy the (4,2)-dimensional G-analyticity conditions, for instance,
\[ D_{\pm}^1 \bar{w}^{12} = D_{3\pm} \bar{w}^{12} = D_{2\pm}^2 \bar{w}^{12} + [a_{2\pm}^2, \bar{w}^{12}] = 0 \] (4.27)
and the non-Abelian H-analyticity conditions
\[ (\nabla_2^1, \nabla_3^2, \nabla_3^1) \bar{w}^{12} = 0 . \] (4.28)
The (4,2)-analytic superspaces have been considered earlier in Refs. [13, 10].

It should be noted that function $d_3^1$ (or $f_3^1$) is an auxiliary quantity in the framework of the analytic basis, since all harmonic and Grassmann connections and tensors in this basis can be expressed by means of superfields $b^1$ and $\bar{b}_3$ only. Nevertheless, the construction of $d_3^1$ is important for the transition to the central basis.
5 Conclusions and discussion

Our method combines the light-cone gauge conditions for the \( N = 3 \) gauge superfields with the \( SU(3) \)-harmonic superspace approach. We have described the harmonic transform of the \( N = 3 \) SYM-equations of motion in the standard superspace into the Grassmann integrability conditions for the harmonized connections. All Grassmann and harmonic connections have been constructed via the nilpotent superfield bridge matrix.

The nilpotency of nonlinear terms in the basic harmonic differential SYM-equations simplifies drastically the iterative procedure of solving these equations. Using the partial decomposition of the basic (4,4)-superfields in terms of the Grassmann coordinates of negative helicities we have obtained the finite set of solvable linear (4,0)-equations which can be used for the explicit construction of SYM-solutions.

The dimensional reduction of our harmonic equations simplify the construction the \( D < 4 \) solutions of the supersymmetric models with 12 supercharges. Since the \( N = 2 \) superfield SYM-equations of motion have the similar G-analyticity representation \[6\] the existence of regular solutions with eight supercharges is not excluded. It will be interesting to analyze the formal reduction of the scalar and fermion degrees of freedom in the exact constructions of supersymmetric solutions in order to estimate possibilities of our supersymmetric methods in solving the non-supersymmetric Yang-Mills equations.

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A Appendices

A.1 Reduced-symmetry form of \( N = 3 \) harmonic superspace

The \( SU(3)/U(1) \times U(1) \) harmonics \[3, 15\] parametrize the corresponding 6-dimensional coset space. They form an \( SU(3) \) matrix \( u_I^i \) and are defined modulo \( U(1) \times U(1) \)

\[
u_1^i = u_i^{(1,0)}, \quad u_2^i = u_i^{(-1,1)}, \quad u_3^i = u_i^{(0,-1)}, \quad (A.1)
\]

where \( i \) is the index of the triplet representation of \( SU(3) \), and the upper indices \( I = 1, 2, 3 \) correspond to different combinations of the \( U(1) \)-charges.

The complex conjugated harmonics \( u_I^i \) have opposite \( U(1) \) charges and the upper index \( i \) of the anti-triplet representation of \( SU(3) \)

\[
u_1^i = u_i^{(-1,0)}, \quad u_2^i = u_i^{(1,-1)}, \quad u_3^i = u_i^{(0,1)}. \quad (A.2)
\]

These harmonics satisfy the following relations:

\[
u_i^j u_j^i = \delta_j^i, \quad u_i^j u_i^k = \delta^k_i, \quad \varepsilon^{ikl} u_i^1 u_k^2 u_l^3 = 1. \quad (A.3)
\]
The $SU(3)$-invariant harmonic derivatives act on the harmonics

$$\partial^I J^I = \delta^I_J u^J, \quad \partial^I J^J = -\delta^I_J u^J,$$

(A.4)

and

$$[\partial^I, \partial^K] = \delta^I_J \partial^K_L - \delta^K_I \partial^J_L.$$

(A.5)

The operators of the $U(1)$ charges on the harmonics are

$$h = \partial_1^1 - \partial_2^2, \quad h' = \partial_2^2 - \partial_3^3.$$

(A.6)

One can consider the triplet of harmonic derivatives which annihilate two harmonics

$$(\partial_1^1, \partial_2^2, \partial_3^3)(u^1_j, u^3_k) = 0.$$

(A.7)

These harmonics are connected with the G-analyticity conditions.

We can define the real analytic harmonic superspace $H(4,6|4,4)$ with 6 coset harmonic dimensions $u^I_i$ and the following set of 4 even and $(4+4)$ odd coordinates:

$$\zeta = (X^\pm, X^\equiv, Y, \bar{Y}) \bigg| \theta^\pm 1, \theta^\equiv 1, \theta^{1\pm}, \theta^{2\pm}) , \quad X^\pm = x^\mp + i(\theta_3^2 \bar{\theta}^{3\mp} - \theta_1^2 \bar{\theta}^{1\mp}),$$

$$X^\equiv = x^\equiv + i(\theta_3^2 \bar{\theta}^{3\equiv} - \theta_1^2 \bar{\theta}^{1\equiv}), \quad Y = y + i(\theta_3^2 \bar{\theta}^{3\equiv} - \theta_1^2 \bar{\theta}^{1\equiv}),$$

$$\bar{Y} = \bar{y} + i(\theta_3^2 \bar{\theta}^{3\equiv} - \theta_1^2 \bar{\theta}^{1\equiv}).$$

(A.8)

This superspace is covariant with respect to the $N = 3$ supersymmetry transformations.

The special $SU(3)$-covariant tilde-conjugation of harmonics

$$u^1_1 \leftrightarrow u^3_3, \quad u^1_3 \leftrightarrow u^1_3, \quad u^2_2 \leftrightarrow -u^2_2,$$

(A.9)

is compatible with conditions (A.3). On the harmonic derivatives of an arbitrary harmonic function $f(u)$ this conjugation acts as follows

$$\tilde{\partial}^I J^I = -\partial_1^I \tilde{f}, \quad \tilde{\partial}^I J^I = \partial_2^I \tilde{f}.$$

(A.10)

The tilde-conjugation of the odd analytic coordinates has the following form:

$$\theta_1^\pm \leftrightarrow \bar{\theta}^{3\pm}, \quad \theta_3^\pm \leftrightarrow \bar{\theta}^{1\pm}, \quad \theta_2^\pm \leftrightarrow -\bar{\theta}^{2\pm}.$$

(A.11)

Coordinates $X^\pm$ and $X^\equiv$ are real and $\bar{Y} = Y$.

The corresponding CR-structure involves the derivatives

$$D^1_1, \bar{D}^3_3, D^1_2, D^2_3, D^1_3 = [D^1_2, D^2_3],$$

(A.12)

which have the following explicit form in these coordinates:

$$D^1_1 \equiv D^1_1 = \partial_1^1 \equiv \partial/\partial \theta_1^\pm, \quad \bar{D}^3_3 \equiv D^3_3 = \partial_3^\pm \equiv \partial/\partial \bar{\theta}^{3\pm},$$

$$D^2_2 \equiv D^2_2 = \partial_2^2 + i\theta_2^3 \bar{\partial}^{3\equiv} + i\theta_3^3 \bar{\partial}^{3\equiv} \partial_2^2 + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 - \theta_2^3 \bar{\theta}^{3\equiv} \partial_2^2 + \bar{\theta}^{3\equiv} \partial_2^2,$$

$$D^2_2 \equiv D^2_2 = \partial_2^2 + i\theta_3^3 \bar{\partial}^{3\equiv} + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 - \theta_2^3 \bar{\theta}^{3\equiv} \partial_2^2 + \bar{\theta}^{3\equiv} \partial_2^2,$$

$$D^2_2 \equiv D^2_2 = \partial_2^2 + i\theta_3^3 \bar{\partial}^{3\equiv} + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 + i\theta_1^3 \bar{\partial}^{1\equiv} \partial_2^2 - \theta_2^3 \bar{\theta}^{3\equiv} \partial_2^2 + \bar{\theta}^{3\equiv} \partial_2^2,$$

(A.13)

$D^1_3 \equiv D^1_3 = \partial_3^1 + 2i\theta_3^2 \bar{\partial}^{2\equiv} + 2i\theta_2^3 \bar{\partial}^{3\equiv} \partial_3^1 + 2i\theta_2^3 \bar{\partial}^{3\equiv} \partial_3^1 + 2i\theta_3^3 \bar{\partial}^{3\equiv} \partial_3^1 = -\theta_3^2 \bar{\theta}^{3\equiv} \partial_3^1 + \bar{\theta}^{3\equiv} \partial_3^1,$$

(A.14)
where \( \partial_+ = \partial / \partial X^+ \), \( \partial_- = \partial / \partial X^- \), \( \partial_\nu = \partial / \partial Y \) and \( \bar{\partial}_\nu = \partial / \partial Y \).

One can construct also all harmonic derivatives in these coordinates

\[
D^{(-2,1)} \equiv D_1^2 = \bar{\partial}_\nu^2 - i\theta_1^+ \bar{\partial}_\nu^2 - \bar{\theta}_1^+ \bar{\partial}_\nu^2 - i\theta_1^- \bar{\partial}_\nu^2 - i\theta_1^- \bar{\partial}_\nu^2 - \bar{\theta}_1^- \bar{\partial}_\nu^2, \quad (A.15)
\]

\[
D^{(1,-2)} \equiv D_2^2 = \bar{\partial}_\nu^2 - i\theta_2^+ \bar{\partial}_\nu^2 - \bar{\theta}_2^+ \bar{\partial}_\nu^2 - i\theta_2^- \bar{\partial}_\nu^2 - i\theta_2^- \bar{\partial}_\nu^2 - \bar{\theta}_2^- \bar{\partial}_\nu^2, \quad (A.16)
\]

\[
D^{(-1,-1)} \equiv D_3^3 = \bar{\partial}_\nu^2 - i\theta_1^+ \bar{\partial}_\nu^2 - 2i\theta_1^+ \bar{\partial}_\nu^2 - 2i\theta_1^- \bar{\partial}_\nu^2 - 2i\theta_1^- \bar{\partial}_\nu^2 - \bar{\theta}_1^+ \bar{\partial}_\nu^2 - \bar{\theta}_1^- \bar{\partial}_\nu^2, \quad (A.17)
\]

\[
H = [D_2^1, D_1^1] = h - \theta_1^+ \bar{\partial}_1^1 - \theta_1^- \bar{\partial}_1^1 + \theta_2^+ \bar{\partial}_2^2 + \theta_2^- \bar{\partial}_2^2 + \theta_3^+ \bar{\partial}_3^3 + \theta_3^- \bar{\partial}_3^3 + \bar{\partial}_2^+ \bar{\partial}_2^- + \bar{\partial}_3^+ \bar{\partial}_3^-, \quad (A.18)
\]

\[
H' = [D_3^2, D_2^2] = h' - \theta_2^+ \bar{\partial}_2^2 - \theta_2^- \bar{\partial}_2^2 + \theta_3^+ \bar{\partial}_3^3 + \theta_3^- \bar{\partial}_3^3 + \bar{\partial}_2^+ \bar{\partial}_2^- + \bar{\partial}_3^+ \bar{\partial}_3^-, \quad (A.19)
\]

and the additional Grassmann derivatives

\[
D_+^2 = \bar{\partial}_\nu^2 + i\theta^2_+ \bar{\partial}_\nu^2 + i\theta^2_- \bar{\partial}_\nu^2, \quad D_-^2 = \bar{\partial}_\nu^2 + i\theta^2_+ \bar{\partial}_\nu^2 + i\theta^2_- \bar{\partial}_\nu^2, \quad (A.20)
\]

\[
D_+^3 = \bar{\partial}_\nu^2 + 2i\theta^3_+ \bar{\partial}_\nu^2 + 2i\theta^3_- \bar{\partial}_\nu^2, \quad D_-^3 = \bar{\partial}_\nu^2 + 2i\theta^3_+ \bar{\partial}_\nu^2 + 2i\theta^3_- \bar{\partial}_\nu^2, \quad (A.21)
\]

\[
\bar{D}_+^1 = \bar{\partial}_\nu^1 + 2i\theta^1_+ \bar{\partial}_\nu^1 + 2i\theta^1_- \bar{\partial}_\nu^1, \quad \bar{D}_-^1 = \bar{\partial}_\nu^1 + 2i\theta^1_+ \bar{\partial}_\nu^1 + 2i\theta^1_- \bar{\partial}_\nu^1, \quad (A.22)
\]

\[
\bar{D}_+^2 = \bar{\partial}_\nu^2 + i\theta^2_+ \bar{\partial}_\nu^2 + i\theta^2_- \bar{\partial}_\nu^2, \quad \bar{D}_-^2 = \bar{\partial}_\nu^2 + i\theta^2_+ \bar{\partial}_\nu^2 + i\theta^2_- \bar{\partial}_\nu^2. \quad (A.23)
\]

The tilde-conjugation of harmonic derivatives \( D_2^1, D_3^2 \) and \( D_3^1 \) is defined by analogy with relations \( [A.14] \), and conjugation of the Grassmann derivatives has the following form:

\[
D_+^1 F \leftrightarrow - (\bar{D}_-^1 \bar{F}) \quad , \quad D_-^1 F \leftrightarrow (\bar{D}_+^1 \bar{F}) \quad , \quad D_+^2 F \leftrightarrow (\bar{D}_+^1 \bar{F}) \quad , \quad D_-^2 F \leftrightarrow - (\bar{D}_-^1 \bar{F}), \quad (A.24)
\]

where \( F \) is some superfield.

### A.2 Analysis of iterative HSS equations

Let us analyze the analytic equations of Sec. 3 for the basic fermionic \((4,4)\) matrices \( b^1 \) and \( c_3 \)

\[
D_1^1 b^1 = -\theta_5^- (b^1)^2, \quad (D_1^1)^2 b^1 = (D_2^2)^2 b^1 = 0, \quad (A.25)
\]

\[
D_3^3 b^1 = \bar{\theta}^2 -(b_3)^2, \quad (D_3^3)^2 b^1 = (D_3^2)^2 b^1 = 0, \quad (A.26)
\]

\[
\theta_5^- D_3^3 b^1 + \bar{\theta}^2 D_3^1 b_3 = \theta_5^- \theta^2 - \{b^1, \bar{b}_3\}, \quad (A.27)
\]

\[
\theta_5^+ D_3^3 b^1 + \bar{\theta}^2 D_3^1 b_3 = \theta_5^+ \theta^2 - \{b^1, \bar{b}_3\}, \quad (A.28)
\]

\[
\theta_5^- D_3^3 b^1 + \bar{\theta}^2 D_3^1 b_3 = -\theta_5^- \theta^2 - \{b^1, \bar{b}_3\} + \theta_5^- \theta^2 - \{b^1, \bar{b}_3\}. \quad (A.29)
\]

The nonlinear terms in these equations contain the negative-helicity Grassmann coordinates \( \theta^-_2, \theta^+ _5, \theta^- _1 \) and \( \bar{\theta}^- _5 \), so a partial decomposition in terms of these coordinates is very useful for the iterative analysis of solutions. Equations for auxiliary matrix \( d_3^3 \) do not give additional restrictions on \( b^1 \) and \( \bar{b}_3 \).
Consider first the decomposition of harmonic derivatives (A.14) and define the harmonic derivatives on the (4,0) analytic functions depending on the analytic Grassmann variables $\theta^+_2$, $\theta_3^+$, $\theta^{1+}$ and $\theta^{2+}$

$$
\hat{D}^1_2 = \partial^1_2 + i\theta^+_2 \bar{\partial}^{1+} \partial^+_2 - \theta^+_2 \partial^+_4 + \bar{\theta}^{1+} \bar{\partial}^+_2 ,
$$

$$
\hat{D}^2_3 = \partial^2_3 + i\theta^+_3 \bar{\partial}^{2+} \partial^+_4 - \theta^+_3 \partial^+_4 + \bar{\theta}^{2+} \bar{\partial}^+_3 ,
$$

$$
\hat{D}^3_4 = \partial^3_4 + 2i\theta^+_3 \bar{\partial}^{1+} \partial^+_4 - \theta^+_3 \partial^+_4 + \bar{\theta}^{1+} \bar{\partial}^+_3 .
$$

(A.30)

The (4,0) decomposition of the (4,4) analytic matrix function has the following form:

$$
b^1 = \beta^1 + \theta^{-}_2 B^{12} + \theta^{-}_3 B^{13} + \bar{\theta}^{1-} B^0 + \bar{\theta}^{2-} B^1 + \theta^+_2 \theta^+_3 \beta^0 + \theta^+_2 \bar{\theta}^{1-} \beta^2 + \theta^+_3 \bar{\theta}^{2-} \beta^{13}
$$

$$
+ \theta^{-}_3 \bar{\theta}^{1-} \beta^3 + \bar{\theta}^{1-} \bar{\theta}^{2-} \beta_2 + \theta^+_2 \bar{\theta}^{2-} \eta^1 + \theta^+_2 \theta^+_3 \bar{\theta}^{1-} B^{23} + \theta^+_3 \bar{\theta}^{2-} B^2 + \theta^+_2 \theta^+_3 \bar{\theta}^{2-} C^{13}
$$

$$
+ \theta^- _3 \bar{\theta}^{2-} C^0 + \theta^+_2 \theta^+_3 \bar{\theta}^{1-} \bar{\theta}^{2-} \eta^3 .
$$

(A.31)

The analogous decompositions can be written for $\hat{b}_3$ and $d^1_3$.

It is easily to show that a part of the (4,0) coefficients can be constructed as the algebraic functions of the basic set of independent (4,0) matrices.

$$
B^1_3 = -\hat{D}^1_2 B^0 - i\theta^+_2 \partial \beta^1 ,
$$

(A.32)

$$
B^{12} = \hat{D}^1_2 B^{13} + i\bar{\theta}^{2+} \bar{\partial} \beta^1 ,
$$

(A.33)

$$
\beta^2 = \hat{D}^2_3 \beta^3 - i\bar{\theta}^{2+} \bar{\partial} B^0 ,
$$

(A.34)

$$
\eta^1 = -\hat{D}^1_2 \beta^2 + i\bar{\theta}^{1+} \bar{\partial} B^0 + i\theta^+_2 \partial \beta^1 + [\beta^1, B^0] ,
$$

(A.35)

$$
\beta^{13}_3 = -\hat{D}^1_2 \beta^3 + i\theta^+_2 \partial B^{13} ,
$$

(A.36)

$$
C^{13} = -\hat{D}^1_2 B^{23} - i\bar{\theta}^{1+} \bar{\partial} \beta^3 - \{\beta^1, \beta^3\} - [B^{13}, B^1_2] ,
$$

(A.37)

$$
C^0 + \bar{C}^0 = \hat{D}^1_2 B^1_2 + \hat{D}^2_3 B^2_3 + i\partial \{\bar{B}^0 - B^0\} + i\bar{\theta}^{2+} \bar{\partial} \beta_2 - i\theta^+_3 \partial \beta^3 - i\bar{\theta}^{1+} \bar{\partial} \beta_1
$$

$$
- i\theta^+_2 \partial \beta^2 - \{\beta_1, \beta^1\} - \{\beta_3, \beta^3\} + [B^0, B^0] + [B_{13}, B^{13}] .
$$

(A.38)

Thus, the independent (4,0) matrix functions are

$$
B^0, B^{13}, B^{23}, (C^0 - \bar{C}^0), B^2_3, \beta^1, \beta_2, \beta^3, \beta^0, \eta^3 .
$$

(A.39)

The (4,0) matrix of dimension $l = -1/2$ satisfies the linear equations

$$
\hat{D}^1_2 \beta^1 = \hat{D}^2_3 \beta^1 = 0 .
$$

(A.40)

The equations for the $l = -1$ matrices $B^{13}$ and $B^0$ are

$$
\hat{D}^1_2 B^{13} = 0 , \quad (\hat{D}^2_3)^2 B^{13} = 0 ,
$$

$$
\hat{D}^1_2 B^{13} = -2i\bar{\theta}^{1+} \bar{\partial} \beta^1 - (\beta^1)^2 ,
$$

(A.41)

$$
\hat{D}^2_3 B^0 = 0 , \quad (\hat{D}^1_2)^2 B^0 = 0 ,
$$

$$
\hat{D}^3_4 (B^0 - \bar{B}^0) = i\theta^+_3 \partial \beta^1 + i\bar{\theta}^{1+} \bar{\partial} \beta_3 + \{\beta^1, \beta_3\} .
$$

(A.42)

The inhomogeneous linear equations for $B^{13}$ and $B^0$ contain sources with functions $\beta^1$ and $\beta_3 = (\beta^1)^1$ calculated on the previous stage.
The iterative equations for the $(4,0)$ matrices with $l < -1$ can be analyzed analogously. Each independent iterative equation is manifestly resolved in terms of the harmonic derivatives of the corresponding function and the sources of these equations can be calculated on the previous stage of iteration.

Thus, it can be shown easily that the basic $N = 3$ harmonic equations for the $(4,4)$ analytic functions with the nilpotent nonlinear terms are equivalent to the finite number of the linear iterative $(4,0)$ equations which contain non-Abelian sources constructed from the solutions of the previous step of iteration.

Note that all iterative equations are simplified essentially for the two-dimensional solutions which do not depend on the variables $Y$ and $\bar{Y}$.

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