UNIQUENESS OF THE MEASURE OF MAXIMAL ENTROPY FOR GEODESIC FLOWS ON CERTAIN MANIFOLDS WITHOUT CONJUGATE POINTS

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Abstract. We prove that for closed surfaces $M$ with Riemannian metrics without conjugate points and genus $\geq 2$ the geodesic flow on the unit tangent bundle $T^1M$ has a unique measure of maximal entropy. Furthermore, this measure is fully supported on $T^1M$ and the flow is mixing with respect to this measure. We formulate conditions under which this result extends to higher dimensions.

1. Introduction

Let $X$ be a compact metric space and $F = \{f_t : X \to X\}_{t \in \mathbb{R}}$ a continuous flow. The complexity of the flow can be quantified by the topological entropy

$$h_{\text{top}}(F) = \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \Lambda_t(X, F, \epsilon),$$

where $\Lambda_t(X, F, \epsilon)$ is the maximum cardinality of a $(t, \epsilon)$-separated set – that is, a set $E \subset X$ such that for every $x, y \in X$, there is $s \in [0, t]$ such that $d(f_sx, f_sy) \geq \epsilon$. The variational principle [Wal82, Theorem 8.6] says that

$$h_{\text{top}}(F) = \sup \{h_{\mu}(f_1) : \mu \in \mathcal{M}_F(X)\},$$

where $\mathcal{M}_F(X)$ is the space of flow-invariant Borel probability measures on $X$, and $h_{\mu}(f_1)$ is measure-theoretic entropy. A measure $\mu \in \mathcal{M}_F(X)$ that achieves the supremum is called a measure of maximal entropy (MME).

If $M$ is a smooth closed Riemannian manifold of negative sectional curvature, then by classical work of Bowen the geodesic flow $F = \{f_t\}_{t \in \mathbb{R}}$ on the unit tangent bundle $T^1M$ has a unique measure of maximal entropy. Moreover, the unique MME is fully supported on $T^1M$ and is the limiting distribution of closed orbits of the geodesic flow [Bow73, Bow74].

The corresponding result for rank 1 manifolds with nonpositive sectional curvatures was proved by the second author in [Kni98]; for manifolds with no focal points, see [GR19] [LWW18].
We consider the more general setting of manifolds without conjugate points, where under the additional assumption of expansivity the uniqueness of the MME was proved by Aurélien Bosché in his thesis [Bos18]. Our first result removes the assumption of expansivity in the two-dimensional case.

**Theorem 1.1.** Let $M$ be a smooth closed surface of genus $\geq 2$, and let $g$ be a Riemannian metric on $M$ without conjugate points. Then the associated geodesic flow on $T^1M$ has a unique measure $\mu$ of maximal entropy. The measure $\mu$ has full support, is Bernoulli, hence mixing, and is the limiting distribution of closed orbits. In particular, the set of closed orbits is dense in $T^1M$.

In fact, Theorem 1.1 is a specific case of a more general result (Theorem 1.2), which establishes uniqueness for a broad class $\mathcal{H}$ of Riemannian manifolds with no conjugate points in arbitrary dimension. The key properties of a manifold $M \in \mathcal{H}$ are:

- $M$ admits a negatively curved “background” Riemannian metric;
- geodesics emanating from a common point on the universal covering eventually diverge;
- the fundamental group is residually finite;
- all invariant measures of “nearly maximal” entropy must have support on the expansive set, see (2.10).

See §3 for precise definitions, an explanation of why $\mathcal{H}$ contains every surface of genus $\geq 2$, and a discussion of how restrictive these conditions are.

The proofs of uniqueness given in [Kni98, LWW18, Bos18] use Patterson–Sullivan measures; our proof of Theorem 1.1 is closer to the approach in [GR19], where uniqueness is derived from a result of Bowen [Bow75] and Franco [Fra77] on expansive systems with specification. To remove the assumption of expansivity, we use a generalization of Bowen’s result due to the first author and D.J. Thompson [CT16]; see Theorem 2.25.

This approach gives uniqueness and ergodicity of the MME. Bernoullicity in Theorem 1.1 requires a result by Ledrappier, Lima, and Sarig [LLS16], which only applies when dim $M = 2$. In higher dimensions, we can still obtain mixing by proving that the Patterson–Sullivan construction gives the unique MME (once uniqueness is known). For nonpositively curved closed manifolds of rank 1, Patterson–Sullivan measures and their geometric applications were studied in [Kni97]. The properties of the Patterson–Sullivan construction were then used by Babillot [Bab02] to prove mixing of the flow with respect to the unique measure of maximal entropy. For manifolds in the class $\mathcal{H}$, we use the uniqueness result to demonstrate that the construction in [Kni98] gives the MME, and thus obtain mixing as in [Bab02, Theorem 2].

**Theorem 1.2.** Let $M$ be a Riemannian manifold with no conjugate points that lies in the class $\mathcal{H}$. Then the associated geodesic flow on $T^1M$ has a unique measure of maximal entropy $\mu$. The measure $\mu$ has full support, is
mixing, and is the limiting distribution of closed orbits. In particular, the set of closed orbits is dense in $T^1M$.

**Remark 1.3.** All closed surfaces of genus at least 2 are in the class $\mathcal{H}$. On the other hand we do not know of any examples of closed manifolds without conjugate points that admit a metric of negative curvature which are not contained in $\mathcal{H}$; see $\S$ 3 for more details. Thus Theorem 1.2 gives the uniqueness and mixing property of the MME for all the known examples of manifolds without conjugate points supporting a metric of negative curvature.

**Remark 1.4.** Our proof that the Patterson–Sullivan construction gives a measure of maximal entropy which is fully supported on $T^1M$ only uses that $(M,g)$ is a Riemannian manifold without conjugate points with the divergence property of geodesic rays as stated above and the existence of a background metric of negative curvature (see $[5]$). However, the proof of the uniqueness of the MME relies on the remaining properties which guarantee nonuniform expansivity and specification properties as in $[CT16]$. This approach has also been used to study equilibrium states for nonzero potentials in nonpositive curvature $[BCFT18]$ and no focal points $[CKP18]$. An important difference between these works and the present paper is that we do not establish strong hyperbolicity estimates for any natural class of orbit segments; such estimates are used in $[BCFT18, CKP18]$ to establish specification at small scales and to deal effectively with nonzero potentials. The difference in our current setting is that we no longer have certain nice properties such as convexity of horospheres and the flat strip theorem, which hold for nonpositive curvature and even for no focal points, but fail for no conjugate points $[Bur92]$. Instead, we rely on ‘coarse’ hyperbolicity properties coming from the existence of an negatively curved metric on $M$ via the Morse Lemma: these allow us to establish shadowing properties in the universal cover at an a priori very large scale $R$. The results from $[CT16]$ still apply when specification is only known at a fixed scale, provided expansivity is controlled at an even larger scale. Since $R$ may be larger than the diameter of $M$, we deal with this by passing to a finite cover of $M$ for which the injectivity radius is much larger than $R$.

The proof of mixing in Theorem 1.2 goes along the same lines as in $[Bab02$, Theorem 2] for rank 1 manifolds where there is an abstract result for mixing provided that the length spectrum is not *arithmetic*, i.e., is not a discrete subgroup of $\mathbb{R}$. However, in our case, there are some technicalities related to the fact that the flow is not expansive and the expansive set is not open in general. More precisely, in $[Bab02]$, the continuity of the cross-ratio function that implies the non-arithmeticity of the length spectrum is established by considering its restriction to the expansive set and using uniform hyperbolicity for the recurrent subset $[Kn98$, Proposition 4.1]. See $\S6.2$ and especially Lemma 6.7 for the results that play an analogous role in our setting. Theorem 1.2 can be considered as a starting point on whether one
could adapt the Margulis estimates about counting closed orbits which are conjectured in nonpositive curvature \cite{BK85}. We remark that the cross-ratio has an analogue in the context of contact Anosov flows, the so called temporal function – see \cite[Figure 2]{Liv04} – whose $C^2$ regularity was key in the proof of exponential mixing for contact Anosov flows.

**Structure of the paper.** In §2.1 we give definition and properties of manifolds without conjugate points and state the Morse lemma in Theorem 2.3. In §2.2 we give definition of specification and state the general results for uniqueness in Theorem 2.25. In §2.3 we discuss the property of fundamental group being residually finite which is enough to have a finite cover of arbitrary large injectivity radius.

In §3 we give a precise definition of the class $\mathcal{H}$ of manifolds to which Theorem 1.2 applies, and we prove Theorem 1.1 under Theorem 1.2. §4 is devoted to the proof of Theorem 1.2. In §5 we prove that the measure of maximal entropy in Theorem 1.2 is given by a Patterson–Sullivan construction as in \cite{Kni98}. This construction is then used in §6 to prove that the flow is mixing with respect to the measure of maximal entropy which completes the proof of Theorem 1.2. Certain technical proofs are given in the appendices.

2. Background

2.1. Geometry of manifolds without conjugate points.

2.1.1. Geodesic flows. Given a smooth closed $n$-dimensional Riemannian manifold $(M,g)$, we write $F = \{f_t : T^1M \to T^1M\}_{t \in \mathbb{R}}$ for the geodesic flow on the unit tangent bundle defined by $f_t(v) = \dot{c}_v(t)$, where $c_v$ is the unique geodesic on $M$ with $\dot{c}_v(0) = v$. It is convenient for us to use the metric $d_1(v, w) = \max_{t \in [0,1]} d(c_v(t), c_w(t))$ on $T^1M$ where $d$ is the metric on $M$ induced by the Riemannian metric.

Throughout the paper, we will assume that $(M, g)$ is a smooth closed Riemannian manifold without conjugate points. Such manifolds are characterized by the fact that the exponential map $\exp_p : T_pM \to M$ is not singular for all $p \in M$ or equivalently, each nontrivial orthogonal Jacobi field vanishes at most at one point. The following relationships between this property and other conditions that give some kind of hyperbolic behavior are straightforward:

- **nonpositive sectional curvature $\Rightarrow$ no focal points $\Rightarrow$ no conjugate points.**

The converse implications all fail in general.

The Cartan–Hadamard Theorem says that the universal cover $\tilde{M}$ of $(M, g)$ is diffeomorphic to $\mathbb{R}^n$ via the exponential map and therefore for every pair of distinct points $p, q \in \tilde{M}$, there is a unique geodesic segment $c : [a, b] \to \tilde{M}$ such that $c(a) = p$ and $c(b) = q$: geodesics are globally minimizing in $\tilde{M}$.
The group of deck transformations $\Gamma$ is isomorphic to the fundamental group $\pi_1(M)$ and acts isometrically on $\tilde{M}$. In particular, $M$ is isometric to the quotient $\tilde{M}/\Gamma$.

2.1.2. **Coarse hyperbolicity using a background metric.** An important strategy throughout the paper will be to compare the geometric properties of $M$ with respect to two different Riemannian metrics $g$ and $g_0$, where $g$ is the original metric we are given, and $g_0$ is a ‘background’ metric which we will always assume to have negative sectional curvatures.

**Remark 2.1.** Existence of a negatively curved background metric places genuine topological restrictions on $M$. In particular, as stated above the universal covering is diffeomorphic to $\mathbb{R}^n$ and by Preissmann’s theorem each abelian subgroup of the fundamental group is infinite cyclic. Therefore, in dimension two it forces the genus of $M$ to be at least 2 and in higher dimensions excludes examples such as Gromov’s graph manifolds of nonpositive curvature for which uniqueness of the MME is known [Kn98, §6]. On the other hand in this paper we do not impose any local assumptions such as restrictions on the curvature.

We will write $d, d^0$ for the distance functions associated to $g, g_0$ on both $M$ and $\tilde{M}$. The first crucial observation is that by compactness of $M$ and the equivalence of the quadratic forms of $g$ and $g_0$, there exists a constant $A > 0$ such that for every $p, q \in \tilde{M}$, we have

\[ A^{-1} \cdot d^0(p, q) \leq d(p, q) \leq A \cdot d^0(p, q). \]  

This has an important consequence for topological entropy. Generalizing an earlier result of Manning [Man79] in nonpositive curvature, it was shown by Freire and Mañé [FM82] that on a closed manifold without conjugate points, the volume growth in the universal cover is equal to the topological entropy of the geodesic flow:

\[ h_{\text{top}}(F) = \lim_{r \to \infty} \frac{1}{r} \log \text{vol}(B(p, r)), \]

where $p$ is any point in $\tilde{M}$ and $B(p, r) \subset \tilde{M}$ is the ball centered at $p$ of radius $r$. We say that $g$ has positive topological entropy if its geodesic flow $F$ has $h_{\text{top}}(F) > 0$. The following is an immediate consequence of (2.1) and (2.2).

**Lemma 2.2.** If $M$ admits a metric $g_0$ without conjugate points that has positive topological entropy, then every metric $g$ without conjugate points on $M$ has positive topological entropy. In particular, this occurs if $M$ admits a metric $g_0$ with negative sectional curvatures.

From now on we consider a closed Riemannian manifold $(M, g)$ without conjugate points which admits a background metric $g_0$ of negative curvature, and thus has positive topological entropy. This lets us deduce certain coarse
hyperbolicity properties, for which we recall that Hausdorff distance $d_H$ between two subsets $C_1, C_2 \subset \tilde{M}$ (with respect to $g$) is defined by
\[ d_H(C_1, C_2) := \inf \{ r > 0 : C_1 \in T_r(C_2), C_2 \in T_r(C_1) \} \]
where $T_r(C) := \{ p \in \tilde{M} : d(p, C) \leq r \}$. We denote by $d^0_H$ the Hausdorff distance with respect to $d^0$. The following result goes back to Morse [Mor24] in dimension two, and Klingenberg [Kni71] in higher dimensions. We follow the statement from [GKOS14, Theorem 3.3]; see [Kni02, Lemma 2.7] for a detailed proof.

**Theorem 2.3** (Morse Lemma). If $g, g_0$ are two metrics on $M$ such that $g$ has no conjugate points and $g_0$ has negative curvature, then there is a constant $R_0 = R_0(g, g_0) > 0$ such that if $c : [a, b] \to \tilde{M}$ and $\alpha : [a_0, b_0] \to \tilde{M}$ are minimizing geodesic segments with respect to $g, g_0$, respectively, joining $c(a) = \alpha(a_0)$ to $c(b) = \alpha(b_0)$, then $d_H(c[a, b], \alpha[a_0, b_0]) \leq R_0$.

We prove the following consequence in Appendix A.

**Lemma 2.4.** Let $g$ be a metric on $M$ without conjugate points. If $M$ admits a metric of negative curvature then for every $R_1 > 0$ there is $R_2 > 0$ such that for every $T > 0$ if $c_1, c_2 : [0, T] \to \tilde{M}$ are two geodesics with $d(c_1(0), c_2(0)) \leq R_1$ and $d(c_1(T), c_2(T)) \leq R_1$, then $d(c_1(t), c_2(t)) \leq R_2$ for all $t \in [0, T]$.

**Remark 2.5.** When $g$ itself has nonpositive curvature, Lemma 2.4 follows easily from the convexity of the distance function $t \mapsto d(c_1(t), c_2(t))$ between two geodesics. In our more general setting we rely on the background metric of negative curvature and use the Morse Lemma.

### 2.1.3. Busemann functions and horospheres

Given $v \in T^1\tilde{M}$, recall that $c_v : \mathbb{R} \to \tilde{M}$ denotes the geodesic with $c_v(0) = v$. For each $t > 0$, consider the function on $T^1\tilde{M}$ defined by $b_{v, t}(p) := d(p, c_v(t)) - t$.

**Lemma 2.6** ([Ese77, Proposition 1]). For every $v \in T^1\tilde{M}$ and $p \in \tilde{M}$, the limit $b_v(p) := \lim_{t \to \infty} b_{v, t}(p)$ exists and defines a $C^1$ function on $\tilde{M}$. Moreover, $\text{grad} b_v(p) = \lim_{t \to \infty} \text{grad} b_{v, t}(p)$.

Existence of the limit is essentially due to the fact that geodesics on $\tilde{M}$ are globally minimizing. The limiting function $b_v$ is called a Busemann function, and was shown in [Kni86] to be in fact $C^{(1,1)}$.

Observe that if $t \geq \tau$, then $b_{v, t}(c_v(\tau)) = d(c_v(\tau), c_v(t)) - t = (t - \tau) - t = -\tau$, so we have
\[ b_v(c_v(t)) = -t \text{ and } -\text{grad} b_v(c_v(t)) = \dot{c}_v(t) = f_t(v). \]

Given $v \in T^1\tilde{M}$, the stable and unstable horospheres $H^s(v)$ and $H^u(v)$ are the subsets of $\tilde{M}$ defined by
\[ H^s(v) := \{ p \in \tilde{M} : b_v(p) = 0 \} \quad \text{and} \quad H^u(v) := H^s(-v). \]
We refer to \( c_v(\infty) \) as the center of the stable horosphere \( H^s(v) \). Similarly, \( c_v(-\infty) := c_v(-\infty) \) is the center of \( H^u(v) \), where we write \( c_v(t) = c_v(-t) \).

We also consider the (weak) stable and unstable manifolds, which are the subsets of \( \tilde{T}_1 \tilde{M} \) defined by

\[
W_s(v) := \{ -\text{grad } b_v(p) | p \in \tilde{M} \} \quad \text{and} \quad W^u(v) := -W_s(-v).
\]

Some justification for this terminology will be given in the next section. For the moment we observe that \( W_s(v) \) is the union of the unit normal vector fields to horospheres centered at \( c_v(\infty) \), and that it is \( F \)-invariant by (2.3).

The regularity of the Busemann function implies that the horospheres are \( C^{(1,1)} \) manifolds and the stable and unstable manifolds are Lipschitz.

We let \( \pi : T^1 \tilde{M} \to \tilde{M} \) be the standard projection.

2.1.4. Manifolds of hyperbolic type and the boundary at infinity. We say that \((\tilde{M},g)\) has the divergence property if any pair of geodesics \( c_1 \neq c_2 \) in \((\tilde{M},g)\) with \( c_1(0) = c_2(0) \) diverge, i.e.,

\[
\lim_{t \to \infty} d(c_1(t),c_2(t)) = \infty.
\]

Remark 2.7. Every surface without conjugate points has the divergence property [Gre56]. In higher dimensions it is unknown whether this condition always holds.

The following definition and theorem are due to Eberlein [Ebe72].

Definition 2.8. A simply connected Riemannian manifold \( \tilde{M} \) without conjugate points is a (uniform) visibility manifold if for every \( \epsilon > 0 \) there exists \( L > 0 \) such that whenever a geodesic \( c: [a,b] \to \tilde{M} \) stays at distance at least \( L \) from some point \( p \in \tilde{M} \), then the angle sustained by \( c \) at \( p \) is less than \( \epsilon \), that is

\[
\angle_p(c) = \sup_{a \leq s,t \leq b} \angle_p(c(s),c(t)) < \epsilon.
\]

Theorem 2.9. Let \((M,g)\) be a closed manifold without conjugate points which admits a background metric \( g_0 \) of negative curvature. Then \((\tilde{M},g)\) is a visibility manifold if and only if \((M,g)\) has the divergence property.

Remark 2.10. We remark that Ruggiero [Rug03] obtained this result without assuming the existence of a negatively curved background metric. Instead he assumed the weaker condition that \( \tilde{M} \) is hyperbolic in the sense of Gromov [Gro87].

Definition 2.11. We say that a closed manifold \((M,g)\) without conjugate points is of hyperbolic type provided it carries a background metric \( g_0 \) of negative curvature and it satisfies the divergence property.

Remark 2.12. By Remark 2.7 and the classification of surfaces, every surface of genus \( \geq 2 \) without conjugate points is of hyperbolic type.
Now we assume that \((M, g)\) is of hyperbolic type, and describe a compactification of \(\tilde{M}\) following Eberlein \cite{Ebe72}. Two geodesic rays \(c_1, c_2: [0, \infty) \to \tilde{M}\) are called asymptotic if \(d(c_1(t), c_2(t))\) is bounded for \(t \geq 0\). This is an equivalence relation; we denote by \(\partial \tilde{M}\) the set of equivalence classes and call its elements points at infinity. We denote the equivalence class of a geodesic ray (or geodesic) \(c\) by \(c(\infty)\). The following construction is useful.

**Lemma 2.13.** Given \(p \in \tilde{M}\) and \(v \in T_p^1 \tilde{M}\), for each \(t > 0\) let \(c_t\) be the geodesic from \(p\) to \(c_t(0) = p\). Then the limit \(w := \lim_{t \to \infty} c_t(0) \in T_p^1 M\) exists and has the property that \(w = -\nabla b_s(p)\) and \(c_w(\infty) = c_v(\infty)\).

**Proof.** Lemma 2.6 gives existence of the limit and the claim regarding \(b_v\). To show that \(c_w(\infty) = c_v(\infty)\), we apply Lemma 2.4 with \(R_1 = d(p, \pi(v))\) to the geodesics \(c_t, c_v: [0, t] \to \tilde{M}\), which gives \(d(c_t(s), c_v(s)) \leq R_2\) for all \(s \in [0, t]\) where \(R_2\) depends only on \(p\) and \(v\). Since \(c_w(s) = \lim_{t \to \infty} c_t(s)\) for all \(s > 0\), we conclude that \(d(c_w(s), c_v(s)) \leq R_2\), and thus \(c_w(\infty) = c_v(\infty)\). \(\square\)

**Lemma 2.14.** Given any \(p \in \tilde{M}\) and \(\xi \in \partial \tilde{M}\), there is a unique geodesic ray \(c: [0, \infty) \to \tilde{M}\) with \(c(0) = p\) and \(c(\infty) = \xi\). Equivalently, the map \(f_p: T_p^1 \tilde{M} \to \partial \tilde{M}\) defined by \(f_p(v) = c_v(\infty)\) is a bijection.

**Proof.** Surjectivity follows from Lemma 2.13. Injectivity is an immediate consequence of the divergence property. \(\square\)

Following Eberlein we equip \(\partial \tilde{M}\) with a topology that makes it a compact metric space homeomorphic to \(S^{n-1}\). Fix \(p \in \tilde{M}\) and let \(f_p: T_p^1 \tilde{M} \to \partial \tilde{M}\) be the bijection \(v \mapsto c_v(\infty)\) from Lemma 2.14. The topology (sphere-topology) on \(\partial \tilde{M}\) is defined such that \(f_p\) becomes a homeomorphism. Since for all \(q \in \tilde{M}\) the map \(f_q^{-1}f_p: T_p^1 \tilde{M} \to T_q^1 \tilde{M}\) is a homeomorphism, see \cite{Ebe72}, the topology is independent on the reference point \(p\).

The topologies on \(\partial \tilde{M}\) and \(\tilde{M}\) extend naturally to \(\text{cl}(\tilde{M}) := \tilde{M} \cup \partial \tilde{M}\) by requiring that the map \(\varphi: B_1(p) = \{v \in T_p^1 \tilde{M}: ||v|| \leq 1\} \to \text{cl}(\tilde{M})\) defined by

\[
\varphi(v) = \begin{cases} 
\exp_p \left( \frac{v}{1-||v||} \right) & ||v|| < 1 \\
 f_p(v) & ||v|| = 1 
\end{cases}
\]

is a homeomorphism. This topology, called the cone topology, was introduced by Eberlein and O’Neill \cite{EO73} in the case of Hadamard manifolds and by Eberlein \cite{Ebe72} in the case of visibility manifolds. In particular, \(\text{cl}(\tilde{M})\) is homeomorphic to a closed ball in \(\mathbb{R}^n\). The relative topology on \(\partial \tilde{M}\) coincides with the sphere topology, and the relative topology on \(\tilde{M}\) coincides with the manifold topology.

**Remark 2.15.** The isometric action of \(\Gamma = \pi_1(M)\) on \(\tilde{M}\) extends to a continuous action on \(\partial \tilde{M}\). Since by \cite{Ebe72} the geodesic flow is topologically transitive, every \(\Gamma\)-orbit in \(\partial \tilde{M}\) is dense, i.e. the action on \(\partial \tilde{M}\) is minimal.
Definition 2.16. Given $p \in \tilde{M}$ and $\xi \in \partial \tilde{M}$, let $v \in T_p\tilde{M}$ be the unique unit tangent vector at $p$ such that $c_p(\infty) = \xi$. We call $b_p(q, \xi) := b_v(q)$ the Busemann function based at $\xi$ and normalized by $p$, i.e. $b_p(p, \xi) = 0$.

The following important property of visibility manifolds is due to Eberlein [Ebe72 Proposition 1.14].

Proposition 2.17. Let $\tilde{M}$ be a visibility manifold and
\[ A = \{(q, z) \in \tilde{M} \times \cl(\tilde{M}) \mid q \neq z\}. \]
Consider the map $V: A \to T^1\tilde{M}$ such that $V(q, z) \in T^1_q\tilde{M}$ is the unique vector with $c_V(q, z)(d(q, z)) = z$. Then $V: A \to T^1\tilde{M}$ is continuous with respect to the topology defined above.

Corollary 2.18. For $p, q, z \in \tilde{M}$ define $b_p(q, z) = d(q, z) - d(p, z)$. Then for all $p \in \tilde{M}$, compact subsets $K \subset \tilde{M}$, $\xi \in \partial \tilde{M}$ and all $\epsilon > 0$ there exists an open set $U \subset \cl(\tilde{M})$ such that
\[ |b_p(q, z) - b_p(q, \xi)| < \epsilon \]
for all $q \in K$ and $z \in U$. In particular, we have
\[ \lim_{z \to \xi} d(q, z) - d(p, z) = -b_p(q, \xi) \text{ for every } p, q \in \tilde{M} \text{ and } \xi \in \partial \tilde{M}. \]

Proof. For $p \in \tilde{M}$ and $z \in \cl(\tilde{M})$ consider the function $q \mapsto b_p(q, z)$. Then for $q \neq z$ we have grad $b_p(q, z) = -V(q, z)$. For a compact set $K \subset \tilde{M}$ define
\[ B(K) = \cl\{q \in [0, 1] \mid c : [0, 1] \to \tilde{M} \text{ geodesic with } c(0) = p, c(1) \in K\} \]
which is compact as well. Choose $r > 0$ such that $K \subset B(p, r)$ and $\epsilon > 0$. Since $V : A \to T^1\tilde{M}$ is continuous it is uniformly continuous on compact subsets. In particular, for $\xi \in \partial \tilde{M}$ there exists a neighborhood $U \subset \cl(\tilde{M})$ such that $U \cap B(K) = \emptyset$ and $\|V(q, z) - V(q, \xi)\| < \frac{\epsilon}{r}$ for all $q \in B(K)$. For $q \in K$ and $z \in U$ and the geodesic $c : [0, 1] \to \tilde{M}$ with $c(0) = p$ and $c(1) = q$. we obtain
\[ |b_p(q, z) - b_p(q, \xi)| = \left|\int_0^1 \frac{d}{dt}(b_p(c(t), z) - b_p((c(t), \xi))dt\right| \]
\[ = \left|\int_0^1 \langle\grad b_p(c(t), z) - \grad b_p(c(t), \xi), \dot{c}(t)\rangle dt\right| < \frac{\epsilon}{r} \int_0^1 \|\dot{c}(t)\| dt \leq \epsilon \]
which yields the claim made in the corollary.

Corollary 2.19. Given $p, p', q \in \tilde{M}$ and $\xi \in \partial \tilde{M}$, we have $b_{p'}(q, \xi) = b_p(q, \xi) - b_p(p', \xi)$. In particular, all Busemann functions based at $\xi$ coincide up to an additive constant and $b_{p'}(q, \xi) = -b_q(p', \xi)$.

Proof. Given $p, p', q, z \in \tilde{M}$ we obtain $b_{p'}(q, z) = b_p(q, z) - b_p(p', z)$ and taking the limit $z \to \xi$ corollary 2.18 yields the claim.\qed
The following result justifies the terminology ‘stable manifold’ for $W^s(v)$.

**Lemma 2.20.** Let $(M,g)$ be a closed Riemannian manifold without conjugate points and of hyperbolic type. Then for each $v \in T^1\tilde{M}$, we have

$$W^s(v) = \{ w \in T^1\tilde{M} : c_w \text{ is asymptotic to } c_v \}.$$  

**Proof.** By Lemma 2.14 both the left- and right-hand sides of (2.8) contain exactly one point from each $T^1\tilde{p}\tilde{M}$. By Lemma 2.13 each $T^1\tilde{p}\tilde{M}$ contains a $w$ that lies in both the left- and right-hand sides. The result follows. □

We say that a geodesic $c : \mathbb{R} \to \tilde{M}$ connects two points at infinity $\eta, \xi \in \partial\tilde{M}$ if $c(-\infty) := c^-(\infty) = \eta$ and $c(\infty) = \xi$, where $c^-(t) = c(-t)$.

**Lemma 2.21 ([Kli71]).** For every $\eta, \xi \in \partial\tilde{M}$ with $\eta \neq \xi$, there exists a geodesic $c$ connecting $\eta$ and $\xi$.

The geodesic $c$ in Lemma 2.21 is not always unique; there may be multiple geodesics connecting $\eta$ and $\xi$, in which case $\eta$ and $\xi$ are in some sense ‘conjugate points at infinity’; we allow such points in $\partial\tilde{M}$ even though we forbid conjugate points in $\tilde{M}$. In the more restrictive setting of no focal points (in particular, if $M$ has nonpositive curvature), any two distinct geodesics connecting $\eta \neq \xi \in \partial\tilde{M}$ must bound a flat strip in $\tilde{M}$, but this is no longer the case in our setting.

Given $v, w \in T^1\tilde{M}$, observe that $c_v(\pm\infty) = c_w(\pm\infty)$ if and only if $w \in W^s(v) \cap W^u(v)$ by Lemma 2.20 and so $c_v$ is the unique geodesic joining $c_w(\pm\infty)$ if and only if

$$(2.9) \quad W^s(v) \cap W^u(v) = \{ \dot{c}_v(t) : t \in \mathbb{R} \} = \{ f_t v : t \in \mathbb{R} \}.$$  

We can also give a characterization in terms of the horospheres: $c_v$ is the unique geodesic joining $c_w(\pm\infty)$ if and only if for every $x \in H^s(v) \cap H^u(v)$ we have $\operatorname{grad} b_v(x) = v$ then $x = \pi(v)$.

In the following we call

$$(2.10) \quad \mathcal{E} := \{ v \in T^1\tilde{M} : W^s(v) \cap W^u(v) = \{ \dot{c}_v(t) : t \in \mathbb{R} \} \}$$

the expansive set, which we will use in the definition of $\mathcal{H}$ in §3. From the discussion above, we have $v \in \mathcal{E}$ if and only if $c_v$ is uniquely determined up to parametrization by $c_v(\pm\infty)$.

### 2.2. General results about equilibrium states.

#### 2.2.1. A uniqueness result using specification.**We use an approach to uniqueness of the measure of maximal entropy that goes back to Bowen [Bow75]; see Franco [Fra77] for the flow case. This approach relies on the properties of expansiveness and specification. We will use versions of these properties that differ slightly from those used by Bowen and Franco, but which keep the essential features; we describe these now, together with a general uniqueness
result proved by the first author and D.J. Thompson [CT16] that extends Bowen’s result to a more nonuniform setting. Let \((X, F)\) be a continuous flow on a compact metric space.

**Definition 2.22.** Given \(\epsilon > 0\), say that a point \(x \in X\) is expansive at scale \(\epsilon\) if there is \(s > 0\) such that

\[
\{ y \in X : d(f_t x, f_t y) < \epsilon \text{ for all } t \in \mathbb{R} \} \subset \{ f_t x : t \in [-s, s] \}.
\]

Let \(\text{NE}(\epsilon)\) be the set of points in \(X\) that are not expansive at scale \(\epsilon\). The entropy of obstructions to expansivity at scale \(\epsilon\) is

\[
h_{\text{exp}}^\perp(\epsilon) = \sup \{ h_\mu(f_1) : \mu \in \mathcal{M}_F(X), \mu(\text{NE}(\epsilon)) = 1 \}.
\]

**Definition 2.23.** Given \(\delta > 0\), we say that the flow has specification at scale \(\delta > 0\) if there exists \(\tau = \tau(\delta)\) such that for every \((x_1, t_1), \ldots, (x_N, t_N) \in X \times [0, \infty)\) there exist a point \(y \in X\) and times \(T_1, \ldots, T_N \in \mathbb{R}\) such that

\[
f_{T_j}(y) \in B_{t_j}(x_j, \delta) := \left\{ y \in X : \sup_{s \in [0, t_j]} d(f_s x_j, f_s y) < \delta \right\}
\]

for every \(j \in \{1, \ldots, N\}\). It is convenient to also use the notation \(s_j = T_j + t_j\) for the time at which the orbit of \(y\) stops shadowing the orbit of \(x_j\), and \(\tau_j = T_{j+1} - s_j\) for the time it takes to transition from the orbit of \(x_j\) to the orbit of \(x_{j+1}\). See Figure 2.1 for the relationship between the various times.

**Figure 2.1.** Book-keeping in the specification property.

**Remark 2.24.** It is well-known that the specification property holds for transitive Anosov flows, including the geodesic flow on a smooth negatively curved closed Riemannian manifold; see [Bow72] and also Appendix B.

The following result is proved in [CT16, Theorem 2.9].

**Theorem 2.25.** Let \((X, F)\) be a continuous flow on a compact metric space, and suppose that there are \(\delta, \epsilon > 0\) with \(\epsilon > 40 \delta\) such that \(h_{\text{exp}}^\perp(\epsilon) < h_{\text{top}}(f_1)\) and the flow has specification at scale \(\delta\). Then \((X, F)\) has a unique measure of maximal entropy.

**Remark 2.26.** The result proved in [CT16] is more general (and more complicated) in two ways: it applies to nonzero potential functions, and it only requires specification to hold for a (sufficiently large) collection of orbit segments. The version stated here follows from [CT16, Theorem 2.9] by putting \(\phi = 0, \mathcal{G} = \mathcal{D} = X \times \mathbb{R}^+, \) and \(\mathcal{P} = \mathcal{S} = \emptyset\).
2.2.2. Limit distribution along periodic orbits. Let $X$ be a compact metric space and $f_t : X \to X$ a continuous flow. For $\delta, T > 0$, let $\mathcal{P}(T, \delta)$ be a maximal $\delta$-separated set of periodic orbits of period less than $T$ and $P(T, \delta) := \text{card} \mathcal{P}(T, \delta)$. Let $\mu_T$ be the invariant probability measure defined by

$$
\int_X \varphi \, d\mu_T = \frac{1}{P(T, \delta)} \sum_{\gamma \in \mathcal{P}(T, \delta)} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} \varphi(\gamma(s)) \, ds
$$

for each $\varphi \in C^0(X)$. Following the arguments in [Wal82, Theorem 9.10] or [Kni98, Proposition 6.4], we have the following.

**Proposition 2.27.** If there exists a sequence $T_k \to \infty$ and $\delta > 0$ such that

$$
\lim_{k \to \infty} \frac{\log P(T_k, \delta)}{T_k} = h_{\text{top}}(f_1) > 0
$$

then every accumulation point of $\{\mu_{T_k}\}$ with respect to the weak* topology is a measure of maximal entropy.

**Definition 2.28.** We say that a flow-invariant probability measure $\mu$ is the limiting distribution of periodic orbits if there exists $\delta > 0$ such that $\mu_T \to \mu$ in the weak* topology. If this occurs in the case when $X = T^1M$ and $f_t$ is the geodesic flow, we also say that $\mu$ is the limiting distribution of closed geodesics.

2.3. Residually finite fundamental groups.

**Definition 2.29.** A group $G$ is residually finite if the intersection of its finite index subgroups is trivial.

For surfaces we have the following result which was first proved by Baumslag [Bau62] and then Hempel [Hem72] gave an alternative proof.

**Theorem 2.30.** Every surface has residually finite fundamental group.

Later on, Hempel [Hem87] proved that fundamental groups of three manifolds are residually finite. It is an open problem whether every manifold supporting a negatively curved metric has a residually finite fundamental group [Arz14].

For our purposes we need the following implication of a manifold having residually finite fundamental group.

**Proposition 2.31.** Let $M$ be a smooth Riemannian manifold and suppose that $\pi_1(M)$ is residually finite. Then for every $R > 0$ there is a smooth Riemannian manifold $N$ and a locally isometric covering map $p : N \to M$ such that the injectivity radius of $N$ is at least $R$.

**Proof.** Fix a point $x$ in the universal cover $\tilde{M}$, and consider the finite set $Z = \{ \gamma \in \pi_1(M) \setminus \{e\} : \gamma(x) \in B(x, 2R) \}$. Since $\pi_1(M)$ is residually finite, for each $\gamma \in Z$ there is a finite index subgroup $G_\gamma < \pi_1(M)$ such that
γ /∈ Gγ. Then G = ∩γ∈Z Gγ < π1(M) is a finite index subgroup such that
d(x, γ(x)) ≥ 2R for all nontrivial γ ∈ G. In particular, N = \tilde{M}/G defines a
compact manifold that is a finite cover of M has injectivity radius at least R. To see this, we can consider without loss of generality that x ∈ D where
D is a fundamental domain corresponding to the covering \tilde{M} → M. Then
there is a fundamental domain of \tilde{M} → N that contains all γD where γ ∈ Z and
by the definition of Z, this domain contains a ball of radius ≥ R. □

3. A CLASS OF MANIFOLDS WITH UNIQUE MEASURE OF MAXIMAL
ENTROPY

Now we can define the class \( \mathcal{H} \) of manifolds to which Theorem 1.2 applies.
First we recall that if \( \mu \) is a probability measure on \( T^1M \), then there is
a unique σ-finite measure \( \tilde{\mu} \) on \( T^1\tilde{M} \) that is \( \pi_1(M) \)-invariant and whose
restriction to every fundamental domain projects to \( \mu \) under the universal
covering map. We refer to \( \tilde{\mu} \) as the lift of \( \mu \), and note that it is flow-invariant
if and only if \( \mu \) is.

Definition 3.1. Let \( \mathcal{H} \) denote the class of closed smooth Riemannian man-
ifolds \((M, g)\) without conjugate points such that the following conditions are
satisfied.

(H1) \( M \) supports a Riemannian metric \( g_0 \) for which all sectional curva-
tures are negative;
(H2) \( (M, g) \) has the divergence property;
(H3) the fundamental group \( \pi_1(M) \) is residually finite;
(H4) \( \sup\{h_\mu(f_1) : \mu \in \mathcal{M}_F(T^1M), \tilde{\mu}(E) = 0\} < h_{top}(F), \) where \( E \) is the
expansive set defined in (2.10);

We show below that every closed surface without conjugate points and
genus ≥ 2 satisfies (H1)(H4); this is the key to deducing Theorem 1.1
from Theorem 1.2

Remark 3.2. As observed in Remark 2.1, there are rank 1 manifolds of non-
positive curvature for which (H1) fails, but which have unique measures of
maximal entropy by [Kni98]. Thus this condition places a genuine topologi-
ical restriction on the class of manifolds contained in \( \mathcal{H} \).

In higher dimensions, the status of the other conditions is less clear; we
are not currently aware of any examples for which (H1) holds but any
of (H2) (H4) fails. It is known that (H3) holds whenever \( \dim M \leq 3 \)
[Bau62, Hem72, Hem87], and it is an open problem in geometric group
theory to determine whether (H1) implies (H3) in general [Arz14].

Following similar arguments to those in [BCPT18 §8], condition (H4)
can be verified under the following assumptions (we omit the proof):

1. Conditions (H1) (H3) hold;
2. the expansive set \( E \) has non-empty interior;
(3) the finite cover \( N \) of \( M \) constructed in the next section has the property that its geodesic flow is entropy-expansive at scale \( 10\delta \), where \( \delta \) is given in in (4.1) below.

**Proof of Theorem 1.1 assuming Theorem 1.2.** Let \( M \) be a closed surface of genus \( \geq 2 \), and \( g \) a metric on \( M \) with no conjugate points. We claim that conditions (H1)–(H4) are satisfied. Indeed, (H1) is a standard result; (H3) is Theorem 2.30; and (H2) was proved in [Gre56]. The proof of (H4) is a consequence of the following proposition.

**Proposition 3.3.** Let \( M \) be a surface of genus \( \geq 2 \) without conjugate points, and \( \mu \in \mathcal{M}_F(T^1M) \) a ergodic measure with \( h_\mu(f_1) > 0 \). Then \( \tilde{\mu}(T^1M \setminus \mathcal{E}) = 0 \).

**Proof.** Let \( \mu \in \mathcal{M}_F(T^1M) \) be a ergodic measure with \( h_\mu(f_1) > 0 \). This implies by Ruelle’s inequality that \( \mu \)-a.e. \( v \in T^1M \) has nonzero Lyapunov exponents, and hence by Pesin theory \( v \) has transverse stable and unstable leaves. Using Lemma 2.20 the stable and unstable manifolds of \( \mu \)-a.e. \( v \) correspond to normal fields of the stable and unstable horospheres, and thus these horospheres intersect in a single point, so \( \mu \)-a.e. \( v \in \mathcal{E} \). \( \square \)

Finally, the topological entropy of the geodesic flow is positive by Lemma 2.2, so Proposition 3.3 establishes (H4). Thus Theorem 1.2 applies to every surface of higher genus with no conjugate points, giving a unique measure of maximal entropy \( \mu \), which is ergodic and is the limiting distribution of closed geodesics. It follows from [LLS16] that \( \mu \) is Bernoulli which concludes the proof of Theorem 1.1. \( \square \)

### 4. Proof of uniqueness

In this section we prove the first part of Theorem 1.2 by using Theorem 2.25 to establish uniqueness of the MME, and the fact that it is the limiting distribution of closed geodesics, when \((M, g)\) is a smooth closed Riemannian manifold without conjugate points satisfying (H1)–(H4).

The first step is to pass to an appropriate finite cover. Let \( A \) be given by (2.1), and \( R_0 \) by the Morse lemma. Fix \( R_1 > 3AR_0 \) and let \( R_2 \) be given by Lemma 2.4 observe that the proof in Appendix \( \Delta \) gives \( R_2 = 4R_0 + (6A^2 + 1)R_1 \). Let
\[(4.1) \quad \delta = R_2 + 2 = 4R_0 + (6A^2 + 1)R_1 + 2.\]

and fix \( \epsilon > 40\delta \). By (H3) and Proposition 2.31 there is a finite cover \( N \) of \( M \) whose injectivity radius exceeds \( 3\epsilon \). Observe that the covering map \( p: N \to M \) naturally extends to a finite-to-1 semi-conjugacy from the geodesic flow on \( T^1N \) to the geodesic flow on \( T^1M \) which implies that

**Lemma 4.1.** The pushforward map \( p_*: \mathcal{M}_F(T^1N) \to \mathcal{M}_F(T^1M) \) is surjective and entropy-preserving.
It follows from Lemma 4.1 that if the geodesic flow on $T^1N$ has a unique measure of maximal entropy, then so does the geodesic flow on $T^1M$. Ergodicity follows from uniqueness because otherwise every ergodic component will be an MME. Similarly, if the unique MME on $T^1N$ is the limit distribution of periodic orbits, then the flow on $T^1N$ satisfies $\lim_{T \to \infty} \frac{1}{T} \log P(T, \delta) = h_{\text{top}}(f_1)$, and since the semi-conjugacy is finite-to-1, the same is true of the geodesic flow on $T^1M$, so Proposition 2.27 gives the corresponding result for $T^1M$.

Thus prove the claims in Theorem 1.2 regarding uniqueness and closed geodesics, it suffices to prove them for geodesic flow on $T^1N$, which we will do using Theorem 2.25. From now on we consider $X = T^1N$ and let $F$ be the geodesic flow.

**Lemma 4.2.** If $v \in \text{NE}(\epsilon) \subset T^1N$, then any lift $\tilde{v}$ of $v$ to $T^1\tilde{N} = T^1\tilde{M}$ has the property that $\tilde{v} /\in E$. In particular, if $\mu \in \mathcal{M}_F(T^1N)$ is such that $\mu(\text{NE}(\epsilon)) = 1$, then $\tilde{\mu}(E) = 0$.

**Proof.** If $v \in \text{NE}(\epsilon)$, then for every $s > 0$, there is $w \in T^1N$ such that $w /\in \{f_t v : t \in [-s, s]\}$, but $d(f_t v, f_t w) < \epsilon$ for all $t \in \mathbb{R}$. Because $\epsilon$ is smaller than the injectivity radius of $N$, for every lift $\tilde{v} \in T^1\tilde{N} = T^1\tilde{M}$ there is some $\tilde{w} \in T^1\tilde{N}$ with the same property. In particular, taking $s > \epsilon$ we conclude that $\tilde{v}, \tilde{w}$ are tangent to distinct geodesics between the same points on $\partial\tilde{M}$, and thus $\tilde{v} /\in E$. The claim regarding $\mu$ and $\tilde{\mu}$ follows immediately. \qed

It follows from Lemma 4.2 and Condition (H4) that

$$h^+_{\exp}(\epsilon) \leq \sup\{h_\mu(f_1) : \mu \in \mathcal{M}_F(T^1M), \tilde{\mu}(E) = 0\} < h_{\text{top}}(F),$$

which verifies the entropy gap condition that is needed for Theorem 2.25.

**Proposition 4.3.** The geodesic flow of $(N, g)$ has specification at scale $\delta$.

**Proof.** We defer the complete proof to Theorem B.1 where we prove a slightly stronger property. The basic idea is to use the Morse Lemma to go from orbit segments for $F$ to orbit segments for $F^0$, then use the specification property for $F^0$ to find a single $g_0$-geodesic that shadows each of these in turn, and finally to use the Morse Lemma again to show that the $g$-geodesic with the same endpoints in $\tilde{M}$ shadows the original sequence of orbit segments. \qed

Once specification at scale $\delta$ has been proved, Theorem 2.25 implies that the geodesic flow on $T^1N$ has a unique measure of maximal entropy $\mu$. To prove that $\mu$ is the limiting distribution of closed geodesics, we use the following special version of a result which has been obtained in [CK02] which was partially based on ideas in [Kni83].

**Theorem 4.4.** Let $(M, g)$ be a closed Riemannian manifold (not necessarily without conjugate points) admitting a metric of negative curvature. Let $P(t)$
be the number of free homotopy classes containing a closed geodesic of period less than $T$. Then there exists positive constants $A, B$ and $t_0$ such that

$$A \frac{e^{ht}}{t} \leq P(t) \leq Be^{ht}$$

for all $t \geq t_0$, where

$$h = \lim_{r \to \infty} \frac{1}{r} \log \text{vol}(B(p, r)),$$

is exponential volume growth (volume entropy) on the universal covering $\widetilde{M}$.

**Remark 4.5.** The result above was formulated in [CK02] in the general context of Gromov hyperbolic metric spaces and since $\widetilde{M}$ is Gromov hyperbolic the result applies. As remarked in 2.2 in case of no conjugate points the volume entropy coincides with the topological entropy of the geodesic flow. Since the set of pairwise non homotopic closed geodesic are $\delta$-separated for some $\delta > 0$ proposition 2.27 implies that the limiting distribution of closed geodesics is a measure of maximal entropy.

## 5. Patterson–Sullivan measure and the MME

In this section we assume that $M$ is a closed Riemannian manifold without conjugate points having the divergence property of geodesic rays and admitting a metric of negative sectional curvature, i.e. we are only assuming that conditions (H1) and (H2) in Definition 3.1 are satisfied. We will show that under this assumption the Patterson–Sullivan measure can be used to define a measure of maximal entropy which is fully supported on $T^1 M$. If we add the conditions (H3) and (H4) from Definition 3.1 we obtain uniqueness as was shown in §4.5.

### 5.1. Poincaré series and the Patterson–Sullivan measure

If $\Gamma$ denotes the group of deck transformations, for $p, q \in \widetilde{M}$ and $s \in \mathbb{R}$, we consider the Poincaré series

$$P(s, p, q) = \sum_{\gamma \in \Gamma} e^{-sd(p, \gamma q)}.$$  

Since $\widetilde{M}$ is Gromov hyperbolic it follows from [Coo93] that the series converges for $s > h$ and diverges for $s \leq h$, where $h$ is the topological entropy. For $x \in \widetilde{M}$ the set $\Lambda(\Gamma)$ of accumulation points of the orbit $\Gamma x$ in $\widetilde{M}$ is called the limit set. Since $\widetilde{M}$ is cocompact we have $\Lambda(\Gamma) = \partial \widetilde{M}$. Fix $x \in \widetilde{M}$, $s > h$ and consider for each $p \in X$ the measure

$$\nu_{p, x, s} = \frac{1}{P(s, x, x)} \sum_{\gamma \in \Gamma} e^{-sd(p, \gamma x)} \delta_{\gamma x} \tag{5.1}$$

where $\delta_y$ is the Dirac mass associated to $y \in \widetilde{M}$. Using the fact that $e^{-s\delta(p,x)}e^{-s\delta(x,\gamma x)} \leq e^{-s\delta(p,\gamma x)} \leq e^{s\delta(p,x)}e^{-s\delta(x,\gamma x)}$ for every $x, p \in X$ and
γ ∈ Γ, we see that
\begin{equation}
\label{eq:5.2}
e^{-sd(p,x)} \leq \nu_{p,x,s}(\text{cl}(\widetilde{M})) \leq e^{sd(p,x)};
\end{equation}
in particular, the ν_{p,x,s} are all finite. Moreover, we clearly have
\begin{equation}
\label{eq:5.3}
\Gamma x \subset \text{supp} \nu_{p,x,s} \subset \Gamma x.
\end{equation}
Now choose for a fixed p ∈ \widetilde{M} and a weak limit \( \lim_{k \to \infty} \nu_{p,x,s_k} =: \nu_p \).

The divergence of the series \( P(s,x,x) \) for \( s = h \) and the discreteness of \( \Gamma \) yields that the support of \( \nu_p \) is contained in the limit set. Moreover, one obtains:

**Proposition 5.1.** There is a sequence \( s_k \to h \) as \( k \to \infty \) such that for every \( p \in \widetilde{M} \) the weak limit \( \lim_{k \to \infty} \nu_{p,x,s_k} =: \nu_p \) exists. The family of measures \( \{ \nu_p \}_{p \in \widetilde{M}} \) has the following properties.

(a) \( \{ \nu_p \}_{p \in \widetilde{M}} \) is \( \Gamma \)-equivariant: for all Borel sets \( A \subset \partial \widetilde{M} \), we have
\[
\nu_{\gamma p}(\gamma A) = \nu_p(A).
\]

(b) \( \frac{d\nu_p}{d\nu_q}(\xi) = e^{-hb(q,\xi)} \) for almost all \( \xi \in \partial \widetilde{M} \), where \( b_p(q,\xi) \) is as in Definition 2.16.

(c) \( \text{supp} \nu_p = \partial \widetilde{M} \) for all \( p \in X \).

**Proof.** We first observe that given a compact subset \( K \subset \widetilde{M} \), there is a sequence \( s_k \to h \) as \( k \to \infty \) such that for every \( p \in K \) the weak limit \( \lim_{k \to \infty} \nu_{p,x,s_k} =: \nu_p \) exists. On the other hand, the definition of \( \nu_{p,x,s} \) immediately gives \( \nu_{\gamma p,x,s}(\gamma A) = \nu_{p,x,s}(A) \) for all \( p, x, s \). Thus, since \( M \) is a compact quotient we have the same sequence \( s_k \to h \) as \( k \to \infty \) such that for every \( p \in \widetilde{M} \) the weak limit \( \lim_{k \to \infty} \nu_{p,x,s_k} =: \nu_p \) exists. This gives the first assertion.

The second assertion follows immediately from (2.7). The third assertion follows from the fact that \( \Gamma \) acts minimally on \( \partial \widetilde{M} \) (see remark 2.15). Suppose \( \xi \notin \text{supp} \nu_p \) then \( \gamma \xi \notin \text{supp} \nu_p \) for all \( \gamma \in \Gamma \), since given an open neighborhood \( U \subset \widetilde{M} \) of \( \xi \) with \( \nu_p(U) = 0 \) we have \( 0 = \nu_{\gamma p}(\gamma \xi) = \nu_p(\gamma(U)) \) by (a) and (b) since the orbit \( \Gamma \xi \) is a dense and the \( \text{supp} \nu_p \) is closed we obtain \( \text{supp} \nu_p = \emptyset \) which contradicts the non-triviality of the Patterson-Sullivan measure. \( \square \)

**Remark 5.2.** Since \( |b_p(q,\xi)| \leq d(p,q) \) for all \( p, q, \xi \) property (b) implies that for every \( p, q \in \widetilde{M} \) and any measurable subsets \( A \subset \partial \widetilde{M} \) that \( \nu_p(A) \leq e^{hd(p,q)} \).

For \( \xi \in \partial \widetilde{M} \) and \( p \in \widetilde{M} \) consider the projections
\[
pr_\xi: \widetilde{M} \to \partial \widetilde{M} \quad \text{and} \quad pr_p: \widetilde{M} \setminus \{ p \} \to \partial \widetilde{M}
\]
along geodesics emanating from \( \xi \) and \( p \), respectively. That is, \( pr_\xi(x) = c_{\xi,x}(\infty) \), where \( c_{\xi,x} \) is the geodesic with \( c_{\xi,x}(-\infty) = \xi, c_{\xi,x}(0) = x \) and...
Lemma 5.3. There exists $R > 0$ such that for all $x \in \text{cl}(\tilde{M})$ and $p \in \tilde{M}$, the shadow set $\text{pr}_x B(p, R)$ of the open geodesic ball $B(p, R)$ with center $p$ and radius $R$ contains an open set in $\partial \tilde{M}$.

Proof. For $x \in \text{cl}(\tilde{M})$ and $p \in \tilde{M}$, let $v = V(p, x)$ be given by Proposition 2.17. By the definition of the topology on $\partial \tilde{M}$, for every $v \in T_p \tilde{M}$ and $\varepsilon > 0$ we have $A_{\varepsilon}(v) := \{ c_w(\infty) : \angle_p(v, w) < \varepsilon \}$ is open in $\partial \tilde{M}$. For every $\eta \in A_{\varepsilon}(v)$ there exists a unique geodesic $c_{\varepsilon, \eta}$ with respect to the metric of negative curvature joining $\xi := c_v(-\infty)$ and $\eta$; every such geodesic stays at a bounded distance to $p$ and this distance can be made arbitrary small by choosing $\varepsilon$ arbitrary small. By the Morse Lemma, every geodesic $c_{\varepsilon, \eta}$ corresponds to at least one geodesic $c_{\xi, \eta}$ which stay at $R_0$ distance (see Theorem 2.3). By choosing $\varepsilon$ small enough and $R$ large, we can guarantee $d(p, c_{\xi, \eta}) < R$ which implies that $A_{\varepsilon}(v) \subset \text{pr}_x B(p, R)$. \qed

Using Lemma 5.3 we obtain:

Proposition 5.4. Let $\{\nu_p\}_{p \in \tilde{M}}$ be the Patterson–Sullivan measures.

(a) For all $\rho \geq R$ there exists $\ell > 0$ such that for every $x \in \text{cl}(\tilde{M})$, we have

$$\nu_p(\text{pr}_x B(p, \rho)) \geq \ell.$$  

(b) There is a constant $b = b(\rho)$ such that for all $x \in \tilde{M}$ and $\xi = c_{p, x}(-\infty)$,

$$\frac{1}{b} e^{-hd(p, x)} \leq \nu_p(\text{pr}_x B(x, \rho)) \leq be^{-hd(p, x)}.$$  

(c) A similar estimate holds if we project from $p \in \tilde{M}$, namely there is a constant $a = a(\rho) > 0$ such that for all $p \in \tilde{M}$,

$$\frac{1}{a} e^{-hd(p, x)} \leq \nu_p(\text{pr}_p B(x, \rho)) \leq ae^{-hd(p, x)}.$$  

Proof. The last two estimates follow from (a) and the defining properties of $\nu_p$. To see this, observe that given $A \subset \partial M$, Proposition 5.1(b) gives

$$\nu_p(A) = \int_A e^{-hb_x(p, \eta)} d\nu_x(\eta).$$

If $A = \text{pr}_x B(x, \rho)$ or $\text{pr}_p B(x, \rho)$ then corollary 2.18 implies that $|b_x(p, \eta) - d(p, x)|$ is bounded by a constant for all $\eta \in A$, which yields (b) and (c).

The first estimate is a consequence of the following steps.

Step 1: $\text{supp} \, \nu_p = \partial M$ for one and, hence, for all $p \in \tilde{M}$ using Proposition 5.1(c).
For $v \in T^1_x \widetilde{M} = \pi^{-1}(x)$ and $\epsilon > 0$ consider the open neighborhood
\[ C_\epsilon(v) = \{ w \in T^1_x \widetilde{M} \mid \angle(v, w) < \epsilon \} \subset \partial \widetilde{M}. \]
Fix a compact set $K \subset \widetilde{M}$ such that $\bigcup_{\gamma \in \Gamma} \gamma(K) = \widetilde{M}$ and a reference point $x_0 \in K$. Then it follows:

**Step 2:** For all $\rho \geq R$ there exists $\epsilon > 0$ such that for all $p \in K$ and $x \in \text{cl}(\widetilde{M})$
\[ C_\epsilon(v) \subset \text{pr}_x(B(p, \rho)) \]
for some $v \in T^1_{x_0} \widetilde{M}$.

Suppose Step 2 is false. Then there exists $\rho \geq R$ and sequences $p_n \in K$, $x_n \in \text{cl}(\widetilde{M})$ such that
\[ C_{1/n}(v) \not\subseteq \text{pr}_{x_n}(B(p_n, \rho)) \]
for all $v \in T^1_{x_0} \widetilde{M}$. We can assume after choosing a subsequence that $x_n \to \xi \in \text{cl}(\widetilde{M})$ and $p_n \to p \in K$. Since $\text{pr}_\xi(B(p, \rho))$ contains some open set in $\partial \widetilde{M}$ there exists $\epsilon > 0$ and $v_0 \in T^1_{x_0} \widetilde{M}$ such that $C_\epsilon(v_0) \subset \text{pr}_\xi(B(p, \rho))$.

The continuity of the projection implies the existence of $n_0$ such that for all $n \geq n_0$ we have: $C_{1/2}(v_0) \subset \text{pr}_{x_n}(B(p_n, \rho))$. But this contradicts the choice of the sequence. Then Step 2 is true.

**Step 3:** For all $\epsilon > 0$ there exists a constant $\ell = \ell(\epsilon) > 0$ such that
\[ \nu_p(C_\epsilon(v)) > \ell \]
for all $v \in T^1_{x_0} \widetilde{M}$ and $p \in K$. This is a consequence of the following facts:
- each $\nu_p$ is fully supported (Step 1);
- $\sup \{ b_p(q, \xi) : \xi \in \partial \widetilde{M}, p, q \in K \} < \infty$ by compactness and continuity; and there is a finite collection of open sets in $\partial \widetilde{M}$ such that each $C_\epsilon(v)$ contains an element of this collection.

Now consider $x \in \text{cl}(\widetilde{M})$ and $p \in \widetilde{M}$. Choose $\gamma \in \Gamma$ such that $\gamma p \in K$. Since $\nu_p(\text{pr}_x(B(p, \rho))) = \nu_{\gamma p}(\text{pr}_{\gamma x} B(\gamma p, \rho))$ the estimate (a) follows from Steps 2 and 3.

**5.2. Construction of the measure of maximal entropy using the Patterson-Sullivan measure.** Now we construct an invariant measure for the geodesic flow using the Patterson-Sullivan measures $\nu_p$, following the approach in [Kni98], which was originally carried out in negative curvature in [Kai90].

Let $\mathcal{G}$ be the space of nonparametrized geodesics in $\widetilde{M}$. By Lemma 2.21 we have
\[ \{(c(-\infty), c(+\infty)) \mid c \in \mathcal{G} \} = (\partial \widetilde{M} \times \partial \widetilde{M}) \setminus \text{diag}. \]

By Proposition 5.1(b) $\nu_p$ is $\Gamma$-quasi-invariant with Radon-Nikodym cocycle
\[ f(\gamma, \xi) = e^{-h_{\gamma^{-1} p}(\gamma^{-1} p, \xi)} = \frac{d\nu_{\gamma^{-1} p}}{d\nu_p}(\xi). \]
For \((\xi,\eta) \in (\partial\wt M \times \partial M) \setminus \text{diag}\) consider

\begin{equation}
\beta_p(\xi,\eta) = -(b_p(q,\xi) + b_p(q,\eta)),
\end{equation}

where \(q\) is a point on a geodesic \(c\) connecting \(\xi\) and \(\eta\). In geometrical terms \(\beta_p(\xi,\eta)\) is the length of the segment \(c\) which is cut out by the horoballs through \((p,\xi)\) and \((p,\eta)\). Since \(\text{grad}_p b_p(q,\xi) = -\text{grad}_p b_p(q,\eta)\) for all points on geodesics connecting \(\xi\) and \(\eta\), this number is independent of the choice of \(q\). An easy computation using (5.4), see [Kni98, Lemma 2.4], shows:

**Lemma 5.5.** For \(p \in \wt M\), the measure \(\wt \mu\) on \((\partial\wt M \times \partial M) \setminus \text{diag}\) defined by

\[d\wt \mu(\xi,\eta) = e^{\beta_p(\xi,\eta)} d\nu_p(\xi) d\nu_p(\eta)\]

is \(\Gamma\)-invariant.

Let \(P: T^1\wt M \to (\partial\wt M \times \partial M) \setminus \text{diag}\) be the projection given by \(P(v) = (c_v(-\infty), c_v(\infty))\), where \(c_v\) is the geodesic with \(\dot c_v(0) = v\). Then \(\wt \mu\) induces a \(f_t\)-invariant measure \(\wt \mu\) on \(T^1\wt M\) by setting for each Borel subset \(A \subset T^1\wt M\)

\[\wt \mu(A) = \int_{(\partial\wt M \times \partial M) \setminus \text{diag}} \sup\{\lambda_1(l(v,A)) \mid v \in P^{-1}(\xi,\eta)\} d\wt \mu(\xi,\eta),\]

where \(l(v,A) = \{t \in \mathbb{R} \mid f_t(v) \cap A \neq \emptyset\}\) and \(\lambda_1\) is the Lebesgue measure of \(\mathbb{R}\). We observe if \(A\) is in a compact set then the integrand in the above definition is uniformly bounded. Indeed, this follows from the observation that given \(\xi,\eta \in \partial M\) the quantity \(\beta_p(\xi,\eta)\) blows up to infinity only when \(\xi\) and \(\eta\) are approaching each other; therefore if \(\xi\) and \(\eta\) are chosen with respect to a given compact set then \(\beta_p(\xi,\eta)\) is uniformly bounded. Thus \(\wt \mu\) is finite on every compact subset of \(\wt M\). By the \(\Gamma\)-invariance of \(\wt \mu\) this measure descends to a finite flow-invariant measure on \(T^1 M\). We normalize it so that it becomes a probability measure denoted by \(\mu\), i.e., \(\mu(T^1 M) = 1\).

**Remark 5.6.** Since \(\text{supp} \mu_p = \partial\wt M\) we obtain \(\text{supp} \wt \mu = T^1\wt M\) and therefore \(\text{supp} \mu = T^1 M\). Indeed, if \(U \subset T^1\wt M\) is open the projection \(P(U) \subset (\partial\wt M \times \partial M) \setminus \text{diag}\) is by the definition of the topology on \(\partial\wt M\) open as well and therefore, \(\wt \mu(P(U)) > 0\). Finally, the definition of \(\wt \mu\) implies \(\wt \mu(U) > 0\).

In order to compute the entropy of the measure \(\mu\) we consider a measurable partition \(A = \{A_1, \ldots, A_m\}\) of \(T^1 M\) such that the diameters of all elements in \(A\) are less than \(\epsilon\) with respect to the metric \(d_1\) defined at the beginning of §2.1.1. If \(v \in \alpha \in A_{f}^{(n)}\), then, by the definition of \(A\), it follows that

\[\alpha \subset \bigcap_{k=0}^{n-1} f_{-k} B_{d_1}(\phi^k v, \epsilon)\]

**Lemma 5.7.** Let \(0 < \epsilon < \min\{R, \text{inj}(M)\}\), where \(\text{inj}(M)\) is the injectivity radius of \(M\). Then there is a constant \(a > 0\) such that

\[\mu(\alpha) \leq e^{-hn} a\]
for all \( n \in \mathbb{N} \) and \( \alpha \in \mathcal{A}_f^{(n)} \).

**Proof.** Since \( \alpha \subset \bigcap_{k=0}^{n-1} f_{-k}B_{d}(f_{k}v,\epsilon) \) if \( v \in \alpha \) we have for all \( w \in \alpha \) that 
\[
d(c_{0}(t),c_{w}(t)) \leq \epsilon \text{ for all } t \in [0,n].
\]
Let \( p \in \tilde{M} \) be the reference point used in the definition of the measure \( \tilde{\mu} \) and \( \tilde{v} \in T^{1}\tilde{M} \) be a lift of \( v \) such that 
\[
d(\pi \tilde{v},p) \leq \text{diam } M = r_{0}.
\]
Since \( \epsilon < \text{inj } (M) \) we can lift the set \( \alpha \) to a set \( \tilde{\alpha} \in T^{1}\tilde{M} \) such that for all \( \tilde{w} \in \tilde{\alpha} \) we have 
\[
d(c_{\tilde{w}}(t),c_{\tilde{w}}(t)) \leq \epsilon \text{ for all } t \in [0,n].
\]
Let \( c_{\tilde{w}}(n) = x \) and \( \xi = c_{\tilde{w}}(-\infty) \). Let \( c_{\xi,x} \) be the geodesic connecting \( \xi \) and \( x \) such that \( c_{\xi,x}(n) = x \). Then,
\[
d(c_{\xi,x}(0),p) \leq d(c_{\xi,x}(0),\pi(\tilde{w})) + d(\pi(\tilde{w}),p) \leq 2\epsilon + r_{0},
\]
i.e., \( \xi \in pr_{x}(B(p,r_{0} + 2\epsilon)) \). Therefore, if \( P : T^{1}\tilde{M} \to \mathcal{G} \) denotes the endpoint projection as in the paragraph following Lemma 5.5, we have
\[
P(\tilde{\alpha}) \subset \bigcup_{\eta \in pr_{x}(B(p,r_{0} + 2\epsilon))} \{ \eta \} \times pr_{\eta}(B(x,\epsilon)).
\]
For each \( \eta \in pr_{x}(B(p,r_{0} + 2\epsilon)) \) choose a point \( q \in B(p,r_{0} + 2\epsilon) \) that lies on the geodesic \( c_{\eta,x} \). Then, using the transformation rule for the Patterson-Sullivan measure, Proposition 5.4(b) and the estimate
\[
d(q,x) \geq d(x,\pi \tilde{v}) - d(\pi \tilde{v},p) - d(p,q) \geq n - r_{0} - (2\epsilon + r_{0}),
\]
we obtain
\[
u_{p}(pr_{\eta}(B(x,\epsilon))) \leq e^{hd(q,p)}\nu_{q}(pr_{\eta}(B(x,\epsilon))) \leq e^{h(r_{0} + 2\epsilon)}b e^{-hd(q,x)} \leq b e^{-hn}
\]
for a constant \( b = be^{h(3r_{0} + 4\epsilon)} > 0 \), where the first inequality follows from Remark 5.2. Since \( \nu_{p}(pr_{x}(B(p,r_{0} + 2\epsilon))) \leq \nu_{\mu}(\partial \tilde{M}) < \infty \) the requires follows from the definition of \( \tilde{\mu} \). Since \( \tilde{\mu} \) and \( \mu \) agree up to normalization constant the proof follows. \( \square \)

**Theorem 5.8.** The measure \( \mu \) is a measure of maximal entropy for the geodesic flow, i.e.,
\[
h_{\mu}(f_{1}) = h_{\text{top}}(f_{1}) = h.
\]
Furthermore, if follows from Remark 5.6 that \( \mu \) has full support in \( T^{1}M \).

**Proof.** Choose a partition \( \mathcal{A} \) as above. Then by Lemma 5.7
\[
H(\mathcal{A}_f^{(n)}) = \sum_{\alpha \in \mathcal{A}_f^{(n)}} \mu(\alpha) (- \log \mu(\alpha)) \geq (hn - \log a) \sum_{\alpha \in \mathcal{A}_f^{(n)}} \mu(\alpha) = hn - \log a.
\]
Hence, \( h(f,\mathcal{A}) \geq h \). The theorem follows since \( h \geq h_{\mu}(f) \geq h(f,\mathcal{A}) \geq h \). \( \square \)
6. ON MIXING OF THE MEASURE OF MAXIMAL ENTROPY

In this section we prove a similar result to [Bab02, Theorem 2] namely that the MME constructed in the above section under the assumptions (H1), (H2) and (H4) is mixing provided the measure is ergodic. However, under the assumptions (H1), (H4) the MME is unique and therefore in particular ergodicity is guaranteed.

Theorem 6.1. Let $M$ be a smooth closed Riemannian manifold without conjugate points satisfying conditions (H1)–(H4). Then $F$ is mixing with respect to the unique measure of maximal entropy $\mu$.

As in [Bab02, Theorem 2], the proof of Theorem 6.1 is based on three key properties of the flow that are derived from the assumptions of Theorem 1.2:

- the product structure properties of the measure of maximal entropy measure that is given by the Patterson-Sullivan construction in §5;
- the continuity of the cross-ratio function which is discussed in §6.1;
- there is enough hyperbolicity for typical point of the MME that allows to run a version of Hopf argument; this is the content of Lemma 6.7.

6.1. The cross-ratio function. Most of the definitions and properties below are given in [Dal99] which are inspired by similar concepts in [Ota92]. However, since in [Dal99], the case of negative curvature is considered, for completeness, we include all the proofs.

Given two points $\xi, \xi' \in \partial \tilde{M}$, $(\xi, \xi')$ denotes a geodesic, which is not necessary unique, joining $\xi$ and $\xi'$. Given $p \in \tilde{M}$ and $\xi \in \partial \tilde{M}$, $H_p(\xi)$ denotes the horosphere centered at $\xi$ containing $p$. Observe that $H_p(\xi) = \{ q \in \tilde{M} : b_p(q, \xi) = 0 \} = H^*(v)$, where $v$ is the unique unit tangent vector at $p$ such that $c_p(\infty)$ (see Lemma 2.14 and Definition 2.16).

The following result follows from [Rug07, Lemma 4.11].

Proposition 6.2. Let $(M, g)$ be a compact Riemannian manifold without conjugate points. If $M$ is of hyperbolic type then for every $p \in \tilde{M}$, the map $\xi \mapsto H_p(\xi)$ is continuous where $\{ H_p(\xi), \xi \in \partial M \}$ is equipped with the compact open topology.

Lemma 6.3 (Definition). For $p \in \tilde{M}$, the length of the segment in $(\xi, \xi')$ with end points in $H_p(\xi) \cap (\xi, \xi')$ and $H_p(\xi') \cap (\xi, \xi')$ does not depend on the choice of the geodesic $(\xi, \xi')$. In particular the Gromov product $(\xi|\xi')_p$ is well defined as the length of that segment, see Figure 6.1.

Proof. Let $v, w \in T^1\tilde{M}$ associated to two different geodesics joining $\xi$ and $\xi'$ as in Figure 6.1. Then the proof follows since the flow on $\tilde{M}$ tangent to $\grad b_v$ takes horospheres to horospheres. \qed

Remark 6.4. Observe that $(\xi|\xi')_p = |\beta_p(\xi, \xi')|$, recall (5.5).
Fixing a reference point $p \in \tilde{M}$, the cross-ratio of four points $\xi, \xi', \eta, \eta' \in \partial \tilde{M}$ is defined by

$[\xi, \xi', \eta, \eta'] := ((\xi|\eta')_p + (\xi'|\eta)_p) - ((\xi|\eta)_p + (\xi'|\eta')_p)$

We remark that from the continuity of the map $\xi \to H_p(\xi)$ (Proposition 6.2), the cross-ratio function is continuous in $(\partial \tilde{M})^4$. Moreover as in [Bou96], we observe that for $q \in \tilde{M}$,

$(6.1) \quad (\xi|\xi')_p - (\xi|\xi')_q = b_p(q, \xi) + b_p(q, \xi')$

This implies that the cross ratio does not depend on the reference point $p$. Using Corollary 2.19 and the fact that $\Gamma$ acts on $\tilde{M}$ by isometries, we have:

$(6.2) \quad b_p(q, \xi) = b_p(p', \xi) + b_{p'}(q, \xi) \quad \forall p, p', q \in \tilde{M}$ and $\forall \xi \in \partial \tilde{M}$,

$(6.3) \quad b_{\gamma(p)}(\gamma(q), \gamma(\xi)) = b_p(q, \xi) \quad \forall p, q \in \tilde{M}, \forall \gamma \in \Gamma$ and $\forall \xi \in \partial \tilde{M}$.

Given $\xi, \xi', \eta, \eta' \in \partial \tilde{M}$, we fix $v \in (\xi, \eta)$. Let $v_1 := (\xi', \eta) \cap H^s(v)$, $v_2 := (\xi', \eta') \cap H^s(v_1)$, $v_3 := (\xi, \eta') \cap H^s(v_2)$, $v_4 := (\xi, \eta) \cap H^s(v_3)$. We have the following

**Lemma 6.5.** $v_4 = f_{[\xi, \xi', \eta, \eta']}(v)$. 

*Figure 6.1. Gromov product.*
Lemma 6.5 is due to Otal [Ota92] in the case of negative curvature and it shows the analogy between the cross-ratio function and the temporal function in [Liv04, Figure 2].

Proof. Let $\xi, \xi', \eta, \eta' \in \partial \widetilde{M}$ and $p \in \widetilde{M}$. We refer to Figure 6.2 where the dotted horospheres $H_p(\xi), H_p(\xi'), H_p(\eta), H_p(\eta)$ are referred to calculate the cross-ratio $[\xi, \xi', \eta, \eta']$ as defined above using the Gromov product, the vectors $v'_i$ whose foot points are $q'_i$ are defined above and there corresponding horospheres. By the definition of the horospherical distance we have

$$b_{q_3}(p, \eta') = b_{q_4}(p, p') = (\xi'|\eta)_p - (\xi'|\eta')_p.$$

On the other hand we also have

$$b_{q_3}(q_2, \eta') = b_{q_4}(p', \xi) - (\xi'|\eta')_p = b_{q_4}(q, \xi) + (\xi|\eta)_p - (\xi'|\eta')_p.$$

Combining these two equalities we conclude that

$$b_{q_4}(q, \xi) = [\xi, \xi', \eta, \eta']$$
which finishes the proof.

6.2. **Asymptotic convergence.** The aim of this section is to prove some hyperbolic estimate for almost every point with respect to the MME.

Given \( v \in T^1 M \), let \( \tilde{v} \) be a lift of \( v \) to \( T^1 M \), and let

\[
W^{ss}(\tilde{v}) = W^s(\tilde{v}) \cap \pi^{-1}H^s(\tilde{v}) \subset T^1 \tilde{M}.
\]

Given \( R > 0 \) and \( \tilde{v} \in T^1 M \), let

\[
W^{ss}_R(\tilde{v}) = \{ \tilde{w} \in \pi^{-1}H^s(\tilde{v}) : d(f_t \tilde{w}, f_t \tilde{v}) \leq R \text{ for all } t \geq 0 \}.
\]

Observe that \( W^{ss}(\tilde{v}) = \bigcup_{R > 0} W^{ss}_R(\tilde{v}) \). Define \( W^{uu}(\tilde{v}) \) and \( W^{uu}_R(\tilde{v}) \) similarly.

Recall that

\[
E = \{ v \in T^1 M : W^{ss}(\tilde{v}) \cap W^{uu}(\tilde{v}) = \{ \tilde{v} \} \}.
\]

Fix \( R > 0 \) and consider for each \( v \in T^1 M \) and \( t > 0 \) the following value:

\[
\varphi_t(v) = \sup\{ d(\tilde{w}, \tilde{v}) : f_{-t} \tilde{w} \in W^{ss}_R(f_{-t} \tilde{v}) \}
\]

**Lemma 6.6.** If \( v \in E \), then \( \varphi_t(v) \downarrow 0 \) monotonically as \( t \to \infty \).

**Proof.** Monotonicity follows from the fact that \( f_{-t'} \tilde{w} \in W^{ss}_R(f_{-t} \tilde{v}) \) implies \( f_{-t'} \tilde{w} \in W^{ss}_R(f_{-t'} \tilde{v}) \) for all \( t' \leq t \), so the sets in the definition of \( \varphi_t \) are nested decreasing as \( t \) increases. For convergence to 0, suppose \( v \in T^1 M \) is such that \( \varphi_t(v) \not\rightarrow 0 \); then there are \( \delta > 0 \), \( t_n \to \infty \), and \( \tilde{w}_n \in f_{t_n} W^{ss}_R(f_{-t_n} \tilde{v}) \) such that \( d(\tilde{w}_n, \tilde{v}) \geq \delta \) for all \( n \). Since \( d(\tilde{w}_n, \tilde{v}) \leq R \), we can replace \( \tilde{w}_n \) with a convergent subsequence that has \( \tilde{w}_n \to \tilde{w} \), and observe that \( f_{-t} \tilde{w} \in W^{ss}_R(f_{-t} \tilde{v}) \) for all \( t > 0 \), so \( \tilde{w} \in W^{ss}_R(f_t \tilde{v}) \cap W^{uu}_R(\tilde{v}) \); moreover, \( d(\tilde{w}, \tilde{v}) \geq \delta \), so \( \tilde{w} \neq \tilde{v} \), and thus \( v \notin E \). \( \square \)

Let \( \mu \) be a flow-invariant probability measure on \( T^1 M \) such that \( \mu(E) = 1 \). Since each \( \varphi_t \) is measurable and bounded, and \( \mu \) is finite, the monotone convergence theorem implies that \( \varphi_t \to 0 \) in the \( L^1 \) norm. Observing that the function \( \varphi_t \circ f_t \) satisfies

\[
\| \varphi_t \circ f_t \|_1 = \int \varphi_t \circ f_t d\mu = \int \varphi_t d\mu = \| \varphi_t \|_1
\]

by flow-invariance of \( \mu \), we conclude that \( \varphi_t \circ f_t \to 0 \) in \( L^1 \). Thus for any sequence \( t_n \to \infty \), there is a subsequence \( t_{n_k} \to \infty \) such that \( \varphi_{t_{n_k}} \circ f_{t_{n_k}} \to 0 \) \( \mu \)-a.e. Note that

\[
\varphi_t(f_{t} v) = \sup\{ d(f_t \tilde{w}, f_t \tilde{v}) : \tilde{w} \in W^{ss}_R(\tilde{v}) \};
\]

thus we can prove the following.

**Lemma 6.7.** Let \( \mu \) be a flow-invariant probability measure on \( T^1 M \) such that \( \mu(E) = 1 \). For every \( t_n \to \infty \) there is a subsequence \( t_{n_k} \to \infty \) such that

\[
d(f_{t_{n_k}} \tilde{w}, f_{t_{n_k}} \tilde{v}) \to 0 \text{ for } \mu \text{-a.e. } v \in T^1 M \text{ and every } \tilde{w} \in W^{ss}(\tilde{v}).
\]
Proof. The preceding discussion shows that for each $R > 0$ and every $t_n \to \infty$, there is a subsequence $t_{n_k} \to \infty$ such that $(6.5)$ holds with $W^{ss} (\tilde{v})$ replaced by $W^{ss}_R (\tilde{v})$. Applying this with $R = 1, 2, 3, \ldots$ gives a nested family of subsequences, and the usual diagonal argument gives a subsequence that works for every $R$. Since $W^{ss} (\tilde{v}) = \bigcup_{R \geq 0} W^{ss}_R (\tilde{v})$, this proves the lemma. \hfill \Box

6.3. Proof of mixing. Now we prove that the unique MME $\mu$ for geodesic flow on $T^1 M$ is mixing, using its product structure to run a version of the Hopf argument due to Babillot [Bab02].

Suppose for a contradiction that $\mu$ is not mixing. Then there is a continuous function $\varphi$ on $T^1 M$ such that $\varphi \circ f_t$ does not converge weakly to 0 in $L^2 (\mu)$. Now we need the following lemma.

**Lemma 6.8 ([Bab02] Lemma 1).** Let $(X, B, m, (T_t)_{t \in A})$ be a measure preserving dynamical system, where $(X, B)$ is a standard Borel space, $m$ a (possibly unbounded) Borel measure on $(X, B)$ and $(T_t)_{t \in A}$ an action of a locally compact second countable abelian group $A$ on $X$ by measure preserving transformations. Let $\varphi \in L^2 (X, m)$ be a real-valued function on $X$ such that $\int \varphi \, dm = 0$ if $m$ is finite.

If there exists a sequence $(t_n)$ going to infinity in $A$ such that $\varphi \circ T_{t_n}$ does not converge to 0 in the weak-$L^2$ topology, then there exist a sequence $(s_n)$ going to infinity in $A$ and a non-constant function $\psi$ in $L^2 (X, m)$ such that

$$\varphi \circ T_{s_n} \to \psi \quad \text{and} \quad \varphi \circ T_{-s_n} \to \psi \quad \text{in the weak-$L^2$ topology.}$$

We conclude that there is $s_n \to \infty$ and a non-constant $\psi \in L^2 (T^1 M, \mu)$ such that $\varphi \circ f_{s_n} \to \psi$ in the weak-$L^2$ topology as $n \to \infty$. Applying Lemma 6.7 we can replace $s_n$ with a subsequence such that for $\mu$-a.e. $v \in T^1 M$, we have

$$\lim_{n \to \infty} d (f_{s_n} \tilde{w}^s, f_{s_n} \tilde{v}) = 0 \quad \text{for all } \tilde{w}^s \in W^{ss} (\tilde{v}),$$

(6.6) $$\lim_{n \to \infty} d (f_{-s_n} \tilde{w}^u, f_{-s_n} \tilde{v}) = 0 \quad \text{for all } \tilde{w}^u \in W^{uu} (\tilde{v}).$$

**Lemma 6.9 ([Bab02]).** Let $(\varphi_n)$ be a sequence converging weakly in $L^2 (X, B, m)$ to some function $\psi$. Then there is a subsequence $(\varphi_{n_k})$ such that the Cesaro averages

$$A_{K^2} = \frac{1}{K^2} \sum_{k=1}^{K^2} \varphi_{n_k}$$

converge almost surely to $\psi$.

Proof. In [Bab02] this is quoted as a consequence of the proof of the Banach–Saks threm (see p. 80 of Riesz–Sz. Nagy 1968), which gives a subsequence such that the square of the $L^2$-norm of $A_K - \psi$ is $O(1/K)$, and then almost-sure convergence of $(A_{K^2})$ follows from Borel–Cantelli. \hfill \Box

Thus there is a set $R \subset T^1 M$ such that $\mu (R) = 1$ and a subsequence $s_{n_k} \to \infty$ such for every $v \in R$, the following are true:

1. the convergence statements in (6.6) hold;
Let \( \tilde{\psi} \) be a lift of \( \psi \) to \( T^1\tilde{M} \), and smooth \( \tilde{\psi} \) along the flow by replacing it with \( v \mapsto \int_0^1 \tilde{\psi}(f_tv) \, dt \). By choosing \( \epsilon \) small enough, \( \tilde{\psi} \) is not constant. By continuity of \( \varphi \) and the two properties just listed, we see that

if \( v, w \in R \) and \( \tilde{w} \in W^{ss}(\tilde{v}) \) or \( \tilde{w} \in W^{uu}(\tilde{v}) \), then \( \psi(v) = \psi(w) \).

There is a set \( R_0 \) of full \( \mu \)-measure such that for every \( v \in R_0 \), the function \( t \mapsto \tilde{\psi}(f_tv) \) is well-defined and continuous at all real \( t \); in particular, the set of periods of this function is a closed subgroup of \( \mathbb{R} \). This subgroup only depends on the geodesic: \( v \) and \( f_tv \) have the same subgroup for all \( t \in \mathbb{R} \). By ergodicity of \( \mu \), there is a single subgroup that works for \( \mu \)-a.e. \( v \). This subgroup is not all of \( \mathbb{R} \) since \( \tilde{\psi} \) is not constant, and now the remaining parts of the proof can be carried out exactly as in \([\text{Bab02}]\):

Because \( \nu_p \times \nu_p \) is a product measure, there is a set \( E \subset (\partial\tilde{M} \times \partial\tilde{M}) \setminus \text{diag} \) of full \( \nu_p \times \nu_p \) measure, a real number \( a > 0 \), and a \( \Gamma \)-invariant function \( \tilde{\psi} \) defined \( \tilde{\mu} \)-a.e. on \( T^1\tilde{M} \) such that for every \( (x, y) \in E \), the group of periods of \( \tilde{\psi} \) restricted to \( c_{x,y} \) is exactly \( a\mathbb{Z} \).

Next step (page 69 of \([\text{Bab02}]\)): for \( \nu_p^4 \)-a.e. quadrilateral, the cross-ratio belongs to \( a\mathbb{Z} \). Since \( \nu_p \) is fully supported on \( \partial\tilde{M} \), every cross-ratio of a quadrilateral belongs to \( a\mathbb{Z} \).

Since the cross-ratio of \( (x, x, y, y) \) is 0, the same is true of any nearby quadrilateral, which leads to a contradiction; choose \( q \) on \( c_{x,y} \) and let \( x', y' \) be such that the corresponding geodesic is regular, passes through \( p \), and is sufficiently close to \( c_{x,y} \), then we have a quadrilateral with strictly positive cross-ratio (‘Fact’ on page 72 of \([\text{Bab02}]\)), a contradiction.

**Appendix A. Morse**

**Proof of Lemma 2.4** Let \( R_1 > 0 \) and \( c_1, c_2 : [0, T] \to \tilde{M} \) be two geodesics with

\[
\left(d(c_1(0), c_2(0)) \leq R_1 \quad \text{and} \quad d(c_1(T), c_2(T)) \leq R_1 \right).
\]

For \( i = 1, 2 \), let \( \alpha_i : [0, T_i] \to \tilde{M} \) be a \( g_0 \)-geodesic such that \( \alpha_i(0) = c_i(0) \) and \( \alpha_i(T_i) = c_i(T) \). By Theorem 2.3, we have \( d_H(c_1, \alpha_i) \leq R_0 \) for \( i = 1, 2 \), where \( R_0 \) depends only on \( g \) and \( g_0 \). Without loss of generality we assume that \( T_1 \leq T_2 \). Then the triangle inequality gives

\[
d^0(\alpha_2(0), \alpha_2(T_2)) \leq d^0(\alpha_2(0), \alpha_1(0)) + d^0(\alpha_1(0), \alpha_1(T_1)) + d^0(\alpha_1(T_1), \alpha_2(T_2)).
\]

Note that \( d^0(\alpha_i(0), \alpha_i(T_i)) = T_i \), and that (2.4) gives

\[
d^0(\alpha_2(0), \alpha_1(0)) = d^0(c_2(0), c_1(0)) \leq Ad(c_2(0), c_1(0)) \leq AR_1,
\]

with a similar bound on \( d^0(\alpha_1(T_1), \alpha_2(T_2)) \). We deduce that

(A.1)

\[
T_2 \leq T_1 + 2AR_1,
\]

and consequently

\[
d^0(\alpha_1(T_1), \alpha_2(T_1)) \leq d^0(\alpha_1(T_1), \alpha_2(T_2)) + |T_2 - T_1| \leq 3AR_1.
\]
Since $g_0$ is negatively curved, the function $t \mapsto d^0(\alpha_1(t), \alpha_2(t))$ is convex, and therefore achieves its maximum on $[0, T_1]$ at an endpoint; since $d^0(\alpha_1(0), \alpha_2(0)) \leq AR_1$, we conclude that
\begin{equation}
(A.2) \quad d^0(\alpha_1(t), \alpha_2(t)) \leq 3AR_1 \quad \text{for all } t \in [0, T_1].
\end{equation}

Since $d_H(c_i, \alpha_i) \leq R_0$, for every $t \in [0, T]$, there exist $t_0, t' \in [0, T]$ such that $d(c_1(t_0), \alpha_1(t_0)), d(c_2(t'), \alpha_2(t_0)) \leq R_0$. Using the triangle inequality via $\alpha_1(t_0), \alpha_2(t_0), c_2(t')$ together with $(A.1)$ and $(A.2)$, this gives
\begin{equation}
(A.3) \quad d(c_1(t), c_2(t)) \leq 2R_0 + 3A^2R_1 + |t - t'|.
\end{equation}

Mimicking the argument that gave us $(A.1)$, we use the triangle inequality via $c_1(0)$ and $c_1(t)$ to deduce
\[ t' = d(c_2(0), c_2(t')) \leq R_1 + t + (2R_0 + 3A^2R_1), \]
and a symmetric argument gives
\[ |t' - t| \leq 2R_0 + (3A^2 + 1)R_1. \]

Together with $(A.2)$, this completes the proof of the lemma by putting $R_2 := 4R_0 + (6A^2 + 1)R_1$. \hfill \Box

**Appendix B. Specification**

Let $M$ be a smooth closed Riemannian manifold without conjugate points satisfying $(H1)$ and $(H2)$. Let $R_1, R_2, \delta$ be as given at the beginning of §4. In this section we prove the following version of the specification property.

**Theorem B.1.** There exist $\tau, \tau' > 0$ such that given $(v_1, t_1), \ldots, (v_k, t_k) \in T^1M \times (0, \infty)$ and $T_1, \ldots, T_k \in \mathbb{R}$ with $T_{j+1} - T_j \geq t_j + \tau$ for all $1 \leq j < k$, there are $\tilde{T}_j \in [T_j - \tau', T_j]$ and $w \in T^1M$ such that for all $1 \leq j \leq k$, we have $f_{\tilde{T}_j}w \in B_{t_j}(v_j, \delta)$.

**Remark B.2.** The conclusion of Theorem B.1 is a mild strengthening of the specification property from Definition 2.23. To deduce that property from this one, observe that the property here remains true if $\tau, \tau'$ are replaced by $\max(\tau, \tau')$, and then choosing $T_{j+1} = T_j + t_j + \tau$, the times $\tilde{T}_j$ satisfy $T_{j+1} - (\tilde{T}_j + t_j) \leq T_{j+1} - (T_j - \tau + t_j) \leq 2\tau$.

Note that it suffices to prove Theorem B.1 in the case when $T_{j+1} = T_j + t_j + \tau$; to reduce the general case to this one, replace $t_j$ by $T_{j+1} - T_j - \tau \geq t_j$ and observe that this does not weaken the condition on $w$.

Let $F^0 = \{f_t^0\}$ denote the geodesic flow in the background (negatively curved) Riemannian metric. We identify $T^1M$ for $g_0$ with $T^1M$ for $g$ in the natural way. To establish specification for $F$, we use the hyperbolicity properties of $F^0$ as in [BCFT18, §4] together with the Morse Lemma.

Let $W^s$ and $W^u$ denote the strong stable and unstable foliations of $T^1M$ for the background flow $F^0$. (We follow the notation in [BCFT18] rather than that used in §2 where this notation referred to the weak foliations; in this section we will not use any of the foliations for the flow $F$.) Equip
the leaves of these foliations with the intrinsic metrics \( d^s \) and \( d^u \) defined by pulling back the metrics that the horospheres inherit from the Riemannian metric. Let \( W^{cs} \) denote the foliation whose leaves have the form \( \{ f_t^L(W^s(v)) : t \in \mathbb{R} \} \), and define \( d^{cs} \) on each leaf of \( W^{cs} \) locally by \( d^{cs}(v,w) = |t| + d^u(f_t v,w) \), where \( t \) is such that \( f_t v \in W^s(w) \). Given \( \rho > 0 \), let \( W_\rho^u(v) \) denote the \( d^u \)-ball in \( W^u(v) \) of radius \( \rho \) centered at \( v \), and similarly for \( W_\rho^{cs} \).

The foliations \( W^{cs} \) and \( W^u \) have the following local product structure property: there are \( \kappa \geq 1 \) and \( \delta > 0 \) such that for every \( \varepsilon \in (0,\delta) \) and all \( w_1, w_2 \in B(v,\varepsilon) \), the intersection \( W_\kappa^u(w_1) \cap W_\kappa^{cs}(w_2) \) contains a single point, which we denote by \([w_1, w_2]\), and this point satisfies

\[
\begin{align*}
  d^u(w_1, [w_1, w_2]) &\leq \kappa d_1(w_1, w_2), \\
  d^{cs}(w_2, [w_1, w_2]) &\leq \kappa d_1(w_1, w_2).
\end{align*}
\]

**Proposition B.3.** For every \( \rho > 0 \), there exists \( T > 0 \) such that for every \( v, w \in T^1M \), we have \( f_t^L(W_\rho^u(v)) \cap W_\rho^{cs}(w) \neq \emptyset \) for every \( t \geq T \).

**Proof.** Since every leaf of \( W^u \) is dense in \( T^1M \), a simple compactness argument as in [CFT88] Lemma 8.1 gives \( R > 0 \) such that \( W_\rho^0(f_t v) \) is \( \rho/\kappa \)-dense in \( T^1M \) for every \( v \in T^1M \) and \( t \in \mathbb{R} \). By the local product structure, we have \( W_\rho^0(f_t v) \cap W_\rho^{cs}(w) \neq \emptyset \). Since leaves of \( W^u \) are uniformly expanded by \( f_t^0 \), there exists \( T > 0 \) such that \( f_t(W^u_\rho(v)) \supset W_\rho^0(f_t v) \) for every \( v \in T^1M \) and \( t \geq T \), which completes the proof. \( \square \)

Since leaves of \( W^u \) are uniformly expanded by \( f_t^0 \), there is \( \lambda \in (0,1) \) such that whenever \( v, w \) lie in the same leaf of \( W^u \), we have

\[
\begin{align*}
  (B.1) \quad d^u(f_{-t}^{-1} v, f_0^0 w) &\leq \lambda d^u(v,w) \quad \text{for all } t \geq 1.
\end{align*}
\]

Let \( \rho' > 0 \) be sufficiently small that \( R_1 > 3A(R_0 + \rho') \), and let

\[
\begin{align*}
  (B.2) \quad \rho = \rho'(1 - \lambda)/2.
\end{align*}
\]

By Proposition B.3, there exists \( T \geq 1 \) such that \( f_t(W^u_\rho(v)) \) intersects \( W^{cs}_\rho(w) \) whenever \( \tau \geq T \). We will prove Theorem B.1 with \( \tau = AT \) and \( \tau' = 2\tau_0 \), where \( \tau_0 = 2\tau + 7AR_0 + 4A\rho' \).

**Definition B.4** (Correspondence between orbit segments). The following procedure defines a map \( E : T^1M \times (0,\infty) \to T^1M \times (0,\infty) \) with the property that if \( (v,t) \) represents an \( F \)-orbit segment, then \( E(v,t) \) represents an \( F^0 \)-orbit segment that shadows it to within the scale given by the Morse Lemma.

1. Given \( (v,t) \in T^1M \times (0,\infty) \), the corresponding \( F \)-orbit segment projects to the \( g \)-geodesic segment \( c_v([0,t]) \).
2. Let \( x,y \in \widetilde{M} \) be the endpoints of a lift of \( c_v([0,t]) \) to \( \widetilde{M} \).
3. Let \( s = d^0(x,y) \), and let \( \alpha : [0,s] \to \widetilde{M} \) be a \( g_0 \)-geodesic that lifts to a segment running from \( x \) to \( y \).
4. Let \( E(v,t) = (\tilde{\alpha}(0),s) \).
Fix \((v_1, t_1), \ldots, (v_k, t_k) \in T^1 M \times (0, \infty)\) and \(T_1, \ldots, T_k, s_1, \ldots, s_k \in \mathbb{R}\) with \(T_1 = 0\) and

\[ s_j = T_j + t_j, \quad T_{j+1} = s_j + \tau. \]

We define sequences \(\hat{t}_i, v_i, t_i', w_i, s_i, T_i, \hat{s}_i, \Delta_i\) recursively (see Figure B.1).

- Let \(\hat{t}_1 = t_1, \hat{s}_1 = s_1,\) and \(\Delta_1 = T'_1 = 0.\)
- Let \((w'_1, s'_1) = (v'_1, t'_1) = E(v_1, \hat{t}_1).\)

Now fix \(j \geq 2\) and suppose all terms have been defined for \(1 \leq i < j.\)

- Let \(\hat{t}_j = t_j + \Delta_{j-1}\) and \((v'_j, t'_j) = E(v_j, \hat{t}_j).\)
- Let \(T'_j = s'_{j-1} + T\) and \(s'_j = T'_j + t'_j.\)
- Use Proposition B.3 to get \(w'_j\) such that

\[ f_{T_j}^0 (w'_j) \in f_T^0 (W^u_{\rho} (s'_{j-1}, w'_{j-1})) \cap W^{cs}_{\rho} (v'_j). \]

- Put \((w_j, \hat{s}_j) = E^{-1} (w'_j, s'_j)\) and \(\Delta_j = s_j - \hat{s}_j.\)

Observe that \(d^{cs}(f_{T_j}^0, w'_j, v'_j) \leq \rho\) and for all \(1 \leq i < j \leq k,\)

\[ d^u(f_{s'_i}^0, w'_j, f_{s'_i}^0, w'_{j-1}) \leq \lambda^{j-i-1} \rho. \]
For each $1 \leq i < k$, summing (B.3) over $j$ from $i$ to $k - 1$ gives

$$d^u(f_{s^i_k}^0 w_k, f_{s^i_i}^0 w_i^j) \leq \sum_{j=i+1}^{k} \lambda^{i-j} \rho < \rho(1 - \lambda)^{-1}.$$ 

Let $\alpha_j: [0, s^i_j] \to \tilde{M}$ be a $g_0$-geodesic corresponding to $(w^i_j, s^i_j)$. Then for each $i$, there is a $g_0$-geodesic $\beta_i: [0, t^i_t] \to \tilde{M}$ corresponding to $(v^i_t, t^i_t)$ such that for every $1 \leq i \leq j \leq k$ and every $0 \leq t \leq t^i_j$, we have

$$(B.4) \quad d^0(\beta_i(t), \alpha_j(T^i_j + t)) \leq d^0(\beta_i(t^i_j), \alpha_j(t^i_j)) + d^0(f_{s^i_k}^0 w_k, f_{s^i_i}^0 w_i^j) < \rho'. $$

In particular, writing $x_i = \beta_i(0), y_i = \beta_i(t^i_j) \in \tilde{M}$, we have

$$(B.5) \quad d^0(x_i, \alpha_j(T^i_i)) < \rho' \quad \text{and} \quad d^0(y_i, \alpha_j(s^i_j)) < \rho' \quad \text{for every} \quad 1 \leq i \leq j \leq k. $$

Let $b_i: [0, \hat{t}_i] \to \tilde{M}$ be the $g$-geodesic connecting $x_i$ and $y_i$; note that $b_i$ corresponds to the $F$-orbit segment $(v_i, \hat{t}_i)$. Let $c_j: \mathbb{R} \to \tilde{M}$ be the $g$-geodesic with the property that

$$(B.6) \quad c_j(0) = \alpha_j(0) \quad \text{and} \quad c_j(\hat{s}_j) = \alpha_j(s^i_j). $$

We prove Theorem [B.1] by showing that $\hat{c}_j(0)$ has the desired shadowing properties.

First note that by the Morse Lemma, for each $1 \leq i < j \leq k$ there are $\tilde{T}^i_j, \tilde{s}^i_j \in \mathbb{R}$ such that

$$(B.7) \quad d(\alpha_j(T^i_i), c_j(\tilde{T}^i_j)) \leq AR_0 \quad \text{and} \quad d(\alpha_j(s^i_j), c_j(\tilde{s}^i_j)) \leq AR_0. $$

We can take $\tilde{T}^i_0 = 0$ and $\tilde{s}^i_0 = \hat{s}_j$. Using (B.5) gives

$$(B.8) \quad d(x_i, c_j(\tilde{T}^i_i)) \leq A(R_0 + \rho') \quad \text{and} \quad d(y_i, c_j(\tilde{s}^i_j)) \leq A(R_0 + \rho'). $$

This, together with Lemma [2.4] will establish the necessary shadowing properties once we have proved the following lemmas relating $\tilde{s}^i_j, \hat{s}_i, s_i$ and $\tilde{T}^i_j, T_i$.

**Lemma B.5.** For every $1 \leq i < j \leq k$, we have $|\tilde{s}^i_j - \hat{s}_i| \leq A(R_0 + 2\rho')$.

**Lemma B.6.** For every $1 \leq i < j \leq k$, we have

$$A^{-1}T - 2AR_0 \leq \tilde{T}^i_i - \tilde{s}^i_{i-1} \leq \tau + 2AR_0, $$

and

$$|\tilde{s}^i_i - (\tilde{T}^i_i + \hat{t}_i)| \leq 2A(R_0 + \rho'). $$

**Lemma B.7.** For every $1 \leq i \leq k$, we have $|\tilde{s}^i_i - s_i| \leq \tau + 5AR_0 + 4A\rho'$ and $|\tilde{T}^i_i - T_i| \leq 2\tau + 7AR_0 + 4A\rho'$.

In what follows, we will repeatedly use the following elementary consequence of the triangle inequality.

**Lemma B.8.** If $(X, d)$ is a metric space, then for every $w, x, y, z \in X$ we have

$$|d(w, x) - d(y, z)| \leq d(w, y) + d(x, z).$$
Proof. The triangle inequality gives \( d(w, x) \leq d(w, y) + d(y, z) + d(z, x) \), so \( d(w, x) - d(y, z) \leq d(w, y) + d(z, x) \), and the other inequality is similar. □

Before proving Lemmas \[ \text{B.5} \] \[ \text{B.7} \], we demonstrate how they complete the proof of Theorem \[ \text{B.1} \]. Since \( s_i^k - T_i^k = d(c_k(T_i^k), c_k(s_i^k)) \) and \( t_i = d(x_i, y_i) \), we can apply Lemma \[ \text{B.8} \] to these four points to get
\[
|s_i^k - T_i^k - t_i| \leq d(c_k(T_i^k), x_i) + d(c_k(s_i^k), y_i) \leq 2A(\rho_0 + \rho'),
\]
where the last inequality uses \[ \text{B.8} \]. Thus
\[
d(c_k(T_i^k + t_i), y_i) \leq |s_i^k - (T_i^k - t_i)| + d(c_k(s_i^k), y_i) \leq 3A(\rho_0 + \rho') < R_1,
\]
so Lemma \[ \text{2.4} \] gives
\[
R_2 = 4R_0 + (6A^2 + 1)R_1.
\]
Taking \( \hat{T}_i = \hat{T}_i^k \) gives the desired shadowing property, with \( \hat{T}_i = [T_i - \tau_0, T_i + \tau_0] \) by Lemma \[ \text{B.7} \], where \( \tau_0 = 2\tau + 7AR_0 + 4A\rho' \). Replacing \( T_i \) by \( \hat{T}_i \) and \( \tau \) gives the form of the property stated in the theorem.

Now we prove Lemmas \[ \text{B.5} \] \[ \text{B.7} \].

Proof of Lemma \[ \text{B.5} \]. Since \( s_i^j = d(c_j(0), c_j(s_i^j)) \) and \( s_i = d(c_i(0), c_i(s_i)) \), we can apply Lemma \[ \text{B.8} \] to these four points to get
\[
|s_i^j - s_i| \leq d(c_j(0), c_j(0)) + d(c_i(s_i), c_j(s_i^j))
\]
\[
= d(\alpha_i(0), \alpha_i(s_i)) + d(\alpha_i(s_i), c_j(s_i^j))
\]
\[
\leq A\rho' + d(\alpha_i(s_i'), \alpha_j(s_i')) + d(\alpha_j(s_i'), c_j(s_i^j)) \leq 2A\rho' + AR_0,
\]
where the last inequality uses \[ \text{B.7} \]. □

Proof of Lemma \[ \text{B.6} \]. Since \( \hat{T}_i^j - \hat{s}_{i-1}^j = d(c_j(s_i^j - 1), c_j(\hat{T}_i^j)) \), we apply Lemma \[ \text{B.8} \] to these two points and \( \alpha_j(s_i^j - 1), \alpha_j(T_i^j) \) to get
\[
|\hat{T}_i^j - \hat{s}_{i-1}^j - d(\alpha_j(s_i^j - 1), \alpha_j(T_i^j))| \leq d(c_j(s_i^j - 1), \alpha_j(s_i^j - 1)) + d(c_j(T_i^j), \alpha_j(T_i^j)) \leq 2AR_0.
\]

Since \( d(\alpha_j(s_i^j - 1), \alpha_j(T_i^j)) = T_i^j - s_i^j - 1 = T \), \[ \text{(2.1)} \] gives us the bounds \( A^{-1}T \leq d(\alpha_j(s_i^j - 1), \alpha_j(T_i^j)) \leq AT = \tau \), which proves \[ \text{B.9} \].

To prove \[ \text{B.10} \], we observe that \( \hat{s}_i^j - \hat{T}_i^j = d(c_j(\hat{T}_i^j), c_j(s_i^j)) \) and \( d(x_i, y_i) = \hat{t}_i \), so we apply Lemma \[ \text{B.8} \] to these four points, obtaining
\[
|(\hat{s}_i^j - \hat{T}_i^j) - \hat{t}_i| \leq d(c_j(\hat{T}_i^j), x_i) + d(c_j(\hat{s}_i^j), y_i),
\]
and then \[ \text{B.8} \] completes the proof. □

Proof of Lemma \[ \text{B.7} \]. Recall that \( \Delta_i = s_i - \hat{s}_i \) and \( \hat{t}_i = t_i + \Delta_i - 1 \), so \( \hat{t}_i - t_i = \Delta_i - 1 = s_i - 1 - \hat{s}_i - 1 \). From Lemma \[ \text{B.6} \] we have
\[
|s_i^k - (\hat{s}_{i-1}^k + \tau + \hat{t}_i)| \leq |s_i^k - (T_i^k + \hat{t}_i)| + |T_i^k - (\hat{s}_{i-1}^k + \tau)| \leq \tau + 4AR_0 + 2A\rho'.
\]
Recalling that \( s_i = s_{i-1} + \tau + t_i \), we have

\[
|\tilde{s}_i^k - s_i| \leq |\tilde{s}_i^k - (\tilde{s}_{i-1}^k + \tau + \hat{t}_i)| + |(\tilde{s}_{i-1}^k + \tau + \hat{t}_i) - (s_{i-1} + \tau + t_i)| \\
\leq \tau + 4AR_0 + 2A\rho' + |\tilde{s}_{i-1}^k - s_{i-1} + \hat{t}_i - t_i| \\
= \tau + 4AR_0 + 2A\rho' + |\tilde{s}_{i-1}^k s_{i-1} + (s_{i-1} - \tilde{s}_{i-1})| \\
\leq \tau + 4AR_0 + 2A\rho' + A(R_0 + 2\rho'),
\]

where the last inequality uses Lemma [B.5]. One further application of Lemma [B.6] gives

\[
|\tilde{T}_i^k - T_i| \leq |\tilde{T}_i^k - (s_{i-1}^k + \tau)| + |(s_{i-1}^k + \tau) - (s_{i-1} + \tau)| \\
\leq \tau + 2AR_0 + (\tau + 5AR_0 + 4A\rho'),
\]

which proves the lemma.

\[\square\]

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