ENVELOPING ALGEBRA $U(gl(3))$ AND ORTHOGONAL POLYNOMIALS IN SEVERAL DISCRETE INDETERMINATES

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Abstract. Let $A$ be an associative algebra over $\mathbb{C}$ and $L$ an invariant linear functional on it (trace). Let $\omega$ be an involutive antiautomorphism of $A$ such that $L(\omega(a)) = L(a)$ for any $a \in A$. Then $A$ admits a symmetric invariant bilinear form $\langle a, b \rangle = L(a \omega(b))$. For $A = U(sl(2))/m$, where $m$ is any maximal ideal of $U(sl(2))$, Leites and I have constructed orthogonal basis whose elements turned out to be, essentially, Chebyshev and Hahn polynomials in one discrete variable.

Here I take $A = U(gl(3))/m$ for the maximal ideals $m$ which annihilate irreducible highest weight $gl(3)$-modules of particular form (generalizations of symmetric powers of the identity representation). In this way we obtain multivariable analogs of Hahn polynomials.

This paper appeared in: Duplij S., Wess J. (eds.) Noncommutative structures in mathematics and physics, Proc. NATO Advanced Reserch Workshop, Kiev, 2000. Kluwer, 2001, 113–124; I just want to make it more accessible.

§1. Background

1.1. Lemma. Let $A$ be an associative algebra generated by a set $X$. Denote by $[X,A]$ the set of linear combinations of the form $\sum [x_i, a_i]$, where $x_i \in X$, $a_i \in A$. Then $[A,A] = [X,A]$.

Proof. Let us apply the identity ([Mo], p.561)

\[ [ab,c] = [a,bc] + [b,ca]. \]  

Namely, let $a = x_1 \ldots x_n$; let us induct on $n$ to prove that $[a,A] \subset [X,A]$. For $n = 1$ the statement is obvious. If $n > 1$, then $a = x_1 a_1$, where $x \in X$ and due to (1.1.1) we have

\[ [a,c] = [xa_1,c] = [x,a_1c] + [a_1,cx]. \]

1.2. Lemma. Let $A$ be an associative algebra and $a \mapsto \omega(a)$ be its involutive antiautomorphism (transposition for $A = \text{Mat}(n)$). Let $L$ be an invariant functional on $A$ (like trace, i.e., $L([A,A]) = 0$) such that $L(\omega(a)) = L(a)$ for any $a \in A$. Define the bilinear form on $A$ by setting

$\langle u, v \rangle = L(u \omega(v))$ for any $u, v \in A$. \hfill (1.2.1)

Then

i) $\langle u, v \rangle = \langle v, u \rangle$;

ii) $\langle xu, v \rangle = \langle u, \omega(x)v \rangle$;

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iii) $\langle ux, v \rangle = \langle u, v_\omega(x) \rangle$;
iv) $\langle [x, u], v \rangle = \langle u, [\omega(x), v] \rangle$.

**Proof.** (Clearly, iii) is similar to ii)).

i) $\langle u, v \rangle = L(u\omega(v)) = L(\omega(u\omega(v))) = L(v\omega(u)) = \langle v, u \rangle$.

ii) $\langle xu, v \rangle = L(xu\omega(v)) = L(u\omega(v)x) = L(u\omega(\omega(x)v)) = \langle u, \omega(x)v \rangle$.

Proof. Let $z \in Z(\mathfrak{g})$, then $\langle u, v \rangle = \langle xu, v \rangle - \langle ux, v \rangle$

$\langle u, \omega(x)v \rangle - \langle u, v \omega(x) \rangle = \langle u, [\omega(x), v] \rangle$.

\[ \square \]

1.3. **Traces and forms on $U(\mathfrak{g})$.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $Z(\mathfrak{g})$ the center of $(U(\mathfrak{g}), W$ the Weyl group of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. The following statements are proved in [3].

1.3.1. **Proposition.** i) $U(\mathfrak{g}) = Z(\mathfrak{g}) \oplus [U(\mathfrak{g}), U(\mathfrak{g})]$.

ii) Let $\sharp : Z(\mathfrak{g}) \oplus [U(\mathfrak{g}), U(\mathfrak{g})] \to Z(\mathfrak{g})$ be the natural projection. Then $(uv)^\sharp = (vu)^\sharp$ and $(zv)^\sharp = z(v)^\sharp$ for any $u, v \in U(\mathfrak{g})$ and $z \in Z(\mathfrak{g})$.

iii) $U(\mathfrak{g}) = S(\mathfrak{h})^W \oplus [U(\mathfrak{g}), U(\mathfrak{g})]$.

iv) Let $\lambda$ be the highest weight of the irreducible finite dimensional $\mathfrak{g}$-module $L^\lambda$ and $\varphi$ the Harish-Chandra homomorphism. Then

\[ \varphi(u^\sharp)(\lambda) = \frac{\text{tr}(u|_{L^\lambda})}{\dim L^\lambda}. \]

1.3.2. On $U(\mathfrak{g})$, define a form with values in $Z(\mathfrak{g})$ by setting

\[ \langle u, v \rangle = (u\omega(v))^\sharp, \]

where $\omega$ is the Chevalley involution in $U(\mathfrak{g})$.

**Lemma.** The form $(\ast)$ is nondegenerate on $U(\mathfrak{g})$.

**Proof.** Let $\langle u, v \rangle = 0$ for any $v \in U(\mathfrak{g})$. By Proposition 1.3.1

\[ \text{tr}(u\omega(v)) = \varphi((u\omega(v))^\sharp)(\lambda) \cdot \dim L(\lambda) = \varphi(\langle u, v \rangle)(\lambda) \cdot \dim L(\lambda) = 0; \]

hence, $u = 0$ on $L(\lambda)$ for any irreducible finite dimensional $L(\lambda)$, and, therefore, $u = 0$ in $U(\mathfrak{g})$.

1.3.3. **Lemma.** For any $\lambda \in \mathfrak{h}^*$ define a $\mathbb{C}$-valued form on $U(\mathfrak{g})$ by setting

\[ \langle u, v \rangle_\lambda = \varphi(\langle u, v \rangle)(\lambda). \]

The kernel of this form is a maximal ideal in $U(\mathfrak{g})$.

**Proof.** The form $\langle \cdot, \cdot \rangle_\lambda$ arises from a linear functional $L(u) = \varphi(u^\sharp)(\lambda)$; hence, by Lemma 1.2 its kernel is a twosided ideal $I$ in $U(\mathfrak{g})$. On $A = U(\mathfrak{g})/I$, the form induced is nondegenerate. If $z \in Z(\mathfrak{g})$, then

\[ \langle z, v \rangle_\lambda = L(z\omega(v)) = L(z)L(\omega(v)); \]

hence, $z - L(z) \in I$. Therefore, the only $\mathfrak{g}$-invariant elements in $A$ are those from $\text{Span}(1)$. 
Let $J$ be a twosided nontrivial ($\neq A, 0$) ideal in $A$ and $J = \bigoplus J^\mu$ be the decomposition into irreducible finite dimensional $g$-modules (with respect to the adjoint representation). Since $J \neq A$, it follows that $J^0 = 0$. Hence, $L(J) = 0$ and $\langle J, A \rangle$. Thus, $J = 0$.

1.4. Gelfand–Tsetlin basis and transvector algebras. (For recapitulation on transvector algebras see [Zh].)

Let $E_{ij}$ be the matrix units. In $\mathfrak{gl}(3)$, we fix the subalgebra $\mathfrak{gl}(2)$ embedded into the left upper corner and let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{gl}(3) = \text{Span}(E_{ii} : i = 1, 2, 3)$.

There is a one-to-one correspondence between finite dimensional irreducible representations of $\mathfrak{gl}(3)$ and the sets

$$(\lambda_1, \lambda_2, \lambda_3) \text{ such that } \lambda_1 - \lambda_2, \lambda_2 - \lambda_3 \in \mathbb{Z}_+.$$  

Such sets are called highest weights of the corresponding irreducible representation whose space is denoted $L^\lambda$. With each such $\lambda$ we associate a Gelfand–Tsetlin diagram $\Lambda$:

$$
\begin{array}{ccc}
\lambda_3 & \lambda_2 & \lambda_1 \\
\lambda_2 & \lambda_{11} & \\
\lambda_{11} & \\
\end{array}
$$

(1.4.1)

where the upper line coincides with $\lambda$ and where “betweenness” conditions hold:

$$
\lambda_{k,i} - \lambda_{k-1,i} \in \mathbb{Z}_+; \quad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}_+ \text{ for any } i = 1, 2; \quad k = 2, 3.
$$  

(1.4.2)

Set

$$
\begin{align*}
z_{21} &= E_{21}; \quad z_{12} = E_{12}; \quad z_{13} = E_{13}; \quad z_{32} = E_{32}; \\
z_{31} &= (E_{11} - E_{22} + 2)E_{31}; \quad z_{23} = (E_{11} - E_{22} + 2)E_{23} - E_{21}E_{13}.
\end{align*}
$$

(1.4.3)

Set $(L^\lambda)^+ = \text{Span}(u : u \in L^\lambda, E_{12}u = 0)$.

1.4.1. Theorem. (see [Mo]) Let $v$ be a nonzero highest weight vector in $L^\lambda$, and $\Lambda$ a Gelfand–Tsetlin diagram. Set

$$
v_\Lambda = z_{21}^{\lambda_{21} - \lambda_{11}}z_{31}^{\lambda_{31} - \lambda_{21}}z_{32}^{\lambda_{32} - \lambda_{22}}v,
$$

and let $l_{ki} = \lambda_{ki} - i + 1$. Then

i) The vectors $v_\Lambda$ parametrized by Gelfand–Tsetlin diagrams form a basis in $L^\lambda$.

ii) The $\mathfrak{gl}(3)$-action on vectors $v_\Lambda$ is given by the following formulas

$$
\begin{align*}
E_{11}v_\Lambda &= \lambda_{11}v_\Lambda; \\
E_{22}v_\Lambda &= (\lambda_{21} + \lambda_{22} - \lambda_{11})v_\Lambda; \\
E_{33}v_\Lambda &= (\sum_{i=1}^{3} \lambda_{3i} - 2 \sum_{j=1}^{2} \lambda_{2j})v_\Lambda; \\
E_{12}v_\Lambda &= -(l_{11} - l_{21})(l_{11} - l_{22})v_{\Lambda + \delta_{11}}; \\
E_{21}v_\Lambda &= v_{\Lambda - \delta_{11}}; \\
E_{23}v_\Lambda &= -\frac{(l_{21} - l_{31})(l_{21} - l_{32})(l_{21} - l_{33})}{(l_{21} - l_{22})}v_{\Lambda + \delta_{11}} - \frac{(l_{22} - l_{31})(l_{22} - l_{32})(l_{22} - l_{33})}{(l_{22} - l_{21})}v_{\Lambda + \delta_{22}}; \\
E_{32}v_\Lambda &= \frac{(l_{21} - l_{11})}{(l_{21} - l_{22})}v_{\Lambda - \delta_{11}} + \frac{(l_{22} - l_{11})}{(l_{22} - l_{21})}v_{\Lambda - \delta_{22}};
\end{align*}
$$
where \( \Lambda \pm \delta_{ki} \) is obtained from \( \Lambda \) by replacing \( \lambda_{ki} \) with \( \lambda_{ki} \pm 1 \) and we assume that \( v_\Lambda = 0 \) if \( \Lambda \) does not satisfy conditions on GTs-diagrams.

iii) The vectors \( v_\Lambda \) corresponding to the GTs-diagrams with \( \lambda_{21} = \lambda_{11} \) form a basis of \( (L^\Lambda)^+ \).

1.5.1. Twisted generalized Weyl algebras. Recall definition of twisted generalized Weyl algebras introduced in [MT]. Let \( R \) be a commutative algebra over \( \mathbb{C} \), \( \Gamma \) a finite nonoriented tree with \( \Gamma_0 \) being the set of its vertices and \( \Gamma_1 \) that of edges. Let also \( \{ \sigma_i : i \in \Gamma_0 \} \) be a set of pairwise commuting automorphisms of \( R \) and \( \{ t_i : i \in \Gamma_0 \} \) the set of nonzero elements from \( R \) satisfying the following conditions:

\[
t_{it_j} = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i) \text{ if } (i, j) \in \Gamma_1,
\]

\[
\sigma_i(t_j) = t_j \text{ if } (i, j) \notin \Gamma_1. \tag{1.5.1}
\]

Lemma. ([MT]) Let \( \mathfrak{A}' \) be the algebra generated by \( X_i, Y_i : i \in \Gamma_0 \) subject to the relations (for any \( r \in R \))

1) \( X_i r = \sigma_i(r)X_i \),
2) \( Y_i r = \sigma_i^{-1}(r)Y_i \),
3) \( X_i Y_j = Y_j X_i \) if \( i \neq j \),
4) \( Y_i X_i = t_i \),
5) \( X_i Y_i = \sigma_i(t_i) \).

Then \( \mathfrak{A}' \neq 0 \), \( \mathfrak{A}' \) is \( \mathbb{Z} \)-graded, and among homogeneous (with respect to the grading) twosided ideals of \( \mathfrak{A}' \) whose intersection with \( R \) is trivial is a maximal one, \( I \).

The quotient \( \mathfrak{A} = \mathfrak{A}'/I \) is called the twisted generalized Weyl algebra.

1.5.2. Example. Let \( \gamma \) be the Dynkin graph for the root system \( A_{n-1} \). Let \( V \) be an \( n \)-dimensional vector space with basis \( e_1, \ldots, e_n \), and let \( \varepsilon_1, \ldots, \varepsilon_n \) be the dual basis of \( V^* \). Let \( T = \{ \varepsilon_i - \varepsilon_j \} \) be the root system of \( A_{n-1} \) in \( V^* \) and \( \alpha_1, \ldots, \alpha_{n-1} \) the system of simple roots. Set \( R = S(V) \) and for \( h \in V \) define:

\[
\sigma_i(h) = h - \alpha_i(h) \text{ for } i = 1, 2, \ldots, n - 1
\]

and having extending \( \sigma_i \) to an automorphism of \( R \). Clearly, the \( \sigma_i \) pairwise commute for \( i \in \Gamma_0 \). In \( V \), select vectors \( h_1, \ldots, h_{n-1} \) such that \( \alpha_i(h_j) = \delta_{ij} \) and for an arbitrary collection \( f_1, \ldots, f_{n-1} \) of polynomials in one indeterminate set

\[
t_1(v) = f_1(h_1) f_2(h_2 - h_1), \quad t_2(v) = f_2(h_2 - h_1 + 1) f_3(h_3 - h_2), \ldots,
\]

\[
t_{n-1}(v) = f_{n-1}(h_{n-1} - h_{n-2} + 1) f_n(h_{n-1}).
\]

It is not difficult to verify that conditions (1.5.1) are satisfied.

§2. Formulations of main results

2.1. Modules \( S^\alpha(V) \). Let \( \mathfrak{g} = \mathfrak{gl}(3) \) be the Lie algebra of \( 3 \times 3 \) matrices over \( \mathbb{C} \). For any \( \alpha \in \mathbb{C} \) denote by \( S^\alpha(V) \) the irreducible \( \mathfrak{g} \)-module with highest weight \( (\alpha, 0, 0) \).

If \( \alpha \in \mathbb{Z}_+ \), then \( S^\alpha(V) \) is the usual \( \alpha \)-th symmetric power of the identity \( \mathfrak{g} \)-module \( V \). Namely:

\[
S^\alpha(V) = \text{Span}(x_1^{k_1}x_2^{k_2}x_3^{k_3} : k_1 + k_2 + k_3 = \alpha; \ k_1, k_2, k_3 \in \mathbb{Z}_+).
\]
For \( \alpha \not\in \mathbb{Z}_+ \) we have (like in semi-infinite cohomology of Lie superalgebras)
\[
S^\alpha(V) = \text{Span}(x_1^{k_1}x_2^{k_2}x_3^{k_3} : k_1 + k_2 + k_3 = \alpha; \ k_2, k_3 \in \mathbb{Z}_+).
\]

Remark. The expression \( x^k \) for \( k \in \mathbb{C} \) is understood as a formal one, satisfying \( \frac{\partial x^k}{\partial x} = k x^{k-1} \).

On \( S^\alpha(V) \) the \( \mathfrak{g} = \mathfrak{gl}(3) \)-action is given by \( E_{ij} \mapsto x_i \frac{\partial}{\partial x_j} \).

2.2. Theorem. i) \( S^\alpha(V) \) is an irreducible \( \mathfrak{g} \)-module for any \( \alpha \).

ii) The kernel \( J^\alpha \) of the corresponding to \( S^\alpha(V) \) representation of \( \mathfrak{g}(\mathfrak{g}) \) is a maximal ideal if \( \alpha \not\in \mathbb{Z}_{<0} \).

Set \( \mathfrak{A}^\alpha = \mathfrak{U}(\mathfrak{g})/J^\alpha \) and let \( \theta \) be the highest weight of the adjoint representation of \( \mathfrak{g} \). Now consider \( \mathfrak{A}^\alpha \) as \( \mathfrak{g} \)-module with respect to the adjoint representation.

iii) \( \mathfrak{A}^\alpha = \bigoplus_{k=0}^\infty L^{k\theta} \) if \( \alpha \not\in \mathbb{Z}_{<0} \).

iv) \( \mathfrak{A}^\alpha = \bigoplus_{k=0}^\infty L^{k\theta} \) if \( \alpha \in \mathbb{Z}_{>0} \).

v) The algebra \( \mathfrak{A}^\alpha \) for \( \alpha \not\in \mathbb{Z}_{>0} \) is isomorphic to the following twisted generalized Weyl algebra
\[
R = \mathbb{C}[E_{11}, E_{22}, E_{33}]/(E_{11} + E_{22} + E_{33} - \alpha),
\]
\[
\sigma_1(E_{11}) = E_{11} - 1, \quad \sigma_1(E_{22}) = E_{22} + 1, \quad \sigma_1(E_{33}) = E_{33},
\]
\[
\sigma_2(E_{11}) = E_{11}, \quad \sigma_2(E_{22}) = E_{22} - 1, \quad \sigma_2(E_{33}) = E_{33} + 1,
\]

vi) The form \( \langle u, v \rangle_\alpha = \varphi(\omega(v)\varphi)(\alpha, 0, 0) \) is nondegenerate on \( \mathfrak{A}^\alpha \) for \( \alpha \not\in \mathbb{Z}_{<0} \).

2.3. Let \( \mathfrak{h} = \text{Span}(E_{11}, E_{22}, E_{33}) \) be Cartan subalgebra in \( \mathfrak{g} \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) the dual basis of \( \mathfrak{h}^* \). Let \( Q = \{ \sum k_i \varepsilon_i : \sum k_i = 0 \} \) be the root lattice of \( \mathfrak{g} \). For any \( \mu \in Q \) define
\[
(\mathfrak{A}^\alpha)_\mu = \{ u \in \mathfrak{A}^\alpha : [h, u] = \gamma(h)u \text{ for any } h \in \mathfrak{h} \}.
\]

Clearly, \( \mathfrak{A}^\alpha \) is \( Q \)-graded:
\[
\mathfrak{A}^\alpha = \bigoplus_{\mu \in Q} (\mathfrak{A}^\alpha)_\mu.
\]

Theorem 2.4 below shows that \( (\mathfrak{A}^\alpha)_\mu = R u_\mu \), where \( u_\mu \in \mathfrak{A}^\alpha \) is defined uniquely up to a constant factor and \( R = \mathbb{C}[E_{11}, E_{22}, E_{33}]/(E_{11} + E_{22} + E_{33} - \alpha) \).

Denote by \( (\mathfrak{A}^\alpha)^+ \) the subalgebra of \( \mathfrak{g} \) consisting of vectors highest with respect to the fixed \( \mathfrak{gl}(2) \):
\[
(\mathfrak{A}^\alpha)^+ = \{ u \in \mathfrak{A}^\alpha : [E_{12}, u] = 0 \}.
\]

The algebra \( (\mathfrak{A}^\alpha)^+ \) also admits \( Q \)-grading:
\[
(\mathfrak{A}^\alpha)^+ = \bigoplus_{\nu \in Q} (\mathfrak{A}^\alpha)^+._\nu.
\]

Denote: \( Q^+ = \{ \nu \in Q : (\mathfrak{A}^\alpha)^+._\nu \neq 0 \} \).

Theorem 2.4 below shows that \( (\mathfrak{A}^\alpha)^+._\nu = \mathbb{C}[E_{33}] u^\nu_\nu \), where \( \nu \in Q^+ \). For \( f, g \in \mathbb{C}[E_{33}] \) and \( \nu \in Q^+ \) set
\[
\langle f, g \rangle^+_\nu = \langle f u^\nu, g u^\nu \rangle_\alpha.
\]

For \( f, g \in R \) and \( \mu \in Q \) set
\[
\langle f, g \rangle_\mu = \langle f u^\mu, g u^\mu \rangle_\alpha.
\]
For $k \geq 0$ and $\nu \in Q^+$ set

$$f_{k,\nu}(E_{33})u_\nu = \begin{cases} (\text{ad}z_{31})^k(u_{\nu + k(e_1 - e_3)}) & \text{for } \nu(E_{33}) \leq 0 \\ (\text{ad}z_{23})^k(u_{\nu + k(e_2 - e_3)}) & \text{for } \nu(E_{33}) \geq 0 \end{cases}$$

(2.3.6) (2.3.7)

For $k, l \geq 0$ and $\nu \in Q$ set

$$f_{l, k}(E_{11}, E_{22}, E_{33})u_\nu = \begin{cases} (\text{ad}z_{21})^l(\text{ad}z_{31})^k(u_{\nu + k(e_1 - e_3) + l(e_1 - e_2)}) & \text{for } \nu(E_{33}) \leq 0 \\ (\text{ad}z_{21})^l(\text{ad}z_{33}^k(u_{\nu + k(e_3 - e_2) + l(e_1 - e_2)}) & \text{for } \nu(E_{33}) \geq 0 \end{cases}$$

(2.3.8) (2.3.9)

2.4. Theorem . 0) $(\mathfrak{A}^\alpha)^+ = \mathbb{C}[E_{33}]u_\nu^+$, where $u_\nu$ is determined uniquely up to a constant factor.

1) $\langle (\mathfrak{A}^\alpha)^+, (\mathfrak{A}^\alpha)^+ \rangle_\alpha = 0$ for $\nu \neq \mu$.

2) The polynomials $f_{k,\nu}(E_{33})$ are orthogonal relative $\langle \cdot, \cdot \rangle_\nu$.

3) The polynomials $f_{k,\nu}(E_{33})$ satisfy the difference equation

$$(E_{33} - \nu(E_{33}) + 1)(E_{33} + \nu(E_{11}) - \alpha)\Delta f - E_{33}(E_{33} + \nu(E_{22}) - \alpha - 2)\nabla f = k(k + 2\nu(E_{11} + 2)f \text{ if } \nu(E_{33}) < 0$$

$$(E_{33} + 1)(E_{33} + \nu(E_{11}) - \alpha)\Delta f - (E_{33} - \nu(E_{33}))(E_{33} + \nu(E_{22}) - \alpha - 2)\nabla f = k(k - 2\nu(E_{11} + 2)f \text{ if } \nu(E_{33}) \geq 0$$

4) Explicitely, $f_{k,\nu}(E_{33})$ is of the form

$$f_{k,\nu}(E_{33}) = \text{const} \times {}_3F_2 \left( \begin{array} {c} -k, k + 2k_1 + 2, -E_{33} \\ 1 - k_3k_1 - \alpha \end{array} \right | 1 \right)$$

where

$$\begin{align*}
\sum_{i=0}^{\infty} \frac{(\alpha_1)_i(\alpha_2)_i(\alpha_3)_i}{(\beta_1)_i(\beta_2)_i} \frac{z^i}{i!}
\end{align*}$$

is a generalized hypergeometric function, $(\alpha)_0 = 1$ and $(\alpha)_i = \alpha(\alpha + 1) \ldots (\alpha + i - 1)$ for $i > 0$.

2.5. Theorem . 0) $(\mathfrak{A}^\alpha)_\nu = \mathbb{C}[E_{11}, E_{22}, E_{33}]u_\nu$, where $u_\nu$ is determined uniquely up to a constant factor.

1) $\langle (\mathfrak{A}^\alpha)_\nu, (\mathfrak{A}^\alpha)_\nu \rangle_\alpha = 0$ for $\nu \neq \mu$.

2) The polynomials $f_{l, k}(E_{11}, E_{22}, E_{33})$ form an orthogonal basis of $R$ relative $\langle \cdot, \cdot \rangle_\nu$.

3) The polynomials $w(f_{l, k}(E_{11}, E_{22}, E_{33}))$ for $w \in W$ form an orthogonal basis of $R$ relative $\langle \cdot, \cdot \rangle_{w(\nu)}$ provided polynomials $f_{l, k}(E_{11}, E_{22}, E_{33})$ form an orthogonal basis of $R$ relative $\langle \cdot, \cdot \rangle_\nu$. 

4) The polynomials \( f_{t_k}^\nu(E_{11}, E_{22}, E_{33}) \) for \( \nu \in \mathbb{Q}^+ \) and \( \nu(E_{33}) \leq 0 \) satisfy the system of two difference equations

\[
\begin{align*}
[f(H_1 + 2, H_2) - f(H_1, H_2)] & \cdot \frac{1}{2}(H_1 - H_2 + \alpha + 1)(H_1 + H_2 - \alpha) - \\
[f(H_1, H_2) - f(H_1 - 2, H_2)] & \cdot \frac{1}{2}(H_1 - H_2 + \alpha - \nu(E_{11}))(H_1 + H_2 - \alpha - 1 + \nu(E_{22})) = \\
[l^2 + l(\nu(E_{11}) + \nu(E_{22}) + 1)) + \nu(E_{22}) - \nu(E_{11})]f, \\
\end{align*}
\]

\[
\begin{align*}
[2\alpha - \nu(H_2)(\alpha + 2 + \nu(H_2)) + H_2(2\alpha + 1 + 2\nu(H_2)) - 2H_2^2]f(H_1, H_2) & - \\
\frac{1}{2}(H_2 + 1 - \nu(H_2))(H_1 - H_2 + \alpha - 2\nu(E_{11}))f(H_1 - 1, H_2 + 1) & - \\
\frac{1}{2}(H_2 + 1 - \nu(H_2))(\alpha - H_1 - H_2)f(H_1 + 1, H_2 + 1) & - \\
\frac{1}{2}H_2(\alpha - H_1 - H_2 + 2 - 2\nu(E_{22}))f(H_1 - 1, H_2 - 1) & = \\
[2k^2 + 4kl + 4k(1 + \nu(E_{11})) + 2l(1 + \nu(E_{11}) - \nu(E_{22})) + \\
\nu(E_{11})^2 - \nu(E_{22})^2 + 4\nu(E_{11})]f(H_1, H_2).
\end{align*}
\]

\section*{§3. Proof of Theorem 2.2}

i) The module \( S^A(V) \) is irreducible if and only if it has no vacuum vectors (i.e., vectors annihilated by \( E_{12} \) and \( E_{23} \). This is subject to a direct verification.

ii) Follows from Exercise 858 of Ch. 8 of [Di].

iii) Let \( A_3 \) be the Weyl algebra (i.e., it is generated by the \( p_i \) and \( q_i \) for \( i = 1, 2, 3 \) satisfying

\[
p_ip_j - p_jp_i = q_iq_j - q_jq_i = 0; \quad p_ip_j - q_jp_i = -\delta_{ij}. \quad (3.1)
\]

Setting \( E_{ij} \mapsto p_iq_j \) we see that the homomorphism \( \varphi : U(\mathfrak{g}) \rightarrow \text{End}(S^A(V)) \) factors through \( A_3 \) and \( A_3 \) acts on \( S^A(V) \) so that \( p_i \mapsto x_i \) and \( q_i \mapsto \frac{\partial}{\partial x_i} \). Let us describe the image of \( \varphi \). To this end, on \( A_3 \), introduce a grading by setting

\[
\deg p_i = 1 \quad \deg q_i = -1 \quad \text{for} \quad i = 1, 2, 3. \quad (3.2)
\]

Now it is clear that \( \text{Im} \varphi \) is the algebra \( B_3 \) of elements of degree 0.

To describe highest weight elements in \( B_3 \), it suffices to describe same in \( S^k(V) \otimes S^k(V^*) \). Let us identify \( S^k(V) \otimes S^k(V^*) \) with \( \text{End}(S^k(V)) \), let \( u \in \text{End}(S^k(V)) \) commutes with the action of \( E_{12} \) and \( E_{23} \) on \( S^k(V) \). Then \( u \) is uniquely determined by its value on the lowest weight vector \( x_3^k \in S^k(V) \); moreover, \( E_{12}x_3^k = 0 \). Hence,

\[
u(x_3^k) = a_0x_3^k + \sum_{i=0}^k a_i x_1^i x_3^{k-i},
\]

so

\[
u(x_3^k) = \frac{1}{k} a_0 \bigg( \sum_{i=0}^k x_1^i \frac{\partial}{\partial x_1}\bigg) x_3^k + \sum_{i=0}^k \frac{(k-i)!}{k!} a_i(x_1\frac{\partial}{\partial x_3})^i x_3^k.
\]

This shows that the algebra of highest weight vectors in \( B_3 \) is generated by \( p_1q_3 \) and \( z = p_1q_1 + p_2q_2 + p_3q_3 \). If \( \alpha \notin \mathbb{Z}_{\geq 0} \), then \( \mathfrak{A}_\alpha \) is the quotient of \( B_3 \) modulo \( (z - \alpha) \). This proves iii).
iv) In this case $\mathfrak{A}_\alpha = \text{End}(S^k(V))$ and the proof follows from the arguments at the end of the above paragraph.

v) For the canonical homomorphism $\varphi : U(\mathfrak{g}) \rightarrow \mathfrak{A}_\alpha$ and any $H \in R$ set

$$X_1 = \varphi(E_{12}), \ X_2 = \varphi(E_{23}), \ Y_1 = \varphi(E_{21}), \ Y_2 = \varphi(E_{32});$$

$$R = \mathbb{C}[E_{11}, E_{22}, E_{33}]/(z - \alpha);$$

$$\sigma_1(H) = H - (\varepsilon_1 - \varepsilon_2)(H), \ \sigma_2(H) = H - (\varepsilon_2 - \varepsilon_3)(H);$$

$$t_1 = E_{22}(E_{11} + 1), \ t_2 = E_{33}(E_{22} + 1).$$

Now, as is easy to verify, all the relations of sec. 1.5.1 are satisfied. If $I$ is a twosided ideal of $\mathfrak{A}_\alpha$, then thanks to iii) and iv) it contains $L^\omega$ for some $k \geq 0$ and, therefore, elements of weight 0 with respect to the adjoint action of Cartan subalgebra of $\mathfrak{g}$. Hence, $I \cap R \neq 0$. Hence, there exists a surjection $\psi : \mathfrak{A}' \rightarrow \mathfrak{A}_\alpha$.

vi) By 1.3.3 the kernel of $\langle \cdot, \cdot \rangle_\alpha$ in $U(\mathfrak{g})$ is a maximal ideal. But $\mathfrak{A}_\alpha = U(\mathfrak{g})/J^\alpha$, where $J^\alpha$ is maximal due to i). So $J^\alpha$ coincides with the kernel of $\langle \cdot, \cdot \rangle_\alpha$ in $U(\mathfrak{g})$ and the form is nondegenerate on $\mathfrak{A}_\alpha$.

§4. Proof of Theorem 2.4

0) Direct computations show that the set of elements from $A_3$ commuting with $E_{12}$ is a subalgebra generated by $p_1$, $q_2$, $p_3$, $q_3$ and $z = p_1 q_1 + p_2 q_2 + p_3 q_3$. So this algebra is the linear span of the elements of the form

$$u = p_1^{k_1} q_2^{k_2} p_3^{k_3} q_3^{k_4} z^{k_5}.$$ 

If $u \in B_3$, then $k_1 + k_3 = k_2 + k_4$, so

$$u = \begin{cases} p_1^{k_1} q_2^{k_2} p_3^{k_3} q_3^{k_4} z^{k_5} & \text{if } k_3 \geq k_4, \\ p_1^{k_1} q_2^{k_2} p_3^{k_3} q_3^{k_4} z^{k_5} & \text{if } k_3 \leq k_4. \end{cases} \quad (4.1)$$

Hence, setting for $\nu = \sum k_i \varepsilon_i$, such that $\sum k_i = 0$, $k_1 \geq 0$ and $k_2 \leq 0$

$$u_\nu^+ = \begin{cases} p_1^{k_1} q_2^{k_2} p_3^{k_3} & \text{if } k_3 \geq 0, \\ p_1^{k_1} q_2^{k_2} q_3^{k_3} & \text{if } k_3 \leq 0. \end{cases} \quad (4.2)$$

we obtain the statement desired.

1) Let $u \in (\mathfrak{A}_\alpha^+)_{\mu}$, $v \in (\mathfrak{A}_\alpha^+)_{\nu}$, and $h \in \mathfrak{h}$. Then by heading iv) of Lemma 1.2 we obtain:

$$\langle [h, u], v \rangle = \mu(h) \langle u, v \rangle = \langle u, [h, v] \rangle = \nu(h) \langle u, v \rangle.$$ 

So $\langle u, v \rangle = 0$ if $\mu \neq \nu$.

2) Let $\nu(E_{33}) \leq 0$. We have:

$$f_{k, \nu} u_\nu^+ = (\text{ad} z_3^1)^k (u_{\nu + (k - 1) \varepsilon_1 - \varepsilon_3}) = \text{ad} z_3^1 (\text{ad} z_3^1)^{k - 1} (u_{\nu + (k - 1) (\varepsilon_1 - \varepsilon_3) + \varepsilon_1 - \varepsilon_3}) = (\text{ad} z_3^1)^k_{k - 1, \nu + (\varepsilon_1 - \varepsilon_3)} u_{\nu + (\varepsilon_1 - \varepsilon_3)}^+.$$ 

Direct verification shows that

$$\langle \text{ad} z_3^1 (f u_\nu^+), \nu(E_{33}) \rangle = \{(E_{33} - \nu(E_{33}))[\nu(E_{33}) - \alpha] \nu(E_{11}) + (\nu(E_{11}) - 1) \nu(E_{11}) \} f(h) - E_{33}[(E_{33} - \alpha) \nu(E_{11}) - (\nu(E_{22}) + 2) \nu(E_{11})] f(h - 1) \} u_{\nu - (\varepsilon_1 - \varepsilon_3)}. \quad (4.3)$$
It easily follows from Lemma 1.2 that for any $z \in U(\mathfrak{g})$ we have

$$\langle (ad z)(u), v \rangle = \langle u, (ad \omega(z))(v) \rangle,$$

but

$$\omega(z_{31}) = \omega((E_{11} - E_{22} + 2)E_{31} + E_{21}E_{32}) = E_{13}(E_{11} - E_{22} + 2) + E_{23}E_{12}.$$  

Since $fu_\nu$ is a highest weight vector with respect to the fixed $\mathfrak{gl}(2)$, it follows that

$$(ad \omega(z_{31}))(fu_\nu) = (ad(E_{13}(E_{11} - E_{22} + 2))(fu_\nu) =$$

$$(\nu(E_{11}) - \nu(E_{22}) + 2)\Delta f \cdot (fu_\nu)(_{e_1 - e_3}).$$

Now, let us induct on $k$. For $k = 0$ the statement is obvious. For $k > 0$ and $\deg g < k$ we have

$$\langle f_{k, \nu + \epsilon_1 - e_3}, (\nu(E_{11}) - \nu(E_{22}) + 2)\Delta g \rangle_{_{e_1 - e_3}} = 0$$

by inductive hypothesis.

The case $\nu(E_{33}) \geq 0$ is similar. 

3) Observe that $z = E_{13}E_{31} + E_{23}E_{32}$ belongs to the centralizer of $\mathfrak{gl}(2)$ in $U(\mathfrak{g})$. Let $\nu(E_{33}) \leq 0$. Then $u_\nu = p_1^{k_1} q_2^{k_2} q_3^{k_3}$ as in (4.1.2). Having applied $ad z$ to $fu_\nu$ we obtain:

$$(ad z)(fu_\nu) = E_{13}E_{31}fu_\nu + f_{u_\nu}E_{13}E_{31} - E_{13}fu_{u_\nu}E_{31} - E_{31}fu_\nu E_{13} +$$

$E_{23}E_{32}fu_\nu + f_{u_\nu}E_{32}E_{23} - E_{23}fu_{u_\nu}E_{32} - E_{32}fu_\nu E_{23} =$$

$E_{11}(E_{33} + 1)fu_\nu + f_{u_\nu}E_{33}(E_{11} + 1) - f(E_{33} + 1)u_\nu E_{11}(E_{33} + 1) - f(E_{33} - 1)E_{33}(E_{11} + 1)u_\nu +$$

$E_{22}(E_{33} + 1)fu_\nu + f_{u_\nu}E_{33}(E_{22} + 1) - f(E_{33} + 1)E_{22}u_\nu (E_{33} + 1) - f(E_{33} - 1)E_{33}u_\nu (E_{22} + 1) =$$

$(E_{11} + E_{22})(E_{33} + 1)fu_\nu + (E_{33} - \nu(E_{33}))(E_{11} + 1 - \nu(E_{11}) + E_{22} + 1 - \nu(E_{22}))fu_\nu -$$

$f(E_{33} + 1) \cdot (E_{33} + 1 - \nu(E_{33}))(E_{11} + E_{22} - \nu(E_{11}))u_\nu - f(E_{33} - 1)E_{33}(E_{11} + E_{22} - \nu(E_{22}) + 2)u_\nu =$$

$[f(E_{33} + 1) \cdot (E_{33} + 1 - \nu(E_{33}))(E_{33} - \alpha + \nu(E_{11})) + f(E_{33} - 1)E_{33}(E_{33} + \nu(E_{22}) - \alpha - 2) -$$

(E_{33} - \alpha)(E_{33} + 1)f - (E_{33} - \nu(E_{33}))E_{33} + \nu(E_{11}) + \nu(E_{22}) - \alpha - 2)f]u_\nu.$

This gives us the right hand side of the first equation of heading 3).

Since $ad z$ commutes with the $\mathfrak{gl}(2)$-action and preserves the degree of polynomial $f$, it follows that $(ad z)(fu_\nu) = c \cdot (fu_\nu)$. Counting the constant factor, we arrive to the first equation of heading 3).

The proof of the second equation is similar.

§5. PROOF OF THEOREM 2.5

0) Recall that $B_3$ is the subalgebra of $A_3$ of the elements of degree 0 relative grading (3.2).

For $k \in \mathbb{Z}$ set $\gamma_i^k = \begin{cases} p_i^k & \text{if } k \geq 0 \\ q_i^k & \text{if } k \leq 0 \end{cases}$. For $\gamma = \sum k_i\epsilon_i$, where $\sum k_i = 0$, set

$$u_\gamma = \gamma_1^{k_1} \gamma_2^{k_2} \gamma_3^{k_3}.$$
Clearly, $B_3$ is the linear span of the elements of the form

$$p_1^{m_1}q_1^{l_1}p_2^{m_2}q_2^{l_2}p_3^{m_3}q_3^{l_3},$$

where $m_1 + m_2 + m_3 = l_1 + l_2 + l_3$.

It is also clear that each such element can be represented in the form

$$f(E_{11}, E_{22}, E_{33})r_1^{k_1}r_2^{k_2}r_3^{k_3}.$$  

This completes the proof of heading 0).

1) Proof is similar to that from sec. 4.2.

2) Let $\nu(E_{33}) \leq 0$. By setting $H_1 = E_{11} - E_{22}$, $H_2 = E_{22} - E_{33}$ we identify $R = \mathbb{C}[E_{11}, E_{22}, E_{33}]/(E_{11} + E_{22} + E_{33} - \alpha)$ with $\mathbb{C}[H_1, H_2]$. Let $\Lambda$ is a Gelfand–Tsetlin diagram of the following form:

$$\begin{array}{ccc}
\nu(E_{11}) + k + l & 0 & -(\nu(E_{11}) + k) \\
\nu(E_{11}) + l & -(\nu(E_{11}) + l) & \\
\nu(E_{11}) & & \\
\end{array}$$

From the explicit formula for $f_{k,l}^\nu$ we derive that

$$f_{k,l}^\nu u_\nu = v_\Lambda.$$  

Now, consider the following operators from the maximal commutative subalgebra of $U(g)$:

$$\begin{align*}
E_{11}, E_{22}, \\
\Omega_2 &= E_{11}^2 + E_{22}^2 + E_{11} - E_{22} + 2E_{21}E_{12}, \\
\Omega_3 &= E_{11}^2 + E_{22}^2 + E_{33}^2 + E_{11} - E_{22} + E_{11} - E_{33} + E_{22} - E_{33} + 2E_{21}E_{12} + 2E_{31}E_{13} + 2E_{32}E_{23}. \\
\end{align*}$$

Then we have:

$$\begin{align*}
E_{11}v_\Lambda &= \nu(E_{11})v_\Lambda; \\
E_{22}v_\Lambda &= -\nu(E_{22})v_\Lambda; \\
\Omega_2 v_\Lambda &= [2l^2 + 2l(\nu(E_{11}) + \nu(E_{22}) + 1) + \\
&\nu(E_{11})^2 + \nu(E_{22})^2]v_\Lambda; \\
\Omega_3 v_\Lambda &= 2(\nu(E_{11}) + k + l)(\nu(E_{11}) + k + l + 2)v_\Lambda.
\end{align*}$$

It is easy to check that the operators (5.2) satisfy

$$\omega(E_{11}) = E_{11}; \quad \omega(E_{22}) = E_{22}; \quad \omega(\Omega_2) = \Omega_2; \quad \omega(\Omega_3) = \Omega_3$$

and, therefore, they are selfadjoint relative the form $\langle \cdot, \cdot \rangle$. Formula (5.3) makes it manifest that operators (5.2) separate the vectors $v_\Lambda$, hence, these vectors are pairwise orthogonal.

Moreover, it is easy to see that $f_{k,l}^\nu$ is of the form

$$f_{k,l}^\nu = H_1^lH_2^k + \ldots,$$

where the dots designate the summands of degrees $\leq k + l$ of the form $H_1^aH_2^b$, where $(a, b) < (l, k)$ with respect to the lexicographic ordering. Thus, the $f_{l,k}^\nu$ constitute a basis of $\mathbb{C}[H_1, H_2]$.

3) The statement follows from the fact that the Weyl group acts on $\mathfrak{A}_\alpha$ and preserves the form $\langle \cdot, \cdot \rangle$. 

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4) Since the polynomials $f_{l,k}^{\nu}u_{\nu}$ are elements of a Gelfand-Tsetlin basis, they are eigenvectors for $\Omega_2$ and $\Omega_3$ with respect to the adjoint action of $g = \mathfrak{gl}(3)$ on $\mathfrak{a}_\alpha$. As we have shown in sec 5.2, we have

$$\Omega_2 f_{l,k}^{\nu}u_{\nu} = [2l^2 + 2l(\nu(E_{11}) + \nu(E_{22}) + 1) + \nu(E_{11})^2 + \nu(E_{22})^2]f_{l,k}^{\nu}u_{\nu}$$

$$\Omega_3 f_{l,k}^{\nu}u_{\nu} = 2(\nu(E_{11}) + k + l)(\nu(E_{11}) + k + l + 2) f_{l,k}^{\nu}u_{\nu}.$$ 

To derive the corresponding equations, we have to explicitly compute the actions of $\Omega_2$ and $\Omega_3$ on $fu_{\nu}$. Let $(\nu(E_{33}) \leq 0)$; then

$$(\Omega_3 - \Omega_2) f_{l,k}^{\nu}u_{\nu} = (E_{33}^2 - E_{11} - E_{22} + 2E_{33} + 2E_{31}E_{13} + 2E_{32}E_{23}) f_{l,k}^{\nu}u_{\nu} =$$

$$(E_{33}^2 - E_{11} - E_{22} + 2E_{33} + E_{13}E_{31} + E_{23}E_{32}) f_{l,k}^{\nu}u_{\nu} =$$

$$(\nu(E_{33})^2 - \nu(E_{11}) - \nu(E_{22}) + 2\nu(E_{33}) f_{l,k}^{\nu}u_{\nu} +$$

$$(E_{11} + E_{22})(E_{33} + 1)f(H_1, H_2)u_{\nu} +$$

$$(E_{33} - \nu(E_{33})(E_{11} + E_{22} + 2 - \nu(E_{11}) - \nu(E_{22}))f(H_1, H_2)u_{\nu} -$$

$f(E_{11} - 1, E_{33} + 1)(E_{11} - \nu(E_{11}))(E_{33} + 1 - \nu(E_{33}))u_{\nu} -$$

$f(E_{11} + 1, E_{33} - 1)E_{33}(E_{11} + 1)u_{\nu} -$$

$f(E_{22} - 1, E_{33} + 1)E_{22}(E_{33} + 1 - \nu(E_{33}))u_{\nu} -$$

$f(E_{22} + 1, E_{33} - 1)E_{33}(E_{22} + 1 - \nu(E_{22}))u_{\nu} =$$

$[(\alpha - H_2)(H_2 + 1)f(H_1, H_2) + (H_2 - \nu(H_2))(\alpha - H_2 + 2 + \nu(H_2))f(H_1, H_2) -$$

$f(H_1 - 1, H_2 + 1)(H_2 + 1 - \nu(H_2))\frac{1}{2}(H_1 - H_2 + \alpha - 2\nu(E_{11})) -$$

$f(H_1 + 1, H_2 + 1)H_2^\frac{1}{2}(H_1 - H_2 + \alpha + 2) -$$

$f(H_1 + 1, H_2 + 1)\frac{1}{2}(\alpha - H_1 - H_2)(H_2 + 1 - \alpha - \nu(H_2)) -$$

$f(H_1 - 1, H_2 - 1)H_2^\frac{1}{2}(\alpha - H_1 - H_2 + 2 - 2\nu(E_{22}))u_{\nu}.$

This implies the second equation.

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