Iterating the Big-Pieces operator and larger sets

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Abstract
We show that if an Ahlfors–David regular set $E$ of dimension $k$ has Big Pieces of Big Pieces of Lipschitz Graphs (denoted usually by $BP(BP(LG))$), then $E \subset \tilde{E}$ where $\tilde{E}$ is Ahlfors–David regular of dimension $k$ and has Big Pieces of Lipschitz Graphs (denoted usually by $BP(LG)$). Our results are quantitative and, in fact, are proven in the setting of a metric space for any family of Ahlfors–David regular sets $\mathcal{F}$ replacing $LG$. As an example corollary is the stability of the BP operator after two iterations. This was previously only known in the Euclidean setting for the case $\mathcal{F} = LG$ with substantially more complicated proofs.

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1 | INTRODUCTION

A closed set $E$ (with more than one point) in a metric space $\mathcal{X}$ is said to be $k$-Ahlfors–David regular if there is a constant $C > 1$ such that for all $r \in (0, \text{diam}(E))$ and $x \in E$ we have $C^{-1}r^k < \mathcal{H}^k(E \cap B(x, r)) < Cr^k$. For some given class $\mathcal{F}$ of $k$-Ahlfors–David regular subsets (of a metric space $\mathcal{X}$), we define $BP(\mathcal{F})$ as follows: $F \in BP(\mathcal{F})$ if $F$ is a $k$-Ahlfors–David regular set for which there exists a constant $\vartheta > 0$ such that for any $x \in F$ and $R > 0$, there is a set $G_{x,R} \in \mathcal{F}$ such that

$$\mathcal{H}^k(B(x, R) \cap F \cap G_{x,R}) \geq \vartheta \mathcal{H}^k(B(x, R) \cap F).$$

Conditions involving $BP(\mathcal{F})$ for various classes of sets $\mathcal{F}$ play an important role in the theory of uniformly rectifiable sets in $\mathbb{R}^n$ developed by David and Semmes (see, for example, [4, 6]). While the original motivation was the study of singular integral operators, the study of such conditions has taken on a life of its own.

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In the context of singular integrals, the condition $BP(F)$ is important because it allows the uniform boundedness of a family of Singular Integral Operators given by convolution with “nice” kernels over sets in $F$ to be transported to sets in $BP(F)$. In particular, one can define successively weaker conditions $BP_j(F)$ for all $j > 0$ which all imply boundedness given that the SIOs are bounded on $F$; the initial case David and Semmes considered [5] used Lipschitz graphs as the base class, that is, $E \in BP^1(LG)$. This raised a natural question: how do the collections $BP_j(LG)$ behave as $j$ grows? It turned out that for $j \geq 2$ the collections $BP_j(LG)$ are all the same and their elements are called Uniformly Rectifiable sets. We refer the reader to [5, 6], specifically to [5, Proposition 2.2 on page 97] and [5, Theorem 2.29 on page 336]. For $n \in [k, 2d + 1)$ one also needs [1] to show that $BPBI$ implies $BP(BP(LG))$, but this is not where most of the work goes — the proofs by David and Semmes of that stability (for $j \geq 2$) are quite sophisticated and rely on a Euclidean ambient space.

There has recently been interest in other families $F$, in particular for the purpose of studying Parabolic Uniform Rectifiability. See, for example, the work in [2], where the questions about Uniform Rectifiability in the metric setting are discussed for this purpose. In fact, we refer to [2] for a great introduction on contemporary applications of the idea of Big Pieces.

An immediate corollary of the main result contained in this essay (Theorem 2.1) is that stabilization of the operator $BP_j$ occurs in the setting of metric spaces for $j \geq 2$ as well. Our proof is both simple and direct.

## A THEOREM

**Theorem 2.1.** Let $F$ be a class of (closed) $k$–Ahlfors–David regular sets in a metric space $\mathbb{X}$. Let $E \subseteq \mathbb{X}$ be a $k$-Ahlfors–David regular set with $E \in BP(BP(F))$. Then there exists a set $F \subset \mathbb{X}$ such that

(i) $E \subseteq F$,
(ii) $F$ is $k$-Ahlfors–David regular,
(iii) $F \in BP(F)$.

The constants in the conclusion are quantitative with dependance on the constants in the assumptions.

**Corollary 2.2.** Let $F$ be a class of closed $k$–Ahlfors–David regular sets in a metric space $\mathbb{X}$. For any $j > 2$, and any constants $\theta_1, \ldots, \theta_j > 0$ defining $BP_j(F)$, there are $\theta'_1, \theta'_2 > 0$ such that the family $BP(BP(F))$ defined using $\theta'_1, \theta'_2$ is equal to $BP_j(F)$ defined using $\theta_1, \ldots, \theta_j$.

**Proof of Corollary 2.2.** Let $E \in BP^3(F)$. Then for any $x \in E$ and $R < \text{diam}(E)$ we have a set $E'_{x,R} \in BP^2(F)$ such that $H^k(B(x, R) \cap E) \leq H^k(B(x, R) \cap E \cap E'_{x,R})$. By Theorem 2.1, there is a set $F'_{x,R} \in BP(F)$ so that $F'_{x,R} \supset E'_{x,R}$. Clearly $H^k(B(x, R) \cap E) \leq H^k(B(x, R) \cap E \cap F_{x,R})$. We have shown $E \in BP^2(F)$. This gives for any $j \geq 3$ that $BP_j(F) = BP^{j-1}(F)$, and so we are done by induction. □

**Proof of Theorem 2.1 for the case diam $E < \infty$.** We suppose that diam $E < \infty$. In order to construct the set $F$, we first fix a dyadic cube decomposition of $E$ denoted by $\Delta = \Delta(E)$ with root cube $\text{root}(\Delta) = Q_0 = E$. By construction, for each cube $Q \in \Delta$ there exists a point $c(Q) \in Q$ which we call the center of $Q$ satisfying

$$\text{dist}(B(c(Q), c_1 \text{ diam } Q), E \setminus Q) \geq c_2 \text{ diam } Q.$$  

(2.1)
for some constants \( c_1, c_2 > 0 \) (see, for example, [3]). From now on, define \( B_{c(Q)} = B(c(Q), c_1 \text{diam}(Q)) \). We construct the set \( F \) desired in the theorem inductively. At stage 0, use the fact that \( E \in \text{BP}(\text{BP}(F)) \) to find a closed set \( F_{Q_0} \in \text{BP}(F) \) such that \( F_{Q_0} \subseteq B_{c(Q_0)} \) and

\[
\mathcal{H}^k(B_{c(Q_0)} \cap E \cap F_{Q_0}) \gtrsim_{\delta_1, c_1, c_2} \mathcal{H}^k(B_{c(Q_0)} \cap E) \gtrsim_{c_1, c_2} \text{diam}(Q_0)^k.
\]

We define

\[
F_0 = F_{Q_0}.
\]

We continue the construction by defining a dyadic decomposition \( Q_1 \) of the set \( E \setminus F_{Q_0} \). Indeed, since \( F_{Q_0} \) is closed, \( E \setminus F_{Q_0} \) is relatively open in \( E \) and for any \( x \in E \setminus F_{Q_0} \), there exists some dyadic cube \( Q \ni x \) of maximal diameter such that \( \text{dist}(Q, F_{Q_0}) > \text{diam} Q \). We call the disjoint family of all such maximal cubes \( Q_1 \), so that we have

\[
E \setminus F_{Q_0} = \bigcup_{Q \in Q_1} Q.
\]

We now give stage 1 of the construction of \( F \). For each \( Q \in Q_1 \), again find a closed set \( F_Q \in \text{BP}(F) \) such that

\[
\mathcal{H}^k(B_{c(Q)} \cap E \cap F_Q) \gtrsim_{\delta_1, \epsilon_1, \epsilon_2} \mathcal{H}^k(Q).
\]  

(2.2)

We define

\[
F_1 = F_{Q_0} \cup \bigcup_{Q \in Q_1} F_Q.
\]

Continue the construction inductively. Given the construction completed up to stage \( m \), we define the set \( Q_{m+1} \) to be the collection of dyadic cubes with maximal diameter contained in \( E \setminus F_m \) such that \( Q \in Q_{m+1} \) satisfies

\[
\text{dist}(Q, F_m) > \text{diam}(Q). \tag{2.3}
\]

\( Q_{m+1} \) is a disjoint decomposition of \( E \setminus F_m \) so that

\[
E \setminus F_m = \bigcup_{Q \in Q_{m+1}} Q. \tag{2.4}
\]

Given such a \( Q \), let \( F_Q \in \text{BP}(F) \) with \( F_Q \subseteq B_{c(Q)} \) be such that (2.2) holds and define

\[
F_{m+1} = F_m \cup \bigcup_{Q \in Q_{m+1}} F_Q = F_{Q_0} \cup \bigcup_{Q \in Q_1} F_Q \cup \cdots \cup \bigcup_{Q \in Q_{m+1}} F_Q. \tag{2.5}
\]

Finally, set

\[
F = \bigcup_{m=0}^{\infty} F_m. \tag{2.6}
\]
and define $Q = \cup_m Q_m$. Now that we have constructed the set $F$, we note two of its simple properties. First, given any $Q \neq Q' \in Q$, equality (2.3) implies

$$\text{dist}(F_Q, F_{Q'}) > \min\{\text{diam}(Q), \text{diam}(Q')\}. \tag{2.7}$$

Second,

$$\lim F \subseteq E \cup \bigcup_{m=0}^{\infty} F_m,$$

where $\lim F$ denotes the set of limit points of $F$. Indeed, suppose $x \in \lim F$ with $x_j \to x$, $x_j \in F_{Q_j}$. If the set $\{Q_j\}$ is finite, then (2.7) implies that the sequence $F_{Q_j}$ is eventually constant, say $F_{Q_j} \to F_{Q_i}$, meaning $x \in F_{Q_i}$ since $F_{Q_i}$ is closed. If instead $\{Q_j\}$ is infinite, then consider a subsequence $x_{k_j} \to x$ such that $Q_{k_j} \neq Q_{k_i}$ for any $i, j$. The fact that $x_{k_j}$ converges combined with (2.7) and then implies $Q_j \to 0$. Since $\text{dist}(F_{Q_j}, E) \leq \text{diam} Q_j$, we have $\text{dist}(x, E) = 0$ which implies $x \in E$. (In particular, we will soon see that this implies $\mathcal{H}^k(\lim F \setminus \cup_m F_m) = 0$.)

**We begin with proving claim (i).** Notice that for any $N \in \mathbb{N}$,

$$\mathcal{H}^k(E \setminus F) \leq \mathcal{H}^k\left(E \setminus \bigcup_{m=0}^{\infty} F_m\right) \leq \mathcal{H}^k(E \setminus F_N)$$

because the sets $F_m$ are increasing. Letting $0 < c_0 < 1$ be the constant implicit in inequality (2.2), we can write

$$\mathcal{H}^k(E \setminus F_N) = \mathcal{H}^k\left(E \setminus F_{N-1} \setminus \bigcup_{Q \in Q_N} F_Q\right) = \mathcal{H}^k\left(\bigcup_{Q \in Q_N} Q \setminus \bigcup_{Q \in Q_N} F_Q\right)$$

$$= \sum_{Q \in Q_N} \mathcal{H}^k(Q \setminus F_Q) \leq (1 - c_0) \sum_{Q \in Q_N} \mathcal{H}^k(Q) = (1 - c_0) \mathcal{H}^k(E \setminus F_{N-1}),$$

where we used the fact that $F_Q \cap F_{Q'} = \emptyset$ for $Q, Q' \in Q_N$. Since this holds for any $N$, we can iterate this inequality to get

$$\mathcal{H}^k(E \setminus F_N) \leq (1 - c_0)^N \mathcal{H}^k(E)$$

from which we conclude $\mathcal{H}^k(E \setminus F) = 0$. To finish the proof of (i), let $x \in E$ be arbitrary. Since $E$ is Ahlfors–David regular, for any $R > 0$, $\mathcal{H}^k(B(x, R)) > 0$ so that $F \cap B(x, r) \neq \emptyset$. This means that $x$ is a limit point of $F$, implying $x \in F$ because $F$ is closed.

**We now prove (ii).** Fix any point $x \in F$ and some $R < \text{diam} F$. If $x \in F \setminus \cup_m F_m$, then we can find a particular $F_Q$ with $\text{dist}(x, F_Q) < \frac{R}{100}$ and $\text{dist}(x, F_Q) = \text{dist}(x, z)$ for $z \in F_Q$. Then, we have $B(z, R/2) \subseteq B(x, R) \subseteq B(z, 2R)$, and substitute the first ball or final ball for $B(x, R)$ in the proofs of lower and upper regularity, respectively. Hence, we can assume $x \in \cup_m F_m$. By definition, there exists $Q_m \in Q_m$ such that $x \in F_{Q_m}$ for some $m \in \mathbb{N}$. Write

$$\mathcal{H}^k(B(x, R) \cap F) = \sum_{F_Q \cap B(x, R) \neq \emptyset, \text{diam } Q > 10R} \mathcal{H}^k(B(x, R) \cap F_Q) + \sum_{F_Q \cap B(x, R) \neq \emptyset, \text{diam } Q \leq 10R} \mathcal{H}^k(B(x, R) \cap F_Q). \tag{2.8}$$
We will first show that $F$ is upper regular. Let $Q_I$ be the collection of cubes summed over in the first term of (2.8). By (2.7), we have that for any $Q, Q' \in Q_I$, $\text{dist}(F_Q, F_{Q'}) > 10R$. This means that $Q_I$ has at most one element. Given such a $Q$, choose $y \in B(x, R) \cap F_Q$ and write

$$H^k(B(x, R) \cap F_Q) \leq H^k(F_Q \cap B(y, 2R)) \lesssim R^k$$

using the fact that $F_Q$ is itself $k$-Ahlfors–David regular. This proves that the first sum in (2.8) has the appropriate upper bound. Let $Q_{II}$ be the collection of cubes summed over in the second term of (2.8). Since $\text{diam}(Q) < 10R$, any $Q \in Q_{II}$ satisfies $Q \subseteq B(x, 20R)$. We first prove a lemma.

**Lemma 2.3.** Let $Q \in Q$, and let $D(Q)$ be the descendants of $Q$ in $Q$. Then

$$H^k \left( \bigcup_{Q' \in D(Q)} F_{Q'} \right) = \sum_{Q' \in D(Q)} H^k(F_{Q'}) \lesssim c_1, c_2 \ H^k(Q).$$

**Proof of Lemma 2.3.** Suppose for simple notation that $Q = Q_0$. Using the regularity of each $F_Q$, we have

$$H^k \left( \bigcup_{Q \in D(Q_0)} F_Q \right) = \sum_{m=0}^{\infty} \sum_{Q \in D(Q_0) \cap Q_m} H^k(F_Q) \leq C \sum_{m=0}^{\infty} \sum_{Q \in D(Q_0) \cap Q_m} H^k(Q).$$

In analogy to (2.4), $Q_0 \setminus F_{m-1} = \bigcup_{Q \in D(Q_0) \cap Q_m} Q$ holds so that

$$\sum_{Q \in D(Q_0) \cap Q_m} H^k(Q) = H^k(Q_0 \setminus F_{m-1}) = H^k \left( Q_0 \setminus F_{m-2} \setminus \bigcup_{Q \in D(Q_0) \cap Q_{m-1}} F_Q \right)$$

$$= H^k \left( \bigcup_{Q \in D(Q_0) \cap Q_{m-1}} Q \setminus \bigcup_{Q \in D(Q_0) \cap Q_{m-1}} F_Q \right),$$

$$\leq \sum_{Q \in D(Q_0) \cap Q_{m-1}} H^k(Q \setminus F_Q)$$

$$\leq (1 - c_0) \sum_{Q \in D(Q_0) \cap Q_{m-1}} H^k(Q)$$

where $c_0$ was defined as the implicit constant in (2.2). Iterating this inequality, we find

$$H^k \left( \bigcup_{Q \in D(Q_0)} F_Q \right) \leq C \sum_{m=0}^{\infty} (1 - c_0)^m H^k(Q_0) \lesssim c_0 \ H^k(Q_0).$$

Using this lemma, we can write

$$\sum_{F_Q \cap B(x, R) \neq \emptyset} H^k(B(x, R) \cap F_Q) \leq \sum_{Q \text{ maximal}} \sum_{Q' \in D(Q)} H^k(F_{Q'}) \lesssim \sum_{Q \text{ maximal}} H^k(Q).$$

$$\leq H^k(E \cap B(x, 20R)) \lesssim R^k.$$
This proves the desired bound for the second sum in (2.8), proving the upper regularity of \( F \). Now we show that \( F \) is lower regular. If \( R < 100 \text{diam } Q_m \), then the claim follows immediately from the lower regularity of \( F_Q \). If \( 100 \text{diam } Q_m \leq R < \text{diam } F \), then since \( F_Q \cap Q_m \neq \emptyset \), there exists \( z \in Q \) (and thus, \( z \in E \)) with \( B(x, R) \supseteq B(z, R/2) \) and
\[
\mathcal{H}^k(B(x, R) \cap F) \geq \mathcal{H}^k(B(z, R/2) \cap E) \geq R^k
\]
using the fact that \( E \subseteq F \). This completes the proof of lower regularity, hence of (ii) as well.

**Finally, we prove (iii).** Fix \( x \in F_Q \) and \( R > 0 \) as in the proof of (ii). Fix a constant \( \alpha > 10 \) to be chosen later. If \( R < \alpha \text{diam } Q_m \), then since \( F_Q \in \text{BP}(F) \), there exists \( G_{x,R} \in F \) such that
\[
\mathcal{H}^k(B(x, R) \cap F_{Q_m} \cap G_{x,R}) \geq \delta_2 \mathcal{H}^k(B(x, R) \cap F_{Q_m}) \geq C' \alpha^k R^k \mathcal{H}^k(B(x, R) \cap F), \tag{2.9}
\]
where \( C' \) is the regularity constant for \( F_{Q_m} \) and \( C'' \) is the regularity constant for \( F \). Now, suppose that \( \alpha \text{diam } Q_m \leq R < \text{diam } F \). Since \( x \in F_Q \), there exists a chain of cubes \( Q_i \in Q_i \), \( 0 \leq i \leq m \) such that
\[
Q_m \subseteq Q_{m-1} \subseteq \ldots \subseteq Q_1 \subseteq Q_0.
\]
Next, notice that for any choice of \( \alpha > 10 \), there exists a smallest cube \( Q_j \) in the above chain such that \( R < \alpha \text{diam } Q_j \) since for all admissible \( R, R < 10 \text{diam } Q_0 \). Choose the constant \( \alpha \) such that for any \( y \in E \setminus F_i \), the cube \( Q_{i+1} \ni y \) satisfies
\[
\text{dist}(Q_{i+1}, F_{Q_i}) < \frac{\alpha}{10} \text{diam } Q_{i+1}. \tag{2.10}
\]
In general, \( \alpha \) will depend on the constants used in the construction of \( \Delta \), as it may be the case that all of the children of the cube \( Q_i \) are small relative to \( Q_i \) with bounds given in terms of these constants. With such an \( \alpha \) chosen, let \( Q_j \) be the smallest cube in the above chain for \( x \) such that \( R < \alpha \text{diam } Q_j \). This means that \( R \geq \alpha \text{diam } Q_{j+1} \) so that \( (2.10) \) implies that there exists \( y \in F_{Q_j} \) such that \( B(y, R/2) \subseteq B(x, R) \). We can now repeat the argument of \( (2.9) \) with \( Q_j \) in place of \( Q_m \) to finish the proof. This completes the proof of Theorem 2.1 for the case \( \text{diam } E < \infty \).

Before we turn to the case \( \text{diam } E = \infty \), we need the following lemma. It says, roughly, that finite diameter subsets of \( E \) can be made regular by extending them slightly. This extension also preserves the \( \text{BP}(F) \) property.

**Lemma 2.4.** Let \( E \subseteq X \) be a \( k \)-Ahlfors–David regular set and suppose that \( G \subseteq E \) satisfies \( \text{diam } G = D < \infty \). For any \( A \geq 1 \), there exists a set \( \tilde{G} \subseteq E \) such that
\[
\begin{align*}
(i) & \quad G \subseteq \tilde{G} \subseteq B(G, \frac{3D}{A}) \cap E = \{ x \in E : d(x, G) < \frac{3D}{A} \}, \\
(ii) & \quad \tilde{G} \text{ is } k\text{-Ahlfors–David regular with constant } C(k, C_E, A).
\end{align*}
\]
Furthermore, if \( E \in \text{BP}(F) \) with constant \( \vartheta_E \) for some class of \( k \)-Ahlfors–David regular sets, then \( \tilde{G} \in \text{BP}(F) \) with constant \( \vartheta(k, \vartheta_E, A) \).

**Proof of Lemma 2.4.** We define an “interior” of the set \( G \subseteq E \) by
\[
I_A(G) = \left\{ x \in G : d(x, E \setminus G) \geq \frac{D}{A} \right\}.
\]
The corresponding “boundary” is then
\[ G \setminus I_A(G) = \left\{ x \in G : d(x, E \setminus G) < \frac{D}{A} \right\}. \]

We will construct the set \( \tilde{G} \) inductively. In the first stage, we will take a maximal net of appropriate size inside \( G \setminus I_A(G) \) and add in balls around each net point to \( G \). In the second step, we consider a smaller “boundary” of this new set and repeat the above process with a finer net and smaller balls. If we continue this process indefinitely while adding balls of exponentially decreasing radii, we get the desired set by taking a closure. We now give this construction explicitly.

Let \( G_0 = G \) and let \( X_1 \) be a maximal \( \frac{D}{A} \)-net for the set \( G \setminus I_A(G) \subseteq E \). Define
\[ G_1 = G \cup \bigcup_{x \in X_1} B\left(x, \frac{2D}{A}\right) \cap E. \]

Given the set \( G_n \), we define \( X_{n+1} \) to be a maximal \( 4^{-n} \frac{D}{A} \)-net for \( G_n \setminus I_{4^n A}(G_n) \) and we let
\[ G_{n+1} = G_n \cup \bigcup_{x \in X_{n+1}} B\left(x, 4^{-n} \frac{2D}{A}\right) \cap E. \]

Finally, define
\[ \tilde{G} = \bigcup_{n=0}^{\infty} G_n. \]

We will now show that \( \tilde{G} \) satisfies the desired properties in the statement of the lemma.

**We begin by proving (i).** The maximal distance of a point \( x \in \tilde{G} \) from \( G \) is just given by the sum of the radii of the balls added in each step:
\[ d(x, G) \leq 2D \sum_{n=0}^{\infty} 4^{-n} = \frac{8}{3} D < \frac{3D}{A}. \]

**We now prove (ii).** First, we observe that since \( \tilde{G} \subseteq E \), we immediately have, for all \( x \in \tilde{G}, R > 0 \),
\[ H^k(B(x, R) \cap \tilde{G}) \leq H^k(B(x, R) \cap E) \leq C_E R^k. \]

Hence, \( \tilde{G} \) is upper \( k \)-Ahlfors–David regular with constant \( C_E \). We will now show that \( \tilde{G} \) is lower regular. In order to do so, we will first prove that there exists a constant \( 0 < c < 1 \) dependent only on \( A \) such that
\[ \forall x \in \tilde{G}, \forall R, 0 < R < \text{diam} \tilde{G}, \exists y \in E \text{ such that } B(y, cR) \cap E \subseteq B(x, R) \cap \tilde{G}. \quad (2.11) \]

We note that (ii) will follow from this since for any relevant pair \( (x, R) \), we get the existence of \( y \in E \) such that
\[ H^k(B(x, R) \cap \tilde{G}) \geq H^k(B(y, cR) \cap E) \geq \frac{c^k R^k}{C_E}. \]
by the lower regularity of \( E \). We now prove (2.11). We begin by using the constant \( c' = \frac{1}{10^{4^4} \cdot A} \) (we will only need to decrease it by a factor of \( \frac{1}{2} \) at the end of the proof). Let \( x \in \tilde{G} \) and assume \( x \in G_m \) for some \( m \). There exists some minimal \( n \) such that \( x \in I_{4^m A}(G_n) \) because \( x \in G_m \setminus I_{4^m A}(G_m) \) implies \( x \in I_{4^{m+1} A}(G_{m+1}) \) by the triangle inequality. Indeed, let \( t \in X_{m+1} \) be a nearest net point to \( x \) and let \( z \in E \setminus G_{m+1} \). We can calculate

\[
d(x, z) \geq d(t, z) - d(t, x) \geq \frac{4^{-m} 2D}{A} - \frac{4^{-m} D}{A} = \frac{4^{-m} D}{A} > \frac{4^{-m-1} D}{A}.
\]

Therefore, \( d(x, E \setminus G_{m+1}) > \frac{4^{-m-1} D}{A} \) so that \( x \in I_{4^{m+1} A}(G_{m+1}) \). Suppose first that \( n \leq 4 \). In this case, we will take \( y = x \), and we must show the inclusion of the balls given in (2.11) for any admissible value of \( R \). For \( 0 < R \leq \frac{4^{-4} D}{A} \), note that \( x \in I_{4^4 A}(G_4) \) implies

\[
d(x, E \setminus \tilde{G}) \geq d(x, E \setminus G_4) > \frac{D}{4^4 A}.
\]

so that \( B(x, R) \cap \tilde{G} = B(x, R) \cap E \). If instead \( \frac{4^{-4} D}{A} < R < \text{diam} \tilde{G} < D + \frac{6D}{A} < 10D \),

\[
c'R = \frac{R}{10 \cdot 4^4 \cdot A} < \frac{10D}{10 \cdot 4^4 \cdot A} = \frac{4^{-4} D}{A}.
\]

Which shows that

\[
B(x, c'R) \cap E \subseteq B\left(x, \frac{4^{-n} D}{A}\right) \cap E = B\left(x, \frac{4^{-n} D}{A}\right) \cap \tilde{G}
\]

by (2.12). Now, suppose \( n > 4 \). This means \( x \in I_{4^n A}(G_n) \setminus I_{4^{n-1} A}(G_{n-1}) \). Hence, if \( R < \frac{4^{-n} D}{A} \), then we can take \( y = x \) and note that \( B(x, R) \cap \tilde{G} = B(x, R) \cap E \) in analogy to (2.12). Now, suppose \( \frac{4^{-m} D}{A} \leq R < \frac{4^{-m+1} D}{A} \) for \( 0 \leq m \leq n - 3 \). There exist net points \( x_p \in X_p \) for \( m + 3 \leq p \leq n \) such that

\[
d(x, x_p) \leq \frac{4^{-n} 2D}{A},
\]

\[
d(x_{p+1}, x_p) \leq \frac{4^{-p} 2D}{A}.
\]

Hence, the triangle inequality implies

\[
d(x, x_{m+3}) \leq \frac{2D}{A} \sum_{p=m+2}^{n} 4^{-p} \leq \frac{2D}{A} \left(4^{-m-2} \cdot 2\right) = \frac{4^{-m-1} D}{A}.
\]

(2.13)

In this case, we choose \( y = x_{m+3} \). We calculate

\[
B(y, c'R) = B\left(x_{m+3}, \frac{R}{10 \cdot 4^4 \cdot A}\right) \subseteq B\left(x_{m+3}, \frac{4^{-(m+3)} D}{10A^2}\right) \subseteq B\left(x, \frac{4^{-m} D}{A}\right) \subseteq B(x, R)
\]
using (2.13) and the fact that $4^{-m} \frac{D}{A} \leq R < 4^{-m+1} \frac{D}{A}$. In the case when $\frac{D}{A} < R < 10D$, choose $y = x_3$, the nearest net point in $X_3$ and observe that

$$B(y, c'R) = B\left(x_3, \frac{R}{10 \cdot 4^i \cdot A}\right) \subseteq B\left(x_{m+3}, 4^{-4} \frac{D}{A}\right) \subseteq B\left(x, \frac{D}{A}\right) \subseteq B(x, R)$$

again using (2.13). This proves (2.11) for all $x \in G_n$ for some $n$. If $x \notin G_n$ for all $n$, then given any admissible $R > 0$, there is a net point $t \in X_N$ for arbitrarily large $N$ such that $d(x, t) < \frac{R}{4}$ so that $B(t, \frac{R}{2}) \subseteq B(x, R)$ and, applying (2.11) to $B(t, \frac{R}{2})$, we get a point $y \in B(t, \frac{R}{2})$ such that $B(y, c'R) \subseteq B(t, \frac{R}{2}) \subseteq B(x, R)$. Take $c = \frac{c'}{2}$ and $B(y, cR) \subseteq B(x, R)$ so that (2.11) holds with $c = \frac{1}{20 \cdot 4^i \cdot A}$.

**Proof that $\tilde{G} \in \text{BP}(F)$.** This follows from (2.11). Indeed, for any admissible pair $(x, R)$, choose $y$ as given by (2.11). Applying the BP($F$) condition for $E$ in the ball $B(y, cR)$ gives a set $H_{y,cR} \subseteq F$ such that

$$H^k(B(x, R) \cap \tilde{G} \cap H_{y,cR}) \geq H^k(B(y, cR) \cap E \cap H_{y,cR}) \geq A, \theta_E, k \cdot R^k \geq C \cdot H^k(B(x, R) \cap \tilde{G}).$$

This concludes the proof of the lemma. \hfill \Box

**Proof of Theorem 2.1 for the case $\text{diam } E = \infty$.** Fix $x_0 \in E$. Let $A > 1$ and, for $n \geq 0$, set

$$B_n = B(x_0, A^n),$$

where the constant $A$ is sufficiently large in terms of $C_E, k$, and $\theta_E$, the BP constant. Let $E_n$ be the Ahlfors–David regular extension of the set $E \cap B_n$ with constant $A$ in Lemma 2.4 replaced with 100 so that $E_n \subseteq B(E \cap B_n, A^n)$, $E_n$ satisfies the hypotheses of the finite diameter case of the theorem, so apply the theorem to get a regular set $F_n \in \text{BP}(F)$ satisfying

$$E_n \subseteq F_n \subseteq B\left(x_0, \frac{5A^n}{4}\right).$$

In order to ensure bounded overlap, we then define $\tilde{F}_0 = F_0$ and $\tilde{F}_n$ for $n \geq 1$ to be the regular extension of $F_n \setminus \frac{1}{2}B_{n-1}$ given by the lemma with constant $A$ there replaced by $100A$ here. By construction, $\tilde{F}_n \subseteq B(F_n, A^{n-1})$ so that $\tilde{F}_n \cap \frac{1}{4}B_{n-1} = \emptyset$ and $\tilde{F}_n \subseteq B(x_0, 2A^n)$. We also have $\tilde{F}_n \in \text{BP}(F)$ with constant $\tilde{\theta}_F$ independent of $n$. We now define

$$F = \bigcup_{n=0}^{\infty} \tilde{F}_n$$

and claim that $F$ satisfies conditions (i)–(iii).

**Proof of (i).** By definition, $E \cap (B_n \setminus \frac{1}{2}B_{n-1}) \subseteq \tilde{F}_n$ so $E = \bigcup_{n=0}^{\infty} E \cap (B_n \setminus \frac{1}{2}B_{n-1}) \subseteq F$.

**Proof of (ii).** For any $n$, $\tilde{F}_n$ is regular with some constant $C_F(A, C_E, k)$ independent of $n$. Lower regularity of $F$ with constant $C_F$ follows immediately, so we only need to show that $F$ is upper regular. Let $x \in \tilde{F}_n$ for some $n$. Observe that, for $j \geq 2$

$$d(x, F_{n+j}) \geq d\left(F_n, \frac{1}{4}B_{n+j-1}\right) \geq \frac{1}{4} A^{n+j-1} - 2A^n > A^{n+j-2},$$

and claim that $F$ satisfies conditions (i)–(iii).
provided that we choose $A$ sufficiently large. Hence, if $R \leq A^{n-2}$, then $B(x, R) \cap \tilde{F}_j = \emptyset$ for $|n - j| \geq 2$. In this case,

$$\mathcal{H}^k(B(x, R) \cap F) = \sum_{j=-1}^{1} \mathcal{H}^k(B(x, R) \cap \tilde{F}_{n+j}) \lesssim_{C_F} R^k$$

independent of $n$ because $F_{n+j}$ is regular with constant independent of $n$. Now, suppose $A^j < R \leq A^{j+1}$ for $j \geq n - 2$. We can write

$$\mathcal{H}^k(B(x, R) \cap F) = \sum_{i=0}^{j+2} \mathcal{H}^k(B(x, R) \cap \tilde{F}_i) \leq \sum_{i=0}^{j+2} \mathcal{H}^k(\tilde{F}_i) \leq \tilde{C}_F \sum_{i=0}^{j+2} \text{diam}(\tilde{F}_i)^k$$

$$\leq \tilde{C}_F \sum_{i=0}^{j+2} (4A)^{ik} \leq 2\tilde{C}_F (4A)^{(j+2)k} \leq (4A)^{2k+1} \tilde{C}_F (4R)^k.$$

This proves upper regularity and finishes the proof of (ii). From now on, let $C_F = C_F(C_E, A, k)$ be the regularity constant for $F$.

**Proof of (iii)** Let $x \in \tilde{F}_n$ and $R > 0$. Suppose first that $0 < R \leq A^{n+2}$. Because $F_n \in \text{BP}(F)$ by the lemma with constant $\tilde{\theta}_F(\theta_E, A, k)$ independent of $n$, we get the existence of a set $G_{x,R} \in F$ such that

$$\mathcal{H}^k(B(x, R) \cap F \cap G_{x,R}) \geq \mathcal{H}^k(B(x, R) \cap \tilde{F}_n \cap G_{x,R}) \geq_{\delta_F, A} \mathcal{H}^k(B(x, R) \cap \tilde{F}_n)$$

$$\geq_{C_F} R^k \geq_{C_F} \mathcal{H}^k(B(x, R) \cap F).$$

using the fact that $\tilde{F}_n$ is regular. Now, suppose $A^j < R \leq A^{j+1}$ for $j \geq n + 2$. Because $x \in \tilde{F}_n$, $\frac{1}{4} A^{n-1} \leq d(x, x_0) \leq 2A^n$ so that

$$\tilde{F}_{j-2} \subseteq B(x_0, 2A^{j-2}) \subseteq B(x, 2A^{j-2} + 2A^n) \subseteq B(x, A^{j-1}) \subseteq B(x, R).$$

Using the above containment and the fact that $\tilde{F}_{j-2} \in \text{BP}(F)$, there exists a set $G_{x,R} \in F$ with both $G_{x,R} \subseteq B(x, R)$ and

$$\mathcal{H}^k(G_{x,R} \cap \tilde{F}_{j-2}) \geq_{\delta_F} \text{diam}(\tilde{F}_{j-2})^k \geq A^{(j-2)k} \geq_{A,k} R^k.$$

Hence, we have $\mathcal{H}^k(B(x, R) \cap G_{x,R} \cap F) \geq_{\delta_F, A,k} R^k \geq_{C_F} \mathcal{H}^k(B(x, R) \cap F)$ as desired. □

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