A new optical field state as an output of diffusion channel when the input being number state

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Abstract

We theoretically propose a new optical field state

$$\rho_{\text{new}} = \lambda (1 - \lambda)^l \cdot L_k \left( \frac{-\lambda^2 a^\dagger a}{1 - \lambda} \right) e^{-\lambda a^\dagger a} :$$

(here :: denotes normal ordering symbol) which is named Laguerre-polynomial-weighted chaotic field. We show that such state can be implemented, i.e., when a number state enters into a diffusion channel, the output state is just this kind of state. We solve the master equation describing the diffusion process by using the summation method within ordered product of operators and the entangled state representation. The solution manifestly shows how a pure state evolves into a mixed state. The physical difference between the diffusion and the amplitude damping is pointed out.

1 Introduction

In quantum optics there are some typical states, e.g., number state, coherent state, and squeezed state, these are pure states; there are also some mixed states, the typical one is the chaotic state described by

$$\rho_c = (1 - e^{-\lambda}) \exp \left( -\lambda a^\dagger a \right),$$

where $a$ and $a^\dagger$ are photon annihilation and creation operators, obeying $[a, a^\dagger] = 1$, $tr\rho_c = 1$. The normally ordered form of $\rho_c$ is $\rho_c = (1 - e^{-\lambda}) \exp \left[ (e^{-\lambda} - 1) a^\dagger a \right] :$ where the symbol :: denotes normal ordering symbol. In this work we shall report that there exists another important mixed state which appears in normally ordered form

$$\rho_{\text{new}} = \lambda (1 - \lambda)^l \cdot L_l \left( \frac{-\lambda^2 a^\dagger a}{1 - \lambda} \right) e^{-\lambda a^\dagger a} :$$

Here $L_l$ is the $l$-th Laguerre polynomial, $tr\rho_{\text{new}} = 1$ (see Appendix 1). We show that this mixed state will appear experimentally as it represents the output state of a diffusion process with the input state being a pure number state.

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We begin with introducing the thermo entangled state [6]. Solution of Eq. (3) obtained by entangled state representation and the technique of integration within an ordered product (IWOP) of operators [4-5] to realize our goal. We shall first obtain \( \rho(t) \) by deriving its infinite operator-sum form

\[
\rho(t) = \sum_{i,j} M_{i,j} \rho_0 M_{i,j}^\dagger,
\]

where \( M_{i,j} \) in general is named Kraus operator [3], whose concrete form will be derived for this diffusion problem, and then we examine how \( \rho_0 = |l\rangle \langle l| \) evolves through the relation (4). We will employ the thermo entangled state representation and the technique of integration within an ordered product (IWOP) of operators [4-5] to realize our goal.

2 Solution of Eq. (3) obtained by entangled state representation and IWOP technique

We begin with introducing the thermo entangled state [6]

\[
|\eta\rangle = \exp \left[ -\frac{1}{2} |\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a} \right] |0\rangle,
\]

where \( \tilde{a}^\dagger \) is a fictitious mode accompanying the real mode \( a^\dagger \), \([\tilde{a}, a^\dagger] = 1\). \(|\eta\rangle\) obeys the eigenvector equations

\[
(a - \tilde{a}^\dagger) |\eta\rangle = \eta |\eta\rangle, \quad (a^\dagger - \tilde{a}) |\eta\rangle = \eta^* |\eta\rangle,
\]

\[
|\eta\rangle \langle a^\dagger - \tilde{a} \rangle = \eta^* |\eta\rangle, \quad |\eta\rangle \langle a - \tilde{a}^\dagger \rangle = |\eta\rangle.
\]

Using the normal ordering form of vacuum projector \( |0\rangle \langle 0| = e^{-a^\dagger a - \tilde{a}^\dagger \tilde{a}} \), and the IWOP technique we can show the orthonormal and completeness relation

\[
\langle \eta' | \eta \rangle = \frac{\pi}{\pi} \delta (\eta' - \eta) \delta (\eta'^* - \eta^*),
\]

\[
1 = \int \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta|\rangle,
\]

Let

\[
|\eta = 0\rangle = e^{a^\dagger \tilde{a}^\dagger} |0\rangle = |I\rangle,
\]

we have

\[
a |I\rangle = \tilde{a}^\dagger |I\rangle, \quad a^\dagger |I\rangle = \tilde{a} |I\rangle, \quad (a^\dagger a)^n |I\rangle = (\tilde{a}^\dagger \tilde{a})^n |I\rangle.
\]

Operating the two-sides of (3) on \( |I\rangle \), noting that the real field \( \rho \) is independent of the fictitious mode, \([\rho, \tilde{a}] = 0\), \([\rho, a^\dagger] = 0\), and using (12) we have

\[
\frac{d}{dt} \rho |I\rangle = -\kappa (a^\dagger a \rho - a \tilde{a} \rho - a^\dagger \tilde{a} \rho + \tilde{a}^\dagger \rho) |I\rangle.
\]
Letting \( \rho |I\rangle = |\rho\rangle \), we see
\[
\frac{d}{dt} |\rho\rangle = -\kappa (a^\dagger - \hat{a}) (a - \hat{a}^\dagger) |\rho\rangle ,
\]
its formal solution is
\[
|\rho\rangle = \exp \left[-\kappa t (a^\dagger - \hat{a}) (a - \hat{a}^\dagger)\right] |\rho_0\rangle ,
\]
where \( |\rho_0\rangle = \rho_0 |I\rangle \). Projecting this equation onto the entangled state representation \( \langle \eta \rangle \) and using the eigenvalue equation (7) we have
\[
\langle \eta | \rho \rangle = \langle \eta | \exp \left[-\kappa t (a^\dagger - \hat{a}) (a - \hat{a}^\dagger)\right] |\rho_0\rangle = e^{-\kappa t|\eta|^2} \langle \eta | \rho_0\rangle .
\]
Multiplying the two-sides of (15) by \( \int \frac{d^2 \eta}{\pi} |\eta\rangle \) and using the completeness relation (10) as well as the IWOP technique we obtain
\[
|\rho\rangle = \int \frac{d^2 \eta}{\pi} e^{-\kappa t|\eta|^2} |\eta\rangle \langle \eta |\rho_0\rangle 
= \int \frac{d^2 \eta}{\pi} : e^{-(\kappa t+\eta)^2} |\eta\rangle \langle \eta |\rho_0\rangle 
= \frac{1}{1 + \kappa t} : \exp \left[\kappa t \left(a^\dagger \hat{a}^\dagger + a\hat{a} - a^\dagger a - \hat{a}\right)\right] : |\rho_0\rangle 
= \frac{1}{1 + \kappa t} e^{\frac{\kappa t}{1 + \kappa t} a^\dagger \hat{a}^\dagger} \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a + \hat{a}^\dagger a 
\]
where we have noticed
\[
: \exp \left[\frac{-\kappa t}{1 + \kappa t} (a^\dagger a + \hat{a}^\dagger a)\right] = \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a + \hat{a}^\dagger a
\]
Using \( [\hat{a}, \rho_0] = 0, \hat{a} |I\rangle = a^\dagger |I\rangle \) we have
\[
e^{\frac{\kappa t}{1 + \kappa t} a^\dagger a} \rho_0 |I\rangle 
= \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\kappa t}{1 + \kappa t}\right)^n a^n \rho_0 a^n |I\rangle 
= \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\kappa t}{1 + \kappa t}\right)^n a^n \rho_0 a^n |I\rangle ,
\]
After substituting (18) into (16) and then using the property that \( \hat{a}^\dagger a \) is commutable with all real field operators and \( f(a^\dagger a) |I\rangle = f(\hat{a}^\dagger a) |I\rangle \), we can put Eq.(10) into the following form
\[
|\rho\rangle = \frac{1}{1 + \kappa t} e^{\frac{\kappa t}{1 + \kappa t} a^\dagger \hat{a}^\dagger} \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a + \hat{a}^\dagger a \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\kappa t}{1 + \kappa t}\right)^n a^n \rho_0 a^n |I\rangle
= \frac{1}{1 + \kappa t} \sum_{m=0}^\infty \frac{1}{m!} \left(\frac{\kappa t}{1 + \kappa t}\right)^m a^\dagger m \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a
\times \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\kappa t}{1 + \kappa t}\right)^n a^n \rho_0 a^n \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a |I\rangle .
\]
Finally, using \( \hat{a}^\dagger m |I\rangle = a^n |I\rangle \) we obtain
\[
\rho \langle t |I\rangle = \sum_{m,n=0}^\infty \frac{(\kappa t)^{m+n}}{m! (\kappa t + 1)^{m+n+1}} a^\dagger m \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a
\times a^n \rho_0 a^n \left(\frac{1}{1 + \kappa t}\right)^a^\dagger a |I\rangle .
\]
It then follows the infinite sum form

\[ \rho(t) = \sum_{m,n=0}^{\infty} \frac{(\kappa t)^{m+n}}{m!n!(\kappa t + 1)^{m+n+1}} a^m t^m \left( \frac{1}{1 + \kappa t} \right)^{a^1} \left( \frac{1 + \kappa t}{1 + \kappa t} \right)^{a^1} a^n t^n \]  

\[ \equiv \sum_{m,n=0}^{\infty} M_{m,n} \rho_0 \rho^{\dagger}_{m,n} \]  

where

\[ M_{m,n} = \sqrt{\frac{1}{m!n!}} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}} a^m t^m \left( \frac{1}{1 + \kappa t} \right)^{a^1} \left( \frac{1 + \kappa t}{1 + \kappa t} \right)^{a^1} a^n t^n, \]  

satisfying \( \sum_{m,n=0}^{\infty} M_{m,n} \rho_0 M_{m,n} = 1 \), which is trace conservative (see Appendix 2). Thus we have employed the entangled state representation to analytically derive the infinitive sum form of \( \rho(t) \).

3 Diffusion of a number state

We now consider the case that a number state undergoes the diffusion channel, i.e., let \( \rho_0 \) in Eq. (21) be \( |l\rangle \langle l| \) and we begin with considering the part of summation over \( n \) in Eq. (21)

\[ J = \sum_{n=0}^{\infty} \frac{1}{n!(\kappa t + 1)^n} \left( \frac{1}{1 + \kappa t} \right)^{a^1} \left( \frac{1 + \kappa t}{1 + \kappa t} \right)^{a^1} \]  

\[ \equiv \sum_{n=0}^{\infty} \frac{1}{n!(\kappa t + 1)^n} \left( \frac{1}{1 + \kappa t} \right)^{a^1} \left( \frac{1 + \kappa t}{1 + \kappa t} \right)^{a^1} a^n t^n. \]  

Using the definition of the two-variable Hermite polynomials

\[ H_{m,n}(x,y) = \sum_{l=0}^{\min(m,n)} \frac{m!n!(-1)^l}{l!(m-l)!(n-l)!} x^{m-l} y^{n-l}, \]  

and \( |0\rangle \langle 0| = e^{-a^1} \), we see

\[ J = \frac{1}{l!} \left( \frac{-\kappa t}{l+1} \right)^l H_{l,l} \left( \frac{ia^1}{\sqrt{l\kappa t(l+1)}}, \frac{ia^1}{\sqrt{l\kappa t(l+1)}} \right) e^{-a^1} \]  

then inserting (25) into (21) and using the summation method within ordered product of operators yields

\[ \rho(t) = \frac{1}{l!} \left( \frac{-\kappa t}{l+1} \right)^l \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(l+1)^{m+1}} \]  

\[ \times a^m t^m H_{l,l} \left[ \frac{ia^1}{\sqrt{l\kappa t(l+1)}}, \frac{ia^1}{\sqrt{l\kappa t(l+1)}} \right] e^{-a^1} \]  

\[ = \frac{(-\kappa t)^l}{l!(l+1)^{l+1}} : e^{-a^1} a^l H_{l,l} \left[ \frac{ia^1}{\sqrt{l\kappa t(l+1)}}, \frac{ia^1}{\sqrt{l\kappa t(l+1)}} \right] :. \]  

Using the definition of Laguerre-polyominal

\[ L_l(x) = \sum_{k=0}^{l} \left( \begin{array}{c} l \cr l-k \end{array} \right) \frac{(-x)^k}{k!} \]  

(27)
and
\[ L_t(xy) = \frac{(-1)^i}{i!} H_{i,l}(x,y), \quad (28) \]
then (20) becomes
\[ \rho(t) = \frac{(\kappa t)^l}{(\kappa t + 1)^{l+1}} L_t \left( \frac{-a^\dagger a}{\kappa t (\kappa t + 1)} \right) e^{\frac{-1}{\kappa t} a^\dagger a}. \quad (29) \]

Noting \( e^{\frac{-1}{\kappa t} a^\dagger a} \) represents a chaotic photon field, so \( \rho(t) \) is a Laguerre-polynomial-weighted chaotic field. Thus we see \( l \langle l \rangle \) evolves into the mixed state (29), so this diffusion process manifestly embodies quantum decoherence.

As Eq. (29) is just in the type of Eq. (2), so we can confirm the state described by Eq. (2) indeed exists as a quantum optical field.

Before we check \( Tr\rho(t) = 1 \) for Eq. (29), let us present an integration formula
\[ \int \frac{d^2\alpha}{\pi} e^{\lambda|\alpha|^2} |\alpha|^2 = \left( \frac{\partial}{\partial \lambda} \right) \int \frac{d^2\alpha}{\pi} e^{\lambda|\alpha|^2} = k! \left( \frac{-1}{\lambda} \right)^{k+1}, \quad (30) \]
then we introduce the completeness relation of coherent state
\[ \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| = 1 \quad (31) \]
here \( |\alpha\rangle = \exp \left[ a^\dagger |\alpha\rangle - |\alpha|^2/2 \right] |0\rangle \). Due to
\[ a |\alpha\rangle = \alpha |\alpha\rangle, \langle \alpha| : f (a^\dagger, a) : |\alpha\rangle = f (|\alpha|^2, \alpha), \quad (32) \]
we see
\[ \langle \alpha| : L_t \left( \frac{-a^\dagger a}{\kappa t (\kappa t + 1)} \right) e^{\frac{-1}{\kappa t} a^\dagger a} : |\alpha\rangle = e^{\frac{-1}{\kappa t} |\alpha|^2} \sum_{k=0}^{l} \frac{l! (-1)^k}{k! ((l-k)!)^2} \left( \frac{-|\alpha|^2}{\kappa t (\kappa t + 1)} \right)^{l-k}. \quad (33) \]
Substituting (33) into \( tr\rho(t) = \int \frac{d^2\alpha}{\pi} (\alpha | \rho(t) | \alpha) \) we should calculate
\[ tr\rho(t) = \int \frac{d^2\alpha}{\pi} \langle \alpha | \rho(t) | \alpha \rangle \]
\[ = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| : \frac{(\kappa t)^l}{(\kappa t + 1)^{l+1}} e^{\frac{-1}{\kappa t} a^\dagger a} L_t \left( \frac{-a^\dagger a}{\kappa t (\kappa t + 1)} \right) : |\alpha\rangle \]
\[ = \frac{(\kappa t)^l}{(\kappa t + 1)^{l+1}} \sum_{k=0}^{l} \frac{l!}{k! ((l-k)!)^2} \int \frac{d^2\alpha}{\pi} e^{\frac{-1}{\kappa t} |\alpha|^2} \left( \frac{|\alpha|^2}{\kappa t (\kappa t + 1)} \right)^{l-k}. \quad (34) \]
By setting \( \frac{\alpha}{\sqrt{\kappa t (\kappa t + 1)}} = \alpha' \) we reform the integration as
\[ \int \frac{d^2\alpha}{\pi} e^{\frac{-1}{\kappa t} |\alpha|^2} \left( \frac{|\alpha|^2}{\kappa t (\kappa t + 1)} \right)^{l-k} \]
\[ = \kappa t (\kappa t + 1) \int \frac{d^2\alpha'}{\pi} e^{-\kappa t |\alpha'|^2} (|\alpha'|^2)^{l-k} \]
\[ = \kappa t (\kappa t + 1) (l-k)! \left( \frac{1}{\kappa t} \right)^{l-k+1} . \quad (35) \]
Substituting (33) into (34) we see
\[ tr\rho(t) = \frac{(\kappa t)^{l+1}}{(\kappa t + 1)^l} \sum_{k=0}^{l} \frac{l!}{k! ((l-k)!)^2} \left( \frac{1}{\kappa t} \right)^{l-k+1} = 1 \quad (36) \]
so it is trace conservative.
4 The photon number in the mixed state

Then we evaluate the photon number for Eq. (29)

\[
Tr \left[ \rho \left( t \right) a^\dagger a \right] = Tr \left[ \rho \left( t \right) aa^\dagger \right] - 1
\]

\[
= \frac{\left( \kappa t \right)^l}{\left( \kappa t + 1 \right)^l} Tr \left[ : a^\dagger a e^{\kappa t a^\dagger a} L_t \left( - \frac{a^\dagger a}{\kappa t \left( \kappa t + 1 \right)} \right) : \right] - 1. \tag{37}
\]

By using the coherent state representation we have

\[
Tr \left[ : a^\dagger a e^{\kappa t a^\dagger a} L_t \left( - \frac{a^\dagger a}{\kappa t \left( \kappa t + 1 \right)} \right) : \right]
\]

\[
= \int \frac{d^2 \alpha}{\pi} e^{\frac{\alpha}{\kappa t} \left| \alpha \right|^2} \left| \alpha \right|^2 L_t \left( - \left| \alpha \right|^2 \right) \frac{\left( - \left| \alpha \right|^2 \right)}{\kappa t \left( \kappa t + 1 \right)} \frac{l!}{k! l - k!} \int \frac{d^2 \alpha'}{\pi} e^{-\kappa t \left| \alpha' \right|^2} \left( \left| \alpha' \right|^2 \right)^{k+1}
\]

\[
= \left( \kappa t + 1 \right)^2 \sum_{k=0}^{l} \frac{l!}{k! (l - k)!} \left( k + 1 \right)^k \left( \frac{1}{\kappa t} \right)^k, \tag{38}
\]

where

\[
\sum_{k=0}^{l} \frac{l!}{k! (l - k)!} \left( \frac{1}{\kappa t} \right)^k
\]

\[
= \frac{l}{\kappa t} \sum_{k=1}^{l} \frac{(l - 1)!}{(k - 1)! (l - k)!} \left( \frac{1}{\kappa t} \right)^{k-1} = \frac{l}{\kappa t} \left( \frac{\kappa t + 1}{\kappa t} \right)^{l-1}, \tag{39}
\]

and

\[
\sum_{k=0}^{l} \frac{l!}{k! (l - k)!} \left( \frac{1}{\kappa t} \right)^k = \left( \frac{\kappa t + 1}{\kappa t} \right)^l. \tag{40}
\]

Substituting (39)-(40) into (38) we obtain

\[
Tr \left[ : a^\dagger a e^{\kappa t a^\dagger a} L_t \left( - \frac{a^\dagger a}{\kappa t \left( \kappa t + 1 \right)} \right) : \right]
\]

\[
= \left( \kappa t + 1 \right)^2 \left( \frac{\kappa t + 1}{\kappa t} \right)^{l-1} \frac{l + \kappa t + 1}{\kappa t}. \tag{41}
\]

Then substituting (41) into (37) we see

\[
Tr \left[ \rho \left( t \right) a^\dagger a \right] = l + \kappa t. \tag{42}
\]

which tells that the photon number \( l \rightarrow l + \kappa t \).

At the end of this work we point out that a diffusion process is quite different from the process in the amplitude dissipative channel (ADC) described by the following master equation [7]

\[
\frac{d}{dt} \rho' = \gamma \left( 2a \rho' a^\dagger - a^\dagger a \rho' - \rho' a^\dagger a \right) \tag{43}
\]
where $\gamma$ is the rate of dissipation. The solution to Eq. (43) is [8]

$$\rho' = \sum_{m=0}^{\infty} \frac{(1 - e^{-2\gamma t})^n}{n!} e^{-\gamma t} a^n \rho_0 a^n e^{-\gamma t} a.$$

(44)

In ADC an initial pure number state $|l\rangle \langle l|$ will evolve into a binomial state as shown in [9]

$$\sum_{m=0}^{l} \left( \begin{array}{c} l \\ m \end{array} \right) e^{-2\gamma m t} (1 - e^{-2\gamma t})^{l-m} |m\rangle \langle m| \equiv \rho_k$$

(45)

with photon number decaying $\text{tr}(a^\dagger a \rho') = te^{-2\gamma t}$. Comparing the diffusion master equation (3) with the dissipation Eq. (43) we realize that the term $a^\dagger a$ may be responsible for diffusion.

In summary, we theoretically propose a new optical field state

$$\rho_{\text{new}} = \lambda (1 - \lambda)^{l} : L_k \left( -\frac{\lambda^2 a^\dagger a}{1 - \lambda} \right) e^{-\lambda a^\dagger a} :$$

(46)

which is named Laguerre-polynomial-weighted chaotic field. We show that such state can be implemented, i.e., when a number state enters into a diffusion channel, the output state is just this kind of states. We solve the master equation describing the diffusion process by using the summation method within ordered product of operators and the entangled state representation. The solution manifestly shows how a pure state evolves into a mixed state. The physical difference between the diffusion and the amplitude damping is pointed out.

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**Appendix 1**

For $\rho_{\text{new}}$ in Eq. (2) we prove $\text{tr} \rho_{\text{new}} = 1$. In fact, using the coherent state representation we have

$$\text{tr} \rho_{\text{new}} = \int \frac{d^2 \alpha}{\pi} \langle \alpha | \lambda (1 - \lambda)^{l} : L_k \left( -\frac{\lambda^2 a^\dagger a}{1 - \lambda} \right) e^{-\lambda a^\dagger a} : | \alpha \rangle$$

$$= \lambda (1 - \lambda)^{l} \int \frac{d^2 \alpha}{\pi} e^{-\lambda |\alpha|^2} L_k \left( -\frac{\lambda^2 |\alpha|^2}{1 - \lambda} \right)$$

$$= \lambda (1 - \lambda)^{l} \int_{0}^{\infty} dx \left( \frac{\lambda - 1}{\lambda^2} x \right) e^{-\frac{\lambda^2}{\lambda^2-1}} L_k (x) = 1,$$

(A1)

where we have used

$$\int_{0}^{\infty} e^{-bx} L_n (x) = (b - 1)^{l} b^{-l-1}.$$  

(A2)

**Appendix 2**

For $M_{m,n}$ in Eq. (22) we now prove $\sum_{m,n=0}^{\infty} M_{m,n}^{\dagger} M_{m,n} = 1$. Because

$$\langle a^\dagger a \rangle = \frac{1}{1 - x} e^{a^\dagger a \ln \frac{1}{1 - x}},$$

(A3)

where $\langle \rangle$ denotes anti-normal ordering, we have

$$\sum_{m,n=0}^{\infty} \frac{1}{m! (m+1)!} \langle a^\dagger a \rangle^{m} a^m a^{m+1}$$

$$= \exp \left[ \frac{\kappa t}{\kappa t + 1} a^\dagger a \right]$$

$$= (\kappa t + 1) e^{a^\dagger a \ln (\kappa t + 1)}.$$

(A4)
Substituting it into the sum representation of $\rho(t)$ yields

$$\sum_{m,n=0}^{\infty} M_{m,n}^\dagger M_{m,n} = \sum_{m,n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}} a_\dagger^n \left( \frac{1}{1 + \kappa t} \right)^{m+n+1} a_\dagger^m a_\dagger^m \left( \frac{1}{1 + \kappa t} \right) a_\dagger^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^n}{(\kappa t + 1)^n} a_\dagger^n \left( \frac{1}{1 + \kappa t} \right)^n 2 a_\dagger a \ln(\kappa t + 1) a_\dagger^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^n}{(\kappa t + 1)^n} a_\dagger^n e^{a_\dagger a \ln(\kappa t + 1)} 2 a_\dagger a \ln \frac{2}{1 + \kappa t} a_\dagger^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^n}{(\kappa t + 1)^n} a_\dagger^n e^{a_\dagger a \ln(\kappa t + 1) + 2 \ln \frac{2}{1 + \kappa t}} a_\dagger^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^n}{(\kappa t + 1)^n} a_\dagger^n e^{a_\dagger a \ln(\frac{2}{1 + \kappa t})} a_\dagger^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\kappa t)^n}{(\kappa t + 1)^n} a_\dagger^n e^{a_\dagger a \ln(\frac{1}{1 + \kappa t}) - 1} a_\dagger^n$$

$$= e^{a_\dagger a \frac{1}{1 + \kappa t} - 1} e^{a_\dagger a \ln(\frac{1}{1 + \kappa t})} = 1. \quad (A5)$$

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