A very simple remark concerning a link between the notions of Kolmogorov-Sinai entropy and of Renormalization Group is performed.
I. INTRODUCTION

Kadanoff-Wilson's \(^1\) renormalization group \(^2\), and the Kolmogorov-Sinai's entropy \(^4\) have one similarity: they involve a sequence of partitions of the underlying probability space respectively decreasing and increasing with respect to the coarse-graining ordering relation.

In the introduction of \(^1\) one, indeed, reads that among the applications of renormalization-group there is the analysis of the onset of chaos in dynamical systems.

Both therein and, as far as as I know, elsewhere, anyway (also taking into account M.J. Feigenbaum's stuff \(^5\)), a structural analysis concerning the inter-relation between Renormalization Group and the Kolmogorov-Sinai entropy is, at least as far as I know, still lacking.

These brief notes are intended (to try) to make a (very little) step in such a direction.

---

\(^1\) As it often happens the attribution of paternity is a subtle matter. In particular a little dispute exists as to the contribution by C. Di Castro and G. Jona-Lasinio. As it is more appropriate in these cases one has to listen both the viewpoints \(^1\), \(^2\).
II. COARSE GRAINING FLOWS, REFINEMENTS’ FLOWS AND THEIR LIMIT POINTS

Let \((X, \sigma, \mu)\) be a classical probability space and let us introduce the following:

**Definition II.1**

**PARTITIONS OF** \((X, \sigma, \mu)\):

\[ P(X, \sigma, \mu) := \left\{ P = \{A_i\}_{i=1}^{n(P)} : n(P) \in \mathbb{N}_+, A_i \in \sigma i = 1, \ldots, n(P), A_i \cap A_j = \emptyset, i, j = 1, \ldots, n(P) : i \neq j, \mu(X - \bigcup_{i=1}^{n(P)} A_i) = 0 \right\} \]  

**(2.1)**

**Remark II.1**

Beside its abstract, mathematical formalization, the definition II.1 has a precise operational meaning.

Given the classical probability space \((X, \sigma, \mu)\) let us suppose to make an experiment on the probabilistic universe it describes using an instrument whose resolutive power is limited in that it is not able to distinguish events belonging to the same atom of a partition \(P = \{A_i\}_{i=1}^{n(P)} \in P(X, \sigma, \mu)\).

Consequentially the outcome of such an experiment will be a number

\[ r \in \{1, \ldots, n\} \]  

specifying the observed atom \(A_r\) in our coarse-grained observation of \((X, \sigma, \mu)\).

We will call such an experiment an operational observation of \((X, \sigma, \mu)\) through the partition \(P\) or, more concisely, a \(P\)-experiment.

The probabilistic structure of the operational observation of \((X, \sigma, \mu)\) through a partition \(P \in P(X, \sigma, \mu)\) is enclosed in the following:

**Definition II.2**

**PROBABILITY MEASURE OF THE P-EXPERIMENT:**

\[ \mu_P := \mu|_{\sigma(P)} \]

where \(\sigma(P) \subset \sigma\) is the \(\sigma\)-algebra generated by \(P\).

Given \(P_1, P_2 \in P(X, \sigma, \mu)\):

**Definition II.3**

\(P_1\) is a coarse-graining \(P_2\) \((P_1 \leq P_2)\):

\[ \text{every atom of } P_1 \text{ is the finite union of atoms of } P_2 \]

**Definition II.4**

coarsest refinement of \(A = \{A_i\}_{i=1}^{n} \text{ and } B = \{B_j\}_{j=1}^{m} \in P(X, \sigma, \mu):\)

\[ A \vee B \in P(X, \sigma, \mu) \]

\[ A \vee B := \{ A_i \cap B_j : i = 1, \ldots, n, j = 1, \ldots, m \} \]  

**(2.3)**

One has that:

**Theorem II.1**

\(\leq\) is an ordering relation over \(P(X, \sigma, \mu)\)

Let us now introduce the following:

**Definition II.5**
ENTROPY OF $P = \{A_i\}_{i=1}^{n(P)} \in \mathcal{P}(X, \sigma, \mu)$:

$$H(P) := -\sum_{i=1}^{n(P)} \mu_P(A_i) \log_2 \mu_P(A_i)$$ (2.4)\

Remark II.2

The entropy $H(P)$ of the partition $P$ measures the amount of information that one acquires realizing the $P$-experiment.

Definition II.6

d : $\mathcal{P}(X, \sigma, \mu) \times \mathcal{P}(X, \sigma, \mu) \to [0, +\infty)$:

d$(P_1, P_2) := |H(P_1) - H(P_2)|$ (2.5)\

Remark II.3

Let us observe that $d$ is not a metric over $\mathcal{P}(X, \sigma, \mu)$ since $d(P_1, P_2) = 0 \nRightarrow P_1 = P_2$.

Let us introduce the following:

Definition II.7

course-graining flow over $(X, \sigma, \mu)$:
a sequence $\{P_n\}_{n \in \mathbb{N}}$ such that:

$$P_n \in \mathcal{P}(X, \sigma, \mu) \text{ and } P_{n+1} \leq P_n \ \forall n \in \mathbb{N}$$ (2.6)\

Definition II.8

refinements’ flow over $(X, \sigma, \mu)$:
a sequence $\{P_n\}_{n \in \mathbb{N}}$ such that:

$$P_n \in \mathcal{P}(X, \sigma, \mu) \text{ and } P_n \leq P_{n+1} \ \forall n \in \mathbb{N}$$ (2.7)\

Given an arbitrary sequence of mathematical objects $\{a_n\}_{n \in \mathbb{N}}$ let us introduce the following:

Definition II.9

reverse of $\{a_n\}_{n \in \mathbb{N}}$:

$$\text{reverse}(\{a_n\}_{n \in \mathbb{N}}) = \{b_{-n}\}_{n \in \mathbb{N}} : b_{-n} := a_n$$ (2.8)\

One has clearly that:

Proposition II.1

1. $\{P_n\}_{n \in \mathbb{N}}$ is a coarse-graining flow $\Rightarrow$ reverse($\{P_n\}_{n \in \mathbb{N}}$) is a refinements’ flow
2. $\{P_n\}_{n \in \mathbb{N}}$ is a refinements’ flow $\Rightarrow$ reverse($\{P_n\}_{n \in \mathbb{N}}$) is a coarse-graining flow

Given a refinements’ flow or a coarse-graining flow $\{P_n\}_{n \in \mathbb{N}}$ and a partition $P_0 \in \mathcal{P}(X, \sigma, \mu)$:

Definition II.10

$P_0$ IS A LIMIT POINT OF $\{P_n\}_{n \in \mathbb{N}}$:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : d(P_n, P_0) < \epsilon \forall n > N$$ (2.9)\

Let us observe that:

Proposition II.2

1. $\{P_n\}_{n \in \mathbb{N}}$ is a refinements’ flow $\Rightarrow H(P_n) \leq H(P_{n+1}) \ \forall n \in \mathbb{N}$
2. $\{P_n\}_{n \in \mathbb{N}}$ is a coarse-graining flow $\Rightarrow H(P_{n+1}) \leq H(P_n) \ \forall n \in \mathbb{N}$
III. KOLMOGOROV-SINAI ENTROPY VERSUS REFINEMENTS’ FLOWS

Let us start from the following:

**Definition III.1**

*classical dynamical system*:

a couple \(((X, \sigma, \mu), T)\) such that:

- \((X, \sigma, \mu)\) is a classical probability space
- \(T : X \mapsto X\) is such that:

\[
\mu \circ T^{-1} = \mu
\]  

(3.1)

Given a classical dynamical system \(CDS := ((X, \sigma, \mu), T)\), the \(T^{-1}\)-invariance of \(\mu\) implies that the partitions \(P = \{A_i\}_{i=1}^n \in \mathcal{P}(X, \sigma, \mu)\) and \(T^{-1}P\) have equal probabilistic structure. Consequentially the \(P\)-experiment and the \(T^{-1}P\)-experiment are replicas, not necessarily independent, of the same experiment made at successive times.

In the same way the \(\bigvee_{k=0}^{n-1} T^{-k}P\)-experiment is the compound experiment consisting in \(n\) replications \(P, T^{-1}P, \ldots, T^{-(n-1)}P\) of the experiment corresponding to \(P \in \mathcal{P}(X, \sigma, \mu)\).

The amount of classical information for replication we obtain in this compound experiment is clearly:

\[
\frac{1}{n} H(\bigvee_{k=0}^{n-1} T^{-k}P)
\]

It may be proved (cfr. e.g. the second paragraph of the third chapter of [6]) that when \(n\) grows this amount of classical information acquired for replication converges, so that the following quantity:

\[
h(P, T) := \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{k=0}^{n-1} T^{-k}P)
\]

(3.2)

exists.

In different words, we can say that \(h(P, T)\) gives the asymptotic rate of production of classical information for replication of the \(P\)-experiment.

**Definition III.2**

\[
h_{CDS} := \sup_{P \in \mathcal{P}(X, \sigma, \mu)} h(A, T)
\]

(3.3)

By definition we have clearly that:

\[
h_{CDS} \geq 0
\]

(3.4)

**Definition III.3**

*CDS IS CHAOTIC:*

\[
h_{CDS} > 0
\]

(3.5)

By construction we have the following:

**Lemma III.1**

**HP:**

\[
P \in \mathcal{P}(X, \sigma, \mu)
\]

**TH:**
\{\bigvee_{k=0}^{n-1} T^{-k} P\}_{n \in \mathbb{N}} \text{ is a refinements’ flow.}

from which it follows that:

**Theorem III.1**

**HP:**

\[ P \in \mathcal{P}(X, \sigma, \mu) \]
\[ \{\bigvee_{k=0}^{n-1} T^{-k} P\}_{n \in \mathbb{N}} \text{ has a limit point} \]

**TH:**

\[ h(P, T) = 0 \]

**PROOF:**

If \( \{\bigvee_{k=0}^{n-1} T^{-k} P\}_{n \in \mathbb{N}} \) has a limit point, the rate of information gaining for replication of the P-experiment at a certain point tends to zero. ■
IV. A BRIEF INTRODUCTION TO THE RENORMALIZATION GROUP

Let us introduce briefly the Kadanoff-Wilson’s Renormalization Group in a simple setting, for instance a system of spins $S_i = \pm 1$ living on the sites of a D-dimensional finite cubic lattice $(\{-N, \cdots 0, \cdots , N - 1, N\})^D$ and having dimensionless hamiltonian:

$$
\mathcal{H} := \beta H := -K_0 \sum_i S_i - K_1 \sum_{<i,j>} S_i S_j - K_2 \sum_{<i,j>} S_i S_j - \cdots - K_n \sum_{<i,j>} S_i S_j - \cdots - K_{\infty,1} \sum_{<i,j>} S_i S_j S_k
$$

$$
- K_{\infty,2} \sum_{<i,j,k>} S_i S_j S_k - \cdots - K_{\infty,n} \sum_{<i,j,k>} S_i S_j S_k - \cdots - K_{\infty,\infty,1} \sum_{<i,j,k,r>} S_i S_j S_k S_r - \cdots - K_{\infty,\infty,\infty,\cdots} \sum_{<i,j,k,r>} S_i S_j S_k S_r - \cdots
$$

(4.1)

(\text{where } \beta := \frac{1}{k_B T}, k_B \text{ being Boltzmann’s constant and } T \text{ being the temperature}) where $< \cdots >_n$ denotes spins having distance $n$.

Let us assume that the interaction decreases enough quickly at large distances so that the vector $K$ of the coupling-constants belongs to the space $l_2(\mathbb{R}) := \{ \{ x_n \}_{n \in \mathbb{N}} : x_n \in \mathbb{R}, \sum_{n=0}^{\infty} |x_n|^2 < +\infty \}$.

Let us now analyze how the dimensionless hamiltonian $\mathcal{H}(K)$ changes under a transformation which coarse-grains the short-distance degrees of freedoms.

At this purpose let us divide the lattice $(\{a\{-N, -N+1, \cdots, 0, \cdots, N - 1, N\}\})^D$ in cubic blocks of linear dimension $l a$ (with $l \ll N$), the generic block $B$ containing consequentially $l^D$ spins, and let us associate to each block $B$ a block variable:

$$
S'_B := f(\{S_i\}_{i \in B})
$$

(4.2)

with, for instance:

$$
f(\{S_i\}_{i \in B}) := \begin{cases} 
\text{sign}(\sum_{i \in B} S_i), & \text{if } \text{sign}(\sum_{i \in B} S_i) \neq 0; \\
S_B, & \text{otherwise.}
\end{cases}
$$

(4.3)

Introduced the function:

$$
P\{S', S\} := \prod_B \delta_{\text{Kronecker}}[S'_B, f(\{S_i\}_{i \in B})]
$$

(4.4)

the partition function of the system can then be expressed as:

$$
Z_N(K) := \sum_{i \in (\{-N, -N+1, \cdots, 0, \cdots, N - 1, N\})^D} \exp(-\mathcal{H}(K), \{S_i\}) = Z^\mathbb{P}(K') := \sum_B \exp(-\mathcal{H}(K', \{S'_B\})) := \sum_{i \in (\{-N, -N+1, \cdots, 0, \cdots, N - 1, N\})^D} \sum_B P\{S', S\} \exp(-\mathcal{H}(K), \{S_i\})
$$

(4.5)

The passage from $Z_N(K)$ to $Z^\mathbb{P}(K')$ corresponds to a map into the space $l_2(\mathbb{R})$ of the coupling constants:

$$
K' = R_l(K)
$$

(4.6)

called a renormalization of the coupling constants.

---

2 The Renormalization Group applies to any model of Classical Statistical Mechanics and in particular to the situation in which the order-parameter lives on a space with cardinality greater than $\aleph_0$; in this case one often speaks about “Statistical Field Theory” [8], [9], [10]. Since according to the Osterwalder-Schrader axiomatization [11] (or according to the less rigorous vulgata of Euclidean Field Theory [12]) Quantum Field Theory reduces to Classical Statistical Mechanics (axiomatization affected by the irreducibility of noncommutative probability spaces to classical probability spaces and the related superiority of the Haag-Kastler axiomatization with respect to the Osterwalder-Schrader one) the application of the Kadanoff-Wilson Renormalization Group to Quantum Field Theory, resulting in the RG equation for the un-renormalized $\Gamma_n$ (where $\Gamma_n(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma_n(x_1, \cdots, x_n) \phi(x_1) \cdots \phi(x_n)$ is the Legendre transform of the logarithm of the partition function), is nothing but a particular case of its general Statistical Mechanics’ framework.
The renormalizations of the coupling constants form a semigroup:

\[ R_{l_1l_2}(K) = R_{l_1}(K) \cdot R_{l_2}(K) \]  

(4.7)

usually called the renormalization group.

Adhering to the usual terminology we will also refer to a renormalization of the coupling constants as to a renormalization group transformation.

Iterating a transformation \( R_l \) one performs a discrete-time dynamics in the coupling-constants’ space \( l_2(\mathbb{R}) \) to which the whole conceptual apparatus of Classical Dynamical Systems’ Theory applies (such as the theory of fixed-points and their basins of attraction):

given \( K_\ast \in l^2(\mathbb{R}) \):

**Definition IV.1**

\( K_\ast \) is a fixed point of the renormalization-group transformation \( R_l \):

\[ R_l(K_\ast) = K_\ast \]

Given a fixed point \( K_\ast \) of the renormalization-group transformation \( R_l \):

**Definition IV.2**

basin of attraction of \( K_\ast \) with respect to \( R_l \):

\[ \mathcal{B}_l(K_\ast) := \{ K \in l_2(\mathbb{R}) : \lim_{n \to +\infty} R^n_l(K) = K_\ast \} \]

where \( R^n_l \) denotes the \( n^{th} \)-iterate of \( R_l \).

**Remark IV.1**

Since each renormalization group transformation \( R_l \) corresponds to a reduction of the number of degrees of freedom of a factor \( l^D \) one could think that a renormalization group flow necessarily terminates with the elimination of all the degrees of freedom.

Taking the thermodynamic limit \( N \to +\infty \) it follows that an infinite number of iterations of a renormalization group transformation \( R_l \) is required in order to eliminate all the degrees of freedom.

It is only under the thermodynamical limit that singularities in the free-energy \( F := -\frac{1}{\beta} \log Z \) or its derivatives can occur.

**Remark IV.2**

According to Ehrenfest’s classification a critical point of \( n^{th} \) order is a point on which the free-energy is differentiable \( n - 1 \) times, but not \( n \) times. Following the usual terminology \([7]\) we will call a transition of second order any critical point of Ehrenfest-ordering greater or equal than two. The phenomenon of Universality of the long-distance behavior in the phase transitions of second order is owed to the fact that different physical systems correspond to different points of a same basin of attraction \( \mathcal{B}_l(K_\ast) \). Such a basin of attraction is then also called a universality class.

---

\[ \text{Introduced the following inner product over } l_2(\mathbb{R}): \]

\[ \left( \{ x_n \}_{n \in \mathbb{N}}, \{ y_n \}_{n \in \mathbb{N}} \right) := \sum_{n=0}^{\infty} x_n y_n \]  

(4.8)

one has that \( (l_2(\mathbb{R}), \langle \cdot, \cdot \rangle) \) in an Hilbert space over \( \mathbb{R} \) so that the norm \( \| \{ x_n \}_{n \in \mathbb{N}} \| := \sqrt{\langle \{ x_n \}_{n \in \mathbb{N}}, \{ x_n \}_{n \in \mathbb{N}} \rangle} \) induces the metric \( d(\{ x_n \}_{n \in \mathbb{N}}, \{ y_n \}_{n \in \mathbb{N}}) := \| \{ x_n \}_{n \in \mathbb{N}} - \{ y_n \}_{n \in \mathbb{N}} \| \) than can be used to define the notion of limit in the usual way \([12]\).
V. REINORMALIZATION GROUP AS A PARTICULAR KIND OF COARSE-GRAINING FLOW

Let us consider the classical probability space \( ((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D, \mathcal{B}_{\text{Borel}}, \mu) \) where:

\[
d\mu([S_i]_{i\in(a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D}) := N \exp[-\mathcal{H}(K, [S_i])] \prod_{i\in(a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D} \delta(S_i^2 - 1)dS_i \tag{5.1}
\]

where \( N \) is a normalization constant.

The coarse-graining underlying the renormalization group transformation \( R_t \) may be represented by the partition \( P \in \mathcal{P}((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D, \mathcal{B}_{\text{Borel}}, \mu) \) whose atoms are the different blocks \( B \) by which \((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D \) has been divided.

The iteration of \( R_t \) corresponds to a coarse-graining flow \( \{P_n^{(l)}\}_{n\in\mathbb{N}} \) over the classical probability space \((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D, \mathcal{B}_{\text{Borel}}, \mu) \) to which corresponds the flow of probability measures \( \{\mu_{P_n^{(l)}}\}_{n\in\mathbb{N}} \).

Let us now suppose that the initial condition \( K \) of the renormalization group flow belongs to the basin of attraction \( B_l(K_*) \) of a fixed point \( K_* \).

It follows that the coarse-graining flow \( \{P_n^{(l)}\}_{n\in\mathbb{N}} \) has a limit point (according to the definition \[\text{II.10}\]).

Let us now introduce:

\[
\{\hat{P}_n^{(l)}\}_{n\in\mathbb{N}} := \text{reverse}(\{P_n^{(l)}\}_{n\in\mathbb{N}}) \tag{5.2}
\]

By Proposition \[\text{II.12}\], \( \{\hat{P}_n^{(l)}\}_{n\in\mathbb{N}} \) is a refinements’ flow.

Let us now consider a \( \mu \)-preserving map \( T_1 : ((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D \rightarrow ((a\{-N,-N+1,\cdots,0,\cdots,N-1,N\})^D \) such that:

\[
\bigwedge_{k=0}^{n-1} T^{-k} \hat{P}_0^{(l)} = \hat{P}_{-n}^{(l)} \quad \forall n \in \mathbb{N} \tag{5.3}
\]

If our knowledge of the renormalization group flow allowed us to know that also \( \{\hat{P}_n^{(l)}\}_{n\in\mathbb{N}} \) has a limit point we could use theorem \[\text{II.13}\] to infer that:

\[
h(\hat{P}_0^{(l)}; T_1) = 0 \tag{5.4}
\]

Example V.1

Let us consider the simplest possibility, i.e. the one dimensional Ising model corresponding to the assumptions that the only coupling constants different from zero are \( K_0 \) and \( K_1 \) and, obviously, that \( D = 1 \).

Let us impose periodic boundary conditions \( S_{N+1} := S_1 \)

The N-spin partition function can be written as:

\[
Z_N = Tr T^N = \lambda_+^N + \lambda_-^N \tag{5.5}
\]

where \( T \) is the transfer matrix:

\[
T := \begin{pmatrix}
\exp(K_0 + K_1) & \exp(-K_1) \\
\exp(-K_1) & \exp(K_0 - K_1)
\end{pmatrix} \tag{5.6}
\]

and where \( \lambda_\pm \) are its eigenvalues:

\[
\lambda_\pm = \exp(K_1)[\cosh(K_0) \pm \sqrt{\sinh^2(K_0) + \exp(-4 K_1)}] \tag{5.7}
\]

Let us consider as blocks couples of nearest neighbors spins. One has that:

\[
Z_\pm(K^*) = Tr T^\pm \tag{5.8}
\]

where clearly:

\[
T' = T^2 \tag{5.9}
\]
Let us impose that, up to a multiplicative constant, $T'$ has the same form as $T$:

$$T' = c \left( \frac{\exp(K' + K'_1)}{\exp(-K'_1)} \right) \left( \frac{\exp(-K'_1)}{\exp(K'_0 - K'_1)} \right)^{-1} \left( \frac{\exp(K'_0)}{\exp(-K'_1)} \right)$$

(5.10)

It is useful to parametrize the coupling-constant's space introducing the vector:

$$V := (V_0, V_1)$$

(5.11)

where:

$$V_i := \exp(-K_i) \quad i = 0, 1$$

(5.12)

The renormalization group transformation $K' = R_2(K)$ induces an analogous map $V' = \hat{R}_2(V)$ where:

$$V' := (V'_0, V'_1)$$

(5.13)

$$V'_i := \exp(-K'_i) \quad i = 0, 1$$

(5.14)

The map $\hat{R}_2$, obtained comparing eq. 5.9 with eq. 10, is explicitly given by:

$$V'_0 = \left( \frac{V_1^4 + V_0^2}{V_1^4 + V_0^2} \right)^{\frac{1}{2}}$$

(5.15)

$$V'_1 = \left( \frac{V_0 + \frac{1}{V_0}}{V_1^4 + V_0^2 + \frac{1}{V_0^2}} \right)^{\frac{1}{2}}$$

(5.16)

$$c = \left( V_0 + \frac{1}{V_0} \right)^{\frac{1}{2}} \left( V_1^4 + \frac{1}{V_1^4} + V_0^2 + \frac{1}{V_0^2} \right)^{\frac{1}{2}}$$

(5.17)

Let us now construct the coarse-graining flow $\{ P_n \} \in \mathbb{N}$.

One has clearly that:

$$P_0^{(2)} = \{-aN, -a(N+1), \ldots, a(N-3), a(N-1), aN\}$$

(5.18)

$$P_1^{(2)} = \{-aN, -a(N+1), -a(N+2), -a(N+3), \ldots, a(N-3), a(N-1), aN\}$$

(5.19)

$$P_2^{(2)} = \{-aN, -a(N+1), -a(N+2), -a(N+3), \ldots, a(N-3), a(N-1), aN\}$$

(5.20)

Introduce the refinements’ flow:

$$\{ \hat{P}_n \}_{n \in \mathbb{N}} := \text{reverse}(\{ P_n \}_{n \in \mathbb{N}})$$

(5.21)

Let us suppose to have a $\mu$-preserving map $T_2 : \mathbb{Z} \to \mathbb{Z}$ such that:

$$\forall k \in \mathbb{N}, T_k \hat{P}_0 = \hat{P}_n \quad \forall n \in \mathbb{N}$$

(5.22)

Let us now analyze the structure of the renormalization flow:

performing in inverted sense the basin of attraction of any fixed point $V_{\lambda} := (\lambda, 1), \lambda \in (0, 1)$ one sees that it is a sequence converging to $V_* := (1, 0)$.

So it follows that the associated refinements’ flow $\{ \hat{P}_n \}_{n \in \mathbb{N}}$ has a limit point.

Hence, by theorem we can infer that:

$$h(\hat{P}_0, T_1) = 0$$

(5.23)
VI. ACKNOWLEDGEMENTS

I acknowledge funding related to a Marie Curie post-doc fellowship of the EU network on "Quantum Probability and Applications in Physics, Information Theory and Biology" contract HPRNT-CT-2002-00279 (prolonged of two months). I would like to thank prof. A. Khrennikov for stimulating discussions; of course he has no responsibility of any error contained in these pages.
[1] G. Benfatto G. Gallavotti. *Renormalization Group*. Princeton University Press, Princeton, 1995.

[2] M.E. Fisher. Renormalization Group Theory: its Basis and Formulation in Statistical Physics. In T.Y. Cao, editor, *Conceptual Foundations of Quantum Field Theory*, pages 89–135. Cambridge University Press, Cambridge, 1999.

[3] J.J. Binney N.J. Dowrick A.J. Fisher M.E. Newman. *The Theory of Critical Phenomena. An Introduction to the Renormalization Group*. Oxford University Press, Oxford, 1992.

[4] Ya. G. Sinai. *Topics in Ergodic Theory*. Princeton University Press, Princeton, 1994.

[5] P. Cvitanovic. *Universality in Chaos*. Institute of Publishing, Bristol, 1984.

[6] I.P. Kornfeld Y.G. Sinai. General Ergodic Theory of Groups of Measure Preserving Transformations. In Y.G. Sinai, editor, *Dynamical Systems, Ergodic Theory and Applications*. Springer Verlag, Berlin, 2000.

[7] G. Parisi. *Statistical Field Theory*. Perseus Books, Reading (Massachusetts), 1988.

[8] C. Itzykson J.M. Drouffe. *Statistical Field Theory. Vol.1: from Brownian motion to renormalization and lattice gauge theory*. Cambridge University Press, Cambridge, 1989.

[9] J. Glimm A. Jaffe. *Quantum Physics*. Springer-Verlag, New York, 1987.

[10] J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Oxford University Press, New York, 1993.

[11] M. Reed B. Simon. *Methods of Modern Mathematical Physics: vol.1 - Functional Analysis*. Academic Press, 1980.

[12] K. Huang. *Statistical Mechanics*. John Wiley and Sons, New York, 1987.