MORE ON (4,4) SUPERMULTIPLETS IN $SU(2) \times SU(2)$ HARMONIC SUPERSPACE

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Abstract

We define the $SU(2) \times SU(2)$ harmonic superspace analogs of tensor and non-linear (4,4), 2D supermultiplets. They are described by constrained analytic superfields and provide an off-shell formulation of a class of torsionful (4,4) supersymmetric sigma models with abelian translational isometries on the bosonic target. We examine their relation to (4,4) twisted multiplets and discuss different types of (4,4) dualities associated with them. One of these dualities implies the standard abelian $T$-duality relations between the bosonic targets in the initial and dual sigma model actions. We show that $N = 4$, 2D superconformal group admits a simple realization on the superfields introduced, and present a new superfield form of the (4,4) $SU(2) \times U(1)$ WZNW action.
1 Introduction

For unambiguous construction of models with 2D supersymmetry, in particular, supersymmetric string models, it is of crucial importance to have the full list of off-shell representations of 2D supersymmetry, as well as to know various interrelations between them, e.g., via duality transformations. Working with off-shell supermultiplets, especially in the superfield approach, allows one to keep supersymmetry manifest at each step and provides simple general rules of the model-building.

Two important off-shell multiplets of $N = 2, 4D$ supersymmetry are the tensor [1] and nonlinear [2] ones. They were primarily used as compensators breaking $N = 2$ conformal supergravity down to some off-shell versions of Einstein $N = 2$ supergravity [2]. Later on, they were exploited to construct a subclass of $N = 2, 4D$ supersymmetric sigma models and to explicitly compute the relevant bosonic hyper-Kähler metrics [3, 4]. Upon reduction $N = 2, 4D \rightarrow N = (4, 4), 2D$, such models can provide some string backgrounds, and this is the main reason of recent revival of interest to these multiplets (and some their further generalizations) in the context of string theory [5, 6, 7]. In particular, in [7] it has been proposed to utilize the nonlinear multiplet from this point of view.

Sigma models associated with these multiplets yield no torsion in the bosonic part of the action, the relevant target manifolds are hyper-Kähler 1. On the other hand, generic string backgrounds possess a nontrivial torsion. The basic aim of the present paper is to propose a generalization of these $(4, 4)$ multiplets, such that the relevant sigma models actions contain the torsion terms which cannot be removed by any duality transformation with preserving manifest $(4, 4)$ supersymmetry.

The natural off-shell description of torsionless $(4, 4)$ supersymmetric sigma models is achieved within the $2D$ version of $SU(2)$ harmonic superspace (HSS) [8]. In refs. [2, 11, 12], the tensor and nonlinear multiplets were formulated as $SU(2)$ harmonic analytic superfields with a restricted dependence on $SU(2)$ harmonics. As follows from the results of ref. [12], general off-shell interactions of the tensor and some other $(4, 4)$ multiplets with finite sets of auxiliary fields are equivalent, via a superfield duality transformation, to some particular classes of self-interaction of the ultimate off-shell $(4, 4)$ hypermultiplet, the unconstrained harmonic analytic superfield $q^+$ with infinite number of auxiliary fields. Though the case of nonlinear multiplet was missed in ref. [12], the previous statement is true for it as well, and in Sect. 2 we will demonstrate this.

As was argued in refs. [13, 14, 15], the appropriate framework for off-shell description of $(4, 4)$ supersymmetric sigma models with torsion is provided by a generalization of $SU(2)$ HSS, viz., the doubly extended $SU(2) \times SU(2)$ harmonic superspace 2. So, in order to generalize the tensor and nonlinear multiplets to the case with torsion it is natural to look for the $SU(2) \times SU(2)$ HSS analogs of the superfields by which these multiplets are represented in the $SU(2)$ HSS.

We start in Sect. 2 by recalling the formulation of tensor and nonlinear $(4, 4)$ supermultiplets in $SU(2)$ HSS. Then, in Sect. 3, we generalize to $SU(2) \times SU(2)$ HSS the defining constraints of these multiplets in $SU(2)$ HSS. The relevant $SU(2) \times SU(2)$ harmonics

\footnotesize
1\textsuperscript{To be more precise, this is entirely true only in the dual representation of the relevant action through hypermultiplets, see Sect. 2.}

2\textsuperscript{$SU(2) \times SU(2)$ HSS is akin to the $(4, 4)$ projective superspace which was earlier introduced in [16].}
monic analytic superfields propagate (16 + 16) physical fields, as distinct from their \( SU(2) \) harmonic prototypes which propagate (4 + 4) such fields. We demonstrate that general self-interactions of these new superfields are off-shell equivalent to particular classes of self-interactions of four twisted chiral (4, 4) multiplets. These subclasses are distinguished in that they possess abelian translational isometries. The superfields defined here seem to be most appropriate for describing this type of torsionful (4, 4) sigma models. In Sect. 4 we discuss some peculiarities of the relevant superfield sigma model actions. In particular, we point out the existence of different dual formulations of them. In one of these formulations the dual bosonic lagrangian is related to the original one by the standard T-duality relations \[17\]. We also present the realization of the world-sheet \( N = 4, \, SU(2) \) superconformal group on the superfields introduced, as well as a new superfield form of the (4, 4) supersymmetric \( SU(2) \times U(1) \) WZNW action.

2 Tensor and nonlinear multiplets in \( SU(2) \) harmonic superspace

In the 2D version of \( SU(2) \) HSS approach \[^8\] the tensor multiplet is represented by the superfield \( L^{(++)} \) which (i) lives on the harmonic analytic (4, 4), 2D superspace

\[ \{ \zeta^M, u \} \equiv \{ x^{\pm \pm}, \theta^{(+)} \pm, \bar{\theta}^{(+)} \pm, u^{(+)} \pm, u^{(-)} \pm \}, \quad L^{(++)} \equiv L^{(++)}(\zeta, u) \, , \quad (2.1) \]

(ii) is real

\[ L^{(++)} = \tilde{L}^{(++)} \]  

(2.2)

and (iii) obeys the following constraint \[^9\]

\[ D^{(++)} L^{(++)} = 0 \, . \quad (2.3) \]

In these formulas, the indices \( \pm \) without and with parentheses are, respectively, the 2D Lorentz and harmonic \( U(1) \) charge ones (this \( U(1) \) charge is assumed to be strictly preserved), the quantities \( u^{(+)} \pm, u^{(-)} \pm \),

\[ u^{(+)} \pm u^{(-)} \pm = 1, \quad u^{(+)} \pm u^{(-)} \pm - u^{(+)} \pm u^{(-)} \pm = -\epsilon^{\pm k}, \]

are harmonic variables parametrizing the group \( SU(2)_A \), one of the diagonal \( SU(2) \)’s in the full (4, 4) supersymmetry automorphism group \( SO(4)_L \times SO(4)_R \), the symbol \( \sim \) means a generalized involution with respect to which the superspace (2.1) is real, and \( D^{(++)} \) is the analyticity-preserving harmonic derivative

\[ D^{(++)} = u^{(+)} \pm \frac{\partial}{\partial u^{(-)} \pm} + i\theta^{(+)} + \bar{\theta}^{(+)} \pm \partial_{++} + i\theta^{(-)} - \bar{\theta}^{(-)} \pm \partial_{--} \, . \quad (2.4) \]

More details of the harmonic superspace approach can be found, e.g., in refs. \[^8\] - \[^11\]. Recall that unconstrained analytic harmonic superfields contain an infinite tail of auxiliary fields arising from the harmonic expansion on the two-sphere \( S^2 \sim SU(2)_A/U(1)_A \) (the expansion just on \( S^2 \) instead of the whole group \( SU(2)_A \) comes out as the result of the preservation of harmonic \( U(1) \) charge in the harmonic superspace formalism). The role of
the constraint (2.3) is to reduce this infinite tail to the standard (8+8) off-shell component content of the tensor multiplet.

The characteristic feature of the superfield \( L^{(++)} \) is that one of its physical bosonic degrees of freedom is supplied by the 2D vector \( V_{\pm\pm} \) ("notoph") subjected to the constraint

\[
\partial_+ V_- + \partial_- V_+ = 0 \tag{2.5}
\]

that is implied by the superfield one (2.3) (the component fields \( V_{\pm\pm} \) enter \( L^{(++)} \) as the coefficients of the \( \theta \) monomials \( \theta^{(+)} + \bar{\theta}^{(+)} \), \( \theta^{(+)} - \bar{\theta}^{(+)} \), respectively). Eq. (2.5) can be solved as

\[
V_{\pm\pm} = \pm i \partial_{\pm\pm} \phi(x), \quad (\phi^\dagger = \phi) \tag{2.6}
\]

thus introducing the fourth bosonic scalar field.

The general \( L^{(++)} \) action reads \[9\]

\[
S_L = \frac{1}{\kappa^2} \int \mu^{(-4)} \tilde{F}^{(++)}(u, L^{(++)}) \tag{2.7}
\]

Here \( \tilde{F}^{(++)} \) is an arbitrary function of its arguments with the appropriate flat part

\[
\tilde{F}^{(++)} = -L^{(++)}L^{(++)} + O(L^2) ,
\]

\( \kappa \) is the dimensionless sigma model coupling constant and \( \mu^{(-4)} \) is the analytic superspace integration measure

\[
\mu^{(-4)} = d^2x a d^2\theta^{(+)} + d^2\bar{\theta}^{(+)} - [du],
\]

([du] denotes the integration over two-sphere \( S^2 \)). An extension to the case of several \( L^{(++)} \) is obvious.

The general distinguishing property of this action is the abelian translational isometry realized as a shift of the \( SU(2) \)-singlet field \( \phi(x) \) coming out as the solution to the notoph constraint (2.3); as the result, the corresponding bosonic metrics do not depend on this field. The constraint (2.3) can be implemented in the action with the help of the analytic superfield lagrange multiplier \( \omega \) to yield a dual \( \omega \) formulation of the action \[12\]

\[
S_{L,\omega} = S_L + \frac{1}{\kappa^2} \int d\varsigma^{(-4)} \omega D^{(++)}L^{(++)} \tag{2.8}
\]

It also possesses an \( U(1) \) isometry, this time realized as shifts of \( \omega \). The dual \( \omega \) form of the general \( L^{(++)} \) action (2.7) can be obtained by eliminating \( L^{(++)} \) by its algebraic equation of motion. The dual action yields in the bosonic sector the most general four-dimensional hyper-Kähler metric with one translation isometry (in the case of \( n \) copies of \( L^{(++)} \), the most general \( 4n \) dimensional hyper-Kähler metric with \( n \) mutually commuting \( U(1) \) isometries) \[11\]. The action (2.7) in its own right produces, along with the metric, also a non-zero torsion; these both are related to the dual hyper-Kähler metric by the well-known Buscher’s formulas \[17\].

Duality transformation in HSS has been firstly introduced in \[18\] and later on has been used to show that the off-shell actions of various matter multiplets of \( N = 2, 4D \) ((4, 4), 2D) supersymmetry with finite numbers of auxiliary fields are duality-equivalent to particular classes of the general action of the analytic \( q^{(+)} \) hypermultiplet with an infinite
number of auxiliary fields \([9, 12]\). The basic feature of this kind of duality transformation is the preservation of manifest \(N = 2\) supersymmetry (or \((4, 4)\) supersymmetry in the two-dimensional case) at each step.

Combining the superfields \(\omega\) and \(L^{(++)}\) into the single unconstrained analytic superfield \(q^{(+)i}\)

\[
q^{(+)i} \equiv u^{(-)i}L^{(++)} - \frac{1}{2}u^{(+)}\omega, \quad L^{(++)} = u^{(+)}q^{(+)}_i, \quad \omega = 2u^{(-)i}q^{(+)i},
\]

(2.9)

where we have made use of the property of completeness of the harmonics, the action (2.8) can be indeed rewritten as a particular representative of actions of the superfield \(q^{(+)i}\). This superfield is “ultimate” for torsionless \((4, 4)\) sigma models, in the sense that its most general self-interactions yield most general hyper-Kähler metrics in the bosonic sector \([19]\). This way, the theorem about the relationship between \((4, 4), 2D\) worldsheet supersymmetry \((N = 2\) in four dimensions) and bosonic target manifolds of the related torsionless sigma models \([20]\) is visualized. General actions of \(q^{(+)i}\) possess no any isometries and do not admit a duality transformation to the form with a finite number of auxiliary fields.

Let us turn to discussing the nonlinear multiplet. As no a systematic treatment of it has been given so far in the literature on the HSS approach, we dwell on this subject in some more detail.

In the SU(2) 2D HSS the nonlinear multiplet is described by the real analytic superfield \(N^{(++)}(\zeta, u)\) subjected to the constraint \([10, 11]\)

\[
D^{(++)}N^{(++)} + (N^{(++)})^2 = 0.
\]

(2.10)

Once again, the role of this constraint is to reduce an infinite tail of the fields appearing in the harmonic decomposition with respect to the variables \(u\) to the standard off-shell component content of nonlinear multiplet which is \((8+8)\) as in the case of tensor multiplet. The equivalence of the analytic superspace description of the nonlinear multiplet to the one in the conventional \((4, 4), 2D\) superspace \([2]\) can be easily demonstrated \([21, 11]\).

Taking into account the constraint (2.10), one can show that the most general action for \(k\) independent superfields \(N^{(++)}_\alpha (\alpha = 1, \ldots, k)\) reads

\[
S^k_N = \frac{1}{\kappa'^2} \int \mu^{(-4)} F^{(++)}_N (N^{(++)}_1, \ldots, N^{(++)}_k, u) .
\]

(2.11)

The action is particularly simple for just one \(N^{(++)}\):

\[
S^1_N = \frac{1}{\kappa'^2} \int d\zeta^{(-4)} N^{(++)}(\zeta)c^{(++)}(u^\pm),
\]

(2.12)

where \(c^{(++)}\) is an arbitrary function of the harmonics. Any power of \(N^{(++)}\) can be reduced to a term linear in \(N^{(++)}\) with making use of the constraint (2.10) and integrating by parts with respect to harmonic variables (harmonic integrals of \(D^{++}\) applied on anything vanish).

We wish to point out that the actions for \(N^{(++)}\) involve explicitly only 3 out of the 4 physical scalars of the on-shell matter multiplet. The fourth scalar, as in the
case of tensor multiplet, is supplied by the constrained vector field $V_{\pm \pm}(x)$ ($N^{(++)} = i\theta^{(+)+\theta^{(+)}} + V_{++}(x) + i\theta^{(+)-\theta^{(+)}} - V_{--}(x) + \ldots$). This constraint follows from (2.10):

$$\partial_{++} V_{--} + \partial_{--} V_{++} + 2 V_{++} = 0 \quad (2.13)$$

(neglecting contributions from other fields). Unlike the notoph constraint (2.5), eq. (2.13) cannot be solved explicitly. It seems that the only reasonable way to deal with (2.13) is to implement it in the action with a scalar Lagrange multiplier. The latter becomes the fourth bosonic degree of freedom upon elimination of $V_{\pm \pm}$ and the action of the nonlinear multiplet acquires the standard sigma model form. This naturally comes about within the dual description of nonlinear multiplet in terms of unconstrained analytic superfields.

In [12] the case of nonlinear multiplet was missed. Here we fill this gap. For simplicity we will consider the case of one $N^{(++)}$.

To obtain the dual action in this case, we insert the constraint (2.10) into (2.12) with the help of suitable Lagrange multiplier:

$$S^1_N = \frac{\kappa^2}{2} \int d\zeta (-4) \left\{ N^{(++)}(\zeta, u)c^{(++)}(u) + \omega \left[ D^{(++)}N^{(+)} + (N^{(++)})^2 \right] \right\}. \quad (2.14)$$

Varying this action with respect to $\omega$, we come back to the constraint (2.10) and action (2.12). On the other hand, varying with respect to $N^{(++)}$, we get

$$2N^{(++)}\omega = D^{(++)}\omega - c^{(++)}. \quad (2.15)$$

Assuming that $\omega$ starts with a constant (i.e. that we can divide by $\omega = 1 + \ldots$), redefining it as

$$\omega = \hat{\omega}^2 \quad (2.16)$$

(this is a canonical redefinition in virtue of the previous assumption) and substituting all this back into (2.14), we obtain the dual $\omega$ representation of the latter in the following form

$$S_{\omega}^{\text{dual}} = \frac{\kappa^2}{2} \int \mu (-4) \left[ -(D^{(++)}\hat{\omega})^2 - \frac{1}{4} \frac{(c^{(++)})^2}{\hat{\omega}^2} - \ln \hat{\omega} D^{(++)}c^{(++)} \right]. \quad (2.17)$$

In the particular case

$$D^{(++)}c^{(++)} = 0 \Rightarrow c^{(++)} = c^{(ik)}u^{(+)}_i u^{(+)}_k, \quad (2.18)$$

the action (2.17) takes a form which highly resembles the $\omega$ representation of the action of (4, 4) sigma model with the Eguchi-Hanson manifold as the bosonic target [22].

$$S_{\omega,N}^{\text{dual}} = \frac{\kappa^2}{2} \int \mu (-4) \left[ -(D^{(++)}\hat{\omega})^2 - \frac{1}{4} \frac{(c^{(++)})^2}{\hat{\omega}^2} \right]. \quad (2.19)$$

The only difference is in the sign of the second term, so in the present case we obtain the EH metric with the wrong sign of the “mass”-parameter (which is just $(c^{12})^2$ in the fixed $SU(2)_A$ frame). The $N = (2, 2)$ superfield form of the same action has been given in [1]. Note that the invariance group of the action (2.19) is $SU(2) \times U(1)$, just as in the case of the standard EH (4, 4) sigma model [22], $SU(2)$ being a kind of Pauli-Gürsey group.

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commuting with (4, 4) supersymmetry while $U(1)$ a part of the (4, 4) supersymmetry automorphism group. On the bosonic target manifold this $SU(2) \times U(1)$ is realized as isometries of the target metric, respectively as “translational” and “rotational” ones, in agreement with the fact that the EH metric possesses such isometries [23]. The explicit realization of the $SU(2)$ factor of the isometry group in the action (2.14) coincides with that given in ref. [22]. Of course, the original $N^{(+)}$ action (2.12) with the specific $c^{(+)}$ (2.18) and constraint (2.10) respects the same invariance group properly realized on $N^{(+)}$.

Note that it is easy to rewrite the $\omega, N^{(+)}$ action (2.14) as a subclass of general actions of the ultimate analytic $q^{(+)}$ hypermultiplet, in accord with the statement that the general $q^{(+)}$ action corresponds to the most general hyper-Kähler off-shell (4, 4) supersymmetric sigma model. The original $\omega, N^{(+)}$ representation (as well as the $\omega, L^{(+)}$ representation of the tensor multiplet action) turns out to be more preferable for generalizing to the case with torsion.

Finally, we make two comments. Firstly, as is seen from (2.17), even in the case when the original action (2.12) is zero ($c^{(+)} = 0$), its dual (2.17) is non-trivial and describes a free hypermultiplet. This subtlety has been also noticed and discussed in [7] in the framework of (2, 2) superfield formalism. It can be traced to the above-mentioned fact that the actions (2.11), (2.12) as they stand admit no standard sigma model interpretation which becomes possible only after passing to the dual description (the assumption that the lagrange multiplier superfield $\omega$ contains a non-zero “classical” constant part is important for self-consistency of such a description).

Another comment concerns the relation to the (4, 4), 2D tensor multiplet superfield $L^{(+)}$. Its defining constraint (2.3) can be regarded as a degenerate limit of (2.10) (one rescales $N^{(+)} = \gamma L^{(+)}$, substitutes this into (2.10), divide by $\gamma$ and finally put $\gamma$ equal to zero), however the actions of $L^{(+)}$ are radically differ from those of $N^{(+)}$ and cannot be related to the latter by any limiting procedure. In contrast to the case of nonlinear multiplet, in the dual action of $L^{(+)}$ (2.8) the lagrange multiplier term alone (with $\tilde{F}^{(+4)} = 0$) produces no non-trivial action: varying $L^{(+)}$ yields $D^{(+)} \omega = 0 \Rightarrow \omega = \text{const.}$

3 Generalizations to the $SU(2) \times SU(2)$ harmonic superspace

The (4, 4) $SU(2) \times SU(2)$ HSS is an extension of the standard real (4, 4) 2D superspace by two independent sets of harmonic variables $u_i^{\pm 1}$ and $v_a^{\pm 1}$ associated with two commuting automorphism groups $SU(2)_L$ and $SU(2)_R$ of the left and right sectors of (4, 4) supersymmetry [13, 16]. The $SU(2) \times SU(2)$ HSS formalism enables one to keep both these $SU(2)$ symmetries manifest at each step and to control their breakdown.

In what follows we will be interested in an analytic subspace of the $SU(2) \times SU(2)$ HSS. It is presented by the following set of coordinates

$$(\zeta, u, v) = (x_1^{++}, x_2^{--}, \theta^{1,0}, \theta^{0,1}, u_i^{\pm 1}, v_a^{\pm 1})$$

and is closed under the (4, 4) supersymmetry transformations. The pairs of superscripts “$n, m$” in (3.1) stand for the values of two independent harmonic $U(1)$ charges which,
like in the case of $SU(2)$ HSS, are assumed to be strictly conserved. As the result of this requirement, all superfields defined on (3.1), the $SU(2) \times SU(2)$ analytic (4, 4) superfields, are expanded in the double harmonic series on the product $SU(2)_L/U(1)_L \otimes SU(2)_R/U(1)_R$. Extra doublet indices $\dot{q}, \underline{a}$ of Grassmann coordinates in (3.1) refer to two additional $SU(2)$ automorphism groups of (4, 4) supersymmetry which, together with $SU(2)_L$ and $SU(2)_R$, constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of the latter. We omit the 2D Lorentz indices of Grassmann coordinates, keeping in mind that the first and second $\theta$’s in (3.1) carry, respectively, the indices $+$ and $−$.

In the present case one can define two harmonic derivatives preserving the analyticity, the left and right ones

$$D^{2,0} = \partial^{2,0} + i\theta^{1,0}i\partial_{++}^{0,0} \quad , \quad D^{0,2} = \partial^{0,2} + i\theta^{0,1}a\partial_{--}^{0,1} \ . \quad (3.2)$$

Their very important property having no analogs in the $SU(2)$ HSS case is their commutativity

$$[D^{2,0}, D^{0,2}] = 0 \ . \quad (3.3)$$

As we will see, it places severe restrictions on the possible form of the constraints one can impose on the $S(2) \times SU(2)$ analytic superfields in order to cut an infinite tail of auxiliary fields in their $u, v$ harmonic expansions and thus to get (4, 4) multiplets with finite sets of fields.

Our further aim will be to discuss possible generalizations of the $SU(2)$ harmonic constraints (2.3), (2.10) to the $SU(2) \times SU(2)$ case. The natural primary requirements are (i) these constraints involve first degrees of $D^{2,0}, D^{0,2}$; (ii) they do not give rise to any dynamical equation for the component fields, i.e. are purely kinematic.

We start with discussing $SU(2) \times SU(2)$ analogs of the linear constraint (2.3) as the simplest one. One of such sets has been already presented in [13], it is the constraints defining (4, 4) twisted multiplet

$$D^{2,0}q^{1,1} = D^{0,2}q^{1,1} = 0 \ , \quad (3.4)$$

where $q^{1,1}(\zeta, u, v)$ is an analytic superfield. Like (2.3) in application to $L^{++}$, they leave in $q^{1,1}$ (8 + 8) independent components, including 4 physical boson fields. However, as was noticed in [13, 14], the mechanism of achieving this irreducible content is different for (2.3) and (3.4). While the fourth bosonic field in $L^{(++)}$ is supplied by a divergenceless 2D vector, (3.4) amount to purely algebraic relations between the components of $q^{1,1}$, so that all four physical bosons appear on equal footing as the components of the $4 \times 4$ matrix $q^{ia}(x)\,$, $q^{1,1} = q^{ia}(x)u^i_1v^a_1 + \ldots$. No any constrained vectors are present. Also it is easy to see that (3.4) do not admit a nonlinear extension like (2.10). Indeed, without allowing for extra harmonic charged constants it is impossible to construct nonlinear addings to the l.h.s. of eqs. (3.4) out of $q^{1,1}$, so that they possess the harmonic charges (3.1) and (1, 3) [8]. For further reference we present the most general action of $n$ copies of $q^{1,1}$ multiplet

$$S_{q^{1,1}} = \int \mu^{-2,2}L^{2,2}(q^{1,1M}, u, v) \ , \ \det \frac{\partial^2 L^{2,2}}{\partial q^{1,1M} \partial q^{1,1N}}\big|_{q^{1,1}=0} \neq 0 \ \ (M, N = 1, \ldots n) \ . \quad (3.5)$$

\footnote{Even with such constants included, any nonlinear modification of (3.4) is reduced to (3.4) via a canonical redefinition of $q^{1,1}$ [14, 15]. This is a consequence of the commutativity condition (3.3).}
Here $\mu^{-2,-2} = d^2xd^2\theta^{1.0}d^2\theta^{0.1}[du][dv]$ is the analytic superspace integration measure.

As another possible generalization of (2.3) which was not discussed so far, we introduce two $SU(2) \times SU(2)$ analytic superfields $q^{2.0}$, $q^{0.2}$ subjected to the constraints

$$D^{2.0}q^{2.0} = 0, \quad D^{0.2}q^{0.2} = 0. \hspace{1cm} (3.6)$$

The appearance of just two superfields is necessary in order to be able to construct the relevant free action which is given by

$$S_{free} \propto \int \mu^{-2,-2}q^{2.0}q^{0.2}. \hspace{1cm} (3.7)$$

No any meaningful action can be constructed out of $q^{2.0}$ or $q^{0.2}$ alone.

The constraints (3.6) do not restrict the $v$ dependence in $q^{2.0}$ and the $u$ dependence in $q^{0.2}$. Besides, they put no any relation between these superfields. If one solves (3.6) and substitutes the solution into (3.7), no reasonable component action still arises. One could fix the $v$ and $u$ dependence of $q^{2.0}$ and $q^{0.2}$ by imposing the extra constraints

$$D^{0.2}q^{2.0} = D^{2.0}q^{0.2} = 0. \hspace{1cm} (3.8)$$

However, from the explicit structure of $D^{2.0}$, $D^{0.2}$ it immediately follows that

$$\partial_{++}V_-(x) = \partial_{--}V_{++}(x) = 0, \hspace{1cm} (3.9)$$

where $V_+(x)$ and $V_-(x)$ enter the $\theta$ expansion of $q^{2.0}$ and $q^{0.2}$ as the coefficients of the monomials $\theta^{1.0}\bar{\theta}^{1.0}$ and $\theta^{0.1}\bar{\theta}^{0.1}$, respectively (more precisely, they are first components in the bi-harmonic decomposition of these coefficients). Thus the constraints (3.8) lead to the dynamical equations-of-motion-type conditions, and so are unacceptable.

The following relaxation of (3.8) proves to provide a reasonable extension of the constraints (3.6)

$$D^{0.2}q^{2.0} - D^{2.0}q^{0.2} = 0, \hspace{1cm} (3.10)$$

(the sign minus here is a convention, one is at liberty to make arbitrary independent rescalings of $q^{2.0}$, $q^{0.2}$). It is a simple exercise to see that the set (3.6), (3.10) does not entail any dynamical constraints and leaves (32 + 32) components in $q^{2.0}$, $q^{0.2}$, 16 bosonic fields being physical and the remaining 16 auxiliary. One of the physical fields is presented, like in the case of (2.3), by the conserved vector

$$\partial_{++}V_-(x) - \partial_{--}V_{++}(x) = 0 \Rightarrow V_\pm(x) = i\partial_\pm q(x). \hspace{1cm} (3.11)$$

The remaining 15 bosonic fields are collected in the $\theta$ independent parts of $q^{2.0}$, $q^{0.2}$

$$q^{2.0} = q^{(ik)}(x)u_i^1u_k^1 + q^{(ik)(ab)}(x)u_i^1u_k^1v_a^1u_b^1 + \ldots$$

$$q^{0.2} = q^{(ab)}(x)v_a^1v_b^1 + q^{(ik)(ab)}(x)u_i^1u_k^1v_a^1v_b^1 + \ldots. \hspace{1cm} (3.12)$$

Substituting these $q^{2.0}$, $q^{0.2}$ (with all the components included) into (3.7), taking into account (3.11) and eliminating auxiliary fields, one is left with the standard free $\left(4,4\right)$ supersymmetric 2D action for 16 free bosonic fields and $\left(16 + 16\right)$ fermionic fields of both light-cone chiralities. It is straightforward to get this action, so we do not quote it here.
The most general action of the superfields $q^{2,0}, q^{0,2}$, by analogy with (2.7), can be taken in the form
\[ S = \int \mu^{-2,-2} L^{2,2}(q^{2,0}, q^{0,2}, u, v), \quad \frac{\partial^2 L^{2,2}}{\partial q^{2,0} \partial q^{0,2}} |_{q^{2,0}=q^{0,2}=0} \neq 0. \tag{3.13} \]

Note that we would also include into the lagrangian arbitrary powers of one independent

where we have written down the having been partially solved in this way, the set (3.6), (3.10) is reduced to

Firstly, (2.3) leaves in $L^{(++)}$ 4 physical bosonic fields while (3.6), (3.10) leave in $q^{2,0}, q^{0,2}$ the set of 16 ones. This means that the action (3.13) actually propagates 4 on-shell scalar (4, 4) multiplets, in contradistinction to the action (2.7) which propagates only one multiplet. Below we will see that this reducibility extends off shell.

Another difference is that the constraint (3.10) can be explicitly solved in terms of scalar analytic superfield $q(\zeta, u, v)$
\[ q^{2,0} = D^{2,0}q, \quad q^{0,2} = D^{0,2}q, \tag{3.14} \]

thus generalizing the solution (3.11) to the full superfield level $(q(\zeta, u, v) = q(x) + \ldots)$. Note that (2.3) can be solved only through some non-analytic prepotential \[ 9. \]

After having been partially solved in this way, the set (3.6), (3.10) is reduced to
\[ (D^{2,0})^2 q = (D^{0,2})^2 q = 0 \quad \Rightarrow \quad \tag{3.15} \]
\[ q(\zeta, u, v) = q(x) + q^{(ik)}(x) u_1^{i} u_k^{-1} + q^{(ab)}(x) v_a^{1} v_b^{-1} + q^{(ik)(ab)}(x) u_1^{i} u_k^{-1} v_a^{1} v_b^{-1} + \ldots, \tag{3.16} \]

where we have written down the $\theta$ independent part of $q$ which now collects all 16 physical bosonic fields. We see that the action (3.13) is a particular representative of the general $q$ action
\[ S_q = \int \mu^{-2,-2} L^{2,2}(q, D^{2,0}q, D^{0,2}q, u, v) = \int \mu^{-2,-2} (-D^{2,0}q D^{0,2}q + \ldots). \tag{3.17} \]

Here we singled out the free part (the sign minus is needed to have the standard form of kinetic terms for physical fields) and took into account that the possible terms with $D^{2,0}D^{0,2}q$ can be reduced to those present in (3.17) after integrating by parts and exploiting the constraints (3.13). The action (3.13) corresponds to neglecting the explicit dependence on $q$ in (3.17) and leaving only harmonic derivatives of $q$. This means that (3.13) is invariant under arbitrary constant shifts of $q$ that is a clear symmetry of the constraints (3.13) as well. The corresponding bosonic metric always has one translational $U(1)$ isometry, while this is not the case for the general $q$ action (3.17).\[ ^{4}\]

These constraints are invariant under more general shift $q \rightarrow q + \alpha_1 + \alpha_2 u_1^{i} u_k^{-1} + \alpha_3 v_a^{1} v_b^{-1}. $
Let us now demonstrate that the action (3.17) and the constraints (3.19) are actually another form of the general $q^{1,1}$ action (3.5) and the constraints (3.4) for the case of 4 independent $q^{1,1}$ superfields $q^{1,1\alpha\bar{\alpha}}$, ($\alpha, \bar{\alpha} = 1, 2$) (we have split the extra vector $SO(4)$ index into the pair of the doublet $SU(2) \times SU(2)$ ones). To avoid a confusion, let us point out that this extra $SO(4)$ commutes with (4, 4) supersymmetry and so has nothing to do with the automorphism $SO(4)$'s. Rather, it is an analog of the Pauli-Gürsey $SU(2)$ known in the $SU(2)$ harmonic superspace formalism.

As the harmonics $u$ and $v$ satisfy the completeness conditions, we can decompose $q^{1,1\alpha\bar{\alpha}}$ over these complete sets. We get

$$q^{1,1\alpha\bar{\alpha}} = q^{1,1\alpha\bar{\alpha}} = q^{2,0} u_{\alpha} \bar{v}_{\bar{\alpha}} - q^{0,2} u_{\bar{\alpha}} \bar{v}_{\alpha} + q^{2,2} u_{\alpha} \bar{v}_{\bar{\alpha}}$$

(3.18)

where, anticipating the result, we denoted some harmonic projections of $q^{1,1\alpha\bar{\alpha}}$ by the same letters as the superfields introduced earlier. The $q^{1,1}$ constraints (3.4) in this new basis can be equivalently rewritten as the following systems

$$(a) \ D^{2,0} q = q^{2,0}, \ (b) \ D^{2,0} q^{2,0} = 0, \ (c) \ D^{2,0} q^{0,2} = q^{2,2}, \ (d) \ D^{2,0} q^{2,2} = 0$$

(3.20)

$$\begin{align*}
(a) & \ D^{0,2} q = q^{0,2}, \ (b) \ D^{0,2} q^{0,2} = 0, \ (c) \ D^{0,2} q^{2,0} = q^{2,2}, \ (d) \ D^{0,2} q^{2,2} = 0.
\end{align*}$$

(3.21)

One sees that eqs. (a) and (c) in both systems are algebraic and serve to express the projections $q^{2,0}, q^{0,2}$ and $q^{2,2}$ in terms of the harmonic derivatives of $q$

$$q^{2,0} = D^{2,0} q, \ q^{0,2} = D^{0,2} q, \ q^{2,2} = D^{2,0} q^{0,2} = D^{0,2} q^{2,0} = D^{2,0} D^{0,2} q.$$  

(3.22)

Then eqs. (b) become just the constraints (3.15), while eqs. (d) are satisfied as a consequence both of the latter and the expression for $q^{2,2}$ in (3.22). So they do not imply any new restriction for the remaining superfield $q$. After substituting the expressions (3.22) into the general action (3.5) for $q^{1,1\alpha\bar{\alpha}}$ in the basis (3.18), (3.19), we recover the general $q$ action (3.17). Note that the free part of the $q^{1,1}$ superfield Lagrangian

$$L_{q^{1,1}}^{\text{free}} \propto q^{1,1\alpha\bar{\alpha}} q^{1,1\bar{\alpha}\alpha} = 2 \left( q^{2,2} - q^{2,0} q^{0,2} \right)$$

(3.23)

after integrating by parts is reduced, up to a numerical coefficient, to

$$- D^{2,0} q D^{0,2} q,$$

(3.24)

as should be.

Thus we have shown that the model associated with the $SU(2) \times SU(2)$ analytic superfield $q$ subjected to the constraints (3.15) is a disguised form of the theory of four self-interacting twisted hypermultiplets. The system of superfields $q^{2,0}$ and $q^{0,2}$ subjected to the constraints (3.4), (3.10) and described by the action (3.13) corresponds to a particular class of such self-interactions, with the lagrangian in (3.4) bearing no dependence on $q = q^{1,1\alpha\bar{\alpha}} u_{\alpha} \bar{v}_{\bar{\alpha}}$. This property in terms of the original field variables $q^{1,1\alpha\bar{\alpha}}$ can be expressed as the condition

$$u_{\alpha} v_{\bar{\alpha}} \partial L^{2,2}(q^{1,1\beta\bar{\beta}}, u, v) = 0,$$

(3.25)
which means the invariance of the given class of actions under the shift

\[ q^{1,1 \alpha \dot{\alpha}} \Rightarrow q^{1,1 \alpha \dot{\alpha}} + \alpha_1 u^1{}^\alpha v^1{}_{\dot{\alpha}}. \]

(3.26)

Having at our disposal the general formulas for the bosonic target metric and torsion in the case of general \( q^{1,1} \) action (3.4) \[13\], it is of course a matter of direct calculation to obtain them for the given particular case. We do not present them here.

Our next subject will be searching for a reasonable \( SU(2) \times SU(2) \) HSS generalization of the nonlinear multiplet constraint (2.10). Once again, requiring the constraints not to lead to the equations of motion fixes their form up to several undetermined constants

\[ D^{2,0}Q^{2,0} + \beta_1 Q^{2,0}Q^{2,0} = 0, \quad D^{0,2}Q^{0,2} + \beta_2 Q^{0,2}Q^{0,2} = 0, \]

(3.27)

\[ D^{2,0}Q^{0,2} - \beta_3 D^{0,2}Q^{2,0} + \beta_4 Q^{2,0}Q^{0,2} = 0, \]

(3.28)

where we use the capital \( Q \) for the involved superfields in order to distinguish this case from the previous one. Further we separately consider the option when at least one of two free parameters in (3.27) equals zero, and the option when they both are non-vanishing. Heavily exploiting the commutativity condition (3.3), one can show that in the first case the only possibility is the set of linear constraints (3.6), (3.10). In the second case, up to non-zero rescalings of the involved superfields, the set (3.27), (3.28) can be cast into the form

\[ D^{2,0}Q^{2,0} + Q^{2,0}Q^{2,0} = 0, \quad D^{0,2}Q^{0,2} + Q^{0,2}Q^{0,2} = 0 \]

(3.29)

\[ D^{2,0}Q^{0,2} - D^{0,2}Q^{2,0} = 0 \]

(3.30)

(notice the surprising fact that the nonlinear term in (3.28) proves to be non-compatible with the self-consistency condition (3.3)).

What concerns the most general form of the invariant action, the constraints (3.29), (3.30) turn out to be more restrictive than the linear ones (3.6), (3.10), though not so severe as their \( SU(2) \) HSS prototype (2.10). Though the superfield lagrangian could involve the harmonic derivative \( D^{2,0}Q^{0,2} \) (or \( D^{0,2}Q^{2,0} \)), it is easy to prove that, like in the previous case of linear constraints, all the derivatives can be removed from the action by integrating by parts and repeatedly exploiting (3.29), (3.30). Then, inspecting the structure of the lagrangian as a function of \( Q^{2,0}, Q^{0,2} \) and explicit harmonics and, once again,

\[ ^5 \text{Generally speaking, the invariance under (3.26) implies putting a full harmonic derivative } D^{2,0}Q^{0,2} + D^{0,2}Q^{2,0} \text{ in the r.h.s. of (3.27), where the functions } \Lambda_{\dot{a}} \text{ a priori can bear an arbitrary dependence on } q^{1,1}, u, v. \text{ However, it is easy to show that, up to full harmonic derivatives, the general solution to such a modified condition is} \]

\[ L^{2,2} = L^{2,2} + u^{-1}_{\alpha} v_{\dot{\alpha}} q^{1,1 \alpha \dot{\alpha}} \left( D^{2,0}Q^{0,2} + D^{0,2}Q^{2,0} \right) \]

where the quantities with \( \sim \) satisfy the condition (3.25) on their own. Then, integrating by parts, one brings \( L^{2,2} \) into the form

\[ L^{2,2} = L^{2,2} - u^1_{\alpha} v^{-1}_{\dot{\alpha}} q^{1,1 \alpha \dot{\alpha}} \Lambda^{2,0} - u^{-1}_{\alpha} v_{\dot{\alpha}} q^{1,1 \alpha \dot{\alpha}} \Lambda^{2,0} \]

in which it satisfies (3.25). Thus, without loss of generality, one can choose as the invariance condition just eq. (3.25).
making use of the defining constraints, one can show that the most general superfield lagrangian in the present case is reducible to the form

$$ L_{\text{nonl}}^{2,2}(Q^{2,0}, Q^{0,2}, u, v) = Q^{2,0} C^{0,2}(u, v) + \sum_{n=1}^{\infty} (Q^{2,0} Q^{0,2})^n C^{-2(n-1),-2(n-1)}(u, v). \quad (3.31) $$

The coefficient functions in (3.31) can involve an arbitrary dependence on harmonics (it should be of course compatible with their harmonic $U(1)$ charges). Note that the first term in (3.31) is non-vanishing only provided the coefficient $C^{0,2}$ reveals a non-trivial dependence on both sets of harmonic variables.

Let us return to examining the constraints (3.29), (3.30). The last of them is linear, therefore in the present case one bosonic physical field is supplied by the vector $V_{\pm \pm}$ still subjected to the linear constraint (3.1). So the main characteristic feature of the $SU(2)$ nonlinear multiplet, the nonlinear constraint (2.13), does not generalize to the $SU(2) \times SU(2)$ case. Then one may suspect that the nonlinearity in the first two superfield constraints (3.29) is also fake. This is indeed so, and now we wish to show that there exists a change of the superfield variables which brings (3.29), (3.30) into the linear form (3.6), (3.10).

This can be done in two equivalent ways. On can, e.g., firstly solve the constraint (3.30) similarly to the previously discussed linear case

$$ Q^{2,0} = D^{2,0} \hat{Q}, \quad Q^{0,2} = D^{0,2} \hat{Q}, \quad (3.32) $$

and, redefining $\hat{Q}$ as

$$ \hat{Q} = \ln(1 + \hat{q}) , \quad (3.33) $$

reduce the remaining constraints to the form of eqs. (3.13)

$$ (D^{2,0})^2 \tilde{q} = (D^{0,2})^2 \tilde{q} = 0. \quad (3.34) $$

After this one could proceed like in the discussion of the meaning of (3.13): embed (3.34) into the linear set of constraints for four twisted superfields $\tilde{q}^{1,1 \alpha \dot{\alpha}}$ and thus demonstrate that in the given case we again deal with a particular class of their self-interactions.

Another way is to embed (3.29), (3.30), before solving them, into some extended set of nonlinear constraints with the superfield content $Q, Q^{2,0}, Q^{0,2}, Q^{2,2}$ characteristic of the projected form of some $Q^{1,1 \alpha \dot{\alpha}}$. Consistency with the commutativity condition (3.3) dictates the following unique form of such an extension (up to unessential constant rescalings)

$$ (a) \ D^{2,0} Q - Q^{2,0}(1 - Q) = 0 , \quad (b) \ D^{2,0} Q^{2,0} + Q^{2,0} Q^{2,0} = 0 , $$
$$ (c) \ D^{2,0} Q^{0,2} - Q^{0,2}(1 - Q) + Q^{2,0} Q^{0,2} = 0 , \quad (d) \ D^{2,0} Q^{2,2} = 0 , \quad (3.35) $$

$$ (a) \ D^{0,2} Q - Q^{0,2}(1 - Q) = 0 , \quad (b) \ D^{0,2} Q^{0,2} + Q^{0,2} Q^{0,2} = 0 , $$
$$ (c) \ D^{0,2} Q^{2,0} - Q^{2,0}(1 - Q) + Q^{2,0} Q^{0,2} = 0 , \quad (d) \ D^{0,2} Q^{2,2} = 0 . \quad (3.36) $$

Like in the system (3.20), (3.21), some of these equations, namely, (a) and (c) in both sets, are algebraic and serve to express $Q^{2,0}, Q^{0,2}, Q^{2,2}$ in terms of $Q$. Also, the constraints for
$Q^{2,2}$ are a consequence of the remainder. The constraint (3.30) is satisfied automatically as a consequence of eqs. (c). All this is visualized by passing to the new superfields (as before we assume that this change of superfield variables is invertible)

$$q = \frac{Q}{1 - Q}, \quad q^{2,0} = \frac{Q^{2,0}}{1 - Q}, \quad q^{0,2} = \frac{Q^{0,2}}{1 - Q}, \quad q^{2,2} = Q^{2,2}. \quad (3.37)$$

In terms of them eqs. (3.35), (3.36) become precisely the linear constraints (3.20), (3.21). So, $q, q^{2,0}, q^{0,2}$ and $q^{2,2}$ can be unified according to formulas (3.18), (3.19) into the twisted superfield $q^{1,1\alpha\dot{\alpha}}$ with the linear constraint (3.4). It is easy to see that $\tilde{q}$ appearing in (3.33) coincides with

$$\tilde{q} = q, \quad \hat{Q} = \ln(1 + q). \quad (3.38)$$

Note that the set of equations (3.35), (3.36) can be rewritten as a nonlinear version of the constraints (3.4) for four twisted superfields $Q^{1,1\alpha\dot{\alpha}}$ composed of $Q, Q^{2,0}, Q^{0,2}, Q^{2,2}$ according to eqs. (3.18), (3.19). The possibility to bring these nonlinear constraints into the linear form (3.4) by passing to $q^{1,1\alpha\dot{\alpha}}$ which is composed in the same way from the superfields (3.37), reflects the fact that any nonlinear modification of (3.4) is reducible to the original linear form by means of some redefinition of the superfields $q^{1,1\alpha\dot{\alpha}}$. The proof is based on the consistency conditions following from (3.3).

It is instructive to see how the lagrangian (3.31) looks in new variables

$$L_{nonl}^{2,2} = \frac{q^{2,0}}{1 + q} C^{0,2} + \sum_{n=1}^{\infty} (1 + q)^{-2n} (q^{2,0} q^{0,2})^n C^{-2(n-1),-2(n-1)}. \quad (3.39)$$

Among all possible actions of $q, q^{2,0}, q^{0,2}$ it is distinguished in having the scaling isometry

$$\delta q = \alpha(1 + q), \quad \delta q^{2,0} = \alpha q^{2,0}, \quad \delta q^{0,2} = \alpha q^{0,2}, \quad \delta q^{2,2} = \alpha q^{2,2}, \quad (3.40)$$

which, in the original variables, affects only $Q$ and $Q^{2,2}$

$$\delta Q = \alpha(1 - Q), \quad \delta Q^{2,2} = \alpha Q^{2,2}, \quad (3.41)$$

leaving $Q^{2,0}, Q^{0,2}$ intact. It is easy to check the covariance of (3.33), (3.36) or (3.20), (3.21) under these transformations. Note that this isometry is different from the pure shifting one which is inherent to the action (3.13). In the language of the superfield $q^{1,1\alpha\dot{\alpha}}$ it is represented as

$$\delta q^{1,1\alpha\dot{\alpha}} = \alpha (q^{1,1\alpha\dot{\alpha}} + u^{1,\alpha} v^{1,\dot{\alpha}}), \quad (3.42)$$

that is to be compared with eq. (3.20). On the superfield $\hat{Q} = \ln(1 + q)$ it is realized by pure translations, however $\hat{Q}$ satisfies nonlinear constraints. On the other hand, the set of linear constraints (3.20), (3.21) reveals invariance under shifts

$$q \rightarrow q + \alpha', \quad \alpha' = const,$$

however the original nonlinear set of constraints (3.29), (3.30) is not closed under such transformations ($Q^{2,0}, Q^{0,2}$ transform through a superfield $Q$ which is not explicitly present in (3.29), (3.30)). Correspondingly, the lagrangians (3.31), (3.39) do not respect this
second isometry. In other words, the linear and nonlinear \( SU(2) \times SU(2) \) multiplets are adapted for describing different subclasses of general self-interactions of \( q^{1,1,0\dot{a}} \).

Note that the first term in (3.39), after substituting \( q^{2,0} = D^{2,0} q \) and integrating by parts, is reduced to

\[
- \ln(1 + q) \ D^{2,0} C^{0,2}.
\]

It produces a non-standard kinetic term for \( q \)

\[
\sim \left( D^{2,0} D^{0,2} Q D^{2,0} D^{0,2} q \right) C^{-2,-2} ; \quad C^{0,2} \equiv D^{2,0} (D^{0,2})^2 C^{-2,-2} .
\]

Its diagonalization implies a complicated redefinition of physical fields by the components of the \( SU(2) \) breaking tensor \( C^{-2,-2} = C^{(ib)(ab)} u_i^{-1} u_k^{-1} v_a^{-1} v_b^{-1} \). On the other hand, the second sequence of self-interactions in (3.39) contains the standard kinetic term. It comes from the first term in the sum

\[
L_{2(1)}^{2,2} = (1 + q)^{-2} q^{2,0} q^{0,2} C^{0,0} = -q^{2,0} q^{0,2} + \ldots .
\]  

\( 4 \) Discussion

Despite the fact that the constrained \( SU(2) \times SU(2) \) harmonic analytic superfields defined above are basically equivalent to four twisted (4,4) superfields, the use of these off-shell representations has some merits which we wish to briefly outline here.

One of these advantages is related to the use of \( SU(2) \times SU(2) \) harmonic analog of nonlinear multiplet. It turns out that 2D, \( N = 4 \) superconformal group admits a simple realization in terms of this superfield, and it becomes easy to construct the corresponding invariant action.

As was discussed in [13], in the \( SU(2) \times SU(2) \) analytic HSS one can realize two different “small” \( N = 4 \), \( SU(2) \) superconformal groups (in each light-cone sector), having as their closure the “large” \( N = 4 \), \( SO(4) \times U(1) \) superconformal group. One of these \( N = 4 \), \( SU(2) \) groups does not affect harmonic variables, the superfields \( q^{1,1} \) and any their harmonic projections behave as scalars with respect to it. The analytic superspace integration measure is also invariant. So all the actions considered above trivially enjoy invariance under this superconformal group. Another \( N = 4 \), \( SU(2) \) superconformal group affects the harmonic variables [24, 13]

\[
\delta u_i^1 = \Lambda^{2,0} u_i^{-1} , \quad \delta u_i^{-1} = 0 ; \quad \delta v_a^1 = \Lambda^{0,2} v_a^{-1} , \quad \delta v_a^{-1} = 0 ,
\]

\[
\delta D^{2,0} = - \Lambda^{2,0} D_u^0 , \quad \delta D^{0,2} = - \Lambda^{0,2} D_v^0 ,
\]

\[
D^{2,0} \Lambda^{2,0} = D^{0,2} \Lambda^{2,0} = D^{0,2} \Lambda^{0,2} = D^{2,0} \Lambda^{0,2} = 0 ,
\]

where \( D_u^0 \), \( D_v^0 \) are the left and right harmonic \( U(1) \) charge operators. It will be essential for us that the parameter superfunctions \( \Lambda^{2,0} (\Lambda^{0,2}) \) depend only on the coordinates \( z_{++}, \theta^{1,0} z, u_i^+ \) (\( z_{--}, \theta^{0,1} z, v_a^+ \)) and satisfy the harmonic constraints just written. The realization of this group on other analytic superspace coordinates besides the harmonic ones can be found in [13]. The analytic superspace integration measure is invariant in this case.
as well. Also we will need the fact that the superfield $q^{1,1}$ transforms with an analytic weight under this group

$$
\delta q^{1,1,\alpha} = (\Lambda_L + \Lambda_R)q^{1,1,\alpha}, \quad \Lambda^{2,0} \equiv D^{2,0}\Lambda_L, \quad \Lambda^{0,2} \equiv D^{0,2}\Lambda_R, \quad (4.4)
$$

$$
D^{0,2}\Lambda_L = D^{2,0}\Lambda_R = 0 \quad (4.5)
$$

(this transformation law unambiguously follows from requiring the covariance of the harmonic constraint $[3,4]$).

It is not so easy to construct the action of $q^{1,1}$ invariant under this second $N = 4$, $SU(2)$ superconformal group. For one $q^{1,1}$, as shown in [13], the unique invariant action is that of $N = 4$ $SU(2) \times U(1)$ WZNW sigma model

$$
S_{wznw} = \frac{1}{\kappa^2} \int \mu^{-2,-2}\dot{q}^{1,1,1}(\frac{\ln(1+X)}{X^2} - \frac{1}{(1+X)^2}), \quad (4.6)
$$

where

$$
\dot{q}^{1,1} \equiv q^{1,1} - c^{1,1}, \quad X \equiv e^{-c^{1,1}q^{1,1}}, \quad c^{\pm,\pm} = \epsilon^{ia}_i u^{\pm1}_a, \quad c^{\pm} = const, \quad c^{ia}c_{ia} = 2.
$$

Despite the presence of an extra quartet constant $c^{\alpha}$ in the analytic superfield lagrangian, the action $[1,6]$ does not depend on $c^{\alpha}$, as it is invariant under arbitrary rescalings and $SU(2) \times SU(2)$ rotations of this constant. The invariance of $[1,6]$ under the second $N = 4$ superconformal group which is realized on $\dot{q}^{1,1}$ as

$$
\delta \dot{q}^{1,1} = (\Lambda_L + \Lambda_R)(\dot{q}^{1,1} + c^{1,1}) - \Lambda^{2,0}\dot{c}^{-1,1} - \Lambda^{0,2}\dot{c}^{1,1} \quad (4.7)
$$

is not manifest. It is a tedious though straightforward exercise to check this invariance.

It turns out that for $q^{1,1,\alpha}$ one can construct the action almost manifestly invariant under this superconformal group. This action belongs to the subclass of actions $[3,31]$, $[3,39]$ associated with the $SU(2) \times SU(2)$ nonlinear multiplet" $Q^{2,0}, Q^{0,2}$ defined by the constraints $[3,29], [3,30]$. Using the transformation rules $[1,2]$ it is easy to check that these constraints are consistent with the following transformation properties of the involved superfields

$$
\delta Q^{2,0} = \Lambda^{2,0}, \quad \delta Q^{0,2} = \Lambda^{0,2}. \quad (4.8)
$$

Then the particular representative of the lagrangians $[3,31]$,

$$
L^{2,2}_{conf} = Q^{2,0}Q^{0,2}C = \frac{q^{2,0}_0q^{0,2}_0}{(1+q)^2}C = D^{2,0}\ln(1+q)D^{0,2}\ln(1+q)C, \quad C = const, \quad (4.9)
$$

is obviously shifted by a full harmonic derivative under $[4,8]$ as a consequence of the structure of $\Lambda^{2,0}, \Lambda^{0,2}$ $[1,4]$ and the harmonic constraints $[4,3], [3,30]$. Ascribing to the superfields defined by eqs. $[3,37]$ the following superconformal transformation properties

$$
\delta q^{2,0} = (\Lambda_L + \Lambda_R)q^{2,0} + \Lambda^{2,0}(1+q), \quad \delta q^{0,2} = (\Lambda_L + \Lambda_R)q^{0,2} + \Lambda^{0,2}(1+q),
$$

$$
\delta q^{} = (\Lambda_L + \Lambda_R)(1+q), \quad \delta q^{2,2} = (\Lambda_L + \Lambda_R)q^{2,2} + \Lambda^{2,0}q^{0,2} + \Lambda^{0,2}q^{2,0}, \quad (4.10)
$$

we see that they are consistent with $[4,8]$ and the standard transformation properties of $q^{1,1,\alpha}$

$$
\delta q^{1,1,\alpha} = (\Lambda_L + \Lambda_R)q^{1,1,\alpha}, \quad (4.11)
$$
provided the following identification has been made
\[ q^{1,1\alpha\dot{\alpha}}u_{\alpha}^{1}v_{\dot{\alpha}}^{-1} \equiv 1 + q, \quad q^{1,1\alpha\dot{\alpha}}u_{\alpha}^{1}v_{\dot{\alpha}}^{-1} \equiv q^{2,0}, \quad q^{1,1\alpha\dot{\alpha}}u_{\alpha}^{1}v_{\dot{\alpha}}^{-1} \equiv q^{0,2}, \quad q^{1,1\alpha\dot{\alpha}}u_{\alpha}^{1}v_{\dot{\alpha}}^{-1} \equiv q^{2,2}. \] (4.12)

The transformation properties of the additional superfields \( Q \) and \( Q^{2,2} \) entering the extended set of nonlinear constraints (3.35), (3.36) can be deduced directly from the relations (3.37). It is straightforward to check the covariance of this set (as well as of its linear counterpart for the superfields \( q, q^{2,0}, ... \)) under the superconformal transformations. It would be interesting to examine in detail the component content of the action (4.3). Note that the above transformation properties can be easily extended to the full conformal \((4,4)\) supergravity in \( SU(2) \times SU(2) \) harmonic superspace \([23]\). Then the multiplet \( Q^{2,0}, Q^{0,2} \) can serve as a compensator reducing this supergravity to a kind of off-shell Einstein \((4,4)\) one.

Our next comment concerns the dual formulations of the superfield actions presented. As was shown in \([13]\), the twisted multiplet constraints (3.4) can be implemented in the action with the help of appropriate analytic superfield lagrange multipliers to yield a new off-shell representation of the \( q^{1,1} \) action in terms of unconstrained analytic superfields with an infinite number of auxiliary fields. For the case of \( q^{1,1\alpha\dot{\alpha}} \), the action with the lagrange multipliers terms added reads
\[ L_{\omega,q}^{2,2} = \omega^{1,1\alpha\dot{\alpha}}D^{2,0}q_{\alpha\dot{\alpha}}^{1,1} + \omega^{1,1\alpha\dot{\alpha}}D^{0,2}q_{\alpha\dot{\alpha}}^{1,1} + L^{2,2}(q^{1,1}, u, v). \] (4.13)

Eliminating \( q^{1,1\alpha\dot{\alpha}} \) by their equations of motion at expense of \( \omega \) superfields, one expresses the action in terms of the latter and so gets another off-shell representation of this action. The basic feature of this type of duality transformation is the gauge invariance
\[ \delta\omega^{1,1\alpha\dot{\alpha}} = D^{2,0}\sigma^{-1,1\alpha\dot{\alpha}}, \quad \delta\omega^{-1,1\alpha\dot{\alpha}} = -D^{0,2}\sigma^{-1,1\alpha\dot{\alpha}}, \] (4.14)
with \( \sigma^{-1,1\alpha\dot{\alpha}} \) being unconstrained analytic superfield parameters. It serves to reduce the set of physical fields in the \( \omega \) superfields just to the on-shell content of twisted multiplet, thus ensuring the on-shell equivalence of the original and dual formulations of the latter.

This kind of duality is crucially different from the duality associated with tensor multiplet discussed in the context of \( SU(2) \) HSS in Sect. 2. Due to the presence of constrained 2D vector in the superfield \( L^{(++)} \), one of the bosonic on-shell degrees of freedom in this case is represented in essentially different ways in the initial and dual formulations. As the result, the relevant duality transformation not only changes the off-shell content of the theory, but also affects the on-shell structure of the action, yielding sigma model with a different target space geometry, in accord with the general concept of abelian T-duality in 2D sigma models \([14, 24]\). The presence of abelian isometry in both the \( L^{(++)} \) action and its dual is most essential for the related duality to fall in this general class.

On the contrary, the duality associated with the \( q^{1,1} \) action is not “genuine” in the sense that it merely changes the off-shell structure of the action. After fixing an appropriate gauge with respect to (4.14) and eliminating an infinite tail of auxiliary fields, the \( \omega \) action gives rise to the precisely same component on-shell sigma model action as the original constrained \( q^{1,1} \) action. Such a duality exists irrespective of whether the \( q^{1,1} \) action possesses any isometry. The obvious reason why the standard T-duality machinery
does not apply to this case is the absence of constrained vectors in the superfield $q^{1,1}$ subjected to constraints (3.4).

The superfield systems which we discussed in Sect. 3 correspond to particular classes of $q^{1,1}$ actions with abelian translational isometries. This matches with the fact that the sets of constraints (3.4), (3.10) and (3.29), (3.30) by which we originally defined these superfields imply the existence of constrained vectors among their irreducible components. Thus we can expect the existence of the “genuine” duality transformation in these cases, along with the standard $q^{1,1}$ duality related to treating (3.4), (3.10) or (3.29), (3.30) as a subclass of twisted multiplet constraints (3.4). In other words, when working with the harmonic projections of $q^{1,1\dot{\alpha}}$, we are at freedom either to include into the action the whole set of constraints, and this amounts to the standard $q^{1,1}$ duality, or to implement with lagrange multipliers only part of them, deducing the remainder by solving these few basic constraints. Both procedures preserve manifest (4.4) supersymmetry, however lead to essentially different dual actions.

Let us apply to the system associated with the action (3.13) and constraints (3.4), (3.10). It is the simplest one because the relevant $U(1)$ isometry is realized in this case as a pure shift of the superfield $q$. We could implement the extended set of constraints (3.20), (3.21) in the action with lagrange multipliers; what we would obtain in this case is the same lagrange multiplier term as in (4.13) but written in terms of the bi-harmonic projections of $q^{1,1\dot{\alpha}}$. On the other hand, we can do the same trick with the original set of constraints (3.6), (3.10) without explicitly solving (3.10) through the superfield projections of $q$. Thus we can expect the existence of the “genuine” duality transformation in these cases, subjected to constraints (3.4).

A subclass of twisted multiplet constraints (3.4). In other words, when working with the standard basic constraints. Both procedures preserve manifest (4.4) supersymmetry, however lead to essentially different dual actions.

Let us apply to the system associated with the action (3.13) and constraints (3.4), (3.10). It is the simplest one because the relevant $U(1)$ isometry is realized in this case as a pure shift of the superfield $q$. We could implement the extended set of constraints (3.20), (3.21) in the action with lagrange multipliers; what we would obtain in this case is the same lagrange multiplier term as in (4.13) but written in terms of the bi-harmonic projections of $q^{1,1\dot{\alpha}}$. On the other hand, we can do the same trick with the original set of constraints (3.6), (3.10) without explicitly solving (3.10) through the superfield $q$. We get in this way the following superfield action

$$S = \int \mu^{-2,-2} \left[ L^{2,2}(q^{2,0}, q^{0,2}, u, v) + \omega^{-2,-2} D^{2,0} q^{2,0} + \omega^{2,-2} D^{0,2} q^{0,2} \right.$$

$$\left. + \omega(D^{2,0} q^{0,2} - D^{0,2} q^{2,0}) \right]. \quad (4.15)$$

Varying with respect to $\omega$’s gives the set (3.6), (3.10) which, after solving eq. (3.10), leads to the set (3.13) or the equivalent one (3.21). As the result, we end up with a particular $U(1)$ invariant class of the $q^{1,1\dot{\alpha}}$ actions. On the other hand, varying with respect to $q^{2,0}$, $q^{0,2}$ which are now unconstrained, we get

$$q^{2,0} = -D^{2,0} \omega - D^{0,2} \omega^{2,-2} + ... , \quad q^{0,2} = D^{0,2} \omega - D^{2,0} \omega^{2,-2} + ... , \quad (4.16)$$

where dots stand for the terms of higher order in superfields. After substituting this back into the action (4.15) one obtains the dual of (3.13) in terms of unconstrained analytic superfields $\omega, \omega^{2,-2}, \omega^{-2,2}$. The number of physical fields in both formulations can be checked to coincide due to the invariance of the action (4.13) and expressions (4.16) under the gauge transformations (cf. (4.14))

$$\delta \omega = -D^{2,0} D^{0,2} \sigma^{-2,-2} , \quad \delta \omega^{2,-2} = (D^{2,0})^2 \sigma^{-2,-2} , \quad \delta \omega^{-2,2} = -(D^{0,2})^2 \sigma^{-2,-2} , \quad (4.17)$$

with $\sigma^{-2,-2}$ being an unconstrained analytic superfield parameter. However, one isoscalar bosonic degree of freedom is represented in different ways in both formulations: in the original setting by the vector $V_{\pm \pm}$ subjected to the constraint (3.4) which can be solved through a field $q(x)$, and in the dual formulation by the first component $\omega_0(x)$ of the superfield $\omega$

$$\omega(\zeta, u, v) = \omega_0(x) + ... . \quad (4.18)$$
Thus we are facing the situation quite similar to the interplay between the original constrained and dual formulations of the \((4,4)\), 2D tensor multiplet action in the \(SU(2)\) HSS.

To give a feeling of the basic geometric features of this interplay in the present case, let us briefly describe some component results concerning the structure of the bosonic part of the action (4.13). After eliminating an infinite tower of auxiliary fields coming from \(q^{2,0}, q^{0,2}\) and the lagrange multipliers (with fixing an appropriate WZ gauge with respect to the gauge freedom (4.17)), this part can be written entirely in terms of the fields \(q^{(ik)}(x), q^{(ab)}(x), q^{(ik)(ab)}(x)\) defined in eq. (3.12), the 2D vector field \(V_{\pm\pm}(x)\) and the scalar field \(\omega_0(x)\)

\[
\tilde{S}_{bos} = \int d^2x \left[ i\omega_0 (\partial_{++} V_- - \partial_- V_{++}) + \int [dudv] L_{bos}(q^{(ik)}, q^{(ab)}, q^{(ik)(ab)}, V_{\pm\pm}, u, v) \right]. \tag{4.19}
\]

For the time being we do not explicitly specify the lagrangian \(L_{bos}\). It involves harmonic integrals of the three functions

\[
A^{-2,2} = \frac{\partial^2 L^{2,2}}{\partial q^{2,0} \partial q^{2,0}}, \quad A^{2,-2} = \frac{\partial^2 L^{2,2}}{\partial q^{0,2} \partial q^{0,2}}, \quad A = -\frac{\partial^2 L^{2,2}}{\partial q^{0,2} \partial q^{0,2}} \tag{4.20}
\]

multiplied by appropriate monomials of the harmonics \(u, v\) (hereafter, the symbol \(|\) means restricting to the \(\theta\) independent parts of the superfields and, for \(q^{2,0}, q^{0,2}\), also keeping only the physical parts (3.12)). Note that we can choose the WZ gauge so that

\[
\omega| = \omega_0(x) + \hat{\omega}_1(x, u) + \hat{\omega}_2(x, v), \tag{4.21}
\]

where the objects with “hat” start with monomials \(u^1_k^1 u^1_1, v^1_k^1 v^1_1\), respectively. Then the non-dynamical harmonic equations of motion

\[
\frac{\partial L^{2,2}}{\partial q^{2,0}} - \partial^{2,0} \omega^{-2,2} + \partial^{0,2} \omega_2 = 0, \quad \frac{\partial L^{2,2}}{\partial q^{0,2}} - \partial^{0,2} \omega^{2,-2} - \partial^{2,0} \omega_1 = 0, \tag{4.22}
\]

express the \(\theta\) independent parts of the lagrange multipliers through the fields \(q^{(ik)}(x), q^{(ab)}(x), q^{(ik)(ab)}(x)\) and simultaneously establish equivalence relations between these fields and the appropriate lowest isospin components in the bi-harmonic expansions of \(\omega^{-2,2}|, \omega^{2,-2}|\) and \(\omega|\):

\[
\hat{\omega}_1(x, u) = -q^{(ik)}(x) u^1_i u^{-1}_k + \ldots, \quad \hat{\omega}_2(x, v) = q^{(ab)}(x) v^1_a v^{-1}_b + \ldots,
\]

\[
\omega^{-2,2}(x, u, v) = -\frac{1}{2} q^{(ik)(ab)}(x) u^1_i u^1_k v^{-1}_a v^{-1}_b + \ldots,
\]

\[
\omega^{2,-2}(x, u, v) = -\frac{1}{2} q^{(ik)(ab)}(x) u^{-1}_i u^{-1}_k v^1_a v^1_b + \ldots. \tag{4.23}
\]

These relations justify the choice of the same fields \(q^{(ik)}(x), q^{(ab)}(x), q^{(ik)(ab)}(x)\) to represent the appropriate physical bosonic degrees of freedom in the original and dual formulations.

The bosonic on-shell action in terms of 16 fields \(q(x), q^{(ik)}(x), q^{(ab)}(x), q^{(ik)(ab)}(x)\) corresponding to the initial formulation comes out if one varies with respect to \(\omega_0(x)\)
in (4.19) and substitutes back the purely gradient solution (3.12) for $V_{\pm \pm}$. On the other hand, varying with respect to $V_{\pm \pm}$ yields the dual action with the 16th scalar represented by the Lagrange multiplier field $\omega_0$. So the on-shell geometry in both formulations is not the same; the relevant geometric quantities, viz. the components of the metric and torsion potential, are interrelated according to the general abelian $T$-duality formulas [17, 26].

In order to precisely see how this occurs, let us consider a simplified situation when all the bosonic fields except for $V_{\pm \pm}(x), q^{(ab)}(x), \omega_0(x)$ are put equal to zero

$$q^{(ik)} = q^{(ik)(ab)} = 0$$

and the functions defined in (4.20) are assumed to bear no dependence on these fields. In this reduced case the second term in the integrand in (4.19) is expressed through one function

$$A = A(q^{(ab)} v^1_a v^1_b, u, v) = 1 + \ldots$$

and is given by the following expression

$$\int [dudv] L_{bos} = -A \left( V_{++} V_{--} - \frac{1}{2} \partial_{++} q^{(ab)} \partial_{--} q^{(ab)} \right)$$

$$-i A_{(ab)} \left( V_{--} \partial_{++} q^{(ab)} - V_{++} \partial_{--} q^{(ab)} \right),$$

with

$$A \equiv \int [dudv] A, A_{(ab)} \equiv \int [dudv] A v^1_a v^1_b.$$

After varying with respect to $\omega_0$ and solving the resulting constraint as in eq. (3.12), one gets the sigma model action with torsion

$$\tilde{S}_{bos}^{red} = \int d^2x \left[ G_{00} \left( \partial_{++} q_0 \partial_{--} q_0 + \frac{1}{2} \partial_{++} q^{(ab)} \partial_{--} q^{(ab)} \right) ight.$$

$$+ B_{0(ab)} \left( \partial_{++} q_0 \partial_{--} q^{(ab)} - \partial_{--} q_0 \partial_{++} q^{(ab)} \right) \right],$$

$$G_{00} = A, B_{0(ab)} = -A_{ab}. \quad (4.29)$$

Varying with respect to $V_{\pm \pm}$ and substituting the result into the action entirely eliminate the torsion term from the latter, and the dual form of (4.28) proves to be as follows

$$\tilde{S}_{dual}^{red} = \int d^2x \left[ (A)^{-1} \left( \partial_{++} \omega_0 + \partial_{++} q^{(ab)} A_{(ab)} \right) \left( \partial_{--} \omega_0 + \partial_{--} q^{(ab)} A_{(ab)} \right) ight.$$

$$+ \frac{1}{2} A \partial_{++} q^{(ab)} \partial_{--} q^{(ab)} \right].$$

What we have got is the general hyper-Kähler 4 dimensional sigma model with one translational isometry. Indeed, in accord with the general parametrization of such metrics [23], the function $A$ by construction satisfies the Laplace’s equation

$$\partial^{(ab)} \partial_{(ab)} A = 0, \quad \partial_{(ab)} \equiv \frac{\partial}{\partial q^{(ab)}}, \quad (4.31)$$

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and is none other than the twistor transform for the general solution of the latter, while \( \mathcal{A}_{(ab)} \) is related to \( \mathcal{A} \) by the well-known equation

\[
\partial_{(ab)} \mathcal{A}_{(cd)} - \partial_{(cd)} \mathcal{A}_{(ab)} = \frac{1}{2} \left( \partial_{(ca)} \mathcal{A} \, \epsilon_{db} + \partial_{(db)} \mathcal{A} \, \epsilon_{ca} \right) .
\] (4.32)

The relation between the quantities entering the torsionless hyper-Kähler sigma model action (4.30) and those present in the action (4.28) is given by the standard T duality relations. In the same way, another reduction

\[
q^{(ab)} = q^{(ik)(ab)} = 0
\] (4.33)

also yields a general 4 dimensional hyper-Kähler manifold with one translational isometry.

Thus we have proven that the general action of four twisted superfields \( q^{1,1 \alpha \dot{\alpha}} \) with one purely translational isometry leads in the bosonic sector to the torsionful sigma model, such that the relevant 16 dimensional bosonic manifold admits two reductions to the 4 dimensional hyper-Kähler submanifolds with the same isometry. The hyper-Kähler nature of these submanifolds is visualized by the duality transformation which can be formulated in a manifestly \((4, 4)\) supersymmetric way, based upon the “master action” (4.15). Note that in the case of one twisted supermultiplet no reduction to hyper-Kähler manifolds exists. Let us also point out that the above duality transformation certainly does not eliminate all the torsion terms from the original non-reduced sigma model action, it is impossible to remove the components of the torsion potential \( B_{(ab)(ik)(cd)} \) and others. This demonstrates that the torsion is intrinsically inherent to the \((4, 4)\) models we consider, in contrast, e.g., to the “fake” torsion in the reduced action (4.28).

Our last remark will be on another, somewhat puzzling relation of the multiplets \( q^{2,0}, q^{0,2} \) and \( Q^{2,0}, Q^{0,2} \) to a single twisted superfield \( q^{1,1} \).

After some algebra which involves integrating by parts with respect to harmonics and making use of the relation

\[
c_{1,1} c_{-1,1} - c_{1,-1} c_{-1,1} = 1,
\]

one can cast the \( N = 4, SU(2) \times U(1) \) WZNW action (4.3) into the following suggestive form

\[
S_{wznw} = \frac{1}{\kappa^2} \int \mu^{-2,-2} \left( \tilde{q}^{1,1} \right)^2 \frac{c_{-1,1} c_{1,1}}{(1 + X)^2}
\]

\[
= \frac{1}{\kappa^2} \int \mu^{-2,-2} D^{2,0} \ln(1 + X) D^{0,2} \ln(1 + X) .
\] (4.34)

In this form the action looks literally as (4.9) with the following identifications

\[
\tilde{Q}^{2,0} = \frac{c_{1,-1} \tilde{q}^{1,1}}{(1 + X)} , \quad \tilde{Q}^{0,2} = \frac{c_{1,-1} \tilde{q}^{1,1}}{(1 + X)} , \quad \tilde{Q} = \frac{X}{1 + X}
\] (4.35)

\[
q^{2,0} = c_{1,-1} q^{1,1} , \quad q^{0,2} = c_{1,-1} q^{1,1} , \quad \bar{q} = X .
\] (4.36)

It is a simple exercise to check that these objects obey, respectively, the sets of constraints (3.29), (3.30) and (3.6), (3.10) as a consequence of the \( q^{1,1} \) constraint (3.4) and the relations

\[
D^{2,0} c_{-1,1} = c_{1,1} , \quad D^{0,2} c_{1,-1} = c_{1,1} .
\]
Then the relations \((3.35), (3.36)\) are precisely eqs. \((3.37)\).

This correspondence between \(\hat{q}^{1,1}\) and the \(Q\) and \(\bar{q}\) superfields can be understood as follows.

We will specialize to the \(q^{2,0}, q^{0,2} \leftrightarrow q^{1,1}\) correspondence, because the existence of an analogous one between \(q^{1,1}\) and the \(\dot{Q}\) superfields automatically follows from the map \((3.37)\). Let us consider the following extension of the constraints \((3.6), (3.10)\)

\[
D^{2,0} q^{2,0} = 0, \quad D^{0,2} q^{0,2} = 0, \quad D^{2,0} \hat{q}^{0,2} - D^{0,2} q^{2,0} = 0, \quad (4.37)
\]

\[
c^{1,-1} q^{0,2} - c^{-1,1} q^{2,0} = 0, \quad (4.38)
\]

where the charged constants \(c^{\pm 1, \mp 1}\) are of the same type as above. Defining

\[
\hat{q}^{1,1} \equiv c^{-1,-1} D^{2,0} q^{0,2} - c^{-1,1} q^{2,0}, \quad (4.39)
\]

it is easy to check that this object satisfies the constraints \((3.4)\),

\[
D^{2,0} \hat{q}^{1,1} = D^{0,2} \hat{q}^{1,1} = 0
\]

as a consequence of \((4.37), (4.38)\). Also, one can check that \(q^{2,0}\) and \(q^{0,2}\) satisfying \((4.37), (4.38)\) are expressed through \(q^{1,1}\) \((4.39)\) by the relations \((4.36)\). Thus we have proven one-to-one correspondence between the superfields \(q^{2,0}, q^{0,2}\) subjected to the constraints \((4.37), (4.38)\) and the twisted superfield \(q^{1,1}\). Quite analogously, we reveal one-to-one correspondence between \(q^{1,1}\) and the superfields \(Q^{2,0}, Q^{0,2}\) subjected to nonlinear constraints \((3.29), (3.30)\) and the same additional constraint \((4.38)\)

\[
c^{1,-1} Q^{0,2} - c^{-1,1} Q^{2,0} = 0. \quad (4.40)
\]

In this case \(\hat{q}^{1,1}\) is defined by

\[
\hat{q}^{1,1} = \frac{A^{1,1}}{1 - c^{-1,-1} A^{1,1}}, \quad A^{1,1} = c^{-1,-1} \left( D^{2,0} Q^{0,2} + Q^{2,0} Q^{0,2} \right) - c^{-1,1} Q^{2,0}. \quad (4.41)
\]

Thus we have found a new description of \(q^{1,1}\) in terms of the superfields \(q^{2,0}, q^{0,2}\) or \(Q^{2,0}, Q^{0,2}\) with one additional algebraic constraint \((4.38), (4.40)\). Note that the latter is not covariant with respect to the simple realization of \(N = 4\) superconformal group given by the transformation laws \((4.8), (4.10)\). At the same time, the extended set of constraints \((4.37), (4.38)\) and its \(Q\) counterpart are covariant under a more complicated realization of this group which is induced on \(Q^{2,0}, Q^{0,2}\) and \(q^{2,0}, q^{0,2}\) by the transformation law \((4.7)\) of \(\hat{q}^{1,1}\) through the correspondence \((4.35), (4.36)\). The interplay between these two realizations of \(N = 4\) superconformal group remains to be understood.

## 5 Conclusion

In this paper we have constructed the \(SU(2) \times SU(2)\) HSS analogs of the standard tensor and nonlinear off-shell \((4, 4)\) multiplets. These new \((4, 4)\) multiplets are represented by the properly constrained \(SU(2) \times SU(2)\) analytic harmonic superfields \(q^{2,0}, q^{0,2}\) and \(Q^{2,0}, Q^{0,2}\), comprise \((32 + 32)\) component fields (one of their 16 physical bosonic fields is supplied
by a constrained vector) and yield (4, 4) sigma models with torsion. The relevant actions
can be equivalently given in terms of four twisted (4, 4) superfields $q^{1,1\alpha\dot{\alpha}}$ and constitute
particular classes in the general variety of actions of the latter. Their distinguishing feature is the presence of abelian translational isometries. The description of these actions in terms of the superfields introduced here makes the isometries manifest and allows to
construct, in the manifestly (4, 4) supersymmetric way, the dual actions the bosonic sectors of which are related to those of the original actions via the familiar abelian $T$ duality. The dual formulation allows to reveal two non-trivial reductions of the relevant bosonic manifold which yield most general 4-dimensional hyper-Kähler manifolds with one translational isometry. We also presented a new sigma model action possessing the $N = 4$, $SU(2)$ superconformal symmetry non-trivially realized on harmonic variables. For the off-shell superfield action of $N = 4 SU(2) \times U(1)$ WZNW sigma model we found a new representation in terms of the superfields $q^{2,0}, q^{0,2}$ or $Q^{2,0}, Q^{0,2}$ on which one additional algebraic constraint is imposed.

It would be interesting to construct analogous torsionful generalizations of some other
off-shell (4, 4) multiplets with finite number of auxiliary fields, e.g., of the relaxed tensor multiplet [27], and to find out possible stringy applications of all such generalized multiplets.

Finally, let us recall that the models considered here admit a formulation in terms of twisted superfields $q^{1,1}$ and so belong to the particular class of torsionful (4, 4) sigma models possessing mutually commuting left and right complex structures on the bosonic target [28]. It was argued in [14, 15] that the true analog of the “ultimate” $q^{(+)}$ hypermultiplet in the case with torsion is the $SU(2) \times SU(2)$ analytic superfield triple consisting of $q^{1,1}$ and the lagrange multipliers $\omega^{1,-1}, \omega^{-1,1}$. One can expect that an off-shell formulation of general (4, 4) sigma models with torsion can be achieved using this triple. A generalization of the dual $q^{1,1}$ lagrangian (4.13) was constructed, such that it does not admit a formulation solely in terms of $q^{1,1}$ and corresponds to a more general case with non-commuting complex structures. It would be of interest to generalize another dual action (4.15) along similar lines and to inquire the relevant target space geometry.

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