A study is presented to observe the effect of residual stress on waves in an incompressible, hyper-elastic, thick and hollow cylinder of infinite length. The problem is based on the non-linear theory of infinitesimal deformations occurring after a finite deformation. A prototype model of strain energy function is used which adequately includes the effects of residual stress and deformation. The expressions for internal pressure and the axial load are calculated and graphical illustrations are presented. Analysis of infinitesimal wave propagation is carried for the axisymmetric case in the considered cylinder. Numerical solution is obtained in the undeformed configuration and analyzed for the two-point boundary-value problem. Dispersion curves are plotted for varying choice of parameters.

1. Introduction. The study of waves in solids is motivated by numerous fields such as (but not limited to) ultrasonics, borehole testing, medicine and technology, nondestructive testing of materials, composite materials etc., to name a few. A solid material may not be completely free of stresses in its reference state. The manufacture and construction of a material may introduce stresses that remain locked inside the material after when the source of producing stress is removed. This stress is now termed in literature as an initial stress. This initial stress is used along with the linear theory of elasticity by [5, 6] who explored the propagation of elastic waves and presented specialized results in geographical context. Here, we use the term initial stress to represent the cases of both the prestress or the residual stress. A pre-stress is the stress produced in the elastic material due to a finite deformation which results from an applied load. An initial stress is a residual stress when it is the result of a process other than a finite deformation. For example, as the case of growth in tissues or tectonic events in rocks, etc. This concept of residual stress is credited to [10]. For further study on residual stress and development of basic constitutive equations for residually-stressed materials, we refer to [10, 11, 12, 13, 14, 15, 16]. It was found that the mechanical properties of elastic materials are inhomogeneous and anisotropic in the presence of inhomogeneous residual stress. The reader is also referred to [32] and the references therein. [17] followed the work in [10] and presented generalized results which relate to much earlier work of [6].

In this paper, a study on the propagation of small amplitude waves in a residually-stressed incompressible (hollow) cylinder is presented. The theoretical formulation
is followed from [28]. To illustrate various results numerically, a prototype strain energy function is used which is a function of initial stress and the finite deformation through their respective second rank tensors. The theory of small (static or time dependent) deformations, which are usually referred to as *incremental deformations*, superimposed on large or finite deformations is used to analyze the effect of the initial stress on the speed of wave in the material. The components of (the fourth order) elasticity tensor appearing in this theoretical formulation depends on the residual stress tensor components and the components of deformation gradient tensor (See [27]). The approach acquired by [5, 6, 32] is different from this more generalized approach since they used the linearized theory and a very specialized assumed form of the constitutive equation. Also, [17] assumed the initial stress to be small in magnitude so that the terms are linear in the initial stress and made use of a different form of elasticity tensor in calculations. More recently, the reader is referred to the work done by [3, 24, 33, 4].

Basic equations for a residually-stressed elastic material are presented and also the discussion on the elasticity tensor for residually-stressed materials is carried out. The focus is on development of the constitutive law for an initially stressed material that has no intrinsic material symmetry, i.e. its response relative to the undeformed configuration is considered isotropic in the absence of initial stress. The constitutive law of the material is based on a strain-energy function that depends on the combined invariants of the initial stress tensor and the right Cauchy–Green deformation tensor. For an incompressible material in the three-dimensional case, 9 such independent invariants exist. Expressions for the Cauchy stress and nominal stress tensors and the elasticity tensor are given in general forms for incompressible materials are given. These are then specialized for a strain energy model. In the later part of the paper, problem of wave propagation is analyzed for a hollow cylindrical tube with inhomogeneous residual stress. Numerical solution to the boundary value problem is presented and behavior of wave speed in the presence of residual stress is presented through graphical illustrations. Various results are specialized to match the results in the linear theory for isotropic materials.

2. Problem formulation: Constitutive laws and elasticity tensor in terms of invariants. In this section, the reader is generally referred to ([28], [29]) for details on the basics of the theory and various expressions involved in the problem formulation.

In this problem, an incompressible material with an initial stress is considered. Let the elastic response of this material is measured by the strain energy function \( P(C, \zeta) \). Here, \( C = \Lambda^T \Lambda \) is the usual right Cauchy-Green tensor and \( \zeta \) is the initial stress tensor and \( \Lambda \) is the deformation gradient tensor. Let \( \zeta_{ii}, (i = 1, 2, 3) \) and \( \zeta_{ij}, i \neq j, (i, j \in \{1, 2, 3\}) \) are the normal and shear components of the initial stress, respectively. Let \( \lambda_1, \lambda_2, \lambda_3 \) are the principal stretches corresponding the principal axes \( x_1, x_2 \) and \( x_3 \), respectively, when the material is subjected to a pure homogeneous pre-strain. The strain energy is considered invariant under an orthogonal rotation if it depends on the following invariants

\[
\begin{align*}
I_1 &= \text{tr}(C), \quad I_2 = \frac{1}{2} [I_1^2 - \text{tr}(C^2)], \quad I_3 = \text{det}(C), \\
I_4 &= \{I_{41}, I_{42}, I_{43}\} = \{\text{tr}(\zeta), \frac{1}{2} [I_1^2 - \text{tr}(\zeta^2)], \text{det}(\zeta)\}, \\
I_5 &= \text{tr}(C\zeta), \quad I_6 = \text{tr}(C^2\zeta), \quad I_7 = \text{tr}(C\zeta^2), \quad I_8 = \text{tr}(C^2\zeta^2),
\end{align*}
\]  

(1)
which reduce to

\[ I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = \{I_{41}, I_{42}, I_{43}\} = \{\text{tr}(\zeta), \frac{1}{2}[I_{41}^2 - \text{tr}(\zeta^2)], \det(\zeta^2)\} \]

\[ I_5 = I_6 = \text{tr}(\zeta), \quad I_7 = I_8 = \text{tr}(\zeta^2). \]

in the undeformed configuration. For further details on invariants of tensors, we refer to [28, 30].

Considering that the strain energy function \( P \) is a function of the above mentioned ten invariants, we have

\[ \frac{\partial P}{\partial \Lambda} = \sum_{r=1,2,3,...,8} \mathcal{P}_r \frac{\partial I_r}{\partial \Lambda}, \]

where \( \mathcal{P}_r = \partial P/\partial I_r \) (See [28]).

For the considered material (with \( I_3 = 1 \)), nominal stress tensor, denoted \( \mathbf{S} \), is

\[ \mathbf{S} = 2P_1 \Lambda^T + 2P_2(I_1 \Lambda^T - \Lambda^T \mathbf{B}) + 2P_5 \zeta \Lambda^T + 2P_6(\zeta \mathbf{C} \Lambda^T + \mathbf{C} \zeta \Lambda^T) + 2P_7 \zeta^2 \Lambda^T + 2P_8(\zeta^2 \mathbf{C} \Lambda^T + \mathbf{C} \zeta^2 \Lambda^T - p \Lambda^{-1}), \]

and Cauchy stress, denoted \( \mathbf{T} \) is

\[ \mathbf{T} = 2P_1 \mathbf{B} + 2P_2 \mathbf{B}^* + 2P_5 \Xi + 2P_6(\Xi \mathbf{B} + \mathbf{B} \Xi) + 2P_7 \Xi^{-1} \Xi \]

\[ + 2P_8(\Xi \mathbf{B}^{-1} \mathbf{B} + \mathbf{B} \Xi \mathbf{B}^{-1} \Xi) - p \mathbf{I}, \]

where \( p \) is the Lagrange multiplier associated with the constraint and \( \Xi = \Lambda \zeta \Lambda^T \) and \( \mathbf{B}^* = I_1 \mathbf{B} - \mathbf{B}^2 \). In the undeformed configuration, Eq. (5) gives

\[ \zeta = (2P_1 + 4P_2 - p_0)I + (2P_5 + 4P_6)\zeta + (2P_7 + 4P_8)\zeta^2, \]

where \( p_0 \) is the value of \( p \) evaluated with respect to reference configuration. This suggests

\[ 2P_1 + 4P_2 - p_0 = 0, \quad 2P_5 + 4P_6 = 1, \quad P_7 + 2P_8 = 0, \]

where the derivatives of \( P \) in Eq. (7) are evaluated in reference configuration.

Considering the principal axes of \( \mathbf{B} \), \( \Xi_{ij} = \lambda_i \lambda_j \zeta_{ij} \) and therefore Eq. (5) in its component form is

\[ T_{ij} = 2P_1 \lambda_i^2 \delta_{ij} + 2P_2(I_1 - \lambda_i^2) \lambda_i^2 \delta_{ij} + 2P_5 \lambda_i \lambda_j \zeta_{ij} + 2P_6(\lambda_i^2 + \lambda_j^2) \lambda_i \lambda_j \zeta_{ij} + 2P_7 \lambda_i \lambda_j (\zeta^2)_{ij} + 2P_8 \lambda_i \lambda_j (\lambda_i^2 + \lambda_j^2)(\zeta^2)_{ij} - p \delta_{ij}. \]

with no summation on the repeated indices on the right hand side.

The (updated) elasticity tensor is given by

\[ A_{0pqij} = J^{-1} \left( \sum_{r=1}^{10} \mathcal{P}_r \Xi_{\alpha \beta} \Xi_{\gamma \delta} \frac{\partial^2 W}{\partial \Xi_{\alpha \beta} \partial \Xi_{\gamma \delta}} + \sum_{r,s=1}^{10} \mathcal{P}_{rs} \Xi_{\alpha \beta} \Xi_{\gamma \delta} \frac{\partial I_r}{\partial \Xi_{\alpha \beta}} \frac{\partial I_s}{\partial \Xi_{\gamma \delta}} \right), \]

where \( \mathcal{P}_{rs} = \partial^2 W/\partial I_r \partial I_s \). For an incompressible material, Eq. (9) gives

\[ A_{0pqij} = 2P_1 B_{pq} \delta_{ij} + 2P_2(I_1 B_{pq} \delta_{ij} - B_{iq} B_{jp} + 2B_{pq} B_{ij} - \delta_{ij}(\mathbf{B}^2)_{pq}) + 2P_5 \zeta \Xi_{pq} \delta_{ij} + 2P_6(\Xi_{pq} B_{ij} + (\Xi \mathbf{B})_{pq} \delta_{ij}) + (\mathbf{B} \Xi)_{pq} \delta_{ij} + \Xi_{ij} B_{pq} + \Xi_{pq} B_{ij} + \Xi_{qi} B_{jq} + 2P_{11} B_{i1} B_{j1} + 2P_{22} B_{i2} B_{j2} \]
\[\lambda\]

\[\sum_{i=1}^{n} \frac{\partial^2 L}{\partial v_i^2} = 0\]

Here are the components of \(B\) where \(i \neq j\):

\[B^{iij} = 2 \lambda_i^2 + 2 \lambda_j^2 - \lambda_i^2 \lambda_j^2 - \lambda_i^4 + 2 \lambda_j \delta_{ij} \]

\[B^{ijj} = -2 \lambda_i^2 \lambda_j^2 2 \lambda_i \delta_{ij} + \lambda_i^4 \]

For brevity, the dependence of \(P\) on \(I_7\) and \(K\) is omitted here. It can be assumed that \(\Xi_{ij} = 0, i \neq j\) and this results in the following non-vanishing components:

\[A_{0ijij} = 2 \lambda_i^2 + 2 \lambda_j^2 (I_1 \lambda_i^2 - \lambda_i^2 \lambda_j^2 - \lambda_i^4) + 2 \lambda_j \delta_{ij} \]

\[A_{0ijji} = -2 \lambda_i^2 \lambda_j^2 2 \lambda_i \delta_{ij} + \lambda_i^4 \]

For the initial configuration, Eq. (10) reduces to:

\[C_{pqij} = \alpha_1 \delta_{pq} \delta_{ij} + \alpha_2 \delta_{pq} \delta_{ij} + \alpha_3 \delta_{pq} \delta_{ij} + \alpha_4 \delta_{pq} \delta_{ij}\]

while Eq. (7) are being satisfied. Here:

\[\alpha_1 = 2 (P_1 + P_2), \quad \alpha_2 = 2 P_6, \quad \alpha_3 = 4 (P_{55} + 4 P_{56} + 4 P_{66}), \quad \alpha_4 = 4 (P_{15} + 2 P_{16} + 2 P_{25} + 4 P_{26}).\]

If the initial stress is assumed to vanish, Eq. (13) gives:

\[C_{pqij} = \alpha_1 \delta_{pq} \delta_{ij}\]

The classical isotropic strain energy function, in terms of invariants, is:

\[P(I_1, I_2, I_3) = \frac{\mu}{4} [I_1^3 - 2(I_1 + I_2) + 3] + \frac{\lambda}{8} [I_1 - 3]^2\]

where \(\lambda\) and \(\mu\) are the Lamé moduli. From Eq. (16) we have:

\[A_{iqij} = A_{ijij} = \alpha_1 = \mu \quad i \neq j,
\]

where \(\mu\) is the shear modulus in \(B_r\). The difference between the expressions in Eq. (17) and any alternative expression is absorbed by \(p\), which is the incremental Lagrange multiplier.
From Eq. (13), $\lambda_i = 1$, $i = \{1, 2, 3\}$ and $\Xi_{ij} = \zeta_{ij}$, $i \neq j \neq k \neq i$, in the undeformed configuration, we have

\[
\begin{align*}
C_{iii} &= \alpha_1 + (1 + 4\alpha_2 + 2\alpha_4)\zeta_{ii} + \alpha_3\zeta_{ii}^2, \\
C_{ijj} &= \alpha_4(\zeta_{ii} + \zeta_{jj}) + \alpha_3\zeta_{ii}\zeta_{jj}, \\
C_{ijj} &= \alpha_1 + (1 + \alpha_2)\zeta_{ii} + \alpha_2\zeta_{jj} + \alpha_3\zeta_{ij}\zeta_{ij}, \\
C_{ijj} &= \alpha_2(\zeta_{ii} + \zeta_{jj}) + \alpha_3\zeta_{ij}\zeta_{ij}, \\
C_{iiij} &= (2\alpha_2 + \alpha_4)\zeta_{ij} + \alpha_3\zeta_{ii}\zeta_{ij}, \\
C_{ijj} &= \alpha_3\zeta_{ij}\zeta_{ij} + \alpha_3\zeta_{ii}\zeta_{ij}, \\
C_{iikj} &= \zeta_{ij} + \zeta_{jj} = \alpha_4\zeta_{ij} + \alpha_3\zeta_{ii}\zeta_{jk}, \\
C_{ijji} &= \zeta_{ij} + \zeta_{ij} = \alpha_4\zeta_{ij} + \alpha_3\zeta_{ii}\zeta_{ik}, \\
C_{ikjk} &= \zeta_{ik} + \zeta_{jk} = \alpha_3\zeta_{ij} + \alpha_3\zeta_{ik}\zeta_{jk}.
\end{align*}
\]

(18) (19) (20) (21) (22) (23) (24) (25) (26)

3. Elastic waves in an incompressible hollow cylinder with residual stress.

3.1. Radial inflation and axial extension. In this section, a thick walled circular hollow cylinder is considered. The geometry in the undeformed configuration is

\[A \leq R \leq B, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq Z \leq L,\]

(27)

where $A$ and $B$ are the internal and outer radii, respectively, and $L$ is the length measured in the coordinate system $R, \Theta, Z$. Let the cylinder be finitely deformed so that the circular cylindrical shape is maintained and the new coordinates are $r, \theta, z$. Under the finite deformation, the basis vectors \{\textbf{e}_R, \textbf{e}_\theta, \textbf{e}_Z\} transform to \{\textbf{e}_r, \textbf{e}_\theta, \textbf{e}_z\}. Therefore, the geometry in the deformed configuration is

\[a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq l,\]

(28)

where $a$ and $b$ are the internal and outer radii, respectively, $l$ is the length after deformation. In this case, we consider the deformation

\[r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \zeta = \zeta, \quad z = \lambda_z Z,\]

(29)

where $\lambda_z$ denotes the uniform axial stretch.

Let the unit basis vectors associated with coordinates $\zeta, z, r$ be $\textbf{e}_1, \textbf{e}_2, \textbf{e}_3$, respectively whereas $\lambda_1, \lambda_2, \lambda_3$ denote the corresponding principal stretches. The incompressibility condition ($\lambda_1\lambda_2\lambda_3 = 1$) and Eq. (29) give

\[
\begin{align*}
\lambda_1 &= \frac{r}{R} = \lambda, \quad \lambda_2 = \lambda_z, \quad \lambda_3 = \lambda^{-1}\lambda_z^{-1},
\end{align*}
\]

(30)

where $\lambda$ is the azimuthal stretch which, from Eq. (29), is a function of $r$ (or $R$).

It follows from Eqs. (29) and (30)

\[
\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_a^2\lambda_z - 1),
\]

(31)

where

\[
\lambda_a = a/A, \quad \lambda_b = b/B.
\]

(32)

Therefore, for fixed $\lambda_z$, we have

\[
\lambda_a \geq \lambda \geq \lambda_b,
\]

(33)

which is satisfied during inflation of the tube. Equality holds in Eq. (33), for $A \leq R \leq B$, if and only if $\lambda = \lambda_z^{-1/2}$, which is the case of simple tension.
It is convenient to assume that residual stress has only non-zero diagonal components with the chosen cylindrical polar axes. Let the principal residual Cauchy stresses are denoted $\zeta_1, \zeta_2, \zeta_3$. Then, from Eq. (1), the invariants can be rewritten as

$$ I_1 = \lambda^2 + \lambda^2 + \lambda^{-2}\lambda_z^{-2}, \quad I_2 = \lambda^2\lambda_z^2 + \lambda^{-2} + \lambda^{-2}, \quad I_3 = 1, $$

$$ I_4 = \zeta_1 + \zeta_2 + \zeta_3, \quad I_5 = \zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3, \quad I_6 = \zeta_1\zeta_2\zeta_3, $$

$$ I_7 = \lambda^2\zeta_1 + \lambda^2\zeta_2 + \lambda^{-2}\lambda_z^{-2}\zeta_3, \quad I_8 = \lambda^4\zeta_1 + \lambda^4\zeta_2 + \lambda^{-4}\lambda_z^{-4}\zeta_3, $$

$$ I_9 = \lambda^2\zeta_1^2 + \lambda^2\zeta_2^2 + \lambda^{-2}\lambda_z^{-2}\zeta_3^2, \quad I_{10} = \lambda^4\zeta_1^2 + \lambda^4\zeta_2^2 + \lambda^{-4}\lambda_z^{-4}\zeta_3^2. \quad (34) $$

Considering $\mathcal{P}$ to be independent of $I_5, I_6, I_9$ and $I_{10}$, the principal components of Cauchy stress are

$$ T_{11} = -p + 2\lambda^2\zeta_1 + 2\lambda^2\zeta_2 + 2\lambda^2\zeta_3 + 4\lambda^4\zeta_1\zeta_2\zeta_3, $$

$$ T_{22} = -p + 2\lambda^2\zeta_1 + 2\lambda^2\zeta_2 + 2\lambda^2\zeta_3 + 4\lambda^4\zeta_1\zeta_2\zeta_3, $$

$$ T_{33} = -p + 2\lambda^2\zeta_1 + 2\lambda^2\zeta_2 + 2\lambda^2\zeta_3 + 4\lambda^4\zeta_1\zeta_2\zeta_3. \quad (35) $$

$$ T_{12} = -p + 2\lambda^2\zeta_1 + 2\lambda^2\zeta_2 + 2\lambda^2\zeta_3 + 4\lambda^4\zeta_1\zeta_2\zeta_3, $$

$$ T_{23} = -p + 2\lambda^2\zeta_1 + 2\lambda^2\zeta_2 + 2\lambda^2\zeta_3 + 4\lambda^4\zeta_1\zeta_2\zeta_3, $$

Considering $\zeta_{ij} = 0, i \neq j$ and using the relation

$$ \frac{\partial \mathcal{P}}{\partial \lambda_i} = \sum_{k=1,2,7,8} \frac{\partial \mathcal{P}}{\partial I_k} \frac{\partial I_k}{\partial \lambda_i}, \quad \{i = 1, 2, 3\}, \quad (38) $$

it can be easily deduced that

$$ T_{ii} = I_i = \lambda_i \frac{\partial \mathcal{P}}{\partial \lambda_i} - p, \quad i = \{1, 2, 3\}. \quad (39) $$

Using Eq. (34) and rewriting $\mathcal{P}$ as a function of $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_z$ and $\zeta_1, \zeta_2, \zeta_3$ as

$$ \mathcal{P}(\lambda, \lambda_z, \zeta_1, \zeta_2, \zeta_3) = \mathcal{P}(\lambda, \lambda_z, \lambda^{-1}\lambda_z^{-1}, \zeta_1, \zeta_2, \zeta_3), \quad (40) $$

It may be noted that generally $\mathcal{P}$ is not symmetric in $\lambda, \lambda_z$. Using Eq. (39), we have

$$ T_1 - T_3 = \lambda \frac{\partial \mathcal{P}}{\partial \lambda}, \quad T_2 - T_3 = \lambda_z \frac{\partial \mathcal{P}}{\partial \lambda_z}. \quad (41) $$

or

$$ \lambda \frac{\partial \mathcal{P}}{\partial \lambda} = T_1 - T_3 = 2(\lambda^2 - \lambda^{-2}\lambda_z^{-2})\zeta_1 + 2(\lambda^2\lambda_z^2 - \lambda^{-2})\zeta_2 + 2\lambda^2\zeta_1 + (\lambda^2 + 2\lambda^2)\zeta_3 $$

$$ -2\lambda^{-2}\lambda_z^{-2}\zeta_1\zeta_2\zeta_3 (\lambda^2 + 2\lambda^2)\zeta_3, $$

$$ \lambda_z \frac{\partial \mathcal{P}}{\partial \lambda_z} = T_2 - T_3 = 2(\lambda^2 - \lambda^{-2}\lambda_z^{-2})\zeta_1 + 2(\lambda^2\lambda_z^2 - \lambda^{-2})\zeta_2 + 2\lambda^2\zeta_2 + (\lambda^2 + 2\lambda^2)\zeta_3 $$

$$ -2\lambda^{-2}\lambda_z^{-2}\zeta_1\zeta_2\zeta_3 (\lambda^2 + 2\lambda^2)\zeta_3. \quad (42) $$

For brevity, we can assume $\zeta_2 = 0$ whereas the non-zero residual stresses satisfy (since $\text{Div}\zeta = 0$)

$$ \frac{d\zeta_3}{dR} + \frac{1}{R}(\zeta_3 - \zeta_1) = 0, \quad (44) $$

with

$$ \zeta_3 = 0 \quad \text{on} \quad R = A \quad \text{and} \quad R = B. \quad (45) $$
which are the boundary conditions in the undeformed state of the cylinder. After

the deformation occurs, Eq. (44) becomes

\[ \frac{dT_3}{dr} + \frac{1}{r} (T_3 - T_1) = 0, \]

with

\[ T_3 = \begin{cases} -P & \text{on } r = a, \\ 0 & \text{on } r = b, \end{cases} \]

as the boundary conditions. Here \( P \) is the internal pressure for inflating the tube.

After some rearrangements and using Eqs. (29), (30) and (31), we have

\[ r \frac{d\lambda}{dr} = -\lambda \left( \lambda^2 \lambda_z - 1 \right). \]

Integrating Eq. (46) and incorporating the boundary conditions (47), we get

\[ P = \int_a^b \frac{\lambda}{r} \frac{\partial \hat{P}}{\partial \lambda} dr = \int_{\lambda_a}^{\lambda_b} \left( \lambda^2 \lambda_z - 1 \right)^{-1} \frac{\partial \hat{P}}{\partial \lambda} d\lambda, \]

where Eq. (48) is being used to change the independent variable \( r \) with \( \lambda \).

Here \( P \) is regarded as a function of \( \lambda_z \) and \( \lambda_a \) due to Eq. (31). Differentiating

Eq. (49) with respect to \( \lambda_a \), we have

\[ \lambda_a^{-1}(\lambda_a^2 - 1) \frac{\partial P}{\partial \lambda_a} = \frac{1}{\lambda_a} \frac{\partial \hat{P}(\lambda_a, \lambda_z)}{\partial \lambda} - \frac{1}{\lambda_b} \frac{\partial \hat{P}(\lambda_b, \lambda_z)}{\partial \lambda}. \]

Using Eq. (41) in Eq. (46) and integrating, we get

\[ T_3(r) = \int_a^r \frac{\lambda}{r} \frac{\partial \hat{P}}{\partial \lambda} dr = \int_{\lambda_a}^{\lambda_b} \left( \lambda^2 \lambda_z - 1 \right)^{-1} \frac{\partial \hat{P}}{\partial \lambda} d\lambda, \]

where Eq. (48) is being used. For a specific strain energy function, the above expression gives the radial Cauchy stress component.

Integrating Eq. (44), we have

\[ R \zeta_3(R) = \int_A^R \zeta_1(R) dR. \]

Using (45), we must have

\[ \int_A^B \zeta_1(R) dR = 0. \]

This implies that \( \zeta_1(R) \) is positive or positive for different values of \( R \). The possibility of \( \zeta_1(R) \equiv 0 \) is discarded as it renders \( \zeta_3(R) \equiv 0 \).

In [22] and [23], a uniform circumferential stress is assumed. The results show that the radial residual stress (here \( \zeta_3 \)) is small in magnitude. It is negative except at the boundaries where its value is zero due to Eq. (45). The circumferential stress (here \( \zeta_1 \)) is of tensile nature at the outer radius and compressive at the inner radius. The reader can find a similar discussion in [7], [31] and [25]. However, in [21], uniform circumferential strain distribution and uniform circumferential stress are considered which result in opposite signs of the residual radial and residual circumferential stresses as compared to those in [22] and [23]. Further, the results in [22] and [23] match those presented in [26] for a particular choice of the circumferential growth stretch ratio. This is when growth is considered in addition to the uniform circumferential stress. [8] have considered a different strain energy function and the
growth in the material results in the development of residual stresses. The results thus obtained very similar results to those in [23].

In view of the above cited papers, for the problem considered here, a particular behavior of residual circumferential stress is expected inside a residually-stressed thick-walled tube. For simplicity only, let $\zeta_1(R)$ is linear in $R$ given by

$$\zeta_1(R) = k_1(R - A) - k_2(B - A), \quad (54)$$

where $k_1$ and $k_2$ are constants. Since $B > A$, it can be easy to infer that

$$\zeta_1(A) = -k_2(B - A) < 0 \text{ if } k_2 > 0,$$
$$\zeta_1(B) = (k_1 - k_2)(B - A) > 0 \text{ if } k_1 > k_2 > 0, \quad (55)$$

which is in accordance with the expected behavior of the residual circumferential stress that is negative(positive) on the inner(outer) boundary. From Eq. (53), we have

$$\int_A^B \zeta_1 dR = (k_1 - 2k_2)(B - A)^2 = 0. \quad (56)$$

which implies $k_1 = 2k_2$. Integration of Eq. (52) gives

$$R\zeta_3 = k_2(R - A)(R - B), \quad (57)$$

which vanishes at $R = A$ and $R = B$.

Equations (54) and (57) can be rewritten as

$$\zeta_1 = k_2 A(2R/A - 1 - B/A), \quad (58)$$
$$\zeta_3 = k_2 A(1 - A/R)(R/A - B/A). \quad (59)$$

For a fixed tube thickness, the behavior of these stress components (in dimensionless form) is shown in Figure 1. The behaviors are very similar to those observed in [23], [22], [31], [25], [26] and [8].

![Figure 1](image_url)

**Figure 1.** Plot of $\zeta_1$ (continuous graph) from Eq. (58) and $\zeta_3$ (dashed graph) from Eq. (59) for $B/A = 1.2$.
A simple non-linear model. In this section, a prototype model is chosen to understand the effect of a non-homogeneous residual stress on wave propagation in an incompressible elastic cylinder. A finite deformation is considered from a configuration that is subject to a non-homogeneous initial stress. Following [28], a simple strain-energy function of the form

\[ \mathcal{P} = \frac{\mu}{2} (I_1 - 3) + \frac{\bar{\mu}}{2} (I_7 - I_{41})^2 + \frac{1}{2} (I_7 - I_{41}), \]  

(60)

where \( \mu \) and \( \bar{\mu} \) are material constants. Eq. (34) gives the invariants \( I_1, I_{41} \) and \( I_7 \). Rewriting Eq. (60) in terms of \( \lambda_1, \lambda_2, \lambda_3 \) and \( \zeta_1, \zeta_2, \zeta_3 \) as

\[
\mathcal{P} = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\bar{\mu}}{2} [(\lambda_1^2 - 1)\zeta_1 + (\lambda_2^2 - 1)\zeta_2 + (\lambda_3^2 - 1)\zeta_3]^2 
+ \frac{1}{2} [(\lambda_1^2 - 1)\zeta_1 + (\lambda_2^2 - 1)\zeta_2 + (\lambda_3^2 - 1)\zeta_3].
\]  

(61)

Using (60), for the deformed configuration of an incompressible material, various required derivatives are

\[
\mathcal{P}_1 = \frac{\mu}{2}, \quad \mathcal{P}_4 = -\bar{\mu} I_7, \quad \mathcal{P}_7 = \bar{\mu} (I_7 - I_4) + \frac{1}{2}, \quad \mathcal{P}_{77} = \bar{\mu},
\]  

(62)

which reduce to

\[
\mathcal{P}_1 = \frac{\mu}{2}, \quad \mathcal{P}_4 = -\bar{\mu} I_4, \quad \mathcal{P}_7 = \frac{1}{2}, \quad \mathcal{P}_{77} = \bar{\mu}.
\]  

(63)

in the undeformed configuration.

3.2. Pressure inside a thick-walled tube with residual stress. In this section, an explicit expression for calculating the pressure \( P \) inside a thick-walled tube is presented in terms of the principal stretches and principal residual stress components.

Equations (30) and (60) give

\[
W = \frac{\mu}{2} (\lambda^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\bar{\mu}}{2} [(\lambda^2 - 1)\zeta_1 + (\lambda_2^2 - 1)\zeta_2 + (\lambda_3^2 - 1)\zeta_3]^2 
+ \frac{1}{2} [(\lambda^2 - 1)\zeta_1 + (\lambda_2^2 - 1)\zeta_2 + (\lambda_3^2 - 1)\zeta_3].
\]  

(64)

and Eq. (64) therefore gives

\[
\lambda \frac{\partial \mathcal{P}}{\partial \lambda} = \mu (\lambda^2 - \lambda^{-2}\lambda_3^{-2}) + 2\bar{\mu} [(\lambda^4 - \lambda^2\lambda_3^{-2} - \lambda_3^2)\zeta_1 \zeta_3 
+ \lambda_2^2 (\lambda_2^2 - 1)\zeta_1 \zeta_2 - \lambda^{-2}\lambda_2^{-2} (\lambda_2^2 - 1)\zeta_2 \zeta_3 
- (\lambda^{-4}\lambda_2^{-4} - \lambda^{-2}\lambda_2^{-2})\zeta_3^2] 
+ \lambda_2^2 \zeta_1 - \lambda^{-2}\lambda_2^{-2} \zeta_3, \tag{65}
\]

\[
\lambda_2 \frac{\partial \mathcal{P}}{\partial \lambda_2} = \mu (\lambda_2^2 - \lambda^{-2}\lambda_3^{-2}) + 2\bar{\mu} [(\lambda_2^2 - 1)\zeta_1 \zeta_2 - \lambda^{-2}\lambda_2^{-2} (\lambda_2^2 - 1)\zeta_2 \zeta_3 
+ \lambda_2^2 (\lambda_2^2 - 1)\zeta_1 \zeta_2 + (\lambda^{-2}\lambda_2^{-2} - \lambda_2^2) \zeta_2 \zeta_3] 
+ \lambda_2^2 \zeta_2 - \lambda^{-2}\lambda_2^{-2} \zeta_3, \tag{66}
\]

which reduces to

\[
\frac{\partial \mathcal{P}}{\partial \lambda} = \zeta_1 - \zeta_3, \quad \frac{\partial \mathcal{P}}{\partial \lambda_2} = \zeta_2 - \zeta_3, \tag{67}
\]

in the undeformed configuration.
Also, from Eq. (61), we have
\[ \lambda_3 \frac{\partial P^*}{\partial \lambda_3} = \lambda_3^2 (\mu + \zeta_3) + 2\hat{\mu} \lambda_3 \zeta_3 \lambda_3^2 [\lambda_3^2 - 1] \zeta_1 + (\lambda_3^2 - 1) \zeta_2 + (\lambda_3^2 - 1) \zeta_3, \] (68)
which, in the undeformed configuration, reduces to
\[ \frac{\partial W}{\partial \lambda_3} = \mu + \zeta_3. \] (69)
Making use Eq. (31) in Eqs. (58) and (59), \( \zeta_1 \) and \( \zeta_3 \) can be rewritten as
\[ \hat{\zeta}_1 = \beta_2 (2\sqrt{\frac{\lambda_3^2 \lambda_3 - 1}{\lambda_3^2 \lambda_3 - 1} - 1} - \frac{B}{A}), \hat{\zeta}_3 = \beta_2 (1 - \sqrt{\frac{\lambda_3^2 \lambda_3 - 1}{\lambda_3^2 \lambda_3 - 1}}) \] (70)
where the dimensionless quantity
\[ \beta_2 = k_2 A/\mu, \] (71)
is being introduced. It may be noted here \( \lambda \) serves as a variable of integration and the residual stress is not explicitly dependent on the stretches.

It can be written from Eq. (50)
\[ \frac{dP^*}{\lambda_a} = \frac{\lambda_a}{\lambda_a^2 \lambda_3 - 1} \left[ 1 - \lambda_a^4 \lambda_3^{-2} + 2\beta_1 (\lambda_a^2 - 1) (1 - B/A)^2 + \beta_2 (1 - B/A) 
\right. \\
- (1 - \lambda_a^4 \lambda_3^{-2}) - 2\beta_1 (\lambda_a^2 - 1) (B/A - 1)^2 - \beta_2 B/A \] (72)
where
\[ \beta_1 = \beta_0 \beta_2 = \hat{\mu} k_2^2 A^2 / \mu^2, \] (73)
and \( \beta_0 = \mu \hat{\mu} \). The value of \( \beta_2 \) is given by Eq. (71).

The dimensionless pressure is calculated using Eqs. (64) and (70) in (49) and is given by
\[ P^* = P/\mu = \lambda_3 \lambda_a \int_{\lambda_a}^{\lambda_3} \frac{\lambda - \lambda^{-3} \lambda_a^{-2} d\lambda}{\lambda_a^2 \lambda_3 - 1} + 2\beta_0 \int_{\lambda_a}^{\lambda_3} \frac{\lambda (\lambda^2 - 1) \zeta_2^2 d\lambda}{\lambda_a^2 \lambda_3 - 1} \\
+ \int_{\lambda_a}^{\lambda_3} \frac{\lambda^3 \lambda^{-2} \hat{\zeta}_1 \zeta_3 d\lambda}{\lambda^2 \lambda_3 - 1} - \int_{\lambda_a}^{\lambda_3} \frac{\lambda^3 \lambda^{-2} \hat{\zeta}_3 d\lambda}{\lambda^2 \lambda_3 - 1} \] (74)
where \( \beta_0 \) is defined above.

Figure 2 is the plot of \( \frac{dP^*}{\lambda_a} \) versus \( \lambda_a \). After using Eqs. (31) and (49), it may be noted in Fig. 3 that for \( \lambda_a = \lambda_a^{-1/2} \), the pressure is zero at \( \lambda_a = \lambda_a^{-1/2} \) in the absence of residual stress. With increase in the value of \( \lambda_a \), the pressure tends to remain constant. This is a similar behavior as observed in the case of neo-Hookean or Mooney-Rivlin material models. This behavior is obvious in plots (a) and (b) in Fig. 2. The plots (c) and (d) in Fig. 2 are for the derivative of pressure with nonzero residual stress. It is observed that the vanishing of the pressure takes place at a shifted value of \( \lambda_a \) (for fixed axial stretch) depending on the values of \( \beta_1 \), \( \beta_2 \) and the ratio \( B/A \). Also, with variation in the values of the parameters, the pressure is expected to increases or decreases. The plot in Fig. 3 is the behavior of pressure for zero residual stress whereas Figs. 4–6 show the graphs for various values of \( \lambda_3, B/A, \beta_1 \) and \( \beta_2 \) with non-zero residual stress. It is observed that for \( \beta_1 > 0, \beta_2 > 0 \), the pressure increases whereas for \( \beta_1 < 0, \beta_2 < 0 \), the pressure
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decreases with increasing values of $\lambda_a$. For fixed axial stretch and wall thickness, Fig. 6 illustrates the increasing or decreasing trend of pressure for different combinations of $\beta_1$ and $\beta_2$ characterizing the presence of residual stress.

\[ \frac{\partial P}{\partial \lambda_a} = \lambda_a \lambda_z^{1/2} \]

![Figure 2](image)

**Figure 2.** $\frac{1}{\mu} \frac{dP}{d\lambda_a}$ versus $\lambda_a$, (a) $\beta_1 = 0 = \beta_2$, $\lambda_z = 1.3 = B/A$, (b) $\beta_1 = 0 = \beta_2$, $\lambda_z = 1.3$, $B/A = 1.5$, (c) $\beta_1 = 8$, $\beta_2 = 1$, $\lambda_z = 1.3 = B/A$, (d) $\beta_1 = 8$, $\beta_2 = 1$, $\lambda_z = 1.3$, $B/A = 1.5$

\[ P^* \] versus $\lambda_a$ for different wall thickness $B/A$ and zero residual stress with ($\lambda_z = 1.3$).

![Figure 3](image)

**Figure 3.** $P^*$ versus $\lambda_a$ for different wall thickness $B/A$ and zero residual stress with ($\lambda_z = 1.3$).

3.3. Axial load for a thick-walled tube with residual stress. An axial load, say $N$, must be applied to the ends of the tube to keep the axial stretch $\lambda_z$ fixed. The value of $N$ is given by

\[ N = 2\pi \int_a^b T_2rdr. \]

Using Eqs. (33) and (41)–(48), the above expression gives

\[ N/\pi A^2 = (\lambda^2_0 \lambda_z - 1) \int_{\lambda_0}^{\lambda_z} (\lambda^2 \lambda_z - 1)^{-2} (2\lambda_z \frac{\partial \hat{P}}{\partial \lambda_z} - \lambda \hat{P}) \lambda d\lambda + P\lambda_z^2. \]
Figure 4. $P^*$ versus $\lambda_a$ for different $B/A$ with $\beta_1 = 2 = \beta_2$ and $\lambda_z = 1.3$.

Figure 5. $P^*$ versus $\lambda_a$ for different $B/A$ with $B/A$, $\beta_1 = -2 = \beta_2$ and $\lambda_z = 2$.

Figure 6. $P^*$ versus $\lambda_a$ with $B/A = 1.2, \lambda_z = 1.2$ and (a) $\beta_1 = 0 = \beta_2$, (b) $\beta_1 = 0.2, \beta_2 = 0.3$, (c) $\beta_1 = 0.7, \beta_2 = 0.3$, (d) $\beta_1 = 0.5, \beta_2 = 0.5$, (e) $\beta_1 = 0.5, \beta_2 = -0.5$, (f) $\beta_1 = 2, \beta_2 = 0.5$, (g) $\beta_1 = 0.5, \beta_2 = 2$. 
Using Eqs. (65) and (66) in Eq. (76), we have

\[
N/A' = (\lambda_a^2 \lambda_z - 1) \left[ \int_{\lambda_b}^{\lambda_a} \frac{2 \lambda_z^2 \lambda - \lambda^3 - \lambda^{-1} \lambda^{-2}}{(\lambda^2 \lambda_z - 1)^2} d\lambda \right.
\]

\[
-2\beta_0 \int_{\lambda_b}^{\lambda_a} \lambda^{-3} \lambda_z^{-4} (1 - \lambda^2 \lambda_z^2) \dot{\zeta}_3 d\lambda
\]

\[
+ \frac{3 \lambda^{-1} \lambda_z^{-2} (\lambda^2 - 1) \ddot{\zeta}_1 \lambda_z^{-4} \dot{\zeta}_3 d\lambda}{(1 - \lambda^2 \lambda_z^2)^2}
\]

\[
+ \frac{\lambda^3 \ddot{\zeta}_1 + \lambda^{-1} \lambda_z^{-2} \ddot{\zeta}_3 d\lambda}{(1 - \lambda^2 \lambda_z^2)^2} + P \lambda_a^2,
\]

where \( A' = \pi \mu A^2 \). Figure 7 is a plot of axial load versus \( \lambda_a \) for various wall-thicknesses and values of parameters. The load tends to remain constant for increasing values of \( \lambda_a \) for zero residual stress. This behavior is similar to that of neo-Hookean materials and Mooney-Rivlin materials. Figure 8 shows the effect of residual stress on the axial load for a given wall-thickness. With increasing values of \( \lambda_a \), the axial load may decrease or increase which also depends on the values of the parameters that specify the magnitude of residual stress.

\[\text{Figure 7. } N/A' \text{ versus } \lambda_a \text{ for } \lambda_z = 1.2 \text{ and (a) } B/A = 1.2, \beta_1 = 0 = \beta_2, \text{ (b) } B/A = 1.5, \beta_1 = 0 = \beta_2, \text{ (c) } B/A = 2, \beta_1 = 0 = \beta_2, \text{ (d) } B/A = 1.2, \beta_1 = -0.5, \beta_2 = 0.8, \text{ (e) } B/A = 1.4, \beta_1 = -0.5, \beta_2 = 0.8, \text{ (f) } B/A = 1.5, \beta_1 = -0.8, \beta_2 = 1.5, \text{ (g) } B/A = 2, \beta_1 = 0.3, \beta_2 = 0.8\]

4. Effect of residual stress on an infinitesimal wave: Axisymmetric case of a thick-walled tube. The problem of propagation of an infinitesimal wave thick-walled cylindrical tube is considered subject to radial inflation and finite axial extension. The tube is supposed to be residually-stressed from its reference configuration. The basis vectors are such that \( e_1 = e_\zeta, e_2 = e_z, e_3 = e_r \) in this case. Accordingly, let \( \zeta_1, \zeta_2, \zeta_3 \) denote the azimuthal, axial and radial (residual) stress components, respectively.
A material point has the displacement vector

\[ \mathbf{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X}, \]  

(78)

where \( \mathbf{x} \) and \( \mathbf{X} \) are the positions of the particle in the deformed and the undeformed configurations, respectively. Let the gradient of displacement be

\[ \mathbf{G} = \text{Grad} \mathbf{u}, \]  

(79)

such that

\[ \Lambda = \mathbf{I} + \mathbf{G}. \]  

(80)

Consider now infinitesimal deformations (which are also time dependent) in the cylindrical tube after it undergoes a finite deformation. It can therefore be inferred from Eq. (80) that

\[ \dot{\Lambda} \approx \dot{\mathbf{G}} \approx \Gamma, \]  

(81)

for small \( \mathbf{G} \).

For small \( \mathbf{G} \) and no body forces, the incremental updated form of equilibrium equation [28], is

\[ \text{div}\dot{\mathbf{S}}_0 \equiv \text{div}[\mathcal{A}_0(\mathbf{G}) - \dot{p}\mathbf{I} + \rho\mathbf{u},tt] = \rho\mathbf{u},tt, \]  

(82)

where \( \dot{\mathbf{S}} \) is the updated nominal stress tensor. Equation 82 can be rewritten as

\[ \dot{\mathbf{S}}_{\text{ijkl}} + \dot{\mathbf{S}}_{\text{kij}} \mathbf{e}_j \cdot \mathbf{e}_{k,j} + \dot{\mathbf{S}}_{\text{ijk}} \mathbf{e}_i \cdot \mathbf{e}_{k,j} = \rho\mathbf{u},tt, \quad i = \{1, 2, 3\}, \]  

(83)

and \( \{j, k\} = \{1, 2, 3\} \) and these subscripts following a comma denote the derivatives \( \partial/\partial \zeta, \partial/\partial z, \partial/\partial r \), respectively. The (non-zero) components of \( \mathbf{e}_i \cdot \mathbf{e}_{k,j} \) are

\[ \mathbf{e}_3 \cdot \mathbf{e}_{1,1} = -1/r, \quad \mathbf{e}_1 \cdot \mathbf{e}_{3,1} = 1/r. \]  

(84)

If \( \mathbf{u} = v\mathbf{e}_\zeta + w\mathbf{e}_z + u\mathbf{e}_r \), we have

\[ \begin{bmatrix} \mathbf{G} \end{bmatrix} = \begin{bmatrix} \text{grad} \mathbf{u} \end{bmatrix} = \begin{pmatrix} (u + v_\zeta)/r & v_z & v_r \\ \mathcal{P}_\zeta/r & \mathcal{P}_z & \mathcal{P}_r \\ (u_\zeta - v)/r & u_z & u_r \end{pmatrix}, \]  

(85)
where \([G]\) is the matrix of components of \(G\) and \((r, \zeta, z)\) in the subscripts denote the partial derivatives. Since the axisymmetric case is under consideration, Eq. (85) gives

\[
G = \begin{pmatrix}
    u/r & 0 & 0 \\
    0 & P_z & P_r \\
    0 & u_z & u_r \\
\end{pmatrix},
\]

where the subscripts represent the derivative according to the respective variable.

Also, the incompressibility condition, \(G_{pp} = 0\), gives

\[
u/r + P_z + u_r = 0,
\]

or

\[
(ru)_r + (rw)_z = 0.
\]

Equation (88) thus allows a potential function \(\phi = \phi(r, z)\), such that

\[
u = \phi_z \Rightarrow (ru)_r = \phi_{rr}, \quad rw = -\phi_r \Rightarrow (rw)_z = -\phi_{rz}.
\]

For \(i = 2\) and \(i = 3\) From Eq. (83), we have respectively

\[
\dot{S}_{022,j} + \frac{1}{r} \dot{S}_{032} = \rho \ddot{w},
\]

\[
\dot{S}_{033,j} + \frac{1}{r} \dot{S}_{033} - \frac{1}{r} \dot{S}_{011} = \rho \ddot{u}.
\]

In the expanded form, Eqs. (90) and (91) give

\[
\dot{p}_z + \rho \ddot{p}_tt = A_{03232}P_{rr} + A_{02222}P_{zz} + (rA'_0A_{03232} + A_{03232})P_r/r + (A_{02233} + A_{03233})u_{rz}
\]

\[
+ (rA'_{03223} + A_{03223} + A_{01122} + rp')u_z/r,
\]

\[
\dot{p}_r + \rho u_{tt} = (rA'_{01133} - A_{01111})u/r^2(rA'_{03333} + A_{03333} + rp')u_r/r + A_{03333}u_r + A_{02323}u_{zz} + (rA'_{02233} + A_{02233} - A_{01122})P_{rr}/r
\]

\[
+ A_{02222} - A_{02233} - A_{03233})P_{zz} - A_{01122} + A_{03233} + A_{03233} - A_{03333}u_{rrz}
\]

\[
+ (rA'_{02222} + A_{02222} + A_{02233} + A_{01122} - A_{03333})u_{zz} + r^3(A_{02222} + A_{02222} + A_{02233} - A_{03333})u_{rz}
\]

\[
+ r^2(A_{02222} + A_{02222} + A_{02233} + A_{01122} + rA'_{01122} + r^2 p') - A_{03233} - A_{01122} - \rho A'_{01133} + A_{01111})u_z = \rho r^4(P_{tt} - u_{zz}).
\]

From Eq. (89), we have

\[
u = \frac{1}{r} \phi_z, \quad w = -\phi_r.
\]
Using the above expressions and their various derivatives in Eq. (94), we get
\[ A_{03232}(r^4 \phi_{rrrr}) + [r^2 A_{03232}' - 3r A_{03232}' + 3A_{03232}][r^2 \phi_{rr}] - [r^2 A_{03232}' - 3r A_{03232}' + 3A_{03232}](r \phi_r) + [A_{02222} + A_{03333} - 2A_{02222} - 2A_{03232}][r^4 \phi_{rrzz}] + A_{02222}(r^4 \phi_{zzzz}) + [r A_{02222} + r A_{03333} - 2r A_{02222} - 2r A_{03232}][r^3 \phi_{zzz}] - [r^2 A_{03232}' + r^2 p'' + r A_{01122} + r A_{03333} - r A_{01122} - r A_{03333} - A_{01111} + A_{01111}][r^2 \phi_{zz}] = \rho r^3[r \phi_{ztt} + r \phi_{rrt} - \phi_{r tt}]. \] (96)

Considering a solution for \( \phi \) of the form
\[ \phi(r, z) = F(r)e^{i(kz - \omega t)}, \] (97)
where \( \omega \) is the frequency and \( k \) is the wave number. Using Eqs. (97) and Eq. (96), we get
\[ A_{03232}r^4 F^{iii} + 2[r A_{03232}' - A_{03232}][r^3 F^{'''} + [r^2 A_{03232}' - 3r A_{03232}' + 3A_{03232} + \rho \omega^2 r^2][r^2 F^{'''} - [r^2 A_{03232}' - 3r A_{03232}' + 3A_{03232} + \rho \omega^2 r^2][r F^]' + k^2 r^2[2A_{02222} + 2A_{03232} - A_{02222} - A_{03333}][r^2 F^{'''} + 2r A_{02222} + 2A_{03232} - r A_{02222} - r A_{03333} - 2A_{02222} - 2A_{03232} - A_{02222} + A_{03333} - A_{01112} - A_{03333} + A_{01111}][r^2 \phi_{zz}] = \rho r^3[r \phi_{ztt} + r \phi_{rrt} - \phi_{r tt}]. \] (98)

Let
\[ \gamma_1 = A_{03232}, \quad \gamma_2 = 2A_{02222} + 2A_{03232} - A_{02222} - A_{03333}, \]
\[ \gamma_3 = A_{03232}, \quad \gamma_4 = A_{01112} + A_{03333} - A_{01113} + A_{03223} - A_{02223}, \]
\[ \gamma_5 = 2A_{02222} - 2A_{01112} - A_{03333} + A_{01111}, \quad \gamma_6 = A_{02323}. \] (99)

Therefore, Eq. (98) becomes
\[ \gamma_1 r^4 F^{iii} + 2[r \gamma_1' - \gamma_1][r^3 F^{'''} + [r^2 \gamma_1'' - 3r \gamma_1' + 3\gamma_1 + \rho \omega^2 r^2][r^2 F^{'''} - [r^2 \gamma_1'' - 3r \gamma_1' + 3\gamma_1 + \rho \omega^2 r^2][r F^]' + k^2 r^2[2\gamma_1 r^2 F^{'''} + (r \gamma_1' - \gamma_1) r F^]' + (r^2 \gamma_3' + r^2 p'' + r \gamma_6' + \gamma_5 - \rho \omega^2 r^2) F] + k^4 r^4 \gamma_6 F = 0. \] (100)

Equation (39) can be used for \( i = 3 \) with Eq. (46) to find the values of \( p' \) and \( p'' \) as
\[ p'(r) = \frac{d}{dr}(\lambda_3 \frac{\partial \mathcal{P}}{\partial \lambda_3}) - \frac{\lambda}{r} \frac{\partial \mathcal{P}}{\partial \lambda}, \] (101)
\[ p''(r) = \frac{d^2}{dr^2}(\lambda_3 \frac{\partial \mathcal{P}}{\partial \lambda_3}) - \frac{1}{r} \frac{d\lambda}{dr} \frac{\partial \mathcal{W}}{\partial \lambda} + \lambda \frac{d}{dr}(\frac{\partial \mathcal{W}}{\partial \lambda}) + \frac{1}{r^2} \frac{\partial \mathcal{W}}{\partial \lambda}. \] (102)
The pressure loading boundary condition in the undeformed configuration is

\[ \mathbf{S}^T \mathbf{N} = -P \mathbf{A}^{-T} \mathbf{N}, \]  

(103)

where \( P \) is the pressure on the boundary per unit area of the deformed configuration and \( \mathbf{N} \) is the unit normal to the area. In its incremental form, Eq. (103) (after updating to the deformed configuration), we have

\[ \dot{\mathbf{S}}_0^T \mathbf{n} = \mathbf{PH}^T \mathbf{n} - \dot{P} \mathbf{n}. \]  

(104)

Equation (104) can now be specialized for an infinite cylindrical tube with the outer boundary as traction free and the inner boundary subject to pressure \( P \). For \( i = 2, 3 \), in Eq. (104) with \( \dot{P} = 0 \), the boundary conditions are

\[ \dot{S}_{10i} = \begin{cases} \mathbf{PH}_{3i} & \text{on } r = a \\ 0 & \text{on } r = b. \end{cases} \]  

(105)

which for the considered case, with Eq. (87), gives on \( r = a, b \)

\[ (\mathbf{A}_{03333} - \mathbf{A}_{02233} + \lambda_3 \partial \mathcal{P} / \partial \lambda_3) u_r + (\mathbf{A}_{01133} - \mathbf{A}_{02233}) u / r - \dot{\rho} = 0, \]  

(106)

\[ \mathcal{P}_r + u_z = 0. \]  

(107)

Using Eqs. (10), (97), (39) and (47), Eq. (106) becomes

\[ \begin{align*}
\mathbf{A}_{03333} r^2 F'' + [r \mathbf{A}_{03232} - \mathbf{A}_{03232}] r^2 F'' - [r \mathbf{A}_{03232} - \mathbf{A}_{03232} - \rho r^2 \omega^2] r F' - r^2 k^2 [(\mathbf{A}_{03333} + \mathbf{A}_{02232} - 2 \mathbf{A}_{02233} - \mathbf{A}_{03232} + \lambda_3 \partial \mathcal{P} / \partial \lambda_3) F] \\
- (r \mathbf{A}_{03232} + \mathbf{A}_{01122} + \mathbf{A}_{03333} - \mathbf{A}_{01133} - \mathbf{A}_{02233} + r p' + \lambda_3 \partial \mathcal{P} / \partial \lambda_3) F = 0,
\end{align*} \]  

(108)

and

\[ r^2 F'' - r F' + r^2 k^2 F = 0, \]  

(109)

respectively.

4.1. **Analysis for a special model.** The theory developed in the previous section may be used now for the special model in Eq. (60). The principal residual stress components from Eqs. (58) and (59) using Eq. (29) in the deformed configuration, are

\[ \begin{align*}
\hat{\gamma}_1 &= \zeta_1 / \mu = \beta_2 (2 \sqrt{1 + \lambda_2 \lambda_3^2 (\hat{R}^2 - 1)} - 1 - \sqrt{1 + \lambda_2 \lambda_3^2 (b^2 / a^2 - 1)}), \\
\hat{\gamma}_3 &= \zeta_3 / \mu = \beta_2 (1 - 1 / \sqrt{1 + \lambda_2 \lambda_3^2 (\hat{R}^2 - 1)}) \\
&\times (\sqrt{1 + \lambda_2 \lambda_3^2 (\hat{R}^2 - 1)} - \sqrt{1 + \lambda_2 \lambda_3^2 (\hat{B}^2 - 1)}),
\end{align*} \]  

(110)

(111)

where

\[ \hat{R} = r / a, \quad \hat{B} = b / a, \]  

(112)

and \( \beta_2 = k_3 A / \mu \). Using these notations, Eq. (100), in the deformed configuration, becomes

\[ \begin{align*}
\hat{\gamma}_1 \hat{R}^4 F''' + 2 (\hat{R} \hat{\gamma}_1' - \hat{\gamma}_1) \hat{R}^3 F'' + (\hat{R}^2 \hat{\gamma}_1'' - 3 \hat{R} \hat{\gamma}_1' + 3 \hat{\gamma}_1 + \hat{\omega}^2 \hat{R}^2) \hat{R}^2 F'' \\
- (\hat{R}^2 \hat{\gamma}_1'' + 3 \hat{R} \hat{\gamma}_1' + 3 \hat{\gamma}_1 + \hat{\omega}^2 \hat{R}^2) \hat{R} F' + k^2 \hat{R}^2 [\hat{\gamma}_2 \hat{R}^2 F'' + (\hat{\gamma}_2 - \hat{\gamma}_2) \hat{R} F'] \\
+ (\hat{R}^2 (p'' / \mu) + \hat{\gamma}_4' + \hat{\gamma}_5 - \hat{\omega}^2 \hat{R}^2) F] + \hat{k}^4 \hat{R}^4 \hat{\gamma}_6 F = 0,
\end{align*} \]  

(113)
where 
\[ \dot{\omega} = \sqrt{\frac{\rho}{\mu}} a \omega, \quad \dot{k} = k a, \quad \gamma_i(\hat{R}) = \gamma_i(\hat{R})/\mu, \quad i \in \{1, 2, 4, 5, 6\}. \] (114)

Rewriting Eq. (113) as
\[ \dot{f}_4(\hat{R}) \hat{R}^4 \frac{d^4 F}{d\hat{R}^4} + \dot{f}_3(\hat{R}) \hat{R}^3 \frac{d^3 F}{d\hat{R}^3} + \dot{f}_2(\hat{R}) \hat{R}^2 \frac{d^2 F}{d\hat{R}^2} + \dot{f}_1(\hat{R}) \hat{R} \frac{d F}{d\hat{R}} + \dot{f}(\hat{R}) F = 0, \] (115)

where
\[ \dot{f}_4(\hat{R}) = \gamma_1, \quad \dot{f}_3(\hat{R}) = 2(\hat{R} \gamma_1' - \gamma_1), \]
\[ \dot{f}_2(\hat{R}) = \hat{R}^2 \gamma_1'' - 3 \hat{R} \gamma_1' + 3 \gamma_1 + \dot{\omega}^2 \hat{R}^2 + \dot{k}^2 \hat{R}^2 \gamma_2, \]
\[ \dot{f}_1(\hat{R}) = \hat{R}^2 \gamma_1'' - 3 \hat{R} \gamma_1' + 3 \gamma_1 + \dot{\omega}^2 \hat{R}^2 - \dot{k}^2 \hat{R}^2 (\hat{R} \gamma_2' - \gamma_2), \]
\[ \dot{f}(\hat{R}) = \dot{k}^2 \hat{R}^2 (\hat{R}^2 (p''/\mu) + \hat{R} \gamma_4' + \gamma_5 - \dot{\omega}^2 \hat{R}^2 + \dot{k}^2 \hat{R}^2 \gamma_6), \] (116)

where, by using Eqs. (18)–(26) with \( \zeta_{ij} = 0, i \neq j \), we get from Eq. (99)
\[ \gamma_1(\hat{R}) = \lambda^{-2} \lambda_z^{-2}[1 + \hat{\zeta}_3 + 2 \beta_0 (I_7 - I_4) \hat{\zeta}_3/\mu], \] (117)
\[ \gamma_2(\hat{R}) = -(1 + \hat{\zeta}_2) \lambda_z^{-2} - (1 + \hat{\zeta}_3) \lambda^{-2} \lambda_z^{-2} - 2 \beta_0 [(I_7 - I_4) (\lambda_z^2 \hat{\zeta}_2 + \lambda^{-2} \lambda_z^2 \hat{\zeta}_3)/\mu + 2 (\lambda_z^2 \hat{\zeta}_2 - \lambda^{-2} \lambda_z^2 \hat{\zeta}_3^2)], \] (118)
\[ \gamma_3(\hat{R}) = 0, \] (119)
\[ \gamma_4(\hat{R}) = (1 + \hat{\zeta}_4) \lambda_z^{-2} \lambda_z^{-2} + 2 \beta_0 \lambda^{-2} \lambda_z^{-2} \hat{\zeta}_3 [(I_7 - I_4)/\mu - 2 \lambda^2 \hat{\zeta}_1 - 2 \lambda^2 \hat{\zeta}_3 + 2 \lambda^{-2} \lambda_z^{-2} \hat{\zeta}_2] + 4 \beta_0 \lambda^2 \lambda_z^2 \hat{\zeta}_1 \hat{\zeta}_3, \] (120)
\[ \gamma_5(\hat{R}) = (1 + \hat{\zeta}_1) \lambda_z^2 - (1 + \hat{\zeta}_3) \lambda^{-2} \lambda_z^{-2} + 2 \beta_0 (\lambda_z^2 \hat{\zeta}_1 - \lambda^{-2} \lambda_z^2 \hat{\zeta}_3) [(I_7 - I_4)/\mu + 2 \lambda^2 \hat{\zeta}_1 + 2 \lambda^{-2} \lambda_z^{-2} \hat{\zeta}_3 - 4 \lambda^2 \hat{\zeta}_2], \] (121)
\[ \gamma_6(\hat{R}) = \lambda_z^2 + 2 \beta_0 (I_7 - I_4) \lambda_z^2 \hat{\zeta}_2/\mu + \lambda_z^2 \hat{\zeta}_2, \] (122)

where \( \beta_0 = \frac{\mu}{\mu} \). Also, in this case
\[ (I_7 - I_4)/\mu = (\lambda^2 - 1) \hat{\zeta}_1 + (\lambda_z^2 - 1) \hat{\zeta}_2 + (\lambda^{-2} \lambda_z^{-2} - 1) \hat{\zeta}_3, \] (123)

which vanishes in the undeformed configuration.

The boundary conditions (108) and (109), at \( \hat{R} = 1 \) and \( \hat{R} = \hat{B} \), give
\[ \gamma_1 \hat{R}^3 F''' + [\hat{R} a \gamma_1' - \gamma_1] \hat{R}^2 F'' - [\hat{R} a \gamma_1' - \gamma_1 - \hat{\omega}^2 \hat{R}^2] \hat{R} F' = 0, \]
\[ \hat{R}^2 \hat{k}^2 ([\gamma_2 + \lambda_3 \partial P/\partial \lambda_3(\hat{R})]) \hat{R} F' - (\gamma_4 + \hat{R} (p'/\mu)) + \lambda_3 \partial P/\partial \lambda_3(\hat{R}) F = 0, \] (124)

and
\[ \hat{R}^2 F'' - \hat{R} F' + \hat{R}^2 \hat{k}^2 F = 0, \] (125)

where
\[ \lambda_3 \partial P/\partial \lambda_3(\hat{R}) = \lambda^{-2} \lambda_z^{-2}, \] (126)
as \( \hat{\zeta}_3 \) is zero at the boundaries. Both Eqs. (124) and (125) hold at \( \hat{R} = 1 \) and \( \hat{R} = \hat{B} \).

Due to its complexity, we seek a numerical solution of Eq. (115). Introducing the notation
\[ F(\hat{R}) = z_1(\hat{R}), \quad F'(\hat{R}) = z_2(\hat{R}), \quad F''(\hat{R}) = z_3(\hat{R}), \quad F'''(\hat{R}) = z_4(\hat{R}), \] (127)
which, from Eq. (115) gives
\[
\frac{dz_1(\hat{R})}{d\hat{R}} = z_2(\hat{R}), \quad \frac{dz_2(\hat{R})}{d\hat{R}} = z_3(\hat{R}), \quad \frac{dz_3(\hat{R})}{d\hat{R}} = z_4(\hat{R}),
\]
\[
\frac{dz_4(\hat{R})}{d\hat{R}} = -\hat{f}_4 \hat{R}^{-1} z_4(\hat{R}) - \hat{f}_2 \hat{R}^{-2} z_3(\hat{R}) + \frac{\hat{f}_1}{\hat{R}} \hat{R}^{-3} z_2(\hat{R}) - \frac{\hat{f}}{\hat{R}} \hat{R}^{-4} z_1(\hat{R}),
\]
along with the four boundary conditions
\[
\hat{\gamma}_1(z_4(1) + [\hat{\gamma}_1(1) - \hat{\gamma}_1(1)] z_3(1) - [\hat{\gamma}_1(1) - \hat{\omega}^2] z_2(1) - \hat{k}^2 ([\hat{\gamma}_2(1) + \lambda^2 \lambda^2] z_2(1) - (\hat{\gamma}_4(1) + (p'/\mu) + \lambda^2 \lambda^2) z_1(1)] = 0,
\]
\[
z_3(1) - z_2(1) + \hat{k}^2 z_1(1) = 0,
\]
\[
\hat{\gamma}_1(\hat{B}) \hat{B}^2 z_4(\hat{B}) + [\hat{\gamma}_1(\hat{B}) - \hat{\gamma}_1(\hat{B})] z_3(\hat{B}) - [\hat{\gamma}_1(\hat{B}) - \hat{\omega}^2] z_2(\hat{B}) - \hat{B}^2 \hat{k}^2 [\hat{\gamma}_2(\hat{B}) + \lambda^2 \lambda^2] z_2(\hat{B}) - (\hat{\gamma}_4(\hat{B}) + \hat{B} (p'/\mu) + \lambda^2 \lambda^2) z_1(\hat{B})] = 0,
\]
\[
\hat{B}^2 z_3(\hat{B}) - \hat{B}^2 \hat{B} z_2(\hat{B}) + \hat{B}^2 \hat{k}^2 z_1(\hat{B}) = 0.
\]

4.2. Numerical solution of the boundary value problem for an infinite residually-stressed thick-walled cylindrical tube. A numerical approach is adopted to find the solution in the undeformed configuration of an infinite thick-walled cylindrical tube. Using Eqs. (117)-(122) in Eq. (100) and (108), we get the differential equations to be solved with the special model (60). For brevity, it is assumed that \( \zeta_2 = 0 \) in the undeformed configuration. For so, the stretches are equal to unity and also the derivative with respect to \( r \) vanishes. Therefore, from Eqs. (117)-(122),
\[
\hat{\gamma}_1 = 1 + \hat{\zeta}_3, \quad \hat{\gamma}_2 = -2 - \hat{\zeta}_3 - 4 \beta_0 \hat{\zeta}_3^2,
\]
\[
\hat{\gamma}_3 = 0, \quad \hat{\gamma}_4 = 1 + \hat{\zeta}_3 - 4 \beta_0 \hat{\zeta}_3 (\hat{\zeta}_1 - \hat{\zeta}_3),
\]
\[
\hat{\gamma}_5 = \hat{\zeta}_1 - \hat{\zeta}_3 + 4 \beta_0 (\hat{\zeta}_1^2 - \hat{\zeta}_3^2), \quad \hat{\gamma}_6 = 1,
\]
where the principal (residual) stresses are
\[
\hat{\zeta}_1 = \beta_2 (2R/A - 1 - B/A), \quad (134)
\]
\[
\hat{\zeta}_3 = \beta_2 (1 - A/R)(R/A - B/A), \quad (135)
\]
in the dimensionless form. Using Eq. (102), the expression for \( p''_0(R) \), in the undeformed configuration, is
\[
p''_0/\mu = \frac{d^2 \hat{\zeta}_3}{d\hat{R}^2} - \frac{\hat{R}}{\hat{R}} \left( \frac{d \hat{\zeta}_1}{d\hat{R}} - \frac{d \hat{\zeta}_3}{d\hat{R}} \right) + \frac{1}{\hat{R}^2} (\hat{\zeta}_1 - \hat{\zeta}_3), \quad (136)
\]
which is evaluated using Eq. (70) and we find that \( p''_0 = 0 \) for this special case.

In order to have dimensionless expressions, the following notations are introduced,
\[
\hat{R} = R/A, \quad \hat{\beta} = B/A, \quad \hat{k} = kA,
\]
\[
\hat{\omega} = \sqrt{(p/\mu)A} \omega, \quad \hat{\zeta}_i(\hat{R}) = \gamma_i(\hat{R})/\mu, \quad i \in \{1, 2, 4, 5, 6\}. \quad (137)
\]
Using these notations and Eqs. (100), (133)-(136), the equation of motion for the special model in the undeformed configuration is
\[
\hat{f}_4(\hat{R}) R^4 \frac{d^4 F}{dR^4} + \hat{f}_3(\hat{R}) R^3 \frac{d^3 F}{dR^3} + \hat{f}_2(\hat{R}) R^2 \frac{d^2 F}{dR^2} - \hat{f}_1(\hat{R}) R \frac{dF}{dR} + \hat{f}(\hat{R}) F = 0, \quad (138)
\]
respectively, and both the above equations hold at \( \hat{\zeta} \) respectively.

\[
\begin{align*}
\dot{f}_4(\mathcal{R}) &= \hat{\gamma}_1, \\
\dot{f}_4(\mathcal{R}) &= 2(\mathcal{R}\hat{\gamma}_1' - \hat{\gamma}_1), \\
\dot{f}_2(\mathcal{R}) &= \mathcal{R}^2\hat{\gamma}_1'' - 3\mathcal{R}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\mathcal{R}^2 + \hat{k}^2\mathcal{R}^2\hat{\gamma}_2, \\
\dot{f}_1(\mathcal{R}) &= \mathcal{R}^2\hat{\gamma}_1'' - 3\mathcal{R}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\mathcal{R}^2 - \hat{k}^2\mathcal{R}^2(\hat{\gamma}_2' - \hat{\gamma}_2), \\
\dot{f}(\mathcal{R}) &= \hat{k}^2\mathcal{R}^2(\hat{\gamma}_1' + \hat{\gamma}_5 - \hat{\omega}^2\mathcal{R}^2 + \hat{k}^2\mathcal{R}^2\hat{\gamma}_6),
\end{align*}
\]

(139)

and from Eqs. (133), we get

\[
\begin{align*}
\hat{\gamma}_1(\mathcal{R}) &= 1 + \beta_2(1 - 1/\mathcal{R})(\hat{\mathcal{R}} - \hat{\beta}), \\
\hat{\gamma}_2(\mathcal{R}) &= -2 - \beta_2(1 - 1/\mathcal{R})(\hat{\mathcal{R}} - \hat{\beta}) - 4\beta_1(1 - 1/\mathcal{R})(\hat{\mathcal{R}} - \hat{\beta})^2, \\
\hat{\gamma}_4(\mathcal{R}) &= 1 + \beta_2(1 - 1/\mathcal{R})(\hat{\mathcal{R}} - \hat{\beta}) - 4\beta_1(1 - 1/\mathcal{R})(\hat{\mathcal{R}} - \hat{\beta})(\hat{\mathcal{R}} - \hat{\beta}), \\
\hat{\gamma}_5(\mathcal{R}) &= (\hat{\mathcal{R}} - \hat{\beta}/\mathcal{R})[\beta_2 + 4\beta_1(3\hat{\mathcal{R}} - 3 - 2\hat{\beta} + \hat{\beta}/\mathcal{R})], \\
\hat{\gamma}_6(\mathcal{R}) &= 1,
\end{align*}
\]

(140)

(141)

(142)

(143)

and the derivatives are

\[
\begin{align*}
\hat{\gamma}_1'(\mathcal{R}) &= \beta_2(1 - \hat{\beta}/\mathcal{R}^2), \\
\hat{\gamma}_1''(\mathcal{R}) &= 2\beta_2/\mathcal{R}^3, \\
\hat{\gamma}_2'(\mathcal{R}) &= -(1 - \hat{\beta}/\mathcal{R}^2)[\beta_2 + 8\beta_1(1 - 1/\mathcal{R})(\mathcal{R} - \hat{\beta})], \\
\hat{\gamma}_4'(\mathcal{R}) &= \beta_2(1 - \hat{\beta}/\mathcal{R}^2) - 4\beta_1[(1 - \hat{\beta}/\mathcal{R}^2)(\mathcal{R} - \hat{\beta}/\mathcal{R}) \\
&\quad + (1 - 1/\mathcal{R})(\mathcal{R} - \hat{\beta})(1 + \beta/\mathcal{R}^2)],
\end{align*}
\]

(144)

(145)

(146)

(147)

respectively.

Also, \( p_0/\mu \), in reference configuration, is

\[
p_0/\mu = \hat{\gamma}_1' - (\hat{\zeta}_1 - \hat{\zeta}_3)/\mathcal{R},
\]

(148)

which vanishes in this special case. As \( \zeta_3 \) is at the boundary, Eqs. (108) and (109), appropriately made dimensionless, specialize to

\[
\begin{align*}
\mathcal{R}^3F''' + (\mathcal{R}\hat{\gamma}_1' - 1)\mathcal{R}^2F'' - (\mathcal{R}\hat{\gamma}_1' - 1 - \hat{\omega}^2\mathcal{R}^2 + 3k^2\mathcal{R}^2)\mathcal{R}F' \\
+ 2k^2\mathcal{R}^2F &= 0, \\
\mathcal{R}^2F'' - \mathcal{R}F' + \mathcal{R}^2\hat{k}^2F &= 0,
\end{align*}
\]

(149)

(150)

respectively, and both the above equations hold at \( \hat{\mathcal{R}} = 1 \), and \( \hat{\beta} \).

In order to find a numerical solution of Eq. (138), the problem is transformed to a system of first order linear ordinary differential equations. Suppose

\[
F'(\mathcal{R}) = y_1(\mathcal{R}), \quad F'(\mathcal{R}) = y_2(\mathcal{R}), \quad F'(\mathcal{R}) = y_3(\mathcal{R}), \quad F''(\mathcal{R}) = y_4(\mathcal{R}),
\]

(151)

which, from Eq. (149) give

\[
\begin{align*}
\frac{dy_1}{d\mathcal{R}} &= y_2(\mathcal{R}), \quad \frac{dy_2}{d\mathcal{R}} = y_3(\mathcal{R}), \quad \frac{dy_3}{d\mathcal{R}} = y_4(\mathcal{R}), \\
\frac{dy_4}{d\mathcal{R}} &= -\frac{f_3}{f_4}\mathcal{R}^{-1}y_4 - \frac{f_2}{f_4}\mathcal{R}^{-2}y_3 + \frac{f_1}{f_4}\mathcal{R}^{-3}y_2 - \frac{f}{f_4}\mathcal{R}^{-4}y_1.
\end{align*}
\]

(152)

Equations (149) and (150) on \( \mathcal{R} = 1 \) and \( \mathcal{R} = \hat{\beta} \), give the boundary equations

\[
\begin{align*}
y_1(1) + (\hat{\gamma}_1' - 1)y_2(1) + (1 + \hat{\omega}^2 - 3\hat{k}^2)y_2(1) + 2\hat{k}^2y_3(1) &= 0, \\
y_3(1) - y_2(1) + \hat{k}^2y_1(1) &= 0,
\end{align*}
\]

(153)

(154)
\[
\dot{\beta}^3 y_4(\dot{\beta}) + (\dot{\beta}^2 \gamma_1(\dot{\beta}) - 1)\ddot{\beta}^2 y_2(\ddot{\beta}) + (1 + \omega^2 \dot{\beta}^2) \\
-3k^2 \dot{\beta}^2 y_2(\dot{\beta}) + 2k^2 \beta^2 y_1(\beta) = 0, \quad (155) \\
\beta^2 y_3(\beta) - \beta y_2(\beta) + \beta^2 k^2 y_1(\beta) = 0. \quad (156)
\]

4.3. Isotropy. As a special case it is considered that when the material is not residually stressed, it is isotropic. The boundary value problem equations above is reduced to the to the classical problem in linear elasticity and can be solved analytically. For so, \( \beta_1 = 0 = \beta_2 \). Dropping the notation defined in Eq. (137), we have \( \gamma_1 = 1, \gamma_2 = -2, \gamma_4 = 1, \gamma_5 = 1, \gamma_6 = 0 = \gamma_1 = \gamma_1' = \gamma_2 = \gamma_4 \). The equation of motion (138) therefore becomes

\[
R^4 F^{(4)} - 2R^3 F^{(3)} + (3 + \omega^2 R^2 - 2k^2 R^2) R^2 F'' - (3 + \omega^2 R^2 - 2k^2 R^2) RF' \\
+k^2 R^4(\omega^2 - k^2) F = 0, \quad (157)
\]

with boundary conditions from Eqs. (153)–(156)

\[
F^{(4)} - F^{(3)} + (1 + \omega^2 - 3k^2) F'' + 2k^2 F = 0, \quad \text{on } R = 1, \quad (158) \\
\beta^3 F^{(4)} - \beta^2 F'' + (1 + \omega^2 \beta^2 - 3k^2 \beta^2) \beta F' + 2k^2 \beta^2 F = 0, \quad \text{on } R = \beta, \quad (159) \\
F^{(4)} - F + k^2 F = 0, \quad \text{on } R = 1, \quad (160) \\
\beta^2 F^{(4)} - \beta F' + \beta^2 k^2 F = 0, \quad \text{on } R = \beta. \quad (161)
\]

Equation (157) can be rewritten as

\[
LM[F(R)] = 0, \quad (162)
\]

where

\[
L = r^4\left(\frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} - k^2\right), \quad M = \frac{d^2}{dR^2} - \frac{1}{R} \frac{d}{dR} - k^2 + \omega^2. \quad (163)
\]

The solution of Eq. (162) is given by

\[
F(R) = A_1 RI_1(kR) + A_2 RK_1(kR) \\
+ A_3 RJ_1(\sqrt{\omega^2 - k^2} R) + A_4 RY_1(\sqrt{\omega^2 - k^2} R), \quad (164)
\]

where \( A_1, A_2, A_3, A_4 \) are arbitrary constants which can be determined using Eqs. (158)–(159). Here, \( J_1 \) and \( Y_1 \) are the Bessel functions of first and the second kind (of order one), respectively. The functions \( I_1 \) and \( K_1 \) are the modified Bessel functions of the first and the second kind (of order one), respectively. A detailed discussion on these functions can be found in [1].

Using Eq. (164) in (158)–(161), four algebraic equations in the unknown coefficients \( A_1, A_2, A_3, A_4 \) are obtained. The determinant of the matrix of coefficient, say \( A \) must vanish for a non-trivial solution. The dispersion relation relating \( k \) and \( \omega \) is thus obtained as

\[
\det A = 0. \quad (165)
\]

For a fixed \( \beta \), a simple code in MAPLE is used to solve Eq. (165) and the built-in command ‘implicitplot’ is used to obtain the dispersion curves. Figure 9 illustrates the comparison between the results for linear elasticity for an isotropic material and the results obtained through Eq. (152)–(156) with no residual stress. It is found that the numerical results obtained here are in good accordance with the classical results.
4.4. Numerical results. A two point boundary value problem (BVP) may possess one solution, more than one solution or no solution. In the case of more than a single solution, BVP solvers in softwares require an initial value for a parameter for the solver to converge to the nearest solution. There can be a single set, a finite number of (possible) sets, or an infinite number of possible sets, of parameters associated with the solution of a BVP.

To find a numerical solution of Eq. (152) with Eqs. (153)–(156), a built-in MATLAB function ‘bvp4c’ is used. This input of this function is a system of first order ordinary differential equations and an initial guess for the unknown parameters. For fixed $\hat{\beta}$, $\beta_1$, $\beta_2$ and $\hat{k}$, the solver gives a solution and therefore dispersion curves are obtained which are graphed for illustration purposes. The code is mentioned in the Appendix.

For the case when there is no residual stress, $\beta_1 = 0 = \beta_2$. For this instance, dispersion curves are plotted in Fig. 9 which are in accordance with the analytic results in Section 4.3. These results are also similar to those presented in [18] for variable wall thickness and different frequencies for waves in a (hollow) elastic rod. In [18], the displacements are expanded in a series of orthogonal polynomials which are a function of the radial coordinate and only the earliest terms in the series are retained. The error because of omission of terms is reduced by various adjustment factors for the frequency spectrum to match the exact theory.

In [3], the authors have considered a pre-stretched and an initially stressed (hollow) cylinder. The results are obtained for study of wave propagation with and without the presence of initial stress. Figure 10 shows dispersion curves that are similar to those found in [3] for variable wall thickness when initial stress is absent. Further, in the same paper, plots of dispersion curves are presented to study the influence of pre-stretch and initial stress on the wave speed. In our discussion, the graphs show modes without any more branches in contrast to the graphs in [3] since the value of pre-stretch is assumed as unity. Apart from this, the behavior of curves is similar in the first few modes.

Figures 11–13 illustrate the effect of residual stress for different modes. For zero residual stress, the strain energy function follows the behavior of the neo-Hookean type materials. In this case, first few modes are shown in Fig. 10 and it may
be noted that for small values of dimensionless wave number, wave speed is same with increasing wall thickness. The plots in Fig. 11 show a different trend. With increasing wall thickness and for small \( k \), the first modes in Fig. 11 have different phase speeds. From Fig. 12, for varying parameters and fixed wall thickness, first few modes are shown. The graph 'a' shows the behavior for zero residual stress which thus the case of neo-Hookean type material. It is observed that as \( \beta_1 \) and \( \beta_2 \) increase above zero, the phase speed is reduced from that observed for a neo-Hookean type materials. For decreasing values, say \( \beta_1 < 0 \) or \( \beta_2 < 0 \), as shown in plots (d) and (e), the phase speed increases. First four modes are shown in Fig. 13 in the case when the residual stress is present, shown in the continuous plots, and when the residual stress is zero (dashed plots).

**Figure 10.** First modes from Eqs. (152)–(156) in the absence of residual stress, \( \beta_1 = 0 = \beta_2 \), (a) \( \hat{\beta} = 3 \), (b) \( \hat{\beta} = 2.5 \), (c) \( \hat{\beta} = 2 \), (d) \( \hat{\beta} = 1.5 \).

**Figure 11.** First modes from Eqs. (152)–(156) for \( \beta_1 = 7, \beta_2 = 2 \) and (a) \( \hat{\beta} = 1.5 \), (b) \( \hat{\beta} = 2 \), (c) \( \hat{\beta} = 2.5 \).
5. **Conclusions.** In this paper, nonlinear theory of elasticity is used to study the effect of initial stress on the wave propagation in an incompressible hollow cylinder. The initial stress is considered non-homogeneous and the material bears properties of isotropic materials in the absence of this initial stress. The hollow cylindrical tube is considered to first undergo a finite deformation and later an infinitesimal deformation. The pressure boundary conditions are used and explicit expression for pressure inside the tube is calculated. This pressure is a function of the finite deformation through the stretches and also of the residual stress. Variation in pressure is observed for a given wall thickness is observed for varying parameters.
A prototype strain energy function is used to study the effect of residual stress on wave propagation in the cylindrical tube. The two point boundary value problem is solved numerically and analyzed. It is noted that for small values of dimensionless wave number, wave speed is same with increasing wall thickness. With increasing wall thickness and for small wave number, the first modes for varying values of parameters have different phase speeds. A comparison is also drawn when the residual stress vanishes (a neo-Hookean type material). It is found that for positive values of parameters, the phase speed is reduced from that observed for a neo-Hookean type materials. For decreasing values the phase speed increases for a fixed wall thickness.

Acknowledgments. This work forms a part of the PhD thesis by the author completed at the University of Glasgow and funded through the Faculty development program, Pakistan. Also, the help provided by Dr. Steven Roper, lecturer, University of Glasgow, UK, in software code for the results presented here is highly appreciated.

Appendix. The following file mentions the system of differential equations and the boundary conditions.

```matlab
% Global parameters
function sol_bvp=bvp(b1,b2,A,B, alpha, omega_guess)
N=200;
options=bvpset('RelTol',1e-4);
solguess=[1;1;1;1;0];
solinit=bvpinit(linspace(A,B,N),solguess,omega_guess);
sol_bvp=bvp4c(@sysode,@boundary_conditions,solinit,options);
function out=gamma1(R)
out=1+b2*(1-A./R).*(R/A-B/A);
end
function out=gamma1p(R)
out=b2*(1/A-B./R.^2);
end
function out=gamma1pp(R)
out=2*b2*B./R.^3;
end
function out=gamma2(R)
out=-2-b2*(1-A./R).*(R/A-B/A)-4*b1*((1-A./R).*(R/A-B/A)).^2;
end
function out=gamma2p(R)
out=-(1/A-B./R.^2).*b2+8*b1*(1-A./R).*(R/A-B/A);
end
function out=gamma4(R)
out=1+b2*(1-A./R).*(R/A-B/A)-4*b1*(1-A./R).*(R/A-B/A).
*(R/A-B./R);
end
function out=gamma4p(R)
out=b2*(1/A-B./R.^2)-4*b1*((1/A-B./R.^2).*(R/A-B./R)+...
```

(1−A./R).∗(R/A−B/A).∗(1/A+B./R.^2));
end
function out=gamma5(R)
out=(R/A−B./R).∗(b2+4∗b1∗(3*R/A−3−2*B/A+B./R));
end
function out=gamma6(R)
out=1;
end
function out=f4(R)
out=gamma1(R).∗R.^4;
end
function out=f3(R)
out=2∗R.^3.∗(R.∗gamma1p(R)−gamma1(R));
end
function out=f2(R,omega)
out=R.^2.∗(R.^2.∗gamma1pp(R)−3∗R∗gamma1p(R)+3∗gamma1(R)
+omega^2∗R.^2+ . . .
alpha^2∗R.^2∗gamma2(R));
end
function out=f1(R,omega)
out=R.∗(R.^2∗gamma1pp(R)−3∗R∗gamma1p(R)+3∗gamma1(R)
−omega^2∗R.^2− . . .
alpha^2∗R.^2∗(R.∗gamma2p(R)−gamma2(R)));
end
function out=f0(R,omega)
out=alpha^2∗R.^2∗gamma4p(R)+gamma5(R)
−omega^2∗R.^2+alpha^2∗ . . .
R.^2.∗gamma6(R));
end
function dydR=sysode(R,y,omega)
dydR=[y(2);
y(3);
y(4);
(−f0(R,omega)∗y(1)+f1(R,omega)∗y(2)−f2(R, . . .
omega)∗y(3)− . . .
f3(R)∗y(4))./f4(R);
y(1)^2];
end
function bcs=boundary_conditions(yA,yB,omega)
bcs=[A^2∗yA(3)−A∗yA(2)+A^2∗alpha^2∗yA(1);
B^2∗yB(3)−B∗yB(2)+B^2∗alpha^2∗yB(1);
gamma1(A)∗A^3∗yA(4)+(A∗gamma1p(A)−gamma1(A))∗A^2∗yA(3)
−(A∗ . . .
gamma1p(A)−gamma1(A)−omega^2*A^2)∗A∗yA(2)
−A^2∗alpha^2∗(2∗gamma1(A)∗A∗yA(2)− . . .
\begin{align*}
\text{gamma1}(A) + 2\text{gamma1}(A) + yA(1); \\
\text{gamma1}(B) + B^{3}yB(4) + (B\text{gamma1}(B) - \text{gamma1}(B))B^{2}yB(3) \\
- (B\text{gamma1}(B) - \text{gamma1}(B) - \omega^{2}B^{2})B yB(2) \\
+ (B\text{gamma1}(B) + 2\text{gamma1}(B))yB(1)); \\
yA(5) \\
yB(5) - 1; \\
\%	ext{ The final condition is a normalisation condition as the system is linear the homogeneous BCS only give the solution to within an arbitrary multiple, the normalisation condition is to select one solution.}
\end{align*}

The following file runs the loop with an initial guess value of \( \omega \).

\begin{verbatim}
omega= any real number as initial guess
alvec =1:0.01:5;
omegavec =[];
for al=1:0.01:5,
al
mysol=bvp(0,0,1,2,1,omega);
omega=mysol.parameters;
omegavec=[omegavec,omega];
end
\end{verbatim}

\section*{REFERENCES}

[1] M. Abramowitz and I. A. Stegun, \textit{Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables}, Dover Publications, Inc., New York, 1966.

[2] J. D. Achenbach, \textit{Wave Propagation in Elastic Solids}, North-Holland Series in Applied Mathematics and Mechanics, 16. North-Holland Publishing Co., Amsterdam, 1976.

[3] S. D. Akbarov and A. N. Guz, Axisymmetric longitudinal wave propagation in pre-stressed compound circular cylinders, \textit{Int. J. Eng. Sc.}, 42 (2004), 769–791.

[4] S. D. Akbarov and E. T. Bagirov, Axisymmetric longitudinal wave dispersion in a bi-layered circular cylinder with inhomogeneous initial stresses, \textit{J. Sound Vib}, 450 (2019), 1–27.

[5] M. A. Biot, Non-linear theory of elasticity and the linearized case for a body under initial stress, \textit{Phil. Mag.}, 27 (1939), 468–489.

[6] M. A. Biot, The influence of initial stress on elastic waves, \textit{J. App. Phy.}, 11 (1940), 522–530.

[7] C. J. Chuong and Y. C. Fung, On residual stress in arteries, \textit{J. Biomech. Eng.}, 108 (1986), 189–192.

[8] A. Guillou and R. W. Ogden, Growth in soft biological tissue and residual stress development, \textit{Mechanics of Biological Tissue}, Springer, Berlin Heidelberg, (2006), 47–62.

[9] M. E. Gurtin, \textit{An Introduction to Continuum Mechanics}, Mathematics in Science and Engineering, 158. Academic Press, Inc., New York-London, 1981.

[10] A. Hoger, On the residual stress possible in an elastic body with material symmetry, \textit{Arch. Rat. Mech. Anal.}, 88 (1985), 271–290.

[11] A. Hoger, On the determination of residual stress in an elastic body, \textit{J. Elasticity}, 16 (1986), 303–324.

[12] A. Hoger, Residual stress in an elastic body: A theory for small strains and arbitrary rotations, \textit{J. Elasticity}, 31 (1993), 1–24.
[13] A. Hoger, The constitutive equation for finite deformations of a transversely isotropic hyperelastic material with residual stress, *J. Elasticity*, 33 (1993), 107–118.

[14] B. E. Johnson and A. Hoger, The dependence of the elasticity tensor on residual stress, *J. Elasticity*, 33 (1993), 145–165.

[15] B. E. Johnson and A. Hoger, The use of strain energy function to quantify the effect of residual stress on mechanical behaviour, *Math. and Mech. of Solids*, 4 (1993), 447–470.

[16] A. Hoger, The elasticity tensor of a residually stressed material, *J. Elasticity*, 31 (1991), 219–237.

[17] C. S. Man and W. Y. Lu, Towards an acoustoelastic theory of measurement of residual stress, *J. Elasticity*, 17 (1987), 159–182.

[18] H. D. McNiven, A. H. Shah and J. L. Sackman, Axially symmetric waves in hollow, elastic rods: Part 1, *J. Acous. Soc. Am.*, 40 (1966), 784–792.

[19] R. W. Ogden, Nonlinear elasticity, anisotropy and residual stresses in soft tissue, *Biomechanics of Soft Tissue in Cardiovascular Systems*, Springer, Wien, (2003), 65–108.

[20] R. W. Ogden, *Nonlinear Elastic Deformations*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, Halsted Press, New York, 1984.

[21] R. W. Ogden and C. A. J. Schulze-Bauer, Phenomenological and structural aspects of the mechanical response of arteries, *Mech. Bio.*, 242 (2000), 125–140.

[22] R. W. Ogden, *Nonlinear Elasticity with Application to Material Modelling, Lecture Notes 6, Centre of Excellence for Advanced Materials and Structures*, Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, 2003.

[23] R. W. Ogden, Nonlinear elasticity, anisotropy, material stability and residual stresses in soft tissue, *Biomechanics of Soft Tissue in Cardiovascular Systems*, Springer, Wien, (2003), 65–108.

[24] A. Ozturk and S. D. Akbarov, Torsional wave propagation in a pre-stressed circular cylinder embedded in a pre-stressed elastic medium, *Applied Mathematical Modelling*, 33 (2009), 3630–3649.

[25] A. Rachev and K. Hayashi, Theoretical study of the effects of vascular smooth muscle contraction and strain and stress distribution in arteries, *Ann. Bio. Eng.*, 27 (1999), 459–468.

[26] E. Rodriguez, A. Hoger and A. D. McCulloch, Stress-dependent finite growth in soft elastic tissues, *J. Biomech.*, 27 (1994), 455–467.

[27] M. Shams, Wave Propagation in Residually-Stressed Materials, PhD thesis, University of Glasgow, Glasgow, UK, 2010.

[28] M. Shams, M. Destrade and R. W. Ogden, Initial stresses in elastic solids: Constitutive laws and acoustoelasticity, *J. Wave Motion*, 48 (2011), 552–567.

[29] M. Shams, Reflection of plane waves from the boundary of an initially stressed incompressible half-space, *Mathematics and Mechanics of Solids*, 24 (2019), 406–433.

[30] A. J. M. Spencer, Theory of invariants, *Continuum Physics, Academic Press, New York*, 1 (1971), 239–353.

[31] K. Takamizawa and K. Hayashi, Strain energy density function and uniform strain hypothesis for arterial mechanics, *J. Biomech.*, 20 (1987), 7–17.

[32] S. Tang, Wave propagation in initially-stressed elastic solids, *Acta Mech.*, 61 (1967), 92–106.

[33] X. M. Zhang, Z. H. Niu and J. G. Yu, Effect of initial stress on axisymmetric torsional wave in a unidirectional composite hollow cylinder, *Proceedings of the 2015 Symposium on Piezoelectricity, Acoustic Waves and Device Applications, SPAWDA*, 2015 (2015), 349–352.

Received February 2019; revised July 2019.

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