The gauging of BV algebras

by

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Abstract

A BV algebra is a formal framework within which the BV quantization algorithm is implemented. In addition to the gauge symmetry, encoded in the BV master equation, the master action often exhibits further global symmetries, which may be in turn gauged. We show how to carry this out in a BV algebraic set up. Depending on the nature of the global symmetry, the gauging involves coupling to a pure ghost system with a varying amount of ghostly supersymmetry. Coupling to an $N = 0$ ghost system yields an ordinary gauge theory whose observables are appropriately classified by the invariant BV cohomology. Coupling to an $N = 1$ ghost system leads to a topological gauge field theory whose observables are classified by the equivariant BV cohomology. Coupling to higher $N$ ghost systems yields topological gauge field theories with higher topological symmetry. In the latter case, however, problems of a completely new kind emerge, which call for a revision of the standard BV algebraic framework.
1 Introduction

The Batalin–Vilkovisky (BV) approach [1,2] is the most general and powerful quantization algorithm presently available. It is suitable for the quantization of ordinary gauge theories, such as Yang–Mills theory, as well as more complicated gauge theories with open and/or reducible gauge symmetries. Its main feature consists in the introduction of ghost fields from the outset automatically incorporating in this way BRST symmetry.

The general structure of the BV formalism is as follows [3,4]. Given a classical field theory with gauge symmetries, one introduces an antifield with opposite statistics for each field, including ghost fields, therefore doubling the total number of fields. The resulting field/antifield space $\mathcal{F}$ is equipped with an odd Poisson bracket $\{\cdot, \cdot\}$, called antibracket, and acquires an odd phase space structure, in which fields and antifields are canonically conjugate. At tree level in the quantum theory, the original classical action is extended to a new action $S_0$ defined on the whole content of $\mathcal{F}$ and exhibiting an off shell odd symmetry corresponding to the gauge symmetry of the original field theory. The gauge fixing is carried out by restricting the action $S_0$ to a suitable Lagrangian submanifold $\mathcal{L}$ in $\mathcal{F}$. Gauge independence, that is independence from the choice of $\mathcal{L}$, is ensured if $S_0$ satisfies the classical BV master equation

$$\{S_0, S_0\} = 0.$$ (1.1)

At loop level, quantum corrections modify the action $S_0$ and turn it into a quantum action $S_\hbar$. Gauge independence is then ensured provided $S_\hbar$ satisfies the quantum BV master equation

$$\hbar\Delta S_\hbar + \frac{1}{2}\{S_\hbar, S_\hbar\} = 0,$$ (1.2)

where $\Delta$ is a suitably regularized odd functional Laplacian in $\mathcal{F}$. Violations of this correspond to gauge anomalies.
The observables of the field theory constructed in this way are characterized by having gauge independent correlators. The gauge independence of a correlator $\langle \psi_h \rangle$ is ensured if $\psi_h$ satisfies the equation

$$\delta_h \psi_h := \hbar \Delta \psi_h + \{ S_h, \psi_h \} = 0 \quad (1.3)$$

The solutions $\psi_h$ of (1.3) are called quantum BV observables. The quantum BV operator $\delta_h$ is nilpotent. Therefore, there is a cohomology associated with it, the quantum BV cohomology. Since correlators of BV exact observables vanish, effectively distinct BV observables are in one-to-one correspondence with the BV cohomology classes.

After this very brief review of BV theory, let us come to the topic of the paper. The algebraic structure consisting of the graded algebra of functionals on the field/antifield space $F$, the antibracket $\{ \cdot, \cdot \}$ and the odd Laplacian $\Delta$ is called a BV algebra. It provides the formal framework within which the BV quantization algorithm is implemented. This has motivated a number of mathematical studies of BV algebras [5–7].

The classical field theory originally considered, even if it is a gauge theory, may still have global symmetries. In certain cases, one may wish to gauge these latter. In a BV framework, the gauging of a global symmetry consists in the coupling of the ungauged “matter” field theory and a suitable pure “ghost” field theory corresponding to the symmetry. (Ordinary ghost and gauge fields normally combine in ghost superfields.) Two procedures of concretely working this out are possible in principle.

i) One couples the matter and the ghost field theories at the classical level, by adding suitable interaction terms, obtaining a gauged classical field theory. Then, one quantizes this latter using the BV algorithm, by constructing the appropriate BV algebra and quantum BV master action.

ii) One separately quantizes the matter and the ghost field theories, by con-
structing the appropriate BV algebra and quantum BV master action of each of them. Then, one embeds the matter and ghost BV algebra structures so obtained in a minimal gauged BV algebra structure and constructs a gauged quantum BV master action by adding the matter and ghost actions and suitable interaction terms in a way consistent with the quantum BV master equation.

We call these two approaches *classical gauging* and *BV algebra gauging*, respectively. Superficially, it may look like that classical gauging is more natural: after all, BV theory was devised precisely to quantize classical gauge theories. In fact, in certain cases, BV gauging is more advantageous.

In a prototypical example, one efficient way of generating a sigma model on a non trivial manifold $X$ is the gauging of a sigma model on a simpler manifold $Y$ carrying the action of a Lie group $G$ such that $X \simeq Y/G$ [8,9]. The target space of the gauged model turns out to be precisely $X$. In a BV formulation of the ungauged sigma model, $G$ acts as a group of global symmetries. The gauging of these is performed by coupling the ungauged model to a suitable ghost sigma model, yielding in a natural way a BV formulation of the gauged model [10–12].

The Alexandrov–Kontsevich–Schwartz–Zaboronsky (AKSZ) formalism of ref. [13] is a method of constructing solutions of the classical BV master equation directly, without starting from a classical action with a set of symmetries, as is originally done in the BV framework. When building models with gauged global symmetries in a AKSZ framework, BV algebra gauging is definitely more natural and transparent than classical gauging.

In this paper, we study in great detail the BV algebra gauging of a matter field theory with global symmetries. For a certain global symmetry, the ghost field theory to be coupled to the matter theory may have a varying amount of “ghostly supersymmetry”. Coupling, if feasible, to an $N = 0$ ghost system yields an ordinary gauge field theory. Coupling to an $N = 1$ ghost system leads to a topological gauge field theory. Coupling to higher $N$ ghost systems yields
topological gauge field theories with higher topological supersymmetry. In the latter case, however, problems of a completely new kind show up, which may require a major revision of the standard BV algebraic framework.

Though BV algebra gauging is ultimately carried out within the framework of BV theory, ordinary BV cohomology is not adequate for the classification of observables of the field theories constructed in this way. If \( \mathfrak{g} \) is the global symmetry Lie algebra, \( \mathfrak{g} \)-invariant BV cohomology in the \( N = 0 \) case, \( \mathfrak{g} \)-equivariant BV cohomology in the \( N = 1 \) case and presumably some higher \( \mathfrak{g} \)-equivariant BV cohomologies for larger \( N \) are required.

We shall carry out our analysis of BV gauging in a finite dimensional setting as in [5–7]. This has its advantages and disadvantages. It allows one to focus on the essential features of gauging, especially those of an algebraic and geometric nature, on one hand, but it is of course no substitute for full–fledged field theory, which is essentially infinite dimensional, on the other. Nevertheless, with the due caution, one can presumably extend our considerations to realistic BV field theories. Further, it is known that certain BV field theories have finite dimensional reductions which capture some of their relevant structural features [14–16].

The plan of this paper is as follows. In sect. 2 we review the basics of BV algebra theory and set the notation used in the subsequent sections. In sect. 3 we recall the definition and the main properties of the BV master action and observables. In sects. 4, 5 we illustrate how to carry out the \( N = 0 \) and \( N = 1 \) gauging of BV algebras and identify the relevant versions of BV cohomology. In sect. 6 we tackle the problem of higher \( N \) BV algebra gauging highlighting the conceptual problems arising in this case. In sect. 7 we illustrate a number of examples and applications of the theory developed in the preceding sections, showing in particular its relevance for the finite dimensional reduction of the gauged Poisson sigma model of refs. [10, 11]. In sect. 8 we provide some concluding remarks. Finally, in the appendices, we conveniently collect details
on a few technical issues involved in our analysis.

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Remarks on conventions and notation

In this paper, we use the following notations. All spaces and algebras are over the field $\mathbb{R}$.

a) Let $\mathcal{E}$ be $\mathbb{Z}$–graded vector space. $\mathcal{E}_k$ is the subspace of $\mathcal{E}$ of degree $k \in \mathbb{Z}$. If $x \in \mathcal{E}_k$ for some $k$, $x$ is said homogeneous and $\partial x = k$ is the degree of $x$. If $p \in \mathbb{Z}$, $\mathcal{E}[p]$ is the $\mathbb{Z}$–graded vector space such that $\mathcal{E}[p]_k = \mathcal{E}_{k-p}$. Similar conventions hold for a $\mathbb{Z}$–graded algebra $\mathcal{A}$. In this case, we denote by $\mathcal{A}_v$ the $\mathbb{Z}$–graded vector space underlying $\mathcal{A}$.

b) Let $\mathcal{E}$, $\mathcal{F}$ be $\mathbb{Z}$–graded vector spaces. $\mathcal{E} \otimes \mathcal{F}$ is the $\mathbb{Z}$–graded tensor product of $\mathcal{E}$, $\mathcal{F}$. Its grading is given by $(\mathcal{E} \otimes \mathcal{F})_k = \bigoplus_{l+m=k} \mathcal{E}_l \otimes \mathcal{F}_m$ for $k \in \mathbb{Z}$. The tensor products $\mathcal{A} \otimes \mathcal{B}$ of two $\mathbb{Z}$–graded algebras $\mathcal{A}$, $\mathcal{B}$ is defined in the same fashion with the graded multiplication

$$x \otimes u y \otimes v = (-1)^{\partial y \partial u} xy \otimes uv,$$

for homogeneous $x, y \in \mathcal{A}$, $u, v \in \mathcal{B}$.

c) Let $\mathcal{E}$, $\mathcal{F}$ be $\mathbb{Z}$–graded vector spaces. $\text{Hom}(\mathcal{E}, \mathcal{F})$ is the $\mathbb{Z}$–graded vector space of vector space homomorphisms of $\mathcal{E}$ into $\mathcal{F}$. Its grading is defined so that $X \in \text{Hom}_k(\mathcal{E})$ if, for all $l \in \mathbb{Z}$, $X \mathcal{E}_l \subset \mathcal{F}_{k+l}$. The space $\text{End}(\mathcal{E}) = \text{Hom}(\mathcal{E}, \mathcal{E})$, the set $\text{Iso}(\mathcal{E}, \mathcal{F})$ and the group $\text{Aut}(\mathcal{E}) = \text{Iso}(\mathcal{E}, \mathcal{E})$ are defined accordingly. Similar notions hold for two $\mathbb{Z}$–graded algebras $\mathcal{A}$, $\mathcal{B}$ with the proviso that algebra homomorphisms are concerned.

d) Let $\mathcal{E}$ be a $\mathbb{Z}$–graded vector space. For homogeneous $X, Y \in \text{End}(\mathcal{E})$, the graded commutator of $X, Y$ is given by

$$[X, Y] = XY - (-1)^{\partial X \partial Y} YX.$$
All commutators will always be assumed to be graded, unless otherwise stated.

e) Let $\mathcal{A}$ be a $\mathbb{Z}$–graded algebra. $\text{Der}(\mathcal{A})$ is the graded vector subspace of $\text{End}(\mathcal{A})$ of graded derivations of $\mathcal{E}$. If $k \in \mathbb{Z}$ and $D \in \text{Der}_k(\mathcal{A})$, then

$$D(xy) = Dxy + (-1)^{k\partial_x} xDy,$$

for homogeneous $x, y \in \mathcal{A}$.

f) If $x$ is a formal graded variable, then $\partial_{Lx} = (\partial/\partial x)_L$ and $\partial_{Rx} = (\partial/\partial x)_R$, the subfixes $L, R$ indicating left, right graded differentiation. If $\phi$ is a function of $x$, then $\partial_{Lx}\phi = (-1)^{(\partial\phi + 1)}\partial_x\partial_{Rx}\phi$.

g) A differential space is a pair $(\mathcal{E}, \delta)$, where $\mathcal{E}$ is a $\mathbb{Z}$–graded vector space, $\delta \in \text{End}_1(\mathcal{E})$ and $\delta^2 = 0$. The associated cohomology $H^*(\mathcal{E}, \delta)$ is a space. A differential algebra is a pair $(\mathcal{A}, \delta)$, where $\mathcal{A}$ is a $\mathbb{Z}$–graded algebra, $\delta \in \text{Der}_1(\mathcal{E})$ and $\delta^2 = 0$. The associated cohomology $H^*(\mathcal{A}, \delta)$ is then an algebra.
Batalin–Vilkovisky (BV) algebras are the formal structure underlying the BV quantization algorithm in quantum field theory \cite{1,2}. The BV algebra of a field theory consists of a graded algebra of functions of fields and antifields, an odd Poisson bracket defining the canonical structure of the field theory at the classical level and an odd Laplacian required for implementing the field theory’s quantization. BV algebras however can be treated in a completely formal setting without invoking any concrete field theoretic realization \cite{3,4}.

A BV algebra is a triple \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) consisting of the following elements.

1) A \(\mathbb{Z}\)-graded commutative associative unital algebra \(\mathcal{A}\).

2) A BV Laplacian, i.e. an element \(\Delta \in \text{End}_1(\mathcal{A})\) that is nilpotent,
\[
\Delta^2 = 0. \tag{2.1}
\]

3) A BV antibracket, i.e. an \(\mathbb{R}\)-bilinear map \(\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}\) such that
\[
\Delta(\phi \psi) = \Delta \phi \psi + (-1)^{\partial \phi} \phi \Delta \psi + (-1)^{\partial \phi} \{\phi, \psi\}, \tag{2.2}
\]
\[
\{\phi, \psi \upsilon\} = \{\phi, \psi\} \upsilon + (-1)^{\partial \phi + 1} \partial \psi \{\phi, \upsilon\}, \tag{2.3}
\]
for all homogeneous \(\phi, \psi, \upsilon \in \mathcal{A}\).

We notice that, by (2.2), \(\{\cdot, \cdot\}\) is determined by \(\Delta\). So, the notion of BV algebra could be defined in terms of \(\mathcal{A}, \Delta\) only.

Several properties can be derived from the BV algebra axioms.

a) One has \(\partial \{\phi, \psi\} = \partial \phi + \partial \psi + 1\) and
\[
\{\phi, \psi\} + (-1)^{\partial \phi + 1}(\partial \psi + 1) \{\psi, \phi\} = 0, \tag{2.4}
\]
\[
(-1)^{\partial \phi + 1}(\partial \psi + 1) \{\phi, \{\psi, \upsilon\}\} + (-1)^{\partial \psi + 1}(\partial \phi + 1) \{\psi, \{\upsilon, \phi\}\} \tag{2.5}
\]
\[
+ (-1)^{\partial \upsilon + 1}(\partial \psi + 1) \{\upsilon, \{\phi, \psi\}\} = 0,
\]

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for all homogeneous $\phi, \psi, \upsilon \in \mathcal{A}$. These relations follow from (2.2). $\mathcal{A}$ with the multiplicative structure given by the bracket $\{ \cdot, \cdot \}$ is a $\mathbb{Z}$–graded commutative algebra $\mathcal{A}_G$, called Gerstenhaber (odd Poisson) algebra. The gradings of $\mathcal{A}$, $\mathcal{A}_G$ are such that $\mathcal{A}_{Gv} = \mathcal{A}_v[1]$.

b) On account of (2.2), $\Delta \not\in \text{Der}_1(\mathcal{A})$. The Gerstenhaber bracket $\{ \cdot, \cdot \}$ measures the failure of $\Delta$ being so. However, one has

$$\Delta \{ \phi, \psi \} = \{ \Delta \phi, \psi \} + (-1)^{\partial \phi + 1} \{ \phi, \Delta \psi \},$$  \hspace{1cm} (2.6)

for homogeneous $\phi, \psi \in \mathcal{A}$. This relation follows from combining (2.1), (2.2). Thus, $\Delta \in \text{Der}_1(\mathcal{A}_G)$.

c) A derivation $D \in \text{Der}_k(\mathcal{A})$ such that

$$[D, \Delta] = 0$$  \hspace{1cm} (2.7)

is called a $BV$ derivation. If $D \in \text{Der}_k(\mathcal{A})$, one has $D \not\in \text{Der}_k(\mathcal{A}_G)$ in general. However, if $D$ is a $BV$ derivation, then $D \in \text{Der}_k(\mathcal{A}_G)$ as well. This follows straightforwardly from combining (2.2), (2.7).

d) For $\alpha \in \mathcal{A}_{k-1}$, let us set

$$\text{ad} \alpha \phi = \{ \alpha, \phi \}, \hspace{1cm} \phi \in \mathcal{A}.$$  \hspace{1cm} (2.8)

Then, simultaneously $\text{ad} \alpha \in \text{Der}_k(\mathcal{A})$, $\text{ad} \alpha \in \text{Der}_k(\mathcal{A}_G)$. These properties follow directly from (2.3), (2.5), respectively. A derivation $D \in \text{Der}_k(\mathcal{A})$ is called $BV$ inner, if it is of the form $D = \text{ad} \alpha$ for some $\alpha \in \mathcal{A}_{k-1}$ such that

$$\Delta \alpha = 0.$$  \hspace{1cm} (2.9)

Else, it is $BV$ outer. By (2.6), (2.9), a $BV$ inner $D$ fulfils (2.7) and, so, is $BV$.

A $BV$ algebra $(\mathcal{A}, \Delta, \{ \cdot, \cdot \})$ is a $BV$ subalgebra of a $BV$ algebra $(\mathcal{A}', \Delta', \{ \cdot, \cdot \}')$ if $\mathcal{A}$ is a subalgebra of $\mathcal{A}'$ such that $\Delta' \mathcal{A} \subset \mathcal{A}$, $\{ \mathcal{A}, \mathcal{A} \}' \subset \mathcal{A}$ and $\Delta = \Delta'|_\mathcal{A}$, $\{ \cdot, \cdot \} = \{ \cdot|_\mathcal{A}, \cdot|_\mathcal{A} \}'$. 

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There is a natural notion of homomorphism of BV algebras. Let \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\), \((\mathcal{A}', \Delta', \{\cdot, \cdot\}')\) be BV algebras. A map \(T : \mathcal{A} \to \mathcal{A}'\) is a BV algebra homomorphism, if

1) \(T \in \text{Hom}_0(\mathcal{A}, \mathcal{A}')\).

2) \(T\) intertwines the BV Laplacians \(\Delta, \Delta'\),

\[
T \Delta = \Delta' T.
\]

(2.10)

3) \(T\) intertwines the brackets \(\{\cdot, \cdot\}, \{\cdot, \cdot\}'\),

\[
T\{\phi, \psi\} = \{T\phi, T\psi\}',
\]

(2.11)

for \(\phi, \psi \in \mathcal{A}\).

As a matter of fact, \((2.11)\) is not an independent condition, as it follows from \((2.2), (2.10)\). One can also define a BV algebra monomorphism, epimorphism, isomorphism, endomorphism and automorphism in obvious fashion.

A few properties can be deduced from the BV algebra homomorphism axioms.

a) By \((2.11)\), \(T \in \text{Hom}_0(\mathcal{A}_G, \mathcal{A}_G')\) as well. Indeed, \(T\) is a homomorphism of the odd Poisson structures of \(\mathcal{A}_G, \mathcal{A}_G'\).

b) \(\ker T\) is a subalgebra of \(\mathcal{A}\) such that \(\Delta \ker T \subset \ker T, \{\ker T, \ker T\} \subset \ker T\) and, so, with the BV algebra structure induced by \(\mathcal{A}\), a BV subalgebra of \(\mathcal{A}\). Likewise, \(\text{im} T\) is a subalgebra of \(\mathcal{A}'\) such that \(\Delta' \text{im} T \subset \text{im} T, \{\text{im} T, \text{im} T\}' \subset \text{im} T\) and, so, with the BV algebra structure induced by \(\mathcal{A}'\), a BV subalgebra of \(\mathcal{A}'\). This follows immediately from \((2.10), (2.11)\).

Homomorphisms describe the natural relationships of BV algebras.

a) Let \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) be a BV subalgebra of the BV algebra \((\mathcal{A}', \Delta', \{\cdot, \cdot\}')\). Then, the natural injection \(I : \mathcal{A} \to \mathcal{A}'\) is a BV algebra monomorphism.

b) The automorphisms of a BV algebra \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) represent the symmetries of this latter. Let \(\alpha \in \mathcal{A}_{-1}\) satisfy \((2.9)\). Define a map \(T_\alpha : \mathcal{A} \to \mathcal{A}\) by

\[
T_\alpha = \exp(\text{ad} \alpha),
\]

(2.12)
the right hand side being defined by the usual exponential series. It is assumed
that that either the series terminates after a finite number of terms by algebraic
reasons or it converges in some natural topology of $\text{End}(A)$. Then, $T_\alpha$ is a BV
algebra automorphism. The automorphisms of this type are called $BV$ inner,
since $\text{ad} \alpha$ is a BV inner derivation of $A$. Correspondingly, all other BV algebra
automorphisms are called $BV$ outer.

The set of BV algebras can be organized as a category having BV algebras
homomorphisms as morphisms. One can define natural operations in this cate-
gory. In particular, there is a notion of tensor product of BV algebras that will
be extensively used in the following. Let $(A', \Delta', \{\cdot, \cdot\}')$, $(A'', \Delta'', \{\cdot, \cdot\}'')$ be BV
algebras. Construct a triple $(A, \Delta, \{\cdot, \cdot\})$ as follows.

1) $A = A' \otimes A''$, a tensor product of graded algebras.
2) $\Delta \in \text{End}_1(A)$ is defined by the relation

$$\Delta(\phi' \otimes \phi'') = \Delta'\phi' \otimes \phi'' + (-1)^{\phi'} \phi' \otimes \Delta''\phi'', \quad (2.13)$$

for homogeneous $\phi' \in A'$, $\phi'' \in A''$.

3) $\{\cdot, \cdot\} : A \times A \to A$ is defined by the relation

$$\{\phi' \otimes \phi'', \psi' \otimes \psi''\} = (-1)^{(\delta\psi'+1)\delta\phi''}\{\phi', \psi'\}' \otimes \phi'' \psi'' + (-1)^{(\delta\phi''+1)\delta\psi'} \phi' \psi' \otimes \{\phi'', \psi''\}'', \quad (2.14)$$

for homogeneous $\phi', \psi' \in A'$, $\phi'', \psi'' \in A''$. Then, $(A, \Delta, \{\cdot, \cdot\})$ is a BV algebra.

The verification of the basic relations $(2.1)$–$(2.3)$ is straightforward. $(A, \Delta, \{\cdot, \cdot\})$ is called the tensor product of the BV algebras $(A', \Delta', \{\cdot, \cdot\}')$, $(A'', \Delta'', \{\cdot, \cdot\}'')$.

The maps $I' : A' \to A$, $I'' : A'' \to A$ defined by $I'\phi' = \phi' \otimes 1''$, $\phi' \in A'$,
$I''\phi'' = 1' \otimes \phi''$, $\phi'' \in A''$ are BV algebra monomorphisms. Indeed, $I', I''$ satisfy
$(2.10), (2.11)$ on account of $(2.13), (2.14)$. In this way, $A', A''$ can be considered
as BV subalgebras of $A$.

Examples of BV algebras will be illustrated in the following sections.
Quantum BV master actions and observables

Let \((\mathcal{F}, \Delta, \{\cdot, \cdot\})\) be the BV algebra relevant for a BV quantization problem. In general, quantization can be viewed as the addition to a classical quantity of a quantum correction expressed perturbatively as a formal power series in the Planck constant \(\hbar\). For this reason, BV quantization requires working with the graded algebra \(\mathcal{F}((\hbar))\) of formal power series \(\phi_\hbar = \sum_{k \geq 0} \hbar^k \phi_{(k)}\) with \(\phi_{(k)} \in \mathcal{F}\), where \(\hbar\) is treated as a degree 0 formal parameter. The BV Laplacian \(\Delta\) and antibracket \(\{\cdot, \cdot\}\) extend by formal linearity to \(\mathcal{F}((\hbar))\). \((\mathcal{F}((\hbar)), \Delta, \{\cdot, \cdot\})\) is then also a BV algebra. The natural injection of \(\mathcal{F}\) into \(\mathcal{F}((\hbar))\), defined by \(\phi \mapsto \sum_{k \geq 0} \hbar^k \delta_{k,0} \phi\), is a BV algebra monomorphism and, so, \(\mathcal{F}\) can be viewed as a BV subalgebra of \(\mathcal{F}((\hbar))\). The quantum BV master action \(S_\hbar\) and observables \(\psi_\hbar\) are the solutions of eqs. (1.2) and (1.3) in \(\mathcal{F}((\hbar))\). The corresponding classical approximations \(S\) and \(\psi\) are obtained by truncating \(S_\hbar\) and \(\psi_\hbar\) to their components in \(\mathcal{F}\).

Keeping explicit the \(\hbar\) dependence of the relevant quantities in the following analysis would lead to unnecessary notational complication. For this reason, we shall treat the problems of quantization and classical approximation thereof more formally in the framework of a given BV algebra \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\). It is tacitly understood that, in any physical realization, \(\mathcal{A}\) must be correspondingly interpreted as either \(\mathcal{F}((\hbar))\) or \(\mathcal{F}\) for the relevant BV algebra \(\mathcal{F}\).

In the constructions of the following sections, other endomorphisms \(f \in \text{End}(\mathcal{A})\) will be considered beside \(\Delta\). It is tacitly understood that, in any physical realization, the \(f\) are independent from \(\hbar\).

Let \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) be a BV algebra. An element \(S \in \mathcal{A}_0\) is called a quantum BV master action of the BV algebra if \(S\) satisfies the quantum BV master equation

\[
\Delta S + \frac{1}{2}\{S, S\} = 0,
\]  
(3.1)
An element $\psi \in A$ is a quantum BV observable, if it satisfies the equation

$$\delta \psi = 0, \quad (3.2)$$

where $\delta$ is the quantum BV operator

$$\delta = \Delta + \text{ad} S. \quad (3.3)$$

From the definition, using the master equation (3.1), it can be easily verified that $\delta \in \text{End}_1(A_v)$ and that $\delta$ is nilpotent,

$$[\delta, \delta] = 2\delta^2 = 0. \quad (3.4)$$

Hence, $(A, \delta)$ is a differential space. The associated cohomology is the quantum BV cohomology space $H_{BV}^*(A)$. We note that $\delta$ is not a derivation, as $\Delta$ is not. So, even though $A$ is an algebra, $(A, \delta)$ is only a differential space. Correspondingly, $H_{BV}^*(A)$ is only a cohomology space.

The classical counterpart of the above is as follows. Let $(A, \Delta, \{\cdot, \cdot\})$ be a BV algebra. An element $S \in A_0$ is called a classical BV master action of the BV algebra if $S$ satisfies the classical BV master equation

$$\{S, S\} = 0. \quad (3.5)$$

An element $\psi \in A$ is a classical BV observable, if it satisfies the equation

$$\delta_c \psi = 0, \quad (3.6)$$

where $\delta_c$ is the classical BV operator

$$\delta_c = \text{ad} S. \quad (3.7)$$

From the definition, using the master equation (3.1), it can be easily verified that $\delta_c \in \text{End}_1(A_v)$ and that $\delta_c$ is nilpotent,

$$[\delta_c, \delta_c] = 2\delta_c^2 = 0. \quad (3.8)$$
So, $(\mathcal{A}, \delta_c)$ is a differential algebra. The associated cohomology is the *classical BV cohomology* algebra $H_{cBV}^*(\mathcal{A})$. Recall that, in the quantum case, $(\mathcal{A}, \delta)$ and $H_{BV}^*(\mathcal{A})$ are merely spaces.

Let $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$, $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$ be BV algebras and let $T : \mathcal{A} \to \mathcal{A}'$ be a BV algebra homomorphism (cf. sect. 2). If $S$ be a quantum BV master action of the BV algebra $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$, then

$$S' = TS$$

(3.9)

is a quantum BV master action of the BV algebra $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$. This follows easily from (3.1), (2.10), (2.11). Similarly, if $\psi$ is a quantum BV observable of the BV algebra $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$ and master action $S$, then

$$\psi' = T\psi$$

(3.10)

is a quantum BV observable of the BV algebra $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$ and master action $S'$. This follows from (3.2), (3.3), (2.10), (2.11). In fact, one has

$$T\delta = \delta'T.$$  

(3.11)

$T$ is therefore a chain map of the differential spaces $(\mathcal{A}, \delta)$, $(\mathcal{A}', \delta')$ and, so, it induces a homomorphism of the corresponding cohomology spaces $H_{BV}^*(\mathcal{A})$, $H_{BV}^*(\mathcal{A}')$. Analogous statements hold also in the classical case.

Let $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$, $(\mathcal{A}'', \Delta'', \{\cdot, \cdot\}'')$ be BV algebras and let $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$ be their tensor product (cf. sect. 2). If $S', S''$ are quantum BV master actions of $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$, $(\mathcal{A}'', \Delta'', \{\cdot, \cdot\}'')$, respectively, then

$$S = S' \otimes 1'' + 1' \otimes S''$$

(3.12)

is a quantum BV master action of $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$. This property follows straightforwardly from applying (2.13), (2.14). Analogously, if $\psi', \psi''$ are quantum BV observables of the BV algebras $(\mathcal{A}', \Delta', \{\cdot, \cdot\}')$, $(\mathcal{A}'', \Delta'', \{\cdot, \cdot\}'')$ and actions $S', S''$, 

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respectively, then

\[ \psi = \psi' \otimes 1'' + 1' \otimes \psi'' \]  \hspace{1cm} (3.13)

is a quantum BV observable of \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) and \(S\). The verification of this property is also straightforward. Again, similar statements hold in the classical case.

In quantum field theory, the above construction is simply the adjoining of two field theories with no mutual interaction. From a physical point of view is therefore rather trivial. In interesting models, one requires adding to the non interacting action \(S\) of eq. (3.12) interaction terms in a consistent way, that is without spoiling the quantum BV master equation (3.1). BV gauging of a given field theory, discussed in the next sections, is an important example of this procedure.

Examples of BV master actions will be given in the following sections.
4 \ N=0 BV gauging and N=0 ghost system

Now, we are ready for starting the study of $N = 0$ BV gauging and the $N = 0$ ghost system. This will set the paradigm for $N = 1$ and higher $N$ gaugings.

$N = 0 \ g$–actions

Let $(\mathcal{A}, \Delta, \{\cdot,\cdot\})$ be a BV algebra. Let $g$ be a Lie algebra. An $N = 0 \ g$–action on the BV algebra is a linear map $l : g \to \text{Der}_0(\mathcal{A})$ such that

\[ [l_x, l_y] = l_{[x,y]}, \quad (4.1a) \]
\[ [l_x, \Delta] = 0, \quad (4.1b) \]

for $x, y \in g$. By (4.1b), for $x \in g$, $l_x \in \text{Der}_0(\mathcal{A})$ is a BV derivation (cf. sect. 2).

Let $S$ be a quantum BV master action of the BV algebra $(\mathcal{A}, \Delta, \{\cdot,\cdot\})$. $S$ is said invariant under the $g$–action if

\[ l_x S = 0, \quad (4.2) \]

for all $x \in g$. This condition is compatible with (4.1a), (4.1b) and the quantum BV master equation (3.1). When $S$ is invariant, one has

\[ [\delta, l_x] = 0, \quad (4.3) \]

where $\delta$ is the quantum BV operator (cf. sect. 3, eq. (3.3)). By (3.4), (4.1a), (4.3), $(\mathcal{A}, g, l, \delta)$ is an algebraic structure known as a differential $g$–module [19] (see appendix A for a review of differential Lie modules). By (4.3), it is possible to define a $g$–invariant quantum BV cohomology, that is the cohomology of the differential space $(\mathcal{A}_{\text{inv}}, \delta)$, where $\mathcal{A}_{\text{inv}} = \cap_{x \in g} \ker l_x \subset \mathcal{A}$. The same statements hold also for the classical BV operator and its cohomology.

The $g$–action is called $BV \ Hamiltonian$, if there is a linear map $\lambda : g \to \mathcal{A}_{-1}$, called $BV \ moment \ map$, such that

\[ l_x = \text{ad} \lambda_x, \quad (4.4) \]
with $x \in \mathfrak{g}$, and that
\[
\{\lambda_x, \lambda_y\} = \lambda_{[x,y]}, \quad (4.5a)
\]
\[
\Delta \lambda_x = 0, \quad (4.5b)
\]
with $x, y \in \mathfrak{g}$. (4.4) together with (4.5a), (4.5b) are indeed sufficient for (4.1a), (4.1b) to hold. By (4.4), (4.5b), $l_x \in \text{Der}_0(\mathcal{A})$ is a BV inner derivation (cf. sect. 2). Below, we consider only BV Hamiltonian $\mathfrak{g}$–actions.

If $S$ is a quantum BV master action of the BV algebra $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$ invariant under the $\mathfrak{g}$–action, then
\[
\{\lambda_x, S\} = 0, \quad (4.6)
\]
for all $x \in \mathfrak{g}$. By (4.5b), (4.6), $\lambda_x$ is a cocycle of both the classical and the quantum BV cohomology (cf. sect. 3).

*Gauging of a global $N = 0$ $\mathfrak{g}$–symmetry*

Consider a matter BV algebra $(\mathcal{A}_M, \Delta_M, \{\cdot, \cdot\}_M)$ carrying a BV Hamiltonian $N = 0$ $\mathfrak{g}$–action $l_M$ with BV moment map $\lambda_M$ and a matter quantum BV master action $S_M$ invariant under the $\mathfrak{g}$–action. By (4.6), we may say that $S_M$ enjoys a global $N = 0$ $\mathfrak{g}$–symmetry and that $\lambda_M$ is the corresponding symmetry charge. We want to find a meaningful way of gauging these symmetries.

The gauging proceeds in three steps.

1. We construct an $N = 0$ ghost BV algebra $(\mathcal{A}_{\mathfrak{g}[0]}, \Delta_{\mathfrak{g}[0]}, \{\cdot, \cdot\}_{\mathfrak{g}[0]})$ with a BV Hamiltonian $N = 0$ $\mathfrak{g}$–action $l_{\mathfrak{g}[0]}$ with BV moment map $\lambda_{\mathfrak{g}[0]}$ and an $N = 0$ ghost quantum BV master action $S_{\mathfrak{g}[0]}$ invariant under the $\mathfrak{g}$–action. The construction is canonical, in that it depends solely on $\mathfrak{g}$.

2. We construct an $N = 0$ gauged matter BV algebra $(\mathcal{A}_{\mathfrak{g}[0]M}, \Delta_{\mathfrak{g}[0]M}, \{\cdot, \cdot\}_{\mathfrak{g}[0]M})$ and equip it with an appropriate BV Hamiltonian $N = 0$ $\mathfrak{g}$–action $l_{\mathfrak{g}[0]M}$ with BV moment map $\lambda_{\mathfrak{g}[0]M}$.

3. We construct an $N = 0$ gauged matter quantum BV master action $S_{\mathfrak{g}[0]M}$.
of the gauged matter BV algebra invariant under the $g$–action.

**Step 1.** Given the Lie algebra $g$, define

$$A_{g|0} = \text{Fun}(g^\vee[-2] \oplus g[1]).$$

Denote by $b_i, c^i$, $i = 1, \ldots, \dim g$, the coordinates of $g^\vee[-2], g[1]$, respectively, corresponding to a chosen basis $\{t_i\}$ of $g$. Then, $A_{g|0}$ can be viewed as the $\mathbb{Z}$ graded commutative associative unital algebra of polynomials in the $b_i, c^i$. Define further the 2nd order differential operator

$$\Delta_{g|0} = \partial_L b^i \partial_{Lc^i}$$

and the bilinear brackets

$$\{\phi, \psi\}_{g|0} = \partial_R b^i \phi \partial_{Lc^i} \psi - \partial_R c^i \phi \partial_L b^i \psi, \quad \phi, \psi \in A_{g|0}.$$  

Then, it is simple to check that relations (2.1)–(2.3) are verified. It follows that $(A_{g|0}, \Delta_{g|0}, \{\cdot, \cdot\}_{g|0})$ is a BV algebra, the $N = 0$ ghost BV algebra.

Let $f_{ijk}$ be the structure constants of $g$ with respect to the basis $\{t_i\}$. Set

$$\lambda_{g|0} = f_{jki} b_j c^k.$$  

Since $\lambda_{g|0} \in A_{g|0-1}$, it defines via (4.4) a linear map $l_{g|0} : g \to \text{Der}_0(A_{g|0})$. If the Lie algebra $g$ is *unimodular*, that is

$$f_{jji} = 0,$$

$l_{g|0}$ is a BV Hamiltonian $N = 0$ $g$–action on the ghost BV algebra having $\lambda_{g|0}$ as BV moment map. Indeed, using (4.8), (4.9), (4.10), one finds that (4.5a) is verified and that $\Delta_{g|0} \lambda_{g|0} = f_{jji}$. So, (4.5b) is also verified, if (4.11) holds.

The action of $l_{g|0}$ on $b_i, c^i$ is given by

$$l_{g|0} b_j = f_{kij} b_k,$$

$$l_{g|0} c^j = -f_{ijk} c^k,$$
as follows readily from (4.4), (4.9), (4.10).

The \( N = 0 \) ghost algebra \( A_{\mathfrak{g}|0} \) contains an element \( S_{\mathfrak{g}|0} \in A_{\mathfrak{g}|00} \) given by

\[
S_{\mathfrak{g}|0} = -\frac{1}{2} f^i_{jk} b_i c^j c^k.
\]

(4.13)

\( S_{\mathfrak{g}|0} \) satisfies the classical BV master equation (3.5) and, if \( \mathfrak{g} \) is unimodular, also the quantum BV master equation (3.1). Indeed, using (4.8), (4.9), (4.13), one finds that (3.5) is verified and that \( \Delta_{\mathfrak{g}|0} S_{\mathfrak{g}|0} = -f^i_{ij} c^j \). So, (3.1) is also verified, if (4.11) holds. \( S_{\mathfrak{g}|0} \) is the \( N = 0 \) ghost quantum BV master action. \( S_{\mathfrak{g}|0} \) is invariant under the \( \mathfrak{g} \)-action \( l_{\mathfrak{g}|0} \). (4.9), (4.10), (4.13) indeed imply (4.6).

The action of the quantum BV operator \( \delta_{\mathfrak{g}|0} \) on \( b_i, c^i \) is given by

\[
\delta_{\mathfrak{g}|0} b_i = f^k_{ji} b_k c^j,
\]

(4.14a)

\[
\delta_{\mathfrak{g}|0} c^i = -\frac{1}{2} f^i_{jk} c^j c^k,
\]

(4.14b)
as follows from the definition (3.3) and from (4.8), (4.9), (4.13). Relations (4.14) are also the expressions of the action of the classical BV operator \( \delta_{\mathfrak{g}|0c} \) defined according to (3.7). Recall however that the actions of \( \delta_{\mathfrak{g}|0} \) and \( \delta_{\mathfrak{g}|0c} \) on higher degree polynomials in \( b_i, c^i \) are different because \( \Delta_{\mathfrak{g}|0} \) acts non trivially on them in general.

The fulfillment of the unimodularity condition (4.11) is required by \( l_{\mathfrak{g}|0} \) being a Hamiltonian \( \mathfrak{g} \)-action and \( S_{\mathfrak{g}|0} \) a quantum master action; it is thus crucial in the above BV construction. In full–fledged quantum field theory, (4.11) would be a quantum anomaly cancellation condition.

Step 2. The \( N = 0 \) gauged matter BV algebra \((A_{\mathfrak{g}|0M}, \Delta_{\mathfrak{g}|0M}, \{\cdot, \cdot\}_{\mathfrak{g}|0M})\) is the tensor product of the \( N = 0 \) ghost BV algebra \((A_{\mathfrak{g}|0}, \Delta_{\mathfrak{g}|0}, \{\cdot, \cdot\}_{\mathfrak{g}|0})\) and the matter BV algebra \((A_M, \Delta_M, \{\cdot, \cdot\}_M)\) (cf. sect. 2). Via (4.4), the element \( \lambda_{\mathfrak{g}|0Mi} \in A_{\mathfrak{g}|0M-1} \) given by the expression

\[
\lambda_{\mathfrak{g}|0Mi} = \lambda_{\mathfrak{g}|0i} \otimes 1_M + 1_{\mathfrak{g}|0} \otimes \lambda_{Mi}
\]

(4.15)
defines a linear map $l_{g|0M} : g \rightarrow \text{Der}_0(A_{g|0M})$. If, again, (4.11) is satisfied, $l_{g|0M}$ is a BV Hamiltonian $N = 0$ $g$–action on the gauged matter BV algebra having $\lambda_{g|0M}$ as BV moment map. One just notices that $\lambda_{g|0M}$ satisfies (4.5a), (4.5b) if simultaneously $\lambda_{g|0}$, $\lambda_M$ do, by (2.13), (2.14). The $g$–action $l_{g|0M}$ extends trivially the $g$–actions $l_{g|0}$, $l_M$, in the sense that

$$l_{g|0M}x = l_{g|0x} \otimes 1_M + 1_{g|0} \otimes l_Mx,$$

for $x \in g$.

**Step 3.** The $N = 0$ gauged matter algebra $A_{g|0M}$ contains a distinguished element $S_{g|0M} \in A_{g|0M0}$ given by

$$S_{g|0M} = S_{g|0} \otimes 1_M + 1_{g|0} \otimes S_M + c^i \otimes \lambda_{Mi}.$$  

The first two terms correspond to the trivial non interacting ghost–matter action (3.12). The third term is a genuine ghost–matter interaction term. By explicit calculation, one can verify that, assuming again that (4.11) holds, $S_{g|0M}$ satisfies the quantum BV master equation (3.1). One notice, using systematically (2.13), (2.14), that $S_{g|0M}$ satisfies (3.1), if simultaneously $S_{g|0}$, $S_M$ satisfy (3.1), $S_M$ satisfies (4.6) and $\lambda_M$ satisfies (4.5a), (4.5b). $S_{g|0M}$ is the $N = 0$ gauged matter quantum BV master action. Proceeding in a similar fashion, we find that $S_{g|0M}$ satisfies also (4.6), so that $S_{g|0M}$ is invariant under the $g$–action $l_{g|0M}$.

The coupling of ghosts and matter in the quantum master action $S_{g|0M}$ modifies the action of their respective quantum BV operators: $\delta_{g|0M}$ extends non trivially $\delta_{g|0}$, $\delta_M$, that is $\delta_{g|0M} \neq \delta_{g|0} \otimes 1_M + 1_{g|0} \otimes \delta_M$. One has instead

$$\delta_{g|0M}(b_i \otimes 1_M) = \delta_{g|0}b_i \otimes 1_M + 1_{g|0} \otimes \lambda_{Mi},$$

$$\delta_{g|0M}(c^i \otimes 1_M) = \delta_{g|0}c^i \otimes 1_M,$$ 

$$\delta_{g|0M}(1_{g|0} \otimes \phi) = 1_{g|0} \otimes \delta_M\phi + c^i \otimes l_{Mi}\phi,$$
where $\delta_{g|0}b_i$, $\delta_{g|0}c^i$ are given by (4.14a), (4.14b), respectively.

**Analysis of BV cohomology.**

On physical grounds, not all the observables of the original matter system remain such upon gauging the global symmetry. Only those which are invariant under the symmetry do. They represent classes of the matter invariant quantum BV cohomology $H_{BV_{inv}}^*(\mathcal{A}_M)$. So, it is the invariant BV cohomology that is relevant rather than the ordinary one.

The map $\Upsilon_0 : \mathcal{A}_M \to \mathcal{A}_{g|0M}$ defined by

$$\Upsilon_0 \phi = 1_{g|0} \otimes \phi, \quad \phi \in \mathcal{A}_M,$$

(4.19)
yields a natural embedding of $\mathcal{A}_M$ into $\mathcal{A}_{g|0M}$. It is immediate to check that $\Upsilon_0$ is a monomorphism of BV algebras (cf. sect. 2). Further, we have

$$l_{g|0Mx} \Upsilon_0 = \Upsilon_0 l_{Mx},$$

(4.20)

for $x \in g$, and

$$\delta_{g|0M} \Upsilon_0 = \Upsilon_0 \delta_M + c^i \otimes 1_M \cdot \Upsilon_0 l_{Mi}.$$  

(4.21)

By (4.20), $\Upsilon_0$ maps the matter invariant subalgebra $\mathcal{A}_{M_{inv}}$ into the gauged matter invariant subalgebra $\mathcal{A}_{g|0M_{inv}}$. By (4.21), $\Upsilon_0|_{\mathcal{A}_{M_{inv}}}$ is a chain map of the matter and gauged matter invariant differential spaces $(\mathcal{A}_{M_{inv}}, \delta_M), (\mathcal{A}_{g|0M_{inv}}, \delta_{g|0M})$. So, $\Upsilon_0|_{\mathcal{A}_{M_{inv}}}$ induces a homomorphism of the matter and gauged matter invariant quantum BV cohomology spaces $H_{BV_{inv}}^*(\mathcal{A}_M), H_{BV_{inv}}^*(\mathcal{A}_{g|0M})$. The homomorphism is not a monomorphism in general and, so, $H_{BV_{inv}}^*(\mathcal{A}_M)$ is not naturally embedded in $H_{BV_{inv}}^*(\mathcal{A}_{g|0M})$. This renders the study of the observables in the gauged matter theory a bit problematic. The way out is the following.

From (4.7), we notice that the $N = 0$ ghost algebra $\mathcal{A}_{g|0}$ contains as a subalgebra the Chevalley–Eilenberg algebra

$$CE(g) = \text{Fun}(g[1])$$

(4.22)
$CE(\mathfrak{g})$ is generated by the $c^i$. By (4.12b), $CE(\mathfrak{g})$ is stable under the $\mathfrak{g}$–action $l_{\mathfrak{g}|0}$. By (4.14b), $CE(\mathfrak{g})$ is also stable under the BV operator $\delta_{\mathfrak{g}|0}$. Thus, $(CE(\mathfrak{g}), \mathfrak{g}, l_{\mathfrak{g}|0}, \delta_{\mathfrak{g}|0})$ is a differential $\mathfrak{g}$–module (see appendix A). Inspecting (4.14b), we realize that the BV cohomology $H_{BV}^*(CE(\mathfrak{g}))$ is the Chevalley–Eilenberg cohomology $H^*_{CE}(\mathfrak{g})$ of $\mathfrak{g}$. Similarly, from (4.12b), (4.14b), we see that the invariant BV cohomology $H_{BV}^{inv}(CE(\mathfrak{g}))$ is the invariant Chevalley–Eilenberg cohomology $H^{inv*}_{CE}(\mathfrak{g})$ of $\mathfrak{g}$. $H^*_{CE}(\mathfrak{g})$ is not known in general, but it is under the weak assumption that $\mathfrak{g}$ is reductive, i.e. the direct sum of a semisimple and an Abelian Lie algebra, in which case $H^*_{CE}(\mathfrak{g}) \cong CE(\mathfrak{g})^{inv}$, the invariant subalgebra of $CE(\mathfrak{g})$. We recall that reductive Lie algebras are unimodular. So, this result fits usefully in the theory developed above. $H^{inv*}_{CE}(\mathfrak{g}) \cong CE(\mathfrak{g})^{inv}$ always. We note that the classical and quantum BV operators are equal on $CE(\mathfrak{g})$, since, by (4.8), $\Delta_{\mathfrak{g}|0}$ vanishes on $CE(\mathfrak{g})$ and, so, the classical and the quantum BV cohomologies coincide.

The $N = 0$ gauged matter algebra $\mathcal{A}_{\mathfrak{g}|0M}$ contains as a subalgebra

$$\mathcal{A}^+_{\mathfrak{g}|0M} = CE(\mathfrak{g}) \otimes \mathcal{A}_M.$$  (4.23)

By (4.16), (4.12b), $\mathcal{A}^+_{\mathfrak{g}|0M}$ is stable under the $\mathfrak{g}$–action $l_{\mathfrak{g}|0M}$. Similarly, by (4.18b), (4.18c), (4.14b), $\mathcal{A}^+_{\mathfrak{g}|0M}$ is stable under the BV operator $\delta_{\mathfrak{g}|0M}$. Thus, $(\mathcal{A}^+_{\mathfrak{g}|0M}, \mathfrak{g}, l_{\mathfrak{g}|0M}, \delta_{\mathfrak{g}|0M})$ is a differential $\mathfrak{g}$–module. By (4.18b), (4.18c), (4.14b), the quantum BV cohomology $H_{BV}^*(\mathcal{A}^+_{\mathfrak{g}|0M})$ is the Chevalley–Eilenberg cohomology $H^*_{CE}(\mathfrak{g}, \mathcal{A}_M)$ of $\mathfrak{g}$ with coefficients in the differential space $(\mathcal{A}_M, \delta_M)$. Similarly, by (4.16), (4.12b), (4.18b), (4.18c), (4.14b), the invariant quantum BV cohomology $H_{BV}^{inv*}(\mathcal{A}^+_{\mathfrak{g}|0M})$ is the invariant Chevalley–Eilenberg cohomology $H^{inv*}_{CE}(\mathfrak{g}, \mathcal{A}_M)$ of $\mathfrak{g}$ with coefficients in the differential $\mathfrak{g}$–module $(\mathcal{A}_M, \mathfrak{g}, l_M, \delta_M)$. Unlike for the pure ghost system, the quantum and classical BV operators are generally different in the matter sector and, so, it is necessary to distinguish the classical and the quantum BV cohomologies. Anyway, analogous statements hold
in the classical case, with the proviso that \((\mathcal{A}_M, \delta_{MC})\) is a differential algebra in this case.

Let us now come back to the problem of the cohomological analysis of observables in the gauged matter theory. We notice that the range of the BV algebra homomorphism \(\Upsilon_0 : \mathcal{A}_M \rightarrow \mathcal{A}_{g|0M}\) is contained in \(\mathcal{A}^+_{g|0M}\). By (4.20), (4.21), \(\Upsilon_0|_{\mathcal{A}_{M\text{inv}}}\) is a chain map of the invariant differential spaces \((\mathcal{A}_{M\text{inv}}, \delta_M)\), \((\mathcal{A}^+_{g|0M\text{inv}}, \delta_{g|0M})\). Thus, \(\Upsilon_0|_{\mathcal{A}_{M\text{inv}}}\) induces a homomorphism of the invariant BV cohomology spaces \(H_{BV\text{inv}}^*(\mathcal{A}_M), H_{BV\text{inv}}^*(\mathcal{A}^+_{g|0M})\), which can be shown to be a monomorphism. So, \(H_{BV\text{inv}}^*(\mathcal{A}_M)\) is naturally embedded in \(H_{BV\text{inv}}^*(\mathcal{A}^+_{g|0M})\). In this way, the study of the observables in the gauged matter theory is naturally framed in that of the invariant BV cohomology \(H_{BV\text{inv}}^*(\mathcal{A}^+_{g|0M})\) of \(\mathcal{A}^+_{g|0M}\). In fact, more can be shown [19]. If the Lie algebra \(g\) is reductive, then

\[
H_{BV\text{inv}}^*(\mathcal{A}^+_{g|0M}) \simeq CE(g)_{\text{inv}} \otimes H_{BV\text{inv}}^*(\mathcal{A}_M). \tag{4.24}
\]

A self-contained proof of (4.24) is given in appendix A. (4.24) indicates that the gauged matter algebra \(\mathcal{A}_{g|0M}\) contains objects which may reasonably considered to be observables, but which do not arise from the original matter algebra \(\mathcal{A}_M\). They are the pure gauge theoretic observables.
5 \textbf{N}=1 BV gauging and \textbf{N}=1 ghost system}

\(N=1\) gauging follows in outline the same steps as \(N=0\) gauging, though the \(N=1\) ghost system has a larger amount of ghost supersymmetry than its \(N=0\) counterpart. However, there are some significant differences, the most conspicuous of which is that the unimodularity of the symmetry Lie algebra \(g\) is no longer required for the consistency of the construction.

\(N=1\) g–actions

Let \((A, \Delta, \{\cdot, \cdot\})\) be a BV algebra. Let \(g\) be a Lie algebra. An \(N=1\) g–action on the BV algebra is a pair of linear maps \(i : g \to \text{Der}_{-1}(A), l : g \to \text{Der}_0(A)\) satisfying the commutation relations

\[
[i_x, i_y] = 0, \quad (5.1a)
\]

\[
[l_x, i_y] = i_{[x,y]}, \quad (5.1b)
\]

\[
[l_x, l_y] = l_{[x,y]}, \quad (5.1c)
\]

\[
[i_x, \Delta] = 0, \quad (5.1d)
\]

\[
l_x, \Delta] = 0, \quad (5.1e)
\]

with \(x, y \in g\). Note that an \(N=1\) action is automatically also an \(N=0\) action (cf. sect. 4). By (5.1d), (5.1e), for \(x \in g\), \(i_x \in \text{Der}_{-1}(A), l_x \in \text{Der}_0(A)\) are both BV derivations (cf. sect. 2).

Let \(S\) be a quantum BV master action of the BV algebra \((A, \Delta, \{\cdot, \cdot\})\). \(S\) is said invariant under the \(g\)–action if

\[
l_x = \text{ad} i_x S, \quad (5.2a)
\]

\[
l_x S = 0, \quad (5.2b)
\]

for all \(x \in g\). These conditions are compatible with (5.1a)–(5.1e) and the quantum
BV master equation (3.1). Note that this notion of invariance is more restrictive than the corresponding one of the $N = 0$ case: it is not simply a condition on $S$, but also on $l$. When $S$ is invariant, one has

$$\delta, i_x] = l_x,$$  \hspace{1cm} (5.3a)

$$\delta, l_x] = 0,$$ \hspace{1cm} (5.3b)

where $\delta$ is the quantum BV operator (cf. sect. (3), eq. (3.3)). By (3.4), (5.1a)–(5.1c), (5.3a), (5.3b), $(\mathcal{A}, \mathfrak{g}, i, l, \delta)$ is an algebraic structure known as a $\mathfrak{g}$–operation [19] (see appendix B for a review of Lie operations). By (5.3a), (5.3b), it is possible to define a $\mathfrak{g}$–basic quantum BV cohomology, that is the cohomology of the differential space $(\mathcal{A}_{bas}, \delta)$, where $\mathcal{A}_{bas} = \cap_{x \in \mathfrak{g}} (\ker i_x \cap \ker l_x) \subset \mathcal{A}$. The same statements hold also for the classical BV operator and its cohomology.

The action is called $BV$ Hamiltonian, if there exists a pair of linear maps $\iota : \mathfrak{g} \to \mathcal{A}_{-2}$, $\lambda : \mathfrak{g} \to \mathcal{A}_{-1}$, called below $BV$ premoment and moment map, respectively, such that

$$i_x = \text{ad} \iota_x,$$ \hspace{1cm} (5.4a)

$$l_x = \text{ad} \lambda_x,$$ \hspace{1cm} (5.4b)

with $x \in \mathfrak{g}$ and that

$$\{\iota_x, \iota_y\} = 0,$$ \hspace{1cm} (5.5a)

$$\{\lambda_x, \iota_y\} = \iota_{[x,y]},$$ \hspace{1cm} (5.5b)

$$\{\lambda_x, \lambda_y\} = \lambda_{[x,y]},$$ \hspace{1cm} (5.5c)

$$\Delta \iota_x = 0,$$ \hspace{1cm} (5.5d)

$$\Delta \lambda_x = 0,$$ \hspace{1cm} (5.5e)

with $x, y \in \mathfrak{g}$. (5.4a), (5.4b) together with (5.5a)–(5.5e) are indeed sufficient for
(5.1a)–(5.1c) to hold. By (5.4a), (5.4b), (5.5d), (5.5e), \( i_x \in \text{Der}_{-1}(\mathcal{A}) \), \( l_x \in \text{Der}_0(\mathcal{A}) \) are both BV inner derivations (cf. sect. 2). Note that the underlying \( N = 0 \) \( \mathfrak{g} \)-action is also Hamiltonian (cf. sect. 4). Below, we consider only BV Hamiltonian \( N = 1 \) actions.

A quantum BV master action \( S \) of the BV algebra \((\mathcal{A}, \Delta, \{\cdot, \cdot\})\) is Hamiltonian invariant under the \( \mathfrak{g} \)-action, if \( S \) satisfies

\[
\{\iota_x, S\} = \lambda_x, \tag{5.6a}
\]

\[
\{\lambda_x, S\} = 0, \tag{5.6b}
\]

for all \( x \in \mathfrak{g} \). Hamiltonian invariance is stricter than simple invariance. If \( S \) were simply invariant, (5.6a) would hold only up to a central element of the Gerstenhaber algebra \( \mathcal{A}_G \), as (5.2a), (5.4a), (5.4b) imply only that \( \lambda_x - \{\iota_x, S\} \) is central. In the \( N = 0 \) case, there is no similar distinction between simple and Hamiltonian invariance. (5.5d), (5.6a) combined imply that \( \lambda_x \) is a coboundary of both the classical and the quantum BV cohomology (cf. sect. 3). Recall that in the \( N = 0 \) case, \( \lambda_x \) is a cocycle in general. (5.6b) is not an independent relation; it follows from (5.5d), (5.5e), (5.6a) and the master equation (3.1).

**Gauging of a global \( N = 1 \) \( \mathfrak{g} \)-symmetry**

Consider a matter BV algebra \((\mathcal{A}_M, \Delta_M, \{\cdot, \cdot\}_M)\) carrying a BV Hamiltonian \( N = 1 \) \( \mathfrak{g} \)-action \( i_M, l_M \) with BV (pre)moment maps \( \iota_M, \lambda_M \) and a matter quantum BV master action \( S_M \) invariant under the \( \mathfrak{g} \)-action. By (5.6a), (5.6b), we may say that \( S_M \) enjoys a global \( N = 1 \) \( \mathfrak{g} \)-symmetry and that \( \lambda_M \) is the corresponding symmetry charge, extending, perhaps with some abuse, the terminology of the \( N = 0 \) case. However, here, the symmetry is derived in the sense that (5.6b) is actually a consequence of the more basic relation (5.6a), unlike for \( N = 0 \). We want to gauge this symmetry in a way that reflects this richer structure.

As the in \( N = 0 \) case, the gauging proceeds in three steps.
1. We construct an $N = 1$ ghost BV algebra $(A_{g|1}, \Delta_{g|1}, \{\cdot, \cdot\}_{g|1})$ with a BV Hamiltonian $N = 1$ $g$–action $i_{g|1}, l_{g|1}$ with BV (pre)moment map $\iota_{g|1}, \lambda_{g|1}$ and an $N = 1$ ghost quantum BV master action $S_{g|1}$ invariant under the $g$–action. The construction is canonical, depending on $g$ only.

2. We construct an $N = 1$ gauged matter BV algebra $(A_{g|1M}, \Delta_{g|1M}, \{\cdot, \cdot\}_{g|1M})$ and equip it with an appropriate BV Hamiltonian $N = 1$ $g$–action $i_{g|1M}, l_{g|1M}$ with BV (pre)moment maps $\iota_{g|1M}, \lambda_{g|1M}$.

3. We construct an $N = 1$ gauged matter quantum BV master action $S_{g|1M}$ of the gauged matter BV algebra invariant under the $g$–action.

Step 1. For a Lie algebra $g$, define

$$A_{g|1} = \text{Fun}(g^\vee[-2] \oplus g[1] \oplus g^\vee[-3] \oplus g[2]) \quad \text{(5.7)}$$

Denote by $b_i, c^i, B_i, C^i$, $i = 1, \ldots, \dim g$, the coordinates of $g^\vee[-2], g[1], g^\vee[-3], g[2]$, respectively, corresponding to a chosen basis $\{t_i\}$ of $g$. Then, $A_{g|1}$ can be viewed as the $\mathbb{Z}$ graded commutative associative unital algebra of polynomials in the $b_i, c^i, B_i, C^i$. Define next the 2nd order differential operator

$$\Delta_{g|1} = \partial_L b^i \partial_L c_i - \partial_L b^i \partial_L c_i \quad \text{(5.8)}$$

and the bilinear bracket

$$\{\phi, \psi\}_{g|1} = \partial_R b^i \phi \partial_L c_i \psi - \partial_R c_i \phi \partial_L b^i \psi \quad \text{(5.9)}$$

$$+ \partial_R b^i \phi \partial_L C_i \psi - \partial_R C_i \phi \partial_L b^i \psi, \quad \phi, \psi \in A_{g|1}.$$ 

Then, it is simple to check that relations (2.1)–(2.3) are verified. It follows that $(A_{g|1}, \Delta_{g|1}, \{\cdot, \cdot\}_{g|1})$ is a BV algebra, the $N = 1$ ghost BV algebra.

Let $f^i_{jk}$ be the structure constants of $g$ with respect to the basis $\{t_i\}$. Set

$$\iota_{g|1} = b_i \quad \text{(5.10a)}$$

$$\lambda_{g|1} = f^i_{kj} b_j c^k + f^i_{kj} B_j C^k. \quad \text{(5.10b)}$$
Since $\iota_{\mathfrak{g}|1} \in A_{\mathfrak{g}|1-2}$, $\lambda_{\mathfrak{g}|1} \in A_{\mathfrak{g}|1-1}$, they define via (5.4a), (5.4b) linear maps $i_{\mathfrak{g}|1} : \mathfrak{g} \to \text{Der}_{-1}(A_{\mathfrak{g}|1})$, $l_{\mathfrak{g}|1} : \mathfrak{g} \to \text{Der}_0(A_{\mathfrak{g}|1})$. The pair $i_{\mathfrak{g}|1}$, $l_{\mathfrak{g}|1}$ is a BV Hamiltonian $N = 1$ action on the ghost BV algebra having $\iota_{\mathfrak{g}|1}$, $\lambda_{\mathfrak{g}|1}$ as BV (pre)moment maps. Indeed, by (5.8), (5.9), (5.10a), (5.10b), relations (5.5a)--(5.5e) are verified. The Lie algebra $\mathfrak{g}$ no longer needs to be unimodular (cf. eq. (4.11)), as in the $N = 0$ case, due to the cancellation of the offending terms $f^i_{jk}$ of the $bc$ and $BC$ sectors.

The action of $i_{\mathfrak{g}|1}$, $l_{\mathfrak{g}|1}$ on $b_i$, $c^i$, $B_i$, $C^i$ is given by

\begin{align*}
i_{\mathfrak{g}|1}b_j &= 0, \\
i_{\mathfrak{g}|1}c^j &= \delta_i^j, \\
i_{\mathfrak{g}|1}B_j &= 0, \\
i_{\mathfrak{g}|1}C^j &= 0, \\
l_{\mathfrak{g}|1}b_j &= f^{k}_{ij}b_k, \\
l_{\mathfrak{g}|1}c^j &= -f^i_{jk}c^k, \\
l_{\mathfrak{g}|1}B_j &= f^{k}_{ij}B_k, \\
l_{\mathfrak{g}|1}C^j &= -f^i_{jk}C^k,
\end{align*}

as follows readily from (5.4a), (5.4b), (5.9), (5.10a), (5.10b).

The $N = 1$ ghost algebra $A_{\mathfrak{g}|1}$ contains an element $S_{\mathfrak{g}|1} \in A_{\mathfrak{g}|10}$ given by

\[ S_{\mathfrak{g}|1} = -\frac{1}{2}f^i_{jk}b_ic^j c^k + b_iC^i + f^i_{jk}B_i c^j C^k. \]  

$S_{\mathfrak{g}|1}$ satisfies both the classical and the quantum BV master equation (3.5). This follows from the definition (5.12) using (5.8), (5.9). Again, $\mathfrak{g}$ needs not to be unimodular, as in the $N = 0$ case, due to the cancellation of the anomalous terms.
$f^i_{ij}c^j$ originating in the $bc$ and $BC$ sectors. $S_{g|1}$ is the $N = 1$ ghost quantum BV master action. $S_{g|1}$ is invariant under the $g$–action $i_{g|1}$, $l_{g|1}$. (5.9), (5.10a), (5.10b), (5.12) indeed imply (5.6a), (5.6b).

The action of the quantum BV operator $\delta_{g|1}$ on $b_i$, $c^i$, $B_i$, $C^i$ reads

\begin{align}
\delta_{g|1}b_i &= f^k_{ji}b_kc^j + f^k_{ji}B_kC^j, \\
\delta_{g|1}c^i &= C^i - \frac{1}{2}f^i_{jk}c^j c^k, \\
\delta_{g|1}B_i &= -b_i - f^k_{ji}B_kc^j, \\
\delta_{g|1}C^i &= -f^i_{jk}c^j C^k,
\end{align}

as follows by direct application of the definition (3.3). Relations (5.13) are also the expressions of the action of the classical BV operator $\delta_{g|1}$ defined according to (3.7). Analogously to the $N = 0$ case, the action of $\delta_{g|1}$ and $\delta_{g|1}c$ on higher polynomials in $b_i$, $B_i$, $c^i$, $C^i$ is different because $\Delta_{g|1}$ acts on them non trivially in general.

The $N = 1$ ghost system has an elegant superfield formulation that is illustrated in appendix C.

Step 2. The $N = 1$ gauged matter BV algebra $(A_{g|1M}, \Delta_{g|1M}, \{\cdot, \cdot\}_{g|1M})$ is the tensor product of the $N = 1$ ghost BV algebra $(A_{g|1}, \Delta_{g|1}, \{\cdot, \cdot\}_{g|1})$ and the matter BV algebra $(A_M, \Delta_M, \{\cdot, \cdot\}_M)$ (cf. sect. 2), analogously to the $N = 0$ case. The elements $t_{g|1Mi} \in A_{g|1M-2}$, $\lambda_{g|1Mi} \in A_{g|1M-1}$ given by

\begin{align}
t_{g|1Mi} &= t_{g|1i} \otimes 1_M, \\
\lambda_{g|1Mi} &= \lambda_{g|1i} \otimes 1_M + 1_{g|1} \otimes \lambda_{Mi}
\end{align}

define via (5.4a), (5.4b) linear maps $i_{g|1M} : g \to \text{Der}_- (A_{g|1M})$, $l_{g|1M} : g \to \text{Der}_0 (A_{g|1M})$. The pair $i_{g|1M}$, $l_{g|1M}$ is a BV Hamiltonian $N = 1 g$–action on the gauged matter BV algebra having $t_{g|1M}$, $\lambda_{g|1M}$ as BV (pre)moment maps. One
just notices that \( \iota_{g|1M} \), \( \lambda_{g|1M} \) satisfy (5.5a)–(5.5e) if \( \iota_{g|1} \), \( \lambda_{g|1} \), \( \lambda_M \) do, by (2.13), (2.14). The \( g \)–action \( i_{g|1M} \) extends the \( g \)–actions \( i_{g|1} \), \( l_{g|1} \), \( i_M \), \( l_M \) as

\[
i_{g|1M} = i_{g|1i} \otimes 1_M, \tag{5.15a}
\]

\[
l_{g|1M} = l_{g|1i} \otimes 1_M + 1_{g|1} \otimes l_{Mi} \tag{5.15b}
\]

for \( x \in g \). Unlike the \( N = 0 \) case, this extension is non trivial: in the right hand side of (5.15a), a term \( 1_{g|1} \otimes i_{Mi} \) is absent.

**Step 3.** The \( N = 1 \) gauged matter algebra \( A_{g|1M} \) contains a distinguished element \( S_{g|1M} \in A_{g|1M0} \) given by

\[
S_{g|1M} = S_{g|1} \otimes 1_M + 1_{g|1} \otimes S_M + c^i \otimes \lambda_{Mi} - C^i \otimes \iota_{Mi}. \tag{5.16}
\]

As in the \( N = 0 \) case, the first two terms correspond to the trivial non interacting ghost–matter action (3.12) while the third and fourth terms are genuine ghost–matter interaction terms. By explicit calculation, one can verify that \( S_{g|1M} \) satisfies the quantum BV master equation (3.1), noticing that, by systematic use of (2.13), (2.14), that \( S_{g|1M} \) satisfies (3.1), if simultaneously \( S_{g|1} \), \( S_M \) satisfy (3.1), \( S_M \) satisfies (5.6a), (5.6b), and \( \iota_M \), \( \lambda_M \) satisfy (5.5a)–(5.5e). \( S_{g|1M} \) is the \( N = 1 \) gauged matter quantum BV master action. As \( S_{g|1M} \) satisfies also (5.6a), (5.6b), \( S_{g|1M} \) is invariant under the \( g \)–action \( l_{g|1M} \).

As in the \( N = 0 \) case, the coupling of ghosts and matter in the quantum master action \( S_{g|1M} \) modifies the action of their respective quantum BV operators: \( \delta_{g|1M} \neq \delta_{g|1} \otimes 1_M + 1_{g|1} \otimes \delta_M \) and, so, \( \delta_{g|1M} \) extends non trivially \( \delta_{g|1} \), \( \delta_M \). One has instead

\[
\delta_{g|1M}(b_i \otimes 1_M) = \delta_{g|1}b_i \otimes 1_M + 1_{g|1} \otimes \lambda_{Mi}, \tag{5.17a}
\]

\[
\delta_{g|1M}(c^i \otimes 1_M) = \delta_{g|1}c^i \otimes 1_M, \tag{5.17b}
\]

\[
\delta_{g|1M}(B_i \otimes 1_M) = \delta_{g|1}B_i \otimes 1_M + 1_{g|1} \otimes \iota_{Mi}, \tag{5.17c}
\]
\[ \delta_{g|1M}(C^i \otimes 1_M) = \delta_{g|1} C^i \otimes 1_M, \]  
\[ \delta_{g|1M}(1_{g|1} \otimes \phi) = 1_{g|1} \otimes \delta_M \phi + c^i \otimes l_{M_i} \phi - C^i \otimes i_{M_i} \phi. \]  

(5.17d) 

(5.17e)

where \( \delta_{g|0} b_i, \delta_{g|0} c^i, \delta_{g|0} B_i, \delta_{g|0} C^i \) are given by the expressions (5.13a)–(5.13d), respectively.

**Analysis of BV cohomology**

Before beginning the study of BV cohomology, the following remarks are in order. The construction illustrated above is modeled on topological gauge field theory. In the Mathai–Quillen formulation of topological field theory [20], the computation of a topological correlator is reduced to that of an integral of the form 

\[ \int_{Z(s)} \omega, \]

where \( \omega \) is a closed form of the field space \( M \) and \( Z(s) \) is the submanifold of \( M \) of solutions of a certain field equation \( s = 0 \) (a phenomenon called localization). Now, it turns out that 

\[ \int_{Z(s)} \omega = \int_M \omega \wedge e(E), \]

where \( e(E) \) is a closed form of \( M \) representing the Euler class of an oriented Riemannian vector bundle \( E \) over \( M \), of which \( s \) is a section. It is known that \( e(E) = s^* t(E) \), where \( t(E) \) is a closed form of \( E \) representing the Thom class of \( E \) (a distinguished element of the vertical rapid decrease cohomology of \( E \)). \( t(E) \) in turn yields the closed form \( \pi^* t(E) \) of the natural principal \( G \)-bundle \( \pi : P \times V \to E \), where \( P \) and \( V \) are the oriented orthogonal frame principal bundle and the typical fiber of \( E \), respectively, and \( G \simeq SO(V) \) is the structure group of \( P \). \( P \) is endowed with a canonical \( g \)-operation, where \( g \) is the Lie algebra of \( G \) [19]. The operation allows one to define basic forms of \( P \times V. \pi^* t(E) \) is closed and basic. In this way, the problem of the computation of the original topological correlator can be formulated in terms of the basic cohomology and the closely related equivariant cohomology of the principal bundle \( P \times V \). See [21][22] for up to date reviews of this subject matter.

As observed at the beginning of this section, \( N = 1 \) \( g \)-actions on BV algebras
are instances of $g$–operations. So, it is reasonable to suppose that, in a BV algebraic formulation, a topological field theory should be realized as matter BV algebra with an $N = 1 g$–action and an invariant BV master action. For the reasons explained above, among all the observables of the matter system, those which are basic under the $g$–action have a central role. In certain topological field theories, the relevant basic observables turn out to be non local. The way to restore locality is precisely the gauging of the $N = 1 g$–symmetry.

Let us thus assume that the observables of the original matter theory which are relevant upon gauging are the basic ones. They represent classes of the matter basic quantum BV cohomology $H_{BVbas}^*(A_M)$. So, it is the basic BV cohomology that is relevant rather than the ordinary one.

We now proceed similarly as we did in the $N = 0$ case (cf. sect. 4). The map $\Upsilon_1 : A_M \to A_{g1M}$ defined by

$$\Upsilon_1 \phi = 1_g \otimes \phi, \quad \phi \in A_M,$$

(5.18)
yields a natural embedding of $A_M$ into $A_{g1M}$, which is a monomorphism of BV algebras (cf. sect. 2). Further, we have

$$i_{g1M} x \Upsilon_1 = 0,$$

(5.19a)

$$l_{g1M} \Upsilon_1 = \Upsilon_1 l_M x,$$

(5.19b)

for $x \in g$, and

$$\delta_{g1M} \Upsilon_1 = \Upsilon_1 \delta_M + c^i \otimes 1_M \cdot \Upsilon_1 l_{Mi} - C^i \otimes 1_M \cdot \Upsilon_1 i_{Mi}.$$  

(5.20)

By (5.19a), (5.19b), $\Upsilon_1$ maps the matter basic subalgebra $A_{Mbas}$ into the gauged matter basic subalgebra $A_{g1Mbas}$. By (5.20), $\Upsilon_1|_{A_{Mbas}}$ is a chain map of the matter and gauged matter basic differential spaces ($A_{Mbas}, \delta_M$), ($A_{g1Mbas}, \delta_g$). So, $\Upsilon_1|_{A_{Mbas}}$ induces a homomorphism of the matter and gauged matter basic
quantum BV cohomologies $H_{BVbas}^*(\mathcal{A}_M)$, $H_{BVbas}^*(\mathcal{A}_{g|1M})$. As in the $N = 0$ case, the homomorphism is not a monomorphism in general and, so, $H_{BVbas}^*(\mathcal{A}_M)$ is not naturally embedded in $H_{BVbas}^*(\mathcal{A}_{g|1M})$. As in the $N = 0$ case again, this renders the study of the observables in the gauged matter theory problematic. The way out is similar in spirit.

From (5.7), we observe that the $N = 1$ ghost algebra $\mathcal{A}_{g|1}$ contains as a subalgebra the Weil algebra

$$ W(\mathfrak{g}) = \text{Fun}(\mathfrak{g}[1] \oplus \mathfrak{g}[2]) $$

(5.21)

$W(\mathfrak{g})$ is generated by the $c^i$, $C^i$. By (5.11b), (5.11d), (5.11f), (5.11h), $W(\mathfrak{g})$ is stable under the $\mathfrak{g}$–action $i_{\mathfrak{g}|1}$, $l_{\mathfrak{g}|1}$. By (5.13b), (5.13d), $W(\mathfrak{g})$ is also stable under the BV operator $\delta_{\mathfrak{g}|1}$. Thus, $(W(\mathfrak{g}), \mathfrak{g}, i_{\mathfrak{g}|1}, l_{\mathfrak{g}|1}, \delta_{\mathfrak{g}|1})$ is a $\mathfrak{g}$–operation (see appendix B). Upon inspecting (5.13b), (5.13d), we recognize that the BV cohomology $H_{BV}^*(W(\mathfrak{g}))$ is the Weil algebra cohomology $H_{W}^*(\mathfrak{g})$ of $\mathfrak{g}$. Similarly, from (5.11b), (5.11d), (5.11f), (5.11h), (5.13b), (5.13d), we see that the basic BV cohomology $H_{BVbas}^*(CE(\mathfrak{g}))$ coincides with the basic Weil algebra cohomology $H_{Wbas}^*(\mathfrak{g})$ of $\mathfrak{g}$. It is known that $H_{W}^*(\mathfrak{g}) \simeq \mathbb{R}\delta_{*,0}$, i.e. the Weil cohomology is trivial. $H_{Wbas}^*(\mathfrak{g})$ is instead non trivial and concentrated in even degree, namely $H_{Wbas}^*(\mathfrak{g}) \simeq W(\mathfrak{g})_{bas} = \text{Fun}(\mathfrak{g}[2])_{\text{inv}}$, the basic subalgebra of $W(\mathfrak{g})$. As in the $N = 0$ case, there is no distinction between classical and quantum BV operators, since, by (5.8), $\Delta_{\mathfrak{g}|1}$ vanishes on $W(\mathfrak{g})$ and, so, there is also no distinction between classical and quantum BV cohomologies.

The $N = 1$ gauged matter algebra $\mathcal{A}_{g|1M}$ contains as a subalgebra

$$ \mathcal{A}^+_{g|1M} = W(\mathfrak{g}) \otimes \mathcal{A}_M. $$

(5.22)

By (5.15a), (5.15b), (5.11b), (5.11d), (5.11f), (5.11h), $\mathcal{A}^+_{g|1M}$ is stable under the $\mathfrak{g}$–action $i_{\mathfrak{g}|1M}$, $l_{\mathfrak{g}|1M}$. Similarly, by (5.17b), (5.17d), (5.17e), $\mathcal{A}^+_{g|1M}$ is stable under the BV operator $\delta_{\mathfrak{g}|1M}$. Thus, $(\mathcal{A}^+_{g|1M}, \mathfrak{g}, i_{\mathfrak{g}|1M}, l_{\mathfrak{g}|1M}, \delta_{\mathfrak{g}|1M})$ is a $\mathfrak{g}$–operation.
From (5.17b), (5.17d), (5.17e), we realize that the quantum BV cohomology $H_{BV}^\ast(A^+_{\mathfrak{g}|1M})$ is the Weil cohomology $H_W^\ast(\mathfrak{g}, A_M)$ of $\mathfrak{g}$ with coefficients in the differential space $(A_M, \delta_M)$. Similarly, from (5.15a), (5.15b), (5.11b), (5.11d), (5.11f), (5.11h), (5.17b), (5.17d), (5.17e), we find that the quantum basic BV cohomology $H_{BV}^{bas}^\ast(A^+_{\mathfrak{g}|1M})$ is the basic Weil cohomology $H_{W}^{bas}^\ast(\mathfrak{g}, A_M)$ of $\mathfrak{g}$ with coefficients in the $g$-operation $(A_M, g, i_M, l_M, \delta_M)$. Unlike for the pure ghost system, the quantum and classical BV operators are generally different in the matter sector and, so, it is necessary to distinguish the classical and the quantum BV cohomologies, as in the $N = 0$ case. Analogous statements hold in the classical case, $(A_M, \delta_{M_C})$ being a differential algebra in this case.

Now, we can solve the problem of the cohomological analysis of observables in the gauged matter theory. We notice that the range of the BV algebra homomorphism $\Upsilon_1 : A_M \rightarrow A_{\mathfrak{g}|1M}$ is contained in $A^+_{\mathfrak{g}|1M}$. From (5.19a), (5.19b), (5.20), $\Upsilon_1|_{A_M^{bas}}$ is a chain map of the basic differential spaces $(A_M^{bas}, \delta_M)$, $(A^+_{\mathfrak{g}|1M}^{bas}, \delta_{\mathfrak{g}|1M})$. Thus, $\Upsilon_1|_{A_M^{bas}}$ induces a homomorphism of the basic BV cohomology spaces $H_{BV}^{bas}^\ast(A_M)$, $H_{BV}^{bas}^\ast(A^+_{\mathfrak{g}|1M})$. It can be shown that this is in fact as an isomorphism [19, 22],

$$H_{BV}^{bas}^\ast(A^+_{\mathfrak{g}|1M}) \simeq H_{BV}^{bas}^\ast(A_M) \quad (5.23)$$

under the mild assumption that the $\mathfrak{g}$-operation $A_M$ admits a connection. A self-contained proof of (5.23) is given in appendix B. (5.23) is to be compared with its $N = 0$ counterpart, eq. (4.24), from which it differs qualitatively in two ways. First, (5.23) holds with no restriction on the Lie algebra $\mathfrak{g}$, whilst (4.24) holds provided $\mathfrak{g}$ is reductive. Second, by (5.23), $H_{BV}^{bas}^\ast(A_M)$ is actually naturally isomorphic to $H_{BV}^{bas}^\ast(A^+_{\mathfrak{g}|1M})$, whilst, by (4.24), $H_{BV}^{inv}^\ast(A_M)$ is simply naturally embedded in $H_{BV}^{inv}^\ast(A^+_{\mathfrak{g}|0M})$. The reason for this can be ultimately traced back to the triviality of the Weil cohomology. In this way, the study of the observables in the gauged matter theory is fully reduced to that of the basic BV
cohomology $H_{BVbas}^*(\mathcal{A}_g^{+}|_{1M})$ of $\mathcal{A}_g^{+}|_{1M}$.

$H_{BVbas}^*(\mathcal{A}_g^{+}|_{1M}) \simeq H_{Wbas}^*(\mathfrak{g}, \mathcal{A}_M)$ is known as the $\mathfrak{g}$–equivariant cohomology $H_{\text{equiv}}^*(\mathcal{A}_M)$ of $\mathcal{A}_M$. Equivariant cohomology is defined usually for differential algebras $\mathcal{A}_M$. In our case, $\mathcal{A}_M$ is a differential algebra in the classical but not in the quantum case (cf. sect. 3). However, $H_{\text{equiv}}^*(\mathcal{A}_M)$ can still be defined.

As is well–known, there are three models of equivariant cohomology: the original models of Weil and Cartan of refs. [23–25] and the so-called BRST model of ref. [26]. The three models are in fact equivalent. The most direct and efficient way to show this was found in ref. [27], where the author proves that the Cartan and Weil model can be obtained from the BRST model via reduction of and application of a suitable inner automorphism to the algebra $\mathcal{A}_g^{+}|_{1M}$, respectively. The formal structure of the underlying algebra $\mathcal{A}_g^{+}|_{1M}$, $\mathfrak{g}$–action $i_{\mathfrak{g}|_{1M}}$, $l_{\mathfrak{g}|_{1M}}$ and differential $\delta_{\mathfrak{g}|_{1M}}$ reproduces very closely that of the corresponding objects of the BRST model of equivariant cohomology. Thus, mimicking the treatment of [27], one may try to generate the counterparts of the Cartan and Weil model in the present BV algebraic framework by reduction and action of a suitable $BV$ inner automorphism (cf. sect. 2), respectively.

The Cartan model relies on the algebra $\mathcal{C}_g^{+}|_{1M} = \text{Fun}(\mathfrak{g}[2]) \otimes \mathcal{A}_M$ instead of $\mathcal{A}_g^{+}|_{1M}$. $\mathcal{C}_g^{+}|_{1M}$ is the subalgebra of the elements of $\mathcal{A}_g^{+}|_{1M}$ containing no occurrences of the $c^i$. The Cartan model can be obtained from BRST model by observing that $\mathcal{A}_g^{+}|_{1Mbas} = \mathcal{C}_g^{+}|_{1Minv}$. So, the Cartan model is in a sense an “effective” reduction of the BRST model in which the $c^i$ have been eliminated from the outset.

The Weil model can be derived from the BRST model as follows. Define

$$\alpha_{\mathfrak{g}|_{1M}} = c^i \otimes \iota_{M_i}. \quad (5.24)$$

Clearly, $\alpha_{\mathfrak{g}|_{1M}} \in \mathcal{A}_{\mathfrak{g}|_{1M}-1}$. Further, by (5.5d), (5.8), we have

$$\Delta_{\mathfrak{g}|_{1M}} \alpha_{\mathfrak{g}|_{1M}} = 0. \quad (5.25)$$
Therefore, as shown in sect. 2,

\[ T_{g|1M} = \exp(-\text{ad}_{g|1M}) \]  

is a BV algebra inner automorphism. It is indeed the BV algebra analog of the automorphism defined and exploited in ref. [27] to show the equivalence of the BRST and Weil models. By an elementary calculation, we find

\[ \iota'_{g|1M_i} := T_{g|1M} \iota_{g|1M_i} = \iota_{g|1M_i} \otimes 1_M + 1_{g|1} \otimes \iota_{M_i}, \]  

\[ \lambda'_{g|1M_i} := T_{g|1M} \lambda_{g|1M_i} = \lambda_{g|1M_i} \otimes 1_M + 1_{g|1} \otimes \lambda_{M_i}, \]  

where \( \iota_{g|1M_i}, \lambda_{g|1M_i} \) are given by (5.10a), (5.10b). In this way, the \( g \)-action \( \iota'_{g|1M}, \lambda'_{g|1M} \) resulting from the application of \( T_{g|1M} \) is a trivial extension of the actions \( \iota_{g|1}, \lambda_{g|1} \) and \( \iota_M, \lambda_M \). Another simple calculation shows that

\[ S'_{g|1M} = T_{g|1M} S_{g|1M} = S_{g|1} \otimes 1_M + 1_{g|1} \otimes S_M, \]  

where \( S_{g|1M} \) is given by (5.16). So, the BV master action \( S'_{g|1M} \) resulting from the application of \( T_{g|1M} \) is the trivial non interacting one. Correspondingly, the quantum BV operator \( \delta'_{g|1M} \) yielded by \( T_{g|1M} \) is a trivial extension of the BV operators \( \delta_{g|1}, \delta_M \). The formal structure of the underlying algebra \( A^+_g|1M, g \)-action \( i'_{g|1M}, l'_{g|1M} \) and differential \( \delta'_{g|1M} \) obtained in this way reproduces closely that of the corresponding objects of the Weil model of equivariant cohomology.

In this way, the interaction of the matter and gauge sector can be absorbed by means of an inner BV automorphism. This would seem to trivialize the gauged matter model. However, recall that in the quantum field theoretic realizations of the construction, the automorphism may introduce non locality.
6 Higher N BV gaugings and ghost systems

In sect. 5 we found out that $N = 1$ BV gauging is at the basis of topological gauge field theory. The topological models concerned here have $N = 1$ topological supersymmetry. There are also topological models having $N = 2$ topological supersymmetry, which were first systematically studied by Dijkgraaf and Moore in ref. [28], where they were called balanced. The problem then arises of describing their gauging in a BV framework as done in the $N = 1$ case. However, when attempting this, problems of a new kind show up, as we now explain.

The basic elements of $N = 1$ BV gauging treated in sect. 5 are a BV algebra $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$ equipped a quantum BV operator $\delta$ and a $N = 1$ $\mathfrak{g}$-action $i, l$ organized in an algebraic structure, called a $\mathfrak{g}$-operation in the terminology of [19]. This structure underlies the Mathai–Quillen formulation of $N = 1$ topological field theory [20]. From now on, we shall refer to it as an $N = 1$ $\mathfrak{g}$-operation.

In ref. [28], the authors showed that the Mathai–Quillen formulation can be generalized to $N = 2$ topological field theory. Their construction hinges on an algebraic framework generalizing that of $N = 1$ $\mathfrak{g}$-operation and thus called $N = 2$ $\mathfrak{g}$-operation henceforth.

If we tried to implement $N = 2$ BV gauging following [28] and mimicking the $N = 1$ case, the basic elements would be a BV algebra $(\mathcal{A}, \Delta, \{\cdot, \cdot\})$ equipped with a doublet of quantum BV operator $\delta_A$, $A = 1, 2$ satisfying

$$[\delta_A, \delta_B] = 0. \quad (6.1)$$

In addition, we would have an $N = 2$ $\mathfrak{g}$-action, which is a set of linear maps $j : \mathfrak{g} \to \text{Der}_2(\mathcal{A})$, $i_A : \mathfrak{g} \to \text{Der}_1(\mathcal{A})$, $A = 1, 2$, $l : \mathfrak{g} \to \text{Der}_0(\mathcal{A})$ satisfying the following commutation relations

$$[j_x, j_y] = 0, \quad (6.2a)$$

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\[ [j_x, i_{Ay}] = 0, \quad (6.2b) \]
\[ [i_{Ax}, i_{By}] = \epsilon_{AB} j_{[x,y]}, \quad (6.2c) \]
\[ [l_x, j_y] = j_{[x,y]}, \quad (6.2d) \]
\[ [l_x, i_{Ay}] = i_{A[x,y]}, \quad (6.2e) \]
\[ [l_x, l_y] = l_{[x,y]}, \quad (6.2f) \]

with \( x, y \in \mathfrak{g} \), where \( \epsilon_{AB} \) is the two dimensional antisymmetric symbol. Finally, the derivations \( j_x, i_{Ax}, l_x \) would be related as

\[ [\delta_A, j_x] = i_{Ax}, \quad (6.3a) \]
\[ [\delta_A, i_{Bx}] = -\epsilon_{AB} l_x, \quad (6.3b) \]
\[ [\delta_A, l_x] = 0, \quad (6.3c) \]
with \( x \in \mathfrak{g} \). Relations (6.2a)–(6.2f) and (6.3a)–(6.3c) define an \( N = 2 \mathfrak{g} \)-operation. (Compare with relations (5.1a)–(5.1c) and (5.3a), (5.3b) defining an \( N = 1 \mathfrak{g} \)-operation). \( N = 2 \mathfrak{g} \)-operations were systematically studied in ref. [29]. One of their main properties is the existence of an internal \( \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} \) algebra of automorphisms, an "\( R \)-symmetry" in physical parlance.

In a BV framework, the existence of a doublet of quantum BV operators \( \delta_A \) is intriguing. It apparently implies the corresponding existence of a doublet of quantum BV master actions \( S_A \). However, a relation of the form

\[ \delta_A = \Delta + \text{ad } S_A \quad (6.4) \]

is incompatible with the internal \( \mathfrak{sl}(2,\mathbb{R}) \)-symmetry. This indicates that the ordinary approach based on BV algebras is inadequate for the construction we are attempting. If we wish to remedy this changing as little as possible our BV
framework, a doublet of degree 1 BV Laplacians $\Delta_A$ rather a single one $\Delta$ is required in addition to the bracket $\{\cdot,\cdot\}$. So, instead of a customary BV algebra $(\mathcal{A}, \Delta, \{\cdot,\cdot\})$, we should have some structure of the form $(\mathcal{A}, \Delta_A, \{\cdot,\cdot\})$.

$\Delta_A$ and $\{\cdot,\cdot\}$ should fulfill certain conditions generalizing those defining a BV algebra in natural fashion. Presumably, they are the following. First, the bracket $\{\cdot,\cdot\}$ satisfy the graded Leibniz relation (2.3) and the Gerstenhaber relations (2.4), (2.5). Second, the BV Laplacians $\Delta_A$ are nilpotent and anticommute

$$[\Delta_A, \Delta_B] = 0$$

(compare with (2.1)). Third, the $\Delta_A$ are degree 1 derivations of $\mathcal{A}_G$,

$$\Delta_A\{\phi, \psi\} = \{\Delta_A\phi, \psi\} + (-1)^{\partial \phi+1}\{\phi, \Delta_A\psi\},$$

(6.6)

with $\phi, \psi \in \mathcal{A}$ (compare with (2.6)). There is no extension of relation (2.2).

In the resulting extended BV algebraic framework, (6.4) is improved as

$$\delta_A = \Delta_A + \text{ad} S_A.$$  

(6.7)

In order (6.1) to be satisfied, it is sufficient that

$$\Delta_A S_B + \Delta_B S_A + \{S_A, S_B\} = 0.$$  

(6.8)

This is the resulting generalization of the master equation (3.1). Its field theoretic origin, if any, is not clear at all.

Let us assume that the $N = 2 \mathfrak{g}$-action is BV Hamiltonian in the following sense. There exist linear BV moment maps $\eta : \mathfrak{g} \to \mathcal{A}_{-3}$, $\iota_A : \mathfrak{g} \to \mathcal{A}_{-2}$, $A = 1, 2$, $\lambda : \mathfrak{g} \to \mathcal{A}_{-1}$ such that

$$j_x = \text{ad} \eta_x,$$

(6.9a)

$$i_{Ax} = \text{ad} \iota_{Ax},$$

(6.9b)

$$l_x = \text{ad} \lambda_x,$$

(6.9c)

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and satisfying the relations

\[
\{\eta_x, \eta_y\} = 0, \quad (6.10a)
\]
\[
\{\eta_x, \iota_A y\} = 0, \quad (6.10b)
\]
\[
\{\iota_A x, \iota_B y\} = \epsilon_{AB} \eta_{[x,y]}, \quad (6.10c)
\]
\[
\{\lambda_x, \eta_y\} = \eta_{[x,y]}, \quad (6.10d)
\]
\[
\{\lambda_x, \iota_A y\} = \iota_A \lambda_{[x,y]}, \quad (6.10e)
\]
\[
\{\lambda_x, \lambda_y\} = \lambda_{[x,y]}, \quad (6.10f)
\]
\[
\Delta_A \eta_x = 0, \quad (6.10g)
\]
\[
\Delta_A \iota_B x = 0, \quad (6.10h)
\]
\[
\Delta_A \lambda_x = 0, \quad (6.10i)
\]

with \(x, y \in \mathfrak{g}\). Assuming that (6.9a)–(6.9c) hold, (6.10a)–(6.10f) ensure that (6.2a)–(6.2f) are fulfilled. On account of (6.7), (6.10g)–(6.10i) ensure that (6.3a)–(6.3c) are also fulfilled if the master action doublet \(S_A\) satisfies

\[
\{S_A, \eta_x\} = \iota_A x, \quad (6.11a)
\]
\[
\{S_A, \iota_B x\} = -\epsilon_{AB} \lambda_x, \quad (6.11b)
\]
\[
\{S_A, \lambda_x\} = 0, \quad (6.11c)
\]

with \(x \in \mathfrak{g}\). We may take this as the definition of invariance of the action doublet \(S_A\) under the \(N = 2\) BV Hamiltonian \(\mathfrak{g}\)–action.

We shall not attempt to fully generalize the constructions of sects. 4, 5 to obtain \(N = 2\) BV gauging. The construction is algebraically complicated, on one hand, and its eventual relevance in field theoretic applications still doubtful,
on the other. Moreover, the definitions of the relevant structures do not seem to be unique. We shall limit ourselves to a broad outline of the \(N = 2\) gauging procedure and the structure of the \(N = 2\) ghost system and its coupling to a matter system.

Consider a matter extended BV algebra \((\mathcal{A}_M, \Delta_{MA}, \{\cdot, \cdot\}_M)\) carrying a BV Hamiltonian \(N = 2\) \(\mathfrak{g}\)-action \(j_M, i_{MA}, l_M\) with BV moment maps \(\eta_M, \iota_{MA}, \lambda_M\) and a matter quantum BV master action doublet \(S_{MA}\) invariant under the \(\mathfrak{g}\)-action. We want to gauge the \(\mathfrak{g}\)-symmetry.

The gauging proceeds in three steps, as usual.

1. We construct an \(N = 2\) ghost extended BV algebra \((\mathcal{A}_{\mathfrak{g}|2}, \Delta_{\mathfrak{g}|2A}, \{\cdot, \cdot\}_{\mathfrak{g}|2})\) with a BV Hamiltonian \(N = 2\) \(\mathfrak{g}\)-action \(j_{\mathfrak{g}|2}, i_{\mathfrak{g}|2A}, l_{\mathfrak{g}|2}\) with BV moment maps \(\eta_{\mathfrak{g}|2}, \iota_{\mathfrak{g}|2A}, \lambda_{\mathfrak{g}|2}\) and an \(N = 2\) ghost quantum BV master action doublet \(S_{\mathfrak{g}|2A}\) invariant under the \(\mathfrak{g}\)-action. The construction is canonical, depending on \(\mathfrak{g}\) only.

2. We construct an \(N = 2\) gauged matter extended BV algebra \((\mathcal{A}_{\mathfrak{g}|2M}, \Delta_{\mathfrak{g}|2MA}, \{\cdot, \cdot\}_{\mathfrak{g}|2M})\) and equip it with an appropriate BV Hamiltonian \(N = 2\) \(\mathfrak{g}\)-action \(j_{\mathfrak{g}|2M}, i_{\mathfrak{g}|2MA}, l_{\mathfrak{g}|2M}\) with BV moment maps \(\eta_{\mathfrak{g}|2M}, \iota_{\mathfrak{g}|2MA}, \lambda_{\mathfrak{g}|2M}\).

3. We construct an \(N = 2\) gauged matter action doublet \(S_{\mathfrak{g}|2MA}\) of the gauged matter BV algebra invariant under the \(\mathfrak{g}\)-action.

The \(N = 2\) ghost system was introduced originally in ref. [28] and studied in detail in ref. [29]. It consists of a degree 1 \(\mathfrak{g}\)-valued doublet \(c_A^i\), a degree 2 \(\mathfrak{g}\)-valued singlet \(c^i\), a degree 2 \(\mathfrak{g}\)-valued triplet \(C_{AB}^i\) symmetric in \(A, B\) and a degree 3 \(\mathfrak{g}\)-valued doublet \(C_A^i\). In the extended BV framework, these are conjugated to a degree \(-2\) \(\mathfrak{g}^\vee\)-valued doublet \(b_A^i\), a degree \(-3\) \(\mathfrak{g}^\vee\)-valued singlet \(b_i\), a degree \(-3\) \(\mathfrak{g}^\vee\)-valued triplet \(B_{AB}^i\) symmetric in \(A, B\) and a degree \(-4\) \(\mathfrak{g}^\vee\)-valued doublet \(B_A^i\), respectively. They span a graded algebra \(\mathcal{A}_{\mathfrak{g}|2}\). Apparently, the only consistent choice of the BV Laplacians \(\Delta_{\mathfrak{g}|2A}\) in \(\mathcal{A}_{\mathfrak{g}|2}\) is the trivial one

\[
\Delta_{\mathfrak{g}|2A} = 0. \quad (6.12)
\]
\( \mathcal{A}_{g|2} \) has instead a natural non trivial bracket

\[
\{ \phi, \psi \}_{g|2} = \partial_{RbA}^i \phi \partial_{Lc}^i \psi - \partial_{RC}^A \phi \partial_{LBA}^i \psi
\]

\[+ \partial_{Rb}^i \phi \partial_{Lc}^i \psi - \partial_{Rc}^i \phi \partial_{LB}^i \psi
\]

\[+ \partial_{RA}^i \phi \partial_{LC}^i \psi - \partial_{RC}^A \phi \partial_{LBA}^i \psi, \quad \phi, \psi \in \mathcal{A}_{g|2}.
\]

The construction of the moment maps \( \eta_{g|2}, \iota_{g|2}, \lambda_{g|2} \) of the appropriate Hamiltonian \( N = 2 \) \( g \)–action on \( \mathcal{A}_{g|2} \) and of the correct \( N = 2 \) ghost master action doublet \( S_{g|2A} \) satisfying the invariance conditions (6.11) and the master equation (6.8) is an open problem. A superfield formulation of the \( N = 2 \) ghost system is possible in principle, as in the \( N = 1 \) case.

The \( N = 2 \) gauged matter extended BV algebra \((\mathcal{A}_{g|2M}, \Delta_{g|2MA}, \{\cdot, \cdot\}_{g|2M})\) is the tensor product of the \( N = 2 \) ghost BV algebra \((\mathcal{A}_{g|2}, \Delta_{g|2A}, \{\cdot, \cdot\}_{g|2})\) and the matter BV algebra \((\mathcal{A}_M, \Delta_{MA}, \{\cdot, \cdot\}_M)\). The tensor product of extended BV algebras is defined by a straightforward generalization of the definition of tensor product of ordinary BV algebras given in sect. 2. We expect that, in a BRST model, the appropriate Hamiltonian \( N = 2 \) \( g \)–action of the gauged matter BV algebra to be some non trivial extension of those of its ghost and matter factors, as in the \( N = 1 \) case. The precise definition of the corresponding moment maps \( \eta_{g|2M}, \iota_{g|2MA}, \lambda_{g|2M} \) is a further open problem.

If we tried to generalize (5.16) in the extended BV framework illustrated above, the gauged matter action doublet would be something like

\[
S_{g|2MA} = S_{g|2A} \otimes 1_M + 1_{g|2} \otimes S_{MA} + c_A^i \otimes \lambda_{Mi}
\]

\[- \epsilon^{BC}(C_A^i - \epsilon_{AB} c^i) \otimes \iota_{MCi} + C_A^i \otimes \eta_{Mi},
\]

the last three terms being interaction terms. The fulfilment of the invariance con-
ditions (6.11) and the master equation (6.8) cannot be ascertained as long as the explicit form of the ghost BV action $S_{g|2A}$ is not known.

It is reasonable to expect that the appropriate classification of the observables of a theory described by an extended BV algebra $(\mathcal{A}, \Delta_A, \{\cdot, \cdot\})$ and a quantum BV master action doublet $S_A$ is encoded in the cohomology of the bidifferential space $(\mathcal{A}, \delta_A)$. However, this cohomology cannot have the customary form of a $\mathbb{Z}$–bigraded cohomology. $\mathcal{A}$ has no $\mathbb{Z}$–bigrading such that there are two independent linear combinations of the $\delta_A$ each of which raises one of the underlying $\mathbb{Z}$–gradings by one unit and leaves invariant the other one. Rather, the observables are classified by the cohomology of any non vanishing linear combination of the $\delta_A$, the internal $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ guaranteeing the independence of the cohomology from the choice of the combination.

If the extended BV algebra $(\mathcal{A}, \Delta_A, \{\cdot, \cdot\})$ is equipped with $N = 2 \mathfrak{g}$–action $j$, $i_A$, $l$ under which the BV action doublet $S_A$ is invariant in the sense that (6.3a)–(6.3c) are satisfied, one may define an $N = 2 \mathfrak{g}$–basic quantum BV cohomology. This is the cohomology, as defined in the previous paragraph, of the bidifferential space $(\mathcal{A}_{bas}, \delta_A)$, where $\mathcal{A}_{bas} = \cap_{x \in \mathfrak{g}} (\ker j_x \cap \ker i_{Ax} \cap \ker l_x) \subset \mathcal{A}$. When carrying out the gauging of a matter extended BV algebra with an invariant matter action doublet as outlined above, a corresponding notion of $N = 2 \mathfrak{g}$–equivariant cohomology should appear.

The above analysis presumably generalizes to higher values of $N$. To the best of our knowledge, virtually nothing is known about $N \geq 3 \mathfrak{g}$–operations and ghost systems. However, we expect the inadequacy of the customary BV algebraic framework to emerge again.
7 Applications and examples

In this section, we shall present a few applications of the formalism developed in the preceding sections. Our examples are drawn from Lie algebroid and Poisson geometry, which cover a broad spectrum of cases. We concentrate on the well understood \( N = 0 \) and \( N = 1 \) gauging.

The BV algebra of a Lie algebroid and its gauging

A Lie algebroid is a vector bundle \( E \) over a manifold \( M \) equipped with a bundle map \( \rho_E : E \to TM \), called the anchor, and an \( \mathbb{R} \)-linear bracket \( [\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \) with the following properties.

1) \([\cdot, \cdot]_E \) is a Lie bracket so that \( \Gamma(E) \) is a Lie algebra:

\[
[X, Y]_E + [Y, X]_E = 0,
\]

\[
[X, [Y, Z]_E]_E + [Y, [Z, X]_E]_E + [Z, [X, Y]_E]_E = 0,
\]

for \( X, Y, Z \in \Gamma(E) \).

2) \( \rho \) defines a Lie algebra homomorphism of \( \Gamma(E) \) into \( \Gamma(TM) \):

\[
\rho([X, Y]_E) = [\rho(X), \rho(Y)]_{TM},
\]

for \( X, Y \in \Gamma(E) \), where \( [\cdot, \cdot]_{TM} \) is the usual Lie bracket of vector fields of \( M \).

3) The generalized Leibniz rule holds:

\[
[X, fY]_E = f[X, Y]_E + (\rho(X)f)Y,
\]

for \( f \in C^\infty(M) \) and \( X, Y \in \Gamma(E) \).

The prototype Lie algebroid over \( M \) is the tangent bundle \( TM \): the anchor is the identity \( \text{id}_{TM} \) and the bracket is the usual Lie bracket \( [\cdot, \cdot]_{TM} \). Lie algebroids generalize Lie algebras: a Lie algebra can be viewed as a Lie algebroid over the singleton manifold \( M = \text{pt} \).
Let \( \{ e_r \} \) be a local frame of \( E \). Then, one has

\[
\rho_E(e_r) = \rho^a_r \partial_a, \tag{7.5}
\]

\[
[e_r, e_s]_E = c^t_{rs} e_t. \tag{7.6}
\]

Here, \( a, b, c, \ldots \) are base coordinate indices while \( r, s, t, \ldots \) are fiber coordinate indices. \( \rho^a_r, c^t_{rs} \) are called the anchor and structure functions of \( E \), respectively. From (7.1)–(7.4), they satisfy

\[
c^r_{st} + c^r_{ts} = 0, \tag{7.7a}
\]

\[
c^r_{sv} c^v_{tu} + c^r_{tv} c^v_{us} + c^r_{uv} c^v_{st} + \rho^a_s \partial_a c^r_{tu} + \rho^a_t \partial_a c^r_{us} + \rho^a_u \partial_a c^r_{st} = 0, \tag{7.7b}
\]

\[
\rho^b_r \partial_b \rho^a_s - \rho^b_s \partial_b \rho^a_r - c^t_{rs} \rho^a_t = 0. \tag{7.7c}
\]

(7.7a)–(7.7c) are the structure relations of \( E \).

The Lie algebroid \( E \) is characterized by a natural cohomology. We shall define this conveniently in the language of graded geometry \([30]\). Consider the parity shifted bundle \( E[1] \) and the \( \mathbb{Z} \) graded algebra \( \text{Fun}(E[1]) \) of functions on \( E[1] \). There exists a degree 1 derivation \( d_E \) of \( \text{Fun}(E[1]) \) defined by

\[
d_E = \rho^a_r(x) \xi^r \partial_L a - \frac{1}{2} c^r_{st}(x) \xi^s \xi^t \partial_L r, \tag{7.8}
\]

where \( x^a, \xi^r \) are the base and fiber coordinates of a generic trivialization of \( E[1] \) with degree 0, 1, respectively, and \( \partial_a = \partial/\partial x^a, \partial_r = \partial/\partial \xi^r \). Using the relations (7.7), one checks easily that \( d_E \) is nilpotent and is therefore a differential

\[
d_E^2 = 0. \tag{7.9}
\]

The cohomology of the differential space \( (\text{Fun}(E[1]), d_E) \) is the Lie algebroid cohomology of \( E \), \( H_{LA}^*(E) \). When \( E = TM \), \( d_E \) reduces to the ordinary de Rham differential and \( H_{LA}^*(E) \) reduces to the familiar de Rham cohomology.
With any section \( X \in \Gamma(E) \), there are associated two derivations of \( \text{Fun}(E[1]) \) of degree \(-1, 0\) defined by

\[
i_{EX} = X^r(x)\partial_{Lr} \\
l_{EX} = \rho_r^a X^r(x)\partial_{La} + (\rho_r^a \partial_a X^s + \tau_{rs}^t X^t)(x)\xi^r \partial_{Ls},
\]

\text{(7.10a)}\text{(7.10b)}

They generalize the interior and Lie derivatives of de Rham theory and reduce to those when \( E = TM \).

It is simple to check that the above derivations satisfy

\[
[d_{E}, d_{E}] = 0, \\
[d_{E}, i_{EX}] = l_{EX}, \\
[d_{E}, l_{EX}] = 0, \\
[i_{EX}, i_{EY}] = 0, \\
[l_{EX}, i_{EY}] = i_{E[X,Y]_{E}}, \\
[l_{EX}, l_{EY}] = l_{E[X,Y]_{E}},
\]

\text{(7.11a)}\text{(7.11b)}\text{(7.11c)}\text{(7.11d)}\text{(7.11e)}\text{(7.11f)}

with \( X, Y \in \Gamma(E) \), generalizing the well–known Cartan relations.

Let \( \mathfrak{g} \) be a Lie algebra and let \( \varphi : \mathfrak{g} \to \Gamma(E) \) be a fiducial Lie algebra homomorphism. Then, for \( x \in \mathfrak{g} \), the degree \(-1, 0\) derivations of \( \text{Fun}(E[1]) \)

\[
i_{E_{x}} = i_{E_{\varphi(x)}}, \\
l_{E_{x}} = l_{E_{\varphi(x)}}
\]

\text{(7.12a)}\text{(7.12b)}

are defined. By \text{(7.11a)}\text{–}(7.11f), \( (\text{Fun}(E[1]), \mathfrak{g}, l_{E}, d_{E}) \) is a differential \( \mathfrak{g} \)–module (cf. app. A). The associated invariant cohomology is the invariant Lie algebroid cohomology \( H_{L_{\text{inv}}}^{*}(E) \) of \( E \) \text{[31]}. Analogously, by \text{(7.11a)}\text{–}(7.11f), \( (\text{Fun}(E[1]), \mathfrak{g}, i_{E}, l_{E}, d_{E}) \) is a \( \mathfrak{g} \)–operation (cf. app. B). The basic cohomology as-
associated with it is the basic Lie algebroid cohomology $H_{LAb}^*(E)$ of $E$ \[32\].

The $\mathbb{Z}$-graded algebra $\text{Fun}(T^*[−1]E[1])$ of functions on the parity shifted cotangent bundle $T^*[−1]E[1]$ of $E[1]$ can be given a structure of BV algebra. This BV algebra extends, in an appropriate sense to be specified, the algebra $\text{Fun}(E[1])$ considered above.

$T^*[−1]E[1]$ has the canonical degree $−1$ symplectic structure

$$\omega_E = dy_a dx_a + d\eta_r d\xi^r,$$

where $x^a, \xi^r, y_a, \eta_r$ are the base and fiber coordinates of a generic trivialization of $T^*[−1]E[1]$ with degree $0, 1, −1, −2$, respectively. Let us assume now that the orientation line bundle $Q_E = \wedge^n T^*M \otimes \wedge^q E$, where $n = \text{dim} M$ and $q = \text{rank} E$ is trivial. There then exists a nowhere vanishing section $\gamma \in \Gamma(Q_E)$, which can be used to construct a volume form on $T^*[−1]E[1]$,

$$\mu_{E\gamma} = \gamma^2(x)dx^1 \cdots dx^n d\xi^1 \cdots d\xi^r dy_1 \cdots dy_n d\eta_1 \cdots d\eta_r.$$  \[7.14\]

These geometrical objects allow us to endow $\text{Fun}(T^*[−1]E[1])$ with the structure of BV algebra. The construction is standard and is illustrated in the literature (see e. g. ref. \[7\]). The BV Laplacian $\Delta_{E\gamma}$ is given by

$$\Delta_{E\gamma} = \gamma^{-1}(x) \partial L_a \gamma(x) \partial L^a - \partial L_r \partial L^r$$

\[7.15\]

where $\partial_a = \partial/\partial x^a$, $\partial_r = \partial/\partial \xi^r$, $\partial^a = \partial/\partial y_a$, $\partial^r = \partial/\partial \eta_r$. The BV antibracket has the standard form

$$\{\phi, \psi\}_E = \partial R_a \phi \partial L^a \psi - \partial R^a \phi \partial L_a \psi + \partial R_r \phi \partial L^r \psi - \partial R^r \phi \partial L_r \psi,$$

\[7.16\]

with $\phi, \psi \in \text{Fun}(T^*[−1]E[1])$. It is easy to check that the triple $(\text{Fun}(T^*[−1]E[1]), \Delta_{E\gamma}, \{\cdot, \cdot\}_E)$ satisfies (2.1)–(2.3) and is therefore a BV algebra as announced.
The bundle projection \( \pi_E : T^*[1]E[1] \to E[1] \) induces a degree 0 graded algebra monomorphism \( \pi_E^* : \text{Fun}(E[1]) \to \text{Fun}(T^*[1]E[1]) \). In this way, \( \text{Fun}(E[1]) \) can be viewed as a subalgebra of \( \text{Fun}(T^*[1]E[1]) \). From (7.15), (7.16), one has \( \Delta_E \gamma|_{\text{Fun}(E[1])} = 0 \) and \( \{ \cdot|_{\text{Fun}(E[1])}, \cdot|_{\text{Fun}(E[1])} \}_E = 0 \). It follows that \( \text{Fun}(E[1]) \), equipped with the trivial BV algebra structure, is a BV subalgebra of the BV algebra \( \text{Fun}(T^*[1]E[1]) \) (cf. sect. 2). Indeed, the BV antibracket structure of \( \text{Fun}(T^*[1]E[1]) \) is closely related to the bracket structure of the “big bracket” formulation of Lie algebroid theory \([33–35]\).

With applications of the theory of the preceding sections in mind, we want to equip the BV algebra \( \text{Fun}(T^*[1]E[1]) \) with a quantum BV master action with global symmetries. This is achieved by the following construction.

\( \text{Fun}(T^*[1]E[1]) \) contains the degree 0 element

\[
S_E = \rho^a_r(x) y_a \xi^r + \frac{1}{2} c^r_{st}(x) \xi^s \xi^t \eta_r
\]  

and, for \( X \in \Gamma(E) \), the degree \(-2\), \(-1\) elements

\[
\iota_{EX} = -X^r(x) \eta_r, \quad (7.18a)
\]

\[
\lambda_{EX} = -\rho^a_r X^r(x) y_a - (\rho^a_r \partial_a X^s + c^a_{rt} X^t)(x) \xi^r \eta_s. \quad (7.18b)
\]

By a straightforward calculation, one finds the brackets

\[
\{S_E, S_E\}_E = 0, \quad (7.19a)
\]

\[
\{\iota_{EX}, S_E\}_E = \lambda_{EX}, \quad (7.19b)
\]

\[
\{\lambda_{EX}, S_E\}_E = 0, \quad (7.19c)
\]

\[
\{\iota_{EX}, \iota_{EY}\}_E = 0, \quad (7.19d)
\]

\[
\{\lambda_{EX}, \iota_{EY}\}_E = \iota_{E[X,Y]_E}, \quad (7.19e)
\]
\{\lambda_{EX}, \lambda_{EY}\}_E = \lambda_{E[X,Y]E}.

One also shows that the relations

\[ \Delta_{E\gamma} S_E = 0, \]

\[ \Delta_{E\gamma} \lambda_{EX} = 0, \]

\[ \Delta_{E\gamma} \lambda_{EX} = 0. \]

hold, provided \( \gamma \) satisfies the condition

\[ \partial_a \rho_r^a + \rho_r^a \partial_a \ln \gamma - c^s_{sr} = 0. \]

In general, the chosen \( \gamma \in \Gamma(Q_E) \) does not fulfil (7.21). Note, however, that \( \gamma \) is determined only up to a rescaling by a factor of the form \( e^f \) with \( f \in \text{Fun}(M) \). Hence, if, instead of (7.21), \( \gamma \) satisfies the weaker condition

\[ \partial_a \rho_r^a + \rho_r^a \partial_a \ln \gamma - c^s_{sr} + \rho_r^a \partial_a f = 0 \]

for some function \( f \in \text{Fun}(M) \), then, after redefining \( \gamma \) into \( e^f \gamma \), one can make \( \gamma \) fulfil (7.21). It can be shown that this is the case precisely when the Lie algebroid \( E \) is \textit{unimodular}, i.e. its modular class \( \theta_E \), a distinguished element of the degree 1 cohomology \( H_{LA}^1(E) \), vanishes \[36\]. Indeed, the first three terms in left hand side of (7.22) constitute the local expression of a generic representative of \( \theta_E \) and (7.22) is the statement that this representative is exact. See appendix D for a review of the definition and the main properties of the modular class. The relevance of unimodularity in BV theory has been recently emphasised in ref. \[16\].

When a \( \gamma \in \Gamma(Q_E) \) satisfying (7.21) exists, it may not be unique. We are still free to redefine \( \gamma \) into \( e^f \gamma \) for any function \( f \in \text{Fun}(M) \) such that

\[ \rho_r^a \partial_a f = 0. \]
Note that eq. (7.23) reads compactly as \( d_E f = 0 \). So, its solutions span the degree 0 cohomology \( H_{LA}^0(E) \).

Henceforth, we assume that a nowhere vanishing \( \gamma \in \Gamma(Q_E) \) satisfying (7.21) exists and has been chosen. Naturalness requires that all the relevant BV structures do not depend on this choice, a property that must be carefully checked.

The BV algebra \( \text{Fun}(T^*[−1]E[1]) \) is now equipped with the degree 1 derivation
\[
\overline{d}_E = \text{ad}_E S_E, \tag{7.24}
\]
and, for \( X \in \Gamma(E) \), the degree \(-1, 0\) derivations
\[
\overline{i}_{EX} = \text{ad}_E i_{EX}, \tag{7.25a}
\]
\[
\overline{l}_{EX} = \text{ad}_E \lambda_{EX}, \tag{7.25b}
\]
where \( \text{ad}_E \) is defined according to (2.8). By (7.19a)–(7.19c), \( \overline{d}_E, \overline{i}_{EX}, \overline{l}_{EX} \) satisfy the Cartan relations (7.11a)–(7.11f). Further, by (7.20a)–(7.20c), \( d_E, i_{EX}, l_{EX} \) are BV inner derivations (cf. sect. 2).

Inspecting (7.8), (7.10a), (7.10b), we observe that \( d_E = \overline{d}_E |_{\text{Fun}(E[1])}, i_{EX} = \overline{i}_{EX}|_{\text{Fun}(E[1])}, l_{EX} = \overline{l}_{EX}|_{\text{Fun}(E[1])} \), with \( X \in \Gamma(E) \). Therefore, the derivations \( d_E, i_{EX}, l_{EX} \) extend \( d_E, i_{E}, l_{E} \) from \( \text{Fun}(E[1]) \) to \( \text{Fun}(T^*[−1]E[1]) \).

By (7.19a), (7.20a), \( S_E \) satisfies the quantum BV master equation (3.1) and is therefore a quantum BV master action of the BV algebra \( \text{Fun}(T^*[−1]E[1]) \). The quantum BV operator is \( \delta_{E\gamma} = \Delta_{E\gamma} + \text{ad}_E S_E \) (cf. eq. (3.3)). \( \delta_{E\gamma} \) depends explicitly on \( \gamma \). The quantum BV cohomology \( H_{BV}^*(\text{Fun}(T^*[−1]E[1])) \), conversely, does not up to isomorphism, since, for \( f \in \text{Fun}(M) \) satisfying (7.23), one has \( \delta_{E\gamma} \gamma f = e^{-f} \delta_{E\gamma} e^f \). The classical BV operator is \( \delta_{E\gamma c} = \text{ad}_E S_E = d_E \) (cf. eq. (3.7)). It is manifestly independent from \( \gamma \). Hence, the classical BV cohomology \( H_{cBV}^*(\text{Fun}(T^*[−1]E[1])) \) also is.

Since \( \delta_{E\gamma}|_{\text{Fun}(E[1])} = \delta_{E\gamma c}|_{\text{Fun}(E[1])} = d_E \), the algebra inclusion \( \pi_E^* : \text{Fun}(E[1]) \to \text{Fun}(T^*[−1]E[1]) \) induces a homomorphism of the Lie algebroid cohomology
$H_{LA}^*(E)$ into the quantum BV cohomology $H_{BV}^*(\text{Fun}(T^*[−1]E[1]))$ as well as the classical BV cohomology $H_{cBV}^*(\text{Fun}(T^*[−1]E[1]))$. Thus, each Lie algebroid cohomology class gives rise to a well-defined BV observable.

Let $g$ be a Lie algebra and let $\varphi : g \to \Gamma(E)$ be a fiducial Lie algebra homomorphism. For any $x \in g$, let us define

\begin{equation}
\iota_{Ex} = \iota_{E\varphi(x)},
\end{equation}

\begin{equation}
\lambda_{Ex} = \lambda_{E\varphi(x)},
\end{equation}

and then define derivations $\tilde{\iota}_{Ex}, \tilde{\lambda}_{Ex}$ on $\text{Fun}(T^*[−1]E[1])$ via (7.25a), (7.25b).

Suppose we keep only the Lie derivations $\tilde{\iota}_{Ex}$ and forget about the interior derivations $\tilde{\lambda}_{Ex}$. From (7.25b), (7.19f), (7.20c), it follows immediately that (4.4), (4.5) are satisfied. Hence, the BV algebra $\text{Fun}(T^*[−1]E[1])$ carries an $N = 0$ BV Hamiltonian $g$–action $\tilde{l}_E$ having $\lambda_E$ as BV moment map. Further, by (7.19c), the master action $S_E$ satisfies (4.6) and is thus invariant under the $g$–action. Therefore, we can perform the $N = 0$ gauging of the BV algebra following the scheme described in sect. 3.

The relevant invariant quantum BV cohomology $H_{BV_{inv}}^*(\text{Fun}(T^*[−1]E[1]))$ (cf. sect. 1) is independent from the choice of $\gamma$, like $H_{BV}^*(\text{Fun}(T^*[−1]E[1]))$. In fact, for $f \in \text{Fun}(M)$ satisfying (7.23), one has $\delta_{E\gamma^*} = e^{-f}\delta_{E_\gamma}e^f$ and, for $x \in g$, $\tilde{l}_{Ex} = e^{-f}\tilde{\iota}_{Ex}e^f$, as $l_{Ex}f = 0$. Obviously, the invariant classical BV cohomology $H_{cBV_{inv}}^*(\text{Fun}(T^*[−1]E[1]))$ is independent from the choice of $\gamma$.

Since $\delta_{E\gamma}|_{\text{Fun}(E[1])} = \delta_{E_\gamma}|_{\text{Fun}(E[1])} = d_E$ and, for $x \in g$, $l_{Ex} = \tilde{l}_{Ex}|_{\text{Fun}(E[1])}$, the algebra inclusion $\pi_E^* : \text{Fun}(E[1]) \to \text{Fun}(T^*[−1]E[1])$ induces a homomorphism of the invariant Lie algebroid cohomology $H_{LA_{inv}}^*(E)$ into the invariant quantum BV cohomology $H_{BV_{inv}}^*(\text{Fun}(T^*[−1]E[1]))$ as well as the invariant classical BV cohomology $H_{cBV_{inv}}^*(\text{Fun}(T^*[−1]E[1]))$. Thus, each invariant Lie algebroid cohomology class gives rise to an invariant BV observable.
Suppose we keep both the interior derivations $i_{Ex}$ and Lie derivations $l_{Ex}$. From (7.25a), (7.25b), (7.19d–(7.19f), (7.20b), (7.20c), it follows immediately that (5.4), (5.5) are satisfied. Hence, the BV algebra $\text{Fun}(T^*[-1]E[1])$ carries an $N = 1$ BV Hamiltonian $g$–action $\tilde{i}_E, \tilde{l}_E$ having $i_E, \lambda_E$ as BV (pre)moment maps. Further by (7.19b), (7.19c), the master action $S_E$ satisfies (5.6) and is thus (Hamiltonian) invariant under the $g$–action. Therefore, the $N = 1$ gauging of the BV algebra can be carried out along the lines illustrated in sect. 5.

The relevant basic quantum BV cohomology $H_{BV_{bas}}^*(\text{Fun}(T^*[-1]E[1]))$ (cf. sect. 5) is independent from the choice of $\gamma$, like $H_{BV}^*(\text{Fun}(T^*[-1]E[1]))$, analogously to the $N = 0$ case. In fact, for $f \in \text{Fun}(M)$ satisfying (7.23), one has $\delta_{Ee} f|_{\gamma} = e^{-f} \delta_{Ee} f|_{\gamma}$ and, for $x \in g$, $i_{Ex} = e^{-f} i_{Ex} e^f$, $l_{Ex} = e^{-f} l_{Ex} e^f$, as $i_{Ex} f = l_{Ex} f = 0$. The basic classical BV cohomology $H_{cBV_{bas}}^*(\text{Fun}(T^*[-1]E[1]))$ is of course independent from the choice of $\gamma$.

As $\delta_{Ee}|_{\text{Fun}(E[1])} = \delta_{Ee}|_{\text{Fun}(E[1])} = d_E$ and, for $x \in g$, $i_{Ex} = i_{Ex}|_{\text{Fun}(E[1])}$, $l_{Ex} = l_{Ex}|_{\text{Fun}(E[1])}$, the algebra inclusion $\pi_E^*: \text{Fun}(E[1]) \rightarrow \text{Fun}(T^*[-1]E[1])$ induces a homomorphism of the basic Lie algebroid cohomology $H_{LA_{bas}}^*(E)$ into the basic quantum BV cohomology $H_{BV_{bas}}^*(\text{Fun}(T^*[-1]E[1]))$ as well as the basic classical BV cohomology $H_{cBV_{bas}}^*(\text{Fun}(T^*[-1]E[1]))$, analogously to the $N = 0$ case. So, each basic Lie algebroid cohomology class gives rise to a basic BV observable.

The general construction expounded above exhibits a rich geometry, but, from the point of view of BV gauging, is kind of trivial: one can gauge any Lie algebra $g$ under the mild assumption that a Lie algebra homomorphism $\varphi: g \rightarrow \Gamma(E)$ is available. In physical problems, there virtually always are restrictions on the symmetries that one can gauge. The Poisson Lie algebroid is a special case of the above general construction, in which such restrictions emerge naturally.

The Poisson Lie algebroid BV algebra and its gauging

Suppose that $M$ is a Poisson manifold and that $P \in \Gamma(\wedge^2 TM)$ is its Poisson
bivector \[37\]. Then, \(P\) satisfies the Poisson condition
\[P^{ad}\partial_a P^{bc} + P^{bd}\partial_d P^{ca} + P^{cd}\partial_d P^{ab} = 0.\]  
(7.27)

As is well-known, the Poisson structure of \(M\) endows the cotangent bundle \(T^*M\) of \(M\) with the structure of Lie algebroid. For simplicity, we shall mark all objects referring to this algebroid with a suffix \(P\). The anchor and structure functions of \(T^*M\) are given by \(\rho^{ab} = P^{ab}\) and \(c^{ab} = \partial_a P^{bc}\). The Lie algebroid cohomology of \(T^*M\) is the Poisson–Lichnerowicz cohomology of \(P\).

Proceeding as explained in detail above, we construct the associated BV algebra \((\text{Fun}(\Omega[T^*M]), \Delta_P, \{\cdot, \cdot\}_P)\), the quantum BV master action \(S_P\) and, for \(\alpha \in \Gamma(T^*M)\), the (pre)moments \(i_P\alpha, \lambda_P\alpha\) of the interior and Lie derivations \(\tilde{i}_P\alpha, \tilde{\lambda}_P\alpha\) of \(\text{Fun}(\Omega[T^*M])\).

The master action \(S_P\) of, defined according to (7.17), reads as
\[S_P = P^{ab}(x)y_b \xi_a + \frac{1}{2}\partial_a P^{bc}(x)\xi_b \xi_c \eta^a.\]  
(7.28)

Similarly, for \(\alpha \in \Gamma(T^*M)\), the (pre)moments \(i_P\alpha, \lambda_P\alpha\) defined according to (7.18a), (7.18b), are given by
\[i_P\alpha = -\alpha_a(x)\eta^a,\]  
(7.29a)
\[\lambda_P\alpha = -P^{ab}\alpha_a(x)y_b - (P^{ac}\partial_c \alpha_b + \partial_b P^{ac}\alpha_c)(x)\xi_a \eta^b.\]  
(7.29b)

The orientation line bundle \(Q_P = (\wedge^n T^*M)^{\otimes 2}\) is always trivial. Letting \(\gamma \in \Gamma(Q_P)\) be a nowhere vanishing section, the unimodularity condition (7.21) reads
\[-2\gamma^{-1/2}\partial_b(\gamma^{1/2} P^{ba}) = 0.,\]  
(7.30)

(7.30) determines \(\gamma\) only up to a rescaling by a factor \(e^f\), where \(f \in \text{Fun}(M)\) is a Casimir function of \(P\) (that is \(P^{ab}\partial_b f = 0\), cf. eq. (7.23)). The action \(S_P\) coincides with the reduced action used in the semiclassical computation of the
correlators of quantum observables for the Poisson sigma model on the sphere in ref. [16].

Now, we shall assess whether it is possible to perform a non trivial gauging of the Poisson BV algebra just constructed on the lines of ref. [11]. To this end, we make the following assumptions.

1. A compact connected Lie group $G$ with Lie algebra $\mathfrak{g}$ is given.

2. $M$ carries a smooth effective left $G$–action.

3. The $G$–action is Hamiltonian.

As is well–known, the fundamental vector fields of the $G$–action organize as a section $u \in \Gamma(TM \otimes \mathfrak{g}^\vee)$. $u$ is $G$–equivariant, that is

$$u^b_i \partial_b u^a_j - u^b_j \partial_b u^a_i = f^k_{ij} u^a_k,$$  \hspace{1cm} (7.31)

where $f^k_{ij}$ are the structure constants of $\mathfrak{g}$. As the $G$–action is Hamiltonian, there exists a moment map $\mu \in \Gamma(\mathfrak{g}^\vee)$ of it. $\mu$ is $G$–equivariant, that is

$$u^b_i \partial_b \mu_j = f^k_{ij} \mu_k,$$  \hspace{1cm} (7.32)

and has the property that

$$u^a_i = -P^{ab} \partial_b \mu_i.$$  \hspace{1cm} (7.33)

Being the fundamental vector fields Hamiltonian, they leave the Poisson 2–vector invariant, $l_{Mu_i} P^{ab} = 0$.

Now, define a section $\varphi \in \Gamma(T^*M \otimes \mathfrak{g}^\vee)$ by

$$\varphi_{ia} = \partial_a \mu_i.$$  \hspace{1cm} (7.34)

A simple calculation based on (7.32), (7.33) shows that

$$[\varphi_i, \varphi_j]_{Pa} = P^{bc}(\varphi_{ib} \partial_c \varphi_{ja} - \varphi_{jb} \partial_c \varphi_{ia}) + \partial_a P^{bc} \varphi_{ib} \varphi_{jc} = f^k_{ij} \varphi_{ka}.$$  \hspace{1cm} (7.35)
Therefore, \( \varphi : \mathfrak{g} \to \Gamma(T^*M) \) is a Lie algebra homomorphism. The (pre)moments

\( \iota_{P_i}, \lambda_{P_i} \), defined according to (7.26a), (7.26b), are obtained by substituting \( \varphi_i \) for \( \alpha \) in (7.29a), (7.29b),

\[
\iota_{P_i} = -\partial_a \mu_i (x) \eta^a, \quad (7.36a)
\]

\[
\lambda_{P_i} = P^{ab} \partial_b \mu_i (x) y_a - \partial_b (P^{ac} \partial_c \mu_i) (x) \xi_a \eta^b. \quad (7.36b)
\]

As explained in the first part of this section, we can construct in this way an

\( N = 0 \) and an \( N = 1 \) gauging of the BV algebra \( \text{Fun}(T^*(-1]T^*[1]M) \) with invariant master action \( S_P \). In the \( N = 0 \) case, the invariant Poisson–Lichnerowicz cohomology of \( P \) is contained, in the sense precisely defined above, in the invariant BV cohomology of \( \text{Fun}(T^*(-1]T^*[1]M) \). Similarly, in the \( N = 1 \) case, the basic Poisson–Lichnerowicz cohomology of \( P \) is contained in the basic BV cohomology of \( \text{Fun}(T^*(-1]T^*[1]M) \).

Relation to the Poisson–Weil sigma model

The Poisson–Weil sigma model is a gauged version of the Poisson sigma model. It has been studied in an AKSZ framework in refs. [10][11] and further generalized in ref. [12]. The target space of the model is a Poisson manifold \( M \) with a Hamiltonian effective left \( G \)—action as described above. The fields of the model are de Rham superfields, that is sections of suitable bundles on the parity shifted tangent bundle \( T[1]\Sigma \) of the 2–dimensional world sheet \( \Sigma \). In the simplest version of the model, the field content is as follows

1. \( b \in \Gamma(T[1]\Sigma, \mathfrak{g}^\vee[0]) \).
2. \( c \in \Gamma(T[1]\Sigma, \mathfrak{g}[1]) \).
3. \( B \in \Gamma(T[1]\Sigma, \mathfrak{g}^\vee[-1]) \).
4. \( C \in \Gamma(T[1]\Sigma, \mathfrak{g}[2]) \)
5. \( x \in \text{Map}(T[1]\Sigma, M) \).

6. \( y \in \Gamma(T[1]\Sigma, x^*T^*[1]M) \).

The classical BV master action of the Poisson–Weil sigma model is

\[
S_{PW} = \int_{T[1]\Sigma} \varrho \left[ b_i \left( dc^i - \frac{1}{2} f^i_{jk} c^j c^k + C^i \right) - B_i \left( dC^i - f^i_{jk} C^j C^k \right) \right] + y_a \left( dx^a + u_i^a(x) c^i \right) - \mu_i(x) C^i - \frac{1}{2} P^{ab}(x) y_a y_b \],
\]

where \( \varrho \) is the invariant supermeasure on \( T[1]\Sigma \). It is not known whether \( S_{PW} \) satisfies also the appropriate quantum BV master equation, though it is known this to be the case for the pure Poisson sigma model \[38\].

The Poisson–Weil sigma model has a finite dimensional reduction defined as follows. Denote by \( \mathbf{1} \) and \( \omega \) the unit and a volume form of \( \Sigma \), viewed respectively as a degree 0 element and a nowhere vanishing degree 2 element of \( \text{Fun}(T[1]\Sigma) \). We assume further that \( \omega \) is normalized as

\[
\int_{T[1]\Sigma} \varrho \omega = 1. \tag{7.38}
\]

Take the superfields of the model to be of the form

\[
b_i = b_i \omega, \quad \tag{7.39a}
\]

\[
c^i = c^i \mathbf{1}, \quad \tag{7.39b}
\]

\[
B_i = B_i \omega, \quad \tag{7.39c}
\]

\[
C^i = C^i \mathbf{1}, \quad \tag{7.39d}
\]

\[
x^a = x^a \mathbf{1} - \eta^a \omega, \quad \tag{7.39e}
\]

\[
y_a = \xi_a \mathbf{1} + y_a \omega, \quad \tag{7.39f}
\]

\[1\] In [10][11], \( \mu_i \) and \( P^{ab} \) have opposite sign.
where \((b_i, c^i), (B_i, C^i), (x^a, y_a), (\xi^a, \eta^a)\) are BV conjugate pairs of variables of degrees \((-2, 1), (-3, 2), (0, -1), (1, -2)\), respectively. Substituting (7.39a)–(7.39f) into (7.37), we get a finite dimensional reduction \(S_{PW}^0\) of \(S_{PW}\). This reads as

\[
S_{PW}^0 = S_{g|1} + S_P + c^i \lambda_{P_i} - C^i \nu_{P_i},
\]

(7.40)

where \(S_{g|1}, S_P, \nu_{P_i}, \lambda_{P_i}\) are given by (5.12), (7.28), (7.36a), (7.36b), respectively. Upon comparing with (5.16), we immediately realize that \(S_{PW}^0\) is nothing but the \(N=1\) gauged matter BV master action of the finite dimensional Poisson Lie algebroid model described above

\[
S_{PW}^0 = S_{g|1P}.
\]

(7.41)

(In eq. (7.40), the tensor product symbol \(\otimes\) is omitted.) Presumably, the finite dimensional model can be used to compute correlators of the Poisson–Weil model along the lines described in ref. [16].
In this paper, we have explored certain less known features of BV algebras, which have not been the object of a systematic study so far. We have pointed out that a BV master action may possess global symmetries not directly related to the gauge symmetries which underlie the BV symmetry and which may be interesting to gauge for a variety of reasons. We have seen that the gauging can be carried out in a purely BV framework. The global symmetry of the master action organizes as a Lie algebra action with a varying amount of supersymmetry, which determines directly the amount of ghost supersymmetry and the procedure of gauging. We have found that \( N = 0 \) and \( N = 1 \) gauging correspond to ordinary gauging and to topological gauging, respectively. For higher \( N \), the situation is not clear yet. The ordinary formal structure of BV algebras seems to be inadequate to treat these cases and, though sensible algebraic constructions can be carried out, their eventual field theoretic origin or underpinning is not clear. This may be the object of future investigation.

We feel that the BV algebraic framework is more versatile than it has so far been realized. It would be certainly worth the effort to explore the full range of its applications.
In this appendix, we recall the basic properties of differential Lie modules and their cohomology. We further provide a self-contained proof of the important cohomology isomorphism (A.6). See ref. [19] for background material.

A differential Lie module is a quadruplet \((\mathcal{E}, \mathfrak{g}, l, \delta)\), where \(\mathcal{E}\) is a \(\mathbb{Z}\)-graded vector space, \(\mathfrak{g}\) is a Lie algebra and \(l : \mathfrak{g} \to \text{End}_0(\mathcal{E})\) is a linear map and \(\delta \in \text{End}_1(\mathcal{E})\) satisfying the graded commutation relations

\[
[l_i, l_j] = f^{k}_{ij} l_k, \quad (A.1a)
\]

\[
[\delta, l_i] = 0, \quad (A.1b)
\]

\[
[\delta, \delta] = 0, \quad (A.1c)
\]

with respect to a chosen basis \(\{t_i\}\) of \(\mathfrak{g}\). We note that neither \(\mathcal{E}\) is supposed to be an algebra nor \(l_i, \delta\) are supposed to be graded derivations. If \(\mathcal{F} \subset \mathcal{E}\) is a subspace, \(\mathcal{F}_{\text{inv}} = \mathcal{F} \cap (\cap_i \ker l_i)\) is called the invariant component of \(\mathcal{F}\).

The pairs \((\mathcal{E}, \delta), (\mathcal{E}_{\text{inv}}, \delta)\) are both differential spaces. Their associated cohomologies \(H^\ast(\mathcal{E}) = H^\ast(\mathcal{E}, \delta), H^\ast_{\text{inv}}(\mathcal{E}) = H^\ast(\mathcal{E}_{\text{inv}}, \delta)\) are the ordinary and the invariant cohomology of the differential Lie module \((\mathcal{E}, \mathfrak{g}, l, \delta)\).

With the Lie algebra \(\mathfrak{g}\), there is associated a canonical differential Lie module \((CE(\mathfrak{g}), \mathfrak{g}, l_\mathfrak{g}, \delta_\mathfrak{g})\), called the Chevalley–Eilenberg Lie module. \(CE(\mathfrak{g})\) is

\[
CE(\mathfrak{g}) = \text{Fun}(\mathfrak{g}[1]), \quad (A.2)
\]

the algebra of polynomials of the coordinates \(c^i\) of \(\mathfrak{g}[1]\) with respect to the basis \(\{t_i\}\). \(l_\mathfrak{g}, \delta_\mathfrak{g}\) are defined by the relations

\[
l_\mathfrak{g}c^i = -f^j_{ik}c^k, \quad (A.3a)
\]

\[
\delta_\mathfrak{g}c^i = -\frac{1}{2}f^i_{jk}c^j c^k. \quad (A.3b)
\]
For a given differential Lie module \((\mathcal{E}, g, l, \delta)\), let us set

\[
\mathcal{E}' = C E(g) \otimes \mathcal{E}.
\]  
(A.4)

We can define endomorphisms of \(\mathcal{E}'\) by

\[
l'_i = l_{g_i} \otimes 1 + 1_g \otimes l_i, \quad \quad \text{(A.5a)}
\]

\[
\delta' = \delta_g \otimes 1 + 1_g \otimes \delta + c^i \otimes l_i. \quad \quad \text{(A.5b)}
\]

Then, \((\mathcal{E}', g, l', \delta')\) is a differential Lie module.

If \(g\) is a reductive Lie algebra, then

\[
H_{\text{inv}}^*(\mathcal{E}') \simeq C E(g)_{\text{inv}} \otimes H_{\text{inv}}^*(\mathcal{E}).
\]  
(A.6)

Recall that \(g\) is reductive if \(g\) is the direct sum of an Abelian and a semisimple Lie algebra. The rest of this appendix is devoted to the sketch of the proof of the above result.

To begin with, we note that

\[
\mathcal{E}'_{\text{inv}} = (CE(g) \otimes \mathcal{E})_{\text{inv}}.
\]  
(A.7)

This suggests defining the following subspaces of \(\mathcal{E}'_{\text{inv}}\)

\[
\mathcal{C}_n = (CE(g)_n \otimes \mathcal{E})_{\text{inv}}, \quad \quad \text{(A.8)}
\]

\[
\mathcal{D}_n = \bigoplus_{0 \leq m \leq n} \mathcal{C}_m, \quad \quad \text{(A.9)}
\]

where \(n \geq 0\). Then, \(\mathcal{E}_{\text{inv}} \simeq \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_h = \mathcal{E}'_{\text{inv}}, \ h = \dim g, \) is a filtration of the vector space \(\mathcal{E}'_{\text{inv}}\). One can show the following two properties.

Let \(\{z_{kx}\}\) be a basis of \(CE(g)_{\text{inv}}\) such that \(z_{kx} \in CE_k(g)\) for all \(x\). (Such a basis exists as \(l_{g_i} CE_k(g) \subset CE_k(g)\) for all \(k\).)
i) Let \( \mu \in D_n \) be such that

\[
\delta' \mu = 0. \tag{A.10}
\]

Then, there are \( \nu \in D_{n-1} \), \( \alpha_k^x \in \mathcal{E}_{\text{inv}} \) with \( 0 \leq k \leq n \) such that

\[
\mu = \delta' \nu + \sum_{0 \leq k \leq n} z_{kx} \otimes \alpha_k^x, \tag{A.11}
\]

\[
\delta \alpha_k^x = 0. \tag{A.12}
\]

ii) Let \( \nu \in D_{n-1} \), \( \alpha_k^x \in \mathcal{E}_{\text{inv}} \) with \( 0 \leq k \leq n \) be such that

\[
\delta' \nu + \sum_{0 \leq k \leq n} z_{kx} \otimes \alpha_k^x = 0. \tag{A.13}
\]

Then, there are \( \beta_k^x \in \mathcal{E}_{\text{inv}} \) with \( 0 \leq k \leq n \), such that

\[
\alpha_k^x = \delta \beta_k^x. \tag{A.14}
\]

By \( i, \, ii \), there is a homomorphism \( q : H^*(\mathcal{E}'_{\text{inv}}, \delta') \to C\mathcal{E}(g)_{\text{inv}} \otimes H^*(\mathcal{E}_{\text{inv}}, \delta) \) defined by the expression

\[
q([\mu]) = \sum_{0 \leq k \leq n} z_{kx} \otimes [\alpha_k^x], \tag{A.15}
\]

where \( \mu \) is expressed as in \( (A.11) \). By \( (A.3b) \), as \( \delta_g z_{kx} = \frac{1}{2} c^i l_{gi} z_{kx} = 0 \), one has

\[
\delta' \sum_{0 \leq k \leq n} z_{kx} \otimes \gamma_k^x = \sum_{0 \leq k \leq n} (-1)^k z_{kx} \otimes \delta \gamma_k^x, \tag{A.16}
\]

for \( \gamma_k^x \in \mathcal{E}_{\text{inv}} \). It follows that \( q \) is an isomorphism. Thus, if one shows \( i, \, ii \), \( (A.6) \) is shown as well.

**Proof of \( i, \, ii \).** \( g \) acts on \( C\mathcal{E}(g) \) via \( (A.3a) \) and so, \( C\mathcal{E}(g) \) is a representation of \( g \). From Lie algebra theory, since \( g \) is reductive, this representation is semisimple. Thus, for any \( g \)-stable subspace \( U \subset C\mathcal{E}(g) \) and for any \( g \)-stable subspace \( V \subset U \) there is a \( g \)-stable subspace \( W \subset U \) such that \( U \simeq V \oplus W \). In particular, if \( U \subset \)
CE(\mathfrak{g}) is a \mathfrak{g}–stable subspace, then \( U = U_{\text{inv}} \oplus l_{g}U \), where \( U_{\text{inv}} = U \cap (\cap_{i} \ker l_{gi}) \) and \( l_{g}U = \text{span}_{i} l_{gi}U = U \cap (\cap_{i} \text{im} l_{gi}) \).

Consider \( Z_{n}(CE(\mathfrak{g})) = \ker \delta_{g} \cap CE_{n}(\mathfrak{g}) \). \( Z_{n}(CE(\mathfrak{g})) \) is \( \mathfrak{g} \)–stable and, therefore, \( Z_{n}(CE(\mathfrak{g})) = Z_{n}(CE(\mathfrak{g}))_{\text{inv}} \oplus l_{g}Z_{n}(CE(\mathfrak{g})) \). Since \( \delta_{g} = \frac{1}{2}c^{i}l_{gi} \), by (A.3b), \( CE_{n}(\mathfrak{g})_{\text{inv}} \subset Z_{n}(CE(\mathfrak{g}))_{\text{inv}} \subset CE_{n}(\mathfrak{g})_{\text{inv}} \) and, hence, \( Z_{n}(CE(\mathfrak{g}))_{\text{inv}} = CE_{n}(\mathfrak{g})_{\text{inv}} \).

Now, let \( B_{n}(CE(\mathfrak{g})) = \text{im} \delta_{g} \cap CE_{n}(\mathfrak{g}) \). Since \( l_{gi} = i_{gi} \delta_{g} + \delta_{g}i_{gi} \), where \( i_{gi} \) is the degree \(-1\) derivation of \( CE(\mathfrak{g}) \) defined by \( i_{gi}c^{j} = \delta^{j}_{i} \), and \( \delta_{g} = \frac{1}{2}l_{gi}c^{i} \), by (A.3b), as \( f^{j}_{ji} = 0 \) for a reductive Lie algebra \( \mathfrak{g} \), \( l_{g}Z_{n}(CE(\mathfrak{g})) \subset B_{n}(CE(\mathfrak{g})) \subset l_{g}Z_{n}(CE(\mathfrak{g})) \) and, thus, \( l_{g}Z_{n}(CE(\mathfrak{g})) = B_{n}(CE(\mathfrak{g})) \). In conclusion,

\[
Z_{n}(CE(\mathfrak{g})) = CE_{n}(\mathfrak{g})_{\text{inv}} \oplus B_{n}(CE(\mathfrak{g})).
\] (A.17)

Further, as \( Z_{n}(CE(\mathfrak{g})) \), \( CE_{n}(\mathfrak{g}) \) are \( \mathfrak{g} \)–stable and \( Z_{n}(CE(\mathfrak{g})) \subset CE_{n}(\mathfrak{g}) \),

\[
CE_{n}(\mathfrak{g}) = Z_{n}(CE(\mathfrak{g})) \oplus \widetilde{CE}_{n}(\mathfrak{g}),
\] (A.18)

for some \( \mathfrak{g} \)–stable subspace \( \widetilde{CE}_{n}(\mathfrak{g}) \subset CE_{n}(\mathfrak{g}) \). As a consequence, \( \delta_{g} : \widetilde{CE}_{n}(\mathfrak{g}) \to B_{n+1}(CE(\mathfrak{g})) \) is an isomorphism.

From the above discussion, it follows that, for each \( n \geq 0 \), there is a basis \( \{z_{nx}, r_{nu}, s_{ns}\} \) of \( CE_{n}(\mathfrak{g}) \) such that \( \{z_{nx}\}, \{r_{nu}\}, \{s_{ns}\} \) are bases of \( CE_{n}(\mathfrak{g})_{\text{inv}} \), \( B_{n}(CE(\mathfrak{g})) \), \( \widetilde{CE}_{n}(\mathfrak{g}) \), respectively, with the property that

\[
l_{gi}z_{nx} = 0, \quad l_{gi}r_{nu} = -A_{ni}^{u}r_{nv}, \quad l_{gi}s_{ns} = -B_{ni}^{t}s_{nt},
\] (A.19)

\[
\delta_{g}z_{nx} = 0, \quad \delta_{g}r_{nu} = 0, \quad \delta_{g}s_{ns} = -Q_{n}^{u}s_{r_{n+1}u},
\] (A.20)

where \( A_{ni}, B_{ni} \) and \( Q_{n} \) are square matrices with \( Q_{n} \) invertible. The relation \([l_{gi}, \delta_{g}] = 0\) implies further the matrix relation

\[
A_{n+1}Q_{n} - Q_{n}B_{ni} = 0.
\] (A.21)

Next, combining (A.5a), (A.5b) and the relation \( \delta_{g} = \frac{1}{2}c^{i}l_{gi} \), \( \delta' \) can be cast as

\[
\delta' = -\delta_{g} \otimes 1 + 1_{g} \otimes \delta + c^{i} \otimes 1 \cdot l_{i}'.
\] (A.22)
It follows that, when restricting to $E'_{\text{inv}}$,

$$\delta' = \delta'_1 + \delta'_2,$$  \hfill (A.23)

where $\delta'_1, \delta'_2$ are given by

$$\delta'_1 = -\delta_0 \otimes 1,$$  \hfill (A.24a)

$$\delta'_2 = 1_0 \otimes \delta.$$  \hfill (A.24b)

Next, we have the following result. Let $n \geq 0$. If $\mu_n \in C_n$ is such that

$$\delta'_1 \mu_n = 0,$$  \hfill (A.25)

then $\mu_n$ is of the special form

$$\mu_n = \delta'_1 \nu_{n-1} + z_{nx} \otimes \alpha^x,$$  \hfill (A.26)

for certain $\nu_{n-1} \in C_{n-1}, \alpha^x \in E_{\text{inv}}$. To see this, we write $\mu_n$ as

$$\mu_n = z_{nx} \otimes \alpha^x + r_{nu} \otimes \beta^u + s_{ns} \otimes \gamma^s,$$  \hfill (A.27)

where $\alpha^x, \beta^u, \gamma^s \in E$. By (A.5a), (A.19), the condition $l'_i \mu_n = 0$ implies that

$$l_i \alpha^x = 0, \quad l_i \beta^u = A_{ni} u_i \beta^u, \quad l_i \gamma^s = B_{ni} u_i \gamma^s.$$  \hfill (A.28)

By the first relation (A.28), $\alpha^x \in E_{\text{inv}}$. By the 3rd relation (A.20), one has $r_{nu} = -Q_{n-1-1s} u_1 \delta_s s_{n-1s}$. Hence, on account of (A.24a), one has

$$r_{nu} \otimes \beta^u = \delta'_1 \nu_{n-1},$$  \hfill (A.29)

where $\nu_{n-1}$ is given by

$$\nu_{n-1} = Q_{n-1-1s} u_1 s_{n-1s} \otimes \beta^u.$$  \hfill (A.30)

Using the 2nd relation (A.28), the 3rd relation (A.19) and (A.21), one finds that $l'_i \nu_{n-1} = 0$. Hence, $\nu_{n-1} \in C_{n-1}$. By (A.24a), (A.20) and the invertibility of the matrix $Q_n$ the condition $\delta'_1 \mu_n = 0$ implies that the $\gamma^s$ all vanish,
From (A.27), (A.29), (A.31), we get (A.26). The proof of $i$ proceeds by induction on $n$. Let $\mu \in D_0$ satisfy (A.10). Then, $\mu = 1_g \otimes \alpha$ for some $\alpha \in E_{\text{inv}}$. Further, by (A.5b), one has $\delta \alpha = 0$. Therefore, (A.11), (A.12) hold with $\nu = 0$. So, $i$ holds for $n = 0$. Suppose now $i$ holds for $n - 1$ with $n \geq 1$. Let $\mu \in D_n$ satisfy (A.10). Write $\mu = \mu_n + \tilde{\mu}$, where $\mu_n \in C_n$, $\tilde{\mu} \in D_{n-1}$. Since $\delta_1' C_m \subset C_{m+1}$, $\delta_2' C_m \subset D_m$ for $m \geq 0$, condition (A.10) implies that $\delta_1' \mu_n = 0$. So, $\mu_n$ satisfies (A.25) and, so, by (A.26), there are $\nu_{n-1} \in C_{n-1}$ and $\alpha_n^x \in E_{\text{inv}}$ such that $\mu = \delta_1' \nu_{n-1} + z_{nx} \otimes \alpha_n^x = \delta_1' \nu_{n-1} + z_{nx} \otimes \alpha_n^x - \delta_2' \nu_{n-1}$. Setting $\mu^* = \tilde{\mu} - \delta_2' \nu_{n-1} \in D_{n-1}$, we have then

$$
\mu = \delta_1' \nu_{n-1} + z_{nx} \otimes \alpha_n^x + \mu^*.
$$

(A.32)

Next, since $\delta_1' \mu = 0$ and $\delta_1'(z_{nx} \otimes \alpha_n^x) = (-1)^n z_{nx} \otimes \delta \alpha_n^x$ by (A.23), (A.24) and the 1st relation (A.20), $(-1)^n z_{nx} \otimes \delta \alpha_n^x + \delta_1' \mu^* = 0$. As $\delta_1' \mu^*$ has no components of the form $z_{nx} \otimes \gamma^x$, one has $\delta \alpha_n^x = 0$. Thus, $\delta_1' \mu^* = 0$. So, $\mu^*$ satisfies (A.10), and, so, by the inductive hypothesis, (A.11), (A.12) hold, yielding

$$
\mu^* = \delta_1' \nu^* + \sum_{0 \leq k \leq n-1} z_{kx} \otimes \alpha_k^x,
$$

(A.33)

with $\nu^* \in D_{n-2}$, $\alpha_k^x \in E_{\text{inv}}$ such that $\delta \alpha_k^x = 0$ for $0 \leq k \leq n - 1$. Substituting (A.33) into (A.32) and setting $\nu = \nu_{n-1} + \nu^* \in D_{n-1}$, we find that $\mu$ is of the form (A.11) with (A.12) satisfied. By induction on $n$, $i$ is shown.

The proof of $ii$ also proceeds by induction on $n$. Let $\nu \in D_0$, $\alpha_0, \alpha_1^x \in E_{\text{inv}}$ satisfy (A.13). (Note that $\{z_0^x\} = \{1\}$.) Then, $\nu = -1_g \otimes \beta$, for some $\beta \in E_{\text{inv}}$. Further, by (A.5b), one has $\alpha_0 = \delta \beta$ and $\alpha_1^x = 0$. Hence, (A.14) holds. So, $ii$ holds for $n = 1$. Suppose $ii$ holds for $n - 1$ with $n \geq 2$. Let $\nu \in D_{n-1}$, $\alpha_k^x \in E_{\text{inv}}$ with $0 \leq k \leq n$ satisfy (A.13). Write $\nu = \nu_{n-1} + \tilde{\nu}$, where $\nu_{n-1} \in C_{n-1}$, $\tilde{\nu} \in D_{n-2}$. Since $\delta_1' C_m \subset C_{m+1}$, $\delta_2' C_m \subset D_m$ for $m \geq 0$, condition (A.13) implies

$$
\gamma^s = 0.
$$

(A.31)
that $\delta'_1 \nu_{n-1} + z_{nx} \otimes \alpha_n^x = 0$. As $\delta'_1 \nu_{n-1}$ has no components of the form $z_{nx} \otimes \gamma^x$, one has $\alpha_n^x = 0$. Hence, $\delta'_1 \nu_{n-1} = 0$. So, $\nu_{n-1}$ satisfies (A.25) and, so, by (A.26), there are $\nu_{n-2} \in \mathcal{C}_{n-2}$ and $\beta^x \in \mathcal{E}_{\text{inv}}$ such that $\nu_{n-1} = \delta'_1 \nu_{n-2} + z_{n-1x} \otimes \beta^x$. Then, by (A.23), (A.24), $\delta' \nu_{n-1} = \delta' \nu_{n-1} = (-1)^{n-1} z_{n-1x} \otimes \delta \beta^x - \delta' \delta' \nu_{n-2}$. Setting $\nu^* = \tilde{\nu} - \delta' \nu_{n-2} \in \mathcal{D}_{n-2}$, we have then

$$\delta' \nu = \delta' \nu^* + (-1)^{n-1} z_{n-1x} \otimes \delta \beta^x. \quad (A.34)$$

Substituting (A.34) in (A.13) and recalling that $\alpha_n^x = 0$, we find

$$\delta' \nu^* + \sum_{0 \leq k \leq n-1} z_{kx} \otimes \alpha_k^x = 0, \quad (A.35)$$

where $\alpha_k^x = \alpha_k^x + (-1)^{n-1} \delta_{k,n-1} \delta \beta^x \in \mathcal{E}_{\text{inv}}$. Thus, $\nu^*$, $\alpha_k^x$ satisfy (A.13) and, so, by the inductive hypothesis, $\alpha_k^x = \delta \beta_k^x$ for certain $\beta_k^x \in \mathcal{E}_{\text{inv}}$. Thus, (A.14) holds. By induction on $n$, ii is shown. QED
In this appendix, we recall the basic properties of Lie operations and their cohomology. We further provide a self-contained proof of the important cohomology isomorphism (B.16). See refs. [19, 22], for background material.

A *Lie operation* is a quintuplet \((E, g, i, l, \delta)\), where \(E\) is a \(\mathbb{Z}\)-graded vector space, \(g\) is a Lie algebra and \(i : g \to \text{End}_{-1}(E), l : g \to \text{End}_0(E)\) are linear maps and \(\delta \in \text{End}_1(E)\) satisfying the commutation relations

\[
\begin{align*}
[i_i, i_j] &= 0, \\
[l_i, i_j] &= f_{ij}^k i_k, \\
[l_i, l_j] &= f_{ij}^k l_k, \\
[\delta, i_i] &= l_i, \\
[\delta, l_i] &= 0, \\
[\delta, \delta] &= 0,
\end{align*}
\]

with respect to a chosen basis \(\{t_i\}\) of \(g\). We note that neither \(E\) is supposed to be an algebra nor \(i_i, l_i, \delta\) are supposed to be graded derivations. If \(F \subset E\) is a subspace, \(F_{\text{hor}} = F \cap (\cap_i \ker i_i)\), \(F_{\text{inv}} = F \cap (\cap_i \ker l_i)\) and \(F_{\text{bas}} = F \cap (\cap_i (\ker i_i \cap \ker l_i)) = F_{\text{hor}} \cap F_{\text{inv}}\) are called the horizontal, invariant and basic component of \(F\), respectively.

The pairs \((E, \delta), (E_{\text{bas}}, \delta)\) are both differential spaces. Their associated cohomologies \(H^*(E) = H^*(E, \delta), H_{\text{bas}}^*(E) = H^*(E_{\text{bas}}, \delta)\) are the ordinary and the basic cohomology of the Lie operation \((E, g, i, l, \delta)\).

Lie operations can be equipped with connections. A *connection* of the Lie operation \((E, g, i, l, \delta)\) is a linear map \(\theta : g^\vee \to \text{End}_1(E)\) satisfying the commutation relations
\[ [i_j, \theta^i] = \delta^i_j, \quad (B.2a) \]
\[ [l_j, \theta^i] = -f^i_{jk}\theta^k. \quad (B.2b) \]

The curvature of the connection \( \theta \) is the linear map \( \Theta : \mathfrak{g}^\vee \rightarrow \text{End}_2(\mathcal{E}) \) defined by
\[ \Theta^i = [\delta, \theta^i] + \frac{1}{2} f^i_{jk}\theta^j\theta^k. \quad (B.3) \]

\( \Theta \) satisfies the commutation relations
\[ [i_j, \Theta^i] = 0, \quad (B.4a) \]
\[ [l_j, \Theta^i] = -f^i_{jk}\Theta^k. \quad (B.4b) \]

The Bianchi identities
\[ [\delta, \theta^i] = \Theta^i - \frac{1}{2} f^i_{jk}\theta^j\theta^k, \quad (B.5a) \]
\[ [\delta, \Theta^i] = -f^i_{jk}\theta^j\Theta^k \]
hold.

With the Lie algebra \( \mathfrak{g} \), there is associated a canonical Lie operation \( (W(\mathfrak{g}), \mathfrak{g}, i_{\mathfrak{g}}, l_{\mathfrak{g}}, \delta_{\mathfrak{g}}) \), called the Weil operation. \( W(\mathfrak{g}) \) is
\[ W(\mathfrak{g}) = \text{Fun}(\mathfrak{g}[1] \oplus \mathfrak{g}[2]), \quad (B.6) \]
the algebra of polynomials of the coordinates \( c^i, C^i \) of \( \mathfrak{g}[1], \mathfrak{g}[2] \) with respect to the basis \( \{t_i\} \). \( i_{\mathfrak{g}}, l_{\mathfrak{g}}, \delta_{\mathfrak{g}} \) are defined by the relations
\[ i_{\mathfrak{g}} c^j = \delta^j_i, \quad (B.7a) \]
\[ i_{\mathfrak{g}} C^j = 0, \quad (B.7b) \]
\[ l_{gi} c^j = -f^j_{jk} c^k, \quad (B.7c) \]
\[ l_{gi} C^j = -f^j_{jk} C^k, \quad (B.7d) \]
\[ \delta_g c^i = C^i - \frac{1}{2} f^i_{jk} c^j c^k. \quad (B.7e) \]
\[ \delta_g C^i = -f^i_{jk} c^j C^k. \quad (B.7f) \]

Note that (multiplication by) \( c^i \) defines a connection of the Weil operation having \( C^i \) as its curvature.

For a given Lie operation \((\mathcal{E}, g, i, l, \delta)\), let us set
\[
\mathcal{E}' = \mathcal{E}'' = W(g) \otimes \mathcal{E}. \quad (B.8)
\]

We can define endomorphisms of \( \mathcal{E}' \) by
\[
i'_i = i_{gi} \otimes 1, \quad (B.9a)\]
\[
l'_i = l_{gi} \otimes 1 + 1_g \otimes l_i, \quad (B.9b)\]
\[
\delta' = \delta_g \otimes 1 + 1_g \otimes \delta + c^i \otimes l_i - C^i \otimes i_i. \quad (B.9c)
\]

Then, \((\mathcal{E}', g, i', l', \delta')\) is a Lie operation. By definition, the equivariant cohomology of the operation \((\mathcal{E}, g, i, l, \delta)\) (in the BRST model) is
\[
H_{\text{equiv}}^*(\mathcal{E}) := H_{\text{bas}}^*(\mathcal{E}'). \quad (B.10)
\]

When the operation \((\mathcal{E}, g, i, l, \delta)\) has a connection \( \theta \) with curvature \( \Theta \), we can define endomorphisms of \( \mathcal{E}'' \) by
\[
i''_i = 1_g \otimes i_i, \quad (B.11a)\]
\[
l''_i = l_{gi} \otimes 1 + 1_g \otimes l_i, \quad (B.11b)\]
\[
\delta'' = \delta_g \otimes 1 + 1_g \otimes \delta + l_{gi} \otimes \theta^i - i_{gi} \otimes \Theta^i. \quad (B.11c)\]

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Then, \((\mathcal{E}'', g, i'', l'', \delta'')\) is also a Lie operation. A crucial technical result is that the basic cohomologies of \(\mathcal{E}\) and \(\mathcal{E}''\) are isomorphic

\[
H_{bas}^*(\mathcal{E}) \simeq H_{bas}^*(\mathcal{E}'').
\] (B.12)

We shall give a sketch of its proof momentarily.

The Lie operations \((\mathcal{E}', g, i', l', \delta')\), \((\mathcal{E}'', g, i'', l'', \delta'')\), which we have just constructed are isomorphic, since \(i', l', \delta'\) and \(i'', l'', \delta''\) are related as

\[
i''_i = I^{-1} i'_i I,
\]
\[
l''_i = I^{-1} l'_i I,
\]
\[
\delta'' = I^{-1} \delta'I,
\]

where \(I \in \text{Iso}_0(\mathcal{E}'', \mathcal{E}')\) is given by

\[
I = \exp(e^i \otimes i) \exp(i g_i \otimes \theta^i).
\] (B.14)

(The exponential are well defined as the exponential series terminate after a finite number of terms.) It follows that

\[
H_{bas}^*(\mathcal{E}') \simeq H_{bas}^*(\mathcal{E}'').
\] (B.15)

From (B.10), (B.12), (B.15), we conclude that

\[
H_{equiv}^*(\mathcal{E}) \simeq H_{bas}^*(\mathcal{E}).
\] (B.16)

Thus, if the Lie operation \((\mathcal{E}, g, i, l, \delta)\) admits a connection, the basic and equivariant cohomologies of \(\mathcal{E}\) are equivalent. The above fundamental result hinges on the isomorphism (B.12), whose proof we shall now sketch.

To begin with, we note that, by (B.11a), (B.11b),

\[
\mathcal{E}''_{bas} = (W(g) \otimes \mathcal{E}_{hor})^{\text{inv}}.
\] (B.17)
This suggests defining the following subspaces of $E''_{bas}$

$$C_n = (W(g)_n \otimes E_{hor})_{inv},$$  \hspace{1cm} (B.18)\

$$D_n = \bigoplus_{0 \leq m \leq n} C_m,$$  \hspace{1cm} (B.19)

where $n \geq 0$. Then, $E_{bas} \simeq D_0 \subset D_1 \subset D_2 \subset \ldots E''_{bas}$ is a filtration of the vector space $E''_{bas}$. One can now show the following two properties.

i) Let $\mu \in D_n$ be such that

$$\delta'' \mu = 0.$$  \hspace{1cm} (B.20)

Then, there are $\nu \in D_{n-1}$, $\alpha \in E_{bas}$ such that

$$\mu = \delta'' \nu + 1_g \otimes \alpha,$$  \hspace{1cm} (B.21)\

$$\delta \alpha = 0.$$  \hspace{1cm} (B.22)

ii) Let $\nu \in D_{n-1}$, $\alpha \in E_{bas}$ be such that

$$\delta'' \nu + 1_g \otimes \alpha = 0.$$  \hspace{1cm} (B.23)

Then, there is $\beta \in E_{bas}$ such that

$$\alpha = \delta \beta.$$  \hspace{1cm} (B.24)

By i, ii, there is a homomorphism $q : H^*(E''_{bas}, \delta'') \to H^*(E_{bas}, \delta)$ defined by

$$q([\mu]) = [\alpha],$$  \hspace{1cm} (B.25)

where $\mu$ is expressed as in (B.21). Since

$$\delta''(1_g \otimes \gamma) = 1_g \otimes \delta \gamma,$$  \hspace{1cm} (B.26)

for $\gamma \in E$, by (B.11c), $q$ is an isomorphism. Thus, if one shows i, ii, (B.12) is shown as well.

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Proof of i, ii. By (B.11c), $\delta''$ can be split as
\[ \delta'' = \delta''_1 + \delta''_2, \]  
where $\delta''_1, \delta''_2$ are given by
\[ \delta''_1 = \delta_g \otimes 1, \]  
\[ \delta''_2 = 1_g \otimes \delta + l_{gi} \otimes \theta^i - i_{gi} \otimes \Theta^i. \]  

The following property holds. Let $n \geq 0$. If $\mu_n \in C_n$ is such that
\[ \delta''_1 \mu_n = 0, \]  
then $\mu_n$ is of the special form
\[ \mu_n = \delta''_1 \nu_{n-1} + \delta_{n,0} 1_g \otimes \alpha, \]  
for certain $\nu_{n-1} \in C_{n-1}, \alpha \in E_{\text{bas}}$. To see this, we notice preliminarily that $c^i, \tilde{C}^i := C^i - \frac{1}{2} f^{ijk} c^j c^k$ are generators of $W(g)$ such that $l_{gi} c^j = -f^{ijk} c^k, l_{gi} \tilde{C}^j = -f^{ijk} \tilde{C}^k$ and $\delta_g c^i = \tilde{C}^i, \delta_g \tilde{C}^i = 0$. Now, being $\mu_n \in C_n$, we have
\[ \mu_n = \sum_{p \geq 0, q \geq 0, p+2q = n} c^i_1 \cdots c^i_p \tilde{C}^{j_1} \cdots \tilde{C}^{j_q} \otimes \alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q}, \]  
where $\alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q} \in \mathcal{E}$. By (B.11a), (B.11b), the conditions $i''_{i_p} \mu_n = 0, l''_{i_p} \mu_n = 0$ imply that
\[ i_k \alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q} = 0, \]  
\[ l_k \alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q} - \sum_r f^{l}_{ki_r} \alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q} - \sum_s f^{l}_{ki_s} \alpha^{p,q}_{i_1 \ldots i_p; j_1 \ldots j_q} = 0. \]

By (B.28a), the condition $\delta''_1 \mu_n = 0$ implies further that
\[ \alpha^{p,q}_{i_1 \ldots \{i_{p'; j_1 \ldots j_q}\}} = 0, \quad p \geq 1, \]  
where $\{\ldots\}$ stands for complete symmetrization of the enclosed indices. Let us
define

\[ \nu_{n-1} = \sum_{p \geq 0, q \geq 1, p+2q = n} \frac{(-1)^p q}{p + q} c^{i_1} \cdots c^{i_p} C^{j_{p+1}} \cdots C^{j_q-1} \otimes \alpha^{p, q_{i_1 \cdots i_{p+1} j_{p+1} \cdots j_q}}. \] (B.34)

Then, by (B.11a), (B.11b), (B.32a), (B.32b), we have \( \nu_{n-1} = 0 \), \( \mu_{i, j} \nu_{n-1} = 0 \) so that \( \nu_{n-1} \in \mathcal{C}_{n-1} \). Further, by (B.28a), (B.33),

\[ \delta''_{1, 1} \nu_{n-1} = \mu_n - \delta_{n, 0} 1_g \otimes \alpha, \] (B.35)

where \( \alpha = \alpha^{0, 0} \). By (B.32a), (B.32b), \( \alpha \in \mathcal{E}_{bas} \). (B.30) follows.

The proof of \( i \) proceeds by induction on \( n \). Let \( \mu \in \mathcal{D}_0 \) satisfy (B.20). Then, \( \mu = 1_g \otimes \alpha \) for some \( \alpha \in \mathcal{E}_{bas} \). Using (B.11c), one has \( \delta \alpha = 0 \). Hence, (B.21), (B.22) hold with \( \nu = 0 \). So, \( i \) holds for \( n = 0 \). Suppose now \( i \) holds for \( n - 1 \) with \( n \geq 1 \). Let \( \mu \in \mathcal{D}_n \) satisfy (B.20). Write \( \mu = \mu_n + \mu \), where \( \mu_n \in \mathcal{C}_n \), \( \mu \in \mathcal{D}_{n-1} \). Since \( \delta''_1 \mathcal{C}_m \subset \mathcal{C}_{m+1} \delta''_2 \mathcal{C}_m \subset \mathcal{D}_m \) for \( m \geq 0 \), condition (B.20) implies that \( \delta''_1 \mu_n = 0 \). So, \( \mu_n \) satisfies (B.29) and, so, by (B.30), there is \( \nu_{n-1} \in \mathcal{C}_{n-1} \) such that \( \mu_n = \delta''_1 \nu_{n-1} = \delta''_2 \nu_{n-1} - \delta''_2 \nu_{n-1} \). Thus, \( \mu = \delta''_1 \nu_{n-1} + \mu^* \) with \( \mu^* = \mu - \delta''_2 \nu_{n-1} \in \mathcal{D}_{n-1} \). As \( \delta''_1 \mu = 0 \), \( \delta''_1 \mu^* = 0 \) as well. So, \( \mu^* \) satisfies (B.20) and, so, by the inductive hypothesis, \( \mu^* = \delta''_1 \nu^* + 1_g \otimes \alpha \), with \( \nu^* \in \mathcal{D}_{n-2} \), \( \alpha \in \mathcal{E}_{bas} \) such that \( \delta \alpha = 0 \). Hence, \( \mu = \delta''_1 \nu + 1_g \otimes \alpha \), where \( \nu = \nu_{n-1} + \nu^* \in \mathcal{D}_{n-1} \).

By induction on \( n \), \( i \) is shown.

The proof of \( ii \) also proceeds by induction on \( n \). Let \( \nu \in \mathcal{D}_0 \), \( \alpha \in \mathcal{E}_{bas} \) satisfy (B.23). Then, \( \nu = -1_g \otimes \beta \) for some \( \beta \in \mathcal{E}_{bas} \). Using (B.11c), one has \( \alpha = \delta \beta \). Hence, (B.24) holds. So, \( ii \) holds for \( n = 1 \). Suppose \( ii \) holds for \( n - 1 \) with \( n \geq 2 \). Let \( \nu \in \mathcal{D}_{n-1} \), \( \alpha \in \mathcal{E}_{bas} \) satisfy (B.23). Write \( \nu = \nu_{n-1} + \tilde{\nu} \), where \( \nu_{n-1} \in \mathcal{C}_{n-1} \), \( \tilde{\nu} \in \mathcal{D}_{n-2} \). Since \( \delta''_1 \mathcal{C}_m \subset \mathcal{C}_{m+1} \delta''_2 \mathcal{C}_m \subset \mathcal{D}_m \) for \( m \geq 0 \), condition (B.23) implies that \( \delta''_1 \nu_{n-1} = 0 \). So, \( \nu_{n-1} \) satisfies (B.29) and, so, by (B.30), there is \( \nu_{n-2} \in \mathcal{C}_{n-2} \) such that \( \nu_{n-1} = \delta''_1 \nu_{n-2} = \delta''_2 \nu_{n-2} - \delta''_2 \nu_{n-2} \). Thus, \( \nu = \delta''_1 \nu_{n-2} + \nu^* \) with \( \nu^* = \tilde{\nu} - \delta''_2 \nu_{n-2} \in \mathcal{D}_{n-2} \). By (B.23), \( \delta''_2 \nu_{n-2} + 1_g \otimes \alpha = 0 \). So, \( \nu^* \) satisfies (B.23).
and, so, by the inductive hypothesis, $\alpha = \delta \beta$ for some $\beta \in \mathcal{E}_{\text{bas}}$. By induction on $n$, $ii$ is shown. 

$QED$

*Remark* Although to prove the isomorphism (B.16) we had to make explicit use at several points of a fixed connection $\theta$ of the operation $(\mathcal{E}, \mathcal{g}, i, l, \delta)$, the isomorphism can be shown to be independent from the choice of $\theta$ and is thus canonical.
C Superfield formulation of the $N=1$ ghost system

In this appendix, we show that the $N=1$ ghost system described in sect. 5 has an elegant superfield formulation.

$N=1$ ghost superfields are elements of the algebra $A_{g|1}((\theta))$, where $\theta$ a formal odd variable such that $\partial \theta = -1$. Define the superfields

$$B_i = B_i + \theta b_i,$$
$$c^i = c^i - \theta C^i.$$  \hspace{1cm} (C.1a, C.1b)

We have $B_i \in A_{g|1}((\theta)) - 3$, $c^i \in A_{g|1}((\theta))_1$. In terms of these, the $g$-action \text{5.11} reads succinctly as

$$i_{g|1}B_j = 0,$$
$$i_{g|1}c^j = \delta_i^j,$$
$$l_{g|1}B_j = f_{ij}^k B_k,$$
$$l_{g|1}c^j = -f_{jk}^i c^k.$$  \hspace{1cm} (C.2a, C.2b, C.2c, C.2d)

The superfield expression of ghost master action \text{5.12} is

$$S_{g|1} = - \int d\theta \left[ B_i \partial_\theta c^i + \frac{1}{2} f_{ij}^k B_i c^j c^k \right].$$  \hspace{1cm} (C.3)

Similarly the quantum BV variations \text{5.13} read concisely as

$$\delta_{g|1}B_i = - \partial_\theta B_i - f_{ji}^k B_k c^j,$$
$$\delta_{g|1}c^i = - \partial_\theta c^i - \frac{1}{2} f_{jk}^i c^j c^k.$$  \hspace{1cm} (C.4a, C.4b)

The superfield formalism was originally worked out in ref. [39].
D The modular class of a Lie algebroid

The modular class of a Lie algebroid was first introduced in [36]. Let $E$ be a Lie algebroid over the manifold $M$ with anchor $\rho_E$ and Lie bracket $[\cdot, \cdot]_E$. Then, the real line bundle over $M$

$$Q_E = \wedge^n T^*M \otimes \wedge^q E,$$

where $n = \dim M$ and $q = \text{rank} E$, is defined. $Q_E$ is called the orientation bundle of $E$.

For $\gamma \in \Gamma(Q_E)$, $X \in \Gamma(E)$, we set

$$D_X \gamma = (l_{M\rho_E(X)} \otimes 1_{\wedge^q E} + 1_{\wedge^n T^*M} \otimes l_{EX})\gamma,$$

where $l_{M\rho(X)}$ is the ordinary Lie derivative along the vector field $\rho_E(X)$ and $l_{EX}$ is defined by

$$l_{EX}(Y_1 \wedge \ldots \wedge Y_d) = \sum_{r=1}^{d} Y_1 \wedge \ldots \wedge [X,Y_r]_E \wedge \ldots \wedge Y_d, \quad Y_r \in \Gamma(E).$$

Clearly, $D_X \gamma \in \Gamma(Q_E)$. It can be verified that the map $X \to D_X$ defines a representation of the Lie algebroid $E$ in $Q_E$,

$$D_{fX} \gamma = f D_X \gamma,$$  \hspace{1cm} (D.4a)

$$D_X (f \gamma) = f D_X \gamma + (\rho_E(X)f) \gamma,$$  \hspace{1cm} (D.4b)

$$[D_X, D_Y] \gamma - D_{[X,Y]_E} \gamma = 0,$$  \hspace{1cm} (D.4c)

for $f \in \text{Fun}(M)$ and $X, Y \in \Gamma(E)$, $\gamma \in \Gamma(Q_E)$.

Suppose that the line bundle $Q_E$ is trivial. Then, there is a nowhere vanishing section $\gamma \in \Gamma(Q_E)$. Further, there is a section $\theta_\gamma \in \Gamma(E^*)$ such that

$$D_X \gamma = \theta_\gamma(X)\gamma,$$  \hspace{1cm} (D.5)
for $X \in \Gamma(E)$. It can be shown that $\theta_\gamma$ satisfies

$$d_E \theta_\gamma = 0. \quad \text{(D.6)}$$

Further, if one rescales $\gamma$ into $\gamma' = e^f \gamma$ with $f \in \text{Fun}(M)$, then one has

$$\theta_{\gamma'} = \theta_\gamma + d_E f. \quad \text{(D.7)}$$

Thus, $\theta_\gamma$ is a representative of a well defined cohomology class $\theta_E \in H^1(E)$, the \textit{modular class} of $E$, independent from the choice of $\gamma$. If $Q_E$ is not trivial, the modular class of $E$ can still be defined as follows. One notes that $Q_E^\otimes 2$ is trivial and one repeats a similar construction with $\gamma$ replaced by a nowhere vanishing section $\nu \in \Gamma(Q_E^\otimes 2)$ and $D_X$ replaced by $D_X \otimes 1_{Q_E} + 1_{Q_E} \otimes D_X$. Then, $\theta_E = \frac{1}{2}[\theta_\nu]_{H^1(E)}.$

In general, $E$ is said \textit{unimodular} if $\theta_E = 0$.

A straightforward calculation shows that, when $Q_E$ is trivial, $\theta_\gamma$ is given in any local trivialisation of $E$ by

$$\theta_{\gamma r} = \partial_a \rho_r^a + \rho_r^a \partial_a \ln \gamma - c^s_{sr}, \quad \text{(D.8)}$$

where $\rho_r^a$, $c^r_{rs}$ are the anchor and structure functions of $E$, respectively. Upon rescaling $\gamma$ into $\gamma' = e^f \gamma$ with $f \in \text{Fun}(M)$, one has

$$\theta_{\gamma'r} = \theta_{\gamma r} + \rho_r^a \partial_a f. \quad \text{(D.9)}$$

Therefore, it is possible to chose $\gamma$ in such a way that $\theta_\gamma = 0$ precisely when $E$ is unimodular.
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