Renormalization-group theory for rotating $^4\text{He}$ near the superfluid transition

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The influence of a uniform rotation with frequency $\Omega$ on the critical behavior of liquid $^4\text{He}$ near $T_\lambda$ is investigated. We apply our recently developed approach [R. Haussmann, submitted to Phys. Rev. B] which is a renormalization-group theory based on model $F$ starting with the calculation of the Green’s function in Hartree approximation. We calculate the specific heat $C_T(T,\Omega)$, the correlation length $\xi(T,\Omega)$, and the thermal-resistivity tensor $\rho_T(T,\Omega)$ as functions of the temperature $T$ for fixed values of the rotation frequency $\Omega$. For nonzero $\Omega$ we find that all physical quantities are smooth near $T_\lambda$ so that the superfluid transition is a smooth crossover. We define a frequency-dependent transition temperature $T_\lambda(\Omega)$ by the maximum of the specific heat and predict the shift $T_\lambda(\Omega) - T_\lambda = -1.2 T_\lambda (2m_4 \xi^2 (\Omega/\bar{h})^{1/2}$. For $T < T_\lambda(\Omega)$ we find mutual friction between the superfluid and the normal-fluid component caused implicitly by the motion of vortex lines and calculate the Vinen coefficients $B$ and $B'$.

I. INTRODUCTION

Liquid $^4\text{He}$ in a uniformly rotating container is influenced by the rotation frequency $\Omega$ in two ways. First of all, a rotational flow is created which implies quantized vortices in the superfluid state for temperatures below $T_\lambda$. Secondly, the centrifugal forces of the rotation cause a spatially dependent pressure $P(r)$ which implies an inhomogeneity in the system. While usually the second influence is negligible, the rotational flow and the occurrence of vortices plays an essential role.

In this paper we consider the critical behavior of uniformly rotating liquid $^4\text{He}$ in $d = 3$ dimensions for temperatures near $T_\lambda$ and present a renormalization-group theory based on model $F$ of Halperin, Hohenberg and Siggia. We will show that the rotation frequency $\Omega$ is an external perturbation which drives the system away from criticality. According to Feynman in uniformly rotating superfluid $^4\text{He}$ straight vortex lines are present which are parallel to the rotation axis and which are uniformly distributed in the helium. The circulation around a vortex line is quantized by $2\pi\hbar/m_4$ where $m_4$ is the mass of a $^4\text{He}$ atom. Consequently, the density of vortex lines $L$ with respect to the area perpendicular to the rotation axis is directly related to the rotation frequency by $L = (m_4/\pi\hbar)\Omega$. The absolute value $L = (m_4/\pi\hbar)\Omega$ is the total length of the vortex lines in a unit volume and hence is a measure of how many vortices are present in the helium.

Close to the superfluid transition the critical fluctuations are very large. The correlation length $\xi$ increases strongly if the temperature $T$ approaches $T_\lambda$. However, for nonzero $\Omega$ the mean distance between the vortex lines $L^{-1/2} = (\pi\hbar/m_4)^{1/2}$ is another characteristic length of the system. We will show that the correlation length $\xi$ is bounded by this characteristic length according to $\xi < L^{-1/2}$. Consequently, the thermodynamic quantities like the specific heat are smooth functions of temperature near $T_\lambda$. As a remnant of the critical singularities the quantities exhibit a maximum or an inflection point at a temperature $T_\lambda(\Omega)$ which is located below $T_\lambda$. We will find a phase diagram for uniformly rotating $^4\text{He}$ which is shown qualitatively in Fig. 1. The dashed line represents the transition temperature $T_\lambda(\Omega)$ between the superfluid and the normal-fluid state. This temperature is not sharply defined, because for nonzero $\Omega$ the thermodynamic quantities are nonsingular. The transition is smooth and shifted to lower temperatures by $\Delta T_\lambda(\Omega) = T_\lambda(\Omega) - T_\lambda$. Nevertheless, $(T,\Omega) = (T_\lambda, 0)$ is the critical point in the phase diagram, for which the correlation length $\xi$ diverges and the thermodynamic quantities are singular.

![FIG. 1. The phase diagram of uniformly rotating liquid $^4\text{He}$. The dashed line represents the transition temperature $T_\lambda(\Omega)$ which separates the superfluid and the normal-fluid phase. The critical point is located at $(T,\Omega) = (T_\lambda, 0)$.](image)
The macroscopic quantum coherence of the superfluid component is described by the complex order-parameter field \( \psi(r, t) \). For \( T \gtrsim T_\lambda(\Omega) \) in the normal-fluid phase the order-parameter field \( \psi \) fluctuates around the average value \( \langle \psi \rangle = 0 \). On the other hand, for \( T \lesssim T_\lambda(\Omega) \) in the superfluid phase the order parameter \( \psi = \eta e^{i\varphi} \) is nonzero. The amplitude \( \eta \) fluctuates around a nonzero average value. However, the vortex lines in the uniformly rotating system imply a strong spatial variation of the phase \( \varphi \) over all values between 0 and \( 2\pi \). Since the vortices move due to fluctuations, the phase \( \varphi \) fluctuates strongly so that eventually the average order parameter is \( \langle \psi \rangle = 0 \) also in the superfluid state.

Conventional field-theoretic methods are not applicable in the present case because in the superfluid state symmetry breaking with a nonzero average order parameter \( \langle \psi \rangle = 0 \) is required for the construction of the perturbation theory. However, recently we developed an approach\([1]\) which can handle \( \langle \psi \rangle = 0 \) also in the superfluid state and which includes the effects of vortices in an indirect way. This approach starts with the calculation of the order-parameter Green’s function in self-consistent Hartree approximation and includes the effects of critical fluctuations by renormalization and application of the renormalization-group (RG) theory. In Sec. \([III]\) we apply the approach to uniformly rotating \(^4\text{He}\). We describe the necessary modifications to include the rotation with frequency \( \Omega \). Since we consider also heat-transport phenomena, we additionally include an infinitesimal heat current \( Q \) as an external perturbation of the thermal equilibrium.

Once the approach is set up, we can calculate several physical quantities explicitly.

In Sec. \([IV]\) we calculate the entropy \( S(T, \Omega) \) and the specific heat \( C_Q(T, \Omega) \) at constant rotation frequency \( \Omega \). For nonzero \( T \) we find that \( S \) and \( C_Q \) are smooth functions of the temperature \( T \), so that the superfluid transition is a smooth crossover. From the maximum of the specific heat we obtain the transition temperature \( T_\lambda(\Omega) \) and calculate the shift \( \Delta T_\lambda(\Omega) = T_\lambda(\Omega) - T_\lambda \). We confirm the phase diagram of uniformly rotating liquid \(^4\text{He}\) shown in Fig. \([II]\). The correlation length \( \xi = \xi(T, \Omega) \) is calculated in Sec. \([V]\). We show that \( \xi \) is bounded by the mean distance between the vortex lines \( L^{-1/2} \) and find a maximum at \( T \approx T_\lambda(\Omega) \) as expected.

In Sec. \([VI]\) we consider heat transport in uniformly rotating liquid \(^4\text{He}\) and calculate the thermal-resistivity tensor \( \rho_T(T, \Omega) \). For \( T \lesssim T_\lambda(\Omega) \) in the superfluid state the thermal resistivity is strongly anisotropic and depends on the direction of the heat current related to the rotation axis. In Sec. \([VII]\) we show that the thermal resistivity is caused by the mutual friction of the superfluid and the normal-fluid component due to the motion of the vortex lines, which was first observed by Hall and Vinen\([2]\). We calculate the Vinen coefficients \( B \) and \( B' \) and compare the results with a previous theory for the motion of vortex lines\([3]\). Finally, in Sec. \([VIII]\) we compare the thermal resistivity of uniformly rotating superfluid \(^4\text{He}\) with the thermal resistivity of nonrotating superfluid \(^4\text{He}\) in the presence of a nonzero heat current \( Q \), which was calculated previously in Ref. \([4]\). In this way we obtain the vortex density \( L \) of the turbulent superfluid flow induced by a finite heat current \( Q \) which causes the Gorter-Mellink mutual friction\([5]\).

**II. RENORMALIZATION-GROUP THEORY FOR ROTATING \(^4\text{HE}\)**

**A. The model**

Dynamic critical phenomena in liquid \(^4\text{He}\) close to \( T_\lambda \) are well described by model \( F \) which is given by the Langevin equations for the order parameter \( \psi(r, t) \) and the entropy variable \( m(r, t) \):

\[
\frac{\partial \psi}{\partial t} = -2\Gamma_0 \frac{\delta H}{\delta \psi^*} + ig_0 \psi \frac{\delta H}{\delta m} + \theta_\psi, \tag{2.1}
\]

\[
\frac{\partial m}{\partial t} = \lambda_0 \nabla^2 \frac{\delta H}{\delta m} - 2g_0 \text{Im} \left( \psi^* \frac{\delta H}{\delta \psi^*} \right) + \theta_m, \tag{2.2}
\]

where

\[
H = \int d^4r \left[ \frac{1}{2} \tau_0(r)|\psi|^2 + \frac{1}{2} \left( \nabla - i k \right)|\psi|^2 + \bar{u}_0|\psi|^4 \right.
\]

\[
+ \frac{1}{2} \lambda_0^{-1} m^2 + \gamma_0 m|\psi|^2 - h_0 m \right] \tag{2.3}
\]

is the free energy functional and \( \theta_\psi \) and \( \theta_m \) are Gaussian stochastic forces which incorporate the fluctuations. Because of the viscosity the normal-fluid component is rotating uniformly with velocity \( v_n = \Omega \times r \). In \( (2.3) \) the uniform rotation is incorporated by the wave vector \( k \) which is related to the normal fluid velocity by

\[
k = (m_4/h) v_n = (m_4/h) (\Omega \times r). \tag{2.4}
\]

We assume that the origin of the coordinate system is located on the rotation axis. The gravity and the centrifugal forces of the rotation imply a variation of the pressure in the helium according to

\[
P = P_0 - \rho |g_z - \frac{1}{2}(\Omega \times r)^2| \tag{2.5}
\]

where \( \rho \) is the density of liquid \(^4\text{He}\) and \( g = 981 \text{ cm/s}^2 \) is the gravitational acceleration. Since the critical temperature \( T_\lambda = T_\lambda(P) \) depends on the pressure \( P \), it depends on the space coordinate \( r \) according to

\[
T_\lambda(r) = T_{\lambda 0} - (\partial T_\lambda/\partial P) \rho g_z - \frac{1}{2}(\Omega \times r)^2 \tag{2.6}
\]

where \( T_{\lambda 0} = \rho g (-\partial T_{\lambda 0}/\partial P) = +1.273 \mu\text{K/cm} \). In the energy functional \( (2.3) \) the spatially dependent critical temperature \( (2.6) \) is represented by the parameter \( \tau_0 = 2\chi_0 \gamma_0 \left[ T_1 - T_\lambda(r) \right] \), where \( T_1 \) is an arbitrary constant reference temperature.

Thus, the rotation influences the \(^4\text{He}\) in two different ways. First, in the kinetic term of the energy functional \( (2.3) \) the wave vector \( (2.4) \) related to the normal
fluid velocity generates straight vortex lines in the superfluid component. Secondly, the centrifugal forces imply a slightly space dependent critical temperature \(2.6\). In this paper we consider only the first kind of influence which is first order in \(\Omega\) and which is the direct and most natural influence of the rotation. The second kind of influence, which is indirect via the pressure variation and second order in \(\Omega\), will be neglected. Furthermore, we neglect gravity. In Sec. 11 we will show that these neglections are justified for realistic experiments performed in a microgravity environment in space. The experiments with rotating superfluid \(4\)He to determine the Vinen coefficients in (2.4) were performed with a slow rotation rotation up to two turns per second. For this reason we assume that the rotation frequency is about \(\Omega \approx 2\pi\) s\(^{-1}\).

For the calculation of the entropy and the specific heat in Sec. 11 and of the correlation length in Sec. 11 we may consider the system in thermal equilibrium, where the temperature \(T\) is constant. However, in Sec. 11 we consider heat transport and calculate the thermal-resistivity tensor in linear response. For this reason we must more generally consider the system in nonequilibrium, where an infinitesimal heat current \(Q\) is present which implies an infinitesimal temperature gradient \(\nabla T\). Thus, the temperature \(T(r)\) depends weakly on the space coordinate \(r\).

Model \(F\) is treated usually by field-theoretic means. The field-theoretic perturbation theory and the renormalization-group (RG) theory were developed by Dohm 13. Our approach starts with the calculation of the order-parameter Green’s function in Hartree approximation and is described in detail in our previous paper (the second paper of Ref. 4, respectively). If \(T(r)\) and \(T_\lambda\) are known, then \(r_1, r_0,\) and \(\Delta r_0\) are determined by the self-consistent equations (2.11)-(2.14). Since we neglect the pressure variations by gravity and by centrifugal forces, \(T_\lambda\) is constant. For the calculations of the entropy, the specific heat, and the correlation length in Secs. 11 and 11 the system is assumed to be in thermal equilibrium so that the temperature \(T(r) = T\) is constant, too. Thus, in this case the effective parameters \(r_1, r_0,\) and \(\Delta r_0\) are constant in space.

For the calculation of the thermal-resistivity tensor in Sec. 11 the presence of an infinitesimal heat current \(Q\) is needed. Consequently, there will be an infinitesimal temperature gradient \(\nabla T\) which implies gradients of the effective parameters \(\nabla r_1, \nabla r_0, \) and \(\nabla (\Delta r_0)\). Since \(T_\lambda\) is assumed to be constant, Eqs. (2.13) and (2.14) imply

\[
\nabla r_0 = \nabla (\Delta r_0) = 2\chi_0\gamma_0 \nabla T/T_\lambda .
\]

On the other hand, from (2.11) we obtain

\[
\nabla r_1 = \nabla r_0 + 4\mu_0 \nabla n_s .
\]

Furthermore, we need a relation between the heat current and the gradients. For this purpose we take the average of the entropy equation (2.2) and obtain \(\partial_t \langle m \rangle + \nabla q = 0\) where

\[
q = -\lambda_0 \nabla \left( \frac{\delta H}{\delta m} \right) - g_0 J_s .
\]
we obtain $Z$ and terms. Inserting this Green’s function into (2.12) we obtain

$$\Delta r_0 = 2\chi_0 \gamma_0 \left< \delta H \right>_{\delta m},$$

(2.18)

$$J_s = \left< \text{Im}\left[ \psi^* (\nabla - ik) \psi \right] \right> = \left< \left( \nabla - ik \right) - \left( \nabla' + ik' \right) \right> \frac{G(r, r')}{r = r},$$

(2.19)

respectively. (Note that here the superfluid current is defined with respect to the rotating frame.) Thus, from (2.17) we obtain

$$\frac{Q}{g_0 k_B T} = \frac{\rho}{g_0} = \frac{\lambda_0}{2\chi_0 \gamma_0} - J_s.$$  

(2.20)

If the heat current $Q$ is given, then the gradients $\nabla (\Delta r_0)$, $\nabla r_0$, $\nabla r_1$, and the temperature gradient $\nabla T'$ are determined by the self-consistent equations (2.15), (2.16), and (2.20).

C. Renormalization

The quantities $n_s$, $\nabla n_s$, and $J_s$, which appear in the first order terms of the self-consistent equations (2.11), (2.16), and (2.20), exhibit infrared divergencies at criticality where $r_1 \to 0$ and $\Omega \to 0$. For this reason, the self-consistent equations must be renormalized and the RG theory must be applied to achieve a resummation of the infrared divergencies and a proper treatment of the critical fluctuations. We perform the renormalization in the same way as in Ref. [3] and use the concept of renormalization by minimal subtraction of dimensional poles, which is described for model $F$ in Ref. [3]. The calculations are performed at fixed dimension $d = 4 - \epsilon$ (i.e. no $\epsilon$ expansion). For the renormalization of (2.11) we need the relations

$$r_0 - r_0c = Z_r r,$$

(2.21)

$$u_0 = u Z_u Z_u^{-2} (\mu' / A_d).$$  

(2.22)

In Hartree approximation it is $r_0c = 0$ and the $Z$ factors are given by

$$Z_r = Z_u = 1 / \left[ 1 - 8u / \epsilon \right].$$  

(2.23)

and $Z_u = 1$. Since $r_1$ is not renormalized from (2.11) we obtain

$$r = r_1 \left[ 1 - 8u / \epsilon \right] - 4u (\mu' / A_d) n_s.$$

(2.24)

In the Appendix the Green’s function $G(r, r')$ is evaluated for infinitesimal gradients of the effective parameters. Inserting this Green’s function into (2.12) we obtain

$$n_s = -\frac{2}{\epsilon} A_d \frac{\ell^{-2+\epsilon}}{\Gamma(-1+\epsilon/2)} \mathcal{F}_{-1+\epsilon/2} (r_1 \ell^2)$$

(2.25)

where $\mathcal{F}_p(\zeta)$ is defined by the integral

$$\mathcal{F}_p(\zeta) = \int_0^\infty dv v^{p-1} e^{-v/\zeta}$$

(2.26)

and $\ell = (\hbar / 2m_3 \Omega)^{1/2}$ is the characteristic length related to the mean distance between the vortices $L^{-1/2}$. (The vortex density is $L = 1 / 2\pi \ell^2$.)

Now, inserting (2.23) into (2.24) we obtain

$$r \ell^2 = r_1 \ell^2 \left( 1 + 8u A_1 \right)$$

(2.27)

where

$$A_1 = \frac{1}{\epsilon} \left[ \left( \frac{(\mu')^2}{2} + (r_1 \ell^2)^{-1} \mathcal{F}_{-1+\epsilon/2} (r_1 \ell^2) - 1 \right) \right].$$  

(2.28)

Eq. (2.27) is the renormalized counterpart of (2.11). We have multiplied both sides by $\ell^2$ because $r \ell^2$ and $r_1 \ell^2$ are dimensionless quantities.

Next, by applying $\nabla$ to (2.27) we obtain the renormalized counterpart of (2.16). We find

$$\nabla r_1 \ell^2 = (\nabla r_1 \ell^2) \left( 1 + 8u A_1 \right)$$

(2.29)

where

$$A_1 = \frac{1}{\epsilon} \left[ \left( \frac{(\mu')^2}{2} + (r_1 \ell^2)^{-1} \mathcal{F}_{-1+\epsilon/2} (r_1 \ell^2) - 1 \right) \right].$$  

(2.30)

The factors $\ell^2$ are multiplied on both sides of (2.29) because $\nabla r_1 \ell^2$ and $\nabla r_1 \ell^2$ are dimensionless quantities.

For the renormalization of (2.20) we need the relations

$$\chi_0 \gamma_0 = \gamma (\chi_0 Z_m)^{1/2} Z_r (\mu' / A_d)^{1/2},$$

(2.31)

$$g_0 = g (\chi_0 Z_m)^{1/2} (\mu' / A_d)^{1/2},$$

(2.32)

$$\lambda_0 = \chi_0 Z_A \lambda,$$

(2.33)

and furthermore $\Delta r_0 = Z_r \Delta r$. The entropy current is renormalized by $q = (\chi_0 Z_m)^{1/2} q_{\text{ren}}$. Consequently, the $Z$ factors cancel in the ratio $q / g_0 = (q_{\text{ren}} / g)(\mu' / A_d)^{1/2}$, so that the left-hand side of (2.24) needs not be renormalized. On the right-hand side of (2.24) part of the $Z$ factors cancel. Thus, we obtain

$$\frac{Q}{g_0 k_B T_J} = \frac{A_d \lambda}{\mu' g} \left( \frac{Z_m}{Z_A} \right) - J_s.$$  

(2.34)

The dynamic couplings of model $F$ like $\lambda$ and $g$ occur only in dimensionless combinations like $\lambda / g$. For this reason, we express (2.34) entirely in terms of the dimensionless couplings $w = \Gamma / \lambda$, $F = g / \lambda$, and $f = g^2 / \lambda \Gamma' = F^2 / w$. In Hartree approximation the $Z$ factors are given by

$$Z_m Z_A = 1 / [1 - f / 2 \epsilon].$$  

(2.35)

Thus, we rewrite (2.34) in the form
\[
\frac{Q}{\mu^{d-1}} = -\frac{A_d}{2\gamma F} \left[ 1 - \frac{f}{2\ell} \right] - \frac{J_s}{\mu^{d-1}} .
\]  
(2.36)

The superfluid current \( J_s \) is obtained from (2.14) by inserting the Green’s function \( G(r,r') \), which we have calculated explicitly in the Appendix. We find

\[
J_s = \frac{1}{\ell} A_d \bar{\Gamma}^{\ell} \left\{ \left[ \mathcal{M}_{\ell/2}(r_1 \ell^2) [e_z \times \nabla r_1] - \frac{F}{4\chi \mu^3} \chi \mathcal{M}_{\ell/2}(r_1 \ell^2) e_z \cdot \nabla(r_1 \ell^2) \right] \right\} \left[ \left[ \mathcal{M}_{\ell/2}(r_1 \ell^2) e_z \cdot \nabla(r_1 \ell^2) \right] \right\}
\]

so that (2.44) is written in the simple form

\[
r/\mu^2 = \tau^{-1} |T(r) - T_\lambda|/T_\lambda .
\]
(2.46)

The renormalized counterparts of (2.14) and (2.15) are obtained analogously. We find

\[
\Delta r/\mu^3 = \frac{1}{\tau} (\mu T_\lambda)^{-1} \nabla T ,
\]
(2.48)

respectively.

### D. Application of the RG theory

By the renormalization a characteristic length scale is introduced which is described by the parameter \( \mu \). The RG theory is based on the fact that this length scale is arbitrary and may be changed according to \( \mu \rightarrow \mu/l \), where \( l \) is the RG flow parameter. As a consequence, the renormalized coupling parameters \( u(l), \gamma(l), w(l), F(l), \) and \( f(l) \) depend on \( l \). Furthermore, also the Z factors depend on \( l \). (Note that the RG flow parameter \( l \) must be distinguished from the characteristic length \( \ell = (2m_\lambda /\mu)^{1/2} \). Now, the dimensionless parameter defined in (2.45) reads

\[
\tau = \left( \frac{A_d(\mu/l)^d}{\chi_0 Z_m} \right)^{1/2} \frac{1}{2\gamma(l)} ,
\]
(2.49)

For convenience we will use \( \tau \) as the RG flow parameter instead of \( l \) because \( \tau \) is closely related to the reduced temperature by (2.48) and the renormalized coupling parameters \( u(\tau), \gamma(\tau), w(\tau), F(\tau), \) and \( f(\tau) \) were determined as functions of \( \tau \) in Ref. [13]. We identify \( \mu \ell = \xi^{-1} \) by the correlation length \( \xi = (\xi(\tau), \) which in the asymptotic region is given by \( \xi(\tau) = \xi_0 \tau^{-\nu} \). The identification \( \mu l = \xi^{-1} \) is correct in one-loop order, corrections appear in higher order [14].

Now, we write the self-consistent equations for the effective parameters in a form which is appropriate for the numerical evaluation. From (2.27), (2.29), and (2.44) we obtain

\[
r \ell^2 = r_1 \ell^2 \{ 1 + 8u[\tau] A_1 \} ,
\]
(2.50)

\[
\nabla r \ell^2 = (\nabla r_1 \ell^2) \{ 1 + 8u[\tau] A_1 \} ,
\]
(2.51)

\[
Q_{\xi^{-d-1}} = -\frac{A_d}{2\gamma F} \left\{ [1 + \frac{1}{2} f[\tau] A_1] e_z (e_z \cdot \nabla(\Delta r))^3 \right\} \left\{ [1 + \frac{1}{2} f[\tau] A_1] e_z (e_z \cdot \nabla(\Delta r))^3 \right\} \xi^3
\]

where \( Q_{\xi^{-d-1}} \) is given by the integrals

\[
Q_{\xi^{-d-1}} = \int_0^{\infty} dw \frac{w^{\nu-1}}{\sinh(w/v)} e^{-v^2} ,
\]

(2.52)

Clearly, in this equation the \( Z \) factor \( (\chi_0 Z_m)^{1/2} \) does not cancel. For convenience we define the parameter

\[
\tau = \left( \frac{A_d(\mu/l)^d}{\chi_0 Z_m} \right)^{1/2} \frac{1}{2\gamma(l)} ,
\]
(2.45)
\[
A = \frac{1}{\epsilon} \left[ \frac{(\ell/\xi)^2}{\Gamma(-1+\epsilon/2)} (r_1 \ell^2)^{-1} F_{-1+\epsilon/2}(r_1 \ell^2) - 1 \right], \quad (2.53)
\]
\[
A_1 = \frac{1}{\epsilon} \left[ \frac{-(\ell/\xi)^2}{\Gamma(-1+\epsilon/2)} F_{-1/2}(r_1 \ell^2) - 1 \right], \quad (2.54)
\]
\[
A' = \frac{1}{\epsilon} \left[ \frac{-(\ell/\xi)^2}{\Gamma(-1+\epsilon/2)} \Lambda''_{1/2}(r_1 \ell^2) - 1 \right], \quad (2.55)
\]
\[
A'' = \frac{1}{\epsilon} \left[ \frac{-(\ell/\xi)^2}{\Gamma(-1+\epsilon/2)} \left[ \Lambda''_{1/2}(r_1 \ell^2) \right. \right.
\]
\[
- \left. 4^r [r w'/\tau] F(\tau) M_{1/2}(r_1 \ell^2) \right] / (1 + 8u_1 [\tau] A_1). \quad (2.56)
\]

The functions \( F_p(\zeta), M_a(\zeta), \Lambda''(\zeta), \) and \( \Lambda''(\zeta) \) are defined by the integrals (2.37), (2.38), and (2.40) as before, where however in (2.37) and (2.40), the parameters \( w' \) and \( w'' \) must be replaced by \( w'/\tau \) and \( w''/\tau \). Finally, from (2.40)-(2.48) we obtain the equations
\[
r \ell^2 = \tau^{-1}[T(r) - T_0] / T_\lambda, \quad (2.57)
\]
\[
\Delta r \ell^2 = \tau^{-1}[T(r) - T_0] / T_\lambda, \quad (2.58)
\]
\[
\nabla r \ell^3 = \nabla(\Delta r) \ell^3 = \tau^{-1}(\xi/\ell) \nabla T \quad (2.59)
\]

which relate the effective parameters to the temperature \( T(r) \) and its gradient.

Supposed the temperature \( T \), the heat current \( Q \), and the rotation frequency \( \Omega \) are known quantities, then Eqs. (2.54)-(2.59) are ten equations for the eleven unknown variables \( r_1, r, \Delta r, \nabla r_1, \nabla r, \nabla (\Delta r), A, A_1, A', A'', \) and \( \tau \). Thus, one variable remains undetermined which actually is the RG flow parameter \( \tau \). In the spirit of the RG theory we must choose the flow parameter \( \tau \) so that in the perturbation series the infrared divergencies are resummed in an optimum way. By experience we find that the condition
\[
[r_1 \ell^2 + 1](\xi/\ell)^2 + (32u_1 [\tau]/\epsilon)(1 - \epsilon/2)(\xi/\ell)^2 - 1 = 1 \quad (2.60)
\]
is a good choice for fixing \( \tau \).

The integrals (2.37) are well defined for \( \zeta = r_1 \ell^2 \) in the interval \(-1 < \zeta < +\infty \). For \( \zeta \gg +1 \) and for \( \zeta \rightarrow -1 \) the integrals can be evaluated asymptotically. From (2.26) we obtain
\[
F_p(\zeta) \approx \begin{cases} 
\Gamma(p) \zeta^{-p}, & \text{for } \zeta \gg +1, \\
2\Gamma(p+1)(\zeta+1)^{-(p+1)}, & \text{for } \zeta \rightarrow -1.
\end{cases} \quad (2.61)
\]

Consequently, from (2.50) together with (2.53) we obtain
\[
r \ell^2 = \begin{cases} 
-16u_1 [\tau]/(1 + 8u_1 [\tau] A_1 r_1 \ell^2 + 1) & \text{for } r_1 \ell^2 \gg +1, \\
(16u_1 [\tau]/(1 + 8u_1 [\tau] A_1 r_1 \ell^2 - 1) & \text{for } r_1 \ell^2 \rightarrow +1, \\
-16u_1 [\tau]/(1 + 8u_1 [\tau] A_1 r_1 \ell^2 - 1) & \text{for } r_1 \ell^2 \rightarrow -1.
\end{cases} \quad (2.62)
\]

For temperatures \( T \) well above \( T_\lambda \) the correlation length \( \xi \) decreases so that \( \zeta < \ell \). Eq. (2.60) implies \( r_1 \ell^2 \gg +1 \), so that on the left-hand side of (2.60) the first term dominates while the second term can be neglected. Thus, Eq. (2.60) reduces into \( r_1 \ell^2 = 1 \) and Eq. (2.62) implies \( r \ell^2 = r_1 \ell^2 \). Consequently, for \( T \gg T_\lambda \) the flow-parameter condition (2.60) reduces into
\[
r \ell^2 = 1 \quad (2.63)
\]

On the other hand, for temperatures \( T \) well below \( T_\lambda \) it is \( r_1 \ell^2 \rightarrow -1 \), so that on the left-hand side of (2.60) now the second term dominates while the first term is small and negligible. Again it is \( \zeta < \ell \). Consequently, (2.62) implies that for \( T \ll T_\lambda \) the flow parameter condition (2.60) reduces into
\[
-2r \ell^2 = 1 \quad (2.64)
\]

Actually, (2.63) and (2.64) are the standard flow-parameter conditions of Ref. 13 for \( T > T_\lambda \) and \( T < T_\lambda \), respectively, where \( Q = 0 \) and \( \Omega = 0 \). (Note that our \( r \ell^2 \) is identified by \( r(l)/\mu \ell^2 \) in Ref. 13.) Thus, we have shown that our flow parameter condition (2.60), which is valid for finite \( \Omega \) and infinitesimal small \( Q \), reduces to the standard flow parameter conditions of Ref. 13 for temperatures \( T \) well above and well below \( T_\lambda \). For \( T \) near \( T_\lambda \) it represents an interpolation.

Now, Eqs. (2.54)-(2.60) are eleven equations which determine the eleven unknown variables uniquely. We solve these equations numerically for given temperature \( T \), heat current \( Q \), and rotation frequency \( \Omega \) to determine the effective parameters. From (2.53) we obtain the temperature gradient \( \nabla T \). Once the effective parameters are known, physical quantities can be calculated explicitly. This will be done in the next section. Since we consider liquid \(^4\)He in three dimensions, we set \( d = 3 \) and \( \epsilon = 4 - d = 1 \) in all formulas when performing the numerical calculations.

As an input we need the dimensionless renormalized couplings \( u_1 [\tau], \gamma [\tau], w_1 [\tau] = w' [\tau] + iw'' [\tau], F_1 [\tau], \) and \( f_1 [\tau] \) as functions of \( \tau \) which have been determined by Dohm.\textsuperscript{13} Furthermore, we need the parameter \( g_0 \) which is related to the entropy at \( T_\lambda \). For liquid helium at saturated vapor pressure this parameter is\textsuperscript{14} \( g_0 = 2.164 \times 10^{11} \text{s}^{-1} \). To calculate the correlation length \( \ell(\tau) = \ell_0 \tau^{-\nu} \) as a function of \( \tau \) we use the exponent \( \nu = 0.671 \) and the amplitude \( \ell_0 = 1.45 \times 10^{-8} \text{cm} \) which were determined experimentally in Refs. 13 and 14. There are no adjustable parameters.

### III. Entropy and Specific Heat

In our previous paper\textsuperscript{18} the entropy was considered in Hartree approximation combined with the renormalization-group theory. The following result was obtained:
\[
S = S_\lambda + t \left( B + \tilde{A} \left( [4n/\alpha] + E[u^*] \right) \tau^{-\alpha} \right) \quad (3.1)
\]
where
This formula is valid also for rotating \(^4\)He without alteration. We just insert the effective parameters \(r\) and \(r_1\) and the RG flow parameter \(\tau\) determined in Sec. II. Since the entropy is an equilibrium quantity, the infinitesimal gradients may be discarded so that only the self-consistent equations (2.50), (2.53), and (2.57) must be solved. The specific heat \(C_\Omega\) is then obtained by numerical differentiation with respect to the temperature according to

\[
C_\Omega = T_\lambda \left( \frac{\partial S}{\partial T} \right)_\Omega = \left( \frac{\partial S}{\partial T} \right)_\Omega
\]

(3.4)

where the rotation frequency \(\Omega = \omega_\Omega e_z\) is kept constant.

The formula (3.1) is derived in the asymptotic regime close to \(T_\lambda\) where \(\xi = \xi_0 \tau^{-\nu}\) and \(u(\tau) \approx u^* \approx 0.0362\). The critical exponents \(\nu = 0.671\) and \(\alpha = -0.013\) and the nonuniversal amplitudes \(A = 2.22\) J/mol K and \(B = 456\) J/mol K are obtained by comparing the theoretical specific heat for \(\Omega = 0\) defined by (3.1)-(3.4) with the most recent experimental data\(^4\) which were obtained in a microgravity environment in space.

FIG. 2. The specific heat for rotating \(^4\)He as a function of temperature. The solid line represents our theoretical result for \(\Omega = 2\pi\) s\(^{-1}\). For comparison, the specific heat for \(\Omega = 0\) is shown as dotted line.

Realistic experiments with rotating \(^4\)He were performed for rotations about one turn per second (see Refs. [13],[12]). For this reason, we calculate the specific heat for \(\Omega = 2\pi\) s\(^{-1}\). In Fig. 2 our numerical result is shown as solid line. For comparison the dotted line represents the specific heat at \(\Omega = 0\). Clearly, for nonzero \(\Omega\) the specific heat is a smooth function of temperature with a maximum slightly below \(T_\lambda\). The position of the maximum can be interpreted as the superfluid transition temperature \(T_\lambda(\Omega)\) for rotating \(^4\)He which, however, is not sharply defined. Thus, for nonzero \(\Omega\) the superfluid transition is a smooth crossover. The shift of the transition temperature \(\Delta T_\lambda(\Omega) = T_\lambda(\Omega) - T_\lambda\) is negative. For \(\Omega = 2\pi\) s\(^{-1}\) we find \(\Delta T_\lambda(\Omega) = -25\) nK from Fig. 2. For other rotation frequencies \(\Omega\) a scaling formula for \(\Delta T_\lambda(\Omega)\) can be derived. Scaling requires the relations \(\xi/\ell = a\) and \(\Delta T_\lambda(\Omega)/\Delta T_\lambda = b\tau\) where \(a\) and \(b\) are dimensionless constants. Inserting \(\xi = \xi_0 \tau^{-\nu}\) and \(\ell = (\hbar/2m_4\Omega)^{1/2}\), eliminating \(\tau\), and solving for \(\Delta T_\lambda(\Omega)\) we obtain

\[
\Delta T_\lambda(\Omega) = -M_\lambda T_\lambda(2m_4\xi_0^2\Omega/\hbar)^{1/2\nu}
\]

(3.5)

where \(M_\lambda\) is a dimensionless constant of order unity related to \(a\) and \(b\). From the position of the maximum in Fig. 2 we find \(M_\lambda = 1.2\). The shift of the critical temperature (3.5) is represented in Fig. 2 by the dashed line. Thus, the phase diagram for uniformly rotating liquid \(^4\)He shown in Fig. 1 is confirmed.

Our theory neglects the influence of the centrifugal forces which imply a spatial variation of the critical temperature given by (3.6). For a sample of diameter \(D\) the maximum variation of the critical temperature is

\[
\delta T_{\lambda,\text{max}}(\Omega) = |\partial T_\lambda/\partial P| \rho \Omega^2 D^2/8
\]

(3.6)

To justify the neglect of the influence of the centrifugal forces we must require that \(\delta T_{\lambda,\text{max}}(\Omega)\) is ten times smaller than \(\Delta T_\lambda(\Omega)\) defined in (3.5). Thus, for \(\Omega = 2\pi\) s\(^{-1}\) we require \(\delta T_{\lambda,\text{max}}(\Omega) = 2.5\) nK. Inserting this value into (3.6) and solving for \(D\) we obtain the maximum diameter \(D_{\text{max}}\) of the sample for which the influences of the centrifugal forces may be neglected. For \(\Omega = 2\pi\) s\(^{-1}\) we obtain \(D_{\text{max}} = 0.6\) cm, while from (3.3) and (3.6) we find the frequency dependence \(D_{\text{max}} \sim \Omega^{1/4\nu-1} \sim \Omega^{-0.627}\). Thus, \(D_{\text{max}}\) is of order of realistic sample sizes.\(^1\) If the samples are made smaller so that \(D \lesssim D_{\text{max}}\) is satisfied, then indeed the influence of the centrifugal forces may be neglected.

On the other hand the influences of gravity on earth imply a much larger variation \(\delta T_{\lambda,\text{g}}\) of the critical temperature. From (3.6) we obtain

\[
\delta T_{\lambda,\text{g}} = |\partial T_\lambda/\partial P| \rho g \Delta z
\]

(3.7)

where \(\Delta z\) is the height of the sample. For \(\Delta z \approx 0.6\) cm we obtain \(\delta T_{\lambda,\text{g}} \approx 0.8\) \(\mu\)K which is much larger than the temperature scale in Fig. 2. For this reason, the maximum of the specific heat \(C_\Omega\) induced by the vortices cannot be resolved in an experiment on earth. The experiment must be performed in a microgravity environment in space.

For temperatures well below \(T_\lambda\) a first-order transition between a vortex liquid and a vortex lattice is expected, because for decreasing temperatures the fluctuations decrease. However, our theory based on the Hartree approximation cannot describe this transition. While the effects of vortices are included indirectly, our theory assumes the physical quantities to be homogeneous in space so that the system is always a vortex liquid.
IV. CORRELATION LENGTH

In the asymptotic regime close to $T_\lambda(\Omega)$ the correlation length is given by $\xi = (\mu l)^{-1} = \xi_0 \tau^{-\nu}$. Inserting the RG flow parameter $\tau$ determined in Sec. II, we calculate $\xi = \xi(T, \Omega)$ as a function of temperature $T$ for the rotation frequency $\Omega = 2\pi s^{-1}$. Our numerical result is shown in Fig. 3 as solid line. For comparison, the dotted line represents the correlation length at $\Omega = 0$. Clearly, for nonzero $\Omega$ the correlation length is a smooth function of temperature with a maximum slightly below $T_\lambda$. This maximum is located close to the maximum of the specific heat in Fig. 4.

![Graph showing correlation length](image)

FIG. 3. The correlation length for rotating $^4$He as a function of temperature. The solid line is the correlation length $\xi$ for $\Omega = 2\pi s^{-1}$. The dashed line represents the magnetic length $\ell$. For comparison, the correlation length $\xi$ for $\Omega = 0$ is shown as dotted line.

On the other hand, the characteristic length $\ell = (h/2m_4\Omega)^{1/2}$ is shown in Fig. 3 as dashed line. By definition this length does not depend on the temperature. We find $\xi < \ell$ so that the correlation length is bounded by $\ell$. Near the maximum it is $\xi \approx \ell$. The characteristic length $\ell$ is closely related to the mean distance between the vortex lines $L^{-1/2}$. Because of $L = 1/2\pi \Omega$ it is $L^{-1/2} = (2\pi)^{1/2} \ell \approx 2.5 \ell$. Consequently, the correlation length $\xi$ is bounded by the mean distance between the vortex lines according to $\xi < L^{-1/2}$.

V. THERMAL CONDUCTIVITY AND RESISTIVITY

We eliminate the gradient $\nabla(\Delta r)$ from (2.52) and (2.55) and solve the resulting equation for $Q$. Then we obtain

$$Q = -\lambda_{T,1} e_z \cdot (e_z \cdot \nabla T) + \lambda_T'(e_z \times (e_z \times \nabla T)) - \lambda_T''(e_z \times \nabla T) \quad (5.1)$$

where

$$\lambda_{T,1} = \frac{g_0 k_B a_d}{\tau \xi^{-d-2}} \left[1 + (\tau r/2) A_1 \right]$$

$$\lambda_T' = \frac{g_0 k_B a_d}{2 \tau \xi^{-d-2}} \left[1 + (\tau r/2) A' \right]$$

$$\lambda_T'' = \frac{g_0 k_B a_d}{2 \tau \xi^{-d-2}} \left[1 + (\tau r/2) A'' \right]$$

are three components of the thermal-conductivity tensor. While the dependence on the dimensionality $d$ is needed for the renormalization, the formulas (5.2)-(5.4) are evaluated for $d = 3$. Clearly, Eq. (5.1) is a linear relation between the temperature gradient $\nabla T$ and the heat current $Q$. Solving this equation for $\nabla T$ we obtain

$$\nabla T = -\rho_{T,1} e_z (e_z \cdot Q) + \rho_T' (e_z \times (e_z \times Q)) - \rho_T'' (e_z \times Q) \quad (5.5)$$

where $\rho_{T,1}$, $\rho_T'$, and $\rho_T''$ are three components of the thermal-resistivity tensor which are related to the conductivities (5.2)-(5.4) by

$$\rho_{T,1} = 1/\lambda_{T,1} \quad (5.6)$$

$$\rho_T' = 1/(\lambda_T' + i\lambda_T'') \quad (5.7)$$

Clearly, Eqs. (5.1) and (5.5) indicate that the heat current $Q$ and the temperature gradient $\nabla T$ are not parallel to each other. This fact is the consequence of the symmetry breaking by the rotation frequency $\Omega = \Omega e_z$. In terms of the components the heat transport equation (5.1) can be written as

$$Q_x + iQ_y = -(\lambda_T' + i\lambda_T'') (\partial_x T + i\partial_y T) \quad (5.8)$$

$$Q_z = -\rho_{T,1} \partial_z T \quad (5.9)$$

Analogously, Eq. (5.3) can be written as

$$\partial_x T + i\partial_y T = -(\rho_T' + i\rho_T'') (Q_x + iQ_y) \quad (5.10)$$

$$\partial_z T = -\rho_{T,1} Q_z \quad (5.11)$$

Thus, the heat transport decouples into two independent contributions, parallel and perpendicular to the rotation axis. The parallel heat transport (in the $z$ direction) is very similar to the heat transport in nonrotating $^4$He: the thermal conductivity $\lambda_{T,1}$ defined in (5.2) has the same structure with the same amplitude $A_1$ as in the nonrotating case (compare (5.2) and (2.30) with (5.2) and (4.33) in the second paper of Ref. 4, respectively). The only difference is the function $F_p(\xi)$ which here is defined in a different way than in Ref. 4. On the other hand, the perpendicular heat transport (in the $xy$ plane) shows a completely different nature. In this cases the thermal conductivity $\lambda_T' + i\lambda_T''$ and the thermal resistivity $\rho_T' + i\rho_T''$ are complex. While the real parts $\lambda_T'$ and $\rho_T'$ describe the dissipation of the heat current, the imaginary parts $\lambda_T''$ and $\rho_T''$ imply a coupling between the $x$ and $y$ direction so that the heat current and the related
temperature gradient do not have the same direction in the \( xy \) plane. The real part \( \rho_\parallel \) may be interpreted as the \textit{longitudinal} thermal resistivity, because for a given heat current in the \( xy \) plane it implies an antiparallel temperature gradient. On the other hand, the imaginary part \( \rho''_\parallel \) may be interpreted as the \textit{transversal} thermal resistivity, because it implies a temperature gradient perpendicular to the heat current. The amplitudes \( A' \) and \( A'' \) defined in (2.53) and (2.56) depend on completely different integrals than \( A_1 \), which are defined in (2.38)-(2.40) and in (2.20), respectively, and depend strongly on the rotation frequency \( \Omega \).

The special structures of the conductivities and resistivities reflect the fact that in rotating \( ^4 \text{He} \) there are straight vortex lines present which move and imply dissipation for a perpendicular heat flow but do not move for a parallel heat flow. The existence of a temperature gradient perpendicular to the heat current is the analogy of the Hall effect in electronic systems, where here the rotation frequency \( \Omega \) plays the role of the magnetic field.

We calculate the thermal conductivities numerically by (5.16) and (5.17) for \( d = 3 \) where the effective parameters and their gradients determined in Sec. IV are inserted. From (5.6) and (5.7) we obtain the resistivities for \( \Omega = 0 \) is shown as dotted line.

![Graph](image)

**Fig. 4.** The several thermal resistivities of rotating \( ^4 \text{He} \) as functions of temperature for \( \Omega = 2 \pi \text{ s}^{-1} \). The longitudinal resistivity \( \rho_\parallel \) and the transversal resistivity \( \rho''_\parallel \) for heat transport perpendicular to the rotation axis are shown as solid lines. The dashed line is the resistivity \( \rho_{T,1} \) for heat transport parallel to the rotation axis. For comparison, the resistivity for \( \Omega = 0 \) is shown as dotted line.

For increasing temperature, the transversal conductivity \( \rho''_\parallel \) exhibits a maximum near \( T_\lambda \). The inflection points of \( \log_{10} \rho_{T,1} \) and \( \log_{10} \rho_\parallel \) and the maximum of \( \log_{10} \rho''_\parallel \) in Fig. 4 are located at a temperature below \( T_\lambda \) close to the temperature of the maximum of the specific heat in Fig. 3. Thus, the shift \( \Delta T_\lambda(\Omega) \) of the superfluid transition temperature for nonzero \( \Omega \), defined in (3.3), is observed also in the thermal conductivities where here the constant \( M_\lambda \) is slightly different.

For temperatures \( T \) well above and well below \( T_\lambda \) asymptotic formulas can be derived for the thermal conductivities and resistivities. To do this we must evaluate the integrals \( F_p(\zeta), M_p(\zeta), N'_p(\zeta), \) and \( N''_p(\zeta) \) for \( \zeta \gg +1 \) and for \( \zeta \rightarrow -1 \), respectively. While for \( F_p(\zeta) \) the asymptotic formula is given by (2.61), analogous asymptotic formulas are found for the other integrals.

First, we consider \( T \gg T_\lambda \), where \( \zeta = r_1 / T^2 \gg +1 \). In this case the flow-parameter condition (2.60) reduces to \( r_1 / T^2 = 1 \). Then, for the amplitudes (2.53)-(2.56) we obtain \( A \approx 0, A_1 \approx A' \approx -1/2, \) and \( A'' \approx 0 \). Consequently, from (5.6) and (5.7) together with (5.2)-(5.4) we obtain the thermal resistivities

\[
\rho_{T,1} \approx \rho'_\parallel \approx \frac{\pi^{d-2} T}{g_0 k_p A_d} \frac{2 \gamma[\tau] F[\tau]}{1 - f[\tau]/4},
\]

and \( \rho''_\parallel \approx 0 \) with \( \zeta = \xi_0 \tau^{-\nu} \) and \( \tau = (T - T_\lambda)/T_\lambda \). Because of (2.53) and (2.61) the RG flow parameter \( \tau \) is identified by the reduced temperature. Eq. (5.12) is the well known formula for the thermal resistivity for \( \Omega = 0 \) and \( Q \rightarrow 0 \) with the amplitude functions evaluated up to one-loop order (see Ref. 13). In Fig. 4 this resistivity is shown by the dotted line. Thus, for increasing temperatures \( T \) well above \( T_\lambda \) the thermal resistivities \( \rho_{T,1} \) and \( \rho'_\parallel \) asymptotically approach the resistivity for \( \Omega = 0 \) where the imaginary part \( \rho''_\parallel \) decreases and is negligibly small. This fact is clearly seen in Fig. 4.

Secondly, we consider \( T \ll T_\lambda \) where \( \zeta \equiv r_1 / T^2 \rightarrow -1 \). In this case in the flow-parameter condition (2.60) the first term can be neglected so that

\[
r_1 / T^2 + 1 = \left[ \frac{32 u[\tau]}{\epsilon} \right] (1 - \epsilon/2)(\xi/\ell)^{2-\epsilon} \]

which is very small for \( \xi \ll \ell \). Because of (2.57) and (2.64) the RG flow parameter is related to the reduced temperature by \( \tau = (T_\lambda - T)/T_\lambda \). Inserting the asymptotic formulas of the integrals (2.20) and (2.38)-(2.40) for \( \zeta \rightarrow -1 \) into (2.53)-(2.56) we obtain the amplitudes \( A, A_1, A', \) and \( A'' \). Then, from (5.2)-(5.4) we obtain the thermal conductivities, and from (5.6) and (5.7) we obtain the related resistivities. Eventually, in leading order we obtain the asymptotic formulas

\[
\rho_{T,1} \approx (4/\epsilon)(r_1 / T^2 + 1) \rho'_\parallel,
\]

\[
\rho'_\parallel \approx \frac{\pi^{d-2} \epsilon}{g_0 k_p A_d} \frac{4 \gamma[\tau] w[\tau]}{F[\tau]} 8 u[\tau] (\xi/\ell)^2,
\]

\[
\rho''_\parallel \approx \frac{\pi^{d-2} \epsilon}{g_0 k_p A_d} \frac{4 \gamma[\tau] v[\tau]}{F[\tau]} 8 u[\tau] (\xi/\ell)^2.
\]
where \( \xi = \xi_0 \tau^{-\nu} \) and \( \ell = (\hbar/2m_4 \Omega)^{1/2} \).

For the heat transport perpendicular to the rotation axis for \( T \ll T_\lambda \), the transversal and the longitudinal resistivities are related to each other by

\[
\rho''_T / \rho_T \approx w''[\tau] / w_0[\tau] .
\]

Neglecting the nonsymptotic effects due to the renormalized coupling parameters \( \gamma[\tau], w'[\tau], w''[\tau], \) and \( F[\tau] \), we obtain asymptotically in leading order

\[
\rho''_T \sim \rho''_T \sim \tau \xi^d \ell^{-2} \sim \Omega (T_\lambda - T)^{-1+\alpha}.
\]

where \( \alpha = 2 - d \nu = -0.013 \). Thus, in slowly rotating superfluid \(^4\)He the thermal resistivities for the perpendicular heat flow depend linearly on the rotation frequency. Since the resistivities are caused by vortex lines, this result is plausible: the resistivities must be proportional to the total length of the vortex lines per unit volume \( L \) which is related to the rotation frequency by \( L = 1/(\pi \ell^2) = (m_4/\pi \hbar) \Omega \). For \( \Omega \rightarrow 0 \) the thermal resistivities \( \rho'_T \) and \( \rho''_T \) vanish as expected.

On the other hand, the heat flow parallel to the rotation axis is not directly influenced by the vortex lines, because the heat current flows parallel to the vortex lines. The dissipation of a parallel heat current and the related resistivity \( \rho_{T,1} \) must be caused by fluctuation effects. For this reason \( \rho_{T,1} \) is much smaller than the resistivity \( \rho'_T \) of the perpendicular heat flow. This fact is clearly seen in Fig. 5 and in Eq. (5.14) because the prefactor defined by (5.13) is very small. Neglecting the nonsymptotic effects due to the renormalized coupling parameters, we obtain asymptotically in leading order

\[
\rho_{T,1} \sim \tau \xi^{d-2} (\xi/\ell)^{4/\epsilon} \sim \tau^{1-(d-2+4/\epsilon)} \ell^{-4/\epsilon} .
\]

For \( d = 3 \) dimensions where \( \epsilon = 4 - d = 1 \) we find \( \rho_{T,1} \sim \Omega^2 (T_\lambda - T)^{-1+\nu} \), so that the resistivity for the parallel heat flow is quadratical in the rotation frequency. Again, \( \rho_{T,1} \) vanishes for \( \Omega \rightarrow 0 \) as expected. The temperature dependence of \( \rho_{T,1} \) is very similar to the temperature dependence of the thermal resistivity in nonrotating \(^4\)He for nonzero heat currents \( Q \) (compare the dashed line in Fig. 4 with the solid lines in Figs. 3 and 4 in the second paper of Ref. 3). The related exponents for the approximate power laws for \( T \ll T_\lambda \) are nearly the same.

VI. MUTUAL FRICTION AND THE VIVEN COEFFICIENTS

Mutual friction in rotating superfluid \(^4\)He was investigated first by Hall and Vinen. While originally the attenuation of second sound in resonant cavities was considered\(^1\), the mutual friction implies dissipation and a related resistance for the heat transport in the superfluid state. Here we show that the phenomenological theory of Hall and Vinen implies a heat-transport equation which has the same form as (5.2) obtained within our approach. In this way we show that mutual friction according to Hall and Vinen is derived from model \( F \) by our approach.

Hall and Vinen proposed a mutual-friction force between the superfluid and the normal-fluid component which is implied by the motion of the vortex lines in the presence of a superfluid-normal-fluid counterflow. This mutual-friction force must be added to and subtracted from the two-fluid hydrodynamic equations for the superfluid and the normal-fluid component, respectively. For a stationary homogeneous counterflow with the relative velocity \( v_s - v_n \) a relation for the temperature gradient \( \nabla T \) can be derived from the two-fluid hydrodynamic equations which reads

\[
(\rho/\rho_n) s \nabla T = -B \Omega (e_z \times (e_z \times (v_s - v_n))) + (2 - B') \Omega (e_z \times (v_s - v_n)) .
\]

Here, \( B \) and \( B' \) are the Vinen coefficients which represent the phenomenological parameters of the mutual-friction force. The term with the coefficient 2 arises from the Coriolis force in the rotating frame. In the critical regime close to \( T_\lambda \) it is \( \rho/\rho_n \approx 1 \) and the entropy per mass is \( s \approx s_\lambda = (\hbar/m_4)(g_0/T_\lambda) \) where \( g_0 = 2.164 \times 10^{11} \text{s}^{-1} \) at saturated vapor pressure \( T \).

For a zero net mass current the superfluid-normal-fluid counterflow with relative velocity \( v_s - v_n \) is directly related to the heat current \( Q \). Considering the counterflow as a metastable state, in Ref. 7 the heat current \( Q \) was calculated for nonrotating superfluid \(^4\)He as a function of the counterflow wave vector \( k = (m_4/\hbar)(v_s - v_n) \). The calculation was performed in the critical regime near \( T_\lambda \) using model \( F \) and the RG theory. Since the rotation frequency \( \Omega \) is very small, we may use this result here. For small counterflows and small heat currents the relation between \( Q \) and \( v_s - v_n \) is linear. While in Ref. 7 this relation was calculated in one-loop RG theory, for our purpose the relation in zero-loop RG theory is sufficient which reads

\[
Q = -g_0 k_B T_\lambda A_d m_4 / 8u[\tau] \hbar (v_s - v_n) .
\]

Now, we solve (5.2) for \( v_s - v_n \) and insert the resulting expression into (6.1). Then, we obtain the temperature gradient

\[
\nabla T = \frac{\xi^{d-2}}{g_0 k_B A_d} 8u[\tau] B \Omega (e_z \times (e_z \times Q)) - \frac{\xi^{d-2}}{g_0 k_B A_d} 8u[\tau] (2 - B') \Omega (e_z \times Q) .
\]

This formula has the same structure as (5.3). By comparison we find the thermal resistivities

\[
\rho'_T = \frac{\xi^{d-2}}{g_0 k_B A_d} 8u[\tau] B \Omega ,
\]

\[
\rho''_T = \frac{\xi^{d-2}}{g_0 k_B A_d} 8u[\tau] (2 - B') \Omega .
\]
and $\rho_T = 0$. For the heat flow perpendicular to the rotation axis the resistivities (6.4) and (6.5) are a direct consequence of the mutual friction due to the motion of the vortices. Because of $\Omega = \omega h/m_4$ L the resistivities of the vortex lines are proportional to the vortex density $L$ which precisely is the total length of the vortex lines per volume. Since $\rho^T$ implies a temperature gradient antiparallel to the heat current, the Vinen coefficient $B$ describes the dissipative effects of the vortex lines. On the other hand, $\rho^T_{\kappa}$ implies a temperature gradient perpendicular to the heat current, so that the Vinen coefficient $B^\prime$ and the Coriolis force coefficient $2$ are related to reversible effects. For a heat flow parallel to the rotation axis the vortex lines do not move and hence do not cause dissipation, so that the resistivity $\rho_T = 0$.

Within our approach which is based on the Hartree approximation combined with the RG theory we have derived the formula (6.3) for the temperature gradient. The terms involving the heat current perpendicular to the rotation axis have the same structure as the terms on the right-hand side of (6.3). Thus, we conclude that our approach includes the mutual friction between the superfluid and the normal-fluid component caused by the motion of vortices. However, since the superfluid state is homogeneous in space and vortex lines are not treated explicitly, our approach includes the effect of the vortices indirectly. Furthermore, our approach includes also fluctuation effects, which is reflected by the first term of (6.3) and the nonzero resistivity $\rho_T(\kappa)$ for the parallel heat flow. This term is not present in (6.3).

We determine the Vinen coefficients by solving (6.4) and (6.5) for $B$ and $2 - B^\prime$ inserting the thermal resistivities $\rho_T^\prime$ and $\rho^\prime_{\kappa}$ obtained within our approach. For small rotation frequencies $\Omega$ the thermal resistivities are given by the asymptotic formulas (6.15) and (6.16). Thus, we obtain the Vinen coefficients

$$
B = (2m_4/h)g_0\tau^2 4\gamma[\tau] w''[\tau]/F[\tau] ,
$$

(6.6)

$$
2 - B^\prime = (2m_4/h)g_0\tau^2 4\gamma[\tau] w''[\tau]/F[\tau] ,
$$

(6.7)

which are defined in the limit $\Omega \to 0$. These formulas are well suited for numerical evaluations because all parameters and constants are known. The RG flow parameter is related to the reduced temperature by $\tau = (2T_{\lambda} - T)/T_{\lambda}$.

Simpler expressions for the Vinen coefficients are obtained if we replace $g_0$ by the renormalized counterpart $g(l) = g[\tau]$. From the renormalization equation (2.32) we obtain

$$
g_0 = g(l) (\chi_0 Z_m(l))^{1/2} ((\mu l)^{\gamma}/A_d)^{1/2} .
$$

(6.8)

We solve (2.32) for $(\chi_0 Z_m(l))^{1/2}$ and insert the resulting expression into (6.8). Then we obtain

$$
g_0 = g[\tau]/(\tau^2 2\gamma[\tau])
$$

(6.9)

where $\mu l = \xi^{-1}$ has been identified. Now, we insert this result for $g_0$ into (6.6) and (6.7). Furthermore, we insert the dimensionless renormalized couplings $w''[\tau] = \Gamma[\tau]/\lambda[\tau]$, $w''[\tau] = \Gamma''[\tau]/\lambda[\tau]$, and $F[\tau] = g[\tau]/\lambda[\tau]$.

Since most renormalized couplings cancel, we obtain eventually

$$
B = (4m_4/h)\Gamma[\tau] ,
$$

(6.10)

$$
2 - B^\prime = (4m_4/h)\Gamma''[\tau] .
$$

(6.11)

This result is remarkably simple. Eqs. (6.10) and (6.11) represent the final formulas for the Vinen coefficients obtained within our approach.

The Vinen coefficients were calculated previously for model $F$ within the renormalized mean-field theory. In this previous approach the vortex lines are considered explicitly. The model-$F$ equations are written first in the renormalized form and then solved as mean-field equations. Solutions were obtained which represent a single straight vortex line moving under the influence of a superfluid-normal-fluid counterflow. From the relation between the velocity of the vortex line $v_L$ and the relative velocity of the counterflow $v_s - v_n$ the Vinen coefficients $B$ and $2 - B^\prime$ were extracted. Eventually, the effects of the critical fluctuations are included by application of the RG theory. The theoretical results were compared with the experimental data of Refs. 10, 12 obtained for temperatures $T_{\lambda} - T \gtrsim 3 \times 10^{-4}$ K. For $B$ the agreement between the experiments and the previous theory is very good, while for $2 - B^\prime$ there are some discrepancies. Consequently, the results of the previous theory may be viewed as correct and reliable.

Close to criticality, the leading temperature dependence of the Vinen coefficients obtained from the previous theory is governed by

$$
B + i(2 - B^\prime) \approx (2m_4/h)\Gamma'[\tau]/[c_1 + c_2(\gamma[\tau] F[\tau])^2 - i c_3(\gamma[\tau] F[\tau])] ,
$$

(6.12)

where $c_1 = 0.2511$, $c_2 = 0.391$, and $c_3 = 0.795$ (see Eq. (37) and Table 1 in Ref. 6, the winding number of the vortices is assumed to be $n = 1$). Since the product of the renormalized couplings $\gamma[\tau] F[\tau]$ is small, the term with the coefficient $c_2$ can be neglected so that (6.12) simplifies into

$$
B \approx (4m_4/h)\Gamma'[\tau] \times 1.99 ,
$$

(6.13)

$$
2 - B^\prime \approx (4m_4/h)\Gamma'[\tau] \gamma[\tau] F[\tau] \times 6.30 .
$$

(6.14)

These formulas must be compared with (6.10) and (6.11). For $B$ the leading critical temperature dependence is governed by the renormalized coupling $\Gamma'[\tau]$ in both cases, in the present theory and in the previous theory. Quantitatively, the previous theory predicts a 1.99 times larger result for $B$ than our present theory. On the other hand, for $2 - B^\prime$ the leading critical temperature dependence is governed by different renormalized couplings for the two approaches, which is clearly seen in (6.11) and (6.14).

The critical divergence of $\Gamma'[\tau]$ is much weaker than the critical divergences of $\Gamma'[\tau]$ and of $\Gamma'[\tau] \gamma[\tau] F[\tau]$.

If we generalize model $F$ by replacing the complex order parameter $\psi$ by a complex vector $\Psi = (\psi_1, \ldots, \psi_n)$...
of $n$ components, it turns out that the Hartree approximation is exact in the limit $n \to \infty$. The RG theory can be applied but is not necessary in this case. Furthermore, in the limit $\Omega \to 0$ for $T < T_\lambda$ the approximation reduces into the mean-field theory. Consequently, Eqs. (6.10) and (6.11) may be interpreted as the Vinen coefficients for $n = \infty$ in renormalized mean-field theory. On the other hand, previously we have calculated the Vinen coefficients for $n = 1$ in renormalized mean-field theory, which are given by (6.13) and (6.14) in leading order close to criticality. Thus, the Vinen coefficients allow a direct comparison of the physics for $n = 1$ and $n = \infty$ where the same approximation scheme is applied. While for $B$ the agreement is quite good, for $2 - B'$ we find serious discrepancies.

VII. COMPARISON WITH THE GORTER-MELLINK MUTUAL FRICTION IN NONROTATING $^4$He AT NONZERO HEAT CURRENTS

Our approach was originally developed in Ref. 4 for nonrotating $^4$He in the presence of a nonzero heat current $Q$. For $T < T_\lambda$ we found mutual friction between the superfluid and the normal-fluid component according to Gorter and Mellink and calculated the Gorter-Mellink coefficient $A$. In the superfluid region we found a nonzero thermal resistivity which for temperatures $T$ well below $T_\lambda$ is given by the asymptotic formula

$$\rho_T \approx \frac{\tau \xi^{d-2}}{g_0 k_B} \frac{4 \gamma [\tau] w'[\tau]}{F[\tau]} \sqrt{\tau} \left( \frac{8 u[\tau]}{(-\zeta)^{1/2} A_d} \right)^3 \left( \frac{Q \xi^{d-4}}{g_0 k_B T_\lambda} \right)^{2}$$

(7.1)

where the RG flow parameter $\tau$ is related to the reduced temperature by $\tau = (T_\lambda - T)/T_\lambda$. Furthermore, we found a second correlation length $\xi_1$ which is a dephasing length for the order-parameter field. For $T \ll T_\lambda$ we obtain the asymptotic formula

$$\xi_1 \approx \left[ \frac{4}{3} (-\zeta)^{3/4} \right]^{1/4} \frac{A_d}{8 u[\tau]} \frac{g_0 k_B T_\lambda}{Q \xi^{d-2}}$$

(7.2)

Here $\zeta$ is a dimensionless variable which depends logarithmically on the reduced temperature and varies between $-5$ and $-15$ in the superfluid region. $\zeta$ can be eliminated in favor of the dephasing length $\xi_1$. To do this, we solve (7.2) for $\zeta$ and insert the resulting expression into (7.1). Then, we obtain the thermal resistivity

$$\rho_T \approx \frac{\tau \xi^{d-2}}{g_0 k_B A_d} \frac{4 \gamma [\tau] w'[\tau]}{F[\tau]} \frac{8 u[\tau]}{(-\zeta)} \left( 2 \xi_1 \right)^2$$

(7.3)

Vinen argued that in superfluid $^4$He a nonzero heat current $Q$ implies a turbulent superfluid flow and a tangle of vortex lines. The mutual friction and the dissipation of the heat current by the motion of the vortex lines. The thermal resistivity (7.3) has nearly the same structure as $\rho_T$ in (7.15) where here the dephasing length $\xi_1$ plays the role of the characteristic length $\ell$. Thus, the resistivity (7.3) must be due to dissipation by the motion of vortex lines, which agrees with Vinen’s picture.

The thermal resistivities (5.14) - (5.17) represent the elements of the resistivity tensor for a heat flow in the presence of uniformly distributed straight vortex lines. For a tangle of vortex lines the thermal resistivity is obtained by taking the average over all directions of the vortex lines. Thus, we obtain

$$\rho_T \approx \frac{1}{4} \left( \rho_T' + \rho_T + \rho_{T,1} \right) \approx \frac{1}{4} \rho_T'$$

(7.4)

which is just the trace of the resistivity tensor divided by $d = 3$. Because of $\rho_{T,1} \ll \rho_T$, the resistivity $\rho_T$ of the heat flow parallel to the vortex lines can be neglected. Now, inserting (5.15) into (7.4) we obtain

$$\rho_T \approx 2 \frac{\tau \xi^{d-2}}{3 g_0 k_B A_d} \frac{4 \gamma [\tau] w'[\tau]}{F[\tau]} \frac{8 u[\tau]}{(-\zeta)} \left( \xi/\ell \right)^2$$

(7.5)

The prefactor $2/3$ is due to the assumption that the direction of the vortex lines is isotropically distributed. However, since the isotropy is broken by a homogeneous heat current, the assumption may not be perfectly true. Thus, the correct prefactor will be slightly different from $2/3$.

Now, we compare (7.3) with (7.4). We eliminate $\rho_T'$ and solve the resulting equation for $\xi_1$. Then, we obtain the total length of the vortex lines per volume

$$L = 1/(2 \pi \ell^2) = (3/\pi) \xi_1^2$$

(7.6)

which is the density of the vortex lines induced by the heat current $Q$. This result indicates that the dephasing length $\xi_1$ can be interpreted as the mean distance between the vortex lines. Inserting (7.2) for the dephasing length we find in leading order

$$L \sim Q^2 \sim (v_s - v_n)^2$$

(7.7)

so that the vortex density increases with the square of the counterflow velocity. Eq. (7.7) agrees with the vortex density proposed by Vinen and with the ansatz of Gorter and Mellink for the mutual-friction force. Furthermore, in leading order we find that the temperature dependence of the vortex density is given by $L \sim (T_\lambda - T)^{2(d-2)/d}$.

The average thermal resistivity (7.4) of the vortex tangle can be expressed in terms of the Vinen coefficient $B$ and the vortex density $L$. For this purpose we insert (7.3) into (7.4), replace $\Omega = (\pi h/m_4) L$, and obtain

$$\rho_T = \frac{2}{3} \frac{\xi^{d-2}}{g_0 k_B A_d} \frac{8 u[\tau]}{(-\zeta)} \frac{\pi h}{m_4} BL$$

(7.8)

i.e. $\rho_T \sim BL$. Thus, there are two sources which determine the magnitude of the thermal resistivity: the dissipation effects of a single vortex line described by $B$ and the density of the vortices $L$. 

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In Ref. we found that for a homogeneous heat current \( Q \) in superfluid \( ^4\text{He} \) our approach considerably overestimates the dissipation by the vortex lines. Our theory predicts a Gorter-Mellink coefficient \( A \) which is about 20 times larger than observed in the experiments. This disagreement was confirmed recently by a measurement of the thermal resistivity \( \rho_T \) in superfluid \( ^4\text{He} \) close to \( T_\lambda \). We may now ask the question whether this large discrepancy arises due to the dissipation of the single vortex lines described by \( B \) or by the vortex density \( L \). Rotating superfluid \( ^4\text{He} \) is the well suited system for a test of our approach to answer this question. Since the vortex density \( L = (m_4/\pi\hbar)\Omega \) is fixed and precisely known for this system, any observed discrepancy of the dissipation in rotating superfluid \( ^4\text{He} \) must be due to the Vinen coefficient \( B \). In Sec. we have compared our present result for \( B \) with the result of the previous theory. While our present approach yields the correct critical temperature dependence, the values of \( B \) are smaller by a factor of 2 than expected. Thus, for a single vortex line our approach predicts a two times smaller dissipation than expected. On the other hand, for the heat transport at nonzero currents \( Q \) our approach predicts a 20 times larger dissipation. This large discrepancy cannot be caused by the Vinen coefficient \( B \). Rather, Eq. implies that in Ref. our approach predicts a 40 times larger vortex density \( L \). Thus, we conclude that for superfluid \( ^4\text{He} \) in the presence of a heat current \( Q \) our theory considerably overestimates the density of the vortex lines \( L \), while for the dissipative effect of a single vortex line described by \( B \) the correct order of magnitude is obtained.

The discrepancy may possibly be explained in the following way. The Hartree approximation implies that our approach effectively considers the generalized model \( F \) with \( n = \infty \) complex order parameters. On the other hand, real liquid \( ^4\text{He} \) is described by model \( F \) with \( n = 1 \) complex order parameter. Thus, the discrepancy may be related to the difference of the physics between \( n = \infty \) and \( n = 1 \). For \( n = 1 \) a homogeneous heat current \( Q \) and the related superfluid-normal-fluid counterflow represent a metastable state, which relaxes only by creation of vortices. For this purpose, energy barriers must be overcome, so that the vortex-creation rate, the vortex density \( L \), and hence the dissipation are strongly suppressed. On the other hand for \( n = \infty \) the heat current \( Q \) is unstable and vortices are created without an energy barrier. Consequently, in this case the dissipation, the vortex-creation rate, and the vortex density \( L \) are not suppressed. Thus, the dissipation of the heat current and the vortex density may be considerably larger for \( n = \infty \) (our theoretical approach) than for \( n = 1 \) (real superfluid \( ^4\text{He} \)). The difference may be a large factor because the energy barriers occur in the argument of an exponential function. Thus, the quantitative discrepancy by a factor of about 20 or 40 can be explained in this way.

### VIII. CONCLUSIONS

Our recently developed approach the Hartree approximation combined with the renormalization-group theory for model \( F \), can be applied also to rotating \( ^4\text{He} \) close to the superfluid transition. For nonzero rotation frequencies \( \Omega \) our theory predicts that all physical quantities are smooth and round near \( T_\lambda \). The superfluid transition is a smooth crossover located at a temperature \( T_\lambda(\Omega) \) which is not sharply defined and shifted to lower temperatures by \( \Delta T_\lambda(\Omega) = T_\lambda(\Omega) - T_\lambda = -M_\lambda T_\lambda(2m_4^2\hbar^4/\hbar^4)^{1/2} \) where \( M_\lambda \approx 1.2 \). For the rotation frequencies \( \Omega \approx 2\pi s^{-1} \) of realistic experiments, we find \( \Delta T_\lambda(\Omega) \approx -25 \text{ nK} \) which is very small. Thus, to observe the influence of the rotation on the superfluid transition, experiments must be performed with a temperature resolution of a few nano Kelvins. Since on earth the gravity does not allow this temperature resolution these experiments must be performed in a microgravity environment in space.

In superfluid \( ^4\text{He} \) for \( T < T_\lambda \) the thermal resistivity and the dissipation of the heat current strongly depend on the direction of the heat flow related to the rotation axis. For a perpendicular heat flow dissipation is caused by the motion of the vortex lines in the superfluid-normal-fluid counterflow. On the other hand, for a parallel heat flow the thermal resistivity is considerably smaller and caused by fluctuation effects.

From the thermal resistivities of the perpendicular heat flow we extract the Vinen coefficients \( B \) and \( B' \) in the limit \( \Omega \to 0 \). In this way we find that model \( F \) includes the mutual friction between the superfluid and the normal-fluid component, which originally was proposed by Hall and Vinen on phenomenological grounds. While our theory does not treat the vortices microscopically, we nevertheless obtain dissipation effects due to vortices. Consequently, our approach includes the vortices indirectly.

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### APPENDIX: CALCULATION OF THE GREEN’S FUNCTION

The equal time Green’s function \( G(\mathbf{r}, \mathbf{r}') \) is defined by Eq. For the renormalization we need additionally the relations \( \Gamma_0 = Z_T^{-1} \Gamma \) and \( \psi = Z_T^{1/2} \psi_{\text{ren}} \) where, however, in Hartree approximation it is \( Z_T = 1 \) and \( Z_{\psi} = 1 \). Thus, from Eq. we obtain the renormalized Green’s function

\[
G(\mathbf{r}, \mathbf{r}') = 4\Gamma' \int_0^\infty d\alpha \ e^{\alpha A} e^{\alpha B} \delta(\mathbf{r} - \mathbf{r}') .
\]
By using the relations (2.31), (2.32), and $\Delta r_0 = Z_\rho \Delta r$, in the operators (2.9) and (2.10) we replace the bare parameters by the renormalized counterparts. All $Z$ factors cancel except $Z_\rho$, which, however, is unity here. Thus, from (2.9) and (2.10) we obtain

\[
A = -\{\Gamma[r_1 - (\nabla - i k)^2] - i(g/2\alpha)\Delta r\}, \quad (A2)
\]

\[
B = -\{\Gamma^*[r_1 - (\nabla - i k)^2] + i(g/2\alpha)\Delta r\}, \quad (A3)
\]

In the following we must figure out, how the operators $e^{\alpha A}$ and $e^{\alpha B}$ act on the delta function $\delta(r - r')$ to evaluate the integrand in (A1).

We derive an explicit integral formula for the Green’s function for rotating $^4$He in the presence of an infinitesimal heat current $Q$ and an infinitesimal temperature gradient $\nabla T$. For the effective parameters $r_1$ and $\Delta r$ we use the ansatz

\[
r_1 = a_1 + b_1 r, \quad (A4)
\]

\[
\Delta r = a + b r, \quad (A5)
\]

where $a_1$, $a$ are constants and $b_1$, $b$ are infinitesimal vectors which are related to the temperature gradient $\nabla T$. We evaluate the Green’s function $[A1]$ up to first order in $b_1$ and $b$. For this purpose we decompose

\[
A = A_0 + \Delta A, \quad (A6)
\]

\[
B = B_0 + \Delta B, \quad (A7)
\]

where

\[
A_0 = \{-\{\Gamma[a_1 - (\nabla - i k)^2] - i(g/2\alpha)a\}, \quad (A8)
\]

\[
B_0 = \{-\{\Gamma^*[a_1 - (\nabla - i k)^2] + i(g/2\alpha)a\}, \quad (A9)
\]

and

\[
\Delta A = -[\Gamma b_1 - i(g/2\alpha) b] r, \quad (A10)
\]

\[
\Delta B = -[\Gamma^* b_1 + i(g/2\alpha) b] r. \quad (A11)
\]

We expand the operator $e^{\alpha A} = e^{\alpha A_0 + \alpha \Delta A}$ up to first order in $\Delta A$ according to

\[
e^{\alpha A} \approx \left\{1 + \int_0^1 d\lambda \ e^{\alpha A_0(\alpha \Delta A)} e^{-\lambda \alpha A_0}\right\} e^{\alpha A_0}
\]

\[
= \left\{1 + \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{(n+1)!} [A_0, \ldots [A_0, \Delta A] \ldots ]\right\} e^{\alpha A_0}
\]

(A12)

where the number of operators $A_0$ in the multiple commutator is $n$. Analogously, we expand $e^{\alpha B} = e^{\alpha B_0 + \alpha \Delta B}$ up to first order in $\Delta B$ according to

\[
e^{\alpha B} \approx e^{\alpha B_0} \left\{1 + \int_0^1 d\lambda \ e^{-\lambda \alpha B_0(\alpha \Delta B)} e^{\lambda \alpha B_0}\right\}
\]

\[
= e^{\alpha B_0} \left\{1 + \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{(n+1)!} [\Delta B, B_0, \ldots B_0] \ldots \right\}
\]

(A13)

so that $\Delta A = -\Gamma Sr$ and $\Delta B = -i^*S^*r$. The rotation frequency $\Omega = \Omega e_z$ is incorporated via the wave vector $k$, which is defined by (2.4) and which represents the uniform rotation of the normal-fluid component. Eventually, we obtain the operators

\[
e^{\alpha A} = \{1 - a \Gamma Sr - (\alpha \Gamma)^2(e_z S)(e_z(\nabla - i k))
\]

\[
+ \frac{1}{2} [\text{ch}(2\Gamma \alpha / \ell^2) - 1] \ell^4 (e_z \times (e_z \times S)(\nabla - i k))
\]

\[
- \frac{1}{2} [\text{sh}(2\Gamma \alpha / \ell^2) - (2\Gamma \alpha / \ell^2)] \ell^4 (e_z \times S)(\nabla - i k)\}
e^{\alpha A_0} \quad (A16)
\]

and

\[
e^{\alpha B} = \alpha B_0 \{1 - a \Gamma^* S^* r + (\alpha \Gamma)^2(e_z S^*)(e_z(\nabla - i k))
\]

\[
- \frac{1}{2} [\text{ch}(2\Gamma^* \alpha / \ell^2) - 1] \ell^4 (e_z \times (e_z \times S^*))(\nabla - i k)
\]

\[
- \frac{1}{2} [\text{sh}(2\Gamma^* \alpha / \ell^2) - (2\Gamma^* \alpha / \ell^2)] \ell^4 (e_z \times S^*)((\nabla - i k)\}
\]

(A17)

where $\ell = (h/2m_\Omega)^{1/2}$. Now, we insert these operators into the formula for the Green’s function $[A1]$ and obtain

\[
G(r, r') = 4\Gamma' \int_0^\infty d\alpha \{\ldots \} e^{\alpha A_0} e^{\alpha B_0} \{\ldots \} \delta(r - r') .
\]

(A18)

where the first curved brackets $\{\ldots \}$ are identified by the expressions in the curved brackets in (A16) and the second curved brackets $\{\ldots \}$ are identified by the expressions in the curved brackets in (A17). Because of $[A_0, B_0] = 0$ we find

\[
e^{\alpha A_0} e^{\alpha B_0} = e^{\alpha (A_0 + B_0)} = e^{-\alpha 2\Gamma'[a_1 - (\nabla - i k)^2]} .
\]

(A19)

In (A18) in the second curved brackets the operator $\nabla$ acts directly on the delta function so that we can use the identity

\[
\nabla \delta(r - r') = -\nabla' \delta(r - r') .
\]

(A20)

Analogous identities are valid for $r$ and $k$. Thus, we obtain the Green’s function

\[
G(r, r') = 4\Gamma' \int_0^\infty d\alpha \{\ldots \} \{\ldots \}'
\]

\[
\times e^{-\alpha 2\Gamma'[a_1 - (\nabla - i k)^2]} \delta(r - r')
\]

(A21)

where in the second curved brackets $\{\ldots \}'$ the operators $\nabla$, $r$, and $k$ are replaced by $-\nabla'$, $r'$, and $k'$, respectively.
(K' is defined by (2.4) with r replaced by r'). We evaluate the product \{\cdots\{\cdots\}' up to the terms first order in S and S'. Furthermore, we substitute 2F'α → α in the integral. Then, from (A21) we obtain the Green’s function

\[ G(r, r') = 2 \int_0^\infty \alpha \left\{ 1 - \frac{\alpha}{2\Gamma^*} \right\} \left[ \left( -\frac{\alpha}{2\Gamma^*} \right)^2 (e_z S)(e_z(\nabla - ik)) - \left( -\frac{\alpha}{2\Gamma^*} \right)^2 (e_z S^*)(e_z(\nabla' + ik')) \right] + \frac{1}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S \right) (\nabla - ik) \right] + \frac{1}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S^* \right) (\nabla' + ik') \right] - \frac{i}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S^* \right) (\nabla - ik) \right] + \frac{i}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S^* \right) (\nabla' + ik') \right] \] 

\[ \times e^{-\alpha[1 - (\nabla - ik)^2]} \delta(r - r') \] 

(A22)

Finally, we use the formula \( e^x \approx \{1 + x\} \) for small \( x \) to combine the terms in the curved brackets into an exponential function. Thus, eventually we obtain

\[ G(r, r') = e^{-i\epsilon x}(r \times r')/2\ell^2 \frac{2}{(4\pi\ell^2/2)^{d/2}} \int_0^\infty \alpha/\ell^2 \left\{ 1 - \frac{\alpha}{2\Gamma^*} \right\} \left[ \left( -\frac{\alpha}{2\Gamma^*} \right)^2 (e_z S)(e_z(\nabla - ik)) - \left( -\frac{\alpha}{2\Gamma^*} \right)^2 (e_z S^*)(e_z(\nabla' + ik')) \right] + \frac{1}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S \right) (\nabla - ik) \right] + \frac{1}{2} \left[ \left( \frac{\Gamma}{\Gamma' \ell^2} - 1 \right) \epsilon \left( e_z + e_z S^* \right) (\nabla' + ik') \right] \] 

\[ \times e^{-\alpha[1 - (\nabla - ik)^2]} \delta(r - r') \] 

(A23)

\[ \frac{\ell^2}{2} \left( 1 - \frac{\alpha/\ell^2}{\theta(\alpha/\ell^2)} \right) (e_z \times b_1) + \frac{1}{\theta(\alpha/\ell^2)} \frac{F}{2w'} \left( \left[ \left( \frac{\Gamma}{\ell^2} - \cos \frac{w''}{w^2} \right) (e_z \times (e_z \times b_1)) + \left[ \frac{w''}{w} \left( \frac{\Gamma}{\ell^2} - \sin \frac{w''}{w^2} \right) (e_z \times (e_z \times b_1)) \right] \right) \] 

\[ \times e^{iK(r - r')} \] 

(A24)

where

\[ \tilde{r}_1 = a_1 + b_1(r + r')/2 \] 

(A25)

and

\[ K = \alpha \frac{F}{8\gamma w'} e_z(e_z \cdot b) \]

\[ + \frac{\ell^2}{2} \left( 1 - \frac{\alpha/\ell^2}{\theta(\alpha/\ell^2)} \right) (e_z \times b_1) + \frac{1}{\theta(\alpha/\ell^2)} \frac{F}{2w'} \left( \left[ \left( \frac{\Gamma}{\ell^2} - \cos \frac{w''}{w^2} \right) (e_z \times (e_z \times b_1)) + \left[ \frac{w''}{w} \left( \frac{\Gamma}{\ell^2} - \sin \frac{w''}{w^2} \right) (e_z \times (e_z \times b_1)) \right] \right) \] 

\[ + \frac{w''}{w} \left( \frac{\Gamma}{\ell^2} - \sin \frac{w''}{w^2} \right) (e_z \times (e_z \times b_1)) \] 

(A27)

\( \rho^2 \) and \( \zeta^2 \) are defined below (A23). Eq. (A23) together with (A26) and (A27) is the final result for the Green’s function \( G(r, r') \), which is valid for infinitesimal gradients \( b_1 = \nabla r_1 \) and \( b = \nabla (\Delta r) \) up to first order in \( b_1 \) and \( b \). For \( K = 0 \) and \( \tilde{r}_1 = r_1 = \text{const.} \) Eq. (A23) represents the Green’s function for rotating \( 4\text{He} \) in thermal equilibrium.

The wave vector \( K \) defined in (A27) is the sum of the two terms of different nature which are orthogonal to each other. The first term in (A27) represents the contribution parallel to the rotation axis and does not depend on the rotation frequency \( \Omega \). The second term, which is given by the remaining terms in (A27), is in the \( xy \) plane perpendicular to the rotation axis and depends on \( \Omega \) via the characteristic length \( \ell \). In the limit \( \Omega \to 0 \) and \( \ell \to \infty \) the wave vector reduces to \( K \to \alpha (F/8\gamma w') b \). This wave vector is well known from the theory of heat transport in nonrotating \( 4\text{He} \) (see e.g. (A28) in the second paper of Ref. [4]).
While the scalar and vector products involving $\mathbf{e}_z$ are defined in $d = 3$ dimensions, our formula for $G(\mathbf{r}, \mathbf{r}')$ is valid for arbitrary $d$ dimensions. The scalar and vector products in (A25) and (A27) must be replaced by appropriate generalized expressions. This, however, is straightforward: $-\mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{b})$ is replaced by the projection of $\mathbf{b}$ onto the $xy$ plane and $\mathbf{e}_z (\mathbf{e}_z \cdot \mathbf{b})$ is replaced by the projection of $\mathbf{b}$ onto the subspace perpendicular to the $xy$ plane. Furthermore, $\mathbf{e}_z \times \mathbf{b}_1$ and $\mathbf{e}_z \times \mathbf{b}$ are replaced by the projections of $\mathbf{b}_1$ and $\mathbf{b}$ onto the $xy$ plane combined with a rotation of angle $\pi/2$.

The density $n_s$ and the superfluid current $\mathbf{J}_s$ are obtained by inserting the Green’s function (A25) into (2.12) and (2.19), respectively. We find

$$n_s = \frac{2}{(4\pi)^{d/2}} \int_0^\infty \frac{d\alpha}{\alpha^{d/2}} \frac{\alpha/\ell^2}{\sinh(\alpha/\ell^2)} e^{-\alpha r_1} ,$$  \hspace{1cm} (A28)

and

$$\mathbf{J}_s = \frac{2}{(4\pi)^{d/2}} \int_0^\infty \frac{d\alpha}{\alpha^{d/2}} \frac{\alpha/\ell^2}{\sinh(\alpha/\ell^2)} \mathbf{K} e^{-\alpha r_1} ,$$  \hspace{1cm} (A29)

where $\mathbf{K}$ is defined in (A27). We replace

$$\frac{1}{(4\pi)^{d/2}} = -\frac{1}{\epsilon} \frac{A_d}{\Gamma(-1+\epsilon/2)}$$  \hspace{1cm} (A30)

where $A_d$ is the geometrical factor which occurs in the renormalization equations (2.22), (2.31), and (2.32). Furthermore, we substitute $\alpha/\ell^2 \to v$. Thus, eventually we obtain the formulas (2.25) and (2.37) with the integrals (2.26) and (2.38)-(2.40).

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1. R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge Univ. Press, Cambridge 1991).
2. B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. Lett. **32**, 1289 (1974); Phys. Rev. B **13**, 1299 (1976).
3. R. P. Feynman, in: Progress in Low Temperature Physics 1, edited by C. J. Gorter (North Holland, Amsterdam 1955), Chap. 2.
4. R. Haussmann, J. Low Temp. Phys. **114**, 1 (1999) and submitted to Phys. Rev. B (1998).
5. H. E. Hall and W. F. Vinen, Proc. R. Soc. A **238**, 204 and 215 (1956).
6. R. Haussmann, Z. Phys. B **87**, 247 (1992).
7. C. J. Gorter and J. H. Mellink, Physica **15**, 285 (1949).
8. W. F. Vinen, Proc. R. Soc. A **240**, 114 and 128 (1957); **242**, 493 (1957); **243**, 400 (1958).
9. G. Ahlers, Phys. Rev. **171**, 275 (1968).
10. H. A. Snyder and D. M. Lynekin, Phys. Rev. **147**, 131 (1966).
11. P. Lucas, J. Phys. C **3**, 1180 (1970).
12. P. Mathieu, A. Serra, and Y. Simon, Phys. Rev. B **14**, 3753 (1976).
13. V. Dohm, Z. Phys. B **60**, 61 (1985); Z. Phys. B **61**, 193 (1985); Phys. Rev. B **44**, 2697 (1991).
14. R. Schloms and V. Dohm, Nucl. Phys. B **328**, 639 (1989).
15. W. Y. Tam and G. Ahlers, Phys. Rev. B **32**, 5932 (1985).
16. J. A. Lipa, D. R. Swanson, J. A. Nissen, T. C. P. Chui, and U. E. Israelsson, Phys. Rev. Lett. **76**, 944 (1996).
17. R. Haussmann and V. Dohm, Phys. Rev. B **46**, 6361 (1992).
18. H. Baddar, G. Ahlers, K. Kuehn, and H. Fu, private communication (1998).