Regularity of the extremal solution for some elliptic problems with singular nonlinearity and advection

Xue Luo\textsuperscript{1}, Dong Ye\textsuperscript{2}, Feng Zhou\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, East China Normal University, 200241 Shanghai, P.R. China. E-mail: luoxue0327@163.com
\textsuperscript{2}LMAM, UMR 7122, Universit\'e Paul Verlaine de Metz, 57045 Metz Cedex 1, France. E-mail: dong.ye@univ-metz.fr
\textsuperscript{3}Department of Mathematics, East China Normal University, 200241 Shanghai, P.R. China. E-mail: fzhou@math.ecnu.edu.cn

Abstract

In this note, we investigate the regularity of the extremal solution $u^*$ for the semilinear elliptic equation $-\Delta u + c(x) \cdot \nabla u = \lambda f(u)$ on a bounded smooth domain of $\mathbb{R}^n$ with Dirichlet boundary condition. Here $f$ is a positive nondecreasing convex function, exploding at a finite value $a \in (0, \infty)$. We show that the extremal solution is regular in the low dimensional case. In particular, we prove that for the radial case, all extremal solutions are regular in dimension two.

Keywords: singular nonlinearity, advection, extremal solution, regularity

1. Introduction

We consider the elliptic problem

\[
\begin{aligned}
-\Delta u + c(x) \cdot \nabla u &= \lambda f(u) \quad \text{in } \Omega, \\
 u &> 0 \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega, \\
\end{aligned}
\tag{P_\lambda}
\]

where $\lambda > 0$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 2$), $c(x)$ is a smooth vector field over $\Omega$ and $f : [0, a) \rightarrow \mathbb{R}_+$ with fixed $a \in (0, \infty)$ satisfies the following condition $(H)$:

$f$ is $C^2$, positive, nondecreasing and convex in $[0, a)$ with $\lim_{t \to a^-} f(t) = \infty$.

In the literature, $f$ is referred to as a \textit{singular nonlinearity}. We say that $u$ is a regular solution if $u \in C^2(\overline{\Omega})$, and we also deal with solutions in the following weak sense.

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Definition 1.1. We say that \( u \) is a weak solution of \((P_\lambda)\) if \( 0 \leq u \leq a \) a.e. in \( \Omega \) such that \( f(u) \, d(x, \partial \Omega) \in L^1(\Omega) \) and
\[
- \int_{\Omega} u \Delta \phi - \int_{\Omega} u \text{div}(\phi c) = \lambda \int_{\Omega} f(u) \phi, \quad \forall \phi \in C^2(\overline{\Omega}) \cap H^1_0(\Omega).
\]
Moreover, \( u \) is a weak super-solution of \((P_\lambda)\) if “=” is replaced by “\( \geq \)” for all nonnegative functions \( \phi \in C^2(\overline{\Omega}) \cap H^1_0(\Omega) \).

Clearly, a weak solution is regular if \( \sup_{\Omega} u < a \). For regular solutions, we introduce a notion of stability.

Definition 1.2. A regular solution \( u \) of \((P_\lambda)\) is said to be stable if the principal eigenvalue of the linearized operator \( L_{u,\lambda,c} := -\Delta + c \cdot \nabla - \lambda f'(u) \) is nonnegative in \( H^1_0(\Omega) \).

Exploiting some ideas in [11, 10], the solvability of \((P_\lambda)\) is characterized by a parameter \( \lambda^* \):

Proposition 1.1. There exists \( \lambda^* \in (0, \infty) \) such that

- For \( 0 < \lambda < \lambda^* \), the problem \((P_\lambda)\) has a minimal solution \( u_\lambda \), \( u_\lambda \) is regular and the map \( \lambda \mapsto u_\lambda \) is increasing. Moreover, \( u_\lambda \) is the unique stable solution of \((P_\lambda)\).

- For \( \lambda = \lambda^* \), \((P_{\lambda^*})\) admits a unique weak solution \( u^* := \lim_{\lambda \to \lambda^*} u_\lambda \), called the extremal solution.

- For \( \lambda > \lambda^* \), \((P_\lambda)\) admits no weak solution.

Here the minimal solution means that \( u_\lambda \leq v \) for any solution \( v \) of \((P_\lambda)\). We remark immediately a close similarity between \((P_\lambda)\) and the Emden-Fowler equation with superlinear regular nonlinearity, that is
\[
- \Delta u = \lambda g(u) \quad \text{in} \quad \Omega \subset \mathbb{R}^n; \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
with \( \lambda > 0 \) and \( g : [0, \infty) \to (0, \infty) \) satisfies
\[
g \text{ is } C^2, \text{ nondecreasing, convex and } \lim_{t \to \infty} \frac{g(t)}{t} = \infty.
\]

In fact, there exists also a critical parameter \( \overline{\lambda} \in (0, \infty) \) for (1.1) such that all conclusions in the above proposition hold true by replacing \( \lambda^* \) by \( \overline{\lambda} \) (see [2, 11]). It is well known by classical examples as \( g(u) = (1 + u)^p \) with \( p > 1 \) or \( g(u) = e^u \), the extremal solution \( u^* \) can be either a regular solution or a real weak solution in the distribution sense with \( \sup_{\Omega} u = \infty \).

For general nonlinearity \( g \) satisfying (1.2), the regularity of the extremal solution \( u^* \) to (1.1) is obtained by Nedev [13] for any bounded smooth domain \( \Omega \subset \mathbb{R}^n \) if \( n = 2, 3 \); by Cabré [4] for convex domains in \( \mathbb{R}^4 \); and for radial symmetry case in \( \mathbb{R}^n \) with \( n \leq 9 \) by Cabré & Capella [5]. In [17], it is proved that, under mild condition on \( g \), the extremal solution \( u^* \) is regular for any smooth bounded domain \( \Omega \subset \mathbb{R}^n \) if \( n \leq 9 \).
We can ask the same question about the problem \((P_\lambda)\): For \(f\) verifying \((H)\), is it true that the extremal solution to \((P_\lambda)\) is regular for general vector field \(c\) and general domain \(\Omega \subset \mathbb{R}^n\) with low dimensions \(n\)? We will partly answer this question. It is worthy to mention that for studying the explosion phenomena in a flow, Berestycki et al. [1] have considered the problem \((P_\lambda)\) with a general source \(f\) verifying \((1.2)\).

Without loss of generality, fix \(a = 1\) in the sequel. The problem \((P_\lambda)\) can be linked to equation \((1.1)\) up to the transformation \(v = -\ln(1 - u)\). In fact, let \(u\) solve \((P_\lambda)\), \(v\) verifies then

\[
\begin{cases}
-\Delta v + |\nabla v|^2 + c(x) \cdot \nabla v = \lambda e^v f(1 - e^{-v}) := \lambda g(v) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

\((Q_\lambda)\)

Therefore \(g\) verifies \((1.2)\) and \(v^* = -\ln(1 - u^*)\) is the extremal solution for the problem \((Q_\lambda)\). Thus the regularity of \(u^*\) is equivalent to the boundedness of \(v^*\), however the situation could be very different with the presence of advection terms (see [7, 16]). In last decade, a model describing the steady state of MEMS (Micro-Electro-Mechanical Systems) device given by Pelesko and Bernstein in [14], has drawn many attentions (see [9] and the references therein).

\[
-\Delta u = \frac{\lambda}{(1 - u)^2} \quad \text{in } \Omega \subset \mathbb{R}^n; \quad u = 0 \quad \text{on } \partial \Omega.
\]

More generally, many precise studies have been done for the singular nonlinearities with negative exponent \(f(u) = (1 - u)^{-p} \quad (p > 0)\) in the advection-free situation, i.e. \(c \equiv 0\). In that case, when \(\Omega\) is moreover the unit ball in \(\mathbb{R}^n\), it is known that \(u^*\) is regular if and only if (see [12, 10])

\[
n < n_p := 2 + \frac{4p}{p + 1} + 4\sqrt{\frac{p}{p + 1}},
\]

\((1.3)\)

Tending \(p \to 0^+\) in \((1.3)\), we see that \(n_p \to 2\). Therefore we cannot expect in general better than dimension two to claim the regularity of \(u^*\).

For the radial case of \((P_\lambda)\), equally when \(\Omega\) is a ball and \(c(x)\) is the gradient of a smooth radial function, \(u_\lambda\) is radial by uniqueness of the minimal solution. We obtain the following optimal results which are new even for the advection-free case.

**Theorem 1.1.** Assume that \(n = 2, \Omega = B_1\). Let \(\gamma\) is a smooth radial function and \(c = \nabla \gamma\), then the extremal solution \(u^*\) is regular for any \(f\) satisfying \((H)\).

**Theorem 1.2.** For any \(f\) satisfying \((H)\), \(\Omega = B_1\) and smooth radial function \(\gamma\), there exists \(C > 0\) such that for all \(\lambda \in (0, \lambda^*)\)

\[
|u_\lambda'(r)| \leq \begin{cases}
Cr^{-1} & \text{if } n \geq 10; \\
Cr^{-\frac{n+1}{2}+1+\sqrt{n-1}} & \text{if } 3 \leq n \leq 9;
\end{cases} \quad \forall \ r = |x| \in (0, 1]
\]

where \(|\cdot|\) is the Euclidean norm in \(\mathbb{R}^n\).
Remark 1.1. The above estimates are optimal. In fact, when $f(u) = (1 - u)^{-p}$, $p > 0$, \( \Omega = B_1 \) and $c \equiv 0$, it is well known that $u^*(x) = 1 - r^{2p+1}$ if $n \geq n_p$ with $n_p$ given in (1.3), and we have

\[ n \geq n_p \quad \text{iff} \quad n \geq 10 \text{ or } 3 \leq n \leq 9, \quad \frac{2}{p+1} \leq \frac{n}{2} + 2 + \sqrt{n-1}. \]

But is the extremal solution $u^*$ of (P) regular with general singular nonlinearity $f$ verifying (H), vector field $c$ and smooth bounded domains in \( \mathbb{R}^2 \)? The answer is affirmative under some additional mild condition on $f$.

Theorem 1.3. Assume that $f$ satisfies conditions (H) and the additional conditions,

\[(H1)\quad \limsup_{t \to 1^-} \frac{f(t)}{f'(t)(1-t)\ln^2(1-t)} < 1 \]

and

\[(H2)\quad \liminf_{t \to 1^-} \frac{f(t)f''(t)}{f^2(t)} > 0. \]

Then $u^*$ is regular solution to (P) if $n = 2$, i.e. \( \Omega \subset \mathbb{R}^2 \).

Under more precise conditions on the growth of $f$, the extremal solution can be showed to be regular in some higher dimensions.

Theorem 1.4. Let $f$ verify (H) and $g(v) = e^v f(1 - e^{-v})$. Assume that $g$ satisfies

\[(H3)\quad \liminf_{t \to \infty} \frac{g'(t)}{g(t)} = 1 + \delta > 1 \]

and

\[(H2)\quad \liminf_{t \to \infty} \frac{g''(t)g(t)}{g^2(t)} = \mu > \frac{1}{1 + \delta}. \]

Then $v^* = -\ln(1 - u^*)$ is bounded (so $u^*$ is regular) when

\[ n < 2 + \frac{4\delta}{1 + \delta} + \frac{\sqrt{8\delta(\mu + \mu\delta - 1)}}{1 + \delta}. \]

Consequently, if $\mu\delta > 1$, $u^*$ is regular for all $n \leq 6$. Furthermore, if we can tend $\delta$ to $\infty$, which means $g = o(g')$ near $\infty$, then $u^*$ is regular for $n < 6 + 4\sqrt{\mu}$ with any $\mu > 0$. However, we can never have $\mu > 1$, since otherwise $g$ blows up at finite value and contradicts (1.2), so the best result we can expect is for $n \leq 9$. For example, if $f(u) = e^{\frac{1}{1-\pi}}$, then $g(v) = e^{v + v^*}$ verifies $\delta = \infty$ and $\mu = 1$.

Theorem 1.5. Let $f$ verify (H) and $g(v) = e^v f(1 - e^{-v})$. Assume that $g = o(g')$ near $\infty$. Rewrite $g(t) = g(0) + te^{h(t)}$ in $(0, \infty)$, suppose there exists $t_0 > 0$ such that $t^{2}h'(t)$ is nondecreasing for $t \geq t_0$, then for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ with $n \leq 9$, $u^*$ is a regular solution.
Furthermore, when \( g = o(g') \) near \( \infty \), the condition \((\widetilde{H}2)\) is just equivalent to \((H2)\), since

\[
\frac{f''(t)f(t)}{f'(t)^2} = \left(\frac{g'' - g'}{g' - g}\right)(s) = \left(\frac{g'g - g}{g'}\right) \times \left(1 - \frac{g}{g'}\right)^{-2}(s), \quad \forall \ t = 1 - e^{-s}.
\]

It is also easy to see that \((H3)\) is equivalent to the condition

\[
\liminf_{t \to 1^-} \frac{f'(t)(1-t)}{f(t)} = \delta > 0.
\]

If the equality holds for the whole limit, we have the following optimal result. The case \( f(u) = (1-u)^{-2} \) was obtained in [7] with a different argument.

**Theorem 1.6.** Assume that

\[
\lim_{u \to 1^-} \frac{f'(u)(1-u)}{f(u)} = p > 0. \tag{1.5}
\]

Then \( u^* \) is a regular solution if \( n < n_p \) where \( n_p \) is defined in \((1.3)\).

One of the main difficulties here is due to the vector field \( c(x) \). When \( c \neq 0 \), the operator \(-\Delta + c \cdot \nabla\) is not self-adjoint, we use ideas from [7] to get some energy estimates. However if \( c \) is a gradient, say \( c = -\nabla \gamma \) in \( \Omega \), then \(-\Delta + c \cdot \nabla\) can be rewritten as \( e^{-\gamma}L_\gamma\) where \( L_\gamma = -\text{div}(e^{\gamma}\nabla) \) is a self-adjoint operator. In that case, \((P_\lambda)\) admits a variational structure and we can expect more precise estimates of minimal solutions \( u_\lambda \), as in the radial case.

The paper is organized as follows: In section 2, we prove quickly Proposition 1.1 and show some general consequences of the stability of \( u_\lambda \). The section 3 is devoted to the proof of Theorems 1.3 to 1.6 for general domains. In section 4, we discuss the radial case. The norm \( \| \cdot \|_q \) denotes always the standard \( L^q \) norm for any \( q \in [1, \infty] \). The capital letter \( C \) denotes a generic positive constant independent of \( \lambda \), it could be changed from one line to another.

2. Preliminaries

As mentioned above, \(-\Delta + c \cdot \nabla\) is not a self-adjoint operator for general vector field \( c \). However using Lemma 1 in [7], we have a kind of Hodge decomposition, which tells us that for any vector field \( c \in C^\infty(\overline{\Omega}, \mathbb{R}^n) \), there exist a smooth scalar function \( \gamma \) and a vector field \( b \in C^\infty(\overline{\Omega}, \mathbb{R}^n) \) such that

\[
c = -\nabla \gamma + b \quad \text{and} \quad \text{div}(e^\gamma b) = 0 \quad \text{in} \ \overline{\Omega}. \tag{2.1}
\]

Therefore the problem \((P_\lambda)\) can be rewritten as

\[
-\text{div}(e^\gamma \nabla u) + e^\gamma b \cdot \nabla u = \lambda e^\gamma f(u) \quad \text{in} \ \Omega. \tag{P'_\lambda}
\]

On the other hand, we don’t have a suitable variational characterization in general to use the stability assumption. Fortunately, we can adopt an energy inequality as in [7], which is derived from a generalized Hardy inequality of [8].
Proposition 2.2. Let $u_\lambda$ be minimal solution of $(P_\lambda)$. For any $1 \leq \beta < 2$, we have
\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \psi^2 \leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \psi|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)} \int_{\Omega} e^\gamma \psi^2, \quad \forall \psi \in H^1_0(\Omega). \tag{2.2}
\]
where $b$ is the vector field in (2.1), $\|b\|_\infty = \max_{x \in \Omega} |b(x)|$.

**Proof.** We use a Hardy type inequality given by Theorem 2 in [7], which says that for a positive principal eigenfunction $\varphi$ of $L_{u_\lambda,\lambda}$, for $\beta \in [1, 2)$ and any $\psi \in H^1_0(\Omega)$,
\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \psi^2 \leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \psi|^2 + \int_{\Omega} \left[-\frac{2-\beta}{2} \frac{|\nabla \varphi |^2}{\varphi^2} + \frac{b \cdot \nabla \varphi}{\varphi} \right] e^\gamma \psi^2.
\]
By Cauchy-Schwarz inequality, it is easy to see
\[
-\frac{2-\beta}{2} \frac{|\nabla \varphi |^2}{\varphi^2} + \frac{b \cdot \nabla \varphi}{\varphi} \leq \frac{\|b\|_\infty^2}{2(2-\beta)},
\]
so we are done. \qed

Another main ingredient of our approach is just the transformation $v = -\ln(1-u)$. Let $\phi$ and $\xi$ be nonnegative $C^1$ functions satisfying $\phi(0) = \xi(0) = 0$ and $\xi' = \phi^2$. Define $v_\lambda = -\ln(1-u_\lambda)$ and $g(v_\lambda) = e^{u_\lambda} f(1-e^{-v_\lambda})$. Using $(Q_\lambda)$, we get $-\text{div}(e^\gamma \nabla v_\lambda) + e^\gamma b \cdot \nabla v_\lambda \leq \lambda e^\gamma g(v_\lambda)$ in $\Omega$. Let $\psi = \phi(v_\lambda)$ in (2.2), $\forall \lambda \in (0, \lambda^*)$,
\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(v_\lambda)
\leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \phi(v_\lambda)|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)} \int_{\Omega} e^\gamma \phi^2(v_\lambda)
= \frac{2}{\beta} \int_{\Omega} e^\gamma \nabla \xi(v_\lambda) \nabla v_\lambda + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda)
= -\frac{2}{\beta} \int_{\Omega} \text{div}(e^\gamma \nabla v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda)
\leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) - \frac{2}{\beta} \int_{\Omega} e^\gamma b \cdot \xi(v_\lambda) \nabla v_\lambda + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda)
= \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda).
\]

The last line is due to $\text{div}(e^\gamma b) = 0$. We claim then

**Proposition 2.3.** Let $1 \leq \beta < 2$. For any $\lambda \in (0, \lambda^*)$ and any nonnegative $C^1$ test functions $\phi, \xi$ verifying $\phi(0) = \xi(0) = 0$ and $\xi' = \phi^2$, there hold
\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(v_\lambda) \leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda) \tag{2.3}
\]
and
\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(u_\lambda) \leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma f(u_\lambda) \xi(u_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(u_\lambda). \tag{2.4}
\]
The proof of (2.4) is completely similar to (2.3) but using \((P_\lambda')\) instead of \((Q_\lambda)\).

We also make use the following behavior of \(f\) proved in [18].

**Lemma 2.1.** For any \(f\) verifying \((H)\), we have \(\lim_{t \to 1} f(t)/f'(t) = 0\).

Choose first \(\phi(u) = e^u - 1\) in (2.4), then \(\xi(u) = e^{2u} - 1\) and

\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda) (e^{u_\lambda} - 1)^2 \leq \frac{\lambda}{\beta} \int_{\Omega} e^\gamma f(u_\lambda)(e^{2u_\lambda} - 1) + C\beta \int_{\Omega} e^\gamma (e^{u_\lambda} - 1)^2.
\]

Fix \(\beta \in (1, 2)\). By Lemma 2.1,

\[
\lambda \int_{\Omega} e^\gamma f'(u_\lambda)e^{2u_\lambda} \leq C.
\]

Consequently \(\|f'(u_\lambda)\|_1\) is uniformly bounded, so is \(\|f(u_\lambda)\|_1\). Multiplying \((P_\lambda)\) by \(u_\lambda\),

\[
\int_{\Omega} |\nabla u_\lambda|^2 = \int_{\Omega} \frac{\text{div}(c)}{2} u_\lambda^2 + \lambda \int_{\Omega} f(u_\lambda)u_\lambda \leq C,
\]

which gives

**Proposition 2.4.** The family of minimal solutions \(\{u_\lambda\}_{0 < \lambda < \lambda^*}\) is uniformly bounded in \(H_0^1(\Omega)\).

**Remark 2.1.** As far as we know, it is always an open question whether the similar \(H^1\) energy estimation holds for minimal solutions of (1.1) with general regular nonlinearity satisfying (1.2) and general domain \(\Omega\) when \(n \geq 6\) (see [13] for \(n \leq 5\)). For the advection-free case \(c = 0\), it was proved in [18] that \(u^* \in H^2 \cap H_0^1(\Omega)\) under the condition \((H)\), it is also true for the gradient case \(c = \nabla \gamma\) (see Lemma 4.1).

**Sketches of proof of Proposition 1.1.** We follow the ideas coming from [1, 11, 10]. The main argument is the maximum principle for operators \(-\Delta + c \cdot \nabla\) and \(L_\gamma\) under the Dirichlet boundary condition, we use also the super-sub solution method and monotone iteration.

Let \(w \in H_0^1(\Omega)\) be the regular solution of \(-\Delta w + c \cdot \nabla w = 1\) in \(\Omega\) and fix \(\alpha > 0\) such that \(\alpha \max_{\Omega} w < 1\). It is easy to verify that \(\alpha w\) is a supersolution of \((P_\lambda)\) for \(\lambda > 0\) small enough. As 0 is a subsolution and \(\alpha w > 0\) in \(\Omega\), \((P_\lambda)\) admits a regular solution for \(\lambda > 0\) small enough. As any regular solution \(u\) of \((P_\lambda)\) is also a supersolution for \((P_\mu)\) if \(\mu \in (0, \lambda)\), the set of \(\lambda\) for which \((P_\lambda)\) admits a regular solution is just an interval. Moreover, for these \(\lambda\), using \((H)\) and the monotone iteration \(v_0 = 0\); \(-\Delta v_{n+1} + c \cdot \nabla v_{n+1} = \lambda f(v_n)\) in \(\Omega\) with \(v_{n+1} = 0\) on \(\partial \Omega\) for \(n \in \mathbb{N}\), we get the minimal solution \(u_\lambda = \lim_{n \to \infty} v_n\).

If we suppose that the principal eigenvalue of \(L_{u_\lambda, \lambda, c}\) is negative, we can construct, as in [1] another solution \(v \leq u_\lambda\) using the associated first eigenfunction, this is just impossible by the definition of \(u_\lambda\), hence \(u_\lambda\) is stable. The uniqueness of stable solution comes from Lemmas 2.16 and 2.17 in [8].
Take a positive first eigenfunction $\varphi$ of $L_\gamma$ with the Dirichlet boundary condition, by $(P_\lambda)$,

$$\lambda f(0) \int_\Omega e^\gamma \varphi \leq \int_\Omega \lambda e^\gamma f(u) \varphi = \int_\Omega \lambda_1 (L_\gamma) w \varphi - \int_\Omega \text{div}(e^\gamma b \varphi) u \leq C.$$  

So $\lambda$ is upper bounded. Define the critical threshold $\lambda^*$ as the supremum of $\lambda > 0$ for which $(P_\lambda)$ admits a regular solution, as $u^*$ is the monotone limit of $u_\lambda$ when $\lambda \to \lambda^*$, we deduce that $u^* \in H^1_0(\Omega)$ is a weak solution of $(P_\lambda)$ by Proposition 2.4.

Suppose that $u$ is a weak solution to $(P_\lambda)$. By the monotonicity of $f$, it is easy to verify that for any $\delta > 1$, the function $v = \delta^{-1} u$ is a weak supersolution for $(P_{\lambda/\delta})$, then the monotone iteration will enable us a weak solution $w$ of $(P_{\lambda/\delta})$ satisfying $0 \leq w \leq v \leq \delta^{-1} < 1$. The regularity theory implies then $w$ is a regular solution of $(P_{\lambda/\delta})$. This means that $\lambda/\delta \leq \lambda^*$. Let $\delta$ tend to 1, we get $\lambda \leq \lambda^*$. Therefore, no weak solution exists for $\lambda > \lambda^*$.

The uniqueness of the weak solution can be proved in the very similar way as in [11] using the monotonicity and convexity of $f$ with the strong maximum principle for the operator $-\Delta + c \cdot \nabla$ associated to Dirichlet boundary condition, so we omit the details. □

3. Regularity of $u^*$ for general $c$ and $\Omega$

For proving our results, we will choose suitable functions $\phi$ to apply (2.3) or (2.4). We need also

**Lemma 3.1.** For any $q > n/2$, there exists $C > 0$ such that the solution $v$ of $(Q_\lambda)$ satisfies $0 \leq v \leq C\|g(v)\|_q$ in $\Omega$.

Indeed, let $w$ be the solution of $L(w) := -\Delta w + c \cdot \nabla w = \lambda g(v)$ in $\Omega$ with $w = 0$ on $\partial \Omega$. By regularity theory and Sobolev embedding, $\|w\|_\infty \leq C\|w\|_{W^{2,q}(\Omega)} \leq C'\lambda^*\|g(v)\|_q$ because $q > n/2 \geq 1$. Moreover, as $L(w - v) \geq 0$, the maximum principle implies then $0 \leq v \leq w \leq C\|g(v)\|_q$.

3.1. Proof of Theorem 1.3

For simplicity, we omit the index $\lambda$ for $u_\lambda$ or $v_\lambda$ Let $\phi(u) = \nu = -\ln(1 - u)$ in (2.3), so $\xi(u) = (1 - u)^{-1} - 1$. Fix $\beta \in (1, 2)$ but very close to 2. Repeating the proof of Theorem 2 in [18] with the assumption $(H1)$, there exists $C > 0$ such that

$$\lambda \int_\Omega e^\gamma f(u) \frac{1}{1 - u} < C + CC_\beta \int_\Omega e^\gamma \phi^2(u).$$

As $\phi^2(u) = o(\xi(u)) = o(f\xi)$ when $u \to 1^-$,

$$\lambda \int_\Omega e^\gamma f(u) \frac{1}{1 - u} < C.$$  

Using the equation $(Q_\lambda)$ and $\partial_v v \leq 0$ on $\partial \Omega$,

$$\int_{\Omega} |\nabla v|^2 \geq \lambda \int_{\Omega} e^\nu f(1 - e^{-\nu}) + \int_{\partial \Omega} e^\nu \partial_v d\sigma - \int_{\Omega} c \cdot \nabla v \leq \lambda \int_{\Omega} \frac{f(u)}{1 - u} + C\|\nabla v\|^2 \leq C + C\|\nabla v\|_2.$$
Therefore \( \| \nabla v \|_2 \leq C \), the classical Moser-Trudinger inequality enables us, as \( n = 2 \)
\[
\int_{\Omega} e^{qu} \leq C_q, \quad \forall \ q \geq 1. \tag{3.1}
\]

Take now \( \phi(u) = f(u) - f(0) \) in (2.4), we need to estimate
\[
\zeta(u) := f'(u)\phi(u) - \frac{2}{\beta} \xi(u) = f'(u)\phi(u) - \frac{2}{\beta} \int_0^u f''(s) ds
\]
\[
= f'(u) f(u) - \frac{2}{\beta} \int_0^u f''(s) ds - Cf'(u)
\]
\[
:= I(u) - \frac{2}{\beta} J(u) - Cf'(u).
\]

By \((H2)\), there exists \( \delta > 0 \) such that
\[
I(u) - I(0) = \int_0^u \left[ f''(s) + f''(s)f(s) \right] ds \geq (1 + \delta) J(u) - Cf'(u), \quad \forall \ u \in [0, 1)
\]
Let \( \frac{1}{2+\delta} < \beta < 2 \), we get \( \zeta(u) \geq CI(u) - C \). Asserting this in (2.4),
\[
\lambda \int_{\Omega} e^{\gamma f'(u)f^2(u)} \leq C \int_{\Omega} e^{\gamma f^2(u)} + C.
\]
Consequently, \( \| f'(u)f^2(u) \|_1 \leq C \). By Lemma 2.1 we deduce \( \| f(u) \|_3 \leq C \). Combining with (3.1), \( \| g(v) \|_p \leq C \) for any \( p < 3 \). The proof is completed by Lemma 3.1 as \( n = 2 \). \( \square \)

3.2. Proof of Theorem 1.4

Without loss of generality, we can assume that \( g(0) = 1 \). Let \( \phi(t) = g^\alpha(t) - 1 \) where \( \alpha > 0 \) is a constant to be determined later. Then
\[
\xi(t) = \int_0^t \phi^2(s) ds
\]
\[
= \alpha^2 \int_0^t g^{2\alpha-2}(s)g^2(s) ds
\]
\[
= \frac{\alpha^2}{2\alpha - 1} g^{2\alpha-1}(t)g'(t) - \frac{\alpha^2}{2\alpha - 1} \int_0^t g^{2\alpha-1}(s)g''(s) ds - C_\alpha. \tag{3.2}
\]

The condition \((\tilde{H}2)\) yields: Given any \( \epsilon \in \left( 0, \mu - \frac{1}{1+\delta} \right) \), there exists \( C \geq 0 \) such that \( g(t)g''(t) \geq (\mu - \epsilon)g^2(t) - C \) in \([0, \infty)\). Therefore
\[
- \int_0^t g^{2\alpha-1}(s)g''(s) ds \leq - (\mu - \epsilon) \int_0^t g^{2\alpha-2}(s)g^2(s) ds + C
\]
\[
\leq - \frac{\mu - \epsilon}{\alpha^2} \xi(t) + C. \tag{3.3}
\]
We divide the proof into two cases.

Case 1: $\delta > 1$ and $\mu > \frac{1}{1+\delta}$; or $\delta \leq 1$ with $\mu > \frac{1+\delta}{4\delta}$.

Take $\alpha > \frac{1}{2}$. Combine (3.2) and (3.3),

$$(1 + \frac{\mu - \epsilon}{2\alpha - 1}) \xi(t) \leq \frac{\alpha^2}{2\alpha - 1} g^{2\alpha - 1}(t)g'(t) + C,$$

consequently

$$\xi(t) \leq \frac{\alpha^2}{2\alpha - 1 + \mu - \epsilon} g^{2\alpha - 1}(t)g'(t) + C, \text{ for any } t \geq 0. \quad (3.4)$$

According to $$(H3),$$ for any $0 < \delta' < \delta$, there exists $C > 0$ such that $g'(t) \geq (1+\delta')g(t) - C$ in $[0, \infty)$. Setting these estimates in (2.3), omitting the index $\lambda$ and recalling that $f'(u) = g'(v) - g(v),$

$$\frac{\delta' \lambda}{1 + \delta'} \int_{\Omega} e^{\gamma g'(v)}(g^\alpha(v) - 1)^2 - C \lambda \int_{\Omega} e^{\gamma}(g^\alpha(v) - 1)^2$$

$$\leq \lambda \int_{\Omega} e^{\gamma f'(u)}(g^\alpha(v) - 1)^2$$

$$\leq \frac{2\alpha^2 \lambda}{\beta(2\alpha - 1 + \mu - \epsilon)} \int_{\Omega} e^{\gamma g^2(v)}g'(v) + C \lambda \int_{\Omega} e^{\gamma}g(v) + C \int_{\Omega} e^{\gamma}(g^\alpha(v) - 1)^2.$$

Consequently,

$$\left[ \frac{\delta'}{1 + \delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)} \right] \lambda \int_{\Omega} e^{\gamma g'(v)}g^2(v)$$

$$\leq 2\delta' C \int_{\Omega} e^{\gamma g'(v)}g^\alpha(v) + C \int_{\Omega} e^{\gamma}g(v) + C \int_{\Omega} e^{\gamma}(g^\alpha(v) - 1)^2.$$

Choose $\delta'$ near $\delta$ such that

either $\delta' > 1$ and $\mu > \frac{1}{1+\delta'}$ or $\delta' < 1$ with $\mu > \frac{1+\delta'}{4\delta'}$.

Through direct computations, for $\epsilon > 0$ sufficiently small and $\beta = 2 - \epsilon$, there exists

$$\alpha \in \left( \frac{1}{2}, \frac{\delta'}{1 + \delta'} + \sqrt{\delta'(1 + \delta')(\mu - \epsilon) - \delta'} \right)$$

such that

$$\left[ \frac{\delta'}{1 + \delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)} \right] > 0. \quad (3.5)$$

For such $\alpha$, we obtain

$$\lambda \int_{\Omega} e^{\gamma}g^2(v)g'(v) \leq C, \quad \forall \lambda \in (0, \lambda^*). \quad (3.6)$$
Tending now $\delta'$ to $\delta$ and $\epsilon$ to 0, (3.6) holds true provided that
\[
\alpha < \frac{\delta}{1 + \delta} + \frac{\sqrt{\delta \mu (1 + \delta) - \delta}}{1 + \delta}.
\] (3.7)
Therefore
\[
\int_{\Omega} e^{\gamma} g^{2\alpha + 1}(v) \leq C \int_{\Omega} e^{\gamma} g^{2\alpha}(v) g'(v) + C \leq \tilde{C},
\]
which implies that $\|g(v)\|_{2\alpha + 1} \leq C$ for $\alpha$ verifying (3.7). Applying Lemma 3.1, we conclude that for $n < 2 + 4\alpha$ with $\alpha$ verifying (3.7), $v_\lambda$ is uniformly bounded, hence $u^*$ is a regular solution if $n$ satisfies (1.4).

**Case 2:** $\delta \leq 1$ and $\frac{\beta}{1 + \delta} < \mu \leq \frac{1 + \delta}{4\delta}$.

Now we take $\alpha \in \left(\frac{1}{2}(1 - \mu + \epsilon), \frac{1}{2}\right)$, the formulas (3.2) and (3.3) imply then
\[
\left(1 + \frac{\mu - \epsilon}{2\alpha - 1}\right) \xi(t) \geq \frac{\alpha^2}{2\alpha - 1} g^{2\alpha - 1}(t) g'(t) + C.
\]
The inequality (3.4) still holds true. Proceeding as for Case 1, we see that for $\delta' < \delta$ but nearby, $\epsilon > 0$ small and $\beta = 2 - \epsilon$, there exists
\[
\alpha \in \left(\frac{1 - \mu + \epsilon}{2}, \frac{\delta'}{1 + \delta'} + \frac{\delta'(1 + \delta')(\mu - \epsilon) - \delta'}{1 + \delta'}\right) \subset \left(\frac{1 - \mu + \epsilon}{2}, \frac{1}{2}\right)
\]
such that (3.5) is satisfied. Hence we conclude exactly as in Case 1.

**3.3. Proof of Theorem 1.5**

Without loss of generality, assume again $g(0) = 1$. Take now $\phi(t) = t e^{\alpha h(t)}$, where $\alpha > 0$ is a constant to be determined, then
\[
\xi(t) = \int_{0}^{t} \left[1 + s \alpha h'(s)\right]^2 e^{2\alpha h(s)} ds
= \int_{0}^{t} \left[1 + 2s \alpha h'(s)\right] e^{2\alpha h(s)} ds + \int_{0}^{t} \alpha^2 s^2 h'^2(s) e^{2\alpha h(s)} ds
= t e^{2\alpha h(t)} + K(t).
\]
Thus, for $t \geq t_0$,
\[
\frac{2K(t)}{\alpha} = 2\alpha \int_{t_0}^{t} s^2 h'^2(s) e^{2\alpha h(s)} ds = C + \int_{t_0}^{t} s^2 h'(s) d\left(e^{2\alpha h(s)}\right)
\leq C + t^2 h'(t) e^{2\alpha h(t)} - \int_{t_0}^{t} e^{2\alpha h(s)} d\left(s^2 h'(s)\right),
\]
where the last integration is considered in the sense of Stieltjes. The monotonicity of $s^2 h'$ in $[t_0, \infty)$ yields
\[
K(t) \leq \frac{\alpha}{2} t^2 h'(t) e^{2\alpha h(t)} + C, \quad \forall t \geq t_0.
\]

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So we get
\[ \xi(t) \leq C + \left[ t + \frac{\alpha}{2} t^2 h'(t) \right] e^{2\alpha h(t)}, \quad \forall t \geq 0. \]

Using (2.3) (we drop the index \( \lambda \)),
\[
\int_{\Omega} e^\gamma \left[ e^{h(v)} + vh'(v)e^{h(v)} - ve^{h(v)} - 1 \right] v^2 e^{2\alpha h(v)} \\
\leq \frac{2}{\beta} \int_{\Omega} e^\gamma \left( 1 + ve^{h(v)} \right) \xi(v) + C \int_{\Omega} e^\gamma v^2 e^{2\alpha h(v)} \\
\leq \frac{2}{\beta} \int_{\Omega} e^\gamma \left( 1 + ve^{h(v)} \right) \left[ C + ve^{2\alpha h(v)} + \frac{\alpha}{2} t^2 h'(v)e^{2\alpha h(v)} \right] + C \int_{\Omega} e^\gamma v^2 e^{2\alpha h(v)},
\]

By Young’s inequality,
\[
\left( 1 - \frac{\alpha}{\beta} \right) \int_{\Omega} e^\gamma v^3 h'(v)e^{(2\alpha+1)h(v)} \\
\leq C \int_{\Omega} e^\gamma \left[ 1 + v^2 h'(v)e^{2\alpha h(v)} + v^3 e^{(2\alpha+1)h(v)} \right]. \tag{3.8}
\]

Moreover, \( g = o(g') \) at infinity yields \( \lim_{t \to \infty} h'(t) = \infty \), hence
\[
\frac{t^2 h'(t)e^{2\alpha h(t)} + v^3 e^{(2\alpha+1)h(t)}}{t^3 h'(t)e^{(2\alpha+1)h(t)}} = \frac{1}{g(t) - 1} + \frac{1}{h'(t)} \to 0 \quad \text{as} \quad t \to \infty.
\]

Fix \( \beta \in (\alpha, 2) \), the inequality (3.8) implies
\[
\int_{\Omega} \frac{\|g(v) - 1\|^{2\alpha+1}_{\gamma}}{v^{2\alpha}} = \int_{\Omega} ve^{(2\alpha+1)h(v)} \leq C + \int_{\Omega} v^3 h'(v)e^{(2\alpha+1)h(v)} \leq C.
\]

Recall that \( g \) is superlinear, we obtain \( \|g(v)\|_1 \leq C \). Consider again \( w \) satisfying \( L(w) = \lambda g(v) \) in \( \Omega \) and \( w = 0 \) on \( \partial \Omega \), as \( v \leq w \) in \( \Omega \) by maximum principle,
\[
\int_{\Omega} \frac{(g(v) - 1)^{2\alpha+1}_{\gamma}}{w^{2\alpha}} \leq C.
\]

Following the proof of Lemma 2.1 in [17] (we just need a minor adjustment, say define \( \Omega_1 = \{ x \in \Omega : g(v) > w^T \} \) instead, here \( T > 0 \) is a suitable constant), we can obtain that

if \( 2\alpha + 1 > n/2 \), \( w \) is uniformly bounded in \( L^\infty(\Omega) \), so does \( v \). Taking \( 2 > \beta > \alpha > 7/4 \), the result holds for \( n \leq 9 \). \( \square \)

### 3.4. Proof of Theorem 1.6

Here we choose \( \phi(u) = (1 - u)^{-\alpha} - 1 \) in (2.4). For \( 2\lambda > \lambda^* \) and \( \epsilon > 0 \),
\[
\left( p - \frac{2\alpha^2}{\beta(2\alpha + 1)} - 2\epsilon \right) \int_{\Omega} \frac{e^\gamma}{(1 - u)^{p+2\alpha+1}} \leq C, \quad \forall \beta \in [1, 2).
\]

We have used \( f'(u)(1 - u) \geq (p - \epsilon)f(u) - C \) in \( [0, 1] \) by (1.5). As \( \epsilon > 0 \) is arbitrary,
\[
\int_{\Omega} \frac{1}{(1 - u)^{p+2\alpha+1}} \leq C
\]
provided that
\[ p > \frac{\alpha^2}{2\alpha + 1}, \] i.e. when \( \alpha < p + \sqrt{p(p+1)} \).

Therefore \( \|(1-u)^{-1}\|_q \leq C \) if \( q < 1 + 3p + 2\sqrt{p(p+1)} \). For any \( \epsilon > 0 \), as \( f'(u)(1-u) \leq (p+\epsilon)f(u) + C \), in \([0,1]\) by \( \text{(1.5)} \), we have \( f(u) \leq C(1-u)^{-p-\epsilon} \), consequently
\[ g(v) = e^v f(1-e^{-v}) = \frac{f(u)}{1-u} \leq C(1-u)^{-p-\epsilon} \], hence \( \|g(v)\|_r \leq C \) when \( r < \frac{1+3p+2\sqrt{p(p+1)}}{p+1+\epsilon} \).

According to Lemma 3.1, the proof is done by taking \( \epsilon \to 0^+ \). \( \Box \)

4. Radial case

As we have mentioned, when \( c = -\nabla \gamma \), the equation \( (P_\lambda) \) is rewritten as
\[ -\text{div}(e^\gamma \nabla u) = \lambda e^\gamma f(u). \] (4.1)

With the variational structure, the stability of minimal solutions \( u_\lambda \) is equivalent to
\[ \int_{\Omega} e^\gamma |\nabla \psi|^2 \geq \lambda \int_{\Omega} e^\gamma f'(u_\lambda)\psi^2, \quad \forall \psi \in H_0^1(\Omega). \] (4.2)

Moreover, for any \( C^1 \) functions \( \phi \) and \( \xi \) satisfying \( \phi(0) = \xi(0) = 0 \), \( \xi' = \phi^2 \), the estimate \( \text{(2.4)} \) is replaced by
\[ \int_{\Omega} e^\gamma f'(u_\lambda)\phi^2(u_\lambda) \leq \int_{\Omega} e^\gamma f(u_\lambda)\xi(u_\lambda). \]

Taking now \( \phi(t) = f(t) - f(0) \) and working as for Theorem 1 in \( \text{[18]} \), we have

**Lemma 4.1.** When \( c = \nabla \gamma \), the extremal solution \( u^* \in H^2 \cap H_0^1(\Omega) \). More precisely,
\[ \int_{\Omega} f'(u_\lambda) f(u_\lambda) \leq C, \quad \forall \lambda \in (0, \lambda^*]. \] (4.3)

When \( \Omega = B_1 \) is the unit ball, \( \gamma(x) = \gamma(r) \) with \( r = |x| \), \( u_\lambda \) is radial by uniqueness of the minimal solution and satisfies
\[ -u'' - \frac{n-1}{r} u' - \gamma' u' = \lambda f(u) \quad \text{in } (0,1], \] (4.4)

with \( u'(0) = 0 \) and \( u(1) = 0 \). Our main result in this section is the regularity of the extremal solution \( u^* \) for any \( f \) satisfying \( (H) \) provided \( n = 2 \) and the optimal estimate for \( u' \) claimed in Theorem \( \text{[12]} \).
The method we use is similar to [5, 15], but the uniform boundedness of \( \|u_\lambda\|_{C^1} \) is not enough to claim the regularity of \( u^* \), because a singular \( u^* \) could be Lipschitz in many cases (see Remark 1.1). In fact, the estimate (4.3) is crucial for our proof.

As in [5, 15], since \( u_\lambda'(r) \leq 0 \) by maximum principle or equation (4.4), the boundedness of \( \|u_\lambda\|_{H^1_0} \) implies that for any \( k \in \mathbb{N}, r > 0, \|u_\lambda\|_{C^k(B_i \setminus B_r)} \leq C_{k,r}, \forall \lambda \in (0, \lambda^*]. \) So we concentrate our attention near the origin. Derivating the equation (4.4) or (4.1) with respect to \( r \),

\[
- \text{div} (e^\gamma \nabla u') = e^\gamma u' \left[ \lambda f'(u) - \frac{n-1}{r^2} + \gamma'' \right] \quad \text{in} (0, 1].
\]

Using \( \psi = r\eta(r)u_\lambda'(r) \) as test function in (4.2) with \( \eta \in H^1_0(B_1) \cap C(\overline{B}_1) \), by similar calculation as for Lemma 2.1 in [5], we obtain

\[
\int_{B_r} e^\gamma \left[ |\nabla (r\eta)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 \right] u_\lambda^2 \geq 0, \quad \forall \lambda \in (0, \lambda^*]. \quad (4.5)
\]

4.1. Proof of Theorem 1.1

For simplicity, we drop the index \( \lambda \). All estimates below hold uniformly for \( \lambda \). First as \( u_\lambda \) is radial, by maximum principle, we see that \( u \) is decreasing in \( r \). Since \( f \) and \( f' \) are nondecreasing functions according to (H), the estimate (4.3) implies (as \( n = 2 \))

\[
\pi r^2 f'(u(r)) f(u(r)) \leq \int_{B_r} f'(u) f(u) \leq C, \quad \forall r \in (0, 1].
\]

By Lemma 2.1 we have

\[
f(u(r)) \leq \frac{C}{r} \quad \text{for all} \quad r \in (0, 1]. \quad (4.6)
\]

Let \( r_0 \in (0, \frac{1}{2}] \). Let \( \eta \) be a radial function in \( H^1_0(B_1) \cap C(\overline{B}_1) \) such that

\[
\eta(r) = \begin{cases} 
0 & \text{if} \quad r < r_0; \\
0 & \text{if} \quad r_0 \leq r \leq \frac{1}{2}, 
\end{cases}
\]

and \( \eta \) be a fixed \( C^1 \) function in \( \overline{B}_1 \setminus B_{1/2}, \) independent of \( r_0 \). The direct calculation yields

\[
|\nabla (r\eta)|^2 - \eta^2 + \gamma'' r^2 \eta^2 = \begin{cases} 
\gamma'' r_0^2 \eta^2 & \text{if} \quad r < r_0; \\
\gamma'' r^2 \eta^2 & \text{if} \quad r_0 < r \leq \frac{1}{2}.
\end{cases}
\]

Using (4.5), as \( u \) is uniformly bounded in \( H^1(B_1) \) by Proposition 2.4 and \( r^2 r_0^{-2} \leq 1 \) in \([0, r_0]\), we get

\[
\int_{r_0}^{1/2} \frac{u'(r)^2}{r} dr \leq C.
\]

Tending \( r_0 \) to 0, there holds

\[
\int_0^1 \frac{u'(r)^2}{r} dr \leq C. \quad (4.7)
\]
Consider the following test function used in (15): For any \( r \leq \frac{1}{2} \) and \( 0 < r_0 < r \),

\[
\eta(s) = \begin{cases} 
(rr_0)^{-1} & \text{if } s < r_0; \\
(rs)^{-1} & \text{if } r_0 \leq s < r; \\
s^{-2} & \text{if } r \leq s \leq \frac{1}{2}.
\end{cases}
\]

Applying again (15) and combining with (1.7), we obtain finally (with \( r_0 \to 0 \))

\[
\int_0^r \frac{u'(s)^2}{s} ds \leq Cr^2, \quad \forall r \leq 1. \tag{4.8}
\]

As \( (e^ru')' = -\lambda e^rf(u) \) with \( n = 2 \), so \( e^ru' \) is nonincreasing in \( r \). Then \( u'(s) \leq Cru'(r)/s \) for \( s \in [r, 1] \), hence \( u'(s) \leq Cu'(r) \leq 0 \) for any \( s \in [r, 2r] \) if \( r \leq \frac{1}{2} \). By (4.8), for any \( 0 < r \leq \frac{1}{2} \),

\[
C_1 r^2 \geq \int_0^{2r} \frac{u'(s)^2}{s} ds \geq \int_r^{2r} \frac{u'(s)^2}{s} ds \geq \frac{C_2}{r} \int_r^{2r} u'(r)^2 ds = C_3 u'(r)^2.
\]

That means

\[
|u'(r)| \leq Cr \quad \text{in } [0, 1]. \tag{4.9}
\]

However, we need to consider also \( u''(r) \) as explained above. Let

\[
G(r) = e^ru' \quad \text{and} \quad \Psi(r) = -2G(\sqrt{r}) - M \int_0^r (r - s)f(u(\sqrt{s})) ds
\]

where \( M \) is a constant to be chosen. Using \( G' = -\lambda e^rf(u) \),

\[
\Psi''(r) = \left[ \lambda e^{\gamma(s)} f'(u(s)) \frac{u'(s)}{2s} + \lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - M f(u(s)) \right]_{s=\sqrt{r}}
\]

\[
\leq \left[ \lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - M f(u(s)) \right]_{s=\sqrt{r}}
\]

\[
\leq C_0 f(u(\sqrt{r})) - M f(u(\sqrt{r})).
\]

For the last line, we used \(|\gamma'(s)|/s \leq C \) in \([0, 1]\) since \( \gamma \) is a smooth function (so \( \gamma'(0) = 0 \). Fix \( M > C_0 + 1 \), \( \Psi \) is then concave in \([0, 1]\). On the other hand, by (4.8)

\[
\Psi'(r) = \lambda e^{\gamma(\sqrt{r})} f(u(\sqrt{r})) - M \int_0^r f(u(\sqrt{s})) ds \geq C\lambda f(0) - CM \sqrt{r}.
\]

There exists \( r_1 > 0 \) small enough such that \( \Psi' \geq 0 \) in \([0, r_1]\) with \( \lambda \geq \frac{\lambda}{2} \). Using (1.3), (4.6) and (4.9), for \( \lambda \geq \frac{\lambda}{2} \) and \( r \leq r_1 \),

\[
- e^{\gamma(\sqrt{r})} \left[ u''(\sqrt{r}) + \frac{u'(\sqrt{r})}{\sqrt{r}} + \gamma' u'(\sqrt{r}) \right] - CM \sqrt{r}
\]

\[
\leq \Psi'(r) \leq \frac{\Psi(r)}{r} \leq -2e^{\gamma(\sqrt{r})} \frac{u'(\sqrt{r})}{\sqrt{r}} \leq C.
\]

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Applying one more time (4.9), we see that $u''(\sqrt{r}) \geq -C$ for any $\lambda \geq \frac{\lambda^*}{2}$ and $r \leq r_1$. Otherwise, by (4.4) and (4.9), $u''(r) \leq -u'(r)r^{-1} - \gamma'(r)u'(r) \leq C$, we claim then

$$\|u''\|_{\infty} \leq C, \quad \forall \lambda \geq \frac{\lambda^*}{2}.$$ Combining with (4.4) and (4.9), it means $\|\lambda f(u)\|_{\infty} \leq C$, no singularity will occur. \[\square\]

4.2. Proof of Theorem 1.2

As above, we drop the index $\lambda$ and all estimations hold uniformly for $\lambda$. First, repeating the proof of Theorem 1.8, c) in [5], we obtain $f'(u(r)) \leq Cr^{-2}$ in $(0, 1]$. Using Lemma 2.1 with (4.5), $f(u(r)) \leq Cr^{-2}$ in $(0, 1]$. Consequently, by (4.4), for $n \geq 3$,

$$0 \leq -e^{\gamma} r^{n-1} u'(r) = \int_0^r e^{\gamma(s)} s^{n-1} f(u(s)) ds \leq C \int_0^r s^{n-3} ds \leq Cr^{n-2}.$$ Hence

$$|u'(r)| \leq C/r.$$ \((4.10)\)

Let $\eta$ be a radial function in $H^1_0(B_1) \cap C^0(\overline{B_1})$ such that

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$

in $\overline{B_{r_1}}$ and be a fixed $C^1$ function in $\overline{B_1} \setminus B_{r_1}$, here $r_0$ is any constant in $(0, r_1)$, $r_1 > 0$ is a small constant to be determined. Therefore

$$|\nabla(\eta r)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 = \begin{cases} (\gamma'' r^2 + 2 - n)r_0^{-2\sqrt{n-1}} & \text{if } r < r_0; \\ (\gamma'' r^2 - 2\sqrt{n-1} + 1)r^{-2\sqrt{n-1}} & \text{if } r \in [r_0, r_1]. \end{cases}$$

We fix $r_1 > 0$ small enough such that

$$\max_{r \in [0, r_1]} \{\gamma'' r^2\} < \min (n-2, 2\sqrt{n-1} - 1).$$

By (4.5), as $|\nabla(\eta r)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 \leq 0$ for $r \in [0, r_0]$,

$$\int_{r_0}^{r_1} u^2(r)r^{n-1-2\sqrt{n-1}} dr \leq C.$$ Tending $r_0$ to 0, we have

$$\int_0^{r_1} u^2(r)r^{n-1-2\sqrt{n-1}} dr \leq C.$$ \((4.11)\)

Now we take another test function used in [15],

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}-1} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$
Combining (4.5) and (4.11), we conclude then

\[ \int_0^{r_0} u'^2(r) r^{n-1} dr \leq C r_0^{2+2\sqrt{n-1}}, \quad \forall r_0 \in [0, r_1]. \]

By the monotonicity of \( e^{r^{n-1}u'} \), similarly as for (4.9), it holds

\[ |u'(r)| \leq C r^{-\frac{n}{2}+1+\sqrt{n-1}}, \quad \forall r \in [0, 1]. \]

Finally, combining with (4.10), we are done (in fact, \( -\frac{n}{2} + 1 + \sqrt{n-1} \leq -1 \) for \( n \geq 10 \)). □

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