Games of Artificial Intelligence: A Continuous-Time Approach

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Abstract

This paper studies the strategic interaction of algorithms in economic games. We analyze games where learning algorithms play against each other while searching for the best strategy. We first establish a fluid approximation technique that enables us to characterize the learning outcomes in continuous time. This tool allows to identify the equilibria of games played by Artificial Intelligence algorithms and perform comparative statics analysis. Thus, our results bridge a gap between traditional learning theory and applied models, allowing quantitative analysis of traditionally experimental systems. We describe the outcomes of a social dilemma, and we provide analytical guidance for the design of pricing algorithms in a Bertrand game. We uncover a new phenomenon, the coordination bias, which explains how algorithms may fail to learn dominant strategies.

Keywords: Artificial Intelligence, Learning, Cooperation, Continuous Time

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1 Introduction

After years of improvements in algorithmic prediction models, now widely employed as support tools for decision making, software programs capable of autonomous decisions are becoming increasingly common. Accenture’s solutions.ai claims “AI optimizes pricing levers in real time”, Vendavo offers dynamic pricing with competitive intelligence, and scibids.com “build customizable AI that dramatically improves Paid Media ROI”. Despite the ever-growing popularity of these algorithms however, the interactions of these systems with human and other autonomous agents are not well-understood. In this paper we focus on the latter, and we characterize their behavior in strategic environments through a continuous-time approximation.

Regulators and designers alike are interested in formally understanding the outcomes of Artificial Intelligence (AI) algorithms in order to improve allocations and surplus. Both scholars and practitioners have raised concerns over unintended behaviors of these programs that could cause disruptions and inefficiencies (OECD (2017), Varian (2018), Competition Bureau (2018)). Particularly challenging is the design and analysis of competing algorithms: even when each program is acting in a self-interested fashion, the experimental work of Calvano et al. (2020) shows evidence of systematic collusive behavior. The same evidence supports the claim that competition in pricing by simple AI algorithms leads to supra-competitive outcomes and lower consumer surplus. In light of these results, it is fundamental to develop a formal approach to analysing the interactions between algorithmic systems. Validating the experimental results and forecasting a software’s behavior would help mechanism designers and regulators in improving existing market structures. However, the theory surrounding multi-agent learning is rather scattered and, for the most part, intractable.

To further our understanding of these environments, we study a model in which a stage game is repeatedly played by learning algorithms. We are interested in predicting what outcomes will be reached by the learning dynamics. To this end, we develop a continuous-time approximation, borrowing from and inspired by the fluid approximation literature in applied probability and Operations Research. This technique yields a deterministic dynamical system whose flow and trajectories fully characterize the learning dynamics. While we mainly focus on the workhorse model of Q-learning, our technique can be applied to many Reinforcement Learning algorithms, and more generally to all adaptive algorithms that satisfy some technical conditions. Our toolkit allows to char-

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1Contributions to this literature come mostly from the Computer Science community, of which Bloembergen et al. (2015) is a good survey.
acterize the equilibrium points of learning, and compare them with traditional game-theoretic predictions. We apply our method to Q-learning in games with dominant strategy equilibria. We would expect well-behaved learning algorithms to find the equilibrium, but the literature shows Q-learning fails to converge to dominant strategies. By analyzing the system in continuous time, we find that a coordination bias limits the ability of the algorithms to best-respond. The algorithms maintain estimates of each action's value that are consistently biased towards symmetric play: unless exploitation is sacrificed for sufficient exploration of new strategies, the bias persists through deviations and sustains cooperative/collusive outcomes.

Our contribution has multiple facets. First, our toolkit allows us to produce comparative statics with respect to parameters of the model. This proves particularly useful for understanding phenomena such as collusion, and whether it arises from characteristics of the game or because of algorithmic design, or a combination of both. Decoupling agent design from mechanism design opens the door for further improvements of algorithmic systems along either direction. By providing an analytical counterpart to traditionally experimental systems, we enable a more thorough description of the learning dynamics and a clear interpretation of their outcomes.

Second, we bridge an existing gap in the learning literature in Economics. On the one hand, analytically simple procedures\(^2\) have been thoroughly analyzed and understood, yet they lack general applicability, while on the other hand complex algorithms, often deployed in practical applications, need theoretical attention. The classic model of Fictitious Play, although intuitive and clearly understood, has seen almost no application (see Heinrich et al. (2015)). On the contrary, AI has been deployed to a wide range of applied settings despite its relative recent development and its unpredictable behavior. For example, the ReAgent platform and its competitor AutoML implement Reinforcement Learning algorithms for optimization of large-scale production environments,\(^3\) adaptive experiment designs using AI are key to improving forecasts and causal estimates (Kasy and Sautmann (2021)), recommender systems focusing on life-time value instead of myopic approaches are used on YouTube (Chen et al. (2019)), and automated portfolio management tools incorporate AI designs (Jiang et al. (2017)). An important feature shared by many of these procedures is their relative independence from the environment they are deployed in. This model-free design allows for quick deployment of complex architectures, thereby granting considerable efficiency improvements. As pointed out before, however, these algorithms often lack a theory of convergence, or even a descriptive theory.

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\(^2\)For a comprehensive reference on learning in games we refer to Fudenberg and Levine (1998).

\(^3\)https://reagent.ai/
of their behavior, particularly in multi-agent settings. We provide a theoretical approach to complement applied procedures in order to understand, improve, and regulate AI agents.

Third, our technique can be applied in a wide range of environments. The paradigm that allowed AI researchers such exceptional achievements is known as Reinforcement Learning (RL) and one of the RL procedures that has received the most attention in Economics is Q-learning. By its nature, experimental work must focus on a selected learning procedure, which may hinder the external validity of results. Concerns have been expressed, e.g. by Asker et al. (2022), with respect to algorithmic design and collusion in pricing. As they find, even though the blueprints of popular algorithms are similar, minimal variation in their structure may affect convergence and equilibria. The techniques derived in this paper can be applied to a number of popular AI techniques, possibly to address robustness concerns of reduced-form experiments.

We apply our theoretical results to two instances: first, we consider an experiment with Q-learning in a social dilemma, and then we analyze the pricing game from Asker et al. (2022). In the social dilemma, two algorithms contribute to a public good: there are private incentives to shirk, but coordination on contributing is socially optimal. We observe experimentally that the algorithms’ behavior is parameter-dependent, because cooperation is only sustained when the social value of contribution is large. These results are hard to square with the game-theoretic predictions, since shirking is the dominant-strategy equilibrium for all parameters. Using our framework, we characterize the equilibria of this simple game completely, and uncover a novel relationship between the value of the public good and the exploration procedure of Q-learning: as the public good becomes more valuable, prohibitively intense exploration is necessary for convergence on the dominant-strategy equilibrium for all parameters. Second, we apply our methods to a Bertrand pricing game also featuring Q-learning agents. Asker et al. (2022) consider different levels of information availability: they observe that the competitive outcome is achieved only when the agents are able to assess the value in hindsight of every price in their action space. We give theoretical foundations to this claim, and highlight that convergence to supra-competitive prices can be taken as an example of insufficient exploration, too.

The remainder of this section is devoted to a review of relevant literature. Section 2 describes the Reinforcement Learning paradigm and introduces our experiment. In Section 3 we present our results for Q-learning, and we leverage them to explain the results of the experiment. In Section 4 we advance sufficient conditions on learning algorithms that allow to use our toolbox. In Section 5 we characterize equilibria in a Bertrand game, and then we conclude.
1.1 Literature Review

Q-learning and Artificial Intelligence have recently sparked some interest in Economics, often through experimental work (see e.g. Klein (2021), Hansen et al. (2021), Banchio and Skrzypacz (2022)) or empirical work (see e.g. Brown and MacKay (2021), Musolff (2021), Assad et al. (2021)). The work of Calvano et al. (2020) draws attention to strategies as a proxy for collusion: they argue that simply looking at outcomes of learning might be insufficient, as collusion might arise as a “mistake” by poorly designed algorithms. Our work allows to delve deeper into the dynamics of learning, and through comparative statics and convergence analysis determine whether collusion is systemic. Particularly relevant is the work by Asker et al. (2022), which analyzes the impact of algorithm design on collusion in a Bertrand pricing game. We contribute to these questions with analytical tools that enable to ascertain the components of the system that drive the results. Contributions from the bandit literature include Aouad and Van den Boer (2021), who prove tacit collusion schemes arise with classical multi-armed bandits algorithms.

Some work has examined more generally Reinforcement Learning in games, for example Erev and Roth (1998) or Mertikopoulos and Sandholm (2016), but with some notable differences. Firstly, many have analyzed systems experimentally (Erev et al. (1999), Lerer and Peysakhovich (2017)). Our approach is complementary: with the aid of our framework, one can tell apart experimental findings from agent design considerations. On the other hand, there is some theoretical work on convergence of learning procedures. For example, learning through reinforcement has been associated with evolutionary game theory by Börgers and Sarin (1997). Others have formally analyzed some of the simpler models, as Hopkins and Posch (2005). These results have then been extended to more complex systems, but with a focus on the connection with replicator dynamics as their core. Our approach is particularly relevant because instead it examines the workhorse model in applied work, $\epsilon$-greedy Q-learning. We obtain a tool valuable for regulation and design, trading this off against a complete understanding of a simpler system. Conveniently, the approach described in Section 4 includes these earlier results under a general structure: Examples 1 and 2 show how our results apply to Fictitious Play and a well-known implementation of regret minimization.

The idea of approximating Q-learning in continuous time is not new, particularly in the single-agent setting. Related to ours is the work of Tuyls et al. (2005): the authors examine a continuous-time approximation of multi-agent Q-learning with Boltzmann exploration, and show a link with the Replicator Dynamics from the Evolutionary Game Theory (EGT) literature. Both Gomes and Kowalczyk (2009) and Wunder et al. (2010)
propose a continuous-time approximation of Q-learning in a multi-agent setting with \( \varepsilon \)-greedy algorithms. Their approximations are mutually inconsistent and require additional conditions on the parameter. Most importantly, those approximations remain model-dependent. Our approach can recover their results, but with a novel technique relying on fluid approximation. Our method applies to general adaptive algorithms, as we show in Section 4. The result is a recipe to analyze equilibria through the lens of dynamical systems, abstaining from heuristic modeling choices. The work of Benaim (1993) often serves as a foundation for stochastic approximations in learning. With respect to our approach, this work has two limitations. First, the analysis is constrained to limit points and sets, thus abstracting away from the dynamics of learning, which we recover instead. Second, our approximation technique is able to accommodate more general learning procedures, including recently developed AI techniques that fall under the umbrella of Temporal Difference learning.

Fluid models and other forms of approximation have enjoyed great popularity in the fields of applied probability and Operations Research. Starting from the seminal contributions of Iglehart (1965), Kurtz (1970), Halfin and Whitt (1981) and Harrison and Reiman (1981), approximations enabled formal analysis of systems otherwise deemed intractable: see, e.g., Wein (1992) for a classic application to inventory management. More recently, Mitzenmacher (2001) applied fluid models to the analysis of parallel computing systems, while Wager and Xu (2021) employ diffusion approximations to study sequential experiments. Taken together, this literature shows the power of approximate models for deriving insights about intractable settings.

2 Reinforcement Learning

In this Section we present a brief overview of the theory of Reinforcement Learning and Q-learning. The reader interested in more details is referred to Sutton and Barto (2018) and references therein.

Reinforcement Learning (RL) is a learning paradigm based on the tenet that experience alone can be responsible for learning optimal behavior. Agents’ actions are evaluated through the rewards collected in their interaction with the environment. In contrast with other learning paradigms, the aim of RL is to let algorithms learn the optimal behavior, or a characterization thereof, to deal with an unknown environment, and not to provide support tools for other decision makers.

In a typical RL problem an agent interacts repeatedly with the environment by taking an action from a given action set and receiving a reward that depends on the action taken
and, possibly, some state of the environment. The objective is to learn a policy maximizing the discounted sum of rewards. Learning occurs on-line and therefore RL algorithms strive to balance exploration to seek new data and exploitation of data and information already acquired.

Formally, the RL paradigm is most naturally implemented in Markov Decision Processes (MDPs). An agent faces an environment, described by a set of states $S$, an action set $A$ and stochastic rewards and state transitions. Rewards and future states are drawn according to the probability kernel $\nu(r,s' \mid s,a)$ when the current state is $s \in S$ and the action taken is $a \in A$. The objective of the agent is to learn on the fly a policy $\pi$ that maximizes the expected $4$ $\gamma$-discounted sum of rewards

$$E_\pi \left[ \sum_{t=0}^{\infty} \gamma^t r_t \right].$$

When the distribution $\nu$ is common knowledge, the problem can be solved with standard Dynamic Programming (DP) techniques. RL, instead, assumes that the distribution is unknown, and therefore the DP-optimal policy can only be approximated.

2.1 Q-learning

Q-learning was introduced by Watkins (1989) in the context of MDPs with finite state and action spaces. Essentially, it consists of a procedure for approximating the optimal value function $V^*(s)$ and its associated optimal policy. The optimal Q-function $5$ is defined as the solution to the system of equations given by

$$Q^*(s,a) = E_\nu \left[ r + \gamma \max_{a' \in A} Q^*(s',a') \mid s,a \right] \quad \forall s \in S, \forall a \in A,$$

where $s'$ denotes the state to which the system transitions. It is related to the optimal value function through the identity $V^*(s) = \max_a Q^*(s,a)$, which shows that the policy $\pi^*(s) \in \text{argmax}_a Q^*(s,a)$ achieves optimality. Hence, we can think of $Q^*(s,a)$ as the value of taking action $a$ in state $s$ and deciding optimally afterwards.

In Q-learning the agent maintains an estimate $Q_t(s,a)$ of the optimal Q-function, which is approximated by iterated updates: starting from a given $q_0 \in \mathbb{R}^{|S| \times |A|}$, at times $t = 0, 1, 2, \ldots$ some action $a_t$ is taken, the reward is collected, and the system transitions to

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$4$ Expectation is taken with respect to $\nu$ and the policy $\pi$.

$5$ Also called optimal “action-value function”.

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the next state, upon which the current estimate of the Q-function gets updated as

$$Q_{t+1}(s,a) = \begin{cases} Q_t(s,a) + \alpha [r_t + \gamma \max_{a'} Q_t(s_{t+1},a') - Q_t(s,a)] & \text{if } (s,a) = (s_t,a_t) \\ Q_t(s,a) & \text{else.} \end{cases} \quad (1)$$

Besides the initialization value $q_0$, the other hyperparameter in this procedure is $\alpha \in [0,1]$: it is called learning rate and plays a role complementary to that of $\gamma$. While the discount factor measures how much the agent cares about the future\textsuperscript{6}, the learning rate measures how fast agents “forget” about the past: the higher $\alpha$, the quicker bygones are forgotten.

We refer to the quantity in brackets in Equation (1) as temporal difference (TD), and we write:

$$TD_t = \begin{cases} r_t + \gamma \max_{a'} Q_t(s_{t+1},a') - Q_t(s,a) & \text{if } (s,a) = (s_t,a_t) \\ 0 & \text{else.} \end{cases} \quad (2)$$

Intuitively, the TD expresses how “unexpected” the reward just obtained was: a positive TD denotes “good news”, while a negative one the opposite. By steering the value of the Q-function in the direction of the TD, Q-learning balances between knowledge obtained so far and fresh information. This qualifies Q-learning as a member of the broad class of TD-based algorithms. Algorithms based on these premises have shown the power of the RL paradigm for developing artificially intelligent agents.\textsuperscript{7} In general, the advantage of TD-based methods is that they do not require knowledge of the environment to be employed, i.e., they are model-free algorithms. This implies that the same algorithm can be employed, with little to no modification, to a wide variety of environments.

**Exploration rules** A salient feature of practical implementations of Q-learning is the resolution of the trade-off between exploration and exploitation. A simple policy — greedy Q-learning — according to which the action selected is always one maximizing $Q_t(s_t,\cdot)$, completely neglects exploration in favour of exploitation.\textsuperscript{8} At the contrary, random Q-learning, in which actions are always randomly chosen, prescribes the opposite. Between these two extremes, a commonly employed and particularly easy to implement exploration rule is $\varepsilon$-greedy: with probability $1-\varepsilon$ the action chosen is $a_t \in \arg\max_a Q(s,a)$, and with probability $\varepsilon$ it is randomly sampled from the whole action set. We focus on this

\textsuperscript{6}The terminology “effective time horizon” is sometimes used in the RL literature.

\textsuperscript{7}Moreover, it has been argued that TD learning also bears some resemblance to the learning patterns observed in psychology, and that its physiological base can be founded in neuro-scientific research. Cfr. Chapters 14-15 in Sutton and Barto (2018).

\textsuperscript{8}See Dong et al. (2021) for an instance in which exploration is achieved for a greedy Q-learning agent by means of optimism, i.e., a choice of $q_0$ very large.
policy in Section 3, but in Section 4 we will show how our methods extend to other rules and procedures as well.

2.2 Multi-Agent Learning

We now extend the previous setting to one with multiple agents. Consider an environment described by: a finite set of players $I$, each endowed with an action set $A_i$; a finite set of states $S$; and a probability kernel $\nu$, which defines the probability distribution over the profile of rewards $r = \{r^i\}_{i \in I}$ and the next state $s'$, given by $\nu(r, s'| s, a)$, where now $a = \{a^i\}_{i \in I} \in A$ denotes the action profile played. Time $t = 0, 1, \ldots$ is discrete and the horizon infinite.

Each agent maintains a Q-function $Q^i_t \in \mathbb{R}^{S \times A}$ and takes action following an $\epsilon$-greedy policy. Therefore, each is endowed with a uniform randomization device $E_i$, whose realizations are independent of those of the other players. In state $s$, if the realization of $E_i$ is greater than $1 - \epsilon$ the agent chooses an action in $\arg\max_a Q^i_t(s, a)$; otherwise, they select one randomly. Subsequently, each player obtains a reward, the state transitions to $s_{t+1}$, and the agents update their Q-function for the state-action pair played according to:

$$Q^i_{t+1}(s_t, a^i_t) = Q^i_t(s_t, a^i_t) + \alpha \left[ r^i_t + \gamma \max_{a^i_{t+1}} Q^i_t(s_{t+1}, a^i_{t+1}) - Q^i_t(s_t, a^i_t) \right],$$

while for all other state-action pairs the Q-function remains the same. We will denote the collection of all players Q-functions at time $t$ as $Q_t \in \mathbb{R}^{I \times S \times A}$. For the sake of simplicity we will take the hyperparameters to be the same for all players.

States introduce additional flexibility to the framework: they can either model some situation of the environment external to the players, or they can be used to introduce memory of past plays, or both. For example, in Calvano et al. (2020) they encode the previous moves of the players. The variety of settings encompassed by this model explains the popularity of Q-learning in applications.

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9The approach adopted here entails that each player forms their own estimate of the Q-function, and is known in the Computer Science literature as independent Q-learning (Tan (1993)). Other approaches are possible, but they have not received the same attention as the one we adopt (cfr. Canese et al. (2021)). Additionally, the independent approach has the advantage of requiring little opponent-awareness — many collaborative AI instead rely heavily on their knowledge of the opponent.
2.3 Social Dilemma: Experiments

In this section we introduce the Social Dilemma game that will serve as workhorse example for the rest of the paper. We document the results of some experiments where two algorithmic agents play in that environment. We observe some rather unexpected outcomes, hard to rationalize from the primitives of the game. Because experimental evidence turns out to be inconclusive, in Section 3.2 we show how our techniques can help shed light over these outcomes.

Consider two independent $\epsilon$-greedy Q-learning agents, $A$ and $B$, learning to interact in a social dilemma over time. In every interaction they have a choice between contributing (C) or not contributing (NC) to a common fund from their own endowment of 2 dollars. The sum of the contributions is increased by a fixed growth factor $g \in (1, 2)$, and the resulting amount is then equally shared between the players: Figure 1 shows the game in matrix form. Each agent’s payoff is the sum of the funds received from the common pool and their remaining budget. For simplicity, assume they can only contribute the entirety of their endowment each time they interact. The only Nash equilibrium in pure strategies in this game is the profile $\{NC, NC\}$, which is also a dominant-strategy equilibrium.

For this experiment we adopt parameters $\epsilon = 0.1$, $\alpha = 0.05$ and $\gamma = 0.9$, and initialize agent symmetrically. Despite the simplicity of the game, when we simulate the interaction we observe diverging results. As shown in Figure 2a, for values of $g$ close to the upper bound the agents learn to cooperate, that is they coordinate on the strategy profile $\{C, C\}$, while for values of the growth factor closer to 1 the strategies of the agents converge to $\{NC, NC\}$.

Even though the non-cooperative action strictly dominates the cooperative one, for larger values of $g$ we observe a form of collusion, or coordination, between the two algorithms. However, collusion is not systemic: if the growth factor is too small, then agents fail to coordinate. What drives this difference? The parameter $g$ should have no effect on strategic decision, and instead it leads to stark differences in outcomes. A market designer might be interested in achieving the cooperative outcome at the lowest out-of-pocket cost for her: what is the lowest $g$ that guarantees cooperation? Moreover, Figure 2b shows that, even when the learned profile is $\{C, C\}$, collusion consists of “cycles” between cooperation
(a) Proportion of times the agents learn to play the cooperative profile \{C, C\}.

Figure 2: We initialized 100 independent runs of Q-learning with over \( T = 100,000 \) time steps each, and we consider as learned profile the one played most frequently in the last 100 time periods of each simulation.

and non-cooperation: since agents in our experiment are not allowed to condition on the past action of the opponent, it is hard to impute these cycles to “retaliatory” strategies, e.g., tit-for-tat. What element(s) of the experiment can cause this pattern?

After laying out the necessary toolbox, in Section 3.2 we will be able to answer these questions and derive additional insights that can be applied to more general forms of strategic interaction.

### 3 Main Results

While algorithmic learning quickly became the industry standard, our experiment shows that its behaviour remains hard to predict, even in simple environments. In this instance, complexities arise because understanding the behaviour of multiple learning agents is equivalent to analyzing the stability of discrete stochastic dynamical systems. In fact, it amounts to predicting the evolution of \( Q_t \), as defined by Equation (3). Updates are asynchronous and probabilistic — most of the rules that govern the exploration-exploitation trade-off entail randomization. Moreover, the direction of the updates is stochastic as well, and in strategic environments transitions and rewards are correlated across agents. Thus, obtaining an analytic description of the system is unrealistic even in simple examples. Nonetheless, understanding the role of game forms and parameters on the learning
procedure is of central importance to produce effective regulation (Calvano et al. (2021)).

3.1 A Continuous-Time Approximation

We show that there exists a continuous-time approximation to learning algorithms that abstracts away from the issues outlined above, but retains the structure of the problem and allows us to characterize the learning outcomes. Essentially, we approximate the problem with a system of Ordinary Differential Equations, easy to analyze with canonical tools from calculus. Should closed form solutions not be available, a qualitative analysis of the system remains simple and delivers the correct insights into the learning process.

More formally, define the system of ODEs:

\[
\frac{dQ_t}{dt} = \alpha \mathbb{E}[TD_t]
\]

where \(TD_t\) was defined in Equation (2). The expectation is taken with respect to all uncertainty: the uncertainty in the exploration rule, in the rewards, and in the state transitions. The following theorem clarifies how we approximate the process with \(Q_t\).

**Theorem 1** (Informal). There exists a sequence of continuous-time processes \(\{Q^n_t\}_{n \in \mathbb{N}}\), each identical in expectation to the Q-learning algorithm, that converges in probability to the solution \(Q\) of the system (4) of Ordinary Differential Equations.

Intuitively, the Theorem guarantees that there exists a sequence of processes, close to the original learning algorithm, that converges in probability to the solution of a system of ODEs. Such ODEs are defined as the expected temporal difference update in period \(t\). However, in order for the Theorem to work, we need to ensure that the temporal difference is Lipschitz-continuous. Recall that there is a maximum operator in Equation (2): in order to guarantee continuity we will focus on the open sets \(\omega_j\) where each maximum is constant and unique.\(^{11}\) Our Theorem then does not guarantee existence of a global solution: each solution only applies to one of the subset \(\omega_j\) just defined. The problem arises on the boundaries of these sets, where the vector field of \(Q\) is not yet defined. When possible, continuous pasting provides a solution for transitioning between two sets; however, if the trajectories from all sides point towards the boundary we cannot handle the system with traditional techniques. To define a global solution, we introduce the theory of differential inclusions (Filippov (1988)), and we show that such a solution always exists.

\(^{10}\)For a formal statement and proof, see Appendix A.

\(^{11}\)Since we are assuming finiteness of \(A_i\) and of \(S\), existence of the maximizing action is not an issue.
Theorem 2. Let $F_j$ be the field $\alpha \mathbb{E}[TD_t]$ over $\omega_j$ for all $j \in J$. There exists a global solution in the sense of Filippov to the differential inclusion

$$
\frac{dQ_t}{dt} = F_j(Q_t) \quad \text{over } \omega_j, \forall j
$$

$$
\frac{dQ_t}{dt} \in \text{co}\{F_k(Q_t) \mid \forall k \in K \subset J\}
$$

when $Q_t \in \bigcap_{k \in K} \bar{\omega}_k, \forall K \subset J$

where $\text{co}\{\cdot\}$ denotes the convex hull of a set.

This solution can be shown to be unique in numerous cases, with a technique termed sigmoid blending; further analysis of this falls out of the scope of the current paper.\textsuperscript{12} We will show in Section 3.2 that it is possible to analytically examine the dynamical system thus defined; moreover we will solve for the learning outcomes and obtain comparative statics useful for regulation and design. While in this Section we present the result for the simple and popular Q-learning algorithm, in Section 4 we will provide general conditions on learning algorithms under which appropriate versions of the two theorems above apply. This approximation approach can yield powerful insight into a variety of AI procedures, informing the choices of algorithm and market designers alike.

Sketch of the Proofs. In the following sketch of the proof, we will work within a single $\omega_j$. This implies that all the maxima within the temporal difference definition are well-defined and unique.

We can divide the proof of Theorem 1 in two main steps:

1. Finding the correct continuous-time embedding of Q-learning;
2. Identifying a scaling that guarantees limits are well-defined.

As far as the first step is concerned, let us define a Poisson process $N^0$ of rate $\lambda^0 = 1$. We write the process $Q^0$ as

$$
dQ^0_t = \alpha TD_t dN^0_t
$$

using the formalism of Stochastic Differential Equations. This is a compound Poisson process with the same updates as the original Q-learning algorithm. However, updates now occur at random with arrival rate of 1. On average, Q-learning is equal to its continuous time equivalent $Q^0$.

\textsuperscript{12}See Alexander and Seidman (1998) for a thorough discussion of the non-uniqueness issues arising and a description of sigmoid blending.
We now want to increase the pace of the updates while retaining the same uncertainty in expectation. Intuitively, we can “speed up” the Poisson arrivals, but we also need to “dampen” the jumps accordingly, otherwise the process will diverge to infinity. Formally, let $N^*_t$ be a Poisson point process of rate $\lambda^* = n$, and define

$$dQ^n_t = \frac{\alpha}{n} TD_t dN^n_t.$$  

In this new process, arrivals are on average more frequent but each increment is only a fraction of the original update. We will show in the Appendix that this new process remains close to the original $Q^0$ — see Lemma 1. By accelerating time, we observe more and more realizations of the random quantities within any given time interval. Taking the speed to infinity, we expect a law of large numbers to guarantee that the instantaneous update corresponds exactly to its expectation. Additionally, while Q-learning updates are originally asynchronous, in the limit all components update simultaneously, and the rates of increase collapses into their expected update rate.

Formally, let $E_i$ be the random variable associated with agent $i$’s exploration rule, and $r$ the random variable associated with rewards. Consider then the measure $\mu^n(x,dz)$ of updates at $x$, with

$$\mu^n(x,dz) = \mathbb{P}\{Q^n(\tau^n) \in dz | Q^n(0) = x\}$$

where $\tau^n$ is the first exit time of $Q^n$ from $x$. We define the component-wise function

$$F^n(x)_m = \lambda^n \int (z_m - x_m) \mu^n(x,dz),$$  

which intuitively describes the expected jump of $Q^n$ from $x$ along the $m$-th component over one unit of time.\(^\text{13}\) In fact, $F^n$ can be rewritten as

$$F^n(x)_m = n \int \frac{\alpha}{n} TD_t \mu = \int \alpha TD_t \mu.$$  

We chose the scaling in such a way that the function $F^n$ is independent of $n$. This shows that each process $Q^n$ is identical in expectation to the Q-learning algorithm:

**Lemma 1.** For all $n$ and for all $t$,

$$\mathbb{E}[Q^n_t - Q^n_{t-}] = \mathbb{E}[Q^0_t - Q^0_{t-}].$$

Let $F(x) := \lim_{n \to \infty} F^n(x)$. It follows from Lemma 1 that $F(x) = F^n(x)$ for all $n$.

\(^\text{13}\)Notice that $dQ^n_t = \frac{F^n(Q^n_t)}{\lambda^n} dN^n_t$. 

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the function $F(x)$ is Lipschitz in every component. This, along with technical conditions verified in Lemma 2 of Appendix A, implies, following a result in Kurtz (1970), that if the system

$$
\begin{align*}
\frac{dQ_t}{dt} &= F(Q_t) \\
Q_0 &= q_0 \in \omega_j
\end{align*}
$$

(6)

admits a solution, the sequence of processes $Q^n_t$ converges in probability to $Q_t$ for any $t$, in each $\omega_j$ separately.

In terms of Stochastic Differential Equations, the new process $Q$ can be written as

$$
dQ_t = a \mathbb{E}[TD_t] dt.
$$

(7)

As advanced at the beginning, this is a deterministic process with all uncertainty collapsed into the drift component. In the case of Q-learning, the system can be solved explicitly within each region $\omega_j$.

Recall that we were working within a set $\omega_j$, where the maxima in the temporal difference are uniquely defined. For each $\omega_j$ we obtain a different $F_j$, and the continuous counterpart of the discrete learning procedure we obtained is still defined piece-wise: the behaviour at the boundaries of the sets $\omega_j$ is not yet determined. Following the theory of piece-wise smooth dynamical systems, we define the system of Theorem 2 as

$$
\frac{dQ_t}{dt} = F_j(Q_t) \quad \text{over } \omega_j, \forall j
$$

$$
\frac{dQ_t}{dt} \in \text{co}\left\{F_k(Q_t) \mid \forall k \in K \subset J\right\} \quad \text{when } Q_t \in \bigcap_{k \in K} \overline{\omega}_k, \forall K \subset J,
$$

where $\overline{\omega}_k$ is the closure of the set $\omega_k$, and $\text{co}\{\cdot\}$ denotes the convex hull of a set. In essence, for any point in the boundary of a single $\omega_j$, the system extends continuously and differentiably. When a point lies on multiple boundaries, the system is overspecified: we say that the derivative lies in the set of convex combinations of the differentiable extensions at that point. Hence the derivative of $Q_t$ is not uniquely defined at the intersections of the boundaries: this generalization of ODE systems is known as differential inclusion, and we refer the interested reader to Filippov (1988). The above system is an autonomous differential inclusion of the form $\dot{x} \in G(x)$, where $G(x)$ is non-empty, bounded, closed, convex, and upper-semicontinuous for all $x$. Thus, a Theorem from Filippov (1988) guar-

---

14A good introduction to the field is di Bernardo et al. (2008)
15The theory of differential inclusions outlined by Filippov covers the case of discontinuous derivatives. For more general discontinuities, see the treatment in Brogliato et al. (2014)
antees existence of a global solution in the sense of Filippov to the system for any initial condition.

### 3.2 Social Dilemma: Theory

We can now show how the machinery introduced in the previous section characterizes equilibria of the game of Section 2.3. Let us consider again that experiment: two identical, but independent, \(\varepsilon\)-greedy Q-learning agents, denoted \(A\) and \(B\), face an environment where they interact according to the social dilemma of Figure 1 after symmetric initialization. By assuming the state space is a singleton, this setting can be embedded in the model of Section 2.2.\(^{16}\) Since \(A = \{C, NC\}\), we have that \(Q_i\) is a vector in \(\mathbb{R}^4\) for all \(t\). We partition the space in the four regions \(\omega_{C,C}\), \(\omega_{C,NC}\), \(\omega_{NC,C}\) and \(\omega_{NC,NC}\): the subscripts indicate which action is the one currently (strictly) preferred by agents \(A\) and \(B\). By symmetry, the system will always be either in \(\omega_{C,C}\) or in \(\omega_{NC,NC}\), and therefore we concentrate on these. In particular, let us derive the fluid approximation within \(\omega_{C,C}\); the other case follows the same logic.

Over \(\omega_{C,C}\) the greedy action for both players is \(C\), which means that in every period an agent will cooperate with probability \(1 - \frac{\varepsilon}{2}\) and \(NC\) with probability \(\frac{\varepsilon}{2}\).\(^{17}\) Hence, with probability \((1 - \frac{\varepsilon}{2})^2\) the profile \(\{C, C\}\) is played and reward \(2g\) is collected — and similarly for other profiles. According to Theorem 1 and Equation (7), the dynamics of the Q-functions are well-approximated by an ODE whose solution has derivative equal to the expected temporal difference. Recalling the payoffs from Figure 1, we have that for agent \(i\) the limit process evolves as

\[
\begin{aligned}
\frac{dQ_i^t}{dt}(C) &= \alpha (1 - \frac{\varepsilon}{2}) \left[ (1 - \frac{\varepsilon}{2}) 2g + \frac{\varepsilon}{2}g + (\gamma - 1)Q_i^t(C) \right] \\
\frac{dQ_i^t}{dt}(NC) &= \alpha \frac{\varepsilon}{2} \left[ (1 - \frac{\varepsilon}{2})(2 + g) + 2\frac{\varepsilon}{2} + \gamma Q_i^t(C) - Q_i^t(NC) \right]
\end{aligned}
\]

(8)

The same argument can be applied to \(\omega_{NC,NC}\). Finally, following Filippov (1988), we can define the vector field on the boundary between \(\omega_{C,C}\) and \(\omega_{NC,NC}\), i.e. \(\partial\omega_{C,C} \cap \partial\omega_{NC,NC}\). The so-called sliding field is a convex combination of the vector fields on \(\omega_{C,C}\) and \(\omega_{NC,NC}\), with weights such that the component normal to the boundary of the resulting field vanishes.\(^{18}\)

\(^{16}\)The state can only transition to itself from \(t\) to \(t + 1\).

\(^{17}\)I.e., \(C\) is selected with probability \(1 - \varepsilon\) if the randomization device instructed the agent to be greedy, and with probability \(\frac{\varepsilon}{2}\) if the agent was instructed to play an action at random.

\(^{18}\)Formally, the boundary can be divided in three regions. A crossing region occurs where the normal components have the same sign, so that there is no need to define a field on the boundary. A repulsive region
Analysis of ODE

The stability points of this system are also the learning outcomes of Q-learning in discrete time. It can be verified that the only stationary point of $Q_t$ lying within a $\omega_j$ is located in $\omega_{NC,NC}$; denote it by $q_{NC}^{'eq}$. This confirms that the dominant strategy equilibrium of the social dilemma is indeed an equilibrium point of the learning dynamics, and a stable one. However, for large values of the growth factor $g$ there exists also another stationary point, located on the boundary $\overline{\omega}_{C,C} \cap \overline{\omega}_{NC,NC}$, which we denote by $q_{C}^{'eq}$. 

In Figures 3a, 3b and 3c we show the phase diagrams of this system for three different values of the growth factor $g$. These figures draw the vector fields in the $(Q^i(NC), Q^i(C))$-plane, with $Q^i(NC)$ and $Q^i(C)$ on the horizontal and vertical axis, respectively. The upper triangular section belongs to $\omega_{C,C}$ while the lower triangular one is $\omega_{NC,NC}$. While for $g = 1.8$ and $g = 1.4$ we recover both $q_{NC}^{'eq}$ and $q_{C}^{'eq}$, only the former survives in the case of $g = 1.1$. As displayed in Figure 3, the two equilibria draw closer in the phase space as the parameter $g$ decreases: as a result, randomness in the discrete Q-learning model may lead to convergence on the lower equilibrium. The equilibrium value of $q_{C}^{'eq}$ can be analytically calculated by equating to 0 the field on the boundary: we obtain that

$$q_{C}^{'eq} = \frac{1 + g + \sqrt{(g-1)(g-1 - \varepsilon g + \frac{\varepsilon^2 g}{2})}}{1 - \gamma}.$$ 

This is the continuous-time equivalent of the discrete system’s cooperative equilibrium, and there are a number of point to be noted in its regard.

Existence of cooperative equilibrium

First, notice that for values of $g$ close to 1 (see Figure 3c), the above equation has no real solution: the cooperative outcome disappears completely. These facts rationalize, at least partially, the results of the experiment from Section 2.3 given in Figure 2a, where we observed that, for low values of the public good there is never cooperation. Let $g(\varepsilon)$ denote the smallest $g$ such that the cooperative equilibrium exists, given $\varepsilon$; $g$ is strictly increasing in the exploration rate $\varepsilon$, and therefore we can also define $\varepsilon(g)$ as the largest exploration rate for which $q_{C}^{'eq}$ is well-defined. This one-to-one correspondence between payoffs of the game and the exploration rate reveals a fundamental driver of the learned outcome: when the exploration rate is too small, the algorithms too often play a symmetric profile of actions. We call this effect the coordi-
Figure 3: Stationary points are marked with a red dot. The domain of attraction of the cooperative outcome is green-shaded, the one for the non-cooperative outcome is blue-shaded.
nation bias. If both agents play the contribute action, their estimate of the value of contributing will be correct conditional on the opponent contributing as well. However, upon sufficient experimentation, one of the agents will stop contributing. The opponent will experience a sharp decrease in his own estimate of the value of contribution, which will lead him to stop contributing as well. Now neither agent is contributing, but the value of contribution remains biased: this is where the coordination bias comes into play. After avoiding contribution for some time, agents return to cooperation almost synchronously, thus reinforcing the coordination bias. Effectively, agents realize too slowly the downside from contributing to the pool when the opponent is playing NC, and not fast enough the benefit of non-contributing when the opponent cooperates. These biases can be overcome with large experimentation parameters.

![Figure 4: Maximum exploration rate $\epsilon$ to support the cooperative equilibrium, as a function of the growth rate. For all $\epsilon > \epsilon(g)$ there does not exist a steady-state where both algorithms learn to cooperate.](image)

Figure 4 shows how the maximum exploration rate $\epsilon$ varies with the value of the public good: as mentioned, it is an increasing function. The intuition is that as the growth rate increases, the relative benefit of non-contributing decreases (and vanishes completely when $g = 2$), so more and more exploration is needed to realize that NC is a dominant action. Notice also that for $g = 1.8$, as in Figure 3a, the exploration rate required to guarantee convergence on $\{NC, NC\}$ is about 0.7, which is considerably larger than standard exploration rates employed in practice.\(^{19}\)

\(^{19}\)The literature on Q-learning in games usually employs $\epsilon = 0.1$ or smaller, either fixed or decreasing
Sustaining cooperation  Note that the system of ODE, when simulated with small but discrete time steps, closely mimics the path of play of the discrete Q-learning. This behavior is clearly observed in Figure 5a and can be compared with Figure 2b. The dynamic provides some intuition on how the equilibrium $q^C_{eq}$ can be sustained: both agents cooperate and contribute to the public good, but with probability $\frac{\varepsilon}{2}$ an agent takes the non-contribute action and realizes its benefit. Over time, these random deviations make non-cooperation the preferred action. However, once an agent defects the opponent will also defect almost immediately after, which will then decrease the value of $NC$ for both players and reinforce the value of joint cooperation. However, if cooperation is not particularly valuable, the decrease in value of non-cooperation is less sharp: incentives to revert back to cooperation are weaker, and disappear completely for small $g$. Similarly, when the exploration rate is large enough the opponent contributes more often, and the decline in the value of $NC$ is slowed down.

![Figure 5](image-url)

(a) Cycles of play around the cooperative equilibrium in the discretized ODE system.  
(b) Proportion of time spent playing the cooperative outcome in $q^C_{eq}$.

These cycles disappear in the continuous-time system. The equilibrium lies on a boundary, which intuitively means that the forces on either side need to be balanced. Agents spend some “local time” playing cooperation, where the local time is related to the weight assigned to the field on $\omega_{C,C}$ on the boundary\(^\text{20}\). The rest of the time is spent playing non-cooperation, balancing out the infinitesimal forces around the equilibrium.

\(^\text{20}\)This intuition can be made formal using the idea of hysteresis loop around the boundary; see di Bernardo et al. (2008)
Our framework allows to characterize this local time; the proportion of time spent cooperating is given by

\[ \tau_C = \frac{\varepsilon^2g + \varepsilon - 2 - q_{eq}^C(\gamma - 1)(1 - \varepsilon)}{2(\varepsilon - 1)(1 + g + (\gamma - 1)q_{eq}^C)}. \]

Figure 5b plots this quantity and shows that the proportion of time spent playing the cooperative profile at equilibrium varies with the growth factor. If \( g \) is above \( g \), the fraction of time the agents cooperate is steeply increasing in the growth factor until \( g = 2 \), at which the agents are always cooperating; in this case the outcome is perfectly stable also in the discrete setting. For smaller growth factors, the agents do not contribute all the times, and the extent to which they do is linked to the width of the cycles of Figure 5a. This is also reflected in the equilibrium value \( q_{eq}^C \): it can be verified that, for all growth factors above \( g \), the value lies between the long-run value of always cooperating and that of never cooperating, and that as \( g \) increases \( q_{eq}^C \) increases as well.

This Section demonstrated the efficacy of fluid approximations in studying strategic interaction of algorithms. In fact, through comparative statics and equilibrium equations, we could explain the dependence of cooperation on the growth factor. We discovered a new effect, the coordination bias, in games played by Q-learning algorithms and we rationalized the presence of cycles in the discrete system through its continuous counterpart. In Section 5 we show how the coordination bias plays its role in a Bertrand game well-known in the literature, and how algorithmic design may overcome this bias.

## 4 General Learning Algorithms

Although popular in applications and research, Q-learning is a simple algorithm with its limitations. The approximation technique introduced when dealing with Q-learning can however be applied much more generally.

**Definition 1.** An algorithm is adaptive if it maintains a state \( \theta_t \) and it is updated incrementally in every step as

\[ \theta_t = \theta_{t-1} + \gamma_t T(\theta_{t-1}, X_t, t), \]

where \( \theta \in K \subset \mathbb{R}^d, \gamma_t \in \mathbb{R} \) and \( X_t \) is a random state in \( \mathbb{R}^e \).

This definition is rather general\(^{21}\): most of TD-based algorithms fall under this category, as well as many popular procedures such as Fictitious Play (Brown (1951)) and

\(^{21}\)The state \( \theta_t \) should not be confused with states of MDPs, and it can be taken to represent almost anything an AI agent is trying to learn. To emphasize this distinction the RL literature sometimes uses the term *epistemic state* in reference to \( \theta_t \): see Lu et al. (2021) for additional details.
the reinforcement model of Erev and Roth (1998). The state $X$ can be stochastic, and we denote its distribution over $\mathbb{R}^e$ by $\nu(X)$. In order to prove an analogous of Theorem 1, we introduce one additional assumption.

**Assumption A1.** The function $\gamma_t T(\theta, X, t)$ is Lipschitz continuous in all its arguments.

We can now formally state the following:

**Theorem 3.** Let $\theta^0_t = \theta^0_{t-1} + \gamma_t T(\theta^0_{t-1}, X_t, t)$ be an adaptive algorithm that satisfies Assumption A1, and let $K \subset \mathbb{R}^d$ be a compact set. The Cauchy problem

\[
\begin{cases}
\frac{d\Theta_t}{dt} = \gamma_t \int_{\mathbb{R}^e} T(\Theta_t, X_t, t) d\nu(X_t) \\
\Theta_0 = y_0
\end{cases}
\]

has a solution over $K$ for all $y_0 \in K$, and there exists a sequence of processes $\{\theta^n_t\}_{n \in \mathbb{N}}$ such that

\[
\lim_{n \to \infty} P\left\{\sup_{t \leq T} |\theta^n_t - \Theta_t| > \eta\right\}
\]

for all $T \geq 0$ and $\eta > 0$ such that $\{\Theta_t\}_{t \leq T} \subset K$.

The process $\Theta_t$, which evolves according to the expected adaptive component of the algorithm, is a continuous-time approximation of the original process.\(^{22}\) The proof of this result follows the steps outlined in Section 3. One important difference is that we allow for non-markovian algorithms through direct time dependence. This type of dependence is particularly useful when representing algorithms such as Fictitious Play or exploration rules such as the Upper Confidence Bound. We walk through the proof in Appendix B. In the examples that follow, we recover well-known results from the learning literature by applying the results from Theorem 3.

**Example 1.** Fictitious Play is a learning procedure that requires agents to best respond to the empirical distribution of the opponent’s strategies. Suppose there are two players, $A$ and $B$, and let $\beta(\cdot)$ be the multi-valued best-response function. In our language, the empirical distribution is the relevant algorithmic statistic, and its evolution for player $A$ is guided by the following:

\[
\theta^A_t = \theta^A_{t-1} + \frac{\beta(\theta^B_{t-1}) - \theta^A_{t-1}}{t}.
\]

\(^{22}\)Briefly, notice here the difference between this approximation technique and the traditional approximation methods developed by Benaim (1993): instead of focusing on the limiting behavior of the process, our work characterizes the dynamics of learning while collapsing uncertainty, with the intuitive interpretation of accelerating time. Additionally, for our original application of Q-learning, no good approximation technique applies: the results in Benveniste et al. (1990) only provide probabilistic bounds.
We can restrict attention to values of the θ such that the best response is unique, and for those values Theorem 3 yields the differential equations

\[
\begin{align*}
\frac{d\Theta^A_t}{dt} &= \beta(\Theta^B_t) - \Theta^A_t \\
\frac{d\Theta^B_t}{dt} &= \beta(\Theta^A_t) - \Theta^B_t \tag{9}
\end{align*}
\]

This system of ODE is known in the literature as Continuous-Time Fictitious Play: Harris (1998) shows that its rate of convergence is \( \frac{1}{t} \), and relates it to discrete-time fictitious play convergence. The derivation is intuitive: suppose that the game is played at a faster pace, but the players update only by a fraction of the original update. This high-frequency learning produces a smoother version of the original learning path. Notice that Theorem 3 would be able to accommodate also a continuous-time approximation of stochastic fictitious play (cfr. Fudenberg and Kreps (1993)), where the high frequency of updates collapses all uncertainty in its expected value.

We observe that Equation (9) for the social dilemma analyzed earlier is very simple: \( \beta(\Theta)_{NC} = 1 \) and \( \beta(\Theta)_{C} = 0 \) for any \( \Theta \). Therefore, for any values \( (\Theta^A_{NC}, \Theta^C_{NC}) \) the ODE system of each player \( i \) evolves as

\[
\begin{align*}
\frac{d\Theta^A_{NC}_t}{dt} &= \frac{1 - \Theta^A_{NC}}{t} \\
\frac{d\Theta^C_{NC}_t}{dt} &= -\frac{\Theta^C_{NC}}{t}
\end{align*}
\]

which implies that the only equilibrium of the system is \( \Theta^A_{NC} = 1, \Theta^C_{NC} = 0 \). This squares with the results in the literature that show Fictitious Play converges to Nash Equilibrium in games solvable by iterated elimination of dominated strategies.

**Example 2.** A simple adaptive implementation of regret minimization is the Polynomial Weights algorithm proposed by Cesa-Bianchi et al. (2005). The algorithm maintains weights \( w^i_t \) for each action \( i \) and updates the action chosen according to

\[
w^i_t = w^i_{t-1} - \lambda l^i_t w^i_{t-1},
\]

where \( l^i_t = r^*_t - r^i_t \) is the regret from obtaining payoff \( r^i \) instead of the optimal payoff \( r^* \) in period \( t \) and \( \lambda \) is a normalization parameter. The weights are then aggregated in a probability of choosing action \( i \) in each period with \( x^i_t = \frac{w^i_t}{\sum_j w^j_t} \).

Theorem 3 suggest that we look at an incremental form of \( x^i_t \). Some algebra shows that
one can write the update of the probabilities as
\[ x_i^t = x_i^{t-1} + \frac{x_i^{t-1}(1 - \lambda l_i^{t})}{\sum_j x_j^{t-1}(1 - \lambda l_j^{t})} - x_i^{t-1}, \]
and therefore analyze the system of ODEs given by
\[ \frac{dx_i^t}{dt} = \frac{x_i^t(1 - \lambda l_i^t)}{\sum_j x_j^t(1 - \lambda l_j^t)} - x_i^t \quad \text{for all } i. \]

Klos et al. (2010) show that such a procedure shares the structure of the Replicator Dynamics from the evolutionary game theory literature.

In the social dilemma above, it is clear that the regret in any period will be \( l_{NC}^t = 0 \). The regret for the coordination action differs depending on the strategy of the opponent, but it is always positive. Let \( \lambda \) be such that \( \lambda \max l_{NC}^t = 1 \). We can show that
\[ \frac{dx_{NC}^t}{dt} = \frac{x_{NC}^t}{x_{NC}^t + x_{C}^t(1 - \lambda l_{C}^t)} - x_{NC}^t \]
\[ = x_{NC}^t \left( \frac{1}{1 - x_{C}^t(1 - \lambda l_{C}^t) - 1} \right) \geq 0 \]
and similarly \( \frac{dx_{C}^t}{dt} \leq 0 \). The only equilibrium of this system is \( x_{NC} = 1, x_{C} = 0 \): the no-regret algorithms defect in each period.

These two examples show the strength of our method. Not only do we recover well-known approximation from the learning literature, but in a simple game, we can characterize outcomes for a wide class of algorithms and compare their behavior. While it is simple to exclude collaborative outcomes for certain procedures, more complicated systems can effectively sustain cooperation. This type of analysis is robust to the parameters of the algorithms, thanks to the analytic form of the quantities involved. This opens the door to comparisons of learning systems and game forms, to the benefit of regulators and designers.

5 Betrand Pricing Game

In previous work on AI and pricing, questions about algorithmic design and their market implications have been consistently brought up: Is collusion algorithm-specific? If yes, is it possible to limit collusion through design choices? Asker et al. (2022) contribute to
this issue in a Bertrand pricing game. Their experiments show that, while asynchronous Q-learning algorithm learn to collude on supra-competitive prices, the synchronous variants of the same algorithms compete close to perfectly. We formally prove the same result using the continuous-time approximation derived in Section 4. We are able to analytically characterize the equilibria of the two systems in this Bertrand game, and we observe an intuitive pattern of short-sightedness that does not depend on the parameters of the model.

There are two firms, firm $A$ and firm $B$, which face a common demand for their product. We fix the market demand at $3 - \min\{p_A, p_B\}$, and if the two firms charge the same price they split demand equally. Suppose that each firm has 0 marginal cost, and for simplicity let the firms choose only between two prices: $p \in \{0.5, 2\}$. Profits equal price times individual demand. This Bertrand game has only one static Nash equilibrium: the profile $(0.5, 0.5)$. Asker et al. (2022) consider two variations of the Q-learning algorithm, both of which are greedy, i.e., the action taken is always the one with the highest estimated value.

(i) Asynchronous Greedy Q-learning: the algorithm updates only the Q-value of the action taken in each period;

(ii) Synchronous Greedy Q-learning: the algorithm updates all Q-values in each period, with the return that he could have obtained had he played the other action instead, but holding the opponent’s action fixed.

We plot both systems in Figure 6 for $\gamma = 0$. The figures depict the vector fields governing the dynamics of the continuous-time analogous of the Q-function, with the value of the competitive price on the vertical axis and that of the collusive price on the horizontal axis.

**Asynchronous Learning.** The fields when both firms play asynchronous Q-learning and both choose 0.5 and 2, respectively, are:

\[
\begin{align*}
\frac{dQ_{0.5}}{dt} &= \alpha \left(\frac{5}{8} + (\gamma - 1)Q_{0.5}\right) & \text{on } \omega_{0.5,0.5} \\
\frac{dQ_{2}}{dt} &= 0 & \text{on } \omega_{0.5,0.5} \\
\frac{dQ_{0.5}}{dt} &= 0 & \text{on } \omega_{0.5,0.5} \\
\frac{dQ_{2}}{dt} &= \alpha \left(1 + (\gamma - 1)Q_{2}\right) & \text{on } \omega_{0.5,0.5}
\end{align*}
\]

Figure 6a plots the fields within their domains. As shown in the picture, there are two equilibrium regions. For values of $Q_2 \leq Q_{0.5} = \frac{5}{8}$ the algorithms converge on the competitive outcome. However, for $Q_{0.5} \leq Q_2 = 1$ the algorithms collude. Notice also that the do-

\[\text{\footnotesize 23The choice of } \gamma = 0 \text{ reflects the specification of Asker et al. (2022), but we provide the vector fields for general values of } \gamma, \text{ and notice that the same intuitions apply.}\]
(a) Asynchronous updating.

(b) Synchronous updating.

Figure 6: The green-shaded area denotes the domain of attraction of the competitive outcome, while the blue-shaded area is the domain of attraction of the collusive outcome. The red dot and red lines are the equilibria of the systems.

mains of attraction of these equilibria are profoundly unbalanced: in order to converge on competition, Q-learning needs to be initialized with a strong bias towards competition. Otherwise, the collusive outcome is an attractor. The algorithms show a bias towards collusion: the coordination bias introduced in Section 3.2 again drives learning. Because most observations of the returns from a supra-competitive price are obtained when colluding, the estimates of returns from charging a price of 2 are biased during a competitive phase. The persistence of this bias amounts to insufficient exploration of alternative strategies.

Synchronous Learning. Instead, if both firms adopt Synchronous Q-learning, the system is described by the following fields:

\[
\begin{align*}
\frac{dQ_{0.5}}{dt} &= \alpha \left( \frac{3}{8} + (\gamma - 1)Q_{0.5} \right) \quad \text{on } \omega_{0.5,0.5} \\
\frac{dQ_{2}}{dt} &= \alpha \left( \gamma Q_{0.5} - Q_{2} \right) \quad \text{on } \omega_{0.5,0.5} \\
\frac{dQ_{0.5}}{dt} &= \alpha \left( \frac{3}{8} + \gamma Q_{2} - Q_{0.5} \right) \quad \text{on } \omega_{2.2} \\
\frac{dQ_{2}}{dt} &= \alpha \left( 1 + (\gamma - 1)Q_{2} \right) \quad \text{on } \omega_{2.2}
\end{align*}
\]

There is only one equilibrium, at \( Q_{0.5} = \frac{5}{8}, Q_{2} = 0 \). The plot of Figure 6b shows a clear pattern: the two firms can only converge on competition. There is no sliding along the boundary, everywhere the trajectories move from colluding to competing. When the two firms are colluding, all arrows point upward: the intuition is that they overestimate the
value of undercutting the opponent, because they do not internalize the effect that defecting from a collusive outcome will have on returns in the future. The counterfactual rewards do not account for the losses that will stem from a change in equilibrium. Once the firms start competing it is then impossible to revert back to collusion: the counterfactual return of a deviation is zero. This contrasts with the asynchronous case, where once agents begin to compete, they almost immediately revert back to collusion. The bias present in the asynchronous algorithm disappears in the synchronous version: actions are updated according to counterfactual returns, therefore the value of joint collusion is short-lived after competition begins. These intuitions are independent of the parameter $\gamma$: even with high discount factors, counterfactual returns do not account for the future losses generated by a short-run defection. Similarly, the parameter $\alpha$ only increases the rate of convergence without meaningfully affecting the results.

This example highlights how the information provided to algorithms, and the use they make of it, are points of great importance both in the design of automated systems and in analyzing the outcomes of their interaction. The synchronous algorithm requires information about the price charged by the other firm, as well as the demand schedule, even in the case she doesn’t serve any customer. By providing this information to the algorithms, it may be possible to avoid supra-competitive pricing. With our continuous-time approximation we provide the theoretical tools necessary to understand the trade-offs involved with such design choices. Finally, note that the system we analyze is small, but the intuitions carry over to larger and more complex environments.

6 Concluding Remarks

This paper studies the interaction of Artificial Intelligence algorithms in a strategic environment. Motivated by the growing experimental and empirical literature that documents unanticipated outcomes resulting from the interaction of AI agents, we propose a continuous-time approximation to these algorithms. Our approach simplifies the analysis of such complex systems to the study of a system of ODEs, which can often be solved analytically. Moreover, we shed light on unexpected experimental outcomes by showing that they result from dynamics that can only be understood from the system of ODEs. We first present our results in the context of Q-learning, a popular Reinforcement Learning algorithm, and then apply them to a social dilemma modeled as a two-agent game. Within this setting, we prove that the continuous-time limit well approximates the original dynamics, and we characterize the possible learning outcomes.

We also show that the scope of our continuous-time toolbox can be broadened to cover
a range of very general algorithms: the approach we derive can tackle some forms of non-markovian environments, while maintaining its transparency. Conveniently, for some better known algorithms we retrieve approximations that have been know to the literature, albeit previously derived in a more ad-hoc fashion.

We view our paper as contributing to the emerging literature on the interaction between Artificial Intelligence and economic systems. In particular, we provide an approach that can be employed in the theoretical analysis of applied algorithms, which in turn opens to further research towards more effective design and regulation of markets where AI agents are actual participants, and not only support tools.

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Appendix A Omitted Proofs of Section 3

Lemma 2. Let $D$ be a compact ball in $\mathbb{R}^{|S| \times |S| \times |A|}$. For every $\omega_i$, there exists a sequence $\{\varepsilon^n\}_n > 0$ with $\lim_{n \to \infty} \varepsilon^n = 0$ such that

$$\lim_{n \to \infty} \sup_{x \in \omega_i \cap D} \lambda^n \int_{|z-x| > \varepsilon^n} |z-x| |\mu^n(x, dz) = 0$$

Moreover,

$$\sup_n \sup_{x \in \omega_i \cap D} \lambda^n \int_{\omega_i \cap D} |z-x| |\mu^n(x, dz) < \infty$$

Proof. Fix $\omega_i$. Since $\overline{\omega_i} \cap D$ is compact and within $\overline{\omega_i} \lambda^n T D_t$ is dominated along each dimension by the continuous function

$$|r_t + \gamma \max_{a'} Q_t(s_{t+1}, a') - Q_t(s, a)|$$

$TD_t$ must be bounded for all $t$. Let $M = \sup_{t, x \in \omega_i \cap D} TD_t$ across dimension, and note that $M < +\infty$. Let then $\{\frac{M}{n}\}_n$ be a sequence satisfying the assumptions of the Lemma, and notice that $\int_{|z-x| > \varepsilon^n} |z-x| |\mu^n(x, dz) = 0$ for all $x$ and $n$. We thus proved the first claim. The second claim follows a simple observation: $|z-x| |\mu^n(x, dz) \leq \frac{M}{n}$ for all $x$. Since $\lambda^n = n$, we obtain that

$$\sup_n \sup_{x \in \omega_i \cap D} \lambda^n \int_{\omega_i \cap D} |z-x| |\mu^n(x, dz) = M < \infty$$

which concludes the proof. \qed

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Theorem (1, Formal). Let $\Omega = \{\omega_j\}_{j \in J}$ be an appropriately defined collection of subsets of $\mathbb{R}^{|I| \times |S| \times |A|}$. For every $j \in J$ there exists:

(i) A sequence of continuous-time processes $\{Q^n\}_{n \in \mathbb{N}}$, where $Q^0$ is defined by Equation (3), and $\mathbb{E}[Q^n_t - Q^n_{t-1}] = \mathbb{E}[Q^0_t - Q^0_{t-1}]$;

(ii) A pair $(A_j, b_j)$ where $A_j$ is a matrix and $b_j$ is a vector for every $\omega_j$;

such that the Cauchy problem

$$
\begin{cases}
\frac{dQ_t}{dt} = A_j Q_t + b_j \\
Q_0 = q_0 \in \omega_j
\end{cases}
$$

has a solution over $\omega_j$, and its solution satisfies

$$
\lim_{n \to \infty} P\left\{ \sup_{t \leq T} |Q^n_t - Q_t| > \eta \right\} = 0,
$$

for all $T \geq 0$ and for all $\eta > 0$ over $\omega_j$.

Proof. Fix a compact ball $D \subset \mathbb{R}^{|I| \times |S| \times |A|}$. Let $\Omega$ be defined as the collection of sets

$$
\overline{\omega}_{a_1, \ldots, a_I} = \left\{ Q \in \mathbb{R}^{|I| \times |S| \times |A|} \text{ such that } a_i \in \text{argmax} Q^i(a,s) \text{ for all } i \right\}
$$

We define the sequence of processes $\{Q^n\}_n$ over the measurable state space $\omega_j \cap D$ with the Borel intersection-sigma algebra as $dQ^n_t = \frac{\alpha}{n} TD_t dN^n_t$ where $N^n_t$ is a Poisson point process of rate $\lambda^n = n$. The processes $Q^n_t$ are pure jump Markov processes with right-continuous sample paths, and their exit distribution is $\mu^n(x, Z)$ measurable for any Borel set $Z$. Finally, let the pair $(A_j, b_j)$ over $\omega_j$ be defined as the pair such that $A_j Q + b_j = \mathbb{E}[TD_t]$. The Cauchy problem

$$
\begin{cases}
\frac{dQ_t}{dt} = A_j Q_t + b_j \\
Q_0 = q_0 \in \omega_j
\end{cases}
$$

has a solution over its domain because it is a linear autonomous system, and as such it is Lipschitz. To prove the second part of the Theorem, we make use of the following theorem from Kurtz (1970).

Theorem 2.11. Suppose there exists $E \subset \mathbb{R}^k$, a function $F: E \to \mathbb{R}^k$ and a constant $M$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in E$ and

$$
\lim_{n \to \infty} \sup_{x \in E_n \cup E} |F_n(x) - F(x)| = 0
$$
Let $X(t, x_0), 0 \leq t \leq T, x_0 \in E$ satisfy

$$X(0, x_0) = x_0, \quad \dot{X}(t, x_0) = F(X(t, x_0))$$

Suppose additionally that the sequence $F^n$ satisfies the conditions of Lemma 2, then for every $\eta > 0$

$$\lim_{n \to \infty} P\left\{ \sup_{t \leq T} |X^n(t) - X(t, x_0)| \geq \eta \right\} = 0$$

Let $E = \omega_i \cap D$. If $X(t, x_0) = Q$, we can verify that the assumptions of the Theorem hold:

- since $F^n = F$ is linear, it is clear that $|F(x) - F(y)| \leq M|x - y|$ holds,
- $\lim_n \sup_{x \in \omega_j} |F_n(x) - F(x)| = 0$ is satisfied by definition of $F^n = F$,
- the conditions of Lemma 2 are verified.

Then, Theorem 2.11 implies that for every $\eta > 0$

$$\lim_{n \to \infty} P\left\{ \sup_{t \leq T} |Q^n_l - Q_l| \geq \eta \right\} = 0$$

**Proof of Theorem 2.** This result relies on the following theorem from Filippov (1988):

**Theorem 1, Chapter 2, Section 7.** Let $D$ be a compact domain. Let $G(t, x)$ be a nonempty, bounded, closed, convex set-valued function that is upper semicontinuous in $t, x$ for all $(t, x) \in D$. Then for any point $(t_0, x_0) \in D$ there exists a solution of the problem

$$\dot{x} \in G(t, x), \quad x(t_0) = x_0$$

and if the domain $D$ contains a cylinder $Z(t_0 \leq t \leq t_0 + a, \ |x - x_0| \leq b)$, the solution exists at least on the interval

$$t_0 \leq t \leq t_0 + d, \quad d = \min \left\{ a; \frac{b}{m} \right\}, \quad m = \sup_Z |G(t, x)|$$

Let $G(t, x)$ be defined as

$$F_j(x) \quad \text{over } \omega_j \forall j$$

$$\text{co} \left\{ F_k(x) \mid \forall k \in K \subset J \right\} \quad \text{over } \bigcap_{k \in K} \omega_k \forall K \subset J$$
where $F_j(x) = F^\|_j(x)$ is the function defined in Equation (5) on $\omega_j$, and notice that $G$ is time-invariant. and let $D$ be a compact ball in $\mathbb{R} \times \mathbb{R}^{[I] \times [S] \times [A]}$. We show that $G$ satisfies all the conditions:

- it is nonempty over $\mathbb{R} \times \mathbb{R}^{[I] \times [S] \times [A]}$,
- it is bounded everywhere, since each individual $F_j$ is bounded over $\omega_j \cup D$,
- it is closed and convex, as it is clear from the definition above,
- it is upper semicontinuous: to see this, select a convergent sequence in the domain. If the sequence is entirely contained in an $\omega_j$, then continuity is clear. Instead, suppose that the sequence lies in an $\omega_j$ but its limit lies on $\partial \omega_j$. Upper semicontinuity is guaranteed by the definition of $G$ as a convex combination of $F$ over the overlapping boundaries. Finally, suppose that the sequence lies within a $\bigcap_{k \in K} \overline{\omega_k}$ for some $K \subset J$, but the limit lies in a different intersection $\bigcap_{h \in H} \overline{\alpha_h}$. Since the sequence is convergent, it has to be the case that $K \subset H$, therefore upper semicontinuity is satisfied.

Additionally, note that the domain is any compact ball, therefore we can find a $D$ that contains any cylinder $Z$ and a solution to this differential inclusion is global within any compact subset of $\mathbb{R}^{[I] \times [S] \times [A]}$.

### Appendix B  Omitted Proofs of Section 4

We follow the argument made in Section 3. First, we define a process $k^0_t$ as a compound Poisson process with rate $\lambda = 1$ whose increment is 1. This process essentially counts the ticks of the Poisson clock. Then, we define a process $X^0_t = X_{k^0_t}$, and note that this process is Poisson. We can now embed the process $\theta^0$ in continuous time: we will denote by $\theta^0_t$ a compound Poisson process of rate $\lambda$ whose jump at time $t$ is defined by $\gamma_k T(\theta^0_t, X^0_t, k_t)$.

From this definition it is easy to extrapolate what the sequence of processes should look like: first, $k^n_t$ is a compound Poisson process with rate $\lambda^n = n$ whose increment is $\frac{1}{n}$. Then, we define $\theta^n_t$ as the compound Poisson process of the same rate with update $\frac{1}{n} \gamma_k T(\theta^n_t, X^n_t, k^n_t)$, where clearly $X^n_t = X_{k^n_t}$. This is, as we discussed in the main body, an acceleration of the original process. Updates arrive more frequently but each update affects the state $\theta^n_t$ by a smaller fraction. This ensures that in the limit all uncertainty will wash away. We can define the expected update for these processes as $\mathbb{E}_X [\frac{1}{n} \gamma_k T(\theta^n_t, X^n_t, k^n_t)]$
where the expectation is taken with respect to the stochastic state $X_t^n$. The expected jump over one unit of time can be defined as

$$F^n(y) = \lambda^n \mathbb{E}_X \left[ \frac{1}{n} \gamma_t^n T(y, X_t^n, k_t^n) \right]$$

for the main process $\theta_t^n$ and similarly for $k_t^n$. Focus on $\lim_{n \to \infty} F^n(y) = F(y)$ for all $n$ and notice that $F(y) = \gamma_t \mathbb{E}[T(y, X_t, t)]$, which is the derivative of $\Theta_t$. In fact, $\lim_{n \to \infty} k_t^n = t$, and $X_t$ is distributed according to $\nu$. Assumption A1 guarantees that the function $F(y)$ is Lipschitz continuous. The proof of Lemma 2 did not depend on the specific form of the algorithm, and the same argument applies to this setting: the function $F(y)$ is dominated on its compact domain by a continuous function, therefore it is bounded and the two technical conditions are satisfied. Finally, again by Assumption A1, the system

$$\begin{cases}
  \frac{d\Theta_t}{dt} = \gamma_t \int_\mathbb{R} T(\Theta_t, X_t, t) d\nu(X_t) \\
  \Theta_0 = y_0
\end{cases}$$

has a solution for every $y_0$ in its domain. Therefore, we can apply Theorem 2.11 to show that

$$\lim_{n \to \infty} P\left\{ \sup_{t \leq T} |\theta_t^n - \Theta_t| > \eta \right\}.$$ 

### Appendix C  Analysis of the Social Dilemma

Notice that, because the field is symmetric and pairwise independent in its coordinates, we can work with the 2-dimensional vector field of a single agent. The boundary is defined as $\Sigma = \{ q \in \mathbb{R}^2 : c \cdot q = 0 \}$ where $c = (1, -1)$ and $\cdot$ denotes the usual dot product. Using the Filippov convention, we can further divide $\Sigma$ in three regions:

- a crossing region, $\Sigma^c = \{ q : (c \cdot (A_C q + b_C))(c \cdot (A_{NC} q + b_{NC})) > 0 \}$
- a repulsive region, $\Sigma^r = \{ q : c \cdot (A_C q + b_C) > 0, c \cdot (A_{NC} q + b_{NC}) < 0 \}$
- a sliding region, $\Sigma^s = \{ q : c \cdot (A_C q + b_C) < 0, c \cdot (A_{NC} q + b_{NC}) > 0 \}$

We can define the sliding solution as the field $\frac{dQ}{dt} = F^s(Q)$ over the sliding region where

$$F^s(Q) = \frac{c \cdot (A_{NC} Q + b_{NC})(A_C Q + b_C) - c \cdot (A_C Q + b_C)(A_{NC} Q + b_{NC})}{c \cdot (A_{NC} Q + b_{NC}) - c \cdot (A_C Q + b_C)}$$

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The relative time spent on $\omega_{C,C}$ at point $Q$ is defined as

$$\tau_C = \frac{c \cdot (A_{NC}Q + b_{NC})}{c \cdot (A_{NC}Q + b_{NC}) - c \cdot (A_CQ + b_C)}$$

The sliding vector field becomes

$$\frac{dQ_j}{dt} = \alpha \left( \frac{\frac{1}{2} \varepsilon g(2 - \varepsilon)(g - 1) + (2g + (\gamma - 1)Q_j)(2 + (\gamma - 1)Q_j))}{2(1 + g + (\gamma - 1)Q_j)} \right)$$

for every direction $j$. By setting the field equal to zero and solving for $Q_j$, we find that there is an equilibrium at

$$q_{eq}^{C,j} = \frac{1 + g - \sqrt{(g - 1)(g - 1 - \varepsilon g + \frac{\varepsilon^2 g}{2})}}{(\gamma - 1)}$$

for all $j$. For the values of the parameters we chose in our simulation, for example $\varepsilon = 0.1$, $\gamma = 0.9$, and $g = 1.8$, we get $q_{eq}^{C} = (35.09, 35.09)$. 