Bethe ansatz for the one-dimensional small-polaron model with open boundary conditions

Heng Fan\textsuperscript{a,b}, Xi-Wen Guan\textsuperscript{a,c}
\textsuperscript{a} CCAST(World Laboratory)
P.O.Box 8730,Beijing 100080,China
\textsuperscript{b} Institute of Modern Physics,Northwest University
P.O.Box 105,Xian,710069,China\textsuperscript{*}
\textsuperscript{c} Center for Theoretical Physics and Department of Physics,
Jilin University, Changchun 130023, China

March 24, 2022

Abstract
The one-dimensional small-polaron model with open boundary conditions is considered in the framework of the quantum inverse scattering method. The spin model which is equivalent to the small-polaron model is the Heisenberg XXZ spin chain in an external magnetic field. For spin model, the solutions of the reflection equation (RE) and the dual RE are obtained. The eigenvalues and eigenvectors problems are solved by using the algebraic Bethe ansatz method. The case of the fermion model is also studied, the commuting of the transfer matrix can be proved, and the eigenvalues and the Bethe ansatz equations for fermion model are obtained.

PACS: 7510Jm, 0520, 0530.
Keywords: One-dimensional small-polaron model, Heisenberg XXZ spin chain, Yang-Baxter equation, reflection equation.

\textsuperscript{*}Mailing address
1 Introduction

Recently, much more attention has been paid to study the integrable models with open boundary conditions [1-10]. This was initiated by Sklyanin [1], who proposed a systematic approach to handle integrable models with reflecting boundary conditions in which the so-called RE plays a key role.

Sklyanin assume that the R-matrix of the vertex models are $P$ and $T$ symmetric $R_{12} = PR_{12}P = R_{21}, R_{12}^{t_1t_2} = R_{12}$, where $t_i, i = 1, 2$ means transposition in the $i$-th space, $P$ is the permutation operator $Px \otimes y = y \otimes x$. Because only few models satisfy these properties, Mezincescu et al [3] extended Sklyanin’s formalism for constructing integrable open chains to the case where the R matrix is only $PT$-invariant $PR_{12}P = R_{12}^{t_1t_2}$. It is later realized that the unitarity and cross-unitarity relations of the R-matrix are enough to prove integrable open boundary conditions.

In the framework of the quantum inverse scattering method (QISM) [11], a lot of models with open boundary conditions are solved by using the algebraic Bethe ansatz method [2-7]. And the boundary cross-unitarity is also proposed for open boundary conditions problems [8].

In this paper, we will study the one-dimensional small-polaron model which describes the motion of an additional electron in a polar crystal [12]. Via the Jordan-Wigner transformation, the one-dimensional small-polaron model is equivalent to the Heisenberg $XXZ$ spin chain with an external magnetic field parallel to the $z$ direction [13]. We will consider the open boundary conditions for this Heisenberg $XXZ$ spin chain. The R-matrix satisfy the unitarity and the cross-unitarity relations, so we can formulate the RE and the dual RE. With the help of these relations we can prove the commuting of the transfer matrix with open boundary conditions. The commuting transfer matrix can give an infinite set of conserved quantities, it is general in this sense that the model under consideration is integrable. It is well known that the first non-trivial conserved quantities is the Hamiltonian. Solving the RE and the dual RE, we find the diagonal solutions $K$ matrices of those REs. From the transfer matrix with open boundary conditions, we obtain the Hamiltonian which includes the boundary terms in the Heisenberg $XXZ$ spin chain with an external magnetic field parallel to the $z$ direction. Those boundary terms are determined by the boundary $K$ matrices. Using the algebraic Bethe ansatz method, we find the eigenvalues and eigenvectors of the transfer matrix. So the eigenvalues of the Hamiltonian can also be obtained.

Using similar procedure, the one-dimensional small-polaron model with open boundary conditions is studied. The eigenvalues of the transfer matrix are obtained by using the algebraic Bethe ansatz method.

The paper is organized as follows: In section 2, we will introduce the model and
formulate the RE and dual RE. We then prove the commuting of the transfer matrix with open boundary conditions in section 3. In section 4, we use the algebraic Bethe ansatz method to solve the eigenvalues and eigenvectors problems. Section 5 will be devoted to the fermion case, and the eigenvalues and the corresponding Bethe ansatz equations are obtained for the one-dimensional small-polaron model with open boundary conditions. Finally, we will give a summary and discussions.

2 The model and the RE and dual RE

As mentioned in the introduction, the one-dimensional small-polaron model describes the motion of an additional electron in a polar crystal [12]. The Hamiltonian takes the form

$$H = W \sum_{j=1}^{N} n_j - J \sum_{j=1}^{N} (a_{j+1}^+ a_j + a_j^+ a_{j+1}) + V \sum_{j=1}^{N} n_j n_{j+1}, \tag{1}$$

where $a_j^+, a_j$ are fermion creation and annihilation operators at lattice site $j$, and satisfy the anticommutation relations

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \{a_i, a_j^+\} = \delta_{ij}, \tag{2}$$

$n_j = a_j^+ a_j$ is the occupation number operator. $W, J$ and $V$ can be found in the paper of Fedyanin et al [12].

Applying the Jordan-Wigner transformation for $a_j, a_j^+$ and $n_j$

$$a_j = \exp \left( i\pi \sum_{l=1}^{j-1} \sigma_l^+ \sigma_l^- \right) \sigma_j^-,$$

$$a_j^+ = \exp \left( i\pi \sum_{l=1}^{j-1} \sigma_l^+ \sigma_l^- \right) \sigma_j^+,$$

$$n_j = \frac{(1 + \sigma_j^z)}{2}, \tag{3}$$

where $\sigma_j^z = \frac{1}{2}(\sigma_j^x \pm i \sigma_j^y)$ and $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are Pauli operators at lattice site $j$. The Hamiltonian of the one-dimensional small-polaron model changes to the Heisenberg $XXZ$ spin chain with an external magnetic field parallel to $z$ direction.

$$H = -\sum_{j=1}^{N} \left( J(\sigma_j^+ \sigma_{j+1}^+ + \sigma_j^- \sigma_{j+1}^-) - \frac{V}{4} \sigma_j^z \sigma_{j+1}^z \right) + \frac{W + V}{2} \sum_{j=1}^{N} \sigma_j^z + \text{cost.} \tag{4}$$

3
Here the model is considered in the periodic boundary conditions.

From this Hamiltonian, the local monodromy matrix \( L \) operator is found to be [13]

\[
L_j(\lambda) = \left( \frac{a'_+b'_+ + a'_-b'_-}{c\sigma^z_j} \right) \left( \frac{a'_+b'_+ - a'_-b'_-}{2} \right)
\]

(5)

where we have used the notations

\[
a'_+ : b'_+ : a'_- : b'_- : c = \xi(\lambda) \sin(\lambda + 2\eta) : \xi^{-1}(\lambda) \sin(\lambda) : \xi^{-1}(\lambda) \sin(\lambda + 2\eta) : \xi(\lambda) \sin(\lambda) : \sin(2\eta),
\]

(6)

with

\[
\xi(\lambda) = \sec(\alpha) \cos(\lambda + \alpha) \sec(\lambda).
\]

(7)

And \( J, V \) and \( W \) are given by

\[
J : -\frac{V}{2} : (W + V) = 1 : \cos(2\eta) : 2 \sin(2\eta) \tan(\alpha).
\]

(8)

It is well known that for periodic boundary condition problems, the Yang-Baxter relation [14,15] plays an essential role which takes the form

\[
R_{12}(\lambda, \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda, \mu),
\]

(9)

where the indices of \( L \) represent the auxiliary spaces, and the R-matrix can be found to be:

\[
R_{12}(\lambda, \mu) \equiv \begin{pmatrix}
    a_+(\lambda, \mu) & 0 & 0 & 0 \\
    0 & b_-(\lambda, \mu) & c(\lambda, \mu) & 0 \\
    0 & c(\lambda, \mu) & b_+(\lambda, \mu) & 0 \\
    0 & 0 & 0 & a_-(\lambda, \mu)
\end{pmatrix},
\]

(10)

the non-zero elements of the R-matrix are defined as

\[
a_+(\lambda, \mu) = \xi(\lambda)\xi^{-1}(\mu) \sin(\lambda - \mu - 2\eta), \quad a_-(\lambda, \mu) = \xi^{-1}(\lambda)\xi(\mu) \sin(\lambda - \mu - 2\eta),
\]

\[
b_-(\lambda, \mu) = \xi^{-1}(\lambda)\xi^{-1}(\mu) \sin(\lambda - \mu), \quad b_+(\lambda, \mu) = \xi(\lambda)\xi(\mu) \sin(\lambda - \mu),
\]

\[
c(\lambda, \mu) = \sin(2\eta),
\]

(11)

here \( \eta \) is the anisotropic parameter. In the framework of the QISM, The row-to-row monodromy matrix is defined as:

\[
T(\lambda) = L_N(\lambda) \cdots L_1(\lambda)
\]

(12)
Here we know that the indices of the operator $L$ mean the quantum spaces. It is known that using repeatedly the Yang-Baxter relation (9), we have

$$R_{12}(\lambda, \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda, \mu). \quad (13)$$

For periodic boundary conditions, we define the transfer matrix as trace of monodromy matrix $T$ in the auxiliary space. Here, to avoid confusion, we represent the auxiliary space by $\bar{0}$.\n
$$t_{peri.}(\lambda) = \text{tr}_{\bar{0}}T_0(\lambda) = \text{tr}_{\bar{0}}L_{N0}(\lambda)\cdots L_{10}(\lambda). \quad (14)$$

The Hamiltonian (4) can be obtained from this transfer matrix

$$H = -\sin(2\eta)\frac{d}{d\lambda} \ln t_{peri.}(\lambda)|_{\lambda=0}. \quad (15)$$

What we study in this paper is the reflecting open boundary conditions. For open boundary condition case, besides the Yang-Baxter relation, we also need the RE and the dual RE. Generally, we write the RE as the following form

$$R_{12}(\lambda, \mu)K_1(\lambda)R_{21}(\mu, -\lambda)K_2(\mu) = K_2(\mu)R_{12}(\lambda, -\mu)K_1(\lambda)R_{21}(-\mu, -\lambda). \quad (16)$$

The form of the dual RE depends on the unitarity and the cross-unitarity relations of the R-matrix. For the model considering in this paper, the unitarity and cross-unitarity relations of the R-matrix are written as

$$R_{12}(\lambda, \mu)R_{21}(\mu, \lambda) = \sin(2\eta + \lambda - \mu)\sin(2\eta - \lambda + \mu) \equiv \rho(\lambda - \mu), \quad (17)$$

$$X_{12}(\lambda, \mu + 4\eta)R_{21}(\mu, \lambda) = -\sin(\lambda - \mu)\sin(\lambda - \mu - 4\eta) \equiv \tilde{\rho}(\lambda - \mu). \quad (18)$$

Here, for convenience, we have introduced the notations

$$X_{12}(\lambda, \mu) \equiv \frac{\xi(\mu)}{\xi(\mu - 4\eta)}M_1(\mu)R_{12}(\lambda, \mu)M_1(\mu),$$

$$M(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \xi(\lambda - 4\eta)\xi^{-1}(\lambda) \end{pmatrix}. \quad (19)$$

It is also convenient to write here another form of the cross-unitarity relation:

$$R_{12}^t(\lambda, \mu)Y_{12}(\mu - 4\eta, \lambda) = \tilde{\rho}(\lambda - \mu + 4\eta),$$

$$Y_{12}(\lambda, \mu) = \frac{\xi(\lambda + 4\eta)}{\xi(\lambda)}M_2(\lambda)R_{12}(\lambda, \mu)M_2(\lambda). \quad (20)$$
Thus, we take the dual RE as the following form:

\[
R_{12}(-\lambda, -\mu)K_1^+(\lambda)Y_{21}(-\mu - 4\eta, \lambda)K_2^+(\mu) = K_2^+(\mu)X_{12}(-\lambda, \mu + 4\eta)K_1(\lambda)R_{21}(\mu, \lambda). \tag{21}
\]

Following the method of Sklyanin, we define the double-row-monodromy matrix as:

\[
T(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda). \tag{22}
\]

Using the Yang-Baxter relation (13) and the RE, it can be proved that the double-row-monodromy matrix also satisfy the RE:

\[
R_{12}(\lambda, \mu)T_1(\lambda)R_{21}(\mu, -\lambda)T_2(\mu) = T_2(\mu)R_{12}(\lambda, -\mu)T_1(\lambda)R_{21}(-\mu, -\lambda). \tag{23}
\]

We define the transfer matrix for open boundary conditions as:

\[
t(\lambda) = trK^+(\lambda)T(\lambda). \tag{24}
\]

### 3 Integrable open boundary conditions

In this section, we will prove that the transfer matrix defined above for open boundary conditions constitute a commuting family. And we know that it is generally in this sense we mean the model is integrable. Now, let us prove the transfer matrix commute for different spectral parameters.

\[
t(\lambda)t(\mu) = tr_1K_1^+(\lambda)T_1(\lambda)tr_2K_2^+(\mu)T_2(\mu) = tr_{12}K_1^+(\lambda)K_2^+(\mu)T_1^t(\lambda)T_2^t(\mu),
\]

here we take transposition \(t_1\) in space 1. Now we insert the cross-unitarity relation (18), we have

\[
\cdots = \frac{1}{\tilde{\rho}(-\lambda - \mu)} tr_{12}K_1^+(\lambda)K_2^+(\mu)X_{12}^t(-\lambda, \mu + 4\eta)R_{21}(\mu, -\lambda)T_1^t(\lambda)T_2(\mu),
\]

\[
= \frac{1}{\tilde{\rho}(-\lambda - \mu)} tr_{12}\{K_1^+(\lambda)K_2^+(\mu)X_{12}^t(-\lambda, \mu + 4\eta)\}T_1(\lambda)R_{21}(\mu, -\lambda)T_2(\mu),
\]

\[
= \frac{1}{\tilde{\rho}(-\lambda - \mu)} tr_{12}\{K_2^+(\mu)X_{12}(-\lambda, \mu + 4\eta)K_1^+(\lambda)\}T_1(\lambda)R_{21}(\mu, -\lambda)T_2(\mu).
\]

In the above calculation, we take transposition in the first space. Insert the unitarity relation of the R-matrix (17), and use the RE (23) and the dual RE (21), we have

\[
\cdots = \frac{1}{\tilde{\rho}(-\lambda - \mu)\rho(\lambda - \mu)} tr_{12}\{K_2^+(\mu)X_{12}(-\lambda, \mu + 4\eta)K_1^+(\lambda)R_{21}(\mu, \lambda)\}
\]
\[ \{ R_{12}(\lambda, \mu) T_1(\lambda) R_{21}(\mu, -\lambda) T_2(\mu) \} = \frac{1}{\tilde{\rho}(-\lambda - \mu) \rho(\lambda - \mu)} tr_{12} \{ R_{12}(-\lambda, -\mu) K_1^{+}(\lambda) Y_{21}(-\mu - 4\eta, \lambda) K_2^{+}(\mu) \} \{ T_2(\mu) R_{12}(\lambda, -\mu) T_1(\lambda) R_{21}(-\mu, -\lambda) \}. \] (25)

Use again the unitarity relation (17) and the cross-unitarity relation (20), and considering the properties of the function

\[ \rho(\lambda - \mu) = \rho(\mu - \lambda), \]
\[ \tilde{\rho}(-\lambda - \mu) = \tilde{\rho}(\lambda + \mu + 4\eta), \] (26)

we have

\[
\cdots = \frac{1}{\tilde{\rho}(-\lambda - \mu)} tr_{12} \{ K_1^{+}(\lambda) Y_{21}(-\mu - 4\eta, \lambda) K_2^{+}(\mu) \} \{ T_2(\mu) R_{12}(\lambda, -\mu) T_1(\lambda) \}.
\]
\[
= \frac{1}{\tilde{\rho}(-\lambda - \mu)} tr_{12} \{ Y_{21}^{t}(\lambda) K_1^{+}(\lambda) Y_{21}^{t}(\mu) \} \{ T_2(\mu) T_1^{t}(\lambda) R_{12}^{t}(\lambda, -\mu) \}.
\]
\[
= tr_2 K_2^{+}(\mu) T_2(\mu) tr_1 K_1^{+}(\lambda) T_1(\lambda)
\]
\[
= t(\mu) t(\lambda)
\] (27)

Thus we have proved that the transfer matrix \( t(\lambda) \) forms a commuting family which gives an infinite set of conserved quantities.

\[ t(\lambda) t(\mu) = t(\mu) t(\lambda) \] (28)

One solution of the RE and dual RE corresponds to one transfer matrix. We have seen that in the proof of the integrability, the RE and dual RE are independent of each other.

By solving directly the RE and the dual RE, we find the following diagonal solutions to the RE and the dual RE:

\[ K(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\xi(-\lambda)}{\xi(\lambda)} \frac{\sin(\psi + \lambda)}{\sin(\psi - \lambda)} \end{pmatrix}, \] (29)
\[ K^{+}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\xi(\lambda)}{\xi(-\lambda)} \frac{\sin(\psi^{+} - \lambda - 2\eta)}{\sin(\psi^{+} + \lambda + 2\eta)} \end{pmatrix}, \] (30)

where \( \psi \) and \( \psi^{+} \) are boundary parameters. We noticed that this diagonal solutions are also the general solutions of the RE (16) and the dual RE (21).

Explicitly, the transfer matrix can be written as:

\[ t(\lambda) = tr_0 K_{0}^{+}(\lambda) L_{0\lambda} \cdots L_{10}(\lambda) K_0(\lambda) L_{10}(\lambda) \cdots L_{N0}(\lambda). \] (31)
Here we denote $\mathcal{L}(\lambda) \equiv L^{-1}(-\lambda)$. Up to a whole factor, we take the inverse of the $L$ operator as:

$$
\mathcal{L}_j(\lambda) = \left( \frac{a''_i + b''_i}{2} \frac{c \sigma^+_j}{2} + \frac{a''_i - b''_i}{2} \sigma^+_j \frac{c \sigma^+_j}{2} - \frac{a''_i - b''_i}{2} \sigma^+_j \right).
$$

(32)

where

$$
a''_+(\lambda) : b''_+(\lambda) : a''_-(\lambda) : b''_-(\lambda) : c
= \xi^{-1}(-\lambda) \sin(\lambda + 2\eta) : \xi(-\lambda) \sin(\lambda) : \xi(-\lambda) \sin(\lambda + 2\eta) : \xi^{-1}(-\lambda) \sin(\lambda) : \sin(2\eta),
$$

(33)

We define the Hamiltonian for open boundary conditions $H_{OB}$ as follows:

$$
H_{OB} \equiv -\frac{1}{2} \sin(2\eta) \frac{d}{d\lambda} \ln t(\lambda)|_{\lambda=0}
= -\sum_{j=1}^{N-1} H_{j,j+1} - \frac{1}{2} \sin(2\eta)K'_{1}(0) - \frac{tr_{0}K^{+}_{0}(0)L'_{N0}(0)P_{N0}}{trK^{+}(0)}. 
$$

(34)

Here

$$
H_{j,j+1} = L'_{j,j+1}(0)P_{j,j+1} = P_{j,j+1}\mathcal{L}'(0).
$$

(35)

And the relation

$$
tr_{0}K^{+}_{0}(0)L'_{N0}(0)P_{N0} = tr_{0}K^{+}_{0}(0)L_{N0}(0)\mathcal{L}_{N0}(0)
$$

(36)

has been used.

So, the open Heisenberg $XXZ$ spin chain with an external magnetic field parallel to $z$ direction is as follows:

$$
H_{OB} = -\sum_{j=1}^{N-1} \left( J(\sigma^+_j \sigma^-_{j+1} + \sigma^-_j \sigma^+_{j+1}) - \frac{V}{4} \sigma^+_j \sigma^-_{j+1} \right) + \frac{W + V}{2} \sum_{j=1}^{N-1} \sigma^+_j
+ \frac{1}{2} \sin(2\eta) \frac{\cos \psi}{\sin \psi} \sigma^+_1 + \frac{1}{2} \left( V + W - \sin(2\eta) \frac{\cos \psi^+}{\sin \psi^+} \right) \sigma^+_N + \text{cost.}
$$

(37)

4 Algebraic Bethe ansatz method

In this section, we will use the algebraic Bethe ansatz method to obtain the eigenvalues and eigenvectors of the transfer matrix with open boundary conditions. As
mentioned above, the double-row-monodromy matrix is defined as \( \mathcal{T}(\lambda) = T(\lambda) \times K(\lambda)T^{-1}(-\lambda) \), and we denote it as follows:

\[
\mathcal{T}(\lambda) = \left( \begin{array}{cc}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{array} \right) = T(\lambda)K(\lambda)T^{-1}(-\lambda)
\]

\[
= \left( \begin{array}{cc}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
\xi(\lambda) \sin(\psi + \lambda) & \xi(\lambda) \sin(\psi - \lambda)
\end{array} \right) \left( \begin{array}{cc}
\bar{A}(-\lambda) & \bar{B}(-\lambda) \\
\bar{C}(-\lambda) & \bar{D}(-\lambda)
\end{array} \right) \tag{38}
\]

We know that one can find easily the pseudovacuum state \(|0\rangle\), and acting the elements of the monodromy matrix \(T(\lambda)\) and \(T^{-1}(-\lambda)\) on this pseudovacuum state, we have

\[
C(\lambda)|0\rangle > = 0, \quad \bar{C}(-\lambda)|0\rangle > = 0,
\]
\[
B(\lambda)|0\rangle > \neq 0, \quad \bar{B}(-\lambda)|0\rangle > \neq 0,
\]
\[
A(\lambda)|0\rangle > = \alpha(\lambda)|0\rangle >, \quad \bar{A}(-\lambda) = \bar{\alpha}(-\lambda)|0\rangle >,
\]
\[
D(\lambda)|0\rangle > = \delta(\lambda)|0\rangle >, \quad \bar{D}(-\lambda) = \bar{\delta}(-\lambda)|0\rangle >, \tag{39}
\]

where

\[
\alpha(\lambda) = (\xi(\lambda) \sin(\lambda + 2\eta))^N,
\]
\[
\delta(\lambda) = (\xi(\lambda) \sin(\lambda))^N,
\]
\[
\bar{\alpha}(-\lambda) = \left(\xi^{-1}(-\lambda) \sin(\lambda + 2\eta)\right)^N,
\]
\[
\bar{\delta}(-\lambda) = \left(\xi^{-1}(-\lambda) \sin(\lambda)\right)^N. \tag{40}
\]

From the Yang-Baxter relation (13), one can obtain

\[
T^{-1}_2(\mu)R_{12}(\lambda, \mu)T_1(\lambda) = T_1(\lambda)R_{12}(\lambda, \mu)T^{-1}_2(\mu). \tag{41}
\]

Let \(\mu = -\lambda\), use the result \(C(\lambda)|0\rangle > = 0\), we have

\[
C(\lambda)\bar{B}(-\lambda)|0\rangle > = \frac{\xi(-\lambda)}{\xi(\lambda)} \frac{\sin(2\eta)}{\sin(2\lambda + 2\eta)} \bar{A}(-\lambda)A(\lambda)|0\rangle >
\]
\[
- \frac{\xi(-\lambda)}{\xi(\lambda)} \frac{\sin(2\eta)}{\sin(2\lambda + 2\eta)} D(\lambda)\bar{D}(-\lambda)|0\rangle >. \tag{42}
\]

For convenience, we introduce here a transformation

\[
\mathcal{D}(\lambda) = \mathcal{D}(\lambda) + \frac{\sin(2\eta)}{\sin(2\lambda + 2\eta)} \frac{\xi(-\lambda)}{\xi(\lambda)} A(\lambda). \tag{43}
\]
Then for double-row-monodromy matrix $\mathcal{T}$, acting its elements on the pseudovacuum state, and considering the above transformation, we have

$$\mathcal{C}(\lambda)|0> = 0, \quad \mathcal{B}(\lambda)|0> \neq 0,$$

$$\mathcal{A}(\lambda)|0> = \alpha(\lambda)\tilde{\alpha}(\lambda)|0>,$$

$$\tilde{\mathcal{D}}(\lambda)|0> = \frac{\xi(-\lambda)\sin(2\lambda)\sin(\psi + \lambda + 2\eta)}{\xi(\lambda)\sin(2\lambda + 2\eta)\sin(\psi - \lambda)}\delta(\lambda)\bar{\delta}(-\lambda)|0>.$$ (44)

With the help of the transformation (43), the transfer matrix for open boundary conditions can be written as:

$$t(\lambda) = \mathcal{A}(\lambda) + \frac{\xi(\lambda)}{\xi(-\lambda)\sin(\psi^+ + \lambda + 2\eta)}\mathcal{D}(\lambda)$$

$$= \frac{\sin(2\lambda + 4\eta)\sin(\psi^+ + \lambda)}{\sin(2\lambda + 2\eta)\sin(\psi^+ + \lambda + 2\eta)}\mathcal{A}(\lambda) + \frac{\xi(\lambda)}{\xi(-\lambda)\sin(\psi^+ + \lambda + 2\eta)}\tilde{\mathcal{D}}(\lambda).$$ (45)

We know that the double-row-monodromy matrix $\mathcal{T}$ satisfy the RE, from the relation (23), we have the commutation relations:

$$\mathcal{A}(\lambda)\mathcal{B}(\mu) = \frac{\sin(\lambda - \mu - 2\eta)\sin(\lambda + \mu)\xi^2(-\lambda)}{\sin(\lambda - \mu)\sin(\lambda + \mu + 2\eta)\xi^2(\lambda)}\mathcal{B}(\mu)\mathcal{A}(\lambda)$$

$$+ \frac{\sin(2\eta)\sin(2\mu)}{\sin(\lambda - \mu)\sin(2\mu + 2\eta)}\frac{\xi(-\lambda)\xi(-\mu)}{\xi(\mu)}\mathcal{B}(\lambda)\mathcal{A}(\mu)$$

$$- \frac{\sin(2\eta)}{\sin(\lambda + \mu + 2\eta)}\frac{\xi(-\lambda)}{\xi(\mu)}\mathcal{B}(\lambda)\tilde{\mathcal{D}}(\mu).$$ (46)

$$\tilde{\mathcal{D}}(\lambda)\mathcal{B}(\mu) = \frac{\sin(\lambda - \mu + 2\eta)\sin(\lambda + \mu + 4\eta)\xi^2(-\lambda)}{\sin(\lambda - \mu)\sin(\lambda + \mu + 2\eta)\xi^2(\lambda)}\mathcal{B}(\mu)\tilde{\mathcal{D}}(\lambda)$$

$$- \frac{\sin(2\eta)\sin(2\lambda + 4\eta)}{\sin(\lambda - \mu)\sin(2\lambda + 2\eta)\xi(\lambda)\xi(\mu)}\mathcal{B}(\lambda)\tilde{\mathcal{D}}(\mu)$$

$$+ \frac{\sin(2\eta)\sin(2\mu)}{\sin(2\lambda + 2\eta)\sin(2\mu + 2\eta)}\frac{\xi^2(-\lambda)\xi(-\mu)}{\xi(\lambda)\xi^2(\mu)}\mathcal{B}(\lambda)\mathcal{A}(\mu),$$

$$\mathcal{B}(\lambda)\mathcal{B}(\mu) = \frac{\xi^2(\mu)\xi^2(-\lambda)}{\xi^2(\lambda)\xi^2(-\mu)}\mathcal{B}(\mu)\mathcal{B}(\lambda).$$ (47)

Using the standard algebraic Bethe ansatz method, we assume the eigenvectors take the form

$$|\mu_1, \ldots, \mu_n> = \mathcal{B}(\mu_1) \cdots \mathcal{B}(\mu_n) F(\mu_1, \ldots, \mu_n)|0>.$$ (48)
where $F$ is a non-vanishing function. Acting the transfer matrix on this eigenvector, we obtain the eigenvalues of the transfer matrix as:

$$\Lambda(\lambda, \mu_1, \cdots, \mu_n) = \frac{\sin(2\lambda + 4\eta) \sin(\psi + \lambda)}{\sin(2\lambda + 2\eta) \sin(\psi + \lambda + 2\eta)}$$

$$\times \alpha(\lambda) \bar{\alpha}(-\lambda) \prod_{i=1}^{n} \left\{ \frac{\sin(\lambda - \mu_i - 2\eta) \sin(\lambda + \mu_i)}{\sin(\lambda - \mu_i) \sin(\lambda + \mu_i + 2\eta)} \right\}$$

$$+ \frac{\sin(\psi - \lambda - 2\eta) \sin(2\lambda) \sin(\psi + \lambda + 2\eta)}{\sin(\psi + \lambda + 2\eta) \sin(\psi - \lambda)} \delta(\lambda) \bar{\delta}(-\lambda)$$

$$\times \prod_{i=1}^{n} \left\{ \frac{\sin(\lambda - \mu_i + 2\eta) \sin(\lambda + \mu_i + 4\eta)}{\sin(\lambda - \mu_i) \sin(\lambda + \mu_i + 2\eta)} \right\},$$

$$\text{(49)}$$

the parameters $\mu_i$, $i = 1, \cdots, n$ should satisfy the following Bethe ansatz equations:

$$\frac{\sin(\mu_j + 2\eta)^{2N}}{\sin(\mu_j)^{2N}} = \frac{\sin(\psi - \mu_j - 2\eta) \sin(\psi + \mu_j + 4\eta)}{\sin(\psi^+ + \mu_j) \sin(\psi - \mu_j)}$$

$$\prod_{i=1, i \neq j}^{n} \left\{ \frac{\sin(\mu_j - \mu_i + 2\eta) \sin(\mu_j + \mu_i + 4\eta)}{\sin(\mu_j - \mu_i) \sin(\mu_j + \mu_i)} \right\},$$

$$j = 1, 2, \cdots, n.$$  

$$\text{(50)}$$

The Bethe ansatz equations ensure that the unwanted terms vanish. On the other hand, the Bethe ansatz equations also ensure that the eigenvalues of the transfer matrix are entire functions.

From the definition of the Hamiltonian (34), we thus can obtain the eigenvalues $E$ of the Hamiltonian

$$E = -N \cos(2\eta) - \sum_{i=1}^{n} \frac{\sin^2(2\eta)}{\sin(\mu_j) \sin(\mu_j + 2\eta)} + (2n - N) \sin(2\eta) \tan(\alpha)$$

$$- \frac{\sin^2(2\eta)}{2 \sin(\psi^+) \sin(\psi^+ + 2\eta)} + \text{cost}.$$  

$$\text{(51)}$$

### 5 Integrable model for fermion case

In this section, we will study the corresponding fermion model. The Lax representation for the Hamiltonian (1) is found to be [13]

$$L_j^F(\lambda) = \begin{pmatrix} b'_+ - (b'_+ - ia'_+) n_j & c a_j \\ -ica'_j & a'_- - (a'_- + ib'_-) n_j \end{pmatrix}.$$  

$$\text{(52)}$$
Here $a_\pm', b_\pm', c$ have already be defined in the above sections. As for the spin model, we first write out the graded Yang-Baxter relation,

$$R_{12}^F(\lambda, \mu) L_1^F(\lambda) L_2^F(\mu) = L_2^F(\mu) L_1^F(\lambda) R_{12}^F(\lambda, \mu),$$

(53)

we should notice that the meaning of some notations here is different from that of spin models presented above. Here

$$L_1^F(\lambda) = L^F(\lambda) \otimes_s 1,$$

$$L_2^F(\lambda) = 1 \otimes_s L^F(\lambda).$$

(54)

$\otimes_s$ denotes the super tensor product

$$[A \otimes_s B]_{ij; kl} = (-1)^{p(j)[p(i)+p(k)]} A_{ik} B_{jl}$$

(55)

with parity $p(1) = 0$, $p(2) = 1$. The R-matrix for the fermion model is defined as:

$$R_{12}^F(\lambda, \mu) \equiv \begin{pmatrix}
  a_+(\lambda, \mu) & 0 & 0 & 0 \\
  0 & -ib_-(\lambda, \mu) & c(\lambda, \mu) & 0 \\
  0 & c(\lambda, \mu) & ib_+(\lambda, \mu) & 0 \\
  0 & 0 & 0 & -a_-(\lambda, \mu)
\end{pmatrix}. $$

(56)

We can prove that this R-matrix satisfy the following unitarity and cross-unitarity relations

$$R_{12}^F(\lambda, \mu) R_{21}^F(\mu, \lambda) = \sin(2\eta + \lambda - \mu) \sin(2\eta - \lambda + \mu) = \rho(\lambda - \mu),$$

$$X_{12}^{F \text{st1}}(\lambda, \mu + 4\eta - \pi) R_{21}^{F \text{st1}}(\mu, \lambda) = \sin(\lambda - \mu) \sin(\lambda - \mu - 4\eta) = -\tilde{\rho}(\lambda - \mu),$$

$$R_{21}^{F \text{st1}}(\lambda, \mu) Y_{21}^{F \text{st1}}(\mu - 4\eta + \pi, \lambda) = -\tilde{\rho}(\lambda - \mu + 4\eta),$$

(57)

(58)

where we have used the notations:

$$X_{12}^F(\lambda, \mu) \equiv \frac{\xi(\mu)}{\xi(\mu - 4\eta + \pi)} M_1^F(\mu) R_{12}^F(\lambda, \mu) M_1^F(\mu),$$

$$Y_{12}^F(\lambda, \mu) = \frac{\xi(\lambda + 4\eta - \pi)}{\xi(\lambda)} M_2^F(\lambda) R_{12}^F(\lambda, \mu) M_2^F(\lambda).$$

$$M^F(\lambda) = \begin{pmatrix}
  1 & 0 \\
  0 & \xi(\lambda - 4\eta + \pi)\xi^{-1}(\lambda)
\end{pmatrix}. $$

(59)

Here,

$$R_{21}^F = P^F R_{12}^F P^F$$

(60)
$P^F$ is super permutation operator which is defined as:
\[
P^F_{ij,kl} = (-1)^{p(i)p(j)}\delta_{ii}\delta_{jk}
\]  
(61)

$st$ means super transposition defined as
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A & -C \\ B & D \end{pmatrix}^{st}.
\]
(62)

We can also write the RE and dual RE for fermion models, here we take the solutions of RE and dual RE $K^F$ and $K^{+F}$ to be diagonal. It is trick to deal with non-diagonal K-matrix, because there are grassmann odd numbers in the non-diagonal positions. One can write the graded RE and dual graded RE for fermion model as:
\[
R_{12}^F(\lambda, \mu)K_1^F(\lambda)R_{21}^F(\mu, -\lambda)K_2^F(\mu) = K_2(\mu)^F R_{12}^F(\lambda, -\mu)K_1^F(\lambda)R_{21}^F(-\mu, -\lambda).
\]
(63)

\[
R_{12}^F(-\lambda, -\mu)K_1^{+F}(\lambda)Y_{21}^F(-\mu - 4\eta + \pi, \lambda)K_2^{+F}(\mu) = K_2^{+F}(\mu)X_{12}^F(-\lambda, \mu + 4\eta - \pi)K_1^{+F}(\lambda)R_{21}^F(\mu, \lambda).
\]
(64)

By solving the graded RE and the dual graded RE, we find respectively the solutions of the graded RE and dual graded RE as:
\[
K^F(\lambda) = \begin{pmatrix} \xi(\lambda) \sin(\psi - \lambda) \\ \xi(-\lambda) \sin(\psi + \lambda) \\ 0 \\ 1 \end{pmatrix},
\]
(65)

\[
K^{+F}(\lambda) = \begin{pmatrix} \xi(-\lambda) \sin(\psi + \lambda + 2\eta) \\ \xi(\lambda) \sin(\psi - \lambda - 2\eta) \\ 0 \\ -1 \end{pmatrix}.
\]
(66)

One can find $K^F$ is similar as the spin model, but there appear a $-\pi$ in $K^{+F}$ compared with spin model. The transfer matrix for fermion models is defined as:
\[
t^F(\lambda) = strK^{+F}(\lambda)T^F(\lambda)
\]
(67)

Here, $str$ means super trace defined as $strA = (-1)^{p(i)}A_{ii}$. And
\[
T^F(\lambda) = L_N^F(\lambda) \cdots L_1^F(\lambda)K^F(\lambda)L_1^F(\lambda)^{st} \cdots L_N^F(\lambda)
\]
(68)

satisfy the graded RE. We have used the notation $L^F(\lambda) \equiv L^{F-1}(-\lambda)$, which is defined as:
\[
L_j^F(\lambda) = \begin{pmatrix} -b_+^\sigma + (b_+^\sigma + ia_+^\sigma)n_j \\ -ca_+^\sigma \\ -a_+^\sigma + (a_+^\sigma - ib_+^\sigma)n_j \end{pmatrix}.
\]
(69)
The definition for \( a', b', \) and \( c \) can be found in the above sections. The Hamiltonian \( \mathcal{H}_{OB} \) for small-polaron model with open boundary conditions is as follows:

\[
\mathcal{H}_{OB} = W \sum_{j=1}^{N-1} n_j - J \sum_{j=1}^{N-1} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + V \sum_{j=1}^{N-1} n_j n_{j+1},
\]

\[
+ \left( \frac{V}{2} + \sin(2\eta) \frac{\cos \psi}{\sin \psi} \right) n_1 + \left( \frac{V}{2} + W - \sin(2\eta) \frac{\cos \psi^+}{\sin \psi^+} \right) n_N + \text{cost.}
\]

(70)

The commuting of the transfer matrix for fermion model with open boundary conditions can be proved similarly as that for the spin model, here we should notice that in the proof of the integrability, besides the super transposition \( st \), we also need a inverse of the super transposition \( \bar{st} \) with \( A^{st \bar{st}} = A \).

We denote

\[
\mathcal{T}^F(\lambda) = \begin{pmatrix} A^F(\lambda) & B^F(\lambda) \\ C^F(\lambda) & D^F(\lambda) \end{pmatrix},
\]

thus we can write the transfer matrix explicitly as

\[
t^F(\lambda) = \frac{\xi(-\lambda) \sin(\psi^+ + \lambda + 2\eta)}{\xi(\lambda) \sin(\psi^- - \lambda - 2\eta)} A^F(\lambda) + D^F(\lambda)
\]

\[
= \frac{\xi(-\lambda) \sin(\psi^+ + \lambda + 2\eta)}{\xi(\lambda) \sin(\psi^- - \lambda - 2\eta)} \tilde{A}^F(\lambda)
\]

\[
+ \frac{\sin(2\lambda + 4\eta) \sin(\psi^+ - \lambda)}{\sin(2\lambda + 2\eta) \sin(\psi^- - \lambda - 2\eta)} D^F(\lambda).
\]

(72)

In above, as for the case of spin model, we have introduced the following transformation for convenience,

\[
\mathcal{A}^F(\lambda) = \tilde{A}^F(\lambda) + \frac{\xi(\lambda)}{\xi(-\lambda)} \frac{\sin(2\eta)}{\sin(2\lambda + 2\eta)} D^F(\lambda).
\]

(73)

Following the standard algebraic Bethe ansatz method, we define the pseudovacuum state for fermion model as \( |0 >^F \) with \( a_j |0 >^F = 0 \). One can find

\[
\tilde{A}^F(\lambda)|0 >^F = \frac{\xi(\lambda) \sin(2\lambda) \sin(\psi - \lambda - 2\eta)}{\xi(-\lambda) \sin(2\lambda + 2\eta) \sin(\psi + \lambda)} \alpha^F(\lambda)|0 >^F,
\]

\[
\mathcal{D}^F(\lambda)|0 >^F = \delta^F(\lambda)|0 >^F,
\]

\[
\mathcal{B}^F(\lambda)|0 >^F = 0, \quad \mathcal{C}^F(\lambda)|0 >^F \neq 0.
\]

(74)
Where we have
\[ \alpha^F(\lambda) = \left( -\frac{\xi(-\lambda)}{\xi(\lambda)} \right)^N \sin^2(\lambda), \delta^F(\lambda) = \left( -\frac{\xi(-\lambda)}{\xi(\lambda)} \right)^N \sin^2(\lambda + 2\eta). \] (75)

Because the double-row-monodromy matrix for fermion model satisfy the graded RE, we can obtain the commutation relations which are necessary for the algebraic Bethe ansatz method.
\[
\begin{align*}
\mathcal{C}^F(\lambda)\mathcal{C}^F(\mu) &= \frac{\xi^2(\lambda)\xi^2(-\mu)}{\xi^2(\mu)\xi^2(-\lambda)} \mathcal{C}^F(\mu)\mathcal{C}^F(\lambda), \\
\tilde{\mathcal{A}}^F(\lambda)\mathcal{C}^F(\mu) &= \frac{\sin(\lambda - \mu + 2\eta)\sin(\lambda + \mu + 4\eta)}{\sin(\lambda - \mu)\sin(\lambda + \mu + 2\eta)} \frac{\xi^2(\lambda)}{\xi^2(-\lambda)} \mathcal{C}^F(\mu)\tilde{\mathcal{A}}^F(\lambda) \\
&- \frac{\sin(2\eta)}{\sin(\lambda - \mu)\sin(2\lambda + 4\eta)} \frac{\xi(\lambda)\xi(\mu)}{\xi^2(-\lambda)} \mathcal{C}^F(\lambda)\tilde{\mathcal{A}}^F(\mu) \\
&+ \frac{\sin(2\lambda + 2\eta)}{\sin(2\lambda + 4\eta)} \frac{\sin(2\mu + 2\eta)}{\sin(\lambda + \mu + 2\eta)} \frac{\xi(\lambda)\xi(\mu)}{\xi^2(-\lambda)} \mathcal{C}^F(\lambda)\tilde{\mathcal{A}}^F(\mu) \\
&\times \frac{\xi(\lambda)\xi^2(\mu)}{\xi^2(-\lambda)\xi(-\mu)} \mathcal{D}^F(\lambda)\mathcal{D}^F(\mu), \\
\mathcal{D}^F(\lambda)\mathcal{C}^F(\mu) &= \frac{\sin(\lambda - \mu - 2\eta)\sin(\lambda + \mu)}{\sin(\lambda - \mu)\sin(\lambda + \mu + 2\eta)} \frac{\xi^2(\lambda)}{\xi^2(-\lambda)} \mathcal{C}^F(\mu)\mathcal{D}^F(\lambda) \\
&+ \frac{\sin(2\eta)}{\sin(\lambda - \mu)\sin(2\mu + 2\eta)} \frac{\xi^2(\mu)}{\xi(-\lambda)\xi(-\mu)} \mathcal{C}^F(\lambda)\mathcal{D}^F(\mu) \\
&- \frac{\sin(2\eta)}{\sin(\lambda + \mu + 2\eta)} \frac{\xi(-\lambda)}{\xi(-\mu)} \mathcal{C}^F(\lambda)\tilde{\mathcal{A}}^F(\mu). (76)
\end{align*}
\]

Those commutation relations are only slightly different from the relations for spin case. Assume the eigenvectors take the form \( \mathcal{C}^F(\mu_1) \cdots \mathcal{C}^F(\mu_n)F(\mu_1, \cdots, \mu_n)|0 >^F, \) where \( F(\mu_1, \cdots, \mu_n) \) are non-vanishing functions. We can find the eigenvalues for the fermion transfer matrix with open boundary conditions are:
\[
\begin{align*}
\Lambda^F(\lambda, \mu_1, \cdots, \mu_n) &= \frac{\sin(\psi^+ + \lambda + 2\eta)\sin(2\lambda)\sin(\psi - \lambda - 2\eta)}{\sin(\psi^+ - \lambda - 2\eta)\sin(2\lambda + 2\eta)\sin(\psi + \lambda)} \alpha^F(\lambda) \\
&\times \prod_{i=1}^{n} \left\{ \frac{\sin(\lambda - \mu_i + 2\eta)\sin(\lambda + \mu_i + 4\eta)}{\sin(\lambda - \mu_i)\sin(\lambda + \mu_i + 2\eta)} \xi^2(\lambda) \right\} \\
&\times \frac{\sin(2\lambda + 4\eta)\sin(\psi^+ - \lambda)}{\sin(2\lambda + 2\eta)\sin(\psi^+ - \lambda - 2\eta)} \delta^F(\lambda) \\
&\times \prod_{i=1}^{n} \left\{ \frac{\sin(\lambda - \mu_i - 2\eta)\sin(\lambda + \mu_i)}{\sin(\lambda - \mu_i)\sin(\lambda + \mu_i + 2\eta)} \xi^2(\lambda) \right\}. (77)
\end{align*}
\]
The parameters $\mu_1, \cdots, \mu_n$ are restricted by the Bethe ansatz equations:

$$
\frac{\sin^{2N}(\mu_j + 2\eta)}{\sin^{2N}(\mu_j)} = \frac{\sin(\psi^+ + \mu_j + 2\eta) \sin(\psi - \mu_j - 2\eta)}{\sin(\psi^+ - \mu_j) \sin(\psi + \mu_j)} \prod_{i=1, i \neq j}^{n} \left\{ \frac{\sin(\mu_j - \mu_i + 2\eta) \sin(\mu_j + \mu_i + 4\eta)}{\sin(\mu_j - \mu_i - 2\eta) \sin(\mu_j + \mu_i)} \right\},
$$

\(j = 1, 2, \cdots, n\). \hspace{1cm} (78)

The eigenvalue of the Hamiltonian for fermion model is:

$$
E^F = -N \cos(2\eta) - \sum_{i=1}^{n} \frac{\sin^2(2\eta)}{\sin(\mu_j) \sin(\mu_j + 2\eta)} + (2n - N) \sin(2\eta) \tan(\alpha) \\
- \frac{\sin^2(2\eta)}{2 \sin(\psi^+) \sin(\psi^+ - 2\eta)} + \text{cost.} \hspace{1cm} (79)
$$

6 Summary and discussions

We formulated in this paper the RE and the dual RE for small-polaron model whose spin chain equivalent is the Heisenberg $XXZ$ chain with an external magnetic field parallel to $z$ direction. We found solutions to the RE and dual RE for both spin and fermion models. Using the algebraic Bethe ansatz method, we find the eigenvalues of the transfer matrix with open boundary conditions.

It is interesting to use some results of this paper to calculate some physical quantities, such as the surface free energy and finite-size corrections, by using the thermodynamic Bethe ansatz method. The Lax pair method for open boundary conditions for this model is also an interesting problem.

**Acknowledgements:** We would like to thank Prof. B.Y.Hou and Prof. K.J.Shi for useful discussions. This work is supported in part by the Natural Science Foundation of China.
References

[1] E.K.Sklyanin, J.Phys. A21 (1988)2375; I.V.Cherednik, Theor.Math.Phys. 17 (1983)77; 61 (1984)911.

[2] H.J.de Vega, Int.J.Mod.Phys. A4(1989)2371; C.Destri, H.J.de Vega, Nucl.Phys. B361(1992)361; Nucl. Phys. B374 (1992) 692; H.de Vega, A.Gonzalez-Ruiz, Nucl.Phys. B417 (1994)553.

[3] L.Meinzescu, R.I.Nepomechie, V.Rittenberg, Phys. Lett. A147 (1990)70; L.Meinzescu, R.I.Nepomechie, J.Phys. A24(1991)L19; Mod.Phys.Lett. A6 (1991)2497; Nucl.Phys. B372 (1992)597; Int.J.Mod.Phys. A6(1991)5231.

[4] A.Foerster, M.Karowski, Nucl.Phys. B408(1994)512.

[5] M.T.Batchelor, Y.K.Zhou, Phys.Rev.Lett. 76 (1996)2826; Y.K.Zhou, Nucl.Phys. B466 [FS] (1996)488; Nucl.Phys. B487 [FS](1997)779; Nucl.Phys. B453 [FS] (1995)619; Nucl.Phys. B458 [FS](1996)504; R.E.Behrend, P.A.Pearce, J.Phys. A29 (1996)7828;

[6] P.Fendley, S.Saleur, N.P.Warner, Nucl. Phys. B430 (1995)577; B428 (1994) 681; A.Leclair, G.Mussardo, H.Saleur, S.Skorik, Nucl.Phys.B453[FS] (1995)581; E.Corrigan, P.E.Dorey, R.H.Rietdijk, R.Sasaki, Phys.Lett. B333(1994)83.

[7] H.Fan, B.Y.Hou, K.J.Shi, Z.X.Yang, Nucl.Phys.B478 (1996)723; R.H.Yue, H.Fan, B.Y.Hou, Nucl. Phys. B462 (1996)167; H.Fan, Nucl. Phys. B488 (1997)409; H.Fan, B.Y.Hou, K.J.Shi, Nucl. Phys. B496[PM](1997)551.

[8] S.Ghoshal, A.Zamolodchikov, Int.J.Mod.Phys. A9(1994)3841.

[9] M.Shiroishi, M.Wadati, J.Phys.Soc.Jpn. 66, No.1 (1997)1; J.Phys.Soc.Jpn. 66, No.8 (1997)1; H.Q.Zhou, Phys. Rev. B54(1996)41.

[10] X.W. Guan, M.S. Wang, S.D. Yang, Nucl. Phys. B485 (1997)685; J.Phys. A30 (1997) 4161.

[11] L.A.Takhtajan, L.D.Faddeev, Russ.Math.Surv. 34 (1979)11; V.E.Korepin, G.Izergin, N.M.Gogolinov, Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press, 1992).
[12] V.K.Fedyanin, V.Yushankhay, Teor. Mat. Fiz. 35 (1978) 240;
    V.G.Makhankov, V.K.Fedyanin, Phys. Rep. 104 (1984) 1.

[13] F.C.Pu, B.H.Zhao, Phys. Lett. A118 (1986) 77;
    H.Q.Zhou, Z.Xiong, L.J.Jiang, J. Phys. A21 (1988) 3385;
    H.Q.Zhou, L.J.Jiang, J.G.Tang, J. Phys. A23 (1990) 213.

[14] C.N.Yang, C.P.Yang, Phys. Rev. 105 (1966) 321, 322, 151 (1966) 258.

[15] R.J.Baxter, ”Exactly Solved Models in Statistical Mechanics” , Academic Press,
    London, 1982.