A New Relation between post and pre- optimal measurement states.

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When an optimal measurement \( (S_z, S_y, S_x) \) is made on a qubit, and what we call a Mutually Unbiased Mixture - MUM of the resulting ensembles is taken, then the post measurement density matrix is shown to be related to the pre-measurement density matrix through a simple linear and universal relation. It is shown that such a relation holds only when the measurements are made in Mutually Unbiased Bases - MUB \([3–5]\). For Spin - 1/2 systems it is also shown explicitly that non-orthogonal measurements fail to give such a linear relation no matter how the ensembles are mixed. The result has been proved to be true for arbitrary quantum mechanical systems of finite dimensional Hilbert spaces. The result is true irrespective of whether the initial state is pure or mixed.

I. THE RELATION FOR SPIN - 1/2

Consider an ensemble \((N \text{ copies}-N \text{ very large})\) of Spin - 1/2 particles and dividing it into 3 equal sub-ensembles. Three independent observables are measured and the resulting mixed subensembles are put together to form the post-measurement density matrix.

Suppose the measured observables are \(S_1, S_2\) and \(S_3\) along three orthonormal directions. Let \(p_1\) be the probability for the outcome \(|+\rangle_1\), \(p_2\) the probability for \(|+\rangle_2\) and \(p_3\) for \(|+\rangle_3\). Further let \(P_{+,i}\) be the projection operator for the i-th outcome etc.

Now, the post-measurement density matrices of the individual measurements are respectively:

\[
\rho_i = p_i P_{+,i} + (1 - p_i) P_{-,i}
\]

(1)

The spectral representation for \(S_i\)

\[
S_i = \frac{1}{2}(P_{+,i} - P_{-,i})
\]

(2)

along with the completeness relation

\[
P_{+,i} + P_{-,i} = I
\]

(3)

yields

\[
\rho_i = \frac{1}{2} + 2 < S_i > S_i
\]

(4)

with \(< S_i >= 2p_i - 1\). An equal mixture of these three leads to the post-measurement density matrix:

\[
\rho_{msmt} = \frac{1}{3} \left[ \frac{1}{2} + \sum_i 2 < S_i > S_i \right]
\]

(5)

Since this is a complete measurement, the initial density matrix can be completely determined and is

\[
\rho_{ini} = \frac{1}{2} + \sum_i 2 < S_i > S_i
\]

(6)

Clearly, there is a linear and universal relation between \(\rho_{msmt}\) and \(\rho_{ini}\):

\[
\rho_{msmt} = (1/3) (I + \rho_{ini})
\]

(7)

This new relationship between the pre-and post-measurement states is the main result of this paper.

Though the relation (7) was shown to be true for an initial pure state, it is straightforward to see that it holds even when the initial state is mixed. To see this let

\[
\rho_{ini}^{mix} = \sum_i c_i \rho_i^{(0)}; \quad (\sum_i c_i = 1)
\]

with \(\rho_i^{(0)}\) all being pure states each of which leads to \(\rho_{msmt,i}\). Thus we have

\[
\rho_{msmt} = \sum_i c_i \rho_{msmt,i} = (I + \rho_{ini}^{mix})/3
\]

When the initial state is pure, the eigenvalues of \(\rho_{ini}\) are \((1,0)\). It then follows that the eigenvalues of \(\rho_{msmt}\) are \((2/3, 1/3)\) allowing us to write the spectral decomposition

\[
\rho_{msmt} = \frac{2}{3} |l\rangle \langle l| + \frac{1}{3} |s\rangle \langle s|
\]

(8)

where \(|l\rangle, |s\rangle\) are the corresponding eigenstates, Their completeness

\[
|l\rangle \langle l| + |s\rangle \langle s| = I
\]

(9)

and eqn(7) lead to the interesting result

\[
\rho_{ini} = |l\rangle \langle l|
\]

(10)

i.e the eigenstate of \(\rho_{msmt}\) with the largest eigenvalue is the original pure state itself.
A. Measurements along non-orthogonal directions.

At first it might appear that the relationship eqn(7) is only a consequence of the measurement being complete. Now, we show that such a universal relation does not hold if the measurements are made along three non-collinear directions which also constitute a complete measurement. Let \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) be three unit vectors along which measurements are made such that \( \langle \hat{n}_1, \hat{n}_2 \times \hat{n}_3 \rangle \neq 0 \). Let \( \hat{S} \, \hat{n}_i \) be the three spin-components being measured. Let \( |+\rangle_{\hat{n}_i} \) and \(-|_{\hat{n}_i} \) be the eigen-vectors of \( \hat{S} \, \hat{n}_i \) and \( \hat{P}(\hat{n}_i, \pm) \) be the corresponding projectors. As before,

\[
P(\hat{n}_i, +) + \hat{P}(\hat{n}_i, -) = 1
\]

and

\[
\hat{S} \, \hat{n}_i = \frac{1}{2}(\hat{P}(\hat{n}_i, +) - \hat{P}(\hat{n}_i, -))
\]

Hence the post measurement state if the three mixed states are further mixed with weights \( x_i \) (\( \sum x_i = 1 \)) is:

\[
\rho_{\text{msmt}} = \frac{1}{2} + \sum_i x_i \langle \hat{S} \, \hat{n}_i \rangle \, \hat{S} \, \hat{n}_i
\]

Now,

\[
\rho_{\text{ini}} = \frac{1}{2} + \sum \langle \hat{S} \, \hat{n}_i \rangle \, \hat{S} \, \hat{n}_i
\]

and \( \langle \hat{S} \rangle \) must be expressed in terms of the observed components: Let,

\[
\langle \hat{S} \rangle = \sum c_i \hat{n}_i; \quad \langle \hat{S} \, \hat{n}_i \rangle = \sum_j \hat{n}_i, \hat{n}_j c_j
\]

The matrix \( \hat{n}_i, \hat{n}_j \) is invertible. Inverting the matrix eqn(14) one gets

\[
c_i = \sum_{ij} d_{ij} \langle \hat{S} \, \hat{n}_j \rangle; \quad \langle \hat{S} \rangle = \sum_{ij} d_{ij} \langle \hat{S} \, \hat{n}_j \rangle \hat{n}_i
\]

where \( d_{ij} \) are functions of \( \{ \hat{n}_i, \hat{n}_j \} \) only and have no dependence on \( \langle \hat{S} \, \hat{n}_i \rangle \). Finally,

\[
\rho_{\text{ini}} = \frac{1}{2} + \sum_{ij} 2d_{ij} \langle \hat{S} \, \hat{n}_j \rangle \hat{S} \, \hat{n}_i
\]

Consider the average of \( \hat{S} \, \hat{n}_i \) in \( \rho_{\text{msmt}} \); on using eqn(13) one gets

\[
\langle \hat{S} \, \hat{n}_i \rangle_{\rho_{\text{msmt}}} = \sum_j x_j \hat{n}_i, \hat{n}_j \langle \hat{S} \, \hat{n}_j \rangle_{\rho_{\text{ini}}}
\]

where \( \hat{n}_i, \hat{n}_j \neq 0 \). Also, \( x_i \neq 0 \) and are independent of the \( \langle \hat{S} \, \hat{n}_i \rangle \). This immediately leads to a contradiction because if we assumed \( \rho_{\text{msmt}} = \alpha \rho_{\text{ini}} + \beta I \) (where \( \alpha \) and \( \beta \) are constants) then,

\[
\langle \hat{S} \, \hat{n}_i \rangle_{\rho_{\text{msmt}}} = \alpha \langle \hat{S} \, \hat{n}_i \rangle_{\rho_{\text{ini}}}
\]

which is clearly not of the form of (17)

II. GENERALIZATION TO ARBITRARY \( \mathcal{H} \):

We now state our result as a theorem:

**Theorem:** For a quantum system with Hilbert space of dimensionality \( N \) (complex), only the post-measurement state resulting from complete measurements made in Mutually Unbiased Basis (MUB) and a Mutually Unbiased Mixture (MUM) have a simple linear and universal relation with the pre-measurement or initial state: \( \rho_{\text{msmt}} = (I + \rho_{\text{ini}})/(N + 1) \).

**Proof:** We first show that it is sufficient to have MUB with MUM for the result to hold. Let \( \mathcal{H} \) be the Hilbert Space of the considered quantum system and let \( \dim(\mathcal{H}) = N \). Every basis of this space has \( N \) components of each of components. A density operator describing such a quantum system is an \( N \times N \) Hermitian matrix with unit trace and having \( (N^2 - 1) \) independent real parameters in general (but fewer for pure states). Any measurement will yield \( (N - 1) \) independent real numbers. Therefore, \( \frac{N^2 - 1}{N} = N + 1 \) independent observables are required to be measured to make a complete measurement. However, certain observable sets are more useful than others and in particular observables whose eigenstates form the so-called Mutually Unbiased Bases [4] have been shown to yield what are called ‘optimal’ measurements which minimise the error matrix. These have also been shown to have important information theoretic properties [8]. MUB bases satisfy the property \( \langle \alpha, \beta \rangle = 1/\sqrt{N} \) where \( \alpha \) and \( \beta \) are vectors belonging to different basis of an \( N \)-dimensional space. It has also been shown that there are exactly \( (N + 1) \) bases that are MUB.

Let \( X^{(\alpha)} \) be the set of observables whose eigenstates are \( |k^{(\alpha)}\rangle \). These observables are independent so that their measurement would constitute a complete measurement. Therefore the set label \( \alpha \) takes on \( N + 1 \) values and the state label \( k \) takes \( N \) values. Let the eigenstate corresponding to the outcome \( k \) when \( X^{(\alpha)} \) is measured be denoted \( |j^{(\alpha)}\rangle \) and \( P_j^{(\alpha)} \) be the associated projector. Further, let \( P_j^{(\alpha)} \) be the corresponding probability. So one has

\[
P_j^{(\alpha)} = |\langle j^{(\alpha)} | \langle j^{(\alpha)} \rangle | P_j^{(\alpha)} = \delta_{ij} P_j^{(\alpha)} \quad \text{tr} P_j^{(\alpha)} = 1
\]

The corresponding post-measurement density matrix is given by

\[
\rho^{(\alpha)} = \sum_{i=1}^{N} P_i^{(\alpha)} P_i^{(\alpha)}
\]

Let the initial state be of the form,

\[
\rho_{\text{ini}} = \sum_{\alpha=1}^{N+1} \sum_{i=1}^{N} c_i^{(\alpha)} P_i^{(\alpha)}
\]

where \( c_i^{(\alpha)} \) are real parameters (there are only \( N^2 - 1 \) independent ones). The unit trace condition leads to
Consider the expectation value of the operator \( P^{(a)} \) in the initial state \( \rho_{ini} \)

\[
tr(\rho_{ini}) = \sum_{\alpha=1}^{N+1} \sum_{i=1}^{N} c_i^{(a)} = 1 \tag{20}
\]

where we used eqn(20) along with the MUB property \( tr \rho_i^{(a)} P_j^{(\beta \neq \alpha)} = \frac{1}{N} \) for all \((i,j,\alpha,\beta \neq \alpha)\). Taking \( \rho_{msmt} \) to be an Unbiased Mixture of all \( \rho^{(a)} \)’s one gets,

\[
\rho_{msmt} = \frac{1}{N+1} \sum_{\alpha=1}^{N+1} \rho^{(a)} = \frac{1}{N+1} \sum_{i=1}^{N} p_i^{(a)} P_i^{(a)} \tag{22}
\]

Substituting for \( p_i^{(a)} \) from eqn(21) one gets

\[
\rho_{msmt} = \frac{1}{N+1} \sum_{i=1}^{N} P_i^{(a)} [c_i^{(a)} + \frac{1}{N}(1 - \sum_{m} c_m^{(a)})] = \frac{1}{N+1} + \frac{1}{N+1} \rho_{ini} \tag{23}
\]

where we used eqn(19) as well as \( \sum_i P_i^{(a)} = 1 \) and \( \sum_\alpha (1 - \sum_m c_m^{(a)}) = N \). In proving the latter use has been made of eqn(20). This completes the proof.

However, in this proof assumptions were made in (22) of the post-measurement density matrix being an Unbiased Mixture of all the \( \rho^{(a)} \)’s; the set of observables \( X^{(a)} \) were also assumed to be such that their eigenfunctions form MUB’s. We shall now show that it is unnecessary to make these assumptions and in fact it can be proved that Mutually Unbiased Bases as well as Mutually Unbiased Mixtures are the only ones that can lead to the result in question.

Let us now consider the expectation value of \( P_k^{(a)} \) in \( \rho_{msmt} \) with unequal weights instead. It then follows that

\[
tr \rho_{msmt} P_k^{(a)} = \sum_{j=1}^{N} c_j^{(\beta)} p_j^{(\beta)} tr P_k^{(a)} P_j^{(\beta)} = c_k^{(a)} P_k^{(a)} + \sum_{j, \beta \neq \alpha} c_j^{(\beta)} p_j^{(\beta)} tr P_k^{(a)} P_j^{(\beta)} \tag{24}
\]

On the other hand, if a relation of the type

\[
\rho_{msmt} = \frac{\lambda}{N} I + (1 - \lambda) \rho_{ini} \tag{25}
\]

were to hold with \( \lambda = \text{const} \) we should also have

\[
tr \rho_{msmt} P_k^{(a)} = \frac{\lambda}{N} + (1 - \lambda) p_k^{(a)} \tag{26}
\]

Now eqns(24,26) subject to the constraint \( \sum_i p_i^{(a)} = 1 \) must be true for arbitrary \( p_j^{(a)} \). Using Lagrange multipliers \( \mu^{(a)} \) for the constraints we get

\[
0 = \sum_{\alpha} \mu^{(a)} (\sum_i p_i^{(a)} - 1) + c^{(a)} p_k^{(a)} + \sum_{j, \beta \neq \alpha} c_j^{(\beta)} p_j^{(\beta)} tr P_k^{(a)} P_j^{(\beta)}
\]

\[
= \frac{\lambda}{N} - (1 - \lambda) p_k^{(a)} \tag{27}
\]

which can be solved to yield

\[
c^{(a)} = 1 - \lambda \]

\[
tr P_k^{(a)} P_j^{(\beta \neq \alpha)} = -\frac{\mu^{(\beta)}}{c^{(\beta)}} \tag{29}
\]

The second of these equations says that \( tr P_k^{(a)} P_j^{(\beta \neq \alpha)} \) is independent of \((j,k)\). But on remembering \( \sum_j P_j^{(\gamma)} = I \) one sees that this is possible iff

\[
tr P_k^{(a)} P_j^{(\beta \neq \alpha)} = \frac{1}{N} \tag{30}
\]

In other words, the bases spanned by the eigenstates of \( X^{(a)} \) are MUB. Furthermore, there is nothing special about the value of \( \alpha \) used above which means \( c^{(a)} \) are the same for all values of \( \alpha \) and each equal to \( \frac{1}{\sqrt{N+1}} \), which proves that the mixture has to be mutually unbiased. Finally, these considerations fix \( \lambda = \frac{N}{N+1} \).

We end our calculations by formulating the criterion for getting the special relationship in terms of the mixture weights and the bases alone without any reference to the initial state or the post-measurement state. For this, let \( \rho_{ini} \) be represented, in one element of the set of \( N+1 \) basis vectors, say, \( |i^{(0)}\rangle \), be

\[
\rho_{ini} = \sum_{k,l} |k^{(0)}\rangle \langle l^{(0)}| \rho_{kl}^{ini} \tag{31}
\]

Denoting the probability of getting the eigenvalue labelled by \( j \) upon a measurement of \( X^{(a)} \) by \( p_j^{(a)} \) one has

\[
p_j^{(a)} = tr \rho_{ini} P_j^{(a)}
\]

\[
= \sum_{k,l} \rho_{kl}^{ini} (j^{(a)} |k^{(0)}\rangle \langle l^{(0)}| j^{(a)})
\]

\[
= \sum_{k,l} \rho_{kl}^{ini} c_{jl}^{(a)} c_{jk}^{(a)} \tag{32}
\]
where
\[ C^{(\alpha)}_{ij} = \langle j | (0) | i^{(\alpha)} \rangle \]  
(33)

The mixed state that results after this measurement is:
\[ \rho^{(\alpha)} = \sum_j p^{(\alpha)}_j P^{(\alpha)}_j \]  
(34)

Let these be mixed with weights \( c^{(\alpha)} \) with \( \sum \alpha c^{(\alpha)} = 1 \). The resulting post-measurement density matrix is given by
\[ \rho_{msmt} = \sum_{i\alpha} c^{(\alpha)} P^{(\alpha)}_i \]  
(35)

We express this in the same basis \( |i^{(0)}\rangle \) in which we had expressed \( \rho_{ini} \)
\[ \rho_{msmt}^{kl} = \sum_{i\alpha} c^{(\alpha)} P^{(\alpha)}_i \langle k^{(0)} | i^{(\alpha)} \rangle \langle i^{(\alpha)} | l^{(0)} \rangle \]
\[ = \sum_{i\alpha} c^{(\alpha)} P^{(\alpha)}_i C_{ik}^{(\alpha)*} C_{il}^{(\alpha)} \]  
(36)

Substituting the expression for \( P^{(\alpha)}_i \) from eqn(32) one has
\[ \rho_{msmt}^{kl} = \sum_{i\alpha} c^{(\alpha)} C_{ik}^{(\alpha)*} \sum_{pq} \rho_{ini}^{pq} C_{ip}^{(\alpha)*} C_{iq}^{(\alpha)} \]
\[ = \sum_{pq} d_{kl,pq} \rho_{ini}^{pq} \]  
(37)

where
\[ d_{kl,pq} = \sum_{i\alpha} c^{(\alpha)} C_{ik}^{(\alpha)*} C_{ip}^{(\alpha)*} C_{iq}^{(\alpha)} \]  
(38)

Since we are looking for a relationship of the type
\[ \rho_{msmt} = \frac{\lambda}{N} \mathbf{I} + (1 - \lambda) \rho_{ini} \]
we must have
\[ d_{kl,pq} = \frac{\lambda}{N} \delta_{kl} \delta_{pq} + (1 - \lambda) \delta_{kp} \delta_{lq} \]  
(39)

Equating eqn(38) to eqn(39) we get the required criterion to be
\[ \sum_{i\alpha} c^{(\alpha)} C_{ik}^{(\alpha)*} C_{ip}^{(\alpha)*} C_{iq}^{(\alpha)} = \frac{\lambda}{N} \delta_{kl} \delta_{pq} + (1 - \lambda) \delta_{kp} \delta_{lq} \]  
(40)

This can be taken as an alternate definition of MUB.

Now we establish the equivalent of eqn(10) for the generalised case. Again note that \( \rho_{msmt} \) has one eigenvalue \( \frac{2}{N+1} \) and \( N - 1 \) smaller eigenvalues \( \frac{1}{N+1} \) leading to the spectral decomposition
\[ \rho_{msmt} = \frac{2}{N+1} |l\rangle \langle l| + \frac{1}{N+1} \sum_{i=1}^{N-1} |s_i\rangle \langle s_i| \]  
(41)

Now using eqn(23) one finds, as before,
\[ \rho_{ini} = |l\rangle \langle l| \]  
(42)

We finally comment on a very similar looking relation first derived by Audretsch et al [9]. Their eqn(20) looks remarkably like our eqn(23) but these equations and the respective contexts are very different. Firstly their eqn(20) is only for fideleties while our result is for density matrices. Secondly the pre and post quantities appear oppositely to our equation. Finally, their results are derived in the context of so called generalised measurements while our considerations are in the context of projective measurements. As generalised measurements are more general than projective measurements, the precise connection between these equations is interesting to pursue.

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[3] J.Schwinger, Proc. Nat. Acad. Sci. U.S.A. 46(1960), 570.
[4] W.K. Wootters and B.C. Fields, Ann. of Phys. 191, 363 (1989).
[5] W.K. Wootters, quant-ph/0406032; Thomas Durt, quant-ph/0401046; A. Klappenecker and M. Roetteler, quant-ph/0309120; P.K. Aravind, quant-ph/0306119, Z. Naturforsch. 58a, 682(2003); S. Chaturvedi, quant-ph/0109003, Phys. Rev. A65, 044301(2002); S.Bandyopadhyay, P. Oscar Boykin, V. Roychowdhury and F.Vatan, quant-ph/0103162.
[6] A.Peres, Quantum Theory: Concepts and Methods, Kluwer Academic Publishers,1993.
[7] M.A. Nielsen and I.L.Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2002.
[8] K.Kraus, Phys. Rev. D 35 (1987), 3070.
[9] Estimating the post-measurement state: Jurgen Audretsch, Lajos Diosi and Thomas Konrad, quant-ph/0210205, Phys.Rev.A6834302(2003).