TOEPLITZ OPERATORS DEFINED BY SESQUILINEAR FORMS: FOCK SPACE CASE

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ABSTRACT. The classical theory of Toeplitz operators in spaces of analytic functions deals usually with symbols that are bounded measurable functions on the domain in question. A further extension of the theory was made for symbols being unbounded functions, measures, and compactly supported distributions, all of them subject to some restrictions.

In the context of a reproducing kernel Hilbert space we propose a certain framework for a ‘maximally possible’ extension of the notion of Toeplitz operators for a ‘maximally wide’ class of ‘highly singular’ symbols. Using the language of sesquilinear forms we describe a certain common pattern for a variety of analytically defined forms which, besides covering all previously considered cases, permits us to introduce a further substantial extension of a class of admissible symbols that generate bounded Toeplitz operators.

Although our approach is unified for all reproducing kernel Hilbert spaces, for concrete operator consideration in this paper we restrict ourselves to Toeplitz operators acting on the standard Fock (or Segal-Bargmann) space.

1. INTRODUCTION

The classical theory of Toeplitz operators in spaces of analytic functions (Hardy, Bergman, Fock, etc spaces) deals usually with symbols that are bounded measurable functions on the domain in question. As it was observed later, in the case of the Bergman and Fock spaces, for certain classes of unbounded symbols one still can reasonably define Toeplitz operators that prove to be bounded.

Further natural extensions towards even more general symbols of Toeplitz operators were developed for the first time (to the best of our knowledge) in [17], for symbols being measures, and in [1], for symbols being compactly supported distributions. The main idea behind such...
extensions was to enrich the class of Toeplitz operators, and, in particular, to turn into Toeplitz many of operators that failed to be Toeplitz in the above classical sense.

The aim of this paper is to propose a certain framework for a ‘maximally possible’ extension of the notion of Toeplitz operators for a ‘maximally wide’ class of ‘highly singular’ symbols, including the ones that involve derivatives of infinite order of measures. In the general context of a reproducing kernel Hilbert space we propose a certain common pattern which, besides covering all previously considered extensions, enables us to introduce a further substantial extension of a class of ‘highly singular’ but still admissible symbols that generate bounded Toeplitz operators.

Our approach is based upon the extensive use of the language of sesquilinear forms. Let us describe shortly its main ingredients (for more details, examples, and proofs see the main text). Let $\mathcal{H}$ be a Hilbert space of functions defined in a domain $\Omega \subseteq \mathbb{R}^d$ or $\Omega \subseteq \mathbb{C}^d$, and let $\mathcal{A}$ be its closed subspace having a reproducing kernel $k_z(w) \in \mathcal{A}$, so that the orthogonal projection $P$ of $\mathcal{H}$ onto $\mathcal{A}$ has the form

$$(Pu)(z) = \langle u(\cdot), k_z(\cdot) \rangle.$$ 

Then, given a bounded sesquilinear form $F(\cdot, \cdot)$ on $\mathcal{A}$, the Toeplitz operator $T_F$ defined by the sesquilinear form $F$ is the operator which acts on $\mathcal{A}$ as

$$(T_F f)(z) = F(f(\cdot), k_z(\cdot)).$$ 

The next step in our study is to describe a certain common pattern for a variety of analytically defined forms. Throughout the paper we consider various concrete classes of sesquilinear forms generated by quite different analytical objects (functions, measures, distributions, etc). For Toeplitz operators defined by such specific sesquilinear forms we say that they have corresponding function, measure, distribution, etc symbols. We mention as well that different analytical expressions may define the same sesquilinear form and thus define the same Toeplitz operator.

Our common pattern in defining sesquilinear forms and the corresponding Toeplitz operators is as follows. We start with a (complex) linear topological space $X$ (not necessarily complete). Let $X'$ be its dual space (the set of all continuous linear functionals on $X$), and denote by $\Phi(\phi) = (\Phi, \phi)$ the intrinsic pairing of $\Phi \in X'$ and $\phi \in X$. Let finally $\mathcal{A}$ be a reproducing kernel Hilbert space with $k_z(\cdot)$ being its reproducing kernel function.
By a continuous \( X \)-valued sesquilinear form \( G \) on \( A \) we mean a continuous mapping

\[
G(\cdot, \cdot) : A \oplus A \rightarrow X,
\]
which is linear in the first argument and anti-linear in the second one. Then, given a continuous \( X \)-valued sesquilinear form \( G \) and an element \( \Phi \in X' \), we define the sesquilinear form \( F_{G,\Phi} \) on \( A \) by

\[
F_{G,\Phi}(f, g) = \Phi(G(f, g)) = (\Phi, G(f, g)).
\]

Being continuous, this form is bounded, and thus defines a bounded Toeplitz operator

\[
(T_{G,\Phi}f)(z) := (T_{F_{G,\Phi}}f)(z) = (\Phi, G(f, \mathbf{k}z)).
\]

Having such an extended approach to Toeplitz operators, we immediately gain, for example, the following very important result (Theorem 4.1): The set of Toeplitz operators is a \( \ast \)-algebra. Recall in this connection that classical Toeplitz operators do not possess such a nice property.

Although our approach is unified for all reproducing kernel Hilbert spaces, for concrete considerations in this paper we restrict ourselves to Toeplitz operators acting in the standard Fock (or Segal-Bargmann) space. We show that all previously introduced extensions of the notion of Toeplitz operators to unbounded, measure, and distributional symbols fit perfectly our pattern. Moreover, we describe several further extensions to even more singular symbols.

We introduce and study symbols being Fock-Carleson measures for derivatives, for which we characterize boundedness and compactness of corresponding Toeplitz operators (Proposition 6.1, Proposition 6.4, and Corollary 6.3). In particular, we show (Theorem 6.13) that given any bounded linear operator on the Fock space, any finite truncation of the infinite matrix representation of this operator with respect to the standard basis in \( \mathcal{F}^2(\mathbb{C}) \) is a Toeplitz operator with distributional symbol supported at \( \{0\} \), or (in other terms) with symbol being the sum of derivatives of certain Fock-Carleson measures.

In Section 3 we collect various examples of operators that fail to be Toeplitz ones in the classical sense with bounded, unbounded (in a certain class), and even distributional symbols. All of them, and many more, become Toeplitz operators in our extended sense.

Our study is essentially based upon some new sharp pointwise estimates for derivatives of functions \( f \in \mathcal{F}^2(\mathbb{C}) \), with explicitly shown dependence of the constants on the order of the differentiation (Proposition 5.1, Proposition 5.7), and a more sharp local estimate, where
the $F^2$-norm is replaced by an integral, involving $f$, over a certain disk (Proposition 5.2).

We note finally that the results of the paper can be easily extended, with just minor technical changes in the proofs, to the case of operators acting on the multidimensional Fock space $F^2(C^n)$, with $n > 1$.

2. General operator theory approach to Toeplitz operators

To proceed with our plan we need to start from the very beginning and discuss anew the basic notions and definitions.

Let $H$ be a Hilbert space with $H_0$ being its closed subspace. We denote by $P$ the orthogonal projection of $H$ onto $H_0$. In the most general setting, given a linear bounded operator $A$ acting in $H$, the Toeplitz operator $T_A$ associated with $A$ and acting in $H_0$ (≡ the compression of $A$ onto $H_0$, or the angle of the operator $A$) is defined as

$$T_A : x \in H_0 \mapsto P(Ax) \in H_0, \quad i.e., \quad T_A = PA|_{H_0}. \quad (2.1)$$

The interrelation between $A$ and $T_A$ is thus very simple: the latter operator is a compression of the former, while the former is a dilation of the latter.

In such a general setting, different operators $A'$ and $A''$ can obviously generate the same Toeplitz operator. Indeed, let

$$A' = \begin{pmatrix} A_{1,1}' & A_{1,2}' \\ A_{2,1}' & A_{2,2}' \end{pmatrix} \quad \text{and} \quad A'' = \begin{pmatrix} A_{1,1}'' & A_{1,2}'' \\ A_{2,1}'' & A_{2,2}'' \end{pmatrix}$$

be the matrix representations of $A'$ and $A''$ in $H = H_0 \oplus H_0^\perp$. Then $T_{A'} = T_{A''}$ if and only if $A_{1,1}' = A_{1,1}''$.

Thus, to have a substantial theory, one should consider some natural and important subclasses of operators on $H$.

One of the most known and classical examples here is represented by Toeplitz operators with bounded measurable symbols, acting in the Fock space. In this example $H$ is $L_2(\mathbb{C}, d\nu)$, i.e., the Hilbert space of functions on $\mathbb{C}$, square-integrable with respect to the Gaussian measure

$$d\nu(z) = \omega(z)dV(z), \quad \text{where} \quad \omega(z) = \pi^{-1} e^{-z \cdot \bar{z}},$$

and $dV(z) = dx dy$ is the Lebesgue plane measure on $\mathbb{C} = \mathbb{R}^2$. Then $H_0 = F^2(\mathbb{C})$ is its Fock space [6, 15] (or Segal–Bargmann [3, 26]) subspace consisting of all functions analytical in $\mathbb{C}$, and $P$ is the orthogonal projection of $L_2(\mathbb{C}, d\nu)$ onto $F^2(\mathbb{C})$. Given a function $a = a(z) \in L_\infty(\mathbb{C})$, the Toeplitz operator $T_a$ with symbol $a$ is the compression
onto $F^2(\mathbb{C})$ of the multiplication operator $(M_a f)(z) = a(z)f(z)$ on $L_2(\mathbb{C}, d\nu)$:

$$T_a : f \in F^2(\mathbb{C}) \mapsto P(af) \in F^2(\mathbb{C}).$$

It is well known (see, for example, [4]) that in this case, i.e., if we restrict the class of defining operators to the above multiplication operators, the (function) symbol $a$ is uniquely defined by the Toeplitz operator.

The aim of this paper is to extend the above setting to possibly most general symbols, admitting unboundedness and various types of singularities. Although the main results of the paper are given for the Toeplitz operators acting in the Fock space $F^2(\mathbb{C})$, our approach is applicable in a more general setting of a rather arbitrary Hilbert space with reproducing kernel.

We recall that if $H$ is a Hilbert space of functions defined in a domain $\Omega \subseteq \mathbb{R}^d$ or $\Omega \subseteq \mathbb{C}^d$, then its closed subspace $\mathcal{A}$ is called a reproducing kernel subspace if any evaluation functional $\mathcal{A} \ni f \mapsto f(z)$ is well defined and bounded. Most typically such subspaces consist of $L_2$-solutions of elliptic equations or systems. For any fixed $z \in \Omega$, let $k_z(w) \in \mathcal{A}$ be the element in $\mathcal{A}$ realizing by the Riesz theorem this evaluation functional, i.e.,

$$f(z) = \langle f(w), k_z(w) \rangle, \quad \text{for all } f \in \mathcal{A},$$

so that the orthogonal projection $P$ of $H$ onto $\mathcal{A}$ has the form

$$P : u(z) \in H \mapsto \langle u(w), k_z(w) \rangle \in \mathcal{A},$$

or

$$(Pu)(z) = \langle u(\cdot), k_z(\cdot) \rangle. \quad (2.2)$$

Here and further on, by $\langle \cdot, \cdot \rangle$ we denote the scalar product in the Hilbert space under consideration. The function $k_z(w)$ is called the reproducing kernel for $\mathcal{A}$.

Recall that a bounded sesquilinear form $F(\cdot, \cdot)$ on $\mathcal{A}$ is a mapping

$$F(\cdot, \cdot) : \mathcal{A} \oplus \mathcal{A} \rightarrow \mathbb{C},$$

which is linear in the first argument and anti-linear in the second one and, additionally, satisfies the boundedness condition: there exists a constant $C \geq 0$ such that

$$|F(f, g)| \leq C\|f\| \cdot \|g\|, \quad \text{for all } f, g \in \mathcal{A}.$$  

By the Riesz theorem for bounded sesquilinear forms, for a given form $F(\cdot, \cdot)$, there exists a unique bounded linear operator $T$ in $\mathcal{A}$ such that

$$F(f, g) = \langle Tf, g \rangle, \quad \text{for all } f, g \in \mathcal{A}. \quad (2.3)$$
In this paper we adopt the following vocabulary. Given a bounded sesquilinear form $F(\cdot,\cdot)$ on $\mathcal{A}$, the Toeplitz operator $T_F$ defined by the sesquilinear form $F$ is the operator which acts on $\mathcal{A}$ as
\[(T_F f)(z) = F(f(\cdot), k_z(\cdot)).\] (2.4)
The terminology “Toeplitz” is consistent with the general definition (2.1) since (2.2) and (2.3) imply that
\[(T_F f)(z) = \langle Tf, k_z \rangle = (Tf)(z) = (TAf)(z)\]
for any dilation $A$ of the operator $T$ on $\mathcal{A}$ to some Hilbert space $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^\perp$.

Note that although the dilation $A$ is not unique, the operator $T_F$ is uniquely defined by the form $F$. We mention as well that quite different analytic expressions may define the same sesquilinear form and thus define the same Toeplitz operator. Examples for this effect will be presented further on.

**Remark 2.1.** Throughout the paper we consider various concrete classes of sesquilinear forms generated by different analytic objects (functions, measures, distributions, etc). For Toeplitz operators defined by such specific sesquilinear forms we will say that they have corresponding function, measure, distribution, etc symbols.

We illustrate now the above approach to Toeplitz operators by specifying some classes of sesquilinear forms. In this paper the enveloping Hilbert space $\mathcal{H}$ will always be $L_2(\mathbb{C},d\nu)$ and its reproducing kernel subspace $\mathcal{A}$ will always be the Fock space $\mathcal{F}^2(\mathbb{C})$ consisting of analytical functions. Recall that in this case
\[k_z(w) = e^{wz},\]
so that the orthogonal projection $P$ from $L_2(\mathbb{C},d\nu)$ onto $\mathcal{F}^2(\mathbb{C})$ has the form
\[(P u)(z) = \int_\mathbb{C} u(w)e^{z\overline{w}}d\nu(w) = \frac{1}{\pi} \int_\mathbb{C} u(w)e^{(z-w)\overline{w}}dV(w).\]
It is important to keep in mind that the standard orthonormal monomial basis in $\mathcal{F}^2(\mathbb{C})$ has the form
\[e_k(z) = \frac{1}{\sqrt{k!}} z^k, \quad k \in \mathbb{Z}_+.\] (2.5)

**Example 2.2.** Classical Toeplitz operators on the Fock space.
We start with an arbitrary bounded linear functional $\Phi \in L_1^*(\mathbb{C},d\nu)$. 
As well known, such a functional is uniquely defined by a function \(a \in L_\infty(C)\) and has the form

\[
\Phi(u) = \Phi_a(u) = \int_C a(z)u(z)d\nu(z),
\]

with \(\|\Phi_a\| = \|a\|_{L_\infty}\). For any \(f, g \in \mathcal{F}^2(C)\) the product \(fg\) belongs to \(L_1(C,d\nu)\), and finite linear combinations of such products are dense in \(L_1(C,d\nu)\). We define the sesquilinear forms \(F_a\) as

\[
F_a(f, g) = \Phi_a(fg) = \int_C a(w)f(w)g(w)d\nu(w).
\]

This form is obviously bounded:

\[
|F_a(f, g)| \leq \|\Phi_a\|\|fg\|_{L_1} \leq \|a\|_{L_\infty}\|f\|\|g\|.
\]

Then

\[
(T_{F_a}f)(z) = F_a(f, k_z) = \Phi_a(fk_z) = \int_C a(w)f(w)k_z(w)d\nu(w) = \int_C a(w)f(w)e^{\overline{z}w}d\nu(w) = (T_a f)(z),
\]

i.e, the Toeplitz operator \(T_{F_a}\) generated by the sesquilinear form \(F_a\) coincides with the classical Toeplitz operator having the function symbol \(a\). Moreover, all Toeplitz operators with bounded measurable symbols can be obtained starting with a proper functional \(\Phi\) in \(L_1^*(C,d\nu)\), which defines in turn the form (2.6).

It was already mentioned (in this setting) that the \(L_\infty\)-symbol \(a(z)\) is uniquely defined by a Toeplitz operator. If we admit unbounded symbols \(a(z)\) then the corresponding sesquilinear forms \(F_a\) in (2.6) are not bounded in \(\mathcal{F}^2(C)\), in general. However, for some classes of unbounded \(a\), this sesquilinear form can still be well defined and bounded. Folland [11, p. 140] extended the above uniqueness result to the class of unbounded symbols that satisfy the inequality

\[
|a(z)| \leq Ce^{\delta|z|^2}, \quad \text{for some} \quad \delta < 1.
\]

On the other hand, the further extension to wider classes of unbounded symbols may lead to the failure of the uniqueness of symbols: there exist nontrivial symbols \(a\) for which \(T_a = 0\). See [14, Theorem 3.4] and [2, Proposition 4.6] for concrete examples of such \(a\).

**Example 2.3.** Toeplitz operators defined by Fock-Carleson measures.

Recall (see, for example, [27, Section 3.4]) that a finite positive Borel measure \(\mu\) on \(C\) is called a Fock-Carleson measure (FC measure) for
the space $\mathcal{F}^2(\mathbb{C})$ if there exists a constant $C > 0$ such that
\[
\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, d\mu(z) \leq C \int_{\mathbb{C}} |f(z)|^2 \, d\nu(z), \text{ for all } f \in \mathcal{F}^2(\mathbb{C}). \quad (2.8)
\]
We define the sesquilinear form $F_{\mu}$ as
\[
F_{\mu}(f, g) = \int_{\mathbb{C}} f(z)g(z)e^{-|z|^2} \, d\mu(z),
\]
which is obviously bounded by the Cauchy inequality and (2.8). Then
\[
(T_{F_{\mu}}f)(z) = \int_{\mathbb{C}} f(w)k_z(w)e^{-|w|^2} \, d\mu(w) = \int_{\mathbb{C}} f(w)e^{(z-w)\overline{\omega}} \, d\mu(w) = (T_\mu f)(z),
\]
i.e., the Toeplitz operator $T_{F_{\mu}}$ generated by the sesquilinear form $F_{\mu}$ is nothing but the (bounded) Toeplitz operator defined by the Fock-Carleson measure $\mu$.

**Example 2.4.** Toeplitz operators with compactly supported distributional symbols.
Any distribution $\Phi$ in $\mathcal{E}'(\mathbb{C})$ has finite order, and thus can be extended to a continuous functional in the space of functions with finite smoothness,
\[
|\Phi(h)| \leq C(\Phi)\|h\|_{C^N(\mathbb{K})}, \quad h \in C^N(\mathbb{K}),
\]
for some compact set $\mathbb{K} \subset \mathbb{C}$ containing the support of $\Phi$. It follows from the Cauchy formula that the $C^N(\mathbb{K})$-norm of the product $h(z) = e^{-z} f(z)g(z)$, $f, g \in \mathcal{F}^2(\mathbb{C})$, is majorated by the product of the $\mathcal{F}^2(\mathbb{C})$-norms of $f$ and $g$. Therefore, the sesquilinear form
\[
F_{\Phi}(f, g) = \Phi \left( \omega(\cdot)f(\cdot)\overline{g(\cdot)} \right)
\]
is bounded in $\mathcal{F}^2(\mathbb{C})$ and thus defines a bounded operator.

The operator
\[
(T_{F_{\Phi}}f)(z) = \Phi \left( \omega(\cdot)f(\cdot)k_z(\cdot) \right) = (T_\Phi f)(z) \quad (2.9)
\]
generated by the sesquilinear form $F_{\Phi}$ coincides with the standardly defined [1, 22] Toeplitz operator $T_\Phi$ with distributional symbol $\Phi$. By the structure theorem for distributions with compact support (see, e.g., [12, Section 4.4]), $\Phi$ admits a representation as a finite sum
\[
\Phi = \sum_q D^q u_q, \quad (2.10)
\]
where $q = (q_1, q_2)$, $D = (D_1, D_2)$ is the distributional gradient, and $u_q$ are continuous functions that can be chosen as having support in an
arbitrarily close neighborhood of the support of $\Phi$. Rearranging the derivatives, we can rewrite (2.10) as the finite sum

$$\Phi = \sum_q \partial^q_1 \partial^2_2 v_q,$$

again with certain continuous functions $v_q$, having support in an arbitrarily close neighborhood of $\text{supp} \Phi$. Then (2.9) transforms to

$$(T_\Phi f)(z) = \sum_q (-1)^{q_1+q_2} \frac{1}{\pi} \int_C v_q(w) \cdot \partial^q_1 \partial^2_2 [e^{(z-w)^2} f(w)] dV(w).$$

(2.12)

We present now several examples of the action of such operators $T_\Phi$ with $\Phi \in \mathcal{E}'(\mathbb{C})$. By $\delta$ we denote here the usual $\delta$-distribution in $\mathbb{R}^2 = \mathbb{C}$, centered at 0.

Let $\Phi_{p,q} = \partial^p w \partial^q w \delta$. Then, by (2.9), we have

$$T_{\Phi_{p,q}} e_k = \left\{ \begin{array}{ll}
\frac{(-1)^{p-q} \sqrt{p!q!}}{\pi} e_{q-p+k}, & \text{if } \max(0, p-q) \leq k \leq p, \\
0, & \text{otherwise}.
\end{array} \right.$$  

(2.13)

For $q = p$ we write $\Phi_p = \Phi_{p,p}$, and in this case

$$T_{\Phi_p} e_k = \left\{ \begin{array}{ll}
\frac{(-1)^{p-k} [p!]^2}{\pi} \frac{1}{(p-k)!} e_k, & \text{if } 0 \leq k \leq p, \\
0, & \text{otherwise}.
\end{array} \right.$$  

(2.14)

Thus, the Toeplitz operator $T_{\Phi_p}$ is a linear combination of the rank one projections $P_k = \langle \cdot, e_k \rangle e_k$, for $k = 0, 1, ..., p$. Vice versa, the orthogonal projection $P_n$ is a linear combination of the Toeplitz operators $T_{\Phi_p}$, for $p = 0, 1, ..., n$, i.e., $P_n$ is a Toeplitz operator with a certain distributional symbol having the one-point support $\{0\}$.

Let now $p \neq q$. If $p < q$, then the Toeplitz operator $T_{\Phi_{p,q}}$ is a linear combination of the rank one operators $P_{n,m} = \langle \cdot, e_n \rangle e_m$ with $n = k$, $m = q - p + k$, and $k = 0, 1, ..., p$. Vice versa: the rank one operator $P_{p,q}$ is a linear combination of the Toeplitz operators $T_{\Phi_{p-n,q-n}}$, with $n = 0, 1, ..., p$.

If $p > q$, then the Toeplitz operator $T_{\Phi_{p,q}}$ is a linear combination of the rank one operators $P_{p-q+n,n}$ with $n = 0, 1, ..., q$. Vice versa: the rank one operator $P_{p,q}$ is a linear combination of the Toeplitz operators $T_{\Phi_{p-n,q-n}}$, with $n = 0, 1, ..., q$. 
We note that for $p \neq q$, again, each rank one operator $P_{p,q}$ is a Toeplitz operator with a certain distributional symbol having the one-point support $\{0\}$.

The above representation of the operators $P_{p,q}$, $p, q \in \mathbb{Z}^+$, does not look quite satisfactory: a rank one operator corresponds to several terms in the distributional symbol. It is more convenient and aesthetic to use another, more straightforward, form: the distributional symbol

$$
\Psi_{p,q} = \frac{(-1)^{p+q}}{\sqrt{p!q!}} \omega^{-1}(w) \partial_w^p \partial_{\bar{w}}^q \delta.
$$

(2.16)

Then, by (2.9), we have

$$
T_{\Psi_{p,q}} e_k = \begin{cases} e_q, & \text{if } k = p, \\ 0, & \text{otherwise.} \end{cases}
$$

Thus $P_{p,q} = T_{\Psi_{p,q}}$, for all $p, q \in \mathbb{Z}^+$.

Either of the above representations of $P_{p,q}$ implies now the following lemma.

**Lemma 2.5.** For each $p, q \in \mathbb{Z}^+$ the rank one operator $P_{p,q}$ is a Toeplitz operator with a certain distributional symbol having the one-point support $\{0\}$.

We mention also that finite linear combinations of the rank one Toeplitz operators $P_{p,q}$, where $p, q \in \mathbb{Z}^+$, form a norm dense subset both in the set of all finite rank and all compact operators on $\mathcal{F}^2(\mathbb{C})$.

We improve the understanding of this effect further on, in Section 6.

## 3. Non-Toeplitz operators in $\mathcal{F}^2(\mathbb{C})$

In this section we present examples of operators in the Fock space $\mathcal{F}^2(\mathbb{C})$, that are bounded but are not Toeplitz operators with classical functional or even distributional symbols. Even certain classes of unbounded symbols cannot generate these operators. We begin with their definitions.

### 3.1. Classes of unbounded symbols. Class $\mathcal{D}_{1-}$

This class consists of all (possibly unbounded) measurable functions $a$ satisfying the condition

$$
a(\cdot) k_z(\cdot) \in L_2(\mathbb{C}, d\nu), \quad \text{for all } z \in \mathbb{C}.
$$

Such functions were already used in [5], while the notation was introduced in [7].
Class $D_c$. This class consists of all (possibly unbounded) measurable functions $a$ satisfying the condition

$$\exists \, d > 0 \quad \text{such that} \quad |a(z)| \leq d e^{c|z|^2} \quad \text{a.e.} \quad z \in \mathbb{C}.$$ 

Such functions were already used in [11] (see also (2.7)), while the notation was introduced in [2, Section 2].

Class $L_1^\infty(\mathbb{R}_+, e^{-r^2})$. This class was introduced in [14, Section 3] and consists of all (possibly unbounded) measurable radial functions $a = a(r), r = |z|$, satisfying the condition

$$\int_{\mathbb{R}_+} |a(r)| e^{-r^2} r^n \, dr < \infty, \quad \text{for all} \quad n \in \mathbb{Z}_+.$$ 

3.2. Radial operators.

Example 3.1. Let $(J \varphi)(z) = \varphi(-z)$ be the reflection operator in $\mathcal{F}^2(\mathbb{C})$. It is obviously bounded and acts in the standard monomial basis (2.5) of $\mathcal{F}^2(\mathbb{C})$ as follows:

$$(Je_k)(z) = (-1)^k e_k(z). \quad (3.1)$$

That is, $J$ is a diagonal operator with respect to the above basis (and thus radial, see [28]), and its eigenvalue sequence, ordered in accordance with (3.1), has the form $\gamma_J = \{(-1)^k\}_{k \in \mathbb{Z}_+}$.

By [5, Theorem 17] the operator $J$ cannot be a Toeplitz operator with $L_\infty$-symbol and even with any (unbounded) symbol $a \in \mathcal{D}_{1,-}$. Moreover, $J$ cannot be norm approximated by Toeplitz operators with symbols in $\mathcal{D}_{1,-}$: by [5, Theorem 17] for any symbol $a \in \mathcal{D}_{1,-}$, the norm $\|J - T_a\|$ is at least 1.

On the other hand, if we allow unbounded radial symbols in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, i.e., with just slightly weaker growth restrictions, then by [14, Theorem 3.7] there exists a symbol $a_J$ in this class such that $J = T_{a_J}$. Note that the proof of Theorem 3.7 in [14] gives an algorithm of constructing such a symbol.

Example 3.2. Consider in $\mathcal{F}^2(\mathbb{C})$ the orthogonal projection $P_0 f = \langle f, e_0 \rangle e_0$ onto the one-dimensional subspace generated by $e_0$. It is a diagonal, and thus radial, operator having the eigenvalue sequence $\gamma_{P_0} = (1, 0, 0, \ldots)$. Let us show now that $P_0$ cannot be a Toeplitz operator with a bounded measurable symbol and even with an unbounded one in the class $\mathcal{D}_c$, for any $c \in (0, 1)$. Indeed, let us suppose that $P_0 = T_{a_0}$ with $a_0 \in \mathcal{D}_c$, and $c \in (0, 1)$. Since $P_0$ is a projection, we have

$$0 = P_0(I - P_0) = T_{a_0}(I - T_{a_0}) = T_{a_0}T_{1-a_0}.$$
where both symbols $a_0$ and $1-a_0$ are radial and belong to $\mathcal{D}_c$. However, by [2, Theorem 4.4], if the product of two Toeplitz operators with $\mathcal{D}_c$-symbols is zero then one of them must be zero. This means that either $a_0 = 0$ or $a_0 = 1$, which is impossible for any non-trivial projection.

The same argument implies the following statement.

**Proposition 3.3.** Neither non-trivial radial projection can be represented as a Toeplitz operator with symbol from $\mathcal{D}_c$, for some $c \in (0,1)$. However, any such projection is a Toeplitz operator with $L^\infty_1(\mathbb{R}_+, e^{-r^2})$-symbol.

On the other hand, the operator $P_0$ can be represented as a Toeplitz operator with just slightly more singular symbol, moreover such a representation is not unique. The first such representation of $P_0$ is as follows. Consider the distributional symbol $\Phi_0 = \frac{1}{\pi} \delta$. Then, by (2.9), for the Toeplitz operator $T_{\Phi_0}$ we have

$$T_{\Phi_0} e_k = \begin{cases} 1, & k = 0, \\ 0, & \text{otherwise}. \end{cases}$$

The Toeplitz operator $T_{\Phi_0}$, thus defined, is nothing but the above rank one projection $P_0$. The second representation of $P_0$ comes via symbols in $L^\infty_1(\mathbb{R}_+, e^{-r^2})$. By [14, Theorem 3.7] there exists a symbol $a_{\Phi_0} \in L^\infty_1(\mathbb{R}_+, e^{-r^2})$ such that $P_0 = T_{a_{\Phi_0}}$.

We mention also that, although the operator $P_0$ cannot be represented as a Toeplitz operator with a bounded measurable radial symbol, it can be norm approximated by Toeplitz operators with such symbols:

$$P_0 = \lim_{n \to \infty} T_{a_n},$$

where $a_n(r) = (1 + n)e^{-nr^2}$.

Indeed, the eigenvalue sequence of the operator $T_{a_n}$ [14, Theorem 3.1] has the form

$$\gamma_{a_n}(k) = \frac{1}{k!} \int_{\mathbb{R}_+} (1 + n)e^{-nr}e^{-r} r^k dr = \frac{1}{k!} \int_{\mathbb{R}_+} \frac{1}{(1 + n)^k} e^{-s} s^k ds = \frac{1}{(1 + n)^k}.$$

Thus

$$\|T_{a_n} - P_0\| = \|\gamma_{a_n} - \gamma_0\|_{\ell^\infty} = \gamma_{a_n}(1) = \frac{1}{1 + n},$$

which implies the desired: $P_0 = \lim_{n \to \infty} T_{a_n}$. 
3.3. **Finite rank operators.** Quite recently, a number of results concerning the characterization of finite rank operators in the Fock space were obtained, see [1, 7, 23, 24]. These results produce more examples of operators which are not Toeplitz ones, with functional or distributional symbols.

By the main theorem in [7], any finite rank Toeplitz operator with symbol-function in the class $D_{1,-}$ must be zero. This leads us to the following statement.

**Example 3.4.** Let $T$ be a nonzero finite rank operator in the space $\mathcal{F}^2(\mathbb{C})$. Then $T$ is not a Toeplitz operator with $D_{1,-}$-symbol.

Finite rank operators with distributional symbols have been considered in [1, 24]. By the main theorem in [1], for any finite rank Toeplitz operator in $\mathcal{F}^2(\mathbb{C})$ with symbol $\Phi \in \mathcal{E}'(\mathbb{C})$, i.e., $\Phi$ being a distribution with compact support, this symbol must be a finite linear combination of $\delta$-distributions and their derivatives. Relations (2.4), (2.12) show that for such a distribution, the range of the Toeplitz operator may contain only linear combinations of finitely many functions of the form $f_{w_j,k_j}(z) = z^{k_j}e^{z w_j}$, with some integers $k_j$ and numbers $w_j \in \mathbb{C}$.

**Example 3.5.** Let $T$ be a finite rank operator in $\mathcal{F}^2(\mathbb{C})$ such that at least one function in the range of $T$ is not a finite linear combination of functions $f_{w_j,k_j}(z)$. Then $T$ is not a Toeplitz operator with distributional symbol in $\mathcal{E}'(\mathbb{C})$. In particular, neither finite rank operator, with at least one function having super-exponential growth in its range, can be Toeplitz with a distributional symbol in $\mathcal{E}'(\mathbb{C})$. On the other hand, a finite rank operator with at least one non-polynomial function in the range, having a sub-exponential growth, cannot be a Toeplitz operator in $\mathcal{F}^2$ with symbol in $\mathcal{E}'(\mathbb{C})$ either. Examples for both cases can be easily constructed using the Weierstrass product formula for entire functions with prescribed zeros.

One more, very simple and transparent illustration of the above effect is as follows. Consider two functions in $\mathcal{F}^2(\mathbb{C})$:

$$e_n(z) = \frac{z^n}{\sqrt{n!}} \quad \text{and} \quad \phi(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^{k+1} z} = \sum_{k=0}^{\infty} \frac{1}{(k!)^k} e_k(z).$$

The function $\phi$ cannot be represented as a finite linear combination of functions $f_{w_j,k_j}$ since its Fourier coefficients (in the basis $\{e_k\}$) tend to zero much faster then those of any finite linear combination of functions $f_{w_j,k_j}$. Thus the rank one operator $P_{e_n,\phi} f = \langle f, e_n \rangle \phi$ is not a Toeplitz operator with distributional symbol in $\mathcal{E}'(\mathbb{C})$. 
It is important to note that the operator $P_{e_n, \phi}$ can be norm approximated by Toeplitz operators with distributional symbol in $\mathcal{E}'(\mathbb{C})$. Indeed, let
\[
\phi_m = \sum_{k=0}^{m} \frac{1}{(k!)^k} e_k \quad \text{and} \quad S_m = P_{e_n, \phi_m} = \sum_{k=0}^{m} \frac{1}{(k!)^k} P_{n,k},
\]
then
\[
\|P_{e_n, \phi} - S_m\| = \|\phi - \phi_m\|_{\mathcal{F}^2(\mathbb{C})} \to 0 \quad \text{as} \quad m \to \infty,
\]
and the statement follows from Lemma 2.5.

In [24], a similar finite rank theorem was proved for a wider class $\Lambda'_{q,m}$ of symbols being distributions without compact support condition but with certain growth restriction. Not repeating the definition of these classes, we just refer to the main theorem in [24].

**Example 3.6.** Let $T$ be a finite rank operator in $\mathcal{F}^2(\mathbb{C})$ such that at least one function in the range of $T$ is not a finite linear combination of functions $f_{w_j,k_j}(z)$. Then $T$ is not a Toeplitz operator with distributional symbol in $\Lambda'_{q,m}$.

The operator $P_{e_n, \phi}$ in the previous example may also serve as an illustration for this case.

# 4. General sesquilinear forms

Our aim in this paper is to consider Toeplitz operators with possibly most general defining sesquilinear forms, so that they include all the above examples and more. Having this aim in view, we introduce the following construction.

Let $X$ be a (complex) linear topological space (not necessarily complete). We denote by $X'$ its dual space (the set of all continuous linear functionals on $X$), and denote by $\Phi(\phi) = (\Phi, \phi)$ the intrinsic pairing of $\Phi \in X'$ and $\phi \in X$. Let finally $\mathcal{A}$ be a reproducing kernel Hilbert space with $k_z(\cdot)$ being its reproducing kernel function.

By a continuous $X$-valued sesquilinear form $G$ on $\mathcal{A}$ we mean a continuous mapping
\[
G(\cdot, \cdot) : \mathcal{A} \oplus \mathcal{A} \to X,
\]
which is linear in the first argument and anti-linear in the second one.

Then, given a continuous $X$-valued sesquilinear form $G$ and an element $\Phi \in X'$, we define the sesquilinear form $F_{G, \Phi}$ on $\mathcal{A}$ by
\[
F_{G, \Phi}(f, g) = \Phi(G(f, g)) = (\Phi, G(f, g)). \quad (4.1)
\]
Being continuous, this form is bounded, and thus defines a bounded Toeplitz operator

$$(T_{G,\Phi}f)(z) := (T_{FG,\Phi}f)(z) = (\Phi(G(f,k_z))). \quad (4.2)$$

As it was already mentioned (and will be shown explicitly in the examples of this section), the sesquilinear form that defines a Toeplitz operator may have several quite different analytic expressions (involving different spaces $X$ and functionals $\Phi$), producing, nevertheless, the same Toeplitz operator.

The algebraic operations with the above defined Toeplitz operators can also be described using the language of sesquilinear forms. We start with two Toeplitz operators $T_1 = T_{G_1,\Phi_1}$ and $T_2 = T_{G_2,\Phi_2}$, with certain (complex) linear topological spaces $X_k, G_k$ being continuous $X_k'$-valued sesquilinear forms, and $\Phi_k \in X'_k, k = 1, 2$.

Then (in one of suitable representations of the sesquilinear form)

- the sum $T_1 + T_2$ is the Toeplitz operator $T = T_{G,\Phi}$ defined by the following data
  
  $X = X_1 \times X_2, \quad G = (G_1, G_2), \quad \Phi = (\Phi_1, \Phi_2) \in X'$;

- the product $T_1T_2$ is the Toeplitz operator $T = T_{G,\Phi}$ defined by the following data
  
  $X = X_1, \quad G(f,g) = G_1((\Phi_2, G_2(f,k_z)), g), \quad \Phi = \Phi_1$.

We note as well that if an operator $T$ is defined by a bounded sesquilinear form $F(f,g)$ then its adjoint $T^*$ is defined by the transposed form

$$F^*(f,g) = \overline{F(g,f)},$$

that is, the adjoint operator to (4.2) is defined by

$$(T_{FG,\Phi}^*(f)) (z) = (\Phi(G(k_z,f))).$$

In particular, the operator (4.2) is self-adjoint if and only if its defining form (4.1) is Hermitian symmetric.

Summing up the above we arrive at the following statement.

**Theorem 4.1.** The set of Toeplitz operators of the form (4.2) is a $*$-algebra.

Note that although all our definitions were given in the context of any reproducing kernel Hilbert space, we apply here this construction only for the operators acting in the Fock space $\mathcal{F}^2(\mathbb{C})$. We return thus to the Fock space $\mathcal{F}^2(\mathbb{C})$ and to the examples given in the introductory part of the paper, and show that all of them (and many more) fit into the above pattern.
Example 4.2. Classical Toeplitz operators. 
In this case \( X = L_1(\mathbb{C}, d\nu) \), \( X' = L_\infty(\mathbb{C}) \), \( G(f,g) = \int \overline{g(z)}d\nu \), \( \Phi = \Phi_\alpha = a \in L_\infty(\mathbb{C}) \), so that 
\[
F_{G,\Phi_\alpha}(f,g) = \int_\mathbb{C} a(z)f(z)\overline{g(z)}d\nu.
\]

Example 4.3. The reflection operator \( J \). In this case \( X = L_1(\mathbb{C}, d\nu) \), \( X' = L_\infty(\mathbb{C}) \), \( G(f,g)(z) = f(-z)\overline{g(z)} \), \( \Phi = \Phi_1 = 1 \in L_\infty(\mathbb{C}) \), so that 
\[
F_{G,\Phi_1}(f,g) = \int_\mathbb{C} f(-z)\overline{g(z)}d\nu.
\]

A natural extension of this example is as follows.

Example 4.4. Composition operators. They are the operators of the form \( \mathcal{C}_\varphi f = f \circ \varphi \), where \( \varphi(z) \) is an analytic map of \( \varphi : \mathbb{C} \to \mathbb{C} \). By the results of [8], the operator \( \mathcal{C}_\varphi \) is bounded if and only if \( \varphi \) is a linear mapping, \( \varphi(z) = az + b \) with \( a, b \in \mathbb{C} \) and either \( |a| = 1 \) and \( b = 0 \) or \( |a| < 1 \). Moreover, it is only in the latter case that the operator \( \mathcal{C}_\varphi \) is compact.

The corresponding sesquilinear form is the following: \( X = L_1(\mathbb{C}, d\nu) \), \( X' = L_\infty(\mathbb{C}) \), \( G(f,g)(z) = f(az + b)\overline{g(z)} \), \( \Phi = \Phi_1 = 1 \in L_\infty(\mathbb{C}) \), so that 
\[
F_{G,\Phi_1}(f,g) = \int_\mathbb{C} f(az + b)\overline{g(z)}d\nu.
\]

Note that if \( \varphi = e^{i\theta}z \) then the composition operator \( \mathcal{C}_\varphi : f(z) \mapsto f(e^{i\theta}z) \) is radial with the eigenvalue sequence \( \gamma_{\mathcal{C}_\varphi} = \{e^{in\theta}\}_{n \in \mathbb{Z}_+} \).

Example 4.5. Toeplitz operators defined by FC measures.
In this case \( X = L_1(\mathbb{C}, d\mu) \), \( X' = L_\infty(\mathbb{C}, d\mu) \), \( G(f,g)(z) = f(z)\overline{g(z)}e^{-|z|^2} \), \( \Phi = \Phi_1 = 1 \in L_\infty(\mathbb{C}, d\mu) \), so that 
\[
F_{G,\Phi_1}(f,g) = \int_\mathbb{C} f(z)\overline{g(z)}e^{-|z|^2}d\mu.
\]

A natural generalization of this situation is to admit a complex valued Borel measure \( \mu \) such that its variation \( |\mu| \) is an FC measure. In such a case \( X = L_1(\mathbb{C}, d\mu) := L_1(\mathbb{C}, d|\mu|) \), \( X' = L_\infty(\mathbb{C}, d\mu) := L_\infty(\mathbb{C}, d|\mu|) \) with the same formulas for \( G(f,g) \) and \( \Phi \) as before. The corresponding Toeplitz operators have the form 
\[
(T_{F_{G,\Phi}} f)(z) = F_{G,\Phi}(f,k_z) = \int_\mathbb{C} f(w)e^{(z-w)\overline{\nu}}d\mu(w).
\]

In particular, this includes the case of a positive FC measure \( \mu \), \( X = L_1(\mathbb{C}, d\mu) \), \( X' = L_\infty(\mathbb{C}, d\mu) \), \( G(f,g)(z) = f(z)\overline{g(z)}e^{-|z|^2} \), \( \Phi = \Phi_\alpha = a \in \mathbb{C} \).
$L_\infty(\mathbb{C}, d\mu)$, so that

$$F_{G, \Phi_a}(f, g) = \int_\mathbb{C} a(z)f(z)\overline{g(z)} e^{-|z|^2} d\mu.$$ 

and

$$(T_{F_{G, \Phi_a}} f)(z) = \int_\mathbb{C} a(w)f(w) e^{(z-w)^2} d\mu(w).$$

**Example 4.6.** Toeplitz operators with compactly supported distributional symbols.

In this case $X = \mathcal{E}(\mathbb{C})$, $X' = \mathcal{E}'(\mathbb{C})$, $G(f, g) = \omega f \overline{g}$, $\Phi \in \mathcal{E}'(\mathbb{C})$, so that

$$F_{G, \Phi}(f, g) = \Phi(\omega(\cdot)f(\cdot)\overline{g(\cdot)}) = (\Phi, \omega f \overline{g}).$$

**Example 4.7.** Finite rank operators.

We start with the rank one operators $P_{p,q}f = \langle f, e_p \rangle e_q$, $p, q \in \mathbb{Z}_+$. They can be defined at least by the following two sesquilinear forms.

For the first one, $X = \mathcal{E}(\mathbb{C})$, $X' = \mathcal{E}'(\mathbb{C})$, $G(f, g) = \omega f \overline{g}$, and $\Phi = \Phi_{p,q} \in \mathcal{E}'(\mathbb{C})$ is given by (2.16), so that

$$F_{G, \Phi_{p,q}}(f, g) = (\Phi_{p,q}, \omega f \overline{g}).$$

In the second case, $X = \mathbb{C}$, $X' = \mathbb{C}$, $G(f, g) = \langle f, e_p \rangle \langle e_q, g \rangle$, $\Phi = \Phi_1 = 1 \in \mathbb{C}$, so that

$$F_{G, \Phi_1}(f, g) = \langle f, e_p \rangle \langle e_q, g \rangle.$$

Any finite rank operator in the Fock space has the form

$$Rf = \sum_{j=1}^K \langle f, \phi_j \rangle \psi_j,$$

where $\phi_j$, $\psi_j$, $j = 1, \ldots, K$, are functions in $\mathcal{F}^2(\mathbb{C})$. To define the operator $R$ via a sesquilinear form, we set $X = \mathbb{C}^K$, $X' = (\mathbb{C}^K)'$, identified with $\mathbb{C}^K$ by means of the standard pairing. Further on, we set $G(f, g) = (\langle f, \phi_1 \rangle \langle \psi_1, g \rangle, \ldots, \langle f, \phi_K \rangle \langle \psi_K, g \rangle)$, $\Phi = \Phi_1 = (1, \ldots, 1) \in \mathbb{C}^K$, so that

$$F_{G, \Phi_1}(f, g) = \sum_{j=1}^K \langle f, \phi_j \rangle \langle \psi_j, g \rangle.$$

Another representation of this form can be obtained by setting $X = (\mathcal{F}^2(\mathbb{C}) \otimes \overline{\mathcal{F}^2(\mathbb{C})})^K$, $X' = (\mathcal{F}^2(\mathbb{C}) \otimes \overline{\mathcal{F}^2(\mathbb{C})})^K$ with the Hilbert space pairing, $G(f, g) = f(\zeta)\overline{g(\eta)(1, \ldots, 1)}$, $\Phi = (\phi_1(\zeta)\overline{\psi_1(\eta)}, \ldots, \phi_K(\zeta)\overline{\psi_K(\eta)})$, which gives

$$F_{G, \Phi}(f, g) = \sum_{j=1}^K \langle f, \phi_j \rangle \langle \psi_j, g \rangle.$$
In the special case, when $\phi_j = k\varsigma_j$ and $\psi_j = k\zeta_j$, for some points $\varsigma_j$ and $\zeta_j$ in $\mathbb{C}$, we have

$$G(f, g) = (f(\varsigma_1)g(\zeta_1), ..., f(\varsigma_K)g(\zeta_K)),$$

so that

$$F_{G, \Phi}(f, g) = \sum_{j=1}^{K} f(\varsigma_j)g(\zeta_j).$$

Another representation of this form can be obtained by setting

$$X = (\mathcal{E}(\mathbb{C}) \otimes \mathcal{E}(\mathbb{C}))^K, X' = (\mathcal{E}'(\mathbb{C}) \otimes \mathcal{E}'(\mathbb{C}))^K,$$

$$G(f, g) = \omega(\xi)f(\xi)\omega(\eta)g(\eta)(1, ..., 1),$$

$$\Phi = \omega^{-1}(\xi)\omega^{-1}(\eta)(\delta(\xi - \varsigma_1)\delta(\eta - \zeta_1), ..., \delta(\xi - \varsigma_K)\delta(\eta - \zeta_K)),$$

which gives

$$F_{G, \Phi}(f, g) = \sum_{j=1}^{K} f(\varsigma_j)g(\zeta_j).$$

5. Estimates for derivatives of functions in the Fock space

In order to extend the set of admissible distributional symbols beyond distributions with compact support we establish, in this section, some additional properties of functions in the Fock space $\mathcal{F}^2(\mathbb{C})$.

5.1. Fock-Carleson measures for derivatives; norm estimates.

It is well known, see, e.g., [27], that functions in the Fock space satisfy the growth estimate:

$$|f(z)| \leq e^{|z|^2/2}\|f\|_{\mathcal{F}^2}, \ f \in \mathcal{F}^2.\ (5.1)$$

Certain estimate for derivatives of $f \in \mathcal{F}^2(\mathbb{C})$, generalizing (5.1), has been established in [20]:

$$\|(1 + |z|)^{-k}f^{(k)}e^{-|z|^2}\|_{L_2} \leq C_k\|f\|_{\mathcal{F}^2},$$

with some, non-specified constants $C_k$. We need, however, pointwise estimates for the derivatives for $f$, with explicitly shown dependence of the constants on the order $k$ of the differentiation.

**Proposition 5.1.** Let $f \in \mathcal{F}^2$. Then

$$|f^{(k)}(z)| \leq Ck!(1 + |z|)^{k}e^{\frac{|z|^2}{2}}\|f\|_{\mathcal{F}^2}, \ (5.2)$$

with some constant $C$ not depending on $k$.

**Proof.** For a given $z \in \mathbb{C}$, we fix some $s = s(z)$, to be determined later, and write the Cauchy formula

$$f^{(k)}(z) = k!(2\pi i)^{-1}\int_{|z-\zeta|=s} (z - \zeta)^{-k-1} f(\zeta)d\zeta.\ (5.3)$$
If $|z| \leq 1$, we take $s = s(z) = 1$, and, by (5.3), obtain
\[ |f^{(k)}(z)| \leq k! \max_{|z| \leq 2} |f(z)| \leq k! \|f\|_{\mathcal{F}^2} e^2. \tag{5.4} \]

Now let $|z| > 1$. We apply the Cauchy formula with $s = s(z) = |z|^{-1}$. With this choice, (5.3) gives
\[ |f^{(k)}(z)| \leq k! |z|^k \max_{|\zeta| \leq |z|+|z|^{-1}} |f(\zeta)||z|^k. \]

We substitute here estimate (5.1), which leads to the inequality we need:
\[ |f^{(k)}(z)| \leq C k! (1 + |z|)^k e^{(|z|+|z|^{-1})^2/2} \|f\|_{\mathcal{F}^2} \leq k! (1 + |z|)^k e^{|z|^2/2} \|f\|_{\mathcal{F}^2}. \]

We complement now Proposition 5.1 by a sharper local estimate, where the $\mathcal{F}^2$-norm on the right hand side in (5.2) is replaced by an integral, involving $f$, over a certain disk.

**Proposition 5.2.** Let $f$ be an entire analytical function. For any $r > 1$, with some constant $C(r)$, depending only on $r$,
\[ |f^{(k)}(z)|^2 \leq k!^2 C(r) e^{|z|^2/2} (1 + |z|)^{2k} \int_{B(z,r)} |f(\zeta)e^{-|\zeta|^2/2}|^2 dV(\zeta). \]

**Proof.** For small $|z|$, the estimate follows directly from the first of inequalities in (5.4) which, actually, does not require $f \in \mathcal{F}^2(\mathbb{C})$ and holds for any analytical function $f$. So, let $|z| > 1$. By (5.3),
\[ |f^{(k)}(z)e^{-|z|^2/2}|^2 \leq k!^2 (1 + |z|)^{2k} e^{-|z|^2} \max_{|w-z| \leq s} |f(w)|^2, \tag{5.5} \]
and, due to our choice of $s = |z|^{-1}$, (5.5) leads to
\[ |f^{(k)}(z)e^{-|z|^2/2}|^2 \leq e^2 k!^2 (1 + |z|)^{2k} \max_{|w-z| \leq s} \left( e^{-|w|^2} |f(w)|^2 \right). \tag{5.6} \]
Now we apply Lemma 2.1 in [16], which gives an estimate of the quantity on the right in (5.6):
\[ |f(w)e^{-|w|^2/2}|^2 \leq C(r) \int_{B(w,r)} |f(\zeta)e^{-|\zeta|^2/2}|^2 dV(\zeta). \tag{5.7} \]
For $|w-z| \leq s$, the disk $B(w, r)$ lies inside the disk $B(z, r+s)$, therefore
\[ |f^{(k)}(z)e^{-|z|^2/2}|^2 \leq e^2 C(r) k!^2 \int_{B(z,r+s)} |f(\zeta)e^{-|\zeta|^2/2}|^2 dV(\zeta). \tag{5.8} \]

The results above lead to the notion of Fock-Carleson measures for derivatives.
Definition 5.3. A complex valued measure $\mu$ on $\mathbb{C}$ is called Fock-Carleson measure for derivatives of order $k$ ($k$-FC measure, in short), if, for some constant $\varpi_k(\mu)$, for any function $f \in F^2(\mathbb{C})$, the following inequality holds
\[
\int_{\mathbb{C}} |f^{(k)}(z)|^2 e^{-|z|^2} d|\mu|(z) \leq \varpi_k(\mu) \|f\|_{F^2}^2,
\] (5.9)
where, recall, $|\mu|$ denotes the variation of the measure $\mu$.

An explicit description of such measures follows immediately from Proposition 5.1:

Theorem 5.4. A measure $\mu$ is a $k$-FC measure if and only if, for some (and, therefore, for any) $r > 0$, the quantity
\[
C_k(\mu, r) = (k!)^2 \sup_{z \in \mathbb{C}} \{|\mu|(B(z, r))(1 + |z|^2)^k\}
\] (5.10)
is finite. For a fixed $r$, the constant $\varpi_k(\mu)$ in (5.9) can be taken as $\varpi_k(\mu) = C(r)C_k(\mu, r)$, with some coefficient $C(r)$ depending only on $r$.

In other words, Theorem 5.4 states that $\mu$ is a $k$-FC measure if and only if $(1 + |z|^2)^k \mu$ is a FC measure. If $k = 0$, then any 0–FC measure is just a FC measure.

Further on, the parameter $r$ will be fixed (say, being equal to $\sqrt{2}$), and the dependence on $r$ will be suppressed in the notations. The quantity $\varpi_k(\mu)$ in (5.9) will be called the $k$-FC norm of the measure $\mu$.

Proof. The proof of the sufficiency follows mainly the one for Fock-Carleson (0-FC) measures in [16], with the replacement of Lemma 2.1 there by our Proposition 5.1. So, suppose that (5.10) is satisfied, i.e.,
\[
|\mu|(B(z))(1 + |z|^2)^k \leq C_k
\] (5.11)
for all disks $B(z)$ (with radius $\sqrt{2}$).

Consider the lattice $\mathbb{Z}^2 \subset \mathbb{C}$ consisting of points with both real and imaginary parts being integer. The disks $B_n = B(n_1 + i n_2)$, $n = n_1 + i n_2 \in \mathbb{Z}^2$, form a covering of $\mathbb{C}$ with multiplicity 9. Denoting the left-hand side in (5.9) by $I_k(f)$, we have
\[
I_k(f) \leq \sum_{n \in \mathbb{Z}^2} \int_{B_n} |f^{(k)}(w)|^2 e^{-|w|^2} d|\mu|(w.)
\]
By Proposition 5.1 and the triangle inequality, there exists a constant $M$, not depending on $k$, such that
\[
|f^k(w)|^2 e^{-|w|^2} \leq Mk!^2(1 + |w|)^{2k} \int_{\tilde{B}_n} |f(\zeta)|^2 e^{-|\zeta|^2} dV(\zeta),
\]
for all $w \in B_n$, where $\widetilde{B}_n$ denotes the concentric with $B_n$ disk, twice as large. Thus, by (5.11),

$$I_k(f) \leq MC_k \sum_{n \in \mathbb{Z}^2} \int_{\widetilde{B}_n} |f(u)|^2 e^{-|w|^2} dA(u).$$

Now, since the disks $\widetilde{B}_n$ form a covering of $\mathbb{C}$ with finite multiplicity, the last inequality implies the required estimate for $I_k(f)$ via the Fock norm of the function $f$. The necessity part of the theorem is proved exactly as in [16], by means of evaluating $I_k(f)$ on the normalized reproducing kernels. \qed

It is convenient to extend the notion of $k$-FC measures to half-integer values of $k$, defining these measures as those for which the relation (5.11) holds.

**Corollary 5.5.** For any integers $p \in \mathbb{Z}_+$ and integer or half-integer $k$ a measure $\mu$ is a $k$-FC measure if and only if the measure $\mu_p = (1 + |z|^2)^{(k-p)} \mu$ is a $p$-FC measure, moreover, for integer $k$.

$$C_p(\mu, r) \asymp C_k(\mu_p, r) \quad (5.12)$$

Note, in particular, that any measure with compact support is a $k$-FC measure for any $k$.

**Remark 5.6.** Estimates similar to the ones derived in this section have been obtained in [20], however the results there are weaker than ours: the inequality similar to (5.9) was proved only under the condition that the measure $\mu$ has the form $(1 + |z|)^{-2k} \phi(z) dV(z)$, with locally bounded function $\phi$, moreover, the dependence of the constant on $k$ was not explicitly found. Our results can also be considered as a kind of dual to the estimates for Fock-Sobolev spaces in [9].

### 5.2. Sharper estimates.

We finish this section by obtaining a sharper form of the the above estimates, which provide a better constant for large values of $|z|$.

**Proposition 5.7.** Let $f$ be an analytical function if $\mathbb{C}$, $k \in \mathbb{Z}_+$. We denote by $\gamma(k, z)$ the quantity

$$\gamma(k, z) = \begin{cases} 
  k!, & \text{for } |z| \leq k, \\
  k^{-\frac{3}{2}}, & \text{for } |z| > k.
\end{cases} \quad (5.13)$$

Then

$$|f^{(k)}(z)| \leq C \gamma(k, z)(1 + |z|)^k e^{\frac{|z|^2}{2}} \|f\|_{F^2}, \quad (5.14)$$

for a constant $C$ not depending on $k$. Thus, compared with (5.2), the constant in the estimate for the growth of derivatives of $f$ is considerably improved.
Proof. The proof follows the one of Proposition 5.1; the improvement is achieved by a better choice of $s$ in (5.3) for $|z| \geq k$: we take here $s = k/|z| \leq 1$. Substitution in (5.3) gives
\[
|f^{(k)}(z)| \leq \frac{k!k^{-k}|z|^k e^{(|z|+k|z|^{-1})^2/2}}{f_{f^2}} \leq \frac{Ck!(k/e)^{-k}|z|^{k e^{(|z|^2)/2}}}{\|f\|_{F^2}}.
\] (5.15)

By the Stirling formula, the quantity $k!(k/e)^{-k}$ is majorated by $Ck^{-1/2}$, which proves the required estimate. □

In a similar way, Proposition 5.2 and Theorem 5.4 can be sharpened. We give here only the formulations; the proofs follow the proofs of Proposition 5.2 and Theorem 5.4, only with the choice $s = |z|^{-1}$ replaced by $s = k|z|^{-1}$, for $|z| \geq k$.

Proposition 5.8. Let $f$ be an entire analytical function. For any $r > 1$, with some constant $C(r)$ depending only on $r$,
\[
|f^{(k)}(z)|^2 \leq \gamma(k, z)^2 e^{(1 + |z|)^2 k} \int_{B(z, r)} |f(\zeta)e^{-|\zeta|^2/2}|^2 dV(\zeta). \] (5.16)

Theorem 5.9. A measure $\mu$ is a $k$-FC if and only if, for some (and therefore for any) $r > 0$, the quantity
\[
\tilde{C}_k(\mu, r) = \sup_{z \in \mathbb{C}} \{\mu(B(z, r)) \gamma(k, z)(1 + |z|^2)^k\}
\] (5.17)
is finite. For a fixed $r$, the constant $\omega_k(\mu)$ in (5.9) can be chosen as $C(r)\tilde{C}_k(\mu, r)$.

It is important to remark that the finiteness of the expression (5.17) is equivalent to the finiteness of (5.10), since the behavior of the functions in these formulas as $|z| \to \infty$ is the same. However, if the support of measure $\mu$ lies far away from the origin, in the domain $|z| > k$, Theorem 5.9 can give a better value of the constant in inequality (5.9).

6. Sesquilinear forms of infinite differential order and corresponding Toeplitz operators

Having the estimates of Fock functions and their derivatives at hand, we can introduce sesquilinear forms involving derivatives, first of a finite, and then of the infinite order.

6.1. Forms of a finite order.

Proposition 6.1. Let $\mu$ be a $k$-FC measure, with integer or half-integer $k$. With $\mu$ we associate the sesquilinear form
\[
F(f, g) = \int_{\mathbb{C}} f^{(\alpha)}(z)\overline{g^{(\beta)}(z)} e^{-|z|^2} d\mu(z), \quad f, g \in F^2(\mathbb{C}),
\] (6.1)
for some $\alpha, \beta$ with $\alpha + \beta = 2k$. This form is bounded in $F^2(\mathbb{C})$, moreover
\[
|F(f, g)| \leq C(F)\|f\|_{F^2}\|g\|_{F^2}, \quad \text{with} \quad C(F) \leq (\varpi_\alpha(\mu)\varpi_\beta(\mu))^{\frac{1}{2}}. \quad (6.2)
\]

Proof. Since $\alpha + \beta = 2k$, by the Cauchy inequality, we have
\[
|F(f, g)| \leq \left( \int_\mathbb{C}|f^{(\alpha)}(z)|^2e^{-|z|^2}(1 + |z|^2)^{k-\alpha}d|\mu|(z) \right)^{\frac{1}{2}} \times \left( \int_\mathbb{C}|g^{(\beta)}(z)|^2e^{-|z|^2}(1 + |z|^2)^{k-\beta}d|\mu|(z) \right)^{\frac{1}{2}}. \quad (6.3)
\]

By Corollary 5.5, $\mu_\alpha = (1 + |z|^2)^{k-\alpha}\mu$ is an $\alpha$-FC measure, $\mu_\beta = (1 + |z|^2)^{k-\beta}\mu$ is a $\beta$-FC measure, and thus we arrive at the required estimate
\[
|F(f, g)| \leq C(F)\|f\|_{F^2}\|g\|_{F^2}, \quad \text{with a constant} \quad C(F) \leq \varpi_k(\mu). \quad \square
\]

As usual, for any norm estimate for the operator defined by a symbol, the boundedness result is accompanied by a compactness result.

**Definition 6.2.** A measure $\mu$ is called **vanishing $k$-FC measure** if
\[
\lim_{|z| \to \infty} (|\mu|(B(z, r))(1 + |z|^2)^k) = 0.
\]

**Corollary 6.3.** Let $\mu$ be a vanishing $k$-FC measure, with integer or half-integer $k$. Then the operator in $F^2(\mathbb{C})$ defined by the form (6.1), with $\alpha + \beta = 2k$ is compact.

Proof. It goes by the usual pattern, see, e.g., [16, Theorem 2.4]. For a given $\varepsilon > 0$, we find $R$ so large that $|\mu|(B(z, r))(1 + |z|^2)^k \leq \varepsilon$ for $|z| > R$. We split the measure $\mu$ into the sum $\mu = \mu_R + \mu'_R$, with $\mu_R$ supported in the disk $|z| \leq R$, and $\mu'_R$ outside this disk, so that $\varpi_k(\mu'_R) \leq \varepsilon$. This splitting leads to the corresponding splitting of the sesquilinear form: $F = F_R + F'_R$. The form $F_R$, corresponding to the symbol with compact support, generates a compact operator. On the other hand, the estimate (6.2), applied to $F'_R$, implies the smallness of the operator defined by this sesquilinear form. These two observations prove the compactness of the operator in question. \square

Note, again, that any measure with compact support is a vanishing $k$-FC measure for any $k$, and therefore, for any $\alpha, \beta$ the operator in $F^2(\mathbb{C})$ generated by the form (6.1) with such measure, is compact.

Now we introduce a class of distributions generated by derivatives of FC measures.

Here and further on, we will simultaneously use the scalar product in the weighted $L^2$-space, and the action of the related distribution, where the action, traditionally, is induced by the (bilinear!) pairing
with respect to the Lebesgue measure. Recall that the scalar product in a Hilbert space is always denoted by \( \langle \cdot, \cdot \rangle \), while the action of the distribution on a smooth function is always denoted by \( (\cdot, \cdot) \).

Two related distributions will be denoted by the same alphabetic symbol, one of them regular, the other one boldfaced, say, \( \Phi \) and \( \Phi^\Phi \). The relation between them is given by

\[
\Phi^\Phi = \omega \Phi,
\]

with \( \omega(z) = \pi^{-1} e^{-|z|^2} \), so that, in the case when \( \Phi \) is a function,

\[
(\Phi^\Phi, h) = \langle \Phi, \bar{h} \rangle.
\]

Let \( \Phi \) be a distribution in \( \mathcal{D}'(\mathbb{C}) \). By \textit{coderivative} of \( \Phi \) we mean the distribution

\[
\partial \Phi = \omega(z)^{-1} \partial(\omega(z)\Phi) = \omega(z)^{-1} \partial(\Phi), \tag{6.4}
\]

thus,

\[
(\partial \Phi, h) = (\Phi, \bar{h}), \tag{6.5}
\]

as soon as the the expressions in (6.5) make sense.

Let \( \mu \) be a \( k \)-FC measure, and \( \alpha + \beta = 2k \). The coderivative \( \partial^\alpha \partial^\beta \mu \) is defined in accordance with (6.4), so that, for a function \( h(z) = f(z)\bar{g}(z), \ f, g \in \mathcal{F}^2 \),

\[
(\partial^\alpha \partial^\beta \mu, h) = (-1)^{\alpha+\beta} (\omega\mu, \partial^\alpha \partial^\beta h) = (-1)^{\alpha+\beta} (\omega\mu, \partial^\alpha f \partial^\beta g),
\]

again, provided the right-hand side makes sense.

**Proposition 6.4.** If \( \mu \) is a \( k \)-FC measure, and \( \alpha + \beta = 2k \), then the sesquilinear form in \( \mathcal{F}^2(\mathbb{C}) \)

\[
F(f, g) = F_{\mu, \alpha, \beta}(f, g) = \langle \partial^\alpha \partial^\beta \mu, f \bar{g} \rangle \tag{6.6}
\]

is bounded in \( \mathcal{F}^2(\mathbb{C}) \),

\[
|F(f, g)| \leq C(\omega_a(\mu) \omega_b(\mu))^{\frac{1}{2}} \|f\|_{\mathcal{F}^2} \|g\|_{\mathcal{F}^2}. \tag{6.7}
\]

**Proof.** The proof follows immediately from the definition of the sesquilinear form and Proposition 6.1. \( \square \)

We show now how the form (6.6) fits the construction of Section 4.

**Example 6.5.** Toeplitz operators defined by the derivatives of \( k \)-FC measures.

The sesquilinear form that corresponds to the derivative \( \partial^\alpha \partial^\beta \mu \) of a \( k \)-FC measure \( \mu \) admits the following description: \( X = L_1(\mathbb{C}, d\mu) \),
For $X' = L_{\infty}(\mathbb{C}, d\mu)$, $G(f, g)(z) = f^{(\alpha)}(z)\overline{g^{(\beta)}(z)}\omega(z)$, $\Phi = (-1)^{\alpha+\beta} \in L_{\infty}(\mathbb{C}, d\mu)$, so that

$$F_{G, \Phi}(f, g) = \int_{\mathbb{C}} (-1)^{\alpha+\beta} f^{(\alpha)}(z)\overline{g^{(\beta)}(z)}\omega(z)\, d\mu.$$ \hfill (6.8)

And

$$(T_{G, \Phi} f)(z) = F_{G, \Phi}(f, k_z) = \frac{(-1)^{\alpha+\beta}}{\pi} \int_{\mathbb{C}} z^\beta f^{(\alpha)}(w)e^{(z-w)\overline{\pi}d\mu(w)}.$$ \hfill (6.9)

We give now illustrative examples to the above notions.

**Example 6.6.** Consider the measure $\mu$ supported in the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2 = \mathbb{C}$: $\mathbb{Z}^2$ consists of points with both co-ordinates integer. Suppose that the measure $\mu$ of the node $n = (n_1, n_2) = n_1 + in_2$ of the lattice satisfies the condition $|\mu(n)| \leq C(|n_1| + |n_2|)^{-2k}$. Then, due to Theorem 5.4, $\mu$ is a $k$-CF measure, and, for $\alpha + \beta = 2k$, the Toeplitz operator $T_{\partial_\alpha\overline{\partial_\beta} \mu}$ is bounded. By (6.9), this operator acts as

$$(T_{\partial_\alpha\overline{\partial_\beta} \mu} f)(z) = \frac{(-1)^k}{\pi} \sum_{n \in \mathbb{Z}^2} f^{(\alpha)}(n)e^{(z-n)\overline{\pi}\mu(n)}.$$**

**Example 6.7.** Given $k, \alpha, \beta \in \mathbb{Z}_+$, we introduce the $k$-FC measure $d\mu_k = (1 + |z|^2)^{-k} dV(z)$, and the corresponding form (6.8)

$$F_{\alpha, \beta, k}(f, g) = (-1)^{\alpha+\beta} \int_{\mathbb{C}} f^{(\alpha)}(z)\overline{g^{(\beta)}(z)}e^{-|z|^2}/\pi(1 + |z|^2)^k \, dV(z).$$

Direct calculations give

$$F_{\alpha, \beta, k}(e_n, e_m) = (-1)^{\alpha+\beta} \int_{\mathbb{C}} e^{(\alpha)}(n)\overline{e^{(\beta)}(m)}(z)e^{-|z|^2}/\pi(1 + |z|^2)^k \, dV(z)$$

$$= (-1)^{\alpha+\beta} \frac{\sqrt{n!m!}}{(n-\alpha)!\overline{m}!(m-\beta)!} \int_{\mathbb{C}} z^{n-\alpha}\overline{z}^{m-\beta} e^{-|z|^2}/\pi(1 + |z|^2)^k \, dV(z)$$

$$= (-1)^{\alpha+\beta} \frac{\sqrt{n!m!}}{(n-\alpha)!\overline{m}!(m-\beta)!} \int_{\mathbb{R}^+} r^{n-\alpha}e^{-r^2}e^{-r^2}/(1 + r^2)^k \, dr$$

$$\times \int_0^{2\pi} e^{i\theta[(n-\alpha)+(m-\beta)]} \frac{d\theta}{\pi}.$$
This expression is always equal to 0 if \( n < \alpha \) or \( n - \alpha \neq m - \beta \). Taking now \( n \geq \alpha \) and \( n - \alpha = m - \beta \), we have

\[
F_{\alpha,\beta,k}(e_n, e_{n-\alpha+\beta}) = \frac{(-1)^{\alpha+\beta} \sqrt{n!(n - \alpha + \beta)!}}{|(n - \alpha)!|^2} \int_{\mathbb{R}_+} \frac{r^{2(n-\alpha)}e^{-r^2}}{(1 + r^2)^k} 2rdr
\]

\[
= \frac{(-1)^{\alpha+\beta} \sqrt{n!(n - \alpha + \beta)!}}{|(n - \alpha)!|^2} \int_{\mathbb{R}_+} \frac{s^{n-\alpha}e^{-s}}{(1 + s)^k} ds
\]

\[
= \gamma_{\alpha,\beta,k}(n).
\]

In this way, the form \( F_{\alpha,\beta,k} \) defines via (6.9) a densely defined (unbounded, in general) Toeplitz operator \( T_{\partial_{\alpha} \bar{\partial}_{\beta} \mu_k} \), whose domain contains all standard basis elements \( e_n(z), n \in \mathbb{Z}_+ \), and

\[
T_{\partial_{\alpha} \bar{\partial}_{\beta} \mu_k} e_n = \begin{cases} \gamma_{\alpha,\beta,k}(n) e_{n-\alpha+\beta}, & \text{if } n \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}
\]

For generic \( k, \alpha, \beta \in \mathbb{Z}_+ \) the exact formula for \( \gamma_{\alpha,\beta,k}(n) \) is rather complicated, but its asymptotic behavior for large \( n \) is quite simple. For \( n > \alpha + k \), we have

\[
\gamma_{\alpha,\beta,k}(n) = (-1)^{\alpha+\beta} \frac{\sqrt{n!(n - \alpha + \beta)!}}{|(n - \alpha)!|^2} (n - \alpha - k)! \left( 1 + O \left( \frac{1}{n} \right) \right),
\]

or, by [13, Formula 8.328.2],

\[
\gamma_{\alpha,\beta,k}(n) = (-1)^{\alpha+\beta} \frac{(n - \alpha)^{\frac{\alpha+\beta}{2}}}{(n - \alpha - k)^k} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

This shows that the Toeplitz operator \( T_{\partial_{\alpha} \bar{\partial}_{\beta} \mu_k} \) is bounded if and only if \( \alpha + \beta \leq 2k \); if \( \alpha + \beta < 2k \) the operator is even compact. If \( \alpha = \beta = k \) the operator \( T_{\partial_{\alpha} \bar{\partial}_{\beta} \mu_k} \) is a compact perturbation of the identity operator \( I \).

**Remark 6.8.** It is interesting to observe that for the specific cases \( \alpha = 0, \beta = 1, k = 0 \) and \( \alpha = 1, \beta = 0, k = 0 \) (the operators are unbounded), the corresponding Toeplitz operators, considered on the natural domain, coincide with the classical creation and annihilation operators in the Fock space,

\[
a^\dagger = zI, \quad a = \frac{\partial}{\partial z},
\]

respectively.

We will call the derivatives of \( k \)-FC measures and the corresponding sesquilinear forms ‘symbols of finite type’.
6.2. Symbols of weak almost-finite type. In this subsection we introduce a class of symbols, more general than the derivatives of \(k\)-FC measures, defined previously. They might be considered as a kind of hyperfunctions, however, this would be not quite consequential, so we use the term ‘symbols of weak almost-finite type’ instead.

Definition 6.9. Let \(F(f, g)\) be a bounded sesquilinear form on the Fock space \(\mathcal{F}^2(\mathbb{C})\). We say that this form is a symbol of weak almost-finite type if there exist a collection \(\mathcal{M} = \{\mu_{\alpha,\beta}\}_{\alpha,\beta=0,1,2,...}\) of \((\alpha + \beta)/2\)-FC measures such that, for each \(f, g \in \mathcal{F}^2\), the series

\[
\sum_j \sum_{\max(\alpha,\beta)=j} F_{\alpha,\beta}(f, g) \equiv \sum_j \sum_{\max(\alpha,\beta)=j} (\partial^{\alpha} \bar{\partial}^{\beta} \mu_{\alpha,\beta}, f \bar{g}), \tag{6.10}
\]

converges to \(F(f, g)\).

It is convenient to consider \(\mathcal{M}\) as the formal sum

\[
\mathcal{M} \simeq \sum_{\alpha,\beta} \partial^{\alpha} \bar{\partial}^{\beta} \mu_{\alpha,\beta}.
\]

The Banach-Steinhaus theorem (more exactly, its version for bilinear mappings, see, e.g., Sect. 7.7 in [10]) implies that the condition of boundedness of the sesquilinear form \(F(f, g)\) in this definition is superfluous, it follows automatically from (6.10).

The condition (6.10) can be in an obvious way formulated in another form, involving Toeplitz operators corresponding to \(F_j = \sum_{\max(\alpha,\beta)=j} F_{\alpha,\beta}\).

Proposition 6.10. Let \(F(f, g)\) be a symbol of weak almost-finite type. Then the sequence of Toeplitz operators \(T_j = \sum_{s \leq j} T_{F_s}\) converges weakly to the Toeplitz operator \(T_F\).

It follows, in particular, that for symbols of weak almost-finite type, the point-wise convergence takes place:

Proposition 6.11. If \(F\) is a symbol of weak almost-finite type, then, for any \(f \in \mathcal{F}^2(\mathbb{C})\) and any \(z \in \mathbb{C}\), the sequence \((T_j f)(z)\) converges to \((T_F f)(z)\).

The proof follows immediately from the relation \((T_F f)(z) = F(f, k_z)\) and similar relations for \(F_j\).

Moreover, the partially converse statement is correct.

Proposition 6.12. If the operators \(T_j\) are uniformly bounded and the sequence \(T_j f\) converges to \(T f\) point-wise: \((T_j f)(z)\) converges to \((T_F f)(z)\) for all \(f \in \mathcal{F}^2\) and \(z \in \mathbb{C}\), then \(T_j\) converges to \(T\) weakly.
The (rather standard) proof goes the following way. A given \( g \in \mathcal{F} \) satisfies \( g(z) = \int k_z(w)g(w)\omega(w)dw \), therefore \( g \) can be norm approximated, with error norm less than \( \varepsilon \), by a finite linear combination \( g_\varepsilon \) of \( k_{z_m} \). On such combinations the convergence of \( (T_j f, g_\varepsilon) \) follows by linearity, and the possibility of passing to the limit follows from the uniform boundedness.

As the following theorem shows, the notion we have just introduced is too general.

**Theorem 6.13.** Any bounded operator in \( \mathcal{F}^2(\mathbb{C}) \) is an operator with symbol of weak almost finite type.

**Proof.** Let \( T \) be a bounded operator in \( \mathcal{F}^2(\mathbb{C}) \). We consider the representation of the operator \( T \) as an infinite matrix in the orthogonal basis \( e_\alpha \) defined in (2.5):

\[
T = \sum_{\alpha,\beta} T_{\alpha\beta} = \sum_{\alpha,\beta} P_\alpha TP_\beta,
\]

where \( P_\alpha = P_{\alpha\alpha} \) is the orthogonal projection onto the one-dimensional subspace spanned by \( e_\alpha \). As described in Example 2.4, each operator \( T_{\alpha\beta} = P_\alpha TP_\beta \), being a rank one operator, with the source subspace spanned by \( e_\beta \) and the range spanned by \( e_\alpha \), is in fact the operator \( \sigma_{\alpha\beta} P_{\alpha\beta} \), with numerical coefficient \( \sigma_{\alpha\beta} = (Te_\alpha, e_\beta) \) and the rank one operators \( P_{\alpha\beta} \) described in (2.15). Now, by (2.13), (2.14), the operator \( \sigma_{\alpha\beta} P_{\alpha\beta} \) is the Toeplitz operator associated with the distributional symbol \( \sigma_{\alpha\beta} \Phi_{\alpha\beta} \in \mathcal{E}'(\mathbb{C}) \). Moreover, any such symbol is, by (2.11), a collection of derivatives of continuous functions with compact support, and the latter correspond to \( k \)-FC measures for any \( k \). Therefore, any symbol \( \sigma_{\alpha\beta} \Phi_{\alpha\beta} \) is the sum of derivatives of order not greater than \( \alpha + \beta \) of \( k \)-FC measures. Finally, we establish the convergence of the sesquilinear forms, required by the theorem. Denote by \( P_j \) the projection \( P_j = \sum_{\alpha \leq j} P_\alpha \). Thus, the operator

\[
P_j TP = \sum_{\alpha,\beta \leq j} T_{\alpha\beta}
\]

is the Toeplitz operator with symbol being the sum of coderivatives of \( j \)-FC measures. On the other hand, the projections \( P_j \) converge strongly to the identity operator as \( j \to \infty \), therefore for any \( f, g \in \mathcal{F}^2(\mathbb{C}) \), we have

\[
\langle P_j TP_j f, g \rangle = \langle TP_j f, P_j g \rangle \to \langle Tf, g \rangle.
\]

**Remark 6.14.** The theorem implies, in particular, that given any bounded linear operator on the Fock space, any finite truncation of the infinite matrix representation with respect to the standard basis in
\( \mathcal{F}^2(\mathbb{C}) \) is a Toeplitz operator with symbol being the sum of derivatives of certain \( k \)-FC measures.

Consider now a special case of the above theorem when the operator \( T \) is compact. Then the sequence \( \{ P_j T P_j \}_{j \in \mathbb{Z}^+} \) converges to \( T \) in norm,
\[
\lim_{j \to \infty} P_j T P_j = T.
\]
The sesquilinear form of the operator \( T \) is the (norm) limit of the sequence of forms of Toeplitz operators \( P_j T P_j \), each one of which, as it was stated in the proof of Theorem 6.13, is a sum of derivatives of order not greater than \( \alpha + \beta \) of \( k \)-FC measures. Note that the differential order of these forms may tend to infinity as \( k \to \infty \). Thus the limiting form will have, in general, an infinite differential order. We will identify the sesquilinear form of the operator \( T \) with the sequence \( \mu \), where each \( \mu \) is the form of the finite rank Toeplitz operator \( P_j T P_j \). Thus we come to the following important result.

**Theorem 6.15.** Each compact operator on \( \mathcal{F}^2(\mathbb{C}) \) is a Toeplitz operator defined, in general, by a sesquilinear form of an infinite differential order.

### 6.3. Symbols of norm almost-finite type.

Let \( \mu = \{ \mu_{\alpha,\beta} \}_{\alpha,\beta=0,1,2,...} \) be a collection of \( (\alpha + \beta)/2 \)-FC measures. We say that this collection is a symbol of norm almost finite type if
\[
||\mu|| = \sum_{\alpha,\beta} \omega_{\alpha+\beta}(\mu_{\alpha,\beta}) < \infty. \tag{6.12}
\]

**Theorem 6.16.** Let \( \mu \) be a symbol of norm-almost finite type. Denote by \( T_{\alpha,\beta}(\mu) \) the operator defined by the sesquilinear form \( F_{\alpha,\beta}(f,g) = (\partial^\alpha \overline{\partial}^\beta \mu_{\alpha,\beta}, f \overline{g}) \), as in (6.6). Then the operator series \( \sum_{\alpha,\beta} T_{\alpha,\beta}(\mu) \) converges in the norm sense.

The sum
\[
T(\mu) = \sum_{\alpha,\beta} T_{\alpha,\beta}(\mu) \tag{6.13}
\]
will be called the Toeplitz operator with symbol \( \mu \), which we associate with the formal sum,
\[
\mu \simeq \sum_{\alpha,\beta} \partial^\alpha \overline{\partial}^\beta \mu_{\alpha,\beta}. \tag{6.14}
\]

**Proof.** By (6.7), the norm of the operator \( T_{\alpha,\beta}(\mu) \) is estimated by \( \omega_{\alpha+\beta}(\mu_{\alpha,\beta}) \). Thus, the convergence of the series in (6.12) implies the norm convergence of the operator series. \( \Box \)
Note that if the measures $\mu_{\alpha,\beta}$ are vanishing $k$-CF measures then each of the operators $T_{\alpha,\beta}(\mu)$ is compact and the norm-convergent series of such operators has automatically a compact sum. It stands to reason that this remark concerns in particular the case when all measures $\mu_{\alpha,\beta}$ have compact support.

Now we present several examples of the above symbols.

**Example 6.17.** Let $h_{\alpha,\beta}(z)$, $\alpha, \beta = 0, 1, \ldots$, be bounded functions in $\mathbb{C}$ such that

$$\sup_{z \in \mathbb{C}} \{|h_{\alpha,\beta}(z)|(1 + |z|)^{\alpha+\beta}\} \leq (\alpha! \beta!)^{-1} c_{\alpha,\beta}$$

with $\sum c_{\alpha,\beta} < \infty$. With each function $h_{\alpha,\beta}(z)$ we associate the measure $\mu_{\alpha,\beta}$ that has the density $h_{\alpha,\beta}(z)$ with respect to the Lebesgue measure, $\mu_{\alpha,\beta}(E) = \int_E h_{\alpha,\beta}(z) dV(z)$, and consider the symbol

$$\mu \simeq \sum_{\alpha,\beta} \partial^\alpha \partial^\beta \mu_{\alpha,\beta},$$

By (6.6), (6.7), the convergence condition (6.12) is satisfied, and therefore, the formal series (6.16) can serve as a symbol of a bounded Toeplitz operator $T\mu$.

**Example 6.18.** Here we modify the previous example in order to handle discrete measures and their derivatives; at the same time we make use of the sharp norm estimate (5.17). Let $\delta_n$, $n = n_1 + in_2$, be the delta-measure placed at the point $n$. With each point $n$ we associate the distribution

$$\theta_n = \partial^{\alpha_n} \partial^{\beta_n} \delta_n,$$

with some $\alpha_n$, $\beta_n$. Being a compactly supported distribution, $\theta_n$ defines a compact Toeplitz $T_n$ operator in $\mathcal{F}^2$. By Theorem 5.4, the norm of this Toeplitz operator is not greater than $p_n = C \alpha_n! \beta_n!(1 + |n|)^{\alpha_n+\beta_n}$, with some absolute constant $C$. Thus, the formal sum $\mu = \sum A_n \theta_n$ of these distributions taken with coefficients $A_n$ defines a norm convergent series of Toeplitz operators $\sum T_n$ as soon as $\sum |A_n| p_n < \infty$. So, we obtain a bounded Toeplitz operator corresponding to a distributional symbol involving derivatives of unbounded order and supported in the integer lattice.

Suppose now that the orders of derivatives $\partial^{\alpha_n} \partial^{\beta_n}$ of $\delta$-measures are sufficiently large,

$$\alpha_n, \beta_n > |n|.$$

Then, for estimating the norm the operator with symbol (6.17) we can use Theorem 5.9. Under the condition (6.18), we have

$$\|T_n\| \leq C q_n \equiv C(1 + |n|)^{\alpha_n+\beta_n} (\alpha_n \beta_n)^{-\frac{1}{2}}.$$
Therefore, as soon as $\sum |A_n|q_n < \infty$ converges, the operator series $\sum T_n$ converges in the norm, which produces the Toeplitz operator. Note, that the latter convergence condition is much weaker than the former one.

We conclude with an example of operators with symbol supported at one point.

**Example 6.19.** Fix a point $z_0 = x_0 + iy_0 \in \mathbb{C}$. We denote by $\delta_{z_0}$ the delta-distribution at the point $z_0$. Consider the formal sum

$$\Theta = \sum_{\alpha,\beta} A_{\alpha,\beta} \partial^\alpha \overline{\partial}^\beta \delta_{z_0}$$

with some coefficients $A_{\alpha,\beta}$. Each of the distributions $\vartheta_{\alpha,\beta} \simeq \partial^\alpha \overline{\partial}^\beta \delta_{z_0}$ defines a bounded (and compact) Toeplitz operator $T_{\alpha,\beta}$ with norm estimate $\|T_{\alpha,\beta}\| \leq C (1 + |z_0|)^{\alpha+\beta} \alpha! \beta!$. Therefore, if the coefficients $A_{\alpha,\beta}$ decay so rapidly that the series $\sum |A_{\alpha,\beta}|(1 + |z_0|)^{\alpha+\beta} \alpha! \beta!$ converges, the corresponding operator series $\sum A_{\alpha,\beta} T_{\alpha,\beta}$ converges in the norm and thus defines a Toeplitz operator with symbol supported at the point $z_0$ involving derivatives of all orders.

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