Three-Dimensional Integrable Models
and Associated Tangle Invariants

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Abstract

In this paper we show that the Boltzmann weights of the three-dimensional Baxter-Bazhanov model give representations of the braid group, if some suitable spectral limits are taken. In the trigonometric case we classify all possible spectral limits which produce braid group representations. Furthermore we prove that for some of them we get cyclotomic invariants of links and for others we obtain tangle invariants generalizing the cyclotomic ones.
1 Introduction

In paper [1] Baxter and Bazhanov introduced a particularly interesting three-di-
dimensional integrable model with N local states. It is one of the few solvable three-
dimensional models and seems to be highly non-trivial.

The Baxter-Bazhanov model is a generalization of the Zamolodchikov model [2, 3], which is the particular case N=2. Kashaev, Mangazeev and Stroganov [4, 5] proved that the Boltzmann weights of the Baxter-Bazhanov model satisfy the tetra-
hedron equations [4, 6, 7]. This is a generalization of the result obtained by Bax-
ter in [8] for the Zamolodchikov model. They use the symmetry properties [4] of
the Boltzmann weights, which have been found independently also by Baxter and
Bazhanov [9].

One of the most important features [1] of the Baxter-Bazhanov model is that
apart from a modification of the boundary conditions it can be obtained as a three-
dimensional interpretation of the generalized sl(n)-chiral Potts model [10, 11, 12].

Given a two-dimensional integrable model, which has Boltzmann weights satisfying
the Yang-Baxter equation, it is an interesting question to ask which braid group
representations and hence which link invariants arise therefrom. Akutsu, Deguchi
and Wadati [13, 14] invented a general procedure to study this problem and ob-
tained link invariants from most two-dimensional integrable models. Date, Jimbo,
Miki and Miwa [13] studied the braid group representations and the corresponding
(cyclotomic) link invariants arising from the sl(n)-chiral Potts model in the trigono-
metric limit. Following a suggestion made by Jones [16], they generalized the results
of Kobayashi, Murakami and Murakami [17] for the sl(2)-chiral Potts model. The
connection of such invariants with the Seifert matrix has been studied by Gold-
schmidt and Jones in [18].

Following a similar scheme we study the three-dimensional integrable Baxter-
Bazhanov model from the point of view of the link theory. We generalize the re-
sults of ref. [15] to the R-matrix with spectral parameters associated to the Baxter-
Bazhanov model. We show that choosing suitable limits of the spectral parameters
this matrix gives cyclic representations of the braid group. In the trigonometric case
we classify all possible spectral limits which produce braid group representations.
We prove that for some of them we get cyclotomic link invariants, while for other
limits of the rapidity variables (spectral parameters) the R-matrix of the Baxter-
Bazhanov model gives tangle invariants. Such invariants are generalizations of the
cyclotomic invariants previously mentioned.

\footnote{This work is mainly based on the thesis of B. L. Cerchiai,(B. L. Cerchiai, "Modelli di Baxter-
Bazhanov e di Potts chirale e teoria dei nodi", academic year 1992-93) in fulfilment of the require-
ments for the degree (laurea) in Physics.}
2 The 3-Dimensional Baxter-Bazhanov Model and its 2-Dimensional Reduction

The Baxter-Bazhanov model \([1]\) is an integrable three-dimensional IRF (=interaction-round-a-face) model. This means that it is defined on a simple cubic lattice \(L\) and that a spin variable \(\sigma\) is placed at each site of \(L\). From the point of view of statistical mechanics the Baxter-Bazhanov model depends on two integer parameters \(N(N \geq 2)\) and \(n\). \(N\) is the number of values that each spin \(\sigma\) can take, while \(n\) is one of the lattice dimensions (=number of elementary cubes in a fixed direction, e.g. in front-to-back direction).

The elementary cube of \(L\) is shown in the following figure:

![Elementary cell](image)

Figure 1: Elementary cell

In order to define the Boltzmann weight of the elementary cell shown in figure \([1]\) it is necessary to introduce some notation first.

Let \(x\) be a complex parameter and \(k, l, m\) three integers, \(0 \leq k, l, m \leq N - 1\). Let \(\omega\) be a primitive \(N\)-th root of unity

\[
\omega = e^{\frac{2\pi i}{N}}
\]

and

\[
\omega^{1/2} = e^{\frac{\pi i}{N}}.
\]

Let \(\Phi\) and \(s\) be the functions defined by

\[
\Phi(x) = (\omega^{1/2})^{N(N+1)}
\]

\[
s(k, l) = \omega^{kl}.
\]

Notice that

\[
s(k + N, l) = s(k, l + N) = s(k, l)
\]

\[
s(k + l, m) = s(k, m)s(l, m)
\]

\[
\Phi(k + l) = \Phi(k)\Phi(l)s(k, l)
\]
Moreover, let \( w(x, l) \) be the function defined by

\[
\frac{w(x, l)}{w(x, 0)} = (\Delta(x))^{1/l} \prod_{k=1}^{l} (1 - \omega^k x)^{-1}
\]  
(2.8)

where

\[
\Delta(x) = (1 - x^N)^{1/N}.
\]  
(2.9)

With this definition the function \( w(x, l) \) is fixed up to the overall normalization factor \( w(x, 0) \), while \( \Delta \) is fixed up to the choice of a phase when taking the root. In particular it is possible to impose the following condition on \( w(x, 0) \)

\[
w(x, 0) = w(\omega^k x, 0).
\]  
(2.10)

Applying this condition (2.10) to the definition of \( w \) (2.8), it follows immediately

\[
w(x, 0)w(x, l + k) = w(x, k)w(\omega^k x, l).
\]  
(2.11)

Having introduced all this notation, following ref. [1] the Boltzmann weight of the elementary cube shown in figure 1 is constructed as

\[
W(a | e, f, g | b, c, d | h) = \sum_{\sigma=0}^{N-1} v_\sigma(a | e, f, g | b, c, d | h)
\]  
(2.12)

with

\[
v_\sigma(a | e, f, g | b, c, d | h) = \frac{w(\ell_{p'}, e - c - d + h)}{w(\ell_{p'}, a - g - f + b)} s(c - h, d - h)s(g, a - g - f + b)
\]  
(2.13)

\[
\times \left\{ \frac{w(\ell_q, d - h - \sigma)w(\ell_{p'}, s - f + b)w(\ell_{q'}, a - g - \sigma)}{w(\ell_q, e - c - \sigma)(\Phi(a - g - \sigma))^{-1}} s(\sigma, a - c - f + h) \right\}.
\]

The parameters \( p, p', q, q' \) are the so-called spectral parameters. To stress the dependence of \( W \) on these parameters, it would be more correct to write

\[
W = W[p, p', q, q'].
\]

Notice that the spins are seen as elements of \( \mathbb{Z}_N \) and that \( W \) depends only on the pairwise differences of adjacent spins. This means that it is consistent to assume the following equivalence relation between the spins

\[
\alpha \sim \beta \iff \alpha_i - \alpha_{i+1} = \beta_i - \beta_{i+1} \forall i = 1, \ldots, n
\]  
(2.14)

In the expressions (2.12), (2.13) \( \sigma \) can be interpreted as a spin at the centre of the cube. The elementary interactions are shown in figure 2.
In this equation $W$ is integrable. The tetrahedron equation \([8, 4, 5]\), which guarantees that the model obtained in this way is an Ising type model. Thus it turns out that (up to an overall normalization factor, a site-type, edge-type and face-type equivalence transformation) $W$ satisfies the tetrahedron equation \(\text{[3, 4, 5]}\), which guarantees that the model is integrable

\[
\sum_d W(a_4 | c_2, c_1, c_3 | b_1, b_3, b_2 | d) \cdot W'(c_1 | b_2, a_3, b_1 | c_4, d, c_6 | b_4) \\
\times W''(b_1 | d, c_4, c_3 | a_2, b_3, b_4 | c_5) \cdot W'''(d | b_2, b_4, b_3 | c_5, c_2, c_6 | a_1)
\]

\[
= \sum_d W''(b_1 | c_1, c_4, c_3 | a_2, a_4, a_3 | d) \cdot W''(c_1 | b_2, a_3, a_4 | d, c_2, c_6 | a_1) \\
\times W'(a_4 | c_2, d, c_3 | a_2, b_3, a_1 | c_5) \cdot W(d | a_1, a_3, a_2 | c_4, c_5, c_6 | b_4).
\] (2.15)

In this this equation $W = W(P), W' = W(P'), W'' = W(P''), W''' = W(P''')$, where

\[
P = (x_1, x_2, x_3, x_4), \quad P' = (x'_1, x'_2, x_3, x'_4), \\
P'' = (x''_1, x''_2, x''_3, x''_4), \quad P''' = (x'''_1, x'''_2, x'''_3, x'''_4)
\] (2.16)

with $(x_1, x_2, x_3, x_4) = (q, q', p, p')$ and the primes are added to the $x$’s in correspondence with primes of the $P$’s. Defining further

\[
x_{ij} = x_i \Delta(x_j/x_i),
\] (2.17)

the tetrahedron equations \(\text{(2.13)}\) hold provided the points $P, P', P'', P'''$ are constrained to satisfy

\[
\begin{align*}
x_2/x_1 &= x'_2/x'_1, \quad &x_{12}/x_1 &= x'_{12}/x'_1, \quad &x_3/\omega x_4 &= x'''_3/x'''_4, \\
x_{34}/\omega^{1/2} x_4 &= x''_{12}/x_{12} x_{24}, \quad &x_{14}/x_{32} &= x''_{14}/x_{14} x_{32} = x'''_{12}/x'''_{12} x'''_{24}.
\end{align*}
\]
At this point it is useful to consider also the Boltzmann weight $S$ of a parallelepiped $\mathcal{P}$ formed by a whole line of $n$ cubes in front-to-back direction with periodic boundary conditions. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \beta = (\beta_1, \ldots, \beta_n),$$
$$\gamma = (\gamma_1, \ldots, \gamma_n), \quad \delta = (\delta_1, \ldots, \delta_n)$$

(2.19)

denote the spins on the edges of $\mathcal{P}$.

Figure 3: Parallelepiped $\mathcal{P}$

Then

$$S(\alpha, \beta, \gamma, \delta) = \prod_{i \in \mathbb{Z}_n} W(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1})$$

(2.20)

Further, following Baxter and Bazhanov, let’s introduce also a slightly modified model. Let’s substitute the variable $\sigma$ with the difference of two new spins in front-to-back direction.

$$\sigma = \mu - \mu'.$$

(2.21)

This means that considering a row of $n$ cubes in front-to-back direction the following constraint is imposed on the variable $\sigma$

$$\sum_{i \in \mathbb{Z}_n} \sigma_i = 0 \pmod{N}$$

(2.22)
The model obtained with this change of boundary conditions is called by Baxter and Bazhanov the "modified model". The Boltzmann weight of the parallelepiped \( P \) formed by a line of cubes in front-to-back direction is denoted \( S_0 \).

\[
S_0(\alpha, \beta, \gamma, \delta) = \prod_{i=0}^{N-1} \sum_{\mu_i=0}^{\mu_{i+1}} v_{\mu_i-\mu_{i+1}}(\delta_i | \alpha_i, \gamma_i, \delta_{i+1} | \gamma_{i+1}, \alpha_{i+1}, \beta_i | \beta_{i+1})
\]

with

\[
v_{\mu_i-\mu_{i+1}}(\delta_i | \alpha_i, \gamma_i, \delta_{i+1} | \gamma_{i+1}, \alpha_{i+1}, \beta_i | \beta_{i+1}) = \\
\frac{w(p_{\mu_i-\mu_{i+1}}, \alpha_i - \alpha_{i+1} - \beta_i + \beta_{i+1})}{w(p_{\mu_i-\mu_{i+1}}, \delta_i - \delta_{i+1} - \gamma_i + \gamma_{i+1})} s(\alpha_{i+1} - \beta_{i+1}, \beta_i - \beta_{i+1})
\]

\[
\times \frac{w(q_{\mu_i-\mu_{i+1}}, \beta_i - \beta_{i+1} - \mu_i + \mu_{i+1})}{w(q_{\mu_i-\mu_{i+1}}, \alpha_i - \alpha_{i+1} - \mu_i + \mu_{i+1}(\Phi(\delta_i - \delta_{i+1} - \mu_i + \mu_{i+1})^{-1})} \times \\
\frac{w(p_{\mu_i-\mu_{i+1}}, \delta_i - \delta_{i+1} - \mu_i + \mu_{i+1})}{w(q_{\mu_i-\mu_{i+1}}, \alpha_i - \alpha_{i+1} - \mu_i + \mu_{i+1})} s(\mu_i - \mu_{i+1}, \delta_i - \alpha_{i+1} - \gamma_i + \beta_{i+1})
\]

The key idea of ref. [1] is to describe the Baxter-Bazhanov model as an integrable generalized chiral Potts model [10, 11, 12] in the IRF presentation by the following prescription.

![Figure 4: Reduction procedure](image)

For this aim, one starts from an edge of the bidimensional lattice on which the chiral Potts model is defined. This edge is extended in a third additional dimension perpendicular to the plane of the two-dimensional lattice to form a rectangle consisting of \( n \) squares. The two spins \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) located at the vertices of the two-dimensional lattice are placed on the edges of the rectangle, as shown in figure 4. Cyclic boundary conditions are assumed in the new dimension, considering the spins \( \alpha_1, \beta_1 \) as next to \( \alpha_n, \beta_n \) respectively. Doing this construction
for all edges of the two-dimensional lattice, it becomes the three-dimensional cubic lattice \( \mathcal{L} \) with \( N \)-valued spins at each site.

Baxter and Bazhanov have proved that the weight function \( W_{pq}(\alpha, \beta) \) of the chiral Potts model associated to an edge can be written in the following form

\[
W_{pq}(\alpha, \beta) = \prod_{i=1}^{n} \left\{ \omega_{\beta_i-\beta_{i+1}+\beta_{i+1}} \omega^l(\beta_i - \beta_{i+1}, \alpha_{i+1} - \beta_{i+1}) \right\}, \tag{2.25}
\]

Notice that the rapidity variables in (2.25) form a \( n \)-vector \( p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \) exactly as the spins do. In the three-dimensional interpretation the weight \( W_{pq} \) is associated to the whole rectangle constructed in figure 4. This three-dimensional re-interpretation of the two-dimensional statistical model is allowed by the factorization property (2.25) of the Boltzmann weight: the \( i \)-th term in the product depends only on the four spins \( \alpha_i, \beta_i, \alpha_{i+1}, \beta_{i+1} \) located at the vertices of the \( i \)-th elementary square in the rectangle. Notice that not all two-dimensional integrable models have this factorization property.

Let us now consider the star of figure 5.

![Elementary star](image)

**Figure 5: Elementary star**

Corresponding to this configuration we define the star-Boltzmann weight \( W_{\text{star}}^{(1)} \) of the IRF chiral Potts model

\[
W_{\text{star}}^{(1)}(p, p', q, q' \mid \alpha, \beta, \gamma, \delta) = \sum_{\mu} \frac{W_{\mu \mu}(\alpha, \beta) W_{pq}(\beta, \mu) W_{q' \mu}(\mu, \gamma) W_{p' q'}(\delta, \mu)}{W_{p' p}(\delta, \gamma) W_{q' q}(\alpha, \mu)} \tag{2.26}
\]

whose \( W_{ij}, (i, j = p, p', q, q') \) are the edge-Boltzmann weights defined in (2.23). It turns out that \( W_{\text{star}}^{(1)} \) satisfy the Yang-Baxter equation \([11, 12]\). It

\[
\sum_{\sigma} W_{\text{star}}^{(1)}(p, p', q, q' \mid \alpha, \beta, \gamma, \sigma) = \sum_{\sigma} W_{\text{star}}^{(1)}(q, q', r, r' \mid \sigma, \gamma, \delta, \epsilon)
\]

\[
W_{\text{star}}^{(1)}(q, q', r, r' \mid \alpha, \sigma, \epsilon, \kappa) = \sum_{\sigma} W_{\text{star}}^{(1)}(q, q', r, r' \mid \beta, \gamma, \delta, \sigma)
\]

\[
W_{\text{star}}^{(1)}(p, p', r, r' \mid \alpha, \beta, \sigma, \kappa) = \sum_{\sigma} W_{\text{star}}^{(1)}(p, p', q, q' \mid \kappa, \sigma, \delta, \epsilon). \tag{2.27}
\]
The connection between the chiral Potts model and the Baxter-Bazhanov model arises, because it turns out that the Boltzmann weight of the row of cubes in front-to-back direction $P$ in the modified model exactly coincides with $W^{(1)}_{star}$.

$$S_0(\alpha, \beta, \gamma, \delta) = W^{(1)}_{star}(\alpha, \beta, \gamma, \delta).$$

(2.28)

Then in order to construct (cyclic) representations of the braid group, the usual procedure [11, 15] is to map by a Wu-Kadanoff-Wegener like transformation the IRF R-matrix defined by $W^{(1)}_{star}$ to a vertex-type one, and hence to show that it is an intertwiner of the (cyclic) representations of the quantum group $U_q(\hat{gl}_n)$. The main result of this paper is to show in the next sections that by choosing some suitable limits of the spectral parameters characterizing the IRF-R-matrix (2.26) of the three-dimensional Baxter-Bazhanov model, one may obtain directly cyclic representations of the braid group, similarly to the two-dimensional case [13, 14].

3 The Cyclotomic Invariants

In order to construct cyclic representations of the braid group and the related cyclotomic invariants, the starting point is the construction of a $\mathbb{C}^*$-algebra $A(c)$ and of a functor $F$ from the category of the uniform oriented tangles $\mathcal{T}_M$ to $A(c)$ [13]. Let us introduce some notations first. Let $L$ be a free $\mathbb{Z}_N$-module of rank $n - 1$ and suppose it is given by the exact sequence

$$0 \to \mathbb{Z}_N \text{Ker} \pi = \mathbb{Z}_N(1, \ldots, 1) \to \mathbb{Z}_N^n \to L \to 0$$

This means that it is possible to write the elements of $L$ as

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

with the equivalence relation (2.14), which implements the $\mathbb{Z}_N^{n-1}$ symmetry of the Baxter-Bazhanov model. Next let’s introduce the non-singular bilinear form $B$ on $L$

$$B(\alpha, \beta) = -\sum_{i \in \mathbb{Z}_n} \alpha_i (\beta_i - \beta_{i+1})$$

(3.1)

which corresponds to the $n \times n$ matrix

$$B_{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } i = j - 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

(3.2)

Let $A(\alpha, \beta)$ be twice the skew-symmetric part of $B(\alpha, \beta)$

$$A(\alpha, \beta) = B(\alpha, \beta) - B(\beta, \alpha)$$

(3.3)
These definitions are consistent, since $B$ respects the equivalence relation (2.14).

Next we assign a configuration $c$. By this we mean that we give a map $c : \{1, \ldots, M\} \mapsto \{\pm 1\}$ that we write $c = (c(1), \ldots, c(M))$. We interpret $c$ as an object of the category $\mathcal{T}_M^M$ of the uniform oriented tangles [19, 20].

The algebra $\mathcal{A}(c)$ is a $C$-algebra with a unit 1. If $M = 0$ or $M = 1$, $\mathcal{A}(c)$ is simply $C$. If $M \geq 2$, $\mathcal{A}(c)$ is the algebra with generators $x_k^\alpha = x_k^\alpha(c)$ where $1 \leq k \leq M - 1$ and $\alpha \in \mathbb{L}$. We impose the relations

$$
\begin{align*}
x_k^0 & = 1, \\
x_k^\alpha x_k^\beta & = \omega^{A(\beta, \alpha)/2} x_k^{\alpha + \beta} \quad \text{if} \quad (c(k), c(k + 1)) = (1, 1) \\
& = \omega^{A(\alpha, \beta)/2} x_k^{\alpha + \beta} \quad \text{if} \quad (c(k), c(k + 1)) = (-1, -1) \\
& = x_k^{\alpha + \beta} \quad \text{if} \quad (c(k) \neq c(k + 1)) \\
x_k^\alpha x_{k+1}^\beta & = \omega^{B(\beta, \alpha)} x_{k+1}^{\beta} x_k^{\alpha} \quad \text{if} \quad (c(k + 1) = -1 \\
& = \omega^{B(\alpha, \beta)} x_{k+1}^{\beta} x_k^{\alpha} \quad \text{if} \quad (c(k + 1) = 1 \\
& = x_k^\beta x_k^\alpha \quad \text{if} \quad |k - k'| \geq 2
\end{align*}
$$

(3.4)

On the algebra $\mathcal{A}(c)$ there is a linear involution, which is defined by its action on the generators

$$(x_k^\alpha)^* = x_k^{-\alpha}.$$ 

(3.5)

In terms of the operators $x_k^\alpha$ it is possible to define the operators describing the images of the functor $F$ of the elementary tangles as

$$
\begin{align*}
T_k(c) & = \frac{1}{D} \sum_{\alpha \in \mathbb{L}} \omega^{-B(\alpha, \alpha)/2} x_k^\alpha(c) \quad \text{if} \quad c(k) = c(k + 1) \\
& = \frac{1}{D} \sum_{\alpha, \beta \in \mathbb{L}} \omega^{B(\beta, \beta) + B(\alpha, \beta)} x_{k+1}^\alpha x_k^\beta(c) \quad \text{if} \quad (c(k) \neq c(k + 1)) \\
E_k(c) & = \frac{1}{D} \sum_{\alpha \in \mathbb{L}} x_k^\alpha(c) \quad \text{if} \quad (c(k) \neq c(k + 1))
\end{align*}
$$

(3.6)

where $D = N^{n-1}$ and $E_k$ is defined only when $c(k) \neq c(k + 1)$. Remind that the functor $F$ from the category $\mathcal{T}_M^M$ of the uniform oriented tangles is constructed as follows [13]. First the morphisms of $\mathcal{T}_M^M$ are generated by the elementary tangles shown in figure 6.

![Figure 6: Elementary tangles](image)

As a consequence of the defining relations (3.6) of the morphisms $T_k, E_k$ and using the commutation relations (3.4) of the $x_k^\alpha$, it is possible to verify that the elements (3.6) satisfy the following relations (3.7), in which the strings should be
oriented in all possible ways

\[
T_k^* = T_k^{-1} \\
E_k^* = E_k \\
T_k T_{k'} = T_{k'} T_k \text{ if } |k - k'| \geq 2 \\
E_k T_{k'} = T_{k'} E_k \text{ if } |k - k'| \geq 2 \\
E_k^2 = E_k \\
E_k E_{k+1} E_k = E_k \\
T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1} \\
T_k E_{k+1} E_k = T^{-1}_{k+1} E_k \\
T_k^{2N} = (\text{scalar})1. 
\]  

(3.7)

Notice that the first relation means that \( T_k \) is unitary, while the third and the seventh are the braid group relations. The (3.7) are some of the defining relations of the class of the morphisms \( h(c, c) \) in the category \( \mathcal{T}_M^M \). Date, Jimbo, Miki and Miwa [15] have shown that the other relations defining this class are satisfied too. This means that the functor \( F \) mapping \( \mathcal{T}_M^M \) to \( \mathcal{A}(c) \) defined by

\[
F(c) = \mathcal{A}(c), \\
F(\sigma_k^\pm) = \mathcal{M}(T_k(c)^\pm), \\
F(\theta_k^\pm) = \mathcal{M}(E_k(c)^\pm). 
\]  

(3.8)

is well-defined. Here \( \mathcal{M}(a) \in \text{End}(\mathcal{A}(c)) \) denotes the left multiplication by \( a \epsilon \mathcal{A} \). Moreover, let \( \text{Tr} \) denote the usual matrix trace on \( \text{End}(\mathcal{A}(c)) \) normalized as \( \text{Tr} (1) = 1 \). Notice that if \( c(k) = 1, \forall k \), or \( c(k) = -1, \forall k \), then the tangles \( T_k \) give a representation of the ordinary braid group. In that case it is possible to consider also the right multiplication by elements of \( \mathcal{A}(c) \) and we obtain a right-regular representation of the braid group, not only a left-regular one. Then if \( n=2 \) and \( c(k) = 1, \forall k \), or \( c(k) = -1, \forall k \), one finds the "generalized skein relations"

\[
T_k^l = \sum_{i=0}^{l-1} A_i T_k^i 
\]  

(3.9)

In this equation the order of the skein relation is given by

\[
l = \begin{cases} 
\frac{N+2}{2} & \text{if } N \text{ even} \\
\frac{N+1}{2} & \text{if } N \text{ odd}
\end{cases}
\]  

(3.10)

Formula (3.10) is valid when the equation

\[
x^2 = 1 \pmod{N} \]  

(3.11)

has only the two solutions \( x = 1 \pmod{N} \) and \( x = N-1 \pmod{N} \). In particular it is valid for the prime numbers. It shows that for \( n = 2, N = 2, 3 \) the algebra
defined by the operators $T_k$ can be expressed in terms of the Hecke algebra \cite{13, 14}. More generally operators satisfying generalized skein relations like (3.9) can be obtained with a "cabling procedure" starting from the generators of the Hecke algebra. In equation (3.9) the coefficients $A_i$ are the solutions of the linear system

\[
\frac{1}{\beta_{1,0}} \sum_{\alpha(2),...,\alpha(l) \in L} \prod_{i=1}^{l-1} \omega^{B(\alpha(i+1), \alpha(i)-\alpha(i+1))} + \sum_{r=1}^{l-1} \frac{A_r}{\beta_{r,r}} \sum_{\alpha(2),...,\alpha(r) \in L} \prod_{i=1}^{r-1} \omega^{B(\alpha(i+1), \alpha(i)-\alpha(i+1))} = \delta_{\alpha_(1),0},
\]

(3.12)

With all this preliminaries an invariant of oriented links can be constructed as follows

\[
\tau(\hat{t}) = D^{(M-1)/2} \text{Tr} (\mathcal{M}(T_{k_1}^{\epsilon(1)} \cdots T_{k_m}^{\epsilon(m)})) \text{ for } t \text{hom}(c, c)
\]

(3.13)

if $\hat{t}$ denotes the link obtained by closing the tangle $t$ and

\[
\mathcal{F}(t) = \mathcal{M}(T_{k_1}^{\epsilon(1)} \cdots T_{k_m}^{\epsilon(m)}).
\]

(3.14)

In equation (3.14) $\epsilon(i)$ denotes the orientation of the $i$-th generator $T_{k_i}$ which appears in the expression of $\mathcal{F}(t)$. Notice that the Tr can be defined by its action on the generators

\[
\text{Tr} (x_k^{\alpha_1} \cdots x_k^{\alpha_n}) = \begin{cases} 
1 & \text{if } \alpha_1 = \cdots = \alpha_n = 0 \\
0 & \text{otherwise}
\end{cases}
\]

(3.15)

The quantity (3.13) is invariant under the Markov moves and gives a cyclotomic invariant of the tangle. Here by cyclotomic invariant we mean a link invariant defined through a cyclic representation of the braid group. In the case that the tangles are associated to braids, Date et al. have shown that if $\hat{b}$ is a closure of a braid $b \in B_M$ with $v$ crossings, then (3.13) becomes

\[
\tau(\hat{b}) = N^{n(M-v-1)/2} \sum_{\alpha} Z_N^\epsilon^{v-M+1} Z_N^\epsilon \omega^{Q(\alpha, \alpha)/2},
\]

(3.16)

with $b \in B_M$.

Here $Q(\alpha, \alpha)$ is the bilinear form determined by the matrix

\[
Q = S \otimes B + S^T \otimes B^T
\]

(3.17)

where $S$ is a $(v-M+1) \times (v-M+1)$ Seifert matrix for $\hat{b}$ and $T$ denotes the transposition of a matrix.

Now (3.17) has a topological meaning, since $Q$ is a presentation matrix for the $\mathbb{Z}$-module $H_1(M_n, \mathbb{Z}_N)$. Here $M_n$ is the $n$-th cyclic covering of $S^3$ branched along the link $\hat{b}$. This means that for $N$ an odd prime number the module of $\tau(\hat{b})$ can be written as

\[
|\tau(\hat{b})| = N^{\beta_n}/2
\]

(3.18)

where $\beta_n$ is the first Betti number of $M_n$ relatively to the homology group $H_1(M_n, \mathbb{Z}_N)$. Hence, if the quadratic form $B$ is non-singular $\tau$ can be expressed in terms of products of Alexander-Conway polynomials associated to the link $\hat{b}$.
4 The Spectral Limits of the IRF-2D-Reduced Baxter-Bazhanov Model R-Matrix and the Tangle Invariants

In this section we shall show that it is possible to obtain directly the cyclotomic invariants from the Boltzmann weights $S$ of the 3D-Baxter-Bazhanov model (see section 2), after taking some suitable limits of the spectral parameters $(p, p', q, q')$. Furthermore, we shall show that taking other limits of the spectral parameters it is possible to obtain generalizations of the cyclotomic invariants from the Boltzmann weights $S_0$ of the modified Baxter-Bazhanov model. The first step is to define the Yang-Baxter operators

$$(Y_k(p, p', q, q'))^{\alpha'(1)\ldots\alpha'(M-1)}_{\alpha(1)\ldots\alpha(M-1)} =$$

$$\frac{1}{D(\prod_{l \neq k} \delta_{\alpha(l)\alpha'(l)})} S_0(\alpha(k-1), \alpha(k), \alpha(k+1), \alpha'(k))$$

$$(Y_0(p, p', q, q'))^{\alpha'(1)\ldots\alpha'(M-1)}_{\alpha(1)\ldots\alpha(M-1)} =$$

$$\frac{1}{D(\prod_{l \neq k} \delta_{\alpha(l)\alpha'(l)})} S(\alpha(k-1), \alpha(k), \alpha(k+1), \alpha'(k))$$

These operators act on a subspace $(\mathcal{W}^{(0)})^{\otimes M-1} \subset \mathcal{W}^{\otimes M-1}$ where $\mathcal{W} = (C^N)^{\otimes n}$ and $\mathcal{W}^{(0)}$ is the subspace generated by the elements of $\mathcal{W}$

$$\xi_{\alpha} = \sum_k \varepsilon_{\alpha_1+k} \otimes \cdots \otimes \varepsilon_{\alpha_n+k}$$

if $\varepsilon_i$ is the canonical base of $C^N$ and $\alpha e L$. The subspace $\mathcal{W}^{(0)}$ has dimension $D = N^{n-1}$, while $\mathcal{W}$ has dimension $N^n$. But this restriction is necessary in order to implement the $\mathbb{Z}_N^{n-1}$-symmetry of the Baxter-Bazhanov model and hence the equivalence relation (2.14).

The Yang-Baxter operators depend on the spectral parameters $(p, p', q, q')$. In analogy with the standard procedure established by, e.g., Akutsu, Deguchi, Wa- dati [13, 14], the operators $Y_k$ and $Y_{k0}$ give a matrix representation of the braid group $B_M$ if some spectral limits on $(p, p', q, q')$ are taken. It turns out that in these limits $Y_k$ goes either to the left-regular or to the right-regular representation of the operators $T_k(c)^{\pm 1}$ with $c(k) = 1, \forall k = 1, \ldots, M$ or $c(k) = -1, \forall k = 1, \ldots, M$. To find braid group representations the first thing to look for the values of the spectral parameters where the model is critical. This means that we must consider the trigonometric limit, in which all the elementary cubes in the parallelepiped $\mathcal{P}$ considered in section 2 have the same spectral parameters. This assumption guarantees that the model is homogeneous. Then we have found the following limits in which
we obtain the left-regular representation of the operators $T_k^{±1}, k = 1, \ldots, M − 1$:

\begin{align*}
Ia) & \quad p \ll q \ll p' = q' : Y_k \mapsto T_k(c), \quad c(k) = 1, \forall k = 1, \ldots, M \\
Ib) & \quad q \ll p \ll p' = q' : Y_k \mapsto T_k^{-1}(c), \quad c(k) = 1, \forall k = 1, \ldots, M \\
IJa) & \quad p' \ll q' \ll p = q : Y_k \mapsto T_k^{-1}(c), \quad c(k) = -1, \forall k = 1, \ldots, M \\
IIb) & \quad q' \ll p' \ll p = q : Y_k \mapsto T_k^{-1}(c), \quad c(k) = -1, \forall k = 1, \ldots, M
\end{align*}

(4.3)

To see this, let’s choose the following base of the algebra $A(c)$

$$\{y(c) = \omega \frac{1}{2} \sum_{k=1}^{M-2} B^c(k+1)(a(k), a(k+1)) x_1^a(1) \cdots x_{M-1}^a(M-1)\}_{a(i) \in L},$$

(4.4)

where

$$B^c(a, b) = \begin{cases} B(a, b) & \text{if } c = 1 \\ B(b, a) & \text{if } c = -1 \end{cases}$$

(4.5)

The map

$$\rho : A(c) \mapsto (W(0))^{\otimes M-1}$$

(4.6)

defined by

$$\rho(y(c)) = \xi_{a(1)} \otimes \cdots \otimes \xi_{a(M-1)}$$

(4.7)

is an isomorphism of $C^*$-algebras. Let’s prove that Ia) is right. The matrix elements of the operators $\rho T_k \rho^{-1}$ in the case $c(k) = 1, \forall k$, omitting $\rho$ can be written as

$$
(T_k)^{a(1)} \cdots (a(M-1)} = (\prod_{l \neq k} \delta_{a(l) a(l)} \omega_{B^c(a(k), a(k))} \times \frac{1}{\sqrt{D}} \omega_{[2(B(a(k) - a(k)), a(k+1)) - B(a(k-1), a(k) - a(k))) - B(a(k), a(k)) - B(a(k), a(k)))})
$$

(4.8)

where $T_k \sim (something)$ means $\rho T_k \rho^{-1} = (something)$. The Yang-Baxter operator $Y_k$ in the limit Ia) gives the same matrix operators, provided a similarity transformation is made. To obtain this result, let’s calculate the limits of the function $w$ defined in (2.8). It results

$$\frac{w(x, l)}{w(x, 0)} = \begin{cases} \Phi(l)^{-1} & \text{if } x \to \infty \\
\delta_{l_0} & \text{if } x \to 1 \\
1 & \text{if } x \to 0 \end{cases}$$

(4.9)

Using these limits it is possible to show that

$$\frac{W_{pq}^c(a, b)}{W_{pq}^c(0, 0)} = \begin{cases} \omega_{B^c(a, a - b)} & \text{if } p/q \to \infty \\
\delta_{a, b} & \text{if } p/q \to 1 \\
\omega_{B^c(b, a - b)} & \text{if } p/q \to 0 \end{cases}$$

(4.10)

From (2.28) it follows immediately that in the limit Ia)

$$S(a, b, c, d) = S_0(a, b, c, d) = \omega_{B^c(d, d) - B(a, b)}$$

(4.11)

To obtain (4.8) from (4.11) we multiply $S$ by the factor

$$\sqrt{D} \omega_{\frac{1}{2}B^c(a, a - b) + B(a, a - b)}.$$
It is a site-type, edge-type, face-type equivalence transformation and does not change the factorization properties nor the partition function of the model. With the same tools it is possible to see that Ib, IIa, IIb) holds, provided that $S$ is multiplied by the factor $(IIc)$ in the case IIb), and by the factor

$$\sqrt{\Omega_{\alpha}^{1}[B(\beta,\beta) - B(\delta,\delta) - B(\beta - \delta,\alpha) - B(\gamma,\beta - \delta)]}$$

in the cases IIa) and IIb). Further, by the same arguments, it is possible to prove that there are other limits giving the $T_k^{\pm 1}$ in the right-regular representation, obtained from the left-regular one by transposing the matrices. These limits are given by

$$IIIa)\quad p = q < q' < q : Y_k \mapsto T_k(c), \quad c(k) = 1, \forall k = 1, \ldots, M$$

$$IIIb)\quad p = q < q' < p' : Y_k \mapsto T_k^{-1}(c), \quad c(k) = 1, \forall k = 1, \ldots, M$$

$$IVA)\quad p' = q' < p < q : Y_k \mapsto T_k(c), \quad c(k) = -1, \forall k = 1, \ldots, M$$

$$IVb)\quad p' = q' < q < p : Y_k \mapsto T_k^{-1}(c), \quad c(k) = -1, \forall k = 1, \ldots, M$$

At this point a question arises: is it possible to get other kinds of braid group representations and hence other link invariants starting from the Yang-Baxter equation of the Baxter-Bazhanov model? We fix the configuration of 2M-1 strings where $c(2k-1) = -1, c(2k) = 1, \forall k = 1, \ldots, M - 2, c(2M - 1) = -1$. We obtain the following picture for $k = 1, \ldots, M - 3$

$$VA)\quad p' < q' < p < q : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))T_{2k}(s_{2k+1}(c))T_{2k+2}(s_{2k+1}(c))T_{2k+1}(c)^{-1}$$

$$VB)\quad q' < p' < q < p : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))T_{2k}(s_{2k+1}(c))^{-1}T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)^{-1}$$

$$VIa)\quad p < q < q' < p' : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))^{-1}T_{2k}(s_{2k+1}(c))^{-1}T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)$$

$$VIIa)\quad q < p < q' < p' : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))^{-1}T_{2k}(s_{2k+1}(c))^{-1}T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)$$

$$VIIIa)\quad q' < p' < q < p : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))T_{2k}(s_{2k+1}(c))T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)^{-1}$$

$$VIIIb)\quad p' < q' < p < q : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))^{-1}T_{2k}(s_{2k+1}(c))T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)^{-1}$$

$$IXa)\quad p < p' < q < q' : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))T_{2k}(s_{2k+1}(c))T_{2k+2}(s_{2k+1}(c))T_{2k+1}(c)$$

$$IXb)\quad q < q' < p < p' : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))^{-1}T_{2k}(s_{2k+1}(c))^{-1}T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)$$

$$Xa)\quad p' < p < q' < q : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))T_{2k}(s_{2k+1}(c))T_{2k+2}(s_{2k+1}(c))T_{2k+1}(c)$$

$$Xb)\quad q' < q < p' < p : Y_k \mapsto T_{2k+1}(s_{2k+1}(c))^{-1}T_{2k}(s_{2k+1}(c))^{-1}T_{2k+2}(s_{2k+1}(c))^{-1}T_{2k+1}(c)^{-1}$$
Now we must explain the meaning of the products
\[ T_{\pm 2k+1}(s_{2k+1}(c))T_{\pm 2k}(s_{2k+1}(c))T_{\pm 2k+2}(s_{2k+1}(c))T_{\pm 2k+1}(c) \]
where \( T_{\pm 1k} = T_{\pm 1k}^+ \). We construct a representation \( \mathcal{R} \) of \( A(c) \) on \( (\mathcal{W}(0))^\otimes M - 1 \) for the configuration \( c(2k-1) = -1, c(2k) = 1 \) for \( 1 \leq k \leq M - 2 \), \( c(2M-1) = -1 \) in the following way. Notice that adjacent strings have the opposite directions. Following reference [13], we introduce the following operators acting on \( \mathcal{W}(0) \)
\[ Z_i = 1 \otimes \cdots \otimes Z \otimes \cdots \otimes 1 \]
(4.16)
\[ X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1 \]
(4.17)
where \( X \) and \( Z \) act on the \( i \)-th factor \( C_N \) in \( \mathcal{W}(0) \) and are defined by
\[ Z_{kl} = \delta_{k,l+1} \]
(4.18)
\[ X_{kl} = \omega^k \delta_{k,l} \]
(4.19)
for \( k, l \in \mathbb{Z}_N \). Moreover, using the operators (4.16), (4.17) we define for \( \alpha \in L \)
\[ X^\alpha = X_1^\alpha \cdots X_n^\alpha \]
(4.20)
\[ Z^\alpha = Z_1^\alpha \cdots Z_n^\alpha \]
(4.21)
Then the representation \( \mathcal{R} \) is given by
\[ \mathcal{R}(x^\alpha_{2k-1}) = 1 \otimes \cdots \otimes Z^\alpha \otimes \cdots \otimes 1 \]
(4.22)
\[ \mathcal{R}(x^\alpha_{2k}) = 1 \otimes \cdots \otimes X^{B\alpha} \otimes X^{-B\alpha} \otimes \cdots \otimes 1 \]
(4.23)
where the action of \( Z^\alpha \) in (4.22) is on the \( k \)-th space, while the action of \( X^{B\alpha} \otimes X^{-B\alpha} \) in (4.23) is on the \( k \)-th and \( k+1 \)-th space. Notice that it is possible to multiply the operators \( T_{k+1}(s_k(c))T_k(c) \) relative to configurations which differ by a permutation, because the algebras \( A(c) \) arising from configurations of the same type are canonically isomorphic (see [15]). Thus, as a consequence, the matrix elements of the Yang-Baxter operators \( Y_k \) in the limits \( V) - X) \) are exactly the matrix elements of the products \( T_{\pm 2k+1}(s_{2k+1}(c))T_{\pm 2k}(s_{2k+1}(c))T_{\pm 2k+2}(s_{2k+1}(c))T_{\pm 2k+1}(c) \) in the representation \( \mathcal{R} \), where in (4.15) we have omitted to write the label \( \mathcal{R} \).
Moreover, we have verified that the trace on the braid group representation given by the operators \( Y_{k0} \) in the limits \( V) \) and \( VI) \) enjoys the Markov properties. This can be verified immediately, by observing that in the representation \( \mathcal{R} \) the trace has the properties (3.15). Let’s show, e.g., the invariance under the Markov move 2 in the case \( Va) \). We define
\[ \pi : B_{M-2} \rightarrow (\mathcal{W}(0))^{M-1} \]
\[ \pi(b_k) = Y_{k0}, \quad k = 1, \ldots, M - 3 \]
(4.24)
where the \( b_k \) are the braid group generators satisfying the relations
\[
\begin{align*}
    b_k b_{k'} &= b_{k'} b_k \\
    b_k b_{k+1} b_k &= b_{k+1} b_k b_{k+1} \\
    b_k &= b_k \quad \text{for } k, k' = 1, \ldots, M - 3, \ | k - k' | \geq 2 \\
    b_{k+1} b_k &= b_{k+1} b_k \quad \text{for } k = 1, \ldots, M - 4 
\end{align*}
\]
and
\[
\tau'(\hat{b}) = D^{M-2} \text{Tr} (\pi(b))
\]
where the trace is normalized as \( \text{Tr} (1) = 1 \). Indeed, omitting to write the configuration \( c \) on which the operators act, by applying repeatedly first and second Markov moves, as well as the braid group relations, we obtain
\[
\tau'(\hat{g}) = D^{M-1} \text{Tr} (\pi(g)T_{2M-2} T_{2M-2} T_{2M-2}^{-1}) =
\]
\[
D^{M-1} \text{Tr} (T_{2M-1}^{-1} \pi(g)T_{2M-1} T_{2M-2} T_{2M}) =
\]
\[
D^{M-2} \text{Tr} (T_{2M-1}^{-1} \pi(g)T_{2M-1} T_{2M-2}^{-1}) =
\]
\[
\tau'(\hat{y})
\]
where \( g \epsilon B_{M-1}, b_{M-1} \epsilon B_M \). To summarize, we have shown that the ordinary trace on the \( Y_{k0} \) is invariant under the Markov moves 1 and 2, and hence provides tangle invariants (The tangles are in correspondence with the Yang-Baxter operators). We shall collect the results of this section in table 1.

5 Generalizations

In the previous section we have shown that the 3-D-Baxter-Bazhanov model can be related to the cyclotomic knot invariants generated by the limits I)-IV) of the associate Yang-Baxter operators \( Y_k \). Under the other limits V)-X) one obtains products like
\[
T_{\pm 2k+1}(s_{2k+1}(c))T_{\pm 2k}(s_{2k+1}(c))T_{\pm 2k+2}(s_{2k+1}(c))T_{\pm 2k+1}(c)
\]
It is intriguing to think that the products \( (5.1) \) give a ”cabling” representation of the braid group, analogously to the procedure established by Akutsu, Wadati, Deguchi \([13, 14]\) to construct higher-dimensional braid group representation of the Hecke algebra of \( B_M \). However some observations are in order:
i) The cabling here is, perhaps, related to higher-dimensional representations of $U_q(gl(n))$ with $q^N = 1$.

ii) Probably we must give up the orientation, and hence the invariants are of non-oriented type.

iii) The single $T_k$ are related to a representation of the Temperley-Lieb algebra [17, 18] only for $N=2,3, n=2$. Therefore only in these cases one may think to generalize the construction implemented in [13, 14].

Work along this direction is in progress.

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|    | \(\frac{\omega}{p} \) | \(\frac{\omega}{q} \) | \(\frac{\omega}{k} \) | \(\frac{\omega}{t} \) | Rapidities (Spectral Limits) | Boltzmann Weights (IRF-type R-matrix) | Yang-Baxter Operators (Braid Group Generators) |
|----|-----------------|---------------|-----------------|---------------|-------------------------|----------------------------------------|----------------------------------|
| Ia) | 0   | \(\infty\) | 1   | \(\infty\) | \(p \ll q \ll p' = q'\) | \(\omega B(\delta, \beta) - B(\delta, \beta) + B(\alpha, \delta - \beta)\) | \(T_k c(k) = 1 \forall k\) |
| Ib) | \(\infty\) | \(\infty\) | 1   | \(\infty\) | \(q \ll p \ll p' = q'\) | \(\omega B(\delta, \beta) + B(\delta, \beta) + B(\alpha, \delta - \beta)\) | \(T_k^{-1} c(k) = 1 \forall k\) |
| IIa) | 1   | 0   | 0   | 0   | \(p' \ll q' \ll q = p\) | \(\omega B(\delta, \beta) - B(\delta, \beta) - B(\gamma, \delta - \beta)\) | \(T_k c(k) = -1 \forall k\) |
| IIB) | 1   | 0   | \(\infty\) | 0   | \(q' \ll p' \ll p = q\) | \(\omega B(\delta, \beta) + B(\delta, \beta) - B(\gamma, \delta - \beta)\) | \(T_k^{-1} c(k) = 1 \forall k\) |
| IIIa) | 1   | \(\infty\) | \(\infty\) | \(\infty\) | \(p = q \ll p' \ll q'\) | \(\omega B(\delta, \beta) - B(\delta, \beta) + B(\delta - \beta, \gamma)\) | \(T_k c(k) = 1 \forall k\) |
| IIIb) | 1   | \(\infty\) | \(\infty\) | \(\infty\) | \(p = q \ll q' \ll p'\) | \(\omega B(\delta, \beta) + B(\delta, \beta) + B(\delta - \beta, \gamma)\) | \(T_k^{-1} c(k) = 1 \forall k\) |
| IVa) | 0   | 0   | 1   | 0   | \(p' = q' \ll p \ll q\) | \(\omega B(\delta, \beta) - B(\delta, \beta) - B(\delta - \beta, \alpha)\) | \(T_k c(k) = -1 \forall k\) |
| IVb) | \(\infty\) | 0   | 1   | 0   | \(p' = q' \ll q \ll p\) | \(\omega B(\delta, \beta) + B(\delta, \beta) - B(\delta - \beta, \alpha)\) | \(T_k^{-1} c(k) = -1 \forall k\) |
| Va)  | 0   | 0   | 0   | 0   | \(p' \ll q' \ll q \ll p\) | \(\sum \mu L \omega B(\delta, \gamma, \alpha - \beta)\) | \(T_{2k+1+T_k} T_{2k+2} T_{2k+1}^{-1}\) |
| Vb)  | \(\infty\) | 0   | \(\infty\) | 0   | \(q' \ll p' \ll q \ll p\) | \(\sum \mu L \omega B(\delta, \gamma, \alpha - \beta)\) | \(T_{2k+1+T_k} T_{2k+2} T_{2k+1}^{-1}\) |
| VIa) | 0   | \(\infty\) | \(\infty\) | \(\infty\) | \(p \ll q \ll p' \ll q'\) | \(\sum \mu L \omega B(\delta, \gamma, \alpha - \beta)\) | \(T_{2k+1+T_k} T_{2k+2} T_{2k+1}^{-1}\) |
| VIb) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(q \ll p \ll q' \ll p'\) | \(\sum \mu L \omega B(\delta, \gamma, \alpha - \beta)\) | \(T_{2k+1+T_k} T_{2k+2} T_{2k+1}^{-1}\) |
| VIIa) | 0   | \(\infty\) | \(\infty\) | \(\infty\) | \(p \ll q \ll q' \ll p'\) | \(\frac{\prod_{i=1}^{k-1} \delta_{\beta_{i+1}-\gamma_{i+1}, \alpha_{i} - \delta_{i}}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| VIIb) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(q \ll p \ll p' \ll q'\) | \(\frac{\prod_{i=1}^{k-1} \delta_{\beta_{i+1}-\gamma_{i+1}, \alpha_{i} - \delta_{i}}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| VIIIa) | 0   | 0   | \(\infty\) | 0   | \(q' \ll p' \ll q \ll p\) | \(\frac{\prod_{i=1}^{k-1} \delta_{\beta_{i+1}-\gamma_{i+1}, \alpha_{i} - \delta_{i}}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| VIIIb) | \(\infty\) | 0   | 0   | 0   | \(p' \ll q' \ll q \ll p\) | \(\frac{\prod_{i=1}^{k-1} \delta_{\beta_{i+1}-\gamma_{i+1}, \alpha_{i} - \delta_{i}}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| IXa) | 0   | \(\infty\) | 0   | 0   | \(p \ll p' \ll q \ll q'\) | \(\frac{T_{2k+1} T_{2k} T_{2k+1}^{-1}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| Xa)  | \(\infty\) | 0   | \(\infty\) | \(\infty\) | \(p' \ll p \ll q \ll q'\) | \(\frac{T_{2k+1} T_{2k} T_{2k+1}^{-1}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| IXb) | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(q \ll q' \ll p \ll p'\) | \(\frac{T_{2k+1} T_{2k} T_{2k+1}^{-1}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |
| Xb)  | \(\infty\) | \(\infty\) | \(\infty\) | \(\infty\) | \(q' \ll q \ll p \ll p'\) | \(\frac{T_{2k+1} T_{2k} T_{2k+1}^{-1}}{\omega B(\delta, \gamma, \alpha - \beta)\}\) | \(T_{2k+1} T_{2k} T_{2k+1}^{-1}\) |

Table 1: The spectral limits and the resulting Boltzmann weights and Yang-Baxter operators.