ADM For Solving Linear Second-Order Fredholm Integro-Differential Equations

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Abstract. In this paper, we apply Adomian Decomposition Method (ADM) as numerically analyse linear second-order Fredholm Integro-differential Equations. The approximate solutions of the problems are calculated by Maple package. Some numerical examples have been considered to illustrate the ADM for solving this equation. The results are compared with the existing exact solution. Thus, the Adomian decomposition method can be the best alternative method for solving linear second-order Fredholm Integro-Differential equation. It converges to the exact solution quickly and in the same time reduces computational work for solving the equation. The result obtained by ADM shows the ability and efficiency for solving these equations.

1. Introduction
An integro-differential equation is the equations which are known to be appear mostly in both derivatives and anti-derivatives of a function. It has been used in various fields of studies including engineering, chemistry, physics and etc. It can be classified into two types, i.e, Fredholm and Volterra equations. Volterra equations have the upper bound limit as a variable while Fredholm an equation has fixed bound of limits. Previous updated work by Adil Al-Rammahi used power series to solve Fredholm Integro-Differential of second-Order which also look intoits relation with Banach fixed point theorem [1]. Parandin, Chenari and Heidari used numerical solution to solve the Fredholm Integro-Differential equations which consider equation solution from finite Bessel polynomials [2]. However, there is some other method used by the researcher to solve Fredholm Integro-Differential equationwith the same objective which is to look up into the convergence of approximationto the exact solution. The method that is used are Conjugate Gradient method [3], Legendre-Galerkin method [4], Tau method [5], Generalize Minimal Residual [6], Trigonometric Scaling Function [7], and Taylor Polynomial [8]. In this study, we only focus on solving linear Fredholm Integro-differential equations in second-order using ADM.

This study will examine the way in which the ADM is solving the linear second-order Fredholm Integro-Differential equations. There are some of physical problems model in terms of second-order linear Fredholm Integro-differential equation, the numerical solutions for those equations has been reviewed by several researchers. However, in this paper, ADM will be used to solve these equations.

A system of second-order linear Fredholm integro-differential equations can be defined as:

\[ P(x)h''(x) + Q(x)h'(x) + R(x)h(x) = g(x) + \int_a^b K(x,t)h(t)dt \]  (1)
Initial condition:
\[ h(x) = p h(x) = q \]

As, the functions:
\[ P(x), Q(x), R(x) \] - Constant matrices.
\[ g(x) \] - Function.
\[ K(x,t) \] - Separable kernel.
\[ h(x) \] - Solution to be determined.

The purpose of this paper is to solve linear second-order Fredholm Integro-differential equation by using ADM. We organized the paper as follows. In Sec.2, we review linear second-order Fredholm Integro-Differential equation and shows ADM as a method for solving. In Sec. 3, we construct an example of equation and shows the result using ADM will converge to the exact solution.

### 2. Preliminaries

In this section we recapitulate general form of linear second-order Fredholm Integro-Differential equation and shows application of ADM to this equation. General form of linear second-order Fredholm Integro-Differential equation in the form of:

\[ h''(x) = g(x) + \lambda \int_a^b K(x,t)h(t)dt \]  \hspace{1cm} (2)

Seperable kernel can be denoted as a finite sum of the form:

\[ K(x,t) = \sum_{k=1}^{n} m_k(x)r_k(t) \]  \hspace{1cm} (3)

Without losing the generality, the analysis of term kernel \( K(x, t) \) in form of

\[ K(x,t) = m(x)r(t) \]  \hspace{1cm} (4)

The ADM is a powerful method that can determined a series solution to the linear Fredholm Integro-Differential equation by assuming standard form given,

\[ h^{(n)}(x) = g(x) + \int_0^1 K(x,t)h(t)dt, \quad h^{(k)}(0) = b_k, \quad 0 \leq k \leq (n - 1) \]  \hspace{1cm} (5)

Where \( h^{(n)}(x) \) indicates the nth derivative of \( h(x) \) with respect tox and \( b_k \) are constants that will gives the initial conditions. Substitute (2.3) into (2.4) yields,

\[ h^{(n)}(x) = g(x) + m(x) \int_1^t r(t)h(t)dt \]  \hspace{1cm} (6)

In this study, we will used nth derivative in second-order, \( n = 2 \). The definite integral in Integro-Differential (6) will easily be observed it involved an integrand which depends on the variable t. In operator form, equation (6) can be written as,

\[ Lh(x) = g(x) + m(x) \int_0^1 r(t)h(t)dt \]  \hspace{1cm} (7)

where, L is given by,

\[ L = \frac{d^n}{dx^n} \]  \hspace{1cm} (8)

L is an invertible operator, and we apply \( L^{-1} \) on both sides of (7) and become equation (9),

\[ h(x) = b_0 + b_1 x + \frac{1}{2!} b_2 x^2 + \ldots + \frac{1}{(n-1)!} b_{n-1} x^{n-1} + L^{-1}(g(x)) + \left( \int_0^1 r(t)h(t)dt L^{-1}(m(x)) \right) \]

The equation (9) is a standard form of Fredholm Integro-Differential equation. The series solution of \( h(x) \) in ADM is,
The unknown function \( h(x) \) is determined by the above mentioned relation, where,

\[
h_0(x) + h_1(x) + h_2(x) + \ldots = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(g(x)) + \left( \int_0^1 r(t)h_0(t)dt L^{-1}(m(x)) \right) + \\
\left( \int_0^1 r(t)h_1(t)dt L^{-1}(m(x)) \right) + \left( \int_0^1 r(t)h_2(t)dt L^{-1}(m(x)) \right) + \ldots
\]

The calculation of \( h_0(x), h_1(x), h_2(x), h_3(x) \) of solution \( h(x) \) of equation (5) can be written in recursive relation by,

\[
h_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(g(x))h_{n-1}(x) = \left( \int_0^1 r(t)h_n(t)dt L^{-1}(m(x)) \right), \quad n \geq 0
\]

Component that we are already determined will be the solution of \( h(x) \). The series obtained for \( h(x) \) frequently provides the exact solution in closed form. Sometimes the closed form is not very easy to determine.

As a solution, we utilize the arrangement frame gotten to rough the arrangement. A couple terms of the arrangement determined by decomposition method more often than not give the higher request exactness of the estimated arrangement.

The advantage of ADM is to avoid extra computational work and difficulties as compared to other methods. It means, that the computational work can be minimized. In addition to it, we can eliminate the noise terms in the series. Further, we will discuss some examples which will illustrate the ADM for solving linear second-order Fredholm Integro-Differential equation and the phenomenon of eliminating noise terms as well.

### 3. Application and Results

In this section we shall apply ADM to solve linear second-order Fredholm Integro-Differential equations. The result we should obtain must converge approximately to the exact solution.

#### 3.1. Example 1

Consider linear second-order Fredholm Integro-Differential equation of fractional order,

\[
h''(x) = x - \sin x - \int_0^x x h(t)dt,
\]

with the initial conditions:

\[
h(0) = 0, \quad h'(0) = 1
\]

Exact solution is given by,

\[
h(x) = \sin x
\]

Integrating both sides twice, from 0 to \( x \) and using the given condition given,
\[ h(x) = \sin x + \frac{1}{3!} x^3 - \frac{1}{3!} x^3 \int_0^\pi t h(t) dt \]

Using ADM, we set,
\[ h_0(x) = \sin x + \frac{1}{3!} x^3 \]
\[ h_1(x) = -\frac{1}{3!} x^3 \int_0^\pi t h_0(t) dt \]
\[ h_2(x) = -\frac{1}{3!} x^3 \int_0^\pi t h_1(t) dt \]
\[ h_3(x) = -\frac{1}{3!} x^3 \int_0^\pi t h_2(t) dt \]
\[ h_4(x) = -\frac{1}{3!} x^3 \int_0^\pi t h_3(t) dt \]

We shall calculate value of \( h_1(x), h_2(x), h_3(x), h_4(x), h_5(x), \ldots \) and the solution as shown in Table 1.

| \( h_n(x) \) | Solution |
|---------------|----------|
| \( h_0(x) \) | \( \sin x + \frac{1}{3!} x^3 \) |
| \( h_1(x) \) | \(-\frac{1}{3!} x^3[1.31877]\) |
| \( h_2(x) \) | \(-\frac{x^3}{6}[-0.42038]\) |
| \( h_3(x) \) | \(-\frac{x^3}{6}[0.134]\) |
| \( h_4(x) \) | \(-\frac{x^3}{6}[-0.04272]\) |
| \( h_5(x) \) | \(-\frac{x^3}{6}[0.01362]\) |

**Table 1.** Table of calculation Example 1.

\[ h(x) = h_0(x) + h_1(x) + h_2(x) + h_3(x) + h_4(x) + h_5(x) + \ldots \]

\[ h(x) \cong \sin x \]

Eliminating the noise terms in \( h_0(x), h_1(x), h_2(x), h_3(x), h_4(x), h_5(x), \ldots, h_n(x) \) and justifying the non-cancelled term of \( u_0(x) \) satisfies the integral equation gives

\[ h(x) = \sin x \]

In figure 1, we plot a graph to compare the exact and ADM solution. We clearly could see that for this equation in example 1, the answer obtained by ADM is converge to the exact solution very fast. The absolute error between ADM and exact solution is extremely small. Therefore, the answer we get using ADM is approximately the same with exact solution.
3.2. Example 2

Consider the linear second-order Fredholm Integro-Differential equation,

\[ h''(x) = -\sin x + \cos x + \left(2 - \frac{\pi}{2}\right) x - \int_0^x x t h(t) dt \]

with the initial conditions:

\[ h(0) = -1, \quad h'(0) = 1 \]

exact solution is given by,

\[ h(x) = \sin x - \cos x \]

integrating both sides twice, from 0 to \( x \) with the conditions given,

\[ h(x) = \sin x - \cos x + \left(2 - \frac{\pi}{2}\right) \frac{x^3}{6} - \frac{x^3}{6} \int_0^x t h(t) dt \]

using ADM, we set,

\[ h_0(x) = \sin x - \cos x + \left(2 - \frac{\pi}{2}\right) \frac{x^3}{6} \]

\[ h_1(x) = -\frac{x^3}{6} \int_0^x t h_0(t) dt \]

\[ h_2(x) = -\frac{x^3}{6} \int_0^x t h_1(t) dt \]

\[ h_3(x) = -\frac{x^3}{6} \int_0^x t h_2(t) dt \]

\[ h_4(x) = -\frac{x^3}{6} \int_0^x t h_3(t) dt \]

We shall calculate value of \( h_1(x), h_2(x), h_3(x), h_4(x), \ldots \) and the solution showed in Table 2,
$$\begin{array}{|c|c|} 
\hline
h_n(x) & \text{Solution} \\
\hline
h_0(x) & \sin x - \cos x + \left(2 - \frac{\pi}{2}\right) \frac{x^3}{6} \\
\hline
h_1(x) & -\frac{x^3}{6} [0.56602] \\
\hline
h_2(x) & -\frac{x^3}{6} [-0.18043] \\
\hline
h_3(x) & -\frac{x^3}{6} [0.05752] \\
\hline
h_4(x) & -\frac{x^3}{6} [-0.01834] \\
\hline
\end{array}$$

Table 2. Table of calculation Example 2

![Figure 2](image)

Figure 2. Graph for example 2 between ADM and exact solution.

\( h(x) = h_0(x) + h_1(x) + h_2(x) + h_3(x) + h_4(x) + \ldots \)

\[ h(x) \approx \sin x - \cos x \]

Sum up the \( h_n(x) \) terms, we shall get the solution of \( u(x) \) is converge to \( \sin x - \cos x \). It proven by the plotted graph in figure 2 that tell us the ADM is converging towards the exact solution. This showed that for this example 2, ADM is finely work to solve the equation which is converge to exact solution fast. It justifying the solution will converge to exact solution after cancelling the noise terms in our calculation. Therefore, the answer we get using ADM also converge to exact solution.

3.3. Example 3.
Consider the linear second-order Fredholm Integro-Differential equation,

\[ h^{(2)}(x) = -e^x + \frac{1}{2}x + \int_0^1 x t h(t) dt \]
subject to initial condition:
\[ h(0) = 0, \quad h'(0) = -1 \]

exact solution given by,
\[ h(x) = 1 - e^x \]

integrating both side twice, from 0 to x with the condition given,
\[ h(x) = 1 - e^x + \frac{x^3}{12} + \frac{x^3}{6} \int_0^1 t h(t) dt \]

using ADM, we get,
\[
\begin{align*}
    h_0(x) &= 1 - e^x + \frac{x^3}{12} \\
    h_1(x) &= \frac{x^3}{6} \int_0^1 t h_0(t) dt \\
    h_2(x) &= \frac{x^3}{6} \int_0^1 t h_1(t) dt \\
    h_3(x) &= \frac{x^3}{6} \int_0^1 t h_2(t) dt \\
    h_4(x) &= \frac{x^3}{6} \int_0^1 t h_3(t) dt
\end{align*}
\]

we shall calculate value of \( h_1(x), h_2(x), h_3(x), h_4(x), \ldots \) and the solutionshowed in Table 3,

| \( h_n(x) \) | Solution |
|--------------|----------|
| \( h_0(x) \) | \( 1 - e^x + \frac{x^3}{12} \) |
| \( h_1(x) \) | \( \frac{x^3}{6} \left[ \frac{29}{60} \right] \) |
| \( h_2(x) \) | \( \frac{x^3}{6} \left[ \frac{29}{1800} \right] \) |
| \( h_3(x) \) | \( \frac{x^3}{6} \left[ \frac{29}{54000} \right] \) |
| \( h_4(x) \) | \( \frac{x^3}{6} \left[ \frac{29}{162000} \right] \) |

**Table 3.** Table of calculation Example 3

\[ h(x) = h_0(x) + h_1(x) + h_2(x) + h_3(x) + h_4(x) + \ldots \]
\[ h(x) \converge 1 - e^x \]

From table 3, by adding the \( h_n(x) \) terms, we obtain the solution of \( h(x) \)whichconverge to 1 – \( e^x \). It is proven by the illustrated graph in figure 3, that the ADM is converging towards the exact solution. The convergence is moving towards the exact solution fast and error is extremely small. Furthermore, after removing the noise terms in table 3, the solution by ADM is converge to the exact solution. Therefore, the answer we get for example 3 approximately close to the exact solution.
3.4. Example 4.
Consider the linear second-order Fredholm Integro-Differential equation,

\[ h''(x) + xh'(x) - xh(x) = e^x - 2 \sin x + \sin x \int_{-1}^{1} e^{-t} h(t) dt \]

subject to initial condition:

\[ h(-1) = e^{-1}, \quad h(1) = e \]

exact solution given by,

\[ h(x) = e^x \]

integrating both side twice, from 0 to \( x \) with the condition given,

\[ h(x) = e^x + 2 \sin x - \sin x \int_{-1}^{1} e^{-t} h(t) dt \]

using ADM, we get,

\[
\begin{align*}
    h_0(x) &= e^x + 2 \sin x \\
    h_1(x) &= - \sin x \int_{-1}^{1} e^{-t} h_0(t) dt \\
    h_2(x) &= - \sin x \int_{-1}^{1} e^{-t} h_1(t) dt \\
    h_3(x) &= - \sin x \int_{-1}^{1} e^{-t} h_2(t) dt \\
    h_4(x) &= - \sin x \int_{-1}^{1} e^{-t} h_3(t) dt \\
\end{align*}
\]

we shall calculate value of \( h_1(x), h_2(x), h_3(x), h_4(x), h_5(x) \) ... and the solution as shown in Table 4.
\begin{center}
\begin{tabular}{|c|c|}
\hline
\textbf{$u_n(x)$} & \textbf{Solution} \\
\hline
\textbf{$u_0(x)$} & $e^x + 2 \sin x$ \\
\textbf{$u_1(x)$} & $- \sin x \ [0.67301]$ \\
\textbf{$u_2(x)$} & $- \sin x \ [0.44654]$ \\
\textbf{$u_3(x)$} & $- \sin x \ [0.29628]$ \\
\textbf{$u_4(x)$} & $- \sin x \ [0.19658]$ \\
\textbf{$u_5(x)$} & $- \sin x \ [0.13043]$ \\
\textbf{$u_6(x)$} & $- \sin x \ [0.08654]$ \\
\textbf{$u_7(x)$} & $- \sin x \ [0.05742]$ \\
\textbf{$u_8(x)$} & $- \sin x \ [0.03810]$ \\
\textbf{$u_9(x)$} & $- \sin x \ [0.02528]$ \\
\textbf{$u_{10}(x)$} & $- \sin x \ [0.01677]$ \\
\hline
\end{tabular}
\end{center}

Table 4. Table of calculation Example 4

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example4_graph.png}
\caption{Graph for example 4 between ADM and exact solution.}
\end{figure}

Here, by adding $h_n(x)$, the solution is also converging to $\text{ex}$. Based on table 4, it showed that the solution converges to the exact solution after eliminating the noise terms. But for this equation the convergence is quite slow, the ADM polynomial we calculate till $h_{10}(x)$. The graph plotted in figure 4 is the reflection of long series of ADM calculated for this example 4. Therefore, the convergence rate
depends on the complexity of the equation. Since we get very close approximate solution using ADM, this equation is also applicable and less computational works required to solve it.

Conclusion
In this study, ADM with the initial conditions have been solved. As it is obvious from the example, those exact solutions of the problems are calculated by using ADM. All the linear second-order Fredholm integro-differential equations are solved in the solution by cancellation of the noise terms in the Adomian polynomial which converge to the exact solution. This demonstrate that the ADM is an efficient method to determine the solution in close form. In addition to this, the method is simple and the results obtained quickly.

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