A THERMODYNAMIC APPROACH TO QUANTUM MEASUREMENT AND QUANTUM PROBABILITY

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Abstract

A simple model of quantum particle is proposed in which the particle in a macroscopic rest frame is represented by a microscopic d-dimensional oscillator, \( s=(d-1)/2 \) being the spin of the particle. The state vectors are defined simply by complex combinations of coordinates and momenta. It is argued that the observables of the system are Hermitian forms (corresponding uniquely to Hermitian matrices). Quantum measurements transforms the equilibrium state obtained after preparation into a family of equilibrium states corresponding to the critical values of the measured observable appearing as values of a random quantity associated with the observable. Our main assumptions state that: i) in the process of measurement the measured observable tends to minimum, and ii) the mean value of every random quantity associated with an observable in some state is proportional to the value of the corresponding observable at the same state. This allows to obtain in a very simple manner the Born rule.

1 Introduction

In a recent paper C.Fuchs [1] has written: "Until we can explain quantum theory’s essence to a junior-high-school or high-school student and have them walk with a deep lasting memory, we will not understand a thing about the quantum foundations." The Born rule [6] is the heart of quantum foundations and the aim of the present work is to make some step in its understanding using only minimum initial information about mechanics and probability. To do that we confine ourselves with particles at rest and assume that the rest frame has necessarily a macroscopic nature. This means that speaking about a quantum particle at rest we mean that the particle accomplishes a microscopic motion around its rest position. The simplest model of a particle at rest is given by a classical \( d \)-dimensional linear oscillator whose (generalized) coordinates measure the deviation of the particle from its rest position, and whose dimension is related to the spin of the particle. The state of the particle is understood in a purely classical sense and described by coordinates and momenta, or equivalently, by their complex combinations forming the state vector of the system. The energy of the particle is equal to the energy of the particle at rest so that the phase
space trajectory is a fixed ellipsoid with microscopic sizes comparable with the Planck scale. Thus our considerations are in agreement with the ’t Hooft statement [3] that Planck scale is “the most logical domain where one may expect quantum mechanics to be replaceble by a more deterministic scenario”. As in classical theory observables are real functions on phase space. The latter generate canonical transformations conserving the energy surface. The macroscopic smallness of the phase space coordinates allows to decompose observables in series retaining only the quadratic terms. It turns out that the latter are Hermitian forms [2, 4, 8]. (In general we would come to the Weinberg theory [7].) The whole weirdness of quantum mechanics is hidden into the process of measurement. Our device measuring some observable has one input (into which the particles come in) and as many outlets as the critical (i.e. minimal or maximal) values [4] of the measured observable are. A particle entering into device can came out only from one of the channels having the corresponding critical value of the observable. This occurs at random with some probability. Following [1] we consider the probabilities in a Bayesian sense, i.e. as degrees of belief determining our decisions in the face of uncertainty. Then the simple (and reasonable) condition that one could determine the value of the observable making statistical measurements leads directly to the Born rule (cf. [6]).

2 The model system-states and observables

As a model of a quantum particle at rest we consider a $d$-dimensional linear harmonic oscillator with mass $m$, frequency $\omega$ and energy $E = \hbar \omega$ in accordance with de Broglie formula. Our motivation is similar to those of Schrödinger [10] (see also Hestenes [11]) introducing the notion of Zitterbewegung. Intuitively we can speak about a particle at rest only on a macroscopic level; in fact the particle accomplishes microscopic motion around its rest position. In our model the phase space trajectory of this motion is given by the equation

$$\sum_{n=1}^{d} \left( \frac{p_n^2}{2m\omega} + \frac{m\omega}{2} q_n^2 \right) = \hbar$$

where the coordinates $q_n$ measure the deviation from the rest position. This is a $d$-dimensional ellipsoid intersecting the plane $(q_n, p_n), n = 1, 2, \ldots, n$ into an ellipse with area $2\pi \hbar$. When the particle is confined to this plane we call that it has then-th configuration. (This terminology will be useful in our consideration of measurements.) Introducing the complex coordinates

$$\psi_n = (m\omega/2)^{1/2} q_n + i(2m\omega)^{-1/2} p_n$$

we can rewrite Eq.(1) in the form

$$\sum_n \left| \psi_n \right|^2 = \hbar$$
The states of the particle are understood in a classical sense and represented in matrix form by ket-vectors

\[ |\psi\rangle = (\psi_1, \psi_2, \ldots, \psi_d)^T \] (4)

or by bra-vectors

\[ \langle \psi | = (\psi_1^*, \psi_2^*, \ldots, \psi_d^*) \] (5)

Thus our \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) is not vector from some abstract (e.g. Hilbert) space but simply another (complex) form of the phase space coordinates. We call it a state vector, or simply a state. In just so introduced symbols Eq.(3) looks as follows

\[ \langle \psi | \psi \rangle = \hbar \] (6)

and can be considered as an equation of a real 2d-dimensional sphere.

As in classical mechanics the observables are (smooth) real functions on phase space, the latter being identified with the sphere (6). Taking in view that macroscopically \( \hbar \) is very small we shall decompose an arbitrary observable \( A(\psi, \psi^*) \) in series up to the quadratic terms:

\[ A(\psi, \psi^*) = A_0 + \sum_n A_n \psi_n + \sum_n A_n^* \psi_n^* + \sum_{n,m} A_{nm} \psi_n \psi_m + \sum_{n,m} B_{nm} \psi_n^* \psi_m^* + \sum_{n,m} B_{nm}^* \psi_n \psi_m \] (7)

Here

\[ A_{nm}^* = A_{mn} \] (8)

(since the observable is a real function). This means that the matrix \( \hat{A} \) with elements \( A_{nm} \) is a Hermitian one. Every observable \( A = A(.) \) generates a canonical transformation which (in infinitesimal form) looks as follows

\[ \dot{\psi} = \psi - i\epsilon \frac{\partial A}{\partial \psi^*} \] (9)

where \( \epsilon \) is an infinitesimal parameter and the indices are omitted. Using that \( \psi' = \psi + \epsilon \dot{\psi} \) we can write the differential equation

\[ \dot{\psi} = -i \frac{\partial A}{\partial \psi^*} \] (10)

which will be called a generalized equation of motion associated with \( A \). (The dot in \( \dot{\psi} \) denotes differentiation with respect some parameter adjoint to \( A \); for example when \( A = H \) is the energy the corresponding parameter is time.) Now, taking in view that \( A(0) = 0 \) (at rest our observable vanish), and that the canonical transformations should preserve the constraint (6) we come to the relation

\[ A(\psi, \psi^*) = \sum_{n,m} A_{nm} \psi_n^* \psi_m = \langle \psi | \hat{A} | \psi \rangle \] (11)

3
Such observables will be called quantum observables. Let us note that confining ourselves with quantum observables only we should identify some of the vectors representing states: vectors differing by a phase factor describe the same state. With this in mind further we continue to call state vectors simply states. Note that quantum observables are Hermitian forms uniquely determined by their matrices. Moreover one can introduce a Lie algebra structure in the set of all observables determined by the Poisson bracket as a Lie bracket:

\[
\{A, B\}_\psi = i \left( \frac{\partial A}{\partial \psi} \frac{\partial B^*}{\partial \psi^*} - \frac{\partial A^*}{\partial \psi^*} \frac{\partial B}{\partial \psi} \right)
\] (12)

For quantum observables \(A\) and \(B\) one has

\[
\{A, B\}_\psi = i \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle
\] (13)

where the squared brackets stand for the commutator of matrices. The Hermitian matrices form a Lie algebra with the commutator (factored by \(i\)) as a Lie bracket. This equation shows that there exists an isomorphism between the Lie algebra of quantum observables and the Lie algebra of Hermitian matrices. This allows to identify the two algebras, and therefore allows to call Hermitian matrices quantum observables too.

As an example we shall cite the configuration observable \([12]\)

\[
Q = \sum_n n |n\rangle \langle n|
\] (14)

where \(|n\rangle = (0, \ldots, 0, 1, 0, \ldots, 0)^T\), the unit being at \(n\)-th place.

### 3 Quantum measurements

Let us consider a particle prepared in a state \(\psi\). We want to measure the observable \(A(\cdot)\). To do that we bring it into the input of the corresponding measuring device. Then the initial state of the particle is destroyed, and the particle interacting with its own environment would become in another state. What is that state? To answer this question we need some hypothesis about the interaction. We assume that the latter is similar to the process of thermalization of a thermodynamic system interacting with a reservoir (i.e. a big system). As known in such a process the temperature of the system becomes equal to the temperature of the reservoir. In the equilibrium thermodynamics processes like that are described considering the behaviour of an appropriate thermodynamic potential (energy, entropy, free energy, Gibbs potential etc.): excluding entropy (which tends to maximum) all the others tend to minimum when the system is going to equilibrium. Guided by this analogy we formulate our first axiom for the quantum measurement.
In the process of measurement the measured observable tends to minimum. In other words we consider the measured observable (considered as a function on phase space) as a kind of thermodynamic potential. It is easy to see that every critical (i.e. minimal or maximal) value of an observable \( A(\psi, \psi^*) = \langle \psi | \hat{A} | \psi \rangle \) on the sphere \( S = \{ \psi | \langle \psi | \psi \rangle = \hbar \} \) coincides with some eigenvalue of the matrix \( \hat{A} \). (For that it is sufficient to solve the corresponding extremality constraint problem.) Supposing for simplicity that this matrix is non-degenerate we can arrange its eigenvalues as follows: \( a_1 < a_2 < \ldots < a_d \). Denoting the corresponding eigenvectors by \( |a_n\rangle \), \( (n = 1, 2, \ldots, d) \) we have
\[
\hat{A} |a_n\rangle = a_n |a_n\rangle
\] (15)
From the linear algebra we know that the eigenvectors form an orthonormal base in the linear space of all ket-vectors:
\[
\langle a_n | a_m \rangle = \delta_{nm}, \sum_n |a_n\rangle \langle a_n| = 1
\] (16)
where 1 denotes the unit matrix. Hence the spectral decomposition follows:
\[
\hat{A} = \sum_n a_n |a_n\rangle \langle a_n|
\] (17)
One can prove that [13]
\[
a_n = \min \{ A(\psi, \psi^*) | \psi \in S_n \}
\] (18)
where
\[
S_n = \{ \psi \in S | \langle a_m | \psi \rangle = 0, m < n \}
\] (19)
This can be easily interpreted. Namely, if in the process of interaction with the measuring device all states \( \psi \in S \) are admissible, the particle is going in state \( |a_1\rangle \), and the observable \( A(\psi) \) takes the minimal value \( a_1 \). However not all states in \( S \) could be admissible (this depends on the local environment in which the particle moves), and when the set of admissible states is \( S_n, (n > 1) \) the particle is going in state \( |a_n\rangle \) and the observable takes the value \( a_n \). After that the particle come out from the device in the corresponding outlet. (To every eigenvalue there corresponds a unique outlet.)
Obviously every such transition is a random event, and the best what one can do in order to make some prediction about the outcome of the experiment, is to associate some probability to every possible outcome. Hence it is natural to define a random quantity \( A \) with values \( a_1, a_2, \ldots, a_d \). Denote the probability that \( A = a_n \) by \( p_n(\psi) \) when the particle (just before the measurement) is in state \( \psi \). Then the mean value of \( A \) in state \( \psi \) is
\[
\langle A \rangle_\psi = \sum_n a_n p_n(\psi)
\] (20)
We call the so defined random quantity $A$ the *measured quantity* associated to the observable $A(.)$. The problem now is to find the appropriate rule which would allow us to determine these probabilities. Following Fucks [1] we consider the probability of some outcome in Bayesian sense, i.e. as a degree of belief that as a result of the measurement the observable would assume just that value. At first sight there exist many possible assignments, and therefore some subjectivity in our decision, but one can suggest a natural criterion eliminating this subjectivity. This is our second axiom:

**(QM1)** The mean value of the measured quantity associated with some observable in some state is proportional to the value of the observable at that state

\[
\langle A \rangle_\psi = \frac{\hbar}{\psi^*} A(\psi, \psi^*)
\]  

(21)

In other words what is measured corresponds to what is in reality! Now all is quite easy. Using Eqs. (17), (20), (21) we readily obtain

\[
p_n(\psi) = \frac{\hbar}{\psi^*} |\langle a_n | \psi \rangle|^2
\]  

(22)

In particular the probability that the configuration observable $Q$ is equal to $n$ in state $\psi$ is

\[
p_n(\psi) = \frac{\hbar}{\psi^*} |\psi_n|^2
\]  

(23)

We obtained the Born rule.

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