THE EQUIVARIANT COARSE BAUM-CONNES CONJECTURE FOR ACTIONS BY A-T-MENABLE GROUPS

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ABSTRACT. The equivariant coarse Baum-Connes conjecture was firstly introduced by Roe [29] as a unified way to approach both the Baum-Connes conjecture and its coarse counterpart. In this paper, we prove that if an a-T-menable group $\Gamma$ acts properly and isometrically on a bounded geometry metric space $X$ with controlled distortion such that the quotient space $X/\Gamma$ is coarsely embeddable, then the equivariant coarse Baum-Connes conjecture holds for this action. This answers a question posed in [8] affirmatively.

1. Introduction

Over the last century, one of the most significant achievements in mathematics is the Atiyah-Singer index theorem [2]. It provides a topological formula for Fredholm indices of elliptic pseudodifferential operators on closed manifolds, and has profound applications in geometry and topology (see, e.g., [24]).

To obtain finer topological and geometric information of the underlying manifold, people consider lifted operators on the universal cover. These operators are by definition equivariant under deck transformations, and hence can be cooked up to provide an element (called the higher index [7]) in the $K$-theory of the reduced group $C^*$-algebra of the fundamental group [13, 30]. Higher indices reveals more information of the underlying manifolds, and hence it is important to compute $K$-theories of reduced group $C^*$-algebras. The famous Baum-Connes conjecture ([4], see also [5]) provides a practical and systematic approach, which has fruitful applications in analysis, geometry and topology (see, e.g., [3, 32]). It has been verified for a large classes of groups, including a-T-menable groups (i.e., groups with the Haagerup property) [16] and hyperbolic groups [23].

On the other hand, Roe introduced a coarse geometric approach in his pioneering work on higher index theory for elliptic differential operators on open manifolds [27], and formulated a coarse version to the Baum-Connes conjecture [28] (see also [18]). Using Higson’s descent principle [15], the coarse Baum-Connes conjecture also found significant applications in geometry and topology (see, e.g., [29, 34]), for example the Novikov conjecture concerning homotopy invariance of higher signatures and the Gromov-Lawson conjecture concerning positive scalar curvature on manifolds. Over the last three decades, a lot of results on the coarse Baum-Connes conjecture have been achieved [6, 11, 21, 22, 33] after Yu’s groundbreaking work for coarsely embeddable metric spaces [36].

As indicated in [29, Section 5] (see also [34, Section 7]), the Baum-Connes conjecture and its coarse counterpart can be formulated in a unified way using the language of group actions. To be more precise, let us assume that $\Gamma$ is a...
countable discrete group acting on a proper metric space $X$ properly (not necessarily cocompactly) by isometries. There is an equivariant index map (see Section 2.5 for precise definitions)

$$\text{Ind}^\Gamma : \lim_{r \to \infty} K^r_\Gamma(P_r(X)) \longrightarrow K_\ast(C^r_r(X)^\Gamma)$$

for $\ast = 0, 1$ where $K^r_\Gamma(P_r(X))$ is the $\Gamma$-equivariant $K$-homology group of the Rips complex $P_r(X)$ of $X$ with scalar $r$, and $K_\ast(C^r_r(X)^\Gamma)$ is the $K$-theory of the $\Gamma$-equivariant Roe algebra of $X$. The equivariant coarse Baum-Connes conjecture asserts that the equivariant index map $\text{Ind}^\Gamma$ is an isomorphism.

Noted by Roe [29], the equivariant Roe algebra $C^r_r(X)^\Gamma$ is Morita equivalent to the reduced group $C^r$-algebra $C_r^r(\Gamma)$ when the action is cocompact, and hence the equivariant index map $\text{Ind}^\Gamma$ being an isomorphism is just a reformulation of the classic Baum-Connes conjecture. A typical example is the fundamental group $\Gamma$ of a closed manifold $M$ acting on its universal cover $\tilde{M}$ via deck transformations. As mentioned above, a lifted differential operator (from $M$ to $\tilde{M}$) is $\Gamma$-equivariant and hence its higher index sits naturally in the $K$-theory of the equivariant Roe algebra $C^r_r(\tilde{M})^{\Gamma}$, which coincides with the $K$-theory of $C_r^r(\Gamma)$. On the other hand when $\Gamma$ is trivial, $\text{Ind}^\Gamma$ being an isomorphism is nothing but the coarse Baum-Connes conjecture. Therefore, the equivariant coarse Baum-Connes conjecture can be regarded as an “interpolation” between the Baum-Connes conjecture for groups and the coarse Baum-Connes conjecture for metric spaces.

The equivariant coarse Baum-Connes conjecture was studied by couples of mathematicians, and recently has attracted more and more attention. In [31], Shan proved the equivariant index map is injective for torsion free groups acting on simply connected complete Riemannian manifolds with non-positive scalar curvature. Later in [9], the first author together with Wang showed that the equivariant coarse Baum-Connes conjecture holds for group actions on metric spaces with bounded geometry which admit equivariant coarse embeddings into Hilbert space, serving as an equivariant analogue of Yu’s celebrated work [36].

However, a recent ingenious example due to Arzhantseva and Tessera [1] illuminates that a metric space with a group action might not admit a coarse embedding into Hilbert space even if both the group and the quotient space are coarsely embeddable. Inspired by their example, the first author together with Wang and Yu [10] managed to show that the equivariant index map remains injective for group actions with bounded distortion when both the group and the quotient space admit coarse embeddings into Hilbert space.

Along this way, very recently the first author together with Deng and Wang [8] proved that the equivariant coarse Baum-Connes conjecture holds for amenable groups acting on metric spaces such that all orbits are equivariantly uniformly coarsely equivalent and quotient spaces are coarsely embeddable. They also asked a question whether the result remains true when the involved group is a-T-menable, inspired by Higson and Kasparov’s significant result that the Baum-Connes conjecture holds for a-T-menable groups [16]. However, the techniques in [8] are no longer applicable for a-T-menable groups due to an obstruction in coarse $K$-amenability (see [8, Theorem 1.2]).

In this paper, we use a different approach to bypass the issue of coarse $K$-amenability and answer the question posed in [8] affirmatively. More precisely, we
show that the equivariant coarse Baum-Connes conjecture holds for a-T-menable groups acting on metric spaces with controlled distortion such that quotient spaces are coarsely embeddable. To state our main result, let us introduce some notions first (see Section 2 for more details).

For a metric space \((X, d)\) with a \(\Gamma\)-action, a subset \(D \subseteq X\) is called a fundamental domain if \(X\) can be decomposed into the disjoint union of \(\Gamma\)-orbits for points in \(D\). An action is said to have controlled distortion if there exists a fundamental domain \(D \subseteq X\) such that the family of orbit maps \(\{O_x : \Gamma \rightarrow \Gamma x, \gamma \mapsto \gamma x\}_{x \in D}\) are uniformly coarsely equivalent.

The following is the main result of this paper:

**Theorem 1.1.** Let \(\Gamma\) be a countable discrete group acting properly and isometrically on a discrete metric space \(X\) with bounded geometry. If the action has controlled distortion, the quotient space \(X/\Gamma\) admits a coarse embedding into Hilbert space and \(\Gamma\) is a-T-menable, then the equivariant index map

\[
\text{Ind}^\Gamma : \lim_{r \to \infty} K^\Gamma_r(P_r(X)) \rightarrow K_*(C^*(X)^\Gamma)
\]

is an isomorphism for \(* = 0, 1\).

Note that when \(\Gamma\) is trivial, Theorem 1.1 recovers Yu’s celebrated result [36] on the coarse Baum-Connes conjecture for coarsely embeddable metric spaces. On the other hand, note that when the action is cocompact then it automatically has controlled distortion, and hence Theorem 1.1 recovers Higson and Kasparov’s famous result [16] on the Baum-Connes conjecture for a-T-menable groups. Therefore, Theorem 1.1 can be regarded as a combination of [16] and [36] in virtue of the language of group actions.

Readers might already notice that the hypothesis of Theorem 1.1 is slightly different from that of [8, Theorem 1.1] except for the requirement on the group. More precisely, in Theorem 1.1 we assume that the action has controlled distortion, while [8, Theorem 1.1] requires that all orbits are equivariantly uniformly coarsely equivalent. As revealed in Section 2.2 having controlled distortion can be deduced from equivariantly uniformly coarse equivalence of orbits, which means that our hypothesis is (at least formally) weaker than the one used in [8, Theorem 1.1]. In Section 2.2 we also provide some analysis on the relations between our hypothesis of Theorem 1.1 and those used in [9, 10].

The proof of Theorem 1.1 follows the machinery from [34, Chapter 12], which was originally invented by Yu in [36]. Roughly speaking, we use an equivariant version of the coarse Mayer-Vietoris argument to chop the space and decompose the associated algebras preserving group actions. This leads to the reduction of the proof for Theorem 1.1 to the case of sequences of cocompact actions with block-diagonal operators thereon (Corollary 3.8). Then we construct equivariant twisted Roe and localisation algebras (Definition 4.9 and 4.10), and index maps in terms of the Bott-Dirac operators (Proposition 5.1). Finally, we prove an equivariant version of the local isomorphism (Theorem 6.1) thanks to (a family version of) the Baum-Connes conjecture with coefficients verified for a-T-menable groups [16], and conclude the proof via a canonical diagram chasing argument.

\[\text{In fact, we think that controlled distortion with other assumptions are also sufficient to prove [8, Theorem 1.1].}\]
We would like to highlight the difference between our approach and the one used in [8]. Although both of them consult Yu’s machinery, the constructions of equivariant twisted algebras and index maps are different. More precisely in the approach of [8], index maps (usually called the Bott and Dirac maps) are defined on dense subalgebras in the form of asymptotic morphisms, and therefore one needs a coarse K-amenability result to extend these maps. While in the current paper, we use a coarse Mayer-Vietoris argument to chop the space and decompose the associated algebras at first, which allows us to construct index maps directly on the K-theories of the whole algebras (see Proposition 5.7). An ingenious “space for speed” argument from [34] is crucial to bypass the issue of coarse K-amenability and accomplish the job.

The paper is organised as follows. In Section 2, we collect necessary preliminaries and provide comparisons for different geometric hypotheses used in [8, 9, 10]. Section 3 is devoted to the reduction of the proof for Theorem 1.1 to the case of sequences of cocompact actions. We introduce equivariant twisted algebras in Section 4, and construct index maps in Section 5. In Section 6, we prove that the equivariant twisted Roe and localisation algebras have the same K-theory using a family version of the Baum-Connes conjecture with coefficients for a-T-menable groups essentially from [16], and finally conclude the proof in Section 7.

We also provide an appendix on a precise statement of the family version of the Baum-Connes conjecture with coefficients for a-T-menable groups which we need to conclude Theorem 6.1. This result is well-known to experts, while we cannot find an explicit statement or proof in literature. For convenience to readers and also for completeness, we provide a detailed proof based on ideas from [16] and [36].

2. Preliminaries

In this section, we recall some notions and definitions.

2.1. Notions from coarse geometry. Here we collect several basic notions.

**Definition 2.1.** Let $(X, d)$ be a metric space and $R > 0$.

1. For $x \in X$, denote the closed $R$-ball $B(x, R) := \{y \in X : d(x, y) \leq R\}$. For $A \subseteq X$, denote its $R$-neighbourhood by $N_R(A) := \{y \in X : d(y, A) \leq R\}$.
2. A subset $A \subseteq X$ is called bounded if its diameter $\sup\{d(x, y) : x, y \in A\}$ is finite; $A$ is called a net in $X$ if there exists a $c > 0$ such that $N_c(A) = X$.
3. $(X, d)$ is called proper if any bounded closed subset in $X$ is compact.
4. If $(X, d)$ is discrete, we say that $(X, d)$ has bounded geometry if $\sup_{x \in X} \#B(x, R)$ is finite for any $R > 0$. In the general case, we say that $(X, d)$ has bounded geometry if it contains a discrete net with bounded geometry.

**Definition 2.2.** A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is called a coarse embedding if there exist two proper non-decreasing functions $\rho_+, \rho_- : [0, \infty) \to [0, \infty)$ such that

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

holds for any $x, y \in X$. The map $f$ is called a coarse equivalence if it is a coarse embedding and the image of $f$ is a net in $Y$. We say that $X$ and $Y$ is coarsely equivalent if there exists a coarse equivalence $f : X \to Y$. 

A family of maps \( \{f_i : X_i \to Y_i\}_{i \in I} \) between metric spaces are called \textit{uniformly coarsely equivalent} if \( f_i \) is a coarse equivalence with the same control functions \( \rho_+ \) and \( \rho_- \), and there exists a \( c > 0 \) such that the \( c \)-neighbourhood of the image of \( f_i \) coincides with \( Y_i \) for each \( i \in I \).

For a proper function \( \rho : [0, \infty) \to [0, \infty) \) (which is not necessarily monotonically increasing), we denote
\[
\rho^{-1}(R) := \sup\{S \in [0, \infty) : \rho(S) \leq R\}, \quad \text{for } R \in [0, \infty).
\]

Now we move to the case of group actions.

\textbf{Definition 2.3.} Let \( \Gamma \) be a countable discrete group acting on a proper metric space \((X, d)\). The action is called \textit{proper} if for any compact set \( K \subseteq X \), the set \( \{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\} \) is finite. The action is called \textit{isometric} if \( d(x, y) = d(\gamma x, \gamma y) \) for all \( \gamma \in \Gamma \) and \( x, y \in X \).

To simplify the notation, we say that a proper metric space \((X, d)\) is a \( \Gamma \)-space if the \( \Gamma \)-action is proper and isometric. For a \( \Gamma \)-space \((X, d)\) and \( x \in X \), the \textit{orbit map} at \( x \) is defined to be \( \Gamma x \) and the \textit{orbit map at} \( x \) is defined by:
\[
O_x : \Gamma \to \Gamma x, \quad \gamma \mapsto \gamma x \quad \text{for} \quad \gamma \in \Gamma.
\]

\textbf{Definition 2.4.} Let \( \Gamma \) be a countable discrete group. A map \( f : X \to Y \) between two \( \Gamma \)-spaces is called \textit{equivariant} if for any \( x \in X \) and \( \gamma \in \Gamma \), we have \( f(\gamma x) = \gamma f(x) \). The map \( f \) is called an \textit{equivariantly coarse equivalence} if \( f \) is equivariant and coarsely equivalent. A family of maps \( \{f_i : X_i \to Y_i\}_{i \in I} \) between \( \Gamma \)-spaces is called an \textit{equivariantly uniformly coarse equivalence} if each \( f_i \) is \( \Gamma \)-equivariant and the family \( \{f_i\}_{i \in I} \) is a uniformly coarse equivalence.

A family of \( \Gamma \)-spaces \( X_i \) for \( i \in I \) are called \textit{equivariantly uniformly coarsely equivalent} if for any \( i, j \in I \) there exists an equivariant map \( f_{ij} : X_i \to X_j \) such that the family \( \{f_{ij} : X_i \to X_j\}_{i, j \in I} \) is a uniformly coarse equivalence.

Recall that for a \( \Gamma \)-space \((X, d)\), a subset \( D \subseteq X \) is called a \textit{fundamental domain} if \( X \) can be decomposed into the disjoint union of \( \Gamma \)-orbits of points in \( D \). An action is said to have \textit{controlled distortion} if there exists a fundamental domain \( D \subseteq X \) such that the family of orbit maps \( \{O_x : x \in D\} \) are uniformly coarsely equivalent. In this case, we also say that the action has controlled distortion with respect to \( D \).

Recall from [10] that for a \( \Gamma \)-space \((X, d)\), the action is said to have \textit{bounded distortion} if there exists a fundamental domain \( D \) such that for any \( \gamma \in \Gamma \), we have
\[
\sup_{y \in D} d(\gamma y, y) < +\infty.
\]

In this case, we also say that the action has bounded distortion with respect to \( D \). It follows directly from definitions that having controlled distortion implies bounded distortion.

2.2. Comparing different geometric hypotheses. In this subsection, we would like to compare the geometric hypothesis of Theorem [11] with those used in the main results of [8, 9, 10]. For convenience to readers, we record these hypotheses chronologically as follows. Assume that \((X, d)\) is a \( \Gamma \)-space.

1. \textit{hypothesis of [9] Theorem 1.2}: \( X \) admits a \( \Gamma \)-equivariant coarse embedding into Hilbert space.


(2) hypothesis of [10] Theorem 1.2: both $\Gamma$ and $X/\Gamma$ are coarsely embeddable, and the action has bounded distortion.

(3) hypothesis of [8] Theorem 1.1: $\Gamma$ is amenable and $X/\Gamma$ is coarsely embeddable, and the orbit spaces are equivariantly uniformly equivalent.

(4) hypothesis of Theorem 1.1 (the current paper): $\Gamma$ is a-T-menable and $X/\Gamma$ is coarsely embeddable, and the action has controlled distortion.

Note that the conclusion of [10] Theorem 1.2 says that the equivariant index map is injective, while the rest conclude isomorphisms.

The following lemma shows that our hypothesis of Theorem 1.1 is (at least formally) weaker than the one of [8] Theorem 1.1.

Lemma 2.5. For a $\Gamma$-space $(X, d)$, if all $\Gamma$-orbits are equivariantly uniformly coarsely equivalent, then the action has controlled distortion.

Proof. Fix a point $x_0 \in X$ and a fundamental domain $D' \subseteq X$ with $x_0 \in D'$. Since the action is proper, it follows from the Milnor-Švarc lemma that the orbit map $O_{x_0} : \Gamma \rightarrow \Gamma x_0$ is a coarsely equivalent. Now for any $y \in D'$, by assumption there exists an equivariant map $\psi_y : \Gamma x_0 \rightarrow \Gamma y$ such that $\{\psi_y : y \in D'\}$ is a uniformly coarse equivalence. Setting $\hat{y} := \psi_y(x_0)$, the equivariance of $\psi_y$ implies that $\psi_y$ has the following form:

$$\psi_y : \Gamma x_0 \rightarrow \Gamma \hat{y}, \quad \gamma x_0 \mapsto \gamma \hat{y} \quad \text{for} \, \gamma \in \Gamma.$$ 

Taking $D := \{\hat{y} : y \in D'\}$, it is clear that $D$ is also a fundamental domain. Moreover, for $\hat{y} \in D$ the orbit map $O_{\hat{y}}$ coincides with $\psi_y \circ O_{x_0}$. Hence we conclude the proof. \qed

Now we turn to the relation with [9] and [10]. The general situation is still unclear while we try to offer some analysis in special cases.

To compare with [10], note that having controlled distortion implies bounded distortion. We show in the following that for certain special cases, these two notions are “almost” the same.

Example 2.6. Let $N, Q$ be countable discrete groups with an action $\alpha : Q \rightarrow \text{Aut}(N)$. Equip the associated semi-direct product $N \rtimes Q$ with a proper length function $\ell$, which derives a left-invariant proper metric $d$. Consider the action of $N$ on $N \rtimes Q$ by left multiplication, which is clearly isometric and proper.

Assume that the action has bounded distortion with respect to the fundamental domain $Q$. Then obviously there exists a proper function $\rho_+: [0, \infty) \rightarrow [0, \infty)$ such that $d(hq, q) \leq \rho_+(\ell(h))$ for any $h \in N$ and $q \in Q$. Hence for any $h \in N$ and $q \in Q$, we also have:

$$\ell(h) = d(q^{-1}hq \cdot q^{-1}, q^{-1}) \leq \rho_+(\ell(q^{-1}hq)) = \rho_+(d(hq, q)),$$

which implies that the action has controlled distortion. Finally we remark that it is unclear to the authors that whether the same result holds if the action has bounded distortion with respect to an arbitrary fundamental domain.

To compare with [9], recall that a recent ingenious example due to Arzhantseva and Tessera [1] illuminates that a $\Gamma$-space might not admit a coarse embedding into Hilbert space even if both the group and the quotient space are coarsely embeddable. However as we show below, the situation often gets better under the hypothesis of controlled distortion.
Lemma 2.7. Let \((X, d)\) be a \(\Gamma\)-space with the quotient map \(\pi : X \to X/\Gamma\). Equip \(\Gamma\) with a proper left-invariant metric \(d_\Gamma\) and \(X/\Gamma\) with the quotient metric \(d_q\). Assume that there exists a fundamental domain \(D \subseteq X\) such that \(\pi|_D : D \to \pi(D)\) is a coarse equivalence and the action has controlled distortion with respect to \(D\). Then \(X\) is coarsely equivalent to the product metric \(\Gamma \times (X/\Gamma)\) equipped with the product metric \(d_\Gamma \times d_q\).

Consequently, if additionally both \(\Gamma\) and \(X/\Gamma\) are coarsely embeddable, then so is \(X\).

Proof. Assume that \(D = \{x_\lambda : \lambda \in \Lambda\}\) and \(\pi|_D\) is a coarse equivalence with controlled functions \(\rho_+\) and \(\rho_-\). Also assume that the orbits maps \(\{O_{\lambda_\gamma} : \lambda \in \Lambda\}\) are uniformly coarsely equivalent with the same controlled functions. For any two points \(\gamma_1x_{\lambda_1}\) and \(\gamma_2x_{\lambda_2}\) in \(X\), note that

\[
d(\gamma_1x_{\lambda_1}, \gamma_2x_{\lambda_2}) = d(x_{\lambda_1}, \gamma_1^{-1}\gamma_2x_{\lambda_2}) \leq d(x_{\lambda_1}, x_{\lambda_2}) + d(\gamma_1^{-1}\gamma_2x_{\lambda_2}, x_{\lambda_2})
\]

\[
\leq \rho_+^{-1}(\rho_+(\pi(x_{\lambda_1})), \pi(x_{\lambda_2})) + \rho_+(d(\gamma_1, \gamma_2)).
\]

On the other hand, given \(R > 0\) and assume that \(d(\gamma_1x_{\lambda_1}, \gamma_2x_{\lambda_2}) \leq R\). Then

\[
d_q(\pi(x_{\lambda_1})), \pi(x_{\lambda_2})) = d_q(\pi(\gamma_1x_{\lambda_1}), \pi(\gamma_2x_{\lambda_2})) \leq R,
\]

which implies that \(d(x_{\lambda_1}, x_{\lambda_2}) \leq \rho_+(R)\). Moreover, we have

\[
d(x_{\lambda_2}, \gamma_1^{-1}\gamma_2x_{\lambda_2}) \leq d(x_{\lambda_1}, x_{\lambda_2}) + d(\gamma_1^{-1}\gamma_2x_{\lambda_2}, x_{\lambda_2}) \leq \rho_+(R) + R,
\]

which implies that \(d_\Gamma(\gamma_1, \gamma_2) \leq \rho_+^{-1}(\rho_+(R) + R)\).

Combining the above two paragraphs, we conclude the proof. \(\square\)

Remark 2.8. It is unclear to the authors whether Lemma 2.7 holds or not without the assumption that \(\pi|_D\) is a coarse equivalence. On the other hand, note that this condition holds for many examples, including the semi-direct product case studied in Example 2.6.

More precisely, it is obvious that the quotient map \(N \rtimes Q \to Q\) restricted to the fundamental domain \(Q\) is a coarse equivalence. As a consequence to Lemma 2.7, we obtain that \(N \rtimes Q\) is coarsely embeddable if both \(N\) and \(Q\) are coarsely embeddable and the action of \(N\) on \(N \rtimes Q\) by left multiplication has bounded distortion with respect to \(Q\). Finally, note that the example constructed in [11] has the form of semi-direct products. It follows by straightforward calculations that their example does not have bounded distortion, alternatively by combining the fact that it is not coarsely embeddable together with the analysis above.

2.3. (Equivariant) Roe algebras. Now we introduce the notion of Roe algebras and their equivariant counterparts.

For a proper metric space \((\mathbb{Z}, d)\), recall that a \(\mathbb{Z}\)-module is a non-degenerate \(\ast\)-representation \(\phi : C_0(\mathbb{Z}) \to \mathcal{B}(\mathcal{H}_Z)\) where \(\mathcal{H}_Z\) is some infinite-dimensional separable Hilbert space. We also say that \(\mathcal{H}_Z\) is a \(\mathbb{Z}\)-module if the representation is clear from the context, and simply write \(f\) as a bounded linear operator on \(\mathcal{H}_Z\) instead of \(\phi(f)\) for \(f \in C_0(\mathbb{Z})\). A \(\mathbb{Z}\)-module is called ample if no non-zero element of \(C_0(\mathbb{Z})\) acts as a compact operator on \(\mathcal{H}_Z\).

Definition 2.9. Let \(\mathcal{H}_Z\) be an ample module of a proper metric space \((\mathbb{Z}, d)\).

1. For \(T \in \mathcal{B}(\mathcal{H}_Z)\), its support \(\text{supp}(T)\) is defined to be the complement of the set of points \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) for which there exist \(f, g \in C_0(\mathbb{Z})\) satisfying:

\[gTf = 0, \quad f(x) \neq 0 \quad \text{and} \quad g(y) \neq 0.\]
To simplify the notation, we call an ample covariant locally free admissible $\mathbb{Z}$-module. e.g. we introduce the notion of equivariant Roe algebras (see, Definition 2.12).

For a proper metric space $Z$ and an ample $Z$-module $\mathcal{H}_Z$, the algebraic Roe algebra $C[\mathcal{H}_Z]$ of $\mathcal{H}_Z$ is defined to be the $\ast$-algebra of locally compact finite propagation operators on $\mathcal{H}_Z$, and the Roe algebra $C'(\mathcal{H}_Z)$ of $\mathcal{H}_Z$ is defined to be the norm-closure of $C[\mathcal{H}_Z]$ in $\mathcal{B}(\mathcal{H}_Z)$.

It is a standard result that the Roe algebra $C'(\mathcal{H}_Z)$ does not depend on the chosen ample module $\mathcal{H}_Z$ up to $\ast$-isomorphisms, hence denoted by $C(Z)$ and called the Roe algebra of $Z$. Furthermore, $C'(Z)$ is a coarse invariant of the metric space $Z$ (up to non-canonical $\ast$-isomorphisms), and their $K$-theories are coarse invariants up to canonical isomorphisms (see, e.g., [28]).

Now we move to the equivariant case. Here we follow the setting of [34, Section 4.5 and 5.2], which is in fact equivalent to those in [8, 9, 29].

For a proper $\Gamma$-space $(Z, d)$, we define a $\Gamma$-action on $C_0(Z)$ by 

$$(\gamma \cdot f)(x) = f(\gamma^{-1}x)$$

for all $\gamma \in \Gamma$ and $f \in C_0(Z)$. For a unitary representation $U : \Gamma \to \mathcal{U}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, we denote the adjoint

$${\text{Ad}}_{U_{\gamma}}(T) = U_{\gamma}TU_{\gamma}^* =: \gamma \cdot T$$

for $\gamma \in \Gamma$ and $T \in \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called $(\Gamma)$-invariant if $\gamma \cdot T = T$ for any $\gamma \in \Gamma$. Denote the set of all $\Gamma$-invariant operators by $\mathcal{B}(\mathcal{H})^\Gamma$.

The following notion essentially comes from [34, Section 4.5].

**Definition 2.11.** Let $(Z, d)$ be a proper $\Gamma$-space, and $(\mathcal{H}_Z, \phi)$ be an ample $Z$-module.

1. $\mathcal{H}_Z$ is called covariant if it is equipped with a unitary representation $U : \Gamma \to \mathcal{U}(\mathcal{H}_Z)$ such that for any $\gamma \in \Gamma$, we have:

$$\phi(\gamma \cdot f) = \gamma \cdot \phi(f) \left(= U_{\gamma}\phi(f)U_{\gamma}^* \right).$$

2. $\mathcal{H}_Z$ is called locally free if for any finite subgroup $F$ of $\Gamma$ and any $F$-invariant Borel subset $E$ of $Z$, there exists a Hilbert space $\mathcal{H}_E$ equipped with the trivial representation of $F$ such that $\chi_E\mathcal{H}_Z$ and $\ell^2(F) \otimes \mathcal{H}_E$ are isomorphic as $F$-representations.

To simplify the notation, we call an ample covariant locally free $Z$-module an admissible $Z$-module.

It follows from [34, Lemma 4.5.5] that admissible modules always exist. Now we introduce the notion of equivariant Roe algebras (see, e.g., [34, Definition 5.2.1]).

**Definition 2.12.** Let $(Z, d)$ be a proper $\Gamma$-space, and $(\mathcal{H}_Z, \phi)$ be an admissible $Z$-module. The algebraic equivariant Roe algebra $C[\mathcal{H}_Z]^\Gamma$ of $\mathcal{H}_Z$ is defined to be the $\ast$-algebra of locally compact finite propagation $\Gamma$-invariant operators on $\mathcal{H}_Z$, and the equivariant Roe algebra $C'(\mathcal{H}_Z)^\Gamma$ of $\mathcal{H}_Z$ is defined to be the norm-closure of $C[\mathcal{H}_Z]^\Gamma$ in $\mathcal{B}(\mathcal{H}_Z)$. 
Analogous to the case of Roe algebras, it follows from [34, Theorem 5.2.6] that the equivariant Roe algebra $C^*(\mathcal{H}_Z)^\Gamma$ does not depend on the chosen admissible module $\mathcal{H}_Z$ up to $\ast$-isomorphisms, hence denoted by $C^*(Z)^\Gamma$ and called the equivariant Roe algebra of $Z$. Moreover, equivariant Roe algebras are invariant under equivariant coarse equivalences (up to non-canonical $\ast$-isomorphisms), and their $K$-theories are equivariant coarse invariants up to canonical isomorphisms.

We would also like to point out an important observation from [29, Lemma 5.14] that the equivariant Roe algebra is compact (see also [34, Theorem 5.3.2]). This allows us to include the setting of the classical Baum-Connes conjecture to the scope of equivariant coarse Baum-Connes conjecture which will be introduced below.

2.4. Equivariant $K$-homology groups and index maps. Let us briefly recall the notion of equivariant $K$-homology groups introduced by Kasparov [20].

**Definition 2.13.** Let $(Z,d)$ be a proper $\Gamma$-space. For $i = 0$ and 1, the equivariant $K$-homology group $K^i_0(Z) = KK^i_1(C_0(Z), C)$ is generated by certain cycles modulo a certain equivalence relation:

1. Each cycle for $K^i_0(Z)$ is a triple $(\mathcal{H}, \phi, F)$, where $\phi : C_0(Z) \to B(\mathcal{H})$ is a covariant $\ast$-representation and $F \in B(\mathcal{H})$ such that $\phi(f)F - F\phi(f)$, $\phi(f)(FF^* - I)$ and $\phi(f)(F^*F - I)$ are compact operators for all $f \in C_0(Z)$ and $\gamma \in \Gamma$;
2. Each cycle for $K^i_1(Z)$ is a triple $(\mathcal{H}, \phi, F)$, where $\phi : C_0(Z) \to B(\mathcal{H})$ is a covariant $\ast$-representation and $F \in B(\mathcal{H})$ is self-adjoint such that $\phi(f)(F^2 - I)$ and $\phi(f)F - F\phi(f)$ are compact for all $f \in C_0(Z)$ and $\gamma \in \Gamma$.

In both cases, the equivalence relation on cycles is given by homotopy of the operator $F$.

Now we define the equivariant index map for a $\Gamma$-space $Z$:

$$\text{Ind}^\Gamma : K^i_0(Z) \to K_*(C^*(Z)^\Gamma).$$

Note that every class in $K^i_0(Z)$ can be represented by a cycle $(\mathcal{H}, \phi, F)$ such that $(\mathcal{H}, \phi)$ is an admissible Z-module (see, e.g., [22, Section 3]).

Consider such a cycle $(\mathcal{H}, \phi, F)$. Take a locally finite, $\Gamma$-equivariant and uniformly bounded open cover $\{U_i\}_{i \in I}$ of $Z$, and let $\{\psi_i\}_{i \in I}$ be a $\Gamma$-equivariant partition of unity subordinate to $\{U_i\}_{i \in I}$. Define

$$F' = \sum_{i \in I} \phi(\sqrt{\psi_i})F\phi(\sqrt{\psi_i}),$$

where the sum converges in strong topology. It is clear that the cycle $(\mathcal{H}, \phi, F')$ is equivalent to $(\mathcal{H}, \phi, F)$ in $K^i_0(Z)$ and $F'$ has finite propagation. Hence $F'$ is a multiplier of $C^*(Z)^\Gamma$ and invertible modulo $C^*(Z)^\Gamma$. Applying the boundary map, $F'$ gives rise to an element in $K_0(C^*(Z)^\Gamma)$, denoted by $\partial([F'])$. We define the equivariant index map

$$\text{Ind}^\Gamma : K^i_0(Z) \to K_0(C^*(Z)^\Gamma)$$

by

$$\text{Ind}^\Gamma([(\mathcal{H}, \phi, F)]) = \partial([F'])$$
with the notation as above. Similarly, we can define the equivariant index map for the $K_1$-case:

$$\text{Ind}^\Gamma : K^\Gamma_1(Z) \longrightarrow K_1(C^*(Z)^\Gamma).$$

2.5. The equivariant coarse Baum–Connes conjecture. In this subsection, we shall recall the equivariant coarse Baum–Connes conjecture for a discrete metric space with bounded geometry.

Let $(X, d)$ be a discrete metric space with bounded geometry. For $r > 0$, the Rips complex $P_r(X)$ with scale $r$ is the simplicial complex with vertices $X$ such that a finite subset $Y = \{x_0, x_1, \cdots, x_n\} \subseteq X$ spans a simplex if and only if $\text{diam}(Y) \leq r$.

The Rips complex $P_r(X)$ is endowed with the following spherical metric. On each connected component of $P_r(X)$, the spherical metric is the maximal metric whose restriction on each simplex $\Delta := (\sum_{i=0}^n t_i x_i : t_i \geq 0, \sum_{i=0}^n t_i = 1)$ is the metric obtained by identifying $\Delta$ with $S^n_+$ via the map

$$\sum_{i=0}^n t_i x_i \mapsto \left(\frac{t_0}{\sqrt{\sum_{i=0}^n t_i^2}}, \frac{t_1}{\sqrt{\sum_{i=0}^n t_i^2}}, \cdots, \frac{t_n}{\sqrt{\sum_{i=0}^n t_i^2}}\right),$$

where $S^n_+ := \{(s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1} : s_i \geq 0, \sum_{i=0}^n s_i^2 = 1\}$ is endowed with the standard Riemannian metric. If $y_0$ and $y_1$ belong to two different connected components $Y_0$ and $Y_1$ of $P_r(X)$, respectively, we define

$$d(y_0, y_1) = \min \{d(y_0, x_0) + d_X(x_0, x_1) + d(x_1, y_1) : x_0 \in X \cap Y_0, x_1 \in X \cap Y_1\}.$$

Assume further that $X$ is a $\Gamma$-space. Then each Rips complex $P_r(X)$ also admits a $\Gamma$-action defined by

$$\gamma \cdot \sum_{i=1}^k c_i x_i = \sum_{i=1}^k c_i (\gamma x_i)$$

for all $\sum_{i=1}^k c_i x_i \in P_r(X)$ and $\gamma \in \Gamma$. It is clear that the action is proper and isometric, and hence $P_r(X)$ is a proper $\Gamma$-space as well.

The following is the main conjecture we study in this paper:

Conjecture 2.14 (The equivariant coarse Baum–Connes conjecture). Let $X$ be a discrete metric space with bounded geometry and $\Gamma$ be a countable discrete group acting on $X$ properly by isometries. Then the equivariant index map

$$\text{Ind}^\Gamma : \lim_{r \to \infty} K^\Gamma_1(P_r(X)) \longrightarrow \lim_{r \to \infty} K_1(C^*(P_r(X))^\Gamma) \cong K_1(C^*(X)^\Gamma)$$

is an isomorphism for $* = 0, 1$.

As noticed in Introduction, the equivariant coarse Baum–Connes conjecture includes both the classical Baum–Connes conjecture for groups and the coarse Baum–Connes conjecture for metric spaces as special cases. More precisely, it coincides with the Baum–Connes conjecture when the action is cocompact, and when the action is trivial it is nothing but the coarse Baum–Connes conjecture (see, e.g., [34, Section 7]).

We reformulate the following significant result due to Higson and Kasparov [16]. Recall that a discrete group $\Gamma$ is said to be $a$-$T$-menable [12] (or has the Haagerup
property) if it admits a metrically proper action on some Hilbert space by affine isometries.

**Proposition 2.15** ([16]). Let $\Gamma$ be a countable discrete $a$-T-menable group acting properly and cocompactly on a proper metric space $X$ with bounded geometry by isometries. Then the equivariant index map

$$\text{Ind}^\Gamma : \lim_{r \to \infty} K_\Gamma^r(P_r(X)) \to K_\Gamma(C^*(X))$$

is an isomorphism for $* = 0, 1$.

### 2.6. Equivariant localisation algebras

Here we introduce another version on the $K$-homology group, which was originally introduced by Yu [35].

**Definition 2.16.** Let $(Z, d)$ be a proper $\Gamma$-space. The **equivariant localisation algebra** $C_\Gamma^L(Z)$ is the supremum norm closure of the algebra of all bounded and uniformly norm-continuous functions $f : [1, +\infty) \to C^*(Z)$ such that $\text{prop}(f(t)) \to 0$ and $t \to \infty$.

We recall that for a proper $\Gamma$-space $Z$, there exists an equivariant local index map which originally comes from [35] (with an equivariant formulation from [10]):

$$\text{Ind}^\Gamma_L : K^\Gamma(Z) \to K_\Gamma(C_\Gamma^L(Z))$$

Moreover, we have the following result which is an equivariant analogue of [35, Theorem 3.2] and [26, Theorem 3.4].

**Proposition 2.17.** Let $Z$ be a proper $\Gamma$-space. Then the equivariant local index map

$$\text{Ind}^\Gamma_L : K^\Gamma(Z) \to K_\Gamma(C_\Gamma^L(Z))$$

is an isomorphism for $* = 0, 1$.

On the other hand, note that for a proper $\Gamma$-space $Z$ there is a natural evaluation-at-one map $\text{ev} : C_\Gamma^L(Z) \to C^*(Z)$ given by

$$\text{ev}(f) = f(1)$$

for all $f \in C_\Gamma^L(Z)$. This is clearly a $*$-homomorphism, and thus induces a homomorphism $\text{ev}_* : K_\Gamma(C_\Gamma^L(Z)) \to K_\Gamma(C^*(Z))$ on $K$-theories. Furthermore, we have the following commutative diagram:

$$
\begin{array}{ccc}
K_\Gamma^L(Z) & \xrightarrow{\text{Ind}^\Gamma_L} & K_\Gamma(C_\Gamma^L(Z)) \\
\text{ev}_* \downarrow & & \downarrow \text{ev}_* \\
& & K_\Gamma(C^*(Z)).
\end{array}
$$

Consequently, for a discrete $\Gamma$-space $X$ with bounded geometry, in order to prove that equivariant coarse Baum-Connes conjecture holds for $X$ it suffices to show that the map

$$\text{ev}_* : \lim_{r \to \infty} K_\Gamma(C_\Gamma^L(P_r(X))) \to \lim_{r \to \infty} K_\Gamma(C^*(P_r(X))) \cong K_\Gamma(C^*(X))$$
induced by the evaluation-at-one map is an isomorphism for \( \ast = 0,1 \).

3. Reduction

The aim of this section is to reduce the proof of Theorem \([14]\) to the case of sequences of metric spaces with proper and cocompact group actions. Let us recall the following notion:

**Definition 3.1 (\([19]\))**. Let \( Z \) be a proper metric space and \( A, B \) be closed subsets in \( Z \) such that \( A \cup B = Z \). We say that \((A, B)\) is \( \omega \)-excisive if for any \( r > 0 \) there exists \( s > 0 \) such that

\[
\mathcal{N}_s(A) \cap \mathcal{N}_s(B) \subseteq \mathcal{N}_s(A \cap B).
\]

Throughout the rest of this section, let us assume that \((X, d)\) is a proper \( \Gamma \)-space with bounded geometry. Denote by \( \pi : X \to X/\Gamma \) the quotient map and equip \( X/\Gamma \) with the quotient metric \( \bar{d} \).

**Lemma 3.2.** For any \( Z \subseteq X/\Gamma \) and \( r > 0 \), we have \( \mathcal{N}_r(\pi^{-1}(Z)) = \pi^{-1}(\mathcal{N}_r(Z)) \).

**Proof.** For any \( x \in \mathcal{N}_r(\pi^{-1}(Z)) \), there exists \( y \in \pi^{-1}(Z) \) such that \( d(x, y) \leq r \). Hence we have \( \bar{d}(\pi(x), \pi(y)) \leq d(x, y) \leq r \), which implies that \( x \in \pi^{-1}(\mathcal{N}_r(Z)) \).

On the other hand, given \( x \in \pi^{-1}(\mathcal{N}_r(Z)) \) we have \( \pi(x) \in \mathcal{N}_r(Z) \). Hence there exists \( z \in Z \) such that \( \bar{d}(\pi(x), z) \leq r \). Assume that \( z = \pi(y) \) for some \( y \in X \). Hence,

\[
r \geq \bar{d}(\pi(x), z) = \inf_{\gamma, \gamma' \in \Gamma} d(\gamma x, \gamma' y) = \inf_{\gamma} \bar{d}(x, \gamma y)
\]

where we use that the action is by isometries in the third equality. Since \( X \) has bounded geometry and \( \gamma y \in \pi^{-1}(Z) \) for each \( \gamma \in \Gamma \), we obtain that \( x \in \mathcal{N}_r(\pi^{-1}(Z)) \). \( \square \)

**Lemma 3.3.** Suppose \((A, B)\) is an \( \omega \)-excisive closed cover of \( X/\Gamma \). Then \((\pi^{-1}(A), \pi^{-1}(B))\) is an \( \omega \)-excisive closed cover of \( X \).

**Proof.** Given \( r > 0 \) there exists \( s > 0 \) such that \( \mathcal{N}_s(A) \cap \mathcal{N}_s(B) \subseteq \mathcal{N}_s(A \cap B) \). Hence by Lemma 3.2 we have:

\[
\mathcal{N}_r(\pi^{-1}(A)) \cap \mathcal{N}_r(\pi^{-1}(B)) = \pi^{-1}(\mathcal{N}_s(A)) \cap \pi^{-1}(\mathcal{N}_s(B)) = \pi^{-1}(\mathcal{N}_s(A \cap B)) \subseteq \pi^{-1}(\mathcal{N}_s(A \cap B)) = \mathcal{N}_s(\pi^{-1}(A \cap B)) = \mathcal{N}_s(\pi^{-1}(A) \cap \pi^{-1}(B)),
\]

which concludes the proof. \( \square \)

Now for the quotient space \( X/\Gamma \), we fix a basepoint \( w_0 \in X/\Gamma \). For each \( n \in \mathbb{N} \cup \{0\} \), we set

\[
W_n := \{ w \in X/\Gamma : n^3 - n \leq \bar{d}(w, w_0) \leq (n + 1)^3 + (n + 1) \}.
\]

Let \( A := \bigsqcup_{\text{even}} W_n \) and \( B := \bigsqcup_{\text{odd}} W_n \). It is obvious that \((A, B)\) is an \( \omega \)-excisive cover of \( X/\Gamma \). Hence Lemma 3.3 implies that \((\pi^{-1}(A), \pi^{-1}(B))\) is an \( \omega \)-excisive closed cover of \( X \) such that both \( \pi^{-1}(A) \) and \( \pi^{-1}(B) \) are \( \Gamma \)-invariant. Denote \( A' = \pi^{-1}(A) \) and \( B' = \pi^{-1}(B) \).

Applying the Mayer-Vietoris sequence arguments to the \( \Gamma \)-invariant \( \omega \)-excisive cover \((A', B')\) for the \( K \)-theory of equivariant Roe algebras and the equivariant...
coarse $K$-homology (which is similar to $[31]$ and $[34]$ Section 7.5)], we obtain the following commutative diagram of long-exact sequences:

$$
\begin{array}{cccc}
\ldots & \lim_{r \to 0} K^r_c(P_r(X')) & \lim_{r \to 0} K^r_c(P_r(A')) & \lim_{r \to 0} K^r_c(P_r(B')) & \lim_{r \to 0} K^r_c(P_r(X)) & \ldots \\
\downarrow & & & & & \\
\ldots & K_r(C^*(A' \cap B')) & K_r(C^*(A')) & K_r(C^*(B')) & K_r(C^*(X)) & \ldots.
\end{array}
$$

Hence it suffices to prove Theorem [1.1] for the spaces $A'$, $B'$ and $A' \cap B'$. Therefore, we obtain the following:

**Lemma 3.4.** To prove Theorem [1.1], it suffices to prove the result for a discrete metric space $(X, d)$ of bounded geometry and having the form $X = \bigsqcup_{n=1}^{\infty} X_n$ with $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ (for $n \neq m$) such that $\Gamma$ acts on each $X_n$ properly and cocompactly by isometries.

Our next aim is to further reduce the proof of Theorem [1.1] to the case of block-diagonal operators. First note that the operations of taking direct limit and direct sum commute. Hence as a consequence of Proposition [2.15] we obtain the following:

**Corollary 3.5.** With the same notation as above and assuming that $\Gamma$ is $a$-T-menable, the following map induced by the equivariant index map

$$
\lim_{r \to 0} \bigoplus_n K^r_c(P_r(X_n)) \longrightarrow \bigoplus_n K_r(C^*(X_n)^\Gamma)
$$

is an isomorphism for $* = 0, 1$.

Now for each $r > 0$, we consider the following $C^*$-subalgebras in $C^*(P_r(X))^\Gamma$ and $C^*_L(P_r(X))^\Gamma$, respectively:

$$
\mathcal{A}_r := \lim_{n \to \infty} C^*(P_r(\bigsqcup_{k \leq n} X_k))^\Gamma = \bigcup_{n=1}^{\infty} C^*(P_r(\bigsqcup_{k \leq n} X_k))^\Gamma
$$

and

$$
\mathcal{A}_r := \lim_{n \to \infty} C^*_L(P_r(\bigsqcup_{k \leq n} X_k))^\Gamma = \bigcup_{n=1}^{\infty} C^*_L(P_r(\bigsqcup_{k \leq n} X_k))^\Gamma.
$$

The following lemma is straightforward, hence we omit the proof.

**Lemma 3.6.** With the same notation as above and for $r > 0$, we have the following:

$$
C^*(P_r(X))^\Gamma = \mathcal{A}_r + \left(C^*(P_r(X))^\Gamma \cap \prod_{n=1}^{\infty} C^*(P_r(X_n))^\Gamma\right),
$$

$$
\bigoplus_{n=1}^{\infty} C^*(P_r(X_n))^\Gamma = \mathcal{A}_r \cap \left(C^*(P_r(X))^\Gamma \cap \prod_{n=1}^{\infty} C^*(P_r(X_n))^\Gamma\right).
$$

The same holds for the case of equivariant localisation algebras.

Now we are able to reduce the proof of Theorem [1.1] to the case of block diagonal operators.

**Proposition 3.7.** Let $(X, d)$ be a discrete metric space of bounded geometry and having the form $X = \bigsqcup_{n=1}^{\infty} X_n$ with $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ (for $n \neq m$) such that $\Gamma$ acts on each $X_n$ properly and cocompactly by isometries. To prove that

$$
\text{Ind}^\Gamma : \lim_{r \to 0} K^r_c(P_r(X)) \longrightarrow K_r(C^*(X)^\Gamma)
$$
is an isomorphism for \(* = 0, 1\), it suffices to prove the map

\[
ev_* : \lim_{r \to \infty} K_*(C_1^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C_1^r(P_r(X_n))^F) \longrightarrow K_*(C(P_r(X))^F \cap \prod_{n=1}^{\infty} C(P_r(X_n))^F)
\]

induced by the evaluation-at-one map is an isomorphism for \(* = 0, 1\).

**Proof.** First note that for each \(r > 0\), we have the following commutative diagram:

\[
\cdots \longrightarrow \bigoplus_{n=1}^{\infty} K_*(C_1^r(P_r(X_n))^F) \longrightarrow K_*(C_1^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C_1^r(P_r(X_n))^F) \longrightarrow K_*(\frac{C_1^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C_1^r(P_r(X_n))^F}{\bigoplus_{n=1}^{\infty} C_1^r(P_r(X_n))^F}) \longrightarrow \cdots
\]

\[
\cdots \longrightarrow \bigoplus_{n=1}^{\infty} K_*(C^r(P_r(X))^F) \longrightarrow K_*(C^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C^r(P_r(X_n))^F) \longrightarrow K_*(\frac{C^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C^r(P_r(X_n))^F}{\bigoplus_{n=1}^{\infty} C^r(P_r(X_n))^F}) \longrightarrow \cdots,
\]

where all vertical lines are induced by evaluation-at-one maps and horizontal lines are exact sequences. When \(r \to \infty\), it follows from Corollary 3.5 and the assumption that the left and middle vertical lines are isomorphisms. Hence we obtain that the right vertical line is also an isomorphism when \(r \to \infty\).

From Lemma 3.6, we obtain the following for each \(r > 0\):

\[
\frac{C^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C^r(P_r(X_n))^F}{\bigoplus_{n=1}^{\infty} C^r(P_r(X_n))^F} \cong \frac{C^r(P_r(X))^F}{\mathcal{A}_r},
\]

and

\[
\frac{C_1^r(P_r(X))^F \cap \prod_{n=1}^{\infty} C_1^r(P_r(X_n))^F}{\bigoplus_{n=1}^{\infty} C_1^r(P_r(X_n))^F} \cong \frac{C_1^r(P_r(X))^F}{\mathcal{A}_{L,r}}.
\]

Hence the above implies that

\[
\lim_{r \to \infty} K_*(\frac{C_1^r(P_r(X))^F}{\mathcal{A}_{L,r}}) \longrightarrow \lim_{r \to \infty} K_*(\frac{C^r(P_r(X))^F}{\mathcal{A}_r})
\]

is an isomorphism for \(* = 0, 1\).

On the other hand, we have the following commutative diagram for each \(r > 0\):

\[
\cdots \longrightarrow K_*(\mathcal{A}_{L,r}) \longrightarrow K_*(C_1^r(P_r(X))^F) \longrightarrow K_*(\frac{C_1^r(P_r(X))^F}{\mathcal{A}_{L,r}}) \longrightarrow \cdots
\]

\[
\cdots \longrightarrow K_*(\mathcal{A}_r) \longrightarrow K_*(C^r(P_r(X))^F) \longrightarrow K_*(\frac{C^r(P_r(X))^F}{\mathcal{A}_r}) \longrightarrow \cdots,
\]

where all vertical lines are induced by evaluation-at-one maps and horizontal lines are exact sequences. The analysis above implies that the third vertical line is an isomorphism when \(r \to \infty\). Hence to finish the proof, it suffices to show that

\[
\lim_{r \to \infty} K_*(\mathcal{A}_{L,r}) \longrightarrow \lim_{r \to \infty} K_*(\mathcal{A}_r)
\]

is an isomorphism for \(* = 0, 1\). Consider the following commutative diagram:

\[
\lim_{r \to \infty} K_*(\mathcal{A}_{L,r}) \cong \lim_{r \to \infty} K_*(C_1^r(P_r(\bigsqcup_{k=1}^{n} X_k))^F) \cong \lim_{r \to \infty} K_*(C^r(P_r(\bigsqcup_{k=1}^{n} X_k))^F)
\]

\[
\lim_{r \to \infty} K_*(\mathcal{A}_r) \cong \lim_{r \to \infty} K_*(C(\bigsqcup_{k=1}^{n} X_k))^F \cong \lim_{r \to \infty} K_*(C(P_r(\bigsqcup_{k=1}^{n} X_k))^F).
\]
where all vertical lines are induced by evaluation-at-one maps. Note that the right two horizontal lines are isomorphisms due to the continuity of $K$-theory and the following isomorphisms

$$\lim_{r \to \infty} \lim_{n \to \infty} C^r \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma \equiv \lim_{n \to \infty} \lim_{r \to \infty} C^r \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma$$

and

$$\lim_{r \to \infty} \lim_{n \to \infty} C' \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma \equiv \lim_{n \to \infty} \lim_{r \to \infty} C' \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma$$

(note that both limits are just increasing unions of subalgebras). For each $n \in \mathbb{N}$, it follows from Proposition 2.15 that

$$\text{ev}_* : \lim_{r \to \infty} K_c \left( C^r \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma \right) \rightarrow \lim_{r \to \infty} K_c \left( C' \left( P_r \left( \bigsqcup_{k \leq n} X_k \right) \right)^\Gamma \right)$$

is an isomorphism. Hence we conclude the proof. □

Consequently, we obtain the following:

**Corollary 3.8.** To prove Theorem 1.1, it suffices to prove the following: Let $\Gamma$ be a countable discrete group and $(X, d)$ be a discrete metric space of bounded geometry and having the form $X = \bigsqcup_{n=1}^\infty X_n$ with $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ ($n \neq m$) such that $\Gamma$ acts on each $X_n$ properly and cocompactly by isometries. Assume that the action of $\Gamma$ on $X$ has controlled distortion, the quotient space $X/\Gamma$ admits a coarse embedding into Hilbert space and $\Gamma$ is a-$T$-menable, then the following map

$$(3.1) \quad \text{ev}_* : \lim_{r \to \infty} K_c \left( C^r (P_r(X))^\Gamma \cap \prod_{n=1}^\infty C^r (P_r(X_n))^\Gamma \right) \rightarrow K_c \left( C' (P_r(X))^\Gamma \cap \prod_{n=1}^\infty C' (P_r(X_n))^\Gamma \right)$$

induced by the evaluation-at-one map is an isomorphism for $* = 0, 1$.

4. **Equivariant twisted algebras**

In this section, we define equivariant twisted Roe and localisation algebras, which follows the constructions in [34, Section 12.6] and originally comes from [36, Section 5].

4.1. **The Bott-Dirac operators on Euclidean spaces.** Let us start by recalling the Bott-Dirac operators. Here we only list necessary notions and facts, and guide readers to [34, Section 12.1] for details.

Let $E$ be a real Hilbert space (also called a *Euclidean space*) with even dimension $d \in \mathbb{N}$. The *complexified Clifford algebra* of $E$, denoted by $\text{Cliff}(E)$, is the universal unital complex algebra containing $E$ as a real subspace and subject to the multiplicative relations $x \cdot x = ||x||_E^2$ for all $x \in E$. It is natural to treat $\text{Cliff}(E)$ as a graded Hilbert space (see for example [34, Example E.2.12]), and in this case we denote it by $\mathcal{H}_E$.

Denote $L^2_E$ the graded Hilbert space of square integrable functions from $E$ to $\mathcal{H}_E$ where the grading is inherited from $\mathcal{H}_E$, and $\mathcal{S}_E$ the dense subspace consisting of Schwartz class functions from $E$ to $\mathcal{H}_E$. Fix an orthonormal basis $\{e_1, \ldots, e_d\}$ of $E$ and let $x_1, \ldots, x_d : E \to \mathbb{R}$ be the corresponding coordinates. Recall that the Bott
operator $C$ and the Dirac operator $D$ are unbounded operators on $L^2_E$ with domain $\mathcal{S}_E$ defined as:

$$(Cu)(x) = x \cdot u(x), \quad \text{and} \quad (Du)(x) = \sum_{i=1}^{d} \hat{e}_i \cdot \frac{\partial u}{\partial x_i}(x)$$

for $u \in \mathcal{S}_E$ and $x \in E$, where $\hat{e}_i : \text{Cliff}(E) \to \text{Cliff}(E)$ is the operator determined by $\hat{e}_i(w) = (-1)^{\omega_i} w \cdot e_i$ for any homogeneous element $w \in \text{Cliff}(E)$.

**Definition 4.1.** The Bott-Dirac operator is the unbounded operator $B := D + C$ on $L^2_E$ with domain $\mathcal{S}_E$.

Given $x \in E$, the left Clifford multiplication operator associated to $x$ is the bounded operator $c_x$ on $L^2_E$ defined as the left Clifford multiplication by the fixed vector $x$, and the translation operator associated to $x$ is the unitary operator $V_x$ on $L^2_E$ defined by $(V_x u)(y) := u(y - x)$. Given $s \in [1, \infty)$, the shrinking operator associated to $s$ is the unitary operator $S_s$ on $L^2_E$ defined by $(S_s u)(y) := s^{-1/2} u(sy)$.

**Definition 4.2.** For $s \in [1, \infty)$ and $x \in E$, the Bott-Dirac operator associated to $(x, s)$ is the unbounded operator $B_{s,x} := s^{-1}D + C - c_x$ on $L^2_E$ with domain $\mathcal{S}_E$.

Note that $B_{1,0} = B$ and $B_{s,x} = s^{-1/2} V_x S_s \sqrt{B} S_s^* V_x$. It is also known that for each $s \in [1, \infty)$ and $x \in E$, the operator $B_{s,x}$ is unbounded, odd, essentially self-adjoint and maps $\mathcal{S}_E$ to itself (see, e.g., [34], Corollary 12.1.4).

**Definition 4.3.** Let $s \in [1, \infty)$, $x \in E$ and $B_{s,x}$ be the Bott-Dirac operator associated to $(x, s)$. Define a bounded operator on $L^2_E$ by:

$$F_{s,x} := B_{s,x}(1 + B_{s,x})^{-1/2}.$$

We list several important properties of the operator $F_{s,x}$. For simplicity, denote $\Lambda_{x,R} := \chi_{B_{s,x}}(R)$ for $x \in E$ and $R \geq 0$.

**Proposition 4.4 ([34], Proposition 12.1.10).** For each $\varepsilon > 0$ there exists an odd function $\Psi : \mathbb{R} \to [-1, 1]$ with $\Psi(t) \to 1$ as $t \to +\infty$, satisfying the following:

1. For all $s \in [1, \infty)$ and $x \in E$, we have $\|F_{s,x} - \Psi(B_{s,x})\| < \varepsilon$.
2. There exists $R_0 > 0$ such that for all $s \in [1, \infty)$ and $x \in E$, we have $\text{prop}_E(\Psi(B_{s,x})) \leq s^{-1}R_0$.
3. For all $s \in [1, \infty)$ and $x \in E$, $\Psi(B_{s,x})^2 - 1$ is compact.
4. For all $s \in [1, \infty)$ and $x, y \in E$, $\Psi(B_{s,x}) - \Psi(B_{s,y})$ is compact.
5. For all $s \in [1, \infty)$ and $x, y \in E$, $\|F_{s,x} - F_{s,y}\| \leq 3\|x - y\|_E$. And there exists $c > 0$ such that for all $s \in [1, \infty)$ and $x, y \in E$, we have $\|\Psi(B_{s,x}) - \Psi(B_{s,y})\| \leq c\|x - y\|_E$.
6. For all $x \in E$, the function $[1, \infty) \to \mathcal{C}(L^2_E)$, $s \mapsto \Psi(B_{s,x})$

is strong-$*$ continuous.
7. The family of functions $[1, \infty) \to \mathcal{C}(L^2_E)$, $s \mapsto \Psi(B_{s,x})^2 - 1$

is norm equi-continuous as $x$ varies over $E$ and $s$ varies over any fixed compact subset of $[1, \infty)$.
(8) For any \( r \geq 0 \), the family of functions
\[ [1, \infty) \to \mathfrak{B}(L^2_E), \; s \mapsto \Psi(B_{s,x}) - \Psi(B_{s,y}) \]
is norm equi-continuous as \((x, y)\) varies over the elements of \( E \times E \) with \(|x - y| \leq r\), and \( s \) varies over any fixed compact subset of \([1, \infty)\).

(9) For any \( \varepsilon_1 > 0 \), there exists \( R_1 > 0 \) such that for all \( R \geq R_1 \), \( s \geq 2d \) and \( x \in E \), we have
\[ \|\Psi(B_{s,x})^2 - 1(1 - \chi_{x,R})\| < \varepsilon_1. \]

(10) For any \( \varepsilon_2 > 0, r > 0 \) there exists \( R_2 > 0 \) such that for all \( R \geq R_2 \), \( s \geq 2d \) and \( x, y \in E \) with \(|x - y|_E \leq r \), we have
\[ \|\Psi(B_{s,x}) - \Psi(B_{s,y})(1 - \chi_{x,R})\| < \varepsilon_2. \]

Moreover, we can require that the function \( \Psi \), constants \( R_0 \) in (2), \( c \) in (5), \( R_1 \) in (9) and \( R_2 \) in (10) are independent of the dimension \( d \) of the Euclidean space \( E \).

The strong-* topology used in (6) above is defined as follows: A net \((T_i)\) of bounded operators converges to a bounded operator \( T \) in the strong-* topology if all \( v \) in the underlying Hilbert space, \( T[v] \to T[v] \) and \( T[v] \to T[v] \). We need the following elementary result:

**Lemma 4.5** ([34, Lemma 12.3.5]). Let \( S \) and \( T \) be norm bounded sets of operators on a Hilbert space \( \mathcal{H} \) such that \( T \) consists only of compact operators. Equip \( S \) with the strong-* topology and \( T \) with the norm topology. Then the product maps
\[ S \times T \to S \quad \text{and} \quad T \times S \to S \]
are jointly continuous where \( S \) denotes the compact operators on \( \mathcal{H} \).

### 4.2. Equivariant twisted algebras

Throughout the rest of this paper, thanks to Section 3, we always assume that \((X, d)\) is a discrete metric space of bounded geometry and having the form \( X = \bigsqcup_{n=1}^{\infty} X_n \) with \( d(X_n, X_m) \to \infty \) as \( n + m \to \infty \) (\( n \neq m \)) such that the a-T-menable group \( \Gamma \) acts on each \( X_n \) properly and cocompactly by isometries. Furthermore, assume that all \( \Gamma \)-orbit maps are uniformly coarsely equivalent and the quotient space admits a coarse embedding \( \xi : X/\Gamma \to \mathcal{H} \) into some Hilbert space \( \mathcal{H} \).

Denoting by \( E \) the underlying Euclidean space of \( H \), it is clear that the map \( \xi : X/\Gamma \to E \) is also a coarse embedding. Note that each \( X_n/\Gamma \) is finite, hence the image of \( \xi \) restricted to \( X_n/\Gamma \) sits inside a finite-dimensional Euclidean space, denoted by \( E_n \). Without loss of generality, we assume that the dimension \( d_n \) of \( E_n \) is even. For \( n \in \mathbb{N} \), denote \( f_n : \xi : \pi|_{X_n} : X_n \to E_n \), where \( \pi : X \to X/\Gamma \) is the quotient map.

For \( n \in \mathbb{N} \) and \( r > 0 \), denote the Rips complex of \( X_n \) with scale \( r \) by \( P_{r,n} := P_r(X_n) \). The \( \Gamma \)-action on \( X_n \) induces a proper and cocompact \( \Gamma \)-action on \( P_{r,n} \) by isometries as explained in Section 2.5.

For \( x \in X_n \), we consider the open star neighbourhood \( B_{x,r,n} := St_{n,r}(x) \) of \( x \) in the barycentric subdivision of \( P_{r,n} \). Note that \( B_{x,r,n} \cap B_{y,r,n} = \emptyset \) for \( x \neq y \in X_n \), \( \gamma \cdot B_{x,r,n} = B_{\gamma x,r,n} \) for \( \gamma \in \Gamma \), and \( B_{y,r,n} \subseteq B_{x,s,r,n} \) for each \( r \leq s \). Taking \( Y_{r,n} := \bigcup_{x \in X_n} B_{x,r,n} \), it is clear that \( Y_{r,n} \) is \( \Gamma \)-invariant (since the action is by isometries) and dense in \( P_{r,n} \). Moreover, let \( Z_{r,n} \) be the collection of points \( \sum c_i x_i \) in \( Y_{r,n} \) such that all the coefficients \( c_i \) take rational values. It is clear that \( Z_{r,n} \) is a \( \Gamma \)-invariant countable subset which is also dense in \( P_{r,n} \).
For $n \in \mathbb{N}$ and $r > 0$, we extend the above $f_n : X_n \to E_n$ to $f_{r,n} : Y_{r,n} \to E_n$ by sending all points in $B_{x,r,n}$ to $f_n(x)$ for $x \in X_n$. It is clear that $f_{r,n}(\gamma \cdot y) = f_{r,n}(y)$ for any $\gamma \in \Gamma$ and $y \in Y_{r,n}$.

Let us fix an infinite-dimensional separable Hilbert space $\mathcal{H}$. For $r > 0$ and $n \in \mathbb{N}$, consider the following Hilbert spaces:

$$\mathcal{H}_{r,n} := \ell^2(Z_{r,n}) \otimes \mathcal{H} \otimes \ell^2(\Gamma) \quad \text{and} \quad \mathcal{H}_{r,n,E} := \ell^2(Z_{r,n}) \otimes \mathcal{H} \otimes \ell^2(\Gamma) \otimes \mathcal{L}^2_{E_n}.$$ 

The group $\Gamma$ acts on $\mathcal{H}_{r,n}$ (respectively, $\mathcal{H}_{r,n,E}$) as follows: for any $\gamma \in \Gamma$,

$$U_{\gamma} : \delta_z \otimes \xi \otimes \delta_y \mapsto \delta_{\gamma z} \otimes \xi \otimes \delta_{\gamma y},$$

(respectively, $U_{\gamma} : \delta_z \otimes \xi \otimes \delta_y \otimes u \mapsto \delta_{\gamma z} \otimes \xi \otimes \delta_{\gamma y} \otimes u$).

It is clear that $\mathcal{H}_{r,n}$ is an admissible $P_{r,n}$-module under the amplified multiplication representation, and similarly $\mathcal{H}_{r,n,E}$ is both an admissible $P_{r,n}$-module and an ample $E_n$-module. We use these modules to the build equivariant Roe algebras $C^*(\mathcal{H}_{r,n})^\Gamma$ and $C^*(\mathcal{H}_{r,n,E})^\Gamma$ of $P_{r,n}$, and the equivariant localisation algebras $C^*_L(\mathcal{H}_{r,n})^\Gamma$ and $C^*_L(\mathcal{H}_{r,n,E})^\Gamma$ of $P_{r,n}$. Moreover, for $T \in \mathcal{B}(\mathcal{H}_{r,n,E})$, we write $\text{prop}_{P}(T)$ and $\text{prop}_{E}(T)$ for the propagation of $T$ with respect to the $P_{r,n}$-module structure and the $E_n$-module structure, respectively. Also denote the Hilbert spaces

$$\mathcal{H}_r := \bigoplus_{n=1}^{\infty} \mathcal{H}_{r,n} \quad \text{and} \quad \mathcal{H}_{r,E} := \bigoplus_{n=1}^{\infty} \mathcal{H}_{r,n,E},$$

and use these modules to build the equivariant Roe algebras $C^*(\mathcal{H}_r)^\Gamma$ and $C^*(\mathcal{H}_{r,E})^\Gamma$ of $P_r := \bigsqcup_{n=1}^{\infty} P_{r,n}$, and the equivariant localisation algebras $C^*_L(\mathcal{H}_r)^\Gamma$ and $C^*_L(\mathcal{H}_{r,E})^\Gamma$ of $P_r$.

For $r \leq s$, the canonical inclusion $Z_r \to Z_s$ induces isometric inclusions of Hilbert spaces

$$\mathcal{H}_{r,n} \to \mathcal{H}_{s,n} \quad \text{and} \quad \mathcal{H}_{r,n,E} \to \mathcal{H}_{s,n,E},$$

and further implies inclusions of $C^*$-algebras

$$C^*(\mathcal{H}_{r,n}) \to C^*(\mathcal{H}_{s,n}), \quad C^*(\mathcal{H}_{r,n,E}) \to C^*(\mathcal{H}_{s,n,E})$$

and

$$C^*_L(\mathcal{H}_{r,n}) \to C^*_L(\mathcal{H}_{s,n}), \quad C^*_L(\mathcal{H}_{r,n,E}) \to C^*_L(\mathcal{H}_{s,n,E}).$$

For $r > 0$ and $n \in \mathbb{N}$, also note that although $\{B_{x,r,n} : x \in X_n\}$ does not cover $P_{r,n}$ we still have the following decomposition of Hilbert spaces:

$$\mathcal{H}_{r,n} = \bigoplus_{x \in X_n} \chi_{B_{x,r,n}} \mathcal{H}_{r,n} \quad \text{and} \quad \mathcal{H}_{r,n,E} = \bigoplus_{x \in X_n} \chi_{B_{x,r,n}} \mathcal{H}_{r,n,E}$$

since $\{B_{x,r,n} : x \in X_n\}$ covers $Z_{r,n}$. Write

$$\mathcal{H}_{x,r,n} := \chi_{B_{x,r,n}} \mathcal{H}_{r,n} \quad \text{and} \quad \mathcal{H}_{x,r,n,E} := \chi_{B_{x,r,n}} \mathcal{H}_{r,n,E}.$$ 

We can represent a bounded linear operator $T$ on $\mathcal{H}_{r,n}$ (respectively, $\mathcal{H}_{r,n,E}$) as an $X_n$-by-$X_n$ matrix $(T_{x,y})_{x,y \in X_n}$, where each $T_{x,y}$ is a bounded linear operator $\mathcal{H}_{y,r,n} \to \mathcal{H}_{x,r,n}$ (respectively, $\mathcal{H}_{y,r,n,E} \to \mathcal{H}_{x,r,n,E}$). Moreover, $T$ is $\Gamma$-invariant if and only if the
following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}_{y,r,n} & \xrightarrow{T_{x,y}} & \mathcal{H}_{x,r,n} \\
\downarrow{u_y} & & \downarrow{u_y} \\
\mathcal{H}_{y',r,n} & \xrightarrow{T_{x',y'}} & \mathcal{H}_{x',r,n}
\end{array}
\]

for any \( \gamma \in \Gamma \) and \( x, y \in X_r \), i.e., \( T_{x',y'} = U_y T_{x,y} U_y^* =: \gamma \cdot T_{x,y} \).

It was pointed out in [34] Inequality (12.11) that for any \( r > 0 \), there exists a proper non-decreasing function \( \varphi_r : [0, \infty) \to [0, \infty) \) such that for any \( T \in \mathfrak{C}[\mathcal{H}_{r,n,E}]^\Gamma \) we have:

\[
(4.1) \quad \text{prop}_p(T) - 2 \leq \sup\{d(x, y) : T_{x,y} \neq 0 \text{ where } x, y \in X_r \} \leq \varphi_r(\text{prop}_p(T)).
\]

We also record the following elementary result (see, e.g., [34] Lemma 12.2.4) for later use:

**Lemma 4.6.** For \( s, r \geq 0 \), there exists an \( N \in \mathbb{N} \) such that for any \( n \in \mathbb{N} \) and any bounded operator \( T = (T_{x,y})_{x,y \in X} \) on \( \mathcal{H}_{r,n,E} \) with \( P \)-propagation at most \( s \), we have

\[
\|T\| \leq N \cdot \sup_{x,y \in X} \|T_{x,y}\|.
\]

To introduce the twisted algebras, we recall the following construction from [34] Definition 12.3.1:

**Definition 4.7.** Given \( r > 0 \), \( n \in \mathbb{N} \) and \( T \in \mathfrak{B}(L_{E_n}^2) \), define a bounded linear operator \( T^\Gamma \) on \( \mathcal{H}_{r,n,E_n} = \ell^2(Z_{r,n}) \otimes \mathcal{H} \otimes \ell^2(\Gamma) \otimes L_{E_n}^2 \) by the formula

\[
T^\Gamma : \delta_z \otimes \xi \otimes \delta_g \otimes u \mapsto \delta_z \otimes \xi \otimes \delta_g \otimes V_{f_r(g)} T_{f_r(g)} u
\]

for \( z \in Z_{r,n}, \xi \in \mathcal{H}, g \in \Gamma \) and \( u \in L_{E_n}^2 \), where \( V_{f_r(g)} \) is the translation operator defined in Section 4.1.

Writing in the matrix representation, we have

\[
T_{x,y}^\Gamma = \begin{cases} 
\text{Id}_{\mathcal{H}_{r,n}} \otimes V_{f_r(x)} T_{f_r(x)} \quad & y = x; \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( f_r(\gamma z) = f_r(z) \) for any \( z \in Z_{r,n} \) and \( \gamma \in \Gamma \), it is easy to see that \( \gamma \cdot T_{x,y} = T_{\gamma x,\gamma y} \).

In other words, \( T^{\Gamma} \) is invariant. Hence we obtain:

**Lemma 4.8.** For any \( T \in \mathfrak{B}(L_{E_n}^2) \), we have \( T^{\Gamma} \in \mathfrak{C}[\mathcal{H}_{r,n,E}]^\Gamma \).

Now we introduce the notion of twisted algebras, which is slightly different from [34] Section 12.6.

**Definition 4.9.** Fix an \( r > 0 \). Denote \( \prod_{n \in \mathbb{N}} C_b([1, \infty), C^*(\mathcal{H}_{r,n,E})^\Gamma) \) the product \( C^* \)-algebra of all bounded continuous functions from \([1, \infty)\) to \( C^*(\mathcal{H}_{r,n,E})^\Gamma \) with supremum norm. Write elements of this \( C^* \)-algebra as a collection \( (T_{n,s})_{n \in \mathbb{N}, s \in [1, \infty)} \) for \( T_{n,s} = (T_{n,s,x,y})_{x,y \in X_n} \in C^*(\mathcal{H}_{r,n,E})^\Gamma \), whose norm is

\[
\|(T_{n,s})\| = \sup_{n \in \mathbb{N}, s \in [1, \infty)} \|T_{n,s}\|_{\mathfrak{B}(\mathcal{H}_{r,n,E})}.
\]

Let \( \mathfrak{A}^r(X; E)^\Gamma \) denote the \( * \)-subalgebra of \( \prod_{n \in \mathbb{N}} C_b([1, \infty), C^*(\mathcal{H}_{r,n,E})^\Gamma) \) consisting of elements satisfying the following conditions:
(1) \( \sup_{s \in [1, \infty], n \in \mathbb{N}} \text{prop}_p(T_{n,s}) < \infty; \)

(2) for each \( n \in \mathbb{N}, \lim_{s \to \infty} \text{prop}_p(T_{n,s}) = 0; \)

(3) \( \lim_{R \to \infty} \sup_{s \in [1, \infty], n \in \mathbb{N}} \| \chi_R T_{n,s} - T_{n,s} \| = \lim_{R \to \infty} \sup_{s \in [1, \infty], n \in \mathbb{N}} \| T_{n,s} \chi_R - T_{n,s} \| = 0; \)

(4) for each \( n \in \mathbb{N} \) and \( x, y \in X_n \), the map \( s \mapsto T_{n,s,x,y} \) belongs to the subalgebra \( \mathcal{S}(\mathcal{H}_{y,n} \cap \mathcal{H}_{x,n}) \) of \( \mathcal{C}_b([1, \infty), \mathcal{S}(\mathcal{H}_{y,n} \cap \mathcal{H}_{x,n} \cap \mathcal{S}(L^2_{E_n}))) \).

The \textit{equivariant twisted Roe algebra} \( A'(X; E) \) of \( X \) is defined to be the norm-closure of \( A'(X; E) \) in \( \prod_{n \in \mathbb{N}} \mathcal{C}_b([1, \infty), C('H_{y,n}E)) \).

**Definition 4.10.** Define \( A'_t(X; E) \) to be the collection of uniformly continuous bounded functions \( T : [1, \infty) \to A'(X; E) \) satisfying the following: writing \( T = (T_t)_{t \in [1, \infty)} = (T_{t,n,s})_{t \in [1, \infty), n \in \mathbb{N}} \) then

(1) \( \lim_{t \to \infty} \sup_{t \in [1, \infty), n \in \mathbb{N}} \text{prop}_p(T_{t,n,s}) = 0; \)

(2) \( \lim_{t \to \infty} \sup_{t \in [1, \infty), n \in \mathbb{N}} \| \chi_R T_{t,n,s} - T_{t,n,s} \| = \lim_{t \to \infty} \sup_{t \in [1, \infty), n \in \mathbb{N}} \| T_{t,n,s} \chi_R - T_{t,n,s} \| = 0. \)

The \textit{equivariant twisted localisation algebra} \( A^*_t(X; E) \) of \( X \) is defined to be the completion of \( A'_t(X; E) \) for the norm \( \|(T_t)\| := \sup_{t} \|T_t\|_{A'(X; E)'} \).

**Remark 4.11.** Readers might already notice that the above definition is slightly different from [34] Definition 12.6.2. More precisely, we weaken condition (2) to level-wise convergence and restrict the living space of each matrix entry (see condition (4) above) for later use. On the other hand, the following lemma shows that the original condition (4) in [34] Definition 12.6.2 can be recovered.

**Lemma 4.12.** Given \( n \in \mathbb{N}, t > 0 \) and \( T \in \mathcal{C}(\mathcal{H}_{x,n}E) \), we have

\[
\lim_{t \to \infty} \|p^T - T\| = \lim_{t \to \infty} \|TP^T - T\| = 0
\]

where \( \{p^T\}_{t \in \mathbb{R}} \) is the net of finite rank projections on \( L^2_{E_n} \).

**Proof.** We fix an operator \( T \in \mathcal{C}(\mathcal{H}_{x,n}E) \) with \( P \)-propagation at most \( R' \), and a point \( x_0 \in X_n \). For \( x, y \in X_n \) with \( T_{x,y} \neq 0 \), we have \( d(x, y) \leq \varphi_r(R') =: R \) where the function \( \varphi_r \) comes from Inequality (4.1). Let \( N \) be the number from Lemma 4.6 for the parameters \( R' \) and \( r, i.e., for any bounded operator \( \tilde{T} = (\tilde{T}_{x,y})_{x,y \in X_n} \) on \( \mathcal{H}_{x,n}E \) with \( P \)-propagation at most \( R' \), then

\[
\|\tilde{T}\| \leq N \cdot \sup_{x,y \in X_n} \|\tilde{T}_{x,y}\|
\]

Recall that the \( \Gamma \)-action on \( X_n \) is cocompact, hence there exists an \( S > 0 \) such that \( \Gamma \cdot B(x_0, S) = X_n \).

We consider a finite set of operators:

\[
\mathcal{B} := \{ T_{x,y} : x \in B(x_0, S) \text{ and } y \in B(x, R) \}.
\]

Since \( T \) is locally compact, each \( T_{x,y} \) is a compact operator from \( \mathcal{H}_{y,n}E \) to \( \mathcal{H}_{x,n}E \).

Note that for any projection \( p \in \mathcal{B}(L^2_{E_n}) \) and \( x, y \in X_n \), we have:

\[
(p^T T - T)_{x,y} = (\text{Id}_{\mathcal{H}_{x,n}} \otimes V_{f_{x,y}}(x)p^T V^*_{f_{x,y}}(x) - \text{Id}_{\mathcal{H}_{x,n}E}) \cdot T_{x,y}
\]
and the net
\[ \{ \text{Id}_{V_{\mathcal{E},r}} \otimes V_{\mathcal{E},r}(x)pV_{\mathcal{E},r}(x) - \text{Id}_{V_{\mathcal{E},r}} : p \text{ is a finite rank projection in } \mathcal{B}(L^2_{\mathcal{E},r}) \} \]
converges to 0 in the strong operator topology. Hence for any \( \varepsilon > 0 \) there exists a finite rank projection \( p_\varepsilon \in \mathcal{B}(L^2_{\mathcal{E},r}) \) such that for any finite rank projection \( p \geq p_\varepsilon \) and \( T_{x,y} \in \mathcal{F} \), we have
\[
\| (p^V T - T)_{x,y} \| < \frac{\varepsilon}{N}.
\]

Now for any \( x', y' \in X_r \) with \( d(x', y') \leq R \), there exists \( r \in \Gamma \) such that \( r x' \in B(x_0, S) \). Letting \( x = r x' \) and \( y = r y' \), we have \( d(x, y) \leq R \) and hence \( T_{x,y} \in \mathcal{F} \).

Moreover, we obtain:
\[
\| (p^V T - T)_{x',y'} \| = \| (p^V T - T)_{y^{-1}x,y^{-1}y} \| = \| r^{-1} \cdot (p^V T - T)_{x,y} \| = \| (p^V T - T)_{x,y} \| < \frac{\varepsilon}{N}.
\]

Note that the \( P \)-propagation of \( p^V T - T \) is again at most \( R' \). Hence from Lemma 4.6 we obtain:
\[
\| p^V T - T \| \leq N \cdot \sup_{x,y \in X_r : d(x,y) \leq R} \| (p^V T - T)_{x,y} \| < N \cdot \frac{\varepsilon}{N} = \varepsilon.
\]

Therefore, we obtain that \( \lim_{i \in I} \| p^V_i T - T \| = 0 \). Finally taking adjoints, we obtain \( \lim_{i \in I} \| Tp_i^V - T \| = 0 \) as well and conclude the proof. \( \square \)

Finally, we introduce the following operators:

**Definition 4.13** ([34, Section 12.3 and 12.6]). Fix an \( r > 0 \). For each \( n \in \mathbb{N}, s \in [1, \infty) \) and \( x \in E_r \), Definition 4.3 provides a bounded linear operator \( F_{n,s} \in \mathcal{B}(L^2_{E_r}) \), also denoted by \( F_{n,s,x} \). Applying Definition 4.7, we obtain an operator
\[
F_{n,s} := F_{n,s+2d_n,0}^\text{F} \in \mathcal{B}(H_{n,r,E}^\Gamma)
\]
where \( d_n \) is the dimension of \( E_r \). Let \( F_s := (F_{n,s})_{n \in \mathbb{N}} \) be the block diagonal operator in \( \prod_n \mathcal{B}(H_{n,r,E}^\Gamma) \subseteq \mathcal{B}(H_{r,E}^\Gamma) \). Finally, we define \( F \) to be an element in \( \prod_n \mathcal{B}(L^2([1, \infty); H_{r,E})) \subseteq \mathcal{B}(L^2([1, \infty); H_{r,E})) \) defined by \((F(u))(s) := F_s u(s)\).

Similarly given \( \varepsilon > 0 \), let \( \Psi \) be a function as in Proposition 4.4 and set \( F_{n,s,x}^\Psi := \Psi(B_{n,s,x}) \). Let \( F_s^\Psi \) be the bounded diagonal operator on \( H_{r,E} \), defined by:
\[
F_s^\Psi := (F_{n,s}^\Psi)_{n \in \mathbb{N}} \quad \text{where} \quad F_{n,s}^\Psi := (F_{n,s+2d_n,0}^\text{F})^\Psi \in \mathcal{B}(H_{n,r,E}^\Gamma).
\]

We also define \( F_s^\Psi \) to be the element in \( \prod_n \mathcal{B}(L^2([1, \infty); H_{r,E})) \subseteq \mathcal{B}(L^2([1, \infty); H_{r,E})) \) by \((F_s^\Psi(u))(s) := F_s^\Psi u(s)\).

5. The index maps

In this section, we construct equivariant index maps (with the same notation as in Section 4.2):

\[
(5.1) \quad \text{Ind}_F : K_*\big(C^*(H_r)^\Gamma \cap \prod_{n=1}^\infty C^*(H_{n,r})^\Gamma\big) \to K_*\big(A_r^r(X; E)^\Gamma\big)
\]

and
\[
(5.2) \quad \text{Ind}_{F_L} : K_*\big(C_r^\ell(H_r)^\Gamma \cap \prod_{n=1}^\infty C_r^\ell(H_{n,r})^\Gamma\big) \to K_*\big(A_r^\ell_r(X; E)^\Gamma\big),
\]
where $F$ is the operator from Definition 4.13. We use these maps to transfer $K$-theoretic information from equivariant Roe and localisation algebras to their twisted counterparts, which allow us to prove Theorem 1.1 via local isomorphisms. This approach is mainly based on [34, Section 12.3 and 12.6] in the case of coarse embedding, while several changes are needed to involve group actions.

The main result of this section is the following:

**Proposition 5.1.** Fix an $r > 0$. With notation as in Section 4.2 for each $s \in [1, \infty)$ the composition

$$K_n \left( C^*(\mathcal{H}_s) \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{rn}) \right) \xrightarrow{\text{Ind}_r} K_n \left( A'(X; E) \right) \xrightarrow{\iota^*} K_n \left( C^*(\mathcal{H}_{rn}) \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{rn,E}) \right)$$

is an isomorphism, where $\iota^* : A'(X; E) \to C^*(\mathcal{H}_{rn,E}) \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{rn,E})$ is the evaluation map at $s$. The analogous statement holds for the equivariant localisation algebras. Moreover, we have the following commutative diagram:

$$K_n \left( C^*(\mathcal{H}_s) \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{rn}) \right) \xrightarrow{\text{Ind}_r} K_n \left( A'_L(X; E) \right) \xrightarrow{\iota^*} K_n \left( C^*(\mathcal{H}_{rn,E}) \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{rn,E}) \right)$$

where all vertical lines are induced by the evaluation-at-one maps.

We follow the same notation as in Section 4.2. Recall that $(X, d)$ is a discrete metric space of bounded geometry and having the form $X = \bigsqcup_{n=1}^\infty X_n$ with $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ $(n \neq m)$ such that $\Gamma$ acts on each $X_n$ properly and cocompactly by isometries. For each $n \in \mathbb{N}$, we have a map $f_n : X_n \to E_n$ coming from the uniformly coarse embedding of $X_n$ by some Euclidean space $E_n$ of even dimension $d_n$.

To construct the index map $\text{Ind}_r$, we need a series of lemmas. Let us fix an $r > 0$ throughout the rest of this section.

**Lemma 5.2.** The operator $F$ is a self-adjoint, norm one, odd operator in the multiplier algebra of $A'(X; E)$.

**Proof.** The operator $F$ is self-adjoint, norm one and odd since each $F_{n,s,x}$ is. Given $\varepsilon > 0$, let $\Psi : \mathbb{R} \to [-1, 1]$ be a function as in Proposition 4.4 for this $\varepsilon$. Then Proposition 4.4(1) implies:

$$\|F - F^\Psi\| \leq \sup_{n \in \mathbb{N}, x \in [1, \infty]} \|F_{n,s,0}^\Psi - \Psi(B_{n,s,0})^\Psi\| \leq \sup_{n \in \mathbb{N}, x \in [1, \infty]} \sup_{x \in X_n} \|F_{n,s,f_n(x)} - \Psi(B_{n,s,f_n(x)})\| \leq \varepsilon.$$}

Hence it suffices to show that $(T_{n,s})F^\Psi = (T_{n,s}F_{n,s}^\Psi)$ belongs to $A'(X; E)$ for any $(T_{n,s}) \in A'(X; E)$.

We first claim that for each $n \in \mathbb{N}$ the map $s \mapsto T_{n,s}F_{n,s}^\Psi$ is norm continuous. In fact, this can be proved using the same argument as in the proof of [34, Lemma 12.3.6] thanks to Lemma 4.12. Here we provide a direct proof.

Fix $n \in \mathbb{N}$ and $x_0 \in X_n$. Assume that each $T_{n,s}$ has $P$-propagation at most $R'$, and set $R := \varphi_p(R')$ where the function $\varphi_p$ comes from Inequality (4.1). Since the $\Gamma$-action on $X_n$ is cocompact, there exists an $S > 0$ such that $\Gamma \cdot \overline{B}(x_0, S) = X_n$. For
each \(x, y \in X_n\), it follows from Proposition 4.4(6) and Lemma 4.5 that the map 
\(s \mapsto T_{n,s,x,y} \cdot F_{n,s}^{\psi} = (T_{n,s,F_{n,s}^{\psi}})_{x,y}\) is norm continuous. On the other hand, for any \(x, y \in X_n\) with \(d(x, y) \leq R\), there exists \(\gamma \in \Gamma\) such that \(\bar{x} := \gamma x \in B(x_0, S)\) and hence \(\bar{y} := \gamma y \in B(\bar{x}, R)\). Moreover, we have 
\(\left(T_{n,s,F_{n,s}^{\psi}}\right)_{x,y} = (T_{n,s,F_{n,s}^{\psi}})_{\bar{x},\bar{y}} = \gamma^{-1} \cdot (T_{n,s,F_{n,s}^{\psi}})_{\bar{x},y}\).

Note that the set \(\{(\bar{x}, \bar{y}) : \bar{x} \in B(x_0, S)\) and \(\bar{y} \in B(\bar{x}, R)\) is finite, hence the family 
\(s \mapsto (T_{n,s,F_{n,s}^{\psi}})_{x,y} : x, y \in X_n\) is uniformly norm continuous. This concludes the claim due to Lemma 4.6.

For conditions (1)-(3) in Definition 4.9 it suffices to note that the \(P\)-propagation of \(F_{n,s}^{\psi}\) is 0 while the \(E\)-propagation of \(F_{n,s}^{\psi}\) is uniformly bounded (in both \(s\) and \(n\)) and (uniformly) tends to 0 (in \(n\)) as \(s \to \infty\) by Proposition 4.4(2). Finally for condition (4), note that for each \(n \in \mathbb{N}\) and \(x, y \in X_n\) we have 
\(s \mapsto (T_{n,s,F_{n,s}^{\psi}})_{x,y} = T_{n,s,x,y} \cdot \left(\text{Id}_{\mathcal{H}_{x,y}} \otimes F_{n,s+2d_{x,y}}\right)\).

Since the map \(s \mapsto T_{n,s,x,y}\) belongs to \(\mathcal{R}(\mathcal{H}_{x,y}) \otimes \mathcal{C}_b([1, \infty), \mathcal{R}(L_2^E))\), it follows from Proposition 4.4(6) and Lemma 4.5 again that the map \(s \mapsto (T_{n,s,F_{n,s}^{\psi}})_{x,y}\) belongs to \(\mathcal{R}(\mathcal{H}_{x,y}) \otimes \mathcal{C}_b([1, \infty), \mathcal{R}(L_2^E))\) as well. Hence we conclude the proof. 

**Lemma 5.3.** Considered as represented on \(L^2([1, \infty)) \otimes \mathcal{H}_E\) via the amplification of identity, \(C^*(\mathcal{H}_E)^{\Gamma} \cap \prod_{n=1}^\infty C^*(\mathcal{H}_{x_n})^{\Gamma}\) is a subalgebra of the multiplier algebra of \(\mathcal{A}'(X; E)^{\Gamma}\).

**Proof.** It suffices to show that \(S_n T_{n, s} \in \mathcal{A}'(X; E)^{\Gamma}\) for any \((T_{n, s}) \in \mathcal{A}'(X; E)^{\Gamma}\) and \((S_n) \in C[\mathcal{H}_E] \cap \prod_{n=1}^\infty C[\mathcal{H}_{x_n}]\). It is clear that the map \(s \mapsto S_n T_{n, s}\) is norm-continuous and bounded for each \(n \in \mathbb{N}\).

For conditions (1) and (2) in Definition 4.9 it suffices to note that \(S_n\) has uniformly finite \(P\)-propagation (in \(n\)) and \(E\)-propagation 0. Condition (4) follows from the fact that \(S_n\) is constant in \(s\). Finally for condition (3), it is clear that 
\[
\lim_{R \to \infty} \sup_{s \in [1, \infty), n \in \mathbb{N}} \|S_n T_{n, s} \cdot \chi_{0, R}^V - S_n T_{n, s}\| = 0.
\]

On the other hand, set 
\(R_0 := \sup_n \left\{\|f_n(x) - f_n(y)\| : x, y \in X_n\right\}\).

This is clear that \(R_0\) is finite. For any \(n \in \mathbb{N}\), \(x, y \in X_n\) and \(R \geq R_0\), we have:
\[
(\chi_{0, R}^V \cdot S_n \cdot \chi_{0, R- R_0}^V)_{x,y} = \chi_{f_n(x),R} \cdot S_n, x,y \cdot \chi_{f_n(y),R-R_0} = S_n, x,y \cdot \chi_{f_n(x),R} \cdot \chi_{f_n(y),R-R_0} = S_n, x,y \cdot \chi_{f_n(y),R-R_0} = (S_n \cdot \chi_{0, R- R_0}^V)_{x,y}.
\]

In other words, we obtain 
\[\chi_{0, R}^V \cdot S_n \cdot \chi_{0, R- R_0} = S_n \cdot \chi_{0, R- R_0}\]
which implies that 
\[
\|\chi_{0, R}^V S_n T_{n,s} - S_n T_{n,s}\| \leq \|\chi_{0, R}^V \cdot S_n \cdot (\chi_{0, R- R_0}^V T_{n,s} - T_{n,s})\| + \|S_n \cdot (\chi_{0, R- R_0} T_{n,s} - T_{n,s})\|.
\]

This tends to 0 as \(R \to \infty\) uniformly (in \(s\) and \(n\)) by assumption. 

**Lemma 5.4.** For any \((S_n) \in C[\mathcal{H}_E] \cap \prod_{n=1}^\infty C[\mathcal{H}_{x_n}]\), we have \([(S_n), F] \in \mathcal{A}'(X; E)^{\Gamma}\).
Proof. Fix an \((S_n) \in C[\mathcal{H}] \cap \prod_{n}^{\infty} C^*(\mathcal{H},n)\). From Proposition 4.4(1), it suffices to show that
\[
(s \mapsto [S_n, F^\Psi_{n,s}])_n
\]
belongs to \(A'(X;E)\) for any \(\Psi\) as in Proposition 4.4.

First note that for any \(n \in \mathbb{N}\) and \(x, y \in X_n\), we have
\[
([S_n, F^\Psi_{n,s}])_{x,y} = S_{n,x,y} \otimes (F^\Psi_{n,s+2d_n,fs}(y) - F^\Psi_{n,s+2d_n,fs}(x)).
\]
Hence for each \(n \in \mathbb{N}\), the function \(s \mapsto [S_n, F^\Psi_{n,s}]\) is bounded and norm continuous by Proposition 4.4(8) and Lemma 4.6. Also condition (4) in Definition 5.6 holds by Proposition 4.4(4) and (8). Moreover, condition (1) and (2) holds since \(F^\Psi_{n,s}\) has \(P\)-propagation 0, and \(S_n\) has \(E\)-propagation 0 together with Proposition 4.4(2).

Finally, for condition (3), note that
\[
(\chi^V_{0,R} \cdot [S_n, F^\Psi_{n,s}])_{x,y} = S_{n,x,y} \otimes \chi_{fs}(x,R)(F^\Psi_{n,s+2d_n,fs}(y) - F^\Psi_{n,s+2d_n,fs}(x)).
\]
Hence condition (3) follows from Proposition 4.4(10), finite \(P\)-propagation of \((S_n)\) and Lemma 4.6. \( \square \)

Lemma 5.5. For any projection \((p_n) \in C^*(\mathcal{H}) \cap \prod_{n}^{\infty} C^*(\mathcal{H},n)\), the element
\[
(s \mapsto (p_n F_{n,s} - p_n))_{n \in \mathbb{N}}
\]
is in \((p_n)A'(X;E)\)(\(p_n\)).

Proof. From Lemma 5.4, it suffices to show that \((s \mapsto p_n(F_{n,s} - p_n))_n\) is in \(A'(X;E)\). Moreover, we only need to show that
\[
(s \mapsto q_n(F^\Psi_{n,s} - q_n))_n
\]
is in \(A'(X;E)\) for any \(\Psi\) as in Proposition 4.4, where \((q_n)\) is a finite propagation approximation to \((p_n)\). For each \(n \in \mathbb{N}\), it follows from Proposition 4.4(7) and Lemma 4.6 that the function \(s \mapsto q_n(F^\Psi_{n,s} - q_n)\) is bounded and continuous. It is routine to check condition (1)-(4) in Definition 4.9 for this map using Proposition 4.4(3) and (9), the finite propagation of \((q_n)\) together with Lemma 4.6. \( \square \)

Now we are in the position to construct the index map \(\text{Ind}_F\) in (5.1). It follows from a standard construction in K-theories (see, e.g., [44, Definition 2.8.5]):

Definition 5.6. Let \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^\perp\) be a graded Hilbert space with grading operator \(U\) (i.e., \(U\) is a self-adjoint unitary operator in \(\mathcal{B}(\mathcal{H})\) such that \(\mathcal{H}^\pm\) coincides with the \((\pm 1)\)-eigenspace of \(U\)), and \(A\) be a \(C\)-subalgebra of \(\mathcal{B}(\mathcal{H})\) such that \(U\) is in the multiplier algebra of \(A\). Let \(F \in \mathcal{B}(\mathcal{H})\) be an odd operator of the form
\[
F = \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix}
\]
for some operators \(V : \mathcal{H}^\perp \to \mathcal{H}^+\) and \(W : \mathcal{H}^+ \to \mathcal{H}^\perp\). Suppose \(F\) satisfies:

- \(F\) is in the multiplier algebra of \(A\);
- \(F^2 - 1\) is in \(A\).

Then we define the index class \(\text{Ind}[F] \in K_0(A)\) of \(F\) to be
\[
\text{Ind}[F] := \begin{pmatrix} (1 - VW)^2 & V(1 - WV) \\ W(2 - VW)(1 - VV) & WV(2 - WV) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Combining Lemma 5.2 and Lemma 5.5 we obtain that for each projection \((p_n) \in C'(\mathcal{H}_t)^F \cap \bigoplus_{n=1}^\infty C'(\mathcal{H}_{r,n})^F\) the operator \((p_n)F(p_n)\) is an odd self-adjoint operator on the graded Hilbert space \(\bigoplus_n p_n(L^2([1, \infty), \mathcal{H}_{r,n,E})\) satisfying:

- \((p_n)F(p_n))\) is in the multiplier algebra of \((p_n)A'(X; E)^F(p_n)\);
- \((p_n)F(p_n))^2 - (p_n)\) is in \((p_n)A'(X; E)^F(p_n)\).

Hence Definition 5.6 produces an index class in \(K_0((p_n)A'(X; E)^F(p_n))\). Composing with the \(K_0\)-map induced by the inclusion \((p_n)A'(X; E)^F(p_n) \hookrightarrow A'(X; E)^F\), we get an element in \(K_0(A'(X; E)^F)\), denoted by \(\text{Ind}_F[(p_n)]\). An argument analogous to [34 Lemma 12.3.11] provides a well-defined homomorphism

\[
\text{Ind}_F : K_0(C'(\mathcal{H}_t)^F) \cap \bigoplus_{n=1}^\infty \bigoplus_{n=1}^\infty C'(\mathcal{H}_{r,n})^F) \rightarrow K_0(A'(X; E)^F).
\]

Passing to suspension and applying the above construction pointwise, we can also define \(\text{Ind}_F\) on the level of \(K_1\)-groups. Similarly, we can also deal with the localisation case by applying the above construction pointwise in the parameter \(t\). Consequently, we obtain the following:

**Proposition 5.7.** The process above provides well-defined homomorphisms:

\[
\text{Ind}_F : K_0(C'(\mathcal{H}_t)^F) \cap \bigoplus_{n=1}^\infty \bigoplus_{n=1}^\infty C'(\mathcal{H}_{r,n})^F) \rightarrow K_0(A'(X; E)^F)
\]

and

\[
\text{Ind}_{F_L} : K_0(C'(\mathcal{H}_t)^F) \cap \bigoplus_{n=1}^\infty \bigoplus_{n=1}^\infty C'(\mathcal{H}_{r,n})^F) \rightarrow K_0(A'(X; E)^F)
\]

for \(s = 0, 1\), which are called the equivariant index maps associated to \(F\).

Finally, we prove Proposition 5.1.

**Proof of Proposition 5.1** The proof follows the outline of [34 Proposition 12.3.13 and Proposition 12.6.3]. We will only focus on the case of equivariant (twisted) Roe algebras, while the localisation case follows from the same argument applied pointwise. Throughout the proof, we fix an \(s \in [1, \infty)\).

For each \(n \in \mathbb{N}\), we define a map \(\kappa_n : E_n \rightarrow E_n\) by

\[
\kappa_n(x) = \begin{cases} \frac{x}{|x|}(|x| - 1), & \text{if } |x| \geq 1; \\ 0, & \text{otherwise} \end{cases}
\]

and a sequence of maps:

\[
F^{(k)}_n : \mathcal{H}_{r,n,E} \rightarrow \mathcal{H}_{r,n,E}, \quad \delta_z \otimes \xi \otimes \delta_g \otimes u \mapsto \delta_z \otimes \xi \otimes \delta_g \otimes F_{n,s+2d_n,k_n^\infty}(r_n(z))^u
\]

for \(k \in \mathbb{N} \cup \{\infty\}\) and \(z \in Z_{r,n}\), where \(k_n^\infty(v) := 0\) for all \(v \in E_n\). Denote

\[
F^{(s)} := (F^{(k)}_n)_n \in \prod_n \mathcal{B}(\mathcal{H}_{r,n,E})^F \subseteq \mathcal{B}(\mathcal{H}_{r,E})^F.
\]

We note that for the fixed \(s\) at the beginning, we have that \(F^{(0)} = F_s\) from Definition 4.13 and \(F^{(\infty)} = (\text{Id}_{\mathcal{H}_n} \otimes F_{n,s+2d_n,0})_n\). Quite analogous to the construction in
Proposition 5.7, we obtain:

$$\text{Ind}_{F^{(0)}} : K_i\left(C^* (H_r)^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n})^T \right) \to K_i\left(C^* (H_{r, E})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E})^T \right)$$

for each $k \in \mathbb{N} \cup \{\infty\}$. It is clear that $\text{Ind}_{F^{(0)}} = \iota_k^* \circ \text{Ind}_f$. In the following, we will show that $\text{Ind}_{F^{(0)}} = \text{Ind}_{F^{(\infty)}}$ and that $\text{Ind}_{F^{(\infty)}}$ is an isomorphism to conclude the proof. We will only focus on the case of $K_0$, and the case of $K_1$ can be handled using a standard suspension argument.

First we show that $\text{Ind}_{F^{(0)}} = \text{Ind}_{F^{(\infty)}}$. Let

$$H_{r, n, E, \infty} := (H_{r, n, E})^{\infty} = H_{r, n} \otimes (L_{1_0})^{\infty} \text{ for all } n \in \mathbb{N} \text{ and } H_{r, E, \infty} := \bigoplus_n H_{r, n, E, \infty}.$$ Denote $C^* (H_{r, E, \infty})^T$ the corresponding equivariant Roe algebra. It is known that the “top-left corner inclusion”:

$$\iota : C^* (H_{r, E})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E})^T \to C^* (H_{r, E, \infty})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E, \infty})^T, \quad T \mapsto \begin{pmatrix} T & 0 & \ldots \\ 0 & 0 & \vdots \end{pmatrix}$$

induces an isomorphism on $K$-theory. To show that $\text{Ind}_{F^{(0)}} = \text{Ind}_{F^{(\infty)}}$, it suffices to show that $\iota_* \circ \text{Ind}_{F^{(0)}} = \iota_* \circ \text{Ind}_{F^{(\infty)}}$.

Without loss of generality, it suffices to show that $\iota_* \circ \text{Ind}_{F^{(\infty)}}[(p_n)] = \iota_* \circ \text{Ind}_{F^{(\infty)}}[(p_n)]$ for any projection $(p_n) \in C^* (H_{r, E})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E})^T$. For each $k \in \mathbb{N} \cup \{\infty\}$, it follows from Definition 5.7 that the class $\text{Ind}_{F^{(\infty)}}[(p_n)]$ can be represented by a concrete difference of projections, say

$$[(p_n^{(k)})] - [(q_n)],$$

where $(q_n)$ is independent of $k$ and we have

$$(p_n^{(k)}) - (q_n) \in M_2 \left( (p_n) \cdot \left( C^* (H_{r, E})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E})^T \right) \cdot (p_n) \right).$$

Now we consider the projections

$$\begin{pmatrix} (p_n^{(0)}) & 0 & 0 & \cdots \\ 0 & (p_n^{(1)}) & 0 & \cdots \\ 0 & 0 & (p_n^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (p_n^{(\infty)}) & 0 & 0 & \cdots \\ 0 & (p_n^{(\infty)}) & 0 & \cdots \\ 0 & 0 & (p_n^{(\infty)}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

in the multiplier algebra of $M_2 \left( C^* (H_{r, E, \infty})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E, \infty})^T \right)$. For any compact subset $K \subseteq P_r$ there exists $k \in \mathbb{N}$ such that $k^\epsilon (f_{r, n}(K \cap P_{r, n})) = 0$ for any $n \in \mathbb{N}$ (in fact there are only finitely many $n \in \mathbb{N}$ satisfying $K \cap P_{r, n} \neq \emptyset$). Hence it is easy to see that the difference of these projections is in $M_2 \left( C^* (H_{r, E, \infty})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E, \infty})^T \right)$. Therefore the formal difference

$$\begin{pmatrix} (p_n^{(0)}) & 0 & 0 & \cdots \\ 0 & (p_n^{(1)}) & 0 & \cdots \\ 0 & 0 & (p_n^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} (p_n^{(\infty)}) & 0 & 0 & \cdots \\ 0 & (p_n^{(\infty)}) & 0 & \cdots \\ 0 & 0 & (p_n^{(\infty)}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

defines a class in $K_0 \left( C^* (H_{r, E, \infty})^T \cap \prod_{n=1}^{\infty} C^* (H_{r, n, E, \infty})^T \right)$, denoted by $a$. 

\[ \text{Ind}_{F^{(\infty)}} \]
On the other hand, there is a class $b \in K_0(C^*(\mathcal{H}_{r,E,\infty})^F \cap \prod_{n=1}^{\infty} C^*(\mathcal{H}_{r,n,E,\infty})^F)$ defined by the formal difference

$$
\begin{pmatrix}
(p_n^{(0)}) & 0 & 0 & \cdots \\
0 & (p_n^{(1)}) & 0 & \cdots \\
0 & 0 & (p_n^{(2)}) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
- \begin{pmatrix}
(p_n^{(0)}) & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

We claim that $a + b = a$, whence $b = 0$. This implies that $\text{Ind}_{F^{(0)}}[p_n] = \text{Ind}_{F^{(\infty)}}[p_n]$. In fact, for each $k \in \mathbb{N}$ we consider the path $(F^{(k),r})_{r \in [0,1]} = ((F^{(k),r})_{r \in [0,1]}$ in $\mathcal{B}(\mathcal{H}_{r,E})^F$ defined by

$$
F^{(k),r}_n : \delta_{\infty} \otimes \xi \otimes \delta_{h} \otimes u \mapsto \delta_{\infty} \otimes \xi \otimes \delta_{h} \otimes F_{n,s+2d_n,(1-r)\kappa_{h}^{*}(f_{n}(2)) r\kappa_{h}^{*1}(f_{n}(2))^{-1}u \quad \text{on each} \quad \mathcal{H}_{r,n,E}.
$$

It follows from Proposition 4.4(5) that $\|F^{(k),r} - F^{(k),r'}\| \leq 3|r - r'|$ for each $n \in \mathbb{N}$. Hence $a$ is homotopic to

$$
\begin{pmatrix}
(p_n^{(1)}) & 0 & 0 & \cdots \\
0 & (p_n^{(2)}) & 0 & \cdots \\
0 & 0 & (p_n^{(3)}) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
- \begin{pmatrix}
(p_n^{(0)}) & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

which implies that $a + b = a$.

Finally we explain that $\text{Ind}_{F^{(\infty)}}$ is an isomorphism. For each $n \in \mathbb{N}$, note that $F^{(\infty)}_n = \text{Id}_{\mathcal{H}_{n,E}} \otimes F_{n,s+2d_n,0}$. Let $p_{n,0}$ be the projection onto the kernel of $F_{n,s+2d_n,0}$, which is one-dimensional. Recall that $F_{n,s+2d_n,0} = g((s+2d_n)^{-1}D + C)$ where $g(x) = x(1+x^2)^{-1/2}$. Now consider the path $(F^{(l)}_{n,s+2d_n,0})_{l \in [0,1]}$ defined by

$$
F^{(l)}_{n,s+2d_n,0} := g(l \left( \frac{D}{s + 2d_n} + C \right)).
$$

This defines a homotopy between $F_{n,s+2d_n,0}$ and $F^{(\infty)}_{n,s+2d_n,0}$ (for each $n \in \mathbb{N}$), which decomposes with respect to the grading as

$$
F^{(\infty)}_{n,s+2d_n,0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Hence for each $n$, this homotopy provides that

$$
\text{Ind}_{F^{(\infty)}}[p_n] = [p_n \otimes p_{n,0}],
$$

for any projection $(p_n) \in C^*(\mathcal{H}_r)^F \cap \prod_{n=1}^{\infty} C^*(\mathcal{H}_{r,n})^F$. Since the above homotopies are not uniformly continuous with respect to $n$, we need an extra argument (which is called a “stacking argument” in the proof of Proposition 3.4) to conclude that

$$
\text{Ind}_{F^{(\infty)}}[p_n] = [(p_n \otimes p_{n,0})].
$$

For the convenience to readers, here we provide more details. Without loss of generality, for any projection $(p_n) \in C^*(\mathcal{H}_r)^F \cap \prod_{n=1}^{\infty} C^*(\mathcal{H}_{r,n})^F$ we assume that

$$
\text{Ind}_{F^{(\infty)}}[p_n] = [(q_n) - (q_n^{(0)})],
$$

where $(q_n)$ and $(q_n^{(0)})$ are projections in $C^*(\mathcal{H}_{r,E})^F \cap \prod_{n=1}^{\infty} C^*(\mathcal{H}_{r,n,E})^F$. For each $n \in \mathbb{N}$, the path $F^{(l)}_{n,s+2d_n,0}$ above provides a homotopy of projections $[H_{n,0}]_{l \in [0,1]}$ such that $H_{n,0} = q_n$. Hence for each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that for any $l, s \in [0,1]$
with $|t - s| < \delta_n$, we have $\|H_{n,t} - H_{n,s}\| < \frac{1}{2}$. For each $n \in \mathbb{N}$, take an $K_n \in \mathbb{N}$ such that $\frac{1}{K_n} < \delta_n$ and an isometry
\[ V_n : \mathcal{H}_{r,n,E} \to \mathcal{H}_{r,K_n,E}, \quad v \mapsto (v, 0, \ldots, 0). \]
They give rise to an isometry
\[ V = (V_n)_n : \mathcal{H}_{r,E} \to \bigoplus_{n=1}^{\infty} \mathcal{H}_{r,K_n,E}. \]
Note that $V$ equivariantly covers the identity map on $P_r$, hence the following homomorphism
\[ \Phi := \text{Ad}_V : C^*(\mathcal{H}_{r,E})^\Gamma \cap \bigoplus_{n=1}^{\infty} C^*(\mathcal{H}_{r,K_n,E})^\Gamma \to C^*(\mathcal{H}_{r,E})^\Gamma \cap \bigoplus_{n=1}^{\infty} \mathcal{B}(\mathcal{H}_{r,K_n,E})^\Gamma \]
gives rise to an isomorphism on $K$-theories (see, e.g., [34] Theorem 5.2.6) for more details). Therefore, it suffices to show that
\[ \Phi_*([\text{Ind}_{F(0)}((p_n))] = \Phi_*([p_n \otimes p_{n,0}]). \]
Note that for each $n \in \mathbb{N}$, we have
\[
\begin{pmatrix}
q_n & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
H_{n,0} & 0 & \cdots & 0 \\
0 & H_{n,K_n} & \cdots & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & H_{n,1}
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & H_{n,0} & \cdots & 0 \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & H_{n,1}\n\end{pmatrix}.
\]
On the other hand, note that
\[
\left\| \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & H_{n,K_n} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ddots & H_{n,1}
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & H_{n,0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ddots & H_{n,1}\n\end{pmatrix} \right\| < \frac{1}{2}
\]
Hence for each $n \in \mathbb{N}$ there exists a homotopy connecting these two matrices, which are uniformly continuous with respect to $n$. Following by a rotation homotopy between
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & H_{n,0} & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & H_{n,K_n-1}\n\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
H_{n,0} & 0 & \cdots & 0 \\
0 & H_{n,K_n} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ddots & H_{n,1}\n\end{pmatrix},
\]
we obtain that
\[
\begin{pmatrix}
q_n & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
H_{n,1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}.
\]
Note that these homotopies are uniformly continuous with respect to $n$, hence from the construction of $\Phi$ we obtain that
\[ \Phi_*([\text{Ind}_{F(0)}((p_n))] = \Phi_*([p_n \otimes p_{n,0}]), \]
which concludes the proof. \qed
6. Isomorphisms between $K$-theories of equivariant twisted algebras

In this section, we study the $K$-theories of equivariant twisted algebras $A'((X,E)\Gamma$ and $A'_L((X,E)\Gamma$ introduced in Section 4.2. First recall that for $r > 0$, there is the evaluation-at-one homomorphism
\[ \text{ev} : A'_L((X,E)\Gamma \rightarrow A'((X,E)\Gamma, \quad (T_t) \mapsto T_1. \]
Also recall that for $r \leq s$, the inclusion $Z_r \rightarrow Z_s$ induces isometric embeddings of Hilbert spaces
\[ \mathcal{H}_{r,n} \rightarrow \mathcal{H}_{s,n} \quad \text{and} \quad \mathcal{H}_{r,n,E} \rightarrow \mathcal{H}_{s,n,E} \]
for each $n \in \mathbb{N}$. These maps give rise to the following commutative diagram:
\[
\begin{array}{ccc}
A'_L((X,E)\Gamma & \xrightarrow{\text{ev}} & A'((X,E)\Gamma \\
\downarrow & & \downarrow \\
A'_L((X,E)\Gamma & \xrightarrow{\text{ev}} & A'((X,E)\Gamma \\
\end{array}
\]
for each $r \leq s$, which further induces a homomorphism between direct limits when taking $r \rightarrow +\infty$.

The main result of this section is the following:

**Theorem 6.1.** The evaluation-at-one map induces the following isomorphism in $K$-theories:
\[ \text{ev}_*: \lim_{r \rightarrow \infty} K_s(A'_L((X,E)\Gamma) \rightarrow \lim_{r \rightarrow \infty} K_s(A'((X,E)\Gamma) \]
for $* = 0, 1$.

The proof follows the outline of that in [34, Section 12.4], and the main ingredient is to use appropriate Mayer-Vietoris arguments for twisted algebras (Proposition 6.5). This allows us to chop the space into easily-handled pieces, on which we apply the result from [16] on the Baum-Connes conjecture with coefficients for a-T-menable groups.

Let us start with some more notions. By saying that $(F_n)_{n \in \mathbb{N}}$ is a sequence of closed subsets in $(E_n)_{n \in \mathbb{N}}$, we mean that each $F_n$ is a closed subset of $E_n$. Firstly, we define the following subalgebras associated to $(F_n)$:

**Definition 6.2.** Fix an $r > 0$. For a sequence of closed subsets $(F_n)$ in $(E_n)$, we define $A'_{L,(F_n)}((X,E)\Gamma$ to be the set of elements $(T_{n,s}) \in A'((X,E)\Gamma$ satisfying: for each $n$ and $\varepsilon > 0$ there exists $s_{n,\varepsilon} > 0$ such that for $s \geq s_{n,\varepsilon}$ we have:
\[ \text{supp}_E(T_{n,s}) \subseteq N_{s}(F_n) \times N_{s}(F_n). \]
Denote by $A'_{L,(F_n)}((X,E)\Gamma$ the $C^*$-algebra of the norm closure of $A'_{L,(F_n)}((X,E)\Gamma$ in $A'((X,E)\Gamma$.

Denote by $A'_{L,(F_n)}((X,E)\Gamma$ the set of elements $(T_t)$ in $A'_{L,(F_n)}((X,E)\Gamma$ satisfying that $T_t \in A'_{L,(F_n)}((X,E)\Gamma$ for each $t \in [1,\infty)$. Define $A'_{L,(F_n)}((X,E)\Gamma$ to be the $C^*$-algebra of the completion of $A'_{L,(F_n)}((X,E)\Gamma$ with respect to the norm $\|T_t\| = \sup_t \|T_t\|$. Equivalently, the algebra $A'_{L,(F_n)}((X,E)\Gamma$ consists of elements $(T_t)$ in $A'_{L,(F_n)}((X,E)\Gamma$ such that each $T_t$ belongs to $A'_{L,(F_n)}((X,E)\Gamma$.

The following lemma is straightforward, hence we omit the proof.
Lemma 6.3. With the same notation as above, \( A'_{(F_n)}(X; E)^\Gamma \) is an ideal in \( A'(X; E)^\Gamma \) and \( A'_{L(F_n)}(X; E)^\Gamma \) is an ideal in \( A'_L(X; E)^\Gamma \).

Moreover, we have the following.

Lemma 6.4. Fix an \( r > 0 \). Let \((F_n)\) and \((G_n)\) be two sequences of compact subsets in \((E_n)\). Then

\[
A'_{(F_n)}(X; E)^\Gamma \cap A'_{(G_n)}(X; E)^\Gamma = A'_{(F_n \cap G_n)}(X; E)^\Gamma
\]

and

\[
A'_{(F_n)}(X; E)^\Gamma + A'_{(G_n)}(X; E)^\Gamma = A'_{(F_n \cup G_n)}(X; E)^\Gamma.
\]

The same holds for the localisation case.

Proof. We first prove the case of equivariant twisted Roe algebras. For the first equation, note that \( A'_{(F_n \cap G_n)}(X; E)^\Gamma \subseteq A'_{(F_n)}(X; E)^\Gamma \cap A'_{(G_n)}(X; E)^\Gamma \) holds trivially. The converse comes from a \( C^*\)-algebraic fact that the intersection of two ideals coincides with their product together with a basic fact for metric space: For a compact metric space \( K \), a closed cover \((C, D)\) of \( K \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( N_\delta(C) \cap N_\delta(D) \subseteq N_\varepsilon(C \cap D) \).

For the second equation, note that \( A'_{(F_n)}(X; E)^\Gamma + A'_{(G_n)}(X; E)^\Gamma \subseteq A'_{(F_n \cup G_n)}(X; E)^\Gamma \) holds trivially. As for the converse, we fix a \((T_{n,s}) \in \mathfrak{A}'_{(F_n \cup G_n)}(X; E)^\Gamma\). For \( n \in \mathbb{N} \) there is a strictly increasing sequence \((s_{n,k})_{k \in \mathbb{N}}\) in \([1, \infty)\) tending to infinity such that for \( s \geq s_{n,k} \), we have

\[
\text{supp}_E(T_{n,s}) \subseteq N_{\frac{1}{n+1}}(F_n \cup G_n) \times N_{\frac{1}{n+1}}(F_n \cup G_n).
\]

For each \( n \in \mathbb{N} \) and \( s \in [1, \infty) \), we construct an operator \( W_{n,s} \in \mathfrak{B}(\mathcal{H}_{r,n,E})^\Gamma \) as follows:

\[
W_{n,s} = \begin{cases} \chi_{N_1(F_n)} & \text{if } 1 \leq s \leq s_{n,1}; \\ \frac{s_{n,k+1}-s}{s_{n,k+1}-s_{n,k}} \chi_{N_{\frac{1}{n+1}}(F_n)} + \frac{s-s_{n,k}}{s_{n,k+1}-s_{n,k}} \chi_{N_{\frac{1}{n+1}}(F_n)} & \text{for } s_{n,k} \leq s \leq s_{n,k+1} \text{ where } k \in \mathbb{N}. \end{cases}
\]

Then the map \( s \mapsto W_{n,s} \) is in \( C_0([1, \infty), \mathfrak{B}(\mathcal{H}_{r,n,E})^\Gamma) \). Moreover, it is clear that \((s \mapsto W_{n,s})_n\) is also in the multiplier algebra of \( A'(X; E)^\Gamma \). Now we consider:

\[
(T_{n,s}) = (W_{n,s})(T_{n,s}) + (1 - W_{n,s})(T_{n,s}) \\
= (W_{n,s})(T_{n,s}) + (1 - W_{n,s})(T_{n,s})(W_{n,s}) + (1 - W_{n,s})(T_{n,s})(1 - W_{n,s}).
\]

It is clear that \((W_{n,s})(T_{n,s})\) and \((1 - W_{n,s})(T_{n,s})(W_{n,s})\) are in \( A'_{(F_n)}(X; E)^\Gamma \). Also note that from the construction above, for each \( n \in \mathbb{N} \) and \( s \geq s_{n,k} \) we have:

\[
\text{supp}_E((1 - W_{n,s})T_{n,s}(1 - W_{n,s})) \subseteq N_{\frac{1}{n+1}}(G_n) \times N_{\frac{1}{n+1}}(G_n).
\]

Hence we obtain that \( A'_{(F_n)}(X; E)^\Gamma + A'_{(G_n)}(X; E)^\Gamma \) is dense in \( A'_{(F_n \cup G_n)}(X; E)^\Gamma \), which concludes the proof.

For the localisation case, we apply the above argument pointwise to obtain the first equation. Concerning the second, note that the above \((W_{n,s})_n\) might not vary continuously with respect to the parameter \( t \). To get around the issue, we need an approximation argument.

Fix a \((T_t) = (T_{t,0}) \in \mathfrak{A}'_{L(F_n \cup G_n)}(X; E)^\Gamma\). Note that the map \( t \mapsto T_t \) is uniformly continuous, hence for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for any \( t', t'' \in [1, \infty) \) with \( |t' - t''| \leq \delta \), we have \( \|T_{t'} - T_{t''}\| \leq \varepsilon \). For each \( p \in \mathbb{N} \) we set \( t_p = 1 + p\delta \), and
also set \( t_{n,1} = 1 \). We construct another element \((T^e_i) = (T^e_{i,p,s}) \in \mathcal{A}'_{L,(F_n \cup G_n)}(X; E)^T \) by setting \( T^e_{i,p} = T^e_p \) for each \( p \in \mathbb{N} \) and do linear combination between \( t_p \) and \( t_{p+1} \). It is clear that \(||(T^e_i) - (T^e_i)|| \leq 2\varepsilon\). Since \( \varepsilon \) is arbitrarily chosen, it suffices to show that each \((T^e_i) \in \mathcal{A}'_{L,(F_n)}(X; E)^T + \mathcal{A}'_{L,(G_n)}(X; E)^T \).

For each \( n \in \mathbb{N} \) and \( p \in \mathbb{N} \cup \{0\} \), there is a strictly increasing sequence \((s_{n,k,p})_{k \in \mathbb{N}}\) in \([1, \infty)\) tending to infinity such that for \( s \geq s_{n,k,p} \) and \( t = t_{p-1}, t_p, t_{p+1} \) we have

\[
\text{supp}_E(T_{i,p,s}) \subseteq \mathcal{N}_{\text{rel}}(F_n \cup G_n) \times \mathcal{N}_{\text{rel}}(F_n \cup G_n).
\]

For each \( n \in \mathbb{N} \), \( p \in \mathbb{N} \cup \{0\} \) and \( s \in [1, \infty) \), we construct an operator \( W_{i,p,n,s} \in \mathcal{B}(\mathcal{H}_{t,n,E})^T \) as follows:

\[
W_{i,p,n,s} = \begin{cases} \chi_{\mathcal{N}_i(F_n)} & \text{if } 1 \leq s \leq s_{n,1,p}; \\ \frac{s_{n,k+1}-s}{s_{n,k+1}-s_{n,k}} \chi_{\mathcal{N}_i(F_n)} + \frac{s-s_{n,k}}{s_{n,k+1}-s_{n,k}} \chi_{\mathcal{N}_i(F_n)}, & \text{for } s_{n,k,p} \leq s \leq s_{n,k+1,p} \text{ where } k \in \mathbb{N}. \end{cases}
\]

Then the map \( s \mapsto W_{i,p,n,s} \) is in \( C_p([1, \infty), \mathcal{B}(\mathcal{H}_{t,n,E})^T) \). Moreover, it is clear that for each \( p \in \mathbb{N} \cup \{0\} \), \((s \mapsto W_{i,p,n,s})_{p} \) is also in the multiplier algebra of \( \mathcal{A}'(X; E)^T \). Now we define a path \((W_i)_{t \in [1, \infty)} \) by setting \( W_i = (W_{i,p,n,s}) \) for each \( p \in \mathbb{N} \cup \{0\} \), and do linear combination between \( t_p \) and \( t_{p+1} \). It is clear that \( t \mapsto W_t \) is \( (W_{i,p,n,s}) \) is uniformly continuous and bounded, and hence it is in the multiplier algebra of \( \mathcal{A}'_{L}(X; E)^T \). Now consider:

\[
(T^e_{i,p,s}) = (W_{l,n,s})(T^e_{l,n,s}) + (1 - W_{l,n,s})(T^e_{l,n,s})
= (W_{l,n,s})(T^e_{l,n,s}) + (1 - W_{l,n,s})(W_{l,n,s})(1 - W_{l,n,s}).
\]

It is clear that \( t \mapsto W_t \cdot T^e_i \) and \( t \mapsto (1 - W_t) \cdot T^e_i \) \( W_t \) are in \( \mathcal{A}'_{L,(F_n)}(X; E)^T \). Note that for any \( p \in \mathbb{N} \cup \{0\} \) and \( s \leq s_{n,k+1,p} \), we have

\[
\text{supp}_E(1 - W_{i,p,s}) \subseteq (\mathcal{N}_{\text{rel}}(F_n)^T) \times (\mathcal{N}_{\text{rel}}(F_n)^T).
\]

Setting \( s'_{n,k,p} \) to be the minimal of \( s_{n,k,p} \) and \( s_{n,k,p+1} \), it follows that for any \( t \in [t_p, t_{p+1}] \) and \( s \in [s'_{n,k,p}, s'_{n,k+1,p}] \), we have

\[
\text{supp}_E(1 - W_{i,s}) = \text{supp}_E(1 - (\lambda W_{i,p,s} + (1 - \lambda) W_{i,p+1,s})) \subseteq (\mathcal{N}_{\text{rel}}(F_n)^T) \times (\mathcal{N}_{\text{rel}}(F_n)^T)
\]

and

\[
\text{supp}_E((T^e_{i,p,s}) = \text{supp}_E(\lambda T_{i,p,s} + (1 - \lambda) T_{i,p+1,s}) \subseteq \mathcal{N}_{\text{rel}}(F_n \cup G_n) \times \mathcal{N}_{\text{rel}}(F_n \cup G_n),
\]

where \( \lambda \) is the combinatorial parameter determined by \( t \in [t_p, t_{p+1}] \). These inequalities imply that

\[
\text{supp}_E((1 - W_{i,s})T^e_{i,s}(1 - W_{i,s})) \subseteq \mathcal{N}_{\text{rel}}(G_n) \times \mathcal{N}_{\text{rel}}(G_n).
\]

Finally note that the sequence \((s'_{n,k,p})_{k \in \mathbb{N}}\) tends to infinity, hence we obtain that \((T^e_i) \in \mathcal{A}'_{L,(F_n)}(X; E)^T + \mathcal{A}'_{L,(G_n)}(X; E)^T \) as required.

Consequently, we obtain the following Mayer-Vietoris sequences for twisted algebras:
Proposition 6.5. Let \((F_n)\) and \((G_n)\) be two sequences of compact subsets in \((E_n)\) and fix an \(r > 0\). Then we have the following six-term exact sequence:

\[
\begin{align*}
K_0\left(A'_r(F_n \cap G_n)(X; E)^\Gamma\right) &\to K_0\left(A'_r(F_n)(X; E)^\Gamma\right) \\
&\quad \oplus K_0\left(A'_r(G_n)(X; E)^\Gamma\right) \to K_0\left(A'_r(F_n \cup G_n)(X; E)^\Gamma\right) \\
K_1\left(A'_r(F_n \cap G_n)(X; E)^\Gamma\right) &\to K_1\left(A'_r(F_n)(X; E)^\Gamma\right) \\
&\quad \oplus K_1\left(A'_r(G_n)(X; E)^\Gamma\right) \to K_1\left(A'_r(F_n \cup G_n)(X; E)^\Gamma\right)
\end{align*}
\]

The same holds in the case of equivariant twisted localisation algebras. Furthermore, we have the following commutative diagram:

\[
\begin{array}{cccc}
\cdots & \to K_0\left(A'_r(F_n \cap G_n)(X; E)^\Gamma\right) & \to K_0\left(A'_r(F_n)(X; E)^\Gamma\right) & \to K_0\left(A'_r(G_n)(X; E)^\Gamma\right) & \to K_0\left(A'_r(F_n \cup G_n)(X; E)^\Gamma\right) & \cdots \\
& & \downarrow & & \downarrow & \\
\cdots & \to K_1\left(A'_r(F_n \cap G_n)(X; E)^\Gamma\right) & \to K_1\left(A'_r(F_n)(X; E)^\Gamma\right) & \to K_1\left(A'_r(G_n)(X; E)^\Gamma\right) & \to K_1\left(A'_r(F_n \cup G_n)(X; E)^\Gamma\right) & \cdots \\
\end{array}
\]

where the vertical maps are induced by the evaluation-at-one homomorphisms.

Proposition 6.5 allows us to chop the space into small pieces, on which we have the following “local isomorphism” result. Recall that a family \([Z_i]_{i \in I}\) of subspaces in a metric space \(Z\) is mutually \(R\)-separated for some \(R > 0\) if \(d(Z_i, Z_j) > R\) for \(i \neq j\).

**Proposition 6.6.** Let \((F_n)\) be a sequence of closed subsets in \((E_n)\) such that \(F_n = \bigsqcup_{j=1}^{\infty} F_j^{(n)}\) for a mutually 3-separated family \(\{F_j^{(n)}\}_j\), and there exist \(R > 0\) and \(y_j^{(n)} \in X_i/\Gamma\) such that \(F_j^{(n)} \subseteq B(y_j^{(n)}; R)\). Then the homomorphism induced by the evaluation-at-one map

\[
\text{ev. : } \lim_{r \to \infty} K_*\left(A'_r(F_n)(X; E)^\Gamma\right) \to \lim_{r \to \infty} K_*\left(A'_r(G_n)(X; E)^\Gamma\right)
\]

is an isomorphism for \(* = 0, 1\).

The proof of Proposition 6.6 is technical and divided into several steps. Before we present the detailed proof, let us first use it to conclude the proof of Theorem 6.1. To achieve that, we need an extra lemma from [34, Lemma 12.4.5]:

**Lemma 6.7.** For any \(s > 0\), there exist \(M \in \mathbb{N}\) and decompositions

\[
X_n/\Gamma = Y_{n,1} \sqcup Y_{n,2} \sqcup \cdots \sqcup Y_{n,M}\]

for all \(n \in \mathbb{N}\), such that the family \(\left\{B(y_j^{(n)}; s)\right\}_{y_j \in Y_{n,i}}\) is mutually 3-separated for each \(n \in \mathbb{N}\) and \(i = 1, 2, \ldots, M\).

**Proof of Theorem 6.1** Given \(s > 0\), let \(M \in \mathbb{N}\) and \(\{Y_{n,i}\}_{n \in \mathbb{N}, 1 \leq i \leq M}\) be provided by Lemma 6.7. Setting \(W_n^s := N_s(y_j^{(n)}) = N_s(f_n(X_n))\) and \(W_{n,j}^s := \bigsqcup_{y_j \in Y_{n,i}} B(y_j^{(n)}; s)\), we have \(W_n^s = \bigcup_{i=1}^{M} W_{n,i}^s\). For each \(i\) applying Proposition 6.6 to the sequence of subsets \((W_{n,i}^s)_{n}\), we obtain that the homomorphism induced by the evaluation-at-one map

\[
\text{ev. : } \lim_{r \to \infty} K_*\left(A'_r(W_{n,i}^s)(X; E)^\Gamma\right) \to \lim_{r \to \infty} K_*\left(A'_r(W_{n,i}^s)(X; E)^\Gamma\right)
\]

is an isomorphism for each \(i\). Applying the Mayer-Vietoris sequence from Proposition 6.5 \((M - 1)\)-times (and Proposition 6.6 again to deal with the intersection)
together with the Five Lemma, we obtain that the homomorphism induced by the evaluation-at-one map

\[ \text{ev.} : \lim_{r \to \infty} K_r(A_{t(L(E))}^r(X/E)^\Gamma) \longrightarrow \lim_{r \to \infty} K_r(A_{t(L(E))}^r(X/E)^\Gamma) \]

is an isomorphism. Finally note that condition (3) in Definition 4.9 and condition (2) in Definition 4.10 imply that for each \( r > 0 \), we have

\[ A'(X; E)^\Gamma = \lim_{s \to 0} A_{t(L(E))}^r(X/E)^\Gamma \quad \text{and} \quad A'_L(X; E)^\Gamma = \lim_{s \to 0} A'_{L(L(E))}^r(X/E)^\Gamma. \]

Hence we conclude the proof. \( \Box \)

The rest of this section is devote to the proof of Proposition 6.6. First let us introduce some more notation. Fix an \( r > 0 \). Let \( (F_n) \) and \( (G_n) \) be sequences of closed subsets in \( (E_n) \). Define:

\[ A'(X; (G_n))^\Gamma := \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \operatorname{Id}_{H_n} \otimes \chi_{G_n} \right) \cdot A'(X; E)^\Gamma \cdot \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \operatorname{Id}_{H_n} \otimes \chi_{G_n} \right) \]

and

\[ A'_{(F_n)}(X; (G_n))^\Gamma := A'(X; (G_n))^\Gamma \cap A'_{(F_n)}(X; E)^\Gamma. \]

Moreover, given a sequence of \( \Gamma \)-invariant subspaces \( O_n \subseteq X_n \) \((n \in \mathbb{N})\) we define:

\[ A'((O_n); (G_n))^\Gamma := \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \chi_{P_n(O_n)} \otimes \operatorname{Id}_{L^2_{F_n}} \right) \cdot A'(X; (G_n))^\Gamma \cdot \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \chi_{P_n(O_n)} \otimes \operatorname{Id}_{L^2_{F_n}} \right) \]

and

\[ A'_{(F_n)}((O_n); (G_n))^\Gamma := \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \chi_{P_n(O_n)} \otimes \operatorname{Id}_{L^2_{F_n}} \right) \cdot A'_{(F_n)}(X; (G_n))^\Gamma \cdot \left( \operatorname{Id}_{L^2([1, \infty])} \otimes \chi_{P_n(O_n)} \otimes \operatorname{Id}_{L^2_{F_n}} \right). \]

Define the \( C^* \)-algebras \( A'_{L}(X; (G_n))^\Gamma \) and \( A'_{L}((O_n); (G_n))^\Gamma \) to be those consisting of elements \((T_i) \in A'_{L}(X; E)^\Gamma\) such that each \( T_i \) belongs to \( A'(X; (G_n))^\Gamma \) and \( A'((O_n); (G_n))^\Gamma \), respectively. Also define the \( C^* \)-algebras \( A'_{L(F_n)}(X; (G_n))^\Gamma \) and \( A'_{L(F_n)}((O_n); (G_n))^\Gamma \) to be those consisting of elements \((T_i) \in A'_{L(F_n)}(X; E)^\Gamma\) such that each \( T_i \) belongs to \( A'_{(F_n)}(X; (G_n))^\Gamma \) and \( A'_{(F_n)}((O_n); (G_n))^\Gamma \), respectively.

On the other hand, we define

\[ A'_0(X; E)^\Gamma := \left\{ (T_{n,s}) \in A'(X; E)^\Gamma : \lim_{n,s} T_{n,s} = 0 \text{ for each } n \in \mathbb{N} \right\} \]

and

\[ A'_0(X; (G_n))^\Gamma := A'_0(X; E)^\Gamma \cap A'(X; (G_n))^\Gamma \]

It is clear that \( A'_0(X; (G_n))^\Gamma \) is an ideal in \( A'(X; (G_n))^\Gamma \), and also an ideal in \( A'_{0(F_n)}(X; (G_n))^\Gamma \) for any sequence of closed subsets \( (F_n) \). Define the algebras \( A'_{L,0}(X; E)^\Gamma \) and \( A'_{L,0}(X; (G_n))^\Gamma \) to be those consisting of elements \((T_i) \in A'_{L}(X; E)^\Gamma\) such that each \( T_i \) belongs to \( A'_0(X; E)^\Gamma \) and \( A'_0(X; (G_n))^\Gamma \), respectively.

**Lemma 6.8.** Fix an \( r > 0 \). Then we have:

\[ K_r(A'_0(X; E)^\Gamma) = 0 \quad \text{and} \quad K_r(A'_0(X; (G_n))^\Gamma) = 0. \]

And we also have:

\[ K_r(A'_{L,0}(X; E)^\Gamma) = 0 \quad \text{and} \quad K_r(A'_{L,0}(X; (G_n))^\Gamma) = 0. \]
Proof. Here we only prove $K_*(A'_0(X; E)^\Gamma) = 0$ since the other is similar, and the case of localisation holds by using the same argument pointwise. Let
\[
\mathcal{H}_{r,n,E,\infty} := (\mathcal{H}_{r,n,E})^{\oplus \infty} \quad \text{and} \quad \mathcal{H}_{r,E,\infty} := \bigoplus_n \mathcal{H}_{r,n,E,\infty}.
\]
Using these admissible modules we construct the equivariant twisted Roe algebra, denoted by $A'_{\infty}(X; E)^\Gamma$, and similarly construct $A'_{0,\infty}(X; E)^\Gamma$. For each $n \in \mathbb{N}$, define an isometry
\[
V_n : \mathcal{H}_{r,n,E} \to \mathcal{H}_{r,n,E,\infty}, \quad v \mapsto (v, 0, 0, \ldots).
\]
Then $(V_n)$ induces the “top-left corner inclusion”
\[
\text{Ad}_{(V_n)} : A'_0(X; E)^\Gamma \to A'_{0,\infty}(X; E)^\Gamma, \quad (T_{n,s}) \mapsto (T_{n,s} \oplus \bigoplus_{k=1}^\infty 0),
\]
which induces the identity map on the $K$-groups.

It is also easy to check that the formula
\[
(\alpha_n) : (T_{n,s})_{n,s} \to \left(0 \oplus \bigoplus_{k=1}^\infty T_{n,s+k}\right)_{n,s}, \quad \text{where} \ \alpha_n : (T_{n,s})_{n,s} \to \left(0 \oplus \bigoplus_{k=1}^\infty T_{n,s+k}\right)_{n,s}
\]
and
\[
(\beta_n) : (T_{n,s})_{n,s} \to \left(\bigoplus_{k=0}^\infty T_{n,s+k+1}\right)_{n,s}, \quad \text{where} \ \beta_n : (T_{n,s})_{n,s} \to \left(\bigoplus_{k=0}^\infty T_{n,s+k+1}\right)_{n,s}
\]
give two well-defined homomorphisms from $A'_0(X; E)^\Gamma$ to $A'_{0,\infty}(X; E)^\Gamma$. Let
\[
S_n : \mathcal{H}_{r,n,E,\infty} \to \mathcal{H}_{r,n,E,\infty}, \quad (v_0, v_1, v_2, \ldots) \mapsto (0, v_0, v_1, \ldots)
\]
be the shift operator. It is clear that $(S_n)_n$ is in the multiplier algebra of $A'_{0,\infty}(X; E)^\Gamma$, which builds a conjugation between $(\alpha_n)$ and $(\beta_n)$. More precisely, for each $n \in \mathbb{N}$ we have
\[
\alpha_n(T_{n,s}) = S_n \cdot \beta_n(T_{n,s}) \cdot S_n^*.
\]
Applying [34, Proposition 2.7.5], we have
\[
(\alpha_n)_* = (\beta_n)_*.
\]

On the other hand, notice that
\[
(\text{Ad}_{(V_n)} + (\alpha_n))(T_{n,s})_{n,s} = \left(\bigoplus_{k=0}^\infty T_{n,s+k}\right)_{n,s}.
\]
Since $\text{Ad}_{(V_n)}$ and $(\alpha_n)$ have orthogonal images and $(\text{Ad}_{(V_n)} + (\alpha_n))(T_{n,s})_{n,s}$ is homotopic to $(\beta_n)(T_{n,s})_{n,s}$, we have
\[
(\beta_n)_* = (\text{Ad}_{(V_n)} + (\alpha_n))_* = (\text{Ad}_{(V_n)})_* + (\alpha_n)_*.
\]
Consequently we obtain that $(\text{Ad}_{(V_n)})_* = 0$, which concludes the proof. \qed

Now we move back to Proposition [6.6] Fix an $r > 0$. Recall that $F_n = \bigsqcup_{j=1}^{\infty} F_j^{(n)}$ where $\{F_j^{(n)}\}_j$ is a mutually 3-separated family, and there exist $R > 0$ and $y_j^{(n)} \in X_n/\Gamma$ such that $F_j^{(n)} \subseteq B(\xi(y_j^{(n)}); R)$. Also recall that the $\Gamma$-action on $\bigsqcup_n X_n$ has controlled distortion with certain fundamental domain $\mathcal{D}$. For each $n$ and $j$, denote $x_j^{(n)}$ the
unique point in $X_n \cap D$ such that $\pi(x_j^{(n)}) = y_j^{(n)}$. Since each $X_n/\Gamma$ is finite, then for each $n \in \mathbb{N}$ there are only finitely many $j$'s such that $F_j^{(n)}$ is non-empty. Denote the set of such $j$ by $J_n$. Taking $G_j^{(n)} = N_1(F_j^{(n)})$ for each $n$ and $j$, we define the “restricted product”:

$$\prod_{j}^{res} A^r_{(F_j^{(n)})}(X; (G_j^{(n)})) : = \left( \prod_{j} A^r_{(F_j^{(n)})}(X; (G_j^{(n)})) \right) \cap A^r_0(X; E)$$

and

$$\prod_{j}^{res} A^r_{L,0}(X; (G_j^{(n)})) : = \left( \prod_{j} A^r_{L,(F_j^{(n)})}(X; (G_j^{(n)})) \right) \cap A^r_{L,0}(X; E)$$

have trivial K-theories.

The following lemma is a key step in the proof of Proposition 6.6.

Lemma 6.10. With the same notation as above, the inclusions

$$t : \prod_{j}^{res} A^r_{(F_j^{(n)})}(X; (G_j^{(n)})) \hookrightarrow A^r_0(X; E)$$

and

$$t_L : \prod_{j}^{res} A^r_{L,(F_j^{(n)})}(X; (G_j^{(n)})) \hookrightarrow A^r_{L,0}(X; E)$$

induce isomorphisms in K-theory.

Proof. We only prove the first, and the second can be proved using the same argument pointwise. The proof follows the outline of [34, Theorem 6.4.20].

Consider the following quotient algebras:

$$A^r_{(F_n)}(X; E) : = \frac{A^r_0(X; E)}{A^r_0(X; E)} \quad \text{and} \quad \prod_{j}^{res} A^r_{(F_j^{(n)})}(X; (G_j^{(n)})) : = \frac{\prod_{j}^{res} A^r_{(F_j^{(n)})}(X; (G_j^{(n)}))}{\prod_{j}^{res} A^r_0(X; (G_j^{(n)}))}.$$
We also define a map \( \iota : X \to \mathbb{C} \), i.e., \( \iota \). We claim that the composition \( \gamma : X \to \mathbb{C} \) induces isomorphisms in \( K \)-theories.

It follows from Lemma 6.8 and 6.9 that the quotient maps

\[
A_{(a)}^{r}((X; E) \to A_{(a)}^{r}(X; E)^{\Gamma} \quad \text{and} \quad \prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma} \to \prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma})
\]

induce isomorphisms in \( K \)-theories.

We also define a map

\[
\gamma : A_{(a)}^{r}(X; E)^{\Gamma} \to \prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma}) \quad \text{by} \quad (T_{n, s}) \mapsto \prod_{j} (\chi_{G_{j}^{(n)}}, T_{n, s} \chi_{G_{j}^{(n)}}),
\]

which induces a \( \ast \)-homomorphism

\[
\gamma_{\ast} : A_{(a)}^{r}(X; E)^{\Gamma} \to \prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma}).
\]

We claim that the composition \( \iota \circ \gamma \) equals to the identity map on the quotient algebra, i.e., the map \( \iota \circ \gamma = \text{identity map} \mod A_{0}^{r}(X; E)^{\Gamma} \). Given \( (T_{n, s}) \in A_{(a)}^{r}(X; E)^{\Gamma} \), then for each \( n \in \mathbb{N} \) there exists \( s_{n} \in [1, \infty) \) such that for any \( s > s_{n} \) we have \( \text{supp}_{E}(T_{n, s}) \subseteq \bigcup_{j} (G_{j}^{(n)} \times G_{j}^{(n)}) \). Hence for \( s > s_{n} \), we obtain:

\[
\sum_{j} \chi_{G_{j}^{(n)}}, T_{n, s} \chi_{G_{j}^{(n)}} - T_{n, s} = 0,
\]

which implies that

\[
(T_{n, s}) - \sum_{j} (\chi_{G_{j}^{(n)}}, T_{n, s} \chi_{G_{j}^{(n)}}) \in A_{0}^{r}(X; E)^{\Gamma}.
\]

On the other hand, it is clear that \( \gamma \circ \iota \) is the identity map and descends to identity map \( \gamma \circ \iota \). Hence \( \iota \circ \gamma \) induces an isomorphism in \( K \)-theory, which implies that the inclusion \( \iota \) induces an isomorphism in \( K \)-theory. \( \square \)

Using the same notation as above, we consider the following commutative diagram:

\[
\begin{array}{ccc}
A_{(a)}^{r}(X; E)^{\Gamma} & \xrightarrow{ev} & A_{(a)}^{r}(X; E)^{\Gamma} \\
\prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma} & \xrightarrow{ev} & \prod_{j} A_{(b)^{n}}^{r}(X; (G_{j}^{(n)})^{\Gamma}).
\end{array}
\]
It follows from Lemma 6.10 that vertical maps induce isomorphisms in K-theory. Also note that condition (3) in Definition 4.9 implies that

\[
\prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}(X; (G_{(n)}^{(j)}))^\Gamma = \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}((\pi^{-1}(B(y_{j}^{(n)}, m))); (G_{(n)}^{(j)}))^\Gamma
\]

\[
= \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma
\]

and condition (2) in Definition 4.10 implies that

\[
\prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}(X; (G_{(n)}^{(j)}))^\Gamma = \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}((\pi^{-1}(B(y_{j}^{(n)}, m))); (G_{(n)}^{(j)}))^\Gamma
\]

\[
= \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma.
\]

On the other hand, note that

\[
\lim_{r \to \infty} \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma = \lim_{m \to \infty} \lim_{r \to \infty} \prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma
\]

and

\[
\lim_{r \to \infty} \lim_{m \to \infty} \prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma = \lim_{m \to \infty} \lim_{r \to \infty} \prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma
\]

since these limits are just direct union of increasing subalgebras.

Consequently, in order to conclude Proposition 6.6, it suffices to prove the following:

**Proposition 6.11.** For each fixed m, the evaluation-at-one map

\[
\text{ev} : \lim_{r \to \infty} \prod_{j}^{\text{res}} A_{(L,(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma \longrightarrow \lim_{r \to \infty} \prod_{j}^{\text{res}} A_{(F_{(n)}^{(j)})}^{r}((\mathcal{N}_{m}(\Gamma \cdot x_{j}^{(n)})); (G_{(n)}^{(j)}))^\Gamma
\]

induces an isomorphism in K-theory.

Roughly speaking, Proposition 6.11 follows from a family version of the result that the Baum-Connes conjecture with coefficients holds for a-T-menable groups due to Higson and Kasparov [16]. For convenience to readers, we provide more details here.

From now on, let us fix an m ≥ 0. To simplify the notation, denote an index set \( \Lambda := \{(n, j) : n \in \mathbb{N}, j \in J\} \). For each \( \lambda = (n, j) \in \Lambda \), denote \( x_{\lambda} = x_{j}^{(n)}, X_{\lambda} := \mathcal{N}_{m}(\Gamma \cdot x_{\lambda}), \)

\( F_{\lambda} := F_{j}^{(n)}, G_{\lambda} := G_{j}^{(n)} \) and \( L_{\lambda}^{2} := L^{2}(G_{\lambda}, \text{Cliff}(E_{n})) \). Also denote the Hilbert spaces

\( \mathcal{H}_{\rho,\lambda} := \ell^{2}(Z_{\rho,\tau} \cap P_{r}(X_{\lambda})) \otimes \mathcal{H} \otimes \ell^{2}(\Gamma) \)

and

\( \mathcal{H}_{\rho,\lambda,E} := \ell^{2}(Z_{\rho,\tau} \cap P_{r}(X_{\lambda})) \otimes \mathcal{H} \otimes \ell^{2}(\Gamma) \otimes L_{\lambda}^{2} \)

for \( \lambda = (n, j) \in \Lambda \). Note that both \( \mathcal{H}_{\rho,\lambda} \) and \( \mathcal{H}_{\rho,\lambda,E} \) are \( \Gamma \)-invariant.
It is clear that \( \mathcal{H}_{r,\lambda} \) is an admissible \( P_r(X_\lambda) \)-module, and \( \mathcal{H}_{r,\lambda,E} \) is both an admissible \( P_r(X_\lambda) \)-module and an ample \( G_\lambda \)-module. We use them to build the equivariant Roe algebras \( C^*(\mathcal{H}_{r,\lambda}) \) and \( C^*(\mathcal{H}_{r,\lambda,E}) \) of \( P_r(X_\lambda) \).

For \( x \in X_\lambda \), write \( B_{x,r,\lambda} = B_{x,r,n} \cap P_r(X_\lambda) \) and

\[
\mathcal{H}_{x,r,\lambda} := \chi_{B_{x,r,\lambda}} \mathcal{H}_{r,\lambda}, \quad \mathcal{H}_{x,r,\lambda,E} = \chi_{B_{x,r,\lambda}} \mathcal{H}_{r,\lambda,E}.
\]

Again we represent a bounded operator \( T \) on \( \mathcal{H}_{r,\lambda} \) (respectively, \( \mathcal{H}_{r,\lambda,E} \)) as an \( X_\lambda \)-by-\( X_\lambda \) matrix \((T_{x,y})_{x,y \in X_\lambda}\) where each \( T_{x,y} \) is a bounded operator \( \mathcal{H}_{y,r,\lambda} \to \mathcal{H}_{x,r,\lambda} \) (respectively, \( \mathcal{H}_{y,r,\lambda,E} \to \mathcal{H}_{x,r,\lambda,E} \)).

**Definition 6.12.** Fix an \( r > 0 \). Let \( \prod_{\lambda \in \Lambda} C_b([1, \infty), C^*(\mathcal{H}_{r,\lambda,E})^\Gamma) \) denote the product \( C^* \)-algebra of all bounded continuous functions from \([1, \infty)\) to \( C^*(\mathcal{H}_{r,\lambda,E})^\Gamma \) with supremum norm. Write elements of this \( C^* \)-algebra as a collection \( (T_{\lambda,s})_{\lambda \in \Lambda, s \in [1, \infty)} \) for \( T_{\lambda,s} = (T_{\lambda,s,x,y})_{x,y \in X_\lambda} \in C^*(\mathcal{H}_{r,\lambda,E})^\Gamma \), whose norm is

\[
\| (T_{\lambda,s}) \| = \sup_{\lambda \in \Lambda, s \in [1, \infty)} \| T_{\lambda,s} \|.
\]

Let \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) denote the \(*\)-subalgebra of \( \prod_{\lambda \in \Lambda} C_b([1, \infty), C^*(\mathcal{H}_{r,\lambda,E})^\Gamma) \) consisting of elements satisfying the following conditions:

1. \( \sup_{s \in [1, \infty), \lambda \in \Lambda} \text{prop}_r(T_{\lambda,s}) < \infty; \)
2. for \( \lambda \in \Lambda \), \( \lim_{s \to \infty} \text{prop}_r(T_{\lambda,s}) = 0; \)
3. for \( \lambda \in \Lambda \) and \( x, y \in X_\lambda \), the map \( s \mapsto T_{\lambda,s,x,y} \) belongs to the subalgebra \( \mathcal{R}(\mathcal{H}_{y,r,\lambda}, \mathcal{H}_{x,r,\lambda}) \otimes C_b([1, \infty), \mathcal{R}(L^2_\lambda)); \)
4. for \( \varepsilon > 0 \) and \( \lambda \in \Lambda \), there exists an \( s_{\lambda,\varepsilon} \in [1, \infty) \) such that for any \( s > s_{\lambda,\varepsilon} \) we have:

\[
\text{supp}_r(T_{\lambda,s}) \subseteq \mathcal{N}_c(F_\lambda) \times \mathcal{N}_c(F_\lambda).
\]

Denote \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) the norm-closure of \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) in \( \prod_{\lambda \in \Lambda} C_b([1, \infty), C^*(\mathcal{H}_{r,\lambda,E})^\Gamma) \).

Also define \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) to be the collection of uniformly continuous bounded functions \((T_t)\) from \([1, \infty)\) to \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) such that the \( P \)-propagation of \( (T_t) = (T_{\lambda,s,x,y}) \) tends to zero as \( t \to +\infty \). Denote \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) the completion of \( C^*[X_\lambda]; (F_\lambda)]^\Gamma \) with respect to the norm \( \| (T_t) \| := \sup_t \| T_t \| \).

For later use, let us record the following lemma. The proof follows directly from the bounded geometry of \( X_\lambda \), hence omitted.

**Lemma 6.13.** Condition (3) in Definition 6.12 is equivalent to the following: for \( \lambda \in \Lambda \) and bounded Borel subsets \( K_1, K_2 \subseteq P_r(X_\lambda) \), the map \( s \mapsto \chi_{K_1} T_{\lambda,s} \chi_{K_2} \) belongs to the algebra \( \mathcal{R}(\mathcal{H}_{\lambda,E}) \otimes C_b([1, \infty), \mathcal{R}(L^2_\lambda)); \)

The following result is our motivation to introduce the above algebras:

**Lemma 6.14.** We have the following natural isomorphisms between \( C^* \)-algebras:

1. \( \prod_j^r \mathcal{A}_{(f_j)}^r \left( (\mathcal{N}_m(\Gamma \cdot \chi_j^{(n)})); (G_j^{(n)}) \right)^\Gamma \cong C^*[X_\lambda]; (F_\lambda)]^\Gamma; \)
2. \( \prod_j^r \mathcal{A}_{(f_j)}^r \left( (\mathcal{N}_m(\Gamma \cdot \chi_j^{(n)})); (G_j^{(n)}) \right)^\Gamma \cong C^*[X_\lambda]; (F_\lambda)]^\Gamma. \)
Proof. We only prove the first isomorphism, while the second follows by the same argument pointwise. Since \(J_n\) is finite, \(s \mapsto T_{n,s}\) is norm-continuous if and only if \(s \mapsto T_{\lambda,s}\) is norm-continuous for \(\lambda = (n,j) \in \Lambda\) where \(T_{\lambda,s} = \chi_G T_{n,s} \chi_G\).

For \((T_{n,s}) \in \prod_{j}^{(\text{res})} A_{(\xi_j)}((N_m(\Gamma \cdot x^{(n)}_j));(G_j^{(m)})^\Gamma),\) condition (3) in Definition 4.9 says that
\[
\lim_{R' \to \infty} \sup_{s \in [1,\infty)} \|\lambda_{R',s}^{(V)} T_{\lambda,s} - T_{\lambda,s}\| = \lim_{R' \to \infty} \sup_{s \in [1,\infty),s \in \Lambda} \|T_{\lambda,s} \lambda_{R',s}^{(V)} - T_{\lambda,s}\| = 0.
\]

Since \(f_n((N_m(\Gamma \cdot x^{(n)}_j))) \subseteq B(\xi(y^{(n)}_j),\rho_+(m))\) where \(\rho_+\) is from the coarse embedding \(\xi,\)
(6.1) is equivalent to the following:
\[
\lim_{R' \to \infty} \sup_{s \in [1,\infty),s \in \Lambda} \|\lambda_{R',s}^{(V)} T_{\lambda,s} - T_{\lambda,s}\| = \lim_{R' \to \infty} \sup_{s \in [1,\infty),s \in \Lambda} \|T_{\lambda,s} \lambda_{R',s}^{(V)} - T_{\lambda,s}\| = 0.
\]

On the other hand, note that for any \(\lambda \in \Lambda\) and \(s \in [1,\infty)\) we have:
\[
\text{supp}_E(T_{\lambda,s}) \subseteq G_\lambda \times G_\lambda \subseteq B(\xi(y^{(n)}_j), R + 1) \times B(\xi(y^{(n)}_j), R + 1)
\]
where \(R\) is the constant given in the assumption of Proposition 6.6. Hence (6.1) holds for \((T_{n,s}),\) which concludes the proof.

Consequently, to prove Proposition 6.11 it suffices to show that the evaluation-at-one map:
\[
\text{ev} : \lim_{r \to \infty} C_{s}(\{(X_{\lambda});(F_{\lambda})\})^\Gamma \rightarrow \lim_{r \to \infty} C_{s}((X_{\lambda});(F_{\lambda}))^\Gamma
\]
induces an isomorphism in \(K\)-theory. To achieve, we need an extra version of the twisted algebras built on Hilbert modules.

For \(\lambda \in \Lambda\), denote the \(C\)-algebra \(B_\lambda\) as the norm closure in \(C_b([1,\infty), \mathcal{R}(L^2_\lambda))\) consisting of operators \(T' = (T_s)\) satisfying the following:

(1) \(\lim_{s \to \infty} \text{prop}_E(T_s) = 0;\)

(2) for \(\varepsilon > 0\) there exists an \(s_{\lambda,\varepsilon} \in [1,\infty)\) such that for any \(s > s_{\lambda,\varepsilon}\) we have:
\[
\text{supp}_E(T_s) \subseteq N_\varepsilon(F_\lambda) \times N_\varepsilon(F_\lambda).
\]

Consider the Hilbert \(B_\lambda\)-module:
\[
\mathcal{H}_{\rho,\lambda} \otimes B_\lambda = \ell^2(Z_{\rho,\lambda} \cap P_r(X_{\lambda})) \otimes \mathcal{H} \otimes \ell^2(\Gamma) \otimes B_\lambda,
\]
and denote the \(C\)-algebra of adjointable morphisms on \(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda\) by \(\mathcal{L}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda)\). For \(T \in \mathcal{L}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda)\), we define its \(P\)-propagation as in Definition 2.9 and also denote by \(\text{prop}_P(T)\). Denote \(\mathcal{R}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda)\) the \(C\)-algebra of compact morphisms on \(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda\). The \(\Gamma\)-action on \(\mathcal{H}_{\rho,\lambda}\) extends to a \(\Gamma\)-action on \(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda\) by adjointable morphisms. Denote the set of \(\Gamma\)-invariant morphisms by \(\mathcal{L}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda)^\Gamma\).

We consider the following:

**Definition 6.15.** With the same notation as above, define \(\mathcal{L}_{\ast}((X_{\lambda});(B_{\lambda})^\Gamma)\) to be the \(*\)-subalgebra in \(\prod_{\lambda \in \Lambda} \mathcal{L}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda)^\Gamma\) consisting of elements \(T = (T_{\lambda})\) satisfying the following conditions:

(1) \(\sup_{\lambda \in \Lambda} \text{prop}_P(T_{\lambda}) < \infty;\)

(2) for \(\lambda \in \Lambda, T_{\lambda}\) is locally compact in the sense that for any bounded Borel subset \(K \subseteq P_r(X_{\lambda}),\) both \(\chi_K T_{\lambda}\) and \(T_{\lambda} \chi_K\) belong to \(\mathcal{R}(\mathcal{H}_{\rho,\lambda} \otimes B_\lambda) \cong \mathcal{R}(\mathcal{H}_{\rho,\lambda}) \otimes B_\lambda.\)
Denote \( \Psi'(X_\lambda); (B_\lambda))^\Gamma \) the norm closure of \( \Psi'(X_\lambda); (B_\lambda))^\Gamma \) in \( \prod_{\lambda \in \Lambda} L(H_{r,\lambda})^\Gamma \).

Also define \( \Psi_f'(X_\lambda); (B_\lambda))^\Gamma \) to be the closure of the collection of uniformly continuous bounded functions \( (T_i) \) from \([1, \infty)\) to \( \Psi'(X_\lambda); (B_\lambda))^\Gamma \) such that the \( P \)-propagation of \( (T_i) \) tends to zero as \( t \to +\infty \).

We provide the following matrix version of elements in \( \Psi'(X_\lambda); (B_\lambda))^\Gamma \) for later use. For \( \lambda \in \Lambda \), we have the following decomposition of Hilbert \( B_\lambda \)-modules:

\[
\mathcal{H}_{r,\lambda} \otimes B_\lambda = \bigoplus_{x \in X_\lambda} \mathcal{H}_{x,\lambda} \otimes B_\lambda.
\]

Given an \( X_\lambda \)-by-\( X_\lambda \) matrix \( S = (S_{x,y})_{x,y \in X_\lambda} \) with \( S_{x,y} \in \mathcal{R}(H_{y,\lambda}, H_{x,\lambda}) \otimes B_\lambda \) and finite propagation, we consider the map (using the same notation)

\[
S : \mathcal{H}_{r,\lambda} \otimes B_\lambda \to \mathcal{H}_{r,\lambda} \otimes B_\lambda
\]

by matrix multiplication. It is easy to check that \( S \) is an adjointable morphism on \( \mathcal{H}_{r,\lambda} \otimes B_\lambda \) which is locally compact. Also it is obvious that any locally compact \( S \in L(\mathcal{H}_{r,\lambda} \otimes B_\lambda) \) with finite propagation comes from such an \( X_\lambda \)-by-\( X_\lambda \) matrix.

In conclusion, elements in \( \Psi'(X_\lambda); (B_\lambda))^\Gamma \) can be written in the form of \( S = (S_\lambda) \) where \( S_\lambda = (S_{x,y})_{x,y \in X_\lambda} \) is an \( X_\lambda \)-by-\( X_\lambda \) matrix with matrix entry \( S_{x,y} \in \mathcal{R}(H_{y,\lambda}, H_{x,\lambda}) \otimes B_\lambda \) such that \( (S_\lambda)_\lambda \) has uniformly finite propagation. The converse holds as well.

The following lemma allows us to turn Proposition 6.11 into the setting of Hilbert modules.

**Lemma 6.16.** For each \( r > 0 \), we have natural \( C^* \)-isomorphisms:

\[
C^r\Gamma(X_\lambda); (F_\lambda))^\Gamma \cong \Psi'(X_\lambda); (B_\lambda))^\Gamma \text{ and } C^r\Gamma(X_\lambda); (F_\lambda))^\Gamma \cong \Psi'_r(X_\lambda); (B_\lambda))^\Gamma.
\]

**Proof.** First we define a map

\[
\Theta : C^r\Gamma(X_\lambda); (F_\lambda))^\Gamma \to \prod_{\lambda \in \Lambda} \mathcal{B}(L^2([1, \infty)) \otimes \mathcal{H}_{r,\lambda} \otimes L^2_\lambda)^\Gamma
\]

by

\[
(\Theta((T_{\lambda,s}))_\lambda(f \otimes \xi))(s) = T_{\lambda,s}(f(s)\xi),
\]

where \( f \in L^2([1, \infty)) \) and \( \xi \in \mathcal{H}_{r,\lambda} \otimes L^2_\lambda \). It is clear that \( \Theta \) is an injective \( * \)-homomorphism. Combining with the following isomorphism

\[
\mathcal{B}(L^2([1, \infty)) \otimes \mathcal{H}_{r,\lambda} \otimes L^2_\lambda)^\Gamma \cong \mathcal{B}(\mathcal{H}_{r,\lambda} \otimes L^2([1, \infty)), L^2_\lambda)^\Gamma
\]

for \( \lambda \in \Lambda \), the image of \( \Theta \) consists of elements \((T_{\lambda})_\lambda \in \prod_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_{r,\lambda} \otimes L^2([1, \infty)), L^2_\lambda)^\Gamma\) satisfying the following:

1. \( \sup_{\lambda \in \Lambda} \text{prop}_p(T_{\lambda}) < \infty \);
2. for \( \lambda \in \Lambda \) and \( x, y \in X_\lambda \), the matrix entry \( T_{\lambda,x,y} \) belongs to \( \mathcal{R}(H_{y,\lambda}, H_{x,\lambda}) \otimes B_\lambda \).

Note that here we use the fact that for each \( \lambda \in \Lambda \), the operator \( T_\lambda \) can be determined by finitely many \( \chi_K T_{\lambda,K'} \) where \( K \) and \( K' \) are bounded Borel subsets in \( P_r(X_\lambda) \) since \( T_\lambda \) is \( \Gamma \)-equivariant and the action on \( P_r(X_\lambda) \) is cocompact (using an argument similar to that in the proof of Lemma 4.12).
Noting that the image of $\Theta$ coincides with elements in $\Psi[(X,\lambda);(B,\lambda)]$ using the matrix form introduced above, we obtain a $\ast$-isomorphism
\[ \hat{\Theta} : C^\lambda((X,\lambda);(F,\lambda)) \to \Psi[(X,\lambda);(B,\lambda)]. \]
Also note that $B,\lambda$ can be faithfully represented on $L^2(1,\infty), L^2,\lambda)$. Hence it follows from the Hilbert module theory that $\hat{\Theta}$ is also isometric, which can be extended to a $C^\ast$-isomorphism (using the same notation)
\[ \hat{\Theta} : C^\ast((X,\lambda);(F,\lambda)) \cong \Psi((X,\lambda);(B,\lambda)). \]
Applying $\hat{\Theta}$ pointwise, we obtain a required isomorphism between twisted localisation algebras. Therefore, we conclude the proof. 

Therefore, to prove Proposition 6.11, it suffices to prove the following:

**Proposition 6.17.** The homomorphism
\[ \text{ev.} : \lim_{r \to \infty} K_* \left( \Psi^\Gamma((X,\lambda);(B,\lambda)) \right) \to \lim_{r \to \infty} K_* \left( \Psi^\Gamma((X,\lambda);(B,\lambda)) \right) \]
induced by the evaluation-at-one map is an isomorphism for $\ast = 0, 1$.

Readers might already notice that Proposition 6.17 is just a reformulation of a family version of the Baum-Connes conjecture with coefficients for the a-T-menable group $\Gamma$ (see, e.g., [22, Section 3]), which holds thanks to a “uniform version” of the proof by Higson and Kasparov [16]. This is well-known to experts, however, we cannot find an explicit proof in literature. For convenience to readers and also for completeness, we provide a detailed proof in Appendix A using an approach slightly different from the original one for a single space [16].

Consequently, we finish the proof of Proposition 6.11 and hence conclude Theorem 6.1.

### 7. Proof of Theorem 1.1

In this final section, we finish the proof of the main result.

**Proof of Theorem 1.1** Consider the following commutative diagram
\[
\begin{array}{ccc}
\lim_{r \to \infty} K_* \left( C_L^\lambda(H^\Gamma_r) \cap \prod_{n=1}^\infty C_L^\lambda(H^\Gamma_{r,n}) \right) & \to & \lim_{r \to \infty} K_* \left( C^\Gamma(H^\Gamma_r) \cap \prod_{n=1}^\infty C^\Gamma(H^\Gamma_{r,n}) \right) \\
\downarrow \text{Ind}_{H_r} & & \downarrow \text{Ind}_{H} \\
\lim_{r \to \infty} K_* \left( A^\lambda_L(X;E) \right) & \to & \lim_{r \to \infty} K_* \left( A^\Gamma(X;E) \right) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\lim_{r \to \infty} K_* \left( C^\lambda(H^\Gamma_{r,E}) \cap \prod_{n=1}^\infty C^\lambda(H^\Gamma_{r,n,E}) \right) & \to & \lim_{r \to \infty} K_* \left( C^\Gamma(H^\Gamma_{r,E}) \cap \prod_{n=1}^\infty C^\Gamma(H^\Gamma_{r,n,E}) \right),
\end{array}
\]
where the vertical maps come from Proposition 5.1 and all horizon maps are induced by evaluation-at-one maps. From Proposition 5.1 again, the compositions of vertical maps are isomorphisms. The middle horizon map is an isomorphism by Theorem 6.1 and it is clear that the upper horizon map identifies with the bottom horizon one. Therefore we obtain that both of the upper and bottom
horizon maps are isomorphisms using diagram chasing. Finally combining with Corollary 3.8, we conclude the proof.

□

Appendix A. Proof of Proposition 6.17

In this appendix, we prove a family version of the Baum-Connes conjecture with coefficients for a-T-menable groups, and equivalently concludes the proof of Proposition 6.17. Here we use a slightly different approach from the original proof for a single space due to Higson and Kasparov [16], which also involves the localisation technique introduced in [35] and [36] (see also [22] and [9]).

Throughout this appendix, let us fix another separable infinite-dimensional Hilbert space $H$ (which is a different notation from the fixed Hilbert space $\mathcal{H}$ in Section 4.2). Also fix a left-invariant proper metric $d_\Gamma$ on the group $\Gamma$.

Let us start with some notation. First recall that by definition, $\Gamma$ being a-T-menable means that $\Gamma$ admits a metrically proper action on $H$ by isometries. Using the Mazur-Ulam Theorem [25], there exists a unitary representation $\pi : \Gamma \to U(H)$ and a 1-cocycle $b : \Gamma \to H$ such that $\gamma \cdot v = \pi(\gamma)v + b(\gamma)$ and $\lim_{\gamma \to \infty} \|b(\gamma)\| = +\infty$. Here $b$ is a 1-cocycle means that $b(\gamma_1\gamma_2) = \pi(\gamma_1)b(\gamma_2) + b(\gamma_1)$ for any $\gamma_1, \gamma_2 \in \Gamma$. It is easy to see that the map $b : \Gamma \to H$ is a coarse embedding.

For each $\gamma \in \Gamma$ and $k \in \mathbb{N}$, we define a finite-dimensional Euclidean affine subspace $W_k(\gamma)$ in $H$ as follows:

$$ (A.1) \quad W_k(\gamma) := b(\gamma) + \text{span}_\mathbb{C} \left\{ b(\gamma') - b(\gamma) : d_\Gamma(\gamma', \gamma) \leq k^2 \right\}. $$

It is straightforward to check that $\gamma' \cdot W_k(\gamma) = W_k(\gamma' \gamma)$. Moreover, we define:

$$ W(\gamma) := \bigcup_{k \in \mathbb{N}} W_k(\gamma). $$

Since $b : \Gamma \to H$ is a coarse embedding, the space $V := W(\gamma)$ is independent of $\gamma \in \Gamma$. Note that $b(1_\Gamma) = 0$, hence $V$ is a $\Gamma$-invariant countably infinite-dimensional linear subspace in $H$. Without loss of generality, we assume that $V$ is dense in $H$.

We recall an algebra associated to $V$ introduced by Hison, Kasparov and Trout in [17]. Let $V_{a \cdot}$ be a finite-dimensional affine subspaces of $V$. Denote by $V_{a \cdot}^0$ the finite-dimensional linear subspace of $V$ consisting of differences of elements in $V_{a \cdot}$. Let $\text{Cliff}(V_{a \cdot}^0)$ be the complexified Clifford algebra of $V_{a \cdot}^0$, and $C(V_{a \cdot}) := C_0(V_{a \cdot}, \text{Cliff}(V_{a \cdot}^0))$ be the graded $C^*$-algebra of continuous functions from $V_{a \cdot}$ to $\text{Cliff}(V_{a \cdot}^0)$ which vanish at infinity. Let $S := C_0(\mathbb{R})$, graded according to odd and even functions. Define the graded tensor product

$$ \mathcal{A}(V_{a \cdot}) := S \otimes C(V_{a \cdot}). $$

For two finite-dimensional affine subspaces $V_{a \cdot} \subseteq V_b$ in $V$, we have a decomposition $V_b = V_{b a}^0 + V_{a \cdot}$, where $V_{b a}^0$ is the orthogonal complement of $V_{a \cdot}$ in $V_{b \cdot}^0$. For each $v_b \in V_{b \cdot}$, we have a corresponding decomposition $v_b = v_{b a} + v_a$, where $v_{b a} \in V_{b a}^0$ and $v_a \in V_{a \cdot}$. Every function $h$ on $V_{a \cdot}$ can be extended to a function $\tilde{h}$ on $V_b$ by the formula $\tilde{h}(v_{b a} + v_a) = h(v_a)$.

**Definition A.1.** For $V_{a \cdot} \subseteq V_b$, denote by $C_{V_{b \cdot}, V_{a \cdot}} : V_b \to \text{Cliff}(V_{b \cdot}^0)$ the function $v_b \mapsto v_{b a} \in \text{Cliff}(V_{b a}^0)$, where $v_{b a}$ is regarded as an element in $\text{Cliff}(V_{b a}^0)$ via the inclusion $V_{b a}^0 \subset \text{Cliff}(V_{b a}^0)$. Let $X$ be the unbounded multiplier of $S$ with degree
one given by the function \( t \mapsto t \). Define a ∗-homomorphism \( \beta_{V_b, V_a} : \mathcal{A}(V_a) \to \mathcal{A}(V_b) \) by the formula
\[
\beta_{V_b, V_a}(g \hat{\otimes} h) = g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{V_b, V_a})(h \hat{\otimes} 1)
\]
for \( g \in S \) and \( h \in C(V_a) \), where \( g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{V_b, V_a}) \) is defined by functional calculus.

The maps \( \beta_{V_b, V_a} \) in Definition [A.1] make the collection \( \{ \mathcal{A}(V_a) \} \) into a directed system as \( V_a \) ranges over the set of all finite-dimensional affine subspaces of \( V \). We define the C∗-algebra \( \mathcal{A}(V) \) as the associated direct limit:
\[
\mathcal{A}(V) := \lim_{\longrightarrow} \mathcal{A}(V_a).
\]
We denote by \( \beta_{V_b, V_a} : \mathcal{A}(V_a) \to \mathcal{A}(V) \) the associated homomorphism.

Endow the set \( \mathbb{R}_+ \times H \) with the weakest topology for which the projection \( \mathbb{R}_+ \times H \to H \) is weakly continuous and the function \((t, h) \mapsto t^2 + ||h||^2 \) on \( \mathbb{R}_+ \times H \) is continuous. It is clear that \( \mathbb{R}_+ \times H \) is a locally compact Hausdorff space with this topology. Also note that for \( v \in H \) and \( r > 0 \), the ball
\[
B(v, r) := \{(t, h) \in \mathbb{R}_+ \times H : t^2 + ||h - v||^2 < r^2 \}
\]
is open. For each finite-dimensional affine subspace \( V_a \subset V \), the center of \( \mathcal{A}(V_a) \) contains \( C_0(\mathbb{R}_+ \times V_a) \) (by even extension). For \( V_a \subset V_b \), the map \( \beta_{ba} \) takes \( C_0(\mathbb{R}_+ \times V_a) \) into \( C_0(\mathbb{R}_+ \times V_b) \). It is clear that the C∗-algebra \( \lim_{\rightarrow} C_0(\mathbb{R}_+ \times V_a) \) is ∗-isomorphic to \( C_0(\mathbb{R}_+ \times H) \), where the direct limit is over the directed set of all finite-dimensional affine subspaces \( V_a \subset V \). Hence the center of \( \mathcal{A}(V) \) contains \( C_0(\mathbb{R}_+ \times H) \).

The \( (\mathbb{R}_+ \times H) \)-support of an element \( a \in \mathcal{A}(V) \), denoted by \( \text{supp}_{\mathbb{R}_+ \times H}(a) \), is defined to be the complement of all \((t, h) \in \mathbb{R}_+ \times H \) for which there exists \( g \in C_0(\mathbb{R}_+ \times H) \) such that \( ag = 0 \) and \( g(t, h) \neq 0 \).

Now we consider actions on these algebras induced by the \( \Gamma \)-action on \( H \). Let \( V_a \) be a finite-dimensional affine subspace in \( V \). For any \( \gamma \in \Gamma \), the unitary \( \pi(\gamma) \) naturally induces a homomorphism \( \text{Cliff}(V_a^0) \to \text{Cliff}(\pi(\gamma)V_a^0) \), also denoted by \( \pi(\gamma) \). This induces a homomorphism
\[
\gamma : C(V_a) \to C(\gamma V_a)
\]
by
\[
\gamma(h)(v) := \pi(\gamma)h(\gamma^{-1}v)
\]
for \( h \in C(V_a) \) and \( v \in \gamma V_a \), which further induces a homomorphism
\[
\gamma : \mathcal{A}(V_a) \to \mathcal{A}(\gamma V_a)
\]
by \( g \hat{\otimes} h \mapsto g \hat{\otimes} \gamma(h) \).

Recall from [9] Lemma 4.6] that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{A}(V_a) & \xrightarrow{\beta_{V_b, V_a}} & \mathcal{A}(V_b) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\mathcal{A}(\gamma V_a) & \xrightarrow{\beta_{\gamma V_b, \gamma V_a}} & \mathcal{A}(\gamma V_b).
\end{array}
\]

Consequently, we obtain a \( \Gamma \)-action on \( \mathcal{A}(V) \), which makes \( \mathcal{A}(V) \) into a \( \Gamma \)-C∗-algebra. Moreover, the \( \Gamma \)-C∗-algebra \( \mathcal{A}(V) \) is proper in the following sense.

**Definition A.2** [14]. A \( \Gamma \)-C∗-algebra \( \mathcal{A} \) is called proper if there exists a second countable, locally compact, proper \( \Gamma \)-space \( Z \), and an equivariant ∗-homomorphism from \( C_0(Z) \) into the center of the multiplier algebra of \( \mathcal{A} \) such that \( C_0(Z) \mathcal{A} \) is dense in \( \mathcal{A} \). In this case, we also say that \( \mathcal{A} \) is proper over \( Z \).
Lemma A.3 ([16 Proposition 4.9]). The \( \Gamma \)-\( \mathcal{C} \)-algebra \( \mathcal{A}(V) \) is proper over \( \mathbb{R}_+ \times H \).

Before we introduce the twisted algebras with \( \mathcal{A}(V) \)-coefficients, let us simplify some notation. Denote

\[
\beta_{k',k}(\gamma) := \beta_{W_k(\gamma),W_{k'}(\gamma)} : \mathcal{A}(W_k(\gamma)) \rightarrow \mathcal{A}(W_{k'}(\gamma))
\]

for \( k < k' \) and \( \gamma \in \Gamma \), and

\[
\beta_k(\gamma) := \beta_{W_k(\gamma)} : \mathcal{A}(W_k(\gamma)) \rightarrow \mathcal{A}(V).
\]

We also write

\[
\beta(\gamma) := \beta_0(\gamma) : S \otimes \mathbb{C} \cong \mathcal{A}(W_0(\gamma)) \rightarrow \mathcal{A}(V).
\]

It follows from [9 Lemma 4.6] that Diagram (A.2) commutes, which implies the following:

Lemma A.4. For \( k \in \mathbb{N} \) and \( \gamma', \gamma \in \Gamma \), we have \( \gamma'\beta_{k,0}(\gamma) = \beta_{k,0}(\gamma'\gamma) \). Hence we obtain that \( \gamma'\beta(\gamma) = \beta(\gamma'\gamma) \).

Now let us introduce twisted algebras for the \( \mathcal{C} \)-algebras \( \mathcal{W}'((X_\Lambda);(B_\lambda))^F \) and \( \mathcal{W}'_r((X_\Lambda);(B_\lambda))^F \) introduced in Definition 6.15. We follow the same notation from Section 6. For each \( r > 0 \), recall that we have the Hilbert \( B_\lambda \)-module:

\[
\mathcal{H}_{r,\lambda} \otimes B_\lambda = \ell^2(Z_{r,\lambda} \cap P_r(X_\lambda)) \otimes \mathcal{H} \otimes \ell^2(\Gamma) \otimes B_\lambda.
\]

We also consider the following Hilbert \( B_\lambda \otimes \mathcal{A}(V) \)-module:

\[
\mathcal{H}_{r,\lambda} := (\mathcal{H}_{r,\lambda} \otimes B_\lambda) \otimes \mathcal{A}(V) = \mathcal{H}_{r,\lambda} \otimes (B_\lambda \otimes \mathcal{A}(V)),
\]

with the extended \( \Gamma \)-action. For \( T \in \mathcal{L}(\mathcal{H}_{r,\lambda}) \), its \( P \)-propagation can be defined as before and also denoted by \( \text{prop}_P(T) \).

For \( x \in X_\Lambda \), recall that \( B_{r,x,\lambda} = B_{r,x,\lambda} \cap P_r(X_\lambda) \) and we write

\[
\mathcal{H}_{x,\lambda} := \chi_{B_{x,\lambda}} \mathcal{H}_{r,\lambda} = \mathcal{H}_{r,\lambda} \otimes (B_\lambda \otimes \mathcal{A}(V)).
\]

Under this decomposition, any operator \( S \in \mathcal{L}(\mathcal{H}_{r,\lambda}) \) can be written in the matrix form \( S = (S_{x,y})_{x,y \in X_\lambda} \), where \( S_{x,y} \in \mathcal{R}(\mathcal{H}_{r,\lambda} \mathcal{H}_{x,\lambda}) \otimes (B_\lambda \otimes \mathcal{A}(V)) \). We can also define the \( (\mathbb{R}_+ \times H) \)-support for each \( S_{x,y} \) as above and denote by \( \text{supp}_{\mathbb{R}_+ \times H}(S_{x,y}) \).

Recall that for each \( \lambda \), we already fixed a point \( x_\lambda \in D \cap X_\lambda \) where \( D \) is the fundamental domain from the assumption of controlled distortion. Consider the amplified orbit map at \( x_\lambda \) (which is the composition of the orbit map with an inclusion)

\[
\psi_\lambda : \Gamma \rightarrow \Gamma x_\lambda \hookrightarrow X_\lambda, \quad \gamma \mapsto \gamma \cdot x_\lambda.
\]

By the assumption of controlled distortion, the family \( \{\psi_\lambda\}_{\lambda \in \Lambda} \) is uniformly coarsely equivalent. For each \( \lambda \in \Lambda \), we choose a coarse inverse \( \phi_\lambda : X_\lambda \rightarrow \Gamma \) to \( \psi_\lambda \) such that the family \( \{\phi_\lambda\}_\lambda \) is uniformly coarsely equivalent as well.

Definition A.5. Define \( \mathcal{W}'((X_\Lambda);(B_\lambda \otimes \mathcal{A}(V)))^F \) to be the *-subalgebra in \( \prod_{\lambda \in \Lambda} \mathcal{L}(\mathcal{H}_{r,\lambda})^F \) consisting of elements \( T = (T_\lambda)_{\lambda} \) (writing \( T_\lambda = (T_{\lambda,x,y})_{x,y \in X_\lambda} \)) satisfying the following conditions:

1. \( \sup_{\lambda \in \Lambda} \text{prop}_P(T_\lambda) < \infty \);

2. for \( \lambda \in \Lambda \) and bounded Borel subset \( B \subseteq P_r(X_\lambda) \), both \( \chi_B T_\lambda \) and \( T_\lambda \chi_B \) belong to \( \mathcal{R}(\mathcal{H}_{r,\lambda}) \);
(3) there exists a compact subset $K$ of $\mathbb{R}_+ \times H$ such that $\text{supp}_{\mathbb{R}_+ \times H}(T_{\Lambda,t,x,y})$ is contained in $\phi_\lambda(x) \cdot K$ for any $\lambda \in \Lambda$ and $x, y \in X_A$;

(4) there exists an integer $N \in \mathbb{N}$ such that
\[
T_{\Lambda,t,x,y} \in (\text{Id} \otimes \beta_N(\phi_\lambda(x)))(\mathcal{R}(\mathcal{H}_{y,r,\lambda}, \mathcal{H}_{x,t,\lambda}) \otimes B_\lambda) \otimes \mathcal{A}(W_N(\phi_\lambda(x)))
\]
for any $\lambda \in \Lambda$ and $x, y \in X_A$;

(5) there exists $c > 0$ such that if $T_{\Lambda,t,x,y} = (\text{Id} \otimes \beta_N(\phi_\lambda(x)))(T'_{\Lambda,t,x,y})$ then for any $Y = (s, h) \in \mathbb{R} \times W_N(\phi_\lambda(x))$ with norm 1, the derivative $D_Y(T'_{\Lambda,t,x,y})$ in the direction of $Y$ of the function
\[
T'_{\Lambda,t,x,y} : \mathbb{R} \times W_N(\phi_\lambda(x)) \to \left(\mathcal{R}(\mathcal{H}_{y,r,\lambda}, \mathcal{H}_{x,t,\lambda}) \otimes B_\lambda\right) \otimes \text{Cliff}(W_N(\phi_\lambda(x)))
\]
exists and $\|D_Y(T'_{\Lambda,t,x,y})\| \leq c$ for all $x, y \in X_A$.

Define $\mathcal{W}((X_A); (B_\lambda \otimes \mathcal{A}(V))^\Gamma)$ to be the norm closure of $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V))]^\Gamma$ in $\prod_{\lambda \in \Lambda} \mathcal{L}(\mathcal{H}_{x,t,\lambda})$.

Also define $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma$ to be the collection of uniformly continuous bounded functions $(T_t)$ from $[1, \infty)$ to $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V))]^\Gamma$ such that the $P$-propagation of $(T_t)$ tends to zero as $t \to +\infty$ and the parameters in condition (3), (4) and (5) above can be chosen to be independent of $t$. Define $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma$ to be the norm closure of $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma$.

We aim to prove the following result:

**Proposition A.6.** The homomorphism
\[
\text{ev.} : \lim_{r \to \infty} K_r\left(\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma\right) \longrightarrow \lim_{r \to \infty} K_r\left(\mathcal{W}^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma\right)
\]
induced by the evaluation-at-one map is an isomorphism for $* = 0, 1$.

The proof, has its root in [36, Section 6], relies on a Mayer-Vietoris argument on the proper $C^*$-algebra $\mathcal{A}(V)$ to chop the twisted algebras into smaller pieces. First we need to introduce some notation.

Given a $\Gamma$-invariant open subset $O \subseteq \mathbb{R}_+ \times H$, we denote
\[
\mathcal{A}(V)_O := C_0(O) \mathcal{A}(V),
\]
which is a two-sided $*$-ideal in $\mathcal{A}(V)$. Define $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V)_O)]^\Gamma$ to be the subalgebra of $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V))]^\Gamma$ consisting of $T = (T_{\Lambda,t,x,y})$ such that $\text{supp}_{\mathbb{R}_+ \times H}(T_{\Lambda,t,x,y}) \subseteq O$, i.e., $T_{\Lambda,t,x,y} \in \mathcal{R}(\mathcal{H}_{y,r,\lambda}, \mathcal{H}_{x,t,\lambda}) \otimes B_\lambda \otimes \mathcal{A}(V)_O$ for any $\lambda \in \Lambda$ and $x, y \in X_A$. Define $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V)_O)]^\Gamma$ to be the norm closure of $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V)_O)]^\Gamma$. It is clear that $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V)_O)]^\Gamma$ is a two-sided $*$-ideal in $\mathcal{W}[(X_A); (B_\lambda \otimes \mathcal{A}(V))]^\Gamma$.

Similarly, define $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)_O))^\Gamma$ to be the norm closure of the subalgebra in $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma$ consisting of elements $T = (T_{t,A,t,x,y})$ such that $\text{supp}_{\mathbb{R}_+ \times H}(T_{t,A,t,x,y}) \subseteq O$ for any $\lambda \in \Lambda$, $t \in [1, \infty)$ and $x, y \in X_A$. It is also clear that $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)_O))^\Gamma$ is a two-sided $*$-ideal in $\mathcal{W}_L^\Gamma((X_A); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma$.

Concerning the proper $\Gamma$-action on $\mathbb{R}_+ \times H$, it is known (see, e.g., [34, Appendix A.2]) that for each $(t, v) \in \mathbb{R}_+ \times H$, there exists an open precompact neighbourhood
Lemma A.7. For a slice $O = \Gamma \cdot U$, we have that
\[
\text{ev}_*: \lim_{r \to \infty} K_* \left( \mathcal{W}^\Gamma \left( (X_A); (B_\lambda \hat{\otimes} \mathcal{A}(V)_O) \right) \right) \longrightarrow \lim_{r \to \infty} K_* \left( \mathcal{W}^\Gamma \left( (X_A); (B_\lambda \hat{\otimes} \mathcal{A}(V)_O) \right) \right)
\]
is an isomorphism for $* = 0, 1$.

Proof. By assumption, we write
\[
O = \Gamma \cdot U = \bigsqcup_{i \in I} \gamma_i \cdot U
\]
where $U$ is an open precompact subset in $\mathbb{R}_+ \times H$, $F$ is a finite subgroup such that
\[
F \cdot U = U, \quad \text{and} \quad \gamma_i \text{ runs over a set of representatives of the left cosets } \{ \gamma F : \gamma \in \Gamma \}.
\]
Fix $i_0 \in I$ such that $\gamma_{i_0} \cdot U = U$ (i.e., $\gamma_{i_0} \in F$).

Given $T = (T_{\lambda,x,y}) \in \mathcal{W}^\Gamma ((X_A); (B_\lambda \hat{\otimes} \mathcal{A}(V)_O))$, we write $T_{\lambda,x,y} = (T_{\lambda,x,y,i})_{i \in I}$ where $T_{\lambda,x,y,i} = T_{\lambda,x,y} \cdot \gamma_i U$. For $\gamma \in \Gamma$, we have $\gamma \cdot T_{\lambda,x,y} = T_{\lambda,\gamma x,\gamma y}$. Moreover, each $\gamma$ induces a permutation $\sigma_\gamma : I \to I$ such that $\gamma \gamma_i U = \gamma_{\sigma_\gamma(i)} U$. Hence we obtain:
\[
T_{\lambda,x,y,\sigma_\gamma(i)} = \gamma^{-1} \cdot T_{\lambda,\gamma x,\gamma y, i} \quad \text{for } i \in I.
\]
Therefore, $T_{\lambda,x,y}$ can be determined by the set $\{ T_{\lambda,x,y,\sigma_\gamma(i)} : \gamma \in \Gamma \}$.

Recall that there exists a compact $K \subset \mathbb{R}_+ \times H$ such that $\text{supp}_{\mathbb{R}_+ \times H} (T_{\lambda,x,y})$ is contained in $\phi_\lambda(x) \cdot K$. Since $K$ is bounded and the action is proper, there exists a finite subset $I_0 \subset I$ such that $K \cap (\gamma_i U) = \emptyset$ only for $i \notin I_0$. Hence
\[
\text{supp}_{\mathbb{R}_+ \times H} (T_{\lambda,x,y}) \subseteq \bigsqcup_{i \notin I_0} \phi_\lambda(x) \gamma_i \cdot U \quad \text{for any } x, y \in X_A.
\]

Assuming that $T_{\lambda,x,y,\sigma_\gamma,i_0} \neq 0$, then there exists $i \in I_0$ such that $\gamma_{i_0} \in \phi_\lambda(\gamma x) \gamma_i F$. Recall that $\phi_\lambda(\gamma x)x_\lambda$ is uniformly close to $\gamma x$ and $T_{\lambda}$ has uniformly finite propagation, hence there exists $R > 0$ such that $\gamma x, \gamma y \in B(x_\lambda, R)$. In conclusion, we obtain that $T_{\lambda,x,y}$ can be determined by the set $\{ T_{\lambda,x',y',i_0} : x', y' \in B(x_\lambda, R) \}$.

For $R > 0$, define $\mathcal{W}^\Gamma ((F \cdot B(x_\lambda, R)); (B_\lambda \hat{\otimes} \mathcal{A}(V)_U))$ similar to those in Definition A.6 except that there we only require operators are $F$-invariant (instead of $\Gamma$-invariant) and their $(\mathbb{R}_+ \times H)$-supports are in $U$. Also define a map
\[
\Psi_R : \mathcal{W}^\Gamma ((F \cdot B(x_\lambda, R)); (B_\lambda \hat{\otimes} \mathcal{A}(V)_U))^F \longrightarrow \mathcal{W}^\Gamma ((X_A); (B_\lambda \hat{\otimes} \mathcal{A}(V)_O))^F
\]
by setting the image of $T = (T_{\lambda,x',y'})$ to be
\[
\Psi_R(T)_{\lambda,x,y,i} := \begin{cases}
\gamma^{-1} \cdot T_{\lambda,\gamma x,\gamma y}, & \text{if } \exists \gamma \in \Gamma : \gamma x, \gamma y \in F \cdot B(x_\lambda, R) \text{ and } \sigma_\gamma(i) = i_0; \\
0, & \text{otherwise},
\end{cases}
\]
inspired by Equation (A.3). We claim that $\Psi_R(T)$ is well-defined. In fact, if there exists another $\gamma' \in \Gamma$ such that $\sigma_{\gamma'}(i) = i_0$ and $\gamma'x, \gamma'y \in F \cdot B(x_\lambda, R)$, then it follows from the definition that

$$\gamma\gamma_i \cdot U = U = \gamma'\gamma_i \cdot U,$$

which implies that $\gamma' = g\gamma$ for some $g \in F$. Note that in this case $\gamma'x \in F \cdot B(x_\lambda, R)$ implies $\gamma'x \in F \cdot B(x_\lambda, R)$ as well. Moreover, the $F$-invariance of $T$ implies that

$$(\gamma')^{-1} \cdot T_{\lambda,y'x'\gamma'y} = \gamma^{-1} g^{-1} \cdot T_{\lambda,y'x'y} = \gamma^{-1} \cdot T_{\lambda,y'x'y}.$$

Hence we conclude the claim.

Now we check that condition (1)-(5) in Definition [A.5] holds for $\Psi_R(T)$. First note that $\Psi_R(T)_{\lambda,x,y} \neq 0$ implies that there exists $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in F \cdot B(x_\lambda, R)$. This implies that $\Psi_R(T)$ has uniformly bounded propagation. For the $\Gamma$-invariance property, we need to check that

$$\hat{\gamma} \cdot \Psi(T)_{\lambda,x,y,\sigma_{\gamma}^{-1}(i)} = \Psi(T)_{\lambda,\hat{\gamma}x,\gamma y, i} \text{ for any } \hat{\gamma} \in \Gamma, x, y \in X_\lambda \text{ and } i \in I.$$

By definition, taking $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in F \cdot B(x_\lambda, R)$ and $\sigma_{\gamma}(\sigma_{\hat{\gamma}}^{-1}(i)) = i_0$ then

$$\Psi(T)_{\lambda,x,y,\sigma_{\gamma}^{-1}(i)} = \gamma^{-1} \cdot T_{\lambda,y'x'y}.$$

Note that $\sigma_{\gamma}(\sigma_{\hat{\gamma}}^{-1}(i)) = i_0$ implies that $\gamma\gamma_i^{-1} \gamma_i \in F$. Since $T$ is $F$-invariant, we obtain

$$(A.4) \quad \gamma_i^{-1} \hat{\gamma} \cdot \Psi(T)_{\lambda,x,y,\sigma_{\gamma}^{-1}(i)} = \gamma_i^{-1} \hat{\gamma} \gamma_i^{-1} \cdot T_{\lambda,y'x'y} = T_{\lambda,y'x'y}.$$

On the other hand, $\gamma_i^{-1} \hat{\gamma} x = (\gamma_i^{-1} \hat{\gamma})^{-1} \gamma i \gamma x \in F \cdot B(x_\lambda, R)$ and similarly $\gamma_i^{-1} \hat{\gamma} y \in F \cdot B(x_\lambda, R)$. Also note that $\sigma_{\gamma}(i_0) = i$, hence we have:

$$(A.5) \quad \Psi(T)_{\lambda,x,y,\gamma^{-1}i} = \gamma_i \cdot T_{\lambda,y'x'y}.$$

Combining Equation (A.4) and (A.5), we conclude that each $\Psi(T)_\lambda$ is $\Gamma$-invariant.

Condition (2) follows from the fact that for each $\lambda, x, y$, there exist only finitely many $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in F \cdot B(x_\lambda, R)$. This implies that there exists only finitely many $i \in I$ such that $\Psi_R(T)_{\lambda,x,y,i} \neq 0$, hence provides condition (2). Condition (4) and (5) hold trivially due to the construction, and finally we check condition (3).

Note that $\Phi_R(T)_{\lambda,x,y,i} \neq 0$ implies that there exists $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in F \cdot B(x_\lambda, R)$ and $\sigma_{\gamma}(i) = i_0$. Hence we have $\gamma x, \gamma y \in F$. Also note that $\gamma_\phi(x)x_\lambda$ is uniformly close to $\gamma x$, which is uniformly close to $x_\lambda$. Thanks to the uniformly coarse embedding of orbit maps, we obtain that $\gamma_\phi(x)$ is uniformly close to the identity, which further implies that $\gamma i$ is uniformly close to $\phi(x)$. Hence we obtain:

$$\text{supp}_{\mathbb{R} \times H}(\Psi_R(T)_{\lambda,x,y,i}) \subseteq B(\phi(x), R') \cdot U$$

for some uniform constant $R' > 0$. Since $U$ is precompact, there exists a compact subset $K \subset \mathbb{R} \times H$ such that $B(\phi(x), R') \cdot U \subseteq \phi(x) \cdot K$ for any $x \in X_\lambda$, which concludes condition (3).

In conclusion, we obtain that $\Psi_R(T)$ is a well-defined map. It is also straightforward to check that $\Psi_R$ is a $*$-homomorphism (details are omitted here). Furthermore, we claim that $\Psi_R$ is isometric. Indeed, given $T \in \Psi([F \cdot B(x_\lambda, R)]; (B_\lambda \otimes \mathcal{A}(V)_{U}))^F$ we have

$$\|\Psi_R(T)\| = \sup_{\lambda \in \Lambda, i \in I} \|\Psi_R(T)_{\lambda,x,y,i}\|.$$
Direct calculation shows that

$$\Psi_R(T)_{i,x,y,i} = \begin{cases} \gamma_i \cdot T_{\lambda, y_i^{-1} x_i y_i} & \text{if } x, y \in F(x_1, R); \\ 0, & \text{otherwise} \end{cases}$$

It follows that $$\|\Psi_R(T)_{i,x,y,i}\| = \|T_{\lambda}\|$$ for each $$i \in I$$, which implies that $$\|\Psi_R\|$$ is isometric. Therefore, the map $$\Psi_R$$ can be extended to an isometric $$\ast$$-homomorphism (still denoted by $$\Psi_R$$)

$$\Psi_R : \mathcal{U}(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \rightarrow \mathcal{U}(X_1; (B_1 \hat{\otimes} \mathcal{A}(V)_O))^F.$$  

Taking direct limits, we obtain a homomorphism

$$\Psi : \lim_{R \to \infty} \mathcal{U}(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \rightarrow \mathcal{U}(X_1; (B_1 \hat{\otimes} \mathcal{A}(V)_O))^F.$$  

The analysis at the beginning shows that $$\Psi$$ is surjective, hence a $$\ast$$-isomorphism. Similarly on the localisation level, we also have a $$\ast$$-isomorphism

$$\Psi_L : \lim_{R \to \infty} \mathcal{U}(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \rightarrow \mathcal{U}_L(X_1; (B_1 \hat{\otimes} \mathcal{A}(V)_O))^F.$$  

Note that

$$\lim_{r \to 0} \lim_{R \to \infty} \mathcal{U}(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F = \lim_{R \to \infty} \lim_{r \to 0} \mathcal{U}(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F$$  

and

$$\lim_{r \to 0} \lim_{R \to \infty} \mathcal{U}_L(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F = \lim_{R \to \infty} \lim_{r \to 0} \mathcal{U}_L(F \cdot B(x_1, R); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F,$$

since all of these limits are closures of unions of increasing algebras. Hence it suffices to prove that for any $$R > 0$$, the following is an isomorphism for $$* = 0, 1$$:

$$\text{ev}_* : \lim_{r \to 0} K_* \left( \mathcal{U}_L^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right) \rightarrow \lim_{r \to 0} K_* \left( \mathcal{U}^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right).$$

Also note that for sufficiently large $$r$$, the Rips complex $$P_r(F \cdot B(x_1, R))$$ is just a simplex which is $$F$$-invariant, denoted by $$\Delta_1$$. Hence it suffices to prove

$$\text{ev}_* : \lim_{r \to 0} K_* \left( \mathcal{U}_L^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right) \rightarrow \lim_{r \to 0} K_* \left( \mathcal{U}^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right)$$

is an isomorphism for $$* = 0, 1$$. Since $$\{\Lambda_1\}$$ is $$F$$-equivariantly uniformly coarsely equivalent to $$\{x_1^0\}_1$$ where $$x_1^0$$ is the barycenter of $$\Delta_1$$, we obtain

$$K_* \left( \mathcal{U}^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right) \cong K_* \left( \mathcal{U}^r((x_1^0); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right).$$

Also $$\{\Lambda_1\}$$ is $$F$$-equivariantly uniformly strongly Lipschitz homotopic equivalent to $$\{x_1^0\}_1$$, hence a similar argument as for [35], Proposition 3.7 shows that

$$K_* \left( \mathcal{U}_L^r((\Lambda_1); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right) \cong K_* \left( \mathcal{U}_L^r((x_1^0); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right).$$

Therefore, it remains to check that

$$\text{ev}_* : \lim_{r \to 0} K_* \left( \mathcal{U}_L^r((x_1^0); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right) \rightarrow \lim_{r \to 0} K_* \left( \mathcal{U}^r((x_1^0); (B_1 \hat{\otimes} \mathcal{A}(V)_U))^F \right)$$

is an isomorphism. This follows directly from [34], Lemma 12.4.3, hence we conclude the proof.
Lemma A.9. For each $t \geq 1$, we define a map
\[ \beta_t : S \otimes \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B})^\Gamma \longrightarrow \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B})^\Gamma \]
by setting
\[ \beta_t \left( g \otimes (T_{\lambda,x,y}) \right)_{\lambda,y} = \frac{1}{|\Gamma_x|} \sum_{\gamma' \in \Gamma_x} T_{\lambda,y} \otimes \beta(\gamma')(g) \]
for $x \in D_\lambda, y \in X_\lambda$ and $\gamma' \in \Gamma$, where $\Gamma_x$ is the stabiliser at $x$.

Lemma A.9. For each $t \geq 1$, the map $\beta_t$ is well-defined.

Proof of Proposition A.6. First we claim that for any bounded open subset $B \subset \mathbb{R}_+ \times H$, the map
\[ \text{ev.} : \lim_{r \to \infty} K_r \left( \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B})^\Gamma \right) \longrightarrow \lim_{r \to \infty} K_r \left( \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B})^\Gamma \right) \]
is an isomorphism for $* = 0, 1$. In fact, we can use finitely many slices to cover $\Gamma \cdot B$ since $\bar{B}$ is compact. For two slices $W_1 = \Gamma \times F_1 U_1$ and $W_2 = \Gamma \times F_2 U_2$, note that
\[ W_1 \cap W_2 = \left( \bigcup_{i \in I} \gamma_i \cdot U_1 \right) \cap W_2 = \left( \bigcup_{i \in I} (\gamma_i U_1 \cap W_2) \right) = \gamma_i \cdot (U_1 \cap W_2). \]
On the other hand for any $\gamma \in F_1$, we have
\[ \gamma \cdot (U_1 \cap W_2) = \gamma U_1 \cap \gamma W_2 = U_1 \cap W_2. \]
Hence we obtain that
\[ W_1 \cap W_2 = \Gamma \times F_1 (U_1 \cap W_2), \]
which is also a slice. Therefore, Lemma A.7 together with a Mayer-Vietoris argument (similar to [36, Lemma 6.3]) concludes the claim.

Finally, for each $n \in \mathbb{N}$ we set $B_n$ to be the open ball in $\mathbb{R}_+ \times H$ with center $(0, 0)$ and radius $n$. From condition (3) in Definition A.5, we obtain:
\[ \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma = \lim_{n \to \infty} \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B_n})^\Gamma \]
and
\[ \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V)))_1^\Gamma = \lim_{n \to \infty} \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V))_{\Gamma,B_n})_1^\Gamma. \]
Hence we finish the proof thanks to the claim above. \qed

Our next aim is to construct the Bott and Dirac maps. First let us introduce the Bott maps. For each $\lambda \in \Lambda$, we choose a fundamental domain $D_\lambda \subseteq X_\lambda$ for the $\Gamma$-action on $X_\lambda$ such that $x_\lambda \in D_\lambda$ and
\[ D = \sup_{\lambda \in \Lambda} \text{diam}(D_\lambda) < +\infty. \]
Note that $D_\lambda$ might not coincide with $D \cap X_\lambda$ where $D$ is the fixed fundamental domain from the condition of controlled distortion\footnote{We would like to point out that $D$ can be modified to satisfy that $\sup_{\lambda \in \Lambda} \text{diam}(D \cap X_\lambda)$ is finite, whence we can choose $D_\lambda$ to be $D \cap X_\lambda$. However, this is not necessary for us to continue.}

Definition A.8. For $t \geq 1$, we define a map
\[ \beta_t : S \otimes \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma \longrightarrow \mathcal{W}((X_\lambda); (B_\lambda \otimes \mathcal{A}(V)))^\Gamma \]
by setting
\[ \beta_t \left( g \otimes (T_{\lambda,x,y}) \right)_{\lambda,y} = \frac{1}{|\Gamma_x|} \sum_{\gamma' \in \Gamma_x} T_{\lambda,y} \otimes \beta(\gamma')(g) \]
for $x \in D_\lambda, y \in X_\lambda$ and $\gamma' \in \Gamma$, where $\Gamma_x$ is the stabiliser at $x$.\footnote{We would like to point out that $D$ can be modified to satisfy that $\sup_{\lambda \in \Lambda} \text{diam}(D \cap X_\lambda)$ is finite, whence we can choose $D_\lambda$ to be $D \cap X_\lambda$. However, this is not necessary for us to continue.}
Since $\beta \gamma \gamma$, on the other hand, it is straightforward to check that

$$Hence the $(\rightarrow \infty)$

$$

Proof. If $\gamma_1 x = \gamma_2 x$, then there exists $\gamma \in \Gamma_x$ such that $\gamma_2 = \gamma_1 \cdot \gamma$. Hence:

$$\beta_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, y} = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} T_{\lambda, \gamma x, y} \hat{\otimes} \beta(\gamma \gamma')(g_1) = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} T_{\lambda, \gamma x, y} \hat{\otimes} \beta(\gamma_1 \gamma')(g_1)$$

$$= \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} T_{\lambda, \gamma x, y} \hat{\otimes} \beta(\gamma_1 \gamma')(g_1) = \beta_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, y}.$$ 

Hence $\beta_1$ is a well-defined map.

We need to check that the range of $\beta_1$ is contained in $\mathcal{F}((X_1); (B, \hat{\otimes} \mathcal{A}(V))^\mathcal{F}$. Given $g \hat{\otimes} (T_{\lambda, x, y}) \in \mathcal{S} \hat{\otimes} \mathcal{F}((X_1); (B, \hat{\otimes} \mathcal{A}(V))^\mathcal{F}$, we take a sequence of bounded smooth functions $g_n$ on $\mathbb{R}$ with bounded derivatives and a sequence of even smooth functions $h_n$ on $\mathbb{R}$ with range in $[0, 1]$, support in $[-n, n]$ and $\|h_n\| \leq 1$ such that $\|g_n h_n - g\|_\infty \to 0$ as $n \to \infty$. We define $\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))$ by setting

$$\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, y} := \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} T_{\lambda, \gamma x, y} \hat{\otimes} \beta(\gamma \gamma')(g_n h_n)$$

for $x \in D_\lambda, y \in X_\lambda$ and $\gamma \in \Gamma$. Thanks to the finite propagation of $(T_{\lambda, x, y})$, we have

$$\left\| \beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y})) - \beta_1 (g \hat{\otimes} (T_{\lambda, x, y})) \right\| \to 0 \quad \text{as} \quad n \to \infty.$$ 

Hence it suffices to show that $\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))$ belongs to $\mathcal{F}((X_1); (B, \hat{\otimes} \mathcal{A}(V))^\mathcal{F}.$

Let us fix an $n \in \mathbb{N}$. It is clear that the $P$-propagation of $\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))$ is the same as that of $(T_{\lambda, x, y})$, which deduces condition (1). For $\gamma, \gamma' \in \Gamma$, we have:

$$\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, \gamma y} = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} T_{\lambda, \gamma x, \gamma y} \hat{\otimes} \beta(\gamma \gamma')(g_n h_n)$$

$$= \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \gamma \cdot T_{\lambda, \gamma x, \gamma y} \hat{\otimes} \gamma \cdot (\beta(\gamma \gamma')(g_n h_n))$$

$$= \gamma \cdot (\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, y})$$

where we use Lemma [A.4] in the second equation. Hence each $\beta^{(n)}_1 (g \hat{\otimes} (T_{\lambda, x, y}))$ is $\Gamma$-invariant. It is also clear that condition (2) holds.

Concerning condition (3): note that $\beta(\gamma \gamma')(g_n h_n) = \beta(\gamma \gamma')(g_n h_n) \cdot \beta(\gamma \gamma')(h_n).$ Hence the $(\mathbb{R} \times H)$-support of $\beta_1 (g \hat{\otimes} (T_{\lambda, x, y}))_{\lambda, \gamma x, y}$ is contained in that of $\beta(\gamma \gamma')(h_n).$ Since $h_n$ is even, it follows from direct calculation that the $(\mathbb{R} \times H)$-support of $\beta(\gamma \gamma')(h_n)$ is contained in:

$$[-nt, nt] \times \{h \in H : \|h - b(\gamma \gamma')\| \leq nt\}.$$ 

On the other hand, it is straightforward to check that $\gamma \gamma'$ and $\phi_\lambda(\gamma x)$ are uniformly close. Hence there exists a constant $K_{n,t} > 0$ such that

$$[-nt, nt] \times \{h \in H : \|h - b(\gamma \gamma')\| \leq nt\} \subseteq \phi_\lambda(\gamma x) \cdot B(0, K_{n,t}),$$

where $B(0, K_{n,t}) = \{(s, h) \in \mathbb{R} \times H : s^2 + \|h\|^2 < K_{n,t}^2\}$ with compact closure. Hence we conclude condition (3).
For condition (4), it follows from the above paragraph that \( \gamma' \) is uniformly close to \( \phi(\gamma x) \). Hence there exists a constant \( N > 0 \) such that \( b(\gamma' \gamma) \) belongs to \( W_N(\phi(\gamma x)) \). It follows that
\[
\beta(\gamma' \gamma)((g_n h_n)_i) = \beta_N(\phi(\gamma x)) \circ \beta_{W_N(\phi(\gamma x)), W_0(\beta(\gamma' \gamma))}(g_n h_n)_i,
\]
which concludes condition (4). Finally condition (5) follows from the choice of \( g_n \) and \( h_n \), hence we finish the proof. \( \square \)

Applying \( \beta_i \), pointwise, we obtain a map on the localisation level:
\[
\beta_{i,i} : S \otimes \mathcal{U}((X_i); (B_i)) \Gamma \longrightarrow \mathcal{U}_i((X_i); (B_i) \otimes \mathcal{A}(V)) \Gamma.
\]

The following lemma was originally from [36, Lemma 7.6], and an equivariant version was proved in [9, Lemma 5.8]. Hence we omit the proof here.

**Lemma A.10.** The maps \( \beta_i, \beta_{i,i} \) can be extended to asymptotic morphisms
\[
\beta : S \otimes \mathcal{U}'((X_i); (B_i)) \Gamma \longrightarrow \mathcal{U}'((X_i); (B_i) \otimes \mathcal{A}(V)) \Gamma
\]
and
\[
\beta_{i,i} : S \otimes \mathcal{U}_i((X_i); (B_i)) \Gamma \longrightarrow \mathcal{U}'_i((X_i); (B_i) \otimes \mathcal{A}(V)) \Gamma
\]

Now we move on to the Dirac map. Let \( H \) and \( V \) be as above. For a finite-dimensional affine subspace \( V_a \) in \( V \), denote \( \mathcal{L}^2(V_a) = \mathcal{L}^2(V_a; \text{Cliff}(V_a^0)) \) the graded infinite-dimensional Hilbert space of square-integrable maps from \( V_a \) to the complex Clifford algebra of \( V_a^0 \).

For finite-dimensional affine subspaces \( V_a \subseteq V_b \) in \( V \), recall that we have the orthogonal decomposition
\[
V_{ba}^0 = V_b^0 \oplus V_a^0.
\]

Define a unit vector \( \xi_0 \in \mathcal{L}^2(V_a^0) \) by
\[
\xi_0(w) = \pi^{-\frac{\dim V_a}{4}} \exp(-\frac{1}{2} ||w||^2).
\]

Then we regard \( \mathcal{L}^2(V_a) \) as a subspace of \( \mathcal{L}^2(V_b) \) via the isometric inclusion:
\[
\mathcal{L}^2(V_a) \rightarrow \mathcal{L}^2(V_{ba}^0) \otimes \mathcal{L}^2(V_a) \cong \mathcal{L}^2(V_b), \quad \xi \mapsto \xi_0 \otimes \xi.
\]

It is easy to check that the collection
\[
\{ \mathcal{L}^2(V_a) \mid V_a \subseteq V \text{ is a finite dimensional affine subspace} \}
\]
forms a directed system. We define
\[
\mathcal{L}^2(V) = \lim \mathcal{L}^2(V_a),
\]
where the limit is taken over the above directed system.

Let \( \mathcal{S}(V_a) \subseteq \mathcal{L}^2(V) \) be the subspace of Schwartz functions from \( V_a \) to \( \text{Cliff}(V_a^0) \). Choosing an orthonormal basis \( \{ e_1, e_2, \cdots, e_n \} \) for \( V_a^0 \), let \( \{ x_1, x_2, \cdots, x_n \} \) be its dual coordinates. The **Dirac operator** \( D_{V_a} \) is an unbounded operator on \( \mathcal{L}^2(V_a) \) with domain \( \mathcal{S}(V_a) \) defined by:
\[
D_{V_a} \xi = \sum_{i=1}^n (-1)^{\deg \xi} \frac{\partial \xi}{\partial x_i} e_i.
\]
where \( e_i \) acts by Clifford multiplication. Given a vector \( v \in V_a \), the Clifford operator \( C_{V_a,v} \) is an unbounded operator on \( L^2(V_a) \) with domain \( \mathcal{S}(V_a) \) defined by:

\[
(C_{V_a,v})_{\xi}(w) = (w - v) \cdot \xi(w).
\]

For each \( \gamma \in \Gamma \), we define a Schwartz subspace of \( L^2(V) \) by

\[
\mathcal{S}(\gamma) = \lim_{\rightarrow} \mathcal{S}(W_k(\gamma)),
\]

where \( W_k(\gamma) \) is defined in (A.1). Set \( V_0(\gamma) = W_1(\gamma) \) and \( V_k(\gamma) = W_{k+1}(\gamma) \ominus W_k(\gamma) \) if \( k \geq 1 \). Then we have an algebraic decomposition:

\[
V = V_0(\gamma) \oplus V_1(\gamma) \oplus \cdots \oplus V_n(\gamma) \oplus \cdots.
\]

For each \( n \in \mathbb{N} \) and \( t \geq 1 \), we define an unbounded operator \( B_{n,t}(\gamma) \) on \( L^2(V) \) (associated to the above decomposition) given by

\[
B_{n,t}(\gamma) = \sum_{k=0}^{n-1} (1 + kt^{-1})D_k + \sum_{k=n}^{\infty} (1 + kt^{-1})(D_k + C_k)
\]

where \( D_k = D_{V_k(\gamma)} \), \( C_0 = C_{V_0(\gamma),b(\gamma)} \) and \( C_k = C_{V_k(\gamma),0} \). The operator \( B_{n,t}(\gamma) \) is well-defined on the Schwartz space \( \mathcal{S}(\gamma) \), which is taken to be its domain.

The \( \Gamma \)-action on \( H \) induce an action on \( L^2(V) \) by unitaries. It is easy to compute that

\[
B_{n,t}(\gamma)g = g \cdot B_{n,t}(\gamma) \cdot g^{-1},
\]

where \( \gamma \in \Gamma \) maps the domain \( \mathcal{S}(\gamma) \) of \( B_{n,t}(\gamma) \) to the domain \( \mathcal{S}(\gamma g) \) of \( B_{n,t}(\gamma g) \). This implies the following:

**Lemma A.11** ([9, Lemma 5.1]). For any \( g \in C_0(\mathbb{R}) \) and \( \gamma \in \Gamma \), we have \( g(B_{n,t}(\gamma g)) = \gamma g(B_{n,t}(\gamma)) \) for each \( \gamma' \in \Gamma \).

**Definition A.12.** We define \( \Psi'[([X_{\lambda}]; (B_{\lambda} \hat{\otimes} \mathcal{S}(L^2(V))))] \) to be the \( * \)-subalgebra in \( \prod_{\lambda \in \Lambda} \mathcal{L}((\mathcal{H}_{\lambda} \otimes B_{\lambda}) \hat{\otimes} \mathcal{S}(L^2(V))) \) consisting of elements \( T = (T_{\lambda})_{\lambda} \) satisfying the following conditions:

1. \( \sup_{\lambda \in \Lambda} \text{prop}_{\lambda}(T_{\lambda}) < \infty \);

2. for \( \lambda \in \Lambda \) and bounded Borel subset \( K \subseteq P_t(X_\lambda) \), both \( \chi_K T_{\lambda} \) and \( T_{\lambda} \chi_K \) belong to \( (\mathcal{S}(\mathcal{H}_{\lambda} \otimes B_{\lambda}) \hat{\otimes} \mathcal{S}(L^2(V))) \).

Denote \( \Psi'((X_{\lambda}); (B_{\lambda} \hat{\otimes} \mathcal{S}(L^2(V)))) \) the norm closure of \( \Psi'((X_{\lambda}); (B_{\lambda} \hat{\otimes} \mathcal{S}(L^2(V)))) \). Define \( \Psi'_w((X_{\lambda}); (B_{\lambda} \hat{\otimes} \mathcal{S}(L^2(V)))) \) to be the closure of the collection of uniformly continuous bounded functions \( (T_t) \) from \([1, \infty) \) to \( \Psi'((X_{\lambda}); (B_{\lambda} \hat{\otimes} \mathcal{S}(L^2(V)))) \) such that the \( P \)-propagation of \( (T_t) \) tends to zero as \( t \to +\infty \).

For every non-negative integer \( n \) and \( \gamma \in \Gamma \), we define

\[
\theta^n_\gamma(\gamma) : \mathcal{A}(W_n(\gamma)) \longrightarrow \mathcal{S}(L^2(V))
\]

by the formula

\[
\theta^n_\gamma(\gamma)(g \otimes h) = g_t(B_{n,t}(\gamma))M_{h_t}
\]

for \( g \in \mathcal{S} \) and \( h \in C_0(W_n(\gamma), \text{Cliff}(W_n(\gamma))) \), where \( h_t(v) = h(b(\gamma) + t^{-1}(v - b(\gamma))) \) for \( v \in W_n(\gamma) \) and \( M_{h_t} \) is the pointwise multiplication operator on \( L^2(V) \) (in the sense that if \( W \supseteq W_n(\gamma) \) is a finite-dimensional affine subspace of \( V \), then we regard:

\[
(M_{h_t})_{\xi}(v + w) = h_t(v)\xi(v + w)
\]
for all $\xi \in \mathcal{L}^2(W)$, $v \in W_n(y)$ and $w \in W \otimes W_n(y)$. It follows from [17, Lemma 5.8] that the image of $\theta^\mu(y)$ indeed sits inside $\mathfrak{R}(\mathcal{L}^2(V))$. Moreover, it follows from the proof of [2, Lemma 5.3] that

(A.7) \[ \theta^\mu(y') = \gamma \cdot \theta^\mu(y') \cdot \gamma^{-1} \]

for any $\gamma, \gamma' \in \Gamma$.

Given $T = (T_{\lambda,x,y}) \in \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathcal{A}(V)))^\Gamma$, let $N' \in \mathbb{N}$ be such that for every $x, y \in X_{\lambda}$ there exists $T'_{\lambda,x,y} \in \mathfrak{R}(\mathfrak{H}_{x,r},\mathfrak{A}_{x,r}) \otimes \mathcal{A}(W_{N'}(\phi_{\lambda}(x)))$ satisfying

\[ T_{\lambda,x,y} = (\text{Id} \otimes \beta_N(\phi_{\lambda}(x)))(T'_{\lambda,x,y}), \]

Taking $N \in \mathbb{N}$ such that for every $x \in D_{\lambda}, y' \in \Gamma_x$ and $\gamma \in \Gamma$ we have $W_{N}(\phi_{\lambda}(y'x)) \subseteq W_{N}(y'\gamma')$. Hence for each $x \in D_{\lambda}, x' \in \Gamma_x$ and $y' \in \Gamma$, we have:

\[ T_{\lambda,y,x,y'} = (\text{Id} \otimes \beta_N(\phi_{\lambda}(y'x)))(T'_{\lambda,y,x,y'}) = (\text{Id} \otimes \beta_N(\phi_{\lambda}(y'x)))(T'_{\lambda,y,x,y'}) \]

where $T'_{\lambda,y,x,y} = (\text{Id} \otimes \beta_{W_N(\phi_{\lambda}(y)\Gamma_x)}(\phi_{\lambda}(y)))(T'_{\lambda,y,x,y})$.

**Definition A.13.** For $t \geq 1$, we define a map

\[ \alpha_t : \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathcal{A}(V)))^\Gamma \rightarrow \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathfrak{R}(\mathcal{L}^2(V))))^\Gamma \]

by setting

\[ \alpha_t(T_{\lambda,x,y})_{\lambda,x,y} = \frac{1}{|\Gamma_{\lambda}|} \sum_{\gamma' \in \Gamma_{\lambda}} (\text{Id} \otimes \theta^\mu_{\lambda}(y'\gamma'))(T'_{\lambda,y,x,y'}) \]

for $x \in D_{\lambda}, y \in X_{\lambda}$ and $\gamma \in \Gamma$, where $T'_{\lambda,y,x,y} = (\text{Id} \otimes \beta_N(\phi_{\lambda}(y'\gamma'))(T'_{\lambda,y,x,y})$.

Similar to (but much easier than) the proof of Lemma [A.9] we obtain:

**Lemma A.14.** For each $t \geq 1$, the map $\alpha_t$ is well-defined.

Applying $\alpha_t$ pointwise, we obtain a map on the localisation level:

\[ \alpha_{t,\lambda} : \mathfrak{W}'_{\lambda}((X\lambda); (B\lambda \otimes \mathcal{A}(V)))^\Gamma \rightarrow \mathfrak{W}'_{\lambda}((X\lambda); (B\lambda \otimes \mathfrak{R}(\mathcal{L}^2(V))))^\Gamma. \]

The following lemma was originally from [36, Lemma 7.2], and an equivariant version was proved in [9, Lemma 5.5]. Hence we omit the proof here.

**Lemma A.15.** The maps $\alpha_t, \alpha_{t,\lambda}$ can be extended to asymptotic morphisms

\[ \alpha : \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathcal{A}(V)))^\Gamma \rightarrow \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathfrak{R}(\mathcal{L}^2(V))))^\Gamma \]

and

\[ \alpha_{\lambda} : \mathfrak{W}'_{\lambda}((X\lambda); (B\lambda \otimes \mathcal{A}(V)))^\Gamma \rightarrow \mathfrak{W}'_{\lambda}((X\lambda); (B\lambda \otimes \mathfrak{R}(\mathcal{L}^2(V))))^\Gamma. \]

Following the proof of [36, Proposition 7.7] (see also [9, Proposition 5.9]), we have:

**Proposition A.16.** The compositions

\[ \alpha \circ \beta_* : K_* \left( \mathfrak{W}'((X\lambda); (B\lambda))^\Gamma \right) \rightarrow K_* \left( \mathfrak{W}'((X\lambda); (B\lambda \otimes \mathfrak{R}(\mathcal{L}^2(V))))^\Gamma \right) \]
and
\[
(\alpha_L) \circ (\beta_L) : K_* \left( \Psi_L^r ((X)_r); (A_r) \right)^\Gamma \longrightarrow K_* \left( \Psi_r^\alpha ((X)_r); (A_r) \right)^\Gamma
\]
equals the identity homomorphisms, respectively.

Finally, we are in the position to prove Proposition 6.17.

**Proof of Proposition 6.17** Consider the following commutative diagram:
\[
\begin{array}{ccc}
\lim_{r \to \infty} K_* \left( \Psi_L^r ((X)_r); (A_r) \right)^\Gamma & \xrightarrow{\text{ev.}} & \lim_{r \to \infty} K_* \left( \Psi_r^\alpha ((X)_r); (A_r) \right)^\Gamma \\
\psi_L & & \psi_r \\
\lim_{r \to \infty} K_* \left( \Psi_L^r ((X)_r); (A_r \otimes \mathcal{A}(V)) \right)^\Gamma & \xrightarrow{\text{ev.}} & \lim_{r \to \infty} K_* \left( \Psi_r^\alpha ((X)_r); (A_r \otimes \mathcal{A}(V)) \right)^\Gamma \\
(\alpha_r) & & (\alpha_r)
\end{array}
\]

The middle horizontal line is an isomorphism due to Proposition A.6, while the compositions of both of the vertical lines are isomorphisms due to Proposition A.16. Consequently, the result follows from a diagram chasing.  

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