Spectral Analysis for Non-Hermitian Matrices and Directed Graphs

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Abstract

We generalize classical results in linear algebra and spectral graph theory from the case where the underlying matrix is Hermitian to the case where it is non-Hermitian. New admissibility conditions are introduced to replace the Hermiticity condition. Variational estimates of the Rayleigh quotient in the case that the matrix is non-Hermitian are proved. As an application, a new Hoffman-type bound on the size of an independent set in a directed graph is developed. Our techniques consist in quantifying the impact of breaking a matrix’s Hermitian symmetry and are broadly applicable.

1 Introduction

The eigendecomposition is among the most powerful tools for analyzing Hermitian matrices $B \in \mathbb{C}^{n \times n}$, i.e. matrices satisfying $B^* = B$, where $(B^*)_{j\ell} = \overline{B_{\ell j}}$. Several classical results in linear algebra can be derived from decomposing a Hermitian matrix $B \in \mathbb{C}^{n \times n}$ as $B = U \text{diag}(\lambda(B)) U^*$, where $U$ is unitary and $\lambda(B) = (\lambda_{\ell}(B))_{0 \leq \ell < n} \subset \mathbb{R}$. In particular, under the assumption that $B$ is Hermitian, variational estimates on the Rayleigh quotient [10] can be stated in terms of $\lambda(B)$:

$$\forall f \in \mathbb{C}^{n \times 1}, \min \{ \lambda(B) \} \leq \frac{f^* B f}{f^* f} \leq \max \{ \lambda(B) \}. \quad (1)$$

In case that $B$ is non-Hermitian, the eigenvalues of $B$ may be complex, or even worse the eigendecomposition may not exist at all.

When the eigendecomposition does not exist, one may consider instead a diagonalization $B = P \text{diag}(\lambda(B)) P^{-1}$ for some invertible matrix $P$ and scalars $\lambda(B)$. Compared to the

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U of the spectral decomposition, the columns of P need not form an orthonormal basis. A different decomposition that is available to all matrices is the singular value decomposition (SVD): \( B = U \text{diag}(\sigma(B))V^* \), where the singular values \( \sigma(B) = (\sigma_\ell(B))_{0 \leq \ell < n} \) are non-negative and \( U, V \) are unitary matrices. However, it need not be the case that \( UV^* = U^*V = I \), which is the primary contrast with the eigendecomposition.

The powerful tools of linear algebra can be applied to graphs via spectral graph theory [Chu97]. Indeed, let \( \mathcal{G} = (V, B) \) be a graph, where \( V \) is the set of vertices and \( B \in \{0, 1\}^{|V| \times |V|} \) an adjacency matrix such that \( B_{ij} = 1 \) if there is an edge between nodes \( x_i \) and \( x_j \). By analyzing the spectral properties of \( B \), a variety of mathematical ideas may be adapted to \( \mathcal{G} \), including notions of geometry [Moh89] [MACO91], Fourier and wavelet analysis [CM06] [HVG11], SRV16], random diffusion processes [CLL+05, CL06], and clusters [SM00] [NJW02]. While these tools have contributed to a renaissance in the analysis of data, spectral graph methods almost uniformly require the underlying graph \( \mathcal{G} \) to be undirected, or in linear algebra terms, \( B \) must be Hermitian, i.e. symmetric. This is a severe limitation in practice, as a variety of real data does not lend itself to representation as an undirected graph, for example social networks [KLPM10], models for the spread of contagious disease in a heterogenous population [KW92], and predator-prey relationships [YA73].

### 1.1 Summary of Contributions

This article develops new approaches for the analysis of non-Hermitian matrices. The primary contributions are twofold. First, we prove a generalized version of the classical variational estimates on the Rayleigh quotient. New admissibility conditions are introduced to replace the Hermiticity condition. Second, the Hoffman bound on the size of independent sets in undirected graphs is generalized to the directed setting. Our major tool consists in quantifying the discrepancy between Hermitian and non-Hermitian matrices, and adding additional hypotheses in the theorems to address this discrepancy. The gap between Hermiticity and non-Hermiticity is made precise, and moreover in the case of the Hoffman bound, the gaps between \( B \) being Hermitian, diagonalizable (with potentially complex eigenvalues) and \( B \) arbitrary is considered by analyzing the SVD in context of the Hoffman bound. Our proof methods are flexible, and may be applicable to settings not considered in the present article.

### 1.2 Related Work

Spectral graph theory has been attempted for operators defined on directed graphs in a variety of contexts, including for the graph Laplacian [AC05] [Chu05] [But07] [Bau12] [BS13] [ZS16] [FAFS18] [CMZ18] and for nonreversible Markov chains [Fil91]. Combinatorial results for directed graphs have also been studied [Bru10] [KS15]. These results do not, however, develop precise characterizations of the ways in which classical results can be
modified in the non-Hermitian setting. In particular, the admissibility conditions proposed in this article explicitly illustrate what is lost when a matrix is perturbed to deviate from Hermiticity, and suggests how to compensate for the loss of Hermiticity. Moreover, the proposed generalized Hoffman bound makes no assumptions of normality of the adjacency matrix of the underlying graph.

1.3 Notation

Throughout, bold typography is used to denote matrices and vectors from scalars. For an \( n \times n \) square matrix \( B \), \( B^0 = I_n \) the \( n \times n \) identity matrix. For a collection of points \( \{ \alpha_\ell \}_{0 \leq \ell < n} \subset \mathbb{C} \), let \( \text{diag}(\{ \alpha_\ell \}_{0 \leq \ell < n}) \) denote the \( n \times n \) diagonal matrix with \( \ell \)-th diagonal entry \( \alpha_\ell \). \( I_{m \times n} \) denotes the \( m \times n \) matrix of all 1’s. \( B[j,:]: \) and \( B[:,\ell] \) denote the \( j \)-th row and \( \ell \)-th column of the matrix \( B \), respectively.

2 Generalized Rayleigh Quotient Estimation

2.1 Rayleigh Quotient for Complex Diagonalizable Matrices

Let \( B \in \mathbb{C}^{n \times n} \). There corresponds to \( B \) a (in general non-symmetric) bilinear form \( \langle f, g \rangle_B \rightarrow f^* B g \), defined for \( f, g \in \mathbb{C}^{n \times 1} \). The behavior of this bilinear form can be analyzed in a scale-invariant manner through the Rayleigh quotient. When \( B \) is Hermitian, (1) states that the Rayleigh quotient \( \langle f, f \rangle_B / \| f \|_2^2 \) is controlled by the largest and smallest eigenvalues of \( B \). We extend this result to the case when \( B \) has complex eigenvalues.

Theorem 2.1. Let \( B \in \mathbb{C}^{n \times n} \) be decomposed as

\[
B^k = U \text{diag} \{ \lambda(B) \}^k V^*, \; \forall k \in \{0, 1\}
\]

with \( \lambda(B) \subset \mathbb{C} \). Write \( \lambda_\ell(B) = |\lambda_\ell(B)| \exp(i \theta_\ell) \) for \( \theta_\ell \in [0, 2\pi) \). Suppose \( f, g \in \mathbb{C}^{n \times 1} \) satisfy the admissibility condition of admitting expansions of the form

\[
f = \sum_{0 \leq \ell < n} f_\ell V[:,\ell], \quad g = \sum_{0 \leq \ell < n} g_\ell U[:,\ell] \; \text{s.t.} \; \forall \; 0 \leq \ell < n, \; (f_\ell g_\ell \exp(i \theta_\ell)) \in \mathbb{R}. \; \text{(2)}
\]

Then

\[
\min_{0 \leq \ell < n} |\lambda_\ell(B)| \text{sgn}(f_\ell g_\ell \exp(i \theta_\ell)) \leq \sum_{0 \leq \ell < n} |f_\ell g_\ell| \leq \max_{0 \leq \ell < n} |\lambda_\ell(B)| \text{sgn}(f_\ell g_\ell \exp(i \theta_\ell)).
\]

Proof. By the spectral decomposition of \( B \),

\[
f^* B g = f^* U \text{diag} \{ \lambda(B) \} V^* g
\]
\[
\sum_{0 \leq \ell < n} f^* U [\cdot, \ell] \lambda_\ell (B) V^* [\ell, \cdot]) g = 
\sum_{0 \leq \ell < n} \left( \sum_{0 \leq j < n} f_j V [\cdot, j] \right)^* U [\cdot, \ell] \lambda_\ell (B) V^* [\ell, \cdot] \left( \sum_{0 \leq k < n} g_k U [\cdot, k] \right) = 
\sum_{0 \leq \ell < n} f_\ell \lambda_\ell (B) g_\ell.
\]

Writing \( \lambda_\ell (B) = |\lambda_\ell (B)| \exp (i \theta_\ell) \) and applying the admissibility condition (2), this yields
\[
\min_{0 \leq \ell < n} \{ |\lambda_\ell (B)| \text{sgn}(f_\ell g_\ell \exp (i \theta_\ell)) \} \sum_{0 \leq \ell < n} |f_\ell g_\ell| 
\leq \sum_{0 \leq \ell < n} f_\ell \lambda_\ell (B) g_\ell 
\leq \max_{0 \leq \ell < n} \{ |\lambda_\ell (B)| \text{sgn}(f_\ell g_\ell \exp (i \theta_\ell)) \} \sum_{0 \leq \ell < n} |f_\ell g_\ell|.
\]

The result follows by algebraic manipulation. \( \square \)

Several remarks are in order. In the case that \( B \) is Hermitian, \( V = U \) are unitary, so that in particular the eigenvalues of \( B \) are real. If the eigenvalues are moreover positive (\( \theta_\ell = 0 \) for all \( \ell \)) then in the case that \( f_\ell g_\ell \geq 0 \) for all \( \ell \),
\[
\sum_{0 \leq \ell < n} |f_\ell g_\ell| = \sum_{0 \leq \ell < n} f_\ell g_\ell = \langle f, g \rangle
\]
since \( U = V \) is unitary. In the special case that \( B \) is Hermitian and \( f = g \),
\[
\sum_{0 \leq \ell < n} |f_\ell f_\ell| = ||f||^2_{L^2}, \\
\min_{0 \leq \ell < n} |\lambda_\ell (B)| \text{sgn}(f_\ell g_\ell \exp (i \theta_\ell)) = \min \{ \lambda (B) \}, \\
\max_{0 \leq \ell < n} |\lambda_\ell (B)| \text{sgn}(f_\ell g_\ell \exp (i \theta_\ell)) = \max \{ \lambda (B) \},
\]
which recovers (1). In the general case, \( f, g \) must interact in a particular way for Theorem 2.1 to hold, as quantified by the admissibility condition.

### 2.2 Examples of Admissible Vectors

A non-trivial class of admissible vectors can be constructed using the following Lemma, which is easily follows by a computation.
Lemma 2.2. Let $M$ be an arbitrary Hermitian matrix and let $B \in \mathbb{C}^{n \times n}$ be defined in terms of the entries of $M$ as $B[n - 1 - i, j] = M[j, i]$, $\forall 0 \leq i, j < n$. Then for all $f, g \in \mathbb{C}^{n \times 1}$,

$$f^*Bg = g^\top M \left( \sum_{0 \leq i < n} I_n[;i] I_n[n - 1 - i, :] \right) \bar{f}.$$ 

We remark that the matrix $B$ in Theorem 2.2 is a persymmetric matrix [GL12]. In particular, for a fixed $x$,

$$f = \left( \sum_{0 \leq i < n} I_n[;i] I_n[n - 1 - i, :] \right) \underline{x}, \ g = \underline{x},$$

then

$$f^*Bg = g^\top M \left( \sum_{0 \leq i < n} I_n[;i] I_n[n - 1 - i, :] \right) \bar{f} = g^\top Mg \in \mathbb{R}.$$ 

Let $V, U, \{\theta_i\}_{i=0}^{n-1}$ be as in Theorem 2.1. The following vectors are thus admissible:

$$f = \left( \sum_{0 \leq i < n} I_n[;i] I_n[n - 1 - i, :] \right) e^{-i\theta_0} V[;0], \ g = e^{-i\theta_0} V[;0].$$

Moreover, $\min \{\lambda(M)\} = f^*Bg$. Similarly,

$$f = \left( \sum_{0 \leq i < n} I_n[;i] I_n[n - 1 - i, :] \right) e^{-i\theta_{n-1}} V[;n - 1], \ g = e^{-i\theta_{n-1}} V[;n - 1],$$

are admissible, and moreover, $\max \{\lambda(M)\} = f^*Bg$. This quantifies the change in magnitude of the extreme eigenvalues after changing a Hermitian matrix $M$ to a non-Hermitian matrix $B$ by defining $B[n - 1 - i, j] = M[j, i]$, $\forall 0 \leq i, j < n$.

3 Estimating Independent Set Cardinalities in Directed Graphs

As an application of our method to graph theory, we develop estimates on the size of the maximal independent set in certain directed graphs. We will consider the inner product on $n \times n$ matrices $\langle B, M \rangle = \text{tr}(B^*M)$, which has associated norm

$$\|B\| = \|B\|_{\text{Fro}} = \sqrt{\sum_{0 \leq i, j < n} |B[i, j]|^2}.$$
Definition 3.1. Let $G = (V, B)$ be an undirected, unweighted graph with adjacency matrix $B \in \{0, 1\}^{n \times n}$. $G$ is said to be $d$-regular if for all $j$, $\sum_{\ell \neq j} B[j, \ell] = d$. A set $S \subset V$ is an independent set if $B[j, \ell] = 0$, $\forall j, \ell \in S$.

The Hoffman bound [Hof03] is a classical estimate on the cardinality of the largest independent set of an undirected, unweighted, $d$-regular graph in terms of the spectrum of its adjacency matrix. We state it and provide a proof for completeness.

Theorem 3.2. (Undirected Hoffman Bound). Let $B \in \{0, 1\}^{n \times n}$ be the adjacency matrix of an undirected $d$-regular graph $G$ on $n$ vertices with eigenvalues $\lambda(B) \subset \mathbb{R}$. Let $I$ be the indices of an independent set in $G$. Then

$$\frac{|I|}{n} \leq -\min \{\lambda(B)\} \left( d - \min \{\lambda(B)\} \right).$$

(3)

Proof. Note that $B = d \frac{1_{n \times n}}{n} + \left( I_n - \frac{1_{n \times n}}{n} \right) B$. Thus, for all $f \in \mathbb{C}^{n \times 1}$,

$$\langle B, ff^* \rangle = d \langle \frac{1_{n \times n}}{n}, ff^* \rangle + \left\langle \left( I_n - \frac{1_{n \times n}}{n} \right) B, ff^* \right\rangle.$$

Let $B = U \text{diag}\{\lambda(B)\} U^*$ where $UU^* = I$. We analyze the second term on the right-hand side as follows:

$$\left\langle \left( I_n - \frac{1_{n \times n}}{n} \right) B, ff^* \right\rangle = f^* \left( I_n - \frac{1_{n \times n}}{n} \right) B f$$

$$= f^* \left( I_n - \frac{1_{n \times n}}{n} \right) \left( \sum_{0 \leq \ell < n} \lambda_{\ell}(B) U[:\ell] U^*[:\ell] \right) f$$

$$= f^* \left( \sum_{0 \leq \ell < n} \lambda_{\ell}(B) U[:\ell] U^*[:\ell] - \frac{1}{n} \sum_{0 \leq \ell < n} \lambda_0(B) 1_{n \times n} U[:\ell] U^*[:\ell] \right) f$$

$$= f^* \left( \sum_{1 \leq \ell < n} \lambda_{\ell}(B) U[:\ell] U^*[:\ell] - \frac{1}{n} \lambda_0(B) 1_{n \times n} U[:0] U^*[:0] \right) f$$

$$= f^* \left( \sum_{1 \leq \ell < n} \lambda_{\ell}(B) U[:\ell] U^*[:\ell] \right) f$$

$$= \sum_{1 \leq \ell < n} \lambda_{\ell}(B) (f^* U[:\ell])^2$$

$$\geq \min \{\lambda(B)\} \sum_{1 \leq \ell < n} (f^* U[:\ell])^2$$
\[
\min\{\lambda(B)\} \sum_{1 \leq \ell < n} f^* U[:, \ell] U^* [\ell, :) f
\]
\[
= \min\{\lambda(B)\} \left\langle \left( I_n - \frac{1_{n \times n}}{n} \right), f f^* \right\rangle.
\]

Hence,
\[
\langle B, f f^* \rangle \geq d \left\langle \frac{1_{n \times n}}{n}, f f^* \right\rangle + \min\{\lambda(B)\} \left\langle \left( I_n - \frac{1_{n \times n}}{n} \right), f f^* \right\rangle
\]
\[
= \frac{d}{n} \|f\|_{\ell^2}^4 + \min\{\lambda(B)\} \left( \|f\|_{\ell^2}^2 - \frac{1}{n} \|f\|_{\ell^2}^4 \right).
\]

To every independent set \( I \), there is a corresponding indicator vector \( f = 1_I \) for which by definition \( \langle B, f f^* \rangle = f^* B f = 0 \). For such an indicator vector \( f \), it follows that
\[
0 \geq \frac{d}{n} \|f\|_{\ell^2}^4 - \min\{\lambda(B)\} \|f\|_{\ell^2}^2 + \min\{\lambda(B)\} \|f\|_{\ell^2}^2.
\]

Noting that \(|I| = \|f\|_{\ell^2}^2\), it follows that \[\frac{|I|}{n} \leq \frac{-\min\{\lambda(B)\}}{d - \min\{\lambda(B)\}}\] as desired.

The condition on the maximum size on an independent set may be characterized as the maximum value \( \|1_{I}\|_{\ell^2}^2 \), where \( 1_{I}^T B 1_{I} = 0 \) for some \( I \subset V \). The spectral decomposition of \( B \) decouples the rank-one matrix \( 1_{n \times n} \) corresponding to the leading eigenvalue \( \lambda_0(B) = d \), from whence the analysis flows. Note that the Hoffman bound is sharp for the complete bipartite graph having \( 2n \) vertices in each partition, since in this case \( \max_{I \text{ independent}} |I| = n, \min\{\lambda(B)\} = -n, d = n \).

### 3.1 Directed Hoffman Bound

We now consider directed graphs such that each vertex has both in-degree and out-degree equal to \( d \).

**Definition 3.3.** Let \( G = (V, B) \) be a directed, unweighted graph with adjacency matrix \( B \in \{0, 1\}^{n \times n} \). \( G \) is said to be \( d \)-regular if for all \( j \), \( \sum_{\ell \neq j} B[j, \ell] = \sum_{k \neq j} B[k, j] = d \).

We develop Hoffman-type bounds based on the spectral decomposition and the singular value decomposition of the (non-Hermitian) adjacency matrix.

**Theorem 3.4.** Let \( B \in \{0, 1\}^{n \times n} \) denote the adjacency matrix of a directed, \( d \)-regular graph \( G \) on \( n \) vertices. Let \( B \) be decomposed as \( B^k = U \text{ diag}\{\lambda(B)\}^k V^* \), \( k \in \{0, 1\} \). Let
\( \lambda_\ell(B) = \alpha_\ell \exp(i\theta_\ell), \alpha_\ell \in \mathbb{R}, \theta_\ell \in [0, 2\pi) \) be a polar decomposition of \( \lambda(B) \). Let \( \alpha(B) = (\alpha_1, \ldots, \alpha_n) \). Suppose vectors \( f, g \in \{0, 1\}^{n \times 1} \) satisfy the admissibility condition

\[
    f = \sum_{0 \leq \ell < n} f_\ell V[:, \ell], \quad g = \sum_{0 \leq \ell < n} g_\ell U[:, \ell] \quad \text{such that} \quad \forall 0 \leq \ell < n, \quad f_\ell g_\ell \exp(i\theta_\ell) \geq 0. \tag{4}
\]

Then if \( f \) and \( g \) denote respectively indicator vectors for rows and columns associated with a rectangular 0 block in \( B \),

\[
    -\min\{\alpha(B)\} \sum_{0 \leq \ell < n} f_\ell \exp(i\theta_\ell) g_\ell \geq \frac{\|f\|_F^2 \|g\|_F^2}{d - \min\{\alpha(B)\}}.
\]

**Proof.** By the \( d \)-regularity of \( G \) we have that \( U[:, 0] = V[:, 0] = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1) \). Hence, \( B = \frac{d}{n} 1_{n \times n} + \left( I_n - \frac{1_{n \times n}}{n} \right) B \). Thus, for all \( f, g \in \mathbb{C}^{n \times 1} \) subject to (1),

\[
    \langle B, g f^* \rangle = d \left\langle \frac{1_{n \times n}}{n}, g f^* \right\rangle + \left\langle \left( I_n - \frac{1_{n \times n}}{n} \right) B, g f^* \right\rangle.
\]

We analyze the second term of the right hand side as follows:

\[
    \left\langle \left( I_n - \frac{1_{n \times n}}{n} \right) B, g f^* \right\rangle = f^* \left( I_n - \frac{1_{n \times n}}{n} \right) B g
    = f^* \left( I_n - \frac{1_{n \times n}}{n} \right) \left( \sum_{0 \leq \ell < n} \lambda_\ell(B) U[:, \ell] V^*[:, \ell] \right) g
    = f^* \left( \sum_{0 \leq \ell < n} \lambda_\ell(B) U[:, \ell] V^*[:, \ell] - \frac{1}{n} \sum_{0 \leq \ell < n} \lambda_0(B) 1_{n \times n} U[:, \ell] V^*[:, \ell] \right) g
    = f^* \left( \sum_{0 \leq \ell < n} \lambda_\ell(B) U[:, \ell] V^*[:, \ell] \right) g
    = f^* \left( \sum_{1 \leq \ell < n} \lambda_\ell(B) U[:, \ell] V^*[:, \ell] \right) \sum_{0 \leq k < n} g_k U[:, k]
    = \sum_{1 \leq \ell < n} f_\ell \lambda_\ell(B) g_\ell.
\]
$$= \sum_{1 \leq \ell < n} f_{\ell} \alpha_{\ell} \exp(i\theta_{\ell}) g_{\ell}$$

$$\geq \min\{\alpha(B)\} \sum_{1 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}$$

$$= \min\{\alpha(B)\} \left(-f_0 g_0 + \sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}\right)$$

$$= \min\{\alpha(B)\} \left(-\frac{1}{n} \|f\|_{L^2}^2 \|g\|_{L^2}^2 + \sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}\right).$$

Note that $f_0 = \frac{1}{\sqrt{n}} \|f\|_{L^1} = \frac{1}{\sqrt{n}} \|f\|_{L^2}^2$ follows from $f$ takes values in $\{0, 1\}$, $f = \sum_{0 \leq \ell < n} \overline{f_{\ell}} V[; \ell, \ell, U[; 0] = \frac{1}{\sqrt{n}} (1, \ldots, 1)^\top$ and that $V^*[\ell, \ell, U[; 0] = 0, \ell \neq 0$ and $V^*[0, \ell, U[; 0] = 1$. That $g_0 = \frac{1}{\sqrt{n}} \|g\|_{L^2}^2$ follows similarly. Hence,

$$\langle B, g f^* \rangle \geq d \left(\frac{1_{n \times n}}{n}, g f^*\right) + \min\{\alpha(B)\} \left(-\frac{1}{n} \|f\|_{L^2}^2 \|g\|_{L^2}^2 + \sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}\right)$$

$$= d \left(\frac{1_{n \times n}}{n}, g \right) + \min\{\alpha(B)\} \left(-\frac{1}{n} \|f\|_{L^2}^2 \|g\|_{L^2}^2 + \sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}\right).$$

Since $f$ and $g$ are indicator vectors for rows and columns associated with a rectangular $0$ block in $B$, $f^* \cdot B g = 0$. It follows that

$$0 \geq d \left(\frac{1}{n}, g \right) + \min\{\alpha(B)\} \left(-\frac{1}{n} \|f\|_{L^2}^2 \|g\|_{L^2}^2 + \sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell}\right).$$

The desired result follows by rearranging.

Several remarks are in order. First, note that the quantity bounded in Theorem 3.3 may be interpreted as the geometric average of the size of independent set with respect to “in” and “out” nodes. Indeed, if $f = 1_{\text{row}}, g = 1_{\text{col}}$ are the indicator functions for the in and out vertices of a directed independent set, then $\|f\|_{L^2} \|g\|_{L^2} = \sqrt{|I_{\text{row}}| |I_{\text{col}}|}$.

Second, if $B$ is Hermitian and $f = g$, the admissibility condition (1) is always satisfied as $\theta_{i} = 0, i = 0, \ldots, n - 1$. Indeed, in this case,

$$\sum_{0 \leq \ell < n} f_{\ell} \exp(i\theta_{\ell}) g_{\ell} = \|f\|_{L^2}^2,$$
so that the conclusion of Theorem 3.2 holds. Hence, Theorem 3.4 is a strict generalization of the classical Hoffman inequality. Moreover, in the case that \( A \) is Hermitian but \( f \neq g \), it follows that
\[
\sum_{0 \leq \ell < n} f_\ell \exp(i\theta_\ell)g_\ell = \langle f, g \rangle.
\]

### 3.1.1 Tightness of Directed Hoffman Bound

When \( n \) is a multiple of 4, adjacency matrices of the form
\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \otimes 1_{\frac{n}{4} \times \frac{n}{4}}
\]
show Theorem 3.3 is tight. Indeed, when \( n = 4 \), this non-Hermitian matrix may be decomposed as
\[
\begin{pmatrix}
-1/2 & -i/2 & i/2 & -1/2 \\
-1/2 & -i/2 & -i/2 & -1/2 \\
1/2 & 1/2 & 1/2 & -1/2 \\
1/2 & -1/2 & -1/2 & -1/2
\end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & -i/2 & i/2 & -1/2 \\ -1/2 & -i/2 & -i/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix}^*.
\]

In this case, the largest independent set has size 2, corresponding to the zero block on the upper left and lower right of the matrix. Let \( f = g = (1, 1, 0, 0)^T \), so that \((f_0, f_1, f_2, f_3) = (g_0, g_1, g_2, g_3) = (-1, 0, 0, -1)\). Decomposing the first and fourth eigenvalues as \( \alpha_0 = -1, \alpha_3 = 1, \theta_0 = \theta_3 = 0 \), it is seen that the admissibility condition is satisfied, and that \( \min\{\alpha\} = -1 \). Moreover,
\[
\sum_{0 \leq \ell < 4} f_\ell g_\ell \exp(i\theta_\ell) = 2
\]
so that the estimate of Theorem 3.4 is
\[
\frac{\|f\|_2^2 \|g\|_2^2}{4} \leq 1,
\]
which is tight since \( \|f\|_2 \|g\|_2 = 2 \). A similar argument holds for the block corresponding to indicator functions \( f = g = (0, 0, 1, 1)^T \). Together, this shows the maximal independent set of this directed graph is tightly estimated by Theorem 3.4.
Consider the singular value decomposition of $B \in \{0,1\}^{n \times n}$ expressed by

$$B = U \text{diag}(\sigma(B))V^* \quad \text{s.t.} \quad UU^* = I_n = VV^*,$$

where each element of $\sigma(B)$ is positive. Theorem 3.5 provides a Hoffman estimate on the size of the independent set using the SVD, which holds for all matrices, not just diagonalizable ones.

**Theorem 3.5.** Let $B$ be the $n \times n$ adjacency matrix of a directed, $d$-regular graph. Suppose $B$ has a decomposition $B = U \text{diag} (\sigma) V^*$ such that $VV^* = UU^* = I$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$. Let $\sigma_{\text{min}} = \min_i \sigma_i$. Suppose that $f, g \in \{0,1\}^n$ correspond to the indicator sets for row and column indices respectively of a rectangular zero block, and admit decompositions of the form

$$f = \sum_{0 \leq i < n} f_i U[:, i], \quad g = \sum_{0 \leq i < n} g_i V[:, i] \quad \text{and} \quad \forall 0 \leq i < n, f_i g_i \geq 0. \quad (5)$$

Then

$$\frac{\|f\|_2^2 \|g\|_2^2}{n} \leq \frac{-\sigma_{\text{min}} \sum_{0 \leq \ell < n} f_{\ell} g_{\ell}}{d - \sigma_{\text{min}}}.$$

**Proof.** By the SVD and by $d$-regularity,

$$B = d \frac{1_{n \times n}}{n} + \left( I_n - \frac{1_{n \times n}}{n} \right) B.$$ 

Analyzing the second term for all $f, g \in \mathbb{C}^{n \times 1}$, subject to (5),

$$\left\langle \left( I_n - \frac{1_{n \times n}}{n} \right) B, g^* f \right\rangle = f^* \left( I_n - \frac{1_{n \times n}}{n} \right) \left( \sum_{0 \leq \ell < n} \sigma_{\ell} U[:, \ell] V^*[:, \ell] \right) g$$

$$= f^* \left( \sum_{1 \leq \ell < n} \sigma_{\ell} U[:, \ell] V^*[:, \ell] \right) g$$

$$= \left( \sum_{0 \leq i < n} f_i U[:, i] \right)^* \left( \sum_{1 \leq \ell < n} \sigma_{\ell} U[:, \ell] V^*[:, \ell] \right) \left( \sum_{0 \leq j < n} g_j V[:, j] \right)$$

$$= \sum_{1 \leq \ell < n} \sigma_{\ell} f_{\ell} g_{\ell} \geq \sigma_{\text{min}} \sum_{1 \leq \ell < n} f_{\ell} g_{\ell}.$$
\[ \begin{align*}
= & \sigma_{\min} \left( -f_0 g_0 + \sum_{0 \leq \ell < n} f_\ell g_\ell \right) \\
= & \sigma_{\min} \left( -\frac{1}{n} \|f\|_2^2 \|g\|_2^2 + \sum_{0 \leq \ell < n} f_\ell g_\ell \right). 
\end{align*} \]

Note that \( f_0 = \frac{1}{\sqrt{n}} \|f\|_1 = \frac{1}{\sqrt{n}} \|f\|_2^2 \) follows from \( f = \sum_{0 \leq \ell < n} f_\ell \mathbf{U}[; \ell] \) and that fact that \( \mathbf{U}[; 0] = \frac{1}{\sqrt{n}} (1, \ldots, 1)^\top \) and that \( \mathbf{U} \) has orthogonal columns; \( g_0 = \frac{1}{\sqrt{n}} \|g\|_2^2 \) follows similarly.

\[ \langle \mathbf{B}, \mathbf{g} f^* \rangle = d \left( \frac{1}{n} \sum_{0 \leq \ell < n} \mathbf{g}_\ell \mathbf{f}_\ell \right) + \left( \langle \mathbf{I}_n - \frac{1}{n} \mathbf{I}_{n \times n}, \mathbf{B}, \mathbf{g} f^* \rangle \right) \]

\[ \geq d \left( \frac{1}{n} \sum_{0 \leq \ell < n} \mathbf{g}_\ell \mathbf{f}_\ell \right) + \sigma_{\min} \left( -\frac{1}{n} \|f\|_2^2 \|g\|_2^2 + \sum_{0 \leq \ell < n} f_\ell g_\ell \right) \]

\[ = d \frac{\|f\|_2^2 \|g\|_2^2}{n} + \sigma_{\min} \left( -\frac{1}{n} \|f\|_2^2 \|g\|_2^2 + \sum_{0 \leq \ell < n} f_\ell g_\ell \right). \]

Every zero block \( I \) is specified by a pair of indicator vectors \( f = \mathbf{1}_{I_L} \) and \( g = \mathbf{1}_{I_R} \) such that \( f^* \mathbf{B} g = 0 \). For such an \( f, g \) pair also subject to the admissibility condition (5) we have

\[ 0 \geq d \left( \frac{\|f\|_2^2 \|g\|_2^2}{n} \right) - \sigma_{\min} \left( \frac{\|f\|_2^2 \|g\|_2^2}{n} \right) + \sigma_{\min} \sum_{0 \leq \ell < n} f_\ell g_\ell, \]

from whence the result follows by algebraic manipulation.

\[ \square \]

Theorem 3.5 requires a decomposition which bears resemblance to the SVD in the fact that \( \mathbf{B} = \mathbf{U} \text{diag}(\sigma) \mathbf{V}^* \), where \( \mathbf{U}^* \mathbf{U} = \mathbf{I}_n = \mathbf{V} \mathbf{V}^* \), but without the condition that \( \sigma_i \geq 0 \) for all \( i \). Note that if \( \sigma_i \mapsto -\sigma_i \), and \( \mathbf{U}[:, i] \mapsto -\mathbf{U}[:, i] \) or \( \mathbf{V}^*[:, i] \mapsto -\mathbf{V}^*[:, i] \), this still expresses such a decomposition for \( \mathbf{B} \). In this sense, there are \( 2^n \) decompositions to consider in Theorem 3.5 corresponding to the \( 2^n \) possible sign assignments. Thus, one can think of the decomposition in Theorem 3.5 as a (non-unique) signed SVD, and the condition (5) as an admissibility condition with respect to this decomposition.

### 4 Discussion and Future Research

This article proposes generalizations of classical linear algebraic and spectral graph theoretic results to the case in which the underlying matrix \( \mathbf{B} \) is non-Hermitian. This is done by
constraining certain vectors to satisfy admissibility conditions. When $B$ is Hermitian, these admissibility conditions hold and the classical results are recovered. The admissibility condition take slightly different forms, depending on which decomposition is used in place of the spectral decomposition into an orthonormal eigenbasis.

In Theorems 2.1, 3.4, $B$ is assumed diagonalizable as $B = U \text{diag}(\lambda(B))V^*$ where $\lambda(B)$ may be complex and $U, V$ need not be unitary, merely inverses: $UV^* = V^*U = I$. The analysis of $f^*Bg$ proceeds by assuming $f$ admits an expansion in terms of the rows of $V$ and $g$ an expansion in terms of the rows of $U$. Of course, when $U = V$ these conditions are the same, and when $f = g$, this condition always holds. On the other hand, Theorem 3.5 takes advantage of the singular value decomposition $B = U \text{diag}(\sigma(B))V^*$ where $UU^* = VV^* = I$ but $U, V^*$ are not inverses. The analysis of $B$ in this situation requires a different condition on $f, g$, namely that $f$ has an admissible decomposition with respect to the rows of $U$, and $g$ with respect to the rows of $V$. We remark that in all of these cases, the crucial property is that for $d$-regular graphs, the first eigenvector or singular vector (both left and right) is the vector $\frac{1}{\sqrt{n}}(1, 1, \ldots, 1) \in \mathbb{R}^n$ with corresponding eigenvalue or singular value $d$.

All subsequent analysis is downstream from this observation.

Intuitively, as $B$ deviates from being Hermitian, the admissibility conditions will still hold for a large class of vectors $f, g$. A topic of future research is to develop a rigorous perturbation theory of Hermitian matrices that quantifies how likely the admissibility conditions are to hold in a probabilistic sense. That is, if $B$ is Hermitian, then the admissibility condition holds automatically for all $f = g$. As $f$ deviates from $g$ and $B$ deviates from Hermiticity, it is of interest to determine which vectors (or, what proportion of them in a probabilistic sense) satisfy the admissibility condition.

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