HYDRODYNAMIC LIMITS OF THE KINETIC SELF-ORGANIZED MODELS

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Abstract. The self-organized hydrodynamic models can be derived from the kinetic version of the Vicsek model. The formal derivations and local well-posedness of the macroscopic equations are done by Degond and his collaborators. In this paper, we rigorously justify this hydrodynamic limit.

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1. Introduction

It has been an active research area in recent years on the modeling of interacting agent systems arising in nature, such as bird flocks, fish schools, herds of mammals, etc. They provide fascinating examples of self-organized systems which are able to produce large scale stable coherent structures. Among these models, the Vicsek model [12] has received particular attention due to its simplicity and the universality of its qualitative features. This model is a discrete particle model which consists of a time-discretized set of ordinary differential equations for the particle position and velocities. The velocities are assumed to be of constant norm and updated according to an alignment rule, i.e. each agent tries to align its velocity to that of its neighbors in some sensing region. Some angular noise is added to account for stochastic fluctuations. A time-continuous version of this model and its kinetic formulation are available in [3, 8]. From the the time-continuous Vicsek model to this kinetic model is rigorously justified in [1].

In [8], Degond-Motsch derived formally the hydrodynamic limit of the time-continuous Vicsek model through an asymptotic analysis of its kinetic version [1]. The resulting model is a system of balanced equations for the density and mean velocity orientation (or polarization vector). This model was later called the Self-Organized Hydrodynamics (SOH). Furthermore, in [7], Degond-Liu-Motsch-Panferov derived the modifications of the SOH model by the introduction of the attraction-repulsion force, different scaling assumptions about the size of the sensing region which involve a higher level of non-locality. As proved in [8], the strict combination of alignment and noise results in the appearance of a pressure term in the SOH model. Introducing an attraction-repulsion force and spanning various scaling assumptions on this force and on the size of the sensing region, they obtain in [7] a variety of effects that are not encompassed in [8].

In [7], besides the derivation of the macroscopic equations, they prove a local-in-time existence theorem in the 2D case for the viscous model (when the non-local effects are retained) and in the 3D case for the inviscid case (when the non-local effects are omitted). Both proofs are based on a suitable symmetrization of the system and on the energy method.

The main concern of the current paper is on the rigorous justification of the hydrodynamic limit from the Self-Organized Kinetic (SOK) system to the Self-Organized Hydrodynamics (SOH). The main challenge is the lack of conservation laws. To overcome this difficulty, the Generalized Collision Invariants (GCI) are employed to derive the macroscopic equations. We start from the solution constructed in [7] and proved that there exists a class of the solutions

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of SOK uniformly on a time interval on which the solutions of the SOH are constructed, such that their hydrodynamic limits are the SOH. We employ the methods developed in the fluid limits of the Boltzmann equations, such as [2] and [11].

The paper is organized as follows. In section 2, we introduce the self-organized models: the time-continuous Vicsek model, the kinetic model and the formal hydrodynamic limits. The main results are stated in section 3. Some mathematical preliminaries are introduced in section 4. In section 5, we estimate the term appear in the expansion, and the main estimates for the remainder equation is presented in section 6. In the last section, building on the estimates in the previous two sections, the proof of the main theorem is completed.

2. The Self-organized Model

2.1. Self-propelled particles interacting through alignment. Let \( X_k(t) \in \mathbb{R}^n \) and \( V_k(t) \in S^{n-1} \) be the position and velocity of the \( k \)-th particle at time \( t \). The time-continuous version of the Vicsek model is written as follow.

\[
(2.1) \quad \dot{X}_k(t) = cX_k(t),
\]

\[
(2.2) \quad \mathrm{d}V_k(t) = P_{V_k} \circ \left( \nu \tilde{V}_k(t) \mathrm{d}t + \sqrt{2D} \, \mathrm{d}B^k_t \right),
\]

\[
(2.3) \quad \tilde{V}_k(t) = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j, |X_j - X_k| \leq R} V_j,
\]

where for \( V \in S^{n-1} \), \( P_{V_k} = I_d - V \otimes V \) is the orthogonal projection onto the plane orthogonal to \( V \). The equation (2.2) takes the form of a stochastic differential equation (SDE). The projection operator \( P_{V_k} \) ensures that the resulting solution \( \tilde{V}_k(t) \) stays on the unit sphere, provided that the SDE is taken in the Stratonovich sense (which is indicated by the symbol \( \circ \)). The first term inside the bracket is the interaction. It corresponds to a force acting in the direction \( \tilde{V}_k \) of intensity \( \nu \). The second term is a white noise consisting of independent Brownian motion \( B^k_t \) in \( \mathbb{R}^2 \) of intensity \( \sqrt{2D} \). Then, let \( N \to \infty \), a mean field model is obtained. This model is described as follows.

2.2. Mean-field model and scaling. The mean-field model describes the evolution of the one-particle distribution function \( f(t,x,v) \) at position \( x \in \mathbb{R}^n \), with velocity \( v \in S^{n-1} \) at time \( t \geq 0 \). The model is written as:

\[
(2.4) \quad \partial_t f + cv \cdot \nabla_x f = -\nabla_v \cdot (F_f f) + D \Delta_v f,
\]

\[
(2.5) \quad F_f(t,x) = \nu P_{v^\perp} \tilde{v}_f(t,x), \quad \tilde{v}_f(t,x) = \frac{J_f(t,x)}{|J_f(t,x)|},
\]

\[
(2.6) \quad J_f(t,x) = \int_{(y,\omega)\in \mathbb{R}^n \times S^{n-1}} K\left(\frac{|y-x|}{R}\right)f(t,y,\omega) \mathrm{d}y \mathrm{d}\omega,
\]

where the constants \( c, \nu, D \) are the same as in (2.1) and (2.2). The equation (2.4) is a Fokker-Planck equation. The left-hand side expresses the rate of change of \( f \) due to the spatial transport of the particle with velocity \( cv \) while the first term at the right-hand side denotes the transport in velocity space due to the interaction force \( F_f \). The last term at the right-hand side is a velocity diffusion term which arises as a consequence of the Brownian noise in particle velocities. Note that since \( v \) lies on the sphere, \( \Delta_v \) stands for the Laplace-Beltrami operator on the sphere. The derivation of the mean-field model (2.4), (2.5) and (2.6) from the discrete system (2.1), (2.2) and (2.3) has been justified in [1].

After nondimensionalization process, (for details, see [6]), we can write down the scaled self-organized kinetic (SOK) model,

\[
(2.7) \quad \partial_t f + v \cdot \nabla_x f + \eta_0 \nabla_v \cdot (P_{v^\perp} D_f f) = \frac{1}{\varepsilon} \left( -\nabla_v \cdot (P_{v^\perp} \Omega f) + d\Delta_v f \right),
\]
where the parameter $\varepsilon$ denotes the mean free path, i.e., the distance needed by a particle to make a finite change in direction of motion due to the interaction force, and $\eta_0 = 0, 1$ indicate the inviscid and viscous case, respectively, and $d > 0$ is a constant. Furthermore, the local current density $j_f$, the local mean direction $\Omega_f$, and the quantity $D_f$ are defined respectively as follows:

\[(2.8) \quad j_f(t, x) \overset{\text{def}}{=} \int_{S^{n-1}} v f \, dv, \]
\[(2.9) \quad \Omega_f(t, x) \overset{\text{def}}{=} \frac{j_f}{|j_f|} \in S^{n-1}, \]
\[(2.10) \quad D_f(t, x) \overset{\text{def}}{=} k P_{\Omega_f} \frac{\Delta_x j_f}{|j_f|}, \quad k > 0. \]

### 2.3. Basic Properties of the SOK Model

In this subsection, we list some basic properties of the self-organized kinetic (SOK) model, and we refer the readers to \cite{4, 6, 9} and references therein for more details.

First we propose a hypothesis: $\Omega_f = 0$, if $|j_f| = 0$. Define the collision operator $Q$ by

\[(2.11) \quad Q(f) = -\nabla_v \cdot (P_{\nu \perp} \Omega_f f) + d \Delta_v f. \]

We mention that $Q(f) = d \Delta_v f$ in the case $|j_f| = 0$ by the above hypothesis.

Next we describe the equilibria of $Q$, which are expressed by the von Mises-Fisher (VMF) distributions with respect to the local mean orientation $\Omega \in S^{n-1}$, namely,

\[\mathcal{E} = \{f \mid Q(f) = 0\} = \{v \mapsto \rho M_\Omega(v), \forall \rho \in \mathbb{R}_+, \Omega \in S^{n-1}\},\]

where the VMF distribution is defined as

\[(2.12) \quad M_\Omega(v) = Z_d^{-1} \exp(\frac{v \cdot \Omega}{d}) \]

with a constant $Z_d = \int_{v \in S^{n-1}} \exp(\frac{v \cdot \Omega}{d}) \, dv$ independent of $\Omega$. The VMF distribution enjoys the following properties:

- i) $M_\Omega(v)$ is a probability density (i.e., $\int_{v \in S^{n-1}} M_\Omega(v) \, dv = 1$);
- ii) The first moment of $M_\Omega(v)$ satisfies

\[\int_{v \in S^{n-1}} \Omega M_\Omega(v) \, dv = c_1 \Omega, \quad c_1(d) = \frac{\int_{v \in S^{n-1}} (v \cdot \Omega) \exp(\frac{v \cdot \Omega}{d}) \, dv}{\int_{v \in S^{n-1}} \exp(\frac{v \cdot \Omega}{d}) \, dv}, \]

where the coefficient $c_1 \in [0, 1]$ denotes the order parameter in the study of phase transitions.

Note that the formula $P_{\nu \perp} \Omega = \nabla_v (v \cdot \Omega)$ ensures that the collision operator $Q$ can be rewritten as

\[Q(f) = d \nabla_v \cdot \left( M_{\Omega_f} \nabla_v \left( \frac{f}{M_{\Omega_f}} \right) \right), \]

which results in a dissipation relation

\[(2.13) \quad \int_{v \in S^{n-1}} Q(f) \left( \frac{f}{M_{\Omega_f}} \right) \, dv = -d \int_{v \in S^{n-1}} \left| \nabla_v \left( \frac{f}{M_{\Omega_f}} \right) \right|^2 M_{\Omega_f} \, dv \leq 0. \]

This implies that $Q(f) = 0$ is equivalent to $f \in \mathcal{E}$.

One of the main difficulties to derive the macroscopic equations of the SOK model is that it obeys only the conservation law of mass. To recover the missing momentum conservation related to the quantity $\Omega(t, x)$, Degond-Motsch introduce the concept of the “Generalized Collision Invariants” (GCI) in \cite{8}, see also \cite{4, 9, 6} its applications.
The leading order is
\( O(\frac{1}{\varepsilon}) \): The leading order is
\( 0 = dL_{\Omega_0} f_0 = Q(f_0) \).

Recalling the equivalence between \( Q(f) = 0 \) and \( f \in \mathcal{E} \), the equation (2.20) implies
\( f_0(t, x, v) = \rho_0(t, x) M_{\Omega_0(t, x)}(v) \),
for some function \( \rho_0 = \rho_0(t, x) \). In the rest of the paper, we use the notation \( M_0 = M_{\Omega_0} \).

Order \( O(1) \): To determine the equations satisfied by the macroscopic variables \( (\rho_0, \Omega_0) \), we consider the order \( O(1) \):
\( dL_{\Omega_0} f_1 \overset{\text{def}}{=} \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \, \nabla_v \cdot (P_{\nu^\perp} D_0 f_0) \).
We require the part of \( f_1 \) in \( \mathcal{N}(L_{\Omega_0}) \) vanishes, and can solve the part in \( \mathcal{N}^1(\mathcal{L}_{\Omega_0}) \),

\[
(2.23) \quad f_1 = \frac{1}{d} \mathcal{L}_{\Omega_0}^{-1}(\partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla_v \cdot (P_{\nu} D_0 f_0)),
\]

under the solvability condition that the right-hand side of (2.22) lies in \( \mathcal{N}^1(\mathcal{L}_{\Omega_0}) \) (for details, see section 5). This gives the following macroscopic equations satisfied by \( (\rho_0, \Omega_0) \):

\[
(\text{SOH}) \quad \begin{cases} 
\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0, \\
\rho(\partial_t \Omega + c_2 \Omega \cdot \nabla_x \Omega) + d P_{\Omega^1} \nabla_x \rho = c_3 P_{\Omega^1} \Delta_x (\rho \Omega), \\
|\Omega| = 1,
\end{cases}
\]

with the coefficients

\[
c_1 = \int_0^\pi \cos \theta \exp(\frac{\cos \theta}{\varepsilon}) \sin^{n-2} \theta \, d\theta, \quad c_2 = \int_0^\pi \cos \theta h(\cos \theta) \exp(\frac{\cos \theta}{\varepsilon}) \sin^n \theta \, d\theta, \\
c_3 = \eta_0 k[(n-1)d + c_2].
\]

This is the hydrodynamic model, which we call self-organized hydrodynamic (SOH) system. We refer to [4, 6, 9] for the derivation of the SOH system and omit the details here. Now the equation of the remainder \( \tilde{f}_2 \) is

\[
(2.24) \quad \partial_t \tilde{f}_2 + v \cdot \nabla_x \tilde{f}_2 + \eta_0 \nabla_v \cdot (P_{\nu} D_0 \tilde{f}_2) - \frac{d}{\varepsilon} \mathcal{L}_{\Omega_0} \tilde{f}_2 \\
= - \{ \partial_t f_1 + v \cdot \nabla_x f_1 + \eta_0 \nabla_v \cdot (P_{\nu} D_0 f_1) \}.
\]

For notational simplicity, we set \( d = 1 \) in the rest of the paper. By setting \( \tilde{f}_2 = f_\varepsilon M_0 \), the equation (2.24) is reduced to the equation of \( f_\varepsilon \):

\[
(2.25) \quad \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \eta_0 \nabla_v \cdot \left( P_{\nu} D_0 f_\varepsilon \right) + \frac{1}{\varepsilon} \mathcal{L}_0 f_\varepsilon = h_0 f_\varepsilon + \frac{1}{\varepsilon} h_1,
\]

where

\[
(2.26) \quad \mathcal{L}_0 f = -\frac{1}{M_0} \mathcal{L}_{\Omega_0}(M_0 f) = -\frac{1}{M_0} \nabla_v \cdot (M_0 \nabla_v f),
\]

\[
(2.27) \quad h_1 = -\frac{1}{M_0} \left[ \partial_t f_1 + v \cdot \nabla_x f_1 + \eta_0 \nabla_v \cdot (P_{\nu} D_0 f_1) \right],
\]

and

\[
(2.28) \quad h_0 = -\frac{1}{M_0} \left[ \partial_t M_0 + v \cdot \nabla_x M_0 + \eta_0 \nabla_v \cdot (P_{\nu} D_0 M_0) \right] \\
= - \left[ v \cdot \partial_t \Omega_0 + v \otimes v : \nabla_x \Omega + \eta_0 D_0 \cdot P_{\nu} \Omega_0 \right].
\]

In the rest of the paper, we work on the remainder equation (2.25).

### 3. Main results

In this section, we state our main result. We first introduce the existence result of Degond-Liu-Motsch-Panferov, on which our result is built. First, we introduce the Cauchy problem of self-organized hydrodynamic (SOH) system:

\[
(\text{SOH}) \quad \begin{cases} 
\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0, \quad x \in \mathbb{T}, \ t > 0, \\
\rho(\partial_t \Omega + c_2 \Omega \cdot \nabla_x \Omega) + d P_{\Omega^1} \nabla_x \rho = c_3 P_{\Omega^1} \Delta_x (\rho \Omega), \\
|\Omega| = 1, \\
|\rho|_{t=0} = \rho_{\text{in}} \geq c_0 > 0, \ \Omega|_{t=0} = \Omega_{\text{in}}, \ |\Omega_{\text{in}}| = 1.
\end{cases}
\]

Here \( \mathbb{T} \) denotes \( \mathbb{T}^2 \) or \( \mathbb{T}^3 \).

On the one hand, the SOH system evidently bears many similarities with the isentropic compressible Navier-Stokes (NS) system. And on the other hand, it also has some different properties. The first important difference is that the SOH system obeys the geometric
constraint $|\Omega| = 1$ which requires the velocity $\Omega$ to be of unit norm. The second important
difference is that, generally speaking, the coefficients $c_i$ ($i = 1, 2$) are different. Indeed, we
have $0 < c_2 \leq c_1$ (the equality holds iff $d = 0$), see [8].

The existence and uniqueness of the solution to the Cauchy problem of SOH system have
been established in [7], we quote the results as follows,

**Theorem 3.1 ([7]).** Let $T = \mathbb{T}^n$, given $\rho^{in} \geq c_0 > 0$, $\Omega^{in} \in S^{n-1}$, i.e.
\begin{itemize}
    \item for $n = 2$, $\Omega^{in} = (\cos \varphi^{in}, \sin \varphi^{in}), \varphi^{in} \in [0, 2\pi]$;
    \item for $n = 3$, $\Omega^{in} = (\sin \theta^{in} \cos \varphi^{in}, \sin \theta^{in} \sin \varphi^{in}, \cos \theta^{in}), \theta^{in} \in [0, \pi], \varphi^{in} \in [0, 2\pi]$.
\end{itemize}
Furthermore, assume $\sin \theta^{in} > 0$, and $c_3 = 0$, (i.e. $\eta_0 = 0$.)
with $(\rho^{in}, \varphi^{in}, \theta^{in}) \in H^{m}(\mathbb{T}), m > \frac{n}{2} + 1$. Then, there exists $T > 0$, such that the Cauchy problem
of SOH system with initial data $(\rho^{in}, \Omega^{in})$ has a unique solution $(\rho_0, \Omega_0) \in L^\infty([0, T], H^m(\mathbb{T})) \cap H^1([0, T], H^{m-1}(\mathbb{T}))$, and $\rho_0 > 0$.

Our hydrodynamic limit result builds on the above theorem, so make the same assumptions
on the initial data, namely,

**Assumption (A):**

1. In 2D case, we consider $c_3 \geq 0$ including both viscous and inviscid cases. The initial
data $(\rho^{in}, \Omega^{in}) \in H^m(\mathbb{T})$ are smooth enough as required. Besides, $\rho^{in}$ has a positive
low bound.
2. In 3D case, we consider only the inviscid case $c_3 = 0$, which arises from the coefficient
$\eta_0 = 0$. (Indeed, $c_3 = \eta_0 c_3$). Besides, $\rho^{in}$ has a positive low bound and $\Omega^{in} \neq (0, 0, 1)$
corresponding to the above constraint $\sin \theta^{in} > 0$).

Now we state the main result of this paper on the hydrodynamic limit from the self-organized kinetic (SOK) equation to the self-organized hydrodynamic (SOH).

**Theorem 3.2.** Let $(\rho_0, \Omega_0) \in L^\infty([0, T], H^m(\mathbb{T})) \cap H^1([0, T], H^{m-1}(\mathbb{T}))$ ($m > 13$) be the solutions provided by Theorem 3.1 to the Cauchy problem of the self-organized hydrodynamic system [SOH] with initial datum $(\rho^{in}, \Omega^{in})$ satisfying the assumption (A). Let $f_0 = \rho_0 M_\Omega_0$, and $f_1(t, x, v) \in N^1(L_{\Omega_0})$ be the unique solution given in (2.23).

Furthermore, assume $f^{\varepsilon, in}(x, v) = \rho^{in}(x) M_{\Omega^{in}(x)}(v) + \varepsilon f_1(0, x, v) + \varepsilon^2 f^2_{\varepsilon}(x, v)$ with the bound $\|f^2_{\varepsilon}(x, v)\|_{H^2_{x,v}} \leq C$.

Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the self-organized kinetic equation
(2.16) admits a unique solution $f^\varepsilon(t, x, v) \in C([0, T]; H^2(\mathbb{T} \times \mathbb{S}^n))$ in the class $j_{\varepsilon} = j_{f_0}$, of the form
\begin{equation}
    f^\varepsilon(t, x, v) = \rho_0(t, x) M_{\Omega_0(t,x)}(v) + \varepsilon f_1(t, x, v) + \varepsilon^2 f^2(t, x, v),
\end{equation}
with $f^2(t, x, v)$ satisfying
\begin{equation}
    \varepsilon^{\frac{1}{2}}\|f^2\|_{L^2_{x,v}} + \varepsilon\|f^2\|_{H^1_{x,v}} + \varepsilon^\frac{3}{2}\|f^2\|_{H^2_{x,v}} \leq C, \tag{3.2}
\end{equation}
where the constant $C$ is independent of $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T]$.

**Remark 3.1.** Indeed, we can get a more generic result about the higher order diffusion expansion,
\begin{equation}
    f^\varepsilon = \rho_0 M_{\Omega_0} + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots + \varepsilon^{n-1} f_{n-1} + \varepsilon^n f_n^\varepsilon,
\end{equation}
if it holds initially with the bound $\|f_n^{\varepsilon, in}\|_{H^2_{x,v}} \leq C$. For $i = 1, 2, \cdots, n - 1$, $f_i$ are determined by the equation
\begin{equation}
    \mathcal{L}_{\Omega_0} f_i = \partial_t f_{i-1} + v \cdot \nabla x f_{i-1} + \eta_0 \nabla v \cdot (P_{v^\perp} D_0 f_{i-1}),
\end{equation}
and the $n$-order remainder $f_n^\varepsilon$ satisfies
\begin{equation}
    \sum_{0 \leq s \leq n} \varepsilon^{\frac{s+1}{2}} \|f^s_n\|_{H^2_{x,v}} \leq C, \tag{3.4}
\end{equation}
with the constant $C$ independent of $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T]$. Proving this high order result works exactly the same as for Theorem $3.2$ and relies heavily on a high order a priori estimate (6.21), see the discussions in Remarks 6.1 and 5.1 below.

4. Preliminaries

In the following context, we will do estimates in the weighted spaces $L^2(M_0 \, dv \, dx)$. For the sake of simplicity, we use the notations $\| \cdot \|$ and $\| \cdot \|_{M_0}$ to denote the norms $\| \cdot \|_{L^2(dv \, dx)}$ and $\| \cdot \|_{L^2(M_0 \, dv \, dx)}$, respectively. Note that the two norms are equivalent since $M_0$ is bounded from up and below. Similarly, the notations $\langle \langle \cdot, \cdot \rangle \rangle$ and $\langle \langle \cdot, \cdot \rangle \rangle_{M_0}$ are adapted to stand for the inner products $\langle \cdot, \cdot \rangle_{L^2(dv \, dx)}$ and $\langle \cdot, \cdot \rangle_{L^2(M_0 \, dv \, dx)}$, respectively.

We will use frequently the following facts on the sphere, for a constant vector $V \in \mathbb{R}^n$,

$$\nabla_v (v \cdot V) = (Id - v \otimes v)V = P_{v^\perp} V,$$

$$\nabla_v \cdot ((Id - v \otimes v)V) = -(n - 1)v \cdot V,$$

where $\nabla_v$ and $\nabla_v \cdot$ are used to denote the tangential gradient (and, divergence) operator on the sphere. Moreover, we have some other useful formulas,

$$\int \nabla_v f \, dv = -(n - 1) \int vf \, dv,$$

$$\int v \nabla_v \cdot A(v) \, dv = - \int A(v) \, dv,$$

with $A(v)$ any smooth tangent vector field.

Now define

$$\langle \langle g \rangle \rangle = \iint g \, dv \, dx, \quad \langle g \rangle = \int g \, dv;$$

$$\langle \langle g \rangle \rangle_{M_0} = \iint gM_0 \, dv \, dx, \quad \langle g \rangle_{M_0} = \int gM_0 \, dv.$$

We give a lemma about the Poincaré inequality on the sphere.

**Lemma 4.1** (Appendix of ([3])). **We have the following weighted Poincaré inequality, for** $g \in H^1(\mathbb{S}^{n-1})$,

$$\langle \langle \nabla_v g \rangle \rangle_{M_0} \geq \Lambda \langle (g - \langle g \rangle_{M_0})^2 \rangle_{M_0},$$

**where $\Lambda$ is the Poincaré constant independent of $\Omega_0$.**

Due to the compatibility condition of the Hilbert expansion $j_\varepsilon = j_{j_0}$, $\Omega_{\varepsilon} = \Omega_{j_0}$, we have

$$\langle \langle f_1 \rangle \rangle_{M_0} = \langle f_1 \rangle_{M_0} = 0,$$

$$\langle \langle f_1 \rangle \rangle_{M_0} = \langle v \cdot f_1 \rangle_{M_0} = \langle v \cdot f_1 \rangle_{M_0} = 0;$$

$$\langle \langle f_2 \rangle \rangle_{M_0} = \langle f_2 \rangle_{M_0} = 0,$$

$$\langle \langle f_2 \rangle \rangle_{M_0} = \langle v \cdot f_2 \rangle_{M_0} = \langle v \cdot f_2 \rangle_{M_0} = 0.$$

Hence it is convenient to introduce the mean free spaces $\bar{L}^2_{M_0}(\mathbb{S}^{n-1}) \subset L^2(\mathbb{S}^{n-1})$ composed of functions $g$ satisfying $\langle g \rangle_{M_0} = 0$.

As pointed out in [3], the linearized operator $L_0$ is a self-adjoint operator under the scalar product $\langle f_1, f_2 \rangle_{L^2(M_0 \, dv)} = \int f_1 f_2 M_0 \, dv$, since we have

$$\langle f_1, L_0 f_2 \rangle_{L^2(M_0 \, dv)} = \langle \nabla_v f_1, \nabla_v f_2 \rangle_{L^2(M_0 \, dv)}.$$

We quote the definition of an operator $L_0^\gamma$ and an equivalent Sobolev norm on the sphere by spectral decomposition, as in [3],

$$\| g \|_{H^2_{M_0}(\mathbb{S}^{n-1})}^2 \overset{\text{def}}{=} \langle g, L_0^\gamma g \rangle_{L^2(M_0 \, dv).}$$

Then we have the following lemma.
Lemma 4.2 (Lemma 2 of [4]). For \( g \in \dot{H}^s_{M_0}(\mathbb{S}^{n-1}) = H^s(\mathbb{S}^{n-1}) \cap \hat{L}^2(\mathbb{S}^{n-1}) \) and \( s \geq 0 \), we have
\[
\|g\|_{\dot{H}^s_{M_0}(\mathbb{S}^{n-1})}^2 \sim \|g\|_{H^s(\mathbb{S}^{n-1})}^2.
\]

Here, we use the notation \( A \sim B \) to denote there exists a universal constant \( C > 0 \), such that \( C^{-1}B \leq A \leq CB \).

Furthermore, given \( g \in \dot{H}^s_{M_0}(\mathbb{S}^{n-1}) \), there exists a constant \( C \) such that,
\[
\|L_0^s g, \nabla_v g\|_{L^2(M_0 \omega \nu)} \leq C \|g\|_{\dot{H}^s_{M_0}(\mathbb{S}^{n-1})}^2.
\]

5. Estimates of \( f_1 \)

Recalling the equation (6.22) of \( f_1 \),
\[
L_{\Omega_0} f_1 = \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla_v \cdot (P_{v \perp} D_0 f_0),
\]
we introduce a new function \( \tilde{f}_1 \) satisfying \( \tilde{f}_1 = f_1 M_0 \), then the formula
\[
\frac{1}{M_0} L_{\Omega_0} f_1 = \frac{1}{M_0} \nabla_v \cdot (M_0 \tilde{f}_1) = \frac{1}{M_0} \nabla_v \cdot (M_0 \tilde{f}_1) = -L_0 \tilde{f}_1
\]
gives the equation (the symbol \( \sim \) has been dropped for brevity):
\[
L_0 f_1 = -\frac{1}{M_0} [\partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla_v \cdot (P_{v \perp} D_0 f_0)].
\]

From the self-adjoint property of \( L_0 \), the existence and uniqueness of \( f_1 \) can be easily established by combining the Lax-Milgram theorem and the Poincaré inequality, we omit the details. Next we give a lemma stating the boundedness of the function \( f_1 \).

Lemma 5.1. There exists a constant \( C > 0 \) depending on \( \|\rho_0\|_{L^\infty(0,T;H^m(\mathbb{T}))} \) and \( \|\Omega_0\|_{L^\infty(0,T;H^m(\mathbb{T}))} \) with \( m > 13 \), such that for any \( t \in [0,T] \),
\[
\|f_1\|_{H^3_{M_0}(t)} + \|\partial_t f_1\|_{H^3_{M_0}(t)} \leq C.
\]

Proof. We split the proof into four steps.

Step I: Estimation for \( \|\nabla_v f_1\|_{M_0} \) (hence for \( \|f_1\|_{M_0} \) by the Poincaré inequality on the sphere).

To deal with the first two terms on the right hand side of the equation (5.1), it follows from the Poincaré inequality on the sphere that,
\[
\langle \langle M_0^{-1} (\partial_t f_0 + v \cdot \nabla_x f_0) , f_1 \rangle \rangle_{M_0} = \langle \partial_t f_0 + v \cdot \nabla_x f_0 , f_1 \rangle_{L^2(\omega \nu \, dx)} \\
\leq \|\partial_t f_0 + v \cdot \nabla_x f_0\|_{M_0} \|f_1\|_{M_0} \\
\leq \frac{1}{4} \|\nabla_v f_1\|_{M_0}^2 + C \|\partial_t f_0 + v \cdot \nabla_x f_0\|^2.
\]

As for the third term, we have
\[
\eta_0 \langle \langle M_0^{-1} \nabla_v \cdot (P_{v \perp} D_0 f_0) , f_1 \rangle \rangle_{M_0} \leq \eta_0 \|M_0^{-1} \nabla_v \cdot (P_{v \perp} D_0 f_0)\|_{M_0} \|f_1\|_{M_0} \\
\leq \eta_0 \|\nabla_v \cdot (P_{v \perp} D_0 f_0)\|_{M_0} \|\nabla_v f_1\|_{M_0} \\
\leq \frac{1}{4} \|\nabla_v f_1\|_{M_0}^2 + \eta_0 C \|\nabla_v \cdot (P_{v \perp} D_0 f_0)\|^2.
\]

Recall that the linearized operator \( L_0 \) is nonnegative, i.e.,
\[
\langle L_0 f_1 , f_1 \rangle_{M_0} = \|\nabla_v f_1\|_{M_0}^2,
\]
then combining the above three estimates, we get
\[
\frac{1}{2} \|\nabla_v f_1\|_{M_0}^2 \leq \|\partial_t f_0 + v \cdot \nabla_x f_0\|^2 + \eta_0 \|\nabla_v \cdot (P_{v \perp} D_0 f_0)\|^2.
\]
The fact \( f_0 = \rho_0 M_0 \) yields that
\[
\begin{align*}
\partial_t f_0 &= M_0 \partial_t \rho_0 + \rho_0 M_0 v \cdot \partial_t \Omega_0, \\
\nabla_x f_0 &= M_0 \nabla_x \rho_0 + \rho_0 M_0 v \cdot \nabla_x \Omega_0,
\end{align*}
\]

Together with the assumption for the (SOH) model \((\rho_0, \Omega_0) \in L^\infty(0, T; H^m(\mathbb{T}))\) with \(m > 4\), the above two equalities result in
\[
\| \partial_t f_0 + v \cdot \nabla_x f_0 \|^2 \leq C,
\]
where the constant \( C \) depends on the value of \( \| \rho_0 \|_{L^\infty(0, T; H^m(\mathbb{T}))} \) and \( \| \Omega_0 \|_{L^\infty(0, T; H^m(\mathbb{T}))} \).

Similar argument applied to the term \( \nabla v \cdot (P_{v^+} D_0 f_0) = - (n - 1) D_0 v f_0 + D_0 \nabla v f_0 \) shows
\[
\eta_0 \| \nabla v \cdot (P_{v^+} D_0 f_0) \|^2 \leq C.
\]

Thus we are lead to the conclusion
\[(5.4)\quad \| f_1 \|^2_{M_0} \lesssim \| \nabla_v f_1 \|^2_{M_0} \leq C.\]

**Step II**: Applying the operator \( \nabla_x \) to the equation \((5.1)\), and taking \( L^2(M_0 dv dx) \) scalar product with \( \nabla_x f_1 \), we can get the estimation for \( \| \nabla_v \nabla_x f_1 \|^2_{M_0} \) (and hence, for \( \| \nabla_x f_1 \|^2_{M_0} \)). We only collect the controls for both sides of the equation \((5.1)\) as follows,
\[
\begin{align*}
\left\langle \langle \nabla_x L_0 f_1, \nabla_x f_1 \rangle \right\rangle_{M_0} &= \left\langle \langle \nabla_0 \nabla_x f_1, \nabla_x f_1 \rangle \right\rangle_{M_0} - \left\langle \langle \nabla_x \Omega_0 \nabla_v f_1, \nabla_x f_1 \rangle \right\rangle_{M_0} \\
&\geq \| \nabla_v \nabla_x f_1 \|^2_{M_0} - \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_v f_1 \|_{M_0} \| \nabla_x f_1 \|_{M_0} \\
&\geq \frac{3}{4} \| \nabla_v \nabla_x f_1 \|^2_{M_0} - C \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_v f_1 \|^2_{M_0} \\
&\leq \left\langle \langle \nabla_x \{ M_0^{-1} \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla v \cdot (P_{v^+} D_0 f_0) \} \cdot \nabla_x f_1 \right\rangle \right\rangle_{M_0} \\
&\leq \frac{1}{4} \| \nabla_v \nabla_x f_1 \|^2_{M_0} + C \| \nabla_x \{ M_0^{-1} \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla v \cdot (P_{v^+} D_0 f_0) \} \|_{M_0} \| \nabla_x f_1 \|_{M_0}^2.
\end{align*}
\]

The above two inequalities together implies
\[(5.5)\quad \frac{1}{2} \| \nabla_v \nabla_x f_1 \|^2_{M_0} \lesssim \| \nabla_x \Omega_0 \|^2_{L^\infty} \| \nabla_v f_1 \|^2_{M_0} \\
+ \| \nabla_x \{ M_0^{-1} \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla v \cdot (P_{v^+} D_0 f_0) \} \|^2_{M_0},
\]

Combining with the fact \( \| \nabla_v f_1 \|^2_{M_0} \leq C \) from step I and the assumption of \((\rho_0, \Omega_0) \in L^\infty(0, T; H^m(\mathbb{T}))\) with \(m > 4\), this gives the result
\[(5.6)\quad \| \nabla_x f_1 \|^2_{M_0} \lesssim \| \nabla_v \nabla_x f_1 \|^2_{M_0} \leq C.
\]

**Step III**: The estimations for \( \| \nabla_v \partial_t f_1 \|_{M_0} \) (and \( \| \partial_t f_1 \|^2_{M_0} \)) is similar to that of step II, and we finally get
\[(5.7)\quad \| \partial_t f_1 \|^2_{M_0} \lesssim \| \nabla_v \partial_t f_1 \|^2_{M_0} \\
\lesssim \| \nabla_v f_1 \|^2_{M_0} + \| \partial_t \{ M_0^{-1} \partial_t f_0 + v \cdot \nabla_x f_0 + \eta_0 \nabla v \cdot (P_{v^+} D_0 f_0) \} \|^2_{M_0} \\
\lesssim C,
\]

where the constant \( C \) depends on \( \| \rho_0 \|_{L^\infty(0, T; H^m(\mathbb{T}))} \) and \( \| \Omega_0 \|_{L^\infty(0, T; H^m(\mathbb{T}))} \) with \(m > 4\).

**Step IV**: Performing a similar scheme, we can get the boundedness of \( \| \nabla_v \nabla_x \partial_t f_1 \|_{M_0} \) (and \( \| \nabla_x \partial_t f_1 \|_{M_0} \)) with \(m > 7\), which yields that, together with the previous estimates,
\[(5.8)\quad \| f_1 \|_{H^2_{M_0}} + \| \partial_t f_1 \|_{H^1_{M_0}} \leq C.
\]

Proving the lemma for the estimates in spaces \( H^2_{M_0} \) (with \(m > 10\)) and \( H^3_{M_0} \) (with \(m > 13\)) proceeds similarly as above, here we omit the details. \(\square\)
Remark 5.1. Actually, we can get the similar results for higher order derivatives of $f_1$.

$$\sup_{t \in [0,T]} (\|f_1\|_{H^N_{M_0}}(t) + \|\partial_t f_1\|_{H^N_{M_0}}(t)) \leq C,$$

where $N \geq 2$ and $C$ depends on $H^m(T)$ ($m > 4N$) norm of $(\rho_0, \Omega_0)$.

6. Energy Estimates for Remainder Equation

The proof of Theorem 3.2 relies crucially on the following a priori estimate for the remainder equation (2.25).

$$\partial_t (\varepsilon f) + v \cdot \nabla_x (\varepsilon f) + \eta_0 \nabla_v \cdot (P_{v\perp} D_0(\varepsilon f)) + L_0 f = h_1 + \varepsilon h_0.$$

We first introduce some instant energy functionals,

$$\mathcal{E} = \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_2, \quad \mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2, \quad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2,$$

where the terms $\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i$ ($i = 0, 1, 2$) are defined as follows,

$$\mathcal{F}_0 = \varepsilon \|f\|_{M_0}^2, \quad \mathcal{G}_0 = \|\nabla_v f\|_{M_0}^2, \quad \mathcal{H}_0 = \|h_1\|_{M_0}^2,$$

$$\mathcal{F}_1 = \varepsilon^2 \|\nabla_x f\|_{M_0}^2 + \varepsilon \|\nabla_v f\|_{M_0}^2, \quad \mathcal{G}_1 = \varepsilon \|\nabla_v \nabla_x f\|_{M_0}^2 + \|\nabla_v f\|_{M_0}^2,$$

$$\mathcal{H}_1 = \varepsilon \|\nabla_x h_1\|_{M_0}^2 + \|\nabla_v h_1\|_{M_0}^2,$$

$$\mathcal{F}_2 = \varepsilon^2 \|\nabla_v \nabla_x f\|_{M_0}^2 + \varepsilon \|\nabla_v \nabla_x f\|_{M_0}^2 + \varepsilon \|\nabla_v f\|_{M_0}^2,$$

$$\mathcal{G}_2 = \varepsilon^2 \|\nabla_v \nabla_x f\|_{M_0}^2 + \varepsilon \|\nabla_v \nabla_x f\|_{M_0}^2 + \|\nabla_v f\|_{M_0}^2,$$

$$\mathcal{H}_2 = \varepsilon \|\nabla_v h_1\|_{M_0}^2 + \|\nabla_v \nabla_x h_1\|_{M_0}^2 + \|\nabla_v h_1\|_{M_0}^2.$$

We shall give a uniform (in $\varepsilon$) estimate for the instant energy $\mathcal{E}$, which plays the most important role in the process of Hilbert expansion.

Lemma 6.1. As for the above instant energy functional, there exists an $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{d}{dt} \mathcal{E} + \mathcal{G} \leq \widetilde{C} (\mathcal{E} + \mathcal{H}),$$

where the constant $\widetilde{C}$ depends on the value of $\|\rho_0\|_{L^\infty(0,T;H^m(T))}$ and $\|\Omega_0\|_{L^\infty(0,T;H^m(T))}$ (with $m > 13$).

Proof of Lemma 6.1. We shall give three type energy estimates in what follows: $L^2_{M_0}$, $H^1_{M_0}$, $H^2_{M_0}$ estimates, corresponding to estimates for energy functionals $\mathcal{F}_0, \mathcal{F}_2, \mathcal{F}_2$, respectively, and then close them to finish the proof. We also give two remarks, one of which concerns the $H^N_{M_0}$ estimates (for large integer $N$) and the other discusses the role of $\mathcal{H}$.

6.1. Claim 1: ($L^2_{M_0}$ estimates). Taking $L^2(M_0 \, dv \, dx)$ inner product with $f$, we get from the equation (6.1) that

$$\varepsilon \langle \langle \partial_t f, f \rangle \rangle_{M_0} = \frac{1}{2} \frac{d}{dt} \varepsilon \|f\|_{L^2_{M_0}}^2 - \frac{1}{2} \varepsilon \iint |f|^2 \partial_t M_0 \, dv \, dx$$

with

$$\iint |f|^2 \partial_t M_0 \, dv \, dx = \iint |f|^2 \partial_t \Omega_0 \cdot v M_0 \, dv \, dx \leq \|\partial_t \Omega_0\|_{L^\infty_x} \|f\|_{L^2_{M_0}}^2.$$

The second term on the left-hand side can be controlled by

$$\langle \langle v \cdot \nabla_x f, f \rangle \rangle_{M_0} = \frac{1}{2} \int \int v \cdot \nabla_x (f^2 M_0) \, dv \, dx - \frac{1}{2} \int \int v \cdot \nabla_x M_0 |f|^2 \, dv \, dx$$

$$= 0 - \frac{1}{2} \int \int v \otimes v : \nabla_x \Omega_0 |f|^2 M_0 \, dv \, dx,$$

hence

$$\varepsilon \langle \langle v \cdot \nabla_x f, f \rangle \rangle_{M_0} \leq \varepsilon \|\nabla_x \Omega_0\|_{L^\infty_x} \|f\|_{L^2_{M_0}}^2.$$
As for the third term, it follows that
\[-\eta_0 \langle \langle \nabla_v \cdot (P_\nu D_0 f), f \rangle \rangle_{M_0} = \eta_0 \langle \langle D_0 f, \nabla_v f \rangle \rangle_{M_0} = \eta_0 \int \int D_0 \frac{1}{2} \nabla_v f^2 M_0 \, dv \, dx \]
\[= \frac{1}{2} \eta_0 (n - 1) \int \int D_0 v f^2 M_0 \, dv \, dx - \frac{1}{2} \eta_0 \int \int D_0 f^2 \nabla_v M_0 \, dv \, dx, \]
combining with the facts that \( \nabla_v M_0 = P_\nu \Omega_0 M_0 \) and \( |P_\nu \Omega_0| \leq 1 \), this gives
\[|\eta_0 \langle \langle \nabla_v \cdot (P_\nu D_0 f), f \rangle \rangle_{M_0} | \lesssim \| D_0 \|_{L^\infty} \| f \|_{M_0}^2. \]

On the other hand, the nonnegativity of the linearized operator \( L_0 \) implies that
\[\langle \langle L_0 f, f \rangle \rangle_{M_0} = \| \nabla_v f \|_{M_0}^2. \]

The Poincaré inequality and the Hölder inequality gives the following estimate,
\[\langle \langle h_1, f \rangle \rangle_{M_0} \leq \| h_1 \|_{M_0} \| f \|_{M_0} \lesssim \| h_1 \|_{M_0} \| \nabla_v f \|_{M_0} \leq \frac{1}{2} C \| h_1 \|_{M_0}^2 + \frac{1}{2} \| \nabla_v f \|_{M_0}^2. \]

Notice that the last term on the right hand of equation (6.1) can be bounded as follows,
\[\epsilon \langle \langle h_0 f, f \rangle \rangle_{M_0} \leq \| h_0 \|_{M_0} \epsilon \| f \|_{M_0}^2 \lesssim (\| \partial_t \Omega_0 \|_{L^\infty} + \| \nabla_x \Omega_0 \|_{L^\infty} + \eta_0 \| D_0 \|_{L^\infty}) \epsilon \| f \|_{M_0}^2, \]
then all these above estimates together gives that
\[\frac{1}{2} \frac{d}{dt} \epsilon \| f \|_{M_0}^2 + \frac{1}{2} \| \nabla_v f \|_{M_0}^2 \lesssim (\| \partial_t \Omega_0 \|_{L^\infty} + \| \nabla_x \Omega_0 \|_{L^\infty} + \eta_0 \| D_0 \|_{L^\infty}) \epsilon \| f \|_{M_0}^2 + \| h_1 \|_{M_0}^2. \]

Denote
\[C_0 = \| \partial_t \Omega_0 \|_{L^\infty} + \| \nabla_x \Omega_0 \|_{L^\infty} + \eta_0 \| D_0 \|_{L^\infty}, \]
then there exists a universal constant \( C \) such that
\[\frac{1}{2} \frac{d}{dt} F_0 + \frac{1}{2} G_0 \leq C (C_0 F_0 + H_0). \]

6.2. Claim 2: \( (H^1_{M_0}) \) estimates.

**Step I:** Taking \( L^2(M_0 \, dv \, dx) \) scalar product with \( L_0 f \) to control the quantity \( \epsilon \| \nabla_v f \|_{M_0}^2 \), we have the following estimates for the first two terms on the left-hand side,
\[\epsilon \langle \langle \partial_t f, L_0 f \rangle \rangle_{M_0} = \epsilon \langle \langle \partial_t \nabla_v f, \nabla_v f \rangle \rangle_{M_0} = \frac{1}{2} \frac{d}{dt} \epsilon \| \nabla_v f \|_{M_0}^2 - \frac{1}{2} \epsilon \int \partial_t M_0 |\nabla_v f|^2 \, dv \, dx \]
\[= \frac{1}{2} \frac{d}{dt} \epsilon \| \nabla_v f \|_{M_0}^2 - \frac{1}{2} \epsilon \int v \cdot \partial_t \Omega_0 \, M_0 |\nabla_v f|^2 \, dv \, dx, \]
\[\epsilon \langle \langle v \cdot \nabla_x f, L_0 f \rangle \rangle_{M_0} = \epsilon \langle \langle v \cdot \nabla_x f, \nabla_v f \rangle \rangle_{M_0} + \epsilon \langle \langle v \cdot \nabla_x f, \nabla_v f \rangle \rangle_{M_0} \]
\[= \frac{1}{2} \epsilon \int v \cdot \nabla_x (|\nabla_v f| M_0) \, dv \, dx - \frac{1}{2} \epsilon \int v \cdot \nabla_x M_0 |\nabla_v f|^2 \, dv \, dx \]
\[+ \epsilon \langle \langle \nabla_x f, \nabla_v f \rangle \rangle_{M_0} \]
\[= 0 - \frac{1}{2} \epsilon \int v \cdot \nabla_x \Omega_0 \, M_0 |\nabla_v f|^2 \, dv \, dx + \epsilon \| \nabla_x f \|_{M_0} \| \nabla_v f \|_{M_0} \]
\[\leq \epsilon \| \nabla_x \Omega \|_{L^\infty} \| \nabla_v f \|_{M_0}^2 + C \epsilon^2 \| \nabla_x f \|_{M_0}^2 + \frac{1}{4} \| \nabla_v f \|_{M_0}^2, \]
where we have used the Hölder inequality and the Poincaré inequality on the sphere in the last line.

As for the third term, we write
\[\epsilon \eta_0 \langle \langle \nabla_v \cdot (P_\nu D_0 f), L_0 f \rangle \rangle_{M_0} = \epsilon \eta_0 \langle \langle - (n - 1) D_0 v f + D_0 \nabla_v f, L_0 f \rangle \rangle_{M_0}. \]
It is an easy matter to calculate,
\[
\langle vf, \mathcal{L}_0 f \rangle_{L^2(M_0 \, dv)} = \langle \nabla_v (vf), \nabla_v f \rangle_{L^2(M_0 \, dv)} = \langle f + v \nabla_v f, \nabla_v f \rangle_{L^2(M_0 \, dv)}
\]
\[
= \frac{1}{2} \int (\nabla_v f^2) M_0 \, dv + \int v |\nabla_v f|^2 M_0 \, dv
\]
\[
= \frac{n-1}{2} \int v f^2 M_0 \, dv - \frac{1}{2} \int \nabla_v M_0 f^2 \, dv + \int v |\nabla_v f|^2 M_0 \, dv.
\]

By Lemma 4.2 we get the fact
\[
(L_0 f, \nabla_v f)_{L^2(M_0 \, dv)} \leq C \|\nabla_v f\|_{L^2(M_0 \, dv)}^2.
\]
(6.9)

Combining with the Poincaré inequality on the sphere, the two above inequalities yield that
\[
|\varepsilon \eta_0 \langle \nabla_v \cdot (P_{v \perp} D_0 f), \mathcal{L}_0 f \rangle_{M_0}| \leq \varepsilon \eta_0 |D_0|_{L^\infty} \|\nabla_v f\|_{M_0}^2 + \varepsilon \eta_0 \frac{(n-1)^2}{2} \|D_0\|_{L^\infty} \|f\|_{M_0}^2
\]
\[
+ \varepsilon \eta_0 \varepsilon_0 \frac{(n-1)}{2} \|D_0\|_{L^\infty} \|f\|_{M_0}^2 + \varepsilon \eta_0 (n-1) \|D_0\|_{L^\infty} \|\nabla_v f\|_{M_0}^2
\]
\[
\leq \varepsilon \eta_0 |D_0|_{L^\infty} \|\nabla_v f\|_{M_0}^2.
\]

On the other hand, we can control the terms on the right-hand side of equation (6.11) as follows,
\[
\langle \mathcal{L}_0 f, \mathcal{L}_0 f \rangle_{M_0} = \|\nabla_v f\|_{M_0}^2,
\]
\[
\langle h_1, \mathcal{L}_0 f \rangle_{M_0} = \langle \nabla_v h_1, \nabla_v f \rangle_{M_0} \leq \|\nabla_v h_1\|_{M_0} \|\nabla_v f\|_{M_0} \leq C \|\nabla_v h_1\|_{M_0}^2 + \frac{1}{4} \|\nabla_v f\|_{M_0}^2,
\]
\[
\varepsilon \langle h_0 f, \mathcal{L}_0 f \rangle_{M_0} = \varepsilon \langle \nabla_v h_0 f, \nabla_v f \rangle_{M_0} + \varepsilon \langle f \nabla_v h_0, \nabla_v f \rangle_{M_0}
\]
\[
\leq \varepsilon \|h_0\|_{L^\infty_{x,v}} \|\nabla_v f\|_{M_0}^2 + \varepsilon \|\nabla_v h_0\|_{L^\infty_{x,v}} \|f\|_{M_0} \|\nabla_v f\|_{M_0},
\]
where we have used the Poincaré inequality again.

Accordingly, we have
\[
\varepsilon \langle \delta_x, \mathcal{L}_0 f \rangle_{M_0} = \frac{1}{2} \frac{d}{dt} \varepsilon \|\nabla_v f\|_{M_0}^2 + \frac{1}{2} \|\nabla_v f\|_{M_0}^2.
\]
\[
\leq \|\nabla_v h_1\|_{M_0}^2 + \varepsilon \|\nabla_v f\|_{M_0}^2 + \|\nabla_v \Omega_0\|_{L^\infty_{x,v}} + \varepsilon \eta_0 \|D_0\|_{L^\infty_{x,v}} \varepsilon \|\nabla_v f\|_{M_0}^2
\]
\[
+ \varepsilon \|h_0\|_{L^\infty_{x,v}} \|\nabla_v f\|_{M_0}^2 + \varepsilon \|\nabla_v h_0\|_{L^\infty_{x,v}} \|f\|_{M_0} \|\nabla_v f\|_{M_0}.
\]

Step II: Apply the operator \( \nabla_x \) to the equation (6.11), and take \( L^2(M_0 \, dv \, dx) \) scalar product with \( \varepsilon \nabla_x f \), then we get
\[
\varepsilon^2 \langle \nabla_x (v \cdot \nabla_x f), \nabla_x f \rangle_{M_0} = \frac{1}{2} \frac{d}{dt} \varepsilon \|\nabla_x f\|_{M_0}^2 - \frac{1}{2} \varepsilon^2 \int \partial_t M_0 |\nabla_x f|^2 \, dv \, dx,
\]
\[
\varepsilon^2 \langle \nabla_x (v \cdot \nabla_x f), \nabla_x f \rangle_{M_0} = \frac{1}{2} \varepsilon^2 \int \nabla_x (|\nabla_x f|^2 M_0) \, dv \, dx
\]
\[
- \frac{1}{2} \varepsilon^2 \int \nabla_x M_0 |\nabla_x f|^2 \, dv \, dx
\]
\[
= - \frac{1}{2} \varepsilon^2 \int v \otimes v : \nabla_x \Omega_0 M_0 |\nabla_x f|^2 \, dv \, dx.
\]

The third term can be decomposed into two terms,
\[
- \varepsilon^2 \eta_0 \langle \nabla_x \nabla_v \cdot (P_{v \perp} D_0 f), \nabla_x f \rangle_{M_0}
\]
\[
= - \varepsilon^2 \eta_0 \langle \nabla_v \cdot (P_{v \perp} D_0 \nabla_x f), \nabla_x f \rangle_{M_0} - \varepsilon^2 \eta_0 \langle \nabla_v \cdot (P_{v \perp} D_0 f), \nabla_x f \rangle_{M_0}
\]
\[
= \varepsilon^2 \eta_0 (I_1 + I_2).
\]
Some straightforward calculations enable us to get
\[ I_1 = \langle D_0 \nabla f, \nabla \nabla (\nabla f M_0) \rangle \]
\[ = \langle D_0 \nabla f, \nabla \nabla (\nabla f M_0 + \nabla x f \nabla v M_0) \rangle \]
\[ = \frac{1}{2} \iint D_0 (\nabla v | \nabla f |^2) \ M_0 \ dv \ dx + \iint D_0 | \nabla x f |^2 \nabla v M_0 \ dv \ dx \]
\[ = \frac{1}{2} \iint D_0 (\nabla v | \nabla f |^2) \ M_0 \ dv \ dx + \frac{1}{2} \iint D_0 | \nabla x f |^2 M_0 P_{v+} \Omega_0 \ dv \ dx \]
\[ = \frac{n-1}{2} \iint D_0 v | \nabla f |^2 M_0 \ dv \ dx + \frac{1}{2} \iint D_0 | \nabla x f |^2 M_0 P_{v+} \Omega_0 \ dv \ dx \]
\[ \leq \frac{n}{2} \| D_0 \|_{L^\infty} \| \nabla x f \|^2_{M_0}, \]
and by the Poincaré inequality,
\[ I_2 = \langle (n-1) \nabla x D_0 v f - \nabla x D_0 \nabla v f, \nabla x f \rangle_{M_0} \]
\[ \leq (n-1) \| \nabla x D_0 \|_{L^\infty} \| f \|_{M_0} \| \nabla x f \|_{M_0} + \| \nabla x D_0 \|_{L^\infty} \| \nabla v f \|_{M_0} \| \nabla x f \|_{M_0} \]
\[ \leq n \| \nabla x D_0 \|_{L^\infty} \| \nabla v f \|_{M_0} \| \nabla x f \|_{M_0}, \]
hence the third term can be bounded by
\[ |^{2} \eta_0 \left| \langle \nabla x \nabla v \cdot (P_{v+} D_0 f), \nabla x f \rangle_{M_0} \right| = \varepsilon^{2} \eta_0 |I_1 + I_2| \]
\[ \leq \varepsilon^{2} \eta_0 \| D_0 \|_{L^\infty} \| \nabla x f \|^2_{M_0} + \varepsilon^{2} \eta_0 \| \nabla x D_0 \|_{L^\infty} \| \nabla v f \|_{M_0} \| \nabla x f \|_{M_0} \].

Next we turn to consider the terms on the right-hand side of equation (6.1). Recalling the nonnegativity property of the linearized operator \( L_0 = -(\Delta_v + \Omega_0 \cdot \nabla v) \), we have
\[ \langle \nabla x L_0 f, \nabla x f \rangle_{M_0} = \varepsilon \langle \langle \nabla \nabla f, \nabla x f \rangle \rangle_{M_0} - \varepsilon \langle \langle \nabla x f \nabla v f, \nabla x f \rangle \rangle_{M_0} \]
\[ \geq \varepsilon \| \nabla v \nabla f \|^2_{M_0} - \varepsilon \| \nabla x \Omega_0 \|_{L^\infty} \| \nabla v f \|_{M_0} \| \nabla x f \|_{M_0} \]
\[ \geq \frac{3}{4} \varepsilon \| \nabla v \nabla f \|^2_{M_0} - C \varepsilon \| \nabla \Omega_0 \|^2_{L^\infty} \| \nabla v f \|^2_{M_0}, \]
where we have used the Poincaré inequality on the sphere \( \| \nabla x f \|^2_{M_0} \leq \| \nabla v \nabla f \|^2_{M_0} \). Note the Poincaré inequality also yield that
\[ \langle \nabla x h_1, \nabla x f \rangle_{M_0} \leq \varepsilon \| \nabla x h_1 \|_{M_0} \| \nabla x f \|_{M_0} \leq C \varepsilon \| \nabla x h_1 \|^2_{M_0} + \frac{1}{4} \varepsilon \| \nabla v \nabla f \|^2_{M_0}, \]
\[ \varepsilon^{2} \langle \nabla x (h_0 f), \nabla x f \rangle_{M_0} = \varepsilon^{2} \langle \langle \nabla x f h_0, \nabla x f \rangle \rangle_{M_0} + \varepsilon^{2} \langle \langle f \nabla x h_0, \nabla x f \rangle \rangle_{M_0} \]
\[ \leq \varepsilon^{2} \| h_0 \|_{L^\infty} \| \nabla x f \|^2_{M_0} + \varepsilon^{2} \| \nabla x h_0 \|_{L^\infty} \| f \|_{M_0} \| \nabla x f \|_{M_0} \],
which ensure that, together with the previous estimates,
\[ \frac{1}{2} \frac{d}{dt} \left( \varepsilon^{2} \| \nabla x f \|^2_{M_0} + \varepsilon \| \nabla v \nabla f \|^2_{M_0} \right) \]
\[ \leq \varepsilon \| \nabla x \Omega_0 \|^2_{L^\infty} \| \nabla v f \|^2_{M_0} + \varepsilon \| \nabla x h_1 \|^2_{M_0} \]
\[ + \left( \| \partial \Omega_0 \|_{L^\infty} + \| \nabla x \Omega_0 \|_{L^\infty} + \eta_0 \| D_0 \|_{L^\infty} \right) \varepsilon^{2} \| \nabla x f \|^2_{M_0} \]
\[ + \varepsilon^{2} \eta_0 \| \nabla x D_0 \|_{L^\infty} \| \nabla v f \|_{M_0} \| \nabla x f \|_{M_0} \]
\[ + \left( \varepsilon^{2} \| h_0 \|_{L^\infty} \| \nabla x f \|^2_{M_0} + \varepsilon^{2} \| \nabla x h_0 \|_{L^\infty} \| f \|_{M_0} \| \nabla x f \|_{M_0} \right). \]

Step III: Combining the two inequalities (6.10) and (6.11) gives that
\[ \frac{1}{2} \frac{d}{dt} \left( \varepsilon^{2} \| \nabla x f \|^2_{M_0} + \varepsilon \| \nabla v f \|^2_{M_0} \right) + \frac{1}{2} \left( \varepsilon \| \nabla v \nabla x f \|^2_{M_0} + \| \nabla v f \|^2_{M_0} \right) \]
\[ \leq \left( \varepsilon^{2} \| \nabla x f \|^2_{M_0} + \| \nabla x \Omega_0 \|_{L^\infty} \| \nabla v f \|^2_{M_0} \right) \]
\[
+ \left( \| \partial_t \Omega \|_{L^\infty_x} + \| \nabla_x \Omega \|_{L^\infty_x} + \eta_0 \| D_0 \|_{L^\infty_x} \right) \left( \varepsilon^2 \| \nabla_x f \|_{M_0}^2 + \varepsilon \| \nabla v f \|_{M_0}^2 \right) \\
+ \eta_0 \| \nabla_x D_0 \|_{L^\infty_x} \varepsilon \| \nabla_x f \|_{M_0} \| \nabla_x f \|_{M_0} \\
+ \left\{ \| h_0 \|_{L^\infty_{x,v}} \varepsilon \| \nabla_x f \|_{M_0}^2 + \varepsilon \| \nabla_v f \|_{M_0}^2 \right\} + \| \nabla_x h_0 \|_{L^\infty_{x,v}} \varepsilon \| f \|_{M_0} \| \nabla_x f \|_{M_0} \\
+ \| \nabla_v h_0 \|_{L^\infty_{x,v}} \varepsilon \| f \|_{M_0} \| \nabla_v f \|_{M_0} \right\}.
\]

Denote
\[
C_1 = \sup \left\{ C_0^2, \| \nabla_x h_0 \|_{L^\infty_{x,v}}, \| \nabla_v h_0 \|_{L^\infty_{x,v}}, \eta_0 \| \nabla_x D_0 \|_{L^\infty_x} \right\},
\]
with \( C_0 = \| \partial_t \Omega \|_{L^\infty_x} + \| \nabla_x \Omega \|_{L^\infty_x} + \eta_0 \| D_0 \|_{L^\infty_x} \) defined as before, then we can conclude that, up to a constant \( C' > 0 \),
\[
(6.13) \quad \frac{1}{2} \frac{d}{dt} \mathcal{F}_1 + \frac{1}{2} \mathcal{G}_1 \leq CC_1 (1 + \varepsilon^2)(\mathcal{F}_1 + \mathcal{F}_0) + C\mathcal{H}_1,
\]
where it should be pointed out that
\[
\varepsilon^2 \| \nabla_v f \|_{M_0} \| \nabla_x f \|_{M_0} = \frac{1}{2} (\varepsilon^2 \| \nabla_v f \|_{M_0})(\varepsilon \| \nabla_x f \|_{M_0}^2) \leq \varepsilon^2 \varepsilon^2 \frac{1}{2} \mathcal{F}_1 \mathcal{F}_1^2.
\]

6.3. Claim 3: \( H_{M_0}^2 \) estimates.

**Step 1:** Apply the operator \( \nabla_x^2 \) to the equation (6.11), and take \( L^2(M_0 dv dx) \) scalar product with the quantity \( \varepsilon^2 \nabla_x^2 f \), then we can get immediately,
\[
\varepsilon^3 \langle \langle \partial_t \nabla_x^2 f, \nabla_x^2 f \rangle \rangle \leq \frac{1}{2} \frac{d}{dt} \varepsilon^3 \| \nabla_x^2 f \|_{M_0}^2 - \frac{1}{2} \varepsilon^3 \int \partial_t M_0 \| \nabla_x f \|_{dM_0}^2 dv dx,
\]
\[
\varepsilon^3 \langle \langle \nabla_x^2 (v \cdot \nabla_x f), \nabla_x^2 f \rangle \rangle \leq \frac{1}{2} \varepsilon^3 \int \int v \cdot \nabla_x (\| \nabla_x f \|_{M_0}^2) M_0 dv dx
\]
\[
= 0 - \frac{1}{2} \varepsilon^3 \int \int v \cdot \nabla_x M_0 \| \nabla_x f \|_{M_0}^2 dv dx.
\]

Similar as before, we split the third term into two terms,
\[
- \varepsilon^3 \eta_0 \langle \langle \nabla_x^2 \nabla v \cdot (P_{v+} D_0 f), \nabla_x^2 f \rangle \rangle_{M_0}
\]
\[
= - \varepsilon^3 \eta_0 \langle \langle \nabla v \cdot [P_{v+} (\nabla_x^2 D_0 f + 2 \nabla_x D_0 \nabla_x f)], \nabla_x^2 f \rangle \rangle_{M_0}
\]
\[
- \varepsilon^3 \eta_0 \langle \langle \nabla v \cdot (P_{v+} D_0 \nabla_x^2 f), \nabla_x^2 f \rangle \rangle_{M_0}
\]
\[
= \varepsilon^3 \eta_0 (II_1 + II_2).
\]

It follows by calculation that,
\[
II_1 = \langle \langle (n - 1) \nabla_x^2 D_0 v f - \nabla_x^2 D_0 \nabla v f, \nabla_x^2 f \rangle \rangle_{M_0}
\]
\[
+ \langle \langle (n - 1) \nabla_x D_0 v \nabla_x f - \nabla_x D_0 \nabla v \nabla_x f, \nabla_x^2 f \rangle \rangle_{M_0}
\]
\[
\leq \| \nabla_x^2 D_0 \|_{L^\infty_x} \left( (n - 1) \| f \|_{M_0} + \| \nabla v f \|_{M_0} \right) \| \nabla_x^2 f \|_{M_0}
\]
\[
+ 2 \| \nabla_x D_0 \|_{L^\infty_x} \left( (n - 1) \| \nabla_x f \|_{M_0} + \| \nabla v \nabla_x f \|_{M_0} \right) \| \nabla_x^2 f \|_{M_0}
\]
\[
\leq \| \nabla_x^2 D_0 \|_{L^\infty_x} \| \nabla v f \|_{M_0} \| \nabla_x^2 f \|_{M_0} + \| \nabla_x D_0 \|_{L^\infty_x} \| \nabla v \nabla_x f \|_{M_0} \| \nabla_x^2 f \|_{M_0},
\]
\[
II_2 = \langle \langle D_0 \nabla_v \nabla_x f, \nabla v \nabla_x^2 f \rangle \rangle_{M_0} = \frac{1}{2} \int \int D_0 \nabla_v (| \nabla_x f |^2) M_0 dv dx
\]
\[
= \frac{n - 1}{2} \int \int D_0 v | \nabla_x^2 f |^2 M_0 dv dx - \frac{1}{2} \int \int D_0 \nabla v M_0 | \nabla_x^2 f |^2 dv dx
\]
\[
\leq \| D_0 \|_{L^\infty_x} \| \nabla_x f \|_{M_0}^2,
\]
which give the estimate

\[
\left| \varepsilon^3 \eta_0 \left\langle \langle \nabla_x^2 \nabla_v \cdot (P_v \cdot D_0 f) , \nabla_x^2 f \rangle \right\rangle_{\partial M_0} \right|
\leq \varepsilon^3 \eta_0 \| D_0 \|_{L^\infty} \| \nabla_x^2 f \|^2_{\partial M_0} + \varepsilon^3 \eta_0 \| \nabla_x D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
+ \varepsilon^3 \eta_0 \| \nabla_x^2 D_0 \|_{L^\infty} \| \nabla_v f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}.
\]

As for the right-hand side of the equation (6.1), we collect their estimates, by the non-negativity property of the linearized operator \(L_0\) and the Poincaré inequality on the sphere \(\| \nabla_x^2 f \|_{\partial M_0} \leq \| \nabla_v \nabla_x^2 f \|_{\partial M_0}\), as follows,

\[
\langle \langle \nabla_x^2 L_0 f , \varepsilon^2 \nabla_x^2 f \rangle \rangle_{\partial M_0}
= \varepsilon^2 \langle \langle L_0 \nabla_x^2 f , \nabla_x^2 f \rangle \rangle_{\partial M_0} - \varepsilon^2 \langle \langle \nabla_x^2 \Omega_0 \nabla_v f + 2 \nabla_x \Omega_0 \nabla_v \nabla_x f , \nabla_x^2 f \rangle \rangle_{\partial M_0}
\geq \varepsilon^2 \| \nabla_v \nabla_x^2 f \|^2_{\partial M_0} - \left[ \| \nabla_x^2 \Omega_0 \|_{L^\infty} \| \nabla_v f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
+ 2 \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0} \right]
\geq \frac{3}{4} \varepsilon^2 \| \nabla_v \nabla_x^2 f \|^2_{\partial M_0} - C \varepsilon^2 \left( \| \nabla_x^2 \Omega_0 \|_{L^\infty} \| \nabla_v f \|^2_{\partial M_0} + \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|^2_{\partial M_0} \right),
\]

\[
\varepsilon^2 \langle \langle \nabla_x^2 \Omega_0 , \nabla_x^2 f \rangle \rangle_{\partial M_0}
\leq \varepsilon^2 \| \nabla_x^2 \Omega_0 \|_{L^\infty} \| \nabla_x \nabla_x^2 f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
\leq C \varepsilon^2 \| \nabla_x^2 h_1 \|^2_{\partial M_0} + \frac{1}{4} \varepsilon^2 \| \nabla_v \nabla_x^2 f \|^2_{\partial M_0},
\]

\[
\varepsilon^3 \langle \langle \nabla_x^2 \Omega_0 , \nabla_x^2 f \rangle \rangle_{\partial M_0}
\leq \varepsilon^3 \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_x h_0 \|_{L^\infty} \| \nabla_x f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
+ \varepsilon^3 \| \nabla_x^2 h_0 \|_{L^\infty} \| f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}.
\]

Then it follows from the above estimates that,

\[
\frac{1}{2} \frac{d}{dt} \varepsilon^3 \| \nabla_x^2 f \|^2_{\partial M_0} + \frac{1}{2} \varepsilon^2 \| \nabla_v \nabla_x^2 f \|^2_{\partial M_0}
\leq \varepsilon^2 \left( \| \nabla_x^2 \Omega_0 \|_{L^\infty} \| \nabla_v f \|^2_{\partial M_0} + \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|^2_{\partial M_0} \right) + \varepsilon^2 \| \nabla_x^2 h_1 \|^2_{\partial M_0}
+ \left( \| \partial_t \Omega_0 \|_{L^\infty} + \| \nabla_x \Omega_0 \|_{L^\infty} + \eta_0 \| D_0 \|_{L^\infty} \right) \varepsilon^3 \| \nabla_x^2 f \|^2_{\partial M_0}
+ \varepsilon^3 \| \nabla_x \Omega_0 \|_{L^\infty} \| \nabla_x \nabla_x f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0} + \varepsilon^3 \eta_0 \| \nabla_x D_0 \|_{L^\infty} \| \nabla_v f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
+ \left( \| h_0 \|_{L^\infty} \varepsilon^3 \| \nabla_x^2 f \|^2_{\partial M_0} + \| \nabla_x h_0 \|_{L^\infty} \varepsilon^3 \| \nabla_x f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}
+ \varepsilon^3 \| \nabla_x^2 h_0 \|_{L^\infty} \varepsilon^3 \| f \|_{\partial M_0} \| \nabla_x^2 f \|_{\partial M_0}. \right)
\]

**Step II:** To control the quantity \(\varepsilon^2 \| \nabla_v \nabla_x f \|^2_{\partial M_0}\), we apply the operator \(\nabla_x\) to the equation (6.1), and take \(L^2(M_0 dv dx)\) scalar product with \(\varepsilon L_0 \nabla_x f\), then it follows

\[
\varepsilon^2 \langle \langle \partial_t \nabla_x f , L_0 \nabla_x f \rangle \rangle_{\partial M_0}
= \varepsilon^2 \langle \langle \partial_t \nabla_v \nabla_x f , \nabla_v \nabla_x f \rangle \rangle_{\partial M_0}
= \frac{1}{2} \frac{d}{dt} \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{\partial M_0} - \frac{1}{2} \varepsilon^2 \int \partial_t M_0 |\nabla_v \nabla_x f|^2 dv dx,
\]

\[
\varepsilon^2 \langle \langle \nabla_v \nabla_x^2 f , L_0 \nabla_x f \rangle \rangle_{\partial M_0}
= \varepsilon^2 \langle \langle \nabla_v (\nabla_x^2 f) , \nabla_v \nabla_x f \rangle \rangle_{\partial M_0}
\]
\[ \frac{1}{2} \varepsilon^2 \int v \cdot \nabla_x (|\nabla_v \nabla_x f|^2 M_0) \, dv \, dx - \frac{1}{2} \varepsilon^2 \int v \cdot \nabla_x M_0 |\nabla_v \nabla_x f|^2 \, dv \, dx \\
+ \varepsilon^2 \left\langle \langle \nabla_x^2 f, \nabla_v \nabla_x f \rangle \right\rangle_{M_0} \leq 0 + \frac{1}{2} \varepsilon^2 \| \nabla_x \Omega \|_{L^\infty}^2 \| \nabla_v \nabla_x f \|_{M_0}^2 + C \varepsilon^3 \| \nabla^2 f \|_{M_0}^2 + \frac{1}{6} \varepsilon \| \nabla_v^2 \nabla_x f \|_{M_0}^2, \]

where we have used the Poincaré inequality on the sphere \( \| \nabla_v \nabla_x f \|_{M_0} \lesssim \| \nabla^2 \nabla_x f \|_{M_0} \).

The third term can be estimated by a similar decomposition as before,

\[ - \varepsilon^2 \eta_0 \left\langle \langle \nabla_x \nabla_v \cdot (P_v D_0 f), \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} = - \varepsilon^2 \eta_0 \left\langle \langle \nabla_v \cdot (P_v D_0 \nabla_x f + P_v \nabla_v \nabla_x f), \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ = \varepsilon^2 \eta_0 \left\langle \langle (n-1)D_0 \nabla_x f - D_0 \nabla_v \nabla_x f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} + \varepsilon^2 \eta_0 \left\langle \langle (n-1)D_0 \nabla_x f - D_0 \nabla_v \nabla_x f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ = \varepsilon^2 \eta_0 (II_3 + II_4). \]

Combining Lemma 4.2 and the Poincaré inequality on the sphere ensures that

\[ II_3 = (n-1) \left\langle \langle D_0 \nabla_v (v \cdot \nabla_x f), \nabla_v \nabla_x f \right\rangle \right\rangle_{M_0} - \left\langle \langle D_0 \nabla_v \nabla_x f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ \leq (n-1) \| D_0 \|_{L^\infty} (\| \nabla_x f \|_{M_0} + \| \nabla_v \nabla_x f \|_{M_0}) \| \nabla_v \nabla_x f \|_{M_0} \]

\[ + \int D_0 \mathcal{L}_0 \nabla_x f \nabla_v \nabla_x f M_0 \, dv \, dx \]

\[ \lesssim (n-1) \| D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0}^2 + \| D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0}^2 \]

\[ \leq n \| D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0}^2, \]

and

\[ II_4 = (n-1) \left\langle \langle \nabla_x D_0 \nabla_v (v \cdot f), \nabla_v \nabla_x f \right\rangle \right\rangle_{M_0} - \left\langle \langle \nabla_x D_0 \nabla^2 f, \nabla_v \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ \leq (n-1) \| \nabla_x D_0 \|_{L^\infty} (\| f \|_{M_0} + \| \nabla_v f \|_{M_0}) \| \nabla_v \nabla_x f \|_{M_0} \]

\[ + n \| \nabla_x \nabla_v f \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0} \]

\[ \lesssim n \| \nabla_x D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0} \]

therefore we get

\[ \| \nabla_v \nabla_x f \|_{M_0} \lesssim \varepsilon^2 \eta_0 \left\langle \langle \nabla_x \nabla_v \cdot (P_v D_0 f), \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ \lesssim \varepsilon^2 \eta_0 \| D_0 \|_{L^\infty} \| \nabla_v \nabla_x f \|_{M_0}^2 + \varepsilon^2 \eta_0 \| \nabla_x D_0 \|_{L^\infty} \| \nabla^2 f \|_{M_0} \| \nabla_v \nabla_x f \|_{M_0}. \]

Similar discussions as before enable us to deal with the terms on the right-hand side, as follows,

\[ \varepsilon \left\langle \langle \nabla_x \mathcal{L}_0 f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} = \varepsilon \left\langle \langle \mathcal{L}_0 \nabla_x f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} - \varepsilon \left\langle \langle \nabla_x \Omega \nabla_v f, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ \geq \varepsilon \| \nabla_v^2 \nabla_x f \|^2_{M_0} - \varepsilon \| \nabla_x \Omega \|_{L^\infty} \| \nabla^2 f \|_{M_0} \| \nabla_v \nabla_x f \|_{M_0} \]

\[ \geq \frac{5}{6} \varepsilon \| \nabla_v^2 \nabla_x f \|^2_{M_0} - C \varepsilon \| \nabla_x \Omega \|_{L^\infty} \| \nabla^2 f \|_{M_0}^2, \]

\[ \varepsilon \left\langle \langle \nabla_x h_1, \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} = \varepsilon \left\langle \langle \nabla_v \nabla_x h_1, \nabla_v \nabla_x f \right\rangle \right\rangle_{M_0} \leq C \varepsilon \| \nabla_v \nabla_x h_1 \|_{M_0}^2 + \frac{1}{6} \varepsilon \| \nabla^2 \nabla_x f \|_{M_0}^2, \]

\[ \varepsilon^2 \left\langle \langle \nabla_x (h_0 f), \mathcal{L}_0 \nabla_x f \right\rangle \right\rangle_{M_0} = \varepsilon^2 \left\langle \langle \nabla_v \nabla_x (h_0 f), \nabla_v \nabla_x f \right\rangle \right\rangle_{M_0} \]

\[ \lesssim \varepsilon^2 \sum_{|\alpha_1| + |\alpha_2| = 1 \atop |\beta_1| + |\beta_2| = 1} \left\| \nabla_v^{\alpha_1} \nabla_x^{\beta_1} h_0 \right\|_{L^\infty} \left\| \nabla_v^{\alpha_2} \nabla_x^{\beta_2} f \right\|_{M_0} \| \nabla_v \nabla_x f \|_{M_0}. \]
All the above estimates together give that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} + \frac{1}{2} \varepsilon \| \nabla_v \nabla_x f \|^2_{M_0} \\
\lesssim \varepsilon \| \nabla_x \Omega_0 \|^2_{L^\infty_x} + \| \nabla_v^2 f \|^2_{M_0} + \| \nabla_v \nabla_x h_1 \|^2_{M_0} + \| \nabla_v^2 f \|^2_{M_0} \\
+ \left( \| \partial_t \Omega_0 \|^2_{L^\infty_x} + \| \nabla_x \Omega_0 \|^2_{L^\infty_x} + \eta_0 \| D_0 \|^2_{L^\infty_x} \right) \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} \\
+ \varepsilon^2 \sum_{|\alpha_1|+|\alpha_2|=1} \| \nabla_v^{\alpha_1} \nabla_v^{\beta_2} h_0 \|_{L^2_{x,v}} \| \nabla_v^{\alpha_2} \nabla_v^2 f \|_{M_0} \| \nabla_v \nabla_x f \|_{M_0}.
\end{equation}

**Step III:** By taking \( L^2(M_0 \text{d}v \text{d}x) \) scalar product with \( \mathcal{L}_0^3 f \), we will get from the equation (6.11) the control for \( \varepsilon \| \nabla_v^2 f \|^2_{M_0} \) in a similar process as before. Here we omit the details and only list the estimates,

\[ \varepsilon \langle \langle \partial_t f, \mathcal{L}_0^3 f \rangle \rangle_{M_0} = \langle \langle \partial_t \nabla_v^2 f, \nabla_v^2 f \rangle \rangle_{M_0} \]

\[ = \frac{1}{2} \frac{d}{dt} \varepsilon \| \nabla_v^2 f \|^2_{M_0} - \frac{1}{2} \varepsilon \int_0^t \partial_t M_0 \| \nabla_v^2 f \|^2 \text{d}v \text{d}x, \]

\[ \varepsilon \langle \langle v \cdot \nabla_x f, \mathcal{L}_0^3 f \rangle \rangle_{M_0} = \varepsilon \langle \langle v \cdot \nabla_x \nabla_v^2 f, \nabla_v^2 f \rangle \rangle_{M_0} + 2 \varepsilon \langle \langle \nabla_v \nabla_x f, \nabla_v^2 f \rangle \rangle_{M_0} \]

\[ = \frac{1}{2} \varepsilon \int_0^t \int v \cdot \nabla_x \| \nabla_v^2 f \|^2 M_0 \text{d}v \text{d}x \]

\[ + \varepsilon \langle \langle \nabla_v \nabla_x f, \nabla_v^2 f \rangle \rangle_{M_0} \]

\[ \leq \varepsilon \| \nabla_x \Omega_0 \|^2_{L^\infty_x} \| \nabla_v^2 f \|^2_{M_0} + C \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} + \frac{1}{4} \| \nabla_v^2 f \|^2_{M_0}, \]

\[ \| \nabla_v \cdot (P_v \cdot D_0 f), \mathcal{L}_0^3 f \|_{M_0} \leq \varepsilon \| \nabla_v \cdot (P_v \cdot D_0 f), \mathcal{L}_0^3 f \|_{M_0} \]

\[ \lesssim \varepsilon \| D_0 \|_{L^\infty_x} \| \nabla_v^2 f \|^2_{M_0}, \text{ (Using Lemma 4.2)} \]

\[ \langle \langle \mathcal{L}_0 f, \mathcal{L}_0^3 f \rangle \rangle_{M_0} = \| \nabla_v^2 f \|^2_{M_0}, \]

\[ \langle \langle h_1, \mathcal{L}_0^3 f \rangle \rangle_{M_0} \leq \| \nabla_v^2 h_1 \|_{M_0} \| \nabla_v^2 f \|_{M_0} \leq C \| \nabla_v^2 h_1 \|^2_{M_0} + \frac{1}{4} \| \nabla_v^3 f \|^2_{M_0}, \]

\[ \varepsilon \langle \langle h_0 f, \mathcal{L}_0^3 f \rangle \rangle_{M_0} = \varepsilon \langle \langle \nabla_v \left( (h_0 f) \right), \nabla_v^2 f \rangle \rangle_{M_0} \]

\[ \lesssim \varepsilon \sum_{|\alpha_1|+|\alpha_2|=2} \| \nabla_v^{\alpha_1} h_0 \|_{L^2_{x,v}} \| \nabla_v^{\alpha_2} f \|_{M_0} \| \nabla_v^2 f \|_{M_0} \]

\[ \lesssim \varepsilon \sum_{0 \leq |\alpha_1| \leq 2} \| \nabla_v^{\alpha_1} h_0 \|_{L^2_{x,v}} \| \nabla_v^2 f \|^2_{M_0}. \]

Combining these above estimates implies that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \varepsilon^2 \| \nabla_v^2 f \|^2_{M_0} + \frac{1}{2} \| \nabla_v^3 f \|^2_{M_0} \\
\lesssim \| \nabla_v^2 h_1 \|^2_{M_0} + \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} + \sum_{0 \leq |\alpha_1| \leq 2} \| \nabla_v^{\alpha_1} h_0 \|_{L^2_{x,v}} \varepsilon \| \nabla_v^2 f \|^2_{M_0} \\
+ \left( \| \partial_t \Omega_0 \|^2_{L^\infty_x} + \| \nabla_x \Omega_0 \|^2_{L^\infty_x} + \eta_0 \| D_0 \|^2_{L^\infty_x} \right) \varepsilon^2 \| \nabla_v^2 f \|^2_{M_0},
\end{equation}

**Step IV:** Summing up the inequalities (6.14), (6.15) and (6.16), we get

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \varepsilon^3 \| \nabla_x^2 f \|^2_{M_0} + \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} + \varepsilon \| \nabla_v^2 f \|^2_{M_0} \right)
\end{equation}
+ \frac{1}{2} \left( \varepsilon^2 \| \nabla_v \nabla^2_v f \|_{M_0}^2 + \varepsilon \| \nabla^3_v f \|_{M_0}^2 \right) \\
 \lesssim \varepsilon^2 (\| \nabla^2 \Omega_0 \|_{L_\infty^2}^2 \| \nabla_v f \|_{M_0}^2 + \| \nabla_x \nabla v \|_{L_\infty^2}^2 \| \nabla_v \nabla x f \|_{M_0}^2) + \varepsilon \| \nabla_x \Omega_0 \|_{L_\infty^2}^2 \| \nabla^2_v f \|_{M_0}^2 \\
 + \left( \varepsilon^2 \| \nabla_x h_1 \|_{M_0}^2 + \varepsilon \| \nabla_v \nabla x h_1 \|_{M_0}^2 + \| \nabla^2 h_1 \|_{M_0}^2 \right) + \varepsilon \| \nabla^2 f \|_{M_0}^2 + \varepsilon \| \nabla_v \nabla x f \|_{M_0}^2 \\
 + \left( \| \partial_t \Omega_0 \|_{L_\infty^2} + \| \nabla_x \Omega_0 \|_{L_\infty^2} + \varepsilon \| D_0 \Omega_0 \|_{L_\infty^2}(\varepsilon^2 \| \nabla^2 f \|_{M_0}^2 + \varepsilon \| \nabla_v \nabla x f \|_{M_0}^2) + \varepsilon \| \nabla^2 f \|_{M_0}^2 \right) \\
 + \varepsilon \| \nabla_x D_0 \|_{L_\infty^2} \| \nabla_v \nabla x f \|_{M_0} + \varepsilon \| \nabla^2 f \|_{M_0} + \varepsilon \| \nabla_v \nabla x f \|_{M_0}^2 \\
 + \varepsilon^2 \| \nabla^2 \Omega_0 \|_{L_\infty^2}^2 \| \nabla_v f \|_{M_0}^2 + \varepsilon^2 \| \nabla^2 \Omega_0 \|_{L_\infty^2}^2 \| \nabla_v f \|_{M_0}^2 \\
 + \varepsilon^3 \sum_{|\beta_1| + |\beta_2| = 2} \left( \| \nabla^{\beta_1} h_0 \|_{L_\infty^2}^2 \| \nabla^{\beta_2} f \|_{M_0} + \| \nabla^2 f \|_{M_0} \right) \\
 + \varepsilon^2 \sum_{|\alpha_1| + |\alpha_2| = 1, |\beta_1| + |\beta_2| = 1} \left( \| \nabla^{\alpha_1} \nabla^{\beta_1} h_0 \|_{L_\infty^2}^2 \| \nabla^{\alpha_2} \nabla^{\beta_2} f \|_{M_0} + \| \nabla_v \nabla x f \|_{M_0} \right) \\
 + \varepsilon \sum_{0 \leq |\alpha_1| \leq 2} \left( \| \nabla^{\alpha_1} h_0 \|_{L_\infty^2} \| \nabla^2 f \|_{M_0} \right).

Denote 
\[ C_2 = \sup \left\{ C_0^2, \| \nabla^2 \Omega_0 \|_{L_\infty^2}^2, \; \eta_0 \| \nabla^2 D_0 \|_{L_\infty^2}^2, \sum_{|\alpha| + |\beta| \leq 2} \| \nabla^{\alpha} \nabla^{\beta} h_0 \|_{L_\infty^2}^2 \right\}, \]
then the above equation can be rewritten as
\[
\frac{1}{2} \frac{d}{dt} F_2 + \frac{1}{2} G_2 \leq \frac{CC_2(1 + \varepsilon + \varepsilon^{\frac{1}{2}} + \eta_0 \varepsilon)}{2} F_2 + CH_2 \\
+ CC_2(1 + \varepsilon^{\frac{1}{2}}) F_2 \frac{1}{2} G_2 + CC_2(1 + \varepsilon^{\frac{1}{2}} + \varepsilon) F_2 \frac{1}{2} F_0^2 + CC_2(1 + \varepsilon^{\frac{1}{2}}) F_2 \frac{1}{2} F_1 + CC_2(1 + \varepsilon^{\frac{1}{2}}) F_0 + CH_2 \\
\leq 2CC_2(F_2 + F_1 + F_0) + CH_2.
\]

6.4. Closing the Estimates. Noticing the definition of \( E = F_0 + F_1 + F_2 \), we collect the inequalities (6.8), (6.13), and (6.18) to get that
\[
\frac{d}{dt} E + G \leq \tilde{C}(E + H),
\]
holds for any \( \varepsilon \in (0, \varepsilon_0) \) and \( t \in [0, T] \). Obviously, the value of \( \tilde{C} \) depends upon the value of \( \sup_{t \in [0, T]} \{ C_0, C_1, C_2 \} \), which is depending eventually upon the value of \( \| \rho_0 \|_{L_\infty^2(0, T; H^m(\Omega))} \) and \( \| \Omega_0 \|_{L_\infty^2(0, T; H^m(\Omega))} \) (with \( m > 13 \)). This completes the whole proof of Lemma 6.1. \( \square \)

Remark 6.1. Supposed that the initial data \( (\rho_0^0, \Omega_0^0) \) are sufficiently smooth, then we can consider high order energy estimates for the remainder equation (6.1). Here we only give the result about the \( H^{N} \) energy estimates because the proof is similar as before. Specifically speaking,
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| + |\beta| = N} \varepsilon^{|\beta|+1} \left\| \nabla^\alpha \nabla^\beta f \right\|_{M_0}^2 + \frac{1}{2} \sum_{|\alpha| + |\beta| = N} \varepsilon^{|\beta|} \left\| \nabla^{\alpha+1} \nabla^\beta f \right\|_{M_0}^2 \\
\lesssim \sum_{|\alpha| + |\beta| = N} \sum_{\beta_1 + \beta_2 = \beta} \sum_{1 \leq |\beta| \leq N} C_{\beta_1} \left\| \nabla^\beta \nabla_0 \Omega_0 \right\|_{L_\infty^2}^2 \left\| \nabla^{\alpha+1} \nabla^\beta f \right\|_{M_0}^2 \varepsilon^{|\beta|}.
\]
Let it follow from the above inequality that

\[ F_N = \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta} \left\| \nabla_{\alpha} \nabla_{\beta} h_1 \right\|^2_{M_0} + \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta+2} \left\| \nabla_{\alpha-1} \nabla_{\beta+1} f \right\|^2_{M_0} \]

\[ + \left( \| \partial_t \Omega_0 \|_{L^\infty_x} + \| \nabla_x \Omega_0 \|_{L^\infty_x} + \eta_0 \| D_0 \|_{L^\infty_x} \right) \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta+1} \left\| \nabla_{\alpha} \nabla_{\beta} f \right\|^2_{M_0} \]

\[ + \eta_0 \sum_{|\alpha|+|\beta|=N\beta_1+\beta_2=\beta} \sum_{|\beta_1|\geq 1} \alpha C_{\alpha} \nabla_{\alpha} \nabla_{\beta} h_0 \left\| \nabla_{\alpha-1} \nabla_{\beta+1} f \right\|^2_{M_0} \]

Denote

\[ F_N = \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta} \left\| \nabla_{\alpha} \nabla_{\beta} f \right\|^2_{M_0}, \quad G_N = \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta} \left\| \nabla_{\alpha} \nabla_{\beta} f \right\|^2_{M_0}, \]

\[ H_N = \sum_{|\alpha|+|\beta|=N} \varepsilon^{\beta} \left\| \nabla_{\alpha} \nabla_{\beta} h_1 \right\|^2_{M_0}, \]

\[ C_N = \sup \left\{ \sum_{|\beta_1|\geq 1} \sum_{|\beta_1|=N} \sum_{|\beta_2|=\beta_1+\beta_2=\beta} \alpha C_{\alpha} \nabla_{\alpha} \nabla_{\beta} h_0 \left\| \nabla_{\alpha-1} \nabla_{\beta+1} f \right\|^2_{M_0}, \sum_{|\alpha_1|+|\alpha_2|=\alpha} \sum_{|\beta_1|=N} \sum_{|\beta_1|=N} \sum_{|\beta_2|=\beta_1+\beta_2=\beta} \nabla_{\alpha_1} \nabla_{\alpha_2} \nabla_{\beta_1} h_0 \left\| \nabla_{\alpha_1} \nabla_{\alpha_2} \nabla_{\beta_1} h_0 \right\|^2_{M_0} \right\}, \]

then it follows from the above inequality that

\[ (6.21) \quad \frac{1}{2} \frac{d}{dt} F_N + \frac{1}{2} G_N \leq C_N (N F_N + (N-1) \varepsilon F_{N-1} + \cdots + 2 \varepsilon^{N-2} F_2 + \varepsilon^{N-1} F_1) \]

\[ + H_N + F_N + C_N F_N \]

\[ \quad \quad + \eta_0 C_N F_N \frac{1}{\varepsilon} \left( N \varepsilon F_N + (N-1) \varepsilon F_{N-1} + \cdots + \varepsilon^{N-1} F_1 + \varepsilon^N F_0 \right) \]

\[ \leq C C_N (F_N + F_{N-1} + \cdots + F_1 + F_0) + C H_N. \]

**Remark 6.2.** In fact, recall the definition

\[ h_1 = - \frac{1}{M_0} [\partial_t f_1 + v \cdot \nabla_x f_1 + \eta_0 \nabla v \cdot (P_v D_0 f_1)] \]

\[ = - \frac{1}{M_0} [\partial_t f_1 + v \cdot \nabla_x f_1 - \eta_0 ((n-1) D_0 v f_1 - D_0 \nabla v f_1)], \]

then by simple calculations, we can infer from Lemma 7.4 that

\[ H_0(t) = \| h_1 \|^2_{M_0} \leq C (\| \partial_t f_1 \|_{M_0}, \| \nabla x f_1 \|_{M_0}, \| v \nabla f_1 \|_{M_0}) \]

\[ \leq C (\| \rho_0 \|_{L^\infty(0,T;H^m(\Omega))}, \| \Omega_0 \|_{L^\infty(0,T;H^m(\Omega))}), \]

\[ H_1(t) = \varepsilon \| \nabla x h_1 \|^2_{M_0} + \| v \nabla v h_1 \|^2_{M_0} \leq C (\| \rho_0 \|_{L^\infty(0,T;H^m(\Omega))}, \| \Omega_0 \|_{L^\infty(0,T;H^m(\Omega))}), \]

\[ H_2(t) = \varepsilon^2 \| \nabla x h_1 \|^2_{M_0} + \varepsilon \| v \nabla x h_1 \|^2_{M_0} + \| \nabla v h_1 \|^2_{M_0} \]

\[ \leq C (\| \rho_0 \|_{L^\infty(0,T;H^m(\Omega))}, \| \Omega_0 \|_{L^\infty(0,T;H^m(\Omega))}), \]

for any \( t \in [0, T] \) and \( m > 13 \).
As a consequence, by choosing a new constant \( \hat{C} \defeq \sup_{t \in [0,T]} \{ C_0, C_1, C_2, \mathcal{H} \} \), we can get a refined formulation of (6.6),

\[
\frac{d}{dt} \mathcal{E} + \mathcal{G} \leq \hat{C}(\mathcal{E} + 1),
\]

which can lead to the same result. Here we keep the original formulation with the term \( \mathcal{H} \) in order to make the contributions of \( f_1 \) more clear.

7. Completion of the Proof of the Main Theorem

7.1. Local Existence of the Remainder Equation. Based on the a priori estimates (Lemma 6.1), we can get the local existence for the remainder equation (6.1) by a standard iteration scheme.

Lemma 7.1. Given the initial datum \( f(0, x, v) \in H^2(\mathbb{T} \times \mathbb{S}^{n-1}) \), then there exist \( T_* > 0 \) such that the remainder equation (6.1) admits a unique solution \( f(t, x, v) \) on \( C([0, T_*]; H^2(\mathbb{T} \times \mathbb{S}^{n-1})) \), and moreover, there exists a constant \( E > 0 \),

\[
\sup_{t \in [0, T_*]} \mathcal{E}(f(t)) \leq E,
\]

provided that the initial datum satisfies \( \mathcal{E}(f(0)) \leq \frac{E}{2} \).

Observing that the equation (6.1) is linear with respect to \( f \), proving the lemma can be established by a standard iteration scheme and a straightforward compactness justification, hence we will only sketch the proof. Here we consider the following iteration scheme:

\[
(P_n) \quad \begin{cases}
\partial_t (\varepsilon f^{n+1}) + v \cdot \nabla_x (\varepsilon f^{n+1}) + \eta_0 \nabla_v \cdot (P_{\varepsilon} D_0(\varepsilon f^{n+1})) + \mathcal{L}_0 f^{n+1} = h_1 + \varepsilon f^nh_0, \\
 f^{n+1}(0, x, v) = f(0, x, v).
\end{cases}
\]

starting with \( f^0(0, x, v) = f(0, x, v) \).

We remark that under the hypothesis of Theorem 3.2 there exists \( \varepsilon_0 > 0 \) such that,

\[
\mathcal{E}(f^{n+1}(0)) = \mathcal{F}_0^{n+1}(0) + \mathcal{F}_1^{n+1}(0) + \mathcal{F}_2^{n+1}(0) \leq \frac{E}{2}
\]

holds for the above \( E > 0 \) and all \( \varepsilon \in (0, \varepsilon_0) \).

To complete the proof of Lemma 7.1 it suffices to get a uniform (in \( n \)) estimate for \( \mathcal{E}^{n+1}(t) \).

Lemma 7.2. There exists \( T_* > 0 \), such that if \( \sup_{t \in [0, T_*]} \mathcal{E}(f^n(t)) \leq E, \) then

\[
\sup_{t \in [0, T_*]} \mathcal{E}(f^{n+1}(t)) \leq E.
\]

Proof. Following exactly the same lines as the proof of the a priori estimates, we can get finally

\[
\begin{align*}
\frac{d}{dt} \mathcal{F}_0^{n+1} + \mathcal{G}_0^{n+1} & \lesssim C_0(\mathcal{F}_0^{n+1} + \mathcal{F}_0^n) + \mathcal{H}_0, \\
\frac{d}{dt} \mathcal{F}_1^{n+1} + \mathcal{G}_1^{n+1} & \lesssim C_1(\mathcal{F}_1^{n+1} + \mathcal{F}_1^n + \mathcal{F}_0^n) + \mathcal{H}_1, \\
\frac{d}{dt} \mathcal{F}_2^{n+1} + \mathcal{G}_2^{n+1} & \lesssim C_2(\mathcal{F}_2^{n+1} + \mathcal{F}_2^n + \mathcal{F}_1^n + \mathcal{F}_0^n) + \mathcal{H}_2,
\end{align*}
\]

hence a constant \( \tilde{C} > 0 \) depending on \( \sup_{t \in [0, T_*]} \{ C_0, C_1, C_2 \} \) exists such that,

\[
\frac{d}{dt} \mathcal{E}(f^{n+1}) + \mathcal{G}(f^{n+1}) \leq \tilde{C}(\mathcal{E}(f^{n+1}) + \mathcal{E}(f^n) + \mathcal{H}).
\]
Integrating from 0 to \( t \) with respect to the time variable, we have

\[
\mathcal{E}(f^{n+1}(t)) + \int_0^t \mathcal{G}(f^{n+1}(s)) \, ds \\
\leq \mathcal{E}(f^{n+1}(0)) + C \epsilon \left( \sup_{s \in [0,t]} \mathcal{E}(f^{n+1}(s)) + \sup_{s \in [0,t]} \mathcal{H}(s) \right).
\]

Take sufficiently small \( T_* \) such that,

\[
\mathcal{C}T_* \leq \frac{1}{6}, \quad \mathcal{C}T_\ast \sup_{s \in [0,t]} \mathcal{H}(s) \leq \frac{1}{6} \mathcal{E},
\]

which together with the assumptions \( \sup_{t \in [0,T_\ast]} \mathcal{E}(f^n(t)) \leq \mathcal{E} \) and \( \mathcal{E}(f^{n+1}(0)) \leq \frac{1}{2} \mathcal{E} \) enables us to get the desired result

\[
\sup_{t \in [0,T_\ast]} \mathcal{E}(f^{n+1}(t)) \leq \mathcal{E}.
\]

Then we are left to prove the convergence. Set \( w^n = f^{n+1} - f^n \), for which the iteration scheme \( (P_n) \) leads to

\[
\begin{cases}
\partial_t (\epsilon w^n) + \nabla_x (\epsilon w^n) + \eta \nabla_v \cdot (P_v \cdot D_0(\epsilon w^n)) + \mathcal{L}_0 w^n = \epsilon w^{n-1} h_0, \\
w^n(0,x,v) = 0.
\end{cases}
\]

The same computations as used for \( (P_n) \) give that

\[
\mathcal{E}(w^n(t)) + \int_0^t \mathcal{G}(w^n(s)) \, ds \leq \mathcal{C}T_* \left( \sup_{s \in [0,T_\ast]} \mathcal{E}(w^n(s)) + \sup_{s \in [0,T_\ast]} \mathcal{E}(w^{n-1}(s)) \right).
\]

Take \( T_* \) small enough so that \( \mathcal{C}T_* \leq \frac{1}{7} \), then we obtain

\[
\sup_{t \in [0,T_\ast]} \mathcal{E}(f^{n+1}(t) - f^n(t)) \leq \frac{1}{2} \sup_{t \in [0,T_\ast]} \mathcal{E}(f^n(t) - f^{n-1}(t)).
\]

Consequently, it can be proved easily the sequence \( \{f^n\} \) is convergent in the space \( C(0,T_*; H^2(T \times \mathbb{S}^{n-1})) \) and so is its limit \( f \). This completes the proof of Lemma 7.1.

### 7.2. Completion for the Main Theorem

We are now in a position to complete the proof of our main Theorem 3.2. Lemma 7.1 shows that there exists a maximal time \( T_\epsilon > 0 \) such that the solution \( f_\epsilon^2 \) to the remainder equation (6.1) satisfies the a priori estimate stated in Lemma 6.1.

\[
\frac{d}{dt} \mathcal{E} + \mathcal{G} \leq \mathcal{C}(\mathcal{E} + \mathcal{H}),
\]

for any \( \epsilon \in (0,\epsilon_0) \) and \( t \in [0,T_\epsilon] \).

Observing the hypothesis of Theorem 3.2 \( \| f_\epsilon^2 \|_{H^2_{\mathbb{S}^{n-1}}} (0) \leq C \) implies that \( \mathcal{E}(0) \leq \mathcal{E} \). Define

\[
E_2 = e^{\mathcal{C}T_0} (E(0) + \mathcal{C}T_\ast \sup_{t \in [0,T_\ast]} \mathcal{H}(t)),
\]

\[
T_0 = \sup\{ t \in [0,T_\epsilon] \mid \mathcal{E}(t) \leq E_2 \}.
\]

We claim that \( T_\epsilon \geq T_0 \). Indeed, applying the Grönwall inequality, we infer from the above a priori estimate, for \( t \in [0,T_\epsilon] \),

\[
\mathcal{E}(t) \leq e^{\mathcal{C}T_0} (E(0) + \mathcal{C}t \sup_{s \in [0,t]} \mathcal{H}(s)).
\]

If \( T_\epsilon < T_0 \), then it follows that

\[
\mathcal{E}(t) \leq E_2,
\]
which in turn yields that $T_\varepsilon = T_0$, and furthermore, the a priori estimate are adapted to the time $t = T_\varepsilon$ so that the solution may be continued beyond $T_\varepsilon$. This stands in contradiction to the maximal property of $T_\varepsilon$. Thus $T_\varepsilon \geq T_0$, and $E(t) \leq E_2$ holds for $t \in [0, T]$.

Recalling the definition of $E$

\[ E = E_0 + E_1 + E_2 \]

\[ = \varepsilon \| f \|^2_{M_0} + (\varepsilon^2 \| \nabla_x f \|^2_{M_0} + \varepsilon \| \nabla_v f \|^2_{M_0}) + (\varepsilon^3 \| \nabla^2_x f \|^2_{M_0} + \varepsilon^2 \| \nabla_v \nabla_x f \|^2_{M_0} + \varepsilon \| \nabla^2_v f \|^2_{M_0}) \]

and the equivalence between the two norms $\| \cdot \|_{M_0}$ and $\| \cdot \|$, we actually have proved the result $E(t)$ satisfied for $t \in [0, T]$. 

(7.6)

\[ \varepsilon^{\frac{1}{2}} \| f_2 \|_{L_2, \nu} + \varepsilon \| f_2^2 \|_{H_2, \nu} + \varepsilon^{\frac{3}{2}} \| f_2^3 \|_{H_3, \nu} \leq C, \]

with $C$ is independent of $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T]$.

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