BASIC REPRESENTATIONS OF QUANTUM CURRENT ALGEBRAS IN HIGHER GENUS

B. ENRIQUEZ, S. PAKULIAK, AND V. RUBTSOV

To the memory of Joseph Donin

Abstract. We construct level 1 basic representations of the quantized current algebras associated to higher genus algebraic curves using one free field. We also clarify the relation between the elliptic current algebras of the papers [EF] and [JKOS].

1. Introduction

The recent decade has been a period of interest to different quantizations of current algebras associated with algebraic curves. Special attention was paid to elliptic quantum current algebras. The first paper where their main ingredients were introduced is probably [L95]. The screening currents of a scaling elliptic algebra appear there for the first time. The algebra of screening currents was later studied in [KLP99, LeKP].

The first relation between elliptic algebras and quasi-Hopf algebras ([D90]) appeared in the papers [HGQG, F97]. In the first paper, elliptic algebras are expressed using total currents, in the framework of Drinfeld’s comultiplication. It became later clear that applications to integrable models require the introduction of half-currents and Gauss coordinates of $L$-operators. The first paper which used the formalism of $L$-operators in the context of elliptic current algebras was [FIJKMY], but the quasi-Hopf nature of these algebras was not observed there.

The paper [EF] used the quasi-Hopf constructions of [HGQG, ER99] and introduced a decomposition of total currents into half-currents, which were identified with the Gauss coordinates of $L$-operators (in analogy with [DF]), and the resulting current algebra was written in terms of $L$-operators and identified with the central extension of Felder’s elliptic quantum group $E_{\tau,\eta}(\widehat{sl}_2)$.

Precise relations between some elliptic current algebras in the quasi-Hopf framework (starting directly from the $L$-operator formulation) were obtained in the papers [ABRR97, JKOS97]. The corresponding algebras also satisfy the dynamical Yang-Baxter equations; in this case, the dynamical shift not only affects the dynamical variable by an element of the Cartan algebra like in the usual $L$-operator formalism, but it also changes the elliptic modulus by a central element. Such deformations, leading to shifts in the moduli of curves, are known only in the case of genus one (elliptic curves).

This paper is a new contribution to the study of quantum current algebras associated with algebraic curves. Namely, we construct level 1 basic representations for these algebras using one free field. This is the analogue of the Frenkel-Jing construction [FJ] (see Frenkel-Kac [FK] in the classical case). Recall that zero-level representations of these algebras were studied in [HGQG]. The next step of our program is the construction of a bundle of level 1 vertex operators, a natural connection on this bundle and an analogue of the qKZB discrete connection. We hope to return to these questions elsewhere.

Date: June 4, 2018.
The paper is organized as follows. In Section 2, we recall the general construction of the papers [HGQG, ER99] of quantum current algebras on algebraic curves. We recall the main ingredients of this construction: the Green function and the function \( q(z, w) \), and we give some examples is low genus. The next section is devoted to the main result of the paper: the construction of level one basic representations of the quantum current algebra associated with \( \mathfrak{sl}_1 \) and arbitrary algebraic curves. This construction resembles the approach developed in the papers [Kh, KLP96]. The last section contains a comparison of the total currents decomposition into half-currents in the elliptic quantum group \( E_{\tau, \eta}(\mathfrak{sl}_2) \) and in the algebra \( U_{q,p}(\mathfrak{sl}_2) \) of the paper [JKOS]. (As Jimbo explained to us, the decomposition in [EF] was a starting point for the paper [JKOS], in which the algebras have the advantage of being graded.) We present some arguments that these algebras are elliptic generalizations of two presentations of the centrally extended Yangian double given in the paper [KLP99]. In the conclusion, we discuss several open problems.

2. Green kernels and the function \( q(z, w) \). Description and examples.

2.1. Notations and conventions. Let \( \Sigma \) be a smooth connected complex compact curve and \( \omega \) be a nonzero meromorphic differential on \( \Sigma \). Let \( S_\omega \subset \Sigma \) denote the set of poles and zeros of \( \omega \). If \( K_s \) denotes the local field at the point \( s \in S_\omega \) and \( K := \bigoplus_{s \in S_\omega} K_s \), then the ring \( R \) of functions regular outside \( S_\omega \) can be viewed as a subring in \( K \). We also choose a subspace \( \Lambda \) in \( K \) which is a Lagrangian complement to \( R \) with respect to the bilinear form \( (f, g)_K = \sum_{s \in S_\omega} \text{res}_s(f g \omega) \), commensurable to \( \bigoplus_{s \in S_\omega} O_f \) (\( O_f \) is the local ring at \( s \)). Finally, we define a derivation \( \partial \) of \( K \) by \( \partial f = \frac{df}{f} \).

We will adopt here the following notations from our earlier works: \( \hbar \) is a formal variable, \( q = e^{\hbar} \). The operator \( \exp(\hbar \partial) \) is an automorphism of \( K[[\hbar]] \) preserving \( R[[\hbar]] \). We will choose a local coordinate \( z_s \) on \( \Sigma \) at \( s \in S_\omega \) and we will set \( V((z)) = \prod_{s \in S_\omega} V[[z_s]][z_s^{-1}] \) for any vector space \( V \); this is a completion in \( V \otimes K \). The ring \( \mathbb{C}[[F, \frac{dF}{F}]] = \prod_{(\infty, \infty) \in S_\omega \times S_\omega} \mathbb{C}[[F^-_{\infty}, F^+_{\infty}]] \) is a completion of \( K \otimes K \).

We will denote by \( f \to f^{(21)} \) the permutation of factors in \( \mathbb{C}[[F, \frac{dF}{F}]] [[F^{-\rho}, F^{\rho}] \). If \( f = \sum_k f_k^{+} f_k^{-} \in R \otimes R \), we set \( f(z, w) = \sum_k f_k^{+}(z) f_k^{-}(w) \), a complex function in \( (\Sigma \setminus S_\omega) \times (\Sigma \setminus S_\omega) \). In this case \( f^{(21)}(z, w) = f(w, z) \). For any \( f \in K \) we will always denote by \( (f)R \) (resp., \( (f)_{\Lambda} \)) the projections of \( f \) on \( R \) parallel to \( \Lambda \) (resp., on \( \Lambda \) parallel to \( R \)). Finally, we will choose \( r_i, i \geq 0 \) and \( \lambda_i, i \geq 0 \) two dual bases of \( R \) and \( \Lambda \). The following notation for the group commutator will be used below:

\[
(a, b) := aba^{-1}b^{-1}
\]

2.2. Green kernels. Let us define the Green kernel by \( G(z, w) = \sum_{i \geq 0} r_i(z) \lambda_i(w) \in \mathbb{C}((F))((\frac{1}{z})) \).

We summarize its main properties:

**Theorem 2.1.** (see [ER99])

\[
((\partial \otimes \text{id})G)(z, w) = -G(z, w)^2 + \gamma,
\]

where \( \gamma \in R^\otimes 2 \).

Let \( \phi, \psi \) be the elements of \( hR^\otimes 2[[\hbar]] \) satisfying the differential equations

\[
\partial_{\hbar} \phi = (\partial \otimes \text{id}) \psi - 1 - \gamma \psi^2, \quad \partial_{\hbar} \psi = (\partial \otimes \text{id}) \phi - \gamma \psi.
\]

Then \( \psi = -\hbar + o(\hbar), \ \phi = (1/2)\hbar^2 \gamma + o(\hbar^2) \). Then the following identity takes place:

\[
\sum_{i \in N} \frac{e^{\hbar \partial} - 1}{\partial} \lambda_i(z) r_i(w) = -(\phi(\hbar) + \log(1 - G^{(21)}(\psi(\hbar))))(z, w).
\]
The element $\sum_{i \geq 0} r^i \otimes (\frac{e^{\hbar \partial} - e^{-\hbar \partial}}{\partial}) \lambda_i) \in R^i \otimes \mathbb{C}$ belongs to $S^2(R[[\hbar]])$. Let us fix an element $\tau$ in $R^{\otimes 2}[[\hbar]]$ such that
\begin{equation}
\hbar (\tau + \tau^{(21)}) + \sum_{i \geq 0} r^i \otimes \left(\frac{e^{\hbar \partial} - e^{-\hbar \partial}}{\partial} \lambda_i\right) = 0. \tag{4}
\end{equation}

We define $\delta(z, w) = \sum_i e^i(z) e_i(w)$, where $(e^i)$ and $(e_i)$ are dual bases for $K$ for $(\cdot, \cdot)_K$. Then $\delta(z, w) = (G + G^{(21)})(z, w)$. We have $(\partial \otimes \text{id} + \text{id} \otimes \partial)(\delta(z, w)) = 0$. One can view the Green kernel as the collection of expansions in $w$ at the vicinity of $s \in S_\omega$ of a rational function on $\Sigma \times \Sigma$ antisymmetric in $z$ and $w$, regular except the poles where $z(w) = s$ for some $s \in S_\omega$ and with a simple pole on diagonal.

2.3. The function $q(z, w)$. Let us define a function $q(z, w) \in \mathbb{C}((w))((z))[[\hbar]]$ by the formula:
\begin{equation}
q(z, w) = \exp \left(\sum_{i} \left(\frac{e^{\hbar \partial} - e^{-\hbar \partial}}{\partial} \lambda_i\right) \otimes r^i\right) \exp(\hbar \tau)(z, w). \tag{5}
\end{equation}

Let us denote by $\mathbb{C}[(\Sigma \setminus S_\omega)^2 \setminus \text{diag}]$ the ring of regular functions on $(\Sigma \setminus S_\omega)^2 \setminus \text{diagonal}$. Theorem 2.1 implies that $q(z, w)$ is the expansion, for $z \in S_\omega$, of an element $\tilde{q}(z, w) \in \mathbb{C}[[\Sigma \setminus S_\omega]^2 \setminus \text{diag}][[\hbar]]$, of the form $1 + O(\hbar)$ and such that $\tilde{q}(z, w)q(w, z) = 1$.

We will also use another property of $q(z, w)$. Assume that $\text{card}(S_\omega) = 1$, then there exists a series $i(z, w) \in \mathbb{C}[[z, w]][z^{-1}][w^{-1}][[\hbar]]$ of the form $1 + O(\hbar)$, such that $q(z, w) = i(z, w)(z - q^0 w)/(q^0 z - w);$ we have $i(z, w) i(w, z) = 1$.

2.4. Examples of the function $q$.

2.4.1. $g = 0, \omega = dz$. Assume $\Sigma = \mathbb{C}P^1, \omega = dz, S_\omega = \{\infty\}$. The local coordinate is $z_\infty = z^{-1}$. We have $R = \mathbb{C}[[t]]$ and take as a complement $\Lambda = z^{-1} \mathbb{C}[[z^{-1}]]$. Then the Green kernel $G(z, w)$ is given by the expansion of $\frac{1}{w-z}$ for $z \ll w$ and the function $q(z, w, \hbar) = \frac{1}{w-z} \exp(\hbar \tau)$ for $w \ll z$.

Remark 1. This example is a particular $(N = 1)$ case of the following more general construction (see [ER-MS]): assume $\Sigma = \mathbb{C}P^1, \omega = z^{N-1} dz, S_\omega = \{\infty\}$.

If $N$ is odd and we take $N = 2n + 1$ then $R = z^{-n-1} \mathbb{C}[z^{-1}]$ and $\Lambda = z^{-n} \mathbb{C}[[z]]$. The dual bases are $r^i = z^{-n-i-1}$ and $\lambda_i = z^i$ for $i \geq 0$. Then the Green kernel $G = \sum_{i \geq 0} z^{-n-i-1} \otimes z^i$ expands at $w = 0$.

2.4.2. $g = 0, \omega = dz$. Assume $\Sigma = \mathbb{C}P^1, \omega = dz, S_\omega = \{0, \infty\}$. We pose $\Lambda = \{(\lambda_0; \lambda_\infty) \in \mathbb{C}[[z]] \times \mathbb{C}[[z^{-1}]] | \lambda_0(0) + \lambda_\infty(\infty) = 0\}$. Dual bases for $R$ and for $\Lambda$ are $\{r^i = z^i, i \in \mathbb{Z}\}$ and $\lambda_i = (z^{-i}; 0)$ for $i < 0$, $\lambda_i = (0; -z^{-i})$ for $i > 0$ and $\lambda_0 = (1; -1)$. We can compute easily
\begin{equation}
\text{exp} \left(\sum_i \left(\frac{e^{\hbar \partial} - e^{-\hbar \partial}}{\partial} \right) \lambda_i \otimes r^i\right)(z, w) = \frac{e^{\hbar z - w} - e^{\hbar w - z}}{w - z}. \tag{6}
\end{equation}

2.4.3. $g = 1, \omega = dz$. Assume $\Sigma = \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau \mathbb{Z}$ for a fixed complex number $\tau, \Im \tau > 0$. We choose a coordinate $z$ on $\Sigma$ and $\omega = dz$. In our case $K$ is the completed local field $\mathbb{C}((z))$ at $z = 0$ and $R = \mathbb{C}[[z]]$. It is clear that $R$ is a maximal subring in $K$ isotropic with respect to $(f, g)_K = \text{res}_{z=0}(fgdz)$.

Let $\theta$ be the theta-function, defined to be holomorphic in $\mathbb{C}$ with only zeros at the points of the lattice $\Gamma$, such that $^1 \theta(z + 1) = -\theta(z), \theta(z + \tau) = -e^{i\pi(2z+\tau)} \theta(z)$ and $\theta'(0) = 1$. Define a complementary subspace $\Lambda(\lambda), \lambda \in \mathbb{C}$ to $\mathcal{R}$:
\begin{equation}
\Lambda(\lambda) = \oplus_{k \geq 0} \mathbb{C} \cdot \partial^k \zeta(z) e^{2i\pi R(\lambda) z}, \lambda \in \Gamma, \tag{6}
\end{equation}

1 We denote $i = \sqrt{-1}$, leaving $i$ for indices.
Here we set
\[ \zeta(z) = \frac{d}{dz}(\log(\theta(z))) = \frac{\theta'}{\theta}(z). \]

Remark 2. \( \Lambda(0) \) is a maximal isotropic subspace in \( \mathcal{K} \), consisting of all \( \Gamma \)-periodic functions, holomorphic on \( \mathbb{C}\backslash \Gamma \) and such that the \( \oint_{\gamma_\epsilon} f(z)dz = 0 \), where \( \epsilon > 0 \) and the closed cycle \( \gamma_\epsilon = (i\epsilon, 1 + i\epsilon) \).

The Green kernel \( G(z, w) \) in this case is \( \zeta(z - w) - \zeta(z) + \zeta(w) \) and \( q(z, w, h) = \frac{\theta(z-w+h)}{\theta(z-w-h)} \).

3. Quantum current algebra associated with an algebraic curve

We define \( U_R(\mathfrak{g}) \) as the topologically free \( \mathbb{C}[[\hbar]] \)-algebra with generators \( K, h^+[r], h^-[\lambda], x^\pm[\epsilon] \), where \( r \in R, \lambda \in \Lambda, \epsilon \in \mathcal{K} \) and relations: each map \( \varphi \mapsto a[\varphi] \) is linear, \( K \) is central, and if we set \( h^+(z) = \sum_i h^+[r_i] \lambda_i(z), h^-(z) = \sum_i h^-[\lambda_i] r_i(z), \)
\[ K^+(z) = e^{h(T + U)h^+}(z), \quad K^-(z) = e^{hUz}(z), \quad x^\pm(z) = \sum_i x^i \epsilon_i(z), \quad (x = e, f), \]
where \( U : \Lambda \rightarrow R \) is defined by \( U(\lambda) = \langle \tau, \text{id} \otimes \lambda \rangle \), then relations are
\[ (e(z)e(w) - q(z,w)e(w)e(z))\alpha(z,w) = 0, \]
\[ (f(z)f(w) - q(w,z)f(w)f(z))\alpha(w,z) = 0, \]
\[ K^+(z)e(w)K^+(z)^{-1} = q(z,w)e(w), \quad K^-(z)e(w)K^-(z)^{-1} = q(w, e^{-hK\partial} z)e(w), \]
\[ K^+(z)f(w)K^+(z)^{-1} = q(z,w)^{-1}f(w), \quad K^-(z)f(w)K^-(z)^{-1} = q(w,z)^{-1}f(w) \]
\[ (K^+(z), K^+(w)) = 1 \]
\[ (K^+(z), K^-(w)) = q(z,w)q(z, e^{-hK\partial} w)^{-1}, \]
\[ (K^-(z), K^-(w)) = q(e^{-hK\partial} z, e^{-hK\partial} w)q(z, w)^{-1}, \]
\[ [e(z), f(w)] = \frac{1}{\hbar} (\delta(z,w)K^+(z) - \delta(z, e^{-hK\partial} w)K^-(w)^{-1}). \]

Here \( q(z,w) \) is as above, and \( \alpha(z,w) \) runs over all elements of \( \mathbb{C}[[z,w]][z^{-1}, w^{-1}][[\hbar]] \) such that \( \alpha(e^{-h\partial} w, w) = 0 \).

For the function
\[ q(z,w) = \frac{z-w+\hbar}{z-w-\hbar} \]
this algebra was considered in [Kh] and it coincides with a centrally extended Yangian double associated to \( \mathfrak{sl}_2 \). For the function
\[ q(z,w) = \frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)} \]
this algebra was considered in [EF] and after proper factorization of the currents \( e(z) \) and \( f(z) \) was identified in this paper with the elliptic quantum group constructed in [F].
4. Basic representations of $U_h(\mathfrak{g})$

In this section, we assume that $\text{card}(S_\lambda) = 1$. Let us fix a local coordinate $z$ on $\Sigma$, so that $\mathcal{K} = \mathbb{C}((\ell))$. We define $a_0 = d \log z/\omega(z) \in \mathcal{K}$. We define $\widetilde{\mathcal{K}} := \mathcal{K} \oplus \mathbb{C} \log \ell$. We then define $\partial : \widetilde{\mathcal{K}} \to \mathcal{K}$ as the extension of $\partial$, taking $\log z$ to $a_0$. We also define $(\partial^{-1}) : \mathcal{K} \to \widetilde{\mathcal{K}}$ as the unique linear map, taking $f \in \mathcal{K}$ such that $(f, 1) = 0$ to the element $g \in \mathcal{K}$ such that $\partial(g) = f$ and $(g, a_0) = 0$, and taking $a_0$ to $\log z$. Then $\partial \circ (\partial^{-1}) = \text{id}_\mathcal{K}$ and $(\text{id} - (\partial^{-1}) \circ \partial)(f) = (f, a_0)1$ for $f \in \mathcal{K}$.

Let us define a Heisenberg algebra $H$ as the topologically free $\mathbb{C}[[\hbar]]$-algebra with generators $A[\epsilon], c (\epsilon \in \mathcal{K})$ and relations: $\epsilon \mapsto A[\epsilon]$ is linear,

$$[A[r], A[r']] = 0, \quad [A[r], A[\lambda]] = \hbar^{-1}((1 - e^{-\hbar\partial})(r), \lambda),$$

$$\quad [A[\lambda], A[\lambda']] = \hbar^{-1} \left( (e^{-\hbar\partial} \otimes e^{-\hbar\partial} - 1) \left( \sum_i (T + U)(\lambda_i) \otimes r^i \right), \lambda \otimes \lambda' \right);$$

$$[c, A[r]] = e^{-\hbar\partial} - \frac{1}{\hbar\partial}(r, \alpha_0), \quad [c, A[\lambda]] = (e^{-\hbar\partial} - \frac{1}{\hbar\partial}) (T + U)((e^{\hbar\partial})\lambda_i), \lambda_0 \right) (r, r' \in R, \lambda, \lambda' \in \Lambda)$; here $T = (e^{\hbar\partial} - e^{-\hbar\partial})/\hbar\partial).$ Note that $(e^{-\hbar\partial} \otimes e^{-\hbar\partial} - 1)(\sum_i (T + U)(\lambda_i) \otimes r^i) \in \wedge^2(R)[[\hbar]].$

The subalgebra of $H$ generated by the $A[\epsilon]$ identifies with the Cartan currents subalgebra of $U_h(\mathfrak{g})$ generated by the $h^+ [r], h^- [\lambda]$, where $K = 1$.

Let us set $A_R(z) = \sum_i A[r^i](T + U)(\lambda_i)(z), \ A_\Lambda(z) = \sum_i A[\lambda_i]r^i(z).$

We then set

$$a_\Lambda(z) := \frac{\hbar\partial}{1 - q^\partial}(\partial^{-1})(A_\Lambda(z)), \quad a_R(z) := \frac{\hbar\partial}{q^\partial - 1}(\partial^{-1})(A_R(z)).$$

We set

$$a(z) := q^\partial a_R(z) + c + a_\Lambda(z), \quad b(z) := a_R(z) + c + a_\Lambda(z).$$

Note that $a(z) - b(z) = \hbar A_R(z), \ a(z) - b(q^\partial z) = -\hbar A_\Lambda(q^\partial z).$

If $f(z) = \bar{f} \log(z) + \sum_{n \in \mathbb{Z}} f_i z^i$, we set $\bar{f}_+(z) = \sum_{i \geq 0} f_i z^i$ and $\bar{f}_-(z) = \bar{f} \log z + \sum_{i < 0} f_i z^i$.

We then split $a(z) = a_+(z) + a_-(z), \ b(z) = b_+(z) + b_-(z)$.

We define special functions $v(z), v'(z), u_R(z)$ and $u_A(z)$ as follows.

We first set $j(z, w) := \frac{z - q^\partial w}{q^{-\partial}z - w}$. We have $j(z, w) \in \mathbb{C}[[w]][z^{-1}, w^{-1}][[\hbar]], \ j = 1 + O(\hbar)$.

We then set $\phi(z, w) := \left( \frac{1}{2} \log(j^{21})_{++} + \frac{1}{2} \log(j^{21})_{--} \right)(z, w), \ then \ \phi(z, w) \in h\mathbb{C}[[w]][z^{-1}, w^{-1}][[\hbar]]$.

There is a unique map $R^{\otimes 2} \to K, \ a(z)b(w) \mapsto a_-(z)b_+(z)$, which we denote by $f(z, w) \mapsto f(z, w)_{--}$. We then set $\alpha(z) = \phi(z, z)$ and $\beta(w) = \phi(q^{-\partial}w, w) + \left( (q^{-\partial} \otimes q^{-\partial} - 1) \log q(z, w) \right)_{++}$.

Recall that $hK[[\hbar]] = hR[[\hbar]] \oplus (T + U)(h\Lambda[[\hbar]])$. If $\rho \in hK[[\hbar]]$, we denote by $\rho = \rho_R + \rho_\Lambda$ the corresponding decomposition.

We then set

$$u_R(z) := e^{\alpha_R(z)}, \quad u_\Lambda(z) := e^{-\beta_\Lambda(w)}.$$
These formulas induce a functor \( \{ \text{topological } H\text{-modules such that } A[1] \text{ is diagonalizable with eigenvalues in } \mathbb{Z} \} \rightarrow \{ \text{level } 1 \text{ modules over } U_{\mathfrak{h}}(\mathfrak{g}) \} \). Here topological means that for any \( v \in V \), \( \langle A_R(z), z^n \rangle v \) tends h-adically to 0 when \( n \rightarrow +\infty \).

**Proof.** The images of \( K^\pm(z) \) have the same functional properties as \( K^\pm(z) \), hence the morphism is well-defined.

The relations between \( A_R(z), A_R(w) \) are such that the relations between Cartan currents are preserved. One checks that
\[
\hbar[A_R(z), a(w)] = \log q(z, w) + (1 + q^{-\partial z}) \log \frac{w - z}{z - w},
\]
\[
\hbar[A_R(z), b(w)] = \log q(z, w) - (1 + q^{\partial z}) \log \frac{w - z}{z - w},
\]
(recall that \( \log(q, w) = \sum h(T + U)(\lambda_i)(z^i(w)) \), therefore the relations between the Cartan currents and the currents \( e(w), f(w) \) are preserved.

These relations imply the relations
\[
[a(z), a(w)] = \log q(z, w) + (1 + q^{-\partial z}) \log \frac{w - z}{z - w}, \tag{18}
\]
\[
[b(z), b(w)] = -\log q(z, w) + (1 + q^{\partial z}) \log \frac{w - z}{z - w}, \tag{19}
\]
\[
[a(z), b(w)] = (1 + q^{\partial z}) \log \frac{w - z}{z - w}. \tag{20}
\]

Here \( \log(w - z)/(z - w) = \log(w) - \log(z) - \sum_{n\neq 0}(z/w)^n/n \).

Let us prove (18). We have
\[
\partial_z[a(z), a(w)] = \frac{\hbar\partial_z}{1 - q^{-\partial z}}[A_R(z) + A_A(z), a(w)] = \frac{\partial_z}{1 - q^{-\partial z}}(\log q(z, w) + \log q(w, q^{-\partial z}))
\]
\[
= \frac{\partial_z}{1 - q^{-\partial z}}(\log q(z, w) - \log q^{-\partial z}, w) + \frac{\partial_z}{1 - q^{-\partial z}}(\log q(q^{-\partial z}, w) + \log q(w, q^{-\partial z})).
\]

Now
\[
\log q(q^{-\partial z}, w) + \log q(w, q^{-\partial z}) = \log \frac{q^{-\partial z} - q^{\partial z}}{z - w} + \log \frac{w - z}{q^{\partial z} - q^{-\partial z}} = (1 - q^{-2\partial z}) \log \frac{w - z}{z - w},
\]
so
\[
\partial_z[a(z), a(w)] = \partial_z \left( \log q(z, w) + (1 + q^{-\partial z}) \log \frac{w - z}{z - w} \right),
\]
therefore the difference of both sides of (18) has the form \( f(z) \). Since this difference is also antisymmetric in \( z, w \), it is zero.

Then (19) follows from the addition of (18) with the formula expressing \( [hA_+(z), b(w)] \). The derivation of (20) is similar.

Let us now show that relation (8) is preserved. We have
\[
E(z)E(w) = \exp(\alpha(z, w)) : E(z)E(w) :,
\]
where : \( E(z)E(w) := v(z)v(w)\exp(a_+(z)+a_+(w))\exp(-z)+a_-(w)) \) and \( \alpha(z, w) := [a_+(z), a_+(w)] + \frac{1}{2}[a_+(z), a_+(w)] + \frac{1}{2}[a_-(z), a_-(w)] \).

Recall that we defined \( j(z, w) := z - q^{\partial z}/q^{-\partial z}, i(z, w) := q(z, w)(q^{\partial z} - w)/(z - q^{\partial z}w) \). Then
\[
[a(z), a(w)] = \log(i(z, w)) + \log \frac{z - q^{\partial z}}{q^{\partial z} - w} + \log \frac{w - z}{z - w} + \log \frac{w - q^{\partial z}}{q^{-\partial z}z - w}.
\]
where:

$$E$$

Therefore

$$[a_+(z), a_+(w)] = \log(i)_{++}(z, w) - (\log(q^\frac{1}{2} z - w))_{++} - (\log(q^{-\frac{1}{2}} z - w))_{++}$$

$$= \log(i)_{++}(z, w) - \left(\frac{q^\frac{1}{2} z - w}{z - q^{-\frac{1}{2}} w}\right)_{++} - \left(\log\frac{q^{-\frac{1}{2}} z - w}{q^\frac{1}{2} z - w}\right)_{++}$$

$$= \log(i)_{++}(z, w) + \log(j_{j^{21}})_{++}(z, w),$$

$$[a_-(z), a_-(w)] = \log(i)_{--}(z, w) + (\log(z - q^\frac{1}{2} w))_{--} + (\log(w - q^{-\frac{1}{2}} z))_{--}$$

$$= \log(i)_{--}(z, w) + \left(\log\frac{w - q^{-\frac{1}{2}} z}{q^\frac{1}{2} w - z}\right)_{--} + \left(\log\frac{w - q^{-\frac{1}{2}} z}{q^\frac{1}{2} w - z}\right)_{--}$$

$$= \log(i)_{--}(z, w),$$

$$[a_+(z), a_-(w)] = \log(i)_{+-}(z, w) + \log(w - z) + (\log(w - z^{-\frac{1}{2}} z))_{+-}$$

$$= \log(i)_{+-}(z, w) - (\log(w - q^{-\frac{1}{2}} z))_{--} + \log(w - z) + \log(w - z^{-\frac{1}{2}} z)$$

$$= \log(i)_{+-}(z, w) - \left(\log\frac{w - q^{-\frac{1}{2}} z}{q^\frac{1}{2} w - z}\right)_{--} + \log(w - z) + \log(w - z^{-\frac{1}{2}} z)$$

$$= \log(i)_{+-}(z, w) + \log(j)_{--}(z, w) + \log(w - z)(w - q^{-\frac{1}{2}} z).$$

Here we decompose an element $\beta$ of (a completion of) $\tilde{K}^\otimes 2$ as $\sum_{\epsilon, \epsilon'}\beta_{\epsilon, \epsilon'}$ according to the decomposition of $K$.

Finally,

$$\alpha(z, w) = \left(\log(i)_{++} + \frac{1}{2}\log(i)_{++} + \frac{1}{2}\log(i)_{--} + \log(j)_{--} + \frac{1}{2}\log(j_{j^{21}})\right)(z, w) + \log(w - z)(w - q^{-\frac{1}{2}} z)$$

$$= \gamma(z, w) + \log(w - z)(w - q^{-\frac{1}{2}} z),$$

where $\gamma(z, w) \in \mathbb{H}[\mathbb{C}[z, w]](z^{-1}, w^{-1})[[\mathbb{H}]]$. One checks that $\exp(\gamma(z, w) - \gamma(w, z)) = i(z, w)j(z, w)/j(w, z)$: this follows from the fact that $(\log(j_{j^{21}})_{++} = 0$, because

$$\log(j_{j^{21}})(z, w) = \log(z - q^\frac{1}{2} w)(z - q^{-\frac{1}{2}} w) - \log(q^{-\frac{1}{2}} z - w)(q^\frac{1}{2} z - w).$$

Now we have

$$(w - q^\frac{1}{2} z)E(z)E(w) = e^{\gamma(z, w)}(w - q^\frac{1}{2} z)(w - q^{-\frac{1}{2}} z) : E(z)E(w) :$$

$$= i(z, w)(q^\frac{1}{2} w - z)e^{\gamma(w, z)}(w - z)(q^{-\frac{1}{2}} w - z) : E(z)E(w) := i(z, w)(q^\frac{1}{2} w - z)E(w)$$

(assuming the fact that $E(z)E(w) ::= E(w)E(z) :$, so relation (8) is preserved.

One proves similarly that relation (9) is preserved.

Let us now show that (15) is preserved. We have $E(z)F(w) = \exp(\lambda(z, w)) : E(z)F(w) :$,

where $E(z)F(w) := v(z)v'(w)\exp(a_-(z) - b_-(w))\exp(a_+(z) - b_+(w))$ and $\lambda(z, w) = -[a_+(z), b_-(w)] - \frac{1}{2}[a_+(z), b_+(w)] - \frac{1}{2}[a_-(z), b_-](w)\exp(\mu(z, w)) : E(z)F(w) :$, where $\mu(z, w) = -[b_+(w), a_-(z)] - \frac{1}{2}[b_+(w), a_+(z)] - \frac{1}{2}[b_-(w), a_-(z)].$

So

$$E(z), F(w) = (\exp(\lambda(z, w)) - \exp(\mu(z, w)) : E(z)F(w) :.$$

Now

$$[a(z), b(w)] = \log(w - z) + \log(w - q^\frac{1}{2} z) - \log(z - w) - \log(q^\frac{1}{2} z - w),$$

Calculating as above, we find

$$[a_+(z), b_+(w)] = -\log(j_{j^{21}})_{++}(z, w),$$

$$[a_+(z), b_-(w)] = \log(w - z)(w - q^\frac{1}{2} z) - \log(j_{j^{21}})_{--}(z, w)$$

$$[a_-(z), b_+(w)] = -\log(z - w)(q^\frac{1}{2} z - w) + \log(j_{j^{21}})_{++}(z, w),$$
Recall that 
\[ \delta(z,w) = \begin{cases} (z-w) & \text{if } q^\beta z \neq w \\ 0 & \text{else} \end{cases} \]

and 
\[ \mu(z,w) = \left( \frac{1}{2} \log(j^{21}++) + \frac{1}{2} \log(j^{21}--) \right)(z,w) - \log(z-w)(q^\theta z - w). \]

Then 
\[ [E(z), F(w)] = e^{\phi(z,w)} \left( \frac{1}{(w-z)(w-q^\theta z)} - \frac{1}{(z-w)(q^\theta z - w)} \right) \cdot E(z)F(w) : 
\]
\[ = \frac{r_0(w)}{z-q^\theta z} \left( \delta(z,w) - \delta(w-q^\theta z) \right) : E(z)F(w) : 
\]
\[ = \frac{r_0(w)}{(z-q^\theta z)/\hbar} \left( e^{\phi(z,w,z)} \delta(z,w) : E(z)F(z) : -e^{\phi(q^{-\theta}w,w)} \delta(q^\theta z,w) : E(q^{-\theta}w)F(w) : \right). \]

Here \( \delta(z,w) = 1/(z-w) + 1/(w-z) \). We have \( \delta(z,w) = \delta(z-w)/r_0(w) \) (recall that \( w(z) = r_0(z)dz \)).

Now 
\[ [A_R(z), A_R(w)] = 0, \text{ hence } [(A_R)_+(z), (A_R)_-(w)] = 0, \text{ which implies : } E(z)F(z) := v(z)v'(z)K^+(z). \]

On the other hand,
\[ E(q^{-\theta}w)F(w) := v(q^{-\theta}w)v'(w)e^{q^{-\theta}a-b}(w) \cdot v(q^{-\theta}w)v'(w)e^{q^{-\theta}a-b}(w) = v(q^{-\theta}w)v'(w)e^{\frac{1}{\hbar}(A_R)_-(z),h(A_R)_+(w)]u_{\Lambda}(w)K^{-1}(w). \]

Now \([hA_{\Lambda}(z),hA_{\Lambda}(w)] = (q^{-\theta} \otimes q^{-\theta} - 1) \log q(z,w) \in \hbar^2 \wedge^2 (R)[[\hbar]]. \]

Then we get
\[ E(q^{-\theta}z)F(z) := v(q^{-\theta}z)v'(z)e^{(q^{-\theta} \otimes q^{-\theta} - 1) \log q(z,w)] + \log u_{\Lambda}(z)K^{-1}(z), \]

with \((q^{-\theta} \otimes q^{-\theta} - 1) \log q(z,w) \in \hbar^2 K[[\hbar]]. \)

Finally,
\[ [E(z), F(w)] = v(z)v'(w) \cdot \frac{r_0(w)}{(z-q^\theta z)/\hbar} \times \]
\[ \frac{1}{\hbar} \left( \delta(z,w)e^{\phi(z,w)}u_{\Lambda}(z)^{-1}K^+(z) - \delta(z,q^{-\theta}w)e^{\phi(q^{-\theta}w,w)}e^{(q^{-\theta} \otimes q^{-\theta} - 1) \log q(w,z)] + \log u_{\Lambda}(z)K^{-1}(w). \right) \]

Recall that \( \alpha(z) = \phi(z,w) \) and \( \beta(w) = \phi(q^{-\theta}w,w) + \left((q^{-\theta} \otimes q^{-\theta} - 1) \log q(w,z) \right)_{-z=z}. \)

Then
\[ \left[ \frac{(z-q^\theta z)/\hbar}{v(z)} \cdot E(z), e^{-\alpha(w)-\beta_R(w)r_0(w)}v'(w)F(w) \right] = \]
\[ = \frac{1}{\hbar} \left( \delta(z,w)e^{\alpha_\Lambda(z)u_{\Lambda}(z)^{-1}K^+(z)} - \delta(z,q^{-\theta}w)e^{\beta_\Lambda(w)}u_{\Lambda}(w)K^{-1}(w). \right) \]

Now since \( v(z) = (z-q^\theta z)/\hbar, v'(w) = r_0(w)e^{-\alpha(w)-\beta_R(w)}, u_{\Lambda}(z) = e^{\alpha_R(z)} \) and \( u_{\Lambda}(w) = e^{-\beta_\Lambda(w)} \), relation (15) is preserved.
5. **Double of elliptic quantum group $E_{\tau,q}(\hat{sl}_2)$ versus elliptic algebra $U_{q,p}(\hat{sl}_2)$**

It was shown in [EF] that the algebra given by the commutation relations (8) to (15) with the function $q(z,w) = \frac{\theta(z-w+h)}{\theta(z-w-h)}$ given in Subsection 2.4.3 is isomorphic to the centrally extended double of the elliptic quantum group $E_{\tau,q}(\hat{sl}_2)$. This isomorphism is an analogue of the Ding-Frenkel isomorphism [DF]. The basic representations considered here provide level one representations of this central extended double of the elliptic quantum group.

Let us write a basis in $K$ in the case of the data given in Subsubsection 2.4.3. For $\lambda \neq 0$ it is

| $i$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | ... |
|-----|------|------|------|----|----|----|-----|
| $e^{i,\lambda}(z)$ | ... | $z^2$ | $z^1$ | $z^0$ | $\frac{\theta(z+\lambda)}{\theta(z)\theta(\lambda)}$ | $\frac{1}{1!} \left( \frac{\theta(z+\lambda)}{\theta(z)\theta(\lambda)} \right)'$ | $\frac{1}{2!} \left( \frac{\theta(z+\lambda)}{\theta(z)\theta(\lambda)} \right)''$ | ... |
| $\epsilon_{i,\lambda}(w)$ | ... | $\frac{1}{2!} \left( \frac{\theta(\lambda+w)}{\theta(w)\theta(\lambda)} \right)''$ | $-1 \left( \frac{\theta(\lambda+w)}{\theta(w)\theta(\lambda)} \right)'$ | $\frac{\theta(\lambda+w)}{\theta(w)\theta(\lambda)}$ | $(-w)^0$ | $(-w)^1$ | $(-w)^2$ | ... |

and for $\lambda = 0$

| $i$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | ... |
|-----|------|------|------|----|----|----|-----|
| $e^{i,0}(z)$ | ... | $z^2$ | $z^1$ | $z^0$ | $(\ln \theta(z))^0$ | $\frac{1}{1!} (\ln \theta(z))^0$ | $\frac{1}{2!} (\ln \theta(z))^0$ | ... |
| $\epsilon_{i,0}(w)$ | ... | $\frac{1}{2!} (\ln \theta(w))''$ | $-1 (\ln \theta(w))'$ | $(\ln \theta(w))'$ | $(-w)^0$ | $(-w)^1$ | $(-w)^2$ | ... |

The total currents $e(z)$ and $f(z)$ can be decomposed into modes with respect to the basis given in the first table, namely

$$ e(z) = \sum_{i \in \mathbb{Z}} e[i;\lambda] e^{i,\lambda}(z), \quad f(z) = \sum_{i \in \mathbb{Z}} f[i;\lambda] e^{i,\lambda}(z) \quad (21) $$

Using a duality of the bases in $K$: $\oint dw e^{i,\lambda}(w) \epsilon_{j,\lambda}(w) = \delta_{i,j}$, we can write the modes as the contour integrals

$$ e[i,\lambda] = \oint_{C^+_0} dw \ e(w) \ e_{i,\lambda}(w), \quad f[i,\lambda] = \oint_{C^+_0} dw \ f(w) \ e_{i,\lambda}(w) \quad (22) $$

where $C^+_0$ is a contour encircling 0 counterclockwise. Define half currents as the sums

$$ e^\tau_{\lambda}(z) = \mp \sum_{i \geq 0} e[i;\mp \lambda] e^{i,\lambda}(z), \quad f^\tau_{\mu}(z) = \mp \sum_{i \geq 0} f[i;\mp \mu] e^{i,\mu}(z) \quad (23) $$

we may easily deduce that they can be written in the integral forms

$$ e^\tau_{\lambda}(z) = \oint_{C^Ζ_0} \frac{\theta(u-z-\lambda)}{\theta(u-z)\theta(-\lambda)} e(u) \ du, \quad e^\tau_{\lambda}(z) = \oint_{C^+_{0,z}} \frac{\theta(u-z-\lambda)}{\theta(u-z)\theta(-\lambda)} e(u) \ du \quad (24) $$

and

$$ f^-_{\lambda}(z) = \oint_{C^-_0} \frac{\theta(u-z+\lambda)}{\theta(u-z)\theta(\lambda)} f(u) \ du, \quad f^+_{\lambda}(z) = \oint_{C^+_{0,z}} \frac{\theta(u-z+\lambda)}{\theta(u-z)\theta(\lambda)} f(u) \ du \quad (25) $$
with $C_{0,z}^{+}$ a contour encircling the points 0 and $z$ clockwise and $C_{0,z}^{-}$ a contour encircling 0 clockwise. In order to obtain positive and negative half-currents (24) and (25) we used the Taylor expansion of the kernel $\frac{\theta(z-w+\lambda)}{\theta(z-w)\theta(\lambda)}$ in powers of $z$ and $w$ respectively. Note that the expansion in powers of $z$ is defined in the domain $z \ll w$ and in the domain $w \ll z$ with respect to powers of $w$. This explains the difference between the contours of integrations in (24) and (25). The same reason is behind the appearance of the $\delta$-function in the sum

$$\sum_{i \geq 0} e^{i\lambda(z)} \epsilon_{i,\lambda}(w) + \sum_{i < 0} e^{-i\lambda(z)} \epsilon_{i,-\lambda}(w) = \frac{\theta(z-w+\lambda)}{\theta(z-w)\theta(\lambda)} + \frac{\theta(w-z-\lambda)}{\theta(w-z)\theta(\lambda)} = \delta(z-w)$$

(26)

Besides the Taylor expansion, the kernel $\frac{\theta(z-w+\lambda)}{\theta(z-w)\theta(\lambda)}$ has the following Fourier mode expansion\(^2\)

$$\frac{\theta(z+\lambda)}{\theta(z)\theta(\lambda)} = \sum_{n \in \mathbb{Z}} e^{2\pi i \frac{z(n+\lambda)}{\tau}} = \sum_{n \in \mathbb{Z}} e^{2\pi i \frac{z(n+\lambda)}{\tau} - \frac{1}{\tau}}$$

(27)

which is valid for $0 < \Im(z/\tau) < \Im(1/\tau)$ and $0 < \Im(\lambda/\tau) < \Im(1/\tau)$. Formula (27) leads to the temptation to write instead of (23) half-currents in the form

$$e_{\lambda}^{\pm}(z) = \sum_{n \in \mathbb{Z}} e[n] e^{2\pi i \frac{zn}{\tau}} e^{\frac{2\pi i \frac{zn}{\tau}}{1 - e^{2\pi i \frac{zn}{\tau}}}}$$

$$f_{\mu}^{\pm}(z) = \sum_{n \in \mathbb{Z}} f[n] e^{2\pi i \frac{zn+\mu}{\tau}} e^{\frac{2\pi i \frac{zn+\mu}{\tau}}{1 - e^{2\pi i \frac{zn+\mu}{\tau}}}}$$

(28)

where $e[n]$ are $f[n]$ are $\lambda$-independent modes of the total currents

$$e(z) = \sum_{n \in \mathbb{Z}} e[n] e^{2\pi i \frac{zn}{\tau}}$$

$$f(z) = \sum_{n \in \mathbb{Z}} f[n] e^{2\pi i \frac{zn}{\tau}}$$

such that

$$e(z) = (e_{\lambda}^{+}(z) + e_{\lambda}^{-}(z)) e^{-2\pi i \frac{z\lambda}{\tau}}$$

$$f(z) = (f_{\mu}^{+}(z) + f_{\mu}^{-}(z)) e^{-2\pi i \frac{z\mu}{\tau}}$$

The definition of the currents of type (28) was used in the paper [JKOS] to build the elliptic algebra $U_{q,p}(sl_2)$. The resulting algebra of half-currents appears to be completely different from the algebra investigated in [EF]. In particular, it acquires a dynamical behaviour not only with respect to the parameter $\lambda$, but also with respect to the elliptic parameter $\tau$. Moreover, because of the formal relation between positive and negative half-currents of the form $e_{\lambda}^{+}(u+1) = -e_{\lambda}^{-}(u)$ only one $L$-operator, say $L_{\lambda}^{+}(u)$, is sufficient to describe the whole algebra. All these phenomena were described in detail in the paper [KLP98] in the example of the scaling limit of the elliptic algebra. A classical version of this scaled elliptic algebra was investigated in the paper [KLPST] where, in particular, it was proved that on the classical level, the central extension of the classical currents should be done by the cocycle which is defined by a differentiation with respect to moduli of the elliptic curve. The decomposition into Fourier modes becomes in the scaling limit the decomposition into continuous Fourier modes and the summations on $n \in \mathbb{Z}$ become integrals along the real axis. The dynamical behaviour with respect to the parameter $\lambda$ disappears in the scaling limit. The resulting algebra is a current algebra on the infinite cylinder and it was used in the papers [KLP98, LeKP] to explain the dynamical symmetries of the Sine-Gordon model.

This situation, when different decompositions of the total currents lead to different algebras for half-currents, was investigated also in the paper [KLP99] in the case of the centrally extended Yangian double. Elliptic algebras considered in the papers [EF] and [JKOS] are in fact elliptic generalizations of two constructions of the centrally extended Yangian double given in [KLP99].

\(^2\)An analogous formula exists for the elliptic current algebra associated with Baxter’s 8-vertex $R$-matrix, see [KLPST].
6. Conclusion

To conclude we would like to mention that the existence of the level zero representations in both elliptic algebras formulated in the papers [EF] and [JKOS] is based on the Fay identity
\[ \theta(a + c) \theta(b + d) \theta(a - c) \theta(b - d) = \theta(a + b) \theta(c + d) \theta(a - b) \theta(c - d) \]
\[- \theta(a + d) \theta(c + b) \theta(a - d) \theta(c - b) \]
which can be written in the form [JKOS]
\[ \frac{\theta(u_1 + t)}{\theta(u_1)} \eta_{s,t}(u_2) = \frac{\theta(u_1 - u_2 + t)}{\theta(u_1 - u_2)} \eta_{s+t,t}(u_2) + \eta_{s,t}(u_2 - u_1) \eta_{s+t,t}(u_1) \]
where
\[ \eta_{s,t}(u) = \frac{\theta(u + s)\theta(t)}{\theta(u)\theta(s)} \]

The Fay identity can be written for higher genus algebraic curve and it is interesting to apply this identity for investigation of higher genus current algebras. It is clear that formulas like (27) does not exist in higher genus, so the decompositions of total currents which lead to dynamical shifts in modules of the curve are impossible. But “non-symmetric” decompositions of the type (23) are always possible. It is interesting to combine the Fay identity and these decompositions to investigate some current algebras associated with higher genus algebraic curves. The work in this direction is in progress.

Acknowledgements

The research by S.P. was supported in part by the grants INTAS-OPEN-03-51-3350, Heisenberg-Landau program, RFBR grant 03-02-17373. V.R. greatly acknowledged the support of the RFBR grant 03-02-17554 and of franco-ukrainian Collaboration Program “Dnipro” during his visit to Kiev. He thanks Bogolyubov ITP of NAN Ukraine for hospitality. S.P. and V.R. are grateful to the RFBR grant to support scientific schools NSSh-1999.2003.2.

This work was started when S.P. enjoyed a CNRS visitor position in Angers. He acknowledges the CNRS support and the Mathematical Departments of the Angers and Strasbourg Universities for hospitality. Part of the work was done during the visit of two authors (S.P. and V.R.) at the Max Planck Institut für Mathematik (Bonn). They wish to thank the Institut for hospitality and for its stimulating scientific atmosphere.

B.E. would also like to thank M. Jimbo for explanations about the relations between [EF] and [JKOS].

References

[L95] Lukyanov, S. Free field representation for massive integrable models. Commun. Math. Phys. 167 (1995) 183–226.

[D90] Drinfeld, V. Quasi-Hopf algebras. Leningrad Math. J. 1 (1990) 1419–1457.

[ER-MS] Enriquez, B., Rubtsov, V. Some examples of quantum groups associated with higher genus algebraic curves, "Moscow Seminar in Mathematical Physics", 33–65, Amer. Math. Soc. Transl. Ser. 2, 191, Amer. Math. Soc., Providence, RI, 1999.

[F97] Frønsdal, C. Quasi-Hopf deformations of quantum groups. Lett. Math. Phys. 40 (1997) 117–134.

[FIJKMY] Foda, O., Iohara, K., Jimbo, M., Kedem, R., Miwa, T., Yan, H. An elliptic quantum algebra for \( sl_4 \). Lett. Math. Phys. 32 (1994) 259–268.

[ABRR97] Arnaudon, D., Buffenoir, E., Ragoucy, E., Roche, P. Universal solution of quantum dynamical Yang-Baxter equations. Preprint q-alg/9712037.

[JKOS97] Jimbo, M., Konno, H., Odake, S., Shiraishi, J. Quasi-Hopf twistors for elliptic quantum groups. Preprint q-alg/9712029.
[EF] Enriquez, B., Felder, G. Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$. *Commun. Math. Phys.* **195** (1998) 651–689.

[F] Felder, G. Conformal field theory and integrable systems associated to elliptic curves, Proc. ICM Zürich 1994, 1247-55, Birkhäuser (1994); Elliptic quantum groups, Proc. ICMP Paris 1994, 211-8, International Press (1995).

[HGQG] Enriquez, B., Rubtsov, V. Quantum groups in higher genus and Drinfeld’s new realizations method ($\mathfrak{sl}_2$ case), *Ann. Sci. Éc. Norm. Sup.* **30** (1997), 821-846.

[ER99] Enriquez, B., Rubtsov, V. Quasi-Hopf algebras associated with $\mathfrak{sl}_1$ and complex curves. *Israel Journal of Mathematics* **112** (1999) 61–108.

[DF] Ding, J., Frenkel, I.B., Isomorphism of two realizations of quantum affine algebras $U_q(\mathfrak{gl}(n))$. *Commun. Math. Phys.* **156** (1993), 277-300.

[Kh] Khoroshkin, S. Central extension of the Yangian double, preprint q-alg/9602031.

[FJ] Frenkel, I.B., Jing, N. Vertex representations of quantum affine algebras. *Proc. Nat. Acad. Sci. U.S.A.* **85** (1988), no. 24, 9373–9377.

[FK] Frenkel, I.B., Kac, V., Basic representations of affine Lie algebras and dual resonance models. *Invent. Math.* **62** (1980/81), no. 1, 23–66.

[JKOS] Jimbo, M., Konno, H., Odake, S., Shiraishi, J. Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$: Drinfeld currents and vertex operators. *Commun. Math. Phys.* **199** (1999) 605–647.

[KLP99] Khoroshkin, S., Lebedev, D., Pakuliak, S. *Yangian algebras and classical Riemann problem*. "Moscow Seminar in Mathematical Physics", 163–198, Amer. Math. Soc. Transl. Ser. 2, **191**, Amer. Math. Soc., Providence, RI, 1999

[KLP98] Khoroshkin, S., Lebedev, D., Pakuliak, S. Elliptic algebra in the scaling limit. *Commun. Math. Phys.* **190** (1998), no. 3, 597-627

[KLPST] Khoroshkin, S., Lebedev, D., Pakuliak, S., Stolin, A., Tolstoy, V. Classical limit of the scaled elliptic algebra. *Compositio Mathematica* **115** (1999), no. 2, 205–230

[LeKP] LeClair, A., Khoroshkin, S., Pakuliak, S. Angular quantization of the Sine-Gordon model at the free fermion point. *Adv. Theor. Math. Phys.* **3** (1999), no. 5, 1227-1287

[KLP96] Khoroshkin, S., Lebedev, D., Pakuliak, S. Intertwining Operators for the Central Extension of the Yangian Double. *Phys. Lett. A* **222** (1996), no. 6, 381–392.

IRMA, Université Louis Pasteur, Strasbourg, 7, rue René Descartes, F-67084 Strasbourg, France

Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow reg., Russia

Département de Mathématiques, Université d’Angers, 2 Bd. Lavoisier, 49045 Angers, France and

ITEP Theory Department, 25, Bol. Tcheremushkinskaya, 117259, Moscow, Russia