THE CRITICAL ENDPOINT OF
BOOTSTRAP AND LATTICE QCD MATTER

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Abstract

The critical sector of strong interactions at high temperatures is explored in the frame of two complementary approaches: Statistical Bootstrap for the hadronic phase and Lattice QCD for the Quark-Gluon partition function. A region of thermodynamic instability of hadronic matter was found, as a direct prediction of Statistical Bootstrap. As a result, critical endpoint solutions for nonzero chemical potential were traced in the phase diagram of strongly interacting matter. These solutions are compared with recent lattice QCD results and their proximity to the freeze-out points of experiments with nuclei at high energies is also discussed.

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1. Introduction

Quantum Chromodynamics is unquestionably the fundamental theory of strong interaction. However, the nonperturbative aspects of QCD belong still to a field of intense investigation. In fact, most of the properties of the hadronic world cannot be extracted yet from first QCD principles and the partition function of interacting hadrons produced and thermalized in high-energy collisions can only be determined within specific models of varying degree of limitation.

In this paper we employ Statistical Bootstrap, including volume corrections for the finite size of hadrons, in order to describe the hadronic phase. The virtue of this description is associated with the fact that, among Statistical models of hadrons, it is the only one which predicts the onset of a phase transition in strongly interacting matter and therefore it is compatible with the properties of the QCD vacuum at high temperatures [1]. The aim of this work is to pursue the search for this compatibility with QCD even further, asking whether Statistical Bootstrap of hadrons is consistent with the existence of a critical endpoint in strongly interacting matter, at high temperature and finite (nonzero) baryonic chemical potential. The existence of such a critical point in the phase diagram is required by QCD in extreme conditions [2,3] and it is the remnant of chiral QCD phase transition [2,3].

The basic ingredients in our approach are (a) the hadronic partition function extracted from the equations of Statistical Bootstrap and (b) the equation of state of the quark-gluon phase given by recent QCD studies on the lattice [4-7]. Our principal aim from the matching of these two descriptions is to trace the formation of a critical point in the quark-hadron phase transition with a mechanism compatible both with Statistical Bootstrap and Lattice QCD.

In [8,9] a similar scheme was pursued but with the quark-gluon partition function mediated by the MIT bag model. This model is a crude approximation leading either to an ideal gas of quarks and gluons [8] near the critical line or to a gas of weakly interacting quarks and gluons with perturbative terms in the sector of two light flavours [9]. The only nonperturbative effect in this model comes from the pressure of the vacuum through the phenomenological bag constant. In order to have an adequate description of QCD matter in this approach the use of lattice equation of state, derived from first principles is an essential
improvement. To this end we have employed, in this work, a realistic partition function of QCD matter which has become available recently as a result of a breakthrough achieved in lattice QCD studies allowing for computations at nonzero chemical potential \[4,7\]. As far as the hadronic phase is concerned we continue to use in this work the pressure partition function but with the volume Laplace conjugate variable \(\xi\) as an active thermodynamic variable and not fixed at the zero value. These calculations, within a three flavour bootstrap scheme, are used here for the first time. Despite all these alterations we shall show that the existence of a critical point at finite chemical potential persists and we shall try to narrow the area of its location.

In Section 2 the principles and basic equations of Statistical Bootstrap, leading to the partition function of hadronic phase, are briefly summarized and the appearance, within this framework, of a thermodynamic instability is discussed in detail. This instability is the origin of the formation of a critical point in the bootstrap matter. In Section 3 the partition function of the quark-gluon phase is extracted from recent lattice calculations of the pressure of QCD matter \[4\]. In Section 4 the above two descriptions of strongly interacting matter are exploited in a search for a critical point in the phase diagram, the location of which is fixed by a set of equations (29-33). Finally, in Section 5 our conclusions are presented whereas in the Appendix certain technical points are discussed concerning the evaluation of the quark-gluon partition function (part A) and the complete form that acquires the equation of maximum hadronic pressure (part B).

2. Thermodynamic Instability in Statistical Bootstrap

The main attribute of the Statistical Bootstrap (SB) \[10-13\] is that it describes thermodynamically interacting hadronic systems. Generally as far as its thermodynamic behaviour is concerned a system of strongly interacting entities may be considered as a collection of particles with the explicit form of the complex interaction among themselves. Another way to deal with the problem is to assume that the strongly interacting entities form a greater entity with certain mass. Then, if this mass is known, the greater entities may be treated as non-interacting and the simple thermodynamic description of an ideal system of particles may be applied. The SB adopts the second approach and assumes that the strongly
interacting hadrons form greater clusters, called “fireballs”. The problem then is moved to determine the mass of these clusters or to evaluate their mass spectrum \( \tilde{\tau}(m^2)dm^2 \), which is equal to the number of discrete hadronic states in the mass interval \( \{m, m + dm\} \). The solution is based on the assumption that the fireballs may consist of smaller fireballs with the same mass spectrum or “input” particles which may not be divided further. The integral bootstrap equation then reads \([14-17]\)

\[
\tilde{B}(p^2)\tilde{\tau}(p^2, b, s, \ldots) = g_{b,s,\ldots} \tilde{B}(p^2)\delta_0(p^2 - m_{b,s,\ldots}^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4 \left(p - \sum_{i=1}^{n} p_i\right) \cdot 
\]

\[
\sum_{\{b\}} \delta_K \left(b - \sum_{i=1}^{n} b_i\right) \sum_{\{s_i\}} \delta_K \left(s - \sum_{i=1}^{n} s_i\right) \cdots \prod_{i=1}^{n} \tilde{B}(p_i^2)\tilde{\tau}(p_i^2, b_i, s_i, \ldots)d^4p_i ,
\]

where \( \delta_0(p^2 - m_{b,s,\ldots}^2) = \delta(p^2 - m_{b,s,\ldots}^2)\theta(p_0) \). Equation (1) also imposes conservation of four-momentum \( p \) and any kind of existing additive quantum number (baryon number \( b \), strangeness \( s \), etc.), through the Kronecker function \( \delta_K \), between a fireball and its constituent fireballs. The masses \( m_{b,s,\ldots} \) are the “input” particle masses which constitute the smaller fireballs that can be formed and \( g_{b,s,\ldots} \) are degeneracy factors due to the spin of the “input” particles. The correct counting of states of a fireball involves, apart from its mass spectrum which is of dynamical origin, a kinematical term \( \hat{B}(p^2) \). This term is related to the appropriate volume of a fireball which is considered to be carried with it, \( V^\munow = Vmp^\mu \). The imposition of the requirement that the sum of volumes of the constituent fireballs has to be equal to the volume of the large fireball, as well as, momentum conservation lead to the fact that all fireballs possess the same volume to mass ratio. This ratio can be connected to the MIT bag constant, \( \frac{V}{m} = \frac{1}{4B} \). The term \( \hat{B}(p^2) \), appearing in eq. (1) is then

\[
\hat{B}(p^2) = \frac{2V^\mu p_\mu}{(2\pi)^3} = \frac{2Vm}{(2\pi)^3} .
\]

Eq. (2) corresponds to a specific choice for \( \hat{B} \). Since in (1) the mass spectrum \( \tilde{\tau} \) is accompanied by the term \( \hat{B} \), the bootstrap equation (apart from the input term) remains unchanged if \( \tilde{\tau} \) and \( \hat{B} \) are redefined in such way so

\[
\tilde{B}\tilde{\tau} = \tilde{B}'\tilde{\tau}'
\]

Every such transformation leads to different thermodynamic description of the system of fireballs, since the relevant quantity for this description is only the mass spectrum \( \tilde{\tau}' \) and not
the term $\tilde{B}'$. These transformations are not uniquely determined in the general bootstrap scheme, allowing for various versions of the model [15].

The bootstrap equation can be simplified after performing a series of Laplace transformations to acquire the form

$$\varphi(\beta, \{\lambda\}) = 2G(\beta, \{\lambda\}) - \exp[G(\beta, \{\lambda\})] + 1. \tag{4}$$

In the last equation $G(\beta, \{\lambda\})$ is the Laplace transform of the mass spectrum with the accompanying kinematical term

$$G(\beta, \{\lambda\}) = \sum_{b'=-\infty}^{\infty} \lambda_b^{b'} \sum_{q'=-\infty}^{\infty} \lambda_q^{q'} \sum_{s'=-\infty}^{\infty} \gamma_s^{s'} \int e^{-\beta p^2} p^2 \tilde{B}'(p^2, b', q', s', |s'|) dp^4,$$

$$= \frac{2\pi}{\beta} \int_0^{\infty} m\tilde{B}'(m^2) \tilde{\tau}'(m^2, \{\lambda\}) K_1(\beta m) dm^2 \tag{5}$$

and $\varphi(\beta, \{\lambda\})$ the Laplace transform of the input term

$$\varphi(\beta, \{\lambda\}) = \sum_{b'=-\infty}^{\infty} \lambda_b^{b'} \sum_{q'=-\infty}^{\infty} \lambda_q^{q'} \sum_{s'=-\infty}^{\infty} \gamma_s^{s'} \int e^{-\beta p^2} g_{b'q's'} |s'| \tilde{B}'(p^2) \delta_0(p^2 - m_{b'q's'}^2) dp^4. \tag{6}$$

The masses $m_{bqss|s|}$ correspond to the masses of the input particles, which in this paper will be all the known hadrons with masses up to 2400 MeVs, the $g_{bqss|s|}$ are degeneracy factors due to spin and the $\lambda$'s are the fugacities of the input particles. Here we have used the extended version of SB [17], where the states are characterised by the set of fugacities relevant to baryon number $\lambda_b$, electric charge $\lambda_q$, strangeness $\lambda_s$ and partial strangeness equilibrium $\gamma_s$ (in the following we shall refer to this set of fugacities as $\{\lambda\}$, for short). The last fugacity $\gamma_s$ is related to the number of strange quarks plus strange antiquarks [18], whereas the fugacity $\lambda_s$ is related to the number of strange quarks minus strange antiquarks. The introduction of $\gamma_s$ is necessary for the accurate theoretical prediction of the experimentally measured hadronic multiplicities.

The bootstrap equation defines the boundaries of the hadronic phase since it exhibits a singularity at the point

$$\varphi(\beta, \{\lambda\}) = \ln 4 - 1. \tag{7}$$

From the physical point of view this singularity is connected with the behaviour of the
mass spectrum as the mass tends to infinity

\[ \tilde{\tau}'(m^2, \{\lambda\}) \xrightarrow{m \to \infty} 2C(\{\lambda\})m^{-\alpha-1} \exp[m\beta^* (\{\lambda\})], \]  

(8)

where \( \beta = T^{-1} \) and \( \beta^* \) corresponds to the inverse maximum temperature. After a certain point, as temperature rises, it is more preferable for the system to use the given energy in producing more hadronic states (since their number rises exponentially) than in increasing the kinetic energy of the already existing states. For our specific choice of \( \tilde{B} \) in (3) the exponent \( \alpha = 4 \).

In order to turn to the thermodynamics it is necessary to consider the fireball states in an external volume \( V^{ext} \). This volume must be distinguished from the physical volume which is carried by each fireball. The partition function of the pointlike interacting hadrons is then

\[
\ln Z_{pt}(V^{ext}, \beta, \{\lambda\}) = \sum_{b'=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} \sum_{s'=-\infty}^{\infty} \sum_{|s'|=0}^{\infty} \gamma_{s'} |s'| \int \frac{2V^{ext}_{\mu} p^{\mu}}{(2\pi)^3} \tilde{\tau}'(p^2, b', q', s', |s'|) e^{-\beta \mu p^\mu} dp^4 \equiv \int \frac{2V^{ext}_{\mu} p^{\mu}}{(2\pi)^3} \tilde{\tau}'(p^2, \{\lambda\}) e^{-\beta \mu p^\mu} dp^4. \]  

(9)

Every choice for the mass spectrum in (3) (which leads to a certain exponent \( \alpha \) in (8)) leads to a different partition function and so to a different physical behaviour of the system. The usual SB choice was \( \alpha = 2 \), but more advantageous is our choice \( \alpha = 4 \). With this choice a better physical behaviour is achieved as the system approaches the hadronic boundaries. Quantities like pressure, baryon density and energy density, even for point-like particles, no longer tend to infinity, as the system tends to the bootstrap singularity. It also allows for the bootstrap singularity to be reached in the thermodynamic limit [19], a necessity imposed by the Lee-Yang theory. Another point in favour of the choice \( \alpha = 4 \) comes from the extension of SB to include strangeness [14,15]. The strange chemical potential equals zero in the quark-gluon phase. With this particular choice of \( \alpha \), \( \mu_s \) acquires smaller positive values as the hadronic boundaries are approached. With the choice \( \alpha = 4 \) the partition function can be written down and for point-like particles it assumes the form

\[
\ln Z_{pt}(V^{ext}, \beta, \{\lambda\}) = \frac{4BV}{\beta^3} \int_{\beta}^{\infty} x^3 G(x, \{\lambda\}) dx \equiv V f(\beta, \{\lambda\}). \]  

(10)
For $\alpha = 4$ the input term acquires the form

$$\varphi(\beta, \{\lambda\}) = \frac{1}{2\pi^2 \beta B} \sum_a (\lambda_a(\{\lambda\}) + \lambda_a(\{\lambda\})^{-1}) \sum_i g_{ai} m_{ai}^3 K_1(\beta m_{ai}) \ ,$$  
(11)

where the index “a” runs to all groups of hadrons, each of which is characterised by the same set of fugacities (e.g. Kaons with electric charge $Q = 1$, $N$ and $\Delta$ Baryons with $Q = 0$, etc.), “i” runs to all hadrons in the same group with different masses and $B$ is the energy density of the vacuum (MIT bag constant).

By including corrections due to the finite size of hadrons (Van der Waals volume corrections) the repulsive part of the interaction is taken into account. The partition function for the real hadron gas is written [19,20] as follows

$$Z(V_{\text{ext}}, \beta, \{\lambda\}) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} \left\{ \frac{2(V_{\text{ext}} - p/4B) \mu p_i^\mu}{(2\pi)^3} \right\}_+ \tau(p_i^2, \{\lambda\}) e^{-\beta \mu p_i^\mu} dp_i^4 \ .$$  
(12)

In the above relation the subscript + indicates that each single bracket has to be non-negative, avoiding, thus, the negative contributions to the volume. The four momentum is $p^\mu = \sum_i p_i^\mu$ and because of its presence the integrations no longer factorize. The factorization property can be recovered through the grand canonical pressure partition function [19,20]

$$\pi(\beta, \xi, \{\lambda\}) = \int_0^\infty dV e^{-\xi V} Z(V, \beta, \{\lambda\}) \ ,$$  
(13)

which is the Laplace transformed partition function with respect to volume. Provided that the thermodynamic limit $\lim_{V \to \infty} (1/V) \ln Z(\beta, V, \{\lambda\})$ exists, the integral in (13) converges for the values

$$\xi > \xi_0(\beta, \{\lambda\}) \equiv \lim_{V \to \infty} \left[ \frac{1}{V} \ln Z(\beta, V, \{\lambda\}) \right] \ .$$  
(14)

If we are constrained to values $\xi > \xi_0$ there is no need to employ Gaussian regularization [20] to evaluate (13) for $\xi \leq \xi_0$. Then the pressure partition function acquires the form

$$\pi(\xi, \beta, \{\lambda\}) = \frac{1}{\xi - f(\beta + \xi/(4B), \{\lambda\})} \ ,$$  
(15)

where $f = \ln Z_{pt}/V$. From (15) it is evident that the value of $\xi$ at the thermodynamic limit (14) can be obtained from

$$\xi_0 = f(\beta + \xi_0/(4B), \{\lambda\}) \ .$$  
(16)
We have, also, to note that for \( \xi \neq 0 \) the critical temperature of the hadronic state at zero density, \( T_{0,HG} \), does not depend only on the MIT bag constant, \( B \), as is the case in [14-17], but on \( \xi \), as well.

The density and the pressure \( P \) of the thermodynamic system can be obtained through the pressure grand canonical partition function (15)

\[
\nu_{HG}(\xi, \beta, \{\lambda\}) = \lambda \frac{\partial f(\beta + \xi/(4B), \{\lambda\})}{\partial \lambda} \left[ 1 - \frac{1}{4B} \frac{\partial f(\beta + \xi/(4B), \{\lambda\})}{\partial \beta} \right]^{-1},
\]

where \( \lambda \) is the fugacity corresponding to the particular density, and

\[
P_{HG}(\xi, \beta, \{\lambda\}) = \frac{1}{\beta} f(\beta + \xi/(4B), \{\lambda\}) \left[ 1 - \frac{1}{4B} \frac{\partial f(\beta + \xi/(4B), \{\lambda\})}{\partial \beta} \right]^{-1}.
\]

Though volume is no longer an active variable of the system it can be calculated for given baryon density and \( \nu_B \) (evaluated through (17)) and baryon number \(< B >\) which is a conserved quantity. The volume would be retrieved through the relation

\[
<V> = \frac{<B>}{\nu_B}.
\]

With the use of SB in order to describe interacting hadronic systems we can trace the possibility of a phase transition. The study of the pressure-volume isotherm curve is then necessary. When this curve is calculated a region of instability is revealed. In fact, this curve has a part (near the boundaries of the hadronic domain) where pressure decreases while volume decreases also (see Fig. 1). Such a behaviour is a signal of a first order phase transition which in turn can be mended with the use of a Maxwell construction.

This behaviour is due to the formation of bigger and bigger clusters as the system tends to its boundaries in the phase diagram. In that way the effective number of particles is reduced, resulting, thus, to a decrease of pressure. This is the basic mechanism that will produce a first order transition at lower temperatures and a critical point at finite density. To show that this instability in the \( P - V \) curve is the result of the attractive part of the interaction included in the SB we shall calculate a similar curve using the Ideal Hadron Gas (IHG) model with Van der Waals volume corrections (repulsive part of interaction). The logarithm of the partition function of IHG (corresponding to (10)) is

\[
\ln Z_{pt\,IHG}(V, \beta, \{\lambda\}) \equiv V f_{pt\,IHG}(\beta, \{\lambda\}) = \frac{V}{2\pi^2 \beta} \sum_a [\lambda_a(\{\lambda\}) + \lambda_a(\{\lambda\})^{-1}] \sum_i g_{ai}m_{ai}K_2(\beta m_{ai}),
\]

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where \( g_{ai} \) are degeneracy factors due to spin and the index \( a \) runs to all groups of hadrons described by the same set of fugacities. This function can be used in eq. (15) to calculate the Ideal Hadron Gas (IHG) pressure partition function in order to include Van der Waals volume corrections. The result is that the pressure is always found to increase as volume decreases, for constant temperature, exhibiting no region of instability and so no possibility of a phase transition.

The comparison of SB with the IHG (with volume corrections) is displayed in Fig. 1, where \( \nu_0 \) is the normal nuclear density \( \nu_0 = 0.14 \text{ fm}^{-3} \). In both cases (SB or IHG) the constraints \( < S > = 0 \) (zero strangeness) and \( < B > = 2 < Q > \) (isospin symmetric system, i.e. the net number of \( u \) and \( d \) quarks are equal) have been imposed. Also strangeness is fully equilibrated which accounts to setting \( \gamma_s = 1 \).

3. The partition function of Quark Matter

Having a description of the hadronic phase at hand, it is necessary to proceed with the thermodynamic behaviour of the quark-gluon phase. The QCD equation of state at finite temperature and baryon density calls for non-perturbative methods of approach. Lattice calculations have been performed but the power of such methods is limited by the sign and the overlap problems [21]. The overlap problem is treated with the reweighting method, called the Glasgow algorithm [22]. In [23] the method of imaginary chemical potential is used to solve the sign problem. In [24-26] various new methods have been used to tackle with the sign and/or the overlap problems. Especially in [26] the overlap problem is eliminated completely. In [27] lattice Taylor expansion is used around \( \mu_B = 0 \), allowing the exploration of the phase transition of QCD at the low density regime.

Lattice calculations of the pressure of the quark-gluon state have been performed at finite chemical potential in [4,6], using an improved reweighting technique. These publications include calculations for rather heavy \( u, d \) quark masses. The mass of the \( u, d \) quarks is about 65 MeV and the strange \( s \) quark 135 MeV [4,6]. The calculated pressure of the quark-gluon phase \( (P/T^4) \) at \( \mu_B = 0 \) is plotted against the ratio of temperature to the transition
temperature of quark matter at zero baryon chemical potential $T/T_c$ in Fig. 2 of [4]. The temperature $T_c$ will be denoted as $T_{0QGP}$ in the following. The results of this graph are extrapolated to the continuum limit by multiplying the raw lattice results with a factor $c_p = 0.518$ [4].

The lattice calculations for finite chemical potential are summarised in Fig. 3 of [4], where the difference of pressure at non-zero chemical potential and the pressure at zero chemical potential ($\Delta p/T^4 = [P(\mu \neq 0, T) - P(\mu = 0, T)]/T^4$) is plotted against $T/T_c$. Again the results of this graph are extrapolated to the continuum limit by multiplying the raw lattice results with a factor $c_\mu = 0.446$ [4]. In this graph five curves are plotted which correspond to baryon chemical potential of 100, 210, 330, 410, 530 MeV.

With the use of Figs. 2, 3 in [4], it is possible to calculate in principle the pressure of the quark-gluon phase at any temperature and baryon chemical potential. The pressure is important, because knowledge of the pressure is equivalent to the knowledge of the partition function of the system in the grand canonical ensemble

$$\ln Z_{QGP}(V, T, \mu_B) = V \frac{T}{T} P(T, \mu_B)$$  \hspace{1cm} (21)

In order to have a complete description of the dependence of the pressure on the temperature and the chemical potential we use two sets of fitting functions. For constant chemical potential the pressure as a function of $T/T_c$ is fitted through

$$f(x) = \frac{a_1}{x^{c_1} \left[ \exp \left( \frac{b_1}{x^{d_1}} \right) - 1 \right]^f_1} + \frac{a_2}{x^{c_2} \left[ \exp \left( \frac{b_2}{x^{d_2}} \right) - 1 \right]^f_2},$$  \hspace{1cm} (22)

where $a_i, b_i, c_i, d_i, f_i$ ($i = 1, 2$) depend on $\mu_B$, while for constant temperature the corresponding fit of the pressure as a function of $\mu_B$ is given by

$$g(x) = a + b \exp(c x^d),$$  \hspace{1cm} (23)

where $a, b, c, d$ depend on the temperature ratio $T/T_c$. The fitting procedure has to be performed in a self-consistent way and subsequently it is straightforward to evaluate the partition function, as well as its derivatives with respect to $\mu_B$ and $T$ at any given point $(T_1, \mu_B)$. In particular, to evaluate physical observables connected with the partition function and drive numerical routines the partial derivatives of the pressure up to second order
with respect to temperature and fugacity have to be evaluated. These derivatives are then
given in part A of the Appendix.

In Fig. 2 we have reproduced the quark-gluon pressure as a function of the temperature
for constant baryon chemical potential. The squares are points directly measured from the
graphs of Fodor et. al and the lines represent the calculation with the fits which has been
performed on these points, via eq. (22). Fig. 3 is a graph similar with Fig. 2, but we have
focused on the area which is useful for our calculations, the area where the matching with
the hadronic phase will be performed. Fig. 4 is a reproduction of the Fodor et. al quark-
gluon pressure as a function of the baryon chemical potential for constant temperature. The
necessary fits have been performed with the use of eq. (23).

4. The critical point in the phase diagram

After developing the necessary tools to handle the thermodynamic description for the
quark-gluon phase as it is produced from the lattice, we can search for the possibility of a
quark-hadron phase transition.

First we have to deal with the free parameters that exist in our models. In [4,6] where
the results of the pressure is presented the transition temperature of the quark state at zero
density $T_0$ (called from now on $T_{0\, QGP}$) is not fixed. In [28], however, the QCD critical
point is studied with quark mass input values closer to the physical ones and a zero-density
temperature $T_{0\, QGP} = 164 \pm 3$ MeV. Therefore in what follows, in order to limit the free
parameters existing in our scheme, we choose

$$T_{0\, QGP} = 164 \text{ MeV}. \quad (24)$$

As far as the hadronic phase is concerned, an upper bound for the parameter $T_{0\, HG}$ can
be fixed at the value 183 MeV. This temperature allows for the best matching of the strange
chemical potential $\mu_s$ between the hadronic and the QGP phase [15]. So we shall set

$$T_{0\, HG} \leq 183 \text{ MeV}. \quad (25)$$

The fact that $T_{0\, HG}$ and $T_{0\, QGP}$ acquire different values does not imply a contradiction. At
\( \mu_B = 0 \) the strongly interacting system belongs to the crossover regime where the quark and hadron phases are indistinguishable. Therefore, the zero-density parameters \((T_0_{QGP}, T_0_{HG})\) do not correspond physically to a distinct transition between the two phases; they simply define the zero-density intercept of the extrapolated critical line to small values of the chemical potential, beyond the critical endpoint of the system at nonzero density. As a result, in the region \((\mu_B = 0, \ 164 \text{ MeV} \leq T \leq 183 \text{ MeV})\) of the phase diagram, defined by the boundary values (24) and (25) a sharp thermodynamic separation between the bootstrap and QCD phase may not exist. Although the physics of the system in the crossover regime is not yet fully understood, it is a plausible assumption (implied by (24), (25)) that the two phases are distinguishable only in the domains: \( \mu_B = 0, \ T \ll 164 \text{ MeV} \) (hadrons) and \( T \gg 183 \text{ MeV} \) (quarks, gluons).

Turning now our attention to \( \xi \), we adopt (14) in order to have always a real pressure partition function. For simplicity we also choose to have \( \xi = \text{const.} \) for every set of \((T, \{\lambda\})\). This means that (14) has to be valid for every set of thermodynamic variables. Since the value of \( \xi \) at the thermodynamic limit, \( \xi_0 \), depends on the choice of these variables we have to locate the specific set that gives us the highest value of \( \xi_0 \). For this reason we calculate \( \xi_0 \) on isotherms for different values of \( \mu_B \) (fixing the remaining chemical potentials in order to fulfil the constraints \( <B> = 2 <Q> \) and \( <S> = 0 \)). It is found that for constant temperature \( \xi_0 \) rises as a function of \( \mu_B \). Then we calculate \( \xi_0 \) on the maximum value of \( \mu_B \) allowed for each temperature, i.e. on the critical curve (fixing accordingly the rest of the chemical potentials). It is found that on the bootstrap critical line \( \xi_0 \) rises as the temperature is increased. Thus the greatest value of \( \xi_0 \) corresponds to \( T = T_{0HG} \) and consequently \( \{\lambda\} = \{1\} \). So in order to have a real pressure partition function for a constant value of \( \xi \) all over the space of our thermodynamic variables it suffices to require

\[
\xi > \xi_0(T_{0HG}, \{\lambda\} = \{1\}) . \tag{26}
\]

Finally as a consistency requirement on the thermodynamics of lattice QCD [28,29] and bootstrap matter we impose the constraint

\[
T_{cr.p.} < T_{0_{QGP}} . \tag{27}
\]

Then, if the values for the free parameters are chosen, within the above constraints, one
may calculate for a specific temperature the pressure isotherms of Hadron Gas and QGP. Assuming that the baryon number is a conserved quantity to both phases, the equality of volumes is equivalent to the equality of baryon densities and so the connection of the isotherms of the two phases is possible through the relation

\[ < V_{HG} > = < V_{QGP} > \iff \frac{< B_{HG} >}{\nu_{B\ HG}} = \frac{< B_{QGP} >}{\nu_{B\ QGP}} \iff \nu_{B\ HG} = \nu_{B\ QGP} \tag{28} \]

The graph of the pressure-volume isotherm can be drawn using the plot of pressure against the inverse baryon density (see eq. (19)). Then at the point where the isotherms of the two phases meet we have equal volumes for equal pressures.

Tracing the point where the isotherms of two phases meet, we find that at a low temperature the intersection of QGP and SB pressure-volume isotherms takes place at a location where the Hadron Gas pressure is decreasing while volume decreases. The resulting pressure-volume curve includes an unstable part which has to be repaired through a suitable Maxwell construction. This curve includes a region where a first-order transition takes place.

As the temperature increases, there exists a value for which the QGP and SB isotherms meet at a point where the Hadron Gas pressure has a maximum. In that case no Maxwell construction is needed and since this point is located at finite volume or not zero baryon density (equivalently not zero chemical potential) it can be associated with the QCD critical point. As temperature rises more, the resulting pressure-volume isotherm also increases while volume decreases without the need of a Maxwell construction and the situation belongs to the crossover region.

A graph that summarises the situations met in the pressure volume isotherms of hadronic and quark systems in the neighbourhood of the critical point is Fig. 5. In this figure the hadronic isotherms have been calculated for parameters that fulfil the constraints (25)-(27) \((T_{0\ HG} = 172\ MeV, \ MIT\ bag\ constant\ \mathcal{B}^{1/4} \simeq 136\ MeV, \ \xi = 177.1 \cdot 10^4\ MeV^3)\) while the quark-gluon isotherms are related to \(T_{0\ QGP} = 164\ MeV\). For the lower temperature isotherm a Maxwell construction is needed to remove the instability from the resulting curve. The horizontal line defines a partition with two equal surfaces (shaded) and represents the final pressure-volume curve after the completion of the Maxwell construction. At the temperature \(T = 162.06\ MeV\) the two curves meet at the point of maximum hadronic pressure and so
a critical point is formed at finite volume. At a greater temperature the plotted pressure-volume isotherm corresponds to the crossover.

To locate the critical point numerically with the use of the lattice partition function, for given parameters $\xi, T_{0\ HG}$ and $T_{0\ QGP}$, the conditions have to be determined for which the SB pressure is equal to the QGP pressure at the same volume, corresponding to the maximum SB pressure. Setting the factor of partial strangeness equilibrium $\gamma_s = 1$ a hadronic state is characterised by the set of thermodynamic variables $(T, \lambda_u, \lambda_d, \lambda_s)$, while a quark-gluon state evaluated on the lattice [4] is characterised by the two variables $(T, \lambda'_q)$. The $u$ and $d$ quarks are characterised by the same fugacity $\lambda'_u = \lambda'_d = \lambda'_q = \lambda_B^{n/3}$. To evaluate the unknown variables we have to solve the following set of non-linear equations

\begin{align}
\nu_{SB}(T, \lambda_u, \lambda_d, \lambda_s) &= \nu_{QGP}(T, \lambda'_q) \tag{29} \\
P_{SB}(T, \lambda_u, \lambda_d, \lambda_s) &= P_{QGP}(T, \lambda'_q) \tag{30} \\
\frac{dP_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s)}{dV} &= 0 \tag{31} \\
\langle S \rangle_{SB}(T, \lambda_u, \lambda_d, \lambda_s) &= 0 \tag{32} \\
\langle B \rangle_{SB}(T, \lambda_u, \lambda_d, \lambda_s) - 2 \langle Q \rangle_{SB}(T, \lambda_u, \lambda_d, \lambda_s) &= 0 \tag{33}
\end{align}

Eq. (29) accounts for the equality of the baryon densities of the two phases which is equivalent to the equality of volumes, since the baryon number is a conserved quantity. Eq. (30) is the equality of pressures of Hadron Gas and QGP. Eqs. (29), (30) determine the point where the two pressure curves meet. Eq. (31) requires that the meeting point of the two phases for a certain temperature is equal to the point which maximizes the Hadron Gas pressure and so this meeting point is the critical point. The form of this equation is discussed in detail in the Appendix. Eq. (32) imposes strangeness neutrality in the hadronic phase. Eq. (33) imposes isospin symmetry to the hadronic system, demanded in order to compare with the lattice studies.

The area in the $(T, \mu_B)$ plane which gives solutions for the critical point compatible with the constraints (24)-(27) is depicted in Fig. 6. The line $\xi = \xi_{0\ max}$ represents solutions for the critical point with the requirement that $\xi$ is set to the thermodynamic limit value at $T = T_{0\ HG}$. Since the value of $\xi$ at thermodynamic limit is calculated through (16), we first
determine $\xi$ and the MIT bag constant $B$, for given $T_{0\,HG}$, by solving the following two equations

$$\xi = f\left(1/T_{0\,HG} + \xi/4B, \{\lambda\} = \{1\}; B\right). \quad (34)$$

$$\varphi\left(1/T_{0\,HG} + \xi/4B, \{\lambda\} = \{1\}; B\right) = \ln 4 - 1. \quad (35)$$

The last equation determines the critical hadronic curve at zero density. We, then, insert the calculated values of $\xi$ and $B$, for the given value of $T_{0\,HG}$, in the system of eqs. (29)-(33), which is solved to extract the thermodynamic variables that locate the critical point on the phase diagram. All the points on the right of the $\xi = \xi_{0\,\text{max}}$ curve on the $T - \mu_B$ plane are compatible with the requirement (26).

The compatible area is also limited by the curve of constant $T_{0\,HG}$; according to (25). Setting $T_{0\,HG}$ and choosing a value of $\xi$ that fulfils (26) we first solve (35) to determine $B$. Then we solve the system (29)-(33) to draw the line of constant $T_{0\,HG}$.

In Fig. 6 we have also drawn the line $T = 164$ MeV, which excludes solutions that do not fulfil (27). The shaded area represents all the points that qualify as a solution for the critical point, compatible with eqs. (24)-(27).

Recent lattice QCD studies offer, apart from the quark-gluon pressure which has been a basic ingredient in our approach, important results on the existence and location of the critical point itself. In [29] the critical point is found to reside at $T_{cr,p.} = 160 \pm 3.5$ MeV and $\mu_B = 725 \pm 35$ MeV, with $T_{0\,QGP} = 172 \pm 3$ MeV. These calculations have the drawback that have been performed with $u$ and $d$ quark mass which has a value about four times the physical value. Improved calculations have been performed in [28], where the light quark masses have decreased by a factor of 3 down their physical values. The critical point is found now (with $T_{0\,QGP} = 164 \pm 3$ MeV) to be at $T_{cr,p.} = 162 \pm 2$ MeV and $\mu_B = 360 \pm 40$ MeV, a value which is considerably reduced with respect to the previous one. This point is depicted on Fig. 6 with the full star and it falls completely inside the compatible domain of the critical point, according to our calculations. In [27] the critical point has also been determined using Taylor expansion around $\mu_B = 0$ of the 3-flavour QCD equation of state. The critical point is found to reside at $\mu_B = 420$ MeV. In [30] a method is exhibited about how to locate the critical point using imaginary values of $\mu_B$. This study also shows the sensitivity of the critical point on the strange quark mass.
In Figs. 7-8 we illustrate a representative solution for the critical point as well as for

the QGP-hadron transition line from those which are included in the shaded area of Fig. 6

on the \((T, \mu_B)\) plane. The chosen critical point is associated with \(T_{0\,HG} = 172\) MeV, while

the rest of the parameters are those of Fig. 5. It is located at \(T_{\text{cr.p.}} = 162.1\) MeV and

\(\mu_B = 218.7\) MeV. The full circle represents our solution for the critical point. This circle is

the endpoint of the solid thick line representing the bootstrap calculation of the maximum

hadronic pressure which is close to the first order critical line.

In Fig. 7 we also compare our solutions for the critical point with other calculations. The

full stars represent solutions of lattice QCD. The points depicted as “lattice reweighting I and

II” are the points from [29] and [28] respectively and the one depicted as “Taylor expansion”

is taken from [27]. The solid thin line is the first order transition line evaluated in [29]

with large \(u, d\) quark masses. The dotted line represents the crossover in this calculation.

The two slashed lines represent the magnitude of the slope \(d^2T/d\mu^2\) taken from lattice

Taylor expansion and which indicate the phase transition line according to [27]. Theoretical

predictions on the location of the critical point from various models [3,31-36] are depicted

as open squares in Fig. 7. The labels adopted are those of Table I in [21]. The open circle

in Fig. 7 is the prediction of the critical point within the statistical bootstrap \((\xi = 0)\) from

[8], where a simplified partition function (MIT bag) for the quark phase was used.

In Figs. 6 and 8 we compare our solutions for the critical point with the freeze-out

points from different experiments. In Fig. 6 we depict freeze-out points from NA49 at

158 AGeV [37] for systems of different size \((C + C, Si + Si, Pb + Pb)\) that fall inside

the compatible domain of the critical point, if the errors in their determination are taken

into account. These experiments are interesting since they could trace critical fluctuations

associated with a critical point of second order [44]. In Fig. 8 the comparison is extended to

a larger number of experiments [38-43]. On the same graph we have also depicted the curve

of \(<E>/<N>=1\) GeV [45] (reproduced from [46]) that fits freeze-out points which

are spread to a wide region of the phase diagram. It is evident that our calculations set the

critical point to a location easily accessible by experiments, especially by CERN/SPS.
5. Conclusions

Statistical Bootstrap presents a more accurate description of the hadronic phase than the ideal Hadron Gas, since it includes in a self consistent way the attractive part of the interaction among hadrons through the mass spectrum, as well as the repulsive part through Van der Waals volume corrections. This interaction is crucial to investigate critical phenomena in connection with the state of quark-gluon plasma. Among the predictions of the bootstrap model is the limitation of the hadronic phase and the forming of an instability in the pressure-volume isotherm near the hadronic boundaries. This instability is connected to a first order quark-hadron phase transition and the existence of a critical endpoint in the strongly interacting matter.

A more realistic pressure diagram of the quark-gluon phase is available from lattice calculations, despite the fact that unphysical values of the light quark masses are still involved in these studies. From these results the lattice partition function of the quark state, as well as, all the necessary derivatives can be calculated, allowing the evaluation of any physical observable.

The joining of the SB and the lattice partition function for the hadronic and the quark state respectively, allows for the determination of a critical point at finite baryon chemical potential which can be related to the critical point of QCD.

More recent lattice calculations [28] drive the position of the critical point to smaller values of baryon chemical potential as the values of \( u \), \( d \) quark masses approach their physical values. It is interesting that the current location is situated in the \((T, \mu_B)\) plane in a region easily accessible by the freeze-out conditions of experiments at the CERN/SPS.

Setting the free parameters in our model in a way to fulfil certain constraints we are left with a compatible domain in the \((T, \mu_B)\) plane for the location of the critical point. Recent lattice calculations [28] drive the critical point within the domain of our solutions.

In a previous work [9,8] a similar solution was found with the use of a simplified partition function for the quark-gluon system, based on the MIT bag model. Therefore, the basic mechanism in our approach for the formation of a critical point in the strongly interacting matter is not associated with the details of the quark-gluon partition function but mainly with the instability of hadronic matter revealed by Statistical Bootstrap. In particular the
local maximum of pressure in the $P - V$ diagram (Fig. 5) of hot hadronic matter lies in the origin of the formation mechanism of the critical endpoint.

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**Appendix**

**A.** The fitting procedures on curves of constant temperature and chemical potential allow us to evaluate the derivatives with respect to $T$ for constant $\mu_B$ and with respect to $\mu_B$ for constant $T$. Physical observables, however, are given as derivatives with respect to temperature for constant fugacity or with respect to $\lambda_B$ for constant $T$. The evaluation of the latter (for the pressure) is given by

$$
\frac{\partial P}{\partial \lambda_B} \bigg|_T = \frac{\partial P}{\partial \mu_B} \bigg|_T \frac{T}{\lambda_B},
$$

(A.1)

$$
\frac{\partial P}{\partial T} \bigg|_{\lambda_B} = \frac{\partial P}{\partial \mu_B} \bigg|_{\mu_B} + \frac{\partial P}{\partial \lambda_B} \bigg|_T \frac{\lambda_B \mu_B}{T^2} = \frac{\partial P}{\partial \mu_B} \bigg|_{\mu_B} + \frac{\partial P}{\partial \mu_B} \bigg|_T \frac{\mu_B}{T}.
$$

(A.2)

$$
\frac{\partial^2 P}{\partial \lambda_B^2} \bigg|_T = -\frac{\partial P}{\partial \mu_B} \bigg|_{\mu_B} \frac{\lambda_B}{{T^2}} + \frac{\partial^2 P}{\partial \mu_B \partial T} \bigg|_T \frac{T^2}{\lambda_B^2},
$$

(A.3)

$$
\frac{\partial^2 P}{\partial T^2} \bigg|_{\lambda_B} = \frac{\partial^2 P}{\partial \mu_B^2} \bigg|_{\mu_B} + 2 \frac{\partial \mu_B}{\partial \lambda_B} \bigg|_T \frac{\mu_B^2}{T^2} + \frac{\partial^2 P}{\partial \lambda_B \partial T} \bigg|_T \frac{\mu_B}{T^2}.
$$

(A.4)

$$
\frac{\partial^2 P}{\partial \lambda_B \partial T} \bigg|_T = \frac{\partial^2 P}{\partial \mu_B \partial \lambda_B} \bigg|_B + \frac{\partial^2 P}{\partial \mu_B \partial T} \bigg|_T \frac{\mu_B}{\lambda_B} + \frac{\partial P}{\partial \mu_B} \bigg|_T \frac{1}{\lambda_B}.
$$

(A.5)

In eqs. (A.4), (A.5) where the 2nd partial derivative with respect to two different variables of $P$ appears, the pressure is considered as a function of these two variables.

**B.** After choosing specific values for the parameters $\xi$ and $B$, the requirement of eq. (31) has to be fulfilled for a certain temperature $T = T_1$ and in the presence of two constraints $g_1 \equiv \langle S \rangle_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s) = 0$ and $g_2 \equiv \langle B \rangle_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s) - 2 \langle Q \rangle_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s) = 0$.
0. Since the maximum of pressure is found for a certain isotherm, the temperature \( T_1 \) may not be considered in the following as an active variable. The same is true for \( \xi \) and \( B \), since the maximum pressure is evaluated for constant values of these parameters. With the above considerations and eq. (19) we may write

\[
\frac{dP_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s)}{dV} = 0 \Rightarrow -\frac{<B>}{\nu_B^2} \frac{dP_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s)}{d\nu_B} = 0 \Rightarrow \\
\frac{dP_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s)}{d\nu_B} = 0 \Rightarrow dP_{SB}(T_1, \lambda_u, \lambda_d, \lambda_s) = 0 \Rightarrow \\
\frac{\partial P_{SB}}{\partial \lambda_u} d\lambda_u + \frac{\partial P_{SB}}{\partial \lambda_d} d\lambda_d + \frac{\partial P_{SB}}{\partial \lambda_s} d\lambda_s = 0 \Rightarrow \frac{\partial P_{SB}}{\partial \lambda_u} \frac{\partial \lambda_u}{\partial \lambda_d} + \frac{\partial P_{SB}}{\partial \lambda_d} \frac{d\lambda_d}{d\lambda_u} + \frac{\partial P_{SB}}{\partial \lambda_s} \frac{d\lambda_s}{d\lambda_u} = 0 
\]

(B.1)

As far the constraint \( g_1 \) is concerned, we have

\[
g_1(T_1, \lambda_u, \lambda_d, \lambda_s) = 0 \Rightarrow dg_1 = 0 \Rightarrow \frac{\partial g_1}{\partial \lambda_u} d\lambda_u + \frac{\partial g_1}{\partial \lambda_d} d\lambda_d + \frac{\partial g_1}{\partial \lambda_s} d\lambda_s = 0 \Rightarrow \\
\frac{\partial g_1}{\partial \lambda_u} \frac{d\lambda_u}{d\lambda_d} + \frac{\partial g_1}{\partial \lambda_d} \frac{d\lambda_d}{d\lambda_u} = -\frac{\partial g_1}{\partial \lambda_u} 
\]

(B.2)

Similarly for the constraint \( g_2 \) we have

\[
\frac{\partial g_2}{\partial \lambda_u} \frac{d\lambda_u}{d\lambda_d} + \frac{\partial g_2}{\partial \lambda_s} \frac{d\lambda_s}{d\lambda_u} = -\frac{\partial g_2}{\partial \lambda_u} 
\]

(B.3)

Eqs. (B.2) and (B.3) may considered as a system of two equations which can be solved to determine \( d\lambda_d/d\lambda_u \) and \( d\lambda_s/d\lambda_u \)

\[
\frac{d\lambda_d}{d\lambda_u} = \frac{1}{D} \left( \frac{\partial g_1}{\partial \lambda_u} \frac{\partial g_2}{\partial \lambda_d} - \frac{\partial g_2}{\partial \lambda_u} \frac{\partial g_1}{\partial \lambda_d} \right), \quad (B.4) \\
\frac{d\lambda_s}{d\lambda_u} = \frac{1}{D} \left( \frac{\partial g_1}{\partial \lambda_u} \frac{\partial g_2}{\partial \lambda_s} - \frac{\partial g_2}{\partial \lambda_u} \frac{\partial g_1}{\partial \lambda_s} \right), \quad (B.5)
\]

with

\[
D = \frac{\partial g_1}{\partial \lambda_d} \frac{\partial g_2}{\partial \lambda_s} - \frac{\partial g_2}{\partial \lambda_d} \frac{\partial g_1}{\partial \lambda_s}. \quad (B.6)
\]

Eqs. (B.4) and (B.5) may now be inserted to eq. (B.1) to give

\[
\frac{\partial P_{SB}}{\partial \lambda_u} + \frac{\partial P_{SB}}{\partial \lambda_d} \left( \frac{\partial g_1}{\partial \lambda_u} \frac{\partial g_2}{\partial \lambda_d} - \frac{\partial g_2}{\partial \lambda_u} \frac{\partial g_1}{\partial \lambda_d} \right) \frac{1}{D} + \frac{\partial P_{SB}}{\partial \lambda_s} \left( \frac{\partial g_1}{\partial \lambda_u} \frac{\partial g_2}{\partial \lambda_s} - \frac{\partial g_2}{\partial \lambda_u} \frac{\partial g_1}{\partial \lambda_s} \right) \frac{1}{D} = 0 \quad (B.7)
\]

Eq. (B.7) is the form of eq. (31) for the maximum hadron pressure in a certain isotherm. The main contribution comes from the term \( \partial P_{SB}/\partial \lambda_u \), as it is verified by comparing the maximum pressure found in the present work using the full equation (B.7) with the solution found using only the first term in [9,8].
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Figure Captions

**Fig. 1** Isotherm pressure-volume curve for SB and IHG (both with Van der Waals volume corrections using the pressure ensemble with the same value of $\xi$). The SB curve is drawn for $T_{0 \ HG} = 172$ MeV ($B^{1/4} \simeq 136$ MeV).

**Fig. 2** The pressure of the quark-gluon state divided by $T^4$ versus the ratio $T/T_{\ QGP}$, for constant baryon chemical potential. The lines from bottom to top correspond to gradually increasing values of $\mu_B$. The squares represent direct measurement from the Figs. 2, 3 in [4] which depict the lattice calculation and the lines indicate our fits to these points.

**Fig. 3** Similar graph with Fig. 2. The pressure of the quark-gluon state is divided by $T_{0 \ QGP}^{4}$. The plot focuses on the region where the matching with the hadronic state will take place.
**Fig. 4** The pressure of the quark-gluon state divided by $T_{0, QGP}^4$ versus the baryon chemical potential, for constant values of the ratio $T/T_{0, QGP}$. The squares represent direct measurement from Figs. 2, 3 in [4] which depict the lattice calculation. The lines indicate our fits to these points.

**Fig. 5** Three isotherm pressure-volume curves for Hadron Gas (using SB) and QGP phase (using the lattice pressure of [4]). The low temperature isotherm needs Maxwell construction, the middle temperature isotherm develops a critical point and the high temperature isotherm corresponds to crossover.

**Fig. 6** Domain in the $(T, \mu_B)$ plane (shaded area) which is compatible with solutions for the position of the critical point, after imposing the constraints (24)-(27) on the free parameters. There are, also, displayed the freeze-out points from NA49 at 158 AGeV [37] and the lattice calculated critical point from [28].

**Fig. 7** The critical point domain of Fig. 6 and a representative solution of the critical point (full circle-“SB/II”) at the $(T, \mu_B)$ plane, as well as the first order part of the quark-hadron transition (solid thick line). The stars represent the lattice calculated critical points [27,28,29]. The line labelled “lat. rew.” represents the phase boundaries from [29] and the lines labelled “Tayl. exp.” represent the phase boundaries from [27]. The open squares are theoretical calculations for the critical point [3,31-36]. The open circle (“SB/I”) is the calculation of critical point using bootstrap model in [8].

**Fig. 8** Comparison of the positions of our critical point solutions presented in Figs. 6-7, with the freeze-out points from different experiments [38-43]. It is, also, displayed, with slashed line the $< E > / < N > = 1$ GeV freeze-out curve of [45], reproduced from [46].
$B^{1/4} = 135.9$ MeV
$T_{0HG} = 172$ MeV
$\xi = 177.1 \times 10^4$ MeV$^3$
$T = 162.06$ MeV
$\mu_B=0, 100, 210, 330, 410, 530$ MeV

$P_{QGP}/T_0$ vs $T/T_0$ QGP
\[ B^{1/4} = 135.9 \text{ MeV} \]
\[ T_0_{\text{HG}} = 172 \text{ MeV} \]
\[ \xi = 177.1 \times 10^4 \text{ MeV}^3 \]
\[ T_0_{\text{QGP}} = 164 \text{ MeV} \]
