On the energy current for harmonic crystals

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Abstract

We consider a $d$-dimensional harmonic crystal, $d \geq 1$, and study the Cauchy problem with random initial data. We assume that the random initial function is close to different translation-invariant processes for large values of $x_1, \ldots, x_k$ with some $k \in \{1, \ldots, d\}$. The distribution $\mu_t$ of the solution at time $t \in \mathbb{R}$ is studied. We prove the convergence of correlation functions of the measures $\mu_t$ to a limit for large times. The explicit formulas for the limiting correlation functions and for the energy current density (in mean) are obtained in the terms of the initial covariance. We give the application to the case of the Gibbs initial measures with different temperatures. In particular, we find stationary states in which there is a constant non-zero energy current flowing through the harmonic crystal. Furthermore, the weak convergence of $\mu_t$ to a limit measure is proved. We also study the initial boundary value problem for the harmonic crystal with zero boundary condition and obtain the similar results.

Key words and phrases: harmonic crystal, Cauchy problem, initial boundary value problem, random initial data, weak convergence of measures, mixing condition, correlation matrices, Gibbs measures, energy current density, Second Law

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1 Introduction

We study the Cauchy problem for a harmonic crystal in $d$ dimensions with $n$ components, $d, n \geq 1$. We assume that the initial datum $Y_0(x)$, $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, of the problem is a random element of the Hilbert space $\mathcal{H}_\alpha$ consisting of real sequences, see Definition 2.1 below. The distribution of $Y_0(x)$ is a probability measure $\mu_0$ with zero mean value. We assume that the covariance $Q_0(x, y)$ of $\mu_0$ decreases like $|x - y|^{-N}$ as $|x - y| \to \infty$ with some $N > d$. Furthermore, we impose the condition S3 (see formulas (2.12)–(2.14) below) which means roughly that $Y_0(x)$ is close to different translation-invariant processes $Y_n(x)$ with distributions $\mu_n$ as $(-1)^{n_j}x_j \to +\infty$ for all $j = 1, \ldots, k$, with some $k \in \{1, \ldots, d\}$. Here $n$ stands for the vector $n = (n_1, \ldots, n_k)$, where all $n_j \in \{1, 2\}$. Given $t \in \mathbb{R}$, denote by $\mu_t$ the probability measure that gives the distribution of the solution $Y(x, t)$ to dynamical equations with the random initial datum $Y_0$. We study the asymptotics of $\mu_t$ as $t \to \infty$. The first objective is to prove the convergence of the correlation functions of $\mu_t$ to a limit,

$$Q_t(x, y) \equiv \int_{\mathcal{H}_\alpha} (Y_0(x) \otimes Y_0(y)) \mu_t(dY_0) \to Q_\infty(x, y), \quad t \to \infty, \quad x, y \in \mathbb{Z}^d. \quad (1.1)$$

The explicit formulas for the limit covariance $Q_\infty$ are given in (2.17)–(2.22). They allow us to derive the expression for the limiting mean energy current density $\mathbf{J}_\infty$ in the terms of the initial covariance $Q_0(x, y)$.

We apply our results to a particular case when $\mu_n$ are Gibbs measures with different temperatures $T_n > 0$. Therefore, our model can be considered as a “system + 2$^k$ reservoirs”, where “reservoirs” consist of the crystal particles lying in $2^k$ regions of a form $\{x \in \mathbb{Z}^d : (-1)^{n_j}x_j > a\}$ for all $j = 1, \ldots, k$, where $n_j = 1$ or 2} with some $a > 0$, and the “system” is the remaining part of the crystal. At $t = 0$, the reservoirs have Gibbs distributions with corresponding temperatures $T_n$, $n = (n_1, \ldots, n_k)$. (In the case of $d = 1$, the similar model was studied by Spohn and Lebowitz [23].) We show that the energy current density $\mathbf{J}_\infty$ is a constant vector satisfying formulas (1.4) and (1.5). Furthermore, under additional symmetry conditions on the harmonic crystal, the coordinates of the energy current $\mathbf{J}_\infty \equiv (J_{\infty}^1, \ldots, J_{\infty}^d)$ are of a form

$$J_{\infty}^l = \begin{cases} 
-c_l \sum' \left( T_n \big|_{n_l=2} - T_n \big|_{n_l=1} \right) & \text{for } l = 1, \ldots, k, \\
0 & \text{for } l = k + 1, \ldots, d,
\end{cases} \quad (1.2)$$

with some constants $c_l > 0$. Here the summation $\sum'$ is taken over all $n_j$ with $j \neq l$.

Our second result gives the (weak) convergence of the measures $\mu_t$ on the Hilbert space $\mathcal{H}_\alpha$ with $\alpha < -d/2$ to a limit measure $\mu_\infty$,

$$\mu_t \to \mu_\infty, \quad t \to \infty. \quad (1.3)$$

This means the convergence of the integrals

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as } t \to \infty,$$

for any bounded continuous functional $f$ on $\mathcal{H}_\alpha$. Furthermore, the limit measure $\mu_\infty$ is a translation-invariant Gaussian measure on $\mathcal{H}_\alpha$ and has the mixing property.
For infinite one-dimensional (1D) chains of harmonic oscillators, similar results have been established by Boldrighini, Pellegrinotti and Triolo \cite{1} and by Spohn and Lebowitz \cite{23}. In earlier investigations, Lebowitz et al. \cite{22,4}, Nakazawa \cite{19} analyzed the stationary energy current through the finite 1D chain of harmonic oscillators in contact with external heat reservoirs at different temperatures. For \(d \geq 1\), the convergence \(1.3\) has been obtained for the first time by Lanford and Lebowitz \cite{18} for initial measures which are absolutely continuous with respect to the canonical Gaussian measure. We consider more general class of initial measures with the mixing condition and do not assume the absolute continuity. For the first time the mixing condition has been introduced by Dobrushin and Suhov for the ideal gas \cite{6}. Using the mixing condition, we have proved the convergence for the wave and Klein–Gordon equations (see \cite{9} and references therein) for non translation invariant initial measures \(\mu_0\). For many-dimensional crystals, the results \(1.1\) and \(1.3\) were obtained in \cite{7} for translation invariant measures \(\mu_0\). The present paper develops our previous work \cite{8}, where \(1.1\)–\(1.3\) were proved in the case of \(k = 1\).

In this paper, we also study the initial boundary value problem for the harmonic crystal in the half-space \(\mathbb{Z}_+^d = \{x \in \mathbb{Z}^d : x_1 \geq 0\}\) with zero boundary condition (as \(x_1 = 0\)) and obtain the results similar to \(1.1\) and \(1.3\). This generalizes the results of \cite{10} on the more general class of the initial measures. For this model, we calculate the limiting energy current density \(J_{+,\infty}(x_1)\), see formulas \(6.15\)–\(6.18\). In particular, if \(d = 1\), then \(J_{+,\infty}(0) = 0\). For \(d \geq 2\) and \(x_1 > 0\), the coordinates of \(J_{+,\infty}(x_1)\) have a form similar to \(1.2\), but with positive functions \(c_l = c_l(x_1)\) if \(l = 2, \ldots, k\), and vanish if \(l = 1, k+1, \ldots, d\). Furthermore, \(J_{+,\infty}(x_1)\) tends to a limit as \(x_1 \to +\infty\) (see formula \(6.19\)). For the 1D infinite chain of harmonic oscillators on the half-line with non-zero boundary condition, we prove the results \(1.1\) and \(1.3\) in \cite{11} and show that there is a negative limiting energy current at origin (see \cite{11} Remark 2.11).

There are a large literature devoted to the study of return to equilibrium, convergence to non-equilibrium states and heat conduction for nonlinear systems, see \cite{2,25} for an extensive list of references. For instance, ergodic properties and long time behavior were studied for weak perturbation of the infinite chain of harmonic oscillators as a model of 1D harmonic crystals with defects by Fidaleo and Liverani \cite{14} and for the finite chain of anharmonic oscillators coupled to a single heat bath by Jakšić and Pillet \cite{15}. A finite chain of nonlinear oscillators coupled to two heat reservoirs has been studied by Eckmann, Rey-Bellet and others \cite{12,13,21}. For such system the existence of non-equilibrium states and convergence to them have been investigated in \cite{12,21}. In \cite{13}, Eckmann, Pillet, and Rey-Bellet showed that heat (in mean) flows from the hot reservoir to the cold one. Fourier’s law for a harmonic crystal with stochastic reservoirs was proved by Bonetto, Lebowitz and Lukkarinen \cite{3}. In the present paper, we find stationary non-equilibrium states in which there is a non-zero energy current flowing through the infinite \(d\)-dimensional harmonic crystal.

The paper is organized as follows. In Sec. 2, we impose the conditions on the model and on the initial measures \(\mu_0\) and state the main results. In Sec. 3, we construct examples of random initial data satisfying all assumptions imposed. The application to Gibbs initial measures and the derivation of the formula \(1.2\) are given in Sec. 4. In Sec. 5.1, the uniform bounds for covariance of \(\mu_k\) are obtained, and the proof of \(1.3\) is discussed. The asymptotics \(1.1\) is proved in Sec. 5.2. In Sec. 6, we study the initial-boundary value problem for harmonic crystals in the half-space and prove the results similar to \(1.1\)–\(1.3\).
2 Main results

2.1 Model

We consider a Bravais lattice in $\mathbb{R}^d$ with a unit cell which contains a finite number of atoms. For notational simplicity, the lattice is assumed to be simple hypercubic. Let $u(x)$ be the field of displacements of the crystal atoms in cell $x$ ($x \in \mathbb{Z}^d$) from the equilibrium position. In the harmonic approximation, the field $u(x)$ is governed by the equations of a type (see, e.g., [20, 18])

\[
\begin{cases}
\ddot{u}(x,t) = -\sum_{y \in \mathbb{Z}^d} V(x-y)u(y,t), \quad x \in \mathbb{Z}^d, \quad t \in \mathbb{R}, \\
u_{t=0} = u_0(x), \quad \dot{u}_{t=0} = v_0(x).
\end{cases}
\]

(2.1)

Here $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$, $u_0 = (u_{01}, \ldots, u_{0n})$, $v_0 = (v_{01}, \ldots, v_{0n}) \in \mathbb{R}^n$, $V(x)$ is the real interaction (or force) matrix, $(V_{kl}(x))$, $k, l = 1, \ldots, n$. Physically $n = d \times$ (number of atoms in the unit cell). Here we take $n$ to be an arbitrary positive integer. The dynamics (2.1) is invariant under lattice translations.

Let us denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (Y^0_0, Y^1_0) \equiv (u_0(\cdot), v_0(\cdot))$. Then (2.1) takes the form of an evolution equation

\[
\dot{Y}(t) = A(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0.
\]

(2.2)

Formally, this is a linear Hamiltonian system, since

\[
A(Y) = J \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} Y = J \nabla H(Y), \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

(2.3)

Here $V$ is a convolution operator with the matrix kernel $V$, $I$ is unit matrix, and $H$ is the Hamiltonian functional

\[
H(Y) := \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle u, Vu \rangle, \quad Y = (u, v),
\]

(2.4)

where the kinetic energy is given by $(1/2)\langle v, v \rangle = (1/2) \sum_{x \in \mathbb{Z}^d} |v(x)|^2$ and the potential energy by $(1/2)\langle u, Vu \rangle = (1/2) \sum_{x,y \in \mathbb{Z}^d} \left( u(x), V(x-y)u(y) \right)$, $(\cdot, \cdot)$ stands for the real scalar product in the Euclidean space $\mathbb{R}^n$ (or in $\mathbb{R}^d$).

We assume that the initial datum $Y_0$ belongs to the phase space $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$.

**Definition 2.1** $\mathcal{H}_\alpha$ is the Hilbert space of pairs $Y \equiv (u(x), v(x))$ of $\mathbb{R}^n$-valued functions of $x \in \mathbb{Z}^d$ endowed with the norm

\[
\|Y\|^2_\alpha \equiv \sum_{x \in \mathbb{Z}^d} \langle x \rangle^{2\alpha} \left( |u(x)|^2 + |v(x)|^2 \right) < \infty, \quad \langle x \rangle := \sqrt{1 + |x|^2}.
\]

(2.5)

We impose the following conditions $E1$–$E6$ on the matrix $V$.

**E1** There exist positive constants $C$ and $\gamma$ such that $\|V(x)\| \leq Ce^{-\gamma|x|}$ for $x \in \mathbb{Z}^d$, $\|V(x)\|$ denoting the matrix norm.
Let \( \hat{V}(\theta) \) be the Fourier transform of \( V(x) \), with the convention
\[
\hat{V}(\theta) = F_{x \to \theta}[V(x)] = \sum_{x \in \mathbb{Z}^d} e^{i(x, \theta)} V(x), \quad \theta \in \mathbb{T}^d,
\]
where \( \mathbb{T}^d \) denotes the \( d \)-torus \( \mathbb{R}^d/(2\pi \mathbb{Z})^d \).

**E2** \( V \) is real and symmetric, i.e., \( V_{lk}(-x) = V_{kl}(x) \in \mathbb{R}, \ k, l = 1, \ldots, n, \ x \in \mathbb{Z}^d. \)

The conditions **E1** and **E2** imply that \( \hat{V}(\theta) \) is a real-analytic Hermitian matrix-valued function in \( \theta \in \mathbb{T}^d. \)

**E3** The matrix \( \hat{V}(\theta) \) is non-negative definite for every \( \theta \in \mathbb{T}^d. \)

Let us define the Hermitian non-negative definite matrix,
\[
\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0.
\]
(2.6)

\( \Omega(\theta) \) has the eigenvalues (“dispersion relations”) \( 0 \leq \omega_1(\theta) < \omega_2(\theta) < \ldots < \omega_s(\theta), \ s \leq n, \) and the corresponding spectral projections \( \Pi_\sigma(\theta) \) with multiplicity \( r_\sigma = \text{tr} \Pi_\sigma(\theta). \)

**Lemma 2.2** (see [7, Lemma 2.2]). Let the conditions **E1** and **E2** be fulfilled. Then there exists a closed subset \( C_* \subset \mathbb{T}^d \) of the zero Lebesgue measure such that the following assertions hold. (i) For any point \( \Theta \in \mathbb{T}^d \setminus C_* \), there exists a neighborhood \( \mathcal{O}(\Theta) \) such that each band function \( \omega_\sigma(\theta) \) can be chosen as the real-analytic function in \( \mathcal{O}(\Theta). \) (ii) The eigenvalue \( \omega_\sigma(\theta) \) has constant multiplicity in \( \mathbb{T}^d \setminus C_* \). (iii) The spectral decomposition holds,
\[
\Omega(\theta) = \sum_{\sigma=1}^s \omega_\sigma(\theta) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_*,
\]
(2.7)

where \( \Pi_\sigma(\theta) \) is the orthogonal projection in \( \mathbb{R}^n. \) \( \Pi_\sigma \) is a real-analytic function on \( \mathbb{T}^d \setminus C_* \).

Below we suggest that \( \omega_\sigma(\theta) \) denote the local real-analytic functions from Lemma 2.2 (i).

The next condition on \( V \) is the following:

**E4** For each \( l = 1, \ldots, d \) and \( \sigma = 1, \ldots, s, \) \( \partial_\theta \omega_\sigma(\theta) \) does not vanish identically on \( \mathbb{T}^d \setminus C_* \).

To prove the convergence (1.3), we need a stronger condition **E4’**.

**E4’** For each \( \sigma = 1, \ldots, s, \) the determinant of the matrix of second partial derivatives of \( \omega_\sigma(\theta) \) does not vanish identically on \( \mathbb{T}^d \setminus C_* \).

Write
\[
C_0 = \{\theta \in \mathbb{T}^d : \det \hat{V}(\theta) = 0\}, \ C_\sigma = \bigcup_{l=1}^d \{\theta \in \mathbb{T}^d \setminus C_* : \partial_\theta \omega_\sigma(\theta) = 0\}, \ \sigma = 1, \ldots, s.
\]
(2.8)

Then the Lebesgue measure of \( C_\sigma \) vanishes, \( \sigma = 0, 1, \ldots, s \) (see [7, Lemma 2.3]).

**E5** For each \( \sigma \neq \sigma’ \), the identities \( \omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm, \ \theta \in \mathbb{T}^d \setminus C_*, \) do not hold with \( \text{const}_\pm \neq 0. \)
The condition $\mathbf{E5}$ can be weakened to the condition $\mathbf{E5}'$, see Remark 2.7 below.

**E6** $\|\hat{V}^{-1}(\theta)\| \in L^1(\mathbb{T}^d)$.

**Example 2.3** For any $d, n \geq 1$ we consider the nearest neighbor crystal for which

$$
\langle u, V u \rangle = \sum_{l=1}^{n} \sum_{x \in \mathbb{Z}^d} \left( \sum_{i=1}^{d} \kappa_l |u_i(x + e_i) - u_i(x)|^2 + m_l^2 |u_i(x)|^2 \right), \quad \kappa_l > 0, \quad m_l \geq 0, \quad (2.9)
$$

where $e_i = (\delta_{i1}, \ldots, \delta_{id})$. Then

$$
V_{kl}(x) = 0 \quad \text{for} \quad k \neq l, \quad V_{ll}(x) = \begin{cases} 
-\kappa_l & \text{for} \ |x| = 1, \\
2d\kappa_l + m_l^2 & \text{for} \ |x| = 0, \ l = 1, \ldots, n.
\end{cases}
$$

Hence, the eigenvalues of $\hat{V}(\theta)$ are

$$
\tilde{\omega}_l(\theta) = \sqrt{2\kappa_l (1 - \cos \theta_1) + \ldots + 2\kappa_l (1 - \cos \theta_d) + m_l^2}, \quad l = 1, \ldots, n. \quad (2.10)
$$

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 2.2. Clearly the conditions $\mathbf{E1} - \mathbf{E5}$ hold and $\mathcal{C}_* = \emptyset$. If all $m_l > 0$, then the set $\mathcal{C}_0$ is empty and the condition $\mathbf{E6}$ is fulfilled. Otherwise, if $m_l = 0$ for some $l$, then $\mathcal{C}_0 = \{0\}$. In this case, $\mathbf{E6}$ is equivalent to the condition $\omega_l^{-2}(\theta) \in L^1(\mathbb{T}^d)$ that holds if $d \geq 3$. Therefore, the conditions $\mathbf{E1} - \mathbf{E6}$ hold for (2.9) provided either i) $d \geq 3$, or ii) $d = 1, 2$ and all $m_l > 0$.

The following Proposition 2.4 is proved in [18, p.150], [1, p.128].

**Proposition 2.4** Let the conditions $\mathbf{E1}$ and $\mathbf{E2}$ hold, and choose some $\alpha \in \mathbb{R}$. Then for any $Y_0 \in \mathcal{H}_\alpha$ there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_\alpha)$ to the Cauchy problem (2.2); the operator $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}_\alpha$.

We assume that $Y_0$ in (2.2) is a measurable random function and denote by $\mu_0$ a Borel probability measure on $\mathcal{H}_\alpha$ giving the distribution of $Y_0$. Expectation with respect to $\mu_0$ is denoted by $\mathbb{E}$. We impose the following conditions $\mathbf{S1} - \mathbf{S3}$ on the initial measure $\mu_0$.

**S1** $\mu_0$ has zero expectation value, $\mathbb{E}(Y_0(x)) \equiv \int (Y_0(x)) \mu_0(dx) = 0$, $x \in \mathbb{Z}^d$.

**S2** The initial correlation functions $Q_0^{ij}(x, y) := \mathbb{E} \left( Y_0^i(x) \otimes Y_0^j(y) \right)$, $x, y \in \mathbb{Z}^d$, satisfy the bound

$$
|Q_0^{ij}(x, y)| \leq h(|x - y|), \quad \text{where} \quad r^{d-1} h(r) \in L^1(0, +\infty). \quad (2.11)
$$

Here for $a, b, c \in \mathbb{C}^n$, we denote by $a \otimes b$ the linear operator $(a \otimes b)c = \sum_{j=1}^{n} b_j c_j$.

**S3** Choose some $k \in \{1, \ldots, d\}$. The initial covariance $Q_0(x, y) = (Q_0^{ij}(x, y))_{i,j=0,1}$ depends on difference $x_l - y_l$ for all $l = k + 1, \ldots, d$, i.e.,

$$
Q_0(x, y) = q_0(\bar{x}, \bar{y}, \bar{x} - \bar{y}), \quad (2.12)
$$

where $x = (x_1, \ldots, x_d) \equiv (\bar{x}, \bar{x})$, $\bar{x} = (x_1, \ldots, x_k)$, $\bar{x} = (x_{k+1}, \ldots, x_d)$. Write

$$
\mathcal{N}^k := \{ n = (n_1, \ldots, n_k), \quad \text{where all} \ n_j \in \{1, 2\} \}. \quad (2.13)
$$
Suppose that $\forall \varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $\bar{y} \in \mathbb{Z}^k$: $(-1)^n y_j > N(\varepsilon)$ for each $j = 1, \ldots, k$, the following bound holds

$$|q_0(\bar{y} + \bar{z}, \bar{y}, \bar{z}) - q_0(z)| < \varepsilon \quad \text{for any fixed } z = (\bar{z}, \bar{z}) \in \mathbb{Z}^d \quad \text{and } n \in \mathbb{N}^k. \quad (2.14)$$

Here $q_n(z), n \in \mathbb{N}^k$, are the correlation matrices of some translation-invariant measures $\mu_n$ with zero mean value in $\mathcal{H}_\alpha$.

In particular, if $k = 1$, then the condition S3 means that $Q_0(x, y) = q_0(x, y, \bar{x} - \bar{y})$, where $x = (x_1, \bar{x}), \bar{x} = (x_2, \ldots, x_d)$, and

$$q_0(y_1 + z_1, y_1, \bar{z}) \rightarrow \begin{cases} q_1(z) & \text{as } y_1 \rightarrow -\infty, \\ q_2(z) & \text{as } y_1 \rightarrow +\infty, \\ z = (z_1, \bar{z}) \in \mathbb{Z}^d. \end{cases} \quad (2.15)$$

A measure $\mu$ is called translation invariant if $\mu(Bh) = \mu(B)$ for $B \in \mathcal{B}(\mathcal{H}_\alpha)$ and $h \in \mathbb{Z}^d$, where $T_hY(x) = Y(x - h), x \in \mathbb{Z}^d$, $\mathcal{B}(\mathcal{H}_\alpha)$ stands for the Borel $\sigma$-algebra in $\mathcal{H}_\alpha$. Note that the initial measure $\mu_0$ is not translation-invariant if $q_n \neq q_n'$ for some $n \neq n'$. The examples of $\mu_0$ satisfying the conditions S1–S3 are given in Sec. 3.

2.2 Convergence of correlations functions

**Definition 2.5** $\mu_t$ is a Borel probability measure in $\mathcal{H}_\alpha$ which gives the distribution of $Y(t), \mu_t(B) = \mu_0(U(-t)B), \forall B \in \mathcal{B}(\mathcal{H}_\alpha), t \in \mathbb{R}$. The correlation functions of the measure $\mu_t$ are defined by

$$Q_t^{ij}(x, y) = \mathbb{E}(Y^i(x, t) \otimes Y^j(y, t)), \quad i, j = 0, 1, \quad x, y \in \mathbb{Z}^d. \quad (2.16)$$

Here $Y^i(x, t)$ are the components of the random solution $Y(t) = (Y^0(\cdot, t), Y^1(\cdot, t))$.

Denote by $Q_t$ the quadratic form with the matrix kernel $(Q_t^{ij}(x, y))_{i,j=0,1}$,

$$Q_t(\Psi, \Psi) = \int |\langle Y, \Psi \rangle|^2 \mu_t(dY) = \sum_{i,j=0,1} \sum_{x,y \in \mathbb{Z}^d} (Q_t^{ij}(x, y), \Psi^i(x) \otimes \Psi^j(y)), \quad t \in \mathbb{R},$$

$$\Psi = (\Psi^0, \Psi^1) \in \mathcal{S} := S \oplus S, \quad S := S(\mathbb{Z}^d) \otimes \mathbb{R}^n,$$ where $S(\mathbb{Z}^d)$ denotes a space of real quickly decreasing sequences, $\langle Y, \Psi \rangle = \sum_{i=0,1} \sum_{x \in \mathbb{Z}^d} \langle Y^i(x), \Psi^i(x) \rangle$.

Let us introduce the limiting correlation matrix $Q_\infty(x, y) = (Q_\infty^{ij}(x, y))_{i,j=0}^1$ as follows

$$Q_\infty(x, y) = q_\infty(x - y), \quad x, y \in \mathbb{Z}^d. \quad (2.17)$$

Here $q_\infty(x)$ has a form (in the Fourier transform)

$$\hat{q}_\infty(\theta) = \sum_{\sigma = 1}^S \Pi_\sigma(\theta)(M_{k,\sigma}^+(\theta) + i M_{k,\sigma}^-(\theta)) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_s, \quad (2.18)$$

where $\Pi_\sigma(\theta)$ is the spectral projection from Lemma 2.2 (iii),

$$M_{k,\sigma}^+(\theta) = \frac{1}{2k} \sum_{n \in \mathbb{N}^k} L_1^+(\hat{q}_n(\theta)) \left[ 1 + S_{k,n}^\text{even}((\omega_\sigma(\theta)) \right],$$

$$M_{k,\sigma}^-(\theta) = \frac{1}{2k} \sum_{n \in \mathbb{N}^k} L_2^-((\hat{q}_n(\theta)) S_{k,n}^\text{odd}((\omega_\sigma(\theta))). \quad (2.19)$$
For $d = n = 1$, formulas (2.23) were obtained in [11, p.139]. For any $d, n \geq 1$ and $k = 1$, these formulas were derived in [8].

Note that $\hat{q}_{\infty} \in L^1(\mathbb{T}^d)$ by Lemma 5.1 and the condition E6. Moreover, by (2.18) – (2.22), the matrix $\hat{q}_{\infty}(\theta)$ satisfies the “equilibrium condition”, i.e., $\hat{q}^{01}_{\infty}(\theta) = \hat{V}(\theta)\hat{q}^{00}_{\infty}(\theta)$, $\hat{q}^{10}_{\infty}(\theta) = -\hat{q}^{01}_{\infty}(\theta)$. Also, $(\hat{q}^{0i}_{\infty}(\theta))^* = \hat{q}^{i0}_{\infty}(\theta) \geq 0$, $i = 0, 1$, $(\hat{q}^{1i}_{\infty}(\theta))^* = -\hat{q}^{i0}_{\infty}(\theta)$.

The first result of the paper is the following theorem.

**Theorem 2.6** Let $d, n \geq 1$, $\alpha < -d/2$, and assume that the conditions E1–E6 and S1–S3 hold. Then the convergence (1.7) is true, where $Q_{\infty}$ is defined in (2.17) – (2.22).

In Sec. 6 we study the initial boundary value problem for harmonic crystals with zero boundary condition and obtain the results similar to Theorem 2.6 see Theorem 6.4 below.

**Remark 2.7** The condition E5 on the matrix $V$ could be weakened. Namely, it suffices to impose the following restriction.

**E5’** If for some $\sigma \neq \sigma'$, $\omega_{\sigma}(\theta) + \omega_{\sigma'}(\theta) \equiv \text{const}_+$ with $\text{const}_+ \neq 0$, then $p^{11}_{n,\sigma\sigma'}(\theta) - \omega_{\sigma}(\theta)\omega_{\sigma'}(\theta) p^{00}_{n,\sigma\sigma'}(\theta) \equiv 0$ and $\omega_{\sigma}(\theta) p^{01}_{n,\sigma\sigma'}(\theta) + \omega_{\sigma'}(\theta) p^{10}_{n,\sigma\sigma'}(\theta) \equiv 0$. If for some $\sigma \neq \sigma'$, $\omega_{\sigma}(\theta) - \omega_{\sigma'}(\theta) \equiv \text{const}_+$ with $\text{const}_- \neq 0$, then $p^{11}_{n,\sigma\sigma'}(\theta) + \omega_{\sigma}(\theta)\omega_{\sigma'}(\theta) p^{00}_{n,\sigma\sigma'}(\theta) \equiv 0$ and $\omega_{\sigma}(\theta) p^{01}_{n,\sigma\sigma'}(\theta) - \omega_{\sigma'}(\theta) p^{10}_{n,\sigma\sigma'}(\theta) \equiv 0$. Here

$$p^{ij}_{n,\sigma\sigma'}(\theta) := \Pi_{\sigma}(\theta) q^{ij}_{\sigma}(\theta) \Pi_{\sigma'}(\theta), \quad \theta \in \mathbb{T}^d, \quad \sigma, \sigma' = 1, \ldots, s, \quad i, j = 0, 1, \quad n \in \mathbb{N}.$$  (2.24)

This condition holds, for instance, for the canonical Gibbs measures $\mu_\mathcal{C}$ considered in Sec. 4.2.

**Examples 2.8** We rewrite the formulas for $q_{\infty}$ in some particular cases.

(i) In the case when the initial covariance is translation invariant, i.e., $Q_0(x, y) = q_0(x - y)$, the matrix $\hat{q}_{\infty}$ is of a form

$$\hat{q}_{\infty}(\theta) = \sum_{\sigma=1}^s \Pi_{\sigma}(\theta) L^\perp_1(\hat{q}(\theta)) \Pi_{\sigma}(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_s.$$  (2.25)
(ii) Let the initial covariance $Q_0$ satisfy a stronger condition than (2.14). Namely, assume that $Q_0$ has a form (2.12) and for any $z = (\tilde{z}, \tilde{z}) \in \mathbb{Z}^d$, \( \lim_{|\tilde{y}| \to \infty} q_0(y + \tilde{z}, \tilde{y}, \tilde{z}) = q_*(z) \). Then the condition (2.14) is fulfilled with $q_n(z) = q_* (z)$ for any $n \in \mathbb{N}^k$. In this case, Theorem 2.6 holds, and $\hat{q}_\infty$ is of a form (2.25) with $\hat{q}_*$ instead of $\hat{q}_0$. Therefore, Theorem 2.6 generalizes the result of [7, Proposition 3.2], where the convergence (1.1) was proved in the case when $Q_0(x, y) = q_0(x - y)$.

2.3 Weak convergence of measures

To prove the convergence (1.3) of the measures $\mu_t$, we impose a stronger condition $S4$ on $\mu_0$ than the bound (2.11). To formulate this condition, let us denote by $\sigma(A)$, $A \subset \mathbb{Z}^d$, the $\sigma$-algebra in $\mathcal{H}_\alpha$ generated by $Y_0(x)$ with $x \in A$. Define the Ibragimov mixing coefficient of a probability measure $\mu_0$ on $\mathcal{H}_\alpha$ by the rule (cf [16, Definition 17.2.2])

\[
\varphi(r) \equiv \sup_{A, B \subset \mathbb{Z}^d} \sup_{A \in \sigma(A), B \in \sigma(B)} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)} \quad \text{for} \quad \text{dist}(A, B) \geq r, \quad \mu_0(B) > 0
\]

Definition 2.9 The measure $\mu_0$ satisfies a strong uniform Ibragimov mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

$S4$ The initial mean “energy” density is uniformly bounded:

\[
\mathbb{E}[|u_0(x)|^2 + |v_0(x)|^2] = \text{tr} Q_0^{10}(x, x) + \text{tr} Q_0^{11}(x, x) \leq e_0 < \infty, \quad x \in \mathbb{Z}^d. \tag{2.26}
\]

Moreover, $\mu_0$ satisfies the strong uniform Ibragimov mixing condition and

\[
\int_0^\infty r^{d-1}\varphi^{1/2}(r) \, dr < \infty.
\]

Remark 2.10 By [16, Lemma 17.2.3], the conditions $S1$ and $S4$ imply the bound (2.11) with $h(r) = Ce_0\varphi^{1/2}(r)$, where $e_0$ is a constant from the bound (2.26).

For a probability measure $\mu$ on $\mathcal{H}_\alpha$ we denote by $\hat{\mu}$ the characteristic functional (Fourier transform), $\hat{\mu}(\Psi) \equiv \int \exp(i(Y, \Psi)) \mu(dY)$, $\Psi \in \mathcal{S}$. A measure $\mu$ is called Gaussian (of zero mean) if its characteristic functional has the form $\hat{\mu}(\Psi) = \exp\{-Q(\Psi, \Psi)/2\}$, where $Q$ is a real nonnegative quadratic form in $\mathcal{S}$.

Theorem 2.11 Let $d, n \geq 1$, $\alpha < -d/2$, and assume that the conditions $E1$–$E3$, $E4'$, $E5'$, $E6$, $S1$, $S3$, and $S4$ be fulfilled. Then the following assertions hold.

(i) The measures $\mu_t$ weakly converge in the Hilbert space $\mathcal{H}_\alpha$,

\[
\mu_t \to \mu_\infty \quad \text{as} \quad t \to \infty. \tag{2.27}
\]

The limit measure $\mu_\infty$ is a Gaussian translation-invariant measure on $\mathcal{H}_\alpha$. The characteristic functional of $\mu_\infty$ is of a form $\hat{\mu}_\infty(\Psi) = \exp\{-\frac{1}{2}Q_\infty(\Psi, \Psi)\}$, $\Psi \in \mathcal{S}$, where $Q_\infty$ is the quadratic form with the matrix kernel $Q_\infty(x, y)$ defined in (2.17).
Lemma 2.14

(ii) The measure \( \mu_\infty \) is time stationary, i.e., \( [U(t)]^* \mu_\infty = \mu_\infty \), \( t \in \mathbb{R} \).

(iii) The flow \( U(t) \) is mixing with respect to the measure \( \mu_\infty \), i.e., for any \( f, g \in L^2(\mathcal{H}_\alpha, \mu_\infty) \),

\[
\lim_{t \to -\infty} \int f(U(t)Y) g(Y) \mu_\infty(dY) = \int f(Y) \mu_\infty(dY) \int g(Y) \mu_\infty(dY)
\]

In particular, the flow \( U(t) \) is ergodic with respect to the measure \( \mu_\infty \).

For harmonic crystals in the half-space, the convergence (2.27) also holds. For details, see Sec. 6. The assertion (i) of Theorem 2.11 follow from Propositions 2.12 and 2.13.

Proposition 2.12 Let the conditions E1 – E3, E6, S1 and S2 hold. Then the measures family \( \{\mu_t, t \in \mathbb{R}\} \) is weakly compact in \( \mathcal{H}_\alpha \) with any \( \alpha < -d/2 \), and the following bounds hold

\[
\sup_{t \in \mathbb{R}} \mathbb{E}\|U(t) Y_0\|^2_\alpha < \infty.
\] (2.28)

Proposition 2.13 Let the conditions E1 – E3, E4’, E5’, E6, S1, S3, and S4 hold. Then for every \( \Psi \in \mathcal{S} \), the characteristic functionals of \( \mu_t \) converge to a Gaussian one,

\[
\hat{\mu}_t(\Psi) := \int e^{i(Y,\Psi)} \mu_t(dY) \to \exp\{-\frac{1}{2} Q_\infty(\Psi, \Psi)\}, \quad t \to \infty.
\]

Proposition 2.12 (Proposition 2.13) provides the existence (resp. the uniqueness) of the limit measure \( \mu_\infty \). Proposition 2.12 is proved in Sec. 5.1. Proposition 2.13 can be proved using the technique from [8]. The assertion (ii) of Theorem 2.11 follows from (2.27) since the group \( U(t) \) is continuous in \( \mathcal{H}_\alpha \) by Proposition 2.4. The ergodicity and mixing of the limit measures \( \mu_\infty \) follow by the same arguments as in [7].

Lemma 2.14 Let the conditions E1 – E4, E5’, and E6 hold. Assume that the initial measure \( \mu_0 \) is Gaussian and satisfies the conditions S1 – S3. Then all assertions of Theorem 2.11 remain valid.

This lemma follows from Theorem 2.6 and Proposition 2.12

3 Examples of initial measures

Now we construct Gaussian initial measures \( \mu_0 \) satisfying the conditions S1 – S3. For \( k = 1 \) (see the condition S3), the example of \( \mu_0 \) is given in [8]. For any \( k \geq 1 \), the measure \( \mu_0 \) can be constructed by a following way. At first, for simplicity, we assume that \( \{u_0, v_0\} \in \mathbb{R}^1 \) and define the correlation functions \( \hat{q}^{ij}_{\mathbf{n}}(x-y) \), \( \mathbf{n} \in \mathbb{N}^k \), which are zero for \( i \neq j \), while for \( i = 0, 1 \),

\[
\hat{q}^{ii}_{\mathbf{n}}(\theta) := F_{z \to \theta}[q^{ii}_{\mathbf{n}}(z)] \in L^1(\mathbb{T}^d), \quad \hat{q}^{ii}_{\mathbf{n}}(\theta) \geq 0.
\] (3.1)

Then, by the Minlos theorem [5], there exist Borel Gaussian measures \( \mu_\mathbf{n} \) on \( \mathcal{H}_\alpha \), \( \alpha < -d/2 \), with the correlation functions \( q^{ii}_{\mathbf{n}}(x-y) \), because

\[
\int \|Y\|^2_\alpha \mu_\mathbf{n}(dY) = \sum_{x \in \mathbb{Z}^d} \langle x \rangle^{2\alpha} \text{tr} (q^{00}_{\mathbf{n}}(0) + q^{11}_{\mathbf{n}}(0)) = C(\alpha, d) \int_{\mathbb{T}^d} \text{tr} (q^{00}_{\mathbf{n}}(\theta) + q^{11}_{\mathbf{n}}(\theta)) d\theta < \infty.
\]
Further, we take the functions $\overline{\zeta}_n \in C(\mathbb{Z}^k)$ such that
\[
\overline{\zeta}_n(\bar{x}) = \zeta_{n_1}(x_1) \cdots \zeta_{n_k}(x_k), \quad \bar{x} = (x_1, \ldots, x_k), \quad n = (n_1, \ldots, n_k), \quad n_j \in \{1, 2\},
\]
where the sequences $\zeta_1(x)$ and $\zeta_2(x)$, $x \in \mathbb{Z}$, are defined by the rule
\[
\zeta_1(x) = \begin{cases} 1 & \text{for } x < -a, \\ 0 & \text{for } x > a, \end{cases} \quad \zeta_2(x) = \begin{cases} 1 & \text{for } x > a, \\ 0 & \text{for } x < -a, \end{cases}
\]
with some $a > 0$. \hfill (3.2)

Finally, define a Borel probability measure $\mu_0$ as a distribution of the random function
\[
Y_0(x) = \sum_{n \in N^k} \overline{\zeta}_n(\bar{x}) Y_n(x), \quad x = (\bar{x}, \bar{y}) \in \mathbb{Z}^d, \quad \bar{x} = (x_1, \ldots, x_k), \quad \bar{y} = (y_1, \ldots, y_k), \quad \bar{x} = (x_{k+1}, \ldots, x_d).
\]
where $Y_n(x)$ are Gaussian independent functions in $\mathcal{H}_\alpha$ with distributions $\mu_n$. Then correlation matrix of $\mu_0$ is of a form
\[
Q_0(x, y) = \sum_{n \in N^k} \overline{\zeta}_n(\bar{x}) \overline{\zeta}_n(\bar{y}) q_n(x - y),
\]
where $x = (\bar{x}, \bar{y})$, $y = (\bar{y}, \bar{y}) \in \mathbb{Z}^d$, and $q_n(x - y)$ are the correlation matrices of the measures $\mu_n$. Hence, $Q_0(x, y) = q_0(\bar{x}, \bar{y}, \bar{x} - \bar{y})$, and for every $z = (\bar{z}, \bar{z}) \in \mathbb{Z}^d$,
\[
q_0(\bar{y} + \bar{z}, \bar{y}, \bar{z}) = q_n(z) \quad \text{if } (-1)^{n_j}y_j > a + |z_j|, \quad \forall j = 1, \ldots, k, \quad n = (n_1, \ldots, n_k).
\]
Therefore, the measure $\mu_0$ satisfies the conditions S1 and S3. If
\[
|q_n^i(z)| \leq h(|z|), \quad \text{where } r^d - 1 h(r) \in L^1(0, +\infty),
\]
then $\mu_0$ satisfies S2 by (3.4). Now we give examples of $q_n^i$ satisfying (3.1) and (3.5).

Example 3.1 Put $q_n^i(z) = f(z_1)f(z_2) \cdots f(z_d)$ and construct sequences $f(z)$, $z \in \mathbb{Z}$, such that conditions (3.1) and (3.5) hold.

(i) Let $f(z) = N_0 - |z|$ for $|z| \leq N_0$ and $f(z) = 0$ for $|z| > N_0$ with some $N_0 > 0$. Then $q_n^i(z) = 0$ for $|z| \geq r_0 \equiv N_0 \sqrt{d}$, and $f(\theta) = (1 - \cos N_0 \theta)/(1 - \cos \theta)$, $\theta \in T^1$. Hence, (3.1) and (3.5) are fulfilled. Furthermore, the condition S4 also follows with $\varphi(r) = 0$ for $r \geq r_0$. This example of the sequence $f$ can be generalized as follows.

Let $f$ be an even nonnegative sequence such that $f \in \ell^1$ and $\Delta_L f(z) \geq 0$ for any $z \geq 1$. Then $\hat{f}(\theta) \geq 0$ by [17, Theorems 4.1 and 2.7], and (3.1) follows. If, in addition, $|f(z)| \leq C(1 + |z|)^{-N}$ with $N > d$, then (3.5) holds.

(ii) Let $f(z) = (a + b|z|)^{\gamma |z|}$, $z \in \mathbb{Z}$, with $\gamma \in (0, 1)$, $a > 0$ and $b \geq 0$. Hence, (3.5) is fulfilled with $h(r) = C(1 + r)^{-N} \quad (N > d)$. If $a \geq 2b\gamma/(1 - \gamma)$, then $\Delta_L f(z) \geq 0$ for any $z \geq 1$ and $\hat{f}(\theta) \geq 0$ (see the case (i)). If $2b\gamma/(1 - \gamma) \leq a < 2b\gamma/(1 - \gamma)$, then $\Delta_L f(1) < 0$. However, in this case, (3.1) also holds because
\[
\hat{f}(\theta) = \frac{a(1 - \gamma)^2}{1 - 2\gamma \cos \theta + \gamma^2} + \frac{2b\gamma ((1 + \gamma^2) \cos \theta - 2\gamma)}{(1 - 2\gamma \cos \theta + \gamma^2)^2}
\]
\[
= \frac{[a(1 - \gamma^2) + 2b\gamma \cos \theta]((1 - \gamma)^2 + [a(1 - \gamma^2) - 2b\gamma](1 - \cos \theta)2\gamma}}{(1 - 2\gamma \cos \theta + \gamma^2)^2} \geq 0.
\]
Remark 3.3 Suppose that the initial covariance has a particular form: and the conditions (3.1) and (3.5) are fulfilled with $h$ of some translation-invariant measure in $H$

Example 3.2 Let $u_0, v_0 \in \mathbb{R}^n$ with any $n \geq 1$, and $q^{00}_n(z) = T_n F^{-1}_{\theta \rightarrow z}[\hat{V}^{-1}(\theta)]$, $q^{11}_n(z) = T_n I$, $z \in \mathbb{Z}^d$, with some constants $T_n > 0$. Assume, in addition, that $C_0 = \emptyset$ (see (2.23)), i.e.

$$\det \hat{V}(\theta) \neq 0, \quad \forall \theta \in \mathbb{T}^d.$$  

Hence,

$$|q^{00}_n(z)| = T_n |F^{-1}_{\theta \rightarrow z}[\hat{V}^{-1}(\theta)]| \sim (1 + |z|)^{-N}, \quad \forall N \in \mathbb{N},$$

and the conditions (3.1) and (3.5) are fulfilled with $h(r) = (1 + r)^{-N}$, where $N > d$.

Remark 3.3 Suppose that the initial covariance has a particular form:

$$Q_0(x, y) = T(\bar{x} + \bar{y})r(x - y) \quad \text{or} \quad Q_0(x, y) = \sqrt{T(\bar{x})T(\bar{y})}r(x - y),$$

where $T(\bar{x})$ is a bounded nonnegative sequence on $\mathbb{Z}^k$, $r(x) = (r^{ij}(x))$ is a correlation matrix of some translation-invariant measure in $\mathcal{H}_\alpha$ with zero mean value, $|r^{ij}(x)| \leq h(|x|)$, where $r^{d-1}h(r) \in L^1(0, +\infty)$. Then, the condition $S2$ is fulfilled. For every $n = (n_1, \ldots, n_k) \in \mathcal{N}^k$, we assume that $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}$ such that for any $\bar{x} \in \mathbb{Z}^k$: $(-1)^{n_j}x_j > N(\varepsilon)$ with any $j = 1, \ldots, k$, $|T(\bar{x}) - T_n| < \varepsilon$. Hence, the condition $S3$ is fulfilled with $q_n(x) := T_n r(x)$.

4 Energy current

4.1 Non-equilibrium states

At first, we derive the expression for the energy current density of the finite energy solutions $u(x, t)$ (see (2.4)). For the half-space $\Omega_t := \{x \in \mathbb{Z}^d : x_1 \geq 0\}$, we define the energy in the region $\Omega_t$ (cf. (2.4)) as

$$\mathcal{E}_t(t) := \frac{1}{2} \sum_{x \in \Omega_t} \left\{ |\dot{u}(x, t)|^2 + \sum_{y \in \mathbb{Z}^d} \left( u(x, t), V(x - y)u(y, t) \right) \right\}, \quad l = 1, \ldots, d.$$  

Introduce new variables: $x = x' + me_l$, $y = y' + pe_l$, where $x', y' \in \mathbb{Z}^d$ with $x'_l = y'_l = 0$, $e_l = (\delta_{l1}, \ldots, \delta_{ld})$, $l = 1, \ldots, d$. Using Eqn (2.1), we obtain $\mathcal{E}_t(t) = \sum_{x'} J^l(x', t)$. Here $J^l(x', t)$ stands for the energy current density in the direction $e_l$:

$$J^l(x', t) := \frac{1}{2} \sum_{y'} \left\{ \sum_{m \leq -1, p \geq 0} \left( \dot{u}(x' + me_l, t), V(x' + me_l - y' - pe_l)u(y' + pe_l, t) \right) 
- \sum_{m \geq 0, p \leq -1} \left( \dot{u}(x' + me_l, t), V(x' + me_l - y' - pe_l)u(y' + pe_l, t) \right) \right\},$$

where $x', y' \in \mathbb{Z}^d$ with $x'_l = y'_l = 0$. Further, let $u(x, t)$ be the random solution of Eqn (2.1) with the initial measure $\mu_0$ satisfying $S1$–$S3$. The convergence (1.1) yields

$$\mathbb{E} \left( J^l(x', t) \right) \rightarrow J^l_\infty := \frac{1}{2} \sum_{y'} \left( \sum_{m \leq -1, p \geq 0} \text{tr} \left[ q^{10}_\infty(x' - y' + (m-p)e_l) V^T(x' - y' + (m-p)e_l) \right] 
- \sum_{m \geq 0, p \leq -1} \text{tr} \left[ q^{10}_\infty(x' - y' + (m-p)e_l) V^T(x' - y' + (m-p)e_l) \right] \right) 
= \frac{1}{2} \sum_{z \in \mathbb{Z}^d} z_l \text{tr} \left[ q^{10}_\infty(z) V^T(z) \right] \quad \text{as} \ t \to \infty.$$  

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Applying Fourier transform and the equality $\hat{V}^*(\theta) = \hat{V}(\theta)$, we obtain

$$J_l^\infty = \frac{-(2\pi)^{-d}}{2} \int_{\mathbb{T}^d} i \text{tr} \left[ \hat{q}_{\infty}^{10}(\theta) \hat{\partial}_\theta \hat{V}(\theta) \right] d\theta, \quad l = 1, \ldots, d.$$ 

Since $\Pi_{\sigma}(\theta)$ are orthogonal projections, then $\Pi_{\sigma}(\theta) (\hat{\partial}_\theta \Pi_{\sigma}(\theta)) \Pi_{\sigma}(\theta) = 0$ for any $\sigma, \sigma' = 1, \ldots, s$ and $l = 1, \ldots, d$. Hence, applying the formula (2.18) and the decomposition of $\hat{V}(\theta)$, $\hat{V}(\theta) = \sum_{\sigma=1}^s \Pi_{\sigma}(\theta) \omega_{\sigma}(\theta)$, we obtain that $\text{tr} \left[ \hat{q}_{\infty}^{10}(\theta) \sum_{\sigma=1}^s \omega_{\sigma}(\theta) \hat{\partial}_\theta \Pi_{\sigma}(\theta) \right] = 0$ and

$$J_l^\infty = -i(2\pi)^{-d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} \text{tr} \left[ \Pi_{\sigma}(\theta) \left( M_{k,\sigma}^+(\theta) + i M_{k,\sigma}^-(\theta) \right) \right] \omega_{\sigma}(\theta) \hat{\partial}_\theta \omega_{\sigma}(\theta) d\theta$$

$$= -(2\pi)^{-d} \frac{1}{2k} \sum_{\sigma=1}^s \sum_{n \in \mathbb{N}^d} \left\{ \frac{1}{2} \int_{\mathbb{T}^d} \text{tr} \left[ \omega_{\sigma}^2(\theta) p_{n,\sigma,\sigma}^{00}(\theta) + p_{n,\sigma,\sigma}^{11}(\theta) \right] \right\}$$

$$+ \int_{\mathbb{T}^d} \text{Im} \left( \text{tr} p_{n,\sigma,\sigma}^{01}(\theta) \right) \left( 1 + S_{k,n}(\omega_{\sigma}) \right) \omega_{\sigma}(\theta) \hat{\partial}_\theta \omega_{\sigma}(\theta) d\theta \right\}, \quad l = 1, \ldots, d, \quad (4.1)$$

where $p_{n,\sigma,\sigma'}^{ij}$ are introduced in (2.24). Here we use the equality $\text{tr} p_{n,\sigma,\sigma}^{ij}(\theta) = \text{tr} p_{n,\sigma,\sigma}^{ji}(\theta)$.

**Remark 4.1** Let, for simplicity, all functions $\text{tr}[p_{n,\sigma,\sigma}^{ij}(\theta)]$ and $\omega_{\sigma}(\theta)$ be even for every variable $\theta_1, \ldots, \theta_d$. Then, $J_l^\infty = 0$ for $l > k$, and $J_l^\infty = C_{l1}^\infty - C_{l2}^\infty$ for $l = 1, \ldots, k$, where

$$C_{l1} := (2\pi)^{-d} \frac{1}{2k} \sum_{\sigma=1}^s \sum_{n \in \mathbb{N}^d} \left\{ \frac{1}{2} \int_{\mathbb{T}^d} \text{tr} \left[ \omega_{\sigma}^2(\theta) p_{n,\sigma,\sigma}^{00}(\theta) + p_{n,\sigma,\sigma}^{11}(\theta) \right] \right\} |\hat{\partial}_\theta \omega_{\sigma}(\theta)| d\theta, \quad j = 1, 2.$$ 

Here the summation $\sum_{n}'$ is taken over $n_1, \ldots, n_{l-1}, n_{l+1}, \ldots, n_k \in \{1, 2\}$. In particular, for the initial correlation matrix $Q_0$ of the form (3.8), $p_{n,\sigma,\sigma}^{ij}(\theta) = T_{n,p_{\sigma,\sigma}^{ij}}(\theta)$, where, by definition, $p_{\sigma,\sigma}^{ij}(\theta) := \Pi_{\sigma}(\theta) \hat{r}^{ij}(\theta) \Pi_{\sigma}(\theta)$. In this case, $J_{l,\infty}$ is of the form (1.2) with

$$c_l = \sum_{\sigma=1}^s (2\pi)^{-d} \frac{1}{2} \int_{\mathbb{T}^d} \text{tr} \left[ \omega_{\sigma}^2(\theta) p_{\sigma,\sigma}^{00}(\theta) + p_{\sigma,\sigma}^{11}(\theta) \right] |\hat{\partial}_\theta \omega_{\sigma}(\theta)| d\theta.$$ 

We see that one can choose numbers $T_n$ such that $J_{l,\infty} \neq 0$ for some $l = 1, \ldots, k$.

Below we simplify the formula (1.1) in the case when $\mu_n$ are Gibbs measures corresponding to positive temperatures $T_n$. Furthermore, under the additional symmetry conditions on the interaction matrix $V$, we derive the formula (1.2) for $J_{l,\infty}$. Thus, there exist stationary non-equilibrium states (in fact, Gaussian measures $\mu_{\infty}$) in which there is a non-zero constant energy current passing through the points of the crystal.

### 4.2 Energy current for the Gibbs measures

Formally, Gibbs measures $g_{\beta}$ are

$$g_{\beta}(dY) = \frac{1}{Z} e^{-\beta H(Y)} \prod_{x \in \mathbb{Z}^d} dY(x),$$

where $Z$ is the normalization constant.
where \( H(Y) \) is defined in (2.4). \( Z \) is normalization factor, \( \beta = T^{-1}, \ T > 0 \) is a corresponding absolute temperature. We introduce the Gibbs measures \( g_\beta(dY) \) as the Gaussian measures in \( \mathcal{H}_\alpha, \ \alpha < -d/2 \), with zero mean and with the correlation matrices defined by their Fourier transform,

\[
\hat{q}_\beta^{(0)}(\theta) = T \hat{V}^{-1}(\theta), \quad \hat{q}_\beta^{(1)}(\theta) = TI, \quad \hat{q}_\beta^{(l)}(\theta) = 0, \quad (4.2)
\]

where \( I \) denotes unit matrix in \( \mathbb{R}^n \times \mathbb{R}^n \). By the Minlos theorem [5], the Borel probability measures \( g_\beta \) exist in the spaces \( \mathcal{H}_\alpha \). Indeed,

\[
\int \| Y \|_\alpha^2 g_\beta(dY) = \sum_{x \in \mathbb{Z}^d} \langle x \rangle^{2\alpha} \text{tr} [q_\beta^{(0)}(0) + q_\beta^{(1)}(0)] < \infty,
\]

since \( \alpha < -d/2 \) and

\[
\text{tr} [q_\beta^{(0)}(0) + q_\beta^{(1)}(0)] = (2\pi)^{-d} \int_{\mathbb{T}^d} \text{tr} [\hat{q}_\beta^{(0)}(\theta) + \hat{q}_\beta^{(1)}(\theta)] d\theta = T(2\pi)^{-d} \int_{\mathbb{T}^d} \text{tr} \hat{V}^{-1}(\theta) d\theta + Tn < \infty.
\]

The last bound is obvious if \( C_0 = \emptyset \) and follows from the condition \( \text{E6} \) if \( C_0 \neq \emptyset \).

Let \( \mu_0 \) be a Borel probability measure in \( \mathcal{H}_\alpha \) giving the distribution of the random function \( Y_0 \) constructed in Sec. 3 (see formula (3.3)) with Gibbs measures \( \mu_n \equiv g_{\beta_n} \ (\beta_n = 1/T_n, \ T_n > 0) \) which have correlation matrices \( q_n(x) \equiv q_{\beta_n}(x) \), where the matrix \( q_\beta = (q_{\beta})_{i,j=0,1} \) is defined by (4.2). We impose, in addition, the condition (3.6). Then, the conditions S1–S3 hold (see Example 3.2). We check that in the case of the Gibbs measures \( \mu_n \equiv g_{\beta_n} \), the condition \( \text{E5'} \) is fulfilled (see Remark 2.7). Indeed, by (4.2) we have

\[
p_{n,\sigma,\sigma'}^{(0)}(\theta) = \Pi_\sigma(\theta) q_n^{(0)}(\theta) \Pi_{\sigma'}(\theta) = T_n \omega \omega^{-2}(\theta) \Pi_{\sigma}(\theta) \delta_{\sigma,\sigma'}, \quad \sigma, \sigma' = 1, \ldots, s, \quad (4.3)
\]

and \( p_{n,\sigma,\sigma'}^{ij}(\theta) = 0 \) for \( i \neq j \). Therefore, the convergence (2.27) holds by Lemma 2.14.

Now we rewrite the limit covariance \( \hat{q}_\infty(\theta) \) and the limit mean energy current \( J_\infty \) in the case when \( \mu_n = g_{\beta_n} \) are Gibbs measures. Applying (2.18), (2.19) and (2.21) we obtain

\[
\hat{q}_\infty^{(1)}(\theta) = \hat{V}(\theta) \hat{q}_\infty^{(0)}(\theta) = \sum_{\sigma = 1}^{s} \Pi_\sigma(\theta) \frac{1}{2^k} \sum_{n \in \mathbb{N}^k} T_n \left[ 1 + S_{k,n}^{(\text{even})}(\omega_\sigma(\theta)) \right],
\]

\[
\hat{q}_\infty^{(l)}(\theta) = -\hat{q}_\infty^{(l)}(\theta) = -i \sum_{\sigma = 1}^{s} \Pi_\sigma(\theta) \omega^{-1}(\theta) \frac{1}{2^k} \sum_{n \in \mathbb{N}^k} T_n S_{k,n}^{(\text{odd})}(\omega_\sigma(\theta)),
\]

where the functions \( S_{k,n}^{(\text{even})}(\omega_\sigma) \) and \( S_{k,n}^{(\text{odd})(\omega_\sigma)} \) are defined in (2.20). Substituting \( p_{n,\sigma,\sigma'}^{ij}(\theta) \) from (4.3) in the r.h.s. of (4.1), we obtain

\[
J_\infty^l = -\frac{1}{(2\pi)^d} \frac{1}{2^k} \sum_{\sigma = 1}^{s} \sum_{n \in \mathbb{N}^k} \int_{\mathbb{T}^d} r_\sigma T_n S_{k,n}^{(\text{odd})(\omega_\sigma(\theta))} \partial_{\theta} \omega_\sigma(\theta) d\theta \]

\[
= -\frac{1}{2^k} \sum_{n \in \mathbb{N}^k} T_n \left( \sum_{\text{odd } m \in \{1, \ldots, k\}} \sum_{(p_1, \ldots, p_m) \in P_m(k) \} c_{p_1 \ldots p_m} (-1)^{n_{p_1} + \ldots + n_{p_m}} \right), \ l = 1, \ldots, d, \quad (4.4)
\]

where \( r_\sigma = \text{tr}[\Pi_\sigma(\theta)] \) is multiplicity of the eigenvalue \( \omega_\sigma \) (see Lemma 2.2), the numbers \( c_{p_1 \ldots p_m} \) are defined as follows

\[
c_{p_1 \ldots p_m} := \frac{1}{(2\pi)^d} \frac{1}{2^k} \sum_{\sigma = 1}^{s} \int_{\mathbb{T}^d} r_\sigma \text{sign} \left( \frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_1}} \right) \ldots \text{sign} \left( \frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_m}} \right) \frac{\partial \omega_\sigma(\theta)}{\partial \theta_l} d\theta. \quad (4.5)
\]
Under the additional symmetry conditions (SC) on the interaction matrix \( V \), the formulas (4.4) and (4.5) can be simplified.

**SC** Suppose that one of the following conditions on \( \omega_\sigma \), \( \sigma = 1, \ldots, s \), holds.

(a) Each \( \omega_\sigma(\theta) \) is even for every variable \( \theta_{k+1}, \ldots, \theta_d \), and, in addition, if \( k \geq 2 \), then each \( \omega_\sigma(\theta) \) is even for some \( k-1 \) variables from the set \( \{\theta_1, \ldots, \theta_k\} \).

(b) Each \( \omega_\sigma(\theta) \) is even for every variable \( \theta_1, \ldots, \theta_k \).

(c) For every \( \sigma \) holds.

For instance, the conditions (a), (b), and (c) hold for the nearest neighbor crystal, see (2.10). Under these restrictions on \( \omega_\sigma \), all numbers \( c_{p_1\ldots p_m}^l \) in (4.3) are equal to zero except for the case when \( m = 1 \) and \( l = p_1 \in \{1, \ldots, k\} \). Write

\[
c_l \equiv c_l^1 = \frac{1}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{R}^d} r_\sigma \left| \frac{\partial \omega_\sigma(\theta)}{\partial \theta_l} \right| d\theta > 0, \quad l = 1, \ldots, k.
\] (4.6)

Therefore,

\[
J_\infty^l = \begin{cases} 
- c_l \frac{1}{2\pi} \sum_{n \in \mathcal{N}^k} (-1)^n T_n = - c_l \frac{1}{2\pi} \sum' (T_n|_{n_l=2} - T_n|_{n_l=1}), & l = 1, \ldots, k, \\
0, & l = k+1, \ldots, d,
\end{cases}
\] (4.7)

where the summation \( \sum' \) is taken over \( n_1, \ldots, n_{l-1}, n_{l+1}, \ldots, n_k \in \{1, 2\} \). In particular, if \( k = 1 \), the limiting energy current density is

\[
J_\infty = -\frac{1}{2} \left( c_1 (T_2 - T_1), 0, \ldots, 0 \right), \quad c_1 > 0.
\] (4.8)

In this case, our model can be considered as a “system + two reservoirs”, where by “reservoirs” we mean two parts of the crystal consisting of the particles with \( x_1 \leq -a \) and with \( x_1 \geq a \), where \( a > 0 \), and by a “system” the remaining (“middle”) part (cf [23, Sec.3]). At \( t = 0 \) the reservoirs are in thermal equilibrium with temperatures \( T_1 \) and \( T_2 \). Therefore, the formula (4.8) corresponds to the Second Law (see, for instance, [2], [23], i.e., the heat flows (on average) from the “hot reservoir” to the “cold” one.

If \( k = 2 \), then our model can be considered as a “system + four reservoirs”, where reservoirs consist of the particles with \( \{x_1, x_2 \leq -a\} \), \( \{x_1 \leq -a, x_2 \geq a\} \), \( \{x_1 \geq a, x_2 \leq -a\} \), and \( \{x_1, x_2 \geq a\} \). The initial states of the reservoirs are distributed according to Gibbs measures with corresponding temperatures \( T_{11}, T_{12}, T_{21}, \) and \( T_{22} \). The formula (4.7) becomes

\[
J_\infty = -\frac{1}{4} \left( c_1 (T_{21} - T_{11} + T_{22} - T_{12}), c_2 (T_{12} - T_{11} + T_{22} - T_{21}), 0, \ldots, 0 \right), \quad c_1, c_2 > 0.
\] (4.9)

For any \( k \), our model is a “system + \( 2^k \) reservoirs”, where “reservoirs” are the crystal particles with position \( \{x \in \mathbb{Z}^d : (-1)^n x_j > a \ \forall j = 1, \ldots, k\} \), and at \( t = 0 \) these reservoirs are assumed to be in thermal equilibrium with temperatures \( T_n \), \( n = (n_1, \ldots, n_k) \in \mathcal{N}^k \).
Remark 4.2 The limiting "kinetic" temperature (average kinetic energy) is
\[ K_\infty = \lim_{t \to \infty} \mathbb{E}|\dot{u}(x,t)|^2 = \text{tr} Q^{11}_{\infty}(x,x) = \text{tr} q^{11}_{\infty}(0), \]
by (2.19). In the case when \( \mu_n \) are Gibbs measures with temperatures \( T_n \), \( K_\infty \) equals
\[ K_\infty = \frac{1}{(2\pi)^d} \int_\mathbb{T}^d \text{tr} \hat{q}^{11}(\theta) \, d\theta = \frac{1}{2^k} \sum_{n \in \mathbb{N}^k} T_n \left( \sum_{\sigma=1}^s \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} r_\sigma \left( 1 + S_{k,n}^{\text{even}}(\omega_\sigma(\theta)) \right) \, d\theta \right) \]
by (4.3). If all \( \omega_\sigma(\theta) \) are even for each \( \theta_j \) with \( j = 1, \ldots, k \), then \( \int_{\mathbb{T}^d} S_{k,n}^{\text{even}}(\omega_\sigma(\theta)) \, d\theta = 0 \), and \( K_\infty = n 2^{-k} \sum_{n \in \mathbb{N}^k} T_n \). For instance, if \( k = 1 \), then \( K_\infty = n(T_1 + T_2)/2 \).}

5 Convergence of covariance

5.1 Bounds of correlation matrices

By \( l_1 \equiv l_1(\mathbb{Z}^d) \otimes \mathbb{R}^n \), \( p, d, n \geq 1 \), we denote the space of sequences \( f(x) = (f_1(x), \ldots, f_n(x)) \) endowed with norm \( \|f\|_{l_p} = \left( \sum_{x \in \mathbb{Z}^d} |f(x)|^p \right)^{1/p} \).

Lemma 5.1 (i) Let the conditions S1 and S2 hold. Then for any \( \Phi, \Psi \in l^2 \),
\[ |\langle Q_0(x,y), \Phi(x) \otimes \Psi(y) \rangle| \leq C \|\Phi\|_2 \|\Psi\|_2. \quad (5.1) \]
(ii) Let the conditions S1–S3 hold. Then \( q_n^{ij} \in l^1 \). Hence, \( \hat{q}_n^{ij} \in C(\mathbb{T}^d) \), \( i, j = 0, 1 \).

Proof (i) It follows from the bound (2.11) that \( \sum_{y \in \mathbb{Z}^d} |Q_0^{ij}(x,y)| \leq \sum_{x \in \mathbb{Z}^d} h(|z|) < \infty \). Similarly, \( \sum_{x \in \mathbb{Z}^d} |Q_0^{ij}(x,y)| \leq C < \infty \) for all \( y \in \mathbb{Z}^d \). This implies the bound (5.1) by the Shur lemma.

(ii) The bound (2.11) and condition (2.14) imply the same bound for \( q_n^{ij}(z) \), i.e., \( |q_n^{ij}(z)| \leq h(|z|) \), where \( r^{d-1}h(r) \in L^1(0, +\infty) \). Hence, \( q_n^{ij} \in l^1 \).

Lemma 5.2 Let the conditions E1–E3, E6, S1, S2 hold, and \( \alpha < -d/2 \). Then the bound (2.28) is true.

This lemma can be proved by a same way as in [9]. We repeat the proof since some notations and technical bounds obtaining in the proof we apply in Sec. 5.2 below.

Proof Note first that
\[ \mathbb{E}\|Y(\cdot, t)\|^2_\alpha = \sum_{x \in \mathbb{Z}^d} \langle x \rangle^{2\alpha} \left( \text{tr} Q^0_t(x,x) + \text{tr} Q^1_t(x,x) \right), \quad (5.2) \]
where \( \alpha < -d/2 \). Hence, to prove (2.28) it suffices to check that
\[ \sup_{t \in \mathbb{R}} \sup_{x,y \in \mathbb{Z}^d} \|Q_t(x,y)\| \leq C < \infty. \quad (5.3) \]
Applying the Fourier transform to (2.2) we obtain
\[
\hat{Y}(t) = \hat{A}(\theta)\hat{Y}(t), \quad t \in \mathbb{R}, \quad \hat{Y}(0) = \hat{Y}_0.
\] (5.4)

Here we denote \(\hat{A}(\theta) = \begin{pmatrix} 0 & 1 \\ -\bar{V}(\theta) & 0 \end{pmatrix}\), \(\theta \in \mathbb{T}^d\). Therefore, the solution \(\hat{Y}(\theta, t)\) of (5.4) admits the representation \(\hat{Y}(\theta, t) = \hat{G}_t(\theta)\hat{Y}_0(\theta)\) with \(\hat{G}_t(\theta) := \exp\left(\hat{A}(\theta)t\right)\). In the coordinate space, we have
\[
Y(x, t) = \sum_{x' \in \mathbb{Z}^d} G_t(x - x')Y_0(x'), \quad x \in \mathbb{Z}^d.
\] (5.5)

The Green function \(G_t(x)\) has a form (in Fourier transform)
\[
\hat{G}_t(\theta) = \begin{pmatrix} \cos \Omega t & \sin \Omega t & \Omega^{-1} \\ -\sin \Omega t & \cos \Omega t \end{pmatrix},
\] (5.6)
where \(\Omega = \Omega(\theta)\) is the Hermitian matrix defined by (2.6). Then
\[
\hat{G}_t(\theta) = \cos \Omega t I + \sin \Omega t C(\theta),
\] (5.7)
where \(C(\theta)\) is defined by (2.22). The representation (5.5) gives
\[
Q_{ij}^k(x, y) = \mathbb{E}\left(Y_i(x, t) \otimes Y_j(y, t)\right) = \sum_{x', y' \in \mathbb{Z}^d} \sum_{k, l = 0, 1} G_{kl}^t(x-x')Q_0^{kl}(x', y')G_{ij}^t(y-y')
\]
\[
= \langle Q_0(x', y'), \Phi_{ik}^t(x', t) \otimes \Phi_{ij}^t(y', t) \rangle,
\] (5.8)
where
\[
\Phi_{ik}^t(x', t) := \left(\varrho_{ik}^t(x-x'), \varrho_{ij}^{tt}(x-x')\right), \quad x' \in \mathbb{Z}^d, \quad i = 0, 1.
\]

Note that the Parseval identity, (5.6) and the condition E6 imply
\[
\|\Phi_{ik}^t(\cdot, t)\|_2^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |\Phi_{ik}^t(\theta, t)|^2 d\theta = (2\pi)^{-d} \int_{\mathbb{T}^d} \left( |\hat{G}_t^{00}(\theta)|^2 + |\hat{G}_t^{11}(\theta)|^2 \right) d\theta \leq C_0 < \infty.
\]

Then the bound (5.1) gives
\[
|Q_{ij}^k(x, y)| = |\langle Q_0(x', y'), \Phi_{ik}^t(x', t) \otimes \Phi_{ij}^t(y', t) \rangle| \leq C \|\Phi_{ik}^t(\cdot, t)\|_2 \|\Phi_{ij}^t(\cdot, t)\|_2 \leq C_1 < \infty,
\] (5.9)
where the constant \(C_1\) does not depend on \(x, y \in \mathbb{Z}^d, \ t \in \mathbb{R}\).

Proposition 2.12 follows from the bound (2.28) by the Prokhorov Theorem [24, Lemma II.3.1] using the method of [24, Theorem XII.5.2], since the embedding \(\mathcal{H}_\alpha \subset \mathcal{H}_\beta\) is compact if \(\alpha > \beta\).

### 5.2 Proof of Theorem 2.6

To prove Theorem 2.6 it suffices to check that for all \(\Psi \in \mathcal{S}\),
\[
Q_t(\Psi, \Psi) \to Q_\infty(\Psi, \Psi), \quad t \to \infty.
\] (5.10)

In the cases when \(k = 0\) and \(k = 1\), the convergence (5.10) was proved in [7] and [8], respectively. We derive (5.10) for any \(k \geq 1\).
Definition 5.3  
(i) The critical set is \( \mathcal{C} := \mathcal{C}_{\ast} \cup \mathcal{C}_{0} \cup \mathcal{C}_{\sigma} \) with \( \mathcal{C}_{\ast} \) as in Lemma 2.2 and sets \( \mathcal{C}_{0} \) and \( \mathcal{C}_{\sigma} \) defined by (2.8).

(ii) \( \mathcal{S}^{0} := \{ \Psi \in \mathcal{S} : \Psi(\theta) = 0 \text{ in a neighborhood of } \mathcal{C} \} \).

Note first that \( \text{mes} \mathcal{C} = 0 \). This fact can be proved by a similar way as Lemmas 2.2 and 2.3 in [7] since \( \mathcal{C} \neq \mathbb{T}^{d} \). Secondly, we write the scalar product \( \langle Y(\cdot, t), \Psi \rangle \) in a form

\[
\langle Y(\cdot, t), \Psi \rangle = \langle Y_{0}, \Phi(\cdot, t) \rangle, \quad \text{where } \Phi(x, t) := F_{\gamma \to x}^{-1}[\hat{\Psi}_{t}(\theta)\Psi(\theta)].
\]

Therefore,

\[
Q_{t}(\Psi, \Psi) = \mathbb{E}|\langle Y(\cdot, t), \Psi \rangle|^{2} = \langle Q_{0}(x, y), \Phi(x, t) \otimes \Phi(y, t) \rangle,
\]

where the Parseval identity and (5.6) yield

\[
\|\Phi(\cdot, t)\|_{2}^{2} = (2\pi)^{-d} \int_{\mathbb{T}^{d}} \|\hat{\Phi}_{t}(\theta)\|^{2} |\Psi(\theta)|^{2} d\theta \leq C \int_{\mathbb{T}^{d}} (1 + \|V^{-1}(\theta)\|) |\hat{\Psi}(\theta)|^{2} d\theta =: C\|\Psi\|_{\mathcal{L}^{2}}^{2}.
\]

By (5.1), (5.11) and (5.12), the uniform bounds hold, \( \sup_{t \in \mathbb{R}} |Q_{t}(\Psi, \Psi)| \leq C\|\Psi\|_{\mathcal{L}^{2}}^{2}, \quad \Psi \in \mathcal{S} \).

Therefore, it suffices to prove the convergence (5.10) for \( \Psi \in \mathcal{S}^{0} \) only.

We define a matrix \( Q_{s}(x, y), \ x, y \in \mathbb{Z}^{d} \), as follows

\[
Q_{s}(x, y) = \frac{1}{2^{k}} \sum_{n \in \mathbb{N}^{k}} q_{n}(x - y) \left( 1 + (-1)^{n_{1}} \text{sign } y_{1} \right) \ldots \left( 1 + (-1)^{n_{k}} \text{sign } y_{k} \right)
\]

\[
= \frac{1}{2^{k}} \sum_{n \in \mathbb{N}^{k}} q_{n}(x - y) \left[ 1 + \sum_{m=1}^{k} \sum_{(p_{1}, \ldots, p_{m}) \in \mathcal{P}_{m}(k)} (-1)^{n_{p_{1}} + \ldots + n_{p_{m}}} \text{sign } y_{p_{1}} \ldots \text{sign } y_{p_{m}} \right],
\]

with the matrices \( q_{n}(x) \) introduced in the condition S3. For instance, for \( k = 1 \),

\[
Q_{s}(x, y) = \frac{1}{2} \left( q_{1}(x - y) + q_{2}(x - y) \right) + \frac{1}{2} \left( q_{2}(x - y) - q_{1}(x - y) \right) \text{sign } y_{1}.
\]

Note that \( Q_{s}(x, y) = q_{n}(x - y) \) in every region \( \{(x, y) \in \mathbb{Z}^{2d} : (-1)^{n_{1}} y_{1} > 0, \ldots, (-1)^{n_{k}} y_{k} > 0\} \), \( n = (n_{1}, \ldots, n_{k}) \in \mathbb{N}^{k} \). Denote \( Q_{s}(x, y) = Q_{0}(x, y) - Q_{s}(x, y) \). Therefore, the convergence (5.10) follows from (5.11) and the following proposition.

Proposition 5.4  For any \( \Psi \in \mathcal{S}^{0} \), the following assertions hold.

(a) \( \lim_{t \to \infty} \langle Q_{s}(x, y), \Phi(x, t) \otimes \Phi(y, t) \rangle = \langle q_{\infty}(x - y), \Psi(x) \otimes \Psi(y) \rangle \).

(b) \( \lim_{t \to \infty} \langle Q_{s}(x, y), \Phi(x, t) \otimes \Phi(y, t) \rangle = 0 \).

At first, we prove the auxiliary lemma.

Lemma 5.5  Let \( q(x) = (q^{ij}(x))_{i,j=0,1}, \ x \in \mathbb{Z}^{d} \), be \( 2n \times 2n \) matrix with \( n \times n \) entries \( q^{ij}(x) \) satisfying the bound \( |q^{ij}(x)| \leq h(|x|) \), where \( r^{d-1}h(r) \in L^{1}(0, +\infty) \). Assume that either the condition E5 holds or the condition E5′ is fulfilled with the matrices \( \hat{q}^{ij}(\theta) \) instead of \( \hat{q}_{n}^{ij}(\theta) \).

Then for any \( \Psi \in \mathcal{S}^{0} \),

\[
\lim_{t \to \infty} \langle q(x - y), \Phi(x, t) \otimes \Phi(y, t) \rangle = \langle q_{\infty}(x - y), \Psi(x) \otimes \Psi(y) \rangle.
\]
where \( q^0_{\infty}(\theta) = \sum_{\sigma=1}^{s} \Pi_{\sigma}(\theta) L_1^+ (\hat{q}(\theta)) \Pi_{\sigma}(\theta), \ \theta \in \mathbb{T}^d \setminus \mathcal{C}_* \). Moreover, for any \( k \in \{1, \ldots, d\} \),

\[
\lim_{t \to \infty} \langle q(x - y) \text{sign} y_1 \cdots \text{sign} y_k, \Phi(x, t) \otimes \Phi(y, t) \rangle = \langle q_k^k(x - y), \Psi(x) \otimes \Psi(y) \rangle,
\]

where the matrix \( q_k^k(x) \) has a form (in the Fourier transform)

\[
\hat{q}_k^k(\theta) = \sum_{\sigma=1}^{s} \Pi_{\sigma}(\theta) L_k (\hat{q}(\theta)) \Pi_{\sigma}(\theta) \text{sign}(\partial_{\theta_1} \omega_{\sigma}(\theta)) \cdots \text{sign}(\partial_{\theta_k} \omega_{\sigma}(\theta)), \ \theta \in \mathbb{T}^d \setminus \mathcal{C}_*.
\]

Here

\[
L_k (\hat{q}(\theta)) = \left\{ \begin{array}{ll}
L_1^+ (\hat{q}(\theta)), & \text{if } k \text{ is even}, \\
L_k^+ (\hat{q}(\theta)), & \text{if } k \text{ is odd}, \end{array} \right.
\]

(5.14)

where the expressions \( L_1^+ \) and \( L_2^- \) are introduced in (2.21).

**Proof** Using the Fourier transform, we have

\[
I_t := \langle q(x - y) \text{sign} y_1 \cdots \text{sign} y_k, \Phi(x, t) \otimes \Phi(y, t) \rangle = (2\pi)^{-d} \int_{\mathbb{T}^d} \left( F_{x \to y} \left[ q(x - y) \text{sign} y_1 \cdots \text{sign} y_k \right], \Phi(\theta, t) \otimes \Phi(\theta', t) \right) d\theta d\theta'.
\]

Note that \( F_{x \to y}(\text{sign} y) = i \text{PV} \left( 1/tg(\theta/2) \right), \ \theta \in \mathbb{T}^1, \ y \in \mathbb{Z}^1, \) where PV stands for the Cauchy principal part. Hence,

\[
F_{x \to y} \left[ q(x - y) \text{sign} y_1 \cdots \text{sign} y_k \right] = (2\pi)^d \delta(\hat{\theta} - \hat{\theta}') \hat{q}(\theta) \times
\]

\[
\times i^k \text{PV} \left( \frac{1}{tg((\theta_1 - \theta'_1)/2)} \right) \cdots \text{PV} \left( \frac{1}{tg((\theta_k - \theta'_k)/2)} \right),
\]

where \( \hat{\theta} = (\theta_{k+1}, \ldots, \theta_d) \). We choose a finite partition of unity

\[
\sum_{m=1}^{M} g_m(\theta) = 1, \ \theta \in \text{supp} \hat{\Psi},
\]

(5.15)

where \( g_m \) are nonnegative functions from \( C_0^\infty(\mathbb{T}^d) \), which vanish in a neighborhood of the set \( \mathcal{C} \) introduced in Definition 5.3 (i). Using equality \( \hat{\Phi}(\theta, t) = \hat{G}^*_{t}(\theta) \hat{\Psi}(\theta), \) formula (5.7), decomposition (2.7), and partition (5.15), we obtain

\[
I_t = (2\pi)^{-d-k} i^k \text{PV} \int_{\mathbb{T}^{d+k}} \frac{1}{tg((\theta_1 - \theta'_1)/2)} \cdots \frac{1}{tg((\theta_k - \theta'_k)/2)} \times
\]

\[
\times \left( \hat{G}_{t}(\theta) \hat{q}(\theta) \hat{G}_{t}^*(\theta'), \overline{\Psi}(\theta) \otimes \hat{\Psi}(\theta') \right) \bigg|_{\theta = (\theta, \hat{\theta})} d\theta d\theta' d\hat{\theta}
\]

\[
= (2\pi)^{-d-k} i^k \sum_{m,m' \sigma,\sigma'} \text{PV} \int_{\mathbb{T}^{d+k}} g_m(\theta) g_{m'}(\theta') \frac{1}{tg((\theta_1 - \theta'_1)/2)} \cdots \frac{1}{tg((\theta_k - \theta'_k)/2)} \times
\]

\[
\times \left( \Pi_{\sigma}(\theta) \hat{G}_{t,\sigma}(\theta) \hat{q}(\theta) \hat{G}_{t,\sigma}^*(\theta') \Pi_{\sigma'}(\theta'), \overline{\Psi}(\theta) \otimes \hat{\Psi}(\theta') \right) \bigg|_{\theta = (\theta, \hat{\theta})} d\theta d\theta' d\hat{\theta}.
\]

(5.16)
Here
\[ \hat{G}_{t,\sigma}(\theta) = \cos \omega_{\sigma}(\theta) t I + \sin \omega_{\sigma}(\theta) t C_{\sigma}(\theta), \quad C_{\sigma}(\theta) = \begin{pmatrix} 0 & 1/\omega_{\sigma}(\theta) \\ -\omega_{\sigma}(\theta) & 0 \end{pmatrix}. \tag{5.17} \]

By Lemma 2.2 we can choose the supports of \( g_m \) so small that the eigenvalues \( \omega_{\sigma}(\theta) \) and
the matrices \( \Pi_{\sigma}(\theta) \) are real-analytic functions inside the \( \text{supp} \, g_m \) for every \( m \). (We do not label
the functions by the index \( m \) to not overburden the notations.) Changing variables
\( \theta_j' \to \xi_j = \theta_j' - \theta_j, \, j = 1, \ldots, k \), in the inner integrals in the r.h.s. of (5.16), we obtain
\[ I_t = (2\pi)^{-d-k} (-i)^k \sum_{m, m', \sigma, \sigma'} \int_{\mathbb{T}^d} \left( g_m(\theta) \overline{\Psi}(\theta) \Pi_{\sigma}(\theta) \hat{G}_{t,\sigma}(\theta) \hat{q}(\theta) \right) \times \]
\[ \times \text{PV} \int_{\mathbb{T}^k} \frac{1}{\text{tg}(\xi_1/2)} \cdot \ldots \cdot \frac{1}{\text{tg}(\xi_k/2)} g_{m'}(\theta') \hat{G}_{t,\sigma'}^*(\theta') \Pi_{\sigma'}(\theta') \hat{\Psi}(\theta') \bigg| _{\theta'=(\theta+\xi, \hat{\theta})} d\xi \, d\theta. \tag{5.18} \]

It follows from Definition 5.3 that \( \partial_{\theta'} \omega_{\sigma'}(\theta') \neq 0 \) for \( \theta' \in \text{supp} \, g_{m'} \subset \text{supp} \hat{\Psi} \). The next lemma follows from [1, Proposition A.4 i), ii)].

**Lemma 5.6** Let \( \chi(\theta) \in C^1(\mathbb{T}^d) \) and \( \partial_{\theta_1} \omega_{\sigma}(\theta) \neq 0 \) for \( \theta \in \text{supp} \chi \). Then for \( \theta \in \text{supp} \chi \),
\[ P_{\sigma}(\theta, t) := \text{PV} \int_{\mathbb{T}^d} \frac{e^{\pm i\omega_{\sigma}(\theta_1+\xi, \hat{\theta}) t}}{\text{tg}(\xi/2)} \chi(\theta_1 + \xi, \hat{\theta}) \; d\xi \]
\[ = \pm 2\pi i \chi(\theta) e^{\pm i\omega_{\sigma}(\theta_1 t)} \text{sign}(\partial_{\theta_1} \omega_{\sigma}(\theta)) + o(1) \quad \text{as} \quad t \to +\infty, \tag{5.19} \]
where \( \hat{\theta} = (\theta_2, \ldots, \theta_d) \). Moreover, \( \sup_{t \in \mathbb{R}, \theta \in \mathbb{T}^d} |P_{\sigma}(\theta, t)| < \infty \). Furthermore, using (5.17), we have
\[ \text{PV} \int_{\mathbb{T}^k} \frac{1}{\text{tg}(\xi/2)} \hat{G}_{t,\sigma}^*(\theta_1 + \xi, \hat{\theta}) \chi(\theta_1 + \xi, \hat{\theta}) \; d\xi \]
\[ = 2\pi \chi(\theta) C_{\sigma}^*(\theta) \hat{G}_{t,\sigma}(\theta) \text{sign}(\partial_{\theta_1} \omega_{\sigma}(\theta)) + o(1) \]
as \( t \to +\infty \).

Applying Lemma 5.6 to the inner integrals w.r.t. \( \xi_1, \ldots, \xi_k \) in (5.18), we obtain
\[ I_t = (2\pi)^{-d} (-i)^k \sum_{m, \sigma, \sigma'} g_m(\theta) \left( \Pi_{\sigma}(\theta) R_{t,\sigma}(\theta) \Pi_{\sigma'}(\theta), \hat{\Psi}(\theta) \otimes \overline{\Psi}(\theta) \right) d\theta + o(1), \tag{5.20} \]
where we denote \( R_{t,\sigma}(\theta)_{\sigma' \sigma} := \hat{G}_{t,\sigma}(\theta) \hat{q}(\theta) (C_{\sigma'}(\theta))^k \hat{G}_{t,\sigma'}^*(\theta) \). Note that \( (C_{\sigma'}(\theta))^k = (-1)^l \) if \( k = 2l \), and \( (C_{\sigma'}(\theta))^k = (-1)^l C_{\sigma'}(\theta) \) if \( k = 2l + 1 \) (with any \( l \geq 0 \)). Using (5.17), we have
\[ R_{t,\sigma}(\theta)_{\sigma' \sigma} = \begin{cases} (-1)^l \sum_{\pm} \left( \cos \left( \omega_{\sigma' \sigma}(\theta) t \right) L_1^\pm \hat{q} + \sin \left( \omega_{\sigma' \sigma}(\theta) t \right) L_2^\pm \hat{q} \right), & k = 2l, \\ (-1)^l \sum_{\pm} \left( \pm \cos \left( \omega_{\sigma' \sigma}(\theta) t \right) L_1^\pm \hat{q} + \mp \sin \left( \omega_{\sigma' \sigma}(\theta) t \right) L_2^\pm \hat{q} \right), & k = 2l + 1, \end{cases} \tag{5.21} \]
where \( \omega_{\sigma' \sigma}(\theta) \equiv \omega_{\sigma}(\theta) \pm \omega_{\sigma'}(\theta) \). The oscillatory integrals in (5.20) with \( \omega_{\sigma' \sigma}(\theta) \neq \text{const} \) vanish
as \( t \to \infty \) by the Lebesgue–Riemann Theorem, since all integrands in (5.20) are summable
by Lemma 5.4 (ii). Furthermore, the identities \( \omega_{\sigma' \sigma}(\theta) \equiv \text{const}_{\pm} \) with the \( \text{const}_{\pm} \neq 0 \) are
impossible by E5. If we impose the condition E5’ (with \( \hat{q}^{ij}(\theta) \) instead of \( \hat{q}_n^{ij}(\theta) \)), then the case
\[ \omega_{\sigma \sigma'}(\theta) = \text{const}_+ \] (with \( \text{const}_\pm \neq 0 \)) is possible. However, in this case, \( \Pi_\sigma(\theta) L^T_q(\hat{q}(\theta)) \Pi_{\sigma'}(\theta) \equiv 0 \) and \( \Pi_{\sigma}(\theta) L^T_2(\hat{q}(\theta)) \Pi_{\sigma}(\theta) \equiv 0 \), which implies that \( \Pi_\sigma(\theta) R_\sigma^k(\theta) \Pi_{\sigma'}(\theta) \equiv 0 \). Thus, only the integrals with \( \omega_{\sigma \sigma'}(\theta) \equiv 0 \) contribute to the limit, since \( \omega_{\sigma \sigma'}(\theta) \equiv 0 \) would imply \( \omega_{\sigma}(\theta) \equiv \omega_{\sigma'}(\theta) \equiv 0 \) which is impossible by \textbf{E4}. Therefore, using (5.20) and (5.21), we obtain

\[
I_t = (2\pi)^{-d} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{N} \int_{\mathbb{T}^d} g_m(\theta) \left( \Pi_{\sigma}(\theta) L_k(\hat{q}(\theta)) \Pi_{\sigma}(\theta), \hat{\Psi}(\theta) \otimes \overline{\hat{\Psi}(\theta)} \right) d\theta + o(1), \quad t \to \infty,
\]

where \( L_k \) is defined in (5.14). Lemma 5.5 is proved.

Now Proposition 5.4 follows from the decomposition (5.13), formulas (2.18)–(2.21) and Lemma 5.5 with the matrices \( q(x) \equiv q_\mu(x) \). We prove Proposition 5.4 (b) using the methods of \cite{1} p.140 and \cite{8}. At first, note that

\[
\langle Q_r(x, y), \Phi(x, t) \otimes \Phi(y, t) \rangle =: \sum_{z \in \mathbb{Z}^d} F_t(z), \quad (5.22)
\]

where

\[
F_t(z) := \sum_{y \in \mathbb{Z}^d} \left( Q_r(y + z, y), \Phi(y + z, t) \otimes \Phi(y, t) \right). \quad (5.23)
\]

The estimates (2.11) and (2.14) imply the estimate for \( Q_r(x, y) : |Q_r(x, y)| \leq h(|x - y|) \). Hence, the Cauchy–Schwartz inequality and (5.12) give

\[
|F_t(z)| \leq \sum_{y \in \mathbb{Z}^d} |Q_r(y + z, y)||\Phi(y + z, t)||\Phi(y, t)| \leq h(|z|) \sum_{y \in \mathbb{Z}^d} |\Phi(y, t)| \leq C_1 h(|z|) \|

\]

\[
\sum_{y \in \mathbb{Z}^d} |\Phi(y, t)| \leq C_1 \|
\]

where \( \|

\]

\[
\sum_{y \in \mathbb{Z}^d} |\Phi(y, t)| \leq C_1 < \infty,
\]

and the series (5.22) converges uniformly in \( t \). Therefore, it suffices to prove that

\[
\lim_{t \to \infty} F_t(z) = 0 \quad \text{for each } z \in \mathbb{Z}^d. \quad (5.26)
\]

Let us prove (5.26). By (2.12) and (5.13), \( Q_r(\bar{x}, \bar{y}) = q_r(\bar{x}, \bar{y} - \bar{y}) \). By (2.14), \( \forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N} \) such that \( \forall \bar{y} \in \mathbb{Z}^k : |y_j| > N(\varepsilon), \forall j = 1, \ldots, k, |q_r(\bar{y} + \bar{z}, \bar{y})| < \varepsilon \) for any fixed \( \bar{z} = (\bar{z}, \bar{z}) \in \mathbb{Z}^d \). Hence, by (5.12) and the condition \textbf{E6},

\[
\left| \sum_{y \in \mathbb{Z}^d : |y_j| > N, \forall j = 1, \ldots, k} \left( Q_r(y + z, y), \Phi(y + z, t) \otimes \Phi(y, t) \right) \right| \leq \varepsilon \sum_{y \in \mathbb{Z}^d} |\Phi(y, t)| \leq \varepsilon C(\Psi),
\]

uniformly on \( t \in \mathbb{R} \). Let us fix \( N = N(\varepsilon) \). Using (5.23), we obtain

\[
|F_t(z)| \leq \varepsilon C(\Psi) + C \sum_{j=1}^{k} \sum_{|y_j| < N} \sum_{y' \in \mathbb{Z}^d - 1} |\Phi(y + z, t)||\Phi(y, t)|
\]

\[
\leq \varepsilon C(\Psi) + C \sum_{j=1}^{k} \sum_{|y_j| < N} \sqrt{\sum_{y' \in \mathbb{Z}^d - 1} |\Phi(y + z, t)|^2} \sqrt{\sum_{y' \in \mathbb{Z}^d - 1} |\Phi(y, t)|^2},
\]

where \( L_k \) is defined in (5.14). Lemma 5.5 is proved.
where, for simplicity, we write \( y = (y_j, y_j') \), \( y' = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_d) \). To prove (5.26), we fix \( j \in \{1, \ldots, k\} \) and verify that for any fixed values of \( y_j \in \mathbb{Z} : |y_j| < N \) and \( z \in \mathbb{Z}^d \),

\[
\sum_{y' \in \mathbb{Z}^{d-1}} |\Phi(y + z, t)|^2 \bigg|_{y=(y_j, y')} \to 0 \quad \text{as} \quad t \to \infty.
\]

Without loss of generality, put \( j = 1 \), \( y = (y_1, y_j') \), \( y' = (y_2, \ldots, y_d) \). By the Parseval identity,

\[
\sum_{y' \in \mathbb{Z}^{d-1}} |\Phi(y + z, t)|^2 = (2\pi)^{-2d+2} \int_{\mathbb{T}^{d-1}} |F_{y' \mapsto \theta'}[\Phi(y + z, t)]|^2 \, d\theta'.
\] (5.28)

It remains to prove that the integral in the r.h.s. of (5.28) tends to zero as \( t \to \infty \) for fixed \( z \in \mathbb{Z}^d \) and \( |y_1| < N \). Note first that the integrand in (5.28) satisfies the following uniform bound,

\[
|F_{y' \mapsto \theta'}[\Phi(y + z, t)]|^2 \leq G(\theta'), \quad t \geq 0,
\]

where \( G(\theta') \in L^1(\mathbb{T}^{d-1}) \). (5.29)

Indeed, let us rewrite the function \( F_{y' \mapsto \theta'}[\Phi(y + z, t)] \) in the form

\[
F_{y' \mapsto \theta'}[\Phi(y + z, t)] = \frac{1}{2\pi} \int_{\mathbb{T}^1} e^{-i\theta_1 y_1} e^{-i(\theta,z)} \hat{\Phi}(\theta, t) \, d\theta_1 = \frac{1}{2\pi} \int_{\mathbb{T}^1} e^{-i\theta_1 y_1} e^{-i(\theta,z)} \hat{G}_{\theta}^* (\theta) \hat{\Psi}(\theta) \, d\theta_1.
\]

Therefore,

\[
|F_{y' \mapsto \theta'}[\Phi(y + z, t)]|^2 \leq C \left( \int_{\mathbb{T}^1} \| \hat{G}_{\theta}^* (\theta) \| \| \hat{\Psi}(\theta) \| \, d\theta_1 \right)^2 \leq C_1 \int_{\mathbb{T}^1} \| \hat{G}_{\theta}^* (\theta) \|^2 \| \hat{\Psi}(\theta) \|^2 \, d\theta_1 
\]

\[
\leq C_2 \int_{\mathbb{T}^1} \| (1 + \| \hat{\Psi}(\theta) \|) \| \hat{\Psi}(\theta) \| \| \hat{\Psi}(\theta) \| \, d\theta_1 := G(\theta')
\]

and (5.29) follows from the condition E6. Therefore, it suffices to prove that the integrand in the r.h.s. of (5.28) tends to zero as \( t \to \infty \) for a.a. fixed \( \theta' \in \mathbb{T}^{d-1} \). We use the finite partition of unity (5.15), formulas (5.7) and (2.7) and split the function \( F_{y' \mapsto \theta'}[\Phi(y + z, t)] \) into the sum of the integrals:

\[
F_{y' \mapsto \theta'}[\Phi(y + z, t)] = \sum_{m} \sum_{\pm} \sum_{\sigma=1}^{s} \int_{\mathbb{T}^1} g_m(\theta) e^{-i\theta_1 y_1} e^{-i(\theta,z)e^{\pm i\omega(\theta)\theta}} a_{\sigma}^\pm (\theta) \hat{\Psi}(\theta) \, d\theta_1, \quad \Psi \in \mathcal{S}^0,
\] (5.30)

where

\[
a_{\sigma}^\pm (\theta) := \frac{1}{2} \left( \frac{1}{\mp i \omega(\theta)} \pm i \omega(\theta) \right) \Pi_{\sigma}(\theta).
\]

The eigenvalues \( \omega(\theta) \) and the matrices \( a_{\sigma}^\pm (\theta) \) are real-analytic functions inside the supp \( g_m \) for every \( m \). It follows from Definition 5.3 (i) and the conditions E4, E6 that mes \( \{ \theta_1 \in \mathbb{T}^1 : \partial_{\theta_1} \omega(\theta) = 0 \} = 0 \) for a.a. fixed \( \theta' \in \mathbb{T}^{d-1} \). Hence, the integrals in (5.30) vanish as \( t \to \infty \) by the Lebesgue–Riemann theorem.

6 Harmonic crystals in the half-space

In this section, we consider the dynamics of the harmonic crystals in the half-space \( \mathbb{Z}^d_+ = \{ x \in \mathbb{Z}^d : x_1 > 0 \} \), \( d \geq 1 \),

\[
\ddot{u}(x,t) = -\sum_{y \in \mathbb{Z}^d_+} (V(x-y) - V(x-y_-)) u(y,t), \quad x \in \mathbb{Z}_+, \quad t \in \mathbb{R},
\] (6.1)
imposed on $V$, some $\alpha$.

**Definition 6.1** $H$ space $U$.

In particular, $Q$ and with the initial data (as $t = 0$)

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad x \in \mathbb{Z}_d^d. \quad (6.3)$$

The matrix $V(x)$ satisfies the conditions $E1–E4$. In addition, we assume that

$$V(x) = V(x). \quad (6.4)$$

This condition is fulfilled, for instance, for the nearest neighbor crystal [29]. The condition $E6$ imposed on $V(x)$ in Sec. 2.1 can be weakened as follows.

$E6'$ $\int_{\mathbb{T}^d} \sin^2(\theta_1)\|\hat{V}^{-1}(\theta)\|d\theta < \infty$

Assume that the initial date $Y_0 = (u_0, v_0)$ of the problem (6.1)–(6.3) belongs to the phase space $H_{\alpha,+}$, $\alpha \in \mathbb{R}$.

**Definition 6.1** $H_{\alpha,+}$ is the Hilbert space of $\mathbb{R}^n \times \mathbb{R}^n$-valued functions of $x \in \mathbb{Z}_d^d$ endowed with the norm $\|Y\|^2_{H_{\alpha,+}} = \sum_{x \in \mathbb{Z}_d^d}(x)2^{2n}\|Y(x)\|^2 < \infty$.

To coordinate the boundary and initial conditions we suppose that $u_0(x) = v_0(x) = 0$ for $x_1 = 0$.

**Lemma 6.2** (see [17], Corollary 2.4) Let the conditions (6.4), $E1$, and $E2$ hold, and choose some $\alpha \in \mathbb{R}$. Then for any $Y_0 \in H_{\alpha,+}$, there exists a unique solution $Y(t) \in C(\mathbb{R}, H_{\alpha,+})$ to the mixed problem (6.1)–(6.3). The operator $U_+(t): Y_0 \mapsto Y(t)$ is continuous in $H_{\alpha,+}$.

Below we assume that $\alpha < -d/2$ if the condition $E6$ holds and $\alpha < -(d + 1)/2$ if the condition $E6'$ holds.

We suppose that $Y_0$ is a measurable random function with values in $(H_{\alpha,+}, \mathcal{B}(H_{\alpha,+}))$ and denote by $\mu_0^+$ a Borel probability measure on $H_{\alpha,+}$ giving the distribution of $Y_0$. Let $E_+$ stand for the integral w.r.t. $\mu_0^+$, and denote by $Q_0^+(x, y)$ the initial covariance of $\mu_0^+$,

$$Q_0^+(x, y) = E_+(Y_0(x) \otimes Y_0(y)) = \int (Y_0(x) \otimes Y_0(y)) \mu_0^+(dY_0), \quad x, y \in \mathbb{Z}_d^d.$$

In particular, $Q_0^+(x, y) = 0$ for $x_1 = 0$ or $y_1 = 0$. We assume that $\mu_0^+$ satisfies the conditions $S1$ and $S2$ stated in Sec. 2.1. The condition $S3$ needs in some modification.

**S3** Choose some $k \in \{1, \ldots, d\}$. The initial covariance $Q_0^+(x, y)$ has a form

$$Q_0^+(x, y) = q_0^+(x, y) = q_0^+(\bar{x}, \bar{y}, \bar{z} - \bar{y}), \quad x, y \in \mathbb{Z}_d^d,$$

where $x = (\bar{x}, \bar{x}), \quad \bar{x} = (x_1, \ldots, x_k), \quad \bar{z} = (x_{k+1}, \ldots, x_d)$. Write (cf [2.13])

$$\mathcal{N}_+^k := \{n = (n_1, n_2, \ldots, n_k), \quad n_1 = 2, \quad n_j \in \{1, 2\} \text{ for all } j = 2, \ldots, k\}.$$ Suppose that $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}$ such that for any $\bar{y} = (y_1, \ldots, y_k) \in \mathbb{Z}^k$:

$$y_1 > N(\varepsilon) \quad \text{and} \quad (-1)^{n_j}y_j > N(\varepsilon), \quad \forall j = 2, \ldots, k,$$

the following bound holds (cf [2.14])

$$|q_0^+(\bar{y} + \bar{z}, \bar{y}, \bar{z}) - q_n(z)| < \varepsilon \quad \text{for any fixed } z = (\bar{z}, \bar{z}) \in \mathbb{Z}_d^d.$$

Here $q_n(z)$, $n \in \mathcal{N}_+^k$, are the correlation matrices of some translation-invariant measures $\mu_n$ with zero mean value in $H_\alpha$. 

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In particular, if \( k = 1 \), then \( Q^+_0(x, y) = q^+_0(x_1, y_1, \tilde{x} - \tilde{y}) \), \( \tilde{x} = (x_2, \ldots, x_d) \), and (cf (2.15))

\[
q^+_0(y_1 + z_1, y_1, \tilde{z}) \to q_2(z) \quad \text{as} \quad y_1 \to +\infty.
\]

(6.7)

**Example 6.3** The example of \( \mu_0^+ \) satisfying the conditions S1–S3 can be constructed by a similar way as for \( \mu_0 \) in Sec. 3. Indeed, define a Borel probability measure \( \mu_0 \) as a distribution of the random function (cf (3.3))

\[
Y_0(x) = \sum_{n \in \mathcal{N}^k_+} \zeta_n(\tilde{x})Y_n(x), \quad x = (\tilde{x}, \bar{x}) \in \mathbb{Z}^d_+, \quad \bar{x} = (x_1, \ldots, x_k), \quad \tilde{x} = (x_{k+1}, \ldots, x_d),
\]

where \( \zeta_n(\tilde{x}) = \zeta_2(x_1)\zeta_{n_2}(x_2) \cdots \zeta_{n_k}(x_k) \) for \( \tilde{x} = (x_1, \ldots, x_k) \), the functions \( \zeta_n \) are defined in (3.2), \( Y_n(x), \ n \in \mathcal{N}^k_+ \), are Gaussian independent vectors in \( \mathcal{H}_{\alpha,+} \) with distributions \( \mu_n \).

We define \( \mu_1^+, \ t \in \mathbb{R} \), as a Borel probability measure in \( \mathcal{H}_{\alpha,+} \) which gives the distribution of the random solution \( Y(t) \), \( \mu^+_1(B) = \mu_0(U_{\alpha,-}(t)B), \ B \in \mathcal{B}(\mathcal{H}_{\alpha,+}) \), \( t \in \mathbb{R} \). Denote by \( Q^+_1(x, y) = \int (Y(x) \otimes Y(y)) \mu^+_1(dY_0), \ x, y \in \mathbb{Z}^d_+ \), the covariance of \( \mu^+_1 \). The mixing condition S4 (see Sec. 2.2) for \( \mu_0^+ \) is formulated as for the measure \( \mu_0 \) but with sets \( \mathcal{A} \) and \( \mathcal{B} \) from \( \mathbb{Z}^d_+ \) instead of \( \mathbb{Z}^d \).

Introduce the limiting correlation matrix \( Q^+_\infty(x, y) \). It has a form

\[
Q^+_\infty(x, y) = q^+_\infty(x - y) - q^+_\infty(x_--y_+) - q^+_\infty(x_+ - y_-), \quad x, y \in \mathbb{Z}^d_+.
\]

(6.8)

Here \( q^+_\infty(x) \) is defined as \( q_\infty(x) \) (see formulas (2.18)–(2.22)) but with \( \mathcal{N}^k_+ \) instead of \( \mathcal{N}^k \). For example, if \( k = 1 \), then \( \hat{q}^+_{\infty}(\theta) \) has a form (2.18) with matrices (cf (2.23))

\[
M^+_{1,\sigma}(\theta) = \frac{1}{2} L^+_1(\hat{q}_2(\theta)), \quad M^-_{1,\sigma}(\theta) = \frac{1}{2} L^-_2(\hat{q}_2(\theta)) \text{ sign}(\partial_1 \omega_\sigma(\theta)),
\]

where \( \hat{q}_2(\theta) \) is the Fourier transform of the matrix \( q_2(z) \) introduced in (6.7).

**Theorem 6.4** Assume that \( \alpha < -d/2 \) if the condition E6 holds and \( \alpha < -(d+1)/2 \) if the condition E6’ holds. Then the following assertions are valid.

(i) Let the conditions (6.4), E1–E4, E5’, E6’, and S1–S3 be fulfilled. Then for any \( x, y \in \mathbb{Z}^d \), \( Q^+_1(x, y) \to Q^+_\infty(x, y) \) as \( t \to \infty \), where \( Q^+_\infty \) is defined in (6.8).

(ii) Let the conditions (6.4), E1–E3, E4’, E5’, E6’, S1, S3, and S4 be fulfilled. Then the measures \( \mu_t \) weakly converge in the Hilbert space \( \mathcal{H}_{\alpha,+} \) as \( t \to \infty \). The limiting measure \( \mu^+_\infty \) is a Gaussian measure on \( \mathcal{H}_{\alpha,+} \) with the covariance \( Q^+_\infty(x, y) \) defined in (6.8).

The second assertion of Theorem 6.4 can be proved by a similar way as Theorem 2.11. The proof of the first assertion has some features in comparison with the proof of Theorem 2.6 see Sec. 6.1 below.

### 6.1 The proof

**Lemma 6.5** (cf Lemma 6.2) Let the conditions (6.4), E1–E3, E6’, S1, and S2 be fulfilled. Then the uniform bound holds, \( \sup_{t \in \mathbb{R}} \mathbb{E}_+ (\|Y(t)\|_{\alpha,+}^2) < \infty \).
Proof By \( l_+^2 \equiv l^2(\mathbb{Z}_+^d) \otimes \mathbb{R}^n \), \( d, n \geq 1 \), we denote the Hilbert space of sequences \( f(x) = (f_1(x), \ldots, f_n(x)) \) endowed with norm \( \|f\|_{l_+^2} = \sqrt{\sum_{x \in \mathbb{Z}_+^d} |f(x)|^2} \). Let \( \langle \cdot, \cdot \rangle_+ \) stand for the inner product in \( l_+^2 \) (or in \( l_+^2 \times l_+^2 \)). At first, by the conditions \( S1 \) and \( S2 \), we have (cf (5.1))
\[
\langle Q^+_0(x, y), \Phi(x) \otimes \Psi(y) \rangle_+ \leq C\|\Phi\|_{l_+^2} \|\Psi\|_{l_+^2} \quad \text{for any } \Phi, \Psi \in l_+^2 \times l_+^2.
\] (6.9)
Second, the solutions of the problem (6.1)–(6.3) have a form
\[
Y(x, t) = \sum_{x' \in \mathbb{Z}_+^d} G_{t,+}(x, x') Y_0(x'), \quad \text{where} \quad G_{t,+}(x, x') = G_t(x-x') - G_t(x-x_-),
\] (6.10)
with \( G_t(x) \) defined in (5.6). Similarly to (5.8), we have
\[
(Q^+_i(x, y))^{ij} = \langle Q^+_0(x', y'), \Phi^i(x', t) \otimes \Phi^j(y', t) \rangle_+, \quad x, y \in \mathbb{Z}_+^d,
\] (6.11)
where \( \Phi^i(x', t) := (G^0_{t,+}(x', x'), G^1_{t,+}(x', x')) \), \( i = 0, 1 \). By the Parseval identity, formula (5.6), the condition \( E6' \) and Fejér’s theorem, we have
\[
\|\Phi^i(\cdot, t)\|_{l_+^2}^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} \|\hat{\Phi}^i(x, t)\|^2 \, d\theta = (2\pi)^{-d}4 \int_{\mathbb{T}^d} \sin^2(\theta_1 x_1) \left( |\hat{G}^0_0(\theta)|^2 + |\hat{G}^1_1(\theta)|^2 \right) \, d\theta \leq C_3 + C_4 |x_1|,
\] (6.12)
where constants \( C_3 \) and \( C_4 \) do not depend on \( t \in \mathbb{R} \) and \( x \in \mathbb{Z}_+^d \), and \( C_4 = 0 \) if the condition \( E6 \) holds. Hence, (6.9), (6.11) and (6.12) imply
\[
| (Q^+_i(x, y))^{ij} | \leq C\|\Phi^i(\cdot, t)\|_{l_+^2} \|\Phi^j(\cdot, t)\|_{l_+^2} \leq C \sqrt{C_3 + C_4 |x_1|} \sqrt{C_3 + C_4 |y_1|}, \quad x, y \in \mathbb{Z}_+^d.
\]
Therefore, the choice of \( \alpha \) implies the following bound
\[
E_+(\|Y(\cdot, t)\|_{l_+^2}^2) = \sum_{x \in \mathbb{Z}_+^d} \langle x \rangle^{2\alpha} \text{tr} \left( (Q^+_i(x, x))^0 + (Q^+_i(x, x))^{11} \right) 
\leq C \sum_{x \in \mathbb{Z}_+^d} \langle x \rangle^{2\alpha} (C_3 + C_4 |x_1|) < \infty. \quad \blacksquare
\]

Proof of Theorem 6.4 (i): At first, using (6.10), we decompose the covariance \( Q^+_i(x, y) \) into a sum of four terms:
\[
Q^+_i(x, y) = \sum_{x', y' \in \mathbb{Z}_+^d} G_{t,+}(x, x') Q^+_0(x', y') G_{t,+}^T(y, y') = R_t(x, y) - R_t(x, y_-) - R_t(x_-, y) + R_t(x_-, y_-),
\]
where \( ()^T \) denotes matrix transposition,
\[
R_t(x, y) := \sum_{x', y' \in \mathbb{Z}_+^d} G_t(x-x') Q^+_0(x', y') G_t^T(y-y').
\]
Therefore, Theorem 6.4 (i) follows from the following convergence
\[
R_t(x, y) \to q^+_\infty(x-y) \quad \text{as} \quad t \to \infty, \quad x, y \in \mathbb{Z}_+^d.
\] (6.13)
To prove (6.13), let us define \( \bar{Q}_0^+(x,y) \) to be equal to \( Q_0^+(x,y) \) for \( x,y \in \mathbb{Z}_+^d \), and to 0 otherwise. Denote by \( Q_*^+(x,y) \) the matrix which is defined as \( Q_*(x,y) \) (see (5.13)) but with the summation over \( N_*^k \) instead of \( N^k \). Put \( Q_+^*(x,y) = \bar{Q}_0^+(x,y) - Q_*^+(x,y) \). Then (6.13) follows from the following assertions. For any \( x,y \in \mathbb{Z}_+^d \),

\[
\sum_{x',y' \in \mathbb{Z}_+^d} G_t(x-x')Q_*^+(x',y')G_t^T(y-y') \to q_+^+(x-y), \quad t \to \infty,
\]

\[
\sum_{x',y' \in \mathbb{Z}_+^d} G_t(x-x')Q_+^*(x',y')G_t^T(y-y') \to 0, \quad t \to \infty.
\]

The proof of these assertions are similarly to the proof of Proposition 5.4.

\[\blacktriangleleft\]

### 6.2 Energy current in the half-space

Here we calculate the limiting energy current density \( \mathbf{J}_{+\infty} = (J_{+\infty}^1, \ldots, J_{+\infty}^d) \).

**Lemma 6.6** If \( d = 1 \), then \( \mathbf{J}_{+\infty} = 0 \). If \( d \geq 2 \), the coordinates of the energy current density \( \mathbf{J}_{+\infty} \equiv \mathbf{J}_{+\infty}(x_1) \), \( x_1 \geq 0 \), are

\[
J_{+\infty}^l(x_1) = 0, \quad J_{+\infty}^l(x_1) = -\frac{2i}{(2\pi)^d} \int_{T^d} \sin^2(\theta_1 x_1) \frac{d}{d\theta_1} \left[ (\tilde{q}_+^l(\theta))^T \partial_\theta \hat{V}(\theta) \right] d\theta, \quad l = 2, \ldots, d, (6.14)
\]

with \( q_+^l \) from (6.8). In particular, \( \mathbf{J}_{+\infty}(0) = 0 \).

To prove (6.14), we first derive formally the expression of the energy current for the finite energy solutions \( u(x,t) \). For the region \( \Omega_l := \{ x \in \mathbb{Z}_+^d : x_1 \geq 0 \}, l \geq 1 \), we define the energy in \( \Omega_l \) as

\[
\mathcal{E}_l^+(t) := \frac{1}{2} \sum_{x \in \Omega_l} \left\{ |\partial_x u(x,t)|^2 + \sum_{y \in \mathbb{Z}_+^d} (u(x,t), (V(x-y) - V(x-y_-))u(y,t)) \right\}.
\]

Then, using Eqn (6.11), (6.4) and E2, we obtain

\[
\mathcal{E}_l^+(t) = 0, \quad \mathcal{E}_l^+ = \sum_{x' \in \mathbb{Z}_+^d} J_{+\infty}^l(x', t), \quad l = 2, \ldots, d.
\]

Here \( J_{+\infty}^l(x', t) \) stands for the energy current density in the direction \( e_l = (0, \delta_2, \ldots, \delta_d) \),

\[
J_{+\infty}^l(x', t) := \frac{1}{2} \sum_{y' \in \mathbb{Z}_+^d} \left\{ \sum_{m \leq -1, p \geq 0} A_{mp}^l(x', y', t) - \sum_{m \geq 0, p \leq -1} A_{mp}^l(x', y', t) \right\},
\]

where \( A_{mp}^l(x', y', t) := (\partial_x u(x,t), (V(x-y) - V(x-y_-))u(y,t)) \) for \( x \equiv x'+me_l, y \equiv y'+pe_l, x', y' \in \mathbb{Z}_+^d \) with \( x_l' = y_l' = 0 \), \( l = 2, \ldots, d \).

Let \( u(x,t) \) be the random solution to problem (6.1)–(6.3) with the initial measure \( \mu_0^+ \) satisfying S1–S3. Using Theorem 6.4 (i), we have

\[
\mathbb{E} \left( J_{+\infty}^l(x', t) \right) \to J_{+\infty}^l := \frac{1}{2} \sum_{y' \in \mathbb{Z}_+^d} \left\{ \sum_{m \leq -1, p \geq 0} B_{mp}^l(x', y') - \sum_{m \geq 0, p \leq -1} B_{mp}^l(x', y') \right\} \quad \text{as} \quad t \to \infty,
\]

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where \( B_{mp}^l(x', y') := \text{tr} \left[ Q_{\infty}^{10}(x, y) \left( V^T(x - y) - V^T(x - y_-) \right) \right] \), \( x \equiv x' + me_l \), \( y \equiv y' + pe_l \), \( x', y' \in \mathbb{Z}^d_+ \) with \( x'_l = y'_l = 0 \). Applying (6.8), we obtain
\[
J_{+,\infty}^l = -\frac{1}{2} \sum_{y \in \mathbb{Z}^d} y_t \left( (q_{\infty}^+(x' + y))^10 - (q_{\infty}^+(x'_- + y))^10 \right) (V^T(x' + y) - V^T(x'_- + y)).
\]

Using the equality \( V(x) = V(x_-) \) and applying the Fourier transform, we obtain (6.14). Lemma 6.6 is proved.

Let \( \mu_n = g_{\beta_n}, \ n \in \mathcal{N}_k^l \), be the Gibbs measures constructed in Sec. 4.2 with temperatures \( T_n > 0 \). The correlation matrices of \( \mu_n \) are \( q_n(x - y) \equiv q_{\beta_n}(x - y) \), see (4.2). We impose, in addition, the condition (3.13) on the matrix \( V \), which implies the bound (3.7) for \( q_{\mu_n}^{00} \). Then, the condition \textbf{S2} is fulfilled. Since
\[
(q_{\infty}^+(\theta))^10 = -i \sum_{\sigma=1}^s \omega^{-1}_\sigma(\theta) \Pi_\sigma(\theta) \left( \frac{1}{2k} \sum_{n \in \mathcal{N}_k^l} T_n S_{k,n}^{\text{odd}}(\omega(\theta)) \right),
\]
where the function \( S_{k,n}^{\text{odd}}(\omega) \) is defined in (2.20), then for \( l = 2, \ldots, d \) (cf (4.4))
\[
J_{+,\infty}^l(x_1) = -\frac{4}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} r_\sigma \sin^2(\theta_1 x_1) \left( \frac{1}{2k} \sum_{n \in \mathcal{N}_k^l} T_n S_{k,n}^{\text{odd}}(\omega(\theta)) \right) \frac{\partial \omega(\theta)}{\partial \theta_l} d\theta
\]
\[
= -\sum_{\text{odd } m \in \{1, \ldots, k\}} \sum_{(p_1, \ldots, p_m) \in \mathcal{P}_m(k)} c_{l_1 \ldots l_m}(x_1) \frac{1}{2k-1} \sum_{n \in \mathcal{N}_k^l} (-1)^{n_1 + \ldots + n_m} T_n, \quad (6.15)
\]
where the functions \( c_{l_1 \ldots l_m}(x_1), \ x_1 \in \mathbb{Z}^d_+ \), are defined as follows (cf (4.5))
\[
c_{l_1 \ldots l_m}(x_1) := \frac{2}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} r_\sigma \sin^2(\theta_1 x_1) \text{sign} \left( \frac{\partial \omega(\theta)}{\partial \theta_{l_1}} \right) \ldots \text{sign} \left( \frac{\partial \omega(\theta)}{\partial \theta_{l_m}} \right) \frac{\partial \omega(\theta)}{\partial \theta_l} d\theta. \quad (6.16)
\]
Write
\[
c_l(x_1) \equiv c_l^l(x_1) = \frac{2}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} r_\sigma \sin^2(\theta_1 x_1) \left| \frac{\partial \omega(\theta)}{\partial \theta_l} \right| d\theta > 0, \quad l = 2, \ldots, k. \quad (6.17)
\]
Applying the condition \textbf{SC} to \( \omega(\theta) \), we obtain
\[
J_{+,\infty}^l(x_1) = \left\{ \begin{array}{ll}
-c_l(x_1) \frac{1}{2k-1} \sum_{n \in \mathcal{N}_k^l} T_n \big| T_n \big|_{n_l=2} - T_n \big|_{n_l=1}, & l = 2, \ldots, k, \\
0, & l = 1, l = k + 1, \ldots, d.
\end{array} \right. \quad (6.18)
\]
where the summation \( \sum' \) is taken over \( n_2, \ldots, n_{l-1}, n_{l+1}, \ldots, n_k \in \{1, 2\} \). Using the formula
\[
2 \sin^2(\theta_1 x_1) = 1 - \cos(2\theta_1 x_1)
\]
and the Lebesgue–Riemann theorem, we see that \( c_l(x_1) \rightarrow c_l \) as \( x_1 \rightarrow +\infty \), where the positive constant \( c_l \) is defined in (4.3). Hence, for \( l = 2, \ldots, k \),
\[
J_{+,\infty}^l(x_1) \rightarrow -c_l \frac{1}{2k-1} \sum' T_n \big| T_n \big|_{n_l=2} - T_n \big|_{n_l=1} \quad \text{as } x_1 \rightarrow +\infty. \quad (6.19)
\]
Consider the particular cases of (6.15) and (6.18).
Example 6.7 Let \( k = 1 \) and \( \mu_0^+ \) satisfy the condition \( \textbf{S3} \) with a Gibbs measure \( \mu_2 \equiv g_\beta \), \( \beta = 1/T_2 \). For instance, the initial datum \( Y_0 \) has a form \( Y_0(x) = \zeta_2(x_1)Y_2(x) \), where \( \zeta_2 \) is defined in (3.2). \( Y_2 \) has the Gibbs distribution \( g_\beta \). Hence, \( J^1_{+,\infty} \equiv 0 \), and
\[
J^1_{+,\infty}(x_1) = -\frac{2T}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} r_\sigma \sin^2(\theta_1 x_1) \left( \frac{\partial \omega_\sigma(\theta)}{\partial \theta_1} \right) \frac{\partial \omega_\sigma(\theta)}{\partial \theta_2} d\theta, \quad l = 2, \ldots, d.
\]
Suppose that the eigenvalues \( \omega_\sigma(\theta) \) satisfy the symmetry conditions \( \textbf{SC} \). Then \( J_{+,\infty}(x_1) = 0 \) for any \( x_1 \geq 0 \).

Example 6.8 Let \( k = 2 \) and \( \mu_0^+ \) satisfy the condition \( \textbf{S3} \) with Gibbs measures \( \mu_n \equiv g_{\beta_n} \), \( \beta_n = 1/T_n \), \( n = (n_1, n_2) \in \mathcal{N}_+^2 = \{(2, 1); (2, 2)\} \). For instance, the initial datum \( Y_0 \) is of a form
\[
Y_0(x) = \zeta_2(x_1) \left( \zeta_1(x_2)Y_{21}(x) + \zeta_2(x_2)Y_{22}(x) \right), \quad x \in \mathbb{Z}_+^d,
\]
where \( \zeta_n(x) \) is defined in (3.2), \( Y_{21}(x) \) and \( Y_{22}(x) \) are independent vectors in \( \mathcal{H}_\alpha \) with Gibbs distributions \( \mu_{21} \) and \( \mu_{22} \), corresponding positive temperatures \( T_{21} \) and \( T_{22} \), respectively. Therefore, \( J^1_{+,\infty}(x_1) \equiv 0 \), and
\[
J^1_{+,\infty}(x_1) = -\frac{1}{(2\pi)^d} \sum_{\sigma=1}^s \int_{\mathbb{T}^d} r_\sigma \sin^2(\theta_1 x_1) \left[ \frac{\partial \omega_\sigma(\theta)}{\partial \theta_1} \right] (T_{21} + T_{22}) \left[ \frac{\partial \omega_\sigma(\theta)}{\partial \theta_2} \right] d\theta, \quad l = 2, \ldots, d.
\]
Under the additional conditions \( \textbf{SC} \) on the eigenvalues \( \omega_\sigma(\theta) \), we obtain
\[
J_{+,\infty}(x_1) = -\frac{1}{2} \left( 0, c_2(x_1)(T_{22} - T_{21}), 0, \ldots, 0 \right)
\]
with \( c_2(x_1) \) introduced in (6.17). Note that (cf (4.8), (4.9))
\[
J_{+,\infty}(x_1) \to -\frac{1}{2} \left( 0, c_2(T_{22} - T_{21}), 0, \ldots, 0 \right) \text{ as } x_1 \to +\infty,
\]
where the positive constant \( c_2 \) is defined in (4.6).

Remark 6.9 In [11], we consider the 1D chain of harmonic oscillators on the half-line with \textit{nonzero} boundary condition and study the following initial boundary value problem:
\[
\begin{aligned}
\ddot{u}(x, t) &= (\Delta_L - m^2)u(x, t), \quad x \geq 1, \quad t > 0, \\
\dot{u}(0, t) &= -\kappa u(0, t) - m^2 u(0, t) - \gamma \dot{u}(0, t), \quad \dot{u}(0, t) = \dot{u}(0, 0) + u(1, t) - u(0, t), \quad t > 0, \\
u(x, 0) &= u_0(x), \quad \dot{u}(x, 0) = \dot{u}_0(x), \quad x \geq 0.
\end{aligned}
\]
Here \( u(x, t) \in \mathbb{R}, \quad m \geq 0, \quad \gamma \geq 0, \Delta_L \) denotes the second derivative on \( \mathbb{Z} \). We impose some restrictions on the coefficients \( m, \kappa, \gamma \) of the system. In particular, if \( \gamma > 0 \), then either \( m > 0 \) or \( \kappa > 0 \). If \( \gamma = 0 \), then \( \kappa \in (0, 2) \). We obtain the results similar to (1.1) and (1.3). Furthermore, the limiting energy current at the origin equals \( J_\infty := -\gamma \lim_{t \to \infty} \mathbb{E}(\dot{u}(0, t))^2 \).

Hence, in the case when \( \gamma > 0 \), \( J_\infty \neq 0 \) (cf Example 6.7) if \( \int (Y_1(0))^2 \mu_\infty(dY) \neq 0 \) (the limit measures \( \mu_\infty \) satisfying the last condition are constructed in [11]).

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