PROOF OF THE EXACT OVERLAPS CONJECTURE FOR SYSTEMS WITH ALGEBRAIC CONTRACTIONS

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Abstract. We establish the exact overlaps conjecture for iterated functions systems on the real line with algebraic contractions and arbitrary translations.

1. Introduction

1.1. Background. Let \( m \geq 1 \) and \( \Phi = \{ \varphi_j(x) = \lambda_j x + t_j \}_{j=0}^{m} \) be a finite set of contracting similarities of \( \mathbb{R} \), so that \( 0 \neq \lambda_j \in (-1,1) \) and \( t_j \in \mathbb{R} \) for each \( 0 \leq j \leq m \). Such a collection \( \Phi \) is called a self-similar iterated function system (IFS). It is well known that there exists a unique nonempty compact \( K \subset \mathbb{R} \), called the attractor of \( \Phi \), which satisfies the relation,

\[
K = \bigcup_{j=0}^{m} \varphi_j(K).
\]

The set \( K \) is said to be self-similar.

Suppose additionally that \( p = (p_j)_{j=0}^{m} \) is a probability vector with strictly positive coordinates. Then there exists a unique Borel probability measure \( \mu = \mu(\Phi, p) \) on \( \mathbb{R} \) such that,

\[
\mu = \sum_{j=0}^{m} p_j \cdot \varphi_j \mu,
\]

where \( \varphi_j \mu \) is the push-forward of \( \mu \) by \( \varphi_j \). Its support is equal to \( K \), it is the unique stationary probability measure for the random walk moving from \( x \in \mathbb{R} \) to \( \varphi_j(x) \) with probability \( p_j \), and it is called the self-similar measure corresponding to \( \Phi \) and \( p \).

The dimension theory of self-similar measures is a central area of research in fractal geometry. It was proven by Feng and Hu [FH] that \( \mu \) is always exact dimensional. That is, there exists a value \( \dim \mu \in [0, 1] \), called the dimension of \( \mu \), such that,

\[
\dim \mu = \lim_{\delta \downarrow 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta} \quad \text{for \( \mu \)-a.e. \( x \in \mathbb{R} \).}
\]

As proven in [FLR], \( \dim \mu \) agrees with the value given to \( \mu \) by other commonly used notions of dimension, such as the Hausdorff, packing and entropy dimensions.

It turns out that in most cases \( \dim \mu \) satisfies a certain formula in terms of \( p \) and the contractions vector \( \lambda = (\lambda_j)_{j=0}^{m} \). Denote by \( H(p) \) the entropy of \( p \) and by \( \chi \) the Lyapunov exponents corresponding to \( p \) and \( \lambda \). That is,

\[
H(p) = -\sum_{j=0}^{m} p_j \log p_j \quad \text{and} \quad \chi = -\sum_{j=0}^{m} p_j \log |\lambda_j|,
\]

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where here and everywhere else in this paper the base of the log function is 2. Set,
\[ \beta = \beta(\Phi, p) = \min\{1, H(p)/\chi\}, \]
then it is not hard to show that \( \beta \) is always an upper bound for \( \dim \mu \) and that it is equal to \( \dim \mu \) whenever the union in (1.1) is disjoint. Moreover, it was proven by Jordan, Pollicott and Simon [JPS] that if \( \lambda \) is kept fixed and \( |\lambda_j| \in (0, \frac{1}{2}) \) for each \( 0 \leq j \leq m \), then \( \dim \mu = \beta \) for Lebesgue almost every selection of the translations \( (t_j)_{j=0}^m \in \mathbb{R}^{m+1} \). A version of this result for sets was first established by Falconer [Fa].

On the other hand there are cases in which it is obvious that dimension drop occurs, i.e. that \( \dim \mu \) is strictly less than \( \beta \). Denote the index set \( \{0, \ldots, m\} \) by \( \Lambda \).

For \( n \geq 1 \) and a word \( j_1 \ldots j_n = w \in \Lambda^n \) set,
\[ \varphi_w = \varphi_{j_1} \circ \ldots \circ \varphi_{j_n} \text{ and } \lambda_w = \lambda_{j_1} \cdot \ldots \cdot \lambda_{j_n}. \]
The IFS \( \Phi \) is said to have exact overlaps if the semigroup generated by its elements is not free. Since the members of \( \Phi \) are contractions, this is equivalent to the existence of \( n \geq 1 \) and distinct words \( w_1, w_2 \in \Lambda^n \) with \( \varphi_{w_1} = \varphi_{w_2} \). It is not difficult to see that \( \dim \mu < \beta \) whenever \( \Phi \) has exact overlaps and \( \dim \mu < 1 \). The following folklore conjecture says that these are the only circumstances in which dimension drop can occur. A version of it for sets was stated, probably for the first time, by Simon [Si].

**Conjecture 1.** Suppose that \( \dim \mu < \beta \) then \( \Phi \) has exact overlaps.

A major step towards the verification of Conjecture 1 was achieved by Hochman [Ho1]. For \( n \geq 1 \) set,
\[ \Delta_n = \min \{|\varphi_{w_1}(0) - \varphi_{w_2}(0)| : w_1, w_2 \in \Lambda^n, w_1 \neq w_2 \text{ and } \lambda_{w_1} = \lambda_{w_2}\}. \]
It always holds that \( \Delta_n \to 0 \) at a rate which is at least exponential, and that \( \Delta_n = 0 \) for some \( n \geq 1 \) if and only if \( \Phi \) has exact overlaps. The main result in [Ho1] says that if \( \dim \mu < \beta \) then \( \Delta_n \to 0 \) super-exponentially, that is
\[ \lim_{n} \frac{1}{n} \log \Delta_n = -\infty. \]
A version of this for \( L^q \) dimensions was recently obtained by Shmerkin [Sh, Theorem 6.6].

Two applications of Hochman’s result are especially relevant to the present paper. It is not hard to see that if \( \lambda_0, \ldots, \lambda_m, t_0, \ldots, t_m \) are all algebraic numbers and \( \Delta_n \to 0 \) super-exponentially, then in fact \( \Phi \) must have exact overlaps. Relaying on this observation, Conjecture 1 is established in [Ho1] Theorem 1.5] for the case of algebraic parameters. The second application verifies a conjecture of Furstenberg regarding projections of the one-dimensional Sierpinski gasket (see e.g. [PS, Question 2.5]). Stated with the notation introduced above, it is proven in [Ho1, Theorem 1.6] that Conjecture 1 is valid when \( m = 2 \) and
\[ \lambda = p = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \]
Another important step towards Conjecture 1 was recently achieved by Varjú [Var]. He has shown that if \( \mu \) is a Bernoulli convolution, that is if in the notation above
\( m = 1, \lambda_0 = \lambda_1 > 0, t_0 = -1 \) and \( t_1 = 1, \)
then \( \dim \mu = \beta \) whenever \( \lambda_0 \) is transcendental. Together with the result mentioned above regarding systems with algebraic parameters, this verifies Conjecture \(^1\) for the family of Bernoulli convolutions.

Given Hochman’s and Shmerkin’s results, it is natural to ask whether \( \Phi \) has exact overlaps whenever \( \Delta_n \to 0 \) super-exponentially. Recently, examples have been constructed by Baker \(^{[Ba]}\) and independently by Bárány and Käenmäki \(^{[BK]}\), which show that this is not necessarily true. In Baker’s construction the maps in the IFS all contract by a rational number, and so it is especially relevant to the present paper. In a joint work with P. Varjú \(^{[RV]}\) we will treat a family of self-similar measures which is closer to the example from \(^{[BK]}\).

1.2. Results. The following theorem is our main result. It verifies Conjecture \(^1\) for the case of algebraic contractions and arbitrary translations.

**Theorem 2.** Let \( m \geq 0 \) and \( \Phi = \{ \varphi_j(x) = \lambda_j x + t_j \}_{j=0}^m \) be a self-similar IFS on \( \mathbb{R} \). Suppose that \( \lambda_0, \ldots, \lambda_m \) are all algebraic numbers and that \( \Phi \) has no exact overlaps. Let \( p = (p_j)_{j=0}^m \) be a probability vector and denote by \( \mu \) the self-similar measure corresponding to \( \Phi \) and \( p \). Then \( \dim \mu = \beta \), where \( \beta \) is as defined in (1.3).

A version for sets of the conjecture follows directly from the last theorem in the case of algebraic contractions. Given an IFS \( \Phi \) as above denote by \( \dim_s \Phi \) its similarity dimension, that is \( \dim_s \Phi \) is the unique \( s \geq 0 \) which satisfies the equation,

\[
\sum_{j=0}^m |\lambda_j|^s = 1.
\]

It is not hard to see that \( \min\{1, \dim_s \Phi\} \) is always an upper bound for \( \dim_H K \), where \( K \) is the attractor of \( \Phi \) and \( \dim_H \) stands for Hausdorff dimension. Moreover, the equality

\[
(1.6) \quad \dim_H K = \min\{1, \dim_s \Phi\}
\]

is satisfied when the union in (1.1) is disjoint or, more generally, if \( \Phi \) satisfies the so-called open set condition (see for instance \(^{[BP]}\) Chapter 2.1). The version for sets of Conjecture \(^1\) says that (1.6) holds whenever \( \Phi \) has no exact overlaps.

**Corollary 3.** Let \( m \geq 0 \) and \( \Phi = \{ \varphi_j(x) = \lambda_j x + t_j \}_{j=0}^m \) be a self-similar IFS on \( \mathbb{R} \). Suppose that \( \lambda_0, \ldots, \lambda_m \) are all algebraic numbers and that \( \Phi \) has no exact overlaps. Let \( K \) be the attractor of \( \Phi \), then (1.6) is satisfied.

**Proof.** Write \( s = \dim_s \Phi \) and denote by \( p \) the probability vector \( (|\lambda_j|^s)_{j=0}^m \). Let \( \beta \) be as defined in (1.3), then \( \beta = \min\{1, s\} \). Let \( \mu \) be the self-similar measure corresponding to \( \Phi \) and \( p \). By Theorem \(^2\) we have \( \dim \mu = \min\{1, s\} \). Additionally, since \( \mu \) is supported on \( K \) it follows that \( \dim_H K \geq \dim \mu \). This completes the proof of the corollary. \( \square \)

For our next application suppose that \( \beta < 1 \). As mentioned in Section \(^1\) if \( \Phi \) has exact overlaps then necessarily \( \dim \mu < \beta \). If instead it only holds that \( \Delta_n \to 0 \) super-exponentially, then in general it is not clear whether there is dimension drop or not. In fact, until now there was no single example in which \( \dim \mu \) was analysed in such a situation. Combining our result with Baker’s construction we are able to obtain such an example. We say that an IFS is homogeneous if all of its maps have the same contraction part.
Corollary 4. Let \((\epsilon_n)_{n \geq 1}\) be an arbitrary sequence of positive real numbers. Then there exists a homogeneous self-similar IFS \(\Phi = \{\varphi_j(x) = \lambda x + t_j\}_{j=0}^m\) on \(\mathbb{R}\) such that,

1. \(\lambda\) is a rational number;
2. \(\Phi\) has no exact overlaps;
3. \(\Delta_n \leq \epsilon_n\) for all \(n \geq 1\), where \(\Delta_n\) is as defined in (1.3);
4. \(\dim_s \Phi < 1\), and in particular \(\beta(\Phi, p) < 1\) for every probability vector \(p = (p_j)_{j=0}^m\);
5. \(\dim \mu(\Phi, p) = \beta(\Phi, p)\) for every probability vector \(p = (p_j)_{j=0}^m\), where recall that \(\mu(\Phi, p)\) is the self-similar measure corresponding to \(\Phi\) and \(p\);
6. \(\dim_H K = \dim_s \Phi\), where \(K\) is the attractor of \(\Phi\).

Proof. The existence of a homogeneous IFS \(\Phi\) which satisfies the first four properties follows from [Ba], Theorem 1.3 and Remark 2.2. The last two properties follow from [Ba] and Corollary 3. □

1.3. About the proof. We derive Theorem 5 from the following more general statement, which concerns also some self-similar measures in higher dimensions. This enables us to prove it by backward induction on the dimension of the ambient space.

Theorem 5. Let \(m \geq 2\) be an integer, \(\lambda = (\lambda_j)_{j=0}^m\) be algebraic numbers in \((-1, 1)\) \(\setminus \{0\}\) and \(p = (p_j)_{j=0}^m\) be a probability vector with strictly positive coordinates. Then for every \(1 \leq d < m\) the following holds. Let \((t_j)_{j=0}^m = t \in (\mathbb{R}^d)^m\) be such that,

\[
\text{span}\{t_1, \ldots, t_m\} = \mathbb{R}^d,
\]

and let \(\Phi_t = \{\varphi_{t,j}\}_{j=0}^m\) be the self-similar IFS on \(\mathbb{R}^d\) with,

\[
\varphi_{t,0}(x) = \lambda_0 x \quad \text{and} \quad \varphi_{t,j}(x) = \lambda_j x + t_j \quad \text{for each} \quad x \in \mathbb{R}^d \quad \text{and} \quad 1 \leq j \leq m.
\]

Denote by \(\mu_t\) the self-similar measure on \(\mathbb{R}^d\) which corresponds to \(\Phi_t\) and \(p\). Suppose that \(\Phi_t\) has no exact overlaps, then

\[
\dim \mu_t \geq \min\{1, H(p)/\chi\},
\]

where \(H(p)\) and \(\chi\) are as defined in (1.2).

Proof of Theorem 5 given Theorem 4. When \(m = 0\) the theorem is trivial. Suppose that \(m = 1\), then we may assume, by conjugating the maps in \(\Phi\) by an appropriate invertible affine map, that \(t_0 = 0\) and \(t_1 = 1\). Thus, the theorem in this case follows from Hochman’s result mentioned above regarding systems with algebraic parameters. When \(m \geq 2\) the theorem follows directly from Corollary 4 with \(d = 1\), and by conjugating the maps in the IFS \(\Phi\) so that \(t_0 = 0\). □

Let us give an informal sketch for the proof of Theorem 5. Given \(d \geq 1\) and \(t \in (\mathbb{R}^d)^m\) let \(\Phi_t = \{\varphi_{t,j}\}_{j=0}^m\) and \(\mu_t\) be as in the statement of the theorem. Recall that \(\Lambda = \{0, \ldots, m\}\), and for \(n \geq 1\) and \(w_1, w_2 \in \Lambda^n\) set

\[
L_{w_1, w_2}(t) = \varphi_{t, w_1}(0) - \varphi_{t, w_2}(0) \quad \text{for} \quad t \in \mathbb{R}^m,
\]

where \(\varphi_{t, w_1}\) and \(\varphi_{t, w_2}\) are as defined in (1.2). It is easy to see that \(L_{w_1, w_2}\) is a linear functional on \(\mathbb{R}^m\). Moreover, if \(\{f_1, \ldots, f_m\}\) is the dual basis of the standard basis
of $\mathbb{R}^m$, then

\begin{equation}
L_{w_1, w_2} = \sum_{j=1}^{m} P_j(\lambda)f_j \text{ for some } P_0, \ldots, P_m \in \mathcal{P}(1, n),
\end{equation}

where $\mathcal{P}(1, n)$ is the set of all $P \in \mathbb{Z}[X_0, \ldots, X_m]$ with $\deg P < n$ and coefficients $\pm 1$ or 0. Denote by $\mathcal{L}_n$ the collection of all $L_{w_1, w_2}$ with $w_1, w_2 \in \Lambda^n$, $w_1 \neq w_2$ and $\lambda_{w_1} = \lambda_{w_2}$.

As mentioned above the proof is carried out by backward induction on $d$. Thus, let $1 \leq d < m$ and assume that the theorem holds for all $d < d' < m$. Suppose that

\begin{equation}
((t_j^d)_{j=1}^{m}) = (t_{j})_{j=1}^{m} = t \in (\mathbb{R}^d)^m
\end{equation}

is such that (1.7) is satisfied but (1.8) does not hold. Our objective is to show that $\Phi_t$ has exact overlaps.

Let $\delta > 0$ be small with respect to all previous parameters. The starting point of the argument is a theorem of Hochman from [Ho2] regarding self-similar measures in $\mathbb{R}^d$. By applying that theorem we deduce that for all sufficiently large $n \geq 1$ there exists a collection $\mathcal{A}_n \subset \mathcal{L}_n$, which is large in a sense to be made precise during the actual proof, such that

\begin{equation}
|L((t_j^d)_{j=1}^{m})| \leq \delta^n \text{ for each } 1 \leq l \leq d \text{ and } L \in \mathcal{A}_n.
\end{equation}

Denote by $r_n$ the dimension of the linear span of the members of $\mathcal{A}_n$. By (1.9) and (1.10), by assuming that $\delta$ is sufficiently small with respect to $\lambda$ and $t$, and since $\lambda_0, \ldots, \lambda_m$ are algebraic, it is not hard to show that $r_n \leq m - d$.

Now there are two options to consider. First assume that $r_n = m - d$ for all sufficiently large $n$. In this case we are able to show that $\Phi_t$ has exact overlaps by extending the argument from [Ho1] used in the proof of Furstenberg’s conjecture mentioned above. Denote the annihilator of $\mathcal{A}_n$ by $V_n$, that is

$$V_n = \{ x \in \mathbb{R}^m : L(x) = 0 \text{ for all } L \in \mathcal{A}_n \}.$$ 

By (1.10) it follows that $V_n$ and $V_{n+1}$ are both $\delta^n$-close to $U = \text{span}((t_j^d)_{j=1}^{m} : 1 \leq l \leq d)$, which implies that they are $\delta^n$-close to each other. From this and the information on the coefficients of members of $\mathcal{L}_n$, it follows that in fact we must have $V_n = V_{n+1}$. This gives $U = V_n$, from which it follows that $L((t_j^d)_{j=1}^{m}) = 0$ for all $1 \leq l \leq d$ and $L \in \mathcal{A}_n$. By the definition of $\mathcal{L}_n$ this implies that $\Phi_t$ has exact overlaps.

In the second case we assume that there exist $1 \leq r < m - d$ and an increasing sequence $\{n_k\}_{k \geq 1}$ such that $r_{n_k} = r$ for all $k \geq 1$. Set $d' = m - r$ and note that $d < d' < m$. For every $k \geq 1$ we choose

\begin{equation}
((s_{k,j}^d)_{j=1}^{m}) = (s_{k,j})_{j=1}^{m} = s_k \in ([-1,1]^d)^m,
\end{equation}

such that $(s_{k,j})_{j=1}^{m}$ contains the standard basis of $\mathbb{R}^d$ and,

\begin{equation}
L((s_{k,j})_{j=1}^{m}) = 0 \text{ for all } 1 \leq l \leq d' \text{ and } L \in \mathcal{A}_{n_k}.
\end{equation}

By moving to a subsequence we may assume that there exists $s \in (\mathbb{R}^d)^m$ such that $s_k \overset{k}{\rightarrow} s$. Now by (1.11), since the collections $\mathcal{A}_{n_k}$ are sufficiently large and by the lower semi-continuity of the dimension of self-similar measures, it follows that $\dim \mu_s < \beta$. From this and the induction hypothesis we get that $\Phi_s$ has exact overlaps. In the last part of the proof we show that this, together with (1.9)
and (1.10), implies that \( \Phi_t \) has exact overlaps, which completes the proof of the theorem.

It is interesting to consider Baker’s construction in the context of the proof just described. Since in this example \( \Delta_n \to 0 \) super-exponentially, where \( \Delta_n \) is defined in (1.6), the collections \( \mathcal{A}_n \) are necessarily nonempty. On the other hand, since there are no exact overlaps, the preceding argument shows that \( \liminf r_n < m - d \) (where \( d = 1 \)) and that the collections \( \mathcal{A}_n \) cannot be large in a manner resulting in a dimension drop.

**Structure of the paper.** In the next section we make some preparations for the proof of Theorem 5. In Section 3 we carry out the proof.

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2. **Preparations for the proof of Theorem 5**

2.1. **Some notations.** For the rest of this paper fix an integer \( m \geq 2 \), a probability vector \( p = (p_j)_{j=0}^m \) with strictly positive coordinates and a vector \( \lambda = (\lambda_j)_{j=0}^m \) such that \( \lambda_j \) is an algebraic number in \((-1, 1) \setminus \{0\}\) for each \( 0 \leq j \leq m \). Recall that the base of the log function is always 2 and set,

\[
H(p) = -\sum_{j=0}^m p_j \log p_j, \quad \chi = -\sum_{j=0}^m p_j \log |\lambda_j| \quad \text{and} \quad \beta = \min\{1, H(p)/\chi\}.
\]

For an integer \( d \geq 1 \) we shall often write \( \mathbb{R}^{dm} \) in place of \( (\mathbb{R}^d)^m \) when there is no risk of confusion. Given \( (t_j)_{j=1}^m = t \in \mathbb{R}^{dm} \) let \( \Phi_t = \{\varphi_{t,j}\}_{j=0}^m \) be the self-similar IFS on \( \mathbb{R}^d \) with,

\[
\varphi_{t,0}(x) = \lambda_0 x \quad \text{and} \quad \varphi_{t,j}(x) = \lambda_j x + t_j \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{and} \quad 1 \leq j \leq m.
\]

Denote by \( K_t \) the attractor of \( \Phi_t \), that is \( K_t \) is the unique nonempty compact subset of \( \mathbb{R}^d \) with,

\[
K_t = \bigcup_{j=0}^m \varphi_{t,j}(K_t).
\]

Write \( \mu_t \) for the self-similar measure corresponding to \( \Phi_t \) and \( p \), i.e. \( \mu_t \) is the unique Borel probability measure on \( \mathbb{R}^d \) with,

\[
\mu_t = \sum_{j=0}^m p_j \cdot \varphi_{t,j} \mu_t.
\]

By [FH] Theorem 2.8 it follows that \( \mu_t \) is exact dimensional. That is there exists a number \( \dim \mu_t \in [0, d] \) such that,

\[
\dim \mu_t = \lim_{\delta \downarrow 0} \frac{\log \mu(B(x, \delta))}{\log \delta} \quad \text{for} \quad \mu\text{-a.e.} \quad x \in \mathbb{R}^d,
\]

where \( B(x, \delta) \) is the open ball in \( \mathbb{R}^d \) with centre \( x \) and radius \( \delta \).

Denote the set \( \{0, \ldots, m\} \) by \( \Lambda \). Given \( n \geq 1 \), \( j_1 \ldots j_n = w \in \Lambda^n \) and \( t \in (\mathbb{R}^d)^m \) we shall write \( \varphi_{t,w}, p_w \) and \( \lambda_w \) in place of,

\[
\varphi_{t,j_1} \circ \ldots \circ \varphi_{t,j_n}, p_{j_1} \cdot \ldots \cdot p_{j_n} \quad \text{and} \quad \lambda_{j_1} \cdot \ldots \cdot \lambda_{j_n}.
\]
The IFS $\Phi_t$ is said to have exact overlaps if there exist $n \geq 1$ and distinct words $w_1, w_2 \in \Lambda^n$ with $\varphi_{t,w_1} = \varphi_{t,w_2}$.

For $d \geq 1$ denote by $G_d$ the group of all affine transformations $\psi : \mathbb{R}^d \to \mathbb{R}^d$ for which there exist $0 \neq \lambda' \in \mathbb{R}$ and $t' \in \mathbb{R}^d$ with $\psi(x) = \lambda'x + t'$ for all $x \in \mathbb{R}^d$. Given such a $\psi \in G_d$ we sometimes write $\lambda_\psi$ for $\lambda'$ and $t_\psi$ for $t'$. For $t \in (\mathbb{R}^d)^n$ and $n \geq 1$ set,

$$\nu_t^{(n)} = \sum_{w \in \Lambda^n} p_w \delta_{\varphi_{t,w}},$$

where $\delta_{\varphi_{t,w}}$ is the Dirac measure at $\varphi_{t,w}$. Thus $\nu_t^{(n)}$ is a finitely supported probability measure on $G_d$.

Suppose that $Y_{t,1}, Y_{t,2}, \ldots$ are i.i.d. $G_d$-valued random variables with,

$$\mathbb{P}\{Y_{t,1} = \varphi_{t,j}\} = p_j \text{ for each } j \in \Lambda.$$

Then for each $n \geq 1$ the distribution of $Y_{t,1} \cdot \ldots \cdot Y_{t,n}$ is equal to $\nu_t^{(n)}$. From this it follows that,

$$\nu_t^{(n+k)} = \nu_t^{(n)} * \nu_t^{(k)} \text{ for all } n, k \geq 1,$$

where $\nu_t^{(n)} * \nu_t^{(k)}$ is the convolution of $\nu_t^{(n)}$ with $\nu_t^{(k)}$.

### 2.2. Partitions, entropy and dimension.

Given $d \geq 1$ and $n \geq 0$ write $D_n^d$ for the level-$n$ dyadic partition of $\mathbb{R}^d$, that is

$$D_n^d = \left\{ \left( \frac{k_1}{2^n}, \ldots, \frac{k_n}{2^n} \right) : k_1, \ldots, k_n \in \mathbb{Z} \right\}.$$

Denote by $\mathcal{E}_n^d$ the level-$n$ dyadic partition of $G_d$ according to the translation part, i.e.

$$\mathcal{E}_n^d = \left\{ \psi \in G_d : t_\psi \in D_n^d \right\}.$$

For a real $r \geq 0$ we write $D_r^d$ and $\mathcal{E}_r^d$ instead of $D_n^d$ and $\mathcal{E}_n^d$. Let $F_d$ be the partition of $G_d$ according to the scaling part, that is

$$F_d = \left\{ \psi \in G_d : \lambda_\psi = \lambda' \right\}.$$

When using these notations we shall omit the superscript $d$ whenever it is clear from the context.

Given a measurable space $X$, a measurable partition of it $\mathcal{D}$ and a probability measure $\theta$ on $X$, we write $H(\theta, \mathcal{D})$ for the entropy of $\theta$ with respect to $\mathcal{D}$. That is,

$$H(\theta, \mathcal{D}) = - \sum_{D \in \mathcal{D}} \theta(D) \log \theta(D).$$

When $\mathcal{D}$ is the partition of $X$ into singletons we write $H(\theta)$ instead of $H(\theta, \mathcal{D})$.

The following lemma is well known. Variations of it has appeared before in [Ho1] and [Ho2]. We include a short proof of it here for completeness. During the proof we use freely some basic properties of entropy which can be found for instance in [Ho1] Section 3.1.

**Lemma 6.** Let $d \geq 1$ and $t \in (\mathbb{R}^d)^n$ be given, then

$$\dim \mu_t = \lim_{n \to \infty} \frac{1}{\log n} H(\nu_t^{(n)} \circ \chi_n)$$

where $\chi_n : \mathcal{D}_n^d \to \mathcal{E}_n^d$ is the map which takes the $n$-th dyadic partition $D_n^d$ of $\mathbb{R}^d$ to the level-$n$ dyadic partition $\mathcal{E}_n^d$ of $G_d$. That is, for $D = (D_{ij})$ with $i,j \in \mathbb{Z}$ and $k = (k_i)$ a partition of $\mathcal{E}_n^d$ we write

$$\chi_n(D) = \psi(k) = \left( \frac{k_1}{2^n}, \ldots, \frac{k_n}{2^n} \right).$$
Proof. Let \( \epsilon > 0 \) and let \( n \geq 1 \) be large with respect to \( \epsilon \). Let \( \Pi, \Pi_n : \Lambda^N \to \mathbb{R}^d \) be such that for \( (\omega_k)_{k \geq 0} = \omega \in \Lambda^N \),
\[
\Pi \omega = \lim_k \varphi_{t,\omega_0,\ldots,\omega_k}(0) \quad \text{and} \quad \Pi_n \omega = \varphi_{t,\omega_0,\ldots,\omega_{n-1}}(0).
\]
Denote by \( \xi \) the Bernoulli measure on \( \Lambda^N \) corresponding to \( p \), i.e. \( \xi = p^N \). Note that \( \Pi \xi = \mu_t \) and,
\[
H(\nu^{(n)}_t, \mathcal{E}_\chi_n) = H(\Pi_n \xi, \mathcal{D}_\chi_n).
\]
By the law of large numbers, and by assuming \( n \) is large enough with respect to \( \epsilon \), there exists a Borel set \( E \subset \Lambda^N \) with \( \xi(E) > 1 - \epsilon \) and,
\[
|\frac{1}{n} \log |\omega_0,\ldots,\omega_{n-1}|| + \chi| < \epsilon \text{ for each } \omega \in E.
\]
Thus, \( E \in \Lambda^N \) with \( \xi(E) > 0 \) write \( \xi_E = \frac{\xi(\omega)}{\xi(E)} \), where \( \xi_E \) is the restriction of \( \xi \) to \( E \). By the concavity of entropy and since \( H(\Pi \xi_E, \mathcal{D}_\chi_n) = O_{\lambda,t}(n) \),
\[
H(\mu_t, \mathcal{D}_\chi_n) \geq H(\Pi \xi_E, \mathcal{D}_\chi_n) + \xi(E)H(\Pi \xi_E, \mathcal{D}_\chi_n)
\]
Similarly, by the convexity bound for entropy and since \( H(\Pi \xi_E, \mathcal{D}_\chi_n) = O_{\lambda,t}(n) \) (assuming \( \xi(E) > 0 \)),
\[
H(\Pi \xi_E, \mathcal{D}_\chi_n) = H(\Pi \xi_E, \mathcal{D}_\chi_n) + O_{\lambda,t}(en).
\]
Thus,
\[
H(\mu_t, \mathcal{D}_\chi_n) = H(\Pi \xi_E, \mathcal{D}_\chi_n) + O_{\lambda,t}(en),
\]
and a similar argument gives,
\[
H(\Pi_n \xi, \mathcal{D}_\chi_n) = H(\Pi_n \xi_E, \mathcal{D}_\chi_n) + O_{\lambda,t}(en).
\]
Note that by \( \|\| \cdot \| \| \),
\[
H(\Pi \omega - \Pi_n \omega) = O_{\lambda,t}(2^{-n(\chi - \epsilon)}) \text{ for all } \omega \in E,
\]
From this, \( \|\| \cdot \| \| \) and \( \|\| \cdot \| \| \),
\[
H(\mu_t, \mathcal{D}_\chi_n) = H(\nu_t^{(n)}, \mathcal{E}_\chi_n) + O_{\lambda,t}(en).
\]
Additionally, since \( \mu_t \) is exact dimensional we may assume,
\[
|\frac{1}{\lambda^n} H(\mu_t, \mathcal{D}_\chi_n) | < \epsilon .
\]
Hence,
\[
\frac{1}{\lambda^n} H(\nu_t^{(n)}, \mathcal{E}_\chi_n) = \dim \mu_t + O_{\lambda,t}(\epsilon),
\]
which completes the proof of the lemma. \( \square \)

**Corollary 7.** Let \( d \geq 1 \) and \( t \in (\mathbb{R}^d)^m \) be given, then
\[
\dim \mu_t \leq \frac{1}{\lambda^n} H(\nu_t^{(n)}) \text{ for all } n \geq 1.
\]
Proof. By \([2.1]\) it follows that for every \(n, k \geq 1\),
\[
H(\nu_t^n) = H(\nu_t^n e_1 + \nu_t(k)) \leq H(\nu_t^n) + H(\nu_t(k)) .
\]
Thus by the Fekete lemma for sub-additive sequences,
\[
\lim_{n} \frac{1}{n} H(\nu_t^n) = \inf \frac{1}{n} H(\nu_t^n) .
\]
Now by Lemma 6,
\[
\dim \mu_t = \lim_{n} \frac{1}{\chi n} H(\nu_t^n, \mathcal{E}_{\chi n}) \leq \lim_{n} \frac{1}{\chi n} H(\nu_t^n) ,
\]
and the corollary follows from \((2.6)\).
\(\square\)

2.3. Affine irreducibility. Given \(d \geq 1\) and \(t \in (\mathbb{R}^d)^m\) we say that \(\Phi_t\) is affinely irreducible if there is no proper affine subspace \(V\) of \(\mathbb{R}^d\) with \(\varphi_{t,j}(V) = V\) for all \(0 \leq j \leq m\).

**Lemma 8.** Let \(1 \leq d \leq m\) and \((t_j)_{j=1}^m = t \in \mathbb{R}^{dm}\) be with,
\[
\text{span}\{t_1, ..., t_m\} = \mathbb{R}^d .
\]
Then \(\Phi_t\) is affinely irreducible.

**Proof.** Without loss of generality we may assume that \(t_1, ..., t_d\) are linearly independent. For \(1 \leq j \leq d\) let \(x_j\) be the fixed point of \(\varphi_{t,j}\), then \(x_j \neq 0\) and \(x_j \in K_t \cap \text{span}\{t_j\}\). Since \(\varphi_{t,0}(0) = 0\) we also have \(0 \in K_t\). From these facts it follows that the affine span of \(K_t\) is equal to \(\mathbb{R}^d\).

On the other hand, if \(s \in (\mathbb{R}^d)^m\) is such that \(\Phi_s\) is not affinely irreducible then there exists a proper affine subspace \(V\) of \(\mathbb{R}^d\) with \(\varphi_{s,j}(V) = V\) for all \(0 \leq j \leq m\). From this it clearly follows that \(K_s \subset V\), which implies that the affine span of \(K_s\) is contained in \(V\). Since \(V\) is proper we must have \(t \neq s\), which completes the proof of the lemma. \(\square\)

2.4. A Theorem of Hochman. The following statement follows almost directly from a theorem of Hochman regarding self-similar measures in \(\mathbb{R}^d\). Given two partitions \(\mathcal{C}_1\) and \(\mathcal{C}_2\) of some space \(X\) we denote by \(\mathcal{C}_1 \vee \mathcal{C}_2\) their common refinement, that is
\[
\mathcal{C}_1 \vee \mathcal{C}_2 = \{C_1 \cap C_2 : C_1 \in \mathcal{C}_1 \text{ and } C_2 \in \mathcal{C}_2\} .
\]

**Theorem 9.** Let \(d \geq 1\) and \(t \in (\mathbb{R}^d)^m\) be given. Suppose that \(\Phi_t\) is affinely irreducible and that \(\dim \mu_t < \beta\). Then for every \(q \geq \chi\),
\[
\lim_{n} \frac{1}{\chi n} H(\nu_t^n, \mathcal{E}_{\chi n} \vee F) = \dim \mu_t .
\]

**Proof.** Given a linear subspace \(V\) of \(\mathbb{R}^d\) let \(\pi_V\) be the orthogonal projection onto \(V\). Denote by \(\mu_t^{V} x\) the disintegration of \(\mu_t\) with respect to \(\pi_V\) (\(B\)), where \(V^\perp\) is the orthogonal complement of \(V\) and \(B\) is the Borel \(\sigma\)-algebra of \(\mathbb{R}^d\). By [FJ Theorem 3.1] it follows that \(\mu_t^{V} x\) is exact dimensional for \(\mu_t\)-a.e. \(x \in \mathbb{R}^d\).

Since \(\Phi_t\) is affinely irreducible it follows by [Ho2 Theorem 1.5] that at least one of the following three alternatives is satisfied,
\[
(1) \quad \dim \mu = \min\{d, H(p)/\chi\} ;
(2) \quad \lim_{n} \frac{1}{\chi n} (H(\nu_t^n, \mathcal{E}_{\chi n}) - H(\nu_t^n, \mathcal{E}_{\chi n})) = 0 \text{ for all } q \geq \chi ;
\]
(3) there exists a linear subspace $V$ of $\mathbb{R}^d$ with $\dim V > 0$ and,
\[ \dim \mu^V = \dim V \] for $\mu$-a.e. $x \in \mathbb{R}^d$.
Since $\dim \mu_t < \beta$ the first option is clearly not possible. By [FJ Corollary 3.1] it follows that,
\[ \dim \mu_t \geq \dim \mu^V \] for $\mu$-a.e. $x \in \mathbb{R}^d$.
From this and $\beta \leq 1$ it follows that the third alternative is also not possible. Thus the second alternative must hold, and so by Lemma [6] it follows that for every $q \geq \chi$,
\begin{equation}
\lim_{n} \frac{1}{\chi^n} H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) = \lim_{n} \frac{1}{\chi^n} H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) = \dim \mu_t.
\end{equation}
Additionally, for each $n \geq 1$ the cardinality of the set $\{ \lambda_w : w \in \Lambda^n \}$ is at most $n^{m+1}$. Hence,
\[ H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) \leq H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) + H(\nu^{(n)}_t, \mathcal{F}) \leq H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) + \log n^{m+1}, \]
which implies,
\[ \lim_{n} \frac{1}{\chi^n} \left( H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) - H(\nu^{(n)}_t, \mathcal{E}_{\mathcal{F}}) \right) = 0. \]
This together with (2.7) completes the proof of the theorem. $\square$

2.5. Lower semicontinuity of dimension. The following theorem follows directly from [6] Corollary 1.8, where it is proven in much greater generality.

**Theorem 10.** For every $d \geq 1$ the function which takes $t \in (\mathbb{R}^d)^m$ to $\dim \mu_t$ is lower semi-continuous.

2.6. Notations for polynomials and an exponential lower bound. Let $\mathbb{N}$ be the set of non-negative integers. For a multi-index $(\alpha_0, ..., \alpha_m) = \alpha \in \mathbb{N}^{m+1}$ write,
\[ X^\alpha = X_0^{\alpha_0} \cdot ... \cdot X_m^{\alpha_m}, \quad \lambda^\alpha = \lambda_0^{\alpha_0} \cdot ... \cdot \lambda_m^{\alpha_m} \quad \text{and} \quad |\alpha| = \alpha_0 + ... + \alpha_m, \]
where $X_0, ..., X_m$ are formal variables and $\lambda = (\lambda_j)_{j=0}^{m}$ was fixed in Section 2.1.
Given a polynomial
\[ \sum_{\alpha \in \mathbb{N}^{m+1}} c_\alpha X^\alpha = P(X) \in \mathbb{Z}[X_0, ..., X_m] \]
denote by $\deg(P)$ the total degree of $P$, that is
\[ \deg(P) = \max\{|\alpha| : \alpha \in \mathbb{N}^{m+1} \text{ and } c_\alpha \neq 0\}. \]
Also set,
\[ \ell_\infty(P) = \max_{\alpha \in \mathbb{N}^{m+1}} |c_\alpha| \quad \text{and} \quad \ell_1(P) = \sum_{\alpha \in \mathbb{N}^{m+1}} |c_\alpha|. \]
For $l, n \geq 1$ write,
\[ \mathcal{P}(l, n) = \{ P \in \mathbb{Z}[X_0, ..., X_m] : \ell_\infty(P) \leq l \text{ and } \deg(P) < n \}. \]
Note that for $P_1, ..., P_k \in \mathbb{Z}[X_0, ..., X_m]$,
\[ \ell_\infty(\Pi_{j=1}^k P_j) \leq \ell_1(\Pi_{j=1}^k P_j) \leq \Pi_{j=1}^k \ell_1(P_j). \]
From this and since $\ell_1(P) \leq ln^{m+1}$ for $P \in \mathcal{P}(l, n)$,
\begin{equation}
\Pi_{j=1}^k P_j \in \mathcal{P}([\ell_1(n^{m+1})^k, kn]) \quad \text{for every } P_1, ..., P_k \in \mathcal{P}(l, n).
\end{equation}
The following lemma will be used many times during the proof of Theorem 1. It is the only place in which the fact that $\lambda_0, ..., \lambda_m$ are algebraic is used.
Lemma 11. There exists a constant $M = M(m, \lambda) > 1$ such that for every $l, n \geq 1$ and $P \in \mathcal{P}(l, n)$ we have $P(\lambda) = 0$ or $|P(\lambda)| \geq l^{-M} M^{-n}$.

Proof. Given an algebraic $\eta \in \mathbb{C}$ write $H(\eta)$ for its height, as defined in [Mas Chapter 14]. Write,

$$H = \max_{0 \leq j \leq m} H(\lambda_j),$$

and denote by $D$ the degree of the field extension $\mathbb{Q}[\lambda_0, \ldots, \lambda_m]/\mathbb{Q}$.

Let $l, n \geq 1$ and $P \in \mathcal{P}(l, n)$ be given. Since $P \in \mathcal{P}(l, n)$ we have $\ell_1(P) \leq ln^{m+1}$. Thus by [Mas Proposition 14.7],

$$H(P(\lambda)) \leq \ell_1(P) \prod_{j=0}^{m} H(\lambda_j)^n \leq ln^{m+1} \cdot H^{(m+1)n}.$$ 

Since $P(\lambda) \in \mathbb{Q}[\lambda_0, \ldots, \lambda_m]$ it follows by [Mas Proposition 14.13] that $P(\lambda) = 0$ or,

$$|P(\lambda)| \geq H(P(\lambda))^{-D} \geq (ln^{m+1} \cdot H^{(m+1)n})^{-D}.$$

The lemma now follows by taking $M$ to be sufficiently large with respect to $m$, $H$ and $D$. \hfill \Box

2.7. A family of linear functionals. Recall that according to the notation introduced in Section 2.1 we have that $\Phi_t = \{\varphi_{t, j}\}_{j=0}^{m}$ is an IFS on $\mathbb{R}$ for each $t \in \mathbb{R}^m$.

Given $n \geq 1$ and $w_1, w_2 \in \Lambda^n$ let $L_{w_1, w_2} : \mathbb{R}^m \to \mathbb{R}$ be such that,

$$L_{w_1, w_2}(t) = \varphi_{t, w_1}(0) - \varphi_{t, w_2}(0) \text{ for each } t \in \mathbb{R}^m.$$ 

Note that $L_{w_1, w_2}$ is a linear functional on $\mathbb{R}^m$. Set,

$$\mathcal{L}_n = \{L_{w_1, w_2} : w_1, w_2 \in \Lambda^n, w_1 \neq w_2 \text{ and } \lambda_{w_1} = \lambda_{w_2}\}.$$

Lemma 12. Let $n \geq 1$ and $L \in \mathcal{L}_n$ be given. Then there exist $P_1, \ldots, P_m \in \mathcal{P}(1, n)$ such that,

$$L(t) = \sum_{j=1}^{m} P_j(\lambda) t_j \text{ for each } (t_1, \ldots, t_m) = t \in \mathbb{R}^m.$$ 

Proof. Given a word $q_0 \ldots q_{n-1} = w \in \Lambda^n$ write $\alpha(w)$ for the multi-index $(\alpha_0, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$ with,

$$\alpha_j = \#\{0 \leq k < n : q_k = j\} \text{ for each } 0 \leq j \leq m.$$ 

Let $q_0 \ldots q_{n-1} = w_1 \in \Lambda^n$ and $l_0 \ldots l_{n-1} = w_2 \in \Lambda^n$ be with $L = L_{w_1, w_2}$. Let $(t_1, \ldots, t_m) = t \in \mathbb{R}^m$ and write $t_0 = 0$, then

$$\varphi_{t, w_1}(0) = \sum_{k=0}^{n-1} \lambda_{q_0 \ldots q_{k-1}} t_k = \sum_{j=1}^{m} t_j \sum_{k=0}^{n-1} 1_{\{q_k = j\}} \Lambda^{\alpha(q_0 \ldots q_{k-1})},$$

and similarly for $\varphi_{t, w_2}(0)$ with $l_k$ in place of $q_k$. Thus,

$$L(t) = \varphi_{t, w_1}(0) - \varphi_{t, w_2}(0) = \sum_{j=1}^{m} t_j \sum_{k=0}^{n-1} \left(\Lambda^{\alpha(q_0 \ldots q_{k-1})} 1_{\{q_k = j\}} - \Lambda^{\alpha(l_0 \ldots l_{k-1})} 1_{\{l_k = j\}}\right).$$

Now since for each $1 \leq j \leq m$,

$$\sum_{k=0}^{n-1} \left(\Lambda^{\alpha(q_0 \ldots q_{k-1})} 1_{\{q_k = j\}} - \Lambda^{\alpha(l_0 \ldots l_{k-1})} 1_{\{l_k = j\}}\right) \in \mathcal{P}(1, n),$$
3. Proof of Theorem 5

Let us recall the statement of Theorem 5

**Theorem.** For every $1 \leq d < m$ the following holds. Let $(t_j)_{j=1}^m = t \in \mathbb{R}^{dm}$ be such that,

$$\text{span}\{t_1, ..., t_m\} = \mathbb{R}^d.$$  

Supposed that $\Phi_t$ has no exact overlaps, then $\dim \mu_t \geq \beta$.

**Proof.** The proof is carried out by backward induction on $d$. Let $1 \leq d < m$ and suppose that the theorem has been proven for each $d < d' < m$. Let $(t_j)_{j=1}^m = t \in \mathbb{R}^{dm}$ be such that,

$$\text{span}\{t_1, ..., t_m\} = \mathbb{R}^d \text{ and } \dim \mu_t < \beta.$$

We need to show that $\Phi_t$ has exact overlaps.

Let $\{e_1, ..., e_d\}$ be the standard basis of $\mathbb{R}^d$. For every invertible linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$,

$$\Phi_{T(t)} = \{T \circ \varphi_{t,j} \circ T^{-1}\}_{j=0}^m.$$  

Thus, without loss of generality we may assume that,

$$(3.1) \quad \{e_1, ..., e_d\} \subset \{t_1, ..., t_m\}. $$

There are two cases to consider during the proof. The containment $(3.1)$ will only be used when dealing with the first case.

Let $0 < \epsilon < \frac{1}{2}(\beta - \dim \mu_t)$ and let $\delta > 0$ be small with respect to $m$, $p$, $\lambda$ and $t$. In particular, $\delta$ is assumed to be small with respect to the constant $M = M(m, \lambda) > 1$ obtained in Lemma 11. By Lemma 8 it follows that $\Phi_t$ is affinely irreducible. Since $\delta$ is small with respect to $p$ and $\lambda$ we may assume that $-\log \delta > \chi$. Thus, by Theorem 4 there exists $N \geq 1$ such that,

$$(3.2) \quad \frac{1}{\chi^n} H(\nu_1^{(n)}, \mathcal{E}_{[-\log \delta]n} \varn \mathcal{F}) < \dim \mu_t + \epsilon \text{ for all } n \geq N.$$  

For every $1 \leq j \leq m$ let $t_1^j, ..., t_d^j \in \mathbb{R}$ be such that $t_j = (t_j^j)_{j=1}^m$. For $1 \leq l \leq d$ set $t^l = (t_j^j)_{j=1}^m$. Given $n \geq N$ let $A_n$ be the collection of all $L \in \mathcal{L}_n$ with $|L(t^l)| \leq \delta^n$ for each $1 \leq l \leq d$. By $(3.2)$ and since,

$$\chi n (\dim \mu_t + \epsilon) < \chi n (\beta - \epsilon) < n \dim(p),$$

there exist $w_1, w_2 \in A_n$ such that $w_1 \neq w_2$ but $\varphi_{t,w_1}$ and $\varphi_{t,w_2}$ belong to the same atom of the partition $\mathcal{E}_{[-\log \delta]n} \varn \mathcal{F}$. Thus $\lambda_{w_1} = \lambda_{w_2}$ and for each $1 \leq l \leq d$,

$$|L_{w_1, w_2}(t^l)| = |\varphi_{t,w_1}(0) - \varphi_{t,w_2}(0)| \leq \delta^n.$$  

It follows that $L_{w_1, w_2} \in A_n$ and in particular that $A_n$ is nonempty.

Write $q_n$ for $|A_n|$ and let $(L_{n,i})_{i=1}^q$ be an enumeration of $A_n$. By Lemma 12 for every $1 \leq i \leq q_n$ there exist $P_{n,i}^1, ..., P_{n,i}^m \in \mathcal{P}(1, n)$ such that,

$$L_{n,i}(x) = \sum_{j=1}^m P_{n,i}^j(\lambda) x_j \text{ for every } (x_1, ..., x_m) = x \in \mathbb{R}^m.$$  

For $1 \leq j \leq m$ set $a_{n,i}^j = P_{n,i}^j(\lambda)$, then $a_{n,i}^j = O_{\lambda, n}(1)$ since $\lambda_0, ..., \lambda_m \in (-1, 1)$ and $P_{n,i}^j \in \mathcal{P}(1, n)$. Write $a_{n,i} = (a_{n,i}^1, ..., a_{n,i}^m)$ and $B_n = \{1, ..., q_n\} \times \{1, ..., m\}$. Denote
by $r_n$ the rank of the matrix $(a_{n,i}^j)_{i,j \in \mathbb{B}_n}$. For each $1 \leq i \leq q_n$ and $1 \leq l \leq d$ set $\rho_{n,i}^l = L_{n,i}(t^l)$, then $|\rho_{n,i}^l| \leq \delta^n$ by the definition of $A_n$.

If $a_{n,i} = 0$ for some $n \geq N$ and $1 \leq i \leq q_n$ then $L_{n,i}(t^l) = 0$ for each $1 \leq l \leq d$. By the definition of $L_n$ this clearly implies that $\Phi_i$ has exact overlaps. Thus we may assume that $a_{n,i} \neq 0$, and in particular that $1 \leq r_n \leq m$. In fact we can get a better upper bound.

Claim. For $n \geq N$ we have $r_n \leq m - d$.

Proof. Without loss of generality we may assume that $\{a_{n,i}\}_{i=1}^{r_n}$ are linearly independent. Write $V = \text{span}\{a_{n,1}, \ldots, a_{n,r_n}\}$ and let $V^\perp$ be the orthogonal complement of $V$ in $\mathbb{R}^m$. Denote the orthogonal projections onto $V^\perp$ by $\pi_{V^\perp}$.

Given $k \geq 1$ and vectors $x_1, \ldots, x_k \in \mathbb{R}^m$ write $G(x_1, \ldots, x_k)$ for their Gram determinant. That is,

$$G(x_1, \ldots, x_k) = \det \left( \langle x_i, x_j \rangle : 1 \leq i, j \leq k \right),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^m$. Note that $G(x_1, \ldots, x_k)$ is equal to the square of the $k$-dimensional volume of the parallelootope spanned by $x_1, \ldots, x_k$. Since $\{a_{n,i}\}_{i=1}^{r_n}$ are linearly independent we have $G(a_{n,1}, \ldots, a_{n,r_n}) \neq 0$. Thus for every $x \in \mathbb{R}^m$,

$$\|\pi_{V^\perp}x\|^2 = \frac{G(a_{n,1}, \ldots, a_{n,r_n}, x)}{G(a_{n,1}, \ldots, a_{n,r_n})}. \tag{3.3}$$

Recall that $a_{n,i}^j = P_{n,i}^j(\lambda)$ with $P_{n,i}^j \in \mathcal{P}(1, n)$ for each $(i, j) \in \mathbb{B}_n$. Thus, by (2.8) it follows that for every $1 \leq i_1, i_2 \leq r_n$ there exists $P \in \mathcal{P}(mn^{2(m+1)}, 2n)$ with $\langle a_{n,i_1}, a_{n,i_2} \rangle = P(\lambda)$. Since $r_n \leq m$ another application of (2.8) shows that there exists,

$$D \in \mathcal{P}(m^{2m}n^{2m(m+1)}, 2mn),$$

such that,

$$G(a_{n,1}, \ldots, a_{n,r_n}) = D(\lambda).$$

From this, $G(a_{n,1}, \ldots, a_{n,r_n}) \neq 0$ and Lemma 11 we obtain,

$$G(a_{n,1}, \ldots, a_{n,r_n}) \geq (m^{2m}n^{2m(m+1)})^{-M} \cdot M^{-2mn}.$$

Thus by assuming that $\delta$ is small enough with respect to $\lambda$ and $m$,

$$G(a_{n,1}, \ldots, a_{n,r_n}) \geq \delta^n/2. \tag{3.4}$$

Since $a_{n,i}^j = O_{\lambda, m}(1)$ for $(i, j) \in \mathbb{B}_n$,

$$\langle a_{n,i_1}, a_{n,i_2} \rangle = O_{\lambda, m}(1) \text{ for each } 1 \leq i_1, i_2 \leq r_n. \tag{3.5}$$

For every $1 \leq l \leq d$ and $1 \leq i \leq r_n$,

$$\delta^n \geq |L_{n,i}(t^l)| = \sum_{j=1}^m a_{n,i}^j t_j^l = \|\langle a_{n,i}, t^l \rangle\|.$$

From this and (3.5),

$$G(a_{n,1}, \ldots, a_{n,r_n}, t^l) = \|t^l\|^2 \cdot G(a_{n,1}, \ldots, a_{n,r_n}) + O_{\lambda, m}(\delta^n).$$

Hence from (3.4) and (3.3),

$$\|\pi_{V^\perp}t^l\|^2 = \|t^l\|^2 + O_{\lambda, m}(\delta^n/2).$$
Thus there exist vectors \( u_1, \ldots, u_d \in V^\perp \) with \( \|t_l - u_l\| = O_{\lambda, m}(\delta^{n/4}) \) for each \( 1 \leq l \leq d \). From \( \text{span}\{t_1, \ldots, t_m\} = \mathbb{R}^d \) it follows that \( \{t_l\}_{l=1}^d \) are linearly independent. Hence by assuming that \( \delta \) is small enough with respect to \( \lambda, m \) and \( t \), we get that \( \{u_l\}_{l=1}^d \) are also linearly independent. From this it follows that,
\[
d \leq \dim V^\perp = m - \dim V = m - r_n,
\]
which completes the proof of the claim. \( \square \)

Now there are two cases to consider.

**First case.** Suppose first that \( \liminf r_n = m - d \). By the last claim and by increasing \( N \) without changing the notation we may assume that \( r_n = m - d \) for all \( n \geq N \). Recall that \( \{e_j\}_{j=1}^d \subset \{t_j\}_{j=1}^m \), and so we may assume for simplicity that,
\[
(3.6) \quad t_j = e_j \text{ for each } 1 \leq j \leq d.
\]
The assumption (3.6) will only be made when dealing with the present case, in which it will be clear that it makes no difference.

Set \( J = \{d+1, \ldots, m\} \) and \( I = \{1, \ldots, m - d\} \). Without loss of generality we may assume that \( \{a_{n,i} : i \in I\} \) are linearly independent for each \( n \geq N \). Given \( J_0 \subset \{1, \ldots, m\} \) with \( |J_0| = m - d \) write \( P_n^{J_0}(X) \) for the determinant of the matrix \((P_n^{(i)}(X))_{(i,j) \in I \times J_0}\). Note that \( P_n^{J_0}(X) \in \mathbb{Z}[X_0, \ldots, X_m] \) and that \( P_n^{J_0}(\lambda) \) is equal to the determinant of \((a_{n,i}^j)_{(i,j) \in I \times J_0}\).

**Claim.** For every \( n \geq N \) we have \( P_n^{J}(\lambda) \neq 0 \).

**Proof.** Assume by contradiction that there exists \( n \geq N \) for which the claim is false. Write \( k \) for the rank of the matrix \((a_{n,i}^j)_{(i,j) \in I \times J} \). Since the claim is false for \( n \) we have \( k < m - d \). Let \( F \subset \{1, \ldots, m\} \) be such that \( |F| = m - d \), \( |F \cap J| = k \) and \( P_n^F(\lambda) \neq 0 \).

Let \( 1 \leq l \leq d \) be with \( l \in F \). For \( j \in J \setminus F \) write \( F_j = (F \setminus \{l\}) \cup \{j\} \). From \( |F \cap J| = k \) it follows that,
\[
(3.7) \quad P_n^{F_j}(\lambda) = 0 \text{ for all } j \in J \setminus F.
\]

Let \( D \) be the determinant of the matrix obtained by replacing the column vector \((a_{n,i}^j)_{i=1}^m \) with the column vector \((\rho_{n,i}^j)_{i=1}^{m-d} \) in the matrix \((a_{n,i}^j)_{(i,j) \in I \times J} \). Since \( \rho_{n,i}^j \leq \delta^n \) for each \( 1 \leq i \leq q_n \) and \( a_{n,i}^j = O_{\lambda, m}(1) \) for each \( (i,j) \in B_n \) it follows that \( D = O_{\lambda, m}(\delta^n) \).

By (3.6) it follows that \( t_l^j = 0 \) for each \( j \in \{1, \ldots, d\} \setminus \{l\} \). Thus for every \( 1 \leq i \leq m - d \),
\[
\rho_{n,i}^j = L_{n,i}(t_l^j) = \sum_{j=1}^m a_{n,i}^j t_l^j = \sum_{j \in F} a_{n,i}^j t_j^l + \sum_{j \in J \setminus F} a_{n,i}^j t_j^l.
\]

By solving these equations in \( \{t_l^j\}_{j \in F} \) and applying Cramer’s rule we get,
\[
1 = t_l^j = \frac{1}{P_n^F(\lambda)} \left[ D - \sum_{j \in J \setminus F} t_j^j P_n^{F_j}(\lambda) \right].
\]

From this, (3.7) and \( D = O_{\lambda, m}(\delta^n) \) it follows that \( P_n^F(\lambda) = O_{\lambda, m}(\delta^n) \).
On the other hand, by (2.8) and since $P^j_{n,i} \in P(1, n)$ for each $(i, j) \in B_n,$
\[ P^F_n \in P((m!)n^{m(m+1)}, mn). \]
From this, $P^F_n(\lambda) \neq 0$ and Lemma 11
\[ |P^F_n(\lambda)| \geq ((m!)n^{m(m+1)})^{-M} \cdot M^{-mn}. \]
Thus, if $\delta$ is taken to be small enough with respect to $\lambda$ and $m$ then $P^F_n(\lambda) = O_{\lambda,m}(\delta^n)$ is not possible. This contradiction completes the proof of the claim. □

For $j \in J$ and $1 \leq l \leq d$ write $J_{j,l} = (J \setminus \{j\}) \cup \{l\}.$

Claim. Let $k \in J$ and $1 \leq l \leq d$ be given, then $t^l_k = -P^{j_{k,i}}_n(\lambda)/P^J_n(\lambda)$.

Proof. Let $n \geq N$ and write $D$ for the determinant of the matrix obtained by replacing the column vector $(\alpha^k_{n,i})_{i=1}^{m-d}$ with the column vector $(\rho^l_{n,i})_{i=1}^{m-d}$ in the matrix $(\alpha^l_{n,i})_{(i,j) \in I \times J}$. Since $|\rho^l_{n,i}| \leq \delta^n$ for each $1 \leq i \leq q_n$ and $\alpha^l_{n,i} = O_{\lambda,m}(1)$ for each $(i, j) \in B_n$, it follows that $D = O_{\lambda,m}(\delta^n)$. By (3.6) it follows that for every $1 \leq i \leq m - d$,
\[ \rho^l_{n,i} = L_{n,i}(t^l_i) = \sum_{j=1}^{m} \alpha^j_{n,i} t^l_i = \alpha^l_{n,i} + \sum_{j=d+1}^{m} \alpha^j_{n,i} t^l_i. \]

By solving these equations in $\{t^l_i\}_{j \in J}$ and applying Cramer’s rule, we get
\[ t^l_i = \frac{1}{P^J_n(\lambda)} (D - P^{j_{k,i}}_n(\lambda)). \]

By (2.8) and since $P^J_{n,i} \in P(1, n)$ for each $(i, j) \in B_n,$
\[ P^J_n \in P((m!)n^{m(m+1)}, mn). \]
Thus, from $P^J_n(\lambda) \neq 0$ and Lemma 11
\[ |P^J_n(\lambda)| \geq ((m!)n^{m(m+1)})^{-M} \cdot M^{-mn}. \]
From this, $D = O_{\lambda,m}(\delta^n)$ and by assuming that $\delta$ is small enough with respect to $\lambda$ and $m$, we get $|D/P^J_n(\lambda)| \leq \delta^{n/2}$. Hence from (2.8) it follows that for every $n \geq N$,
\[ \left| t^l_i + \frac{P^{j_{k,i}}_n(\lambda)}{P^J_n(\lambda)} \right| \leq \delta^{n/2}, \]
and so,
\[ \left| \frac{P^{j_{k,i}}_n(\lambda)}{P^J_n(\lambda)} \right| \leq \delta^{n/2}. \]
From this and $P^J_n(\lambda), P^J_{n+1}(\lambda) = O_{\lambda,m}(1)$, we get
\[ \left| P^{j_{k,i}}_n(\lambda) P^J_{n+1}(\lambda) - P^{j_{k,i}}_{n+1}(\lambda) P^J_n(\lambda) \right| = O_{\lambda,m}(\delta^{n/2}). \]
Now set,
\[ Q_n(X) = P^{j_{k,i}}_n(X) P^J_{n+1}(X) - P^{j_{k,i}}_{n+1}(X) P^J_n(X), \]
then $Q_n(\lambda) = O_{\lambda,m}(\delta^{n/2}).$ Moreover, by (2.8) and since,
\[ P^J_n, P^{j_{k,i}}_n, P^J_{n+1}, P^{j_{k,i}}_{n+1} \in P((m!)n^m(n+1)^{m(m+1)}, mn+1)), \]
we have,
\[ Q_n \in P(m^{5(m+1)}(n+1)^{4m(m+1)}, 2m(n+1)). \]
Thus from Lemma \[11\] it follows that \( Q_n(\lambda) = 0 \) or,
\[
|Q_n(\lambda)| \geq (m^{5(m+1)}(n + 1)^{4m(m+1)})^{-M} \cdot M^{-2m(n+1)}.
\]
From this, \( Q_n(\lambda) = O_{\lambda,m}(\delta^{n/2}) \) and by assuming that \( \delta \) is small enough with respect to \( \lambda \) and \( m \), it follows that we must have \( Q_n(\lambda) = 0 \). We have thus shown that for every \( n \geq N \),
\[
\left| t_k + \frac{P_n^{(i,j)}(\lambda)}{P_n^{(i)}(\lambda)} \right| \leq \delta^{n/2} \text{ and } \frac{P_n^{(i,j)}(\lambda)}{P_n^{(i)}(\lambda)} = \frac{P_{n+1}^{(i,j)}(\lambda)}{P_{n+1}^{(i)}(\lambda)}
\]
which clearly completes the proof of the claim. \( \square \)

From the last claim and \[3.6\] it follows that for every \( 1 \leq l \leq d \) and \( 1 \leq i \leq m-d \),
\[
L_{N,i}(t^l) = \sum_{j=1}^{m} a_{N,i}^{(i,j)} = a_{N,i}^{(i)} - \sum_{j=d+1}^{m} a_{N,i}^{(i,j)} \frac{P_n^{(i,j)}}{P_n^{(i)}(\lambda)}
\]
\[
= a_{N,i}^{(i)} - \sum_{j=d+1}^{m} a_{N,i}^{(i,j)} \det \left( (a_{N,i}^{(0)})_{(0,j) \in I \times J_i} \right) \det \left( (a_{N,i}^{(0)})_{(0,j) \in I \times J} \right)
\]
Thus by Cramer’s rule we have \( L_{N,i}(t^l) = 0 \). Since \( L_{N,1} \in \mathcal{L}_N \) there exist \( w_1, w_2 \in \Lambda^N \) such that \( w_1 \neq w_2 \), \( \lambda_{w_1} = \lambda_{w_2} \) and \( L_{N,1} = L_{w_1,w_2} \). For each \( 1 \leq l \leq d \),
\[
0 = L_{w_1,w_2}(t^l) = \varphi_{t^l,w_1}(0) - \varphi_{t^l,w_2}(0),
\]
which together with \( \lambda_{w_1} = \lambda_{w_2} \) implies \( \varphi_{t,w_1} = \varphi_{t,w_2} \). We have thus shown that when
\[
\liminf_{n} r_n = m - d
\]
the IFS \( \Phi_t \) has exact overlaps. This completes the treatment of the first case.

**Second case.** Suppose next that for
\[
r = \liminf_{n} r_n
\]
we have \( 1 \leq r < m-d \) (recall that \( r_n \geq 1 \) for all \( n \geq N \)). Let \( \{n_k\}_{k \geq 1} \) be an increasing sequence of positive integers with \( n_1 \geq N \) and \( r_{n_k} = r \) for all \( k \geq 1 \). Write \( I = \{1, \ldots, r\} \), then without loss of generality we may assume that the row vectors \( \{a_{n_k,i}\}_{i \in I} \) are linearly independent for all \( k \geq 1 \).

Write \( d' = m - r \) and \( J = \{d' + 1, \ldots, m\} \), and note that \( d < d' < m \). Given \( J_0 \subset \{1, \ldots, m\} \) with \( |J_0| = r \) write \( P_k^{(i)}(X) \) for the determinant of the matrix \( (P_{n_k,i}^{(i)})_{(i,j) \in I \times J_0} \). Since we no longer assume \[3.6\], we may without loss of generality assume that for infinitely many integers \( k \geq 1 \),
\[
(3.9) \quad |P_k^{(i)}(\lambda)| \geq |P_k^{(j_0)}(\lambda)| \text{ for all } J_0 \subset \{1, \ldots, m\} \text{ with } |J_0| = r.
\]
By moving to a subsequence without changing the notation we may suppose that \[3.9\] holds for all \( k \geq 1 \). Note that since \( \{a_{n_k,i}\}_{i \in I} \) are independent this implies that \( P_k^{(i)}(\lambda) \neq 0 \).

For every \( j \in J \) and \( 1 \leq l \leq d' \) set \( J_{j,l} = (J \setminus \{j\}) \cup \{l\} \). By moving to a subsequence without changing the notation we may assume that there exist numbers
$s_{0,j}^l \in [-1, 1]$ such that,

$$s_{0,j}^l = \lim_{k \to \infty} \frac{P_k^{j,i}(\lambda)}{P_k^j(\lambda)}$$

for all $j \in J$ and $1 \leq l \leq d'$.  

For $1 \leq j \leq d'$ and $1 \leq l \leq d'$ set $s_{0,j}^l = \delta_{j,l}$, where $\delta_{j,l}$ is the Kronecker delta.  For each $1 \leq j \leq m$ write $s_{0,j} = (s_{0,j}^l)_{l=1}^{d'}$, and note that $(s_{0,j}^l)_{l=1}^{d'}$ is the standard basis of $\mathbb{R}^{d'}$.  Set $s_0 = (s_{0,j}^l)_{j=1}^m$, so that $s_0 \in \mathbb{R}^{d'm}$.  For $1 \leq l \leq d'$ write $s_0^l = (s_{0,j}^l)_{j=1}^m$.

For $k \geq 1$, $1 \leq j \leq d'$ and $1 \leq l \leq d'$ write $s_{k,j}^l = s_{0,j}^l$.  For $j \in J$ and $1 \leq l \leq d'$ write, 

$$s_{k,j}^l = \frac{-\lambda^{l,j,i}(\lambda)}{P_k^j(\lambda)}.$$ 

For each $1 \leq j \leq m$ set $s_{k,j} = (s_{k,j}^l)_{l=1}^{d'}$ and write $s_k = (s_{k,j})_{j=1}^m \in \mathbb{R}^{d'm}$.  Since $(s_{k,j})_{j=1}^m$ is the standard basis of $\mathbb{R}^{d'}$ for every $i \in I$ we have,

$$L_{n,k,i}(s_k^l) = \sum_{j=1}^m a_{nk,i}^j s_{k,j}^l = a_{nk,i}^l - \sum_{j=1}^m a_{nk,i}^j \frac{P_k^{j,i}(\lambda)}{P_k^j(\lambda)}.$$

Thus by Cramer’s rule,

$$L_{n,k,i}(s_k^l) = 0$$

for each $i \in I$ and $1 \leq l \leq d'$.  

Since $\{a_{nk,i}\}_{i \in I}$ are independent and $r_{nk} = r$,  

$$\{a_{nk,i}\}_{i=1}^{q_{nk}} \subset \text{span}\{a_{nk,i}\}_{i \in I}.$$ 

Hence by (3.11),

$$L_{n,k,i}(s_k^l) = 0$$

for all $1 \leq i \leq q_{nk}$ and $1 \leq l \leq d'$.  

Let $w_1, w_2 \in \Lambda^n_k$ be such that $w_1 \neq w_2$ but $\varphi_{t,w_1}$ and $\varphi_{t,w_2}$ belong to the same atom of the partition $\mathcal{E}_{[-\log \delta]} \vee \mathcal{F}$.  By the the definition of $\mathcal{F}$ we have $\lambda_{w_1} = \lambda_{w_2}$ and by the definition of $\mathcal{E}_{[-\log \delta]} \vee \mathcal{F}$,

$$|L_{w_1,w_2}(s_k^l)| = |\varphi_{t,w_1}(0) - \varphi_{t,w_2}(0)| \leq \delta_{nk}$$

for each $1 \leq l \leq d$.  This shows that $L_{w_1,w_2} \in \mathcal{A}_{nk}$ and so by (3.12),

$$\varphi_{s_{k,w_1}^l}(0) - \varphi_{s_{k,w_2}^l}(0) = L_{w_1,w_2}(s_k) = 0$$

for each $1 \leq l \leq d'$.  Since $\lambda_{w_1} = \lambda_{w_2}$ it follows that $\varphi_{s_{k,w_1}} = \varphi_{s_{k,w_2}}$.

We have thus shown that $\varphi_{s_{k,w_1}} = \varphi_{s_{k,w_2}}$ for every $w_1, w_2 \in \Lambda^n_k$ such that $\varphi_{t,w_1}$ and $\varphi_{t,w_2}$ belong to the same atom of $\mathcal{E}_{[-\log \delta]} \vee \mathcal{F}$.  This clearly implies that,

$$H(\mu_{s_{k,w_1}^{(nk)}}) \leq H(\mu_{s_{k,w_2}^{(nk)}}) = H(\mu_{s_{k}^{(nk)}})$$

From this and (3.2),

$$\frac{1}{\lambda_{nk}} H(\mu_{s_{k}^{(nk)}}) < \dim \mu_{t} + \epsilon < \beta - \epsilon.$$ 

Hence $\dim \mu_{s_{k}} < \beta - \epsilon$ by Corollary [?].
Note that by \[3.10\] and the definitions of \(s_0\) and \(\{s_k\}_{k \geq 1}\) it follows that \(s_k \to s_0\). Thus from Theorem \[11\] we get that \(\dim \mu_{s_0} \leq \beta - \epsilon\). Also, since \((s_0,j)_{j=1}^{d'}\) is the standard basis of \(\mathbb{R}^{d'}\),
\[
\text{span}\{s_0,1,...,s_0,m\} = \mathbb{R}^{d'}.
\]
Now because \(d < d' < m\) we may use the induction hypothesis in order to conclude that \(\Phi_{s_0}\) has exact overlaps. It remains to show that this implies that \(\Phi_t\) also has exact overlaps.

**Claim.** For every \(1 \leq j \leq m\) we have \(t_j = \sum_{l=1}^{d'} s_{0,j}^{l} t_l\).

**Proof.** For every \(1 \leq j \leq d'\),
\[
\sum_{l=1}^{d'} s_{0,j}^{l} t_l = \sum_{l=1}^{d'} \delta_{j,l} t_l = t_j.
\]
Thus the claim holds for \(1 \leq j \leq d'\).

Let \(j_0 \in J, 1 \leq l \leq d\) and \(k \geq 1\) be given. Write \(D_k\) for the determinant of the matrix obtained by replacing the column vector \((a_{j_0,k})_{i=1}^{n}\) with the column vector \((\rho_{j_0,i})_{i=1}^{n}\) in the matrix \((a_{j_0,i})_{(i,j) \in J \times J}\). Since \(|\rho_{j_0,i}| \leq \delta^{n_k}\) for each \(1 \leq i \leq q_{n_k}\) and \(a_{j_0,i} = O_{\lambda,m}(1)\) for each \((i,j) \in P_{n_k}\), it follows that \(D_k = O_{\lambda,m}(\delta^{n_k})\). By \[2.8\] and since \(P_{n_k,i} \in \mathcal{P}(1,n_k)\) for each \((i,j) \in B_{n_k}\),
\[
P^j_{k} \in \mathcal{P}((m!)n_k^{m(m+1)/2}, mn_k).
\]
From this, \(P^j_{k}(\lambda) \neq 0\) and Lemma \[11\]
\[
|P^j_{k}(\lambda)| \geq ((m!)n_k^{m(m+1)/2})^{-M} M^{-mn_k}.
\]
Thus, by taking \(\delta\) to be small enough with respect to \(\lambda\) and \(m\) we may assume that,
\[
|D_k/P^j_{k}(\lambda)| \leq \delta^{n_k/2}.
\]
For every \(1 \leq i \leq r\),
\[
\rho_{j_0,i}^j = L_{n_k,i}(t^j) = \sum_{j=1}^{d'} a_{j_0,i}^j t_j^j.
\]
Thus by solving these equations in \(\{t^j_j\}_{j \in J}\) and applying Cramer’s rule,
\[
t^j_{j_0} = \frac{1}{P^j_{k}(\lambda)}(D_k - \sum_{j=1}^{d'} t^j_j P^j_{k}(\lambda)).
\]
From this, \(|D_k/P^j_{k}(\lambda)| \leq \delta^{n_k/2}\) and \[3.10\],
\[
t^j_{j_0} = -\lim_{k \to \infty} \sum_{j=1}^{d'} \frac{t^j_j P^j_{k}(\lambda)}{P^j_{k}(\lambda)} = \sum_{j=1}^{d'} s_{0,j_0}^j t_j^j.
\]
Since this holds for all \(1 \leq l \leq d\) it follows that \(t_{j_0} = \sum_{j=1}^{d'} s_{0,j_0}^j t_j\), which completes the proof of the claim. \(\square\)
Now since $\Phi_{t, s}$ has exact overlaps there exist $n \geq 1$ and $w_1, w_2 \in \Lambda^n$ such that $w_1 \neq w_2$, $\lambda_{w_1} = \lambda_{w_2}$ and $L_{w_1, w_2}(s^l_0) = 0$ for all $1 \leq l \leq d'$. Write $L = L_{w_1, w_2}$ and let $c_1, \ldots, c_m \in \mathbb{R}$ be such that,

$$L(x) = \sum_{j=1}^{m} c_j x_j$$

for $(x_j)_{j=1}^{m} = x \in \mathbb{R}^m$.

Since $L(s^l_0) = 0$ for each $1 \leq l \leq d'$ and by the last claim,

$$0 = \sum_{l=1}^{d'} L(s^l_0) t_l = \sum_{j=1}^{m} c_j \sum_{l=1}^{d'} s^l_{0,j} t_l = \sum_{j=1}^{m} c_j t_j.$$

This implies that $L(t^l) = 0$ for each $1 \leq l \leq d$, which shows that $\varphi_{t, w_1} = \varphi_{t, w_2}$.

We have thus shown that also in the second case $\Phi_t$ has exact overlaps, which completes the proof of the theorem. \qed

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