A Trotter Product Formula for quantum stochastic flows

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Abstract

We prove an analogue of the Trotter product formula for quantum stochastic flows satisfying quantum stochastic differential equations, under some further hypotheses. Applications for a wide class of classical and quantum stochastic processes are also studied.

1 Introduction.

Several years ago K.R.Parthasarathy and the third-named author obtained a stochastic Trotter Product formula for unitary operator-valued evolutions, constituted from independent increments of classical Brownian motion [11] and more recently in [7], this was extended to those constituted from the fundamental quantum processes, satisfying Hudson-Parthasarathy type [10, 14] quantum stochastic differential equations (q.s.d.e for short) with bounded operator coefficients. In this article, we extend the Trotter product formula to quantum stochastic flows on C*- or von Neumann algebra $A$. Each of these constituent flows satisfy q.s.d.e.’s of the Evans-Hudson type [10, 14] with respective (possibly unbounded) structure maps satisfying a structure relation on a suitable *-subalgebra $A_0$ of $A$. It is known that each of these are completely positive, contractive (CPC) cocycles, and two of them are composed over a dyadic decomposition of an interval after shifting in each sub-interval to $[0, 2^{-n}]$. The first set of main results (in section 4) proven here says that if the vacuum expectation semigroups are analytic and if the sum of the two associated generators is a pre-generator of a contractive semigroup (this part is the usual assumption for a Trotter-Kato type of product formula for semigroups), then the composed sequence weakly converges to a CPC flow satisfying a q.s.d.e. with its structure maps made up of the constituent pair in a specific manner. It is clear that since each member-map of the weakly convergent sequence is a homomorphism, the convergence is strong if and only if the limit-map itself is homomorphic. This is achieved in section 5 by giving a new proof of the homomorphism property of the limit (in section 3) under some further hypothesis on the dependence of the CPC flow on the algebra, by mimicking partially the proof of unitarity of the solution of a Hudson-Parthasarathy q.s.d.e with unbounded operator coefficients satisfying the formal unitarity relations [14, 4]. Finally, in section 6, the results of section 4 are applied to obtain a construction of Brownian motion on a compact Lie group and of random walk on discrete groups. These results has also been used to extend the earlier constructions of a quantum stochastic flow on certain UHF algebras [5, 14].

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2 Notation and terminologies.

2.1 Quantum stochastic flows

We shall refer the reader to [10, 3, 14] and references therein for the basics of Evans-Hudson formalism of quantum stochastic calculus, which we very briefly review here. All the Hilbert spaces appearing in this article will be separable and for a Hilbert space $H$ we shall denote by $\Gamma(H), \Gamma^f(H)$ the symmetric and free Fock space over $H$ respectively, and by $\text{Lin}(V, W)$ the space of the linear (possibly defined on a subspace of $V$) maps from a vector space $V$ to another vector space $W$, and by $D(L)$ the domain of a possibly unbounded operator $L$ on a Banach space. The tensor product of Hilbert spaces or of operators will usually be denoted by $\otimes$, and sometimes $\otimes_{\text{alg}}$ is used for the algebraic tensor product. We shall also use the projective tensor product $\otimes_{\gamma}$ of Banach spaces, which will be explained later.

Definition 2.1. We say that a family of completely positive maps $(j_t)_{t \geq 0}$ from a unital $C^*$ or von Neumann algebra $A$ to $A^t \otimes B(\Gamma)$ (where $\Gamma := \Gamma(L^2(\mathbb{R}_+, k_0)))$, is a CPC flow, where $k_0$ is the Hilbert space $(1 \leq \dim k_0 \leq \infty)$ of noise or multiplicity with structure maps $(\theta^t_\mu)$ belonging to $\text{Lin}(A, A)$, where $\mu, \nu \in \{0\} \cup \{1, 2, \ldots, \dim k_0\}$, if the following holds:

(i) There is a dense $*$-subalgebra $A_0$ of $A$ (norm dense for $C^*$ algebra and ultraweakly dense for von-Neumann algebra) such that $A_0$ is contained in the domain of all the maps $\theta^t_\mu$, and $j_t$ is normal, if $A$ is a von-Neumann algebra.

(ii) The family $\{j_t(x)\}_{t \geq 0}$ satisfy a weak q.s.d.e. of the form: for $u, v \in H, f, g \in L^2(\mathbb{R}_+, k_0)$ and $x \in A_0$:

\[
\langle j_t(x)ue(f), ve(g) \rangle = \langle xue(f), ve(g) \rangle + \sum_{\mu, \nu} \int_0^t ds \langle j_s(\theta^t_\mu(x))ue(f), ve(g) \rangle g^\mu(s)f^\nu(s)
\]

or symbolically,

\[
j_t(x) = x + \sum_{\mu, \nu} \int_0^t j_s(\theta^t_\mu(x))\Lambda^\nu_s(ds)
\]

where $f^i(s) = \langle e_i, f(s) \rangle$, $f_i(s) = \bar{f}^i(s)$, $f_0(s) = f^0(s) = 1$, $\{e_i\}_{i=1}^{\dim k_0}$ being an orthonormal basis for the noise space $k_0$ with respect to which the structure maps $\theta^t_{ij}(i, j) \neq (0, 0))$ are given, and where $\Lambda^\nu_s$ are the fundamental integrators [10, 14].

(iii) The structure maps $\theta^t_\mu$ satisfy: for $x \in A_0$,

\[
\theta^t_\mu(xy) = \theta^t_\mu(x)y + x\theta^t_\mu(y) + \sum_{i=1}^{\dim k_0} \theta^t_\mu(x)\theta^t_\mu(y), \quad \theta^t_\mu(x)^* = \theta^t_{\mu^*}(x^*).
\]

The CPC flow $(j_t)_{t \geq 0}$ is called a quantum stochastic flow if furthermore $j_t$ is a * homomorphism.
Often it is convenient to associate a matrix, called the structure matrix, with the structure maps $θ^\mu_\nu$ as follows:

$$
\left( \begin{array}{cc}
\mathcal{L} & \delta^\dagger \\
\delta & \sigma
\end{array} \right),
$$

where $σ := \sum_{i,j} θ^i_j(x) \otimes |e_j><e_i|$, $δ(x) := \sum_i θ^i_0(x) \otimes e_i$, $δ^\dagger(x) := δ(x^*)^*$, and $\mathcal{L}(x) = θ^0_0(x)$, for $x \in \mathcal{A}_0$.

In the above, $\{Λ^\mu_\nu(\mu,\nu)\}_{(\mu,\nu)\neq(0,0)}$ are the fundamental martingales satisfying the quantum-Itô formula, (see page 127 of [14]):

$$
dΛ_\alpha^\beta(t)dΛ_\mu^\nu(t) = \hat{δ}^\alpha_\nu dΛ_\mu^\beta(t)
$$

for $α, β = 0, 1, 2, 3, ...$, and

$$
\hat{δ}^\alpha_\beta := 0 \text{ if } α = 0 \text{ or } β = 0 \\
:= δ^α_β \text{ otherwise,}
$$

(4)

$δ^α_β$ being usual Kronecker delta. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $A$ belongs to $\text{Lin}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$, with domain $\mathcal{D}$. For each $f \in \mathcal{H}_2$, we define a linear operator $⟨A,f⟩$ with domain $\mathcal{D}$ and taking values in $\mathcal{H}_1$ such that,

$$
\langle ⟨A,f⟩u,v \rangle = \langle Au,v \otimes f \rangle
$$

for $u \in \mathcal{D}$, $v \in \mathcal{H}_1$. We shall denote by $⟨A,f⟩$ the adjoint of $⟨f,A⟩$, whenever it exists.

For $T \in \text{Lin}(D_0 \otimes \text{alg} V_0, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where $D_0$ and $V_0$ are subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and $f \in V_0, g \in \mathcal{H}_2$, define partial trace of $T$ with respect to the rank one operator $|f><g|$, denoted by $⟨g,Tf⟩$ as:

$$
⟨g,Tf⟩ : D_0 \rightarrow \mathcal{H}_1 \text{ by } \langle g,Tf⟩(u,v) = \langle T(u \otimes f),v \otimes g \rangle, \ u \in D_0, v \in \mathcal{H}_1.
$$

2.2 Tensor product of Banach spaces

Here we collect a few facts about the projective tensor product of Banach spaces which is an important technical tool, needed to prove our main result. Recall that (see [16]) for two Banach spaces $E_1, E_2$, the projective tensor product $E_1 \otimes_γ E_2$ is the completion of the algebraic tensor product $E_1 \otimes_{\text{alg}} E_2$ under the cross-norm $\| \cdot \|_γ$ given by $\|X\|_γ = \inf \sum_i \|x_i\|\|y_i\|$, where infimum is taken over all possible expressions of $X$ of the form $X = \sum_{i=1}^n x_i \otimes y_i$.

Lemma 2.2. Suppose $T_j \in B(E_j, F_j)$ where $E_j, F_j$, for $j = 1, 2$ are Banach spaces. Then $T_1 \otimes_{\text{alg}} T_2$ extends to a bounded operator

$$
T_1 \otimes_γ T_2 : E_1 \otimes_γ E_2 \rightarrow F_1 \otimes_γ F_2
$$

with bound

$$
\|T_1 \otimes_γ T_2\| \leq ||T_1||||T_2||.
$$

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Proof. The proof is an easy consequence of the estimate:

\[ \| (T_1 \otimes_{alg} T_2)(\sum_{i=1}^{k} x_i \otimes y_i) \|_{\gamma} \leq \sum_{i=1}^{k} \| T_1(x_i) \otimes T_2(y_i) \|_{\gamma} \]

\[ \leq \sum_{i=1}^{k} \| T_1(x_i) \|_{F_1} \| T_2(y_i) \|_{F_2} \leq \| T_1 \| \| T_2 \| \sum_{i=1}^{k} \| x_i \|_{E_1} \| y_i \|_{E_2}. \]  \hspace{1cm} (6) \]

\[ \square \]

Lemma 2.3. Suppose \( T_t \) and \( S_t \) are two \( C_0 \) semigroups of bounded operators on \( E_1 \) & \( E_2 \) with generators \( L_1 \) and \( L_2 \) respectively. Then \( T_t \otimes_{\gamma} S_t \) becomes a \( C_0 \) semigroup of operators on \( E_1 \otimes_{\gamma} E_2 \) whose generator is the closed extension of the operator \( L_1 \otimes_{alg} 1 + 1 \otimes_{alg} L_2 \), defined on \( D(L_1) \otimes_{alg} D(L_2) \) in the space \( E_1 \otimes_{\gamma} E_2 \).

Proof. Since \( (T_t \otimes_{\gamma} S_t) \circ (T_s \otimes_{\gamma} S_s) = (T_{t+s} \otimes_{\gamma} S_{t+s}) \) (in \( E_1 \otimes_{alg} E_2 \)), both sides being continuous in \( E_1 \otimes_{alg} E_2 \), and as \( E_1 \otimes_{alg} E_2 \) is dense in \( E_1 \otimes_{\gamma} E_2 \), the above identity extends by Lemma (2.2) to \( E_1 \otimes_{\gamma} E_2 \) leading to the semigroup property

\[ (T_t \otimes_{\gamma} S_t) \circ (T_s \otimes_{\gamma} S_s) = (T_{t+s} \otimes_{\gamma} S_{t+s}). \]

Also similar reasoning gives us \( (T_t \otimes_{\gamma} 1) \circ (1 \otimes_{\gamma} S_t) = T_t \otimes_{\gamma} S_t \) and thus the strong continuity of \( T_t \otimes_{\gamma} 1 \) as well as of \( 1 \otimes_{\gamma} S_t \) as a function of \( t \) yields the strong continuity of \( T_t \otimes_{\gamma} S_t \). Hence \( (T_t \otimes_{\gamma} S_t)_{t \geq 0} \) is a \( C_0 \) semigroup. Moreover \( T_t \otimes_{\gamma} S_t \) keeps \( D(L_1) \otimes_{alg} D(L_2) \) \( \gamma \)-invariant, which is dense in \( E_1 \otimes_{\gamma} E_2 \). Thus \( D(L_1) \otimes_{alg} D(L_2) \) is a core for the generator of \( T_t \otimes_{\gamma} S_t \) (see [2]) which is the closure of the operator \( L_1 \otimes_{alg} 1 + 1 \otimes_{alg} L_2 \) denoted by \( L_1 \otimes_{\gamma}, 1 + 1 \otimes_{\gamma} L_2 \) \( \square \)

We state without proof the following corollary:

Corollary 2.4. The operator \( L_1 \otimes_{alg} I \) defined on \( D(L_1) \otimes_{alg} E_2 \) is closable in \( E_1 \otimes_{\gamma} E_2 \). Similar results hold for \( I \otimes_{alg} L_2 \) on \( E_1 \otimes_{alg} D(L_2) \). We denote the respective closures by \( L_1 \otimes_{\gamma}, I \) and \( I \otimes_{\gamma} L_2 \).

We conclude this section with the following result about operators on Banach spaces, which will play a crucial role in the proof of homomorphism property in section 3.

Lemma 2.5. Let \( E \) be a Banach space, and let \( A \) and \( B \) belong to \( Lin(E, E) \) with dense domains \( D(A) \) and \( D(B) \) respectively. Suppose there is a total set \( T \subset D(A) \cap D(B) \) with the properties :

(i) \( A(T) \) is total in \( E \),  \( (ii) \| B(x) \| < \| A(x) \| \) for all \( x \in D. \)

Then \( (A + B)(T) \) is also total in \( E. \)

Proof. First note that if \( A(T) \subseteq (A + B)(T) \), then \( F \equiv span\{ (A + B)(T) \} \) is dense in \( E \). Therefore without loss of generality we suppose \( F \neq E \), or that there is a non-zero \( y_0 \) in \( A(T) \), such that \( y_0 \notin (A + B)(T) \), and let \( y_0 = A(x_0) \), for some \( x_0 \in D \). Then by Hahn-Banach theorem, there exists \( \Lambda \in E^* \), the topological dual of \( E \), such that \( \| \Lambda \| = 1 \), \( |\Lambda(y_0)| = \| y_0 \| \) as well as \( \Lambda((A + B)(T)) = 0 \). Then \( \| y_0 \| = |\Lambda(A(x_0))| \) and \( |\Lambda(A(x_0))| = |\Lambda(B(x_0))| \). But \( |\Lambda(B(x_0))| \leq \| B(x_0) \| < \| A(x_0) \| = \| y_0 \| \) which leads to a contradiction. Therefore \( T = E. \) \( \square \)
3 A proof of homomorphism property.

Let $\mathcal{A}$ be a $C^*$ or von-Neumann algebra, equipped with a semifinite, faithful, lower-semicontinuous (also normal in case $\mathcal{A}$ is a von-Neumann algebra) trace $\tau$, and let $\mathcal{A}_0$ be a dense (as given in definition 2.1) $*$-subalgebra of $\mathcal{A}$ which is also dense in $h(\equiv L^2(\mathcal{A}, \tau))$ in the $L^2$- topology. Assume that $j_t, t \geq 0$ is a CPC flow as given in 2.1 and let $(T_t)_{t \geq 0}$ be the vacuum expectation semigroup of $j_t$, i.e. $\langle u, T_t(x)v \rangle = \langle ue(0), j_t(x)e(0) \rangle = \langle u, j_t^{0,0}(x)v \rangle$ for $u, v \in h$, $x \in \mathcal{A}$. We assume that the vacuum expectation semigroup $j_t^{0,0}$ is a $C_0$ (in the norm or ultraweak topology according as $\mathcal{A}$ is $C^*$ or von-Neumann algebra) semigroup. Furthermore, we make the following assumptions:

**A(i)** For each $t \geq 0$, $T_t$ extends as a bounded operator (which we again denote by $T_t$) on the Hilbert space $h$ such that $(T_t)_{t \geq 0}$ is a contractive, analytic $C_0$-semigroup of operators in the Hilbert space $h$. We shall denote by $\mathcal{L}_2$ the generator of $((T_t)_{t \geq 0})$ in $h$.

**A(ii)** Suppose that $\mathcal{A}_0 \subseteq D(\mathcal{L}) \cap D(\mathcal{L}_2)$, and that $T_t$ leaves $\mathcal{A}_0$ invariant.

**A(iii)** The map $\pi$ defined by

$$\pi(x) = \sigma(x) + x \otimes 1_k,$$

is a $*$-homomorphism (normal if $\mathcal{A}$ is a von-Neumann algebra) belonging to $\text{Lin}(\mathcal{A}, \mathcal{A} \otimes B(k_0))$, and the map $\delta$ is a well defined $\pi$-derivation belonging to $\text{Lin}(\mathcal{A}_0, \mathcal{A} \otimes k_0)$, i.e.

$$\delta(xy) = \delta(x)y + \pi(x)\delta(y),$$

for $x, y \in \mathcal{A}_0$.

**A(iv)** For $x, y$ in $\mathcal{A}_0$, the following second order co-cycle relation holds:

$$\delta(x)^*\delta(y) = \mathcal{L}(x^*y) - \mathcal{L}(x^*)y - x^*\mathcal{L}(y). \quad (7)$$

**A(v)** For $x \in \mathcal{A}_0$, $\mathcal{L}(x^*x) \in \mathcal{A} \cap L^1(\tau)$ and $\tau(\mathcal{L}(x^*x)) \leq 0$ (a kind of weak dissipativity).

**Remark 3.1.** If $j_t$ is a $*$-homomorphism, the assumptions A(iii) and A(iv) hold (see [10]).

**Remark 3.2.** A(ii) implies $\mathcal{A}_0$ is a core for both $\mathcal{L}$ and $\mathcal{L}_2$. Furthermore observe that because of analyticity in A(i), A(iv) and A(v), the real part of the operator $(-2\mathcal{L}_2)$ exists as an operator and is non-negative (see pages 322 and 336 of [10]). Moreover it also follows from these assumptions that $\delta(x) \in h \otimes k_0$ for $x \in \mathcal{A}_0$.

**Remark 3.3.** It can be shown that if $(T_t)_{t \geq 0}$ is symmetric with respect to $\tau$, that is if $\tau(T_t(x)y) = \tau(xT_t(y))$, then A(i) follows. Also if we assume furthermore that $T_t$ is conservative i.e.

$T_t(I) = I \forall t \geq 0$ and A(ii) is valid, then A(v) also follows.

**Remark 3.4.** Consider a typical diffusion process in $\mathbb{R}$ whose generator is of the form:

$$\mathcal{L} = \frac{1}{2} \frac{d}{dx}a^2(x) \frac{d}{dx} + b(x) \frac{d}{dx}.$$
Let Remark 3.6.

for which \( L \) easily conclude that \( p \) has a measure on \( \mathbb{R} \).

\[
\text{Proof.}
\]

Next let \( \hat{L} = L_2 \otimes_\gamma 1 + 1 \otimes_\gamma \hat{L}_2, \ C = (-2\text{Re}(L_2))^\frac{1}{2}, \) and let

\[
C \otimes_\gamma C := (C \otimes_\gamma 1) \circ (1 \otimes_\gamma C) = (1 \otimes_\gamma C) \circ (C \otimes_\gamma 1) \text{ in } h \otimes_\gamma h.
\]

Moreover we set \( F := A_0 \otimes_{alg} A_0, \) and \( Y := \{(\lambda - \hat{L})^{-1}(x \otimes y) | \ x, y \in A_0\}. \)

The next two lemmas set the stage for the application of lemma 2.5 to our problem, leading to the main result of this section, viz. the proof of the homomorphism property of \( j_t \) under an additional hypothesis.

**Lemma 3.7.** For \( x \in A_0, \ x \neq 0, \)

\[
\int_0^\infty e^{-\lambda t} \|C(T_t(x))\|^2 < \|x\|^2.
\]

**Proof.** For \( x \) in \( A_0, \)

\[
\frac{d}{dt} \|T_t(x)\|^2 = \langle L_2(T_t(x)), T_t(x) \rangle + \langle T_t(x), L_2(T_t(x)) \rangle = -\|C \circ T_t(x)\|^2. \tag{8}
\]
We get for $\lambda > 0$,
\[
\int_0^\infty e^{-\lambda t} \|C(T_t(x))\|^2 = -\int_0^\infty e^{-\lambda t} \frac{d}{dt} \|T_t(x)\|^2 \\
= \left\{ \|x\|^2 - \lambda \int_0^\infty e^{-\lambda t} \|T_t(x)\|^2 \right\}.
\]
(9)

Thus if $\lambda \int_0^\infty dt e^{-\lambda t} \|T_t(x)\|^2 = 0$, for some $\lambda > 0$, then we have $\|T_t(x)\| = 0$ for almost all $t$ and by virtue of the continuity of $\|T_t(x)\|$ as a function of $t$, this leads to the contradiction $x = 0$. Thus $\lambda \int_0^\infty dt e^{-\lambda t} \|T_t(x)\|^2 > 0$ for all $\lambda > 0$ which gives us the required strict inequality. $\Box$

**Lemma 3.8.** $\|(C \otimes C)(X)\|_\gamma \leq \|(\lambda - \hat{\lambda})(X)\|_\gamma$ for all $X$ in $D(\hat{\lambda})$ and we have strict inequality if $X$ is in $\mathcal{Y}$.

**Proof.** Let $X \in \mathcal{F}$, such that $X = \sum_{i=1}^k x_i \otimes y_i$. It is obvious that $(1 \otimes C)(\mathcal{F}) \subset D((C \otimes 1))$. So using lemma [5.7]
\[
\int_0^\infty dt e^{-\lambda t} \|C \otimes C(T_t(x))\|_\gamma \\
= \int_0^\infty dt e^{-\lambda t} \sum_{i=1}^k \|C(T_t(x_i)) \otimes C(T_t(y_i))\|_\gamma \\
\leq \sum_{i=1}^k \int_0^\infty dt e^{-\lambda t} \|C(T_t(x_i))\|^2 \frac{1}{2} \left( \int_0^\infty dt e^{-\lambda t} \|C(T_t(y_i))\|^2 \right) < \sum_{i=1}^k \|x_i\| \|y_i\|.
\]
Equation(10) and Remark [3.6] together yields:
\[
\|(C \otimes C)((\lambda - \hat{\lambda})^{-1}(X))\|_\gamma \leq \|X\|_\gamma.
\]
Thus $(C \otimes C)((\lambda - \hat{\lambda})^{-1}$ is a contraction. As a consequence, $C \otimes C$ extends to $D(\hat{\lambda})$ and we have the required inequality. Now with $X = x \otimes y$, where $x, y \in \mathcal{A}_0$, the above equations give
\[
\|(C \otimes C)((\lambda - \hat{\lambda})^{-1}(x \otimes y))\|_\gamma < \|x\| \|y\| = \|x \otimes y\|_\gamma \\
or \|(C \otimes C)(Y)\|_\gamma < \|(\lambda - \hat{\lambda})(Y)\|_\gamma \text{ for } Y \in \mathcal{Y}.
\]
$\Box$

The assumptions $A(iv)$ and $A(v)$ lead to
\[
\|\theta^j_0(x)\|_h^2 \leq \sum_{j=1}^\infty \|\theta^j_0(x)\|_h^2 \leq \|C(x)\|_h^2 \leq \|(C + \epsilon)(x)\|_h^2.
\]
(12)
for all $x$ in $\mathcal{A}_0$, $\epsilon > 0$. This implies $\theta^j_0$ can be extended to $D(C)$. Now define a map $B$ belonging to $Lin(D(C) \otimes_{alg} D(C), L^2(\tau) \otimes_{\gamma} L^2(\tau))$ by
\[
B(x \otimes y) = \sum_{i \geq 1} \theta^i_0(x) \otimes \theta^i_0(y),
\]
7
and extend linearly. This operator is well-defined because: using (12)
\[
\sum_{i \geq 1} \|\theta_0^i(x)\| \|\theta_0^i(y)\| \leq \left\{ \left( \sum_{i \geq 1} \|\theta_0^i(x)\|^2 \right)^2 \left( \sum_{i \geq 1} \|\theta_0^i(y)\|^2 \right)^2 \right\}^{\frac{1}{2}}
\]
\[
\leq \|(C + \epsilon)x\| \|(C + \epsilon)y\| < \infty.
\]
Thus we have
\[
\|B\{(C + \epsilon)^{-1} \otimes \gamma (C + \epsilon)^{-1}\}(x \otimes y)\|_\gamma \leq \|x \otimes y\|_\gamma,
\]
which implies that $B\{(C + \epsilon)^{-1} \otimes \gamma (C + \epsilon)^{-1}\}$ extends to a contraction on $h \otimes h$, and hence
\[
\|B(X)\|_\gamma \leq \|(C + \epsilon) \otimes \gamma (C + \epsilon)(X)\|_\gamma
\]
for all $X \in D(C) \otimes_{alg} D(C)$. Letting $\epsilon \to 0$, we see that
\[
\|B(X)\|_\gamma \leq \|(C \otimes \gamma C)(X)\|_\gamma
\]
for all $X$ in $D(C) \otimes_{alg} D(C)$. By Lemma 3.8, $C \otimes \gamma C$ extends to $D(\hat{L})$ and thus we can also extend $B$ to $D(\hat{L})$. So we have
\[
\|B(X)\| \leq \|(C \otimes \gamma C)(X)\|_\gamma \leq \|(\lambda - \hat{L})(X)\|_\gamma \text{ for all } X \in D(\hat{L}).
\]
Now $\text{span}\{Y\} \subseteq D(\hat{L})$, and in particular for $Y$ in $Y$,
\[
\|B(Y)\|_\gamma \leq \|(C \otimes \gamma C)(Y)\|_\gamma < \|(\lambda - \hat{L})(Y)\|_\gamma.
\]

\[\square\]

**Theorem 3.9.** Assume that the flow $j_t$ satisfies **A(i)** - **A(v)** and suppose that the following hold:

**A(vi)** There exists a total subset $W$ of $L^2(\mathbb{R}_+; k_0)$, such that for $f, g$ in $W$, $x \in A \cap L^1(\tau)$ and $u, v$ in $L^\infty(\tau) \cap L^2(\tau)$, we have:
\[
\sup_{0 \leq s \leq t} |\langle u f^\otimes m, j_t(x) v g^\otimes n \rangle| \leq C(u, v, f, g, m, n, t)\|x\|_1,
\]

\[
\text{such that for fixed } u, v, f, g, m, n, \text{ } C(u, v, f, g, m, n, t) = O(e^{\beta t}) \text{ for some } \beta \geq 0.
\]

Then $j_t$ is a $*$-homomorphism.

**Proof.** For brevity, we adopt Einstein’s summation convention in the proof.

For $f, g$ in $W$, the flow equation (2) leads to:
\[
\langle j_t(x) u(e(f), v(e(g)) = \langle x u(e(f), v(e(g)) + \int_0^t ds \langle j_s(\theta_x^u(s)) u(e(f), v(e(g)) g^u(s) f_v(s). \]

Using the quantum Ito formula we get:
\[
\langle j_t(x) u(e(f), j_t(y) v(e(g))
\]
\[
= \langle x u(e(f), y v(e(g)) + \int_0^t ds \langle j_s(\theta_x^u(s)) u(e(f), j_s(y) v(e(g)) g^u(s) f_v(s)
\]
\[
+ \langle j_s(x) u(e(f), j_s(\theta_y^v(s)) v(e(g)) f_u(s) g^v(s)
\]
\[
+ \langle j_s(\theta_x^u(s)) u(e(f), j_s(\theta_y^v(s)) v(e(g)) f_u(s) g^v(s).
\]

}\]
For fixed \( u, v \) in \( A \cap h, f, g \) in \( W \), we define for each \( t \geq 0 \), \( \phi_t : A_0 \times A_0 \to \mathbb{C} \) by

\[
\phi_t(x, y) := \langle j_t(x) u e(f), j_t(y) v e(g) \rangle - \langle j_t(y^* x) u e(f), v e(g) \rangle.
\]

Using (20) and (21), we get:

\[
\phi_t(x, y) = \int_0^t ds [\phi_s(\theta^0_0(x), y) + \phi_s(x, \theta^0_0(y))] + \phi_s(\theta^0_0(x), \theta^0_0(y))
+ g^i(s) \phi_s(\theta^0_0(x), y) + g^i(s) \phi_s(x, \theta^0_0(y)) + f_i(s) \phi_s(\theta^0_0(x), y)
+ f_i(s) \phi_s(x, \theta^0_0(y)) + g^i(s) f_j(s) \phi_s(\theta^0_0(x), y) + g^j(s) f_j(s) \phi_s(x, \theta^0_0(y))
+ g^i(s) g^{i+n}(s) \phi_s(\theta^0_0(x), \theta^0_0(y)) + f_j(s) g^j(s) \phi_s(\theta^0_0(x), \theta^0_0(y)].
\]

Next we follow the ideas indicated in the pages 178-181 in [14] and define for \( m,n \) in \( \mathbb{N} \cup 0 \),

\[
\phi_t^{m,n}(x, y) := \frac{1}{(m! n!)^{\frac{1}{2}}} \left[ \langle j_t(x) u f^m, j_t(y) v g^n \rangle - \langle j_t(y^* x) u f^m, v g^n \rangle \right]
= \frac{1}{m! n!} \frac{\partial^m}{\partial \rho^m} \frac{\partial^n}{\partial \eta^n} \left[ \langle j_t(x) u e(\rho f), j_t(y) v e(\eta g) \rangle - \langle j_t(y^* x) u e(\rho f), v e(\eta g) \rangle \right]|_{\rho=\eta=0}.
\]

Differentiating (23) with respect to \( \rho \) and \( \eta \) and setting \( \rho = \eta = 0 \), we get a recursive integral relation amongst \( \phi_t^{m,n}(x, y) \) as follows:

\[
\phi_t^{m,n}(x, y) = \int_0^t ds [\phi_s^{m,n}(\theta^0_0(x), y) + \phi_s^{m,n}(x, \theta^0_0(y))] + \phi_s^{m,n}(\theta^0_0(x), \theta^0_0(y))
+ g^i(s) \phi_s^{m-1,n}(\theta^0_0(x), y) + g^i(s) \phi_s^{m,n-1}(x, \theta^0_0(y))
+ f_i(s) \phi_s^{m-1,n}(\theta^0_0(x), y) + f_i(s) \phi_s^{m,n-1}(x, \theta^0_0(y))
+ g^i(s) f_j(s) \phi_s^{m-1,n-1}(\theta^0_0(x), \theta^0_0(y)) + g^j(s) f_j(s) \phi_s^{m-1,n-1}(x, \theta^0_0(y))
+ g^i(s) g^{i+n}(s) \phi_s^{m-1,n-1}(\theta^0_0(x), \theta^0_0(y)) + f_j(s) g^j(s) \phi_s^{m-1,n-1}(\theta^0_0(x), \theta^0_0(y))]
\]

where \( \phi_t^{-1}(x, y) := \phi_t^{m,n}(x, y) := 0 \) for all \( m, n \) and \( x, y \). We set in (25), \( m = n = 0 \) to get

\[
\phi_t^{0,0}(x, y) = \int_0^t ds [\phi_s^{0,0}(\theta^0_0(x), y) + \phi_s^{0,0}(x, \theta^0_0(y))] + \phi_s^{0,0}(\theta^0_0(x), \theta^0_0(y))]
\]

and if we can show that the hypothesis of this theorem and (26) imply that \( \phi_t^{0,0}(x, y) = 0 \), then we can embark on our induction hypothesis as

\[
\phi_t^{k,l}(x, y) = 0 \text{ for } k + l \leq m + n - 1.
\]

Under the induction hypothesis, (25) reduces to

\[
\phi_t^{m,n}(x, y) = \int_0^t ds [\phi_s^{m,n}(\theta^0_0(x), y) + \phi_s^{m,n}(x, \theta^0_0(y))] + \phi_s^{m,n}(\theta^0_0(x), \theta^0_0(y))]
\]
for $x, y \in A_0$, which is an equation similar to (26) leading to $\phi_t^{m,n}(x, y) = 0$, as earlier and this will complete the induction process. Thus it only remains to show that the assumptions of this theorem lead to a trivial solution of equation of the type (26). Omitting the indices $m,n$, define a map $\psi_t$ belonging to $\text{Lin}(A_0 \otimes_{\text{alg}} A_0, \mathbb{C})$ by:

$$\psi_t(x \otimes y) = \phi_t^{m,n}(x, y),$$

and extend linearly. Thus equation (26) leads to:

$$\psi_t(X) = \int_0^t ds[\psi_s(\vartheta_0^t 1 + 1 \otimes \theta_0^t + \sum_i (\theta_0^t \otimes_{\text{alg}} \theta_0^t)(X))], \text{ for } X \in \mathcal{F}. \quad (28)$$

The complete positivity of the map $j_t$ implies that

$$|\langle j_t(x)\xi, j_t(x)\xi \rangle| \leq \langle j_t(x^*x)\xi, \xi \rangle \leq O(e^{\beta t})\|x\|_2\|y\|_2. \quad (30)$$

The assumptions $A(vi)$, Cauchy-Schwartz inequality and (30) together yields

$$|\psi_t(X)| \leq O(e^{\beta t})\|X\|_\gamma, \text{ for } X \in \mathcal{F}, \quad (31)$$

which proves (by virtue of denseness of $\mathcal{F}$ in $h \otimes \Gamma$) that $\psi_t$ extends as a bounded map from $h \otimes \gamma$ to $\mathbb{C}$. If we let $G = \hat{\mathcal{L}} + B$, then for $X \in \mathcal{F}$, the equation (28) becomes:

$$\psi_t(X) = \int_0^t \psi_s(G(X))ds.$$ 

Note that by (31), $\int_0^\infty dt e^{-\lambda t}|\psi_t(X)| < \infty$ for $\lambda \geq \beta$ and thus

$$\int_0^\infty dt e^{-\lambda t}|\psi_t(X)| = \int_0^\infty dt e^{-\lambda t} \int_0^t ds\psi_s(G(X)),$$

which on an integration by parts leads to

$$\int_0^\infty dt e^{-\lambda t}\psi_t((G - \lambda)(X)) = 0, \text{ for } X \in \mathcal{F}. \quad (32)$$

Now for $Y \in \text{span}\{Y\}$, let $\{X_n \in \mathcal{F}\}$ be a sequence such that $G(X_n)$ goes to $G(Y)$ (this happens because of the fact that $\mathcal{F}$ is a core for $\hat{\mathcal{L}}$ and the inequality in (18)). Thus we have

$$\int_0^\infty dt e^{-\lambda t}\psi_t((G - \lambda)(Y)) = 0,$$
by an application of the dominated convergence theorem. With $A$ in Lemma [2.5] to be $(\mathcal{L} - \lambda)$, $D = \mathcal{Y}$, and because of the inequality [18], Lemma [2.5] applies and the denseness of $(G - \lambda)(\text{span}\{\mathcal{Y}\})$ follows. Therefore the last equation and (31) lead to
\[ \int_0^\infty dt e^{-\lambda t} \psi_t(X) = 0 \text{ for all } X \in h \otimes \gamma h, \text{ for } \lambda > \beta. \]
This implies that $\psi_t(X) = 0$ which in turn proves that $\phi^m_{t}(x,y) = 0$ for $x, y \in \mathcal{A}_0$, $t \geq 0$. □

**Corollary 3.10.** Suppose the trace $\tau$ on the algebra is finite. Assume $A(i)$ through $A(v)$, and replace the assumption of analyticity in condition $A(i)$ by the following: $\mathcal{A}_0 \subseteq D(\mathcal{L}_2) \cap D(\mathcal{L}_2^*)$. Then the conclusion of Theorem 3.9 remains valid.

**Proof.** Define a symmetric form $q(x, y) = \langle \mathcal{L}_2(x), y \rangle - \langle x, \mathcal{L}_2(y) \rangle$ for all $x, y \in \mathcal{A}_0$, with domain $D(q) = \mathcal{A}_0$. This form is non-negative by $A(iv)$ and $A(v)$. Since $q(x, y) = \langle x, (-\mathcal{L}_2 - \mathcal{L}_2^*)y \rangle$, $\forall x, y \in D(q)$, the standard proof for the Friedrich extension (see [13], vol-II, page-177) is valid and we get a positive self-adjoint operator $Z$ with $D(q) \subseteq D(Z)$ such that $q(x, y) = \langle x, Z(y) \rangle$. Set $C = Z^2$. Observe that by the form extension and hypothesis $A(v)$, we have
\[ \frac{d}{dt} \|T_t(x)\|^2 = \langle \mathcal{L}_2(T_t(x)), T_t(x) \rangle + \langle T_t(x), \mathcal{L}_2(T_t(x)) \rangle = -\|C \circ T_t(x)\|^2. \] (33)
Thus the rest of the proofs of Lemma 3.8 and of Theorem 3.9 remains valid. □

4 The Weak Trotter Product Formulae for q.s.d.e. with unbounded co-efficients.

**Definition 4.1.** The time shift operator $\theta_t$, $\theta_t: L^2(\mathbb{R}_+) \rightarrow L^2([t, \infty))$ is defined as
\[ \theta_t(f)(s) = 0 \quad \text{if } s < t \]
\[ = f(s - t) \quad \text{if } s \geq t. \] (34)

Let $\Gamma(\theta_t)$ denotes its second quantization, that is $\Gamma(\theta_t)(e(g)) = e(\theta_t(g))$, for $g$ in $L^2(\mathbb{R}_+, k_0)$ and extended linearly as an isometry on whole $\Gamma(L^2(\mathbb{R}_+, k_0))$. For $X \in \mathcal{A} \otimes B(\Gamma_{[r,s]})$,
\[ \Gamma(\theta_t)(X \otimes I_{\Gamma_s}) \Gamma(\theta_t^*) = P_{12}(|\Omega_t > < \Omega_t| \otimes 1_{\Gamma_{r+t}} \otimes \hat{X} \otimes I_{\Gamma^{++}}) P_{12}^{*}, \]
where $P_{12}: \Gamma_t \otimes h \otimes \Gamma' \rightarrow h \otimes \Gamma_t \otimes \Gamma' (\cong h \otimes \Gamma)$ is the unitary flip between first and second tensor components. Let $\xi_t: B(h \otimes \Gamma_t^*) \rightarrow B(h \otimes \Gamma_t^{++})$ be given by:
\[ \xi_t(X) = \hat{X}. \]

**Definition 4.2.** A CPC flow $j_t$ is called a cocycle if
\[ j_{s+t}(x) = j_s \circ \xi_t \circ j_t(x), \text{ for } x \in \mathcal{A}. \]

Henceforth, all the CPC flows considered are assumed to be cocycles.
Lemma 4.3. For a CPC cocycle flow \( j_t \), \( j_t^{c,d}(x) \) defined by \( \langle e(c1_{[0, t]}), j_t(x)e(d1_{[0, t]}) \rangle \) is a \( C_0 \) semigroup on \( A \). Furthermore the restriction of the generator of \( j_t^{c,d}(x) \) to \( A_0 \) is \( \mathcal{L} + \langle c, \delta \rangle + \delta_t^d + \langle c, \sigma_d \rangle + \langle c, d \rangle \) id.

Proof. Let \( s < t \), the semigroup property follows as:

\[
j_{s+t}^{c,d}(x) = \langle e(c1_{[0, s+t]}), j_{s+t}(x)e(d1_{[0, s+t]}) \rangle
= \langle e(c1_{[0, s]}), \Gamma(\theta_s), j_t(x)\Gamma(\theta_s^*)e(d1_{[0, s]}) \rangle \otimes e(d1_{[s, s+t]})
= \langle e(c1_{[0, s]}), \Gamma(\theta_s), j_t(x)\Gamma(\theta_s^*)e(d1_{[s, s+t]}) \rangle
= j_s^{c,d}(\langle e(c1_{[0, t]}), j_t(x)e(d1_{[0, t]}) \rangle)
= j_s^{c,d} \circ j_t^{c,d}(x).
\]

(35)

\( C_0 \) property can be proved as follows: for a vector \( c \in k_0 \), we write \( \alpha_t \) for \( c1_{[0, t]} \).

\[
\begin{align*}
| \langle u(e(c_1), j_t(x)v(d_1)) - (u, xv) \rangle | &
\leq | \langle u(e(c_1) - e(0), j_t(x)v(0)) \rangle | + | \langle u(e(0), j_t(x)v(0)) - (u, xv) \rangle |
+ | \langle u(e(c_1), j_t(x)v(d_1) - e(0)) \rangle | \\
& \leq \|u\|\|v\|\|x\| \left( e^{\frac{\|c\|^2}{4}} \sqrt{e^{\|c\|^2} - 1} + e^{\frac{\|c\|^2}{2}} \sqrt{e^{|d|^2} - 1} \right) + \|u, (j_{t}^{0,0}(x) - x)v \|
\end{align*}
\]

(36)

If \( j_t^{0,0} \) is \( C_0 \) in the norm topology of \( A \) i.e. if \( A \) is a \( C^* \) algebra, then the above estimates implies that \( \langle u, j_t^{c,d}(x)v \rangle \rightarrow (u, xv) \), uniformly over the unit ball of \( h \). On the other hand, if \( j_t^{0,0} \) is \( C_0 \) in the ultraweak topology of \( A \), in case \( A \) is a von-Neumann algebra, then it follows from the above estimates that \( \langle u, j_t^{c,d}(x)v \rangle \rightarrow (u, xv) \) for a given \( u, v \in h \).

Now for \( x \in A_0 \) and \( u, v \in h \), we have

\[
\begin{align*}
\langle u(e(c1_{[0, s]}), j_s(x)v(d1_{[0, s]})) \rangle &= \langle u, xv \rangle e^{\langle c, d \rangle s}
+ \int_0^s dt \left \langle u(e(c1_{[0, s]}), j_t(\mathcal{L}(x)), c, \delta(x)) + \delta^t(x)d + \langle c, \sigma_d(x)d \rangle \right \rangle v(d1_{[0, s]})
\end{align*}
\]

i.e.

\[
\begin{align*}
\langle u, e^{-s(c,d)}j_s^{c,d}(x)v \rangle &= \langle u, xv \rangle + \int_0^s dt \left \langle u, e^{-t(c,d)}j_t^{c,d}(\mathcal{L}(x)), c, \delta(x) + \delta^t(x)d + \langle c, \sigma(x)d \rangle \right \rangle v(d1_{[0, s]})
\end{align*}
\]

(37)

and from this the conclusion follows. □

Remark 4.4. If \( \theta_t^c \) are all bounded maps, then uniqueness of the solution of q.s.d.e. with bounded coefficients implies that \( j_t \) is a cocycle.

Lemma 4.5. Suppose the CPC flow \( (j_t)_{t \geq 0} \) satisfies A(i)-A(v) and that for \( x \in A \cap L^1(\tau) \),

\[
\||j_t^{c,d}(x)||1 \leq \exp(tM)\|x\|1
\]

(38)

for \( c, d \) in \( k_0 \), where \( M \) depends only on \( \|c\|, \|d\| \). Then the condition A(vi) and hence the conclusion of Theorem [7.9] holds.
Proof. Note that for a partition $0 = s_0 < s_1 < s_2 < \ldots < s_n = t$, and for functions of the form $f = \sum_j 1_{[s_j, s_{j+1})} c_j$, $g = \sum_j 1_{[s_j, s_{j+1})} d_j$, ($c_j, d_j \in k_j$ for $j = 1, 2$) we get using the cocycle property of $j_t(\cdot)$ and (28) that

$$\| \langle e(f), j_t(x)e(g) \rangle \|_1 \leq \exp(tM) \|x\|_1,$$

where $M = \max_j(M_j)$, where each $M_j$ depends only on $\|c_j\|$ and $\|d_j\|$. Let $\Lambda(z) := \langle u e(zf), j_t(x)e(g) \rangle$ and hence $\| \langle u e(zf), j_t(x)e(g) \rangle \| \leq \exp(tM) \|x\|_1$, for $|z| = 1$. Clearly $\Lambda$ is entire in $z$ since $\{z \mapsto e(zf)\}$ is strongly entire, and by considering a unit disc around zero and applying Cauchy’s estimate for this function, we obtain

$$(m!)^{1/2} \| \langle u f^{\otimes m}, j_t(x)e(g) \rangle \| \leq \|u^* v\|_\infty \exp(tM) \|x\|_1,$$ (39)

for $u, v \in A \cap L^2(\tau)$ and $x \in A \cap L^1(\tau)$. Doing a similar calculation to the function $\beta(z) := \langle u f^{\otimes m}, j_t(x)e(zg) \rangle$, we get:

$$\| \langle u f^{\otimes m}, j_t(x)e(zg) \rangle \| \leq \|u^* v\|_\infty \exp(tM) \|x\|_1,$$ (40)

which proves that the CPC flow $j_t$ satisfies $A(vi)$, if we take $C(f, g, m, n, t) := \|u^* v\|_\infty \exp(tM) \|x\|_1$. □

Corollary 4.6. For a CPC flow $(j_t)_{t \geq 0}$ on a type-I von-Neumann algebra with atomic centre, the conditions $A(i)$ through $A(v)$ imply $A(vi)$ and hence also imply that $j_t$ is a * homomorphism.

Proof. Observe that in a type-I algebra with atomic centre, we have for $x \in L^1(\tau)$,

$$\|x\|_\infty \leq \|x\|_1.$$

As $j_t$ is a contractive flow, we have that for $x \in L^1(\tau)$,

$$\sup_{0 \leq s \leq t} \| \langle u f^{\otimes m}, j_t(x)e(zg) \rangle \| \leq \|x\|_\infty \|f^{\otimes m}\| \|g^{\otimes n}\| \|u\|_2 \|v\|_2 \leq \|x\|_1 \|f^{\otimes m}\| \|g^{\otimes n}\| \|u\|_2 \|v\|_2,$$ (41)

from which the required estimate $A(vi)$ follows. □

Let $A$ be a $C^*$ or von-Neumann algebra which is equipped with a faithful, semifinite and lower-semicontinuous trace $\tau$. Suppose we are given two quantum stochastic flows

$$j_t^{(1)} : A \rightarrow A'' \otimes B(\Gamma(L^2(\mathbb{R}_+, k_1))),$$

and

$$j_t^{(2)} : A \rightarrow A'' \otimes B(\Gamma(L^2(\mathbb{R}_+, k_2))),$$

which satisfy two quantum stochastic differential equations of the type (2) with coefficients $(\mathcal{L}^{(1)}, \delta^{(1)}, \sigma^{(1)})$ and $(\mathcal{L}^{(2)}, \delta^{(2)}, \sigma^{(2)})$ respectively. In the following, we assume that the hypothesis in the definition (2.1) is true for both sets of structure maps with the same $A_0$. Let $\Gamma_1 := \Gamma(L^2(\mathbb{R}_+, k_1))$ and
Theorem 4.8. The (weak) Trotter product formula-I:

For $\{ c_j, d_j \} \in k_j$, $j = 1, 2$, define $j_t^{c(j),d(j)} = j_t^{c(j)}j_t^{d(j)}$. We now define the Trotter product of these two flows:

For $x \in A$, define $\eta_t : A \rightarrow A \otimes B(\Gamma_1 \otimes \Gamma_2)$ by:

$$\eta_t(x) = (j_t^{(1)} \otimes \text{id}_B(\Gamma_2)) \circ j_t^{(2)}(x).$$

Take a dyadic partition of the whole real line $\mathbb{R}$ and consider the part of the partition in $[s, t]$ for large $n$, described in the picture below:

$$\cdots \cdots |2^n s| 2^{-n} - - - - - - [s - - - - |(2^n s + 1) - 2^{-n} - - - - |2^n t| 2^{-n} - t \cdots \cdots |(2^n t + 1) - 2^{-n} - - - - ,$$

where $[t] = \text{integer} \leq t$ for real $t$.

Definition 4.7. Set

$$\phi_t^{(n)}_{[s, t]} = [\xi_s \circ \eta_{[2^n s + 1), 2^n s - 2^{-n}}] \circ \left\{ \prod_{j = [2^n s + 1]}^{[2^n t] - 1} (\xi_j 2^{-n} \circ \eta_{2^{-n}} \otimes 1_{B(\Gamma_j 2^{-n})}) \right\}.$$ 

$$\left\{ (\xi_{[2^n t] - 2^{-n}} \circ \eta_{-2^{-n}} \otimes 1_{B(\Gamma_{[2^n t] - 2^{-n}})}) \right\}.$$ 

Set $\phi_t^{(n)} := \phi_t^{(n)}_{[0, t]}$. The map $\phi_t^{(n)}$ will be called the n-fold Trotter product of the flows $j_t^{(1)}$ and $j_t^{(2)}$.

Clearly this map $\phi_t^{(n)}$ is a $*$-homomorphism for each $n$ and being compositions of cocycles, $\phi_t^{(n)}$ itself is a cocycle. Let $(e_i)_{i \geq 1}$ be an orthonormal basis for $k_1$ and $(l_i)_{i \geq 1}$ be an orthonormal basis for $k_2$ so that the set $G = \{ (\lambda e_i, 0), (0, \beta l_j) | \lambda, \beta \in \mathbb{C}, i, j \geq 1 \}$ is total in $k_1 \otimes k_2$. Let $M$ be the set of step functions $f$ supported over intervals with dyadic end points and taking values in $G$.

It is known [15] that $\{ e(f) | f \in M \}$ is total in $\Gamma(L^2(\mathbb{R}_+, k_1 \otimes k_2))$.

Now we state the weak version of the Random Trotter Product Formula for quantum stochastic flows, leaving the strong one for the next section.

Theorem 4.8. The (weak) Trotter product formula-I:

Suppose $A$ is a $C^*$-algebra and that for each $c_j, d_j$ belonging to $k_j$, $j = 1, 2$, the closure of the operator $\sum_{j=1}^{2} \Big( L^{(j)} + \langle c_j, \delta^{(j)} \rangle + \delta_d^{(j)} + \langle c_j, \sigma_d \rangle + \langle c_j, d_j \rangle \Big)$ generates a $C_0$ contractive semigroup in $A$.

Then $\phi_t^{(n)}(x)$ as defined above converges in the weak operator topology of $h \otimes \Gamma^1 \otimes \Gamma^2$ to $j_t(x)$ where $j_t$ is another CPC flow satisfying a q.s.d.e. with structure matrix

$$
\begin{pmatrix}
L^{(1)} & L^{(2)} & \delta^{(1)} & \delta^{(2)} \\
\delta^{(1)} & \sigma^{(1)} & 0 & \delta^{(2)} \\
\delta^{(2)} & 0 & \sigma^{(2)}
\end{pmatrix}.
$$
Proof of Theorem 4.8:

Since \( \| \phi_t^{(n)}(x) \|_\infty \leq \| x \|_\infty \), it is enough to prove the weak convergence of this sequence of maps for \( u, v \) belonging to some dense subset of \( L^2(\tau) \) and \( f, g \) belonging to some total subset \( L^2(\mathbb{R}_+, k_1 + k_2) \). Let \( f \in \mathcal{M} \) be of the form:

\[
\begin{align*}
    f(x) &= 0 \quad \text{if } x < \frac{[2^m s]}{2^n} \text{ or } x > \frac{[2^m t] + 1}{2^n} \\
    &= c_0^{(1)} \oplus c_0^{(2)} \quad \text{if } x \in \left[s, \frac{[2^m s] + 1}{2^n}\right) \\
    &= c_j^{(1)} \oplus c_j^{(2)} \quad \text{if } x \in \left[\frac{[2^m s] + j}{2^n}, \frac{[2^m s] + j + 1}{2^n}\right)
\end{align*}
\]

(44)

Similarly, let \( g \) be of the form (44) with \( c_j^{(1)} \oplus c_j^{(2)} \) replaced by \( d_j^{(1)} \oplus d_j^{(2)} \). For an interval \( [a, b] \subseteq \left[\frac{[2^m s] + j}{2^n}, \frac{[2^m s] + j + 1}{2^n}\right) \), let

\[
\Sigma_{[a, b]}^{(1),d(s)} = \left( j_{\frac{2^n}{2^n}}^{(2)} \circ j_{\frac{2^n}{2^n}}^{(1)} \circ j_{\frac{2^n}{2^n}}^{(2)} \right)
\]

For \( m \), sufficiently larger than \( n \), for \( x \in \mathcal{A} \) and considering \( f \) and \( g \) as above, we have:

\[
\langle \phi_t^{(n)}(x) u e(f), v e(g) \rangle
\]

(45)

So it is enough to prove the strong convergence of the operators \( \Sigma_{[a, b]}^{(p)}(x) \) for a single interval \( [a, b] \). So let \( c = c^{(1)} \oplus c^{(2)} \) and \( d = d^{(1)} \oplus d^{(2)} \) belong to \( \mathcal{G} \). Now for \( f = (c^{(1)} \oplus c^{(2)}) \chi_{[a, b]}, g = (d^{(1)} \oplus d^{(2)}) \chi_{[a, b]} \), we have from (43) that

\[
\begin{align*}
    \langle \Sigma_{[a, b]}^{(m)}(x) u e(f), v e(g) \rangle
    &= \left( j_{\frac{2^n}{2^n}}^{(1),d(s)} \circ j_{\frac{2^n}{2^n}}^{(2),d(s)} \right) \circ \left( j_{\frac{2^n}{2^n}}^{(1),d(s)} \circ j_{\frac{2^n}{2^n}}^{(2),d(s)} \right) \circ \left( j_{\frac{2^n}{2^n}}^{(1),d(s)} \circ j_{\frac{2^n}{2^n}}^{(2),d(s)} \right)(x) u, v \\
    &= \langle Q_1 \circ \{ Q_2 \}^{2^n s - 2^n a} \circ Q_3(x) u, v \rangle.
\end{align*}
\]

(46)

We note that the semigroups \( j_{t}^{(j),d(j)} \) (discussed in page 14) are \( C_0 \) semigroups for \( j = 1, 2 \), and that \( \frac{[nt]}{n} \to t \) as \( n \to \infty \). Thus the maps \( Q_1 \) and \( Q_3 \) strongly converge to \( 1_{\mathcal{A}} \). As for \( Q_2 \), we get

\[
\begin{align*}
    Q_2^{2^n b - 2^n a - 1} = (j_{\frac{2^n}{2^n}}^{(1),d(s)} \circ j_{\frac{2^n}{2^n}}^{(2),d(s)})^{2^n b - 2^n a - 1} \\
    = (j_{\frac{2^n}{2^n}}^{(1),d(s)} \circ j_{\frac{2^n}{2^n}}^{(2),d(s)})^{2^n b - 2^n a - 1}
\end{align*}
\]

(47)

15
which converges strongly by the Trotter product formula for semigroups on Banach spaces, since the generator of \( (j_t^{(l)}, d_t^{(l)}) \) restricted to \( \mathcal{A}_0 \) is
\[
\mathcal{L}^{(l)} + \langle c_l, \delta^{(l)} \rangle + \delta^{(l)}_d + \langle c_j, \sigma_d \rangle + \langle c_j, d_j \rangle \text{id},
\]
for \( l = 1, 2 \) and by the assumption of the theorem, the closure of \( \sum_{j=1}^2 \left( \mathcal{L}^{(j)} + \langle c_j, \delta^{(j)} \rangle + \delta^{(j)}_d + \langle c_j, \sigma_d \rangle + \langle c_j, d_j \rangle \text{id} \right) \) generates a \( C_0 \) contractive semigroup in \( \mathcal{A} \).

On \( \mathcal{A}_0 \), the semigroups \( j_t^{c,d} \) satisfies the following:
\[
j_t^{c,d}(x) - j_0^{c,d}(x) = \int_0^t ds j_s^{c,d} \circ (\mathcal{L}^{(1)}(x) + \mathcal{L}^{(2)}(x) + \left( c^{(1)}(x) \oplus c^{(2)}(x), \delta^{(1)}(x) \oplus \delta^{(2)}(x) \right)(x) + \\
(\delta^{(1)}(x) \oplus \delta^{(2)}(x))_{d^{(1)} \oplus d^{(2)}}(x) + \left( \sigma^{(1)}(x) \oplus \sigma^{(2)}(x) \right)_{d^{(1)} \oplus d^{(2)}}(x) + \left( c^{(1)}, c^{(2)}, d^{(1)} \oplus d^{(2)} \right)(x)).
\]

Thus there exists a contractive map \( j_{s,t} : \mathcal{A}_0 \to \mathcal{A}'' \otimes B(\Gamma_{s,t}) \) such that \( \phi^{(n)}_{s,t}(x) \) converges in the weak operator topology to \( j_{s,t}(x) \) in \( h \otimes \Gamma \). Clearly by density of \( \mathcal{A}_0 \) in \( \mathcal{A} \), \( j_{s,t} \) extends to the whole of \( \mathcal{A} \). Thus \( j_t \) satisfies a weak q.s.d.e. in \( h \otimes \Gamma_1 \otimes \Gamma_2 \cong h \otimes \Gamma (L^2(\mathbb{R}, k)) \), with the structure matrix:
\[
\begin{pmatrix}
\mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{(1)} & \delta^{(2)} \\
\delta^{(1)} & \sigma^{(1)} & 0 \\
\delta^{(2)} & 0 & \sigma^{(2)}
\end{pmatrix},
\]
where \( k := k_1 \oplus k_2 \) being the noise space.

It is clear that \( j_t \) is a cocycle and since \( j_t \) is contractive, \( j_t \) also satisfies the strong q.s.d.e. with the above structure matrix. □

**Theorem 4.9. The (Weak) Trotter product formula-II** :
Let \( \mathcal{A} \) be a \( C^* \) or von-Neumann algebra, and \( \tau \) be a trace on it. Furthermore assume that:

(a) in the structure matrices associated with \( j_t^{(1)} \) and \( j_t^{(2)} \), \( \sigma^{(j)} = 0 \) for \( j = 1, 2 \),

(b) the closure of \( \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} \) generates a \( C_0 \), contractive, analytic semigroup in \( L^2(\tau) \).

Then \( \phi^{(n)}_{t}(x) \) as defined above converges in the weak operator topology of \( h \otimes \Gamma_1 \otimes \Gamma_2 \) to \( j_t(x) \) for all \( x \) in \( \mathcal{A} \), where \( j_t \) is a CPC flow satisfying the q.s.d.e. with structure matrix
\[
\begin{pmatrix}
\mathcal{L}^{(1)} + \mathcal{L}^{(2)} & \delta^{(1)} & \delta^{(2)} \\
\delta^{(1)} & 0 & 0 \\
\delta^{(2)} & 0 & 0
\end{pmatrix}.
\]

**Proof of Theorem 4.9** : Set \( \theta_0^{(1)}(x) := \langle e_i, \delta^{(1)} \rangle(x) \), \( \theta_0^{(2)}(x) := \langle l_i, \delta^{(2)} \rangle(x) \), for \( i \geq 1 \). For \( x \in \mathcal{A}_0 \) and every positive integer \( n \), we have
\[ \| \delta^{(j)}(x) \|_2^2 = \sum_i \| \theta_0^{(j)}(x) \|_2 \leq 2 \| L_2^{(j)}(x) \|_2 \| x \|_2 \]
\[ \leq 2 \frac{1}{\sqrt{2} \ n} \| L_2^{(j)}(x) \|_2 \frac{1}{\sqrt{2}} n \| x \|_2 \]
\[ \leq \left\{ \frac{1}{\sqrt{2} \ n} \| (L_2^{(j)}(x)) \|_2 + n \| x \|_2 \right\}^2 \]
\text{for } j = 1, 2. \tag{49}

Thus the operators \( \theta_0^{(j)} \) are relatively bounded with respect to \( L_2^{(j)} \) with arbitrarily small bound. Similar calculations hold for \( \theta_0^{(j)}(x)(= \theta_0^{(j)}(x)^*), j = 1, 2 \). Since \( L_2^{(j)} \) are the pre-generators of contractive analytic semigroups in \( L^2(\tau) \), we see that the operators

\[ \theta_0^{(j)} + \theta_k^{(j)} + L_2^{(j)}, \]

for \( j = 1, 2 \) are pre-generators of \( C_0 \) semigroups (see [6] Theorem 2.4 and Corollary 2.5, p 497-498).

A similar proof as above yields that for \( x \in A_0, c, d \in \mathcal{G} \),

\[ \| \left\langle c, \delta^{(1)} \oplus \delta^{(2)} \right\rangle (x) \|_2 \leq \left\{ \frac{1}{\sqrt{2} \ n} \| (L_2^{(1)} + L_2^{(2)})(x) \|_2 + \frac{n}{\sqrt{2}} \| x \|_2 \right\} \| c \|, \]
\[ \| \left\langle \delta^{(1)} \oplus \delta^{(2)}, d \right\rangle (x) \|_2 \leq \left\{ \frac{1}{\sqrt{2} \ n} \| (L_2^{(1)} + L_2^{(2)})(x) \|_2 + \frac{n}{\sqrt{2}} \| x \|_2 \right\} \| d \|. \]

Thus because of the hypothesis that \( L_2^{(1)} + L_2^{(2)} \) generates an analytic \( C_0 \) semigroups in \( L^2(\tau) \), we see that for \( c, d \in \mathcal{G} \), the operator

\[ \left\langle c, \delta^{(1)} \oplus \delta^{(2)} \right\rangle + \left\langle \delta^{(1)} \oplus \delta^{(2)}, d \right\rangle + L_2^{(1)} + L_2^{(2)} + \langle c, d \rangle \]

generates a \( C_0 \) semigroup in \( L^2(\tau) \). The rest of the proof proceeds as that of the Theorem 4.8. \( \square \)

**Remark 4.10.** As can be noticed in the proof of the theorems 4.8 and 4.9, the convergence of \( \phi_t^{(m)}(x) \) is actually in a topology stronger than the weak operator topology in \( h \otimes \Gamma_1 \otimes \Gamma_2 \); it is in the product topology of strong operator topology in \( \Gamma_1 \otimes \Gamma_2 \).

**Proposition 4.11.** Let \( j_t^{(k)} \) \( (k = 1, 2) \) be two CPC cocyle flows satisfying all the assumptions for Theorem 4.8 such that the weak limit of the Trotter product \( \phi_t^{(n)} \) exists. Assume furthermore that \( j_t^{(k)} \) satisfy (28) for each \( k \). Then \( j_t \) also satisfies (28).

**Proof.** Let \( \Delta_0 = [s, \frac{[2^n]s + 1}{2^n}], \Delta_j = [\frac{2^n}s + j, \frac{2^n}s + j + 1] \) for

\[ 1 \leq j \leq [2^n]t - [2^n]s - 1 \quad \text{and} \quad \Delta' = [\frac{[2^n]t}{2^n}, t]. \]

Let \( \chi_0 = 1_{\Delta_0}, \chi_j = 1_{\Delta_j} \) and \( \chi' = 1_{\Delta'}. \) Then for \( c = c_1 \oplus c_2, d = d_1 \oplus d_2 \) and \( x \in L^1 \cap \mathcal{A} \), define \( \eta_t^{c,d}(x) = \langle e(c \chi_0, \tau), \eta_t(x) e(d \chi_0, \tau) \rangle \).

\[ \langle e(c_{1[s,t]}, \phi_{[s,t]}^{(n)})(x) e(d_{1[s,t]}) \rangle \]
\[ = \langle e(c \chi_0) \otimes x e(c \chi_j) \otimes e(c \chi'), \phi_{[s,t]}^{(n)}(x) e(d \chi_0) \otimes x e(d \chi_j) \otimes e(d \chi') \rangle \]
\[ = \eta_{[2^n]t - [2^n]s - 1}^{c,d} \circ (\eta_{[2^n]t}^{c,d})^{-1} \circ \eta_{[2^n]t}^{c,d} \]
\[ = \eta_{[2^n]t}^{c,d} \circ (\eta_{[2^n]t}^{c,d})^{-1} \circ \eta_{[2^n]t}^{c,d}. \]

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Now \( \eta^{c,d}_r(x) = j^{c_1,d_1}_r \circ j^{c_2,d_2}_r(x) \). Thus
\[
\| \eta^{c,d}_r(x) \|_1 \leq e^{rM_1} \| j^{c_2,d_2}_r(x) \|_1
\]
\[
\leq e^{r(M_1+M_2)} \| x \|_1 \quad \text{(where } M_j \text{ depends on } \| c_j \|, \| d_j \| \text{ for } j = 1, 2.)
\]
Thus
\[
\bigg\| \left\langle e(c_1[s,t]), j^{(n)}_t(x) e(d_1[s,t]) \right\rangle \bigg\|_1 \leq e^{(t-s)(M_1+M_2)} \| x \|_1
\]
and from this the conclusion follows. \( \square \)

5 The Strong Trotter product formula.

The theorems 4.8 and 4.9 have established that \( \phi^{(n)}_t \) converges weakly to \( j_t \) (a CPC cocycle flow) on \( h \otimes \Gamma_1 \otimes \Gamma_2 \cong h \otimes \Gamma \). Clearly since \( \phi^{(n)}_t \) is a *-homomorphism from \( A \to A'' \otimes B(\Gamma) \), the above convergence is strong if and only if \( j_t \) itself is a *-homomorphism. The next theorem, using the crucial results of Lemma 4.5 and Proposition 4.11, exactly does that.

**Theorem 5.1.** The (strong) Trotter product formula-III:

(i) Suppose \( A \) is a C*-algebra. Let \( j^{(1)}_t \) and \( j^{(2)}_t \) be two quantum stochastic flows satisfying the condition of Theorem 4.8. Suppose furthermore that these two flows satisfy the following:

(a) For \( x \in A \cap L^1(\tau) \), \( j^{(c_j,d_j)}(x) \) satisfies (38) or \( A(\text{vi}) \) for \( c_j, d_j \in k_j, j = 1, 2 \);
(b) \( \tau(L^{(j)}(x^*x)) \leq 0 \) for \( j = 1, 2 \);
(c) each of the semigroups generated by \( L^{(1)} \) and \( L^{(2)} \) as well as their Trotter product limit have analytic \( L^2(\tau) \) extensions as semigroups.

Then \( \phi^{(n)}_t(x) \) as defined above converges in the strong operator topology of \( h \otimes \Gamma_1 \otimes \Gamma_2 \) to \( j_t(x) \) where \( j_t \) is another quantum stochastic flow which satisfy a q.s.d.e. with the structure matrix
\[
\begin{pmatrix}
L^{(1)} + L^{(2)} & \delta^{(1)} & \delta^{(2)} \\
\delta^{(1)} & \sigma^{(1)} & 0 \\
\delta^{(2)} & 0 & \sigma^{(2)}
\end{pmatrix}.
\]

(ii) Let \( A \) be a von-Neumann algebra. Suppose the two quantum stochastic flows \( (j^{(j)}_t)_{t \geq 0} \), for \( j = 1, 2 \), satisfy the conditions of theorem 4.9 and conditions (a) and (b) of part (i) of the statement above. Then the same conclusion as in part (i) above holds.

**Proof of Theorem 5.1:**

It suffices to prove that the limiting CPC flow \( j_t \) is a *-homomorphism. Condition (38) implies \( A(\text{vi}) \) and thus by Proposition 4.11 and Theorem 3.9 \( j_t \) is a *-homomorphism. \( \square \)

**Remark 5.2.** Assumption (c) of Theorem 5.1 can be replaced by the assumption that each of the maps \( L^{(1)}_2 \) and \( L^{(2)}_2 \) satisfy the condition of Corollary 3.10.
6 Applications.

6.1 Construction of classical and non-commutative stochastic processes:

We shall now illustrate how to construct various multidimensional processes as random Trotter-Kato limits of the corresponding “marginals”. Our examples will include Brownian Motion on Lie groups and random walk on discrete groups.

Let \( G \) be a second countable locally compact group. Let \( \{ \phi_i \}_{i=1}^\infty \) constitute a countable family of functions from \( C_0(G) \) which separates points on \( G \). Define a metric \( \rho \) as follows:

\[
\rho(g, g') := \sum_{n=1}^\infty \frac{1}{2^n (1 + |\phi_n(g) - \phi'_n(g)|)}.
\]

It can be shown that this metric gives the same topology on \( G \) and also that \( G \) is complete, and thus in particular, \( G \) is a Polish group. Let \( \mathcal{A} \) be a \( C^* \) algebra equipped with a faithful, semifinite, lower-semicontinuous trace \( \tau \). As before, we imbed \( \mathcal{A} \) in \( B(h) \), where \( h = L^2(\tau) \), and extend \( \tau \) as a normal semifinite trace on \( \mathcal{A}' \). Assume furthermore that there is a strongly continuous, \(*\)-automorphic \( G \)-action \( \alpha_g \) on \( \mathcal{A} \) which is also \( \tau \)-preserving i.e. \( \tau(\alpha_g(a)) = \tau(a) \). This allows us to extend \( \alpha_g \) to a unitary operator \( U_g \) on \( L^2(\tau) \), and we extend \( \alpha \) to \( \mathcal{A}' \) as a normal \(*\) automorphism given by \( \alpha_g(\cdot) = U_g \cdot \cdot U_g^* \).

Lemma 6.1. Let \( (X_n)_n \) be a \( G \)-valued random variable on some space \( (\Omega, \mathcal{F}, P) \), and suppose that for all \( \psi \) in \( L^2(G) \) and for all \( \phi \) in \( C_0(G) \),

\[
\int_G dg \int_\Omega d\omega |\psi(g)|^2 |\phi(g.X_n) - \phi(g.X_m)|^2 \rightarrow 0
\]
as \( n, m \rightarrow \infty \), where \( dg \) is the left-invariant Haar measure on \( G \). Then there exists a random variable \( X : \Omega \rightarrow G \) such that \( X_n \rightarrow X \) in probability.

Proof. We choose and fix some \( \psi \) in \( L^2(G) \) with \( \|\psi\|_2 = 1 \), and let \( d\mathbb{P}(\omega, g) := dP(\omega) \otimes |\psi(g)|^2 dg. \) Since \( \int_G dg |\psi(g)|^2 \int_\Omega d\omega |\phi_1(g.X_n) - \phi_2(g.X_m)|^2 \rightarrow 0 \), for every \( i \), it follows by setting \( Y_n(g, \omega) = g.X_n(\omega) \), and using the dominated convergence theorem for \( g \in G, \omega \in \Omega \), that for all \( \epsilon > 0 \),

\[
\mathbb{P}(\rho(Y_n, Y_m) \geq \epsilon) \leq \frac{1}{C_2} \mathbb{E}_\mathbb{P} E^\mathbb{P}(\rho(Y_n, Y_m)) \rightarrow 0,
\]
as \( m, n \rightarrow \infty \). Thus there is a \( G \)-valued random variable \( Y \), defined on \( \Omega \), such that \( Y_n \overset{P}{\rightarrow} Y \).

So

\[
\mathbb{P}(\rho(Y_n, Y) > \epsilon) = \int_G dg |\psi(g)|^2 P(\rho(g.X_n, Y) \geq \epsilon) \equiv \int_G dg |\psi(g)|^2 f_n(g) \rightarrow 0,
\]

where \( f_n(g) = P(\rho(g.X_n, Y) \geq \epsilon) \). By Egoroff’s theorem, there exists a measurable set say \( \Delta \) of positive Haar-measure such that for all \( g \) in \( \Delta \), we have \( f_n(g) \rightarrow 0 \), and the proof of the theorem is complete by taking \( X(\omega) = g^{-1}Y(g, \omega) \), for any fixed \( g \) in \( \Delta \).
Lemma 6.2. Let \((g_t)_{t \geq 0}\) be a G-valued Levy process, defined on some probability space \((\Omega, \mathcal{F}, P)\). Define \(j_t : \mathcal{A}' \to L^\infty(\Omega, \mathcal{A}') \subseteq B(L^2(\tau) \otimes L^2(\Omega)), \) by \(j_t(x)(\omega) := \alpha_{g_t(\omega)}(x)\), and \(T_t(x) = \mathbb{E}((\alpha_{g_t(\omega)}(x))\).

(i) Then for \(f, g \in L^2(\mathbb{R}_+)\), \(x \in \mathcal{A} \cap L^1(\tau)\), we have
\[
\tau(||\langle e(f), j_t(x)e(g)\rangle||) \leq ||e(f)||_2||e(g)||_2||x||_1.
\] (53)

(ii) \((T_t)_{t \geq 0}\) is a normal QDS on \(\mathcal{A}'\), and if its restriction on \(\mathcal{A}\) leaves \(\mathcal{A}\) invariant, it is a QDS on \(\mathcal{A}\) with respect to the norm-topology. Furthermore, if \(g_t\) and \((g_t)^{-1}\) have the same distribution for each \(t\), then the semigroup is \(\tau\)-symmetric.

Proof. To prove (i), it suffices to show the inequality for positive \(x \in \mathcal{A}\). For such \(x\), we have
\[
\tau(||\langle e(f), j_t(x)e(g)\rangle||) \leq \mathbb{E}\tau\left(\exp\left[\int_0^\infty (\tau + g)d\omega - \frac{1}{2} \int_0^\infty (\tau^2 + g^2)dt\right] j_t(x)(\omega)\right)
\] (54)
from which the result follows. (ii) From the defining property of Levy processes, the semigroup property of \(T_t\) follows; while the normality of \(T_t\) is a consequence of the fact that \(j_t\) is implemented by an automorphism of \(\mathcal{A}'\). Moreover, if \(g_t\) and \((g_t)^{-1}\) have the same distribution, we have
\[
\tau(T_t(a)b) = \tau[\mathbb{E}\{\alpha_{g_t(\omega)}(a)b\}]
= \tau[\mathbb{E}\{\alpha_{g_t(\omega)}(a\alpha_{g_t(\omega)}^{-1}(b))\}]
= \mathbb{E}[\tau\{\alpha_{g_t(\omega)}(\alpha_{g_t(\omega)}^{-1}(b))\}]
= \mathbb{E}\tau[\{\alpha_{g_t(\omega)}^{-1}(b))\}]
= \tau[\alpha\mathbb{E}\{\alpha_{g_t(\omega)}(b)\}] = \tau(aT_t(b)).
\] (55)

After these two lemmas, we now give a few concrete examples.

(A) Classical and non-commutative Brownian motion: Assume now that \(G\) be a compact Lie group (of dimension \(k\)) acting smoothly on a \(C^*\) algebra \(\mathcal{A}\) and \(\tau\) is a lower semicontinuous, faithful, finite trace on \(\mathcal{A}\). Denote by \(\mathcal{A}^\infty\), the \(*\)-subalgebra of \(\mathcal{A}\) generated by elements \(x\) such that \(g \to \alpha_g(x)\) is norm-smooth, where \(g \to \alpha_g\) is the group action. Let \(\{\chi_i\}_{i=1}^k\) be a basis for the Lie algebra of \(G\) and let \(G_t\) be the one-parameter subgroup \(\exp(t\chi_i)\), \(t \in \mathbb{R}\). Define
\[j_t(\mathcal{A}) : \mathcal{A} \to \mathcal{A}' \otimes B(L^2(W(t))) \cong \mathcal{A}' \otimes B(\Gamma(L^2(\mathbb{R}_+)))))\), by
\[j_t(x)(\omega) := \alpha_{\exp(W_t(\omega)\chi_i}(x)\), where \(W_t(\omega)\) is the standard Brownian motion on \(\mathbb{R}\). Then by (i) of Lemma 6.2 replacing \(G\) by \(G_t\), estimate (55) or equivalently the condition (a) of Theorem 5.1 (i) is verified for \(j_t\). Furthermore, since \(W_t(\omega)\) and \(\bar{W}_t(\omega)\) have the same distribution, (ii) of Lemma 6.2 applies. Combining this with Remark 5.2 for \(\mathcal{A}_0 = \mathcal{A}^\infty\), we see that condition (b) of Theorem 5.1 (i) holds. For applying Theorem 5.1 (i), we now need
to check only that $\sum_{i=1}^{k} L_{2}^{(\ell)}$ is the pregenerator of a $C_{0}$ semigroup. For this we proceed as follows: 
As $\alpha_{g}$ is a trace preserving automorphism, it extends to a unitary operator $U_{\alpha}$ in the Hilbert space $L^{2}(\tau)$. Let $\delta_{\ell}$ be the norm generator of the automorphism group $(U_{\exp(t\lambda g)})_{t \in R}$. Then $\delta_{\ell}$ extends to an unbounded, densely defined skew-adjoint operator in $L^{2}(\tau)$, which generates the unitary group $(U_{\exp(t\lambda g)})_{t \in R}$. By an abuse of notation, we again denote this extension by $\delta_{\ell}$. Note that $L_{2}^{(\ell)} = \frac{1}{2} \delta_{\ell}^{2}$, on $A^{\infty}$, for all $\ell$. Thus $\sum_{\ell=1}^{k} L_{2}^{(\ell)} = \frac{1}{2} \sum_{\ell=1}^{k} \delta_{\ell}^{2}$ is a densely defined, negative and symmetric operator. By Nelson’s analytic vector theorem for the representations of the Lie algebra [9, Theorem 3, p. 591] and [13, Theorem X.39], $\sum_{\ell=1}^{k} L_{2}^{(\ell)}$ is essentially selfadjoint in $L^{2}(\tau)$, and hence its’ closure generates a $C_{0}$ contraction semigroup. Thus condition (c) of Theorem 5.1-(i) holds. So Theorem 5.1-(i) applies. Specializing this to the case when $A = C(G)$, and by Lemma 6.1 we get the convergence in probability of the following sequence of random variables:

$$X_{t}^{(n)} := \prod_{i=1}^{k} \prod_{l=0}^{2^{n}t} \exp((W_{(2^{n+l}t)}^{(i)} - W_{2^{n}t}^{(i)})\chi_{i}),$$

where the limiting random variable is clearly a Brownian motion on $G$, giving a result similar to that in [11].

In case of $G = \mathbb{T}^{2}$ and $A$ the irrational-rotational $C^{*}$ algebra $A_{\theta}$ (see page 254 of [14]), the quantum Brownian motion described in page-275 of [14] can be constructed using the method described here.

(B) Random walk in discrete group: Let $G$ be a discrete, finitely generated group, generated by a symmetric set of torsion free generators, say $\{g_{1}, g_{2}, \ldots, g_{2k}\}$, let $e$ be the identity element of $G$ and $g_{1}g_{1}^{1} = e$, and $\alpha_{g}$ for each $g$ be the automorphism obtained by the action of $G$ on itself. Take $A = C_{0}(G)$, $\tau$ to be the trace with respect to the counting measure. Consider $2k$ mutually independent Poisson-processes $(N_{t}^{(i)}_{l})_{t \geq 0}$, $i = 1, \ldots, 2k$, on $\mathbb{N} \cup \{0\}$, with intensity parameter $(\lambda_{i})_{l=1}^{2k}$, respectively. Let $Z_{l}^{(i)} := N_{l}^{(i)} - N_{l}^{(k+i)}$, $i = 1, 2, \ldots, k$. Define $j_{l}^{(i)} : A \rightarrow L^{\infty}(G) \otimes B(L^{2}(N_{t}^{(l)}, N_{t}^{(l+k)}))$, $(l = 1, 2, \ldots, k)$, by $j_{l}^{(i)}(\phi)(\omega) = \alpha_{Z_{l}^{(i)}(\tau)}(\phi)$. Since the generator $L^{(i)}$ of the vacuum semigroup associated with $j_{l}^{(i)}$ is bounded, so are the other structure maps. This implies that all the hypotheses of the Theorem 4.8 are satisfied and we do not need the homomorphism theorem of section 3 because homomorphism property follows from the fact that $j_{l}$, the limiting flow satisfies a q.s.d.e. with bounded structure maps [3]. So by Lemma 6.1 we have convergence in probability of the following sequence of random variables

$$X_{t}^{(n)} := \prod_{i=1}^{k} \prod_{l=0}^{2^{n}t} g_{l}^{(i)}(G_{l}^{(i)}(2^{n}t)), \quad \text{where} \quad g_{l}^{(i)}(\omega) := g_{l}^{(i)}(\omega).$$

The limit is a random variable $X_{t}$ which is a time homogeneous continuous time simple random walk.
6.2 An Application to a class of Stochastic Processes on a UHF algebra:

We recall here that a quantum stochastic flow \( j_t \) is called a quantum stochastic dilation of the associated vacuum semigroup \( T_t := j_t^{0.0} \). In this subsection, we want to apply the results obtained in section 4 to construct quantum stochastic dilation (in the sense of \( \mathbb{G} \)) to a class of QDS on uniformly hyperfinite (UHF for short) algebras. Let \( \mathcal{A} \) be the UHF \( C^* \) algebra generated by the infinite tensor product of finite dimensional matrix algebras \( M_N(\mathbb{C}) \), i.e the \( C^* \)-completion of \( \bigotimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C}) \) where \( N \) and \( d \) are two fixed positive integers. The unique normalized trace \( tr \) on \( \mathcal{A} \) is given by \( tr(x) = \frac{1}{N^n} Tr(x) \) for \( x \in M_N^n(\mathbb{C}) \). For a simple tensor \( a \in \mathcal{A} \), let \( a(j) \) be the \( j^{th} \) component of \( a \). We define support of \( a \) to be the set:

\[ \{ j \in \mathbb{Z}^d \mid a(j) \neq 1 \}. \]

For a general element \( a = \sum_{n=1}^{\infty} c_n a_n \), we define support of \( a \) to be

\[ \cup_{n \geq 1} supp(a_n) \]

Let \( \mathcal{A}_{loc} \) be the \(*\)-algebra generated by finitely supported simple tensors in \( \mathcal{A} \). Clearly \( \mathcal{A}_{loc} \) is dense in \( \mathcal{A} \). For \( k \in \mathbb{Z}^d \), the translation \( \tau_k \) on \( \mathcal{A} \) is an automorphism determined by \( \tau_k(x(j)) = x(j+k) \).

Note that \( M_N(\mathbb{C}) \) is generated by a pair of non-commutative representatives of the finite discrete group \( \mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\} \) such that \( U^N = V^N = 1 \in M_N(\mathbb{C}) \) and \( UV = \omega VU \) where \( \omega \) is the \( N^{th} \) root of unity. Using this, we get a unitary representation of \( G = \prod_{j \in \mathbb{Z}^d} G \) where \( G = \mathbb{Z}_N \times \mathbb{Z}_N \), in \( L^2(tr) \) given by

\[ G \ni g \to U_g = \prod_{j \in \mathbb{Z}^d} U^{(j)^\alpha_j} V^{(j)^\beta_j} \in \mathcal{A}, \]

where \( g = \prod_{j \in \mathbb{Z}^d}(\alpha_j, \beta_j), \alpha_j, \beta_j \in \mathbb{Z}_N \). For a given CP map \( \psi \) on \( \mathcal{A} \), formally we define the Linbladian \( \mathcal{L} = \sum_{k \in \mathbb{Z}^d} \mathcal{L}_k \), where \( \mathcal{L}_k(x) = \tau_k \mathcal{L}_0(\tau_{-k}x), x \in \mathcal{A}_{loc} \), with \( \mathcal{L}_0(x) = -\frac{1}{2}(\psi(1)x + x\psi(1)) + \psi(x) \). Consider the Linbladian \( \mathcal{L} \) for the CP map \( \psi(x) = \sum_{i=1}^p r^{(i)x} t^{(i)x}, x \in \mathcal{A} \), where each \( r^{(i)} \) belongs to a suitable class. For \( g^{(j)} = (\alpha_j, \beta_j) \in G, j \in \mathbb{Z}^d \), we set \( W_{g^{(j)}}(x) = U^{(j)^\alpha_j} V^{(j)^\beta_j} \in \mathcal{A}_{loc} \). Next let \( \|x\|_1 = \sum_{j,g} \|W_{g^{(j)}}(x)W_{g^{(j)}}^* - x\| \) and let \( C^1(A) = \{x \in \mathcal{A} : \|x\|_1 < \infty\} \). Matsui (\( \mathbb{G} \)) proved that the Lindbladian \( \mathcal{L} \) is well-defined on \( C^1(A) \), is closable, the closure generates a QDS (to be denoted by \( T_t^\psi \)) on \( \mathcal{A} \) and \( C^1(A) \) is invariant under \( T_t^\psi \).

In \( \mathbb{G} \), the authors considered the problem of constructing quantum stochastic dilation of such QDS. However they constructed the associated quantum stochastic process for only those semigroups for which the CP map \( \psi \) is of the form: \( \psi(x) = r^x x r \), when \( r = \sum_{g \in \prod_{j \in \mathbb{Z}^d} \mathbb{Z}_N} c_g W_g \), \( W_g = \prod_{j \in \mathbb{Z}^d} (U^{a_j} V^{b_j})^{\alpha_j} \) for \( g = \prod_{j \in \mathbb{Z}^d} \alpha_j \), \( \sum_{g} |c_g| |g|^2 < \infty \) and fixed \( a, b \in \mathbb{Z}_N \); where \( |g| := \# \{ j | (\alpha_j, \beta_j) \neq (0,0) \} \). Here we will generalize this result. First of all we will prove a result by considering \( \psi \) of the form given earlier, viz. \( \psi(x) = \sum_{m=1}^p r^{(m)x} x r^{(m)} \) with the following assumptions:

(i) There exists E-H dilation say \( j_t^{(m)} \) for the QDS \( T_t^{(m)} \) corresponding to the CP map \( \psi^{(m)}(x) = r^{(m)x} x r^{(m)*} \) for each \( m = 1, 2, \ldots, p \).
(ii) \( r^{(m)} \in \mathcal{A}_\text{loc} \) and

(iii) \( [r^{(m)}, r^{(m)*}] \leq 0 \), for \( m = 1, 2, \ldots p \).

Before proving the main theorem of this section, we first prove few lemmas:

**Lemma 6.3.** Suppose \((T_t)_{t \geq 0}\) is a QDS on a C*-algebra \( \mathcal{A} \) and let \( \tau \) be a finite trace on it. Furthermore, let \( \tau(\mathcal{L}(y)) \leq 0 \) for all \( y \geq 0 \), \( y \in \mathcal{A}_0 \), \( \mathcal{L} \) being the norm generator of \((T_t)_{t \geq 0}\). Then \((T_t)_{t \geq 0}\) has contractive \( L^1 \)-extension.

**Proof.** In what follows, \( L^1_\mathbb{R} \) denotes the real Banach space, obtained by taking the \( L^1 \) closure of the real Banach space of self adjoint elements of \( \mathcal{A} \), whereas \( L^1 \) denotes the complex Banach space obtained by taking the \( L^1 \) closure of \( \mathcal{A} \). Let \( f(t) = \tau(T_t(y)) \), \( y \geq 0 \) and \( y \in \mathcal{A}_0 \). Then \( f(t) = \tau(\mathcal{L}(T_t(y))) \leq 0 \). Thus \( f(t) \) is monotone decreasing and so \( f(t) \leq f(0) \), i.e.

\[
\|T_t(y)\|_1 \leq \|y\|_1 \quad , y \geq 0 \quad , y \in \mathcal{A}_0 .
\]

(57)

Let \( y \in \mathcal{A} \), \( y \geq 0 \), so that \( y = x^*x \), for some \( x \in \mathcal{A} \). Since \( \mathcal{A}_0 \) is dense in \( \mathcal{A} \), there is a sequence \((x_n)_n \in \mathcal{A}_0 \), such that \( x_n \to x \) in \( \|\cdot\|_\infty \), leading to the conclusion that \( x^*_n x_n \to x^*x \) in \( \|\cdot\|_\infty \). Since \( \tau \) is finite, \( \|\cdot\|_1 \leq \|\cdot\|_\infty \), and a positive element of \( \mathcal{A} \) can be approximated by a sequence of positive elements from \( \mathcal{A}_0 \) in \( \|\cdot\|_\infty \). Thus the same result follows in \( \|\cdot\|_1 \), which proves that the inequality (57) extends to all the positive elements of \( \mathcal{A} \). Since every self-adjoint \( x \) in \( \mathcal{A} \) can be decomposed as \( x = x_+ - x_- \) such that \( |x| = x_+ + x_- \), we have

\[
\|T_t(x)\|_1 = \|T_t(x_+) - T_t(x_-)\|_1 \leq \|T_t(x_+)\|_1 + \|T_t(x_-)\|_1 \leq \|x_+\|_1 + \|x_-\|_1 = \|x\|_1 .
\]

(58)

Thus \( T_t \) extends as a contractive map on the real Banach space \( L^1_\mathbb{R} \). We denote this extension by \( T_t^\text{sa} \). We consider its complexification \( T_t^\prime : L^1_\mathbb{R} \to L^1_\mathbb{R} \) given by \( T_t^\prime(x) = T_t^\text{sa}(\text{Re}(x)) + iT_t^\text{sa}(\text{Im}(x)) \).

As \( \|\text{Re}(x)\|_1 \leq 2\|x\|_1 \) and \( \|\text{Im}(x)\|_1 \leq 2\|x\|_1 \), \( T_t^\prime \) is a bounded (not necessarily contractive) map on \( L^1_\mathbb{R} \). It follows that the dual map \( T_t^{\prime*} : L^\infty \to L^\infty \) is a weak-* continuous map (i.e. ultraweakly continuous in this case). Moreover observe that for positive \( x \in \mathcal{A} \) and positive \( y \in \mathcal{A} \cap L^1 = \mathcal{A} \), we have, using the positivity of \( T_t \) on \( \mathcal{A} \), that

\[
\tau(T_t^{\prime*}(x)y) = \tau(xT_t^{\prime}(y)) = \tau(xT_t(y)) \geq 0 ,
\]

(59)

hence \( T_t^{\prime*}(x) \geq 0 \), i.e. \( T_t^{\prime*} \) is a positive map, which implies,

\[
\|T_t^{\prime*}\|_1 = \|T_t^{\prime*}(1)\|_\infty = \|T_t^{\text{sa}*}(1)\|_\infty = \sup_{\|\rho\|_1 \leq 1, \rho \in L^1_\mathbb{R}} |\tau(T_t^{\text{sa}*}(1)\rho)|
\]

\[
= \sup_{\|\rho\|_1 \leq 1, \rho \in L^1_\mathbb{R}} |\tau(T_t^{\text{sa}}(\rho))| \leq 1 .
\]

(60)

Thus \( T_t^{\prime*} \) is contractive on \( L^\infty \), and hence so is its predual \( T_t^\prime \) on \( L^1 \). The semigroup property and strong continuity of \((T_t)_{t \geq 0}\) on \( L^1 \) follows from the similar properties of \( T_t = T_t^\prime |_{L^\infty} \) with respect to the \( L^\infty \)-norm and the fact that \( \|\cdot\|_1 \leq \|\cdot\|_\infty \). \( \Box \)

**Lemma 6.4.** Let \( \mathcal{L}^{(m)} \) be the generator of the QDS \((T_t^{(m)})_{t \geq 0}\) corresponding to the CP map \( \psi^{(m)}(x) := r^{(m)}x_+r^{(m)} \), with \( r^{(m)} \) satisfying conditions (ii) and (iii) above. Then the QDS \((T_t^{(m)})_{t \geq 0}\) extends to \( L^1(\text{tr}) \) as a contractive \( C_0 \) semigroup.
Proof. For simplicity, we will drop the index $m$. Let $r_k := \tau_k(r)$ and let $A_0 := C^1(A)$. Suppose $y \in A_0$ and $y \geq 0$. A simple computation yields $tr(L(y)) \leq 0$. Thus by Lemma 6.3, $(T_k^m)_{t \geq 0}$ has contractive $L^1$ extension.

Lemma 6.5. Each of the QDS $(T_k^m)_{t \geq 0}$ for $m = 1, 2, \ldots, p$ has $L^2$-extensions.

Proof. Let $y \in A_0$. Then using contractivity of $T_t$ and the result of lemma 6.3, we have that
\[
|T_t(y)|_2^2 = tr(T_t(y^*)T_t(y)) \leq tr(y^*y) = |y|_2^2,
\]
since $tr(L(x)) \leq 0$ for $x \geq 0$, $x \in A_0$. The conclusion now follows, since $A_0$ is dense in $L^2(tr)$ and $\|\cdot\|_2 \leq \|\cdot\|$. □

Lemma 6.6. Let $(j_t^m)_{t \geq 0}$ be the quantum stochastic dilation of the QDS generated by the Lindbladian $L^m$ corresponding to the CP map $\psi^m(x) = r^m(x)r^m_t$ ($j_t^m$ exists by assumption (i)). Then $j_t^m$ satisfies the conditions (a),(b) of Theorem 5.1 and the condition of Remark 5.2.

Proof. Observe that $[r^m, r^m_t] \leq 0$ implies condition (b) of Theorem 5.1. Let $\delta_t^m(x) := [x, \tau_j^m(r^m)]$, $x \in A_0$. Then
\[
||\delta_t^m(x)||_1 \leq 2||\tau_j^m(r^m)||_\infty ||x||_1.
\]
Thus $\delta_t^m$ extends to a bounded operator on $L^1$. Similar result holds for $\delta_j^m$. So we obtain condition (a) of Theorem 5.1 for the quantum stochastic flow $j_t^m$. Condition (b) of Theorem 5.1 holds for each of the flows $j_t^m$ as $[r^m, r^m_t] \leq 0$. The condition of remark 5.2 also holds which can be shown as follows: Since the computations are identical for different $m$’s, we drop the index $m$ and see that formally
\[
L^m_2(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} (r_k[x, r_k^*] + [r_k, x]r_k^* + [r_k, r_k^*]) + [r_k, r_k^*]x. \quad \text{Let } x \in A_0(= C^1(A)). \quad \text{Then we have:}
\]
\[
\|L^m_2(x)\| \leq \frac{\|r\|}{2} \sum_{k \in \mathbb{Z}^d} \{||\delta_t^m(x)|| + ||\delta_k(x)||\} + ||x|| \|\mathbf{1}\| \sum_{k \in \mathbb{Z}^d} \|r_k, r_k^*\|.
\]
But $\sum_{k \in \mathbb{Z}^d} \{||\delta_t^m(x)|| + ||\delta_k(x)||\} < \infty$ (see [28] and [29]) and thus it suffices to show the convergence of the third series in (63). For this we proceed as follows: As $r = \sum_{g \in G} c_g \prod_{j \in \mathbb{Z}^d} U^{(j)\alpha_j}V^{(j)\beta_j}$,
\[
\|L^m_2(x)\| \leq \frac{\|r\|}{2} \sum_{k \in \mathbb{Z}^d} \left\{ \prod_{j \in \mathbb{Z}^d} U^{(j)\alpha_j}V^{(j)\beta_j} \prod_{j \in \mathbb{Z}^d} V^{(j)\beta_j}U^{(j)\alpha_j'} \right\} = \sum_{k \in \mathbb{Z}^d} \left\{ \prod_{j \in \mathbb{Z}^d} U^{(j)\alpha_j+k}V^{(j)\beta_j+k} \prod_{j \in \mathbb{Z}^d} V^{(j)\beta_j}U^{(j)\alpha_j'} \right\} = \sum_{k \in \mathbb{Z}^d} \left\{ \prod_{j \in \mathbb{Z}^d} U^{(j)\alpha_j-k}V^{(j)\beta_j-k} \prod_{j \in \mathbb{Z}^d} V^{(j)\beta_j}U^{(j)\alpha_j'} \right\}.
\]
Since $\alpha_{j-k} = \beta_{j-k} = \alpha'_{j-k} = \beta'_{j-k} = 0 \in \mathbb{Z}_N$ for $|k| \geq M$ by assumption (ii), $[r_k, r_k^*] = 0$ for such $k$. Thus the series is actually finite and hence $\|L^m_2(x)\| < \infty$, i.e. $A_0 \subseteq D(L_2) \cap D(L_2^2)$. □
Remark 6.7. Note that if we assume normality for each \( r^{(m)}, m = 1, 2, \ldots, p \), then we may drop the assumption that \( r^{(m)} \in \mathcal{A}_{loc} \). This is because then \( [r_k, r_k^*] = \tau_k \{r, r^*\} = 0 \).

Now we prove the main theorem of this section.

Theorem 6.8. Assume the hypotheses of Lemma 6.6. Then the QSDE:

\[
d_{\mathbf{t}}(x) = \sum_{j \in \mathbb{Z}^d} \sum_{m=1}^{p} j_t(\delta_j^{(m)}(x))da_j^{(m)}(t) + \sum_{j \in \mathbb{Z}^d} \sum_{m=1}^{p} j_t(\delta_j^{(m)}(x))da_j^{(m)}(t) + j_t(\mathcal{L}^{(m)})dt, \tag{65}
\]

admits a \(*\)-homomorphic unique solution \( j_t \), where \( j_t \) is the E-H dilation of the vacuum semigroup \( (T_t^{(m)})_{t \geq 0} \).

Proof. Let \( j_t^{(m)}, m = 1, 2, \ldots, p \), be the \(*\)-homomorphic quantum stochastic flow with the structure maps \((\mathcal{L}^{(m)}, \delta^{(m)}, \delta^{(m)}_\dagger)\) respectively. By Lemma 6.6, we are in the set up for applying Theorem 5.1-(i) and hence the present theorem follows. \( \square \)

Corollary 6.9. Let \( r^{(m)} = \sum_{g \in \mathcal{N}_c} c_g W_g \) for each \( m \), where \( W_g = \prod_{j \in \mathbb{Z}^d} (U^a V^b)^{\alpha_j} \) for \( g = \prod_{j \in \mathbb{Z}^d} \alpha_j \) (as in [5]). Then the hypothesis of Theorem 6.8 are satisfied and the same conclusion follows, which generalizes the dilation result obtained in [5].

Proof. Since \( r^{(m)} = \sum_{g \in \mathcal{N}_c} c_g W_g \), assumption (i) is satisfied (by the dilation result in [5]). It can be verified that \( r^{(m)} \) is normal for each \( m \). Thus the results of [5], assumption (i) is satisfied and by Remark 6.7, all the hypotheses of Theorem 6.8 are verified as well, and hence the result. \( \square \)

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References

[1] Accardi, L.; Kozyrev, S. V. On the structure of Markov flows. Irreversibility, probability and complexity (Les Treilles/Clausthal, 1999). Chaos Solitons Fractals 12 (2001), no.14-15, 2639–2655.

[2] Davies, E.B. One-parameter semigroups. London Mathematical Society Monographs, 15. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1980.

[3] Evans, M. P. Existence of quantum diffusions. Probab. Theory Related Fields 81 (1989), no.4, 473–483.

[4] Fagnola, F; Quantum Markov Semigroups and Quantum Flows. Proyecciones 18, n.3(1999), 1-144.

[5] Goswami, Debashish; Sahu, Lingaraj; Sinha, Kalyan B. Dilation of a class of quantum dynamical semigroups with unbounded generators on UHF algebras. Inst. H. Poincare-Probab. Statist. 41 (2005), no.3, 505–522.
[6] Kato, Tosio. Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[7] Lindsay, J.M., Sinha, Kalyan B. A Quantum Stochastic Lie-Trotter Product Formula, to appear in the Platinum Jubilee volume of the Indian Journal of Pure and Applied Mathematics, INSA, New Delhi, 2009.

[8] Matsui, Taku. Markov semigroups on UHF algebras. Rev. Math. Phys. 5 (1993), no.3, 587–600.

[9] Nelson, Edward. Analytic vectors. Ann. of Math. (2) 70 1959 572–615.

[10] Parthasarathy, K. R. An introduction to quantum stochastic calculus. Monographs in Mathematics, 85. Birkhser Verlag, Basel, 1992.

[11] Parthasarathy, K. R.; Sinha, Kalyan B. A random Trotter Kato product formula. Statistics and probability: essays in honor of C. R. Rao, pp.553–565, North-Holland, Amsterdam-New York, 1982.

[12] Parthasarathy, K. R; Sunder, V. S. Exponentials of indicator functions are total in the boson Fock space $\Gamma(L^2[0, 1])$. (English summary) Quantum probability communications, 281–284.

[13] Reed, M; Simon, B. Methods of modern mathematical physics, vol-I and vol-II. Academic press, 1972.

[14] Sinha, Kalyan B; Goswami, Debashish. Quantum stochastic processes and noncommutative geometry. Cambridge Tracts in Mathematics, 169. Cambridge University Press, Cambridge, 2007.

[15] Skeide, M. Indicator functions of intervals are totalizing in the symmetric Fock space $\Gamma(L^2(R^+))$, in (Accardi, L, Kuo, H.-H, Obata, N., Saito, K., Si Si, Streit, L., eds.) Trends in Contemporary Infinite Dimensional Analysis and Quantum Probability, Volume in honour of Takeyuki Hida, Istituto Italiano di Cultura (ISEAS), Kyoto 2000 (Rome, Volterra-Preprint 1999/0395).

[16] Takesaki, M. Theory of operator algebras. I. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Springer-Verlag, Berlin, 2002.

[17] Yosida, Kosaku. Functional analysis. Reprint of the sixth (1980) edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
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