INCLUSION AND MAJORIZATION PROPERTIES OF CERTAIN SUBCLASSES OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR.

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Abstract. The object of the present paper is to study certain properties and characteristics of the operator \( Q_{p,\alpha}^{\alpha,\beta} \) defined on \( p \)-valent analytic function by using technique of differential subordination. We also obtained result involving majorization problems by applying the operator to \( p \)-valent analytic function. Relevant connection of the the result are presented here with those obtained by earlier worker are pointed out.

1. Introduction and preliminaries

Let \( A_p(n) \) denote the class of functions of the form
\[
f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p, n \in \mathbb{N} = \{1, 2, \ldots \})
\]
which are analytic and \( p \)-valent in the open unit disk \( U = \{z \in \mathbb{C} : |z| < 1\} \). For convenience, we write \( A_p(1) = A_p, A_1(n) = A(n) \) and \( A_1(1) = A \).

For the function \( f \), given by (1.1) and the function \( g \) defined in \( U \) by
\[
g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k},
\]
the Hadamard product (or convolution) of \( f \) and \( g \) is given by
\[
(f \ast g)(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g \ast f)(z) \quad (z \in U).
\]

For \( p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha \in \mathbb{R}, \beta > 0 \) with \( \alpha + p\beta > 0 \), Swamy \cite{2}( see also \cite{1}) introduced and studied a linear operator \( I_{p,\alpha,\beta}^m : A_p \rightarrow A_p \) defined as follows:
\[
I_{p,\alpha,\beta}^0 f(z) = f(z), \quad I_{p,\alpha,\beta}^1 f(z) = I_{p,\alpha,\beta} f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + p\beta},
\]
and, in general
\[
I_{p,\alpha,\beta}^m f(z) = I_{p,\alpha,\beta} \left( I_{p,\alpha,\beta}^{m-1} f(z) \right) \quad (m \in \mathbb{N}_0; z \in U).
\]

If \( f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in A_p \), then it follows from the definition of the operator \( I_{p,\alpha,\beta}^m \) that
\[
I_{p,\alpha,\beta}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{\alpha + k\beta}{\alpha + p\beta} \right)^m a_k z^k \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U).
\]

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Since $\alpha + k\beta > \alpha + p\beta > 0$ for $k \geq p + 1$, the operator can be defined for negative integral values of $m$ as

$$I_{p,a,\beta}^{-m}f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{\alpha + p\beta}{\alpha + k\beta} \right)^m a_k z^k \quad (m \in \mathbb{N}_0; z \in \mathbb{U}).$$

We, further observe that

$$I_{p,a,\beta}^{-1}f(z) = \frac{\alpha + p}{z^{\beta}} \int_0^z \frac{t^{\beta-1}f(t)dt}{(1-z)^{p}} = I_{p,a,\beta}^{-1} \left( \frac{z^p}{1-z} \right) * f(z) \quad (z \in \mathbb{U})$$

so that

$$I_{p,a,\beta}^{-m}f(z) = I_{p,a,\beta}^{-1} \left( \frac{z^p}{1-z} \right) * I_{p,a,\beta}^{-1} \left( \frac{z^p}{1-z} \right) * \ldots * I_{p,a,\beta}^{-1} \left( \frac{z^p}{1-z} \right) * f(z) \quad (z \in \mathbb{U}).$$

Thus, in view of (1.3) and (1.4), we define a linear operator

$$\Theta_p^m(n; \alpha, \beta) : A_p(n) \rightarrow A_p(n)$$

by

$$\Theta_p^m(n; \alpha, \beta)f(z) = z^p + \sum_{k=p+n}^{\infty} \left( \frac{\alpha + k\beta}{\alpha + p\beta} \right)^m a_k z^k \quad (z \in \mathbb{U})$$

where $p, n \in \mathbb{N}, m \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}, \alpha \in \mathbb{R}$ and $\beta > 0$ with $\alpha + p\beta > 0$. From (1.5), it is easily seen that

$$\beta \ z (\Theta_p^m(n; \alpha, \beta)f)'(z) = (\alpha + p\beta)\Theta_p^{m+1}(n; \alpha, \beta)f(z) - \alpha \Theta_p^m(n; \alpha, \beta)f(z) \quad (m \in \mathbb{Z}; z \in \mathbb{U}).$$

For convenience we write

$$\Theta_p^m(1; \alpha, \beta) f(z) = \Theta_p^m(\alpha, \beta) f(z) \quad (z \in \mathbb{U}).$$

Further, by suitably specializing the parameters $p, \alpha$ and $\beta$ in (1.7), we obtain the following linear operators studied earlier by various authors.

(i) $\Theta_1^m(\alpha, \beta)f(z) = T_{\alpha,\beta}^m f(z) \quad (f \in A; m \in \mathbb{N}_0)$ (see Swamy [1]);

(ii) $\Theta_p^m(\alpha, \beta)f(z) = I_{p,a,\beta}^{-m}f(z) \quad (f \in A_p, m \in \mathbb{N}_0)$ (see Swamy [2]);

(iii) $\Theta_p^m(\ell + p - p, \alpha, \beta)f(z) = J_{p}^m(\alpha, \ell) \quad (f \in A_p, \ell \geq 0, \alpha > 0; m \in \mathbb{Z})$ (see Cátaş [6]);

(iv) $\Theta_p^m(\alpha, 1)f(z) = T_{p}^m(\alpha)f(z) \quad (f \in A_p, \alpha > -p; m \in \mathbb{N}_0)$ (see Aghalary [3], Shivaprasad et al. [15], Srivastava et al. [16]);

(v) $\Theta_p^m(0, \beta)f(z) = D_{p}^m f(z) \quad (f \in A_p, \beta > 0; m \in \mathbb{N}_0)$ (see Aouf et al. [5], Kamali et al. [9], Orhan et al. [10]);

(vi) $\Theta_1^m(\ell, 1)f(z) = I_{\ell}^m f(z) \quad (\ell > 0, m \in \mathbb{N}_0)$ (see Cho and Kim [7], Cho and Srivastava [8]);

(vii) $\Theta_1^m(1 - \lambda, \lambda)f(z) = D_{\lambda}^m f(z) \quad (\lambda \geq 0, m \in \mathbb{N}_0)$ (see Al-Oboudi [4]), which yields the operator $D_{\lambda}^{m}$ studied by Salagean [14], for $\lambda = 0$.

Using the operator $\Theta_p^m(n; \alpha, \beta)$, we now define

**Definition 1.1.** For fixed parameters $A, B (-1 \leq B < A \leq 1), \beta > 0, \mu \geq 0$ and $\alpha + p\beta > 0$, we say that a function $f \in A_p(n)$ is in the class $S_{p,n}^m(\alpha, \beta, \mu, A, B)$, if it satisfies the following subordination condition:

$$\left(1 - \mu \right) \Theta_p^{m+1}(n; \alpha, \beta)f(z) + \mu \Theta_p^{m+2}(n; \alpha, \beta)f(z) < \frac{1 + Az}{1 + Bz} \quad (m \in \mathbb{Z}; z \in \mathbb{U}).$$

(1.8)
For ease of notation, we write

(i) \( S_{p,1}^m(\ell + p - p\lambda, \lambda, \mu; A, B) = S_p^m(\lambda, \ell, \mu, A, B) \), the class of functions \( f \in A_p \) satisfying

the subordination condition:

\[
(1 - \mu) \frac{f^{m+1}(\lambda, \ell)f(z)}{f_m(\lambda, \ell)f(z)} + \mu \frac{f^{m+2}(\lambda, \ell)f(z)}{f_m^{m+1}(\lambda, \ell)f(z)} < \frac{1 + Az}{1 + Bz} \quad (\lambda > 0, \ell > -p, m \in \mathbb{Z}; z \in \mathbb{U});
\]

(ii) \( S_{p,1}^m(\alpha, \beta, 1, A, B) = S_p^m(\alpha, \beta, A, B) \), the class of functions \( f \in A_p \) satisfying the subordination condition:

\[
\frac{\Theta^{m+2}(\alpha, \beta)f(z)}{\Theta^{m+1}(\alpha, \beta)f(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U});
\]

(iii) \( S_{p,1}^m(\alpha, \beta, \mu, A, B) = S_p^m(\alpha, \beta, \mu, A, B) \), the class of functions \( f \in A_p \) satisfying the subordination condition \( (1.3) \);

(iv) \( S_{p,1}^m(\alpha, \beta, \mu, 1 - 2\rho, -1) = S_p^m(\alpha, \beta, \mu, \rho) \), the class of functions \( f \in A_p \) satisfying

\[
\text{Re}\left\{ (1 - \mu) \frac{\Theta^{m+1}(\alpha, \beta)f(z)}{\Theta^{m}(\alpha, \beta)f(z)} + \mu \frac{\Theta^{m+2}(\alpha, \beta)f(z)}{\Theta^{m+1}(\alpha, \beta)f(z)} \right\} > \rho \quad (0 \leq \rho < 1; z \in \mathbb{U});
\]

(v) \( S_{p,1}^m(\alpha, \beta, 0, 1 - 2\rho, -1) = S_p^{m-1}(\alpha, \beta, 1 - 2\rho, -1) = S_p^m(\alpha, \beta; \rho) \), the class of functions \( f \in A_p \) satisfying

\[
\text{Re}\left\{ \frac{\Theta^{m+1}(\alpha, \beta)f(z)}{\Theta^{m}(\alpha, \beta)f(z)} \right\} > \rho \quad (0 \leq \rho < 1; z \in \mathbb{U});
\]

(vi) \( S_{p,1}^{m-1}(\alpha, \beta, 1, \frac{A\beta(p - \eta) + B(\alpha + \beta\eta)}{\alpha + p\beta}, B) = S_{p,1}^m(\alpha, \beta, 0, \frac{A\beta(p - \eta) + B(\alpha + \beta\eta)}{\alpha + p\beta}, B) = S_{p,\alpha,\beta}^m(\eta; A, B) \)

The class \( S_{p,\alpha,\beta}^m(\eta; A, B) \) of functions \( f \in A_p \) satisfying the subordination condition:

\[
\frac{1}{p - \eta} \left\{ \frac{z(\Theta^{m}(\alpha, \beta)f)^{\prime}(z)}{\Theta^{m}(\alpha, \beta)f(z)} - \eta \right\} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]

This class was introduced and studied by Swamy [2], which in turn yields the class \( S_{\alpha,\beta}^m(\eta; A, B) \) studied in [1] for \( p = 1 \).

(vii) \( S_{p,1}^0(0, \beta, 0, A, B) = S_p^0(A, B) \), the class of functions \( f \in A_p \) satisfying the subordination condition:

\[
\frac{zf'(z)}{f(z)} < \frac{p(1 + Az)}{1 + Bz} \quad (z \in \mathbb{U}).
\]

(viii) \( S_{p,1}^0(0, \beta, 1, A, B) = S_p^1(0, \beta, 0; A, B) = C_p(A, B) \), the class of functions \( f \in A_p \) satisfying

the subordination condition:

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{p(1 + Az)}{1 + Bz} \quad (z \in \mathbb{U}).
\]

In the present investigation, we introduce a subclass \( S_{p,n}^m(\alpha, \beta, \mu, A, B) \) of \( A_p(n) \). We derive certain inclusion relationships, some useful characteristics and majorization properties for the class \( S_{p,n}^m(\alpha, \beta, \mu, A, B) \). The results obtained here in addition to generalizing some of the work of Patel et al. [13] and MacGregor [23] improves the corresponding work of Swamy [2]. We also obtain a number of new results for functions belonging to this class in terms of subordination and the various subclasses obtained as special cases of the class \( S_{p,n}^m(\alpha, \beta, \mu, A, B) \).
2. Preliminary Lemmas

Lemma 2.1. [22], see also [21] p.71: Let $h$ be an analytic and convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$ and $\phi$ be given by

$$
\phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \quad (n \in \mathbb{N}; \ z \in \mathbb{U}). \tag{2.1}
$$

If

$$
\phi(z) + \frac{z\phi'(z)}{\gamma} < h(z) \quad (\text{Re}(\gamma) \geq 0, \gamma \neq 0; \ z \in \mathbb{U}), \tag{2.2}
$$

then

$$
\phi(z) < \psi(z) = \frac{\gamma}{n} z^{-\frac{1}{n}} \int_0^z t^\frac{n-1}{n} h(t) dt < h(z) \quad (z \in \mathbb{U})
$$

and the function $\psi$ is the best dominant of (2.2).

We recall the definition of the class $\mathcal{P}(\gamma)$ ($0 \leq \gamma < 1$) (cf., Section 1.2) consisting of all functions of the form

$$
\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{2.3}
$$

such that $\text{Re}\{\phi(z)\} > \gamma$ in $\mathbb{U}$. We have

Lemma 2.2. [20]: If the function $\phi$, given by (2.3) belongs to the class $\mathcal{P}(\gamma)$, then

$$
\text{Re}\{\phi(z)\} \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leq \gamma < 1; \ z \in \mathbb{U}).
$$

Lemma 2.3. [21]: If $-1 \leq B < A \leq 1, \beta^* > 0$ and the complex number $\gamma^*$ is constrained by $\text{Re}(\gamma^*) \geq -\beta^*(1-A)/(1-B)$, then the following differential equation:

$$
q(z) + \frac{zq'(z)}{\beta^*q(z) + \gamma^*} = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})
$$

has a univalent solution in $\mathbb{U}$ given by

$$
q(z) = \begin{cases} 
\frac{z^{\beta^*+\gamma^*}(1+Bz)^{\beta^*(A-B)/B}}{\beta^*} + \frac{\gamma^*}{\beta^*}, & B \neq 0 \\
\frac{\gamma^*}{\beta^*}, & B = 0.
\end{cases} \tag{2.4}
$$

If the function $\phi$, given by (2.3) is analytic in $\mathbb{U}$ and satisfies the following subordination:

$$
\phi(z) + \frac{z\phi'(z)}{\beta^*\phi(z) + \gamma^*} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{2.5}
$$

then

$$
\phi(z) < q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})
$$

and $q$ is the best dominant of (2.5).

Lemma 2.4. [19]: Let $\nu$ be a positive measure on $[0,1]$. Let $h(z,t)$ be a complex-valued function defined on $\mathbb{U} \times [0,1]$ such that $h(\cdot,t)$ is analytic in $\mathbb{U}$ for each $t \in [0,1]$ and that $h(\cdot,t)$ is $\nu$-integrable on $[0,1]$ for all $z \in \mathbb{U}$. In addition, suppose that $\text{Re}\{h(\cdot,t)\} > 0, h(-r,t)$ is real and

$$
\text{Re} \left\{ \frac{1}{h(z,t)} \right\} \geq \frac{1}{h(-r,t)} \quad (|z| \leq r < 1, \ t \in [0,1]).
$$
If the function $H$ is defined in $U$ by

$$H(z) = \int_0^1 h(z, t) d\nu(t),$$

then

$$\text{Re} \left\{ \frac{1}{H(z)} \right\} \geq \frac{1}{H(-r)}.$$

Each of the identities given below are well-known (see Whittaker et al. [18, Chapter 14]) for the hypergeometric function $2F_1$.

**Lemma 2.5.** [18] For real or complex numbers $a, b$ and $c$ ($c \neq 0, -1, -2, \ldots$), we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} 2F_1(a, b; c; z) \quad (\text{Re}(c) > \text{Re}(b) > 0); \quad (2.6)$$

$$2F_1(a, b; c; z) = (1-z)^{-a} 2F_1\left(a, c-b; c; \frac{z}{z-1}\right); \quad (2.7)$$

$$2F_1(a, b; c; z) = 2F_1(b, a; c; z); \quad (2.8)$$

$$(a + 1) 2F_1(1, a; a + 1; z) = (a + 1) + az 2F_1(1, a + 1; a + 2; z). \quad (2.9)$$

### 3. Inclusion Relationships

Unless otherwise mentioned, we shall assume throughout the sequel that $\alpha \in \mathbb{R}$, $\beta > 0$, $\alpha + p\beta > 0$, $\mu > 0$, $m \in \mathbb{Z}$, $-1 \leq B < A \leq 1$ and the powers are understood as principal values.

In this section, we establish some inclusion relationships involving the class $S_{p,n}^m(\alpha, \beta, \mu, A, B)$.

**Theorem 3.1.** If $f \in S_{p,n}^m(\alpha, \beta, \mu; A, B)$, then

$$\frac{\Theta_B^m(\alpha, \beta)f(z)}{\Theta_p^m(\alpha, \beta)f(z)} < \frac{\mu\beta}{(\alpha + p\beta)\mu\beta}Q(z) = q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U), \quad (3.1)$$

where

$$Q(z) = \begin{cases} \int_0^1 t^{\alpha+p\beta-1} \frac{(1 + Btz)^{(\alpha+p\beta)(A-B)}}{1 + Bz} \, dt, & B \neq 0 \\ \int_0^1 t^{\alpha+p\beta-1} \exp\left(\frac{(\alpha + p\beta)}{\mu\beta} A(t-1)z\right) \, dt, & B = 0 \end{cases} \quad (3.2)$$

and $q$ is the best dominant of (3.1). Furthermore, if

$$A \leq -\frac{\mu\beta}{\alpha + p\beta} B \quad \text{with} \quad -1 \leq B < 0,$$

then

$$S_{p,n}^m(\alpha, \beta, \mu, A, B) \subset S_{p,n}^m(\alpha, \beta, \rho), \quad (3.3)$$

where

$$\rho = \left\{ 2F_1\left(1, \frac{\alpha + p\beta}{\mu\beta} \left(\frac{B - A}{B}\right); \frac{\alpha + p\beta}{\mu\beta} + 1; \frac{B}{B - 1}\right) \right\}^{-1}.$$  

The result is the best possible.
Proof. Let \( f \in S^m_p(\alpha, \beta, \mu; A, B) \). Consider the function \( g \) defined by

\[
g(z) = z \left( \Theta^m_p(\alpha, \beta) f(z) \right)^{\frac{\beta}{\alpha + p\beta}}
\]

and let \( r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\} \). Then \( g \) is single-valued and analytic in \(|z| < r_1\). Taking logarithmic differentiation in (3.4) and using the identity (1.6) in the resulting equation, it follows that the function \( \phi \) given by

\[
\phi(z) = \frac{zg'(z)}{g(z)} = \frac{\Theta^{m+1}_p(\alpha, \beta) f(z)}{\Theta^m_p(\alpha, \beta) f(z)}
\]

is analytic in \(|z| < r_1\) and \( \phi(0) = 1 \). Carrying out logarithmic differentiation in (3.5), followed by the use of the identity (1.6) and (1.8), we deduce that

\[
\phi(z) + \frac{\mu \beta z \phi'(z)}{(\alpha + p\beta) \phi(z)} < \frac{1 + Az}{1 + Bz} \quad (|z| < r_1).
\]

Hence, by using Lemma 2.3 we find that

\[
\phi(z) < \frac{\mu \beta}{(\alpha + p\beta) Q(z)} = q(z) < \frac{1 + Az}{1 + Bz} \quad (|z| < r_1),
\]

where \( q \) is the best dominant of (3.1) and is given by (2.4) with \( \beta^* = (\alpha + p\beta)/\mu \beta \) and \( \gamma^* = 0 \). For \(-1 \leq B < A \leq 1\), it is easy to see that

\[
\Re \left( \frac{1 + Az}{1 + Bz} \right) > 0 \quad (z \in U),
\]

so that by (3.7), we have

\[
\Re \{ \phi(z) \} > 0 \quad (|z| < r_1).
\]

Now, (3.5) shows that the function \( g \) is starlike (univalent) in \(|z| < r_1\). Thus, it is not possible that \( g \) vanishes on \(|z| = r_1\), if \( r_1 < 1 \). So, we conclude that \( r_1 = 1 \) and the function \( \phi \) given by (3.5) is analytic in \( U \). Hence, in view of (3.7), we have

\[
\phi(z) \prec q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U).
\]

This proves the assertion (3.1). To prove (3.3), we need to show that

\[
\inf_{z \in U} \{ \Re(q(z)) \} = q(-1).
\]

If we set

\[
a = \frac{\alpha + p\beta}{\mu \beta} \left( \frac{B - A}{B} \right), \quad b = \frac{\alpha + p\beta}{\mu \beta} \quad \text{and} \quad c = \frac{\alpha + p\beta}{\mu \beta} + 1,
\]

then \( c > b > 0 \). From (3.2), by using (2.6) to (2.9), we see that for \( B \neq 0 \),

\[
Q(z) = (1 + Bz)^a \int_0^1 t^{b-1}(1 + Btz)^{-a} dt
\]

\[
= \frac{\mu \beta}{\alpha + p\beta} \:_2F_1 \left( 1, \frac{\alpha + p\beta}{\mu \beta} \left( \frac{B - A}{B} \right) ; \frac{\alpha + p\beta}{\mu \beta} + 1; \frac{Bz}{1 + Bz} \right).
\]

To prove (3.5), it suffices to show that

\[
\Re \left( \frac{1}{Q(z)} \right) \geq \frac{1}{Q(-1)} \quad (z \in U).
\]

Since

\[
A \leq -\frac{\mu \beta}{\alpha + p\beta} B \quad \text{with} \quad -1 \leq B < 0
\]
implies that $c > a > 0$, by using (2.4), we find from (3.9) that
$$Q(z) = \int_0^1 h(z, t) \, dt,$$
where
$$h(z, t) = \frac{1 + Bz}{1 + B(1 - t)z} \quad (0 \leq t \leq 1) \quad \text{and} \quad dv(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c - a)} t^{a - 1} e^{-a - 1} \, dt$$
which is a positive measure on $[0, 1]$. For $-1 \leq B < 0$, it may be noted that $\text{Re}\{h(z, t)\} > 0$ and $h(-r, t)$ is real for $0 \leq |z| \leq r < 1$ and $t \in [0, 1]$. Therefore, by using Lemma 2.4 we obtain
$$\text{Re} \left( \frac{1}{Q(z)} \right) \geq \frac{1}{Q(-r)} \quad (|z| \leq r < 1)$$
which, upon letting $r \to 1^-$ yields
$$\text{Re} \left( \frac{1}{Q(z)} \right) \geq \frac{1}{Q(-1)}.$$
Further, by taking $A \to (-\mu \beta B/(\alpha + p\beta))^+$ for the case $A = -\mu \beta B/(\alpha + p\beta)$ and using (3.1), we get (3.3). The result is the best possible as the function $q$ is the best dominant of the subordination (3.1). This evidently completes the proof of Theorem 3.1.

Letting $\mu = 1$ in Theorem 3.1, we obtain the following result which, in turn yields the corresponding work of Patel et al. [12, Corollary 1] for $m = \alpha = 0$.

**Corollary 3.1.** If $A \leq -\beta B/(\alpha + p\beta)(-1 \leq B < 0)$, then
$$S_p^{m-1} (\alpha, \beta, A, B) \subset S_p^m (\alpha, \beta, \tilde{\rho}),$$
where
$$\tilde{\rho} = \left\{ \left. 2F1 \left( 1, \frac{\alpha + p\beta}{\beta} \left( \frac{B - A}{B} \right); \frac{\alpha + p\beta}{\beta} + 1; \frac{B}{B - 1} \right) \right\}^{-1}.$$
The result is the best possible.

Setting $m = -1$, $\mu = \beta = 1$ and $\tilde{A} = \{(p + \alpha)A - \alpha B\}/p$ in Theorem 3.1, we deduce the following result obtained earlier by Patel et al. [12, Corollary 2].

**Corollary 3.2.** If $\alpha > -p, -1 \leq B < 0$ and
$$\tilde{A} \leq \min \left\{ 1 + \frac{\alpha(1 - B)}{p}, -\frac{(\alpha + 1)B}{p} \right\} \leq 1,$$
then for $f \in S_p^m (\tilde{A}, B)$, we have
$$\text{Re} \left( \frac{z^\alpha f(z)}{\int_0^1 t^\alpha - 1 f(t) \, dt} \right) > (p + \alpha) \left\{ 2F1 \left( 1, p \left( \frac{B - \tilde{A}}{B} \right); \alpha + p + 1; \frac{B}{B - 1} \right) \right\}^{-1}.$$
The result is the best possible.

Setting $\alpha = p + \ell - p\lambda$ and $\beta = \lambda$ in Theorem 3.1, we get

**Corollary 3.3.** If $p > -\ell, -1 \leq B < 0, A \leq \mu \beta/(p + \ell)$ and $f \in S_p^m (\lambda, \ell, \mu, A, B)$, then
$$\text{Re} \left( \frac{\mathcal{J}_p^{m+1} (\lambda, \ell, f(z))}{\mathcal{J}_p^m (\lambda, \ell) f(z)} \right) > \left\{ 2F1 \left( 1, \frac{(p + \ell)(B - A)}{\mu \lambda B}; \frac{p + \ell}{\mu \lambda} + 1; \frac{B}{B - 1} \right) \right\}^{-1}.$$
The result is the best possible.
Theorem 3.2. If $\kappa = (1 - A)/(1 - B)$ and $f \in S_p^m(\alpha, \beta; \kappa)$, then

$$f \in S_p^m(\alpha, \beta, \mu, 1 - 2\kappa, -1) \text{ for } |z| < R = R(p, \alpha, \beta, \mu, \kappa),$$

where

$$R = \begin{cases} 
\frac{(\alpha + p\beta)(1 - \kappa) + \mu\beta - \sqrt{(\alpha + p\beta)\kappa - \mu\beta}^2 + 2\mu\beta(\alpha + p\beta)}{(\alpha + p\beta)(1 - 2\kappa)}, & \kappa \neq \frac{1}{2} \\
\frac{1}{2}\{(\alpha + p\beta)(1 - \kappa) + \mu\beta\}, & \kappa = \frac{1}{2}.
\end{cases} \quad (3.10)
$$

The result is the best possible.

Proof. Since $f \in S_p^m(\alpha, \beta, \kappa)$, we have

$$\frac{\Theta_p^{m+1}(\alpha, \beta)f(z)}{\Theta_p^m(\alpha, \beta)f(z)} = \kappa + (1 - \kappa)\phi(z) \quad (z \in \mathbb{U}) \quad (3.11)$$

where $\phi$, given by (2.3) is analytic and has a positive real part in $\mathbb{U}$. Taking logarithmic differentiation in (3.11), and using (1.6) in the resulting equation followed by simplifications, we deduce that

$$\text{Re} \left\{ \left(1 - \mu\right)\frac{\Theta_p^{m+1}(\alpha, \beta)f(z)}{\Theta_p^m(\alpha, \beta)f(z)} + \mu\frac{\Theta_p^{m+2}(\alpha, \beta)f(z)}{\Theta_p^{m+1}(\alpha, \beta)f(z)} \right\} - \kappa
\geq (1 - \kappa) \left\{ \text{Re}(\phi(z)) - \frac{\mu\beta}{\alpha + p\beta} \frac{|z\phi'(z)|}{\kappa + (1 - \kappa)\phi(z)} \right\}. \quad (3.12)$$

Now, by using the well-known [23] estimates

(i) $|z\phi'(z)| \leq \frac{2n^m}{1 - r^{2n}} \text{Re}(\phi(z))$ and (ii) $\text{Re}\{\phi(z)\} \geq \frac{1 - r^n}{1 + r^n} \quad (|z| = r < 1) \quad (3.13)$

with $n = 1$ in (3.12), we obtain

$$\text{Re} \left\{ \left(1 - \mu\right)\frac{\Theta_p^{m+1}(\alpha, \beta)f(z)}{\Theta_p^m(\alpha, \beta)f(z)} + \mu\frac{\Theta_p^{m+2}(\alpha, \beta)f(z)}{\Theta_p^{m+1}(\alpha, \beta)f(z)} \right\} - \kappa
\geq (1 - \kappa)\text{Re}\{\phi(z)\} \left[ 1 - \frac{2\mu\beta r}{(\alpha + p\beta)\{\kappa(1 - r^2) + (1 - \kappa)(1 - r^2)\}} \right]$$

which is certainly positive, if $r < R$, where $R$ is given by (3.10).

It is easily seen that the bound $R$ is the best possible for the function $f \in \mathcal{A}_p$, defined by

$$\frac{\Theta_p^{m+1}(\alpha, \beta)f(z)}{\Theta_p^m(\alpha, \beta)f(z)} = \frac{1 + (1 - 2\kappa)z}{1 - z} \quad \left(\kappa = \frac{1 - A}{1 - B}; z \in \mathbb{U}\right).$$

□

A special case of Theorem 3.2 when $m = \alpha = 0$, $A = 1 - (2\rho/p)$ and $B = -1$ we have.

Corollary 2.3.4. If $0 \leq \rho < p$ and $f \in S_p^\rho(\rho)$, then

$$\text{Re} \left\{ \left(1 - \mu\right)\frac{zf'(z)}{f(z)} + \mu\left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right\} > \rho \quad \text{for } |z| < R(p, \mu, \rho),$$

where

$$R(p, \mu, \rho) = \begin{cases} 
\frac{(p + \mu - \rho) - \sqrt{\rho^2 + 2\mu(p - \rho) + \mu^2}}{p - 2\rho}, & \rho \neq \frac{p}{2} \\
\frac{p}{2(p + \mu - \rho)}, & \rho = \frac{p}{2}.
\end{cases}$$
The result is the best possible.

For a function \( f \in \mathcal{A}_p(n) \), we define the integral operator \( \mathcal{F}_{\delta,p} : \mathcal{A}_p(n) \to \mathcal{A}_p(n) \) by

\[
\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta-1} f(t) \, dt \quad (\delta > -p; z \in \mathbb{U}).
\] (3.14)

If \( f \) is defined by (1.1), then

\[
\mathcal{F}_{\delta,p}(f)(z) = z^p + \sum_{k=n}^{\infty} \frac{\delta + p}{\delta + p + k} a_{p+k} z^{p+k} \quad (z \in \mathbb{U})
\] (3.15)

where

\[
= z^p \, 2 \, \binom{1}{\delta + p; \delta + p + 1; z} * f(z) = \Theta_p^{-1}(n, \delta, 1)f(z) \quad (z \in \mathbb{U}).
\]

It follows from (1.5) and (3.15) that for \( f \in \mathcal{A}_p(n) \) and \( \delta > -p \),

\[
z \left( \Theta_p^{m}(n; \alpha, \beta) \mathcal{F}_{\delta,p}(f)(z) \right)' = (\delta + p) \Theta_p^{m}(n; \alpha, \beta)f(z) - \delta \Theta_p^{m}(n; \alpha, \beta) \mathcal{F}_{\delta,p}(f)(z) \quad (z \in \mathbb{U}).
\] (3.16)

Now we have

**Theorem 3.3.** Let \( \delta \) be a real number satisfying the condition

\[
\delta \geq \frac{\alpha(A - B) - p\beta(1 - A)}{\beta(1 - B)}.
\]

(i) If \( f \in \mathcal{S}_p^{m}(\alpha, \beta, A, B) \), then the function \( \mathcal{F}_{\delta,p}(f) \) given by (3.14) belongs to the class \( \mathcal{S}_p^{m}(\alpha, \beta, A, B) \).

Furthermore,

\[
\frac{\Theta_p^{m+1}(\alpha, \beta) \mathcal{F}_{\delta,p}(f)(z)}{\Theta_p^{m}(\alpha, \beta) \mathcal{F}_{\delta,p}(f)(z)} < \frac{1}{\alpha + p\beta} \left( \frac{\beta}{Q(z)} - (\delta\beta - \alpha) \right) = q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),
\]

where

\[
Q(z) = \begin{cases} 
\int_0^1 t^{\delta+p-1} \left( \frac{1 + Btz}{1 + Bz} \right)^{\frac{(\alpha + p\beta)(A - B)}{\beta A}} \, dt, & B \neq 0 \\
\int_0^1 t^{\delta+p-1} \exp \left( \frac{(\alpha + p\beta)}{\beta} A(t - 1) \right), & B = 0
\end{cases}
\] (3.17)

and \( q \) is the best dominant.

(ii) If \(-1 \leq B < 0\) and

\[
\delta \geq \max \left\{ \frac{\alpha + p\beta}{\beta} \left( \frac{B - A}{B} \right) - p - 1, \frac{\alpha}{\beta} - \frac{(\alpha + p\beta)(1 - A)}{\beta(1 - B)} \right\},
\]

then

\[
f \in \mathcal{S}_p^{m}(\alpha, \beta, A, B) \implies \mathcal{F}_{\delta,p}(f) \in \mathcal{S}_p^{m}(\alpha, \beta; \tau),
\]

where

\[
\tau = \frac{1}{\alpha + p\beta} \left[ \beta(\delta + p) \left\{ 2F_1 \left( 1, \frac{(\alpha + p\beta)(B - A)}{\beta B}; \delta + p + 1; \frac{B}{B - 1} \right) \right\}^{-1} - (\delta\beta - \alpha) \right].
\]

The result is the best possible.

**Proof.** Setting

\[
g(z) = z \left( \frac{\Theta_p^{m}(\alpha, \beta) \mathcal{F}_{\delta,p}(f)(z)}{z^p} \right)^{\frac{\delta}{\alpha + p\beta}}
\] (3.18)
and \( r_1 = \sup\{r : g(z) \neq 0, 0 < |z| \leq r < 1\} \), we see that \( g \) is single-valued and analytic in \( |z| < r_1 \). By taking the logarithmic differentiation in (3.18) and using the identity (3.16) for the function \( \mathcal{F}_{\delta,p}(f) \), it follows that

\[
\phi(z) = zg'(z) = \frac{\Theta^{m+1}_p(\alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{\Theta^m_p(\alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)} \quad (3.19)
\]

is analytic in \( |z| < r_1 \) and \( \phi(0) = 1 \). Again, by making use of the identity (1.6) and (3.16), we deduce that

\[
\frac{\Theta^m_p(\alpha, \beta)(f)(z)}{\Theta^m_p(\alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)} = \frac{(\alpha + p\beta)\phi(z) + (\delta\beta - \alpha)}{\beta(\delta + p)} \quad (|z| < r_1). \quad (3.20)
\]

Since \( f \in S^m_p(\alpha, \beta, A, B) \), it is clear that \( \Theta^m_p(\alpha, \beta)(f)(z) \neq 0 \) in \( 0 < |z| < 1 \). So, in view of (3.20), we have

\[
\frac{\Theta^m_p(\alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{\Theta^m_p(\alpha, \beta)(f)(z)} = \frac{\beta(\delta + p)}{(\alpha + p\beta)\phi(z) + (\delta\beta - \alpha)} \quad (|z| < r_1). \quad (3.21)
\]

Now, by carrying out logarithmic differentiation in both sides of (3.21) followed by the use of the identity (1.6), (3.16) and (3.19) in the resulting equation, we obtain

\[
\frac{\Theta^{m+1}_p(\alpha, \beta)(f)(z)}{\Theta^m_p(\alpha, \beta)(f)(z)} = \phi(z) + \frac{z\phi'(z)}{\left(\frac{\alpha + p\beta}{\beta}\right)\phi(z) + \left(\frac{\delta - \alpha}{\beta}\right)} < \frac{1 + Az}{1 + Bz} \quad (|z| < r_1). \quad (3.22)
\]

Thus, by making use of Lemma 2.3 with \( \beta^* = (\alpha + p\beta)/\beta \) and \( \gamma^* = (\delta\beta - \alpha)/\beta \) in (3.22), we get

\[
\phi(z) = \frac{1}{\alpha + p\beta} \left(\frac{\beta}{Q(z)} - (\delta\beta - \alpha)\right) = q(z) < \frac{1 + Az}{1 + Bz} \quad (|z| < r_1), \quad (3.23)
\]

where \( Q \) is given by (3.17), and \( q \) is the best dominant.

Since for \( -1 \leq B < A \leq 1 \),

\[
\text{Re}\left(\frac{1 + Az}{1 + Bz}\right) > 0 \quad (z \in \mathbb{U}),
\]

by (3.22), we have \( \text{Re}\{\phi(z)\} > 0 \) in \( |z| < r_1 \). Now, in view of (3.19) the function \( g \) is univalent in \( |z| < r_1 \). Thus, it is not possible that the function \( g \) vanishes on \( |z| = r_1 \), if \( r_1 < 1 \). So, we conclude that \( r_1 = 1 \) and the function \( \phi \) is analytic in \( \mathbb{U} \). From (3.19) and (3.23), we prove the assertion (i) of Theorem 3.3.

Following the same technique as in the proof of Theorem 3.1, we can prove the assertion (ii) of Theorem 3.3. The result is the best possible as \( q \) is the best dominant. \( \square \)

**Remark 3.1.** If, in Theorem 3.3 with \( A = 1 - (2\eta/p) \) \( (0 \leq \eta < p) \), \( B = -1 \), we set \( m = \alpha = 0 \) and \( m = 1 \), \( \alpha = 0 \), we shall obtain the corresponding results by Patel et al. [13, Remark 2].

### 4. Properties involving the operator \( \Theta^m_p(\alpha, \beta) \)

In this section, we derive certain properties and characteristics of functions in \( \mathcal{A}_p \) involving operator \( \Theta^m_p(\alpha, \beta)f(z) \)

**Theorem 4.1.** Let \( 0 < \mu < 1 \), \( 0 < \gamma \leq 1 \) and \( A \leq 1 - \mu(1 - B) \). If \( f \in \mathcal{A}_p \) satisfies the following subordination condition

\[
(1 - \mu)\left(\frac{\Theta^m_p(\alpha, \beta)f(z)}{z^p}\right)^{\frac{1}{p}} + \mu\frac{\Theta^{m+1}_p(\alpha, \beta)f(z)}{\Theta^m_p(\alpha, \beta)f(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \quad (4.1)
\]
\[
\left(\frac{\Theta_p^m(\alpha, \beta)f(z)}{z^p}\right)^{1/\gamma} \leq \frac{\mu \beta \gamma}{(1 - \mu)(\alpha + p\beta)} \left(\frac{1}{Q(z)}\right) = q(z) < \frac{1 + A - \mu B}{1 + B} z \quad (z \in \mathbb{U}),
\]

where
\[
Q(z) = \begin{cases} 
\int_0^1 t \frac{(1 - \mu)(\alpha + p\beta) - 1}{\mu \beta \gamma} \left(1 + B t z\right)^{\frac{\alpha + p\beta}{\mu \beta \gamma}} \left(\frac{A - B}{B} t z\right)^{\frac{1 + p\beta}{\mu \beta \gamma}} dt, & B \neq 0 \\
\int_0^1 \frac{(1 - \mu)(\alpha + p\beta) - 1}{\mu \beta \gamma} \exp\left(\frac{\alpha + p\beta}{\mu \beta \gamma} A(t - 1)\right) dt, & B = 0
\end{cases}
\]

and \( q \) is the best dominant of \( \Theta_p^m(\alpha, \beta)_f(z) \). Furthermore, if
\[
A \leq \min \left\{ 1 - \mu(1 - B), -\mu \left(\frac{\beta \gamma}{\alpha + p\beta} - 1\right) B \right\} \quad \text{with} \quad -1 \leq B < 0,
\]
then
\[
\Re \left(\frac{\Theta_p^m(\alpha, \beta)f(z)}{z^p}\right) > \xi \quad (z \in \mathbb{U}),
\]
where
\[
\xi = \left\{ \frac{2}{\Gamma_2} \left(1, \frac{1}{\mu \beta \gamma} \left(\frac{B - A}{B} ; \frac{1}{\mu \beta \gamma} (1 + B) + 1; \frac{B}{B - 1}\right) \right) \right\}^{-1}.
\]
The result is the best possible.

**Proof.** Setting
\[
\phi(z) = \left(\frac{\Theta_p^m(\alpha, \beta)f(z)}{z^p}\right)^{1/\gamma} \quad (\gamma > 0, \ z \in \mathbb{U}),
\]
we note that the function \( \phi \) of the form \( (2.3) \) and it is analytic in \( \mathbb{U} \). Taking logarithmic differentiation in both sides of \( (4.3) \) and using \( (1.6) \) in the resulting equation, we deduce that
\[
\psi(z) + \frac{z\psi'(z)}{\beta \psi(z) + \gamma} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),
\]
where \( \psi(z) = \mu + (1 - \mu)\phi(z) \), \( \beta^* = (\alpha + p\beta)/\mu \beta \gamma \) and \( \gamma^* = -(\alpha + p\beta)/\beta \gamma \). Applying Lemma \( 2.3 \) in \( (4.4) \) and following the lines of proof of Theorem \( 3.1 \) we shall obtain the assertion of Theorem \( 4.1 \) \( \square \)

Letting \( m = 1, \alpha = 0, \beta = 1, A = 1 - (2\eta/p) \) and \( B = -1 \) in Theorem \( 4.1 \) we get

**Corollary 4.1.** If \( \max \left\{ p\mu, \frac{p + (p - 1)\mu}{2} \right\} \leq \eta < p \) and \( f \in A_p \) satisfies
\[
\Re \left\{ (1 - \mu) \frac{f'(z)}{z^{p - 1}} + \mu \left(1 + z \frac{f''(z)}{f'(z)}\right) \right\} > \eta \quad (z \in \mathbb{U}),
\]
then
\[
\Re \left\{ \frac{f'(z)}{z^{p - 1}} \right\} > p \left\{ 2 F_1 \left(1, \frac{2(p - \eta)}{\mu} ; \frac{p(1 - \mu)}{\mu} + 1; \frac{1}{2}\right) \right\}^{-1} \quad (z \in \mathbb{U}).
\]
The result is the best possible.

Setting
\[
\left(\frac{\Theta_p^m(\alpha, \beta)f(z)}{z^p}\right)^{1/\gamma} = \kappa + (1 - \kappa)\phi(z) \quad (0 < \gamma \leq 1, \kappa = \frac{1 - A}{1 - B} ; z \in \mathbb{U}),
\]
where \( \phi \) is of the form \( (2.1) \), using the estimates \( (3.13) \) and following the lines of proof of Theorem \( 3.2 \) we obtain
Theorem 4.2. Let $0 < \mu < 1$, $0 < \gamma \leq 1$ and $f \in A_p(n)$ satisfies the following subordination condition
\[
\Theta_p^m(n; \alpha, \beta) f(z) < \left( \frac{1 + Az}{1 + Bz} \right)^\gamma \quad (z \in U),
\]
then
\[
\Re \left\{ (1 - \mu) \left( \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} \right)^{1/t} + \mu \frac{\Theta_p^{m+1}(n; \alpha, \beta) f(z)}{z^p} \right\} > \mu + (1 - \mu) \kappa
\]
for $|z| < R \equiv R(p, n, \mu, \alpha, \beta, \gamma, \kappa)$, where $R$ is the smallest positive root of the equation
\[
(1 - \mu)(\alpha + p\beta)(1 - 2\kappa)r^{2n} - 2((1 - \mu)(\alpha + p\beta)(1 - \kappa) + n\mu\beta\gamma)r^n + (1 - \mu)(\alpha + p\beta) = 0.
\]

The result is the best possible for the function $f \in A_p(n)$ defined by
\[
\left( \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} \right)^{1/n} = \left( \frac{1 + (1 - 2\kappa)z^n}{1 - z^n} \right)^\gamma \quad (0 < \gamma \leq 1, \kappa = \frac{1 - A}{1 - B}; z \in U).
\]

Next, we derive the following result.

Theorem 4.3. If $\mu > 0$ and $f \in A_p(n)$ satisfies the following subordination condition:
\[
(1 - \mu) \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} + \mu \frac{\Theta_p^{m+1}(n; \alpha, \beta) f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in U),
\]
then for $z \in U$,
\[
\Theta_p^m(n; \alpha, \beta) f(z) = \left\{ \begin{array}{ll}
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} 
\binom{2F1}{1, 1; \frac{\alpha + p\beta}{\mu\beta n} + 1; \frac{Bz}{1 + Bz}}, & B \neq 0 \\
1 + \frac{A}{\alpha + p\beta + \mu\beta n} A, & B = 0,
\end{array} \right.
\]

Further,\[
\Re \left\{ \left( \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} \right)^{1/t} \right\} > \sigma^{1/t} \quad (t \in N; z \in U),
\]
where
\[
\sigma = \left\{ \begin{array}{ll}
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} 
\binom{2F1}{1, 1; \frac{\alpha + p\beta}{\mu\beta n} + 1; \frac{B}{B - 1}}, & B \neq 0 \\
1 - \frac{A}{\alpha + p\beta + \mu\beta n} A, & B = 0,
\end{array} \right.
\]
The result is the best possible.

Proof. For $f \in A_p(n)$, we write
\[
\phi(z) = \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} \quad (f \in A_p(n); z \in U).
\]

Then, $\phi$ is of the form (2.1) and it is analytic in the unit disk $U$. On differentiating both the sides of (4.8), using the identity (1.6) in the resulting equation followed by the use of (4.5), we get
\[
\phi(z) + \frac{z\phi'(z)}{(\alpha + p\beta)/\mu\beta} < \frac{1 + Az}{1 + Bz} \quad (z \in U).
\]

Now, by an application of Lemma (2.1) with $\gamma = (\alpha + p\beta)/\mu\beta$ in (4.9), we obtain
\[
\frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} < Q(z) = \frac{\alpha + p\beta}{\mu\beta n} z^{-\frac{\alpha + p\beta}{\mu\beta n}} \int_0^z \left( \frac{1 + At}{1 + Bt} \right) dt \quad (z \in U).
\]
which yields (4.6) by change of variables followed by the use of the identities (2.6) to (2.9) (with \( a = 1, b = (\alpha + p\beta)/\mu\beta n \) and \( c = b + 1 \)). This proves the assertion (4.6) of Theorem 4.3.

To prove (4.7), it suffices to show that
\[
\inf_{z \in U} \{\text{Re}(Q(z))\} = Q(-1).
\] (4.10)

Indeed, for \( |z| \leq r < 1 \),
\[
\text{Re} \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}.
\]

Setting
\[
G(s, z) = \frac{1 + A sz}{1 + Bsz} \quad (0 \leq s \leq 1) \quad \text{and} \quad d\nu(s) = \frac{\alpha + p\beta}{\mu\beta n} \frac{s^{\alpha + p\beta} - 1}{s} ds,
\]
which is a positive measure on \([0, 1]\), we get
\[
Q(z) = \int_0^1 G(s, z)d\nu(s),
\]
so that
\[
\text{Re}\{Q(z)\} \geq \int_0^1 \frac{1 - A sz}{1 - Bsz} d\nu(s) = Q(-r) \quad (|z| \leq r < 1).
\]

Upon letting \( r \to 1^- \) in the above inequality, we obtain the assertion (4.9). Now, with the aid of the elementary inequality:
\[
\text{Re} \left( \frac{\omega^{1/t}}{t} \right) \geq (\text{Re}(\omega))^{1/t} \quad (\text{Re}(\omega) > 0; t \in \mathbb{N}),
\]
the estimate (4.7) follows from (4.10).

The estimate in (4.7) is the best possible as the function \( Q \) is the best dominant of (4.6). \(\square\)

Putting \( \alpha = p + \ell - p\lambda, \beta = \lambda \) and \( t = 1 \) in Theorem 4.3, we get the following result.

**Corollary 4.2.** If \( f \in A_p \) satisfies
\[
(1 - \mu) \frac{\mathcal{J}_p^m(\lambda, \ell, f(z))}{z^p} + \mu \frac{\mathcal{J}_p^{m+1}(\lambda, \ell, f(z))}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in U),
\]
then
\[
\text{Re} \left\{ \frac{\mathcal{J}_p^m(\lambda, \ell, f(z))}{z^p} \right\} > \varrho \quad (z \in U),
\]
where
\[
\varrho = \begin{cases} 
\frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 - B)^{-1} _2F_1 \left( 1, 1; \frac{p + \ell}{\mu\lambda n} + 1; \frac{B}{B - 1} \right), & B \neq 0 \\
1 - \frac{p + \ell}{p + \ell + \mu\lambda n} A, & B = 0,
\end{cases}
\]
The result is the best possible.

Setting \( m = -1, \alpha = \delta, \beta = 1, A = 1 - 2\eta \quad (0 \leq \eta < 1) \) and \( B = -1 \) in Theorem 4.3 we obtain

**Corollary 4.3.** If \( f \in A_p(n) \) satisfies
\[
(1 - \mu) \frac{\mathcal{J}_p^m(f(z))}{z^p} + \mu \frac{f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (\mu > 0, \delta > -p; z \in U),
\]
then
\[
\text{Re} \left( \frac{\mathcal{J}_p^m(f(z))}{z^p} \right) > \eta + (1 - \eta) \left\{ _2F_1 \left( 1, 1; \frac{\delta + p}{\mu n} + 1; \frac{1}{2} \right) - 1 \right\} \quad (z \in U).
\]
The result is the best possible.
**Theorem 4.4.** If \( \kappa = \frac{1 - A}{1 - B} \) and \( f \in A_p(n) \) satisfies the subordination condition:

\[
\frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),
\]

then

\[
\Re \left\{ (1 - \mu) \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} + \mu \frac{\Theta_p^{m+1}(n; \alpha, \beta) f(z)}{z^p} \right\} > \kappa
\]

for \( |z| < R = R(p, \alpha, \beta, \mu, n) \),

where

\[
R = \left[ \sqrt{\left( \frac{\alpha + p\beta}{\alpha + p\beta} \right)^2 + \left( \frac{\mu \beta n}{\alpha + p\beta} \right)^2} - \mu \beta n \right] \frac{1}{n}.
\]

The result is the best possible.

**Proof.** From (4.11), we note that

\[
\Re \left\{ \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} \right\} > \kappa \quad (z \in \mathbb{U})
\]

so that

\[
\frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} = \kappa + (1 - \kappa) \phi(z) \quad (z \in \mathbb{U}),
\]

where \( \phi \), given by (2.1) is analytic and has a positive real part in \( \mathbb{U} \). Taking logarithmic differentiation in both sides of (4.12), and using (1.6) in the resulting equation, we deduce that

\[
\Re \left\{ (1 - \mu) \frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} + \mu \frac{\Theta_p^{m+1}(n; \alpha, \beta) f(z)}{z^p} \right\} - \kappa
\]

\[
\geq (1 - \kappa) \left\{ \Re(\phi(z)) - \frac{\mu \beta n}{\alpha + p\beta} |z\phi'(z)| \right\} \quad (z \in \mathbb{U}).
\]

By using the estimate (i) of (3.13) in the above inequality and following the lines of proof of Theorem 3.2, we get the required assertion of Theorem 4.4.

It is easily seen that the bound \( R \) is the best possible for the function \( f \in A_p(n) \) defined by

\[
\frac{\Theta_p^m(n; \alpha, \beta) f(z)}{z^p} = \frac{1 + (1 - 2\kappa)z^n}{1 - z^n} \quad \left( \kappa = \frac{1 - A}{1 - B} ; z \in \mathbb{U} \right).
\]

\[\Box\]

Putting \( A = 1 - 2\eta, B = -1, m = -1, \alpha = \delta \) and \( \beta = 1 \) in Theorem 4.4 we get

**Corollary 4.4.** If \( \mu > 0, \delta > -p \) and \( f \in A_p(n) \) satisfies

\[
\Re \left( \frac{\mathcal{F}_\delta f(z)}{z^p} \right) > \eta \quad (0 \leq \eta < 1; z \in \mathbb{U}),
\]

then

\[
(1 - \mu) \frac{\mathcal{F}_\delta f(z)}{z^p} + \mu \frac{f(z)}{z^p} > \eta \quad \text{for} \quad |z| < R(p, \delta, \mu, n),
\]

where

\[
R(p, \delta, \mu, n) = \left[ \frac{\sqrt{(\delta + p)^2 + (\mu n)^2} - \mu n}{\delta + p} \right]^{\frac{1}{n}}.
\]
The bound $\tilde{R}(p, \delta, \mu, n)$ is the best possible for the function $f \in \mathcal{A}_p(n)$ defined by

\[
\frac{\mathcal{F}_{\delta,p}(f)(z)}{z^p} = \frac{1 + (1 - 2\eta)z^n}{1 - z^n} \quad (0 \leq \eta < 1, \delta > -p; z \in \mathbb{U}).
\]

**Theorem 4.5.** Let $\mu > 0$ and $\delta > -p$. Suppose that $f \in \mathcal{A}_p(n)$ and $\mathcal{F}_{\delta,p}(f)$ is given by (3.14). If

\[
(1 - \mu)\frac{\Theta_p^m(n; \alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{z^p} + \mu\frac{\Theta_p^m(n; \alpha, \beta)f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),
\]

then

\[\text{Re} \left\{ \left( \frac{\Theta_p^m(n; \alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{z^p} \right)^{1/t} \right\} > \xi^{1/t} \quad (t \in \mathbb{N}; z \in \mathbb{U}),\]

where

\[
\xi = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} \binom{1}{1; \frac{\delta + p}{\mu n} + 1} \left(1, \frac{B}{B - 1}\right), & B \neq 0 \\
1 - \frac{\delta + p}{\delta + p + \mu n}A, & B = 0,
\end{cases}
\]

The result is the best possible.

**Proof.** If we let

\[
\phi(z) = \frac{\Theta_p^m(n; \alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{z^p} \quad (z \in \mathbb{U}),
\]

then $\phi$ is of the form (2.1) and is analytic in $\mathbb{U}$. On differentiating both the sides of (4.14) and using (3.16) in conjunction with (4.13), we deduce that

\[
\phi(z) + \frac{z\phi'(z)}{(\delta + p)/\mu} = (1 - \mu)\frac{\Theta_p^m(n; \alpha, \beta)\mathcal{F}_{\delta,p}(f)(z)}{z^p} + \mu\frac{\Theta_p^m(n; \alpha, \beta)f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]

The remaining part of the proof of Theorem 4.5 is similar to that of Theorem 4.3 and we omit the details. \qed

Letting $A = 1 - 2\eta$, $B = -1$, $m = -1$, $\alpha = \delta$ and $\mu = \beta = t = 1$ in Theorem 4.4, we obtain

**Corollary 4.5.** Suppose that $f \in \mathcal{A}_p(n)$ and $\mathcal{F}_{\delta,p}(f)$ is given by (3.14). If

\[\text{Re} \left\{ \frac{\mathcal{F}_{\delta,p}(f)(z)}{z^p} \right\} > \eta \quad (0 \leq \eta < 1, \delta > -p; z \in \mathbb{U}),\]

then

\[\text{Re} \left( \int_0^z t^{\delta - 1} \frac{\mathcal{F}_{\delta,p}(f)(t)dt}{z^{\delta + p}} \right) > \eta + (1 - \eta) \left\{ \binom{1.5}{1; \frac{\delta + p}{\mu n} + 1; \frac{1}{2}} - 1 \right\} \quad (z \in \mathbb{U}).\]

The result is the best possible.

**Theorem 4.6.** Let $\mu > 0$ and $-1 \leq B_j < A_j \leq 1$ ($j = 1, 2$). If the functions $f_j \in \mathcal{A}_p$ satisfy the subordination condition (4.5) and

\[
\mathcal{H}(z) = \Theta_p^m(\alpha, \beta)(f_1 * f_2)(z) \quad (z \in \mathbb{U}),
\]

then

\[\text{Re} \left\{ (1 - \mu)\frac{\Theta_p^m(\alpha, \beta)\mathcal{H}(z)}{z^p} + \mu\frac{\Theta_p^{m+1}(\alpha, \beta)\mathcal{H}(z)}{z^p} \right\} > \eta \quad (0 \leq \eta < 1; z \in \mathbb{U}),
\]

(4.15)
where

\[ \eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ 1 - \frac{1}{2} \binom{1,1}{2} F_1 \left( \frac{\alpha + p\beta}{\mu\beta} + 1; \frac{1}{2} \right) \right\}. \]

The result is the best possible when \( B_1 = B_2 = -1 \).

**Proof.** On setting

\[ \phi_j(z) = (1 - \mu) \frac{\Theta_p^m(\alpha, \beta)f_j(z)}{z^p} + \mu \frac{\Theta_p^{m+1}(\alpha, \beta)f_j(z)}{z^p} \quad (j = 1, 2; z \in \mathbb{U}) \quad (4.16) \]

and using (4.5), we note that \( \phi_j \in \mathcal{P}(\gamma_j) \), where \( \gamma_j = (1 - A_j)/(1 - B_j) \) for \( j = 1, 2 \). Now, by making use of the identity (1.6) and (4.16), we deduce that

\[ \Theta_p^m(\alpha, \beta)f_j(z) = \frac{p + \alpha\beta}{\mu\beta} z^{p - \frac{p + \alpha\beta}{\mu\beta}} \int_0^z t^{\frac{\alpha + p\beta}{\mu\beta} - 1} \phi_j(t) dt \quad (j = 1, 2). \quad (4.17) \]

Thus, by making use of (4.17) followed by simple calculations, we obtain

\[ \Theta_p^n(\alpha, \beta)\mathcal{H}(z) = \frac{p + \alpha\beta}{\mu\beta} z^{p - \frac{p + \alpha\beta}{\mu\beta}} \int_0^z t^{\frac{\alpha + p\beta}{\mu\beta} - 1} \phi_0(t) dt \quad (z \in \mathbb{U}), \quad (4.18) \]

where

\[
\phi_0(z) = (1 - \mu) \frac{\Theta_p^m(\alpha, \beta)\mathcal{H}(z)}{z^p} + \mu \frac{\Theta_p^{m+1}(\alpha, \beta)\mathcal{H}(z)}{z^p} \\
= \frac{p + \alpha\beta}{\mu\beta} z^{p - \frac{p + \alpha\beta}{\mu\beta}} \int_0^z t^{\frac{\alpha + p\beta}{\mu\beta} - 1} (\phi_1 * \phi_2)(t) dt \quad (z \in \mathbb{U}). \quad (4.19)
\]

Since \( \phi_j \in \mathcal{P}(\gamma_j) \) for \( j = 1, 2 \), it follows from (4.17) that

\[ (\phi_1 * \phi_2) \in \mathcal{P}(\gamma_3) \quad (\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2)). \]

and the bound \( \gamma_3 \) is the best possible. Hence, by using Lemma 2.2 in (4.19), we deduce that

\[
\text{Re}\{\phi_0(z)\} = \frac{p + \alpha\beta}{\mu\beta} \int_0^1 s^{\frac{p + \alpha\beta}{\mu\beta} - 1} \text{Re}\{(\phi_1 * \phi_2)(sz)\} ds \\
\geq \frac{p + \alpha\beta}{\mu\beta} \int_0^1 s^{\frac{p + \alpha\beta}{\mu\beta} - 1} \left( 2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + s|z|} \right) ds \\
> \frac{\alpha + p\beta}{\mu\beta} \int_0^1 s^{\frac{p + \alpha\beta}{\mu\beta} - 1} \left( 2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + s} \right) ds \\
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left( 1 - \frac{\alpha + p\beta}{\mu\beta} \int_0^1 s^{\frac{p + \alpha\beta}{\mu\beta} - 1} ds \right) \\
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left\{ 1 - \frac{1}{2} \binom{1,1}{2} F_1 \left( \frac{\alpha + p\beta}{\mu\beta} + 1; \frac{1}{2} \right) \right\} \\
= \eta \quad (z \in \mathbb{U}).
\]

When \( B_1 = B_2 = -1 \), we consider the functions \( f_j \in A_p \) satisfying the hypothesis (4.5) and defined by

\[ \Theta_p^m(\alpha, \beta)f_j(z) = \frac{p + \alpha\beta}{\mu\beta} z^{p - \frac{p + \alpha\beta}{\mu\beta}} \int_0^z t^{\frac{\alpha + p\beta}{\mu\beta} - 1} \left( \frac{1 + A_1t}{1 - t} \right) dt \quad (j = 1, 2; z \in \mathbb{U}). \]
It follows from (4.19) and Lemma 2.2 that
\[
\phi_0(z) = \frac{p + \alpha \beta}{\mu \beta} z^p \frac{z^{\alpha + \beta}}{\alpha + \beta - 1} \int_0^1 t^{\alpha + \beta - 1} \left( 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - sz} \right) ds
\]
\[
= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \binom{2}{1,1} \left( 1; \frac{\alpha + p \beta}{\mu \beta} + 1; \frac{z}{z - 1} \right)
\]
\[
\to 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \binom{2}{1,1} \left( 1; \frac{\alpha + p \beta}{\mu \beta} + 1; \frac{1}{2} \right)
\]
as \(z \to -1\), which evidently completes the proof of Theorem 4.6.

5. Majorization Properties

In this section, we establish majorization properties of functions belonging to the class \(S_p^m(\alpha, \beta, A, B)\).

Theorem 5.1. Let the function \(f \in A_p\), and suppose that the function \(g \in S_p^m(\alpha, \beta, A, B)\). If
\[
\Theta_p^{m+1}(\alpha, \beta) f(z) \preceq \Theta_p^{m+1}(\alpha, \beta) g(z) \quad (z \in \mathbb{U}),
\]
then
\[
|\Theta_p^{m+1}(\alpha, \beta) f(z)| \leq |\Theta_p^{m+1}(\alpha, \beta) g(z)| \quad (|z| < \tilde{r}),
\]
where \(\tilde{r} = \tilde{r}(p, \alpha, \beta, A, B)\) is the smallest positive root of the equation
\[
(\alpha + p \beta)|A|r^3 - ((\alpha + p \beta) + 2\beta |B|)r^2 - ((\alpha + p \beta)|A| + 2\beta r + (\alpha + p \beta) = 0.
\]

Proof. Since \(g \in S_p^m(\alpha, \beta, A, B)\), it follows that
\[
\Theta_p^{m+1}(\alpha, \beta) g(z) = \left( \frac{1 + B w(z)}{1 + A w(z)} \right) \Theta_p^{m+2}(\alpha, \beta) g(z) \quad (z \in \mathbb{U}),
\]
where \(w\) is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| \leq 1\) for all \(z \in \mathbb{U}\). Thus,
\[
|\Theta_p^m(\alpha, \beta) g(z)| \leq \frac{1 + |B||z|}{1 - |A||z|} |\Theta_p^{m+1}(\alpha, \beta) g(z)|.
\]

From (5.2) with the aid of (5.1), we get
\[
\Theta_p^{m+1}(\alpha, \beta) f(z) = \varphi(z) \Theta_p^{m+1}(\alpha, \beta) g(z) \quad (z \in \mathbb{U}),
\]
where \(\varphi\) is analytic in \(\mathbb{U}\) and \(|\varphi(z)| \leq 1\) in \(\mathbb{U}\). Differentiating both the sides of (5.5) and using the identity (1.6) for the functions \(f\) and \(g\) in the resulting equation, we deduce that
\[
|\Theta_p^{m+2}(\alpha, \beta) f(z)| \leq |\varphi(z)| \left| \Theta_p^{m+2}(\alpha, \beta) g(z) \right| + \frac{\beta |z|}{\alpha + p \beta} \left| \varphi'(z) \right| \left| \Theta_p^{m+1}(\alpha, \beta) g(z) \right|.
\]

Now, by using the following estimate [23]
\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U})
\]
and (5.4) in (5.6), we get
\[
|\Theta_p^{m+2}(\alpha, \beta) f(z)| \leq \left\{ |\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \frac{\beta (1 + |B||z|)|z|}{(\alpha + p \beta)(1 - |A||z|)} \right\} \left| \Theta_p^{m+2}(\alpha, \beta) g(z) \right|
\]
which upon setting \(|z| = r\) and \(|\phi(z)| = x (0 \leq x \leq 1)\) yields the inequality
\[
|\Theta_p^{m+1}(\alpha, \beta) f(z)| \leq \frac{\Psi(x)}{(1 - r^2)((\alpha + p \beta)(1 - |A|r))} \left| \Theta_p^{m+1}(\alpha, \beta) g(z) \right|,
\]
where
\[
\Psi(x) = -\beta r(1 + |B|r)x^2 + (\alpha + p\beta)(1 - r^2)(1 - |A|r)x + \beta r(1 + |B|r).
\]
We note that the function \(\Psi\) takes its maximum value at \(x = 1\) with \(\tilde{r} = \tilde{r}(p, \alpha, \beta, A, B)\) the smallest positive root of the equation (5.3). Furthermore, if \(0 \leq y \leq \tilde{r}(p, \alpha, \beta, A, B)\), then the function
\[
\Psi(x) = -\beta y(1 + |B|y)x^2 + (\alpha + p\beta)(1 - y^2)(1 - |A|y)x + \beta y(1 + |B|y)
\]
increases in the interval \(0 \leq x \leq 1\), so that
\[
\Psi(x) \leq \Psi(1) = (\alpha + p\beta)(1 - y^2)(1 - |A|y).
\]
Thus, in view of the above fact and (5.7), we get the assertion (5.2) of Theorem 5.1 \(\Box\)

Letting \(\alpha = p + \ell - p\lambda\) and \(\beta = \lambda\) in Theorem 5.1 we have

**Corollary 5.1.** Let the function \(f \in A_p\) and the function \(g \in S^m_p(\alpha, \beta, A, B)\) satisfies
\[
\Re \left\{ \frac{\mathcal{J}_p^{m+2}(\lambda, \ell)g(z)}{\mathcal{J}_p^{m+1}(\lambda, \ell)g(z)} \right\} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).
\]
If \(f(z) \ll g(z)\) in \(\mathbb{U}\), then
\[
|\mathcal{J}_p^{m+2}(\lambda, \ell)f(z)| \leq |\mathcal{J}_p^{m+2}(\lambda, \ell)g(z)| \quad \text{for} \quad |z| \leq \tilde{r}(p, \ell, \lambda, A, B),
\]
where \(\tilde{r}(p, \ell, \lambda, A, B)\) is the smallest positive root of the equation
\[
(p + \ell)|A|r^3 - (p + \ell + \lambda|B||r^2 - ((p + \ell)|A| + 2\lambda)r + p + \ell = 0.
\]

**Remark 5.1.** Setting \(A = 1 - (2\eta/p), B = -1\) and \(m = -1\alpha = 0\) in Theorem 5.1 equation (5.3) becomes
\[
|p - 2\eta|r^3 - (p + 2)r^2 - (|p - 2\eta| + 2)r + p = 0.
\]
It is easily seen that \(r = -1\) is a solution of the above equation and the other two roots can be obtained by solving
\[
|p - 2\eta|r^2 - (|p - 2\eta| + p + 2)r + p = 0. \quad (5.8)
\]
So, we can easily find the smallest positive root of (5.8).

In view of the above remark, we deduce the following result which, in turn, yields the corresponding work of MacGregor [26] p.96, Theorem 1B] for \(p = 1\) and \(\eta = 0\).

**Corollary 5.2.** Let the functions \(f \in A_p\) and \(g \in S^*_p(\eta)\) (\(0 \leq \eta < p\)) be such that
\[
f(z) \ll g(z) \quad (z \in \mathbb{U}),
\]
then
\[
|f'(z)| \leq |g'(z)| \quad \text{for} \quad |z| \leq \tilde{r}(p, \eta),
\]
where
\[
\tilde{r}(p, \eta) = \begin{cases} 
\frac{p + |p - 2\eta| + 2 - \sqrt{(p + |p - 2\eta| + 2)^2 - 4p|p - 2\eta|}}{2|p - 2\eta|}, & \eta \neq \frac{p}{2} \\
\frac{p}{p + 2}, & \eta = \frac{p}{2}.
\end{cases}
\]
Letting $m = -1$, $A = 1 - (2\eta/p)$ and $B = -1$ in Corollary 5.2, it follows that for $(p - 1)/2 \leq \eta < p$

\[ \mathcal{C}_p(\eta) \subset S_p^*(\rho), \quad (5.9) \]

where

\[ \rho = p \left\{ \frac{2F_1 \left( 1, 2(p - \eta); p + 1; \frac{1}{2} \right)}{2F_1 \left( 1, 2(p - \eta); p + 1; \frac{1}{2} \right)} \right\}^{-1}. \quad (5.10) \]

Thus, in view of (5.9) and Corollary 5.2, we obtain the following result which, in turn, give the corresponding work by MacGregor [26, p.96, Theorem 1C], for $p = 1$ and $\eta = 0$.

**Corollary 5.3.** Let $(p - 1)/2 \leq \eta < p$. If $f \in A_p$ and $g \in \mathcal{C}_p(\eta)$ satisfy

\[ f(z) \ll g(z) \quad (z \in \mathbb{U}), \]

then

\[ |f'(z)| \leq |g'(z)| \quad \text{for} \quad |z| < \tilde{r}(p, \rho), \]

where

\[ \tilde{r}(p, \rho) = \begin{cases} \frac{p + |p - 2\rho| + 2 - \sqrt{(p + |p - 2\rho| + 2)^2 - 4p|p - 2\rho|}}{2|p - 2\rho|}, & \rho \neq \frac{p}{2} \\ \frac{p}{p + 2}, & \rho = \frac{p}{2} \end{cases} \]

and $\rho$ is given by (5.10).

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