The fractional quantum Hall effect in infinite layer systems

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Stacked two dimensional electron systems in transverse magnetic fields exhibit three dimensional fractional quantum Hall phases. We analyze the simplest such phases and find novel bulk properties, e.g., irrational braiding. These phases host “one and a half” dimensional surface phases in which motion in one direction is chiral. We offer a general analysis of conduction in the latter by combining sum rule and renormalization group arguments, and find that when interlayer tunneling is marginal or irrelevant they are chiral semi-metals that conduct only at $T > 0$ or with disorder.

Low dimensional electron systems exhibit striking examples of strongly correlated behavior. In the search for higher dimensional analogs, a tempting strategy is to couple such systems weakly in the hope of achieving a “dimensional continuation” of the strong correlation physics $^1$. In this letter we report our results on several aspects of the continuation of the fractional quantum Hall effect to a three dimensional setting. The central idea behind such a continuation is this: in the absence of interlayer tunneling, an infinite stack of two dimensional electron systems can exhibit “multi-component” quantum Hall behavior which generalizes the possibility already known to be realized in bilayers or systems where spin is important $^2$. Such behavior will be accompanied by a gap which will then allow a weak tunneling to be turned on without destroying continuity. This route to three dimensional quantum Hall phases comes with an added bonus, namely that the chiral edge states existing in each layer will hybridize and yield a family of “one and a half” dimensional phases that will live on the surfaces of the three dimensional systems and exhibit interesting transport in the direction transverse to the layers.

In the context of the integer effect, this possibility was first explored experimentally $^3$ and has more recently been studied systematically by both theory $^4$ and experiment $^5$ leading not only to a demonstration of three dimensional quantum Hall behavior but also of the formation of a chiral metal at its surface via hybridization of the (Fermi liquid) edge states of the individual layers. The extension of fractional QH behavior has attracted less attention. Balents and Fisher $^6$ commented on the edge dynamics of uncorrelated fractional layers but more central to many of our concerns is the very early work of Qiu, Joynt, and MacDonald (QJM) $^7$ on the possibility of interlayer correlated states and the evaluation of their energetic stability.

We begin by describing the states that we study, and note some unusual features of the bulk physics such as non-trivial quasiparticle structure, irrationally distributed charge and a corresponding statistics. Next we formulate the edge theory for these states and show that in the clean limit a very general sum rule argument can be used to exactly determine the conductivity along the stack (the $z$-axis), $\sigma^{zz}$. This argument implies metallic behavior at finite temperature and disorder in cases where the tunneling is relevant and semi-metallic behavior in cases where it is marginal or irrelevant. This connection relies on a rigid correspondence between RG relevance and the form of the ground state which is specific to chiral systems. We illustrate the general argument by computations of the conductivity at weak tunneling in which the disorder is treated exactly. In much of the analysis we will be particularly interested in the QJM “131” state (defined below) which is perhaps the simplest correlated infinite layer state. Technical details of most assertions and related material on other chiral many-body systems will follow in a separate publication $^8$.

**States:** Consider a system which consists of $N$ parallel layers of 2DEGs in a strong perpendicular magnetic field that we assume freezes out any interesting spin dynamics as well. We will be interested in the generalized Laughlin-Halperin states,

$$
\Psi_\nu(\{z_{i\alpha}\}) = \prod_{i=1}^{N} \prod_{\alpha<\beta} (z_{i\alpha} - z_{i\beta})^{K_{ii}} \times \prod_{i<j} \prod_{\alpha=1}^{N_i} \prod_{\beta=1}^{N_j} (z_{i\alpha} - z_{j\beta})^{K_{ij}} e^{-\sum_{\alpha,i} |z_{i\alpha}|^2/4}.
$$

(1)

Here $z_{i\alpha}$ is the coordinate of electron $\alpha$ in layer $i$, and $N_i$ is the number of electrons in layer $i$. The exponents are specified by a symmetric $N \times N$ matrix $K$, which we will take to be tridiagonal $K_{ij} = m \delta_{ij} + n(\delta_{i,j-1} + \delta_{i,j+1})$ in the interests of simplicity and plausibility. Clearly, the diagonal elements determine the intralayer and the off-diagonal elements specify the interlayer correlations. In the large $N$ limit of interest, it is convenient to assume periodic boundary conditions in the $z$-direction $^8$ whereupon the filling factor in each layer of this “nnnn” state is $\nu = 1/(m+2n)$. Evidently, more than one state can occur at the same filling. The competition between the states 050 and 131 at filling $\nu = 1/5$ per layer is especially interesting as the simplest example of a potential...
transition between interlayer uncorrelated and correlated states. We note that QJM found that 050 gives way to 131 as the interlayer separation was decreased. In what follows we will impose one further restriction in order to obtain the simplest edge dynamics—we will require that the states give rise only to co-propagating edge modes, which is equivalent to requiring that $K$ posses only positive eigenvalues or that $n < m/2$.

**Quasiparticles:** By analogy to the single-layer quantum Hall effect we can construct a state with a single quasihole at point $\xi$ in layer $j$ as:

$$
\Psi_{(j,\xi)}(\{z_{i\alpha}\}) = \prod_{\alpha=1}^{N_j} (z_{j\alpha} - \xi) \Psi_0(\{z_{i\alpha}\}).
$$

A standard plasma screening argument yields a total charge of $Q = \sum_k Q_k^{(j)} = \sum_k (K^{-1})_{jk} = \nu_j = 1/(m+2n)$ for the quasiholes, where $Q_k^{(j)}$ is the charge in layer $k$ due to the quasiholes in layer $j$. The distribution of this charge, first noted by QJM, is rather more interesting:

$$
Q_k^{(j)} = \frac{1}{\sqrt{m^2 - 4n^2}} \left( \frac{\sqrt{m^2 - 4n^2} - m}{2n} \right)^{|k-j|}.
$$

For interlayer correlated states ($n > 0$), the individual charges in the layers are irrational. This result, for an isolated quasihole, is stable to the inclusion of weak tunneling.

A closer look at the plasma screening computation yields an interesting structure for the quasiholes in that the charge in layers farther from the nominal location of the quasiholes, though smaller, is spread over larger areas. For example, for the 131 state, the mean squared radius goes as $\langle r_j^2 \rangle = 4 (3 + \sqrt{5})/5$ for a quasihole centered in layer 0. Finally, we also readily obtain the braiding phase for a quasihole in layer $k$ as it encircles the planar position of a quasihole in layer $j$,

$$
\gamma_{B}^{(stat)} = -2\pi Q_k^{(j)},
$$

which is again irrational.

**Edge Theory:** The edge theory of the $N$-layer quantum Hall state contains $N$ chiral bosons $u_i(x)$ whose commutation relations are determined by the same matrix $K$ which specifies the correlation exponents in the bulk wavefunction:

$$
[u_i(x), u_j(x')] = i\pi K_{ij} \text{sgn}(x - x').
$$

In the absence of tunneling the low-energy effective Hamiltonian of the edge theory is

$$
\mathcal{H}_0 = \int_{-L/2}^{L/2} dx \frac{1}{4\pi} V_{ij} \partial_x u_i \partial_x u_j,
$$

where $L$ is the length of the edge and $V$ is a symmetric, positive definite, $N \times N$ matrix which depends on the interactions and confining potentials at the edge. The electron annihilation operator at the edge of layer $i$ is $\Psi_i(x) \propto e^{iu_i(x)}$, and the Hamiltonian density for tunneling between layers $i$ and $i+1$ is $\lambda \Psi_i(x) \Psi_{i+1}(x) + \text{h.c.}$, where $\lambda$ is the tunneling amplitude which we take to be uniform along the edge. The lowest-order perturbative RG flow of the tunneling amplitude for the $nnn$ state is

$$
\frac{d\lambda}{dl} = (2 - m + n) \lambda.
$$

Combining this result with the condition for maximum chirality, $n < m/2$, we obtain the diagram in Fig. 1. There are only two maximally chiral multilayer states for which interlayer electron tunneling is not irrelevant: 010 with relevant tunneling and 131 with marginal tunneling. (Also of interest are the bosonic 020 state with marginal tunneling and the states 121 and 242 which have positive semi-definite $K$ matrices and hence a non-trivial mixing of edge and bulk excitations [9].)

**FIG. 1.** The $m$-$n$ plane. States below the solid line are maximally chiral. The dashed line separates states with irrelevant tunneling (below) from states with relevant tunneling (above). The shaded region contains all maximally-chiral states with non-irrelevant tunneling: 010 and 131.

**Sum Rule and $\sigma^{zz}$:** As we mentioned above, the case of relevant tunneling, 010, has been previously studied, so we will concentrate on the case of marginal tunneling, 131. First consider the multilayer edge theory in the absence of disorder. By evaluating the double commutator of the full Hamiltonian (including tunneling) with the $z$-axis Fourier transform of the charge density operator $\rho_i(x) = (K^{-1})_{ij} \partial_x u_j(x)/2\pi$ integrated over all $x$, we can derive the following exact sum rule

$$
\int d\omega \Re \sigma^{zz}(\omega) = -\frac{\pi d}{NL} (\mathcal{H}_\lambda),
$$

where $d$ is the interlayer separation and $\mathcal{H}_\lambda$ is the tunneling part of the Hamiltonian.

In the absence of tunneling, the ground state of the Hamiltonian $\mathcal{H}_0$ is known, and the expectation value on the r.h.s. of the sum rule is zero. The marginality of the tunneling perturbation for the 131 state implies that there exists a finite range of tunneling amplitudes $\lambda$ for which the ground state is stable, which can be understood as follows. The tunneling Hamiltonian $\mathcal{H}_\lambda$ com-
mutes with $H_0$, and therefore first-order degenerate perturbation theory is exact, and the eigenvalues of $H_0 + H_\lambda$ are linear functions of $\lambda$. The case of marginal tunneling corresponds to a dimensionless $\lambda$, and hence the energy of the first excited state behaves like $E_1(\lambda) - E_1(0) \sim \lambda/L$. If $E_0(\lambda)$ is the ground state energy, then $E_1(\lambda) - E_0(\lambda)$ must be $\sim 1/L$ since the edge theory is gapless, and we can conclude that the first level crossing occurs at $\lambda \sim 1$. Therefore, for a finite range of $\lambda$ the ground state for $\lambda \neq 0$ is identical to the ground state for $\lambda = 0$. In contrast, for the case of relevant tunneling, $\lambda$ has the dimensions of an energy and an analogous line of reasoning shows that the first level crossing occurs at a value of $\lambda$ that approaches zero as the system size $L$ is taken to infinity, indicating the ground state is not stable. Note that these arguments are non-perturbative in the tunneling amplitude. This reasoning can be made precise, and the region of stability in $\lambda$ bounded.

Combining the stability of the ground state, the sum rule, and the general condition that $\Re \sigma^{zz}(\omega)$ is positive semi-definite, we can conclude that at $T = 0$: $\sigma^{zz}(\omega) = 0$ for all $\omega$ [11]. In the absence of disorder the surface of the 131 state exhibits insulating behavior in the direction perpendicular to the layers. The presence of either a finite temperature or disorder would make the expectation value on the r.h.s. of the sum rule non-zero, and therefore lead to a non-vanishing conductivity. This property, in particular that adding disorder increases the conductivity, is reminiscent of a semi-metal and hence the surface of the 131 state may be considered a “chiral semi-metal.”

**$\sigma^{zz}$ with disorder:** To illustrate the above claim, consider adding disorder to the multilayer edge theory in the form of random scalar potentials $V_i(x)$ which couple to the edge charge density $\rho_i(x)$ in each layer. We assume $V_i(x)$ is a Gaussian random variable uncorrelated between different layers, i.e., $\langle V_i(x)V_j(x') \rangle = \delta_{ij}Z(x-x')$, where the overbar denotes disorder averaging. The conductivity is evaluated via a Kubo formula. Since the Hamiltonian with tunneling is a theory of $N$ interacting chiral bosons, the calculation is performed perturbatively in $\lambda$, but the disorder is treated exactly. We expect that for the disordered case the lowest-order result in $\lambda$ is reliable [1]. To illustrate our results it is especially convenient to use a specific short-ranged disorder potential correlator $Z(x)$ characterized by a single energy $\Delta$ which determines the strength of the disorder and the correlation length of the random potential along the edges $\ell_0 \sim 1/\Delta$. It is also useful to choose the matrix $V$ appearing in $H_0$ proportional to $K^{-1}$. Physically, this corresponds to a specific non-zero value of $g$, the nearest-neighbor density-density coupling between the layers. With these choices we find,

$$\Re \sigma^{zz}(\omega) = \frac{\lambda^2 d^2}{48\pi^2} \frac{\omega^2 + 4\pi^2 T^2}{\Delta \cosh(\pi\omega/2\Delta)}. \quad (9)$$

This result is shown in Fig. 2 at various temperatures.

At zero temperature the conductivity vanishes in the DC limit. At low temperatures the maximum occurs at a finite frequency, while at higher temperatures the conductivity is peaked around $\omega = 0$. It can be shown that the large and small frequency asymptotics of the above result for $\sigma^{zz}(\omega)$ are not significantly modified by perturbing the matrix $V$ away from this point [7].

**FIG. 2.** The real part of the vertical conductivity for the 131 multilayer with disorder [11]. The horizontal axis is frequency measured in units of the disorder parameter $\Delta$. The temperatures of the curves, starting with the uppermost, are: $T/\Delta = 0.25, 0.2, 0.15, 0.1, 0.01$.

**Clean limit:** The 131 state in the clean ($\Delta = 0$) limit is characterized by two dimensionless parameters, $\lambda$ and $g$, and is therefore quantum critical. Its finite temperature conductivity is thus of particular interest as there are very few cases where the transport in such a regime is reliable [7]. To illustrate our results it is necessary to define the question about the finiteness of the DC conductivity beyond $O(\lambda^2)$, we have proven that at higher orders in $\lambda$ the conductivity cannot vanish at all $\omega \neq 0$. The proof proceeds by showing that $\sigma^{zz}(\omega)$ is zero for all $\omega \neq 0$ if and only if the current operator $I^z$ commutes with the Hamiltonian, and then demonstrating that for the 131 state this commutator is non-vanishing [1]. On general grounds we expect that the $\delta(\omega)$ factor in Eq. (10) is broadened by both higher-order corrections in the tunneling $\lambda$, and nearest-neighbor density-density interactions $g$, but the exact form remains to be found.

**Irrelevant Cases:** The perturbative calculation of the $z$-axis conductivity in the presence of tunneling and disorder can be extended to states with irrelevant tunneling,
i.e., $nmn$ states with $m > n + 2$. These states exhibit similar behavior to the marginal case of the 131 state, in particular, in the absence of disorder they are perfectly insulating in the $z$-direction at $T = 0$. In the presence of disorder we find $\bar{\sigma}_{zz} \sim \omega^\alpha$ as $\omega \to 0$ at zero temperature and $\bar{\sigma}_{zz} \sim T^\alpha$ as $T \to 0$ at zero frequency, where the exponent $\alpha = 2(m - n - 1)$. For the case of states without interlayer correlations, $n = 0$, this temperature scaling of the DC conductivity is different from that found by Balents and Fisher [4] who considered the case where interlayer correlations, $n \neq 0$, this temperature scaling depends on the dimensionful parameter $\Delta$. 

**050 versus 131:** Note that in the presence of disorder the $z$-axis DC conductivity scales as $T^8$ for the 050 state, while for the 131 state it scales as $T^2$. These states occur at the same electron density, $\nu = 1/5$ per layer. In their numerical work, QJM found a phase transition between 050 versus 131: the dimensionful parameter $\Delta$. The significant difference in the temperature scaling arises because in the clean case the scattering is determined by a dimensionless interaction strength $g$, while in the disordered case the scattering depends on the dimensionful parameter $\Delta$.

**Spectrum:** In our analysis of the surface conduction we have been fortunate to make progress without detailed knowledge of the excitation spectrum. The latter would clearly be desirable in the cases where tunneling is not irrelevant. We digress to remark that in the simplest non-trivial case of $N = 2$ layers, the edge theory in the presence of tunneling and disorder can be solved exactly via fermionization [2]. In the bilayer case the interesting experimental probe is not the $z$-axis conductivity but rather the collective mode structure, and hence their solution would appear to be (half) a step up from the solution of strictly one dimensional problems.

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