A Class of Special Matrices and Quantum Entanglement

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Abstract

We present a kind of construction for a class of special matrices with at most two different eigenvalues, in terms of some interesting multiplicators which are very useful in calculating eigenvalue polynomials of these matrices. This class of matrices defines a special kind of quantum states — $d$-computable states. The entanglement of formation for a large class of quantum mixed states is explicitly presented.

Keywords: Entanglement of formation, Generalized concurrence, $d$-computable states

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1 Introduction

Quantum entangled states are playing an important role in quantum communication, information processing and quantum computing [1], especially in the investigation of quantum teleportation [2, 3], dense coding [5], decoherence in quantum computers and the evaluation of quantum cryptographic schemes [6]. To quantify entanglement, a number of entanglement measures such as the entanglement of formation and distillation [7, 8, 9], negativity [10, 11], relative entropy [9, 12] have been proposed for bipartite states [6, 8] [11-13]. Most of these measures of entanglement involve extremizations which are difficult to handle analytically. For instance the entanglement of formation [7] is intended to quantify the amount of quantum communication required to create a given state. The entanglement of formation for a
A pair of qubits can be expressed as a monotonically increasing function of the “concurrence”, which can be taken as a measure of entanglement in its own right [14]. From the expression of this concurrence, the entanglement of formation for mixed states of a pair of qubits is calculated [14]. Although entanglement of formation is defined for arbitrary dimension, so far no explicit analytic formulae for entanglement of formation have been found for systems larger than a pair of qubits, except for some special symmetric states [15].

For a multipartite quantum system, the degree of entanglement will neither increase nor decrease under local unitary transformations on a quantum subsystem. Therefore the measure of entanglement must be an invariant of local unitary transformations. The entanglements have been studied in the view of this kind of invariants and a generalized formula of concurrence for high dimensional bipartite and multipartite systems is derived from the relations among these invariants [16]. The generalized concurrence can be used to deduce necessary and sufficient separability conditions for some high dimensional mixed states [17]. However in general the generalized concurrence is not a suitable measure for \( N \)-dimensional bipartite quantum pure states, except for \( N = 2 \). Therefore it does not help in calculating the entanglement of formation for bipartite mixed states.

Nevertheless in [18] it has been shown that for some class of quantum states with \( N > 2 \), the corresponding entanglement of formation is a monotonically increasing function of a generalized concurrence, and the entanglement of formation can be also calculated analytically. Let \( \mathcal{H} \) be an \( N \)-dimensional complex Hilbert space with orthonormal basis \( e_i, i = 1, ..., N \). A general bipartite pure state on \( \mathcal{H} \otimes \mathcal{H} \) is of the form,

\[
|\psi> = \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j, \quad a_{ij} \in \mathbb{C}
\]  

(1)

with normalization \( \sum_{i,j=1}^{N} a_{ij} a_{ij}^* = 1 \). The entanglement of formation \( E \) is defined as the entropy of either of the two sub-Hilbert spaces [8],

\[
E(|\psi>) = -\text{Tr} (\rho_1 \log_2 \rho_1) = -\text{Tr} (\rho_2 \log_2 \rho_2),
\]  

(2)

where \( \rho_1 \) (resp. \( \rho_2 \)) is the partial trace of \( |\psi><\psi| \) over the first (resp. second) Hilbert space of \( \mathcal{H} \otimes \mathcal{H} \). Let \( A \) denote the matrix with entries given by \( a_{ij} \) in (1). \( \rho_1 \) can be expressed as \( \rho_1 = AA^\dagger \).
The quantum mixed states are described by density matrices $\rho$ on $\mathcal{H} \otimes \mathcal{H}$, with pure-state decompositions, i.e., all ensembles of states $|\psi_i\rangle$ of the form (1) with probabilities $p_i \geq 0$, $\rho = \sum_{i=1}^{l} p_i |\psi_i\rangle \langle \psi_i|$, $\sum_{i=1}^{l} p_i = 1$ for some $l \in \mathbb{N}$. The entanglement of formation for the mixed state $\rho$ is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of $\rho$, $E(\rho) = \min \sum_{i=1}^{l} p_i E(|\psi_i\rangle)$.

For $N = 2$ equation (2) can be written as $E(|\psi\rangle)|_{N=2} = h((1 + \sqrt{1 - C^2})/2)$, where $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. $C$ is called concurrence, $C(|\psi\rangle) = 2|a_{11}a_{22} - a_{12}a_{21}|$ [14]. $E$ is a monotonically increasing function of $C$ and therefore $C$ can be also taken as a kind of measure of entanglement. Calculating $E(\rho)$ is then reduced to the calculation of the corresponding minimum of $C(\rho) = \min \sum_{i=1}^{M} p_i C(|\psi_i\rangle)$, which simplifies the problems, as $C(|\psi_i\rangle)$ has a much simpler expression than $E(|\psi_i\rangle)$.

For $N \geq 3$, there is no such concurrence $C$ in general. The concurrences discussed in [16] can be only used to judge whether a pure state is separable (or maximally entangled) or not [17]. The entanglement of formation is no longer a monotonically increasing function of these concurrences.

Nevertheless, for a special class of quantum states such that $AA^\dagger$ has only two non-zero eigenvalues, a kind of generalized concurrence has been found to simplify the calculation of the corresponding entanglement of formation [18]. Let $\lambda_1$ (resp. $\lambda_2$) be the two non-zero eigenvalues of $AA^\dagger$ with degeneracy $n$ (resp. $m$), $n + m \leq N$, and $D$ the maximal non-zero diagonal determinant, $D = \lambda_1^n \lambda_2^m$. In this case the entanglement of formation of $|\psi\rangle$ is given by $E(|\psi\rangle) = -n \lambda_1 \log_2 \lambda_1 - m \lambda_2 \log_2 \lambda_2$. It is straightforward to show that $E(|\psi\rangle)$ is a monotonically increasing function of $D$ and hence $D$ is a kind of measure of entanglement in this case. In particular for the case $n = m > 1$, we have

$$E(|\psi\rangle) = n \left(-x \log_2 x - \left(\frac{1}{n} - x\right) \log_2 \left(\frac{1}{n} - x\right)\right),$$

where

$$x = \frac{1}{2} \left(\frac{1}{n} + \sqrt{\frac{1}{n^2(1 - d^2)}}\right)$$

and

$$d \equiv 2nD^{1/n} = 2n \sqrt{\lambda_1 \lambda_2}.$$ 

$d$ is defined to be the generalized concurrence in this case. Instead of calculating $E(\rho)$
directly, one may calculate the minimum decomposition of $D(\rho)$ or $d(\rho)$ to simplify the calculations. In [18] a class of pure states (1) with the matrix $A$ given by

$$A = \begin{pmatrix} 0 & b & a_1 & b_1 \\ -b & 0 & c_1 & d_1 \\ a_1 & c_1 & 0 & -e \\ b_1 & d_1 & e & 0 \end{pmatrix},$$

(5)
a_1, b_1, c_1, d_1, b, e \in \mathbb{C}, is considered. The matrix $AA^\dagger$ has two eigenvalues with degeneracy two, i.e., $n = m = 2$ and $|AA^\dagger| = |b_1c_1 - a_1d_1 + be|^4$. The generalized concurrence $d$ is given by $d = 4|b_1c_1 - a_1d_1 + be|$. Let $p$ be a $16 \times 16$ matrix with only non-zero entries $p_{1,16} = p_{2,15} = -p_{3,14} = p_{4,10} = p_{5,12} = p_{6,11} = p_{7,13} = -p_{8,8} = -p_{9,9} = p_{10,4} = p_{11,6} = p_{12,5} = p_{13,7} = -p_{14,3} = p_{15,2} = p_{16,1} = 1$. $d$ can be further written as

$$d = |\langle \psi | p \psi^* \rangle|.$$

(6)

Let $\Psi$ denote the set of pure states (1) with $A$ given as the form of (5). Consider all mixed states with density matrix $\rho$ such that its decompositions are of the form

$$\rho = \sum_{i=1}^{M} p_i |\psi_i\rangle \langle \psi_i|, \quad \sum_{i=1}^{M} p_i = 1, \quad |\psi_i\rangle \in \Psi.$$

(7)

All other kind of decompositions, say decomposition from $|\psi'_i\rangle$, can be obtained from a unitary linear combination of $|\psi_i\rangle$ [14, 18]. As linear combinations of $|\psi_i\rangle$ do not change the form of the corresponding matrices (5), once $\rho$ has a decomposition with all $|\psi_i\rangle \in \Psi$, all other decompositions, including the minimum decomposition of the entanglement of formation, also satisfy that $|\psi'_i\rangle \in \Psi$. Then the minimum decomposition of the generalized concurrence is [18]

$$d(\rho) = \Lambda_1 - \sum_{i=2}^{16} \Lambda_i,$$

(8)

where $\Lambda_i$, in decreasing order, are the square roots of the eigenvalues of the Hermitian matrix $R \equiv \sqrt{\rho pp^* p} \sqrt{\rho}$, or, alternatively, the square roots of the eigenvalues of the non-Hermitian matrix $\rho pp^* p$.

2 Entanglement of formation for a class of high dimensional quantum states

An important fact in obtaining the formula (8) is that the generalized concurrence $d$ is a quadratic form of the entries of the matrix $A$, so that $d$ can be expressed in the form of
In terms of a suitable matrix $p$. Generalizing to the $N$-dimensional case we call an pure state (1) $d$-computable if $A$ satisfies the following relations:

\[ |AA\dagger| = ([A][A]^*)^{N/2}, \]
\[ |AA\dagger - \lambda I d_N| = (\lambda^2 - \|A\| \lambda + [A][A]^*)^{N/2}, \]

where $[A]$ and $\|A\|$ are quadratic forms of $a_{ij}$, $I d_N$ is the $N \times N$ identity matrix. We denote $\mathcal{A}$ the set of matrices satisfying (9), which implies that for $A \in \mathcal{A}$, $AA\dagger$ has at most two different eigenvalues, each one has order $N/2$ and $d$ is a quadratic form of the entries of the matrix $A$.

In the following we give a kind of constructions of high dimensional $d$-computable states. For all $N^2 \times N^2$ density matrices with decompositions on these $N$-dimensional $d$-computable pure states, their entanglement of formations can be calculated with a similar formula to (8) (see (31)).

We first present a kind of construction for a class of $N$-dimensional, $N = 2^k$, $2 \leq k \in \mathbb{N}$, $d$-computable states. Set

\[ A_2 = \begin{pmatrix} a & -c \\ c & d \end{pmatrix}, \]

where $a, c, d \in \mathbb{C}$. For any $b_1, c_1 \in \mathbb{C}$, a $4 \times 4$ matrix $A_4 \in \mathcal{A}$ can be constructed in the following way,

\[ A_4 = \begin{pmatrix} B_2 & A_2 \\ -A_2^t & C_2^t \end{pmatrix} = \begin{pmatrix} 0 & b_1 & a & -c \\ -b_1 & 0 & c & d \\ -a & -c & 0 & -c_1 \\ c & -d & c_1 & 0 \end{pmatrix}, \]

where

\[ B_2 = b_1 J_2, \quad C_2 = c_1 J_2, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

$A_4$ satisfies the relations in (9):

\[ |A_4A_4^\dagger| = [(b_1c_1 + ad + c^2)(b_1c_1 + ad + c^2)\ast]^2 = ([A_4][A_4]^*)^2, \]
\[ |A_4A_4^\dagger - \lambda I d_4| = (\lambda^2 - (b_1 b_1^\ast + c_1 c_1^\ast + aa^\ast + 2cc^\ast + dd^\ast) \lambda \\
+ (b_1c_1 + ad + c^2)(b_1c_1 + ad + c^2)^\ast)^2 \\
= (\lambda^2 - \|A_4\| \lambda + [A_4][A_4]^*)^2, \]
where
\[ [A_4] = (b_1 c_1 + ad + c^2), \quad \|A_4\| = b_1 b_1^* + c_1 c_1^* + aa^* + 2cc^* + dd^*. \] (11)

\( A_8 \in \mathcal{A} \) can be obtained from \( A_4 \),
\[
A_8 = \begin{pmatrix} B_4 & A_4 \\ -A_4^t & C_4^t \end{pmatrix},
\] (12)

where
\[
B_4 = b_2 J_4, \quad C_4 = c_2 J_4, \quad J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad b_2, c_2 \in \mathcal{C}.
\] (13)

For general construction of high dimensional matrices \( A_{2^k+1} \in \mathcal{A}, 2 \leq k \in \mathbb{N} \), we have
\[
A_{2^{k+1}} = \begin{pmatrix} B_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A_{2^k}^t & C_{2^k}^t \end{pmatrix} \equiv \begin{pmatrix} b_k J_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A_{2^k}^t & c_k J_{2^k} \end{pmatrix},
\] (14)

\[
J_{2^{k+1}} = \begin{pmatrix} 0 & J_{2^k} \\ (-1)^{\frac{k(k+1)(k+2)}{2}} J_{2^k}^t & 0 \end{pmatrix},
\] (15)

where \( b_k, c_k \in \mathcal{C}, B_{2^k} = b_k J_{2^k}, C_{2^k} = c_k J_{2^k} \). We call \( J_{2^{k+1}} \) multipliers. Before proving that \( A_{2^k+1} \in \mathcal{A} \), we first give the following lemma.

**Lemma 1.** \( A_{2^{k+1}} \) and \( J_{2^{k+1}} \) satisfy the following relations:

\[
J_{2^{k+1}}^t J_{2^{k+1}} = J_{2^{k+1}} J_{2^{k+1}}^t = Id_{2^{k+1}},
\] (16)

\[
J_{2^{k+1}}^t J_{2^{k+1}} = J_{2^{k+1}} J_{2^{k+1}} = (-1)^{\frac{k(k+1)(k+2)}{2}} Id_{2^{k+1}},
\] (17)

\[
J_{2^{k+1}}^t = J_{2^{k+1}}^t, \quad J_{2^{k+1}}^t = (-1)^{\frac{k(k+1)(k+2)}{2}} J_{2^{k+1}}^t,
\]

\[
A_{2^{k+1}}^t = (-1)^{\frac{k(k+1)}{2}} A_{2^{k+1}}, \quad A_{2^{k+1}} = (-1)^{\frac{k(k+1)}{2}} A_{2^{k+1}}^t.
\]

**Proof.** One easily checks that relations in (16) hold for \( k = 1 \). Suppose (16) hold for general \( k \). We have

\[
J_{2^{k+1}}^t J_{2^{k+1}} = \begin{pmatrix} 0 & (-1)^{\frac{k(k+1)(k+2)}{2}} J_{2^k}^t \\ J_{2^k} & 0 \end{pmatrix} \begin{pmatrix} 0 & J_{2^k} \\ (-1)^{\frac{k(k+1)(k+2)}{2}} J_{2^k}^t & 0 \end{pmatrix} = \begin{pmatrix} (-1)^{\frac{k(k+1)(k+2)}{2}} J_{2^k} J_{2^k}^t & 0 \\ 0 & J_{2^k} J_{2^k} \end{pmatrix} = Id_{2^{k+1}}
\]
and
\[
J_{2k+1}^t J_{2k+1}^* = \begin{pmatrix}
0 & (-1)^{(k+1)(k+2)/2} J_{2k}^t \\
J_{2k} & 0
\end{pmatrix} \begin{pmatrix}
0 & (-1)^{(k+1)(k+2)/2} J_{2k}^* \\
J_{2k} & 0
\end{pmatrix}
= \begin{pmatrix}
(-1)^{(k+1)(k+2)/2} J_{2k}^t J_{2k} & 0 \\
0 & (-1)^{(k+1)(k+2)/2} J_{2k} J_{2k}^t
\end{pmatrix} = (-1)^{(k+1)(k+2)/2} I d_{2k+1}.
\]

Therefore the relations for \(J_{2k+1}^t J_{2k+1}\) and \(J_{2k+1}^* J_{2k+1}^t\) are valid also for \(k + 1\). The cases for \(J_{2k+1}^t J_{2k+1}\) and \(J_{2k+1} J_{2k+1}^t\) can be similarly treated.

The formula \(J_{2k+1}^t = (-1)^{(k+1)(k+2)/2} J_{2k+1}^t\) in (17) is easily deduced from (16) and the fact \(J_{2k+1}^t = J_{2k+1}^t\).

The last two formulae in (17) are easily verified for \(k = 1\). If it holds for general \(k\), we have then,
\[
A_{2k+1}^t = \begin{pmatrix}
B_{2k}^t & (-1)^{(k+1)}/2 A_{2k} \\
A_{2k}^t & C_{2k}
\end{pmatrix} = \begin{pmatrix}
(-1)^{(k+1)/2} B_{2k}^t & (-1)^{(k+1)/2} A_{2k} \\
A_{2k}^t & (-1)^{(k+1)/2} C_{2k}
\end{pmatrix} = (-1)^{(k+1)/2} A_{2k+1}^t,
\]
i.e., it holds also for \(k + 1\). The last equality in (17) is obtained from the conjugate of the formula above.

**Lemma 2.** The following relations can be verified straightforwardly from Lemma 1,
\[
B_{2k}^t = (-1)^{(k+1)/2} B_{2k}, \quad C_{2k}^t = (-1)^{(k+1)/2} C_{2k}, \quad (18)
\]
\[
B_{2k+1}^t B_{2k+1} = B_{2k+1} B_{2k+1}^t = b_k b_{k+1}^t I d_{2k+1}, \quad C_{2k+1}^t C_{2k+1} = C_{2k+1} C_{2k+1}^t = c_k c_{k+1}^t I d_{2k+1}, \quad (20)
\]
For any \(A_{2k+1} \in \mathcal{A}, k \geq 2\), we define
\[
||A_{2k+1}|| =: b_k b_k + c_k c_k + ||A_{2k}||, \quad (21)
\]
\[
[A_{2k+1}] =: (-1)^{(k+1)/2} b_k c_k - [A_{2k}].
\]

**Lemma 3.** For any \(k \geq 2\), we have,
\[
(A_{2k+1} J_{2k+1})(J_{2k+1} A_{2k+1})^t = (A_{2k+1} J_{2k+1})^t (J_{2k+1} A_{2k+1})
\]
\[
= ((-1)^{(k+1)/2} b_k c_k - [A_{2k}]) I d_{2k+1} = [A_{2k+1}] I d_{2k+1}, \quad (22)
\]
\[
(A_{2k+1}^* J_{2k+1})(J_{2k+1} A_{2k+1}^*)^t = (A_{2k+1}^* J_{2k+1})^t (J_{2k+1} A_{2k+1}^*) = [A_{2k+1}]^* I d_{2k+1}.
\]
Proof. One can verify that Lemma 3 holds for \( k = 2 \). Suppose it is valid for \( k \), we have

\[
(A_{2k+1} J_{2k+1})(J_{2k+1} A_{2k+1})^t
\]

\[
= \begin{pmatrix}
-1 \frac{(k+1)(k+2)}{2} A_{2k} J_{2k}^t & B_{2k} J_{2k} \\
-1 \frac{(k+1)(k+2)}{2} C_{2k} J_{2k}^t & A_{2k} J_{2k}^t
\end{pmatrix}
\begin{pmatrix}
-1 \frac{k(k+1)}{2} J_{2k} A_{2k}^t \\
-1 \frac{(k+1)(k+2)}{2} J_{2k}^t B_{2k} \\
-1 \frac{(k+1)(k+2)}{2} J_{2k}^t A_{2k}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e_{11} & e_{12} \\
e_{21} & e_{22}
\end{pmatrix},
\]

where

\[
e_{11} = \left( -1 \frac{(k+1)(k+2) + k(k+1)}{2} A_{2k} J_{2k}^t A_{2k} J_{2k}^t + (-1) \frac{k(k+1)}{2} b_k c_k Id_{2k} \right)
\]

\[
= \left( -1 \frac{k(k+1)}{2} b_k c_k - [A_{2k}] \right) Id_{2k},
\]

\[
e_{12} = b_k A_{2k} J_{2k}^t + (-1) \frac{(k+1)(k+2) + k(k+1)}{2} b_k A_{2k} J_{2k}^t,
\]

\[
= b_k A_{2k} J_{2k}^t (1 + (-1) \frac{(k+1)(k+2) + k(k+1)}{2}) = 0,
\]

\[
e_{21} = (-1) \frac{k(k+1)}{2} c_k A_{2k} J_{2k}^t + (-1) \frac{2(k+1)}{2} c_k A_{2k} J_{2k} = 0,
\]

\[
e_{22} = \left( -1 \frac{k(k+1)}{2} b_k c_k Id_{2k} + (-1) \frac{(k+1)(k+2) + k(k+1)}{2} A_{2k} J_{2k} A_{2k} J_{2k}^t \right)
\]

\[
= \left( -1 \frac{k(k+1)}{2} b_k c_k Id_{2k} + (-1) \frac{(k+1)(k+2) + k(k+1)}{2} (A_{2k} J_{2k})(J_{2k} A_{2k})^t \right)
\]

\[
= \left( -1 \frac{k(k+1)}{2} b_k c_k - [A_{2k}] \right) Id_{2k}.
\]

Hence

\[
(A_{2k+1} J_{2k+1})(J_{2k+1} A_{2k+1})^t = \left( -1 \frac{k(k+1)}{2} b_k c_k - [A_{2k}] \right) Id_{2k+1} = [A_{2k+1}] Id_{2k+1}.
\]

Similar calculations apply to \((A_{2k+1} J_{2k+1})^t(J_{2k+1} A_{2k+1})\). Therefore the Lemma holds for \( k+1 \).

The last equation can be deduced from the first one.

**Theorem 2.** \( A_{2k} \) satisfies the following relation:

\[
|A_{2k+1} A_{2k+1}^t| = ([A_{2k+1}] [A_{2k+1}]^*)^{2k} = \left( (-1) \frac{k(k+1)}{2} b_k c_k - [A_{2k}] \right) \left( (-1) \frac{k(k+1)}{2} b_k^* c_k^* - [A_{2k}]^* \right)^{2k}.
\]

(23)
Proof. By using Lemma 1-3, we have

\[
|A_{2k+1}| = \begin{vmatrix}
B_{2k} & A_{2k} \\
\frac{1}{k(k+1)} & A^t_{2k} \\
(-1) & C_{2k}^t
\end{vmatrix}
= \begin{vmatrix}
Id_{2k} & A^t_{2k} (C_{2k}^t)^{-1} \\
0 & Id_{2k}
\end{vmatrix}
\begin{vmatrix}
B_{2k} & A_{2k} \\
\frac{1}{k(k+1)} & A^t_{2k} \\
(-1) & C_{2k}^t
\end{vmatrix}
= \left| B_{2k} - \left(-1\right)^{\frac{k(k+1)}{2}} A_{2k} (C_{2k}^t)^{-1} A^t_{2k} \right| \\
\left(-1\right)^{\frac{k(k+1)}{2}} A^t_{2k} \\
C_{2k}^t
\right|
= |b_kc_k J_{2k} J^t_{2k} - \left(-1\right)^{\frac{k(k+1)}{2}} C_k^t A_{2k} C_k A^t_{2k}|
= |b_kc_k Id_{2k} - \left(-1\right)^{\frac{k(k+1)}{2}} (A_{2k} J_{2k}) (J_{2k} A_{2k})^t|
= \left| (-1)^{\frac{k(k+1)}{2}} b_kc_k Id_{2k} - [A_{2k}] Id_{2k} \right| = \left| (-1)^{\frac{k(k+1)}{2}} b_kc_k - [A_{2k}] \right|^{2k}.
\]

Therefore

\[
|A_{2k+1} A^t_{2k+1}| = (|A_{2k+1}|^*)^{2k}.
\]

Lemma 4. \(A_{2k+1}\) and \(J_{2k+1}\) satisfy the following relations:

\[
(A_{2k+1} J_{2k+1})(J_{2k+1} A_{2k+1})^t + (J_{2k+1} A^*_{2k+1})(J_{2k+1} A_{2k+1})^t
= A_{2k+1} A^t_{2k+1} + J_{2k+1} A^*_{2k+1} A^t_{2k+1} J^t_{2k+1} = |A_{2k+1}| Id_{2k+1},

(A_{2k+1} J_{2k+1})^t (A^*_{2k+1} J_{2k+1}) + (J_{2k+1} A_{2k+1})^t (J^*_{2k+1} A^*_{2k+1})
= A^t_{2k+1} A_{2k+1} + J^t_{2k+1} A^*_{2k+1} A^t_{2k+1} J_{2k+1} = |A_{2k+1}| Id_{2k+1}.
\]

Proof. It can be verified that the first formula holds for \(k = 2\), if it holds for \(k\), we have

\[
(A_{2k+1} J_{2k+1})(A_{2k+1} J_{2k+1})^t + (J_{2k+1} A^*_{2k+1})(J_{2k+1} A_{2k+1})^t
= \begin{pmatrix}
(-1)^{\frac{(k+1)(k+2)}{2}} A_{2k} J^t_{2k} & B_{2k} J_{2k} \\
(-1)^{\frac{(k+1)(k+2)}{2}} C_{2k}^t J^t_{2k} & (-1)^{\frac{(k+1)(k+2)}{2}} A^t_{2k} J_{2k}
\end{pmatrix}
\begin{pmatrix}
(-1)^{\frac{(k+1)(k+2)}{2}} J_{2k} A_{2k}^t & (-1)^{\frac{(k+1)(k+2)}{2}} J_{2k} C_{2k}^t \\
J^t_{2k} B_{2k} & (-1)^{\frac{(k+1)(k+2)}{2}} J^t_{2k} A^t_{2k}
\end{pmatrix}
= \begin{pmatrix}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{pmatrix},
\]
where, by using Lemma 1 and 2,

\[ f_{11} = f_{22} = A_{2k}^* A_{2k} + J_{2k}^\dagger A_{2k} J_{2k}^* + BB^\dagger + J_{2k} C^\dagger C J_{2k}^* \]

= \( A_{2k}^2 A_{2k} + J_{2k} (-1)^{\frac{k+1}{2}} A_{2k}^2 (-1)^{\frac{k+1}{2}} A_{2k}^t J_{2k}^* + (b_k b_k^* + c_k c_k^*) I d_{2k} \)

= ((|A_{2k}| + b_k b_k^* + c_k c_k^*) I d_{2k} = |A_{2k+1}| I d_{2k},

\[ f_{12} = A_{2k}^* C_{2k}^* + (-1)^{\frac{k+1}{2}} B_{2k} A_{2k}^* + (-1)^{\frac{k+1}{2} + (k+1)(k+2)} b_k J_{2k}^* A_{2k}^t + (-1)^{\frac{k+1}{2} + (k+1)(k+2)} c_k^* A_{2k}^t J_{2k}^* \]

= \((-1)^{\frac{k+1}{2}} (B_{2k} A_{2k}^* - (-1)^{\frac{k+1}{2} + (k+1)(k+2)} B_{2k} A_{2k}^t) \]

\[ f_{21} = C_{2k}^t A_{2k}^t + (-1)^{\frac{k+1}{2}} J_{2k}^* B_{2k}^* + (-1)^{\frac{k+1}{2} + (k+1)(k+2)} b_k^* A_{2k}^t J_{2k}^* + (-1)^{\frac{k+1}{2} + (k+1)(k+2)} c_k J_{2k}^* A_{2k}^* \]

= \((-1)^{\frac{k+1}{2}} (b_k^* A_{2k}^t J_{2k}^* - (-1)^{\frac{k+1}{2} + (k+1)(k+2)} b_k A_{2k}^t J_{2k}^* ) \]

Hence the first formula holds also for k + 1. The second formula can be verified similarly. ■

**Lemma 5.** Matrices \( B_{2k}, A_{2k} \) and \( C_{2k} \) satisfy the following relations:

\[ ((-1)^{\frac{k+1}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^*)((-1)^{\frac{k+1}{2}} A_{2k}^* B_{2k} + C_{2k} A_{2k})^t = F(A_{2k+1}) I d_{2k}, \]

where

\[ F(A_{2k+1}) = c_k^2 [A_{2k}] + b_k^2 [A_{2k}]^* + (-1)^{\frac{k+1}{2}} b_k c_k^* |A_{2k}|, \]

**Proof.** By using Lemma 3 and 4, we have

\[ ((-1)^{\frac{k+1}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^*)((-1)^{\frac{k+1}{2}} A_{2k}^* B_{2k} + C_{2k} A_{2k})^t \]

= \( b_k^2 (J_{2k} A_{2k}^* A_{2k}^* J_{2k}) + c_k^2 (A_{2k} J_{2k})(J_{2k} A_{2k})^t \]

\[ + (-1)^{\frac{k+1}{2}} b_k c_k^* (A_{2k} J_{2k} A_{2k})^t + (J_{2k} A_{2k})^t J_{2k} A_{2k})^t \]

= \( (c_k^2 [A_{2k}] + b_k^2 [A_{2k}]^* + (-1)^{\frac{k+1}{2}} b_k c_k^* |A_{2k}|) I d_{2k} = F(A_{2k+1}) I d_{2k}. \]

**Lemma 6.** \( A_{2k} \) and \( J_{2k} \) satisfy the following relation:

\[ |A_{2k}| J_{2k} A_{2k}^* A_{2k} J_{2k}^* = [A_{2k}] [A_{2k}]^* I d_{2k} + J_{2k} A_{2k}^* A_{2k} J_{2k}^* J_{2k}^*. \]
Proof. From (24) we have the following relation:

\[ F(A_{2k+1}) J_{2k}^t A_{2k}^* A_{2k} J_{2k}^t \]

\[ = \left( -1 \right)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^* \left( -1 \right) \frac{k(k+1)}{2} b_k (A_{2k}^* J_{2k})^t J_{2k} A_{2k} A_{2k}^* J_{2k}^t \]

\[ + \left( -1 \right)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^* c_k^* (J_{2k}^* A_{2k})^t J_{2k} A_{2k} A_{2k}^* J_{2k}^t \]

\[ = \frac{1}{2} b_k \left( -1 \right)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^* \left( (A_{2k}^* J_{2k})^t (J_{2k} A_{2k})^t \right) A_{2k}^* J_{2k}^t \]

\[ + c_k^* \left( -1 \right)^{\frac{k(k+1)}{2}} b_k (J_{2k}^* A_{2k}^*)^t A_{2k}^* J_{2k}^t + c_k^* k A_{2k} J_{2k}^t A_{2k} J_{2k} + c_k^* k A_{2k} J_{2k} A_{2k}^* J_{2k}^t \]

\[ = \left( -1 \right)^{\frac{k(k+1)}{2}} b_k \left( -1 \right)^{\frac{k(k+1)}{2}} b_k (J_{2k}^* A_{2k}^*)^t (J_{2k} A_{2k})^t A_{2k}^* + c_k^* k A_{2k} J_{2k}^t A_{2k} J_{2k} + c_k^* k A_{2k} J_{2k} A_{2k}^* J_{2k}^t \]

Using (25) we have

\[ \|A_{2k}\| J_{2k}^* A_{2k}^* A_{2k} J_{2k}^t = [A_{2k}][A_{2k}]^* J_{2k}^t A_{2k} A_{2k}^* + J_{2k} A_{2k}^* A_{2k} A_{2k}^* J_{2k}^t . \]

\[ \]

**Theorem 3.** The eigenvalue polynomial of \( A_{2k+1} A_{2k+1}^\dagger \) satisfies the following relations:

\[ |A_{2k+1} A_{2k+1}^\dagger - \lambda I d_{2k+1}| = (\lambda^2 - ||A_{2k+1}|| \lambda + [A_{2k+1}] [A_{2k+1}]^*)^{2k}, \]

\[ |A_{2k+1}^\dagger A_{2k+1} - \lambda I d_{2k+1}| = (\lambda^2 - ||A_{2k+1}|| \lambda + [A_{2k+1}] [A_{2k+1}]^*)^{2k}. \]

(27)

**Proof.** Let

\[ \Lambda_k = -[(c_k c_k^* - \lambda) I d_{2k} + A_{2k}^* A_{2k}^+] \left( -1 \right)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^* \right]^{-1}. \]
where

\[ I = (-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}^* )((-1)^{\frac{k(k+1)}{2}} B_{2k}^* A_{2k} + A_{2k}^* C_{2k})^t \]

\[ = (-1)^{\frac{k(k+1)}{2}} b_k c_k [A_{2k}]^* I d_{2k} + (-1)^{\frac{k(k+1)}{2}} b_k^* c_k [A_{2k}] Id_{2k} + b_k b_k^* J_{2k} A_{2k}^* A_{2k}^* J_{2k}^* + c_k c_k A_{2k} A_{2k}^* \]

and, by using Lemma 5,

\[ II = -((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k}^* C_{2k}) A_k [(b_k b_k^* - \lambda) I d_{2k} + A_{2k} A_{2k}^*] \]

\[ = [(c_k c_k^* - \lambda)((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}) + ((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k} C_{2k}) A_{2k} A_{2k}^*] \]

\[ = (b_k b_k^* - \lambda)(c_k c_k^* - \lambda) Id_{2k} + (b_k b_k^* - \lambda)((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k}^* C_{2k})^t \]

\[ + A_{2k} C_{2k}^* A_{2k}^* ((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k}^* C_{2k})^{-1} + (c_k c_k^* - \lambda) A_{2k} A_{2k}^* \]

\[ + ((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k}^* C_{2k}) A_{2k}^* A_{2k}^* ((-1)^{\frac{k(k+1)}{2}} B_{2k} A_{2k}^* + A_{2k}^* C_{2k})^{-1} A_{2k} A_{2k}^* \]

\[ = (b_k b_k^* - \lambda)(c_k c_k^* - \lambda) Id_{2k} + (c_k c_k^* - \lambda) A_{2k} A_{2k}^* + \frac{b_k b_k^* - \lambda}{F(A_{2k+1})} III + \frac{1}{F(A_{2k+1})} III A_{2k} A_{2k}^* , \]
where

\[
III = (1) \frac{k(k+1)}{2} B_{2k} A_{2k}^* + A_{2k} C_{2k} A_{2k}^* (1) \frac{k(k+1)}{2} A_{2k}^* B_{2k} + C_{2k} A_{2k}^*)^t
\]

= \((-1) \frac{k(k+1)}{2} B_{2k} A_{2k}^* + A_{2k} C_{2k} A_{2k}^* A_{2k} J_{2k}^t J_{2k} A_{2k}^* J_{2k}^t A_{2k}^* J_{2k}^t A_{2k}^* J_{2k}^t A_{2k}^* J_{2k}^t (1) \frac{k(k+1)}{2} A_{2k}^* B_{2k} + C_{2k} A_{2k}^*)^t
\]

= \((-1) \frac{k(k+1)}{2} b_k (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t + c_k^i (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t
\]

= \((-1) \frac{k(k+1)}{2} b_k (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t + c_k^i (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t
\]

From Lemma 6, we get

\[
III = (1) \frac{k(k+1)}{2} b_k (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t + c_k^i (J_{2k} A_{2k}^*) (J_{2k} A_{2k}^*)^t
\]

= F(A_{2k+1}) J_{2k} A_{2k}^* A_{2k}^* J_{2k}^t.

From Lemma 3 we also have

\[
III A_{2k} A_{2k}^t = III A_{2k} J_{2k}^t A_{2k}^t
\]

= F(A_{2k+1}) J_{2k} A_{2k}^* (J_{2k} A_{2k}^*)^t (A_{2k} J_{2k}) (A_{2k}^* J_{2k})^t = F(A_{2k+1}) [A_{2k}]^t A_{2k} J_{2k}^t.

Therefore,

\[
\begin{align*}
|A_{2k+1} A_{2k}^t - \lambda I d_{2k+1}| &= |I + III| \\
&= | - \lambda^2 I d_{2k} + \lambda (b_k b_k^* c_k c_k^* + ||A_{2k}||) I d_{2k} - (b_k b_k^* c_k c_k^* - (1) \frac{k(k+1)}{2} b_k^* c_k^* [A_{2k}]^t
\]

- (1) \frac{k(k+1)}{2} b_k c_k [A_{2k}]^* + [A_{2k}] [A_{2k}^*]^t I d_{2k})
\]

= (\lambda^2 - ||A_{2k+1}|| \lambda + [A_{2k+1}] [A_{2k+1}^*]^{2k},
\]

where the first formula in Lemma 4 is used. The second formula in Theorem 3 is obtained from the fact that $A_{2k+1} A_{2k+1}^*$ and $A_{2k+1} A_{2k+1}^*$ have the same eigenvalue set.

From Theorem 2 and 3 the states given by (14) are $d$-computable. In terms of (4) the generalized concurrence for these states is given by

\[
d_{2k+1} = 2^{k+1}||[A_{2k+1}]|| = 2^{k+1} |b_k c_k + b_{k-1} c_{k-1} + \ldots + b_1 c_1 + ad + c^2|.
\]

Let $p_{2k+1}$ be a symmetric anti-diagonal $2^{2k+2} \times 2^{2k+2}$ matrix with all the anti-diagonal elements 1 except for those at rows $2^{k+1} - 1$ and $2^{k+1} + s(2^{k+2} - 2), 2^{k+1} + s(2^{k+2} - 2), 2^{k+2} - 1$.

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s(2^{k+2} - 2), 2^{k+2} + s(2^{k+2} - 2), s = 0, ..., 2^{k+1} - 1, which are \(-1\). \(d_{2k+1}\) can then be written as

\[
d_{2k+1} = |\langle \psi_{2k+1} | p_{2k+1} | \psi_{2k+1}^* \rangle|,
\]

where

\[
|\psi_{2k+1}\rangle = \sum_{i,j=1}^{2k+1} (A_{2k+1})_{ij} e_i \otimes e_j.
\]

Let \(\Phi\) denote the set of pure states with the form (29). For mixed states with density matrices such that their decompositions are of the form

\[
\rho_{2k+2} = \sum_{i=1}^{M} p_i |\psi_i\rangle \langle \psi_i|, \quad \sum_{i=1}^{M} p_i = 1, \quad |\psi_i\rangle \in \Phi,
\]

their entanglement of formations, by using a similar calculation in obtaining formula (8) [18], are then given by \(E(d_{2k+1}(\rho_{2k+2}))\), where

\[
d_{2k+1}(\rho_{2k+2}) = \Omega_1 - \sum_{i=2}^{2k+2} \Omega_i,
\]

and \(\Omega_i\), in decreasing order, are the the square roots of the eigenvalues of the matrix \(\rho_{2k+2}p_{2k+1}\rho_{2k+2}^*p_{2k+1}\). Here again due to the form of the so constructed matrix \(A_{2k+1}\) in (14), once \(\rho\) has a decomposition with all \(|\psi_i\rangle \in \Phi\), all other decompositions of \(|\psi_i'\rangle\) also satisfy \(|\psi_i'\rangle \in \Phi\). Therefore from high dimensional \(d\)-computable states \(A_{2k+1}\), \(2 \leq k \leq N\), the entanglement of formation for a class of density matrices whose decompositions lie in these \(d\)-computable quantum states can be obtained analytically.

3 Remarks and conclusions

Besides the \(d\)-computable states constructed above, from (10) we can also construct another class of high dimensional \(d\)-computable states given by \(2^{k+1} \times 2^{k+1}\) matrices \(A_{2k+1}\), \(2 \leq k \in \mathbb{N}\),

\[
A_{2k+1} = \begin{pmatrix} B_k & A_k \\ -A_k^t & C_k \end{pmatrix} \equiv \begin{pmatrix} b_k I_{2k} & A_{2k} \\ -A_{2k}^t & c_k I_{2k} \end{pmatrix},
\]

where \(b_k, c_k \in \mathbb{C}\), \(I_4 = J_4\),

\[
I_{2k+1} = \begin{pmatrix} 0 & I_{2k} \\ -I_{2k} & 0 \end{pmatrix}
\]
for $k + 2$ mode $4 = 0$,
\[
I_{2^{k+1}} = \begin{pmatrix} 0 & I_{2^k} \\ I_{2^k} & 0 \end{pmatrix}
\]  
(34)
for $k + 2$ mode $4 = 1$,
\[
I_{2^{k+1}} = \begin{pmatrix} 0 & 0 & 0 & I_{2^{k-1}} \\ 0 & 0 & -I_{2^{k-1}} & 0 \\ 0 & I_{2^{k-1}} & 0 & 0 \\ -I_{2^{k-1}} & 0 & 0 & 0 \end{pmatrix}
\]  
(35)
for $k + 2$ mode $4 = 2$, and
\[
I_{2^{k+1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{2^{k-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{2^{k-2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{2^{k-2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{2^{k-2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{2^{k-2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]  
(36)
for $k + 2$ mode $4 = 3$.

One can prove that the matrices in (32) also give rise to $d$-computable states:
\[
|A_{2^{k+1}}A_{2^{k+1}}^\dagger| = [(c^2 + ad - \sum_{i=1}^{k} b_i c_i)(c^2 + ad - \sum_{i=1}^{k} b_i c_i)^*]^{2^k},
\]
\[
|A_{2^{k+1}}A_{2^{k+1}}^\dagger - \lambda I_{2^{k+1}}| = [\lambda^2 - (aa^* + 2cc^* + dd^* + \sum_{i=1}^{k} b_i b_i^* + \sum_{i=1}^{k} c_i c_i^*)\lambda + (c^2 + ad - \sum_{i=1}^{k} b_i c_i)(c^2 + ad - \sum_{i=1}^{k} b_i c_i)^*]^{2^k}.
\]
The entanglement of formation for a density matrix with decompositions in these states is also given by a formula of the form (31).

In addition, the results obtained above may be used to solve linear equation systems, e.g., in the analysis of data bank, described by $Ax = y$, where $A$ is a $2^k \times 2^k$ matrix, $k \in \mathbb{N}$, $x$ and $y$ are $2^k$-dimensional column vectors. When the dimension $2^k$ is large, the standard methods such as Gauss elimination to solve $Ax = y$ could be not efficient. From our Lemma
3, if the matrix $A$ is of one of the following forms: $A_{2^k}$, $B_{2^k}A_{2^k}$, $A_{2^k}^t$ or $A_{2^k}B_{2^k}^t$, the solution $\mathbf{x}$ can be obtained easily by applying the matrix multipicators. For example, $A_{2^k}\mathbf{x} = \mathbf{y}$ is solved by

$$\mathbf{x} = \frac{1}{[A_{2^k}]} (A_{2^k}J_{2^k})^t J_{2^k} \mathbf{y}.$$ 

The solution to $B_{2^k}A_{2^k}\mathbf{x} = \mathbf{y}$ is given by

$$\mathbf{x} = \frac{1}{b_k[A_{2^k}]} (A_{2^k}J_{2^k})^t J_{2^k} \mathbf{y}.$$ 

We have presented a kind of construction for a class of special matrices with at most two different eigenvalues. This class of matrices defines a special kind of $d$-computable states. The entanglement of formation for these $d$-computable states is a monotonically increasing function of a the generalized concurrence. From this generalized concurrence the entanglement of formation for a large class of density matrices whose decompositions lie in these $d$-computable quantum states is obtained analytically. Besides the relations to the quantum entanglement, the construction of $d$-computable states has its own mathematical interests.

References

[1] See, for example, D.P. DiVincenzo, *Science* **270**, 255 (1995).

[2] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).

[3] S. Albeverio and S.M. Fei, *Phys. Lett. A* **276**, 8 (2000).

[4] S. Albeverio and S.M. Fei and W.L. Yang, *Commun. Theor. Phys.* **38**, 301 (2002); *Phys. Rev. A* **66**, 012301 (2002).

[5] C.H. Bennett and S.J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).

[6] See, for example, C.A. Fuchs, N. Gisin, R.B. Griffiths, C-S. Niu, and A. Peres, *Phys. Rev., A* **56**, 1163 (1997) and references therein.
[7] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin and W.K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).

[8] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).

[9] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997); V. Vedral, M.B. Plenio, K. Jacobs and P.L. Knight, *Phys. Rev. A* **56**, 4452 (1997); V. Vedral and M.B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).

[10] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[11] K. Życzkowski and P. Horodecki, Phys. Rev. A **58**, 883 (1998).

[12] B. Schumacher and M.D. Westmoreland, *Relative entropy in quantum information theory*, quant-ph/0004045.

[13] M. Horodecki, P. Horodecki and R. Horodecki, *Phys. Rev. Lett.* **80**, 5239 (1998).

[14] S. Hill and W.K. Wootters, *Phys. Rev. Lett.* **78**, 5022 (1997). W.K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).

[15] B.M. Terhal, K. Gerd and K.G.H. Vollbrecht, Phys. Rev. Lett. **85**, 2625 (2000).

[16] A.Uhlmann, *Phys. Rev. A* **62**, 032307 (2000).

[17] S. Albererio and S.M. Fei, *J. Opt. B: Quantum Semiclass. Opt.* **3**, 1 (2001).

[18] S.M. Fei, X.H. Gao, X.H. Wang, Z.X. Wang and K. Wu, *Phys. Lett. A* **300**, 559 (2002); *Int. J. Quant. Inform.* **1**, 37 (2003).