Recursive representation of the torus 1-point conformal block

Leszek Hadasz†, Zbigniew Jaskólski‡ and Paulina Suchanek‡

† M. Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland,

‡ Institute of Theoretical Physics, University of Wrocław
pl. M. Borna, 50-204 Wrocław, Poland.

Abstract

The recursive relation for the 1-point conformal block on a torus is derived and used to prove the identities between conformal blocks recently conjectured by Poghossian in [1]. As an illustration of the efficiency of the recurrence method the modular invariance of the 1-point Liouville correlation function is numerically analyzed.

PACS: 11.25.Hf, 11.30.Pb

1 e-mail: hadasz@th.if.uj.edu.pl
2 e-mail: jask@ift.uni.wroc.pl
3 e-mail: paulina@ift.uni.wroc.pl
1 Introduction

An interesting duality between certain class of $N = 2$ superconformal field theories in four dimensions and the two-dimensional Liouville field theory has been recently discovered by Alday, Gaiotto and Tachikawa [2]. An essential part of this relation actively studied in a number of papers [3–11] is an exact correspondence between instanton parts of Nekrasov partition functions in $N=2$ SCFT and conformal blocks of the 2-dimensional CFT. This in particular concerns two basic objects of CFT: the 4-point conformal block on a sphere and the 1-point conformal block on a torus. These special cases has been recently analyzed by Poghossian [1] who applied the recursive relation for the 4-point blocks on a sphere discovered long time ago by Alexei Zamolodchikov in CFT [12–14] to the instanton part of the Nekrasov partition function with four fundamentals. He also proposed and verified to certain order a recursive relation for the Nekrasov function with one adjoint hypermultiplet. It was observed that by the AGT duality this yields previously unknown recursive relation for conformal blocks in 2-dimensional CFT.

The aim of the present paper is to provide a complete derivation of the recursive relation for 1-point conformal blocks on a torus conjectured by Poghossian [1]. Our method is based on the algebraic properties of the 3-point conformal blocks and the analytic structure of the inverse of the Gram matrices in Verma modules [15]. It parallels to large extend Zamolodchikov’s original derivation for the 4-point blocks [12–14]. We set our notation in Section 2 and present the derivation in Section 3.

As an application of the recurrence relations we prove two important identities between conformal blocks conjectured by Poghossian [1]. The first one relates the 1-point block on a torus with certain 4-point elliptic blocks on a sphere. The second relates 4-point elliptic blocks for different values of central charges, external weights and the elliptic variable $q$.

It had been motivated by the intriguing relation between the Liouville 1-point correlation function on the torus and the Liouville 4-point correlation function on the sphere recently proposed by Fateev, Litvinov, Neveu and Onofri [16]. The derivation of both relations is presented in Section 4. They in particular imply\(^4\):

\[ H_{c,\Delta_\alpha}^\lambda (q^2) = H_{c',\Delta_{\alpha'}}^\lambda \left[ b_{\frac{\sqrt{2}}{2b'}} \lambda \right] (q), \ b' = \frac{b}{\sqrt{2}}, \ \alpha' = \sqrt{2}\alpha, \]

(1)

and

\[ H_{c,\Delta_\alpha}^\lambda (q^2) = H_{c',\Delta_{\alpha'}}^\lambda \left[ b_{\frac{\sqrt{2}}{2b'}} \lambda \right] (q), \ b' = \sqrt{2}b, \ \alpha' = \sqrt{2}\alpha. \]

(2)

\(^4\)See the next section for our notation conventions.
These identities, which can be seen as chiral versions of the identity for the Liouville correlation functions proposed in [16], constitute the most interesting results of the present work. They provide for instance a new tool for analyzing virtually untouched problem of modular invariance of nontrivial torus 1-point functions in CFT. We hope to report on some of their consequences in the forthcoming paper.

Beside theoretical applications the recursive relation for 1-point block on a torus gives a very efficient numerical method of analyzing the modular bootstrap in CFT. As an illustration of this method we present in Section 5 some numerical checks of modular bootstrap in the Liouville theory.

2 1-point functions

Consider a primary field $\phi_{\lambda,\bar{\lambda}}$ with the conformal weights $(\Delta_\lambda, \bar{\Delta}_\lambda)$ for which the following parametrization is assumed:

$$\Delta_\lambda = \frac{1}{4}(Q^2 - \lambda^2), \quad c = 1 + 6Q^2, \quad Q = b + b^{-1}.$$  

In terms of the CFT on the complex plane the 1-point correlation function of $\phi_{\lambda,\bar{\lambda}}$ on a torus takes the form:

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle = \text{Tr} \left( e^{-(\text{Im}\tau)\hat{H} + i(\text{Re}\tau)\hat{P}} \phi_{\lambda}(1,1) \right) = (q\bar{q})^{-\frac{c}{24}} \text{Tr} \left( q^{L_0} \bar{q}^{\bar{L}_0} \phi_{\lambda}(1,1) \right),$$  

where $\tau$ is the torus modular parameter and

$$\hat{H} = 2\pi(L_0 + \bar{L}_0) - \frac{\pi c}{6}, \quad \hat{P} = 2\pi(L_0 - \bar{L}_0), \quad q = e^{2\pi i \tau}.$$  

The trace can be calculated in the standard bases of Verma modules:

$$\nu_{\Delta, M} = L_{-M_1} \nu_{\Delta} \equiv L_{-m_j} \ldots L_{-m_1} \nu_{\Delta},$$

where $M = \{m_1, m_2, \ldots, m_j\} \subset \mathbb{N}$ stands for an arbitrary ordered set of indices $m_j \leq \ldots \leq m_2 \leq m_1$, and $\nu_{\Delta} \in \mathcal{V}_{\Delta}$ is the highest weight state. This yields:

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle = (q\bar{q})^{-\frac{c}{24}} \sum_{(\Delta, \bar{\Delta})} \sum_{\nu_{\Delta, M} = \nu_{\Delta}}^{\infty} q^{\Delta+n} \bar{q}^{\bar{\Delta}+n} \sum_{n=|M|=|\bar{N}|}^{\infty} \left[ B_{\nu,\Delta}^n \right]_{\nu_{\Delta}, M}^{M, N} \langle \nu_{\Delta, M} \otimes \nu_{\Delta, \bar{N}} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta, \bar{N}} \otimes \nu_{\Delta, N} \rangle$$

\[ \text{(3)} \]
where the sum over all weights \((\Delta, \bar{\Delta})\) from the spectrum of the theory is assumed, \(|M| = m_1 + \ldots + m_j\) and \([B^n_{c,\Delta}]^{MN}_{M,N}\) is the inverse of the Gram matrix

\[
[B^n_{c,\Delta}]^{MN}_{M,N} = \langle \nu_{\Delta,N} | \nu_{\Delta,M} \rangle, \quad |M| = |N| = n,
\]

at the level \(n\).

The 3-point correlation function in (3) can be expressed as a product of the 3-point (“left” and “right”) conformal blocks and a structure constant:

\[
\langle \nu_{\Delta,M} \otimes \bar{\nu}_{\Delta,M} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta,N} \otimes \bar{\nu}_{\Delta,N} \rangle = \rho(\nu_{\Delta,N}, \nu_{\lambda}, \nu_{\Delta,M}) \rho(\nu_{\Delta,N}, \nu_{\bar{\lambda}}, \nu_{\Delta,M}) C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}},
\]

\[
C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}} = \langle \nu_{\Delta} \otimes \nu_{\Delta} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta} \otimes \nu_{\Delta} \rangle.
\]

Introducing the 1-point conformal block:

\[
\mathcal{F}^{\lambda}_{c,\Delta}(q) = q^{\Delta - \frac{c}{24}} \sum_{n=0}^{\infty} q^n F^{\lambda,n}_{c,\Delta},
\]

\[
F^{\lambda,n}_{c,\Delta} = \sum_{n=|M|=|N|} \rho(\nu_{\Delta,N}, \nu_{\lambda}, \nu_{\Delta,M}) [B^n_{c,\Delta}]^{MN}_{M,N},
\]

one can write the 1-point correlation function on the torus in the following form:

\[
\langle \phi_{\lambda,\bar{\lambda}} \rangle = \sum_{(\Delta,\bar{\Delta})} \mathcal{F}^{\lambda}_{c,\Delta}(q) \mathcal{F}^{\lambda}_{c,\bar{\Delta}}(\bar{q}) C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}}.
\]

3 Recursive relations

For arbitrary vectors \(\xi_i \in \mathcal{V}_{\Delta_i}\) the 3-point block \(\rho(\xi_i, \xi_2, \xi_1)\) is a polynomial function of the weights \(\Delta_i\), completely determined by the conformal Ward identities [15] and the normalization condition \(\rho(\nu_3, \nu_2, \nu_1) = 1\). It follows from (4) that the block coefficients \(F^{\lambda,n}_{c,\Delta}\) are polynomials in the external weight \(\Delta_{\lambda}\) and rational functions of the central charge \(c\) and \(\bar{\Delta}\) with the locations of poles determined by the zeroes of the determinant of the Gram matrix \([B^n_{c,\Delta}]^{MN}_{M,N}\).

In the generic case the Verma module \(\mathcal{V}_{\Delta_{r,s}+rs}\) is not reducible and the only singularities of \([B^n_{c,\Delta}]^{MN}_{M,N}\) as a function of \(\Delta\) are simple poles at

\[
\Delta_{rs}(c) = \frac{Q^2}{4} - \frac{1}{4} \left( rb + \frac{s}{b} \right)^2
\]

where \(r, s \in \mathbb{Z}, \quad r \geq 1, \quad s \geq 1, \quad 1 \leq rs \leq n\).
As a function of $c$ the inverse Gram matrix at the level $n$ has simple poles at the locations

\begin{align*}
c_{rs}(\Delta) &= 1 + 6 \left( b_{rs}(\Delta) + \frac{1}{b_{rs}(\Delta)} \right)^2 , \\
b_{rs}^2(\Delta) &= \frac{1}{1 - r^2} \left( rs - 1 + 2 \Delta + \sqrt{(r - s)^2 + 4 (rs - 1) \Delta + 4 \Delta^2} \right).
\end{align*}

where $r, s \in \mathbb{Z}$, $r \geq 2$, $s \geq 1$, $1 \leq rs \leq n$.

The block’s coefficient $F_{\lambda,n}^{\lambda,n}$ can be expressed either as a sum over the poles in the intermediate weight:

\begin{equation}
F_{\lambda,n}^{\lambda,n} = h_{\lambda,n}^{\lambda,n} + \sum_{1 \leq rs \leq n} \mathcal{R}_{\lambda,n}^{\lambda,n}_{\Delta,rs}(c),
\end{equation}

or in the central charge:

\begin{equation}
F_{\lambda,n}^{\lambda,n} = f_{\lambda,n}^{\lambda,n} + \sum_{1 < rs \leq n} \mathcal{R}_{\lambda,n}^{\lambda,n}_{\Delta,rs}. \end{equation}

The residues at $\Delta_{rs}(c)$ and $c_{rs}(\Delta)$ are simply related:

\begin{align*}
\mathcal{R}_{\lambda,n}^{\lambda,n}_{\Delta,rs} &= - \frac{\partial c_{rs}(\Delta)}{\partial \Delta} \mathcal{R}_{\lambda,n}^{\lambda,n}_{\Delta,rs}, \\
\frac{\partial c_{rs}(\Delta)}{\partial \Delta} &= 4 \frac{c_{rs}(\Delta) - 1}{(r^2 - 1) (b_{rs}(\Delta))^4 - (s^2 - 1)}.
\end{align*}

In order to calculate the residue at $\Delta = \Delta_{rs}$ it is useful to choose a specific basis in the Verma module $\mathcal{V}_{\lambda}$. Let us introduce the state:

\begin{equation}
\chi_{\Delta,rs}^{\lambda,n} = \sum_{|M| = rs} \chi_{rs}^M L_{-M} \nu_{\Delta}
\end{equation}

where $\chi_{rs}^M$ are the coefficients of the singular vector $\chi_{rs}$ in the standard basis of $\mathcal{V}_{\Delta,rs}$:

\begin{equation}
\chi_{rs} = \sum_{|M| = rs} \chi_{rs}^M L_{-M} \nu_{\Delta,rs}.
\end{equation}

The family of states $\{L_{-N} \chi_{rs}^\Delta\}_{|N| = n - rs}$ can be completed to a full basis in the Verma module $\mathcal{V}_{\Delta}$ at the level $n > rs$. Working in this basis one gets [15]:

\begin{equation}
\mathcal{R}_{\lambda,n}^{\lambda,n}_{\Delta,rs} = \lim_{\Delta \to \Delta_{rs}} (\Delta - \Delta_{rs}(c)) F_{\lambda,n}^{\lambda,n}_{\Delta,rs} = A_{rs}(c) \sum_{n-rs=|M|=|N|} \rho(L_{-N} \chi_{rs}, \nu_{\lambda}, L_{-M} \chi_{rs}) \left[ B_{n-rs}^{n-rs}_{\lambda,rs} \right]^{MN},
\end{equation}

where

\begin{equation}
A_{rs}(c) = \lim_{\Delta \to \Delta_{rs}} \left( \frac{\langle \chi_{rs}^\Delta | \chi_{rs}^\Delta \rangle}{\Delta - \Delta_{rs}(c)} \right)^{-1}.
\end{equation}
It is convenient to normalize the singular vector \( \chi_{rs} \) such that the coefficient in front of \((L_{-1})^{rs}\) equals 1. For this normalization the exact form of the coefficient \( A_{rs}(c) \) was first proposed by Al. Zamolodchikov in [12] and then justified in [17]. It reads:

\[
A_{rs}(c) = \frac{1}{2} \prod_{p=1-r}^{r} \prod_{q=1-s}^{s} \left( pb + \frac{q}{b} \right)^{-1},
\]

Using the factorization formula:

\[
\rho(L_{-N}\chi_{rs}, \nu_{\lambda}, L_{-M}\chi_{rs}) = \rho(L_{-N}\nu_{\Delta_{rs}+rs}, \nu_{\lambda}, L_{-M}\nu_{\Delta_{rs}+rs}) \rho(\chi_{rs}, \nu_{\lambda}, \chi_{rs})
\]

one can show that the residue is proportional to the lower order block coefficient:

\[
R_{\nu_{c,rs}}^{\lambda,n} = A_{rs}(c) \rho(\chi_{rs}, \nu_{\lambda}, \chi_{rs}) F_{\nu_{c,rs}}^{\lambda,n-rs}.
\]

For the normalized singular vector \( \chi_{rs} \) one gets [15]:

\[
\rho(\chi_{rs}, \nu_{\lambda}, \chi_{rs}) = \rho(\chi_{rs}, \nu_{\lambda}, \nu_{\Delta_{rs}+rs}) \rho(\nu_{\Delta_{rs}+rs}, \nu_{\lambda}, \chi_{rs}) = P_{c}^{\lambda}_{rs} \rho \left( \chi_{rs}, \nu_{\lambda}, \nu_{\Delta_{rs}+rs} \right) P_{c}^{\lambda}_{rs} \rho \left( \nu_{\Delta_{rs}+rs}, \nu_{\lambda}, \chi_{rs} \right)
\]

where the fusion polynomials are defined by:

\[
P_{c}^{\lambda}_{rs} \left[ \frac{\Delta\lambda}{\Delta_{rs}+rs} \right] = \prod_{p=1-r}^{r-1} \prod_{q=1-s}^{s-1} \frac{\left( \frac{\lambda_2 + \lambda_1 + pb + qb^{-1}}{2} \right) \left( \frac{\lambda_2 - \lambda_1 + pb + qb^{-1}}{2} \right)}{\prod_{p+r=1 \bmod 2}^{r} \prod_{q+s=1 \bmod 2}^{s} \left( \lambda + kb + lb^{-1} \right) \left( \lambda + kb - lb^{-1} \right) \left( \lambda - kb + lb^{-1} \right) \left( \lambda - kb - lb^{-1} \right)}
\]

and \( \Delta = \frac{1}{4} \left( Q^2 - \lambda_i^2 \right) \). In the case under consideration:

\[
P_{c}^{\lambda}_{rs} \left[ \frac{\Delta\lambda}{\Delta_{rs}+rs} \right] P_{c}^{\lambda}_{rs} \left[ \frac{\Delta\lambda}{\Delta_{rs}} \right] = \prod_{k=1 \bmod 2}^{2r-1} \prod_{l=1 \bmod 2}^{2s-1} \left( \frac{\lambda + kb + lb^{-1}}{2} \right) \left( \frac{\lambda + kb - lb^{-1}}{2} \right) \left( \frac{\lambda - kb + lb^{-1}}{2} \right) \left( \frac{\lambda - kb - lb^{-1}}{2} \right)
\]

The last step in our derivation is to find the non-singular terms in the equations (5), (6). Let us start with the expansion (6). Since \( f_{\lambda,n}^{\Delta} \) does not depend on the central charge it can be calculated from the \( c \to \infty \) limit:

\[
\sum_{n=0}^{\infty} q^{n} f_{\lambda,n}^{\Delta} = \lim_{c \to \infty} q^{-\Delta} f_{\lambda}^{\Delta}(q).
\]

Note that the block’s coefficients (4) depend on \( c \) only via the inverse Gram matrix \( B_{c,\Delta}^{M,N} \). Analyzing the polynomial dependence of the Gram matrix minors on \( c \) one can show [15] that
the only element of the inverse Gram matrix which does not vanish in the limit $c \to \infty$ is the diagonal one corresponding to the state $L^{-1}_n |\nu_{\Delta}\rangle$:

$$
\lim_{c \to \infty} \left[ B_{c,\Delta}^n \right]^{\perp} = \frac{1}{\langle \nu_{\Delta} | P_1 L^{-1}_n |\nu_{\Delta}\rangle} = \frac{1}{n!(2\Delta)_n},
$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. The 3-point block for this state is given by:

$$
\rho(L^{-1}_n \nu_{\Delta}, \nu_{\lambda}, L^{-1}_n \nu_{\Delta}) = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left( \frac{n!}{k!} \right)^2 \frac{\Gamma(2\Delta + n)}{\Gamma(2\Delta + k)} \frac{\Gamma(\Delta + k)}{\Gamma(\Delta - k)}.
$$

Thus the $c \to \infty$ limit of the 1-point block reads:

$$
\lim_{c \to \infty} q^{-\Delta} \hat{F}_{c,\Delta}^\lambda(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{\Gamma(2\Delta)}{\Gamma(2\Delta + k)} \frac{\Gamma(\Delta + k)}{\Gamma(\Delta - k)},
$$

which yields:

$$
\sum_{n=0}^{\infty} q^n f_{c,\Delta}^\lambda = \frac{1}{1-q} \frac{2F_1}{2\Delta,1-\Delta;2\Delta;\frac{q}{q-1}}.
$$

The $\Delta \to \infty$ asymptotics of the block is even easier to obtain and leads to a more convenient recursion. With the help of the Ward identities one easily shows that for $\Delta \to \infty$:

$$
\rho(\nu_{\Delta,N}, \nu_{\lambda}, \nu_{\Delta,M}) = \rho(\nu_{\Delta,N}, \nu_{Q}, \nu_{\Delta,M}) (1 + \mathcal{O}(\Delta^{-1}))
$$

where $\nu_{Q}$ is the highest weight state in the vacuum Verma module ($\Delta_{Q} = 0$). Since (for $|M| = |N| = n$):

$$
\rho(\nu_{\Delta,N}, \nu_{Q}, \nu_{\Delta,M}) = \left[ B_{c,\Delta}^n \right]_{NM}
$$

we get:

$$
\hat{h}_{c,\Delta}^\lambda = \lim_{\Delta \to \infty} F_{c,\Delta}^\lambda = \sum_{|M| = |N| = n} \left[ B_{c,\Delta}^n \right]_{NM} \left[ B_{c,\Delta}^n \right]^{MN} = \sum_{|N| = n} \delta_{N}^N = p(n)
$$

where $p(n)$ denotes the number of ways $n$ can be written as a sum of positive integers. Thus

$$
\lim_{\Delta \to \infty} \left( q^{\frac{c-1}{3} \Delta} \hat{F}_{c,\Delta}^\lambda(q) \right) = \prod_{n=1}^{\infty} (1-q^n)^{-1}.
$$

This suggest the following definition of the elliptic 1-point block on a torus:

$$
\mathcal{H}_{c,\Delta}^\lambda(q) = q^{\frac{c-1}{3} \Delta} \eta(q) F_{c,\Delta}^\lambda(q) = \sum_{n=0}^{\infty} q^n \mathcal{H}_{c,\Delta}^{\lambda,n},
$$

where $\eta(q)$ is the Dedekind eta function. One easily checks that the coefficients $\mathcal{H}_{c,\Delta}^{\lambda,n}$ have essentially the same pole structure as the coefficients $F_{c,\Delta}^{\lambda,n}$ and the following recursive formula holds:

$$
\mathcal{H}_{c,\Delta}^{\lambda,n} = \delta_{0}^n + \sum_{1 \leq r \leq s \leq n} A_{r,s}(c) P_{c}^{rs} \left[ \frac{\Delta_{\lambda}^{\Delta_{\lambda} + rs}}{\Delta - \Delta_{rs}(c)} \right] P_{c}^{rs} \left[ \frac{\Delta_{\lambda}}{\Delta_{rs}} \right] \mathcal{H}_{c,\Delta_{rs} + rs}^{\lambda,n-rs}.
$$

(9)
Let us note that the form of the regular terms in (9) can be also deduced from the limiting case \( \Delta \to 0 \) (\( \lambda \to Q \)) where the fusion polynomials, and therefore all the residua, vanish and the 1-point block become the Virasoro character.

### 4 Poghossian identities

Using (8) one can check the identity:

\[
P_{c}^{rs} \left[ \frac{\Delta_{r+s}}{\Delta_{r+s+rs}} \right] P_{c}^{rs} \left[ \frac{\Delta_{s}}{\Delta_{rs}} \right] = \prod_{p=1}^{r-1} \prod_{q=1}^{s-1} \left( \frac{\lambda}{2} + pb + qb^{-1} + \frac{Q}{2} \right) \times \left( \frac{\lambda}{2} + pb + qb^{-1} - \frac{Q}{2} \right)
\]

The r.h.s. can be identified as a product of fusion polynomials:

\[
P_{c}^{rs} \left[ \frac{\Delta_{r+s}}{\Delta_{r+s+rs}} \right] P_{c}^{rs} \left[ \frac{\Delta_{s}}{\Delta_{rs}} \right] = \prod_{p=1}^{r-1} \prod_{q=1}^{s-1} \left( \frac{\lambda}{2} + pb + qb^{-1} + \frac{Q}{2} \right) \times \left( \frac{\lambda}{2} + pb + qb^{-1} - \frac{Q}{2} \right)
\]

provided that one of the equalities:

\[
(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}) = \left( \frac{Q}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}, \frac{b}{2} \right),
\]

\[
(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}) = \left( \frac{1}{2b}, \frac{\lambda}{2}, \frac{\lambda}{2}, \frac{b}{2} \right),
\]

\[
(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}) = \left( \frac{b}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}, \frac{b}{2} \right),
\]

hold. Let us now recall that in the case of the 4-point elliptic block on the sphere

\[
\mathcal{H}_{c,\Delta} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right] (q) = 1 + \sum_{n=1}^{\infty} (16q)^{n} \mathcal{H}_{c,\Delta}^{n} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right]
\]

the recursive relation takes the form [13, 14]:

\[
\mathcal{H}_{c,\Delta}^{n} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right] (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right] (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right] (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right] (q).
\]

Since the solutions to the recursive formulae (9) and (12) are unique we get by comparing (9), (10) and (12) that

\[
\mathcal{H}_{c,\Delta}^{\lambda, n} = 16^{n} \mathcal{H}_{c,\Delta}^{\lambda, n} \left[ \frac{\Delta_{3}}{\Delta_{4}} \right]
\]

and therefore

\[
\mathcal{H}_{c,\Delta}^{\lambda, n} (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\lambda}{2(b-\frac{1}{b})}, \frac{\lambda}{2(b+\frac{1}{b})} \right] (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\lambda}{2(b-\frac{1}{b})}, \frac{\lambda}{2(b+\frac{1}{b})} \right] (q) = \mathcal{H}_{c,\Delta} \left[ \frac{\lambda}{2(b-\frac{1}{b})}, \frac{\lambda}{2(b+\frac{1}{b})} \right] (q).
\]

(13)
We have thus obtained a simple proof of the first relation proposed in [1].

In the present notation the second relation conjectured in [1] reads:

\[
\mathcal{H}_{c,\Delta} \left[ \frac{1}{r} \sum_{\eta} \right] (q) = \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2)
\]

\[
= \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2)
\]

\[
= \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2),
\]

where:

\[
c' = 1 + 6 \left( b' + \frac{1}{b'} \right)^2, \quad \Delta_{\chi'} = \frac{(b' + \frac{1}{b'})^2}{4} - \frac{(\chi')^2}{4},
\]

\[
b' = \sqrt{2} b, \quad \chi' = \frac{\lambda}{\sqrt{2}}.
\]

There is also another relation of a similar origin with \( b' = \frac{b}{\sqrt{2}} \):

\[
\mathcal{H}_{c,\Delta} \left[ \frac{1}{r} \sum_{\eta} \right] (q) = \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2)
\]

\[
= \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2)
\]

\[
= \mathcal{H}_{c',\Delta'} \left[ \frac{1}{r'} \sum_{\zeta} \right] (q^2),
\]

We shall show that relations of the form:

\[
\mathcal{H}_{c,\Delta} \left[ \frac{\eta}{\eta} \right] (q) = \mathcal{H}_{c',\Delta'} \left[ \frac{\chi_1}{\chi_1} \right] (q^2)
\]

are to large extent unique. Let us first observe that the residua of the coefficients of \( \mathcal{H}_{c,\Delta} \left[ \frac{\eta}{\eta} \right] (q) \) contain the fusion polynomial

\[
P_{c}^{rs[\eta]} = \prod_{p=1}^{r-1} \prod_{q=1}^{s-1} \left( \frac{2\eta + pb + qb^{-1}}{2} \right) \left( \frac{pb + qb^{-1}}{2} \right)
\]

which always vanishes if both \( r \) and \( s \) are odd. Moreover if \( \eta = \frac{1}{2} \) it vanishes for odd \( r \) and all \( s \). Since \( H_{c,\Delta}^{1} \left[ \frac{\eta}{\eta} \right] (q) = 0 \), it follows that for \( \eta = \frac{1}{2} \) all the odd coefficients of \( \mathcal{H}_{c,\Delta} \left[ \frac{\eta}{\eta} \right] (q) \) vanish and the even ones satisfy the recursive relation:

\[
H_{c,\Delta}^{2m} \left[ \frac{\eta}{\eta} \right] = \delta_{1}^{1} + \sum_{k,s \in \mathbb{N}} \sum_{1 \leq k,s \leq m} \frac{R_{c}^{2k,s[\eta\mu]} (\Delta - \Delta_{2k,s}) H_{c,\Delta_{2k,s}}^{2m-2ks[\eta\eta]}}{2k,s},
\]

\[\text{The case } \eta = \frac{1}{2} \text{ leading to the relation (16) can be analyzed in a similar way.}\]
This is to be compared with the recursive relation:

\[
H^{m}_{\nu', \Delta'} \left[ \lambda'_{4}, \lambda'_{3}, \lambda'_{2}, \lambda'_{1} \right] = \delta_{0}^{m} + \sum_{\nu', \Delta' \in \mathbb{N}} \left\{ \begin{array}{c}
\sum_{k,s \leq m} P_{\nu', \Delta'}^{k,s} \left[ \lambda'_{4}, \lambda'_{3}, \lambda'_{2}, \lambda'_{1} \right] \right. \\
\left. H^{m-ks}_{\nu', \Delta'_{ks} + ks} \left[ \lambda'_{4}, \lambda'_{3}, \lambda'_{2}, \lambda'_{1} \right] \right\}.
\]

The numbers of terms in both relations coincide. Moreover for the identification of parameters (15):

\[
(\Delta - \Delta_{2k,s}) = 2 \left[ \frac{1}{4} (\sqrt{2} kb + \frac{s}{\sqrt{2} b})^2 - \frac{\lambda^2}{8} \right] = 2(\Delta' - \Delta'_{ks}),
\]

and:

\[
\Delta_{2k,s} + 2ks = \Delta'_{ks} + ks.
\]

The fusion polynomials can be expressed in the form:

\[
P_{\nu'}^{2k,s} \left[ \frac{\mu}{\sqrt{2} b} \right] P_{\nu'}^{2k,s} \left[ \frac{\mu}{\sqrt{2} b} \right] = 4^{-ks} \prod_{p=1-2k}^{2k-1} \prod_{q=1-s}^{s-1} \left( \frac{p b + (q + 1)b^{-1}}{2} \right) \left( \frac{p b + q b^{-1}}{2} \right)
\]

\[
\times \prod_{p=1-k}^{k-1} \prod_{q=1-s}^{s-1} \left( \frac{\mu}{\sqrt{2} b} + \frac{1}{2b'} + \frac{b'}{2} + pb' + q \right) \left( \frac{1}{2} - \frac{1}{2b'} - \frac{b'}{2} + pb' + \frac{q}{b'} \right)
\]

As in the case of our previous derivation the last two lines can be identified as the fusion polynomials:

\[
P_{\nu'}^{2k,s} \left[ \frac{\mu}{\sqrt{2} b} \right] P_{\nu'}^{2k,s} \left[ \frac{\mu}{\sqrt{2} b} \right] = 4^{-ks} \prod_{p=1-2k}^{2k-1} \prod_{q=1-s}^{s-1} \left( \frac{p b + (q + 1)b^{-1}}{2} \right) \left( \frac{p b + q b^{-1}}{2} \right)
\]

\[
\times (16)^{ks} P_{\nu'}^{ks} \left[ \lambda'_{4} \right] P_{\nu'}^{ks} \left[ \lambda'_{4} \right] \]

where one of the following choices is assumed:

\[
(\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}) = \left( \frac{\mu}{\sqrt{2}} + \frac{1}{2b'} + \frac{b'}{2}, \frac{\mu}{\sqrt{2}} - \frac{1}{2b'} \right)
\]

\[
(\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}) = \left( \frac{\mu}{\sqrt{2}} + \frac{1}{2b'}, \frac{1}{2b'}, \frac{1}{2b'}, \frac{1}{2b'} \right)
\]

\[
(\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}) = \left( \frac{\mu}{\sqrt{2}} + \frac{1}{2b'}, \frac{b'}{2}, \frac{\mu}{\sqrt{2}} - \frac{1}{2b'} \right)
\]

\[
(\lambda'_{1}, \lambda'_{2}, \lambda'_{3}, \lambda'_{4}) = \left( \frac{\mu}{\sqrt{2}} + \frac{1}{2b'} + \frac{b'}{2}, \frac{1}{2b'}, \frac{1}{2b'}, \frac{b'}{2} \right)
\]
Finally calculating the coefficients $A_{rs}$ one gets:

$$A_{2k,s}(c) = 2^{1-2k_s} A_{ks}(c') \prod_{p=1-2k}^{2k-1} \prod_{q=1-s}^{s-1} \left( pb + (q+1)b^{-1} \right)^{-1} \left( pb + qb^{-1} \right)^{-1}, \quad (22)$$

where:

$$A_{ks}(c') = \frac{1}{2} \prod_{p=1-k}^{k} \prod_{q=1-s}^{s} (pb' + \frac{q}{b'})^{-1}. \quad (p,q) \neq (0,0),(k,s)$$

Taking into account formulae (18), (19), (20) and (22) one obtains:

$$\frac{R_{c}^{2k,s} \left[ \begin{array}{c}
\frac{\eta}{2} \\
\frac{\eta'}{2}
\end{array} \right]}{\Delta - \Delta_{rs}} = 16^{-ks} \frac{R_{c'}^{k,s} \left[ \begin{array}{c}
\lambda_1' \\
\lambda_2'
\end{array} \right]}{(\Delta' - \Delta'_{rs})}.$$  

Hence the coefficients $H_{2m,c\Delta}^{\eta,\eta} \left[ \begin{array}{c}
\lambda_1 \\
\lambda_2
\end{array} \right]$ and $(16)^{-m} H_{b',\Delta'}^{m} \left[ \begin{array}{c}
\lambda_1' \\
\lambda_2'
\end{array} \right]$ with the weights (21) satisfy the same recursive relations. This completes our proof of the formula (14). Formula (16) can be derived along the same lines starting with $\eta = \frac{b}{2}$.

5 Modular bootstrap in Liouville theory

The modular invariance of the 1-point function on the torus:

$$\langle \phi_{\lambda,\lambda} \rangle_{-\frac{1}{2}} = (-1)^{\Delta_{\lambda} - \Delta_{\lambda'} - 1} \tau \Delta_{\lambda} \tau \Delta_{\lambda} \langle \phi_{\lambda,\lambda} \rangle_{\tau} \quad (23)$$

along with the crossing invariance of the 4-point function on the sphere form the basic consistency conditions for any CFT on closed surfaces [18]. In the case of the Liouville theory the 1-point function can be expressed in terms of the elliptic blocks as follows:

$$\langle \phi_{\lambda} \rangle_{\tau} = \int \frac{d\alpha}{2\pi} \left| q^{\Delta_{\alpha} - \frac{\Delta_{\lambda}^{2}}{2}} \eta(q)^{-1} H_{c,\Delta_{\alpha}}^{\lambda}(\hat{q}) \right|^{2} C_{\Delta_{\alpha}}^{\lambda},$$

where $q = e^{2\pi i \tau}$. Condition (23) then takes the form:

$$\int \frac{d\alpha}{2\pi} \left| \tilde{q}^{\frac{\Delta_{\lambda}^{2}}{2}} H_{c,\Delta_{\alpha}}^{\lambda}(\hat{\tilde{q}}) \right|^{2} C_{\Delta_{\alpha}}^{\lambda} = \left| \tau \right|^{2\Delta_{\lambda} + 1} \int \frac{d\alpha}{2\pi} \left| q^{\frac{\Delta_{\lambda}^{2}}{2}} H_{c,\Delta_{\alpha}}^{\lambda}(q) \right|^{2} C_{\Delta_{\alpha}}^{\lambda} \quad (24)$$

where and $\tilde{q} = e^{-2\pi i \frac{\alpha}{\tau}}$. The Liouville structure constant reads [19]:

$$C_{\Delta_{\alpha}}^{\lambda} = \left[ \pi \mu \nu (b^{2} - 2b^{2}) \right]^{-\frac{Q_{\lambda}}{2b^{2}}} \times \frac{\mathcal{Y}_{\lambda}(Q + \lambda)\mathcal{Y}(\alpha)\mathcal{Y}(-\lambda)}{\mathcal{Y}^{2} \left( \frac{Q}{2} + \frac{\lambda}{2} \right) \mathcal{Y} \left( \frac{Q}{2} + \frac{\lambda}{2} + \alpha \right) \mathcal{Y} \left( \frac{Q}{2} + \frac{\lambda}{2} - \alpha \right)}. \quad (24)$$

11
Separating the $\alpha$-dependent part

$$C^\lambda_{\Delta_\alpha} = 4 \left[ \pi \mu \gamma (b^2) b^{2-2b^2} \right]^{-\frac{Q+\lambda}{2b^2}} \times \frac{\Upsilon_0 \Upsilon (Q+\lambda)}{\Upsilon^2 \left( \frac{Q}{2} + \frac{1}{2} \right)} r(\alpha)$$

$$r(\alpha) = -\frac{\alpha^2}{4} \exp \int_0^\infty \frac{dt}{t} \left( 1 + b^4 - b^2 (\lambda^2 + 2) \right) e^{-t} + \cosh(\alpha t) \frac{\cosh \left( \frac{\lambda t}{2} \right) - \cosh \left( \frac{(b^2-1)t}{2b} \right)}{\sinh \left( \frac{bt}{2} \right) \sinh \left( \frac{t}{2b} \right)}$$

one can write (24) as:

$$\int_{i\mathbb{R}} \frac{d\alpha}{2i} \left| \frac{\alpha^2}{2} \mathcal{H}^\lambda_{c,\Delta_\alpha} (\bar{q}) \right|^2 r(\alpha) = |\tau|^{2\Delta+1} \int_{i\mathbb{R}} \frac{d\alpha}{2i} \left| q^{-\frac{\alpha^2}{2}} \mathcal{H}^\lambda_{c,\Delta_\alpha} (q) \right|^2 r(\alpha). \quad (25)$$

This relation can be numerically analyzed with the help of the recursion relations derived in Section 3. Due to the rapidly oscillating integrand in (25), the numerical calculation of function $r(t)$ has to be carefully done. We present a sample of the calculations for $c = 2, \lambda = i$ and for the modular parameter $\tau$ along the imaginary axis in the range $[0.2i, 5i]$. The results for the elliptic block expanded up to the term $q^n$, $n = 4, 5, 6$ are presented on Fig.1, where the relative difference of the left and the right side of (25) is plotted.

![Graph](image)

**Fig.1** Numerical check of the modular bootstrap.

**Acknowledgements**

This work was supported by the Polish State Research Committee (KBN) grant no. N N202 0859 33. The work of L.H. was also supported by MNII grant 189/6.PRUE/2007/7.
References

[1] R. Poghossian, *Recursion relations in CFT and N=2 SYM theory*, arXiv:0909.3412 [hep-th].

[2] L. F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, arXiv:0906.3219 [hep-th].

[3] N. Wyllard, *$A_{N−1}$ conformal Toda field theory correlation functions from conformal $N=2$ SU(N) quiver gauge theories*, arXiv:0907.2189 [hep-th].

[4] A. Marshakov, A. Mironov and A. Morozov, *On Combinatorial Expansions of Conformal Blocks*, arXiv:0907.3946 [hep-th].

[5] D. Gaiotto, *Asymptotically free N=2 theories and irregular conformal blocks*, arXiv:0908.0307 [hep-th].

[6] A. Mironov, S. Mironov, A. Morozov and A. Morozov, *CFT exercises for the needs of AGT*, arXiv:0908.2064 [hep-th].

[7] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in N=2 gauge theory and Liouville modular geometry*, arXiv:0909.0945 [hep-th].

[8] A. Marshakov, A. Mironov and A. Morozov, *Zamolodchikov asymptotic formula and instanton expansion in N=2 SUSY $N_f = 2N_c$ QCD*, arXiv:0909.3338 [hep-th].

[9] A. Mironov and A. Morozov, *Proving AGT relations in the large-c limit*, arXiv:0909.3531 [hep-th].

[10] G. Bonelli and A. Tanzini, *Hitchin systems, N=2 gauge theories and W-gravity*, arXiv:0909.4031 [hep-th].

[11] V. Alba and A. Morozov, *Non-conformal limit of AGT relation from the 1-point torus conformal block*, arXiv:0911.0363 [hep-th].

[12] A. Zamolodchikov, *Conformal Symmetry In Two-Dimensions: An Explicit Recurrence Formula For The Conformal Partial Wave Amplitude*, Commun. Math. Phys. 96 (1984) 419.

[13] A. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, Sov. Phys. JETP 63 (1986) 1061.

[14] A. Zamolodchikov, *Conformal symmetry in two-dimensional space: recursion representation of conformal block*, Theor. Math. Phys. 73 (1987) 1088.

[15] L. Hadasz, Z. Jaskólski and P. Suchanek, *Recursion representation of the Neveu-Schwarz superconformal block*, JHEP 03 (2007) 032 [hep-th/0611266].

[16] V. A. Fateev, A. V. Litvinov, A. Neveu and E. Onofri, *Differential equation for four-point correlation function in Liouville field theory and elliptic four-point conformal blocks*, J. Phys. A 42 (2009) 304011 [arXiv:0902.1331 [hep-th]].

[17] A. Zamolodchikov, *Higher equations of motion in Liouville field theory*, Int. J. Mod. Phys. A 1982 (2004) 510 [arXiv:hep-th/0312279].
[18] H. Sonoda, *Sewing Conformal Field Theories*. 2, Nucl. Phys. B 311 (1988) 417.

[19] A. B. Zamolodchikov and Al. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. B 477, 577 (1996) [arXiv:hep-th/9506136].