Leaks identification on a Darcy model by solving Cauchy problem

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Abstract. In this work, we aim to identify leaks at a boundary of homogeneous porous media saturated with fluid. The identification is held on by solving a Cauchy problem on the Darcy equation with pressure formulation.

1. Introduction
Leaks identification have several possible fields of application as petroleum engineering, hydrogeology, agriculture, where boundary data at some edges of an oil or water reservoir is required without possibility of probing. The leaks identification may be also applied to backtracing of pollutants through an aquifer.

Previous works on leaks identification are reported by Belhachmi et al. [4] assuming the leak position and identifying its intensity. They derived an equivalent methodology from the one proposed by Santosa et al. [7] to detect interior cracks in a medium by using measurements of electrostatic field at the boundaries. We propose to identify leaks without any position assumptions, by solving an equivalent Cauchy problem, or data completion problem. Recently this formulation was updated by Andrieux et al. [2, 1, 3] as a minimisation of an energy-like functional, firstly introduced by Kohn and Vogelius [6].

The outline of this paper is as follows: in section two we define the direct Darcy problem, and in section three, the equivalent Cauchy problem is derived. In section four, we define the error functional and present the numerical procedure used to minimize this functional. In section five, we show some results about leaks identification with and without measurement noise and discuss some issues on numerical convergence.

2. Direct Darcy problem
We consider a homogeneous saturated porous media domain denoted by $\Omega$ as shown in figure 1. The direct problem is defined by the following incompressible Darcy equations:

$$\begin{cases}
u + \frac{k}{\mu} \nabla p = f & \text{and} \quad \nabla \cdot u = 0 \text{ in } \Omega \\
p = p_m \text{ on } \Gamma_m, \quad u \cdot n = q_m \text{ on } \Gamma_m
\end{cases}$$

(1)

where: $p$ is the pressure field, $u$ is the velocity field, $k$ is the homogeneous and isotropic permeability coefficient and $\mu$ the dynamic viscosity of the fluid. Without loss of generality, we
assume \( f = 0 \). We denote by \( \Gamma_m = \Gamma_a \cup \Gamma_b \cup \Gamma_c \) the boundary where the data are overspecified and by \( \Gamma_u \) the boundary where the leaks have to be identified.

Figure 1. Geometry and boundary conditions

3. Derived Cauchy problem
The direct problem can be transformed into a classical Laplacian Cauchy problem for the pressure field \( p \), by applying the divergence operator to the first equation of (1) which becomes

\[
\nabla \cdot \left( \frac{k}{\mu} \nabla p \right) = 0 \quad \text{in } \Omega,
\]

\[
p = p_m, \quad -\frac{k}{\mu} \frac{\partial p}{\partial n} = q_m \quad \text{on } \Gamma_m
\]

The problem above is known as Cauchy problem with overspecified data on \( \Gamma_m \). It is known since Hadamard [8] to be ill-posed in the sense that the dependence of \( p \) on the data \((q_m, p_m)\) is known to be not continuous. Solving Cauchy problem can be stated as follows: for all compatible pairs data \((q_m, p_m)\), find \((q_u, p_u)\) on \( \Gamma_u \) such that:

\[
\begin{align*}
\nabla \cdot \left( \frac{k}{\mu} \nabla p_1 \right) &= 0 \quad \text{in } \Omega, \\
p_1 &= p_m, \quad -\frac{k}{\mu} \frac{\partial p_1}{\partial n} = q_m \quad \text{on } \Gamma_m, \quad \text{and } p = p_u, \quad -\frac{k}{\mu} \frac{\partial p_u}{\partial n} = q_u \quad \text{on } \Gamma_u
\end{align*}
\]

4. Solving Cauchy problem by minimizing an error functional
We propose now to use the method developed by Andrieux et al. [2] to solve the Cauchy problem (3) defined on the pressure field \( p \). The idea is to use simultaneously the overspecified data by defining two well posed problem as follows:

\[
\begin{align*}
\nabla \cdot \left( \frac{k}{\mu} \nabla p_1 \right) &= 0 \quad \text{in } \Omega, \\
p_1 &= p_m, \quad -\frac{k}{\mu} \frac{\partial p_1}{\partial n} = \eta \quad \text{on } \Gamma_u
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \left( \frac{k}{\mu} \nabla p_2 \right) &= 0 \quad \text{in } \Omega, \\
p_2 &= \tau, \quad -\frac{k}{\mu} \frac{\partial p_2}{\partial n} = q_m \quad \text{on } \Gamma_m
\end{align*}
\]

Then, an error functional is defined as a function of the pressure fields \( p_1(p_m, \eta) \) and \( p_2(\tau, q_m) \) as follows:

\[
E(p_1, p_2) = \int_{\Omega} \nabla (p_1 - p_2) \cdot \frac{k}{\mu} \nabla (p_1 - p_2) \, d\Omega
\]

Using the properties of the \( p_i \) fields, we easily derive an alternate expression of \( E \):

\[
E(\eta, \tau) = \int_{\Gamma_u} \left( \eta - \nabla p_2 \cdot n \right) \frac{k}{\mu} (p_1 - \tau) + \int_{\Gamma_m} \left( \nabla p_1 \cdot n - q_m \right) \frac{k}{\mu} (p_m - p_2)
\]

The functional \( E \) is always positive and expresses an energy-like error between the two fields \( p_1 \) and \( p_2 \). Assuming that the data \( q_m \) and \( p_m \) are compatible, the functional \( E \) vanishes.
when the pair \((\eta, \tau)\) meets the real data \((q_u, p_u)\). Then the Cauchy problem (3) is solved with \(p_1 = p_2 + \text{Cst}\), for more details see Andrieux et al [2]. To conclude, a Cauchy problem is derived from the Darcy equations, which is solved by minimizing an energy-like error functional. The optimization problem can be written as follows:

\[
\begin{cases}
(q_u, p_u) = \arg \min_{\eta, \tau} E(p_1(\eta, p_m), p_2(q_m, \tau)) \\
\text{with } p_1, p_2 \text{ solutions of (4) and } (\eta, \tau) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)
\end{cases}
\]

(7)

5. Numerical experiments

In the current section, we apply the present method to a hydrogeology problem. We consider an underground aquifer flowed by liquid saturating a porous media. The goal is to identify leaks on an inaccessible part of the boundary by exploiting overspecified measurements on the remaining parts. We assume, for sake of simplicity, that the domain is a rectangle with constant pressure levels imposed on boundary \(\Gamma_a (P_a = -60)\) and \(\Gamma_b (P_b = -30)\) and with impermeable wall on the lower side. The inaccessible upper side may include one or various leaks (see figure 1). We identify a leak zone as a zero plateau on the pressure plot. Following, the measured data are generated numerically by solving the direct problem and we assume \(\frac{\mu}{\mu} = 1\).

5.1. Parametric analysis

Without leaks, the flow is uniform and unidimensional with a linear pressure drop as shown in figure 2 with dashed line. With leaks, we consider three different cases: a large leak, a small leak and and double leaks. Regular meshes are used: a 80 by 16 cells for the first case and a 120 by 16 cells mesh for both last cases. The stopping criterion \(\epsilon_0\) for the minimisation algorithm (Trusted region method) is set to \(10^{-7}\). Pressure and normal pressure gradient profiles on the unknown boundary \(\Gamma_u\) are presented from figures 2 to figure 7.

The figures 2, 4 and 6 show a very good reconstruction of the pressure in the three cases. In figures 3, 5 and 7, the normal pressure gradient curves are plotted and we measure a local volumic flow rate \(q_m = -\frac{\partial p}{\partial n}\) through the leak where the values are non zero. By convention, a
positive pressure gradient corresponds to an outward leak whereas a negative pressure gradient corresponds to an inward leak. The large leak is located between $x=1.75$ and $x=2.25$ (figures 2 and 3) and the small leak is located between $x=1.5$ and $x=1.6$ (figures 4 and 5). In the third case, two leaks of different sizes and positions are identified: they are visible on the pressure curve (figure 6 and 7). We can observe on figure 7 that the two leaks presents positive pressure gradients which correspond to outward flow.
In this last section, we investigate the effect of gaussian noise added to the overspecified data boundaries. First we show in figures 10, 12 and 14 the effect of the added noise on the pressure and the normal pressure gradient on $\Gamma_u$. The pressure footprint is clearly identifiable, although the noise is amplified due to the ill-posed nature of the Cauchy problem. Moreover the normal pressure gradient is rapidly pledge by large oscillations due to the derivation operation very sensitive to short wavelength perturbations.

At last, we investigated the effect of the noise on the numerical convergence for the case of the 80 by 8 cells mesh with 1% of added noise. The convergence of the errors $Err_{\Omega}$, $Err_{\Gamma_u}$ and $Err_{Qv}$ against the stopping criterion is plotted in figures 16 and 17. Contrary to the denoised results, we observe that the method is very sensitive to the stopping criterion and the convergence fails as $\varepsilon_0$ tends to zero. As the stopping criterion $\varepsilon_0$ is lowered, short wavelength modes those containing the noise information are amplified by minimisation algorithm. Generally speaking, the optimal

### Table 1. Computed values of volumic rate flow on $\Gamma_u$ for data completion problem (P1). The relative error is computed against the direct problem (P0) value $Q_v = 9.88$ (outward flow).

| FEM   | Linear | Linear | Linear | Linear | Linear | Quadratic | Quadratic |
|-------|--------|--------|--------|--------|--------|-----------|-----------|
| Mesh  | 80 x 4 | 80 x 8 | 80 x 16| 120 x 8| 120 x 16| 80 x 8    | 120 x 16  |
| $Q_v$ | 8.85   | 9.47   | 9.79   | 9.29   | 9.59   | 9.89      | 9.88      |
| $Err_{\Omega}$ | 2.60 x 10^{-3} | 2.09 x 10^{-3} | 1.81 x 10^{-3} | 1.68 x 10^{-3} | 1.68 x 10^{-3} | 8.55 x 10^{-3} | 6.10 x 10^{-3} |
| $Err_{\Gamma_u}$ | 5.8 x 10^{-3} | 6 x 10^{-3} | 6 x 10^{-3} | 4.9 x 10^{-3} | 5 x 10^{-3} | 3.47 x 10^{-3} | 3.35 x 10^{-3} |
| $Err_{Qv}$ | 4.57 x 10^{-7} | 3.79 x 10^{-7} | 3.03 x 10^{-6} | 1.58 x 10^{-9} | 8.83 x 10^{-7} | 9.43 x 10^{-6} | 5.7 x 10^{-5} |

5.2. Numerical analysis

In this section, sensitivity analysis of the convergence of the inverse problem results is performed in relation with the mesh size and the stopping criterion parameter. The third line of the table 1 presents the global volumic flow rate $Q_v = \int_{\Gamma_u} q_u dx$ evaluated on the direct problem (P0) solution. This value is used as a convergence criterion to the exact numerical solution against the mesh size. The 120 by 16 cells mesh case with the quadratic finite element is choosen as the reference value ($Q_v = 9.88$). Both linear and quadratic finite element methods converge in all cases to the reference solution.

To compare the inverse problem solution to the direct problem solution, we define three error measurements for the data completion problem:

$$Err_{\Omega} = \frac{|p_1 - p_0|_\Omega}{|p_0|_\Omega}, \quad Err_{\Gamma_u} = \frac{|p_1 - p_0|_{\Gamma_u}}{|p_0|_{\Gamma_u}}, \quad Err_{Qv} = \frac{|Q_v - Q_{v0}|_{\Gamma_u}}{|Q_{v0}|_{\Gamma_u}}$$

where $p_0$ and $p_1$ are the pressures with respect to the problems P0 and P1. For the linear finite elements, we observe in table 1 that these errors $Err_{\Omega}$ and $Err_{\Gamma_u}$ decrease with the mesh size proving that the method is convergent to the direct problem solution. In the case of the quadratic finite element, larger errors due to amplification of the numerical interpolation error during the inverse problem solving forbids a satisfactory convergence (figure 8).

In figure 9, the convergence of $Err_{\Omega}$ and $Err_{\Gamma_u}$ against the stopping criterion is very slow. Hence there is no need to carry on the minimisation algorithm beyond $\varepsilon_0 = 10^{-7}$ for a fine or medium mesh and $\varepsilon_0 = 10^{-4}$ for a coarse mesh. The error on $Err_{Qv}$ is negligible in all cases.

5.3. Noise effect analysis

In this last section, we investigate the effect of gaussian noise added to the overspecified data boundaries. First we show in figures 10, 12 and 14 the effect of the added noise on the pressure and the normal pressure gradient on $\Gamma_u$. The pressure footprint is clearly identifiable, although the noise is amplified due to the ill-posed nature of the Cauchy problem. Moreover the normal pressure gradient is rapidly pledge by large oscillations due to the derivation operation very sensitive to short wavelength perturbations.

At last, we investigated the effect of the noise on the numerical convergence for the case of the 80 by 8 cells mesh with 1% of added noise. The convergence of the errors $Err_{\Omega}$, $Err_{\Gamma_u}$ and $Err_{Qv}$ against the stopping criterion is plotted in figures 16 and 17. Contrary to the denoised results, we observe that the method is very sensitive to the stopping criterion and the convergence fails as $\varepsilon_0$ tends to zero. As the stopping criterion $\varepsilon_0$ is lowered, short wavelength modes those containing the noise information are amplified by minimisation algorithm. Generally speaking, the optimal
value of the stopping criterion may be computed, for the purpose of regularization for example, through a discrepancy principle method. Here, we evaluate in a similar manner and heuristically the best stopping criterion, typically $\epsilon_0 = 10^{-1}$, when the errors $\text{Err}_\Omega$ and $\text{Err}_{\Gamma_u}$ reach their minima against the mesh size dimension. In figure 17, the error $\text{Err}_{Q_v}$ remains convergent as $\epsilon_0$ decreases. The computation of the volumic rate flow $Q_v$ is still a good indicator of the leak intensity on $\Gamma_u$.

6. Conclusion and perspectives

In this paper, we presented a application of the Cauchy problem derived from the Darcy equations on leaks identification. Results in several configurations with regard to the size and position the leaks, \textit{a priori} unknown, were compared to the direct problem results: Leaks identification is always achievable by means of pressure fingerprint and volumic flow rate $Q_v$. The local volumic flow rate, i.e normal pressure gradient, is pledged by error with noisy data. A numerical analysis was addressed, showing convergence of the data completion problem to direct problem for linear finite element method. But convergence of the inverse problems with respect to the noise should be looked with care, due to the dependency of the stopping criterion to the noise level.

In the forthcoming works, a thoroughful analysis should be made to understand the issues of higher order finite elements and noise influence on minimisation process and its regularization. On the other hand, application of the Cauchy problem to more complex models (Stokes and Navier-Stokes) and geometries is underway.

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Figure 10. Pressure plots on $\Gamma_u$ for large leak configuration with 0.5% noise

Figure 11. Normal Pressure gradient plots on $\Gamma_u$ for large leak configuration with 0.5% noise

Figure 12. Pressure plots on $\Gamma_u$ for large leak configuration with 1% noise

Figure 13. Normal Pressure gradient plots on $\Gamma_u$ for large leak configuration with 1% noise
Figure 14. Pressure plots on $\Gamma_u$ for large leak configuration with 2% noise.

Figure 15. Normal pressure gradient plots on $\Gamma_u$ for large leak configuration with 2% noise.

Figure 16. Measures of $Err_\Omega$ and $Err_{\Gamma_u}$ against $\epsilon_0$ for 1% noised large leak setup.

Figure 17. Measures of $Err_{Qv}$ against $\epsilon_0$ for 1% noised large leak setup.