Budget Feasible Mechanisms

Christos Papadimitriou∗ Yaron Singer†

Abstract

We study a novel class of mechanism design problems in which the outcomes are constrained by the payments. This basic class of mechanism design problems captures many common economic situations, and yet it has not been studied, to our knowledge, in the past. We focus on the case of procurement auctions in which sellers have private costs, and the auctioneer aims to maximize a utility function on subsets of items, under the constraint that the sum of the payments provided by the mechanism does not exceed a given budget. Standard mechanism design ideas such as the VCG mechanism and its variants are not applicable here. We show that, for general functions, the budget constraint can render mechanisms arbitrarily bad in terms of the utility of the buyer. However, our main result shows that for the important class of submodular functions, a bounded approximation ratio is achievable. Better approximation results are obtained for subclasses of the submodular functions. We explore the space of budget feasible mechanisms in other domains and give a characterization under more restricted conditions.

∗Computer Science Division, University of California at Berkeley, CA, 94720 USA, christos@cs.berkeley.edu
†Computer Science Division, University of California at Berkeley, CA, 94720 USA, yaron@cs.berkeley.edu
1 Introduction

Consider the following familiar problems:

- **Knapsack**: Given a budget $B$ and a set of items $N = \{1, \ldots, n\}$, each with cost $c_i$ and value $v_i$, find a subset of items $S$ which maximizes $\sum_{i \in S} v_i$ under the budget constraint.

- **Matching**: Given a budget $B$ and a bipartite graph, with set of edges $N = \{e_1, \ldots, e_n\}$ each with cost $c_e$ and value $v_e$, find a legal matching $S$ which maximizes $\sum_{e \in S} v_e$ under the budget constraint.

- **Coverage**: Given a budget $B$ and subsets $N = \{T_1, \ldots, T_n\}$ of some ground set, each with a cost $c_i$ find a subset $S$ which maximizes $|\cup_{i \in S} T_i|$ under the budget constraint.

Three much studied, and much solved, optimization problems. However, suppose that the elements of $N$ are not combinatorial objects, but strategic agents with private costs. Then the above problems capture natural economic interactions: Knapsack, for example, models a simple procurement auction, while Coverage may model the problem of maximizing advertising effectiveness under a budget in a social network\[18, 24\]. These are precisely the kinds of economic interactions we wish to study here: reverse auctions with private costs, with the goal of optimizing the auctioneer’s value.

At first glance it may seem that the problem we describe falls within the well understood class of single-parameter domain. However, closer inspection reveals a new dimension of difficulty: the budget constraint applies not to the costs but to the payments the mechanism uses to support truthfulness. We need mechanisms whose sum of payments never exceeds the given budget.

Can we design mechanisms that implement these intended economic interactions in the most favorable way to the auctioneer without the payments exceeding the budget?

Mechanism design is by now a very mature discipline and the recent injection of computational thought has helped develop it even further, and in new and forward-looking directions \[28\]. Procurement auctions, introduced to computer scientists already in \[28\], were at first studied under utilitarian objectives, seeking to optimize social welfare \[28, 14\]. More recently procurement auctions have been studied under the non-utilitarian framework of frugality \[10, 3, 17, 13, 30\] — essentially, payment optimization in reverse auctions.

These situations still fall within classical Mechanism Design Theory, where the set of possible outcomes is a priori fixed and publicly known. By “set of all possible outcomes” here we mean the set of all possible allocations, with payments projected out. In other words, there is a rich class of allocations, independent of payments, each of which is realizable by a truthful mechanism. In the three introductory examples, however, the set of possible outcomes is not fixed or publicly known: It depends crucially on the participants’ private information, ultimately on the mechanism’s payments. It is this peculiarity that makes these three problems difficult, and places them at a blind spot of mechanism design.

**Budget Feasible Mechanisms**

We say a truthful mechanism is budget feasible if its payments do not exceed a given budget. In single parameter domains, where each agent’s private information is a single number, designing truthful mechanisms often reduces to designing monotone allocation rules, since payments can be computed via binary search \[25\]. This no longer holds when the payments are restricted by a

---

1Think of $T_i$ as the set of friends of agent $i$. 

budget: Designing a budget feasible allocation rule requires understanding its payments, which in-turn depend on the allocation rule itself. Not surprisingly, budget feasible mechanisms are very tricky to find.

The VCG mechanism does not work. Consider a simple Knapsack instance where all items have identical values, and except for one item whose cost equals the budget, all items have small costs. The VCG mechanism will choose the \( n - 1 \) small-cost agents, paying the budget to each. Thus, while this mechanism returns the optimal solution with total cost within the budget, the total payment will be way over budget (in fact, \((n - 1)\) times the budget).

In general, nothing can work. Consider a slight variation of the above problem, in which all items have small costs, and identical values as long as a particular item \( i \) is in the solution, and otherwise all have value 0 (for example, think of \( i \) as a corkscrew and the rest of the items as bottles of wine). How well can a budget feasible mechanism do here? If the mechanism has a bounded approximation ratio it must always guarantee to include \( i \) in its solution. This however implies that as long as \( i \) declares a cost that is less than the mechanism’s budget, the mechanism includes her in the solution. A truthful mechanism must therefore surrender its entire budget to \( i \). This of course results in an unbounded approximation ratio.

Our Results

The question, then, is: Which classes of functions have budget feasible mechanisms with good approximation properties? Our main result is a randomized constant factor budget feasible mechanism that is universally truthful for the quite general, and important, class of nondecreasing submodular functions (Theorem 4.5).

It seems that there is not much hope beyond submodular functions. For a slightly broader class, that of fractionally subadditive functions, we show that computational constraints dictate a lower bound. As shown in the simple example above, superadditive functions bring out the clash between truthfulness and the budget constraint. On a positive note, the three problems in the beginning of the section correspond to subclasses (additive, XOS, and coverage) of submodular maximization problems. We show improved approximations for these problems and other special cases. We further explore the space of budget feasible mechanisms, showing several impossibilities as well as a characterization under more restricted conditions.

Related Work

Budgets in auctions. Budgets came under scrutiny in auction theory [11, 9, 15, 8] after observing behavior of bidders in online automated auctions [2], as well as in spectrum auctions where bidding is performed by groups of strategic experts [9]. While these pioneering works highlight the significance and challenges that budgets introduce to mechanism design, they relate to an entirely different concept than the one we study here. While these works study the impact of budgets on strategic bidders, our interest is to explore the budget’s effect on the mechanism. These papers, however, do point out the complexity induced by budget constraints in mechanism design, and the need for approximations.

Frugality. In recent years a theory of frugality has been developed with the goal of providing mechanisms for procurement auctions that admit minimal payments [10, 3, 17, 13, 30]. Budget feasibility and frugality are complementary concepts. Frugality is about buying a feasible solution at minimum cost — there are no preferences among the solutions, and the goal is to minimize
payment. In our setting we have no preferences among payments — as long as they are below the budget — but we do care about the value of the solutions.

**Cost Sharing.** Somewhat conceptually closer to our work is the subject of cost sharing, in which agents have private values for a service, there is a nondecreasing cost for allocating the service to agents, and the goal is to maximize the agents’ valuations under the cost (see [16] for a survey). Here, our goal is non-utilitarian — we aim to maximize the buyer’s demand, which is independent of the agents’ utilities — and this makes a huge difference in the available options.

**Paper Organization**

After the necessary definitions in Section 2, we present a mechanism for the class of symmetric submodular functions (Section 3); this special case simplifies the problem enormously and facilitates the introduction of ideas and intuition for the general submodular case. Our main result for submodular functions (Section 4) is developed in Section 4. Finally, in Section 6, we further discuss the space of budget feasibility, improved approximations, impossibility results and characterization.

**2 The Model**

In a budget-limited reverse auction we have a set of items \([n] = \{1, \ldots, n\}\), and a single buyer. Each item \(i \in [n]\) is associated with a cost \(c_i \in \mathbb{R}_+\), while the buyer has a budget \(B \in \mathbb{R}_+\) and a demand valuation function \(V : 2^n \to \mathbb{R}_+\). In the full information case, costs are common knowledge, and the objective is to maximize the demand function under the budget, i.e. find the subset \(S \in \{T|\sum_{i \in T} c_i \leq B\}\) for which \(V(S)\) is maximized.

We focus on the strategic case, in which each item is held by a unique agent and costs are private. The budget and demand function of the buyer are common knowledge. A solution is a subset and payment vector, and the objective is to maximize the demand function while the payments (not costs) are within the budget. More formally, a mechanism \(M = (f, p)\) consists of an allocation function \(f : \mathcal{R}_+^n \to 2^n\) and a payment function \(p : \mathcal{R}_+^n \to \mathbb{R}_+^n\). The allocation function \(f\) maps a set of \(n\) bids to a subset \(S = f(c_1, \ldots, c_n) \subseteq [n]\). The payment function \(p\) returns a vector \(p_1, \ldots, p_n\) of payments to the agents. We shall often omit the arguments \(c_1, \ldots, c_n\) when writing \(f\) and \(p\).

1. **Truthful,** that is, reporting the true costs is a dominant strategy for sellers. Formally, a mechanism \(M = (f, p)\) is truthful or incentive compatible if for every \(i \in [n]\) with cost \(c_i\) and bid \(c_i'\), and every set of bids by \([n] \setminus \{i\}\) we have \(p_i - s_i \cdot c_i \geq p_i' - s_i' \cdot c_i\), where \((s_i, p_i)\) and \((s_i', p_i')\) are the allocations and payments when the bidding is \(c_i\) and \(c_i'\), respectively. A mechanism that is a randomization over truthful mechanisms is universally truthful.

2. **Computationally Efficient.** The functions \(f\) and \(p\) can be computed in polynomial time. In cases where the demand function requires exponential data to be represented (as in the general submodular case), we take the common “black-box” approach and assume the buyer has access to an oracle which allows evaluating any subset \(S \subseteq [n]\), with polynomially many queries. Such queries are known as value queries. This is a weaker model than ones allowing demand or general queries (see [7] for a definition). Since our main interest here is algorithmic, this strengthens our results.
3. **Budget Feasible.** Importantly, we require that a mechanism’s allocation rule and payments do not exceed the budget: \( \sum_i p_i s_i \leq B \). We call such mechanisms **budget feasible**.

4. **Approximation.** We want the allocated subset to yield the highest possible value for the buyer. For \( \alpha \geq 1 \) we say that a mechanism is \( \alpha \)-approximate if the mechanism allocates to a set \( S \) such that \( V(S^*) \leq \alpha V(S) \), where \( S^* \) is the full information optimum solution. As usual, when dealing with randomization we seek mechanism that yield constant factor approximations in expectation.

This is a **single parameter** mechanism design problem, in that each bidder has only one private value. We shall repeatedly rely on Myerson’s well-known characterization\(^2\).

**Theorem 2.1** (\[26\]). In single parameter domains a normalized mechanism \( M = (f, p) \) is truthful iff:

(i) \( f \) is monotone: \( \forall i \in [n], \text{ if } c'_i \leq c_i \text{ then } i \in f(c_i, c_{-i}) \text{ implies } i \in f(c'_i, c_{-i}) \) for every \( c_{-i} \);

(ii) winners are paid threshold payments: payment to each winning bidder is \( \inf \{ c_i : i \notin f(c_i, c_{-i}) \} \).

3 **Symmetric Submodular Functions**

We now introduce a subclass of submodular functions which is devoid of many of the intricacies of the general case. It will serve as an exposition of the basic ideas, and will help understand the difficulties in the general case.

We say a set function is **symmetric** if it only depends on the cardinality of the set, rather than the identity of the items it holds. Symmetric submodular functions (also called **downward sloping**), were used by Vickrey in his seminal work on multi-unit auctions \[31\]. They have a very simple structure:

**Definition 3.1.** A function \( V : 2^{[n]} \rightarrow \mathbb{R}_+ \) is symmetric submodular if there exist \( r_1 \geq \ldots \geq r_n \geq 0 \), such that \( V(S) = \sum_{i=1}^{\lfloor S \rfloor} r_i \).

Consider the following mechanism \( f_M \): Sort the \( n \) bids so that \( c_1 \leq c_2 \leq \ldots \leq c_n \), and consider the largest \( k \) such that \( c_k \leq B/k \). That is, \( k \) is the place where the curve of the increasing costs intersects the hyperbola \( B/k \). Our mechanism allocates to the set \( \{1, 2, \ldots, k\} \). This is obviously a monotone allocation rule: an agent cannot be excluded when decreasing her bid. The only question is, what is the threshold payment? In the Appendix we show that the threshold payments are \( \theta_i = \min\{B/k, c_{k+1}\} \) for all \( i \leq k \).\(^3\)

Obviously, the payments satisfy the budget constraint. Furthermore, the mechanism is truthful, because the payments were taken to be the threshold payments for this allocation rule, as required by Myerson’s Theorem. Hence the mechanism is budget feasible. Importantly, this is also a good approximation of the optimum solution:

**Theorem 3.2.** The above mechanism has approximation ratio of two.

\(^2\)Note that although there is a budget constraint on the payments, Myerson’s characterization applies to our setting as well. Due to the characterization, we know that the allocation function determines the payment function. The budget constraint can therefore be viewed as a property of the allocation function alone.

\(^3\)It is rather interesting that the second term is needed; we show in the Appendix that the mechanism breaks down in its absence.
Proof. Observe that the optimal solution is obtained by greedily choosing the lowest-priced items until the budget is exhausted. By the downward sloping property, to prove the result it suffices to show that the mechanism returns at least half of the items in the greedy solution. Assume for purpose of contradiction that the optimum solution has $\ell$ items, and the mechanism returns less than $\ell/2$. It follows that $c_{\lceil \ell/2 \rceil} > 2B/\ell$. Note however, that this is impossible since we assume that $c_{\lceil \ell/2 \rceil} \leq \cdots \leq c_\ell$, and $\sum_{i=\lceil \ell/2 \rceil}^{\ell} c_i \leq B$ which implies that $c_{\lceil \ell/2 \rceil} \leq 2B/\ell$, a contradiction. □

We now show that no better approximation ratio is possible. Given the simplicity of the full-information problem this is rather surprising and illustrates the intricacies of budget feasibility.

Proposition 3.3. For symmetric submodular functions, no budget feasible mechanism can guarantee an approximation of $2 - \epsilon$, for any $\epsilon > 0$.

Proof. Suppose that $v(S) = |S|$, and we have $n$ items with costs $c_i = \delta$, and $c_1 = c_2 = \cdots = c_{n-1} = B - \delta$, for some small $\delta > 0$. In this case the optimal solution has value 2. Assume for purpose of contradiction that we have a $2 + \epsilon$ budget feasible approximation, for some $\epsilon > 0$. Note that in that case $i^*$ must be in every winning set. Therefore, according to Myerson’s characterization the threshold payment is the supremum of costs $i^*$ can declare and remain allocated which in our case is $B$. The budget feasibility constraint prevents us from adding another element, leaving us with a 2 approximation, in contradiction to our assumption. □

The above propositions, together with Theorem 3.2 imply the following.

Theorem 3.4. For symmetric submodular functions, there exists a budget feasible 2-approximation mechanism. Furthermore, no budget feasible mechanism can do better. □

4 General Submodular Functions

We now turn to the general case of nondecreasing submodular functions. A demand function $V$ is nondecreasing if $S \subseteq T$ implies $V(S) \leq V(T)$.

Definition 4.1. $V : 2^{[n]} \rightarrow \mathbb{R}_+$ is submodular if $V(S \cup \{i\}) - V(S) \geq V(T \cup \{i\}) - V(T) \forall S \subseteq T$.

In general, submodular functions may require exponential data to be represented. We therefore assume the buyer has access to a value oracle which given a query $S \subseteq [n]$ returns $V(S)$.

From a pure algorithmic perspective, even under a cardinality constraint, maximizing a submodular function is well known to be NP-hard, and an $1 - 1/e$ approximation ratio can be achieved by greedily taking items based on their marginal contribution [27]. When items have costs, variations of greedy on marginal contribution normalized by cost can achieve constant factor approximations, and even the optimal $1 - 1/e$ ratio [19, 20]. For submodular maximization problems that can be expressed as integer programs, rounding solutions of linear and nonlinear programs can, in some cases, achieve the optimal constant approximation ratio [1].

In designing truthful mechanisms for submodular maximization problems, the greedy approach is a natural fit, since it is monotone. Agents are sorted according to their increasing marginal contributions relative to cost, recursively defined by: $i + 1 = \arg\max_{j \in [n]} V_j(S_i)/c_j$ where $S_i = \{1, 2 \ldots, i\}$. To simplify notation we will write $V_i$ instead of $V_i(S_i)$. This sorting, in the presence of submodularity, implies:

$$V_1/c_1 \geq V_2/c_2 \geq \cdots \geq V_n/c_n.$$  \hspace{1cm} (1)

Notice that $V(S_k) = \sum_{i \leq k} V_i$ for all $k$. 5
4.1 The Difficulties

The mechanism of the previous section for symmetric functions can be generalized appropriately to work in certain cases; it is easy to see that it fails in others. One common remedy is to take the maximum between the largest value, call it \( V_{\text{max}} \), and the generalization of the symmetric mechanism. This works, with some minor adjustments, in many problems (e.g. for Knapsack, as shown later). The reason is, it assures us that the natural proportional share threshold payments

\[
\theta' = \min \{ B \cdot V_i / \sum_{i \in S} V_i, c_{k+1} \cdot V_i / V_{k+1} \},
\]

which are budget feasible, are also individually rational: \( \theta'_i \leq c_i \). Here by \( k + 1 \) we denote, as in the symmetric case, the first item not allocated by greedy. But even this fails in the general case.

Marginal Contributions are Affected by Costs. Recall the Coverage problem from the Introduction; it captures many of the difficulties of the problem. In Coverage, it is easy to see that the marginal contribution is not fixed, but depends on the subset allocated by the algorithm in the previous stages. That is, an agent’s marginal contribution depends on its position in the sorting. Therefore although the payments \( \theta' \) above are budget feasible, they cannot induce truthfulness because they rely on the agent’s marginal contribution, which depends on her cost, and hence the proportional share mechanism is hopeless here. Other simple allocation and payment schemes that are independent of the agent’s position in the sorting also fail. Another maneuver, replacing \( V_i \) by another quantity \( V_{i(j)} \) defined later, for some \( j > i \), may result in prices that are not individually rational. Another approach that may seem natural is to replace marginal contribution with Shapley values [16] since they make the proportional contribution of an allocated agent independent of her position in the sorting. This fails as well, since we show that it cannot approximate better than a factor of \( \sqrt{n} \) (see the Appendix).

Non-monotonicity of the Maximum Operator. Bounded approximation ratios for submodular maximization depend crucially on the ability to take the maximum between a greedy solution and the item with highest value. In the general case, as well in the case of Coverage, taking this maximum does not preserve monotonicity: simple examples show that an agent which declares a lower cost may cause the value of the allocated set to decrease (see the Appendix). We emphasize that the difficulty here is not directly related to budget feasibility, but to the interaction between submodular maximization and truthfulness. Similar difficulties were first addressed in the context of social welfare maximization in [25] and later in [6, 5].

4.2 Overview of Our Approach

Our approach is based in three ideas:

• First, we derive an alternative characterization of the threshold payments of the proportional share allocation rule. Since we know that this rule does not work, this may seem futile. However, the characterization implies that payments are “not too far” from the agents’ proportional contributions, and thus budget feasible within a constant factor.

• Our mechanism is designed in the light of this characterization, so that it uses an appropriate fraction of its budget.

• Finally, we divide the set of agents in a particular manner: for Coverage and its variants we are to take the maximum between the maximal element and the nonlinear relaxation of the problem defined on a smaller domain (described in the following section). For the general submodular case, we rely on randomization (thus obtaining a universally truthful mechanism).
4.3 Characterizing Threshold Payments

Let \( f_M(c, B, X) \) denote the allocation rule which sorts items in \( X \) as in \( \text{1} \), and allocates to items \( \{1, \ldots, k\} \) as long as they meet condition \( c_k \leq V_k / \sum_{j \leq k} V_j \). Observe that this condition is met for every \( \{1, \ldots, i\} \) when \( i \leq k \).

Consider running the mechanism without item \( i \), and let \( K_i = f_M(c, X \setminus \{i\}, B) \) denote the allocated set. Let us relabel the agents according to a relative marginal contribution sorting without \( i \): \( j + 1 = \argmax_{j \neq i} V_j(T_j)/c_q \), where \( T_j \) denotes the set \( \{1, \ldots, j\} \) according to this new order, \( T_0 = \emptyset \). Under this sorting we will write \( V_{i(j)} \) instead of \( V_i(T_j) \). For every element \( j \in K_i \), there is a marginal contribution value for \( i \) had she been placed instead of \( j \), denoted \( V_{i(j)} \). Declaring \( \tau_{i(j)} = V_{i(j)} \cdot c_{i(j)}/V_j \) — the cost-per-value of \( j \) normalized according to this value — would place \( i \) ahead of \( j \) in the sorting. Taking the minimum between this value and \( \rho_{i(j)} = B \cdot V_{i(j)}/V(T_{j-1} \cup \{i\}) \), the mechanism’s allocation condition at this stage guarantees \( i \) is allocated. Since \( V_{i(j)} \) monotonically decreases with \( j \) while \( c_{i(j)}/V_j \) increases, \( \tau_{i(j)} \) may have arbitrary behavior as a function of \( j \).

However, as we now show, taking the maximum of these values guarantees payments that support truthfulness.

**Lemma 4.2 (Payments Characterization).** The threshold payment for \( f_M \) is

\[
\theta_i = \max_{j \in [K_i] + 1} \left\{ \min\{\tau_{i(j)}, \rho_{i(j)}\} \right\}.
\]

**Proof.** Let \( r \) be the index for which \( \theta_i = \min\{\tau_{i(r)}, \rho_{i(r)}\} \). Declaring a cost below \( \theta_i \leq \tau_{i(r)} \) guarantees \( i \) to be within the first \( r \leq |K_i| + 1 \) elements in the sorting stage of the mechanism, with \( r - 1 \) items allocated. Since \( \theta_i \leq \rho_{i(r)} \), \( i \) will be allocated.

To see that declaring a higher cost prevents \( i \) from being allocated, consider first the case where \( \tau_{i(r)} \leq \rho_{i(r)} \). A higher cost places \( i \) after \( r \) in the sorting stage of the mechanism. If \( \tau_{i(r)} \) is the maximum over all \( \tau_{i(j)} \), \( j \in [K_i] + 1 \), reporting a higher cost places \( i \) after an element which is not allocated and therefore it will not be allocated. Otherwise, if \( \tau_{i(r)} < \tau_{i(j)} \), for some \( j \leq |K_i| + 1 \), by the maximally of \( r \) it must be the case that: \( \rho_{i(j)} \leq \tau_{i(r)} < \tau_{i(j)} \), and \( i \) will not be allocated as a cost above \( \rho_{i(j)} = B \cdot V_{i(j)}/V(T_{j-1} \cup \{i\}) \) will not meet the allocation condition.

In the second case when \( \tau_{i(r)} > \rho_{i(r)} \), if \( r \) is the index which maximizes \( \rho_{i(j)} \) over all indices in \([K_i] + 1\), reporting a higher cost will not meet the mechanism’s allocation condition at each index in \([K_i] + 1\). Otherwise, if there is some other index \( j \in [K_i] + 1 \) for which this maximum is achieved, then: \( \tau_{i(j)} \leq \rho_{i(j)} < \rho_{i(j)} \), and thus declaring a higher cost in this case places \( i \) after \( j \) in the sorting, and the mechanism will not consider \( i \).

We defer proving that the payments are indeed individually rational to the Appendix.

**Lemma 4.3 (Individual Rationality).** The mechanism is individually rational, i.e., \( c_i \leq \theta_i \).

Given the characterization above, we can now show that the threshold payments are no more than a constant factor away from agents’ proportional share. This is a key property which guides the design of the allocation rule. For a subset of agents \( X \) which will be specified later, our solutions will be of the form \( W = f(c, X \setminus \{i_{\text{max}}\}) \cup \{i_{\text{max}}\} \) where \( i_{\text{max}} = \argmax_{j \in X} V((j)) \).

**Lemma 4.4 (Payment Bounds).** For each agent \( i \), \( \theta_i \leq \left(\frac{6e-2}{e-1}\right) V_i \cdot B/V(W) \).

**Proof.** Let \( W' = K_i \cup \{i_{\text{max}}\} \). We first show that \( \theta_i \leq V_i \cdot B/V(W') \), and then that \( V(W) \leq \left(\frac{6e-2}{e-1}\right) V(W') \) which together imply the lemma. Let \( r \) be the index for which \( \theta_i = \min\{\tau_{i(r)}, \rho_{i(r)}\} \), and \( k' = |K_i| \). If \( r = k' + 1 \):

7
\[ \theta_i \leq \rho_{i(r)} = \frac{B \cdot V_{i(k+1)}}{V(W \cup \{i\})} \leq \frac{B \cdot V_i}{V(W')} \]

where the second inequality is implied from the decreasing marginal utilities property and the fact that \( V_i = V_i(T_{i-1}) \). Otherwise, if \( r \leq k' \), it is an allocated agent, and therefore:

\[ \theta_i \leq \alpha_{i(r)} = \frac{V_{i(r)} \cdot c_r}{V_r} \leq \frac{V_{i(r)} \cdot B}{V(W')} \leq \frac{V_i \cdot B}{V(W')} \]

To see that \( V(W) \leq \left( \frac{6e^2 - 2}{e - 1} \right) V(W') \), let \( OPT_+ \) be the optimal solution and \( \ell \) be the largest index s.t. \( \sum_{i \leq \ell} c_i \leq B \), both when defined on the set \( X \setminus \{i, i_{\text{max}}\} \). We argue that \( \sum_{j=1}^{k'+1} V_j > \sum_{j=k'+1}^{\ell} V_j \).

Assume, for purpose of contradiction that this inequality does not hold. Since the items in \( [\ell] \) do not exceed the budget, and the items are sorted according to their marginal contribution-per-cost, it follows that \( \frac{c_{k'+1}}{V_{k'+1}} \leq \frac{B}{\sum_{j=1}^{\ell} V_j} \). This, together with our assumption implies that \( c_{k'+1} \leq \frac{B \cdot V_{k'+1}}{\sum_{j=1}^{k'+1} V_j} \), which contradicts the maximality of \( k' \).

From submodularity, it can be shown that \( V([\ell+1]) \) is a \( c/(e - 1) \)-approximation of \( OPT_+ \) [19]. From the above we can therefore conclude that \( OPT_+ \leq c/(e - 1) V([\ell+1]) \leq (4e/(e - 1))V(W') \). Since \( V(W) \) is bound from above by the optimal solution, we have \( V(W) \leq OPT_+ + 2V(i_{\text{max}}) \leq \left( \frac{6e^2 - 2}{e - 1} \right) V(W') \).

### 4.4 Main Result

Given all the above, we can now prove our main theorem.

**Theorem 4.5.** For any submodular maximization problem there exists a constant factor approximation randomized mechanism in the value query model which is budget feasible and universally truthful. Furthermore, no budget feasible mechanism can do better than \( 2 - c \), for any fixed \( e > 0 \).

Our analysis shows our mechanism guarantees an approximation ratio of \( (52e^2 - 20e)/(e - 1)^2 \approx 112 \) in expectation. It is possible that tighter analysis can show the mechanism does better. We further discuss these points in Section 7.

**Proof.** Consider the following mechanism:

**A Budget Feasible Approximation Mechanism for Submodular Functions**

**Initialize:** \( X_1 \leftarrow \{i: c_i \leq B/2\} \), \( i_{\text{max}}(1) \leftarrow \arg\max_{i \in X_1} V_i(\emptyset) \)

\[ W \leftarrow \{i_{\text{max}}(1)\} \]

\[ B' \leftarrow \left( \frac{e - 1}{12e - 4} \right) B \]

\[ i \leftarrow \arg\max_{j \in X_1 \setminus \{i_{\text{max}}(1)\}} \frac{V_j(\emptyset)}{c_j} \]

**While** \( c_i \leq V_i \cdot B'/V(S_i) \)

**Do:** \( W \leftarrow W \cup \{i\} \)

\[ i \leftarrow \arg\max_{j \in X_1 \setminus \{i_{\text{max}}(1)\}} \frac{V_j(S_i)}{c_j} \]

**Output:** choose at random between \((W, \theta)\) and \((i_{\text{max}}, B)\)

The payment \( \hat{\theta} \) here is \( \hat{\theta}_{i_{\text{max}}(1)} = B/2 \) and \( \hat{\theta}_i = \min\{\theta_i, B/2\} \) for \( i \neq i_{\text{max}}(1) \), where \( \theta_i \) is as described in lemma 4.2 with budget \( B' = (e - 1)B/(12e - 4) \). Observe this budget is twice the bound from lemma 4.4. Let \( i_{\text{max}} = \arg\max_{i \in [n]} V_i([i]) \), \( X_1 = \{i \neq i_{\text{max}} | c_i \leq B/2\} \), and \( i_{\text{max}}(1) = \arg\max_{i \in X_1} V_i([i]) \). Let \( [k] = f_M(c, X_1 \setminus \{i_{\text{max}}(1)\}, B/\alpha) \), where \( \alpha = (12e - 4)/(e - 1) \). In
case $i_{\text{max}}$ is allocated, $B$ is clearly her threshold payment. If $W = [k] \cup \{i_{\text{max}(1)}\}$ is allocated, from the characterization lemma and the fact that $X_1$ consists only of agents with cost less than $B/2$, $\hat{\theta}$ as described above are clearly the threshold payments. Since $f_M$ runs with budget $B' = B/\alpha$, from lemma 4.3 we can conclude:

$$\sum_i \hat{\theta}_i = B/2 + \sum_{i \neq i_{\text{max}(1)}} \hat{\theta}_i \leq B/2 + \sum_{i \neq i_{\text{max}(1)}} \frac{\alpha V_i \cdot B'}{V(W)} \leq B.$$ 

The mechanism is therefore truthful and budget feasible. Individual rationality and monotonicity were discussed above, and the lower bound from the symmetric case applies here as well.

Similarly to what we have shown in lemma 4.3 one can show that $\alpha \cdot \sum_{j=1}^{\ell} V_j > \sum_{j=k+1}^{\ell} V_j$ where $\ell$ is the maximal index for which $\sum_{j \leq \ell} c_j \leq B$ when items are taken according to their marginal contribution relative to cost on $X_1 \setminus \{i_{\text{max}(1)}\}$. Similar to before, we know that $V([\ell+1]) \geq \left(\frac{e}{e-1}\right)OPT_{X_1}$, and for $W = [k] \cup \{i_{\text{max}(1)}\}$, we have:

$$OPT_{X_1} \leq \left(\frac{2\alpha + 2}{e-1}\right)V(W).$$

One can verify that when choosing at random between $W$ and $i_{\text{max}}$ we obtain the constant factor approximation ratio of $(52e^2 - 20e)/(e - 1)^2$. \qed

5 Improved Approximation for Specific Problems

In this section we discuss improving the upper bound for particular problems in the submodular domain. All the mechanisms we present are deterministic, are truthful and budget feasible.

Improved Approximation for Knapsack

Knapsack (additive functions): Given a budget $B$ and a set of agents $[n]$, each with nonnegative cost $c_i$ and value $V_i$, find a subset $S \in \arg\max_T \sum_{i \in T} c_i \leq B \sum_{i \in T} V_i$.

**Theorem 5.1.** For Knapsack there is a budget feasible 6-approximation mechanism. Also, no budget feasible mechanism can approximate within a factor better than $2 - \epsilon$, for any fixed $\epsilon > 0$.

In proof, consider the mechanism below which generalizes the mechanism for the symmetric submodular case by distributing the budget among the agents proportionally to their contribution to the value of the winning set.

A Budget Feasible Approximation Mechanism for Knapsack

| **Initialize:** Reorder bids s.t. $V_1/c_1 \geq \ldots \geq V_n/c_n$, $W \leftarrow \emptyset$, $k \leftarrow 1$, $i^* \leftarrow \max_{i \in [n]} V_i$ |
| **While** $c_k/V_k \leq B/\sum_{j \leq k} V_j$ |
| **Do:** $W \leftarrow W \cup \{k\}$ $k \leftarrow k + 1$ |
| **If** $\sum_{i \in W \setminus \{i^*\}} V_i > V_{i^*}$ *output:* $(W, \theta)$, else *output:* $(\{i^*\}, \theta^*)$ |

Denote $k$ to be the maximal index $i$ that respects $c_i \leq V_i \cdot B/\sum_{j \leq i} V_j$, and $W = [k]$. The payment here is $\theta_i = \min\{V_i \cdot c_{k+1}/V_{k+1}, V_i \cdot B/\sum_{j \in W} v_j\}$ when $W$ is allocated. (for $\theta^*$ as defined below.) Note that $[k] \subseteq [\ell]$. To explicitly define $\theta^*$, consider the set

$$S_{-i^*} = \left\{ i \in [n] : \frac{c_i}{v_i} \leq \frac{B}{\sum_{j \in [i] \setminus \{i^*\}} v_j} \right\} \cap \left\{ i \in [n] : \sum_{j \in [i] \setminus \{i^*\}} v_j \geq v_{i^*} \right\}$$

9
and let \( r \) denote the minimum index in \( S_{-i^*} \). Then, \( \theta^* \) is defined as follows:

\[
\theta^* = \begin{cases} 
V_i^* \cdot \frac{c_r}{v_r} \quad & S_{-i^*} \neq \emptyset \\
B & \text{otherwise}
\end{cases}
\]

(2)

Proposition 5.2. The mechanism is truthful.

Proof. It is easy to verify that the mechanism is monotone. When \([k]\) is allocated, to witness \( \theta_i \) is the threshold payment, observe that declaring a cost \( c_i' < \theta_i \) places \( i \) before the first element for which the mechanism terminates. Since \( c_i' < \theta_i \leq V_i \cdot B / \sum_{j \leq i} V_i \), it will be allocated. When \( \theta_i = V_i \cdot c_{k+1} / V_{k+1} \) declaring any cost \( c_i' > \theta_i \) places \( i \) after agent \( k + 1 \) and since \( c_{k+1} / v_{k+1} > B / \sum_{j \leq k+1} V_j \), it will not be allocated. When \( \theta_i = V_i \cdot B / \sum_{j \leq k} V_j \) declaring a higher cost will place at least \( k - 1 \) items before \( i \), and it will not be allocated.

When \( \{i^*\} \) is allocated, we have that \( V_i^* > \sum_{j \in [k] \setminus \{i^*\}} V_j \). If \( S_{-i^*} \neq \emptyset \) declaring any value below \( \theta^* \) places \( i^* \) ahead of \( r \). The minimality of \( r \) implies that \( i^* \) will then be allocated. Note that \( c_r / v_r > B / \sum_{j \in [r]} V_j \), as otherwise \( \{i^*\} \) will not be allocated. Therefore, declaring a cost above \( \theta^* \) prevents \( i^* \) from being allocated. In case no such \( r \) exists, this implies that \( i^* \) is allocated by declaring any cost below \( B \).

Proposition 5.3. The mechanism is individually rational and budget feasible.

Proof. To see that the mechanism is individually rational and budget feasible, since

\[
c_1 / V_i \leq \ldots \leq c_k / V_k \leq \min \left\{ \frac{c_{k+1}}{V_{k+1}}, \frac{B}{\sum_{i \leq k} V_i} \right\} \leq \frac{B}{\sum_{i \leq k} V_i}
\]

in the case where \([k]\) is the allocated set, the second inequality from the right implies individual rationality and the rightmost inequality implies budget feasibility. If \( \{i^*\} \) wins, it is either paid \( B \) which is trivially individually rational and budget feasible, or \( V_i^* \cdot c_r / v_r \), which may occur in cases where \( i^* \in \{1, \ldots, k\} \), and \( r \) is the index as defined above. Observe that in such cases it must be that \( r > k \), therefore \( c_r / v_r \leq c_r / v_r \), and the payment is individually rational. Since \( r \) is chosen s.t. \( \sum_{i \in [r] \setminus \{i^*\}} V_i \geq V_i^* \) under the condition that \( c_r / v_r \leq B / \sum_{j \in [r] \setminus \{i^*\}} V_i \), it follows that \( V_i^* \cdot c_r / v_r \) is budget feasible.

Proposition 5.4. The mechanism is a 6-approximation.

Proof. Similarly to lemma 4.4 for \( k \) and the maximal item \( \ell \) for which \( \sum_{i \leq \ell} c_i \leq B \) we can show that \( \sum_{i=1}^{k+1} V_i > \sum_{i=k+1}^{\ell} V_i \). Since \( \sum_{i \leq \ell + 1} V_i \) is an upper bound on the optimal solution, it can easily be derived we have a 4 approximation when \( \sum_{i \leq k} V_i > V_i^* \), and 6 approximation otherwise.

Proposition 5.5. For additive functions, no budget feasible mechanism can guarantee an approximation of \( 2 - \epsilon \), for any fixed \( \epsilon > 0 \).

Proof. Consider a set of items \([n]\) with costs greater than 0, where the value of the optimal subset \( S^o \) is \( V(S^o) = d \) for some \( d \in \mathbb{R}_+ \) and \( \sum_{i \in S^o} c_i < B \). Introducing a new item \( i^* \), with value \( d \) and small cost \( \delta > 0 \), we have that the optimal solution in \([n] \cup \{i^*\}\) is \( 2d \). Any mechanism which approximates within factor \( 2 - \epsilon \), for any \( \epsilon > 0 \), must include \( i^* \), as well as some other element \( j \in [n] \). Therefore, the payment to \( i^* \) must be \( B \). To be individually rational it must be that \( p_j > 0 \), thus contradicting budget feasibility.
Improved Approximation for Matching

Matching (OXS functions): Given a budget $B$, bipartite graph $G = (V, E)$, where each edge $e \in E$ has nonnegative cost $c_e$ and weight $w_e$, find a matching $S$ in the set of all legal matchings $M$ s.t. $S \in \arg \max_{T \in M} \sum_{e \in T} c_e \leq B \sum_{e \in T} w_e$.

Theorem 5.6. For Matching there is a budget feasible $(\frac{5e-1}{e-1})$-approximation mechanism. Also, no budget feasible mechanism can approximate within a factor better than $2 - \epsilon$, for any fixed $\epsilon > 0$.

This problem corresponds with maximizing an OXS function, which contains additive functions and is known to be submodular [22], and the lower bound from Knapsack therefore holds. In this case, an agent’s marginal contribution to all subsets that do not contain a neighboring edge is her weight, and 0 otherwise. When applying the proportional share rule $f_M$ an agent may try to decrease her cost below that of a neighboring edge (since otherwise, if the mechanism allocates to its neighbor first, the agent’s marginal contribution drops to 0). This however, can be easily taken care of by a small adjustment to the threshold payments shown for Knapsack above. Let $\{k\} = f_M(c, [n] \setminus \{i_{\text{max}\}}), B)$ and Let $r = \min \{\min_{d \in N(e)} \{c_d/w_d, c_{k+1}/w_{k+1}\}\}$ where $N(e)$ is the set of all edges that share a vertex with $e$. The payment rule in this case would be:

$$\theta_e = \min \left\{ \frac{w_e \cdot B}{\sum_{j \leq k} w_j}, \frac{w_e \cdot c_r}{w_r} \right\}.$$  

The same reasoning as in Knapsack can be applied here to show these are indeed the threshold payments of $f_M$. An approximation for this case can be derived by taking the maximum between $V([k]) = \sum_{e \leq k} w_e$ and $V(i_{\text{max}}) = w_{i_{\text{max}}}$. Observe that if an agent is allocated, then there are no other agents that share its edge in a solution, and therefore the value of the $f_M$ does not decrease when an agent reduces her cost. Thus, taking the maximum between $i_{\text{max}}$ and $|k|$ preserves monotonicity in this case. Similarly to the cases above, it can be shown that $\sum_{e=1}^{k+1} w_e > \sum_{e=k+1}^{l} w_e$ which, together with submodularity, leads to the conclusion that $OPT \leq OPT_{[n] \setminus \{i_{\text{max}}\}} + V(\{i_{\text{max}}\}) \leq \left(\frac{5e-1}{e-1}\right) V([k])$ when $V([k]) \geq V(\{i_{\text{max}}\})$, and similarly this bound holds for $i_{\text{max}}$ when $V(\{i_{\text{max}}\}) > V([k])$.

The Maximum Operator for Coverage

Note that the randomization used for the general submodular case was required, as taking the maximum in general cases of submodular functions does not preserve truthfulness. In the Appendix we give a simple example where this happens with Coverage functions. Still, for the Coverage problem, one can avoid randomization by using a sophisticated technique that uses relaxation of a nonlinear program.

The following method can be applied to Coverage problems. Our method is inspired by [2] and [21] who display the powerful use of LP relaxations for designing truthful mechanisms. First, we partition the set of agents into three disjoint sets: $X_0 = \{i_{\text{max}}\}, X_1 = \{i \in [n] \setminus \{i_{\text{max}}\} : c_i \leq B/2\}, X_2 = \{i \in [n] \setminus \{i_{\text{max}}\} : c_i > B/2\}$. Let $i_{\text{max}(1)} = \arg \max_{i \in X_1} V(\{i\})$. In [1] Ageev and Svirdenko show that for the budgeted Coverage problem there is a nonlinear programming relaxation which an optimal solution that can be computed in polynomial time and rounded deterministically, via an extension of their “pipage rounding” method. They prove the integrality gap is $e/(e-1)$. We therefore compare the solution of the nonlinear relaxation over $X_1$ and $i_{\text{max}}$, taking $f_M(X_1) \cup \{i_{\text{max}(1)}\}$ as the solution if the relaxation solution is greater, and $i_{\text{max}}$ otherwise.

An important observation is that if $W = f_M(c, X_1) \cup \{i_{\text{max}(1)}\}$ is a $\beta$ approximation of the optimal integral solution over the $X_1$, it is at most $\beta \cdot e/(e-1)$ of the optimal fractional solution. This
guarantees the constant factor approximation ratio when selecting based on comparison between
the relaxation and $i_{\text{max}}$.

Note that this procedure is monotone. Since the nonlinear relaxation find the optimal solution,
decreasing one’s value can only increase the solution of the relaxation, applying the maximum
operator on the relaxation guarantees monotonicity is preserved. In case $i_{\text{max}}$ is allocated her
payment is $B$, and if a subset of $X_1$ is allocated by a monotone allocation rule with payments $\theta$,
adjusting the payments to be $\min\{\theta, B/2\}$ achieves truthfulness.

In [5] Azar and Gamzu suggest an elegant technique that enables monotone approximations for
multiple Knapsack problems in social welfare maximization. The setting works for problems where
there are several monotone mechanisms, each with a good approximation on a subset of items, but
not necessarily on the entire set, and taking the maximum breaks monotonicity. For problems that
can be formulated as integer programs, rather than taking the maximum between two monotone
solutions, the method in [5] compares their linear programming relaxations. When constant factor
integrality gaps exist between the integer programs and their linear relaxations, these methods can
guarantee constant factor approximations.

**Theorem 5.7.** For Coverage there is a deterministic truthful budget feasible constant factor ap-
proximation mechanism.

6 The Space of Budget Feasible Mechanisms

We conclude with exploring the space of budget feasible mechanisms, and address some of the
questions that naturally arise when exploring this setting. Whether our positive results can be
extended to more general functions, the connection between frugality and budget feasibility, and
whether one can maximize the demand surplus $(V(S) - \sum_{i \in S} p_i)$ under the budget. We conclude
with a first step towards characterization of budget feasible mechanisms. We limit our discussion
to deterministic mechanisms.

**Lower Bound on Fractionally Subadditive Functions**

In light of the positive results for submodular functions, one may hope positive results can be
obtained for more general classes. The following lower bound shows that this is unlikely, even for
when we looked at a slightly more relax class of functions.

**Definition 6.1.** A function $V : 2^{[n]} \rightarrow \mathbb{R}$ is called fractionally subadditive if there exists a finite
set of additive valuations $\{a_1, \ldots, a_t\}$ s.t. $V(S) = \max_{i \in [t]} a_i(S)$.

It is known that every submodular function can be represented as a fractionally subadditive
function, and that all fractionally subadditive functions are subadditive [22]. Using a simple reduc-
tion from [23] we show that for mechanisms which use value query oracles, obtaining reasonable
approximations in the case fractionally subadditive demands is hard, regardless of incentive con-
siderations.

**Theorem 6.2.** In the case of fractionally subadditive demands, any algorithm which approximates
within a factor better than $n^{2^{-\epsilon}}$, for any fixed $\epsilon > 0$, requires exponentially many value queries.
This is true even in the setting where all costs are public knowledge.

**Proof.** Our proof relies on the following lemma shown in [23]. We will use $v_S$ to denote the additive
function that assigns the value 1 to an item $j \in S$ and 0 to all other items, and $\overline{V}$ to denote the
function that assigns a value $(1 + \epsilon/2)/(n^{2})$ to every item $j \in [n]$. 

Lemma 6.3 ([23]). There exists a set $T$, $|T| = \sqrt{n}$, for which distinguishing between the functions $V = \max\{v_{S, |S| \leq (1+\epsilon/2)n'}\}$, and $V' = \max\{V, v_T\}$ requires exponentially (in $n'$) many value queries.

Let $c_1 = c_2, \ldots, c_n = 1$ and $B = \sqrt{n}$. For such $T$ as in the lemma, we have that $\sum_{i \in T} c_i = B$, and $V'(T) = \sqrt{n}$ while $V(T) = (1 + \epsilon/2)n'$. Thus, an algorithm that approximates better than the desired approximation ratio must be able to distinguish between these two valuations, which requires exponentially many value queries.

Hiring a Team of Agents

Looking at problems previously studied in procurement, one could also ask whether the “hiring a team of agents” problems [3, 30, 17, 10] have good budget feasible mechanisms. In these problems there is a set of feasible outcomes (e.g. all possible spanning trees or all paths from $s$ to $t$) and the goal is to design a mechanism that yields a feasible solution at minimal cost. Call such a problem nontrivial if all solutions contain more than one element.

Theorem 6.4. There is no budget feasible mechanism with a bounded approximation ratio for any nontrivial “hiring a team of agents” problem.

Proof. Assume for purpose of contradiction, there exists a a bounded approximation ratio mechanism $f$ which is truthful and budget feasible. Let $S$ be a feasible solution, and consider the bid profile in which all agents in $S$ declare $\epsilon > 0$, and all other agents declare $B$. Since the problem is nontrivial, the minimal cost of a feasible solution different than $S$ (if it exists) exceeds the budget. Since $f$ guarantees a bounded approximation ratio, it must allocate to $S$. Therefore every agent in $S$ is in all possible outcomes of the mechanism, and must be paid $B$. Since $S$ is nontrivial, it has more than one element, which contradicts budget feasibility.

Maximizing Demand Surplus

It might seem natural to ask whether one can optimize the function $\bar{V} = V(S) - \sum_{i \in S} p_i$ under the budget. One might argue that in some cases it is more natural to assume that the buyer is not indifferent to the surplus. Since such mechanisms are budget feasible, from what we have discussed thus far it would be natural to seek them for submodular functions. The following simple example shows there is not much hope in optimizing this objective.

Consider an instance where all agents declare cost $1/2$, and the budget is 1. Any budget-feasible mechanism can procure at most 2 agents. Of course, $\bar{V}(\emptyset) = 0$, and we can therefore consider solutions of cardinality one or two. Similar to the above lower bounds, any budget-feasible mechanism that allocates to a single agent must surrender its entire to the agent. In case the mechanism procure two items, individual rationality implies the payments must at least amount to the budget in this case. Thus, in both cases, $\bar{V}(S) = 0$.

Towards a Characterization of Budget Feasible Mechanisms

We have made a first step here by considering mechanisms that respect the two additional conditions of anonymity [4] and weak stability (similar to [12]). Informally, a mechanism is weakly stable if an agent doesn’t hurt the rest when reducing her cost, and anonymous if its allocation rule does not depend on the agents’ identities.

Definition. An allocation rule $f$ satisfies anonymity if $i \in f(c_i, c_j, c_{-ij})$ implies $j \in f(c_i', c_j', c_{-ij})$ when $c_i' = c_j, c_j' = c_i$.
Definition. An allocation rule $f$ satisfies weak stability if for every $i, j \in f(c_i, c_{-i})$, $c'_i \leq c_i$ implies $j \in f(c'_i, c_{-i})$.

Theorem 6.5. Let $f$ be a budget feasible mechanism that is anonymous and weakly stable, and let $S = f(c)$ for some bid profile $c$. Then, for all $i \in S$ it must be that $c_i \leq B/|S|$.

Proof. Assume for purpose of contradiction that there is a bid profile $c = (c_1, \ldots, c_n)$, s.t. $f(c) = S$ and there is some $i \in S$ for which $c_i > B/|S|$. Let $c'$ be the bid profile in which all agents in $S \setminus \{i\}$ bid $c_{\min} = \min_{j \in S} c_j$, and the rest bid as in $c$. Since $f$ is weakly stable, we have that $S \subseteq f(c')$. Let $c''$ the bid profile where $i$ bids $c_{\min}$ as well, and the rest of the agents bid as in $c'$. From monotonicity we have that $i$ is allocated, and again $S \subseteq f(c'')$. We now claim that under the profile $c''$, the threshold price for each agent in $S$ is at least $c_i > B/|S|$. To see this, observe that $i$’s threshold price must be at least $c_i$, since $i \in f(c')$. Since $f$ is anonymous, and all agents in $S$ declare the same price, the threshold price for each agent in $S$ must also be at least $c_i$. Thus, payments to agents in $S$ exceed the budget, contradicting budget feasibility. \qed

It is worth noting that in the case of symmetric submodular functions, where these conditions seem quite reasonable, the characterization suggests we shouldn’t expect much variety beyond what is shown in section 3. The obvious question here is whether all budget feasible mechanisms for the symmetric submodular case take the form of the proportional share allocation rule.

7 Discussion

Is a better approximation ratio possible for submodular functions? We believe so. More importantly, are there more general classes of functions having budget feasible mechanisms with bounded ratio? The richness of the submodular class implies there are many problems for which better approximation ratios are achievable.

Is there a sweeping characterization of budget feasible mechanisms, akin to Roberts’ theorem \[29\] or Myerson’s \[26\]? We have made a first step here by considering mechanisms that respect stricter conditions, though we believe characterizations under weaker conditions are possible. Finally, perhaps the most interesting question is exploring the lower bounds that are dictated by budget feasibility. Here we showed a simple lower bound which is independent of computational assumptions and only uses a single bidder. It would be interesting to find cases in which more sophisticated lower bounds are met.

Acknowledgements

We are grateful to Dave Buchfuhrer, Iftah Gamzu, Arpita Ghosh, Mohammad Mahdian, Amin Saberi, Michael Schapira, and Mukund Sundararajan for meaningful discussions and valuable advice.

References

[1] Alexander A. Ageev and Maxim Sviridenko. Pipage rounding: A new method of constructing algorithms with proven performance guarantee. J. Comb. Optim., 8(3):307–328, 2004.

[2] Gagan Aggarwal, Nir Ailon, Florin Constantin, Eyal Even-Dar, Jon Feldman, Gereon Frahling, Monika Rauch Henzinger, S. Muthukrishnan, Noam Nisan, Martin Pál, Mark Sandler, and Anastasios Sidiropoulos. Theory research at google. SIGACT News, 39(2):10–28, 2008.
[3] Aaron Archer and Éva Tardos. Frugal path mechanisms. *ACM Transactions on Algorithms*, 3(1), 2007.

[4] Itai Ashlagi, Shahar Dobzinski, and Ron Lavi. An optimal lower bound for anonymous scheduling mechanisms. In *ACM Conference on Electronic Commerce*, pages 169–176, 2009.

[5] Yossi Azar and Iftah Gamzu. Truthful unification framework for packing integer programs with choices. In *ICALP (1)*, pages 833–844, 2008.

[6] Liad Blumrosen. Implementing the maximum of monotone algorithms. In *AAAI*, pages 30–35, 2007.

[7] Liad Blumrosen and Noam Nisan. On the computational power of iterative auctions. In *ACM Conference on Electronic Commerce*, pages 29–43, 2005.

[8] Christian Borgs, Jennifer T. Chayes, Nicole Immorlica, Mohammad Mahdian, and Amin Saberi. Multi-unit auctions with budget-constrained bidders. In *ACM Conference on Electronic Commerce*, pages 44–51, 2005.

[9] Jeremy Bulow, Jonathan Levin, and Paul Milgrom. Winning play in spectrum auctions. *Working Paper*.

[10] Matthew Cary, Abraham D. Flaxman, Jason D. Hartline, and Anna R. Karlin. Auctions for structured procurement. In *SODA*, pages 304–313, 2008.

[11] Shahar Dobzinski, Ron Lavi, and Noam Nisan. Multi-unit auctions with budget limits. In *FOCS*, pages 260–269, 2008.

[12] Shahar Dobzinski and Mukund Sundararajan. On characterizations of truthful mechanisms for combinatorial auctions and scheduling. In *ACM Conference on Electronic Commerce*, pages 38–47, 2008.

[13] Edith Elkind, Amit Sahai, and Kenneth Steiglitz. Frugality in path auctions. In *SODA*, pages 701–709, 2004.

[14] Joan Feigenbaum, Christos H. Papadimitriou, Rahul Sami, and Scott Shenker. A bgp-based mechanism for lowest-cost routing. In *PODC*, pages 173–182, 2002.

[15] Jon Feldman, S. Muthukrishnan, Martin Pál, and Clifford Stein. Budget optimization in search-based advertising auctions. In *ACM Conference on Electronic Commerce*, pages 40–49, 2007.

[16] Kamal Jain and Mohammad Mahdian. Cost sharing. In Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani, editors, *Algorithmic Game Theory*. Cambridge University Press, 2007.

[17] Anna R. Karlin, David Kempe, and Tami Tamir. Beyond VCG: Frugality of truthful mechanisms. In *FOCS*, pages 615–626, 2005.

[18] David Kempe, Jon M. Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In *KDD*, pages 137–146, 2003.

[19] Samir Khuller, Anna Moss, and Joseph (Seffi) Naor. The budgeted maximum coverage problem. *Inf. Process. Lett.*, 70(1):39–45, 1999.
[20] Andreas Krause and Carlos Guestrin. A note on the budgeted maximization of submodular functions. In *CMU Technical Report*, pages CMU–CALD–05–103, 2005.

[21] Ron Lavi and Chaitanya Swamy. Truthful and near-optimal mechanism design via linear programming. In *FOCS*, pages 595–604, 2005.

[22] Benny Lehmann, Daniel Lehmann, and Noam Nisan. Combinatorial auctions with decreasing marginal utilities. In *ACM conference on electronic commerce*, 2001.

[23] Vahab S. Mirrokni, Michael Schapira, and Jan Vondrák. Tight information-theoretic lower bounds for welfare maximization in combinatorial auctions. In *ACM Conference on Electronic Commerce*, pages 70–77, 2008.

[24] Elchanan Mossel and Sébastien Roch. On the submodularity of influence in social networks. In *STOC*, pages 128–134, 2007.

[25] Ahuva Mu’alem and Noam Nisan. Truthful approximation mechanisms for restricted combinatorial auctions. *Games and Economic Behavior*, 64(2):612–631, 2008.

[26] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1), 1981.

[27] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher. An analysis of approximations for maximizing submodular set functions ii. *Math. Programming Study 8*, pages 73–87, 1978.

[28] Noam Nisan and Amir Ronen. Algorithmic mechanism design. *Games and Economic Behaviour*, 35:166 – 196, 2001. A preliminary version appeared in STOC 1999.

[29] Kevin Roberts. The characterization of implementable choice rules. pages 321–349, 1979.

[30] Kunal Talwar. The price of truth: Frugality in truthful mechanisms. In *STACS*, pages 608–619, 2003.

[31] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.
Appendix A: Symmetric Submodular Functions

Example (Paying $B/k$ is not truthful). Consider running the mechanism shown in section 3 with payments $B/k$ on three agents with real costs $c_1 = 3, c_2 = 5 - \epsilon, c_3 = 5$ and a budget $B = 10$. If the mechanism were truthful, then it should result in agents with real costs 3 and $5 - \epsilon$ being allocated, and paid 5. Note that in this case it would be in agent 3’s best interest to report a false cost $c_3' = 5 - 2\epsilon$, in which case the allocation would go to her and agent 1, leaving agent 3 with a profit of $\epsilon$. Paying the minimum of the fair share and the cost of the agent that is excluded from the solution solves this.

Proposition 7.1. The mechanism for symmetric submodular functions is truthful.

Proof. The allocation rule is monotone since declaring a lower cost advances an item in the sorting. To see that $\theta_i$ is indeed the threshold payment for all $i \in W$, consider first the case where $c_{k+1} < B/k$. Declaring a cost $c_i' > c_{k+1}$ places $i$ after agent $k + 1$. Since all agents in $(W \setminus \{i\}) \cup \{k + 1\}$ have costs less than $B/k$, as the mechanism reaches agent $i$’s bid, there are already (at least) $k$ agents ahead of $i$. Since $c_i' > c_{k+1} > B/(k+1)$ agent $i$ will not be allocated. Declaring a cost below $c_{k+1}$ places $i$ within the first $k$ items, all with costs less than $B/k$, and thus $i$ will be allocated.

In case $B/k \leq c_{k+1}$, declaring cost $c_i' > B/k$ places at least $k - 1$ items ahead of $i$, since all items in the winning set have cost less than $B/k$. Therefore, even if $i$ will be considered by the mechanism it will not be allocated as it does not meet the mechanism’s allocation condition. Declaring a lower cost ensures that $i$ is placed within the first $k$ items and it will be allocated. The payment rule therefore respects the threshold property and we conclude that the mechanism is indeed truthful.

Appendix B: General Submodular Functions

Example (MAX is not monotone). While a greedy allocation rule that allocates to all items that respect $c_i \leq V_iB/V(S_i)$ (when items are sorted according to marginal contribution relative to cost) is monotone allocating based on taking the maximum value of this allocated set and another solution is not monotone in the case of general submodular problems. To see this, consider an instance to the Coverage problem with a universe of elements, partitioned to the following disjoint subsets $W, X, Y, Z$, with cardinalities $|W| = 7, |X| = 2, |Y| = 2, |Z| = 4$. Let $\{0, 1, 2\}$ be the set of agents, with $T_0 = W, T_1 = X \cup Y, T_2 = X \cup Z$. Set the budget $B = 1$, and costs $c_0 = 7/24$ (a fraction between $1/3$ and $1/4$) and $c_2 = 1/2$. We have that agent 1 appears before 2 in the sorting, both agents are allocated as both satisfy the algorithm’s allocation condition $c_i \leq V_i(S_{i-1})/V(S_{i-1} \cup \{i\})$ and $|X \cup W \cup Z| > |W|$. If 2 declares a lower value, which puts her ahead of 1, she will no longer be allocated: the marginal contribution of 1 will be $|Y| = 2$, and since $c_1 > 2/|X \cup Y \cup Z| = 1/4$, only agent 2 satisfied the condition of the algorithm, and the set covered by the greedy algorithm is therefore $X \cup Z$. Since $|X \cup Z| = 6 \leq |W|$, agent 2 is no longer allocated.

Example. (Shapley Values-Based Mechanism) For the Coverage problem, the proportional share allocation rule can be generalized via Shapley values which are often used in cost sharing [16]. For the Coverage problem with subsets (agents) $\{T_1, \ldots, T_n\}$ of a universe $U$, let $U_j = \cup_{i \in [j]} T_j$ and $\gamma_j(u)$ denote the number of agents in $[j]$ that cover an element $u \in U_j$. In our context, for a set $U_j$ and agent $i$, the Shapley values are:

$$\xi_{i,j} = \frac{1}{\gamma_j(u)} \sum_{u \in U_j \cap T_i} \gamma_j(u)$$
Note that by definition $|U_j| = \sum_{i \leq j} \xi_{i,j}$. The attractive property of Shapley values is that they make the proportional contribution of an agent independent of the stage in which she was selected by the mechanism. While it seems natural to replace the marginal contributions with Shapley values in the proportional share allocation rule presented above, this results in a poor approximation ratio. Under Shapley values, at every stage as an item is added to the solution, the proportional share of the rest can decrease. Individual rationality requires that the mechanism stops at stage $k$ if there is an agent in $\{1 \ldots k - 1\}$ whose Shapley value decreases below her cost. To see this can result in a poor approximation ratio consider the following instance. The set $T_1 = \{u_0, u_1\}$ has cost $1 - \epsilon_1$, and the rest of the sets, all with cost $1 - \epsilon_i$, are of the form $T_i = \{u_{m(i)}, u_i\}$, where $m(i) = i \mod 2$, and $\epsilon_1 > \ldots > \epsilon_n > 0$ are small. We set the budget to be $B = n$. For this instance, the mechanism picks $T_1$ first, and after every odd stage $j$ the value of the solution is $j + 1$, and the elements in $T_1$ are covered by $j/2$ sets. Therefore the Shapley value of agent 1 at stage $j$ is $n/(j^2 + j)$, and thus after $\sqrt{n}$ stage no longer exceeds her the cost. This gives total value of $\sqrt{n}$, while the optimal solution $\{1 \ldots n\}$ has value of $n + 1$.

**Lemma 7.2 (Individual Rationality).** The mechanism $f_M$ is individually rational, i.e., $c_i \leq \theta_i$.

**Proof.** Observe that:

(a) $\forall j$ we have $V_i(j+1) \leq V_i(j)$;

(b) $\forall j < i$, agent $j$ stays in the position when $i$ is removed and $T_j = S_j$;

(c) $V_i(T_{i-1}) = V_i$.

Since the threshold payment is the maximum over all $\{\tau_{i(j)}, \rho_{i(j)}\}$ in $|K_i| + 1$, it is enough to show that $c_i \leq \tau_{i(j)}$ for a certain $j \leq |K_i| + 1$. Since (b) implies that $i \leq |K_i| + 1$ we can consider the is replacement $j$ which appears in the is place in $K_i$. We will show $c_i$ is bounded from above by either value that $\tau_{i(j)}$ that can take. Since $i \in [k]$, and due to (b) and (c) above, we have that

$$c_i \leq \frac{B \cdot V_i}{V((S_{i-1} \cup \{i\})} = \frac{B \cdot V_i(T_{i-1})}{V(T_{i-1} \cup \{i\})}.$$  

In the original sorting, $i$ appears ahead of $j$ (as implied from (b)), and therefore its relative marginal contribution is greater. Thus:

$$c_i \leq \frac{V_i(S_{i-1}) \cdot c_j}{V_j(S_{i-1})} = \frac{V_i(T_{i-1}) \cdot c_j}{V_j(T_{j-1})}.$$  

Therefore $c_i \leq \min \left\{ \frac{B \cdot V_i(T_{i-1})}{V(T_{i-1} \cup \{i\})}, \frac{V_i(T_{i-1}) \cdot c_j}{V_j(T_{j-1})} \right\} \leq \theta_i$.  

18