New representations of epigraphs of conjugate mappings and Lagrange, Fenchel–Lagrange duality for vector optimization problems

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ABSTRACT

In this paper, we concern the vector problem:

\[(VP) \quad \text{WInf}\{F(x) : x \in C, \ G(x) \in -S\},\]

where \(X, Y, Z\) are locally convex Hausdorff topological vector spaces, \(F: X \to Y \cup \{+\infty\} \) and \(G: X \to Z \cup \{+\infty\}\) are proper mappings, \(C\) is a nonempty convex subset of \(X\), and \(S\) is a non-empty closed, convex cone in \(Z\). Several new presentations of epigraphs of composite conjugate mappings associated to \((VP)\) are established under variant qualifying conditions. The significance of these representations is twofold: Firstly, they play a key role in establishing new kinds of vector Farkas lemmas which serve as tools in the study of vector optimization problems; secondly, they pay the way to define Lagrange and two new kinds of Fenchel–Lagrange dual problems for \((VP)\). Strong and stable strong duality results corresponding to these mentioned dual problems of \((VP)\) are established using the new Farkas-type results just obtained. It is shown that in the special case where \(Y = \mathbb{R}\), the Lagrange and Fenchel–Lagrange dual problems for \((VP)\), go back to Lagrange, and Fenchel–Lagrange dual problems for scalar problems, and the resulting duality results cover, and in some cases, extend the corresponding ones for scalar problems in the literature.

1. Introduction

We consider a vector optimization problem of the form:

\[(VP) \quad \text{WInf}\{F(x) : x \in C, \ G(x) \in -S\},\]
where $X$, $Y$, $Z$ are locally convex Hausdorff topological vector spaces, $F: X \rightarrow Y \cup \{+\infty\}$ and $G: X \rightarrow Z \cup \{+\infty\}$ are proper mappings, $C$ is a nonempty subset of $X$, and $S$ is a non-empty closed, convex cone in $Z$. Let $A := C \cap G^{-1}(-S)$ and assume along this paper that $A \cap \text{dom} F \neq \emptyset$. Throughout this paper, we often say that the triple $(F; G, C)$ defined the (VP).

In the special case where $Y = \mathbb{R}$, the problem (VP) reduces to the scalar problem

$$(P) \quad \inf \{f(x) : x \in C, \, G(x) \in -S\},$$

where $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$.

Several dual problems of the (VP) are defined in the literature (see Refs. [1–13]). However, many among the dual problems in these works when specified to the special case where $Y = \mathbb{R}$ result to only Lagrangian dual problem of (P):

$$(D_1) \quad \sup_{\lambda \in S^+} \inf_{x \in C} (f + \lambda G)(x),$$

and consequently, strong duality results also go back to Lagrange strong duality for (P). In Ref. [3, p.338, 373], the author introduced some kind of dual problems for set-valued optimization problems that can reduce (when $Y = \mathbb{R}$) to some kind of Fenchel dual problem for (P). However, just a few dual problems for (VP) in the literature that can result to Fenchel or Fenchel–Lagrange dual problems for (P) (see Refs. [14–17] and reference therein) as

$$(D_2) \quad \sup_{x^* \in X^*} \inf_{\lambda \in S^+} \left[ -f^*(x^*) - (i_C + \lambda G)^*(-x^*) \right],$$

$$(D_3) \quad \sup_{x^*, u^* \in X^*} \inf_{\lambda \in S^+} \left[ -f^*(x^*) - i_C^*(u^*) - (\lambda G)^*(-x^* - u^*) \right],$$

where $i_C$ is the indicator function of $C$, which were proposed in many works such as Refs. [16–22]. The difficulty is that the representation of epigraphs of conjugate mappings (in the case of vector problems being multivalued functions) becomes very complicated and it is so difficult to get some similar forms as for the scalar problem (P).

The aim of this paper is to introduce a way to overcome this difficulty to establish some new ways of representations of epigraphs of conjugate mappings that pay a general way to define some new dual problems for (VP) (also called Lagrange and Fenchel–Lagrange dual problems for (VP)) which extend the dual problems $(D_2)$ and $(D_3)$ to vector setting. In other words, such Lagrange and Fenchel–Lagrange dual problems for (VP) in the case where $Y = \mathbb{R}$ go back to the corresponding dual problems $(D_1)$, $(D_2)$, and $(D_3)$ for (P).

To overcome the difficulty mentioned above in getting representations of epigraphs of conjugate mappings, we need some new notions on the order between subsets of $Y$ generated by $K$, denoted by $\preceq_K$, and operations of subsets
in product spaces and some key results (called basic lemmas) on these subsets. Concretely, we use the weak order in \( Y \) generated by a closed and convex cone \( K \). Such kinds of order are used frequently in the literature such as Refs. [3–7,12]. We introduce the space of nonempty subsets of \( Y \), namely, \( (\mathcal{P}_p(Y) \bowtie, \preceq_K) \), the notion of ‘extended epigraph’ of conjugate mappings and the sum (denoted by \( \text{Epi} F^* \) and \( \boxplus \), respectively) of these extended epigraphs (e.g. \( \text{Epi} F^* \boxplus \text{Epi} G^* \)). Some basic properties of the sum, called basic lemmas, are established (see Section 3). These form an important cornerstone for the main results of this paper. These new tools may have their own meanings with their possible uses in other approaches to vector or set-valued optimization problems.

The novelty of our work is threefold: Firstly, armed with the notions as the order \( \preceq_K \), the structure \( (\mathcal{P}_p(Y)^\infty, \preceq_K, \bowtie) \), the \( \boxplus \)-sum, together with the basic lemmas, three new forms of presentation of epigraphs of conjugate mappings, namely, the representations of the forms: \( \text{epi}(F + I_A)^* = A_i, \ i = 1, 2, 3 \) (\( I_A \) is the indicator mapping of \( A \)), under certain regularity conditions are proved (Theorem 4.1). These representations play key roles in establish new kinds of vector Farkas lemmas. They pay the way to define Lagrange and two new kinds of Fenchel–Lagrange dual problems for the vector problem (VP) (in Section 6) as well. Secondly, from these three new representations, three kinds of Farkas lemmas for vector functions are established (Theorem 5.1, Corollaries 5.2-5.3), among them, the first one goes back to the form of vector Farkas lemma introduced in Ref. [5] and also to the form in Ref. [4] when specified to (VP)\(^1\). However, the other two forms of Farkas lemma for vector functions, expressed in terms of conjugate mappings, to the best knowledge of the authors, are new. In a special case when \( Y = \mathbb{R} \) these go back to/extend (with the stable forms), the Farkas lemmas expressed in terms of conjugate functions appeared in the literature such as Refs. [14,16,17,21]. Thirdly, the new representations of epigraphs \( \text{epi}(F + I_A)^* \) also pay the way to define the Lagrange and two Fenchel–Lagrange dual problems for (VP), for which, the first one goes back to the one defined in Ref. [7] (in the case when the uncertainty set is a singleton), while the other two Fenchel–Lagrange dual problems are new. Stable strong duality for these pairs of primal-dual problems are established based on the vector Farkas lemmas obtained in the previous sections. Coming back to the case where \( Y = \mathbb{R} \), these duality results go back/cover/extend many of the Lagrange and Fenchel–Lagrange duality results for scalar problems in the literature (e.g. see Refs. [14–17,19–22]), showing that for the first time Fenchel–Lagrange duality is extended to vector optimization problems.

The paper is organized as follows: Section 2 gives some preliminaries and some first results, where we recall the conjugate of a vector-valued functions and some properties of their epigraphs. In Section 3, we introduce some new notions and some basic lemmas which are useful for our study through out the paper. In particular, the structure \( (\mathcal{P}_p(Y)^\infty, \preceq_K, \bowtie) \), the sum \( \boxplus \)-sum of extended epigraphs of
conjugate mappings, together with two basic lemmas (Lemmas 3.2 and 3.4) will serve as key tools for our research. Section 4 introduces certain regularity conditions for (VP), under which, several presentations of epigraphs of conjugate mappings are established which serve as a cornerstone for the paper. In Section 5, several versions of vector Farkas results for general systems and convex systems are established. Section 6 introduces Lagrange dual problem and two forms of Fenchel–Lagrange dual problems for vector optimization problem (VP). Strong and stable strong duality results for (VP) corresponding to these dual problems are established. It is worth noticing that in the special case where $Y = R$, these dual problems collapse to the dual problems $(D_1)$, $(D_2)$, and $(D_3)$ mentioned above.

2. Preliminaries, notations and first results

Let $X$, $Y$, $Z$ be locally convex Hausdorff topological vector spaces with their topological dual spaces denoted by $X^*$, $Y^*$ and $Z^*$, respectively. The only topology we consider on dual spaces is the weak*-topology. For a set $U \subset X$, we denote by $\text{int}U$, $\text{cl}U$, $\text{bd}U$, $\text{lin}U$, $\text{aff}U$, and $\text{cone}U$ the interior, the closure, the boundary, the convex hull, the linear hull, the affine hull, and the cone hull of $U$, respectively. For each $x \in X$, $\mathcal{N}_X(x)$ denotes the collection of all neighbourhoods of $x$ in $X$. Assume that $W$ is a topological subspace of $X$. For $A \subset W$, denote by $\text{int}_WA$ the interior of $A$ w.r.t. the topology induced in $W$. Let $K$ be a proper, closed, and convex cone in $Y$ with nonempty interior, i.e. $\text{int}K \neq \emptyset$. It is worth observing that

\[ K + \text{int}K = \text{int}K. \]  

(1)

Weak Ordering Generated by a Convex Cone: Weak Infima and Weak Suprema.

We define a weak ordering in $Y$ generated by $K$ as follows: for all $y_1, y_2 \in Y$,

\[ y_1 <_K y_2 \iff y_1 - y_2 \in -\text{int}K. \]  

(2)

In $Y$ we sometimes also consider an usual ordering generated by the cone $K$, $\leq_K$, which is defined by $y_1 \leq_K y_2$ if and only if $y_1 - y_2 \in -K$, for $y_1, y_2 \in Y$.

We enlarge $Y$ by attaching a greatest element $+\infty_Y$ and a smallest element $-\infty_Y$ w.r.t. $<_K$, which do not belong to $Y$, and we denote $Y^* := Y \cup \{-\infty_Y, +\infty_Y\}$. We assume by convention that $-\infty_Y <_K y <_K +\infty_Y$ for any $y \in Y$ and, for $M \subset Y$,

\[
- (+\infty_Y) = -\infty_Y, \quad -(+\infty_Y) = +\infty_Y, \quad (+\infty_Y) + y = y + (+\infty_Y), \quad \forall y \in Y \cup \{+\infty_Y\},
\]

\[
(-\infty_Y) + y = y + (-\infty_Y) = -\infty_Y, \quad \forall y \in Y \cup \{-\infty_Y\},
\]

\[
M + (-\infty_Y) = \{-\infty_Y\} + M = \{-\infty_Y\}, \quad M + (+\infty_Y) = \{+\infty_Y\} + M = \{+\infty_Y\}.
\]  

(3)
The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

The following notions are the key ones of the paper.

**Definition 2.1** ([3, Definition 7.4.1],[12]): Let $M \subset Y^\cdot$.

(a) An element $\bar{v} \in Y^\cdot$ is said to be a *weakly infimal element* of $M$ if for all $v \in M$ we have $v \not< K \bar{v}$ and if for any $\tilde{v} \in Y^\cdot$ such that $\bar{v} < K \tilde{v}$, then there exists some $v \in M$ satisfying $v < K \tilde{v}$. The set of all weakly infimal elements of $M$ is denoted by $W\text{Inf}M$ and is called the *weak infimum* of $M$.

(b) An element $\bar{v} \in Y^\cdot$ is said to be a *weakly supremal element* of $M$ if for all $v \in M$ we have $\bar{v} \not< K v$ and if for any $\tilde{v} \in Y^\cdot$ such that $\tilde{v} < K \bar{v}$, then there exists some $v \in M$ satisfying $\tilde{v} < K v$. The set of all supremal elements of $M$ is denoted by $W\text{Sup}M$ and is called the *weak supremum* of $M$.

(c) The *weak minimum* of $M$ is the set $W\text{Min}M = M \cap W\text{Inf}M$ and its elements are the *weakly minimal elements* of $M$.

(d) The *weak maximum* of $M$ is the set $W\text{Max}M = M \cap W\text{Sup}M$ and its elements are the *weakly maximal elements* of $M$.

**Proposition 2.1:** Let $\emptyset \neq M \subset Y^\cdot$. One has:

(i) $W\text{Sup}M \neq \{+\infty_Y\}$ if and only if $Y \setminus (M - \text{int } K) \neq \emptyset$.

(ii) For all $y \in Y$, $W\text{Sup}(y + M) = y + W\text{Sup} M$.

Assume further that $M \subset Y$ and $W\text{Sup} M \neq \{+\infty_Y\}$ then it holds:

(iii) $W\text{Sup} M - \text{int } K = M - \text{int } K$.

(iv) The following decomposition\textsuperscript{2} of $Y$ holds

$$Y = (M - \text{int } K) \cup W\text{Sup} M \cup (W\text{Sup} M + \text{int } K).$$

(v) $W\text{Sup} M = \text{cl}(M - \text{int } K) \setminus (M - \text{int } K)$.

(vi) $W\text{Sup}(W\text{Sup} M + W\text{Sup} N) = W\text{Sup}(M + W\text{Sup} N) = W\text{Sup}(M + N)$.

(vii) If $0_Y \in N \subset -K$ then $W\text{Sup}(M + N) = W\text{Sup} M$.

In particular, one has $W\text{Sup}(M - K) = W\text{Sup}(M - \text{bd } K) = W\text{Sup} M$.

**Proof:** The assertions (i)–(v) are quoted from Refs. [5,7], while (vi) from [3, Proposition 7.4.3] (see also [12, Proposition 2.6]).

(vii) If $0_Y \in N \subset -K$ then it is easy to check that $M + N - \text{int } K = M - \text{int } K$. Taking (v) into account, we get $W\text{Sup}(M + N) = W\text{Sup} M$. Moreover, as $0_Y \in -\text{bd } K \subset -K$, taking $N = -\text{bd } K$ and $N = -K$ successively in the previous equality, one gets $W\text{Sup}(M - K) = W\text{Sup}(M - \text{bd } K) = W\text{Sup} M$.
Proposition 2.2: Let $\emptyset \neq M \subset Y^\ast$. Then, only one of three following cases is possible: $\text{WSup} M = \{+\infty_Y\}$, $\text{WSup} M = \{-\infty_Y\}$, and $\emptyset \neq \text{WSup} M \subset Y$.

Proof: Assume that $\text{WSup} M \neq \{+\infty_Y\}$ and $\text{WSup} M \neq \{-\infty_Y\}$. It follows from Definition 2.1(b) that if $M \ni +\infty_Y$ then $\text{WSup} M = \{+\infty_Y\}$ and that $\text{WSup} \{-\infty_Y\} = \{-\infty_Y\}$. So, we can assume that $M \not\ni +\infty_Y$ and $M \neq \{-\infty_Y\}$, which yields $\emptyset \neq M \setminus \{-\infty_Y\} \subset Y$ (recall that $M \neq \emptyset$). On the other hand, it follows again from Definition 2.1(b) that $\text{WSup} M = \text{WSup} M \setminus \{-\infty_Y\}$. So, replacing $M$ by $M \setminus \{-\infty_Y\}$ if necessary, we can assume that $\emptyset \neq M \subset Y$.

Apply Proposition 2.1(iii) and (v) (recall that $\text{WSup} M \neq \{+\infty_Y\}$), one gets $\text{WSup} M \neq \emptyset$ and $\text{WSup} M \subset Y$. $lacksquare$

Remark 2.1: It is worth noting that $\text{WInf} M = -\text{WSup}(-M)$ for all $M \subset Y^\ast$. So, Propositions 2.1 and 2.2 hold true when $\text{WSup}$, $+\infty_Y$, $K$, and $\text{int} K$ are replaced by $\text{WInf}$, $-\infty_Y$, $-K$, and $-\text{int} K$, respectively.

Mappings and Cone of Positive Operators. Let $F: X \to Y^\ast$ be a mapping. The domain, $\text{dom} F$, and the K-epigraph of $F$, $\text{epi}_K F$, are defined respectively by

$$\text{dom} F := \{x \in X : F(x) \neq +\infty_Y\}, \quad \text{epi}_K F := \{(x, y) \in X \times Y : F(x) \leq_K y\}.$$ 

$F$ is said to be proper if $\text{dom} F \neq \emptyset$ and $-\infty_Y \notin F(X)$. It is said to be $K$-convex (resp., $K$-epi closed) if $\text{epi}_K F$ is a convex subset of $X \times Y$ (resp., $\text{epi}_K F$ is a closed subset of the product space $X \times Y$ [14], [23, Definition 5.1]).

The concept $K$-epi closed extends the concept lower semicontinuous (lsc, briefly) of a real-valued function. The mapping $F$ is said to be positively $K$-lsc if $y^\ast \circ F$ is lsc for all $y^\ast \in K^+$ (see Ref. [24], [25, Definition 2.16]). According to [23, Theorem 5.9], every positively $K$-lsc mapping is $K$-epi closed but the converse is not true. Moreover, when $Y = \mathbb{R}$, three notions lsc, positively $\mathbb{R}_+^+$-lsc, and $\mathbb{R}_+^+$-epi closed coincide with each other (as in this case, the second notion means that $\alpha F$ is lsc for all $\alpha \in \mathbb{R}_+^+$ while last one means that the (usual) epigraph of the real-valued function $F$ is closed).

Denote by $\mathcal{L}(X, Y)$ the space of all continuous linear mappings from $X$ to $Y$ equipped with the topology of point-wise convergence, i.e. if $(L_i)_{i \in I} \subset \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(X, Y)$, $L_i \to L$ in $\mathcal{L}(X, Y)$ means $L_i(x) \to L(x)$ in $Y$ for all $x \in X$. The zero element of $\mathcal{L}(X, Y)$ is $0_{\mathcal{L}}$.

Now let $S$ be a non-empty convex cone in $Z$. Recall that the cone of positive operators (see Refs. [26,27]) is defined by $\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \subset K\}$. When $Y = \mathbb{R}$, this cone collapse to the positive dual cone $S^+ := \{z^\ast \in Z^\ast : \langle z^\ast, s \rangle \geq 0, \forall s \in S\}$ of $S$.

Conjugate Mappings of Vector-Valued Functions. The following notion of conjugate mapping of a mapping is specified from the corresponding one for set-valued mappings in [3, Definition 7.4.2] and [12, Definition 3.1].
Definition 2.2: For $F: X \rightarrow Y \cup \{+\infty\}$, the set-valued mapping $F^*: L(X,Y) \Rightarrow Y^*$ defined by $F^*(L) := \text{WSup}[L(x) - F(x) : x \in X]$ is called the conjugate mapping of $F$. The $K$-epigraph and the domain of $F^*$ are, respectively,

$$
epi_K F^* := \{(L,y) \in L(X,Y) \times Y : y \in F^*(L) + K\},$$
$$\text{dom } F^* := \{L \in L(X,Y) : F^*(L) \neq \{+\infty\}\}.$$

It is well-known that if $F: X \rightarrow Y \cup \{+\infty\}$ is a proper mapping then $\text{epi } F^*$ is a closed subset of $L(X,Y) \times Y$ (see [5, Lemma 3.5]). Moreover, the next characterization of $\text{epi } F^*$ (see [28, Theorem 3.1]) is of importance in the sequel:

$$(L,y) \in \text{epi } F^* \iff [F(x) - L(x) + y / \notin \text{int } K, \forall x \in X]. \quad (4)$$

For $D \subset X$, the indicator map $I_D: X \rightarrow Y^*$ is defined by $I_D(x) = 0_Y$ if $x \in D$ and $I_D(x) = +\infty_Y$, otherwise. As the cone $K$ in $Y$ is fixed throughout the paper, for the simplicity, we will write $\text{epi } F$ and $\text{epi } F^*$ instead of $\text{epi } KF$ and $\text{epi } KF^*$, respectively.

3. New tools and basic lemmas

In this section, we firstly introduce the order space $(P_p(Y)^\infty, \preceq_K)$ and the structure $(P_p(Y)^\infty, \preceq_K, \cup)$ with the so-called ‘WS-sum’, new notions of extended epigraphs of conjugate mappings and a new operator on these sets, called the $\boxplus$-sum. Next, we prove two basic lemmas. One among them (Lemma 3.4) establishes the relation between the ‘normal’ epigraph (defined in Section 2) of the conjugate of a sum of two mappings and the $\boxplus$-sum of two extended epigraphs of the conjugates.

3.1. The ordered space $(P_p(Y)^*, \preceq_K)$ and the structure $(P_p(Y)^\infty, \preceq_K, \cup)$

The Ordered Space $(P_p(Y)^*, \preceq_K)$. Let $P_0(Y^*)$ be the collection of all non-empty subsets of $Y^*$. The ordering ‘$\preceq_K$’ on $P_0(Y^*)$ is defined [7] as, for $M, N \in P_0(Y^*)$,

$$M \preceq_K N \iff (v \notin_K u, \forall u \in M, \forall v \in N). \quad (5)$$

Proposition 3.1: The following assertions hold

(i) For all $\emptyset \neq M, N \subset Y$, one has:

$$M \preceq_K N \iff N \cap (M - \text{int } K) = \emptyset \iff M \cap (N + \text{int } K) = \emptyset.$$  

(ii) For all $M \in P_0(Y^*)$, one has $\text{WInf } M \preceq_K M$ and $M \preceq_K \text{WSup } M$. 
(iii) If \( M \subset N \subset Y^* \) then \( WSup M \preceq_K WSup N \).

\textbf{Proof:} (i) and (ii) follow easily from (5) while (iii) is [7, Proposition 1].

A subset \( U \subset Y \) is called a \((Y, K)\)-partition style subset of \( Y \), if the following decomposition of \( Y \) holds
\[
Y = (U - \text{int } K) \cup U \cup (U + \text{int } K).
\]

(6)

Denote by \( \mathcal{P}_p(Y) \) the collection of all \((Y, K)\)-partition style subsets of \( Y \) and set
\[
\mathcal{P}_p(Y)^* := \mathcal{P}_p(Y) \cup \{+\infty_Y, -\infty_Y\}.
\]

It is obvious that if \( M \subset Y^* \) then \( \pm WSup M \in \mathcal{P}_p(Y)^* \), \( \pm WInf M \in \mathcal{P}_p(Y)^* \) and (by (5)), for any \( U \in \mathcal{P}_p(Y) \), one has \( U \preceq_K \{+\infty_Y\} \) and \( \{-\infty_Y\} \preceq_K U \). Moreover, \( (\mathcal{P}_p(Y)^*, \preceq_K) \) is an ordered space [7], i.e. \( \preceq_K \) is a partial order on \( \mathcal{P}_p(Y)^* \) with properties: reflexive, anti-symmetric, and transitive.

**Proposition 3.2:** Let \( U, V \in \mathcal{P}_p(Y)^* \). Then

(i) If \( U \subset V \) then \( U = V \),

(ii) If \( U \in \mathcal{P}_p(Y) \) then
\[
U + K = U \cup (U + \text{int } K), \quad U - K = U \cup (U - \text{int } K).
\]

(7)

\textbf{Proof:} (i) is [7, Proposition 4]. For the proof of (ii), take \( U \in \mathcal{P}_p(Y) \). It is clear that
\[
U + K \supset U \cup (U + \text{int } K).
\]

(8)

Next, we claim that \((U + K) \cap (U - \text{int } K) = \emptyset\). Indeed, assume the contrary that \((U + K) \cap (U - \text{int } K) \neq \emptyset\). Then, there exist \( u, v \in U \), \( k \in K \), and \( k_0 \in \text{int } K \) such that \( u + k = v - k_0 \), and hence, \( u = v - k_0 - k \in U \cap (U - \text{int } K) \), which contradicts the decomposition (6). So, \((U + K) \cap (U - \text{int } K) = \emptyset\), and \( U + K \subset U \cup (U + \text{int } K) \). The first equality in (7) now follows from (8) while that of the second one is similar. 

*The structure \((\mathcal{P}_p(Y)^\infty, \preceq_K, \cup)\).* We now introduce a new kind of ‘sum’ of two sets (called ‘WS-sum’) on the collection \( \mathcal{P}_p(Y)^\infty := \mathcal{P}_p(Y) \cup \{+\infty_Y\} \).

**Definition 3.1:** For \( U, V \in \mathcal{P}_p(Y)^\infty \), the WS-sum of \( U \) and \( V \), denoted by \( U \cup V \), is a set from \( \mathcal{P}_p(Y)^\infty \) and is defined by
\[
U \cup V := WSup(U + V).
\]

(9)
Lemma 3.1: Let \( U, V \in \mathcal{P}_p(Y) \). Then

(i) the following decompositions of \( Y \) holds:

\[
Y = (U - \text{int} K) \cup (U + K) = (U - K) \cup (U + \text{int} K),
\]

(ii) it holds: \( U \preceq_K V \iff V \subseteq U + K \iff U \subseteq V - K \),

(iii) \( \text{WSup} U = \text{WInf} U = U \).

Proof: It is easy to see that (i) is a direct consequence of (6) and Proposition 3.2(ii) while (ii) follows from (i) and Proposition 3.1(i). For (iii), as \( U \in \mathcal{P}_p(Y) \), (6) holds yielding \( Y \setminus (U - \text{int} K) \neq \emptyset \), and hence, by Proposition 2.1(i), \( \text{WSup} U \neq +\infty_Y \). Now, according to Proposition 2.1(v), \( \text{WSup} U = \text{cl}(U - \text{int} K) \setminus (U - \text{int} K) \).

On the other hand, as \( U + \text{int} K \) is open, \( U - K \) is closed (see (i)). We now show that \( U - K \) is the smallest closed subset of \( Y \) containing \( U - \text{int} K \). Indeed, assume that \( M \) is a closed subset containing \( U - \text{int} K \), we will prove that \( M \supseteq U - K \). Take \( w \in U - K \). Then, there are \( u \in U \) and \( k \in K \) such that \( w = u - k \). Pick \( k_0 \in \text{int} K \). It is easy to see that \( w_n := u - k - \frac{1}{n} k_0 \in U - K - \text{int} K = U - \text{int} K \subseteq M \) for all \( n \in \mathbb{N}^* \) and \( w_n \to w \). So, by the closedness of \( M \), \( w \in M \) and \( \text{cl}(U - \text{int} K) = U - K \). Thus, by Proposition 2.1(v),

\[
\text{WSup} U = \text{cl}(U - \text{int} K) \setminus (U - \text{int} K) = (U - K) \setminus (U - \text{int} K).
\]

This, together with (7), yields \( \text{WSup} U = U \) (note that \( U \cap (U - \text{int} K) = \emptyset \)). The proof of the equality \( \text{WInf} U = U \) is similar. 

We now give some properties of the structure \( (\mathcal{P}_p(Y)^\infty, \preceq_K, \cup) \).

Proposition 3.3: Let \( U, V, W, U', V' \in \mathcal{P}_p(Y)^\infty, y \in Y \). One has

(i) \( U \cup (-\text{bd} U) = U \),

(ii) \( U \cup V = V \cup U \) (commutative),

(iii) \( (U \cup V) \cup W = U \cup (V \cup W) \) (associative),

(iv) If \( U \preceq_K V \) then \( U \cup W \preceq_K V \cup W \) (compatible of the sum \( \cup \) with \( \preceq_K \)),

(v) If \( U \preceq_K V \) and \( U' \preceq_K V' \) then \( U \cup U' \preceq_K V \cup V' \).

(vi) \( y \in (U \cup V) + K \) if and only if there exists \( W' \in \mathcal{P}_p(Y) \) such that \( U \preceq_K W' \) and \( y \in W' \cup V \).

Proof: (i) and (ii) follow easily from Definition 3.1 (see Proposition 2.1 (vii)). For (iii), one has, by Proposition 2.1(vi),

\[
(U \cup V) \cup W = \text{WSup}(\text{WSup}(U + V) + W)
\]

\[
= \text{WSup}(U + V + W) = U \cup (V \cup W).
\]
(iv) holds trivially if \( V \uplus W := \text{WSup}(V + W) = \{+\infty \} \). Assume now that \( \text{WSup}(V + W) \subseteq Y \). Consequently, \( V, W \subseteq Y \). As \( U \preceq_{K} V \), one has \( U \subseteq Y \) and by Lemma 3.1(ii), \( U \subseteq V - K \). Then \( U + W \subseteq V + W - K \), and from Proposition 3.1(iii), \( \text{WSup}(U + W) \preceq_{K} \text{WSup}(V + W - K) \). One also has, by Proposition 2.1(vii), \( \text{WSup}(U + V - K) = \text{WSup}(U + V) \). So, \( \text{WSup}(U + W) \preceq_{K} \text{WSup}(V + W) \). In other words, \( U \uplus W \preceq_{K} V \uplus W \).

(v) It follows from (iv) and the transitive property of \( \preceq_{K} \).

(vi) If \( y \in (U \uplus V) + K \) then there is \( k \in K \) such that (see Proposition 2.1(ii))

\[
y \in (U \uplus V) + k = \text{WSup}(U + V) + k = \text{WSup}(U + V + k) = (k + U) \uplus V.
\]

So, if we take \( W' = k + U \) then \( W' \in \mathcal{P}_{p}(Y) \), \( y \in W' \uplus V \), and \( W' \subseteq U + K \), meaning that \( U \preceq_{K} W' \) by Lemma 3.1(ii).

Conversely, if there exists \( W' \in \mathcal{P}_{p}(Y) \) such that \( U \preceq_{K} W' \) then \( U \uplus V \preceq_{K} W' \uplus V \) (by (iv)). So, if further, \( y \in W' \uplus V \) then it means that \( W' \uplus V \neq \{+\infty\} \), and hence, \( W' \uplus V \subseteq (U \uplus V) + K \) (again, by Lemma 3.1(ii)), yielding \( y \in (U \uplus V) + K \).

### 3.2. Extended epigraphs of conjugate mappings and their \( \boxplus \)-Sums

**Definition 3.2:** (a) The \( K\)-extended epigraph of the conjugate mapping \( F^{\ast} \) is

\[
\mathcal{Epi}F^{\ast} := \{(L, U) \in \mathcal{L}(X, Y) \times \mathcal{P}_{p}(Y) : L \in \text{dom } F^{\ast}, (F^{\ast}(L) \preceq_{K} U)\}.
\]

(b) For \( \mathcal{M}_{1}, \mathcal{M}_{2} \subseteq \mathcal{L}(X, Y) \times \mathcal{P}_{p}(Y) \), the \( \boxplus \)-sum of these two sets is defined as:

\[
\mathcal{M}_{1} \boxplus \mathcal{M}_{2} := \{(L_{1} + L_{2}, U_{1} \uplus U_{2}) : (L_{1}, U_{1}) \in \mathcal{M}_{1}, (L_{2}, U_{2}) \in \mathcal{M}_{2}\}.
\]

In particular, if \( F, G : X \rightarrow Y^{\ast} \) then the \( \boxplus \)-sum of \( \mathcal{Epi}F^{\ast} \) and \( \mathcal{Epi}G^{\ast} \) is

\[
\mathcal{Epi}F^{\ast} \boxplus \mathcal{Epi}G^{\ast} = \{(L_{1} + L_{2}, U_{1} \uplus U_{2}) : (L_{1}, U_{1}) \in \mathcal{Epi}F^{\ast}, (L_{2}, U_{2}) \in \mathcal{Epi}G^{\ast}\}.
\]

We can understand simply that the extended epigraph of \( F^{\ast} \), \( \mathcal{Epi}F^{\ast} \), is the ‘epigraph’ of \( F^{\ast} \) which is considered as a single valued-mapping \( F^{\ast} : \mathcal{L}(X, Y) \rightarrow (\mathcal{P}_{p}(Y), \preceq_{K}) \) and the ‘epigraph’ here is understood in the same way as the one of a real-valued function.

It is also worth observing that from the definition of \( \boxplus \)-sum and Proposition 3.3, the \( \boxplus \)-sum is commutative and associative on \( \mathcal{L}(X, Y) \times \mathcal{P}_{p}(Y) \).

Note that \( \mathcal{Epi}F^{\ast} \subseteq \mathcal{L}(X, Y) \times \mathcal{P}_{p}(Y) \) while \( \text{epi}F^{\ast} \subseteq \mathcal{L}(X, Y) \times Y \). We define the set-valued mapping \( \Psi \) as follows:

\[
\Psi : \mathcal{L}(X, Y) \times \mathcal{P}_{p}(Y) \ni \mathcal{L}(X, Y) \times Y
\]

\[
(L, U) \mapsto \Psi(L, U) := \{L\} \times U.
\]
It is easy to verify that for $M, N, Q \subset \mathcal{L}(X, Y) \times \mathcal{P}_p(Y)$, for all $i \in I$ ($I$ is an arbitrary index set), it holds

\begin{align*}
M \subset N & \implies \Psi(M) \subset \Psi(N), \quad (13) \\
M \subset N & \implies M \oplus Q \subset N \oplus Q, \quad (14) \\
\bigcup_{i \in I}(M_i \oplus N) & = \left(\bigcup_{i \in I}M_i\right) \oplus N, \quad \text{and} \\
\Psi\left(\bigcup_{i \in I}M_i\right) & = \bigcup_{i \in I}\Psi(M_i). \quad (16)
\end{align*}

The relation between $\mathcal{Epi}F^*$ and $\text{epi} F^*$ is given in the next proposition.

**Proposition 3.4:** Let $F: X \to Y^*$ be a proper mapping. Then $\Psi(\mathcal{Epi} F^*) = \text{epi} F^*$.

**Proof:** Assume that $(L, y) \in \Psi(\mathcal{Epi} F^*)$. Then, there exists $U \in \mathcal{P}_p(Y)$ such that $(L, U) \in \mathcal{Epi} F^*$ and $y \in U$. As $(L, U) \in \mathcal{Epi} F^*$, one has $F^*(L) \leq_K U$, or equivalently, $U \subset F^*(L) + K$ (by Lemma 3.1(ii)). So, $y \in U \subset F^*(L) + K$, and hence, $(L, y) \in \text{epi} F^*$.

Assume now that $(L, y) \in \text{epi} F^*$. Then $y \in F^*(L) + K$. As $\text{WInf} K = \text{bd} K \subset K$, $y + \text{WInf} K \subset F^*(L) + K$ which yields $(L, y + \text{WInf} K) \in \mathcal{Epi} F^*$. It is clear that $y \in y + \text{WInf} K$ and $y + \text{WInf} K \in \mathcal{P}_p(Y)$. So, $(L, y) \in \Psi(\mathcal{Epi} F^*)$ and we are done. 

**Remark 3.1:** Coming back to the scalar case, when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, one has $\mathcal{P}_p(Y) = \mathbb{R}$ while the order `$\leq_K$’ (see (5)) and the sum `$+$’ (see (9)) become the normal order `$\leq$’ and the usual sum `$+$’ on the set of extended real numbers, respectively. Hence, $\mathcal{L}(X, Y) \times \mathcal{P}_p(Y) = X^* \times \mathbb{R}$ and the $\oplus$-sum defined in (11) collapses to the usual Minkowski sum of two subsets in $X^* \times \mathbb{R}$. In this special case, the mapping $F$ becomes an extended-real-valued function, and the conjugate mapping $F^*$ collapses to the usual conjugate function in the sense of convex analysis. Consequently, both $K$-epigraph and $K$-extended epigraph of conjugate mappings collapse to their usual epigraphs in the sense of convex analysis. The mapping $\Psi$ (defined by (12)) in this case is nothing else but the identical mapping of $X^* \times \mathbb{R}$. In other words, if $M_1, M_2 \subset X^* \times \mathbb{R}$, one has $\Psi(M_1 \oplus M_2) = \Psi(M_1 + M_2) = M_1 + M_2$.

**3.3. Basic Lemmas**

Let $F_1, F_2: X \to Y^*$ be proper $K$-convex mappings. We say that the regularity condition $(C_0)$ holds for $F_1$ and $F_2$ (in this order) if there holds:
(C0) \( \exists \hat{x} \in \text{dom } F_1 : F_2 \) is continuous at \( \hat{x} \).

**Lemma 3.2 (Basic Lemma 1):** Let \( F_1, F_2 : X \to Y^* \) be proper \( K \)-convex mappings, \( M \subset Y \) be nonempty. Assume that the condition (C0) holds for \( F_1 \) and \( F_2 \). If

\[
\tilde{y} \in Y \setminus [M + (\bar{L} - F_1 - F_2)(X) - \text{int } K]
\]

for some \( \bar{L} \in \mathcal{L}(X, Y) \), then there exist \( L_1, L_2 \in \mathcal{L}(X, Y) \) such that \( L_1 + L_2 = \bar{L} \), \( \text{WSup}[M + (L_1 - F_1)(X) + (L_2 - F_2)(X)] \neq (+\infty) \) and

\[
\tilde{y} \notin M + (L_1 - F_1)(X) + (L_2 - F_2)(X) - \text{int } K.
\] (17)

**Proof:** (See Appendix 1). \( \blacksquare \)

**Lemma 3.3:** Let \( F_1, F_2, F_3 : X \to Y^* \) be proper mappings. Then, it holds:

(i) \( \mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^* \subset \mathcal{E}pi(F_1 + F_2)^* \),

(ii) \( \Psi(\mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^*) \subset \text{epi}(F_1 + F_2)^* \),

(iii) \( \Psi(\mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^* \boxplus \mathcal{E}pi F_3^*) \subset \Psi(\mathcal{E}pi(F_1 + F_2)^* \boxplus \mathcal{E}pi F_3^*) \).

**Proof:** (See Appendix 2). \( \blacksquare \)

**Lemma 3.4 (Basic Lemma 2):** Let \( F_1, F_2, F_3 : X \to Y^* \) be proper \( K \)-convex mappings and assume that the condition (C0) holds for \( F_1 \) and \( F_2 \). Then, one has:

(i) \( \text{epi}(F_1 + F_2)^* = \Psi(\mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^*) \),

(ii) \( \Psi(\mathcal{E}pi(F_1 + F_2)^* \boxplus \mathcal{E}pi F_3^*) = \Psi(\mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^* \boxplus \mathcal{E}pi F_3^*) \).

(iii) If, in addition that one of the following conditions holds

\begin{align*}
(C'_0) \quad & \exists \bar{x} \in \text{dom } F_1 \cap \text{dom } F_2 : F_3 \text{ is continuous at } \bar{x}, \\
(C'_0') \quad & \exists \bar{x} \in \text{dom } F_3 : F_1 \text{ and } F_2 \text{ are continuous at } \bar{x},
\end{align*}

then

\[
\text{epi}(F_1 + F_2 + F_3)^* = \Psi (\mathcal{E}pi(F_1 + F_2)^* \boxplus \mathcal{E}pi F_3^*)
\]

\[
= \Psi (\mathcal{E}pi F_1^* \boxplus \mathcal{E}pi F_2^* \boxplus \mathcal{E}pi F_3^*).
\] (18)

**Proof:** (See Appendix 3). \( \blacksquare \)

### 4. Representations of the epigraphs of conjugate mappings

Let \( X, Y, Z \) be locally convex topological vector spaces, and \( K, S \) be non-empty convex cones in \( Y \) and \( Z \), respectively, with \( \text{int } K \neq \emptyset \). Let further that \( (F; G, C) \) be the triple defined the problem (VP) as in Section 1 with the assumption that \( A \cap \text{dom } F \neq \emptyset \), where \( A := C \cap G^{-1} (-S) \) is the feasible set of (VP).
In this section, we establish the main results of the paper: representations of the epigraph of the conjugate of the mapping \(F + I_A\), \(\text{epi}(F + I_A)^*\), in terms of epigraphs of the conjugate mappings of its members \(F\), \(G\), and \(I_C\).

Concerning the triple \((F; G, C)\), let us set

\[
\mathcal{A}_1 := \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^*, \\
\mathcal{A}_2 := \text{epi } F^* \oplus \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(I_C + T \circ G)^*, \\
\mathcal{A}_3 := \text{epi } F^* \oplus \text{epi } I_C^* \oplus \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(T \circ G)^*,
\]

and consider the following sets:

\[
\mathcal{A}_i = \Psi(\mathcal{A}_i), \quad i = 1, 2, 3,
\]

where \(\Psi\) is the mapping defined in (12). It is worth observing that the sets \(\mathcal{A}_1\) is exactly the qualifying set proposed recently in Refs. [5,28] (see Lemma 4.1). Moreover, in the special case where \(Y = \mathbb{R}\), and \(K = \mathbb{R}_+\), the sets \(\mathcal{A}_i, i = 1, 2, 3\) go back to the well-known sets appeared in convex scalar optimization theory (see Remark 4.1).

**Lemma 4.1:** It holds

\[
\text{epi}(F + I_A)^* \supset \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^* \quad \text{and} \quad \mathcal{A}_1 = \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^*. \tag{21}
\]

**Proof:** Note that (20) is [28, Lemma 4.1]. For the proof of (21), observe firstly that by the definitions of \(\mathcal{A}_1, \mathcal{A}_1\), and by (16), one has

\[
\mathcal{A}_1 = \Psi \left[ \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^* \right]
\]

\[
= \bigcup_{T \in \mathcal{L}_+(S, K)} \Psi[\text{epi}(F + I_C + T \circ G)^*]
\]

\[
= \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^*
\]

(the last equality follows from Proposition 3.4), which means that (21) is proved.

**Proposition 4.1:** The next inclusions hold: \(\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \text{epi}(F + I_A)^*\).
Proof: It follows from Lemma 3.3(i) that \( A_3 \subset A_2 \subset A_1 \), and taking (13) into account, we get \( A_3 \subset A_2 \subset A_1 \). On the other hand, by Lemma 4.1,

\[
A_1 = \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^* \subset \text{epi}(F + I_A)^*, \tag{22}
\]

and the proof is complete. \qed

In the rest of this section, we assume that \( F \) is \( K \)-convex, \( G \) is \( S \)-convex, and that \( C \) is a convex subset of \( X \). Consider the following regularity conditions:

\( (C_1) \) \( \exists x_1 \in C \cap \text{dom} \ F : G(x_1) \in -\text{int} \ S \)

\( (C_2) \) \( \exists x_2 \in C \cap \text{dom} \ G : F \) is continuous at \( x_2 \).

\( (C_3) \) \( \exists x_3 \in C : G \) is continuous at \( x_3 \).

**Theorem 4.1 (Representation of \( \text{epi}(F + I_A)^* \)):** The next assertions hold:

(a) If \( (C_1) \) holds then \( \text{epi}(F + I_A)^* = A_1 \),

(b) If \( (C_1) \) and \( (C_2) \) hold then \( \text{epi}(F + I_A)^* = A_2 \),

(c) If \( (C_1) \), \( (C_2) \), and \( (C_3) \) hold then \( \text{epi}(F + I_A)^* = A_3 \).

Proof: (a) follows from Lemma 4.1 and [5, Theorems 4.3].

(b) Firstly, if \( (C_1) \) holds then by (a), one has

\[
\text{epi}(F + I_A)^* = A_1 = \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^*. \tag{23}
\]

If further that \( (C_2) \) holds, then \( (C_0) \) holds with \( F_1 = I_C + T \circ G \) and \( F_2 = F \). It now follows from Basic Lemma 2 (Lemma 3.4(i)) that, for each \( T \in \mathcal{L}_+(S, K) \),

\[
\text{epi}(F + I_C + T \circ G)^* = \Psi[\mathcal{E} \text{epi} F^* \boxplus \mathcal{E} \text{epi}(I_C + T \circ G)^*],
\]

which, together with (23), gives us

\[
\text{epi}(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}(F + I_C + T \circ G)^* = \bigcup_{T \in \mathcal{L}_+(S, K)} \Psi[\mathcal{E} \text{epi} F^* \boxplus \mathcal{E} \text{epi}(I_C + T \circ G)^*] = \Psi\left[\mathcal{E} \text{epi} F^* \boxplus \bigcup_{T \in \mathcal{L}_+(S, K)} \mathcal{E} \text{epi}(I_C + T \circ G)^*\right] = A_2.
\]

(Note that the third equality comes from (15)).
(c) Firstly, if \((C_1)\) and \((C_2)\) hold, we get from (b)

\[
\epi(F + I_A)^* = \mathcal{A}_2 = \bigcup_{T \in \mathcal{L}_+(S,K)} \Psi \left( \epi F^* \boxplus \epi (I_C + T \circ G)^* \right). \tag{24}
\]

Now, for each \(T \in \mathcal{L}_+(S,K)\), if \((C_3)\) holds then \((C_0)\) holds with \(I_C\) and \(T \circ G\) as well. Thus, it follows from Lemma 3.4(ii) that

\[
\Psi \left( \epi F^* \boxplus \epi (I_C + T \circ G)^* \right) = \Psi \left( \epi F^* \boxplus \epi I_C^* \boxplus \epi (T \circ G)^* \right). \tag{25}
\]

Combining (24) and (25), one arrives at

\[
\epi(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+(S,K)} \Psi \left( \epi F^* \boxplus \epi I_C^* \boxplus \epi (T \circ G)^* \right)
= \Psi \left( \bigcup_{T \in \mathcal{L}_+(S,K)} \left[ \epi F^* \boxplus \epi I_C^* \boxplus \epi (T \circ G)^* \right] \right) \quad \text{(by (16))}
= \Psi \left( \epi F^* \boxplus \epi I_C^* \boxplus \bigcup_{T \in \mathcal{L}_+(S,K)} \epi (T \circ G)^* \right) \quad \text{(by (15))}
= \Psi(\mathcal{A}_3) = \mathcal{A}_3,
\]

which completes the proof. \[\blacksquare\]

**Remark 4.1:** It is worth observing that when turning back to the case where \(Y = \mathbb{R}\) and \(K = \mathbb{R}_+\), the cone \(\mathcal{L}_+(S,K)\) reduces to \(S^+\), \(\mathcal{P}_p(Y) = \mathbb{R}\), \(K\)-extended epigraph of conjugate mappings collapse to usual epigraphs of extended real-valued functions, the \(\boxplus\)-sum now is the usual sum of two subsets in \(X^* \times \mathbb{R}\) while the mapping \(\Psi\) is nothing else but the identical mapping of \(X^* \times \mathbb{R}\) (see Remark 3.1). As a result, (in this special case), under the light of Theorem 4.1, (and under some suitable regularity conditions), the sets \(\mathcal{A}_i, i = 1, 2, 3\), go back to the known sets that represent \(\epi(F + I_A)^*\) in the theory of (scalar) convex optimization (see Refs. [3,14,16–18,20–22,29,30], and references therein), and as usual, in this case (\(Y = \mathbb{R}\)), we will use the lowercase letters for the extended real-valued functions (e.g. \(f, i_C, i_{-S}\) instead of \(F, I_C, I_{-S}\)):

\[
\mathcal{A}_1' := \bigcup_{\lambda \in S^+} \epi(f + i_C + \lambda G)^* = \epi(f + i_A)^*,
\]

\[
\mathcal{A}_2' := \epi F^* + \bigcup_{\lambda \in S^+} \epi(i_C + \lambda G)^* = \epi(f + i_A)^*,
\]

\[
\mathcal{A}_3' := \epi F^* + \epi i_C^* + \bigcup_{\lambda \in S^+} \epi(\lambda G)^* = \epi(f + i_A)^*.
\]
In the next two sections, we give some applications of the representations established in this section, firstly to establish characterizations of some equivalent forms of vector inequalities (also called vector Farkas-type results), and secondly to introduce variant forms of duality problems (called Lagrange and Fenchel–Lagrange dual problems) for vector optimization problems and establish dual strong or stable dual strong duality results for these primal–dual pairs of problems.

5. Characterizations of vector inequalities: vector Farkas lemmas

Let \((F; G, C)\) be the triple that defines the problem (VP) with its feasible set \(A\) as in Section 4. For each \((L, y) \in \mathcal{L}(X, Y) \times Y\), we concern the vector inequality:

\[
F(x) - L(x) \not\in K - y, \quad \forall x \in A,
\]

which is equivalent to the inclusion:

\[(\alpha) \quad x \in C, \quad G(x) \in -S \implies F(x) - L(x) + y \notin -\text{int} \ K.\]

We now provide some necessary and sufficient conditions for \((\alpha)\) to hold. Each of such pair of equivalence is often called a version of the vector Farkas lemma. A version of vector Farkas lemma is called stable if such an equivalence pair holds for every \((L, y) \in \mathcal{L}(X, Y) \times Y\). We start firstly with the general case.

5.1. General vector inequalities

Consider the following statements:

\[(\beta_1) \quad \exists T \in \mathcal{L}_+(S, K) : (F + I_C + T \circ G)^* (L) \subset Y \quad \text{and} \quad y - (F + I_C + T \circ G)^* (L) \notin -\text{int} \ K,\]

\[(\beta_2) \quad \exists L' \in \mathcal{L}(X, Y), \exists T \in \mathcal{L}_+(S, K) : F^*(L') \cup (I_C + T \circ G)^* (L - L') \subset Y \text{and} y - F^*(L') - (I_C + T \circ G)^* (L - L') \notin -\text{int} \ K,\]

\[(\beta_3) \quad \exists L', L'' \in \mathcal{L}(X, Y), \exists T \in \mathcal{L}_+(S, K) : F^*(L') \cup I_C^*(L'') \cup (T \circ G)^* (L - L' - L'') \subset Y \text{and} y - F^*(L') - I_C^*(L'') - (T \circ G)^* (L - L' - L'') \notin -\text{int} \ K,\]

**Theorem 5.1 (Characterizations of stable vector Farkas Lemma):** For \(i = 1, 2, 3\), consider the following statements:

\[(a_i) \quad \text{epi}(F + I_A)^* = A_i,\]

\[(b_i) \quad (\alpha) \iff (\beta_i), \quad \forall (L, y) \in \mathcal{L}(X, Y) \times Y.\]

Then, \([a_i] \iff (b_i)\] for all \(i = 1, 2, 3\).
Proof: We will give the proof for the most complicated case, i.e, the conclusion for other cases can be obtained by the same way.

- Take \((L, y) \in \mathcal{L}(X, Y) \times Y\). According to (4),

\[
(\alpha) \iff (L, y) \in \text{epi}(F + I_A)^*.
\] (26)

- Now, we will show that

\[
(\beta_3) \iff (L, y) \in \mathcal{A}_3.
\] (27)

Proof of “⇒". Assume that \((\beta_3)\) holds. Then, there are \(L', L'' \in \mathcal{L}(X, Y)\) and \(T \in \mathcal{L}^+_+(S, K)\) such that \(F^*(L') \cup I^*_C(L'') \cup (T \circ G)^*(L - L' - L'') \subset Y\) and \(y \notin F^*(L') + I^*_C(L'') + (T \circ G)^*(L - L' - L'') - \text{int} K\). On the other hand, by Proposition 2.1(iii),

\[
F^*(L') + I^*_C(L'') + (T \circ G)^*(L - L' - L'') - \text{int} K = \text{WSup}[F^*(L') + I^*_C(L'') + (T \circ G)^*(L - L' - L'')] - \text{int} K
= F^*(L') \cup I^*_C(L'') \cup (T \circ G)^*(L - L' - L'') - \text{int} K.
\] (28)

Thus, \(y \notin F^*(L') \cup I^*_C(L'') \cup (T \circ G)^*(L - L' - L'') - \text{int} K\). It now follows from Lemma 3.1(i), Proposition 3.3(vi), there exists \(V \in \mathcal{P}_p(Y)\) such that \(y \in F^*(L') \cup I^*_C(L'') \cup V\), \((T \circ G)^*(L - L' - L'') \ll_K V\). Letting \(U = F^*(L') \cup I^*_C(L'') \cup V \in \mathcal{P}_p(Y)\), one gets \(y \in U\), and

\[
(L, U) = (L' + L'' + L - L' - L'', F^*(L') \cup I^*_C(L'') \cup V)
= (L', F^*(L')) \boxplus (L'', I^*_C(L'')) \boxplus (L - L' - L''), V
\in \mathcal{Epi} F^* \boxplus \mathcal{Epi} I^*_C \boxplus \mathcal{Epi} (T \circ G)^* \subset \mathcal{A}_3.
\] (29)

Hence, \((L, y) \in \Psi(\mathcal{A}_3) = \mathcal{A}_3\).

Proof of “⇐" in (27). Assume that \((L, y) \in \mathcal{A}_3\). Then there exists \(U \in \mathcal{P}_p(Y)\) such that \((L, U) \in \mathcal{A}_3\) and \(y \in U\). As \((L, U) \in \mathcal{A}_3\), there are \((L', U') \in \mathcal{Epi} F^*, (L'', U'') \in \mathcal{Epi} I^*_C, (L - L' - L'', W) \in \mathcal{Epi} (T \circ G)^*\) such that \(U = U' \cup U'' \cup W\). Then, \(F^*(L') \ll_K U', I^*_C(L'') \ll_K U'', (T \circ G)^*(L - L' - L'') \ll_K W\), and consequently,

\[
M := F^*(L') \cup I^*_C(L'') \cup (T \circ G)^*(L - L' - L'') \ll_K U
\] (30)
(by Proposition 3.3 (v)). This yields \(M \neq \{+\infty_Y\}\), which, together with the fact that \(M\) belong to \(\mathcal{P}_p(Y)^\infty\), shows that \(M \subset Y\). So, according to Proposition 3.1 (i), (30) entails \(U \cap (M - \text{int} K) = \emptyset\) showing that \(y \notin M - \text{int} K\) (as \(y \in U\)) which also means that \((\beta_3)\) holds.
Finally, taking (26) and (27) into account, one gets

\[(a_3) \iff [(\alpha) \iff (\beta_3), \ \forall (L, y) \in \mathcal{L}(X, Y) \times Y] \iff (b_3),\]

which is desired. ■

**Remark 5.1:** According to Lemma 4.1, \((a_1) \iff (b_1)\) is [5, Proposition 5.1] \((a) \iff (c)\), when \(\mathcal{V} = \mathcal{L}(X, Y)\) and \(\mathcal{W} = Y\). To the best knowledge of the authors, the characterizations of vector Farkas lemmas given in Theorem 5.1 with \(i = 2, 3\) are all new. Moreover, taking Remark 4.1 into account, we will see that Theorem 5.1 when specifying to the scalar case \((Y = \mathbb{R} \text{ and } K = \mathbb{R}_+)\), extends some known results in the literature, such as the ones in Refs. [16,17], as shown in the next corollary (see Remark 5.2).

Let us recall that \(A'_i, i = 1, 2, 3\), are the sets defined in Remark 4.1.

**Corollary 5.1:** Let \(f : X \to \mathbb{R}\) be a proper function, \(G : X \to Z \cup \{+\infty\}\) – proper mapping, and \(C \subset X\) nonempty convex, \(A \cap \text{dom } F \neq \emptyset\). Consider the statements:

\[(\alpha') x \in C, \ G(x) \in -S \implies f(x) - x^*(x) \geq r,\]

\[(\beta'_1) \ \exists \lambda \in S^+: (f + iC + \lambda G)^*(x^*) \leq -r,\]

\[(\beta'_2) \ \exists u^* \in X^*, \ \exists \lambda \in S^+: f^*(u^*) + (iC + \lambda G)^*(x^* - u^*) \leq -r,\]

\[(\beta'_3) \ \exists u^*, v^* \in X^*, \ \exists \lambda \in S^+: f^*(u^*) + iC^*(v^*) + (\lambda G)^*(x^* - u^* - v^*) \leq -r.\]

Moreover, let

\[(a'_i) \ \text{epi}(f + iA)^* = A'_i,\]

\[(b'_i) (\alpha') \iff (\beta'_i), \ \forall (x^*, y) \in X^* \times \mathbb{R}.\]

Then, \([(a'_i) \iff (b'_i)]\) for all \(i = 1, 2, 3\).

**Proof:** When \(Y = \mathbb{R}\) and \(K = \mathbb{R}_+\), one has, for each \(i = 1, 2, 3\), \(A_i\) collapses to \(A'_i\) (see Remark 4.1), \((\alpha)\) becomes \((\alpha')\), while \((\beta_i)\) becomes \((\beta'_i)\). The equivalences \([(a'_i) \iff (b'_i)]\), \(i = 1, 2, 3\) now follow immediately from Theorem 5.1. ■

**Remark 5.2:** In the case where \(Y = \mathbb{R}\) and \(K = \mathbb{R}_+\), if we assume further that \(C\) is a closed convex subset of \(X\), that \(f\) is a proper convex and lsc function, and that \(G\) is a proper \(S\)-convex and positively \(S\)-lsc mapping. Then, according to [14, Section 8],

\[\text{epi}(f + iA)^* = \text{cl} \left[\text{epi } F^* + \text{epi } iC^* + \bigcup_{z^* \in S^+} \text{epi}(z^* \circ G)^*\right].\]

So, in this setting, observe firstly that Corollary 5.1 with \(i = 1\) is [31, Corollary 6.14]. Moreover, note also that \((a'_3)\) is equivalent to the condition stating that
the set \( \text{epi} F^* + \text{epi} i_C^* + \bigcup_{z^* \in S^+} \text{epi}(z^* \circ G)^* \) is weak*-closed, which is exactly the condition (CC) introduced in [16,17]. Moreover, by Proposition 4.1, the fulfilment of \((a'_3)\) ensures that both \((a'_1)\) and \((a'_2)\) hold. So, Corollary 5.1 can be considered as extended versions of [16, Corollary 5.2] and [17, Corollary 6.2], and as a result, Theorem 5.1 is vector extension version of the Farkas lemmas just mentioned in Refs. [16,17] in twofolds: Firstly, the Farkas versions in Theorem 5.1 extend the Farkas-type results in the two mentioned paper from scalar systems to systems involving vector-valued functions; secondly, they extend the Farkas-type results to stable Farkas-type ones.

### 5.2. Convex vector inequalities

**Corollary 5.2 (Convex vector Farkas lemma):** Let \( F \) be a proper and \( K \)-convex mapping, \( G \) be a proper and \( S \)-convex mapping, and \( C \) be a nonempty and convex subset of \( X \). The following statements hold:

(i) If \((C'_1)\) holds then \((b'_1)\) holds,
(ii) If \((C'_1)\) and \((C'_2)\) hold then \((b'_2)\) holds,
(iii) If \((C'_1)\), \((C'_2)\) and \((C'_3)\) hold then \((b'_3)\) holds,

where \((b'_i)\), \(i = 1, 2, 3\) are statements in Theorem 5.1.

**Proof:** These are direct consequences of Theorems 4.1 and 5.1.

Turning to the scalar case, i.e. \( Y = \mathbb{R} \) and \( K = \mathbb{R}_+ \), the conditions \((C'_1)\) and \((C'_2)\) collapse, respectively, to the ones below:

\[
\begin{align*}
(C'_1) & \exists x_1 \in C \cap \text{dom } F : G(x_1) \in -\text{int } S \\
(C'_2) & \exists x_2 \in C \cap \text{dom } G : f \text{ is continuous at } x_2.
\end{align*}
\]

The next corollary is a direct sequence of Corollary 5.2.

**Corollary 5.3 (Convex Farkas-type results):** Let \( f : X \to \overline{\mathbb{R}} \) be a proper convex function, \( G : X \to Z \cup \{+\infty\} \) be a proper \( S \)-convex mapping, and \( C \) be a nonempty convex subset of \( X \) such that \( A \cap \text{dom } F \neq \emptyset \) (where \( A := C \cap G^{-1}(-S) \)). The following statements hold:

(i) If \((C'_1)\) holds then \((b'_1)\) holds,
(ii) If \((C'_1)\) and \((C'_2)\) hold then \((b'_2)\) holds,
(iii) If \((C'_1)\), \((C'_2)\) and \((C'_3)\) hold then \((b'_3)\) holds,

where \((b'_i)\), \(i = 1, 2, 3\) are statements in Corollary 5.1.
6. Lagrange and Fenchel–Lagrange duality for vector optimization problems

We retain the notation in Section 4 and consider a vector optimization problem

\[(VP) \quad \text{WInf}\{F(x) : x \in C, \ G(x) \in -S\}\]

with the feasible set \(A := C \cap G^{-1}(-G)\) being non-empty.

The Lagrange dual problem of (CVP) is defined in Ref. [7] as:

\[(VD_1) \quad \text{WSup}_{T \in \mathcal{L}_+(S,K)} \text{WInf}_{x \in C} [F(x) + (T \circ G)(x)],\]

or, equivalently,

\[(VD_1) \quad \text{WSup}_{T \in \mathcal{L}_+(S,K)} -(F + I_C + T \circ G)^*(0_C).\]

We now propose some two new types of ‘Fenchel–Lagrange’ dual problems of (VP):

\[(VD_2) \quad \text{WSup}_{L' \in \mathcal{L}(X,Y), \ T \in \mathcal{L}_+(S,K)} -[F^*(L') \cup (I_C + T \circ G)^*(-L')],\]

\[(VD_3) \quad \text{WSup}_{L',L'' \in \mathcal{L}(X,Y), \ T \in \mathcal{L}_+(S,K)} -[F^*(L') \cup I_C^b(L'') \cup (T \circ G)^*(-L' - L'')].\]

It is worth observing that when going back to the scalar problem, i.e. when \(Y = \mathbb{R}\) and (VP) is a scalar problem (P), the last two dual problems turn back to the Fenchel–Lagrange dual problems (D2) and (D3) of (P) mentioned in the Introduction (Section 1), and this justifies the names of these dual problems.

**Example 6.1:** Let \(X = C = Z = \mathbb{R}, \ Y = \mathbb{R}^2, \ S = \mathbb{R}_+, \ K = \mathbb{R}^2_+\). Then \(\mathcal{L}(X,Y) = \mathcal{L}(Z,Y) = \mathcal{L}(\mathbb{R}, \mathbb{R}^2) \equiv \mathbb{R}^2\) and \(\mathcal{L}_+(S,K) \equiv \mathbb{R}^2_+\). Let further \(F: \mathbb{R} \to \mathbb{R}^2, \ G: \mathbb{R} \to \mathbb{R}\) such that \(F(x) = (x, x^2)\) and \(G(x) = x\) for all \(x \in \mathbb{R}\). Then, the vector optimization problem (VP) reads as

\[(\hat{VP}) \quad \text{WInf}\{(x, x^2) : x \leq 0\}.\]

The first Fenchel–Lagrange dual problem (VD2) now becomes

\[(\hat{VD}_2) \quad \text{WSup}_{(L',T) \in \mathcal{L} \times \mathcal{L}_+^2} \mathcal{H}_{L',T},\]

where, for \(L' = (a, b) \in \mathbb{R}^2\) and \(T = (c, d) \in \mathbb{R}^2_+\),

\[\mathcal{H}_{L',T} = -[F^*(L') \cup (I_C + T \circ G)^*(-L')] = -\text{WSup}[\text{WSup}((L' - F)(\mathbb{R}^2)] + \text{WSup}[(-L' - T \circ G)(0_C)]\]
\[= -\text{WSup} \left[ (L' - F)(\mathbb{R}^2) + (-L' - T \circ G)(C) \right] \]
\[= \text{WInf} \left[ (F - L')(\mathbb{R}^2) + (T \circ G + L')(C) \right] \]
\[= \text{WInf} \left[ \{ (x, x^2) - (ax, bx) : x \in \mathbb{R} \} + \{ (cy, dy) + (ay, by) : y \in \mathbb{R} \} \right]. \]

So, the first Lagrange–Fenchel dual problem can be rewritten as
\[
(VD_2) \quad \begin{align*}
& \text{WSup} \quad \text{WInf} \quad \left[ \{ (1 - a)x, x^2 - bx) : x \in \mathbb{R} \} \\
& \quad + \{ y(c + a, d + b) : y \in \mathbb{R} \}. \end{align*}
\]

**Definition 6.1:** We say that ‘strong duality holds for the pair (VP)–(VD_1)’ if \(\text{WInf}(\text{VP}) = \text{WMax}(\text{VD}_1)\). 

We denote by (VP^L) the problem (VP) perturbed by a linear operator \(L \in \mathcal{L}(X, Y)\),
\[
(VP^L) \quad \text{WInf} \{ F(x) - L(x) : x \in C, \; G(x) \in -S \}. \]

Then the Lagrange dual problem of (VP^L) will be denote by (VD_1^L). We say that the **stable strong duality holds for the pair (VP)–(VD_1)** if the strong duality holds for the pair (VP^L)–(VD_1^L) for any \(L \in \mathcal{L}(X, Y)\). The notions stable strong duality corresponding to the other pairs of primal-dual problems (VP)–(VD_2) and (VP)–(VD_3) will be understood in the same way.

**Lemma 6.1:** \(\text{WSup}(VD_1^L) = \text{WSup} \{ -y : (L, y) \in A_i \} \) for all \(L \in \mathcal{L}(X, Y)\) and \(i \in \{1, 2, 3\}\), where \(A_i\) are the sets defined in (19).

**Proof:** We prove the conclusion for \(i = 3\), i.e. that \(\text{WSup}(VD_3^L) = \text{WSup} \{ -y : (L, y) \in A_3 \}\). The proofs of other cases are similar. On the one hand, one has
\[
\text{WSup}(VD_3^L) \]
\[= \text{WSup} \quad \begin{align*}
& [(F - L)^* \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'')] \\
& + [F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'') - K] \\
& = \text{WSup} \left\{ -y : y \in F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'') + K \right\} \quad \text{for some } L', L'', T \in \mathcal{L}_+(S, K) \]
\]
(where the third equality follows from Proposition 2.1(vii)). On the other hand, as $F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'') \in P_p(Y)$, one has

$$y \in F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'') + K$$

for some $L', L'' \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}_+(S, K)$.

$$\iff y \notin F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'') - \text{int } K$$

for some $L', L'' \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}_+(S, K)$ (by Lemma (3.1)(i))

$$\iff y \notin F^*(L + L') + I_C^*(L'') + (T \circ G)^*(-L' - L'') - \text{int } K$$

for some $L', L'' \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}_+(S, K)$ (by Proposition (2.1)(iii))

$$\iff (\beta_3) \text{ holds}$$

$$\iff (L, y) \in \mathcal{A}_3 \quad (\text{see (27)}).$$

Consequently, $\text{WSup}(VD_3^L) = \text{WSup}\{y : (L, y) \in \mathcal{A}_3\}$ and we are done. \hfill \blacksquare

**Proposition 6.1 (Weak duality):** For any $L \in \mathcal{L}(X, Y)$, it holds

$$\text{WSup}(VD_3^L) \preceq_K \text{WSup}(VD_2^L) \preceq_K \text{WSup}(VD_1^L) \preceq_K \text{WInf}(VP^L).$$

**Proof:** Recall that $(VD_1)$ is nothing else but the problems (DVP) in Ref. [7]; hence, the inequality $\text{WSup}(VD_1^L) \preceq_K \text{WInf}(VP^L)$ was established in [7, Theorem 5] (specialize for the case when the uncertain set is a singleton).

Now, by Lemma 6.1 one has $\text{WSup}(VD_3^L) = \text{WSup}\{y : (L, y) \in \mathcal{A}_i\}$ for each $i = 1, 2, 3$. Moreover, by Proposition 4.1, $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1$, and the conclusion follows as $\text{WSup } M \preceq_K \text{WSup } N$ if $M, N \subset Y^*$ and $M \subset N$ (see Proposition 3.1(iii)). \hfill \blacksquare

The main result of this section on the stable strong duality for (VP) is given in the next theorem.

**Theorem 6.1 (Principle for stable strong duality of (VP)):** For $i = 1, 2, 3$, consider the following statements:

1. $(a_i)$ $\text{epi}(F + I_A)^* = \mathcal{A}_i$,
2. $(c_i)$ The stable strong duality holds for the pair $(VP) - (VD_i)$.

Then, $[(a_i) \iff (c_i)]$ for all $i = 1, 2, 3$.

**Proof:** We prove $[(a_3) \iff (c_3)]$ only. The proofs for the case $i = 1, 2$ are similar. Take $L \in \mathcal{L}(X, Y)$. For each $L', L'' \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}_+(S, K)$, let

$$\mathcal{K}_L(L', L'', T) := -F^*(L + L') \cup I_C^*(L'') \cup (T \circ G)^*(-L' - L'').$$
\[ N_L := \bigcup_{L', L'' \in \mathcal{L}(X, Y) \atop T \in \mathcal{L}_+(S, K)} \mathcal{K}_L(L', L'', T). \]

One then has

\[
\begin{align*}
WSup(VD^I_Y) &= WSup_{L', L'' \in \mathcal{L}(X, Y)} \mathcal{K}_L(L', L'', T) = WSup N_L \\
WMax(VD^I_Y) &= WMax_{L', L'' \in \mathcal{L}(X, Y)} \mathcal{K}_L(L', L'', T) = WMax N_L.
\end{align*}
\] (32)

\[ [(a_3) \Rightarrow (c_3)] \] Assume that \((a_3)\) holds. Take \(L \in \mathcal{L}(X, Y)\), and we will show that

\[ WInf(VP^L) = WMax(VD^I_Y). \] (33)

- **Step 1.** As \((VP)\) is feasible, \(WInf(VP^L) \neq \{+\infty_Y\}\). If \(WInf(VP^L) = \{-\infty_Y\}\) then, by Proposition 6.1, \(WSup(VD^I_Y) = \{-\infty_Y\}\), and so, \(\mathcal{K}_L(L', L'', T) = \{-\infty_Y\}\) for all \(L', L'' \in \mathcal{L}(X, Y)\) and \(T \in \mathcal{L}_+(S, K)\). Consequently, \(WMax(VD^I_Y) = \{-\infty_Y\} = WInf(VP^L)\), and (33) holds.

- **Step 2.** Assume now that \(WInf(VP^L) \subset Y\). As \(WInf(VP^L) \in \mathcal{P}_p(Y)\), one has

\[
\begin{align*}
WInf(VP^L) &= WSup[WInf(VP^L)] \quad \text{(see Lemma 3.1(iii))} \\
&= WSup[WInf[F(x) - L(x) : x \in A]] \\
&= WSup[-(F + I_A)^*(L)] \\
&= WSup[-(F + I_A)^*(L) - K] \quad \text{(see Proposition 2.1(vii))} \\
&= WSup\{-y : (L, y) \in \text{epi}(F + I_A)^*\}.
\end{align*}
\] (34)

On the other hand, as \((a_3)\) holds, \(\text{epi}(F + I_A)^* = A_3\), and by Lemma 6.1, \(WSup(VD^I_Y) = WSup\{-y : (L, y) \in A_3\}\), which, together with (34) and (32), yields

\[ WInf(VP^L) = WSup(VD^I_Y) = WSup N_L. \] (35)

- **Step 3.** We now prove that \(WInf(VP^L) \subset N_L\). Take \(y \in WInf(VP^L) = WInf[(F - L)(X)]\). It then follows from Proposition 2.1(iv) (see also, Remark 2.1) that \(y \notin (F - L)(X) + \text{int } K\), which is equivalent to

\[ F(x) - L(x) - y \notin -\text{int } K, \quad \forall x \in A. \]

This is nothing else but \((\alpha)\) in Section 5 with \(y\) being replaced by \(-y\). As \((a_3)\) holds, it follows from the stable vector Farkas lemma (Theorem 5.1 with \(i = 3\))
that there are $\bar{L}', \bar{L}'' \in \mathcal{L}(X, Y)$ and $\bar{T} \in \mathcal{L}_+(S, K)$ such that

$$F^*(L + L') \cup I^*_C(L'') \cup (T \circ G)^*(-L' - L'') \subset Y$$

(36) 

and $-y \notin F^*(L + \bar{L}') + I^*_C(\bar{L}'') + (\bar{T} \circ G)^*(-\bar{L}' - \bar{L}'') - \text{int} \ K$, which means that

$$-y \notin -K_L(\bar{L}', \bar{L}'', \bar{T}) - \text{int} \ K.$$ 

(37)  

On the other hand, as $y \in \text{WInf}(\text{VP}^L) = \text{WSup} N_L$ (see (35)), one has $y \neq K y'$ for all $y' \in N_L$, yielding

$$y \notin K_L(\bar{L}', \bar{L}'', \bar{T}) - \text{int} \ K.$$  

(38)  

Now, as $K_L(\bar{L}', \bar{L}'', \bar{T}) \in \mathcal{P}_Y(Y)$ (see (36)), the three sets $K_L(\bar{L}', \bar{L}'', \bar{T}) - \text{int} \ K$, $K_L(\bar{L}', \bar{L}'', \bar{T})$, and $K_L(\bar{L}', \bar{L}'', \bar{T}) + \text{int} \ K$ constitute a decomposition of $Y$, which, together with (37) and (38), yields $y \in K_L(\bar{L}', \bar{L}'', \bar{T})$. This shows that $y \in N_L$, and hence, WInf $M_L \subset N_L$.

- **Step 4.** It now follows from Step 2 and Step 3 that (see also (32))

$$\text{WInf}(\text{VP}^L) = N_L \cap \text{WSup} N_L = \text{WMax} N_L = \text{WMax}(\text{VD}^L_3),$$

which is (33).

[(c$_3$) $\Rightarrow$ (a$_3$)] Assume that (c$_3$) holds, i.e. (33) holds for all $L \in \mathcal{L}(X, Y)$. Taking Proposition 4.1 into account, to prove (a$_3$), it suffices to show that

$$\text{epi}(F + I_A)^* \subset A_3.$$  

(39)  

Take $(L, y) \in \text{epi}(F + I_A)^*$. Then

$$y \in \text{WSup} [(L - F)(A)] + K = -\text{WInf}(F - L)(A) + K = -\text{WInf}(\text{VP}^L) + K.$$ 

On the other hand, as (c$_3$) holds, one also has

$$\text{WInf}(\text{VP}^L) = \text{WMax}(\text{VD}^L_3) = \text{WMax} N_L \subset N_L.$$ 

So, $y \in -N_L + K$ and thus, there are $\bar{L}', \bar{L}'' \in \mathcal{L}(X, Y)$ and $\bar{T} \in \mathcal{L}_+(S, K)$ such that

$$y \in -K_L(\bar{L}', \bar{L}'', \bar{T}) + K$$

$$\iff y \in F^*(L + \bar{L}') \cup I^*_C(\bar{L}'') \cup (\bar{T} \circ G)^*(-\bar{L}' - \bar{L}'') + K$$

$$\iff (L, y) \in A_3 \quad (\text{see (31)}).$$ 

Consequently, $(L, y) \in \text{epi}(F + I_A)^*$ then $(L, y) \in A_3$ and (39) follows.
Corollary 6.1 (Stable strong duality I): Assume that $F$ is $K$-convex, that $G$ is $S$-convex, and that $C$ is convex. Then, the following statements are holds true:

(i) If $(C_1)$ holds then the stable strong duality holds for the pair $(VP) - (VD_1)$.
(ii) If $(C_1)$ and $(C_2)$ hold then the stable strong duality holds for pairs $(VP) - (VD_1)$ and $(VP) - (VD_2)$.
(iii) If $(C_1), (C_2)$ and $(C_3)$ hold then the stable strong duality holds for three pairs $(VP) - (VD_i)$, $i = 1, 2, 3$.

Proof: It follows from Theorems 6.1 and 4.1.

Example 6.2 (Example 6.1 revisited): We consider again the problem $(\overline{VP})$ in Example 6.1:

$$(\overline{VP}) \quad \text{WInf}\{(x, x^2) : x \leq 0\}.$$ 

Its Fenchel–Lagrange dual problem is

$$(\overline{VD}_2) \quad \text{WSup}_{(L', T) \in \mathbb{R}^2 \times \mathbb{R}^2_+} \mathcal{H}_{L', T}$$

where, for $L' = (a, b) \in \mathbb{R}^2$, $T = (c, d) \in \mathbb{R}^2_+$,

$$\mathcal{H}_{L', T} = -[F^*(L') \cup (I_C + T \circ G)^*(-L')]$$

$$= \text{WInf} \left[ \mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2 \right],$$

with $\mathcal{H}_{L', T}^1 := \{(1 - a)x, x^2 - bx) : x \in \mathbb{R}\}$, $\mathcal{H}_{L', T}^2 := \mathbb{R}(c + a, d + b)$. Consider the following cases:

- $a = 1$. $\mathcal{H}_{L', T}^1 = \{0\} \times \left[-\frac{b^2}{4}, +\infty\right)$. So, $\mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2$ is the half-plane above the line $(0, -\frac{b^2}{4}) + \mathbb{R}(c + a, d + b)$ (Figure 1). This yields

$$\mathcal{H}_{L', T} = \text{WInf} \left[ \mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2 \right]$$

$$= \begin{cases} 
(0, -\frac{b^2}{4}) + \mathbb{R}(c + 1, d + b), & \text{if } (c + 1)(d + b) \leq 0, \\
(-\infty \mathbb{R}^2), & \text{else}.
\end{cases}$$

- $a > 1$. Then $\mathcal{H}_{L', T}^1$ is the (concave) parabola $y = \varphi_{a,b}(x) := \frac{x^2}{1-a} - \frac{bx}{1-a}$. Observing that $\mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2 \supset \mathcal{H}_{L', T}^1$, and hence, for all $b \in \mathbb{R}$ and $c, d > 0$,

$$\mathcal{H}_{L', T} = \text{WInf} \left[ \mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2 \right] = \{-\infty \mathbb{R}^2\}.$$

- $a < 1$. Then $\mathcal{H}_{L', T}^1$ is the (convex) parabola $(P)$: $y = \varphi_{a,b}(x)$, and hence $\mathcal{H}_{L', T}^1 + \mathcal{H}_{L', T}^2$ is $(P)$ if $c + a = d + b = 0$, and is the half-plane above the tangent $(d)$ (whose direction vector is $(c + a, d + b)$) of $(P)$. So,
Figure 1. Graphical description of $W_{\text{Inf}}(\text{VP})$ and some $H_{L,T}$. 

\[ H_{L',T} = W_{\text{Inf}}(\text{VP}) \]

\[ (x, \varphi_{a,b}(x) : x \leq \frac{b}{2}) \cup \mathbb{R} \]

\[ = + \times \left\{ -\frac{b^2}{4(1-a)} \right\}, \quad \text{if } c + a = d + b = 0, \]

\[ (x_0, y_0) + \mathbb{R}(c + a, d + b), \quad \text{if } (c + a)(d + b) < 0 \text{ or } d + b = 0, \]

\[ \{-\infty, \infty^2\}, \quad \text{else}, \]

\((d)\) is the tangent line to \((P)\) at \((x_0, y_0)\). By this calculation, it is easy to see that $W_{\text{Sup}}(\overline{\text{D}_2}) = (C) = H_{T_0,L_0} = W_{\text{Max}}(\overline{\text{D}_2})$, where $L'_0 = (0,0)$ and $T_0 = (0,0)$. One also can verify that $W_{\text{Inf}}(\overline{\text{P}}) = (C)$. So, $W_{\text{Max}}(\overline{\text{D}_2}) = W_{\text{Inf}}(\overline{\text{P}}) = (C)$. Note that in this example both the conditions $(C_1)$ and $(C_2)$ hold, and the last equalities agree with the conclusion of Corollary 6.1.

Remark 6.1: Theorem 6.1 and Corollary 6.1 for the case $i = 1$ are stable versions of [7, Corollaries 1 and 3] (with the uncertainty set being a singleton). The results for other cases (i.e. $i = 2, 3$), up to the best knowledge of the authors, are new. Consequently, Theorem 6.1 and Corollary 6.1 with $i = 2, 3$, are probably the first version of Fenchel–Lagrange duality results for vector problems which extend the same kinds of duality (see Refs. [14,16], and references therein) for scalar to vector optimization problems (see Corollaries 6.2 and 6.3).

It is worth observing that when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ the problem \((\text{VP})\) becomes \((P)\), while the dual problems \((\text{VD}_i), i = 1, 2, 3\), collapse to the dual problem \((\text{D}_1), (\text{D}_2), \) and \((\text{D}_3)\) in Section 1 (Introduction). Note that \((\text{D}_1)\) and \((\text{D}_2)\)
are \( (D^{CL}) \) and \( (D^{CLC}) \), respectively, in Ref. [14] while \( (D_1) \) and \( (D_3) \) are the problems \( (LDQ) \) and \( (FLDQ) \) proposed in Ref. [16] for the case when \( Y = \mathbb{R} \) and \( K = \mathbb{R}_+ \). The next two corollaries are stable versions (extensions) of the corresponding results in Ref. [14] and in Ref. [16]. They are direct consequences of Theorem 6.1 and Corollary 6.1, respectively.

**Corollary 6.2:** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \), \( A'_i \) be the sets defined in Remark 4.1, and \( i_C \) be the usual indicator function of the set \( C \). For each \( i = 1, 2, 3 \), consider the following statements:

\[
\begin{align*}
(a'_i) \quad & \text{epi}(f + i_A)^* = A'_i, \\
(c'_1) \quad & \inf(f - x^*)(A) = \max_{\lambda \in S^+} \inf(f - x^* + \lambda G)(C), \forall x^* \in X^*, \\
(c'_2) \quad & \inf(f - x^*)(A) = \max_{\lambda \in S^+} \inf_{u^*, v^* \in X^*} [-f^*(u^*) - (i_C + \lambda G)^*(x^* - u^*)], \forall x^* \in X^*, \\
(c'_3) \quad & \inf(f - x^*)(A) = \max_{\lambda \in S^+} \inf_{u^*, v^* \in X^*} [-f^*(u^*) - i_C^*(v^*) - (\lambda G)^*(x^* - u^* - v^*)], \\
& \forall x^* \in X^*.
\end{align*}
\]

Then, \( (a'_i) \Leftrightarrow (c'_i) \) for each \( i = 1, 2, 3 \).

**Corollary 6.3:** Assume that \( f \) is convex, that \( G \) is \( S \)-convex, and that \( C \) is convex. The following statements hold:

\[
\begin{align*}
(i) \quad & \text{If } (C'_1) \text{ holds then } (c'_1) \text{ holds}, \\
(ii) \quad & \text{If } (C'_1) \text{ and } (C'_2) \text{ hold then } (c'_1) \text{ and } (c'_2) \text{ hold}, \\
(iii) \quad & \text{If } (C'_1), (C'_2) \text{ and } (C'_3) \text{ hold then } (c'_1), (c'_2), \text{ and } (c'_3) \text{ hold},
\end{align*}
\]

where \( (c'_1), \ (c'_2), \) and \( (c'_3) \) are statements in Corollary 6.2, and \( (C'_1), \ (C'_2) \) are regularity conditions introduced in Subsection 5.2.

**Remark 6.2:** (a) The condition \( (C'_1) \) and \( (C'_2) \) are nothing but the condition \( (RC^L_1) \) and \( (RC^L_1) \) in Ref. [14], and hence, Corollary 6.3 i)-(ii) extends and can be considered as ‘stable strong duality version’ of the strong duality in [14, Theorems 3.4, 3.6].

(b) Recall that when \( C \) is a closed convex subset of \( X, f \) is a proper convex and lsc function, and \( G \) is a proper \( S \)-convex and positively \( S \)-lsc mapping, \( (a'_3) \) is equivalent to \( (GC) \) in Refs. [16,17], and that \( (a'_1) \) and \( (a'_2) \) hold whenever \( (a'_1) \) holds (see Remark 5.2). So, Corollary 6.2 covers [16, Corollaries 4.5, 4.6, 4.7] and [17, Corollaries 6.4, 6.5] (non-stable version) and [17, Theorem 6.2, 6.3] (stable version).

(c) Under the convex and closedness assumptions as in (b), \( (a'_1) \) is equivalent to the fact that \( \bigcup_{z^* \in S^+} \text{epi}(f + i_C + z^* \circ G)^* \) is weak*-closed, and hence, Corollary 5.1, together with Corollary 6.2, for the case \( i = 1 \) returns to [19, Corollary 5].
(d) Lastly, it is worth emphasizing that (to the best knowledge of the authors) no results in duality for vector problems existed in the literature could cover the ones in \((c'_2), (c'_3)\) (when turning back to the case \(Y = \mathbb{R}\)).

**Notes**

1. The problem considered in Ref. [4] is the vector composite problem which is more general than (VP). The dual problems considered there, however, are all of the Lagrange ones.
2. Here, by the term ‘decomposition’ we mean the sets in the right-hand side of the equality are disjoint.
3. This notion was used in Refs. [14,30] as ‘star K-lower semicontinuous’.
4. For other orderings on \(P_0(Y^*)\), see, e.g. [33].
5. Observe that when \(W\text{Inf}(VP) = W\text{Max}(VD_1)\), one has \(W\text{Sup}(VD_1) = W\text{Max}(VD_1)\) (see Proposition 3.2 (i)), and hence, \(VD_1\) attains any value from \(W\text{Sup}(VD_1)\).

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Appendices

Appendix 1. Proof of the Basic Lemma 1 (Lemma 3.2)

Let us set

\[ \Delta := \bigcup_{x \in \text{dom } F_1} \left[ M + \bar{L}(x) - F_1(x) - F_2(x') - K \right] \times \{x - x'\}. \]  (A1)
Step 1. We firstly prove that there is \( \hat{y} \in Y \) such that \((\hat{y} - \text{int} \ K) \times \{0 \_X\} \subset \text{int} \Delta\). Pick \( \bar{k} \in \text{int} \ K \), it is obvious that \( F_2(\bar{x}) + \bar{k} - K \in \mathcal{N}_Y(\bar{x}) \). So, by (C0), there is \( U \in \mathcal{N}_Y(0 \_X) \) such that \( F_2(\bar{x} + U) \subset F_2(\bar{x}) + \bar{k} - K \) which leads to
\[
F_2(\bar{x}) + \bar{k} \in F_2(\bar{x} + U) + K, \quad \forall u \in U.
\]
It results that if \( \hat{y} := m + \bar{L}(\bar{x}) - F_1(\bar{x}) - F_2(\bar{x}) - \bar{k} \) for some \( m \in M \) then it holds:
\[
\hat{y} \in m + \bar{L}(\bar{x}) - F_1(\bar{x}) - F_2(\bar{x} + u) - K, \quad \forall u \in U
\]
\[
\Rightarrow \hat{y} - k \in m + \bar{L}(\bar{x}) - F_1(\bar{x}) - F_2(\bar{x} + u) - K, \quad \forall k \in \text{int} \ K, \quad \forall u \in U
\]
\[
\Rightarrow (\hat{y} - k, -u) \in \Delta, \quad \forall k \in \text{int} \ K, \quad \forall u \in U \text{ (by (A1))}
\]
\[
\Rightarrow (\hat{y} - \text{int} \ K) \times (-U) \subset \Delta
\]
\[
\Rightarrow (\hat{y} - \text{int} \ K) \times \{0 \_X\} \subset \text{int} \Delta.
\]

Step 2. We prove that \((\hat{y}, 0 \_X) \notin \text{int} \Delta \). Indeed, if we assume on the contrary, then there exists \( V \in \mathcal{N}_Y(0 \_Y) \) such that \((\hat{y} + V) \times \{0 \_X\} \subset \Delta \). Take \( \bar{k} \in V \cap \text{int} \ K \), one gets \((\bar{y} + \bar{k}, 0 \_X) \in \Delta \). This leads to exist \( \bar{x} \in \text{dom} \ F_1 \) and \( \bar{x}' \in \text{dom} \ F_2 \) such that \( \bar{y} + \bar{k} \in \bar{L}(\bar{x}) - F_1(\bar{x}) - F_2(\bar{x}') - K \) and \( \bar{x} = \bar{x}' \), which contradicts the assumption \( \hat{y} \in Y \setminus [M + (\bar{L} - F_1 - F_2)(X) - \text{int} \ K] \).

Step 3. Applying of the convex separation theorem. By the convexity of \( M, F_1 \) and \( F_2 \), it is easy to check that \( \Delta \) is a convex subset of \( Y \times X \). Moreover, it follows from Steps 1 and 2 that \( \text{int} \Delta \neq \emptyset \) and \((\hat{y}, 0 \_X) \notin \text{int} \Delta \). Now, the convex separation theorem ([32, Theorem 3.4]) applying to the convex sets \( \{(\hat{y}, 0 \_X)\} \) and \( \Delta \) yields the existence of a nonzero functional \((\mu_0, u_0) \in Y^* \times X^* \) satisfying
\[
y_0^\mu(\hat{y}) < y_0^\mu(y) + u_0^\mu(u), \quad \forall(y, u) \in \text{int} \Delta, \tag{A2}
\]
and consequently,
\[
y_0^\mu(\hat{y}) \leq y_0^\mu(y) + u_0^\mu(u), \quad \forall(y, u) \in \Delta. \tag{A3}
\]
Next, we show that
\[
y_0^\mu(k') < 0, \quad \forall k' \in \text{int} \ K. \tag{A4}
\]
Take \( k' \in \text{int} \ K \). According to Step 1, there is \( \hat{y} \in Y \) such that \((\hat{y} - \text{int} \ K) \times \{0 \_X\} \subset \text{int} \Delta \). On the other hand, by [4, Lemma 2.1(i)], there is \( \mu > 0 \) such that \( \hat{y} - \mu k' \in \hat{y} - \text{int} \ K \). Hence, \((\hat{y} - \mu k', 0 \_X) \in \text{int} \Delta \), which, together with (A2), leads to \( y_0^\mu(\hat{y}) < y_0^\mu(\hat{y} - \mu k') \), or \( y_0^\mu(k') < 0 \), and (A4) holds.

Step 4. Define \( L_1, L_2 \) and verify (17). Take \( k_0 \in \text{int} \ K \) such that \( y_0^\mu(k_0) = -1 \) (it is possible by (A4)). We define
\[
L_2(u) = u_0^\mu(u)k_0, \quad \forall u \in X \quad \text{and} \quad L_1 = \bar{L} - L_2.
\]
It is easy to see that \( L_1, L_2 \in \mathcal{L}(X, Y), L_1 + L_2 = \bar{L}, \) and \( y_0^\mu \circ L_2 = -u_0^\mu \). Thus, (A3) can be rewritten as \( y_0^\mu(\hat{y}) \leq y_0^\mu(y - L_2(u)) \) for all \((y, u) \in \Delta\), or equivalently, \( y_0^\mu(y - L_2(u) - \bar{y}) \geq 0 \) for all \((y, u) \in \Delta\). So, by (A4), \( y - L_2(u) - \bar{y} \notin \text{int} \ K \), yielding
\[
\hat{y} \notin y - L_2(u) - \text{int} \ K, \quad \forall(y, u) \in \Delta. \tag{A5}
\]
Now, as \((m + \bar{L}(x) - F_1(x) - F_2(x'), x - x') \in \Delta\) for all \( m \in M, x \in \text{dom} F_1, \) and \( x' \in \text{dom} F_2 \), it follows from (A5) that
\[
\hat{y} \notin M + \bar{L}(x) - F_1(x) - F_2(x') - L_2(x - x') - \text{int} \ K, \quad \forall(m, x, x') \in M \times \text{dom} F_1 \times \text{dom} F_2,
\]
which is (17) and the lemma has been proved. \( \square \)
Appendix 2. Proof of the Lemma 3.3

For the proof of (i), take \((L_i, U_i) \in \mathcal{E}pi F_i^\ast\), \(i = 1, 2\), and show that \((L_1 + L_2, U_1 \uplus U_2) \in \mathcal{E}pi(F_1 + F_2)^\ast\), or equivalently,

\[
(F_1 + F_2)^\ast(L_1 + L_2) \preceq_K U_1 \uplus U_2. \tag{A6}
\]

On the one hand, for each \(i = 1, 2\), as \((L_i, U_i) \in \mathcal{E}pi F_i^\ast\), it holds \(F_i^\ast(L_i) \preceq_K U_i\), and hence, \(F_i^\ast(L_1) \uplus F_i^\ast(L_2) \preceq_K U_1 \uplus U_2\) (see Proposition 3.3(v)). On the other hand, we have \((F_1 + F_2)^\ast(L_1 + L_2) = \text{WSup}[(L_1 + L_2 - F_1 - F_2)(X)]\) and

\[
F_i^\ast(L_1) \uplus F_i^\ast(L_2) = \text{WSup}[F_i^\ast(L_1) + F_i^\ast(L_2)]
\]

(\(i\))-th equality follows from Proposition 2.1(vi)). It is clear that \((L_1 + L_2 - F_1 - F_2)(X) \subset (L_1 - F_1)(X) + (L_2 - F_2)(X)\), and consequently, by Proposition 3.1(iii), \((F_1 + F_2)^\ast(L_1 + L_2) \preceq_K F_1^\ast(L_1) \uplus F_2^\ast(L_2)\) and (A6) follows.

The assertion (ii) follows from (i) and Proposition 3.4. Concretely, one has (see (13)) \(\Psi(\mathcal{E}pi F_1^\ast \uplus \mathcal{E}pi F_2^\ast) \subset \Psi(\mathcal{E}pi(F_1 + F_2)^\ast) = \mathcal{E}pi(F_1 + F_2)^\ast.\) Lastly, (iii) follows from (i), taking (13) and (14) into account. □

Appendix 3. Proof of the Basic Lemma 2 (Lemma 3.4)

- **Proof of (i).** By Lemma 3.3(ii), it suffices to show that \(\mathcal{E}pi(F_1 + F_2)^\ast \subset \Psi(\mathcal{E}pi F_1^\ast \uplus \mathcal{E}pi F_2^\ast)\). Take \((\bar{L}, \bar{y}) \in \mathcal{E}pi(F + I_d)^\ast\). Then, by (4),

\[
\bar{y} \notin (\bar{L} - F_1 - F_2)(X) - \text{int} K.
\]

Apply now the Basic Lemma 1 (Lemma 3.2) to the case where \(M = \{0_Y\}\), there exist \(L_1, L_2 \in \mathcal{L}(X, Y)\) such that \(L_1 + L_2 = L\) and

\[
\bar{y} \notin (L_1 - F_1)(X) + (L_2 - F_2)(X) - \text{int} K. \tag{A7}
\]

This, together with Proposition 2.1(i), yields

\[
\text{WSup}\left[(L_1 - F_1)(\text{dom} F_1) + (L_2 - F_2)(\text{dom} F_2)\right] \neq \{+\infty_Y\}. \tag{A8}
\]

And then, according to (iii), and (vi) of Proposition 2.1,

\[
(L_1 - F_1)(X) + (L_2 - F_2)(X) - \text{int} K
\]

\[
= \text{WSup}\left[(L_1 - F_1)(\text{dom} F_1) + (L_2 - F_2)(\text{dom} F_2)\right] - \text{int} K
\]

\[
= \text{WSup}\left[\text{WSup}(L_1 - F_1)(\text{dom} F) + \text{WSup}(L_2 - F_2)(\text{dom} F_2)\right] - \text{int} K
\]

\[
= \text{WSup}\left[F_1^\ast(L_1) + F_2^\ast(L_2)\right] - \text{int} K = F_1^\ast(L_1) \uplus F_2^\ast(L_2) - \text{int} K.
\]

Combine this to (A7), we obtain \(\bar{y} \notin F_1^\ast(L_1) \uplus F_2^\ast(L_2) - \text{int} K\).

As a WS-sum belongs to \(\mathcal{P}_p(Y)^\infty\), (A8) leads to the fact that \(F_1^\ast(L_1) \uplus F_2^\ast(L_2) \in \mathcal{P}_p(Y)\). Consequently, according to Lemma 3.1(i), one gets \(\bar{y} \in F_1^\ast(L_1) \uplus F_2^\ast(L_2) + K\). Hence, by Proposition 3.3(vi), there exists \(\tilde{V} \in \mathcal{P}_p(Y)\) such that \(\bar{y} \in F_1^\ast(L_1) \uplus \tilde{V} \) and \(F_2^\ast(L_2) \preceq_K \tilde{V}\). So, by taking \(\bar{U} = F_1^\ast(L_1) \uplus \tilde{V} \in \mathcal{P}_p(Y)\), one gets \(\bar{y} \in \bar{U}\) and

\[
(\bar{L}, \bar{U}) = (L_1 + L_2, F_1^\ast(L_1) \uplus \tilde{V}) = (L_1, F_1^\ast(L_1)) \uplus (L_2, \tilde{V}) \in \mathcal{E}pi F_1^\ast \uplus \mathcal{E}pi F_2^\ast,
\]

and consequently, \((\bar{L}, \bar{y}) \in \Psi(\mathcal{E}pi F_1^\ast \uplus \mathcal{E}pi F_2^\ast)\) and (i) has been proved.
• Proof of (ii). Due to Lemma 3.3(iii), it suffices to show that

\[ \Psi \left( \text{Epi}(F_1 + F_2)^* \boxplus \text{Epi} F_3^* \right) \subset \Psi \left( \text{Epi} F_1^* \boxplus \text{Epi} F_2^* \boxplus \text{Epi} F_3^* \right). \] (A9)

(a) Take \((\tilde{L}, \tilde{y}) \in \Psi(\text{Epi}(F_1 + F_2)^* \boxplus \text{Epi} F_3^*). Then, there are \((\tilde{L}, \tilde{U}) \in \text{Epi}(F_1 + F_2)^*\) and \((L_3, U_3) \in \text{Epi} F_3^*\) such that

\[ \tilde{L} + L_3 = \tilde{L} \quad \text{and} \quad \tilde{y} \in \tilde{U} \cup U_3. \] (A10)

As \((\tilde{L}, \tilde{U}) \in \text{Epi}(F_1 + F_2)^*\) and \((L_3, U_3) \in \text{Epi} F_3^*, one gets \((F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) \leq_K \tilde{U} \cup U_3, which, together with Proposition 3.3(v), yields

\[ (F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) \leq_K \tilde{U} \cup U_3. \] (A11)

Now as \(\tilde{y} \in \tilde{U} \cup U_3 \) (see (A10)), \(\tilde{U} \cup U_3 \neq [+\infty_Y]\), we get from (A11) that \((F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) \neq [+\infty_Y], and hence, \(\tilde{U} \cup U_3 \subset Y\) and \((F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) \subset Y.\)

It now follows from (A11) and Proposition 3.1(i) that

\[ [(F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) - \text{int } K] \cap [\tilde{U} \cup U_3] = \emptyset. \] (A12)

(\gamma) It follows from (A10), (A12) that

\[ \tilde{y} \notin (F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) - \text{int } K. \] (A13)

On the other hand, from Proposition 2.1 (vi) and (iii) that (see also (A10))

\[ (F_1 + F_2)^* (\tilde{L}) \cup F_3^* (L_3) - \text{int } K = \text{WSup}[(F_1 + F_2)^* (\tilde{L}) + F_3^* (L_3)] - \text{int } K \]

= \text{WSup} \left[ \text{WSup}(\tilde{L} - F_1 - F_2)(\text{dom } F_1 \cap \text{dom } F_2) + \text{WSup}(L_3 - F_3)(\text{dom } F_3) \right] - \text{int } K

= \text{WSup}(\tilde{L} - F_1 - F_2)(\text{dom } F_1 \cap \text{dom } F_2) + \text{WSup}(L_3 - F_3)(\text{dom } F_3) - \text{int } K

= (\tilde{L}_1 - F_1)(\text{dom } F_1 \cap \text{dom } F_2) + (L_3 - F_3)(\text{dom } F_3) - \text{int } K.

This and (A13) yields

\[ \tilde{y} \notin (\tilde{L}_1 - F_1)(\text{dom } F_1 \cap \text{dom } F_2) + (L_3 - F_3)(\text{dom } F_3) - \text{int } K, \text{ or,} \]

\[ \tilde{y} \notin (\tilde{L}_1 - F_1)(X) + (L_3 - F_3)(\text{dom } F_3) - \text{int } K. \] (A14)

(\delta) By Basic lemma 1 (apply to \(M = (L_3 - F_3)(\text{dom } F_3)\)), there exist \(L_1, L_2 \in \mathcal{L}(X, Y)\) such that \(L_1 + L_2 = \tilde{L}\) (note that, together with (A10), \(L_1 + L_2 + L_3 = \tilde{L}\) and

\[ \tilde{y} \notin (L_1 - F_1)(X) + (L_2 - F_2)(X) + (L_3 - F_3)(\text{dom } F_3) - \text{int } K. \]

A similar argument as in the proof of Appendix 3 one has

\[ (L_1 - F_1)(X) + (L_2 - F_2)(X) + (L_3 - F_3)(\text{dom } F_3) - \text{int } K \]

= \text{F}_1^*(L_1) \cup \text{F}_2^*(L_2) \cup \text{F}_3^*(L_3) - \text{int } K,

(note that (A14) ensures \(\text{WSup}[(L_1 - F_1)(X) + (L_2 - F_2)(X) + (L_3 - F_3)(\text{dom } F_3)] \neq [+\infty_Y]\) which yields \(\tilde{y} \notin \text{F}_1^*(L_1) \cup \text{F}_2^*(L_2) \cup \text{F}_3^*(L_3) - \text{int } K.\) By the same argument as in the proof of (29), there exists \(\tilde{V} \in \mathcal{P}_p(Y)\) such that \(y \in \tilde{V}\) and \((\tilde{L}, \tilde{V}) \in \text{Epi} F_1^* \boxplus \text{Epi} F_2^* \boxplus \text{Epi} F_3^*, meaning that \((\tilde{L}, \tilde{y}) \in \Psi(\text{Epi} F_1^* \boxplus \text{Epi} F_2^* \boxplus \text{Epi} F_3^*)\) and (A9) holds.

• Proof of (iii). Firstly, note that from (ii) one gets

\[ \Psi (\text{Epi} F_1^* \boxplus \text{Epi} F_2^* \boxplus \text{Epi} F_3^*) = \Psi (\text{Epi}(F_1 + F_2)^* \boxplus \text{Epi} F_3^*). \] (A15)
Assume now that \((C'_0)\) holds. Then apply (i) to the two maps \(F_1 + F_2\) and \(F_3\) (play the roles of \(F_1\) and \(F_2\), respectively), one gets 
\[
\Psi(\mathcal{Epi}(F_1^* + F_2)^* \boxtimes \mathcal{Epi} F_3^*) = \mathcal{Epi}(F_1 + F_2 + F_3)^*,
\]
which together with \((A15)\), proves (18).

In the case when \((C'_0)\) holds, one applies (i) to the mappings \(F_3\) and \(F_1 + F_2\). The equalities in (18) then follow by the similar argument as above. \(\square\)