CAPACITY THEORY FOR
MONOTONE OPERATORS

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Abstract

If $Au = -\text{div}(a(x, Du))$ is a monotone operator defined on the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 < p < +\infty$, with $a(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$, the capacity $C_A(E, F)$ relative to $A$ can be defined for every pair $(E, F)$ of bounded sets in $\mathbb{R}^n$ with $E \subset F$. We prove that $C_A(E, F)$ is increasing and countably subadditive with respect to $E$ and decreasing with respect to $F$. Moreover we investigate the continuity properties of $C_A(E, F)$ with respect to $E$ and $F$.

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Introduction

Let $A: W^{1,p}(\mathbb{R}^n) \to W^{-1,q}(\mathbb{R}^n)$, $1 < p < +\infty$, $1/p + 1/q = 1$, be a monotone operator of the form

\begin{equation}
Au = -\text{div}(a(x, Du)),
\end{equation}

where $a: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function which satisfies the usual monotonicity, coerciveness, and growth conditions (see (1.5), (1.6), (1.7) below), and $a(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$.

If $E$ and $F$ are bounded sets in $\mathbb{R}^n$, with $E$ closed, $F$ open, and $E \subset F$, the capacity of $E$ in $F$ relative to the operator $A$ is defined as

\begin{equation}
C_A(E, F) = \int_{F \setminus E} (a(x, Du), Du) \, dx,
\end{equation}

where $u$, the $C_A$-potential of $E$ in $F$, is the weak solution of the Dirichlet problem

\begin{equation}
Au = 0 \text{ in } F \setminus E, \quad u = 1 \text{ in } \partial E, \quad u = 0 \text{ in } \partial F.
\end{equation}

This definition is extended to arbitrary bounded sets by giving a suitable meaning to problem (0.3) when $F \setminus E$ is not open (Definition 3.8).

The purpose of this paper is to prove the main properties of the set function $C_A$. In particular we prove that $C_A(E, F)$ is increasing with respect to $E$ (Theorem 4.3) and decreasing with respect to $F$ (Theorem 4.5). Moreover, we show that $C_A(\cdot, F)$ is continuous along all increasing sequences of sets (Theorem 4.7) and along all decreasing sequences of closed sets contained in the interior of $F$ (Theorem 4.8), while $C_A(E, \cdot)$ is continuous along all decreasing sequences of sets (Theorem 4.10) and along all increasing sequences of open sets containing the closure of $E$ (Theorem 4.11). These results allow us to show that

\begin{equation*}
C_A(E, F) = \sup\{C_A(K, U) : K \text{ compact, } K \subset E, \ U \text{ bounded and open, } U \supseteq F\}
\end{equation*}

when $E$ and $F$ are bounded Borel sets (Theorem 5.5), and to prove that $C_A(\cdot, F)$ is countably subadditive (Theorem 5.9).

Finally, we introduce the capacity $C_A(E, F, s)$ with respect to a constant $s \in \mathbb{R}$ by replacing the condition $u = 1$ in $\partial E$ which appears in (0.3) with the condition $u = s$
in \( \partial E \) (Definition 6.3). We prove that the function \( \frac{1}{s} C_A(E, F, s) \) is continuous and increasing with respect to \( s \) (Theorems 6.10 and 6.11).

When \( Au = - \text{div}(|Du|^{p-2} Du) \), the capacity \( C_A \) coincides with the usual capacity \( C_p \) associated with the Sobolev space \( W^{1,p}(\mathbb{R}^n) \) (see Section 1), for which the above mentioned properties are well known and can be obtained easily by using the fact that (0.3) is the Euler equation of a suitable minimum problem, and thus \( C_p(E, F) \) can be defined equivalently as the infimum of \( \int_F |Dv|^p dx \) over the set of all functions \( v \) in \( W^{1,p}_0(F) \) such that \( v \geq 1 \) in a neighbourhood of \( E \). For a monotone operator of the form (0.1) problem (0.3) is, in general, not equivalent to a minimum problem, and this fact forces us to develop a completely new proof.

When the operator \( A \) is linear, the capacity \( C_A \) was introduced in [20], but, to our knowledge, the properties considered above have been established only in [5]. The proof avoids auxiliary minimum problems, but involves the adjoint operator \( A^* \) in an essential way, and, therefore, it can not be adapted to the monotone case.

Our proof is based on an estimate of the \( C_A \)-potentials (Lemmas 4.1 and 4.2), which follows from a standard comparison argument (Theorem 2.11). The main new idea is to deduce the inequalities for the capacity \( C_A \) from the corresponding inequalities for the \( C_A \)-potentials. The tools used in this approach are the notion of \( C_A \)-distribution (Theorem 3.14, Definition 3.15, and Proposition 3.17) and a technical lemma which allows us, under very special conditions, to deduce the inequality \( Au_1 \geq Au_2 \) from the inequality \( u_1 \leq u_2 \) (Lemma 2.5).

For a complete treatment of the problem, we consider also the case when the operator \( A \) is not strictly monotone, and thus (0.3) may have more than one solution. We prove that in this case the capacity \( C_A(E, F) \) defined by (0.2) does not depend on the choice of the \( C_A \)-potential \( u \). The proof is based on a careful investigation, developed in Section 2, of the properties of the set of all solutions of problem (0.3).

Under some natural assumptions on \( a \) the capacities \( C_A \) and \( C_p \) are equivalent (Remark 3.12), i.e., there exist two constants \( \alpha > 0 \) and \( \beta > 0 \) such that

\[
\alpha C_p(E, F) \leq C_A(E, F) \leq \beta C_p(E, F).
\]

Therefore the precise behaviour of the capacity \( C_A \) is not important in all those problems where it is enough to obtain just an estimate of \( C_A(E, F) \), like, e.g., the characterization of regular boundary points for the operator \( A \), which, actually, can be expressed in terms of the capacity \( C_p \).
There are, however, problems where the capacity $C_A$ cannot be replaced by an equivalent capacity. An example is given by the study of the asymptotic behaviour, as $j \to \infty$, of the solutions $u_j$ of the Dirichlet problems

\begin{equation}
Au_j = f \quad \text{in } \Omega_j, \quad u_j = 0 \quad \text{in } \partial \Omega_j,
\end{equation}

where $A$ is a monotone operator of the form (0.1), $f \in W^{-1,q}(\mathbb{R}^n)$, and $(\Omega_j)$ is a sequence of uniformly bounded open sets in $\mathbb{R}^n$. Under some special assumptions on the structure of the sets $\Omega_j$, this problem has been studied by means of the capacities $C_A(E,F,s)$ in [15], [16], [17], [18], and [19], where a rigorous asymptotic development of $u_j$ is expressed in terms of the $C_A$-potentials of suitable sets related to $\Omega_j$.

When $A$ is the differential of a convex functional of the form

$$G(u) = \int_{\mathbb{R}^n} g(x,Du) \, dx,$$

with $g(x,\cdot)$ even and homogeneous of degree $p$, all assumptions on the structure of the sets $\Omega_j$ can be avoided. In this case, given a bounded open set $\Omega$ containing all sets $\Omega_j$, the asymptotic behaviour of $(u_j)$ is determined by the limit, as $j \to \infty$, of the capacities $C_A(E \setminus \Omega_j, \Omega)$ on a sufficiently large class of subsets $E$ of $\Omega$ (see [4]). Since these results depend strongly on the operator $A$, it is clear that in this problem $C_A$ cannot be replaced by an equivalent capacity.

The properties of $C_A$ will be used in a forthcoming paper to extend the results of [4] to the case of an arbitrary monotone operator $A$ of the form (0.1). To this aim we intend to adapt the techniques of [7] to the non-linear case, by using the results of the present paper and the compactness results of [8], [9], and [1].

1. Notation and preliminaries

* Sobolev spaces and $p$-capacity. Let $p$ and $q$ be two real numbers with $1 < p < +\infty$, $1 < q < +\infty$, and $1/p + 1/q = 1$. For every open set $\Omega \subset \mathbb{R}^n$ the Sobolev space $W^{1,p}(\Omega)$ is defined as the space of all functions $u$ in $L^p(\Omega)$ whose first order distribution derivatives $D_i u$ belong to $L^p(\Omega)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)}^p = \int_{\Omega} |Du|^p \, dx + \int_{\Omega} |u|^p \, dx,$$
where $Du = (D_1 u, \ldots, D_n u)$ is the gradient of $u$. The space $W^{1,p}_0(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega)$, and $W^{-1,q}(\Omega)$ is the dual of $W^{1,p}_0(\Omega)$. We shall always identify each function $u$ of $W^{1,p}_0(\Omega)$ with the function $v$ of $W^{1,p}(\mathbb{R}^n)$ such that $v = u$ in $\Omega$ and $v = 0$ in $\Omega^c = \mathbb{R}^n \setminus \Omega$. With this convention the space $W^{1,p}_0(\Omega)$ can be regarded as a closed subspace of $W^{1,p}(\mathbb{R}^n)$.

The lattice operations $\vee$ and $\wedge$ are defined by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. It is well known that, if $u$ and $v$ belong to $W^{1,p}(\Omega)$, then $u \vee v$ and $u \wedge v$ belong to $W^{1,p}(\Omega)$, and that the same property holds for $W^{1,p}_0(\Omega)$. The positive and the negative part of a function $u$ are denoted by $u^+$ and $u^-$.

If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $E \subset \Omega$ is an arbitrary set, the $p$-capacity of $E$ in $\Omega$, denoted by $C_p(E, \Omega)$, is defined as the infimum of $\int_\Omega |Du|^p dx$ over the set of all functions $u$ in $W^{1,p}_0(\Omega)$ such that $u \geq 1$ in a neighbourhood of $E$, with the usual convention $\inf \emptyset = +\infty$. It follows immediately from the definition that

$$C_p(E, \Omega) = \inf\{C_p(U, \Omega) : U \text{ open}, E \subset U \subset \Omega\}. \tag{1.1}$$

Moreover it is possible to prove that the set function $C_p(\cdot, \Omega)$ is increasing and countably subadditive.

We say that a set $N$ in $\mathbb{R}^n$ is $C_p$-null if $C_p(N \cap \Omega, \Omega) = 0$ for every bounded open set $\Omega \subset \mathbb{R}^n$. It is easy to prove that, if $N$ is contained in a bounded open set $\Omega_0$ and $C_p(N, \Omega_0) = 0$, then $N$ is $C_p$-null, i.e., $C_p(N \cap \Omega, \Omega) = 0$ for every other bounded open set $\Omega$.

We say that a property $\mathcal{P}(x)$ holds $C_p$-quasi everywhere (abbreviated as $C_p$-q.e.) in a set $E \subset \mathbb{R}^n$ if it holds for all $x \in E$ except for a $C_p$-null set $N \subset E$. The expression almost everywhere (abbreviated as a.e.) refers, as usual, to the Lebesgue measure.

A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be quasi continuous if for every bounded open set $\Omega \subset \mathbb{R}^n$ and for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $C_p(E, \Omega) \leq \varepsilon$, such that the restriction of $u$ to $\Omega \setminus E$ is continuous.

It is well known that every $u \in W^{1,p}(\mathbb{R}^n)$ has a quasi continuous representative, which is uniquely defined up to a $C_p$-null set. In the sequel we shall always identify $u$ with its quasi continuous representative, so that the pointwise values of a function $u \in W^{1,p}(\mathbb{R}^n)$ are defined $C_p$-quasi everywhere in $\mathbb{R}^n$. We recall that, if a sequence $(u_j)$ converges to $u$ in $W^{1,p}(\mathbb{R}^n)$, then a subsequence of $(u_j)$ converges to $u$ $C_p$-q.e. in $\mathbb{R}^n$.

For all these properties of quasi continuous representatives of Sobolev functions we refer to [10], Section 4.8, [12], Section 4, [14], Section 7.2.4, and [21], Section 3.
Given any set \( E \subset \mathbb{R}^n \) we define the Sobolev space \( W^{1,p}_0(E) \) as the set of all functions \( u \in W^{1,p} (\mathbb{R}^n) \) such that \( u = 0 \) \( C_p \)-q.e. in \( E^c \), where \( E^c \) denotes the complement of \( E \) with respect to \( \mathbb{R}^n \). It is easy to see that \( W^{1,p}_0(E) \) is a closed subspace of \( W^{1,p}(\mathbb{R}^n) \). The space \( W^{-1,q}(E) \) is defined as the dual of \( W^{1,p}_0(E) \) and the duality pairing is denoted by \( \langle \cdot, \cdot \rangle \). The transpose of the imbedding of \( W^{1,p}_0(E) \) into \( W^{1,p}(\mathbb{R}^n) \) defines a natural projection of \( W^{-1,q}(\mathbb{R}^n) \) onto \( W^{-1,q}(E) \), so that all elements of \( W^{-1,q}(\mathbb{R}^n) \) can be regarded as elements of \( W^{-1,q}(E) \). When \( E \) is open, our definitions coincide with the classical definitions considered at the beginning of this section (see [12], Theorem 4.5).

If \( f \) and \( g \) belong to \( W^{-1,q}(E) \), we say that \( f = g \) in \( W^{-1,q}(E) \) if \( \langle f, v \rangle = \langle g, v \rangle \) for every \( v \in W^{1,p}_0(E) \). We say that \( f \leq g \) in \( W^{-1,q}(E) \) if \( \langle f, v \rangle \leq \langle g, v \rangle \) for every \( v \in W^{1,p}_0(E) \) with \( v \geq 0 \) \( C_p \)-q.e. in \( E \). It is easy to see that \( f = g \) in \( W^{-1,q}(E) \) if and only if \( f \leq g \) in \( W^{-1,q}(E) \) and \( g \leq f \) in \( W^{-1,q}(E) \).

The previous definitions allow us to consider the capacity \( C_p(E,F) \) when \( E \) and \( F \) are arbitrary bounded sets in \( \mathbb{R}^n \). In this case we define

\[
(1.2) \quad C_p(E,F) = \min \left\{ \int_F |D u|^p dx : u \in W^{1,p}_0(F), \ u \geq 1 \ \text{C}_p\text{-q.e. in } E \right\},
\]

with the usual convention \( \min \emptyset = +\infty \). When \( F \) is open and \( E \subset F \), this definition agrees with the definition considered at the beginning of the paper (see [11], Section 10, or [12], Corollary 4.13). If \( C_p(E,F) < +\infty \), then the minimum problem (1.2) has a unique minimum point, which is called the \( C_p \)-potential of \( E \) in \( F \).

**Quasi open and quasi closed sets.** We say that a set \( U \) in \( \mathbb{R}^n \) is \( C_p \)-quasi open (resp. \( C_p \)-quasi closed) if for every bounded open set \( \Omega \subset \mathbb{R}^n \) and for every \( \varepsilon > 0 \) there exists an open (resp. closed) set \( V \subset \mathbb{R}^n \) such that \( C_p((U \triangle V) \cap \Omega, \Omega) < \varepsilon \), where \( \triangle \) denotes the symmetric difference of sets.

If a function \( u: \mathbb{R}^n \to \mathbb{R} \) is \( C_p \)-quasi continuous, then for every \( t \in \mathbb{R} \) the sets \( \{ u < t \} = \{ x \in \mathbb{R}^n : u(x) < t \} \) and \( \{ u > t \} = \{ x \in \mathbb{R}^n : u(x) > t \} \) are \( C_p \)-quasi open, while the sets \( \{ u \leq t \} = \{ x \in \mathbb{R}^n : u(x) \leq t \} \) and \( \{ u \geq t \} = \{ x \in \mathbb{R}^n : u(x) \geq t \} \) are \( C_p \)-quasi closed. In particular this property holds for every \( u \in W^{1,p}(\mathbb{R}^n) \).

We shall frequently use the following lemma on the approximation of the characteristic function of a \( C_p \)-quasi open set. We recall that the characteristic function \( 1_E \) of a set \( E \subset \mathbb{R}^n \) is defined by \( 1_E(x) = 1 \), if \( x \in E \), and by \( 1_E(x) = 0 \), if \( x \in E^c \).

**Lemma 1.1.** For every \( C_p \)-quasi open set \( U \) in \( \mathbb{R}^n \) there exists an increasing sequence \( (v_j) \) of non-negative functions of \( W^{1,p}(\mathbb{R}^n) \) which converges to \( 1_U \) \( C_p \)-quasi everywhere in \( \mathbb{R}^n \).
Proof. See [3], Lemma 1.5, or [6], Lemma 2.1.

The following lemma is used in the proof of Theorems 4.8 and 4.11.

**Lemma 1.2.** Let $U$ be the union of an increasing sequence $(U_j)$ of $C_p$-quasi open sets in $\mathbb{R}^n$ and let $F$ be an arbitrary set in $\mathbb{R}^n$. Then for every function $u$ of $W^{1,p}_0(F \cap U)$ there exists a sequence $(u_j)$ which converges to $u$ strongly in $W^{1,p}(\mathbb{R}^n)$ and such that $u_j \in W^{1,p}_0(F \cap U_j)$ for every $j$.

**Proof.** Let $u$ be a function of $W^{1,p}_0(F \cap U)$. It is not restrictive to assume that $u \geq 0$ $C_p$-q.e. in $\mathbb{R}^n$. By Lemma 1.6 of [3] there exists an increasing sequence $(u_j)$ which converges to $u$ strongly in $W^{1,p}(\mathbb{R}^n)$ and such that $u_j \leq u 1_{U_j}$ $C_p$-q.e. in $\mathbb{R}^n$ for every $j$. Then the sequence $(u_j^+)$ converges to $u$ strongly in $W^{1,p}(\mathbb{R}^n)$ and satisfies $0 \leq u_j^+ \leq u 1_{U_j}$ $C_p$-q.e. in $\mathbb{R}^n$ for every $j$. Since $u = 0$ $C_p$-q.e. in $F^c$, we conclude that $u_j^+ = 0$ $C_p$-q.e. in $F^c \cup U_j^c$, hence $u_j^+ \in W^{1,p}_0(F \cap U_j)$ for every $j$. 

The following lemmas show that all $C_p$-quasi open sets and all $C_p$-quasi closed sets can be approximated by an increasing sequence of compact sets.

**Lemma 1.3.** Let $U$ be a $C_p$-quasi open set in $\mathbb{R}^n$. Then there exists an increasing sequence $(K_j)$ of compact subsets of $U$ whose union covers $C_p$-quasi all of $U$.

**Proof.** Since every $C_p$-quasi open set is the union of an increasing sequence of $C_p$-quasi open bounded sets, we may assume that $U$ is bounded. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ containing $U$. Since $U$ is $C_p$-quasi open, for every $k \in \mathbb{N}$ there exists an open set $V_k$, contained in $\Omega$, such that $C_p(U \triangle V_k, \Omega) < 1/k$. By (1.1) there exists an open set $W_k$ such that $U \triangle V_k \subset W_k \subset \Omega$ and $C_p(W_k, \Omega) < 1/k$. This implies, in particular, that $V_k \setminus W_k = U \setminus W_k$. As $V_k$ is open, it is the union of an increasing sequence $(C^j_k)_j$ of compact sets. Let us define

$$K_j = (C^0_k \setminus W_1) \cup \cdots \cup (C^j_k \setminus W_j).$$

Then $K_j$ is compact and contained in $U$. As $C^j_k \subset C^{j+1}_k$, the sequence $(K_j)$ is increasing. Moreover the union $E$ of $(K_j)$ contains $V_k \setminus W_k = U \setminus W_k$ for every $k$. Therefore $C_p(U \setminus E, \Omega) \leq C_p(W_k, \Omega) < 1/k$ for every $k$, and hence $C_p(U \setminus E, \Omega) = 0$. 

**Lemma 1.4.** Let $F$ be a $C_p$-quasi closed set in $\mathbb{R}^n$. Then there exists an increasing sequence $(K_j)$ of compact subsets of $F$ whose union covers $C_p$-quasi all of $F$.

*Proof.* Since every $C_p$-quasi closed set is the union of an increasing sequence of $C_p$-quasi closed bounded sets, we may assume that $F$ is bounded. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ containing $\overline{F}$. Since $F$ is $C_p$-quasi closed, for every $j \in \mathbb{N}$ there exists a compact set $F_j$, contained in $\Omega$, such that $C_p(F \triangle F_j, \Omega) < 2^{-j}$. By (1.1) there exists an open set $U_j$ such that $F \triangle F_j \subset U_j \subset \Omega$ and $C_p(U_j, \Omega) < 2^{-j}$. Let $V_j = U_j \cup U_{j+1} \cup \cdots$, so that $V_{j+1} \subset V_j$ and $C_p(V_j, \Omega) < 2^{1-j}$. Let $K_j$ be the compact set defined by $K_j = F \setminus V_j$. As $F \triangle F_j \subset U_j \subset V_j$, we have $K_j = F \setminus V_j$. This implies that $K_j \subset F$ and that the sequence $(K_j)$ is increasing. Moreover the union $E$ of $(K_j)$ contains $F \setminus V_j$ for every $j$. Therefore $C_p(F \setminus E, \Omega) \leq C_p(V_j, \Omega) < 2^{1-j}$ for every $j$, and hence $C_p(F \setminus E, \Omega) = 0$.

**Lemma 1.5.** Let $E = U \cap F$ be the intersection of a $C_p$-quasi open set $U$ and a $C_p$-quasi closed set $F$. Then there exists an increasing sequence $(K_j)$ of compact subsets of $E$ whose union covers $C_p$-quasi all of $E$.

*Proof.* The assertion follows from Lemmas 1.3 and 1.4.

**Measures.** By a Radon measure on $\mathbb{R}^n$ we mean a continuous linear functional on the space $C_0(\mathbb{R}^n)$ of all continuous functions with compact support in $\mathbb{R}^n$. It is well known that for every Radon measure $\lambda$ there exists a countably additive set function $\mu$, defined on the family of all bounded Borel subsets of $\mathbb{R}^n$, such that $\lambda(u) = \int_{\Omega} u \, d\mu$ for every $u \in C_0(\Omega)$. We shall always identify the functional $\lambda$ with the set function $\mu$.

We say that a Radon measure $\mu$ on $\mathbb{R}^n$ belongs to $W^{-1,q}(\mathbb{R}^n)$ if there exists $f \in W^{-1,q}(\mathbb{R}^n)$ such that

$$
(1.3) \quad \langle f, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, d\mu \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,q}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$. We shall always identify $f$ and $\mu$. Note that, by the Riesz Representation Theorem, for every non-negative functional $f \in W^{-1,q}(\mathbb{R}^n)$ there exists a non-negative Radon measure $\mu$ on $\mathbb{R}^n$ such that (1.3) holds.

We say that a Radon measure $\mu$ on $\mathbb{R}^n$ is $C_p$-absolutely continuous if $\mu(N) = 0$ for every $C_p$-null Borel set $N \subset \mathbb{R}^n$. It is well known that every non-negative Radon
measure \( \mu \) which belongs to \( W_{-1,q}(\mathbb{R}^n) \) is \( C_p \)-absolutely continuous and that, in this case, \( W^{1,p}(\mathbb{R}^n) \subset L^1_\mu(\mathbb{R}^n) \) and (1.3) holds for every \( \varphi \in W^{1,p}(\mathbb{R}^n) \).

The monotone operator. Let \( a: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) be a function which satisfies the usual Carathéodory conditions, i.e., for every \( \xi \in \mathbb{R}^n \) the function \( x \mapsto a(x, \xi) \) is (Lebesgue) measurable on \( \mathbb{R}^n \), and for a.e. \( x \in \mathbb{R}^n \) the function \( \xi \mapsto a(x, \xi) \) is continuous on \( \mathbb{R}^n \). We assume that there exist two constants \( c_1 > 0 \) and \( c_2 > 0 \), and two functions \( b_1 \in L^1(\mathbb{R}^n) \) and \( b_2 \in L^q(\mathbb{R}^n) \), such that

\[
\begin{align*}
(1.4) & \quad a(x, 0) = 0, \\
(1.5) & \quad (a(x, \xi) - a(x, \eta), \xi - \eta) \geq 0, \\
(1.6) & \quad (a(x, \xi), \xi) \geq c_1 |\xi|^p - b_1(x), \\
(1.7) & \quad |a(x, \xi)| \leq c_2 |\xi|^{p-1} + b_2(x)
\end{align*}
\]

for a.e. \( x \in \mathbb{R}^n \) and for every \( \xi, \eta \in \mathbb{R}^n \). Note that (1.4) and (1.5) imply

\[
(1.8) \quad (a(x, \xi), \xi) \geq 0
\]

for a.e. \( x \in \mathbb{R}^n \) and for every \( \xi \in \mathbb{R}^n \). Let \( A: W^{1,p}(\mathbb{R}^n) \to W_{-1,q}(\mathbb{R}^n) \) be the operator defined by \( Au = -\text{div}(a(x, Du)) \), i.e.,

\[
(1.9) \quad \langle Au, v \rangle = \int_{\mathbb{R}^n} (a(x, Du), Dv) \, dx
\]

for every \( u, v \in W^{1,p}(\mathbb{R}^n) \).

**Remark 1.6.** Since \( a \) satisfies the Carathéodory conditions, by (1.7) the operator \( A \) is continuous. Moreover, for every \( E \subset \mathbb{R}^n \) and for every \( u \in W^{1,p}(\mathbb{R}^n) \) we have

\[
(1.10) \quad \langle Au, v \rangle \leq (c_2 \|Du\|_{L^p(E, \mathbb{R}^n)}^{p-1} + \|b_2\|_{L^q(E)}) \|Dv\|_{L^p(E, \mathbb{R}^n)} \quad \forall v \in W_0^{1,p}(E).
\]

From (1.5) we get

\[
(1.11) \quad \langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in W^{1,p}(\mathbb{R}^n),
\]

hence \( A \) is monotone. Inequality (1.8) implies that

\[
(1.12) \quad \langle Au, u \rangle \geq 0 \quad \forall u \in W^{1,p}(\mathbb{R}^n).
\]

By (1.6) for every \( E \subset \mathbb{R}^n \) we have

\[
(1.13) \quad \langle Au, u \rangle \geq c_1 \int_E |Du|^p \, dx - \int_E b_1 \, dx \quad \forall u \in W_0^{1,p}(E).
\]

By Poincaré’s Inequality this implies that \( A \) is coercive on all subspaces of the form \( \{ u \in W^{1,p}(\mathbb{R}^n) : u - \psi \in W_0^{1,p}(E) \} \), with \( E \) bounded and \( \psi \in W^{1,p}(\mathbb{R}^n) \).
2. Some properties of the solutions

In this section we prove some properties of the solutions \( u \) of the Dirichlet problem

\[
\begin{cases}
  u - \psi \in W^{1,p}_0(E), \\
  Au = f \quad \text{in } W^{-1,q}(E),
\end{cases}
\]

where \( E \) is an arbitrary bounded set in \( \mathbb{R}^n \), \( \psi \) is a function in \( W^{1,p}(\mathbb{R}^n) \), and \( f \) belongs to \( W^{-1,q}(E) \).

**Theorem 2.1.** Let \( E \) be a bounded set in \( \mathbb{R}^n \), let \( \psi \in W^{1,p}(\mathbb{R}^n) \), and let \( f \in W^{-1,q}(E) \). Then the Dirichlet problem (2.1) has at least a solution, and the set of all solutions of (2.1) is bounded, closed, and convex in \( W^{1,p}(\mathbb{R}^n) \).

**Proof.** By Remark 1.6 the operator \( A: W^{1,p}(\mathbb{R}^n) \rightarrow W^{-1,q}(\mathbb{R}^n) \) is continuous and monotone. Since \( E \) is bounded, \( A \) is coercive on the set \( \{ u \in W^{1,p}(\mathbb{R}^n) : u - \psi \in W^{1,p}_0(E) \} \). Therefore the properties of the set of the solutions of (2.1) follow from the classical theory of monotone operators (see, e.g., [13], Chapter III).

**Lemma 2.2.** Let \( E \) be a bounded set in \( \mathbb{R}^n \), let \( \psi \in W^{1,p}(\mathbb{R}^n) \), let \( f \in W^{-1,q}(E) \), and let \( u_1 \) and \( u_2 \) be two solutions of (2.1). Then \( u_1 \lor u_2 \) and \( u_1 \land u_2 \) are solutions of (2.1).

**Proof.** Since \( u_1 - u_2 \in W^{1,p}_0(E) \), by (2.1) we have

\[
\int_E (a(x,Du_1) - a(x,Du_2), Du_1 - Du_2) \, dx = 0.
\]

By (1.5) we have \( (a(x,Du_1) - a(x,Du_2), Du_1 - Du_2) \geq 0 \) a.e. in \( E \), hence

\[
(2.2) \quad (a(x,Du_1) - a(x,Du_2), Du_1 - Du_2) = 0 \quad \text{a.e. in } E.
\]

Let us fix \( v \in W^{1,p}_0(E) \) with \( v \geq 0 \) \( C_p \)-q.e. in \( E \). For every \( \varepsilon > 0 \) let us define \( v_\varepsilon = (\varepsilon v) \land (u_1 - u_2) \). As \( v_\varepsilon \in W^{1,p}_0(E) \), by (2.1) we have

\[
\int_E (a(x,Du_1), Dv_\varepsilon) \, dx = \langle f, v_\varepsilon \rangle,
\]

\[
\int_E (a(x,Du_2), \varepsilon Dv - Dv_\varepsilon) \, dx = \langle f, \varepsilon v - v_\varepsilon \rangle.
\]
By adding these equalities we obtain
\[
\int_E (a(x, Du_1), Dv_\varepsilon) \, dx + \int_E (a(x, Du_2), \varepsilon Dv - Dv_\varepsilon) \, dx = \varepsilon \langle f, v \rangle.
\]
This implies that
\[
\varepsilon \int_{\{\varepsilon v < u_1 - u_2\}} (a(x, Du_1), Dv) \, dx + \int_{\{u_1 - u_2 \leq \varepsilon v\}} (a(x, Du_1), Du_1 - Du_2) \, dx +
+ \varepsilon \int_{\{u_1 - u_2 \leq \varepsilon v\}} (a(x, Du_2), Dv) \, dx - \int_{\{u_1 - u_2 \leq \varepsilon v\}} (a(x, Du_2), Du_1 - Du_2) \, dx = \varepsilon \langle f, v \rangle.
\]
By (2.2) we have
\[
\int_{\{\varepsilon v < u_1 - u_2\}} (a(x, Du_1), Dv) \, dx + \int_{\{u_1 - u_2 \leq \varepsilon v\}} (a(x, Du_2), Dv) \, dx = \langle f, v \rangle.
\]
Passing to the limit as \(\varepsilon \to 0\) we get
\[
\int_{\{u_1 > u_2\}} (a(x, Du_1), Dv) \, dx + \int_{\{u_1 \leq u_2\}} (a(x, Du_2), Dv) \, dx = \langle f, v \rangle,
\]
which implies
\[
\int_E (a(x, D(u_1 \vee u_2)), Dv) \, dx = \langle f, v \rangle
\]
for every \(v \in W_0^{1,p}(E)\) with \(v \geq 0\) \(C_p\)-q.e. in \(E\). This proves that \(A(u_1 \vee u_2) = f\) in \(W^{-1,q}(E)\). The equality \(A(u_1 \wedge u_2) = f\) in \(W^{-1,q}(E)\) can be proved in a similar way. \(\Box\)

We are now in a position to prove the existence of a maximal and a minimal solution of (2.1).

**Theorem 2.3.** Let \(E\) be a bounded set in \(\mathbb{R}^n\), let \(\psi \in W^{1,p}(\mathbb{R}^n)\), and let \(f \in W^{-1,q}(E)\). Then there exist two solutions \(u_1\) and \(u_2\) of problem (2.1) such that \(u_1 \leq u \leq u_2\) \(C_p\)-q.e. in \(E\) for every other solution \(u\) of (2.1).

**Proof.** Let \(K\) be the set of all solutions of (2.1). By Theorem 2.1 \(K\) is non-empty, bounded, closed, and convex in \(W^{1,p}(\mathbb{R}^n)\). Since this space is separable, there exists a sequence \((v_j)\) in \(K\) which is dense in \(K\). For every \(k \in \mathbb{N}\) let us define
\[
u^k_1 = \inf_{1 \leq j \leq k} v_j, \quad u^k_2 = \sup_{1 \leq j \leq k} v_j, \quad u_1 = \inf_{1 \leq j} v_j, \quad u_2 = \sup_{1 \leq j} v_j.
\]
By Lemma 2.2 both $u_1^k$ and $u_2^k$ belong to $K$, therefore the sequences $(u_1^k)$ and $(u_2^k)$ are bounded in $W^{1,p}(\mathbb{R}^n)$ and converge to $u_1$ and $u_2$ weakly in $W^{1,p}(\mathbb{R}^n)$. As $K$ is weakly closed in $W^{1,p}(\mathbb{R}^n)$, we conclude that $u_1$ and $u_2$ belong to $K$, i.e., they are solutions of (2.1). If $u$ is another solution of (2.1), by the density of $(v_j)$ there exists a subsequence $(v_{j_k})$ which converges to $u$ strongly in $W^{1,p}(\mathbb{R}^n)$ and $C_p$-q.e. in $\mathbb{R}^n$. Since $u_{1k}^j \leq v_{jk} \leq u_{2k}^j$ $C_p$-q.e. in $\mathbb{R}^n$ for every $k$, we conclude that $u_1 \leq u \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$.

**Definition 2.4.** The functions $u_1$ and $u_2$ introduced in the previous theorem are called the *minimal* and the *maximal solution* of problem (2.1).

The following lemma will be fundamental in the proof of the monotonicity properties of the capacity $C_A$ associated with the operator $A$.

**Lemma 2.5.** Let $B$ and $C$ be two sets in $\mathbb{R}^n$, and let $w_1$ and $w_2$ be two functions in $W^{1,p}(\mathbb{R}^n)$ such that $w_1 = w_2$ $C_p$-q.e. in $B$ and $w_1 \leq w_2$ $C_p$-q.e. in $C$. Assume that $Aw_1 = Aw_2$ in $W^{-1,q}(C \setminus B)$. Then $Aw_1 \geq Aw_2$ in $W^{-1,q}(C)$.

**Proof.** Let us fix a function $v$ in $W^{1,p}_0(C)$ with $v \geq 0$ $C_p$-q.e. in $C$. For every $\varepsilon > 0$ let us define $v_\varepsilon = (\varepsilon v) \wedge (w_2 - w_1)^+$. Since $v_\varepsilon$ belongs to $W^{1,p}_0(C \setminus B)$ and $Aw_1 = Aw_2$ in $W^{-1,q}(C \setminus B)$, we have $\langle Aw_1, v_\varepsilon \rangle = \langle Aw_2, v_\varepsilon \rangle$. Therefore by the monotonicity condition (1.5)

\[
\varepsilon \langle Aw_1, v \rangle - \varepsilon \langle Aw_2, v \rangle = \langle Aw_1 - Aw_2, \varepsilon v - v_\varepsilon \rangle = \\
= \varepsilon \int_{C \cap \{w_2 - w_1 \leq \varepsilon v\}} (a(x, Dw_1) - a(x, Dw_2), Dv) \, dx - \\
- \int_{C \cap \{w_2 - w_1 \leq \varepsilon v\}} (a(x, Dw_1) - a(x, Dw_2), Dw_2 - Dw_1) \, dx \geq \\
\geq \varepsilon \int_{C \cap \{w_2 - w_1 \leq \varepsilon v\}} (a(x, Dw_1) - a(x, Dw_2), Dv) \, dx.
\]

Dividing by $\varepsilon$ and passing to the limit as $\varepsilon \to 0$ we obtain

\[
\langle Aw_1, v \rangle - \langle Aw_2, v \rangle \geq \int_{C \cap \{w_2 \leq w_1\}} (a(x, Dw_1) - a(x, Dw_2), Dv) \, dx.
\]

(2.3) $\langle Aw_1, v \rangle - \langle Aw_2, v \rangle \geq \int_{C \cap \{w_2 \leq w_1\}} (a(x, Dw_1) - a(x, Dw_2), Dv) \, dx$.

Since $w_1 \leq w_2$ $C_p$-q.e. in $C$, we have $Dw_1 = Dw_2$ a.e. in $C \cap \{w_2 \leq w_1\} = C \cap \{w_2 = w_1\}$. Therefore the right hand side of (2.3) is equal to 0 and, consequently, $\langle Aw_1, v \rangle \geq \langle Aw_2, v \rangle$, which concludes the proof. \qed
If \( u_1 \) and \( u_2 \) are two solutions of problem (2.1), then \( Au_1 = Au_2 \) in \( W^{-1,q}(E) \). As \( u_1 = u_2 = \psi \) \( C_p \)-q.e. in \( E^c \), we have also \( Au_1 = Au_2 \) in \( W^{-1,q}(E^c) \). The following theorem shows that actually \( Au_1 = Au_2 \) in \( W^{-1,q}(\mathbb{R}^n) \).

**Theorem 2.6.** Let \( E \) be a bounded set in \( \mathbb{R}^n \), let \( \psi \in W^{1,p}(\mathbb{R}^n) \), and let \( f \in W^{-1,q}(E) \). Then there exists \( g \in W^{-1,q}(\mathbb{R}^n) \) such that \( Au = g \) in \( W^{-1,q}(\mathbb{R}^n) \) for every solution \( u \) of problem (2.1).

*Proof.* Let \( u_1 \) and \( u_2 \) be the minimal and the maximal solution of (2.1). If we apply Lemma 2.5 with \( B = E^c \) and \( C = \mathbb{R}^n \), we obtain that \( Au_1 \geq Au \geq Au_2 \) in \( W^{-1,q}(\mathbb{R}^n) \) for every solution \( u \) of (2.1). Therefore it is enough to prove that \( Au_1 \leq Au_2 \) in \( W^{-1,q}(\mathbb{R}^n) \). Let \( \varphi \in C^\infty_0(\mathbb{R}^n) \) with \( \varphi \geq 0 \) in \( \mathbb{R}^n \) and let \( c = \|\varphi\|_{L^\infty(\mathbb{R}^n)} \). Let us fix a function \( \chi \) in \( C^\infty_0(\mathbb{R}^n) \) such that \( \chi = c \) in \( E \cup \{ \varphi \neq 0 \} \) and \( \chi \geq 0 \) in \( \mathbb{R}^n \). Then \( \chi - \varphi \) belongs to \( C^\infty_0(\mathbb{R}^n) \) and \( \chi - \varphi \geq 0 \) in \( \mathbb{R}^n \). As \( Au_1 \geq Au_2 \) in \( W^{-1,q}(\mathbb{R}^n) \), we have

\[
\langle Au_1, \chi - \varphi \rangle \geq \langle Au_2, \chi - \varphi \rangle.
\]

Since \( u_1 = u_2 = \psi \) \( C_p \)-q.e. in \( E^c \), we have \( Du_1 = Du_2 \) a.e. in \( \{ \chi \neq c \} \). As \( D\chi = 0 \) a.e. in \( \{ \chi = c \} \), we have

\[
\langle Au_1, \chi \rangle = \int_{\{\chi \neq c\}} (a(x,Du_1), D\chi) \, dx = \int_{\{\chi \neq c\}} (a(x,Du_2), D\chi) \, dx = \langle Au_2, \chi \rangle.
\]

By (2.4) this implies \( \langle Au_1, \varphi \rangle \leq \langle Au_2, \varphi \rangle \) for every \( \varphi \in C^\infty_0(\mathbb{R}^n) \) with \( \varphi \geq 0 \) and, by density, this implies \( Au_1 \leq Au_2 \) in \( W^{-1,q}(\mathbb{R}^n) \). 

**Corollary 2.7.** Let \( E \) be a bounded set in \( \mathbb{R}^n \), let \( \psi \in W^{1,p}(\mathbb{R}^n) \), let \( f \in W^{-1,q}(E) \), and let \( u_1 \) and \( u_2 \) be the minimal and the maximal solution of (2.1). Let \( B \) be a subset of \( E \). Then \( u_1 \) coincides with the minimal solution \( v_1 \) of the Dirichlet problem

\[
\begin{aligned}
\begin{cases}
 v - u_1 \in W^{1,p}_0(B), \\
 Av = f & \text{in } W^{-1,q}(B),
\end{cases}
\end{aligned}
\]

and \( u_2 \) coincides with the maximal solution \( v_2 \) of the Dirichlet problem

\[
\begin{aligned}
\begin{cases}
 v - u_2 \in W^{1,p}_0(B), \\
 Av = f & \text{in } W^{-1,q}(B).
\end{cases}
\end{aligned}
\]
Proof. It is clear that $u_1$ is a solution of (2.5). Let $v_0$ be another solution of (2.5). We have to prove that $u_1 \leq v_0$ $C_p$-q.e. in $B$. By Theorem 2.6, applied to problem (2.5), there exists $g \in W^{-1,q}(\mathbb{R}^n)$ such that $Av = g$ in $W^{-1,q}(\mathbb{R}^n)$ for every solution $v$ of (2.5). In particular we have $Av_0 = Au_1$ in $W^{-1,q}(\mathbb{R}^n)$. Since $u_1$ is a solution of (2.1), we have $Au_1 = f$ in $W^{-1,q}(E)$. This implies that $Av_0 = f$ in $W^{-1,q}(E)$, thus $v_0$ is a solution of (2.1) too. By the minimality of $u_1$ we conclude that $u_1 \leq v_0$ $C_p$-q.e. in $E$, and, therefore, $u_1$ is the minimal solution of (2.5). The proof for $u_2$ is analogous. \[\]

Corollary 2.8. Let $E$ be a bounded set in $\mathbb{R}^n$, let $\psi \in W^{1,p}(\mathbb{R}^n)$, let $f \in W^{-1,q}(\mathbb{R}^n)$, and let $u_1$ and $u_2$ be two solutions of (2.1). Then $\langle Au_1 - f, u_1 \rangle = \langle Au_2 - f, u_2 \rangle$.

Proof. Since $u_1 - u_2$ belongs to $W^{1,p}_0(E)$, by (2.1) we have $\langle Au_1 - f, u_1 - u_2 \rangle = 0$, hence $\langle Au_1 - f, u_1 \rangle = \langle Au_1 - f, u_2 \rangle$. By Theorem 2.6 we have $Au_1 = Au_2$ in $W^{-1,q}(\mathbb{R}^n)$, hence $\langle Au_1 - f, u_2 \rangle = \langle Au_2 - f, u_2 \rangle$.

The following lemma will be used in the proof of the Comparison Principle.

Lemma 2.9. Let $E$ be a bounded open set in $\mathbb{R}^n$, let $\psi \in W^{1,p}(\mathbb{R}^n)$, and let $f \in W^{-1,q}(E)$. Assume that $Aw \leq f$ in $W^{-1,q}(E)$ and that $(w - \psi)^+ \in W^{1,p}_0(E)$, i.e., $w \leq \psi$ $C_p$-q.e. in $E^c$. Then there exists a solution $u$ of (2.1) such that $u \geq w$ $C_p$-q.e. in $E$.

Proof. Let $K$ be the set of all functions $v$ in $W^{1,p}(\mathbb{R}^n)$ such that $v = \psi$ $C_p$-q.e. in $E^c$ and $v \geq w$ $C_p$-q.e. in $E$. As $(w - \psi)^+$ belongs to $W^{1,p}_0(E)$, the function $\psi + (w - \psi)^+$ belongs to $K$, so that $K$ is non-empty. By Remark 1.6, using the classical theory of monotone operators (see, e.g., [13], Chapter III), we can find a solution $u$ of the variational inequality

\[
\begin{cases}
  u \in K,
  \\
  \langle Au, v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in K.
\end{cases}
\]

We want to prove that $u$ is a solution of (2.1). If $z$ belongs to $W^{1,p}_0(E)$ and $z \geq 0$ $C_p$-q.e. in $E$, then $v = u + z$ belongs to $K$, hence

\[
\langle Au, z \rangle \geq \langle f, z \rangle.
\]
In order to prove the opposite inequality, for every \( \varepsilon > 0 \) we consider the function \( z_\varepsilon = (\varepsilon z) \wedge (u - w) \). Since \( u = \psi \geq w \) and \( z = 0 \) \( C_p \)-a.e. in \( E^c \), we have \( z_\varepsilon = 0 \) \( C_p \)-a.e. in \( E^c \), hence \( z_\varepsilon \in W_0^{1,p}(E) \). As \( u - z_\varepsilon \geq u - (u - w) = w \) \( C_p \)-a.e. in \( E \), the function \( u_\varepsilon = u - z_\varepsilon \) belongs to \( K \). Therefore (2.7) yields \( \langle Au, z_\varepsilon \rangle \leq \langle f, z_\varepsilon \rangle \). This implies

\[
\varepsilon \int_{\{\varepsilon z < u - w\}} (a(x, Du), Dz) \, dx + \int_{\{\varepsilon z \geq u - w\}} (a(x, Du), Du - Dw) \, dx \leq \langle f, z_\varepsilon \rangle.
\]

Since \( Aw \leq f \) in \( W^{-1,q}(E) \) and \( \varepsilon z - z_\varepsilon \geq 0 \) \( C_p \)-a.e. in \( E \), we have also \( \langle Aw, \varepsilon z - z_\varepsilon \rangle \leq \langle f, \varepsilon z - z_\varepsilon \rangle \), which gives

\[
\varepsilon \int_{\{\varepsilon z \geq u - w\}} (a(x, Dw), Dz) \, dx - \int_{\{\varepsilon z \geq u - w\}} (a(x, Dw), Du - Dw) \, dx \leq \langle f, \varepsilon z - z_\varepsilon \rangle.
\]

By adding this inequality to (2.9) we obtain

\[
\varepsilon \int_{\{\varepsilon z < u - w\}} (a(x, Du), Dz) \, dx + \varepsilon \int_{\{\varepsilon z \geq u - w\}} (a(x, Dw), Dz) \, dx + \\
+ \int_{\{\varepsilon z \geq u - w\}} (a(x, Du) - a(x, Dw), Du - Dw) \, dx \leq \varepsilon \langle f, z \rangle.
\]

By the monotonicity condition (1.5) we get

\[
\int_{\{\varepsilon z < u - w\}} (a(x, Du), Dz) \, dx + \int_{\{\varepsilon z \geq u - w\}} (a(x, Dw), Dz) \, dx \leq \langle f, z \rangle,
\]

and taking the limit as \( \varepsilon \to 0 \) we obtain

\[
(2.10) \quad \int_{\{u > w\}} (a(x, Du), Dz) \, dx + \int_{\{u \leq w\}} (a(x, Dw), Dz) \, dx \leq \langle f, z \rangle.
\]

As \( u \geq w \) \( C_p \)-a.e. in \( \mathbb{R}^n \), we have \( Du = Dw \) a.e. in \( \{u \leq w\} = \{u = w\} \), so that (2.10) gives \( \langle Au, z \rangle \leq \langle f, z \rangle \). Together with (2.8) this implies \( Au = f \) in \( W^{-1,q}(E) \). As \( u \in K \), we have also \( u - \psi \in W_0^{1,p}(E) \) and \( u \geq w \) \( C_p \)-a.e. in \( E \) as required. \( \square \)

**Remark 2.10.** If in Lemma 2.9 we assume that \( Aw \geq f \) in \( W^{-1,q}(E) \) and that \( (w - \psi)^- \in W_0^{1,p}(E) \), i.e., \( w \geq \psi \) \( C_p \)-a.e. in \( E^c \), then we can prove that there exists a solution \( u \) of (2.1) such that \( u \leq w \) \( C_p \)-a.e. in \( E \).

We are now in a position to prove the Comparison Principle.
Theorem 2.11. Let $E$ be a bounded set in $\mathbb{R}^n$, let $\varphi$, $\psi \in W^{1,p}(\mathbb{R}^n)$, and let $f$, $g \in W^{-1,q}(E)$. Assume that $f \leq g$ in $W^{-1,q}(E)$ and $(\varphi - \psi)^+ \in W_{0}^{1,p}(E)$, i.e., $\varphi \leq \psi$ $C_p$-q.e. in $E^c$. Let $u$ and $v$ be two solutions of the Dirichlet problems

$$
\begin{align*}
&\begin{cases}
u - \varphi \in W_0^{1,p}(E), \\
Au = f \quad \text{in } W^{-1,q}(E),
\end{cases} \\
&\begin{cases}
v - \psi \in W_0^{1,p}(E), \\
Av = g \quad \text{in } W^{-1,q}(E),
\end{cases}
\end{align*}
$$

let $u_1$ and $v_1$ be the minimal solutions, and let $u_2$ and $v_2$ be the maximal solutions. Then $u_1 \leq v_1 \leq v$ and $u \leq u_2 \leq v_2$ $C_p$-q.e. in $E$.

Proof. Since $Au_2 = f \leq g$ in $W^{-1,q}(E)$ and $u_2 = \varphi \leq \psi$ $C_p$-q.e. in $E^c$, by Lemma 2.9 there exists a solution $v_0$ of the second problem in (2.11) such that $u_2 \leq v_0$ $C_p$-q.e. in $E$. Since $u_2$ and $v_2$ are the maximal solutions, we have $u \leq u_2$ and $v_0 \leq v_2$ $C_p$-q.e. in $E$, hence $u \leq u_2 \leq v_2$ $C_p$-q.e. in $E$. The other inequalities can be proved in the same way by using Remark 2.10.

3. Capacity, capacitary potentials, and capacitary distributions

In this section we introduce the capacity $C_A$ associated with the operator $A$, and the related notions of $C_A$-potential and $C_A$-distributions.

Definition 3.1. We say that two bounded sets $E$ and $F$ are $C_p$-compatible if there exists a function $\psi$ in $W^{1,p}(\mathbb{R}^n)$ such that $\psi = 1$ $C_p$-q.e. in $E$ and $\psi = 0$ $C_p$-q.e. in $F^c$.

Remark 3.2. It is clear that $E$ and $F$ are $C_p$-compatible if and only if there exists a function $\psi$ in $W^{1,p}(\mathbb{R}^n)$ such that $0 \leq \psi \leq 1$ $C_p$-q.e. in $\mathbb{R}^n$, $\psi = 1$ $C_p$-q.e. in $E$, and $\psi = 0$ $C_p$-q.e. in $F^c$.

Remark 3.3. If $\overline{E} \subset \hat{F}$, then $E$ and $F$ are $C_p$-compatible. The converse is not true. For instance, for every function $u$ in $W^{1,p}(\mathbb{R}^n)$ the sets $E = \{u > t\}$ and $F = \{u > s\}$ are $C_p$-compatible for $s < t$, with $\psi = (t-s)^{-1}((u \wedge t) - s)^+$, but, in general, $\overline{E}$ is not contained in $\hat{F}$.

Remark 3.4. If $E_1$ and $E_2$ are $C_p$-compatible with $F$, so is $E_1 \cup E_2$. For the proof, let us consider two functions $\psi_1$ and $\psi_2$ as in Remark 3.2 with $E = E_1$ and $E = E_2$. Then $\psi_1 \vee \psi_2$ satisfies the same conditions with $E = E_1 \cup E_2$. Similarly we can prove that, if $E$ is $C_p$-compatible with $F_1$ and $F_2$, then $E$ is $C_p$-compatible with $F_1 \cap F_2$. 

Definition 3.5. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $\psi$ be a function as in Definition 3.1. Every solution $u$ of the Dirichlet problem

$$
\begin{cases}
    u - \psi \in W^{1,p}_0(F \setminus E), \\
    Au = 0 \quad \text{in } W^{-1,q}(F \setminus E),
\end{cases}
$$

is called a $C_A$-potential of $E$ in $F$. The maximal and the minimal solutions of (3.1) are called the maximal and minimal $C_A$-potentials of $E$ in $F$.

Remark 3.6. Clearly the definition of $C_A$-potential does not depend on the choice of $\psi$. By the definition of the space $W^{1,p}_0(F \setminus E)$ and by the properties of $\psi$ we have that every $C_A$-potential $u$ of $E$ in $F$ satisfies $u = 1$ $C_p$-q.e. in $E$ and $u = 0$ $C_p$-q.e. in $F^c$.

Remark 3.7. Let $u_1$ and $u_2$ be the minimal and the maximal $C_A$-potentials of $E$ in $F$. Since $a(x,0) = 0$ by (1.4), the Comparison Principle (Theorem 2.11) implies that $0 \leq u_2$ and $u_1 \leq 1$ $C_p$-q.e. in $F \setminus E$.

Definition 3.8. Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. If $E$ and $F$ are $C_p$-compatible, the capacity of $E$ in $F$ relative to the operator $A$ is defined as

$$
C_A(E,F) = \langle Au, u \rangle = \int_{F \setminus E} (a(x,Du), Du) \, dx,
$$

where $u$ is any $C_A$-potential of $E$ in $F$. By Corollary 2.8 this definition is independent of the choice of $u$. If $E$ and $F$ are not $C_p$-compatible, we define $C_A(E,F) = +\infty$.

Remark 3.9. By (1.12) we have $C_A(E,F) \geq 0$, and by (1.4) we have $C_A(\emptyset,F) = 0$. By (3.1) we have also $C_A(E,F) = \langle Au, v \rangle$ for every $C_A$-potential $u$ of $E$ in $F$ and for every function $v$ in $W^{1,p}(\mathbb{R}^n)$ such that $v = 1$ $C_p$-q.e. in $E$ and $v = 0$ $C_p$-q.e. in $F^c$.

Remark 3.10. It follows immediately from the definitions that if $E_1$, $E_2$, $F_1$, $F_2$ are bounded sets in $\mathbb{R}^n$ and $F_1 \Delta F_2$ and $E_1 \Delta E_2$ are $C_p$-null sets, then $C_A(E_1,F_1) = C_A(E_2,F_2)$ and the $C_A$-potentials are the same.

If $Au = -\text{div}(|Du|^{p-2}Du)$, then $C_A = C_p$. In the general case the relationship between $C_A$ and $C_p$ is given by the following proposition.
Proposition 3.11. Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. Then

(3.2) \[ C_A(E, F) \geq c_1 C_p(E, F) - \|b_1\|_{L^1(F)}, \]

(3.3) \[ C_A(E, F) \leq k_1 C_p(E, F) + k_2(F) C_p(E, F)^{1/p}, \]

with

(3.4) \[ k_1 = \frac{(4c_2)^p}{p(qc_1)^{p-1}}, \quad k_2(F) = \frac{4c_2}{c_1^{1/q}} \|b_1\|^{1/q}_{L^1(F)} + 4\|b_2\|_{L^q(F)}, \]

where $c_1$, $c_2$ and $b_1$, $b_2$ are the constants and the functions which appear in (1.6) and (1.7). If $b_1$ and $b_2$ belong to $L^\infty(F)$, then

(3.5) \[ C_A(E, F) \leq (k_1 + k_3(F)) C_p(E, F), \]

with

(3.6) \[ k_3(F) = 2^{p+1} \left( \frac{c_2}{c_1^{1/q}} \|b_1\|^{1/q}_{L^\infty(F)} + \|b_2\|_{L^\infty(F)} \right) \operatorname{diam}(F)^{p-1}, \]

where $\operatorname{diam}(F)$ is the diameter of the set $F$.

Proof. To prove (3.2) we may assume that $C_A(E, F) < +\infty$. Let $u$ be a $C_A$-potential of $E$ in $F$. By (1.13) we have

(3.7) \[ C_A(E, F) = \langle Au, u \rangle \geq c_1 \int_F |Du|^p dx - \int_F b_1 dx. \]

Since by (1.2)

\[ C_p(E, F) \leq \int_F |Du|^p dx, \]

from (3.7) we obtain (3.2).

To prove (3.3) and (3.5) we may assume that $C_p(E, F) < +\infty$. Let $w$ be the $C_p$-potential of $E$ in $F$, let $v = (2w - 1)^+$, and let $G = \{v > 0\} = \{w > \frac{1}{2}\}$. Since $w = 1$ $C_p$-q.e. in $E$ and $w = 0$ $C_p$-q.e. in $F^c$, we have $v = 1$ $C_p$-q.e. in $E$ and $v = 0$ $C_p$-q.e. in $F^c$. From (1.10) and from Remark 3.9 we obtain

(3.8) \[ C_A(E, F) = \langle Au, v \rangle \leq (c_2 \|Du\|_{L^p(G, \mathbb{R}^n)}^{p-1} + \|b_2\|_{L^q(G)}) \|Dv\|_{L^p(G, \mathbb{R}^n)}. \]

By (1.13) we have

(3.9) \[ c_1^{1/q} \|Du\|_{L^p(F, \mathbb{R}^n)}^{p-1} \leq C_A(E, F)^{1/q} + \|b_1\|^{1/q}_{L^1(F)}, \]
while the definition of $v$ gives

$$\|Dv\|_{L^p(G, \mathbb{R}^n)} = 2 \|Dw\|_{L^p(G, \mathbb{R}^n)} \leq 2 C_p(E, F)^{1/p}. \tag{3.10}$$

From (3.8), (3.9), and (3.10) we get

$$C_A(E, F) \leq 2 \left( \frac{c_2}{c_1^{1/q}} C_A(E, F)^{1/q} + \frac{c_2}{c_1^{1/q}} \|b_1\|_{L^1(G)}^{1/q} + \|b_2\|_{L^q(G)} \right) C_p(E, F)^{1/p}, \tag{3.11}$$

Using Young’s Inequality we obtain

$$C_A(E, F) \leq \frac{(4c_2)^p}{p(qc_1)^{p-1}} C_p(E, F) + \left( \frac{4c_2}{c_1^{1/q}} \|b_1\|_{L^1(G)}^{1/q} + 4\|b_2\|_{L^q(G)} \right) C_p(E, F)^{1/p}, \tag{3.12}$$

which implies (3.3). If $b_1$ and $b_2$ belong to $L^\infty(F)$, then

$$\|b_1\|_{L^1(G)}^{1/q} \leq \text{meas}(G)^{1/q} \|b_1\|_{L^\infty(G)}^{1/q},$$

$$\|b_2\|_{L^q(G)} \leq \text{meas}(G)^{1/q} \|b_2\|_{L^\infty(G)}.$$

As $G = \{ w > \frac{1}{2} \}$, by Poincaré’s Inequality we have

$$\text{meas}(G) \leq 2^p \|w\|_{L^p(F)}^p \leq 2^p \text{diam}(F)^p \|Dw\|_{L^p(F)}^p = 2^p \text{diam}(F)^p C_p(E, F).$$

Therefore (3.12) implies

$$\|b_1\|_{L^1(G)}^{1/q} \leq \|b_1\|_{L^\infty(G)}^{1/q} 2^{p-1} \text{diam}(F)^p C_p(E, F)^{1/q},$$

$$\|b_2\|_{L^q(G)} \leq \|b_2\|_{L^\infty(F)}^{1/q} 2^{p-1} \text{diam}(F)^p C_p(E, F)^{1/q},$$

which, together with (3.11), yields (3.5).

\[\square\]

**Remark 3.12.** If $b_1 = 0$ a.e. in $F$ and $b_2 \in L^\infty(F)$, then for every bounded set $E$ we have

$$c_1 C_p(E, F) \leq C_A(E, F) \leq (k_1 + k_3(F)) C_p(E, F),$$

where $c_1$, $k_1$, and $k_3(F)$ are defined in (1.6), (3.4), and (3.6).

The following lemma will be useful in the proof of Theorems 3.14, 4.7, 4.8, and 6.10.
Lemma 3.13. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $u$ be a $C_A$-potential of $E$ in $F$. Then $Au \geq 0$ in $W^{-1,q}(F)$ and $Au \leq 0$ in $W^{-1,q}(E^c)$.

Proof. Let $u_1$ and $u_2$ be the minimal and the maximal $C_A$-potentials of $E$ in $F$. By Remark 3.7 we have $u_1 \leq 1$ $C_p$-q.e. in $\mathbb{R}^n$. If we apply Lemma 2.5 with $B = E$, $C = F$, $w_1 = u_1$, and we choose as $w_2$ any function in $W^{1,p}(\mathbb{R}^n)$ which is equal to 1 $C_p$-q.e. in $F$, we obtain $Au_1 \geq Aw_2 = 0$ in $W^{-1,q}(F)$. Since $Au = Au_1$ in $W^{-1,q}(\mathbb{R}^n)$ by Theorem 2.6, we conclude that $Au \geq 0$ in $W^{-1,q}(F)$.

By Remark 3.7 we have $u_2 \geq 0$ $C_p$-q.e. in $\mathbb{R}^n$. If we apply Lemma 2.5 with $B = F^c$, $C = E^c$, $w_1 = 0$, and $w_2 = u_2$, we obtain $Au_2 \leq A0 = 0$ in $W^{-1,q}(E^c)$. Since $Au = Au_2$ in $W^{-1,q}(\mathbb{R}^n)$ by Theorem 2.6, we conclude that $Au \leq 0$ in $W^{-1,q}(E^c)$.

By Theorem 2.6 there exists $g \in W^{-1,q}(\mathbb{R}^n)$ such that $Au = g$ in $W^{-1,q}(\mathbb{R}^n)$ for every $C_A$-potential $u$ of $E$ in $F$. The following theorem gives a precise description of $g$.

Theorem 3.14. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$. Then there exists a unique pair $(\lambda, \nu)$ of non-negative Radon measures on $\mathbb{R}^n$ such that:

(a) $\lambda$ and $\nu$ are mutually singular;
(b) for every $C_A$-potential $u$ of $E$ in $F$ and for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have
\[
\langle Au, \varphi \rangle = \int_{\mathbb{R}^n} \varphi d\lambda - \int_{\mathbb{R}^n} \varphi d\nu.
\]

Moreover, the following properties hold:
(c) the measures $\lambda$ and $\nu$ are bounded and $C_p$-absolutely continuous;
(d) condition (b) holds also for every $\varphi \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$;
(e) the measure $\lambda - \nu$ belongs to $W^{-1,q}(\mathbb{R}^n)$;
(f) $\text{supp} \lambda \subset \partial E$ and $\text{supp} \nu \subset \partial F$;
(g) $v \in W_0^{1,p}(E) \cup W_0^{1,p}(E^c) \implies v = 0$ $\lambda$-a.e. in $\mathbb{R}^n$;
(h) $v \in W_0^{1,p}(F) \cup W_0^{1,p}(F^c) \implies v = 0$ $\nu$-a.e. in $\mathbb{R}^n$;
(i) $\lambda(U) = 0$ whenever $U \cap E$ and $U \cap E^c$ are $C_p$-quasi open;
(j) $\nu(U) = 0$ whenever $U \cap F$ and $U \cap F^c$ are $C_p$-quasi open;
(k) $\lambda(F^c) = 0$ and $\nu(E) = 0$. 
Proof. By Theorem 2.6 there exists \( g \in W^{-1,q}(\mathbb{R}^n) \) such that \( Au = g \) in \( W^{-1,q}(\mathbb{R}^n) \) for every \( C_A \)-potential \( u \) of \( E \) in \( F \). We want to prove that there exists a unique pair \((\lambda, \nu)\) of mutually singular non-negative Radon measures on \( \mathbb{R}^n \) such that

\[
(g, \varphi) = \int_{\mathbb{R}^n} \varphi \, d\lambda - \int_{\mathbb{R}^n} \varphi \, d\nu \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),
\]

and that \( \lambda \) and \( \nu \) satisfy properties (c)–(k).

Let us fix a function \( \psi \) as in Remark 3.2. By Lemma 3.13 we have \( g \geq 0 \) in \( W^{-1,q}(F) \). Since \( \psi \varphi \) belongs to \( W^{1,p}_0(F) \) for every \( \varphi \) in \( C_0^\infty(\mathbb{R}^n) \), we conclude that

\[
(g, \psi \varphi) \geq 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 0.
\]

By the Riesz Representation Theorem there exists a non-negative Radon measure \( \lambda \) on \( \mathbb{R}^n \) such that

\[
(g, \varphi) = \int_{\mathbb{R}^n} \varphi \, d\lambda \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).
\]

In order to construct \( \nu \), we recall that by Lemma 3.13 we have \( g \leq 0 \) in \( W^{-1,q}(E^c) \). Since \((1 - \psi)\varphi \) belongs to \( W^{1,p}_0(E^c) \) for every \( \varphi \) in \( C_0^\infty(\mathbb{R}^n) \), we conclude that

\[
(g, (1 - \psi) \varphi) \leq 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 0.
\]

By the Riesz Representation Theorem there exists a non-negative Radon measure \( \nu \) on \( \mathbb{R}^n \) such that

\[
(g, (1 - \psi) \varphi) = -\int_{\mathbb{R}^n} \varphi \, d\nu \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).
\]

From (3.14) and (3.15) we obtain

\[
(g, \varphi) = (g, \psi \varphi) + (g, (1 - \psi) \varphi) = \int_{\mathbb{R}^n} \varphi \, d\lambda - \int_{\mathbb{R}^n} \varphi \, d\nu,
\]

which proves (3.13) and hence (b). Property (e) follows from (b) and from the fact that \( Au \) belongs to \( W^{-1,q}(\mathbb{R}^n) \).

Let us prove (c). As \( g = 0 \) in \( W^{-1,q}(F^c) \) by (1.4), using (3.14) and (3.15) we obtain

\[
\int_{\mathbb{R}^n} \varphi \, d\lambda = \int_{\mathbb{R}^n} \varphi \, d\nu = 0.
\]
for every \( \varphi \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp} \varphi \subset F^c \). Therefore the supports of \( \lambda \) and \( \nu \) are contained in the compact set \( \overline{F} \). This implies that the measures \( \lambda \) and \( \nu \) are bounded. It remains to show that \( \lambda \) and \( \nu \) vanish on all \( C_p \)-null sets. To this aim, it is sufficient to prove that \( \lambda(C) = \nu(C) = 0 \) for every \( C_p \)-null compact set \( C \subset \mathbb{R}^n \). In this case it is possible to construct a sequence \( (\varphi_j) \) of functions in \( C_\infty^\infty(\mathbb{R}^n) \) such that \( 0 \leq \varphi_j \leq 1 \) in \( \mathbb{R}^n \), \( \varphi_j = 1 \) in \( C \), and \( (\varphi_j) \) converges to 0 strongly in \( W^{1,p}(\mathbb{R}^n) \). Then by (3.14) for every \( j \) we have

\[
\lambda(C) \leq \int_{\mathbb{R}^n} \varphi_j \, d\lambda = \langle g, \psi \varphi_j \rangle.
\]

Since \( (\psi \varphi_j) \) converges to 0 strongly in \( W^{1,p}(\mathbb{R}^n) \), passing to the limit as \( j \to \infty \) we obtain \( \lambda(C) = 0 \). In the same way we prove that \( \nu(C) = 0 \).

Since the measures \( \lambda \) and \( \nu \) are bounded and \( C_p \)-absolutely continuous, every function of \( W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) belongs to \( L^1_\lambda(\mathbb{R}^n) \) and to \( L^1_\nu(\mathbb{R}^n) \), and thus, by an easy approximation argument, from (3.14) and (3.15) we obtain

\[
\langle g, \psi v \rangle = \int_{\mathbb{R}^n} v \, d\lambda, \quad \langle g, (1-\psi)v \rangle = -\int_{\mathbb{R}^n} v \, d\nu
\]

for every \( v \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), which implies (d).

By considering separately the positive and the negative part of \( v \), it is enough to prove (g) when \( v \) is non-negative. Let us fix \( v \in W^{1,p}_0(E) \cup W^{1,p}_0(E^c) \) with \( v \geq 0 \) \( C_p \)-q.e. in \( \mathbb{R}^n \). If \( v \) belongs to \( W^{1,p}_0(E) \), then \( v = \psi v \) \( C_p \)-q.e. in \( \mathbb{R}^n \). Since \( g = 0 \) in \( W^{-1,q}(E) \) by (1.4), from (3.16) we obtain

\[
\int_{\mathbb{R}^n} v \, d\lambda = \langle g, v \rangle = 0.
\]

If \( v \) belongs to \( W^{1,p}_0(E^c) \), then \( \psi v \) belongs to \( W^{1,p}_0(F \setminus E) \). As \( g = 0 \) in \( W^{-1,q}(F \setminus E) \) by (3.1), it follows from (3.16) that

\[
\int_{\mathbb{R}^n} v \, d\lambda = \langle g, \psi v \rangle = 0.
\]

In both cases \( \int_{\mathbb{R}^n} v \, d\lambda = 0 \). Since \( v \) and \( \lambda \) are non-negative, this implies \( v = 0 \) \( \lambda \)-a.e. in \( \mathbb{R}^n \).

Similarly, it is enough to prove (h) when \( v \) is non-negative. Let us fix \( v \in W^{1,p}_0(F) \cup W^{1,p}_0(F^c) \) with \( v \geq 0 \) \( C_p \)-q.e. in \( \mathbb{R}^n \). If \( v \) belongs to \( W^{1,p}_0(F) \), then \( (1-\psi)v \) belongs to \( W^{1,p}_0(F \setminus E) \). As \( g = 0 \) in \( W^{-1,q}(F \setminus E) \) by (3.1), it follows from (3.16) that

\[
\int_{\mathbb{R}^n} v \, d\nu = -\langle g, (1-\psi)v \rangle = 0.
\]
If $v$ belongs to $W^{1,p}_0(F^c)$, then $v = (1 - \psi)v$ $C_p$-q.e. in $\mathbb{R}^n$. Since $g = 0$ in $W^{-1,q}(F^c)$ by (1.4), from (3.16) we obtain

$$\int_{\mathbb{R}^n} v \, d\nu = -\langle g, v \rangle = 0.$$  

In both cases $\int_{\mathbb{R}^n} v \, d\nu = 0$. Since $v$ and $\nu$ are non-negative, this implies $v = 0$ $\nu$-a.e. in $\mathbb{R}^n$.

It is enough to prove (i) for every $C_p$-quasi open set $U$ such that either $U \subset E$ or $U \subset E^c$. In both cases by Lemma 1.1 there exists an increasing sequence $(v_j)$ of functions of $W^{1,p}_0(E) \cup W^{1,p}_0(E^c)$, with $0 \leq v_j \leq 1_U$ $C_p$-q.e. in $\mathbb{R}^n$, which converges to $1_U$ $C_p$-q.e. in $\mathbb{R}^n$. By (g) we have $\int_{\mathbb{R}^n} v_j \, d\lambda = 0$ for every $j$, and passing to the limit as $j \to \infty$ we get $\lambda(U) = 0$. The proof of (j) is similar.

Since $(\partial E)^c \cap E$ and $(\partial E)^c \cap E^c$ are open sets, by (i) we have $\lambda((\partial E)^c) = 0$, hence $\text{supp} \lambda \subset \partial E$. Similarly we prove that the inclusion $\text{supp} \nu \subset \partial F$ follows from (j).

Since $E$ is $C_p$-compatible with $F$, the set $E \setminus F$ is $C_p$-null. Consequently, by Remark 3.10, the $C_A$-potentials do not change if we replace $E$ by $E \cap F$. Therefore in the rest of the proof we may assume that $E \subset F$.

Let $\chi$ be a function in $C^\infty_0(\mathbb{R}^n)$ such that $\chi = 1$ in $F$. Since $\chi - \psi \in W^{1,p}_0(E^c)$, by (f) and (g) we have $1 - \psi = \chi - \psi = 0$ $\lambda$-a.e. in $\mathbb{R}^n$. Since $\psi \in W^{1,p}_0(F)$, by (h) we have $\psi = 0$ $\nu$-a.e. in $\mathbb{R}^n$. These facts imply that $\lambda$ is concentrated in the set $\{\psi = 1\}$ while $\nu$ is concentrated in the set $\{\psi = 0\}$ and prove that $\lambda$ and $\nu$ are mutually singular.

Since $\psi = 1$ $\lambda$-a.e. in $\mathbb{R}^n$ and $\psi = 0$ $C_p$-q.e. in $F^c$, by (c) we have $\lambda(F^c) = 0$. Similarly, as $\psi = 0$ $\nu$-a.e. in $\mathbb{R}^n$ and $\psi = 1$ $C_p$-q.e. in $E$, by (c) we have $\nu(E) = 0$.

Finally, condition (b) determines uniquely the signed measure $\lambda - \nu$. Since $\lambda$ and $\nu$ are non-negative and mutually singular, by the uniqueness of the Hahn Decomposition we have $\lambda = (\lambda - \nu)^+$ and $\nu = (\lambda - \nu)^-$, thus the pair $(\lambda, \nu)$ is uniquely determined by conditions (a) and (b). In particular $\lambda$ and $\nu$ do not depend on the function $\psi$ used in the proof. \hfill $\Box$

**Definition 3.15.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$. The measures $\lambda$ and $\nu$ introduced in the previous theorem are called the *inner* and the *outer* $C_A$-distributions of $E$ in $F$.

**Remark 3.16.** If $\overline{E} \subset \hat{F}$, it is easy to see that the $C_A$-distributions $\lambda$ and $\nu$ belong to $W^{-1,q}(\mathbb{R}^n)$. This is not true, in general, when $E$ and $F$ are only $C_p$-compatible. For a counterexample we refer to the Appendix of [5].
Proposition 3.17. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $\lambda$ and $\nu$ be the inner and the outer $C_A$-distributions of $E$ in $F$. Then

$$C_A(E, F) = \lambda(\partial E) = \lambda(\mathbb{R}^n) = \lambda(F) = \nu(\partial F) = \nu(\mathbb{R}^n) = \nu(E^c).$$

Proof. By properties (f) and (k) of Theorem 3.14 we have $\lambda(\partial E) = \lambda(\mathbb{R}^n) = \lambda(F)$ and $\nu(\partial F) = \nu(\mathbb{R}^n) = \nu(E^c)$.

Since $E$ is $C_p$-compatible with $F$, the set $E \setminus F$ is $C_p$-null. Consequently, by Remark 3.10, the inner an the outer $C_A$-distributions do not change if we replace $E$ by $E \cap F$. Therefore it is not restrictive to assume that $E \subset F$. Let $u$ be a $C_A$-potential of $E$ in $F$, let $\psi$ be as in Remark 3.2, and let $\chi$ be a function in $C^\infty_0(\mathbb{R}^n)$ such that $\chi = 1$ in $F$. Since $\psi \in W^{1,p}_0(F)$ and $\chi - \psi \in W^{1,p}_0(E^c)$, by properties (f), (g), and (h) of Theorem 3.14 we have $\psi = 0$ $\nu$-a.e. in $\mathbb{R}^n$, $\chi = 1$ $\nu$-a.e. in $\mathbb{R}^n$, and $\psi = \chi = 1$ $\lambda$-a.e. in $\mathbb{R}^n$. By Theorem 3.14(d) this implies

$$\langle Au, \psi \rangle = \int_{\mathbb{R}^n} \psi \, d\lambda = \lambda(\mathbb{R}^n),$$

$$\langle Au, \chi - \psi \rangle = -\int_{\mathbb{R}^n} (\chi - \psi) \, d\nu = -\nu(\mathbb{R}^n).$$

By Remark 3.9 we have $C_A(E, F) = \langle Au, \psi \rangle$. Since $Du = 0$ a.e. in $F^c$ and $D\chi = 0$ a.e. in $F$, by (1.4) we have $\langle Au, \chi - \psi \rangle = -\langle Au, \psi \rangle$. Therefore (3.17) implies that $C_A(E, F) = \lambda(\mathbb{R}^n) = \nu(\mathbb{R}^n)$. \hfill $\Box$

4. Monotonicity and continuity along monotone sequences

In this section we study the monotonicity and continuity properties of $C_A(E, F)$ with respect to $E$ and $F$. These results are based on the fundamental inequality proved in Lemma 2.5, on the properties of the $C_A$-distributions discussed in Section 3, and on the properties of the minimal and maximal $C_A$-potentials proved in the following lemmas.

Lemma 4.1. Let $E_1$, $E_2$, $F$ be three bounded sets in $\mathbb{R}^n$. Assume that $E_1 \subset E_2$ and that $E_2$ and $F$ are $C_p$-compatible. If $u_1$ and $u_2$ are the minimal $C_A$-potentials of $E_1$ and $E_2$ in $F$, then $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$. 
Proof. By Corollary 2.7 $u_1$ coincides with the minimal solution $u$ of the problem
\[
\begin{align*}
\begin{cases}
u - u_1 & \in W^{1,p}_0(F \setminus E_2), \\
Au & = 0 \text{ in } W^{-1,q}(F \setminus E_2).
\end{cases}
\end{align*}
\]
By Remark 3.7 we have $u_1 \leq 1$ $C_p$-q.e. in $E_2$. Therefore the Comparison Principle (Theorem 2.11) implies that $u_1 \leq u_2$ $C_p$-q.e. in $F \setminus E_2$. Since $u_1 \leq 1 = u_2$ $C_p$-q.e. in $E_2$ and $u_1 = 0 = u_2$ $C_p$-q.e. in $F^c$, we conclude that $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$. 

**Lemma 4.2.** Let $E, F_1, F_2$ be three bounded sets in $\mathbb{R}^n$. Assume that $F_1 \subset F_2$ and that $E$ and $F_1$ are $C_p$-compatible. If $u_1$ and $u_2$ are the maximal $C_A$-potentials of $E$ in $F_1$ and $F_2$, then $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$.

Proof. By Corollary 2.7 $u_2$ coincides with the minimal solution $u$ of the problem
\[
\begin{align*}
\begin{cases}
u - u_2 & \in W^{1,p}_0(F_1 \setminus E), \\
Au & = 0 \text{ in } W^{-1,q}(F_1 \setminus E).
\end{cases}
\end{align*}
\]
By Remark 3.7 we have $u_2 \geq 0$ $C_p$-q.e. in $F_1^c$. Therefore the Comparison Principle (Theorem 2.11) implies that $u_1 \leq u_2$ $C_p$-q.e. in $F_1 \setminus E$. Since $u_1 = 1 = u_2$ $C_p$-q.e. in $E$ and $u_1 = 0 \leq u_2$ $C_p$-q.e. in $F_1^c$, we conclude that $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$.

We prove now that the set function $C_A(\cdot, F)$ is increasing.

**Theorem 4.3.** Let $E_1, E_2, F$ be three bounded sets in $\mathbb{R}^n$ such that $E_1 \subset E_2$. Then $C_A(E_1, F) \leq C_A(E_2, F)$.

**Proof.** Since the inequality is trivial when $E_2$ and $F$ are not $C_p$-compatible, the conclusion follows from Proposition 3.17 and from the following lemma.

**Lemma 4.4.** Let $E_1, E_2, F$ be three bounded sets in $\mathbb{R}^n$. Assume that $E_1 \subset E_2$ and that $E_2$ and $F$ are $C_p$-compatible. Let $\nu_1$ and $\nu_2$ be the outer $C_A$-distributions of $E_1$ and $E_2$ in $F$. Then $\nu_1(B) \leq \nu_2(B)$ for every Borel set $B \subset \mathbb{R}^n$.

**Proof.** Let $u_1$ and $u_2$ be the minimal $C_A$-potentials of $E_1$ and $E_2$ in $F$. Then $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$ by Lemma 4.1. If we apply Lemma 2.5 with $B = F^c$, $C = E_2^c$, $w_1 = u_1$,
and $w_2 = u_2$, we obtain $Au_1 \geq Au_2$ in $W^{-1,q}(E_2^c)$. By (d) and (g) of Theorem 3.14 we have
\[ \int_{\mathbb{R}^n} v d\nu_1 = -\langle Au_1, v \rangle \leq -\langle Au_2, v \rangle = \int_{\mathbb{R}^n} v d\nu_2 \]
for every $v \in W_0^{1,p}(E_2^c)$ with $v \geq 0$ $C_p$-q.e. in $\mathbb{R}^n$. By Lemma 1.1 this implies $\nu_1(V) \leq \nu_2(V)$ for every $C_p$-quasi open set $V$ contained in $E_2^c$. As $\{u_2 > 0\}$ is $C_p$-quasi open, by Theorem 3.14(j) we have $\nu_1(\{u_2 \geq 1\}) \leq \nu_1(\{u_2 > 0\}) = 0$ and $\nu_2(\{u_2 \geq 1\}) \leq \nu_2(\{u_2 > 0\}) = 0$. For every open set $U \subset \mathbb{R}^n$ the set $U \cap \{u_2 < 1\}$ is $C_p$-quasi open and is contained in $E_2^c$ (up to a $C_p$-null set). Therefore $\nu_1(U) = \nu_1(U \cap \{u_2 < 1\}) \leq \nu_2(U \cap \{u_2 < 1\}) = \nu_2(U)$. Since $\nu_1$ and $\nu_2$ are Radon measures, this implies that $\nu_1(B) \leq \nu_2(B)$ for every Borel set $B \subset \mathbb{R}^n$.

The following theorem shows that $C_A(E, \cdot)$ is decreasing.

**Theorem 4.5.** Let $E$, $F_1$, $F_2$ be three bounded sets in $\mathbb{R}^n$ such that $F_1 \subset F_2$. Then $C_A(E, F_1) \geq C_A(E, F_2)$.

**Proof.** Since the inequality is trivial when $E$ and $F_1$ are not $C_p$-compatible, the conclusion follows from Proposition 3.17 and from the following lemma.

**Lemma 4.6.** Let $E$, $F_1$, $F_2$ be three bounded sets in $\mathbb{R}^n$. Assume that $F_1 \subset F_2$ and that $E$ and $F_1$ are $C_p$-compatible. Let $\lambda_1$ and $\lambda_2$ be the inner $C_A$-distributions of $E$ in $F_1$ and $F_2$. Then $\lambda_1(B) \geq \lambda_2(B)$ for every Borel set $B \subset \mathbb{R}^n$.

**Proof.** Let $u_1$ and $u_2$ be the maximal $C_A$-potentials of $E$ in $F_1$ and $F_2$. Then $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$ by Lemma 4.2. If we apply Lemma 2.5 with $B = E$, $C = F_1$, $w_1 = u_1$, and $w_2 = u_2$, we obtain $Au_1 \geq Au_2$ in $W^{-1,q}(F_1)$. By (d) and (h) of Theorem 3.14 we have
\[ \int_{\mathbb{R}^n} v d\lambda_1 = \langle Au_1, v \rangle \geq \langle Au_2, v \rangle = \int_{\mathbb{R}^n} v d\lambda_2 \]
for every $v \in W_0^{1,p}(F_1)$ with $v \geq 0$ $C_p$-q.e. in $\mathbb{R}^n$. By Lemma 1.1 this implies $\lambda_1(V) \leq \lambda_2(V)$ for every $C_p$-quasi open set $V$ contained in $F_1$. As $\{u_1 < 1\}$ is $C_p$-quasi open, by Theorem 3.14(i) we have $\lambda_1(\{u_1 \leq 0\}) \leq \lambda_1(\{u_1 < 1\}) = 0$ and $\lambda_2(\{u_1 \leq 0\}) \leq \lambda_2(\{u_1 < 1\}) = 0$. For every open set $U \subset \mathbb{R}^n$ the set $U \cap \{u_1 > 0\}$ is $C_p$-quasi open and is contained in $F_1$ (up to a $C_p$-null set). Therefore $\lambda_1(U) = \lambda_1(U \cap \{u_1 > 0\}) \geq \lambda_2(U \cap \{u_1 > 0\}) = \lambda_2(U)$. Since $\lambda_1$ and $\lambda_2$ are Radon measures, this implies that $\lambda_1(B) \geq \lambda_2(B)$ for every Borel set $B \subset \mathbb{R}^n$. 

\[ \square \]
The following theorem shows that the set function $C_A(\cdot, F)$ is continuous along all increasing sequences.

**Theorem 4.7.** Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. If $E$ is the union of an increasing sequence of sets $(E_j)$, then

$$C_A(E, F) = \lim_{j \to \infty} C_A(E_j, F) = \sup_j C_A(E_j, F).$$

**Proof.** Let $S = \sup_j C_A(E_j, F)$. By monotonicity (Theorem 4.3) we have $S \leq C_A(E, F)$. It remains to prove the opposite inequality when $S < +\infty$, and hence each set $E_j$ is $C_p$-compatible with $F$. Let $u$ and $u_j$ be the minimal $C_A$-potentials of $E$ and $E_j$ in $F$. As $S < +\infty$, by (1.13) the sequence $(u_j)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, and by (1.10) the sequence $(Au_j)$ is bounded in $W^{-1,q}(\mathbb{R}^n)$. Passing, if necessary, to a subsequence, we may assume that $(u_j)$ converges weakly in $W^{1,p}(\mathbb{R}^n)$ to some function $w \in W^{1,p}_0(F)$ and that $(Au_j)$ converges weakly in $W^{-1,q}(\mathbb{R}^n)$ to some element $f$ of $W^{-1,q}(\mathbb{R}^n)$.

We want to prove that $Au = f$ in $W^{-1,q}(F)$ and that $w = u$ $C_p$-a.e. in $\mathbb{R}^n$. From the monotonicity condition (1.11) for every $j$ we obtain

$$\langle A v, v - u_j \rangle \geq \langle A u_j, v - u_j \rangle \quad \forall v \in W^{1,p}(\mathbb{R}^n). \quad (4.1)$$

If $j \leq i$, by Lemma 4.1 we have $u_j \leq u_i \leq u$ $C_p$-a.e. in $\mathbb{R}^n$, hence $u_j \leq w \leq u$ $C_p$-a.e. in $\mathbb{R}^n$ for every $j$. Since $Au_j \geq 0$ in $W^{-1,q}(F)$ (Lemma 3.13), we have

$$\langle Au_j, v - u_j \rangle \geq \langle Au_j, v - w \rangle \quad \forall v \in W^{1,p}_0(F),$$

which, together with (4.1), gives

$$\langle A v, v - u_j \rangle \geq \langle A u_j, v - w \rangle \quad \forall v \in W^{1,p}_0(F).$$

Passing to the limit as $j \to \infty$ we get

$$\langle A v, v - w \rangle \geq \langle f, v - w \rangle \quad \forall v \in W^{1,p}_0(F).$$

Putting $v = w + \varepsilon z$, with $z \in W^{1,p}_0(F)$ and $\varepsilon > 0$, and dividing by $\varepsilon$ we obtain

$$\langle A(w + \varepsilon z), z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F).$$
Passing to the limit as $\varepsilon \to 0$ we get

$$\langle Aw, z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F),$$

hence $Aw = f$ in $W^{-1,q}(F)$. As $u_j \leq u \leq w$ are $C_p$-q.e. in $\mathbb{R}^n$ and $u_j = u = 1$ $C_p$-q.e. in $E_j$ (Remark 3.6), we have $w = 1$ $C_p$-q.e. in $E_j$ for every $j$, hence $w = 1$ $C_p$-q.e. in $E$. Since $w \in W^{1,p}_0(F)$, we have also $w = 0$ $C_p$-q.e. in $F^c$. This shows that $w$ satisfies the first condition in (3.1).

It remains to prove that $Aw = 0$ in $W^{-1,q}(F \setminus E)$. If $v \in W^{1,p}_0(F \setminus E)$, then $v \in W^{1,p}_0(F \setminus E_j)$ and the definition of $u_j$ implies that $\langle Au_j, v \rangle = 0$ for every $j$. Since $(Au_j)$ converges to $f$ weakly in $W^{-1,q}(\mathbb{R}^n)$ and $Aw = f$ in $W^{-1,q}(F)$, we conclude that

$$\langle Aw, v \rangle = \langle f, v \rangle = \lim_{j \to \infty} \langle Au_j, v \rangle = 0 \quad \forall v \in W^{1,p}_0(F \setminus E),$$

hence $Aw = 0$ in $W^{-1,q}(F \setminus E)$. This proves that $w$ is a $C_A$-potential of $E$ in $F$. Since $w \leq u$ $C_p$-q.e. in $\mathbb{R}^n$, by the minimality of $u$ we obtain $w = u$ $C_p$-q.e. in $\mathbb{R}^n$.

By Remark 3.9 we have $C_A(E_j, F) = \langle Au_j, u \rangle$ for every $j$. Using again the fact that $(Au_j)$ converges to $f$ weakly in $W^{-1,q}(\mathbb{R}^n)$ and that $Au = f$ in $W^{-1,q}(F)$ we obtain

$$C_A(E, F) = \langle Au, u \rangle = \langle f, u \rangle = \lim_{j \to \infty} \langle Au_j, u \rangle = \lim_{j \to \infty} C_A(E_j, F),$$

which concludes the proof of the theorem. 

For the continuity of the set function $C_A(\cdot, F)$ along decreasing sequences $(E_j)$ we need two additional assumptions: the sets $E_j$ must be $C_p$-quasi closed and $C_p$-compatible with $F$.

**Theorem 4.8.** Let $F$ be a bounded set in $\mathbb{R}^n$, let $(E_j)$ be a decreasing sequence of $C_p$-quasi closed bounded sets, and let $E$ be their intersection. If $E_1$ and $F$ are $C_p$-compatible, then

$$C_A(E, F) = \lim_{j \to \infty} C_A(E_j, F) = \inf_{j} C_A(E_j, F).$$

**Proof.** By Theorem 4.3 we have $C_A(E_j, F) \leq C_A(E_1, F) < +\infty$ for every $j$. Let $u_j$ be the minimal $C_A$-potential of $E_j$ in $F$. By (1.13) the sequence $(u_j)$ is bounded in
$W^{1,p}(\mathbb{R}^n)$, and by (1.10) the sequence $(Au_j)$ is bounded in $W^{-1,q}(\mathbb{R}^n)$. Passing, if necessary, to a subsequence, we may assume that $(u_j)$ converges weakly in $W^{1,p}(\mathbb{R}^n)$ to some function $u \in W^{1,p}_0(F)$ and that $(Au_j)$ converges weakly in $W^{-1,q}(\mathbb{R}^n)$ to some element $f$ of $W^{-1,q}(\mathbb{R}^n)$.

We want to prove that $Au = f$ in $W^{-1,q}(F)$ and that $u$ is a $C_A$-potential of $E$ in $F$. From the monotonicity condition (1.11) for every $j$ we obtain

\[(4.2) \quad \langle Av, v - u_j \rangle \geq \langle Au_j, v - u_j \rangle \quad \forall v \in W^{1,p}(\mathbb{R}^n).\]

If $j \geq i$, by Lemma 4.1 we have $u_j \leq u_i$ $C_p$-q.e. in $\mathbb{R}^n$. Since $Au_j \geq 0$ in $W^{-1,q}(F)$ (Lemma 3.13), we have

\[\langle Au_j, v - u_j \rangle \geq \langle Au_j, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F),\]

which, together with (4.2), gives

\[\langle Av, v - u_j \rangle \geq \langle Au_j, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F)\]

whenever $j \geq i$. Passing to the limit as $j \to \infty$ we obtain

\[\langle Av, v - u \rangle \geq \langle f, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F),\]

and as $i \to \infty$ we get

\[\langle Av, v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in W^{1,p}_0(F).\]

Putting $v = u + \varepsilon z$, with $z \in W^{1,p}_0(F)$ and $\varepsilon > 0$, and dividing by $\varepsilon$ we obtain

\[\langle A(u + \varepsilon z), z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F).\]

Passing to the limit as $\varepsilon \to 0$ we get

\[\langle Au, z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F),\]

hence $Au = f$ in $W^{-1,q}(F)$. Since $u_j = 1$ $C_p$-q.e. in $E$ and $u_j = 0$ $C_p$-q.e. in $F^c$ for every $j$ (Remark 3.6), we have $u = 1$ $C_p$-q.e. in $E$ and $u = 0$ $C_p$-q.e. in $F^c$. This shows that $u$ satisfies the first condition in (3.1).

It remains to prove that $Au = 0$ in $W^{-1,q}(F \setminus E)$. Let us fix $v \in W^{-1,q}(F \setminus E)$. Since the sets $E_j$ are $C_p$-quasi closed, by Lemma 1.2 there exists a sequence $(v_j)$ which
converges to \( v \) strongly in \( W^{1,p}(\mathbb{R}^n) \) and such that \( v_j \in W^{1,p}_0(F \setminus E_j) \) for every \( j \). As the sequence \( (E_j) \) is decreasing, we have \( v_i \in W^{1,p}_0(F \setminus E_j) \) for every \( j \geq i \). By the definition of \( u_j \) we have \( \langle Au_j, v_i \rangle = 0 \) for every \( j \geq i \). Since \( (Au_j) \) converges to \( f \) in \( W^{-1,q}(\mathbb{R}^n) \) and \( Au = f \) in \( W^{-1,q}(F \setminus E) \), as \( j \to \infty \) we get
\[
\langle Au, v_i \rangle = \langle f, v_i \rangle = \lim_{j \to \infty} \langle Au_j, v_i \rangle = 0.
\]

Passing to the limit as \( i \to \infty \) we obtain \( \langle Au, v \rangle = 0 \), hence \( Au = 0 \) in \( W^{-1,q}(F \setminus E) \).

By Remark 3.9 we have \( C_A(E, F) = \langle Au, u_1 \rangle \) and \( C_A(E_j, F) = \langle Au_j, u_1 \rangle \) for every \( j \). Using again the fact that \( (Au_j) \) converges to \( f \) weakly in \( W^{-1,q}(\mathbb{R}^n) \) and that \( Au = f \) in \( W^{-1,q}(F) \) we obtain
\[
C_A(E, F) = \langle Au, u_1 \rangle = \langle f, u_1 \rangle = \lim_{j \to \infty} \langle Au_j, u_1 \rangle = \lim_{j \to \infty} C_A(E_j, F),
\]
which concludes the proof of the theorem. \( \square \)

**Remark 4.9.** Elementary examples in the case \( p = 2 \) and \( Au = -\Delta u \) show that the conclusion of Theorem 4.8 does not hold if the sets \( E_j \) are not \( C_p \)-quasi closed. The assumption that \( E_1 \) and \( F \) are \( C_p \)-compatible is automatically satisfied if \( F \) is open and the sets \( E_j \) are compact and contained in \( F \).

We consider now the continuity properties with respect to \( F \). The following theorem shows that the set function \( C_A(E, \cdot) \) is continuous along all decreasing sequences.

**Theorem 4.10.** Let \( E \) and \( F \) be two bounded sets in \( \mathbb{R}^n \). If \( F \) is the intersection of a decreasing sequence of sets \( (F_j) \), then
\[
C_A(E, F) = \lim_{j \to \infty} C_A(E, F_j) = \sup_j C_A(E, F_j).
\]

**Proof.** It is enough to repeat the proof of Theorem 4.7 with obvious modifications. For instance we have to replace the minimal \( C_A \)-potentials by the maximal \( C_A \)-potentials, \( W^{1,p}_0(F) \) by \( \{ u \in W^{1,p}(\mathbb{R}^n) : u = 1 \text{ \( C_p \)-q.e. in } E \} \), and \( W^{-1,q}(F) \) by \( W^{-1,q}(F^c) \). \( \square \)

For the continuity of the set function \( C_A(E, \cdot) \) along increasing sequences, we need additional assumptions.
Theorem 4.11. Let $E$ and $F$ be two bounded set in $\mathbb{R}^n$. Assume that $F$ is the union of an increasing sequence $(F_j)$ of $C_p$-quasi open sets such that $E$ and $F_1$ are $C_p$-compatible. Then

$$C_A(E, F) = \lim_{j \to \infty} C_A(E, F_j) = \inf_j C_A(E, F_j).$$

Proof. It is enough to modify the proof of Theorem 4.8 as in the proof of Theorem 4.10.

\[\square\]

5. Approximation properties and subadditivity

In this section we prove that, if $E$ and $F$ are bounded Borel sets, then $C_A(E, F)$ can be approximated by $C_A(K, U)$, with $K$ compact, $K \subset E$, and $U$ bounded and open, $U \supset F$. Finally we prove that $C_A(E, F)$ is countably subadditive with respect to $E$.

We begin with the problem of the approximation of $C_A(E, F)$ by $C_A(K, F)$, with $K$ compact, $K \subset E$.

Lemma 5.1. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$. If $E$ is a Borel set, then

$$(5.1) \quad C_A(E, F) = \sup\{C_A(K, F) : K \text{ compact, } K \subset E\}.$$ 

Proof. Let $\psi$ be a function in $W_0^{1,p}(F)$ such that $\psi = 1$ $C_p$-q.e. in $E$, let $H = \{\psi \geq 1\}$, and let $\alpha$ be the set function defined by $\alpha(B) = C_A(B \cap H, F)$ for every $B \subset \mathbb{R}^n$. Since $H$ is $C_p$-quasi closed and $C_p$-compatible with $F$, the set function $\alpha$ satisfies the following properties:

(i) if $B \subset C$, then $\alpha(B) \leq \alpha(C)$ (Theorem 4.3);

(ii) if $B$ is the union of an increasing sequence of sets $(B_j)$, then $\alpha(B) = \sup_j \alpha(B_j)$ (Theorem 4.7);

(iii) if $K$ is the intersection of a decreasing sequence of compact sets $(K_j)$, then $\alpha(K) = \inf_j \alpha(K_j)$ (Theorem 4.8).
Therefore $\alpha$ is an abstract capacity in the sense of Choquet. By the Capacitability Theorem ([2], Theorem 1) for every Borel set $B \subset \mathbb{R}^n$ we have

\begin{equation}
\alpha(B) = \sup\{\alpha(K) : K \text{ compact}, K \subset B\}.
\end{equation}

Since $\psi = 1$ $C_p$-q.e. in $E$, we have $\alpha(B) = C_A(B, F)$ for every $B \subset E$ (Remark 3.10). Consequently (5.2) implies (5.1).

**Theorem 5.2.** Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. If $E$ is a Borel set, then

\begin{equation}
C_A(E, F) = \sup\{C_A(K, F) : K \text{ compact}, K \subset E\}.
\end{equation}

**Proof.** Let $\Omega$ be a bounded open set containing $E$ and $F$, let $D$ be a countable dense subset of $W^{1,p}_0(F)$, and let $F_0$ be the union of the sets $\{v \neq 0\}$ for $v \in D$. If $u \in W^{1,p}_0(F)$, then there exists a sequence $(v_j)$ in $D$ which converges to $u$ strongly in $W^{1,p}(\mathbb{R}^n)$. Since $v_j = 0$ $C_p$-q.e. in $F_0^c$, we have $u = 0$ $C_p$-q.e. in $F_0^c$. Therefore, if $B$ is a bounded set in $\mathbb{R}^n$ such that $C_p(B \setminus F_0, \Omega) > 0$, then $B$ and $F$ are not $C_p$-compatible. We may assume that all functions $v \in D$ are Borel functions, so that $F_0$ is a Borel set.

If $C_p(E \setminus F_0, \Omega) > 0$, then $C_A(E, F) = +\infty$ by the previous remark. If we apply Choquet’s Capacitability Theorem ([2], Theorem 1) to the capacity $C_p(\cdot, \Omega)$, we obtain that there exists a compact set $K$ contained in $E \setminus F_0$ such that $C_p(K \setminus F_0, \Omega) = C_p(K, \Omega) > 0$. This implies that $C_A(K, F) = +\infty$ and proves (5.3) in this case.

If $C_p(E \setminus F_0, \Omega) = 0$, then $C_A(E, F) = C_A(E \cap F_0, F)$ (Remark 3.10) and $E \cap F_0$ is the union of the sets $E \cap \{|v| > \frac{1}{k}\}$ for $k \in \mathbb{N}$ and $v \in D$. Since all these sets are $C_p$-compatible with $F$ (with $\psi = (k|v|) \wedge 1$), so are their finite unions (Remark 3.4). Therefore $E \cap F_0$ is the union of an increasing sequence $(E_j)$ of sets which are $C_p$-compatible with $F$. By Theorem 4.7 we have

$$C_A(E, F) = C_A(E \cap F_0, F) = \sup_j C_A(E_j, F),$$

and (5.3) follows from the fact that

$$C_A(E_j, F) = \sup\{C_A(K, F) : K \text{ compact}, K \subset E_j\}$$

by Lemma 5.1. \qed
We consider now the problem of the approximation of $C_A(E,F)$ by $C_A(E,U)$, with $U$ bounded and open, $U \supseteq F$.

**Lemma 5.3.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$. If $F$ is a Borel set, then

\[(5.4) \quad C_A(E,F) = \sup \{ C_A(E,U) : U \text{ bounded and open, } U \supseteq F \}.\]

**Proof.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ containing $F$, let $\psi$ be a function in $W_0^{1,p}(F)$ such that $\psi = 1$ $C_p$-q.e. in $E$, let $V = \{ \psi > 0 \}$, and let $\beta$ be the set function defined by $\beta(B) = C_A(E,(\Omega \setminus B) \cup V)$ for every $B \subset \mathbb{R}^n$. Since $V$ is $C_p$-quasi open and $E$ is $C_p$-compatible with $V$, the set function $\beta$ satisfies the following properties:

(i) if $B \subset C$, then $\beta(B) \leq \beta(C)$ (Theorem 4.5);

(ii) if $B$ is the union of an increasing sequence of sets $(B_j)$, then $\beta(B) = \sup_j \beta(B_j)$ (Theorem 4.10);

(iii) if $K$ is the intersection of a decreasing sequence of compact sets $(K_j)$, then $\beta(K) = \inf_j \beta(K_j)$ (Theorem 4.11).

Therefore $\beta$ is an abstract capacity in the sense of Choquet. By the Capacitability Theorem ([2], Theorem 1) for every Borel set $B \subset \mathbb{R}^n$ we have

\[(5.5) \quad \beta(B) = \sup \{ \beta(K) : K \text{ compact, } K \subset B \}.\]

Since $\psi = 0$ $C_p$-q.e. in $F^c$, we have $\beta(B) = C_A(E,\Omega \setminus B)$ for every $B \subset F^c$ (Remark 3.10). In particular $\beta(F^c) = C_A(E,F)$. Consequently (5.5) gives

$$C_A(E,F) = \sup \{ C_A(E,\Omega \setminus K) : K \text{ compact, } K \subset F^c \},$$

which implies (5.4). \qed

**Theorem 5.4.** Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. If $F$ is a Borel set, then

\[(5.6) \quad C_A(E,F) = \sup \{ C_A(E,U) : U \text{ bounded and open, } U \supseteq F \}.\]

**Proof.** Let $\Omega$ be a bounded open set containing $\overline{E}$ and $\overline{F}$, let $D$ be a countable dense subset of $H = \{ v \in W^{1,p}(\mathbb{R}^n) : v \geq 1 \text{ } C_p\text{-q.e. in } E \}$, and let $E_0$ be the intersection of
the sets \( \{ v \geq 1 \} \) for \( v \in D \). If \( u \in H \), then there exists a sequence \( (v_j) \) in \( D \) which converges to \( u \) strongly in \( W^{1,p}(\mathbb{R}^n) \). Since \( v_j \geq 1 \) \( C_p \)-q.e. in \( E_0 \), we have \( u \geq 1 \) \( C_p \)-q.e. in \( E_0 \). Therefore, if \( B \) is a bounded set in \( \mathbb{R}^n \) such that \( C_p(E_0 \setminus B, \Omega) > 0 \), then \( E \) and \( B \) are not \( C_p \)-compatible. We may assume that all functions \( v \in D \) are Borel functions, so that \( E_0 \) is a Borel set. As \( \overline{E} \subset \Omega \), the set \( E_0 \setminus \Omega \) is \( C_p \)-null, thus we may assume that \( E_0 \subset \Omega \).

If \( C_p(E_0 \setminus F, \Omega) > 0 \), then \( C_A(E, F) = +\infty \) by the previous remark. If we apply Choquet’s Capacitability Theorem ([2], Theorem 1) to the capacity \( C_p(\cdot, \Omega) \), we obtain that there exists a compact set \( K \) contained in \( E_0 \setminus F \) such that \( C_p(K, \Omega) > 0 \). As \( C_p(E_0 \setminus (\Omega \setminus K), \Omega) = C_p(K, \Omega) > 0 \), we obtain that \( C_A(E, \Omega \setminus K) = +\infty \) and (5.6) is proved.

If \( C_p(E_0 \setminus F, \Omega) = 0 \), then \( C_A(E, F) = C_A(E, F \cup E_0) \) (Remark 3.10) and \( F \cup E_0 \) is the intersection of the sets \( F \cup \{ v > 1 - \frac{k}{k+1} \} \) for \( k \in \mathbb{N} \) and \( v \in D \). Since \( E \) is \( C_p \)-compatible with all these sets (with \( \psi = (kv - k + 1)^+ \)), \( E \) is \( C_p \)-compatible with their finite intersections (Remark 3.4). Therefore \( F \cup E_0 \) is the intersection of a decreasing sequence \( (F_j) \) of sets such that \( E \) is \( C_p \)-compatible with \( F_j \) for every \( j \). By Theorem 4.10 we have

\[
C_A(E, F) = C_A(E, F \cup E_0) = \sup_j C_A(E, F_j),
\]

and (5.6) follows from the fact that

\[
C_A(E, F_j) = \sup\{ C_A(E, U) : U \text{ bounded and open}, U \supset F_j \}
\]

by Lemma 5.3. \( \square \)

We are now in a position to prove the main approximation theorem for \( C_A \).

**Theorem 5.5.** Let \( E \) and \( F \) be two bounded Borel sets in \( \mathbb{R}^n \). Then

\[
C_A(E, F) = \sup\{ C_A(K, U) : K \text{ compact}, K \subset E, U \text{ bounded and open}, U \supset F \}.
\]

**Proof.** The conclusion follows from Theorems 5.2 and 5.4. \( \square \)

We consider now the problem of the approximation of \( C_A(E, F) \) by \( C_A(U, F) \), with \( U \) bounded and \( C_p \)-quasi open, \( U \supset E \).
Proposition 5.6. Let $E$ and $F$ be two bounded sets in $\mathbb{R}^n$. Then

(5.7) \[ C_A(E, F) = \inf \{ C_A(U, F) : U \text{ bounded and } C_p\text{-quasi open, } U \supseteq E \}. \]

Proof. Let $I$ be the right hand side of (5.7). By monotonicity we have $C_A(E, F) \leq I$. Let us prove the opposite inequality when $C_A(E, F) < +\infty$, and hence $E$ and $F$ are $C_p$-compatible. Let $u$ be the minimal $C_A$-potential of $E$ in $F$. For every $k \in \mathbb{N}$ let $E_k = \{ u \geq 1 - \frac{1}{2^k} \}$, $U_k = \{ u > 1 - \frac{1}{2^k} \}$, and let $E_0 = \{ u = 1 \}$. By Remark 3.7 we can write $E_0 = \{ u \geq 1 \}$, hence $E_0$ is the intersection of the decreasing sequence $(E_k)$. It is easy to see that $u$ is a $C_A$-potential of $E_0$ in $F$, hence

\[ C_A(E_0, F) = \int_F (a(x, Du), Du) \, dx = C_A(E, F). \]

Since $E_1$ and $F$ are $C_p$-compatible, with $\psi = (2u) \wedge 1$, and all sets $E_k$ are $C_p$-quasi closed, by Theorem 4.8 we have

\[ C_A(E, F) = C_A(E_0, F) = \inf_k C_A(E_k, F). \]

Since the sets $U_k$ are $C_p$-quasi open and $E \subset U_k \subset E_k$ (up to a $C_p$-null set), we have

\[ C_A(E, F) = \inf_k C_A(E_k, F) \geq \inf_k C_A(U_k, F) \geq I, \]

which concludes the proof.

We are now in a position to prove the subadditivity of the capacity $C_A(\cdot, F)$.

Theorem 5.7. Let $E_1$, $E_2$, $F$ be three bounded sets in $\mathbb{R}^n$. Then

(5.8) \[ C_A(E_1 \cup E_2, F) \leq C_A(E_1, F) + C_A(E_2, F). \]

Proof. The inequality is trivial if $C_A(E_1, F) = +\infty$ or $C_A(E_2, F) = +\infty$. Therefore we may assume that $E_1$ and $E_2$ are $C_p$-compatible with $F$. By Proposition 5.6 for every $\varepsilon > 0$ there exist two $C_p$-quasi open sets $U_1$ and $U_2$ such that $E_1 \subset U_1$, $E_2 \subset U_2$, and

(5.9) \[ C_A(U_1, F) + C_A(U_2, F) \leq C_A(E_1, F) + C_A(E_2, F) + \varepsilon. \]
By Lemma 1.5 there exist two increasing sequences \((K_1^j)\) and \((K_2^j)\) of compact sets, contained in \(U_1\) and \(U_2 \setminus U_1\) respectively, whose unions cover \(C_p\)-quasi all of \(U_1\) and \(U_2 \setminus U_1\). By Remark 3.10 and by Theorems 4.3 and 4.7 we have

\[
C_A(E_1 \cup E_2, F) \leq C_A(U_1 \cup U_2, F) = \lim_{j \to \infty} C_A(K_1^j \cup K_2^j, F).
\]

Since by monotonicity (Theorem 4.3)

\[
C_A(K_1^j, F) + C_A(K_2^j, F) \leq C_A(U_1, F) + C_A(U_2, F),
\]

in view of (5.9) and (5.10) it is enough to prove that for every \(j\) we have

\[
C_A(K_1^j \cup K_2^j, F) \leq C_A(K_1^j, F) + C_A(K_2^j, F).
\]

Let us fix \(j\) and let \(K_1 = K_1^j\) and \(K_2 = K_2^j\). As the compact sets \(K_1\) and \(K_2\) are disjoint, there exist two disjoint open set \(V_1\) and \(V_2\) such that \(K_1 \subset V_1\) and \(K_2 \subset V_2\).

Let \(u_1\), \(u_2\), and \(u\) be the minimal \(C_A\)-potentials of \(K_1\), \(K_2\), and \(K_1 \cup K_2\) in \(F\), and let \(\lambda_1\), \(\lambda_2\), and \(\lambda\) be the corresponding inner \(C_A\)-distributions. We want to prove that

\[
\lambda(B \cap K_1) \leq \lambda_1(B) \quad \text{and} \quad \lambda(B \cap K_2) \leq \lambda_2(B) \quad \text{for every Borel set } B \subset \mathbb{R}^n.
\]

By Lemma 4.1 we have \(u_1 \leq u\) and \(u_2 \leq u\) \(C_p\)-q.e. in \(\mathbb{R}^n\). If we apply Lemma 2.5 with \(B = K_1\), \(C = V_1 \cap F\), \(w_1 = u_1\), \(w_2 = u\), we obtain \(Au \leq Au_1\) in \(W^{-1,q}(V_1 \cap F)\). By properties (d) and (h) of Theorem 3.14 we have

\[
\int_{\mathbb{R}^n} v \, d\lambda = \langle Au, v \rangle \leq \langle Au_1, v \rangle = \int_{\mathbb{R}^n} v \, d\lambda_1
\]

for every \(v \in W_0^{1,p}(V_1 \cap F)\) with \(v \geq 0\) \(C_p\)-q.e. in \(\mathbb{R}^n\). By Lemma 1.1 this implies \(\lambda(V) \leq \lambda_1(V)\) for every \(C_p\)-quasi open set \(V\) in \(V_1 \cap F\). In particular, since \(\{u_1 > 0\}\) is \(C_p\)-quasi open and is contained in \(F\), we have \(\lambda(U \cap V_1 \cap \{u_1 > 0\}) \leq \lambda_1(U \cap V_1 \cap \{u_1 > 0\})\) for every open set \(U\) in \(\mathbb{R}^n\). As \(\lambda\) and \(\lambda_1\) are Radon measures, this implies that \(\lambda(B) \leq \lambda_1(B)\) for every Borel set \(B \subset V_1 \cap \{u_1 > 0\}\). Since \(u_1 = 1\) \(C_p\)-q.e. in \(K_1\) and \(K_1 \subset V_1\), we have \(\lambda(B \cap K_1) \leq \lambda_1(B \cap K_1) \leq \lambda_1(B)\) for every Borel set \(B \subset \mathbb{R}^n\), which proves the first inequality in (5.12). The other inequality is proved in a similar way.

Since \(\text{supp } \lambda \subset K_1 \cup K_2\) (Theorem 3.14(f)), by (5.12) we have \(\lambda(\mathbb{R}^n) = \lambda(K_1) + \lambda(K_2) \leq \lambda_1(\mathbb{R}^n) + \lambda_2(\mathbb{R}^n)\), which gives (5.11) by Proposition 3.17. \(\square\)
When $\Omega$ is a bounded open set in $\mathbb{R}^n$ and $E \subset \Omega$, then $C_A(E, \Omega)$ can be approximated by $C_A(U, \Omega)$, with $U$ open, $E \subset U \subset \Omega$.

**Proposition 5.8.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and let $E \subset \Omega$. Then
\begin{equation}
C_A(E, \Omega) = \inf\{C_A(U, \Omega) : U \text{ open, } E \subset U \subset \Omega\}.
\end{equation}

**Proof.** Let $I$ be the right hand side of (5.13). Since $C_A(E, F) \leq I$ by monotonicity, we have only to prove the opposite inequality. By Theorem 4.3 and by Proposition 5.6 for every $\varepsilon > 0$ there exists a $C_p$-quasi open set $V$ such that $E \subset V \subset \Omega$ and
\begin{equation}
C_A(V, \Omega) \leq C_A(E, \Omega) + \varepsilon.
\end{equation}
Since $V$ is $C_p$-quasi open, there exists an open set $U$ contained in $\Omega$ such that $C_p(U \triangle V, \Omega) < \varepsilon$, and by (1.1) there exists an open set $W$ contained in $\Omega$ such that $U \triangle V \subset W$ and $C_p(W, \Omega) < \varepsilon$. As $V \cup W = U \cup W$, the set $V \cup W$ is open. By subadditivity (Theorem 5.7) we have
\begin{equation*}
I \leq C_A(V \cup W, \Omega) \leq C_A(V, \Omega) + C_A(W, \Omega).
\end{equation*}
By (3.3) and (5.14) we have
\begin{equation*}
I \leq C_A(V \cup W, \Omega) \leq C_A(E, \Omega) + (1 + k_1)\varepsilon + k_2(\Omega)\varepsilon^{1/p},
\end{equation*}
where $k_1$ and $k_2(\Omega)$ are the constants defined in (3.4). Since $\varepsilon > 0$ is arbitrary, we obtain $I \leq C_A(E, \Omega)$.

We conclude by proving that $C_A(\cdot, F)$ is countably subadditive.

**Theorem 5.9.** Let $E$ and $F$ be a bounded set in $\mathbb{R}^n$ and let $(E_j)$ be a sequence of bounded sets in $\mathbb{R}^n$. If $E$ is contained in the union of the sequence $(E_j)$, then
\begin{equation}
C_A(E, F) \leq \sum_{j=1}^{\infty} C_A(E_j, F).
\end{equation}

**Proof.** For every $k \in \mathbb{N}$ let $B_k = E \cap E_k$ and let $G_k = B_1 \cup \ldots \cup B_k$. By Theorems 4.3 and 5.7 for every $k$ we have
\begin{equation*}
C_A(G_k, F) \leq \sum_{j=1}^{k} C_A(B_j, F) \leq \sum_{j=1}^{\infty} C_A(E_j, F).
\end{equation*}
Since $E$ is the union of the increasing sequence $(G_k)$, the continuity along increasing sequences (Theorem 4.7) implies (5.15).
6. Capacity relative to a constant

In this section we define the capacity $C_A(E, F, s)$ with respect to a constant $s$ by replacing the condition $u = 1$ in $\partial E$ which appears in (0.3) with the condition $u = s$ in $\partial E$.

**Definition 6.1.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$, let $s$ be a real number, and let $\psi$ be a function in $W^{1, p}(\mathbb{R}^n)$ such that $\psi = 1$ $C_p$-q.e. in $E$ and $\psi = 0$ $C_p$-q.e. in $F^c$. Every solution $u$ of the Dirichlet problem

$$
\begin{cases}
    u - s\psi \in W^{1, p}_0(F \setminus E), \\
    Au = 0 \quad \text{in } W^{-1, q}(F \setminus E),
\end{cases}
$$

is called a $C_A$-potential of $E$ in $F$ relative to the constant $s$. The maximal and the minimal solutions of (6.1) are called the maximal and minimal $C_A$-potentials of $E$ in $F$ relative to the constant $s$.

**Remark 6.2.** Clearly the previous definition does not depend on the choice of $\psi$. By the definition of the space $W^{1, p}_0(F \setminus E)$ and by the properties of $\psi$ we have that every $C_A$-potential $u$ of $E$ in $F$ relative to the constant $s$ satisfies $u = s$ $C_p$-q.e. in $E$ and $u = 0$ $C_p$-q.e. in $F^c$.

**Definition 6.3.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $s \in \mathbb{R}$. The capacity of $E$ in $F$ relative to the operator $A$ and to the constant $s$ is defined as

$$
C_A(E, F, s) = \langle Au, u \rangle = \int_{F \setminus E} (a(x, Du), Du) \, dx,
$$

where $u$ is any $C_A$-potential of $E$ in $F$ relative to the constant $s$. By Corollary 2.8 this definition is independent of the choice of $u$.

**Remark 6.4.** Let $s \neq 0$ and let $a_s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by

$$
a_s(x, \xi) = s \, a(x, s\xi).
$$

Then $a_s$ satisfies conditions (1.4)–(1.7) with $c_1$, $c_2$, and $b_2$ replaced by $|s|^p c_1$, $|s|^p c_2$, and $|s|^b_2$. Let $A_s$ be the operator defined by (1.9) with $a$ replaced by $a_s$. Then $u$ is a
$C_A$-potential of $E$ in $F$ relative to the constant $s$ if and only if $u/s$ is a $C_{A_s}$-potential of $E$ in $F$ according to Definition 3.5, and

$$C_A(E, F, s) = C_{A_s}(E, F).$$

This shows that all properties of $C_A(E, F)$ proved in Sections 3–5 are still valid for $C_A(E, F, s)$, with some obvious modifications, for every $s \in \mathbb{R}$.

**Remark 6.5.** From Lemma 3.13 and Remark 6.4 we obtain that, if $u$ is a $C_A$-potential of $E$ in $F$ relative to a constant $s > 0$, then $Au \geq 0$ in $W^{-1,q}(F)$ and $Au \leq 0$ in $W^{-1,q}(E^c)$, whereas the inequalities are reversed when $s < 0$. If $s = 0$, by (1.4) the function 0 is a $C_A$-potential of $E$ in $F$ relative to the constant 0. If $u$ is another $C_A$-potential of $E$ in $F$ relative to the constant 0, then $Au = 0$ in $W^{-1,q}(\mathbb{R}^n)$ by Theorem 2.6.

If $Au = -\text{div}(|Du|^{p-2}Du)$, then $C_A(E, F, s) = |s|^p C_p(E, F)$. In the general case the relationship between $C_A(E, F, s)$ and $C_p(E, F)$ is given by the following proposition, which follows immediately from Proposition 3.11 and Remark 6.4.

**Proposition 6.6.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $s \in \mathbb{R}$. Then

\[(6.2) \quad C_A(E, F, s) \geq |s|^p c_1 C_p(E, F) - \|b_1\|_{L^1(F)}, \]

\[(6.3) \quad C_A(E, F, s) \leq |s|^p k_1 C_p(E, F) + |s| k_2(F) C_p(E, F)^{1/p}, \]

where $c_1$, $k_1$ and $k_2(F)$ are the constants which appear in (1.6) and (3.4). If $b_1$ and $b_2$ belong to $L^\infty(F)$, then

$$C_A(E, F, s) \leq (|s|^p k_1 + |s| k_3(F)) C_p(E, F),$$

where $k_3(F)$ is defined in (3.6).

The following lemma is an immediate consequence of the Comparison Principle (Theorem 2.11).

**Lemma 6.7.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$, let $s_1$ and $s_2$ be two real numbers, and let $u_1$ and $u_2$ be the maximal (or minimal) $C_A$-potentials of $E$ in $F$ relative to the constants $s_1$ and $s_2$ respectively. If $s_1 \leq s_2$, then $u_1 \leq u_2$ $C_p$-q.e. in $\mathbb{R}^n$. 
Definition 6.8. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $s \in \mathbb{R}$. We define
\[
\hat{C}_A(E,F,s) = \begin{cases} 
\frac{1}{s}C_A(E,F,s), & \text{if } s \neq 0, \\
0, & \text{if } s = 0.
\end{cases}
\]

Remark 6.9. By (6.1) we have $\hat{C}_A(E,F,s) = \langle Au, v \rangle$ for every $C_A$-potential of $E$ in $F$ relative to the constant $s$ and for every function $v$ in $W^{1,p}(\mathbb{R}^n)$ such that $v = 1$ $C_p$-a.e. in $E$ and $v = 0$ $C_p$-a.e. in $F^c$ (see Remark 6.5 for the case $s = 0$).

We prove now that $\hat{C}_A(E,F,s)$ depends continuously on $s$.

Theorem 6.10. Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$. Then the function $s \mapsto \hat{C}_A(E,F,s)$ is continuous on $\mathbb{R}$.

Proof. Let $\mathbb{R}_+ = \{ s \in \mathbb{R} : s \geq 0 \}$ and $\mathbb{R}_- = \{ s \in \mathbb{R} : s \leq 0 \}$. We prove only that $s \mapsto \hat{C}_A(E,F,s)$ is continuous on $\mathbb{R}_+$, the proof for $\mathbb{R}_-$ being analogous. We begin by proving the right continuity on $\mathbb{R}_+$. Let us fix $s \geq 0$ and let $(s_j)$ be a decreasing sequence in $\mathbb{R}$ converging to $s$. Let $u$ and $u_j$ be the maximal $C_A$-potentials of $E$ in $F$ relative to the constants $s$ and $s_j$ respectively. As $C_p(E,F) < +\infty$, by (6.3) and (1.13) the sequence $(u_j)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, and by (1.10) the sequence $(Au_j)$ is bounded in $W^{-1,q}(\mathbb{R}^n)$. Passing, if necessary, to a subsequence, we may assume that $(u_j)$ converges weakly in $W^{1,p}(\mathbb{R}^n)$ to some function $w \in W^{1,p}_0(F)$ and that $(Au_j)$ converges weakly in $W^{-1,q}(\mathbb{R}^n)$ to some element $f$ of $W^{-1,q}(\mathbb{R}^n)$. By Lemma 6.7 we have $u \leq u_j \leq u_i$ $C_p$-a.e. in $\mathbb{R}^n$ for every $j \geq i$, hence $u \leq w \leq u_i$ $C_p$-a.e. in $\mathbb{R}^n$. Since $u = s$ and $u_i = s_i$ $C_p$-a.e. in $E$ (Remark 6.2), as $i \to \infty$ we obtain that $w = s$ $C_p$-a.e. in $E$.

We want to prove that $f = Aw$ in $W^{-1,q}(F)$ and that $w = u$ $C_p$-a.e. in $\mathbb{R}^n$. From the monotonicity condition (1.11) for every $j$ we obtain
\[
(6.4) \quad \langle Av, v - u_j \rangle \geq \langle Au_j, v - u_j \rangle \quad \forall v \in W^{1,p}(\mathbb{R}^n).
\]

If $j \geq i$, by Lemma 6.7 we have $u_j \leq u_i$ $C_p$-a.e. in $\mathbb{R}^n$. Since $Au_j \geq 0$ in $W^{-1,q}(F)$ (Remark 6.5), we have
\[
(6.5) \quad \langle Au_j, v - u_j \rangle \geq \langle Au_j, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F),
\]
which, together with (6.4), gives

$$\langle Av, v - u_j \rangle \geq \langle Au_j, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F)$$

whenever \(j \geq i\). Passing to the limit as \(j \to \infty\) we obtain

$$\langle Av, v - w \rangle \geq \langle f, v - u_i \rangle \quad \forall v \in W^{1,p}_0(F),$$

and as \(i \to \infty\) we get

(6.6) $$\langle Av, v - w \rangle \geq \langle f, v - w \rangle \quad \forall v \in W^{1,p}_0(F).$$

Putting \(v = w + \varepsilon z\), with \(z \in W^{1,p}_0(F)\) and \(\varepsilon > 0\), and dividing by \(\varepsilon\) we obtain

$$\langle A(w + \varepsilon z), z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F).$$

Passing to the limit as \(\varepsilon \to 0\) we get

$$\langle Aw, z \rangle \geq \langle f, z \rangle \quad \forall z \in W^{1,p}_0(F),$$

hence \(Aw = f\) in \(W^{-1,q}(F)\).

By the definition of \(u_j\) we have \(\langle Au_j, v \rangle = 0\) for every \(v \in W^{1,p}_0(F \setminus E)\) and for every \(j\). As \(j \to \infty\) we obtain

$$\langle Aw, v \rangle = \langle f, v \rangle = \lim_{j \to \infty} \langle Au_j, v \rangle = 0 \quad \forall v \in W^{1,p}_0(F \setminus E),$$

hence \(w\) is a \(C_A\)-potential of \(E\) in \(F\) relative to the constant \(s\). Since \(u \leq w\) \(C_p\)-q.e. in \(\mathbb{R}^n\), by the maximality of \(u\) we obtain \(u = w\) \(C_p\)-q.e. in \(\mathbb{R}^n\).

Let \(\psi\) be a function in \(W^{1,p}(\mathbb{R}^n)\) such that \(\psi = 1\) \(C_p\)-q.e. in \(E\) and \(\psi = 0\) \(C_p\)-q.e. in \(F^c\). Since \((Au_j)\) converges to \(f\) weakly in \(W^{-1,q}(\mathbb{R}^n)\) and \(Au = f\) in \(W^{-1,q}(F)\), by Remark 6.9 we have

$$\hat{C}_A(E, F, s) = \langle Au, \psi \rangle = \langle f, \psi \rangle = \lim_{j \to \infty} \langle Au_j, \psi \rangle = \lim_{j \to \infty} \hat{C}_A(E, F, s_j).$$

This proves that the function \(s \mapsto \hat{C}_A(E, F)\) is right continuous on \(\mathbb{R}_+\). For the proof of the left continuity on \(\mathbb{R}_+\), we fix \(s > 0\) and an increasing sequence \((s_j)\) converging to \(s\). We may assume that \(s_j > 0\) for every \(j\). Then we use the same arguments as in the first part of the proof, with the only difference that now we use the minimal \(C_A\)-potentials instead of the maximal \(C_A\)-potentials. As \((s_j)\) is increasing, the sequence \((u_j)\) is increasing, and, consequently, \(u_i\) must be replaced by \(w\) in (6.5) and we obtain directly (6.6). The final part of the proof remains unchanged. \(\square\)
We prove now that $\hat{C}_A(E, F, s)$ is increasing with respect to $s$.

**Theorem 6.11.** Let $E$ and $F$ be two $C_p$-compatible bounded sets in $\mathbb{R}^n$ and let $s_1$ and $s_2$ be two real numbers with $s_1 < s_2$. Then $\hat{C}_A(E, F, s_1) \leq \hat{C}_A(E, F, s_2)$.

**Proof.** Let $u_1$ and $u_2$ be two $C_A$-potentials of $E$ in $F$ relative to the constants $s_1$ and $s_2$ respectively, let $t$ be a real number such that $0 < t < s_2 - s_1$, and let $v$ be the function of $W^{1,p}(\mathbb{R}^n)$ defined by $v = \frac{1}{t}((u_2 - u_1) \wedge t)$. By Remark 6.2 we have $v = 1$ $C_p$-a.e. in $E$ and $v = 0$ $C_p$-a.e. in $F^c$. Therefore, Remark 6.9 implies that

$$\hat{C}_A(E, F, s_2) - \hat{C}_A(E, F, s_1) = \langle Au_2 - Au_1, v \rangle = \frac{1}{t} \int_{\{u_2 - u_1 < t\}} (a(x, Du_2) - a(x, Du_1), Du_2 - Du_1) \, dx,$$

and the conclusion follows from the monotonicity condition (1.5).

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