EXTREMAL METRICS ON BLOW UPS

C. AREZZO, F. PACARD, AND M. SINGER

1. Introduction

In this paper we study the problem of constructing extremal Kähler metrics on blow ups at finitely many points of Kähler manifolds which already carry an extremal metric.

In [8], [9] Calabi has proposed, as best representatives of a given Kähler class $[\omega]$ of a complex compact manifold $(M,J)$, a special type of metrics baptized extremal. These metrics are critical points of the $L^2$-square norm of the scalar curvature $s$. The corresponding Euler-Lagrange equation reduces to the fact that

$$\Xi_s := J \nabla s + i \nabla s$$

is a holomorphic vector field on $M$. In particular, the set of extremal metrics contains the set of constant scalar curvature Kähler ones. Calabi’s intuition of looking at extremal metrics as canonical representatives of a given Kähler class has found a number of important confirmations and also (unfortunately) nontrivial constraints. Calabi himself proved that an extremal Kähler metric must have the maximal possible symmetry allowed by the complex manifold $M$, and, as observed by LeBrun and Simanca [17], this symmetry group can be fixed in advance. More precisely, the identity component of the isometry group of any extremal metric $g$ must be a maximal compact subgroup of $\text{Aut}_0(M,J)$, the identity component of the group $\text{Aut}(M,J)$ of biholomorphic maps of $M$ to itself. This group thus contains the complexification of the isometry group, but may be strictly larger (the blow-up of $\mathbb{P}^2$ at a point is the simplest example of such a situation). Moreover Lebrun and Simanca [15] have proved that the set of Kähler classes having an extremal representative is an open subset of $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$ and Chen and Tian [10] have proved the uniqueness of such metrics in a given Kähler class up to automorphisms. Also, the important relationship between the existence of extremal metrics and various stability notions of the corresponding polarized manifolds (algebraic if the class is rational, analytic otherwise) has been deeply investigated for example by Tian [31], Mabuchi [21] and Székelyhidi [29]. Yet, a complete understanding of the existence theory for extremal metrics is still missing. Given this last fact, the first two authors have started in [1] and [2] to develop a perturbation theory for constant scalar curvature Kähler metrics, giving sufficient conditions for the existence of constant scalar curvature Kähler metrics on the blow up at finitely many points of a manifold which already carries a constant scalar curvature Kähler metric. The aim of the present paper is to extend these results to the framework of extremal metrics.

1
2. Statement of the result

Let \((M, J, \omega)\) be a Kähler manifold with complex structure \(J\) and Kähler form \(\omega\) and let \(g\) denote the metric associated to the Kähler form \(\omega\), so that \(\omega(X, Y) = g(JX, Y)\).

Further assume that \(g\) is an extremal metric. Since the automorphism group of any blow up of \(M\) can be identified with a subgroup of \(\text{Aut}(M, J)\), and in light of the above mentioned result of Calabi-LeBrun-Simanca about the isometry group of any extremal metric, our strategy is to fix \textit{a priori} a compact subgroup \(K\) of \(\text{Isom}(M, g)\) and work \(K\)-equivariantly. Such a \(K\) will then be contained in the isometry group of the extremal metric we are seeking on the blow up of \(M\) at any set of points \(p_1, \ldots, p_n \in M\) in the \(\text{Fix}(K_0)\), the fixed locus of the identity component \(K_0\) of \(K\). We will denote by \(k\) the Lie algebra associated to identity component of \(K\). Observe that elements of \(k\) vanish at the points \(p_1, \ldots, p_n\) to be blown up and hence these vector fields can be lifted to the blown up manifold.

In order to produce extremal metrics on the blown up manifold, we have to identify, among all \(C^\infty\) functions on the blown up manifold, those who generate real-holomorphic vector fields, since these can arise as scalar curvatures of extremal metrics. To this aim, we define \(h\) to be the vector space of \(K\)-invariant hamiltonian real-holomorphic vector fields on \(M\) or equivalently, the Lie algebra of the group \(H\) of exact simplectomorphisms commuting with \(K\). The correspondence between real-holomorphic vector fields and the scalar functions on \(M\) can be encoded in a compact way in a moment map \(\xi_\omega: M \to h^*\)

\[\xi_\omega: M \to h^*\]

for the action of \(H\) uniquely determined by imposing to have mean zero. More explicitly the function \(f := \langle \xi_\omega, X \rangle\) associated to the vector field \(X \in h\) is defined to be the unique solution of

\[-df = \omega(X, -)\]

whose mean value over \(M\) is 0.

Using this setup, our main result reads:

**Theorem 2.1.** Let \((M, J, \omega)\) be a compact \(m\)-dimensional Kähler manifold whose associated Kähler metric \(g\) is extremal, and let \(K\) be a compact subgroup of \(\text{Isom}(M, g)\) whose Lie algebra contains the vector field \(J\nabla s\) as well as any element of \(h\). Let \(K_0\) denote the identity component of \(K\).

Given \(p_1, \ldots, p_n \in \text{Fix}(K_0)\) and \(a_1, \ldots, a_n > 0\) such that \(a_{j_1} = a_{j_2}\) if \(p_{j_1}\) and \(p_{j_2}\) are in the same \(K\)-orbit, there exists \(\varepsilon_0 > 0\) and, for all \(\varepsilon \in (0, \varepsilon_0)\), there exists a \(K\)-invariant extremal Kähler metric \(\omega_\varepsilon\) on \(\tilde{M}\), the blow up of \(M\) at \(p_1, \ldots, p_n\), such that its associated Kähler form \(\omega_\varepsilon\) lies in the class

\[\pi^*[\omega] - \varepsilon^2 \left( \frac{1}{a_1} \text{PD}[E_1] + \ldots + \frac{1}{a_n} \text{PD}[E_n] \right)\]

where \(\pi: \tilde{M} \to M\) is the standard projection map, the \(\text{PD}[E_i]\) are the Poincaré duals of the \((2m - 2)\)-homology classes of the exceptional divisors of the blow up at \(p_j\).
Finally, the sequence of metrics \((g_\varepsilon)\varepsilon\) converges to \(g\) (in smooth topology) on compacts, away from the exceptional divisors.

It is important to stress that our analytical construction does not give one extremal metric but a family converging to the starting metric on the base manifold. For such a construction to work, it is necessary to have \(p_1, \ldots, p_n \in \text{Fix}(K_0)\) and \(J \nabla s \in \mathfrak{t}\). On the other hand the condition \(\mathfrak{h} \subset \mathfrak{t}\), while often satisfied in important examples such as toric manifolds with \(K\) giving the torus action, is certainly far from being necessary. We give a simple geometric sufficient condition for it to hold in Proposition 7.4.

In the general case when \(\mathfrak{h}\) is not included in \(\mathfrak{t}\), there is a natural decomposition \(\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''\), where \(\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{t}\) is the subspace of \(K\)-invariant real-holomorphic vector fields in \(\mathfrak{t}\). The previous result then appears as a special case of the more general:

**Theorem 2.2.** Assume that \((M, J, \omega)\) is a compact Kähler manifold whose associated Kähler metric \(g\) is extremal, and let \(K\) be a compact subgroup of Isom\((M, g)\) whose Lie algebra contains the vector field \(J \nabla s\). Let \(K_0\) denote the identity component of \(K\). We decompose the space of \(K\)-invariant Hamiltonian real-holomorphic vector fields \(\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''\) where \(\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{t})\). Given \(p_1, \ldots, p_n \in \text{Fix}(K_0)\) such that:

(i) there exists \(a_1, \ldots, a_n > 0\) satisfying

\[\sum_j a_j \xi_\omega(p_j) \in \mathfrak{h}'^*\]

and \(a_1, \ldots, a_n > 0\) such that \(a_{j_1} = a_{j_2}\) if \(p_{j_1}\) and \(p_{j_2}\) are in the same \(K\)-orbit,

(ii) the projections of \(\xi_\omega(p_1), \ldots, \xi_\omega(p_n)\) over \(\mathfrak{h}''^*\) span \(\mathfrak{h}''^*\),

(iii) there are no nontrivial elements of \(\mathfrak{h}''\) that vanishes at \(p_1, \ldots, p_n\),

there exists \(\varepsilon_0 > 0\) and, for all \(\varepsilon \in (0, \varepsilon_0)\), there exists a \(K\)-invariant extremal Kähler metric \(g_\varepsilon\) on \(\tilde{M}\), the blow up of \(M\) at \(p_1, \ldots, p_n\), whose associated Kähler form \(\omega_\varepsilon\) lies in the class

\[\pi^* [\omega] - \varepsilon^2 \left( \frac{1}{a_1^{m-1}} PD[E_1] + \ldots + \frac{1}{a_n^{m-1}} PD[E_n] \right)\]

where \(\pi: \tilde{M} \to M\) is the standard projection map, the \(PD[E_i]\) are the Poincaré duals of the \((2m-2)\)-homology classes of the exceptional divisors of the blow up at \(p_j\).

Finally, the sequence of metrics \((g_\varepsilon)\varepsilon\) converges to \(g\) (in smooth topology) on compacts, away from the exceptional divisors.

When \(\mathfrak{h} \subset \mathfrak{t}\), \(\mathfrak{h}' = \mathfrak{h}\) and \(\mathfrak{h}'' = \{0\}\), hence (i), (ii) and (iii) become vacuous and Theorem 2.2 reduces to Theorem 2.1. In §4 we explain how conditions (i)-(iii) arise in our analytical approach.

**Remark 2.1.** Condition (iii) can be removed if we leave some freedom on the weights of the exceptional divisor on the blown up manifold. More precisely, Theorem 2.2 still holds without
assuming (iii) but in this case, the only information we have about \([\omega_\varepsilon]\) reads
\[
\omega_\varepsilon \in \pi^*[\omega] - \varepsilon^2 \left( \frac{1}{\tilde{a}_1^{m-1}} PD[E_1] + \ldots + \frac{1}{\tilde{a}_n^{m-1}} PD[E_n] \right)
\]
where \(\tilde{a}_1, \ldots, \tilde{a}_n > 0\) depend on \(\varepsilon\) and satisfy
\[
|\tilde{a}_j - a_j| \leq c \varepsilon^{2m+1}.
\]
In other words, by removing (iii) we slightly loose the control on the Kähler classes.

Theorem 2.2 is a generalization of the constructions given in [1] and [2]. Indeed, in [1] is treated the case where \(g\) is a constant scalar curvature Kähler metric \(K = \{Id\}\) and where \(\mathfrak{h} = \{0\}\), while in [2] is treated the case where \(g\) is a constant scalar curvature Kähler metric, \(K\) is a discrete subgroup of \(\text{Isom}(M,g)\), \(\mathfrak{h}' = \{0\}\), and \(\mathfrak{h}''\) is not necessarily trivial.

The choice of the symmetry groups \(K\) is a really delicate problem. Indeed, given the fact that the blown up points have to be chosen in \(\text{Fix}(K_0)\) it is rather natural to choose \(K_0\) to be fairly small, so that its fixed-point set is large. However, the smaller \(K_0\) the larger \(\mathfrak{h}\) and hence the harder it is to fulfill the requirement that \(\mathfrak{h} \subset \mathfrak{k}\) in Theorem 2.1 or the requirement that conditions (i) and (ii) in Theorem 2.2 are fulfilled. Conditions (i) and (ii) are of course difficult to check given the fact that the moment map \(\xi_\omega\) is in general hard to analyze. Nevertheless, there are large classes of manifolds, notably all toric ones, for which computations can be done. Concrete examples listed in section §11 will illustrate how delicate this issue is.

Once the necessary notations and definitions are introduced, in §4 we will show how conditions (i)-(iii) naturally arise in our construction of the converging family of extremal metrics \(g_\varepsilon\).

Remark 2.2. Condition (i) and (ii) should be related to Mabuchi’s \(T\)-stability [21] and to Szekelyhidi’s relative \(K\)-stability [29] in the same way the analogue conditions for constant scalar curvature metrics are related to the asymptotic Chow semi-stability along the line of ideas described by Thomas in [30] (pages 27 and 28). Indeed, if instead of fixing the group \(K\) a priori, we fix the set of points \(\{p_1, \ldots, p_n\}\) to be blown up, a natural choice of \(K\) would be any maximal torus in the subgroup of \(\text{Isom}(M,g)\) fixing each \(p_j\). We believe that with these choices, conditions (i) and (ii) should be equivalent to the relative \(K\)-stability of the blown up manifold, when the resulting Kähler classes are rational (which in turn should be equivalent to a relative GIT stability of the configurations of points in \(M^n\) [12]). Moreover let us observe that one can apply our construction to any extremal representative of the class \([\omega]\). While loosing control on the explicit shape of the metric \(g_\varepsilon\), this clearly allows to find families of extremal representatives in \([\omega_\varepsilon]\), and gives some flexibility on the choice of points and weights for which our construction works for some representative in \([\omega]\). This flexibility is indeed connected to the above mentioned stability question, and it will be investigated in detail in [3]. A first simple appearance of this freedom will be used below in the case of projective spaces.

If the initial manifold has constant scalar curvature, it might well be that the extremal metrics we obtain are in fact constant scalar curvature metrics. There is a simple criterion involving the points \(p_1, \ldots, p_n\) and the parameters \(a_1, \ldots, a_n\), which ensures that this is not the case.
Proposition 2.1. Under the assumptions of Theorem 2.2 (or Theorem 2.1), if \( \sum_j a_j \xi(p_j) \neq 0 \) then the metrics we obtain on \( \tilde{M} \) are extremal with nonconstant scalar curvature.

We now emphasize the consequences of the above results for projective spaces and more generally for toric varieties.

When \((M, \omega)\) is \(\mathbb{P}^m\) endowed with the Kähler form \(\omega_{FS}\) associated to a Fubini-Study metric, we let \((z^1, \ldots, z^{m+1})\) be complex coordinates in \(\mathbb{C}^{m+1}\) and let us fix for the rest of the paper the convention that \(\omega_{FS} = PD[\mathbb{P}^{m-1}]\), where \(\mathbb{P}^{m-1} \subset \mathbb{P}^m\) is a linear subspace. This is particularly relevant when getting quantitative estimates on the Kähler classes reachable by our constructions.

We consider the group \(K := S^1 \times \cdots \times S^1\), the maximal compact subgroup of \(PGL(m+1)\), whose action is given by

\[
K \times \mathbb{P}^m \rightarrow \mathbb{P}^m
\]

\[
((\alpha_1, \ldots, \alpha_{m+1}), [z^1, \ldots, z^{m+1}]) \mapsto [\alpha_1 z^1, \ldots, \alpha_{m+1} z^{m+1}]
\]

and we consider the set of fixed points of \(K\)

\[
p_1 := [1 : 0 : \ldots : 0], \quad \ldots \quad p_{m+1} := [0 : \ldots : 0 : 1]
\]

In this case, the space \(\mathfrak{h}\) is spanned by vector fields of the form

\[
\Re (z^j \partial_{z^j} - z^k \partial_{z^k})
\]

and we have \(\mathfrak{k} = \mathfrak{h} = \mathfrak{h}'\) and \(\mathfrak{h}'' = \{0\}\). As a consequence of the result of Theorem 2.1, we obtain extremal Kähler metrics on the blow up of \(\mathbb{P}^m\) at the points \(p_1, \ldots, p_n\), for any \(n = 1, \ldots, m+1\).

It is worth emphasizing that the special structure of the points which can be blown up on \(\mathbb{P}^m\) has its origin in the fact that we are starting from a specific choice of a Fubini-Study metric and hence, away from the blow up points the extremal Kähler metric \(\omega_{\varepsilon}\) is close to \(\omega_{FS}\). This example shows well the riemannian nature of our results (see Remark 2.2). Now, if \(q_1, \ldots, q_n \in \mathbb{P}^m\) are linearly independent one can find extremal metrics on the blow up of \(\mathbb{P}^m\) at \(q_1, \ldots, q_n\) but this time the metric will be close to \(\psi^* \omega_{FS}\) away from the blow up points, where \(\psi\) is an automorphism of the projective space such that

\[
\psi(p_j) = q_j.
\]

Yet, since \([\psi^* \omega_{FS}]\) is independent of \(\psi\) and of the choice of the Fubini-Study metric, we have obtained the following Kählerian version of Theorem 2.1 for \(\mathbb{P}^m\):

Corollary 2.1. Fix \(1 \leq n \leq m+1\). Given \(q_1, \ldots, q_n \in \mathbb{P}^m\) linearly independent points and \(a_1, \ldots, a_n > 0\), there exists \(\varepsilon_0 > 0\) and for all \(\varepsilon \in (0, \varepsilon_0)\) there exists an extremal Kähler metric \(g_\varepsilon\) on the blow up of \(\mathbb{P}^m\) at \(q_1, \ldots, q_n\) whose associated Kähler form \(\omega_{\varepsilon}\) lies in the class

\[
\pi^* [\omega_{FS}] - \varepsilon^2 \left( a_1^{\frac{1}{n-m}} PD[E_1] + \ldots + a_n^{\frac{1}{n-m}} PD[E_n] \right)
\]

In addition, the Kähler metrics \(g_\varepsilon\) do not have constant scalar curvature unless \(n = m+1\) and \(a_1 = \ldots = a_{m+1}\).
The conditions \( n = m + 1 \) and \( a_1 = \ldots = a_{m+1} \) being necessary and sufficient to get constant scalar curvature metrics among our family of extremal ones fits exactly with the more familiar picture of the Futaki invariants. Calabi has in fact proved that an extremal metric has constant scalar curvature iff its Futaki invariant vanishes [9], and we will show in §11, using Mabuchi’s result [20] relating the Futaki invariant to the coordinates of the barycenter of the convex polytope of a toric variety, that the above conditions are indeed equivalent to the vanishing of the Futaki invariants for blow ups of \( \mathbb{P}^m \).

The case corresponding to \( n = 1 \) in Corollary 2.1 was already obtained by Calabi in more generality (i.e. for all Kähler classes) [8] and the case where \( \mathbb{P}^m \) is blown up at \( m + 1 \) linearly independent points \( q_1, \ldots, q_{m+1} \) and \( a_1 = \ldots = a_{m+1} \) was already studied in [2] where constant scalar curvature metrics were obtained.

In the case where \( \mathbb{P}^m \) is blown up at more than \( m + 1 \) points in general position the resulting manifolds do not have nonzero holomorphic vector fields, hence extremal metrics are forced to have constant scalar curvature and the existence of some constant scalar curvature Kähler metrics follows from [2] and [26].

The previous Corollary can be understood as a special case of the the existence of extremal metrics on the blow up of toric varieties, which in fact leads to a more general result as we will see below even for \( \mathbb{P}^m \). If \((M, J, \omega)\) is a \( m \)-dimensional toric variety whose associated metric is extremal, one can take \( K \) to be the torus \( T^m \) giving the torus action. It then follows from Proposition [7.4] that \( h = \mathfrak{k} \), the Lie algebra associated to \( K \). One can apply Theorem 2.1 to get:

**Corollary 2.2.** Assume that \((M, J, \omega)\) is a toric variety whose associated metric is extremal, and let \( K \) be the torus \( T^m \) giving the torus action. Given \( p_1, \ldots, p_n \in \text{Fix}(K) \) and \( a_1, \ldots, a_n > 0 \), there exists \( \varepsilon_0 > 0 \) and for all \( \varepsilon \in (0, \varepsilon_0) \) there exists an extremal Kähler metric \( g_\varepsilon \) on the blow up of \( M \) at \( p_1, \ldots, p_n \) whose associated Kähler form \( \omega_\varepsilon \) lies in the class \( \pi^* [\omega] - \varepsilon^2 \left( a_1 \overline{PD}[E_1] + \ldots + a_n \overline{PD}[E_n] \right) \).

In other words, one can blow up any set of points contained in the fixed-point set of the torus action and the weights \( a_j > 0 \) can be chosen arbitrarily.

Since blowing up a toric variety at such points preserves the toric structure, one can apply inductively the last Corollary. Therefore, we obtain extremal metrics on any such iterated blow up. Beside these applications this last Corollary can be applied to can be applied to any toric Kähler-Einstein manifold, the classification of which has been completed in dimension \( m = 2, 3 \) and 4, in [21, 24] and all the symmetric examples have been found by Batyrev and Selivanova [5] and to the one parameter family of extremal metrics found by Calabi of the blow up of \( \mathbb{P}^m \) at one point, producing then a wealth of open subsets of classes in the Kähler cone which have extremal representatives.

For example our result applied to \( \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \) and \( Bl_p \mathbb{P}^2 \) as base manifolds leads to the following:

**Corollary 2.3.**

1. If \( M = Bl_{p_1, p_2} \mathbb{P}^2 \) then the following Kähler classes have extremal representatives

\[
\pi^*[\omega_{FS}] - (a_1 \overline{PD}[E_1] + \varepsilon^2 a_2 \overline{PD}[E_2]), \quad a_1 < 1
\]
$\pi^*[\omega_{FS}] - \frac{a_1}{a_1+a_2+\varepsilon^2} PD[E_1] - \frac{a_2}{a_1+a_2+\varepsilon^2} PD[E_2],$

(2) If $M = Bl_{p_1,p_2,p_3}\mathbb{P}^2$ and the points do not lie on a complex line, then the following Kähler classes have extremal representatives

$\pi^*[\omega_{FS}] - (a_1 PD[E_1] + \varepsilon a_2 PD[E_2] + \varepsilon^2 a_3 PD[E_3]), \quad a_1 < 1$

$\pi^*[\omega_{FS}] - \frac{a_1-a_2^{\varepsilon}}{a_1+a_2-\varepsilon^2} PD[E_1] - \frac{a_2-a_3^{\varepsilon}}{a_1+a_3-\varepsilon^2} PD[E_2] - \varepsilon a_4 PD[E_3],$

$\pi^*[\omega_{FS}] - \frac{1-a^2(a_1+a_2)}{2-a^2(a_1+a_2+a_3)} PD[E_1] - \frac{1-a^2(a_1+a_3)}{2-a^2(a_1+a_2+a_3)} PD[E_2] - \frac{1-a^2(a_2+a_3)}{2-a^2(a_1+a_2+a_3)} PD[E_3]$

where $a_j + a_k < 1$ for all $j,k$, and $a_1 + a_2 + a_3 < 2.$

(3) If $M = Bl_{p_1,p_2,p_3}\mathbb{P}^2$ and the points lie on a complex line, then the following Kähler classes have extremal representatives

$\pi^*[\omega_{FS}] - (a_1 PD[E_1] + \varepsilon a_2 PD[E_2] + \varepsilon^2 a_3 PD[E_3]), \quad a_1 < 1$

$\pi^*[\omega_{FS}] - \varepsilon^2 (a PD[E_1] + b PD[E_2] + b PD[E_3]), \quad b < a.$

Remark 2.3. This last family of examples is interesting also because it has been shown by A. Della Vedova [12], building on Szekelyhidi’s work, that the above Kähler classes do not have extremal representatives for $b > 2a$, giving then an explicit upper bound for our construction to work.

In all the above cases, the first families of classes are immediately obtained by our direct construction applied once or twice to $Bl_p\mathbb{P}^2$ with a Calabi’s metric. The other classes are obtained by applying our result either to $\mathbb{P}^1 \times \mathbb{P}^1$ with a product of Fubini-Study metrics, or by using some classical algebraic constructions which will be recalled in §11.

We should also recall that when we blow up three not aligned points, the Kähler classes $\pi^*[\omega_{FS}] - (a_1 PD[E_1] + a_2 PD[E_2] + a_3 PD[E_3]),$ where all the $a_j$ are sufficiently close to $\frac{1}{3},$ also have extremal representatives, thanks to the existence of a Kähler-Einstein metric on the resulting manifold, as shown by Siu-Tian-Yau, and the recalled deformation theory of Lebrun and Simanca [17].

3. Notation and conventions

The following conventions are used throughout. If $(M, J)$ is a complex manifold, we write $\text{Aut}(M, J)$ for the group of biholomorphic maps $M \to M.$ If $(M, \omega)$ is a symplectic manifold, we write $\text{Exact}(M, \omega)$ for the group of exact symplectomorphisms; that is, those that are generated by hamiltonian vector fields. Finally if $(M, g)$ is a riemannian manifold, we write $\text{Isom}(M, g)$ for the group of isometries of $(M, g).$ We denote by a subscript 0 the identity-component of these groups (even though the group of exact symplectomorphisms is already connected).
The metric $g$, Kähler form $\omega$ and complex structure $J$ are related by
\begin{equation}
g(JX,Y) = \omega(X,Y), \quad \text{or equivalently} \quad \omega(X,JY) = g(X,Y).
\end{equation}

The action of $J$ commutes with the musical isomorphisms, but if $\alpha$ is a 1-form and $X$ is a vector field, we have
\begin{equation}
J\alpha(X) = -\alpha(JX).
\end{equation}
Then $T^{1,0}$ corresponds to the $+i$-eigenspace of $J$ while $\Lambda^{1,0}$ corresponds to the $-i$-eigenspace of $J$. In particular, we have
\begin{equation}
\bar{\partial} f = \frac{1}{2} (df - iJdf), \quad Jdf = i(\bar{\partial} f - \partial f)
\end{equation}
and so on.

Recall that a vector field $X$ is said to be a Hamiltonian vector field if there exists a smooth real valued function $f$ satisfying
\begin{equation}
X = J\nabla f.
\end{equation}
In this case we will write $X = X_f$. Using (1) we see that this equation is always equivalent to
\begin{equation}
\omega(X_f, -) = -\partial f.
\end{equation}
or, using (3), is also equivalent to
\begin{equation}
\frac{1}{2} \omega(\Xi_f, -) = -\bar{\partial} f.
\end{equation}
when $\bar{\partial} f = \frac{1}{2} (df - iJdf), Jdf = i(\bar{\partial} f - \partial f)$.

Let us now define the second order operator
\begin{equation}
P_\omega : C^\infty(M) \rightarrow \Lambda^{0,1}(M, T^{1,0}),
\end{equation}
\begin{equation}
f \mapsto \frac{1}{4} \bar{\partial} \Xi_f
\end{equation}
so that the null-space of $P_\omega$ (beside the constant function) corresponds to holomorphic vector fields with zeros. Observe that the operator $P_\omega$ depends on the Kähler metric $\omega$. Also, with this definition, a metric $\omega$ is extremal if and only if $P_\omega(s(\omega)) = 0$.

Clearly, any smooth, complex valued function $f$ solution of
\begin{equation}
P_\omega^* P_\omega f = 0
\end{equation}
on $M$ gives rise to a holomorphic vector field $\Xi_f$ defined by (9) since by integration over $M$ implies that $\|\bar{\partial} \Xi_f\|_{L^2(M)} = 0$. We recall the following important result which shows that the converse is also true:

\textbf{Proposition 3.1.} \[9, 17\] $\Xi \in T^{1,0}$ is a holomorphic vector field with zeros if and only if there exists a complex valued function $f$ solution of $P_\omega^* P_\omega f = 0$ such that $\frac{1}{2} \omega(\Xi, -) = -\bar{\partial} f$.

In addition, we have the following result which follows from a theorem of Lichnerowicz (see Besse [4], Corollary 2.125 and [17]):

\begin{footnote}{To help the reader connecting this notation with the existing literature, let us remark that $\Xi_f = 2i \bar{\partial}^\# f = 2i \partial^\# f$, the (1,0) part of the gradient of $f$, in the notations used in [9] and [17].}
Proposition 3.2. [4], [17] A vector field $X$ is a Killing vector field with zeros if and only if there exists a real valued function $f$ solution of $P_\omega^*P_\omega f = 0$ such that $\omega(X, -) = -df$.

In other words, if $\Xi$ is a holomorphic vector field and $f$ the function given in Proposition 3.1, then $f$ can be chosen to be real valued when $X = \Re \Xi$ is a Killing vector field. Also, any Killing vector field is automatically real-holomorphic.

Observe that, in particular, if $X$ is a Killing vector field with zeros, then $\Xi = X - iJX$ is a holomorphic vector field. We recall the:

Definition 3.1. A vector field $X$ is real-holomorphic if and only if $X - iJX$ is a holomorphic section of $T^{1,0}M$.

4. Equivariant set-up

We fix a compact subgroup $K$ of $\text{Isom}(M, g)$ and we assume that the Lie algebra of $K$ contains $X_s = J \nabla s$. We do not insist that $K$ be connected.

Let us denote by $\mathfrak{h}$ the Lie algebra of real-holomorphic vector fields which are $K$-invariant and are hamiltonian. Note that, since $\omega$ is $K$-invariant, $X_s$ certainly lies in $\mathfrak{h}$ for any choice of $K$.

There is a large flexibility in the choice of $K$ varying from the two extreme cases where $X_s$ happens to generate a closed subgroup of $\text{Isom}_0(M, g)$, then this will be a circle-subgroup $S$ contained in the center of $\text{Isom}_0(M, g)$ and one could choose $K = S$ and the opposite situation where we choose $K = \text{Isom}_0(M, g)$.

With slight abuse of notations, we will identify elements of $\mathfrak{h}$ with the real-holomorphic vector fields corresponding to the infinitesimal action of $H$ on $M$. For any Kähler metric $\omega$, denote by $\xi_\omega$ the moment map for the action of $H$, uniquely determined by requiring to have mean zero

$$\xi_\omega : M \to \mathfrak{h}^*$$

Recall that this is defined as follows. If $X \in \mathfrak{h}$, then the function $f = \langle \xi_\omega, X \rangle$ on $M$ is a hamiltonian for the vector field $X$, namely, a solution of :

$$\omega(X, -) = -df$$

normalized by

$$\int_M f \omega^m = 0.$$

Observe that according to [19] we also have

$$\frac{1}{2} \omega(\Xi, -) = -\bar{\partial} \langle \xi_\omega, X \rangle$$

where $\Xi = X - iJX$ is the holomorphic vector field associated to $X$.

This is just an invariant way of introducing the potentials corresponding to hamiltonian, real-holomorphic vector fields with zeros.

Remark 4.1. Notice that as the Kähler form varies (among $K$-invariant forms the moment map varies. In section 4, we will explicitly study the dependence of $\xi_\omega$ on $\omega$. 
For the blow-up problem, we note that a vector field $X$ lifts to $\tilde{M}$, the blow up of $M$ at $p_1, \ldots, p_n$, if and only if it vanishes at each of the points $p_j$. If we have fixed the isometry group to contain $K$, it follows that we only stand a chance of blowing up points which are fixed by every element of $K$. So, we suppose that

(10) For all $j = 1, \ldots, n$, $p_j \in \text{Fix}(K)$.

Now, if $\tilde{\omega}$ is a putative extremal Kähler metric on $\tilde{M}$ its scalar curvature must be a sum of $K$-invariant potentials corresponding to vector fields that vanish at the $p_j$ and are $K$-invariant and hence they have to correspond to vector fields which are in $h'$. Thus we introduce the lie algebra $h'$ that is given by

$$h' = \mathfrak{k} \cap h$$

We denote by $h''$ the orthogonal complement of $h'$ in $h$ with respect to the scalar product

$$(X, \tilde{X})_h := \int_M \langle \xi_\omega, X \rangle \langle \xi_\omega, \tilde{X} \rangle \, d\text{vol}_g.$$  

Informally, potentials of the form $\langle \xi_\omega, X' \rangle$ (for $X' \in h'$) will the good potentials corresponding to vector fields vanishing at the $p_j$, and hence lifting them to $\tilde{M}$ they can be used to deform the scalar curvature of the Kähler form $\tilde{\omega}$. The potentials $\langle \xi_\omega, X'' \rangle$ (for $X'' \in h''$) will be the bad potentials corresponding to vector fields that do not necessarily lift to $\tilde{M}$ but, in any case, these are potentials that will not be used in the deformation of the scalar curvature of $\tilde{\omega}$.

To apply a perturbation argument, as in [2], we shall need to solve two linear problems. First we need to find a function $\Gamma$, a constant $\lambda$ and a vector field $Y' \in h'$ solutions of

(11) \[ \frac{1}{2} P^*_\omega P_\omega \Gamma + \langle \xi_\omega, Y' \rangle + \lambda = c_m \sum_{j=1}^n a_j \delta_{p_j} \]

where the masses $a_j$ are positive and $c_m > 0$ is a positive constant only depending on the dimension $m$. The solvability of this problem comes down to the relative moment condition:

(12) \[ \sum_{j=1}^n a_j \xi_\omega(p_j) \in h'^* \text{ for some } a_j > 0 \]

Using this, we first consider a first perturbation of $\omega$, away from the points to be blown up. This perturbed Kähler form we consider is given explicitly by

$$\tilde{\omega}_\varepsilon := \omega + i \partial \bar{\partial}(\varepsilon^{2m-2} \Gamma)$$

where $\varepsilon > 0$ is a small parameter. This Kähler form is well defined away from the points $p_j$ (provided $\varepsilon$ is chose small enough) and, as will follow from the analysis in the next section, has scalar curvature given by

$$s(\tilde{\omega}_\varepsilon) = s(\omega) + \varepsilon^{2m-2} \left( \langle \xi_\omega + i \partial \bar{\partial}(\varepsilon^{2m-2} \Gamma), Y' \rangle + \lambda \right) + O(\varepsilon^{4m-2})$$

The final task will be to perturb this Kähler metric into an extremal metric. To this aim, given any (smooth) function $f$, we need to be able to find a function $\phi$, a constant $\nu$, a vector
field $Z' \in \mathfrak{h}'$ and masses $b_j \in \mathbb{R}$ solutions of

$$\frac{1}{2} P^*_{\omega} P_{\omega} \phi + \nu + \langle \xi_\omega, Z' \rangle + c_m \sum_{j=1}^n b_j \delta_{p_j} = f$$

The solvability of this problem is precisely equivalent to the *genericity condition*:

$$\text{The projections of } \xi_\omega(p_1), \ldots, \xi_\omega(p_n) \text{ span } \mathfrak{h}''^*.$$

The core of the paper is to show that these conditions are indeed *sufficient* conditions to guarantee the existence of extremal metrics in the appropriate classes.

5. **Linear Operators**

The linearization of the mapping

$$f \mapsto s(\omega + i\bar{\partial}f)$$

is given by the formula

$$\mathbb{L} := -\frac{1}{2} \Delta_g^2 - \text{Ric}_g \cdot \nabla_g^2$$

where Ric$_g$ stands for the Ricci tensor of the metric $g$ associated to $\omega$. On the other hand,

$$P^*_{\omega} P_{\omega} = \Delta_g^2 + 2 \text{Ric}_g \cdot \nabla_g^2 - J X_s + iX_s.$$

where $P$ is the operator defined in (7) and $X_s$ is the hamiltonian vector field associated with $s$.

Hence we have:

$$\mathbb{L} = -\frac{1}{2} P^*_{\omega} P_{\omega} - \frac{1}{2} J X_s + \frac{i}{2} X_s$$

Working equivariantly with respect to a compact group $K$ whose lie algebra contains $X_s$ has the important effect of making the last term in (18) disappearing, leaving a real operator on $K$-invariant functions.

Consider the map

$$F : \mathfrak{h} \times \mathcal{C}^\infty(M)^K \rightarrow \mathcal{C}^\infty(M)^K,$$

$$(X,f) \mapsto s(\omega + i\bar{\partial}f) - \langle \xi_{\omega+i\bar{\partial}f}, X \rangle.$$

Here the superscripts $K$ denote the $K$-invariant part of the function space.

The following is due to Calabi and LeBrun–Simanca.

**Proposition 5.1.** Assume that $\omega$ is extremal and $X_s \in \mathfrak{h}$, then $D_f F |_{(X_s,0)}$, the linearization of $F$ with respect to $f$ at $(X_s,0)$, is equal to $-\frac{1}{2} P^*_{\omega} P_{\omega}$.

**Proof.** We already know the linearization of the scalar curvature map, so we only need to know the linearization of

$$f \mapsto \xi_{\omega+i\bar{\partial}f}$$

with respect to $f$. Take any $X \in \mathfrak{h}$. Since $f$ is $K$-invariant, $X$ is a Killing vector field (with zeros) for the Kähler form $\omega + i\bar{\partial}f$. Hence, using the analysis of §4, we can write

$$\frac{1}{2} (\omega + i\bar{\partial}f)(\Xi, -) = -\bar{\partial}(\xi_{\omega+i\bar{\partial}f}, X)$$
where $\Xi := X - i J X$, and we see immediately that $\dot{\xi}$, the first variation of $f \mapsto - \iota_\xi \omega + i \partial \bar{\partial} f$ with respect to $f$ computed at $f = 0$, satisfies
\[
\frac{i}{2} \partial \bar{\partial} f(\Xi, -) = - \bar{\partial} \langle \dot{\xi}, X \rangle.
\]
Working in local coordinates, the left hand side of this expression is equal to
\[
\frac{i}{2} d\bar{z}^j \frac{\partial}{\partial \bar{z}^j} (\Xi_k \frac{\partial f}{\partial z_k})
\]
because $\Xi$ is holomorphic. Hence we see that
\[
\langle \dot{\xi}, X \rangle = - \frac{i}{2} \Xi f.
\]
Now, we apply this analysis when $\omega$ is extremal, with extremal vector field $X_s \in \mathfrak{h}$. We obtain for any smooth function $f$
\[
D_f F_{|X_s,0}(f) = \mathbb{L} f + \frac{1}{2} \Xi f \quad \text{with} \quad \Xi := X_s - i J X_s.
\]
Hence
\[
D_f F_{|X_s,0}(f) = - \frac{1}{2} P_\omega^* P_\omega f - \frac{1}{2} J X_s f + \frac{1}{2} X_s f + \frac{1}{2} \Xi f = - \frac{1}{2} P_\omega^* P_\omega f + i X_s f.
\]
Remembering that when $f$ is $K$-invariant and $X_s \in \mathfrak{k}$, we have $X_s f = 0$ we conclude that $D_f F_{|X_s,0}(f) = - \frac{1}{2} P_\omega^* P_\omega f$. This completes the proof. \hfill \square

6. Burns-Simanca’s metric on the blow up of $\mathbb{C}^m$ at the origin

We describe a scalar flat Kähler form $\eta$ defined on $\tilde{\mathbb{C}}^m$, the blow up at the origin of $\mathbb{C}^m$. This metric is $U(m)$ invariant and was found by Burns [16], when $m = 2$, and Simanca [27], when $m \geq 3$, following a method introduced in [8]. Away from the exceptional divisor, the Kähler form $\eta$ is given by
\[
\eta = i \partial \bar{\partial} E_m(v)
\]
where $v = (v^1, \ldots, v^m)$ are complex coordinates in $\mathbb{C}^m \setminus \{0\}$ and where the function $E_m$ is explicitly given, in dimension $m = 2$, by
\[
E_2(v) := \frac{1}{2} |v|^2 + \log |v|^2
\]
while in dimension $m \geq 3$, even though there is no explicit formula for $E_m$ we have the following expansion
\[
E_m(v) = \frac{1}{2} |v|^2 - |v|^{4-2m} + \mathcal{O}(|v|^{2-2m})
\]
as $|v|$ tends to $\infty$. Observe that $E_m$ is defined up to a constant. Details can be obtained either in [8], or [27] or even [2].

It is important to stress that the metric $\eta$ is defined in terms of a choice of coordinates $(v^1, \ldots, v^m)$, and any choice of local coordinates around the point $p_j$ gives rise to a preferred $\eta$. On the other hand the geometry of extremal metrics, and in particular our choice of the group $K$, points to a preferred choice as we will see in the next section.
7. $K$-invariance and Extensions on the Blow Up

We discuss the crucial question of the lifting of objects (such as the action of $K$, holomorphic vector fields, associated potential, . . . ) all of which are defined on $M$, to the blown up manifold.

Recall that, blowing up a $m$-dimensional complex manifold at a point can be understood as a connected sum construction which can be performed by excising a small ball in complex normal coordinates around the point we want to blow up and replacing it by a large neighborhood of the exceptional divisor in $\tilde{C}^m$, the blow up of $C^m$ at the origin, keeping some compatibility between metrics and complex structures on the different summands.

Now, $M$ is endowed with a $K$ invariant Kähler form $\omega$ and $\tilde{C}^m$ will be equipped with a suitable multiple of the scalar flat Kähler form $\eta$ defined in the previous section. Since we want the action of the group $K$ to lift to an isometric action also for the new metrics we have to impose the condition that $K \subset U(m)$ on the neighborhood of the point $p$ which will be blown up, since $U(m)$ is the isometry group of any Burns-Simanca’s metric $\eta$. This is accomplished by linearizing on a small neighborhood of $p$ the action of $K$, which is ensured by the following classical result [6] :

Proposition 7.1. Let $D$ be a domain of a complex manifold and $G \subset Aut(D,J)$ be a compact subgroup with a fixed point $p \in D$. In a neighborhood of $p$, there exist complex coordinates centered at $p$ such that in these coordinates the action of $G$ is given by linear transformations.

We will refer to these coordinates as $G$-linear coordinates. The following proposition, proved in [2], shows that one can find $G$-linear coordinates which are also normal coordinates about $p$, a fixed point of $G$.

Proposition 7.2. Assume that $G \subset Isom(M, \omega)$ is compact. Then there exist $(z^1, \ldots, z^m)$, $G$-linear coordinates centered at $p \in Fix(G)$ such that

$\omega = i \partial \bar{\partial} (\frac{1}{2} |z|^2 + \varphi).$

where the function $\varphi$ is $G$ invariant and $\varphi = O(|z|^4)$.

In our construction of extremal Kähler metrics on blow ups, this proposition will be used in the following way. We apply the previous result to $G = K$ close to a fixed point $p \in Fix(K)$. We obtain normal coordinates, in $D$ a neighborhood of $p$ for which the action of $K$ is linear. Given $X \in \mathfrak{k}$ a Killing vector field vanishing at a point $p$ we can lift this vector field as a vector field $\tilde{X}$ on $\tilde{D}$ the blow up of $D$ at the point $p$. If $D$ is endowed with $\alpha \eta$, a multiple of Burns-Simanca’s metric, then $\tilde{X}$ will still be a Killing vector field of $(\tilde{D}, J, \alpha \eta)$. In addition, $\tilde{X}$ still vanishes at some point on the exceptional divisor over $p$ as is shown in the following :

Proposition 7.3. Let $X$ be a real-holomorphic vector field on $M$, and let $p$ be any point in $M$ such that $X(p) = 0$. We denote by $\tilde{X}$ the lift of $X$ to the blow up of $M$ at $p$. Then, there exists a point $q$ on the exceptional divisor over $p$ such that $\tilde{X}(q) = 0$.

Proof. For simplicity, we give the proof in the case where $m = 2$. Given $z := (z^1, z^2)$ complex coordinates centered at $p$, we write

$X - iJX = X^1 \partial_{z^1} + X^2 \partial_{z^2},$
with $X^i(0) = 0$. Let $(u^1, u^2)$ be complex coordinates on $\tilde{C}^2$ such that $z^1 = u^1 u^2$ and $z^2 = u^2$, covering an affine chart of the exceptional divisor over $p$. Then, $\tilde{X} - iJ\tilde{X}$, the lift of the vector field $X - iJX$, is given by

$$\tilde{X} - iJ\tilde{X} = \frac{z^2 X^1 - z^1 X^2}{(z^2)^2} \partial_{u^1} + X^2 \partial_{u^2}.$$ 

We can always write

$$X^1 = a z^1 + b z^2 + O(|z|^2), \quad \text{and} \quad X^2 = c z^1 + d z^2 + O(|z|^2)$$

for $z$ close to 0.

We consider the point of the exceptional divisor corresponding to the line $z^1 = \lambda z^2$ (i.e. $(u^1, u^2) = (\lambda, 0)$). Obviously, we have

$$\lim_{z^2 \to 0} X^2(\lambda z^2, z^2) = 0$$

for all $\lambda \in \mathbb{C}$ and

$$\lim_{z^2 \to 0} \frac{z^2 X^1 - z^1 X^2}{(z^2)^2} (\lambda z^2, z^2) = -c\lambda^2 + (a - d)\lambda + b,$$

Unless $c = 0, d = a, b \neq 0$, in which case the point at infinity of $\mathbb{C} = \{\lambda\}$ annihilates $\tilde{X}$, the equation $-c\lambda^2 + (a - d)\lambda + b = 0$ has always a root $\lambda$ which corresponds to a zero of $\tilde{X}$. \hfill \Box

Let us end this section with a simple but useful result. Since the obstruction for our construction to be successful is essentially contained in $h''$, it is interesting to have some geometric efficient property which implies that $h'' = 0$.

To state this condition precisely, let us fix $p \in M$ and denote by $\rho: K \to \text{Gl}(T_p M)$ the representation of $K$ induced on $T_p M$ by the action of $K$ on $M$.

**Proposition 7.4.** If there exists $p \in \text{Fix}(K)$ s.t. the maximal torus $T_k$ of $\rho(K)$ has dimension equal to $\dim C M$, then $h'' = 0$.

**Proof.** We need to show that $h \subset t$. If $p \in \text{Fix}(K)$ we can use Proposition linearize the action of $T_k^C$ near $p$ in such a way that $\rho(K)$ can be written as $z_j \mapsto e^{i\theta_j} z_j$ in suitable complex coordinates on $T_p M$. The condition $\dim_{\mathbb{R}} T_k = \dim C M$ then implies that $\theta_j \neq \theta_l$ for $j \neq l$. This immediately implies that the elements of $H$ are also diagonal, and hence the result follows. \hfill \Box

Observe that the above condition is easily satisfied by any toric manifold with $K$ the maximal compact torus giving the torus action.

8. Mapping properties

For all $r > 0$, we agree that

$$B_r := \{z \in \mathbb{C}^m : |z| < r\},$$

denotes the open ball of radius $r > 0$ in $\mathbb{C}^m$, $\tilde{B}_r$ denotes the corresponding closed ball and

$$\tilde{B}_r^* := B_r - \{0\}$$

the punctured closed ball. We will also define

$$C_r := \mathbb{C}^m - B_r \quad \text{and} \quad \tilde{C}_r := \mathbb{C}^m - B_r$$
to be respectively the complement in $\mathbb{C}^m$ of the closed the ball and the open ball of radius $r > 0$.

8.1. **Operators defined on** $M - \{p_1, \ldots, p_n\}$. Assume that we are given $n$ distinct points $p_1, \ldots, p_n \in M$. For each $j = 1, \ldots, n$, we can choose complex coordinates $z := (z^1, \ldots, z^m)$ in a neighborhood of 0 in $\mathbb{C}^m$, to parameterize a geodesic ball of radius $r$ centered at $p_j$ in $M$. Furthermore, as explained in the previous section, these coordinates can be chosen to be normal at $p_j$ and to be $K$-linear. In order to distinguish between the different neighborhoods and coordinate systems, we agree that, for all $r$ small enough, say $r \in (0, r_0)$, $B_{j,r}$ (resp. $\bar{B}_{j,r}$ and $\bar{B}_{j,r}^*$) denotes the open ball (resp. the closed and closed punctured ball) of radius $r$ in the coordinates $z$ parameterizing a fixed neighborhood of $p_j$.

We fix $r_0$ small enough so that $\bar{B}_{j,r}$ are disjoint for all $r \leq 4r_0$. We set

$$M_r := M - \bigcup_j B_{j,r}.$$ 

The weighted space for functions defined on the noncompact manifold

$$M^* := M - \{p_1, \ldots, p_n\},$$

is then defined as the set of functions whose decay or blow up near any $p_j$ is controlled by a power of the distance to $p_j$. More precisely, we have the :

**Definition 8.2.** Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define the weighted space $C^{\ell,\alpha}_\delta(M^*)$ to be the space of functions $f \in C^{\ell,\alpha}_\delta(M)$ for which the following norm is finite

$$\|f\|_{C^{\ell,\alpha}_\delta(M^*)} := \|f\|_{C^{\ell,\alpha}_\delta(M)} + \sum_{j=1}^n \left( \sup_{r \leq r_0} \left( r^{-\delta} \|f|_{B_{j,r_0}^*}(r)\|_{C^{\ell,\alpha}_\delta(\bar{B}_{j,r} - B_1)} \right) \right).$$

We are interested in the mapping properties of the operator

$$L_{\omega} := -\frac{1}{2} D_{\omega}^* D_{\omega},$$

which has been defined in §6. We define $B_{j,r_0}^*$ the function $G_j$ by

$$G_j(z) = -\log |z|^2 \quad \text{when} \quad m = 2 \quad \text{and} \quad G_j(z) = |z|^{4-2m} \quad \text{when} \quad m \geq 3.$$ 

Observe that, unless the metric $\omega$ is the Euclidean metric, these functions are not solutions of the homogeneous equation associated to $L_{\omega}$, however they can be perturbed into $\tilde{G}_j$ solutions of the homogeneous problem $L_{\omega} \tilde{G}_j = 0$. Indeed, reducing $r_0$ if this is necessary, we know from [2] that there exist functions $\tilde{G}_j$ which are solutions of $L_{\omega} \tilde{G}_j = 0$ in $B_{j,r_0}^*$ and which are asymptotic to $G_j$ in the sense that $\tilde{G}_j - G_j \in C^{4,\alpha}_\delta(B_{j,r_0}^*)$ when $m \geq 4$ and $\tilde{G}_j - G_j \in C^{4,\alpha}_\delta(\bar{B}_{j,r_0}^*)$ for any $\delta < 6 - 2m$ when $m = 2, 3$. The rational behind these constructions is that the Kähler metric $\omega$ osculates the Euclidean metric to order 2 and hence

$$L_{\omega} G_j \in C^{0,\alpha}_2(\bar{B}_{j,r_0}^*).$$

With the functions $\tilde{G}_j$ at hand, we define the **deficiency spaces**

$$D_0 := \text{Span}\{\chi_1, \ldots, \chi_n\}, \quad \text{and} \quad D_1 := \text{Span}\{\chi_1 \tilde{G}_1, \ldots, \chi_n \tilde{G}_n\},$$
where \( \chi_j \) is a cutoff function which is identically equal to 1 in \( B_{j,r_0/2} \) and identically equal to 0 in \( M - B_{j,r_0} \). Furthermore, we assume that \( \chi_j \) is \( K \)-invariant.

When \( m \geq 3 \), we fix \( \delta \in (4 - 2m, 0) \) and when \( m = 2 \) we choose \( \delta \in (0, 1) \). We define the operator

\[
\mathcal{L}_\delta : (C^{4,\alpha}_0(M^* K) \oplus D) \times h' \times \mathbb{R} \to C^{0,\alpha}_{4-4}(M^* K)
\]

\[
(f, X', \mu) \mapsto -\frac{1}{2} P_\omega^* P_\omega f - \langle \xi, X' \rangle - \mu,
\]

where \( D = D_1 \) when \( m \geq 3 \) and \( D = D_0 \oplus D_1 \) when \( m = 2 \). The main result of this section reads:

**Proposition 8.1.** Assume that the points \( p_1, \ldots, p_n \in M \) are chosen so that. Then, the operator \( \mathcal{L}_\delta \) defined above is surjective and has a kernel of dimension \( n + 1 + \dim h' \).

**Proof.** The proof of this result follows from the general theorem described in [23] and [22], nevertheless, we choose here to describe an (almost) self contained proof. Recall that, when working equivariantly with respect to a group \( K \), the kernel of \( L \) is spanned by the functions of the form \( \langle \xi, X \rangle + \mu \) where \( X \in h \) and \( \mu \in \mathbb{R} \). Also recall that, by assumption \( \langle \xi, X \rangle \) has mean 0.

Observe that \( C^{4,\alpha}_0(M^* K) \subset L^1(M) \) precisely when \( \delta > 4 - 2m \). We use the fact that \( L \) is self adjoint and hence, for \( h \in L^1(M) \), the problem

\[
\frac{1}{2} P^* P f + \langle \xi, X' \rangle + \mu + \sum_{j=1}^{n} b_j \delta_{p_j} = h
\]

is solvable in \( W^{3,q}(M) \) for all \( q \in [1, \frac{2m}{2m-1}] \) if and only if

\[
\mu \, \text{Vol}_g(M) + \sum_{j=1}^{n} b_j = \int_M h \, d\text{vol}_g
\]

(remember the \( \langle \xi, X \rangle \) is normalized to have mean 0) and, for all \( X'' \in h'' \)

\[
\sum_{j=1}^{n} b_j \langle \xi(p_j), X'' \rangle = \int_M h \langle \xi, X'' \rangle \, d\text{vol}_g
\]

(remember that \( h' \) and \( h'' \) are constructed so that

\[
(X', X'')_h := \int_M \langle \xi, X' \rangle \langle \xi, X'' \rangle \, d\text{vol}_M = 0,
\]

for all \( X' \in h' \) and all \( X'' \in h'' \)). The first equation gives the value of \( \mu \) in terms of the \( b_j \)'s and the function \( h \). While, the second system in \( b_j \)'s is solvable since we have assumed that the projection of \( \xi(p_1), \ldots, \xi(p_n) \) over \( h' \) spans \( h'' \).

To complete the proof, we simply invoke regularity theory which implies that \( f \in C^{4,\alpha}_0(M^* K) \oplus D \). The estimate of the dimension of the kernel is left to the reader since it will not be used in the paper. 
\[\square\]
8.3. Operators defined on \( \tilde{C}^m \). We choose coordinates \( u := (u^1, \ldots, u^m) \) to parameterize \( \tilde{C}^m \), the blow up of \( C^m \) at the origin, away from the exceptional divisor.

We start with the:

**Definition 8.4.** Given \( \ell \in \mathbb{N}, \alpha \in (0, 1) \) and \( \delta \in \mathbb{R} \), we define the weighted space \( C^{\ell, \alpha}_{\delta}(\tilde{C}^m) \) to be the space of functions \( w \in C^{\ell, \alpha}_{\delta}(\tilde{C}^m) \) for which the following norm is finite

\[
\|f\|_{C^{\ell, \alpha}_{\delta}(\tilde{C}^m)} := \|f\|_{C^{\ell, \alpha}((\tilde{C}^m) - C_1)} + \sup_{r \geq R_0} r^{-\delta} \|f|_{C^{\ell, \alpha}_{\delta}(\tilde{B}_2 - B_1)}\).
\]

Given \( \delta \in \mathbb{R} \), we define the operator

\[
\tilde{L}_\delta : C^{4, \alpha}_{\delta}(\tilde{C}^m)^K \rightarrow C^{0, \alpha}_{\delta-4}(\tilde{C}^m)^K
\]

\[
f \mapsto -\frac{1}{2} P^* P f
\]

and recall the following result which is borrowed from [2]

**Proposition 8.2.** Assume that \( \delta \in (0, 1) \). Then the operator \( \tilde{L}_\delta \) defined above is surjective and has a one dimensional kernel spanned by a constant function.

8.5. Bi-harmonic extensions. The following results are concerned with Bi-harmonic extensions either on the complement of the unit ball or on the unit ball of \( C^m \) of boundary data defined on the unit sphere. Here \( \Delta \) denotes the Laplacian on \( C^m \).

**Proposition 8.3.** Given \( h \in C^{4, \alpha}(\partial B_1), k \in C^{2, \alpha}(\partial B_1) \) such that

\[
\int_{\partial B_1} (4 m h - k) = 0
\]

there exists a function \( W^i(= W^i_{h, k}) \in C^{4, \alpha}_{1}(\tilde{B}_1^*) \) such that

\[
\Delta^2 W^i = 0 \quad \text{in} \quad B_1 \quad W^i = h \quad \text{and} \quad \Delta W^i = k \quad \text{on} \quad \partial B_1.
\]

Moreover,

\[
\|W^i\|_{C^{4, \alpha}_{1}(\tilde{B}_1^*)} \leq c (\|h\|_{C^{4, \alpha}(\partial B_1)} + \|k\|_{C^{2, \alpha}(\partial B_1)})
\]

Given \( h \in C^{4, \alpha}(\partial B_1), k \in C^{2, \alpha}(\partial B_1) \) such that

\[
\int_{\partial B_1} k = 0
\]

there exists a function \( W^o(= W^o_{h, k}) \in C^{4, \alpha}_{3-2m}(C^m - B_1) \) such that

\[
\Delta^2 W^o = 0 \quad \text{in} \quad C^m - B_1 \quad W^o = h \quad \text{and} \quad \Delta W^o = k \quad \text{on} \quad \partial B_1.
\]

Moreover,

\[
\|W^o\|_{C^{4, \alpha}_{3-2m}(C_1)} \leq c (\|h\|_{C^{4, \alpha}(\partial B_1)} + \|k\|_{C^{2, \alpha}(\partial B_1)})
\]
Let us briefly comment on the assumption. To this aim, let us concentrate on the case where both $h$ and $k$ are constant functions in which case their bi-harmonic extensions $W^i$ and $W^o$ are given explicitly by
\[ W^i(z) = a + b|z|^2 \]
and
\[ W^o(z) = c|z|^{4-2m} + d|z|^{2-2m} \]
when $m \geq 3$ and
\[ W^o(z) = c \log|z| + d|z|^{-2} \]
when $m = 2$. It is easy to see that $W^i \in C_4\alpha(B_1)$ if and only if $a = 0$. While $W^o \in C^{2,\alpha}B_1$ if and only if $c = 0$. These conditions lead to the constraints on the function $h$ and $k$ as in the statement of the result.

9. Nonlinear perturbation results

9.1. Perturbation of $\omega$. The first perturbation we will perform is concerned with the perturbation of the extremal Kähler form $\omega$ which is defined on the manifold $M$. We keep the notations which have been introduced in §6.

The Kähler metric $\omega$ being extremal, we have
\[ s(\omega) = \langle \xi, X_s \rangle + \mu_s \]
for some Killing field $X_s \in \mathfrak{h}$

and some constant $\mu_s \in \mathbb{R}$.

Given $n$ points $p_1, \ldots, p_n \in M$ and real parameters $a_1, \ldots, a_n > 0$. The points $p_j$ are precisely the points where the manifold $M$ will be blown up and the parameters $a_j > 0$ will be closely related to the Kähler classes and also the volume of the exceptional divisor on the blow up manifold, once the construction will be complete.

We recall here the crucial assumption on the choice of the points $p_j$ and the parameters $a_j$, namely, we assume that
\[ \sum a_j \xi(p_j) \in \mathfrak{h}^* \]
This condition is precisely the one which allows one to find a function $\Gamma$, a vector field $Y' \in \mathfrak{h}'$ and a constant $\lambda \in \mathbb{R}$ such that
\[ \frac{1}{2} P_\omega P_\omega \Gamma + \langle \xi, Y' \rangle + \lambda = c_m \sum_{j=1}^{n} a_j \delta_{p_j} \]
where the constant $c_m$ is defined by
\[ c_m := 4(m-1)(m-2)|S^{2m-1}| \quad \text{when } m \geq 3 \quad \text{and} \quad c_2 := 2|S^3| \]
Indeed, the existence of $\Gamma$ depends on the ability to choose $Y' \in \mathfrak{h}'$ and $\lambda \in \mathbb{R}$ so that
\[ \langle \xi, Y' \rangle + \lambda - c_m \sum_{j=1}^{n} a_j \delta_{p_j} \]
is "orthogonal" to the kernel of \(-\frac{1}{2}P^*P_\omega\). Since this kernel is precisely spanned by the functions of the form \(\langle \xi, X \rangle + \alpha\) where \(X \in h\) and \(\alpha \in \mathbb{R}\), we see that (24) is a necessary and sufficient condition for the existence of \(\Gamma\). We also have the relation

\[
\lambda = c_m \sum_{j=1}^{n} a_j
\]

and we know that the Killing field \(Y' \in h'\) has to be chosen so that

\[
\int_M \langle \xi_\omega, Y' \rangle \omega d\text{vol}_g - \sum_{j=1}^{n} a_j \xi_\omega(p_j) \in h''^*
\]

It is not hard to check that the function \(\Gamma\) has a nice expansion near each \(p_j\). Indeed, we have

**Lemma 9.1.** Near \(p_j'\,\), the following expansions hold

\[
\Gamma(z) = -a_j |z|^{4-2m} + \mathcal{O}(|z|^{6-2m})
\]

when \(m \geq 4\),

\[
\Gamma(z) = -a_j |z|^{-2} + b_j \log |z| + c_j + \mathcal{O}(|z|)
\]

for some \(b_j, c_j \in \mathbb{R}\) when \(m = 3\), while

\[
\Gamma(z) = a_j \log |z| + b_j + c_j \cdot z + \mathcal{O}(|z|^2 (- \log |z|))
\]

for some \(b_j \in \mathbb{R}\) and \(c_j \in \mathbb{C}^m\), when \(m = 2\). Here \(\mathcal{O}(|z|^q(- \log |z|)^q)\) denotes a smooth function defined away from 0, whose partial derivatives, when taken with respect to the vector fields \(|z| \partial_z\) and \(|z| \partial_{\overline{z}}\), are bounded by a constant (depending on the number of derivatives) times \(|z|^q(- \log |z|)^q\).

We fix

\[
r_\varepsilon := \varepsilon^{\frac{2m-1}{2m+1}}
\]

This corresponds to the radius of the balls centered at the points \(p_j\) which will be excised from \(M\). Recall that, for \(r > 0\) small enough we have defined

\[\tilde{M}_r := M - \bigcup_j B_{r_j, r}\]

On each of the boundaries of \(\tilde{M}_{r_\varepsilon}\), we will need some small boundary data. Hence, we assume that we are given

\[h_j \in C^{4,\alpha}(\partial B_1)^K\quad \text{and} \quad k_j \in C^{2,\alpha}(\partial B_1)^K\]

for \(j = 1, \ldots, n\), satisfying

\[
\|h_j\|_{C^{4,\alpha}(\partial B_1)} + \|k_j\|_{C^{2,\alpha}(\partial B_1)} \leq \kappa r_{\varepsilon}^2,
\]

where \(\kappa > 0\) will be fixed later on. The subscript \(K\) in the definition of the spaces is meant to remind the reader that the functions \(h_j\) and \(k_j\) are invariant under the action of \(K\). We further assume that

\[
\int_{\partial B_1} k_j = 0
\]
so that the second half of the result of Proposition 8.3 applies. To keep notation short we set
\[ h := (h_1, \ldots, h_n) \quad \text{and} \quad k := (k_1, \ldots, k_n). \]

In particular, we can define the function \( W_{\varepsilon,h,k} \) which is identically equal to 0 in \( \bar{M}_{2r_0} \) and, for \( j = 1, \ldots, n \) is equal to
\[ W_{\varepsilon,h,k} := \chi_j \chi_{h,j,k}(\cdot/r_2), \]
in \( B_{j,2r_0} \). Here \( \chi_j \) is a cutoff function which is identically equal to 1 in \( B_{j,r_0} \) and identically equal to 0 in \( M - B_{j,2r_0} \). Observe that the function \( W_{\varepsilon,h,k} \) depends (linearly) on \( h \) and \( k \) and is \( K \) invariant.

This being understood, we have the :

**Proposition 9.1.** Given \( \delta \in (4-2m, 5-2m) \) when \( m \geq 3 \) or \( \delta \in (0, 2/3) \) when \( m = 2 \), there exists \( \varepsilon_\kappa > 0 \) and \( c_\kappa > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_\kappa) \) one can find a function \( \phi_{\varepsilon,h,k} \in \mathcal{C}^{4,\alpha}(\bar{M}_\varepsilon) \), a vector field \( Y'_{\varepsilon,h,k} \in h' \) and a constant \( \lambda_{\varepsilon,h,k} \in \mathbb{R} \) such that the scalar curvature of the Kähler form
\[ \omega_{\varepsilon,h,k} = \omega + i \partial \bar{\partial} (\varepsilon^{2m-2} \Gamma + W_{\varepsilon,h,k} + \phi_{\varepsilon,h,k}) \]
defined on \( \bar{M}_\varepsilon \), satisfies
\[ -ds(\omega_{\varepsilon,h,k}) = \omega_{\varepsilon,h,k}(X_\varepsilon + \varepsilon^{2m-2} \bar{Y}' + Y'_{\varepsilon,h,k}, -) \]
with
\[
\frac{1}{|\partial B_j, r_2|} \int_{\partial B_j, r_2} s(\omega_{\varepsilon,h,k}) - s(\omega_{\varepsilon,h,k}) = \lambda_{\varepsilon,h,k}
\]
In addition, we have the estimate
\[
\| Y'_{\varepsilon,h,k} \|_{L^\infty} + r_2^{2m-4} \sup_{j=1,\ldots,n} \| \phi_{\varepsilon,h,k} \|_{\bar{B}_{j,2r_2}-B_{j,r_2}(r_2 \cdot)} \|_{\mathcal{C}^{4,\alpha}(\bar{B}_2-B_1)} \leq c_\kappa (r_2^{2m+1} + \varepsilon^{4m-4} r_2^6 \varepsilon^{-4m-\delta}),
\]
\[ |\lambda_{\varepsilon,h,k}| \leq c \varepsilon^{2m-2} \]
And we also have
\[ |\lambda_{\varepsilon,h,(1),h(2),k(1),k(2)}(1) - Y'_{\varepsilon,h,(1),h(2),k(1),k(2)}(1) - Y'_{\varepsilon,h,(2),h(2),k(2)}(1) - Y'_{\varepsilon,h,(1),h(2),k(2)}(1)|_{L^\infty} \]
\[ + r_2^{2m-4} \sup_{j=1,\ldots,n} \| (\phi_{\varepsilon,h,(1),h(2),k(1),k(2)}(\cdot/\bar{B}_2-B_1) \|_{\mathcal{C}^{4,\alpha}(\bar{B}_2-B_1)} \]
\[ \leq c_\kappa (r_2^{2m+3} + \varepsilon^{2m-2} r_2^{2m-2-\delta}) \| (h(1) - h(2), k(1) - k(2)) \|_{\mathcal{C}^{4,\alpha} \times \mathcal{C}^{2,\alpha}}, \]

provided the components of \( h, h(1), h(2), k, k(1), k(2) \) satisfy (32) and (33).

The remaining of this section is devoted to the proof of this result.

**Proof.** To begin with, using the analysis of §6, we can expand :
\[ s(\omega + i\partial \bar{\partial} f) = s(\omega) - \frac{1}{2} P^*_\omega P_\omega f - \frac{1}{2} J X_\omega + \frac{1}{2} J X_\omega f + Q_\omega (\nabla^2 f) \]
The structure of the nonlinear operator $Q_\omega$ is quite complicated but in each $\bar{B}_{j,\vec{r}_0}$, this operator enjoys the following decomposition

\[
Q_\omega(\nabla^2 f) = \sum_q B_{q,4,2}(\nabla^4 f, \nabla^2 f) C_{q,4,2}(\nabla^2 f) + \sum_q B_{q,3,3}(\nabla^3 f, \nabla^3 f) C_{q,3,3}(\nabla^2 f) + |z| \sum_q B_{q,3,2}(\nabla^3 f, \nabla^2 \varphi) C_{q,3,2}(\nabla^2 f) + \sum_q B_{q,2,2}(\nabla^2 f, \nabla^2 f) C_{q,2,2}(\nabla^2 f)
\]

(33)

where the sum over $q$ is finite, the operators $(U, V) \mapsto B_{q,a,b}(U, V)$ are bilinear in the entries and have coefficients which are smooth functions on $\bar{B}_{j,\vec{r}_0}$. The nonlinear operators $W \mapsto C_{q,a,b}(W)$ have Taylor expansions (with respect to $W$) whose coefficients are smooth functions on $\bar{B}_{j,\vec{r}_0}$. These facts follow at once from the expression of the scalar curvature of $s(\omega_0 + i \partial \bar{\partial} f)$ in local coordinates $[1]$.

The equation we would like to solve in $\bar{M}_r$, reads

\[
s(\omega + i \partial \bar{\partial} f) - \zeta - \mu_\alpha - \mu = 0
\]

where

\[
\zeta = \langle \xi_\omega, X_\alpha + X' \rangle - \frac{1}{2} J(X_\alpha + X') f - \frac{i}{2} (X_\alpha + X') f
\]

Observe that, using the analysis of section 5, we find that

\[
ad\zeta = (\omega + i \partial \bar{\partial} f)(X_\alpha + X', -)
\]

Using the above expansion together with (23), we can rewrite this equation as

\[
-\frac{1}{2} P^*_\omega P_\omega f + i X_\alpha f + Q_\omega(\nabla^2 f) - \langle \xi_\omega, X' \rangle + \frac{1}{2} X' f + \frac{1}{2} J X' f - \mu = 0
\]

Assuming that $X' \in \mathfrak{h}'$ and keeping in mind that we work equivariantly so that the function $f$ is now assumed to be $K$ invariant, this simplifies into

\[
\frac{1}{2} P^*_\omega P_\omega f + \langle \xi_\omega, X' \rangle + \mu = \frac{1}{2} J X' f + Q_\omega(\nabla^2 f)
\]

We set

\[
f := \varepsilon^{2m-2} \Gamma + W + \phi
\]

\[
X' := \varepsilon^{2m-2} Y' + Z'
\]

\[
\mu := \varepsilon^{2m-2} \lambda + \nu
\]

where $\Gamma, Y' \in \mathfrak{h}'$ and $\nu$ are defined in (25), (26) and (27) and $W = W_{\varepsilon, h, k}$ is defined in (31).

It is easy to see that the system of equations which remains to be solved on $\bar{M}_r$ can be formally written as

\[
\frac{1}{2} P^*_\omega P_\omega \phi + \langle \xi_\omega, Z' \rangle + \nu = Q(\phi, Z')
\]

where the operator $Q = Q(\varepsilon, h, k; \cdot, \cdot)$ is defined by

\[
Q(\varepsilon, h, k; \phi, Z') := -\frac{1}{2} P^*_\omega P_\omega W + \frac{1}{2} J(\varepsilon^{2m-2} Y' + Z') (\varepsilon^{2m-2} \Gamma + W + \phi)
\]

\[
+ Q_\omega(\nabla^2 (\varepsilon^{2m-2} \Gamma + W + \phi))
\]

We need the:
Definition 9.2. Given $\bar{r} \in (0, r_0)$, $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\delta \in \mathbb{R}$, the weighted space $C^{\ell, \alpha}_{\delta, \bar{r}}(\bar{M}_{\bar{r}})$ is defined to be the space of functions $f \in C^{\ell, \alpha}(\bar{M}_{\bar{r}})$ endowed with the norm

$$\|f\|_{C^{\ell, \alpha}_{\delta, \bar{r}}(\bar{M}_{\bar{r}})} := \|f\|_{C^{\ell, \alpha}(\bar{M}_{\bar{r}})} + \sum_{j=1}^{n} \sup_{\bar{r} \leq r \leq r_0} r^{-\delta} \|f|_{(\bar{B}_j, r_0) \setminus (\bar{B}_j, \bar{r})}(r \cdot)\|_{C^{\ell, \alpha}(\bar{B}_2 - \bar{B}_1)}$$

Next, we consider an extension (linear) operator $E_{\bar{r}, \delta'} : C^{0, \alpha}_{\delta'}(\bar{M}_{\bar{r}}) \rightarrow C^{0, \alpha}(M^*)$,

which is defined as follows:

(i) In $M_{\bar{r}}$, $E_{\bar{r}, \delta'}(f) = f$.

(ii) In each $\bar{B}_j, \bar{r} - B_j, \bar{r}/2$,

$$E_{\bar{r}, \delta'}(f)(z) = \frac{2|z - \bar{r}|}{\bar{r}} f \left( \frac{z}{|z|} \right).$$

(iii) In each $\bar{B}_j, \bar{r}/2$, $E_{\bar{r}, \delta'}(f) = 0$.

It is easy to check that there exists a constant $c = c(\delta') > 0$, independent of $\bar{r} \in (0, r_0)$, such that

$$\|E_{\bar{r}, \delta'}(f)\|_{C^{0, \alpha}(M^*)} \leq c \|f\|_{C^{0, \alpha}(\bar{M}_{\bar{r}})}.$$  \hfill (34)

The equation we would like to solve can now be written as

$$\frac{1}{2} P_{\omega}^* P_{\omega} \phi + (\xi, Y') + \nu + E_{r, \delta - 4} \circ Q_{\epsilon}(\phi, Y') = 0$$  \hfill (35)

We fix $\delta \in (4 - 2m, 5 - 2m)$. If $G_{\delta}$ denotes a right inverse for $L_{\delta}$ which is provided by Proposition 8.1, we just need to solve

$$(\phi, Z', \nu) = N(\epsilon, h, k, \phi, Z')$$

where $\phi \in C^{4, \alpha}_\delta(M^*)^K \oplus D$, $Z' \in h'$, $\nu \in \mathbb{R}$ and where the nonlinear operator $N(\epsilon, h, k; \cdot, \cdot)$ is defined by

$$N(\epsilon, h, k; \cdot, \cdot) := G_{\delta} \circ E_{r, \delta - 4} \circ Q$$

We set

$$\mathcal{F} := (c^{4, \alpha}_\delta(\bar{M}_{\bar{r}})^K \oplus D) \times h' \times \mathbb{R}$$

which is endowed with the product norm.

The existence of a solution to this system depends is a simple consequence of the fixed point theorem for contraction mappings once the following estimates are proved.

Lemma 9.2. There exists $c > 0$, $c_\kappa > 0$ and there exists $\epsilon_\kappa = \epsilon(\kappa) > 0$ such that, for all $\epsilon \in (0, \epsilon_\kappa)$

$$||N(\epsilon, h, k; 0, 0)||_{\mathcal{F}} \leq c_\kappa (r_\epsilon^{2m+1} + \epsilon^{4m-4} r_\epsilon^{6-4m-\delta}),$$  \hfill (36)
and
\[ \| \mathcal{N}(\varepsilon, h, k; \phi^{(1)}, Z') - \mathcal{N}(\varepsilon, h, k; \phi^{(2)}, Z'') \|_F \]
\[ \leq c_\kappa \varepsilon^{2m-2} r_\varepsilon^{6-4m-\delta} \| \phi^{(1)} - \phi^{(2)}, Z' - Z''_\varepsilon \|_F \]
(37)

Finally,
\[ \| \mathcal{N}(\varepsilon, h^{(1)}, k^{(1)}; \phi, Z') - \mathcal{N}(\varepsilon, a, h^{(2)}, k^{(2)}; \phi, Z') \|_F \]
\[ \leq c_\kappa \varepsilon^{2m-3} + \varepsilon^{2m-2} r_\varepsilon^{2-2m-\delta} \| (h^{(1)} - h^{(2)}, k^{(1)} - k^{(2)}) \|_{(C^4, a)^n} \]
(38)

provided \((\phi, Z', 0), (\phi^{(1)}, Z^{(1)}', 0), (\phi^{(2)}, Z^{(2)}', 0) \in \mathcal{F}\) satisfy
\[ \| (\phi, Z', 0) \|_F + \| (\phi^{(1)}, Z^{(1)}', 0) \|_F + \| (\phi^{(2)}, Z^{(2)}', 0) \|_F \leq 6 c_\kappa \varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{6-4m-\delta}, \]
and the components of \(h, h^{(1)}, h^{(2)}, k, k^{(1)}, k^{(2)}\) satisfy (37) and (29).

Proof. The proof of these estimates can be follows what is already done in [1] and [2] with minor modifications. We briefly recall how the proof of the first estimate is obtained and leave the proof of the second and third estimates to the reader. First, we use the result of Proposition 8.3 to estimate
\[ \| W \|_{C^{4, a}_{1-2m}(\bar{M}_s)} \leq c_\kappa r_\varepsilon^{2m+1}. \]
(39)

Now observe that, by construction, \(\Delta^2 W = 0\) in each \(\bar{B}_{r_0/2} - B_{r_\varepsilon}\) (here \(\Delta\) is the Euclidean Laplacian), hence
\[ L \omega W = (L \omega + \frac{1}{2} \Delta^2) W \]
in this set. Making use of the fact that the coordinates near \(p_j\) are chosen to be normal, we get the existence of a constant \(c_\kappa > 0\) such that
\[ \| L \omega W \|_{C^{4, a}_{1-2m}(\bar{M}_s)} \leq c_\kappa r_\varepsilon^{2m+1}. \]
Note that this is where we implicitly use the fact that \(\delta < 5 - 2m.\)

Next, we estimate the norm of \(\varepsilon^{2m-2} J Y' (\varepsilon^{2m-2} \Gamma + W)\) by
\[ \| \varepsilon^{2m-2} J Y' (\varepsilon^{2m-2} \Gamma + W) \|_{C^{0, a}_{4-4}(\bar{M}_s)} \leq c \varepsilon^{4m-4} \]
where the constant \(c_\delta\) does not depend on \(\kappa\) provided \(\varepsilon\) is chosen small enough, say \(\varepsilon \in (0, \varepsilon_\kappa).\)

Finally, we use the structure of the nonlinear operator \(Q_\omega\) as described in (33) together with the estimate (39) to get
\[ \| Q_\omega(\nabla^2 (\varepsilon^{2m-2} \Gamma + W)) \|_{C^{0, a}_{4-4}(\bar{M}_s)} \leq c \varepsilon^{4m-4} r_\varepsilon^{6-4m-\delta} \]
for some constant \(c_\delta > 0\) which does not depend on \(\kappa\) provided \(\varepsilon\) stays small enough. The first estimate then follows at once. \(\square\)

Reducing \(\varepsilon_\kappa > 0\) if necessary, we can assume that,
\[ c_\kappa \varepsilon^{2m-2} r_\varepsilon^{6-4m-\delta} \leq \frac{1}{2} \]
(40)
for all \( \varepsilon \in (0, \varepsilon_\kappa) \). Then, the estimates (36) and (37) in the above Lemma are enough to show that

\[
(\phi, X', \nu) \mapsto N(\varepsilon, h, k; \phi, X')
\]

is a contraction from

\[
\{(\phi, X', \nu) \in \mathcal{F} : \| (\phi, X', \nu) \|_{\mathcal{F}} \leq 2c_\kappa \left( r_\varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{6-4m-\delta} \right) \}
\]

into itself and hence has a unique fixed point \((\phi_\varepsilon, h_\varepsilon, k_\varepsilon, Y_\varepsilon', h_\varepsilon, k_\varepsilon, \nu_\varepsilon)\) in this set. This fixed point yields a solution of (35) in \( \bar{M}_{r_\varepsilon} \) and hence provides an extremal Kähler form on \( \bar{M}_{r_\varepsilon} \). The estimates in Proposition 9.1 follow at once from the estimates in Lemma 9.2 increasing the value of \( c_\kappa \) and reducing \( \varepsilon_\kappa \) if this is necessary. \( \square \)

9.3. Perturbation of \( \eta \). We perform an analysis similar to the one we have done in the previous section starting with \( \tilde{C}^m \), the blow up of \( C^m \) at 0, endowed with the Burns-Simanca’s metric \( \eta \).

Given \( a > 0 \), we now consider on \( \tilde{C}^m \), the perturbed Kähler form

\[
\tilde{\eta} = a^2 \eta + i \partial \bar{\partial} f,
\]

Everything we will do will be uniform in \( a \) as long as this parameter remains both bounded from above and bounded away from 0. In fact, we will apply our analysis when \( a \) satisfies

\[
a_{\min} \leq a \leq a_{\max},
\]

Given \( \nu \in \mathbb{R} \) and a \( K \)-invariant killing field \( X \in \mathfrak{k} \), we would like to solve the equation

\[
s(a^2 \eta + i \partial \bar{\partial} f) = \varepsilon^2 \zeta
\]

in \( N_{R_\varepsilon/a} \) where

\[
\zeta = \tilde{\zeta} - \frac{1}{2} J X f - \frac{i}{2} X f
\]

and \( \tilde{\zeta} \) is the solution of

\[
- d\tilde{\zeta} = a^2 \eta(X, -)
\]

normalized so that the mean of \( \zeta \) over \( \partial B_{R_\varepsilon/a} \) is prescribed by

\[
\frac{1}{|\partial B_{R_\varepsilon/a}|} \int_{\partial B_{R_\varepsilon/a}} \zeta = \nu
\]

Observe that, using the analysis of section 5, we find that

\[
- d\zeta = (a^2 \eta + i \partial \bar{\partial} f)(X, -)
\]

Also, working with \( K \)-invariant functions, since \( X \in \mathfrak{k} \), we have \( X f = 0 \).

It will be convenient to denote

\[
\tilde{N}_R := \tilde{C}^m - C_R
\]

We fix

\[
R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}
\]

Assume we are given \( h \in \mathcal{C}^{4,\alpha}(\partial B_1)^K \) and \( k \in \mathcal{C}^{2,\alpha}(\partial B_1)^K \) satisfying

\[
\| h \|_{\mathcal{C}^{4,\alpha}(\partial B_1)} + \| k \|_{\mathcal{C}^{2,\alpha}(\partial B_1)} \leq \kappa R_\varepsilon^{3-2m},
\]
satisfying
\[ \int_{\partial B^1} (4mh - k) = 0 \]
where \( \kappa > 0 \) will be fixed later on. We define in \( N_{R_{\epsilon}/a} \) the function
\[ W_{\epsilon,a,h,k} := \tilde{\chi}(W^h_{h,k}(a \cdot /R_{\epsilon})), \]
where \( \tilde{\chi} \) is a cutoff function which is identically equal to 1 in \( C^2 \) and identically equal to 0 in \( N_1 \) and where \( W^h_{h,k} \) has been defined in Proposition 8.3.

We will assume that the parameter \( \nu \) and the vector field \( X \) are uniformly bounded in \( N_{R_{\epsilon}/a} \).

To be more specific, we agree that
\[ |\nu| + \epsilon^{-1} \|X\|_{L^\infty(N_{R_{\epsilon}/a})} \leq c \]

Following the analysis already performed in the previous section, we have the :

**Proposition 9.2.** Given \( \delta \in (0,1) \), there exist \( c > 0 \) (independent of \( \kappa \)) and \( \epsilon_{\kappa} = \epsilon(\kappa) > 0 \) such that, for all \( \epsilon \in (0,\epsilon_{\kappa}) \) there exists a function \( \phi_{\epsilon,a,\nu,X,h,k} \in C^{4,\alpha}(\tilde{N}_{R_{\epsilon}/a}) \) such that the scalar curvature of the Kähler form
\[ \eta_{\epsilon,a,\nu,X,h,k} := a^2 \eta + i \partial \bar{\partial} (W_{\epsilon,a,h,k} + \phi_{\epsilon,a,\nu,X,h,k}), \]
defined on \( \tilde{N}_{R_{\epsilon}/a} \), is given by
\[ -ds(\eta_{\epsilon,a,\nu,X,h,k}) = \epsilon^2 \eta_{\epsilon,a,\nu,X,h,k}(X, -) \]
with
\[ -\frac{1}{|\partial B_{R_{\epsilon}/a}|} \int_{\partial B_{R_{\epsilon}/a}} s(\eta_{\epsilon,a,\nu,X,h,k}) = \epsilon^2 \nu \]
Moreover
\[ \|\phi_{\epsilon,a,\nu,X,h,k}(R_{\epsilon} \cdot /a)\|_{C^{4,\alpha}(\tilde{B}_{1 - B_{1/2}})} \leq c R_{\epsilon}^{3-2m}, \]
for some constant \( c > 0 \) independent of \( \kappa \). In addition, we have
\[ \|\phi_{\epsilon,a,\nu,X,h,k}(R_{\epsilon} \cdot /a) - \phi_{\epsilon,a,\nu',X',h',k'}(R_{\epsilon} \cdot /a')\|_{C^{4,\alpha}(<B_{1 - B_{1/2}})} \leq c_{\kappa} (R_{\epsilon}^{3-1} \|\eta - \eta'\|_{C^{4,\alpha}x\mathcal{L}^2}\alpha + R_{\epsilon}^{3-2m} (|\nu - \nu'| + \|X - X'\|_{L^\infty} + |a - a'|)). \]

Again the rest of the section is devoted to the proof of this result.

**Proof.** Using the fact that
\[ s(a^2 \eta + i \partial \bar{\partial} f) = s(a^2 (\eta + a^{-2} \partial \bar{\partial} f)) = a^{-2} s(\eta + a^{-2} \partial \bar{\partial} f) \]
We see that the scalar curvature of \( \tilde{\eta} \) can be expanded as
\[ s(a^2 \eta + i \partial \bar{\partial} f) = -\frac{1}{2} a^{-2} P_\eta P_\eta f + a^{-2} Q_\eta(a^{-2} \nabla^2 f), \]
since the scalar curvature of \( \eta \) is identically equal to 0. Again, the structure of the nonlinear operator \( Q_\eta \) is quite complicated but away from the exceptional divisor, it enjoys a decomposition
similar to the one described in the previous section. Indeed, we know from [2] that we can decompose
\[
Q_\eta(\nabla^2 f) = \sum_q B_{q,4,2}(\nabla^4 f, \nabla^2 f) C_{q,4,2}(\nabla^2 f) + \sum_q B_{q,3,3}(\nabla^3 f, \nabla^3 f) C_{q,3,3}(\nabla^2 f) + \sum_q |u|^{1-2m} B_{q,3,2}(\nabla^3 f, \nabla^2 f) C_{q,3,2}(\nabla^2 f) + \sum_q |u|^{-2m} B_{q,2,2}(\nabla^2 f, \nabla^2 f) C_{q,2,2}(\nabla^2 f)
\]
where the sum over \(q\) is finite, the operators \((U,V) - \rightarrow B_{q,j,j'}(U,V)\) are bilinear in the entries and have coefficients which are bounded functions in \(C_0(\bar{C}_1)\). The nonlinear operators \(W - \rightarrow C_{q,a,b}(W)\) have Taylor expansion (with respect to \(W\)) whose coefficients are bounded functions on \(C_0,\alpha(\bar{C}_1)\).

We set
\[
L_\eta := -\frac{1}{2} P_\eta^* P_\eta
\]
Replacing in (47) the function \(f = W + \phi\), where \(W = W_{\varepsilon,h,k}\) has been defined in (45), we see that (42) can be written as
\[
(48) L_\eta \phi + Q_\eta(\varepsilon^{-2} \nabla^2 (W + \phi)) = \varepsilon^2 \alpha^2 \zeta,
\]
which we would like to solve in \(\bar{N}_{\varepsilon,\alpha}\). Remember that the function \(\zeta\) solves
\[
-d\zeta = (a^2 \eta + i \partial \bar{\partial} (W + \phi))(X,-)
\]
with
\[
\frac{1}{|\partial B_{R_\varepsilon}|} \int_{\partial B_{R_\varepsilon}} \zeta = \nu
\]
We will need the :

**Definition 9.4.** Given \(\bar{R} > 1, \ell \in \mathbb{N}, \alpha \in (0,1)\) and \(\delta \in \mathbb{R}\), the weighted space \(C^\ell,\alpha_\delta(\bar{N}_{\bar{R}})\) is defined to be the space of functions \(f \in C^\ell,\alpha(\bar{N}_{\bar{R}})\) endowed with the norm
\[
\|f\|_{C^\ell,\alpha_\delta(\bar{N}_{\bar{R}})} := \|f\|_{C^\ell,\alpha(\bar{N}_{\bar{R}})} + \sup_{1 \leq R \leq \bar{R}} R^{-\delta} \|f(R\cdot)\|_{C^\ell,\alpha(B_{1}-B_{1/2})}
\]
For each \(\bar{R} \geq 1\), will be convenient to define an "extension" (linear) operator
\[
\tilde{\mathcal{E}}_{\bar{R}} : C^{0,\alpha}_{\delta}(N_{\bar{R}}) \rightarrow C^{0,\alpha}_{\delta}(\mathbb{C}^m),
\]
as follows :

(i) In \(\bar{N}_{\bar{R}}\), \(\tilde{\mathcal{E}}_{\bar{R}}(f) = f\),

(ii) in \(C_{2\bar{R}} - C_{\bar{R}}\)
\[
\tilde{\mathcal{E}}_{\bar{R}}(f)(u) = \frac{2 R - |u|}{R} f \left( \frac{\bar{R} u}{|u|} \right),
\]

(iii) in \(C_{2\bar{R}}\), \(\tilde{\mathcal{E}}_{\bar{R}}(f) = 0\).
It is easy to check that there exists a constant $c = c(\delta') > 0$, independent of $\hat{R} \geq 2$, such that
\begin{equation}
\| \hat{L}(f) \|_{C^{0,\alpha}(\hat{\Omega}_m)} \leq c \| f \|_{C^{0,\alpha}(N_R)},
\end{equation}
and furthermore, one can arrange easily for $\hat{L}$ to depend smoothly on $\hat{R}$.

The equation we will solve can be rewritten as
\begin{equation}
L_\eta \phi = \hat{L}_{R_\eta/a} \left( (\varepsilon^2 a^2 \zeta - Q_\eta(a^{-2} \nabla^2(W + \phi)) - L_\eta W ) \right).
\end{equation}
We fix $\delta \in (0,1)$ and use the result of Proposition 8.2. This provides a right inverse $\tilde{G}_\delta$ for the operator $L_\eta$.

We can now rephrase the solvability of (50) as a fixed point problem.
\begin{equation}
\phi = \tilde{N}(\varepsilon,a,\nu,X,h,k;\phi)
\end{equation}
where the nonlinear operator $\tilde{N}$ is defined by
\begin{equation}
\tilde{N}(\varepsilon,a,\nu,X,h,k;\phi) := \tilde{G}_\delta \circ \hat{L}_{R_\eta/a} \left( (Q_\eta(a^{-2} \nabla^2(W + \phi)) - L_\eta \hat{H}_{h,k} - \varepsilon^2 a^2 (\nu + \zeta) \right)
\end{equation}
To keep notations short, it will be convenient to define
\begin{equation}
\tilde{F} := C^{4,\alpha}_\delta(\hat{\Omega}_m)
\end{equation}

The existence of a fixed point for $\tilde{N}$ will follow from the :

**Lemma 9.3.** There exists $c > 0$ (independent of $\kappa$), $c_\kappa > 0$ and there exists $\varepsilon_\kappa = \varepsilon(\kappa) > 0$ such that, for all $\varepsilon \in (0,\varepsilon_\kappa)$
\begin{equation}
\| \tilde{N}(\varepsilon,a,\nu,X,h,k;0) \|_{\tilde{F}} \leq c R^{3 - 2m - \delta}_\varepsilon,
\end{equation}
Moreover, we have
\begin{equation}
\| \tilde{N}(\varepsilon,a,\nu,X,h,k;\phi) - \tilde{N}(\varepsilon,a,\nu,X,h,k;\phi') \|_{\tilde{F}} \leq c_\kappa R^{3 - 2m - \delta}_\varepsilon \| \phi - \phi' \|_{\tilde{F}}
\end{equation}
and
\begin{equation}
\| \tilde{N}(\varepsilon,a,\nu,X,h,k;\phi) - \tilde{N}(\varepsilon,a',\nu',X',h',k';\phi) \|_{\tilde{F}} \leq c_\kappa \left( R^{-1}_\varepsilon \| (h - h', k - k') \|_{C^{4,\alpha} \times C^{2,\alpha}} + R^{3 - 2m - \delta}_\varepsilon (|\nu' - \nu| + \|X - X'||_{L^\infty} + |a' - a|) \right)
\end{equation}
provided $\phi, \phi' \in \tilde{F}$, satisfy
\begin{equation}
\| \phi \|_{\tilde{F}} + \| \phi' \|_{\tilde{F}} \leq 4 c R^{3 - 2m - \delta}_\varepsilon,
\end{equation}
and $h, h'$ and $k, k'$ satisfy [44].

The proof of these estimates being identical to the one in [2], we omit it. Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,
\begin{equation}
c_\kappa R^{3 - 2m - \delta}_\varepsilon \leq \frac{1}{2}
\end{equation}
for all $\varepsilon \in (0,\varepsilon_\kappa)$. Then, the estimates (52) and (53) in the above Lemma are enough to show that
\begin{equation}
\phi \mapsto \tilde{N}(\varepsilon,a,\nu,X,h,k;\phi)
\end{equation}
is a contraction from
\begin{equation}
\{ \phi \in \tilde{F} : \| \phi \|_{\tilde{F}} \leq 2 c R^{3 - 2m - \delta}_\varepsilon \},
\end{equation}
into itself and hence has a unique fixed point \( \phi_{\varepsilon,a,\nu,X,h,k} \) in this set. This fixed point is a solution of (38) in \( \tilde{N}_{R_\varepsilon} \) and hence provides a constant scalar curvature Kähler form on \( \tilde{N}_{R_\varepsilon} \). The estimates in Proposition 9.2 follow at once from the estimates in Lemma 9.3 increasing the value of \( c_\varepsilon \) and reducing \( \varepsilon_n \) if this is necessary.

\[ \Box \]

10. Gluing the pieces together

Building on the analysis of the previous sections we complete the proof of Theorem 2.2. As far as technicalities are concerned the proof is identical to the one in [2] therefore, we shall only emphasize the differences in the present framework.

Before we proceed, a word about notations. In this section \( O_{C^{t,a}}(A) \) will refer to a function whose \( C^{t,a} \)-norm is bounded by \( A \) times a constant independent of \( \varepsilon \) and also independent of \( \kappa \) provided \( \varepsilon \) is chosen small enough (but which might depend on \( m, \omega \), the points \( p_j \) and the coefficients \( a_j \)). In general this function will be a nonlinear operator of the data.

We first exploit the result of Proposition 9.1. Given boundary data

\[ h := (h_1, \ldots, h_n), \quad k := (k_1, \ldots, k_n) \]

so that

\[ \int_{\partial B^2_j} k_j = 0 \]

we can apply the result of Proposition 9.1 to define on \( M_{\varepsilon,s} \) a Kähler form \( \omega_{\varepsilon,h,k} \) which can be written as

\[ \omega_{\varepsilon,h,k} = i \partial \bar{\partial} (\frac{1}{2} |z|^2 + \varphi^j (z) + \varepsilon^{2m-2} \Gamma_{\varepsilon,h,k} + W_{\varepsilon,h,k} + \phi_{\varepsilon,h,k}) \]

in \( B_{j,2R_\varepsilon} - B_{j,R_\varepsilon} \) where the function \( \varphi^j (z) = O(|z|^4) \) is the one which appears in Proposition 7.2 so that

\[ \omega = i \partial \bar{\partial} (\frac{1}{2} |z|^2 + \varphi^j (z)) \]

near \( p_j \). We define the function

\[ \psi^{a,j} := (\varphi^j + \varepsilon^{2m-2} \Gamma_{\varepsilon,h,k} + W_{\varepsilon,h,k} + \phi_{\varepsilon,h,k})(r_\varepsilon \cdot) \]

in \( B_2 - B_1 \). Collecting the result of Proposition 9.1, the definition of \( W_{\varepsilon,h,k}^a \) given in (31) and the expansion of \( \Gamma \) given in Lemma 9.1 we find that the function \( \psi^{a,j} \) can be expanded as

\[ \psi^{a,j} = -\frac{1}{m-2} a_j \varepsilon^{2m-2} \Gamma_{\varepsilon,h,k} + W_{\varepsilon,h,k} + \phi_{\varepsilon,h,k} \]

in dimension \( m \geq 3 \) while, in dimension \( m = 2 \), in view of the expansion of \( \Gamma \) Lemma 9.1 we have

\[ \psi^{a,j} - \varepsilon^2 (a_j \log r_\varepsilon + \varepsilon^2 b_j) = a_j \varepsilon^2 \log |\cdot| + W_{h_j,k_j}^a + \mathcal{O}_{C^{t,a}} (r_\varepsilon^4) \]

We will replace \( \psi^{a,j} \) by \( \psi^{a,j} - \varepsilon^2 (a_j \log r_\varepsilon + b_j) \) and there is no loss of generality in doing so since changing the potential by some constant function does not alter the corresponding Kähler forms.

According to Proposition 9.1, the scalar curvature of the Kähler form \( \omega_{\varepsilon,h,k} \) is given by

\[ s(\omega_{\varepsilon,h,k}) = s(\omega) + \langle \xi_{\omega_{\varepsilon,h,k}}, X_{\varepsilon,h,k} \rangle + \mu_s + \varepsilon^{2m-2} \lambda + \lambda_{\varepsilon,h,k} \]

where we have defined

\[ X_{\varepsilon,h,k} := X_s + \varepsilon^{2m-2} Y' + Y'_{\varepsilon,h,k} \in \mathfrak{h}' \]

(56)
We now exploit the result of Proposition 9.2. We choose boundary data
\[ \tilde{h} := (\tilde{h}_1, \ldots, \tilde{h}_n), \quad \tilde{k} := (\tilde{k}_1, \ldots, \tilde{k}_n) \]
whose components satisfy
\[ \int_{\partial B_1} (4m\tilde{h}_j - \tilde{k}_j) = 0 \]
as well as real positive parameters \( \tilde{a} := (\tilde{a}_1, \ldots, \tilde{a}_n) \). For each \( j = 1, \ldots, n \), we apply the result of Proposition 9.2 and define on \( \tilde{N}_{R_\varepsilon}/\tilde{a}_j \) the Kähler form
\[ \varepsilon^2 \eta_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j} \]
where
\[ \tilde{a}_j := \frac{\tilde{a}_j}{(m-1)} \]
and
\[ \nu_j := \frac{1}{|\partial B_1|} \int_{\partial B_1} s(\omega_{\varepsilon, h, k})(r_\varepsilon) \]
Moreover the scalar curvature of \( \varepsilon^2 \eta_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j} \) satisfies
\[ -d s(\varepsilon^2 \eta_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j}) = \eta_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j}(X_j, -) \]
with
\[ \int_{\partial B_1} \zeta(R_{\varepsilon \cdot /\tilde{a}_j}) = \nu_j \]
and \( X_j \) is the lift to \( \tilde{N}_{R_\varepsilon}/\tilde{a}_j \) of the holomorphic vector field \( X_{\varepsilon, h, k} \) defined in \( B_{\varepsilon, h, k} \). As explained at the end of section 7, this lifting can be performed in normal \( K \)-linear coordinates so that the vector field \( X_j \) is a \( K \)-invariant Killing vector field for the metric \( \tilde{a}_j^2 \eta \).

According to the analysis of section 6 and the result of Proposition 9.2, the Kähler form \( \eta \) can be written as
\[ \varepsilon^2 \eta_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j} = i \partial \bar{\partial} \left( \varepsilon^2 \tilde{a}_j^2 E_m(u) + \varepsilon^2 W_{\varepsilon, \tilde{a}_j, \tilde{h}_j, \tilde{k}_j} + \varepsilon^2 \phi_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j} \right) \]
in \( B_{R_\varepsilon}/\tilde{a}_j - B_{R_\varepsilon}/2\tilde{a}_j \). We define the function
\[ \psi^{i,j} := \left( \varepsilon^2 \tilde{a}_j^2 (E_m - \frac{1}{2} | \cdot |^2) + \varepsilon^2 W_{\varepsilon, \tilde{a}_j, \tilde{h}_j, \tilde{k}_j} + \varepsilon^2 \phi_{\varepsilon, \tilde{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j} \right) (R_{\varepsilon \cdot /\tilde{a}_j}), \]
defined in \( B_1 - B_{1/2} \). Using the analysis of section 6 as well as the result of Proposition 9.2, we have the expansion
\[ \psi^{i,j} = -\tilde{a}_j \varepsilon^{2m-2} r_\varepsilon^{4-2m} | \cdot |^{4-2m} + \varepsilon^2 W_{\tilde{h}_j, \tilde{k}_j} + \mathcal{O}_{C^{4,\alpha}}(r_\varepsilon^4) \]
in \( B_1 - B_{1/2} \), when \( m \geq 3 \) while we have
\[ \psi^{i,j} - \tilde{a}_j \varepsilon^2 \log \varepsilon = \tilde{a}_j \varepsilon^2 \log | \cdot | + \varepsilon^2 W_{\tilde{h}_j, \tilde{k}_j} + \mathcal{O}_{C^{4,\alpha}}(r_\varepsilon^4) \]
when \( m = 2 \). Again, in dimension \( m = 2 \) we will replace \( \psi^{i,j} \) by \( \psi^{i,j} - \tilde{a}_j \varepsilon^2 \log \varepsilon \) since this does not affect the definition of the corresponding Kähler metric.
The proof now follows verbatim the proof in [2]. We first describe the connected sum construction. By construction,
\[ M_\varepsilon := M \sqcup_{p_1, \varepsilon} N_1 \sqcup_{p_2, \varepsilon} \cdots \sqcup_{p_n, \varepsilon} N_n, \]
is obtained by connecting \( M_\varepsilon \) with the truncated spaces \( N_{R_\varepsilon/\hat{a}_1}, \ldots, N_{R_\varepsilon/\hat{a}_n} \). The identification of the boundary \( \partial B_{j, \varepsilon} \) in \( M_\varepsilon \) with the boundary \( \partial N_{R_\varepsilon/\hat{a}_j} \) of \( N_{R_\varepsilon/\hat{a}_j} \) is performed using the change of variables
\[ (z^1, \ldots, z^m) = \varepsilon \hat{a}_j (u^1, \ldots, u^m), \]
where \((z^1, \ldots, z^m)\) are the coordinates in \( B_{j, \varepsilon} \) and \((u^1, \ldots, u^m)\) are the coordinates in \( C_1 \).

The problem is now to determine these boundary data and parameters in such a way that, for each \( j = 1, \ldots, n \) the functions \( \psi^{\alpha j} \) and \( \psi^{i j} \) have their partial derivatives up to order 3 which coincide on \( \partial B_1 \).

In fact, we shall solve the following system of equations
\[ \psi^{\alpha j} = \psi^{i j}, \quad \partial_r \psi^{\alpha j} = \partial_r \psi^{i j}, \quad \Delta \psi^{\alpha j} = \Delta \psi^{i j}, \quad \partial_r \Delta \psi^{\alpha j} = \partial_r \Delta \psi^{i j}, \]
on \( \partial B_1 \) where \( r = |v| \) and \( v = (v^1, \ldots, v^m) \) are coordinates in \( \mathbb{C}^m \).

Let us assume that we have already solved this problem. The first identity in (59) implies that \( \psi^{\alpha j} \) and \( \psi^{i j} \) as well as all their \( k \)-th order partial derivatives with respect any vector field tangent to \( \partial B_1 \), with \( k \leq 4 \), agree on \( \partial B_1 \). The second identity in (59) then shows that \( \partial_r \psi^{\alpha j} \) and \( \partial_r \psi^{i j} \) as well as all their \( k \)-th order partial derivatives with respect any vector field tangent to \( \partial B_1 \), with \( k \leq 3 \), agree on \( \partial B_1 \). Using the decomposition of the Laplacian in polar coordinates, it is easy to check that the third identity implies that \( \partial^2_r \psi^{\alpha j} \) and \( \partial^2_r \psi^{i j} \) as well as all their \( k \)-th order partial derivatives with respect any vector field tangent to \( \partial B_1 \), with \( k \leq 2 \), agree on \( \partial B_1 \). And finally, the last identity in (59) implies that \( \partial^3_r \psi^{\alpha j} \) and \( \partial^3_r \psi^{i j} \) as well as all their first order partial derivatives with respect any vector field tangent to \( \partial B_1 \), agree on \( \partial B_1 \).

Moreover, the scalar curvature of the Kähler form
\[ \omega^{\alpha j} := i \partial \bar{\partial} \left( \frac{1}{2} |v|^2 + \psi^{\alpha j} \right), \]
defined in \( \bar{B}_2 - B_1 \) and the scalar curvature of the Kähler form
\[ \omega^{i j} := i \partial \bar{\partial} \left( \frac{1}{2} |v|^2 + \psi^{i j} \right), \]
defined in \( \bar{B}_1 - B_{1/2} \), match on \( \partial B_1 \) to produce a \( C^2 \) function on \( \bar{B}_2 - B_{1/2} \). To see this observe that both scalar curvature functions have the same mean value on \( \partial B_1 \) (this was precisely the purpose of (58)) and they satisfy
\[ -ds = \tilde{\omega}(X, -) \]
for the same vector field \( X \). Since the Kähler form is already \( C^1 \), we find that the right hand side is \( C^1 \) and hence the scalar curvature function is \( C^2 \).

This then implies that any \( k \)-th order partial derivatives of the functions \( \psi^{\alpha j} \) and \( \psi^{i j} \), with \( k \leq 4 \), coincide on \( \partial B_1 \).

Therefore, we conclude that the function \( \psi \) defined by \( \psi := \psi^{\alpha j} \) in \( \bar{B}_2 - B_1 \) and \( \psi := \psi^{i j} \) in \( \bar{B}_1 - B_{1/2} \) is \( C^4 \) in \( \bar{B}_2 - B_{1/2} \) and is a solution of the nonlinear elliptic partial differential equation
\[ s \left( i \partial \bar{\partial} \left( \frac{1}{2} |v|^2 + \psi \right) \right) = f. \]
where $f$ is defined by

$$-df = i\partial \bar{\partial}(\frac{1}{2} |v|^2 + \psi)(X, -)$$

and hence is a nonlocal first order differential operator in $\psi$. It then follows from elliptic regularity theory together with a bootstrap argument that the function $\psi$ is in fact smooth.

Hence, by gluing the Kähler metrics $\omega_{\epsilon, h, k}$ defined on $M_{\epsilon}$, with the metrics $\epsilon^2 \eta_{\epsilon}, \psi_{\epsilon}, X_j, \tilde{h}_j, \tilde{k}_j$ defined on $\tilde{N}_{\epsilon}/\tilde{h}_j$, we will produced a Kähler metric on $M_\epsilon$ which has constant scalar curvature. This will end the proof of Theorem 2.2.

It remains to explain how to find the boundary data $h = (h_1, \ldots, h_n)$, $k = (k_1, \ldots, k_n)$, $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n)$ and $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_n)$ as well as the parameters $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$.

We change the boundary data functions $h_j$ and $k_j$ into $h'_j$ and $k'_j$ defined by

$$h'_j := (\tilde{a}_j - a_j) r_\epsilon^4 - 2m \epsilon^{2m-2} + h_j$$
$$k'_j := 4 (m-2)(a_j - \tilde{a}_j) \epsilon^{2m-2} r_\epsilon^4 - 2m + k_j$$

when $m \geq 3$ and

$$h'_j := h_j$$
$$k'_j := 4 (a_j - \tilde{a}_j) \epsilon^2 + k_j$$

when $m = 2$. Recall that the functions $k_j$ are assumed to have mean 0 while the functions $k'_j$ are not assumed to satisfy such a constraint anymore. The role of the scalar $\tilde{a}_j - a_j$ is precisely to recover this lost degree of freedom in the assignment of the boundary data.

If $k$ is a constant function of $\partial B_1$, we extend the definition of $W_{0,k}^\alpha$ by setting

$$W_{0,k}^\alpha := \frac{k}{4(m-2)} (|z|^{2m-2} - |z|^{4m-2})$$

when $m \geq 3$ and by

$$W_{0,k}^\alpha = \frac{k}{4} \log |z|^2$$

when $m = 2$.

We also do not assume anymore that $\tilde{h}_j$ and $\tilde{k}_j$ satisfy (57) anymore. If $h$ is a constant function of $\partial B_1$, we extend the definition of $W_{0,k}^\alpha$ by setting

$$W_{h,k}^\alpha := h$$

Finally, we set

$$\tilde{h}'_j := \epsilon^2 \tilde{h}_j$$
$$\tilde{k}'_j := \epsilon^2 \tilde{k}_j$$

With these new variables, the expansions for both $\psi^{0,j}$ and $\psi^{i,j}$ can now be written as

$$\psi^{0,j} = -\tilde{a}_j r_\epsilon^4 - 2m \epsilon^{2m-2} | \cdot |^{4m-2} + W_{h'_j, k'_j}^\alpha + O_{c^{4,\alpha}}(r_\epsilon^4)$$
$$\psi^{i,j} = -\tilde{a}_j r_\epsilon^4 - 2m \epsilon^{2m-2} | \cdot |^{4m-2} + W_{h'_j, k'_j}^\alpha + O_{c^{4,\alpha}}(r_\epsilon^4).$$
when \( m \geq 3 \) and
\[
\psi_{m,j} = \tilde{a}_j \varepsilon^2 \log | \cdot | + W_{\tilde{h}^j,\tilde{k}^j}^\alpha + \mathcal{O}_{C^{4,\alpha}}(r_{z}^4)
\]
\[
\psi_{i,j} = a_j \varepsilon^2 \log | \cdot | + W_{h^i,k^j}^\alpha + \mathcal{O}_{C^{4,\alpha}}(r_{z}^4).
\]
when \( m = 2 \). Observe that, since \( \tilde{h}_j \) and \( \tilde{k}_j \) are not assumed to satisfy (57) anymore, this has changed the value of \( \psi_{i,j} \) by some constant which will not be relevant for the computation of the corresponding Kähler form.

The system (59) we have to solve can now be written as: For all \( j = 1, \ldots, n \)
\[
W_{h^j,k^j} = W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{4,\alpha}}(r_{z}^4)
\]
\[
\partial_r W_{h^j,k^j} = \partial_r W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{2,\alpha}}(r_{z}^4)
\]
\[
\Delta W_{h^j,k^j} = \Delta W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{4,\alpha}}(r_{z}^4)
\]
\[
\partial_r \Delta W_{h^j,k^j} = \partial_r \Delta W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{1,\alpha}}(r_{z}^4)
\]
on \( \partial B_1 \).

By definition of \( W_{h,k}^\alpha \) and \( W_{h,k}^i \), the first equations and third equations reduce to
\[
h^j = \tilde{h}^j + \mathcal{O}_{C^{4,\alpha}}(r_{z}^4)
\]
\[
k^j = \tilde{k}^j + \mathcal{O}_{C^{2,\alpha}}(r_{z}^4)
\]
Inserting these into the second and third sets of equations and using the linearity of the mapping \( (h, k) \mapsto W_{h,k}^\alpha \) and \( (h, k) \mapsto W_{h,k}^i \), the second and third equations become
\[
\partial_r W_{h^j,k^j} = \partial_r W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{3,\alpha}}(r_{z}^4)
\]
\[
\partial_r \Delta W_{h^j,k^j} = \partial_r \Delta W_{\tilde{h}^j,\tilde{k}^j} + \mathcal{O}_{C^{1,\alpha}}(r_{z}^4)
\]
for all \( j = 1, \ldots, n \). We now make use of the following result whose proof can be found in [1]:

**Lemma 10.1.** The mapping
\[
\mathcal{P} : C^{4,\alpha}(\partial B_1) \times C^{2,\alpha}(\partial B_1) \rightarrow C^{4,\alpha}(\partial B_1) \times C^{1,\alpha}(\partial B_1)
\]
\[
(h, k) \mapsto (\partial_r (W_{h,k}^\alpha - W_{h,k}^i), \partial_r \Delta (W_{h,k}^i - W_{h,k}^\alpha))
\]
is an isomorphism.

Using Lemma 10.1 (62) reduces to
\[
h^j \rightarrow \mathcal{O}_{C^{4,\alpha}}(r_{z}^4)
\]
\[
k^j \rightarrow \mathcal{O}_{C^{2,\alpha}}(r_{z}^4)
\]
for all \( j = 1, \ldots, n \). This, together with (61), yields a fixed point problem which can be written as
\[
(h', \tilde{h}', k', \tilde{k}') = S_{\varepsilon}(h', \tilde{h}', k', \tilde{k}),
\]
and we know from \((31)\) and \((63)\) that the nonlinear operator \(S_\varepsilon\) satisfies
\[
\|S_\varepsilon(h', \tilde{h}', k', \tilde{k}')\|_{(C^{3,\alpha})^{2n} \times (C^{2,\alpha})^{2n}} \leq c_0 \varepsilon^4,
\]
for some constant \(c_0 > 0\) which does not depend on \(\kappa\), provided \(\varepsilon \in (0, \varepsilon_\kappa)\). We finally choose
\[
\kappa = 2c_0,
\]
and \(\varepsilon \in (0, \varepsilon_\kappa)\). We have therefore proved that \(S_\varepsilon\) is a map from
\[
A_\varepsilon := \left\{(h', \tilde{h}', k', \tilde{k}') \in (C^{4,\alpha})^{2n} \times (C^{2,\alpha})^{2n} : \|\langle h', \tilde{h}', k', \tilde{k}'\rangle\|_{(C^{4,\alpha})^{2n} \times (C^{2,\alpha})^{2n}} \leq \kappa \varepsilon^4 \right\},
\]
into itself. It follows from \((32)\) and \((46)\) that, reducing \(\varepsilon_\kappa\) if this is necessary, \(S_\varepsilon\) is a contraction mapping from \(A_\varepsilon\) into itself for all \(\varepsilon \in (0, \varepsilon_\kappa)\). Therefore, \(S_\varepsilon\) has a fixed point in this set. This completes the proof of the existence of a solution of \((59)\).

The proof of the existence on \(M_{\varepsilon}^{Y}\) of a Kähler form \(\omega_{\varepsilon}\) which has constant scalar curvature is therefore complete. Observe that the scalar curvature of \(\omega\) and \(\omega_{\varepsilon}\) are close since the estimate
\[
|s(\omega_{\varepsilon}) - s(\omega)| \leq c \varepsilon^{2m-2}
\]
follows directly from the construction. We also have the estimates
\[
|\tilde{a}_j - a_j| \leq c \varepsilon^{2m} \varepsilon^{2m-2} = c \varepsilon^{2m+1}
\]
which is the last estimate which appears in the statement of Theorem \(2.2\).

Since the construction of the Kähler form \(\omega_{\varepsilon}\) is performed using fixed point theorems for contraction mappings, it should be clear that \(\omega_{\varepsilon}\) depends continuously on the parameters of the construction (such as the Kähler form \(\omega\), the points \(p_j\) and the coefficients \(a_j\)). In particular, when \(\mathfrak{h}'' = \{0\}\), conditions (i), (ii) and (iii) in the statement of Theorem \(2.2\) is void and in constructing \(\omega_{\varepsilon}\) one is free to prescribe the parameters \(a_j\). A simple degree argument then shows that given \(A \subset (\mathbb{R}^+)^n\) the image of mapping
\[
(a_1, \ldots, a_n) \mapsto (\tilde{a}_1, \ldots, \tilde{a}_n)
\]
contains \(A\) provided \(\varepsilon\) is chosen small enough. This completes the proof of the last remark at the end of the statement of Theorem \(2.2\). In the case where \(\mathfrak{h}'' \neq \{0\}\), one needs to apply a modified version of the analysis of \([2]\) which guarantees that the set of weights is open (keeping the required symmetries) provided no nontrivial element of \(\mathfrak{h}''\) vanishes at all points we blow up.

The proof of Proposition \(2.1\) also follows from the construction itself. Indeed, when \(\omega\) is a constant scalar curvature Kähler form, then \(X_0 = 0\). However, in the expansion of Proposition \(9.1\) one directly sees that the scalar curvature of \(\omega_{\varepsilon}\) will not be constant if the vector field \(Y'\) is not zero. Now \(Y' = 0\) if and only if \(\sum_j a_j \xi_\omega(p_j) = 0\). This completes the proof of Proposition \(2.1\).

11. Examples and Comments

11.1. Toric varieties. If \(M^m\) is a toric variety, then we can take \(K = T^m\) and \(\mathfrak{h} = \mathfrak{t}\) in virtue of Proposition \(7.4\). Theorem \(2.1\) asserts that one can blow up any set of points contained in the fixed-point set of the torus-action and the weights \(a_j > 0\) can be chosen arbitrarily. This is because the algebra of good vector fields that extends to the blown up manifold is precisely the
Lie algebra of the torus in this case, so $\mathfrak{h}'' = \{0\}$ and conditions (i) and (ii) become vacuous in this case.

This type of examples leads naturally to some observations:

(i) As mentioned in the introduction, our procedure can be iterated since blowing up a toric variety at such points preserves the toric structure. Therefore, we obtain extremal metrics on any such iterated blow up. Among this type of manifolds Donaldson [11] has studied one particular iterated blow up of $\mathbb{P}^2$, where all successive blow ups take place at fixed points of the torus action at the previous step. The number of iteration cannot be explicitly determined, yet these manifolds fall into the category to which our result applies. Nonetheless Donaldson analysis shows if on these manifolds we take Kähler classes sufficiently far from the boundary of the Kähler cone no extremal representative exist. This shows that even for these deceptively simple examples the understanding of the maximal range of application of our result (namely the determination of the optimal value of $\varepsilon_0$) is far from been trivial.

(ii) It is known that not every manifold admits an extremal metric but Tian [31] has conjectured that every Kähler manifold degenerates in a suitable sense to a manifold with an extremal metric.

We know two types of manifolds which do not carry any extremal metric: The projectivization of unstable rank two vector bundles over compact Riemann surfaces [7] (which verify Tian’s conjecture by their very construction) and some special iterated blow up of $\mathbb{P}^2$ constructed by Levine [19].

Our result can be used to shows how to fit these examples in Tian’s conjecture. For example, one observes that in Levine’s examples all the blow ups are of the type allowed by our construction except the last one where two points which do not correspond to vertices of the polytope are blown up. Nevertheless, this last manifold degenerates on the manifold obtained by blowing up one further vertex and then another point on the last exceptional divisor and, by our main theorem extremal metrics do exist on this limit manifold.

Besides these general results, we now look at the effect of our construction in specific cases.

Firstly, since $\mathbb{P}^n$ is a toric manifold the construction of the new extremal metrics on its blow ups at $n \leq m + 1$ linearly independent points follows. The fact that the resulting extremal metrics have constant scalar curvature iff we blow up $m + 1$ points with equal weights follows from Proposition 2.1 and from our preceding work [2]. Nonetheless this last result can be easily obtained directly. Let us in fact look at the Futaki invariants of the resulting manifolds. Let us recall Mabuchi’s result [20] stating that the Futaki invariant of a Kähler manifold vanishes if and only if the barycenter of the associated polytope lies in the origin. It is in general a hard combinatorial task, knowing the polytope, to determine its barycenter (as the reader of [25] can easily glance) and this prevents us to state general results, yet restricting ourselves to projective spaces we can get a clear picture.
The polytope associated to \( \mathbb{P}^m \) with its Kähler-Einstein metric is well known to be the simplex in \( \mathbb{R}^m \) with vertices

\[
p_1 = (1, 0, \ldots, 0) \ldots p_m = (0, \ldots, 0, 1) \quad p_{m+1} = (-1, \ldots, -1)
\]

and the vertices are exactly the images of the fixed points of the torus action. Let us recall also the effect of blowing up one of these points. For all the toric geometry we will use we refer to [15]. Given a vertex \( p_j \), the polytope associated to the manifold \( (Bl_p M, \pi^*[\omega] - aPD[E]) \) is the one of \( M \) where the vertex \( p_j \) is substituted by \( m \) vertices \( q^k_j, k = 1, \ldots m + 1, k \neq j \), where \( q^k_j = p_j + a(p_j - p_k) \). Blowing up a vertex with weight \( a > 0 \) has then the effect of cutting the starting polytope removing a simplex of “size” \( a \).

In general it is possible to get that the barycenter stays in the origin even blowing up fewer than the whole set of vertices and with uncontrollable weights. For example, let us take a rectangle with vertices \((\alpha, \beta)\), \((1, 0)\), \((0, 1)\), \((-1, 0)\), \((0, -1)\), and \((-1, -1)\). The existence of an Einstein (constant scalar curvature) metric in this class is a result of Sin [28] and Tian-Yau [22]. Blowing up two pairs of vertices symmetric with respect to the origin (hence 4 points) with equal pairwise weights, we get new constant scalar curvature metrics on the blow up.

On the other hand, in the case of \( \mathbb{P}^m \), plugging in the above numbers, it is an elementary calculation to get that the barycenter of the blow up is still in the origin iff \( n = m + 1 \) and \( a_1 = \cdots = a_{m+1} \). Calabi’s result [3], stating that an extremal metric in a Kähler class whose Futaki invariant vanishes is of constant scalar curvature, gives a different proof of our characterization of Kähler constant scalar curvature metrics among our new extremal ones.

Once we know that all manifolds obtained by blowing up any number of points in general position in \( \mathbb{P}^m \) admit an extremal metric (for \( n \geq m + 2 \) these are in fact constant scalar curvature metrics [2]), let us focus on the Kähler classes we can reach for \( m = 2 \).

Recall that \( \mathbb{P}^1 \times \mathbb{P}^1 \) is itself a toric variety whose polytope corresponding to the Kähler class \( \alpha[\omega_{FS}] + \beta[\omega_{FS}], \alpha, \beta > 0 \), where \( \omega_{FS} \) is half the Kähler-Einstein metric on the \( j \)-th factor, is the rectangle with vertices \((\alpha, \beta), (-\alpha, \beta), (\alpha, -\beta), (-\alpha, -\beta)\). Let us also briefly recall the following classical construction: take \( p_1, p_2 \in \mathbb{P}^2 \) and consider \( M = Bl_{p_1, p_2} \mathbb{P}^2 \). \( M \) contains three \((-1)\)-curves, the two exceptional divisors \( E_1, E_2 \) and the proper transform \( L \) of the line in \( \mathbb{P}^2 \) passing through \( p_1 \) and \( p_2 \). \( M \) is in fact biholomorphic to \( Bl_q(\mathbb{P}^1 \times \mathbb{P}^1) \) for some (hence any) choice of \( q \in \mathbb{P}^1 \times \mathbb{P}^1 \). In fact, contracting (“blowing down”) \( L \) we get a manifold biholomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) where the rulings correspond to the pencils of lines through \( p_1 \) and \( p_2 \).

Now, having called \( A_1 = [(\mathbb{P}^1 \times \{pt\}], A_2 = [(\{pt\} \times \mathbb{P}^1], \) and \( E \) the exceptional divisor in \( Bl_q(\mathbb{P}^1 \times \mathbb{P}^1) \), it is easy to check the correspondence of classes

\[
\alpha PD[A_1] + \beta PD[A_2] - \lambda PD[E] \leftrightarrow (\alpha + \beta - \lambda)\pi^*[\omega_{FS}] - (\alpha - \lambda)PD[E_1] - (\beta - \lambda)PD[E_2].
\]

Hence our new extremal metrics on \( Bl_q(\mathbb{P}^1 \times \mathbb{P}^1) \), which lie in the classes \( \alpha PD[A_1] + \beta PD[A_2] - \varepsilon^2 \lambda PD[E] \), give extremal metrics in the classes of \( Bl_{p_1, p_2} \mathbb{P}^2 \)

\[
\pi^*[\omega_{FS}] - \frac{\alpha - \varepsilon^2 \lambda}{\alpha + \beta - \varepsilon^2 \lambda} PD[E_1] - \frac{\beta - \varepsilon^2 \lambda}{\alpha + \beta - \varepsilon^2 \lambda} PD[E_2].
\]
Hence we have a whole neighborhood of the boundary line \( a + b = 1 \) in the Kähler cone of \( Bl_{p_1,p_2} \mathbb{P}^2 \), where \( \pi^*[\omega_{FS}] - aPD[E_1] - bPD[E_2], a + b \leq 1, a, b > 0 \). This proves Corollary 2.3 (1).

Another similar construction, known as Cremona transformation (see e.g. [13] pages 397-399), allows us to prove Corollary 2.3 (2). We wish to thank M. Abreu for bringing it to our attention.

This time we construct an automorphism of \( Bl_{p_1,p_2,p_3} \mathbb{P}^2 \), when the points do not lie on a complex line, in the following way: call \( l_{jk} \) the lines in \( \mathbb{P}^2 \) through \( p_j \) and \( p_k \) their proper transforms. The key observation this time is that blowing down \( L_{jk} \) we are left with a new copy of \( \mathbb{P}^2 \), where the new coordinate lines are the exceptional divisors of the original blow up. The resulting automorphism of \( Bl_{p_1,p_2,p_3} \mathbb{P}^2 \) has the following action in cohomology, where we indicate by \( F_j \) the exceptional divisors in the new copy of \( Bl_{p_1,p_2,p_3} \mathbb{P}^2 \):

\[
\begin{align*}
\pi^*[\omega_{FS}] - PD[E_1] - PD[E_2] & \leftrightarrow PD[F_1], \\
\pi^*[\omega_{FS}] - PD[E_1] - PD[E_3] & \leftrightarrow PD[F_2], \\
\pi^*[\omega_{FS}] - PD[E_2] - PD[E_3] & \leftrightarrow PD[F_3], \\
\pi^*[\omega_{FS}] & \leftrightarrow 2\pi^*[\omega_{FS}] - PD[F_1] - PD[F_2] - PD[F_3].
\end{align*}
\]

Hence

\[
\pi^*[\omega_{FS}] - aPD[E_1] - bPD[E_2] - cPD[E_3]
\]

corresponds to

\[
(2 - a - b - c)\pi^*[\omega_{FS}] - (1 - a - b)PD[F_1] - (1 - a - c)PD[F_2] - (1 - b - c)PD[F_3]
\]

This gives the sought Kähler classes as claimed.

11.2. Sporadic examples. In the case where the manifold we study is the projective space, we can also study the existence existence of extremal metrics on the blown up manifold when the position of the blown up points is not generic, hence leaving the toric world. For example in [2] it was shown how to construct Kähler constant scalar curvature metrics on \( \mathbb{P}^2 \) blown up at 4 points, 3 of which were lying on a line. Considering extremal metrics instead on constant scalar curvature metrics, one can do more: for example, consider the points

\[
p_1 = [0 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}}], \quad p_2 = [0, \frac{\alpha}{\sqrt{[\alpha]^2 + [\beta]^2}} : \frac{\beta}{\sqrt{[\alpha]^2 + [\beta]^2}}], \quad p_3 = [0 : \frac{\beta}{\sqrt{[\alpha]^2 + [\beta]^2}} : \frac{\alpha}{\sqrt{[\alpha]^2 + [\beta]^2}}]
\]

and the group \( K = S^1 \) whose action on \( \mathbb{P}^2 \) is given by

\[
S^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \\
(\theta, [z^1 : z^2 : z^3]) \quad \longrightarrow \quad [\theta^{-2} z^1 : \theta z^2 : \theta z^3]
\]

Of course \( p_1, p_2 \) and \( p_3 \) are fixed under the action of \( K \), but we want to impose more symmetries working equivariantly with respect to a discrete group \( A \) of permutations of the last two coordinates. It is easy to check that the space of vector fields of \( \mathfrak{h} \) which are invariant under the action of \( A \) is given by

\[
\mathfrak{h}_A = \text{Span}\{ \Re(z^2 \partial_{z^1} + z^3 \partial_{z^2}), \Re(2 z^1 \partial_{z^1} - z^2 \partial_{z^2} - z^3 \partial_{z^3}) \}
\]
It is immediate to see that
\[ h'_A = \text{Span} \{ \Re (2z_1 \partial z_1 - z_2 \partial z_2 - z_3 \partial z_3) \} \]
Observe that all points belong to the zero set of the vector fields in \( h' \). Condition (ii) in Theorem 2.2 is fulfilled provided
\[ \Re (\alpha \bar{\beta}) < 0 \]
with \( a_1 = -\frac{2\Re (\alpha \bar{\beta})}{|\alpha|^2 + |\beta|^2} \) and \( a_2 = a_3 = 1 \), while condition (iii) holds since \( \Re (z_2 \partial z_3 + z_3 \partial z_2) \) does not vanish at \( p_2 \) and at \( p_3 \). It is important to observe that \( a_1 \) automatically satisfies the upper bound \( a_1 \leq 1 \). In fact, it has been shown by A. Della Vedova [12] that for \( a_1 > 2 \) the corresponding polarized manifold is not relative K-stable, hence forbidding the existence of an extremal metric in the corresponding Kähler class thanks to Szekelehidi’s result [29].

This gives the construction of extremal Kähler metrics on the blow up \( \mathbb{P}^2 \) at \( p_1, p_2 \) and \( p_3 \). The fact that the corresponding metrics do not have constant scalar curvature follows directly from Proposition 2.1 or implicitly observing that the resulting manifold does not satisfy the Matsushima-Licherovitz obstruction so in fact it does not admit constant scalar curvature metrics in any Kähler class.

The same remark we observed after Corollary 2.1 also holds for this example. Indeed, the position of three (even aligned) points in \( \mathbb{P}^2 \) can be \textit{a priori} prescribed (just leaving them on some line) with the use of an appropriate automorphism. Therefore the above calculation implies that \( \text{any} \) set of three aligned points can be blow up and extremal metrics on the blown up manifold can be found even if in the initial coordinates the symmetries of the above example are not present. Of course the change of coordinates required to put the initial set of points into the above one might change the base Fubini-Study metric we work with, but this does not change its Kähler class. This discussion can be summarized in the:

**Corollary 11.1.** Given three aligned points \( p_1, p_2, p_3 \) in \( \mathbb{P}^2 \) and weights \( a_1, a_2, a_3 \) satisfying \( a_j = a_k \) for some \( j, k \) and \( a_l \leq a_j \) for all \( l, j \), there exists \( \varepsilon_0 > 0 \) and for all \( \varepsilon \in (0, \varepsilon_0) \) there exists an extremal Kähler form of non constant scalar curvature \( \omega_\varepsilon \) on the blow up of \( \mathbb{P}^2 \) at \( p_1, p_2, p_3 \) with \( \omega_\varepsilon \in \pi^* [\omega_{FS}] - \varepsilon^2 (a_1 PD[E_1] + a_2 PD[E_2] + a_3 PD[E_3]) \)

As expected, adding points to be blown up, also on the same line, makes things even simpler. For example let us work out the situation where 4 aligned points are to be blown up. In this case we can avoid using extra symmetries and we can work directly with a connected group of isometries. Therefore, we now consider the points
\[ p_1 = [0 : 1 : 0], \quad p_2 = [0 : \frac{1}{\sqrt{3}} : \frac{1+i}{\sqrt{6}}], \quad p_3 = [0 : \frac{1}{\sqrt{5}} : \frac{2-i}{\sqrt{6}}], \quad \text{and} \quad p_4 = [0 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}}] \]
and the group \( K = S^1 \), whose action on \( \mathbb{P}^2 \) given by
\[ S^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \]
\[ (\alpha, [z^1, z^2, z^3]) \rightarrow [\alpha z^1, z^2, z^3] \]
Of course $p_1, \ldots, p_4$ are fixed by the action of $K$. It is easy to check that $h$ is now given by
\[ h = \text{Span}\{\Re(z^2\partial z_3 + z^3\partial z_2), \Re(i(z^2\partial z_3 - z^3\partial z_2)), \Re(2z^1\partial z_1 - z^2\partial z_2 - z^3\partial z_3), \Re(z^2\partial z_2 - z^3\partial z_3)\} \]

We choose
\[ h' = \text{Span}\{\Re(2z^1\partial z_1 - z^2\partial z_2 - z^3\partial z_3)\} \]

Observe that all points belong to the zero set of the vector fields in $h'$ and that none of the nontrivial elements of $h''$ vanish at all the $p_j$. Our construction then gives extremal (non constant scalar curvature) Kähler metrics on the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_4$ with weights $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 2$.

These examples can be easily extended to projective spaces of any dimension.

References

[1] C. Arezzo and F. Pacard, Blowing up and Desingularizing Kähler orbifolds with constant scalar curvature, math.DG/0412405, to appear in Acta Math.
[2] C. Arezzo and F. Pacard, Blowing up Kähler manifolds with constant scalar curvature II, math.DG/0504115.
[3] C. Arezzo, F. Pacard and M. Singer, On the Kähler classes of extremal metric on blow ups, in preparation.
[4] A. Besse, Einstein manifolds. Springer-Verlag, Berlin, 1987.
[5] V.V. Batyrev and E. N. Selinova, Einstein-Kählermetrics on symmetric toric manifolds, J. Reine und Angew. Math. 512 (1999), 225-236.
[6] S. Bochner and W.T. Martin, Several complex variables, Princeton Univ. Press, 1948.
[7] D. Burns and P. De Bartolomeis, Stability of vector bundles and extremal metrics, Invent. Math. 92 (1988), 403-407.
[8] E. Calabi, Extremal Kähler metrics, Annals of Math. Studies 102, Princeton Univ. Press, 1982, 269-290.
[9] E. Calabi, Extremal Kähler metrics II, in Differ. Geometry and its Complex Analysis, edited by I. Chavel and H.M. Farkas, Springer, 1985.
[10] X.X. Chen and G. Tian, Geometry of Kähler metrics and holomorphic foliation by discs, math.DG/0507148.
[11] S.K. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), 289-349.
[12] A. Della Vedova, Relative stability of points and relative $K$-stability, in preparation.
[13] R. Hartshorne, Algebraic geometry, GTM 52, Springer, 1977.
[14] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley-Interscience Publ., 1978.
[15] V. Guillemin, Moment maps and combinatorial invariants of Hamiltonian $T^n$-spaces, Progress in Mathematics 122, Birkhäuser, 1994.
[16] C. LeBrun, Counter-examples to the generalized positive action conjecture Comm. Math. Phys. 118 (1988), 591-596.
[17] C. LeBrun and S. Simanca, Extremal Kähler metrics and complex deformation theory, Geom. Funct. Anal. 4 (1994), 298-336.
[18] C. LeBrun and S. Simanca, On the Kähler classes of extremal metrics, in Geometry and Global Analysis (Sendai, 1993), Tohoku Univ., 255-271.
[19] M. Levine, A remark on extremal Kähler metrics, J. Differential Geom. 21 (1985), 73-77.
[20] T. Mabuchi, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties. Osaka J. Math. 24 (1987), no. 4, 705-737.
[21] T. Mabuchi, Stability of extremal Kähler manifolds. Osaka J. Math. 41 (2004), no. 3, 563-582.
[22] R. Mazzeo, Elliptic theory of edge operators I, Comm. in PDE, 10 (1991) 1616-1664.
[23] R. Melrose, The Atiyah-Patodi-Singer index theorem, Research notes in Math. 4 (1993).
[24] Y. Nakagawa, Classification of Einstein-Kähler toric Fano fourfolds, Tohoku Math. J. 46 (1994), 125-133.
[25] Y. Nakagawa, Combinatorial formulae for Futaki characters and generalized Killing forms of toric Fano orbifolds, Monogr. Geom. Topol. 25 (1998), Intern. Press.
[26] Y. Rollin and M. Singer, *Non-minimal scalar-flat Kaehler surfaces and parabolic stability*, math.DG/0404423, to appear in Invent. Math.

[27] S. Simanca, *Kähler metrics of constant scalar curvature on bundles over CP^{n-1}*, Math. Ann. 291 (1991), 239-246.

[28] Y.T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitably symmetry group*, Ann. of Math 127 (1988), 585-627.

[29] G. Szekelyhidi, *Extremal metrics and K-stability*, math.AG/0410401

[30] R. Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*, math.AG/0512411

[31] G. Tian, *Extremal metrics and geometric stability*, Houston J. Math. 28 (2002), 411-432.

[32] G. Tian and S.T. Yau, *Kähler-Einstein metrics on complex surfaces with C_1(M) > 0*, Comm. Math. Phys. 112 (1987), 175-203.