A Theory of Flavor

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Abstract. Understanding flavor is one of the fundamental puzzles of the standard model. The discovery of neutrino masses in the past decade, exhibiting mixing and mass patterns so very different from the quark sector has added an extra dimension to this problem. In this paper, I discuss a proposal that appears to provide a promising way to resolve this puzzle in a unified framework incorporating both quarks and leptons.

1. Introduction
The discovery of the Higgs boson at the LHC has solved the problem of the origin of mass for the quarks and charged leptons. This is a major accomplishment. There however remains the problem of why there are three generations of quarks and leptons and the pattern of masses and mixings between generations. In the quark sector the masses as well as mixings exhibit a hierarchical pattern i.e. for masses $m_{u,d} \ll m_{c,s} \ll m_{t,b}$; for mixing angles $V_{ub} \ll V_{cb} \ll V_{cd} \ll V_{ud,cs,tb}$. This is known as the flavor puzzle for quarks. Unravelling this puzzle has long been recognized as a challenge for physics beyond the standard model. The puzzle of flavor got more challenging after the discovery of neutrino masses and mixings. Unlike the quark sector, lepton mixings do not exhibit a hierarchical pattern. For example, neutrino mixings between generations, denoted by $\theta_{ij}$, are given by $\theta_{23} \sim 45^\circ$, $\theta_{12} \simeq 35^\circ$ and $\theta_{13} \sim 9^\circ$ (as against $\theta_{23} \sim 2.5^\circ$ and $\theta_{12} \sim 13^\circ$ and $\theta_{13} \simeq 0.03^\circ$). The neutrino masses also do not exhibit as strong as strong a hierarchy as the quarks or charged leptons i.e. for a normal hierarchy for neutrinos, $m_2/m_3 \simeq V_{us} \gg m_\mu/m_\tau$. An important question now is: what kind of new physics beyond the standard model can provide a unified explanation of these observations?

There are two possible approaches to the flavor problem: (i) treat quarks and leptons separately and understand each sector on the basis of new physics or (ii) consider an unified approach to both flavor problems in the same framework. The first approach has led to the consideration of many new discrete flavor symmetries in the lepton sector. We take the second approach in this talk and discuss the flavor problem within a grand unified framework, since such a framework is known to unify quarks and leptons. There are also other reasons for considering grand unified theories, which have many appealing features such as unification of forces, quantization of electric charges etc. More importantly for us, understanding small neutrino masses may also be pushing us in that direction, as we see below. The reason is that neutrinos are known to have very tiny masses compared to charged fermions. One popular way to understand this has been to realize that neutrinos are electrically neutral particles and could therefore be their own anti-particles, the so-called Majorana fermions and adopt a framework...
known as the seesaw mechanism\[1\], where this Majorana nature plays a critical role. In the seesaw mechanism, one introduces a right handed neutrino (\(N\)) to the standard model and give it a large Majorana mass, \(M_N\). The right handed neutrino of course mixes with the familiar left-handed one (\(\nu\)). The neutrino mass matrix then involves both the (\(\nu, N\)). Diagonalizing the (\(\nu, N\)) mass matrix then immediately gives the formula for the light neutrino mass as \(m_\nu \simeq \frac{m^2_\nu}{M_N}\).

First this predicts that the neutrinos are their own anti-particles and secondly, if we assume that \(M_N \gg v_{ew}\), it explains why \(m_\nu \ll m_{e,\mu}\).

The question then arises as to what new physics gives \(N\) a large Majorana mass? Since the Majorana mass of \(N\) breaks B-L symmetry of the standard model, seesaw mechanism is indicative of a new symmetry of nature. Secondly, it is more appealing to consider local \(B-L\) symmetry rather than a global one. What we now need to know is: what is the scale of B-L symmetry breaking (or the value of the \(N\) Majorana mass) ? Secondly: what the associated new physics is. Naive analysis of the seesaw formula implies that \(M_N \simeq 10^{14}\) GeV or thereabouts, although TeV scale \(M_N\) can be entertained if neutrino Yukawa couplings are chosen to be of the same order as the electron Yukawas. Since the high scale seesaw mass 10\(^{14}\) GeV is tantalizingly close to the typical grand unification scale of 10\(^{16}\) GeV, one may suspect that GUT physics may be manifesting via the neutrino mass. In fact seesaw mechanism becomes truly natural within a grand unified theory suggesting that this framework which also unifies quarks and leptons may provide a logical choice while considering a unified approach to flavor.

Clearly, the problem of quark-lepton masses and mixings becomes specially puzzling in grand unified theories due to quark-lepton unification at very high scales. However, we argue that if proper approach and a proper class of grand unified model is chosen, this problem can be resolved. Thus the hope is that cracking this code may not only move us forward in our attempt at understanding the flavor problem but it may also provide a hint of some really new exciting underlying physics. We give an example of this below.

The proposal for flavor unification described here was made in a paper by Dutta and Mimura and this author in 2010\[2\] where it was found that it is realizable in the framework of SO(10) grand unified theories. We discuss the various elements of the proposal and its realization below.

As a prelude to this discussion, let us realize that one way to understand the quark mass hierarchy is to start with a rank one mass matrix for up, down quarks and charged leptons in the leading order i.e.

\[
M_{u,d,l} = m_{t,b,\tau} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

and consider corrections coming from non-leading operators. Within such a framework, the third generation masses would come from the leading order terms and the others from the nonleading terms. Second clue for our ansatz is the observation that small quark mixings could be due to the following: if we wrote the up and down quark mass matrices as a sum a “big” matrix plus a small matrix with the “big” matrix part for both sectors being proportional to each other, then in the leading order the CKM angles would vanish e.g.

\[
M_{u,d} = M^0_{u,d} + \delta_{u,d}
\]

with \(M^0_{u,d} = r M^0_{u,d}\) being the “big” matrix and \(\delta_{u,d}\) being the smaller part. As just mentioned, the proportionality of the large parts of the mass matrices guarantees that the mixing angles will necessarily be small since the diagonalizing matrix for the large parts are “parallel” or “aligned” and the nontrivial CKM matrix represents the “small” misalignment between the two matrices determined by the “smaller” parts of the mass matrix.

We can therefore now state our ansatz\[2\] which consists of two parts:

\[
\]
(i) The quark and lepton mass matrices have the following general feature:

\[
M_u = M_0 + \delta_u; \\
M_d = rM_0 + \delta_d; \\
M_l = rM_0 + \delta_l; \\
M_\nu = f v_L
\]

(ii) \(M_0\) has rank one.

Note that by a choice of the lepton basis, we can make \(f\) diagonal without loss of generality. It is then clear from the first part of the ansatz (item one) that for an arbitrary form of the matrices \(M_0\) and \(\delta\) as long as \(\delta_{ij} \ll M_{0,ij}\), the lepton mixing angles are large whereas the quark mixing angles are small. The second rank one property than guarantees that quark and charged lepton masses are hierarchical whereas since \(f\) matrix is arbitrary, any hierarchy in the neutrino sector is likely to be milder. Incidentally, rank one property to understand mass hierarchy has been used in the past; see for instance[3].

2. Gauge group required to implement the ansatz

The question that arises next is how to implement our ansatz with a gauge model framework. To implement the first part, it is important to notice that the up and down quark mass matrices must be proportional to each other. Such relations do not emerge from the standard model since the \(u_R\) and \(d_R\) fields are separate fields and their Yukawa couplings responsible for the up and down quark mass matrices are therefore independent of each other. The situation however changes once we expand the gauge group to the left-right symmetric group \(SU(2)_L \times SU(2)_R \times U(1)_{B-L}\) group since the \((u_R, d_R)\) form a doublet of the \(SU(2)_R\) group and thus relate the up and down Yukawa matrices[4]. Since quark-lepton unifications arises naturally within grand unification theories, the obvious group to consider is the SO(10) group as we do in the next section, although the basic conditions of the ansatz could also be realized with less predictive power within the \(SU(2)_L \times SU(2)_R \times SU(4)_c\) partial unification groups.

3. SO(10) realization of flavor unification ansatz

As is well known, in SO(10) models, the matter fermions belong to 16-dim. spinor representations. To get fermion masses, we will consider SO(10) models with 10, 126 plus possibly another 10 or 120 Higgs fields where fermion masses are generated by renormalizable Yukawa couplings [5] only and where type II seesaw[6] is responsible for neutrino masses [7]. To implement our idea, we require that one of the 10 Yukawa couplings is the dominant one contributing to up, down and charged lepton masses and has rank one with other smaller couplings providing neutrino masses as well as most of the quark lepton flavor hierarchy. We postpone the discussion of how to get rank one till later. Let us see if this model does indeed give us our ansatz. It is well known that in these models[5], we have the following form for all fermion masses:

\[
Y_u = h + r_2 f + r_3 h', \\
Y_d = r_1(h + f + h'), \\
Y_e = r_1(h - 3f + c_e h'), \\
Y_{\nu D} = h - 3r_2 f + c_\nu h',
\]

where \(Y_a\) are mass matrices divided by the electro-weak vev \(v_{ew}\) and \(r_i\) and \(c_{e,\nu}\) are the mixing parameters which relate the \(H_{u,d}\) to the doublets in the various GUT multiplets. More precisely,
the matrices $h$, $f$ and $h'$ in $Y_a$ are multiplied by the Higgs mixing parameters when they appear in the fermion mass matrices.

Furthermore, we use the type II seesaw formula[6] for getting neutrino masses gives [7].

\[ M_\nu = f v_L. \] (5)

Note that $f$ is the same coupling matrix that appears in the charged fermion masses in Eq. (4), up to factors from the Higgs mixings and the Clebsch-Gordan coefficients. This helps us to connect the neutrino parameters to the quark-sector parameters. The equations (4) and (5) are the key equations in our unified approach to addressing the flavor problem and obviously satisfy our flavor unification ansatz.

4. Implementing rank one strategy

The rank one Yukawa coupling with 10 Higgs field generates the features of flavor hierarchy, and rank 1 matrices can often appear in various ways (flavor symmetry, discrete symmetry, and string models). In this section, we give an SO(10) model, where the rank one ansatz used in our discussion of flavor emerges from extra vector like spinors above the GUT scale as well as a discrete symmetry.

When the direct couplings of chiral fermions with a Higgs field are forbidden by the chosen discrete symmetry, and the effective Yukawa couplings are generated by propagating vector-like matter fields, the rank of the effective Yukawa matrix depends on the number of the vector-like fields. Actually, when there are only one pair of vector-like matter fields as a flavor singlet, the effective Yukawa matrix is rank 1.

To illustrate this in a warm-up example, we consider a model which has one extra vector-like pair of matter fields to start with with mass slightly above the GUT scale contributing to the 10 coupling (denoted by $\psi_V \equiv 16_V \oplus \bar{\psi}_V \equiv \overline{16}_V$) and three gauge singlet fields $Y_a$. We add a $Z_4$ discrete symmetry to the model under which the fields $\psi_a \rightarrow i \psi_a$, and $Y_a \rightarrow -i Y_a$. The 10-Higgs field $H$ is invariant under this symmetry. The gauge invariant Yukawa superpotential under this assumption is given by

\[ W = \psi_V H \lambda \psi_V + M_V \psi_V \bar{\psi}_V + \bar{\psi}_V \sum_a Y_a \psi_a. \] (6)

When we give vevs $\langle Y_a \rangle \neq 0$, $\psi_V$ and $\psi_a$ are mixed. The heavy vector-like fields, $\bar{\psi}_V$ and a linear combination of $\psi_V$ and $\psi_a$ (i.e. $M_V \psi_V + \sum_a Y_a \psi_a$), and the effective operator below its scale and at the GUT scale is given by:

\[ \mathcal{L}_{\text{eff}} = \frac{\lambda}{M_V^2 + \sum_a Y_a^2} \left[ \sum_a Y_a \psi_a \right] H \left[ \sum_b Y_b \psi_b \right]. \] (7)

This gives rise to a rank one $h$ coupling. We note that it does not contradict the $O(1)$ top Yukawa coupling, when $M_V^2 \sim \sum_a Y_a^2$ (or $M_V^2 < \sum_a Y_a^2$).

If we let the $126 \overline{126}$ Higgs field transform like $-1$ under $Z_4$, it can induce the $f$ coupling with rank three. Our final model given below builds on this but differs in details e.g. it has got two vectorlike spinor multiplets instead of one etc.

5. Making the model predictive

Simply using the above ansatz in the context of an SO(10) model with mass relations in Eq. (5), turns out to reproduce the qualitative features of the quark and lepton spectra quite well. For example in the context of a two generation model (involving the second and third generation), this simple ansatz predicts $V_{cb} \simeq (m_s/m_b + e^{\sigma_m} m_c/m_t) \cot \theta$, where $\theta$ is the atmospheric mixing
angle. This relation is in rough agreement with observations to leading order. In addition, we have at GUT scale $m_0 \sim m_r$ as well as $m - \mu \sim -3m_s$ also in rough agreement with observations.

Encouraged by these results, we can be more ambitious and start using this ansatz in combination of other ideas to make as many predictions as possible. To this end, we note that in the limit vanishing Yukawa couplings, the standard model has $[U(3)]^5$ global symmetry. It is therefore quite possible that in the final understanding of flavor, a subgroup of this large symmetry does play an important role\cite{8}, specially subgroups which have three dimensional representation to fit three generations. In order to exploit this observation, one may replace all Yukawa couplings by flavon fields which transform as three dimensional representations of a subgroup of $[SU(3)]^5$ and consider the minima of the flavon theory in flavon space as determining the values of the Yukawa couplings. It turns out that there are nontrivial examples where this program is realized. In the second paper of \cite{2}, we presented an $S_4$ subgroup example. Below I briefly recapitulate this example.

6. The $SO(10) \times S_4 \times Z_n$ model of flavor
Recall that the $S_4$ group is a 24 element group describing permutations of four distinct objects and has five irreducible representations with dimensions $3_1 \oplus 3_2 \oplus 2 \oplus 1_2 \oplus 1_1$. The distinction between the representations with subscripts 1 and 2 is that the later change sign under the transformation of group elements involving the odd number of permutations of $S_4$.

We assign the three families of $16$-dim. matter fermions $\psi$ to $3_2$-dim. representation of $S_4$ and the Higgs field $H$, $\Delta$ and $H'$ to $1_1, 1_2$, and $1_1$ reps, respectively. We then choose three $SO(10)$ singlet flavons $\phi_i$ transforming as $3_2, 3_1, 3_2$ reps of $S_4$ and one gauge and $S_4$ singlet fields $s_1, s_2$ transforming as $1_2$ and $1_1$ respectively. We further assume that at a scale slightly above the GUT scale, there are two $S_4$ singlet vectorlike pairs of $16 \oplus \overline{16}$ fields denoted by $\psi_V$ and $\overline{\psi}_V$. In order to get the desired Yukawa couplings naturally from this high scale theory, we supplement the $S_4$ group by an $Z_n$ group with all the above fields belonging to representations given in the Table 1. The fields and representations to generate the desired Yukawa couplings.

The most general high scale Yukawa superpotential involving matter fields invariant under this symmetry is given by:

$$W = (\phi_1 \psi) \overline{\psi}_V V_1 + \psi V_1 \psi V_1 H + M_1 \overline{\psi}_V V_1 \psi V_1 + \frac{1}{M_P} s_1 \psi V_2 \psi V_2 \Delta + \frac{1}{M_P} s_2 (\phi_3 \psi \overline{\psi}) \overline{\Delta} + \frac{1}{M_P} (\phi_2 \psi \psi) H',$$

where the brackets stand for the $S_4$ singlet contraction of flavor index. The singlet field $s_i$ can have large vev as follows: consider its $Z_n$ charge to be such that the only polynomial term involving the $s_i$ in the superpotential has the form $s_i^{k_i} / M_P^{k_i-3}$ (in order to describe the essential potential, we ignore a possible $s_1^{k_1} s_2^{k_2}$ term). The dominant part of the potential in the presence
of SUSY breaking has the form:

\[
V(s_i) = -m_{s_i}^2 |s_i|^2 + k \frac{s_i^{2k_i-2}}{M_P^{2k_i-6}} + \cdots .
\]  

Minimizing this leads to \( \langle s_i \rangle \sim \left[ m_{s_i}^2 M_P^{2k_i-6} \right]^{\frac{1}{2k_i-4}} \), which is above GUT scale for larger values of the integer \( k_i \) (which in turn is determined by the \( Z_n \) symmetry charge of \( s_i \)). One could also have large vevs for \( s_1, s_2 \) by using anomalous \( U(1) \) charges for them using \( D \)-terms to break the \( U(1) \) symmetry.

The effective theory below the scales \( M_{1,2} \) and \( \langle s_i \rangle \) of the vector-like pair masses and the \( s_i \)-vevs respectively is given by:

\[
W = (\phi_1 \psi)(\phi_1 \psi) H + (\phi_2 \psi)(\phi_2 \psi) \Delta + (\phi_3 \psi \psi) \Delta + (\phi_2 \psi \psi) H',
\]  

where we have omitted the dimensional coupling constants to make it simple for the purpose of writing. The discrete symmetries prevent \( \phi^2/M^2 \) corrections to these terms. So our predictions based on this effective superpotential do not receive large corrections. We note that the non-renormalizable terms in Eq.(8) can also be obtained from renormalizable couplings if we introduce further \( S_4 \)-triplet vectorlike fields. Here, however we use only \( S_4 \)-singlet vectorlike fields to get rank 1 contribution to \( h \) and \( f \) Yukawa couplings and that is why we need the non-renormalizable terms to be present in Eq.(8).

In order to get fermion masses, we have to find the alignment [9] of the vevs of the flavon fields \( \phi_{1,2,3} \). We show below that the following choice of vevs are among the minima of the flavon superpotential provided the couplings of mixed terms between different \( \phi_i \)'s are small compared to other couplings:

\[
\phi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]  

Clearly, there are other vacua for the flavon model that we do not choose. What is however nontrivial is that the alignments are along quantized directions. This is a consequence of supersymmetry combined with discrete symmetries in the theory. Given these vev, we find from Eq. (10) that the Yukawa coupling matrices \( h, f, h' \) have the form:

\[
h \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(12)

\[
f \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},
\]

(13)

\[
h' \propto \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

(14)

and the charged fermion mass matrices can then be inferred. The neutrino mass matrix in this basis has the form:

\[
\mathcal{M}_\nu = \begin{pmatrix} 0 & c & c \\ c & a & c-a \\ c & c-a & a \end{pmatrix},
\]

(15)
where $c/a = \lambda \ll 1$. It is diagonalized by the tri-bi-maximal matrix

$$
U_{TB} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}}
\end{pmatrix}.
$$

This is however not the full PMNS matrix which will receive small corrections from diagonalization of the charged lepton matrix, which not only make small contributions to the $\theta_{\text{atm}}$ and $\theta_{\text{C}}$ but also generate a small $\theta_{13}$.

The neutrino masses are given by $m_{\nu 3} = 2a - c$; $m_{\nu 2} = 2c$ and $m_{\nu 1} = -c$. To fit observations, we require $\lambda = c/a \simeq \sqrt{\Delta m^2_{\odot}/\Delta m^2_{\text{atm}}} \sim 0.2$, which fixes the neutrino masses $m_{\nu 3} \simeq 0.05$ eV, $m_{\nu 2} \simeq 0.01$ eV, and $m_{\nu 1} \simeq 0.005$ eV. We will see below that $\lambda$ is also the Cabibbo angle substantiating our claim that neutrino mass ratio and Cabibbo angle are related.

For the charged lepton, up and down quark mass matrices, we have:

$$
M_e = \frac{r_1}{\tan \beta} \begin{pmatrix}
0 & -3m_1 + \delta & -3m_1 - \delta \\
-3m_1 + \delta & -3m_0 & 3m_0 - 3m_1 \\
-3m_1 - \delta & 3m_0 - 3m_1 & -3m_0 + M
\end{pmatrix},
$$

$$
M_d = \frac{r_1}{\tan \beta} \begin{pmatrix}
0 & m_1 + \delta & m_1 - \delta \\
- m_1 + \delta & m_0 & -m_0 + m_1 \\
- m_1 - \delta & -m_0 + m_1 & m_0 + M
\end{pmatrix},
$$

$$
M_u = \begin{pmatrix}
r_2 m_1 + r_3 \delta & r_2 m_0 & -r_2 m_0 + r_3 m_1 \\
r_2 m_1 + r_3 \delta & r_2 m_0 & r_2 m_0 + r_3 m_1
\end{pmatrix},
$$

where $\tan \beta$ is a ratio of $H_u,d$ vevs. Note that $m_1/m_0 = \lambda \sim 0.2$ and of course $m_0 \ll M$. A quick examination of these mass matrices leads to several immediate conclusions:

(i) The model predicts that at GUT scale $m_b \simeq m_\tau$.

(ii) Since $(M_d)_{11} \rightarrow 0$, we get $V_{us} \simeq \sqrt{m_d/m_s}$.

(iii) The empirically satisfied relation $m_\mu/m_\tau \simeq m_sm_d$ can be obtained by the choice of parameters $-3m_1 + \delta = (m_1 + \delta)e^{i\sigma}$, where $\sigma$ is a phase. Solving this equation, we find that $\delta = m_1(1 + i \cot \sigma/2)$. We obtain $V_{us} \simeq (1 - r_3/r_2)\delta/m_0$, thereby relating Cabibbo angle to the neutrino mass ratio $m_\odot/m_{\text{atm}} \simeq \lambda$.

(iv) $m_\mu \sim -3m_s$.

(v) $V_{tb} \sim m_s/m_b \cot \theta_{\text{atm}}$.

(vi) The masses of up and charm quarks are given by the parameters $r_{2,3}$ and are therefore not predictions of the model.

(vii) CP violation in quark sector can put in by making the parameters $r_{2,3}$ complex.

(viii) The model predicts a small amplitude for neutrino-less double beta decay from light neutrino mass: $m_{ee} \sim c \sin \theta_{12} \simeq 0.3$ meV.

The first four relations are fairly well satisfied by observations; the fifth prediction (i.e. that for $U_{e3}$) can be tested in upcoming reactor and long baseline experiments. Note that the deviation from tri-bi-maximal mixing pattern coming from the charged lepton mass diagonalization could be thought of as a small perturbation of the neutrino mass matrix except that we predict the form of the perturbation from symmetry considerations. The sixth prediction
gives a smaller value for $V_{cb}$ (0.02 as against observed GUT scale value of 0.03) if one uses GUT scale extrapolated value of the known $b$ mass. However, in the MSSM there are threshold corrections to the $b-s$ quark mass mixing from gluino and wino exchange one-loop diagrams; by choosing this contribution, one could obtain the desired $V_{cb}$.

Note that in this model, the top quark Yukawa coupling at GUT scale arises from an effective higher dimensional operator. We have showed the effective operator in Eq. (10) by expanding $\phi/M_1$. The more precise form for the top Yukawa coupling is $\phi^2/(M_1^2 + \phi^2) h_{\psi V} V_{33}$, where $h_{\psi V} V_{33}$ is a coupling of $\psi_V \psi V H$ term, and $\phi$ is the vev of $\phi_1$ multiplied by $\phi_1 \psi \psi V$ coupling. This is simply because the low energy third generation field is a linear combination of the form $\cos \alpha \psi_3 - \sin \alpha \psi_V$ with the mixing angle $\sin \alpha \simeq \phi/\sqrt{M_1^2 + \phi^2}$.

Therefore, in general, there is no gross contradiction to the fact that the top Yukawa coupling is order 1. However, in our case, if $\phi/M_1$ becomes close to 1, the atmospheric mixing shifts from the maximal angle. Given the error in the determination of the atmospheric mixing angle, this is consistent with data and as this measurement sharpens, this is going to provide a test of this particular model. The desired smallness of the effective $f$ and $h'$ couplings however are more naturally obtained due to the presence of the Planck mass in the denominator. In order to make the $f$-coupling dominate over the $h'$, we have to choose a small coupling for the $H'$ Higgs field in Eq. (4). Similarly the $\lambda$ term in Eq. (9) is assumed to be small compared to the coefficient of the first matrix.

Thus within these set of assumptions, this model is in good phenomenological agreement with observations. In a more complete theory, these assumptions need to be addressed. We however find it remarkable that despite these shortcomings, the model provides a very useful unification strategy of the diverse quark-lepton mixing patterns.

7. Vev alignment as minima of flavon theories

In this section, we give examples of how the minima of flavon theories can determine the Yukawa couplings of the fermions and lead to predictive flavor models. We discuss the specific case of the $S_4$ model at hand. This mechanism is of course applicable to any general group.

We start our discussion by giving some simple examples and discussing the flavon alignment as a prelude to the more realistic example. First thing to note is that $S_4^3$ is invariant under $S_4$, but $S_2^3$ is not. Denoting $\phi = (x, y, z)$, we see that in the first case, the singlet of $\phi^3 = xyz$. The superpotential for a $S_3$ flavon field $\phi$ can therefore be written as

$$W = \frac{1}{2} m\phi^2 - \lambda \phi^3 = \frac{1}{2} m(x^2 + y^2 + z^2) - \lambda xyz.$$  \hspace{1cm} (18)

The solution of $F$-flat vacua ($\phi \neq 0$) are

$$\phi = \frac{m}{\lambda} \{ (1,1,1) \text{ or } (1,-1,-1) \text{ or } (-1,1,-1) \text{ or } (-1,-1,1) \}. \hspace{1cm} (19)$$

These vacua break $S_4$ down to $S_3$ and in the process determine the Yukawa couplings.

On the other hand, when $S_2$ flavon is used (or the cubic term is forbidden by a discrete symmetry), quartic term involving the triplet is crucial for the $F$-flat vacua. The invariant quartic term $\phi^4$ gives two linear combinations of the form $x^4 + y^4 + z^4$ and $x^2y^2 + y^2z^2 + z^2x^2$. This is because they have to be symmetric homogenous terms and invariant under the Klein’s group, which is $\pi$ rotation around the $x, y, z$ axes.

Thus, the superpotential term for $S_2$ field $\phi$ is

$$W = \frac{1}{2} m\phi^2 - \frac{\kappa^{(1)}}{M} \phi^4_1 - \frac{\kappa^{(2)}}{M} \phi^4_2 \hspace{1cm} (20)$$

$$= \frac{1}{2} \left( x^4 + y^4 + z^4 \right) - \frac{\kappa^{(1)}}{4M} (x^2y^2 + y^2z^2 + z^2x^2).$$
The nontrivial $F$-flat vacua ($\phi \neq 0$) are

$$\phi = \sqrt{\frac{m M}{\kappa(1)}} \vec{a}, \sqrt{\frac{m M}{\kappa(1) + 2\kappa(2)}} \vec{b}, \sqrt{\frac{m M}{\kappa(1) + \kappa(2)}} \vec{c},$$  \hspace{1cm} (21)

where $\vec{a} = (0, 0, \pm 1), (0, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$, $\vec{b} = (\pm 1, \pm 1, \pm 1)$, and $\vec{c} = (0, \pm 1, \pm 1), (\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1)$. We note that these vectors correspond to the axes of the regular hexahedron. The vacua break $S_4$ down to $Z_4, Z_3$, and $Z_2$, respectively. More importantly, the vacuum states in Eq. (12) used in the analysis of fermion masses in the previous section are a subset of the above vacua.

Note that if we add a $\phi^4$ term to the superpotential involving the $3_1$ flavon field, $\vec{a}$ vacuum is possible, in addition to the original $\vec{b}$ vacua. However, $\vec{c}$ vacuum is absent.

A detailed numerical analysis of the model subsequent to the measurement of $\theta_{13}$ [10] and it turns out that in the strictly minimal vev alignment discussed above predicts $\theta_{13} \approx 5^0$; however, a slight change in the vev alignment without any extension of the particle content of the model leads to the experimentally observed value[11].

8. Comments
A complete understanding of flavor is clearly a very ambitious task. Our proposal should be considered as a simple beginning towards a final theory. It should be noted that even though we have considered on SO(10) group, our general unification ansatz (without as much predictivity) in $SU(2)_L \times SU(2)_R \times SU(4)_c$ theories as well and perhaps other groups such as $E_6$. Similarly, one should explore other flavor models.

A second point of importance is that while we have kept only leading order terms, one should clearly consider higher order corrections to our predictions systematically. In the above mode, we have checked next order corrections and found them to be absent due to the discrete symmetries.

9. Conclusion
In summary, I have discussed a recently proposed ansatz that has the potential to provide a unified description of the diverse quark and lepton flavor patterns. This could provide the first opening into a very difficult problem of particle physics- the problem of flavor. A simple realization of this ansatz is shown to occur within a grand unified SO(10) model with type II seesaw describing the neutrino masses. The successes of that model are that it seems to provide an understanding of several observed quark-lepton mass relations such as bottom-tau mass unification, strange quark-muon mass ratio $(1/3)$ etc. and predicts a value for $\theta_{13}$ in agreement with observations and the atmospheric mixing angle different from the maximal value. The model like most grand unified theories of neutrinos predicts a normal hierarchy and observation of inverted hierarchy will therefore rule out this model (as well as most simple grand unified theories). Under certain reasonable approximations, this also seems to explain why $m_{\text{solar}}/m_{\text{atm}} \sim \theta_C$. It also predicts a value of 0.3 meV for the effective neutrino mass in neutrinoless double beta decay experiments.

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