Seymour’s Second Neighborhood Conjecture
for orientations of (pseudo)random graphs

Fábio Botler\textsuperscript{1} Phablo F. S. Moura\textsuperscript{2} Tássio Naia\textsuperscript{3}

November 15, 2022

\textsuperscript{1}Programa de Engenharia de Sistemas e Computação
Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa em Engenharia
Universidade Federal do Rio de Janeiro, Brasil
fbotler@cos.ufrj.br

\textsuperscript{2}Departamento de Ciência da Computação
Instituto de Ciências Exatas
Universidade Federal de Minas Gerais, Brasil
phablo@dcc.ufmg.br

\textsuperscript{3}Departamento de Ciência da Computação
Instituto de Matemática e Estatística
Universidade de São Paulo, Brasil
tnaia@member.fsf.org

Abstract
Seymour’s Second Neighborhood Conjecture (SNC) states that every oriented graph contains a vertex whose second neighborhood is as large as its first neighborhood. We investigate the SNC for orientations of both binomial and pseudo random graphs, verifying the SNC asymptotically almost surely (a.a.s.) (i) for all orientations of $G(n,p)$ if $\limsup_{n \to \infty} p < 1/4$; and (ii) for a uniformly-random orientation of each weakly $(p,A\sqrt{n}p)$-bijumbled graph of order $n$ and density $p$, where $p = \Omega(n^{-1/2})$ and $1 - p = \Omega(n^{-1/6})$ and $A > 0$ is a universal constant independent of both $n$ and $p$.

We also show that a.a.s. the SNC holds for almost every orientation of $G(n,p)$. More specifically, we prove that a.a.s.
(iii) for all $\varepsilon > 0$ and $p = p(n)$ with $\limsup_{n \to \infty} p \leq 2/3 - \varepsilon$, every orientation of $G(n,p)$ with minimum outdegree $\Omega_{\varepsilon}(\sqrt{n})$ satisfies the SNC; and (iv) for all $p = p(n)$, a random orientation of $G(n,p)$ satisfies the SNC.

1 Introduction

An oriented graph $D$ is a digraph obtained from a simple graph $G$ by assigning directions to its edges (i.e., $D$ contains neither loops, nor parallel arcs, nor directed cycles of length 2); we also call $D$ an orientation of $G$. Given $i \in \mathbb{N}$, the $i$-th neighborhood of $u \in V(D)$, denoted by $N^i(u)$, is the set of vertices $v$ for which a shortest directed path from $u$ to $v$ has precisely $i$ arcs. A Seymour vertex (see [14]) is a vertex $u$ for which $|N^2(u)| \geq |N^1(u)|$. Seymour conjectured the following (see [6]).

Conjecture 1. Every oriented graph contains a Seymour vertex.
Conjecture 1, known as Seymour’s Second Neighborhood Conjecture (SNC), is a notorious open question (see, e.g., [3, 7, 9, 14]). In particular, it was confirmed for tournaments (orientations of cliques) by Fisher [8] and (with a purely combinatorial argument) by Havet and Thomassé [10]; it was also studied by Cohn, Godbole, Harkness and Zhang [4] for the random digraph model in which each ordered pair of vertices is picked independently as an arc with probability \( p < 1/2 \). Throughout the paper, we denote by \( S \) the set of graphs \( \{G : \text{all orientations of } G \text{ contain a Seymour vertex}\} \).

Our contribution comes from considering this combinatorial problem in a random and pseudorandom setting (see, e.g., [5, 13]). More precisely, we explore Conjecture 1 for orientations of the binomial random graph \( G(n, p) \), defined as the random graph with vertex set \( \{1, \ldots, n\} \) in which every pair of vertices appears as an edge independently and with probability \( p \).

We say that an event \( \mathcal{E} \) holds asymptotically almost surely (a.a.s.) if \( \Pr[\mathcal{E}] \to 1 \) as \( n \to \infty \). If \( G = G(n, p) \) is very sparse (say, if \( np \leq (1 - \varepsilon) \ln n \) for large \( n \) and fixed \( \varepsilon > 0 \)), then a.a.s. \( G \) has an isolated vertex, which clearly is a Seymour vertex. Our first result extends this observation to much denser random graphs.

**Theorem 2.** Let \( p : \mathbb{N} \to (0, 1) \). If \( \limsup_{n \to \infty} p < 1/4 \), then a.a.s. \( G(n, p) \in S \).

If we impose restrictions on the orientations, requiring, for example, somewhat large minimum outdegree, the range of \( p \) can be further increased.

**Theorem 3.** For every \( \beta > 0 \), there exists \( C = C(\beta) \) such that the following holds for all \( p : \mathbb{N} \to (0, 1) \): if \( \limsup_{n \to \infty} p \leq 2/3 - \beta \), then a.a.s. every orientation of \( G(n, p) \) with minimum degree at least \( Cn^{1/2} \) contains a Seymour vertex.

For an even larger range of \( p \), we show that most orientations of \( G(n, p) \) contain a Seymour vertex; i.e., Conjecture 1 holds for almost every (labeled) oriented graph.

**Theorem 4.** Let \( p : \mathbb{N} \to (0, 1) \) and let \( G = G(n, p) \). If \( D \) is chosen uniformly at random among the \( 2^{e(G)} \) orientations of \( G \), then a.a.s. \( D \) has a Seymour vertex.

In fact, we prove a version of Theorem 4 in a more general setting, namely orientations of pseudorandom graphs (see Section 4).

**Theorem 5.** There exists an absolute constant \( C > 1 \) such that the following holds. Let \( G \) be a weakly \((p, A\sqrt{mp})\)-bijumbled graph of order \( n \), where \( \varepsilon^3 np^2 \geq A^2 C \) and \( p < 1 - 15\sqrt{\varepsilon} \). If \( D \) is chosen uniformly at random among the \( 2^{e(G)} \) possible orientations of \( G \), then a.a.s. \( D \) has a Seymour vertex.

This paper is organized as follows. In Section 2 we prove Conjecture 1 for wheel-free graphs, which implies the particular case of Theorem 2 when \( n^2 p^3 \to 0 \). In Section 3 we complete the proof of Theorem 2 and prove Theorems 3 and 4 using a set of standard properties of \( G(n, p) \). These properties are collected in Definition 9 and Lemma 10 (proved in Appendix A). In Section 4, we introduce bijumbled graphs and prove Theorem 5. We make a few further remarks in Section 5.

To avoid uninteresting technicalities, we omit floor and ceiling signs. If \( A \) and \( B \) are sets of vertices, we denote by \( e(A, B) \) the number of arcs directed from \( A \) to \( B \), by \( e(A) \) the number of edges or arcs with one vertex in each set, and by \( e(A) \) the number of edges or arcs with both vertices in \( A \). The (underlying) neighborhood of a vertex \( u \) is denoted by \( N(u) \), and the codegree of vertices \( u, v \) is \( \deg(u, v) = |N(u) \cap N(v)| \).
We remark that Theorem 2 and a weaker version of Theorem 3 appeared in the extended abstracts [1, 2].

2 Wheel-free graphs

A wheel is a graph obtained from a cycle $C$ by adding a new vertex adjacent to all vertices in $C$. Firstly, we show that $G(n, p)$ is wheel-free when $p$ is small; then prove that all wheel-free graphs satisfy Conjecture 1.

Lemma 6. If $p \in (0, 1)$ and $n^4p^6 < \varepsilon/16$, then $\Pr[G(n, p) \text{ is wheel-free}] \geq 1 - \varepsilon$.

Proof. We can assume $\varepsilon < 1$. Since $n^4p^6 < \varepsilon/16$, we have that

$$np^2 < (\varepsilon p^2/16)^{1/4} < 1/2. \quad (1)$$

Let $X = \sum_{k=4}^{n} X_k$, where $X_k$ denotes the number of wheels of order $k$ in $G(n, p)$. By the linearity of expectation,

$$\mathbb{E} X = \sum_{k=4}^{n} \mathbb{E} X_k = \sum_{k=4}^{n} \left(\frac{n}{k}\right) \frac{(k-1)!}{2(k-1)} p^{2(k-1)}$$

$$< n \sum_{k=4}^{n} (np^2)^{k-1} = n^4p^6 \sum_{k=0}^{n-4} (np^2)^k < \frac{n^4p^6}{1 - np^2} \overset{(1)}{<} 2n^4p^6 < \frac{\varepsilon}{8} < \varepsilon. \quad (2)$$

Where in (2) we use the formula $\sum_{i=0}^{\infty} r^i = (1 - r)^{-1}$ for the geometric series (G.S.) of ratio $r = np^2 < 1$. Markov’s inequality then yields $\Pr[X \geq 1] \leq \mathbb{E} X < \varepsilon$. \hfill \square

To show that every orientation of a wheel-free graph has a Seymour vertex, we prove a slightly stronger result. A digraph is locally cornering if the outneighborhood of each vertex induces a digraph with a sink (i.e., a vertex of outdegree 0). The next proposition follows immediately by noting that, in a locally cornering digraph, each vertex of minimum outdegree is a Seymour vertex.

Proposition 7. Every locally cornering digraph has a Seymour vertex.

Lemma 6 and Proposition 7 immediately yield the following corollary.

Corollary 8. If $p \in (0, 1)$, and $n^4p^6 < \varepsilon/16$, then $\Pr[G(n, p) \in \mathcal{S}] \geq 1 - \varepsilon$.

Proof. Note that every orientation of a wheel-free graph is locally cornering, since the (out)neighborhood of each vertex is a forest, and every oriented forest has a vertex with outdegree 0. Hence the result follows by Lemma 6 and Proposition 7. \hfill \square

3 Typical graphs

In this section we prove that if $\lim \sup_{n \to \infty} p < 1/4$, then a.a.s. $G(n, p) \in \mathcal{S}$. We use a number of standard properties of $G(n, p)$, stated for convenience in Definition 9.
**Definition 9.** Let $p \in (0, 1)$. A graph $G$ of order $n$ is *$p$-typical* if the following hold.

(i) For every $X \subseteq V(G)$, we have $$|e(X) - \left(\frac{|X|}{2}\right) p| \leq |X|\sqrt{3np(1-p)} + 2n.$$  

(ii) If $n'\ln n \leq n'' \leq n$ or $n' = n'' = n$, then all $X, Y \subseteq V(G)$ with $|X|, |Y| \leq n'$ satisfy $$|e(X, Y) - |X||Y|p| \leq \sqrt{6n''p(1-p)||X||Y|} + 2n''.$$  

(iii) For every $v \in V(G)$, we have $$|\deg(v) - np| \leq \sqrt{6np(1-p)\ln n} + 2\ln n.$$  

(iv) For every distinct $u, v \in V(G)$, we have $$|\deg(u, v) - (n - 2)p^2| \leq \sqrt{6np^2(1-p^2)\ln n} + 2\ln n.$$  

It can be shown, using standard Chernoff-type concentration inequalities, that $G(n, p)$ is $p$-typical with high probability (see Appendix A).

**Lemma 10.** For every $p : N \to (0, 1)$, a.a.s. $G = G(n, p)$ is $p$-typical.

We also use the following property of graphs satisfying Definition 9 (i).

**Lemma 11.** Let $G$ be a graph of order $n$ which satisfies Definition 9 (i), and fix $a \in \mathbb{N}$. If $D$ is an orientation of $G$ and $B = \{v \in V(D) : \deg_D^+(v) < a\}$, then $$|B| \leq \frac{2}{p}(a - 1) + 1 + \sqrt{\frac{12n(1-p)}{p}} + \frac{4n}{|B|}.$$  

**Proof.** The lemma follows by multiplying all terms in the inequality below by $2/|B|p$. $$|B|(a - 1) \geq e(G[B]) \geq \left(\frac{|B|}{2}\right)p - |B|\sqrt{3np(1-p)} - 2n. \quad \square$$

### 3.1 Proof of Theorem 2

Let us outline the proof of Theorem 2. Firstly, we find a vertex $w$ whose out-neighborhood contains many vertices with large outdegree. Then, we note that $|N^1(w)| = O(np)$ and that $N^1(w) \cup N^2(w)$ cannot be too dense. Finally, since many outneighbors of $w$ have large outdegree, we conclude that $N^1(w) \cup N^2(w)$ must contain at least $2|N^1(w)|$ vertices, completing the proof. This yields the following.

**Lemma 12.** Fix $0 < \alpha < 1/4$ and $\varepsilon > 0$. There is $n_1 = n_1(\alpha, \varepsilon)$ such that $S$ contains all $p$-typical graphs of order $n$ such that $n \geq n_1$ and $\varepsilon n^{-2/3} \leq p \leq 1/4 - \alpha$.

Lemma 12 is our last ingredient for proving Theorem 2. Indeed, fix $\varepsilon > 0$, set $\alpha = 1/4 - \limsup_{n \to \infty} p(n)$ and let $n_0$ be large enough so that $p(n) \leq 1/4 - \alpha$ and so that $G(n, p)$ is $p$-typical with probability at least $1 - \varepsilon$ for all $n \geq n_0$ (this is Lemma 10). Now either $p < \varepsilon n^{-2/3}$ or $\varepsilon n^{-2/3} \leq p(n) \leq 1/4 - \alpha$. In the former case we use Corollary 8, and in the latter case Lemma 12, concluding either way that $$\Pr[G(n, p) \in S] \geq 1 - \varepsilon.$$
Proof of Lemma 12. We may and shall assume (choosing $n_1$ accordingly) that $np$ is large enough whenever necessary. Fix an arbitrary orientation of $G$. For simplicity, we write $G$ for both the oriented and underlying graphs. Let

$$S = \{v \in V(G) : \deg^+(v) < (1-\alpha)np/2\}$$

and $T = V(G) \setminus S$. Firstly, we show that $|T| \geq \alpha n/2$. This is clearly the case if $|S| < \alpha n$ (since $\alpha < 1/4 < 1-\alpha$); let us show that this also holds if $|S| \geq \alpha n$. Indeed, since $p \geq \varepsilon n^{-2/3}$, from Lemma 11 with $a = (1-\alpha)np/2$ we obtain

$$|S| \leq \frac{2(a-1)}{p} + 1 + \sqrt{\frac{12n(1-p)}{p} + \frac{4n}{|S|}} < (1-\alpha)n + o(n) < \left(1 - \frac{\alpha}{2}\right)n.$$ 

Therefore $|T| = n - |S| \geq \alpha n/2$ as desired. Recall that $np$ is large and $p \leq 1/4$. Then $\sqrt{3np(1-p)} \geq 4/\alpha$, and hence, from Definition 9 (i), we get

$$e(T) \geq \left(\frac{|T|}{2}\right)p - |T|\sqrt{3np(1-p)} - 2n > \left(\frac{|T|}{2}\right)p - 2|T|\sqrt{n/p} > \frac{|T|^2p}{3},$$

and therefore, by averaging, there exists $w \in T$ satisfying

$$\deg^+(w) \geq \frac{e(T)}{|T|} \geq \frac{\alpha np}{6}. \quad (4)$$

We next show that $w$ is a Seymour vertex. Let $X = N_G^1(w)$ and $Y = N_G^2(w)$, and suppose, for a contradiction, that $|Y| < |X|$. From 9 (iii) and $p + \alpha \leq 1/4$, we have

$$|X| \leq np + \sqrt{6np \ln n + 2 \ln n} < n \left(p + \frac{\alpha}{2}\right) \leq \frac{n}{4} \leq \frac{n}{2}(1 - 2\alpha - 2p). \quad (5)$$

Moreover,

$$|X| = \deg^+(w) \leq np + \sqrt{6np \ln n + 2 \ln n} < 2np. \quad (6)$$

Recall that $w \in T$ and let $N = X \cap T$ be the set of outneighbors of $w$ in $T$. By the definition of $N$ and (4) we have

$$|N| \geq \frac{\alpha np}{6}. \quad (7)$$

Note that $\bar{e}(N, X)$ counts arcs induced by $N$ precisely once (as $N \subseteq X$), and if the arc $u \rightarrow v$ is counted by $\bar{e}(N, X)$, then $v$ is a common neighbor of $w$ and $u \in N$. Hence, by Definition 9 (iv), we have that

$$\bar{e}(N, X) + e(N) \leq |N|(np^2 + \sqrt{6np^2 \ln n + 2 \ln n}).$$

Since vertices in $T$ (and hence in $N$) have at least $(1-\alpha)np/2$ outneighbors, we have

$$\bar{e}(N, Y) \geq |N|\frac{(1-\alpha)np}{2} - \bar{e}(N, X) - e(N) \geq |N|\frac{(1-\alpha)np}{2} - |N|(np^2 + \sqrt{6np^2 \ln n + 2 \ln n}). \quad (8)$$

The following estimate will be useful.
Claim 13. It holds that $2 \ln n + \sqrt{6np^2 \ln n} + \sqrt{6|Y|np/|N|} = o(np)$.

Proof. We prove that each term in the sum above is $o(n)$ when divided by $p$. Clearly, $\sqrt{6np^2 \ln n}/p = o(n)$. Recall that $p \geq \varepsilon n^{-2/3}$ and thus $(2\ln n)/p = o(n)$. Also,

$$\sqrt{|Y|/|N|} \leq \sqrt{|Y|/(\alpha p)^2} < \sqrt{|X|/(\alpha p)^2} < \frac{72n}{\alpha p} = o(n).$$

\[\Box\]

We divide the remainder of the proof in two cases. Fix $\gamma \in (1/2, 2/3)$.

Case 1. Suppose firstly that $p > n^{\gamma-1}/2$. Using Definition 9 (ii) we obtain

$$\bar{e}(N, Y) \leq |N||Y|p + \sqrt{6np|N||Y|} + 2n. \quad (9)$$

Thus, combining (8) and (9), we have

$$\frac{(1-\alpha)np}{2} - (np^2 + \sqrt{6np^2 \ln n + 2\ln n}) \leq |Y|p + \sqrt{\frac{6np|Y|}{|N|}} + 2n. \quad (10)$$

Also note that since $p > n^{\gamma-1}/2$ and $\gamma > 1/2$, we can estimate

$$\frac{2n}{|N|p} \leq \frac{12}{\alpha p^2} < \frac{24}{\alpha n^{2\gamma-2}} = o(n). \quad (11)$$

Finally, we conclude that $w$ is a Seymour vertex, since (10) becomes

$$|Y| \geq \frac{(1-\alpha-2p)n}{2} - \sqrt{6n \ln n} - \sqrt{\frac{6n|Y|}{|N|}} - \frac{2n}{|N|p} - \frac{2 \ln n}{p} \overset{(*)}{\approx} \frac{n}{2} (1-2\alpha-2p) > |X|,$$

where inequality $(*)$ follows from Claim 13 and (11).

Case 2. Suppose now that $p \leq n^{\gamma-1}/2$. In this case (6) implies $|X| \leq n^\gamma$. Since $N \subseteq X$ and $|Y| < |X|$, Definition 9 (ii) (with $n' = n^\gamma$ and $n'' = n^\gamma \ln n$) yields

$$\bar{e}(N, Y) \leq |N||Y|p + \sqrt{6(n^\gamma \ln n)p|N||Y|} + 2n^\gamma \ln n$$

$$< |N||Y|p + \sqrt{6np|N||Y|} + 2n^\gamma \ln n. \quad (12)$$

Now, from (8) and (12), we obtain the following inequality, which is analogous to (10), but with the term $2n/|N|$ replaced by $2n^\gamma \ln n/|N|$.

$$\frac{(1-\alpha)np}{2} - (p^2n + \sqrt{6np^2 \ln n + 2\ln n}) \leq |Y|p + \sqrt{\frac{6np|Y|}{|N|}} + \frac{2n^\gamma \ln n}{|N|}. \quad (13)$$

We claim that $2n^\gamma \ln n/|N| = o(np)$. Indeed, since $p \geq \varepsilon n^{-2/3}$ and $\gamma < 2/3$, we have

$$\frac{2n^\gamma \ln n}{|N|p} \leq \frac{12n^\gamma \ln n}{\alpha np^2} = \frac{12n^\gamma-1 \ln n}{\alpha p^2} \leq \frac{12n^\gamma+1/3 \ln n}{\alpha \varepsilon^2} = o(n), \quad (14)$$

We complete the proof of Case 2 by solving (13) for $|Y|$ as in Case 1 (using Claim 13 and (14) to estimate $2n^\gamma \ln n/|N|$). \[\Box\]
3.2 Proof of Theorem 4

We are now in a position to prove Theorem 4, which we restate for convenience.

**Theorem 4.** Let \( p : \mathbb{N} \to (0, 1) \), and let \( G = G(n, p) \). If \( D \) is chosen uniformly at random among the \( 2^e(G) \) orientations of \( G \), then a.a.s. \( D \) has a Seymour vertex.

**Proof of Theorem 4.** Let \( G = G(n, p) \). If \( p < 1/5 \), then \( \Pr [G \in S] = 1 - o(1) \) by Theorem 2. On the other hand, if \( p \geq 1/5 \), then standard concentration results for binomial random variables (e.g., Chernoff-type bounds) yield that every ordered pair \((u, v)\) of distinct vertices of \( G \) satisfies, say \( \deg(u, v) \geq n/50 \), and hence with probability \( 1 - o(1) \) every such pair is joined by a directed path of length 2. This is because building a random orientation of \( G(n, p) \) is equivalent to first choosing which edges are present and then choosing the orientation of each edge uniformly at random, with choices mutually independent for each edge. In other words, with probability \( 1 - o(1) \), for all \( u \in V(G) \) we have \( V(G) = \{u\} \cup N^1(u) \cup N^2(u) \). Finally, by averaging outdegrees, we can find a vertex \( z \in V(D) \) with outdegree at most \((n - 1)/2\), because \( \sum_{e \in V(D)} \deg^+(v) = e(G) \leq n(n - 1)/2 \). Such \( z \) is a Seymour vertex as desired.

3.3 Orientations with large minimum outdegree

Our last result in this section yields yet another class of orientations of \( p \)-typical graphs which must always contain a Seymour vertex. In fact, we consider a larger class of underlying graphs, showing that if a graph \( G \) satisfies items (i) and (ii) of Definition 9, then every orientation \( D \) of \( G \) with minimum outdegree \( \delta^+(D) = \Omega(n^{1/2}) \) contains a Seymour vertex. This may be useful towards extending the range of \( p \) for which a.a.s. \( G(n, p) \in S \).

**Lemma 14.** Fix \( \beta > 0 \). There exist a constant \( C = C(\beta) \) and \( n_0 = n_0(\beta) \) such that the following holds for all \( n \geq n_0 \) and \( p \leq 2/3 - \beta \). If \( G \) is a graph of order \( n \) that satisfies items (i) and (ii) of Definition 9, then every orientation \( D \) of \( G \) for which \( \delta^+(D) \geq Cn^{1/2} \) has a Seymour vertex.

Note that Lemma 14 and Lemma 10 immediately imply Theorem 3.

**Proof of Lemma 14.** Since \( (1 - 3p/2) \geq 3\beta/2 \), we may fix \( C \geq 4 \) so that

\[
\left(1 - \frac{3p}{2}\right)C - \left(\sqrt{3p(1-p)} + \sqrt{6p(1-p)}\right) \geq \frac{3\beta C}{2} - 4 \geq 1.
\]

Fix \( v \in V(D) \) with \( \deg^+(v) = \delta^+(D) \), let \( X = N^1(v) \) and \( Y = N^2(v) \). We shall prove that \( |X| \leq |Y| \). Suppose to the contrary that \( |Y| < |X| \). By Definition 9 (i),

\[
\tilde{e}(X, Y) = \sum_{a \in X} \deg^+(a) - e(X) \geq |X|^2 - \left(\frac{|X|^2p}{2} + |X|\sqrt{3np(1-p) + 2n}\right) = \left(1 - \frac{p}{2}\right)|X|^2 - \left(|X|\sqrt{3np(1-p) + 2n}\right), \tag{15}
\]

and by Definition 9 (ii) (with \( n' = n'' = n \)) we have

\[
\tilde{e}(X, Y) \leq e(X, Y) \leq |X||Y|p + \sqrt{6np(1-p)|X||Y|} + 2n < |X|^2p + |X|\sqrt{6np(1-p) + 2n}. \tag{16}
\]
Since $|X| \geq Cn^{1/2} \geq n^{1/2}$, combining (15) and (16) yields the following contradiction.

$$4n > \left(1 - \frac{3p}{2}\right)|X|^2 - |X|\left(\sqrt{3np(1-p)} + \sqrt{6np(1-p)}\right) \geq Cn \left(1 - \frac{3p}{2}\right)C - \left(\sqrt{3np(1-p)} + \sqrt{6p(1-p)}\right) \geq 4n. \quad \square$$

4 Typical orientations of bijumbled graphs

In this section, we focus on a well-known class of pseudorandom graphs (that is, deterministic graphs which embody many properties of $G(n,p)$), and argue that almost all of their orientations contain a Seymour vertex. The following results concern graphs of order $n$ and density $p$, where $Cn^{-1/2} \leq p \leq 1 - \varepsilon$, and $C = C(\varepsilon) > 0$ depends only on the constant $\varepsilon > 0$.

**Definition 15** – $(p, \alpha)$-bijumbled. Let $p$ and $\alpha$ be given. We say that a graph $G$ of order $n$ is weakly $(p, \alpha)$-bijumbled if, for all $U, W \subset V(G)$ with $U \cap W = \emptyset$ and $1 \leq |U| \leq |W| \leq np|U|$, we have

$$|e(U, W) - p|U||W|\| \leq \alpha \sqrt{|U||W|}. \quad (17)$$

If (17) holds for all disjoint $U, W \subset V(G)$, then we say that $G$ is $(p, \alpha)$-bijumbled.

We note that the random graph is a.a.s. bijumbled.

**Theorem 16** – Lemma 3.8 in [11]. For any $p : \mathbb{N} \to (0, 1]$, the random graph $G(n, p)$ is a.a.s. weakly $(p, A\sqrt{np})$-bijumbled for a certain absolute constant $A \leq e^2\sqrt{6}$.

In what follows, $A$ shall always denote the constant from Theorem 16. A simple double-counting argument shows the following.

**Fact 17.** If $G$ is weakly $(p, \alpha)$-bijumbled, then for every $U \subset V(G)$ we have

$$\left|e(G[U]) - p\left(\frac{|U|}{2}\right)\right| \leq \alpha|U|. \quad (18)$$

We also use the following result, whose simple proof we include for completeness.

**Lemma 18.** There exists a universal constant $C > 1$ such that if $A \geq 2$ and $\varepsilon, p \in (0, 1)$ are such that $\varepsilon^3np^2 \geq A^2C$, then every weakly $(p, A\sqrt{np})$-bijumbled graph $G$ of order $n$ satisfies the following properties.

(i) $|\{v \in V(G) : |\deg(v) - np| > \varepsilon np\}| \leq \varepsilon n$.

(ii) $|\{(u, v) \in V(G)^2 : \deg(u, v) \leq (1 - \varepsilon)np^2\}| \leq \varepsilon n^2$.

(iii) For every orientation of $G$ and every integer $d$, we have

$$|\{v \in V(G) : \deg^+(v) < d\}| \leq 2\frac{d - 1}{p} + 2A\sqrt{\frac{n}{p}} + 1$$

**Proof.** Let $G$ be as in the statement. We may and shall assume that $C$ is large enough so that the required inequalities hold. Throughout this proof, $W$ denotes the set of vertices with degree strictly below $(1 - 2\varepsilon/3)np$. Firstly, we prove (i). We
Analogously, we have
\[ e(W') \geq p\left(\frac{\varepsilon n/2}{3}\right)^2 - A\sqrt{np}(\varepsilon n/2) = p\left(\frac{\varepsilon n/2}{3}\right)^2 \left(1 - \sqrt{\frac{36A^2}{\varepsilon^2 np}}\right) \]
\[ > p\left(\frac{\varepsilon n/2}{4}\right)^2 = \frac{\sqrt{2}}{16} \varepsilon np\sqrt{\frac{\varepsilon}{2}(1 - \varepsilon/2)n^3p} \]
\[ \geq A\frac{\varepsilon n}{2}(1 - \varepsilon/2)n = A\sqrt{np}|W'|(|n - |W'||). \quad (19) \]

Now, note that \(|V(G) \setminus W'| \leq n < A^2 Cn/\varepsilon^2 p \leq \varepsilon n^2 p = np|W'|\), but
\[ e(W', V(G) \setminus W') < |W'| \cdot (1 - 2\varepsilon/3)np - 2e(W') \]
\[ < |W'| \cdot (1 - \varepsilon/2)np - 2e(W') \]
\[ = p|W'|(n - |W'|) - 2e(W') \]
\[ \leq p|W'|(n - |W'|) - A\sqrt{np}|W'|(|n - |W'||), \]
which contradicts the weak bijumbledness of \(G\).

Similarly, we show that the set \(Z\) of vertices having degree strictly greater than \((1 + 2\varepsilon/3)p)n\) satisfies \(|Z| < \varepsilon n/2\), which together with the argument above proves (i). More precisely, suppose \(|Z| \geq \varepsilon n/2\), fix \(Z' \subseteq Z\) with \(|Z'| = \varepsilon n/2\). We claim that \(A\sqrt{np}|Z'|\) and \(A\sqrt{np}|Z'|(n - |Z'|)\) are both small (constant) fractions of \(p|Z'|^2\). Indeed, as \(|Z'|^2 < |Z'|(n - |Z'|) < |Z'|n\), it follows that
\[ \frac{A\sqrt{np}|Z'|}{p|Z'|^2} < \frac{A\sqrt{np}|Z'|(n - |Z'|)}{p|Z'|^2} < \frac{A\sqrt{np}|Z'|^2}{p|Z'|^2} = \sqrt{\frac{A^2 n^2}{p|Z'|^3}} = \sqrt{\frac{A^2 n^2}{p|Z'|^3}} \leq \sqrt{\frac{8p}{C}}, \]
where (2) is due to \(\varepsilon^3 np^2 \geq C A^2\). Fact 17 and the previous inequalities imply
\[ e(Z') < \frac{p|Z'|^2}{2} + A\sqrt{np}|Z'| \]
\[ < p|Z'|^2 \left(\frac{1}{2} + \sqrt{\frac{8p}{C}}\right) \]
\[ < p|Z'|^2 \left(\frac{1}{2} + \sqrt{\frac{32p}{C}}\right) - A\sqrt{np}|Z'|(n - |Z'|) \]
\[ < p|Z'|^2 - A\sqrt{np}|Z'|(n - |Z'|). \]

Analogously, we have \(|V(G) \setminus Z'| < np|Z'|\), but
\[ e(Z', V(G) \setminus Z') \geq (1 + 2\varepsilon/3)np|Z'| - 2e(Z') \]
\[ \geq p|Z'|(n - |Z'|) + \left(\frac{1}{2} + \frac{2}{3}\right) \varepsilon np|Z'| - 2e(Z') \]
\[ > p|Z'|^2(n - |Z'|) + 2p|Z'|^2 - 2e(Z') \]
\[ > p|Z'|^2(n - |Z'|) + A\sqrt{np}|Z'|(n - |Z'|), \]
which is again a contradiction to Definition 15. This concludes the proof of (i).

We next prove (ii). For each \(u \in V(G)\), let \(B(u)\) be the set of vertices that have fewer than \((1 - \varepsilon)np^2\) common neighbors with \(u\). By definition, for any vertex \(u\) and set
We claim that there exists an absolute constant oriented and underlying graphs. Let \( p < 1 \) for a contradiction, that \( u \in V(G) \setminus W \) and \( |B(u)| \geq \varepsilon n / 2 \). Let \( N' \subset N(u) \) be a set of size precisely \( (1 - 2 \varepsilon / 3)np \), and let \( B' \subseteq B(u) \) be a set of size precisely \( \varepsilon n / 2 \). Since \( \varepsilon^3 np^2 \geq A^2 C \), we have

\[
\frac{\varepsilon np^2 |B'|}{3} = \frac{\varepsilon^2 n^2 p^2}{6} > \frac{1}{6} \sqrt{\frac{\varepsilon^4 n^4 p^4 (1 - 2 \varepsilon / 3)}{2}} > A \sqrt{np} |N'| |B'|. \tag{20}
\]

We claim that \( |B'| \leq np |N'| \). Indeed, \( |N'| \leq np \leq \varepsilon n^2 p / 2 = np |B'| \) because \( \varepsilon n / 2 > 1 \), and \( |B'| = \varepsilon n / 2 \leq A^2 C n / (3 \varepsilon^3) \leq n^2 p^2 / 3 \leq np |N'| \) because \( \varepsilon^3 np^2 \geq A^2 C \) and \( \varepsilon < 1 \).

Hence, since \( G \) is weakly b(j)unbled, we reach the following contradiction

\[
p |N'| |B'| - A \sqrt{np} |N'| |B'| \leq e(N', B') < (1 - \varepsilon) np^2 |B'|
\]

\[
= \left( 1 - \frac{2 \varepsilon}{3} \right) np^2 |B'| - \varepsilon np^2 |B'| \leq \frac{20}{p} |N'| |B'| - A \sqrt{np} |N'| |B'|.
\]

Hence \( |B(u)| \leq \varepsilon n / 2 \) for all \( u \in V(G) \setminus W \). Note that if \( |N(u) \cap N(v)| < np^2 (1 - \varepsilon) \) for distinct \( u, v \in V(G) \), then either \( u \in W \) or \( v \in B(u) \). We conclude that there are at most \( |W| n + n(\varepsilon n / 2) < \varepsilon n^2 \) such pairs, as desired.

To prove (iii), fix an orientation \( D \) of \( G \) and put \( X = \{ v \in V(G) : \text{deg}_D^+(v) < d \} \). Fact 17 then yields the desired inequality:

\[
|X|(d - 1) \geq e(G[B]) \geq \left( \frac{|X|}{2} \right) p - A \sqrt{np} |X|.
\]

4.1 Almost all orientations of b(j)unbled graphs

In this section we show that almost every orientation of a weakly b(j)unbled graph contains a Seymour vertex.

**Theorem 5.** There exists an absolute constant \( C > 1 \) such that the following holds.

Let \( G \) be a weakly \((p, A \sqrt{np})\)-b(j)unbled graph of order \( n \), where \( \varepsilon^3 np^2 \geq A^2 C \) and \( p < 1 - 15 \sqrt{\varepsilon} \). If \( D \) is chosen uniformly at random among the \( 2^{e(G)} \) possible orientations of \( G \), then a.a.s. \( D \) has a Seymour vertex.

**Proof.** We may and shall assume that \( A^2 C \) is larger than any given absolute constant. Let \( V = V(G) \). For each \( u \in V \), let \( B(u) = \{ v \in V : \text{deg}(u, v) \leq (1 - \varepsilon)np^2 \} \). Also, let \( \text{BAD}_1 = \{ u \in V : |B(u)| \geq \sqrt{\varepsilon n} \} \). Lemma 18 (ii) guarantees that \( |\text{BAD}_1| \leq \sqrt{\varepsilon n} \) and, by definition, \( |B(u)| < \sqrt{\varepsilon n} \) for each \( u \notin \text{BAD}_1 \).

Fix an arbitrary orientation of \( G \). For simplicity, we write \( G \) for both the oriented and underlying graphs. Let \( \text{BAD}_2 = \{ v \in V(G) : \text{deg}_D^+(v) < 2 \sqrt{\varepsilon np} \} \). By Lemma 18 (iii), we must have

\[
|\text{BAD}_2| \leq \frac{2(2 \sqrt{\varepsilon np} - 1)}{p} + 2 A \sqrt{\frac{\pi}{p}} + 1 < 5 \sqrt{\varepsilon n}.
\]

Let \( \text{BAD} = \text{BAD}_1 \cup \text{BAD}_2 \) and put \( U = V \setminus \text{BAD} \), and note that \( |\text{BAD}| \leq 6 \sqrt{\varepsilon n} \).

**Claim 19.** There exists \( w \in U \) such that

\[
\text{deg}_G^+(w) < n / 2 - \sqrt{\varepsilon n}.
\]
Proof. Recall that \( p < 1 - 15\sqrt{\varepsilon} \). Hence \( \varepsilon < 15^{-2} < 1 \) and
\[
\frac{(1 + \varepsilon)p + 6\sqrt{\varepsilon}}{2} < \frac{(1 + \varepsilon)(1 - 15\sqrt{\varepsilon})}{2} + 6\sqrt{\varepsilon} < \frac{1 - 2\sqrt{\varepsilon}}{2}
\]  
(21)
Note also that \( \varepsilon^2np^2 \geq A^2C \) yields \( A \leq \sqrt{\varepsilon^3np^2/C} \). Hence,
\[
A\sqrt{n\bar{p}} \leq \varepsilon np\sqrt{\frac{\varepsilon p}{C}} < \frac{\varepsilon np}{2}.
\]  
(22)
By Fact 17, we have
\[
e(G[U])/|U| \leq \frac{p}{|U|} \left( \frac{|U|}{2} \right) + A\sqrt{n\bar{p}} \leq \frac{p|U|}{2} + A\sqrt{n\bar{p}} \leq \frac{(1 + \varepsilon)np}{2}.
\]  
(23)
Owing to (23), averaging the outdegrees of vertices in \( U \) yields that some \( w \in U \) satisfies \( \deg_{G[U]}^+(w) \leq e(G[U])/|U| < (1 + \varepsilon)np/2 \). Hence,
\[
\deg_{G[U]}^+(w) \leq \deg_{G[U]}^+(w) + |\text{BAD}|
\]
\[
< \frac{(1 + \varepsilon)np}{2} + 6\sqrt{\varepsilon n} \leq \frac{(1 - 2\sqrt{\varepsilon})n}{2}.
\]
\( \Box \)

Note that since we picked an arbitrary orientation of \( G \), the vertex \( w \) given by Claim 19 exists for any such orientation. To conclude the proof, we next show that in a random orientation of \( G \) almost surely every vertex in \( U \) is an \( (1 - 2\sqrt{\varepsilon}) \)-king, where a vertex \( v \) is said to be a \( \lambda\)-king if the number of vertices \( z \) for which there exists a directed path of length 2 from \( v \) to \( z \) is at least \( \lambda n \).

Claim 20. In a random orientation of \( G \), a.a.s. for each \( X \subseteq V(G) \) with \( |X| = 2\sqrt{\varepsilon}np \) we have \( |N^1(X)| \geq (1 - 2\sqrt{\varepsilon})n \), where \( N^1(X) = \bigcup_{x \in X} N^1(x) \).

Proof. Note that for all \( X, Y \subseteq V(G) \), there exist \( X' \subseteq X \) and \( Y' \subseteq Y \) such that \( X' \cap Y' = \emptyset \) and \( |X'| = |X|/2 \) and \( |Y'| = |Y|/2 \). Fix \( X \subseteq V(G) \) with \( |X| = 2\sqrt{\varepsilon}np \).
If we choose \( Y \) such that \( |Y| = 2\sqrt{\varepsilon}n \), then \( |X'| \leq |Y'| = \sqrt{\varepsilon}n \leq \sqrt{\varepsilon}n^2p^2 = np|X'| \) because \( np^2 \geq A^2C/\varepsilon^3 \geq 1 \). Hence, as \( G \) is weakly bi-jumbled,
\[
e(X, Y) \geq e(X', Y') \geq \frac{p|X||Y|}{2} - A\sqrt{np|X||Y|}.
\]  
(24)
Let \( \mathcal{E}_X \) denote the ‘bad’ event that \( |N^1(X)| < (1 - 2\sqrt{\varepsilon})n \), so \( \mathcal{E}_X \) occurs if and only if there exists \( Y \subseteq V(G) \) with \( |Y| = 2\sqrt{\varepsilon}n \) such that \( e(X, Y) = 0 \). For any \( X \) such that \( |X| = 2\sqrt{\varepsilon}np \), summing over all \( Y \) of size \( 2\sqrt{\varepsilon}n \) yields
\[
\Pr[\mathcal{E}_X] \leq \sum_{Y} 2^{-e(X, Y)} \leq \left( \frac{n}{2\sqrt{\varepsilon}n} \right)^{\operatorname{exp}\left( -(\ln 2)(\varepsilon n^2p^2 - A\sqrt{\varepsilon}n^3p^2) \right)} \leq \exp\left( 2n\sqrt{\varepsilon} \ln \left( \frac{e}{2\sqrt{\varepsilon}} \right) - (\ln 2)\varepsilon n^2p^2\left( 1 - \frac{\varepsilon}{C} \right) \right) \leq \exp\left( 2n\sqrt{\varepsilon} \ln \left( \frac{e}{2\sqrt{\varepsilon}} \right) - (\ln 2)\varepsilon n^2p^2\left( 1 - \frac{\varepsilon}{C} \right) \right) \leq \exp(-(2\ln 2)n) \]
(25)
using that \( \varepsilon np^2 \geq A^2C\varepsilon^{-2} \geq 12 \) and that \( \varepsilon/\sqrt{C} \leq C^{-1/2} < 1/2 \) because \( \varepsilon < 1 \) and \( C \) is a large constant. Taking a union bound over all \( X \) of size \( 2\sqrt{\varepsilon}np \), we see that no bad event occurs is with high probability, since
\[
\sum_{X} \Pr[\mathcal{E}_X] \leq 2^\alpha \exp(-(2\ln 2)n) = o(1),
\]
and the claim holds as required. \( \Box \)
We conclude showing that $w$ is a Seymour vertex. Indeed, since $w \notin \text{BAD}_2$, we have $\deg^+(w) \geq 2\sqrt{\varepsilon}np$. Now, Claim 20 implies that $|N^2_G(w)| \geq (1 - 2\sqrt{\varepsilon})n$, and thus, by Claim 19, we have $\deg^+(v) < (1 - 2\sqrt{\varepsilon})n/2$, which implies

$$|N^2_G(w)| \geq (1 - 2\sqrt{\varepsilon})n - \deg^+_G(w) > \frac{(1 - 2\sqrt{\varepsilon})n}{2} > \deg^+_G(w).$$

\(\square\)

5 Concluding remarks

In this paper we confirmed Seymour’s Second Neighborhood Conjecture (SNC) for a large family of graphs, including almost all orientations of (pseudo)random graphs. We also prove that this conjecture holds a.a.s. for arbitrary orientations of the random graph $G(n, p)$, where $p = p(n)$ lies below $1/4$. Interestingly, this range of $p$ encompasses both sparse and dense random graphs.

The main arguments in our proofs lie in finding a vertex $w$ of relatively low outdegree whose outneighborhood contains many vertices of somewhat large outdegree. Since outneighbors of $w$ cannot have small common outneighborhood, we conclude that $|N^2_G(w)|$ must be large.

Naturally, it would be interesting to extend further the range of densities for which arbitrary orientations of $G(n, p)$ satisfy the SNC.

It is seems likely that other classes of graphs, such as $(n, d, \lambda)$-graphs, are susceptible to attack using this approach. Theorem 4 is also a small step towards the following weaker version of Conjecture 1.

**Question 21.** Do most orientations of an arbitrary graph $G$ satisfy the SNC?

Acknowledgments

The authors thank Yoshiharu Kohayakawa for useful discussions, in particular for suggesting we consider bijumbled graphs.

This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil – CAPES – Finance Code 001. F. Botler is supported by CNPq (423395/2018-1) and by FAPERJ (211.305/2019 and 201.334/2022). P. Moura is supported by FAPEMIG (APQ-01040-21). T. Naia is supported by CNPq (201114/2014-3) and FAPESP (2019/04375-5, 2019/13364-7, 2020/16570-4). FAPEMIG, FAPERJ and FAPESP are, respectively, Research Foundations of Minas Gerais, Rio de Janeiro and São Paulo. CNPq is the National Council for Scientific and Technological Development of Brazil.

References

[1] F. Botler, P. Moura, and T. Naia. Seymour’s Second Neighborhood Conjecture in arbitrary orientations of a random graph. In Discrete Mathematics Days 2022, volume 263, page 58. Ed. Universidad de Cantabria, 2022.
[2] F. Botler, P. Moura, and T. Naia. Seymour’s Second Neighborhood Conjecture on sparse random graphs. In Anais do VII Encontro de Teoria da Computação, pages 37–40. SBC, 2022.
[3] G. Chen, J. Shen, and R. Yuster. Second neighborhood via first neighborhood in digraphs. Ann. Comb., 7(1):15–20, 2003.
[4] Z. Cohn, A. Godbole, E. Wright Harkness, and Y. Zhang. The number of Seymour vertices in random tournaments and digraphs. Graphs Combin., 32(5):1805–1816, 2016.
[5] D. Conlon and W. T. Gowers. Combinatorial theorems in sparse random sets. Ann. of Math. (2), 184(2):367–454, 2016.
A Proof that $G(n,p)$ is $p$-typical (Lemma 10)

In this section, we show that $G(n,p)$ satisfies the standard properties of Definition 9.

To simplify this exposition, we make use of Lemma 22 below. Let $B \sim \mathcal{B}(N,p)$ denote a binomial random variable corresponding to the number of successes in $N$ mutually independent trials, each with success probability $p$.

**Lemma 22.** For all $N \in \mathbb{N}$, all $p \in (0,1)$ and all positive $x$, if $B \sim \mathcal{B}(N,p)$ then

$$\Pr[|B - Np| > \sqrt{6Np(1-p)x + 2x}] < 2 \exp(-3x).$$

Lemma 22 follows from the following Chernoff inequality (see [12, Lemma 2.1]).

**Lemma 23.** Let $X \sim \mathcal{B}(N,p)$ and $\sigma^2 = Np(1-p)$. For all $t > 0$ we have

$$\Pr[|X - \mathbb{E}X| > t] < 2 \exp\left(-\frac{t^2}{2(\sigma^2 + t/3)}\right).$$

**Proof of Lemma 22 using Lemma 23.** Let $\sigma^2 = Np(1-p)$ and $t = \sqrt{x^2 + 6x\sigma^2 + x}$. Since $(t - x)^2 = x^2 + 6x\sigma^2$, we have $t^2 = 2tx + 6x\sigma^2 = 6x(\sigma^2 + t/3)$. By Lemma 23,

$$\Pr[|B - \mathbb{E}B| > t] < 2 \exp\left(-\frac{t^2}{2(\sigma^2 + t/3)}\right) = 2 \exp(-3x). \quad (26)$$

Since $t \leq \sqrt{6\sigma^2 x + 2x}$, we have

$$\Pr[|B - \mathbb{E}B| > \sqrt{6\sigma^2 x + 2x}] \leq \Pr[|B - \mathbb{E}B| > t] \leq 2 \exp(-3x). \quad \Box$$

We next show that $G(n,p)$ is $p$-typical. The properties in Definition 9 follow by choosing $x$ in Lemma 22 so as to make the appropriate union bound small.

**Lemma 10.** For every $p : \mathbb{N} \to (0,1)$, a.a.s. $G = G(n,p)$ is $p$-typical.

**Proof.** We will show that a.a.s. (i)–(iv) of Definition 9 hold. Given a random variable $Z$ and $x > 0$, let $\mathbb{1}(Z,x)$ be the indicator variable of the ‘bad’ event

$$|Z - \mathbb{E}Z| > \sqrt{6x \text{Var}(Z) + 2x},$$
where \( \text{Var}(Z) \) is the variance of \( Z \). By Lemma 22, if \( Z \sim \mathcal{B}(N, p) \) then

\[
E(1(Z, x)) = \Pr[1(Z, x) = 1] < 2 \exp(-3x). \tag{27}
\]

Firstly, we show that a.a.s. \((i)\) holds. For each \( X \subseteq V(G) \), let \( Z_X = e(X) \) and let

\[
Z^* = \sum_{X \subseteq V(G)} 1(Z_X, n),
\]

taking \( x = n \). Note that \( Z_X \sim \mathcal{B}((\lvert X \rvert \lvert 2 \rvert), p) \) for all \( X \). By linearity of expectation,

\[
E[Z^*] = \sum_{X \subseteq V(G)} E(1(Z_X, n)) < \sum_{X \subseteq V(G)} 2 \exp(-3n) < 2^{n+1} \exp(-3n) = o(1).
\]

Since \( Z^* \geq 0 \) (it is the sum of indicator random variables), we may use Markov’s inequality, obtaining \( \Pr[Z^* \geq 1] \leq E[Z^*] = o(1) \).

A similar calculation, considering in turn \( \text{deg}(v) \) or \( N(u) \cap N(v) \) instead of \( e(X) \), proves that each of the items \((iii)\) and \((iv)\) fails to hold with probability \( o(1) \), taking \( x = \ln n \) in both cases, and taking union bounds over \( n \) or \( \binom{n}{2} \) events respectively. Hence \( G(n, p) \) satisfies properties \((i), (iii)\) and \((iv)\) with probability \( 1 - o(1) \).

The strategy to prove \((ii)\) is similar to the above, but calculating the number of events in the union bound is slightly more involved. If \( n' = n'' = n \), then (as above) we consider \( e(X, Y) \) in place of \( e(X) \), let \( x = n \) and take a union bound over \( 2^{2n} \) events. Otherwise, if \( 1 \leq n'\ln n \leq n'' \leq n \), then let \( \Omega \) be the set of pairs \( \{X, Y\} \) with \( X, Y \in V(G) \) and \( |X|, |Y| \leq n' \), and note that \( |\Omega| \leq 1 + \left( \sum_{i=1}^{n''} \binom{n}{i} \right)^2 \). Since \( i \leq n' < n/2 \) for sufficiently large \( n \), we have \( \binom{n}{i} \leq \binom{n}{n'} \leq (\frac{en}{n'})^{n''} \) and therefore

\[
|\Omega| \leq 1 + \left( \sum_{i=1}^{n'} \binom{n}{n'} \right)^2 < \left( \binom{n''}{n'} \right)^2 < \left( \binom{en}{n'} \right)^2 < \exp(2n'(1 + \ln n))
\]

By Lemma 22, for each \( \{X, Y\} \in \Omega \) we have \( \Pr[1(e(X, Y), n'')] < 2 \exp(-3n'') \). Applying Markov’s inequality to \( Z^* = \sum_{\{X, Y\} \in \Omega} 1(e(X, Y), n'') \), we obtain

\[
\Pr[Z^* \geq 1] \leq E[Z^*] \leq \exp(2n'(1 + \ln n)) \cdot 2 \exp(-3n'') \leq 2 \exp(-n''/2) = o(1),
\]

where we use that \( \ln n \leq n' \ln n \leq n'' \). \( \square \)