On the fractional heat semigroup and product estimates in Besov spaces and applications in theoretical analysis of the fractional Keller-Segel system

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Abstract

This paper is concerned with the fractional Keller-Segel system in the temporal and spatial variables. We consider fractional dissipation for the physical variables including a fractional dissipation mechanism for the chemotactic diffusion, as well as a time fractional variation assumed in the Caputo sense. We analyze the fractional heat semigroup obtaining time decay and integral estimates of the Mittag-Leffler operators in critical Besov spaces, and prove a bilinear estimate derived from the nonlinearity of the Keller-Segel system, without using auxiliary norms. We use these results in order to prove the existence of global solutions in critical homogeneous Besov spaces employing only the norm of the natural persistence space, including the existence of self-similar solutions, which constitutes a persistence result in this framework. In addition, we prove a uniqueness result without assuming any smallness condition of the initial data.

Keywords: Fractional Keller-Segel system, Besov spaces, bilinear estimate, global existence, uniqueness.

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1 Introduction

We consider the Keller-Segel system which describes the movement of living organisms towards higher concentration regions of chemical attractants. This system is composed of two coupled parabolic equations describing the interaction between the density of cells and the concentration of the chemotactrant, which reads as follows:

\[
\begin{align*}
\eta_t - D_\eta \Delta \eta &= -\chi \nabla \cdot (\eta \nabla v), & \text{in } \mathbb{R}^n \times (0, \infty), \\
v_t - D_v \Delta v &= -\gamma v + \kappa \eta, & \text{in } \mathbb{R}^n \times (0, \infty), \\
\eta(x, 0) &= \eta_0(x), & \text{in } \mathbb{R}^n, \\
v(x, 0) &= v_0(x), & \text{in } \mathbb{R}^n,
\end{align*}
\] (1.1)

where \( n \geq 1 \). In (1.1), \( \eta, v \) are the unknowns denoting the density of cells and the chemical concentration, respectively. The parameters \( D_\eta \) and \( D_v \) represent the corresponding diffusion coefficients for \( \eta \) and \( v \), while \( \chi, \gamma \) and \( \kappa \) are nonnegative parameters denoting the chemotactic sensitivity, and the decay and production rates, respectively. The issues of existence and long-time asymptotic behaviour of solutions for the Keller-Segel system have attracted the attention of many authors (see for instance [1, 2, 3, 4, 5, 6, 7] and references therein). In particular, for \( n = 1 \) it is known the existence of global solutions, and the blow-up is entirely ruled out. In two and three dimensions, there exist global solutions for small data. In the two-dimensional case, solutions of (1.1) with total mass of class \( m < 4\pi \) remain bounded for all times, while for \( \epsilon > 0 \), there exist unbounded solutions with total mass of cells \( m < 4\pi + \epsilon \). For \( n \geq 3 \), system (1.1) has unbounded solutions for arbitrarily small mass of cells [7].

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The classical Keller-Segel model assumes that the density diffusion is not affected by the nonlocal behaviour of the organisms. However, in many situations found in nature, organisms develop alternative search strategies, particularly when chemoattractants, food, or other targets are sparse or rare. Then, the trajectories of the population of organisms are better described by the so-called Lévy flights than Brownian motion (see [8, 9]). Lévy flights behaviour has been suggested in numerous biological contexts, including immune cells, ecology, and human populations (c.f. [10] and references for a deeper discussion). This consideration motivates the substitution of the classical diffusion in the Keller-Segel system (1.1) by a fractional diffusion. On the other hand, regarding to the flux by chemotaxis, it is also relevant to consider that the attraction force be replaced by a less singular interaction kernel. This last consideration has been point out relevant in the analysis of the propagation of chaos for some aggregation-diffusion models (11). In addition, taking into account that the behavior of most biological systems has memory properties, which are neglected when an integer-order time derivative is assumed, we also assume a time variation in a fractional framework. This introduces a nonlocal delay in time for the moving population (12). Based on observations such as those mentioned, we are interested in the theoretical analysis of the following Keller-Segel system in the fractional setting

\[
\begin{cases}
  \frac{D_t^\alpha}{c} \eta + D_\eta (-\Delta)^{\theta/2} \eta = -\chi \nabla \cdot (f(v) \eta), & \text{in } \mathbb{R}^n \times (0, \infty), \\
  \frac{D_t^\alpha}{c} v + D_v (-\Delta)^{\theta/2} v = -\gamma v + \kappa \eta, & \text{in } \mathbb{R}^n \times (0, \infty), \\
  \eta(x, 0) = \eta_0(x), \quad v(x, 0) = v_0(x), & \text{in } \mathbb{R}^n,
\end{cases}
\]

(1.2)

where $D_t^\alpha$ denotes the time fractional derivative operator of order $\alpha \in (0, 1)$ in the Caputo sense.

We recall that if $f \in L^1(0, T; X)$, $T > 0$, and $X$ is a Banach space, the Riemann-Liouville fractional integral of order $\alpha$ of $f$ is defined by

\[ I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau, \quad t \in [0, T]. \]

In addition, if $f \in C([0, T]; X)$, $0 < T \leq \infty$, is such that $I_t^1 f \in W^{1,1}(0, T; X)$, the Caputo fractional derivative of order $\alpha$ of $f$ is defined by

\[ D_t^\alpha f(t) := \frac{d}{dt} \left( I_t^1 (f(t) - f(0)) \right) = \frac{d}{dt} \left\{ \int_0^t (t - \tau)^{-\alpha} [f(\tau) - f(0)] d\tau \right\}. \]

In (1.2), $(-\Delta)^{\theta/2}$, $\theta \in (0, 2]$, denotes the fractional laplacian operator of order $\theta/2$ defined by $(-\Delta)^{\theta/2} f(x) = \mathcal{F}^{-1}(|\xi|^{\theta} \hat{f}(\xi))(x)$, where $\hat{f}(\xi) = \mathcal{F}(f)(\xi)$ and $\mathcal{F}^{-1}(f)(\xi)$ denote the Fourier transform and the inverse Fourier transform of $f$, respectively. In addition, $G(x)$ is also a nonlocal term defined by

\[ G(v)(x) = \nabla \left( (-\Delta)^{-\theta_1/2} v \right)(x), \quad x \in \mathbb{R}^n, \]

for $\theta_1 \in [0, n)$, which can be alternatively represented by $G(v) f = K(x) * f$, $K(x) \sim |x|^{-\theta_1}$. The case $\theta = 2$ and $\theta_1 = 0$ corresponds to the classical Keller-Segel system (1.1). For $\theta_1 = 0$, $\alpha = 1$ and $n = 2$, in (13) the authors proved a result of local existence and uniqueness of solution for (1.2) in homogeneous Besov spaces by using some estimates of the linear dissipative equation in the framework of mixed temporal-spatial spaces, the Chemin mono-norm methods, the Fourier localization and the Littlewood-Paley theory. Later, in (14), the author proved the existence, uniqueness and stability of solutions for (1.2) in critical Besov spaces under smallness condition on the initial data. The results of (14), are based on the $L^p$-$L^q$ time decay for the semigroup $e^{-t(-\Delta)^{\theta/2}}$ in Besov spaces, which leads to use auxiliar norms of Besov type and Kato-time-weighted norms. Some results of global existence and blow-up for the particular case of (1.2) with $\alpha = 1$ and without considering the $v$-equation, have been obtained in (15) and some references therein.

On the other hand, the time fractional Keller-Segel system has not been extensively studied. In (17) the authors studied the global existence and long time behaviour of solutions for the particular case of (1.2) assuming $\theta = 2, \theta_1 = 0$ and $\alpha \in (0, 1)$, with small initial data in the Besov-Morrey space $\mathcal{N}_{r,A}^{-b} \times B_{\infty, \infty}$ with $n \geq 2, 0 \leq \lambda \leq n - 2, b = 2 - \frac{n - \lambda}{r}$ and $\frac{2b}{n - \lambda} < r < n - \lambda$, in the same spirit of the results of (13) for the classical Keller-Segel system (1.1). Some regularity properties of solution for (1.2) assuming $\theta = 2, \theta_1 = 0$ and $\alpha \in (0, 1)$, with initial data in $L^n \cap L^{n/2} \cap L_\infty \times B_{\infty, \infty}$ where recently obtained in (13). It is worthwhile to observe that the existence space in (17) includes auxiliary norms $a la$ Kato, like in (14). In that approach, the fixed point argument is applied by considering a suitable time-dependent $X$ whose norm is given by the sum of a norm $L^\infty(0, \infty; X_1)$ and a norm of kind $\sup_{t > 0} t^a ||w||_{X_2}$, for some $a \neq 0$. With this type of norm is possible to deal with the bilinear term coming from the cross-diffusion term; however, the uniqueness in
Motivated by the above considerations, the aim of this paper is to analyze the existence, uniqueness and persistence of global solutions for the spatio-temporal fractional Keller-Segel system \((1.2)\) in the framework of critical Besov spaces without using auxiliary norms. In order to get this aim, we first derive time decay and integral estimates of the Mittag-Leffler operators in critical Besov spaces, and prove a bilinear estimate derived from the nonlinearity of the Keller-Segel system employing only the norm of the natural persistence space. In order to estimate the bilinear operator, in addition to dealing with the action of the fractional heat semigroup, is necessary to prove a product estimate in the homogeneous Besov setting.

In order to establish the main result, we start by recalling the mild formulation of \((1.2)\) in the fractional setting. According to Duhamel’s principle, the system \((1.2)\) is formally equivalent to the following integral formulation:

\[
\begin{aligned}
\{ \eta(t) &= E_\alpha(-t\alpha(-\Delta)^{\theta/2})\eta_0 - \int_0^t (t - \tau)^{-\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-(t - \tau)\alpha(-\Delta)^{\theta/2})(\eta G(v))(\tau) d\tau, \\
v(t) &= E_\alpha(-t\alpha(-\Delta)^{\theta/2} - \gamma))v_0 + \int_0^t (t - \tau)^{-\alpha-1} E_{\alpha,\alpha}(-(t - \tau)\alpha(-\Delta)^{\theta/2} - \gamma))\eta(\tau) d\tau.
\end{aligned}
\]  

(1.3)

Here \(\{E_\alpha(-t\alpha(-\Delta)^{\theta/2})\}_{t \geq 0}\) and \(\{E_{\alpha,\alpha}(-t\alpha(-\Delta)^{\theta/2})\}_{t \geq 0}\) denote the Mittag-Leffler families defined by

\[
\begin{aligned}
E_\alpha(-t\alpha(-\Delta)^{\theta/2}) &= \int_0^\infty M_\alpha(\tau)U_\theta(\tau t^\alpha) d\tau, \\
E_{\alpha,\alpha}(-t\alpha(-\Delta)^{\theta/2}) &= \int_0^\infty \alpha \tau M_\alpha(\tau)U_\theta(\tau t^\alpha) d\tau,
\end{aligned}
\]

where \(U_\theta(t)\) is the fractional heat semigroup defined in Fourier variables as \(\hat{U_\theta(t)} f = e^{-t|\xi|^{\theta}} \hat{f}\), and \(M_\alpha : \mathbb{C} \to \mathbb{C}\) is the Mainardi function which is defined by

\[
M_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(1 - \alpha(1 + n))}.
\]

In the classical case \(\alpha = 1\), according to Duhamel’s principle, the system \((1.1)\) is formally equivalent to the following integral formulation:

\[
\begin{aligned}
\{ \eta(t) &= U_\theta(t)\eta_0 - \int_0^t \nabla \cdot U_\theta(t - \tau)(\eta G(v))((\tau) d\tau, \\
v(t) &= \tilde{U_\theta}(t)v_0 + \int_0^t \tilde{U_\theta}(t - \tau)\eta(\tau) d\tau,
\end{aligned}
\]  

(1.4)

where \(\tilde{U_\theta}(t) = e^{-\gamma t}U_\theta(t)\).

A solution \([\eta, v]\) of the integral system \((1.3)\) is called a mild solution of the differential system \((1.2)\). In the rest of this work, we will denote the bilinear and linear operators appearing in \((1.3)\) as:

\[
\begin{aligned}
B_\theta(\eta, v)(t) &= -\int_0^t (t - \tau)^{-\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-(t - \tau)\alpha(-\Delta)^{\theta/2})(\eta G(v))(\tau) d\tau, \\
T_\theta(\eta)(t) &= \int_0^t (t - \tau)^{-\alpha-1} E_{\alpha,\alpha}(-(t - \tau)\alpha(-\Delta)^{\theta/2} - \gamma)\eta(\tau) d\tau.
\end{aligned}
\]  

(1.5) \quad (1.6)

Thus, we rewrite system \((1.3)\) as follows

\[
\begin{aligned}
\{ \eta(t) &= E_\alpha(-t\alpha(-\Delta)^{\theta/2})\eta_0 + B_\theta(\eta, v)(t), \\
v(t) &= E_\alpha(-t\alpha((-\Delta)^{\theta/2} - \gamma))v_0 + T_\theta(\eta)(t).
\end{aligned}
\]  

(1.7)

Note that if \(\gamma = 0\), the system \((1.2)\) has a scaling property. Indeed, it is not difficult to check that if \([\eta, v]\) is a regular solution of \((1.2)\) (with \(\gamma = 0\)), then the pair \([\eta_\sigma, v_\sigma]\) defined by

\[
\begin{aligned}
\eta_\sigma(x, t) &= \sigma^{\theta+\theta_1-2}\eta\left(\sigma x, \sigma^{\frac{\theta}{\theta_1}} t\right) \quad \text{and} \quad v_\sigma(x, t) := \sigma^{\theta+\theta_1-2}v\left(\sigma x, \sigma^{\frac{\theta}{\theta_1}} t\right),
\end{aligned}
\]  

(1.8)
is also a solution of \([1.2]\). In this case, the map

\[ [\eta, v] \mapsto [\eta_\sigma, v_\sigma], \]

(1.9)
is called the scaling of \([1.2]\), and solutions invariant by the scaling, this is, solutions \([\eta, v]\) such that \([\eta, v] = [\eta_\sigma, v_\sigma]\) for all \(\sigma > 0\), are called self-similar solutions. Note that if \([\eta, v]\) is a self-similar solution, the initial data \([\eta_0, v_0]\) must be invariant by the scaling

\[ [\eta_0, v_0] \mapsto [\eta_{0\sigma}, v_{0\sigma}], \]

(1.10)
and from \([1.8]\) it must have

\[ \eta_0(x) = \sigma^{2\theta + \theta_1 - 2} \eta_0(\sigma x) \quad \text{and} \quad v_0(x) = \sigma^{\theta + \theta_1 - 2} v_0(\sigma x), \]

this is, a necessary condition to obtain self-similar solutions is that the data \(\eta_0\) and \(v_0\) be homogeneous functions of degrees \(2 - 2\theta - \theta_1\) and \(2 - \theta - \theta_1\), respectively.

In the case \(\gamma \neq 0\) the system \([1.2]\) has not a scaling property: however, we can use the “intrinsic scaling” \([1.9]\) in order to choose the function spaces of initial data. Explicitly, we consider the following class of initial data (see the notations in Section 2):

\[ \eta_0 \in \dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty} \quad \text{and} \quad v_0 \in \dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}. \]

Now, we are in position to establish the main results of this paper.

**Theorem 1.1.** (Product estimate) Let \(n \geq 1\), \(\theta_1 \in (0, n)\), \(\frac{6n}{5n+\theta_1} < p \leq q \leq p'\), \(\max \left\{ 1, 1 - \frac{n}{2} - \frac{\theta_1}{2} + \frac{n}{p} \right\} < \theta < 1 + \frac{n-\theta_1}{3}\), and \(\rho_1, \rho_2 \geq 0\) small enough. Then, for \(f \in \dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}+\rho_1}_{p,\infty}\) and \(g \in \dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}+\rho_2}_{q,\infty}\), we have that \(fG(g) \in \dot{B}^{3-3\theta-\theta_1+\frac{\alpha}{p}+\rho_1+\rho_2}_{p,\infty}\) and

\[ \|fG(g)\|_{\dot{B}^{3-3\theta-\theta_1+\frac{\alpha}{p}+\rho_1+\rho_2}_{p,\infty}} \leq C \|f\|_{\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}+\rho_1}_{p,\infty}} \|g\|_{\dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}+\rho_2}_{q,\infty}}. \]

(1.11)

**Theorem 1.2.** (Bilinear estimate) Let \(n \geq 1\), \(0 < T \leq \infty\), \(\theta_1 \in (0, n)\), \(\alpha > 0\), \(\frac{6n}{5n+\theta_1} < p \leq q \leq p'\) and \(\max \left\{ 1, 1 - \frac{n}{2} - \frac{\theta_1}{2} + \frac{n}{p} \right\} < \theta < 1 + \frac{n-\theta_1}{3}\). Then, there exists a constant \(K > 0\) (independent of \(T\)) such that

\[ \|B(\eta, v)\|_{L^{\infty}\left((0,T);\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}\right)} \leq K \||\eta||_{L^{\infty}\left((0,T);\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}\right)} \|v\|_{L^{\infty}\left((0,T);\dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}\right)}, \]

(1.12)
for all \(\eta \in L^{\infty}\left((0,T);\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}\right)\) and \(v \in L^{\infty}\left((0,T);\dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}\right)\).

**Theorem 1.3.** (Well-posedness) Let \(n \geq 1\), \(\theta_1 \in (0, n)\), \(\alpha > 0\), \(\frac{6n}{5n+\theta_1} < p \leq q \leq p'\) and \(\theta\) such that \(\max \left\{ 1, 1 - \frac{n}{2} - \frac{\theta_1}{2} + \frac{n}{p} \right\} < \theta < 1 + \frac{n-\theta_1}{3}\). There exist \(\varepsilon > 0\) and \(\delta > 0\) such that, if

\[ \|\eta_0\|_{\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}} < \varepsilon \quad \text{and} \quad \|v_0\|_{\dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}} < \varepsilon, \]

then there exists a unique mild solution \([\eta, v]\) for \([1.2]\) such that

\[ \|\eta\|_{L^{\infty}\left((0,\infty);\dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}\right)} < \delta \quad \text{and} \quad \|v\|_{L^{\infty}\left((0,\infty);\dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}\right)} < \delta. \]

**Corollary 1.4.** (Self-similarity) Assume the hypotheses of Theorem 1.3 with \(\gamma = 0\), and consider \(\eta_0 \in \dot{B}^{2-2\theta-\theta_1+\frac{\alpha}{p}}_{p,\infty}\) and \(v_0 \in \dot{B}^{2-\theta-\theta_1+\frac{\alpha}{q}}_{q,\infty}\) being homogeneous functions with degrees \(2 - 2\theta - \theta_1\) and \(2 - \theta - \theta_1\), respectively. Then the solution \([\eta, v]\) obtained through Theorem 1.3 is self-similar.

**Remark 1.5.** 1. Theorem 1.2 plays a central role in the persistence part of Theorem 1.3, and is also central to the proof of Theorem 1.6 below. Moreover, to our knowledge, this type of bilinear estimate for the Keller-Segel system is new in the context of critical spaces.
2. Theorem 1.3 additionally to existence and uniqueness, establishes a persistence result because we do not use auxiliary norms in the solution spaces as it is used in previous works (cf. [14, 17]). In particular, considering \( \theta \neq 2 \) and \( \theta_1 \neq 0 \), Theorem 1.3 complements the existence and uniqueness result in [17], as well as for \( \alpha \neq 1 \), Theorem 1.3 complements the existence and uniqueness result in [17].

3. The proof of Theorem 1.3 in the case \( \alpha = 1 \) is carried out taking into account the mild formulation (1.4). If we denote by \([\eta_{\alpha}, v_{\alpha}]\), the mild solution of (1.2) in the sense of (1.3) for \( \alpha \in (0, 1) \), and \([\eta_1, v_1]\) the mild solution of (1.2) in the sense of (1.4) for \( \alpha = 1 \), it is not clear if \( \lim_{\alpha \to 1^-} [\eta_{\alpha}, v_{\alpha}] = [\eta_1, v_1] \). This is an open question that beyond being raised in this model, can be formulated in general parabolic problems (cf. [19]).

4. The analysis carried out in the proof of Theorem 1.3 allows us to include negative values for the parameter \( \theta_1 \), namely, \( \theta_1 \in (-2n, 0) \) (cf. Lemma 1.1 and Remark 3.9). However, we do not know the possible physical meaning in the description of the model.

Using the estimates developed in the proof of Theorem 1.3 we prove the following uniqueness theorem without assuming any smallness condition of the initial data. The existence of solutions for arbitrary large initial data is an open problem. This uniqueness result seems new for chemotaxis problems in the context of critical spaces, including the classical Keller-Segel system (1.1). This issue has been raised in the context of Navier-Stokes equations (see [20] and some references therein).

In general, given a Banach space \( X \) we denote by \( \tilde{X} \) the maximal closed subspace of \( X \) in which the family of operators \( \{ E_\alpha ( -t^n ( -\Delta )^{\theta/2} ) \} \) is continuous.

**Theorem 1.6 (Uniqueness).** Let \( n \geq 1 \), \( 0 < T \leq \infty \), \( \theta_1 \in [0, n) \), \( \alpha > 0 \), \( \frac{6n}{5n+\theta_1} < p \leq q \leq p' \) and \( \max \left\{ 1, 1 - \frac{\alpha}{p}, \frac{n}{2} - \theta_1 + \frac{\alpha}{p} \right\} < \theta < 1 + \frac{n-\theta_1}{3} \). If \([\eta^1, v^1]\) and \([\eta^2, v^2]\) are two mild solutions of (1.2) in \( C \left( [0, T); B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \right) \times C \left( [0, T); B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \right) \) with the same initial data \([\eta_0, v_0]\) in \( B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \times B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \), then \([\eta^1(t), v^1(t)] = [\eta^2(t), v^2(t)] \) in \( B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \times B_{p,q}^{2-\theta-\theta_1+\frac{\alpha}{p}} \) for all \( t \in [0, T) \).

The rest of this paper is organized as follows. In Section 2 we give some preliminaries about Besov spaces. Section 3 is devoted to the proof of the linear and nonlinear estimates; in particular, we prove the product and bilinear estimates established. Finally, in Section 4 we prove our results about existence and uniqueness of mild solutions (1.3).

## 2 Preliminaries

Briefly we recall some preliminaries about Besov spaces. In what follows \( \varphi \) denotes a radially symmetric function such that

\[
\varphi \in C^\infty_c \left( \mathbb{R}^n \setminus \{0\} \right), \quad \supp \varphi \subset \left\{ x; \frac{3}{4} \leq |x| \leq \frac{8}{3} \right\},
\]

and

\[
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{where } \varphi_j(\xi) := \varphi \left( \xi 2^{-j} \right).
\]

Recall the localization operators \( \Delta_j \) and \( S_k \) defined by

\[
\Delta_j f = \varphi_j (D)f = (\varphi_j)' \ast f \quad \text{and} \quad S_k f = \sum_{j \leq k} \Delta_j f.
\]

One can check easily the identities

\[
\Delta_j \Delta_k f = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-2} \Delta f) = 0 \quad \text{if } |j - k| \geq 5.
\]

Moreover, we have the Bony’s decomposition (see [21])

\[
f g = T_f g + T_g f + R(f g),
\]

(2.1)
where

\[ Tfg = \sum_{j \in \mathbb{Z}} S_{j-2} f \Delta_j g, \quad R(fg) = \sum_{j \in \mathbb{Z}} \Delta_j f \tilde{\Delta}_j g \quad \text{and} \quad \tilde{\Delta}_j g = \sum_{|j-j'| \leq 1} \Delta_{j'} g. \]

We also denote \(\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}\) and \(\tilde{D}_j = D_{j-1} \cup D_j \cup D_{j+1}\) where \(j \in \mathbb{Z}\) and \(D_j = \{ x : \frac{3}{4} 2^j \leq |x| \leq \frac{5}{4} 2^j \}\).

Notice that \(\tilde{\varphi}_j = 1\) on \(D_j\).

**Lemma 2.1. (Bernstein inequality)** \(\text{(2.2)}\) Assume that \(1 \leq q \leq p \leq \infty\). Then

\[ \| f \|_{L^p(\mathbb{R}^n)} \leq C 2^{q (\frac{n}{p} - 1)} \| f \|_{L^q(\mathbb{R}^n)}, \]

for all \(f \in L^q(\mathbb{R}^n)\) such that \(\text{supp} \tilde{f} \subset D_j\).

**Definition 2.2.** Let \(1 \leq p, r \leq \infty\) and \(s \in \mathbb{R}\). The homogeneous Besov space \(\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^n)\) is defined as

\[ \dot{B}^s_{p,r} = \left\{ f \in S'(\mathbb{R}^n)/P ; \| f \|_{\dot{B}^s_{p,r}} < \infty \right\}, \]

where

\[ \| f \|_{\dot{B}^s_{p,r}} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{L^r}^p \right)^{\frac{1}{p}} & \text{if } r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^r} & \text{if } r = \infty. \end{cases} \]

**Lemma 2.3.** \(\text{(2.3)}\) Let \(1 \leq p \leq \infty, 1 \leq r, r_0, r_1 \leq \infty\) and \(s, s_0, s_1 \in \mathbb{R}\) be such that \(s = (1 - \theta) s_0 + \theta s_1\) with \(\theta \in (0, 1)\). Then

\[ \left( \dot{B}^{s_0}_{p,r_0}, \dot{B}^{s_1}_{p,r_1} \right)_{\theta,r} = \dot{B}^s_{p,r}. \]

**Lemma 2.4.** \(\text{(2.4)}\) Let \(1 < p \leq \infty, 1 < r \leq \infty\) and \(s \in \mathbb{R}\). Then

\[ \left( \dot{B}^{-s}_{p',r'} \right)' = \dot{B}^s_{p,r}. \]

### 3 Key estimates

#### 3.1 Time decay estimates of Mittag-Leffler operators

In order to estimate the bilinear operator \(B_\theta \cdot \cdot \cdot \cdot \cdot \cdot \) and the linear operator \(T_\theta \cdot \cdot \cdot \cdot \cdot \cdot \) introduced in \(\text{(1.5)}\) and \(\text{(1.6)}\), we need to deal with the action of the operators \(E_\alpha(-t^\alpha (-\Delta)^{\theta/2})\) and \(E_{\alpha,\alpha}(-t^\alpha (-\Delta)^{\theta/2})\) in Besov spaces.

**Lemma 3.1.** \(\text{(1.4)}\) Let \(\theta > 0\) and \(\zeta \geq 0\) and consider the fractional heat operator \(U_\theta(t)\) defined in Fourier variables as \(\hat{U}_\theta(t)f = e^{-t|\xi|^\theta} \hat{f}\). If \(s_1 \leq s_2, 1 \leq p_1 \leq p_2 \leq \infty\) and \(1 \leq r \leq \infty\), then the following inequality holds

\[ \| (-\Delta)^{s/2} U_\theta(t)f \|_{\dot{B}^s_{p_2,r}} \leq C t^{-\frac{\alpha - s_1 + \zeta}{\alpha r_1} - \frac{n}{p_2} \left( s_2 - s_1 \frac{\alpha}{n} \right)} \| f \|_{\dot{B}^{s_1}_{p_1,r}}. \]

**Lemma 3.2.** \(\text{(1.9)}\) Let \(\alpha \in (0, 1)\) and \(-1 < r < \infty\). Then \(M_\alpha(t) \geq 0\) for all \(t \geq 0\) and

\[ \int_0^\infty t^\alpha M_\alpha(t) dt = \frac{\Gamma(r + 1)}{\Gamma(\alpha r + 1)}. \]

**Lemma 3.3.** Let \(\theta > 0\) and \(\zeta \geq 0\).

If \(s_1 \leq s_2, 1 \leq p_1 \leq p_2 \leq \infty\), \(1 \leq r \leq \infty\) and \(\frac{\alpha}{p_1} - \frac{n}{p_2} < 1\), then the following inequality holds

\[ \| (-\Delta)^{s/2} E_\alpha(-t^\alpha (-\Delta)^{\theta/2})f \|_{\dot{B}^s_{p_2,r}} \leq C t^{-\frac{\alpha - s_1 + \zeta}{\alpha r_1} - \frac{n}{p_2} \left( s_2 - s_1 \frac{\alpha}{n} \right)} \| f \|_{\dot{B}^{s_1}_{p_1,r}}. \]
Moreover, if \( s_1 \leq s_2, \ 1 \leq p_1 \leq p_2 \leq \infty, \ 1 \leq r \leq \infty \) and \( \frac{1}{\theta}(s_1 - s_2 + \zeta + \frac{\alpha}{p_1} - \frac{\alpha}{p_2}) < 2 \), then the following inequality holds

\[
\left\| (-\Delta)^{\zeta/2} E_{\alpha,\alpha}(-t^\alpha(-\Delta)^{\theta/2})f \right\|_{\dot{B}^1_{p_2,r}} \leq C t^{-\frac{\alpha}{\theta}(s_2-s_1+\zeta)-\frac{\alpha}{\theta} \left( \frac{n}{p_1} - \frac{n}{p_2} \right)} \| f \|_{\dot{B}^1_{p_1,r}}.
\]  

(3.3)

Proof. Let \( f \in \dot{B}^1_{p_1,r} \). From Lemma 3.2 and estimate (3.1) in Lemma 3.1 it holds

\[
\left\| (-\Delta)^{\zeta/2} E_{\alpha}(-t^\alpha(-\Delta)^{\theta/2})f \right\|_{\dot{B}^1_{p_2,r}} \leq \int_0^\infty M_\alpha(\tau) \left\| (-\Delta)^{\zeta/2} U_\theta(\tau t^\alpha)f \right\|_{\dot{B}^1_{p_2,r}} d\tau
\]

\[
\leq C \left[ \int_0^\infty M_\alpha(\tau) t^{-\frac{\alpha}{\theta}(s_2-s_1+\zeta)-\frac{\alpha}{\theta} \left( \frac{n}{p_1} - \frac{n}{p_2} \right)} d\tau \right] t^{-\frac{\alpha}{\theta}(s_2-s_1+\zeta)-\frac{\alpha}{\theta} \left( \frac{n}{p_1} - \frac{n}{p_2} \right)} \| f \|_{\dot{B}^1_{p_1,r}}, \ t > 0.
\]

Which prove (3.2). The proof of (3.3) follows analogously. \( \square \)

3.2 Integral Estimates

In order to estimate the integral terms in (1.3) we present a version in Besov spaces of the Yamazaki estimate obtained in \([24]\) in the context of Lorentz spaces \( L^{(p,\theta)} \). In the context of Besov-Lorentz-Morrey spaces and working in the non fractional case, a related estimate was proved in \([20]\). We remark that the estimate presented here is more general and we do not need (although it is possible) to use Lorentz spaces as base space for Besov, and therefore our way of prove is different to that presented in \([20]\).

Lemma 3.4. Let \( 1 \leq p \leq \infty, \ \zeta \geq 0, \ \alpha > 0, \ \theta > \zeta \) and \( s_0, s \in \mathbb{R} \) be such that \( -s + \theta - \zeta = -s_0 \). Then, there exists a constant \( C > 0 \) such that

\[
\int_0^\infty \left\| \tau^{\alpha-1}(-\Delta)^{\zeta/2} E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})f \right\|_{\dot{B}_{p,1}^{-r_0}} d\tau \leq C \| f \|_{\dot{B}_{p,1}^{-r_0}},
\]

for all \( f \in \dot{B}_{p,1}^{-s} \).

Proof. Let \( f \in \dot{B}_{p,1}^{-s} \) and define the function \( h_f \) by

\[
h_f(\tau) = \left\| \tau^{\alpha-1}(-\Delta)^{\zeta/2} E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})f \right\|_{\dot{B}_{p,1}^{-r_0}}.
\]

Thus, for \( -s_i \leq -s_0 \) (\( i = 1, 2 \)) and using Lemma 3.3 we have

\[
h_f(\tau) = \left\| \tau^{\alpha-1}(-\Delta)^{\zeta/2} E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})f \right\|_{\dot{B}_{p,1}^{-r_0}} \leq C \tau^{-\frac{\alpha}{\theta}(-s_0-s_1+\zeta+(1-\alpha)\theta)} \| f \|_{\dot{B}_{p,1}^{-s_i}}.
\]

Taking, for example, \( -s_1 = -s - \varepsilon \) and \( -s_2 = -s + \varepsilon \) for \( \varepsilon \) small enough, we have that \( -s_i < -s_0 \), and defining \( \frac{1}{\varepsilon} = \frac{\alpha}{\theta}(-s_0 + s_i + \zeta) \) we obtain

\[
\frac{1}{\varepsilon} = \frac{\alpha(-s_0 + s_1 + \zeta + (1-\alpha)\theta)}{\theta} - \frac{\alpha(-s_0 + s + \varepsilon + \zeta + (1-\alpha)\theta)}{\theta} = \frac{\alpha}{\theta} \left( \theta + \varepsilon + \frac{(1-\alpha)\theta}{\alpha} \right) = 1 + \frac{\alpha}{\theta} \varepsilon > 1,
\]

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and

\[
0 < \frac{1}{z_2} = \frac{\alpha(-s_0 + s + \zeta + \frac{(1-\alpha)}{\alpha})}{\theta} = \frac{\alpha(-s_0 + s - \varepsilon + \zeta + \frac{(1-\alpha)}{\alpha})}{\theta} = \frac{\alpha}{\theta} \left( \theta - \varepsilon + \frac{(1-\alpha)}{\alpha} \right) = 1 - \frac{\alpha}{\theta} \varepsilon < 1.
\]

Therefore \(0 < z_1 < 1 < z_2 < \infty\), and for \(\phi = 1/2\) we have \(1 = \frac{\phi}{z_2} + \frac{1-\phi}{z_2}\) and \(-s = (1-\phi)(-s_1) + \phi(-s_2)\). Thus, for \(i = 1, 2\), it follows that \(h_f \in L^{z_i,\infty}(0, \infty)\) with the estimate \(\|h_f\|_{L^{z,\infty}(0, \infty)} \leq C \|f\|_{\dot{B}^{-s}_{p,1}}\) and we can use interpolation in Lorentz spaces and Lemma 2.3 in order to get

\[
\|h_f\|_{L^{1}(0,\infty)} \leq C \|f\|_{\dot{B}^{-s}_{p,1}},
\]

which finishes the proof.

\[\square\]

**Remark 3.5.** Under the same conditions in Lemma 3.4, we also have

\[
\int_0^\infty \left\| \tau^{\alpha-1}(-\Delta)^{\zeta/2}E_{\alpha,\alpha}(-t^\alpha((-\Delta)^{\theta/2} - \gamma))f \right\|_{\dot{B}^{-s}_{p,1}}^2 \, d\tau \leq C \|f\|_{\dot{B}^{-s}_{p,1}}^2,
\]

for all \(f \in \dot{B}^{-s}_{p,1}\).

The next lemma concerns with an estimate for the operator \(B\) defined by

\[
B(f) := \int_0^\infty \tau^{\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})f(\cdot, \tau) \, d\tau.
\]

**Lemma 3.6.** Let \(1 < p \leq \infty\), \(\alpha > 0\), \(\theta > 1\) and \(s_0, s \in \mathbb{R}\) be such that \(-s + \theta - 1 = -s_0\). Then, there exists \(C > 0\) such that

\[
\|B(f)\|_{\dot{B}^s_p, \infty} \leq C \sup_{t > 0} \|f(t)\|_{\dot{B}^{s_0}_{p,1}},
\]

for all \(f \in L^\infty(0, \infty; \dot{B}^{s_0}_{p,1})\).

**Proof.** Using duality and Lemma 3.4 with \(\zeta = 1\) we have

\[
\|B(f)\|_{\dot{B}^s_p, \infty} = \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \|\langle B(f), h \rangle\| \leq C \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \int_0^\infty \left| \langle \tau^{\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})f(\cdot, \tau), h \rangle \right| \, d\tau
\]

\[
\leq C \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \int_0^\infty \left| \left\langle f(\cdot, \tau), \tau^{\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})h \right\rangle \right| \, d\tau
\]

\[
\leq C \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \int_0^\infty \|f(\cdot, \tau)\|_{\dot{B}^{s_0}_{p,1}} \left\| \tau^{\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})h \right\|_{\dot{B}^{-s_0}_{p,1}} \, d\tau
\]

\[
\leq C \sup_{\|f\|_{\dot{B}^{s_0}_{p,1}} \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \int_0^\infty \left\| \tau^{\alpha-1} \nabla \cdot E_{\alpha,\alpha}(-\tau^\alpha(-\Delta)^{\theta/2})h \right\|_{\dot{B}^{-s_0}_{p,1}} \, d\tau
\]

\[
\leq C \sup_{\|f\|_{\dot{B}^{s_0}_{p,1}} \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \|h\|_{\dot{B}^{-s}_{p,1}} \sup_{\|f\|_{\dot{B}^{s_0}_{p,1}} \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \|f\|_{\dot{B}^{s_0}_{p,1}}}
\]

\[
\leq C \sup_{\|f\|_{\dot{B}^{s_0}_{p,1}} \sup_{\|h\|_{\dot{B}^{s_0}_{p,1}} = 1} \|f\|_{\dot{B}^{s_0}_{p,1}}. \tag{3.5}
\]

\[\square\]

Now we consider the operator \(T\) defined by \(T(\eta) := \int_0^\infty s^{\alpha-1} E_{\alpha,\alpha}(-s^\alpha((-\Delta)^{\theta/2} - \gamma))\eta(s) \, ds\). We have the following lemma.
Lemma 3.7. Let $1 < p \leq q$, $\alpha > 0$ and $\theta > 0$, then
\[
\|T(\eta)\|_{B^p_q,\infty} \leq C \sup_{t \geq 0} \|\eta(t)\|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}},
\]
for all $\eta \in L^\infty((0, \infty); B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty})$.

Proof. Using duality and Lemma 3.4 with $\zeta = 0$ we have
\[
\begin{align*}
\|T(\eta)\|_{B^p_q,\infty} & = \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} |\langle T(\eta), h \rangle| \\
& = \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \int_0^\infty \left| \left\langle \tau^{\alpha-1} E_{\alpha,\alpha} (\tau^{-\alpha} (-\Delta)^{\theta/2}) \eta(\tau), h \right\rangle \right| d\tau \\
& = \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \int_0^\infty \left| \left\langle \eta(\tau), \tau^{\alpha-1} E_{\alpha,\alpha} (\tau^{-\alpha} (-\Delta)^{\theta/2}) h \right\rangle \right| d\tau \\
& \leq C \sup_{\tau > 0} \|\eta(\tau)\|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}} \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \int_0^\infty \left| \tau^{\alpha-1} E_{\alpha,\alpha} (\tau^{-\alpha} (-\Delta)^{\theta/2}) h \right|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}} d\tau \\
& \leq C \sup_{\tau > 0} \|\eta(\tau)\|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}} \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \|h\|_{B^{q',1}_{q',1}} \\
& \leq C \sup_{\tau > 0} \|\eta(\tau)\|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}} \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \|h\|_{B^{q',1}_{q',1}} \\
& \leq C \sup_{\tau > 0} \|\eta(\tau)\|_{B^{2p-2\theta-1+\frac{n}{q}}_{p,\infty}} \sup_{\|h\|_{B^{q',1}_{q',1}} = 1} \|h\|_{B^{q',1}_{q',1}}.
\end{align*}
\]
(3.6)

Remark 3.8. In the previous proof we have used that $B^{-2(2\theta-\theta_1+\frac{n}{q})}_{q',1} \hookrightarrow B^{-2(2\theta-\theta_1+\frac{n}{q})}_{p',1}$ which is a direct consequence of Lemma 2.7.

3.3 Product Estimate. Proof of Theorem 1.1

Proof. For this proof denote $s_1 = 2 - 2\theta - \theta_1 + \frac{n}{p}$, $s_2 = 3 - 3\theta - \theta_1 + \frac{n}{p}$ and $s_2 = 2 - \theta - \theta_1 + \frac{n}{q}$.

From the decomposition (2.1), we obtain
\[
\begin{align*}
\Delta_j(fG(g)) &= \sum_{|k-j| \leq 4} \Delta_j(S_{k-2}f \Delta_k G(g)) + \sum_{|k-j| \leq 4} \Delta_j(S_{k-2}G(g) \Delta_k f) + \sum_{k \geq j - 2} \Delta_j \left( \Delta_k \tilde{\Delta}_k G(g) \right) \\
& = I_1^j + I_2^j + I_3^j.
\end{align*}
\]
(3.7)

In order to estimate $I_1^j$, let $p^*$ such that $\frac{1}{p} = \frac{1}{p^*} + \frac{1}{q}$, then
\[
\left\| I_1^j \right\|_{L^p} \leq C \sum_{|k-j| \leq 4} \| S_{k-2} f \|_{L^p} \| \Delta_k G(g) \|_{L^q} \leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} \| \Delta_m f \|_{L^p} \right) \| G(\Delta_k g) \|_{L^q}
\]
\[
\leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{\theta}{4} - \frac{\theta}{q} \right) \| \Delta_m f \|_{L^p} \right) 2^k \left( 1 - \theta \right) \| \Delta_k g \|_{L^q}
\]
\[
\leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( -2 + 2\theta + \theta_1 - \frac{\theta}{q} - \rho_1 \right) \right) \| \Delta_k g \|_{L^q}
\]
\[
\leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( -2 + 2\theta + \theta_1 - \frac{\theta}{q} - \rho_1 \right) \right) 2^k \left( -1 - \theta - \frac{\theta}{q} - \rho_1 \right).
\]

Note that \(-2 + 2\theta + \theta_1 + \frac{\theta}{q} - \frac{\theta}{p} - \rho_1 > 0\) for some \(\rho_1\) small enough, in fact, since \(\frac{\theta}{q} \geq \frac{\theta}{p}\) we only need to verify that \(-2 + 2\theta + \theta_1 + \frac{\theta}{p} - \frac{\theta}{q} > 0\), which reduces to the condition \(\theta > 1 - \frac{\theta}{p} - \frac{\theta}{q}\). Therefore,
\[
\left\| I_1^j \right\|_{L^p} \leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} 2^{-j(\theta_0 + \rho_1 + \rho_2)}.
\]

In order to estimate \(I_2^j\), we proceed similarly to obtain
\[
\left\| I_2^j \right\|_{L^p} \leq C \sum_{|k-j| \leq 4} \| S_{k-2} G(g) \|_{L^\infty} \| \Delta_k f \|_{L^p} \leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} \| \Delta_m G(g) \|_{L^\infty} \right) \| \Delta_k f \|_{L^p}
\]
\[
\leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{\theta}{4} - \frac{\theta}{q} \right) \| \Delta_m G(g) \|_{L^\infty} \right) \| \Delta_k f \|_{L^p}
\]
\[
\leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{\theta}{q} + 1 - \theta_1 \right) \| \Delta_m g \|_{L^q} \right) \| \Delta_k f \|_{L^p}
\]
\[
\leq C \| g \|_{B^{2+\rho_2}_{q,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( \frac{\theta}{q} + 1 - \theta_1 - 2 + \theta + \theta_1 - \frac{\theta}{q} - \rho_1 \right) \right) \| \Delta_k f \|_{L^p}
\]
\[
\leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( -1 - \theta - \rho_2 \right) \right) 2^k \left( -2 + 2\theta + \theta_1 - \frac{\theta}{q} \right)
\]
\[
\leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m \left( 1 + \theta - \rho_2 \right) \right) 2^k \left( -2 + 2\theta + \theta_1 - \frac{\theta}{q} \right)
\]
\[
\leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} 2^j \left( -1 + \theta - \frac{\theta}{q} - \rho_1 - \rho_2 \right).
\]

Here we use that \(\theta > 1\) which implies that \(-1 + \theta > 0\) and so \(-1 + \theta - \rho_2 > 0\) for \(\rho_2\) small enough. The previous inequality reduces to
\[
\left\| I_2^j \right\|_{L^p} \leq C \| f \|_{B^{1+\rho_1}_{p,1}} \| g \|_{B^{2+\rho_2}_{q,\infty}} 2^{-j(\theta_0 + \rho_1 + \rho_2)}.
\]

(3.9)
Now we turn to $I^4_1$. Note that in the given conditions we have $-3+3\theta+\theta_1-n<0$ and so $-3+3\theta+\theta_1-n-\rho_1-\rho_2<0$; then we have the estimate
\[
\left\| I^4_1 \right\|_{L^1} \leq C \sum_{k \geq j=2} \left\| \Delta_k f \Delta_k G(g) \right\|_{L^1} \leq C \sum_{k \geq j=2} \left\| \Delta_k f \right\|_{L^p} \left\| \Delta_k G(g) \right\|_{L^{p'}}
\]
\[
\leq C \left\| f \right\|_{L^1} \sum_{k \geq j=2} 2^{k(-2+2\theta+\theta_1-\frac{\theta}{p}-\rho_1)} \left\| \Delta_k G(g) \right\|_{L^{p'}}
\]
\[
\leq C \left\| f \right\|_{L^1} \sum_{k \geq j=2} 2^{k(-2+2\theta+\theta_1-\frac{\theta}{p}-\rho_1)} 2k(1-\theta_1) \left\| \Delta_k G \right\|_{L^{p'}}
\]
\[
\leq C \left\| f \right\|_{L^1} \sum_{k \geq j=2} 2^{k(-2+2\theta+\theta_1-\frac{\theta}{p}-\rho_1)} 2k(1-\theta_1) 2^{3-3\theta+\theta_1-n-\rho_1-\rho_2}
\]
\[
\leq C \left\| f \right\|_{L^1} \sum_{k \geq j=2} 2^{k(-3+3\theta+\theta_1-n-\rho_1-\rho_2)}
\]
So
\[
\left\| I^4_1 \right\|_{L^p} \leq C 2^j \left( 1+\frac{s}{p} \right) \left\| I^4_1 \right\|_{L^1}
\]
\[
\leq C 2^j \left( 1+\frac{s}{p} \right) \left\| f \right\|_{L^1} \left\| g \right\|_{L^{p}} 2^{j(-3+3\theta+\theta_1-n-\rho_1-\rho_2)}
\]
\[
\leq C \left\| f \right\|_{L^1} \left\| g \right\|_{L^{p}} 2^{j(-3+3\theta+\theta_1-n-\rho_1-\rho_2)}.
\]
Computing the norm $\left\| \cdot \right\|_{L^p}$ in (3.7) and considering the estimates (3.8), (3.9) and (3.10), we get the result. \hfill \Box

**Remark 3.9.** In the proof of Lemma 3.11 the condition $\eta_1(\theta_1+5n) < p$ is not directly used. This condition is imposed in order to guarantee that the interval $\left( 1-\frac{\theta}{p}+\theta_1, 1+\frac{n-\theta}{p} \right)$ for $\theta$ is nonempty. Also, the condition $\theta_1 \in (0,n)$ guarantees that the interval $\left( 1, 1+\frac{n-\theta}{p} \right)$ for $\theta$ is nonempty.

### 3.4 Bilinear estimate. Proof of Theorem 1.2

**Proof.** Let $0 < T \leq \infty$ and $t \in (0,T)$. The bilinear term $B_\theta(\eta, v)$ can be written as
\[
B_\theta(\eta, v)(t) = \int_0^t (t-\tau)^{-1} \nabla \cdot E_{\alpha,\alpha} (-\Delta)^{\theta/2} (\eta G(v)) d\tau = B(f_t),
\]
where $f_t(x, \tau)$ is defined by
\[
f_t(x, \tau) = (\eta G(v))(x, t-\tau), \quad \text{a.e. } \tau \in (0, t),
\]
\[
f_t(x, \tau) = 0, \quad \text{a.e. } \tau \in (t, \infty).
\]
From Lemma 3.6 (with $s = 2-2\theta-\theta_1+\frac{\theta}{p}$ and $s_0 = 3-3\theta-\theta_1+\frac{\theta}{p}$) we get
\[
\left\| B(f_t) \right\|_{L^{2-2\theta-\theta_1+\frac{\theta}{p}}} \leq C \sup_{\tau>0} \left\| f_t(\tau) \right\|_{L^{3-3\theta-\theta_1+\frac{\theta}{p}}}.
\]
Using Lemma 3.1 with $\rho_1 = \rho_2 = 0$, we can estimate
\[
\sup_{0 < \tau < T} \left\| f_t(\tau) \right\|_{L^{3-3\theta-\theta_1+\frac{\theta}{p}}} = \sup_{0 < \tau < T} \left\| (\eta G(v))(x, t-\tau) \right\|_{L^{3-3\theta-\theta_1+\frac{\theta}{p}}}
\]
\[
\leq C \sup_{0 < \tau < T} \left\| \eta \left( x, t-\tau \right) \right\|_{L^{2-2\theta-\theta_1+\frac{\theta}{p}}} \left\| v \left( x, t-\tau \right) \right\|_{L^{2-\theta-\theta_1+\frac{\theta}{p}}}
\]
\[
\leq C \sup_{0 < \tau < T} \left\| \eta \left( x, t-\tau \right) \right\|_{L^{2-2\theta-\theta_1+\frac{\theta}{p}}} \sup_{0 < \tau < T} \left\| v \left( x, t-\tau \right) \right\|_{L^{2-\theta-\theta_1+\frac{\theta}{p}}}
\]
\[
\leq C \sup_{0 < \tau < T} \left\| \eta \left( x, \tau \right) \right\|_{L^{2-2\theta-\theta_1+\frac{\theta}{p}}} \sup_{0 < \tau < T} \left\| v \left( x, \tau \right) \right\|_{L^{2-\theta-\theta_1+\frac{\theta}{p}}}
\]
Thus, we can conclude that

\[ \sup_{0 < t < T} \| B_\theta(\eta, v)(t) \|_{B_{2p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}} \leq K \sup_{0 < t < T} \| \eta(t) \|_{B_{2p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}} \sup_{0 < t < T} \| v(t) \|_{B_{2q, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}} \]  

\[ \square \]

**Lemma 3.10.** Let \( 1 < p \leq q, \alpha > 0 \) and \( \theta > 0 \). Then, there exists a constant \( K > 0 \) (independent of \( T \)) such that

\[ \| T_\theta(\eta) \|_{L^\infty((0, T); B_{q, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}})} \leq C \| \eta(t) \|_{L^\infty((0, T); B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}})} \]

for all \( \eta \in L^\infty((0, T); \dot{B}_{p, \infty}^{\alpha}) \).

**Proof.** Let \( 0 < T \leq \infty \) and \( t \in (0, T) \). Note that the operator \( T(\eta) \) can be written as

\[ T_\theta(\eta) = \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}((-t - \tau)^{\alpha}((-\Delta)^{\theta/2} - \gamma)) \eta(\tau) d\tau = -T(f_t), \]

where \( f_t(x, \tau) \) is defined by

\[ f_t(\cdot, \tau) = \eta(\cdot, t - \tau), \quad \text{a.e.} \tau \in (0, t), \]

\[ f_t(\cdot, \tau) = 0, \quad \text{a.e.} \tau \in (t, \infty). \]

From Lemma 3.7 we obtain

\[ \| T(f_t) \|_{B_{q, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}} \leq C \sup_{\tau > 0} \| f_t(\tau) \|_{B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}}, \]

moreover

\[ \sup_{0 < \tau < T} \| f_t(\tau) \|_{B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{p}}} = \sup_{0 < \tau < T} \| \eta(t - \tau) \|_{B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{p}}} \leq \sup_{0 < \tau < T} \| \eta(\cdot, \tau) \|_{B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{p}}}. \]

Thus, we arrive at

\[ \sup_{0 < t < T} \| T_\theta(\eta) \|_{B_{q, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}} \leq K \sup_{0 < t < T} \| \eta(\cdot, \tau) \|_{B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}}. \]

\[ \square \]

### 4 Existence and uniqueness of global solutions

The aim of this section is to prove the existence and uniqueness of global mild solution of system (1.2), which will be carried out through an iterative approach.

#### 4.1 Proof of Theorem 1.3

To simplify the notation, we denote

\[ X = L^\infty((0, T); B_{p, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}) \quad \text{and} \quad Y = L^\infty((0, T); \dot{B}_{q, \infty}^{2-\theta - \theta_1 + \frac{\theta}{q}}). \]

In order to prove Theorem 1.3 we consider the following iterative system:

\[ \eta^1 := E_\alpha(-t^\alpha(-\Delta)^{\theta/2})\eta_0, \quad v^1 := E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0, \]

and for \( n \geq 1 \)

\[ \eta^{n+1} := \eta^n + B_\theta(\eta^n, v^n), \]

\[ v^{n+1} := v^n + T_\theta(\eta^n). \]
From Lemma 3.3 it follows that

\[ \|\eta^1\|_X = \|E_\alpha(-t^\alpha(\Delta)^{\theta/2})\eta_0\|_X \leq C_1\|\eta_0\|_X, \]

and

\[ \|v^1\|_Y = \|E_\alpha(-t^\alpha(-(\Delta)^{\theta/2} - \gamma))v_0\|_Y \leq C_2\|v_0\|_Y. \]

Additionally, using Lemmas 1.2 and 3.10 we have

\[ \|\eta^{n+1}\|_X = \|\eta^1 + B_\theta(\eta^n, v^n)\|_X \leq \|\eta^1\|_X + \|B_\theta(\eta^n, v^n)\|_X \leq C_1\|\eta_0\|_X + K\|\eta^n\|_X\|v^n\|_Y, \]

\[ \|v^{n+1}\|_Y = \|v^1 + T_\theta(\eta^{n+1})\|_Y \leq \|v^1\|_Y + \|T_\theta(\eta^{n+1})\|_Y \leq C_2\|v_0\|_Y + C\|\eta^{n+1}\|_X. \]

Let \(0 < \varepsilon < \frac{\theta}{2K}\) and \(\eta_0, v_0\) such that \(C_1\|\eta_0\|_X \leq \frac{\varepsilon}{4C} < \frac{\varepsilon}{2C}\) and \(C_2\|v_0\|_Y \leq \frac{\varepsilon}{2} < \varepsilon\), then

\[ \|\eta^2\|_X < \frac{\varepsilon}{4C} + K\frac{\varepsilon}{4C} < \frac{\varepsilon}{4C} + \frac{\varepsilon}{4C} = \frac{\varepsilon}{2C}, \]

\[ \|v^2\|_Y < \frac{\varepsilon}{2} + C\frac{\varepsilon}{2C} = \varepsilon. \]

Proceeding inductively, we prove that

\[ \|\eta^{n+1}\|_X < \frac{\varepsilon}{2C} \quad \text{and} \quad \|v^{n+1}\|_Y < \varepsilon. \]

Now we prove that the sequences \((\eta^n)\) and \((v^n)\) are Cauchy in the respective spaces. In fact, we have that

\[ \eta^{n+1} - \eta^n = B_\theta(\eta^n, v^n) - B_\theta(\eta^{n-1}, v^{n-1}) \]

\[ = B_\theta(\eta^n - \eta^{n-1}, v^n) + B_\theta(\eta^{n-1}, v^n - v^{n-1}), \]

and so

\[ \|\eta^{n+1} - \eta^n\|_X \leq K\left(\|\eta^n - \eta^{n-1}\|_X\|v^n\|_Y + \|\eta^{n-1}\|_X\|v^n - v^{n-1}\|_Y\right) \]

\[ \leq K\left(\varepsilon\|\eta^n - \eta^{n-1}\|_X + \frac{\varepsilon}{2C}\|v^n - v^{n-1}\|_Y\right). \tag{4.1} \]

On the other hand, since

\[ v^{n+1} - v^n = T_\theta(\eta^{n+1} - \eta^n), \]

we have

\[ \|v^{n+1} - v^n\|_Y \leq C\|\eta^{n+1} - \eta^n\|_X. \tag{4.2} \]

Now, using (4.2) in (4.1) we arrive at

\[ \|\eta^{n+1} - \eta^n\|_X \leq K\left(\|\eta^n - \eta^{n-1}\|_X\varepsilon + \frac{\varepsilon}{2C}\|\eta^n - \eta^{n-1}\|_X\right) \]

\[ \leq \frac{3K\varepsilon}{2}\|\eta^n - \eta^{n-1}\|_X \leq C(\varepsilon)\|\eta^n - \eta^{n-1}\|_X. \tag{4.3} \]

Under an additional condition on \(\varepsilon\) (if required) we ensure that \(C(\varepsilon) < 1\), and follows from (4.3) that \((\eta^n)\) is Cauchy. Finally, from (4.2) we also have that \((v^n)\) is Cauchy. Let \(\eta\) and \(v\) be such that \(\eta^n \to \eta\) and \(v^n \to v\). We have that

\[ 0 \leq \|\eta - E_\alpha(-t^\alpha(\Delta)^{\theta/2})\eta_0 - B(\eta, v) - \eta^n + \eta^n\|_X = \|\eta - E_\alpha(-t^\alpha(\Delta)^{\theta/2})\eta_0 - B(\eta, v) - \eta^n + \eta^n\|_X \]

\[ \leq \|\eta - \eta^n\|_X + \|E_\alpha(-t^\alpha(\Delta)^{\theta/2})\eta_0 - B(\eta, v) + \eta^n\|_X \]

\[ \leq \|\eta - \eta^n\|_X + \|B(\eta^{n-1} - \eta, v^{n-1})\|_X \]

\[ \leq \|\eta - \eta^n\|_X + \|B(\eta^{n-1} - \eta, v^{n-1}) + B(\eta, v^{n-1} - v)\|_X \]

\[ \leq \|\eta - \eta^n\|_X + K\left(\|\eta^{n-1} - \eta\|_X\varepsilon + \frac{\varepsilon}{2C}\|v^{n-1} - v\|_Y\right) \]

\[ \to 0, \tag{4.4} \]
and
\[
0 \leq \|v - E_{\alpha}(-t^\alpha ((-\Delta)^{\theta/2} - \gamma))v_0 - T(\eta)\|_Y = \|v - E_{\alpha}(-t^\alpha ((-\Delta)^{\theta/2} - \gamma))v_0 - T(\eta) - v^n + v^n\|_Y \\
\leq \|v - v^n\|_Y + \| - E_{\alpha}(-t^\alpha ((-\Delta)^{\theta/2} - \gamma))v_0 - T(\eta) + v^n\|_Y \\
\leq \|v - v^n\|_Y + \|T(\eta^n - \eta)\|_Y \\
\leq \|v - v^n\|_Y + C\|\eta^n - \eta\|_X \\
\rightarrow 0.
\]

The estimates (4.4) and (4.5) prove that \([\eta, v]\) is a mild solution of (1.2). Finally, to prove the uniqueness suppose that \([\eta, v]\) and \([\hat{\eta}, \hat{v}]\) are two solutions in the same conditions of Theorem 1.3 with the same initial data, then, following the same ideas of the proof of (4.3) we have

\[
\|\eta - \hat{\eta}\|_X \leq C(\varepsilon)\|\eta - \hat{\eta}\|_X,
\]

with \(0 < C(\varepsilon) < 1\), which implies that \(\eta - \hat{\eta} = 0\), this is \(\eta = \hat{\eta}\). From this is obvious that \(v = \hat{v}\).

### 4.2 Proof of Corollary 1.4

First, note that
\[
F(U_\theta(\tau t^\alpha)\eta_0(\sigma \cdot))(\xi) = e^{-\tau t^\alpha|\xi|^{\theta}}F(\eta_0(\sigma \cdot)) = \sigma^{-n}e^{-\tau t^\alpha|\xi|^{\theta}}F(\eta_0)(\xi/\sigma) \\
= \sigma^{-n}e^{-\tau(\sigma^\frac{\theta}{t} \xi/\sigma)^{\alpha}}F(\eta_0)(\xi/\sigma) \\
= F\left(\left(U_\theta\left(\tau \left(\sigma^\frac{\theta}{t}\right)^{\alpha}\right)\eta_0\right)(\sigma \cdot)\right)(\xi).
\]
Thus,
\[
U_\theta(\tau t^\alpha)\eta_0(\sigma \cdot) = \left(U_\theta\left(\tau \left(\sigma^\frac{\theta}{t}\right)^{\alpha}\right)\eta_0\right)(\sigma \cdot),
\]
and
\[
E_{\alpha}(-t^\alpha ((-\Delta)^{\theta/2} - \gamma))\eta_0 = \int_0^\infty M_{\alpha}(\tau)U_\theta(\tau t^\alpha)\eta_0 d\tau = \int_0^\infty M_{\alpha}(\tau)U_\theta(\tau t^\alpha)\eta_0 d\tau \\
= \sigma^{2\theta + \theta_1 - 2}\int_0^\infty M_{\alpha}(\tau)d\tau \left(U_\theta\left(\tau \left(\sigma^\frac{\theta}{t}\right)^{\alpha}\right)\eta_0\right)(\sigma \cdot) d\tau,
\]
this is,
\[
\eta^1(x, t) = \sigma^{2\theta + \theta_1 - 2}\eta^1(x, \sigma^\frac{\theta}{t}).
\]

Using induction, is direct to verify that the members of the sequences \((\eta^n)\) and \((v^n)\) are invariant by the scaling (1.8), this is
\[
\eta^n(x, t) = \sigma^{2\theta + \theta_1 - 2}\eta^n\left(x, \sigma^\frac{\theta}{t}\right) \quad \text{and} \quad v^n(x, t) = \sigma^{\theta + \theta_1 - 2}v^n\left(x, \sigma^\frac{\theta}{t}\right).
\]
Finally, since the solution \([\eta, v]\) is the limit in \(X \times Y\) of the sequence \([\eta^n, v^n]\) and the spaces \(X, Y\) are invariant for the scaling, we can conclude that
\[
\eta(x, t) = \sigma^{2\theta + \theta_1 - 2}\eta\left(x, \sigma^\frac{\theta}{t}\right) \quad \text{and} \quad v(x, t) = \sigma^{\theta + \theta_1 - 2}v\left(x, \sigma^\frac{\theta}{t}\right),
\]
this is, \([\eta, v]\) is a self-similar solution.
4.3 Uniqueness. Proof of Theorem 1.6

With the bilinear estimate (1.12) in hands and the correct use of the product estimate (1.11), the uniqueness follows by adapting an argument due to Meyer [25]. For this proof denote \( s_1 = 2 - 2 \theta - \theta_1 + \frac{2}{p} \), \( s_0 = 3 - 3 \theta - \theta_1 + \frac{2}{p} \), and \( s_2 = 2 - \theta - \theta_1 + \frac{2}{q} \), and let \([\eta^1, v^1]\) and \([\eta^2, v^2]\) be two mild solutions in \( C \left( [0, T); \dot{B}^{s_1}_{p, \infty} \right) \times C \left( [0, T); \dot{B}^{s_2}_{q, \infty} \right) \) with the same initial data \([\eta_0, v_0]\) in \( \dot{B}^{s_1}_{p, \infty} \times \dot{B}^{s_2}_{q, \infty} \). First we prove that there exists \( 0 < T_1 < T \) such that \([\eta^1(t), v^1(t)] = [\eta^2(t), v^2(t)]\) in \( \dot{B}^{s_1}_{p, \infty} \times \dot{B}^{s_2}_{q, \infty} \) for all \( t \in [0, T_1) \). Denoting

\[
\begin{align*}
N &= \eta^1 - \eta^2, \quad V = v^1 - v^2, \\
N_1 &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 - \eta^1, \\
N_2 &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 - \eta^2, \\
V_1 &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 - v^1, \\
V_2 &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 - v^2,
\end{align*}
\]

we have that

\[
V = T_\theta(N). \tag{4.6}
\]

Then,

\[
\sup_{0 < t < T_1} \|V\|_{\dot{B}^{s_2}_{q, \infty}} \leq K \sup_{0 < t < T_1} \|N\|_{\dot{B}^{s_1}_{p, \infty}}. \tag{4.7}
\]

Moreover,

\[
\begin{align*}
\eta^1 G(v^1) - \eta^2 G(v^2) &= NG(v^1) + \eta^2 G(V) \\
&= NG \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 - V_1 \right) + \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 - N_2 \right) G(V) \\
&= NG \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 \right) + \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 \right) G(V) - NG(V_1) - N_2 G(V).
\end{align*}
\]

Thus,

\[
\begin{align*}
\|N\|_{\dot{B}^{s_1}_{p, \infty}} &= \|B(\eta^1, v^1) - B(\eta^2, v^2)\|_{\dot{B}^{s_1}_{p, \infty}} \\
&= \left\| \int_0^t (t - \tau)^{\alpha - 1} \nabla \cdot E_{\alpha, \alpha}(-t - \tau)^\alpha((-\Delta)^{\theta/2}) \left( (\eta^1 G(v^1)) - (\eta^2 G(v^2)) \right) (\tau) d\tau \right\|_{\dot{B}^{s_1}_{p, \infty}} \\
&\leq \left\| \int_0^t (t - \tau)^{\alpha - 1} \nabla \cdot E_{\alpha, \alpha}(-t - \tau)^\alpha((-\Delta)^{\theta/2}) \left( NG(V_1) + N_2 G(V) \right) (\tau) d\tau \right\|_{\dot{B}^{s_1}_{p, \infty}} \\
&\quad + \left\| \int_0^t (t - \tau)^{\alpha - 1} \nabla \cdot E_{\alpha, \alpha}(-t - \tau)^\alpha((-\Delta)^{\theta/2}) (NG \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 \right) + \left( E_\alpha(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 \right) G(V)) d\tau \right\|_{\dot{B}^{s_1}_{p, \infty}} \\
&:= J_1(t) + J_2(t).
\end{align*}
\]

For \( J_1(t) \) we have

\[
J_1(t) \leq K \left( \sup_{0 < t < T_1} \|N\|_{\dot{B}^{s_1}_{p, \infty}} \sup_{0 < t < T_1} \|V_1\|_{\dot{B}^{s_2}_{q, \infty}} \sup_{0 < t < T_1} \|N_2\|_{\dot{B}^{s_1}_{p, \infty}} \sup_{0 < t < T_1} \|V\|_{\dot{B}^{s_2}_{q, \infty}} \right).
\]

Using (4.7) we arrive at

\[
J_1(t) \leq C \sup_{0 < t < T_1} \|N\|_{\dot{B}^{s_1}_{p, \infty}} \left( \sup_{0 < t < T_1} \|V_1\|_{\dot{B}^{s_2}_{q, \infty}} \sup_{0 < t < T_1} \|N_2\|_{\dot{B}^{s_1}_{p, \infty}} \right). \tag{4.8}
\]
On the other hand, for \( J_2(t) \) it follows that

\[
J_2(t) = \left\| \int_0^t (t - \tau)^{\alpha - 1} \nabla \cdot E_{\alpha,\alpha}(-t - \tau)^{\alpha} \left((-\Delta)^{\theta/2}\right)(NG \left(E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2} - \gamma))v_0\right)
+ (E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2}))\eta_0) G(V) \right\|_{B_{p,\infty}^{s+1}},
\]

\[
\leq \int_0^t \left\| (t - \tau)^{\alpha - 1} \nabla \cdot E_{\alpha,\alpha}(-t - \tau)^{\alpha} \left((-\Delta)^{\theta/2}\right)(NG \left(E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2} - \gamma))v_0\right)
+ (E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2}))\eta_0) G(V) \right\|_{B_{p,\infty}^{s+1}} d\tau,
\]

\[
\leq C \int_0^t (t - \tau)^{\alpha - 1} \left(1 - (t - \tau)^{\alpha - 1}\left[\tau_{\tau}(\eta_0)^{\theta/2} + 1\right] NG \left(E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2} - \gamma))v_0\right)
+ (E_{\alpha}(-\tau^\alpha((-\Delta)^{\theta/2}))\eta_0) G(V) \right\|_{B_{p,\infty}^{s+1}} d\tau.
\]

where we have used Lemma 1.1 adequately, the relation (4.7), and the fact that for \( \rho > 0 \) small enough the following integral holds

\[
\int_0^t (t - \tau)^{-1 + \frac{\theta}{2}\rho} \tau^{-\frac{\theta}{2}\rho} d\tau = \int_0^1 (1 - r)^{-1 + (1 - \alpha)} + \frac{\rho}{\theta} r^{-\frac{\theta}{2}\rho} dr = C.
\]

Thus, from the estimates (4.8) and (4.9) we get

\[
\sup_{0 < t < T_1} \| V(t) \|_{B_{p,\infty}^{\theta + 1}} \leq C Z(T_1) \sup_{0 < t < T_1} \| N(t) \|_{B_{p,\infty}^{s+1}},
\]

with

\[
Z(T_1) = \sup_{0 < t < T_1} \| V(t) \|_{B_{p,\infty}^{\theta + 1}} + \sup_{0 < t < T_1} \| N(t) \|_{B_{p,\infty}^{s+1}}
+ \left( \sup_{0 < t < T_1} t^{\frac{\theta}{2}\rho} \| E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 \|_{B_{p,\infty}^{\theta + 1}} + \sup_{0 < t < T_1} t^{\frac{\theta}{2}\rho} \| E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 \|_{B_{p,\infty}^{\theta + 1}} \right).
\]

From hypotheses, it holds \( E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2}))\eta_0, \eta_1, \eta_2 \to \eta_0 \) and \( E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0, v^1, v^2 \to v_0 \) as \( t \to 0^+ \), which implies that

\[
\lim_{t \to 0^+} \sup_{t < T_1} t^{\frac{\theta}{2}\rho} \| E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 \|_{B_{p,\infty}^{\theta + 1}} = 0.
\]

Now we prove that

\[
\limsup_{t \to 0^+} t^{\frac{\theta}{2}\rho} \| E_{\alpha}(-t^\alpha((-\Delta)^{\theta/2}))\eta_0 \|_{B_{p,\infty}^{\theta + 1}} = 0.
\]

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In fact, let \( \eta_{0k} = E_\alpha(- \left( \frac{t}{k} \right)^\alpha (-\Delta)^{\theta/2})\eta_0 \) for all \( k \in \mathbb{N} \). It follows from Lemma 3.1 that \( \eta_{0k} \in \dot{B}^s_{p,\infty} \), moreover, from the hypothesis on \( \eta_0 \), we have that \( \eta_{0k} \to \eta_0 \) in \( \dot{B}^{s_i}_{p,\infty} \) as \( k \to \infty \). Then,

\[
\limsup_{t \to 0^+} t^{\frac{\beta}{p}} \left\| E_\alpha(-t^\alpha (-\Delta)^{\theta/2})\eta_0 \right\|_{\dot{B}^{s_i}_{p,\infty}} \\
\leq \limsup_{t \to 0^+} t^{\frac{\beta}{p}} \left\| E_\alpha(-t^\alpha (-\Delta)^{\theta/2})\eta_0 - \eta_{0k} \right\|_{\dot{B}^{s_i}_{p,\infty}} + \limsup_{t \to 0^+} t^{\frac{\beta}{p}} \left\| E_\alpha(-t^\alpha (-\Delta)^{\theta/2})\eta_{0k} \right\|_{\dot{B}^{s_i}_{p,\infty}} \\
\leq C \| \eta_0 - \eta_{0k} \|_{\dot{B}^{s_i}_{p,\infty}} + C \| \eta_{0k} \|_{\dot{B}^{s_i}_{p,\infty}} \limsup_{t \to 0^+} t^{\frac{\beta}{p}} \\
\leq C \| \eta_0 - \eta_{0k} \|_{\dot{B}^{s_i}_{p,\infty}} \to 0, \text{ as } k \to \infty.
\]

A similar argument is used to show that

\[
\limsup_{t \to 0^+} t^{\frac{\beta}{p}} \left\| E_\alpha(-t^\alpha (-\Delta)^{\theta/2} - \gamma)\eta_0 \right\|_{\dot{B}^{s_i}_{p,\infty}} = 0. \tag{4.13}
\]

Now, using estimates (4.11), (4.12) and (4.13), we can choose \( T_1 > 0 \) such that \( CZ(T_1) < 1 \) and then \( N(t) = 0 \) for all \( t \in [0, T_1] \). From (4.9) we also have that \( V(t) = 0 \) for all \( t \in [0, T_1] \). In order to finish the proof, we will show that \( T_1 \in (0, T] \) can be arbitrary. For that define

\[
T_* = \sup \left\{ \tilde{T} ; 0 < \tilde{T} < T, \eta^1(t) = \eta^2(t) \text{ in } \dot{B}^{s_i}_{p,\infty} \text{ for all } t \in [0, \tilde{T}] \right\}.
\]

If \( T_* = T \) we finish. If not, we have that \( \eta^1(t) = \eta^2(t) \) for \( t \in [0, T_*] \) which implies that \( \eta^1(T_*) = \eta^2(T_*) \) because of time continuity of \( \eta^1, \eta^2 \). In this case we also have that \( v^1(t) = v^2(t) \) for \( t \in [0, T_*] \) which implies that \( v^1(T_*) = v^2(T_*) \) because of time continuity of \( v^1, v^2 \). It follows from the first part of the proof, starting at \( T_* \), that there exists \( \sigma > 0 \) small enough such that \( \eta^1(t) = \eta^2(t) \) for \( t \in [T_*, T_* + \sigma] \), therefore \( \eta^1(t) = \eta^2(t) \) for \( t \in [0, T_* + \sigma] \), which contradicts the definition of \( T_* \).

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