Optimal Quantum Filtering and Quantum Feedback Control

Simon C. Edwards, Viacheslav P. Belavkin
School of Mathematical Sciences
University of Nottingham
Nottingham. NG7 2RD. UK
pmxsce@nottingham.ac.uk
vpb@maths.nottingham.ac.uk

Abstract

Quantum mechanical systems exhibit an inherently probabilistic nature upon measurement. Using a quantum noise model to describe the stochastic evolution of the open quantum system and working in parallel with classical indeterministic control theory, we present the theory of nonlinear optimal quantum feedback control. The resulting quantum Bellman equation is then applied to the explicitly solvable quantum linear-quadratic-Gaussian (LQG) problem which emphasizes many similarities with the corresponding classical control problem.

1 Introduction

With technological advances now allowing the possibility of continuous monitoring and rapid manipulations of systems at the quantum level [1, 2], there is an increasing awareness of the applications and importance of quantum feedback control. Such applications include the engineering of quantum states, stability theory, quantum error correction and substantial applications in quantum computation [3, 4, 5, 6, 7, 8]. This current interest marks quantum control theory as a highly rewarding branch of control theory for study and as such there is a growing number of recent publications on the subject [9, 10, 11, 12, 13, 14]. In particular, [14] contains a useful introduction to quantum probability and along with [15] gives a comprehensive discussion on the comparison of classical and quantum control techniques and we refer the unfamiliar reader to these articles and references within.

The main ingredients of quantum control are essentially the same as in the classical case. One controls the system by coupling to an external control field which modifies the system in a desirable manner. The desired objectives of the control can be encoded into a cost function along with any other stipulations or restrictions on the controls such that the minimization of this cost indicates optimality of the control process. There are two types of dynamical control - open loop (or blind) control where the controls are predetermined at the start of the experiment and closed loop (or feedback) control where controls can be chosen throughout the experiment and thus is preferable for stochastic dynamics. Previous work on the theory of optimal quantum open loop control includes variational techniques on closed qubit systems [10, 17], which was also extended to open (dissipative) quantum systems [18]. However, this approach can only seek locally optimal solutions which can often be improved further with measurement and feedback, since an open quantum system inevitably loses information to its surrounding environment.

Quantum feedback control was formally initiated by Belavkin in a series of papers [15, 20, 21]
in the 80s. This work was developed as a quantum analogy to the classical theories of nonlinear (Stratonovich) filtering and Bellman dynamic programming. In fact, the separation lemma of classical control theory was shown also to hold in the quantum domain. That is, the problem of optimal quantum feedback control is separated into quantum filtering which provides optimal estimates of the stochastic quantum variables (operators) and then an optimal control problem based on the output of the quantum filter. The quantum noise which we filter out comes from the disturbance to the system due to the quantum measurement. Unlike classical systems, this is an unavoidable feature of quantum measurement since the quantum system is not directly observable. The quantum filter describes a classical stochastic process, albeit on the space of quantum states, so Belavkin showed how one can progress using tools from classical feedback control theory when applied to sufficient coordinates of the system \[20\]. However, the lack of urgency for such a theory and the complexity of the mathematical language at the time left this work relatively undiscovered only to be rediscovered recently in the physics and engineering community.

The purpose of this paper is to build on the original work of Belavkin and present an accessible account of the theory of nonlinear optimal quantum feedback control. Firstly we introduce the necessary concepts from modern quantum theory including quantum probability, non-demolition measurement, quantum stochastic calculus and quantum filtering. Next the quantum Bellman equation for optimal feedback control with diffusive non demolition measurement is derived. Often in optimal control problems of this nature, the separation lemma is assumed and the control objectives are defined in terms of posterior sufficient coordinates \[15\]. In this paper, we show how the general Bellman equation is applied with the same effect by application to the many dimensional quantum LQG problem. Next a physical example of LQG control is given and we conclude with a discussion on the results with comparison to the corresponding classical control problem.

2 Optimal quantum measurement and filtering

This section highlights the differences between quantum and classical systems and introduces the problem of quantum measurement. After the appropriate setting is given, the measurement problem is then restated as a problem of optimal estimation of the output of a noisy quantum channel. Finally, the quantum filtering equation describing the dynamical least squares estimator is given.

2.1 Quantum Probability

Quantum physics which deals with the unavoidable random nature of the microworld requires a new, more general, noncommutative probability theory than the classical one based on Kolmogorov’s axioms. It was developed through the 70s and 80s by Accardi, Belavkin, Gardiner, Holevo, Hudson and Parthasarthy \[22\] amongst others.

The essential difference between classical and quantum probability is that classically, Kolmogorov’s probability axioms allow the occurrence of simultaneous events only. This is because the classical events are described by indicator functions \(1_{\Delta}(\omega)\) of the measurable subsets \(\Delta \subseteq \Omega\) on the space of point states \(\Omega\). They are the building blocks for the classical random variables described by measurable functions \(x : \Omega \to \mathbb{R}\) as linear combinations (integrals) of the indicator functions \(1_{\Delta}\). Such classical essentially bounded variables represented by operators of multiplication by the corresponding functions, form an abelian (commutative) von Neumann algebra on the Hilbert space \(L^2(\Omega, \mathbb{P})\) of square-integrable random functions with respect to a probability measure \(\mathbb{P}\).
In quantum probability, there are some events which cannot occur simultaneously, so we must generalize the framework of classical probability to incorporate these features. This is done by considering quantum events as self adjoint orthoprojectors $P^2 = P = P^*$ (where $*$ denotes the Hermitian adjoint) acting in some Hilbert space $\mathcal{H}$ not only by multiplications on the indicator functions $1_{\Delta}(\omega)$. Quantum random variables are also built from events as linear (integral) combinations of their projectors $P$. The events are incompatible if the corresponding projectors do not commute, i.e. $[P_i, P_j] := P_i P_j - P_j P_i \neq 0$ and therefore cannot be represented classically by the indicator functions which always commute.

One can form the non commutative von Neumann algebra $\mathcal{A}$ of bounded quantum random variables generated by the self adjoint projectors $\{P_1, \ldots, P_m\}$. This algebra is equal to its double commutant $\mathcal{A} := \{P_1, \ldots, P_m\}''$ where the commutant of a set $S \subset \mathcal{B}(\mathcal{H})$ in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is defined by $S' := \{X' \in \mathcal{B}(\mathcal{H}) \text{ s.t. } [X, X'] = 0 \ \forall X \in S\}$. The quantum state on $\mathcal{A}$, given by a positive operator $\rho = \rho^* \geq 0$ with unit trace $\text{Tr}[\rho] = 1$, defines all expectations

$$\langle X \rangle = \text{Tr}[\rho X] = \langle \rho, X \rangle$$

for operators $X \in \mathcal{A}$. So we describe a quantum probability space by the pair $(\mathcal{A}, \rho)$. In the case where $\mathcal{A}$ is an abelian von Neumann operator algebra, there is a natural isomorphism $(\mathcal{A}, \rho) \simeq L^\infty(\Omega, \mathcal{F})$ with bounded functions on the classical probability space $(\Omega, \mathcal{F})$ and so we recover the classical statistics.

The incompatibility of quantum events means that after one has observed an event, the state of the system needs to be updated to account for the change to the system or back-action affecting the expectations of all other incompatible events. This state change was traditionally described by the normalized projection postulate

$$\rho \rightarrow \rho_i = \frac{P_i \rho P_i}{\text{Tr}[\rho P_i]}$$

which also ensures instantaneous repeatability of the observed event corresponding to the projection $P_i$. However, it has long been known that this phenomenological description is inadequate, since it fails to describe continuous measurements and experimentally it is not possible to perform a direct measurement of eigenstates of such a quantum operator. Instead we must consider an indirect measurement of operators in a coupled semi-classical field and describe the state change $\rho \rightarrow \rho_i$ by an optimal estimator based on the results of measurements in this field. Let $\mathcal{F}$ denote the Hilbert space of the field, which we view as a noisy measurement channel in the initial vacuum state $\phi$. We only observe compatible events in the channel (corresponding to output meter readings for example). So we describe these events by commuting projectors $\{P_\omega \}_{\omega \in \Omega}$ which can be represented by classical indicator functions and generate the abelian subalgebra $\mathcal{B} \subset \mathcal{B}(\mathcal{F})$ where $\Omega$ is now the space of measurement results (eigenvalues) for these commuting operators. So the field operators $W \in \mathcal{B}$ which are linear combinations of the commuting projectors are in one-to-one correspondence with classical random variables as functions $w : \Omega \rightarrow \mathbb{R}$ on the data space $\Omega$. In the quantum noise model, we consider input quantum noises as quantum random variables represented by operators in the full field algebra $\mathcal{B}(\mathcal{F})$ of bounded operators on $\mathcal{F}$ which perturb the quantum system in such a way to allow a classical correlated output. This interaction between the open quantum system and the semi-classical field is described on the composite system by a unitary operator $U$, which for an initial state $\phi$ of the field gives the state evolution

$$\rho \rightarrow U(\rho \otimes \phi)U^*$$

called the prior state. The reduced conditional evolution can then be described by the nonlinear map

$$\rho \rightarrow \rho_\omega = \frac{\text{Tr}_\mathcal{F}[U(\rho \otimes \phi)U^*(I \otimes P_\omega)]}{\text{Tr}[U(\rho \otimes \phi)U^*(I \otimes P_\omega)]}$$

(3)
called the posterior state which is the Bayes law of conditioning for the measurement result $\omega \in \Omega$, normalized with respect to the output
probabilities $P(\omega) = \text{Tr}[U(\rho \otimes \phi)U^*(I \otimes P_\omega)]$ and $\text{Tr}_\mathcal{F}$ denotes the partial trace over $\mathcal{F}$.

We denote the posterior state as a classical random variable $\rho_* : \Omega \to \mathcal{A}_*,$ taking values $\rho_\omega$ in the space $\mathcal{A}_*$ of states on $\mathcal{A}.$ The posterior state gives the conditional expectation
\[
\mathbb{E}[X'|Y] = \langle \rho_*, X \rangle
\]
which is the least squares estimator of the system operator $X' = U^*(X \otimes I)U$ after interaction, with respect to the output operators $Y := U^*(I \otimes W)U.$ We now describe the appropriate model for the dynamical coupling between the open quantum system and the field.

2.2 The Quantum Vacuum Noise Model and Markov Approximation

The indirect measurement of the quantum system is via a coupled measurement channel, playing the role of a quantum noise bath. It is modelled by the symmetric Fock space $\mathcal{F}$ over the single particle space $L^2(\mathbb{R}_+ \rightarrow \mathcal{G})$ of square integrable functions from $[0, \infty)$ into a Hilbert space $\mathcal{G}$ of the bath degrees of freedom. Having in mind the vacuum noise model of the bath, let $\mathcal{W} := \mathcal{B}(\mathcal{F})$ denote the quantum noise algebra of bounded operators on $\mathcal{F}$ initially in the vacuum state $\phi.$ From the divisibility property of the symmetric Fock space, we can factorize the noise algebra
\[
\mathcal{W} = \mathcal{W}_0^t \otimes \mathcal{W}_t^\infty, \quad \mathcal{F} = \mathcal{F}_{(0,t)} \otimes \mathcal{F}_{(t,\infty)}
\]
for arbitrary $t > 0$ where $\mathcal{W}_0^t = \mathcal{B}(\mathcal{F}_{(a,b)})$ and $\mathcal{F}_{(a,b)}$ is the symmetric Fock space over $L^2([a, b) \rightarrow \mathcal{G})$ for $0 \leq a < b.$ This tensor independence implies compatibility for operators belonging to the disjoint time intervals of the noise algebra. The time evolution of the quantum system and the quantum noise bath (which together form a closed composite quantum system) can be described in the interaction representation by a family $\{U_t\}_{t \in \mathbb{R}_+}$ of unitary operators $U_t : \mathcal{H} \otimes \mathcal{F}_{(0,t)} \to \mathcal{H} \otimes \mathcal{F}_{(t,\infty)}.$ In the weak coupling limit $[22, 26]$ (short bath memory), they describe the Markovian flow $J_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_0^t$ by $J_t(X) := U_t^*(X \otimes I)U_t$ for operators $X \in \mathcal{A}$ (we use the symbol $\mathcal{A} \ni X$ to denote that $X$ is an element of $\mathcal{A},$ or that its spectral projectors belong in $\mathcal{A}$ for the case of unbounded $X$). We complete the description of the joint system and field evolution by introducing the unitary shift operator $S_t : \mathcal{F}_{(0,s)} \to \mathcal{F}_{(t,s+t)}$ which models the free evolution in the field. Thus the combined evolution and interaction on the composite system is given by a family of endomorphisms $\{\gamma_t\}$ on $\mathcal{A} \otimes \mathcal{W}$ such that $\gamma_t(X \otimes W) = U_t^*(X \otimes W)U_t$ for unitaries $U_t := (I \otimes S_t)U_t.$ This gives the cocyle identity $U_{t+s} = S_s U_t S_s$ for the interaction unitaries $\{U_t\}.$ Note that for ease of presentation, we avoid the repetition of tensoring with the identity on $\mathcal{H}$ and $\mathcal{F}_{(t,\infty)}$ and assume the domain of the operators is clear from the context.

We now briefly discuss quantum stochastic calculus, a necessary tool when developing a time-continuous theory of quantum stochastic evolution.

2.3 Quantum Stochastic Calculus

In this paper we consider feedback control based on a homodyne detection scheme. This is the quantum analogue of measurement of the Wiener process in the field and is described by the field quadrature $W_t = A_t + A_t^\dagger$ where $A_t \ni W_0^t$ is called the annihilation operator on $\mathcal{F}.$ The properties of $A_t$ are such that $W_\theta^t := \exp(i\theta)A_t + \exp(-i\theta)A_t^\dagger$ is equivalent to the classical Wiener process for each $\theta \in [0, 2\pi),$ however they do not commute for different $\theta,$ so by considering solely the measurement of $W_t,$ we restrict ourselves to a chosen classical diffusive measurement process corresponding to $\theta = 0.$

Hudson and Parthasarathy $[27, 28]$ developed the theory of quantum stochastic calculus using the annihilation process and its adjoint, the
creation process $A_t^*$ as the fundamental diffusive adapted processes and defined the interaction unitaries $\{U_t\}$ as the unique solutions to the quantum stochastic differential equation which we chose of the simple form

$$dU_t + KU_t \otimes dt = LU_t \otimes dA_t^* - L^*U_t \otimes dA_t, \quad (4)$$

with $U_0 = I$. Here $K = \frac{1}{\hbar}H + \frac{1}{2}L^*L$, $H$ is the Hamiltonian of the quantum system and $L$ is the operator describing the coupling of the system to the measurement channel. The increments $dt$, $dA_t$, $dA_t^*$ are considered as operators acting in $\mathcal{F}_{[t,t+dt)}$ and define stochastic Itô calculus using the product rule

$$d(M_tN_t) = d(M_t)N_t + M_t d(N_t) + d(M_t)d(N_t)$$

for adapted quantum stochastic processes $M_t$, $N_t$ where the quantum Itô correction term (last term) is calculated using the multiplication table

$$\begin{align*}
(dt)^2 &= 0, \quad dt dA_t = 0 = dt dA_t^*, \\
 dA_t^* dA_t &= 0, \quad dA_t dA_t^* = dt.
\end{align*} \quad (5)$$

### 2.4 Quantum Langevin Equations and Non-demolition Measurements

From the quantum Itô formula applied to $X_t = U_t^* (X \otimes I)U_t$ and the quantum Itô multiplication table [8], we obtain the quantum Langevin equation

$$dX_t = \mathcal{L}_t[X_t] \otimes dt + [X_t, L_t] \otimes dA_t^* - [X_t, L_t^*] \otimes dA_t. \quad (6)$$

Here $\mathcal{L}_t[X_t] = j_t(\mathcal{L}[X])$ is the time evolved Lindblad (or Gorini-Kossakowski-Sudarshan) generator [25, 29].

$$\mathcal{L}[X] = \frac{i}{\hbar} [H, X] + \frac{1}{2} (L^* [X, L] + [L^*, X] L) \quad (7)$$

for the semigroup of completely positive maps describing the dissipative evolution in the Markovian limit. The dual $\mathcal{L}^*$ of this map describes the unconditional dissipative evolution of states

$$\frac{d}{dt}\rho^t = -\frac{1}{\hbar} [H, \rho^t] + \frac{1}{2} (L[\rho^t, L^*] + [L, \rho^t] L^*) \quad (8)$$

called the master equation which is the quantum analogue of the Fokker-Plank equation. A time continuous measurement of the field quadrature $W_t$ in the output channel represents an indirect measurement of the evolved generalized coordinate $L_t + L_t^* \equiv A_t$ as can be seen from the quantum Itô formula applied to the output operators $Y_t = U_t^* (I \otimes W_t) U_t$:

$$dY_t = (L_t + L_t^*) \otimes dt + I \otimes dW_t. \quad (9)$$

Note that the output process $Y_t$ is directly observable as it is a commutative family of self-adjoint operators $\{Y_s\}_{s \leq t}$ unitary equivalent to the family $\{W_s\}_{s \leq t}$ for each $t$. This simply follows from the following lemma which was first observed by Belavkin in [19, 30].

**Lemma 1.** The input and output operators satisfy the quantum non-demolition (QND) condition

$$[X_t, Y_s] = 0 \quad [Y_t, Y_s] = 0 \quad \forall 0 \leq s \leq t \quad (10)$$

**Proof.** Let $t = s + r, r > 0$, then from the cocycle identity we get

$$U_{s+r}^* (I \otimes W_s) U_{s+r} = U_s^* (S_s U_r S_s^*) (I \otimes W_s) (S_s U_r S_s^*) U_s = U_s^* (I \otimes W_s) U_s = Y_s$$

where the last step uses the commutativity of $S_s U_r S_s^* \equiv A \otimes W_s^{s+r}$ and $W_s \equiv W_0^s$. So $[X_t, Y_s] = U_t^*[X, W_s] U_t = 0$ and $[Y_t, Y_s] = U_t^*[W_s, W_s] U_t = 0$ follows from the tensor independence of $X$, $W_s$ and $W_t$ for all $s \neq t$. \hfill \Box

### 2.5 Quantum Filtering

Classically, filtering equations are used when we need to estimate the value of dynamical variables about which we have incomplete knowledge due to an indirect observation. For example, the Kalman-Bucy filter [31, 32] gives a continuous least-squares estimator for a Gaussian classical random variable with linear dynamics when we only have access to a correlated, noisy output signal. Since closed quantum
systems are fundamentally unobservable unless they are open, e.g. disturbed by quantum noise processes (c.f. [24, 30]), filtering of quantum noise plays an important role in quantum measurement. Belavkin was the first to realize that an optimal estimation without further disturbance is possible in the Markovian limit and is based on an output nondemolition measurement [19, 31, 21]. He constructed the quantum filtering equation which describes the evolution of the optimal estimate given by the density matrix conditioned on a classical output of the noisy quantum channel. This is used to estimate arbitrary input operators $X_t \in \mathcal{A}_t$ which are driven by environmental quantum noises. The previous lemma shows that the expectation of $X_t$ is not disturbed when we measure $Y_s$ for $0 \leq s \leq t$. This is necessary for the existence of a well-defined conditional expectation of $X_t$ with respect to past measurement results of $Y_s$.

Let $\mathcal{C}_s := \{Y_s\}$ be the abelian von Neumann algebra generated by the output operators $Y^t_s := \{Y_s|s \leq r \leq t\}$ (or their spectral projectors in the case of unbounded $Y_t$). Also let $\mathcal{A}_s := \{X_r|s \leq r \leq t\}$ denote the von Neumann algebra generated by the system operators $X_r \in \mathcal{A}_r$. From the QND condition, $\mathcal{C}_0$ lies in the center of (i.e. it is a subalgebra commuting with the whole of) $\mathcal{B}_T \subset \mathcal{A} \otimes \mathcal{W}_T$, where $\mathcal{B}_T := \mathcal{A}_T \otimes \mathcal{C}_0$ is the smallest von Neumann algebra containing $\mathcal{A}_T$ and $\mathcal{C}_0$ as subalgebras for $0 \leq t \leq T$. This gives the necessary conditions for the existence of a conditional expectation [33], defined as a linear, normcontractive projection $E_0^T : \mathcal{B}_T \rightarrow \mathcal{C}_0$.

The conditional expectation $E[X_t|Y^t_0] := E_0^T[X_t]$ gives the least squares estimator $X_t$ of an operator $X_t \in \mathcal{A}_t$ conditional on the output operators $Y^t_0$ and so is equivalent to a classical random variable on the space of measurement trajectories $\Omega^t_0 := \{\omega_s|0 \leq s \leq t\}$ s.t. $\omega_s$ is an eigenvalue of $Y_s$. This conditional expectation is most conveniently written in the Schrödinger picture $E_0^T[X_t] = \langle \rho^t_s, X_t \rangle$ for the solution $\rho^t_s$ to the classical stochastic nonlinear differential equation

$$d\rho^t_s = L^* [\rho^t_s] dt + \sigma(\rho^t_s)(dY_t - \langle \rho^t_s, L + L^* \rangle dt)$$

(11)

often called the Belavkin quantum filtering equation, where

$$\sigma(\rho^t_s) = \rho^t_s L^* + L \rho^t_s - \langle \rho^t_s, L^* + L \rangle \rho^t_s$$

is the nonlinear fluctuation coefficient.

We can generalize the filtering equation to the case where we couple the open quantum system to $d$ independent measurement channels. If we assume no scattering between the channels, then the family of unitary operators \{${U_t\}_t \in \mathbb{R}_+$\} describing the evolution in the interaction picture $U_t : \mathcal{H} \otimes \mathcal{F}^d_{[0,t]} \rightarrow \mathcal{H} \otimes \mathcal{F}^d_{[0,t]}$ satisfy

$$dU_t + KU_t \otimes dt = \sum_{i=1}^d [L_i U_t \otimes dA^i_{t,t} - L^*_i U_t \otimes dA^i_{t,t}]$$

where $L_i$ describes the coupling to the $i$th channel and $K = \frac{1}{2} H + \frac{1}{2} \sum_{i=1}^d L^*_i L_i$. Throughout this paper we reserve the Roman character $i$ to denote the imaginary unit $i := \sqrt{-1}$, whereas italic $i$ is freely used as an index. Note that we have tensor independence of the annihilation increments $dA^i_{t,t}, dA^j_{t,t}$ for $i \neq j$, so the quantum vacuum noises commute for different channels. The Belavkin filtering equation for a simultaneous diffusive measurement of $Y_{i,t} = U^*_{t}(I \otimes W_{i,t})U_t$ gives

$$d\rho^t_s = L^* [\rho^t_s] dt + \sum_{i=1}^d \sigma_i(\rho^t_s)(dY_{i,t} - \langle \rho^t_s, L + L^* \rangle dt)$$

(12)

for $W_{i,t} = (A^i_{t,t} + A^*_i\_t)$.

3 Optimal Quantum Control

We now couple the system to a control field. If we assume no scattering between the measurement and control fields and assume a weak coupling such that information is not lost into the
control field, then this effectively replaces the Hamiltonian $H$ of the system with a controlled Hamiltonian $H(u_s)$ for admissible real valued control functions $u_s \in \mathbb{R}$ say, at time $s$. This Hamiltonian generates the controlled unitaries $U_t(u^*_0)$ giving the controlled flow

$$j_t(u^*_0)|X⟩ := U^*_t(u^*_0)(X \otimes I)U_t(u^*_0)$$

where $u^*_0 := \{u_s|0 \leq s < t\}$ is the control process over the interval $[0,t)$. The controlled posterior density operator $ρ^*_0(u^*_0)$ can then be obtained from (12) with the controlled Hamiltonian $H(u_t)$ which appears in the controlled Lindblad term $L(u_t)$.

In classical control, we can allow complete observability of the controllable system, so that feedback controls are determined by the system variables $x_t \rightarrow u_t(x_t)$. However, in quantum systems, we do not have the point states $x_t$ due to joint non observability of the system operators $X_t$, so the stochastic feedback controls should be given by a function of the stochastic output process $Y^*_0$ which is associated with a classical random variable $u^*_t$ on $Ω_0^t$. I.e. the measurement trajectory is fed into the control $ω^*_0 \rightarrow u_t(ω^*_0)$. Thus the feedback controlled flow is a map $j_t(u^*_0(Y^*_0))$ from $\mathcal{A}$ to $\mathcal{A}_t \cup C^t_0$.

The optimality of control is judged by the expected cost associated to the admissible control process $u^*_0$ for the finite duration $T$ of the experiment. Admissible control strategies are defined as those $u^*_0$ for which the operator valued cost integral

$$J(u^*_0) = \int_0^T j_s(u^*_0)[C(u_s)]ds + j_T(u^*_0)[S] \quad (13)$$

exists in the strong operator topology for self adjoint positive operators $C(u_s)$, $S \vdash \mathcal{A}$ giving the expected cost by the expectation

$$⟨\rho \otimes φ, J(u^*_0)⟩.$$

An optimal feedback control strategy $u^*_0$ for nondemolition measurements of the output operators $Y^*_0$ is one which minimizes the expected posterior cost-to-go

$$⟨\rho \otimes φ, J(u^*_0)⟩ = \min_{u^*_0(\cdot) ∈ U^*_0(\cdot)} ⟨\rho \otimes φ, J(u^*_0)⟩$$

where $U^*_0(\cdot)$ is the space of admissible stochastic control strategies $u^*_0(\cdot)$. This dynamical optimization problem is considerably simplified by the following Lemma first observed by Bellman.

**Lemma 2 (Principle of Optimality).** If $u^*_0(\cdot)$ is an optimal strategy for the cost function (13) given the initial state $ρ \otimes φ$, then its restriction $u^*_0(\cdot)$ to the interval $[t,T]$ is optimal for the cost-to-go

$$J_t(u^*_0(\cdot)) = \int_t^T j_s(u^*_0(\cdot))[C(u_s(\cdot))]ds + j_T(u^*_0(\cdot))[S] \quad (15)$$

given the state $ρ^*_0(u^*_0(\cdot))$ at time $t$.

We can now reduce the dynamics to the observable output algebra and rewrite the expectation as a conditional one

$$⟨\rho \otimes φ, J_t(u^*_0(Y^*_0)))⟩ = ⟨φ^*_0, E^*_0[J_t(u^*_0(Y^*_0))]⟩$$

where $E^*_0 : \mathcal{B}^*_t → C^t_0$ is the conditional expectation on $\mathcal{B}^*_t = \mathcal{A}^*_t \cup C^t_0$ which defines the feedback controlled posterior density operator by $⟨ρ^*_0(u^*_0(\cdot)), X⟩ = E^*_0 \circ j_t(u^*_0(\cdot))[X]$ at time $t$.

**Theorem 1.** The posterior cost-to-go from state $ρ$ at time $t$ satisfies

$$E^*_0[J_t(u^*_0(\cdot))] = E^*_0[J(t, u^*_t(\cdot), ρ)] \quad (16)$$

where

$$J(t, u^*_t(\cdot), ρ) = \int_t^T j_s(ρ^*_s(u^*_s(\cdot)), C(u_s(\cdot)))ds + ⟨ρ^*_s(u^*_s(\cdot)), S⟩$$

is a random variable on $Ω^t_0$ and $ρ^*_s(u^*_s(\cdot))$ is the solution to the controlled filtering equation for $s ≥ t$ with $ρ = ρ^*_0(u^*_0(\cdot))$.

**Proof.** The ‘quantum’ conditional expectation $E^*_0$ acting on future operators gives

$$E^*_0 \circ j_s(u^*_s(\cdot))[X] = E^*_0[ρ^*_s(u^*_s(\cdot)), X]$$

for $X \vdash \mathcal{A}$, where $E^*_0 : C^t_0 → C^t_0$ is the ‘classical’ conditional expectation on $C^t_0$ satisfying the tower property $E^*_0 ∘ E^*_0 = E^*_0$ for $t ≤ s ≤ T$. □
Let us denote the minimum posterior cost-to-go
\[ S(t, \rho) := \min_{u_t^T(\cdot)} \mathbb{E}_0^t[J(t, u_t^T(\cdot), \rho)]. \tag{17} \]

**Theorem 2.** The minimum posterior cost-to-go satisfies the Bellman equation
\[ \frac{\partial}{\partial t} S(t, \rho) + \frac{1}{2} \sum_{i=1}^d (\sigma_i(\rho) \otimes \sigma_i(\rho), (\delta \otimes \delta) S(t, \rho)) \]
\[ + \min_{u_t^T(\cdot)} \{ (\rho, C(u_t(\cdot))) + \mathcal{L}(u_t(\cdot))[\delta S(t, \rho)] \} = 0 \tag{18} \]
where \( \delta S(t, \rho) = A \) denotes the derivation of \( S(t, \rho) \) with respect to \( \rho \) and \( \sigma_i(\rho) \) is the non-linear fluctuation coefficient in the filtering equation \( \mathcal{L} \).

**Proof.** From the definition of \( S(t, \rho) \) and \( J(t, u_t^T(\cdot), \rho) \), we have
\[ S(t, \rho^t) = \min_{u_t^T(\cdot)} \mathbb{E}_0^t \left\{ \int_t^{t+\epsilon} (\rho^s(\cdot), C(u_s(\cdot))) ds + J(t + \epsilon, u_{t+\epsilon}^T(\cdot), \rho^{t+\epsilon}) \right\} \]
So when \( \epsilon \to 0 \) becomes sufficiently small, we approximate this by
\[ S(t, \rho^t) = \min_{u_t^T(\cdot)} \mathbb{E}_0^t \left\{ \langle \rho^t, C(u_t(\cdot)) \rangle dt + S(t + dt, \rho^{t+dt}) \right\} \tag{19} \]
where we use the tower property of the classical conditional expectation. Assuming that \( S(t, \rho^t) \) is sufficiently differentiable, we use the Taylor expansion
\[ S(t + dt, \rho^{t+dt}) = S(t, \rho^t) + (\frac{\partial}{\partial \rho} S(t, \rho^t)) dt + (d\rho^t, \delta S(t, \rho^t)) + \frac{1}{2} \sum_{i=1}^d (\sigma_i(\rho^t) \otimes \sigma_i(\rho^t), (\delta \otimes \delta) S(t, \rho^t)) dt \]
where \( \delta S(t, \rho) = \frac{\partial}{\partial \rho} S(t, \rho) \) denotes the derivation of \( S(t, \rho) \) with respect to \( \rho \). Using this expansion in \( \mathcal{L} \) gives the Bellman equation \( \mathcal{L} \) when we observe that \( S(t, \rho) + \frac{\partial}{\partial \rho} S(t, \rho) \) does not depend on \( u_t \) and \( \mathbb{E}_0^t[d\tilde{Y}_{i,t}] = 0 \) for the innovation process \( d\tilde{Y}_{i,t} = dY_{i,t} - (\rho^t, L_i + L_i^\dagger) dt \).

### 4 Application of Results to a Linear Quantum Dynamical System

We illustrate the ideas of quantum filtering and control described above by application to the multidimensional quantum LQG control problem. LQG control is well studied in classical control theory and we shall see many similarities between quantum and classical LQG control theory.

#### 4.1 Quantum Filtering of Linear, Gaussian Dynamics

Let \( X \) be the phase space vector of self-adjoint operators \( X^i, i = 1, \ldots, m \) satisfying the canonical commutation relations (CCRs)
\[ [X^i, X^j] = X^i X^j - X^j X^i = i\hbar J^{ij} I \]
for \( i, j = 1, \ldots, m \) where \( I \) is the identity operator on \( \mathcal{H} \). The CCRs can be written in vector form as
\[ [X, X^\dagger] := XX^\dagger - (XX^\dagger)^\dagger = i\hbar J I \]
where \( X^\dagger = (X^1, \ldots, X^m) \) is the row vector transpose of \( X \) and \( J = (J^{ij}) \) is an antisymmetric real valued matrix which is assumed to be nondegenerate for an even \( m = 2d \) say. We couple the open quantum system to \( d \) measurement channels via the operator vector \( L = \Lambda X \), where \( \Lambda \) is a \( d \times m \) matrix of complex-valued coefficients. Let us place it in a controllable potential which is described by the Hamiltonian
\[ H(u_t) = \frac{1}{2} X^\dagger R X + X^\dagger K u_t + u_t^\dagger K^\dagger X \tag{20} \]
for real vector valued control parameters \( u_t \in \mathbb{R}^d \), where \( R \) is a real symmetric \( m \times m \) matrix and \( K \) is a complex \( m \times d \) matrix. We shall use \( \Lambda^* \) to denote complex conjugation \( (\Lambda^*)_{ij} = \Lambda_{ij}^* \) and \( \Lambda^\dagger = (\Lambda^*)^\dagger \) the Hermitian conjugate.

These definitions allow us to calculate the components of the controlled Lindblad generator
from \((\ref{7})\) with the controlled Hamiltonian \((\ref{20})\) which we write here in vector form

\[
\mathcal{L}(u_t)[X] = J(R + h\mathbb{H}(A^\dagger A))X + J(K + K^*)u_t
\]

omitting the identity \(I\) for notational convenience where \(2i\mathbb{H}(A^\dagger A) = \Lambda^\dagger \Lambda - \Lambda^\dagger \Lambda^*\). So from \((\ref{3})\) and \((\ref{9})\) we obtain the following quantum linear Langevin vector equation

\[
dX_t = (AX_t + Bu_t)dt + dV_t \tag{21}
\]

and linear output equation

\[
dY_t = CX_t dt + dW_t \tag{22}
\]

where \(A := J(R + h\mathbb{H}(A^\dagger A)), B := J(K + K^*), C := \Lambda + \Lambda^*\). The quantum noise increments are given by vectors

\[
dV_t = ith(J(A^T dA_t^* - A^\dagger dA_t) \tag{23}
\]

\[
dW_t = dA_t + dA_t^* \tag{23}
\]

for \((A_t)_i = A_{i,t}\) the annihilation operator on the \(i\)th coupled independent measurement channel.

Let us denote the initial mean \(\hat{X}\) of the phase space operator vector by the component wise expectation \((\hat{X})_i^j = \hat{X}_i^j = \langle \rho, X^i_j \rangle\) and symmetric covariance

\[
\Sigma^j := \frac{1}{2}\langle \rho, X^i_j + X^j_i \rangle - \hat{X}_i^j \hat{X}_i^j \tag{23}
\]

which is given by a real positive definite matrix \(\Sigma = (\Sigma^j_i)\) satisfying the Heisenberg uncertainty principle

\[
\Sigma \geq \pm \frac{i\hbar}{2} J \tag{23}
\]

The filtering equation \((\ref{12})\) preserves the Gaussian nature of the posterior state \((\ref{25})\), so the posterior mean \((\hat{X})_t^j = \hat{X}_t^j = \langle \rho^t, X^i_j \rangle\) and symmetric error covariances

\[
\Sigma^j := \frac{1}{2}\langle \rho^t, X^i_j + X^j_i \rangle - \hat{X}_t^j \hat{X}_t^j \tag{23}
\]

form a set of sufficient coordinates for the quantum LQG system and agree with the initial mean and covariance for \(\rho^t = \rho\). Using \((\ref{12})\), the posterior expectation of \(X_t\) for non-demolition measurement of the output operators \(Y_t\) is given in vector form

\[
d\hat{X}_t = (A\hat{X}_t + Bu_t)dt + \tilde{K}_t d\tilde{Y}_t \tag{24}
\]

\[
\tilde{K}_t = \Sigma_t C^T + M \tag{25}
\]

where \(d\tilde{Y}_t = dY_t - C\tilde{X}_t dt\) is the innovating martingale which describes the information gain from measurement of the output vector operator \(Y_t\).

The symmetric error covariance \(\Sigma_t\) satisfies the matrix Ricatti equation

\[
\frac{d}{dt} \Sigma_t = A\Sigma_t + \Sigma_t A^T + N - (\Sigma_t C^T + M)(\Sigma_t C^T + M)^T \tag{26}
\]

where

\[
N = \frac{1}{2} \hbar J(A^\dagger \Lambda + \Lambda^\dagger A^*)J^T
\]

is the intensity (symmetric covariance) matrix of the quantum noise increment \(dV_t\) and

\[
M = \frac{i}{2} \hbar J(\Lambda^T - \Lambda^\dagger)
\]

is the covariance matrix of the noise increments \(dV_t\) and \(dW_t\).

### 4.2 Quantum LQG Control

We aim to control the phase space operator whilst constraining the amplitude of the controlling force for energy considerations. Thus, our control objectives and restraints can be described by the operator valued risk \((\ref{18})\) with quadratic parameters

\[
C(u_s) = X^T F X + X^T G^T u_s + u_s^T G X + u_s^T u_s
\]

and

\[
S = X^T \Omega X
\]

for positive real symmetric \(m \times m\) matrices \(\Omega, F\) and a real \(d \times m\) matrix \(G\).

Since \(\tilde{X}\) and \(\Sigma\) form a set of sufficient coordinates, they describe the full probability distribution given by \(\rho\), so we may consider the derivation of \(S(t, \rho)\) as partial derivatives of \(S(t, X, \Sigma)\).
So from (21) and the Gaussian nature of the system, we obtain

\[
\langle \rho, \mathcal{L}(u_t) | \delta S(t, \hat{X}, \Sigma) \rangle =
\frac{1}{2}(A \hat{X} + B u_t)\mathlarger{\mathlarger{\mathcal{T}}}(\nabla_\hat{X} S + \nabla_\hat{X} S^T (A \hat{X} + B u_t))
+ (A \Sigma + \Sigma A^T + N, \nabla_\Sigma S)
\]

\[
\sum_{j=1}^d \langle \sigma_j(\rho) \otimes \sigma_j(\rho), (\delta \otimes \delta) S(t, \hat{X}, \Sigma) \rangle =
(\Sigma C^T + M)(\Sigma C^T + M)^T, \nabla_\Sigma^2 S - 2\nabla_\Sigma S
\]

where \((D, E) := \text{Tr}[D^T E]\) is the Hilbert-Schmidt inner product on the vector space of complex-valued \(m \times m\) matrices. We denote the partial derivatives by \((\nabla_\Sigma S)_{ij} = \frac{\partial}{\partial \Sigma_{ij}} S(t, \hat{X}, \Sigma)\), \((\nabla_\Sigma S)_{ij} = \frac{\partial}{\partial \Sigma_{ij}} S(t, \hat{X}, \Sigma)\) and \((\nabla_\Sigma S)_{ij} = \frac{\partial}{\partial \Sigma_{ij}} S(t, \hat{X}, \Sigma)\). Inserting into the Bellman equation (27) and minimizing gives \(u_t = -(\frac{1}{2} B^T \nabla_\hat{X} S + G \hat{X})\) where \(S(t, \hat{X}, \Sigma)\) now satisfies the nonlinear partial differential equation

\[
-\frac{\partial}{\partial t} S(t, \hat{X}, \Sigma) =
\frac{1}{2}(\hat{X}^T A^T \nabla_\hat{X} S + \nabla_\hat{X} S^T A \hat{X}) + \hat{X}^T F \hat{X}
+ (A \Sigma + \Sigma A^T + N, \nabla_\Sigma S) + (\Sigma, F)
- (\frac{1}{2} B^T \nabla_\hat{X} S + G \hat{X})^T (\frac{1}{2} B^T \nabla_\hat{X} S + G \hat{X})
+ (\Sigma C^T + M)(\Sigma C^T + M)^T, \frac{1}{2} \nabla_\Sigma^2 S - \nabla_\Sigma S
\]

which is called the Hamilton-Jacobi-Bellman (HJB) equation for this example.

It is well known from classical control theory that LQG control gives a posterior cost-to-go which is quadratic in the posterior mean. So we use the ansatz

\[
S(t, \hat{X}, \Sigma) = \hat{X}^T \Omega_t \hat{X} + \langle \Omega_t, \Sigma \rangle + \alpha_t
\]

in the HJB equation (27). This gives the optimal feedback control strategy

\[
\begin{align*}
    u_t &= -\tilde{L}_t \hat{X}_t \\
    \tilde{L}_t &= B^T \Omega_t + G
\end{align*}
\]

which is linear in the solution to the filtering equation \(\hat{X}_t\) at time \(t\) where \(\Omega_t\) satisfies the backwards matrix Ricatti equation

\[
-\frac{d}{dt} \Omega_t = \Omega_t A + A^T \Omega_t + F
- (B^T \Omega_t + G)^T (B^T \Omega_t + G)
\]

\[
\Omega_T = \Omega
\]

and \(\alpha_t\) satisfies

\[
-\frac{d}{dt} \alpha_t = (B^T \Omega + G)^T (B^T \Omega + G), \Sigma_t)
+ (\Omega_t, N)
\]

\[
\alpha_T = 0.
\]

From this we obtain the total minimal cost

\[
S(0, \hat{X}, \Sigma) = \hat{X}^T \Omega_0 \hat{X} + \text{Tr}[\Omega_0 \Sigma] + \int_0^T \text{Tr}[\Omega_t N] dt + \int_0^T \text{Tr}[(B^T \Omega + G)^T (B^T \Omega + G) \Sigma_t] dt
\]

where \(\Omega_0\) is the solution to (30) at time \(t = 0\).

### 4.3 Duality

The example of the quantum LQG control problem is important since it is one of the few exactly solvable control problems and emphasizes the similarities between the two components of optimal quantum feedback control, namely quantum filtering and optimal control. The duality between the solutions of filtering (28)-(30) and control (24)-(26) is summarized in the duality table

| Filtering | Control | \(\Sigma_t\) | \(\tilde{L}_t\) | \(A\) | \(C\) | \(N\) | \(M\) |
|-----------|---------|--------------|----------------|------|------|------|------|
| \(\Omega_{T-t}\) | \(L_{T-t}\) | \(A^T\) | \(B^T\) | \(F\) | \(G^T\) |

which allows us to formulate and solve the dual control problem given the filtering parameters. The duality can be understood when we examine the nature of each of the methods used. Both methods involve the minimization of a quadratic function for linear, Gaussian systems, (i.e. the least squares error for filtering and the quadratic cost for control). The time reversal in the dual picture is explained by the forward (backward) induction used in the dynamical minimization problem for the filtering (control) problem.
4.4 Optimal feedback control of continuously observed quantum free particle

We give a more physical interpretation of the above results by application to an explicit example of LQG control where the duality between filtering and control is preserved. The example of the complex Gaussian oscillator was given in [36], however we may now use the multidimensional quantum LQG control solutions derived above for application on higher dimensional systems which do not have such complex representation. The optimal control of a continuously observed quantum free particle with quadratic cost is the simplest such example.

Let \( X^\top = (Q, P) \) be the phase space vector operator consisting of the position \( Q \) and momentum \( P \) operators of the free particle having the initial expectations \( \bar{Q} \) and \( \bar{P} \) respectively. Let us also denote the initial dispersions by \( \sigma_Q \) and \( \sigma_P \) respectively and the initial covariance of \( Q \) and \( P \) by \( \sigma_{QP} = \sigma_{PQ} \). We can perform a continuous observation of the particle by coupling the position operator to the measurement channel \( L = Q \) in which we measure the classical Wiener process \( W_t = A_t + \eta_t \) and the particle is controlled using the linear potential \( V(u_t) = -u_t Q \) for \( u_t \in \mathbb{R} \). The Hamiltonian of this simple system is then given by \( H(u_t) = \frac{1}{2} \sigma^2 P^2 - u_t Q \) where \( M \) is the mass of the particle and the corresponding Langevin equations are

\[
\frac{d}{dt} \dot{Q}_t = \frac{1}{M} \dot{P}_t dt + 2\sigma_{Q,t} \dot{Y}_t \tag{36}
\]

\[
\frac{d}{dt} \dot{P}_t = u_t dt + 2\sigma_{Q,P,t} \dot{Y}_t \tag{37}
\]

where the innovation process \( \dot{Y}_t \) describes the gain of information due to measurement of \( Y_t \) given by

\[
\dot{Y}_t = Y_t - \dot{Q}_t.
\]

In practice, for a continuous observation, it is the measurement current \( I_t := dY_t/dt \) which we observe and so we write the filtering equations in the form

\[
\frac{d}{dt} \dot{Q}_t = \frac{1}{M} \dot{P}_t + 2\sigma_{Q,t} (I_t - \dot{Q}_t) \tag{38}
\]

\[
\frac{d}{dt} \dot{P}_t = u_t + 2\sigma_{Q,P,t} (I_t - \dot{Q}_t) \tag{39}
\]

where the error covariances satisfy the Ricatti equations

\[
\frac{d}{dt} \sigma_{Q,t} = \frac{2}{M} \sigma_{Q,P,t} - 4(\sigma_{Q,t})^2 \tag{40}
\]

\[
\frac{d}{dt} \sigma_{Q,P,t} = \frac{1}{M} \sigma_{P,t} = 4\sigma_{Q,t} \sigma_{Q,P,t} \tag{40}
\]

with initial conditions

\[
\sigma_{Q,0} = \sigma_Q, \quad \sigma_{Q,P,0} = \sigma_{QP}, \quad \sigma_{P,0} = \sigma_P.
\]

The Ricatti equations for the error covariance in the filtered free particle dynamics have an exact solution [37], however we will simply comment on the stationary solutions which are the solutions obtained by setting the LHS of (40) to zero, giving the asymptotic behaviour of the posterior dispersions for \( t \to \infty \)

\[
\sigma_{Q,t} \to \frac{1}{2} \sqrt{\frac{\hbar}{M}}, \quad \sigma_{P,t} \to \hbar \sqrt{\frac{\hbar}{M}}, \quad \sigma_{PQ,t} \to \frac{\hbar}{2}.
\tag{41}
\]

This proper treatment dispels the paradoxical quantum Zeno effect which insists that a quantum state is frozen in time by a continuous observation. Instead we can describe the continuous observation as an optimal estimation with
posterior dispersions tending to a finite limit satsifying the Heisenberg uncertainty relation

$$\Delta Q_t \Delta P_t = \sqrt{\sigma_Q^2 \sigma_P^2} \to \hbar / \sqrt{2} \geq \hbar / 2.$$ 

In contrast, for the case without conditioning (where the measurement results are ignored or averaged over) the Ricatti equations for the dispersions become linear which have solutions tending to infinity like $t^3$. This is faster than the $t^2$ spreading of the wavefunction due to the closed evolution described by Schrödinger’s equation as one would expect since the coupled noise bath only serves to increase the dispersion.

The dual optimal control problem can be found by identifying the corresponding dual matrices from the table (33) which give the quadratic control parameters

$$C(u_t) = \beta Q^2 + u_t^2$$

$$S = \omega Q^2 + \omega Q (P + Q) + \omega P^2$$

which for the linear Gaussian system gives the optimal control strategy

$$u_t = -2(\omega_{Q,t} \dot{P}_t + \omega_{P,t} \dot{Q}_t)$$

(42)

where the coefficients are the solutions to the Ricatti equations

$$-\frac{d}{dt} \omega_{P,t} = \frac{2}{T} \omega_{Q,P,t} - 4(\omega_{P,t})^2$$

$$-\frac{d}{dt} \omega_{Q,P,t} = \frac{2}{T} \omega_{Q,Q,t} - 4\omega_{P,t} \omega_{Q,P,t}$$

$$-\frac{d}{dt} \omega_{Q,t} = \beta - 4(\omega_{P,t})^2$$

(43)

with terminal solutions

$$\omega_{P,0} = \omega_P, \quad \omega_{Q,P,0} = \omega_{QP}, \quad \omega_{Q,0} = \omega_Q.$$ 

Note that in this example, as well as identifying the dual matrices by transposition and time reversal according to the duality table (33), one must also interchange the coordinates $P \leftrightarrow Q$. This is because the matrix of coefficients A is non-symmetric and nilpotent, so it is dual to its transpose only when we interchange the coordinates in the dual picture. Thus the optimal coefficients $\{\omega_{P,t}, \omega_{Q,P,t}, \omega_{Q,t}\}$ in the quadratic cost-to-go correspond to the minimal error covariances $\{\sigma_{Q,T-t}, \sigma_{QP,T-t}, \sigma_{P,T-t}\}$ in the dual picture.

The minimal total cost for the experiment can be obtained from (32) by substitution of these solutions

$$5 = \omega Q_0 (Q^2 + \sigma_Q) + 2\omega Q_P (Q + \sigma_Q)$$

$$+ \omega P (P^2 + \sigma_P) + \int_0^T (h^2 \omega_{P,t} + \omega_{P,Q,t}^2 \sigma_{Q,t}) dt$$

$$+ \int_0^T (\omega_{P,Q,t} \sigma_{P,t} + 2\omega_{Q,P,t} \omega_{P,t} \sigma_{P,Q,t}) dt$$

(44)

5 Discussion

We have shown that the optimal quantum feedback control problem reduces to an optimal estimation problem followed by an optimal control problem based on this optimal estimator. The optimal (least-squares) estimator for quantum random variables (operators) given a classical nondemolition output measurement process is the conditional expectation which is given by the result of the filtering equation (12). The resulting optimal control problem is then defined on the output of this filter, which reduces to a classical control problem on the space of quantum states. For cost functions that are linear in the state, the optimal feedback control strategy is given by the solution to the Bellman equation (18).

In the LQG example, the space of quantum states are restricted to the class of Gaussian states so the probability distribution is parameterized by the mean and covariance of the generating operators. However, due to non commutativity of these quantum operators there are many different definitions of the covariance matrices. For direct comparison to classical LQG control theory, we choose the symmetric representation of the covariance matrices, although unlike the classical case, the Heisenberg uncertainty principle places a positive lower bound on the covariances. In particular, this prohibits the common classical assumption of uncorrelated process and measurement noise if the coupling to the noise bath is complex.
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