The General Universal Property of the Propositional Truncation

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Abstract

In a type-theoretic fibration category in the sense of Shulman (representing a dependent type theory with at least 1, Σ, Π, and identity types), we define the type of constant functions $A \rightarrow B$. This involves an infinite tower of coherence conditions, and we therefore need the category to have Reedy limits of diagrams over $\omega^{op}$. Our main result is that, if the category further has propositional truncations and satisfies function extensionality, the type of constant function is equivalent to the type $\parallel alt1 A \parallel \rightarrow B$.

If $B$ is an $n$-type for a given finite $n$, the tower of coherence conditions becomes finite and the requirement of nontrivial Reedy limits vanishes. The whole construction can then be carried out in Homotopy Type Theory and generalises the universal property of the truncation. This provides a way to define functions $\parallel alt1 A \parallel \rightarrow B$ if $B$ is not known to be propositional, and it streamlines the common approach of finding a proposition $Q$ with $A \rightarrow Q$ and $Q \rightarrow B$.

1 Introduction

In Homotopy Type Theory (HoTT), we can truncate (propositionally or (-1)-truncate, to be precise) a type $A$ to get a type $\parallel A \parallel$ witnessing that $A$ is inhabited without revealing an inhabitant [12] Chapter 3.7. This operation roughly corresponds to the bracket types [2] of extensional Martin-Löf Type Theory, and to the squash types [3] of NuPRL.

$\parallel A \parallel$ is always a proposition, meaning that any two of its inhabitants are equal, and its universal property states that functions $\parallel A \parallel \rightarrow B$ correspond to functions $A \rightarrow B$, provided that $B$ is a proposition. In particular, we always have a canonical map $\parallel \parallel A \parallel \parallel A \parallel$ of the subcategory of propositions. Unfortunately, it can be rather tricky to define a function $\parallel A \parallel \rightarrow B$ if $B$ is not known to be propositional.

One possible way to understand the propositional truncation is to think of elements of $\parallel A \parallel$ as anonymous inhabitants of $A$, with the function $\parallel \parallel A \parallel \parallel A \parallel$ hiding the information which concrete element of $A$ one actually has. With this in mind, let us have a closer look at the mentioned universal property of the propositional truncation, or equivalently, at its elimination principles. If we want to find an inhabitant of $\parallel A \parallel \rightarrow B$ and $B$ is a proposition, then a function $f : A \rightarrow B$ is enough. A possible interpretation of this fact is that $f$ cannot take different values for different inputs, because $B$ is propositional, justifying that $f$ does (in a certain sense) not have to “look at” its argument, such that an anonymous argument
allows us to define a function \(A \to B\). From Reedy limits exist, we can formulate the "complete" type of coherently constant functions map is a fibration (projection). We can think of those limits as "infinite contexts". If these truncation.

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Given a function \(f : A \to B\) and a proof \(c : \text{const}_f\) of weak constancy, we can ask whether the paths (identity proofs) that \(c\) gives are well-behaved in the sense that they fit together. Essentially, if we use \(c\) to construct two inhabitants of \(f(a_1) = f(a_2)\), then those inhabitants should be equal. If we know this, we can weaken the condition that \(B\) is a set to the condition that \(B\) is a groupoid (i.e. 1-truncated), and still construct a function \(\|A\| \to B\). This, and the (simpler) case that \(B\) is a set as described above, are presented as examples 2.2 and 2.3 in Section 2. In principle, we could go on and prove the corresponding statement for the case that \(B\) is 2, 3, . . . -truncated, each step requiring one additional coherence assumption.

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A setting in which we can deal nicely with such "towers" of conditions was given by Shulman [9], who makes the idea that type-theoretic contexts (or "nested Σ-types") correspond to diagrams over inverse categories of a certain shape precise. Although we do not make use of the main result (the construction of univalent models and some applications) of [9], we can make use of the framework and technical results. Working in a type-theoretic fibration category in the sense of Shulman, we can further consider the case that this category has Reedy \(\omega\text{-limits},\) that is, limits of infinite sequences \(A_1 \ll A_2 \ll A_3 \ll \ldots\), where every map is a fibration (projection). We can think of those limits as "infinite contexts". If these Reedy limits exist, we can formulate the "complete" type of coherently constant functions from \(A\) to \(B\) (for which we write \(A \xrightarrow{\omega} B\)), and we can show that such a constant function allows us to define a function \(\|A\| \to B\), even if \(B\) is not known to be \(n\)-truncated for any finite \(n\). Even stronger, the type \(A \xrightarrow{\omega} B\) is homotopy equivalent to the type \(\|A\| \to B\), in the same way as \(A \to B\) is equivalent to \(\|A\| \to B\) under the very strict assumption that \(B\) is a proposition.

The existence of Reedy \(\omega\text{-limits} is the only assumption that we make which is not validated in the standard syntactical version of HoTT as it is presented in [12, Appendix A.2]. However, if we consider an \(n\)-truncated type \(B\) for some finite fixed number \(n\), then

1Of course, we only think of internal properties here. When it comes to computation, the term \(f\) can certainly behave differently if applied on different terms of type \(A\).

2This is not completely true: in a type-theoretic fibration category it is natural to also assume stronger \(η\)-rules, but, as explained by Shulman [9, Example 2.9], those are used to simplify the presentation and the construction does not crucially rely on them.
nearly all of the coherence conditions captured by \( A \xrightarrow{\omega} B \) become trivial, and that type can be simplified to a “nested \( \Sigma \)-type” with \( n + 2 \) components, for which we could write \( A \xrightarrow{n+2} B \). It can be formulated in the standard syntactical version of HoTT, where we can then prove that, for any \( A \) and any \( n \)-truncated \( B \), the type \( A \xrightarrow{n+2} B \) is equivalent to \( \|A\| \rightarrow B \). We thereby generalise the usual universal property of the propositional truncation [12, Lemma 7.3.3], because if \( B \) is not only \( n \)-truncated, but propositional, then \( A \xrightarrow{n+2} B \) can be reduced to \( A \rightarrow B \) simply by removing contractible components. From the point of view of the standard syntactical version of HoTT, an application of our construction could therefore be the construction of functions \( \|A\| \rightarrow B \) for the case that \( B \) is not propositional.

The usual approach for this problem is to construct a proposition \( Q \) such that \( A \rightarrow Q \) and \( Q \rightarrow B \) (see [12, Chapter 3.9]). Our construction can be seen as a uniform construction of such a \( Q \), since the equivalence \((A \xrightarrow{n+1} B) = (\|A\| \rightarrow B)\) is proved by constructing a suitable “contractible extension” to \( A \xrightarrow{n+2} B \); the general strategy is to “expand and contract” expression, as we strive to explain with the help of the examples in Section 2.

Nevertheless, we want to stress that we consider the correspondence between \( A \xrightarrow{\omega} B \) and \( \|A\| \rightarrow B \) in a type-theoretic fibration category with Reedy \( \omega \)-op-limits our main result, and the finite special cases described in the previous paragraph simply fall out as a corollary. In fact, we think that Reedy \( \omega \)-op-limits are a somewhat reasonable assumption. Recently, it has been discussed regularly how these or similar concepts can be introduced into syntactical type theory (for example, see the blog posts [10] and [8] with the comments sections, and the discussion on the HoTT mailinglist [11] titled “Infinitary type theory”). A major motivation is the question whether HoTT can serve as its own meta-theory, whether we can write an interpreter for HoTT in HoTT, and related questions such as the definition of semi-isimplicial types [5]. Moreover, a concept that is somewhat similar has been suggested earlier as “very dependent types” [6], even though this suggestion was made in the setting of NuPRL [3].

**Contents**

We first discuss the cases that the codomain \( B \) is a set or a groupoid, as described in the introduction, in Section 2. This provides some intuition for our general strategy of proving a correspondence between coherently constant functions and maps out of propositional truncations. In particular, we describe how the method of “adding and removing contractible components” for proving equivalences can be applied. In Section 3 we briefly review the notion of a type-theoretic fibration category, of an inverse category, and, most importantly, constructions related to Reedy fibrant diagrams, as described by Shulman [9]. Some simple observations about the restriction of diagrams so subsets of the index categories are recorded in Section 4. We proceed by defining the equality diagram over a given type for a given inverse category in Section 5. The special case where the inverse category is \( \Delta^n_{op} \) (the category of nonempty finite sets and strictly increasing functions) gives rise to the equality semi-simplicial type, which is discussed in Section 6. We show that the projection of a full \( n \)-dimensional tetrahedron to any of its horns is a homotopy equivalence. Then, in Section 7 we manually construct a fibrant diagram that more or less represents the exponential of a fibrant and a non-fibrant diagram. We extend the category \( \Delta^n_{op} \) in Section 8 which allows us to make precise how contractible components can be “added and removed” in general. Our main result, namely that the types \( A \xrightarrow{\omega} B \) and \( \|A\| \rightarrow B \) are homotopy equivalent, is shown in Section 9. In the final section 10, discuss the corollaries of the main theorem for truncated \( B \), the definability of the semi-simplicial equality type and its relation to other open problems, and the possibility to extend our result to higher
truncations.

**Notation** We use type-theoretic notation and we assume familiarity with HoTT, in particular with the book [12] and its notations. If \( A \) is a type and \( B \) depends on \( A \), it is standard to write \( \Sigma_{a:A} B(a) \) for the dependent pair (or \( \Sigma \)-) type. However, we think it can sometimes support readability to write \( (a : A) \times B(a) \) instead, especially if we have to talk about nested \( \Sigma \)-types: instead of \( \Sigma_{a:A} \Sigma_{b:B(a)} C(a,b) \) we write \( (a : A) \times (b : B(a)) \times C(a,b) \), or even \( (a : A) \times (b : B(a)) \times (c : C(a,b)) \) if we want to give a name to the last component.

We do not specify the associativity of the operator \( \times \); whether the mentioned expression corresponds to \( \Sigma_{(a,b)} \Sigma_{a:B(a)} C(a,b) \) or \( \Sigma_{a:A} \Sigma_{b:B(a)} C(a,b) \) is not relevant since we are (when talking about types) only interested in the type “up to homotopy equivalence”.

Regarding notation, one potentially dangerous issue is that there are many different notions of equality-like concepts, such as the internal propositional equality of type theory, internal equivalence of types, judgmental equality theoretic expressions, isomorphism of objects in a category, isomorphism or equivalence of categories, and strict equality of morphisms. For this article, we use the convention that *internal* concepts are written using “two-line” symbols, coinciding with the notation of [12]: we write \( a = b \) for the internal equality type \( \text{Id}(a,b) \), and \( A \equiv B \) for the type of equivalences between \( A \) and \( B \). Other concepts are denoted (if at all) using “three-line” symbols: we write \( a \equiv b \) if \( a \) and \( b \) denote two judgmentally equal expressions, and we use \( \equiv \) for other cases of strict equality in the meta-theory. \( x \equiv y \) means we want to express that \( x \) and \( y \) are isomorphic objects of a category. Equality of morphisms (of a category) is sometimes expressed with \( \equiv \), but usually by saying that some diagram commutes, and if we say that some diagram commutes, we always mean that it commutes strictly, not only up to homotopy. Other notions of equality are written out.

If \( C \) is some category and \( x \in C \) an object, we write (as it is standard) \( x/C \) for the co-slice category of arrows \( x \to y \). We do many constructions involving subcategories, but we want to stress that we always and exclusively work with full subcategories (apart from the subcategory of fibrations in Definition 3.1). Thus, we write \( C - x \) for the full subcategory of \( C \) that we get by removing the object \( x \). Further, if \( D \) is a full subcategory of \( C \) (we write \( D \subset C \)) which does not contain \( x \), we write \( D + x \) for the full subcategory of \( C \) that has all the objects of \( D \) and the object \( x \).

Not exactly notation, but in a similar direction, are the following two remarks: First, when we refer to the *distributivity law* of \( \Pi \) and \( \Sigma \), we mean the equivalence

\[
(\Pi_{a:A} \Sigma_{b:B(a)} C(a,b)) \cong (\Sigma_{f:\Pi_{a:A} B(a)} \Pi_{a:A} C(a,f(a)))
\]

which is sometimes referred to as the *type-theoretic axiom of choice* or \( AC_\infty \) (see [12]). Second, if we talk about a *coconut*, or *coconut type*, we mean a type of the form \( \Sigma_{a:A} x = a \) or \( \Sigma_{a:A} x = a \) for a fixed \( x \). As it is well-known, coconuts are contractible.

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3The name comes from a misreading of the expression “coconutstot” in Coq code by Voevodsky (which stands for “co-cone to it”) and came up on the mailing list [11]. Other names that I know of are “singleton type”, but the standard reference uses this synonymously with “contractible type” ([12] Definition 3.11.1), and “path-to type” or “path-from type”, which (although reasonable) are somewhat ambiguous. Therefore, I use “coconut” (which, with some fantasy, can be associated with contractibility), even though it bears the danger of being taken as a joke.
2 A First Few Special Cases

In this section, we want to discuss some simple examples and aim to build up intuition for the general case. We work entirely in standard Homotopy Type Theory as it is specified in [12, Appendix A.2], together with function extensionality (see [12, Appendix A.3.1]) and propositional truncation. To clarify the latter, we assume that, for any type $A$, there is a propositional type $\parallel A \parallel$ such that we have a type equivalence $(A \to B) \simeq (\parallel A \parallel \to B)$ for any propositional type $B$. This characterization is sufficient for us. Due to the “equivalence-style reasoning” nature of our proofs, we can avoid the necessity of any “unpleasant manual computation” such that we would not benefit further from the usual truncation’s judgmental computation rules (other than not having to assume function extensionality explicitly [7]).

We think it is worth mentioning that we actually do not require much of the power of Homotopy Type Theory: we only use 1, $\Sigma$, $\Pi$, identity types, propositional truncations, and assume function extensionality. This will in later sections turn out to be a key feature which enables us to perform the construction in the infinite case (assuming the existence of certain Reedy limits).

Assume we want to construct an inhabitant of $\parallel A \parallel \to B$ and $B$ is an $n$-type, for a fixed given $n$. The case $n \equiv -2$ is trivial. For $n \equiv -1$, the universal property can be applied directly. In this section, we explain the cases $n \equiv 0$ and $n \equiv 1$. The following auxiliary statement will be useful:

**Lemma 2.1.** Let $C_1, C_2, \ldots, C_m$ be types dependent on $A$, possibly with $C_j$ depending on $C_i$ for $i < j$. Then, the types

\[ (\Pi A C_1) \times (\Pi A C_2) \times \ldots \times (\Pi A C_m) \]

and

\[ (\Pi A C_1) \times (\Pi A C_2) \times \ldots \times (\Pi A C_m) \]

are equivalent.

**Proof.** This holds by the usual distributivity law (2) of $\Pi$ (or $\to$) and $\Sigma$, together with the equivalence $\parallel A \parallel \times A \simeq A$. \qed

2.1 Constant Functions into Sets

We consider the case $n \equiv 0$ first; that is, we assume that $B$ is a set. Recall the definition of const given in (1).

**Example 2.2.** Let $B$ be a set and $A$ any type. Then, we have the equivalence

\[ (\parallel A \parallel \to B) \simeq (f : A \to B) \times (\text{const}_f). \]  

Note that, if $B$ is not only a set but even a proposition, the condition $\text{const}_f$ is not only automatically satisfied, but it is actually contractible as a type. By the usual equivalence lemmata, the type on the right-hand side of (5) then simplifies to $(A \to B)$, which exactly is the universal property. Thus, we view (5) as a first generalisation.

**Proof of Example 2.2.** Assume $a : A$ is some point in $A$. In the following, we construct a chain of equivalences. The variable names for certain components might seem somewhat odd: for example, we introduce a point $f_1 : B$. The reason for this choice will become clear...
later. For now, we simply emphasise that \( f_1 \) is “on the same level” as \( f : A \to B \) in the sense that they both give points, rather than for example paths (like, for example, an inhabitant of \( \text{const}_f \)).

\[
\begin{align*}
B \\
(S1) & \simeq (f_1 : B) \times \Pi_{a : A}((b : B) \times (b = f_1)) \\
(S2) & \simeq (f_1 : B) \times (f : A \to B) \times (\Pi_{a : A} f(a) = f_1) \\
(S3) & \simeq (f_1 : B) \times (f : A \to B) \times (\Pi_{a : A} f(a) = f_1) \times (\text{const}_f) \times (f(a_0) = f_1) \\
(S4) & \simeq (f : A \to B) \times (\text{const}_f) \times (f_1 : B) \times (f(a_0) = f_1) \times (\Pi_{a : A} f(a) = f_1) \\
(S5) & \simeq (f : A \to B) \times (\text{const}_f) \times ((f_1 : B) \times (f(a_0) = f_1)) \\
(S6) & \simeq (f : A \to B) \times (\text{const}_f)
\end{align*}
\]

Let us explain the validity of the single steps. In the first step, we add a family of coconuts. In the second step, we apply the distributivity law \(^2\). In the third step, we add two components, and \( B \) being a set ensures that both of them are propositional. But it is very easy to derive both of them from \( \Pi_{a : A} f(a) = f_1 \), showing that both of them are contractible. In the fourth step, we simply reorder some components, and in the sixth step, we use that \( \Pi_{a : A} f(a) = f_1 \) is contractible by an argument analogous to that of the third step. Finally, we can remove two components which form a contractible “path-from”-type.

If we carefully trace the equivalences, we see that the function part

\[
ed : B \to (f : A \to B) \times \text{const}_f
\]

is given by

\[
ed(b) \equiv (\lambda a . b , \lambda a^1 . \text{refl}_b),
\]

not depending on the assumed \( a_0 : A \). But as \( e \) is an equivalence assuming \( A \), it is also an equivalence assuming \( \| A \| \).

As \( \| A \| \to (B \simeq ((f : A \to B) \times \text{const}_f)) \) implies that the types \((\| A \| \to B)\) and \((\| A \| \to ((f : A \to B) \times \text{const}_f))\) are equivalent, the statement follows from Lemma \(^2.1\).

The core strategy of \((S1) - (S6)\) is to add and remove contractible components. This principle of expanding and contracting a type expression can be generalised and, as we will see, even works for the infinite case when \( B \) is not known to be of any finite truncation level. Generally speaking, we use two ways of showing that components are contractible. The first is to group two of them together such that they form a “path-to” type, as we did in \((S1)\) and \((S6)\). The second is to use the fact that \( B \) is truncated, as we did in \((S4)\) and \((S3)\). We consider the first to be the key technique, and in the general (infinite) case of an untruncated \( B \), the second can not be applied at all. We thus view the second method as a tool to deal with single components that lack a “partner” only because the case that we consider is finite, and which is unnecessary in the infinite case.

### 2.2 Constant Functions into Groupoids

The next special case is \( n \equiv 1 \). Assume that \( B \) is a groupoid, that is a 1-type. Let us first clarify which kind of constancy we expect for a map \( f : A \to B \) to be necessary. Not only do we require \( e : \text{const}_f \), we also want this constancy proof (which is in general not propositional
any more) to be coherent: given \( a^1 \) and \( a^2 : A \), we expect that \( c \) only allows us to construct essentially one proof of \( f(a^1) = f(a^2) \). The reason is that we want the data (which includes \( f \) and \( c \)) together to be just as powerful as a map \( \| A \| \to B \), and from such a map, we definitely do not get parallel pairs of paths in \( B \) which cannot be seen to be equal either.

We claim that the required coherence condition is

\[
\text{coh}_{f,c} \equiv \Pi_{a^1,a^2 : A} c(a^1,a^2) \cdot c(a^2,a^3) = c(a^1,a^3).
\] (8)

A first sanity check is to see whether from \( d : \text{coh}_{f,c} \) we can now prove that \( c(a,a) \) is equal to \( \text{refl}_a \), something that should definitely be the case if we do not want to be able to construct possibly different parallel paths in \( B \). To give a positive answer, we only need to see what \( d(a,a,a) \) is telling us.

**Example 2.3** (case \( n = 1 \)). Let \( B \) be a groupoid (1-type) and \( A \) any type. Then, we have

\[
(\| A \| \to B) \simeq ((f : A \to B) \times (e : \text{const}_f) \times (\text{coh}_{f,c})).
\] (9)

Note that Example 2.3 generalises Example 2.2 if \( B \) is a set (as in 2.2), it is also a groupoid and the type \( \text{coh}_{f,c} \) becomes contractible, as it talks about equality of equalities.

**Proof.** Although not conceptually harder, it is already significantly more tedious to write down the chain of equivalences. We therefore choose a slightly different representation. Assume \( a_\approx : A \) as before. We then have:

\[
B
\]

\[
\text{(S1)} \quad \equiv \quad (f_1 : B) \\
\times (f : A \to B) \times (c_1 : \Pi_{a^1} f(a) = f_1) \\
\times (e : \text{const}_f) \times (d_1 : \Pi_{a^2} c(a^1,a^2) \cdot c_1(a^2) = c_1(a^1)) \\
\times (d_2 : f(a_\approx) = f_1) \times (d_3 : c(a_\approx,a_\approx) \cdot c_1(a_\approx) = c_2) \\
\times (d : \text{coh}_{f,c}) \\
\times (d_2 : \Pi_{a^2} c(a_\approx,a^2) \cdot c_1(a_\approx) = c_2)
\]

\[
\text{(S2)} \quad \equiv \quad (f : A \to B) \times (e : \text{const}_f) \times (d : \text{coh}_{f,c}) \\
\times (f_1 : B) \times (e_2 : f(a_\approx) = f_1) \\
\times (c_1 : \Pi_{a^1} f(a) = f_1) \times (d_2 : \Pi_{a^2} c(a_\approx,a^2) \cdot c_1(a_\approx) = c_2) \\
\times (d_1 : \Pi_{a^2} c(a^1,a^2) \cdot c_1(a^2) = c_1(a^1)) \\
\times (d_3 : c(a_\approx,a_\approx) \cdot c_1(a_\approx) = c_2)
\]

\[
\text{(S3)} \quad \equiv \quad (f : A \to B) \times (e : \text{const}_f) \times (d : \text{coh}_{f,c})
\]

In the first step (S1), we add five contractible components, each line (apart from the first, which is \((f_1 : B)\)) representing one.

The first three of the five components are seen to be contractible coconuts (after applying the distributivity law [2]), while the last two are clearly propositional, but also derivable
from the other components. In the second step, we simply re-order some components. Then, in step \((S3)\), we remove several contractible components (again, each but the first line is a contractible component).

We trace the canonical equivalences to see that the function-part of the constructed equivalence is

\[
e : \mathcal{B} \to (f : \mathcal{A} \to \mathcal{B}) \times (c : \text{const}_f) \times (d : \text{coh}_{f,c})
\]

\[
e(b) \equiv (\lambda a.b, \lambda a^1 a^2. \text{refl}_b, \lambda a^1 a^2 a^3. \text{refl}_b).
\]

In particular, \(e\) is independent from the assumed \(a_0 : \mathcal{A}\). As before, this means that \(e\) is an equivalence assuming \(\|A\|\), and, with the help of \ref{2.1}, we derive the claimed equivalence. \(\Box\)

2.3 Outline of the General Idea

At this point, it seems plausible that what we have done for the special cases of \(n \equiv 0\) and \(n \equiv 1\) can be done for any (fixed) \(n < \infty\). Nevertheless, we have seen that the case of groupoids is already significantly more involved than the case of sets. To prove a generalisation, we have to be able to state what it means for a function to be “coherently constant” on \(n\) levels, rather than just the first one or two.

Let us try to specify what “coherently constant” should mean in general. If we have a function \(f : \mathcal{A} \to \mathcal{B}\), we get a point in \(\mathcal{B}\) for any \(a : \mathcal{A}\). A constancy proof \(c : \text{const}_f\) gives us, for any pair of points in \(\mathcal{A}\), a path between the corresponding points in \(\mathcal{B}\). Given three points, \(c\) gives us three paths which form a “triangle”, and an inhabitant of \(\text{coh}_{f,c}\) does nothing else than providing a filler for such a triangle. It does not take much imagination to assume that, on the next level, the appropriate coherence condition should state that the “boundary” of a tetrahedron, consisting of four filled triangles, can be filled.

To gain some intuition, let us look at the following diagram:

All arrows are given by the canonical projections. Consider the category \(D\) with objects the finite ordinals 1, 2 and 3, and arrows the strictly monotonous maps. Then, the left-hand
side and the right-hand side can both be seen as a diagram over $D^{op}$. The data that we need for a “coherently constant function” from $A$ into $B$, if $B$ is a groupoid, can now be viewed as a natural transformation $t$ from the left to the right diagram. On the lowest level, such a natural transformation consists of a function $t_1 : A \to B$, which we called $f$. On the next level, we have $t_2 : A^2 \to \Sigma b_1, b_2 : B \quad b_1 = b_2$, but in such a way that the diagram commutes (strictly, not up to homotopy), enforcing

$$\pi_1(t_2(a^1, a^2)) \equiv (f(a^1), f(a^2))$$

(10)

and thereby making $t_2$ the condition that $t_1$ is naively constant. Finally, $t_3$ yields the coherence condition $\text{coh}$.

In the most general case, where we do not put any restriction on $B$, we certainly cannot expect that a finite number of coherence conditions can suffice. What we then need is what we call the equality semi-simplicial type over $B$, and the infinite tower of coherence conditions will correspond to natural transformations into this diagram.

3 Type Theoretic Fibration Categories, Inverse Diagrams, and Reedy Limits

In his work on *Univalence for inverse diagrams and homotopy canonicity*, Shulman has proved several deep results [9]. Among other, he shows that diagrams over inverse categories can be used to build new models of univalent type theory, and uses this to prove a partial solution to Voevodsky’s homotopy-canonicity conjecture. We do not require those main results; in fact, we do not even assume that there is a universe, an consequently we also do not use univalence! At the same time, what we want to do can be explained nicely in terms of diagrams over inverse diagrams, and we therefore choose to work in the same setting. Luckily, it is possible to do this with only a very short introduction to type-theoretic fibration categories, inverse diagrams and Reedy-limits, and this is what the current section servers for.

**Type-theoretic fibration categories** A type-theoretic fibration category [9, Definition 2.1] is a category with some structure that allows to model dependent type theory with identity types. Let us recall the definition, where we use a lemma by Shulman to give an equivalent (more “type-theoretic”) formulation:

Definition 3.1 (Type-theoretic fibration category [9, Definition 2.1 with Lemma 2.4]). A type-theoretic fibration category is a category $\mathcal{C}$ which has the following structure.

1. A terminal object 1.

2. A (not necessarily full) subcategory $\mathfrak{S} \subset \mathcal{C}$ containing all the objects, all the isomorphisms, and all the morphisms with codomain 1. A morphism in $\mathfrak{S}$ is called a fibration, and written as $A \Rightarrow B$. Any morphism $i$ is called an acyclic cofibration and written $i : X \rightsquigarrow Y$ if it has the left lifting property with respect to all fibrations, meaning that

\[
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow^i & & \downarrow^f \\
Y & \longrightarrow & B \\
\end{array}
\]

every commutative square has a (not necessarily unique) filler $h : Y \to A$ that makes both triangles commute.
3. All pullbacks of fibrations exist and are fibrations.

4. For every fibration \( g : A \to B \), the pullback functor \( g^* : \mathcal{C}/B \to \mathcal{C}/A \) has a partial right adjoint \( \Pi_g \), defined at all fibrations over \( A \), such that its values are fibrations over \( B \).

5. For any fibration \( A \to B \), the diagonal morphism \( A \to A \times_B A \) factors as \( A \to P_B A \to A \times_B A \), with the first map being an acyclic cofibration and the second being a fibration.

6. For any \( A \to B \), there exists a factorization as in (5.) such that in any diagram of the shape

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
A & \sim & P_B A \\
\end{array}
\]

we have the following: if both squares are pullback squares (which implies that \( Y \to Z \) and \( X \to Z \) are fibrations), then \( X \to Y \) is an acyclic cofibration.

The example of a type-theoretic fibration category that we mainly have in mind is \([9, \text{Example 2.9}]\), the category of contexts of a dependent type theory with a unit type, \( \Sigma \)- and \( \Pi \)-types, and identity types. The unit, \( \Sigma \)- and \( \Pi \)-types are required to satisfy judgmental \( \eta \)-rules, which, as pointed out by Shulman, are not strictly necessary but simplify the presentation. Because of these \( \eta \)-rules, we do not need to talk about contexts; we can view every object of the category as a “nested \( \Sigma \)-type” with some finite number of components. Of course, the terminal object is the unit type. The subset of fibrations is the closure of the projections under isomorphisms. One nice property is that the \( \eta \)-rules also imply that we can assume that all fibrations are a projection of the form \( \Sigma_{x : X} Y(x) \to X \). Pullbacks correspond to substitutions, and the partial functor \( \Pi_g \) comes from dependent function types. For any fibration \( f : A \to B \), the factorization in item (5.) can be obtained using the intensional identity type: if \( B \) is the unit type, then the factorization can be written as \( A \to \Sigma_{(x,y) : A \times A} x = y \to A \times A \), and similar otherwise (see \([4]\)). The acyclic cofibration is given by reflexivity.

It is not exactly true that a type-theoretic fibration category has an intensional dependent type theory as its internal language due to the well-known issue that substitution in type theory is strictly functorial. Fortunately, coherence theorems can be applied to solve this problem, and we do not worry about it but simply refer to Shulman’s explanation \([9, \text{Chapter 4}]\). The crux is that the syntactic category of the dependent type theory with unit, \( \Sigma \), \( \Pi \), and identity types is essentially the initial type-theoretic fibration category. This means that we can use type-theoretic constructions freely (as long as they can be performed using unit, \( \Pi \), \( \Sigma \), and identity types); and we will exploit this heavily. For example, the same notion of type equivalence \( A \cong B \) and function extensionality can be defined. Note that an any isomorphism is automatically an acyclic cofibration, and both are automatically a homotopy equivalence.

**Inverse categories and Reedy fibrant diagrams** For objects \( x \) and \( y \) of a category, write \( y \lesssim x \) if \( y \) receives a nonidentity morphism from \( x \) (and \( y \lesssim x \) if \( y \lesssim x \) or \( y \equiv x \)). A category \( \mathcal{J} \) is called an inverse category if the relation \( \lesssim \) is well-founded. In this case, the ordinal rank of an object \( x \) in \( \mathcal{J} \) is defined by

\[
\rho(x) := \sup_{y \lesssim x} (\rho(y) + 1).
\]
As described by Shulman [9 Section 11], diagrams on \( \mathcal{I} \) can be constructed by well-founded induction in the following way. If \( x \) is an object, write \( x \parallel \mathcal{I} \) for the full subcategory of the co-slice category \( x/\mathcal{I} \) which excludes only the identity morphism \( \text{id}_x \). Consider the full subcategory \( \{ y \mid y \prec x \} \subset \mathcal{I} \). There is the forgetful functor \( U : x \parallel \mathcal{I} \to \{ y \mid y \prec x \} \), mapping any \( x \to y \) to the codomain \( y \). If further \( A \) is a diagram in a type-theoretic fibration category \( \mathcal{C} \) that is defined on this full subcategory, if the limit

\[
M^A(x) \equiv \lim_{x \neq y} (A \circ U).
\]

exists, it is called the corresponding matching object. To extend the diagram \( A \) to the full subcategory \( \{ y \mid y \preceq x \} \subset \mathcal{I} \), it is then sufficient to give an object \( A(x) \) and a morphism \( A(x) \to M^A(x) \). The diagram \( A : \mathcal{I} \to \mathcal{C} \) is Reedy fibrant if all matching objects \( M^A(x) \) exist and all the maps \( A(x) \to M^A(x) \) are fibrations. We use the fact that fibrations can be regarded as “one-type projections” in the following way:

**Convention 3.2** (Decomposition in matching object and fibre). If \( A : \mathcal{I} \to \mathcal{C} \) is a Reedy fibrant diagram, we write (as said above) \( M^A(x) \) for its matching objects, and \( F^A(x,m) \) for the fibre over \( m \); that is, we have

\[
A(x) \equiv \Sigma_{m : M^A(x)} F^A(x,m).
\]

There is the more general notion of a Reedy fibration (a natural transformation between two diagrams over \( \mathcal{I} \) with certain properties), so that a diagram is Reedy fibrant if and only if the unique transformation to the terminal diagram is a Reedy fibration. Further, \( \mathcal{C} \) is said to have Reedy \( \mathcal{I} \)-limits if any Reedy fibrant \( A : \mathcal{I} \to \mathcal{C} \) has a limit which behaves in the way one would expect; in particular, if a natural transformation between two Reedy fibrant diagrams is levelwise a homotopy equivalence, then the map between the limits is a homotopy equivalence. We omit the exact definitions as our constructions do not require them (apart from Lemma 3.3, which is only a very plausible auxiliary statement), and refer to [9, Chapter 11] for the details instead.

For later, we record the following:

**Lemma 3.3.** If \( A : I \to \mathcal{C} \) is Reedy fibrant, then so is \( A \circ U : x/I \to \mathcal{C} \).

*Proof.* This is due to the fact that for a (nonidentity) morphism \( k : x \to y \) in \( \mathcal{I} \) the categories \( k/(x \parallel \mathcal{I}) \) and \( y/(x \parallel \mathcal{I}) \) are isomorphic, and it is already used in the proof of [9, Lemma 11.8].

An inverse category \( \mathcal{I} \) is admissible for \( \mathcal{C} \) if \( \mathcal{C} \) has all Reedy \( (x \parallel \mathcal{I}) \)-limits. If \( \mathcal{I} \) is finite, then any type-theoretic fibration category has Reedy \( \mathcal{I} \)-limits by [9, Lemma 11.8]. From the same Lemma, it follows that for all constructions that we are going to do, it will be sufficient if \( \mathcal{C} \) has Reedy \( \omega^{op} \)-limits. Further, in all our cases of interest, all co-slices of \( \mathcal{I} \) are finite, and \( \mathcal{C} \) is automatically admissible.

Because of the above, let us fix the following:

**Convention 3.4.** For the rest of this article, let \( \mathcal{C} \) be a type-theoretic fibration category with Reedy \( \omega^{op} \)-limits, which further satisfies function extensionality. We further say that \( \mathcal{I} \) is a tame inverse category if all co-slices \( x/\mathcal{I} \) are finite (which implies that \( \rho(x) \) is finite for all objects \( x \)) and, for all \( n \), the set of objects at “level” \( n \), that is \( \{ x \in \mathcal{I} \mid \rho(x) \equiv n \} \), is finite. We will not be interested in any non-tame inverse categories.

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4 Subdiagrams

Let \( \mathcal{I} \) be a tame category. We are interested in full subcategories of \( \mathcal{I} \), and we mean “subcategory” in the strict sense that the set of objects is a subset of the set of objects of \( \mathcal{I} \). We say that a full subcategory \( \mathcal{J} \) of \( \mathcal{I} \) is downwards closed if, for any pair \( x, y \) of objects in \( \mathcal{I} \) with \( y \triangleleft x \), if \( x \) is in \( \mathcal{J} \), then so is \( y \). The full downwards closed subcategories of \( \mathcal{I} \) always form a poset \( \text{Sub}(\mathcal{I}) \), with an arrow \( \mathcal{J} \to \mathcal{J}' \) if \( \mathcal{J}' \) is a subcategory of \( \mathcal{J} \). Again, “subcategory” is to be understood in the set-theoretic sense. In particular, we do not identify isomorphic subcategories (since their objects will in general be different, in other words, the isomorphism will not commute with the embeddings into \( \mathcal{I} \)).

It is easy to see that the poset \( \text{Sub}(\mathcal{I}) \) has all limits and colimits. For example, given downwards closed full subcategories \( \mathcal{J} \) and \( \mathcal{J}' \), their product is given by taking the union of their sets of objects. We therefore write \( \mathcal{J} \cup \mathcal{J}' \). Dually, coproducts are given by intersection and we can write \( \mathcal{J} \cap \mathcal{J}' \).

An object \( x \) of \( \mathcal{I} \) generates a subcategory \( \{ y \mid y \triangleleft x \} \), for which we write \( \tau \).

If \( A : \mathcal{I} \to \mathcal{C} \) is a Reedy fibrant diagram and \( \mathcal{C} \) has Reedy \( \mathcal{I} \)-limits, we can consider the functor

\[
\lim_- A : \text{Sub}(\mathcal{I}) \to \mathcal{C}
\]

which maps any downwards closed full subcategory \( \mathcal{J} \subseteq \mathcal{I} \) to \( \lim \mathcal{J} A \), the Reedy limit of \( A \) restricted to \( \mathcal{J} \).

**Lemma 4.1.** Let \( \mathcal{I} \) be a tame category and \( \mathcal{J}, \mathcal{K} \) two downwards closed subcategories of \( \mathcal{I} \). Assume that \( \mathcal{C} \) has Reedy \( \mathcal{J} \cup \mathcal{L} \)-limits. Then, the functor \( \lim_- A \) maps the pullback square

\[
\begin{array}{ccc}
J \cup K & \to & K \\
\downarrow & & \downarrow \\
J & \to & J \cap K
\end{array}
\]

in \( \text{Sub}(\mathcal{I}) \) to a pullback square in \( \mathcal{C} \).

**Proof.** For an object \( X \), a cone \( X \to A|_{J \cup K} \) corresponds to a pair of two cones, \( X \to A|_J \) and \( X \to A|_K \), which coincide on \( J \cap K \).

Note that for the lemma above it is crucial that the subcategories are downwards closed.

**Lemma 4.2.** Under the same assumptions as before, the functor \( \lim_- A \) maps all morphisms to fibrations. In other word, if \( \mathcal{K} \) is a downwards closed subcategory of the inverse category \( \mathcal{J} \), then

\[
\lim J A \to \lim \mathcal{K} A
\]

is a fibration.

**Proof.** We only need to consider the case that \( \mathcal{J} \) has exactly one object that \( \mathcal{K} \) does not have, say \( J \equiv K + x \), because the composition of fibrations is a fibration (this holds true even for “infinite compositions”, with the same short proof as Lemma \( \text{[9,3]} \)). Further, we may assume that all objects of \( \mathcal{J} \) are predecessors of \( x \), i.e. we have \( \tau \equiv J \); otherwise, we could view \( J \to K \) as a pullback of \( \tau \to \tau - x \) and apply Lemma \( \text{[1.1]} \).

The cone \( \lim \mathcal{K} A \to A|_K \) gives rise to a cone \( \lim \mathcal{K} A \to (A \circ U)|_{x \not\to K} \) (the morphism into \( x \not\to y \) is given by the morphism into \( y \)), and we thereby get a morphism \( m : \lim \mathcal{K} A \to M^A(x) \).

If we pull the fibration \( A(x) \to M^A(x) \) back along the morphism \( m \), we get a fibration \( P \to \lim \mathcal{K} A \), and it is easy to see that \( P \cong \lim J A \).

\[ \square \]
**Lemma 5.1.** For all \( \iota \) satisfy the required naturality conditions.

**Proof.** This is due to the fact that

\[
( k : \lim K A) \times F^A(x, m(k)) \to \lim K A.
\]

(16)

This remains true even if not all objects in \( J \) are predecessors of \( x \).

**Remark 1.** From the above proof, we also get a description of how the fibration \( \lim_J A \to \lim K A \) looks like in type-theoretic notation. It can be written as

\[
( k : \lim K A) \times F^A(x, m(k)) \to \lim K A.
\]

(17)

**Remark 2.** The proof of Lemma 4.2 also show that, if \( J \) is admissible for \( \mathcal{C} \) and \( \mathcal{C} \) has Reedy \( J \)-limits, then \( \mathcal{C} \) also has Reedy \( K \)-limits for \( K \subset J \): Given a functor \( A : K \to \mathcal{C} \), we can extend it to a functor \( J \to \mathcal{C} \) by “filling up” the missing components with the corresponding matching objects. Note that the requirement that \( J \) is admissible is necessary. For example, if \( \alpha \) is a limit ordinal, then \( (\alpha + 1)^{op} \) has an initial object and \( \mathcal{C} \) thereby trivially has Reedy \( (\alpha + 1)^{op} \)-limits, while it might not have Reedy \( \alpha^{op} \)-limits.

### 5 Equality Diagrams

Given any tame inverse category \( \mathcal{J} \) and a fixed type \( B \), we can now define the “equality diagram” of \( B \) over \( \mathcal{J} \). We define inductively:

1. A diagram \( \mathcal{E} : \mathcal{J} \to \mathcal{C} \)
2. A cone \( \eta : B \to \mathcal{E} \) (i.e. a natural transformation from the functor that is constantly \( B \) to \( \mathcal{E} \))
3. A diagram \( \mathcal{M}^{\mathcal{E}} : \mathcal{J} \to \mathcal{C} \) (the diagram of matching objects)
4. An auxiliary cone \( \tilde{\eta} : B \to \mathcal{M}^{\mathcal{E}} \).

What we actually want is the “equality diagram” \( \mathcal{E} \), the other components are mainly auxiliary constructions.

Assume that \( i \) is an object in \( \mathcal{J} \) such that \( \mathcal{E} \) and \( \eta \) are defined for all predecessors of \( i \). This is in particular the case if \( i \) has no predecessors. We define the matching object \( \mathcal{M}^{\mathcal{E}}(i) \equiv \lim_{J \to \mathcal{E}} \mathcal{E} \) as discussed in Section 3. For every arrow \( f : i \to j \), we immediately get \( \overline{f} : M(i) \to \mathcal{E}(j) \). The fact that \( \mathcal{M}^{\mathcal{E}}(i) \) is defined to be a limit gives rise to \( \tilde{\eta} \), such that \( \overline{f} \circ \tilde{\eta} \equiv \eta_f \) for every arrow \( f \) in \( \mathcal{J} \).

Let us now assume that \( M \) and \( \tilde{\eta} \) are defined for \( i \) and all predecessors of \( i \). We define

\[
\mathcal{E}(i) \equiv \left( m : \mathcal{M}^{\mathcal{E}}(i) \right) \times \left( x : B \right) \times \left( \tilde{\eta}_i(x) = m \right).
\]

(18)

There is a canonical natural transformation \( \iota : \mathcal{E} \to \mathcal{M}^{\mathcal{E}} \), given by the first projection. For \( f : i \to j \) in \( \mathcal{J} \), we can then set \( \mathcal{E}(f) \equiv \overline{f} \circ \iota_i \) and \( \mathcal{M}^{\mathcal{E}}(f) \equiv \iota_j \circ \overline{f} \). By construction, \( \eta, \tilde{\eta}, \) and \( \iota \) satisfy the required naturality conditions.

**Lemma 5.1.** For all \( i : \mathcal{J} \), the morphism \( \eta_i : B \to \mathcal{E}(i) \) is a homotopy equivalence.

**Proof.** This is due to the fact that

\[
\mathcal{E}(i) \equiv \left( m : \mathcal{M}^{\mathcal{E}}(i) \right) \times \left( x : B \right) \times \left( \tilde{\eta}_i(x) = m \right)
\]

\[
= \left( x : B \right) \times \left( m : \mathcal{M}^{\mathcal{E}}(i) \right) \times \left( \tilde{\eta}_i(x) = m \right)
\]

(19)

where the last step uses that the last two components have the form of a coconut.
The proceeding lemma tells us that $E$ is levelwise homotopy equivalent to the constant diagram. The crux is that, unlike the constant diagram, $E$ is Reedy fibrant by construction. To be precise, $E$ is the Reedy fibrant replacement of the constant diagram.

**Lemma 5.2.** For all morphisms $f$ in the category $I$, the fibration $E(f)$ is a homotopy equivalence.

**Proof.** If $f : i \to j$ is a morphism in $I$, we have $E(f) \circ \eta_i \equiv \eta_j$ due to the naturality of $\eta$. The claim then follows by 5.1 as homotopy equivalences satisfy “2-out-of-3”.

### 6 The Equality Semisimplicial Type

Let $\Delta_+$ be the category of non-zero finite ordinals and strictly increasing maps between them, written $k \rightarrow m$. We can now turn to our main case of interest, which is the tame category $J \equiv \Delta_+^{op}$. In this case, we call $E$ the equality semi-simplicial type of the (given) type $B$. We could write down the first few values of $M^E(n)$ and $E(n)$ explicitly. However, the explicit types would look rather bloated. More revealing might be the presentation in Figure 2.

![Figure 2: The “nicer” formulation of the equality semi-simplicial type](image.png)

For any $n$, the co-slice category $n/\Delta_+^{op}$ is a poset. This is a consequence of the fact that all morphisms in $\Delta_+$ are monic. Further, it is isomorphic to the poset $P_+(n)$ of nonempty subsets of the set $n \equiv \{0, 1, \ldots, n-1\}$, where we have an arrow between two subsets if the first is a superset of the second. The downwards closed full subcategories of $n/\Delta_+^{op}$ correspond to downwards closed subsets of $P_+(n)$. If $S$ is such a downwards closed subset, we write $\lim_S(E \circ U)$ by slight abuse of notation.

Any set $s \subseteq n$ generates such a downwards closed set for which we write $\pi \equiv P_+(s)$. For $k \in s$, we write $\pi_{-k}$ for the set that we get if we remove exactly two sets from $\pi$, namely $s$ itself and the set $s - k$. We call $\lim_{\pi_{-k}}(E \circ U)$ the $k$-th $n$-horn.
Lemma 6.1. For any $n \geq 2$ and $k \in n$, call the fibration from the full $n$-dimensional tetrahedron to the $k$-th $n$-horn a horn-filler fibration. All horn-filler fibrations are homotopy equivalences.

Proof. Fix $n$. We show more generally that, for any $s \subseteq n$ with cardinality $|s| \geq 2$ and $k \in s$, the fibration

$$\lim_{s}(E \circ U) \to \lim_{s\setminus\{k\}}(E \circ U)$$

is an equivalence. For the one-object downwards closed category $\{\{k\}\} \subseteq s$ we have

$$\lim_{\{\{k\}\}}(E \circ U) \cong E(1) \simeq B.$$ 

The inclusion $\{\{k\}\} \subseteq s \setminus k \subseteq s$ gives rise to a triangle

$$\lim_{s}(E \circ U) \to \lim_{s\setminus\{k\}}(E \circ U) \to \lim_{\{\{k\}\}}(E \circ U)$$

of fibrations. The top horizontal fibration is the one which we want to prove of that it is an equivalence. Using “2-out-of-3” and the fact that the left (diagonal) fibration is an equivalence by Lemma 5.2, it is sufficient to show that the right vertical fibration is an equivalence. To do this, we decompose it into $2^{|s|-1}$ fibrations, each of which can be viewed as the pullback of a smaller horn-filler fibration:

Consider the set $\mathcal{P}_s(s - k)$ of those nonempty subsets of $s$ that do not contain $k$. The number of those is $2^{|s|-1} - 1$. We label those sets as $\alpha_1, \alpha_2, \ldots, \alpha_{2^{|s|-1}-1}$, where the order is arbitrary with the only condition that their cardinality is nondecreasing, i.e. $i < j$ implies $|\alpha_i| < |\alpha_j|$.

We further define $2^{|s|-1}$ subsets of $\mathcal{P}_s(s)$, named $S_0, S_1, \ldots, S_{2^{|s|-1}}$. Define $S_0$ to be $\{\{k\}\}$. Then, define $S_i$ to be $S_{i-1}$ with two additional elements, namely $\alpha_i$ and $\alpha_i \cup \{k\}$. In this process, every element of $\mathcal{P}_s(s)$ is clearly added exactly once. In particular, $S_{2^{|s|-1}} \subseteq \mathcal{P}_s(s)$ and $S_{2^{|s|-1}} \equiv \mathcal{P}_s(s)$. Further, all $S_i$ are downwards closed, which is easily seen to be the case by induction on $i$: it is the case for $i \equiv 0$, and in general, $S_i$ contains all proper subsets of $\alpha_i \cup \{k\}$ due to the single ordering condition that we have put on the sequence $(\alpha_i)$.

It is easy to see that

$$S_i \equiv S_{i-1} \cup \alpha_i \cup \{k\}$$

(23)

$$\alpha_i \cup \{k\} \subseteq S_{i-1} \cap \alpha_i \cup \{k\}.$$ 

(24)

By Lemma 4.1, we thus have a pullback square

$$\begin{array}{ccc}
\lim_{S_i}(E \circ U) & \to & \lim_{\alpha_i \cup \{k\}}(E \circ U) \\
\downarrow & & \downarrow \\
\lim_{S_{i-1}}(E \circ U) & \to & \lim_{\alpha_i \cup \{k\} \setminus k}(E \circ U)
\end{array}$$

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For $i \leq 2^{|\mathcal{I}|-1} - 2$, the right vertical morphism is a homotopy equivalence by the induction hypothesis. As “acyclic fibrations” (fibrations that are homotopy equivalences) are stable under pullback, the left vertical morphism is one as well. As the composition of equivalences is an equivalence, we conclude that
\[
\lim_{\mathcal{I}}_{\mathcal{U}} (\mathcal{E} \circ U) \to \lim_{\mathcal{I}_1(\mathcal{I})} (\mathcal{E} \circ U)
\]
(25)
is indeed an equivalence.

\[\square\]

7 The Fibrant Diagram of Natural Transformations

Let us first formalise what we mean by the “type of natural transformations between two diagrams”. If $I$ is an inverse category and $D, E : I \to \mathcal{C}$ are Reedy fibrant diagrams, then the exponential $E^D : I \to \mathcal{C}$ exists and is Reedy fibrant \[9\] Theorem 11.11 and thus has a limit if $\mathcal{C}$ has Reedy $I$-limits. What we are interested in is the more general case that $D$ might not be fibrant, but we also do not need any exponential. (This author conjectures that the exponential $E^D$ exists and is fibrant even if only $E$ is fibrant. This would lead to an alternative representation of the same construction, but I have decided to use the less abstract one presented here as it seems to give a more direct argument.) On a more abstract level, what we want to do can be described as follows. For any downwards closed subcategory of $I$, we consider the exponential of $D$ and $E$ restricted to this subcategory, and take its limit. We basically construct approximations to the “type of natural transformations” from $D$ to $E$ which, in fact, corresponds to the limit of these approximations, should it exist. Fortunately, it is easy to do everything “by hand” on a very basic level.

We write $[I]$ for the underlying partially ordered set of $I$ that we get if we make any two parallel arrows equal (we “truncated” all hom-sets). This makes sense even if $I$ is not inverse, but if it is, then so is $[I]$. There is a canonical functor $|-| : I \to [I]$. As the objects of $I$ are the same as those of $[I]$, we omit this functor when applied to an object, i.e. for $i \in I$ we write $i \in [I]$ instead of $|i|_I \in [I]$.

Given an inverse category $I$, a diagram $D : I \to \mathcal{C}$ and a fibrant diagram $E : I \to \mathcal{C}$ (with $E(i) \equiv (m : M^D(i)) \times F^E(i, m)$) as discussed above, we define a fibrant diagram $N : [I] \to \mathcal{C}$ together with a natural transformation
\[
v : ((N \circ |-|) \times D) \to E
\]
(26)
simultaneously, where $(N \circ |-|) \times D$ is the functor $I \to \mathcal{C}$ that is given by taking the product pointwise.

Assume $i$ is an object in $I$. Assume further that we have defined both $N$ and $v$ for all predecessors of $i$ (i.e. $N$ is defined on $\{ x \in [I] \mid x \prec i \}$ and $v$ is defined on $\{ x \in I \mid x \prec i \}$). $v$ then gives rise to a map
\[
\overline{v} : \lim_{x \in [I]}_{x \prec i} N \times D(i) \to M^E(i).
\]
(27)
Note that we have $\lim_{x \in [I]}_{x \prec i} N \equiv \lim_{x \in [I]} (N \circ U) \equiv M^N(i)$ since $[I]$ is a poset. Now, define $N(i) \equiv (m : M^N(i)) \times F^N(i, m)$ by choosing the fibre over $m$ to be
\[
F^N(i, m) \equiv \Pi_{E(D(i))} F^E(i, \overline{v}(m, d)).
\]
(28)
This definition also gives a canonical morphism $v_i : N(i) \times D(i) \to E(i)$ which extends $v$.
Let us apply this construction to define the type of constant functions between types $A$ and $B$ in the way that we already suggested in Figure 1 on page 8. First, define the functor $A : \Delta^{op}_\ast \to \mathcal{C}$. For objects, it is simply given by $A(i) \equiv A^i$. If we view an element of $A^i$ as a function $i \to A$, for a map $f : j \to i$ we get $A(f) : A^i \to A^j$ by composition with $f$. We then define the functor $\mathcal{N} : [\Delta^{op} \ast] \to \mathcal{C}$ via the above construction as the “fibrant diagram of natural transformations” from $A$ to $\mathcal{E}$. Note that $[\Delta^{op} \ast]$ is isomorphic to $\omega^{op}$. Using the homotopy equivalent formulation of $\mathcal{E}$ stated in \ref{section:homotopy equivalent formulation} and the definitions of $\text{const}$ and $\text{coh}$ of Section \ref{section:coherence condition}, we get $\mathcal{N}(1) \equiv (A \to B)$ as well as $\mathcal{N}(2) \equiv (f : A \to B) \times \text{const}_f$ and $\mathcal{N}(3) \equiv (f : A \to B) \times (c : \text{const}_f) \times \text{coh}_{f,c}$. We want to stress the intuition that we think of functions with an infinite tower of coherence condition by introducing the following notation:

**Convention 7.1** ($A \overset{\omega}{\to} B$). Given types $A$ and $B$, we write $A \overset{\omega}{\to} B$ synonymously for $\lim_{[\Delta^{op}_\ast]N}$.

Our main goal in this article is to show that a constant function $A \overset{\omega}{\to} B$ corresponds exactly to a map $\|A\| \to B$, if the theory supports propositional truncation. For now, let us record that we can get a function $B \to (A \overset{\omega}{\to} B)$. In the following definition, we use the cones $\eta : B \to \mathcal{E}$ and $\bar{\eta} : B \to M\mathcal{E}$ from Section \ref{section:coherence condition}.

**Definition 7.2** (Canonical function $s : B \to (A \overset{\omega}{\to} B)$). Define a cone $\gamma : B \to \mathcal{N}$ which maps $b : B$ to the function that is “judgmentally constantly $b$”, in the following way. First, notice that the matching object $M^{\mathcal{N}}(n)$ is simply $\mathcal{N}(n - 1)$ (due to the fact that $[\Delta^{op} \ast]$ is a total order). Assume we have already defined the component $\gamma_{n-1} : B \to \mathcal{N}(n-1)$ such that $\overline{\mathcal{N}}(\gamma_{n-1}(b), x) \equiv \overline{\eta}_{n}(b)$, with $\overline{\mathcal{N}}$ as in (27), for all $x : A(n)$. We can then define $\gamma_n(b)$ by giving $F^{\mathcal{N}}(n, \gamma_{n-1}(b))$, but that expression evaluates to $\Pi_{x:A(n)} \Sigma_{x:B} \overline{\eta}_{n}(x) = \overline{\eta}_{n}(b)$. Thus, we can take $\gamma_n(b)$ to be

$$\gamma_n(b) \equiv (\gamma_{n-1}(b), \lambda z.(b, \text{refl}_{\overline{\eta}_{n-1}(b)})) .$$

(29)

It is straightforward to check that the condition $\overline{\mathcal{N}}(\gamma_n, x) \equiv \overline{\eta}_{n+1}(b)$ is preserved. Define the function $s : B \to (A \overset{\omega}{\to} B)$ to be $\lim_{[\Delta^{op}_\ast]}\gamma$.

## 8 Extending Semi-Simplicial Types

In this section, we first define the category $\mathfrak{d}_\ast$. We can then view $\mathfrak{d}_\ast^{op}$ as an extension of $\Delta^{op}_\ast$, as $\Delta^{op}_\ast$ can be embedded into $\mathfrak{d}_\ast^{op}$, and this embedding has a retraction $T$ with the property that the co-slice $c\mathfrak{d}_\ast^{op}$ is always isomorphic to $T(c)/\Delta^{op}_\ast$. With the help of this category, we can describe precisely how the components look like that are used for our “expanding and contracting” argument.

**Definition 8.1.** Let $\mathfrak{d}_\ast$ be the following category. For all pairs $(i, j) \in \omega^2$ with $1 \leq i$ and $0 \leq j \leq i$, we have an object $c^i_j$.

Given objects $c^i_1$ and $c^i_2$, we define $\mathfrak{d}_\ast(c^i_1, c^i_2)$ to be the subset of the set of maps $i_1 \to i_2$ that is given by

$$\mathfrak{d}_\ast(c^i_1, c^i_2) \equiv \{ f : i_1 \to i_2 \mid \alpha \} .$$

(30)
where the condition $\alpha$ is defined as

$$\alpha \equiv \begin{cases} 
  f(i) \equiv i \text{ for } 0 \leq i < j_1, \text{ and } f(j_1) > j_1 & \text{if } j_1 < j_2 \\
  f(i) \equiv i \text{ for } 0 \leq i < j_1 & \text{if } j_1 \equiv j_2 \\
  \bot & \text{if } j_1 > j_2.
\end{cases} \quad (31)$$

This definition implies that the cardinality of $\mathcal{D}_+ (c^{i_1}_{j_1}, c^{i_2}_{j_2})$ is

- $(i_2-j_1^{-1})$, if $j_1 < j_2$
- $(i_1-j_1)$, if $j_1 \equiv j_2$
- 0, if $j_1 > j_2$.

What will be useful for us is the opposite category. A part of it, namely the subcategory $\{ c^i_j \in \mathcal{D}^\text{op}_+ \mid i \leq 4 \}$, can be pictured as shown in Figure 3. We only draw the “generating” arrows $c^i_{j+1} \to c^i_j$.

![Figure 3: The category $\mathcal{D}^\text{op}_+$](image)

Clearly, the full subcategory of $\mathcal{D}^\text{op}_+$ which has all the objects $c^0_0$ (the leftmost column in Figure 3) is exactly the category $\Delta^\text{op}_{+}$. Put differently, there is a canonical embedding $\Delta^\text{op}_{+} \to \mathcal{D}^\text{op}_+$, and this embedding has a retraction $T : \mathcal{D}^\text{op}_+ \to \Delta^\text{op}_{+}$ with $T(c^i_j) := i$ (the action on morphisms is trivial). Given $i_1 \leq i_2$ and $j_2 \leq j_2$, any function $f : i_1 \to i_2$ either is the identity on the set $i_1 \cup j_2$, or there is a minimal $j_1 \in i_1 \cup j_2$ such that $f(j_1) > j_1$. This shows the following:

**Lemma 8.2.** For all objects $c$ in $\Delta_+$, the functor that $T$ induces between the co-slice categories $c/\Delta_+ \to T(c)/\Delta_+$ is an isomorphism of categories.

Let us extend the functor $A : \Delta^\text{op}_{+} \to \mathcal{C}$ (see Section 7) to the whole category $\mathcal{D}^\text{op}_+$. Assume a type $A$ is given. Unlike before, we now further assume that we have a specific point $a_0 : A$. We then define the functor $\bar{A} : \mathcal{D}^\text{op}_+ \to \mathcal{C}$ on objects by $\bar{A}(c^i_j) := A^{i-j}$. Note that this makes

18
sense as we have required \( j \leq i \). Given \( c^j_1 \xrightarrow{f} c^j_2 \) in \( \mathcal{A} \), we thus need to define a map \( \overline{\mathcal{A}}(f) : A^{i_2-j_2} \to A^{i_1-j_1} \). As in the definition of \( \mathcal{A} \), the map \( f : i_1 \Rightarrow i_2 \) gives rise to a map \( f : A^{i_2} \to A^{i_1} \) by composition. We define \( \overline{\mathcal{A}}(f) \) as the composite

\[
A^{i_2-j_2} \xrightarrow{\overline{a}} (a_0, a_1, \ldots, a_n, \overline{a}) \quad j_2\text{-times } a_o.
\]

\[
A^{j_2} \times A^{i_2-j_2} \xrightarrow{f} A^{i_1} \times A^{i_1-j_1} \xrightarrow{\pi_2} A^{i_1-j_1}
\]

Further, we write \( \overline{\mathcal{E}} \) for the functor \( \mathcal{E} \circ T : \mathcal{D}_+ \to \mathcal{C} \). This diagram is Reedy fibrant. With the construction of Section 7, we can define \( \overline{\mathcal{N}} : \mathcal{N} \to \mathcal{C} \) to be the “fibrant diagram of natural transformations” from \( \mathcal{A} \) to \( \overline{\mathcal{E}} \). \( \overline{\mathcal{N}} \) extends \( \mathcal{N} \) in the sense that \( \overline{\mathcal{N}}(c^j_0) = \mathcal{N}(i) \). We can picture \( \overline{\mathcal{N}} \) on the subcategory \( c^j \in [\mathcal{D}_+] \mid i \leq 3 \) as shown in Figure 4. For readability, we only write down the values of \( F \overline{\mathcal{E}} \) (i.e. the “new” part) instead of the “full” expression \( \mathcal{E}(i) \equiv (m : M\overline{\mathcal{E}}(i)) \times F\overline{\mathcal{E}}(i, m) \), and we use the homotopy equivalent representation of the values of \( \mathcal{E} \) as shown in Figure 2.

Recall that we have defined a cone \( \gamma : B \to \mathcal{N} \) and an arrow \( s : B \to \lim_{\Delta^{op}} \mathcal{N} \) in Definition 7.2. Exploiting that \( \gamma_n(b) \) was defined in a way that makes it completely independent of the “argument” \( x : \mathcal{A}(n) \), and using Lemma 8.2, we can extend \( \gamma \) to a cone \( \overline{\gamma} : B \to \overline{\mathcal{N}} \), essentially by putting \( \overline{\gamma}_n \equiv \gamma_n \). This gives a morphism

\[
\pi : B \to \lim_{[\mathcal{D}_+]^{op}} \overline{\mathcal{N}}
\]
which extends $s$, in the sense that $s$ is the composition
\[ B \xrightarrow{\sim} \lim_{[\Delta^{op}]} \hat{N} \xrightarrow{pr} \lim_{[\Delta^{op}]} N, \]  
with $pr$ coming from the embedding $[\Delta^{op}]/\approx \subset [\mathcal{D}^{op}]$ and the fact that the restriction of $\hat{N}$ to $\{c^i_j\}$ is $N$. Further, noting that $\hat{N}(c^i_1)$ is canonically equivalent to $B$ (as used in Figure 4), the composition
\[ B \xrightarrow{\sim} \lim_{[\Delta^{op}]} \hat{N} \xrightarrow{pr'} \hat{N}(c^i_1) \sim B \]  
is the identity on $B$.

9 The Main Theorem

In this section, we will be able to show our main result, namely that a constant function $A \xrightarrow{\sim} B$ gives rise to a function $\|A\| \rightarrow B$. We proceed analogously to our arguments for the special cases in Section 2: Lemma 9.1 and Corollary 9.2 show that certain fibrations are homotopy equivalences, i.e. that certain types are contractible. This is then used in Lemma 9.4 to perform the “expanding and contracting” argument, which shows that the function $s$ (from Definition 7.2) is an equivalence.

For the following statement, note that $\hat{N}(c^i_j)$ is the same as $\lim_{\{x \in [\mathcal{D}]\}^\oplus} x \equiv c^i_j - 1$.

Lemma 9.1. The fibration
\[ \hat{N}(c^i_j) \rightarrow \lim_{\{x \in [\mathcal{D}]\}^\oplus} x \equiv c^i_j - 1 \]  
is a homotopy equivalence.

Proof. There is exactly one morphism in $\mathcal{D}^{op}/\{c^i_j, c^i_{j-1}\}$. We write $c^i_j \not\in \mathcal{D}^{op} - c^i_{j-1}$, for the category $\{c^i_j, c^i_{j-1}\}$. We have a natural transformation $v : (\hat{N} \circ \mathcal{D}) \times \mathcal{A} \rightarrow \mathcal{E}$, which gives rise to a morphism
\[ w : (\lim_{\{x \in [\mathcal{D}]\}^\oplus} x \equiv c^i_j - 1) \times \mathcal{A}(c^i_j) \rightarrow \lim_{\mathcal{D}^{op} - c^i_{j-1}} \mathcal{E} \circ U. \]  
Consider the following diagram, in which the five fibrations are given as specified below, and $Q$ is defined to be the pullback.
The fibration labelled $\pi$ comes, of course, from $(c_1^j / D_+^{op} - c_{j-1}^{1-1}) \to (c_1^{j-1}/ D_+^{op})$; we give it a name solely to make referencing it easier. Our goal is to derive a representation of $Q$. As the right square is a pullback square by Lemma 4.1, we clearly must have

$$M^\mathcal{E}(c_j^1) \to (m : M^\mathcal{E}(c_{j-1}^{-1}) \times F^\mathcal{E}(c_{j-1}^{-1}, m))$$

The right part (everything without the leftmost column) of the above diagram comes from applying the functor $\lim \mathcal{E}$ to the diagram

$$c_j^1 / D_+^{op}$$

$$\downarrow$$

$$c_j^1 / D_+^{op} - c_{j-1}^{1-1}$$

$$\downarrow$$

$$c_j^1 / D_+^{op}$$

The fibration labelled $\pi$ comes from $(c_1^j / D_+^{op} - c_{j-1}^{1-1}) \to (c_1^{j-1}/ D_+^{op})$; we give it a name solely to make referencing it easier. Our goal is to derive a representation of $Q$. As the right square is a pullback square by Lemma 4.1, we clearly must have

$$M^\mathcal{E}(c_j^1) \cong (m : \lim_{c_j^1 \in \mathcal{E}_+} c_{j-1}^{1-1}) \times F^\mathcal{E}(c_{j-1}^{-1}, \pi(m)).$$

(37)

Doing so, we can write the top expression of the middle column as

$$(k : M^\mathcal{E}(c_j^1)) \times F^\mathcal{E}(c_j^1, k) \cong (m : \lim_{c_j^1 \in \mathcal{E}_+} c_{j-1}^{1-1}) \times (n : F^\mathcal{E}(c_{j-1}^{-1}, \pi(m))) \times F^\mathcal{E}(c_j^1, (m, n)).$$

(38)

The pullback $Q$ is thus

$$(p : \lim_{x \in \{p\}_+} x < c_j^1 \times c_{j-1}^{1-1}) \times (a : A(c_j^1)) \times (n : F^\mathcal{E}(c_{j-1}^{-1}, \pi(u(p, a)))) \times F^\mathcal{E}(c_j^1, (u(p, a), n)).$$

(39)

The composition of the two vertical fibrations in the middle column is a homotopy equivalence by Lemma 6.1 and Lemma 8.2. As “acyclic fibrations” are stable under pullback, the fibration $Q \to (x : \lim_{c_j^1 \in \mathcal{E}_+} c_{j-1}^{1-1}) \times (a : A(c_j^1))$ is a homotopy equivalence as well.
Function extensionality implies that a family of contractible types is contractible (i.e. that “acyclic fibrations” are preserved by $\Pi$), and we get that

\[
(p : \lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge) \times \Pi_{w, \mathcal{A}(c_j)}(n : F^E(c_{j-1}, \pi(w(p, a)))) \times F^E(c_j, (w(p, a), n))
\]

\[
\lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge \mathcal{N}
\]

is an equivalence as well. The lemma is therefore shown if we can prove that the domain of the above fibration (40), a rather lengthy expression, is equivalent to $N(c_j)$. Our first step is to apply the distributivity law (2) to transform this expression to

\[
\lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge (p : \lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge)
\times (n : \Pi_{w, \mathcal{A}(c_j)} F^E(c_{j-1}, \pi(w(p, a))))
\times (\Pi_{w, \mathcal{A}(c_j)} F^E(c_j, (w(p, a), n(a))))
\]

(41)

When we look at the following square, in which $w$ is the map (36), $w'$ is induced by the natural transformation $v$ in the same way as $w$, and $\pi, \pi'$ come from the restriction to subcategories,

\[
\begin{array}{ccc}
\lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge \mathcal{A}(c_j) & \xrightarrow{w} & \lim_{c_j \not\in \alpha_{\mathcal{A}} - c_{j-1}} E \circ U \\
\pi' \downarrow & & \downarrow \pi \\
\lim\{ x : \{\alpha]\} \mid x < c_j, x \neq c_{j-1}\} \wedge \mathcal{A}(c_{j-1}) & \xrightarrow{w'} & \lim_{c_{j-1} \not\in \alpha_{\mathcal{A}} - E \circ U}
\end{array}
\]

(42)

we can see that it commutes due to the naturality of the natural transformation $v$. In particular, note that $\mathcal{A}$ maps the single morphism $c_j \to c_{j-1}$ to the identity on $A^{\mathcal{A}} - 1$. This is exactly what is needed to see that the second line of (41) corresponds to the “missing” component $\mathcal{N}(c_j)$ in the limit of the first line. Hence, the first and the second line can be “merged” and are equivalent to $\lim\{ x : \{\alpha]\} \mid x < c_j \wedge \mathcal{N}$, in other words, $M^\mathcal{N}(c_j)$. Comparing the third line of (41) with the definition of the “fibrant diagram of natural transformations” (see (28)), we see that (41) is indeed equivalent to $\mathcal{N}(c_j)$, as required.

By pullback (Lemma 4.1 and preservation of homotopy equivalences along pullbacks), we immediately get:

**Corollary 9.2.** Let $D$ be a downwards closed subcategory of $\mathcal{A}$ which does not contain the objects $c_j$ and $c_{j-1}$, and all other predecessors of $c_j$. The full subcategory of $\mathcal{A}$ which has all the objects of $D$ and the objects $c_{j-1}, c_j$ (for which we write $D + c_{j-1} + c_j$) is also downwards closed and the fibration

\[
\lim_{D + c_{j-1}, c_j} \mathcal{N} \Rightarrow \lim_D \mathcal{N}
\]

is a homotopy equivalence.
Corollary 9.2 is the crucial statement that summarizes all of our efforts so far. We can use it to “add and remove” contractible components in the same way as we did it in the motivating examples (Section 2). More precisely, we exploit that we can group together components of $\mathfrak{d}^{op}_{\mathcal{C}}$ in two different ways, as we will explain below. However, we first record the following simple observation:

**Lemma 9.3.** Suppose
\[ F : \equiv F(1) \leftarrow F(2) \leftarrow F(3) \leftarrow \ldots \] is a diagram $F : \omega^{op} \rightarrow \mathcal{C}$. The canonical map $\lim F \rightarrow F(i)$ is a homotopy equivalence.

**Proof.** Consider the diagram that is constantly $F(i)$ apart from a finite part,
\[ G := F(1) \leftarrow F(2) \leftarrow \ldots \leftarrow F(i-1) \leftarrow F(i) \leftarrow F(i) \leftarrow F(i) \ldots. \]

There is a canonical natural transformation $F \rightarrow G$, induced by the arrows in $F$, which is levelwise an acyclic fibration (therefore a Reedy fibration). It follows directly from precise definition of Reedy limits [9, Definition 11.4] that the induced map between the limits $\lim_{\omega^{op}} F \rightarrow A_i$ is a fibration and a homotopy equivalence.

Our main lemma is the following:

**Lemma 9.4.** Given types $A, B$, recall that we have defined $s : B \rightarrow (A \xrightarrow{\sim} B)$ in Definition 7.2. Assume further that we are given a point $a_0 : A$. Then, the function $s$ is a homotopy equivalence.

**Proof.** Using the point $a_0$, we define $\hat{\mathcal{N}}$ and $\pi : B \rightarrow \lim_{[\mathfrak{d}^{op}]\hat{\mathcal{N}}}$ as before in (32), and consider the following:

\[
\begin{array}{ccc}
B & \xrightarrow{\pi} & \lim_{[\mathfrak{d}^{op}]\hat{\mathcal{N}}} \\
\downarrow & & \downarrow \text{pr'} \\
A & \xrightarrow{s} & B
\end{array}
\]

\[
\pi' \xrightarrow{\sim} \hat{\mathcal{N}}(c_1) \xrightarrow{\text{pr}} B
\]

(46)
The commutativity of the triangle on the left is given by (33). Our first goal is to show that the fibration $\text{pr}'$ is a homotopy equivalence.

Consider the set $S := \{ (a, b) \in \mathbb{N}^2 \mid b \text{ is even and } b \leq a \}$. A pair $(a, b)$ is in $S$ if and only if $c^b_a$ is an object in an “odd column” of $\mathfrak{d}^{op}_{\mathcal{C}}$ in Figure 3 on page 18 (where we consider the leftmost column the “first”). Define a total order on $S$ by letting $(a, b) < (c, d)$ if either $a + b < c + d$ or $(a + b) \equiv c + d$ and $b < d$. We represent this total order by an isomorphism $f : \mathbb{N} \rightarrow S$ (where $\mathbb{N}$ are the positive natural numbers) which has the property that $f(n)$ is always smaller than $f(n+1)$. Write $f_1(n)$ and $f_2(n)$ for the first respectively the second component of $f(n)$.

Let us define a sequence $D_0 \subset D_1 \subset D_2 \subset D_3 \subset \ldots$ of full subcategories of $[\mathfrak{d}^{op}]$ by

\[ D_0 \equiv \{ c^1_1 \} \]
\[ D_n \equiv D_{n-1} + c_{f_2(n)}^{f_1(n)} + c_{f_2(n)+1}^{f_1(n)+1}. \]

(47)
(48)

It is easy to see that every object $c^i_j$ is added exactly once, i.e. it is either $c^1_1$ or it is of the form $c_{f_2(n)}^{f_1(n)}$ or of the form $c_{f_2(n)+1}^{f_1(n)+1}$ for exactly one $n$. We have chosen the total order on
\[ S \text{ in such a way that every } D_n \text{ is a downwards closed full subcategory of } \mathcal{N}^{op}. \] Applying Corollary \text{\ref{corollary:equivalence}} we get a sequence
\[
\lim_{\Delta_n} \mathcal{N} \hookrightarrow D_1 \hookrightarrow D_2 \hookrightarrow D_3 \hookrightarrow \ldots \quad (49)
\]
of trivial fibrations. Lemma \text{\ref{lemma:acyclic}} then shows that the canonical map \( \lim_{\Delta_n} \mathcal{N} \rightarrow \lim_{\Delta_n} \mathcal{N} \) is a trivial fibration. As \( \lim_{\Delta_n} \mathcal{N} \) is simply \( \mathcal{N}(c_1^1) \), this proves that \( \mathcal{P} \mathcal{R} \) is indeed a homotopy equivalence.

Next, we want to show the same about \( \mathcal{P} \mathcal{R} \). We proceed very similarly. This time, we define \( S' \equiv \{ (a, b) \in \mathbb{N}^2 \mid b \text{ is odd and } b \leq a \} \). A pair \( (a, b) \) is consequently in \( S' \) if and only if \( c_1^i \) is an object in an “even” column of Figure 4. As before, we define an isomorphism \( f' : \mathbb{N} \rightarrow S' \), and define a sequence \( D_0 \subset D_1 \subset D_2 \subset D_3 \subset \ldots \) of full subcategories of \( \mathcal{N}^{op} \) by
\[
D_0 \equiv \{ c_1^i \} \quad \text{(i.e. the full subcategory corresponding to } \mathcal{N}(\Delta_0^{op})) \quad (50)
\]
\[
D_n \equiv D_{n-1} + c_{f_1(n)^{+}} + c_{f_2(n)^{+}}. \quad (51)
\]
Again, every object \( c_1^i \) is added exactly once, and every \( D_n \) is downwards closed. Corollary \text{\ref{corollary:equivalence}} and Lemma \text{\ref{lemma:acyclic}} then tell us that \( \lim_{\Delta_n} \mathcal{N} \rightarrow \lim_{\Delta_n} \mathcal{N} \) is an acyclic fibration. Hence, \( \mathcal{P} \mathcal{R} \) is indeed a homotopy equivalence, as claimed.

We take another look at the diagram \text{\ref{diagram:composition}}. The composition of the three horizontal arrows is the identity by \text{\ref{equation:identity}}. But homotopy equivalences satisfy “2-out-of-3”, and we can conclude that \( \mathcal{P} \mathcal{R} \) is an equivalence. Using “2-out-of-3” again, we see that \( \mathcal{P} \mathcal{R} \) is an equivalence as well.

As “being an equivalence” is a proposition, the property of the propositional truncation immediately tells us that we can replace the assumption \( A \) in Lemma \text{\ref{lemma:equivalence}} by the weaker assumption \( \| A \| \). Of course, this only makes sense if the theory that we are working in (i.e. \( \mathcal{E} \)) has propositional truncations, or at least an object \( \| A \| \). This allows us to prove our main result:

\textbf{Theorem 9.5} (General universal property of the propositional truncation). Let \( \mathcal{E} \) be a type theoretic fibration category that satisfies function extensionality and has a unit type, \( \Sigma, \Pi \), propositional truncation and Reedy \( \omega^{op} \)-limits. Let \( A \) and \( B \) be two types. In this situation, the type of coherently constant functions from \( A \) to \( B \) is equivalent to the type of functions from \( \| A \| \) to \( B \),
\[
(A \twoheadrightarrow B) \equiv (\| A \| \to B). \quad (52)
\]

\textit{Proof.} As an immediate corollary of Lemma \text{\ref{lemma:equivalence}}, we have \( \| A \| \to (A \twoheadrightarrow B) \equiv (\| A \| \to B) \). Just as in the special cases in Section \textbf{2}, we can conclude
\[
(\| A \| \to (A \twoheadrightarrow B)) \equiv (\| A \| \to B). \quad (53)
\]
This is not yet what we aim for. We need a statement corresponding to the infinite case of Lemma \text{\ref{lemma:infinite}}, i.e. we need to prove that \( \| A \| \to (A \twoheadrightarrow B) \) is equivalent to \( A \twoheadrightarrow B \). To do this, consider the diagram \( \mathcal{P} : [\Delta^{op}_n] \to \mathcal{E} \), defined by \( \mathcal{P}(i) \equiv \| A \| \to \mathcal{N}(i) \). The limit of this diagram is \( \| A \| \to \lim_{\Delta^{op}_n} \mathcal{N}, \) i.e. \( \| A \| \to (A \twoheadrightarrow B) \). Paolo Capriotti has pointed out to the author that the diagram \( \mathcal{P} \) is Reedy fibrant, and this is a crucial observation. The argument for this is that the maps in both directions which are used to prove the distributivity law \text{\ref{equation:law}} are strict inverses, i.e. their compositions (in both orders) are judgmentally equal to

\[ 24 \]
the identities. This means that the two type expressions are isomorphic as objects in \( \mathfrak{C} \), and this generalises to the “distributivity law for \( n \)-tuples”, rather than pairs. In particular, \( \| A \| \to \lim_{\Delta^{op}_n} \mathcal{N} \) is (as object in \( \mathfrak{C} \)) isomorphic to a nested \( \Sigma \)-type, and as fibrations are closed under isomorphisms, \( \mathcal{P}(i + 1) \to \mathcal{P}(i) \) is a fibration.

Because of Lemma 2.1 (and the fact that the equivalence there can be defined uniformly), there is a natural transformation between \( \mathcal{P} \) and \( \mathcal{N} \) which is levelwise a homotopy equivalence. By the definition of \( \mathfrak{C} \) having Reedy \( \omega^{op} \)-limits, the resulting arrow between the two limits is a homotopy equivalence as well, as required.

### 10 Conclusions

We have shown that, assuming the existence of Reedy \( \omega^{op} \)-limits, we can eliminate out of a propositionally truncated type and construct a function \( \| A \| \) by finding a constant function \( A \overset{\omega}{\to} B \). If the type \( B \) is an \( n \)-type for any fixed finite \( n \), then all the fibrations \( \mathcal{N}(i + 1) \to \mathcal{N}(i) \) for \( i \geq n + 2 \) become homotopy equivalences. This is obvious for the representation of \( \mathcal{N} \) and \( \mathcal{E} \) given in the figures 2 and 4, although admittedly not for our actual definition of \( \mathcal{E} \) (and the corresponding definition of \( \mathcal{N} \) and \( \mathcal{N} \)) where it requires a little more thought. The whole proof can then be carried out using only finite parts of the diagrams \( \mathcal{N} \) and \( \mathcal{N} \); to be precise, it is enough to use \( \mathcal{N} \) restricted to \( \{ i \in \Delta^{op}_n \mid i \leq n + 2 \} \), and \( \mathcal{N} \) restricted to \( \{ c^j \in \mathcal{O}^{op}_i \mid i \leq n + 2 \} \), by manipulating type-theoretic expressions as in Section 2. Analogously to the notation \( A \overset{\omega}{\to} B \), we can write \( A \overset{\omega}{\to} B \) for \( \mathcal{N}(n) \) in order to support the intuition that an element of \( \mathcal{N}(n) \) is a “function from \( A \) to \( B \) with \( n \) components”; for example, \( A \overset{2}{\to} B \) is equivalent to \( (f : A \to B) \times (\text{const}_f) \). In the finite case, the issue discussed in the remark under Theorem 9.5 vanishes, and we can derive as a corollary:

**Theorem 10.1 (Finite general universal property of the propositional truncation).** Let \( n \) be a fixed number, \(-2 \leq n < \infty\). For any type \( A \) and an \( n \)-type \( B \) in Homotopy Type Theory, we have

\[
(A \overset{n+2}{\to} B) \simeq (\| A \| \to B).
\]  

(54)

One question is whether the assumptions of Reedy \( \omega^{op} \)-limits is actually needed. It could be possible to define the expression \( \mathcal{N}(n) \) uniformly in HoTT, that is to give a function \( f_{A,B} : \mathbb{N} \to \mathcal{U} \) (where \( \mathcal{U} \) is the lowest universe) such that, for given types \( A, B \), the type \( f_{A,B}(n) \) is equivalent to \( \mathcal{N}(n) \). If the type theory supports coinduction, it should then be possible to actually construct what is intuitively an “infinite \( \Sigma \)-type”, and we could reasonably hope that Theorem 9.5 can be proved in HoTT without any further assumptions. However, we do believe that defining \( \mathcal{N} \) uniformly is impossible. This seems to be very similar to the famous open problem of defining semi-simplicial types internally, but, as far as this author can see, there is neither a reason why it should be easier nor a reason why it should be harder. This means that, while we can prove Theorem 10.1 internally if \( n \) is instantiated with any number, we conjecture that it is impossible to prove it for a variable \( n \). What we think is certainly possible is to write a program in any standard programming language that takes a number \( n \) as input and prints out the formalized statement of Theorem 10.1 (in the syntax of a proof assistant such as Coq or Agda) together with a proof.

One further question is whether statements analogous to the theorems 9.5 and 10.1 can be derived for higher truncations, not only for the propositional one. The answer seems to be positive for the finite case as in Theorem 10.1. Indeed, this author has had long discussions...
with Paolo Capriotti and we believe that the following is true, even though we have not worked out the details so far:

**Conjecture 10.2** (Capriotti and Kraus). For any two types $X, Y$, define the type family \( \text{is-}n\text{-constant} \) (indexed over the type $X \to Y$) to be the fibration $N(n+1) \to N(1)$ (using $N(1) \simeq (X \to Y)$). Given $f : A \to B$, a point $a : A$, and a number $k \geq -1$, we can consider $\text{ap}_{f,a}^{k+1} : \Omega^{k+1}(A, a) \to \Omega^{k+1}(B, f(a))$. If $B$ is a $(k+n)$-type, we claim that the function

$$
(\Sigma_{f : A \to B} H_{\text{is-}n\text{-constant}}(\text{ap}_{f,a}^{k+1})) \to (\parallel A \parallel_k \to B)
$$

is derivable in HoTT (and an equivalence for $k \geq 0$).

We think that this can be proved constructing an appropriate higher inductive type, namely the “universal” higher inductive type over which any map $f$ factors, provided that $\text{ap}_{f,a}^{k+1}$ is always $n$-constant. It uses an idea similar to the construction of the Rezk completion \cite{1}, which already implies the special case with $k \equiv 0$ and $n \equiv 1$.

However, if we ask whether the analogue of Theorem 9.5 can be proved for higher truncations, we are less optimistic. It seems to be possible under very strong assumptions on the theory (intuitively, higher inductive types with an infinite number of constructors), and it has to be examined whether these are reasonable assumption. Our guess is that the general case in which $B$ is untruncated can not be proved in standard HoTT with Reedy $\omega^{op}$-limits, even though it could be possible to use Theorem 9.5 together with a cleverly defined higher inductive type to derive such a result.

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