SOME EXAMPLES OF GENERALIZED REFLECTIONLESS SCHRÖDINGER POTENTIALS

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ABSTRACT. The class of generalized reflectionless Schrödinger operators was introduced by Lundina in 1985. Marchenko worked out a useful parametrization of these potentials, and Kotani showed that each such potential is of Sato-Segal-Wilson type. Nevertheless the dynamics under translation of a generic generalized reflectionless potential is still not well understood. We give examples which show that certain dynamical anomalies can occur.

1. Introduction. The one-dimensional spectral problem

$$\frac{d^2 \phi}{dx^2} + q(x) \phi = \lambda \phi$$

(1.1)

has been systematically studied for nearly two hundred years. The basic Sturm-Liouville theory concerning the eigenvalues and eigenfunctions of (1.1) on a finite interval $a \leq x \leq b$ was worked out in the mid-nineteenth century. Important elements of the spectral theory of (1.1) on the semi-infinite interval $[0, \infty)$ were developed by H. Weyl in a famous paper of 1910 [44]. The fact that (1.1) is of basic significance in the quantum theory of a particle moving under the effect of a potential $q(x)$ spurred new developments when (1.1) is considered on a semi-infinite subinterval of $\mathbb{R}$ or on $\mathbb{R}$ itself. Contributions by many authors have led to a quite detailed understanding of the spectral theory of the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

on $L^2(\mathbb{R})$, when $q$ decays rapidly as $|x| \to \infty$ or when $q(x)$ is a periodic function. The properties of the operators obtained when $L$ is restricted to a half-line $(-\infty, 0]$ or $[0, \infty)$ are also well-understood.

We are interested here in the Schrödinger operator $L$ when the potential $q(x)$ is continuous and bounded on $\mathbb{R}$, but oscillates in a non-periodic way as $|x| \to \infty$. The spectral theory of $L$ remains incomplete in this case, and in particular presents

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great complications as compared to the theory when \( q \) is periodic. Indeed the Bloch analysis of \( L \) when \( q \) is periodic does not even provide appropriate intuitions for developing the spectral theory when \( q(x) \) oscillates aperiodically. This fact became evident in the 1970s and 1980s, when work on random potentials and on almost periodic potentials led to the development of paradigms and tools which are now commonly accepted as being of fundamental significance. For example, it was shown that almost every realization of a random potential gives rise to an operator \( L \) which has dense point spectrum on \( L^2(\mathbb{R}) \). It was realized that almost periodic potentials tend to give rise to operators \( L \) with Cantor spectrum. Moreover, basic objects including the rotation number/integrated density of states, the Lyapunov exponent, and the Floquet exponent were introduced and exploited systematically. See, e.g., [22, 26, 1, 15, 35, 18] concerning this formative period. Generally speaking it was recognized that methods of probability theory and of topological dynamics can play a role in the study of \( L \) when \( q(x) \) is bounded and non-decaying at \( x = \pm \infty \). For example, the Lyapunov exponent and the rotation number are time-averaged quantities; also the resolvent of \( L \) can be described using the concept of exponential dichotomy/hyperbolic splitting for an associated two-dimensional linear differential system.

In this paper, we will consider the set \( \mathcal{R} = \{ q \} \) of generalized reflectionless potentials, to be defined shortly. Every element of \( \mathcal{R} \) is bounded and real-analytic. The set \( \mathcal{R} \) was introduced by Lundina in 1985 [29], who defined it as the compact uniform closure of the set of soliton potentials (see below), and proved that some very natural subsets of \( \mathcal{R} \) are compact. Somewhat later Marchenko [30] showed how to parametrize the elements in \( \mathcal{R} \) using certain Borel regular measures on \( \mathbb{R} \). Kotani [28] then proved that each \( q \in \mathcal{R} \) is of the Sato-Segal-Wilson class [38, 39]; this implies in particular that the Korteweg-de Vries equation

\[
\begin{align*}
\frac{\partial v}{\partial t} &= 6v \frac{\partial v}{\partial x} - \frac{\partial^3 v}{\partial x^3} \\
v(0, x) &= q(x)
\end{align*}
\]

admits a global solution whenever \( q \in \mathcal{R} \). This picture was rounded out in [24], where it was shown that, if a Sato-Segal-Wilson potential is appropriately translated and dilated, then it lies in \( \mathcal{R} \).

Our purpose is to investigate some interesting translation-invariant and compact subsets of \( \mathcal{R} \). They will lie inside one of the compact sets found by Lundina. Our starting point is a preceding paper [25], which was in part motivated by work of Sodin-Yuditskii [42, 43] and Gesztesy-Yuditskii [14], and which contains results related to those of Damanik-Yuditskii [7]. The papers [42, 43, 14] study potentials \( q(x) \) for which the corresponding Schrödinger operator \( L \) has a resolvent set which is a Parreau-Widom subdomain of \( \mathbb{C} [16] \). Many (but not all) of these potentials lie in \( \mathcal{R} \). The main goal in [7] was to construct Schrödinger potentials which do not satisfy the Kotani-Last conjecture. They do this in a tour-de-force which makes use of the theory of character-automorphic Hardy spaces [16]. In particular they relate the validity/non validity of the Kotani-Last conjecture to the validity/non validity of the Direct Cauchy Theorem for an appropriate Parreau-Widom domain. In [25], among other things, we also give counter examples to the Kotani-Last conjecture; we make use of dynamical methods and rely somewhat less on the Hardy-type
theory. Our counterexamples lie in \( \mathcal{R} \), and in fact the properties of \( \mathcal{R} \) are exploited in constructing them.

In any case, [25] offers ample evidence that \( \mathcal{R} \) contains a large stock of interesting examples of compact, translation-invariant sets of Schrödinger potentials. In this paper we will add to that stock of examples. We will also generalize a basic result of [25] concerning the so-called divisor map, and analyze this map in our examples. Below we will explain in a bit more detail what we propose to do. But first it is necessary to introduce some definitions and basic concepts.

First of all, a potential \( q(x) \) is said to be a soliton potential if it has the form

\[
q(x) = -2 \frac{d^2}{dx^2} \ln \det(I + A(x))
\]

where \( A(x) = (A_{ij}(x))_{1 \leq i,j \leq n} \) and

\[
A_{ij}(x) = \frac{\sqrt{L_{ij}}}{\eta_i + \eta_j} e^{-(\eta_i + \eta_j)x}
\]

for positive numbers \( t_1, \ldots, t_n; \eta_1, \ldots, \eta_n \in \mathbb{R} \). It is assumed that the \( \eta_k \) are distinct one from another. Let \( q \) be a soliton potential; then the spectrum of \( L_q = -\frac{d^2}{dx^2} + q(x) \) on \( L^2(\mathbb{R}) \) consists of \([0, \infty)\) together with finitely many negative eigenvalues \(-\eta_1^2, \ldots, -\eta_n^2\). A potential \( q(x) \) is said to be generalized reflectionless if it can be written in the form

\[
q(x) = \lim_{n \to \infty} q_n(x)
\]

where each \( q_n \) is a soliton potential, and the convergence is uniform on compact subsets of \( \mathbb{R} \). We let \( \mathcal{R} \) denote the set of all generalized reflectionless potentials. It turns out that, if \( q \in \mathcal{R} \), then the spectrum \( \Sigma(q) \) of \( L_q \) on \( L^2(\mathbb{R}) \) contains \([0, \infty)\). It further turns out that, if \( c < 0 \), then \( \mathcal{R}_c = \{ q \in \mathcal{R} \mid \text{the spectrum } \Sigma(q) \text{ is contained in } [c, \infty) \} \) is a compact subset of \( \mathcal{R} \). Moreover each \( q \in \mathcal{R}_c \) can be extended to a holomorphic function in an open horizontal strip in \( \mathbb{C} \) which contains \( \mathbb{R} \) and whose width depends only on \( c \) [29, 28]. In the rest of the paper we will choose \( c = -1 \) and write \( \mathcal{R} = \mathcal{R}_{-1} \). That is, we fix the

**Notation 1.1.** We denote by \( \mathcal{R} \) the set of all generalized reflectionless potentials such that the spectrum \( \Sigma(q) \) of \( L_q = -\frac{d^2}{dx^2} + q(x) \) on \( L^2(\mathbb{R}) \) is contained in \([-1, \infty)\).

Next let \( E \subset [-1, \infty) \) be a closed set containing \([0, \infty)\). The complement \( \Omega_E = \mathbb{C} \setminus E \) consists of the upper and lower half-planes \( \mathbb{C}_+ (\mathbb{C}_-) = \{ \lambda \in \mathbb{C} \mid \Im \lambda > (\leq) 0 \} \), together with the open interval \((-\infty, b_{-1})\) where \( b_{-1} \in [-1, 0] \), and the union of at most countably many open intervals \((a_0, b_0), (a_1, b_1), \ldots, (a_j, b_j), \ldots \subset (-1, 0) \). We will always assume that, if \( I \subset \mathbb{R} \) is an open interval, then either \( I \cap E = \emptyset \) or \( I \cap E \) has positive Lebesgue measure. Let us formulate these conditions explicitly in the

**Hypotheses 1.2 (Standing Hypotheses).** We denote by \( E \) a closed subset of \([-1, \infty) \) which contains \([0, \infty) \) and which has locally positive measure: whenever \( I \subset \mathbb{R} \) is an open interval and \( I \cap E \neq \emptyset \), the Lebesgue measure of \( E \cap I \) is positive.

Let \( E \) satisfy the conditions of Hypotheses 1.2. It is interesting to study the set \( \mathcal{Q}_E \) of bounded continuous functions \( q \) which have the property that \( \Sigma(q) = E \), and which are “reflectionless in the sense of Craig”, or simply reflectionless. To define this concept, note that the operator \( L_q - \lambda \) has a bounded inverse on \( L^2(\mathbb{R}) \) for each \( \lambda \in \Omega_E \). Let \( g(q, x, y, \lambda) \) be the Green’s function of \( q \), that is the resolvent kernel
of \((L_q - \lambda)^{-1}\). This function is defined for \(x, y \in \mathbb{R}\) and \(\lambda \in \Omega_E\), is continuous, and is holomorphic in \(\Omega_E\) for fixed values \(x, y \in \mathbb{R}\).

**Definition 1.3.** Let \(q : \mathbb{R} \to \mathbb{R}\) be a bounded continuous function, and let \(\Sigma(q)\) be the spectrum of \(L_q = -\frac{d^2}{dx^2} + q(x)\) on \(L^1(\mathbb{R})\). We say that \(q\) is reflectionless if for Lebesgue almost all \(\lambda \in \Sigma(q)\) and for all \(x \in \mathbb{R}\) one has

\[
\lim_{\varepsilon \to 0^+} \Re q(x, x, \lambda + i\varepsilon) = 0.
\]

It turns out that, if \(E\) satisfies Hypotheses 1.2, then \(Q_E \subset \mathcal{R}\); see Section 2 below. To study the potentials \(q \in Q_E\), it is convenient to introduce the Weyl \(m\)-functions \(m_\pm(q, \lambda)\). We recall some basic facts concerning these objects [3].

Let \(q : \mathbb{R} \to \mathbb{R}\) be a bounded continuous functions, and let \(\lambda\) be an element of the resolvent set \(\mathbb{C} \setminus \Sigma(q)\) of \(L_q\) on \(L^2(\mathbb{R})\). Since \(q\) is in the limit point-case at \(x = \pm \infty\), there are linearly independent solutions \(\varphi_\pm(x)\) of \(L_q \varphi = \lambda \varphi\) such that \(\varphi_\pm(x) \to 0\) exponentially as \(x \to \pm \infty\). These solutions are unique up to constant multiple. Set

\[
m_\pm(q, \lambda) = \frac{\varphi'_\pm(0)}{\varphi_\pm(0)} \quad \lambda \in \mathbb{C} \setminus \Sigma(q).
\]

It is well-known that the two functions \(m_\pm(q, \cdot)\) are holomorphic on the half-planes \(\mathbb{C}_\pm\), and moreover

\[
\text{sgn} \frac{\Im m_\pm(q, \lambda)}{\Im \lambda} = \pm 1 \quad (\Im \lambda \neq 0). \tag{1.3}
\]

Also, if \(I \subset \mathbb{R}\) is an open subinterval of \(\mathbb{C} \setminus \Sigma(q)\), then the functions \(m_\pm(q, \lambda)\) extend meromorphically through \(I\).

The condition (1.3) has a simple but useful consequence. Let \(I \subset \mathbb{R}\) be an open subinterval of \(\mathbb{C} \setminus \Sigma(q)\). Then \(m_\pm(q, \cdot)\) takes on extended real values on \(I\), and is monotone in the sense that \(\frac{dm_\pm(q, \lambda)}{d\lambda} > 0\) whenever \(\lambda \in I\) and \(m_\pm(q, \lambda) \neq \infty\). If \(m_+(q, \lambda_*) = \infty\) then one still has monotonicity in the sense that, if \(\lambda \to \lambda_*\) from the left then \(m_+(q, \lambda) \to \infty\), and if \(\lambda \to \lambda_*\) from the right, then \(m_+(q, \lambda) \to -\infty\). We can describe the situation as follows. Let \(\mathbb{R} \cup \{-\infty, \infty\}\) be the extended real line; identifying \(\pm \infty\) we obtain a topological circle \(\mathbb{P}\). Give \(\mathbb{P}\) the orientation induced by the positive orientation on \(\mathbb{R}\). Then as \(\lambda\) increases in \(I\), \(m_+(q, \lambda)\) moves on \(\mathbb{P}\) in the positive sense. Also, as \(\lambda\) increases on \(I\), \(m_-(q, \lambda)\) moves in the negative sense.

Next let \((a, b) \subset \mathbb{R}\) be a maximal interval in the resolvent set of \(L_q\). Then \(m_\pm(a) = \lim_{\lambda \to a^+} m_\pm(q, \lambda)\) and \(m_\pm(b) = \lim_{\lambda \to b^-} m_\pm(q, \lambda)\) exist as elements of \(\mathbb{P}\). We will say that \(m_\pm(q, \cdot)\) have the glueing property in \(a\) (resp. \(b\)) if \(m_-(a) = m_+(a)\) (resp. \(m_+(b) = m_-(b)\)). Of course there is no a priori reason why either these pairs of limits should be equal.

Now return to set \(E \subset \mathbb{R}\) satisfying Hypotheses 1.2, and to the corresponding set \(Q_E \subset \mathcal{R}\) of reflectionless potentials \(q\) such that \(\Sigma(q) = E\). Let us write as before \(E = [b_{-1}, \infty) \setminus \bigcup_{j=0}^{\infty} (a_j, b_j)\) where the open intervals \((a_0, b_0), \ldots, (a_j, b_j), \ldots\) are contained in \((-1, 0)\) and are pairwise disjoint. Suppose that, for each \(q \in Q_E\), the above glueing property holds for each \(a_j, b_j\) \((j \geq 0)\). In this situation we will say that \(Q_E\) has the glueing property. There exist sets \(E\) for which \(Q_E\) consists entirely of periodic functions, and in this case the glueing property holds. More generally, \(Q_E\) has the glueing property when \(E\) satisfies the “Property P” of [25];
see Section 2. We regard the glueing property to be a mark of regularity of the set of potentials \( Q_E \). One of our objectives in this paper will be to give an example of a set \( E \) which satisfies Hypotheses 1.2, for which there exists \( q \in Q_E \) such that the glueing property breaks down for \( m_\pm(q, \cdot) \), for some \( j \geq 0 \).

A second issue regarding the regularity of the set \( Q_E \) is formulated in terms of the set \( D_E \) of divisors of \( E \). To define \( D_E \), take two copies of each interval \([a_j, b_j]\) and glue them together at the endpoints to obtain a topological circle \( c_j \) 

\((j = 0, 1, \ldots)\). Then define \( D_E \) to be the Cartesian product \( c_0 \times c_1 \times \cdots = \prod_{j=0}^{\infty} c_j \),

with the Tychonov (product) topology. There is a natural way to assign a divisor \( d \in D_E \) to a potential \( q \in Q_E \), which will be discussed in more detail below: roughly speaking, \( d = \{(\mu_0, \varepsilon_0), (\mu_1, \varepsilon_1), \ldots, (\mu_j, \varepsilon_j), \ldots\} \) where \( \mu_j \in [a_j, b_j] \), \( \varepsilon_j \in \{+, -\} \), and \( m_{\varepsilon_j}(\mu_j) = \infty \). Let \( \pi: Q_E \to D_E \) be the divisor map. We will see that, if conditions 1.2 hold, then \( \pi \) is continuous and surjective. This generalizes a result of [25]; the surjectivity was discussed by Kotani in ([27], Theorem 8.2).

It is natural to ask if \( \pi \) is a homeomorphism, i.e., if it is injective. We take the injectivity of \( \pi \) to be second mark of regularity of the set of potentials \( Q_E \). A sufficient condition for the injectivity of \( \pi \) was given in [25]. To state it, recall that, if \( q: \mathbb{R} \to \mathbb{R} \) is bounded and continuous, then the differential expression \( -\frac{d^2}{dx^2} + q(x) \) together with the Dirichlet boundary condition \( \varphi(0) = 0 \) induce self-adjoint operators \( L_q^+ \) resp. \( L_q^- \) on \( L^2[0, \infty) \) resp. \( L^2(-\infty, 0] \). These half-line operators \( L_q^\pm \) can be effectively studied using the Weyl functions \( m_\pm(q, \lambda) \) as we will recall in Section 2. It turns out that, if Property P holds and if, for each \( q \in Q_E \), the half-line operators \( L_q^\pm \) admit purely absolutely continuous spectrum on \( E \), then \( \pi \) is injective.

Another of our objectives in this paper is to find sets \( E \) for which conditions 1.2 are valid and for which \( \pi \) is not injective. We will do this in the following way. We construct a set \( E \) satisfying conditions 1.2 together with an element \( q \in Q_E \) such that \( L_q^+ \) (say) has some singular spectrum, namely a non-isolated eigenvalue \( \lambda = b \). Using Marchenko’s description of \( \mathcal{R} \) in terms of Borel measures on \( \mathbb{R} \), one can then show that \( \pi^{-1}(q) \) contains more than one point. This phenomenon was predicted by Kotani ([27], Theorem 8.3). It turns out that \( \pi^{-1}(q) \) contains potentials \( \tilde{q} \) such that \( b \) is an eigenvalue of the operator \( L_{\tilde{q}} \) on \( L^2(\mathbb{R}) \). The proof that an appropriate \( E \) exists will require some elementary potential theory, together with the dynamical concepts of Lyapunov exponent and of Floquet exponent.

Let \( E \subset [-1, \infty) \) satisfy conditions 1.2. Let us consider the translation (Bebutov) flow \( \{\tau_x \mid x \in \mathbb{R}\} \) defined in \( Q_E \) by \( \tau_x(q)(\cdot) = q(\cdot + x) \). If \( \pi: Q_E \to D_E \) is a homeomorphism, then this flow is transferred to \( D_E \) in the natural way. Now suppose that \( E \) is homogeneous in the sense of Carleson [42]. In the above-mentioned papers of Sodin-Yuditskii [43] and Gesztesy-Yuditskii [14], it was shown that \( \pi \) is a homeomorphism, and that furthermore the flow on \( D_E \) is sent isomorphically via a generalized Abel map to an almost periodic flow on the character group \( \mathcal{F}_E \) of \( \Omega_E \). This has the consequence that each \( q \in Q_E \) is a Bohr almost periodic function.

The point we wish to make here is the following. In our example of a set \( E \) for which \( \pi \) is not injective, it turns out that there cannot exist a flow on \( D_E \) for which \( \pi \) is a flow homomorphism. This means that, for such examples, there can be no hope of using a generalized Abel map to gain information concerning the
recurrence properties of the elements of \( Q_E \). These results may be compared with those of Remling [36] concerning reflectionless Jacobi matrices.

The paper is organized as follows. We state basic definitions and results in Section 2. There we also review the contents of [25]. In Section 3 we study the set \( Q_E \) when \( E \) satisfies the Standing Hypotheses. We show that, in general, the divisor map \( \pi : Q_E \to D_E \) is continuous and surjective. However, we give an explicit example of a set \( E \) for which \( Q_E \) does not have the glueing property. Then, by modifying \( E \) in a simple way, we show that \( \pi \) need not be injective, and that the translation flow on \( Q_E \) need not descend to \( D_E \).

We would like to emphasize the utility and power of dynamical methods in our proofs and examples. In particular, the exponential dichotomy concept will be used to study \( \pi \), and the Lyapunov and Floquet exponents will be of aid in determining examples of sets \( E \) for which \( \pi \) is not injective. We believe that these methods can be a useful complement to others which have been used to study \( Q_E \) for “exotic” sets \( E \), e.g., the powerful theory of character-automorphic Hardy spaces [42, 43, 7].

2. Preliminaries. Let us first review some terminology from topological dynamics. Let \( \Omega \) be a topological space. A (real) flow on \( \Omega \) consists of a family \( \{ \tau_x \mid x \in \mathbb{R} \} \) of homeomorphisms of \( \Omega \) such that: (i) \( \tau_0(\omega) = \omega \) for all \( \omega \in \Omega \); (ii) \( \tau_x \circ \tau_y = \tau_{x+y} \) for all \( x, y \in \mathbb{R} \); (iii) the map \( \tau : \Omega \times \mathbb{R} \to \Omega : (\omega, x) \mapsto \tau_x(\omega) \) is continuous. A flow \( (\Omega, \{ \tau_x \}) \) is said to be minimal or Birkhoff recurrent if \( \Omega \) is a compact metric space and, for each \( \Omega \in \Omega \), the orbit \( \{ \tau_x(\omega) \mid x \in \mathbb{R} \} \) is dense in \( \Omega \). A flow \( (\Omega, \{ \tau_x \}) \) is said to be almost periodic if there is a metric \( d \) on \( \Omega \) (which is compatible with its topology) such that, for each \( x \in \mathbb{R} \) and each pair \( \omega_1, \omega_2 \in \Omega \) there holds \( d(\omega_1, \omega_2) = d(\tau_x(\omega_1), \tau_x(\omega_2)) \). An almost periodic flow with compact phase space \( \Omega \) can be expressed as the union of pairwise disjoint minimal sets [11].

Let \( \Omega \) be a compact metric space, and let \( \{ \tau_x \} \) be a flow on \( \Omega \). A regular Borel probability measure \( \nu \) on \( \Omega \) is said to be \( \{ \tau_x \} \)-invariant if for every Borel subset \( B \subset \Omega \) one has \( \nu(B) = \nu(\tau_x(B)) \) for all \( x \in \mathbb{R} \). An invariant measure \( \nu \) is said to be ergodic if in addition the condition \( \nu(\tau_x(B)\Delta B) = 0 \) \((x \in \mathbb{R})\) implies that \( \nu(B) = 0 \) or \( \nu(B) = 1 \). Here \( \Delta \) denote the symmetric difference of sets. A basic theorem of Krylov and Bogoliubov [33] states that, if \( (\Omega, \{ \tau_x \}) \) is a flow and if \( \Omega \) is a compact metric space, then there exists a \( \{ \tau_x \} \)-ergodic measure \( \nu \) on \( \Omega \).

Next let \( \mathcal{C} \) be the space of bounded continuous functions \( c : \mathbb{R} \to \mathbb{R} \), endowed with the topology of uniform convergence on compact sets. We introduce the Bebutov [2] (translation) flow on \( \mathcal{C} \) as follows: \( \tau_x(c)(\cdot) = c(\cdot + x) \) for \( c \in \mathcal{C}, x \in \mathbb{R} \). One checks that \( (\mathcal{C}, \{ \tau_x \}) \) is indeed a real flow. Let \( \mathcal{Q} \) be a compact, Bebutov-invariant subset of \( \mathcal{C} \). Examples of such sets can be determined as follows: let \( q \in \mathcal{C} \) be a uniformly continuous bounded function; then \( \mathcal{Q} = \text{cls}\{ \tau_x(q) \mid x \in \mathbb{R} \} \) is compact and translation-invariant. Of course such a \( \mathcal{Q} \) is far from being the most general compact, translation-invariant subset of \( \mathcal{C} \).

We will consider families of Schrödinger operators

\[
L_q = -\frac{d^2}{dx^2} + q(x)
\]

where \( q \) ranges over a compact translation-invariant subset \( \mathcal{Q} \) of \( \mathcal{C} \). It has proven to be fruitful to study such families of equations in questions relating to the quantum-mechanical motion of a particle in a one-dimensional medium with certain properties; e.g., randomness [15, 35] or non-periodic recurrence [32, 1]. The operator \( L_q \) acts on \( L^2(\mathbb{R}) \) with domain \( \{ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ and } \varphi' \text{ are absolutely continuous, and} \}

\[
\begin{align*}
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x) \\
0 &= -\frac{d^2}{dx^2} \varphi(x) + q(x) \varphi(x)
\end{align*}
\]
ϕ'' ∈ L^2(\mathbb{R})]; it is self-adjoint and bounded below. Also the differential expression
\[-\frac{d^2}{dx^2} + q(x)\] induces self-adjoint operators on L^2[0, ±∞) by imposing a boundary
condition at x = 0; we will always impose the Dirichlet condition ϕ(0) = 0 and call
the corresponding operators L_q^\pm.

Let λ ∈ \mathbb{C}, and consider the Schrödinger equation
\[L_q \varphi = -\varphi'' + q(x)\varphi = \lambda \varphi.\]

It is convenient to rewrite this equation as a 2-dimensional system via the usual
substitution \[u = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} .\]
If we let q vary over a compact translation-invariant subset \(Q \subset \mathbb{C}\) we obtain a family of ODEs
\[u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{pmatrix} u. \tag{2.1q}\]

Of course this family depends also on λ but we will not explicitly indicate this
dependence and will write \((2.1_q)\) instead of \((4_q, \lambda)\). Fix
λ ∈ \mathbb{C}, and let \(U(x,q)\) be the
matrix solution of \((2.1_q)\) such that
\[U(0,q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (q \in Q).\]
Then \(U\) is jointly
continuous, and satisfies the cocycle relation
\[U(x,\tau y(q))U(y,q) = U(x+y,q).\]

We recall the basic concept of exponential dichotomy (ED for short) for a family
such as \((2.1_q)\) [4].

**Definition 2.1.** The family \((2.1_q)\) has an exponential dichotomy over \(Q\) if there
exist constants κ, γ > 0 together with a continuous projection-valued function q →
P_q = P^q : \mathbb{C}^2 → \mathbb{C}^2 such that
\[|U(x,q)P_y U(y,q)^{-1}| ≤ κe^{-\gamma(x-y)}, \quad x ≥ y\]
\[|U(x,q)(I - P_y)U(y,q)^{-1}| ≤ κe^{\gamma(x-y)}, \quad x ≤ y.\]

Here |·| denotes, say, the Euclidean norm on the set of 2 × 2 complex matrices.

There is a close relation between the ED concept and the spectral theory of the op-
erators \(L_q\) and \(L_q^\pm\). To explain this point we introduce the Weyl functions
\(m^\pm(q,\lambda)\) for each \(q \in Q\). These functions were introduced and discussed in Section 1. The
following result holds [20].

**Theorem 2.2.** Let \(Q \subset \mathbb{C}\) be a compact translation-invariant set. Let \(D = \{ \lambda \in \mathbb{C} \mid\)
equations \((2.1_q)\) admit an ED over \(Q\}\}. Then \(\mathbb{C}_+ \cup \mathbb{C}_- \subset D\) and \(D\) is a subset of
the resolvent set of \(L_q\) for each \(q \in Q\). Let λ ∈ \(\mathbb{C}_+ \cup \mathbb{C}_-\), that is \(\exists \lambda \neq 0.\) Let \(P_q\) be
the dichotomy projection of \((2.1_q)\). Then
\[\text{Im } P_q = \text{Span } \left( \begin{array}{c} 1 \\ m_+(q,\lambda) \end{array} \right), \quad \text{Ker } P_q = \text{Span } \left( \begin{array}{c} 1 \\ m_-(q,\lambda) \end{array} \right).\]

The same relation holds when \(\lambda \in D \cap \mathbb{R}\) if one interprets \(\text{Span } \left( \begin{array}{c} 1 \\ \infty \end{array} \right) = \text{Span } \left( \begin{array}{c} 0 \\ 1 \end{array} \right),\)
i.e., as the “vertical line in \(\mathbb{R}^2\)”. Finally, if \(q \in Q\) has the property that the orbit
\(\{\tau_x(q) \mid x \in \mathbb{R}\}\) is dense in \(Q\), then \(D\) equals the resolvent set of \(L_q\).
At this point we can reinterpret the topological circle $\mathbb{P} = \mathbb{R} \cup \{ \pm \infty \}$ which we introduced in Section 1. Namely, if $\lambda \in \mathcal{D}$, we can regard $\text{Im} P_q$ and $\text{Ker} P_q$ as complex lines in $\mathbb{C}^2$; that is, as elements of the complex one-dimensional projective space $\mathbb{P}^1(\mathbb{C})$. We identify $m_{\pm}(q, \lambda)$ with the line which it parametrizes. Thus if $\lambda \in \mathcal{D} \cap \mathbb{R}$, then $m_{\pm}(q, \lambda)$ is identified with an element of the real projective circle $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. In this way, $\mathbb{P}$ is identified with $\mathbb{P}^1(\mathbb{R})$.

Next we return to the set of generalized reflectionless Schrödinger potentials introduced by Lundina [29]. Consider the closure $\mathcal{R}$ in $\mathcal{C}$ of the set of soliton potentials $q$ having the property that the spectrum of $L_q$ is contained in $[-1, \infty)$. Then $\mathcal{R}$ is compact; it is also connected because it is the closure of a connected set. Introduce the following set of measures:

$$
\Sigma = \left\{ \sigma \mid \sigma \text{ is a Borel regular measure on } [-1,1] \text{ and } \int_{-1}^{1} \frac{\sigma(d\zeta)}{1 - \zeta^2} \leq 1 \right\}.
$$

Give $\Sigma$ the weak-* topology on the set of Borel regular measures on $[-1,1]$. Marchenko ([30]; see also [28]) showed how to set up a homeomorphism from $\mathcal{R}$ onto $\Sigma$. We will not discuss this homeomorphism in detail, but we will write down a consequence of its construction which will be used later. For this, let $q \in \mathcal{R}$, and let $\sigma$ be its Marchenko measure. Let $m_{\pm}(q, \lambda)$ be the Weyl functions of $q$. Then $m_{\pm}(q, \cdot)$ map $\mathbb{C}_+$ to $\mathbb{C}_+$; i.e., are so called Herglotz functions. Moreover

$$
\pm m_{\pm}(q, \lambda) \sim -\sqrt{-\lambda} \quad (\lambda \to -\infty),
$$

so there are unique real constants $c_{\pm}$ and unique Borel regular measures $\sigma_{\pm}$ supported on $[-1, \infty)$ such that

$$
\pm m_{\pm}(q, \lambda) = c_{\pm} + \int_{-\infty}^{\infty} \left[ \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right] \sigma_{\pm}(dt).
$$

The fact that we will need is the following. Let $b \in (-1, 0)$ and let $\beta = \sqrt{|b|}$ be the positive square root of $b$. Suppose that $\sigma_{\pm}$ have Dirac components in $b$ of the form

$$
\alpha_{\pm} \cdot \delta_b
$$

where $\alpha_+ \geq 0$, $\alpha_- \geq 0$ and $\delta_b$ is the unit mass in $b$. Then the corresponding pure point part of $\sigma$ is

$$
\Delta \sigma = \frac{\beta}{2} \{ \alpha_+ \delta_\beta + \alpha_- \delta_\beta \}. \quad (2.2)
$$

Despite the information furnished by the Marchenko homeomorphism, not too much is known concerning the structural characteristics (e.g., recurrence properties) of a generic element of $\mathcal{R}$. In the following lines, we review some of the known facts concerning “what is in $\mathcal{R}$”.

One crucial property which is possessed by the elements of $\mathcal{R}$ is the following: the Weyl functions $m_{\pm}$ satisfy the non-reflection condition

$$
m_{\pm}(q, \lambda + i0) = \overline{m_{\mp}(q, \lambda + i0)} \quad (2.3)
$$

for Lebesgue-a.a. $\lambda \in (0, \infty)$. Here we used the very convenient notation $f(\lambda + i0) = \lim_{\varepsilon \to 0^+} f(\lambda + i\varepsilon)$ ($\lambda \in \mathbb{R}$), which is standard in the context of Herglotz functions $f : \mathbb{C}_+ \to \mathbb{C}_+$. Condition (2.3) permits one to relate the generalized reflectionless potentials to the Sato-Segal-Wilson potentials [38, 39, 28, 24]. It was actually used implicitly above, in the construction of the Marchenko homeomorphism.

The set $\mathcal{R}$ also contains certain algebro-geometric potentials, which we now describe briefly and superficially. For more details see [31, 9]. Let $-1 \leq b_{-1} <
a_0 < b_0 < \cdots < a_q < b_q \leq 0 \text{ be real numbers. Let } q \in \mathcal{C} \text{ be a function such that the spectrum } \Sigma_q \text{ of } L_q \text{ equals } [b_{-1}, a_0] \cup [b_0, a_1] \cup \cdots \cup [b_q, \infty). \text{ We suppose that the non-reflection condition (2.3) holds on each open “spectral interval” } (b_{-1}, a_0), \ldots, (b_q, \infty) \text{ of } \Sigma_q. \text{ One can then prove that the functions } m_{\pm}(q, \cdot) \text{ extend holomorphically through each open spectral interval. Moreover, for each point } \lambda_* \in \{b_{-1}, a_0, \ldots, b_q\} \text{ there is a disc } D \text{ in the (extended) complex plane which is centered at } \lambda_* \text{, such that the functions } m_{\pm}(q, \cdot) \text{ glue together to form branches of a single meromorphic function } M_*, \text{ which is defined in a ramified disc } D_* \text{ centered at } \lambda_* \text{ and lying over } D. \text{ If } z = \sqrt{\lambda - \lambda_*} \text{ resp. } z = \frac{1}{\sqrt{\lambda}} \text{ is a parameter in } D_*, \text{ then } M_* \text{ has at most a simple pole in } z = 0. \text{ See [8]. All these statements are proved using a standard extension of the Schwarz reflection principle [10] and the Riemann theorem on removable singularities. One can now use methods of classical algebraic geometry (hyperelliptic Riemann surfaces, Jacobi varieties, theta functions) to give a quite detailed description of } q(x): \text{ in particular } q(x) \text{ extends to a meromorphic function on the entire complex } x\text{-plane, and is expressible in terms of a theta-function evaluated along a rectilinear winding in a Jacobi variety. Let } Q_q = \{q \in \mathcal{C} \mid \text{ the spectrum of } L_q \text{ equals } \Sigma = [b_{-1}, a_0] \cup \cdots \cup [b_q, \infty), \text{ and the non-reflection condition (2.3) holds on each open interval } (b_{-1}, a_0), \ldots, (b_q, \infty)\}. \text{ Then } Q_q \text{ is a compact, translation-invariant subset of } \mathcal{R}. \text{ It is homeomorphic to a } (g + 1)\text{-dimensional torus. The flow } (Q_q, \{\tau_x\}) \text{ is almost periodic. These facts are now classical: see the above-mentioned references [31, 9].}

Another interesting class of Schrödinger potentials was introduced by Craig [5]. Let } q \in \mathcal{C}, \text{ and suppose that the spectrum } \Sigma_q \text{ of the full-line operator } L_q \text{ has locally positive Lebesgue measure. This means that, for each open interval } I \subset \mathbb{R}, \text{ either } I \cap \Sigma_q \text{ is empty or } I \cap \Sigma_q \text{ has positive Lebesgue measure. Let } g(q, x, y, \lambda) \text{ be the Green's function of } q, \text{ that is the kernel of the resolvent operator } (L_q - I)^{-1} \text{ for } \lambda \in \mathbb{C} \setminus \Sigma_q. \text{ We repeat the Definition 1.3:}

**Definition 2.3.** With notation as above: the potential } q \text{ is said to be reflectionless if for Lebesgue-a.a. } \lambda \in \Sigma_q \text{ and all } x \in \mathbb{R}:

\[ \Re g(q, x, x, \lambda) = 0. \text{ (2.4)} \]

It turns out that condition (2.4) can be rewritten in the form (2.3). In fact, set

\[ g(q, \lambda) = g(q, 0, 0, \lambda) \text{ (} \lambda \in \mathbb{C} \setminus \Sigma_q). \text{ It is well-known that} \]

\[ g(q, \lambda) = [m_{-}(q, \lambda) - m_{+}(q, \lambda)]^{-1} \text{ (2.5)} \]

for all } \lambda \in \mathbb{C} \setminus \Sigma_q. \text{ Using this relation one can show that}

**Proposition 2.4.** Let } q \in \mathcal{C}; \text{ then } q \text{ is reflectionless if and only if (2.3) holds for Lebesgue-a.a. } \lambda \in \Sigma_q.

For a proof of Proposition 2.4, see the Appendix of [42].

We now consider reflectionless potentials which lie in } \mathcal{R}. \text{ It will be convenient to proceed as follows. Let } E \subset [-1, \infty) \text{ be a closed set which satisfies to the Standing Hypotheses 1.2. This amounts to saying that } E \text{ has locally positive measure, and that}

\[ E = [b_{-1}, \infty) \setminus \bigcup_{j=0}^{\infty} (a_j, b_j) \quad b_{-1} \in [-1, 0) \text{ (2.6)} \]
where $-1 \leq a_j < b_j \leq 0$ ($j \geq 0$) and the open intervals $(-\infty, b_{-1}), (a_0, b_0), \ldots$ are pairwise disjoint. We state some simple facts; proofs can be found in [25] and elsewhere.

**Proposition 2.5.** Let $E$ satisfy 1.2, and let $Q_E = \{ q \in \mathbb{Q} \mid \text{the spectrum of } L_q \text{ equals } E, \text{ and } q \text{ is reflectionless} \}$. Then $Q_E$ is compact and translation-invariant over $Q_E$. Let $D = \{ \lambda \in \mathbb{C} \mid \text{the family } \{L_q\} \text{ admits an exponential dichotomy over } Q_E \}$. Then the resolvent set of $L_q$ equals $D$ for all $q \in Q_E$. Suppose that $E_1 \supset E_2 \supset \ldots \supset E_n \supset \ldots$ satisfy the Standing Hypotheses 1.2, and that $E = \bigcap_{n=1}^{\infty} E_n$ also satisfies 1.2. Let $q_n \in Q_{E_n}$; if $q_n \to q$ in $\mathbb{R}$, then $q \in Q_E$.

Suppose that $E$ satisfies conditions 1.2. Let $[a_j, b_j]$ be one of the intervals in (2.6). Let $q \in Q_E$, and let $m_{\pm}(q, \lambda)$ be the corresponding Weyl functions. In Section 1 we introduced the concept of glueing the $m$-functions in $a_j$ resp. $b_j$. In fact, we noted that $m_{+}(q, \cdot)$ and $m_{-}(q, \cdot)$ move monotonically on $\mathbb{P}$ as $\lambda$ increases from $a_j$ to $b_j$, and that moreover $m_{+}(q, \lambda) \neq m_{-}(q, \lambda)$ when $a_j < \lambda < b_j$. Hence $m_{\pm}(a_j) = \lim_{\lambda \to a_j^+} m_{\pm}(q, \lambda)$ and $m_{\pm}(b_j) = \lim_{\lambda \to b_j^-} m_{\pm}(q, \lambda)$ exist in $\mathbb{P}$. One says that $m_{\pm}(q, \lambda)$ glue at $a_j$ resp. $b_j$ if $m_{+}(a_j) = m_{-}(a_j)$ resp. $m_{+}(b_j) = m_{-}(b_j)$. One says that $Q_E$ has the glueing property if this condition holds for all $j \geq 0$, for all $q \in Q_E$.

The following condition is sufficient to guarantee that $Q_E$ has the glueing property. For each $0 \leq j \neq k$, let $\rho_{jk}$ be the distance between $[a_j, b_j]$ and $[a_k, b_k]$.

**Definition 2.6.** Say that the set $E$ has Property P if for each $k = 0, 1, \ldots$ one has

$$s_k = \sum_{j \neq k} \frac{b_j - a_j}{\rho_{jk}} < \infty.$$ 

It is not required that the sequence $\{s_k\}$ be bounded.

In the course of proving Theorem 3.3 in [25], the following result is obtained.

**Proposition 2.7.** Let $E$ satisfy the Standing Hypotheses 1.2 and also Property P. Then $Q_E$ has the glueing property.

We continue to consider a set $E$ which satisfies conditions 1.2. Let $D_E$ be the set of divisors of $E$. This space was defined in Section 1: for each $j = 0, 1, 2, \ldots$ take two copies of the closed interval $[a_j, b_j]$ and identify the corresponding endpoints to obtain a topological circle $c_j$. Define $D_E$ to be the Cartesian product $\prod_{j=0}^{\infty} c_j$ with the Tychonov topology. We denote an element $d \in D_E$ by $d = \{(\mu_0, \varepsilon_0), (\mu_1, \varepsilon_1), \ldots, (\mu_j, \varepsilon_j), \ldots\}$ where $\mu_j \in [a_j, b_j]$, $\varepsilon_j \in \{+, -\}$ and it is understood that $(a_j, +) = (a_j, -)$ and $(b_j, +) = (b_j, -)$ for each $j \geq 0$.

Let $q \in Q_E$. For each $j \geq 0$, exactly one of the following three possibilities holds. (1) There exists exactly one $\mu_j \in (a_j, b_j)$ such that $m_{+}(q, \mu_j) = \infty$ or $m_{-}(q, \mu_j) = \infty$. If $m_{+}(q, \lambda) \neq \infty$ and $m_{-}(q, \lambda) \neq \infty$ for all $\lambda \in (a_j, b_j)$, then either (2) $g(q, \lambda) > 0$ for all $\lambda \in (a_j, b_j)$, or (3) $g(q, \lambda) < 0$ for all $\lambda \in (a_j, b_j)$. This statement follows from (2.5) and some elementary reasoning.
We can now define the divisor map \( \pi : \mathcal{Q}_E \to \mathcal{D}_E \). Let \( q \in \mathcal{Q}_E \). Set \( \pi(q) = d = \{ \delta_0, \delta_1, \ldots, \delta_j, \ldots \} \) where

\[
\begin{cases}
\delta_j = (\mu_j, \varepsilon_j), & \text{if (1) holds and } m_{\varepsilon_j}(\mu_j) = \infty; \\
\delta_j = a_j, & \text{if (2) holds; } \\
\delta_j = b_j, & \text{if (3) holds.}
\end{cases}
\]

In [25] the following result is proved.

**Theorem 2.8.** Suppose that \( E \) satisfies condition 1.2 and has Property P. Then the divisor map \( \pi \) is continuous and surjective. If \( L^+_E \) have purely absolutely continuous spectrum on \( E \) for all \( q \in \mathcal{Q}_E \), then \( \pi \) is also injective.

In Section 3, we will show that the conclusions of Theorem 2.8 hold even if the hypothesis of Property P is omitted. However, if \( E \) does not have Property P, then the map \( \pi \) may have anomalous behavior. We explain this statement. Let \( q \in \mathcal{Q}_E \), and consider the map \( x \mapsto \pi(\tau_x(q)) = \{ (\mu_0(x), \varepsilon_0(x)), \ldots, (\mu_j(x), \varepsilon_j(x)) \} \) from \( \mathbb{R} \) to \( \mathcal{D}_E \). If Property P holds, then it turns out that, for each \( j \geq 0 \), the pole motion \( x \mapsto \mu_j(x) \) passes without rest through \( a_j \) or \( b_j \); that is to say, \( \{ x \in \mathbb{R} \mid \mu_j(x) = a_j \text{ or } b_j \} \) is a discrete set. This need no longer be true if Property P fails, as we will verify by example. In fact \( \{ x \in \mathbb{R} \mid \mu_j(x) = a_j \text{ or } b_j \} \) may contain non-degenerate intervals. See Section 3.

These facts shed light in the nature of the trace formula, which holds for reflectionless potentials [5, 28]:

\[
q(x) = b_{-1} + \sum_{j=0}^{\infty} [a_j + b_j - 2\mu_j(x)]. \tag{2.7}
\]

The fact that a pole \( \mu_j(x) \) can “stop” at \( a_j \) or \( b_j \) seems interesting in view of the analyticity of \( q(x) \).

We finish this section with a brief discussion of the Floquet exponent in the context of reflectionless potentials. Let \( E \subset [-1, \infty) \) satisfy the Standing Hypotheses 1.2, and let \( \nu \) be a \( \{ \tau_x \} \)-ergodic measure on \( \mathcal{Q}_E \).

**Definition 2.9.** The \( \nu \)-Floquet exponent \( w_\nu(\lambda) \) is defined for \( \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \) as follows:

\[
w_\nu(\lambda) = \int_{\mathcal{Q}_E} m_+(q, \lambda)\nu(dq).
\]

It is clear from this definition that \( w_\nu \) is holomorphic on \( \mathbb{C}_+ \cup \mathbb{C}_- \), and that

\[
\text{sgn} \frac{\Im w_\nu(\lambda)}{\Im \lambda} = \pm 1 \quad (\Im \lambda \neq 0).
\]

In fact \( w_\nu(\lambda) = \overline{w_\nu(\lambda)} \) because each function \( m_+(q, \cdot) \) has this symmetry property.

It turns out that the Floquet exponent has very useful properties. We list some of them. To begin, the boundary value

\[
w_\nu(\lambda + i0) = \lim_{\varepsilon \to 0^+} w_\nu(\lambda + i\varepsilon)
\]

exists for all \( \lambda \in \mathbb{R} \) (and not just a.a. \( \lambda \in \mathbb{R} \); see [21]). Let us write

\[
w_\nu(\lambda) = w_\nu(\lambda + i0) = \beta_\nu(\lambda) + i\alpha_\nu(\lambda) \quad (\lambda \in \mathbb{R}).
\]
Then $\beta_\nu(\lambda)$ is the $\nu$-Lyapunov exponent and $\alpha_\nu(\lambda)$ is the $\nu$-rotation number [22]. Here $\beta_\nu(\lambda)$ is defined for $\nu$-a.a. $q$ as follows:

$$\beta_\nu(\lambda) = \lim_{x \to \infty} \frac{1}{x} \ln |\mathcal{U}(x, q)|$$

where $\mathcal{U}(x, q)$ is the fundamental matrix solution of (2.1$q$). Moreover, $\alpha_\nu(\lambda)$ is defined by

$$\alpha_\nu(\lambda) = -\lim_{x \to \infty} \frac{\theta(x)}{x}$$

where (2.1$q$) is written in polar coordinates $(r, \theta)$ and a non-zero solution $u(x)$ of (2.1$q$) gives rise to a motion $x \mapsto (r(x), \theta(x))$. These limits exist in a good sense; see [22]. The rotation number $\alpha_\nu(\cdot)$ is constant on each resolvent interval $(a_j, b_j)$, $j = 0, 1, \ldots$. And further $\alpha_\nu(\lambda) = 0$ for $-\infty < \lambda \leq b_{-1}$. The Lyapunov exponent $\beta_\nu(\lambda)$ may be defined by the same formula as above for all $\lambda \in \mathbb{C}$; it then turns out that $\beta_\nu(\lambda) = \Re w_\nu(\lambda)$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. Furthermore $\beta_\nu(\cdot)$ is harmonic on all of $\Omega_E = \mathbb{C} \setminus E$, and in fact is subharmonic on all of $\mathbb{C}$ [6].

We make some further remarks concerning the Floquet exponent which will not be needed in the sequel. First, the non-reflection condition (2.3) can be used to show that $\beta_\nu$ is a (symmetric) Martin function on $\Omega_E$ [12]. This observation can be used to show that $w_\nu$ defines a conformal mapping of $\mathbb{C}_+$ onto the second quadrant minus horizontal slits with terminal points on the imaginary axis at the numbers $w = i\alpha_j$, $\alpha_j = \alpha_\nu((a_j, b_j))$, $j = 0, 1, \ldots$. The $j$-th slit has length $\max\{\beta_\nu(\lambda) \mid a_j < \lambda < b_j\}$. Finally, $w_\nu(\lambda) \sim -\sqrt{-\lambda}$ as $\lambda \to \infty$, which taken together with the previous conditions on $w_\nu$ is enough to identify it uniquely as a map on $\mathbb{C}_+$. So in fact $w_\nu$ does not depend on the choice of the ergodic measure $\nu$.

3. Results and examples. Our first goal is to study the divisor map $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ when $E$ satisfies the Standing Hypotheses 1.2. We repeat them: $E$ is a closed subset of $[-1, \infty)$ which contains $[0, \infty)$, and $E$ has locally positive measure. We write

$$E = (b_{-1}, \infty) \setminus \bigcup_{j=0}^{\infty} (a_j, b_j)$$

where $-1 \leq b_{-1} < a_j < b_j \leq 0$ for each $j = 0, 1, 2, \ldots$. The intervals $(a_j, b_j)$ are pairwise disjoint and may be finite in number. We know that the set $\mathcal{Q}_E = \{q \in \mathcal{C} \mid q$ is reflectionless, and the spectrum of $L_q = -\frac{d^2}{dx^2} + q(x)$ on $L^2(\mathbb{R})$ equals $E\}$ is contained in $\mathcal{R}$, is compact (in the topology of uniform convergence on compact subsets of $\mathbb{R}$), and is invariant with respect to the Bebutov (translation) flow on $\mathcal{C}$.

Our proof of the following result is based on the theory of exponential dichotomies [4, 37]. Kotani ([27], Theorem 8.2) treats the question of the surjectivity of $\pi$ in a different way. In [25], we proved Theorem 3.1 under the additional hypothesis of Property P. In reality only a little more care is needed to carry out the proof omitting the hypothesis of Property P.

**Theorem 3.1.** If $E$ satisfies conditions 1.2, then the divisor map $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is continuous and surjective. If the half-line operators $L_q^\pm$ have absolutely continuous spectrum on $E$ for each $q \in \mathcal{Q}_E$, then $\pi$ is injective as well.

**Proof.** We first consider the question of the continuity of $\pi$. Suppose that $q_n \to q$ in $\mathcal{Q}_E$. Let $d_n = \pi(q_n)$, $d = \pi(q)$ be the corresponding divisors, so that $d_n =
\{(\mu_{n,j},\varepsilon_{n,j}) \mid j \geq 0\} and \(d = \{(\mu_j,\varepsilon_j) \mid j \geq 0\}\). We must show that, for each \(j \geq 0\), one has \(d_{n,j} \to d_j\); that is, that \(\mu_{n,j} \to \mu_j\) and (if \(\mu_j \in (a_j,b_j)\)) that \(\varepsilon_{n,j} = \varepsilon_j\) for all sufficiently large \(n\).

For this, we use the results concerning exponential dichotomies and spectral theory which are stated in Proposition 2.5. In particular, for each \(\lambda \in \Omega_E = \mathbb{C} \setminus E\), the family of differential systems (2.1a)

\[
\begin{pmatrix}
0 \\
-\lambda + q(x) \\
0
\end{pmatrix}u
\]

admits an ED over \(Q_E\). This implies that the dichotomy projections \(P(q,\lambda)\) are continuous in \((q,\lambda) \in Q_E \times \Omega_E\). Therefore the Weyl functions \(m_{\pm}(q,\lambda)\) are also continuous on \(Q_E \times \Omega_E\), where for real values of \(\lambda\), \(m_{\pm}(q,\lambda)\) is viewed as an element of the circle \(\mathbb{P}\).

Suppose first that \(d_j = (\mu_j,\varepsilon_j)\) has the property that \(a_j < \mu_j < b_j\). Then exactly one of the quantities \(m_{\pm}(q,\mu_j)\) equals \(\infty\), and \(\varepsilon_j = \pm\) accordingly. Suppose for example that \(\varepsilon_j = +\). Choose \(\delta > 0\) such that \(a_j < \mu_j - \delta < \mu_j + \delta < b_j\). Since the quantities \(m_{\pm}(q,\lambda)\) move monotonically on the circle \(\mathbb{P}\) in opposite directions as \(\lambda\) increases in \((a_j,b_j)\), we see that \(m_+(q,\lambda) > 0\) for \(\lambda < \mu_j\), \(m_+(q,\lambda) < 0\) for \(\lambda > \mu_j\) and \(m_+(q,\mu_j) = \infty\) for \(0 < |\lambda - \mu_j| < \delta\), for small enough \(\delta\). For each constant \(c > 0\) we can choose \(\delta\) and \(N\) such that, if \(n \geq N\), then \(m_+(q_n,\lambda) \geq c\) when \(\lambda = \mu_j - \delta\) and \(m_+(q_n,\lambda) \leq -c\) when \(\lambda = \mu_j + \delta\). Using the monotone movement property of \(m_+(q_n,\cdot)\), we see that there is a unique point \(\mu_{n,j} \in (\mu_j - \delta,\mu_j + \delta)\) such that \(m_+(q_n,\mu_{n,j}) = \infty\). This means that \(\mu_{n,j} \to \mu_j\) and that \(\varepsilon_{n,j} = \varepsilon_j\) for \(n \geq N\), so indeed \((\mu_{n,j},\varepsilon_{n,j}) = d_{n,j} \to d_j = (\mu_j,\varepsilon_j)\) in the indicated situation.

It is clear that a similar conclusion holds if \(\varepsilon_j = -\). If \(\mu_j \in \{a_j,b_j\}\) we must use a bit more care because the gluing property of the \(m\)-functions need not hold. Suppose for example that \(\mu_j = b_j\). The Green’s function \(g(q,\lambda) = [m_-(q,\lambda) - m_+(q,\lambda)]^{-1}\) has the property that \(g(q,\lambda) < 0\) for all \(\lambda \in (a_j,b_j)\). It follows that \(m_-(q,\lambda) < m_+(q,\lambda)\) for all \(\lambda \in (a_j,b_j)\), and so \(\infty\) is between \(m_+(q,\lambda)\) and \(m_-(q,\lambda)\) with respect to the orientation on \(\mathbb{P}\). So \(m_-(q,\lambda)\) decreases towards \(-\infty\) and \(m_+(q,\lambda)\) increases towards \(\infty\) as \(\lambda\) increases to \(b_j\).

Let \(m_{\pm} = \lim_{\lambda \to b_j^-} m_{\pm}(q,\lambda)\) where these quantities are interpreted as extended real numbers with \(-\infty\) and \(\infty\) identified, i.e., as elements of \(\mathbb{P}\). Since \(g(q,\lambda) < 0\) on \((a_j,b_j)\), one of the following four possibilities must hold: (i) \(m_+ = m_+ = \infty\); (ii) \(m_+ = \infty, m_+ \in \mathbb{R}\); (iii) \(m_- < m_+ \in \mathbb{R}\); (iv) \(m_- \in \mathbb{R}, m_+ = \infty\). Let us consider the case (iii); the others can be handled similarly. Choose \(\delta > 0\) such that \(b_j - a_j > \delta\), and write \(c_\pm = m_\pm(q,b_j - \delta)\). Using the joint continuity of the Weyl functions, we can determine \(N\) such that, if \(n \geq N\), then \(c_- - 1 \leq m_-(q_n,b_j - \delta) < m_+(q_n,b_j - \delta) \leq c_+ + 1\). By the monotone motion property we must have that \(\mu_{n,j} \in (b_j - \delta,b_j)\) for \(n \geq N\). This proves that \(\mu_{n,j} \to \mu_j = b_j\) when (iii) holds. One now works out the other cases (i), (ii) and (iv), and concludes that \(d_{n,j} \to d_j\) when \(\mu_j = b_j\). Similar arguments can be used to prove that \(d_{n,j} \to d_j\) when \(\mu_j = a_j\). We conclude that \(\pi\) is continuous.

To prove that \(\pi\) is surjective, we use the method of algebro-geometric approximation. Let \(d = \{(\mu_0,\varepsilon_0),\ldots,(\mu_j,\varepsilon_j),\ldots\}\) be an element of \(D_E\). For each \(n = 0,1,2,\ldots\) let \(d_n \in D_E\) be the divisor \(\{(\mu_0,\varepsilon_0),\ldots,(\mu_n,\varepsilon_n),b_{n+1},b_{n+2},\ldots\}\). We associate to \(d_n\) the finite divisor \(d_n^- = \{(\mu_0,0),\ldots,(\mu_n,0)\}\). There is a unique algebro-geometric potential \(q_n\) such that the resolvent set of \(L_{q_n}\) equals
concerning exponential dichotomies \cite{37}, which implies the following statement. Let $R$

where $\pi$ immediately clear that $q$ potential $q$

We must show that $d_j = (\mu_j, \varepsilon_j)$. To do this, we use a basic robustness result concerning exponential dichotomies \cite{37}, which implies the following statement. Let $K \subset \Omega_E = \mathbb{C} \setminus E$ be a compact set; then there is a compact neighborhood $V$ of $\Omega_E$ in $R$ with the following property: if $M \subset V$ is the maximal translation-invariant subset of $V$, then equations (2.1

This shows that $d_j = (\mu_j, \varepsilon_j)$ in the indicated situation.

If $\lambda_j \in \{a_j, b_j\}$ one uses the Sacker-Sell robustness result together with the monotone motion of the Weyl functions to conclude that $\lambda_j = \mu_j$. We omit the details; the conclusion is again that $d_j$ equals the $j$-component of the divisor $d$. This shows that $\pi$ is surjective.

We now know that, if $E$ satisfies the Standing Hypotheses 1.2, then $\pi : \Omega_E \to \mathcal{D}_E$ is continuous and surjective. We want to prove that, under the additional hypothesis that $L^\pm_q$ have purely absolutely continuous (a.c.) spectrum on $E$, the map $\pi$ is also injective, hence defines a homeomorphism of $\Omega_E$ onto $\mathcal{D}_E$.

To do this, we repeat the reasoning of \cite{25, Theorem 3.3}. Fix $q \in \Omega_E$ and let $\pi(q) = d = \{(\mu_0, \varepsilon_0), \ldots, (\mu_j, \varepsilon_j), \ldots\}$ be its divisor. Let $g(q, \lambda)$ be the Green’s function of $q$. One has the following formula for the Green’s function:

$$g(q, \lambda) = \frac{i}{2\sqrt{\lambda - b_{-1}}} \prod_{j=0}^{\infty} \frac{\lambda - \mu_j}{\sqrt{(\lambda - a_j)(\lambda - b_j)}}$$ \tag{3.1}$$

where the product converges for all $\lambda \in \Omega_E$. The relation (3.1) is a particular instance of a quite general result of Gesztesy and Simon concerning their $\xi$-function \cite{13, 41}. For a reflectionless potential such as $q$, the relation (3.1) can actually be proved starting from the following formula for that branch of $\ln g(q, \lambda)$ ($\lambda \in \mathbb{C}_+$) which has imaginary part in the interval $(0, \pi)$:

$$\ln g(q, \lambda) = c + \int_{b_{-1}}^{\infty} \xi(t) dt \left[ \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right]$$

where

$$\xi(t) = \begin{cases} 1, & a_j < t \leq \mu_j \\ 0, & \mu_j < t < b_j \\ 1/2, & t \in E. \end{cases}$$
The factor in front of the product in (3.1) is determined by the asymptotic relation
\[ g(q, \lambda) \sim \frac{i}{2\sqrt{\lambda - b_{-1}}} \text{ as } \lambda \to -\infty. \]

The formula (3.1) shows that \( g(q, \lambda) \) is determined by the divisor \( d \), hence so is the difference \( m_+(q, \lambda) - m_-(q, \lambda) \) for \( \lambda \in \Omega_E \). By the non-reflection condition (2.3), which holds almost everywhere, we have
\[ 3m_+(q, \lambda + i0) = \pm \frac{1}{2} [3m_+(q, \lambda + i0) - 3m_-(q, \lambda + i0)] \]
for a.a. \( \lambda \in E \), which means that the absolutely continuous component \( \sigma_{\pm, ac}(d\lambda) \) of \( \sigma_\pm(d\lambda) \) is completely determined in \( E \) by the divisor \( d \). There remains to consider the discrete (isolated point) part of \( \sigma_\pm(d\lambda) \), which is determined by those components \((\mu_j, \varepsilon_j)\) of \( d \) with \( a_j < \mu_j < b_j \) and \( \varepsilon_j = \pm \), together with the residue of \( m_{\varepsilon_j}(q, \cdot) \) at \( \mu_j \). But this residue is determined by the formula (2.5) for \( g(q, \lambda) \). So in fact \( \sigma_{\pm, ac}(d\lambda) \) is completely determined by \( d \), which implies by the Gel'fand-Levitan theory that \( q|_{[-\infty,0]} \) and \( q|_{[0,\infty]} \) are determined by \( d \). This means that the continuous function \( q \) is uniquely determined by \( d \). This completes the proof of Theorem 3.1.

\[
\square
\]

Remarks 3.2. (a) The proof of Theorem 3.1 shows that the conclusions of Theorem 3.3 of [25] are valid without assuming Property P. On the other hand, if Property P holds, then the Weyl functions glue at \( a_j \) and \( b_j \) \((j \geq 0)\) for each \( q \in \mathcal{Q}_E \).

(b) The examples below lead one to think that Property P may be equivalent to the glueing property of the Weyl functions (for all \( a_j, b_j \) and for all \( q \in \mathcal{Q}_E \)). We offer a conjecture to this effect. They might also lead one to suspect that Property P is equivalent to the condition that \( \pi \) be injective, i.e. a homeomorphism of \( \mathcal{Q}_E \) onto \( \mathcal{D}_E \). However we make no conjecture in either sense concerning this statement.

(c) The injectivity of \( \pi \) actually depends only on the properties of each individual \( q \in \mathcal{Q}_E \). We will see below that \( \pi \) may be injective at some but not all points \( q \in \mathcal{Q}_E \).

We will now consider a class of sets \( E \) which satisfy the conditions 1.2, and for which \( \mathcal{Q}_E \) and \( \pi \) have illuminating properties. One main feature of the examples will be that the glueing property of the \( m \)-functions does not hold for at least some \( q \in \mathcal{Q}_E \). Another will be that the pole motion \( x \mapsto \pi(x(q)) = \{(\mu_j(x), \varepsilon_j(x))\} \) may have anomalous properties. And, for a certain subclass of these sets \( E \), it will turn out that \( \pi \) is not a homeomorphism.

We first forget about \( \mathcal{Q}_E, \pi, g(q, \lambda) \) etc., and concentrate on constructing a set \( E \) and an analytic function \( g_* \) on \( \Omega = \mathbb{C} \setminus E \) which have certain properties. Then, we will use these objects and slight modifications of them to work out our examples. To begin, set \( b_{-1} = -1, a_0 = -3/4, b_0 = -1/2 \). We will write \( a = a_0, b = b_0 \). In the next lines we will define intervals \((a_j, b_j) \ (j \geq 1)\), set \( E = [-1, \infty) \setminus \bigcup_{j=0}^{\infty} (a_j, b_j) \), and define the function \( g_* \).

The starting point of the construction is the following observation. Let \( m \geq 1 \) and set
\[ \lambda_m = -\frac{1}{2} - \frac{1}{4(m+1)}. \]
Then \( \lambda_m - b = \frac{1}{m} \). We see that \( \lambda_1 = -5/8 \) is the midpoint of the interval \( (a, b) = (-3/4, -1/2) \). Moreover \( \{ \lambda_m \} \) increases monotonically to \(-1/2 \) as \( m \to \infty \).

Next, choose \( m_1, m_2, \ldots, m_j, \ldots \) in the following way. Set \( m_1 = 10 - 1 = 9, m_2 = 10^3 - 1 = 999, m_3 = 10^6 - 1 = 999999, \ldots, m_j = 10^{j+2} - 1 - 1 = 10^{j+1}/2 - 1 \). Then set

\[
 I_1 = (a_1, b_1) = (b + \frac{1}{40}, b + \frac{9}{40}) \\
 I_2 = (a_2, b_2) = (b + \frac{1}{300}, b + \frac{99}{300}) \\
 I_3 = (a_3, b_3) = (b + \frac{1}{4000000}, b + \frac{999}{4000000}) \\
 \vdots \\
 I_j = (a_j, b_j) = \left(b + \frac{1}{4(m_j+1)}, b + \frac{10^j-1}{4(m_j+1)}\right)
\]

Observe that \( 0 > b_1 > a_1 > b_2 > a_2 > \cdots > b_j > a_j > \cdots \to -1/2 = b \) as \( j \to \infty \).

Set \( E = [-1, \infty) \setminus \bigcup_{j=0}^{\infty} (a_j, b_j) \) and write \( \Omega_E = \mathbb{C} \setminus E \). Then \( E \) satisfies the conditions 1.2. We abuse notation and write \( \lambda_k = \lambda_{m_k} (k \geq 1) \). The product

\[
\left| \frac{\lambda_k - b}{\lambda_k - a} \right| \prod_{j=1}^{\infty} \frac{\lambda_k - b_j}{\lambda_k - a_j} = \frac{1}{m_k} \prod_{j=1}^{\infty} \frac{b_j - \lambda_k}{a_j - \lambda_k}.
\]

Making elementary estimates we find that this product is greater than

\[
\frac{1}{2} \cdot \frac{9 - 10^{-1}}{10} \cdot \frac{99 - 10^{-2}}{100} \cdots \cdot \frac{(10^k - 1) - 10^{-k}}{10^k},
\]

which is greater than the \( k \)-independent number

\[
C = (2^{2+\sqrt{3}})^{-1} > 0.
\]

Now consider the expression

\[
g_*(\lambda) = \frac{1}{2^{2+\sqrt{3}}} \prod_{j=0}^{\infty} \sqrt[\infty]{\frac{\lambda - b_j}{\lambda - a_j}} \quad \lambda \in \Omega_E. \quad (3.2)
\]

Then \( g_* \) is analytic in \( \Omega_E \). Note that the quantity on the right-hand side of (3.1) is obtained by substituting the divisor \( d_* = (b_0, b_1, \ldots, b_j, \ldots) \) in the right-hand side of (3.2) (but remember: we must still show that \( g_*(\lambda) \) is the Green’s function of \( L_{g_*} \) for some \( g_* \in \mathcal{Q}_E \)). In any case one checks that \( g_*(\lambda) \) is monotone increasing on each \( (a_j, b_j) \), and in particular on \( (a_0, b_0) = (a, b) \). Moreover \( \lim_{\lambda \to a}^* g_*(\lambda) = -\infty \). Let us show that \( \lim_{\lambda \to b^-}^* g_*(\lambda) \) is strictly negative. In fact consider \( \lambda = \lambda_1, \lambda_2, \ldots, \lambda_k, \ldots \in (a, b) \). Then \( \lim_{k \to \infty} \lambda_k = b \). If \( j > k \) then

\[
\sqrt[\infty]{\frac{\lambda_k - b_j}{\lambda_k - a_j}} > 1,
\]

hence

\[
\prod_{j=0}^{\infty} \sqrt[\infty]{\frac{\lambda_k - b_j}{\lambda_k - a_j}} > \sqrt{C} > 0. \quad (3.3)
\]
Taking account of signs and the factor $\frac{i}{2\sqrt{\lambda+1}}$, one concludes that $g_*(\lambda_k) \leq -\sqrt{C}$, for all $k \geq 1$. Since $g_*(\lambda)$ is increasing on $(a,b)$ it has a limit as $\lambda \to b^-$, and since $\lambda_k \to b$ we must conclude that $\lim_{\lambda \to b^-} g_*(\lambda) \leq -\sqrt{C}$.

We have already noted that the set $E$ satisfies the Standing Hypotheses. Let $Q_E$ be the compact, translation-invariant subset of $\mathcal{R}$ which consists of reflectionless potentials $q$ such that $L_q$ has spectrum $E$. If $q \in Q_E$, then the Green’s function $g(q, \lambda)$ is expressed in terms of the divisor $\pi(q) = d = \{(\mu_0, \epsilon_0), \ldots, (\mu_j, \epsilon_j), \ldots\}$ of $q$:

$$g(q, \lambda) = \frac{1}{2\sqrt{\lambda+1}} \prod_{j=0}^\infty \frac{\lambda - \mu_j}{\sqrt{(\lambda-a_j)(\lambda-b_j)}}. $$

Set $d_\ast \{b_0, b_1, \ldots, b_j, \ldots\}$ and let $q_\ast \in Q_E$ be a potential such that $\pi(q_\ast) = d_\ast$. Such a $q_\ast$ exists by Theorem 3.1. By (3.3) we have

$$g(q_\ast, \lambda) \leq -\sqrt{C} \quad a < \lambda < b,$$

and of course $g_*(\lambda)$ equals $g(q_\ast, \lambda)$ for all $\lambda \in \Omega_E$.

Let $m_\pm(\lambda) = m_\pm(q_\ast, \lambda)$ be the Weyl functions of $q_\ast$. We examine their behavior as $\lambda \in (a,b)$ increases to $b$. First of all, neither function assumes the value $\infty$ on the interval $(a,b)$. Taking account of (3.4) and of the monotone motion of $m_\pm$ on $(a,b)$, we see that $m_+(b) > m_-(b)$, and moreover $\infty$ lies between $m_+(b)$ and $m_-(b)$ with respect to the orientation on $\mathbb{P}$.

So in particular, the $m$-functions of $q_\ast$ do not glue at $\lambda = b$. Thus our set $E$ satisfies one of the goals set out at the beginning of the present discussion.

Let us also note that the divisor map $x \mapsto d_x = \pi(\tau_x(q_\ast))$ has a peculiar property, which is due to the fact that the $m$-functions of $q_\ast$ do not glue in $\lambda = b$. We write $d_x = \{(\mu_0(x), \epsilon_0(x)), \ldots, (\mu_j(x), \epsilon_j(x)), \ldots\}$, and note that $\mu_0(x) = b$. Taking account of the continuity and monotonicity properties of the Weyl functions, we can find $\delta > 0$ such that, if $|x| < \delta$, then $\mu_0(x) = b$, that is, $\mu_0(\cdot)$ is constant on an open interval containing $x = 0$. Furthermore, $b > -1 = \inf E$, and so some standard theory can be used to show that the equation

$$-\varphi'' + q_\ast(x)\varphi = b\varphi$$

is oscillatory. This means that $\mu_0(x)$ leaves $b$ at some $x_\ast > 0$, only to return infinitely often as $x$ increases to $\infty$. A similar phenomenon takes place as $x \to -\infty$.

So in fact $\mu_0(x)$ has infinitely many nondegenerate intervals of constancy, both as $x \to \infty$ and as $x \to -\infty$. This behavior seems remarkable in view of the trace formula (2.7).

Let us now consider the divisor map $\pi : Q_E \to \mathcal{D}_E$ for the set $E$ which we have just constructed. Let $q \in Q_E$, $d = \pi(q)$. We want to pick out divisors $d$ such that $\pi^{-1}(d)$ is not a singleton.

Note first that the points $\lambda = -1, a_0, a_1, b_1, \ldots$ are all endpoints of nondegenerate closed intervals lying in $E$. Let $e \in \{-1, a_0, a_1, b_1, \ldots\}$. Using the non-reflection property (2.3) of the $m$-functions of $q$, one can find a disc $D \subset \mathbb{C}$ centered in $e$ such that the Weyl functions $m_+(q, \lambda)$ define branches of a single meromorphic function $M$ of the variable $\zeta = \sqrt{\lambda-e}$ defined in a disc in the $\zeta$-plane which is centered at $\zeta = 0$. The function $M$ admits at most a simple pole at $\zeta = 0$. These things can be proved using a well-known generalization of the Schwarz reflection principle [10] together with the Riemann theorem on removable singularities.
exception of a pure point (Dirac) component in $\lambda$. Multiplying $h_\sigma$ out the possibility that scribbing measures $\sigma$, we must analyze the situation in more detail.

Let $q \in Q_E$, $d = \pi(q)$. Let us write $m_\pm(\lambda) = m_\pm(q, \lambda)$, $m_\pm(b) = \lim_{\lambda \to b^-} m_\pm(\lambda)$, $\sigma_\pm(d\lambda) = \sigma_\pm(q, d\lambda)$. We also write $\sigma_\pm, d(d\lambda)$ for the discrete (isolated point) part of $\sigma_\pm$, and $\delta_b$ for the unit Dirac measure in $b$. Suppose that, e.g., $\sigma_+(d\lambda)$ has a component $r_+\delta_b$ in $b$ with $r_+ > 0$. Then

$$m_+(\lambda) = \frac{r_+}{b - \lambda} + h_+(\lambda)$$

where $h_+(\lambda)$ is determined by the sum of the absolutely continuous and isolated point parts of $\sigma_+$:

$$h_+(\lambda) = c + \int_{-\infty}^\infty [\sigma_{+}, ac(d\lambda) + \sigma_{+}, d(d\lambda)] \left[ \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right].$$

Multiplying $h_+(\lambda)$ by $b - \lambda$ and taking the limit as $\lambda \to b^-$, one finds that $\lim_{\lambda \to b^-} (b - \lambda)h_+(\lambda) = 0$, therefore $m_+(b) = \infty$.

Continuing the discussion, let $r_-\delta_b$ be the component of $\sigma_-(d\lambda)$ in $b$, where $r_- > 0$. Then

$$m_-(\lambda) = \frac{r_-}{\lambda - b} + h_-(\lambda)$$

where $h_-(\lambda)$ is determined by $\sigma_{-, ac}(d\lambda) + \sigma_{-, d}(d\lambda)$. Let us verify that, if $\tilde{r}_+ \geq 0, \tilde{r}_- \geq 0$, and $\tilde{r}_+ + \tilde{r}_- = r_+ + r_-$, then there exists $\tilde{q} \in Q_E$ such that $\pi(\tilde{q}) = d$ and such that the representing measures of $m_\pm(\tilde{q}, \lambda)$ are

$$\tilde{\sigma}_+(d\lambda) = \sigma_{+, ac}(d\lambda) + \sigma_{+, d}(d\lambda) + \tilde{r}_+\delta_b$$

$$\tilde{\sigma}_-(d\lambda) = \sigma_{-, ac}(d\lambda) + \sigma_{-, d}(d\lambda) + \tilde{r}_-\delta_b.$$ 

In fact, this follows from the Marchenko parametrization of $R$; see Section 2. In a bit more detail: let $\sigma$ be the regular Borel measure on $[-1, 1]$ which corresponds to $q$; then

$$\int_{-1}^1 \frac{\sigma(d\zeta)}{1 - \zeta^2} \leq 1. \quad (3.5)$$

Here $\sigma(d\zeta)$ is determined by $\sigma_\pm(q, d\lambda)$ and vice-versa.

The measures $\tilde{\sigma}_\pm(d\lambda)$ give rise via Marchenko’s construction to a measure $\tilde{\sigma}(d\zeta)$ on $[-1, 1]$ which still satisfies (3.5). Therefore, there exists $\tilde{q} \in R$ which corresponds to the measure $\tilde{\sigma}$, and hence gives rise via the Weyl functions $m_\pm(\tilde{q}, \lambda)$ to the representing measures $\tilde{\sigma}_\pm(d\lambda)$. By this construction, the operator $L_\tilde{q}$ has spectrum $E$, and the functions $m_\pm(\tilde{q}, \lambda)$ satisfy the non-reflection condition (2.3) because $\tilde{r}_+ + \tilde{r}_- = r_+ + r_-$. Therefore $\tilde{q} \in Q_E$, and the divisor $\pi(\tilde{q})$ equals $d$.

We see that, if $\sigma_+(q, d\lambda)$ has a pure point component in $\lambda = b$, then there is a 1-parameter family $\{q_s\}$ of elements of $Q_E$ such that $\pi(q_s) = d$, where $s$ takes values in $[0, 1]$ and

$$r_{+, s} = (1 - s)r_+ + sr_-$$

$$r_{-, s} = sr_+ + (1 - s)r_-.$$

We summarize these developments in
Remarks 3.3. 1. Let $E$ be the set constructed above. Then $\pi : Q_E \to D_E$ is continuous and surjective. If $d_\ast$ is the divisor $\{ b_0, b_1, \ldots, b_j, \ldots \}$ and $\pi(q_\ast) = d_\ast$, then the Weyl functions $m_{\pm}(q_\ast, \lambda)$ do not glue in $\lambda = b$, and moreover the pole motion $x \mapsto \mu_0(x)$ has infinitely many nontrivial intervals of constancy.

2. Let $q \in Q_E$ be a potential such that at least one of the representing measures $\sigma_{\pm}(q, d\lambda)$ has a nonzero Dirac component in $\lambda = b$. Let $\pi(q) = d$, then $\pi^{-1}(d)$ contains a homeomorph of a nondegenerate closed interval.

The reader will object that the second remark in 3.3 is as yet insignificant because we have not determined an element $q \in Q_E$ for which $\sigma_{\pm}(q, d\lambda)$ or $\sigma_{-}(q, d\lambda)$ has a Dirac component in $b$. And in fact we do not know how to find such a $q$, for this set $E$. We will show, however, that by modifying $E$ in a simple way, an element $q \in Q_E$ with the above property can be found.

To begin this discussion, let $E$ be the set constructed above. Let us study the point $b = -1/2$ from the point of view of classical potential theory. We claim that $b$ is a regular point of $E$. To see this, we use the well-known Wiener criterion; see, e.g., [17]. Let $\gamma = \frac{1}{10}$, and set

$$E_n = \{ \lambda \in \mathbb{C} \mid \gamma^{n+1} \leq |\lambda - b| \leq \gamma^n \} \quad n = 1, 2, \ldots$$

Then $E_n$ is the union of $\{ b \}$ with a countable family of pairwise disjoint closed intervals. The Wiener criterion states that $b$ is a regular point of $E$ if and only if

$$\sum_{n=1}^{\infty} - \frac{n}{\log \text{cap} E_n} = \infty.$$  \hspace{1cm} \text{(3.6)}$$

Here $\text{cap} E_n$ is the logarithmic capacity of $E_n$. Now the logarithmic capacity of an interval is one-fourth of its length. This permits one to study the quantities $\text{cap} E_n$.

It is convenient to consider the integers $n_j = 1 + 2 + \cdots + j = \frac{j(j+1)}{2}$, $j = 1, 2, \ldots$. From the construction of $E$, one sees that $E_{n_j}$ contains an interval of length $\frac{1}{4} \cdot 10^{-n_{j+1}}$. Therefore

$$\frac{n_j}{\log \text{cap} E_{n_j}} \geq \frac{n_j}{4 \log 2 + n_{j+1} \log 10},$$

and so the series in (3.6) diverges. This proves our claim.

The above analysis has produced a set $E$ which satisfies hypotheses 1.2, together with a point $b \in E$ which is regular in the sense of potential theory and an element $q_\ast$ of $Q_E$ such that the Weyl functions $m_{\pm}(q_\ast, \lambda)$ do not glue in $\lambda = b$. However, the very fact that $b$ is a regular point makes it impossible (for us) to find a $q \in Q_E$ for which at least one of the $\sigma_{\pm}(q, d\lambda)$ admits a Dirac component in $b$. We are led to alter the construction of $E$ in such a way that $b$ becomes irregular in the sense of potential theory; i.e., it does not satisfy (3.6). It is clear that this can be done by suitably enlarging the intervals $(a_1, b_1), \ldots, (a_j, b_j), \ldots$; we leave the detailed construction to the reader. It will turn out that an element $q \in Q_E$ with the desired properties can now be found.

Let us assume, then, that $E$ has been redefined in such a way that $b$ is irregular in the sense of potential theory, and $E \setminus \{ b \}$ is a union of a pairwise disjoint family of closed nondegenerate intervals. To study this situation, we introduce a $\{ \tau \}$-ergodic measure $\nu$ on $E$. The corresponding Floquet exponent $w_\nu(\lambda) = -\beta_\nu(\lambda) + i\alpha_\nu(\lambda)$. 

was introduced in Section 2; here \( \beta_\nu(\lambda) \) and \( \alpha_\nu(\lambda) \) are respectively the Lyapunov exponent and the rotation number of the family \((2.1_q)\) with respect to \( \nu \).

By construction \( E \setminus \{ b \} \) consists of a family of nondegenerate closed intervals. As we have already noted, this means that the functions \( m_\pm(q, \cdot) \) extend holomorphically through each spectral interval \((-1, -3/4), (b_1, 1), (b_2, a_1), \ldots, \) and satisfy the non-reflection condition \( m_+(q, \lambda + i0) = m_-(q, \lambda + i0) \) on each such interval \( (q \in Q_E) \). Thus together with some additional reasoning allows one to prove that \( \beta_\nu(\lambda) = 0 \) for all \( \lambda \in E \setminus \{ b \} \); see, e.g., [8]. Now, if \( \beta_\nu(b) \) were equal to zero, then \( \beta_\nu \) would define a weak harmonic barrier at \( b \), which would imply that \( b \) was a regular point of \( E \) [17]. Since this is not true, we must have \( \beta_\nu(b) > 0 \).

This last property has some interesting dynamical consequences, which are obtained by applying the Oseledets theorem [34] to the family \((2.1_q)\) with ergodic measure \( \nu \). According to this theorem, there exists a \( \{ \tau_x \}\)-invariant subset \( Q_1 \subset Q_E \) such that \( \mu(\tau_1) = 1 \) and such that, if \( q \in Q_1 \), then equation \((2.1_q)\) with \( \lambda = b \) admits linearly independent solutions \( u_\pm(x) = \left( \varphi_\pm(x) \right) \) with the property that \( u_\pm(x) \rightarrow 0 \) exponentially as \( x \rightarrow \pm \infty \). Choose \( q \in Q_1 \), and consider, say, \( \varphi_+(x) \).

Since the equation \(-\varphi''(x) + g(x)\varphi = b\varphi\) is oscillatory, there exists \( x_\ast > 0 \) such that \( \varphi_+(x_\ast) = 0 \). This means that \( \varphi_+ \) is an eigenfunction of the half-line operator \( L^+_q \), where we have written \( q_\ast = \tau_{x_\ast}(q) \). So by the standard theory of half-line operators, there is a non-zero Dirac component \( \tau_\pm \delta_b \) in the representing measure \( \sigma_+(q_\ast, d\lambda) \). Let us also remark that \( m_-(q_\ast, b) \neq 0 \) because \( \varphi_- \) is independent of \( \varphi_+ \), and therefore the Weyl functions \( m_+(q_\ast, \cdot) \) do not glue in \( \lambda = b \).

According to what we said earlier, the preimage \( I = \pi^{-1}\pi(q_\ast) \) contains a homeomorphic image of a nondegenerate closed interval. Let us observe that, if \( \tilde{q} \neq q_\ast \) is an element of \( I \), then the \( m \)-functions \( m_\pm(\tilde{q}, \lambda) \) must glue in \( \lambda = b \), because \( b \) is an eigenvalue of the full-line operator \( L^\pm_{\tilde{q}} \). In fact \( m_+(\tilde{q}, b) = m_-(\tilde{q}, b) = \infty \).

To complete the picture, we note that, if \( q \in Q_E \) and if \( b \) is an eigenvalue of neither \( L^+_{q_\ast} \) nor of \( L^-_{q_\ast} \), then \( \pi^{-1}\pi(q) = \{ q \} \). See the proof of the injectivity statement of Theorem 3.1. So we conclude that \( \pi \) is injective on \( Q_E \) with exception of a certain set \( Q_* \), which can be shown to have \( \nu \)-measure zero. The set \( Q_* \) consists of those \( q \in Q_E \) for which at least one of the operators \( L^+_q \), \( L^-_q \) admits \( b \) as an eigenvalue.

These considerations lead to the further conclusion that the flow \( \{ \tau_x \} \) on \( Q_E \) does not descend via \( \pi \) to a flow on \( D_E \); that is, there is no divisor flow in this example. In fact, let \( q_\ast \in Q_E \) be as above, so that \( m_+(q_\ast, b) = \infty \) and \( m_-(q_\ast, b) \neq \infty \). Then for small \( x \neq 0 \), \( m_\pm(\tau_x(q_\ast), b) \) are unequal real numbers. Let \( q' \in I = \pi^{-1}\pi(q_\ast) \), \( b \) be a point with \( q' \neq q_\ast \). Then for small \( x \neq 0 \), one has that \( m_+(\tau_x(q'), b) = m_-(\tau_x(q'), b) \) and the common value is not \( \infty \). This means that, for small \( x \neq 0 \), the operators \( L^\pm_{\tau_x(q_\ast)} \) have purely a.e. spectrum on \( E \). The same is true for the operators \( L^\pm_{\tau_x(q')} \).

This means that \( \pi \) is injective at \( \tau_x(q_\ast) \) and also at \( \tau_x(q') \), for small \( x \neq 0 \).

We see that, if \( q' \in I \) and \( q' \neq q_\ast \), then \( \pi(\tau_x(q')) \neq \pi(\tau_x(q_\ast)) \) for small \( x \neq 0 \). Thus the images of the orbits \( \{ \tau_x(q_\ast) \mid x \in \mathbb{R} \} \) and \( \{ \tau_x(q') \mid x \in \mathbb{R} \} \) are distinct even though these images have a point in common. This is inconsistent with the existence of a flow on \( D_E \) with respect to which \( \pi : Q_E \rightarrow D_E \) is a flow homomorphism.

We close the discussion with the following observation. When \( \Omega_E \) is a Parreau-Widom domain, one can study the translation flow on \( Q_E \) by descending to \( D_E \) via \( \pi \), then applying a generalized Abel map which takes \( D_E \) surjectively onto the group of characters \( \mathcal{J}_E \) of the domain \( \Omega_E \). The beautiful theory of this generalized Abel map is discussed in [43, 7]. It seems that, for sets \( E \) of the type we have
discussed here, one cannot hope to use any sort of generalized Abel map to study the flow on $Q_E$.

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