Real-potential-driven anti-$\mathcal{PT}$-symmetry breaking in non-Hermitian Su–Schrieffer–Heeger model

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Keywords: anti-$\mathcal{PT}$-symmetry breaking, topological phase, Su–Schrieffer–Heeger model

Abstract

Non-Hermitian Hamiltonians with anti-$\mathcal{PT}$ symmetry show distinct discrepancies compared to $\mathcal{PT}$-symmetric systems. Here we investigate the non-Hermitian Su–Schrieffer–Heeger (SSH) model and reveal the interplay between anti-$\mathcal{PT}$ symmetry and different topological phases, in which the anti-$\mathcal{PT}$ symmetry is triggered by introducing real chemical potential and imaginary nearest-neighbor hopping. Three types of anti-$\mathcal{PT}$-symmetric SSH models with different on-site configurations are investigated, and the effect of each on-site configuration on the energy spectrum of system is analyzed and discussed. We find that the type of on-site configuration affects the energy spectrum of the same topological region significantly and each on-site configuration has distinguishable impacts on the energy spectra of different topological regions. Moreover, the spontaneous anti-$\mathcal{PT}$-symmetry-breaking transitions undergone in both of the topologically trivial and nontrivial regions are also clarified. Furthermore, we address the corresponding eigenstates of some specific eigenvalues and some interesting localized phenomena are observed. Our results enrich the study of non-Hermitian topological phases and broaden the understanding of anti-$\mathcal{PT}$-symmetric systems.

1. Introduction

Since the discovery of topological insulators, the topological states of matter have attracted extensive interest in photonics [1], condensed-matter physics [2, 3], and atomic and molecular physics [4]. As a Frontier, much attention has been paid to the exploration and investigation of these exotic materials [5–7] and many researchers have devoted their efforts to hunting for them. In general, topological insulators are insulated in bulk but support electrical conducting edges or surfaces and are characterized by nonvanishing bulk topological invariants [8–12]. In traditional topological band theory, topological insulators are classified by three discrete symmetries [13–16] and their dimensionality. The extraordinary features of topological insulators are the principle of the bulk-boundary correspondence [17–20] and the existence of robust ‘in-gap’ modes that are immune to local disorder and perturbations, including defects and thermal fluctuations. Specifically, as the simplest topological insulator, the one-dimensional Su–Schrieffer–Heeger (SSH) model [21] exhibits abundant physical phenomena, e.g. topological soliton excitation, fractional charge, and nontrivial gapless zero-energy edge state [22–28]. Furthermore, the topological properties of Bloch bands for the SSH model have been measured experimentally in a dimerized optical superconducting lattice [29].

Contrary to the Hermitian operators in standard quantum mechanics [30], on the other hand, for open systems, it is commonly believed that the non-Hermitian operators possess complex eigenvalues and non-orthogonal eigenfunctions. However, in 1998, Bender and Boettcher [31] demonstrated that the non-Hermitian Hamiltonians with parity-time symmetry, where the parity-time symmetry corresponds to a combined symmetry of parity ($\mathcal{P}$) symmetry and time-reversal ($\mathcal{T}$) symmetry, namely, $[\mathcal{PT}, H] = 0$, can
hold entirely real eigenvalues. More importantly, a spontaneous $PT$-symmetry-breaking transition will occur once the parameter measuring non-Hermitian degree exceeds a definite threshold. The system in the exact $PT$-symmetric phase not only can exhibit an entirely real energy spectrum but also shares common eigenfunctions with the $PT$ operator and all the eigenfunctions satisfy the $PT$ symmetry, whereas the eigenvalues of the Hamiltonian become partially or completely complex if the system is in $PT$-symmetry-breaking phase and some or all of the eigenfunctions are neither the eigenfunctions of the $PT$ operator nor $PT$-symmetric. Motivated by this sensational discovery, various non-Hermitian Hamiltonians with $PT$ symmetry have been theoretically proposed in numerous literatures [32–38] and different $PT$-symmetric systems have been experimentally realized based on diverse platforms [39–44]. Moreover, the topological phases of $PT$-symmetric systems have been analyzed and discussed intensively [45–51] and numerous novel phenomena and applications in $PT$-symmetric topological systems have been reported, such as robust edge state [52–55], high-order topological phase [56–61], topological energy transfer [62], and so on.

Inspired by the idea of $PT$ symmetry, Ge et al proposed the concept of anti-$PT$-symmetric for the first time [63], which implies that the commutator is replaced by the anticommutator or $\{PT, H\} = 0$. As an extension of $PT$ symmetry, the anti-$PT$-symmetric systems exhibit distinct discrepancies compared with the $PT$-symmetric systems [64–66]. On a theoretical level, it turns out that the anti-$PT$-symmetric systems can undergo a spontaneous anti-$PT$-symmetry-breaking transition, which is manifested by a conversion from purely imaginary eigenvalue to complex or entirely real eigenvalues. From an experimental perspective, the $PT$ symmetry generally requires the introduction of balanced gain and loss to be established, while one prominent advantage of the anti-$PT$-symmetric systems is that they can be engineered and fabricated without any gain media, which is very attractive for the construction of non-Hermitian quantum systems. Also, the anti-$PT$-symmetric systems have been realized in optical systems [67–71], cold atoms [72], electrical circuits [73], and dissipatively coupled atomic beams [74]. Even though the subject of topological insulators and the theories of both $PT$ and anti-$PT$ symmetries have made significant progress in their respective domains, the investigation on the interplay between anti-$PT$-symmetry and different topological phases is almost absent, thus it is meaningful to address and reveal this interesting problem.

In this paper, we consider three types of anti-$PT$-symmetric SSH models with different on-site configurations and explore the interplay between anti-$PT$-symmetry and different topological phases. We mainly focus on the effect of each on-site configuration with entirely real chemical potential on the energy spectrum of system under the open boundary condition, and detailedly analyze and discuss the spontaneous anti-$PT$-symmetry-breaking transitions that both of the topologically trivial and nontrivial regions undergo. The results indicate that in both of the topologically trivial and nontrivial phases, the systems exhibit novel and distinguishable behaviors. For the first on-site configuration in which a pair of the on-site configurations and explore the interplay between anti-$PT$-symmetry and different topological phases is almost absent, thus it is meaningful to address and reveal this interesting problem.

The remainder of this paper is organized as follows. In section 2, the Hamiltonians of the three types of anti-$PT$-symmetric SSH models with different on-site configurations are presented. In section 3, the effect of each on-site configuration with entirely real chemical potential on the energy spectrum of system and the spontaneous anti-$PT$-symmetry-breaking transition in both the topologically trivial and nontrivial regions
are analyzed and discussed detailedly. Moreover, the populations of some special eigenstates are also displayed. Finally, a conclusion is given in section 4.

2. Model and Hamiltonian

We consider a non-Hermitian counterpart of the SSH model composed of even $N$ sites, which is characterized by the purely imaginary dimerized nearest-neighbor hopping, as described by the following Hamiltonian \cite{75}

$$H_h = \sum_{n \in \text{odd}} \left( T^f_1 c_{n+1}^\dagger c_n + T^b_1 c_n c_{n+1} \right) + \sum_{n \in \text{even}} \left( T^f_2 c_{n+1}^\dagger c_n + T^b_2 c_n c_{n+1} \right).$$  \hspace{1cm} (1)

Here $c_n^\dagger$ ($c_n$) is the creation (annihilation) operator of particle at site $n$, $T^f_1$ and $T^b_1$ are the forward and backward tunneling amplitudes of intracell (intercell) ($i=1$) or intercell ($i=2$), respectively, $T^f_1 = T^b_1 = iJ_1$, $T^f_2 = T^b_2 = iJ_2$ with $J_1 = J - \lambda \cos (\Phi)$ and $J_2 = J + \lambda \cos (\Phi)$ are the hopping coefficients of the conventional SSH model, where $\lambda$ is the strength of dimerization and $\Phi \in [0, 2\pi]$ is the periodic parameter. Such a model can be mapped to the binary optical waveguide array intimately and the purely imaginary coupling can be realized by incorporating assistant optical waveguides into the bare array and further resorting to adiabatic elimination \cite{64, 76}. For convenience, we set $J = 1$ as the energy unit hereafter. The system will become Hermitian if $T^f_1 = \left(T^f_2\right)^\dagger$. Here, we consider three types of on-site configurations with entirely real chemical potential, which are respectively governed as follows

$$H^i_o = \gamma \left[ c_n^\dagger c_n \right],$$  \hspace{1cm} (2a)

$$H^2_o = \gamma \left[ \sum_{n \in \text{odd}} c_n^\dagger c_n - \sum_{n \in \text{even}} c_n^\dagger c_n \right],$$  \hspace{1cm} (2b)

$$H^3_o = \gamma \left[ \sum_{n \in \left[1, \frac{N}{2}\right]} c_n^\dagger c_n - \sum_{n \in \left[\frac{N}{2}+1, N\right]} c_n^\dagger c_n \right],$$  \hspace{1cm} (2c)

where $H^i_o$ denotes that only the two ends of the lattice have the entirely real chemical potential $\gamma$ (the 1st site) and $-\gamma$ (the $N$th site), $H^2_o$ represents that the entirely real chemical potential $\gamma$ is located at odd site, whereas $-\gamma$ is located at even site, and $H^3_o$ shows that each site has the entirely real chemical potential, indicating that the entirely real chemical potential $\gamma$ is located at one half of the lattice and $-\gamma$ is located at the other half of the lattice. Therefore, the Hamiltonian of the whole system can be written as below

$$H = H_h + H^i_o \hspace{1cm} (i = 1, 2, 3),$$  \hspace{1cm} (3)

and the schematic diagrams of the three models are respectively sketched in figure 1. Furthermore, all the Hamiltonians of the three models meet a combined anti-symmetry of $P$ symmetry and $T$ symmetry, where the operations of $P$ and $T$ operators satisfy $P c_n^\dagger c_n = c_n^\dagger c_{n+1}$ and $T i T = -i$, respectively. It is evident that all the three Hamiltonians satisfy the anti-$P^T$ symmetry $\{P^T, H\} = 0$, namely, $P^T H (P^T)^{-1} = P^T H T P = -H$. Compared with the energy eigenvalues of the $P^T$-symmetric system emerging in conjugate pairs, the anti-$P^T$-symmetry ensures the energy eigenvalues in pairs with identical imaginary part and opposite real part.
In the following we explore the novel and unique properties of all the three types of anti-$\mathcal{PT}$-symmetric SSH models from the perspective of the energy spectrum, respectively, and further reveal the interplay between anti-$\mathcal{PT}$ symmetry and different topological phases.

3. Analyses and discussions

In this section, the energy spectrum of the anti-$\mathcal{PT}$-symmetric SSH model under the open boundary condition is exhibited to study the interplay between anti-$\mathcal{PT}$ symmetry and different topological phases, and the effects of the three types of on-site configurations on the real and imaginary parts of the energy spectrum in different topological phase regions are analyzed and discussed, respectively. Furthermore, we also depict some special eigenstates therein and the analytical results of the edge states in each case are detailed in appendix.

Without loss of generality, we start with the on-site configuration in the absence of entirely real chemical potential. In this case the non-Hermitian SSH model is not only anti-$\mathcal{PT}$-symmetric, but also anti-Hermitian, satisfying $H = -H^\dagger$. As detailed in references [77, 78], the anti-Hermitian topological insulators can merely hold purely imaginary energy eigenvalues and the corresponding eigenstates are all orthogonal to each other. Moreover, they can share the same topological transition points with their Hermitian counterparts. To this end, we plot the real and imaginary parts of the energy spectrum as a function of $\Phi$, as shown in figure 2(a). One can observe that when the strength of the chemical potential $\gamma = 0$, the system exhibits a purely imaginary energy spectrum and the imaginary part of the energy spectrum essentially has the identical band structure with the conventional SSH model, which indicates that the anti-$\mathcal{PT}$ symmetry of the whole system is preserved. Specifically, when $0 \leq \Phi < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \Phi \leq 2\pi$, the system is in the topologically nontrivial phase and characterized by the emergence of the twofold degenerate zero-energy edge states that are localized either on the left or on the right boundary of the system, which means that only the two edge states are dynamically stable. On the contrary, the twofold degenerate zero-energy edge states are not supported within the bulk gap in the region of $\frac{\pi}{2} < \Phi < \frac{3\pi}{2}$ and the system is thus topologically trivial and dynamically unstable. It is visible that the bulk gap closes and reopens at the topological transition points $\Phi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ in conformity to the conventional SSH model.

In order to further demonstrate the topological property of the anti-$\mathcal{PT}$-symmetric SSH model, by employing Fourier transformation $c_{n,j} = \frac{1}{\sqrt{M}} \sum_k e^{i\theta_j} c_{n,k}$ with $M$ being the total number of unit cells and $\alpha$ taking the sublattices $A$ or $B$, the Hamiltonian of the system in momentum space can be written in the form of

$$H_h(k) = \psi_h^\dagger(k) h(k) \psi_h,$$

(4)
where \( \psi_k = (c_{A,k}, c_{B,k})^T \) and
\[
h(k) = \begin{pmatrix} 0 & iJ_1 + iJ_2 e^{ik} \\ iJ_1 + iJ_2 e^{-ik} & 0 \end{pmatrix}.
\] (5)

Diagonalizing equation (5), we can get two bands
\[
E(k) = \pm i \sqrt{J_1^2 + J_2^2 + 2J_1J_2 \cos k}.
\] (6)

Both the bands are purely imaginary but the imaginary parts are consistent with the energy eigenvalues of the conventional SSH model in momentum space, viz, there is a gap between these bands as long as \( J_1 \neq J_2 \) and they touch once \( J_1 = J_2 \). When the two purely imaginary bands are separated, each band is relevant to a Zak phase defined as
\[
\Theta_n = i \int dk \langle \eta_n | \partial_k | \eta_n \rangle,
\] (7)

where \( n = 1 \ (2) \) labels the upper (lower) purely imaginary band, \( |\eta_n \rangle \) and \( |\bar{\eta}_n \rangle \) are the right and left eigenstates from \( h(k) \) and \( h^\dagger(k) \), respectively. We calculate the Zak phase for each band when \( \Phi \neq \frac{\pi}{2} \) and \( \frac{3\pi}{2} \) \( (J_1 \neq J_2) \) and \( \Theta_1 \) as a function of \( \Phi \) is shown in figure 2(b). One can observe that \( \Theta_1 = \pi \) if \( 0 \leq \Phi < \frac{\pi}{2} \) and \( \frac{3\pi}{2} < \Phi \leq 2\pi \) \( (J_1 < J_2) \) and \( \Theta_1 = 0 \) if \( \frac{\pi}{2} < \Phi < \frac{3\pi}{2} \) \( (J_1 > J_2) \). While a vanishing \( \Theta_1 \) indicates a topologically trivial phase, a nonzero \( \Theta_1 \) characterizes a topologically nontrivial one and a pair of zero-energy edge states emerge in the energy spectrum of the system. Furthermore, the profile of \( \Theta_2 \) is identical to \( \Theta_1 \).

3.1. The first on-site configuration

We first consider the on-site configuration that a pair of the positive and negative chemical potentials are located at the two ends of the lattice, respectively. The Hamiltonian of the system under this circumstance is governed by
\[
H = H_h + H_A^\dagger.
\] (8)

Figures 3 and 4 exhibit the numerically calculated energy spectra for different system parameters. The real and imaginary parts of the energy spectrum as a function of \( \Phi \) of the Zak phase for each band when \( \Phi \neq \frac{\pi}{2} \) and \( \frac{3\pi}{2} \) \( (J_1 \neq J_2) \) and \( \Theta_1 \) as a function of \( \Phi \) is shown in figure 2(b). One can observe that \( \Theta_1 = \pi \) if \( 0 \leq \Phi < \frac{\pi}{2} \) and \( \frac{3\pi}{2} < \Phi \leq 2\pi \) \( (J_1 < J_2) \) and \( \Theta_1 = 0 \) if \( \frac{\pi}{2} < \Phi < \frac{3\pi}{2} \) \( (J_1 > J_2) \). While a vanishing \( \Theta_1 \) indicates a topologically trivial phase, a nonzero \( \Theta_1 \) characterizes a topologically nontrivial one and a pair of zero-energy edge states emerge in the mixed anti-\( PT \)-symmetric and broken phases all the time, as shown in figures 3(b)–(e).

On the other hand, the topologically trivial region \( \frac{\pi}{2} < \Phi < \frac{3\pi}{2} \) shows more abundant features with the increase of \( \gamma \), which is obviously different from the topologically nontrivial regions. To be concrete, for weak \( \gamma \), the system is capable of possessing a purely imaginary energy spectrum, as shown in figure 3(a), which implies that the anti-\( PT \) symmetry of this region is unbroken. As \( \gamma \) increases, the purely imaginary energy spectrum can be maintained until \( \gamma \) exceeds a threshold \( \gamma_{c1}(\Phi) \) and four conjugated complex energy eigenvalues with the form of \( \pm a \pm ib \ (a, b > 0) \) will emerge, which indicates a transition from the exact anti-\( PT \)-symmetric phase to the mixed anti-\( PT \)-symmetric and broken phases. The transition initially occurs at \( \Phi = \pi \) with \( \gamma_{c1}(\pi) = 0.506 \) and eventually terminates at the two topological transition points with \( \gamma_{c1}(\frac{\pi}{2}) = \gamma_{c1}(\frac{3\pi}{2}) = 1 \). Definitely, the system has unbroken anti-\( PT \)-symmetric phase in this region when \( 0.506 < \gamma < 1 \) and the whole system covering both the topologically trivial and nontrivial regions is in the mixed anti-\( PT \)-symmetric and broken phases when \( \gamma = 1 \), as shown in figures 3(b) and (c).

Interestingly, a novel phenomenon arises in this region when \( \gamma > 1 \). We find another transition manifested by the bifurcation of the real parts of the four conjugated complex energy eigenvalues until \( \gamma \) is beyond a second threshold \( \gamma_{c2}(\Phi) \), corresponding to the two pairs of conjugated complex energy eigenvalues converting into four entirely real energy eigenvalues with the form of \( \pm a_1 \pm ib_1 \ (a_1, b_1 > 0) \), as shown in figure 3(d). The second transition extends from the two topological transition points to \( \Phi = \pi \) with increasing \( \gamma \) and the energy spectrum in the whole topologically trivial region is composed of four entirely real energy eigenvalues and \( N - 4 \) conjugated purely imaginary energy eigenvalues when \( \gamma > \gamma_{c2}(\pi) = 2.914 \), as shown in figure 3(e). Moreover, the whole system is in the mixed anti-\( PT \)-symmetric and broken phases.

In order to address the interplay between anti-\( PT \) symmetry and different topological phases more profoundly, figure 4 exhibits the real and imaginary parts of the energy spectrum in different topological phase regions versus \( \gamma \). The situations for the topologically nontrivial phase with \( \Phi = 0 \), topological transition points with \( \Phi = \frac{\pi}{2} \), and topologically trivial phase with \( \Phi = \pi \) are shown in figures 4(a)–(c), respectively. What one can see is that two entirely real energy eigenvalues emerge in the topologically
nontrivial regions once $\gamma \neq 0$ and their absolute values will increase with the increase of $\gamma$. Moreover, two pairs of conjugated complex energy eigenvalues in the topologically trivial region will emerge at a threshold $\gamma_{c1}$ before which the system exhibits a purely imaginary energy spectrum and is thus in the exact anti-$\mathcal{PT}$-symmetric phase. As $\gamma$ increases unceasingly, the real parts of the two pairs of conjugated complex energy eigenvalues split into four parts and their imaginary parts shrink into zero at a second threshold $\gamma_{c2}$ after which a pair of entirely real energy eigenvalues tends to zero in the limit of $\gamma \to \infty$. Additionally, when $\Phi$ approaches to the two topological transition points, $\gamma_{c1}$ increases but $\gamma_{c2}$ decreases, which reflects that the initial occurrence of the first transition is at $\Phi = \pi$, whereas the second transition ultimately ends at it indeed, and $\gamma_{c1}$ and $\gamma_{c2}$ will merge at the two topological transition points.

Furthermore, the populations of some special eigenstates belonging to different topological phase regions are displayed in figure 5. Figure 5(a) depicts the populations of eigenstates with the two entirely real energy eigenvalues $\pm a$ emerging in the topologically nontrivial regions when $\Phi = 0$ and $\gamma = 4$, which are

![Figure 3. The real and imaginary parts of energy spectrum of the anti-$\mathcal{PT}$-symmetric SSH model with the first on-site configuration as a function of $\Phi$. The strength of the chemical potential $\gamma = (a) 0.1$, (b) 0.8, (c) 1.2, (d) 2.2, and (e) 3.1, respectively. Here $N = 100$, $\lambda = 0.5$, and the magenta points label the entirely real and conjugated complex energy eigenvalues.](image-url)
Figure 4. The real and imaginary parts of energy spectra in different topological phase regions versus $\gamma$ for the anti-PT-symmetric SSH model with the first on-site configuration. Here $\Phi = (a) 0$, (b) $\pi/2$, and (c) $\pi$, respectively. The magenta points label the entirely real and conjugated complex energy eigenvalues.

symmetrically localized on the left and right boundaries of the system. Figure 5(b) shows the eigenstates of the two pairs of conjugated complex energy eigenvalues emerging in the topologically trivial region with $\Phi = \pi$ and $\gamma = 2$ and the energy eigenvalues $a \pm ib$ correspond to the left-edge-localized states with an identical population, the eigenstates for the energy eigenvalues $-a \pm ib$ are the right-edge-localized states with an identical population. Also, the eigenstates of the four entirely real energy eigenvalues $\pm a_1$ and $\pm a_2$ ($a_1 > a_2$) emerging in the topologically trivial region with $\Phi = \pi$ and $\gamma = 4$ are depicted in figures 5(c) and (d), respectively, all of which describe two pairs of symmetrically edge-localized states.

The preceding analyses and discussions elucidate that although the effects of the first on-site configuration on the topologically trivial and nontrivial regions are considerably different, the whole system is at most in the mixed anti-PT-symmetric and broken phases no matter how large $\gamma$ is.

3.2. The second on-site configuration

We next consider the on-site configuration that the positive and negative chemical potentials are staggered and the Hamiltonian of the system now reads

$$H = H_h + H_o^2.$$  \hspace{1cm} (9)

In figure 6, we show the real and imaginary parts of the energy spectrum as a function of $\Phi$. When $\gamma$ is weak, as shown in figure 6(a), both the topologically nontrivial regions and topological transition points contain two entirely real energy eigenvalues with the form of $\pm a$ ($a > 0$), the topologically trivial region exhibits a purely imaginary energy spectrum nonetheless, which is different from the first configuration. As $\gamma$ increases, the number of the entirely real energy eigenvalues at the two topological transition points begins to increase and the behavior will extend to both the topologically trivial and nontrivial regions, meanwhile, the number of the purely imaginary energy eigenvalues reduces accordingly and the gaps in the imaginary part of the energy spectrum gradually shrink to vanish, as shown in figures 6(b)–(d). As $\gamma$ continues to increase until $\gamma \geq 2$, the whole system can exhibit an entirely real energy spectrum, as shown in figures 6(e) and (f).
Figure 5. The populations of eigenstates for (a) the two entirely real energy eigenvalues $\pm a$ in the topologically nontrivial regions with $\Phi = 0$ and $\gamma = 4$, (b) the two pairs of conjugated complex energy eigenvalues $a \pm ib$ (left) and $-a \pm ib$ (right) in the topologically trivial region with $\Phi = \pi$ and $\gamma = 2$, and the four entirely real energy eigenvalues (c) $\pm a_1$ and (d) $\pm a_2$ ($a_1 > a_2$) in the topologically trivial region with $\Phi = \pi$ and $\gamma = 4$.

We also plot the real and imaginary parts of the energy spectrum in different topological phase regions versus $\gamma$ by respectively choosing $\Phi = 0, \pi/2, \pi$, as shown in figure 7. Figure 7(a) indicates that two entirely real energy eigenvalues emerge in the topologically nontrivial regions for weak $\gamma$ and their corresponding eigenstates with $\gamma = 1$ are depicted in figure 8, which also captures the states symmetrically localized on the left and right boundaries of the system. It is clear that with the increase of $\gamma$, more and more purely imaginary energy eigenvalues convert into entirely real energy eigenvalues, thus the moduli of the purely imaginary energy eigenvalues decrease, whereas the absolute values of the entirely real energy eigenvalues increase. Furthermore, the energy spectrum of the system at the topological transition points are immediately accompanied by the emergence of large-scale entirely real energy eigenvalues, as shown in figure 7(b). Subsequently, the behavior occurs in both the topologically trivial and nontrivial regions, as shown in figures 7(a) and (c). All the purely imaginary energy eigenvalues become entirely real when $\gamma \geq 2$ and there is only an entirely real energy spectrum left.

The second on-site configuration has a similar effect for both of the topologically trivial and nontrivial regions and can induce a transition from the mixed anti-$PT$-symmetric and broken phases to the fully anti-$PT$-broken phase. Consequently, the system holds two different phases in the topologically nontrivial regions, i.e. the mixed anti-$PT$-symmetric and broken phases and the fully anti-$PT$-broken phase, while the system can undergo a transition from the exact anti-$PT$-symmetric phase to the mixed anti-$PT$-symmetric and broken phases further to the fully anti-$PT$-broken phase in the topologically trivial region. In brief, the whole system can be in the fully anti-$PT$-broken phase with the increase of $\gamma$. 
3.3. The third on-site configuration

We now turn to the on-site configuration that the positive and negative chemical potentials are separately distributed on the half of the lattice, the Hamiltonian of the system in this case can be written as

$$H = H_h + H_o^3.$$  \hfill (10)
Figure 7. The real and imaginary parts of the energy spectra in different topological phase regions versus $\gamma$ for the anti-PT-symmetric SSH model with the second on-site configuration. Here $\Phi = (a) \ 0$, (b) $\frac{\pi}{2}$, and (c) $\pi$, respectively. The magenta points label the entirely real energy eigenvalues.

Figure 8. The populations of the eigenstates for the two entirely real energy eigenvalues $\pm a$ in the topologically nontrivial regions with $\Phi = 0$ and $\gamma = 1$.

Analogous to the first two configurations, as $\gamma$ increases, we present the real and imaginary parts of the energy spectrum as a function of $\Phi$ in figure 9. It now turns out that the energy spectrum in the topologically trivial region consists of $\frac{N}{2}$ pairs of conjugated complex energy eigenvalues all the time no matter how large $\gamma$ is and the absolute values of the real parts of these conjugated complex energy eigenvalues become almost equal with the increase of $\gamma$, whereas their imaginary parts are immune to the increase of $\gamma$ and remain unchanged, as shown in figures 9(a)–(e). By contrast, there are some unique properties in both of the topologically nontrivial regions. Concretely, when $\gamma$ is weak, apart from the $N - 4$ conjugated complex energy eigenvalues, two entirely real energy eigenvalues with the form of $\pm a_1$ ($a_1 > 0$) and two purely imaginary energy eigenvalues with the form of $\pm ib$ ($b > 0$) also emerge in both of the regions, as shown in figure 9(a). As $\gamma$ increases up to $\gamma = 0.251$, an interesting phenomenon that the pair of conjugated purely imaginary energy eigenvalues converting into two entirely real energy eigenvalues with the form of $\pm a_2$ ($a_2 > 0$) arises. The behavior initially occurs in the vicinity of the two topological transition points and eventually terminates at $\Phi = 0$ and $2\pi$ with further increasing $\gamma$, as shown in...
Figure 9. The real and imaginary parts of energy spectrum of the anti-\(\mathcal{PT}\)-symmetric SSH model with the third on-site configuration as a function of \(\Phi\). The strength of the chemical potential \(\gamma\) = (a) 0.2, (b) 0.8, (c) 1.415, (d) 4, and (e) 8, respectively. Here \(N = 100\) and \(\lambda = 0.5\). The red points label the entirely real energy eigenvalues and the black points label the purely imaginary energy eigenvalues which will become entirely real with the increase of \(\gamma\).

Figure 9(b). When \(\gamma = 1.415\), the two purely imaginary energy eigenvalues in both of the regions completely become entirely real, as shown in figure 9(c). As \(\gamma\) increases continuously, the absolute values of the two newly emerging entirely real energy eigenvalues increase and the real part of the whole energy spectrum in \(\Phi \in [0, 2\pi]\) becomes narrower, as shown in figures 9(d)–(e).

In figure 10, we plot the real and imaginary parts of the energy spectrum of different topological phase regions for \(\Phi = 0, \frac{\pi}{2}\), and \(\pi\) versus \(\gamma\). It is visible that there are always two entirely real energy eigenvalues and two purely imaginary energy eigenvalues in the topologically nontrivial regions when \(\gamma\) is moderate. As \(\gamma\) increases, when the moduli of the two purely imaginary energy eigenvalues decrease, the absolute values of the real parts of the rest of energy eigenvalues including the two entirely real energy eigenvalues increase and become almost equal. As further increasing \(\gamma\), the two purely imaginary energy eigenvalues coalesce at zero energy followed by the emergence of two extra entirely real energy eigenvalues, as shown in
Figure 10. The real and imaginary parts of energy spectra in different topological phase regions versus $\gamma$ for the anti-$\mathcal{PT}$-symmetric SSH model with the third on-site configuration. We choose $\Phi = (a) \ 0$, (b) $\pi/2$, and (c) $\pi$, respectively. The red points label the entirely real energy eigenvalues and the black points label the purely imaginary energy eigenvalues which will become entirely real with the increase of $\gamma$.

Furthermore, figures 11(a)–(c) display the populations of some special eigenstates belonging to the topologically nontrivial regions with $\Phi = 0$. The populations of eigenstates corresponding to two entirely real energy eigenvalues $\pm a_1$ with $\gamma = 4$ are depicted in figure 11(a), which describe the states symmetrically localized on the left and right boundaries of the system. Nevertheless, the eigenstates of both the two purely imaginary energy eigenvalues $\pm ib$ and two newly emerging entirely real energy eigenvalues $\pm a_2$ correspond to the bound states mainly concentrated in the center of the lattice, as respectively depicted in figures 11(b) and (c) with $\gamma = 0.8$ and 2. It is worth mentioning that a unique distribution is manifested by the eigenstates of these conjugated complex energy eigenvalues in the whole region $\Phi \in [0, 2\pi]$ of the energy spectrum. In other words, when all the corresponding eigenstates are extended, they show the semi-lattice localization behavior depending on the sign of the real parts of these conjugated complex energy eigenvalues solely. When the real parts are positive (negative), the corresponding eigenstates are only localized on the left- (right-) half part of the lattice. In figure 11(d), we show the situation of $\Phi = \pi/2$ and $\gamma = 0.8$ for the energy eigenvalues $0.8 + 1.996i$ (left) and $-0.8 + 1.996i$ (right).

Overall, as $\gamma$ increases, the third on-site configuration mainly affects the topologically nontrivial regions where the conversion of the purely imaginary energy eigenvalues into the entirely real energy eigenvalues for the two bound states leads to a transition from the mixed anti-$\mathcal{PT}$-symmetric and broken phases to the fully anti-$\mathcal{PT}$-broken phase, whereas the topologically trivial region is in the fully anti-$\mathcal{PT}$-broken phase from the outset.
4. Conclusions

In conclusion, we have considered three types of anti-$\mathcal{PT}$-symmetric SSH models and explored the interplay between anti-$\mathcal{PT}$ symmetry and different topological phases. We detailedly analyze and discuss the effect of different on-site configurations on the energy spectrum of the system. Specifically, in the first on-site configuration, with the increase of the strength of chemical potential, the energy spectra in both of the topologically nontrivial regions always consist of two entirely real and $N-2$ purely imaginary energy eigenvalues. While the system initially exhibits a purely imaginary energy spectrum in the topologically trivial region. When the strength of chemical potential continues to increase, two pairs of conjugated complex energy eigenvalues emerge and further convert into four entirely real energy eigenvalues. Ultimately, there are four entirely real and $N-4$ purely imaginary energy eigenvalues in the energy spectrum in the topologically trivial region. As a result, the system in the topologically nontrivial regions is in the mixed anti-$\mathcal{PT}$-symmetric and broken phases all the time but the system in the topologically trivial region undergoes a transition from the exact anti-$\mathcal{PT}$-symmetric phase to the mixed anti-$\mathcal{PT}$-symmetric and broken phases. In the second on-site configuration, when the strength of chemical potential is weak, the energy spectra in both of the topologically nontrivial regions and the topological transition points contain two entirely real energy eigenvalues, whereas the energy spectrum in the topologically trivial region remains purely imaginary. As the strength of chemical potential increases ceaselessly, the energy spectrum of the whole system undergoes a process characterized by large-scale purely imaginary energy eigenvalues converting into entirely real energy eigenvalues. In the end, there is only an entirely real energy spectrum left, which implies that the topologically nontrivial and trivial regions of the system undergo a transition.
from the mixed anti-\(\mathcal{PT}\)-symmetric and broken phases to the fully anti-\(\mathcal{PT}\)-broken phase and from the exact anti-\(\mathcal{PT}\)-symmetric phase to the fully anti-\(\mathcal{PT}\)-broken phase, respectively. In the third on-site configuration, the energy spectrum in the topologically trivial region is consisted of 2\(N\) pairs of conjugated complex energy eigenvalues and there are \(2N-2\) pairs of conjugated complex and two entirely real energy eigenvalues in both of the topologically nontrivial regions as long as the strength of chemical potential is nonvanishing. Moreover, a pair of conjugated purely imaginary energy eigenvalues can also be found in both of the topologically nontrivial regions, which will convert into two entirely real energy eigenvalues once the strength of chemical potential exceeds a threshold. These aforesaid results indicate that the system in the topologically trivial region is in the fully anti-\(\mathcal{PT}\)-broken phase from the outset, and both of the topologically nontrivial regions of the system undergo a transition from the mixed anti-\(\mathcal{PT}\)-symmetric and broken phases to the fully anti-\(\mathcal{PT}\)-broken phase. Furthermore, the populations of some special eigenstates in each case are also addressed and a few interesting localized phenomena are observed. Our work would further deepen the understanding of the interplay between anti-\(\mathcal{PT}\)-symmetry and different topological phases and provide helpful insights for the study of non-Hermitian topological systems.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant Nos. 61822114, 12074330, 11874132, 62071412, and 61575055.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Solutions of the edge states

In this appendix, for all the three cases in the main text, we solve the left and right edge states emerging in both the topological nontrivial regions analytically.

For the sake of clarity, we rewrite the Hamiltonian of the first case as follows

\[
H_h = \sum_j \left( i J_1 a_j^\dagger b_j + i J_2 b_j^\dagger a_{j+1} + i J_1 b_j^\dagger a_j + i J_2 a_j^\dagger b_{j+1} \right),
\]

\[
H_1^o = \gamma \left( a_1^\dagger a_1 - b_M^\dagger b_M \right),
\]

where \(a_j\) (\(b_j\)) and \(a_j^\dagger\) (\(b_j^\dagger\)) are the annihilation and creation operators of particle at site \(A\) (\(B\)) in the \(j\)th unit cell, respectively. Even \(M\) is the total number of unit cells. \(J_1\) and \(J_2\) have the same definition as the main text. The Hamiltonian of the whole system reads

\[
H = H_h + H_1^o.
\]

We resort to the transfer matrix method to calculate the edge states. For a single-particle eigenstate \(\Psi = \sum_{j=1}^{M} \Phi_{j,A} a_j^0 + \Phi_{j,B} b_j^0\) with an energy eigenvalue \(E\), where \(\Phi_{j,A}\) and \(\Phi_{j,B}\) are the wave functions at sites \(A\) and \(B\) in the \(j\)th unit cell, respectively, the Schrödinger equation can be derived as

\[
(E - \gamma) \Phi_{j,A} = i J_1 \Phi_{j-1,B}, \quad j \in (1, M),
\]

\[
E \Phi_{j,A} = i J_1 \Phi_{j,B} + i J_2 \Phi_{j-1,B}, \quad j \in (1, M),
\]

\[
(E + \gamma) \Phi_{j,B} = i J_1 \Phi_{j,A} + i J_2 \Phi_{j+1,A}, \quad j \in [1, M),
\]

\[
(E + \gamma) \Phi_{M,B} = i J_1 \Phi_{M,A}.
\]
Sublattice $A$ in equation (A.3) can be eliminated and we obtain
\[
(E^2 + \gamma E + J_1^2) \Phi_{M,B} = -J_1 J_2 \Phi_{M-1,B},
\]
\[
(E^2 + J_1^2) \Phi_{j,B} = -J_1 J_2 (\Phi_{j-1,B} + \Phi_{j+1,B}), \quad j \in (1, M),
\]
\[
\left(E^2 + \frac{E}{E - \gamma J_1^2 + J_2^2}\right) \Phi_{1,B} = -J_1 J_2 \Phi_{2,B}.
\]

Similarly, we can also eliminate sublattice $B$ in equation (A.3) and obtain
\[
(E^2 - \gamma E + J_1^2) \Phi_{1,A} = -J_1 J_2 \Phi_{2,A},
\]
\[
(E^2 + J_1^2) \Phi_{j,A} = -J_1 J_2 (\Phi_{j-1,A} + \Phi_{j+1,A}), \quad j \in (1, M),
\]
\[
\left(E^2 + \frac{E}{E + \gamma J_1^2 + J_2^2}\right) \Phi_{M,A} = -J_1 J_2 \Phi_{M-1,A}.
\]

By employing transfer matrix method, equations (A.4) and (A.5) can be rewritten as follows
\[
\begin{align*}
(\Phi_{2A} & \quad \Phi_{1A})^T = T_1 (\Phi_{1A} & \quad \Phi_{0A})^T, \\
(\Phi_{j+1,A} & \quad \Phi_{jA})^T = T (\Phi_{jA} & \quad \Phi_{j-1A})^T, \quad j \in (1, M), \\
(\Phi_{M-1,A} & \quad \Phi_{M,A})^T = T_3 (\Phi_{M,A} & \quad \Phi_{M+1,A})^T, \\
(\Phi_{2B} & \quad \Phi_{1B})^T = T_3 (\Phi_{1B} & \quad \Phi_{0B})^T, \\
(\Phi_{j-1,B} & \quad \Phi_{j,B})^T = T_4 (\Phi_{j,B} & \quad \Phi_{j+1,B})^T, \quad j \in (1, M), \\
(\Phi_{M-1,B} & \quad \Phi_{M,B})^T = T_4 (\Phi_{M,B} & \quad \Phi_{M+1,B})^T,
\end{align*}
\]

here, these transfer matrices take the form
\[
T = \begin{pmatrix} \nu & -1 \\ 1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} \mu_1 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
T_2 = \begin{pmatrix} \varphi_1 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \varphi_2 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
T_4 = \begin{pmatrix} \mu_2 & 0 \\ 1 & 0 \end{pmatrix},
\]

where $\nu = -\frac{(E^2 + J_1^2)}{J_1 J_2}$, $\mu_1 = -\frac{(E^2 - \gamma E + J_1^2)}{J_1 J_2}$, $\mu_2 = -\frac{(E^2 + \gamma E + J_1^2)}{J_1 J_2}$, $\varphi_1 = -\frac{(E^2 + E J_1^2)}{J_1 J_2}$, and $\varphi_2 = -\frac{(E^2 + \gamma E J_1^2)}{J_1 J_2}$.

We can diagonalize the transfer matrix $T$, that is, $D = U^{-1}TU$,
\[
D = \begin{pmatrix} \beta_- & 0 \\ 0 & \beta_+ \end{pmatrix}, \quad U = \begin{pmatrix} \beta_- & \beta_+ \\ 1 & 1 \end{pmatrix},
\]
\[
U^{-1} = \frac{1}{\sqrt{\beta^2 - 4}} \begin{pmatrix} -1 & \beta_+ \\ 1 & -\beta_- \end{pmatrix},
\]

where $\beta_\pm = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$.

Therefore, under the open boundary condition, $\Phi_{0A} = \Phi_{0B} = \Phi_{M+1A} = \Phi_{M+1B} = 0$, an arbitrary single-particle eigenstate $\Psi$ satisfies
\[
\begin{align*}
\Phi_{j+1,A} &= \frac{1}{\sqrt{\beta^2 - 4}} U \begin{pmatrix} \beta_+ \beta_{j+1}^{-1} - \mu_1 \beta_j^{-1} \\ \mu_1 \beta_{j+1}^{-1} - \beta_+ \beta_j^{-1} \end{pmatrix} \Phi_{j,A}, \\
\Phi_{j+1,B} &= \frac{1}{\sqrt{\beta^2 - 4}} U \begin{pmatrix} \beta_+ \beta_{j+1}^{-1} - \varphi_1 \beta_j^{-1} \\ \varphi_1 \beta_{j+1}^{-1} - \beta_+ \beta_j^{-1} \end{pmatrix} \Phi_{j,B}.
\end{align*}
\]

(A.9)
According to equation (A.9), in the large $M$ limit, we can conclude that the necessary conditions for the existence of the edge state localized on the left boundary are $\beta_+ = \mu_1 = \varphi_2$ and $|\beta_-| > 1$. The energy eigenvalue of the left edge state is governed by

$$E_L = \left( \frac{\gamma - \frac{\beta_+^2}{4}}{2} \right) + \sqrt{\left( \frac{\gamma - \frac{\beta_+^2}{4}}{2} \right)^2 + 4 \left( J_1^2 - J_1^2 / 2 \right)}.$$

(A.10)

Also, the solution for the energy eigenvalue of the right edge state is

$$E_R = - \left( \frac{\gamma - \frac{\beta_-^2}{4}}{2} \right) + \sqrt{\left( \frac{\gamma - \frac{\beta_-^2}{4}}{2} \right)^2 + 4 \left( J_1^2 - J_1^2 / 2 \right)}.$$

(A.11)

For the second case, the Hamiltonian of the on-site configuration is described by

$$H_0^R = \sum_j \gamma \left( a_j^\dagger a_j + b_j^\dagger b_j \right).$$

(A.12)

We can write the Schrödinger equation as follows

$$(E - \gamma) \Phi_{1,1} = iJ_1 \Phi_{1,2},$$

$$(E - \gamma) \Phi_{1,2} = \Phi_{1,1},$$

$$(E - \gamma) \Phi_{2,1} = iJ_1 \Phi_{2,2},$$

$$(E - \gamma) \Phi_{2,2} = (\gamma - \beta_-^2 / 4) \Phi_{2,1} - (\gamma - \beta_-^2 / 4) \Phi_{2,2}.$$

(A.13)

The energy eigenvalues of the left and right edge states in the large $M$ limit are $E_L = +\gamma$ and $E_R = -\gamma$, respectively. For the edge state localized on the left boundary, the wave functions are $\Phi_{1,2} = 0$ and $\Phi_{2,1} = \left( -\frac{\beta_+^2}{4} \right)^{\frac{1}{2}} \Phi_{1,1}$, the wave functions of the right edge state satisfy $\Phi_{1,1} = 0$ and $\Phi_{2,2} = \left( -\frac{\beta_-^2}{4} \right)^{\frac{1}{2}} \Phi_{1,2}$.

The Hamiltonian of the third on-site configuration reads

$$H_0^R = \sum_{j \in [1, M]} \gamma \left( a_j^\dagger a_j + b_j^\dagger b_j \right).$$

(A.14)

The Schrödinger equation is given by

$$(E - \gamma) \Phi_{1,1} = iJ_1 \Phi_{1,2},$$

$$(E - \gamma) \Phi_{1,2} = \Phi_{1,1},$$

$$(E - \gamma) \Phi_{2,1} = iJ_1 \Phi_{2,2},$$

$$(E - \gamma) \Phi_{2,2} = (\gamma - \beta_-^2 / 4) \Phi_{2,1} - (\gamma - \beta_-^2 / 4) \Phi_{2,2}.$$

(A.15)

When $M$ is large enough, the energy eigenvalues of the left and right edge states are $E_L = +\gamma$ and $E_R = -\gamma$, respectively. The wave functions of the left edge state are $\Phi_{1,2} = 0$, $\Phi_{2,1} = \left( -\frac{\beta_+^2}{4} \right)^{\frac{1}{2}} \Phi_{1,1}$ with $j \in [1, \frac{M}{2}]$, and $\Phi_{1,1} = 0$ with $j \in [\frac{M}{2} + 1, M]$. For the right edge state, the wave functions are $\Phi_{1,1} = 0$, $\Phi_{1,2} = \left( -\frac{\beta_-^2}{4} \right)^{\frac{1}{2}} \Phi_{2,2}$ with $j \in [\frac{M}{2} + 1, M]$, and $\Phi_{2,2} = 0$ with $j \in [1, \frac{M}{2}]$.

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