Projections of Orbital Measures for Classical Lie Groups

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Abstract. In this paper we compute the radial parts of the projections of orbital measures for the compact Lie groups of B, C, and D type, extending previous results obtained for the case of the unitary group by Olshanski and Faraut. Applying the method of Faraut, we show that the radial part of the projection of an orbital measure is expressed in terms of a B-spline with knots located symmetrically with respect to zero.

Key words: orbital measures, B-splines, divided differences, Harish-Chandra–Itzykson–Zuber integral.

1. Introduction. Let $G_n$ be one of the compact Lie groups $\text{SO}(2n+1)$, $\text{Sp}(2n)$, and $\text{O}(2n)$. Consider the adjoint action of $G_n$ on its Lie algebra $\mathfrak{g}_n$ by conjugation. Since $G_n$ is a compact group, on each orbit of the action a unique $G_n$-invariant probability measure is supported. We shall refer to such measures as the orbital measures.

Each orbit can be parameterized by a $n$-tuple $X = (x_1 \leq \cdots \leq x_n)$, $x_1 \geq 0$, of weakly decreasing numbers being the eigenvalues of a matrix in $\mathfrak{g}_n$. Let us denote the set of such $n$-tuples by $\mathcal{X}_n$.

Now we consider the natural projection map $p^n_k : \mathfrak{g}_n \to \mathfrak{g}_k$. Let $\mu_X$ be a $G_n$-orbital measure; then the measure $p^n_k(\mu_X)$ is invariant under the action of the group $G_k$. Each invariant measure can be represented as a continuous combination of orbital measures; indeed, for each Borel subset $S \in \mathfrak{g}_k$, we have

$$p^n_k(\mu_X)(S) = \int_{Y \in \mathcal{X}_k} \mu^{(k)}_Y(S) \cdot \nu_{X,k}(dY),$$

where $\mu^{(k)}_Y, Y \in \mathcal{X}_k$, are $G_k$-orbital measures and $\nu_{X,k}$ is a certain probability measure on the set $\mathcal{X}_k$ of $k$-tuples.

The measure $\nu_{X,k}$ is called the radial part of the measure $p^n_k(\mu_X)$. In the case of the unitary group $\text{U}(n)$, the measure $\nu_{X,k}$ was computed by Olshanski [4] and Faraut [2]. Using the method of Faraut, we compute this measure for the groups $\text{SO}(2n+1)$, $\text{Sp}(2n)$, and $\text{O}(2n)$, expressing the radial parts of orbital measures in terms of determinants of B-splines with knots symmetric with respect to 0.

2. Main result. Before formulating the main result, we give the necessary definitions.

According to Curry and Schoenberg [1], the B-spline with $n$ knots $t_1 < \cdots < t_n$ is a $C^{n-3}$ function $M_n(t_1, \ldots, t_n; t)$ on $\mathbb{R}$ with the following properties:

- $\text{supp}(M_n(t_1, \ldots, t_n; t)) = [t_1, t_n]$;
- the function $M_n(t_1, \ldots, t_n; t)$ is a polynomial in $t$ of degree $n-2$ on each subinterval $(t_i, t_{i+1})$;
- $\int_{\mathbb{R}} M_n(t_1, \ldots, t_n; t) dt = 1$.

Note that the conditions specified above determine the B-spline $M_n(t_1, \ldots, t_n; t)$ uniquely.

Recall also that the divided differences of a function $f$ are defined by induction as follows:

$$f[t_1, t_2] = \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \ldots, f[t_1, \ldots, t_n] = \frac{f[t_1, t_2, \ldots, t_{n-1}] - f[t_2, \ldots, t_{n-1}, t_n]}{t_1 - t_n}.$$

The Hermite–Genocchi formula relates B-splines to divided differences. According to this formula, for a function $f$ with piecewise continuous derivative of degree $n-1$, we have

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\[ f[t_1, \ldots, t_n] = \frac{1}{(n-1)!} \int_\mathbb{R} f^{(n-1)}(t) M_n(t_1, \ldots, t_n; t) \, dt. \]

Using the method of Faraut [2], for the groups SO(2n + 1), Sp(2n), and O(2n), we obtain formulas expressing \( \nu_{X,k} \) in terms of determinants of B-splines (Theorem 1).

Let
\[ V_n(T) := V_n(t_1, \ldots, t_n) = \prod_{1 \leq i < j \leq n} (t_i - t_j) \]
denote the Vandermonde determinant in the variables \( t_1, \ldots, t_n \), and let
\[ v_n(dT) = V_n(T^2) \cdot \prod_{i=1}^n dt_i = V_n(t_1^2, \ldots, t_n^2) \cdot \prod_{i=1}^n dt_i. \]

Given arbitrary \( j, m \in \mathbb{N} \), we shall use the short notation
\[ M_{2m+2}(\pm t_1^{n+j}; t) := M_{2m+2}(-t_{m+j}, \ldots, -t_j, t_1, \ldots, t_{m+j}; t). \]

**Theorem 1.** The radial part \( \nu_{X,k} \) of the projection of a \( G_n \)-orbital measure \( \mu_X \) is given by
\[ \nu_{X,k}(dY) = \frac{c(n,k)}{\prod_{j-i \geq n-k+1} (x_j^2 - x_i^2)} \det [\Delta M_{2n-2k+2}(\pm x_{ij}^{2+n-k}; y_i)]_{i,j=1}^k \hat{\nu}_k(dY), \]
where \( \Delta \) is a differential operator of the form
\[ \Delta = -y \frac{d}{dy} + \xi(n,k) \]
and \( c(n,k) \) and \( \xi(n,k) \) are constants depending on \( n \) and \( k \). If \( G_n = SO(2n+1) \) or \( G_n = Sp(2n) \), then \( \xi(n,k) = 0 \) and
\[ c(n,k) = \frac{(2n-2k)!!}{(2n)!!} \prod_{i=0}^{k-1} \binom{2n-2k+2i+2}{2i+1}. \]

In the case where \( G_n = O(2n) \), we have \( \xi(n,k) = 2(n-k) \) and
\[ c(n,k) = \frac{(2n-2k-1)!!}{(2n-1)!!} \prod_{i=0}^{k-1} \binom{2n-2k+2i+1}{2i}. \]

**Remark.** Note that the derivative of the B-spline \( M_m, m \in \mathbb{N} \), can be expressed as the difference of two splines of order \( m-1 \); namely, for any \( m \) points \( t_1, \ldots, t_m \in \mathbb{R} \), we have (see [5])
\[ \frac{t_m - t_1}{m-1} \cdot \frac{d}{dt} M_m(t_1, \ldots, t_m; t) = M_{m-1}(t_1, \ldots, t_{m-1}; t) - M_{m-1}(t_2, \ldots, t_m; t). \]

3. Proof of Theorem 1

3.1. The Laplace transform of orbital measures. We define the Laplace transform of an orbital measure \( \mu \) on the Lie algebra \( \mathfrak{g}_n \) of the group \( G_n \) as the orbital integral
\[ \hat{\mu}_X(T) = \int_{\text{Orbit}(X)} e^{\text{tr}(TX)} \mu(dX) = \int_{G_n} e^{\text{tr}(T \text{Ad}_g(X))} \, dg, \]
where \( dg \) is the Haar measure on \( G_n \).

In general, the Laplace transform for compactly supported measures is defined on the complexification of the Lie algebra. But notice that the function \( \hat{\mu}_X(T) \) is invariant under the adjoint action of \( G_n \). Thus, we can think of \( T \) as a matrix of canonical form and consider the Laplace transform of an orbital measure as a function on the coordinate space \( \mathbb{C}^n \).

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Using the Harish-Chandra theorem (see [3, Theorem 2]), one can obtain the following formulas for the Laplace transform of an orbital measure $\mu_X$ for $G_n$:

\[
\hat{\mu}_X^{(B_n,C_n)}(t_1, \ldots, t_n) = \frac{(2n-1)!}{V_n(X^2)V_n(T^2/4^n)} \det \left[ \frac{\sin(t_i t_j)}{t_i - t_j} \right]_{i,j=1}^n, \tag{1}
\]

\[
\hat{\mu}_X^{(D_n)}(t_1, \ldots, t_n) = \frac{(2n-2)!}{V_n(X^2)V_n(T^2)} \det \left[ \cosh(t_i t_j) \right]_{i,j=1}^n. \tag{2}
\]

In (1) and (2) $B_n$, $C_n$, and $D_n$ are the standard notations for the classical series of Lie algebras of the groups SO($2n+1$), Sp($2n$), and O($2n$), respectively.

These formulas imply that, up to multiplicative constants, the Laplace transform of an orbital measure $\mu_X$ has the form

\[
D_n(f; T; X) = \frac{\det[f(t_i t_j)]_{i,j=1}^n}{V_n(T^2)V_n(X^2)},
\]

where $f$ is an analytic function with Taylor series

\[
f(tz) = \sum_{m=0}^{\infty} \frac{(zt)^{2m}}{2^{2m} m! (\alpha + 1)_m}. \tag{3}
\]

Here $(\alpha + 1)_m = (\alpha + 1)(\alpha + 2) \cdots (\alpha + m)$ is the Pochhammer symbol and the parameter $\alpha$ is equal to 1/2 in the case of the groups SO($2n+1$) and Sp($2n$) and to $-1/2$ when $G_n = O(2n)$.

3.2. Projections and the Laplace transform. For any Borel measure $\mu$ on $g_n$, the restriction of the Laplace transform to $g_k$ is equal to the Laplace transform of the measure $p_k^*(\mu)$ on $g_k$. Thus, the problem reduces to the computation of the quantities $D_n(f; t_1, \ldots, t_k, 0, \ldots, 0; X)$.

**Lemma 1.** If $f$ is an even analytic function in a neighborhood of 0, then the quantity $D_n(f; t_1, \ldots, t_k, 0, \ldots, 0; X)$ can be expressed in terms of divided differences of the functions $\varphi_i(y) = f(t_i \sqrt{y})$ as follows:

\[
D_n(f; t_1, \ldots, t_k, 0, \ldots, 0; X) = \frac{a_{n-k}(f)}{V_n(X^2)V_k(T^2)(t_1 \cdots t_k)^{2(n-k)}} \prod_{1 \leq j < i \leq n-k} (x_j^2 - x_i^2) \cdot \det[\varphi_i(x_1^2, \ldots, x_j^2, \ldots, x_{j+n-k})]_{i,j=1}^k. \tag{4}
\]

Here $a_n(f) = \prod_{j=0}^{n-1} c_{2j}$, where the $c_{2j}$ are the even coefficients of the Taylor series of $f$.

The proof of this lemma is similar to those of Theorem 4.1 and 5.3 in [2].

3.3. Doubling the knots. Since the functions $\varphi_i$ in Lemma 1 depend on $\sqrt{z}$, formula (4) becomes inconvenient for applying the Hermite–Genocchi formula. But we can overcome this difficulty by doubling the number of knots.

**Lemma 2.** Let $f(z)$ be an even analytic function in a neighborhood of 0, and let $\varphi(z) = f(\sqrt{z})$. Then, given $m$ points $0 < z_1 < \cdots < z_m$, the following equality of divided differences holds:

\[
\varphi[z_1^2, \ldots, z_m^2] = g[-z_m, \ldots, -z_1, z_1, \ldots, z_m],
\]

where $g(z) = zf(z)$.

**Proof.** By the definition of divided differences, we have

\[
\varphi[z_1^2, \ldots, z_m^2] = \sum_{l=1}^m \frac{1}{z_l^2} \prod_{r \neq l} \frac{1}{z_l^2 - z_r^2} = 2 \sum_{l=1}^m \frac{z_l f(z_l)}{2z_l} \prod_{r \neq l} \frac{1}{z_l^2 - z_r^2} = g[-z_m, \ldots, -z_1, z_1, \ldots, z_m],
\]

where $g(z) = zf(z)$. This proves Lemma 2.

3.4. Applying the Hermite–Genocchi formula. Lemmas 1 and 2 reduce the computation of $D_n(f; t_1, \ldots, t_k, 0, \ldots, 0; X)$ to the computation of the determinant of divided differences of the
functions \( g_i(z) = zf(t_i z) \). Applying the Hermite–Genocchi formula, we obtain

\[
g_i[-x_{j+n-k}, \ldots, -x_j, x_j, \ldots, x_{j+n-k}] = \frac{1}{(2n-2k+1)!} \int R \left( g_i(z) \right)^{(2n-2k+1)} M_{2n-2k+2}(-x_{j+n-k}, \ldots, -x_j, x_j, \ldots, x_{j+n-k}; z) \, dz
\]

\[
= \frac{-1}{(2n-2k+1)!} \int R \left( g'_i(z) \right)^{(2n-2k)} M'_{2n-2k+2}(\pm x_j^{j+n-k}; z) \, dz.
\]

Now we want to express \( (g_i(z))^{(2n-2k)} = (zf_i(z))^{(2n-2k)} \) in terms of \( f_i(z) = f(t_i z) \).

According to (3), we have

\[
(g_i(z))^{(2n-2k)} = z t_i^{2n-2k} f(t_i z) \cdot \gamma_{\alpha,m},
\]

where

\[
\gamma_{\alpha,m} = \gamma_{\alpha}(m) := \frac{(m + \frac{3}{2})_{n-k}}{(m + \alpha + 1)_{n-k}}.
\]

The problem now splits into two cases.

(B-C) \( G_n = \text{Sp}(2n) \) or \( \text{SO}(2n+1) \). In this case, \( \alpha = 1/2 \) and \( \gamma_{\alpha,m} = 1 \).

(D) \( G_n = \text{SO}(2n) \). In this case, \( \alpha = -1/2 \) and thus \( \gamma_{\alpha,m} = 1 + 2(n-k)/(2m+1) \).

In both cases, the divided differences of the \( g_i \) can be expressed as

\[
g_i[-x_{j+n-k}, \ldots, -x_j, x_j, \ldots, x_{j+n-k}] = -\frac{t_i^{2n-2k}}{(2n-2k+1)!} \int R f(t_i y) \Delta M_{2n-2k+2}(\pm x_j^{j+n-k}; y) \, dy,
\]

where the operator \( \Delta \) has the form

\[
\Delta = \Delta_{1/2} = -y \frac{d}{dy} \quad \text{(case (B-C))}
\]

or

\[
\Delta = \Delta_{-1/2} = -y \frac{d}{dy} + 2(n-k) \quad \text{(case (D)).}
\]

Now let us apply the Binet–Cauchy identity to (5):

\[
\det[g_i[-x_j, \ldots, -x_{j+n-k}, x_j, \ldots, x_{j+n-k}, \ldots, x_j]]
\]

\[
= \prod_{i=1}^{k} \frac{t_i^{2n-2k-1}}{((2n-2k+1)!)^k} \int_{\mathbb{R}^k_+} \det[f(t_i y_j)]_{i,j=1}^{k} \det[\Delta M_{2n-2k+2}(\pm x_j^{j+n-k}; y_i)]_{i,j=1}^{k} \, dy_1 \cdots dy_k.
\]

To complete the proof of Theorem 1, it suffices to compare the left- and right-hand sides of (4) and apply the inverse Laplace transform to both sides. This completes the proof of Theorem 1.

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**Proofreading remark.** After this work had been finished, the author became aware of an interesting paper of Defosseux (M. Defosseux, Ann. Inst. H. Poincaré Probab. Statist., 46:1 (2010), 209–249), who considered, given a matrix \( X \in \mathfrak{g}_n \), the array \( M(X) \) which is the union of the spectra of \( X \) and of all its images under the projections \( p_k^n \), \( k = 1, \ldots, n-1 \). In turns out that the radial part of the projection of the orbital measure \( \mu_X \) can be calculated as the correlation function of the determinantal point process on \( M(X) \) with kernel given explicitely (see Theorem 6.3 of Defosseux’s paper).

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