Analysis of variant working vacations queue with reneging

P. Vijaya Laxmi∗1 E. Girija Bhavani2 Ch. Lalitha3
1,2,3Department of Applied Mathematics, Andhra University, Visakhapatnam - 530 003, India
E-mail: vijaya_iit2003@yahoo.co.in

Abstract. In this paper, we study a single server queueing system with variant working vacations wherein customers arrive according to a Poisson process. The server takes variant working vacations as soon as the system becomes empty. The service time during regular busy period, working vacation period and vacation times are assumed to be exponentially distributed and are mutually independent. During working vacation customer may renege due to impatience which follows exponential distribution. The steady state probabilities of number in the system are obtained using matrix geometric method. Numerical investigations showing the effect of model parameters on different performance measures are shown through tables and graphs. An optimization of cost function is performed to find the optimal service rate that minimizes the cost function.

Keywords: Queue, Reneging, Variant working vacation, Matrix geometric method.

1. Introduction
Queueing system with server vacations have been investigated extensively due to their wide applications in several areas including computer communication systems, manufacturing and production systems. Vacation models are useful in systems where the server wants to utilize the idle time for different purposes. For more information on this topic we may refer to the papers of Doshi [4], Tian and Zhang [12]. In classical vacation queues, the server is completely idle during the vacation period. However, there are numerous situations where the server remains active during the vacation period which is called working vacation (WV). Servi and Finn introduced this class of semi-vacation policy in an M/M/1 queue with multiple working vacations (MWV’s). Baba [2] analyzed a GI/M/1 queue with MWV. Wu and Takagi [15] generalized Servi and Finn’s [11] M/M1/WV queue to an M/G/1/WV queue. Banik et al. [3] studied a GI/M/1/N WV queue to an M/G/1/WV queue.

In real-life, many queueing situations arise where customers tend to be discouraged by a long queue. As a result, customers after joining the queue depart (renege) without getting service. Palm’s [10] work seems to be the first to analyze the act of impatient customers in an M/M/c queueing systems, where the customers have independent exponential distributed sojourn times. Altman and Yechiali [1] presented the analysis for impatient customers in M/M/1, M/G/1 and M/M/c queueing models with server vacations. Kim and Kim [6] analyzed a multi-server batch arrival MX/M/c queue with reneging. The concept of variant multiple vacation policy is relatively a new one where the server is allowed to take a certain number of consecutive vacations, if the system remains idle at the end of a vacation period. This type of vacation schedule can
be observed in the papers by Ke and Chang [7], Ke et al. [8] and Gao and Yao [5] developed the variant working vacations (VWV) policy for the $M^X/G/1$ queueing system, where the server uses a randomized vacation policy. Yue et al [16] analyzed an $M/M/1$ queueing system with impatient customers and a variant of multiple vacation policy. Vijaya et al [13] studied finite buffer Markovian queue with balking, reneging and working vacations. In this paper, we considered the work of Vijaya Laxmi and Rajesh [14] by considering the concept of VWV with reneging. We have obtained the steady state probability distributions using matrix geometric method. The customers become impatient when the server is in vacation. Variant working policy refers to the phenomena of taking a maximum number, say $K$ of consecutive WV’s, if the system remains empty at the end of a WV. The VWV generates $MWV$ when $K \to \infty$. However, after the end of the $K^{th}$ vacation, the server switches to normal busy period and stays idle or busy depending on the number of customers in the system.

2. Model Description

We consider an $M/M/1$ queue with variant working vacations. Customers arrive according to a Poisson process with rate $\lambda$. The service is provided by a single server with service rate $\mu$ and it is exponentially distributed. If there are no customers in the system (exhaustive service policy) then the server begins a vacation of random length which is exponentially distributed with parameter $\phi$. If the server finds customer at a vacation completion instant then it returns to a regular busy period; otherwise, he takes another vacation and continues to have $K$ vacations sequentially. After $K$ vacations the server switches to regular busy period. During these $K$ vacations the server renders service generally with a slower rate and the service time follows exponential distribution with the parameter $\eta$. This type of vacation policy is referred as variant working vacation(VWV). If $K \to \infty$ then the queuing model is referred to as multiple working vacation model and $K = 1$ gives the results for single working vacation. Reneging of customers occurs during working vacation and it is exponentially distributed with parameter $\alpha$.

3. Steady state equations:

Let $L(t)$ be the number of customers in the system at time $t$ and $J(t)$ denotes the status of the server which is defined as

$$J(t)= \begin{cases} j, & \text{the server is on (j+1)th WV for } j = 0, 1, \ldots, K-1 \\ K, & \text{the server is idle or busy.} \end{cases}$$

The process $\{L(t), J(t), t \geq 0\}$ defines a continuous-time Markov process with state space $\Omega = \{(n,j): n \geq 0, j = 0, 1, \ldots, K\}$. Let $P_{n,j} = \lim_{t \to \infty} P\{L(t) = n, J(t) = j\}$, $n \geq 0, j = 0, 1, \ldots, K$ denotes the steady state probability of the process $\{L(t), J(t), t \geq 0\}$.

Using the Markov theory, the set of balance equations are given below:

$$
(\lambda + \phi)P_{0,0} = (\eta + \alpha)P_{1,0} + \mu P_{1,k}, \quad (1)
$$

$$
(\lambda + \phi + \eta + \alpha)P_{1,0} = \lambda P_{0,0} + (\eta + \alpha)P_{2,0}, \quad (2)
$$

$$
(\lambda + \phi + \eta + \alpha)P_{n,0} = \lambda P_{n-1,0} + (\eta + \alpha)P_{n+1,0}, \quad n \geq 2, \quad (3)
$$

$$
(\lambda + \phi)P_{0,j} = (\eta + \alpha)P_{1,j} + \phi P_{0,j-1}, \quad 1 \leq j \leq K-1, \quad (4)
$$

$$
(\lambda + \phi + \eta + \alpha)P_{1,j} = \lambda P_{0,j} + (\eta + \alpha)P_{2,j}, \quad 1 \leq j \leq K-1, \quad (5)
$$

$$
(\lambda + \phi + \eta + \alpha)P_{n,j} = \lambda P_{n-1,j} + (\eta + \alpha)P_{n+1,j}, \quad 1 \leq j \leq K-1, \quad n \geq 2, \quad (6)
$$

$$
\lambda P_{0,K} = \phi P_{0,K-1}, \quad (7)
$$

$$
(\lambda + \mu)P_{1,K} = \lambda P_{0,K} + \mu P_{2,K} + \phi \sum_{j=0}^{K-1} P_{1,j}, \quad (8)
$$
\[(\lambda + \mu)P_{n,K} = \lambda P_{n-1,K} + \mu P_{n+1,K} + \phi \sum_{j=0}^{K-1} P_{n,j}, \; n \geq 2, \quad (9)\]

Let \(P\) be the steady state probability vector given by

\[P = \{P_0, P_1, P_2, \ldots\}\]

where \(P_n = \{P_{n,0}, P_{n,1}, P_{n,2}, \ldots, P_{n,K}\}\) where \(n \geq 0\).

The system of equations given above can be expressed in matrix form as \(PQ = 0\) where the infinitesimal generator matrix \(Q\) is given by

\[
Q = \begin{pmatrix}
  A_0 & C & C \\
  B_1 & A_1 & C \\
  B_2 & A_1 & C \\
  \vdots & \vdots & \vdots
\end{pmatrix}
\]

and the elements of the matrix \(Q\) are given by

\[
C = \begin{pmatrix}
  \lambda \\
  \lambda \\
  \vdots \\
  \lambda
\end{pmatrix}_{(K+1) \times (K+1)}
\]

\[
A_0 = \begin{pmatrix}
  - (\lambda + \phi) & \phi & \phi & \phi & \cdots \\
  - (\lambda + \phi) & - (\lambda + \phi) & \phi & \cdots \\
  - (\lambda + \phi) & - (\lambda + \phi) & - (\lambda + \phi) & \phi & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
  - (\lambda + \phi) & - (\lambda + \phi) & \cdots & \cdots & \cdots & - (\lambda + \phi) & \phi & - \lambda
\end{pmatrix}_{(K+1) \times (K+1)}
\]

\[
B_1 = \begin{pmatrix}
  \eta + \alpha \\
  \eta + \alpha \\
  \vdots \\
  \eta + \alpha \\
  \mu
\end{pmatrix}_{(K+1) \times (K+1)}
\]

\[
A_i = \begin{pmatrix}
  - (\lambda + \eta + \phi + \alpha) & \phi & \phi & \cdots \\
  - (\lambda + \eta + \phi + \alpha) & - (\lambda + \eta + \phi + \alpha) & \phi & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  - (\lambda + \eta + \phi + \alpha) & \cdots & \cdots & \cdots & - (\lambda + \eta + \phi + \alpha) & \phi & - (\lambda + \mu)
\end{pmatrix}_{(K+1) \times (K+1)}, \; 1 \leq i \leq n
\]
\[
B_i = \begin{pmatrix}
(\eta + \alpha) & \cdots & (\eta + \alpha) \\
\vdots & \ddots & \vdots \\
(\eta + \alpha) & \cdots & \mu
\end{pmatrix}
\]
\[\text{, } 2 \leq i \leq n \]

4. Steady State Solution:
According to the Neuts [9] the necessary and sufficient condition for the existence of probability vector at steady state is

\[\textbf{PCe} < \textbf{PB}_2 \textbf{e}, \quad (10)\]

where \(\textbf{P}\) represents the invariant probability vector of the matrix

\[\textbf{M} = \textbf{B}_2 + \textbf{A}_1 + \textbf{C} \quad (11)\]

The governing system of difference equations can be given as:

\[\textbf{P}_{0}\textbf{A}_0 + \textbf{P}_1\textbf{B}_1 = 0, \quad (12)\]
\[\textbf{P}_{n-1}\textbf{C} + \textbf{P}_n\textbf{A}_1 + \textbf{P}_{n+1}\textbf{B}_2 = 0, \quad n \geq 1, \quad (13)\]

and normalizing condition

\[\sum_{n=0}^{\infty} \textbf{P}_n\textbf{e} = 1. \quad (14)\]

where \(\textbf{e}\) is a column vector of dimension \((k + 1) \times 1\) whose elements are unity. Further, we observe that

\[\textbf{P}_n = \textbf{P}_1\textbf{R}^{n-1}, \quad n \geq 2, \quad (15)\]

where the matrix \(\textbf{R}\) is the unique non-negative solution of the matrix quadratic equation

\[\textbf{C} + \textbf{RA}_1 + \textbf{R}^2\textbf{B}_2 = 0, \quad \text{with spectral radius less than unity.}\]

**Computational algorithm for \(\textbf{R}\):**

**Step 1:** Set \(k = 1\).

**Step 2:** Set \(\textbf{U} = \textbf{A}_1\) and calculate \(\textbf{G} = (\textbf{I} - \textbf{U})^{-1}\textbf{B}_2\).

**Step 3:** Increment \(k\) by 1.

**Step 4:** Replace \(\textbf{U} = \textbf{A}_1 + \textbf{CG}\) and \(\textbf{G} = (\textbf{I} - \textbf{U})^{-1}\textbf{B}_2\).

**Step 5:** Repeat steps 3 and 4 until \(\|\textbf{e} - \textbf{Ge}\|_{\infty} < \epsilon\), where \(\epsilon\) is a stopping tolerance.

**Step 6:** Calculate \(\textbf{R} = \textbf{C}(\textbf{I} - \textbf{U})^{-1}\).

From equations (12), (13), after some mathematical manipulations, we get

\[\textbf{P}_0 = -\textbf{P}_1\textbf{B}_1(\textbf{A}_0^{-1}) = \textbf{P}_1\phi_1, \quad (16)\]
\[\textbf{P}_1(\phi_1\textbf{C} + \textbf{A}_1 + \textbf{RB}_2) = 0, \quad (17)\]

The remaining steady state probabilities can be obtained recursively in terms of \(\textbf{P}_1\) from equation (15). To determine \(\textbf{P}_1\) we make use of the normalization condition and equation (15) to get

\[\sum_{n=0}^{\infty} \textbf{P}_n\textbf{e} = \textbf{P}_1[\phi_1 + \textbf{I} + \textbf{R}(\textbf{I} - \textbf{R})^{-1}]\textbf{e} = 1. \quad (18)\]

Solving equations (17) and (18) we obtain \(\textbf{P}_1\).
5. Performance Measures:
In this section, we give some performance measures which can reflect the behaviour of the model for small variations in the parameters. When the system is in the state \((i,j)\), \(i \geq 1, \ j = 0, 1, 2, \ldots K - 1\) the rate of abandonment of a customer due to impatience is given by

\[
R[a] = \sum_{j=0}^{K-1} \sum_{i=1}^{\infty} \alpha P_{i,j}
\]

Probability that the server is on working vacation is given by

\[
P[wv] = \sum_{j=0}^{k-1} \sum_{i=0}^{n} P_{i,j}.
\]

Expected queue length when the server is on working vacation is

\[
L[wv] = \sum_{j=0}^{K-1} \sum_{i=2}^{\infty} (i - 1)P_{i,j}.
\]

Probability that the server is in regular busy period

\[
P[b] = \sum_{i=0}^{\infty} P_{i,K}.
\]

Expected queue length when the server is in regular busy period

\[
L[b] = \sum_{i=2}^{\infty} (i - 1)P_{i,K}.
\]

Probability that the system is empty when the server is on working vacation is given by

\[
P[e] = \sum_{j=0}^{K-1} P_{0,j}.
\]

Probability that the server is idle during busy period

\[
P[I] = P_{0,K}.
\]

6. Numerical Investigation:
In this section, some numerical illustrations are presented in the form of tables and graphs. The various parameters of the model are assumed as \(\lambda = 1.67, \mu = 2.5, \phi = 1.0, \eta = 0.59, \alpha = 0.62\) and \(K = 4\).

Table 1 shows that the effect of \(\phi\) on performance measures. We can observe that as \(\phi\) increases \(P[wv], L[wv]\) decrease and \(P[e], P[b], L[b]\) increase. As \(\phi\) increases, the mean vacation time decreases and the server spends lesser duration of time in working vacations resulting in the corresponding decrease or increase in the above performance index. Table 2 shows that the effect of \(\mu\) on performance measures. We can see that as \(\mu\) increases, \(P[wv], P[e]\) and \(L[wv]\) increase and \(P[b], L[b]\) decrease. Since as \(\mu\) increases, server spends more duration of time in working vacation.
Table 1. Effect of $\phi$ on performance measures

| $\phi$ | $P[wv]$ | $P[e]$ | $L[wv]$ | $P[b]$ | $L[b]$ |
|-------|---------|--------|---------|--------|--------|
| 1.0   | 0.43704 | 0.204949 | 0.24374 | 0.351304 | 0.668565 |
| 1.3   | 0.41778 | 0.214406 | 0.173982 | 0.39755 | 0.755911 |
| 1.6   | 0.400689 | 0.219731 | 0.131326 | 0.437098 | 0.828268 |
| 1.9   | 0.385038 | 0.222325 | 0.102877 | 0.471672 | 0.889246 |

Percentage variation
-5.20 1.73 -14.08 12.03 22.06

Table 2. Effect of $\mu$ on performance measures

| $\mu$ | $P[wv]$ | $P[e]$ | $L[wv]$ | $P[b]$ | $L[b]$ |
|-------|---------|--------|---------|--------|--------|
| 2.3   | 0.369551 | 0.175909 | 0.203361 | 0.38707 | 0.945011 |
| 2.5   | 0.43704 | 0.204949 | 0.24374 | 0.351304 | 0.668565 |
| 2.7   | 0.491511 | 0.227891 | 0.276851 | 0.321553 | 0.509443 |

Percentage variation
12.19 5.19 7.34 -6.55 -43.55

Figure 1 shows the effect of $\mu$ on $L[wv]$ for different values of $\alpha$. We can see from the figure that as $\mu$ increases, $L[wv]$ increases. Since server spends more time in working vacations and $\eta < \mu$. But as reneging rate $\alpha$ increases, it is obvious that $L[wv]$ decreases.

Figure 2 shows the effect of $\phi$ on $P[b]$ for different values of $\eta$. We can observe from the graph that as $\phi$ increases, $P[b]$ increases. This is because service rate in working vacations is slow, server could find more number of customers at the end of working vacation. But as service times in working vacation ($\eta$) increases, we can see from the graph that $P[b]$ decreases.

Figure 3 presents the effect of $\phi$ and $\eta$ on expected queue length when the server is on regular busy period ($L[b]$). We can see that $L[b]$ decreases with the increase of both $\phi$ and $\eta$.

We have developed a total expected cost function per unit time with an objective to determine the optimum value of $\mu$ that minimizes the expected cost function. Let us define:

- $C_1 \equiv$ Service cost per unit time when the server is busy,
- $C_2 \equiv$ Service cost per unit time when the server is on WV,
- $C_3 \equiv$ Fixed cost per unit time when the server is on WV,
- $C_4 \equiv$ Cost per unit time when the server is idle during WV,
- $C_5 \equiv$ Cost per unit time of every customer in the queue and waiting for service.

Using the definition of each cost element and its corresponding system characteristics, the total expected cost function per unit time is given by

$$ F[\mu] = C_1 \ast \mu \ast P[b] + (C_2 \ast \eta + C_3 \ast \phi)P[wv] + C_4 P[e] + C_5 L[b], $$

Assuming the values of cost functions as $C_1 = 38$, $C_2 = 24$, $C_3 = 19$, $C_4 = 16$, $C_5 = 10$ and the model parameters are assumed to take the values $\lambda = 1.67$, $\eta = 0.59$, $\alpha = 0.62$, $\phi = 1.0$ and $K = 4$. Using direct search method, it is found that the minimum value of the total expected cost function is $F[\mu] = 62.5729$ at $\mu = 2.3$ which is shown graphically in Figure 3.

7. Conclusion:
In this paper, we have studied an M/M/1 queuing system with variant WV’s and customer’s impatience. We used matrix geometric method to find steady state probabilities. The effects of some parameters on the performance measures of the system have been investigated and the
Figure 1. Effect of $\mu$ on $L[wv]$ for different $\alpha$

Figure 2. Effect of $\phi$ on $P[b]$ for different $\eta$

Figure 3. Effect of $\phi$ and $\eta$ on $L[b]$
Figure 4. Effect of $\mu$ on $F'(\mu)$

results are presented in the form of graphs. We have derived the cost model to determine the optimal value of $\mu$.

References
[1] Altman E and Yechiali U 2006 Queueing systems 52 261–279
[2] Baba Y 2005 Operations Research Letters 33 201–209
[3] Banik A D, Gupta U C and Pathak S S 2007 Applied Mathematical Modelling 31 1701–1710
[4] Doshi B T 1986 Queueing systems 1 29–66
[5] Gao S and Yao Y 2014 Int. J. Comp. Mth. 913 368–383
[6] Kim B and Kim J 2014 Operation Research Letters 42 180–185
[7] Ke J C and Chang F M 2009 Computers and Industrial Engineering 57 433–443
[8] Ke J C, Chang K B and Pearn W L 2010 Journal of system science and system Engineering 19 50–71
[9] Marcel F Neuts 1981 (Baltimore: John Hopkins University Press)
[10] Palm C 1953 Tele. 4 189–208
[11] Servi L D and Finn S G 2002 Performance Evaluation 50 41–52
[12] Tian N and Zhang Z G 2006 Springer Science & Business Media 93
[13] Vijaya Laxmi P, Goswami V and Jyothsna K 2013 Inter. J. Math. Oper. Res. 6 505–521
[14] Vijaya Laxmi P and Rajesh P 2015 Opsearch. 53 303–316
[15] Wu D and Takagi H 2006 Performance Evaluation 63 654–681
[16] Yue D, Yue W, Saffer Z and Chen X 2014 J. Ind. Manag. Optim. 10 89–112