Unknotted gropes, Whitney towers, and doubly slicing knots

Jae Choon Cha and Taehee Kim

Abstract. We study the structure of the exteriors of gropes and Whitney towers in dimension 4, focusing on their fundamental groups. In particular we introduce a notion of unknottedness of gropes and Whitney towers in the 4-sphere. We prove that various modifications of gropes and Whitney towers preserve the unknottedness and do not enlarge the fundamental group. We exhibit handlebody structures of the exteriors of gropes and Whitney towers constructed by earlier methods of Cochran, Teichner, Horn, and the first author, and use them to construct examples of unknotted gropes and Whitney towers. As an application, we introduce geometric bi-filtrations of knots which approximate the double sliceness in terms of unknotted gropes and Whitney towers. We prove that the bi-filtrations do not stabilize at any stage.

1. Introduction

In 4-dimensional topology, the disk embedding problem is of primary importance. Since Freedman’s 1983 ICM proceedings paper [Fre84], capped gropes have been used as the most essential ingredient for disk embedding, especially for the non-simply connected case. The book of Freedman and Quinn [FQ90] presents beautiful grope-based treatments of foundational results in dimension 4. It is also natural to consider a closely related notion of Whitney towers together with gropes. Cochran, Orr, and Teichner introduced a framework of the study of knot concordance, which can be viewed as the local case of the general disk embedding problem, in terms of symmetric gropes and Whitney towers [COT03]. For links, Conant, Schneiderman, and Teichner developed a theory of asymmetric grope concordance and Whitney tower concordance, as summarized in [CST11].

To obtain flat embedded disks from capped gropes, the present technology (e.g. see [FQ90] [FT95] [KQ00] [CP16]) requires that the involved fundamental group is, very roughly speaking, “not too large.” Such a group is often called good. The key question, which is still left open, is whether all groups are good. The best known result is that subexponential groups and their iterated extensions and direct limits are good, due to Freedman and Teichner [FT95] and Krushkal and Quinn [KQ00].

Unknotted gropes and Whitney towers. In this paper, motivated by the above, we begin to study the structure of the exteriors of gropes and Whitney towers in 4-manifolds, focusing on their fundamental groups. We also present an application to double slicing of knots.

The following definition formulates the case of the smallest possible fundamental group.

Definition 1.1. A sphere-like capped grope, or Whitney tower, in the 4-sphere is $\pi_1$-unknotted if its complement has infinite cyclic fundamental group.

Precise definitions of sphere-like capped gropes and Whitney towers are given in Section 2.1. (For those who are familiar with the notion of properly immersed capped gropes in [FQ90], we remark that we consider a more general class of capped gropes in 4-manifolds, for which a cap is allowed to intersect body surfaces as well as caps, similarly to, e.g. [CST14].) An infinite cyclic fundamental group is the smallest possible in
the sense that the fundamental group of every sphere-like capped grope/Whitney tower exterior in $S^4$ has an infinite cyclic quotient (see Remark 2.6).

The following observation justifies our terminology: Freedman showed that a flat 2-sphere $S$ embedded in $S^4$ is unknotted in the classical sense if and only if $\pi_1(S^4 \setminus S)$ is infinite cyclic [Fre84]. Thus an embedded 2-sphere is unknotted if and only if it is $\pi_1$-unknotted as a capped grope or Whitney tower. Furthermore, if one performs finger moves on a 2-sphere embedded in $S^4$ and obtains a Whitney tower consisting of the resulting immersed 2-sphere together with the additional Whitney disks introduced by the finger moves, then the Whitney tower is $\pi_1$-unknotted if and only if the original embedded sphere is unknotted (see Lemma 2.9).

We note that if a sphere-like properly immersed capped grope $G$ in $S^4$ has height at least 1.5 and is unknotted, then for any 2-disk $D$ contained in the base surface, the disk embedding result in [CP16, Theorem 3.4] tells us that the subgrope $G \setminus D$ can be replaced with a flat embedded disk. Thus the grope $G$ can be modified, relative to $D$, to a flat embedded 2-sphere.

**Modifications of Whitney towers and gropes.** It turns out that various fundamental operations on capped gropes and Whitney towers do not enlarge the fundamental group of the exterior, and consequently preserve the $\pi_1$-unknottedness. For instance, in Sections 2 and 3, we show that it is the case for the following operations:

1. Regular homotopy of Whitney towers (Definition 2.7 and Lemma 2.9)
2. Taking a subtower of a Whitney tower by removing Whitney disks (Proposition 2.10 and Corollary 2.11)
3. Whitney disk splitting (Proposition 3.3)
4. Symmetric and asymmetric contraction of a capped grope (Proposition 2.13)
5. Pushing an intersection down in a capped grope (Proposition 3.6)
6. Krushkal’s grope splitting (Proposition 3.7)

In [Sch06], Schneiderman presented fundamental procedures which transform a capped grope to a Whitney tower in an arbitrary 4-manifold, and vice versa. Concerning this, we prove the following result on the exteriors:

**Theorem A.** Schneiderman’s transformation [Sch06] from a Whitney tower to a capped grope does not enlarge the fundamental group of the exterior, and for sphere-like ones in the 4-sphere, it preserves the $\pi_1$-unknottedness. The same holds for Schneiderman’s transformation from a capped grope to a Whitney tower.

In Section 3, we state and prove a refined version of Theorem A as Theorem 3.1.

**Analysis of grope constructions.** Cochran and Teichner [CT07], Horn [Hor10], and Cha [Cha14b] developed methods to produce annulus-like symmetric gropes in $S^3 \times I$ cobounded by knots in $S^3 \times 0$ and $S^3 \times 1$, which can be viewed as approximations of ordinary concordance. Together with obstructions from $L^2$-signatures, these constructions have been used as key ingredients in revealing the rich structure of the grope and Whitney tower theory in the context of concordance (e.g. see [Hor11, CP14, Jan17]). Briefly, in these methods, the two knots in $S^3 \times 0$ and $S^3 \times 1$ are related by (iterated) satellite constructions, and annular gropes cobounded by them are constructed by first finding simpler gropes and surfaces in 3-space for companions and patterns of the satellite constructions, and then stacking them in 4-space. A precise formulation of this procedure is described in Sections 4.1 and 4.2 (see Definitions 4.1, 4.3 and 4.6).

We present a handle decomposition of the exterior of a grope in $S^3 \times I$ obtained by these methods. In particular, we show the following:
Theorem B. Suppose $G$ is an annular capped grope in $S^3 \times I$ cobounded by $K \subset S^3 \times 0$ and $K' \subset S^3 \times 1$, which is constructed by the methods of Cochran and Teichner [CT07], Horn [Hor10], and Cha [Cha14b] described in Sections 4.2 and 4.3. Let $E_G$ and $E_K$ be the exteriors of $G$ and $K$. Then $E_G$ has a relative handle decomposition with 2-handles only: $E_G \cong (E_K \times I) \cup (2$-handles).

We state and prove a full version of Theorem B in Section 4.2 (see in particular Theorem 4.27).

One can apply Schneiderman’s method [Sch06] to convert a capped grope constructed by the methods of [CT07, Hor10, Cha14b] to a Whitney tower of the same height. Combining Theorem B with our analysis of Schneiderman’s method (which is used to prove Theorem A), it turns out that the exterior of the resulting Whitney tower has a similar handle decomposition with 2-handles only.

Application to knot double slicing. We use the notion of $\pi_1$-unknottedness and the above handle structure results to study double slicing of knots in terms of gropes and Whitney towers.

Recall that a knot $K$ in $S^3$ is double slice if there is an unknotted flat 2-sphere in $S^4$ which intersects the standard $S^3 \subset S^4$ at $K$. That is, $K$ is a slice in $S^3$ of an unknotted 2-sphere in $S^4$. Inspired from this, we define a knot $K$ in $S^3$ to be height $(m,n)$ grope slice if it is a slice of a $\pi_1$-unknotted sphere-like capped grope $G$ in $S^3$ such that the intersections of $G$ with the upper and lower hemispheres of $S^3$ have height $m$ and $n$ respectively. Here $m$ and $n$ are nonnegative half integers. See Definition 5.2. We also define a height $(m,n)$ Whitney slice knot similarly by replacing the grope $G$ above by a Whitney tower.

We denote by $\mathcal{G}_{m,n}$ and $W_{m,n}$ the collection of height $(m,n)$ grope slice knots and height $(m,n)$ Whitney slice knots, respectively. Each of $\mathcal{G}_{m,n}$ and $W_{m,n}$ is a submonoid of the monoid of knots under connected sum, and $\{\mathcal{G}_{m,n}\}$ and $\{W_{m,n}\}$ are descending (non-increasing) bi-filtrations. It turns out that a height $(m,n)$ grope slice knot is height $(m,n)$ Whitney slice, that is, $\mathcal{G}_{m,n} \subset W_{m,n}$ (see Theorem 5.3).

A doubly slice knot is height $(m,n)$ grope slice and height $(m,n)$ Whitney slice for all $m,n$, and therefore lies in the intersection of all $\mathcal{G}_{m,n}$ and $W_{m,n}$.

It turns out that previously known obstructions to knots being doubly slice are obstructions to lying in low height terms of our bi-filtrations. In Section 7, we show that the double sliceness obstructions by Gilmer-Livingston, Friedl, and Livingston-Meier [GL83, Fri04, LM15] vanish for knots in $\mathcal{G}_{4,4}$ or $W_{4,4}$: it is proven in stronger forms in Propositions 7.2, 7.4, and 7.6.

Our main result on the geometric bi-filtrations $\{\mathcal{G}_{m,n}\}$ and $\{W_{m,n}\}$ is the following.

Theorem C. Let $m,n \geq 3$ be integers.

1. There exists a family of slice knots $\{J^i\}_{i=1,2,...}$ in $\mathcal{G}_{m,m}$ whose arbitrary nontrivial linear combination $\#a_iJ^i$ ($a_i \in \mathbb{Z}$) is not in $W_{m,5,m,5}$.

2. There exists a family of knots $\{J^i_{m,n}\}_{i=1,2,...}$ in $\mathcal{G}_{m,n}$ whose arbitrary nontrivial linear combination $\#a_iJ^i_{m,n}$ ($a_i \in \mathbb{Z}$) is not in $W_{m,5,n} \cup W_{m,5,n}$.

Theorem C is proven in a stronger form in Theorem 6.10. Since $\mathcal{G}_{m,n} \subset W_{m,n}$, it follows from Theorem C that both bi-filtrations $\{\mathcal{G}_{m,n}\}$ and $\{W_{m,n}\}$ are highly nontrivial.

In the proof of Theorem C, we give the knots $J^i$ and $J^i_{m,n}$ using iterated satellite constructions. The main proof consists of two parts: the existence of a $\pi_1$-unknotted capped grope of the given height, and the non-existence of a $\pi_1$-unknotted Whitney tower of larger height.
For the existence, briefly speaking, we employ the methods of \cite{CT07, Hor10, Cha14b} to produce capped gropes which slice our knots. We prove that they are $\pi_1$-unknotted by using handle structures given by Theorem 1. Details can be found in Section 5.2.

For the non-existence, we first relate our geometric filtrations to the solvable bi-filtration, which was defined by the second author \cite{Kim06} as a double slicing analog of Cochran-Orr-Teichner’s solvable filtration \cite{COT03}. Briefly, a knot $K$ is $(m,n)$-solvable if there are 4-manifolds $U$ and $V$ bounded by the zero surgery manifold $M(K)$, each of which approximates a slice disk exterior in terms of a certain duality property over the group ring $\mathbb{Z}[\pi_1(\mathbb{R})/\pi_1(\mathbb{R})^{(n)}]$, where $\pi_1(\mathbb{R})^{(n)}$ designates the $n$th derived subgroup, such that the union $U \cup M(K) V$ has infinite cyclic fundamental group. See Definition 6.4 for a precise description. In Section 6.1 we prove that a knot is $(m,n)$-solvable whenever it is height $(m+2, n+2)$ Whitney slice, that is, $W_{m+2,n+2} \subset F_{m,n}$ (see Theorem 6.4). Then, in Sections 6.2–6.4, we use the amenable signature theorem of \cite{CO12, Cha14a} and combine it with the ideas of \cite{Kim06} to extract obstructions to being $(m,n)$-solvable from the von Neumann-Cheeger-Gromov $\rho(2)$-invariants of the zero surgery manifolds.

A corollary of (the above outlined proof of) Theorem C is that its analog holds for the solvable bi-filtration $\{F_{m,n}\}$ (in place of both $\{G_{m,n}\}$ and $\{W_{m,n}\}$) as well. This consequence, which provides infinitely many linearly independent knots in each stage of the solvable bi-filtration modulo the next stages, generalizes the main result of \cite{Kim06} which gives nontrivial knots in each stage of the solvable bi-filtration modulo the next stages.

**Acknowledgements**

The authors thank Patrick Orson for his helpful comments on the first draft version of this paper. The first named author was supported by NRF grant 2011-0030044. The second named author was supported by NRF grant 2011-0030044 and NRF grant 2015R1D1A1A01056634.

2. Unknotted Whitney towers and gropes

In this section we define the notion of unknotted Whitney towers and gropes in terms of the fundamental group, and discuss some basic properties.

2.1. Preliminaries: Whitney towers and gropes

We begin by recalling the definitions of Whitney towers and gropes. Readers familiar with them may skip to Section 2.2 after reading our conventions given after Definition 2.3.

In the following definitions, we will assume that the readers are familiar with the notion of (framed) immersions of surfaces and Whitney disks; for instance see \cite{FQ90} §1.2, §1.4.

In this paper, we will define and use framed gropes and Whitney towers only. In particular a Whitney disk will always be framed as follows: for a Whitney disk $D$ pairing two intersections of sheets $A$ and $B$, the restriction of the unique framing of $D$ on $\partial D$ is equal to the framing determined by the tangential direction of $A$ on $A \cap \partial D$ and the common normal direction of $B$ and $D$ on $B \cap \partial D$.

In this paper, surfaces are always orientable.

**Definition 2.1** (Whitney tower). Suppose $S$ is a surface. An $S$-like Whitney tower in a 4-manifold $W$ is a 2-complex defined inductively as follows. A properly immersed surface $(S, \partial S) \looparrowright (W, \partial W)$ is an $S$-like Whitney tower in $W$. Suppose $T$ is an $S$-like Whitney tower and $D$ is a framed immersed Whitney disk pairing two intersections of opposite signs between two sheets in $T$, where the interior of $D$ is allowed to transversely intersect
the interior of surfaces/disks of $T$ but is disjoint from the boundary of any surface in $T$. Then $T$ together with $D$ is an $S$-like Whitney tower. We say that $S$ is the base surface of $T$, and $S$ supports $T$. The boundary of $T$ is that of the base surface of $T$.

We will mostly use sphere-like or disk-like Whitney towers.

We often consider symmetric Whitney towers, which are defined to be a Whitney tower with a well-defined height in the following sense. This was introduced by Cochran, Orr and Teichner [COT03]. We remark that the notion of order is defined for any Whitney towers as in, for instance, [CST12].

**Definition 2.2** (Height of a Whitney tower). Suppose $T$ is a Whitney tower. For surfaces/disks and intersections in $T$, the height is defined inductively as follows. The height of the base surface of $T$ is one. If $p$ is an intersection of two surfaces/disks of the same height $k$ in $T$, then we say $p$ has height $k$. If a Whitney disk in $T$ pairs up two intersections of the same height $k$, we say the disk has height $k + 1$. (Note that the height may not be defined for some intersections and for some Whitney disks.)

Let $n$ be a positive integer. We say that $T$ is a Whitney tower of height $n$ if the following hold: (i) for all surfaces/disks in $T$ and their intersections, the height is defined and not greater than $n$, and (ii) all intersections of height $< n$ are paired up by Whitney disks in $T$.

We say that $T$ is a Whitney tower of height $n.5$ if the following hold: (i) for all surfaces/disks in $T$, the height is defined and not greater than $n + 1$, (ii) all intersections between surfaces/disks of height $\leq n$ have well-defined height, and are paired up by Whitney disks in $T$, and (iii) each height $n + 1$ Whitney disk does not meet surfaces/disks of height $< n$ (but allowed to meet surfaces/disks of height $n$ and $n + 1$).

An immersed surface can be viewed as a Whitney tower of height one.

**Definition 2.3** (Grope). A capped surface is a surface $\Sigma$ together with disks attached along $2g$ standard symplectic basis curves on $\Sigma$, where $g$ is the genus of $\Sigma$ [FQ90 §2.1]. The disks are called caps. Suppose $S$ is a surface. A model $S$-like capped grope is a 2-complex defined inductively as follows. A 2-complex obtained from $S$ by replacing finitely many disjointly embedded disk in $S$ by capped surfaces is a model $S$-like capped grope. Its base surface is defined to be the surface we obtain by modifying $S$. If $G$ is a model $S$-like capped grope, then a 2-complex obtained by replacing a cap with a capped surface is a model $S$-like capped grope. Its caps are the unmodified caps of $G$ together with the caps of the attached capped surface. Its base surface is defined to be that of $G$. The body of a model capped grope $G$ is defined to be $G$ with all caps removed. The boundary of $G$ is the boundary of the base surface.

Note that a model capped grope $G$ admits a standard embedding in $\mathbb{R}^3$. Composing it with $\mathbb{R}^3 \hookrightarrow \mathbb{R}^4$, $G$ embeds in $\mathbb{R}^4$. Take a regular neighborhood of $G \subset \mathbb{R}^4$, and then possibly introduce finitely many plumblings between caps/surfaces. An embedding of the result into a 4-manifold $W$ is called an immersed capped grope in $W$.

In this paper we assume that each intersection in an immersed capped grope always involves a cap. That is, there is no intersection between two sheets of body surfaces, while caps are allowed to meet caps and body surfaces.

We remark that this differs from the notion of a properly immersed capped grope [FQ90], which is defined to be an immersed capped grope whose intersections are always between caps. In our case, every intersection can be changed to intersections of a cap and the base surface, by “pushing intersections down” as described in [FQ90 Section 2.5] (see also Section 3.2 of this paper).
Definition 2.4 (Height of a capped grope). Suppose $G$ is a (model) capped grope. The \textit{height} of surfaces/caps in $G$ is defined inductively as follows. The height of the base surface of $G$ is one. The height of a surface/cap in $G$ is $k+1$ if its boundary is attached to a surface of height $k$. If all caps of $G$ have height $n+1$, then we say that $G$ is a \textit{capped grope of height $n$}. If, for each dual pair of curves on the base surface, a height $n$ capped grope is attached along one of them and a height $n-1$ capped grope along the other, then we say that $G$ is a \textit{capped grope of height $n-1$}.

2.2. Unknotted Whitney towers

Definition 2.5. A (union-of-spheres)-like Whitney tower $T$ in $S^4$ is $\pi_1$-unknotted if $\pi_1(S^4 \setminus T)$ is a free group. (If it is the case, the rank of the free group is automatically equal to the number of the base spheres.) In particular, a sphere-like Whitney tower $T$ in $S^4$ is $\pi_1$-unknotted if and only if $\pi_1(S^4 \setminus T) \cong \mathbb{Z}$.

Remark 2.6. (1) For every sphere-like Whitney tower $T$ in $S^4$, the abelianization $H_1(S^4 \setminus T)$ of $\pi_1(S^4 \setminus T)$ is equal to $\mathbb{Z}$ by Alexander duality. In this sense, for the sphere-like case, $T$ is $\pi_1$-unknotted if and only if the complement has the “smallest” fundamental group.

(2) Definition 2.5 is a Whitney tower generalization of the standard notion of an unknotted 2-sphere embedded in the 4-sphere, in light of the following well-known result [FQ90] §1.7A: a locally flat 2-sphere $S$ embedded in $S^4$ is unknotted if and only if $\pi_1(S^4 \setminus S) \cong \mathbb{Z}$. In other words, $S$ is unknotted as a sphere if and only if $S$ is $\pi_1$-unknotted as a Whitney tower.

Recall that a \textit{finger move} [FQ90] §1.5 for surfaces introduces new intersections paired by an embedded Whitney disk; the finger move is reversed by a Whitney move across the disk. For a Whitney tower, we will consider a finger move which introduces new intersections of surfaces/disks of the tower, performed along an arc whose interior is disjoint from the tower. As our convention, the newly introduced Whitney disk is added to the resulting Whitney tower. Similarly, when we apply a Whitney move across an embedded Whitney disk in a Whitney tower, we remove the Whitney disk we used from the tower.

Definition 2.7. A \textit{regular homotopy} for a Whitney tower is a finite sequence of ambient isotopy, finger moves, and Whitney moves (across an embedded Whitney disk).

For brevity, we use the following terminology. If $A$, $B$, and $C$ are subspaces of $X$ satisfying $A \cup B \subseteq C$ and $f: \pi_1(X \setminus A) \rightarrow \pi_1(X \setminus B)$ is a homomorphism such that $f(i) = j$ for the inclusions $i: X \setminus C \rightarrow X \setminus A$ and $j: X \setminus C \rightarrow X \setminus B$, then we say that $f$ is supported by $C$, or $f$ extends the identity on $X \setminus C$.

For a subcomplex $K$ of a manifold $X$, we denote its regular neighborhood by $\nu(K)$, and the exterior $X \setminus \nu(K)$ by $E_K$. When $X$ is a topological manifold, $\nu(K)$ and $E_K$ are defined if a neighborhood of $K$ admits a PL/smooth structure with respect to which $K$ is PL/smooth. In this paper it is always the case. Since we will use the following standard fact repeatedly, we state it as a lemma.

Lemma 2.8. Suppose $(K,L)$ is a subcomplex pair embedded in an $n$-manifold $X$ with $L = K \cap \partial X$. If $K = L \cup e_1 \cup \cdots \cup e_r$ with $e_i$ a $p_i$-cell, then $X = E_K \cup h_1 \cup \cdots \cup h_r$ with $h_i$ an $(n-p_i)$-handle.

Proof. Let $K_0 = L$ and $K_i = L \cup e_1 \cup \cdots \cup e_i$. Then $E_{K_r} = E_K$ and $E_{K_{r-1}}$ is obtained from $E_{K_r}$ by filling in it with $\nu(e_r) = D^{p_r} \times D^{n-p_r}$. Since $\nu(e_i) \cap E_{K_{r-i}} = D^{p_i} \times \partial D^{n-p_i}$, $\nu(e_i)$ is an $(n-p_i)$-handle. \qed
Lemma 2.9 (Regular homotopy preserves unknottedness). Suppose $T'$ is a Whitney tower obtained from another Whitney tower $T$ by a finger move in a 4-manifold $W$. Then the exterior $E_T$ is obtained by attaching a 3-handle to the exterior $E_{T'}$, and thus there is an isomorphism $\pi_1(W \setminus T) \cong \pi_1(W \setminus T')$ supported by a regular neighborhood of the trace of the finger move. Consequently, regular homotopy preserves the $\pi_1$-unknotedness of Whitney towers in $S^4$.

Proof. Consider a finger move which converts a tower $T$ to another tower $T'$. Suppose this is performed along an arc $\gamma$ joining two interior points of surfaces/disks of $T$. See Figure 1. Then $E_T \cong E_T \cup \nu(\gamma)$, that is, $E_T$ is homeomorphic to the exterior of $\gamma$ in $E_T$. Since $\gamma$ is a 1-cell, $E_T \cong E_T \cup (3\text{-handle})$ by Lemma 2.8. The other conclusions follow from this. □

![Figure 1. A finger move performed along $\gamma$.](image)

Proposition 2.10 (Unknottedness of subtowers). Suppose $T$ is a Whitney tower in a 4-manifold $W$, and let $T'$ be a Whitney tower which is obtained from $T$ by removing some Whitney disks. Then $E_{T'} \cong E_T \cup (\text{handles of index} \geq 2)$, and consequently the inclusion induces an epimorphism of $\pi_1(W \setminus T)$ onto $\pi_1(W \setminus T')$. In addition, if $W = S^4$ and $T$ is sphere-like and $\pi_1$-unknotted, then $T'$ is $\pi_1$-unknotted.

Proof. It suffices to consider the case that $T'$ is obtained by deleting a Whitney disk $D$. Note that $D$ is immersed and may intersect other surfaces in general. View $E_{T'} \cap D$ as a 2-complex embedded in $E_T$. Since $E_T$ is equal to the exterior of $E_{T'} \cap D$ in $E_T$, it follows that $E_T \cong E_T \cup (\text{handles of index} \geq 2)$ by Lemma 2.8. Therefore the inclusion-induced homomorphism $\pi_1(W \setminus T) \to \pi_1(W \setminus T')$ is surjective. Finally, if $W = S^4$ and $T$ is a sphere-like Whitney tower which is $\pi_1$-unknotted, then since $\pi_1(S^4 \setminus T) \cong \mathbb{Z} \cong H_1(S^4 \setminus T')$, it follows that $\pi_1(S^4 \setminus T') \cong \mathbb{Z}$. □

An immediate corollary is that we can lower the height of a Whitney tower without losing $\pi_1$-unknottedness.

Corollary 2.11. Suppose $S$ is an immersed 2-sphere in $S^4$ supporting a $\pi_1$-unknotted Whitney tower of height $h \in \frac{1}{2}\mathbb{Z}$. Then for any $h' < h$, $S$ supports a $\pi_1$-unknotted Whitney tower of height $h'$.

Proof. Let $T$ be the given Whitney tower of height $h$. When $h' = n \in \mathbb{Z}$, remove Whitney disks of height $> n$. By Proposition 2.10 the resulting tower is $\pi_1$-unknotted, and has height $n$. When $h' = n.5$, remove Whitney disks of height $> n + 1$. This produces a $\pi_1$-unknotted Whitney tower of height $n + 1$, and by definition it is also a tower of height $n.5$. □
2.3. Unknotted capped gropes

As in the Whitney tower case, we define the notion of unknottedness of a capped grope in terms of the fundamental group:

**Definition 2.12.** A (union-of-spheres)-like capped grope $G$ immersed in $S^4$ is $\pi_1$-unknotted if $\pi_1(S^4 \setminus G)$ is a free group. In particular, a sphere-like capped grope $G$ in $S^4$ is $\pi_1$-unknotted if $\pi_1(S^4 \setminus G) \cong \mathbb{Z}$.

Analog of Proposition 2.10 and its Corollary 2.11 for capped gropes can be formulated in terms of contraction. Suppose $D_1$ and $D_2$ are dual caps of a capped grope $G$, that is, they are attached to the same body surface, say $\Sigma$, along curves intersecting at a single point. Following [FQ90, §2.3], take two parallel copies of $D_1$ and $D_2$ and attach them to a square neighborhood of the point $\partial D_1 \cap \partial D_2$ in $\Sigma$ to obtain a disk. Cut $\Sigma$ along $\partial D_1$ and $\partial D_2$, and then attach this disk. This gives a surface with genus one less than that of $\Sigma$. See the left hand side of Figure 2. We call this symmetric contraction. Applying this to all the dual pairs of caps attached to a top stage surface, we can replace the top stage surface together with its caps by a disk, which may be viewed as a new cap.

There is an “asymmetric” version of the above. Following [KQ00], forget one of the caps, say $D_2$, cut $\Sigma$ along $\partial D_1$, and attach two parallel copies of $D_1$ to obtain a new surface with genus one less than that of $\Sigma$. See the right hand side of Figure 2. We call this operation asymmetric contraction. Similarly to the above, a top stage surface of a capped grope can be changed to a cap by applying this repeatedly.

![Figure 2. Symmetric and asymmetric contraction.](image)

**Proposition 2.13** (Unknottedness of contraction). Suppose $G'$ is a capped grope obtained from another capped grope $G$ in a 4-manifold $W$ by either asymmetric or symmetric contraction. Then $E_{G'} \cong E_G \cup (2$-handles). Consequently, there is an epimorphism of $\pi_1(W \setminus G)$ onto $\pi_1(W \setminus G')$ supported by a regular neighborhood of $G$. In addition, if $W = S^4$ and $G$ is sphere-like and $\pi_1$-unknotted, then $G'$ is $\pi_1$-unknotted.

**Proof.** We assert that $E_G$ is homeomorphic to the exterior $E_L$ of a 2-complex $L$ obtained by attaching $k$ 2-cells to $G'$, where $k$ is the number of intersections of the caps $D_1 \cup D_2$ with other surfaces/caps. All the conclusions follow from this, since $E_{G'} \cong E_L \cup (2$-handles) by Lemma 2.8.

To prove the assertion for the symmetric contraction case, attach to $G'$ $k$ 2-cells which are shown in Figure 3 as hatched rectangles. Denote the resulting complex by $L$. Briefly speaking, the picture tells us that $L$ has the same regular neighborhood as that of $G$, and consequently the exterior of $L$ in $W$ is homeomorphic to that of $G$. 

Figure 3. A 2-complex $L$ with the same exterior as the initial capped grope $G$. This is the case of $k = 3$, that is, there are three arcs representing other surfaces which intersect the caps. The three hatched rectangles are additional 2-cells attached to the contraction.

In what follows we will present a formal approach for the last sentence, since it will also be useful in later sections. It is best described using the language of PL topology since we need to deal with complexes which are not manifolds and their regular neighborhoods.

**Definition 2.14.** Suppose $K$ is a subcomplex of a simplicial complex $X$. If $\Delta$ is a subcomplex of $X$ such that $(\Delta, \Delta \cap K) \cong (D^n, \text{an embedded } (n-1)\text{-ball in } \partial D^n)$ for some $n \geq 1$, then we say that $K \cup \Delta$ is obtained from $K$ by elementary cellular expansion in $X$ (and $K$ is obtained from $K \cup \Delta$ by elementary cellular collapse). We say that $K$ expands cellularly to $K'$ in $X$ (and $K'$ collapses cellularly to $K$) if there is a sequence of elementary cellular expansions in $X$ transforming $K$ to $K'$.

When the ambient complex $X$ is clearly understood, we often omit “in $X$.”

The following is a standard fact. For instance see [RS72, Chapter 3].

**Lemma 2.15.** If $K$ expands cellularly to $K'$ in a manifold $X$, then the regular neighborhoods $\nu(K)$ and $\nu(K')$ in $X$ are isotopic in $X$. Consequently the exteriors $E_K$ and $E_{K'}$ are homeomorphic.

We remark that Lemma 2.15 applies to subcomplexes in a triangulable codimension zero submanifold of a topological manifold, for instance, in a regular neighborhood of an immersed capped grope.

Now, returning to the proof of Proposition 2.13, observe that the hatched rectangles in Figure 3 lie in thickened caps of $G$, and each thickened cap of $G$ cut along the hatched rectangles is a 3-cell. It follows that $L$ expands cellularly to $G$ with thickened caps. By Lemma 2.15 it follows that $E_L$ is homeomorphic to $E_G$.

A similar argument can be carried out for the asymmetric case as well.

**Corollary 2.16.** Suppose $G$ is a $\pi_1$-unknotted sphere-like capped grope of height $h$ in $S^4$, $h \in \frac{1}{2}\mathbb{Z}$. Then for any $h' < h$, there is a $\pi_1$-unknotted capped grope of height $h'$ which is obtained from $G$ by symmetric/asymmetric contraction.

**Proof.** Contract top stage surfaces repeatedly until the capped grope has the desired height. The result is $\pi_1$-unknotted by Proposition 2.13.

3. Transformation between unknotted Whitney towers and gropes

In [Sch06], Schneiderman presents fundamental constructions which convert an immersed $S$-like capped grope to an $S$-like Whitney tower immersed in a neighborhood of the capped
grope, and vice versa. Furthermore, he shows that the corresponding capped gropes and Whitney towers have exactly the same intersection data (which are described precisely in terms of uni-trivalent trees). As a corollary he shows that a capped grope of height \( h \) in a 4-manifold can be transformed to a Whitney tower of height \( h \) with the same boundary [Sch06, Corollary 2].

In this section we show that Schneiderman’s method preserves unknottedness, as stated below:

**Theorem 3.1.**

1. If a capped grope \( G \) is obtained from a Whitney tower \( T \) in a 4-manifold \( W \) by Schneiderman’s construction, then \( E_G \cong E_T \cup (\text{handles of index } \geq 2) \), and thus there is an epimorphism of \( \pi_1(W \setminus T) \) onto \( \pi_1(W \setminus G) \) supported by a regular neighborhood of \( T \). Consequently, if \( T \) is sphere-like and \( \pi_1 \)-unknotted in \( W = S^4 \), then \( G \) is \( \pi_1 \)-unknotted.

2. If a Whitney tower \( T \) is obtained from a capped grope \( G \) in a 4-manifold \( W \) by Schneiderman’s construction, then \( E_T \cong E_G \cup (\text{handles of index } \geq 2) \), and thus there is an epimorphism of \( \pi_1(W \setminus G) \) onto \( \pi_1(W \setminus T) \) supported by a regular neighborhood of \( G \). Consequently, if \( G \) is sphere-like and \( \pi_1 \)-unknotted in \( W = S^4 \), then \( T \) is \( \pi_1 \)-unknotted.

Applying Theorem 3.1 to a grope of height \( h \), we obtain the following:

**Corollary 3.2.** Suppose \( G \) is a \( \pi_1 \)-unknotted sphere-like capped grope of height \( h \) in \( S^4 \). Then, in a regular neighborhood of \( G \), there is a \( \pi_1 \)-unknotted sphere-like Whitney tower of height \( h \) with the same boundary.

**Remark 3.3.** If \( P \) is a planar surface contained in the base surface of a Whitney tower \( T \) and \( P \) is disjoint from non-base Whitney disks, then Schneiderman’s construction produces a capped grope \( G \) such that the base surface of \( G \) contains \( P \) and \( G \setminus P \) is contained in a regular neighborhood of \( T \setminus P \). Conversely, if \( P \) is a planar surface contained in the base surface of a capped grope \( G \) and \( P \) is disjoint from non-base surfaces and caps, then for the Whitney tower \( T \) obtained by Schneiderman’s construction, the base surface of \( T \) contains \( P \) and \( T \setminus P \) is contained in a regular neighborhood of \( G \setminus P \). In particular, this holds for the height \( h \) Whitney tower obtained in Corollary 3.2.

### 3.1. Modifications of Whitney towers and unknottedness

In the proof of Theorem 3.1 we will use that \( \pi_1 \)-unknottedness is preserved under certain modifications of Whitney towers and capped gropes, which are discussed in this and next subsections.

**Splitting.** In [Sch06, §3.7, §3.8], a splitting procedure was introduced to separate intersections in a Whitney tower. Suppose \( D \) is a Whitney disk between two sheets \( A \) and \( B \), and \( \beta \) is a properly embedded arc in \( D \) which joins \( A \cap \partial D \) to \( B \cap \partial D \) and avoids the intersection of \( D \) with other surfaces/disks. Then a finger move on \( A \) along \( \beta \) introduces two additional intersections between \( A \) and \( B \), and divides \( D \) into two new Whitney disks. See Figure 4 (for now ignore the disk \( \Delta \)). Repeated applying this splitting procedure, one eventually obtains a Whitney tower each of whose Whitney disks is embedded and has either a single unpaired intersection or two intersections paired by another Whitney disk.
Proposition 3.4. Suppose a Whitney tower $T'$ is obtained from another Whitney tower $T$ by splitting in a 4-manifold $W$. Then $E_{T'} \cong E_T \cup (2\text{-handles})$. Consequently there is an epimorphism of $\pi_1(W \setminus T)$ onto $\pi_1(W \setminus T')$ supported by a regular neighborhood of $T$. In addition, splitting changes a $\pi_1$-unknotted sphere-like Whitney tower to a $\pi_1$-unknotted Whitney tower in $S^4$.

Proof. Let $\Delta$ be the Whitney disk shown in Figure 4, which pairs the two new intersections of $T'$ introduced by the finger move. Let $L$ be the union of $T'$ and the 3-dimensional trace of the finger move. It is easily seen that $T$ expands cellularly to the 3-complex $L$. Also, the 2-complex $T' \cup \Delta$ expands cellularly to $L$: first thicken $\Delta$ to fill in the top part of the trace of the finger move in Figure 4 and then stretch it down to fill in the remaining part of the trace. By Lemma 2.15 it follows that the exterior $E_{T'} \cong E_{T' \cup \Delta}$. By Lemma 2.8 $E_{T'} \cong E_{T' \cup \Delta} \cup (2\text{-handles}) \cong E_T \cup (2\text{-handles})$. \hfill \Box

Tri-sheet move. The following modification, which we call a tri-sheet move in this paper, was introduced in [Sch06, Lemma 3.6]. Suppose $A$, $B$, and $C$ are surfaces/disks in a Whitney tower. Suppose $D$ is a Whitney disk which pairs two intersections between $A$ and $B$, and $p$ is an intersection between $D$ and $C$, as illustrated in the left hand side of Figure 5. We assume that $p$ is the only intersection of $D$. (By splitting, we may assume this for any unpaired intersection $p$.) If we denote a Whitney disk between $X$ and $Y$ by $(X,Y)$ and denote an intersection between $X$ and $Y$ by $\langle X,Y \rangle$, then we may write $p = \langle (A,B),C \rangle$. Apply an isotopy of $B$ in a regular neighborhood of $D$ and replace $D$ by another Whitney disk $D'$ as shown in the right hand side of Figure 5. This replaces $p = \langle (A,B),C \rangle$ by a new intersection $q = (A,(B,C))$. 
Proposition 3.5. If a Whitney tower $T'$ is obtained from another Whitney tower $T$ by a tri-sheet move in a 4-manifold $W$, then $E_{T'} \cong E_T$, and thus $\pi_1(W \setminus T') \cong \pi_1(W \setminus T)$. Consequently a tri-sheet move changes a $\pi_1$-unknotted (union-of-spheres)-like Whitney tower to a $\pi_1$-unknotted Whitney tower in $S^4$.

Proof. From Figure 5 it is seen that $T'$ cellularly expands to $T$ with thickened $D$. By Lemma 2.15 it follows that $E_{T'}$ is homeomorphic to $E_T$.

3.2. Modification of capped gropes and unknottedness

To prove Theorem 3.1 we also need the following observations on some modifications of capped gropes.

Proposition 3.6. If a capped grope $G'$ is obtained from another capped grope $G$ by pushing an intersection down in a 4-manifold $W$, then $E_{G'} \cong E_G \cup (2$-handle), and thus there is an epimorphism of $\pi_1(W \setminus G)$ onto $\pi_1(W \setminus G')$ supported by a regular neighborhood of $G$. Consequently, by pushing an intersection down, a $\pi_1$-unknotted sphere-like capped grope is changed to a $\pi_1$-unknotted capped grope in $S^4$.

Proof. Let $\Delta$ be the disk shown in the right hand side of Figure 6, and let $L$ be the union of $G$ and the trace of the finger move performed. It is seen from Figure 6 that both $G$ and $G' \cup \Delta$ expand cellularly to the 3-complex $L$. By Lemma 2.15 it follows that $E_{G' \cup \Delta}$ is homeomorphic to $E_G$. By Lemma 2.8 $E_{G'} \cong E_{G' \cup \Delta} \cup (2$-handle) $\cong E_G \cup (2$-handle).

Grope splitting. In [Kru00], Krushkal introduced an operation which splits caps and body surfaces of a grope. The cap splitting is described as follows. Suppose $D_1$ and $D_2$ are dual caps attached to a body surface $S$ of a capped grope immersed in a 4-manifold. Choose an embedded arc $\alpha$ in $D_1$, which is disjoint from intersection points and joins $\partial D_1 \cap D_2$ and another point in $\partial D_1$. Perform tubing (surgery) on $S$ along $\alpha$. The cap $D_1$ is divided into two disks which can be used as caps for the new surface obtained, and two parallel copies of $D_2$ can be used as their dual caps. See Figure 7 (ignore the hatched rectangles $\Delta_1$ for now).
Repeatedly applying this operation, one may assume that each cap has no self intersection and has at most one intersection point. (We may further assume that each cap has exactly one intersection point since we can apply contraction to remove a cap which has no intersection points.) Similarly to the cap case, one can split body surfaces: if $\Sigma_1$ and $\Sigma_2$ are dual surfaces attached to a previous stage surface $S$ in a capped grope $G$, then apply tubing to $S$ along an arc on $\Sigma_1$ and attach two parallel copies of the subgrope supported by $\Sigma_2$. If $\Sigma_1$ has genus greater than one, then by tubing along an appropriate arc, $\Sigma_1$ splits into two surfaces with genera less than that of $\Sigma_1$. Iterating this, we may assume that each non-base surface has genus one. We call such a capped grope a \textit{dyadic capped grope}.

In what follows we assume that a cap of a capped grope is embedded and can intersect the base surface only, by pushing intersections down if necessary.

\begin{proposition}
If a capped grope $G'$ is obtained from another capped grope $G$ by grope splitting in a 4-manifold $W$, then $E_G' \cong E_G \cup (\text{handles of index } \geq 2)$, and thus there is an epimorphism of $\pi_1(W \setminus G)$ onto $\pi_1(W \setminus G')$ supported by a regular neighborhood of $G$. Consequently, grope splitting changes a $\pi_1$-unknotted sphere-like capped grope to a $\pi_1$-unknotted capped grope in $S^4$.
\end{proposition}

\begin{proof}
We assert that there is a 2-complex $L$ in $W$ which contains $G'$ as a subcomplex and expands cellularly to a 3-complex $L'$ which collapses cellularly to $G$. From the assertion it follows that $E_G$ is homeomorphic to $E_L$ by Lemma 2.15. Since $L$ is obtained from $G'$ by attaching cells of dimension $\leq 2$, $E_G' \cong E_L \cup (\text{handles of index } \geq 2)$. This completes the proof.

The remaining part of this proof is devoted to showing the assertion.

The idea is easier to see for the case of cap splitting. If we split $G$ using a cap $D_1$ whose dual cap $D_2$ intersects other surfaces $k$ times, then it can be seen from Figure 7 (illustrated for $k = 2$) that there are $k$ 2-cells $\Delta_1, \ldots, \Delta_k$ shown as hatched rectangles such that the 2-complex $L := G' \cup (\bigcup \Delta_i)$ expands cellularly to $L' := (G$ with thickened $D_2) \cup (\text{solid tube})$: first expand the disk $\bigcup \Delta_i$ to fill in the interior of thickened $D_2$, and then stretch down its bottom to fill in the solid tube. It is obvious that the 3-complex $L'$ collapses cellularly to $G$.

The body surface splitting case could also be understood in a similar way, but this would require a more complicated picture which does not fit into 3-space. Instead, in what follows we present a more formal approach for both cap and body surface cases.

Suppose $\Sigma_1$ and $\Sigma_2$ are dual surfaces (or caps) in $G$ attached to the same body surface $S$, and $G'$ is obtained by splitting $\Sigma_1$. Let $G_i$ be the capped subgrope supported by $\Sigma_i$ for $i = 1, 2$. (If $\Sigma_i$ is a cap, then $G_i$ is $\Sigma_i$ itself.) Each $G_i$ is embedded since the caps
are embedded and can intersect the base surface of $G$ only. Let $V \cong I^3$ be the solid tube used to perform tubing on $S$. See Figure 8. Recall that $G'$ has two parallel copies of $G_2$ instead of $G_2$. Choose a thickening $G_2 \times I$ in such a way that the parallel copies are $G_2 \times 0$ and $G_2 \times 1$ and $(G_2 \times I) \cap V$ is a rectangle $R$ of the form (an interval in $\partial G_2) \times I$. In Figure 8 $R$ is shown as a hatched rectangle.

We choose a tree $K$ embedded in $G_2$ as follows. For brevity, for a body surface or cap $S_1$ dual to another body surface or cap $S_2$, we call the point $\partial S_1 \cap \partial S_2$ the basepoint of $S_1$ (and $S_2$). For the base surface of $G_2$, choose a basepoint in $\partial G_2$. For each intersection $p$ of a cap $D$ in $G_2$ and a sheet in $G$, choose an arc on $D$ which joins $p$ to the basepoint of $D$. For each basepoint $q$ of a surface/cap attached to a body surface $S$ in $G_2$, choose an arc on $S$ joining $q$ to the basepoint of $S$. We assume that the interiors of the arcs chosen above are pairwise disjoint and are disjoint from the boundary of any surface and cap. Finally choose two arcs in $\partial G_2$ whose intersection is the basepoint of the base surface of $G_2$. The union of all the above arcs is a tree $K$ in $G_2$ since $G_2$ is embedded. We regard the above arcs as an edge of $K$, and their endpoints as vertices of $K$. See Figure 9 for an example. In addition, we may assume that $\partial G_2 \setminus K \times I$ is equal to the rectangle $R$.

We choose a tree $K$ which expands cellularly to $G_2$. 

**Figure 8.** The solid tube and thickened subgrope used for grope splitting.

**Figure 9.** A tree $K$ which expands cellularly to $G_2$. 


Assertion 1. The tree $K$ expands cellulary to $G_2$.

Actually it is seen from Figure 8 that one can expand $K$ to fill in the caps, expand further to fill in the top stage body surfaces, and continue to the lower stages to eventually fill in the whole grope $G$. To make it rigorous, we will prove a generalized statement that for any capped subgrope $H$ in $G_2$, $K \cap H$ expands cellulary to $H$, by an induction on the number $n$ of surfaces/caps in $H$. Assertion 4 is the case when $H = G_2$. If $n = 1$, then $H$ is a cap, and since $H$ cut along $K \cap H$ is a disk, $K \cap H$ expands cellulary to $H$. Suppose $n > 1$, that is, $H$ consists of a body surface $S$ and subgropes $H_i$ attached to $S$. By induction, $H_i \cap K$ expands cellulary to $H_i$. Thus it suffices to show that $(\bigcup \partial H_i) \cup (K \cap S) \subset S$ expands cellulary to $S$. It is true since $S$ cut along $(\bigcup \partial H_i) \cup (K \cap S)$ is a disk. This completes the induction and hence the proof of Assertion 1.

Let $L$ be the intersection of $G_2 \times I$ with sheets in the initial capped grope $G$. We may assume that each component of $A$ is a straight arc of the form $v \times I \subset G_2 \times I$ for some vertex $v$ of $K$, since the intersection of a cap of $G_2$ with a sheet is always a vertex of $K$.

Let $L_0 := (G_2 \times \{0,1\}) \cup (K \times I)$. Observe that $L_0$ is the 2-complex obtained from the 2-complex $(G_2 \times \{0,1\}) \cup A$ by attaching 1-cells of the form $v \times I$ for vertices $v$ of $K$ not intersecting any sheet, and attaching 2-cells of the form $e \times I$ for edges $e$ of $K$.

Assertion 2 follows from Assertion 1 by applying the following, which must be regarded as a known fact:

Lemma 3.8. If a subcomplex $K$ of $X$ expands cellulary to $X$, then the subcomplex $(X \times \{0,1\}) \cup (K \times [0,1])$ expands cellulary to $X \times [0,1]$.

Proof. For an elementary cellular expansion of a complex $K$ in $X$ across an $n$-disk $\Delta$ such that $\partial_0 \Delta := \Delta \cap K$ is an $(n-1)$-disk embedded in $\partial \Delta$, $\Delta \times [0,1]$ is an $(n+1)$-disk in $X \times I$ such that

$$(\Delta \times [0,1]) \cap ((X \times \{0,1\}) \cup (K \times [0,1])) = (\Delta \times \{0,1\}) \cup (\partial_0 \Delta \times [0,1])$$

is an $n$-disk embedded in $\partial (\Delta \times [0,1])$. Thus there is an elementary cellular expansion of $(X \times \{0,1\}) \cup (K \times [0,1])$ across $\Delta \times [0,1]$. By repeatedly applying this, the desired conclusion is obtained. \qed

Let $G''$ be $G'$ with $G_2 \times \{0,1\}$ removed. Let $L := G'' \cup L_0$ and $L' := G'' \cup (G_2 \times I) \cup V$. We will verify that $L$ and $L'$ satisfy the properties promised at the beginning of the proof. Obviously $L'$ collapses cellulary to $G$; see Figure 8. Since $G_2 \times \{0,1\} \subset L_0$, $L$ is a 2-complex containing $G'$ as a subcomplex. Since $R = \partial G_2 \times K \times I$, we have

$$(G_2 \times I) \cap G'' = (\partial G_2 \times I) \cap R = (K \cap \partial G_2) \times I = L_0 \cap G'' \subset L_0.$$  

See Figures 8 and 9. It follows that the cellular expansion of $L_0$ to $G_2 \times I$ in Assertion 2 is indeed a cellular expansion of $L = G'' \cup L_0$ to $G'' \cup (G_2 \times I)$. Since

$$(G'' \cup (G_2 \times I)) \cap V = \partial V \setminus \text{bottom face of } V$$

is a disk (see Figure 8), there is a cellular expansion of $G'' \cup (G_2 \times I)$ to $L'$ across the cube $V$. Consequently $L$ expands cellulary to $L'$. This completes the proof. \qed

3.3. Proof of Theorem 3.1

From Whitney towers to capped gropes. Suppose $T$ is a Whitney tower in a 4-manifold $W$. In [Sch06, Section 5.1], a capped grope $G$ is constructed from $T$ by first applying Whitney tower splitting and then applying a sequence of tri-sheet moves and the following operation:
**Tubing on a cap.** Suppose a dyadic capped grope immersed in a 4-manifold is given. Suppose no cap has self intersections while a cap may intersect base surfaces. Suppose the union of caps and base surfaces supports a Whitney tower disjoint from non-base body surfaces. We regard the union of the capped grope and the Whitney tower as a 2-complex, say $P$. (In particular a split grope subtower introduced in [Sch06, Section 4.2] is such a 2-complex; we will not need its precise definition.) Suppose $D$ is a cap of the capped grope which has two intersections paired by a Whitney disk $\Delta$. Perform tubing on $D$ along $\partial \Delta \setminus D$ and attach, as new caps, a meridional disk of the tube and $\Delta$ with a collar neighborhood removed. See Figure 10. This gives us a new 2-complex, say $P'$, which consist of a capped grope and a Whitney tower.

![Figure 10. Tubing on a cap $D$ which has two intersections paired by a Whitney disk $\Delta$.](image)

Observe that the above tubing operation preserves the homeomorphism type of the 2-complex exterior. For, from Figure 10 it is seen that the new 2-complex $P'$ expands cellulary to the union $P''$ of the solid tube and the initial 2-complex $P$, and then $P''$ collapses cellulary to $P$. By Lemma 2.15 the exterior is preserved.

From this and Propositions 3.4 and 3.5, it follows that $\pi_1(W \setminus G)$ is a quotient of $\pi_1(W \setminus T)$. This proves Theorem 3.1(1).

**From capped gropes to Whitney towers.** Suppose $G$ is a capped grope in a 4-manifold $W$. In [Sch06, Section 5.2], a Whitney tower $T$ is constructed from $G$ by first pushing intersections down and applying grope splitting, and then applying a sequence of tri-sheet moves and the following operation:

**Surgery along a cap.** Suppose $P$ is a 2-complex consisting of a dyadic capped grope and a Whitney tower supported by caps and base surfaces as in the case of tubing on a cap. Suppose $D_1$ and $D_2$ are dual caps attached to the same body surface $S$, and $D_1$ intersects exactly one sheet $A$ of the Whitney tower. Replace the subgrope $S \cup D_1 \cup D_2$ by a cap obtained by ambient surgery on $S$ using $D_1$. Note that this new cap has two intersections with the sheet $A$, which are paired by a Whitney disk obtained by attaching to $D_2$ a band contained in $\nu(D_1)$, as shown in Figure 11. Add this disk to the Whitney tower. This gives us a new 2-complex $P'$ which consists of the modified capped grope and the modified Whitney tower.

Although we start with a capped grope, this surgery operation produces a hybrid 2-complex of a capped grope and a nontrivial Whitney tower. Thus the use of a tri-sheet move, which is defined for a Whitney tower, makes sense in the above conversion process.
In the literature constructions of gropes in a 4-manifold bounded by knots were developed for the study of concordance and related 4-dimensional equivalence relations. For instance, see [CT07, Hor10, Cha14b, CP14]. In this section we study the handle decomposition of the exterior of such gropes. We will use this in the next sections to construct certain explicit unknotted gropes and Whitney towers in $S^4$.

4. Handle decomposition of grope exteriors

In the literature constructions of gropes in a 4-manifold bounded by knots were developed for the study of concordance and related 4-dimensional equivalence relations. For instance, see [CT07, Hor10, Cha14b, CP14]. In this section we study the handle decomposition of the exterior of such gropes. We will use this in the next sections to construct certain explicit unknotted gropes and Whitney towers in $S^4$.

4.1. Satellite capped gropes and capped grope concordances

We begin by presenting some definitions, which appeared or are influenced by earlier papers [Cha14b, CT07, Hor10]. From now on all immersed capped gropes are assumed to satisfy that each cap can intersect the base surface only. (Push intersections down as described in Section 3.2 if necessary.)

**Definition 4.1** (Satellite capped grope [Cha14b, Definition 4.2]). Suppose $L$ is a link in $S^3$ and $\alpha$ is an unknotted simple closed curve in $S^3$ which is disjoint from $L$. Let $E_\alpha$ be the exterior of $\alpha \subset S^3$. Let $\lambda_\alpha$ be a zero linking longitude in $\partial E_\alpha$. A satellite capped grope for $(L, \alpha)$ is a disk-like capped grope $G$ immersed in the 4-manifold $E_\alpha \times I$ such that $G$ is bounded by $\lambda_\alpha \times 0$, the body of $G$ is disjoint from $L \times I$, and the caps are transverse to $L \times I$.

If $(L, \alpha)$ is as above, $L$ may be viewed as a pattern in $E_\alpha \cong D^2 \times S^1$ for a satellite construction: for a knot $K$, by attaching $(E_\alpha, L)$ to $E_K$ along an orientation reversing homeomorphism $\partial E_\alpha \cong \partial E_K$ which identifies a meridian and a zero linking longitude of $\alpha$ with a zero linking longitude and a meridian of $K$ respectively, we obtain a satellite link in $E_\alpha \cup_{\partial} E_K \cong S^3$. We denote this link by $L(\alpha, K)$ or $L(\alpha; K)$.

**Definition 4.2** (Product of satellite capped gropes). Suppose $G \subset E_\alpha \times I$ and $H \subset E_\eta \times I$ are satellite capped gropes for $(L, \alpha)$ and $(K, \eta)$ respectively. View $\eta$ as a curve in $E_K \subset E_{L(\alpha, K)}$. The product $G \cdot H$ of $G$ and $H$ is a satellite capped grope for $(L(\alpha, K), \eta)$ described below. Note $E_\eta = E_{\eta; K} \cup E_\alpha$. By an isotopy of caps of $H$ if necessary, we may assume that $H \cap (\partial E_{\eta; K} \times I)$ consists of circles of the form $\mu_K \times t$ where $\mu_K$ is a meridian of $K$ and $0 < t < 1$. For each $t < 1$, take a copy $G_t$ of $G$ in $E_\alpha \times [t, t + \epsilon] \cong E_\alpha \times I$ and attach $G_t$ to $H \cap (E_{\eta; K} \times I)$ along $\mu_K \times t = \partial G_t$. This gives a desired satellite capped grope for $(L(\alpha, K), \eta)$ in $E_\eta \times I = (E_{\eta; K} \cup E_\alpha) \times I$ [Cha14b, Proposition 4.3].
The product $G \cdot H$ is obtained by removing disjoint disks embedded in caps of $H$ and then fill in it with copies of $G$. It follows that if $G$ and $H$ have height $h$ and $k$ respectively, then $G \cdot H$ has height $h + k$.

**Definition 4.4** (Capped grope concordance). A capped grope concordance between two knots $L$ and $L'$ is a (union-of-annuli)-like capped grope immersed in $S^1 \times I$ such that the $i$th component of the base surface is cobounded by that of $L \times 0$ and that of $L' \times 1$.

In the literature, a grope concordance without caps is often considered. A grope concordance can be promoted to a capped grope concordance since $S^3 \times I$ is simply connected.

The following is motivated by [CT07, Theorem 3.8], [Hor10, Theorem 3.4].

**Definition 4.4** (Product of a satellite capped grope and a capped grope concordance). Suppose $G$ is a satellite capped grope for $(L, \alpha)$ and $H$ is a capped grope concordance between two knots $J$ and $J'$. The product $G \cdot H$ is a capped grope concordance between $L(\alpha, J)$ and $L(\alpha, J')$ described below. Fix closed intervals $U \subset S^1$, $V \subset (0, 1) \subset I$, and regard $H$ as the annulus $S^1 \times I$ with the disk $U \times V$ replaced by a disk-like capped grope $B$ with $\partial B = \partial (U \times V)$:

$$H = (S^1 \times I) \setminus (U \times V) \cup_{\partial (U \times V) = \partial B} B \subset S^3 \times I.$$ 

Here caps of $B$ may be plumbed with $(S^1 \times I) \setminus (U \times V)$. A regular neighborhood of $H$ can be written as

$$\nu(H) = (S^1 \times D^2 \times I) \setminus (U \times D^2 \times V) \cup_{S^1 \times D^2} \nu(B) \subset S^3 \times I$$

with plumbings performed. By an isotopy of $L \subset E_\alpha = S^1 \times D^2$, we may assume $L \cap (U \times D^2) = U \times \{p_1, \ldots, p_r\}$ for some $p_i \in D^2$. Then $L \times I \subset S^1 \times D^2 \times I$ intersects the 4-ball $U \times D^2 \times V$ at disks $U \times \{p_1, \ldots, p_r\} \times V$. Choose $r$ parallel copies $B_1, \ldots, B_r$ of $B$ such that $\partial B_i = \partial (U \times p_i \times V)$, and consider the capped grope

$$(L \times I) \setminus U \times \{p_1, \ldots, p_r\} \times V \cup_{\partial (U \times p_i \times V) = \partial B_i} \left( \bigcup B_i \right)$$

in $\nu(G)$. By isotopy, we may assume that caps of $B_i$ intersect $S^1 \times D^2 \times I$ at disks of the form $z_j \times D^2 \times t_j$ where $z_j \in S^1$, $t_j \in (0, 1) \times V$. Choose sufficiently small $\epsilon > 0$, and replace each disk $z_j \times D^2 \times t_j$ in caps of $B_i$ with a copy of the satellite capped grope $G$ in $S^1 \times D^2 \times [t_j, t_j + \epsilon) \cong E_\alpha \times I$. This gives a promised capped grope concordance $G \cdot H$ between $L(\alpha, J)$ and $L(\alpha, J')$.

Similarly to the case of Definition 4.2, if $G$ and $H$ have height $h$ and $k$ respectively, then the product $G \cdot H$ in Definition 4.4 has height $h + k$.

**Remark 4.5.** The product operations defined above are associative. Precisely, if $G_i$ is a satellite capped grope for $(K_i, \alpha_i)$ for $i = 1, 2, 3$, then $(G_1 \cdot G_2) \cdot G_3$ and $G_1 \cdot (G_2 \cdot G_3)$ are isotopic satellite capped gropes for

$$(K_1(\alpha_1, K_2))(\alpha_2, K_3), \alpha_3) \approx (K_1(\alpha_1, K_2(\alpha_2, K_3)), \alpha_3).$$

Also, if $H$ is a capped grope concordance between $J$ and $J'$, then $(G_1 \cdot G_2) \cdot H$ and $G_1 \cdot (G_2 \cdot H)$ are isotopic capped grope concordances between

$$(K_1(\alpha_1, K_2))(\alpha_2, J) \approx K_1(\alpha_1, K_2(\alpha_2, J))$$

and

$$(K_1(\alpha_1, K_2))(\alpha_2, J') \approx K_1(\alpha_1, K_2(\alpha_2, J')).$$

The proofs are straightforward. Since we do not use this in this paper, we omit details.
4.2. Handle structure of grope exteriors

In the literature, satellite capped gropes and grope concordances in a 4-manifold are often obtained by pushing capped gropes in the boundary 3-manifold. For instance see [CT07, Hor10, Cha14b, CP14]. In this subsection we will investigate handle decomposition of the 4-dimensional exteriors of such capped gropes and their iterated products obtained by the constructions described in Definitions 4.1 and 4.2.

To state the result rigorously, we use the following definitions.

**Definition 4.6.**

1. We say that $G$ is a 3D satellite capped grope for $(L, \alpha)$ if $G$ is a capped grope embedded in $S^3$ bounded by an unknotted circle $\alpha$, and $L$ is a link in $S^3$ disjoint from the body of $G$ and transverse to caps of $G$. By pushing $G \subset S^3 = S^3 \times 0$ into $S^3 \times (0, 1)$, a satellite capped grope for $(L, \alpha)$ is obtained.

2. We say that $G$ is a 3D capped grope concordance between two links $L$ and $L'$ in $S^3$ if the following hold: (i) $L'$ is obtained from $L$ by replacing disjoint arcs $\gamma_1, \ldots, \gamma_k \subset L$ with arcs $\gamma'_1, \ldots, \gamma'_k \subset S^3$ such that $\partial \gamma_i = \partial \gamma'_i$, that is, letting $C := L \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$, $L' = C \cup \gamma'_1 \cup \cdots \cup \gamma'_k$; (ii) $G$ is an embedded (union-of-disks)-like capped grope in $S^3$ bounded by the circles $\gamma_i \cup \gamma'_i$ such that the body is disjoint from $C$ and the caps are transverse to $C$. Note that the capped grope

$$(L \times [0, \frac{1}{2}]) \cup (G \times \frac{1}{2}) \cup (L' \times \frac{1}{2}, 1)$$

is a capped grope concordance in $S^3 \times I$ between $L$ and $L'$. We say that it is obtained by pushing $G$ into $S^3 \times I$.

**Theorem 4.7.** Suppose $H$ is a capped grope concordance which is obtained by pushing a 3D capped grope concordance between two links $J$ and $J'$ into $S^3 \times I$. For $i = 1, \ldots, n$, suppose $G_i$ is a satellite capped grope obtained by pushing a 3D satellite capped grope for $(K_i, \alpha_i)$ into $S^3 \times I$, where $K_i$ is a knot. Let $G$ be the iterated product

$$G = G_1 \cdot G_2 \cdots G_n \cdot H$$

which is a capped grope concordance between

$$K := K_1(\alpha_1, \ldots, K_n(\alpha_n, J) \cdots) \quad \text{and} \quad K' := K_1(\alpha_1, \cdots, K_n(\alpha_n, J') \cdots).$$

Then the exterior $E_G$ in $S^3 \times I$ has a handle decomposition

$$E_G \cong (E_K \times I) \cup (2\text{-handles}).$$

In the proof of Theorem 4.7 we will show that the conclusion holds for any choice of parenthesization for the product $G_1 \cdots G_n \cdot H$. (Indeed, the product is well defined up to isotopy by Remark 4.3.)

Our approach may be compared with the standard methods in embedded Morse theory, which is used to construct a handle decomposition of the exterior of an embedded submanifold in $M \times I$ from the critical points of the submanifolds. We will present an analog for the gropes in Theorem 4.7 which are not submanifolds but 2-complexes.

**Near a surface stage (critical level of type $A$).** First we consider a surface stage of a capped grope together with a collar neighborhood of the boundary of next stages attached to it. This is explicitly described as follows. Suppose $\Sigma$ is a surface of genus $g$ with connected nonempty boundary, which is embedded in a 3-manifold $M$. Suppose $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are standard symplectic basis curves on $\Sigma$, that is, they are simple closed curves such that any two of them are disjoint except $\alpha_i \cap \beta_i = \{\text{one point}\}$. Choose a
bicollar $\Sigma \times [-\epsilon, \epsilon]$ of $\Sigma = \Sigma \times 0$ in $M$, and let
\[
\Sigma' := \Sigma \cup \left( \bigcup \alpha_i \times [0, \epsilon] \right) \cup \left( \bigcup \beta_i \times [-\epsilon, 0] \right) \subset M,
\]
\[
\partial_- \Sigma' := \partial \Sigma \subset \Sigma',
\]
\[
\partial_+ \Sigma' := \left( \bigcup \alpha_i \times \epsilon \right) \cup \left( \bigcup \beta_i \times (-\epsilon) \right) \subset \Sigma'.
\]

Now, in the 4-manifold $M \times I$, let
\[
A := (\partial_- \Sigma' \times [0, \frac{1}{2}]) \cup (\Sigma' \times \frac{1}{2}) \cup (\partial_+ \Sigma' \times [\frac{1}{2}, 1]).
\]

Let $E_A = M \times I \setminus \nu(A)$ and $E_{\partial_- \Sigma'} = M \setminus \nu(\partial_- \Sigma')$ be the exteriors.

**Proposition 4.8.** The 4-manifold $E_A$ has a handle decomposition
\[
E_A \cong (E_{\partial_- \Sigma'} \times I) \cup \left( (2g - 1) \text{-handles} \right).
\]

**Proof.** Choose a regular neighborhood $Y := \nu(\Sigma')$ in $M$ and choose regular neighborhood $Y_+ := \nu(\partial_+ \Sigma')$ and $Y_- := \nu(\partial_- \Sigma')$ lying in the interior of $Y$. The exterior $E_A$ is homeomorphic to
\[
M \times I \setminus \left( Y_- \times [0, \frac{1}{3}] \cup Y \times [\frac{1}{3}, \frac{2}{3}] \cup Y_+ \times [\frac{2}{3}, 1] \right).
\]

Fix a point $*$ on $\partial \Sigma$. Choose arcs $\gamma_i$ $(i = 1, \ldots, g)$ on $\Sigma$ joining $*$ to $\alpha_i \cap \beta_i$ in such a way that $\Sigma$ collapses cellularly (and thus is homotopy equivalent) to $K := \bigcup (\alpha_i \cup \beta_i \cup \gamma_i)$. See Figure 12. Since $\Sigma'$ collapses cellularly to $\Sigma$, $Y = \nu(\Sigma')$ is isotopic to a regular neighborhood of $K$.

![Figure 12. A regular neighborhood of the 2-complex $\Sigma'$.](image)

From this it is seen that $Y \setminus Y_+$ is obtained by attaching $(2g - 1)$ 2-handles to a collar of $\partial Y = \partial Y \setminus \partial Y_+$ (in this case $\partial Y \cap \partial Y_+ = \emptyset$); the attaching circles are shown in Figure 12. The proof is completed by applying the following fact, which we state as a lemma for later use as well. \qed
Lemma 4.9. Suppose $Y_+, Y_-, Y$ are compact codimension zero submanifolds in a 3-manifold $M$ such that $Y_+, Y_- \subset Y$. Suppose $Y \setminus Y_+$ is obtained from a regular neighborhood of $\partial Y \setminus \partial Y_+$ in $Y$ by attachments of handles $h_1^{p_1}, \ldots, h_k^{p_k}$ of index $p_i$. That is, there is a decomposition

$$Y = \nu(\partial Y \setminus \partial Y_+) \cup h_1^{p_1} \cup \cdots \cup h_k^{p_k} \cup Y_+.$$ 

Then the 4-manifold

$$(4.1) \quad W := M \times I \setminus \left(Y_- \times [0, \frac{1}{2}] \cup Y \times [\frac{1}{2}, \frac{2}{3}] \cup Y_+ \times [\frac{2}{3}, 1]\right)$$

has a handle decomposition

$$W \cong M \setminus Y_- \times I \cup H_1^{p_1} \cup \cdots \cup H_k^{p_k},$$

where $H_i^{p_i}$ is a handle of index $p_i$.

![Figure 13. The 4-manifold $W$ in Lemma 4.9](image)

Proof. Let $W_t = W \cap M \times [0, t]$ for $t \in I$. We will investigate how $W_t$ changes as $t$ increases. See the schematic picture in Figure 13. For $t \leq \frac{1}{2}$, $W_t = M \setminus Y_- \times [0, t] \cong M \setminus Y_- \times I$ obviously. For $\frac{1}{2} < t < \frac{2}{3}$, we still have $W_t \cong M \setminus Y_- \times I$, since $W_t$ is obtained from $M \setminus Y_- \times [0, t]$ by pushing the codimension zero submanifold $Y \setminus Y_- \times [\frac{1}{2}, t]$ into its complement. (This can be viewed as, for instance, repeatedly applying elementary cellular collapsing across the 4-balls $\Delta^3 \times [\frac{1}{2}, t]$ where $\Delta^3$ is a 3-simplex of $Y \setminus Y_-$. ) For $t > \frac{2}{3}$, fixing $t_0 \in (\frac{1}{2}, \frac{2}{3})$, we have

$$W_t \cong W_{t_0} \cup Y_+ \times [\frac{1}{2}, t_0].$$

The attachment of $Y_+ \times [\frac{1}{2}, t_0]$ to $W_t$ along $\partial Y_+ \setminus \partial Y_+ \times [\frac{1}{2}, t_0]$ is equivalent to attachments of handles $H_i^{p_i} := h_i^{p_i} \times [\frac{1}{2}, t_0]$ of index $p_i$, since $Y \setminus Y_+$ is obtained by attaching the handles $h_i^{p_i}$ to a collar of $\partial Y \setminus \partial Y_+$. This shows that $W$ has the promised handle decomposition.

Near a plumbing point (critical level of type $B$). Suppose $D$ is a 2-disk embedded in a 3-manifold $M$, and $L$ is a 1-submanifold in $M$ which is disjoint from $\partial D$ and meets $D$ at a single transverse intersection point. In the 4-manifold $M \times I$, let

$$B := (L \times I) \cup (\partial D \times [0, \frac{1}{2}]) \cup (D \times \frac{1}{2})$$

Let $E_B = M \times I \setminus \nu(B)$ and $E_{L \cup \partial D} = M \setminus \nu(L \cup \partial D)$ be the exteriors.

Proposition 4.10. The 4-manifold $E_B$ is diffeomorphic to $E_{L \cup \partial D} \times I$. 


Proof. We proceed similarly to the proof of Proposition 4.8. Let \( Y = \nu(L \cup D) \) be a regular neighborhood in \( M \) and choose regular neighborhoods \( Y^+ = \nu(L) \) and \( Y^- = \nu(L \cup \partial D) \) which are contained in the interior of \( Y \). Then \( E_B \) is diffeomorphic to the 4-manifold described in (4.1) of Lemma 4.9. Since \( D \) intersects \( L \) at one point, \( Y \approx \nu(L) \) and \( Y \setminus Y^+ \) is just a collar of \( \partial Y = \partial Y \setminus \partial Y^+ \). By Lemma 4.9 it follows that \( E_B \cong E_{L \cup \partial D} \times I \), without adding any handle.

Near punctured surfaces (critical levels of type \( R \) and \( S \)). We also need the following two cases. First, we consider a “horizontal” punctured disk described as follows. Suppose \( \Gamma \) is a planar surface embedded in a 3-manifold \( M \). Let \( \partial^- \Gamma \) be a boundary component of \( \Gamma \), and \( \partial^+ \Gamma := \partial \Gamma \setminus \partial^- \Gamma \). We assume both \( \partial^+ \Gamma \) and \( \partial^- \Gamma \) are nonempty. Consider the surface
\[
R := (\partial^- \Gamma \times [0, \frac{1}{2}]) \cup (\Gamma \times \frac{1}{2}) \cup (\partial^+ \Gamma \times [\frac{1}{2}, 1])
\]
embedded in \( M \times I \). Let \( E_R = M \times I \setminus \nu(R) \) and \( E_{\partial^- \Gamma} = M \setminus \nu(\partial^- \Gamma) \) be the exteriors.

**Proposition 4.11.** The 4-manifold \( E_R \) has a handle decomposition
\[
E_R \cong (E_{\partial^- \Gamma} \times I) \cup ((k - 1) \text{ 2-handles})
\]
where \( k \) is the number of components of \( \partial^+ \Gamma \).

The proof will be given together with that of Proposition 4.12 which treats a “vertical” punctured sheet described as follows. Suppose \( C_- \) is a 1-submanifold in a 3-manifold \( M \) and \( A \) is an annulus embedded in \( M \) with two boundary circles \( \partial_0 A, \partial_1 A \) such that \( J_- := \partial_0 A \cap C_- \) is an arc and \( A \setminus J_- \) is disjoint from \( C_- \). Let \( J_+ = \partial_0 A \setminus J_- \) and \( C_+ = (C_- \setminus J_-) \cup J_+ \). Let
\[
S = (C_- \times [0, \frac{1}{2}]) \cup ((C_- \cup A) \times \frac{1}{2}) \cup ((C_+ \cup \partial_1 A) \times [\frac{1}{2}, 1]) \subset M \times I.
\]
Note that \( S \) is homeomorphic to \( C_- \times I \) with a disk removed, and embedded in \( M \times I \) in such a way that the boundary of the removed disk appears as a circle \( \partial_1 A \times 1 \). See the schematic picture in Figure 14. Let \( E_S = M \times I \setminus \nu(S) \) and \( E_{C_-} = M \setminus \nu(C_-) \) be the exteriors.

![Figure 14. An embedded punctured sheet.](image)

**Proposition 4.12.** The 4-manifold \( E_S \) has a handle decomposition
\[
E_S \cong (E_{C_-} \times I) \cup (2\text{-handle}).
\]
Proofs of Propositions 4.11 and 4.12. Since $R$ and $S$ are embedded submanifolds, we can apply the standard embedded Morse theory to obtain a handle decomposition of the exteriors (e.g., see [GS99 Proposition 6.2.1]); briefly, in our case, a $p$-handle of the submanifold corresponds to a $(p + 1)$-handle of the exterior. Since $R \cong (\partial^- \Gamma \times I) \cup ((k - 1) 1$-handles), $E_R$ has a handle decomposition with $(k - 1)$ 2-handles as claimed. Since $S \cong (C_ - \times I) \cup (1$-handle), $E_S$ has a handle decomposition with a 2-handle as claimed. Alternatively, one may use Lemma 4.9 similarly to the proofs of Propositions 4.8 and 4.10. The attaching circles for each case are shown in Figure 15. 

![Diagram](https://example.com/diagram.png)

**Figure 15.** Regular neighborhoods of $\Gamma$ and $C_ - \cup A$.

We are now almost ready for the proof of Theorem 4.7. For clarity in the proof we will use the following definition.

**Definition 4.13.** Suppose $M$ is a 3-manifold. We say that a 2-complex $G$ in $M \times I$ is **ABRS-admissible** if for some $0 < t_1 < \cdots < t_r < 1$ and $\epsilon > 0$ the following hold.

1. $G \cap (M \times J)$ is an isotopy of an 1-submanifold in $M$ for each subinterval $J = [0, t_1 - \epsilon], [t_1 + \epsilon, t_2 - \epsilon], \ldots, [t_{r - 1} + \epsilon, t_r - \epsilon]$, and $[t_r + \epsilon, 1].$

2. For each $i$, $G \cap (M \times [t_i - \epsilon, t_i + \epsilon])$ is the disjoint union of $L \times [t_i - \epsilon, t_i + \epsilon]$ and a 2-complex $Z$, where $L$ is a link in $M$ and $Z$ is of the form of either $A, B, R,$ or $S$ in Propositions 4.8, 4.10, 4.11, and 4.12.

We call each $M \times t_i$ a **critical level of type** $A, B, R,$ or $S$.

The following are basic observations on the ABRS-admissibility of capped gropes.

1. For a height $n$ satellite capped grope $G$ in $S^3 \times I$ obtained by pushing a 3D satellite capped grope $G_0$ for $(K, \alpha)$, we may assume that $G \cup (K \times I) \subset E_\alpha \times I$ is ABRS-admissible; for instance, choose sufficiently small $\epsilon > 0$, and push a height $k$ surface $S \subset E_\alpha = E_\alpha \times 0$ in $G_0$ to $(\partial S \times [(k - 1)\epsilon, k\epsilon]) \cup (S \times k\epsilon) \subset E_\alpha \times I$, $1 \leq k \leq n$. For a cap $C$ in $G_0$, which intersects $n$ times $K$ transversely, let $\Gamma$ be the corresponding punctured cap. That is, $\Gamma$ is a planar surface contained in $C$ such that $\Gamma$ does not intersect $K$ and $\partial \Gamma = \partial_+ \Gamma \cup \partial_\ast \Gamma$ where $\partial_+ \Gamma = \partial C$ and $\partial_\ast \Gamma$ are $m$ meridional curves $\cup_{i=1}^m \mu_i$ of $K$. Then push $C$ to $(\partial_+ \Gamma \times [n\epsilon, (n + 1)\epsilon]) \cup (\Gamma \times (n + 1)\epsilon) \cup (\partial_\ast \Gamma \times [(n + 1)\epsilon, (n + 2)\epsilon]) \cup A$ in $E_\alpha \times I$ where

$$A = \bigcup_{i=1}^m [(\mu_i \times [(n + 2\epsilon, (n + 2 + i)\epsilon)] \cup (D_i \times (n + 2 + i)\epsilon))]$$

for a (planar) meridional disk $D_i$ with boundary $\mu_i$. Push the remaining caps further in a similar way. In this case, we use types $A, B,$ and $R.$
(2) We may assume that a capped grope concordance in \(S^3 \times I\) obtained by pushing a 3D capped grope concordance is ABRS-admissible, by a similar isotopy. Critical levels have types \(A, B, R,\) and \(S.\)

(3) If \(G\) and \(H\) are satellite capped gropes for \((K, \alpha)\) and \((J, \beta)\) such that
\[
G \cup (K \times I) \subset E_\alpha \times I \quad \text{and} \quad H \cup (J \times I) \subset E_\beta \times I
\]
are ABRS-admissible, then
\[
(G \cdot H) \cup (K(\alpha, J) \times I) \subset E_\beta \times I
\]
is ABRS-admissible. It is verified straightforwardly by inspecting Definition 5.2.

In this case, we use type \(R\) when we attach gropes to a punctured cap of another grope.

(4) If \(G\) is a satellite capped grope for \((K, \alpha)\) such that \(G \cup \{K \times I\} \subset E_\alpha \times I\) is ABRS-admissible and \(H\) is an ABRS-admissible capped grope concordance in \(S^3 \times I\), then \(G \cdot H\) is an ABRS-admissible capped grope concordance. It is verified straightforwardly by inspecting Definition 4.12. In general, we need all the types \(A, B, R,\) and \(S.\)

Proof of Theorem 4.7. Repeatedly applying the above observations, it follows that the given product \(G = G_1 \cdots G_n \cdot H\) is an ABRS-admissible capped grope concordance in \(S^3 \times I\). This is true for any choice of parenthesization of the product. Near each critical level \(S^3 \times t_i\), apply one of Propositions 1.1, 4.11, 4.12, and 4.12 to obtain 2-handles. (Here, writing \(G \cap (S^3 \times [t_i - \epsilon, t_i + \epsilon]) = (L \times [t_i - \epsilon, t_i + \epsilon]) \cup Z\) as in Definition 4.11, we apply the proposition to \(Z\) in the product \(E_L \times [t_i - \epsilon, t_i + \epsilon].\) Stacking them, we obtain a desired handle decomposition of \(E_G\) with only 2-handles added to \(E_K \times I.\) \(\square\)

5. Knots in \(S^3\) sliced by unknotted Whitney towers and gropes

5.1. Whitney tower and grope bi-filtrations

We begin by recalling the classical notion of doubly slice knots. In what follows, regard \(S^3 \subset S^4\) in the standard way.

Definition 5.1. A knot \(K\) in \(S^3\) is doubly slice if there is an unknotted flat 2-sphere embedded in \(S^4\) which intersects \(S^3\) transversely at \(K.\)

Considering our Whitney tower and grope generalizations of unknotted 2-spheres, we are naturally led to the following generalization. Denote by \(D^4_+\) and \(D^4_-\) the upper and lower hemispheres of \(S^4\) bounded by \(S^3.\)

Definition 5.2. (1) We say that a knot \(K\) in \(S^3\) is a slice of a Whitney tower \(T\) in \(S^4\) if the base sphere of \(T\) intersects \(S^3\) transversely and \(K = T \cap S^3.\) In this case, each \(T_\pm := T \cap D^4_\pm\) is a disk-like Whitney tower bounded by \(K.\)

(2) For half-integers \(m, n \geq 1,\) a knot \(K\) in \(S^3\) is a height \((m, n)\) Whitney slice if it is a slice of a \(\pi_1\)-unknotted sphere-like Whitney tower \(T\) in \(S^4\) such that \(T_+\) and \(T_-\) have height \(m\) and \(n\) respectively. We denote by \(W_{m,n}\) the set of height \((m, n)\) Whitney slice knots. A knot \(K\) is height \(m\) Whitney doubly slice if \(K \in W_{m,m}.\)

Define a height \((m, n)\) grope slice knot and a height \(m\) grope doubly slice knot by replacing Whitney towers with capped gropes. Denote by \(G_{m,n}\) the set of height \((m, n)\) grope slice knots.

Using a Seifert-van Kampen argument one can show that \(W_{m,n}\) and \(G_{m,n}\) are closed under connected sum, that is, \(W_{m,n}\) and \(G_{m,n}\) are submonoids of the monoid of knots. By
Propositions 2.10 and 2.13 we have $W_{k, ℓ} ⊂ W_{m,n}$ and $G_{k, ℓ} ⊂ G_{m,n}$ for $k ≥ m$ and $ℓ ≥ n$. Therefore we obtain bi-filtrations \{W_{m,n}\} and \{G_{m,n}\} of the monoid of knots, which we call the Whitney tower bi-filtration and the grope bi-filtration, respectively.

The Whitney tower and grope cases are related as follows in this context.

\textbf{Theorem 5.3.} A height $(m, n)$ grope slice knot is height $(m, n)$ Whitney slice. That is, $G_{m,n} ⊂ W_{m,n}$.

\textit{Proof.} Suppose $K$ is a slice of a $\pi_1$-unknotted capped grope $G$ in $S^4$ and $G_{±} := G ∩ D^4_{±}$ have height $m$ and $n$ respectively. Apply Theorem 5.1 to transform $G$ to a $\pi_1$-unknotted Whitney tower $T$. Since the disk-like Whitney towers $T_{±}$ are obtained from the capped gropes $G_{±}$ (see Remark 5.3), it follows that $T_{±}$ have height $m$ and $n$ respectively, as in Corollary 5.2. $\square$

\section{Construction of examples}

In this subsection, using the results of Section 4 we construct certain examples of knots which are height $(m, n)$ grope slice, and consequently height $(m, n)$ Whitney slice by Theorem 5.3. In the next section, a particular subfamily of those examples will be shown to be not doubly slice. Furthermore the examples will be used to exhibit the rich structure of the Whitney tower and grope bi-filtrations.

We start with the following input data. Fix nonnegative integers $m ≥ n$. Suppose $J_0$ is a knot, and for each $k = 0, \ldots , m − 2$, suppose $(K_{k}, \eta_{k})$ is a pair of a knot $K_{k}$ and a simple closed curve $\eta_{k}$ in the exterior $E_{K_{k}}$ which is unknotted in $S^3$. Suppose the following:

(G1) There is a 3D capped grope concordance of height 2 between $J_0$ and the unknot.

(G2) The knot $K_{k}$ is ribbon and there is a 3D satellite capped grope of height 1 for $(K_{k}, \eta_{k})$ for each $k$.

We remark that there are numerous examples satisfying the above. We will specify explicit choices in later sections.

Let $R$ be the knot $9_{46}$, and let $α'$ and $β'$ be the curves depicted in Figure 16 (see [Hor10, Figure 7]). Define a knot $J_{k+1}$ inductively for $k ≥ 0$ by $J_{k+1} := K_{k}(\eta_{k}; J_{k})$. Finally define $$J_{m,n} := R(α', β'; J_{m−1}, J_{n−1}) := R(α', J_{m−1})(β', J_{n−1})$$ to be the knot obtained by applying the satellite construction twice, once along $α'$ using $J_{m−1}$ as the companion and then along $β'$ using $J_{n−1}$ as the companion.

\textbf{Theorem 5.4.} If (G1) and (G2) hold, then the knot $J_{m,n}$ is height $(m + 2, n + 2)$ grope slice, and consequently height $(m + 2, n + 2)$ Whitney slice by Theorem 5.3.

\textit{Proof.} Let $J := J_{m,n}$ for brevity. First, we construct a disk-like capped grope of height $m + 2$ with boundary $J$ in $D^4$.

In what follows $U$ designates a trivial knot. Let $$R_{β'} := R(α', β'; U, J_{n−1}) = R(β'; J_{n−1}).$$ Then we can view $J$ as $J = R_{β'}(α'; J_{m−1})$. Let $L_{0} = U$ and $L_{k+1} = K_{k}(\eta_{k}; L_{k})$ for $k ≥ 0$. Let $$R_{±} := R(α', β'; L_{m−1}, J_{n−1}) = R_{β'}(α'; L_{m−1}).$$

Figure 16 depicts a 3D satellite capped grope for $(R, α')$. After the satellite operation along $β'$, it becomes a 3D satellite capped grope of height 1 for $(R_{β'}, α')$. Push this into $S^3 × I$ to obtain a satellite capped grope, which we denote by $H_{m−1}$. Push those given
The knot $R = 9_{46}$ and 3D satellite capped gropes bounded by $\alpha'$ and $\beta'$. The curves $\alpha$ and $\beta$ are dual to the left and right handles of the genus one Seifert surface.

Figure 16. The knot $R = 9_{46}$ and 3D satellite capped gropes bounded by $\alpha'$ and $\beta'$. The curves $\alpha$ and $\beta$ are dual to the left and right handles of the genus one Seifert surface.

This is a capped grope concordance of height $m + 2$ between $J_0$ and $U$, and a height $1$ satellite capped grope $H_k$ of $(K_k, \eta_k)$ for $k = 0, \ldots, m - 2$. Let

$$P_+ := H_{m-1} \cdot H_{m-2} \cdots H_0 \cdot H.$$  

This is a capped grope concordance of height $m + 2$ between $J$ and $R_+$. By Theorem \[\text{1.7}\] the exterior $E_{P_+}$ has a handle decomposition $E_{P_+} \cong (E_J \times I) \cup (2$-handles). Also, the construction of the product (see Section \[\text{4.1}\]) tells us that $\beta \subset E_J$ is isotopic, in $E_{P_+}$, to $\beta \subset E_{R_+}$. The same conclusion holds for (a parallel of) $\beta'$ as well.

Since each $K_k$ is ribbon and $L_0$ is the unknot, it follows that each $L_k$ is ribbon by an induction. Therefore, there is a ribbon concordance, say $Q_+$, from $R_+ = R_{3'}(\alpha'; L_{m-1})$ to $R_{3'} = R_{3'}(\alpha'; U)$.

The obvious genus one Seifert surface for $R$ (see Figure \[\text{16}\]) becomes a genus one Seifert surface for $R_{3'}$ after the satellite operation. If we cut the right band (whose linking circle is the curve $\beta$) of the Seifert surface by attaching a band, $R_{3'}$ becomes a 2-component trivial link. This gives us a genus zero cobordism between $R_{3'}$ and a 2-component trivial link. Cap off the latter to obtain a ribbon disk $\Delta_+$ in $D^4$ for $R_{3'}$. (In this paper a ribbon disk in $D^4$ designates a slice disk obtained by pushing an immersed ribbon disk in $S^3$.) Note that (parallels of) $\beta$ and $\beta'$ in $E_{R_{3'}}$ are null-homotopic in $E_{\Delta_+}$.

Identify a collar of $S^3$ in the upper hemisphere $D^4_+$ with $S^3 \times [0, 2]$, so that $D^4_+ = S^3 \times [0, 2] \cup D^4$ where $S^3 \times 0$ is the boundary of $D^4_+$. View $P_+, Q_+$, and $\Delta_+$ as subsets of $S^3 \times [0, 1]$, $S^3 \times [1, 2]$ and $D^4$ respectively. Let

$$G_+ := P_+ \cup_{R_+} Q_+ \cup_{R_{3'}} \Delta_+ \subset D^4_+.$$  

Then $G_+$ is a disk-like capped grope of height $m + 2$ in $D^4_+$ with boundary $J$. The following hold:
(i) The exterior $E_{G_+}$ is obtained from $E_J \times I$ by attaching handles of index $\geq 2$.

(ii) The curves $\beta$ and $\beta' \subset E_J$ are null-homotopic in $D^4_+ \setminus G_+$.

The condition (i) follows from that $E_{G_+}$ is obtained by stacking $E_{P_1}, E_{Q_1}$, and $E_{\Delta_+}$, each of which has handles of index $\geq 2$ only. The condition (ii) holds since $\beta \subset E_J$ is isotopic to $\beta \subset E_{R_{\sigma'}}$ in $E_{P_1} \cup E_{Q_1}$ and $\beta \subset E_{R_{\sigma'}}$ is null-homotopic in $E_{\Delta_+}$. The same argument works for $\beta'$ as well.

Similarly to the above, construct a disk-like capped grope $G_-$ of height $n + 2$ in $D^-_+$ with boundary $J$ which satisfies the following:

(i) The exterior $E_{G_-}$ is obtained from $E_J \times I$ by attaching handles of index $\geq 2$.

(ii) The curves $\alpha$ and $\alpha' \subset E_J$ are null-homotopic in $D^4_- \setminus G_-$.

Finally, let $G := G_+ \cup J_+ \subset S^4 = D^4_+ \cup S^3 \cup D^4_-$. Then $G$ is a sphere-like capped grope in $S^4$ which intersects $S^3$ at the knot $J$. The only remaining thing to show is that $G$ is $\pi_1$-unknotted, that is, $\pi_1(S^4 \setminus G) \cong \mathbb{Z}$.

By Seifert-van Kampen,

$$\pi_1(S^4 \setminus G) \cong \pi_1(E_{G_-}) \pi_1(E_J) \pi_1(E_{G_+}).$$

From (i), it follows that each inclusion-induced map $\pi_1 E_J \to \pi_1 E_{G_\pm}$ is surjective. Therefore $\pi_1(S^4 \setminus G)$ is a quotient of $\pi_1 E_J$. From (ii), it follows that $\alpha, \alpha', \beta$, and $\beta'$ are null-homotopic in $\pi_1(S^4 \setminus G)$. Therefore $\pi_1(S^4 \setminus G)$ is a quotient of $\pi_1 E_J / (\alpha, \alpha' , \beta, \beta')$ where $(\cdots)$ denotes the normal subgroup generated by $\cdots$.

Recall that $\alpha'$ and $\beta'$ are identified with the meridians of $J_{m-1}$ and $J_{n-1}$ respectively. By Seifert-van Kampen, it follows that

$$\pi_1 E_J / (\alpha', \beta') \cong \pi_1 E_J / (\text{Im}\{\pi_1 E_{j_{m-1}}, \pi_1 E_{j_{n-1}}\}) \cong \pi_1 E_R / (\alpha', \beta').$$

Consequently $\pi_1 E_J / (\alpha, \alpha', \beta, \beta')$ is isomorphic to $\pi_1 E_R / (\alpha, \alpha', \beta, \beta')$, which is a quotient of $\mathbb{Z}$ since $\pi_1 E_R (\alpha, \beta) \cong \mathbb{Z}$.

Therefore, $\pi_1(S^4 \setminus G)$ is a quotient of $\mathbb{Z}$. Since $H_1(S^4 \setminus G) \cong \mathbb{Z}$, it follows that $\pi_1(S^4 \setminus G) \cong \mathbb{Z}$.

\section{Obstructions to grope and Whitney doubly slicing}

In this section, we give examples illustrating the rich structure of the grope and Whitney tower bi-filtrations. The main result of this section is the following:

\begin{theorem}

(1) For any integer $m \geq 3$, there exists a family of slice knots $\{J^i\}_{i=1,2,\ldots}$ such that each $J^i$ is height $m$ grope doubly slice but any nontrivial linear combination of the $J^i$ is not height $m.5$ Whitney doubly slice.

(2) For any integers $m, n \geq 3$, there exists a family of knots $\{J^i_{m,n}\}_{i=1,2,\ldots}$ such that each $J^i_{m,n}$ is height $(m, n)$ grope slice but any nontrivial linear combination of the $J^i_{m,n}$ is not height $(m, n.5)$ Whitney slice.

\end{theorem}

The above theorem is proved in a stronger form in Theorem 6.10. Since height $(m, n)$ grope slice knots are height $(m, n)$ Whitney slice (Theorem 5.3), it follows that both grope and Whitney tower bi-filtrations $\{G_{m,n}\}$ and $\{W_{m,n}\}$ are highly nontrivial.

The examples $J^i$ and $J^i_{m,n}$ in Theorem 6.1 will be obtained by the construction in Section 5.2. Theorem 5.4 will tell us the promised grope/Whitney sliceness.

To show that the examples and their linear combinations are not grope/Whitney slice, we will first relate the grope/Whitney sliceness to the $(m, n)$-solvability studied in [Kim06].
in Section 6.1 and then use amenable $\rho^{(2)}$-invariant techniques developed in [CO12, Cha14a, Cha14b] in Sections 6.2, 6.3, and 6.4.

6.1. $(m, n)$-solvable knots

In [Kim06], the second author introduced the notion of $(m, n)$-solvable knots which is a generalization of a doubly slice knot. Since its definition is lengthy, we will first state the main result of this subsection and then describe necessary terms.

Theorem 6.2. A height $(m + 2, n + 2)$ Whitney slice knot is $(m, n)$-solvable.

To define an $(m, n)$-solvable knot, we need to recall the notion of an $n$-solvable knot defined in [COT03], from which the former was inspired. For a group $G$ and $n \geq 0$, the $n$th derived subgroup $G^{(n)}$ is defined by $G^{(0)} := G$ and $G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$ for $n \geq 1$.

Definition 6.3 ([COT03]). Suppose $n$ is a nonnegative integer, $M$ is a closed 3-manifold with $H_1(M) \cong \mathbb{Z}$, and $W$ is a compact spin 4-manifold with boundary $M$. Let $\pi := \pi_1(W)$.

1. $W$ is an $n$-solution for $M$ if the inclusion induces an isomorphism $H_1(M) \cong H_1(W)$ and there exist $[x_1], \ldots, [x_r]$ and $[y_1], \ldots, [y_r]$ in $H_2(W; \mathbb{Z}/\pi^{(n)})$ with $r = \frac{1}{2} \text{rank}_\mathbb{Z} H_2(W)$ such that the intersection form and self-intersection form

\[
\lambda_n : H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \to \mathbb{Z}[\pi/\pi^{(n)}]
\]

\[
\mu_n : H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \to \mathbb{Z}[\pi/\pi^{(n)}]/\{(a - a)\}
\]

have values $\lambda_n([x_i], [x_j]) = \mu_n([x_i]) = 0$ and $\lambda_n([x_i], [y_j]) = \delta_{ij}$ for all $i$ and $j$.

2. $W$ is an $n, 5$-solution for $M$ if $W$ satisfies (1) and in addition the $[x_i]$ have lifts $\tilde{x}_i \in H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)})]$ such that $\lambda_{n+1}([\tilde{x}_i], [\tilde{x}_j]) = \mu_{n+1}([\tilde{x}_i]) = 0$ for all $i$ and $j$.

3. A knot $K$ is $h$-solvable ($h \in \frac{1}{2} \mathbb{Z}_{\geq 0}$) if the zero surgery $M(K)$ has an $h$-solution.

We denote by $F_n$ the subgroup of the concordance classes of $n$-solvable knots. Since an $m$-solvable knot is $n$-solvable for $m \geq n$, the subgroups $F_n$ with $n \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ form a descending filtration of the knot concordance group.

Definition 6.4 ([Kim06, Definitions 2.2, 2.3, and 6.2]). Suppose $m, n \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ and $M$ is a closed 3-manifold with $H_1(M) \cong \mathbb{Z}$.

1. A pair $(W_1, W_2)$ of compact 4-manifolds $W_i$ bounded by $M$ is an $(m, n)$-solution for $M$ if $W_1$ is an $m$-solution for $M$, $W_2$ is an $n$-solution for $M$, and $\pi_1(W_1 \cup_M W_2)$ is an infinite cyclic group.

2. A knot $K$ is $(m, n)$-solvable if there is an $(m, n)$-solution $(W_1, W_2)$ for the zero surgery $M(K)$.

3. We call an $(m, m)$-solution a double $m$-solution. We also call an $(m, m)$-solvable knot or 3-manifold doubly $m$-solvable.

We denote by $F_{m, n}$ the set of (isotopy classes of) $(m, n)$-solvable knots. By [Kim06, Proposition 2.6] each $F_{m, n}$ is a submonoid under connected sum, and since an $m$-solution for a knot $K$ is an $n$-solution for $m \geq n$, it is obvious that $F_{k, \ell} \subset F_{m, n}$ for $k \geq m$ and $\ell \geq n$. Therefore $\{F_{m, n}\}$ is a bi-filtration of the monoid of knots. We call it the solvable bi-filtration of knots. Theorems 5.3 and 6.2 tell us that $G_{m+2, n+2} \subset W_{m+2, n+2} \subset F_{m, n}$.

Due to [COT03, Theorem 8.11], a height $h + 2$ Whitney tower in $D^3$ bounded by a knot $K$ in $S^3$ can be transformed to an $h$-solution for $K$. Here we show that this can be done in such a way that the fundamental group does not grow.
**Theorem 6.5.** Suppose $T$ is a disk-like Whitney tower of height $h+2$ in $D^4$ bounded by a knot $K$ in $S^3$. Then there is an $h$-solution $W$ for $K$ with an epimorphism $\phi: \pi_1(D^4 \setminus T) \to \pi_1(W)$ making the following diagram commute:

$$
\pi_1(S^3 \setminus K) \quad \quad \quad \quad \pi_1(W)
$$

where $i_*$ and $j_*$ are homomorphisms induced by inclusions.

Our construction of the $h$-solution in Theorem 6.5 is slightly different from that in [COT03] Theorem 8.12.

**Proof.** Write $h = n$ or $n.5$ with $n$ an integer. The top stage disks of $T$ have height $n + 2$ if $h = n$, height $n + 3$ if $h = n.5$.

For each Whitney disk $D_i$ of height $n + 2$, choose a parallel copy of $\partial D_i$ in the interior of $D_i$, and perform surgery on $D^4$ along the parallel copy (using the framing induced by $D_i$). Let $V$ be the 4-manifold obtained by surgery. Using the surgery core disk as an embedded Whitney disk for the Whitney circle $\partial D_i$, do Whitney moves to make height $n + 1$ Whitney disks embedded in $V$. Repeating this inductively for height $n, \ldots, 2, 1$, the base disk of $T$ is changed to an embedded disk $\Delta$ in $V$. Let $W = V \setminus \nu(\Delta)$, the exterior of $\Delta$ in $V$.

Let $T_0$ be the subtower of $T$ which consists of height $\leq n + 1$ Whitney disks, and let $T'$ be the Whitney tower of height $n + 2$ in $V$ obtained from $T_0$ by adding the surgery core disks as height $n + 2$ disks. By Proposition 2.10, $\pi_1(D^4 \setminus T_0)$ is a quotient of $\pi_1(D^4 \setminus T)$. A Seifert-van Kampen argument shows that $\pi_1(D^4 \setminus T_0) \cong \pi_1(V \setminus T')$. By applying Lemma 2.9 repeatedly, $\pi_1(V \setminus T') \cong \pi_1(V \setminus \Delta)$. It follows that $\pi_1(V \setminus \Delta)$ is a quotient of $\pi_1(D^4 \setminus T)$.

We claim that $W$ is an $h$-solution for $K$. The spin structure of $D^4$ gives rise to a spin structure on $W$. Since the base disk of $T$ induces the zero framing on its boundary $K$ and Whitney moves do not alter this framing, we have $\partial W = M(K)$, the zero surgery on $K$ in $S^3$. Since the surgery on $D^4$ is performed along null-homotopic curves, $H_2(V) \cong \mathbb{Z}^{2k}$ where $k$ is the number of height $n + 2$ Whitney disks $D_i$. From the long exact sequence for $(V, W)$, it follows that $H_2(W) \cong H_2(V) \cong \mathbb{Z}^{2k}$. The union of $D_i \setminus (\text{collar of } \partial D_i)$ and the surgery core disk bounded by $\partial D_i$ is an immersed sphere in $W$, which we denote by $S_i$. Choose a Clifford torus $T_i$ around one of the two intersections paired by the Whitney disk $D_i$. Then $\{S_i, T_i\}_{i=1, \ldots, k}$ is a basis for $H_2(W)$. A basis curve on $T_i$ is a meridian $\mu$ of a height $n + 1$ disk $D$. A Clifford torus $T$ around an intersection paired (with another intersection) by $D$ meets $D$ at one point, and the basis curves of $T$ are meridians of height $n$ disks. It follows that $\mu$ is a commutator of meridians of height $n$ disks. Repeating the same argument inductively for height $n, \ldots, 2, 1$, it follows that $\mu$ is an element in the $n$th derived subgroup $\pi^{(n)}$, where $\pi = \pi_1(W)$. Thus the Clifford torus $T_i$ lifts to the $\pi/\pi^{(n)}$-cover of $W$ and represents a homology class $[T_i] \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$.

To complete the proof, we now consider the two cases of $h = n$ and $h = n.5$ separately. For $h = n$, since the tori $T_i$ are framed and pairwise disjoint, $\mu_n([T_i]) = 0$ and $\lambda_n([T_i], [T_j]) = 0$ for any $i, j$. Since $S_i$ meets $T_i$ at one point and disjoint from $T_j$ for $j \neq i$, $\lambda_n([T_i], [S_j]) = \delta_{ij}$.

For $h = n.5$, consider the homology classes $[S_i] \in H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)}])$. Since the sphere $S_i$ are framed, $\mu_n([S_i]) = 0$. Recall that the intersections of the height $n + 2$ disks are paired up by height $n + 3$ disks in $D^4$. Since the height $n + 3$ disks are disjoint...
from height $\leq n + 1$ disks, the height $n + 3$ disks lie in $V$ and pair up intersections of the immersed spheres $S_i$. It follows that $\lambda_{n+1}([S_i], [S_j]) = 0$ for any $i, j$. As above, $\lambda_n([S_i], [T_j]) = \delta_{ij}$. \hfill \Box

Now we are ready for the proof of Theorem 6.2.

**Proof of Theorem 6.2.** Suppose a knot $K$ in $S^3$ is sliced by a $\pi_1$-unknotted Whitney tower $T$ in $S^4$ whose upper half $T_+ = T \cap D^4_1$ and lower half $T_- = T \cap D^4_2$ have height $m + 2$ and $n + 2$ respectively. Apply Theorem 6.5 to $T_+$ and $T_-$ to obtain an $m$-solution $W_+$ and an $n$-solution $W_-$ such that $\pi_1(W_+)$ and $\pi_1(W_-)$ are homomorphic images of $\pi_1(D^4_1 \setminus T_+)$ and $\pi_1(D^4_2 \setminus T_-)$ respectively. Since

$$Z \cong \pi_1(S^4 \setminus T) \cong \pi_1(D^4_1 \setminus T_+) \ast \pi_1(S^3 \setminus K) \ast \pi_1(D^4_2 \setminus T_-),$$

it follows that

$$\pi_1(W_+ \cup_{M(K)} W_-) \cong \pi_1(W_+) \ast \pi_1(M(K)) \ast \pi_1(W_-)$$

is a quotient of $Z$. Since $H_1(W_+ \cup_{M(K)} W_-) = \mathbb{Z}$, it follows that $\pi_1(W_+ \cup_{M(K)} W_-) = \mathbb{Z}$. Therefore $(W_+, W_-)$ is an $(m, n)$-solution. \hfill \Box

### 6.2. Amenable signature theorem and mixed commutator series

We recall some results on the von Neumann-Cheeger-Gromov $\rho(2)$-invariant for a closed 3-manifold and mixed commutator series in [Cha14a].

Let $M$ be a closed 3-manifold and $\phi: \pi_1(M) \to G$ a group homomorphism. Suppose there is a 4-manifold $W$ bounded by $M$ such that $\phi$ extends to $\psi: \pi_1(W) \to G$. Then the von Neumann-Cheeger-Gromov $\rho(2)$-invariant associated to $(M, \phi)$ is defined to be

$$\rho(2)(M, \phi) = \text{sign}^2_G(W) - \text{sign}(W) \in \mathbb{R}$$

where $\text{sign}^2_G(W)$ and $\text{sign}(W)$ denote the $L^2$-signature of $W$ associated to $\psi$ and the ordinary signature of $W$, respectively. We refer the reader to [Cha16] for more about $L^2$-signatures and $\rho(2)$-invariants.

The following lemma gives a universal bound on the $\rho(2)$-invariants.

**Lemma 6.6 ([Cha16, Theorem 1.9]).** If $M$ is the zero surgery on a knot with crossing number $n$, then $|\rho(2)(M, \phi)| \leq 69713280 \cdot n$ for all homomorphisms $\phi$.

The following is the $\rho(2)$-invariant obstruction for a knot to being $n.5$-solvable which we will use in Section 6.3. Although we use the most general known term **amenable group lying in $D(R)$** in the statement, we do not define it here (precise definitions can be found from [Cha14a] as well as [Pat88, Str74]), since we will use Theorem 6.7 only for the class of groups described in Lemma 6.9 below.

**Theorem 6.7 ([Cha14a, Theorem 3.2]).** Let $K$ be an $n.5$-solvable knot. Let $G$ be an amenable group lying in $D(R)$ for $R = \mathbb{Z}_p$ or $\mathbb{Q}$ with $G^{(n+1)} = \{e\}$. Suppose $\phi: \pi_1(M(K)) \to G$ is a homomorphism which extends to an $n.5$-solution for $M(K)$ and sends a meridian to an infinite order element in $G$. Then $\rho(2)(M(K), \phi) = 0$.

**Definition 6.8 ([Cha14a, Definition 4.1]).** Let $G$ be a group and $\mathcal{P} = (R_0, R_1, \ldots)$ be a sequence of rings with unity. The $\mathcal{P}$-mixed-coefficient commutator series $\{\mathcal{P}^kG\}$ is defined inductively by $\mathcal{P}^0G := G$ and

$$\mathcal{P}^{k+1}G := \text{Ker} \left\{ \mathcal{P}^kG \to \frac{\mathcal{P}^kG}{[\mathcal{P}^kG, \mathcal{P}^kG]} \to \frac{\mathcal{P}^kG}{[\mathcal{P}^kG, \mathcal{P}^kG]} \otimes \frac{\mathcal{P}^kG}{[\mathcal{P}^kG, \mathcal{P}^kG]} \right\}. $$
For instance, if \( R_k = \mathbb{Z} \) for all \( k \), then \( \{ \mathcal{P}^kG \} \) is the standard derived series of \( G \), that is, \( \mathcal{P}^kG = G^{(k)} \). If \( R_k = \mathbb{Q} \) for all \( k \), then \( \{ \mathcal{P}^kG \} \) is the rational derived series of \( G \).

The lemma below immediately follows from [Cha14a, Lemma 4.3].

**Lemma 6.9.** Let \( n \) be a positive integer and \( G \) a group. Let \( \mathcal{P} = (R_0, R_1, \ldots, R_n) \) with \( R_k = \mathbb{Q} \) for \( k < n \) and \( R_n = \mathbb{Z}_p \) for a prime \( p \). Then, the group \( G/\mathcal{P}^{n+1}G \) is amenable and lies in \( D(\mathbb{Z}_p) \).

### 6.3. Non-triviality of the bi-filtrations

In this subsection, we prove the following:

**Theorem 6.10.** Let \( m \) and \( n \) be positive integers.

1. There exists a family of slice knots \( \{ J^i \}_{i=1,2,\ldots} \) in \( \mathcal{G}_{m+2,m+2} \) whose arbitrary non-trivial linear combinations are not in \( \mathcal{F}_{m,5,m,5} \).
2. There exists a family of knots \( \{ J_{m,n}^i \}_{i=1,2,\ldots} \) in \( \mathcal{G}_{m+2,n+2} \) whose arbitrary non-trivial linear combinations are not in \( \mathcal{F}_{m,5,m,5} \cup \mathcal{F}_{m,n,5} \).

From this it follows that there are infinitely many linearly independent knots at each level of the grope, Whitney tower, and solvable bi-filtrations of knots, since \( \mathcal{G}_{m+2,n+2} \subset \mathcal{W}_{m+2,n+2} \subset \mathcal{F}_{m,n} \). By Theorems 5.3 and 6.2 in [Cha14a], Theorem 6.10 follows immediately.

We remark that Theorem 6.10 generalizes Theorem 1.1 in [Kim06].

**Proof of Theorem 6.10.** We will prove (2) first. Since \( \mathcal{F}_{k,t} = \mathcal{F}_{t,k} \), we may assume \( m \geq n \).

For each \( i \geq 1 \), construct a knot \( J_{m,n}^i \) as described in Section 6.2 using the knot \( \mathcal{R} = 9_4 \) and the following choices for \( K_k, \eta_k \), and \( J_0 = J_0^0 \): in addition to \((G1)\) and \((G2)\) in Section 6.2, suppose the following.

1. Each \( K_k \) has a cyclic Alexander module generated by \( \eta_k \). For convenience, we take the same \((K_k, \eta_k)\) as \((K_k, \eta_k)\) for all \( k \).
2. For the knot \( K \) in \((N1)\) above, let \( a_K \) be the top coefficient of the Alexander polynomial of \( K \) and let \( C_K \) be a constant such that \( |\rho(2)(M(K), \phi)| < C_K \) for all homomorphisms \( \phi : \pi_1(M) \to G \) given by Lemma 6.6. Then, \( J_0^i \) are knots obtained by Lemma 6.11 below with \( C_0 = (m-1)C_K \) and \( A = a_K \).

**Lemma 6.11 ([Han17] Proposition 3.4).** For any constants \( C_0 \) and \( A \), there exist knots \( J_0^1, J_0^2, \ldots \), and an increasing sequence of primes \( A < p_1 < p_2 \ldots \), which satisfy the following: letting \( \omega_i = e^{2\pi \sqrt{-1}/p_i} \), a primitive \( p_i \)-th root of unity,

1. there is a 3D capped grope concordance of height 2 between \( J_0^1 \) and the unknot,
2. \( \sum_{r=0}^{p_i-1} \sigma_{\mu}^r(\omega_i^r) > C_0 \), and
3. \( \sum_{r=0}^{p_i-1} \sigma_{\mu}^r(\omega_i^r) = 0 \) for \( j < i \).

In Lemma 6.11 (Condition 1) is needed to satisfy \((G1)\). Condition (2) is needed to make each \( J_{m,n}^i \) nontrivial modulo \( \mathcal{F}_{m,5,m,5} \cup \mathcal{F}_{m,n,5} \), and (3) is needed to have any nontrivial linear combination of \( J_{m,n}^i \) not in \( \mathcal{F}_{m,5,m,5} \cup \mathcal{F}_{m,n,5} \).

For instance, take the knot \( 6_1 \) (stevedore’s knot) as \( K = K_k \), and take the curve \( \eta \) shown in Figure 17 as \( \eta = \eta_k \) for all \( k \). In this case \( C_K = 418279680 \). Note that \( K \) is a ribbon knot and has a cyclic Alexander module \( \mathbb{Z}[(t^{\pm 1})/(2t^2 - 5t + 2)] \) generated by \( \eta \), and there is a 3D satellite capped grope of height 1 for \((K, \eta)\) as in Figure 17.
Proof of Lemma 6.11. For each $m \geq 1$, let $P_m$ be the knot in [Hor10, p.756], which generalizes the knot in [CT07, Figure 3.6].

The Levine-Tristram signature function $\sigma_{P_m}$ of the knot $P_m$ is

$$\sigma_{P_m}(e^{2\pi \sqrt{-1} \theta}) = \begin{cases} 2 & \text{if } \theta_m < |\theta| \leq \pi, \\ 0 & \text{if } 0 \leq |\theta| < \theta_m, \end{cases}$$

where $\theta_m$ is uniquely determined by the conditions $0 < \theta_m < \pi$ and $\cos(\theta_m) = \frac{2\sqrt{3m-1}}{\sqrt{3m}}$ (see [Hor10, Section 4]). Since $\{\theta_m\}$ is a decreasing sequence converging to 0, there are integers $m_i$ and primes $p_i$ such that $A < p_i$ and $\theta_{m_i+1} < 2\pi/p_i < \theta_{m_i}$ for all $i$. Choose an integer $N$ such that $N > \frac{C}{4}$ and let $J_k^0$ be a connected sum of $N$ copies of $P_{m_i+1} \# (-P_{m_i})$ for each $i$. Then, $J_k^0$ satisfy the conditions (2) and (3). In [Hor10, Proposition 3.1], it is shown that in $S^3 \times [0,1]$ there is a capped grope concordance of height 2 between $P_m$ and the unknot. In fact, this capped grope concordance is a 3D capped grope concordance (see Figure 5 in [Hor10]). From this observation, one can easily see that there is a 3D capped grope concordance of height 2 between $J_k^0$ and the unknot. \hfill \Box

Now we have knots $J_{m,n}^i$ which satisfy [G1] [G2] [N1] and [N2]. By Theorem 5.4, we have $J_{m,n}^i \in G_{m+2,n+2}$ for each $i$. Let $J = \#_i a_i J_{m,n}^i$ ($a_i \in \mathbb{Z}$) be a nontrivial linear combination of $J_{m,n}^i$. We will show that $J$ is neither $(m,5,n)$-solvable nor $(m,n,5)$-solvable. We may assume that there are finitely many summands $J_{m,n}^i$ and all coefficients $a_i$ are nonzero. By taking $-J$ if necessary, we may assume $a_1 > 0$.

Let $C$ be the standard cobordism between $U_0(M(J_{m,n}^i))$ and $M(J)$ obtained by following the method in [COT04, p.113]. Let $(U, V)$ be an $(m,n)$-solution for $J$. We define

$$U_C = C \amalg_{M(\bar{J})} U,$$
$$V_C = C \amalg_{M(\bar{J})} V.$$
where $P = \langle \alpha_1 \rangle \cong R[t^\pm]/(2t-1)$ and $Q = \langle \beta_1 \rangle \cong R[t^\pm]/(t-2)$. This can be verified by straightforward computation. (Alternatively, one may invoke Proposition 6.14 which is stated in Section 6.4.)

**Lemma 6.12.** Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$ with $p$ an odd prime. Let $(U,V)$ be an $(m,n)$-solution for $J$. Let $U_0$ and $V_0$ be defined as above. Let $i_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(U_0; R[t^\pm])$ and $j_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(V_0; R[t^\pm])$ be inclusion-induced homomorphisms. Then, either \( i_\ast(\alpha_1) \neq 0 \) or \( j_\ast(\alpha_1) \neq 0 \).

The proof of Lemma 6.12 is postponed to Section 6.4.

The next proposition is a crucial technical result. Below we use the convention that $J_{m,n} : = \mathbb{R}(\alpha', \beta'; J_{m-1})$ and $J_{0,n}: = \mathbb{R}(\alpha', \beta'; \text{unknot}, J_{n-1})$.

**Proposition 6.13.** Let $m, n \geq 0$. Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. Let $W$ be a compact 4-manifold with boundary $M(J)$. Let $W_C = C \amalg M(J)$. Let

$$inc_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(W; R[t^\pm])$$

be the inclusion-induced homomorphism. Then, the following hold:

1. If $m \geq 1$ and $inc_\ast(\alpha_1) \neq 0$, then $W$ is not an $m.5$-solution for $J$.
2. If $n \geq 1$ and $inc_\ast(\beta_1) \neq 0$, then $W$ is not an $n.5$-solution for $J$.

We postpone the proof of Proposition 6.13 to Section 6.4.

Suppose $J$ is $(m,5)$-solvable via $(U, V)$. We apply Proposition 6.13 for $W = U$ and $inc_\ast = i_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(U_0; R[t^\pm])$; since $U$ is an $m.5$-solution for $J$, by Proposition 6.13 (1) we have $i_\ast(\alpha_1) = 0$. Then, by Lemma 6.12 we have $j_\ast(\alpha_1) \neq 0$ and $i_\ast(\beta_1) \neq 0$. Since $i_\ast(\beta_1) \neq 0$, by Proposition 6.13 (2), $U$ is not an $n.5$-solution for $J$. Since $m \geq n$, this implies that $U$ is not an $m.5$-solution for $J$, which is a contradiction.

Next, we prove that $J$ is not $(m,5)$-solvable. Since $F_{k,t} = F_{k,t}$ and it has been shown that $J$ is not $(m,5)$-solvable, we may assume $m > n$. Suppose $J$ is $(m, 5)$-solvable via $(U, V)$. By applying Proposition 6.13 (2) for $W = V$ and

$$inc_\ast = j_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(V_0; R[t^\pm]),$$

we have $j_\ast(\beta_1) = 0$. Then, by Lemma 6.12 we have $i_\ast(\beta_1) \neq 0$. Then, by applying Proposition 6.13 (2) for $W = U$ and

$$inc_\ast = i_\ast : H_1(M(J_{m,n}; R[t^\pm])) \to H_1(U_0; R[t^\pm]),$$

it follows that $U$ is not an $n.5$-solution for $J$. But since $m > n$, this contradicts that $U$ is an $n$-solution for $J$. This completes the proof of Theorem 6.10 (2).

To prove Theorem 6.10 (1), we use an argument similar to the proof of Theorem 6.10 (2). Choose $K_k, \eta_k$, and $J_0$ as in the proof of Theorem 6.10 (2). Let $J' = \mathbb{R}(\alpha'; J_{m-1})$ where $\mathbb{R}$, $\alpha'$, and $J_{m-1}$ are those given and defined in Section 5.2. Then each $J'$ is ribbon, in particular slice, since it bounds a ribbon disk, say $\Delta$, obtained by cutting the band dual to $\alpha$. Let $G'$ be a capped grop of height $n+2$ in $D^4$ bounded by $J'$ obtained by replacing a sufficiently small disk in the interior of $\Delta$ by a model disk-like capped grope of height $m+2$. Then there is an epimorphism $\pi_1(S^3 \setminus J') \to \pi_1(D^3 \setminus G')$ induced from the inclusion since $\Delta$ and $G'$ have homeomorphic exteriors in $D^4$. Now by Theorem 5.3 and its proof, where we use $G' \subset D^4$ instead of $G \subset D^4$, we can see $J' \in \mathcal{G}_{m+2,m+2}$.

Suppose $J = \#_{a_1}J'$ is a nontrivial linear combination. We will show that $J$ is not doubly $m.5$-solvable. As before, we may assume that there are finitely many summands $J'$ and all the coefficients $a_i$ are nonzero. By taking $-J$ if necessary, we may assume $a_1 > 0$.
Suppose $J$ is doubly $m.5$-solvable via $(U, V)$. Let $\alpha_1$ be the curve in $M(J^1)$ which is the image of a parallel copy of the curve $\alpha'$ under the satellite construction with the pattern $R$. For $R = \mathbb{Q}$ or $\mathbb{Z}_p$, let $i_*$ and $j_*$ be the inclusion-induced homomorphisms as in the proof of the statement (2). By Lemma 6.12, we have $i_*(\alpha_1) \neq 0$ or $j_*(\alpha_1) \neq 0$. By exchanging $U$ and $V$ if necessary, we may assume that $i_*(\alpha_1) \neq 0$. Note that $J^1 = R(\alpha', \beta'; J^1_{m-1}, \text{unknot}) = J^1_{m,0}$ with $m \geq 1$. Therefore, by Proposition 6.13 (1), $U$ is not an $m.5$-solution, which is a contradiction.

This completes the proof of Theorem 6.10 modulo the proofs of Lemma 6.12 and Proposition 6.13.

### 6.4. Proofs of Lemma 6.12 and Proposition 6.13

In this subsection, we prove Lemma 6.12 and Proposition 6.13. For this purpose we need the following splitting result for doubly 1-solvable knots. Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. For a knot $K$, the Blanchfield form

$$B\ell: H_1(M(K); R[t^{\pm 1}]) \times H_1(M(K); R[t^{\pm 1}]) \longrightarrow R[t]/R[t^{\pm 1}]$$

is defined [Bl65]. For a submodule $P$ of $H_1(M(K); R[t^{\pm 1}])$, define

$$P^\perp := \{x \in H_1(M(K); R[t^{\pm 1}]) \mid B\ell(x, y) = 0 \text{ for all } y \in P\}.$$

We say that $P$ is self-annihilating with respect to $B\ell$ if $P = P^\perp$. We have the following proposition.

**Proposition 6.14 ([Kim96])** Proposition 2.10 for $R = \mathbb{Q}$ and $\mathbb{Z}_p$. Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. Let $K$ be a knot which is doubly 1-solvable via $(W_1, W_2)$. Let $P_i$ be the kernel of the inclusion-induced homomorphism $H_1(M(K); R[t^{\pm 1}]) \rightarrow H_1(W_i; R[t^{\pm 1}])$ for $i = 1, 2$.

Then

$$H_1(M(K); R[t^{\pm 1}]) \cong P_1 \oplus P_2$$

as $R[t^{\pm 1}]$-modules, and each $P_i$ is self-annihilating with respect to the Blanchfield form.

Now we are ready to prove Lemma 6.12.

**Proof of Lemma 6.12.** Recall that $R = \mathbb{Q}$ or $\mathbb{Z}_p$. Using a Mayer-Vietoris sequence, we obtain

$$H_1(C; R[t^{\pm 1}]) \cong H_1(\partial_+ C; R[t^{\pm 1}]) \cong H_1(\partial_+ C; R[t^{\pm 1}])$$

where $\partial_+ C = M(J), \partial_+ C = \cup_n M(J^n_{m,n})$, and the isomorphisms are induced by the inclusions. Therefore, via the isomorphisms we can consider $H_1(M(J^n_{m,n}); R[t^{\pm 1}])$ as a submodule of $H_1(M(J); R[t^{\pm 1}])$. Using Mayer-Vietoris sequences, it follows that $H_1(U; R[t^{\pm 1}]) \cong H_1(U; R[t^{\pm 1}])$ and $H_1(V; R[t^{\pm 1}]) \cong H_1(V; R[t^{\pm 1}])$.

Since $m, n \geq 1$, $(U, V)$ is a double 1-solution for $M(J)$. Therefore, by Proposition 6.14

$$H_1(M(J); R[t^{\pm 1}]) \cong \text{Ker } \iota_*^U \oplus \text{Ker } j_*^V$$

where $\iota_*^U : H_1(M(J); R[t^{\pm 1}]) \rightarrow H_1(U; R[t^{\pm 1}])$ and $j_*^V : H_1(M(J); R[t^{\pm 1}]) \rightarrow H_1(U; R[t^{\pm 1}])$ are inclusion-induced homomorphisms.

Suppose $i_*(\alpha_1) = j_*(\alpha_1) = 0$. Considering $H_1(M(J^n_{m,n}); R[t^{\pm 1}])$ as a submodule of $H_1(M(J); R[t^{\pm 1}])$, this implies that $i_*^U(\alpha_1) = j_*^V(\alpha_1) = 0$. Since $H_1(M(J); R[t^{\pm 1}]) \cong \text{Ker } \iota_*^U \oplus \text{Ker } j_*^V$, $\alpha_1$ is a nontrivial element of $H_1(M(J); R[t^{\pm 1}])$, this is a contradiction. Therefore we cannot have $i_*(\alpha_1) = j_*(\alpha_1) = 0$. With a similar reason, we cannot have $i_*(\beta_1) = j_*(\beta_1) = 0$.

Since $m, n \geq 1$, the knots $J^n_{m,n}$ are 1-solvable. Let $W^i$ be a 1-solution for $M(J^n_{m,n})$ for each $i$. Let $W^i_*$ be a copy of $-W^i$. 


As in Figure 18, we define
\[ U_1 = \left( \prod_{M(J)} C \right) \setminus \left( \prod_{M(J_{m,n})} (a_i - 1)W^1_{i} \cup \bigcup_{i \geq 2} a_i W^i_{i} \right). \]

Then \( \partial U_1 = M(J_{m,n}) \). Let \( i_1; H_2(M(J_{m,n}); R[t^{\pm 1}]) \rightarrow H_2(U_1; R[t^{\pm 1}]) \) be the inclusion-induced homomorphism. Since \( U \) is an \( m \)-solution, it is also a 1-solution. Using Mayer-Vietoris sequences, one can show that \( U_1 \) is a 1-solution for \( M(J_{m,n}) \).

It is known that \( \text{Ker} i_1^* \) is a self-annihilating submodule of \( H_1(M(J_{m,n}); R[t^{\pm 1}]) \) \cite{COT03} Theorem 4.1. Therefore \( \text{Ker} i_1^* \) is a proper submodule. Since \( H_1(M(J_{m,n}); R[t^{\pm 1}]) \) is generated by \( \alpha_1 \) and \( \beta_1 \), it follows that we cannot have \( i_*(\alpha_1) = i_*(\beta_1) = 0 \). Similarly, we cannot have \( j_*(\alpha_1) = j_*(\beta_1) = 0 \).

Now suppose \( i_*(\alpha_1) = 0 \). Since we can have neither \( i_*(\alpha_1) = j_*(\alpha_1) = 0 \) nor \( i_*(\alpha_1) = i_*(\beta_1) = 0 \), it follows \( j_*(\alpha_1) \neq 0 \) and \( i_*(\beta_1) \neq 0 \). Next, suppose \( j_*(\alpha_1) = 0 \). Using similar arguments we can easily see that \( i_*(\alpha_1) \neq 0 \) and \( j_*(\beta_1) \neq 0 \).

Suppose the remaining case \( i_*(\alpha_1) \neq 0 \) and \( j_*(\alpha_1) \neq 0 \). Since we cannot have \( i_*(\beta_1) = j_*(\beta_1) = 0 \), either \( i_*(\beta_1) \neq 0 \) or \( j_*(\beta_1) \neq 0 \). In either case, the conclusion of the theorem holds.

In the proof of Proposition 6.13 we will use the following fact.

**Proposition 6.15 [Cha14a Proposition 4.10].** Suppose the Arf invariant of \( J_0 \) vanishes, and let \( J_{m,n} \) be the knot constructed in Section 5.2. Then there exist an \( m \)-solution \( U \) and an \( n \)-solution \( V \) for \( J_{m,n} \) satisfying the following:

1. If \( \phi: \pi_1 M(J_{m,n}) \rightarrow G \) is a homomorphism which extends to \( U \) and \( G \) is an amenable group lying in \( D(R) \) for some ring \( R \) with unity, then \( \rho(\phi)(M(J_{m,n},\phi)) = \rho(\phi)(M(J_0,\psi)) \) where \( \pi_1 M(J_0) \rightarrow \mathbb{Z}_d \) is a surjection and \( d \) is the order of the image of the meridian of \( J_0 \) under \( \phi \) in \( G(m) \). The same statement holds when \( U \) and \( G(m) \) are replaced by \( V \) and \( G(n) \).
2. For \( R = \mathbb{Q} \) and \( \mathbb{Z}_p \) with \( p \) an odd prime, \( H_1(U; R[t^{\pm 1}]) \cong R[t^{\pm 1}]/(2t-1) \) generated by the curve \( \alpha_1 \) and \( H_1(V; R[t^{\pm 1}]) \cong R[t^{\pm 1}]/(t-2) \) generated by the curve \( \beta_1 \).

The statement (1) follows immediately from [Cha14a Proposition 4.10], and the statement (2) is implicitly proven in the proof of [Cha14a Proposition 4.10]. Now we give a proof of Proposition 6.13.

**Proof of Proposition 6.13.** To prove (1), we will construct a certain 4-manifold with boundary \( M(J_{m,n}^0) \). The building blocks for the construction are as follows.

1. Let \( U^1 \) be the \( m \)-solution for \( M(J_{m,n}^0) \) given in Proposition 6.15.
2. Let \( W'_+ \) be the exterior in \( D^4 \) of \( W_r \) the slice disk for \( R_{\beta'} := R(\beta'; J_{m,n-1}^0) \) in which \( \beta' \) is null-homotopic, where the slice disk is obtained by cutting the band dual to \( \beta \).
Recalling $K_{k+1} = K_k(\eta_k; J_k)$, let $E_k$ be the standard cobordism between $M(J_k^1) \cup M(K_k)$ and $M(K_{k+1})$. That is, letting $E(J_k^1)$ denote the exterior of $J_k^1$ in $S^3$, $E_k = M(J_k^1) \times [0, 1] \sqcup M(K_k) \times [0, 1] / \sim$ where the solid torus $M(J_k^1) \setminus E(J_k^1)$ in $M(J_k^1) = M(J_k^1) \times 0$ is identified with the tubular neighborhood of $\eta_k$ in $M(K_k) = M(K_k) \times 1$ in such a way that the meridian (resp. the 0-linking longitude) of $J_k^1$ is identified with the 0-linking longitude (resp. the meridian) of $\eta_k$ (see [CHL09, p.1429]).

Let $E_{m-1}^+$ be the standard cobordism between $M(J_{m-1}^1) \cup M(R_{\beta'})$ and $M(J_m^1)$ constructed similarly to (3), noting that $J_{m,n}^1 = R_{\beta'}(\alpha'; J_{m-1}^1)$.

Let $C$ be the standard cobordism from $\cup a_i M(J_{m,n}^1)$ to $M(J)$ as defined in the proof of Theorem 6.10. Let $U_r^i$ be a copy of $-U_r^i$. We define

$$W_m = W = \bigcup_{M(J)} C \bigcup_{M(J_{m,n}^1)} \left( (a_i - 1)U_r^i \cup \left( \bigcup_{i \geq 2} a_i U_r^i \right) \right).$$

Then, $\partial W_m = M(J_{m,n}^1)$. We define

$$W_{m-1} = W_m \bigcup_{M(J_{m,n}^1)} E_{m-1}^+ \bigcup_{M(R_{\beta'})} E_+.$$

Then, $\partial W_{m-1} = M(J_{m-1}^1)$. Finally, for $k = m - 2, m - 3, \ldots, 0$, as in Figure 19 we define

$$W_k = E_k \bigcup_{M(J_{k+1}^1)} W_{k+1}$$

$$= E_k \bigcup_{M(J_{k+1}^1)} E_{k+1} \bigcup_{M(J_{k+2}^1)} \ldots \bigcup_{M(J_{m-2}^1)} E_{m-2} \bigcup_{M(J_{m-1}^1)} W_{m-1}.$$

Then $\partial W_k = M(J_k^1) \cup \left( \bigcup_{j=k}^{m-2} M(J_j^1) \right) = M(J_k^1) \cup (m - k - 1)M(K)$.

**Figure 19.** The cobordism $W_k$

Let $P = (R_0, R_1, \ldots, R_m)$ be a sequence of rings where $R_k = \mathbb{Q}$ for $0 \leq k \leq m - 1$ and $R_m = \mathbb{Z}_{p_1}$. Then for a group $G$, we get the $P$-mixed-coefficient commutator series $P^k G$ for $k = 0, 1, \ldots, m + 1$ as in Definition 6.8. The following lemma is essentially due to [Cha14a, Theorem 4.14].

**Lemma 6.16 (Cha14a, Theorem 4.14)].** Rename $J_{m,x}^1 := J_{m,n}^1$ for brevity. Suppose $\operatorname{inc}_{\alpha_1} \neq 0$ as in the hypothesis of Proposition 6.15 (1). For $k = 0, 1, \ldots, m$, the homomorphism

$$\phi_k : \pi_1 W_k \longrightarrow \pi_1 W_k / \mathcal{P}^{m-k+1}\pi_1 W_k$$
sends the meridian of $J^1_k$ into the abelian subgroup $\mathcal{P}^{m-k} \pi_1 W_k / \mathcal{P}^{m-k+1} \pi_1 W_k$. Furthermore, the image of the meridian of $J^1_k$ under $\phi_k$ has order $p_1$ if $k = 0$, and is of infinite order if $k > 0$.

We postpone the proof of Lemma 6.16 to the end of this section and finish the proof of Proposition 6.13. Let $G = \pi_1 W_0 / \mathcal{P}^{m+1} \pi_1 W_0$. In particular, $G^{(m+1)} = \{e\}$. By Lemma 6.9, the group $G$ is amenable and lies in $D(\mathbb{Z}_{p_1})$. Now we have the canonical homomorphism $\phi: \pi_1 W_0 \to G$, and by abuse of notation, we denote by $\phi$ restrictions of $\phi$ to subspaces of $W_0$.

For a 4-manifold $X$ with a homomorphism $\pi_1 X \to G$, we let $S_G(X) = \text{sign}_G^{(2)}(X) - \text{sign}(X)$, the $L^2$-signature defect. Since $\partial W_0 = M(J^1_0) \cup \left( \bigcup_{k=0}^{m-2} M(K_k) \right)$, using the definition of $\rho^{(2)}$-invariants in Section 5.2, we have

$$\rho^{(2)}(\partial W_0, \phi) = \rho^{(2)}(M(J^1_0), \phi) + \sum_{k=0}^{m-2} \rho^{(2)}(M(K_k), \phi) = S_G(W_0).$$

On the other hand, using Novikov additivity, we have

$$S_G(W_0) = S_G(W) + S_G(C) + \sum_{i=1}^{b_1} \sum_{r=1}^{b_i} S_G(U^+_i) + S_G(E^+_{m-1}) + S_G(E_+) + \sum_{k=0}^{m-2} S_G(E_k).$$

Here we use the convention that

$$\sum_{k=0}^{m-2} \rho^{(2)}(M(K_k), \phi) = \sum_{k=0}^{m-2} S_G(E_k) = 0$$

if $m = 1$. We compute each term of these equalities below.

1. $\rho^{(2)}(M(J^1_0), \phi) = \sum_{r=0}^{p_1-1} \sigma_{J^1_0}(\omega^1_r) > 0$ where $\omega^1_r = e^{2\pi \sqrt{-1}/p_1}$ and $\sigma_{J^1_0}$ is the Levine-Tristram signature function for $J^1_0$: by Lemma 6.16, the image of the meridian of $J^1_0$ under $\phi$ has order $p_1$, and in this case it is known that $\rho^{(2)}(M(J^1_0), \phi) = \sum_{r=0}^{p_1-1} \sigma_{J^1_0}(\omega^1_r) > 0$ (see [COT04, Proposition 5.1] and [Fri05, Corollary 4.3]). The last inequality follows from Lemma 6.11(2).

2. $S_G(C) = 0$ by [COT04, Lemma 4.2].

3. $S_G(U^+_i) = 0$ or $-\sum_{r=0}^{p_1-1} \sigma_{U^+_i}(\omega^1_r)$: since $G^{(m+1)} = \{e\}$, the subgroup $G^{(m)}$ is abelian. By our choice of the ring $R_m = \mathbb{Z}_{p_1}$, the group $G^{(m)}$ is a vector space over $\mathbb{Z}_{p_1}$, and hence every element in $G^{(m)}$ has order either 1 or $p_1$. By Proposition 6.13, $S_G(U^+_i) = S_G(-U^+_i) = -\rho^{(2)}(M(J^1_0), \psi)$, where $\psi: \pi_1 M(J^1_0) \to \mathbb{Z}_{p_1}$ is a homomorphism. If $\psi$ is trivial, then $\rho^{(2)}(M(J^1_0), \psi) = 0$, and otherwise $\rho^{(2)}(M(J^1_0), \psi) = \sum_{r=0}^{p_1-1} \sigma_{J^1_0}(\omega^1_r)$ as calculated in (1) above.

4. $S_G(U^+_i) = 0$ for $i > 1$. With the same reason in (3), we have $S_G(U^+_i) = 0$ or $-\sum_{r=0}^{p_1-1} \sigma_{U^+_i}(\omega^1_r)$.

5. $S_G(E_+) = 0$: since $E_+$ is a slice disk exterior, we have $H_2(E_+) = 0$. Therefore $\text{sign}(E_+) = 0$. By [Cha08, Lemma 2.7], $|\text{sign}_G^{(2)}(E_+)| \leq \text{rank}_\mathbb{Z} H_2(E_+) = 0$. Therefore $\text{sign}_G^{(2)}(E_+) = 0$.

6. $S_G(E^+_{m-1}) = S_G(E_k) = 0$ for all $k$ due to [CHL09, Lemma 2.4].

From these computations, we conclude that

$$\epsilon \cdot \sum_{r=0}^{p_1-1} \sigma_{J^1_0}(\omega^1_r) + \sum_{k=0}^{m-2} \rho^{(2)}(M(K_k), \phi) = S_G(W)$$

for some constant $\epsilon \geq 1$. 
Suppose \( W \) is an \( m,5 \)-solution for \( J \). Then, by Theorem \( \text{[Cha14a]} \) we have \( S_G(W) = 0 \). But this leads us to a contradiction: since \( K_k = K \) for all \( k \), and by our choice of \( J^1 \) we have 
\[
\sum_{i=0}^{m-1} \sigma_i J^1(\omega_i^2) > (m-1)C_K.
\]
On the other hand, \( \sum_{k=0}^{m-1} \rho(2)(M(K_k), \phi) \leq (m-1)C_K \). This completes the proof of the statement (1) modulo the proof of Lemma \( \text{[Cha14a]} \).

The statement (2) can be proved using arguments similar to the one for the statement (1).

Now we finish the proof of Proposition \( \text{[Cha14a]} \) by proving Lemma \( \text{[Cha14a]} \).

**Proof of Lemma \( \text{[Cha14a]} \).** The proof is essentially identical to the one of \( \text{[Cha14a]} \) Theorem 4.14. The only difference occurs at the first step where we use \( E_{m-1} / M(R_{\alpha'}) \) instead of \( E_{m-1} \) to construct \( W_{m-1} \) from \( W_m \), and we only need to show that 
\[
\mathcal{P}^1 \pi_1 W_{m-1} / \mathcal{P}^2 \pi_1 W_{m-1} \cong \mathcal{P}^1 \pi_1 W_m / \mathcal{P}^2 \pi_1 W_m.
\]

Let \( W' = W_m \prod_{M(J^1)} E_{m-1} \). Then, using exactly the same argument as in the proof of Assertion 1 in \( \text{[Cha14a]} \) p.4799, one can show that 
\[
\pi_1 W'/\pi_1 W'(2) \cong \pi_1 W_m / \pi_1 W_m(2).
\]

Therefore, we have 
\[
\mathcal{P}^1 \pi_1 W'/\mathcal{P}^2 \pi_1 W' \cong \mathcal{P}^1 \pi_1 W_m / \mathcal{P}^2 \pi_1 W_m.
\]

Since \( W_{m-1} = W' \prod_{M(J^1)} E_{m-1} \) and \( \pi_1 E_+ \cong \pi_1 (M(R_{\alpha'})) \langle \beta' \rangle \) where \( \langle \beta' \rangle \) denotes the subgroup normally generated by \( \beta' \), by Seifert-van Kampen we obtain \( \pi_1 W_{m-1} \cong \pi_1 W'/\langle \beta' \rangle \).

For the moment, suppose \( \beta' \in \mathcal{P}^2 \pi_1 W' \). Then, since \( \mathcal{P}^2 \pi_1 W' \) maps into \( \mathcal{P}^2 \pi_1 W_{m-1} \), we have 
\[
\mathcal{P}^1 \pi_1 W'/\mathcal{P}^2 \pi_1 W' \cong \mathcal{P}^1 \pi_1 W_{m-1} / \mathcal{P}^2 \pi_1 W_{m-1},
\]
and therefore 
\[
\mathcal{P}^1 \pi_1 W_{m-1} / \mathcal{P}^2 \pi_1 W_{m-1} \cong \mathcal{P}^1 \pi_1 W_m / \mathcal{P}^2 \pi_1 W_m.
\]

Therefore, it suffices to show \( \beta' \in \mathcal{P}^2 \pi_1 W' \). Note that \( \beta' \) is isotopic to \( \beta_1 \subset M(J^1_{m,n}) \) in \( E_{m-1} \), and again it suffices to show \( \beta_1 \in \mathcal{P}^2 \pi_1 W_m \).

Using Mayer-Vietoris sequences one can show that \( W_m \) is a 1-solution for \( J^1_{m,n} \). Noting \( R_1 = \mathbb{Z}_p \) if \( m = 1 \) and \( \mathbb{Q} \) if \( m > 1 \), by \( \text{[COT03]} \) Theorem 4.4 it is known that \( \ker i_* \) is a self-annihilating submodule of \( H_1(M(J^1_{m,n}); R_1[\pm 1]) \) where \( i_*: H_1(M(J^1_{m,n}); R_1[\pm 1]) \to H_1(W_m; R_1[\pm 1]) \) is the inclusion-induced homomorphism. Therefore \( \ker i_* = P = \langle \alpha_1 \rangle \) or \( \ker i_* = Q = \langle \beta_1 \rangle \). Recall that 
\[
W_C = C \prod_{M(J)} W \quad \text{and} \quad W_m = W_C \prod_{M(J^1)} \left( (a_i - 1)U_i \cup \bigcup_{i=2} \right),
\]

Since \( U_i = -U^i \) where \( U^i \) has been obtained using Proposition \( \text{[Cha14a]} \) we have \( H_1(U^i; R[\pm 1]) \cong R[\pm 1] / (2t - 1) \). Therefore, using Mayer-Vietoris sequences, one can see that the inclusion-induced homomorphism \( i_*: H_1(W_C; R[\pm 1]) \to H_1(W_m; R[\pm 1]) \) is a surjection whose kernel is \( (t-2) \)-torsion. Since \( \alpha_1 \) is \( (2t-1) \)-torsion in \( H_1(M(J^1_{m,n}); R[\pm 1]) \) and \( \text{inc}_*(\alpha_1) \neq 0 \) in \( H_1(W_C; R[\pm 1]) \) by the hypothesis, \( \text{inc}_*(\alpha_1) \) is not contained in the kernel of the surjection.

Therefore \( i_* = Q = \langle \beta_1 \rangle \), and hence \( i_*(\beta_1) = 0 \).

Since \( \pi_1 W_m / \mathcal{P}^1 \pi_1 W_m \cong \mathbb{Z} = (t) \), the quotient group \( \mathcal{P}^1 \pi_1 W_m / \mathcal{P}^2 \pi_1 W_m \) injects into \( H_1(W_m; R_1[\pm 1]) \). Since \( i_*(\beta_1) = 0 \), it follows that \( \beta_1 = 0 \) in \( \mathcal{P}^1 \pi_1 W_m / \mathcal{P}^2 \pi_1 W_m \). Therefore \( \beta_1 \in \mathcal{P}^2 \pi_1 W_m \). \( \square \)
7. Bi-filtrations and classical obstructions

In this section, we discuss relationships between our bi-filtrations and previously known double sliceness obstructions. We show that the double sliceness obstructions in \cite{GL83} vanish for knots in \( F_{2,2} \) and hence for knots in \( G_{4,4} \) and \( W_{4,4} \). We also show that the obstructions in \cite{Fri04} \cite{LM15} vanish for knots in \( F_{1,5,1.5} \), and hence for knots in \( G_{1,5,1.5} \) and \( W_{4,1.5,3.5} \). Details are given below.

Algebraic double sliceness

Due to Sumners \cite{Sum71}, a doubly slice knot has a hyperbolic Seifert matrix, that is, a Seifert matrix of the form \( \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \) with the zero blocks of the same size. Following \cite{GL83}, we say that a knot \( K \) is algebraically doubly slice if \( K \) has a hyperbolic Seifert matrix. We remark that the following are equivalent:

1. \( K \) has a hyperbolic Seifert matrix.
2. Every Seifert matrix of \( K \) is hyperbolic.
3. \( K \) has a stably hyperbolic Seifert matrix, that is, the orthogonal sum of a Seifert matrix of \( K \) and some other hyperbolic Seifert matrix is hyperbolic.
4. Every Seifert matrix of \( K \) is stably hyperbolic.
5. The Blanchfield form of \( K \) is hyperbolic, that is, the Alexander module is an internal direct sum \( P_1 \oplus P_2 \) for some submodules \( P_1 \) and \( P_2 \) which are self-annihilating with respect to the Blanchfield form.
6. The Blanchfield form of \( K \) is stably hyperbolic, that is, the orthogonal sum of the Blanchfield form of \( K \) and some other hyperbolic Blanchfield form is hyperbolic.

The equivalence (3) \( \iff \) (6) \( \iff \) (4) seems to be a classical fact; an explicit proof can be found in \cite{Ors17}. Equivalences (1) \( \iff \) (3), (2) \( \iff \) (4) and (5) \( \iff \) (6) are recent results of Orson \cite{Ors17}.

By Proposition 6.4 with \( R = \mathbb{Z} \), a doubly 1-solvable knot is algebraically doubly slice. In particular, our examples in Theorem 6.10 are algebraically doubly slice.

It is known that a knot is doubly 0-solvable (respectively doubly 0.5-solvable) if and only if it has vanishing Arf invariant (respectively it is algebraically slice) \cite[Corollary 2.9]{Kim06}.

Gilmer-Livingston obstruction

In \cite{GL83}, Gilmer-Livingston introduced obstructions for a knot to being doubly slice, using the idea that a prime power fold cyclic cover of \( S^3 \) branched over a doubly slice knot embeds in \( S^3 \). To state their result, we need the following notations. For a space \( X \) and a homomorphism \( \phi: H_1(X) \to \mathbb{Z}_m \) where \( m = q^s \) for a prime \( q \), let \( X_m \) denote the associated \( m \)-fold cyclic cover of \( X \) and \( T \) a generator of the group of covering transformations. Let \( \overline{H}_k(X_m) \) be the \( e^{2\pi i/m} \)-eigenspace for the action of \( T \) on \( H_k(X_m; \mathbb{C}) \). We define \( \overline{\beta}_k(X_m) := \dim \overline{H}_k(X_m) \) and \( \rho_k(X) := \dim H_k(X; \mathbb{Z}_m) \).

Suppose \( M \) is a closed 3-manifold and \( \phi: H_1(M) \to \mathbb{Z}_m \) is a homomorphism. Then there is a 4-manifold \( V \) with a homomorphism \( \bar{\phi}: H_1(V) \to \mathbb{Z}_m \) such that \( \partial(V, \bar{\phi}) = r(M, \phi) \) for some nonzero integer \( r \). Let \( \sigma_1(V_m) \) be the signature of the intersection form on \( H_2(V_m; \mathbb{C}) \) restricted to \( \overline{H}_2(V_m) \). We define \( \sigma(M, \phi) := \frac{1}{r}(\bar{\phi}(\sigma_1(V_m) - \sigma_0(V)) \) where \( \sigma_0(V) \) is the ordinary signature of \( V \).

Let \( n \) be a prime power. Let \( L^n \) be the \( n \)-fold cyclic cover of \( S^3 \) branched over a knot \( K \), which is a rational homology 3-sphere.

**Theorem 7.1** (Gilmer-Livingston obstruction \cite{GL83}). If \( K \) is a doubly slice knot, then the following hold.
(1) \( H_1(L^n) \cong G_1 \oplus G_2 \) where \( G_1 \) and \( G_2 \) are metabolizers of the linking form of \( L^n \).

(2) If \( \phi : H_1(L^n) \to \mathbb{Z}_m \) is an epimorphism such that \( \phi(G_1) = 0 \) or \( \phi(G_2) = 0 \), then
\[
|\sigma(L^n, \phi)| + |d - 1 - \beta_1(L^n_m)| \leq d
\]
where \( d = \frac{1}{2} \dim H_1(L^n; \mathbb{Z}_q) \).

**Proposition 7.2.** If a knot \( K \) is doubly 2-solvable, then \( K \) has vanishing Gilmer-Livingston obstructions, that is, the conclusions (1) and (2) of Theorem 7.1 hold.

**Proof.** Suppose \( M(K) \) is doubly 2-solvable via \( (U^+, U^-) \). Let \( V^+ \) be the 4-manifold obtained by adding a 2-handle to \( U^+_n \) along the meridian of the preimage of \( K \) in \( M(K)_n \). We define \( V^- \) in a similar fashion using \( U^- \). Then \( \partial V^\pm = L^n \).

By Proposition 6.14, we have
\[
H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \cong H_1(U^+; \mathbb{Z}[t^{\pm 1}]) \oplus H_1(U^-; \mathbb{Z}[t^{\pm 1}])
\]
Also, each of \( H_1(U^\pm; \mathbb{Z}[t^{\pm 1}]) \) is a self-annihilating submodule of the Blanchfield form. Since \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \) is \( \mathbb{Z} \)-torsion free, so are \( H_1(U^\pm; \mathbb{Z}[t^{\pm 1}]) \). By taking the quotients by the submodule generated by \( t^n - 1 \), it follows that \( H_1(L^n) \cong H_1(V^+) \oplus H_1(V^-) \) and that \( H_1(V^\pm) \) are metabolizers of the linking form of \( L^n \).

For brevity, let \( U := U^+ \) and \( V := V^- \). Suppose that \( \phi : H_1(L^n) \to \mathbb{Z}_m \) is an epimorphism which vanishes on \( H_1(V^-) \). Then the map \( \phi \) factors through \( H_1(V) \). Denote by \( \tilde{\phi} \) the homomorphism \( H_1(V) \to \mathbb{Z}_m \). Now we have \( \partial(V, \tilde{\phi}) = (L^n, \phi) \).

Since \( U \) is a 2-solution, for \( r := \frac{1}{2} \dim H_2(U; \mathbb{Q}) \), we have surfaces \( x_i \) and \( y_i \) (\( 1 \leq i \leq r \)) in \( U \) whose homology classes \( [x_i] \) and \( [y_i] \) satisfy the conditions (1) and (2) in Definition 6.3. Let \( x_i^1 \) and \( y_i^1 \) (\( 1 \leq i \leq r, 1 \leq j \leq n \)) be the lifts of \( x_i \) and \( y_i \) in \( V \). Since \( \chi(U_m) = 2rn \), we have \( \chi(V) = 1 + 2rn \). This implies \( \dim H_2(V; \mathbb{Q}) = 2rn \). It follows that \( [x_i^1] \) and \( [y_i^1] \) generate \( H_2(V; \mathbb{Q}) \), since \( x_i^1 \) and \( y_i^1 \) are dual to each other with respect to the intersection form on \( H_2(V; \mathbb{Q}) \). Since the intersection form on \( H_2(V; \mathbb{Q}) \) vanishes on the span of \( [x_i^1] \), we have \( \sigma_0(V) = 0 \). It follows that \( \sigma(L^n, \phi) = \sigma_1(V_m) - \sigma_0(V) = \sigma_1(V_m) \).

Since \( H_3(V; \mathbb{Z}_2) \cong H^1(V, L^n_m; \mathbb{Z}_q) = 0 \), we have \( \rho_3(V) = 0 \). By [GHS21] Proposition 1.4, which states \( \beta_3(V_m) = \rho_3(V) \), it follows that \( \beta_3(V_m) = 0 \), and hence \( \overline{H}_3(V_m) = 0 \). By duality we have \( \overline{H}_1(V_m, L^n_m) \cong \overline{H}_3(V_m) = 0 \). Then the sequence
\[
\overline{H}_2(V_m) \to \overline{H}_2(V_m, L^n_m) \to \overline{H}_1(L^n_m) \to \overline{H}_1(V_m) \to 0
\]
is exact. Therefore, letting \( N \) be the nullity of the matrix for the intersection form restricted to \( \overline{H}_2(V_m) \), we have \( |\sigma_1(V_m)| + N \leq \beta_2(V_m) = 2rn \). The proof is as follows. Regard \( \mathbb{C} \) as a \( \mathbb{Z}[\mathbb{Z}_m] \)-module where \( T \), a generator of the group of the covering transformations \( \mathbb{Z}_m \), acts on \( \mathbb{C} \) via multiplication by \( e^{2\pi i/m} \). Define \( H_*(V; \mathbb{C}^\times) := H_*(\mathbb{C}, (V_m \otimes \mathbb{Z}[\mathbb{Z}_m]) \mathbb{C}) \), where the coefficient group \( \mathbb{C} \) is twisted as above. Let \( \lambda : H_2(V; \mathbb{C}^\times) \to H_2(V; \mathbb{C}^\times) \to \mathbb{C} \) be the (twisted) intersection form. Since \( \mathbb{C} \) is flat as a \( \mathbb{Z}[\mathbb{Z}_m] \)-module, \( H_*(V; \mathbb{C}^\times) \) is isomorphic to \( H_*(V_m) \otimes \mathbb{Z}[\mathbb{Z}_m] \). Note that \( V_m \) is a metabelian cover of \( U \). Therefore, the surfaces \( x_i^l \) and \( y_i^l \) can be lifted to \( V_m \). Since \( U \) is a 2-solution, by our choice of \( x_i \) and \( y_i \), we have \( \lambda([x_i^k], [y_i^k]) = 0 \) and \( \lambda([x_i^j], [y_i^l]) = \delta_{ik} \delta_{jl} \) where \( 1 \leq i, k \leq r \) and \( 1 \leq j, l \leq n \). Let \( \overline{X} : \overline{H}_2(V_m) \times \overline{H}_2(V_m) \to \mathbb{C} \) be the intersection form on \( H_2(V_m; \mathbb{C}) \) restricted to \( \overline{H}_2(V_m) \). Then, \( H_2(V; \mathbb{C}^\times) \cong \overline{H}_2(V_m) \) and \( \lambda(x, y) = m\overline{X}(x, y) \) (see [CSS20]). It follows that \( [x_i^1] \) and \( [y_i^1] \) span a \( 2rn \)-dimensional subspace of \( \overline{H}_2(V_m) \) such that \( \overline{X} \) vanishes on the span of \( [x_i^1] \), which is a subspace of dimension \( 7rn \). Moreover, this \( 2rn \)-dimensional subspace has trivial intersection with the nullspace of \( \overline{X} \). Therefore, \( |\sigma_1(V_m)| + N \leq \overline{\beta}_2(V_m) = 2rn \), as asserted above.
Combining the above assertion with \( \sigma(L^n, \phi) = \sigma_1(V_m) \) and \( \overline{\beta_1}(L^n_m) = N + \overline{\beta_1}(V_m) \),
we obtain that \( |\sigma(L^n, \phi)| + \overline{\beta_1}(L^n_m) \leq \overline{\beta_2}(V_m) + \overline{\beta_1}(V_m) - 2rn \).
We compute \( \overline{\beta_2}(V_m) \). We have \( \overline{\chi}(V) = 1 + 2rn \), where the first equality
is due to [Gil81, Proposition 1.1]. Since \( \phi \) is not trivial, \( \overline{\beta_0}(V_m) = 0 \). Since \( V_m \) is not
closed, \( \overline{\beta_k}(V_m) = 0 \) for \( k \geq 4 \). We have shown \( \overline{\beta_0}(V_m) = 0 \) above, and therefore
\( \overline{\beta_2}(V_m) = \overline{\beta_1}(V_m) + 1 + 2rn \).
Therefore, we have \( |\sigma(L^n, \phi)| + \overline{\beta_1}(L^n_m) \leq 2\overline{\beta_1}(V_m) + 1 \). Since \( \overline{\beta_1}(V_m, L^n_m) = 0 \), we
have \( \overline{\beta_1}(V_m) \leq \overline{\beta_1}(L^n_m) \). Also,
\[ \overline{\beta_1}(V_m) = \rho_1(V) - 1 = \dim H_1(V; \mathbb{Z}_q) - 1 = \frac{1}{2} \dim H_1(L^n; \mathbb{Z}_q) - 1 = d - 1, \]
where the inequality is due to [Gil81, Proposition 1.5]. It follows that
\[ |\sigma(L^n, \phi)| + \overline{\beta_1}(L^n_m) \leq 2 \min \{ \overline{\beta_1}(L^n_m), d - 1 \} + 1. \]
It is straightforward to verify that this is equivalent to \( |\sigma(L^n, \phi)| + |d - 1 - \overline{\beta_1}(L^n_m)| \leq d \).

\[ \square \]

\textbf{Friedl obstruction}

In [Fri04], Friedl gave double sliceness obstructions using \( \eta \)-invariants. He defined a collection, denoted by \( P^\text{irr}_k(H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \times \mathbb{Z}) \), of certain \( k \)-dimensional unitary
representations of the group \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \times \mathbb{Z} \), and considered \( \eta \)-invariants, denoted by \( \eta(M(K), \alpha) \), of the zero surgery manifold \( M(K) \) associated to \( \alpha \in P^\text{irr}_k(H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \times \mathbb{Z}) \). Since we do not need precise descriptions of them, we omit details. Refer to [Fri04, Sections 3 and 4].

\textbf{Theorem 7.3} (Friedl obstruction [Fri04, Theorem 8.4]). If \( K \) is a doubly slice knot, then there exist self-annihilating submodules \( P_1 \) and \( P_2 \) of \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \) for the Blanchfield form such that

(1) \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \cong P_1 \oplus P_2 \), and

(2) \( \eta(M(K), \alpha) = 0 \) for any \( \alpha \in P^\text{irr}_k(H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \times \mathbb{Z}) \) which vanishes on either \( P_1 \) or \( P_2 \).

\textbf{Proposition 7.4}. If a knot \( K \) is doubly 1.5-solvable, then \( K \) has vanishing Friedl obstructions, that is, the conclusions of Theorem 7.3 hold.

\textbf{Proof}. Suppose that \( K \) is doubly 1.5-solvable via \((W_1, W_2)\). Let \( P_i \) be the kernel of the map \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \to H_1(W_i; \mathbb{Z}[t^{\pm 1}]) \). By Proposition 6.14, each \( P_i \) is a self-annihilating submodule with respect to the Blanchfield form and \( H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \cong P_1 \oplus P_2 \). Therefore the conclusion (1) in Theorem 7.3 holds. Let \( k \) be a prime power and \( \alpha \in P^\text{irr}_k(H_1(M(K); \mathbb{Z}[t^{\pm 1}]) \times \mathbb{Z}) \), following the notation in [Fri04]. Suppose \( \alpha \) vanishes on either \( P_1 \) or \( P_2 \). Let \( M_k \) denote the \( k \)-fold cyclic cover of \( M(K) \) and \( L^k \) the \( k \)-fold cyclic cover of \( S^3 \) branched over \( K \). Let \( m \) be a prime power. By [Fri04, Proposition 5.4], \( \eta(M(K), \alpha) = \eta(M_k, \beta) \) for some 1-dimensional unitary representation \( \beta : \pi_1 M_k \to U(1) \) which is associated to a certain character \( \chi : H_1(L^k) \to \mathbb{Z}_m \) and vanishes on \( P_i / (1 - 1) \). Since \( W_i \) are 1.5-solutions, the Casson-Gordon invariant \( \tau(K, \chi) = 0 \) (see [COT03, Section 9]). By [Fri04, Proposition 5.3], \( \eta(M_k, \beta) = 0 \), and hence \( \eta(M(K), \alpha) = 0 \). Therefore, the conclusion (2) in Theorem 7.3 holds.

\[ \square \]

\textbf{Livingston-Meier obstruction}

In [LM15], Livingston and Meier gave double sliceness obstructions using twisted Alexander polynomials. Let \( n \) be a prime power and \( p \) an odd prime. For a knot \( K \), recall that \( L^n \) denote the \( n \)-fold cyclic cover of \( S^3 \) branched over \( K \). Let \( \zeta_p \) be a primitive \( p \)th root
of unity and $\Gamma_p := \mathbb{Q}(\zeta_p)[t^{\pm 1}]$. For a homomorphism $\rho: H_1(L^n) \to \mathbb{Z}_p$, let $\Delta_{K,\rho}(t)$ be the twisted Alexander polynomial of $K$ associated to $\rho$, which is defined in [KL99].

**Theorem 7.5** (Livingston-Meier obstruction [LM15 Theorem 3.2]). If $K$ is a doubly slice knot, then there exist subgroups $G_1$ and $G_2$ of $H_1(L^n)$ such that

1. $H_1(L^n) \cong G_1 \oplus G_2$ where $G_i$ are invariant under the action of the covering transformations of $H_1(L^n)$,

2. for every homomorphism $\rho: H_1(L^n) \to \mathbb{Z}_p$ which vanishes on $G_1$ or $G_2$, we have $\Delta_{K,\rho}(t) = af(t)f(\overline{t})$ for some unit $a \in \Gamma_p$ and $f(t) \in \Gamma_p$.

**Proposition 7.6.** If a knot $K$ is doubly 1.5-solvable, then $K$ has vanishing Livingston-Meier obstructions, that is, the conclusions of Theorem 7.5 hold.

**Proof.** Suppose $K$ is doubly 1.5-solvable via $(U^+, U^-)$. Using the same arguments as in the proof of Proposition 7.2 we can show that there are subgroups $G_i$ for $i = 1, 2$ such that $H_1(L^n) \cong G_1 \oplus G_2$. Since $G_i$ are obtained by taking a quotient of a $\mathbb{Z}[t^{\pm 1}]$-module by $t^n - 1$, $G_i$ are invariant under the action of the covering transformations. This shows (1).

Suppose that $\rho: H_1(L^n) \to \mathbb{Z}_p$ vanishes on $G_1$ or $G_2$. Using the notations in the proof of Proposition 7.2 observe that $G_1$ and $G_2$ are the torsion subgroups of $H_1(U^+_n)$ and $H_1(U^-_n)$ respectively. Then by [COT03 Theorem 9.11] and its proof, the Casson-Gordon discriminant invariant $K$ associated to $\rho$ vanishes. Then by [KL99 Theorem 6.5], $\Delta_{K,\rho}(t)$ has a desired factorization required in (2). \qed

**References**

[Bla57] Richard C. Blanchfield, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. (2) **65** (1957), 340–356.

[CG86] Andrew Casson and Cameron Gordon, *Cobordism of classical knots*, À la recherche de la topologie perdue, Birkauser Boston, Boston, MA, 1986, With an appendix by P. M. Gilmer, pp. 181–199.

[Cha08] Jae Choon Cha, *Topological minimal genus and $L^2$-signatures*, Algebr. Geom. Topol. **8** (2008), 885–909.

[Cha14a] **Symmetric Whitney tower cobordism for bordered 3-manifolds and links**, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3241–3273.

[Cha14b] **Amenable $L^2$-theoretic methods and knot concordance**, Int. Math. Res. Not. IMRN (2014), no. 17, 4768–4803.

[Cho16] **A topological approach to Cheeger-Gromov universal bounds for von Neumann $\rho$-invariants**, Comm. Pure Appl. Math. **69** (2016), no. 6, 1154–1209.

[CHL09] Tim D. Cochran, Shelly Harvey, and Constance Leidy, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. **13** (2009), no. 3, 1419–1482.

[CO12] Jae Choon Cha and Kent E. Orr, *$L^2$-signatures, homology localization, and amenable groups*, Comm. Pure Appl. Math. **65** (2012), 790–832.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and $L^2$-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 435–519.

[COT04] **Structure in the classical knot concordance group**, Comment. Math. Helv. **79** (2004), no. 1, 105–123.

[CP14] Jae Choon Cha and Mark Powell, *Nonconcordant links with homology cobordant zero-framed surgery manifolds*, Pacific J. Math. **272** (2014), no. 1, 1–33.

[CP16] **Casson towers and slice links**, Invent. Math. **205** (2016), no. 2, 413–457.

[CST11] Jim Conant, Rob Schneiderman, and Peter Teichner, *Higher-order intersections in low-dimensional topology*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8131–8138.

[CST12] James Conant, Rob Schneiderman, and Peter Teichner, *Whitney tower concordance of classical links*, Geom. Topol. **16** (2012), no. 3, 1419–1479.

[CST14] J. Conant, R. Schneiderman, and P. Teichner, *Milnor invariants and twisted Whitney towers*, J. Topol. **7** (2014), no. 1, 187–224.

[CT07] Tim D. Cochran and Peter Teichner, *Knot concordance and von Neumann $\rho$-invariants*, Duke Math. J. **137** (2007), no. 2, 337–379.
UNKNOTTED GROPES, WHITNEY TOWERS, AND DOUBLY SLICING KNOTS

[Fre84] Michael H. Freedman, The disk theorem for four-dimensional manifolds, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 647–663.

[Fri04] Stefan Friedl, Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants, Algebr. Geom. Topol. 4 (2004), 893–934 (electronic).

[Fri05] Stefan Friedl, \(L^2\)-eta-invariants and their approximation by unitary eta-invariants, Math. Proc. Cambridge Philos. Soc. 138 (2005), no. 2, 327–338.

[FT95] Michael H. Freedman and Peter Teichner, 4-manifold topology. I. Subexponential groups, Invent. Math. 122 (1995), no. 3, 509–529.

[GL83] Patrick M. Gilmer and Charles Livingston, On embedding 3-manifolds in 4-space, Topology 22 (1983), no. 3, 241–252.

[GS99] Robert E. Gompf and András I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.

[Horn10] Peter D. Horn, The non-triviality of the grope filtrations of the knot and link concordance groups, Comment. Math. Helv. 85 (2010), no. 4, 751–773.

[Horn11] Peter D. Horn, Higher-order analogues of the slice genus of a knot, Int. Math. Res. Not. IMRN (2011), no. 5, 1091–1106.

[Jang17] Hye Jin Jang, Two-torsion in the grope and solvable filtrations of knots, Internat. J. Math. 28 (2017), no. 4, 1750023, 34.

[Kim06] Taehee Kim, New obstructions to doubly slicing knots, Topology 45 (2006), no. 3, 543–566.

[KL99] Paul Kirk and Charles Livingston, Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants, Topology 38 (1999), no. 3, 635–661.

[KQ00] Vyacheslav S. Krushkal and Frank Quinn, Subexponential groups in 4-manifold topology, Geom. Topol. 4 (1999), 407–430 (electronic).

[Kru00] Vyacheslav S. Krushkal, Exponential separation in 4-manifolds, Geom. Topol. 4 (2000), 397–405 (electronic).

[LM15] Charles Livingston and Jeffrey Meier, Doubly slice knots with low crossing number, New York J. Math. 21 (2015), 1007–1026.

[Ors15] Patrick Orson, Double Witt groups, arXiv:1508.00383, 2015.

[Ors17] Patrick Orson, Double L-groups and doubly slice knots, Algebr. Geom. Topol. 17 (2017), no. 1, 273–329.

[Pat88] Alan L. T. Paterson, Amenability, Mathematical Surveys and Monographs, vol. 29, American Mathematical Society, Providence, RI, 1988.

[Rou72] C. P. Rourke and B. J. Sanderson, Introduction to piecewise-linear topology, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69.

[Sch06] Rob Schreiderman, Whitney towers and gropes in 4-manifolds, Trans. Amer. Math. Soc. 358 (2006), no. 10, 4251–4278 (electronic).

[Str74] Ralph Strebel, Homological methods applied to the derived series of groups, Comment. Math. Helv. 49 (1974), 302–332.

[Sum71] D. W. Sumners, Invertible knot cobordisms, Comment. Math. Helv. 46 (1971), 240–256.

Department of Mathematics, POSTECH, Pohang 37673, Republic of Korea – and – School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea E-mail address: jccha@postech.ac.kr

Department of Mathematics, Konkuk University, Seoul 05029, Republic of Korea E-mail address: tkim@konkuk.ac.kr