On model selection consistency of M-estimators with geometrically decomposable penalties

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July 29, 2013

Abstract

Penalized M-estimators are used in many areas of science and engineering to fit models with some low-dimensional structure in high-dimensional settings. In many problems arising in machine learning, signal processing, and high-dimensional statistics, the penalties are geometrically decomposable, i.e. can be expressed as a sum of support functions. We generalize the notion of irrepresentability and develop a general framework for establishing the model selection consistency of M-estimators with these penalties. We then use this framework to derive results for some special cases of interest in machine learning and high-dimensional statistics.

1 Introduction

The principle of parsimony is used in many areas of science and engineering to promote “simple” models over more complex ones. In machine learning, signal processing, and high-dimensional statistics, this principle motivates the use of sparsity inducing penalties for variable/feature selection and signal recovery from incomplete measurements. In this work, we consider M-estimators of the form:

\[
\minimize_{\theta \in \mathbb{R}^p} \ell^{(n)}(\theta) + \lambda \rho(\theta),
\]

where \(\ell^{(n)}\) is a convex, twice continuously differentiable loss function and \(\rho\) is a penalty function. Many commonly used penalties are geometrically decomposable, i.e. can be expressed as a sum of support functions. We generalize the notion of irrepresentability and develop a general framework to establish consistency and model selection consistency of these penalized M-estimators. When specialized to various statistical models, our framework yields some known and some new model selection consistency results.

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The rest of the paper is organized as follows: First, we review existing work on consistency and model selection consistency of penalized M-estimators. Then, in Section 2 we introduce geometrically decomposable penalties and give two examples from statistical learning. In Section 3 we generalize the notion of irrepresentability and state our main result (Theorem 3.4). We prove our main result in Section 4 and develop a converse result concerning the necessity of the irrepresentable condition in Section 6. Finally, in Section 5 we use our main result to derive consistency and model selection consistency results for two statistical models and verify the consequences of these results empirically.

1.1 Consistency and model selection consistency of penalized M-estimators

The consistency of penalized M-estimators has been studied extensively. The three most well-studied problems are (i) the lasso \cite{6, 3, 31}, (ii) generalized linear models (GLM) with the lasso penalty \cite{13}, and (iii) penalized inverse covariance estimators (equivalent to penalized maximum likelihood for a Gaussian Markov random field) \cite{2, 20, 15, 25}. There are also many results for M-estimators with group and structured variants of the lasso penalty \cite{1, 11, 18, 10}.

Negahban et al. \cite{22} proposed a unified framework for establishing consistency and convergence rates for M-estimators with penalties \( \rho \) that are decomposable with respect to a pair of subspaces \( M, \bar{M} \):

\[
\rho(x + y) = \rho(x) + \rho(y), \quad \text{for all } x \in M, y \in \bar{M}^\perp.
\]

Many commonly used penalties such as the lasso, group lasso, and nuclear norm are decomposable in this sense. Negahban et al. also develop a notion of restricted strong convexity for such penalties and prove a general result that establishes the consistency of M-estimators with these penalties. Using their framework, they derive consistency results for special cases like sparse and group sparse regression.

We focus on model selection consistency of penalized M-estimators. Previous work in this area identified the notion of irrepresentability for the lasso \cite{20, 35, 31} and then generalized to GLM’s with the lasso penalty \cite{11, 24, 33}. These results were later extended to group and structured variants of the lasso penalty \cite{34, 21, 14, 27, 28, 42, 29}. The irrepresentable condition has also been used to obtain model selection consistency results for estimating inverse covariance matrices with the lasso penalty \cite{14, 25}. These methods have been extended to fit discrete graphical models using penalized composite likelihood estimators \cite{9} and generalized covariance matrices \cite{19}.

There is also a rich literature on constrained M-estimators of the form

\[
\min_{\theta \in \mathbb{R}^p} \rho(\theta), \quad \text{subject to } \mathcal{A}(\theta) \in \mathcal{K},
\]

\footnote{Given the extensive work in these areas, our review and referencing is necessarily incomplete.}
where \( A \) is an affine mapping and \( K \) is a convex cone. We do not review this literature except to describe a notion of decomposability proposed by Candés and Recht [7]. A penalty \( \rho \) is decomposable according to Definition 2.2 in [7] if there exists a subspace \( S \) such that 
\[
\partial \rho(\theta^*) = \{ z \in \mathbb{R}^p \mid \rho(z_{S^\perp}) \leq 1 \},
\]
where \( \theta^* \) is the unknown parameters and \( z_{S^\perp} \) is the component of \( z \) in \( S^\perp \). Many commonly used penalties such as the lasso, group lasso, and nuclear norm are also decomposable in this sense. We refer to the introduction in [7] for a review of the work about constrained M-estimators in compressed sensing and low-rank matrix completion.

## 2 Geometrically decomposable penalties

Let \( C \) be a closed convex set in \( \mathbb{R}^p \). Then the support function of \( C \) is
\[
h_C(x) = \sup_y \{ y^T x \mid y \in C \}.
\]
If \( C \) is a norm ball, i.e. \( C = \{ x \mid \|x\| \leq 1 \} \), then \( h_C \) is the dual norm:
\[
\|y\|^* = \sup_x \{ x^T y \mid \|x\| \leq 1 \}.
\]
The support function is a supremum of linear functions, hence
\[
\partial h_C(x) = \{ y \in C \mid y^T x = h_C(x) \}.
\]
The support function (as a function of the convex set \( C \)) is also additive over Minkowski sums, i.e.
\[
h_{C+D}(x) = h_C(x) + h_D(x).
\]
We use this property to express penalty functions as sums of support functions. E.g. if \( \rho \) is a norm and the dual norm ball can be expressed as a (Minkowski) sum of convex sets, then \( \rho \) can be expressed as a sum of support functions.

If a penalty function \( \rho \) can expressed as
\[
\rho(\theta) = h_A(\theta) + h_I(\theta) + h_{S^\perp}(\theta),
\]
where \( A \) and \( I \) are closed convex sets and \( S \) is a subspace, then we say \( \rho \) is a geometrically decomposable penalty function. This form is general; if \( \rho \) can be expressed as a sum of support functions, i.e.
\[
\rho(\theta) = h_{C_1}(\theta) + \cdots + h_{C_k}(\theta),
\]
then we can set \( A \) and \( I \) to be sums of the sets \( C_1, \ldots, C_k \) to express \( \rho \) in geometrically decomposable form (2.1). In many cases of interest, \( A + I \) is a norm ball and \( h_{A+I} = h_A + h_I \) is the dual norm. In our analysis, we assume
1. \(A\) and \(I\) are bounded.

2. \(I\) contains a relative neighborhood of the origin, \(i.e. 0 \in \text{relint}(I)\).

We do not require \(A + I\) to contain a neighborhood of the origin. This generality allows for unpenalized variables.

The notation \(A\) and \(I\) should be as read as “active” and “inactive”: \(\text{span}(A)\) should contain the true parameter vector and \(\text{span}(I)\) should contain deviations from the truth that we want to penalize. \(E.g.\) if we know the sparsity pattern of the unknown parameter vector, then \(A\) should span the subspace of all vectors with the correct sparsity pattern.

The third term enforces a subspace constraint \(\theta \in S\) because the support function of a subspace is the characteristic function of the orthogonal complement:

\[
h_S(x) = 1_{S^\perp}(x) = \begin{cases} 
0 & x \in S^\perp \\
\infty & \text{otherwise}.
\end{cases}
\]

Such subspace constraints arise in many problems, either naturally or after reformulation. We give two examples of M-estimators with geometrically decomposable penalty functions from statistical learning.

2.1 The lasso and group lasso penalties

Two geometrically decomposable penalties are the \textit{lasso} and \textit{group lasso} penalties. Let \(A\) and \(I\) be complementary subsets of \(\{1, \ldots, p\}\). We can decompose the lasso penalty component-wise to obtain

\[
\|\theta\|_1 = h_{B_{\infty, A}}(\theta) + h_{B_{\infty, I}}(\theta),
\]

where \(h_{B_{\infty, A}}\) and \(h_{B_{\infty, I}}\) are support functions of the sets

\[
B_{\infty, A} = \{\theta \in \mathbb{R}^p \mid \|\theta\|_\infty \leq 1 \text{ and } \theta_I = 0\}
\]

\[
B_{\infty, I} = \{\theta \in \mathbb{R}^p \mid \|\theta\|_\infty \leq 1 \text{ and } \theta_A = 0\}.
\]

We can also decompose the group lasso penalty group-wise (\(A\) and \(I\) are now complementary sets of groups) to obtain

\[
\sum_{g \in \mathcal{G}} \|\theta_g\|_2 = h_{B_{(2, \infty), A}}(\theta) + h_{B_{(2, \infty), I}}(\theta).
\]

\(h_{B_{(2, \infty), A}}\) and \(h_{B_{(2, \infty), I}}\) are support functions of the sets

\[
B_{(2, \infty), A} = \{\theta \in \mathbb{R}^p \mid \max_{g \in \mathcal{G}} \|\theta_g\|_2 \leq 1 \text{ and } \theta_g = 0, g \in I\}
\]

\[
B_{(2, \infty), I} = \{\theta \in \mathbb{R}^p \mid \max_{g \in \mathcal{G}} \|\theta_g\|_2 \leq 1 \text{ and } \theta_g = 0, g \in A\}.
\]
The generalized lasso penalty

Another geometrically decomposable penalty is the generalized lasso penalty \[28\]. Let \(D \in \mathbb{R}^{m \times p}\) and \(A\) and \(I\) be complementary subsets of \(\{1, \ldots, m\}\). We can express the generalized lasso penalty in decomposable form:

\[
\|D\theta\|_1 = h_{D^TB_{\infty,A}}(\theta) + h_{D^TB_{\infty,I}}(\theta). \tag{2.2}
\]

\(h_{D^TB_{\infty,A}}\) and \(h_{D^TB_{\infty,I}}\) are support functions of the sets

\[
D^TB_{\infty,A} = \{x \in \mathbb{R}^p \mid x = D^T_A y, \|y\|_{\infty} \leq 1\} \tag{2.3}
\]

and

\[
D^TB_{\infty,I} = \{x \in \mathbb{R}^p \mid x = D^T_I y, \|y\|_{\infty} \leq 1\}. \tag{2.4}
\]

We can also formulate any generalized lasso penalized M-estimator as a linearly constrained, lasso penalized M-estimator. After a change of variables, a generalized lasso penalized M-estimator is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \ell(n)(D^\dagger \theta + \gamma) + \lambda \|\theta\|_1 \\
\text{subject to} & \quad \gamma \in N(D).
\end{align*}
\]

The lasso penalty can be decomposed component-wise to obtain

\[
\|\theta\|_1 = h_{B_{\infty,A}}(\theta) + h_{B_{\infty,I}}(\theta).
\]

The subspace constraint \(\theta \in N(D)\) can be enforced with the support function of \(R(D)\). This yields the optimization problem

\[
\begin{align*}
\text{minimize} & \quad \ell(n)(D^\dagger \theta + \gamma) + \lambda h_{B_{\infty,A}}(\theta) + h_{B_{\infty,I}}(\theta) + h_{N(D)}(\gamma) \\
\text{subject to} & \quad \gamma \in N(D).
\end{align*}
\]

There are many interesting applications of the generalized lasso in signal processing and statistical learning. We refer to Section 2 in \[28\] for some examples.

3 Main result

We assume the unknown parameter vector \(\theta^*\) is contained in the model subspace

\[
M := \text{span}(I)^\perp \cap S. \tag{3.1}
\]

We say an estimate \(\hat{\theta}\) is consistent (in the \(\ell_2\) norm) if the estimation error in the \(\ell_2\) norm decays to zero as sample size grows:

\[
\|\hat{\theta} - \theta^*\|_2 \to 0 \text{ as } n \to \infty.
\]

We say \(\hat{\theta}\) is model selection consistent if the estimator selects the correct model with probability tending to one as sample size grows:

\[
\Pr(\hat{\theta} \in M) \to 1 \text{ as } n \to \infty.
\]
Before we state our main result, we state our assumptions on the problem. These assumptions are stated in terms of the sample Fisher information matrix:

\[ Q^{(n)} = \nabla^2 \ell^{(n)}(\theta^*). \]

We use \( B_r(x) \) to denote the ball (in the \( \ell_2 \) norm) of radius \( r \) centered at \( x \), i.e.

\[ B_r(x) = \{ y \in \mathbb{R}^p \mid \| x - y \|_2 \leq r \}. \]

**Assumption 3.1** (Restricted strong convexity). We assume the loss function \( \ell^{(n)} \) is locally strongly convex with constant \( m \) over the model subspace, i.e.

\[ \ell^{(n)}(\theta_1) - \ell^{(n)}(\theta_2) \geq \nabla \ell^{(n)}(\theta_2)^T (\theta_1 - \theta_2) + \frac{m}{2} \| \theta_1 - \theta_2 \|_2^2 \quad (3.2) \]

for some \( m > 0 \) and all \( \theta_1, \theta_2 \in B_r(\theta^*) \cap M \).

We require this assumption to make the maximum likelihood estimate unique over the model subspace. Otherwise, there is no hope for consistency. This assumption requires the loss function to be curved along certain directions in the model subspace and is very similar to the notion of restricted strong convexity in [22] and compatibility in [4]. Intuitively, this assumption means the “active” predictors are not overly dependent on each other.

We also require \( \nabla^2 \ell^{(n)} \) to be locally Lipschitz continuous, i.e.

\[ \| \nabla^2 \ell^{(n)}(\theta_1) - \nabla^2 \ell^{(n)}(\theta_2) \|_2 \leq L \| \theta_1 - \theta_2 \|_2 \]

for some \( L > 0 \) and all \( \theta_1, \theta_2 \in B_r(\theta^*) \cap M \). This condition automatically holds for all twice-continuously differentiable \( \ell^{(n)} \), hence we do not state this condition as an assumption.

To obtain model selection consistency results, we must first generalize the irrepresentable condition for the lasso penalty to a geometrically decomposable penalty. We use \( P_C \) to denote the orthogonal projector onto \( \text{span}(C) \):

\[ \| P_C x - x \|_2 = \inf_y \{ \| x - y \|_2 \mid y \in \text{span}(C) \}. \]

and \( \gamma_C \) to denote the gauge function of a convex set \( C \) containing the origin:

\[ \gamma_C(x) = \inf_x \{ \lambda \in \mathbb{R}^+ \mid x \in \lambda C \}. \]

If \( C \) is a norm ball \( \{ x \in \mathbb{R}^p \mid \| x \| \leq 1 \} \), then \( \gamma_C \) is the norm, i.e. \( \gamma_C(x) = \| x \| \).

**Assumption 3.2** (Irrepresentability). There exist \( \tau \in (0, 1) \) such that

\[ \sup_z \{ V(P_M (Q^{(n)} P_M^T P_M z - z)) \mid z \in \partial h_A(B_r(\theta^*) \cap M) \} < 1 - \tau, \]

where \( V \) is the infimal convolution of \( \gamma_I \) and \( 1_{S^\perp} \)

\[ V(z) = \inf_u \{ \gamma_I(u) + 1_{S^\perp}(z - u) \}. \]
If \( u_I(z) \) and \( u_S \perp (u) \) achieve \( V(z) = \gamma_I(u_I(z)) \). Thus, if \( V(u) < 1 \), then \( (u_I(z)) \in \text{relint}(I) \). Thus the irrepresentable condition says we can decompose any \( z \in A \) into \( u_I + u_S \perp \), where \( u_I \in \text{relint}(I) \) and \( u_S \perp \in S \perp \).

**Lemma 3.3.** \( V \) is a bounded semi-norm over \( M \perp \), i.e. \( V \) is finite for all \( z, w \in M \perp \) and sublinear.

**Proof.** First, we show \( V \) is positive homogeneous. For any \( \alpha \geq 0 \), we have
\[
V(\alpha z) = \inf_u \{ \gamma_I(u) + 1_{S \perp } (\alpha z - u) \}.
\]
Let \( u = \alpha v \). Then we have
\[
V(\alpha z) = \inf_u \{ \gamma_I(\alpha v) + 1_{S \perp } (\alpha(z - v)) \}
= \inf_u \{ \alpha \gamma_I(v) + 1_{S \perp } (z - v) \}
= \alpha V(z).
\]
\( V \) also satisfies the triangle inequality:
\[
V(z + w) = \inf_u \{ \gamma_I(u) + 1_{S \perp } (z + w - u) \}
= \inf_u \{ \gamma_I(u) + 1_{S \perp } (z + w - u_I(z) - u_S \perp (z)) \}
\leq \gamma_I(u_I(z)) + 1_{S \perp } (u_S \perp (z)) + \inf_u \{ \gamma_I(u - u_I(z))
+ 1_{S \perp } (z + w - u - u_S \perp (z)) \}
\]
\[
= V(z) + 1_{S \perp } (u_S \perp (z)) + \inf_v \{ \gamma_I(u - u_I(z))
+ 1_{S \perp } (z + w - u - u_S \perp (z)) \}.
\]
Let \( v = u - u_I(z) \). Then we have
\[
V(z + w) \leq V(z) + \inf_v \{ \gamma_I(v) + 1_{S \perp } (z + w - v - u_I(z) - u_S \perp (z)) \}
= V(z) + \inf_v \{ \gamma_I(v) + 1_{S \perp } (w - v) \}
= V(z) + V(w).
\]
Thus \( V \) satisfies the triangle inequality. \( \square \)

Intuitively, the irrepresentable condition requires the active predictors to be not overly dependent on the inactive predictors. The irrepresentable condition is a standard assumption for model selection consistency and has been shown to be almost necessary for the sign consistency of the lasso \cite{35, 31}. We generalize their analysis to geometrically decomposable penalties in Section 6.

We also require there to be a finite \( \bar{\tau} \) such that
\[
V(P_M \perp (Q^{(n)}P_M (P_M Q^{(n)}P_M)^\dagger P_M x - x)) \leq \bar{\tau} \|x\|_p.
\]
$V$ is bounded over $M^\perp$, so $V(P_M \ll (Q_n P_M (P_M Q_n)^d P_M x - x))$ is a continuous function of $x$ and attains its supremum over compact sets. Thus $\bar{\tau}$ surely exists, so we do not state this requirement as an assumption.

Finally, we state our main theorem and describe how to use this result. We use $\kappa(\ell_p)$ to denote the compatibility constant between a semi-norm $p$ and the $\ell_2$ norm over the model subspace (3.1):

$$
\kappa(p) := \sup_x \{ p(x) \mid \|x\|_2 \leq 1, x \in M \}.
$$

This constant quantifies how large $p(x)$ can be compared to $\|x\|_2$ for all $x \in M$.

**Theorem 3.4.** Suppose Assumptions 3.1 and 3.2 hold. If we select $\lambda$ such that

$$
\lambda > \frac{2\bar{\tau}}{\bar{\tau}} \frac{\|\nabla \ell(n)(\theta^*)\|_p}{\|\nabla \ell(n)(\theta^*)\|_p} \text{ w.h.p. and } \lambda < \min \left\{ \frac{m^2 \tau^2}{2 \tau \kappa(h_A) (2 \kappa(h_A) + \bar{\tau} \kappa(\ell_p^*))}, \frac{m \tau^2}{2 \kappa(h_A) + \bar{\tau} \kappa(\ell_p^*)} \right\},
$$

where $\ell_p$ and $\ell_p^*$ are dual norms, then the penalized M-estimator is unique, consistent (in the $\ell_2$ norm), and model selection consistent, i.e. the optimal solution to (1.1) satisfies

1. $\|\hat{\theta} - \theta^*\|_2 \leq \frac{\kappa(h_A)}{m} \left( \kappa(h_A) + \frac{\kappa(\ell_p^*)}{2\bar{\tau}} \right) \lambda$,
2. $\hat{\theta} \in M := \text{span}(I^\perp) \cap S$.

**Remark 3.1.** Theorem 3.4 makes a deterministic statement about the optimal solution to (1.1). To use this result to derive consistency and model selection consistency results for a statistical model, we must first verify Assumptions (3.1) and (3.2) are satisfied w.h.p.. Then, we must select $\lambda$ such that

$$
\lambda > \frac{2\bar{\tau}}{\bar{\tau}} \frac{\|\nabla \ell(n)(\theta^*)\|_p}{\|\nabla \ell(n)(\theta^*)\|_p} \text{ w.h.p. and } \lambda < \min \left\{ \frac{m^2 \tau^2}{2 \tau \kappa(h_A) (2 \kappa(h_A) + \bar{\tau} \kappa(\ell_p^*))}, \frac{m \tau^2}{2 \kappa(h_A) + \bar{\tau} \kappa(\ell_p^*)} \right\}.
$$

In Section 5, we use this theorem to derive consistency and model selection consistency results for the generalized lasso and penalized likelihood estimation for exponential families.

**4 Proof of the main result**

We prove Theorem 3.4 by constructing a primal-dual pair for the original problem with the desired properties: consistency and model selection consistency. The proof consists of these steps:

1. Solve a restricted problem (4.1) that enforces the constraint $\theta \in M$ to obtain a restricted primal-dual pair, and show this restricted primal solution $\hat{\theta}$ is consistent (cf. Propositions 1.1).
2. Establish a dual certificate condition that guarantees all solutions to the original problem are also solutions to the restricted problem (cf. Proposition 4.2).

3. Construct a primal-dual pair for the original problem using the restricted primal dual pair that satisfies the dual certificate condition. This means the solution to the restricted problem is also the solution to the original problem.

This strategy is called the dual certificate or primal-dual witness technique [31].

First, we solve the restricted problem
\[
\minimize \theta \in \mathbb{R}^p \ell(n)(\theta) + \lambda (h_A(\theta) + h_{M^\perp}(\theta))
\]
(4.1)
to obtain a restricted primal-dual pair $\hat{\theta}, \hat{v}_A, \hat{v}_{M^\perp}$. This restricted primal-dual pair satisfies the first order optimality condition
\[
\nabla \ell(\hat{\theta}) + \lambda \hat{v}_A + \lambda \hat{v}_{M^\perp} = 0
\]
(4.2)
\[
\hat{v}_A \in \partial h_A(\hat{\theta}), \hat{v}_{M^\perp} \in M^\perp.
\]
(4.3)

We enforce the subspace constraint $\hat{\theta} \in M$, hence $\hat{\theta}$ is model selection consistent. We also show that $\hat{\theta}$ is consistent.

**Proposition 4.1.** Suppose Assumption 3.1 holds. If $\lambda$ is selected s.t.
\[
\frac{2\tau}{\tau} \|\nabla \ell^{(n)}(\theta^*)\|_p < \lambda < \frac{mr}{2\kappa(h_A) + \frac{\tau}{2}\kappa(\ell_p^*)},
\]
Then the solution to the restricted problem (4.1) is unique and satisfies
\[
\|\hat{\theta} - \theta^*\|_2 \leq \frac{2}{m} \left(\kappa(h_A) + \frac{\tau}{2\tau}\kappa(\ell_p^*)\right) \lambda < r.
\]

**Proof.** $\hat{\theta}$ solve the restricted problem, hence
\[
\ell^{(n)}(\hat{\theta}) + \lambda h_A(\hat{\theta}) \leq \ell^{(n)}(\theta^*) + \lambda h_A(\theta^*).
\]
We rearrange to obtain
\[
\ell^{(n)}(\hat{\theta}) - \ell^{(n)}(\theta^*) + \lambda (h_A(\hat{\theta}) - h_A(\theta^*)) \leq 0.
\]
(4.4)
$\hat{\theta} \in M$ and $\ell^{(n)}$ is locally strongly convex over $R$ (and convex in general), hence $\hat{\theta}$ is unique. If $\|\hat{\theta} - \theta^*\|_2 \leq r$, then
\[
\nabla \ell^{(n)}(\theta^*)^T(\hat{\theta} - \theta^*) + \frac{m}{2} \|\hat{\theta} - \theta^*\|_2^2 + \lambda (h_A(\hat{\theta}) - h_A(\theta^*)) \leq 0.
\]
We assume $\|\hat{\theta} - \theta^*\|_2 \leq r$ and verify this assumption later. We take norms to obtain
\[
0 \geq -\|\nabla \ell^{(n)}(\theta^*)\|_p \|\hat{\theta} - \theta^*\|_p + \frac{m}{2} \|\hat{\theta} - \theta^*\|_2^2 - \lambda h_A(\hat{\theta} - \theta^*),
\]
(4.5)
where \( \ell^*_p \) is the dual norm to \( \ell_p \). It is more convenient to bound the estimation error in the \( \ell_2 \) norm, hence

\[
0 \geq -\kappa(\ell^*_p)\|\nabla \ell(n)(\theta^*)\|_p \|\hat{\theta} - \theta^*\|_2 + \frac{m}{2} \|\hat{\theta} - \theta^*\|_2^2 - \lambda h_A(\hat{\theta} - \theta^*). \tag{4.5}
\]

A is bounded, hence there exist \( \kappa(h_A) \) such that

\[
h_A(\hat{\theta} - \theta^*) \leq \kappa(h_A)\|\hat{\theta} - \theta^*\|_2.
\]

We substitute this bound into (4.5) to obtain

\[
0 \geq -\kappa(\ell^*_p)\|\nabla \ell(n)(\theta^*)\|_p \|\hat{\theta} - \theta^*\|_2 + \frac{m}{2} \|\hat{\theta} - \theta^*\|_2^2 - \kappa(h_A)\lambda \|\hat{\theta} - \theta^*\|_2
\]

This means

\[
\|\hat{\theta} - \theta^*\|_2 \leq \frac{2}{m} (\kappa(\ell^*_p)\|\nabla \ell(n)(\theta^*)\|_p + \kappa(h_A)\lambda).
\]

We select \( \lambda \) such that \( \lambda > \frac{2m}{\kappa(\ell^*_p)\|\nabla \ell(n)(\theta^*)\|_p} \), hence \( \frac{\|\nabla \ell(n)(\theta^*)\|_p}{\lambda} \leq \frac{1}{2m} \) and

\[
\|\hat{\theta} - \theta^*\|_2 \leq 2m \left( \kappa(h_A) + \frac{\tau}{2\kappa(\ell^*_p)} \right) \lambda.
\]

Finally, we can verify \( \|\hat{\theta} - \theta^*\|_2 < r \) if \( \lambda \leq \frac{mr}{2\kappa(h_A) + \tau \kappa(\ell^*_p)} \). \( \square \)

**Remark 4.1.** In some cases, this bound on the estimation error can be tightened. E.g., in some special instances of the generalized lasso, we can handle the first term in (4.5) more delicately to obtain a tighter bound. This allows us to use a smaller \( \lambda \) and reduces the sample complexity of the procedure.

Then, we establish a dual certificate condition that guarantees all solutions to the original problem satisfy \( h_I(\theta) = 0 \). Thus all solutions to the original problem are also solutions to the restricted problem.

**Proposition 4.2.** Suppose \( \hat{\theta} \) is a primal solution to (1.1) and \( \hat{u}_A, \hat{u}_I, \hat{u}_{S^\perp} \) are dual solutions, i.e. \( \hat{\theta}, \hat{u}_A, \hat{u}_I, \hat{u}_{S^\perp} \) satisfy

\[
\nabla \ell(\hat{\theta}) + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S^\perp}) = 0 \quad \hat{u}_I \in \partial h_I(\hat{\theta}), \hat{u}_A \in \partial h_A(\hat{\theta}), \hat{u}_{S^\perp} \in S^\perp.
\]

If \( \hat{u}_I \in \text{relint}(I) \), then all primal solutions satisfy \( h_I(\theta) = 0 \).

**Proof.** Suppose there are two primal dual solution pairs, \( \theta_1, u_{A,1}, u_{I,1}, u_{S^\perp,1} \) and \( \theta_2, u_{A,2}, u_{I,2}, u_{S^\perp,2} \), i.e.

\[
\nabla \ell(n)(\theta_1) + \lambda (u_{A,1} + u_{I,1} + u_{S^\perp,1}) = 0 \tag{4.6}
\]

\[
\nabla \ell(n)(\theta_2) + \lambda (u_{A,2} + u_{I,2} + u_{S^\perp,2}) = 0 \tag{4.7}
\]
The original problem \(1.1\) is convex, hence the optimal value is unique
\[
\ell(n)(\theta_1) + P(\theta_1) = \ell(n)(\theta_1) + \lambda(u_{A,1} + u_{I,1} + u_{S_{-1}})^T \theta_1 \\
= \ell(n)(\theta_2) + P(\theta_2) = \ell(n)(\theta_2) + \lambda(u_{A,2} + u_{I,2} + u_{S_{-2}})^T \theta_2.
\]
We subtract \(\lambda(u_{A,1} + u_{I,1} + u_{S_{-1}})^T \theta_2\) from both sides to obtain
\[
\ell(n)(\theta_1) + \lambda(u_{A,1} + u_{I,1} + u_{S_{-1}})^T (\theta_1 - \theta_2) \\
= \ell(n)(\theta_2) + \lambda(u_{A,2} + u_{I,2} + u_{S_{-2}} - u_{A,1} - u_{I,1} - u_{S_{-1}})^T \theta_2
\]
We rearrange this expression to obtain
\[
\ell(n)(\theta_1) - \ell(n)(\theta_2) + \lambda(u_{A,1} + u_{I,1} + u_{S_{-1}})^T (\theta_1 - \theta_2) \\
= \lambda(u_{A,2} + u_{I,2} + u_{S_{-2}} - u_{A,1} - u_{I,1} - u_{S_{-1}})^T \theta_2
\]
We substitute in \(4.6\) to obtain
\[
\ell(n)(\theta_1) - \ell(n)(\theta_2) - \nabla \ell(n)(\theta_1)^T (\theta_1 - \theta_2) \\
= \lambda(u_{A,2} + u_{I,2} + u_{S_{-2}} - u_{A,1} - u_{I,1} - u_{S_{-1}})^T \theta_2
\]
\(\ell(n)\) is convex, hence the left side is non-positive and
\[
(u_{A,2} + u_{I,2} + u_{S_{-2}})^T \theta_2 \leq (u_{A,1} + u_{I,1} + u_{S_{-1}})^T \theta_2.
\]
Both \(\theta_1\) and \(\theta_2\) are in \(S\), hence we can ignore the terms \(u_{S_{-2}}^T \theta_2\) and \(u_{S_{-1}}^T \theta_2\) to obtain
\[
(u_{A,2} + u_{I,2})^T \theta_2 \leq (u_{A,1} + u_{I,1})^T \theta_2.
\]
But we also know
\[
(u_{A,1} + u_{I,1})^T \theta_2 \leq \sup_u \{u^T \theta_2 \mid u \in A\} + \sup_u \{u^T \theta_2 \mid u \in I\} \\
= u_{A,2}^T \theta_2 + u_{I,2}^T \theta_2.
\]
We combine these two inequalities to obtain
\[
(u_{A,2} + u_{I,2})^T \theta_2 = (u_{A,1} + u_{I,1})^T \theta_2 \leq u_{A,2}^T \theta_2 + u_{I,2}^T \theta_2
\]
This simplifies to \(u_{I,2}^T \theta_2 \leq u_{I,1}^T \theta_2\). If \(u_{I,1} \in \text{relint}(I)\), then
\[
u_{I,1}^T \theta_2 = u_{I,2}^T \theta_2 \text{ if } \theta_2 \text{ has no component in } \text{span}(I) \\
u_{I,1}^T \theta_2 < u_{I,2}^T \theta_2 \text{ if } \theta_2 \text{ has a component in } \text{span}(I).
\]
But we also know \(u_{I,2}^T \theta_2 \leq u_{I,1}^T \theta_2\). Thus we deduce \(\theta_2\) has no component in \(\text{span}(I)\) and \(h_I(\theta_2) = 0\).
Finally, we use the restricted primal-dual pair \( \hat{\theta}, \hat{v}_A, \hat{v}_{M^\perp} \) to construct a primal-dual pair for the original problem \( \text{(4.1)} \). The optimality conditions of the original problem are

\[
\nabla \ell(\hat{\theta}) + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S^\perp}) = 0 \tag{4.8}
\]

\[
\hat{u}_I \in \partial h_I(\hat{\theta}), \quad \hat{u}_A \in \partial h_A(\hat{\theta}), \quad \hat{u}_{S^\perp} \in S^\perp. \tag{4.9}
\]

We set \( \hat{u}_I = \arg \min_u \gamma_I(u) + 1_{S^\perp}(P_{M^\perp}v_{M^\perp} - u), \hat{u}_{S^\perp} = P_{M^\perp}v_{M^\perp} - u \) and verify \( \hat{\theta}, \hat{u}_A, \hat{u}_I, \hat{u}_{S^\perp} \) satisfies \( \text{(4.9)} \). Hence the \( \hat{\theta} \) is also a solution to the original problem.

We seek to show the solution to the original problem is unique using Proposition \( \text{(4.2)} \) are satisfied. To do this, we must verify \( \hat{u}_I \) is satisfies the dual certificate condition, i.e. \( \hat{u}_I \in \text{relint}(I) \).

A primal-dual solution \( \hat{\theta}, \hat{v}_A, \hat{v}_{M^\perp} \) for the restricted problem \( \text{(4.1)} \) satisfies \( \text{(4.3)} \) and thus the zero reduced gradient condition:

\[
P_M \nabla \ell(\hat{\theta}) + \lambda P_M \hat{v}_A = 0.
\]

We Taylor expand \( \nabla \ell \) around \( \theta^* \) to obtain

\[
P_M \nabla \ell^{(n)}(\theta^*) + P_M Q^{(n)}P_M(\hat{\theta} - \theta^*) + P_M R^{(n)} + \lambda P_M \hat{v}_A = 0,
\]

where

\[
R^{(n)} = \nabla \ell(\hat{\theta}) - \nabla \ell^{(n)}(\theta^*) - Q^{(n)}(\hat{\theta} - \theta^*)
\]

is the Taylor remainder term. We rearrange to obtain

\[
P_M Q^{(n)}P_M(\hat{\theta} - \theta^*) = -P_M(\nabla \ell^{(n)}(\theta^*) + \lambda \hat{v}_A + R^{(n)}),
\]

\[
P_M Q^{(n)}P_M \text{ is invertible over } M, \text{ hence we can solve for } \hat{\theta} \text{ to obtain}
\]

\[
\hat{\theta} = \theta^* - (P_M Q^{(n)}P_M)^\dagger P_M(\nabla \ell^{(n)}(\theta^*) + \lambda \hat{v}_A + R^{(n)}). \tag{4.10}
\]

We can Taylor expand \( \text{(4.3)} \) to obtain

\[
\nabla \ell^{(n)}(\theta^*) + Q^{(n)}(\hat{\theta} - \theta^*) + R^{(n)} + \lambda (\hat{v}_A + \hat{v}_{M^\perp}) = 0.
\]

We substitute \( \text{(4.10)} \) into this expression to obtain

\[
0 = \nabla \ell^{(n)}(\theta^*) - Q^{(n)}(P_M Q^{(n)}P_M)^\dagger P_M(\nabla \ell^{(n)}(\theta^*) + \lambda \hat{v}_A + R^{(n)})
\]

\[
+ R^{(n)} + \lambda (\hat{v}_A + \hat{v}_{M^\perp}). \tag{4.11}
\]

We can solve for \( \hat{v}_{M^\perp} \) to obtain

\[
\hat{v}_{M^\perp} = \frac{1}{\lambda} \left( Q^{(n)}(P_M Q^{(n)}P_M)^\dagger P_M(\nabla \ell^{(n)}(\theta^*) + \lambda \hat{v}_A + R^{(n)})
\]

\[
- \nabla \ell^{(n)}(\theta^*) - R^{(n)} - \lambda \hat{v}_A \right) \tag{4.12}
\]

\[
= Q^{(n)}P_M(P_M Q^{(n)}P_M)^\dagger P_M \hat{v}_A - \hat{v}_A
\]

\[
+ \frac{1}{\lambda} \left( Q^{(n)}P_M(P_M Q^{(n)}P_M)^\dagger P_M(\nabla \ell^{(n)}(\theta^*) + R^{(n)})
\]

\[
- \nabla \ell^{(n)}(\theta^*) + R^{(n)} \right),
\]

\[
12
\]
where we used the fact that the row space of $P_M Q^{(n)} P_M$ is $M$. We need to show $V(P_M \hat{v}_{M^\perp}) < 1$. Using the facts (i) $V$ is a semi-norm, and (ii) $\hat{v}_{M^\perp} \in M^\perp$, we obtain a bound on $V(P_M \hat{v}_{M^\perp})$:

$$V(P_M \hat{v}_{M^\perp}) \leq V(P_M \hat{v}_{A} - \hat{v}_{A}) + \frac{1}{\lambda} V(P_M \hat{v}_{A} - \hat{v}_{A}) \leq 1 - \tau.$$

We use the irrepresentable condition to bound the first term:

$$V(P_M \hat{v}_{M^\perp}) \leq V(P_M \hat{v}_{M^\perp}) \leq 1 - \tau.$$

$V(P_M \hat{v}_{M^\perp})$ is a semi-norm, hence there exist $\bar{\tau}$ such that

$$V(P_M \hat{v}_{M^\perp}) \leq \bar{\tau} \|x\|_p.$$

We deduce the second term is bounded by $\bar{\tau} \|\nabla \ell^{(n)}(\theta^*)\|_p$. Thus $V(P_M \hat{v}_{M^\perp})$ is bounded by

$$V(\hat{u}_I) \leq 1 - \tau + \bar{\tau} \left( \frac{\|\nabla \ell^{(n)}(\theta^*)\|_p}{\lambda} + \frac{\|R^{(n)}\|_p}{\lambda} \right).$$

We select $\lambda$ such that $\lambda > 2\bar{\tau} \|\nabla \ell^{(n)}(\theta^*)\|_p$, hence $\frac{\|\nabla \ell^{(n)}(\theta^*)\|_p}{\lambda} \leq \frac{\tau}{2\bar{\tau}}$ and

$$V(\hat{u}_I) < 1 - \tau + \frac{\tau}{2} + \frac{\|R^{(n)}\|_p}{\lambda}.$$ (1.13)

To show $\hat{u}_I \in \text{relint}(I)$, we must show $\frac{\|R^{(n)}\|_p}{\lambda} < \frac{\tau}{2\bar{\tau}}$.

**Lemma 4.3.** Suppose $\ell^{(n)}$ is twice continuously differentiable. If the assumptions of Proposition 4.3 hold and we select $\lambda$ such that

$$\lambda < \frac{m^2}{L} \frac{2\bar{\tau} \kappa(\ell_p) (2\kappa(h_A) + \frac{1}{\bar{\tau} \kappa(\ell_p)} )^2}{(2\kappa(h_A) + \frac{1}{\bar{\tau} \kappa(\ell_p)} )^2},$$

then $\frac{\|R^{(n)}\|_p}{\lambda} < \frac{\tau}{2\bar{\tau}}$.

**Proof.** The Taylor remainder term can be expressed as

$$R^{(n)} = \nabla \ell^{(n)}(\hat{\theta}) - \nabla \ell^{(n)}(\theta^*) - Q^{(n)}(\hat{\theta} - \theta^*).$$

According to Taylor’s theorem, there is a point $\hat{\theta}$ on the line segment between $\hat{\theta}$ and $\theta^*$ such that

$$\nabla \ell^{(n)}(\hat{\theta}) = \nabla \ell^{(n)}(\theta^*) + \nabla^2 \ell^{(n)}(\hat{\theta})(\hat{\theta} - \theta^*).$$
We add these two expressions to obtain
\[ R^{(n)} = (\nabla^2 \ell^{(n)}(\hat{\theta}) - Q^{(n)})(\hat{\theta} - \theta^*). \]
\( \nabla \ell^{(n)} \) is continuously differentiable, hence there exists \( L \) such that
\[ \|\nabla^2 \ell^{(n)}(\theta) - Q^{(n)}\|_2 \leq L \|\theta - \theta^*\|_2. \]
for all \( \theta \in M \) in a ball of radius \( r \) at \( \theta^* \). The assumptions of Proposition 4.1 hold, hence \( \|\hat{\theta} - \theta^*\| \leq r \) and
\[ \frac{\|R^{(n)}\|}{\lambda} \leq \frac{\kappa(\ell_p)}{m^2 L \tau^2 \kappa(\ell_p) \left(2\kappa(h_A) + \frac{\tau}{\tau} \kappa(\ell_p^*)\right)^2}. \]
If we select \( \lambda \) such that
\[ \lambda < \frac{m^2}{L} \frac{\tau}{2\tau \kappa(\ell_p) \left(2\kappa(h_A) + \frac{\tau}{\tau} \kappa(\ell_p^*)\right)^2}, \]
then we can verify \( \frac{\|R^{(n)}\|}{\lambda} \leq \frac{\tau}{\tau}. \]

We substitute this bound into (4.13) to obtain
\[ V(\hat{u}_I) < 1 - \frac{\tau}{2} + \frac{\tau}{2} < 1. \]
This means \( \hat{u}_I \in \text{relint}(I) \), and by Proposition 4.2 all solutions to the original problem (1.1) satisfy \( h_I(\theta) = 0 \). Thus \( \theta \) is also the unique solution to the original problem.

5 Examples

We use Theorem 3.4 to establish the consistency and model selection consistency of the generalized lasso and a group lasso penalized likelihood estimator in the high-dimensional setting. Our results are nonasymptotic, i.e. we obtain bounds in terms of sample size \( n \) and problem dimension \( p \) that hold with high probability.

5.1 The generalized lasso

Consider the linear model \( y = X^T \theta^* + \epsilon \), where \( X \in \mathbb{R}^{n \times p} \) is the design matrix, and \( \theta^* \in \mathbb{R}^p \) are unknown regression parameters. We assume the columns of \( X \) are normalized so \( \|x_i\|_2 \leq \sqrt{n} \). \( \epsilon \in \mathbb{R}^n \) is i.i.d., zero mean, sub-Gaussian noise with parameter \( \sigma^2 \).
We seek an estimate of $\theta^*$ with the generalized lasso:

$$
\minimize_{\theta \in \mathbb{R}^p} \frac{1}{2n} \| y - X\theta \|_2^2 + \lambda \| D\theta \|_1,
$$

(5.1)

where $D \in \mathbb{R}^{m \times p}$. The generalized lasso penalty is geometrically decomposable:

$$
\| D\theta \|_1 = h_{D\tau B_{\infty, A}}(\theta) + h_{D\tau B_{\infty, I}}(\theta).
$$

$h_{D\tau B_{\infty, A}}$ and $h_{D\tau B_{\infty, I}}$ are support functions of the sets

$$
D\tau B_{\infty, A} = \{ x \in \mathbb{R}^p \mid x = D\tau y, y_I = 0, \| y \|_{\infty} \leq 1 \}
$$

and

$$
D\tau B_{\infty, I} = \{ x \in \mathbb{R}^p \mid x = D\tau y, y_A = 0, \| y \|_{\infty} \leq 1 \}.
$$

The sample fisher information matrix is $Q(n) = \frac{1}{n} X^T X$. $Q(n)$ does not depend on $\theta$, hence the Lipschitz constant of $Q(n)$ is zero. The restricted strong convexity constant is

$$
m = \lambda_{\text{min}}(Q(n)) = \inf_x \{ x^T Q(n) x \mid \| x \|_2 = 1 \}.
$$

The columns of $X$ are normalized so $x_i^T \epsilon$ is sub-Gaussian and satisfies a Hoeffding-type inequality (cf. Proposition 5.10 in [30]):

$$
\Pr(\| \nabla \ell_n(\theta^*) \| > nt) = \Pr(\| x_i^T \epsilon \| > nt) \leq 2 \exp\left(-\frac{nt^2}{2\sigma^2}\right).
$$

By the union bound over $i = 1, \ldots, p$, we have

$$
\Pr(\| \nabla \ell_n(\theta^*) \|_{\infty} > nt) \leq 2 \exp\left(-\frac{nt^2}{2\sigma^2} + \log p\right).
$$

The model subspace is the set

$$
\text{span}(D\tau B_{\infty, I})^\perp = \mathcal{R}(D\tau I)^\perp = \mathcal{N}(D\tau),
$$

where $I$ indexes the rows of $D$. The compatibility constants $\kappa(\ell_1), \kappa(h_A)$ are

$$
\kappa(\ell_1) = \sup_x \{ \| x \|_1 \mid \| x \|_2 \leq 1, x \in \mathcal{N}(D\tau) \}
$$

and

$$
\kappa(h_A) = \sup_x \{ h_{D\tau B_{\infty, A}}(x) \mid \| x \|_2 \leq 1, x \in M \} \leq \| D_A \|_2 \sqrt{|A|}.
$$

If we select $\lambda > 2\sqrt{2\tau} \frac{\bar{\tau}}{\tau} \sqrt{\frac{\log p}{n}}$, then there exists $c$ such that

$$
\Pr\left( \lambda \geq \frac{2\tau}{\tau} \| \nabla \ell_n(\theta^*) \|_{\infty} \right)
$$

$$
= 1 - \Pr\left( \lambda < \frac{2\tau}{\tau} \| \nabla \ell_n(\theta^*) \|_{\infty} \right)
$$

$$
= 1 - \Pr\left( \| \nabla \ell_n(\theta^*) \|_{\infty} \leq \frac{\lambda \tau}{2\tau} \right)
$$

$$
\leq 1 - 2 \exp\left(-c\lambda^2 n\right).
$$
Thus the assumptions of Theorem 5.4 are satisfied with probability at least $1 - 2\exp(-c\lambda^2 n)$, and we deduce the generalized lasso is consistent and model selection consistent.

**Corollary 5.1.** Suppose $y = X\theta^* + \epsilon$, where $X \in \mathbb{R}^{n \times p}$ is the design matrix, $\theta^*$ are unknown regression parameters, and $\epsilon$ is i.i.d., zero mean, sub-Gaussian noise with parameter $\sigma^2$. If we select

$$\lambda > 2\sqrt{2\sigma \bar{\tau}} \sqrt{\frac{\log p}{n}},$$

then, with probability at least $1 - 2\exp\left(-c\lambda^2 n\right)$, the generalized lasso is unique, consistent, and model selection consistent, i.e. the optimal solution to (5.1) satisfies

1. $\|\hat{\theta} - \theta^*\|_2 \leq \frac{2}{m} \left(\|D_A\|_2 \sqrt{|A|} + \frac{\bar{\tau}}{\sqrt{p}}\kappa(\ell_1)\right)\lambda$ w.h.p.,

2. $D_i \hat{\theta} = 0$ for $i \in I$ w.h.p.

### 5.2 Learning exponential families with redundant representations

Suppose $X$ is a random vector, and let $\phi$ be a vector of sufficient statistics. The exponential family associated with these sufficient statistics is the set of distributions with the form

$$\Pr(x; \theta) = \exp(\theta^T \phi(x) - A(\theta)),$$

where $\theta$ are the natural parameters and $A$ is the log-partition function:

$$A(\theta) = \log \int_x \exp(\theta^T \phi(x))\mu(dx),$$

where $\mu$ is some reference measure. Assuming this integral is finite, $A$ ensures the distribution is normalized. The set of $\theta$ such that $A(\theta)$ is finite is called the domain of this exponential family:

$$\Omega = \{\theta \mid A(\theta) > -\infty\}.$$

If the domain is open, then this is a regular exponential family. In this case, $A$ is an analytic function so its derivatives exist and cannot grow too quickly:

$$\left|\frac{\partial^{|a|} A}{\partial \theta^a}\right| \leq c^{|a|} |a|!.$$

Thus the gradient and Hessian of $A$ are locally Lipschitz continuous, i.e. Lipschitz continuous in a ball of radius $r$ around $\theta^*$:

$$\|\nabla A(\theta_1) - \nabla A(\theta_2)\|_2 \leq L_1 \|\theta_1 - \theta_2\|_2, \quad \theta \in B_r(\theta^*)$$

$$\|\nabla^2 A(\theta_1) - \nabla^2 A(\theta_2)\|_2 \leq L_2 \|\theta_1 - \theta_2\|_2, \quad \theta \in B_r(\theta^*).$$
∇A(θ) and ∇²A(θ) are the (centered) moments of the sufficient statistic:

\[ \nabla A(\theta) = \mathbb{E}_\theta[\phi(X)], \]
\[ \nabla^2 A(\theta) = \mathbb{E}_\theta[\phi(X)^2] - \mathbb{E}_\theta[\phi(X)]^2 = \text{cov}_\theta[\phi(X)], \]

L₁ and L₂ can be expressed in terms of the operator norm of ∇²A and ∇³A:

\[ L_1 = \sup_{\theta} \{ \| \nabla^2 A(\theta) \|_2 \mid \theta \in B_r(\theta^*) \} \]
\[ L_2 = \sup_{\theta} \{ \| \nabla^3 A(\theta) \|_2 \mid \theta \in B_r(\theta^*) \}, \]

where \( \mathbb{E}_\theta \) is the expectation with respect to the distribution with parameters \( \theta \):

\[ \mathbb{E}_\theta[f(X)] = \int_x f(x) \exp(\theta^T \phi(x) - A(\theta)) \mu(dx), \]

Suppose we are given samples \( x^{(1)}, \ldots, x^{(n)} \) drawn i.i.d. from an exponential family with unknown parameters \( \theta^* \in \mathbb{R}^p \). We seek a group lasso penalized maximum likelihood estimate (MLE) of the unknown parameters:

\[ \min_{\theta \in \mathbb{R}^p} \ell_{\text{ML}}^{(n)}(\theta) + \lambda \| \theta \|_{2,1}, \text{ subject to } \theta \in S. \quad (5.2) \]

where \( \ell_{\text{ML}}^{(n)} \) is the (negative) log-likelihood function

\[ \ell_{\text{ML}}^{(n)}(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \log \Pr(x^{(i)}; \theta) = -\frac{1}{n} \sum_{i=1}^{n} \theta^T \phi(x^{(i)}) + A(\theta) \]

and \( \| \theta \|_{2,1} \) is the group lasso penalty

\[ \| \theta \|_{2,1} = \sum_{g \in G} \| \theta_g \|_2. \]

If the exponential family has a redundant representation, then each distribution in this family is associated with an affine subspace of the parameter space. The constraint \( \theta \in S \) makes the solution is unique even when exponential family has a redundant representation.

Many undirected graphical models can be naturally viewed as exponential families. Thus estimating the parameters of exponential families is equivalent to learning undirected graphical models, a problem of interest in many statistical, computational and mathematical fields. We refer to Section 2.4 in [32] for some examples of graphical models.

We can decompose group lasso penalty group-wise to obtain

\[ \| \theta \|_{2,1} = \sum_{g \in G} \| \theta_g \|_2 = h_{B(2,\infty),A}(\theta) + h_{B(2,\infty),I}(\theta), \]
where $h_{B(2,\infty),A}$ and $h_{B(2,\infty),I}$ are support functions of the sets

$$B(2,\infty),A = \{ \theta \in \mathbb{R}^p \mid \max_{g \in G} \| \theta_g \|_2 \leq 1 \text{ and } \| \theta_g \|_2 = 0, g \in I \}$$

$$B(2,\infty),I = \{ \theta \in \mathbb{R}^p \mid \max_{g \in G} \| \theta_g \|_2 \leq 1 \text{ and } \| \theta_g \|_2 = 0, g \in A \}.$$

We enforce the subspace constraint using the support function of $S^\perp$. Thus we can express (5.2) as

$$\min_{\theta \in \mathbb{R}^p} \ell(n)(\theta) + \lambda (h_{B(2,\infty),A}(\theta) + h_{B(2,\infty),I}(\theta) + h_{S^\perp}(\theta)).$$

The sample fisher information matrix is

$$Q^{(n)}(\theta^*) = \nabla^2 \left\{ -\frac{1}{n} \sum_{i=1}^n \phi(x^{(i)})^T \theta + A(\theta) \right\}(\theta^*) = \nabla^2 A(\theta).$$

$Q^{(n)}$ does not depend on the sample, hence if the population Fisher information matrix $Q = \nabla^2 A$ satisfies Assumptions 3.1 and 3.2 then $Q^{(n)}$ also satisfies these assumptions. If the model is identifiable over the feasible subspace $S$, then $Q$ satisfies Assumption 3.1 because $Q$ is strictly convex over $S$, hence strongly convex in a compact subset of $S$.

We select $\lambda$ such that

$$\lambda > \frac{2^\tau}{\tau^2} \max_{g \in \theta} \left\| \nabla \ell^{(n)}_{ML}(\theta^*) \right\|_2 \text{ w.h.p.}$$

First we show that if $\nabla A$ is Lipschitz continuous in $B_r(\theta^*)$, then the components of $\nabla \ell^{(n)}$ are sub-exponential random variables. Thus they satisfy a Bernstein-type inequality (cf. Proposition 5.16 in [30])

$$\Pr \left( \left| \nabla \ell^{(n)}_{ML}(\theta^*) \right| > t \right) \leq 2 \exp \left( -\frac{nt^2}{2L} \right), \quad |t| \leq r. \tag{5.3}$$

**Lemma 5.2.** Suppose $X$ is distributed according to a distribution in the exponential family and $\nabla A$ is Lipschitz continuous with constant $L$ in a ball of radius $r$ around $\theta^*$. Then for $|t| \leq r$,

$$\mathbb{E} \left[ \exp \left( t (\nabla \ell^{(n)}_{ML}(\theta^*))_j \right) \right] \leq \exp \left( \frac{Lt^2}{2} \right).$$

**Proof.** $\nabla \ell^{(n)}(\theta)$ can be expressed as

$$\nabla \ell^{(n)}_{ML}(\theta^*) = \frac{1}{n} \sum_{i=1}^n \phi(x^{(i)}) - \mathbb{E} \left[ \phi(X) \right].$$
hence we can express the m.g.f. of \((\nabla \ell^{(n)}_{\text{ML}}(\theta))_j\) as

\[
\mathbb{E} \left[ \exp \left( t \left( \phi_j(X) - \mathbb{E}[\phi_j(X)] \right) \right) \right] = \int_X \exp \left( t \left( \phi_j(X) - \mathbb{E}[\phi_j(X)] \right) \right) \phi(X)^T \theta^* - A(\theta^*) \, dx
\]

\[
= \exp \left( -A(\theta^*) - t \mathbb{E}[\phi_j(X)] \right) \int_X \exp \left( \left( t \phi_j(X) \right)^T \theta^* \right) \phi(X) \, dx
\]

\[
= \exp \left( -A(\theta^*) - t \mathbb{E}[\phi_j(X)] \right) \exp(\mathbb{E}[A(t \phi_j(X))])
\]

\[
= \exp \left( A(t \phi_j(X)) - A(\theta^*) - t \mathbb{E}[\phi_j(X)] \right)
\]

(5.4)

\[
\nabla A \text{ is Lipschitz continuous in ball of radius } r \text{ around } \theta^* \text{ so if } |t| \leq r, \text{ then }
\]

\[
A(t \phi_j(X) + \theta^*) - A(\theta^*) \leq t(A(\theta^*))_j + \frac{L}{2} t^2.
\]

We substitute this bound into (5.4) to obtain the desired bound on the m.g.f.:

\[
\mathbb{E} \left[ \exp \left( t \left( \phi_j(X) - \mathbb{E}[\phi_j(X)] \right) \right) \right] \leq \exp \left( \frac{L t^2}{2} \right).
\]

By the Bernstein-type inequality (5.3) and Lemma 5.2, we deduce

\[
\text{Pr} \left( \left\| (\nabla \ell^{(n)}_{\text{ML}}(\theta^*))_g \right\|_2 > t \right) \leq |g| \text{Pr} \left( \left\| (\nabla \ell^{(n)}(\theta^*))_g \right\|_\infty > t/|g| \right)
\]

\[
\leq 2|g| \exp \left( -\frac{nt^2}{2L |g|} \right)
\]

\[
\leq 2\left( \max_{g \in \mathcal{G}} |g| \right) \exp \left( -\frac{nt^2}{2L \left( \max_{g \in \mathcal{G}} |g| \right) \log |g|} \right).
\]

We take a union bound over the groups to obtain

\[
\text{Pr} \left( \max_{g \in \mathcal{G}} \left\| (\nabla \ell^{(n)}_{\text{ML}}(\theta^*))_g \right\|_2 > t \right)
\]

\[
\leq 2\left( \max_{g \in \mathcal{G}} |g| \right) \exp \left( -\frac{nt^2}{2L \left( \max_{g \in \mathcal{G}} |g| \right) \log |g|} \right).
\]

If we select

\[
\lambda > \frac{2\sqrt{2L \tau}}{\tau} \sqrt{\frac{\left( \max_{g \in \mathcal{G}} |g| \right) \log |\mathcal{G}|}{n}}, \tag{5.5}
\]

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then there exist $c$ such that
\[
\Pr \left( \lambda \geq \frac{2\tau}{\tau} \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 \right) \\
= 1 - \Pr \left( \lambda < \frac{2\tau}{\tau} \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 \right) \\
= 1 - \Pr \left( \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 > \frac{\lambda \tau}{2\tau} \right) \\
\leq 1 - 2 \left( \max_{g \in \mathcal{G}} |g| \right) \exp(-c\lambda^2 n).
\]

We also require
\[
\lambda < \min \left( \frac{\mu^2}{2\tau}, \frac{2\tau \kappa(\ell_p) (2\kappa(h_A) + \frac{2}{3} \kappa(\ell_p))}{\tau \kappa(\ell_p) + \frac{2}{3} \kappa(h_A)} \right),
\]
hence the sample size $n$ must be larger than
\[
\max \left\{ \frac{32L_1 L_2^2 \tau^2}{m^2 \tau^4} \kappa(\ell_{2,\infty})^2 \left( 2 \kappa(h_B(2,\infty), A) \right) + \frac{2}{3} \kappa(\ell_{2,1}) \right\} \left( \max_{g \in \mathcal{G}} |g| \right) \log |\mathcal{G}| \\
+ \frac{16L_1 L_2^2 \tau^2}{m^2 \tau^4} \left( 2 \kappa(h_B(2,\infty), A) \right) + \frac{2}{3} \kappa(h_A) \left( \max_{g \in \mathcal{G}} |g| \right) \log |\mathcal{G}|.
\]

(5.6)

The model subspace $M$ is the set $\{ \theta \mid \theta_g = 0, g \in I; \theta \in S \}$ and the compatibility constants $\kappa(\ell_{2,\infty}), \kappa(\ell_{2,1}), \kappa(h_A)$ are
\[
\kappa(\ell_{2,\infty}) = \sup \left\{ \max_{g \in \mathcal{G}} \| x_g \|_2 \mid \| x \|_2 \leq 1, x \in M \right\} \leq 1
\]
\[
\kappa(\ell_{2,1}) = \sup \left\{ \sum_{g \in \mathcal{G}} \| x_g \|_2 \mid \| x \|_2 \leq 1, x \in M \right\} \leq \sqrt{|A|}
\]
\[
\kappa(h_A) = \sup \left\{ \sum_{g \in A} \| x_g \|_2 \mid \| x \|_2 \leq 1, x \in M \right\} \leq \sqrt{|A|}.
\]

We substitute these expressions into (5.6) to deduce $n$ must be larger than
\[
\max \left\{ \frac{32L_1 L_2^2 \tau^2}{m^2 \tau^4} (2 + \frac{2}{3}) \left( \max_{g \in \mathcal{G}} |g| \right) |A|^2 \log |\mathcal{G}| \\
+ \frac{16L_1 L_2^2 \tau^2}{m^2 \tau^4} (2 + \frac{2}{3}) \left( \max_{g \in \mathcal{G}} |g| \right) |A| \log |\mathcal{G}|. \right\}
\]

(5.7)

If we select $\lambda$ according to (5.6) and $n$ satisfies (5.7), then the assumptions of Theorem 3.4 are satisfied w.h.p. and we use the theorem to deduce the penalized MLE is consistent and model selection consistent.

**Corollary 5.3.** Suppose we are given samples $x^{(1)}, \ldots, x^{(n)}$ drawn i.i.d. from an exponential family with unknown parameters $\theta^*$. If we select
\[
\lambda > \frac{2\sqrt{2L_1 \tau}}{\tau} \sqrt{\left( \max_{g \in \mathcal{G}} |g| \right) \log |\mathcal{G}|}
\]

then there exist $\xi$ such that
\[
\Pr \left( \lambda \geq \frac{2\tau}{\tau} \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 \right) \\
= 1 - \Pr \left( \lambda < \frac{2\tau}{\tau} \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 \right) \\
= 1 - \Pr \left( \max_{g \in \mathcal{G}} \| (\nabla \ell_{\text{ML}}^{(n)}(\theta^*) )_g \|_2 > \frac{\lambda \tau}{2\tau} \right) \\
\leq 1 - 2 \left( \max_{g \in \mathcal{G}} |g| \right) \exp(-c\lambda^2 n).
\]
and the sample size $n$ is larger than
\[
\max \left( \frac{32L^2 L^2}{m^4 + 1} \left( 2 + \frac{\tau}{\bar{\tau}} \right)^4 \left( \max_{g \in G} |g| \right) \log |G| \right),
\]
then, with probability at least $1 - 2 \left( \max_{g \in G} |g| \right) \exp(-c \lambda^2 n)$, the penalized maximum likelihood estimator is unique, consistent, and model selection consistent, i.e. the optimal solution to (5.2) satisfies
1. $\|\hat{\theta} - \theta^*\|_2 \leq \frac{2}{m} \left( 1 + \frac{\tau}{\bar{\tau}} \right) \sqrt{|A|} \lambda$,
2. $\hat{\theta}_g = 0, \; g \in \mathcal{I}$ and $\hat{\theta}_g \neq 0$ if $\|\theta^*_g\|_2 > \frac{1}{m} \left( 1 + \frac{\tau}{\bar{\tau}} \right) \sqrt{|A|} \lambda$.

6 On necessity of the irrepresentable condition

Although the irrepresentable condition seems cryptic and hard to verify, Zhao and Yu and Wainwright showed the irrepresentable condition is almost necessary for sign consistency of the lasso. We extend their results to M-estimators with geometrically decomposable penalties and discuss some consequence of our result. We state our results in terms of the sampling error
\[
\xi(n) = \nabla \ell(n)(\theta^*) + R(n) - Q(n) P_M(P_M Q(n) P_M)^\dagger P_M(\nabla \ell(n)(\theta^*) + R(n)).
\]

Lemma 6.1. Suppose $\hat{\theta}$ is a primal solution to (1.1) and $\hat{u}_A, \hat{u}_I, \hat{u}_{S^\perp}$ are dual solutions, i.e. $\hat{\theta}, \hat{u}_A, \hat{u}_I, \hat{u}_{S^\perp}$ satisfy
\[
\nabla \ell(\hat{\theta}) + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S^\perp}) = 0 \quad (6.1)
\]
\[
\hat{u}_I \in \partial h_I(\hat{\theta}), \; \hat{u}_A \in \partial h_A(\hat{\theta}), \; \hat{u}_{S^\perp} \in S^\perp. \quad (6.2)
\]

If $\hat{\theta} \in B_r(\theta^*) \cap M$, then we must have
\[
P_M \xi(n) = \lambda P_M (Q(n) P_M(P_M Q(n) P_M)^\dagger P_M \partial h_A(B_r(\theta^*) \cap M) \]
\[
- P_M \partial h_A(B_r(\theta^*) \cap M) + \lambda (I + S^\perp) \quad (6.3)
\]
\[
\text{Proof.} \text{ The optimal solution to (1.1) satisfies}
\]
\[
\nabla \ell(\hat{\theta}) + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S^\perp}) = 0 \quad (6.5)
\]
\[
\hat{u}_I \in \partial h_I(\hat{\theta}), \; \hat{u}_A \in \partial h_A(\hat{\theta}), \; \hat{u}_{S^\perp} \in S^\perp. \quad (6.6)
\]

We project onto $M$ to obtain
\[
P_M \nabla \ell(\hat{\theta}) + \lambda P_M \hat{u}_A = 0.
\]

We Taylor expand $\nabla \ell$ around $\theta^*$ to obtain
\[
P_M (\nabla \ell(n)(\theta^*) + Q(n)(\hat{\theta} - \theta^*) + R(n)) + \lambda P_M \hat{u}_A = 0,
\]
where
\[ R^{(n)} = \nabla \ell(\hat{\theta}) - \nabla \ell^{(n)}(\theta^*) - Q^{(n)}(\hat{\theta} - \theta^*) \]
is the Taylor remainder term. \( \hat{\theta} \in M \), so this is equivalent to
\[ P_M \nabla \ell^{(n)}(\theta^*) + P_M Q^{(n)} P_M (\hat{\theta} - \theta^*) + P_M R^{(n)} + \lambda P_M \hat{u}_A = 0. \]
We rearrange to obtain
\[ P_M Q^{(n)} P_M (\hat{\theta} - \theta^*) = -P_M (\nabla \ell^{(n)}(\theta^*) + \lambda \hat{u}_A + R^{(n)}), \]
\( P_M Q^{(n)} P_M \) is invertible over \( M \), hence we can solve for \( \hat{\theta} - \theta^* \) to obtain
\[ \hat{\theta} - \theta^* = -(P_M Q^{(n)} P_M)^\dagger P_M (\nabla \ell^{(n)}(\theta^*) + \lambda \hat{u}_A + R^{(n)}). \tag{6.7} \]
We can Taylor expand (6.6) around \( \theta^* \) to obtain
\[ \nabla \ell^{(n)}(\theta^*) + Q^{(n)}(\hat{\theta} - \theta^*) + R^{(n)} + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S \perp}) = 0. \]
We substitute (6.7) into this expression to obtain
\[ 0 = \nabla \ell^{(n)}(\theta^*) - Q^{(n)} (P_M Q^{(n)} P_M)^\dagger P_M (\nabla \ell^{(n)}(\theta^*) + \lambda \hat{u}_A + R^{(n)}) \]
\[ + R^{(n)} + \lambda (\hat{u}_A + \hat{u}_I + \hat{u}_{S \perp}). \]
This expression is equivalent to
\[ \xi^{(n)} = \lambda (Q^{(n)} (P_M Q^{(n)} P_M)^\dagger P_M \hat{u}_A - \hat{u}_A) + \lambda (\hat{u}_I + \hat{u}_{S \perp}), \]
where \( \xi^{(n)} \) is the sampling error
\[ \nabla \ell^{(n)}(\theta^*) + R^{(n)} - Q^{(n)} (P_M Q^{(n)} P_M)^\dagger P_M (\nabla \ell^{(n)}(\theta^*) + R^{(n)}). \]
We project onto \( M^\perp \) to obtain
\[ \xi^{(n)} = \lambda (Q^{(n)} (P_M Q^{(n)} P_M)^\dagger P_M \hat{u}_A - \hat{u}_A) + \lambda (\hat{u}_I + \hat{u}_{S \perp}). \]
We substitute (6.7) into this expression to obtained the desired result. \( \square \)

**Remark 6.1.** Lemma 6.1 states a necessary condition for \( \hat{\theta} \in B_r(\theta^*) \cap M \). To use this result to deduce the necessity of the irrepresentable condition, we must show if the irrepresentable condition is violated, then there is \( \delta > 0 \) such that
\[ \Pr(\hat{\theta} \in M^\perp \xi^{(n)} \in \text{right side of (6.4)}) \leq 1 - \delta. \]
Since (6.3) is necessary for \( \hat{\theta} \in B_r(\theta^*) \cap M \), we must have
\[ \Pr(\hat{\theta} \in B_r(\theta^*) \cap M) \leq 1 - \delta. \]
For example, consider the linear model $y = X^T \theta^* + \epsilon$, where $X \in \mathbb{R}^{n \times p}$ is the design matrix, $\theta^* \in \mathbb{R}^p$ are unknown regression parameters, and $\epsilon \in \mathbb{R}^n$ is i.i.d., zero mean Gaussian noise. We seek a generalized lasso estimate of $\theta^*$

$$\minimize_{\theta} \frac{1}{2n} \|y - X\theta\|_2^2 + \lambda \|D\theta\|_1,$$

(6.8)

where $D \in \mathbb{R}^{m \times p}$. Let $Q$ be the sample covariance. The sampling error is

$$\xi^{(n)} = P_{M^\perp} (X^T \epsilon/n - Q^{(n)} P_M (P_M Q^{(n)} P_M)^\dagger P_M X^T \epsilon/n),$$

Since $P_{M^\perp} \xi^{(n)}$ is a zero mean Gaussian, we must have for any convex set not containing a relative neighborhood of the origin

$$\Pr(P_{M^\perp} \xi^{(n)} \in C) \leq \frac{1}{2}.$$

The generalized lasso penalty is geometrically decomposable

$$\|D\theta\|_1 = h_{DTB_{\infty,A}}(\theta) + h_{DTB_{\infty,I}}(\theta).$$

where $DTB_{\infty,A}$ and $DTB_{\infty,I}$ are the sets

$$DTB_{\infty,A} = \{x \in \mathbb{R}^p \mid x = DTy, y_A = 0, \|y\|_\infty \leq 1\}$$

$$DTB_{\infty,I} = \{x \in \mathbb{R}^p \mid x = DTy, y_A = 0, \|y\|_\infty \leq 1\}.$$

For $r$ sufficiently small,

$$\partial h_{DTB_{\infty,A}} (B_r(\theta^* \cap M)) = \partial h_{DTB_{\infty,A}} (\theta^*).$$

(6.9) simplifies to

$$P_{M^\perp} \xi^{(n)} \in \lambda P_{M^\perp} (Q^{(n)} P_M (P_M Q^{(n)} P_M)^\dagger P_M \partial h_{DTB_{\infty,A}} (\theta^*)) + \lambda (I + S^\perp).$$

(6.10)

If the irrepresentable condition is violated, i.e.

$$V(P_{M^\perp} (Q^{(n)} P_M (P_M Q^{(n)} P_M)^\dagger P_M \partial h_{DTB_{\infty,A}} (\theta^*) - \partial h_{DTB_{\infty,A}} (\theta^*)) \geq 1,$$

then the right side of (6.10) is a convex set not containing a relative neighborhood of the origin. We deduce

$$\Pr(P_{M^\perp} \xi^{(n)} \in \text{right side of (6.10)}) \leq \frac{1}{2}.$$

7 Computational experiments

We show some consequences of Corollary 5.3 with experiments on two models from structure learning of networks that are motivated by bioinformatics applications. We select $\lambda$ to be proportional to $\max_{g \in G} |g| \log |G|/n$ and use a proximal Newton-type method [17] to solve the penalized likelihood maximization problem.
7.1 Graphical lasso

Suppose we are given samples drawn i.i.d. from a normal distribution. We seek a penalized MLE of the inverse covariance matrix:

$$\min_{\Theta} \text{tr} (\Sigma \Theta) - \log \det(\Theta) + \lambda \sum_{s,t \in G} \|\Theta_{st}\|_2,$$

(7.1)

where $\Sigma$ denotes the sample covariance matrix. We use a $\ell_1/\ell_2$ penalty to promote block sparse inverse covariance matrices. $\lambda$ is a parameter that trades-off goodness-of-fit and sparsity. This estimator is a group variant of the graphical lasso [8].

We create a group sparse Gaussian MRF with a random group structure (see Figure 1). The nonzero entries of the inverse covariance matrix are drawn i.i.d. (uniform) between 0 and 1. We draw samples and use the grouped graphical lasso to estimate the inverse covariance matrix. In these experiments, we varied the number of variables $p$ from 64 to 225 and the sample size from 100 to 1000.

We estimate the probability of correct model selection using the fraction of 100 trials when the grouped graphical lasso correctly estimates the true group structure. Figure 2 plots the frequency of correct group structure selection versus the sample size $n$ for four graphs with 64, 100, 144, and 225 nodes.

The fraction of correct model selection is small for small sample sizes but grows to one as the sample size increases. Naturally more samples are required to learn a larger model, hence the curves for larger graphs are to the right of curves for smaller graphs. If we plot these curves with the x-axis rescaled by $1/((\max_{g \in G} |g|) \log |G|)$, then the curves align. This is consistent with Corollary 5.3 that say the effective sample size scales logarithmically with $|G|$.

7.2 Learning mixed graphical models

The pairwise mixed graphical model was developed to model data that contain both categorical and continuous features [16] e.g., two features about a person are weight (continuous) and gender (categorical). The model is a natural pairwise extension of the Gaussian MRF and a pairwise discrete MRF:

$$\Pr(x, y; (\beta, \theta, \gamma)) \propto \exp\left(\sum_{s,t} -\frac{1}{2} \beta_{st} x_s x_t + \sum_{s,j} \theta_{sj} y_j x_s + \sum_{j,r} \gamma_{rj} (y_r, y_j)\right),$$

(7.2)

where $x_s$, $s = 1, \ldots, p$ and $y_j$, $j = 1, \ldots, q$’s are continuous and discrete variables and $\beta_{st}, \theta_{sj}, \gamma_{rj}$ are continuous-continuous, continuous-discrete, and discrete-discrete edge potentials. We seek penalized MLE and penalized PLE of the parameters $(\beta, \theta, \gamma)$:

$$\min_{(\beta, \theta, \gamma)} \ell((\beta, \theta, \gamma)) + \lambda \rho((\beta, \theta, \gamma)).$$

(7.3)

We use a $\ell_1/\ell_2$ penalty to promote group sparse estimates:

$$\rho((\beta, \theta, \gamma)) = \sum_{s,t} |\beta_{st}| + \sum_{s,j} \|\theta_{sj}\|_2 + \sum_{j,r} \|\gamma_{rj}\|_F.$$
Figure 1: Group structures used in the graphical lasso experiment. The labeled vertices represent groups of nodes in the graph. Two labeled vertices are connected if the nodes in the two groups are connected.
Figure 2: Fraction of correct model selection versus sample size \( n \) and rescaled sample size \( n/(\max_{g \in G} |g| \log |G|) \) with the grouped graphical lasso. Each point represents the fraction of 100 trials when the grouped graphical lasso correctly estimated the true group structure.

To make sure the model is identifiable, we enforce linear constraints on \( \gamma_{rj} \):

\[
\sum_{x_r, x_j} \gamma_{rj} (x_r, x_j) = 0, \quad j, r = 1, \ldots, q.
\]

We create a mixed model with 10 continuous variables and 10 binary variables (see Figure 3). We estimate the probability of correct model selection using the fraction of 100 trials when the estimator correctly estimates the true group structure. Figure 3 plots the fraction of correct group structure selection versus the sample size \( n \).

The fraction of correct model selection is small for small sample sizes but grows with the sample size. The fraction of correct model selection with the penalized PLE grows to one but the fraction with the penalized MLE stays around 0.9. This can be explained by the penalized MLE violating the irrepresentable condition. We refer to Section 3.1.1 in [25] for a similar example where the irrepresentable condition holds for a neighborhood-selection estimator but fails for the penalized MLE.

8 Conclusion

We proposed the notion of geometric decomposability and generalized the irrepresentable condition to geometrically decomposable penalties. This notion of decomposability builds on those by Negahban et al. [22] and Candés and Recht [7] and includes many common sparsity inducing penalties. This notion of decomposability also allows us to enforce linear constraints.

We developed a general framework for establishing the model selection consistency of M-estimators with geometrically decomposable penalties. Our main result gives deterministic conditions on the problem that guarantee consistency and model selection consistency. We combine our main result with probabilis-
The graph topology used in this experiment. The blue nodes are continuous variables and the red nodes are discrete variables. The actual experiment had 10 continuous and 10 discrete variables.

Figure 3: Fraction of correct model selection versus sample size $n$ of the penalized MLE and PLE on a mixed graphical model. Each point represents the fraction of 100 trials when the grouped graphical lasso correctly estimated the true group structure.

Acknowledgements

We thank Trevor Hastie for his insightful comments. J. Lee was supported by a National Defense Science and Engineering Graduate Fellowship (NDSEG) and an NSF Graduate Fellowship. Y. Sun was supported by the NIH, award number 1U01GM102098-01. J. Taylor was supported by the NSF, grant DMS 1208857, and by the AFOSR, grant 113039.

References

[1] F. Bach. Consistency of the group lasso and multiple kernel learning. J. Mach. Learn. Res., 9:1179–1225, 2008.

[2] P.J. Bickel and E. Levina. Covariance regularization by thresholding. Ann. Statist., 36(6):2577–2604, 2008.

[3] P.J. Bickel, Y. Ritov, and A.B. Tsybakov. Simultaneous analysis of lasso and dantzig selector. Ann. Statist., 37(4):1705–1732, 2009.

[4] P. Bühlmann and S. van de Geer. Statistics for high-dimensional data: Methods, theory and applications. 2011.

[5] F. Bunea. Honest variable selection in linear and logistic regression models via $\ell_1$ and $\ell_1 + \ell_2$ penalization. Electron. J. Stat., 2:1153–1194, 2008.
[6] F. Bunea, A. Tsybakov, and M. Wegkamp. Sparsity oracle inequalities for the lasso. *Electron. J. Stat.*, 1:169–194, 2007.

[7] E. Candès and B. Recht. Simple bounds for recovering low-complexity models. *Math. Prog. Ser. A*, pages 1–13, 2012.

[8] J. Friedman, T. Hastie, and R. Tibshirani. Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441, 2008.

[9] J. Guo, E. Levina, G. Michailidis, and J. Zhu. Asymptotic properties of the joint neighborhood selection method for estimating categorical markov networks. *arXiv preprint*.

[10] J. Huang and T. Zhang. The benefit of group sparsity. *Ann. Statis.*, 38(4):1978–2004, 2010.

[11] L. Jacob, G. Obozinski, and J. Vert. Group lasso with overlap and graph lasso. In *Int. Conf. Mach. Learn. (ICML)*, pages 433–440. ACM, 2009.

[12] A. Jalali, P. Ravikumar, V. Vasuki, S. Sanghavi, and UT ECE. On learning discrete graphical models using group-sparse regularization. In *Int. Conf. Artif. Intell. Stat. (AISTATS)*, 2011.

[13] S.M. Kakade, O. Shamir, K. Sridharan, and A. Tewari. Learning exponential families in high-dimensions: Strong convexity and sparsity. In *Int. Conf. Artif. Intell. Stat. (AISTATS)*, 2010.

[14] M. Kolar, L. Song, A. Ahmed, and E. Xing. Estimating time-varying networks. *Ann. Appl. Stat.*, 4(1):94–123, 2010.

[15] C. Lam and J. Fan. Sparsistency and rates of convergence in large covariance matrix estimation. *Ann. Statis.*, 37(6B):4254, 2009.

[16] J.D. Lee and T. Hastie. Learning mixed graphical models. *arXiv preprint arXiv:1205.5012*, 2012.

[17] J.D. Lee, Y. Sun, and M.A. Saunders. Proximal Newton-type methods for minimizing composite functions. In *Adv. Neural Inf. Process. Syst. (NIPS)*, pages 827–835, 2009.

[18] H. Liu and J. Zhang. Estimation consistency of the group lasso and its applications. *Journal of Machine Learning Research–Proceedings Track*, 5:376–383, 2009.

[19] P.L. Loh and M.J. Wainwright. Structure estimation for discrete graphical models: Generalized covariance matrices and their inverses. *arXiv:1212.0478*, 2012.

[20] N. Meinshausen and P. Bühlmann. High-dimensional graphs and variable selection with the lasso. *Ann. Statis.*, 34(3):1436–1462, 2006.
[21] Y. Nardi and A. Rinaldo. On the asymptotic properties of the group lasso estimator for linear models. *Electron. J. Stat.*, 2:605–633, 2008.

[22] S.N. Negahban, P. Ravikumar, M.J. Wainwright, and B. Yu. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. *Statist. Sci.*, 27(4):538–557, 2012.

[23] G. Obozinski, M.J. Wainwright, and M.I. Jordan. Support union recovery in high-dimensional multivariate regression. *Ann. Stat.*, 39(1):1–47, 2011.

[24] P. Ravikumar, M.J. Wainwright, and J.D. Lafferty. High-dimensional ising model selection using $\ell_1$-regularized logistic regression. *Ann. Stat.*, 38(3):1287–1319, 2010.

[25] P. Ravikumar, M.J. Wainwright, G. Raskutti, and B. Yu. High-dimensional covariance estimation by minimizing $\ell_1$-penalized log-determinant divergence. *Electron. J. Stat.*, 5:935–980, 2011.

[26] A.J. Rothman, P.J. Bickel, E. Levina, and J. Zhu. Sparse permutation invariant covariance estimation. *Electron. J. Stat.*, 2:494–515, 2008.

[27] Y. She. Sparse regression with exact clustering. *Electron. J. Stat.*, 4:1055–1096, 2010.

[28] R.J. Tibshirani and J.E. Taylor. The solution path of the generalized lasso. *Ann. Stat.*, 39(3):1335–1371, 2011.

[29] S. Vaiter, G. Peyré, C. Dossal, and J. Fadili. Robust sparse analysis regularization. *IEEE Trans. Inform. Theory*, 59(4):2001–2016, 2013.

[30] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.

[31] M.J. Wainwright. Sharp thresholds for high-dimensional and noisy sparsity recovery using $\ell_1$-constrained quadratic programming (lasso). *IEEE Trans. Inform. Theory*, 55(5):2183–2202, 2009.

[32] M.J. Wainwright and M.I. Jordan. Graphical models, exponential families, and variational inference. *Found. Trends Mach. Learn.*, 1(1-2):1–305, 2008.

[33] E. Yang, P. Ravikumar, G.I. Allen, and Z. Liu. On graphical models via univariate exponential family distributions. *arXiv:1301.4183*, 2013.

[34] M. Yuan and Y. Lin. Model selection and estimation in regression with grouped variables. 68(1):49–67, 2006.

[35] P. Zhao and B. Yu. On model selection consistency of lasso. *J. Mach. Learn. Res.*, 7:2541–2563, 2006.