A polar complex for locally free sheaves

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Abstract

We construct the so-called polar complex for an arbitrary locally free sheaf on a smooth variety over a field of characteristic zero. This complex is built from logarithmic forms on all irreducible subvarieties with values in a locally free sheaf. We prove that cohomology groups of the polar complex are canonically isomorphic to the cohomology groups of the locally free sheaf. Relations of the polar complex with Rost’s cycle modules, algebraic cycles, Cousin complex, and adelic complex are discussed.

1 Introduction

In this paper we are going to study certain complexes where the differential is constructed from a residue map acting on differential forms with logarithmic singularities. To explain the motivations, let us start from the following classical fact: a divisor \( \sum a_i x_i \) with complex coefficients \( a_i \) on a compact Riemann surface is the residue of a logarithmic 1-form if and only if \( \sum a_i = 0 \). A higher-dimensional version of this concerns a \( p \)-dimensional subvariety \( Z \) in a smooth variety \( X \) together with a holomorphic differential form of top degree (that is, a \( p \)-form) \( \alpha \in H^0(Z, \omega_Z) \). The question is then whether there exists a \( (p+1) \)-dimensional subvariety \( Y \) in \( X \) containing \( Z \) and a logarithmic top form \( \beta \in H^0(Y, \omega_Y(Z)) \) such that \( \text{res} \beta = \alpha \). More generally, one can consider formal linear combinations of subvarieties together with logarithmic top forms on them.

When \( X \) is a compact Riemann surface, the sum \( \sum_i a_i \) is an element in the group \( \mathbb{C} \) and one can canonically identify \( \mathbb{C} \) with \( H^1(X, \omega_X) \) by taking integrals of \((1,1)\)-forms over \( X \). In the higher-dimensional case it is shown in [17] that the obstruction for the existence of \( (Y, \beta) \) as above is an element in the group \( H^{d-p}(X, \omega_X) \), where \( d = \dim X \). The aim of this paper is to prove an analogous result for smooth varieties over a field of characteristic zero and forms with coefficients in vector bundles.

Given a smooth variety \( X \) and a locally free sheaf \( \mathcal{F} \) on \( X \), one constructs a (homological type) complex \( \text{Pol} \bullet(X, \mathcal{F}) \), called a polar complex, whose \( p \)-chains are finite
formal sums of pairs \((Z, \alpha)\), where \(Z\) is a \(p\)-dimensional irreducible subvariety in \(X\) and \(\alpha\) is a logarithmic top form on \(Z\) with coefficients in \(\mathcal{F}|_Z\) (see Definitions 3.1 and 3.3). The boundary map in the polar complex is defined by the residue morphism on logarithmic forms. The main result of the paper (Theorem 3.4) is that, for \(\dim X = d\), there is a canonical isomorphism

\[
H^{d-p}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \cong H_p(\text{Pol}_\bullet(X, \mathcal{F})) .
\]  

(1)

This theorem answers the above question about obstructions to being one’s residue. On the other hand it gives a new interpretation of cohomology groups of locally free sheaves in terms of logarithmic forms on subvarieties. The isomorphism (1) has, in particular, the following interpretation. The homology groups of the polar complex, \(H_p(\text{Pol}_\bullet(X, \mathcal{F}))\), admit a canonical pairing with \(H^p(X, \mathcal{F}^\vee)\), the cohomology of a dual to \(\mathcal{F}\). This is described as follows. For a polar \(p\)-cycle \((Z, \alpha)\), \(\dim Z = p\), and an element \(u \in H^p(X, \mathcal{F}^\vee)\), we can define \(u' \in H^p(Z, \mathcal{F}^\vee|_Z)\) as a restriction of \(u\) to \(Z\) and form a natural pairing \(\langle \alpha, u' \rangle\), because \(\alpha\) is an element of \(H^0(Z, \omega_Z \otimes_{\mathcal{O}_Z} \mathcal{F}|_Z)\) (we assume here \(Z\) being smooth for simplicity). Taking a sum over all \(p\)-dimensional subvarieties \(Z\) in \(X\), we obtain a pairing \(H_p(\text{Pol}_\bullet(X, \mathcal{F})) \otimes H^p(X, \mathcal{F}^\vee) \to \mathbb{C}\). The isomorphism (1) amounts to saying that this pairing is non-degenerate (by Serre’s duality).

Our proof of the relation (1) is based on an analogy between cohomology groups of locally free sheaves on algebraic varieties and singular cohomology of local systems on manifolds, see Subsection 3.2. We also use Quillen’s trick in algebraic \(K\)-theory as an algebraic analogue of a tubular neighborhood in topology. Note that methods of the present paper are completely different from those in [17], where the isomorphism (1) is established for \(\mathcal{F} = \mathcal{O}_X\) and \(X\) projective.

One can also find relations of the polar complex with other concepts. In the first place, the polar complex can be regarded as a Gersten type complex\(^1\) associated with a certain cycle module over \(X\). Recall that cycle modules were introduced by M. Rost in [22]. We prove in Theorem 4.4 that the polar complex is locally exact on \(X\) except for its left most term. It is worth noticing that an analogue of Theorem 4.4 is not true for an arbitrary cycle module over a non-trivial \(X\) (different from a point). Nevertheless, the polar complex for a locally free sheaf \(\mathcal{F}\) on \(X\) corresponds to a cycle module over \(X\), which, for a non-trivial \(\mathcal{F}\) (different from \(\mathcal{O}_X\)) cannot be defined over a point. Thus, the polar complex provides a new example of a (non-trivial direct summand in a) cycle module over a non-trivial base such that, nevertheless, the corresponding Gersten complex is locally exact except for its left most term.

It might be also worth mentioning that, the polar complex provides a canonical construction of classes of algebraic \(p\)-cycles in groups \(H^{d-p}(X, \mathcal{O}_X^{d-p})\). On the other hand, the polar complex \(\text{Pol}_\bullet(X, \mathcal{F})\) is canonically a subcomplex in the Cousin complex of \(\omega_X \otimes_{\mathcal{O}_X} \mathcal{F}\). The isomorphism (1) implies that the polar complex is actually quasiisomorphic to the Cousin complex. One can say that the polar complex is a first order pole part of the Cousin complex, providing thus a much smaller, but still quasiisomorphic subcomplex.

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\(^1\)By this we mean a complex whose \(p\)-th term is naturally written as a direct sum over \(p\)-dimensional irreducible subvarieties.
In addition, the polar complex on a smooth complex projective variety fills in a vacant site in a chain of quasiisomorphisms connecting the rational adelic complex (see [19, 2, 14]) to the Dolbeault complex, such that each complex in this chain has an explicit geometric description.

Let us mention one more motivation for the construction of polar complexes with coefficients in locally free sheaves. It comes from a study of a holomorphic analogue of the linking number. Such an analogue is supposed to be a certain function of positions of, say, two complex curves in a three-dimensional complex manifold. In the paper [1], Atiyah considered the case of two lines in a complex projective space $\mathbb{P}^3$. He defined a function (or, rather, a section of a line bundle) on the configuration space of such pairs and exploited it in a construction of Green’s function for a certain Laplace operator. This function depends holomorphically on the positions of the lines and develops a pole on the discriminant of intersecting lines, while the topological linking number is a locally constant function on the configuration space; more on the analogy between topological and holomorphic objects see in Subsection 3.2. Atiyah suggested to regard this function as a holomorphic analogue of the classical Gauss linking number in topology. As the latter can be nicely described in terms of singular chains, one may ask what should be a holomorphic analogue of singular chains appropriate to the context of the above holomorphic linking. The polar complex can be also thought of as an answer to that question. Indeed, one can notice that in the construction of [1] each line $L \subset \mathbb{P}^3$ is implicitly endowed with a section $\alpha \in H^0(L, \omega_L \otimes \mathcal{O}_{\mathbb{P}^3}(2)|_L)$, the latter space being (non-canonically) isomorphic to $\mathbb{C}$. Thus, $(L, \alpha)$ is a polar 1-cycle in $\mathbb{P}^3$ with coefficients in $\mathcal{O}_{\mathbb{P}^3}(2)$. The holomorphic linking number for polar cycles in arbitrary smooth varieties with coefficients in locally free sheaves can be defined in the same way as in [16, 18, 7].

The paper is organized as follows. In Section 2 we recollect general properties of logarithmic forms and fix some notations. In particular, we prove here a version of the Grauert–Riemenschneider theorem (Proposition 2.6). Note that the proof is purely algebraic. Section 3 contains the main definitions and describes the main result (Definitions 3.1, 3.3 and Theorem 3.4). We also give some topological motivations in Subsection 3.2. The proof of the main result in Section 1 is done in several steps. Namely, in Subsection 4.1 we define for each locally free sheaf a certain subsheaf of abelian groups, which we call a polar sheaf (Definition 4.2). The use of polar sheaves allows us to split the main theorem into two assertions about polar sheaves (Theorem 4.4 and Theorem 4.5). The second assertion says that cohomology groups of a locally free sheaf coincide with cohomology groups of its polar subsheaf. Subsections 4.2 and 4.3 contain simple auxiliary lemmas. Quillen’s trick is applied to prove Theorem 4.4 for projective varieties in Subsection 4.4. Theorem 4.5 for projective varieties is proved in Subsection 4.5. Finally, reduction of the general case to the projective one is done in Subsection 4.6. In Section 5 the relations of the polar complex with Rost’s cycle modules, algebraic cycles, Cousin complex, and adelic complex are discussed.

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Throughout the paper we fix a ground field $k$ of characteristic zero. The reader may additionally assume that $k$ is algebraically closed; neither our results nor the proofs will have essential changes in this case.

All varieties and morphisms between them are considered over $k$. For a smooth variety $V$, denote by $\Omega^p_V$ the sheaf of differential $p$-forms. We also set $\omega_V := \Omega^d_V$, where $d$ is the dimension of $V$. For an irreducible variety $V$ (not necessarily smooth), denote by $k(V)$ the field of rational functions on $V$. Let $\Omega^p_{k(V)}$ be the space of rational $p$-forms on $V$ and, correspondingly, $\omega_{k(V)} := \Omega^d_{k(V)}$. Among rational forms, the logarithmic ones (e.g. [11, Sect. 3.5], [24, Sect. 8.2]) will be of most importance for us. Consider a smooth irreducible variety $V$ and a smooth reduced irreducible divisor $W$ in $V$. A rational form $\alpha \in \Omega^p_{k(V)}$ will be called locally logarithmic along $W$ if both $\alpha$ and $d\alpha$ have at most first order poles along $W$. Equivalently, at the generic point of $W$ the form $\alpha$ can be written (not uniquely) as $\alpha = \frac{dt}{t} \wedge \beta$, where $t$ is an equation of $W$ at its generic point, while $\beta$ is a rational $(p - 1)$-form on $V$ which is regular at the generic point of $W$. If now $\alpha_1$ and $\alpha_2$ are forms locally logarithmic along $W$, then the form $\alpha_1 \wedge \alpha_2$ is also locally logarithmic along $W$. Moreover, for any non-zero rational function $f$ on $V$, the rational 1-form $d\log f = df/f$ is locally logarithmic along any $W$.

In the above notations, consider the sheaf

$$\Omega^p_V(\log W) := \text{Ker} \left( \Omega^p_V(W) \to j_* \Omega^p_W(W) \right),$$

where $j : W \hookrightarrow V$. The sheaf $\Omega^p_V(\log W)$ is a locally free sheaf on $V$. A form $\alpha \in H^0(V \smallsetminus W, \Omega^p_V)$ is locally logarithmic along $W$ if and only if $\alpha \in H^0(V, \Omega^p_V(\log W))$. It follows from the exact sequence

$$0 \to j^* \mathcal{O}_V(-W) \to j^* \Omega^1_V \to \Omega^1_W \to 0$$

and a linear algebra concerning wedge powers that there is a well defined morphism of sheaves $j^* \Omega^p_V(\log W) \to \Omega_V^{p-1}$. This gives a residue homomorphism

$$\text{res}_{VW} : H^0(V, \Omega^p_V(\log W)) \to H^0(W, \Omega_W^{p-1}).$$
Locally along $W$ we have $\text{res}_{VW}(\frac{dt}{t} \land \beta) = \beta|_W$.

We call $W \subset V$ a simple normal crossing divisor if $W = \cup W_i$, where each $W_i$ is a smooth reduced irreducible divisor in $V$ and the divisors $W_i$ meet transversely at each point of $V$. Explicitly, for any point $x \in V$, local equations $t_1, \ldots, t_r$ of $W_1, \ldots, W_r$ have linearly independent differentials at $x$, where $W_1, \ldots, W_r$ are components of $W$ that contain $x$. In this case, let $\Omega^p_V(\log W)$ be the sheaf of rational differential forms that are locally logarithmic along all divisors $W_i$ and regular elsewhere on $V$. In particular, we have $\Omega^1_V(\log W) = \omega_V$, where $d$ is the dimension of $V$. Zariski locally at any point $x \in V$, a form $\alpha \in H^0(V, \Omega^p_V(\log W))$ can be written (not uniquely) as

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_l \leq r} \frac{dt_{i_1}}{t_{i_1}} \land \cdots \land \frac{dt_{i_l}}{t_{i_l}} \land \beta_{i_1, \ldots, i_l},$$

where $l$ runs over $0, \ldots, p$, while $t_1, \ldots, t_r$ are local equations of $W_1, \ldots, W_r$ at $x$, and $\beta_{i_1, \ldots, i_l}$ is a regular differential form at $x$ of degree $p - l$. The sheaf $\Omega^p_V(\log W)$ is a locally free sheaf. In what follows, by a logarithmic $p$-form on $V$ we will understand an element of $H^0(V, \Omega^p_V(\log W))$ for some simple normal crossing divisor $W \subset V$.

The properties of logarithmic forms with a smooth divisor of poles described above extend to a more general case of simple normal crossing divisors. Namely, the product of differential forms defines a morphism of sheaves

$$\Omega^p_V(\log W) \otimes_{\mathcal{O}_X} \Omega^q_V(\log W) \to \Omega^{p+q}_V(\log W).$$

Moreover, for any non-zero rational function $f$ on $V$ such that $f$ and $f^{-1}$ are regular on the complement $V \setminus W$, the rational differential form $d\log f = df/f$ belongs to $H^0(V, \Omega^1_V(\log W))$. In the above notations, for any $i$, denote by $W - W_i$ the union of all irreducible components in $W$ except $W_i$. For any $\alpha \in H^0(V, \Omega^p_W(\log W))$, there is a residue

$$\text{res}_{VW_i}(\alpha) \in H^0(W_i \setminus (W_i \cap (W - W_i)), \Omega^{p-1}_{W_i}).$$

Actually, $W'_i := W_i \cap (W - W_i)$ is a simple normal crossing divisor in $W_i$ and the rational form $\text{res}_{VW_i}(\alpha)$ belongs to $H^0(W'_i, \Omega^{p-1}_{W'_i}(\log W'_i))$. Below we recollect some further properties of logarithmic forms that will be needed later. First we show that the logarithmicity of rational differential forms is preserved by pull-backs.

**Lemma 2.1.** Let $f : V' \to V$ be a dominant morphism between smooth varieties and suppose $W \subset V$, $W' \subset V'$ are simple normal crossing divisors such that $f^{-1}(W) \subset W'$. Then for any differential form $\alpha \in H^0(V, \Omega^p_V(\log W))$, the rational differential form $f^*\alpha$ belongs to $H^0(V', \Omega^p_{V'}(\log W'))$.

**Proof.** There is a covering of $V$ by open subsets $U$ such that the restriction $\alpha|_U$ has the shape as in equation (2). Then $f^*\alpha$ is locally logarithmic along any smooth irreducible divisor in $U' := f^{-1}(U)$, because $f^*\alpha|_{U'}$ is a sum of products of $d\log$’s of rational functions and regular differential forms. Since $f^*\alpha$ has poles only along components of a simple normal crossing divisor $W'$, this completes the proof. $\square$
Let $K$ be a finitely generated field over $k$, that is, $K = k(V)$ for some irreducible variety $V$ over $k$, and consider the space of rational $p$-forms $\Omega^p_K$. Let now $K \subset K'$ be a finite extension of fields. Then, the trace map $\text{Tr}_{K'/K} : K' \to K$ induces a map on rational differential forms

$$\text{Tr}_{K'/K} : \Omega^p_{K'} = \Omega^p_K \otimes_K K' \to \Omega^p_K, \quad p \geq 0.$$ 

The trace map commutes with the de Rham differential (but does not respect multiplication of forms) and there is a projection formula involving the trace and the pull-back maps on rational differential forms. If $f : V' \to V$ is a generically finite dominant morphism of irreducible varieties then we denote the trace map $\text{Tr}_{k(V')/k(V)}$ on rational differential forms by $f_*$. When $f$ is finite and unramified, and the varieties $V$ and $V'$ are smooth, the trace of a regular form $\alpha'$ on $V'$ can be explicitly written at a point $x \in V$ as follows:

$$(f_*\alpha')(x) = \sum_{x' \in f^{-1}(x)} ((df_{x'})^\vee)^{-1}(\alpha'(x')).$$

For an arbitrary generically finite dominant morphism $f$ between smooth $V$ and $V'$, this formula determines the trace of a rational differential form on a non-empty open subset in $V$.

If the morphism $f$ is generically finite and proper one can show that the trace $f_*\alpha'$ of a regular differential form $\alpha'$ on $V'$ is regular on $V$. To prove the latter\textsuperscript{2} let us consider an open subset $U \subset V$ such that $f$ is finite over $U$. One can assume that the complement $V \setminus U$ is of codimension at least two in $V$. Therefore, it suffices to show that $f_*\alpha'$ is regular in $U$. Without loss of generality, we can assume further that $f : f^{-1}(U) \to U$ is a Galois covering. Consider now the form $f^*f_*\alpha'$ on $f^{-1}(U)$. This form is equal to a sum of Galois conjugates of $\alpha'$. (This is obviously so over an open subset of $U$, where $f$ is unramified; the equality for the whole of $U$ then follows.) The regularity of $\alpha'$ implies now the regularity of $f^*f_*\alpha'$. The desired assertion follows by noticing that $f^*f_*\alpha'$ is regular if and only if $f_*\alpha'$ is regular (on $U$ and, hence, on $V$).

The next result says that also the logarithmicity of rational differential forms is preserved by the trace map.

**Lemma 2.2.** Let $f : V' \to V$ be a proper generically finite surjective morphism between smooth varieties and suppose $W \subset V$, $W' \subset V'$ are simple normal crossing divisors such that $W' \subset f^{-1}(W)$. Then for any differential form $\alpha' \in H^0(V', \Omega^p_{V'}(\log W'))$, the rational differential form $f_*\alpha'$ belongs to $H^0(V, \Omega^p_V(\log W))$.

**Proof.** The restriction of $f$ defines a proper morphism

$$V' \setminus f^{-1}(W) \to V \setminus W.$$ 

Since $W' \subset f^{-1}(W)$ and $f$ is proper, the form $f_*\alpha'$ is regular on $V \setminus W$. Let $W_i \subset W \subset V$ be an irreducible component of $W$ and choose an equation $t_i$ of $W_i$ at its generic point. Then the forms

$$t_i \cdot f_*\alpha' = f_*((f^*t_i) \cdot \alpha'),$$

\textsuperscript{2}See [10] for an analytic approach.
are regular. Therefore the form $f_\ast \alpha'$ is locally logarithmic along $W_i$, and this completes the proof. \hfill \Box

Let us consider a dominant morphism between smooth varieties, $f : V' \to V$, and suppose $W$ is a smooth irreducible divisor in $V$ such that $W' := f^{-1}(W)$ is a simple normal crossing divisor in $V'$. Let $\{W'_i\}$ be the set of those irreducible components of $W'$ which map dominantly to $W$. Denote by $e_i$ the ramification index of $f$ along $W'_i$ and denote by $f_i : W'_i \to W$ the restriction of $f$. Then for any differential form $\alpha \in H^0(V, \Omega^p_V(\log W))$, a local calculation shows that

$$\text{res}_{W'}(f^* \alpha) = e_i \cdot f_i^*(\text{res}_W(\alpha)). \quad (3)$$

Suppose in addition that $f$ is proper and generically finite. Then for any differential form $\alpha' \in H^0(V', \Omega^p_{V'}(\log W'))$, we have

$$\text{res}_W(f_\ast \alpha') = \sum_i f_i^*(\text{res}_{W'}(\alpha')). \quad (4)$$

The next lemma follows directly from the Bertini theorem (note that the Bertini theorem is valid over an arbitrary infinite field as any dense open subset in the projective space has a rational point):

**Lemma 2.3.** Let $V$ be a smooth quasi-projective variety, $\varphi : V \to \mathbb{P}^N$ a regular morphism, $W \subset V$ a simple normal crossing divisor. Then for a general hyperplane $H \subset \mathbb{P}^N$, the divisor $\varphi^{-1}(H)$ does not contain any component of $W$, and $W \cup \varphi^{-1}(H)$ is a simple normal crossing divisor.

We will also need the following form of the Hironaka theorem (see \[13\]):

**Proposition 2.4.**

(i) For an irreducible variety $V$ and a subvariety $Z \subset V$, there exists a proper birational morphism $\pi : \tilde{V} \to V$ such that $\tilde{V}$ is smooth, $\pi^{-1}(Z)$ is a simple normal crossing divisor in $\tilde{V}$, and the morphism $\pi$ is an isomorphism over the complement $V \setminus (V_{\text{sing}} \cup Z)$, where $V_{\text{sing}}$ is the singular locus of $V$.

(ii) Let $\varphi : V_1 \dashrightarrow V_2$ be a birational map between complete irreducible varieties. Then there is a composition of blow-ups at smooth centers $f : V'_1 \to V_1$ and a regular morphism $\varphi' : V'_1 \to V_2$ such that $\varphi \circ f = \varphi'$.

**Lemma 2.5.** Let $f : V \to X$ and $f' : V' \to X$ be proper morphisms from irreducible varieties to a variety $X$. Let $g : V' \dashrightarrow V$ be a rational map such that $f \circ g = f'$. Then, there exist a smooth irreducible variety $V''$ and proper morphisms $h : V'' \to V$, $h' : V'' \to V'$ such that $h'$ is birational and $g \circ h' = h$.

\[3\] By $\varphi^{-1}(H)$ we denote the set-theoretical preimage of $H$; in particular, $\varphi^{-1}(H)$ is a reduced divisor.

\[4\] By a birational morphism we mean a regular morphism between irreducible varieties that has a rational inverse.
In other words, for $V$ and $V'$ proper over $X$, a rational map $g : V' \to V$ over $X$ can be turned into a proper regular morphism by replacing $V'$ with a suitable birational smooth variety $V''$, which is summarized in the following commutative diagram:

\[
\begin{array}{ccc}
  V'' & \xleftarrow{h'} & V' \\
  \downarrow{h} & & \downarrow{g} \\
  V & & V \\
  \downarrow{f} & \xrightarrow{f'} & X \\
\end{array}
\]

**Proof.** Let $U \subset V'$ be a non-empty open subset such that $g_U := g|_U$ is regular on $U$. Consider the embedding of the graph of $g_U$ followed by an open embedding as follows:

\[
\Gamma_{g_U} \subset U \times_X V \subset V' \times_X V.
\]

Take the closure $\bar{\Gamma}_{g_U}$ of $\Gamma_{g_U}$ in $V' \times_X V$. Note that $\bar{\Gamma}_{g_U}$ contains $\Gamma_{g_U} \cong U$ as an open dense subset and the projections to the two factors in $V' \times_X V$ define proper morphisms from $\bar{\Gamma}_{g_U}$ to $V'$ and $V$. As a smooth $V''$ we can now pick up a resolution of singularities of $\bar{\Gamma}_{g_U}$. \qed

The next result can be considered as a logarithmic version of the Grauert–Riemenschneider theorem [9] (in other words, of the assertion that smooth varieties have rational singularities). Since the authors could not find any reference for this case, the proof is given. For a morphism $\pi : \tilde{X} \to X$ between smooth irreducible quasi-projective varieties of equal dimensions and a locally free sheaf $G$ on $X$, we set:

\[
\pi^!G := \omega_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \pi^*(\omega_X^{-1} \otimes_{\mathcal{O}_X} G).
\]

(5)

We will use that $\pi^!$ extends to a functor between derived categories of coherent sheaves, which is right adjoint to the functor $R\pi_*$ by the Grothendieck duality, provided that $\pi$ is proper (see [12]).

**Proposition 2.6.** Let $\pi : \tilde{X} \to X$ be a proper birational morphism between smooth irreducible quasi-projective varieties. Let $D \subset X$ and $\tilde{D} := \pi^{-1}(D)$ be simple normal crossing divisors. Then, we have that

\[
R\pi_*\omega_{\tilde{X}}(\tilde{D}) = \omega_X(D).
\]

**Proof.** The Hironaka theorem (see Proposition 2.4) implies the existence of a smooth projective closure of all the data in the proposition. The statement of the proposition is local along $X$, while the fibers of the proper morphism $\pi$ over $X$ are not changing under taking a projective closure. Therefore, we may assume that $\tilde{X}$ and $X$ are projective.

By Proposition 2.4 (ii) applied to the birational map $\varphi := \pi^{-1}$, there is a smooth variety $Y$ together with regular birational proper morphisms $f : Y \to X$ and $\tilde{f} : Y \to \tilde{X}$ such that $f = \pi \circ \tilde{f}$ and $f$ is a composition of blow-ups at smooth centers. We set
$C := f^{-1}(D)$, $C' := \tilde{f}^*(\tilde{D})$. By definition of $\tilde{f}^!$, we have that $\tilde{f}^! \omega_{\tilde{X}}(\tilde{D}) = \omega_Y(C')$. Now, since $\tilde{f}^!$ is right-adjoint to $R\tilde{f}_*$, there is a trace morphism

$$R\tilde{f}_*\omega_Y(C') \to \omega_{\tilde{X}}(\tilde{D}).$$

Equation (2) implies the existence of the pull-back map on differential forms, $\omega_{\tilde{X}}(\tilde{D}) \to f_*\omega_Y(C)$, because $\tilde{D}$ is a simple normal crossing divisor. The composition of this pull-back map with the natural morphisms $f_*\omega_Y(C) \to f_*\omega_Y(C')$, for $C \subseteq C'$, and $f_*\omega_Y(C') \to R\tilde{f}_*\omega_Y(C')$, followed by the above trace morphism gives us, in particular, a sequence

$$\omega_{\tilde{X}}(\tilde{D}) \to R\tilde{f}_*\omega_Y(C) \to \omega_{\tilde{X}}(\tilde{D}),$$

which is an identity on $\omega_{\tilde{X}}(\tilde{D})$ (it is enough to check this on an open subset in $\tilde{X}$ over which $\tilde{f}$ is an isomorphism). Thus, there is a splitting

$$R\tilde{f}_*\omega_Y(C) \cong \omega_{\tilde{X}}(\tilde{D}) \oplus N$$

for a certain object $N$ in the derived category of coherent sheaves on $\tilde{X}$. Applying further $R\pi_*$, we obtain a splitting

$$Rf_*\omega_Y(C) \cong R\pi_*\omega_{\tilde{X}}(\tilde{D}) \oplus R\pi_*N.$$ 

On the other hand, we claim that $Rf_*\omega_Y(C) = \omega_X(D)$. To prove this we may assume that $f$ is a blow-up at a smooth center $Z \subset X$. By the projection formula, it is enough to consider only those components of $D$ which contain $Z$, because for the other components, their set-theoretical preimages coincide with their pull-backs. Then, a direct calculation gives the required result.

Thus, we see that $R\pi_*\omega_{\tilde{X}}(\tilde{D})$ is a direct summand in $\omega_X(D)$, which proves immediately the proposition, because an invertible sheaf on a connected variety has no non-trivial direct summands.

The following implication will be useful for our purposes.

**Corollary 2.7.** Let $\pi : \tilde{X} \to X$ be a proper birational morphism between smooth quasi-projective varieties. Let $D \subset X$ and $\tilde{D} := \pi^{-1}(D)$ be simple normal crossing divisors. Then, for any locally free sheaf $\mathcal{G}$ on $X$, and for any $q \geq 0$, we have that

$$H^q(\tilde{X}, \pi^!\mathcal{G}(\tilde{D})) = H^q(X, \mathcal{G}(D)).$$

### 3 Statement of the main result

#### 3.1 The polar complex

Let $Z$ be an irreducible variety and $\mathcal{E}$ a locally free sheaf on $Z$. We are going to define a distinguished subset in $\omega_K \otimes_K \mathcal{E}_K$ where $K := k(Z)$ and $\mathcal{E}_K$ denotes the space of rational sections of $\mathcal{E}$. In particular, its elements will be allowed to have poles of the first order at most.
Definition 3.1. An element $\alpha \in \omega_K \otimes_K \mathcal{E}_K$ is called polar if there exist a smooth variety $V$, a proper birational morphism $f : V \to Z$, a simple normal crossing divisor $W \subset V$, and a section $\alpha_V \in H^0(V, \omega_V(W) \otimes_{O_V} f^*\mathcal{E})$ such that the restriction of $\alpha_V$ to the generic point of $V$ equals $f^*\alpha$. We shall use the notation $\text{Pol}_Z(\mathcal{E}) = \{\text{the set of all polar elements in } \omega_K \otimes_K \mathcal{E}_K\}$.

Remark 3.2. It follows from the Hironaka theorem that, for $Z$ and $\mathcal{E}$ as in Definition 3.1, the set $\text{Pol}_Z(\mathcal{E})$ is a $k$-vector subspace in $\omega_K \otimes_K \mathcal{E}_K$. For more details see Remark 4.9 and Proposition 4.11(i) below.

For an irreducible reduced divisor $B$ on $Z$, there is a canonical residue homomorphism $\text{res}_{ZB} : \text{Pol}_Z(\mathcal{E}) \to \text{Pol}_B(\mathcal{E}|_B)$ induced by taking residues of logarithmic forms. This homomorphism is defined as follows. First, if $Z$ is normal, the generic point $x$ in $B$ is smooth both in $B$ and $Z$ and any form $\alpha \in \text{Pol}_Z(\mathcal{E})$ has at most first order pole along $B$ locally at $x$. Then $\text{res}_{ZB}(\alpha)$ is defined as the usual residue of $\alpha$ at $B$. If $Z$ is not normal, one takes the normalization $f : \tilde{Z} \to Z$ and defines $\text{res}_{ZB}(\alpha)$ as the sum $\sum_i f_*(\text{res}_{\tilde{Z}B_i}(\alpha))$, where $\tilde{B}_i$ runs through irreducible components in the preimage of $B$ in $\tilde{Z}$. By construction, $\text{res}_{ZB}(\alpha)$ is a rational form on $B$ with coefficients in $\mathcal{E}|_B$. Proposition 4.11(ii) below shows (cf. Remark 4.9) that $\text{res}_{ZB}$ maps polar elements to polar ones.

Let now $X$ be a variety and choose a locally free sheaf $\mathcal{F}$ on $X$. Taking residue homomorphisms for all possible pairs $B \subset Z$ of irreducible subvarieties in $X$, we get a homomorphism $\partial : \bigoplus_{Z \in X_{(p)}} \text{Pol}_Z(\mathcal{F}|_Z) \to \bigoplus_{B \in X_{(p-1)}} \text{Pol}_B(\mathcal{F}|_B)$, where $X_{(p)}$ denotes the set of irreducible subvarieties in $X$ of dimension $p$. We have that $\partial^2 = 0$. The proof of this fact is essentially the same as of Theorem 3.9 in [16], cf. Remark 3.5 below.

Definition 3.3. A polar complex of a locally free sheaf $\mathcal{F}$ on a variety $X$ is the complex $\text{Pol}_\bullet(X, \mathcal{F})$ with $\text{Pol}_p(X, \mathcal{F}) = \bigoplus_{Z \in X_{(p)}} \text{Pol}_Z(\mathcal{F}|_Z)$ and with the differential $\partial$. For a chain $\gamma = \oplus \gamma_Z \in \text{Pol}_p(X, \mathcal{F})$, with $\gamma_Z \in \text{Pol}_Z(\mathcal{F}|_Z)$, denote by $\text{supp } \gamma$ its support, that is, the set of irreducible subvarieties $Z \in X_{(p)}$ such that $\gamma_Z \neq 0$.

The main result of the paper is the following statement.

Theorem 3.4. Let $X$ be a smooth irreducible quasi-projective variety of dimension $d$ and suppose $\mathcal{F}$ is a locally free sheaf on $X$. Then, there is a canonical isomorphism $H^{d-p}(X, \omega_X \otimes_{O_X} \mathcal{F}) \cong H_p(\text{Pol}_\bullet(X, \mathcal{F}))$. 

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Remark 3.5. Let $X$ be a smooth projective variety and $\mathcal{F} = \mathcal{O}_X$. The polar complex in this particular case was dealt with in the papers [16, 17], where the groups $H_p(\text{Pol}_\bullet(X, \mathcal{O}_X))$ were denoted as $HP_p(X)$ and called polar homology groups. The above theorem was proved in this case in [17]; that is, $H^{d-p}(X, \omega_X) = HP_p(X)$ in the notation of [17].

Example 3.6. Let us describe the complex $\text{Pol}_\bullet(X, \mathcal{F})$ in the simplest case when $X$ is a smooth curve over $k$ and $\mathcal{F}$ a locally free sheaf on $X$. Then, in homological degree one, we have

$$\text{Pol}_X(\mathcal{F}) = \lim_{\to} H^0(X, (\omega_X \otimes \mathcal{O}_X \mathcal{F})(D)),$$

where the direct limit is taken over all reduced effective divisors $D$ on $X$. In other words, $\text{Pol}_X(\mathcal{F})$ consists of all rational sections of $\omega_X \otimes \mathcal{O}_X \mathcal{F}$ which have only first order poles somewhere on $X$. In homological degree zero, for each $x \in X$, we find that $\text{Pol}_x(\mathcal{F}|_x)$ equals $\mathcal{F}|_x$, the fiber at $x$ of the vector bundle associated with $\mathcal{F}$. The polar complex $\text{Pol}_\bullet(X, \mathcal{F})$ then takes the form

$$0 \to \text{Pol}_X(\mathcal{F}) \xrightarrow{\partial} \bigoplus_{x \in X} \mathcal{F}|_x \to 0.$$ 

This is a direct limit of the complexes

$$0 \to H^0(X, (\omega_X \otimes \mathcal{O}_X \mathcal{F})(D)) \xrightarrow{\partial} \bigoplus_{x \in D} \mathcal{F}|_x \to 0$$

over all reduced effective divisors $i_D : D \hookrightarrow X$. The assertion of Theorem 3.4 is implied in this case by the long exact sequence of cohomology groups associated with the exact sequence of sheaves on $X$:

$$0 \to \omega_X \otimes \mathcal{O}_X \mathcal{F} \to (\omega_X \otimes \mathcal{O}_X \mathcal{F})(D) \to i_D^*(\mathcal{F}|_D) \to 0.$$ 

### 3.2 A topological analogy

The result of Theorem 3.4 was called in the paper [17] the Polar de Rham Theorem by a reason of the following analogy with the topological situation proposed in [16]:

| Smooth real manifold | $\leftrightarrow$ Complex algebraic manifold |
|----------------------|------------------------------------------|
| $d$                  | $\leftrightarrow$ $\bar{\partial}$       |
| de Rham complex      | $\leftrightarrow$ Dolbeault complex      |
| Smooth functions     | $\leftrightarrow$ Smooth functions       |
| Locally constant     | $\leftrightarrow$ Holomorphic functions  |
| Flat bundles         | $\leftrightarrow$ Holomorphic bundles    |
| Orientation          | $\leftrightarrow$ Holomorphic top form   |
| Orientation sheaf    | $\leftrightarrow$ Canonical sheaf        |
| Boundary operator    | $\leftrightarrow$ Residue homomorphism   |
| Borel–Moore homology with coefficients | $\leftrightarrow$ Polar homology with coefficients |
| In a locally constant sheaf | $\leftrightarrow$ In a locally free sheaf |
Thus, Theorem 3.4 can be viewed on as an analog of the following fact (see, e.g., [23, Sect. 8.8 A and D of Ch. 2]):

**Proposition 3.7.** Let $M$ be a smooth real manifold of dimension $d$, suppose $\mathcal{L}$ is a locally constant sheaf on $M$, and denote by $\text{or}_M$ the orientation sheaf on $M$. Then there is a canonical isomorphism

$$H^{d-p}(M, \text{or}_M \otimes Z \mathcal{L}) \cong H^{BM}_p(M, \mathcal{L}).$$

The proof of Theorem 3.4 is given in Section 4 and can be considered as an algebro-geometric analogue of the following proof of Proposition 3.7. Recall that the Borel–Moore homology on the right of the above relation can be defined as homology of the complex $C_\bullet(M, \mathcal{L})$ of locally finite chains with coefficients in $\mathcal{L}$, that is, an element in $C_p(M, \mathcal{L})$ is an infinite formal linear combination of singular $p$-chains with coefficients in $\mathcal{L}$ such that for any compact subset $K \subset M$, there are only finitely many terms whose support meets $K$. Consider a homological complex of flabby sheaves on $M$ given by the formula

$$C_\bullet(M, \mathcal{L})(U) := C_\bullet(M, \mathcal{L}) / C_\bullet(M \setminus U, \mathcal{L}|_{M \setminus U}).$$

for an open subset $U \subset M$. To conclude one shows that this complex of sheaves is quasisomorphic to the sheaf $\text{or}_M \otimes Z \mathcal{L}$ placed in the homological degree $d$. A natural way to prove the latter is to use tubular neighborhoods, which allows to represent locally a singular cycle as a boundary. An analog of tubular neighborhoods in algebraic geometry was invented by Quillen in [21]. This construction is sometimes called “Quillen’s trick” and is often used in various motivic considerations. We also apply this method in our situation, see Section 4.4.

### 4 Proof of the main result

#### 4.1 Polar sheaves

By analogy with what was used above in the proof of Proposition 3.7 consider the following sheafified version of the polar complex:

**Definition 4.1.** For a variety $X$, a locally free sheaf $\mathcal{F}$ on $X$, and $p \geq 0$, let us denote by $\text{Pol}_p(X, \mathcal{F})$ the sheaf of abelian groups on $X$ defined so that, for an open subset $U \subset X$, we have

$$\text{Pol}_p(X, \mathcal{F})(U) := \text{Pol}_p(X, \mathcal{F}) / \text{Pol}_p(X \setminus U, \mathcal{F}|_{X \setminus U}).$$

(6)

More explicitly, we have that

$$\text{Pol}_p(X, \mathcal{F})(U) = \bigoplus_{Z \in \mathcal{F}(p)} \text{Pol}_{Z}(\mathcal{F}|_Z).$$

(7)

The latter expression implies that the sheaves $\text{Pol}_p(X, \mathcal{F})$ are flabby and form a complex $\text{Pol}_\bullet(X, \mathcal{F})$ with a differential $\partial$ induced by that on $\text{Pol}_\bullet$, as defined in Section 3.
We are going to deal with polar sections of $\mathcal{G} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}$, which are distinguished among other rational sections by saying (roughly) that they have at most first order poles, see Definition \[\text{Def.} 4.1\]. Therefore, it looks reasonable to consider also the following subsheaf of abelian groups, $\mathcal{G}_{\text{pol}}$, of the sheaf $\mathcal{G}$. For an open subset $U \subset X$, the group $\mathcal{G}(U)$ consists of rational sections of $\mathcal{G}$ on $X$ which are regular in $U$, and we consider a subset of those, $\mathcal{G}_{\text{pol}}(U)$, characterized by a certain restriction on their singularities at $X \setminus U$. We wished to say here “first order poles at $X \setminus U$”, but a precise formulation is as follows.

**Definition-Proposition 4.2.** Let $\mathcal{G}$ be a locally free sheaf on a smooth irreducible variety $X$ and $U \subset X$ an open subset. Let us choose a smooth variety $\tilde{X}$ and a proper birational morphism $\pi: \tilde{X} \to X$ such that $\pi$ is an isomorphism over $U$ and $\tilde{D} := \pi^{-1}(X \setminus U)$ is a simple normal crossing divisor. Then, the formula (cf. eq. \[\text{(5)}\] in Section 2)

$$
\mathcal{G}_{\text{pol}}(U) = H^0(\tilde{X}, \pi^! \mathcal{G}(\tilde{D}))
$$

defines a sheaf of abelian groups on $X$ which will be called the polar sheaf associated with a locally free sheaf $\mathcal{G}$. The definition does not depend on the choice of $\tilde{X}$.

**Proof.** Suppose that we are given two pairs $(\tilde{X}_1, \pi_1)$ and $(\tilde{X}_2, \pi_2)$ as above. By Lemma \[\text{2.5}\] and the Hironaka theorem, there exists a smooth irreducible variety $\bar{X}$ and proper birational morphisms $\pi: \bar{X} \to X$, $f_i: \bar{X} \to \tilde{X}_i$, $i = 1, 2$, such that $\bar{D} := \pi^{-1}(X \setminus U)$ is a simple normal crossing divisor and $\pi = \pi_1 \circ f_1 = \pi_2 \circ f_2$. By Corollary \[\text{2.7}\] we have

$$
H^0(\tilde{X}_i, \pi_i^! \mathcal{G}(\tilde{D}_i)) = H^0(\bar{X}, \pi^! \mathcal{G}(\bar{D})),
$$

where $\tilde{D}_i = \pi_i^{-1}(X \setminus U)$, $i = 1, 2$. This shows that the definition of the group $\mathcal{G}_{\text{pol}}(U)$ is correct and we obtain a presheaf $\mathcal{G}_{\text{pol}}$. Obviously, there is a canonical injective morphism of presheaves $\mathcal{G}_{\text{pol}} \hookrightarrow \mathcal{G}$. Moreover, we have the following canonical isomorphism of presheaves:

$$
\mathcal{G}_{\text{pol}} \sim \ker \left( \text{Pol}_d(X, \omega_X^{-1} \otimes \mathcal{O}_X \mathcal{G}) \overset{\partial}{\to} \text{Pol}_{d-1}(X, \omega_X^{-1} \otimes \mathcal{O}_X \mathcal{G}) \right),
$$

where $d$ is the dimension of $X$. This shows that $\mathcal{G}_{\text{pol}}$ is a sheaf.

**Remark 4.3.**

(i) As it was noticed in the proof of Proposition \[\text{4.2}\] for a locally free sheaf $\mathcal{F}$ on a smooth irreducible variety $X$ of dimension $d$, there is a canonical isomorphism of sheaves

$$
(\omega_X \otimes \mathcal{O}_X \mathcal{F})_{\text{pol}} \sim \ker \left( \text{Pol}_d(X, \mathcal{F}) \overset{\partial}{\to} \text{Pol}_{d-1}(X, \mathcal{F}) \right).
$$

(ii) Given an open embedding $U \subset X$, the sheaf $(\mathcal{G}_{\text{pol}})|_U$ is a subsheaf in the sheaf $(\mathcal{G}|_U)_{\text{pol}}$, but these two are not equal in general. Indeed, one has $H^0(U, (\mathcal{G}|_U)_{\text{pol}}) = \mathcal{G}(U)$, whereas $H^0(U, (\mathcal{G}_{\text{pol}})|_U)$ consists of those sections in $\mathcal{G}(U)$ which have at most first order poles at $X \setminus U$ (in the precise sense of Definition \[\text{4.2}\]).
(iii) The functor $G \mapsto G_{\text{pol}}$ is exact in a locally free sheaf $G$ (left exactness is obvious, while right exactness follows from the Serre vanishing theorem). Similarly, the functors $\text{Pol}_Z(F|_Z)$ and $\text{Pol}_*(X, F)$ are exact in a locally free sheaf $F$ on $X$.

**Theorem 4.4.** Let $X$ be a smooth irreducible quasi-projective variety of dimension $d$ and $F$ a locally free sheaf on $X$. Then, the complex of sheaves $\text{Pol}_*(X, F)$ on $X$ is a flabby resolution for the sheaf $(\omega_X \otimes_\mathcal{O}_X F)_{\text{pol}}$, that is, the following sequence of sheaves is exact:

$$0 \to (\omega_X \otimes_\mathcal{O}_X F)_{\text{pol}} \to \text{Pol}_d(X, F) \xrightarrow{\partial} \text{Pol}_{d-1}(X, F) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \text{Pol}_0(X, F) \to 0.$$

**Theorem 4.5.** Let $X$ be a smooth irreducible quasi-projective variety and $G$ a locally free sheaf on $X$. Then the canonical injective morphism of sheaves $G_{\text{pol}} \to G$ induces isomorphisms of cohomology groups:

$$H^p(X, G_{\text{pol}}) \cong H^p(X, G), \quad p \geq 0.$$

Theorem 4.4 and Theorem 4.5 are proved first for a projective $X$ in Section 4.4 and Section 4.5, respectively. A reduction of both theorems to the projective case is done in Section 4.6. All together this implies Theorem 3.4.

### 4.2 Auxiliary tools

To prove Theorem 4.4 and Theorem 4.5 we will use the following properties of the polar complex.

**Proposition 4.6.** For a locally free sheaf $F$ on a variety $X$ the following is true:

(i) Let $f : Y \to X$ be a proper morphism. Then, there exists a canonical morphism of complexes

$$f_* : \text{Pol}_*(Y, f^* F) \to \text{Pol}_*(X, F)$$

compatible with the trace of differential forms.

(ii) For $f : Y \to X$ as above, let $U \subset Y$ be an open subset. Then, the quotient complex

$$\text{Pol}_*(Y, f^* F) / \text{Pol}_*(Y \setminus U, f^* F|_{Y \setminus U})$$

canonically depends only on the morphism $f|_U : U \to X$.

(iii) The projection $\pi : X \times \mathbb{P}^1 \to X$ induces a quasiisomorphism of complexes

$$\pi_* : \text{Pol}_*(X \times \mathbb{P}^1, \pi^* F) \to \text{Pol}_*(X, F).$$

(iv) Let $E$ be a vector bundle on $X$ and $\mathbb{P}(E)$ a projectivization of $E$. Then, the projection $\pi : \mathbb{P}(E) \to X$ induces a quasiisomorphism of complexes

$$\pi_* : \text{Pol}_*(\mathbb{P}(E), \pi^* F) \to \text{Pol}_*(X, F).$$
(v) Let $\pi : \tilde{X} \to X$ be the blow-up at a center $R \subset X$ with a locally free normal sheaf. Let $Z$ be any closed subvariety in $X$ and $\tilde{Z}$ be the preimage $\pi^{-1}(Z)$ of $Z$ in $X$. Then, the canonical morphism

$$\pi_* : \text{Pol}_*(\tilde{Z}, \pi^* F|_{\tilde{Z}}) \to \text{Pol}_*(Z, F|_Z)$$

is a quasiisomorphism.

**Proof.** The proofs of (i) and (iii) are essentially the same as of Theorem 3.8 in the paper [16] (see also Remark 3.12 therein) and as of Lemma 3.5 in [17], respectively.

To prove (ii), note first that for any irreducible variety $Z$ and a proper morphism $g : Z \to X$, the group $\text{Pol}_Z(g^* F)$ depends only on the birational class of $g$, in other words, it depends only on the morphism $\text{Spec}(k(Z)) \to X$, the restriction to the generic point. This can be shown with help of Lemma 2.5. The assertion (ii) follows then from the relation (compare to the equations (6, 7)):

$$\text{Pol}_p(Y, f^* F) / \text{Pol}_p(Y \setminus U, f^* F|_{Y \setminus U}) \cong \bigoplus_{Z \subseteq Y(\rho) \cap Z \cap U \neq \emptyset} \text{Pol}_Z(f^* F|_Z).$$

Turning now to (iv), let us first consider the case of a trivial bundle $\pi : X \times \mathbb{P}^n \to X$ and use induction on $n$ as follows. The base of induction is (iii). To deduce $n$ from $n - 1$, notice that the varieties $X \times \mathbb{P}^n$ and $X \times \mathbb{P}^{n-1} \times \mathbb{P}^1$ have open subsets that are isomorphic to $X \times \mathbb{A}^n$. The corresponding complements are $X \times \mathbb{P}^{n-1}$ and $X \times T$, where $T := (\mathbb{P}^{n-1} \times \{\infty\}) \cup (\mathbb{P}^2 \times \mathbb{P}^1)$. Using (ii), we obtain the isomorphism of complexes

$$\text{Pol}_*(X \times \mathbb{P}^n, \pi^* F) / \text{Pol}_*(X \times \mathbb{P}^{n-1}, \pi^* F) \cong \text{Pol}_*(X \times \mathbb{P}^{n-1} \times \mathbb{P}^1, \rho^* F) / \text{Pol}_*(X \times T, \rho^* F),$$

where $\rho : X \times \mathbb{P}^{n-1} \times \mathbb{P}^1 \to X$ is the natural projection. By this isomorphism and the induction hypothesis, it suffices to prove that

$$\text{Pol}_*(X \times T, \rho^* F) \to \text{Pol}_*(X \times \mathbb{P}^{n-1} \times \mathbb{P}^1, \rho^* F)$$

is a quasiisomorphism. By (iii), it is enough to show that the morphism $f : X \times T \to X \times \mathbb{P}^{n-1}$ induces a quasiisomorphism

$$f_* : \text{Pol}_*(X \times T, \rho^* F) \to \text{Pol}_*(X \times \mathbb{P}^{n-1}, \rho^* F).$$

Since $f$ is an isomorphism over $\mathbb{P}^{n-1} \setminus \mathbb{P}^{n-2}$ and the restriction of $f$ to $\mathbb{P}^{n-2}$ is the projection $\mathbb{P}^{n-2} \times \mathbb{P}^1 \to \mathbb{P}^{n-2}$, we conclude by (ii) and (iii).

For the general case, $\pi : \mathbb{P}(E) \to X$, we use Noetherian induction, that is, we assume that the desired quasiisomorphism is proved for all closed subsets $S \subset X$. Let $U \subset X$ be a non-empty open subset where the initial bundle is trivial, $\mathbb{P}(E|_U) \cong U \times \mathbb{P}^n$. Let $S = X \setminus U$ and let $\pi_S : \mathbb{P}(E|_S) \to S$ be the natural projection. Then, by (ii) and (iv) for a trivial bundle, we have a quasiisomorphism

$$\text{Pol}_*(\mathbb{P}(E), \pi^* F) / \text{Pol}_*(\mathbb{P}(E|_S), \pi_S^* (F|_S)) \to \text{Pol}_*(X, F) / \text{Pol}_*(S, F|_S).$$

Thus, we conclude by Noetherian induction.

Finally, (v) is implied by (ii) and (iv), because $\pi$ is an isomorphism over $Z \setminus (Z \cap R)$ and the restriction of $\pi : \tilde{Z} \to Z$ to $Z \cap R$ is a projectivization of a vector bundle on $Z \cap R$. \qed
Up to now we considered irreducible subvarieties in a given variety $X$ and certain rational forms of top degree on them with coefficients in a locally free sheaf $F$ on $X$ (see Definition 3.3). For the sake of the proof of Theorem 4.4 it is convenient to work with irreducible varieties that map to $X$ not necessarily birationally to their image and certain rational forms of not only top degree on them. Thus, Definition 4.8 below can be thought of as an extension of the notion of a polar elements in $\omega_K \otimes_K F$, $K = k(Z)$, $Z \subset X$, given in Definition 3.1, to the case when $Z$ is replaced by an irreducible variety $V$ together with a morphism $V \to X$, and rational top forms are replaced by forms of arbitrary degree $q$ in $\Omega^q_{K}$, $K = k(V)$. The requirement of first order poles is then to be replaced by the requirement that the $q$-forms have logarithmic singularities.

**Definition 4.7.** For a variety $X$, we say that $(K, \varphi)$ is a field over $X$ if $K$ is a field and there is a given morphism $\varphi : \text{Spec}(K) \to X$ such that there exists an irreducible variety $V$ with $k(V) = K$ and a morphism of varieties $f : V \to X$ that agrees at the generic point of $V$ with the given morphism $\varphi$. We also say that $(K', \varphi')$ is an extension of $(K, \varphi)$ over $X$ if $K \subset K'$ and the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Spec}(K') & \to & \text{Spec}(K) \\
\varphi' \downarrow & & \varphi \downarrow \\
X & & \\
\end{array}
$$

In other words, for a given $X$, a field $(K, \varphi)$ over $X$ is a class of birationally equivalent pairs $(V, f : V \to X)$.

**Definition 4.8.** Let $X$ be a variety, $F$ a locally free sheaf on $X$, and $(K, \varphi)$ a field over $X$. We define $M_q(F)(K, \varphi)$ as a set of all elements $\alpha \in \Omega^q_{K} \otimes_K \varphi^*F$ such that there exists a collection $(V, f, W, \alpha_V)$, where $V$ is a smooth irreducible variety with $k(V) = K$, $f : V \to X$ is a proper morphism that agrees at the generic point of $V$ with the given morphism $\varphi$, $W \subset V$ is a simple normal crossing divisor,

$$
\alpha_V \in H^0(V, \Omega^q_{\nu}(\log W) \otimes_{\nu} f^*F),
$$

and $\alpha$ is the restriction of $\alpha_V$ to the generic point of $V$.

**Remark 4.9.** If $Z$ is an irreducible subvariety of dimension $p$ in $X$, and $\varphi_Z : \text{Spec}(k(Z)) \to X$ is the natural morphism, then we have

$$
M_p(F)(k(Z), \varphi_Z) = \text{Pol}_Z(F|_Z).
$$

In Proposition 4.11 below we will recollect various useful properties of $M_q(F)(K, \varphi)$ defined above. The proof of these properties is a direct application of the Hironaka theorem. More precisely, we will use the following facts:

**Lemma 4.10.** Given a field $(K, \varphi)$ over a variety $X$, there exists a smooth irreducible variety $V$ with $k(V) = K$ and a proper morphism $f : V \to X$ that agrees at the generic point of $V$ with the given morphism $\varphi$.
Proof. There exist an affine variety $U$ with $k(U) = K$ and a morphism $f_U : U \to X$ that agrees at the generic point of $U$ with the given morphism $\varphi$. Let $\Gamma_{f_U}$ be the graph of $f_U$. Consider the embedding $U \subset \mathbb{P}^n$ for some $n$. Then,

$$\Gamma_{f_U} \subset U \times X \subset \mathbb{P}^n \times X.$$ 

Take the closure $\bar{\Gamma}_{f_U}$ of $\Gamma_{f_U}$ in $\mathbb{P}^n \times X$. Note that $\bar{\Gamma}_{f_U}$ contains $\Gamma_{f_U}$ as an open dense subset and the projection to $X$ defines a proper (even projective) morphism $\bar{\Gamma}_{f_U} \to X$. We can now get a smooth $V$ by resolving the singularities of $\bar{\Gamma}_{f_U}$. □

Proposition 4.11. Let $\mathcal{F}$ be a locally free sheaf on a variety $X$.

(i) For a field $(K, \varphi)$ over $X$, the set $M_q(\mathcal{F})(K, \varphi)$ is a $k$-vector subspace in $\Omega^q_K \otimes_K \varphi^* \mathcal{F}$.

(ii) If $(K', \varphi')$ is an extension of $(K, \varphi)$ over $X$ (see Definition 4.7) then the pull-back map

$$\Omega^q_K \otimes_K \varphi^* \mathcal{F} \to \Omega^q_{K'} \otimes_{K'} \varphi'^* \mathcal{F}$$

sends $M_q(\mathcal{F})(K, \varphi)$ to $M_q(\mathcal{F})(K', \varphi')$.

(iii) If $(K', \varphi')$ is an extension of $(K, \varphi)$ over $X$, such that $K \subset K'$ is a finite extension then the trace map

$$\Omega^q_{K'} \otimes_{K'} \varphi'^* \mathcal{F} \to \Omega^q_K \otimes_K \varphi^* \mathcal{F}$$

sends $M_q(\mathcal{F})(K', \varphi')$ to $M_q(\mathcal{F})(K, \varphi)$.

(iv) For a field $(K, \varphi)$ over $X$, the product of differential forms defines an associative graded ring structure on $\bigoplus_{q \geq 0} M_q(o_X)(K, \varphi)$ and a graded module structure over this ring on $\bigoplus_{q \geq 0} M_q(\mathcal{F})(K, \varphi)$.

(v) For a field $(K, \varphi)$ over $X$, the image of the homomorphism $d\log: K^* \to \Omega^1_K$ is contained in $M_1(o_X)(K, \varphi)$.

(vi) Given an irreducible variety $V$, a proper morphism $f: V \to X$, and an irreducible divisor $D \subset V$, there is a residue homomorphism

$$\text{res}_{V,D}: M_q(\mathcal{F})(k(V), \varphi) \to M_{q-1}(\mathcal{F})(k(D), \psi), \quad q \geq 1,$$

where $\varphi$ and $\psi$ are the restrictions of $f$ to the generic point of $V$ and $D$, respectively.

(vii) Given an irreducible variety $V$, a morphism $f: V \to X$, an irreducible subvariety $Z \subset V$ of codimension two, and an element $\alpha \in M_q(\mathcal{F})(k(V), \varphi)$ with $\varphi$ being the restriction of $f$ to the generic point of $V$, we have

$$\sum_D (\text{res}_{D,Z} \circ \text{res}_{V,D})(\alpha) = 0$$

where the sum is taken over all irreducible divisors $D$ in $V$ that contain $Z$. 

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(viii) Suppose $f : X' \to X$ is a proper morphism. For a field $(K, \varphi')$ over $X'$ we set 
$\varphi := f \circ \varphi'$. We have then a canonical isomorphism 

$$M_q(F)(K, \varphi) \cong M_q(f^*F)(K, \varphi').$$

**Proof.** Note that Lemma 2.1 and Lemma 2.2 can be easily extended to the case of forms with coefficients in a locally free sheaf on the target variety (denoted by $V$ there). Below we shall use this version of Lemma 2.1 and Lemma 2.2.

It is clear that, for a field $(K, \varphi)$ over $X$, the subset

$$M_q(F)(K, \varphi) \subset \Omega^q_K \otimes_K \varphi^*F$$

is closed under multiplication by elements of $k$. Further, consider two elements $\alpha_1, \alpha_2 \in M_q(F)(K, \varphi)$ and the corresponding collections $(V_1, f_1, W_1, \alpha_{V_1}), (V_2, f_2, W_2, \alpha_{V_2})$ as in Definition 4.3. Since $k(V_1) = k(V_2) = K$, there is a fixed birational morphism $g : V_1 \dashrightarrow V_2$. By Lemma 2.5 there is a smooth variety $V$ with proper birational morphisms $h_i : V \to V_i, i = 1, 2$, such that $f_1 \circ h_1 = f_2 \circ h_2$. Let $f : V \to X$ be any of the latter compositions. By the Hironaka theorem, changing $V$ birationally, we may assume that $W := h_1^{-1}(W_1) \cup h_2^{-1}(W_2)$ is a simple normal crossing divisor in $V$. By Lemma 2.1, the forms $h_i^*\alpha_{V_i}, i = 1, 2$, belong to $H^0(V, \Omega^q_V(\log W) \otimes_{O_V} f^*F)$. The restriction of the form

$$h_1^*\alpha_{V_1} + h_2^*\alpha_{V_2} \in H^0(V, \Omega^q_V(\log W) \otimes_{O_V} f^*F)$$

to the generic point of $V$ is equal to $\alpha_1 + \alpha_2 \in \Omega^q_K \otimes_K \varphi^*F$, which shows (i).

Consider an extension $(K', \varphi')$ of $(K, \varphi)$ over $X$, a form $\alpha$ in $M_q(F)(K, \varphi)$, and a corresponding collection $(V, f, W, \alpha_V)$ as in Definition 4.3. Let us denote by $\alpha'$ the pullback of $\alpha$. Our aim is to show that the rational form $\alpha' \in \Omega^q_{K'} \otimes_{K'} \varphi'^*F$ belongs in fact to $M_q(F)(K', \varphi')$. By Lemma 4.10 there exist a smooth irreducible variety $V'$ with $k(V') = K'$ and a proper morphism $f' : V' \to X$ that agrees at the generic point of $V'$ with the given morphism $\varphi'$. The extension $(K', \varphi')$ of $(K, \varphi)$ gives us a rational map $g : V' \dashrightarrow V$ over $X$ with $f \circ g = f'$. By Lemma 2.5 we may assume now that $g$ is regular. By further changing $V'$ birationally, we may assume according to the Hironaka theorem that $W' := g^{-1}(W)$ is a simple normal crossing divisor in $V'$. By Lemma 2.1 the pull-back $\alpha_{V'} := g^*\alpha_V$ gives an element in $H^0(V', \Omega^q_{V'}(\log W') \otimes_{O_{V'}} f'^*F)$. Since the restriction of $\alpha_{V'}$ to the generic point of $V'$ is equal to $\alpha'$, the assertion (ii) is proved.

Suppose now that the extension $K \subset K'$ is finite, consider a form $\alpha'$ in $M_q(F)(K', \varphi')$ and a corresponding collection $(V', f', W', \alpha_{V'})$ as in Definition 4.3. Let us denote by $\alpha$ the trace of $\alpha'$. Our aim is to show that the rational form $\alpha \in \Omega^q_K \otimes_K \varphi^*F$ belongs in fact to $M_q(F)(K, \varphi)$. By Lemma 4.10 there is a smooth irreducible variety $V$ with $k(V) = K$, a proper morphism $f : V \to X$ that agrees at the generic point of $V$ with the given morphism $\varphi$. The finite extension $K \subset K'$ defines a rational map $g : V' \dashrightarrow V$ such that $f \circ g = f'$. By Lemma 2.5 we may assume that $g$ is regular. The morphism $g$ sends the divisor $W' \subset V'$ to a subvariety $g(W')$ in $V$, which, however, does not have to be a simple normal crossing in $V$. We may now apply the Hironaka theorem to the pair
Lemma 4.12. Let $\tilde{X}$ be a smooth irreducible projective variety of dimension $d$ and $x \in X$ a point there. Let $Z \subset X$, $W \subset X$ be two closed subvarieties such that $Z \neq X$ and any irreducible component of $W$ has codimension at least two in $X$. Then, there exist a blow-up $f : \tilde{X} \to X$ at a finite set not containing $x$ and not intersecting $Z$ and $W$, and a surjective morphism $\pi : \tilde{X} \to \mathbb{P}^{d-1}$ such that $\pi$ is smooth at $\tilde{x} := f^{-1}(x)$, the restriction $\varphi := \pi|_{\tilde{Z}}$ to $\tilde{Z} := f^{-1}(Z)$ is finite, and

$$W \cap \pi^{-1}(\pi(\tilde{x})) \subseteq \{\tilde{x}\},$$

$g(W') \subset V$ and get a proper birational morphism $\pi : \tilde{V} \to V$ such that $\tilde{W} := \pi^{-1}(g(W'))$ is a simple normal crossing divisor in $\tilde{V}$. Simultaneously, this gives us a rational map $\tilde{g} : V' \to \tilde{V}$ and by Lemma 2.2 we may assume that this morphism is regular. By abuse of notation, we shall write now $V$ instead of $\tilde{V}$, $g$ instead of $\tilde{g}$, and $W$ instead of $\tilde{W}$. Thus, we constructed a proper morphism $g : V' \to V$ such that $f \circ g = f'$, $g$ corresponds at the generic point of $V'$ to the given morphism of fields over $X$, and we have that $W' \subset g^{-1}(W)$. By Lemma 2.2, the trace $\alpha_V := g_*\alpha_{V'}$ gives an element in $H^0(V, \Omega^1_V(\log W) \otimes O_V f^*\mathcal{F})$. Since the restriction of $\alpha_V$ to the generic point of $V$ is equal to $\alpha$, the assertion (iii) is proved.

Further, (iv) follows by the same argument as in (i) and the fact that logarithmic forms are closed under products, provided the union of their poles is a simple normal crossing divisor.

The assertion (v) follows from the fact that $d\log$ applied to a non-zero rational function gives a logarithmic function, provided that the support of the divisor of that function is a simple normal crossing divisor. The latter can be achieved by the Hironaka theorem.

Consider $V$, $f$, and $D$ as in (vi). Note that $V$ and $D$ are not necessarily smooth. Take a form $\alpha$ in $M_q(\mathcal{F})(k(V), \varphi)$ and a corresponding collection $(V', f', W', \alpha_{V'})$ as in Definition 4.8. This means in particular that $k(V') \cong k(V)$. By Lemma 2.2 we may assume that the latter isomorphism corresponds to a birational morphism $g : V' \to V$. Changing $V'$ birationally if necessary, we can make $g^{-1}(D) \cup W'$ be a simple normal crossing divisor in $V'$. Let $\{D'_i\}$ be the set of all components of $g^{-1}(D)$ that map dominantly to $D$. Denote by $g_i : D'_i \to D$ the restriction of $g$ to $D'_i$ and by $\psi'_i : \text{Spec}(k(D'_i)) \to X$ the restriction of $f'$ to the generic point of $D'_i$. For each $i$, the form $\text{res}_{V'D'_i}(\alpha_{V'})$ on $D'_i$ defines an element $\beta_i$ in $M_{q-1}(\mathcal{F})(k(D'_i), \psi'_i)$. By (iii) and (i), the rational form $\beta := \sum_i g_i^*(\beta_i)$ belongs to $M_{q-1}(\mathcal{F})(k(D), \psi)$. We set now $\text{res}_{V,D}(\alpha) := \beta$. Since the residue map commutes with the pull-back and the trace map (cf. eqs. (3) and (4) in Section 2; note that the ramification indices are $e_i = 1$ in the present case) the form $\beta$ does not depend on the choices made. This explains (vii).

The proof of (viii) goes without any difference with the proof of Theorem 3.9 in [16].

Finally, (viii) is obvious. $\square$

### 4.3 A geometric lemma

In the next subsection we shall use the following simple geometric fact (it seems that there is no suitable analogue of that fact when $X$ is not projective):

**Lemma 4.12.** Let $g(W') \subset V$ and get a proper birational morphism $\pi : \tilde{V} \to V$ such that $\tilde{W} := \pi^{-1}(g(W'))$ is a simple normal crossing divisor in $\tilde{V}$. Simultaneously, this gives us a rational map $\tilde{g} : V' \to \tilde{V}$ and by Lemma 2.2 we may assume that this morphism is regular. By abuse of notation, we shall write now $V$ instead of $\tilde{V}$, $g$ instead of $\tilde{g}$, and $W$ instead of $\tilde{W}$. Thus, we constructed a proper morphism $g : V' \to V$ such that $f \circ g = f'$, $g$ corresponds at the generic point of $V'$ to the given morphism of fields over $X$, and we have that $W' \subset g^{-1}(W)$. By Lemma 2.2, the trace $\alpha_V := g_*\alpha_{V'}$ gives an element in $H^0(V, \Omega^1_V(\log W) \otimes O_V f^*\mathcal{F})$. Since the restriction of $\alpha_V$ to the generic point of $V$ is equal to $\alpha$, the assertion (iii) is proved.

Further, (iv) follows by the same argument as in (i) and the fact that logarithmic forms are closed under products, provided the union of their poles is a simple normal crossing divisor.

The assertion (v) follows from the fact that $d\log$ applied to a non-zero rational function gives a logarithmic function, provided that the support of the divisor of that function is a simple normal crossing divisor. The latter can be achieved by the Hironaka theorem.

Consider $V$, $f$, and $D$ as in (vi). Note that $V$ and $D$ are not necessarily smooth. Take a form $\alpha$ in $M_q(\mathcal{F})(k(V), \varphi)$ and a corresponding collection $(V', f', W', \alpha_{V'})$ as in Definition 4.8. This means in particular that $k(V') \cong k(V)$. By Lemma 2.2 we may assume that the latter isomorphism corresponds to a birational morphism $g : V' \to V$. Changing $V'$ birationally if necessary, we can make $g^{-1}(D) \cup W'$ be a simple normal crossing divisor in $V'$. Let $\{D'_i\}$ be the set of all components of $g^{-1}(D)$ that map dominantly to $D$. Denote by $g_i : D'_i \to D$ the restriction of $g$ to $D'_i$ and by $\psi'_i : \text{Spec}(k(D'_i)) \to X$ the restriction of $f'$ to the generic point of $D'_i$. For each $i$, the form $\text{res}_{V'D'_i}(\alpha_{V'})$ on $D'_i$ defines an element $\beta_i$ in $M_{q-1}(\mathcal{F})(k(D'_i), \psi'_i)$. By (iii) and (i), the rational form $\beta := \sum_i g_i^*(\beta_i)$ belongs to $M_{q-1}(\mathcal{F})(k(D), \psi)$. We set now $\text{res}_{V,D}(\alpha) := \beta$. Since the residue map commutes with the pull-back and the trace map (cf. eqs. (3) and (4) in Section 2; note that the ramification indices are $e_i = 1$ in the present case) the form $\beta$ does not depend on the choices made. This explains (vii).

The proof of (viii) goes without any difference with the proof of Theorem 3.9 in [16].

Finally, (viii) is obvious. $\square$
where $\tilde{W} := f^{-1}(W)$.

**Remark 4.13.** The last condition means that we are trying to minimize the intersection of $\tilde{W}$ and the fiber of $\pi$ through $\tilde{x}$. If $x \notin W$, then we can achieve that they do not meet at all, otherwise, we can achieve that they intersect only at $\tilde{x}$.

**Proof.** Choose an embedding $X \subset \mathbb{P}^N$. By the Bertini theorem, there exists a linear subspace $L \subset \mathbb{P}^N$ of dimension $N - d$ such that $L \cap X$ is a finite set not containing $x$, $L \cap T_xX$ is a point, $L \cap Z = \emptyset$, and $L \cap C_xW = \emptyset$, where $T_xX \subset \mathbb{P}^N$ is the projective tangent space to $X$ at $x$, while $C_xW$ is the cone over $W$ with the vertex at $x$. Let $f: \tilde{X} \to X$ be the blow-up of $X$ at the finite set $L \cap X$. The projection $\pi_L: \mathbb{P}^N \to \mathbb{P}^{d-1}$ with the center at $L$ defines then a morphism $\pi: \tilde{X} \to \mathbb{P}^{d-1}$ with $\pi = \pi_L|_X \circ f$, which satisfies all the properties needed.

### 4.4 Algebraic tubular neighborhood

In this subsection we prove Theorem 4.4 when $X$ is a smooth irreducible projective variety.

**Proof of Theorem 4.4 for the projective case.** We have already noticed in Remark 4.3(i) that

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}} \sim \ker (\text{Pol}_d(X, \mathcal{F}) \to \text{Pol}_{d-1}(X, \mathcal{F})).$$

It remains thus to show that the complex of sheaves $\text{Pol}_d(X, \mathcal{F})$ is acyclic. In other words, given an arbitrary point $x \in X$ and a chain $\alpha \in \text{Pol}_p(X, \mathcal{F})$, $p < d$, such that $\partial \alpha = 0$ locally at $x$, we have to show that there exists a chain $\beta \in \text{Pol}_{d+1}(X, \mathcal{F})$ such that $\partial \beta = \alpha$ locally at $x$. More precisely, this means that having $x \notin \text{supp} \partial \alpha$, we want to find a chain $\beta$ such that $x \notin \text{supp} (\alpha - \partial \beta)$.

Let $Z := \text{supp} \alpha$ and suppose $\{Z_i\}$ are the irreducible components of $Z$, so that $\alpha = \bigoplus \alpha_i$ with $\alpha_i \in \text{Pol}_{Z_i}(\mathcal{F}|_{Z_i})$. We may assume that $Z_i$ contains $x$ for each $i$.

**Remark 4.14.** The idea of the proof is to use an algebraic version of a tubular neighborhood. Suppose a subvariety $T \subset X$ is a “tubular neighborhood” of $Z$ Zariski locally at $x$. By this we mean that $Z \subset T$, $\dim T = \dim Z + 1 = p + 1$, there is a “contraction” $\rho: T \to Z$, and $T$ is smooth at $x \in Z$ along the normal direction to $Z$. If $\mathcal{F} = \mathcal{O}_X$, then one could take $\beta = \frac{dt}{t} \wedge \rho^*(\alpha)$, where $t$ is an equation of $Z$ in $T$ locally at $x$. The problem is that such a tubular neighborhood $T$ does not always exist. An idea of how to overcome this problem was found by Quillen in [21, Theorem 5.11]. The main point is to use a certain finite surjective map onto $X$ which has a section over $Z$, that is, to solve the problem replacing Zariski topology by Nisnevich topology.

First of all, by the local nature of the problem, we may replace $X$ by its blow-up at a finite set not containing $x$. Hence by Lemma 4.12 we may assume that there exists a morphism $\pi: X \to \mathbb{P}^{d-1}$ such that $\pi$ is smooth at $x$, $\varphi := \pi|_Z$ is finite, and $W \cap \pi^{-1}(\varphi(x)) \subseteq \{x\}$, where $W$ is the union of poles of forms $\alpha_i$ and singularities of $Z_i$ over all $i$. Consider the scheme $Y' = Z \times_{\mathbb{P}^{d-1}} X$, so that we have $Z \xleftarrow{\sim} Y' \xrightarrow{\pi'} X$. Denote by $Y$ the union of all irreducible components of $Y'_{\text{red}}$ which contain an irreducible
component of $\sigma(Z)$, where $\sigma : Z \to Y'$ is a section of $\rho$ induced by the embedding $Z \hookrightarrow X$. By construction, the set $\{ Y_i \}$ of irreducible components of $Y$ is bijective with the set $\{ Z_i \}$ of irreducible components of $Z$ and each morphism $\rho : Y_i \to Z_i$ is surjective. We have thus the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & X \\
\sigma \uparrow \rho & & \downarrow \pi \\
Z & \xrightarrow{\varphi} & \mathbb{P}^{d-1}
\end{array}
$$

Note that the morphism $\psi$ is finite, while the morphism $\rho : Y' \to Z$ is smooth at the finite set $S := \psi^{-1}(x)$, as $\pi$ is smooth at $x$. The latter implies that all the $Y_i$ have the same dimension $p + 1$. Since $\rho$ is smooth at $S$, there exists a local equation $t$ for the section $\sigma(Z)$ in $Y$ locally at $S$.

Denote by $\chi_i : \text{Spec}(k(Y_i)) \to Y$ the natural morphism for each irreducible component $Y_i$ of $Y$. By Proposition 4.11(ii), (iv), (v), (viii) for each $i$, the rational differential form $\gamma_i := \frac{dx}{t} \wedge \rho^* \alpha$, on $Y_i$ belongs to $M_{p+1}(\rho^* F_Z)(k(Y_i), \chi_i) = \text{Pol}_X((\rho^* F_Z)|_{Y_i})$, where $\rho^* F_Z$ is the pull-back to $Y$ of $F_Z$, the restriction to $Z$ of the locally free sheaf $F$ on $X$. Consider the chain

$$
\gamma := \oplus \gamma_i \in \text{Pol}_{p+1}(Y, \rho^* F_Z)
$$

For a point $y \in S$ different from $\sigma(x)$, each $\gamma_i$ is regular at $y$, because $y \notin \sigma(Z)$ and $\rho(y) \notin W$. Besides, locally at $\sigma(x)$ on $Y$, we have $\partial \gamma = \sigma_* \alpha$, where we consider $\alpha$ as an element in $\text{Pol}_p(Z, \mathcal{F}_Z) \subset \text{Pol}_p(X, \mathcal{F})$ and we use the identity $\sigma^*(\rho^* F_Z) = \mathcal{F}_Z$ and the morphism

$$
\sigma_* : \text{Pol}_p(Z, \mathcal{F}_Z) \to \text{Pol}_p(Y, \rho^* F_Z)
$$

from Proposition 4.16(i). Hence, $\partial \gamma = \sigma_* \alpha$ locally at $S$ on $Y$.

Now, from the chain $\gamma \in \text{Pol}_{p+1}(Y, \rho^* F_Z)$, we are going to construct a suitable chain in $\text{Pol}_{p+1}(Y, \psi^* \mathcal{F})$ and, then, take its push-forward by $\psi$ to $X$. This will give us the desired polar chain $\beta$ in $\text{Pol}_{p+1}(X, \mathcal{F})$. For this aim, let us consider a locally free sheaf $\mathcal{H}om(\rho^* F_Z, \psi^* \mathcal{F})$ on $Y$ and a coherent sheaf

$$
\mathcal{K} := \text{Ker}(\mathcal{H}om(\rho^* F_Z, \psi^* \mathcal{F}) \to \sigma_* \sigma^* \mathcal{H}om(\rho^* F_Z, \psi^* \mathcal{F})).
$$

By the Serre vanishing theorem, there exists a very ample invertible sheaf $\mathcal{L}$ on $Y$ such that

$$
H^1(Y, \mathcal{K} \otimes_{\mathcal{O}_Y} \mathcal{L}) = 0.
$$

In this case, the map

$$
H^0(Y, \mathcal{H}om(\rho^* F_Z, \psi^* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L})) \to H^0(Z, \sigma^* \mathcal{H}om(\rho^* F_Z, \psi^* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}))
$$

is surjective. Equivalently, the map

$$
\sigma^* : \text{Hom}(\rho^* F_Z, \psi^* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}) \to \text{Hom}(\mathcal{F}_Z, \mathcal{F}_Z \otimes_{\mathcal{O}_Z} \mathcal{L}|_Z)
$$

is surjective, where we used that

$$
\sigma^*(\rho^* F_Z) = \sigma^*(\psi^* \mathcal{F}) = \mathcal{F}_Z.
$$
Let $D \subset Y$ be the divisor of a general regular section of $\mathcal{L}$, and so $\mathcal{L} \cong \mathcal{O}_Y(D)$. Then, we may assume that $D \cap S = \emptyset$ and $\sigma^{-1}(D)$ is a Cartier divisor on $Z$. The surjectivity of the map $(\mathbf{8})$ implies that there exists a morphism of sheaves $\Phi : \rho^* \mathcal{F}_Z \to \psi^* \mathcal{F}(D)$ such that $\sigma^* \Phi$ coincides with the embedding $\Upsilon : \mathcal{F}_Z \to \mathcal{F}_Z(\sigma^{-1}(D))$. The following chain on $Y$,

$$
\Phi(\gamma) = \oplus \Phi(\gamma_i) \in \text{Pol}_{p+1}(Y, \psi^* \mathcal{F}(D)),
$$

satisfies $\partial(\Phi(\gamma)) = \Phi(\partial \gamma)$, because the polar complex is functorial in locally free sheaves. Recall that $\gamma$ was constructed so that $\partial \gamma = \sigma_* \alpha$ locally at $S$. Hence, we have $\Phi(\partial \gamma) = \Phi(\sigma_* \alpha)$ locally at $S$. Since the push-forward map from Proposition 4.6 $(i)$ is functorial in locally free sheaves on the target, we have that $\Phi(\sigma_* \alpha) = \sigma_* ((\sigma^* \Phi)(\alpha))$. Since $\sigma^* \Phi = \Upsilon$, we have that $\partial(\Phi(\gamma)) = \sigma_* (\Upsilon(\alpha))$ locally at $S$.

On the other hand, note that the space $\text{Pol}_{Y_i}(\psi^* \mathcal{F}|_{Y_i})$ consists of such elements in

$$
\text{Pol}_{Y_i}(\psi^* \mathcal{F}(D)|_{Y_i}) \subset \omega_{k(Y_i)} \otimes_{k(Y_i)} (\psi^* \mathcal{F}(D))_{k(Y_i)} = \omega_{k(Y_i)} \otimes_{k(Y_i)} (\psi^* \mathcal{F})_{k(Y_i)}
$$

that have at most a first order pole along $D$. Consequently, by the Bertini theorem (cf. Lemma 2.3), we may assume that $\Phi(\gamma_i)$ belongs to $\text{Pol}_{Y_i}(\psi^* \mathcal{F}|_{Y_i})$, whence $\Phi(\gamma) \in \text{Pol}_{p+1}(Y, \psi^* \mathcal{F})$ with $\partial \Phi(\gamma) = \sigma_* \alpha$ locally at $S$. Therefore, by Proposition 4.6 $(i)$, the chain

$$
\beta = \psi_* (\Phi(\gamma)) \in \text{Pol}_{p+1}(X, \mathcal{F})
$$

satisfies $\partial \beta = \alpha$ locally at $x \in X$. This completes the proof. \hfill \Box

### 4.5 A comparison with cohomology of locally free sheaves

In this section we prove Theorem 4.5 for the projective case. Thus, everywhere in this subsection we assume that $X$ is a smooth irreducible projective variety, while $\mathcal{G}$ is a locally free sheaf on $X$.

**Definition 4.15.** An open subset $U \subset X$ is called $\mathcal{G}$-small if there exists a smooth variety $\tilde{X}$ together with a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{D} : = \pi^{-1}(X \setminus U)$ is a simple normal crossing divisor in $\tilde{X}$ and for all $p > 0$, we have $H^p(\tilde{X}, \pi^! \mathcal{G}(\tilde{D})) = 0$.

**Remark 4.16.**

1. Suppose $U \subset X$ is a $\mathcal{G}$-small subset and choose any proper birational morphism $\tilde{\pi} : \tilde{X}' \to X$ such that $\tilde{D}' : = \tilde{\pi}^{-1}(X \setminus U)$ is a simple normal crossing divisor in $X'$. Then by Lemma 2.5 and Corollary 2.7 for all $p > 0$, we have that $H^p(\tilde{X}', \pi'^! \mathcal{G}(\tilde{D}')) = 0$.

2. It follows from (i) that if $U \subset X$ is $\mathcal{G}$-small and the complement $D := X \setminus U$ is a simple normal crossing divisor, then $H^p(X, \mathcal{G}(D)) = 0$ for all $p > 0$.

3. Let $U \subset X$ be a $\mathcal{G}$-small subset and $\pi : \tilde{X} \to X$ a proper birational morphism. Then it follows from (i) that the open subset $\pi^{-1}(U) \subset \tilde{X}$ is $\pi'^! \mathcal{G}$-small.
We will use the following notation. Given a finite open covering \( \{U_i\}, i \in I \), of an open subset \( U \subset X \) and a non-empty subset \( S \subset I \) we define \( U_S := \cap_{i \in S} U_i \), \( D_S := X \setminus U_S \), while, for \( S = \emptyset \), we set \( U_{\emptyset} := U \) and \( D_{\emptyset} := X \setminus U \).

First we establish a relation between Čech cohomology of a polar sheaf and cohomology of the underlying locally free sheaf.

**Lemma 4.17.** Let \( U \subset X \) be an open subset and suppose \( U = \cup_{i \in I} U_i \) is a finite open covering such that, for any non-empty subset \( S \subset I \), the open subset \( U_S \) is \( \mathcal{G} \)-small and, for any subset \( S \subset I \) (including \( S = \emptyset \)), the subvariety \( D_S \) is a simple normal crossing divisor in \( X \). Then, for any \( p \geq 0 \), we have

\[
\check{H}^p(\{U_i\}, \mathcal{G}_{\text{pol}}|_U) = H^p(X, \mathcal{G}(D_{\emptyset})),
\]

where \( \check{H} \) denotes the Čech cohomology groups associated with an open covering.

**Proof.** For each \( i \in I \), denote by \( D'_i \) the union of all irreducible components of \( D_i \) that are not contained in \( D_{\emptyset} \). *A priori*, \( \cap_i D'_i \subset D_{\emptyset} \), because \( U = \cup_i U_i \). We claim that, in fact, \( \cap_i D'_i = \emptyset \). Indeed, suppose \( Z \) is an irreducible component of the intersection \( \cap_i D'_i \).

Since the divisor \( \cup_i D'_i \) is a simple normal crossing one, the codimension of \( Z \) in \( X \) equals the number \( c \) of irreducible components in \( \cup_i D'_i \) that contain \( Z \). On the other hand, \( Z \subset D_{\emptyset} \), so there are at least \( c + 1 \) irreducible components in the simple normal crossing divisor \( \cup_i D_i \) that contain \( Z \), provided that \( D_{\emptyset} \) is non-empty. Hence the codimension of \( Z \) in \( X \) is at least \( c + 1 \), whence the contradiction.

Consider the following Koszul type complex of locally free sheaves on \( X \):

\[
\mathcal{K}^\bullet = \{0 \to \bigoplus_{i \in I} \mathcal{O}_X(D'_i) \to \bigoplus_{\{i,j\} \subset I} \mathcal{O}_X(D'_i \cup D'_j) \to \ldots \to \mathcal{O}_X(D'_I) \to 0\},
\]

where \( \mathcal{K}' := \bigoplus_{S \subset I, |S| = t} \mathcal{O}_X(D'_S), 0 \leq t \leq |I| - 1 \), and \( D'_S := \cup_{i \in S} D'_i \) for each subset \( S \subset I \). The differential in this complex is given by an alternating sum over the natural embeddings \( \mathcal{O}_X(D'_S) \to \mathcal{O}_X(D'_T) \) for \( S \subset T \subset I \), \( |T| = |S| + 1 \). Note that all divisors \( D'_S \) are reduced. Since \( \cap_i D'_i = \emptyset \), the only non-zero cohomology sheaf of the complex of sheaves \( \mathcal{K}^\bullet \) sits in degree zero and equals \( \mathcal{O}_X \). Let us form a tensor product of \( \mathcal{K}^\bullet \) with the sheaf \( \mathcal{G}(D_{\emptyset}) \), which yields now a resolution for \( \mathcal{G}(D_{\emptyset}) \). Recall that by condition of the lemma, \( D_{\emptyset} + D'_S = D_S \) and \( U_S \) is \( \mathcal{G} \)-small for any \( S \subset I \), \( S \neq \emptyset \). Hence by Remark 4.16(ii), \( H^p(X, \mathcal{G}(D_S)) = 0 \) for all \( p > 0 \), that is, the sheaves in the resolution \( \mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}(D_{\emptyset}) \) are acyclic and we have:

\[
H^p(X, \mathcal{G}(D_{\emptyset})) = H^p(H^0(X, \mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}(D_{\emptyset}))).
\]

It remains only to notice that the right hand side coincides by definition with the Čech cohomology groups \( \check{H}^p(\{U_i\}, \mathcal{G}_{\text{pol}}|_U) \). \( \square \)

We want to prove now that \( \check{H}^p(\{U_i\}, \mathcal{G}_{\text{pol}}) = H^p(X, \mathcal{G}_{\text{pol}}) \) for any \( \mathcal{G} \)-small open covering \( \{U_i\} \) of \( X \). This will follow once we prove that the sheaf \( \mathcal{G}_{\text{pol}} \) is acyclic on \( \mathcal{G} \)-small open subsets.

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Lemma 4.18. Let \( U \subset X \) be an open \( \mathcal{G} \)-small subset. Then for any \( p > 0 \), we have
\[
H^p(U, \mathcal{G}_{\text{pol}}|_U) = 0.
\]

Proof. We follow the idea of the proof of a well-known theorem of H. Cartan on acyclicity of coverings (see, e.g., [5, Section 1.4.5]). We use induction on \( p \geq 1 \). Consider an element \( \alpha \in H^p(U, \mathcal{G}_{\text{pol}}|_U) \). Any cocycle is locally exact. Thus, there exists a finite open covering \( U = \bigcup_{i \in I} U_i \) such that \( \alpha|_{U_i} = 0 \) for all \( i \in I \). By Theorem 4.4 for the projective case and equation (6) in Section 4.1, we have an isomorphism
\[
H^p(U, \mathcal{G}_{\text{pol}}|_U) \cong H^{d-p}(\text{Pol}_{\bullet}(X, \mathcal{F}) / \text{Pol}_{\bullet}(Z, \mathcal{F}|_Z)),
\]
where \( d \) is the dimension of \( X \), \( Z := X \setminus U \), and \( \mathcal{F} := \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{G} \). Therefore, by the Hironaka theorem and Proposition 4.6, we may suppose that every \( D_i = X \setminus U_i, i \in I \), is a divisor and \( D_I = \bigcup_{i \in I} D_i \) is a simple normal crossing divisor in \( X \). In particular, this implies that for any subset \( S \subset I \), the divisor \( D_S \) is a simple normal crossing divisor in \( X \).

Let \( \mathcal{L} \) be a very ample invertible sheaf on \( X \) such that for any finite subset \( S \subset I \) and for all \( p > 0, m > 0 \), we have that
\[
H^p(X, \mathcal{G}(D_S) \otimes_{\mathcal{O}_X} \mathcal{L}^m) = 0.
\]
By Lemma 4.17, there exist \( d + 1 \) regular sections of \( \mathcal{L} \) with irreducible divisors \( E_j \), \( j = 1, \ldots, d + 1 \) such that \( \cap_j E_j = \emptyset \), none of \( E_j \) is contained in \( D_I \), and the divisor \( D_I \cup (\cup_j E_j) \) is a simple normal crossing divisor in \( X \). In particular, this implies that for any subsets \( S \subset I \) and \( T \subset \{1, \ldots, d + 1\} \), the divisor \( D_S \cup (\cup_{j \in T} E_j) \) is a simple normal crossing divisor in \( X \).

Consider the covering \( U = \bigcup V_{ij} \), where the open sets \( V_{ij} := U_i \setminus E_j \) are \( \mathcal{G} \)-small by construction as well as their intersections. By ascending induction on \( p \), it follows from the Čech spectral sequence that there is an exact sequence
\[
0 \to H^p(\{V_{ij}\}, \mathcal{G}_{\text{pol}}|_U) \to H^p(U, \mathcal{G}_{\text{pol}}|_U) \to \bigoplus_{ij} H^p(V_{ij}, \mathcal{G}_{\text{pol}}|_{V_{ij}}).
\]
By construction, the covering \( \{V_{ij}\} \) of \( U \) satisfies the condition of Lemma 4.17. Hence the left term equals \( H^p(X, \mathcal{G}(D_\emptyset)) \), where \( D_\emptyset = X \setminus U \). By Remark 4.16(ii), since \( U \) is \( \mathcal{G} \)-small, the left term vanishes. On the other hand, the right arrow sends any \( \alpha \in H^p(U, \mathcal{G}_{\text{pol}}|_U) \) to zero. Therefore, \( \alpha = 0 \) and this gives the needed result.

We are able now to prove Theorem 4.5 for the projective case.

Proof of Theorem 4.5 for the projective case. We are going to apply Lemma 4.17 with \( U = X \) and, thus, \( D_\emptyset = \emptyset \). By the same application of Lemma 2.3 as in the proof of Lemma 4.18, there exists an open covering \( \{U_i\} \) of \( X \) that satisfies the conditions of Lemma 4.17. Combining Lemma 4.17 and Lemma 4.18, we get the needed result.
Remark 4.19. It follows from Lemma 4.17 and the proof of Theorem 4.5 for a smooth projective $X$ that for an open subset $U \subset X$ with $D := X \setminus U$ being a simple normal crossing divisor in $X$ and for all $p \geq 0$, we have that

$$H^p(U, G_{pol}|_U) = H^p(X, G(D)).$$

Then, by setting $G := \omega_X \otimes \mathcal{O}_X \mathcal{F}$ and applying Theorem 4.4 for the projective case and Definition 4.1, we obtain that

$$H^{d-p}(X, \omega_X \otimes \mathcal{O}_X \mathcal{F}(D)) = H_p \left( \text{Pol}_\bullet(X, \mathcal{F}) / \text{Pol}_\bullet(D, \mathcal{F}|_D) \right).$$

4.6 A reduction to the projective case

In this section we prove Theorem 4.4 and Theorem 4.5 in full generality, that is, when $X$ is a smooth irreducible quasi-projective variety. As a matter of fact, this more general case can be reduced to the projective case.

The following fact was explained to the authors by D. Orlov.

Lemma 4.20. Let $X$ be a quasi-projective variety and $\mathcal{F}$ a locally free sheaf on $X$. Then there exists a projective variety $\bar{X}$, an open embedding with dense image $X \hookrightarrow \bar{X}$, and a locally free sheaf $\bar{\mathcal{F}}$ on $\bar{X}$ such that $\bar{\mathcal{F}}|_X \cong \mathcal{F}$. If $X$ is smooth, then one can additionally require that $\bar{X}$ is smooth and the complement $D := \bar{X} \setminus X$ is a simple normal crossing divisor.

Proof. Let $\mathcal{O}_X(1)$ be a very ample sheaf on $X$ corresponding to a locally closed embedding $X \hookrightarrow \mathbb{P}^m$ for some $m$ and let $r$ be the rank of $\mathcal{F}$. For some natural numbers $n$ and $N$, there is a surjective morphism $\mathcal{O}_X^\oplus n \to \mathcal{F}(N)$ of sheaves on $X$. The latter defines a locally closed embedding, $g : X \hookrightarrow G$, into the Grassmannian $G := \text{Gr}(r, n)$ of dimension $r$ subspaces in a dimension $n$ vector space. Besides, $\mathcal{F}(N) \cong g^* \mathcal{E}$, where $\mathcal{E}$ is the dual of the tautological bundle on the Grassmannian. Consider the locally closed embedding

$$X \hookrightarrow G \times \mathbb{P}^m$$

and the closure of its image, $\bar{X}$. It remains to set $\bar{\mathcal{F}} := p_1^* \mathcal{E} \otimes p_2^* \mathcal{O}_{\mathbb{P}^m}(-N)$, where $p_1$ and $p_2$ denote the projections from $\bar{X}$ to $G$ and $\mathbb{P}^m$, respectively. If $X$ is smooth, the additional condition on $\bar{X}$ and $D$ can be achieved by the Hironaka theorem. \[\square\]

We also need the following result.

Proposition 4.21. Let $X$ be an open subvariety in a smooth projective variety $\bar{X}$ such that the complement $D := \bar{X} \setminus X$ is a simple normal crossing divisor in $\bar{X}$. Suppose $\bar{\mathcal{F}}$ is a locally free sheaf on $\bar{X}$ such that $\mathcal{F} := \bar{\mathcal{F}}|_X$. Then we have

$$\mathcal{F}_{pol} = \lim_{n \to \infty} \bar{\mathcal{F}}(nD)_{pol}|_X,$$

$$\text{Pol}_\bullet(X, \mathcal{F}) = \lim_{n \to \infty} \text{Pol}_\bullet(X, \mathcal{F}(nD))|_X.$$
Proof. To show the first equality let us choose an arbitrary open subset $U$ in $X$ and compare sections of both sheaves on $U$. By the Hironaka theorem, there exists a smooth projective variety $\bar{Y}$ and a proper birational morphism $\bar{\pi} : \bar{Y} \to \bar{X}$ such that $\bar{\pi}$ is an isomorphism over $U$ and the following divisors in $\bar{Y}$ are simple normal crossing: $\bar{D} := \bar{\pi}^{-1}(D)$, the closure $E$ of $\bar{\pi}^{-1}(X \setminus U)$ in $\bar{Y}$, and $\bar{\pi}^{-1}(X \setminus U) = E \cup \bar{D}$. We set $Y := \bar{\pi}^{-1}(X) = \bar{Y} \setminus \bar{D}$ and $\pi := \bar{\pi}|_Y$. Below, we use that for a variety $V$, a Cartier divisor $W \subset V$, and a coherent sheaf $G$ on $V$, one has that $H^0(V \setminus W, G|_{V \setminus W}) = \lim_{\to} H^0(V, G(nW))$. Thus, we have

$$F_{pol}(U) = H^0(Y, \pi^1 F \otimes O_Y (\pi^{-1}(X \setminus U))) = \lim_{\to} H^0(\bar{Y}, \bar{\pi}^1 \bar{F} \otimes \bar{O}_{\bar{Y}} (E \otimes O_Y (\pi^{-1}(n\bar{D})))) =$$

$$= \lim_{\to} H^0(\bar{Y}, \bar{\pi}^1 \bar{F} \otimes \bar{O}_{\bar{Y}} (n\bar{D}) \otimes \bar{O}_{\bar{Y}} (E \cup \bar{D})) = \lim_{\to} H^0(\bar{Y}, \bar{\pi}^1 \bar{F} \otimes \bar{\pi}^* \bar{O}_\bar{X} (nD) \otimes \bar{O}_{\bar{Y}} (E \cup \bar{D})) =$$

$$= \lim_{\to} H^0(\bar{Y}, \bar{\pi}^1 \bar{F} (nD)) \otimes \bar{O}_{\bar{Y}} (\bar{\pi}^{-1}(\bar{X} \setminus U))) = \lim_{\to} \bar{F}(nD)_{pol}(U),$$

which proves the first equality.

For the second equality it is enough to show that for any irreducible closed subvariety $Z \subset \bar{X}$ with non-empty $Z := X \cap \bar{Z}$, we have

$$\text{Pol}_Z (F|_Z) = \lim_{\to} \text{Pol}_Z (F(nD)|_Z).$$

To prove the latter, recall that, by definition, for each $n$, the space $\text{Pol}_Z (F(nD)|_Z)$ is the union in $\omega_{k(Z)} \otimes_{k(Z)} F_{k(Z)}$ of the subspaces

$$H^0(\bar{V}, \omega_{\bar{V}} (\bar{W}) \otimes_{\bar{O}_{\bar{V}}} \bar{f}^* \bar{F}(nD)|_Z)$$

over all collections $(\bar{V}, \bar{f}, \bar{W})$, where $\bar{V}$ is a smooth variety, $\bar{f} : \bar{V} \to \bar{Z}$ is a proper birational morphism, and $\bar{W} \subset \bar{V}$ is a simple normal crossing divisor in $\bar{V}$. Furthermore, by the Hironaka theorem and Corollary 2.7, we may consider only collections $(\bar{V}, \bar{f}, \bar{W})$ such that $C := \bar{f}^{-1}(\bar{Z} \setminus \bar{Z}) = \bar{f}^{-1}(\bar{Z} \cap D)$ and $C \cup \bar{W}$ are simple normal crossing divisors in $\bar{V}$. On the other hand, by Lemma 2.2 and Corollary 2.7, the group $\text{Pol}_Z (F|_Z)$ is equal to the union in $\omega_{k(Z)} \otimes_{k(Z)} F_{k(Z)}$ of the groups

$$H^0(V, \omega_V (W) \otimes_{O_V} f^* F|_Z)$$

over all collections $(\bar{V}, \bar{f}, \bar{W})$ as above, where $V := \bar{f}^{-1}(Z)$, $f := \bar{f}|_V$, and $W := \bar{W} \cap V$. Analogously to what was done for the first equality, we have

$$H^0(V, \omega_V (W) \otimes_{O_V} f^* F|_Z) = \lim_{\to} H^0(\bar{V}, \omega_{\bar{V}} (\bar{W}) \otimes_{\bar{O}_{\bar{V}}} \bar{f}^* \bar{F}|_Z \otimes_{\bar{O}_{\bar{V}}} \bar{O}_{\bar{V}} (nC)) =$$

$$= \lim_{\to} H^0(\bar{V}, \omega_{\bar{V}} (\bar{W}) \otimes_{\bar{O}_{\bar{V}}} \bar{f}^* \bar{F}(nD)|_Z),$$

which proves the second equality. \qed
Recall that for $X$, $\bar{X}$, $D$ as in Proposition $4.21$ and for any $p \geq 0$, we have

$$H^p(X, F) = \lim_{\to} H^p(\bar{X}, \bar{F}(nD)),$$

because filtered direct limits commute with cohomology of complexes. Therefore, combining Lemma $4.20$, Proposition $4.21$ and Remark $4.19$, we complete the reduction to the projective case and, thus, finally prove Theorems $4.4$ and $4.5$.

5 Further properties of polar chains

5.1 A relation with Rost’s cycle modules

In the paper [22] M. Rost introduced the so-called cycle modules. Roughly speaking, a cycle module over a variety $X$ is a rule that, to each field $(K, \varphi)$ over $X$, associates a graded abelian group $M(K, \varphi)$. The groups $M(K, \varphi)$ should be equipped with pull-back, trace, and residue maps with respect to fields over $X$, which satisfy various natural properties. Let us consider the groups $M_q(F)(K, \varphi)$ introduced in Definition $4.8$ and set

$$M(F)(K, \varphi) := \bigoplus_{q \geq 0} M_q(F)(K, \varphi).$$

Proposition $4.11$ is equivalent to saying that we have in this way obtained a cycle module:

**Proposition 5.1.** Let $F$ be a locally free sheaf on a variety $X$. Then, the correspondence

$$(K, \varphi) \mapsto M(F)(K, \varphi)$$

extends to a cycle module over the variety $X$, where $(K, \varphi)$ runs over all fields over $X$.

Indeed, Proposition $4.11(i)$-$\text{(vi)}$ establishes the existence of all the data needed to define a cycle module over $X$, while Proposition $4.11(\text{vi})$ together with general properties of the pull-back, trace, and residue maps for logarithmic forms (see Section 2) imply Rost’s axioms obeyed by the data above.

Until the end of this subsection we assume that $X$ is smooth and projective. Then by Proposition $4.11(\text{viii})$, the cycle module $M(O_X)$ is the pull-back of the cycle module $M(O_{\text{pt}})$ over a point $\text{pt} = \text{Spec}(k)$: for any field $(K, \varphi)$ over $X$, we have

$$M(O_X)(K, \varphi) = M(O_{\text{pt}})(K, \varphi_K),$$

where $\varphi_K : \text{Spec}(K) \to \text{Spec}(k)$ is the canonical morphism. Recall that Theorem 6.1 in [22] asserts local exactness of the Gersten complex associated with any cycle module over a point. This implies Theorem $4.4$ for this particular case. However, note that an analogue of Theorem $4.4$ is not true for an arbitrary cycle module over $X$.

In order to prove Theorem $4.5$ for $G = \omega_X$ within this approach one uses one more cycle module over a point:

$$H(K, \varphi_K) := \lim_{\to} \bigoplus_{q \geq 0} H^q_{dR}(U),$$

27
Polar chains give a new construction for the classes of algebraic cycles in the groups $\text{H}^{dR}(X)$, where $X$ is a smooth irreducible variety of dimension $p$. This cycle module was previously considered by S. Bloch and A. Ogus in [17], where the authors also proved local exactness of the corresponding Gersten complex for any smooth $X$. Moreover, they constructed a spectral sequence that converges to $\text{H}^{dR}(X)$ with $E^1$-term being formed by homogeneous summands of this Gersten complex.

Below we use some facts from mixed Hodge theory (see [6]). One has a decreasing Hodge filtration $F^pH$ on the cycle module $H$ with

$$F^pH(K, \varphi_K) := \lim_{U \to k} \bigoplus_{q \geq 0} F^{p+q}H^q_{dR}(U),$$

where $r := \dim U$. An important fact is that all (higher) differentials in the Bloch–Ogus spectral sequence are strict with respect to the Hodge filtration. (By Lefschetz principle, one can assume that $k = \mathbb{C}$, in which case de Rham cohomology are endowed with mixed Hodge structures, which form an abelian category with morphisms being strict with respect to the Hodge filtration.)

For $X$ smooth and projective, application of $F^0$ to the Bloch–Ogus spectral sequence kills all lines except one, which gives the complex $\text{Pol}_\bullet(X, \mathcal{O}_X)$. (The reason is that for a smooth affine variety $U$ of dimension $r$, we have that $F^q H^q_{dR}(U) = 0$ unless $q = r$, while $F^q H^q_{dR}(U) = (\omega_U)^{\text{pol}}(U)$, where $U$ is any smooth compactification of $U$.) This proves again Theorem 3.4 for $\mathcal{F} = \mathcal{O}_X$ and, moreover, shows that

$$F^d H^{p+d}_{dR}(X) = H^p(X, (\omega_X)^{\text{pol}}).$$

On the other hand, the degeneration of the Hodge spectral sequence implies that $F^d H^{p+d}_{dR}(X) = H^p(X, (\omega_X))$. Altogether, this implies Theorem 3.4 for $\mathcal{G} = \omega_X$. Thus, cycle modules lead to another proof of Theorem 3.4 for smooth projective $X$ and $\mathcal{F} = \mathcal{O}_X$, that is, the main theorem in [17].

### 5.2 Algebraic cycles

Polar chains give a new construction for the classes of algebraic cycles in the groups $H^q(X, \Omega^p_X)$, where $X$ is a smooth variety. Namely, let $Z$ be an irreducible subvariety of dimension $p$ in a smooth irreducible variety $X$ of dimension $d$ and let $f : V \to X$ be a proper morphism birational onto its image with smooth $V$ and $f(V) = Z$. The differential of $f$ defines a canonical section $\alpha_Z$ of the following bundle on $V$:

$$\text{Hom}(\mathcal{T}_V^p, f^* \mathcal{T}_X^p) = \omega_V \otimes \mathcal{O}_V f^* \mathcal{T}_X^p,$$

where $\mathcal{T}_X^p := \wedge^p \mathcal{T}_X$. The regular section $\alpha_Z$ is thus an element in

$$\text{Pol}_Z(\mathcal{T}_X^p|_Z) \subset \text{Pol}_p(X, \mathcal{T}_X^p).$$

Obviously, $\partial \alpha_Z = 0$. By Theorem 3.4, the homology class of $\alpha_Z$ in the complex $\text{Pol}_\bullet(X, \mathcal{T}_X^p)$ defines an element in $H^{d-p}(X, \Omega^{d-p}_X)$, because $\omega_X \otimes \mathcal{O}_X \mathcal{T}_X^p \cong \Omega^{d-p}_X$ and

$$H_p(\text{Pol}_\bullet(X, \mathcal{T}_X^p)) \cong H^{d-p}(X, \Omega^{d-p}_X).$$
By linearity, this defines an element in $H^{d-p}(X, \Omega_X^{d-p})$ for any algebraic $p$-cycle on $X$ with coefficients in $k$.

It follows that the class of an algebraic cycle $\sum_i c_i Z_i$ on $X$, $c_i \in k$, is trivial in $H^{d-p}(X, \Omega_X^{d-p})$ if and only if the corresponding polar cycle is a polar boundary. The latter means that there exists a chain $\beta \in \text{Pol}_{p+1}(X, T^p_X)$ such that $\partial \beta = \bigoplus (c_i \alpha_{Z_i})$ with $\alpha_{Z_i}$ defined as above. In explicit terms, there exists a collection of $(p+1)$-dimensional irreducible subvarieties $Y_j \subset X$ and polar elements $\beta_j \in \omega_{k(Y_j)} \otimes_{k(Y_j)} (T^p_X|_{Y_j})_{k(Y_j)}$ such that, for each $i$, we have that $\sum_j \text{res}_{Y_j Z_i}(\beta_j) = c_i \alpha_{Z_i}$, where the sum is taken over $Y_j$ such that $Z_i \subset Y_j$.

Furthermore, by Theorem 3.34 an arbitrary element in $H^{d-p}(X, \Omega_X^{d-p})$ can be represented by a polar chain

$$\gamma = \bigoplus \gamma_Z \in \text{Pol}_p(X, T^p_X)$$

with polar elements $\gamma_Z \in \omega_{k(Z)} \otimes_{k(Z)} (T^p_X|_Z)_{k(Z)}$. Then, the class of $\gamma$ in $H^{d-p}(X, \Omega_X^{d-p})$ is represented by an algebraic cycle if and only if $\gamma$ is homologous in the polar complex $\text{Pol}_*(X, T^p_X)$ to a chain $\gamma' = \bigoplus \gamma'_Z$, where each $\gamma'_Z$ is a multiple of a distinguished element $\alpha_Z$ in $\omega_{k(Z)} \otimes_{k(Z)} (T^p_X|_Z)_{k(Z)}$ defined above.

5.3 The Cousin complex

The classical Cousin problem consists in finding a rational (or meromorphic) section of a locally free sheaf with a given principle part. Similarly, the polar complex discussed in this paper corresponds to a problem of finding a logarithmic form (with coefficients in a locally free sheaf) which possesses given residues at its first order poles. In this subsection we expand on a relation between the polar complex and the Cousin complex.

Let $X$ be a smooth irreducible quasi-projective variety of dimension $d$. For an irreducible subvariety $Z \subset X$ and a sheaf of abelian groups $\mathcal{P}$ on $X$, denote by $\gamma_Z \mathcal{P}$ the subsheaf of $\mathcal{P}$ that consists of sections of $\mathcal{P}$ supported on $Z$. Consider $\gamma_Z \mathcal{P}$ as a sheaf on $Z$. The functor $\gamma_Z$ from the sheaves on $X$ to the sheaves on $Z$ is left exact. Let $\eta_Z$ denote the generic point of $Z$. The local cohomology groups $H^i_{\eta_Z}(X, \mathcal{P})$ are defined as fibers at $\eta_Z$ of the right derived sheaves $R^i \gamma_Z \mathcal{P}$:

$$H^i_{\eta_Z}(X, \mathcal{P}) := (R^i \gamma_Z \mathcal{P})_{\eta_Z}.$$ 

To every sheaf $\mathcal{P}$ one can associate a Cousin complex $\text{Cous}_*(X, \mathcal{P})$ (see [12]). We will use the homological form of this, where the terms of the complex receive the following form:

$$\text{Cous}_p(X, \mathcal{P}) = \bigoplus_{Z \in X^{(i)}} H^{d-p}_{\eta_Z}(X, \mathcal{P}).$$

Let $\mathcal{G}$ be a locally free sheaf on $X$. In this case, the groups $H^i_{\eta_Z}(X, \mathcal{G})$ vanish unless $i = d - p$, where $p = \dim Z$, while for $i = d - p$, we have that

$$H^{d-p}_{\eta_Z}(X, \mathcal{G}) = \mathcal{E}_{\gamma_Z}(\omega_X^{-1} \otimes \mathcal{G} |_{\gamma_Z} \otimes \omega_{Z})_{k(Z)}.$$
where \( E_{\mathcal{O}_{X,Z}}(M) \) denotes the injective hull of an \( \mathcal{O}_{X,Z} \)-module \( M \). Moreover, there is a canonical isomorphism (cf. [12]):

\[
H^{d-p}(X, \mathcal{G}) \cong H_p(\text{Cous}_\bullet(X, \mathcal{G})).
\]

(9)

Back to polar complexes, Theorem 4.4 provides a resolution \( \text{Pol}_\bullet(X, \mathcal{F}) \) for the sheaf \((\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}\). The explicit form of \( \text{Pol}_\bullet(X, \mathcal{F}) \) as of a direct sum over irreducible subvarieties allows one to show that for any \( p \)-dimensional irreducible subvariety \( Z \) in \( X \), the local cohomology groups \( H^i_{\eta Z}(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}) \) vanish unless \( i = d - p \), where \( p = \dim Z \), while for \( i = d - p \), we have that

\[
H^{d-p}_{\eta Z}(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}) = \text{Pol}_Z(\mathcal{F}|_Z).
\]

This implies that the Cousin complex of the sheaf \((\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}\) coincides with the polar complex of \( \mathcal{F} \):

\[
\text{Cous}_\bullet(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}) = \text{Pol}_\bullet(X, \mathcal{F}).
\]

(10)

An injective morphism of sheaves, \((\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}} \hookrightarrow (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})\), induces a morphism of the corresponding Cousin complexes, \( \text{Cous}_\bullet(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}) \rightarrow \text{Cous}_\bullet(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \). Thus, by the equation (10), we have an injection,

\[
\text{Pol}_\bullet(X, \mathcal{F}) \hookrightarrow \text{Cous}_\bullet(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}),
\]

(11)

which is assembled from the injective maps

\[
\text{Pol}_Z(\mathcal{F}|_Z) \hookrightarrow (\omega_Z \otimes_{\mathcal{O}_Z} \mathcal{F}|_Z)_{k(Z)} \hookrightarrow E_{\mathcal{O}_{X,Z}}((\omega_Z \otimes_{\mathcal{O}_Z} \mathcal{F}|_Z)_{k(Z)}).
\]

Equations (9) and (10) together with Theorem 3.4 imply now that (11) is a quasiisomorphism and one can say that, for a locally free sheaf, the Cousin complex is quasiisomorphic to its own first order pole part.

As an example consider a smooth curve \( X \) and choose \( \mathcal{F} = \mathcal{O}_X \) (cf. Example 3.6). Then,

\[
\text{Pol}_\bullet(X, \omega_X) = \{0 \rightarrow \text{Pol}(X, \Omega^1_X) \rightarrow \bigoplus_{x \in X} k(x) \rightarrow 0\},
\]

\[
\text{Cous}_\bullet(X, \omega_X) = \{0 \rightarrow \Omega^1_{k(X)} \rightarrow \bigoplus_{x \in X} \Omega^1_{k(X)}/\Omega^1_{\mathcal{O}_{X,z}} \rightarrow 0\},
\]

where \( \text{Pol}(X, \Omega^1_X) \) consists of rational 1-forms that have only first order poles on \( X \) and for each point \( x \in X \), the first order pole part of \( \Omega^1_{k(X)}/\Omega^1_{\mathcal{O}_{X,z}} \) is identified with \( k(x) \) by the residue map.

**Remark 5.2.** It might be possible to prove Theorem 3.4 in the same way as one proves that the Cousin complex \( \text{Cous}_\bullet(X, \mathcal{G}) \) computes cohomology groups \( H^\bullet(X, \mathcal{G}) \) for a locally free sheaf \( \mathcal{G} \) on \( X \) (cf. [12]). This could be achieved if, for each closed subset \( i : Z \hookrightarrow X \), one would be able to replace the “topological” functor \( \gamma_Z \) relevant for \( \text{Cous}_\bullet \) by the “algebro-geometric” functor \( i^! \) relevant for \( \text{Pol}_\bullet \).
5.4 The adelic complex

A. N. Parshin [19] introduced a certain complex $A_{rat}(X, F)^{•}$ for a quasi-coherent sheaf $F$ on a surface $X$, nowadays called the rational adelic complex. Generalizations to higher dimensions were studied in a number of papers, in particular, see [20], [2], and [14]. One of the main properties of the rational adelic complex $A_{rat}(X, F)^{•}$ is that it gives a flabby resolution for any quasi-coherent sheaf $F$ on $X$. In particular, there is canonical isomorphism $H^p(X, F) \cong H^p(A_{rat}(X, F)^{•})$. For a smooth complex algebraic variety, there are several analogies between the adelic resolution and Dolbeault resolution (see, e.g., [15]). Although any two resolutions can be related with help of the canonical Godement construction, this is by no means explicit. In this somewhat sketchy subsection we describe a construction of a chain of quasiisomorphisms between the adelic and Dolbeault resolutions for smooth complex projective varieties, such that each term of the chain is represented by an explicit geometric construction.

A version of adelic complexes for sheaves of abelian groups was defined in [8] (a different definition, appropriate to a more general situation, is given in [3]). Let us briefly describe the shape of this adelic complex. Suppose that $X$ is an irreducible variety and $P$ is a sheaf of abelian groups on $X$. A non-degenerate flag on $X$ is a strictly decreasing chain of irreducible subvarieties in $X$:

$$Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_p.$$  

Each term of the adelic complex $A(X, P)^{•}$ is by definition a certain subgroup,

$$A(X, P)^p \subset \prod_{(Z_0 \ldots Z_p)} P_{\eta_0},$$

in the direct product of fibers $P_{\eta_0}$ of the sheaf $P$, where the product is taken over non-degenerate flags $(Z_0 \ldots Z_p)$ in $X$, while each fiber $P_{\eta_0}$ is taken at the generic point $\eta_0 \in Z_0$ of the largest elements $Z_0$ in a flag. Thus, an element $f \in A(X, P)^p$ is a collection of elements $f_{Z_0 \ldots Z_p} \in P_{\eta_0}$ subject to a certain condition, which we do not specify explicitly here. The differential in the adelic complex is given by the formula

$$d(f)_{Z_0 \ldots Z_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{Z_0 \ldots \hat{Z}_i \ldots Z_{p+1}} \in P_{\eta_0},$$

where the hat means that we omit the corresponding element in a flag.

For example, if $X$ is a smooth curve, we have

$$A(X, P)^{•} = \{ 0 \to P_{\eta} \oplus \prod_{x \in X} P_x \to \prod'_{x \in X} P_{\eta} \to 0 \},$$

where $\eta$ denotes the generic point of $X$ and the symbol $\prod'$ means that we consider the set of collections

$$f = (f_x) \in \prod_{x \in X} P_\eta$$

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such that \( f_x \in \text{Im}(\mathcal{P}_x \to \mathcal{P}_\eta) \) for almost all \( x \in X \). The latter condition explains the name “adelic” for the complex \( \mathbf{A}(X, \mathcal{P})^\bullet \).

It is proved in [8] that for a smooth \( X \) and a wide range of sheaves \( \mathcal{P} \) on \( X \), the adelic complex \( \mathbf{A}(X, \mathcal{P})^\bullet \) gives a flabby resolution of \( \mathcal{P} \). This seems to extend to the case \( \mathcal{P} = \mathcal{G}_{\text{pol}} \), where \( \mathcal{G} \) is a locally free sheaf; to achieve this one should combine methods from [8] with the last part of the proof in Section 4.4. Thus, for a smooth variety \( X \), we may assume that the following isomorphism holds:

\[
H^i(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}}) \cong H^i(\mathbf{A}(X, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pol}})^\bullet).
\]

On the other hand, it follows from the definitions of the rational adelic complex \( \mathbf{A}_{\text{rat}}(X, \mathcal{G})^\bullet \) and the adelic complex \( \mathbf{A}(X, \mathcal{P})^\bullet \) with \( \mathcal{P} = \mathcal{G}_{\text{pol}} \) that there is a canonical injective morphism of complexes

\[
\mathbf{A}(X, \mathcal{G}_{\text{pol}})^\bullet \hookrightarrow \mathbf{A}_{\text{rat}}(X, \mathcal{G})^\bullet,
\]

which agrees with the embedding of sheaves \( \mathcal{G}_{\text{pol}} \hookrightarrow \mathcal{G} \). The above assumption implies that this is a quasi-isomorphism.

For an arbitrary sheaf of abelian groups \( \mathcal{P} \), there is a morphism of complexes from the adelic complex to the Cousin complex. Let us consider the case when \( \mathcal{P} = \mathcal{G}_{\text{pol}} \) as above. Then, for each \( p \), \( 0 \leq p \leq d = \dim X \), there is a surjective map

\[
\nu : \mathbf{A}(X, \mathcal{G}_{\text{pol}})^p \to \text{Cous}_{d-p}(X, \mathcal{G}_{\text{pol}}) = \text{Pol}_{d-p}(X, \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{G})
\]

defined by the formula

\[
\nu(f) = \left(-1\right)^{\frac{p(p+1)}{2}} \sum_{(XZ_1 \cdots Z_{p-1}Z)} (\text{res}_{Z_{p-1}Z} \circ \cdots \circ \text{res}_{XZ_1}) (f_{XZ_1 \cdots Z_{p-1}Z}) \in \text{Pol}_Z((\omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{G})|_Z),
\]

where \( Z \subset X \) is an irreducible subvariety of codimension \( p \) in \( X \) and \( (XZ_1 \ldots Z_{p-1}Z) \) are complete non-degenerate flags between \( X \) and \( Z \). Here we use that \( f_{XZ_1 \cdots Z_{p-1}Z} \) belongs to \( \mathcal{P}_\eta = \text{Pol}_X(\omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{G}) \), where \( \eta \) is the generic point of \( X \). We have in fact a surjective morphism of complexes

\[
\nu : \mathbf{A}(X, \mathcal{G}_{\text{pol}})^\bullet \to \text{Pol}_{d-\bullet}(X, \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{G}).
\]

The above discussion, together with Theorem 3.4, implies that this morphism is a quasi-isomorphism.

On the other hand, if we choose \( k = \mathbb{C} \), we can consider, on a smooth variety \( X \) as on any complex manifold, the following natural complexes constructed with help of the Dolbeault \( \bar{\partial} \)-operator. Let \( \Lambda^{0,p}(X, \mathcal{G}) \) be the space of \( C^\infty \) differential \((0, p)\)-forms on \( X \) with coefficients in the holomorphic vector bundle corresponding to \( \mathcal{G} \). Then, the \( \bar{\partial} \)-operator, \( \bar{\partial} : \Lambda^{0,p}(X, \mathcal{G}) \to \Lambda^{0,p+1}(X, \mathcal{G}) \), gives rise to a complex \( \Lambda^{0,\bullet}(X, \mathcal{G}) \), which computes the cohomology groups of the sheaf associated with \( \mathcal{G} \) in the analytic topology. One can also consider differential forms with distributional coefficients, or currents, \( D^{0,p}(X, \mathcal{G}) \), which
form a complex as well, \( \bar{\partial} : D^{0,p}(X, G) \to D^{0,p+1}(X, G) \). A well-known property of these complexes (see [11]) is that the obvious embedding,

\[ \Lambda^0\dot{\bullet}(X, G) \hookrightarrow D^0\dot{\bullet}(X, G), \]

is in fact a quasiisomorphism. The elements of the space of currents \( D^{0,p}(X, G) \) are, by definition, certain linear functionals on \( \Lambda^0_{c,d-p}(X, \omega_X \otimes \mathcal{O}_X \mathcal{G}^{\vee}) \), the space of \( C^\infty \) differential forms on \( X \) with compact support, where \( \mathcal{G}^{\vee} \) is dual to \( \mathcal{G} \). Recall now that a polar \( r \)-chain \( \alpha \in \text{Pol}_Z(X, F) \subset \text{Pol}_r(X, F) \), where \( Z \subset X \) is an \( r \)-dimensional irreducible subvariety in \( X \), can also be used to define a linear functional on \( \Lambda^0_{c,r}(X, F^{\vee}) \): take a \((0,r)\)-form \( u \in \Lambda^0_{c,r}(X, F^{\vee}) \), restrict it to \( Z \), and integrate the product of \((2\pi i)^{-r}\alpha \) and \( u|_Z \) over \( Z \) (it is important here that \( \alpha \) has only first order poles, for these are locally integrable singularities). This gives us yet another embedding of complexes (considered also in [16]):

\[ \text{Pol}_{d-\bullet}(X, \omega_X^{-1} \otimes \mathcal{O}_X \mathcal{G}) \hookrightarrow D^0\dot{\bullet}(X, G). \]

Collecting all above together, we obtain the following chain of morphisms of complexes:

\[ A_{\text{rat}}(X, G)^\bullet \leftarrow A(X, G_{\text{pol}})^\bullet \rightarrow \text{Pol}_{d-\bullet}(X, \omega_X^{-1} \otimes \mathcal{O}_X \mathcal{G}) \hookrightarrow D^0\dot{\bullet}(X, G) \hookrightarrow \Lambda^0\dot{\bullet}(X, G). \] (12)

In the case when \( X \) is a smooth projective variety, the analytic and Zariski cohomology groups of \( G \) are the same. Therefore, all complexes in (12) have the same cohomology groups and we obtain a chain of quasiisomorphisms connecting the rational adelic complex \( A_{\text{rat}}(X, G)^\bullet \) and the Dolbeault complex \( \Lambda^0\dot{\bullet}(X, G) \).

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