On the role of coherent attacks in a type of strategic problem related to quantum key distribution

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Abstract

We consider a strategic problem of the Evesdropping to quantum key distribution. Evesdropper hopes to obtain the maximum information given the disturbance to the qubits is often For this strategy, the optimized individual attack have been extensively constructed under various conditions. However, it seems a difficult task in the case of coherent attack, i.e., Eve may treat a number of intercepted qubits collectively, including the collective unitary transformations and the measurements. It was conjectured by Cirac and Gisin that no coherent attack can be more powerful for this strategy for BB84 protocol. In this paper we give a general conclusion on the role of coherent attacks for the strategy of maximizing the information given the disturbance. Suppose in a quantum key distribution(QKD) protocol, all the transmitted bits from Alice are independent and only the individual disturbances to each qubits are examined by Alice and Bob. For this type of protocols( so far almost all QKD protocols belong to this type), in principle no coherent attack is more powerful than the product of optimized individual attack to each individual qubits. All coherent attacks to the above QKD protocols can be disregarded for the strategy above.

Since Bennett and Brassard \(\cite{1}\) suggested their quantum key distribution protocol(BB84 protocol) in 1984, the subject has been extensively studied both theoretically and experimentally. The protocol allows two remote parties Alice and Bob to create and share a secret key using a quantum channel and public authenticated communications. The quantum key created in this way is in principle secure because eavesdroppers have no way to tap the quantum channel without disturb it. In the protocol, \(k\) independent qubits( such as photons) \(|Q_A\rangle\) are first prepared by Alice, each one is randomly chosen from a set of states \(V\). In BB84 scheme \(V = \{ |0\rangle, |1\rangle, |\bar{0}\rangle, |\bar{1}\rangle \}\), \(|0\rangle, |1\rangle\) are bases of \(Z\) and \(|\bar{0}\rangle, |\bar{1}\rangle\) are bases of \(X\). According to each

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individual states, she writes down a string of classical bits, $S_{0A}$. She then sends them to Bob. Bob measures each individual qubits in basis randomly chosen from $Z$ or $X$. For those Bob has happened to choose the correct bases, the results should be identical to the corresponding bits in $S_{0A}$. Alice and Bob discard the bit values where Bob measures in a wrong basis. After that, Alice has a string $S_A$ and Bob has a string $S'_A$. The shared secret key for Alice and Bob can be built up based on this. In the case of noiseless channel without Eve, $S_A$ should be identical to $S'_A$ and no third party knows any information about $S_A$. So far there are many new proposals on QKD scheme. For example, the 6 state protocol [2], the $d$--level qubit protocol [3] and so on.

Normally, one can classify Eve’s attack into two classes. In an individual attack, Eve operates the qubits from Alice individually with her ancilla. In such an attack, Eve’s information to each qubit is independent. In a coherent attack, Eve may operate a number of qubits from Alice collectively with her ancilla by all possible unitary transformations and measurements. In this paper, we give a general proof for the Cirac-Giain conjecture [4]. It was shown that, Eve’s total information in average about the raw bits given a fixed disturbance does not increase through 2-qubit coherent attack. No 2--qubit coherent attack can be more powerful than the optimized individual attack which maximizes Eve’s total information about the raw key given the disturbance. It is conjectured [4] that the conclusion can be also correct for a coherent attack on arbitrary number of qubits from Alice. But no strict proof has been given there. So far it is not clear on how Eve’s total information about the raw key is connected to the security of the final key shared by Alice and Bob in the security proofs for the final key [5,6,7,8,9]. However, they could have a relationship by many people’s intuition, since there is indeed a relationship between Eve’s information of raw key and the security of the final key in classical private information. Cirac-Gisin conjecture could be useful in the future when we are clear of the role of Eve’s information about the raw key.

In our proof, we require that Bob measure every received qubit independently. This requirement is used by all QKD protocols proposed so far. However, we don’t add any constraint to Eve in the coherent attack. The quantity of disturbance is measured by the error rate on Bob’s measurement result. Here we assume all errors are caused by the channel noise which includes the action of Eve. We will consider the BB84 protocol in our proof, but the conclusion is obviously correct for all protocols raised so far.

Most generally, we assume Eve first intercepts state $|Q_A\rangle$ from Alice which includes $n$ qubits, she takes a unitary transformation $\hat{U}_{AE}$ on both $|Q_A\rangle$ and her own ancillas state $|E\rangle$. After this transformation, she sends those qubits originally from Alice to Bob and keeps the ancillas. Finally she measures her ancilla to obtain the information about $S_A$ in the future. Here the final measurement $M_E$ can be any type of POVM and not limited to the projective measurement. Basically, there are two types of attack, the individual attack and the coherent attack [10,11,4,16]. In an individual attack, Eve’s attack $\hat{O}$ is the simple product of $\hat{O}_1$. 
More precisely, in an individual measurement, Eve’s operation on the qubits can be described by

\[
\hat{O}|Q_A\rangle_E = M_{E1} \cdot U_{AE1}|Q_A\rangle|E_1\rangle \otimes M_{E2} \cdot U_{AE2}|Q_A\rangle|E_2\rangle \otimes \cdots M_{En} \cdot U_{AEn}|Q_A\rangle|E_n\rangle
\] (0.1)

Where state \(|E_i\rangle\) is the \(i\)th ancilla which is attached to the \(i\)th qubit from Alice \(|Q_Ai\rangle\). \(U_{AEi}\) is the unitary transformation on state \(|Q_Ai\rangle|E_i\rangle\), \(M_{Ei}\) is certain measurement on \(i\)th ancilla. Therefore an individual attack can be defined as

\[
\hat{O}_1 \otimes \hat{O}_2 \otimes \cdots \hat{O}_n = M_{E1}U_{AE1} \otimes M_{E2}U_{AE2} \otimes \cdots M_{En}U_{AEn}.
\] (0.2)

That is to say, in an individual attack, both the unitary transformation \(U_{AE}\) and the measurement \(M_E\) are factorizable. However, in a coherent attack, there is no restriction to either \(\hat{U}_{AE}\) or the measurement \(M_E\). What we shall show is that, given disturbance to \(|Q_A\rangle\), it is enough for Eve to use the individual attack only in order to obtain the maximum amount of information about \(S_A\).

Basically, there are two quantities \(I_{EA}\) and \(D\) in evaluating the security of a protocol under eavesdropper’s attack. Here \(I_{EA}\) is the amount of information about \(S_A\) eavesdropper can obtain after the attack, \(D\) is the disturbance to \(|Q_A\rangle\). Most generally, \(D\) can be defined as the distance between \(Q_A\) and \(\rho_A'\), where \(\rho_A'\) is the state sent to Bob from Eve. Here we assume in the QKD protocols Bob and Alice only examine the individual disturbance to each qubits. For example, Bob takes the independent measurements to each individual qubits, each qubits have the equal probability to be chosen for the check. The disturbance is measured by Bob’s error rate of his measurement results for those qubits which are chosen for the check. So far all QKD protocols work in such a way to estimate the disturbance. Therefore the detectable disturbance is dependent on the average disturbances to each individual qubits, \(\{D_i\}\).

Given the protocol and the attacking scheme, the attacking results are in general different for different initial state \(Q_A\). Here we evaluate an attacking scheme by the average effect on all possible \(|Q_A\rangle\). That is, if eavesdropper chooses the optimized attack over \(n\) qubits intercepted from Alice, the security is evaluated by the quantities averaged over all possible states for the \(n\) independent qubits, each of which are randomly chosen from certain set as required by the specific QKD protocol itself, and all possible actions( such as the independent measurements) Bob may take to the \(n\) qubits received. The ensemble averaged quantity are denoted by \(\langle I_{EA} \rangle\) and \(\{\langle D_i \rangle\} = \{\langle D_1 \rangle, \langle D_2 \rangle \cdots \langle D_k \rangle\}\) for a QKD protocol under certain attack.

In any attacking scheme, to the eavesdropper the information obtained should be the larger the better while the disturbance should be the less the better. Given disturbance \(\{\langle D_i \rangle\}\), we define the optimized attack \(\hat{O}(\{\langle D_i \rangle\})\) as the one by which the eavesdropper gains the maximum information among all possible attacking schemes with the same disturbances.

**Theorem:** No (coherent) attack can be more powerful than the optimized individual attack for Eve’s strategy of maximizing the total information given the disturbance.
Here we put the word *coherent* inside brackets because the theorem holds for all attacks, however, only coherent attacks need a proof.

Specifically, if eavesdropper is attacking $m$ qubits which are being transmitted from Alice to Bob, she can have two different attacks. One is the coherent attack $\hat{O}_{cm}$. After this attack, the disturbance to the $i$th qubit is $\langle D_i \rangle$. The other is the product of individual attack $\hat{O}_m = \hat{O}_0(1) \otimes \hat{O}_0(2) \otimes \cdots \otimes \hat{O}_0(m)$. After this attack, the disturbance to $i$th qubit is also $\langle D_i \rangle$. That is to say, to each individual qubit, the individual attack $\hat{O}_m$ causes the same disturbance as the coherent attack does. $\hat{O}_0(i)$ is the optimized individual attack to qubit $i$ given the disturbance $\langle D_i \rangle$. Explicitly, we give the following definition on optimized individual attack $\hat{O}_0(i)$ with fixed disturbance:

**Definition** for the notation $\hat{O}_0(i)$: When the disturbance $\langle D_i \rangle$ is given, eavesdropper may obtain the maximum information through $\hat{O}_0(i)$, among all individual attacking schemes. There is not any individual attacking scheme by which Eave can obtain more information about the $i$'s bit than that by scheme $\hat{O}_0(i)$.

To show the theorem, we need only show the following Lemma:

**Lemma:** No (coherent) attack $\hat{O}_{cm}$ can help the eavesdropper to obtain more information than the individual attacking scheme $\hat{O}_m$ as defined above.

To show this, we use the following idea:

*Step 1.* When $m = 1$ it is obviously correct.

*Step 2.* Assume the it is true when $m = n - 1$.

*Step 3.* We can then prove it must be also true in the case of $m = n$.

Now we show *Step 3* based on the assumption in *Step 2* and *Step 1*. We shall do it in the following way:

Suppose the phrase in step 3 is not true. Then there must be a coherent attack $\hat{O}_{cn}$ which

outperform the individual attack $\hat{O}_n$. Here $\hat{O}_{cn}$ can include any collective treatments such as the coherent unitary transformations and the coherent measurements to the $n$ qubits Eve has intercepted from Alice (and Eve’s ancilla). Then we can construct a game $G$ which is an individual attack to one qubit from Alice. After the game $G$ we find $\hat{O}_0(i)$ is not the optimized individual attack to a single qubit because it does not work as effectively as game $G$. This conflict shows that the Lemma must be true.

The game $G$ is played in this way: When Alice and Bob is carrying out the QKD programme, eavesdropper asks her friends Clare and David do the same QKD protocol. Eavesdropper intercepts 1 qubit from Alice, and $n - 1$ qubits from Clare. Without loss of generality, eavesdropper may put the qubit from Alice at the $n$th order in this group of qubits. We denote these $n$ qubits as $|Q_{CA}\rangle$. She then carries out the $\hat{O}_{cn}$ to these $n$ qubits. Since the first $n$-1 qubits can be regarded as ancillas attached to the only qubit from Alice, $\hat{O}_{cn}$ here is an individual attack to the qubit from Alice, although it were a coherent attack if all $n$ qubits had been from Alice. After certain operation as required in $\hat{O}_{cn}$, $|Q_{CA}\rangle$ is changed into $\rho'_{CA}$ and then she sends
the \( n \)th bit to Bob and the first \( n-1 \) bits to David. Let David then play the role as Bob to those \( n-1 \) qubits. When she completes everything required in \( \hat{O}_{cn} \), she asks Clare announce the exact information about the first \( n-1 \) bits. We can show that, with this announcement, eavesdropper’s information to the \( n \)th bit, i.e., the only Alice’s bit, is larger than that from the individual attack \( \hat{O}_0(n) \). Note that, to the qubit from Alice, the game \( G \) here is an individual attack. All \( n-1 \) qubits from Clare can be regarded as part of Eve’s ancilla. This shows that we can use game \( G \), a new individual attack to outperform the optimized individual attack \( \hat{O}_0(\cdot) \) given the disturbance \( \langle D_n \rangle \). This is obviously impossible, because we have already assumed that \( \hat{O}_0(n) \) is the optimized individual attack given \( \langle D_i \rangle \).

Now we give the mathematical details of the game \( G \) above. Using the Shannon entropy \[17,18\] we have the following quantity for the (average) degree of uncertainty corresponding to coherent attack \( \hat{O}_{cn} \):

\[
H(\hat{O}_{cn}) = -\sum_{X,x_n} p(X,x_n) \log p(X,x_n).
\]  

(0.3)

\( X = x_1, x_2, \ldots, x_{n-1} \) and \( p(X,x_n) \) is eavesdropper’s probability distribution for the \( n \) bits. With the definition of \( X \), the mathematical symbol \( (X,x_n) \) is nothing but \( (x_1, x_2, \ldots, x_{n-1}, x_n) \). We write it in the form of \( (X,x_n) \) because we will play some tricks to \( x_n \) latter. Note that given different input states \( |Q_A\rangle \) and different measurement basis taken by Bob, the same attacking scheme may lead to different output \( Y \). In general Eve’s probability distribution is dependent on the outcome \( Y \). Here the bar over \( \sum_{X,x_n} p(X,x_n) \log p(X,x_n) \) represents the averaged result over all different \( Y \). Thus the entropy \( H(\hat{O}_{cn}) \) here is the average entropy over all possible \( Y \). This bar average, the average over different outcome for one configuration, is different from the ensemble average, which is the average over different configurations. The formula above shows how uncertain Eve is to the \( n \) bits after she completes her coherent attack \( \hat{O}_{cn} \) on the \( n \) qubits (first \( n-1 \) originally from Clare, the last one from Alice), but before Clare announces the exact information for his \( n-1 \) bits.

We also have the (average) entropy corresponding to the individual attack \( \hat{O}_n \)

\[
H(\hat{O}_n) = \sum_{i=0}^{n} H(\hat{O}_0(i)),
\]  

(0.4)

and

\[
H(\hat{O}_0(i)) = -\sum_{x} p_i(x) \log p_i(x),
\]  

(0.5)

\( p_i(x) \) is eavesdropper’s probability distribution for the value of the \( i \)th the bit through the individual attack \( \hat{O}_0(i) \). Again, \( H(\hat{O}_0) \) is the average entropy for all possible outcome of one configuration. The disturbances caused by the two attacks are same. Therefore if the coherent attack \( \hat{O}_{cn} \) here is more powerful than the individual attack \( \hat{O}_n = \hat{O}_0(1) \otimes \hat{O}_0(2) \otimes \cdots \otimes \hat{O}_0(n) \) we must have the following inequality for the ensemble averaged entropy.
\[ \langle H(\hat{O}_{cn}) \rangle < \sum_{i=0}^{n} H(\hat{O}_0(i)). \] (0.6)

Now eavesdropper asks Clare announce bits information to his \( n - 1 \) bits. With the exact information about the first \( n - 1 \) bits, eavesdropper's new average entropy through game \( G \) is

\[ \langle H'(\hat{O}_{cn}) \rangle = \langle H(\hat{O}_{cn}) \rangle + \langle \sum_X p(X) \log p(X) \rangle. \] (0.7)

Here \( p(X) \) is eavesdropper probability distribution to the first \( n - 1 \) qubits just before Clare's announcement, \( p(X) = \sum_{x_n} p(X, x_n) \).

Since we have assumed the the Lemma to be true in the case \( m = n - 1 \), i.e., eavesdropper's information to the first \( n - 1 \) bits through any coherent attack (and also any other attack) should never be larger than the information obtained through the individual attack \( \hat{O}_{n-1} \) before Clare announces the exact results. Therefore we have

\[ -\langle \sum_X p(X) \log p(X) \rangle \geq \langle \sum_{i=1}^{n-1} H(\hat{O}_0(i)) \rangle. \] (0.8)

Combining this with the eq(0.6) and eq(0.7) we have the following inequality

\[ \langle H'(\hat{O}_{cn}) \rangle < \langle H(\hat{O}_0(n)) \rangle. \] (0.9)

Now eavesdropper has the exact information to the first \( n - 1 \) bits, \( H'(\hat{O}_{cn}) \) can also be interpreted as Eve's entropy of the \( n \)th bit through game \( G \). The inequality(0.9) shows that eavesdropper's information on the single bit initially from Alice through game \( G \) is larger than her information on the same bit through \( \hat{O}_0(n) \). And also we have assumed the disturbance to that bit caused by game \( G \) is equal to that caused by \( \hat{O}_0(n) \). This is to say, game \( G \), which is an individual attack to Alice's qubit, can help eavesdropper to obtain more information to the bit than the optimized individual attack to the bit with same disturbance. This conflicts with our definition about optimized individual attack given the disturbance. Thus the inequality(0.6) must be wrong. Therefore we obtain our theorem.

Thus we draw the following conclusion:

Suppose in in a QKD protocol, all the transmitted bits are independent and the measurements are carried out to each individual qubits independently. To this type of protocols, no coherent attack can be more powerful than the product of optimized individual attack \( \hat{O}_n \) for Eve's strategy of maximizing the total information given the disturbance.

Remarks: Our conclusion is only for the raw key stage. With our result, the conclusions in ref [2], ref [3] and ref [13] on Eve's maximum information through individual attack to the 6 state protocol, \( d \)-level state protocol and 3-level state protocol are also correct in the coherent attack case. It should be interesting to investigate the role of coherent attacks with the error correction and privacy amplification being taken into consideration. We believe our
result here can be useful in the future study of this case. For example, it is generally believed that a QKD can be unconditionally secure after the error privacy amplification, given that Eve's information smaller than Bob's information in the raw key stage. Our result can greatly simplify the estimation of Eve's maximum information, therefore the explicit formula as the criteria of security can be obtained easier. In the cases that the raw key is directly used for the secret communication, coherent attacks can be disregarded.

In the end of this paper, we have to clarify something. Although we have proven the Cirac-Gisin conjecture in a rather general sense, we are not clear on the role of our conclusion in the most important topic of the optimized Eve's attack towards the final key in the subject of quantum key distribution. On the other hand, here the strategic problem is for the total information, however, the total information at that step is not everything in the whole game of quantum key distribution [19]. In the strategic problem above, there is no test for Alice and Bob. This shows we have assumed that the error rate in the test is equivalent to the disturbance caused by Eve. This is in general not true because there can be a statistical deviation and only the subset that passes the test will interest the Evesdropper. However, this issue can be resolved in the case that the number of qubits in the QKD job between Alice and Bob is much larger than the number of qubits intercepted for the coherent attack, i.e. $k \gg n$. This has been illustrated in ref [14].

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