UNIQUE ERGODICITY FOR FOLIATIONS ON COMPACT KÄHLER SURFACES

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ABSTRACT. Let $\mathcal{F}$ be a holomorphic foliation by Riemann surfaces on a compact Kähler surface $X$. Assume that all the singularities are hyperbolic and that the foliation admits no directed positive closed current. Then there exists a unique (up to a multiplicative constant) positive $dd^c$-closed $(1,1)$-current $T$ directed by $\mathcal{F}$. This is a very strong ergodic property of the foliation $\mathcal{F}$. The proof uses an extension of the theory of densities to a class of non-$dd^c$-closed currents.

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1. Introduction

Let \( \mathcal{F} \) be a (possibly singular) holomorphic foliation on a compact Kähler surface \( X \) endowed with a Kähler form \( \omega \).

Recall that the foliation \( \mathcal{F} \) is given by an open covering \( \{ U_i \} \) of \( X \) and holomorphic vector fields \( v_j \in H^0(U_j, TX) \) with isolated singularities (i.e. isolated zeroes) such that

\[
v_i = g_{ij}v_j \quad \text{on} \quad U_i \cap U_j
\]

for some nonvanishing holomorphic functions \( g_{ij} \in H^0(U_i \cap U_j, \Omega_X^1) \). Its leaves are locally integral curves, of these vector fields with isolated zeros. The set of singularities of \( \mathcal{F} \) is precisely the union of the zero sets of these local vector fields. When \( X \) is a compact complex surface, this set is finite. Using rational vector fields, we see that projective complex surfaces admit large families of foliations.

Foliations can be also given locally by a non-zero holomorphic 1-form and the leaves are Riemann surfaces on which these forms vanish. In the case of complex dimension 2 that we consider, there is no integrability condition.

If a holomorphic vector field \( F \) has an isolated zero at some point \( p \), we say that the singularity \( p \) is hyperbolic if the two eigenvalues of the linear part of the vector field at \( p \) have non-real quotient. According to Poincaré, if \( p \) is such a singular point, then there are local holomorphic coordinates centered at \( p \) such that the vector field has the form

\[
z_2 \frac{\partial}{\partial z_2} - \eta z_1 \frac{\partial}{\partial z_1},
\]

where \((z_1, z_2) \in \mathbb{C}^2\), \( \eta = a + ib \), \( a, b \in \mathbb{R} \), and \( b \neq 0 \).

In order to develop an ergodic theory of foliations, in the Riemannian case, Garnett [14] introduced the notion of harmonic measures for nonsingular foliations which are generalizations of the foliation cycles of Sullivan [28]. According to Sullivan [28], the existence of a positive closed current, directed by the foliation, corresponds to the existence of measures on transversals, invariant by the holonomy maps.

In the complex case, it is more fruitful to consider rather the formalism of directed \( dd^c \)-closed currents. This permits to use the interplay between cohomological intersection and geometric intersection. In the present article we use the cohomological properties of tangent currents.

Recall that \( d, d^c \) denote the real differential operators on \( X \) defined by \( d := \partial + \overline{\partial}, \quad d^c := \frac{1}{2\pi i}(\partial - \overline{\partial}) \) so that \( dd^c = \frac{i}{2\pi} \partial \overline{\partial} \). A positive current \( T \) of bi-dimension \((1, 1)\) is directed by the foliation \( \mathcal{F} \) if \( T \wedge \Omega = 0 \) for every local holomorphic 1-form \( \Omega \) defining \( \mathcal{F} \). Let \( B \) be any flow box of \( \mathcal{F} \) outside the singularities and denote by \( V_\alpha \) the plaques of \( \mathcal{F} \) in \( B \) parametrized by \( \alpha \) in some transversal \( \Sigma \) of \( B \). On the flow box \( B \), this current has the form

\[
T|_B = \int_{\alpha \in \Sigma} h_\alpha[V_\alpha]d\mu(\alpha),
\]

where \( h_\alpha \) is a positive harmonic function on \( V_\alpha \), and \([V_\alpha]\) denotes the current of integration on the plaque \( V_\alpha \) (see e.g. [6, Proposition 2.3]). In [1] it is shown that for a foliation \( \mathcal{F} \) by Riemann surfaces with finitely many singular points in a compact complex manifold, there exists a directed positive \( dd^c \)-closed current of mass 1. If \( T \) is a positive \( dd^c \)-closed \((1, 1)\)-current directed by \( \mathcal{F} \), then it has no mass on the singularities of \( \mathcal{F} \) because this set is finite, see e.g. [1] [27].
Our hypothesis that \( \mathcal{F} \) does not admit a positive directed closed current implies that there are no invariant closed curve, and that the leaves are uniformized by the unit disc, i.e leaves are \textit{hyperbolic} (see \cite{4}). The case where there is an invariant closed curve is studied in \cite{8}.

Now we discuss briefly the family of singular holomorphic foliations on \( \mathbb{P}^2 \) with a given degree \( d > 1 \). Recall that the degree is the number of tangencies of the foliation with a generic line. This family can be identified with a Zariski dense open set \( \mathcal{U}_d \) of some projective space. We will say that a property is \textit{typical} for this family if it is valid for \( \mathcal{F} \) in a set of full Lebesgue measure of \( \mathcal{U}_d \). Here are some typical properties of a foliation in \( \mathcal{U}_d \).

1. (Jouanolou \cite{16} and Lins Neto-Soares \cite{20}) all the singularities of \( \mathcal{F} \) are hyperbolic and \( \mathcal{F} \) does not possess any invariant algebraic curve.
2. (Glutsyuk \cite{15} and Lins Neto \cite{19}) \( \mathcal{F} \) is hyperbolic.
3. (Brunella \cite{3}) \( \mathcal{F} \) admits no nontrivial directed positive closed current.

Let \( \mathcal{F} \) be a hyperbolic foliation in a compact complex manifold. Fornaess and the third author in \cite{11} introduced an average on each leaf \( L_w \) which allows us to get another construction of directed positive \( dd^c \)-closed currents.

More precisely, let \( \mathbb{D} \) and \( \mathbb{D}_r \) denote the unit disc and the disc of center 0 and radius \( r \) in \( \mathbb{C} \). Let \( \phi^w : \mathbb{D} \to L_w \) be a universal covering map for the leaf \( L_w \) passing through \( w \) with \( \phi^w(0) = w \). Define the Ahlfors-Shimizu characteristic function for \( \phi^w \) by

\[
T^w(r) := \int_0^r \frac{dt}{t} \int_{\mathbb{D}_t} (\phi^w)^*(\omega),
\]

where we recall that \( \omega \) is the Kähler form on \( X \). Define the Nevanlinna current of index \( r, 0 < r < 1 \), associated with \( L_w \) by

\[
\tau^w_r := \frac{1}{T^w(r)} (\phi^w)_* \left[ \log^+ \frac{r}{|\zeta|} \right] = \frac{1}{T^w(r)} \int_0^r \frac{dt}{t} (\phi^w)_* [\mathbb{D}_t].
\]

Here, \( \log^+ := \max(\log, 0) \) and \( \zeta \) is the standard coordinate of \( \mathbb{C} \) so that the unit disc \( \mathbb{D} \) is equal to \( \{ |\zeta| < 1 \} \). Note that for each \( w \), the map \( \phi^w \) is uniquely defined up to a rotation in \( \mathbb{D} \). So the above definitions do not depend on the choice of \( \phi^w \).

When the singularities of \( \mathcal{F} \) are all isolated (not necessarily hyperbolic), it was shown in \cite{11} (see also \cite{8}) that \( T^w(r) \to \infty \) as \( r \to 1 \). Consequently, the cluster points of \( \tau^w_r \) are \( dd^c \)-closed currents directed by \( \mathcal{F} \). It turns out that a Birkhoff type theorem implies that for a generic foliation all extremal directed positive \( dd^c \)-closed currents of mass 1 can be obtained in this way \cite{6}. General directed positive \( dd^c \)-closed currents are averages of the extremal ones.

Here are the main results of the present paper. The first two theorems deal with singular holomorphic foliations whereas the third one considers bi-Lipschitz laminations.

**Theorem 1.1.** Let \( \mathcal{F} \) be a holomorphic foliation by Riemann surfaces with only hyperbolic singularities in a compact Kähler surface \((X, \omega)\). Assume that \( \mathcal{F} \) admits no directed positive closed current. Then there exists a unique positive \( dd^c \)-closed current \( T \) of mass 1 directed by \( \mathcal{F} \). In particular, if \( \phi^w : \mathbb{D} \to L_w \) is a universal covering map of an arbitrary leaf \( L_w \) as above, then \( \tau^w_r \to T \), in the sense of currents, as \( r \to 1 \).

When \( X = \mathbb{P}^2 \) the theorem was proved by Fornaess and the third author in \cite{13}. In that case according to \cite{3}, if all the singularities of \( \mathcal{F} \in \mathcal{U}_d \) are hyperbolic and \( \mathcal{F} \) does
not possess any invariant algebraic curve, then \( \mathcal{F} \) admits no nontrivial directed positive closed current. So the conclusion of Theorem 1.1 is a typical property of the family \( \mathcal{U}_d \). The proof in [13] is based on two ingredients. The first one is an energy theory for positive \( dd^c \)-closed currents which was previously developed in [11]. The second one is a geometric intersection calculus for these currents. For the second ingredient, the transitivity of the automorphism group of \( \mathbb{P}^2 \) is heavily used. Moreover, the proof is quite technical. The computations needed to estimate the geometric intersections are quite involved. Using these techniques, Pérez-Garrandés [25] has studied the case where \( X \) is a homogeneous compact Kähler surface.

The new idea in the proof of Theorem 1.1 is to replace the geometric intersection calculus by a more flexible tool which is a density theory for the tensor product of positive \( dd^c \)-closed currents. The method allows to bypass the assumption of homogeneity of \( X \). The proof is more conceptual and also far less technical. The strategy is as follows. Given a positive \( dd^c \)-closed current \( T \) on a surface \( X \), we consider the positive current \( T \otimes T \) near the diagonal \( \Delta \) of \( X \times X \), observe that in general \( T \otimes T \) is not \( dd^c \)-closed. We study the tangent currents to \( T \otimes T \) along the diagonal \( \Delta \). As one can expect this is related to the self intersection properties of the current \( T \). It turns out that the geometry of the tangent currents is quite simple. They are positive closed and are the pull-back of a positive measure \( \nu \), living on \( \Delta \). We relate the mass of \( \nu \) to a cohomology class of \( T \) involving the energy of the current.

The foliation enters in the picture to prove that \( \nu \) is zero. This is done using the local properties of the foliation, the local description of \( T \) and in particular, that the singularities are hyperbolic. The vanishing of \( \nu \) gives easily the uniqueness using a kind of Hodge-Riemann relations.

We expect that our results could have numerous applications. Using Theorem 1.1, the second author has very recently shown in [24] that under the assumption of this theorem, if \( X \) is moreover projective, then the Lyapunov exponent of \( \mathcal{F} \) defined in [21, 23] is strictly negative. In particular, when \( X = \mathbb{P}^2 \) the Lyapunov exponent of a typical foliation \( \mathcal{F} \in \mathcal{U}_d \) is equal to \( \frac{d+2}{2(d-1)} \).

**Theorem 1.2.** Let \( \mathcal{F} \) be a holomorphic foliation by Riemann surfaces with only hyperbolic singularities in a compact Kähler surface \((X, \omega)\). Assume that \( \mathcal{F} \) does not admit a closed invariant curve, but admits a directed positive closed current. Then every directed positive \( dd^c \)-closed current \( T \) is closed, the support of \( T \) is disjoint from the singularities of \( \mathcal{F} \), and \( \{T\}^2 = 0 \). If moreover \( X \) is projective, then the class \( \{T\} \) is nef.

The case where \( \mathcal{F} \) admits an invariant curve is treated in [8]. It is shown there under mild cohomological restrictions that all directed positive \( dd^c \)-closed currents are supported by the invariant curves.

Our last main theorem gives a similar result in the context of bi-Lipschitz laminations.

**Theorem 1.3.** Let \( \mathcal{F} \) be a bi-Lipschitz lamination (without singularities) by hyperbolic Riemann surfaces in a compact Kähler surface \((X, \omega)\). If \( \mathcal{F} \) does not admit a transverse measure, then there is only one positive \( dd^c \)-closed current of mass 1, directed by the lamination. If \( \mathcal{F} \) does not admit a closed invariant curve, but admits a transverse measure, then every directed positive \( dd^c \)-closed current \( T \) is closed with \( \{T\}^2 = 0 \). If moreover \( X \) is projective, then the class \( \{T\} \) is nef.
By Sullivan [28], the non-existence of a transverse measure for the holonomy map is equivalent to the non-existence of a directed positive closed current.

The paper is organized as follows. In Section 2, we first give some basic properties of positive \(dd^c\)-closed currents and recall some elements of the theory of \(dd^c\)-closed currents of finite energy developed in [11]. Next, we introduce the theory of densities for the tensor product of such currents which is described in Theorem [2.2]. Then we state Theorem [2.4] dealing with the tensor square power of a positive \(dd^c\)-closed current directed by a foliation. These are the key ingredients in the proof of the Main Theorems which is presented at the end of this section. The proof of Theorem [2.2] occupies Sections [3-4] and [5] below. The last two sections are devoted to the proof of Theorem [2.4].

After we had finished the article, Deroin informed us that independently with Kleptsyn, they had obtained a similar result under stronger hypotheses on the foliation and on the surface.

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2. Theory of densities and strategy for the proof of the Main Theorems

2.1. Positive \(dd^c\)-closed currents and tangent currents to their tensor products. We recall some basic notions on (weakly) positive currents and positive \(dd^c\)-closed currents on a complex manifold, and refer the reader to Berndtsson–Sibony [1] and Skoda [27] for details.

Let \(M\) be a complex manifold of dimension \(k\). A \((p, p)\)-current \(T\) on \(M\) is (weakly) positive if

\[
T \wedge (i\gamma_1 \wedge \bar{\gamma}_1) \wedge \ldots \wedge (i\gamma_k \wedge \bar{\gamma}_k)\]

is a positive measure for any smooth \((1, 0)\)-forms \(\gamma_1, \ldots, \gamma_k, \bar{\gamma}_1, \ldots, \bar{\gamma}_k\) on \(M\).

Let \(T\) be a positive \(dd^c\)-closed current of bidimension \((p, p)\) defined in a neighborhood of a point \(x \in X\). Let \(B(x, r)\) denote the ball of center \(x\) with radius \(r\) in a local holomorphic coordinate \(z\) around \(x\). By Skoda [27],

\[
\nu(T, x, r) := \frac{1}{\pi r^2} \int_{B(x, r)} T \wedge dd^c\|z\|^2
\]

is increasing in \(r\), and the limit

\[
\nu(T, x) := \lim_{r \to 0^+} \nu(T, x, r)
\]

is a nonnegative finite number which is called the Lelong number of \(T\) at \(x\). The notion of Lelong number does not depend on the choice of local holomorphic coordinates. In particular, the function \(X \ni x \mapsto \nu(T, x)\) is upper-semicontinuous. Moreover, positive \(dd^c\)-closed currents of bidimension \((p, p)\) give no mass to sets of \(2p\)-Hausdorff dimension 0 [11 p. 389].

From now on consider a compact Kähler surface \((X, \omega)\). A smooth \((1, 1)\)-forms \(\Omega\) on \(X\) is said to be \(\square\)-harmonic if \((\partial \bar{\partial} + \bar{\partial} \partial)\Omega = 0\), where \(\bar{\partial}\) is the adjoint operator of \(\partial\) (see e.g. [5]). These forms consist of the common null space of \(\partial\) and \(\bar{\partial}\).
Let $T$ be a positive $dd^c$-closed $(1,1)$-current on $X$. By \cite{[11]} it can be represented as
\begin{equation}
T = \Omega + \partial S + \overline{\partial S}
\end{equation}
with $S$ a current of bidegree $(0,1)$, $\Omega$ a $\Box$-harmonic form. Moreover, the $(0,2)$-current $\overline{\partial S}$ is uniquely determined by $T$ and $\partial S \in L^2$. We define the energy $E(T) = E(T,T)$ of $T$ as
\[ E(T) = E(T,T) := \int_X \overline{\partial S} \wedge \partial S. \]

The energy $E(T)$ depends only on $T$, not on the choice of $S$, it is a nonnegative finite number.

Let $\Delta = \{(x,x) : x \in X\}$ be the diagonal of $X \times X$. Consider positive $dd^c$-closed $(1,1)$-currents $T_1$ and $T_2$ on $X$. We will study the density of $T_1 \otimes T_2$ near $\Delta$ via a notion of “tangent cone” to $T_1 \times T_2$ along $\Delta$ that we introduce now.

Denote by $\Tan(X \times X)$ (resp. $\Tan(X)$) be the tangent bundle of $X \times X$ (resp. of $X$) and $\Tan(X \times X)|_{\Delta}$ its restriction to $\Delta$. Let $\Tan(\Delta)$ be the tangent bundle of $\Delta$ which is naturally identified with $\Tan(X)$, and which is also naturally identified with a sub-bundle of $\Tan(X \times X)|_{\Delta}$. The quotient $\Tan(X \times X)|_{\Delta}/\Tan(\Delta)$ is the normal bundle of $\Delta$ in $X \times X$ and will be denoted by $\mathcal{E}$. We identify $\Delta$ with the zero section in each vector bundle $\Tan(X)|_{\Delta}$, $\Tan(\Delta)$ and $\mathcal{E}$. Denote by $\pi : \mathcal{E} \to \Delta$ the canonical projection.

**Definition 2.1.** A **smooth admissible map** is a smooth bijective map $\tau$ from a neighbourhood of $\Delta$ in $X \times X$ to a neighbourhood of $\Delta$ in $\mathcal{E}$ such that

1. The restriction of $\tau$ to $\Delta$ is the identity map on $\Delta$; in particular, the restriction of the differential $d\tau$ to $\Delta$ induces three endomorphisms of $\Tan(X \times X)|_{\Delta}$, $\Tan(\Delta)$ and $\mathcal{E}$ respectively;
2. The differential $d\tau(x,x)$, at any point $(x,x) \in \Delta$, is a $C$-linear endomorphism of the tangent space to $X \times X$ at $(x,x)$;
3. The endomorphism of $\mathcal{E}$, induced by $d\tau$, is the identity map on the fibers.

Note that the dependence of $d\tau(x,x)$ in $(x,x) \in \Delta$ is in general not holomorphic. The construction of a smooth admissible map will be given in Subsection 4.3 below.

For $\lambda \in C$, denote by $A_\lambda$ the map from the bundle $\mathcal{E}$ to itself induced by the fiberwise multiplication by $\lambda$. Let $\tau$ be a smooth admissible map as above. Define
\[(T_1 \otimes T_2)_\lambda := (A_\lambda)_* \tau_*(T_1 \otimes T_2).\]

This is a current of degree 4. Its domain of definition is some open subset of $\mathcal{E}$ containing $\Delta$ which increases to $\mathcal{E}$ when $|\lambda|$ increases to infinity. Note that $(T_1 \otimes T_2)_\lambda$ is not a $(2,2)$-current and we cannot speak of its positivity. Moreover, it is not $dd^c$-closed in general and we cannot speak of its cohomology class. When $T_1$ and $T_2$ are positive closed currents, then $(T_1 \otimes T_2)_\lambda$ is closed and the situation is simpler.

By (2.2) we can write for $j \in \{1,2\}$,
\[ T_j = \Omega_j + \partial S_j + \overline{\partial S_j} \]
with $S_j$ a current of bidegree $(0,1)$, $\Omega_j$ a $\Box$-harmonic form. Moreover, the $(0,2)$-current $\overline{\partial S_j}$ is uniquely determined by $T_j$.

We have the following theorem.
Definition 2.3. Any current tensor square power of directed positive diffuse currents, see [9] for details. Sections 4 and 5 are devoted to the proof of Theorem 2.2.

Theorem 2.2. Let $T_1$ and $T_2$ be positive $dd^c$-closed $(1, 1)$-currents on $X$. Suppose in addition that either $\nu(T_1, x) = 0$ for almost every $x \in X$ with respect to the measure $T_2 \wedge \omega$ or $\nu(T_2, x) = 0$ for almost every $x \in X$ with respect to the measure $T_1 \wedge \omega$. Then

1. the mass of $(T_1 \otimes T_2)_\lambda$ on any given compact subset of $E$ is uniformly bounded. If $S$ is a cluster value of $(T_1 \otimes T_2)_\lambda$ when $\lambda \to \infty$, then it is a positive closed $(2, 2)$-current on $E$ given by $S = \pi^*(\nu)$, where $\nu$ is a positive measure on $\Delta$. Moreover, if $(\lambda_n)$ is a sequence tending to infinity such that $(T_1 \otimes T_2)_{\lambda_n} \to S$, then $S$ may depend on $(\lambda_n)$ but it does not depend on the choice of the map $\tau$.

2. The mass of $\nu$ is

$$\|\nu\| = \int_X \Omega_1 \wedge \Omega_2 - \int_X \bar{\partial}S_1 \wedge \partial S_2 - \int_X \partial \bar{\partial}S_1.$$  

In particular, if $T_1 = T_2 = T$ and $T = \Omega + \partial S + \bar{\partial}S$ as in (2.2), then

$$\|\nu\| = \int_X \Omega^2 - 2 \int_X \partial S \wedge \bar{\partial}S = \int_X \Omega^2 - 2E(T).$$

We can now introduce the following notion.

Definition 2.3. Any current $S$ obtained as in Theorem 2.2 is called a tangent current to $T_1 \otimes T_2$ along the diagonal $\Delta$.

Note that in general $S$ is not unique as this is already the case for positive closed currents, see [9] for details. Sections 4 and 5 are devoted to the proof of Theorem 2.2.

Finally, we state the following result about the vanishing of the tangent currents of the tensor square power of directed positive diffuse $dd^c$-closed currents. Its proof will occupy Sections 6 and 7.

Theorem 2.4. Let $\mathcal{F}$ be either a holomorphic foliation by hyperbolic Riemann surfaces with only hyperbolic singularities, or a laminating by hyperbolic Riemann surfaces in a compact Kähler surface $(X, \omega)$. Then for every positive $dd^c$-closed current $T$ directed by $\mathcal{F}$ which does not give mass to any invariant analytic curve, zero is the unique tangent current to $T \otimes T$ along the diagonal $\Delta$.

Recall that a closed subset $Y$ of a complex manifold $X$ is laminated by Riemann surfaces if it admits an open covering $U_i$ and on each $U_i$, there is a homeomorphism $\varphi_i = (h_i, \lambda_i) : U_i \cap Y \to \mathbb{D}(z_i) \times T_i(t_i)$, where $\mathbb{D}(z_i)$ is as usual the unit disc and $T_i$ is a locally compact metric space, the $\varphi^{-1}$ are holomorphic in $z_i$. Moreover, $\varphi_{ij}(z_i, t_i) = \varphi_j \circ \varphi_{i}^{-1}(z_i, t_i) = (h_{ij}(z_i, t_i), \lambda_{ij}(t_i))$, where the $h_{ij}(z_i, t_i)$ are holomorphic with respect to $z_i$. When $T_i$ is in a Euclidean space and $\varphi_i$ extends to a bi-Lipschitz map, we say that the lamination is bi-Lipschitz.

The theorem expresses that the current $T \otimes T$ is not too singular: regular currents have Lelong number zero.

2.2. Sketch of the proof of the Main Theorems. We recall the following result due to Fornaess and the third author [13]) which is needed for the proof of the Main Theorems.

Theorem 2.5. Let $\mathcal{F}$ be either a holomorphic foliation by hyperbolic Riemann surfaces with only hyperbolic singularities, or a laminating by hyperbolic Riemann surfaces in a compact complex surface $X$. Let $T$ be a positive $dd^c$-closed current directed by $\mathcal{F}$ which does not give
mass to any invariant analytic curve. Then its transverse measure is diffuse, that is, $T$ gives no mass to each single leaf.

The first step of our proof consists in proving the following

**Lemma 2.6.** Let $\mathcal{F}$ be either a holomorphic foliation by hyperbolic Riemann surfaces with only hyperbolic singularities, or a lamination by hyperbolic Riemann surfaces in a compact Kähler surface $(X, \omega)$. Let $T_1$, $T_2$ be positive $dd^c$-closed currents directed by $\mathcal{F}$ such that neither of them gives mass to any invariant analytic curve and that $\int_X T_1 \wedge \omega = \int_X T_2 \wedge \omega = 1$. Then $T_1 - T_2$ is a closed current.

**Proof.** Since both $T_1$ and $T_2$ do not give mass to any invariant analytic curve, it follows from Theorem 2.5 that $\nu(T_1, x) = \nu(T_2, x) = 0$ for all $x$ outside set of the singularities of $\mathcal{F}$. Since $T_1$ and $T_2$ do not give mass to this finite set, we see that $T_1$ and $T_2$ satisfy the assumption of Theorem 2.2.

By (2.2), we write, for $j \in \{1, 2\}$,

$$T_j = \Omega_j + \partial S_j + \overline{\partial S}_j,$$

with $S_j$ a current of bidegree $(0, 1)$ with $\overline{\partial S}_j \in L^2$, $\Omega_j$ a $\square$-harmonic form. By Stokes’ theorem, we get from (2.5) that

$$\int_X \Omega_j \wedge \omega = \int_X T_j \wedge \omega = 1 \quad \text{for} \quad j = 1, 2.$$

Applying Theorem 2.2 and 2.4 to each one of the three positive $dd^c$-closed currents $T_1$, $T_2$ and $T_1 + T_2$ directed by $\mathcal{F}$, we infer that all three currents $T_1 \otimes T_1$, $T_2 \otimes T_2$ and $(T_1 + T_2) \otimes (T_1 + T_2)$, admits zero as the unique tangent current along the diagonal $\Delta$. This, combined with the mass formula (2.4) and the decompositions (2.5), implies that

$$\int_X \Omega_1^2 = 2 \int_X \overline{\partial S}_1 \wedge \partial S_1 \quad \text{and} \quad \int_X \Omega_2^2 = 2 \int_X \overline{\partial S}_2 \wedge \partial S_2,$$

$$\int_X (\Omega_1 + \Omega_2)^2 = 2 \int_X \overline{\partial (S_1 + S_2)} \wedge \partial (\overline{S}_1 + \overline{S}_2).$$

Let $T := T_1 - T_2$, $\Omega := \Omega_1 - \Omega_2$ and $S := S_1 - S_2$. Then we infer from (2.5) and (2.6) that

$$T = \Omega + \partial S + \overline{\partial S} \quad \text{and} \quad \int_X \Omega \wedge \omega = 0.$$

Moreover, it follows from (2.7) that

$$\int_X \Omega^2 = \int_X (\Omega_1 - \Omega_2)^2 \quad \text{for} \quad j = 1, 2$$

$$= 2 \int_X \Omega_1^2 + 2 \int_X \Omega_2^2 - \int_X (\Omega_1 + \Omega_2)^2$$

$$= 2 \int_X \overline{\partial S} \wedge \partial S.$$

On the one hand, since we know by (2.8) that $\int_X \Omega \wedge \omega = 0$, the cohomology class of $\Omega$ is a primitive class of $H^{1,1}(X, \mathbb{R})$. Therefore, it follows from the classical Hodge–Riemann theorem that $\int_X \Omega^2 \leq 0$ and that the equality holds if only if the class of $\Omega$ is zero, that is, $\Omega = 0$ because $\Omega$ is $\square$-harmonic. On the other hand, since $\overline{\partial S}$ is a $(0, 2)$-form, the current $\overline{\partial S} \wedge \partial S = \overline{\partial S} \wedge \overline{\partial S}$ is a positive measure. So the integral $\int_X \overline{\partial S} \wedge \partial S \geq 0$ and...
the equality holds if only if $\partial S = 0$ almost everywhere. Therefore, we deduce from the above equality that $\int_X \Omega^2 = 2 \int_X \partial \Omega \wedge \partial S = 0$. Hence, $\Omega$ the form giving the class of $T$ is zero and $\partial S = 0$ almost everywhere on $X$. Inserting this into (2.8), we see that $dT = 0$. The proof of the lemma is thereby completed. □

End of the proof of Theorem 1.1 (see also [11]). The existence of a positive $dd^c$-closed current directed by $\mathcal{F}$ has been established in [1] Theorem 1.4] (see also [12, Theorem 23]). In order to prove the uniqueness, consider two positive $dd^c$-closed currents $T_1, T_2$ directed by $\mathcal{F}$ with $\int_X T_1 \wedge \omega = \int_X T_2 \wedge \omega = 1$. We need to show that $T_1 = T_2$.

Let $U \simeq \mathbb{B} \times \Sigma$ be a flow box away from the set of singularities $E$ of $\mathcal{F}$. Then

$$T_j = \int_{\Sigma} h_j^\omega [V_\alpha] d\mu_j(\alpha), \quad j = 1, 2.$$ 

Let $\mu(\alpha) = \mu_1(\alpha) + \mu_2(\alpha)$, so $\mu_j = r_j(\alpha) \mu$. Then

$$T_1 - T_2 = \int_{\Sigma} (h_1^\omega r_1(\alpha) - h_2^\omega r_2(\alpha))[V_\alpha] d\mu(\alpha).$$ 

Since we know by Lemma 2.6 that $T_1 - T_2$ is a closed current, it follows that $h_1^\omega r_1(\alpha) - h_2^\omega r_2(\alpha) = c(\alpha)$ is constant $\mu$-almost every $\alpha$. We decompose $c(\alpha) \mu(\alpha)$ on the space of plaques $\Sigma$ and obtain that $c(\alpha) \mu(\alpha) = \nu_1 - \nu_2$ for mutually singular positive measures. Then

$$T_1 - T_2 = [V_\alpha] \nu_1(\alpha) - [V_\alpha] \nu_2(\alpha) = T^+ - T^-$$

for positive closed currents $T^\pm$. These currents fit together to global positive closed currents on $X \setminus E$. Observe that the mass of $T^\pm$ is bounded by the mass of $T_1 + T_2$. As the mass of $T^\pm$ is bounded near $E$ and $E$ is an analytic set, the currents $T^\pm$ extend as closed currents through $E$, see for example [27] or [26]. Consequently, since $\mathcal{F}$ does not admit directed closed currents, we get that $T^+ = T^- = 0$, and hence $T_1 = T_2$ since both currents have no mass on $E$. This completes the proof of the uniqueness part.

Let $T$ be the unique positive $dd^c$-closed current of mass 1. Since every cluster point of $\tau^w_r$ as $r \to 1$ is a positive $dd^c$-closed current of mass 1, it follows that $\tau^w_r \to T$ as $r \to 1$. □

The following result is needed for the proof of Theorems 1.2 and 1.3

**Lemma 2.7.** Let $\mathcal{F}$ be either a singular holomorphic foliation by hyperbolic Riemann surfaces with only hyperbolic singularities, or a lamination by hyperbolic Riemann surfaces in a compact projective surface. Suppose that $T$ is a positive closed current directed by $\mathcal{F}$ and that $T$ is diffuse. Then its class $\{T\}$ is nef.

**Proof.** First we consider the case where $\mathcal{F}$ is a singular holomorphic foliation.

Let $Z$ be a curve in $X$. We have to check that $\{T\} \sim \{Z\} \geq 0$. Let $\alpha$ be a closed smooth $(1, 1)$-form representing $\{Z\}$. So we can find a function $u \leq 0$ on $X$ such that

$$[Z] - \alpha = dd^c u.$$ 

By [6, Theorem 1.1], there is $w \in X \setminus (E \cup Z)$ such that $T = \lim_{r \to 1} \tau^w_r$, where $\tau^w_r$ is defined in (1.1). Since $\phi^w(D) \not\subset Z$, we have $u \circ \phi^w \neq -\infty$. We can assume that $u \circ \phi^w(0) > -\infty$,
that is, \( u(w) \neq -\infty \). So we get that \( \langle T, \alpha \rangle = \lim_{r \to 1} \langle \tau^w_r, \alpha \rangle \) and
\[
\langle \tau^w_r, \alpha \rangle = \frac{1}{T^w(r)} \left( \log^+ \frac{r}{|\xi|} \right) \langle (\phi^w)^*(\alpha) \rangle
\]
\[
= \frac{1}{T^w(r)} \left( \log^+ \frac{r}{|\xi|} \right) \langle (\phi^w)^*(\alpha) + dd^c(u \circ \phi^w) - \frac{1}{T^w(r)} \langle \log^+ \frac{r}{|\xi|} dd^c(u \circ \phi^w) \rangle. \]

Since \( (\phi^w)^*(\alpha) + dd^c(u \circ \phi^w) = (\phi^w)^*[Z] \), the first term in the last line is \( \geq 0 \). By Jensen’s formula, the second term is
\[
- \frac{1}{T^w(r)} \left( \int_{|\zeta|=r} (d^c \log |\zeta|) (u \circ \phi^w) - (u \circ \phi^w)(0) \right).
\]
Observe that \( \frac{\langle u \circ \phi^w \rangle(0)}{T^w(r)} \to 0 \) as \( r \to 1 \) since \( (u \circ \phi^w)(0) > -\infty \) and \( T^w_r \to \infty \) as \( r \to 1 \). So the limit of the second term is \( \lim_{r \to 1} \frac{1}{T^w(r)} \int_{|\zeta|=r} (d^c \log |\zeta|) |u \circ \phi^w| \) which is clearly \( \geq 0 \). Hence, \( \langle T, \alpha \rangle \geq 0 \).

The case where \( \mathcal{F} \) is a lamination can be treated similarly. Indeed, we only need to replace [6, Theorem 1.1] by [6, Theorem 7.1].

This argument is similar to McQuillan [18] and Burns–Sibony [4].

**End of the proof of Theorem 1.2.** Let \( T_0 \) be a directed positive closed current with \( \int_X T_0 \wedge \omega = 1 \). Let \( T \) be a directed positive \( dd^c \)-closed current such that \( \int_X T \wedge \omega = 1 \). By Lemma 2.6, \( T - T_0 \) is a closed current. Hence, \( T \) is closed as claimed.

Since the singularities are hyperbolic and near such a point the holonomy is contracting, if a directed positive closed current passes through such a point, it gives mass to a separatrix, and hence there is a closed invariant curve. Consequently, the support of \( T \) is disjoint from the singularities.

Next, by Lemma 2.7 the class \( \{T\} \) is nef. It remains to show that \( \{T\}^2 = 0 \). By (2.2), write \( T = \Omega + \partial S + \bar{\partial} S \). Since \( T \) is closed, we get that \( \bar{\partial} S = 0 \). Applying Theorem 2.2 and 2.4 to \( T \), we infer that \( T \otimes T \) admits zero as the unique tangent current along the diagonal \( \Delta \). This, combined with the mass formula (2.4) and the equality \( \bar{\partial} S = 0 \), implies that \( 0 = \int_X \Omega^2 \). Hence, \( \{T\}^2 = 0 \).

**End of the proof of Theorem 1.3.** The proof is similar to that of Theorems 1.1 and 1.2 with few modifications. We leave it to the reader.

3. Positive \( dd^c \)-closed currents and tensor products

3.1. **Preliminary results.** Let \( (X, \omega) \) be a compact Kähler surface. Let \( \langle \Omega_1, \Omega_2 \rangle \) be a scalar product on the finite dimensional space of \( \square \)-harmonic \((1, 1)\)-forms. Let \( H_c \) be the real vector space spanned by all positive \( dd^c \)-closed \((1, 1)\)-currents on \( X \). We consider the following inner product and seminorm:
\[
\langle T_1, T_2 \rangle_c := \langle \Omega_1, \Omega_2 \rangle + \frac{1}{2} \int \bar{\partial} S_1 \wedge \partial S_2 + \frac{1}{2} \int \bar{\partial} S_2 \wedge \partial S_1,
\]
\[
\| T \|_c^2 := \langle \Omega, \Omega \rangle + \int \bar{\partial} S \wedge \partial S,
\]
where \( T_j = \Omega_j + \partial S_j + \bar{\partial} S_j \) for \( j \in \{1, 2\} \).

**Proposition 3.1.**
(1) If \( T \in H_c \), then \( \| T \|_c = 0 \) if and only if \( T = dd^c u \) for some real function \( u \in L^1(X) \).
(2) There is a constant $c = c_X$ such that every $T \in \mathcal{H}_e$ can be written as

$$T = \Omega + \partial S + \overline{\partial S} + \partial^\epsilon u,$$

where $S$ and $\Omega$ are as in (2.2) and $u \in L^1(X)$ is a real function such that

$$\|S\|_{L^2}, \|\partial S\|_{L^2}, \|\overline{\partial S}\|_{L^2} \leq c\|T\|_e.$$

Proof. See [11], Props. 2.6, 2.7 and Thm. 2.9. \qed

**Proposition 3.2.** Let $T_1, T_2 \in \mathcal{H}_e$. By Proposition 3.1 we can write

$$T_j = \Omega_j + \partial S_j + \overline{\partial S}_j + \partial^\epsilon u_j \quad \text{for} \quad j \in \{1, 2\}.$$

Then for every closed smooth form $R$ of bidegree $(2, 2)$ on $X \times X$, we have

$$\langle T_1 \otimes T_2, R \rangle = \langle \Omega_1 \otimes \Omega_2, R \rangle + \langle \partial S_1 \otimes \overline{\partial S}_2, R \rangle + \langle \overline{\partial S}_1 \otimes \partial S_2, R \rangle + \langle \partial S_1 \otimes \partial S_2, R \rangle.$$

If moreover $R$ is $\partial^\epsilon$-exact, then

$$\langle T_1 \otimes T_2, R \rangle = \langle \partial S_1 \otimes \overline{\partial S}_2, R \rangle + \langle \overline{\partial S}_1 \otimes \partial S_2, R \rangle = -\langle \partial S_1 \otimes \partial S_2, R \rangle.$$

Proof. Using the above decompositions of $T_1$ and $T_2$, we see that

$$\langle T_1 \otimes T_2, R \rangle = \langle \Omega_1 \otimes \Omega_2, R \rangle + \langle \partial S_1 \otimes \overline{\partial S}_2, R \rangle + \langle \overline{\partial S}_1 \otimes \partial S_2, R \rangle
+ \langle \partial S_1 \otimes \partial S_2, R \rangle + \langle \overline{\partial S}_1 \otimes \overline{\partial S}_2, R \rangle
+ \sum_k \langle \partial P_k \otimes Q_k, R \rangle + \langle \overline{\partial P}_k' \otimes Q'_k, R \rangle,$$

where $\sum$ denotes a finite sum, and $P_k, Q'_k$ are currents of bidegree $(0, 1)$ and $Q_k, Q'_k$ are closed current of bidegree $(1, 1)$. Applying Stokes’ theorem and using a bidegree argument and the assumption on $S$, we infer that

$$\langle \partial S_1 \otimes \partial S_2, R \rangle = 0 \quad \text{and} \quad \langle \overline{\partial S}_1 \otimes \overline{\partial S}_2, R \rangle = 0,$$

$$\langle \partial P_k \otimes Q_k, R \rangle = 0 \quad \text{and} \quad \langle \overline{\partial P}_k' \otimes Q'_k, R \rangle = 0,$$

and that

$$\langle \partial S_1 \otimes \overline{\partial S}_2, R \rangle = -\langle \overline{\partial S}_1 \otimes \partial S_2, R \rangle,$$

$$\langle \overline{\partial S}_1 \otimes \partial S_2, R \rangle = -\langle \partial S_1 \otimes \overline{\partial S}_2, R \rangle.$$
Lemma 3.3. The operator $P$ maps continuously $\mathcal{M}(\mathbb{D}^2)$ into $L^{1+\delta}(\mathbb{D}^2)$, $L^p(\mathbb{D}^2)$ into $L^q(\mathbb{D}^2)$, $L^\infty$ into $cC^0$ respectively, with norm $\leq C$, where $q = \infty$ if $p^{-1} + (1+\delta)^{-1} \leq 1$ and $p^{-1} + (1+\delta)^{-1} = 1 + q^{-1}$ otherwise.

We list here two examples of kernels, satisfying the above condition, that we will use.

Example 3.4. Consider a kernel $K$ with coefficients satisfying the inequality

$$K(x, y) \leq C'\|x - y\|^{-2}$$

for some constant $C' > 0$.

In this case we can choose any $0 < \delta < 1$.

Example 3.5. Consider a family of convolution kernels

$$K_r(x, y) = r^{-d}g_r(x, y)1_{\{\|x - y\| < r\}},$$

where $1_{\{\|x - y\| < r\}}$ is the characteristic function of the set $\{\|x - y\| < r\} \subset \mathbb{D}^2 \times \mathbb{D}^2$ and $(g_r)$ is a family of uniformly bounded $(2, 2)$-forms. In this case $\delta = 0$. Let $P_r$ be the operator with kernel $K_r$, mapping $L^p(\mathbb{D}^2)$ to itself. We see easily that norm of $P_r$ is bounded by a constant independent of $r$.

3.2. Mass estimates. In the rest of this section we analyze the repartition of mass (density) of the tensor product of positive $dd^c$-closed current near a subvariety: the diagonal $\Delta$. The estimates are given in Lemma 3.12 and they will be used in the study of tangent currents. We will use some currents $R_m$ we construct as test forms.

Let $(X, \omega)$ be a compact Kähler surface and let $\Delta := \{(x, x) : x \in X\}$ be the diagonal of $X \times X$. Consider the Kähler form $\hat{\omega} := \pi_1^*\omega + \pi_2^*\omega$ on $X \times X$, where $\pi_i$ is the canonical projection of $X \times X$ onto its $i$-th factor. Let $\Pi : X \times X \rightarrow X \times X$ be the blow-up of $X \times X$ along $\Delta$ and let $\hat{\Delta} := \Pi^{-1}(\Delta)$ be the exceptional hypersurface. By Blanchard’s theorem $[2]$, $\hat{X} \times \hat{X}$ can be endowed with a Kähler form $\hat{\omega}$. The current $\Pi_*(\hat{\omega})$ is positive closed and has positive Lelong number along $\Delta$ and is smooth outside $\Delta$. Multiplying $\hat{\omega}$ by a positive constant allows us to assume that the Lelong number of $\Pi_*(\hat{\omega})$ along $\Delta$ is equal to 1. So

$$\Pi^*(\Pi_*\hat{\omega}) = \hat{\omega} + [\Delta].$$

We can find a negative quasi psh function $\phi$ on $X \times X$ such that $dd^c\phi - \Pi_*(\hat{\omega})$ is a smooth form. Replacing $\phi(x, y)$ with $\frac{1}{2} = \phi(x, y) + \frac{1}{2}\phi(y, x)$ allows us to assume that $\phi$ is symmetric, i.e. invariant under the involution $(x, y) \mapsto (y, x)$. There is a constant $c > 0$ such that $dd^c\phi - \Pi_*(\hat{\omega}) + c\hat{\omega} \geq 0$, and hence

$$\Pi^*dd^c\phi - \hat{\omega} + c\Pi^*\hat{\omega} \geq 0.$$

We cover $\Pi^{-1}(\mathbb{D}^2 \times \mathbb{D}^2)$ with 2 charts $U_i$. We describe only one of them. The other is similar. The chart we consider is denoted by $\mathbb{D}^2 \times \mathbb{D}^2$ and is given with local coordinates

$$w = (w_3, w_4, w_1, w_2)$$

with $|w_1| < 1$, $|w_2| < 1$ and $|w_3| < 2$, $|w_4| < 2$.

and such that

$$\Pi(w) = (w_3, w_3 w_4, w_1, w_2) = (z_1, z_2, w_1, w_2).$$

In this chart the hypersurface $\hat{\Delta}$ is equal to $\{w_3 = 0\}$. Since $\alpha := dd^c(\phi \circ \Pi) - [\Delta]$ is a smooth form and $dd^c\log|w_3| = [\Delta]$ on $\hat{X} \times \hat{X}$, the function $\phi \circ \Pi - \log|w_3|$ is smooth. So
ψ := φ − log ||z|| is bounded in $\mathbb{D}^4$. Now the dilating map $A_\lambda$ for $E$ in $\mathbb{D}^4$ can be expressed as

(3.2) \[ A_\lambda(z_1, z_2, w_1, w_2) := (\lambda z_1, \lambda z_2, w_1, w_2) \text{ for } (z_1, z_2, w_1, w_2) \in \mathbb{D}^4 \text{ and } \lambda \in \mathbb{C}. \]

We now construct some test forms $R_m$ in order to estimate the mass of $T_1 \otimes T_2$ near $\Delta$. Let $\lambda$ be a constant large enough such that $|\phi - log ||z||| \leq M$ on each chart $U_i$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be an increasing convex smooth function such that $\chi(t) = 0$ for $t \leq -3M$, $\chi(t) = t$ for $t \geq 3M$, $0 \leq \chi(t) \leq 1$, and $\chi''(t) \in \left[\frac{1}{(10M)}, \frac{1}{(5M)}\right]$ for $t \in [-M, 2M]$. Fix a constant $A \gg M$ large enough. Define for $m \in \mathbb{N}$

\[ R_m := Add^\ast[\chi(\phi + m)] + A^2 \hat{\omega}. \]

This is clearly a smooth closed $(1, 1)$-form on $X \times X$. We first show that it is positive and has bounded mass. A direct computation gives

(3.3) \[ R_m = A\chi'(\phi + m)dd^\ast\phi + \frac{A}{n}\chi''(\phi + m)i\partial\phi \wedge \bar{\partial}\phi + A^2 \hat{\omega}. \]

The second term is positive. The first term is bounded below by $Ac$\(\chi\) for $C > 0$ large enough. Define $R_m$ to be positive since $A$ is chosen large enough. Now, since $R_m$ is cohomologous to $A^2 \hat{\omega}$, its mass is equal to the mass of $A^2 \hat{\omega}$ and hence is bounded.

The following technical lemmas are needed.

**Lemma 3.6.** The following inequality holds for a constant $c > 0$ independent of $m$:

\[ e^{2m}(idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2) \leq cR_m \text{ on } \{e^{-m-1} \leq |z| \leq e^{-m}, \ |w| \leq 1\}. \]

**Proof.** In the considered domain we have $|\phi + m| \leq 2M$, and therefore, $\chi'(\phi + m) \geq \frac{1}{10}$. Define $\hat{\phi} = \phi \circ \Pi$ and write $\hat{\phi} = \log |w_3| + \hat{\psi}$. So $\hat{\psi}$ is a smooth function on $\mathbb{D}$. On the other hand, on $\Pi^{-1}\{e^{-m-1} \leq ||z|| \leq e^{-m}, \ |w| \leq 1\}$ the form $i\partial\hat{\phi} \wedge \bar{\partial}\hat{\phi}$ is equal to

\[
\begin{align*}
&i\partial(\hat{\psi} + \log |w_3|) \wedge \bar{\partial}(\hat{\psi} + \log |w_3|) \\
&= -3i\partial\hat{\psi} \wedge \bar{\partial}\hat{\psi} + i\partial(2\hat{\psi} + \frac{1}{2} \log |w_3|) \wedge \bar{\partial}(2\hat{\psi} + \frac{1}{2} \log |w_3|) \\
&\quad + \frac{3}{4} i\partial \log |w_3| \wedge \bar{\partial} \log |w_3| \\
&\geq -3i\partial\hat{\psi} \wedge \bar{\partial}\hat{\psi} + \frac{3}{16} e^{2m} 1d\bar{z}_3 \wedge d\bar{w}_3 \text{ since the second term in the last sum is positive.}
\end{align*}
\]

As the first term in the last line is a bounded form, using that $\Pi^*dd^\ast\phi \geq \hat{\omega} - c\Pi^*\hat{\omega}$ we see easily that for $A \gg M \gg 1$ large enough,

\[ \Pi^*R_m \geq \frac{A}{100M}(e^{2m} 1dw_3 \wedge d\bar{w}_3 + \hat{\omega}). \]

On the other hand, we can find bounded forms on $\{e^{-m-1} \leq ||z|| \leq e^{-m}, \ |w| \leq 1\}$ such that

\[ \Pi^*(ie^{2m}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)) = e^{2m} dd^\ast|w_3|^2 + e^m dw_3 \wedge \theta_1 + e^m d\bar{w}_3 \wedge \theta_2 + \theta_3. \]

Cauchy-Schwartz inequality implies that the last sum is bounded above by $2e^{2m} dd^\ast|w_3|^2 + \theta_4$ for some bounded form $\theta_4$. This, combined with the previous estimate for $\Pi^*R_m$, implies the conclusion of the lemma. \qed
Lemma 3.7. For each $0 < r \leq 1$, let $m$ be the integer such that $e^{-m-1} < r \leq e^{-m}$. Then

$$ir^{-2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \leq c \sum_{n=0}^{\infty} e^{2-2n}R_{m+n} \text{ on } \{0 < |z| < r, |w| < 1\},$$

where $c$ is a constant independent of $r$.

**Proof.** Observe that $r^{-2} \leq e^{2m+2}$. Applying Lemma 3.6 yields the estimate. \hfill \Box

Fix a constant $0 < r_0 \ll 1$. For $0 < r < r_0$ and $x \in X$, let

$$\mathbb{B}(x, r) := \{y \in X : \phi(x, y) < \log r\}.$$

In the local coordinates $(z, w)$ of the chart $\mathbb{D}^2 \times \mathbb{D}^2 = \Pi(\mathbb{D}^2 \times \mathbb{D}^2)$, we have for $x - y = z$ and $y = w$ that $\phi(x, y) = \log ||x - y|| + O(1)$. Consequently, there is a constant $c > 1$ independent of $r$ and $x$ such that

$$\mathbb{B}_\omega(x, c^{-1}r) \subset \mathbb{B}(x, r) \subset \mathbb{B}_\omega(x, cr),$$

where $\mathbb{B}_\omega(x, r)$ is the ball with center $x$ and radius $r$ with respect to the metric $\omega$.

Let $T$ be a positive $dd^c$-closed current defined on $X$. Then we have the following special case of Lelong-Jensen identity (see e.g. [27, Prop. 1]):

$$r_2^{-2} \int_{\mathbb{B}(x, r_2)} T \wedge dd_y^c e^{2\phi(x, y)} - r_1^{-2} \int_{\mathbb{B}(x, r_1)} T \wedge dd_y^c e^{2\phi(x, y)} = 2 \int_{\mathbb{B}(x, r_2) \setminus \mathbb{B}(x, r_1)} T \wedge dd_y^c \phi(x, y)$$

for every $0 < r_1 < r_2 < r_0$.

**Lemma 3.8.** Let $T$ be a positive $dd^c$-closed current of mass 1 on $X$. Then there exist a constant $r_0 > 0$ small enough and a constant $c > 0$ large enough depending only on $X$ with the following properties:

1. For all $x \in X$ and $0 < r < r_0$,

$$\nu(T, x, r) = \frac{1}{r^2} \int_{y \in \mathbb{B}(x, r)} T(y) \wedge dd_y^c e^{2\phi(x, y)} \leq c.$$

In particular, $\nu(T, x) \leq c$ for all $x \in X$.

2. For all $x \in X$, we have that

$$\int_{y \in X} T(y) \wedge dd_y^c \phi(x, y) \leq c.$$

**Proof.** Since $X$ is compact, we cover it by a finite number of charts. Let $r_0 > 0$ be a constant small enough such that for all $x \in X$, $\mathbb{B}(x, r_0)$ is contained in such a chart. As there is a constant $c > 0$ such that $dd^c \phi \geq c\tilde{\omega}$, we infer from Lelong-Jensen identity (3.5) that $\nu(T, x, r)$ is essentially increasing in $r \in (0, r_0]$, that is, $\nu(T, x, r_1) \leq \nu(T, x, r_2) + c'$ for $0 < r_1 < r_2 < r_0$, where $c' = \frac{1}{2} \int_X T \wedge \omega$. Assertion (1) follows.

Assertion (2) is a consequence of assertion (1) and Lelong-Jensen identity (3.5). \hfill \Box

Recall that $\tilde{\omega} := \pi^*_x \omega + \pi^*_y \omega$, so $\tilde{\omega}(x, y) = \omega(x) + \omega(y)$ is a Kähler form on $X \times X$.

**Lemma 3.9.** Let $T_1$ and $T_2$ be two positive $dd^c$-closed $(1, 1)$-currents of mass 1 on $X$. Then

$$\langle T_1(x) \otimes T_2(y), dd^c \phi(x, y) \wedge \tilde{\omega}(x, y) \rangle < c,$$

where $c > 0$ is a constant depending only on $X$. 
Proof. A bidegree consideration shows that
\[ \langle T_1(x) \otimes T_2(y), dd^c\phi(x,y) \wedge \tilde{\omega}(x,y) \rangle = \langle T_2(y), \langle T_1(x), dd^c_\nu \phi(x,y) \rangle \omega(y) \rangle + \langle T_1(x), \langle T_2(y), dd^c_\nu \phi(x,y) \rangle \omega(x) \rangle \]

On the other hand, by Lemma 3.8, there is a constant \( c' > 0 \) such that
\[ \langle T_1(x), dd^c_\nu \phi(x,y) \rangle < c' \quad \text{and} \quad \langle T_2(y), dd^c_\nu \phi(x,y) \rangle < c'. \]

Consequently, we infer that
\[ \langle T_1(x) \otimes T_2(y), dd^c\phi(x,y) \wedge \tilde{\omega}(x,y) \rangle \leq c'(\langle T_2(y), \omega(y) \rangle + \langle T_1(x), \omega(x) \rangle) < \infty. \]

\[ \square \]

Lemma 3.10. Let \( T_1 \) and \( T_2 \) be two positive \( dd^c \)-closed \((1,1)\)-currents of mass 1 on \( X \). Then there is a constant \( c > 0 \) such that
\[ \langle T_1 \otimes T_2, R_m \wedge \tilde{\omega} \rangle \leq c \quad \text{for all} \quad m \geq 1. \]

Proof. Since \( \chi'' \) is supported on \([-3M,3M]\), we see that the factor of the form \( i\partial \phi \wedge \overline{\partial} \phi \) in (3.3) is nonzero only if \( |\phi + m| \leq 3M. \) Moreover, we know that \( |\phi - \log ||z||| \leq M. \)

So the above factor is nonzero only if \( |m - \log ||z||| \leq 4M, \) that is, \( z \) belongs to the ring \( \{e^{-m-4M} \leq ||z|| \leq e^{-m+4M}\}. \) Moreover, using that \( \phi = \phi \circ \Pi = \log ||w_3| + \psi, \) where \( \psi \) is a smooth function on \( \hat{D}, \) we see that
\[ i\partial \phi \wedge \overline{\partial} \phi \leq c ||z||^{-2}dd^c ||z||^2 + \tilde{\omega}. \]

This, coupled with the expression of \( R_m \) in (3.3) and (3.4), implies that
\[ \langle T_1 \otimes T_2, R_m \wedge \tilde{\omega} \rangle \lesssim \langle T_1 \otimes T_2, \tilde{\omega}^2 \rangle + \langle T_1(x) \otimes T_2(y), dd^c \phi(x,y) \wedge \tilde{\omega} \rangle \]
\[ + \int e^{-m-4M} \leq ||x-y|| \leq e^{-m+4M} T_1(x) \otimes T_2(y), ||x-y||^{-2}dd^c ||x-y||^2 \wedge \tilde{\omega} \]
\[ = I + II + III, \]

where in the second line we write \( w = y \) and \( z = x-y \) for \( (x,y) \) near \( \Delta. \) It is clear that \( I \)

is finite. By Lemma 3.9 II is also finite. So it remains to show that III is bounded by a constant independent of \( m. \) Setting \( r := e^{-m+4M}, \) since \( ||x-y|| \geq e^{-m}, \) we have that
\[ III \leq e^{8M} \left( \int_{x \in X} (r^{-2} \int_{y \in B(x,r)} T_2(y) \wedge dd^c_\nu ||x-y||^2) T_1(x) \wedge \omega(x) \right) \]
\[ + \int_{y \in X} (r^{-2} \int_{x \in B(y,r)} T_1(x) \wedge dd^c_\nu ||x-y||^2) T_2(y) \wedge \omega(y) \right) \]
\[ \lesssim e^{8M} \left( \int_{x \in X} \nu(T_2, x, r) T_1(x) \wedge \omega(x) + \int_{y \in X} \nu(T_1, y, r) T_2(y) \wedge \omega(y) \right). \]

Applying Lemma 3.8 (1) to the two terms in last line yields that they are bounded. So is III.

\[ \square \]

Lemma 3.11. Let \( T_1 \) and \( T_2 \) be two positive \( dd^c \)-closed \((1,1)\)-currents of mass 1 on \( X. \) Then there is a constant \( c > 0 \) such that
\[ \langle T_1 \otimes T_2, R_m \wedge R_n \rangle \leq c \quad \text{for all} \quad m, n \geq 1. \]
Proof. By Proposition 3.1 we can write
\[ T_j = \Omega_j + \partial S_j + \overline{\partial S_j} + dd^c u_j \quad \text{for} \quad j \in \{1, 2\}. \]
Let \( T_1, T_2 \in \mathcal{H} \). Since \( R_m \wedge R_n \) is a closed smooth form of bidegree \((2, 2)\) on \( X \times X \), it follows from Proposition 3.2 that
\[
\langle T_1 \otimes T_2, R_m \wedge R_n \rangle = \langle \Omega_1 \otimes \Omega_2, R_m \wedge R_n \rangle - \langle \partial S_1 \otimes \overline{\partial S_2}, R_m \wedge R_n \rangle - \langle \partial S_1 \otimes \overline{\partial S_2}, R_m \wedge R_n \rangle
\]
\[
= I_{m,n} + II_{m,n} + III_{m,n}. \]
We need to show that three sequences \( I_{m,n}, II_{m,n} \) and \( III_{m,n} \) are bounded. Since \( \Omega_1 \otimes \Omega_2 \) is closed, it follows that
\[
\langle \Omega_1 \otimes \Omega_2, dd^c \chi (\phi + m) \wedge \omega \rangle = \langle \Omega_1 \otimes \Omega_2, dd^c \chi (\phi + m) \wedge dd^c \chi (\phi + n) \rangle = 0. \]
Therefore, we infer that \( I_{m,n} = A^4 \langle \Omega_1 \otimes \Omega_2, \hat{\omega}^2 \rangle \). Hence, \( I_{m,n} \) is uniformly bounded.

In order to show that the sequence \( II_{m,n} \) and \( III_{m,n} \) are bounded, we only need to prove that for every \( L^2 \)-forms \( f_1, f_2 \) of bidegree \((1, 1)\) on \( X \):
\[
\langle f_1 \otimes f_2, R_m \wedge R_n \rangle \leq c \| f_1 \|_{L^2} \| f_2 \|_{L^2} \quad \text{for a constant } c \text{ independent of } m, n. \tag{3.6} \]
Using (3.3) we see that the factor of \( dd^c \phi \) (resp. of \( i \partial \phi \wedge \overline{\partial \phi} \)) in the expression of \( R_m \) vanishes outside \( \{ e^{-m-4M} \leq \| z \| \leq 1 \} \) (resp. outside \( \{ e^{-m-4M} \leq \| z \| \leq e^{-m+4M} \} \)). Moreover recall that \( \phi - \log \| z \| \) is a bounded function on each \( U_j \). We can localize (3.6), that is, we may assume that \( f_1 \) and \( f_2 \) are both compactly supported in \( V_i \), where we may assume that \( U_i = V_i \times V_i \) for a given \( U_i \). Moreover, we may find a constant \( c > 0 \) and bounded forms \( h_{1,m,n} h_{2,m,n} h_{3,m,n} \) on \( V_i \times V_i \) such that
- \( |h_{1,m,n}| \leq c \| h_{2,m,n} \| \leq c1 \{ e^{-m-4M} \leq \| z \| \leq 1 \}; \quad |h_{3,m,n}| \leq c1 \{ e^{-m-4M} \leq \| z \| \leq e^{-m+4M} \}; \)
- Using that \( \hat{\phi} = \hat{\phi} \circ \hat{\Pi} = \log |w| + \hat{\psi} \), where \( \hat{\psi} \) is a smooth function on \( \hat{\mathbb{D}} \), the following equality holds:
\[
R_m \wedge R_n = h_{1,m,n} + \| z \|^{-2} h_{2,m,n} + \| z \|^{-4} h_{3,m,n}. \]
Consider the integral operator \( P \) defined on forms on \( V_i \times V_i \) with the kernel \( K(x, y) \) given by the right hand side of the last expression. Here we invoke Examples 3.4 and 3.5 by taking into account that \( \| z \| \approx \| x - y \| \) and setting \( r = e^{m+4M} \). So we can write
\[
\langle f_1 \otimes f_2, R_m \wedge R_n \rangle = \langle f_1, P(f_2) \rangle. \]
Applying Lemma 3.3 to \( K \) for \( \delta = 0 \), we get that \( \| P(f_2) \|_{L^2} \leq A \| f_2 \|_{L^2} \). Hence,
\[
\langle f_1 \otimes f_2, R_m \wedge R_n \rangle = \langle f_1, P(f_2) \rangle \leq A \| f_1 \|_{L^2} \| f_2 \|_{L^2}. \]
This completes the proof (3.6).

\[ \square \]

Lemma 3.12. Let \( T_1 \) and \( T_2 \) be two positive \( dd^c \)-closed \((1, 1)\)-currents of mass 1 on \( X \). Suppose that either the measure \( T_2 \wedge \omega \) gives no mass to the set \( \{ \nu(T_1, \cdot) > 0 \} \) or the measure \( T_1 \wedge \omega \) gives no mass to the set \( \{ \nu(T_2, \cdot) > 0 \} \). Then there is a constant \( c > 0 \) such that for each \( r : 0 < r < 1 \), there is a constant \( \epsilon_r > 0 \) depending on \( T_1 \) and \( T_2 \), such that \( \epsilon_r \to 0 \) as \( r \to 0 \) and the following estimates hold. For any continuous function \( f(z) \) with compact support in \( \{ |w| < 1, |z| < r \} \) with \( |f| \leq 1 \), we have
\[
|\langle T_1 \otimes T_2, f(z, w) d\gamma \wedge d\gamma' \wedge d\theta \wedge d\theta' \rangle| \leq \max(\epsilon_r r^k, cr^4). \]
Here \( d\gamma, d\gamma', d\theta \) and \( d\theta' \) may be \( dz_1, dz_2, dw_1, dw_2 \) or their complex-conjugates, \( k \) is the total degree of \( dz_i, dz_i \) in \( d\gamma \wedge d\gamma' \wedge d\theta \wedge d\theta' \).
Proof. Applying Fubini’s theorem and using that $T_1$ and $T_2$ do not give mass to a single point, it follows that $T_1 \otimes T_2$ has no mass on $\Delta$. Therefore, the measure $T_1 \otimes T_2 \land dw_1 \land \bar{d}w_1 \land dw_2 \land \bar{d}w_2$ has no mass on $\Delta$. We infer that its mass on $\{|w| \leq 1, |z| \leq r\}$ tends to 0 as $r \to 0$. Hence,

$$|\langle T_1 \otimes T_2, f(z, w)dw_1 \land \bar{d}w_1 \land dw_2 \land \bar{d}w_2 \rangle| \leq \epsilon_r.$$  

For the estimate $|\langle T_1 \otimes T_2, f(z, w)dz_1 \land \bar{d}z_1 \land dz_2 \land \bar{d}z_2 \rangle| \leq cr^4$, it is enough to consider the case where $f$ is positive. Let $m$ be the integer such that $e^{-m-1} < r \leq e^{-m}$. So $r^{-4}f(z, w)dz_1 \land \bar{d}z_1 \land dz_2 \land \bar{d}z_2$ is a positive form which is bounded from above by a positive constant (depending only on $f$) times $(ir^{-2}(dz_1 \land \bar{d}z_1 + dz_2 \land \bar{d}z_2))^2$. Since $T_1 \otimes T_2$ has no mass on $\Delta$, it follows from Lemma 3.7 that

$$\langle T_1 \otimes T_2, f(z, w)dz_1 \land \bar{d}z_1 \land dz_2 \land \bar{d}z_2 \rangle \leq cr^4 \sum_{n,n'=0} e^{4-2n-2n'} \langle T_1 \otimes T_2, R_{m+n} \land R_{m+n'} \rangle$$

on $\{|z| < r, |w| < 1\}$, where $c$ is a constant independent of $r$. The last sum is bounded because we know by Lemma 3.11 that the terms $\langle T_1 \otimes T_2, R_{m+n} \land R_{m+n'} \rangle$ are uniformly bounded. This proves the second estimate.

We now prove the inequality:

$$\langle T_1 \otimes T_2, f(z, w)dz_m \land \bar{d}z_n \land dw_p \land \bar{d}w_q \rangle| \leq \epsilon_r r^2 \quad \text{for} \quad 1 \leq m, n, p, q \leq 2.$$  

Assume that $T_2 \land \omega$ gives no mass to $\{\nu(T_1, \cdot) > 0\}$. Observe that if we write $w = y$ and $z = x - y$, then

$$r^{-2}\langle T_1 \otimes T_2, f(z, w)dz_m \land \bar{d}z_n \land dw_p \land \bar{d}w_q \rangle|$$

$$\leq \int_{y \in \chi} (r^{-2} \int_{x \in B(y, r)} T_1(x) \land dd_x \|x - y\|^2)T_2(y) \land \omega(y))$$

$$\approx \int_{y \in \chi} \nu(T_1, y, r)T_2(y) \land \omega(y).$$

Applying Lemma 3.8 (1) and Lebesgue’s dominated convergence to the expression in the last line, we see that it converges to the limit $\int_{y \in \chi} \nu(T_1, y)T_2(y) \land \omega(y)$ as $r$ tends 0. By the hypothesis, the above limit is equal to 0. This ends the proof of the third inequality.

We now prove the fourth inequality.

$$\langle T_1 \otimes T_2, f(z, w)dz_m \land \bar{d}z_n \land dw_p \land \bar{d}w_q \rangle| \leq \epsilon_r r^2 \quad \text{for} \quad 1 \leq m, n, p, q \leq 2.$$  

Let $\chi$ be a smooth function with compact support in $\{|w| < 1, |z| < r\}$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ in a neighbourhood of the support of $f$. By the Cauchy-Schwarz inequality we have

$$|\langle T_1 \otimes T_2, f(z, w)dz_1 \land dz_2 \land \bar{d}w_1 \land \bar{d}w_2 \rangle|^2 \leq |\langle T_1 \otimes T_2, \chi^2(z, w)dz_1 \land \bar{d}z_1 \land dz_2 \land \bar{d}z_2 \rangle| \cdot |\langle T_1 \otimes T_2, |f(z, w)|^2 dw_1 \land \bar{d}w_1 \land dw_2 \land \bar{d}w_2 \rangle|.$$  

We deduce the fourth inequality in the lemma from the first and second one by replacing $\epsilon_r$ with $\sqrt{c\epsilon_r}$.

We now prove the fifth inequality

$$\langle T_1 \otimes T_2, f(z, w)dz_m \land dw_n \land \bar{d}w_p \land \bar{d}w_q \rangle| \leq \epsilon_r r \quad \text{for} \quad 1 \leq m, n, p, q \leq 2.$$
Let $\chi$ be as before. We have by the Cauchy-Schwarz inequality
\[|\langle T_1 \otimes T_2, f(z, w)dz_1 \wedge dw_1 \wedge d\bar{w}_1 \wedge d\bar{w}_2 \rangle|^2 \leq |\langle T_1 \otimes T_2, \chi^2(z, w)dz_1 \wedge d\bar{z}_1 \wedge dw_1 \wedge d\bar{w}_1 \rangle| \cdot |\langle T_1 \otimes T_2, |f(z, w)|^2 dw_1 \wedge d\bar{w}_1 \wedge dw_2 \wedge d\bar{w}_2 \rangle|.
\]
We deduce the fifth inequality in the lemma from the third and the first one.

The last two inequalities
\[|\langle T_1 \otimes T_2, f(z, w)dz_m \wedge d\bar{z}_m \wedge d\bar{z}_p \wedge dw_q \rangle| \leq \epsilon r^3, \quad |\langle T_1 \otimes T_2, f(z, w)dz_m \wedge d\bar{z}_m \wedge d\bar{z}_p \wedge dw_q \rangle| \leq \epsilon r^3.
\]
can be obtained in the same way. \hfill $\square$

4. Existence and properties of tangent currents

This section is devoted to the proof of Part (1) of Theorem 2.2. It is divided into two subsections. In the first one we explain how to compute tangent currents using local coordinates. We show the existence of global tangent currents, i.e. Part (1) in Theorem 2.2 modulo a technical lemma. The proof of this lemma will be provided in the last subsection.

4.1. Tangent currents in the local setting. Fix a local holomorphic coordinate system $(z, w) = (z_1, z_2, w_1, w_2)$ of $X \times X$, with $|z| < 3$ and $|w| < 3$, defined on an open set such that $\Delta$ is given there by the equation $z = 0$. Consider the open set $U := \{|z| < 1, |w| < 1\}$ of $X \times X$ and for simplicity, we identify it with the unit polydisc $D^4$ in $\mathbb{C}^4$. So $\Delta \cap D^4$ is identified with $\{0\} \times D^2$. We can also, in a natural way, identify $E$ with $\mathbb{C}^2 \times D^2$ which is an open subset of $\mathbb{C}^4$ containing $D^4$. In these local coordinates, the map $A_\lambda$, introduced above, is given by
\[a_\lambda(z, w) = (\lambda z, w).
\]
We use here the notation $a_\lambda$ in order to avoid the confusion with the global map $A_\lambda$ on $E$. Tangent currents can be computed locally using the following result.

**Proposition 4.1.** Let $S$ and $\lambda_n$ be as in Theorem 2.2. Then in the above local coordinates we have
\[S = \lim_{n \to \infty} (a_{\lambda_n})_*(T) \quad \text{on} \quad \mathbb{C}^2 \times D^2.
\]

Note that the proposition also shows that $S$ does not depend on the choice of $\tau$ because the limit does not involve $\tau$.

**Definition 4.2.** Let $(\alpha_\lambda)$ be a family of $4$-forms in $X \times X$ or in $E$ depending on $\lambda \in \mathbb{C}$ with $|\lambda|$ larger than a constant. We say that this family is **negligible** if the support $\text{supp}(\alpha_\lambda)$ tends to $\Delta$ as $\lambda \to \infty$ and if we have in any local coordinate system as above

1. $\text{supp}(\alpha_\lambda) \cap \{||w|| < 1, |z| < 1\}$ is contained in $\{||z|| \leq A|\lambda|^{-1}\}$ for some constant $A > 0$ independent of $\lambda$;
2. The sup-norm of the coefficient of $dz_1 \wedge dz'_1 \wedge d\theta \wedge d\theta'$ is bounded by $O(\lambda^k)$, where as in Lemma 3.12, $dz_1, dz'_1, d\theta$ and $d\theta'$ may be $dz_1, dz_2, dw_1, dw_2$ or their complex-conjugates, and $k$ is the total degree of $dz_i$, $d\bar{z}_i$ in $dz \wedge dz'_1 \wedge d\theta \wedge d\theta'$;
3. The sup-norm of the coefficient of $dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ is $o(\lambda^d)$.

Note that Property (1) is often easy to check. If we use the coordinates $(\lambda z, w)$ instead of $(z, w)$, then Property (2) is equivalent to the following (2a)

- The sup-norm of the coefficients of $A_\lambda^*(dw \wedge d\bar{w} \wedge dz \wedge d\bar{z})$ is bounded;
• The sup-norm of the coefficient of \( d(\lambda z_1) \wedge d(\lambda \bar{z}_1) \wedge d(\lambda z_2) \wedge d(\lambda \bar{z}_2) \) tends to 0 as \( \lambda \to \infty \).

Property (2a) is often easy to check. So to check that a family is negligible, one often only has to bound the coefficient of \( d\bar{z}_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2 \).

We will need the notion of negligible family in the study of tangent currents. More precisely, we will use the following lemma to establish the properties of tangent currents.

**Lemma 4.3.** Let \((\alpha_\lambda)\) be a negligible family of smooth 4-forms in \(\mathbb{X} \times \mathbb{X}\). Let \(T_1, T_2\) be two positive \(dd^c\)-closed currents satisfying the assumption of Lemma 3.12. Then \(\langle T_1 \otimes T_2, \alpha_\lambda \rangle \to 0\) as \(\lambda \to \infty\).

**Proof.** We can use a partition of unity in order to work in local coordinates. So we can assume that the forms \(\alpha_\lambda\) have supports in \(\{|w| < 1, |z| < 1\}\). Since \(T_1 \otimes T_2\) is of bidegree \((2, 2)\), we only need to test forms of bidegree \((2, 2)\). Lemma 3.12 applied to \(r := A|\lambda|^{-1}\) with \(A\) from Definition 4.2 shows that \(\langle T_1 \otimes T_2, \alpha_\lambda \rangle \to 0\). \(\square\)

To study tangent currents, we need a description of \(\tau\) in local coordinates \((z, w)\) in a neighborhood \(U := \{(0, 0)\}\). Consider the Taylor expansion of order 2 of \(\tau\) in \(z, \bar{z}\) with functions in \(w\) as coefficients. Since \(\tau\) is smooth admissible, when \(z \to 0\), this map and its differential can be written as

\[
(4.1) \quad \tau(z, w) = (z + O(|z|^2), w + a(w)z + O(|z|^2)), \quad z = (z_1, z_2), \quad w = (w_1, w_2),
\]

and

\[
(4.2) \quad d\tau(z, w) = (dz + O^*(|z|^2), dw + O(1)dz + zO(1) + O^*(|z|^2)),
\]

where \(a(w)\) is a \(2 \times 2\) matrix whose entries are smooth functions in \(w\) and \(O^*(|z|^k)\) is any smooth 1-form that can be written as

\[
O^*(|z|^k) = O(|z|^{k-1})dz + O(|z|^{k-1})d\bar{z} + O(|z|^k).
\]

We also have

\[
(4.3) \quad d\tau^{-1}(z, w) = (dz + O^*(|z|^2), dw + O(1)dz + zO(1) + O^*(|z|^2)).
\]

**Lemma 4.4.** If \((\alpha_\lambda)\) is a negligible family of 4-forms on \(\mathbb{E}\), then \((\tau^*(\alpha_\lambda))\) is also a negligible family of 4-forms on \(\mathbb{X} \times \mathbb{X}\).

**Proof.** This is a direct consequence of the above local description of \(d\tau\). \(\square\)

Let \(\varphi : \mathbb{V} \to \mathbb{R}\) and \(s = (s_1, \ldots, s_k) : \mathbb{U} \to \mathbb{C}^k\) be \(\mathcal{C}^2\)-smooth functions, where \(\mathbb{U}, \mathbb{V}\) are open subsets of \(\mathbb{C}^k\). The following formula gives an explicit expression for \(dd^c(\varphi \circ s)\):

\[
\begin{align*}
\overline{\partial}(\varphi \circ s)(\zeta) &= \sum_{m,n=1}^k \frac{\partial^2 \varphi}{\partial \zeta_m \partial \bar{s}_n}(s(\zeta)) \partial \bar{s}_m \wedge \partial s_n + \sum_{m,n=1}^k \frac{\partial^2 \varphi}{\partial \bar{s}_m \partial \zeta_n}(s(\zeta)) \partial \bar{s}_m \wedge \partial s_n \\
&\quad + \sum_{m,n=1}^k \frac{\partial^2 \varphi}{\partial \zeta_m \partial \bar{s}_n}(s(\zeta)) \partial \bar{s}_m \wedge \partial s_n + \sum_{m,n=1}^k \frac{\partial^2 \varphi}{\partial \bar{s}_m \partial \zeta_n}(s(\zeta)) \partial \bar{s}_m \wedge \partial s_n \\
&\quad + \sum_{m=1}^k \frac{\partial \varphi}{\partial \zeta_m}(s(\zeta)) \partial \bar{s}_m + \sum_{m=1}^k \frac{\partial \varphi}{\partial \bar{s}_m}(s(\zeta)) \partial \bar{s}_m.
\end{align*}
\]

**Lemma 4.5.** We use the coordinates \((z, w)\) for \(\mathbb{U}\) and write \(s = (s_1, s_2, s_3, s_4) = a_\lambda \circ \tau\) for \(a\) given \(|\lambda| > 1\). Then the following properties hold.
(1) We have that
\[ a_\lambda^*(dw) = dw \quad \text{and} \quad \tau^*(a_\lambda^*(dw)) = dw + O(1)dz + O(|z|) \]
\[ a_\lambda^*(dz) = \lambda dz \quad \text{and} \quad \tau^*(a_\lambda^*(dz)) = \lambda dz + \lambda O^*|z|^2). \]

(2) For every \( 1 \leq n \leq 4 \),
\[ \partial_{s_n} = \lambda dz_n + O^*|z| \quad \text{if} \quad 1 \leq n \leq 2, \quad \overline{\partial s_n} = O^*|z| \quad \text{if} \quad 3 \leq n \leq 4, \]
\[ dd^c s_n = \sum_{m,k=1}^2 O(\lambda)|dz_m \wedge dz_k| + \sum_{m,k=1}^2 O(1)|dz_m \wedge dw_k| \]
\[ + \sum_{m,k=1}^2 O(1)|dz_m \wedge dw_k| + \sum_{m,k=1}^2 O(\lambda^{-1})|dw_m \wedge dw_k|. \]

(3) For every smooth compactly supported function \( \varphi : U \rightarrow \mathbb{R} \), and for every \( |\lambda| > 1 \), the following estimates hold.

(a) There is a constant \( c \) independent of \( \lambda \) such that \( \|(\varphi \circ a_\lambda \circ \tau)(z,w) - (\varphi \circ a_\lambda)(z,w)\| \leq C\|\varphi\|_{C^1} |\lambda|^{-1} \) for \( (z,w) \in U \) with \( \|z\| \leq |\lambda|^{-1} \).

(b) \( dd^c(\varphi \circ a_\lambda) \) can be written in the form:
\[ \sum_{m,k=1}^2 O(\lambda^2)|dz_m \wedge dz_k| + \sum_{m,k=1}^2 O(\lambda)|dz_m \wedge dw_k| + \sum_{m,k=1}^2 O(1)|dz_m \wedge dw_k| + \sum_{m,k=1}^2 O(\lambda^{-1})|dw_m \wedge dw_k|. \]

(c) \( dd^c(\varphi \circ a_\lambda \circ \tau) - dd^c(\varphi \circ a_\lambda) \) and \( \tau^*(dd^c(\varphi \circ a_\lambda)) - dd^c(\varphi \circ a_\lambda \circ \tau) \) and \( dd^c(\varphi \circ a_\lambda \circ \tau) \) can be written in the form:
\[ \sum_{m,k=1}^2 O(\lambda)|dz_m \wedge dz_k| + \sum_{m,k=1}^2 O(1)|dz_m \wedge dw_k| + \sum_{m,k=1}^2 O(1)|dz_m \wedge dw_k| + \sum_{m,k=1}^2 O(\lambda^{-1})|dw_m \wedge dw_k|. \]

Proof. Using the description of \( \tau \) and \( d\tau \) in (4.1) and (4.1), assertion (1) follows.

Writing \( dz = (ds_1, ds_2) \) and \( dw = (ds_3, ds_4) \), assertion (2) follows from assertion (1).

Now we turn to the proof of assertion (3).

Proof of assertion (3a). Write \( \tau = (\tau_1, \tau_2, \tau_3, \tau_4) \). Since \( \varphi \) is supported in \( U \), we may suppose without loss of generality that \( |z| < |\lambda|^{-1} \). Consequently, we infer from the description of \( \tau \) in (4.1) and the inequality \( |z| < |\lambda|^{-1} \) that
\[ (a_\lambda \circ \tau)(z,w) - a_\lambda(z,w) = (\lambda \tau_1, \lambda \tau_2, \lambda \tau_3, \lambda \tau_4) - (\lambda z, w) = (\lambda O(|z|^2), a(w)z + O(|z|^2)) = O(\lambda^{-1}). \]
This proves assertion (3a).

Proof of assertion (3b). Using \( a_\lambda(z,w) = (\lambda z, w) \) and
\[ \overline{\partial}(\varphi \circ a_\lambda)(z,w) = \frac{\partial^2 \varphi}{\partial \zeta_m \partial \zeta_n}(a_\lambda(z,w))_m \partial(a_\lambda(z,w))_m \wedge (a_\lambda(z,w))_n\),
assertion (3b) follows.
Proof of assertion (3c). To prove the first part of assertion (3c), recall that \( s = a_\lambda \circ \tau \). By formula (4.4), we get that

\[
\begin{align*}
\partial \partial \partial \partial (\varphi \circ a_\lambda \circ \tau)(z, w) &= \partial \partial \partial \partial (\varphi \circ a_\lambda)(z, w) \\
&= \sum_{m,n=1}^{k} \left( \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \partial s_m \wedge \partial s_n - \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(a_\lambda(z, w)) \partial (a_\lambda(z, w))_m \wedge \partial (a_\lambda(z, w))_n \right) \\
&+ \sum_{m,n=1}^{k} \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \partial s_m \wedge \partial s_n + \sum_{m,n=1}^{k} \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \partial \bar{s}_m \wedge \partial s_n \\
&+ \sum_{m,n=1}^{k} \frac{\partial \varphi}{\partial \zeta^m}(s(z, w)) \partial \bar{s}_m + \sum_{m=1}^{k} \frac{\varphi}{\partial \zeta^m}(s(z, w)) \partial \bar{s}_m \\
&= I + II + III.
\end{align*}
\]

To estimate the sum \( I \) we fix \( 1 \leq m, n \leq 4 \) and we make the following two observations. By assertion (3a),

\[
(4.5) \quad \left| \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) - \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(a_\lambda(z, w)) \right| \leq c\|\varphi\|_{C^3} |\lambda|^{-1}.
\]

Moreover, by assertions (1) and (2),

\[
(\partial s_n)(z, w) - \partial (a_\lambda(z, w))_n = \begin{cases} \lambda O^*(|z|^2), & \text{if } 1 \leq n \leq 2, \\
O(1)dz + O(|z|), & \text{if } 3 \leq n \leq 4.
\end{cases}
\]

Using the fact that \( |z| < |\lambda|^{-1} \) (see assertion (1)), we see that

\[
(4.6) \quad (\partial s_n)(z, w) - \partial (a_\lambda(z, w))_n = O(1)dz + O(|z|).
\]

Hence, a straightforward computation shows that \( I \) can be put in the given form.

Next, we show that each term in the sum \( II \) has the described form. We will consider the term \( \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \partial s_m \wedge \partial s_n \). The other ones can be handled similarly. By assertion (2) we have

\[
\partial s_n = O(\lambda)d\bar{z}(1) + O(\lambda)d\bar{z}(2) + O(|z|) \quad \text{if } 1 \leq n \leq 2, \quad \partial s_n = O(1) \quad \text{if } 3 \leq n \leq 4,
\]

\[
\overline{\partial s}_n = O(1)d\bar{z}(1) + O(1)d\bar{z}(2) + O(|z|) \quad \text{if } 1 \leq n \leq 2, \quad \overline{\partial s}_n = O(|z|) \quad \text{if } 3 \leq n \leq 4.
\]

Putting this together with the obvious estimate \( \| \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \|_{L^\infty} \leq \|\varphi\|_{C^2} \), we can check that the term in question has the desired form.

To conclude the proof of the first part of assertion (3c), we need to show that \( \frac{\partial \varphi}{\partial \zeta^m}(s(z, w)) \partial \bar{s}_m \) has the described form. This fact follows readily from the expression for \( dd^c s_m \) given in assertion (2).

Now we prove the second part of assertion (3c). Write (recall that \( s = a_\lambda \circ \tau \))

\[
\tau^*(dd^c(\varphi \circ a_\lambda)) - dd^c(\varphi \circ a_\lambda) = \sum_{m,n=1}^{4} \left( \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(s(z, w)) \partial s_m \wedge \bar{s}_n - \frac{\partial^2 \varphi}{\partial \zeta^m \partial \zeta^n}(a_\lambda(z, w)) \partial (a_\lambda(z, w))_m \wedge \bar{(a_\lambda(z, w))}_n \right).
\]
Using this and (4.6) and (4.5), we can check that the term in question has the described form.

Writing

$$dd^c(\varphi \circ a_\lambda \circ \tau) - \tau^*(dd^c(\varphi \circ a_\lambda)) = (dd^c(\varphi \circ a_\lambda \circ \tau) - dd^c(\varphi \circ a_\lambda)) - (\tau^*(dd^c(\varphi \circ a_\lambda)) - dd^c(\varphi \circ a_\lambda)),$$

the third part of assertion (3c) follows from the combination of the first and the second ones. □

In the following proposition we study properties of tangent currents. The proof is completed after Lemma 4.8.

**Proposition 4.6.** Let $\Phi$ be a continuous 4-form with support in a fixed compact subset $K$ of $\mathbb{E}$. Define $\Phi_\lambda := A^*_\lambda(\Phi)$ and $\Psi_\lambda := \tau^*A^*_\lambda(\Phi)$. The following properties hold.

1. If $\Phi \wedge \pi^*(\Omega) = 0$ for any smooth $(2,2)$-form $\Omega$ on $\Delta$, then the families of $(\Phi_\lambda)$ and $(\Psi_\lambda)$ are negligible.
2. If $\|\Phi\|_\infty \leq 1$, then $\limsup_{\lambda \to \infty} |\langle T_\lambda, \Phi \rangle|$ is bounded by a constant which does not depend on $\Phi$.
3. If $\Phi \wedge \pi^*(\Omega) \geq 0$ for any smooth positive $(2,2)$-form $\Omega$ on $\Delta$, then any limit value of $\langle T_\lambda, \Phi \rangle$, when $\lambda \to \infty$, is non-negative. In particular, the property holds when $\Phi$ is a positive $(2,2)$-form.
4. If $\Phi = dd^c\phi$ for some smooth $(1,1)$-form $\phi$ with compact support in $\mathbb{E}$, then $\langle T_\lambda, \Phi \rangle \to 0$ as $\lambda \to \infty$.

**Proof.** (1) If $(\chi_k)$ is a finite partition of unity for $\Delta$, then $(\chi_k \circ \pi)$ is a finite partition of unity for $\mathbb{E}$. Using such a partition, we can reduce the problem to the case where $\Phi$ has support in $\{|w| < 1/2, |z| < A/2\}$ with $(z,w)$ as above and $A > 0$ a constant. The hypothesis in (1) implies that the coefficient of $(idz_1 \wedge d\bar{z}_1) \wedge (idz_2 \wedge d\bar{z}_2)$ in $\Phi$ vanishes. Then, a direct computation shows that $(\Phi_\lambda)$ is negligible. By Lemma 4.4, the family $(\Psi_\lambda)$ is also negligible.

(2) As above, we can assume that $\Phi$ has support in $\{|w| < 1/2, |z| < A/2\}$. Modulo a negligible family of forms, thanks to the first assertion, we have $\Phi_\lambda \sim f_\lambda(z,w)|\lambda|^4(idz_1 \wedge d\bar{z}_1 \wedge idz_2 \wedge d\bar{z}_2)$, where $f_\lambda$ is a smooth function supported by $\{|w| < 1/2, |z| < A|\lambda|^{-1}/2\}$ and $|f_\lambda|$ is bounded by a constant. Then, we deduce from the above expansion of $d\tau$ that $\Psi_\lambda$ satisfies a similar property with support in $\{|w| < 1, |z| < A|\lambda|^{-1}\}$ when $\lambda$ is large enough. By Lemma 4.3, negligible families of forms do not change the limit we are considering. The second estimate in Lemma 3.12 implies the result.

(3) We can assume that $\Phi$ is as in (2). The hypothesis of (3) implies that the coefficient of $(idz_1 \wedge d\bar{z}_1) \wedge (idz_2 \wedge d\bar{z}_2)$ in $\Phi$ is positive. It follows that $f_\lambda \geq 0$. We also see using the expansion of $\tau$ that $\Psi_\lambda$ is the product of a positive function $g_\lambda$ with $(idz_1 \wedge d\bar{z}_1) \wedge (idz_2 \wedge d\bar{z}_2)$ plus a form in a negligible family. Since $T_1$ and $T_2$ are positive, we have $\langle T_1 \otimes T_2, g_\lambda(idz_1 \wedge d\bar{z}_1) \wedge (idz_2 \wedge d\bar{z}_2) \rangle \geq 0$. The result follows easily.

(4) Using a partition of unity and local coordinates, we can assume that $\phi = udd^c v$, where $u$ and $v$ are smooth functions supported in $\{|w| < 1/2, |z| < A/2\}$. Define $\phi_\lambda := A^*_\lambda \phi = (u \circ A_\lambda)dd^c(v \circ A_\lambda)$ and $\psi_\lambda := (\tau^*A^*_\lambda u)(dd^c\tau^*A^*_\lambda v)$. We have $(A^*_\lambda u)(z,w) = u(\lambda z, w)$ and $\tau^*(A^*_\lambda u) = u(\tau_1(z, w, \lambda \tau_2(z, w))$ and similar expressions for $v$ if we write $\tau = (\tau_1, \tau_2)$.
in the considered coordinates. Write
\[
\tau^* (\ddbar \phi_\lambda) - \ddbar \psi_\lambda = \tau^* \ddbar (u \circ A_\lambda) \wedge \tau^* \ddbar (v \circ A_\lambda) - \ddbar (\tau^* A^*_\lambda u) \wedge \ddbar (\tau^* A^*_\lambda v) \\
= \left( \tau^* \ddbar (u \circ A_\lambda) - \ddbar (u \circ A_\lambda \circ \tau) \right) \wedge \left( \tau^* \ddbar (v \circ A_\lambda) - \ddbar (v \circ A_\lambda) \right) \\
+ \left( \tau^* \ddbar (u \circ A_\lambda) - \ddbar (u \circ A_\lambda) \wedge \ddbar (v \circ A_\lambda) \right) \\
+ \left( \ddbar (u \circ A_\lambda \circ \tau) - \ddbar (u \circ A_\lambda) \wedge \left( \tau^* \ddbar (v \circ A_\lambda) - \ddbar (v \circ A_\lambda \circ \tau) \right) \right) \\
+ \ddbar (u \circ A_\lambda) \wedge \left( \tau^* \ddbar (v \circ A_\lambda) - \ddbar (v \circ A_\lambda \circ \tau) \right) \\
= \lambda I + II + III + IV.
\]

Applying assertions (3b) and (3c) of Lemma 4.5 and using Definition 4.2, we can check that each term among $I, II, III, IV$ forms a negligible family of 4-forms.

Now, it follows from Lemma 4.3 that
\[
\langle (T_1 \otimes T_2)_\lambda, \ddbar \phi \rangle = \langle T_1 \otimes T_2, \tau^* (\ddbar \phi_\lambda) \rangle = \langle T_1 \otimes T_2, \ddbar \psi_\lambda \rangle + o(1)
\]
as $\lambda \to \infty$. On the other hand, by Lemma 4.5 below we have
\[
\langle T_1 \otimes T_2, \ddbar \psi_\lambda \rangle = \langle T_1 \otimes T_2, \ddbar (\tau^* A^*_\lambda u) \wedge \ddbar (\tau^* A^*_\lambda v) \rangle \to 0 \quad \text{as} \quad \lambda \to \infty.
\]
The result follows. \qed

Consider any sequence $(\lambda_n)$ in $\mathbb{C}$ tending to infinity. The second assertion in Proposition 4.6 implies that we can extract a subsequence $(\lambda_n)$ such that $(T_1 \otimes T_2)_{\lambda_n}$ converges to a 4-current $S$ of locally finite mass in $\mathbb{E}$. The first assertion in that proposition shows that in the above local coordinates, if the coefficient of $d\bar{z}_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2$ in $\Phi$ vanishes then $\langle S, \Phi \rangle = 0$. Consequently, $S \wedge dw_j = 0$ and $S \wedge d\bar{w}_j = 0$. Hence, $S$ is a current of bi-degree $(2, 2)$. The third assertion of the last proposition implies that $S$ is positive. Finally, the fourth assertion is equivalent to saying that $S$ is $\ddbar$-closed.

**Lemma 4.7.** There is a positive measure $\nu$ on $\Delta$ such that $S = \pi^* (\nu)$. In particular, $S$ is closed.

**Proof.** We follow the argument in the proof of [3, Lemma 3.7].

Consider the family $\mathcal{G}$ of all positive $\ddbar$-closed $(2, 2)$-currents $R$ on $\mathbb{E}$ which are vertical in the sense that $R \wedge \pi^* (\Omega) = 0$ for any smooth form $\Omega$ of degree $\leq 4$ on $\Delta$.

**Claim.** If $S$ is any current in $\mathcal{G}$ and $u$ is a smooth positive function on $\Delta$, then $(u \circ \pi)S$ also belongs to $\mathcal{G}$.

Indeed, it is clear that $(u \circ \pi)S$ is a positive and vertical $(2, 2)$-current. The only point to check is that $(u \circ \pi)S$ is $\ddbar$-closed. Define $\tilde{u} := u \circ \pi$. We have $\ddbar S = 0$ and since $S$ is vertical, we get that $\tilde{u} \wedge S = 0$, $d\bar{u} \wedge S = 0$, and $\ddbar \tilde{u} \wedge S = 0$. Consequently, a straightforward calculation shows that
\[
\ddbar (\tilde{u} S) = d (d\bar{u} \wedge S) \ddbar (d\bar{u} \wedge S) \ddbar \tilde{u} \wedge S + \tilde{u} \ddbar S = 0,
\]
which completes the proof of the claim.

We infer from the claim that every extremal element in $\mathcal{G}$ is supported by a fiber of $\pi$ which is a complex plane. A positive $\ddbar$-closed current on a complex plane is defined by a positive pluriharmonic function. On the other hand, on each fiber of $\pi$ a positive plurisubharmonic function is necessarily constant. Hence, extremal elements in $\mathcal{G}$ are multiples of currents of integration on fibers of $\pi$. In order to get the lemma, we only need to show that $S$ is an average of those extremal currents.
Finally, consider the convex cone of positive $dd^c$-closed vertical currents $S$ as above. Observe that the set of currents with mass 1 is compact and is a basis of the considered cone. Therefore, Choquet’s representation theorem implies that any current in the cone is an average on the extremal elements. The lemma follows.

Proof of Proposition 4.7. Consider a smooth test 4-form $\Omega$ with compact support in $\mathbb{C}^2 \times \mathbb{D}^2$. Denote by $f(z, w)$ the coefficient of $dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ in $\Omega$. By the definition of $S$ and the above discussion on negligible families of forms, we see that only the component $f(z, w)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ of $\Omega$ matters in computing the limit. So

$$
\langle S, \Omega \rangle = \lim_{n \to \infty} \langle T_1 \otimes T_2, \tau^* A^*_\lambda_n(f(z, w)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2) \rangle
$$

and

$$
\langle T, a^*_\lambda \Omega \rangle = \lim_{n \to \infty} \langle T_1 \otimes T_2, a^*_\lambda_n(f(z, w)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2) \rangle
$$

So it is enough to check that the family

$$
|\lambda_n|^4 f(\lambda_n \tau_1, \tau_2) \tau^*_1 (dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2) - |\lambda_n|^4 f(\lambda_n z, w)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2
$$

is negligible. But this can be easily deduced from the local description of $\tau$ and $d\tau$ and the fact that $|z| \lesssim |\lambda|^{-1}$ in the support of the above form.

4.2. A convergence lemma. The main purpose of this subsection is to prove the following result which completes the proof of Proposition 4.6, i.e., the tangent currents are $dd^c$-closed.

Lemma 4.8. Let $u_1$ and $u_2$ be smooth functions supported in $\{|w| < 1/2, |z| < A/2\}$. Then for positive $dd^c$-closed (1, 1)-currents $T_1, T_2$, we have that

$$
\langle T_1 \otimes T_2, dd^c(\tau^* A^*_\lambda u_1) \wedge dd^c(\tau^* A^*_\lambda u_2) \rangle \to 0 \quad \text{as} \quad \lambda \to \infty.
$$

In order to prove this key lemma, we need to introduce some more auxiliary results and notation. Recall from Subsection 4.1 that we fix a local holomorphic coordinate system $(z_1, z_2, w_1, w_2)$ of $X \times X$, with $|z| < 3$ and $|w| < 3$, defined on an open set such that $\Delta$ is given there by the equation $z = 0$. Consider the open set $U := \{|z| < 1, |w| < 1\}$ of $X \times X$ and for simplicity, we identify it with the unit polydisc $\mathbb{D}^4$ in $\mathbb{C}^4$. So $\Delta \cap \mathbb{D}^4$ is identified with $\{0\} \times \mathbb{D}^2$. Here is a modified version of the $\ast$-norm which has been introduced in [5, Definition 2.1]. It is convenient because bounding by closed positive currents permits to compute cohomologically, avoiding uniform estimates.

Definition 4.9. Let $R$ be a continuous (1, 1)-current on $X \times X$ with no mass outside $U$. We define the $\ast$-norm of $R$ as the infimum of the constants $c \geq 0$ such that the real and imaginary parts of $R$ satisfy

$$
-c(\pi_1^* \omega + \pi_2^* \omega + dd^c \varphi) \leq \text{Re}(R), \text{Im}(R) \leq c(\pi_1^* \omega + \pi_2^* \omega + dd^c \varphi) \quad \text{on} \quad U
$$

for some quasi-psh smooth function $\varphi$ on $X \times X$ satisfying $dd^c \varphi \geq -\pi_1^* \omega - \pi_2^* \omega$ and which vanishes outside $U$. 

Let \( \Gamma \) be a form of bidegree \((1,0)\) or \((0,1)\) on \(X \times X\) with continuous coefficients compactly supported in \(U\). We define the \(*\)-norm of \(\Gamma\) by \(\|\Gamma\|_* := \|i\Gamma \wedge \overline{\Gamma}\|^{1/2}\). If \(\Gamma\) is a continuous 1-form compactly supported in \(U\), we define \(\|\Gamma\|_*\) as the supremum of the \(*\)-norms of its bidegree \((1,0)\) and bidegree \((0,1)\) components. We recall here some basic properties of the \(*\)-norm.

**Lemma 4.10.** ([9, Lemma 2.4]) The map \(\Gamma \mapsto \|\Gamma\|_*\) defines a norm on the space of 1-forms \(\Gamma\) vanishing outside \(U\) such that \(\|\Gamma\|_*\) is finite. If \(\Gamma_1, \Gamma_2\) are such forms, then
\[
\|\Gamma_1 \wedge \overline{\Gamma_2}\|_* \leq \|\Gamma_1\|_* \|\Gamma_2\|_*. 
\]

**Lemma 4.11.** ([9, Lemma 2.12]) Let \(R\) (resp. \(\Gamma\)) be a continuous form compactly supported in \(U\) and of bidegree \((1,1)\) (resp. \((1,0)\) or \((0,1)\)). Assume that their coefficients have modulus smaller or equal to 1. Then the forms \((A_\lambda)^*(R)\) and \((A_\lambda)^*(\Gamma)\) have \(*\)-norms bounded by a constant \(c\) independent of \(R, \Gamma\) and \(\lambda\).

We also need a modified version of the notion of \(*\)-negligible family introduced in [9, Definition 2.7].

**Definition 4.12.** Let \((R_\lambda)\) be a family of \(q\)-currents on \(X \times X\) with \(\lambda \in \mathbb{C}\) and \(|\lambda| \geq 1\). Assume that they are compactly supported in \(U\). We say that \((R_\lambda)\) is \(*\)-negligible if it can be written as a finite sum of families of \(q\)-currents of type
\[
\Gamma_1^\lambda \wedge \cdots \wedge \Gamma_j^\lambda,
\]
where
(a) for each index \(j\), the \(\Gamma_j^\lambda\) are continuous forms of the same bidegree \((1,0)\) or \((0,1)\) with \(*\)-norms bounded uniformly on \(\lambda\);
(b) there is an index \(j_0\) such that the \(*\)-norm of \(\Gamma_{j_0}^\lambda\) is of order \(O(\lambda^{-1})\).

It is easy to see that for a \(*\)-negligible family of forms \((R_\lambda)\), \(R_\lambda \to 0\) weakly as \(\lambda \to \infty\).

Using the coordinates \((z, w)\) of \(U\), we may identify \(U \subset X^2\) with an open set of the vector bundle \(E\) near \(\Delta\). Consequently, we may assume that the function \(u_j, A_\lambda, u_j \circ A_\lambda \circ \tau\) are defined on \(U\). Moreover, \(u_j, u_j \circ A_\lambda \circ \tau\) are compactly supported in \(U\). For \(\lambda\) large enough, set
\[
\phi_\lambda := dd^c(u_1 \circ A_\lambda) \wedge dd^c(u_2 \circ A_\lambda) \quad \text{and} \quad \psi_\lambda := dd^c(u_1 \circ A_\lambda \circ \tau) \wedge dd^c(u_2 \circ A_\lambda \circ \tau).
\]

**Lemma 4.13.** Under the assumption of Lemma 4.8, we have
(1) the family \(\psi_\lambda - \phi_\lambda\) is negligible as well as \(*\)-negligible;
(2) \(\phi_\lambda\) and \(\psi_\lambda\) converge weakly to 0 as \(\lambda \to \infty\).

**Proof.** We start with the the proof of assertion (1). Combining Lemma 4.10 and Lemma 4.11 there is a constant \(c\) independent of \(\lambda\) such that for \(j \in \{1, 2\}\), \(\|dd^c(u_j \circ A_\lambda)\|_* \leq c\). Therefore, there are positive closed smooth currents \(T_j^\pm\) of bidegree \((1,1)\) on \(X \times X\) such that
\[
\|dd^c(u_j \circ A_\lambda)\|_* = \|T_j^\pm\|_1 \quad \text{with} \quad \|T_j^\pm\|_* \leq c.
\]
So
\[
\phi_\lambda = (T_{\lambda, 1}^+ - T_{\lambda, 1}^-) \wedge (T_{\lambda, 2}^+ - T_{\lambda, 2}^-).
\]
Setting
\[
\tilde{\phi}_\lambda := T_{\lambda, 1}^+ T_{\lambda, 2}^+ + T_{\lambda, 1}^- T_{\lambda, 2}^-,
\]
we have that \( \phi_\lambda \leq \tilde{\phi}_\lambda^+ \). Let
\[
\tilde{\phi}_\lambda^- = \phi_\lambda^+ - \phi_\lambda = T_{\lambda,1}^+ \wedge T_{\lambda,2}^- + T_{\lambda,1}^- \wedge T_{\lambda,2}^+.
\]
So \( \phi_\lambda = \tilde{\phi}_\lambda^+ - \tilde{\phi}_\lambda^- \), where \( \tilde{\phi}_\lambda^\pm \) are positive closed currents of bidegree \((2,2)\) on \( X \times X \).
We infer from \( \|T_{\lambda,j}^\pm\| \leq c \) and from a routine cohomological argument that \( \|\tilde{\phi}_\lambda^\pm\| \leq c' \) for some constant \( c' \) independent of \( \lambda \).

Write
\[
\psi_\lambda - \phi_\lambda = (ddc^\infty(u_1 \circ A_\lambda \circ \tau) - ddc^\infty(u_2 \circ A_\lambda)) \wedge (ddc^\infty(u_2 \circ A_\lambda \circ \tau) - ddc^\infty(u_2 \circ A_\lambda))
+ (ddc^\infty(u_1 \circ A_\lambda \circ \tau) - ddc^\infty(u_1 \circ A_\lambda)) \wedge ddc^\infty(u_2 \circ A_\lambda)
+ ddc^\infty(u_1 \circ A_\lambda) \wedge (ddc^\infty(u_2 \circ A_\lambda \circ \tau) - ddc^\infty(u_2 \circ A_\lambda))
= I + II + III.
\]

Applying assertions (3b) and (3c) of Lemma 4.12 to \( I, II \) and \( III \), we can check that each of them is a negligible family according to Definition 4.2. Hence, \( \psi_\lambda - \phi_\lambda \) is a negligible family.

Consequently, the family of forms \( \psi_\lambda - \phi_\lambda \) is also \( \ast\)-negligible. This proves assertion (1).

Since the family of forms \( \psi_\lambda - \phi_\lambda \) is \( \ast\)-negligible, it follows that \( \psi_\lambda - \phi_\lambda \) converges weakly to 0 as \( \lambda \to \infty \). Moreover, as \( \phi_\lambda = \tilde{\phi}_\lambda^- - \tilde{\phi}_\lambda^+ \), where \( \tilde{\phi}_\lambda^\pm \) are positive closed currents with \( \|\tilde{\phi}_\lambda^\pm\| \leq c' \), we deduce from the compactness of the set of positive closed currents with uniformly bounded mass that any cluster value of \( \phi_\lambda \) has the form \( \phi^+ - \phi^- \) for some positive closed \((2,2)\)-currents \( \phi^\pm \). On the other hand, we infer from the expression of \( \phi_\lambda \) that its support tends to \( \Delta \) as \( \lambda \to \infty \). Therefore, there is a real constant \( c \) such that \( \phi^+ - \phi^- = c[\Delta] \). As \( \phi_\lambda \) is always cohomologous to 0, we get that \( c = 0 \). Thus, \( \phi_\lambda \), and hence \( \psi_\lambda \), converge to 0 as \( \lambda \to \infty \). Assertion (2) follows. □

End of the proof of Lemma 4.8

By Lemma 4.13 and Lemma 4.3, we can use \( \phi_\lambda \) instead of \( \psi_\lambda \). By Proposition 3.1, we can write
\[
T_j = \Omega_j + \partial S_j + \bar{\partial} S_j + ddc u_j \quad \text{for} \quad j \in \{1, 2\}.
\]

Let \( T_1, T_2 \in \mathcal{M} \). Since \( \phi_\lambda \) is a closed smooth form of bidegree \((2,2)\) on \( X \times X \), it follows from Proposition 3.2 that
\[
\langle T_1 \otimes T_2, \phi_\lambda \rangle = \langle \Omega_1 \otimes \Omega_2, \phi_\lambda \rangle - \langle \partial S_1 \otimes \partial \bar{S}_2, \phi_\lambda \rangle - \langle \partial \bar{S}_1 \otimes \bar{S}_2, \phi_\lambda \rangle
= I_\lambda + II_\lambda + III_\lambda.
\]

Moreover, we only need to show that \( I_\lambda = 0 \) and \( II_\lambda \to 0, III_\lambda \to 0 \) as \( \lambda \to \infty \). Since \( \phi_\lambda \) is a \( ddc^\infty \)-exact smooth form of bidegree \((2,2)\) on \( X \times X \) and \( \Omega_1, \Omega_2 \) are closed smooth \((1,1)\)-forms on \( X \), Stokes theorem gives that \( I_\lambda = 0 \).

In the next we will prove that that for every \( L^2 \)-forms \( f_1, f_2 \) of bidegree \((1,1)\) on \( X \):
\[
\langle f_1 \otimes f_2, \phi_\lambda \rangle \leq c \|f_1\|_{L^2} \|f_2\|_{L^2} \quad \text{for a constant} \quad c \text{ independent of} \ \lambda.
\]
and
\[
\lim_{\lambda \to \infty} \langle f_1 \otimes f_2, \phi_\lambda \rangle = 0.
\]

Clearly, (4.8) will imply that \( II_\lambda \to 0, III_\lambda \to 0 \) as \( \lambda \to \infty \), which finishes the proof.
For all $\lambda \in \mathbb{C}$ large enough, consider the integral operator $P_\lambda$ defined on forms on $X$ with the kernel $K_\lambda(x, y)$ given by $\phi_\lambda$. Here we invoke Example 3.3 with $r = |\lambda|^{-1}$. Note that $\phi_\lambda = A_\lambda^*(dd^c u \wedge dd^c v)$. So we can write

$$\langle f_1 \otimes f_2, \phi_\lambda \rangle = \langle f_1, P_\lambda(f_2) \rangle.$$  

Applying Lemma 3.3 to $K_\lambda$ with $\delta = 0$, we get that $\|P_\lambda(f_2)\|_{L^2} \leq A\|f_2\|_{L^2}$ for a constant $A$ independent of $\lambda$. Hence,

$$\|\langle f_1 \otimes f_2, \phi_\lambda \rangle\| = \|\langle f_1, P_\lambda(f_2) \rangle\| \leq A\|f_1\|_{L^2}\|f_2\|_{L^2}.$$  

This completes the proof (4.7).

Now we are in the position to prove (4.8). Pick an arbitrary $\epsilon > 0$, and we may assume that $\|f_1\|_{L^2} = \|f_2\|_{L^2} = 1$. Fix two smooth $(1, 1)$-forms $(f_{1, \epsilon})$ and $(f_{2, \epsilon})$ such that $\|f_{1, \epsilon} - f_{2, \epsilon}\|_{L^2} \leq \epsilon/(8c)$. By Lemma 4.13, we may find $\Lambda = \Lambda_\epsilon \gg 1$ such that for all $|\lambda| > \Lambda$,

$$\|\langle f_{1, \epsilon} \otimes f_{2, \epsilon}, \phi_\lambda \rangle\| < \epsilon/8.$$  

On the other hand, by (4.7),

$$\|\langle f_{1, \epsilon} - f_1 \otimes f_{2, \epsilon}, \phi_\lambda \rangle\| \leq c\|f_{1, \epsilon} - f_1\|_{L^2}\|f_{2, \epsilon}\|_{L^2} < \epsilon/4.$$  

Similarly, we have that

$$\|\langle f_1 \otimes (f_{2, \epsilon} - f_2), \phi_\lambda \rangle\| \leq c\|f_1\|_{L^2}\|f_{2, \epsilon} - f_2\|_{L^2} < \epsilon/4.$$  

Putting the last three inequalities together, we infer that for all $|\lambda| > \Lambda$, $\|\langle f_1 \otimes f_2, \phi_\lambda \rangle\| < \epsilon$. Hence, (4.8) follows.  

### 4.3. Construction of an admissible map.

Consider the smooth Kähler metric on $X \times X$ associated with the Kähler form $\omega := \pi_1^*\omega + \pi_2^*\omega$, where $\omega$ is a Kähler form on $X$. For each point $(x, x) \in \Delta$ denote by $N_x$ the orthogonal complement of the tangent space to $\Delta$ at $x$ in the tangent space to $X \times X$ at $(x, x)$, with respect to the considered metric. The union of $N_x$ for $(x, x) \in \Delta$ can be identified with the normal bundle $E$ of $\Delta$ in $X \times X$, but this identification is not a holomorphic map in general. We construct the map $\tau^{-1}$ from a neighbourhood of $\Delta$ in $E$ to a neighbourhood of $\Delta$ in $X \times X$ in the following way: for $y \in N_x$ close enough to $(x, x)$, $\tau^{-1}(y)$ is the image of $y$ by the exponential map, which is defined on the tangent space to $\Delta$ at $(x, x)$. We can check that $\tau$ is well-defined and is smooth admissible with $d\tau(x, x) = \text{id}$ for $(x, x) \in \Delta$ (see [9, Lemma 4.2]).

**Proposition 4.14.** In any local coordinates $(z, w)$ around a point $(x, x) \in \Delta$ such that $\Delta = \{z = 0\}$, we have for $|z| \to 0$,

$$\tau(z, w) = (z + O(1)z_1^2 + O(1)z_2 + O(1)\lambda z^3 + z + O(1)z + O(|z|^2), w + a(w)z + O(|z|^2)), \quad z = (z_1, z_2), \quad w = (w_1, w_2),$$

and

$$d\tau(z, w) = (dz + O^*(|z|^2), dw + O(1)dz + zO(1) + O^*(|z|^2)),$$

where we use the notation in (4.9)–(4.10).

**Proof.** The second assertion is a consequence of the first one. Observe also that the first identity is equivalent to the similar identity for $\tilde{\tau} := \tau^{-1}$. We will prove the last one. Since $d\tilde{\tau}(z, w)$ is the identity when $z = 0$, we have $\tilde{\tau}(z, w) = (z, w + a(w)z + O(|z|^2))$. So if we write

$$\tilde{\tau} = (\tilde{\tau}_{12}, \tilde{\tau}_{34}) = (\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4)$$
in coordinates \((z, w)\), we only have to check that
\[
(4.11) \quad \hat{\tau}_{12} = z + O(1)z_1^2 + O(1)z_1z_2 + O(1)z_2^2 + O(|z|^3).
\]
This property means there are no terms with Taylor expansion of \(\tilde{\tau}_1\) in \(z, \bar{z}\) with functions in \(w\) as coefficients. So it is enough to check it on each complex plane \(\{w\} \times \mathbb{C}^2\). Recall that in the local coordinates \((z, w)\) as above, we identify this complex plane with the fiber of \(E\) over \((0, w)\). We will need to make some changes of coordinates. So we first check that the property does not depend on our choice of coordinates.

Now consider another system of local holomorphic coordinates \((z', w')\) such that \(z' = 0\) on \(\Delta\). We can write \(w' = H(z, w)\) and \(z' = \alpha(w)z + h_{11}(z, w)z_1^2 + h_{12}(z, w)z_1z_2 + h_{22}(z, w)z_2^2\), where \(H\) and \(h_{11}, h_{12}, h_{22}\) are \(2 \times 1\) matrix whose entries are holomorphic functions, and \(\alpha\) is a \(2 \times 2\) matrix whose entries are holomorphic functions. For \(b' = H(0, b)\), the two complex planes \(\mathbb{C}^2 \times \{b\}\) for the coordinates \((z, w)\) and \(\mathbb{C}^2 \times \{b'\}\) for the coordinates \((z', w')\) are both identified with the same fiber of \(E\). The linear map connecting them is \((z, b) \mapsto (\alpha(b)z, b')\). We will keep the notation \(\hat{\tau} = (\hat{\tau}_{12}, \hat{\tau}_{34})\) for the map \(\hat{\tau}\) in coordinates \((z, w)\) and use
\[
\hat{\tau}' = (\hat{\tau}_{12}', \hat{\tau}_{34}') = (\hat{\tau}_1', \hat{\tau}_2', \hat{\tau}_3', \hat{\tau}_4')
\]
for the same map in coordinates \((z', w')\). With these notations, the point \(\hat{\tau}(\alpha(b)^{-1}(a)a', b)\) in coordinates \((z, w)\) and the point \(\hat{\tau}'(a', b')\) in coordinates \((z', w')\) represent the same point of \(X \times X\). It follows that
\[
\begin{align*}
\hat{\tau}'_{12}(a', b') &= \alpha(b)\hat{\tau}_{12}(\alpha(b)^{-1}a', b) + h_{11}(\hat{\tau}(\alpha(b)^{-1}a', b))\hat{\tau}_1(\alpha(b)^{-1}a', b)^2 \\
&+ h_{12}(\hat{\tau}(\alpha(b)^{-1}a', b))\hat{\tau}_1(\alpha(b)^{-1}a', b)\hat{\tau}_2(\alpha(b)^{-1}a', b) + h_{22}(\hat{\tau}(\alpha(b)^{-1}a', b))\hat{\tau}_2(\alpha(b)^{-1}a', b)^2.
\end{align*}
\]

We see that if \(\hat{\tau}_{12}(a, b) = a + O(1)a_1^2 + O(1)a_1a_2 + O(1)a_2^2 + O(|a|^3)\) then \(\hat{\tau}'_{12}(a', b')\) satisfies a similar property.

In the rest of the proof, we show (4.11). Without loss of generality, we will only check the property for \(w = 0\) and \(z = t\zeta\) with \(t \in \mathbb{R}^+\) and \(|\zeta| = 1\). In a neighbourhood of 0, we can write \(w = d\zeta u\) with \(u\) a smooth strictly psh function. Subtracting from \(u\) a pluriharmonic function, we can assume the existence of a positive definite \(4 \times 4\)-matrix \((c_{ij})\) such that
\[
u(z, w) = (z, w)(c_{ij}) \left(\begin{array}{c} \zeta \\ w \end{array} \right) + O(||(z, w)||^3).
\]
We will make changes of coordinates keeping the property \(\Delta = \{z = 0\}\). With a linear change of coordinates \((z, w) \mapsto (\alpha z, \beta w)\), where \(\alpha\) is a \(2 \times 2\)-matrix and \(\beta\) is a \(2 \times 4\)-matrix, we can assume that
\[
u(z, w) = |z|^2 + |w|^2 + O(||(z, w)||^3).
\]

Then, using a change of coordinates of type \(z_j \mapsto z_j + z_k A_{jk}(z, w)\), \(A_{jk}(z, w)\) being a linear form in \(z\) and \(w\), we can assume that the coefficients of all monomials in the last \(O(||(z, w)||^3)\) which can be factored by \(z_j\bar{z}_j\) vanish. Note that since \(u\) is real, when we eliminate the coefficient of a monomial, the coefficient of its complex conjugate is also eliminated. Next, using a change of coordinates of type \(w_j \mapsto \)
\[ w_j + \text{quadratic form in } z \text{ and } w, \text{ we can assume that the coefficients of all monomials } w_1z_3, w_2z_1w_3, w_2z_2w_4, \text{ in the last expression } O(\|z, w\|^2) \text{ and their conjugates vanish. It follows that there remain only monomials } z_1^2, w_2^2, w_jw_kz_i \text{ and their conjugates, that is,} \]

\[
\tilde{\omega} = \sum_{j=1}^{2} i dz_j \wedge d\bar{z}_j + \sum_{j=1}^{2} idw_j \wedge d\bar{w}_j + \sum_{1 \leq j, k \leq 2, j \neq k} O(\|w\|) i dz_j \wedge d\bar{w}_k + \sum_{1 \leq j, k \leq 2, j \neq k} O(\|w\|) idw_j \wedge d\bar{z}_k + O(\|z, w\|^2).
\]

For the rest of the proof, we use real coordinates \( x = (x^1, \ldots, x^8) \) such that \( z_1 = x^1 + ix^2, z_2 = x^3 + ix^4, \) and \( w_1 = x^5 + ix^6, w_2 = x^7 + ix^8. \) Denote by \( v = (v^1, \ldots, v^8) \) the unit tangent vector to \( X \times X \) at \( (0, 0) \) corresponding to \( (\zeta, 0) \in \mathbb{C}^4, \) i.e. \( v^5 = \cdots = v^8 = 0 \) and \( \eta_1 = v^1 + i\eta_2, \eta_2 = v^3 + i\eta_4. \) So \( \tilde{\tau}(t, 0) = 0 \) implies that \( \exp(t\eta), \) where \( \exp \) denotes the exponential map from the tangent space to \( X \times X \) at \( 0 \) to \( X \times X. \) If we write \( \tilde{\tau}(t, 0) = (x^1(t), \ldots, x^8(t)), \) then \( x^j(t) \) satisfy the geodesic equations

\[ \vec{x}^j = \Gamma^j_{kl} \vec{x}^k \vec{x}^l \quad \text{and} \quad \dot{x}^j(0) = v^j, \]

where \( \Gamma^j_{kl} \) are the Christoffel symbols associated with the considered Kähler metric.

We will show in the present setting that \( \tilde{\tau}_{12}(t, \zeta, 0) = t\zeta + O(t^3) \) and we already know that \( \tilde{\tau}_{12}(t, \zeta, 0) = t\zeta + O(t^2) \). This is equivalent to checking that \( \vec{x}^j(0) = 0 \) for \( 1 \leq j \leq 4. \) Note that the property implies that there is no term of order 2 in the Taylor expansion of \( \tilde{\tau}_{12}(z, 0) \) in the latest system of coordinates.

According to the discussion at the beginning of the proof, three terms with \( z_1^2, z_2^2, \) and \( z_1z_2 \) may appear when we come back to the original coordinates. Since \( v^j = 0 \) for \( 5 \leq j \leq 8, \) we only need to show that \( \Gamma^j_{kl}(0) = 0 \) for \( j, k, l \in \{5, \ldots, 8\}. \) Let \( g = (g_{jk}) \) be the Riemannian metric associated with \( \omega. \) The above description of \( \omega \) implies that \( g_{jk} = \delta_{jk} + O(\|x^5\| + \|x^6\| + \|x^7\| + \|x^8\| + \|x\|^2) \) for all \( j, k, \) where \( \delta_{jk} = 1 \) if \( j = k \) and \( 0 \) otherwise. The coefficients of the inverse \((g^{jk})\) of the matrix \((g_{jk})\) satisfy a similar property. Recall that the Christoffel symbols are given by

\[ \Gamma^j_{kl} = \frac{1}{2} g^{jm} \left( \frac{\partial g_{mk}}{\partial x_l} + \frac{\partial g_{ml}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_m} \right). \]

It is now easy to check that \( \Gamma^j_{kl}(0) = 0 \) for \( j, k, l \in \{5, \ldots, 8\}. \) The proposition follows

5. Proof of the Mass Formula

Recall from Subsection 2.1 that \( E \) is the normal bundle of the diagonal \( \Delta \) in \( X \times X \) and that \( \pi : E \to \Delta \) is its canonical projection. Recall also that \( \Delta \) is naturally identified with the zero section of \( E. \) We first give a weak regularization of the current of integration \([\Delta]\) in \( E. \) The mass of \( \nu \) will be computed by applying \( (T \otimes T) \lambda \) to these regularized approximations of \([\Delta]\). So we need to construct regularizations of \([\Delta].\)

Let \( \tilde{E} \) denote the blow-up of \( E \) along \( \Delta \) with the canonical projection \( \Pi : \tilde{E} \to E. \) Then \( \Delta := \Pi^{-1}(\Delta) \) is a compact smooth hypersurface in \( \tilde{E}. \) Let \( \phi \) be a function with support in a small neighborhood of \( \Delta \) in \( \tilde{E} \) such that in a local coordinate system \( w = (w_1, w_2, w_3, w_4) \) near a given point of \( \Delta \) as in Subsection 3.2 the function \( \psi(w) := \tilde{\phi}(w) - \log |w_3| \) is smooth, where \( \Delta \) is equal to \( \{w_3 = 0\}. \) This property does not depend on the choice of such a local coordinate system. Therefore, we can construct easily such a function.
\(\hat{\phi}\) using a partition of unity on a neighborhood of \(\tilde{\Delta}\) in \(\tilde{E}\). Note that \(\tilde{\phi}\) is smooth out of \(\tilde{\Delta}\). Let \(\Theta := [\tilde{\Delta}] - d\tilde{c}^2\hat{\phi}\). This is a smooth \((1,1)\)-form compactly supported in a small neighborhood of \(\tilde{\Delta}\) in \(\tilde{E}\). Let \(\chi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R}\) be an increasing convex smooth function such that \(\chi(t) = 0\) for \(t \in [-\infty, -1]\), \(\chi(t) = t\) for \(t \in [1, \infty]\) and \(0 \leq \chi' \leq 1\). Define, for \(n \in \mathbb{N}\), \(\chi_n(t) := \chi(t+n) - n\) and \(\phi_n := \chi_n \circ \phi\). So \(\hat{\phi}_n = \phi\) outside a tubular neighborhood of \(\tilde{\Delta}\) with radius of order \(e^{-n+1}\) and \(\hat{\phi}_n = 0\) inside a tubular neighborhood of \(\tilde{\Delta}\) with radius of order \(e^{-n-1}\). Moreover, the functions \(\hat{\phi}_n\) are smooth decreasing to \(\hat{\phi}\), and we have
\[
\begin{align*}
\text{dd}^c \hat{\phi}_n &= (\chi_n'' \circ \hat{\phi}) d\hat{\phi} \wedge d^c \hat{\phi} + (\chi_n' \circ \hat{\phi}) d\text{dd}^c \hat{\phi} \\
&\geq (\chi_n' \circ \hat{\phi}) d\text{dd}^c \hat{\phi} = -(\chi_n' \circ \hat{\phi}) \Theta \geq -\Theta',
\end{align*}
\]
where we choose the smooth positive closed form \(\Theta\) big enough such that \(\Theta' - \Theta\) is positive. Define, for \(n \in \mathbb{N}\), the positive closed smooth \((1,1)\)-form on \(\tilde{E}\):
\[
\Theta_n := d\text{dd}^c \hat{\phi}_n + \Theta = [\tilde{\Delta}] + d\tilde{c}^2 \hat{\phi}_n - d\text{dd}^c \hat{\phi}.
\]

Let \(\tilde{\gamma}\) be a closed smooth \((1,1)\)-form on \(\tilde{E}\) which is strictly positive near \(\tilde{\Delta}\). Then \(\Pi_n(\tilde{\gamma} \wedge [\tilde{\Delta}])\) is a positive closed \((2,2)\)-current on \(\tilde{E}\) supported on \(\tilde{\Delta}\). So, it is a multiple of \([\tilde{\Delta}]\). We choose \(\tilde{\gamma}\) so that \(\Pi_n(\tilde{\gamma} \wedge [\tilde{\Delta}]) = [\tilde{\Delta}]\). Define, for \(n \in \mathbb{N}\) and \(|\lambda| \gg 1\),
\[
\tilde{\Phi}_n := \tilde{\gamma} \wedge \Theta_n \quad \text{and} \quad \Phi_n := \Pi_n(\tilde{\Phi}_n) = [\Delta] + d\tilde{c}^2 \Pi_n((\hat{\phi}_n - \hat{\phi}) \tilde{\gamma}) \quad \text{and} \quad \Phi_{n,\lambda} := (A\lambda)^{\ast} \Phi_n.
\]

Observe that \(\tilde{\Phi}_n\) is smooth. Hence, \(\Phi_n\) is a form with coefficients in \(L^1\). Consider the function \(\phi : E \to \mathbb{R} \cup \{-\infty\}\) given by
\[
\phi(x) := \begin{cases} \tilde{\phi}(\Pi^{-1}(x)), & x \in E \setminus \Delta; \\ -\infty, & x \in \Delta. \end{cases}
\]

Consider the bundle \(\mathbb{P}(E \oplus \mathbb{C})\) with the canonical projection \(\pi^\# : \mathbb{P}(E \oplus \mathbb{C}) \to \Delta\). The map \(\pi^\#\) defines a regular fibration over \(\Delta\) with \(\mathbb{P}^2\) fibers. Consider a Hermitian metric \(\| \cdot \|\) on \(E \oplus \mathbb{C}\) and denote by \(\omega'\) the closed \((1,1)\)-form on \(\mathbb{P}(E \oplus \mathbb{C})\) induced by \(d\text{dd}^c \log \|v\|\) with \(v \in E \oplus \mathbb{C}\). The restriction of \(\omega'\) to each fiber of \(\mathbb{P}(E \oplus \mathbb{C})\) is the Fubini-Study form on this fiber. So \(\omega'\) is strictly positive in the fiber direction. It follows that there is a constant \(c > 0\) large enough such that \(\omega_E := c(\pi^\#)^{\ast} (\omega_{\Delta}) + \omega'\) defines a Kähler metric on \(\mathbb{P}(E \oplus \mathbb{C})\). Here \(\omega_{\Delta}\) is a Kähler form on \(\Delta\). The canonical injection \(E \hookrightarrow \mathbb{P}(E \oplus \mathbb{C})\) induces a Kähler form (still denoted by) \(\omega_E\) on \(E\). As in (3.3), we fix a constant \(A \gg M\) large enough and define for \(m \in \mathbb{N}\) the following smooth closed \((1,1)\)-form on \(E\):
\[
R_m^E := A \text{dd}^c[\chi(\phi + m)] + A^2 \omega_E = A \chi'(\phi + m) d\text{dd}^c \phi + \frac{A}{\pi} \chi''(\phi + m) i\partial \phi \wedge \overline{\partial} \phi + A^2 \omega_E.
\]

**Lemma 5.1.**

1. The \((2,2)\)-forms \(\Phi_n, \Phi_{n,\lambda}\) are closed with coefficients in \(L^1\) and smooth out of \(\Delta\) in \(E\). Moreover, \(\Phi_n\) (resp. \(\Phi_{n,\lambda}\)) is supported on a tubular neighborhood of \(\Delta\) with radius of order \(e^{-n+1}\) (resp. \(e^{-n+\log |\lambda|+1}\)).
2. The smooth functions \(\pi_n \Phi_n\) on \(\Delta\) satisfy \(\|\pi_n \Phi_n - 1\| \leq e^{-n}\). In particular, \(\pi_n \Phi_n\) converge uniformly to 1 and \(\Phi_n \to [\Delta]\) weakly.
3. \(\Phi_{n,\lambda} - \Phi_{n+\log |\lambda|} = O(e^{-n})(R_m^E)_{n+\log |\lambda|}\), where the forms \(R_m^E\) are defined in (5.3).

**Proof.** Assertion (1) is clear.
To prove assertion (2), we work out the expression of $\Phi_n$ in the above considered neighborhood of a given point of $\Delta$. Write
\[
\tilde{\Phi}_n = \tilde{\gamma} \wedge \Theta_n = \tilde{\gamma} \wedge (dd^c \tilde{\phi}_n - dd^c \tilde{\psi})
\]
\[= -\tilde{\gamma} \wedge dd^c \tilde{\psi} + \tilde{\gamma} \wedge (\chi_n' \circ \tilde{\phi}) d\tilde{\phi} \wedge d\tilde{c} \tilde{\phi} + (\chi_n' \circ \tilde{\phi}) dd^c \tilde{\phi},
\]
where the last line holds by equality (5.1). Note that $dd^c \tilde{\phi} = dd^c \tilde{\psi} = -\Theta$ and $\partial \tilde{\phi} = \frac{d\tau}{2w_3} + \partial \tilde{\psi}$ outside $\Delta$. Consequently, there is a smooth $(1,1)$-form $\tilde{\psi}$ such that
\[
(5.4)
\]
\[\tilde{\Phi}_n = \tilde{\gamma} \wedge (\Theta - (\chi_n' \circ \tilde{\phi}) \Theta + (\chi_n'' \circ \tilde{\phi}) \frac{\tilde{\psi}}{|w_3|^2}).
\]
Using this and the formula for $\chi_n$, assertion (2) follows.

Now we turn to assertion (3). We infer from (5.4) that assertion (3) will follow if one can show that
\[
|\chi_n' (\tilde{\phi}(\lambda w)) - \chi_n' + \log |\lambda| (\tilde{\phi}(w))| = O(e^{-n}) \quad \text{and} \quad |\chi_n'' (\tilde{\phi}(\lambda w)) - \chi_n'' + \log |\lambda| (\tilde{\phi}(w))| = O(e^{-n}).
\]
We only prove the first inequality since the second one can be proved similarly. Using $\tilde{\phi}(w) = \log |w_3| + \tilde{\psi}(w)$, we have that
\[
\chi_n' (\tilde{\phi}(\lambda w)) - \chi_n' + \log |\lambda| (\tilde{\phi}(w)) = \chi_n' (\log |\lambda w_3| + \tilde{\psi}(\lambda w)) - \chi_n' + \log |\lambda| \log |w_3| + \tilde{\psi}(w))
\]
\[= \chi' (\log |w_3| + \log |\lambda| + n + \tilde{\psi}(\lambda w)) - \chi' (\log |w_3| + \log |\lambda| + n + \tilde{\psi}(w)).
\]
We deduce from this and from the definition of $\chi$ that the above difference is nonzero only if $|\log |w_3| + (n + \log |\lambda|)| < 1$, in which case $|\tilde{\psi}(\lambda w) - \tilde{\psi}(w)| = O(e^{-n})$. This proves the estimate. □

Let $\tau$ be a smooth admissible map (see Definition 2.1). We need estimates for the images by $\tau$. Recall that $\Phi_n := \Pi_{\tau}(\tilde{\Phi}_n)$. For $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, define
\[
\Psi_n := \tau^* \Phi_n \quad \text{and} \quad \Psi_{n,\lambda} := \tau^* \Phi_{n,\lambda},
\]
where $\Phi_n$ and $\Phi_{n,\lambda}$ are given in (5.2). Observe that $\Phi_{n,\lambda}$ is a closed $(2,2)$-form on $\mathbb{E}$ and that $\Psi_n$, $\Psi_{n,\lambda}$ are closed forms of degree 4 on $X \times X$. Note that $\Psi_n$, $\Psi_{n,\lambda}$ are not necessarily of bidegree $(2,2)$ since $\tau$ is in general not holomorphic.

Now we construct closed $(2,2)$-forms $\Psi_n$, $\Psi_{n,\lambda}$ on $X \times X$ such that the difference $\Psi_n - \Psi'_n$ is, in some sense, small of order $O(e^{-n})$ as $n \to \infty$ and such that for a fixed $n$, the difference $\Psi_{n,\lambda} - \Psi'_{n,\lambda}$ is asymptotically small of order $O(e^{-n})$ as $\lambda \to \infty$ (see Lemmas 5.2 and 5.4 below). Since $\tilde{\phi}_n - \tilde{\phi}$ are smooth functions compactly supported on a fixed small neighborhood of $\Delta$ in $\mathbb{E}$, we can find, using a partition of unity, a finite index set $J$ and smooth real-valued functions $\tilde{u}_j$, $\tilde{v}_j$ with $j \in J$ such that all these functions are smooth and are compactly supported on a small neighborhood of $\Delta$ and that
\[
\tilde{\gamma} = \sum_{j \in J} \tilde{u}_j dd^c \tilde{v}_j
\]
on this neighborhood, where $\tilde{\gamma}$ is introduced just before (5.2). Set $\tilde{u}_{jn} := (\tilde{\phi}_n - \tilde{\phi}) \tilde{u}_j$. This is a smooth function compactly supported near $\Delta$. 
Define, for \( j \in J \), \( \tilde{u}_{jn} := \Pi_s u_{jn} \) and \( \tilde{v}_j := \Pi_s v_j \). So these functions are compactly supported near \( \Delta \) on \( \mathbb{E} \) and they are smooth out of \( \Delta \). Hence,

\[
\Phi_n = [\Delta] + dd^c \tilde{\pi}_s((\phi_n - \hat{\phi})\hat{\gamma}) = [\Delta] + \sum_{j \in J} dd^c \tilde{u}_{jn} \wedge dd^c \tilde{v}_j.
\]

For \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \), define

\[
\hat{\Psi}_n := [\Delta] + \sum_{j \in J} dd^c (\tilde{u}_{jn} \circ \tau) \wedge dd^c (\tilde{v}_j \circ \tau),
\]

\[
\hat{\Psi}_{n,\lambda} := [\Delta] + \sum_{j \in J} dd^c (\tilde{u}_{jn} \circ A_\lambda \circ \tau) \wedge dd^c (\tilde{v}_j \circ A_\lambda \circ \tau).
\]

They are closed \((2,2)\)-forms on \( X \times X \) which are smooth out of \( \Delta \) and with \( L^1 \)-coefficients.

Consider the family defined in (3.3) of positive closed currents on \( X \times X \). For a sequence of \((2,2)\)-currents \( Q_{n,\lambda} \) and a sequence of positive constants \( c_n \), we write

\[
Q_{n,\lambda} = O(c_n R_{n+\log |\lambda|}^2)
\]

if there is a constant \( c > 0 \) such that \(-cc_n R_{n+\log |\lambda|}^2 \leq Q_{n,\lambda} \leq cc_n R_{n+\log |\lambda|}^2 \).

**Lemma 5.2.** (1) The \((2,2)\)-forms \( \hat{\Phi}_n \), \( \hat{\Phi}_{n,\lambda} \) are closed with \( L^1 \)-coefficients and smooth out of \( \Delta \) in \( X \times X \). Moreover, \( \hat{\Phi}_n \) (resp. \( \hat{\Phi}_{n,\lambda} \)) is supported in a tubular neighborhood of \( \Delta \) with radius of order \( e^{-n+1} \) (resp. \( e^{-n-log |\lambda|+1} \)).

(2) We have that \( \Psi_{n,\lambda} - \hat{\Psi}_{n,\lambda} = O(e^{-n}) R_{n+\log |\lambda|}^2 \).

**Proof.** To prove assertion (1) we argue as in the proof of Lemma 5.1.

Now we turn to assertion (2). In fact, the smooth positive closed \((1,1)\)-forms \( R_{n+\log |\lambda|}^E \) on \( \mathbb{E} \) are replaced by \( R_{n+\log |\lambda|}^q \) on \( X \times X \). Consequently, by Lemma 5.1(3), we may replace \( \Psi_{n,\lambda} \) by \( \hat{\Psi}_{n,\lambda} \) and \( \hat{\Psi}_{n,\lambda} \) by \( \hat{\Psi}_{n,\lambda} \). Next, we see that \( \Psi_{n,\lambda} - \hat{\Psi}_{n,\lambda} \) is small when \( \lambda \) is large.

Let \( \mathcal{M}(X) \) be the space of Radon measures on \( X \). Consider the integral operators \( \hat{P}_n \) and \( \hat{P}_{n,\lambda} \) on the space of \((1,1)\)-forms with coefficients in \( \mathcal{M}(X) \):

\[
(\hat{P}_n f)(x) := \int_{y \in X} \hat{\Psi}_n(x,y) \wedge f(y) \quad \text{and} \quad (\hat{P}_{n,\lambda} f)(x) := \int_{y \in X} \hat{\Psi}_{n,\lambda}(x,y) \wedge f(y) \quad \text{for} \quad x \in X.
\]

**Lemma 5.3.** Let \( 0 \leq \delta < 1 \).

(1) The operator \( \hat{P}_n \) (resp. \( \hat{P}_{n,\lambda} \)) maps continuously the space of \((1,1)\)-forms with coefficients in \( \mathcal{M}(X) \) into the space of forms with coefficients in \( L^{1+\delta}(X) \), \( L^p(X) \) into \( L^q(X) \), \( L^\infty(X) \) into \( \mathcal{C}(X) \), with norm \( \leq A_\delta \) (in particular, independent of \( n \) and \( \lambda \)), where \( q = \infty \) if \( p^{-1} + (1+\delta)^{-1} \leq 1 \) and \( p^{-1} + (1+\delta)^{-1} = 1 + q^{-1} \) otherwise.

(2) The norm of the operator \( \hat{P}_n - \text{id} \) (resp. \( \hat{P}_{n,\lambda} - \text{id} \)) acting on the space of \((1,1)\)-forms with coefficients in \( L^2(X) \) is of order \( O(e^{-n}) \).

**Proof.** First observe that the kernels \( \hat{\Psi}_n \) is of the type considered in Example 3.4 and the kernel \( \hat{\Psi}_{n,\lambda} \) is of the type considered in Example 3.5 with \( r := e^{-n}|\lambda|^{-1} \). Consequently, assertion (1) follows from Lemma 3.3.

Assertion (2) is a consequence of Lemma 5.1 (2).
Although the closed (2, 2)-form \( \hat{\Psi}_{n,\lambda} \) is asymptotically close to \( \Phi_{n,\lambda} \), it is not regular enough in the sense that the right hand side in the following expected equality

\[
\langle T_1 \otimes T_2, \hat{\Psi}_{n,\lambda} \rangle = \langle T_1, \hat{P}_{n,\lambda}(T_2) \rangle
\]

does not make sense if we apply Lemma 5.3(1). Observe from Lemma 5.3(2) that \( \hat{P}_n \) (resp. \( \hat{P}_{n,\lambda} \)) behaves asymptotically like the identity as \( n, \lambda \to \infty \). Therefore, the idea is to perturb \( \hat{P}_n \) (resp. \( \hat{P}_{n,\lambda} \)) in order to obtain more regular closed (2, 2)-forms \( \Psi'_n \) and \( \Psi'_{n,\lambda} \).

Consider the integral operators \( P_n \) and \( P_{n,\lambda} \) on the space of (1, 1)-forms with coefficients in \( \mathcal{M}(X) \):

\[
P_n := \hat{P}_{8n} \circ \hat{P}_{4n} \circ \hat{P}_{2n} \circ \hat{P}_n \quad \text{and} \quad P_{n,\lambda} := \hat{P}_{8n+8 \log |\lambda|} \circ \hat{P}_{4n+4 \log |\lambda|} \circ \hat{P}_{2n+2 \log |\lambda|} \circ \hat{P}_{n,\lambda}.
\]

The kernel form \( \Psi'_n \) of \( P_n \) is given by

\[
\Psi'_n(x, y) := \int_{(z_1, z_2, z_3) \in X^3} \hat{\Psi}_{8n}(x, z_3) \wedge \hat{\Psi}_{4n}(z_3, z_2) \wedge \hat{\Psi}_{2n}(z_2, z_1) \wedge \hat{\Psi}_n(z_1, y) \quad \text{for} \ (x, y) \in X^2.
\]

Similarly, the kernel form \( \Psi'_{n,\lambda} \) of \( P_{n,\lambda} \) is given for \((x, y) \in X^2\) by

\[
\Psi'_{n,\lambda}(x, y) := \int_{(z_1, z_2, z_3) \in X^3} \hat{\Psi}_{8n+8 \log |\lambda|}(x, z_3) \wedge \hat{\Psi}_{4n+4 \log |\lambda|}(z_3, z_2) \wedge \hat{\Psi}_{2n+2 \log |\lambda|}(z_2, z_1) \wedge \hat{\Psi}_{n,\lambda}(z_1, y).
\]

The properties of \( P_n \) and \( P_{n,\lambda} \) and their kernels \( \Psi'_n \) and \( \Psi'_{n,\lambda} \) are collected in the following result.

**Lemma 5.4.**

1. \( \Psi'_n \), \( \Psi'_{n,\lambda} \) are closed (2, 2)-forms with coefficients in \( L^1(X) \) and smooth out of \( \Delta \) in \( X \times X \). Moreover, \( \Psi'_n \) (resp. \( \Psi'_{n,\lambda} \)) is supported on a tubular neighborhood of radius \( O(e^{-n}) \) (resp. \( O(e^{-n}|\lambda|^{-1}) \)) of \( \Delta \).
2. \( P_n \), \( P_{n,\lambda} \) map continuously \( \mathcal{M}(X) \) into the space of continuous (2, 2)-forms on \( X \) with norm \( \leq A \) for some constant \( A \) independent of \( n \) and \( \lambda \).
3. The norm of the operator \( \hat{P}_{n,\lambda} - \text{id} \) acting on \( L^2(X) \) is of order \( O(e^{-n}) \).
4. We keep the notation of Theorem 2.2. Then \( \langle T_1 \otimes T_2, \Psi'_{n,\lambda} \rangle \) is well-defined and we have

\[
\langle T_1 \otimes T_2, \Psi'_{n,\lambda} \rangle = \langle \Omega_1 \otimes \Omega_2, \Psi'_{n,\lambda} \rangle + \langle \partial S_1 \otimes \overline{\partial S}_2, \Psi'_{n,\lambda} \rangle + \langle \overline{\partial S}_1 \otimes \partial S_2, \Psi'_{n,\lambda} \rangle
\]

\[
= \langle \Omega_1 \otimes \Omega_2, \Psi'_{n,\lambda} \rangle - \langle \overline{\partial S}_1 \otimes \partial S_2, \Psi'_{n,\lambda} \rangle - \langle \partial S_1 \otimes \overline{\partial S}_2, \Psi'_{n,\lambda} \rangle.
\]

5. \( \Psi'_{n,\lambda} - \hat{\Psi}_{n,\lambda} = O(e^{-n})R_2^{n+\log |\lambda|} \).

**Proof.** Assertion (1) follows from the fact that \( \hat{\Psi}_n - [\Delta] \) (resp. \( \hat{\Psi}_{n,\delta} - [\Delta] \)) is a dd\( \bar{\partial} \)-exact (2, 2)-form and from an application of Stokes' theorem.

We infer from Lemma 5.3(1) that \( \hat{P}_{n,\lambda} \) maps continuously \( \mathcal{M}(X) \) into \( L^p(X) \) for every \( 1 < p < 2 \), \( \hat{P}_{2n+2 \log |\lambda|} \) maps continuously \( L^p(X) \) into \( L^q(X) \) for \( 2 < q < \infty \) and \( \frac{2q}{q+2} < p < 1/2 \). Also \( \hat{P}_{4n+4 \log |\lambda|} \) maps continuously \( L^q(X) \) into \( L^\infty(X) \) for every \( 2 < q < \infty \), and \( \hat{P}_{8n+8 \log |\lambda|} \) maps continuously \( L^\infty(X) \) into \( C(X) \). Since \( P_{n,\lambda} := \hat{P}_{8n+8 \log |\lambda|} \circ \hat{P}_{4n+4 \log |\lambda|} \circ \hat{P}_{2n+2 \log |\lambda|} \circ \hat{P}_{n,\lambda} \), assertion (2) for \( P_{n,\lambda} \) follows. Assertion (2) for \( P_n \) can be treated similarly.

Assertion (3) is a consequence of Lemma 5.3(2).

Write

\[
\langle T_1 \otimes T_2, \Psi'_{n,\lambda} \rangle = \langle T_1, P_{n,\lambda}(T_2) \rangle.
\]
By assertion (1), \( P_{n,\lambda}(T_2) \) is a continuous form on \( X \). Hence, \( \langle T_1 \otimes T_2, \Psi'_{n,\lambda} \rangle \) is well-defined. The formula in assertion (4) follows from Proposition \ref{3.2}. Assertion (5) follows from Lemma \ref{5.2} (2).

End of the proof of identity \((2.3)\). Let \( \lambda_n \) be a sequence of complex numbers such that \( \lambda_n \to \infty \) and that \( (A_{\lambda_n})_* \tau_* (T_1 \otimes T_2) \to \pi^* \nu \). Pick an arbitrary \( \epsilon > 0 \). By Lemma \ref{5.1} (2), we may find \( N = N_1 \) such that \( \| \pi_* \Phi_m - 1 \|_\infty < \frac{\epsilon}{3(1+\|\nu\|)} \) for all \( m > N \). We may choose \( N \) large enough such that \( e^{-N} \ll \epsilon/5 \). Hence,

\[
\|\|\nu\| - \langle \pi^* \nu, \Phi_m \rangle \| = \| \langle \nu, 1 - \pi_\ast \Phi_m \rangle \| < \epsilon/5.
\]

Observe that for fixed \( m > N \) and for \( n \) large enough,

\[
\langle \pi^* \nu, \Phi_m \rangle = \lim_{n \to \infty} \langle T_1 \otimes T_2, \tau_* (A_{\lambda_n})^* \Phi_m \rangle = \langle T_1 \otimes T_2, \Psi_{m,\lambda_n} \rangle + O(e^{-m})
\]

\[
= \langle T_1 \otimes T_2, \Psi'_{m,\lambda_n} \rangle + O(e^{-m}),
\]

where the first estimate holds by an application of Lemma \ref{5.2} (2) and Lemma \ref{3.11}, and the second one holds by Lemmas \ref{5.4} (5) and Lemma \ref{3.11}. So we have shown that for every \( m > N \), there is \( n_m \) such that for \( n > n_m \),

\[
\|\|\nu\| - \langle T_1 \otimes T_2, \Psi'_{m,\lambda_n} \rangle \| < \epsilon/4.
\]

By Lemma \ref{5.4} (4), we get

\[
\langle T_1 \otimes T_2, \Psi'_{m,\lambda_n} \rangle = \langle \Omega_1 \otimes \Omega_2, \Psi'_{m,\lambda_n} \rangle - \langle \partial S_1 \otimes \partial \bar{S}_2, \Psi'_{m,\lambda_n} \rangle - \langle \partial \bar{S}_1 \otimes \partial S_2, \Psi'_{m,\lambda_n} \rangle.
\]

Write the right-hand side as

\[
\langle \Omega_1, P_{m,\lambda_n}(\Omega_2) \rangle - \langle \partial S_1, \partial \bar{S}_2 \rangle - \langle \partial \bar{S}_1, \partial S_2 \rangle.
\]

By Lemma \ref{5.4} (3), the above expression is equal to

\[
\langle \Omega_1, \Omega_2 \rangle - \langle \partial S_1, \partial \bar{S}_2 \rangle - \langle \partial \bar{S}_1, \partial S_2 \rangle + O(e^{-m}).
\]

Hence,

\[
\|\|\nu\| - \langle \Omega_1, \Omega_2 \rangle - \langle \partial S_1, \partial \bar{S}_2 \rangle + \langle \partial \bar{S}_1, \partial S_2 \rangle \| < \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrarily chosen, the proof of identity \((2.4)\) is thereby completed. \( \square \)

6. Vanishing of the tangent currents

The purpose of this section is to prove Theorem \ref{2.4}. The proof is given in the first subsection modulo two auxiliary propositions which will be proved in the last two subsections.

6.1. Proof of the vanishing theorem. Let \( \mathcal{F} \) be either a holomorphic foliation by Riemann surfaces with only hyperbolic singularities, or a bi-Lipschitz lamination by hyperbolic Riemann surfaces (without singularities) in a compact Kähler surface \( X \). Consider also a positive \( d\bar{\partial} \)-closed \((1,1)\)-current \( T \) directed by \( \mathcal{F} \). Suppose that \( T \) has no mass on each single leaf of \( \mathcal{F} \). Therefore, we deduce from \((2.1)\) that \( \nu(T, x) = 0 \) for all \( x \) outside the set of the singularities of \( \mathcal{F} \). Note that in case \( \mathcal{F} \) is a bi-Lipschitz lamination by hyperbolic Riemann surfaces, the set of singularities of \( \mathcal{F} \) is simply empty. Moreover, \( T \) does not give mass to these singularities. So by Theorem \ref{2.2}, there is a tangent current \( S = \pi^*(\nu) \) to \( T \otimes T \) along \( \Delta \). Identifying \( X \) with \( \Delta \subset \mathbb{E} \) by the canonical identification
$X \ni x \mapsto (x,x) \in \Delta$, we consider $\nu$ as a positive measure on $X$. The next subsection is devoted to the proof of the following

**Proposition 6.1.** $\nu$ is supported on the singularities of $\mathcal{F}$.

In the last subsection we will prove the following result.

**Proposition 6.2.** $\nu$ does not give mass to any singular point of $\mathcal{F}$.

*End of the proof of Theorem 2.4.* Combining Propositions 6.1 and Proposition 6.2, we conclude that $\nu = 0$ as claimed. □

### 6.2. Vanishing of the tangent currents outside singularities.

The proof of Proposition 6.1 follows the same lines as in Kaufmann’s work [17]. We indicate here the main steps in the argument. Let $\mathbb{B}$ be any flow box of $\mathcal{F}$ outside the singularities. So we can choose holomorphic coordinates $z = (z_1, z_2)$ in $\mathbb{B} \cong \mathbb{D} \times \mathbb{D}$ in which the plaques of $\mathcal{F}$ in $\mathbb{B}$ are given by

$$L_\alpha = \{z_2 = \phi_\alpha(z_1)\},$$

for some uniformly bounded holomorphic functions $\phi_\alpha : \mathbb{D} \to \mathbb{D}$ depending holomorphically on $\alpha = \phi_\alpha(0)$ and $\phi_0 \equiv 0$.

Since $T$ is a diffuse positive $dd^c$-closed current directed by $\mathcal{F}$, we obtain the following decomposition in the flow box $\mathbb{B}$,

$$T = \int h_\alpha [L_\alpha] d\mu(\alpha),$$

where $[L_\alpha]$ denotes the current of integration along the plaque $L_\alpha$, $h_\alpha$ is a positive harmonic function on $L_\alpha$ for $\mu$-almost every $\alpha \in \mathbb{D}$, and $\mu$ is a diffuse positive measure on $\mathbb{D}$. We multiply $\mu$ by the positive function $h_\alpha(\alpha, 0)$ and divide $h_\alpha$ by $h_\alpha(\alpha, 0)$ in order to assume that $h_\alpha(\alpha, 0) = 1$ for $\mu$-almost every $\alpha \in \mathbb{D}$. By Harnack’s inequality, we may find a constant $\kappa > 1$ such that

$$\kappa^{-1} \leq h_\alpha(z) \leq \kappa \quad \text{for } \mu\text{-almost every } \alpha \in \mathbb{D} \text{ and for } z \in L_\alpha. \quad (6.1)$$

Let $\mathcal{F} \times \mathcal{F}$ denote the product foliation on $X \times X$. The coordinates on the flow box $\mathbb{B}$ induce natural holomorphic coordinates $(z, w)$ on $\mathbb{B} \times \mathbb{B}$ in which the plaques of $\mathcal{F} \times \mathcal{F}$ are given by

$$L_{\alpha, \beta} := L_\alpha \times L_\beta = \{z_2 = \phi_\alpha(z_1), \ y_2 = \phi_\beta(y_1)\}.$$ 

The product $T \otimes T$ is a positive current of bidimension $(2, 2)$ on $X \times X$ directed by $\mathcal{F} \times \mathcal{F}$ which is given, on $\mathbb{B} \times \mathbb{B}$, by

$$T \otimes T = \int (h_\alpha \otimes h_\beta)[L_{\alpha, \beta}] d(\mu \otimes \mu)(\alpha, \beta).$$

Since $\mu$ has no atoms, we can prove using Fubini’s theorem that $\mu \otimes \mu$ gives no mass to the diagonal $\Delta \subset X \times X$.

To investigate the tangent currents of $T \times T$ along the diagonal $\Delta = \{z = w\}$, it is convenient to work in new holomorphic coordinates given by $(x, y) = (z, w - z)$ and new parameters given by $a = (a_1, a_2)$, where $a_1 = \alpha$ and $a_2 = \beta - \alpha$.

Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. In this new coordinate system, the diagonal is given by $\Delta = \{y = 0\}$ and $L_{\alpha, \beta}$ transforms to

$$\Gamma_a = \{(x_1, f_a(x_1, y_1), y_1, g_a(x_1, y_1))\},$$ where

$$f_a(x_1, y_1) = a_1(x_1) + a_2(x_1) y_1,$$ and

$$g_a(x_1, y_1) = a_1(x_1) - a_2(x_1) y_1.$$
where $f_a(x_1, y_1) = \phi_{a_1}(x_1)$ and $g_a(x_1, y_1) = \phi_{a_1+a_2}(x_1+y_1) - \phi_{a_1}(x_1)$. Note that $f_a(0, y_1) = a_1$ and $g_a(0, 0) = a_2$. Moreover, $(f_a, g_a)$ depends on $a$ in a bi-Lipschitz way.

The expression of $T \otimes T$ becomes
\[
T \otimes T = \int h_{a_1} \otimes h_{a_1+a_2}[\Gamma_a]d(\mu \otimes \mu)(a).
\]

In the new coordinates $(x, y)$, the dilation $A_\lambda$ in the direction normal to $\Delta$ reads as: $A_\lambda(x, y) = (x, \lambda y)$ for $|\lambda| \geq 1$.

Using the above discussion, we can show as in [17] Lemmas 4.4 and 4.5 that

**Lemma 6.3.** (1) The mass of $(A_\lambda)_*[\Gamma_a]$ on a compact set is bounded uniformly in $(\lambda, a)$ when $|a_2| \leq |\lambda|^{-1}$.

(2) There exists a compact neighborhood $K$ of the origin such that $(A_\lambda)_*[\Gamma_a]$ has no mass on $K$ for every pair $(\lambda, a)$ such that $a_2 > |\lambda|^{-1}$.

**Proof of Proposition 6.1.** We only need to show that any limit of the sequence of $(A_\lambda)_*(T \otimes T)$ is zero in a neighborhood of $0 \in \mathbb{C}^4$. Using estimate (6.1), we see that
\[
(A_\lambda)_*(T \otimes T) \leq \kappa^2 T_\lambda,
\]
where
\[
T_\lambda := \int (A_\lambda)_*[\Gamma_a]d(\mu \otimes \mu)(a).
\]

Write $T_\lambda := T'_\lambda + T''_\lambda$, where
\[
T'_\lambda = \int_{|a_2| \leq |\lambda|^{-1}} (A_\lambda)_*[\Gamma_a]d(\mu \otimes \mu)(a) \quad \text{and} \quad T''_\lambda = \int_{|a_2| > |\lambda|^{-1}} (A_\lambda)_*[\Gamma_a]d(\mu \otimes \mu)(a).
\]

Take a compact neighborhood $K$ of the origin as in Lemma 6.3 (2), so that $T''_\lambda$ is zero on $K$ for every $|\lambda| > 1$. It follows from Lemma 6.3 (1) that there is a constant $M > 0$, independent of $\lambda$, such that the mass of $T'_\lambda$ over $K$ is bounded by $M$ times $(\mu \otimes \mu)(\{|a_2| < |\lambda|^{-1}\})$. The fact that $\mu \otimes \mu$ gives no mass to the set $\{a_2 = 0\}$ shows that $T'_\lambda \to 0$ on $K$ as $|\lambda| \to \infty$. This completes the proof. 

6.3. **Vanishing of the tangent currents near singularities.** The proof of Proposition 6.2 requires properties of $T$ near the singular points that we will discuss below. Let $p$ be a singular point of the foliation. We have to prove that $\nu(\{p\}) = 0$. Since $p$ is hyperbolic, there are local coordinates $z = (z_1, z_2)$ centered at $p$, with $|z_1| < 3, |z_2| < 3$, such that in the bidisc $\mathbb{D}(0, 3)^2 := \{|z_1| < 3, |z_2| < 3\}$, the foliation $\mathcal{F}$ is defined by the vector field
\[
F := z_2 \frac{\partial}{\partial z_2} - \eta z_1 \frac{\partial}{\partial z_1},
\]
where $\eta = a + ib, a, b \in \mathbb{R}$, and $b \neq 0$. Notice that if we flip $z_1$ and $z_2$, we replace $\eta$ by $1/\eta = \bar{\eta}/|\eta|^2 = a/(a^2 + b^2) - ib/(a^2 + b^2)$. We will assume below that the axes are chosen so that $b > 0$. Observe that the two axes of the bidisc $\mathbb{D}(0, 3)^2$ are invariant and are the separatrices of the foliation in the bidisc $\mathbb{D}(0, 3)^2$. Consider the ring $\Lambda$ defined by
\[
\Lambda := \{\alpha \in \mathbb{C}, e^{-2\pi|b|} < |\alpha| \leq 1\}.
\]

Define also the sectors $S$ and $S'$ by
\[
S := \{\zeta = u + iv \in \mathbb{C}, v > 0 \quad \text{and} \quad bu + av > 0\}
\]

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and
\[ S' := \{ \zeta = u + iv \in \mathbb{C}, \; v > -\log 3 \quad \text{and} \quad bu + av > -\log 3 \} . \]

Note that the sector \( S \) is contained in the upper half-plane \( \mathbb{H} := \{ u + iv, \; v > 0 \} \) and that \( S \subset S' \). For \( \alpha \in \mathbb{C}^* \), consider the following manifold \( L_\alpha \) immersed in \( \mathbb{C}^2 \) and defined by

\[ z_1 = \alpha e^{i(\zeta + \log |\alpha|/b)} \quad \text{and} \quad z_2 = e^{i(\zeta + \log |\alpha|/b)} \quad \text{with} \quad \zeta = u + iv \in \mathbb{C} . \]

So we have

\[ |z_1| = e^{-v} \quad \text{and} \quad |z_2| = e^{-bu-av} . \]

Note that the map \( \zeta \mapsto (z_1, z_2) \) is injective because \( \eta \notin \mathbb{R} \). It is easy to check the following properties:

1. \( L_\alpha \) is tangent to the vector field \( F \) and is a submanifold of \( \mathbb{C}^* \).
2. \( L_{\alpha_1} \) is equal to \( L_{\alpha_2} \) if \( \alpha_1/\alpha_2 = e^{2\pi k} \) for some \( k \in \mathbb{Z} \) and they are disjoint otherwise. In particular, \( L_{\alpha_1} \) and \( L_{\alpha_2} \) are disjoint if \( \alpha_1, \alpha_2 \in \mathbb{A} \) and \( \alpha_1 \neq \alpha_2 \).
3. The union of \( L_\alpha \) is equal to \( \mathbb{C}^* \) for \( \alpha \in \mathbb{C}^* \), and then also for \( \alpha \in \mathbb{A} \).
4. The intersection \( L_\alpha := L_{\alpha_1} \cap \mathbb{D}^2 \) of \( L_\alpha \) with the unit bidisc \( \mathbb{D}^2 \) is given by the same equations as in the definition of \( L_\alpha \) but with \( \zeta \in S \). Moreover, \( L_\alpha \) is a connected submanifold of \( \mathbb{D}^2 \). In particular, it is a leaf of \( \mathcal{F} \cap \mathbb{D}^2 \).
5. Similarly, the intersection \( L'_\alpha := L\alpha \cap \mathbb{D}(0, 3)^2 \) is given by the same equations with \( \zeta \in S' \). Moreover, \( L'_\alpha \) is a connected submanifold of \( \mathbb{D}(0, 3)^2 \) and is the leaf of \( \mathcal{F} \cap \mathbb{D}(0, 3)^2 \) which contains \( L_\alpha \).

Since \( T \) is positive \( dd^c \)-closed directed by \( \mathcal{F} \) and has no mass on the separatrices of \( p \), we obtain the following decomposition (see [8, Lemma 4.1]).

**Lemma 6.4.** There is a positive measure \( \mu \) of finite mass on \( \mathbb{A} \) and positive harmonic functions \( h_\alpha \) on \( L'_\alpha \) for \( \mu \)-almost every \( \alpha \in \mathbb{A} \) such that in \( \mathbb{D}(0, 3)^2 \)

\[ T = \int_{\mathbb{A}} T_\alpha d\mu(\alpha), \quad \text{where} \quad T_\alpha := h_\alpha[L'_\alpha]. \]

Moreover, the mass of \( T_\alpha \) in \( \mathbb{D}(0, 2)^2 \) is 1 for \( \mu \)-almost every \( \alpha \in \mathbb{A} \).

Define

\[ H_\alpha(\zeta) := h_\alpha(\alpha e^{i(\zeta + \log |\alpha|/b)}, e^{i(\zeta + \log |\alpha|/b)}). \]

This is a positive harmonic function in \( S' \).

Consider the holomorphic normal bundle \( \mathcal{E} \) over the diagonal \( \Delta \) of \( X \times X \) introduced in Subsection 2.1 and the above local coordinates \( z = (z_1, z_2) \in \mathbb{D}(0, 3)^2 \) for a neighborhood of \( p \) in \( X \). Next, using a copy \( w \) of \( z \), we consider the local coordinates \( (z, w) = (z_1, z_2, w_1, w_2) \in \mathbb{D}(0, 3)^4 \) for a neighborhood of \( (p, p) \) in \( X \times X \). We will investigate \( \mathcal{E} \) near the point \( (p, p) \) using the local coordinates \( (z, w) \). Observe that \( \mathcal{E} \) over \( \Delta \cap \mathbb{D}(0, 3)^4 \) is trivial. Therefore, we may identify a neighborhood of \( \Delta \cap \mathbb{D}(0, 3)^4 \) in \( \mathcal{E} \) with \( \mathbb{D}(0, 3)^4 \), where the fiber at \( (z, z) \) (with \( z \in \mathbb{D}^2(0, 3) \)) in this neighborhood is the set \( \{ (z, w) : w \in \mathbb{D}(0, 3)^2 \} \). Now we will describe the multiplications \( A_\lambda \) on the fibers of \( \mathcal{E} \) over \( \Delta \cap \mathbb{D}(0, 3)^4 \) as follows:

\[ A_\lambda(z, w) := (z + \lambda(w - z)), \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C}^2, \quad \lambda \in \mathbb{C} . \]

\footnote{Note that we used another system of local coordinates for \( A_\lambda \) in (3.2). We use both systems since each one has its own advantages for each problem we deal with.}
Fix a smooth nonnegative function \( \vartheta(z, w) \) in \( \mathbb{C}^4 \) with compact support in the domain
\[
\{(z, w) \in \mathbb{D}(0, 3)^4 : \ 1/2 < |z_1 - w_1| < 2 \quad \text{and} \quad 1/2 < |z_2 - w_2| < 2\}
\]
such that \( \vartheta(z, w) = 1 \) on the subdomain
\[
\{(z, w) \in \mathbb{D}(0, 5/2)^4 : 2/3 < |z_1 - w_1| < 3/2 \quad \text{and} \quad 2/3 < |z_2 - w_2| < 3/2\}.
\]

Recall that \( \nu \) is supported on the set of singularities. So in order to prove that \( \nu(\{p\}) = 0 \), it is enough to check that the mass of the measure
\[
S \wedge \vartheta(z, w)i(dz_1 - dw_1) \wedge (d\bar{z}_1 - d\bar{w}_1) \wedge i(dz_2 - dw_2) \wedge (d\bar{z}_2 - d\bar{w}_2)
\]
in a neighborhood of \( 0 \in \mathbb{C}^4 \) is zero. Indeed, the properties of tangent currents developed in Theorem 2.2 imply that this mass is equal to \( c_0 \) times the mass of \( \nu \) at the singular point \( p \). The following elementary result is needed.

**Lemma 6.5.** On the bidisc \( (x, y) \in \mathbb{D}^2 \), the following identity holds
\[
\int_{m \in \mathbb{D}} [x = m] idm \wedge d\bar{m} = idx \wedge d\bar{x},
\]
where \([x = m]\) denotes the current of integration on the complex line \( \{x = m\} \subset \mathbb{C}^2 \).

**Proof.** Let \( \phi \) be a smooth test form compactly supported in \( \mathbb{D}^2 \). Observe that if \( \phi \) can be factored by either \( dx \) or \( d\bar{x} \), then by the bidegree reason we get that
\[
\int_{m \in \mathbb{D}} \langle [x = m], \phi \rangle idm \wedge d\bar{m} = 0 = \langle idx \wedge d\bar{x}, \phi \rangle.
\]
Consequently, we may assume that \( \phi(x, y) = \theta(x, y) idy \wedge d\bar{y} \), where \( \theta \) is a smooth function compacted supported on \( \mathbb{D}^2 \). Since
\[
\int_{m \in \mathbb{D}} \langle [x = m], \phi \rangle idm \wedge d\bar{m} = \int_{m \in \mathbb{D}} \int_{y \in \mathbb{D}} \theta(m, y) idy \wedge d\bar{y} \wedge idm \wedge d\bar{m} = \langle idx \wedge d\bar{x}, \phi \rangle,
\]
the desired identity follows.

For \( a = (a_1, a_2) \in \mathbb{C}^2 \) let \( \Delta_a \) denote the current of integration on the plane
\[
\{w_1 - z_1 = a_1, w_2 - z_2 = a_2\} \subset \mathbb{C}^4.
\]
The following lemma permits us to use an average on the currents \( \Delta_a \) as a test form.

**Lemma 6.6.** (1) There is a constant \( c > 1 \) depending only on \( \vartheta \) such that
\[
c^{-1} \int_{2/3 < |a_1|, |a_2| < 3/2} \Delta_a \mid_{(z, w) \in \mathbb{D}(0, 2)} ida_1 \wedge d\bar{a}_1 \wedge ida_2 \wedge d\bar{a}_2
\leq \vartheta(z, w)i(dz_1 - dw_1) \wedge (d\bar{z}_1 - d\bar{w}_1) \wedge i(dz_2 - dw_2) \wedge (d\bar{z}_2 - d\bar{w}_2)
\leq c \int_{1/2 < |a_1|, |a_2| < 2} \Delta_a \mid_{(z, w) \in \mathbb{D}(0, 3)} ida_1 \wedge d\bar{a}_1 \wedge ida_2 \wedge d\bar{a}_2.
\]
Proof. Part (1) follows from Lemma 6.5 and the properties of $\vartheta(z, w)$ listed above.

Since $A^*_\lambda \Delta = \Delta_{a/\lambda}$, we apply $A^*_\lambda$ on the three currents in Part (1) and obtain that

$$c^{-1} \int_{|a_1|, |a_2| < 2} \langle T \otimes T, \Delta_{a/\lambda} \rangle \big|_{(z, w) \in \mathbb{D}(0, 2)} i\alpha \wedge i\beta \wedge i\gamma$$

$$\leq c \int_{1/2 < |a_1|, |a_2| < 2} \langle T \otimes T, \Delta_{a/\lambda} \rangle \big|_{(z, w) \in \mathbb{D}(0, 3)} i\alpha \wedge i\beta \wedge i\gamma$$

This, combined with the equality

$$\langle (A^*_\lambda \Delta \Delta_{\alpha/\lambda} \big|_{(z, w) \in \mathbb{D}(0, 3)} i\alpha \wedge i\beta \wedge i\gamma \rangle = \langle T \otimes T, A^*_\lambda \big| \vartheta(z, w)i(dz_1 - dw_1) \wedge (dz_1 - dw_1) \wedge i(dz_2 - dw_2) \wedge (dz_2 - dw_2) \rangle,$$

implies Part (2). \hfill \square

By Lemma 6.6, the study of $\langle (A^*_\lambda \Delta \Delta_{\alpha/\lambda} \big|_{(z, w) \in \mathbb{D}(0, 3)} i\alpha \wedge i\beta \wedge i\gamma \rangle$ is reduced to that of $\langle T \otimes T, \Delta_{a/\lambda} \big|_{(z, w) \in \mathbb{D}(0, 3)} i\alpha \wedge i\beta \wedge i\gamma \rangle$. By Lemma 6.4, the study of the latter integral is reduced to that of $\langle h_\alpha L_\beta \wedge h_\beta L_\beta, \Delta_{a/\lambda} \rangle \big|_{0, e^r} \rangle$, where $h_\alpha$ (resp. $h_\beta$) is the positive harmonic function on $L'_\alpha$ (resp. $L'_\beta$) determined by this lemma.

For $\mu$-almost every $\alpha, \beta \in \Lambda$, for every $a = (a_1, a_2)$ with $1/2 \leq |a_1|, |a_2| \leq 2$, and $r \geq -\log 3$, we are thus naturally led to estimate the integral $\langle h_\alpha L_\beta \wedge h_\beta L_\beta, \Delta_{a/\lambda} \rangle \big|_{0, e^r} \rangle$.

For $z = (z_1, z_2) \in L_\alpha$ and $w = (w_1, w_2) \in L_\beta$, we write using (6.2),

$$z_1 = e^{i(\zeta + \log |a_1|)} \quad \text{and} \quad z_2 = e^{i(\zeta + \log |a_2|)} \quad \zeta \in \mathbb{C};$$

$$w_1 = e^{i(\xi + \log |b_1|)} \quad \text{and} \quad w_2 = e^{i(\xi + \log |b_2|)} \quad \xi \in \mathbb{C}.$$
(6.7), they have different estimates. Define, for $\alpha, \beta \in \mathbb{A}$ and $s \geq 0$ and $0 < \rho \ll 1$ and $r \geq -\log 3$, the following diagonal sum:

$$G_{\alpha,\beta,\rho}^{(0)}(s, r) := \begin{cases} \sum_{m \in \mathbb{Z}}: r \leq m \leq s H_{\alpha}((a^2 + b^2)m - \eta^{-1}m)H_{\beta}((a^2 + b^2)m - \eta^{-1}m), & \text{if } |\alpha - \beta| < \rho; \\ 0, & \text{otherwise.} \end{cases}$$

We also consider two other sums which correspond to the solutions of (6.8) which are close to a quadrant of the lattice of basis $\{1, -\eta^{-1}\}$ in $\mathbb{C}$:

$$G_{\alpha,\beta}^{(1)}(s, r) := \sum_{(m,n) \in \mathbb{Z}^2: m,n \geq r} H_{\alpha}(m - \eta^{-1}s)H_{\beta}(m - \eta^{-1}(n + s)) + \sum_{(m,n) \in \mathbb{Z}^2: m,n \geq r} H_{\beta}(m - \eta^{-1}s)H_{\alpha}(m - \eta^{-1}(n + s)),
$$

$$G_{\alpha,\beta}^{(2)}(s, r) := \sum_{(m,n) \in \mathbb{Z}^2: m,n \geq r} H_{\alpha}(m - \eta^{-1}s)H_{\beta}(m + n - \eta^{-1}s) + \sum_{(m,n) \in \mathbb{Z}^2: m,n \geq r} H_{\beta}(m - \eta^{-1}s)H_{\alpha}(m + n - \eta^{-1}s).$$

Note that $G_{\alpha,\beta}^{(j)} = G_{\beta,\alpha}^{(j)}$ and $G_{\alpha,\beta,\rho}^{(0)} = G_{\beta,\alpha,\rho}^{(0)}$.

**Proposition 6.7.** There is a constant $\gamma > 0$ such that for every $r \geq h - \log 3$ and every $0 < \rho \ll 1$, there are constants $c_r > 1$ and $c_{r,\rho} > 1$ such that the following inequalities hold:

(6.9)

$$G_{\alpha,\beta}^{(1)}(\log |\lambda|, r + h) + G_{\alpha,\beta}^{(2)}(\log |\lambda|, r + h) \leq c_r \langle h_{\alpha}L_{\alpha} \otimes h_{\beta}L_{\beta}, \Delta_{\alpha/\lambda} \rangle_{\mathbb{D}(0,e^{-r}t)},$$

$$\langle h_{\alpha}L_{\alpha} \otimes h_{\beta}L_{\beta}, \Delta_{\alpha/\lambda} \rangle_{\mathbb{D}(0,e^{-r}t)} \leq c_{r,\rho} \left( G_{\alpha,\beta}^{(1)}(\log |\lambda|, r - h) + G_{\alpha,\beta}^{(2)}(\log |\lambda|, r - h) \right) + c_r G_{\alpha,\beta,\rho}^{(0)}(\log |\lambda|, r - h)$$

for $\mu$-almost every $\alpha, \beta \in \mathbb{A}$, and for every $a = (a_1, a_2)$ with $1/2 \leq |a_1|, |a_2| \leq 2$ and $|\lambda| > 1$.

**Proof.** Fix a constant $0 < \rho \ll 1$ small enough. For each $(z_j, w_j)$ in (6.6) with $1 \leq j \leq 2$ consider the following three cases

Case 1: $|z_j| + |w_j| > \rho^{-2}|\lambda|^{-1}$.  

Case 2: $|z_j| + |w_j| \leq \rho^{-2}|\lambda|^{-1}$ and either $|z_j| < \rho|w_j|$ or $|w_j| < \rho|z_j|$.  

Case 3: $\rho \leq |z_j|/|w_j| \leq \rho^{-2}$ and $|z_j| + |w_j| \leq \rho^{-2}|\lambda|^{-1}$.

For $1 \leq k, l \leq 3$, let $\Sigma_{kl}$ be the sum of all terms $H_{\alpha}(\zeta)H_{\beta}(\xi)$ in the right hand side of (6.7) such that $(z_1, w_1)$ in the above Case $k$ and $(z_2, w_2)$ in the above Case $l$, where $(\zeta, \xi) \in \mathbb{C}^2$ is, as usual, related to $(z_1, w_1)$ and $(z_2, w_2)$ by (6.6) and the system (6.8) holds.

We deduce easily from (6.7) and the above definition of $\Sigma_{kl}$ that

$$\Sigma_{12} + \Sigma_{22} \leq \langle h_{\alpha}L_{\alpha} \otimes h_{\beta}L_{\beta}, \Delta_{\alpha/\lambda} \rangle_{\mathbb{D}(0,e^{-r}t)} \leq \Sigma_{k,l=1}^{3}.$$

We will prove the following estimates

(6.10)

$$\Sigma_{11} \leq c_r G_{\alpha,\beta,\rho}^{(0)}(\log |\lambda|, r - h).$$

(6.11)

$$\Sigma_{12} + \Sigma_{22} \leq \min\{\Sigma_{12}, \Sigma_{21}\} \leq \max\{\Sigma_{12}, \Sigma_{21}\} \leq c_{r,\rho} G_{\alpha,\beta}^{(1)}(\log |\lambda|, r - h).$$

(6.12)

$$\Sigma_{13} + \Sigma_{31} \leq c_{r,\rho} G_{\alpha,\beta}^{(1)}(\log |\lambda|, r - h).$$

(6.13)
(6.14) \[ c_{r,p}^{-1}G_{\alpha,\beta}^{(2)}(\log |\lambda|, r + h) \leq \Sigma_{22} \leq c_{r,p}G_{\alpha,\beta}^{(2)}(\log |\lambda|, r - h). \]

(6.15) \[ \Sigma_{23} + \Sigma_{32} \leq c_{r,p}G_{\alpha,\beta}^{(2)}(\log |\lambda|, r - h). \]

(6.16) \[ \Sigma_{33} \leq c_{r,p}G_{\alpha,\beta}^{(j)}(\log |\lambda|, r - h) \quad \text{for} \quad j = 1, 2. \]

Assuming for the moment the estimates (6.11)–(6.16), we will complete the proof of the proposition.

To prove the first inequality in (6.9), we fix \( 0 < \rho < 1 \). Combining the first inequalities in (6.10) and (6.12) and (6.14), the first inequality in (6.9) follows.

The second inequality in (6.9) holds by putting the second inequalities in (6.10), (6.12), (6.14), together with inequalities (6.13), (6.15), (6.16), together with inequality (6.11).

Now we prove these estimates.

**Proof of estimate (6.11).** Observe that \( \Sigma_{11} \) consists of all terms \((z,w)\) such that \(|z_1| + |w_1| > \rho^{-2}|\lambda|^{-1}\) and \(|z_2| + |w_2| > \rho^{-2}|\lambda|^{-1}\). This, combined with (6.8), implies that \( \frac{z_j}{w_j} = 1 + O(\rho^2)\) for \( j = 1 \) and \( 2 \). Consequently, we deduce from (6.6) that there are \( m,n \in \mathbb{Z} \) such that

\[
\begin{align*}
(\zeta - \xi + \frac{\log |\alpha| - \log |\beta|}{b}) = 2m\pi + O(\rho^2)
\end{align*}
\]

(6.17)

We infer from this system that

\[
(\arg \alpha - \arg \beta) - i(\log |\alpha| - \log |\beta|) = 2n\pi + O(\rho^2).
\]

When \( 0 < \rho \ll 1 \), we conclude from the last estimate and the fact that \( \alpha, \beta \in \mathbb{A}, \eta \not\in \mathbb{R} \) that \( m = n = 0 \) and

\[
\begin{align*}
\alpha - \beta = O(\rho^2) \quad \text{and} \quad \zeta - \xi = O(\rho^2).
\end{align*}
\]

(6.18)

We infer from (6.18) and (6.8) that

\[
\begin{align*}
a_1 \lambda z_1 &= \frac{w_1 - z_1}{z_1} = \frac{\beta}{\alpha} \exp \left( i(\zeta - \xi) + \frac{\log |\beta| - \log |\alpha|}{b} \right) - 1 = \frac{\beta}{\alpha} \left( \frac{\alpha}{\beta} \right)^{(ia-b)(\zeta-\xi)} - 1 \\
&= (1 + O(\rho^2))(1 + (ia-b)(\xi - \zeta)) - 1 = (ia-b)(\xi - \zeta) + O(\rho^2),
\end{align*}
\]

\[
\begin{align*}
a_2 \lambda z_2 &= \frac{w_2 - z_2}{z_2} = e^{i(\zeta - \xi) + \frac{\log |\beta| - \log |\alpha|}{b}} - 1 \\
&= (1 + O(\rho^2))e^{i(\zeta - \xi)} - 1 = i(\xi - \zeta) + O(\rho^2).
\end{align*}
\]

Since \( \eta = a + ib \), we deduce from the above equalities that

\[
\begin{align*}
a_1 \lambda z_1 &\cdot a_2 \lambda z_2 = \frac{(ia-b)(\xi - \zeta) + O(\rho^2)}{i(\xi - \zeta) + O(\rho^2)} = \eta + O(\rho^2).
\end{align*}
\]

So \( \frac{a_2}{a_1} = (\eta + O(\rho^2))\frac{a_2}{a_1} \). This, combined with (6.3), implies that

\[
\begin{align*}
b \nu + a \nu - v = \log |\eta \frac{a_2}{a_1}| + O(\rho^2).
\end{align*}
\]

(6.19)

We have shown that there is a constant \( h > 0 \) such that for every \((z,w)\) (resp. \((\zeta,\xi)\)) appearing in the sum \( \Sigma_{11} \), \( z - w = O(\rho^2) \) and there is a point \( \tilde{\zeta} = \tilde{u} + i\tilde{v} \) on the real line \( b\tilde{u} + (a-1)\tilde{v} = 0 \) such that \( |\zeta - \tilde{\zeta}| < h \). Introduce the variable \( m := \tilde{v}/b \). Since \( \tilde{\zeta} = \tilde{u} + i\tilde{v} \)
belongs to the above line, we get that $\tilde{u} = (1 - a)m$, and hence this line is the image of the map $\mathbb{R} \ni m \mapsto (a^2 + b^2)m - \eta^{-1}m$. The condition $(s, t) \in \mathbb{D}(0, e^{-r})^4$ and $b > 0$ implies that $m > r/b - h$. The condition $|z_j| + |w_j| > \rho^{-2}|\lambda|^{-1}$ for $j \in \{1, 2\}$ implies that $e^{-v} > \rho^{-2}|\lambda|^{-1}$, and hence $m < \frac{\log|\lambda|}{b}$.

Next, we will show the following claim: for every point $\tilde{\zeta} = \tilde{u} + i\tilde{v}$ on the above line with $r/b - h \leq m \leq \frac{\log|\lambda|}{b}$, there is at most a finite number of pairs $(z, w)$ (resp. $(\zeta, \xi)$) appearing in the sum $\Sigma_{11}$ such that $|\zeta - \tilde{\zeta}| = O(h)$. Assuming this for the moment, we deduce from the previous paragraph and from Harnack’s inequality that

$$\Sigma_{11} \leq c_r \sum_{m \in \mathbb{Z} : r/b - h \leq m \leq \frac{\log|\lambda|}{b}} H_\alpha((a^2 + b^2)m - \eta^{-1}m)H_\beta((a^2 + b^2)m - \eta^{-1}m) = G^{(0)}_{\alpha, \beta}(\log|\lambda|, r - h)$$

for some constant $c_r$. This will proves estimate (6.11).

To prove the above claim, we consider two points $C = (C_1, C_2) \in L_\alpha$ and $D = (D_1, D_2) \in L_\beta$ given by

$$C_1 = \alpha e^{i(\tilde{\zeta} + \log|\alpha|/b)} \quad \text{and} \quad C_2 = e^{i(\tilde{\zeta} + \log|\beta|/b)};$$

$$D_1 = \beta e^{i(\tilde{\zeta} + \log|\beta|/b)} \quad \text{and} \quad D_2 = e^{i(\tilde{\zeta} + \log|\beta|/b)}.$$

Note that $|C_1|, |C_2|, |D_1|, |D_2| \gtrsim \rho^{-2}$ and $C_1/D_1 = 1 + O(\rho^2)$, $C_2/D_2 = 1 + O(\rho^2)$. We need to show that when $0 < r_0 \ll 1$, the following system of equations with two unknown $(s, t) \in \mathbb{D}(0, r_0)^2$ admits at most one solution

$$\begin{cases}
D_1 e^{-\eta s} - C_1 e^{-\eta t} = a_1/\lambda \\
D_2 e^{-\eta s} - C_2 e^{-\eta t} = a_2/\lambda
\end{cases}$$

If $(s, t)$ is a solution, then by (6.8) we obtain the desired pair $(z, w)$ with $z = (C_1 e^{-\eta t}, C_2 e^t)$ and $w = (D_1 e^{-\eta s}, D_2 e^s)$. We deduce from the second equation that

$$s = f(t) := \log \left( (C_2/D_2) e^t + a_2/(\lambda D_2) \right).$$

So the above system boils down to the equation $g(t) = 0$, where

$$g(t) := D_1 e^{-\eta f(t)} - C_1 e^{-\eta t} - a_1/\lambda.$$

Using the Taylor expansion of $f$, we obtain the Taylor expansion of the order 2 of $g$ that

$$g(t) = -a_1/\lambda + c_0 + c_1 t + t^2 O(1/\lambda),$$

where $c_0, c_1 \approx 1/\lambda$. Consider $h(t) := -a_1/\lambda + c_0 + c_1 t$. Now choose $0 < r_0 \ll 1$ small enough. Observe that

$$\max \left( \min_{|t|=r_0} |h(t)|, \min_{|t|=2r_0} |h(t)| \right) \approx r_0/\lambda.$$

So either for all $t \in \partial \mathbb{D}(0, r_0)$ or for all $t \in \partial \mathbb{D}(0, 2r_0)$, we have that $|h(t)| > |t^2| O(1/\lambda)|$. Since $h(t) = 0$ admits at most one solution $t \in \mathbb{D}(0, r_0)$, it follows from Rouché’s theorem that so does $g(t) = 0$. This completes the proof of the above claim, and hence of estimate (6.11).

**Proof of estimate (6.12).** We only give the proof of the estimates for $\Sigma_{12}$ since the proof of the estimates for $\Sigma_{21}$ is similar. Write $\Sigma_{12} = \Sigma_{12}' + \Sigma_{12}''$, where $\Sigma_{12}', \Sigma_{12}''$ are defined as below.
Estimate of $\Sigma_{12}$: the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| > \rho^{-2}|\lambda|^{-1}$ and $|z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and $|w_2| < \rho|z_2|$. Using Harnack’s inequality and Rouche’s theorem we see that $\Sigma_{12}$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $z_1 - w_1 = 0$ and $z_2 \approx e^{-in^{-1}c\log|\lambda|}$, where $c := \frac{a^2 + b^2}{6} > 0$. The equation $z_1 - w_1$ gives that $\xi - \zeta = 2\pi n^{-1} \lambda n$, and the equation $z_2 \approx e^{-in^{-1}c\log|\lambda|}$ gives that $\xi = 2\pi m + \eta^{-1}c\log|\lambda|$ for some $m, n \in \mathbb{Z}$. So we get that

$$\frac{\zeta}{2\pi} = m + \eta^{-1}c\log|\lambda| \quad \text{and} \quad \frac{\xi}{2\pi} = m + \eta^{-1}c(\log|\lambda| + n) \quad \text{for} \quad m, n \in \mathbb{Z}.$$ 

This, combined with the condition $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, implies that $m, n \geq r$. So $\Sigma'_{12}$ is equivalent to the first sum in $G^{(1)}_{\alpha, \beta}(\log|\lambda|, r)$.

Estimate of $\Sigma''_{12}$: the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| > \rho^{-2}|\lambda|^{-1}$ and $|z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and $|z_2| < \rho|w_2|$. Using Harnack’s inequality and Rouche’s theorem we see that $\Sigma_{12}$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $z_1 - w_1 = 0$ and $w_2 \approx e^{-in^{-1}c\log|\lambda|}$, where $c := \frac{a^2 + b^2}{6} > 0$. The equation $z_1 - w_1$ gives that $\xi - \zeta = 2\pi n^{-1} \lambda n$, and the equation $w_2 \approx e^{-in^{-1}c\log|\lambda|}$ gives that $\xi = 2\pi m + \eta^{-1}c\log|\lambda|$ for some $m, n \in \mathbb{Z}$. So we get that

$$\frac{\zeta}{2\pi} = m + \eta^{-1}c\log|\lambda| \quad \text{and} \quad \frac{\xi}{2\pi} = m + \eta^{-1}c(\log|\lambda| + n) \quad \text{for} \quad m, n \in \mathbb{Z}.$$ 

This, combined with the condition $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, implies that $m, n \geq r$. So $\Sigma''_{12}$ is equivalent to the second sum in $G^{(1)}_{\alpha, \beta}(\log|\lambda|, r)$.

In summary, $\Sigma_{12} \approx G^{(1)}_{\alpha, \beta}(\log|\lambda|, r)$. This completes the proof of estimate (6.12).

Proof of estimate (6.13). We only give the proof of the estimates for $\Sigma_{13}$ since the proof of the estimates for $\Sigma_{41}$ is similar.

Estimate of $\Sigma_{13}$: the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| > \rho^{-2}|\lambda|^{-1}$ and $|z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and $\rho \leq |z_1|/|w_2| \leq \rho^{-1}$. Using Harnack’s inequality and Rouche’s theorem we see that $\Sigma_{13}$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $z_1 - w_1 = 0$ and that $z_2$ and $w_2$ can take only a finite number of values of order $|\lambda|^{-1}$. Arguing as in the estimate of $\Sigma_{12}$ we see easily that

$$\Sigma_{13} \approx G^{(1)}_{\alpha, \beta}(\log|\lambda|, r).$$

Proof of estimate (6.14). Write $\Sigma_{22} = \Sigma''_{22} + \Sigma'_{22}$, where $\Sigma''_{22}, \Sigma'_{22}$ are defined as below.

Estimate of $\Sigma''_{22}$: the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| \leq \rho^{-2}|\lambda|^{-1}$, $|z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and that either the two estimates $|w_1| \leq \rho|z_1|$ and $|z_2| \leq \rho|w_2|$ hold, or the two estimates $|z_1| \leq \rho|w_1|$ and $|w_2| \leq \rho|z_2|$ hold. Write $\Sigma''_{22} := \Sigma_{221} + \Sigma_{222}$. Here $\Sigma_{221}$ (resp. $\Sigma_{222}$) is the partial sum of $\Sigma_{22}$ such that two estimates $|w_1| \leq \rho|z_1|$ and $|z_2| \leq \rho|w_2|$ hold (resp. the two estimates $|z_1| \leq \rho|w_1|$ and $|w_2| \leq \rho|z_2|$ hold).

Using Harnack’s inequality and Rouche’s theorem we see that $\Sigma''_{221}$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $z_1 \approx e^{-\log|\lambda|}$ and $w_2 \approx e^{-in^{-1}c\log|\lambda|}$ with $c := \frac{a^2 + b^2}{6} > 0$. The equation $z_1 \approx e^{-\log|\lambda|}$ gives that $\zeta = 2\pi n^{-1}m - \eta^{-1}c\log|\lambda|$, and the equation $w_2 \approx e^{-in^{-1}c\log|\lambda|}$ gives that $\xi = 2\pi(m + n) + 2\eta^{-1}c\log|\lambda|$ for some $m, n \in \mathbb{Z}$. So we get that

$$\frac{\zeta}{2\pi} = m - \eta^{-1}c\log|\lambda| \quad \text{and} \quad \frac{\xi}{2\pi} = m + n - \eta^{-1}c\log|\lambda| \quad \text{for} \quad m, n \in \mathbb{Z}.$$
This, combined with the condition $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, implies that $m, n \geq r$. So $\Sigma_{121}'$ is equivalent to the first sum in $G_{a,\beta}^{(2)}(\log |\lambda|, r)$.

Similarly, we can show that $\Sigma_{121}'$ is equivalent to the second sum in $G_{a,\beta}^{(2)}(\log |\lambda|, r)$.

In summary, $\Sigma_{22}'$ is equivalent to $G_{a,\beta}^{(2)}(\log |\lambda|, r)$.

**Estimate of $\Sigma_{22}'$:** the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| \leq \rho^{-2}|\lambda|^{-1}, |z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and that either the two estimates $|w_1| \leq \rho|z_1|$ and $|w_2| \leq \rho|z_2|$ hold, or the two estimates $|z_1| \leq \rho|w_1|$ and $|z_2| \leq \rho|w_2|$ hold. Write $\Sigma_{22}'' := \Sigma_{221}' + \Sigma_{222}'$. Here $\Sigma_{221}'$ (resp. $\Sigma_{222}'$) is the partial sum of $\Sigma_{22}'$ such that two estimates $|w_1| \leq \rho|z_1|$ and $|w_2| \leq \rho|z_2|$ hold (resp. the two estimates $|z_1| \leq \rho|w_1|$ and $|z_2| \leq \rho|w_2|$ hold).

Using Harnack’s inequality and Rouché’s theorem we see that $\Sigma_{221}'$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $z_1 \approx e^{-\log |\lambda|}$ and $z_2 \approx e^{-\log |\lambda|}$ with $c := \frac{a^2 + b^2}{b} > 0$. The equation $z_1 \approx e^{-\log |\lambda|}$ gives that $\zeta = 2\pi\eta^{-1}m - \eta^{-1}\log |\lambda|$, and the equation $z_2 \approx e^{-\log |\lambda|}$ gives that $\zeta = 2\pi(m + n) + 2\pi\eta^{-1}\log |\lambda|$ for some $m, n \in \mathbb{Z}$. Since $\{1, -\eta^{-1}\}$ forms a $\mathbb{R}$-basis in $\mathbb{C}$, it follows that when $\rho > 0$ is chosen small enough, the last two equations have no solution $(m, n) \in \mathbb{Z}^2$. Hence, we may assume that $\Sigma_{221}' = 0$.

The proof of estimate (6.14) is thereby completed.

**Proof of estimate (6.15).** We only give the proof of the estimates for $\Sigma_{23}$ since the proof of the estimates for $\Sigma_{42}$ is similar.

**Estimate of $\Sigma_{23}$:** the sum is taken over all $(z, w)$ such that $|z_1| + |w_1| \leq \rho^{-2}|\lambda|^{-1}$ and either $|z_1| < \rho|w_1|$ or $|w_1| < \rho|z_1|$ and that $|z_2| + |w_2| \leq \rho^{-2}|\lambda|^{-1}$ and $\rho \leq |z_2|/|w_2| \leq \rho^{-1}$.

Write $\Sigma_{23} := \Sigma_{231} + \Sigma_{232}$. Here $\Sigma_{231}$ (resp. $\Sigma_{232}$) is the partial sum of $\Sigma_{23}$ such that $|z_1| \leq \rho|w_1|$ (resp. $|w_1| \leq \rho|z_1|$).

Now we estimate $\Sigma_{231}$. Using Harnack’s inequality and Rouché’s theorem we see that $\Sigma_{231}$ is equivalent to the sum taken over all pairs $(\zeta, \xi) \in \mathbb{C}^2$ such that $|z_1|, |z_2|, |w_1|, |w_2| < e^{-r}$, and that $w_1 \approx e^{-\log |\lambda|}$ and that $z_2$ and $w_2$ range over a finite set of complex numbers of order $|\lambda|^{-1}$. Consequently, we can show that

$$\Sigma_{231} \lesssim G_{a,\beta}^{(2)}(\log |\lambda|, r).$$

Similarly, we get that $\Sigma_{232} \lesssim G_{a,\beta}^{(2)}(\log |\lambda|, r)$. Hence, estimate (6.15) follows.

**Proof of estimate (6.16).** We only prove the first inequality in (6.16) since the second one can be done similarly. The sum is taken over all $(z, w)$ such that $|z_j| + |w_j| \leq \rho^{-2}|\lambda|^{-1}$ and $\rho \leq |z_j|/|w_j| \leq \rho^{-1}$ for $j = 1, 2$. We can show that if $(\zeta, \xi)$ satisfies the system (6.6) and the last inequalities, then $|\zeta - \xi| < c_\rho$ for a constant $c_\rho$. Using this estimate and applying Harnack’s inequality and Rouché’s theorem, we next show that $\zeta$ and $\xi$ range over a finite set of complex numbers of order $|\lambda|^{-1}$. So the set of all such $(\zeta, \xi)$ is contained in the sum $G_{a,\beta}^{(4)}(\log |\lambda|, r)$. Hence, the first inequality in (6.16) follows.

Prior to the proof of Proposition 6.2, we take for granted the following two lemmas. Their proofs will be provided in the next section.

**Lemma 6.8.** For every $r \geq -\log 3$ and every $\epsilon > 0$, there exists $0 < \rho_{r,\epsilon} < 1$ such that for every $0 < \rho < \rho_{r,\epsilon}$,

$$\lim_{s \to \infty} \int_{\alpha \in A} \int_{\beta \in A} G_{a,\beta,\rho}^{(0)}(s, r) d(\mu \otimes \mu)(\alpha, \beta) < \epsilon.$$
We need to show that there is a sequence $\lambda_k \to \infty$ such that
\[
\lim_{k \to \infty} \langle (A_{\lambda_k})_*(T \otimes T), \vartheta(z, w) i(dz_1 - dw_1) \wedge (d\bar{z}_1 - d\bar{w}_1) \wedge i(dz_2 - dw_2) \wedge (d\bar{z}_2 - d\bar{w}_2) \rangle = 0.
\]
Choose $\lambda_k := e^{h_k}$, where $s_k$ is the sequence given by Lemma 6.9. Pick an arbitrary $\epsilon > 0$. We need to show that there is $K'_{\epsilon}$ such that for every $k > K'_{\epsilon}$,
\[
\langle (A_{\lambda_k})_*(T \otimes T), \vartheta(z, w) i(dz_1 - dw_1) \wedge (d\bar{z}_1 - d\bar{w}_1) \wedge i(dz_2 - dw_2) \wedge (d\bar{z}_2 - d\bar{w}_2) \rangle < \epsilon.
\]

By Lemma 6.6 (2), we may find a constant $c'_r > 1$ such that
\[
\langle (A_{\lambda_k})_*(T \otimes T), \vartheta(z, w) i(dz_1 - dw_1) \wedge (d\bar{z}_1 - d\bar{w}_1) \wedge i(dz_2 - dw_2) \wedge (d\bar{z}_2 - d\bar{w}_2) \rangle \leq \frac{c'_r}{r} \int e^{-r/6} \langle (T \otimes T, \Delta_{a/\lambda_k}) |(z, w) \in \mathbb{D}(0, e^{-r}), i da_1 \wedge d\bar{a}_1 \wedge i da_2 \wedge d\bar{a}_2 \rangle.
\]

Choose $r := h - \log 3$. We may assume that $h \gg 2 \log 3$. Choose $0 < \rho := \rho_{r, c'_r}$, where $\epsilon' = \epsilon/(2c_r c'_r)$ and $c_r$ is the constant given in Proposition 6.7. The integrand on the right-hand side of the last line can be written as
\[
\langle (T \otimes T, \Delta_{a/\lambda_k}) |(z, w) \in \mathbb{D}(0, e^{-r}), i da_1 \wedge d\bar{a}_1 \wedge i da_2 \wedge d\bar{a}_2 \rangle.
\]

By Proposition 6.7 and the equality $\log \lambda_k = s_k$, the right hand side of the last line is dominated by
\[
\int_{a \in A} \int_{b \in A} \left( c_{r, \rho}(G_{a, \beta}^{(1)}(s_k, -\log 3) + G_{a, \beta}^{(2)}(s_k, -\log 3)) + c_r G_{a, \beta, \rho}^{(0)}(s_k, -\log 3) \right) d(\mu \otimes \mu)(\alpha, \beta).
\]

By Lemma 6.9, we can find $K'_{\epsilon} \gg 1$ such that for every $k > K'_{\epsilon}$,
\[
c_{r, \rho} c'_r \int \int \left( G_{a, \beta}^{(1)}(s_k, -\log 3) + G_{a, \beta}^{(2)}(s_k, -\log 3) \right) d(\mu \otimes \mu)(\alpha, \beta) < \epsilon/2.
\]

Moreover, by Lemma 6.8 and the fact that $\rho = \rho_{r, c'_r}$, we can find $K'_{\epsilon} \geq K'_{\epsilon}$ such that for every $k > K'_{\epsilon}$ we have that
\[
c_{r, \rho} c'_r \int \int G_{a, \beta, \rho}^{(0)}(s_k, -\log 3) d(\mu \otimes \mu)(\alpha, \beta) < \epsilon/2.
\]

On the other hand, since $h \gg 2 \log 3$, it follows that
\[
\int e^{-r/6} \langle (T \otimes T, \Delta_{a/\lambda_k}) |(z, w) \in \mathbb{D}(0, e^{-r}), i da_1 \wedge d\bar{a}_1 \wedge i da_2 \wedge d\bar{a}_2 \rangle < 1.
\]

These estimates imply Proposition 6.2. □
7. **Integral estimates**

The main purpose of this section is to prove Lemmas 6.8 and 6.9. From now on, assume for simplicity that $b > 0$ and hence $S$ is a sector of angle $\arctan(a/b) \in (0, \pi)$ having $\mathbb{R}_+$ in its boundary (otherwise, it has $\mathbb{R}_-$ in the boundary and we need to change slightly the definition of the map $\Phi$ below). Let $\gamma := \frac{\pi}{\arctan(-b/a)}$.

Then the map

$$\Phi : \zeta = u + iv \mapsto \zeta^{\gamma} = (u + iv)^\gamma = U + iV$$

sending $S$ bi-holomorphically to the upper half-plane $\mathbb{H} \subset \mathbb{C}$. The fact that $\gamma > 1$ will be crucial (see Lemma 7.4), this is where the hyperbolicity of singularities is used.

Consider the new variable

$$t = t(\zeta) := bu + av.$$

So identity (6.3) becomes

$$|z_1| = e^{-v} \quad \text{and} \quad |z_2| = e^{-t}.$$

Define $\hat{H}_\alpha := H_\alpha \circ \Phi^{-1}$. This is a positive harmonic function in $\mathbb{H}$ which is continuous up to the boundary because $S' \supset S = \Phi^{-1}(\mathbb{H})$.

The following mass-clustering estimate of Fornaess–Sibony [13] is needed.

**Lemma 7.1.**

1. For $\mu$-almost every $\alpha \in \mathbb{A}$, the following properties hold:
   
   (i) the function $\hat{H}_\alpha$ is the Poisson integral of its boundary values, that is, for $U, V \in \mathbb{R}$ with $V > 0$, we have

   $$\hat{H}_\alpha(U + iV) = \frac{1}{\pi} \int_{x \in \mathbb{R}} \hat{H}_\alpha(x) \frac{V}{V^2 + (x - U)^2} dx;$$

   (ii) $\int_{x \in \mathbb{R}} \hat{H}_\alpha(x)|x|^{1+1/\gamma} dx \leq C$, where $C > 0$ is a constant independent of $\alpha$.

2. If, moreover, $T$ gives no mass to every invariant analytic curve, then $\mu$ is diffuse, that is, $\mu(\alpha) = 0$ for every $\alpha \in \mathbb{A}$.

**Proof.** The first part is proved in [13, Proposition 1] (see also [8, Lemmas 4.2 and 4.3] for a slightly different proof).

When $\mathcal{F}$ has no invariant analytic curve, the second part is proved in [13, Corollary 2]. But that proof still works in the more general context of Part 2) making the obviously necessary changes. □

The following lemma reduces the estimates on $G_{\alpha,\beta,\rho}^{(0)}$ and $G_{\alpha,\beta}^{(1)}$, $G_{\alpha,\beta}^{(2)}$ to some integrals which are easier to handle. We will use the continuous variables $y, y' \in \mathbb{R}^+$ in place of the discrete variables $m, n \in \mathbb{N}$.

**Lemma 7.2.** For $0 \leq j \leq 2$, and $s, r \geq 0$ and $0 < \rho \ll 1$ and $\alpha, \beta \in \mathbb{A}$, we have that

$$G_{\alpha,\beta,\rho}^{(0)}(s, r) \approx \int_{x=-\infty}^{\infty} \int_{x'= -\infty}^{\infty} \int_{x=-\infty}^{\infty} \int_{x'= -\infty}^{\infty} K^{(0)}(s, r, x, x')\hat{H}_\alpha(x)\hat{H}_\beta(x')dxdxa'$,$

$$G_{\alpha,\beta}^{(j)}(s, r) \approx \int_{x=-\infty}^{\infty} \int_{x'= -\infty}^{\infty} \int_{x=-\infty}^{\infty} \int_{x'= -\infty}^{\infty} K^{(j)}(s, r, x, x')\hat{H}_\alpha(x)\hat{H}_\beta(x')dxdxa' \quad \text{for} \quad j = 1, 2.$$

Here:
(1) For $j = 0$: if $|\alpha - \beta| \geq \rho$ then,

$$K^{(0)}(s, r, x, x') := \int_{y=r}^{s} \frac{V}{V' + (x - U)^2} dy;$$

where $U + iv = ((a^2 + b^2)y - \eta^{-1}y)^{\gamma}$, otherwise $K^{(0)}(s, r, x, x') = 0$.

(2) For $j = 1$:

$$K^{(1)}(s, r, x, x') := \left( \int_{y=r}^{s} \int_{y'=r}^{s} \frac{V}{V' + (x - U)^2} dy \right) dy',$$

where in the first double integral $U + iv = (y + \eta^{-1}s)^{\gamma}$ and $U' + iv' = (y + \eta^{-1}(s + y'))^{\gamma}$; in the second double integral $U + iv = (y + \eta^{-1}s)^{\gamma}$ and $U + iv = (y + \eta^{-1}(s + y'))^{\gamma}$.

(3) For $j = 2$:

$$K^{(2)}(s, r, x, x') := \left( \int_{y=r}^{s} \int_{y'=r}^{s} \frac{V}{V' + (x - U)^2} dy \right) dy',$$

where in the first double integral $U + iv = (y + \eta^{-1}s)^{\gamma}$ and $U' + iv' = (y + \eta^{-1}s)^{\gamma}$; in the second double integral $U + iv = (y + \eta^{-1}s)^{\gamma}$ and $U + iv = (y + \eta^{-1}s)^{\gamma}$.

Proof. Using Harnack’s inequality and replacing $m, n \in \mathbb{N}$ by $y, y' \in \mathbb{R}$ respectively, we see that

$$G^{(1)}_{\alpha, \beta}(s, r) \approx \int_{y=r}^{s} \int_{y'=r}^{s} (H_{\alpha}(y - \eta^{-1}s)H_{\beta}(y - \eta^{-1}(s + y')) + H_{\beta}(y - \eta^{-1}s)H_{\alpha}(y - \eta^{-1}(s + y'))) dydy',$$

and that if $|\alpha - \beta| \geq \rho$ then $G^{(0)}_{\alpha, \beta}(s, r) = 0$, otherwise

$$G^{(0)}_{\alpha, \beta, \rho}(s, r) \approx \int_{y=r}^{s} H_{\alpha}((a^2 + b^2)y - \eta^{-1}y)H_{\beta}((a^2 + b^2)y - \eta^{-1}y) dy.$$

Writing for $\zeta, \zeta' \in \mathbb{S}$,

$$\zeta = u + iv, \quad \zeta^{\gamma} = U + iv \quad \text{and} \quad \zeta' = u' + iv', \quad \zeta'^{\gamma} = U' + iv',$$

we see that

$$H_{\alpha}(u + iv) = \tilde{H}_{\alpha}(U + iv) \quad \text{and} \quad H_{\beta}(u' + iv') = \tilde{H}_{\beta}(U' + iv') \quad \text{for} \quad \alpha, \beta \in \mathbb{A}.$$

Consequently, for $\alpha, \beta \in \mathbb{A}$, we have by Lemma 7.1 that

$$H_{\alpha}(u + iv) = \frac{1}{\pi} \int_{x \in \mathbb{R}} \tilde{H}_{\alpha}(x) \frac{V}{V' + (x - U)^2} dx \quad \text{and} \quad H_{\beta}(u' + iv') = \frac{1}{\pi} \int_{x' \in \mathbb{R}} \tilde{H}_{\beta}(x') \frac{V'}{V'^2 + (x' - U')^2} dx'.$$

Inserting these representations into the above-mentioned equivalent expressions for $G^{(1)}_{\alpha, \beta}(s, r)$, $G^{(2)}_{\alpha, \beta}(s, r)$ and $G^{(0)}_{\alpha, \beta, \rho}(s, r)$ gives the expressions in the lemma.

The behavior of the Poisson kernel $\frac{V}{V' + (x - U)^2}$ in terms of $v$ and $t = bu + av$ is summarized in the following

Lemma 7.3. For every $0 < \epsilon < 1/2$ there is a constant $c_\epsilon > 1$ large enough such that the following properties hold for all $t, v \geq 0$ and $x \in \mathbb{R}$ with $|x| \gg 1$. 

(1) If \( \max\{v, t\} \geq (1 + \epsilon)(1 + |x|)^{1/\gamma} \), then
\[
\frac{1}{c_\epsilon} \leq \frac{V}{V^2 + (x - U)^2} : \frac{\min\{v, t\}}{(\max\{v, t\})^{\gamma+1}} \leq c_\epsilon.
\]

(2) If \( \max\{v, t\} \leq (1 - \epsilon)(1 + |x|)^{1/\gamma} \), then
\[
\frac{1}{c_\epsilon} \leq \frac{V}{V^2 + (x - U)^2} : \frac{1}{(\max\{v, t\})^{\gamma-1}(\min\{v, t\})} \leq c_\epsilon.
\]

(3) If \( (1 - \epsilon)(1 + |x|)^{1/\gamma} \leq \max\{v, t\} \leq (1 + \epsilon)(1 + |x|)^{1/\gamma} \) and \( \min\{v, t\} \geq \epsilon(1 + |x|)^{1/\gamma} \), then
\[
\frac{1}{c_\epsilon} \leq \frac{V}{V^2 + (x - U)^2} : \frac{\min\{v, t\}}{(\max\{v, t\})^{\gamma-1}(\min\{v, t\})} \leq c_\epsilon.
\]

(4) If \( \min\{v, t\} \leq \epsilon(1 + |x|)^{1/\gamma} \) and \( (1 - \epsilon)(1 + |x|)^{1/\gamma} \leq \max\{v, t\} \leq (1 + \epsilon)(1 + |x|)^{1/\gamma} \), then
\[
\frac{1}{c_\epsilon} \leq \frac{V}{V^2 + (x - U)^2} : \frac{(1 + |x|)^{\gamma-1}(\min\{v, t\})}{(\min\{v, t\})^{\gamma-1}(\max\{v, t\})} \leq c_\epsilon.
\]

Proof. Parts (1) and (3) are exactly Parts (2) and (4) in [22, Lemma 3.3]. Part (2) holds by combining Parts (1) and (3) in [22, Lemma 3.3].

Now we explain how to deduce Part (4) from from Part (5) in [22, Lemma 3.3]. Indeed, by the latter part, there is a constant \(c_\epsilon > 1\) such that
\[
\frac{1}{c_\epsilon} \leq \frac{V}{V^2 + (x - U)^2} : \frac{(1 + |x|)^{\gamma-1}(\min\{v, t\})}{(\min\{v, t\})^{\gamma-1}(\max\{v, t\}) - \rho^2} \leq c_\epsilon,
\]
where \(\rho\) is a real number which depends only on \(x\) and on \(\min\{v, t\}\) which satisfies \((1 - \epsilon)(1 + |x|)^{1/\gamma} \leq \rho \leq (1 + \epsilon)(1 + |x|)^{1/\gamma}\). In fact, \(\rho := bu(x, v) + av\), where \(u\) is the unique solution of the equation
\[
U = x, \quad \text{where } U + iV = (u + iv)^\gamma
\]
which satisfies \((1 - \epsilon)(1 + |x|)^{1/\gamma} \leq \max\{v, bu(x, v) + av\} \leq (1 + \epsilon)(1 + |x|)^{1/\gamma}\). We deduce easily from this equation that \(\rho = |x|^{1/\gamma} + O(\min\{v, t\}^{1/\gamma})\). Inserting this into (7.4), Part (4) follows.

Note that the roles of \(v\) and \(t\) can be exchanged in the estimates of the Poisson kernel in Lemma [7.3]

In what follows, we fix an \(\epsilon = \epsilon_0 \in (0, 1/2)\) small enough. Let \(c_0 = c_\epsilon\) be the constant given in Lemma [7.3]. Let \(x \in \mathbb{R}\) and fix \(t \geq 0\) (resp. \(v \geq 0\)). Let \(D_{x,t}\) (resp. \(D_{x,v}\)) denote the set of all \(v \geq 0\) (resp. all \(t \geq 0\)) such that \(x, t, v\) satisfies at least one of the items (1)–(2)–(3) in Lemma [7.3]. Let \(D_{x,t}^*\) (resp. \(D_{x,v}^*\)) denote the set of all \(v \geq t\) (resp. all \(t \geq v\)) such that \(x, t, v\) satisfies item (4) in Lemma [7.3]. Let \(D_{x,t}^{**}\) (resp. \(D_{x,v}^{**}\)) denote the set of all \(v \in [0, t]\) (resp. all \(t \in [0, v]\)) such that \(x, t, v\) satisfies item (4) in Lemma [7.3]. So
\[
R_v^+ = D_{x,t} \cup D_{x,t}^* \cup D_{x,t}^{**} \quad \text{and} \quad R_t^+ = D_{x,v} \cup D_{x,v}^* \cup D_{x,v}^{**}.
\]

More specifically, if \(0 \leq t \leq \epsilon_0(1 + |x|)^{1/\gamma}\), then \(D_{x,t}^* = [(1 - \epsilon_0)(1 + |x|)^{1/\gamma}, (1 + \epsilon_0)(1 + |x|)^{1/\gamma}]\), else \(D_{x,t}^* = \emptyset\). If \((1 - \epsilon_0)(1 + |x|)^{1/\gamma} \leq t \leq (1 + \epsilon_0)(1 + |x|)^{1/\gamma}\), then \(D_{x,t}^{**} = [0, \epsilon_0(1 + |x|)^{1/\gamma}]\), else \(D_{x,t}^{**} = \emptyset\). Moreover, \(D_{x,t} = R_v \setminus (D_{x,t}^* \cup D_{x,t}^{**})\). The same equalities hold if we exchange \(v\) and \(t\).
Lemma 7.4. There is a constant $c_1 \gg c_0$ such that for all $x \in \mathbb{R}$, the following inequalities hold.

1. $$\frac{V}{V^2 + (x - U)^2} \leq c_1 \min\{v, t\} \left(\min\{v, t\}\right)^{-(1 + |x|)^{\gamma - 1}}$$ for $v \in D_{x,t}$.

2. For $t \geq 0$,
   $$\int_{D_{x,t}} \frac{V}{V^2 + (x - U)^2} dv \leq c_1 (1 + |x|)^{\gamma - 1}.$$

3. For $0 \leq t \leq c_0 (1 + |x|)^{\gamma}$,
   $$\int_{D_{x,t}} \frac{V}{V^2 + (x - U)^2} dv \leq c_1 (1 + |x|)^{\gamma - 1}.$$

4. For $(1 - \epsilon_0)(1 + |x|)^{\gamma} \leq t \leq (1 + \epsilon_0)(1 + |x|)^{\gamma}$ and for every $0 \leq s \leq c_0 (1 + |x|)^{\gamma}$,
   $$c_1^{-1} \leq \int_{v \in D_{x,t} : v \geq s} \frac{V}{V^2 + (x - U)^2} dv : \log \left(\frac{|x|^{\gamma}}{s + |t - |x|^{\gamma}}\right)^{1 + |x|^{\gamma}} \leq c_1.$$

The same inequalities holds if we exchange $t$ and $v$.

Proof. Assertion (1) holds by a combination of Lemma [7.3](1)-(2)-(3). Assertion (2) follows immediately from assertion (1). Assertions (3) and (4) are consequences of Lemma [7.5](4).

Lemma 7.5. There is a constant $c > 0$ such that for $\alpha, \beta \in \mathbb{A}$ and $r \geq - \log 3$,

$$\sum_{m \in \mathbb{N}} H_\alpha((a^2 + b^2)m - \eta^{-1} m) H_\beta((a^2 + b^2)m - \eta^{-1} m) \leq c \quad \text{and} \quad G^{(0)}_{\alpha,\beta,\rho}(s, r) \leq c.$$

Proof. Using Harnack’s inequality as in the proof of Lemma [7.2](3), we see that the above sum and $G^{(0)}_{\alpha,\beta,\rho}(s, - \log 3)$ is equivalent to

$$\int_{-\infty}^{\infty} \int_{x_0}^{\infty} \lim_{s \to \infty} K^{(3)}_{\alpha,\beta}(s, 0, x, x_0) \tilde{H}_\alpha(x) \tilde{H}_\beta(x) dx dx'.$$

Here

$$\lim_{s \to \infty} K^{(3)}_{\alpha,\beta}(s, 0, x, x_0) = \int_{0}^{\infty} \frac{V}{V^2 + (x - U)^2} \frac{V}{V^2 + (x' - U)^2} dy,$$

where $U + iV = ((a^2 + b^2)y - \eta^{-1}y)^{\gamma}$.

We deduce from this and from Lemma [7.1](1)(ii) that the lemma will follow if one can show that for $x, x_0 \in \mathbb{R}$,

$$\lim_{s \to \infty} K^{(3)}_{\alpha,\beta}(s, 0, x, x_0) \leq c(1 + |x|)^{\gamma - 1}(1 + |x'|)^{(1 + \gamma) - 1}.$$

Observe that if we write $u + iv = (a^2 + b^2)y - \eta^{-1}y$ and $t = bu + av$, then $t = v = by$, and hence when $|y| \gg 1$, we have that $v = t \approx y$. Therefore, $(t, v, x)$ and $(t, v, x')$ satisfy at least one of the items (1)–(2)–(3) in Lemma [7.3]. Consequently, applying Lemma [7.4](1) yields that

$$\int_{0}^{\infty} \frac{V}{V^2 + (x - U)^2} \frac{V}{V^2 + (x' - U)^2} dy \lesssim (c' + \int_{1}^{\infty} \frac{1}{y^2} dy) c(1 + |x|)^{(1 + \gamma) - 1}(1 + |x'|)^{(1 + \gamma) - 1} \lesssim c'(1 + |x|)^{(1 + \gamma) - 1}(1 + |x'|)^{(1 + \gamma) - 1},$$
as desired. □

End of the proof of Lemma 6.8 Let \( \Delta \) be the diagonal \( \{ (\alpha, \alpha) : \alpha \in \mathbb{A} \} \). Since \( \mu \) gives no mass to each single point of \( \mathbb{A} \), Fubini’s theorem implies that \( (\mu \otimes \mu)(\Delta) = 0 \). Now let \( \epsilon > 0 \) be arbitrary. It follows from \( (\mu \otimes \mu)(\Delta) = 0 \) that there is \( \rho_\epsilon > 0 \) such that

\[
\int_{\{ (\alpha, \beta) \in K^2 : |\alpha - \beta| < \rho_\epsilon \}} d(\mu \otimes \mu)(\alpha, \beta) < \epsilon.
\]

By Lemma 7.5, we see that for every \( 0 < \rho < \rho_\epsilon \),

\[
\int_{\alpha \in K} \int_{\beta \in K} G_{\alpha,\beta,\rho}^{(0)}(s, r) d(\mu \otimes \mu)(\alpha, \beta) = \int_{\{ (\alpha, \beta) \in K^2 : |\alpha - \beta| < \rho_\epsilon \}} G_{\alpha,\beta,\rho}^{(0)}(s, r) d(\mu \otimes \mu)(\alpha, \beta)
\]

is dominated by a constant times

\[
\int_{\{ (\alpha, \beta) \in K^2 : |\alpha - \beta| < \rho_\epsilon \}} d(\mu \otimes \mu)(\alpha, \beta),
\]

which is bounded by \( \epsilon \) by the choice of \( \rho_\epsilon \). This completes the proof. □

Define for \( \alpha \in K \) and \( s \geq 1 \) and \( r \geq 0 \),

\[
G_{\beta}(s, r) := \sum_{m \in \mathbb{N}, m \geq r} H_{\beta}(m - \eta^{-1}s).
\]

Lemma 7.6. There is a sequence \( s_k \to \infty \) such that \( \int_{\alpha \in K} G_{\alpha}(s_k, r) d\mu(\beta) \) tends to 0 as \( k \) goes to infinity for every \( r \geq 0 \).

Proof. Observe that \( G_{\alpha}(s, r) \) is dominated by a constant times \( \int_{u + is \in S} H_{\alpha}(u + \frac{ibs}{\alpha + \eta}) du \). By \[8\] Lemmas 5.8, 5.9 and pp. 30-31, we see that the last integral tends to 0 through a sequence \( s_k \to \infty \). This completes the proof. □

Lemma 7.7. There is a function \( (s, x, x') \in \mathbb{R}^+ \times \mathbb{R}^2 \mapsto K^{(3)}(s, x, x') \in \mathbb{R}^+ \) with the following properties.

1. For every \( r \geq 0 \) there is a constant \( c = c_r > 1 \) such that for \( j = 1, 2, \) and \( \alpha, \beta \in K \), and \( s \geq 0 \), we have that

\[
c^{-1} G_{\alpha,\beta}^{(3)}(s) \leq G_{\alpha,\beta}^{(j)}(s, r) \leq c G_{\alpha}(s, r) + c G_{\beta}(s, r) + c G_{\alpha,\beta}^{(3)}(s),
\]

where

\[
G_{\alpha,\beta}^{(3)}(s) := \int_{x = -\infty}^{\infty} \int_{x' = -\infty}^{\infty} K^{(3)}(s, x, x') \cdot \hat{H}_{\alpha}(x) \hat{H}_{\beta}(x')(1 + |x|)^{(1/\gamma) - 1}(1 + |x'|)^{(1/\gamma) - 1} dx dx'.
\]

2. For \( x, x' \in \mathbb{R} \), \( \lim_{s \to \infty} K^{(3)}(s, x, x') = 0 \).

3. If \( 1 - \epsilon_0 \leq \frac{1 + |\gamma|}{1 + |\gamma|} \leq 1 + \epsilon_0 \), then

\[
c^{-1} \leq K^{(3)}(s, x, x') : \log \left( \frac{1 + |x|^{1/\gamma} + |x'|^{1/\gamma}}{s + |x|^{1/\gamma} + |x'|^{1/\gamma}} \right) < c,
\]

otherwise \( K^{(3)}(s, x, x') \leq 1 \).
Proof. First observe that there is a constant $c' > 1$ such that for every $r \geq 0$,
\[
c'^{-1} \leq G_\alpha(s, r) = \int_{y=r}^{\infty} H_\alpha(y - \eta^{-1}s)dy \leq c',
\]
Next, we make the following reduction. We only treat the case $r = 0$ and we only prove the existence of the kernel $K^{(3)}$ satisfying assertion (1) for $j = 1$ and assertions (2)-(3). Indeed, the corresponding case for $j = 2$ can be treated similarly, and if $K^{(3)}$ is a kernel satisfying assertion (1) for $j = 2$ and assertions (2)-(3), then we can check easily that $1/2(K^{(3)} + K^{(3)})$ is the desired kernel satisfying both assertions (1) and (2). Moreover, for the same reason, we may assume that
\[
G^{(1)}_{\alpha, \beta}(s, 0) \approx G^{(1)}_{\alpha, \beta}(s) := \int_{y=0}^{\infty} \int_{y'=0}^{\infty} H_\alpha(y - \eta^{-1}s)H_\beta(y - \eta^{-1}(s + y'))dy'dy.
\]
The other integral
\[
\int_{y=0}^{\infty} \int_{y'=0}^{\infty} H_\beta(y - \eta^{-1}s)H_\alpha(y - \eta^{-1}(s + y'))dy'dy
\]
can be treated similarly. So the lemma boils down to the existence of a constant $c$ and a function $G^{(3)}_{\alpha, \beta}$, with kernel $K^{(3)}(s, x, x')$ as in assertion (1) which also satisfies both assertions (2)-(3) and the following inequality
\[
(7.5) \quad G^{(3)}_{\alpha, \beta}(s) \leq G^{(1)}_{\alpha, \beta}(s) \leq c \int_{y=0}^{\infty} H_\alpha(y - \eta^{-1}s)dy + G^{(3)}_{\alpha, \beta}(s).
\]
Applying Lemma 7.1 (1-i), we get
\[
\int_{y'=0}^{\infty} H_\beta(y - \eta^{-1}(s + y'))dy' = \int_{x'=\infty}^{\infty} \left( \int_{z=0}^{\infty} \frac{V'}{V'^2 + (x' - U')^2}dz \right) \tilde{H}_\beta(x')dx'
\]
with $U' + iV' = (y - \eta^{-1}(s + z))^\gamma$. So
\[
(7.6) \quad G^{(3)}_{\alpha, \beta}(s) = \int_{y=0}^{\infty} \left( \int_{x'=\infty}^{\infty} \left( \int_{z=0}^{\infty} \frac{V'}{V'^2 + (x' - U')^2}dz \right) \tilde{H}_\beta(x')dx' \right)H_\alpha(y - \eta^{-1}s)dy.
\]
Let $c$ be a large enough constant and define
\[
(7.7) \quad G^{(3)}_{\alpha, \beta}(s) := \int_{y'} \left( \int_{x'} \left( \int_{y'=0}^{\infty} \frac{V'}{V'^2 + (x' - U')^2}dy' \right) \tilde{H}_\beta(x')dx' \right)H_\alpha(y - \eta^{-1}s)dy,
\]
where $y \in \mathbb{R}^+$ and $x' \in \mathbb{R}$ are such that
\[
(7.8) \quad \left( \int_{z=0}^{\infty} \frac{V'}{V'^2 + (x' - U')^2}dz \right) > c(1 + |x'|)^{(1/\gamma) - 1}.
\]
Clearly, by definition, $G^{(3)}_{\alpha, \beta}(s) \leq G^{(1)}_{\alpha, \beta}(s)$. So the first inequality in (7.5) holds. Moreover, by (7.6), (7.7) and (7.8),
\[
G^{(1)}_{\alpha, \beta}(s) \leq G^{(3)}_{\alpha, \beta}(s) + \int_{y=0}^{\infty} \left( \int_{x'=\infty}^{\infty} c(1 + |x'|)^{(1/\gamma) - 1} \tilde{H}_\beta(x')dx' \right)H_\alpha(y - \eta^{-1}s)dy.
\]
Applying Lemma 7.1 (1-i) to the inner integral, the second inequality in (7.5) follows. It remains to prove that the kernel $K^{(3)}$ of $G^{(3)}_{\alpha, \beta}$ satisfies assertions (2)-(3). Write
\[
u + iv = y - \eta^{-1}s, \quad t = bv + av \quad \text{and} \quad \nu' + iv' = y - \eta^{-1}(s + z), \quad t' = bu' + av'.
\]
We obtain

\[(7.9)\quad v = \frac{bs}{a^2 + b^2}, \quad t = by \quad \text{and} \quad v' = \frac{b(s + y')}{a^2 + b^2}, \quad t' = by.\]

By Lemma 7.4 (2)-(3), there is a constant \(c > 1\) independent of \(\beta\) such that

\[
\int_{y'} \frac{V'}{V'^2 + (x' - U')^2} dy' = \frac{a^2 + b^2}{b} \int_{y'} \frac{V'}{V'^2 + (x' - U')^2} dv' < c,
\]

where the integral is taken over all \(y' \geq 0\) such that \(v', t'\) related to \(y', b\) by (7.9) satisfies \(v' \in D_{x', t'} \cup D_{x', t'}^*\). Moreover, it follows from (7.9) and \(y' \geq 0\) that \(v' \geq \frac{bs}{a^2 + b^2} - \frac{y'}{\epsilon}\). Recall that \(D_{x', t'}^* = \mathbb{R}_v \setminus (D_{x', t'} \cup D_{x', t'}^*)\) and that if \(t' \in [(1 - \epsilon_0)(1 + |x'|)^{1/\gamma}, (1 + \epsilon_0)(1 + |x'|)^{1/\gamma}]\) then \(D_{x', t'}^* = \emptyset\), else \(D_{x', t'}^* = [0, \epsilon_0(1 + |x'|)^{1/\gamma}]\). So it follows from (7.9) that in (7.7) the integral is taken over all \(x' \in \mathbb{R}\) and \(y \geq 0\) such that

\[(7.10)\quad s \leq \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma} \quad \text{and} \quad (1 - \epsilon_0)(1 + |x'|)^{1/\gamma} \leq 1 \leq (1 + \epsilon_0)(1 + |x'|)^{1/\gamma}.\]

Hence, it follows from Lemma 7.4 (4) that the integral in (7.8) is equivalent to

\[
\int_{v' \in D_{x', t'}^*} \cdot v' \geq \frac{bs}{a^2 + b^2} \frac{V'}{V'^2 + (x' - U')^2} \approx \log \left(\frac{\frac{bs}{a^2 + b^2} + |by|}{H_o(y - \eta^{-1} s)}\right)(1 + |x'|)^{(1/\gamma) - 1}.
\]

Inserting this into (7.7), we get

\[(7.11)\quad G_{\alpha, \beta}^{(3)}(s) \approx \int_{x' \in \mathbb{R}}: s \leq \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma} \quad \text{and} \quad (1 - \epsilon_0)(1 + |x'|)^{1/\gamma} \leq 1 \leq (1 + \epsilon_0)(1 + |x'|)^{1/\gamma} \text{,}\]

Applying Lemma 7.1 (1-i), we see that the kernel of \(G_{\alpha, \beta}^{(3)}\) can be defined as follows. If

\[(7.12)\quad s \leq \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma}, \quad \text{then}\]

\[
K^{(3)}(s, x, x') \approx (1 + |x'|)^{(1/\gamma) - 1} \int_{y' \in \mathbb{R}}: s \leq \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma} \quad \text{and} \quad (1 - \epsilon_0)(1 + |x'|)^{1/\gamma} \leq 1 \leq (1 + \epsilon_0)(1 + |x'|)^{1/\gamma} \text{,}\]

with \(U + iV = (y + \eta^{-1} s, y')\), if \(s > \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma}\), then \(K^{(3)}(s, x, x') := 0\). In particular, assertion (2) follows immediately from (7.12).

In order to prove assertion (3) we use the variable \(v, t\) in (7.9). We choose \(\epsilon_0\) small enough so that if \(s\) and \(y\) satisfies (7.10), then \(v \ll t\). There are three cases to consider.

**Case 1:** \(1 - \epsilon_0 \leq \frac{1 + |y|}{1 + |x'|} \leq 1 + \epsilon_0\).

It follows from Lemma 7.3 (4) and (7.12) and the inequality \(v \ll t\) that

\[
K^{(3)}(s, x, x') \approx (1 + |x'|)^{(1 + |x'|)^{1/\gamma} - 1} \int_{y' \in \mathbb{R}}: s \leq \left(\frac{a^2 + b^2}b\right)\epsilon_0(1 + |x'|)^{1/\gamma} \quad \text{and} \quad (1 - \epsilon_0)(1 + |x'|)^{1/\gamma} \leq 1 \leq (1 + \epsilon_0)(1 + |x'|)^{1/\gamma} \text{,}\]

A straightforward computation shows that the last integral is equivalent to

\[
\log \left(\frac{1 + |x'|^{1/\gamma} + |x'|^{1/\gamma}}{s + |x'|^{1/\gamma} - |x'|^{1/\gamma}}\right).
\]

This proves the first part of assertion (3).
Case 2: $\frac{1+|x|}{1+|x'|} > 1 + \epsilon_0$.

In this case $t = by \leq (1 - \epsilon_0)(1 + |x|)^{1/\gamma}$. Hence, it follows from Lemma 7.3 (2) and (7.12) and the inequality $v \ll t$ that

$$K^3(s, x, x') \approx \int_y \frac{(1 + |x'|)^{1/\gamma}}{y^\gamma \frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \log \left( \frac{|x'|^{1/\gamma}}{\frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \right) dy,$$

Since there is a constant $c' > 0$ such that for $s \in [0, \epsilon_0(1 + |x'|)^{1/\gamma}]$,

$$\frac{bs}{a^2 + b^2} \log \left( \frac{|x'|^{1/\gamma}}{\frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \right) \leq \frac{bs}{a^2 + b^2} \log \left( \frac{|x'|^{1/\gamma}}{\frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \right) \leq c'(1 + |x'|)^{1/\gamma},$$

we infer that there is a constant $c''$ such that $K^3(s, x, x') \leq c''$.

Case 3: $\frac{1+|x|}{1+|x'|} < 1 - \epsilon_0$.

In this case $t = by \geq (1 + \epsilon_0)(1 + |x|)^{1/\gamma}$. Hence, it follows from Lemma 7.3 (1) and (7.12) and the inequality $v \ll t$ that

$$K^3(s, x, x') \approx (1 + |x|)^{(-1/\gamma) + 1} \int_y \frac{(1 + |x'|)^{1/\gamma}}{y^\gamma \frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \log \left( \frac{|x'|^{1/\gamma}}{\frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \right) dy,$$

Since $s \in [0, \epsilon_0(1 + |x'|)^{1/\gamma}]$, we see that the right hand side is smaller than a constant times

$$s(1 + |x|)^{(-1/\gamma) + 1} \int_y \frac{(1 + |x'|)^{1/\gamma}}{y^\gamma \frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \log \left( \frac{|x'|^{1/\gamma}}{\frac{bs}{a^2 + b^2} + |by - |x'||^{1/\gamma}} \right) dy \leq c'' s(1 + |x|)^{(-1/\gamma) + 1} \leq c''',$$

where $c''', c'''$ are constants. Hence, $K^3(s, x, x') \leq c'''$.

In summary, we deduce from Cases 2 and 3 the last part of assertion (3).

Lemma 7.8.

$$\int_{a \in A} \int_{b \in A} \left( \int_{x, x' \in \mathbb{R}: 1 - \epsilon_0 \leq \frac{1+|x|}{1+|x'|} \leq 1 + \epsilon_0} \left( \log \left( \frac{1 + |x|^{1/\gamma} + |x'|^{1/\gamma}}{1 + |x|^{1/\gamma} - |x'|^{1/\gamma}} \right) \right) \right) \times \tilde{H}_a(x)\tilde{H}_b(x') (1 + |x|)^{1/\gamma - 1} (1 + |x'|)^{1/\gamma - 1} dxdx' d(\mu \otimes \mu)(\alpha, \beta) < \infty.$$
End of the proof of Lemma 6.9] Assume without loss of generality that \( r = 0 \). By Lemma 7.7 there is a constant \( c > 1 \) such that for \( j = 1,2 \),
\[
\int_{\alpha \in A} \int_{\beta \in A} G_{\alpha,\beta}^{(j)}(s, 0) d(\mu \otimes \mu)(\alpha, \beta) \leq c(I_s + II_s),
\]
where
\[
I_s := \int_{\alpha \in A} G_{\alpha}(s, 0) d\mu(\alpha) + \int_{\beta \in A} G_{\beta}(s, 0) d\mu(\beta),
\]
and
\[
II_s := \int_{\alpha \in A} \int_{\beta \in A} G_{\alpha,\beta}^{(3)}(s) d(\mu \otimes \mu)(\alpha, \beta).
\]
By Lemma 7.6 there is a sequence \( s_k \rightarrow \infty \) such that \( I_{s_k} \rightarrow 0 \) as \( k \rightarrow \infty \). On the other hand, by Lemma 7.7 (2), we see that
\[
K^{(3)}(s, x, x') \leq M(x, x'),
\]
where
\[
M(x, x') := \begin{cases} 
\log \left( \frac{1+|x|^{1/\gamma}+|x'|^{1/\gamma}}{s+|x|^{1/\gamma}-|x'|^{1/\gamma}} \right), & \text{if } 1-\epsilon_0 \leq \frac{1+|x|}{1+|x'|} \leq 1+\epsilon_0; \\
1, & \text{otherwise}.
\end{cases}
\]
It follows from Lemma 7.8 and Lemma 7.1 (1-ii) that
\[
\int_{\alpha \in A} \int_{\beta \in A} \int_{x=-\infty}^{\infty} \int_{x'= -\infty}^{\infty} M(x, x') \cdot H_\alpha(x) \cdot H_\beta(x')(1+|x|)^{1/\gamma-1}(1+|x'|)^{1/\gamma-1} dx dx'd\mu(\mu)(\alpha, \beta) < \infty.
\]
On the other hand, by Lemma 7.7 (2) \( \lim_{s \rightarrow \infty} K^{(3)}(s, x, x') = 0 \) for \( x, x' \in \mathbb{R} \). Therefore, the Lebesgue’s dominated convergence that \( II_s \rightarrow 0 \) as \( s \rightarrow \infty \). Putting the convergence of \( I_{s_k} \) and \( II_{s_k} \) together, we infer that \( I_{s_k} + II_{s_k} \rightarrow 0 \) as \( k \rightarrow \infty \). The proof is thereby completed. \( \square \)

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