Role of galactic bars in the formation of spiral arms: a study through orbital and escape dynamics—I

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Abstract
In this work, we have developed a three-dimensional gravitational model of barred galaxies, in order to study orbital and escape dynamics of the stars inside their central barred region. Our gravitational model is composed of four components: central bulge, bar, disc and dark matter halo. Furthermore, we have analysed the model for two different types of bar potentials. The study has been carried out for a Hamiltonian system, and thorough numerical investigations has been done in order to explore the regular and chaotic motions of stars. We have seen that escape mechanism has only seen near saddle points \((L_1, L_2)\) of the Hamiltonian system. Orbital structures in the \(x - y\) plane indicate that this escaping motion corresponds to the two ends of the bar. Classifications of orbits are found by calculating maximal Lyapunov exponent of the stellar trajectories corresponding to a specific initial condition vector. Poincaré surface section maps are studied in both the \(x - y\) and \(x - p_x\) planes to get a complete information about the escape properties of the system in the phase space. Also, we studied in detail how the chaotic dynamics varies with the mass, length and nature of the bar. We found that under suitable physical conditions the chaos plays a pivotal role behind the formation of grand design or less prominent spiral patterns for stronger bars and ring structures for weaker bars.

Keywords
Galaxy: kinematics and dynamics · Galaxies: structure · Galaxies: bar · Chaos

1 Introduction
In Hubble’s classification of galaxies, central stellar bar structure is mainly observed in lenticular (e.g. NGC 1460, NGC 1533, NGC 2787) and spiral galaxies (e.g. M58, M91, M95, M109, NGC 1300, NGC 1365, NGC 1512, NGC 2217, NGC 2903, NGC 3953, NGC 4314, NGC 4921, NGC 7541, UGC 12158). Some irregular galaxies like Large Magellanic Cloud (LMC) have off-centred stellar bar (Bekki 2009; Piatti 2017; Monteagudo et al. 2018)
and also a bar-like feature has been observed in Small Magellanic Cloud (SMC) but whether that is a genuine bar or a temporary star burst region is not confirmed yet (Maragoudaki et al. 2001; Gonidakis et al. 2009; Strantzalis et al. 2019). Not all lenticular and spiral galaxies have stellar bars. All lenticulars and spirals are disc-supported systems, and their stellar discs may or may not support stellar bars. Only one-third of the local disc galaxies have determinable type of bars (or strong bars), and another one-third have indeterminable type of bars (or weak bars) (Eskridge et al. 2000; Cheung et al. 2013; Yoon et al. 2019). Fraction of barred galaxies among lenticulars and spirals is strongly dependent on red-shift, stellar mass, colour and bulge prominence (Abraham et al. 1999; Sheth et al. 2008; Nair and Abraham 2010; Simmons et al. 2014).

Galactic bars are one of the robust substructure of the barred galaxies. They are solid, dense stellar bodies rotating around the central core. Pattern speed of the bar is different than that of the disc. There are many theories behind origin of the galactic bar (Bournaud and Combes 2002; Seo et al. 2019; Petersen et al. 2019; Polyachenko and Shukhman 2020). Most evident theory is galactic bars are rotational instabilities, which arises due to density waves radiating outwards from the galactic core. These instabilities influence stellar orbits by redistributing their trajectories. As time goes, these reshaped orbits follow an outward motion, which further creates a self-stabilising stellar structure, in the form of bar (Raha et al. 1991; Sellwood 2016; Bovy et al. 2019; Łokas 2019; Sanders et al. 2019). Barred galaxies mostly have single bar structure embedded inside the bulge, though there are many examples of double barred galaxies also (e.g. NGC 1291, NGC 1326, NGC 1543). In such cases, the inner secondary bar is wrapped inside the larger primary bar (Erwin 2004; Debattista and Shen 2006; de Lorenzo-Cáceres et al. 2019). In case of our Milky Way, there is speculation about the presence of secondary inner bar inside the larger primary bar (Nishiyama et al. 2006).

Galactic bars have many shapes and sizes according to the functional form of potential energy. There are many three-dimensional bar potential models like spherical, homeoidal, triaxial, ellipsoidal, etc. (Ferrers 1877; De Vaucouleurs and Freeman 1972; Long and Murari 1992; Jung and Zotos 2015; Williams and Evans 2017), and all were extensively studied till date. Ferrers’ triaxial potential (Ferrers 1877) is the most used realistic bar potential model, though its functional form is very much complex and also it is computationally very much challenging. Models like three-dimensional homeoidal potential of De Vaucouleurs and Freeman (1972), two-dimensional ad hoc potential of Barbanis and Wolter (1967); Dehnen (2000), etc., have simpler functional forms than Ferrers’ but still they are rigorous to handle numerically. There are some simple realistic bar potential models too like two-dimensional anharmonic mass-model potential of Caranicolas (2002), three-dimensional potential of Jung and Zotos (2015), etc.

Due to the influence of galactic bars, some of the stellar orbits remain trapped inside the potential interior, while others are escaped from that inner boundary during their time evolution. This problem can be studied from the viewpoint of the problem of escape in an open Hamiltonian dynamical system (Contopoulos and Efstathiou 2004; Ernst et al. 2008; Ernst and Peters 2014; Jung and Zotos 2015, 2016). An open Hamiltonian system is a system where for energies above an escape threshold, the energy shell is non-compact, and as a result a part of the stellar orbits explores (here from potential holes to saddles) an infinite part of the position space. Also, the Hamiltonian dynamics is a time reversal invariant (Jung and Zotos 2016). Now for a conservative dynamical system, the Hamiltonian (or the total energy) is a constant of motion. Hence, all the stellar orbits are confined inside the five-dimensional energy hypersurface of the six-dimensional phase space of the Hamiltonian system. These stellar orbits are either regular or chaotic in nature according to their initial condition. Orbits having initial
energy below the escape energy are remain trapped inside the potential interior and exhibit bounded motion (regular or chaotic). Again orbits having initial energy above the escape energy can exhibit either bounded or escaping motion. Escape from the interior are possible along the open zero velocity curves (the escape channels of the potential). For bounded motions, there are chaotic orbits which do not escape within the predefined time interval and eventually escape to infinity. These orbits are known as trapped chaotic orbits. Existence of such orbits makes the stellar dynamics more geometrically complicated in the phase space. For escaping motions, orbits are generally chaotic in nature. There are many dynamical indicators, which can classify these orbits according to their regular or chaotic behaviour. Lyapunov exponent is one such effective dynamical indicator, and its working mechanism is quite simple (Sandri 1996). It calculates the rate of separation of two neighbouring trajectories during the entire time period of evolution. Value of the maximal Lyapunov exponent (MLE) gives us more complete view about dynamical nature of these orbits (regular or chaotic) in the vast sea of initial conditions of the phase space. Geometrically MLE is the highest separation between two neighbouring trajectories starting from the same initial condition in a designated time interval. If the value of MLE is positive, then it indicates that orbits are chaotic in nature, while MLE = 0 indicates that orbits are periodic in nature (Strogatz 1994). To analyse escape properties of the orbits, one need to visualise Poincaré surface section maps (Birkhoff 1927) in different two-dimensional phase planes. Under suitable physical conditions, the orbits escaped through the potential saddles further fuel the formation of spiral arms, which means there is some kind of bar-driven spiral arm formation mechanism in case of barred spiral galaxies. Effect of dynamical chaos of the stellar orbits behind the formation of the spiral arms has extensively studied in the recent past (Lindblad 1947; Lynden-Bell and Kalnajs 1972; Pfenniger 1984; Patsis 2012; Mestre et al. 2020).

Most of the earlier studies were focused on the computation of the chaotic invariant manifolds of hyperbolic orbits around the saddle points, which governs the general dynamics in the central barred region (Romero-Gómez et al. 2006; Voglis et al. 2006; Romero-Gómez et al. 2007; Sanchez-Martin et al. 2016; Efthymiopoulos et al. 2019). Also, these studies discussed the role chaotic invariant manifolds behind possibilities of the subsequent structure formations. Due to chaotic dynamics in the central region, the escaping stars produce tidal trails at the two ends of the bar. These tidal trails have tendency to create different morphologies like ring or spiral arms due to non-axisymmetric perturbations. In that way, the fate of escaping stars further relates with the subsequent structure formations (Di Matteo et al. 2005; Minchev et al. 2010; Quillen et al. 2011; Grand et al. 2012; D’Onghia et al. 2013; Ernst and Peters 2014; Jung and Zotos 2016).

In the present work, we showed the same analogy, i.e. the fate of escaping stars behind the formation of spiral arms but from the viewpoint of the amount of chaos produced therein. Therefore, we primarily focus on detection of the chaotic dynamics in the central barred region of disc galaxies, under suitable realistic bar potential models. We then figure out a comparison between these bar potential models in order to show how these models affect the chaotic dynamics with respect to the mass and length of the bar. We relate these measurements of chaos with subsequent structure formations like spiral arms or rings. Also, we have shown which bar model is more feasible for certain type of structure formations under suitable astrophysical circumstances.

Here we used a four-component three-dimensional gravitational model for the barred galaxies. The model consists of a spherical bulge, a bar embedded inside bulge, a flat disc and a logarithmic dark matter halo. Modelling is done in two parts corresponding to following bar potential models: (i) the three-dimensional extension of the anharmonic mass-model bar potential of Caranicolas (2002) and (ii) the Zotos bar potential (Jung and Zotos 2015).
These two are among the non-arduous potential forms studied till date. For each of these potential models, first we have studied several orbital structures corresponding to different initial conditions. Also, we calculate MLE values for each of the initial condition, which gives us information about dynamical nature of these orbits (regular or chaotic). Then, we draw Poincaré surface section maps in both the $x - y$ and $x - p_x$ subspaces of the phase space. These two surface section maps are important in order to visualise the motion along the galactic plane which contains the disc. Also, these two types of surface section maps are plotted for different escape energy values (the energy of the saddles of our gravitational potential models), in order to gather idea about escape mechanism of the system. Finally, we described how the nature of orbits varies with the bar parameters, e.g. mass and length by calculating the MLE in each of the bar potential models.

Our work is divided into four sections: Sect. 1 describes the introduction part. Section 2 describes the mathematical part of the barred galaxy model and it has three sub-parts. In Sect. 2.1, we have discussed the model for an anharmonic bar potential, and in Sect. 2.2 we have discussed the same model for the Zotos bar potential. Detailed comparisons between both bar models are given in Sect. 2.3. Section 3 consists of numerical analysis part, and it has three sub-parts. In Sect. 3.1, several orbital structures are plotted in the $x - y$ plane. In Sect. 3.2, several Poincaré surface section maps are plotted in both the $x - y$ and $x - p_x$ planes, and in Sect. 3.3 we have discussed how the chaotic dynamics evolved with the mass and length of the bar under influence of the two bar potential models. Finally, interpretations and conclusions are given in Sect. 4.

2 Gravitational theory

2.1 Model 1

We consider a three-dimensional gravitational model of barred galaxies and investigate orbital motions of stars inside their central region. Our gravitational model has four components—(i) central bulge, (ii) bar embedded inside the bulge, (iii) disc and (iv) extended dark matter halo. Here all the modelling and calculations are done in a Cartesian coordinate system. Let $\Phi_t(x, y, z)$ be the total potential of the galaxy. This $\Phi_t(x, y, z)$ consists of four parts, and they are (i) bulge potential—$\Phi_B(x, y, z)$, (ii) bar potential—$\Phi_b(x, y, z)$, (iii) disc potential—$\Phi_d(x, y, z)$ and (iv) dark matter halo potential—$\Phi_h(x, y, z)$. Therefore,

$$\Phi_t(x, y, z) = \Phi_B(x, y, z) + \Phi_b(x, y, z) + \Phi_d(x, y, z) + \Phi_h(x, y, z).$$

Density distribution $\rho_t(x, y, z)$ corresponding to $\Phi_t(x, y, z)$ is given through the Poisson equation,

$$\nabla^2 \Phi_t(x, y, z) = 4\pi G \rho_t(x, y, z),$$

where $G$ is the gravitational constant. Now let $\overline{\Omega}_b \equiv (0, 0, \Omega_b)$ be the constant angular velocity of the bar, which follows a clockwise rotation along $z$-axis. In this rotating reference frame, the effective potential is,

$$\Phi_{\text{eff}}(x, y, z) = \Phi_t(x, y, z) - \frac{1}{2} \Omega^2_b (x^2 + y^2). \quad (1)$$

The potential functions for different substructures are as follows:

1. Bulge: Central bulge is the excess of luminosity from the surrounding galactic disc. The distribution of stars in the galactic bulges is not exponential rather spherically symmetric
and dominated by old red stars. That’s why we choose a potential of the Plummer type (Plummer 1911) to describe the distribution of matter inside the bulge of barred galaxies (Binney and Tremaine 1987; Sofue and Rubin 2001; Halle and Combes 2013). Now density distribution of matter in the bulge for Plummer potential is,

$$\rho_B(x, y, z) = \frac{3M_Bc_B^2}{4\pi} \left( \frac{1}{(x^2 + y^2 + z^2 + c_B^2)^{\frac{5}{2}}} \right),$$

and the associated form of the potential is,

$$\Phi_B(x, y, z) = -\frac{GM_B}{\sqrt{x^2 + y^2 + z^2 + c_B^2}},$$

where $M_B$ is the mass of the bulge and $c_B$ is the scale length of the bulge. Here we consider a massive dense bulge rather than a central supermassive black hole, so that we can exclude all relativistic effects from our model.

2. Bar: Bar is an extended linear non-axisymmetric stellar structure in the central region of a galaxy. For model 1, we choose a strong bar potential, whose density in the central region is very high (i.e. a cuspy type, see Fig. 3a). For this, we consider the three-dimensional extension of the anharmonic mass-model bar potential of Caranicolas (2002), and its density distribution of matter is,

$$\rho_b(x, y, z) = \frac{M_b h^2}{4\pi} \left( \frac{k(x^2 + y^2) + (k + 3\sqrt{h^2 + z^2})(k + \sqrt{h^2 + z^2})^2}{[(x^2 + y^2) + (k + \sqrt{h^2 + z^2})^2]^\frac{5}{2}} (h^2 + z^2)\frac{3}{2} \right),$$

and the associated form of the potential is,

$$\Phi_b(x, y, z) = -\frac{GM_b}{\sqrt{x^2 + (\alpha y)^2 + z^2 + c_b^2}},$$

where $M_b$ is the mass of the bar, $\alpha$ is the bar flattening parameter and $c_b$ is the scale length of the bar.

3. Disc: Disc is the most luminous component of a galaxy. Distribution of matter inside the disc is axisymmetric, flattened and exponentially falls off with galactocentric radius. Structure of the disc can be thought as a flattened spheroid (Binney and Tremaine 1987; Smet et al. 2015; An and Evans 2019). For the disc, we use the gravitational potential model of Miyamoto and Nagai (1975), often termed as Miyamoto and Nagai potential. The density distribution of matter corresponding to Miyamoto and Nagai potential is,

$$\rho_d(x, y, z) = \frac{M_d h^2}{4\pi} \left( \frac{k(x^2 + y^2) + (k + 3\sqrt{h^2 + z^2})(k + \sqrt{h^2 + z^2})^2}{[(x^2 + y^2) + (k + \sqrt{h^2 + z^2})^2]^\frac{5}{2}} (h^2 + z^2)\frac{3}{2} \right),$$

and the associated form of the potential is,

$$\Phi_d(x, y, z) = -\frac{GM_d}{\sqrt{x^2 + y^2 + (k + \sqrt{h^2 + z^2})^2}},$$

where $M_d$ is the mass of the disc and $k$, $h$ are the corresponding horizontal and vertical scale lengths, respectively.
4. Dark matter halo: Dark matter halo is the extended distribution of non-luminous (non-baryonic) matter of a galaxy (Ostriker et al. 1974). Structure of the dark matter haloes in barred galaxies is flattened axisymmetric, as observed rotation curve becomes almost flat at larger galactocentric distances (Binney and Tremaine 1987; Zotos 2012; Ernst and Peters 2014). For the dark matter halo, we use a variant of logarithmic potential. The corresponding density distribution of matter is,

$$\rho_h(x, y, z) = \frac{v_0^2}{4\pi G} \frac{(2\beta^2 - \beta^4)y^2 + (\beta^2 + 2)c_h^2}{(x^2 + \beta^2 y^2 + z^2 + c_h^2)^2},$$

and the associated form of the potential is,

$$\Phi_h(x, y, z) = \frac{v_0^2}{2} \ln(x^2 + \beta^2 y^2 + z^2 + c_h^2),$$

where $v_0$ is the circular velocity of the dark matter halo, $\beta$ is the dark matter halo flattening parameter and $c_h$ is the scale length of the dark matter halo.

In this model, we take the following system of units,

1. Unit of length: 1 kpc,
2. Unit of mass: $2.325 \times 10^7 M_\odot$,
3. Unit of time: $0.9778 \times 10^8$ yr,
4. Unit of velocity: 10 km s$^{-1}$,
5. Unit of angular momentum per unit mass: 10 km s$^{-1}$ kpc$^{-1}$,
6. Unit of energy per unit mass: 100 km$^2$ s$^{-2}$ (Jung and Zotos 2016).

Without loss of any generality, we consider $G = 1$. Values of all physical parameters are taken from Zotos (2012) and Jung and Zotos (2016) and given in Table 1.

Now for a test particle (star) of unit mass, the Hamiltonian ($H$) of the given system is defined as,

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \Phi_t(x, y, z) - \Omega_b L_z = E,$$

where $r \equiv (x, y, z)$ is the position vector of test particle at time $t$, $p \equiv (p_x, p_y, p_z)$ is the corresponding linear momentum vector, $E$ is the total energy and $L_z = xp_y - yp_x$ is the $z$-component of the angular momentum vector $L (= r \times p)$. Hence, Hamilton’s equations of
Fig. 1 Model 1—The isoline contours of $\Phi_{\text{eff}}(x, y, z)$ in the $x-y$ plane at $z = 0$. Locations of the Lagrangian points are marked in red. For $E < E_{L_1}(-100.82313737)$, orbital motions are bounded around $L_3$ and for $E > E_{L_1}$ orbits may escape through the exit channels near $L_1$ and $L_2$.

The nature of stellar orbits in different energy ranges is given as,

1. $E_{L_3} \leq E < E_{L_1}$: In this energy range, stellar orbits exhibit bounded motion inside the barred region.

This autonomous Hamiltonian system has five Lagrangian (or equilibrium) points, namely $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$, which are solutions of the following equations:

\[
\begin{align*}
\dot{x} &= p_x + \Omega_b y, \\
\dot{y} &= p_y - \Omega_b x, \\
\dot{z} &= p_z, \\
\dot{p}_x &= -\frac{\partial \Phi_1}{\partial x} + \Omega_b p_y, \\
\dot{p}_y &= -\frac{\partial \Phi_1}{\partial y} - \Omega_b p_x, \\
\dot{p}_z &= -\frac{\partial \Phi_1}{\partial z},
\end{align*}
\]

(3)

where ‘·’ $\equiv \frac{d}{dt}$. The values of $E$ (or Jacobi integral of motion) at $L_1$ and $L_2$ are identical and that value is $E_{L_1} = -100.82313737 = E_{L_2}$. Similarly, the values of $E$ at $L_4$ and $L_5$ are identical and that value is $E_{L_4} = 20.21395666 = E_{L_5}$. Also, the value of $E$ at $L_3$ is $E_{L_3} = -6630.68464790$. The nature of stellar orbits in different energy ranges is given as,
Table 2—locations and types of the Lagrangian points

| Lagrangian point | Location               | Type         |
|------------------|------------------------|--------------|
| $L_1$            | (20.23113677, 0, 0)    | Index-1 Saddle |
| $L_2$            | (−20.23113677, 0, 0)   | Index-1 Saddle |
| $L_3$            | (0, 0, 0)              | Centre       |
| $L_4$            | (0, −19.49818983, 0)   | Index-2 Saddle |
| $L_5$            | (0, 19.49818983, 0)    | Index-2 Saddle |

2. $E = E_{L_1}$: This is the threshold energy for orbital escapes through the bar ends.
3. $E > E_{L_1}$: In this energy range, escape is possible for stellar orbits from the barred region through the two symmetrical exit channels exist near $L_1$ and $L_2$. Also, orbits are either bounded or escaped depending upon their initial starting point.

2.2 Model 2

In this model, we consider the same three-dimensional gravitational model as discussed in Sect. 2.1, but change only the potential form of galactic bar. Here we choose a comparatively weak bar potential, whose density at the central region is moderate (see Fig. 3a). For this model, we used the bar potential model of Jung and Zotos (2015), often termed as Zotos bar potential. For this potential, the density distribution of matter is,

$$\rho_b(x, y, z) = \frac{M_b c_b^2}{8\pi a} [f(x + a, y, z) - f(x - a, y, z)],$$

where

$$f(x, y, z) = \frac{x(2x^2 + 3y^2 + 3z^2 + 3c_b^2)}{(y^2 + z^2 + c_b^2)(x^2 + y^2 + z^2 + c_b^2)^3},$$

and the associated form of the potential is,

$$\Phi_b(x, y, z) = \frac{GM_b}{2a} \ln \left(\frac{x - a + \sqrt{(x - a)^2 + y^2 + z^2 + c_b^2}}{x + a + \sqrt{(x + a)^2 + y^2 + z^2 + c_b^2}}\right),$$

where $M_b$ is the mass of the bar, $a$ is the length of semi-major axis of the bar and $c_b$ is the scale length of the bar. For this bar potential, we use $a = 10$ (Jung and Zotos 2016) and values of other model parameters due to bulge, bar, disc and dark matter halo remain same as given in Table 1.

Locations and types of the Lagrangian points, namely $L_1'$, $L_2'$, $L_3'$, $L_4'$ and $L_5'$, are given in Fig. 2 and Table 3. The values of $E$ at $L_1'$ and $L_2'$ are identical and that value is $E_{L_1'} = −116.46144116 = E_{L_2'}$. Similarly, the values of $E$ at $L_4'$ and $L_5'$ are identical and that value is $E_{L_4'} = −60.76593330 = E_{L_5'}$. Also, the value of $E$ at $L_3'$ is $E_{L_3'} = −4180.06268050$. The nature of stellar orbits in different energy ranges is the same as discussed in model 1.
Fig. 2 Model 2—The isoline contours of $\Phi_{\text{eff}}(x, y, z)$ in the $x − y$ plane at $z = 0$. Locations of the Lagrangian points are marked in red. For $E < E_{L_1'} (\approx -116.46144116)$, orbital motions are bounded around $L_3'$ and for $E > E_{L_1'}$, orbits may escape through the exit channels near $L_1'$ and $L_2'$.

Table 3 Model 2—locations and types of the Lagrangian points

| Lagrangian point | Location      | Type          |
|------------------|---------------|---------------|
| $L_1'$           | $(20.82978638, 0, 0)$ | Index-1 Saddle |
| $L_2'$           | $(-20.82978638, 0, 0)$ | Index-1 Saddle |
| $L_3'$           | $(0, 0, 0)$    | Centre        |
| $L_4'$           | $(0, -20.36721028, 0)$ | Index-2 Saddle |
| $L_5'$           | $(0, 20.36721028, 0)$ | Index-2 Saddle |

2.3 Comparison between model 1 and model 2

The comparison between model 1 and model 2 in terms of the rotation curve, density distribution of the bar, variation of the radial and tangential force components is shown in Figs. 3a, b and 4a, b, respectively. We use a fast bar (Athanassoula 1992) in both of the bar models.

1. Density distribution of the bar: Variation of the bar density ($\rho_b$) with radius ($R = \sqrt{x^2 + y^2}$) of both models at $z = 0$ is shown in Fig. 3a. Here we observe that the bar density of model 1 is much cusplier than model 2, which justifies the strongness of the bar of model 1. Such strong bars are found in early-type disc galaxies, which show lower star formation activity and older stellar populations in the disc. Due to high efficiency of gas inflow from the disc to the centre, in relatively short timescale such massive bars are formed (Laurikainen et al. 2007).

2. Rotation curve: Rotation curve, i.e. the circular velocity of stars ($V_{\text{rot}} = \sqrt{R \frac{d\Phi}{dR}}$) versus radius ($R = \sqrt{x^2 + y^2}$), of both models is shown in Fig. 3b. From this, we figure out
that for model 1 rotational velocities are higher in the bulge region due to the presence of massive bar than model 2. Higher kinetic energy transport mechanism inside the bulge of massive bar leads to such phenomena.

3. Radial force: Variation of the radial force component \( F_R = \frac{\partial \phi}{\partial R} \) with radius \( R = \sqrt{x^2 + y^2} \) of both models at \( z = 0 \) is shown in Fig. 4a. In Fig. 4a, we observed that the radial force components are significant only inside the bulge region and also these distributions become steeper as \( R \to 0 \). For model 1, the amount of steepness is higher due to the presence of massive bar than model 2. The steepness die out completely as \( R \) crossed the barred region.

4. Tangential force: Variation of the tangential force component \( F_\theta = \frac{1}{R} \frac{\partial \phi}{\partial \theta} \) with radius \( R = \sqrt{x^2 + y^2} \) of both models at \( z = 0 \) is shown in Fig. 4b, where \( R \) and \( \theta \) are parameters associated with polar coordinate system. In Fig. 4b, we observed that for model 1 the tangential force component is only significant near the central bulge region and die out gradually for larger values of \( R \). In case of model 2, the variation of the tangential force component is significantly different than model 1 as the tangential force component is mainly concentrated between the annulus of the outer bulge and the barred region.

3 Numerical results

To study orbital and escape dynamics of stars along the galactic plane (which contains the bar and disc), we put \( z = 0 = p_z \) in both the gravitational models 1 and 2, respectively. Depending upon the initial starting point, stellar orbits are either trapped or escaped from the barred region. We have seen that two symmetrical escape channels exist near \( L_1, L_2 \) and \( L'_1, L'_2 \) (Figs. 1 and 2). Hence, escape mechanism through the bar ends is only relevant near these Lagrangian points. Due to symmetry in the potential studying nature of orbits is ‘The effective potential is symmetric about both the x and y axes. On the other hand orbital escapes are observed only near the Lagrangian points \( (L_1) \) and \( (L_2) \) (for model 1) or \( (L'_1) \) and \( (L'_2) \) (for model 2)’. Now the locations of \( (L_1) \) and \( (L_2) \) (or \( (L'_1) \) and \( (L'_2) \)) are symmetric about the x axis. Hence studying the orbital and escape dynamics near either of \( (L_1) \) or \( (L_2) \) (for model 1) and also near either of \( (L'_1) \) or \( (L'_2) \) (for model 2) is sufficient for our gravitational model. In order to do that, we have investigated the dynamics in the following energy ranges: \( E \geq E_{L_1} \) and \( E \geq E_{L'_1} \). Now for simpler analysis we replace \( E \) with the dimensionless energy parameter \( C \) (Ernst et al. 2008), which is defined as,

\[
\text{model 1: } C = \frac{E_{L_1} - E}{E_{L_1}} = \frac{E_{L_2} - E}{E_{L_2}} \quad (\because E_{L_1} = E_{L_2})
\]

\[
\text{model 2: } C = \frac{E'_{L'_1} - E}{E'_{L'_1}} = \frac{E'_{L'_2} - E}{E'_{L'_2}} \quad (\because E'_{L'_1} = E'_{L'_2}).
\]

For \( C > 0 \), orbits may exhibit escaping motion and escapes are possible through the exit channel near \( L_1 \). To study the nature of escaping motion around \( L_1 \), we have chosen the energy levels higher than the escape threshold (\( C = 0 \)). Our tested energy levels are \( C = 0.01 \) and \( C = 0.1 \). Energy value of \( L_1 \) for different values of \( C \) is given in Table 4. Also in the phase space we consider only initial conditions inside the Lagrange radius, i.e. if \((x_0, y_0)\) is an
(a) Evolution of the bar density ($\rho_b$) with radius ($R = \sqrt{x^2 + y^2}$) at $z = 0$.

(b) Rotation curve.

Fig. 3 The bar density and rotation curve: model 1 versus model 2
(a) Evolution of the radial force component \( F_R \) with radius \( R = \sqrt{x^2 + y^2} \) at \( z = 0 \).

(b) Evolution of the tangential force component \( F_\theta \) with radius \( R = \sqrt{x^2 + y^2} \) at \( z = 0 \).

**Fig. 4** The force components: model 1 versus model 2
Table 4  Tested energy levels of both models

| C      | $E_{L_1}$        | $E_{L'_1}$     |
|--------|------------------|----------------|
| 0.010  | -99.81490600     | -115.29682674  |
| 0.100  | -90.74082363     | -104.81529704  |

initial condition in the $x - y$ plane, then $x_0^2 + y_0^2 \leq r_{L_1}^2$, where $r_{L_1}$ is the radial length of $L_1$. The same formalism is carried out for the Lagrangian point $L'_1$ also.

In our work, regular and chaotic motions have been classified with the help of chaos detector MLE. For a given initial condition vector in phase space, let $\delta x(t_0)$ be the initial separation vector of two neighbouring trajectories, where $t_0$ is the initial time. Also let $\delta x(t)$ be the separation vector at time $t$. Then MLE for that designated initial condition vector is defined as,

$$\text{MLE} = \lim_{t \to \infty} \lim_{|\delta x(t_0)| \to 0} \frac{1}{t} \ln \left| \frac{\delta x(t)}{|\delta x(t_0)|} \right|.$$  

(5)

To follow the evolution of orbits on long times, we choose our integration time as $10^2$ units, which is nearly equivalent to $10^{10}$ (= 10 Gyr) years—typical age of barred galaxies (Dauphas 2005; James and Percival 2017; Sharma et al. 2019). In this vast integration time, orbits starting from an initial condition will reveal their true nature (regular or chaotic). We use a set of MATLAB programs in order to integrate the system of Eqs. (3). The system of differential equations are solved through the ode45 MATLAB package with small time step $(\Delta t) = 10^{-2}$. Here all the calculated values are corrected up to eight decimal places. All the presented graphics are produced in MATLAB − 2015a environment.

3.1 Orbital structures

In order to figure out the orbital and escape dynamics for the saddle point $L_1$ (or $L'_1$), we choose two initial conditions. One is $(x_0, y_0, p_{x_0}) \equiv (5, 0, 15)$, chosen from the vicinity of the saddle point $L_1$ (or $L'_1$) within the barred region and the other is $(x_0, y_0, p_{x_0}) \equiv (-5, 0, 15)$, chosen at a suitable distance from $L_1$ (or $L'_1$) from the vicinity of the saddle point $L_2$ (or $L'_2$) within the barred region. For each of the initial condition, $p_{y_0}$ is calculated from Eq. (2). Any other initial conditions considered from the suitable neighbourhoods of the respective initial points will follow the similar trend of orbital and escape dynamics. The corotation radius of the disc and bar ($r_{L_1}$ or $r_{L'_1}$) is slightly greater than the bar length. Depending upon the bar model (strong or weak), orbits starting within the corotation region of the disc and bar may escape from it. Here the bar length decided the area of the aforesaid neighbourhoods.

- Model 1: In Figs. 5 and 6, stellar orbits in the $x - y$ plane have been plotted for values $C = 0.01$ and 0.1, respectively, with the initial condition $x_0 = 5$, $y_0 = 0$ and $p_{x_0} = 15$. In both figures, we get escaping chaotic orbits. Similarly for Figs. 7 and 8, the initial condition is $x_0 = -5$, $y_0 = 0$ and $p_{x_0} = 15$. In these figures, we get non-escaping retrograde quasi-periodic rosette orbits. $p_{y_0}$ value of each figure is evaluated from Eq. (2) and the corresponding MLE values are calculated from Eq. (5) and listed in Table 5.
- Model 2: Similar as model 1, here also in Figs. 9 and 10 stellar orbits in the $x - y$ plane have been plotted for values $C = 0.01$ and 0.1, respectively, with the initial condition $x_0 = 5$, $y_0 = 0$ and $p_{x_0} = 15$. In both figures, we get non-escaping chaotic orbits.
Fig. 5 Model 1—escaping chaotic orbit for $C = 0.01$ with $(x_0, y_0, p_{x_0}) \equiv (5, 0, 15)$

Fig. 6 Model 1—escaping chaotic orbit for $C = 0.1$ with $(x_0, y_0, p_{x_0}) \equiv (5, 0, 15)$
Fig. 7 Model 1—non-escaping retrograde quasi-periodic rosette orbit for $C = 0.01$ with 
$(x_0, y_0, p_{x0}) \equiv (-5, 0, 15)$

Fig. 8 Model 1—non-escaping retrograde quasi-periodic rosette orbit for $C = 0.1$ with 
$(x_0, y_0, p_{x0}) \equiv (-5, 0, 15)$
Fig. 9 Model 2—non-escaping chaotic orbit for $C = 0.01$ with $(x_0, y_0, p_{x_0}) \equiv (5, 0, 15)$

Fig. 10 Model 2—non-escaping chaotic orbit for $C = 0.1$ with $(x_0, y_0, p_{x_0}) \equiv (5, 0, 15)$
Fig. 11 Model 2—non-escaping retrograde quasi-periodic rosette orbit for $C = 0.01$ with $(x_0, y_0, p_x) \equiv (-5, 0, 15)$

![Graph](image1.png)

Fig. 12 Model 2—non-escaping retrograde quasi-periodic rosette orbit for $C = 0.1$ with $(x_0, y_0, p_x) \equiv (-5, 0, 15)$

![Graph](image2.png)
Table 5  Model 1—MLE for different values of C

| Initial condition | C   | MLE       |
|-------------------|-----|-----------|
| \((x_0, y_0, p_{x_0})\) | 0.01 | 0.18041129 |
| \((-5, 0, 15)\)     | 0.1  | 0.19086893 |
| \((x_0, y_0, p_{x_0})\) | 0.01 | 0.08539470 |
| \((-5, 0, 15)\)     | 0.1  | 0.08948487 |

Table 6  Model 2—MLE for different values of C

| Initial condition | C   | MLE       |
|-------------------|-----|-----------|
| \((x_0, y_0, p_{x_0})\) | 0.01 | 0.16912615 |
| \((-5, 0, 15)\)     | 0.1  | 0.17423968 |
| \((x_0, y_0, p_{x_0})\) | 0.01 | 0.07609452 |
| \((-5, 0, 15)\)     | 0.1  | 0.07688508 |

Similarly for Figs. 11 and 12, the initial condition is \(x_0 = -5, y_0 = 0\) and \(p_{x_0} = 15\). In these figures, we get non-escaping retrograde quasi-periodic rosette orbits. \(p_{y_0}\) value of each figure is evaluated from Eq. (2) and the corresponding MLE values are calculated from Eq. (5) and listed in Table 6.

3.2 Poincaré maps

Poincaré maps are two-dimensional cuts of the six-dimensional hyper-surface in case of our gravitational models. For model 1, Poincaré surface section maps in the \(x - y\) plane are plotted in Fig. 13a and b for different energy values. For this, we consider a \(43 \times 43\) grid of initial conditions with step sizes \(\Delta x = 1\) kpc and \(\Delta y = 1\) kpc. Among them, initial conditions are only considered inside the Lagrange radius, i.e. \(x_0^2 + y_0^2 \leq r_{L1}^2\). Initial conditions for \(p_{x_0}\) and \(p_{y_0}\) are \(p_{x_0} = 0\) and \(p_{y_0} > 0\). Also, for Poincaré maps in \(x - y\) plane surface cross sections are \(p_{x} = 0\) and \(p_{y} \leq 0\) (Ernst and Peters 2014). Again for model 1, Poincaré surface section maps in the \(x - p_x\) plane are plotted in Fig. 13c and d for different energy values. For this, we also consider a \(43 \times 31\) grid of initial conditions with step sizes \(\Delta x = 1\) kpc and \(\Delta p_x = 10\) km s\(^{-1}\). Among them, initial conditions are only considered inside the Lagrange radius as already mentioned. Initial conditions for \(y_0\) and \(p_{y_0}\) are \(y_0 = 0\) and \(p_{y_0} > 0\). Also, for Poincaré maps in the \(x - p_x\) plane surface cross sections are \(y = 0\) and \(p_{y} \leq 0\) (Ernst and Peters 2014). For every Poincaré map, the \(p_{y_0}\) value is evaluated from Eq. (2). Similarly for model 2 Poincaré surface section maps in the \(x - y\) plane are plotted in Fig. 14a and b and Poincaré surface section maps in the \(x - p_x\) plane are plotted in Fig. 14c and d, respectively.

- Model 1: In Fig. 13a, we observed that a primary stability island exists near \(5, 0\) in the \(x - y\) plane for the energy value \(C = 0.01\), which is formed due to the quasi-periodic motions. Again when the energy value is increased to \(C = 0.1\) (see Fig. 13b), then this stability island diminishes because of enhancement in chaoticity in that region. There is also a hint about escaping motions in both Fig. 13a and b as the number of cross-sectional points in the \(x - y\) plane outside the corotation region (near \(L_2\)) is increased with increment of \(C\). Similarly, a primary stability island has been observed in Fig. 13c and d, which exists near \(5, 0\) in the \(x - p_x\) plane. In these figures, the orbital trends are similar as observed in the \(x - y\) plane.
Fig. 13 Model 1—Poincaré surface section maps

(a) Poincaré surface sections of $p_x = 0$ and $p_y \leq 0$ for $C = 0.01$.

(b) Poincaré surface sections of $p_x = 0$ and $p_y \leq 0$ for $C = 0.1$.

(c) Poincaré surface sections of $y = 0$ and $p_y \leq 0$ for $C = 0.01$.

(d) Poincaré surface sections of $y = 0$ and $p_y \leq 0$ for $C = 0.1$.

– Model 2: Same as model 1, here in Fig. 14a and b, we observed a primary stability island which is formed due to quasi-periodic motions and it exists near $(6, 0)$ in the $x - y$ plane. Now as the energy value is increased from $C = 0.01$ to $C = 0.1$, then this stability island diminishes due to the enhancement in chaoticity in that region. Here also a hint about escaping motions in both Fig. 14a and b as the number of cross-sectional points in the $x - y$ plane outside the corotation region (near $L_2'$) is increased with increment of $C$. Similarly in Fig. 14c and d, a primary stability island has formed and it exists near $(6, 0)$ in the $x - p_x$ plane. Also in the $x - p_x$ plane some smaller stability islands are formed for $C = 0.01$, which diminishes as the energy value increased to $C = 0.1$. This feature of smaller stability islands is absent in model 1. This indicates that the bar of model 2 is
Fig. 14 Model 2—Poincaré surface section maps

(a) Poincaré surface sections of $p_x = 0$ and $p_y \leq 0$ for $C = 0.01$.

(b) Poincaré surface sections of $p_x = 0$ and $p_y \leq 0$ for $C = 0.1$.

(c) Poincaré surface sections of $y = 0$ and $p_y \leq 0$ for $C = 0.01$.

(d) Poincaré surface sections of $y = 0$ and $p_y \leq 0$ for $C = 0.1$.

more dynamically stable than model 1. Beside this, the overall orbital trends are similar as in model 1.

### 3.3 Evolution of chaos with respect to the bar parameters

Here we have discussed how the chaotic dynamics (in terms of MLE) evolved over the vast integration time with respect to the mass and length of the bar. In order to do that, we have calculated the MLE for different values of the bar mass and length for each of the two bar potential models. Our main focus is to study the orbital dynamics in the vicinity of the Lagrangian points $L_1$ and $L'_1$, respectively. That is why we restrict the study of this Section...
only for the orbits starting with the initial condition \((x_0, y_0, p_{x0}) \equiv (5, 0, 15)\), where \(p_{y0}\) is evaluated from Eq. (2). In our bar potential models, the total mass of the galaxy \((M) = 30900\) units and total mass of the bar \((M_b) = 3500\) units. The bar mass \((M_b)\) is nearly 11.33\% of the total mass of the galaxy. We studied the variation of \(M_b\) in the range from 3100 to 4000 units. The mass range of the bar spans from nearly 10\% to 13\% of the total mass. Also, the variation of \(\alpha\) and \(a\) is studied in the range from 1 to 10 units.

1. Model 1: In Fig. 15, we have shown how the MLE values vary with the bar flattening parameter \((\alpha)\) and energy parameter \(C\) for model 1. Similarly in Fig. 16, we have shown how the MLE values vary with the bar mass \(M_b\) and energy parameter \(C\).

2. Model 2: In Fig. 17, we have shown how the chaotic dynamics varies with the length of semi-major axis of the bar \((a)\) and energy parameter \(C\) for model 2. Similarly in Fig. 18, we have shown how the chaotic dynamics varies with the bar mass \(M_b\) and energy parameter \(C\).
Fig. 17 Model 2—MLE for different values of $a$ and $C$ with $(x_0, y_0, px_0) \equiv (5, 0, 15)$

Fig. 18 Model 2—MLE for different values of $M_b$ and $C$ with $(x_0, y_0, px_0) \equiv (5, 0, 15)$

4 Interpretations and conclusions

The present work describes the nature of stellar orbits in barred galaxies and the influence of bars along with the development of spiral arms as a result of the escape mechanism.

We have considered the stellar orbits in barred galaxies in the presence of four components, e.g. bulge, bar, disc and dark matter halo. We considered two types of bars, namely (i) anharmonic bar and (ii) Zotos bar. It is clear from Figs. 1 and 2 that the bar area of the latter one is bigger and it is more elongated in the $x$-direction than the first one. Also, the density distribution (see Fig. 3a) for bar in model 1 is very high close to the centre, i.e. the nature is one of cuspy type compared to model 2. So it may be associated with strong bar. On the other hand, in model 2 the density distribution is rather flat and slightly rising in the central region. This kind of distribution may be associated with weak bar.

Regarding the nature of orbits, it depends upon the initial conditions as well as the bar potential. For model 1, the orbit integrated for the initial condition $x_0 = 5, y_0 = 0, px_0 = 15$ is escaping and chaotic with high MLE. Also, the orbit is non-escaping and quasi-periodic with low MLE for $x_0 = -5, y_0 = 0, px_0 = 15$ (see Table 5). Again, the radial and tangential force components are very much high near the galactic centre and eventually die off at larger distances (see Fig. 4a). Hence, the bar potential of model 1 promotes the escape mechanism through the bar ends and escape is possible for suitable choice of initial conditions. Also, escaping orbits may result in spiral arms. On the other hand, for model 2 the orbit integrated
for initial condition $x_0 = 5$, $y_0 = 0$, $p_{x_0} = 15$ is non-escaping and chaotic with high MLE. Also, the orbit is non-escaping and quasi-periodic with low MLE for $x_0 = -5$, $y_0 = 0$, $p_{x_0} = 15$ (see Table 6). In both cases, MLE values are comparatively less than model 1. Again, the radial force component is very high near the galactic centre, but the tangential force component is very high in the annulus between the outer bulge and the barred region (see Fig. 4b). In this model, radial forces near the galactic centre have less strength than model 1 and that’s why it does not promote the escape mechanism through the bar ends. But as the tangential force strength is higher in the annulus between the outer bulge and barred region, such non-escaping orbits tend to form ring type structures within the disc. This is evident in S0 or ring galaxies where spiral arms are more or less absent (Van den Bergh 2009; Querejeta et al. 2015; Sil’chenko et al. 2018).

The second aspect is to study the nature of chaotic orbits of stars and gas in barred galaxies. The understanding of the dynamics of barred galaxies closely relates with chaotic motions of stars and gas in the central region. The presence of chaos is a manifestation of unstable orbits. Chaos propagates further over a long time period and influences the evolution of several structural components, e.g. bar, disc and dark matter halo. Also, the integration of chaotic theory in the orbital and escaping stellar motions helps us to figure out the true shape of these components. In case of barred galaxies, these growing instabilities relate with the formation and strength of spiral arms.

Galactic bars are density waves in the disc. Spiral arms are thought to be the result of these outward density wave patterns. Observational studies of barred spiral galaxies in blue and near IR band confirm that spiral arms are continuations of the bar, i.e. spiral arms are outcomes of the bar-driven mechanisms. Theoretical studies also suggested that there is a correlation between bar pattern speed and that of spiral arms (Elmegreen and Elmegreen 1985; Buta et al. 2009). These density waves and stellar orbits usually have different rotational speeds, but there exists some sort of corotation region inside the disc. Spiral arms emerge from the two ends of the bar in that corotation region. Also, this corotation region is dominated by chaotic dynamics. Theoretical studies confirm that these chaotic orbits in the corotation region are the building blocks of the spiral arms (Contopoulos and Grosbøl 1989; Kaufmann and Contopoulos 1996; Romero-Gómez et al. 2006; Voglis et al. 2006; Romero-Gómez et al. 2007; Sanchez-Martin et al. 2016; Efthymiopoulos et al. 2019). Also a study by Patsis et al. (1997) has shown that these chaotic orbits are the reason behind the characteristic outer boxy isophotes of the nearly face-on bar of the barred spiral galaxy NGC 4314. Under suitable physical circumstances, these spiral arms survived the chaotic dynamics in the corotation region of disc and bar. Nearly 70% of the barred spirals in field have tightly wounded two-armed spiral pattern, while only 30% of the unbarred spirals in field have grand design spiral pattern (Elmegreen and Elmegreen 1982).

Hence, it is clear from the above discussions that the spiral arms might be the continuation of the escaping orbits of stars emerging from the end of bars driven by the chaos. In our case, there are chaotic orbits when the energy exceeds the energy of the Lagrangian point $L_1$ (or $L'_1$), i.e. $E > E_{L_1}$ (or $E > E_{{L'_1}}$). We have computed MLE for various initial conditions as well as bar parameters. The following observations have been found.

(i) From the orbital structures (Figs. 5, 6, 7, 8) of model 1, we conclude that stellar orbits may escape from the disc through the bar ends (i.e. Lagrangian points $L_1$ and $L_2$). Also, the chaoticity of orbits inside the corotation region of the disc and bar is increased with increment in the energy. In the Poincaré surface section maps (Fig. 13) of model 1, trends are similar as observed in the orbital maps.
For model 1, MLE increases with the flattening parameter ($\alpha$) up to a threshold length and again slowly decreases (viz. Fig. 15). This threshold length decreases with increment in the value of $C$. This implies that when the escape energy of $L_1$ is high ($C \sim 0.1$), escape of stars is even encouraged at smaller bar length, e.g. for $C = 0.01$, the threshold value of $\alpha = 8$, whereas for $C = 0.1$, $\alpha = 6$ (viz. Fig. 15). Also, there is a sudden increment in the MLE from $\alpha = 1$ to $\alpha = 2$. For $\alpha = 1$ the bar of model 1 resembles a bulge-like spherical structure. For this reason, chaotic orbits in the central region are comparatively less due to the presence of a central black hole, which suppress the star formation more effectively inside that spherical region. For $\alpha > 1$, the bar becomes flatter and that suppression effect diminishes.

For Fig. 16, we have following observations—(i) the variation of $M_b$ with MLE follows a decaying oscillating pattern, i.e. the amplitude of oscillations gradually diminishes with $M_b$ for lower escape energy value ($C = 0.01$) and (ii) the same variation follows a stable oscillating pattern for higher escape energy value ($C = 0.1$). Now the amount of chaos is very much dependent on the orbital energy value (Zotos 2017). At lower escape energy ($C = 0.01$), a gradual increment of $M_b$ diminishes the chaos, because the stellar orbits spend a significant fraction of their time inside the barred region and do not fill up the phase space homogeneously, which suggests a weak chaotic motion. At higher escape energy ($C = 0.1$), a stable oscillating pattern of MLE is followed along with increment in $M_b$, because stellar orbits fill up the phase space homogeneously, which suggests a strong chaotic motion. Hence for model 1, the formation of prominent spiral arms for smaller values of the escape energy is not favourable but becomes favourable once the escape energy value is increased (may be due to central explosions, shocks, etc.).

So in the presence of heavier (or strong) bar, the formation of prominent spiral arms is more likely in those barred galaxies where violent activities are occurring in the central region. We all know giant spirals harbour supermassive black holes (SMBHs) at their nuclear regions where violent activities are going on (Basu and Kanjilal 1989; Melia and Falcke 2001; Kaviraj et al. 2015; Mondal and Chattopadhyay 2019; Kim et al. 2020). The strength of spiral arms is strongly correlated with the central black hole mass and kinetic energy of random motions inside the bulge. Galaxies with SMBHs have higher kinetic energy transport inside their bulges and spiral arms with small pitch angles, which results in tightly wound grand design spiral patterns (Seigar et al. 2008; Berrier et al. 2013; Al-Baidhany et al. 2014; Davis et al. 2017). Many giant spirals containing SMBH like M83, NGC 628, NGC 3310, NGC 4303, NGC 4258, NGC 5457, NGC 6946 (Pastorini et al. 2007; Cédrés et al. 2013; Frick et al. 2016; Weżgowiec et al. 2016), etc., have grand design spiral patterns. Hence, the bar potential used in model 1 favours the formation of grand design spirals when SMBH is present at the core and spiral arms are emerged from the bar ends.

From the orbital structures (Figs. 9, 10, 11, 12) of model 2, we conclude that stellar orbits are not encouraged to escape from the disc through the bar ends (i.e. Lagrangian points $L_1$ and $L_2$) but remain encapsulated within the disc. Also, the chaoticity of orbits inside the corotation region of the disc and bar is increased with increment in the energy, but this increment in chaoticity is less comparable to model 1. In the Poincaré surface of section maps (Fig. 14) of model 2, trends are similar as observed in the orbital maps.

The MLE increases with semi-major axis of the bar ($a$) (viz. Fig. 17) up to a threshold value ($a = 6$) and does not vary much further. Again this variation is independent of the tested escape energy levels. Hence, from Fig. 17 we concluded that for weaker bars the fate of the escaping orbits does not much rely on the escape energy values. Also, escape of stars through the bar ends is only possible for an optimal bar length and that length does not depend on the escape energy.
(vii) Weaker bars also help an escape mechanism, but the increase of chaos with bar mass is very slow (viz. Fig. 18). In Fig. 18, we also observed that the variation $M_b$ with MLE follows a stable oscillating pattern for both higher and lower escape energy values. Now such stable oscillating pattern is only observed for a higher escape energy value in model 1. Also, the MLE values of model 2 are smaller as compared to model 1. Hence, for model 2 chaotic motions escaped from the bar ends remaining trapped inside the disc and become favourable for the formation of ring type of structures irrespective of the escape energy values.

(viii) Thus, in the presence of weak bar, the formation of ring type structures is more likely. This might be the reason why inner rings are observed in many disc galaxies (Regan and Teuben 2003; Byrd et al. 2006; Proshina et al. 2019). Observational evidences of ring structures have been found in NGC 1326 by Buta (1995). Similarly studies by Sakamoto et al. (1999, 2000) have also identified such ring structures in NGC 5005. Hence, the bar potential used in model 2 favours the formation of ring type of structures, which will emerge from the bar ends.

The earlier studies which are cited in the introduction section are also discussed about the possibilities of formation of spiral arms (or ring structures) from strong bars (or weak bars) as a result of orbital escape. In these studies, conclusions are drawn based upon either observational evidences or through the N-body simulations (via manifolds in the phase space). In this study, we provide a simple analytical model for barred galaxies and our analysis methods are much simpler than earlier studies. We have shown that two simple distinct classes of potentials are sufficient for analysing strong and weak bars. Again, results of this study provide the optimal values of the parameters, which help us to explore the properties of subsequent structure formations. Also, we found that the presence of the central SMBH may be related to such structure formations.

The final conclusions of the present study are as follows:

1. Barred galaxies with massive (or stronger) bar potential (viz. model 1) may lead to the formation of grand design spirals only when there are some kind of violent activities going inside their central region. SMBHs may be one of the reasons for such kind of violent activities. Again galaxies with massive bars but without central SMBHs may lead to the formation of less prominent spiral arms.
2. On the contrary, barred galaxies with weaker bar potential (viz. model 2) may lead to the formation of ring type structures.

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Data Availability Both the authors confirm that the analysed data supporting the findings of this study are available within this article.

References

Abraham, R.G., Merrifield, M.R., Ellis, R.S., Tanvir, N.R., Brinchmann, J.: The evolution of barred spiral galaxies in the Hubble Deep Fields North and South. Mon. Not. R. Astron. Soc. 308, 569–576 (1999)
Al-Baidhany, I., Seigar, M., Treuthardt, P., Sierra, A., Davis, B., et al.: Study of the relation between the spiral arm pitch angle and the kinetic energy of random motions of the host spiral galaxies. A. J. Ark. Acad. Sci. 6(8), 25–36 (2014)
An, J., Evans, N.W.: Self-consistent potential-density pairs of thick discs and flattened galaxies. Mon. Not. R. Astron. Soc. 486, 3915–3926 (2019)
Athanassoula, E.: The existence and shapes of dust lanes in galactic bars. Mon. Not. R. Astron. Soc. 259, 345–364 (1992)
Barbanis, B., Woltjer, L.: Orbits in spiral galaxies and the velocity dispersion of population I stars. Astrophys. J. 150, 461–468 (1967)
Basu, B., Kanjilal, T.: Explosion-triggered star formation in the central region of the galaxy. Astrophys. Space Sci. 152, 203–214 (1989)
Bekki, K.: Formation of the off-centre bar in the large Magellanic cloud: a collision with a dark satellite? Mon. Not. R. Astron. Soc. 393, L60–L64 (2009)
Berrier, J.C., Davis, B.L., Kennefick, D., Kennefick, J.D., Seigar, M.S., et al.: Further evidence for a supermassive black hole mass-pitch angle relation. Astrophys. J. 769, 132 (2013)
Binney, J., Tremaine, S.: Galactic Dynamics. Princeton Univ. Press, New Jersey (1987)
Birkhoff, G.D.: Dynamical Systems. Amer. Math. Soc. Colloq. Publ, Rhode Island (1927)
Bournaud, F., Combes, F.: Gas accretion on spiral galaxies: bar formation and renewal. Astron. Astrophys. 392, 83–102 (2002)
Bovy, J., Leung, H.W., Hunt, J.A., Mackereth, J.T., García-Hernández, D.A., et al.: Life in the fast lane: a direct view of the dynamics, formation, and evolution of the Milky Way’s bar. Mon. Not. R. Astron. Soc. 490, 4740–4747 (2019)
Buta, R.: The catalog of southern ringed galaxies. Astrophys. J. Suppl. Ser. 96, 39–116 (1995)
Buta, R.J., Knapen, J.H., Elmegreen, B.G., Salo, H., Laurikainen, E., et al.: Do bars drive spiral density waves? Astron. J. 137, 4487–4516 (2009)
Byrd, G.G., Freeman, T., Buta, R.J.: The inner resonance ring of NGC 3081. II. Star formation, bar strength, disk surface mass density, and mass-to-light ratio. Astron. J. 131, 1377–1393 (2006)
Caranicolas, N.D.: Connecting global to local parameters in barred galaxy models. J. Astrophys. Astron. 23, 173–183 (2002)
Cedrés, B., Cepa, J., Bongiovanni, Á., Castañeda, H., Sánchez-Portal, M., et al.: Density waves and star formation in grand-design spirals. Astron. Astrophys. 560, A59 (2013)
Cheung, E., Athanassoula, E., Masters, K.L., Nichol, R.C., Bosma, A., et al.: Galaxy zoo: observing secular evolution through bars. Astrophys. J. 779, 162 (2013)
Contopoulos, G., Grosbøl, P.: Orbits in barred galaxies. Astron. Astrophys. Rev. 1, 261–289 (1989)
Contopoulos, G., Efstatiou, K.: Escapes and recurrence in a simple Hamiltonian system. Celest. Mech. Dyn. Astron. 88, 163–183 (2004)
Dauphas, N.: The U/Th production ratio and the age of the Milky Way from meteorites and Galactic halo stars. Nature 435, 1203–1205 (2005)
Davis, B.L., Graham, A.W., Seigar, M.S.: Updating the (supermassive black hole mass)-(spiral arm pitch angle) relation: a strong correlation for galaxies with pseudobulges. Mon. Not. R. Astron. Soc. 471, 2187–2203 (2017)
de Lorenzo-Cáceres, A., Méndez-Abreu, J., Thorne, B., Costantin, L.: Deconstructing double-barred galaxies in 2D and 3D-I. Classical nature of the dominant bulges. Mon. Not. R. Astron. Soc. 484, 665–686 (2019)
De Vaucouleurs, G., Freeman, K.C.: Structure and dynamics of barred spiral galaxies, in particular of the Magellanic type. Vistas Astron. 14, 163–294 (1972)
Debattista, V.P., Shen, J.: Long-lived double-barred galaxies from pseudobulges. Astrophys. J. 654, L127–L130 (2006)
Dehnen, W.: The effect of the outer Lindblad resonance of the galactic bar on the local stellar velocity distribution. Astron. J. 119, 800–812 (2000)
Di Matteo, P., Dolcetta, R.C., Miorchi, P.: Clumpy substructures in globular cluster tidal tails. Celest. Mech. Dyn. Astron. 91, 59–73 (2005)
D’Onghia, E., Vogelsberger, M., Hernquist, L.: Self-perpetuating spiral arms in disk galaxies. Astrophys. J. 766, 34 (2013)
Efthymiopoulos, C., Kyzriopoulos, P.E., Páez, R.I., Zouloumi, K., Gravvanis, G.A.: Manifold spirals, disc–halo interactions, and the secular evolution in N-body models of barred galaxies. Mon. Not. R. Astron. Soc. 484, 1487–1505
Elmegreen, D.M., Elmegreen, B.G.: Flocculent and grand design spiral structure in field, binary and group galaxies. Mon. Not. R. Astron. Soc. 201, 1021–1034 (1982)
Elmegreen, B.G., Elmegreen, D.M.: Properties of barred spiral galaxies. Astrophys. J. 288, 438–455 (1985)
Ernst, A., Just, A., Spurzem, R., Porth, O.: Escape from the vicinity of fractal basin boundaries of a star cluster. Mon. Not. R. Astron. Soc. 383, 897–906 (2008)
Ernst, A., Peters, T.: Fractal basins of escape and the formation of spiral arms in a galactic potential with a bar. Mon. Not. R. Astron. Soc. 443, 2579–2589 (2014)
Erwin, P.: Double-barred galaxies-I. A catalog of barred galaxies with stellar secondary bars and inner disks. Astron. Astrophys. 415, 941–957 (2004)
Eskridge, P.B., Frogel, J.A., Pogge, R.W., Quillen, A.C., Davies, R.L., et al.: The frequency of barred spiral galaxies in the near-infrared. Astron. J. 119, 536–544 (2000)
Ferrers, N.M.: On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae and elliptic rings of variable densities. Q. J. Pure Appl. Math. 14, 1–22 (1877)
Frick, P., Stepahov, R., Beck, R., Sokoloff, D., Shukurov, A., et al.: Magnetic and gaseous spiral arms in M83. Astron. Astrophys. 585, A21 (2016)
Gonidakis, I., Livaniou, E., Kontizas, E., Klein, U., Kontizas, M., et al.: Structure of the SMC-Stellar component distribution from 2MASS data. Astron. Astrophys. 496, 375–380 (2009)
Grand, R.J.J., Kawata, D., Cropper, M.: The dynamics of stars around spiral arms. Mon. Not. R. Astron. Soc. 421, 1529–1538 (2012)
Halle, A., Combes, F.: Influence of baryonic physics in galaxy simulations: a semi-analytic treatment of the molecular component. Astron. Astrophys. 559, A55 (2013)
James, P.A., Percival, S.M.: Star formation suppression and bar ages in nearby barred galaxies. Mon. Not. R. Astron. Soc. 474, 3101–3109 (2017)
Jung, C., Zotos, E.E.: Introducing a new 3D dynamical model for barred galaxies. Publ. Astron. Soc. Aust. 32, e042 (2015)
Jung, C., Zotos, E.E.: Orbital and escape dynamics in barred galaxies-I. The 2D system. Mon. Not. R. Astron. Soc. 457, 2583–2603 (2016)
Kaufmann, D.E., Contopoulos, G.: Self-consistent models of barred spiral galaxies. Astron. Astrophys. 309, 381–402 (1996)
Kaviraj, S., Shabala, S.S., Deller, A.T., Middelberg, E.: Radio AGN in spiral galaxies. Mon. Not. R. Astron. Soc. 454, 1595–1604 (2015)
Kim, M., Choi, Y.Y., Kim, S.S.: Effect of bars on evolution of SDSS spiral galaxies. Mon. Not. R. Astron. Soc. 494, 5839–5850 (2020)
Laurikainen, E., Salo, H., Buta, R., Knapp, J.H.: Properties of bars and bulges in the Hubble sequence. Mon. Not. R. Astron. Soc. 381, 401–417 (2007)
Lindblad, B.: On the dynamics of the barred spiral nebulae. Publ. Astron. Soc. Pac. 59, 305–309 (1947)
Lokas, E.L.: Anatomy of a buckling galactic bar. Astron. Astrophys. 629, A52 (2019)
Long, K., Murali, C.: Analytical potentials for barred galaxies. Astrophys. J. 397, 44–48 (1992)
Lynden-Bell, D., Kalnajs, A.J.: On the generating mechanism of spiral structure. Mon. Not. R. Astron. Soc. 157, 1–30 (1972)
Maragoudaki, F., Kontizas, M., Morgan, D.H., Kontizas, E., Dapergolas, A., et al.: The recent structural evolution of the SMC. Astron. Astrophys. 379, 864–869 (2001)
Melia, F., Falcke, H.: The supermassive black hole at the Galactic Center. Annu. Rev. Astron. Astrophys. 39, 309–352 (2001)
Mestre, M., Llinares, C., Carpineto, D.D.: Effects of chaos on the detectability of stellar streams. Mon. Not. R. Astron. Soc. 439, 4398–4408 (2020)
Minchev, I., Boily, C., Siebert, A., Bienayme, O.: Low-velocity streams in the solar neighbourhood caused by the Galactic bar. Mon. Not. R. Astron. Soc. 407, 2122–2130 (2010)
Miyaomoto, M., Nagai, R.: Three-dimensional models for the distribution of mass in galaxies. Publ. Astron. Soc. Jpn. 27, 533–543 (1975)
Mondal, D., Chattopadhayay, T.: Star formation under explosion mechanism in a magnetized medium. Bulg. Astron. J. 31, 16–29 (2019)
Monteagudo, L., Gallart, C., Monelli, M., Bernard, E.J., Stetson, P.B.: The origin of the LMC stellar bar: clues from the SFH of the bar and inner disc. Mon. Not. R. Astron. Soc. 473, L16–L20 (2018)
Nair, P.B., Abraham, R.G.: On the fraction of barred spiral galaxies. Astrophys. J. Lett. 714, L260–L264 (2010)
Nishiyama, S., Nagata, T.: IRSF/SIRIUS team: is the milky way a double-barred galaxy? J. Phys. Conf. Ser. 54, 62–66 (2006)
Ostriker, J.P., Peebles, P.J.E., Yahil, A.: The size and mass of galaxies, and the mass of the universe. Astrophys. J. 193, L1–L4 (1974)
Pastorini, G., Marconi, A., Capetti, A., Axon, D.J., Alonso-Herrero, A., et al.: Supermassive black holes in the Sbc spiral galaxies NGC 3310, NGC 4303 and NGC 4258. Astron. Astrophys. 469, 405–423 (2007)
Patsis, P.A., Athanassoula, E., Quillen, A.C.: Orbits in the Bar of NGC 4314. Astrophys. J. 483, 731–744 (1997)
Patsis, P.A.: Structures out of Chaos in barred-spiral galaxies. Int. J. Bifurcation Chaos 22, 1230029 (2012)
Petersen, M.S., Weinberg, M.D., Katz, N.: Using torque to understand barred galaxy models. Mon. Not. R. Astron. Soc. 490, 3616–3632 (2019)
Pfenniger, D.: The 3D dynamics of barred galaxies. Astron. Astrophys. 134, 373–386 (1984)
Piatti, A.E.: The real population of star clusters in the bar of the Large Magellanic Cloud. Astron. Astrophys. 606, A21 (2017)
Plummer, H.C.: On the problem of distribution in globular star clusters. Mon. Not. R. Astron. Soc. 71, 460–470 (1911)
Polyachenko, E.V., Shukhman, I.G.: Star-forming rings in lenticular galaxies: origin of the gas. Astron. J. 158, 5 (2019)
Querejeta, M., Eliche-Moral, M.D.C., Tapia, T., Borlaff, A., van de Ven, G., et al.: Formation of S0 galaxies through mergers—explaining angular momentum and concentration change from spirals to S0s. Astron. Astrophys. 579, L2 (2015)
Quillen, A.C., Dougherty, J., Bagley, M.B., Minchev, I., Comparetta, J.: Structure in phase space associated with spiral and bar density waves in an N-body hybrid galactic disc. Mon. Not. R. Astron. Soc. 417, 762–784 (2011)
Raha, N., Sellwood, J.A., James, R.A., Kahn, F.D.: A dynamical instability of bars in disk galaxies. Nature 352, 411–412 (1991)
Regan, M.W., Teuben, P.: The formation of nuclear rings in barred spiral galaxies. Astrophys. J. 582, 723–742 (2003)
Romero-Gómez, M., Masdemont, J.J., Athanassoula, E., García-Gómez, C.: The origin of rR1 ring structures in barred galaxies. Astron. Astrophys. 453, 39–45 (2006)
Romero-Gómez, M., Athanassoula, E., Masdemont, J.J., García-Gómez, C.: The formation of spiral arms and rings in barred galaxies. Astron. Astrophys. 472, 63–75 (2007)
Sakamoto, K., Okumura, S.K., Ishizuki, S., Scoville, N.Z.: Bar-driven transport of molecular gas to galactic centers and its consequences. Astrophys. J. 525, 691–701 (1999)
Sakamoto, K., Baker, A.J., Scoville, N.Z.: Gasdynamics in the linear galaxy NGC 5005: episodic fueling of a nuclear disk. Astrophys. J. 533, 149–161 (2000)
Sanchez-Martin, P., Romero-Gomez, M., Masdemont, J.J.: Warp evidence in precessing galactic bar models. Astron. Astrophys. 588, A76 (2016)
Sanders, J.L., Smith, L., Evans, N.W.: The pattern speed of the Milky Way bar from transverse velocities. Mon. Not. R. Astron. Soc. 488, 4552–4564 (2019)
Sandri, M.: Numerical calculation of Lyapunov exponents. Math. J. 6, 78–84 (1996)
Seigar, M.S., Kennefick, D., Kennefick, J., Lacy, C.H.: Discovery of a relationship between spiral arm morphology and supermassive black hole mass in disk galaxies. Astrophys. J. 678, L93–L96 (2008)
Sellwood, J.A.: Bar instability in disk-halo systems. Astrophys. J. 819, 92 (2016)
Seo, W.Y., Kim, W.T., Kwak, S., Hsieh, P.Y., Han, C., et al.: Effects of gas on formation and evolution of stellar bars and nuclear rings in disk galaxies. Astrophys. J. 872, 5 (2019)
Sharma, S., Stello, D., Bland-Hawthorn, J., Hayden, M.R., Zinn, J.C., et al.: The K2-HERMES survey: age and metallicity of the thick disc. Mon. Not. R. Astron. Soc. 490, 5335–5352 (2019)
Sheth, K., Elmegreen, D.M., Elmegreen, B.G., Capak, P., Abraham, R.G., et al.: Evolution of the bar fraction in COSMOS: quantifying the assembly of the Hubble sequence. Astrophys. J. 675, 1141–1155 (2008)
Sofue, Y., Rubin, V.: Rotation curves of spiral galaxies. Annu. Rev. Astron. Astrophys. 39, 137–174 (2001)
Straitzalis, A., Hatzipimou, D., Zezas, A., Antoniou, V., Lianou, S., et al.: Discrete star formation events in the central bar of the Small Magellanic Cloud. Mon. Not. R. Astron. Soc. 489, 5087–5097 (2019)
Strazewski, M., Eile, M., Beck, R.: Hot gas and magnetic arms of NGC 6946: indications for reconnection heating? Astron. Astrophys. 585, A3 (2016)
Williams, A.A., Evans, N.W.: Models of bars-I. Platfiff profiles for early-type galaxies. Mon. Not. R. Astron. Soc. 469, 4414–4421 (2017)
Yoon, Y., Im, M., Lee, G.H., Lee, S.K., Lim, G.: Observational evidence for bar formation in disk galaxies via cluster-cluster interaction. Nat. Astron. 3, 844–850 (2019)
Zotos, E.E.: Order and chaos in a galactic model with a strong nuclear bar. Res. Astron. Astrophys. 12, 500–512 (2012)
Zotos, E.E.: Distinguishing between order and chaos in a simple barred galaxy model. Astron. Nachr. 338, 614–620 (2017)

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