Numerical study of multiparticle scattering in $\lambda \phi^4$ theory

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Abstract: We study numerically classical collisions of waves in $\lambda \phi^4$ theory. These processes correspond to multiparticle scattering in the semiclassical regime. Parametrizing initial and final wavepackets by energy $E$ and particle numbers $N_i$, $N_f$ we find classically allowed region in the parameter space. We describe properties of the scattering solutions at the boundary of the classically allowed region. We comment on implications of our results for multiparticle production in quantum regime.

Keywords: Classical scattering, Multiparticle production
1 Introduction

Multiparticle production in weakly coupled bosonic field theories has long been of great interest (see e.g. [1, 2] for reviews). Tree level calculations performed in (3 + 1)-dimensional scalar $\lambda \phi^4$ theory show [3–7] that multiparticle amplitudes of $1 \rightarrow N_f$ processes near the threshold grow factorially with the number of produced particles $N_f$. This behaviour persists at one-loop level [8] and violates unitarity at sufficiently large multiplicities $N_f \sim 1/\lambda$. It was observed [9], that corresponding multiparticle cross section of $1 \rightarrow N_f$ processes can be conveniently written in the following functional form

$$\sigma_{1 \rightarrow N_f} \sim \exp \left( \frac{1}{\lambda} F(\lambda N_f, \mathcal{E}) \right)$$

(1.1)

in the limit $\lambda \rightarrow 0$ with $\lambda N_f$ and $\mathcal{E}$ fixed, where $\mathcal{E}$ is the average kinetic energy of the final particles. Inspired by this exponential behaviour several semiclassical techniques for calculation of multiparticle cross sections were developed in [10–14]. They involved singular solutions of classical equations of motion and allowed to extend the observations obtained with perturbative calculations to higher energies. Still the most reliable results were obtained in the regime of small $\lambda N_f$ where corresponding cross sections turns out to be exponentially suppressed. Unitarity bounds [2, 15, 16] indicate that such suppression of the probability (1.1) takes place at all particle numbers. Interesting insights come from studies of an analogue of the scattering amplitudes in
(0 + 1)-dimensional theory, i.e. for the anharmonic oscillator with a quartic potential [17–19], where it was found that the perturbative factorial growth is replaced by exponential suppression for $\lambda N_f \gtrsim 1$. But still behaviour of the probability (1.1) in $\lambda \phi^4$ field theory at large $\lambda N_f$ is unknown even in the weak coupling regime which will be assumed in the paper.

It was long ago understood that the processes $few \rightarrow N_f$ for large $N_f$ can be studied starting from the processes $N_i \rightarrow N_f$ where both initial $N_i$ and final $N_f$ number are large, i.e. of order $1/\lambda$. In this case, one can consider classical counterpart of the quantum scattering processes, i.e. collisions of classical wave packets. If the ingoing and outgoing waves are in the linear regime at $t \rightarrow \pm \infty$ corresponding initial and final field configurations can be associated with coherent states having average energy $E$, initial $N_i$ and final $N_f$ particle numbers. The probability of corresponding quantum scattering in the semiclassical regime is not exponentially suppressed and the whole family of such solutions span a classically allowed region in the space of parameters $E, N_i$ and $N_f$. At a given energy $E_*$ and final particle number $N_f$ one can try to minimize initial particle number $N_i$ with respect to initial conditions. If the minimum goes to zero for some energy then the probability of the processes $2 \rightarrow N_f$ are not exponentially suppressed at $E > E_*$ [16]. Otherwise the existence of nontrivial minimum at $N_i \sim 1/\lambda$ would indicate on exponential suppression of the $2 \rightarrow N_f$ scattering for $E \lesssim E_*$. This idea of exploring classical region was used previously for study of several different processes induced by collisions of particles. Among of them are false vacuum decay [20], the baryon number violating processes in the Standard Model [21, 22] and soliton-antisoliton pair production in (1 + 1)–dimensional scalar field theory [23].

In this paper we study classical solutions describing scattering of wave packets in the unbroken $\lambda \phi^4$ theory. In particular, we try to approach boundary of the corresponding classically allowed region. Classical scattering of waves in relation to multiparticle production was previously studied to some extent in (1 + 3)-dimensional $\phi^4$ model [24], (1 + 1)-dimensional abelian Higgs model [25] and non-abelian gauge theories [26–28]. In particular, authors of Ref. [24] constructed initial wave packet consisting of a few high frequency free field modes and examine its evolution. They found no significant energy transfer to low frequency modes which would indicate to production of many “quanta”. In our study we use stochastic sampling technique to scan numerically over classical solutions describing multiparticle scattering. We limit ourselves to spherically symmetric solutions which reduces the system to a (1 + 1)-dimensional model and makes the problem numerically feasible. We find it is technically more convenient to fix not final $N_f$ but initial particle number $N_i$ and energy $E$ and find the minimal and maximal values of particle number $N_f$ of outgoing wave packets which is obtained by solving classical equations of motion. Both approaches are equivalent due to the time
reversal symmetry. With our numerical methods we find non-trivial classically allowed region for multiparticle scattering processes. Namely, we are able to determine the upper and lower boundaries $N_{f}^{\text{min}}$ and $N_{f}^{\text{max}}$ as functions of energy $E$ for a set of fixed values of $N_i$. We observe that the change in particle number in the scattering reaches values up to 22% for studied energy range. We examine properties of the classical solutions corresponding to the boundaries. Obtained results give indirect indication for exponential suppression of the probability of $\text{few} \rightarrow N_f$ processes for considered energies.

Let us note that recently an interest to the problem of multiparticle production has been renewed [29–39]. In particular, calculations of Refs. [35, 39] performed in $\lambda \phi^4$ theory with spontaneously broken $\mathbb{Z}_2$ symmetry, exhibited that the probability of $2 \rightarrow N_f$ scattering process near threshold is unsuppressed at $\lambda N_f \gg 1$. In view of this result it would be interesting to apply the method of classical solutions used in our study to explore the classically allowed region in $(E, N_i, N_f)$ parameter space in the spontaneously broken $\lambda \phi^4$ theory.

The rest of the paper is organized as follows. Section 2 is devoted to the description of the model and introduction of notations and most relevant quantities. In Section 3 we describe numerical method which we utilize to study solutions to the classical equations of motion describing scattering of wave packets with a particular interest to those “providing” maximal change in their particle numbers. In Section 4 we present our numerical results. Section 5 is reserved for discussion and conclusions.

2 The model

We consider $(3 + 1)$–dimensional model of a scalar field with the action

$$S[\phi] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 \right].$$

(2.1)

The parameter $m^2$ is taken to be positive which corresponds to unbroken phase of this theory. By making rescaling $x^\mu \rightarrow m^{-1} x^\mu$ and $\phi \rightarrow \sqrt{\frac{m^2}{\lambda}} \phi$ this action can be cast into

$$S[\phi] = \frac{1}{\lambda} \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4 \right].$$

(2.2)

In this form $\lambda$ is a semiclassical parameter which does not enter equations of motion. We are interested in a particular type of classical solutions, which describe collisions of wave packets. In what follows we limit ourselves to spherically symmetric field configurations. In this case it is useful to make the following redefinition

$$\phi(t, r) = \frac{1}{r} \chi(t, r)$$

(2.3)
and we obtain the action

\[ S = \frac{4\pi}{\lambda} \int dt dr \left[ \frac{1}{2} \left( \frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \chi}{\partial r} \right)^2 - \frac{\chi^2}{2} - \frac{\chi^4}{4r^2} \right] \]  

(2.4)
of (1 + 1)–dimensional theory on a half-line with the boundary condition \( \chi(t, 0) = 0 \) and position dependent interaction. Corresponding equation of motion reads

\[ \frac{d^2 \chi}{dt^2} - \frac{d^2 \chi}{dr^2} + \chi + \frac{1}{r^2} \chi^3 = 0 \]  

(2.5)
and the energy related to the field configuration \( \chi(t, r) \) is

\[ E = \frac{4\pi}{\lambda} \int_0^\infty dr \left[ \frac{1}{2} \left( \frac{\partial \chi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \chi}{\partial r} \right)^2 + \frac{1}{2} \chi^2 + \frac{1}{4r^2} \chi^4 \right]. \]  

(2.6)
In what follows we solve classical e.o.m. (2.5) numerically. To do this we restrict our solutions to a finite space interval \([0, R]\) where \( R \) is sufficiently large. At \( r = R \) we impose Neumann boundary condition \( \partial_r \chi = 0 \). This restriction allows us to expand field configuration as follows

\[ \chi(t, r) = \sum_{n=0}^\infty c_n(t) \sqrt{\frac{2}{R}} \sin k_n r, \quad \text{where} \quad k_n = \frac{\pi}{2R} (2n + 1), \quad n = 0, 1, \ldots \]  

(2.7)
Inserting this expansion in the action (2.4) we obtain the following equations of motion

\[ \ddot{c}_n + \omega_n^2 c_n + I_n = 0, \quad n = 0, 1, \ldots \]  

(2.8)
where \( \omega_n^2 = k_n^2 + 1 \) and the interaction term reads

\[ I_n = \sqrt{\frac{2}{R}} \int_0^R dr \frac{\chi^3(t, r)}{r^2} \sin k_n r, \quad n = 0, 1, \ldots \]  

(2.9)
Total energy of the solution expressed via \( c_n(t) \) has the form

\[ E = \frac{4\pi}{\lambda} \sum_n \left[ \frac{1}{2} \dot{c}_n^2 + \frac{1}{2} \omega_n^2 c_n^2 + \frac{1}{4} c_n I_n \right]. \]  

(2.10)
Below we consider solutions which linearize at initial and final times. In this case they can be interpreted as describing multiparticle scattering. In linear regime the time-dependent Fourier components \( c_n(t) \) can be conveniently written via positive and negative frequency components as follows

\[ c_n(t) \to \frac{1}{\sqrt{2\omega_n}} \left( a_n e^{-i\omega_n t} + a_n^* e^{i\omega_n t} \right) \quad \text{as} \quad t \to -\infty \]  

(2.11)
and
\[ c_n(t) \to \frac{1}{\sqrt{2\omega_n}} \left( b_n e^{-i\omega_n t} + b_n^* e^{i\omega_n t} \right) \quad \text{as} \quad t \to +\infty. \] (2.12)

Using these representations one can compute the energy of colliding wave packets as
\[ E = \frac{4\pi}{\lambda} \sum_n \omega_n |a_n|^2 = \frac{4\pi}{\lambda} \sum_n \omega_n |b_n|^2. \] (2.13)

Here $|a_n|^2$ and $|b_n|^2$ can be thought as occupation numbers for initial and final states and their sums
\[ N_i = \frac{4\pi}{\lambda} \sum_n |a_n|^2, \quad N_f = \frac{4\pi}{\lambda} \sum_n |b_n|^2 \] (2.14)

are initial and final particle numbers, respectively. For convenience we introduce the following shorthand notations
\[ \bar{E} = \frac{\lambda}{4\pi} E, \quad \bar{N}_i = \frac{\lambda}{4\pi} N_i, \quad \bar{N}_f = \frac{\lambda}{4\pi} N_f. \] (2.15)

For numerical implementation we truncate the expansion (2.7) at $n = N_r$ and solve the evolution equations (2.8) using Bulirsch-Stoer method, see e.g. [40]. Some of the results have been verified with 4-th order Runge-Kutta method with a very small time step. For convenience, we introduce uniform spacial lattice $r_i$, $i = 0, ..., N_r$ with $r_0 = 0$ and $r_{N_r} = R$. In what follows we use several values of $R = 20, 30$ and 50 and $N_r = 400, 600$ and 1000 to study the dependence of our results on the lattice. The time interval is taken to be somewhat smaller than $\sim 2R$. We use FFTW implementation of discrete Fourier transformation [41] to compute field configuration $\chi(t, r)$ as (2.7) and interaction term (2.9).

3 The method

In this Section we describe a method used to explore classical transitions $\bar{N}_i \to \bar{N}_f$ at fixed energy $\bar{E}$. In particular, we are interested in solutions of classical e.o.m. which maximize $|\bar{N}_f - \bar{N}_i|$ at fixed values of $\bar{N}_i$ and $\bar{E}$.

3.1 Initial conditions

We take initial conditions which correspond to a wave packet which is localized well away from the interaction region and which is propagating towards $r = 0$. Technically, we choose an interval $[r_1, r_2]$, where $r_1, r_2$ are chosen points belonging to the spacial lattice, i.e. $r_1 = r_{i_1}$ and $r_2 = r_{i_2}$ for some $i_1 < i_2$, and construct following function
\[ \chi(r) = \begin{cases} \sum_{n=1}^{i_2-i_1} \frac{1}{\sqrt{2\omega_n}} f_n \sin \left( \tilde{k}_n (r - r_{i_1}) \right), & r \in [r_1, r_2] \\ 0, & r \notin [r_1, r_2] \end{cases} \] (3.1)
Here $\tilde{k}_n = \frac{\pi n}{r_2 - r_1}$, $\tilde{\omega}_n = \sqrt{k_n^2 + 1}$, $n = 1, \ldots, i_2 - i_1$ and $f_n$ are initial Fourier amplitudes. By construction the configuration (3.1) nullifies at the boundaries $r = r_{i_1}$ and $r = r_{i_2}$. Next, to make the initial configuration smooth enough we introduce an artificial smearing multiplying (3.1) by the function

$$
\left( \frac{e^{-r_{i_1} + d - a r_a}}{1} + 1 \right)^{-1} \cdot \left( e^{-\frac{r_{i_2} - d - a r_a}{1} + 1} \right)^{-1},
$$

where $d$ and $a$ are chosen parameters. We checked that our results have very mild dependence on precise way of the smearing. Next, we obtain Fourier amplitudes $\tilde{f}_n$ of the smeared configuration by inverting (3.1) and take the initial condition for e.o.m. (2.8) (i.e. field configuration and its first time derivative at initial time $t = t_i$) from

$$
\chi_i(t, r) = \begin{cases} 
\sum_{n=1}^{i_2-i_1} \frac{1}{\sqrt{2} \omega_n} \tilde{f}_n \sin \left( \tilde{k}_n(r - r_1) + \tilde{\omega}_n(t - t_i) \right), & r \in [r_1, r_2] \\
0, & r \notin [r_1, r_2].
\end{cases}
$$

This corresponds to the initial wave packet propagating towards $r = 0$. In what follows we set $t_i = 0$. We consider only solutions whose evolution is linear at initial (and final) times.

### 3.2 Going to the classical boundary

Using the initial field configuration we calculate its energy $\tilde{E}$ and initial particle number $\tilde{N}_i$ by using Eqs. (2.10), (2.11), (2.13) and (2.14). Our aim is to find the classical solutions describing scattering of wave packets in which particle number changes as much as possible at a given energy $\tilde{E}$. Since we solve initial value problem it is more convenient to fix initial particle number $\tilde{N}_i$ and find absolute minimum (or maximum) of $\tilde{N}_f$ with respect to chosen set of initial conditions, i.e. $f_n$ entering (3.1). We fix the initial particle number by respectful normalization of the Fourier amplitudes $\tilde{f}_n$. Final particle number $\tilde{N}_f$ is a highly non-linear function of initial data $\tilde{f}_n$ and it may have several local extrema. To find its absolute minimum (maximum) we use stochastic sampling technique in combination with the simulated annealing method, see [40, 42]. Namely, we generate an ensemble of the classical solutions with fixed initial particle number $\tilde{N}_i$ weighted by probability

$$
p \sim e^{-F},
$$

where

$$
F = \beta \left( \tilde{N}_f + \xi(\tilde{E} - \tilde{E}_s)^2 \right).
$$
If large positive $\beta$ and $\xi$ are taken the ensemble will be dominated by solutions having small $F$ thus driving their distribution towards lower boundary $\tilde{N}_{f}^{\text{min}}(\tilde{E}_{*})$ of classically allowed region. To reach the upper boundary $\tilde{N}_{f}^{\text{max}}(\tilde{E}_{*})$ signs of $\beta$ and $\xi$ should be negative.

To generate the ensemble (3.4) we use Metropolis Monte Carlo algorithm. We start with a solution specifying by a randomly chosen set of $f_{n}$, see (3.1), which have initial particle number $\tilde{N}_{i}$. This solution has an energy $\tilde{E}$ and final particle number $\tilde{N}_{f}$ which is found after solving of classical equations of motion. Next, we choose a few randomly chosen amplitude numbers and change corresponding amplitudes $f_{n}$ by small amounts, $f_{n} \rightarrow f'_{n} = f_{n} + \Delta f_{n}$. We used to take three amplitudes at a time which shows a good performance of our algorithm. The quantities $\Delta f_{n}$ are chosen to be normally distributed with the standard deviation $a/\sqrt{\tilde{\omega}_{n}}$, where $a$ is a small number. In our calculations its value is varied from $10^{-4}$ to 0.1. Modified set of amplitudes $f'_{n}$ is rescaled in a proper way and used to construct initial wave packet having the same initial particle number $\tilde{N}_{i}$ and some value of energy $\tilde{E}'$, see Section 3.1. Then we evolve the system forward in time far enough until it reaches linear regime and we can find final particle number $\tilde{N}_{f}'$ of new solution. We compute $\Delta F \equiv F' - F$ using (3.5). New set of amplitudes $f'_{n}$ is accepted with probability

$$p_{\text{accept}} = \min \left(1, e^{-\Delta F} \right)$$

(3.6) 

and used for next iteration. Typically, we fix $\tilde{N}_{i}$ and $\tilde{E}_{*}$ and perform several runs starting with different $\beta$ and $\xi$. Each run spans of order $10^{3} - 10^{4}$ iterations. The value of $\xi$ remains a constant during a single run, while $\beta$ which is analogue of inverse temperature is gradually increased from its initial value $\beta_{0}$ according [43] to

$$\beta_{i} = \beta_{0} \log \left(1 + i \right),$$

(3.7) 

where $i$ is iteration number. Values of $\beta_{0}$ and $\xi$ are temporarily increased from 10 to $10^{6}$ and from $10^{-4}$ to about 1, respectively. We stop the procedure when relative change in the value of $F$ during a run becomes smaller than $10^{-3}$ for the same $\xi$ and it does not increases with variation of $a$ and $\beta_{0}$.

Let us note, that the stochastic sampling technique was previously used to study classically allowed sphaleron transitions [21, 22] and classical soliton-antisoliton pair production in particle collisions in (1+1)-dimensional scalar field theory [23], where its efficiency in determination of the boundary of classically allowed region was confirmed.
Figure 1: Results of stochastic sampling for $\tilde{N}_i = 1$, $N_r = 400$ and $R = 20$. Each dot in the figure represents a classical solution with an energy $\tilde{E}$ and final particle number $\tilde{N}_f$ obtained using Monte-Carlo technique described in the text. Red (dark) and green (gray) points correspond to positive and negative values of $\beta$ and $\xi$, respectively. The dashed line marks the threshold energy $\tilde{E} = \tilde{N}_f = \tilde{N}_i \equiv 1$. The upper and lower boundaries, $\tilde{N}_f^{max}(\tilde{E})$ and $\tilde{N}_f^{min}(\tilde{E})$ are shown by solid lines.

4 Numerical results

Below we describe regions of the classical solutions for several different values of $\tilde{N}_i = 0.1, 1.0, 10.0$ and 30.0. In Fig. 1 we plot numerical results of the stochastic sampling in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 1$, $N_r = 400$ and $R = 20$. The initial field configurations are set in the interval $[r_1, r_2] \equiv [5.0, 19.0]$. Each dot in the Figure represents a classical solution accepted by our numerical procedure on the way to the boundaries $\tilde{N}_f^{max}(\tilde{E})$ and $\tilde{N}_f^{min}(\tilde{E})$. To reach the boundaries we run the numerical procedure described in the previous Section for a chosen set of $\tilde{E}_s$

$$\frac{\tilde{E}_s}{\tilde{N}_i} = 1.5, 2.0, ..., 9.5, 10.0, 11.0, ..., 15.0,$$  \hspace{1cm} (4.1)

which determines the relevant energy interval for our study. We comment on this choice below. The solutions which seed our Monte Carlo search are randomly chosen with the only condition that they have $\tilde{N}_i = 1$ and linearize at initial and final times. These
solutions typically have very small change in particle number are situated near the line \( \tilde{N}_f = 1 \) on \((\tilde{E}, \tilde{N}_f)\) plane. Initially, we take relatively large value of \( a = 0.1 \) and start our stochastic sampling with small \( \beta_0 \) about 10.0. During single run its value is changed according to (3.7). We repeat the runs with larger starting value of \( \beta \). The value of \( a \) which determines size of the Fourier amplitude changes \( \Delta f_n \) is also decreased after several runs. At the same time the value of \( \xi \) is increased to fix the energy of the solution to be near \( \tilde{E}_* \). The obtained domain of the classical solutions in \((\tilde{E}, \tilde{N}_f)\) plane, see Fig. 1, has smooth envelopes, \( \tilde{N}_f^{\text{min}}(\tilde{E}) \) and \( \tilde{N}_f^{\text{max}}(\tilde{E}) \), which represent the boundary of the classically allowed region of the lattice version of the continuous model. We see that the maximal possible change in the particle number \( |\tilde{N}_f - \tilde{N}_i| \) during classical evolution increases with the energy of solutions but does not exceed 12% in the chosen energy range. Let us stress that due to time reversal symmetry one can always interchange the initial \( \tilde{N}_i \) and final \( \tilde{N}_f \) particle numbers in the discussion of classically allowed region in the Fig. 1.

Let us study the properties of the classical solutions near the boundaries \( \tilde{N}_f^{\text{min}}(\tilde{E}) \) and \( \tilde{N}_f^{\text{max}}(\tilde{E}) \). Although the solutions which seed our Monte Carlo runs for each value of \( \tilde{E}_* \) are chosen at random it is remarkable that the boundary solutions have similar forms for different \( \tilde{E}_* \). At the panels of Fig. 2 we plot initial wave packets of the classical solutions at the upper boundary \( \tilde{N}_f = \tilde{N}_f(\tilde{E}) \) for several values of energy \( \tilde{E} \approx 3.0, 4.5, 8.0 \) and 11.0. The interval \([r_1, r_2]\) is marked by dashed lines. Corresponding final particle number for these solutions can be found in the Fig. 1. Evolution in time of the upper boundary solution with \( \tilde{E} \approx 6.1 \) is presented in Fig. 3 where \( \chi(t, r) = \frac{\phi(t, r)}{r} \) is shown for convenience. For all boundary solutions which we have found with \( \tilde{N}_i = 1 \) the initial wave packet has a relatively sharp part near its end. This part becomes more pronounced with increase of its energy as seen in Fig. 3. The initial wave packet propagates towards interaction region and its spiky parts increases considerably near the point \( r = 0 \). The reflected wave packet have the form very similar to the incoming one. The spiky part of the incoming wave packet is the last to arrive into the interaction region and the first to leave it.

According to the used extremization procedure which involves stochastic sampling, the found classical solutions describe scattering of waves with \( \tilde{N}_i = 1 \) and maximal value of \( |\tilde{N}_f - \tilde{N}_i| \) at a given energy \( \tilde{E} \). It is interesting to study how the particle number changes during the time evolution. In Section 2 we introduced the particle numbers for initial and final wave packets. However, we can calculate instantaneous particle number \( N(t) \) by using Eq. (2.11) (or (2.12)) as a definition of time-dependent positive and negative frequency components at arbitrary time \( t \) and applying formulas similar to (2.14). The quantity \( \tilde{N}(t) \) coincides with \( \tilde{N}_i \) and \( \tilde{N}_f \) at initial and final times where evolution of the field is linear. On the right panel of the Fig. 4 we plot the instantaneous
Figure 2: The field configurations $\phi(t = 0, r)$ of the solutions at the upper boundary with $\tilde{N}_f = \tilde{N}_f(\tilde{E})$ for $\tilde{E} \approx 3.0, 4.5, 8.0$ and $11.0$. The space lattice parameters are $N_r = 400$, $R = 30$. The dashed lines mark the space interval for the initial wave packet.

particle number for the boundary solution shown in Fig. 3. Comparing it with the time evolution of the field presented on Fig. 3 we see that the actual change of the particle number occurs precisely when the sharpest part of the initial wave packet reaches the interaction region. For instance, the deviations of $\tilde{N}$ at time stamps $t = 12.4$ and $t = 21.3$ (see Fig. 3) from its asymptotic values $\tilde{N}_i$ and $\tilde{N}_f$ are less than 0.0005, which are much smaller then $\tilde{N}_f - \tilde{N}_i \approx 0.072$. Such behaviour of $\tilde{N}(t)$ give us support that found extrema of $F$ are independent of choice of the interval $[r_1, r_2]$ which is used to generate initial field configurations as far as it is sufficiently large and include the part of the wave packet responsible for the most of the change in $\tilde{N}(t)$. Also we verified that numerical error in $|\tilde{N}_f - \tilde{N}_i|$ related to smearing is less than $10^{-3}$. In a way similar to $\tilde{N}(t)$ we define linearized energy $\tilde{E}_{lin}(t)$ which is used to check the linearity of the solution by comparing it with the exact energy (2.10). Such comparison is presented in Fig. 4 (left panel) for the solution from Fig. 3. The total energy is conserved on the entire solution with the accuracy less than $10^{-3}$, while it coincides with the linearized energy better than $10^{-4}$ at initial and final times.

Nontrivial change in particle number implies a redistribution of energy between low and high frequency modes. In Fig. 5 we show energy distributions $\epsilon_k$ per wave number
Figure 3: Time evolution of the field $\chi(t, r) = \frac{\phi(t, r)}{r}$ for the upper boundary solution with $\tilde{E} \approx 6.1$; $N_r = 400, R = 20$.

Figure 4: Time evolution of linearized and exact energy (left panel) as well as instantaneous particle number (right panel) for the exemplary solution in Fig. 3.

unit for initial and final wave packets for the same upper boundary solutions in Fig. 2. These distributions are defined as

$$\epsilon_k = \begin{cases} \frac{1}{\Delta k} \omega_n |a_n|^2, & \text{for initial wave packet}, \\ \frac{1}{\Delta k} \omega_n |b_n|^2, & \text{for final wave packet}, \end{cases} \quad (4.2)$$
where $\Delta k = \frac{\pi}{R}$. We observe expected softening of the energy distributions for the final wave packet as compared to the initial one. However the effect is quite small even for boundary solutions which describe processes with maximal change of particle number. Also we see that with increase of colliding energy $\tilde{E}$ modes with larger $k_n$ (i.e. higher frequencies) become filled in and at $\tilde{E} \gtrsim 11.0$ the tail of energy distributions reaches the largest value of $k_n$ for a given lattice size and spacing. This is clear indication that a finer lattice is required to obtain reliable results at these and higher energies. At the same time solutions with energies not far from the threshold energy $\tilde{E}_{th} = \tilde{N}_i$ consists mostly of nonrelativistic modes and larger space interval $R$ is needed to obtain solutions which are linear at initial and final times. For this reason we do not consider solutions with energies lower than 1.5.

Next, we study the dependence of the obtained results for the boundary of the classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 1$ and for the boundary solutions on parameters of the lattice, namely on the spacial cutoff $R$ and on number of modes $N_r$. For comparison to the case with $R = 20$ and $N_r = 400$ we repeat the same numerical procedure to find the boundary of classically allowed region for the case $R = 20, N_r = 600$ with smaller lattice spacing and for the case $R = 30, N_r = 600$ with larger space interval but the same lattice spacing. The results are presented in Fig. 6, where we show lower and upper boundaries $\tilde{N}_f^{\text{min}}(E)$ and $\tilde{N}_f^{\text{max}}(E)$ for different

**Figure 5:** Distribution of energy over the modes $\epsilon_k$ per wave number for initial (red, thick line) and final (blue, thin line) wave packet for the boundary solutions from Fig. 2.
cases. The space intervals for selection of the initial wave packets are taken to be [6.7, 19.2] in the former case and [8.5, 29.8] in the latter. We observe that different boundaries coincide with accuracy better than $4 \cdot 10^{-3}$. A deviation is seen for the case $N_r = 600, R = 30$ for large energies $\tilde{E} \gtrsim 11$ which reflects appearance of new high frequency modes. Examples of the upper boundary solutions for the same set of energies as in Fig. 2 but for the case $N_r = 600, R = 30$ are shown in Fig. 7. Comparing them with the solutions presented in Fig. 2 we see that the difference lies in enlarged soft oscillating part of the solution. We found that the time evolution of the particle number $\tilde{N}(t)$ for these solution is similar to that of presented in Fig. 4. Namely, the actual change in $\tilde{N}$ starts with arrival of the sharp tail of the incoming wave train into the interaction region. Energy distributions over Fourier modes for the solutions with the same energy obtained on different lattices are almost indistinguishable as seen from Fig. 8.

Our numerical results clearly show existence of classically allowed region of a finite size in $(\tilde{E}, \tilde{N}_f)$ plane for fixed value of $\tilde{N}_i$ in the continuous version of the model with $\phi^4$ potential. We suggest existence of the boundary solutions in the continuous limit of the model. The classical equations of motion (2.5) and (2.8) are invariant with respect
Figure 7: The same as in Fig. 2 but for $N_r = 600, R = 30$.

Figure 8: Distribution of energy over the modes $\epsilon_k$ per wave number for the upper boundary solutions for $N_r = 400, R = 20$ (red, thick line) and $N_r = 600, R = 30$ (blue, thin line) Figs. 2 and 7.
to time translations. This symmetry is explicitly broken in our numerical setup by the choice of the initial configuration (3.1) and (3.3) at \( t = t_i \equiv 0 \), which in particular implies
\[
\chi(0, r_1) = \chi(0, r_2) = 0. \tag{4.3}
\]
Still the oscillating form of the initial wave packets (see Figs. 2 and 7) and the time evolution of the particle number indicate that an approximate discrete time-shift symmetry should exists in the lattice version of the model. In terms of initial wave packets this symmetry connects the configurations which are related by time evolution and which satisfy conditions (4.3) as long as the spiky part of the initial wave train lies inside the interval \([r_1, r_2]\). Indeed, with our numerical technique (making several runs at the same energy \( \tilde{E} \) with different seeds) we actually find several branches of boundary solutions which are related with each other by such shifts in time. In fact, for constructing the boundaries in Fig. 1 we selected the solutions whose initial wave packets have the longest soft oscillating part while their sharp edges are placed as far as possible (for given \( R \) and initial space interval \([r_1, r_2]\)) from the origin. In Fig. 9 we plot initial field configurations (red solid lines) corresponding to another branch of the upper boundary solutions at the same values of energy as in Fig. 2. These wavepackets have spiky parts

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{The upper boundary initial field configurations of solutions related by time-shift symmetry for \( \tilde{E} \approx 3.0, 4.5, 8.0 \) and 11.0. The space lattice parameters are \( N_r = 400, R = 20 \). The dashed lines mark the space interval for the initial wave packet.}
\end{figure}
shifted closer to the origin as compared to those in Fig. 2. Also in the Fig. 9 in thin (blue) lines we show the initial field configurations from Fig. 2 evolved forward in time until the positions spiky part of the different solutions coincide. We observe perfect coincidence of the form of that part. As the spiky part of the wavepackets is responsible for the change of the particle number we expect that time-shifted solutions would have similar values of $\tilde{N}_f$. This is indeed the case and in Fig. 6 we show the upper boundary $\tilde{N}_f^{\max}$ obtained for $N_r = 400$ and $R = 20$ with the branch of the time shifted solutions. It coincides with that of for upper boundary solutions with the longer soft oscillating part with accuracy better than $2 \cdot 10^{-3}$.

So far we described classical solutions at the upper part $\tilde{N}_f = \tilde{N}_f^{\max}(E)$ of the boundary. Examples of solutions corresponding to the lower part $\tilde{N}_f = \tilde{N}_f^{\min}(E)$ of the boundary are shown in Fig. 10 for the case $N_r = 600, R = 30$. We observe that they have very similar properties. Let us note that the time reversal symmetry translates a lower boundary solution with initial $\tilde{N}_i$ and final $\tilde{N}_f$ particle numbers into an upper boundary solution with the initial and final particle numbers interchanged.

Now we turn to discussion of numerical results for another values of initial particle number $\tilde{N}_i$. In Fig. 11 we show the upper $\tilde{N}_f(E)$ and lower $\tilde{N}_f(E)$ boundaries of the classically allowed region for $\tilde{N}_i = 0.1$ and relevant energy interval $[1.5\tilde{N}_i, 15.0\tilde{N}_i]$ with the values of $\tilde{E}_s$ defined by (4.1). We see that maximal relative difference of initial

![Figure 10](image)

Figure 10: The same as in Fig. 7 but for lower boundary solutions.
Figure 11: Classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 0.1; N_r = 600, R = 30$. The envelopes $\tilde{N}_f^{\text{min}}$ and $\tilde{N}_f^{\text{max}}$ are composed of the classical solutions with two spiky parts (thick red line) and with three spiky parts (dashed blue line). Dots show other branches of solutions which deliver some local minima to $F$ (or $\tilde{N}_f$).

and final particle numbers, i.e. $|\tilde{N}_f - \tilde{N}_i|/\tilde{N}_i$, is more than two order of magnitude smaller than for the case $\tilde{N}_i = 1$ for energies $\tilde{E} \lesssim 15 \tilde{N}_i$. Numerically, the particle number changes no more than 0.33% for the chosen energy range. In Fig. 12 we show examples of the initial wave packets and instantaneous particle number evolution for upper and lower boundary solutions for $\tilde{E} \approx 0.6$. The space interval for initial wavepacket $[r_1, r_2]$ is shown by dashed lines. Qualitative properties of these solutions are very similar to those obtained for $\tilde{N}_i = 1$. Namely, the solutions have a spiky part whose transition through interaction region results in actual change in particle number and a soft oscillating part. As in the previous case we obtain several branches of boundary solutions which differ by corresponding shifts in time.

Let us now turn to the case $\tilde{N}_i = 10$ in which we obtain numerical results concerning boundary solutions which are qualitatively different from those obtained at smaller values of $\tilde{N}_i = 0.1, 1.0$. In Fig. 13 boundary of the classically allowed region is shown with thick solid (red) and thick dashed (blue) lines for the energy interval $[1.5 \tilde{N}_i, 15.0 \tilde{N}_i]$ with the same set (4.1) of $\tilde{E}_*$. We find that although the form of the boundary is similar to that of for $\tilde{N}_i = 0.1, 1.0$, its upper and lower parts, i.e. $\tilde{N}_f^{\text{max}}(\tilde{E})$
$\tilde{N}(t)$

Figure 12: Initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for boundary solutions with $\tilde{N}_i = 0.1$ and $\tilde{E} \approx 0.6$. Upper and lower panels correspond to upper and lower boundary solutions respectively.

Figure 13: Classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 10.0; N_r = 600, R = 30.$
and $\tilde{N}_{min}^f(E)$, consist of two different branches of classical solutions. At energies lower than about 70 the boundary solutions have two distinct spiky parts. An example of initial field configuration $\phi(t = 0, r)$ corresponding to lower boundary is presented in Fig. 14 (upper left panel) for $\tilde{E} \approx 65$. The evolution of instantaneous particle number $\tilde{N}(t)$ as well as distribution of energy over the modes $\epsilon_k$ for initial and final field configurations are also shown in this figure on left panels. The oscillating pattern in $\epsilon_k$ indicates that the incoming (and outgoing) field configuration looks as a sum of two separated in space wavetrains each having similar smooth Fourier image. In particular, the distance $\delta r$ between the spikes in the initial configuration and the oscillation period $\delta k$ in the initial energy distribution are related approximately by $\delta r \cdot \delta k \approx 2\pi$ for all solutions of this branch. For example, the solution with $\tilde{E} \approx 65$ presented on left panels of Fig. 14 has $\delta r \approx 2.8$ and $\delta k \approx 2.3$. Comparing energy distributions $\epsilon_k$ for the field configurations at initial and final times one observes a small energy transfer from low to high frequency modes. At energies larger than about 70 our numerical results show that the absolute minimum (and maximum) of $\tilde{N}_f$ is delivered by different branch of solutions whose initial (and final) space field configuration contains already three spikes. On the right panels in Fig. 14 we present the initial field configuration $\phi(t = 0, r)$, time evolution of $\tilde{N}(t)$, and energy distributions over the Fourier modes $\epsilon_k$ for the initial and final field configurations for a boundary solution with $\tilde{E} \approx 80$ having three spikes. On this branch of solutions the initial and final field configurations looks as a sum of three separated in space wavetrains with similar Fourier image. As in the cases of smaller $\tilde{N}_i$, the instantaneous particle number $\tilde{N}(t)$ undergoes the most dramatic changes when spiky parts of the field configurations reach the interaction region. We find the two-spike solutions at $\tilde{E} \gtrsim 70$ and three-spike solutions at $\tilde{E} \lesssim 70$ are still represent local minima of $\tilde{N}_f$. To figure that out we take a two-spike lower boundary solution at $\tilde{E} \approx 65$ as a seed for our Monte-Carlo procedure with $E_\ast > 70$. To ensure that the field configuration does not jump to another (three-spike) branch we take relatively small value of $\alpha$ which governs the size of changes in the amplitudes $f_n$. In this way we find that two-spike branch of solutions at $\tilde{E} \gtrsim 70$ which is shown in Fig. 13 by thin red (solid) line. In similar way we find continuation of three-spike branch of the solutions at $\tilde{E} \lesssim 70$ which is shown by this blue (dashed) line in Fig. 13. Upper boundary solutions have very similar properties\(^1\). Let us note that in the chosen energy range the maximal changes in particle number reach values around 20%.

There have been also found another branches of solutions which deliver only local

\(^1\)As previously, for construction of the boundaries $\tilde{N}_f^{min}(E)$ and $\tilde{N}_f^{max}(E)$ we select solutions with the longest oscillating tale, although we obtained also time-shifted field configurations with the same number of the spiky parts.
Figure 14: From top to bottom: a) initial wave packet; b) evolution of the instantaneous particle number $\tilde{N}(t)$; c) distribution of energy over the modes $\epsilon_k$ per wave number for initial field configuration; d) the same as in c) but for final field configuration. Left and right panels correspond to the lower boundary solutions with $\tilde{E} \approx 65$ and 80, respectively.
extrema to $\tilde{N}_f$. In the linear regime these solutions look as a sum of two or three single-spike wavetrains which differ from those already described by somewhat different distances between the wavetrains. In Fig. 15 we show two examples of such solutions. Left panels show a solution of this type near the upper boundary with two spikes and

![Figure 15](https://example.com/figure15.png)

**Figure 15:** Initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for solutions with $\tilde{N}_i = 10$ and $\tilde{E} \approx 80$ corresponding to local minima of $\tilde{N}_f$. Upper and lower panels correspond to $\tilde{N}_f > \tilde{N}_i$ and $\tilde{N}_f < \tilde{N}_i$.

$\tilde{N}_f > \tilde{N}_i$. The distance between the wavetrains in the initial field configuration is larger than that of for the boundary two-spike solution. Similarly, right panels show an example of solution near the lower boundary delivering a local minima to $\tilde{N}_f$ with three spike. Again we observe somewhat larger distance between wavetrains in the initial field configuration, c.f. Fig. 14 (upper right panel). These two branches of solutions are shown in Fig. 13 by dots. Let us note, that for $\tilde{N}_i = 10$ we do not find any field configuration which would deliver (even local) extrema to the functional $\tilde{N}_f(E)$ and whose initial (final) wave packet would have a single spike only. We tried to find it on purpose by taking as a seed some initial field configuration which was obtained for $\tilde{N}_i = 1$ as a boundary solution which is of single-spike type and increasing $\tilde{N}_i$ by small steps to $\tilde{N}_i = 10$. We find that at intermediate values of $\tilde{N}_i$ around 5–6 the initial field configurations of the boundary configurations prefer to be splitted into more than one wavetrains to produce larger change in particle number $|\tilde{N}_f - \tilde{N}_i|$. 
The picture becomes even more complicated at larger $\tilde{N}_i$. In Fig. 16 we show our

![Graph](image)

**Figure 16**: Classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 30.0$. The lattice parameters are $R = 30, N_r = 600$ (solid blue line) and $R = 50, N_r = 1000$ (dashed red line).

numerical results for the boundary of classically allowed region for $\tilde{N}_i = 30$ taking $R = 30, N_r = 600$ (solid blue line). We observe that the difference $|\tilde{N}_f - \tilde{N}_i|$ does not exceed 22% in this case in the chosen energy interval. We find that the corresponding boundary solutions contain already 4–7 spikes in their initial (and final) wavetrains. Examples of the initial field configurations of the boundary solutions are shown in Fig. 17. At the same time we find that number of local extrema of $\tilde{N}_f$ increases dramatically at large $\tilde{N}_i$. Corresponding solutions have similar number of wavetrains but with different distances between them. This greatly complicates the task of finding the boundary of the classically allowed region because, on the one hand, many of our numerical runs get stuck in such local minima and, on the other, the values of $\tilde{N}_f$ at the same $\tilde{E}$ for different branches of near-boundary solutions are at some cases seem to be close to our numerical accuracy. For these reasons at present we are unable to classify in details the boundary solutions as it has been done for the cases with smaller $\tilde{N}_i$. In the Fig. 16 we present only the envelopes $\tilde{N}_f^{\text{min}}(\tilde{E})$ and $\tilde{N}_f^{\text{max}}(\tilde{E})$ of the classically allowed region. We performed an additional check by finding $\tilde{N}_f^{\text{max}}$ and $\tilde{N}_f^{\text{min}}$ for larger space interval $R = 50$ and taking $N_r = 1000$. Corresponding boundary is shown in Fig. 17 by dashed red line. From our numerical results we expect that number of closely
separated spiky parts in the wavepackets of boundary solutions continue to grow with the increase of $\tilde{N}_i$.

It is interesting to compare the classically allowed regions at different $\tilde{N}_i$. Firstly, let us see that if two classical solutions describing scattering of waves with different sets of parameters, i.e. $\tilde{N}_i^{(1)}, \tilde{N}_f^{(1)}, \tilde{E}^{(1)}$ and $\tilde{N}_i^{(2)}, \tilde{N}_f^{(2)}, \tilde{E}^{(2)}$, exists then there should also exist a solution which has energy and particle numbers equal to the following sums

$$N_i = N_i^{(1)} + N_i^{(2)}, \quad N_f = N_f^{(1)} + N_f^{(2)}, \quad E = E^{(1)} + E^{(2)}.$$  \hspace{1cm} (4.4)

Such a solution can be constructed explicitly by taking it as a sum of the individual solutions sufficiently separated in space-time. This observation, in particular, means that the width of the classically allowed region, i.e. $|\tilde{N}_f^{\text{max}} - \tilde{N}_i|$ or $|\tilde{N}_f^{\text{min}} - \tilde{N}_i|$ at fixed ratio $\tilde{E}/\tilde{N}_i$ should grow faster than a linear function with increase of initial particle number $\tilde{N}_i$. On the Fig. 18 we show$^2$ the dependence of quantity $|\tilde{N}_f^{\text{min}} - \tilde{N}_i|$ as a function of $\tilde{N}_i$ for two values of $\tilde{E}/\tilde{N}_i$. For comparison a linear function is shown in thin (blue) line. One can see that the energy dependence of $|\tilde{N}_f^{\text{min}} - \tilde{N}_i|$ tends to be linear at large $\tilde{N}_i$. We found that $|\tilde{N}_f^{\text{max}} - \tilde{N}_i|$ has similar behaviour. This can

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$^2$To plot this dependence we additionally found several extrema at the lower boundary for $\tilde{N}_i = 0.01$. 

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**Figure 17:** Initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for boundary solutions with $\tilde{N}_i = 30$ and $\tilde{E} \approx 200$. Upper and lower panels correspond to $\tilde{N}_f > \tilde{N}_i$ and $\tilde{N}_f < \tilde{N}_i$. 

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be considered as an indication on existence of some limiting boundary of classically allowed region in the plane \((\tilde{E}/\tilde{N}_i, \tilde{N}_f/\tilde{N}_i)\) at large \(\tilde{N}_i\) or in the plane \((\tilde{E}/\tilde{N}_f, \tilde{N}_i/\tilde{N}_f)\) at large \(\tilde{N}_f\) due to the symmetry \(\tilde{N}_i \leftrightarrow \tilde{N}_f\).

5 Discussions and conclusions

In this paper we considered classical scattering of wavepackets in the unbroken \(\lambda\phi^4\) theory. Initial and final field configurations were characterized by energy \(E\), initial \(N_i\) and final \(N_f\) particle numbers. We found that although maximal change of particle number in the scattering grows with energy at fixed value of \(N_i\) (or \(N_f\)), the value of \(|N_f - N_i| / N_i\) does not exceed 22\% for considered energy range \(E \lesssim 10N_f\, m\). In quantum counterpart of the problem the initial and final wavepackets correspond to initial and final coherent states with given average values of energy and particle number. Our classical approach to the multiparticle scattering is valid when occupation numbers are large, i.e. \(E, N_i, N_f \sim \frac{1}{\lambda}\) and \(\lambda \ll 1\). In this study we numerically obtained classically allowed regions of these processes in the \((E, N_f)\) plane at several values of \(N_i\) (or, equivalently, in the \((E, N_i)\) plane at several values of \(N_f\)). They are shown in Figs. 1, 11, 13 and 16. Here we considered spherically symmetric field configurations only. We believe that this simplified consideration has much in common with most general setup.

The scattering processes with the parameters \(E, N_i\) and \(N_f\) outside the classically allowed regions are classically forbidden. The common lore here is that their probability in quantum theory is expected to be exponentially suppressed, i.e. have the form \(\mathcal{A}e^{\frac{1}{\lambda} F}\) with some negative suppression exponent \(F\) and a prefactor \(\mathcal{A}\). At the same time the
probability of the classically allowed processes is not exponentially suppressed. In the most interesting case $2 \rightarrow N_f$ production, with $\lambda N_f \gtrsim 1$, the initial state contains semiclassically small number of particles. Our results show that such processes (as well as $N_f \rightarrow 2$ scattering) lie deeply in the classically forbidden region and therefore their probabilities are expected to be exponentially suppressed at least for $\frac{4\pi}{\lambda} N_f \lesssim 30$ and $E \lesssim 10 N_f$. Limitations of our numerical procedure do not permit us to obtain the boundary at classically allowed region for arbitrarily large values of $N_f$ and $E$. However, we obtain an evidence for existence of a limiting boundary the classically allowed region in $(E/N_i, N_f/N_i)$ plane at large $N_i$.

Although our results indicate that few $\rightarrow N_f$ scattering processes lie in the classically forbidden region, they tell nothing about actual value of the probability of these processes. As we already mentioned in the introduction there exist semiclassical methods to calculate the suppression exponent of the probability which, however, are difficult to apply due to singular nature of corresponding classical solutions. On the one hand, one can start, as we did in this paper, with processes $N_i \rightarrow N_f$ where both initial and final states contain large occupation numbers and apply the same semiclassical techniques. In this case based on the results of the semiclassical studies of baryon number violating processes [44, 45] and soliton pair production [46, 47] in particle collisions one can expect that corresponding classical solutions will be nonsingular. The classical solutions at the boundary of the classically allowed region obtained in the present study are expected to be close to the solutions describing transitions in the forbidden region. Using the Rubakov-Son-Tinyakov conjecture [48] the probability of $2 \rightarrow N_f$ scattering in the leading semiclassical approximation can be obtained from that of $N_i \rightarrow N_f$ process by taking the limit $\lambda N_i \rightarrow 0$ if it exists\(^3\). This procedure can be viewed as a regularization to the semiclassical method of singular classical solutions. We are going to pursue this idea in future work [55].

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\(^3\)This conjecture was checked in several models [49–52] and was used in Ref. [46, 47, 53, 54] for semiclassical calculations of probability of soliton pair production and false vacuum decay induced by particle collisions.
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