Existence of Doubly Periodic Vortices in a Generalized Chern–Simons Model

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Abstract. We establish an existence theorem for the doubly periodic vortices in a generalized self-dual Chern–Simons model. We show that there exists a critical value of the coupling parameter such that there exits self-dual doubly periodic vortex solutions for the generalized self-dual Chern–Simons equation if and only if the coupling parameter is less than or equal to the value. The energy, magnetic flux, and electric charge associated to the field configurations are all specifically quantized. By the solutions obtained for this generalized self-dual Chern–Simons equation we can also construct doubly periodic vortex solutions to a generalized self-dual Abelian Higgs equation.

1 Introduction

Vortices, which arise in spontaneous broken gauge theories in two-space dimension, play important roles in many areas of physics including superconductivity [12,17], optics [3], cosmology [13,15,28], and the quantum Hall effect [22]. In recent years much attention has been devoted to the study of vortices in (2+1)-dimensional Chern–Simons gauge theory. An important feature of such vortices is that they are both magnetically and electrically charged, which is different from the neutral Nielsen–Olesen vortices [20]. In the work of Hong, Kim, and Pac [15] and Jackiw and Weinberg [16], the Yang–Mills (or Maxwell) term is removed from the action Lagrangian density while the Chern–Simons term alone governs electromagnetism, which is physically sensible at large distances and low energies. When the Higgs potential takes a special form as that in the classical Abelian Higgs model [17], the static equations of motion can be reduced from a second-order differential to a Bogolmol’ny type (self-dual) system of first-order equations [4], which enables one to make rigorous mathematical studies of such solutions. In such a setting, topological multivortices with quantized charges [23,29], non-topological multivortics with fractional values of charges [7,9,24] and doubly periodic vortices with quantized charges [6,10,11,19,21,25,26] are all present.

In [5] Burzlaff, Chakrabarti, and Tchrakian proposed a generalized self-dual Chern–Simons–Higgs model and a generalized Abelian Higgs model. The non-topological and topological vortices for the models were established more than ten years ago in [27] and [31], respectively. However, up to now, the existence of doubly periodic vortices for the models is still open. Our purpose of this paper is to establish the existence of doubly periodic multivortices to the generalized self-dual Chern–Simons model. As in [27,31] we first reduce the generalized self-dual Chern–Simons equations into

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a scalar quasilinear elliptic equation with Dirac source terms characterizing the locations of the vortices. Then by a transformation the quasilinear elliptic equation can be simplified further into a semilinear one. We establish an existence theorem by applying a sub-super solution method, which was used by Caffarelli and Yang \[6\] to construct multiple doubly periodic vortex solutions to the Chern–Simons model proposed in \[15,16\].

The rest of our paper is organized as follows. In section 2, we formulate our problem and state our main results. In section 3 we prove the existence of doubly periodic vortices for the generalized self-dual Chern–Simons equation. In section 4 we construct a doubly periodic vortex solution for the generalized self-dual Abelian Higgs model using our results in the previous section.

## 2 Generalized Chern–Simons vortices

We consider the generalized self-dual Chern–Simons equations derived in \[5\] over a doubly periodic domain \(\Omega\) such that the field configurations are subject to the ’t Hooft boundary condition \[14,30,32\] under which periodicity is achieved modulo gauge transformations.

Following \[5\], we derive the generalized self-dual Chern–Simons equations. The Lagrangian density in \((2 + 1)\) dimensions reads

\[
\mathcal{L} = \sqrt{2\kappa\varepsilon^{\mu\nu\alpha}} \left[ A_\alpha - 2i \left( 1 - \frac{1}{2} |\phi|^2 \right) \phi \overline{D_\mu \phi} \right] F_{\mu\nu} + 2(1 - |\phi|^2)^2 |D_\mu \phi|^2 - V, \tag{2.1}
\]

where \(D_\mu = \partial_\mu + iA_\mu\) is the gauge-covariant derivative, \(A_\mu (\mu = 0, 1, 2)\) a 3-vector gauge field, \(\phi\) a complex scalar field called the Higgs field, \(F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\) the induced electromagnetic field, \(\alpha, \beta, \mu, \nu = 0, 1, 2, \kappa > 0\) a constant referred to as the Chern–Simons coupling parameter, \(\varepsilon^{\alpha\beta\gamma}\) the Levi-Civita totally skew-symmetric tensor with \(\varepsilon^{012} = 1\), \(V\) the Higgs potential function, and the summation convention over repeated indices is observed.

Varying \((2.1)\) with respect to \(A_\alpha\) and \(\phi\), we have the Euler-Lagrange equations

\[
\frac{1}{2\sqrt{2}}\varepsilon^{\mu\nu\alpha} \left[ (1 - |\phi|^2) F_{\mu\nu} - i \left( D_\mu \phi \overline{D_\nu \phi} - D_\nu \phi \overline{D_\mu \phi} \right) \right] + i(1 - |\phi|^2) \phi \overline{D_\alpha \phi} = 0, \tag{2.2}
\]

\[
2\sqrt{2}\kappa \varepsilon^{\mu\nu\alpha} D_\mu \left[ (1 - \frac{1}{2} |\phi|^2) \phi \right] F_{\mu\nu} + 2 \partial_\mu \left[ (1 - |\phi|^2)^2 \right] D_\mu \phi + 2(1 - |\phi|^2)^2 D^\mu D_\mu \phi - 4(1 - |\phi|^2) \phi |D_\mu \phi|^2 - \frac{\partial V}{\partial \phi} = 0. \tag{2.3}
\]

In the static limit, the \(\alpha = 0\) component of \((2.2)\) implies

\[
A_0 = -\frac{\kappa}{\sqrt{2}|\phi|^2 (1 - |\phi|^2)} \left[ (1 - |\phi|^2) F_{12} - i(D_1 \phi \overline{D_2 \phi} - D_1 \phi \overline{D_2 \phi}) \right]. \tag{2.4}
\]

From \((2.4)\), we can express the density of electric charge as

\[
\rho = -\sqrt{2}A_0 |\phi|^2 (1 - |\phi|^2) = \kappa \left[ (1 - |\phi|^2) F_{12} - i(D_1 \phi \overline{D_2 \phi} - D_1 \phi \overline{D_2 \phi}) \right]. \tag{2.5}
\]

Note that the energy \(E\) can be expressed as

\[
E = \int_\Omega dx \left[ 2(1 - |\phi|^2)^2 (|D_0 \phi|^2 + |D_1 \phi|^2 + |D_2 \phi|^2) + V \right].
\]
Theorem 2.1
Let $\kappa$ in the static limit we have
\[ \phi, A \]
Consequently, we have
\[ \phi, A \] and the lower bound is saturated if and only if (2.7)
Then we rewrite the energy as
\[ E = \int dx \left\{ \frac{\kappa^2}{|\phi|^2} \left[ (1 - |\phi|^2) F_{12} - i(D_1 \phi \bar{D}_2 \phi - \bar{D}_1 \phi D_2 \phi) \right] - \frac{1}{2\kappa} |\phi|(1 - |\phi|^2)^2 \right\} + 2(1 - |\phi|^2)^2 |D_1 \phi - iD_2 \phi|^2 + (1 - |\phi|^2)^2 F_{12} - 3i(1 - |\phi|^2)^2 (D_1 \phi \bar{D}_2 \phi - \bar{D}_1 \phi D_2 \phi) \}
\[ = \int dx \left\{ \frac{\kappa}{|\phi|} \left[ (1 - |\phi|^2) F_{12} - i(D_1 \phi \bar{D}_2 \phi - \bar{D}_1 \phi D_2 \phi) \right] - \frac{1}{2\kappa} |\phi|(1 - |\phi|^2)^2 \right\} + 2(1 - |\phi|^2)^2 |D_1 \phi - iD_2 \phi|^2 + F_{12} - 3i\varepsilon_{ij} \partial_i \left[ \left( 1 - |\phi|^2 + \frac{1}{3} |\phi|^4 \right) \phi \bar{D}_j \phi \right] \}
Consequently, we have
\[ E \geq \int \Omega F_{12} dx, \quad (2.6) \]
and the lower bound is saturated if and only if $(\phi, A)$ satisfies the self-dual equations
\[ D_1 \phi = iD_2 \phi, \quad (2.7) \]
\[ (1 - |\phi|^2) F_{12} = i(D_1 \phi \bar{D}_2 \phi - \bar{D}_1 \phi D_2 \phi) + \frac{1}{2\kappa^2} |\phi|^2 (1 - |\phi|^2)^2. \quad (2.8) \]
We aim to seek doubly periodic $N$-vortex solutions of (2.7) and (2.8) such that, $\phi$ vanishes at the arbitrarily prescribed points, $p_1, p_2, \ldots, p_m \in \Omega$ with multiplicities $n_1, n_2, \ldots, n_m$, respectively, and $\sum_{i=1}^{m} n_i = N$.
Our main result for the existence of periodic multiple vortices of (2.7) and (2.8) reads as follows.

**Theorem 2.1** Let $p_1, p_2, \ldots, p_m \in \Omega, n_1, n_2, \ldots, n_m$ be some positive integers and $N = \sum_{i=1}^{m} n_i$. There exists a critical value of the coupling parameter, say $\kappa_c$, satisfying
\[ 0 < \kappa_c \leq \sqrt{\frac{\Omega}{27\pi N}}, \]
such that the self-dual equations (2.7) and (2.8) admit a solution $(\phi, A)$ for which $p_1, p_2, \ldots, p_m$ are zeros of $\phi$ with multiplicities $n_1, n_2, \ldots, n_m$, if and only if $0 < \kappa \leq \kappa_c$.

The solution $(\phi, A)$ also satisfies the following properties.

The energy, magnetic flux, and electric charge are given by
\[ E = 2\pi N, \quad \Phi = 2\pi N, \quad Q = 2\kappa \pi N. \quad (2.9) \]

The solution $(\phi, A)$ can be chosen such that the magnitude of $\phi$, $|\phi|$ has the largest possible values.
Let the prescribed data be denoted by \( S = \{p_1, p_2, \ldots, p_m; n_1, n_2, \ldots, n_m\} \), where \( n_i \) may be zero for \( i = 1, \ldots, m \), and denote the dependence of \( \kappa_c \) on \( S \) by \( \kappa_c(S) \). For \( S' = \{p_1, p_2, \ldots, p_m; n'_1, n'_2, \ldots, n'_m\} \), we denote \( S \leq S' \) if \( n_1 \leq n'_1, \ldots, n_m \leq n'_m \). Then \( \kappa_c \) is a decreasing function of \( S \) in the sense that

\[
\kappa_c(S) \geq \kappa_c(S'), \quad \text{if} \quad S \leq S'.
\]

(2.10)

### 3 Existence of doubly periodic vortices

Following [31], we first rewrite the equations (2.7) and (2.8) as a quasilinear elliptic equation with the Dirac source terms.

Using (2.7), we have

\[
i(D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) = -(|D_1 \phi|^2 + |D_2 \phi|^2)
\]

Then we can rewrite (2.8) in the form

\[
(1 - |\phi|^2) F_{12} = -(|D_1 \phi|^2 + |D_2 \phi|^2) + \frac{1}{2\kappa^2} |\phi|^2 (1 - |\phi|^2)^2.
\]

(3.1)

We complexify the variables

\[
z = x^1 + ix^2, \quad A = A_1 + iA_2.
\]

Let

\[
\partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2).
\]

Then by (2.7), we can get, away from the zeros of \( \phi \),

\[
F_{12} = -2\bar{\partial} \partial \ln |\phi|^2 = -\frac{1}{2} \Delta \ln |\phi|^2.
\]

(3.2)

Introduce the real variable \( u = \ln |\phi|^2 \). A direct computation leads to

\[
|D_1 \phi|^2 + |D_2 \phi|^2 = \frac{1}{2} e^u |\nabla u|^2.
\]

(3.3)

Counting all the multiplicities of the zeros of \( \phi \), we write the prescribed zero set as \( Z(\phi) = \{p_1, \ldots, p_N\} \). Inserting (3.3) into (3.2), the equations (2.7) and (2.8) are transformed into the following quasilinear elliptic equation

\[
(1 - e^u) \Delta u - e^u |\nabla u|^2 = -\lambda e^u(e^u - 1)^2 + 4\pi \sum_{s=1}^{N} \delta_{p_s} \quad \text{in} \quad \Omega,
\]

(3.4)

where

\[
\lambda = \frac{1}{\kappa^2},
\]

and \( \delta_p \) is the Dirac distribution centered at \( p \in \Omega \).

Conversely, if \( u \) is a solution of (3.4), we can obtain a solution of (2.7)-(2.8) according to the transformation

\[
\phi(z) = \exp \left( \frac{1}{2} u(z) + i \sum_{s=1}^{N} \text{arg}(z - p_s) \right),
\]

\[
A_1(z) = -2\text{Re}\{i\bar{\partial} \ln \phi\}, \quad A_2(z) = -2\text{Im}\{i\bar{\partial} \ln \phi\}.
\]

(3.5)

(3.6)

Hence it is sufficient to solve (3.4). Indeed we can establish the following existence result for (3.4).
Theorem 3.1 For any prescribed points \( p_1, \ldots, p_N \in \Omega \), there is a critical value of \( \lambda \), say \( \lambda_c \), satisfying
\[
\lambda_c \geq \frac{27\pi N}{|\Omega|},
\]
such that, the equation (3.4) has a negative solution if and only if \( \lambda \geq \lambda_c \). Moreover, there holds the quantized integral
\[
\lambda \int_{\Omega} e^u(e^u - 1)^2 \, dx = 4\pi N. \tag{3.7}
\]

Let the prescribed data be denoted by \( S = \{p_1, p_2, \ldots, p_m; n_1, n_2, \ldots, n_m\} \), where \( n_i \) may be zero for \( i = 1, \ldots, m \), and denote the dependence of \( \lambda_c \) on \( S \) by \( \lambda_c(S) \). For \( S' = \{p_1, p_2, \ldots, p_m'; n'_1, n'_2, \ldots, n'_m\} \), we denote \( S \leq S' \) if \( n_1 \leq n'_1, \ldots, n_m \leq n'_m \). Then \( \lambda_c \) is an increasing function of \( S \) in the sense that
\[
\lambda_c(S) \leq \lambda_c(S'), \quad \text{if} \quad S \leq S'. \tag{3.8}
\]

By Theorem 3.1, to complete the proof of Theorem 2.1, we just need to compute the energy, magnetic flux and electric charge associated to the field configurations \( (\phi, A) \). Let \( u \) be a solution of (3.4) obtained in Theorem 3.1. Then \( (\phi, A) \) defined by (3.5) and (3.6) is a \( N \)-vortex solution of (2.7) and (2.8).

By (2.6), (2.7), and (2.8), we have
\[
E = \Phi = \int_{\Omega} F_{12} \, dx = -\frac{1}{2} \int_{\Omega} \Delta u \, dx = -\frac{1}{2} \lim_{r \to 0} \int_{\Omega \cup \bigcup_{j=1}^N B_r(p_j)} \nabla \cdot \nabla u \, dx
\]
\[
= \frac{1}{2} \sum_{j=1}^N \lim_{r \to 0} \int_{\partial B_r(p_j)} (-\partial_2 u dx^1 + \partial_1 u dx^2).
\tag{3.9}
\]

where \( B_r(p_j) \) is the disc in \( \Omega \) centered at \( p_j \) with radius \( r > 0 \) (\( j = 1, \ldots, N \)).

Noting that near the the point \( p_j \), we have the expression
\[
u(x) = \ln |x - p_j|^2 + f_j(x), \quad f_j \in C^\infty(B_r(p_j)), \quad j = 1, \ldots, N, \tag{3.10}
\]
where \( r > 0 \) is small. Then, plugging (3.10) into (3.9), we can obtain
\[
E = \Phi = 2\pi N. \tag{3.11}
\]

From (2.5), the density of the electric charge can be expressed as
\[
\rho = \kappa \left[ (1 - |\phi|^2) F_{12} + (|D_1 \phi|^2 + |D_2 \phi|^2) \right] = -\frac{\kappa}{2}(1 - e^u)\Delta u + \frac{\kappa}{2}e^u|\nabla u|^2.
\]
Therefore, by (3.10), the electric charge is
\[
Q = \int_{\Omega} \rho \, dx
\]
\[
= -\frac{\kappa}{2} \int_{\Omega} \nabla \cdot [(1 - e^u)\nabla u] \, dx
\]
\[
= \frac{\kappa}{2} \sum_{j=1}^N \lim_{r \to 0} \int_{\partial B_r(p_j)} (1 - e^u)(-\partial_2 u dx^1 + \partial_1 u dx^2) = 2\kappa \pi N. \tag{3.12}
\]

From (3.11) and (3.12), we obtain (2.9), which says that the energy, magnetic flux, and electric charge are all quantized.

In what follows we only need to prove Theorem 3.1. To simplify the problem further, we first derive an a priori estimate for the solutions of (3.4).
Lemma 3.1 If \( u \) is a solution to (3.4), then \( u \) is negative throughout \( \Omega \).

Proof. Denote \( B_\varepsilon(p_j) = \{ x \in \Omega, |x - p_j| < \varepsilon \} \), and
\[
\Omega_\varepsilon = \Omega \setminus \bigcup_{j=1}^{N} B_\varepsilon(p_j). \tag{3.13}
\]
We see that \( u \) is negative on \( \partial \Omega_\varepsilon \) when \( \varepsilon \) is sufficiently small. Noting that
\[
(1 - e^u) \Delta u + \lambda e^u (e^u - 1)^2 = e^u |\nabla u|^2 \geq 0 \quad \text{in} \quad \Omega \setminus \{ p_1, \ldots, p_N \},
\]
by the maximum principle, we obtain \( u < 0 \) in \( \Omega_\varepsilon \). Then we have \( u < 0 \) in \( \Omega \).

Then by Lemma 3.1 to solve (3.4), we just need to consider the negative solutions to (3.4). Since (3.4) is quasilinear, it is difficult to deal with directly. Therefore, as in [27,31], we consider a new dependent variable \( v \) defined by
\[
v = F(u) = 1 + u - e^u. \tag{3.14}
\]
It is easy to see that \( F'(t) = 1 - e^t \), \( F''(t) = -e^t < 0 \). Then \( F(\cdot) \) is increasing and invertible over \( (-\infty,0] \). Denoting the inverse of \( F \) over \( (-\infty,0] \) by \( G \), we see that both \( F \) and \( G \) are \( 1 \sim 1 \) from \( (-\infty,0] \) to itself.

In view of the fact that solutions to the equation (3.4) are all negative, the equation (3.4) is equivalent to the following semilinear equation,
\[
\Delta v = -\lambda e^{G(v)} (e^{G(v)} - 1)^2 + 4\pi N \sum_{j=1}^{N} \delta_{p_j} \quad \text{in} \quad \Omega. \tag{3.15}
\]
Then we just need to seek negative solutions to (3.15).

Let \( v_0 \) be a solution of the equation (see [2])
\[
\Delta v_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{N} \delta_{p_j}. \tag{3.16}
\]
Setting \( v = v_0 + w \), then the equation (3.15) is reduced to the following equation,
\[
\Delta w = -\lambda e^{G(v_0+w)} (e^{G(v_0+w)} - 1)^2 + \frac{4\pi N}{|\Omega|} \quad \text{in} \quad \Omega. \tag{3.17}
\]
In the sequel we just need to consider (3.17).

We easily see that the function \( f(t) = -e^t(e^t - 1)^2, t \in (-\infty,0] \), has a unique minimal value \(-\frac{4}{27}\). If \( w \) is a solution of (3.17), then \( v_0 + w < 0 \). Hence we have
\[
\Delta w \geq -\frac{4}{27}\lambda + \frac{4\pi N}{|\Omega|}. \tag{3.18}
\]
Then integrating (3.18) over \( \Omega \), we can obtain
\[
\lambda \geq \frac{27\pi N}{|\Omega|}, \tag{3.19}
\]
which is a necessary condition for the existence of solutions to \( (3.17) \).

As in [6] or Chapter 5 in [32] we can use a super-sub solution method to establish the existence results for \( (3.17) \).

It is easy to see that \( w^* = -v_0 \) is a supersolution to \( (3.17) \) in the distributional sense.

Then, in order to solve \( (3.17) \), we introduce the following iterative scheme

\[
\begin{cases}
(\Delta - K)w_n = -\lambda e^{G(v_0 + w_{n-1})}(e^{G(v_0 + w_{n-1})} - 1)^2 - K w_{n-1} + \frac{4\pi N}{|\Omega|}, \\
n = 1, 2, \ldots, \\
w_0 = -v_0,
\end{cases}
\]

(3.20)

where \( K \) is a positive constant to be determined.

**Lemma 3.2** Let \( \{w_n\} \) be the sequence defined by (3.20) with \( K > 2\lambda \). Then

\[
w_0 > w_1 > w_2 > \cdots > w_n > \cdots > w^*
\]

(3.21)

for any subsolution \( w_* \) of \( (3.17) \). Therefore, if \( (3.17) \) has a subsolution, the sequence \( \{w_n\} \) converges to a solution of \( (3.17) \) in the space \( C^k(\Omega) \) for any \( k \geq 0 \) and such a solution is the maximal solution of the equation \( (3.17) \).

**Proof.** We prove by (3.21) by induction.

When \( n = 1 \), from (3.20) we have,

\[(\Delta - K)w_1 = K v_0 + \frac{4\pi N}{|\Omega|},\]

which implies \( w_1 \in C^\infty(\Omega) \cap C^\alpha(\Omega) \) for some \( 0 < \alpha < 1 \). Noting that

\[(\Delta - K)(w_1 - w_0) = 0 \quad \text{in} \quad \Omega \setminus \{p_1, p_2, \ldots, p_N\},\]

and \( w_1 - w_0 < 0 \) on \( \partial \Omega_\varepsilon \), where \( \Omega_\varepsilon \) is defined by (3.13) for \( \varepsilon \) sufficiently small, and using the maximum principle, we have \( w_1 - w_0 < 0 \) in \( \Omega_\varepsilon \). Hence we obtain \( w_1 < w_0 \) in \( \Omega \).

Suppose that \( w_0 > w_1 > \cdots > w_k \). It follows from (3.20) and \( K > 2\lambda \) that

\[
\begin{align*}
(\Delta - K)(w_{k+1} - w_k) &= -\lambda \left[ e^{G(v_0 + w_k)}(e^{G(v_0 + w_k)} - 1)^2 - e^{G(v_0 + w_{k-1})}(e^{G(v_0 + w_{k-1})} - 1)^2 \right] - K (w_k - w_{k-1}) \\
&= \left[ \lambda e^{G(v_0 + \xi)}(3e^{G(v_0 + \xi)} - 1) - K \right] (w_k - w_{k-1}) \\
&\geq (2\lambda - K)(v_k - v_{k-1}) \\
&\geq 0,
\end{align*}
\]

where we have used the mean value theorem, \( w_k \leq \xi \leq w_{k-1} \). Applying the maximum principle again, we obtain \( w_{k+1} < v_k \) in \( \Omega \).

Now we prove the lower bound in (3.21) in terms of the subsolution \( w_* \) of \( (3.17) \). That is, \( w_* \in C^2(\Omega) \) and

\[
\Delta w_* \geq -\lambda e^{G(v_0 + w_*)}(e^{G(v_0 + w_*)} - 1)^2 + \frac{4\pi N}{|\Omega|},
\]

(3.22)
Noting that \( w_0 = -v_0 \) and (3.22), we have
\[
\Delta (w^* - w_0) \geq -\lambda e^{G(v_0 + w_0)} \left( e^{G(v_0 + w_0)} - 1 \right)^2
\]
\[
= 2\lambda e^{G(v_0 + w_0)} e^{G(\xi - w_0)} (w^* - w_0)
\]
in \( \Omega \setminus \{ p_1, \ldots, p_N \} \),
where \( \xi \) lies between \( w^* \) and \( w_0 \). If \( \varepsilon > 0 \) is small, we see that \( w^* - w_0 < 0 \) on \( \partial \Omega_\varepsilon \). Then, by the maximum principle, we obtain \( w^* - w_0 < 0 \) in \( \Omega_\varepsilon \). Therefore, \( w^* < w_0 \) throughout \( \Omega \).

Now assume \( w^* < w_k \) for some \( k \geq 0 \). It follows from (3.20), (3.22), and the fact \( K > 2\lambda \) that
\[
\Delta (w^* - w_{k+1}) \geq -\lambda \left[ e^{G(v_0 + w_*)} \left( e^{G(v_0 + w_*)} - 1 \right)^2 - e^{G(v_0 + w_k)} \left( e^{G(v_0 + w_k)} - 1 \right)^2 \right] - K (w^* - w_k)
\]
\[
= \left[ 2\lambda - K \right] (w^* - w_k)
\]
\[
\geq 0,
\]
where \( w_0 \leq \xi \leq w_k \). Using the maximum principle again, we get \( w^* < w_{k+1} \). Then we get (3.21).

Following a standard bootstrap argument, we can obtain the convergence of the sequence \( \{ v_n \} \) in any \( C^k(\Omega) \).

In the sequel we only need to construct a subsolution of (3.17). Indeed, we can establish the following lemma.

**Lemma 3.3** If \( \lambda > 0 \) is sufficiently large, the equation (3.17) admits a subsolution satisfying (3.22).

**Proof.** Take \( \varepsilon > 0 \) sufficiently small such that the balls
\[
B(p_j, 2\varepsilon) = \{ x \in \Omega \mid |x - p_j| < 2\varepsilon \}, \ j = 1, 2, \ldots, N,
\]
verify \( B(p_i, 2\varepsilon) \cap B(p_j, 2\varepsilon) = \emptyset \), if \( i \neq j \). Let \( f_\varepsilon \) be a smooth function defined on \( \Omega \) such that \( 0 \leq f_\varepsilon \leq 1 \) and
\[
f_\varepsilon = \begin{cases} 
1, & x \in B(p_j, \varepsilon), \ j = 1, 2, \ldots, N, \\
0, & x \notin \bigcup_{j=1}^{N} B(p_j, 2\varepsilon), \\
\text{smooth connection}, & \text{elsewhere}.
\end{cases}
\]

Then,
\[
\bar{f}_\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} f_\varepsilon \, dx \leq \frac{4\pi N \varepsilon^2}{|\Omega|}, \quad (3.23)
\]
Define
\[
g_\varepsilon = \frac{8\pi N}{|\Omega|} (f_\varepsilon - \bar{f}_\varepsilon).
\]
It is easy to see that
\[
\int_{\Omega} g_\varepsilon \, dx = 0.
\]
Then we see that the linear elliptic equation
\[
\Delta w = g_\varepsilon \quad \text{in} \quad \Omega, \quad (3.24)
\]
admits a unique solution up to an additive constant.

When \( x \in B(p_j, \varepsilon) \) (\( j = 1, 2, \cdots, N \)), it follows from (3.23) that
\[
g_\varepsilon \geq \frac{4\pi N}{|\Omega|} \left( 2 - \frac{8\pi N \varepsilon^2}{|\Omega|} \right) > \frac{4\pi N}{|\Omega|},
\]
if \( \varepsilon \) is sufficiently small. In the sequel we fix \( \varepsilon \) such that (3.25) is valid.

Now we choose a solution of (3.24), say \( w \), to satisfy
\[
v_0 + w \leq 0, \quad x \in \Omega.
\]
Hence, for any \( \lambda > 0 \), we have
\[
\Delta w = g_\varepsilon > \frac{4\pi N}{|\Omega|} \geq -\lambda e^{G(v_0+w)}(e^{G(v_0+w)} - 1)^2 + \frac{4\pi N}{|\Omega|},
\]
for \( x \in B(p_j, \varepsilon) \), \( j = 1, 2, \cdots, N \).

Finally, set
\[
\begin{align*}
\mu_0 &= \inf \left\{ e^{G(v_0+w)} \bigg| x \in \Omega \setminus \bigcup_{j=1}^{N} B(p_j, \varepsilon) \right\}, \\
\mu_1 &= \sup \left\{ e^{G(v_0+w)} \bigg| x \in \Omega \setminus \bigcup_{j=1}^{N} B(p_j, \varepsilon) \right\}.
\end{align*}
\]
Then \( 0 < \mu_0 < \mu_1 \) and
\[
-e^{G(v_0+w)}(e^{G(v_0+w)} - 1)^2 \leq -\mu_0(1 - \mu_1)^2 \quad \text{for} \quad x \in \Omega \setminus \bigcup_{j=1}^{N} B(p_j, \varepsilon).
\]
Therefore, noting the boundedness of \( g_\varepsilon \), we have
\[
\Delta w = g_\varepsilon \geq -\lambda e^{G(v_0+w)}(e^{G(v_0+w)} - 1)^2 + \frac{4\pi N}{|\Omega|} \quad \text{for} \quad x \in \Omega \setminus \bigcup_{j=1}^{N} B(p_j, \varepsilon),
\]
if we take \( \lambda \) large enough.
Hence from (3.26) and (3.27) we infer that \( w \) is a subsolution to (3.17) if \( \lambda \) is sufficiently large.

**Lemma 3.4** There is a critical value of \( \lambda \), say \( \lambda_c \), satisfying
\[
\lambda_c \geq \frac{27\pi N}{|\Omega|},
\]
such that, for \( \lambda > \lambda_c \), the equation (3.17) has a solution, while for \( \lambda < \lambda_c \), the equation (3.17) has no solution.

**Proof.** Assume that \( w \) is a solution of (3.17). Then \( v = v_0 + w \) satisfies (3.15) and is negative throughout \( \Omega \).

Define
\[
\Lambda = \{ \lambda > 0 \mid \lambda \text{ is such that (3.17) has a solution} \}.
\]
Then we can prove that $\Lambda$ is an interval. To this end, we prove that, if $\lambda' \in \Lambda$, then $[\lambda', +\infty) \subset \Lambda$. Denote by $w'$ the solution of (3.17) at $\lambda = \lambda'$. Noting that $v_0 + w' < 0$, we see that $w'$ is a subsolution of (3.17) for any $\lambda > \lambda'$. By Lemma 3.2, we obtain a solution of (3.17) for any $\lambda > \lambda'$.

Hence $[\lambda', +\infty) \subset \Lambda$.

Set $\lambda_c = \inf \Lambda$. Then, by (3.19), we have $\lambda > \frac{27\pi N}{|\Omega|}$ for any $\lambda > \lambda_c$. Taking the limit $\lambda \to \lambda_c$, we obtain (3.25).

Let $w$ be a solution of (3.17) we have just obtained. Then $v = v_0 + w$ is a solution to (3.15) and $v = G(v)$ is a solution to (3.4). Hence, integrating (3.17) over $\Omega$, we have

$$\lambda \int_{\Omega} e^{G(v_0 + w)}(e^{G(v_0 + w)} - 1)^2 \, dx = 4\pi N,$$

which implies (3.7).

Now we consider the critical case $\lambda = \lambda_c$. We use the method of [25] to deal with this. We first show that the solution of (3.17) is monotonic with respect to $\lambda$.

**Lemma 3.5** The maximum solutions of (3.17), $\{w_\lambda | \lambda > \lambda_c\}$, are a monotone family in the sense that $w_{\lambda_1} > w_{\lambda_2}$ whenever $\lambda_1 > \lambda_2 > \lambda_c$.

**Proof.** Let $w_\lambda$ be a solution of (3.17) obtained. Then we have $v_0 + w_\lambda < 0$. By the equation (3.17) we obtain

$$\Delta w_{\lambda_2} = -\lambda_2 e^{G(v_0 + w_{\lambda_2})}(e^{G(v_0 + w_{\lambda_2})} - 1)^2 + \frac{4\pi N}{|\Omega|} + \int_{\Omega} \Delta w_{\lambda_2} = -\lambda_1 e^{G(v_0 + w_{\lambda_2})}(e^{G(v_0 + w_{\lambda_2})} - 1)^2 + \frac{4\pi N}{|\Omega|} + (\lambda_2 - \lambda_1) e^{G(v_0 + w_{\lambda_2})}(e^{G(v_0 + w_{\lambda_2})} - 1)^2 \geq -\lambda_1 e^{G(v_0 + w_{\lambda_2})}(e^{G(v_0 + w_{\lambda_2})} - 1)^2 + \frac{4\pi N}{|\Omega|}$$

for $\lambda_1 > \lambda_2 > \lambda_c$. Hence $w_{\lambda_2}$ is a subsolution of (3.17) with $\lambda = \lambda_1$. Then by the maximum principle, we have $w_{\lambda_1} > w_{\lambda_2}$ if $\lambda_1 > \lambda_2 > \lambda_c$.

Next we show that solutions to (3.17) are all bounded in $W^{1,2}(\Omega)$. We know that $W^{1,2}(\Omega)$ can be decomposed as

$$W^{1,2}(\Omega) = \mathbb{R} \oplus X,$$

where

$$X = \left\{ v \in W^{1,2}(\Omega) \left| \int_{\Omega} v \, dx = 0 \right. \right\}$$

is a closed subspace of $W^{1,2}(\Omega)$. In other words, for any $v \in W^{1,2}(\Omega)$, there exits a unique number $c \in \mathbb{R}$ and $v' \in X$ such that $v = c + v'$.

**Lemma 3.6** Let $w_\lambda$ be a solution of (3.17). Then $w_\lambda = c_\lambda + w'_\lambda$, where $c_\lambda \in \mathbb{R}$ and $w'_\lambda \in X$. We have

$$\|\nabla w'_\lambda\|_2 \leq C\lambda,$$

where $C$ is a positive constant depending only on the size of the domain $\Omega$. Furthermore, $\{c_\lambda\}$ satisfies the estimate

$$|c_\lambda| \leq C(1 + \lambda + \lambda^2).$$

Especially, $w_\lambda$ satisfies

$$\|w_\lambda\|_{W^{1,2}(\Omega)} \leq C(1 + \lambda + \lambda^2).$$
Proof. Noting that
\[ v_0 + w_\lambda = v_0 + c_\lambda + w'_\lambda < 0, \] (3.32)
then multiplying the equation (3.17) by \( v'_\lambda \), integrating over \( \Omega \), using the Hölder inequality and the Poincaré inequality, we can obtain
\[ \|\nabla w'_\lambda\|_2^2 = \lambda \int_\Omega e^{G(v_0 + w_\lambda)}(e^{G(v_0 + w_\lambda)} - 1)^2 w'_\lambda dx \leq \lambda \int_\Omega |w'_\lambda| dx \leq \lambda |\Omega|^{1/2}\|w'_\lambda\|_2 \leq C\lambda\|\nabla w'_\lambda\|_2, \]
which implies (3.29).

Using (3.32) again, we get an upper bound for \( c_\lambda \),
\[ c_\lambda < -\frac{1}{|\Omega|} \int_\Omega v_0(x)dx. \] (3.33)

Now we show that \( c_\lambda \) is also bounded from below. In view of (3.32), it follows from the equation (3.17) that
\[ \Delta w_\lambda + \lambda e^{G(v_0 + w_\lambda)}(1 - e^{G(v_0 + w_\lambda)}) \geq \frac{4\pi N}{|\Omega|}. \]
Integrating the above inequality over \( \Omega \), we have
\[ \lambda \int_\Omega e^{G(v_0 + w_\lambda)} dx \geq \lambda \int_\Omega e^{2G(v_0 + w_\lambda)} dx + 4\pi N > 4\pi N, \]
which leads to
\[ \lambda \int_\Omega e^{G(v_0 + w_\lambda)} dx > 4\pi N. \] (3.34)

Noting that the function \( G(t) \) is an increasing function which maps \( (-\infty, 0) \) to itself with
\[ \lim_{t \to -\infty} G(t) = -\infty. \]

Then we have
\[ \lim_{t \to -\infty} \frac{G(t)}{t} = \lim_{t \to -\infty} \frac{G'(t)}{t} = \lim_{t \to -\infty} \frac{1}{1 - e^{G(t)}} = 1 \]
hence, there exists a positive constant \( M \) such that
\[ G(t) \leq t + 1 \quad \text{as} \quad t < -M. \] (3.35)

Since \( v_0 + w_\lambda < 0 \) in \( \Omega \), we decompose \( \Omega \) as
\[ \Omega = \Omega_1 \cup \Omega_2, \]
where
\[ \Omega_1 = \{ x \in \Theta \mid v_0 + w_\lambda < -M \}, \quad \Omega_2 = \{ x \in \Theta \mid -M \leq v_0 + w_\lambda < 0 \}. \] (3.36)

Hence, by (3.35), (3.36), the Hölder inequality, and Trudinger–Moser inequality (see [2]),
\[ \int_\Omega e^{v'} dx \leq C_1 \int_\Omega e^{C_2\|\nabla v'\|_2^2} dx, \quad \forall v' \in X, \]
where $C_1$ and $C_2$ are positive constants, we obtain
\[
\int_{\Omega} e^{G(v_0 + w_\lambda)} dx = \int_{\Omega_1} e^{G(v_0 + w_\lambda)} dx + \int_{\Omega_2} e^{G(v_0 + w_\lambda)} dx \\
\leq \int_{\Omega_1} e^{v_0 + w_\lambda - 1} dx + |\Omega| \\
\leq ee^{C_\lambda} \int_{\Omega} e^{v_0 + w_\lambda} dx + |\Omega| \\
\leq ee^{C_\lambda} \left( \int_{\Omega} e^{2v_0} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} e^{2w_\lambda} dx \right)^{\frac{1}{2}} + |\Omega| \\
\leq C_1 e^{C_\lambda} \exp(C_2 \|\nabla w_\lambda\|_2^2) + |\Omega|. \tag{3.37}
\]

Then from (3.34), (3.37), and (3.29), we obtain a lower bound for $c_\lambda$,
\[
c_\lambda \geq -C(1 + \lambda + \lambda^2). \tag{3.38}
\]

Consequently, (3.30) follows from (3.33) and (3.38). Combining (3.29), and (3.30), we obtain (3.31).

**Lemma 3.7** The set of $\lambda$ for which the equation (3.17) has a solution is a closed interval. In other words, at $\lambda = \lambda_c$, (3.17) has a solution as well.

**Proof.** For $\lambda_c < \lambda < \lambda_c + 1$ (say), by Lemma 3.6 the set $\{w_\lambda\}$ is uniformly bounded in $W^{1,2}(\Omega)$. Noting the monotonicity of $\{w_\lambda\}$ with respect to $\lambda$ in Lemma 3.5 we conclude that there exist a function $\tilde{w} \in W^{1,2}(\Omega)$ such that
\[
w_\lambda \to \tilde{w} \text{ weakly in } W^{1,2}(\Omega) \text{ as } \lambda \to \lambda_c,
\]
and
\[
v_0 + \tilde{w} < 0 \text{ in } \Omega. \tag{3.39}
\]

Therefore $w_\lambda \to \tilde{w}$ strongly in $L^p(\Omega)$ for any $p \geq 1$ as $\lambda \to \lambda_c$.

Define
\[
g(t) = e^{G(t)}(e^{G(t)} - 1)^2, \quad t \in (-\infty, 0].
\]

It is easy to see that
\[
g'(t) = -e^{G(t)}(3e^{G(t)} - 1).
\]

Since $G(t) < 0$ for all $t < 0$, we have
\[
|g'(t)| \leq 2 \text{ for all } t < 0. \tag{3.40}
\]

Hence, in view of $v_0 + w_\lambda < 0$, (3.39), (3.40), and the fact that $w_\lambda \to \tilde{w}$ strongly in $L^p(\Omega)$ for any $p \geq 1$ as $\lambda \to \lambda_c$, we infer that
\[
e^{G(v_0 + w_\lambda)}(e^{G(v_0 + w_\lambda)} - 1)^2 \text{ converges to } e^{G(v_0 + \tilde{w})}(e^{G(v_0 + \tilde{w})} - 1)^2,
\]

strongly in $L^p(\Omega)$ for any $p \geq 1$ as $\lambda \to \lambda_c$. Using this result in (3.17) and the elliptic $L^2$-estimates, we see that $\tilde{w} \in W^{2,2}(\Omega)$ and $w_\lambda \to \tilde{w}$ strongly in $W^{2,2}(\Omega)$ as $\lambda \to \lambda_c$. Particularly, taking the limit $\lambda \to \lambda_c$ in (3.17), we obtain that $\tilde{w}$ is a solution of (3.17) for $\lambda = \lambda_c$. 

\[\text{12}\]
Finally we show the last statement of Theorem 3.1.

Denote

\[ S = \{p_1, \cdots, p_m; n_1, n_2, \cdots, n_m\}, \quad S' = \{p_1, \cdots, p_m; n'_1, n'_2, \cdots, n'_m\}. \]

We denote the dependence of \( \lambda_c \) on \( S \) by \( \lambda_c(S) \). Consider the equation

\[
(1 - e^u)\Delta u - e^u|\nabla u|^2 = -\lambda e^u(e^u - 1)^2 + 4\pi \sum_{j=1}^{m} n_j \delta_{p_j} \quad \text{in} \quad \Omega. \tag{3.41}
\]

As before, setting \( v = F(u) = 1 + u - e^u \), the equation (3.41) is equivalent to

\[
\Delta v = -\lambda e^{G(v)}(e^{G(v)} - 1)^2 + 4\pi \sum_{j=1}^{m} n_j \delta_{p_j} \quad \text{in} \quad \Omega. \tag{3.42}
\]

**Lemma 3.8** If \( S \leq S' \), then \( \lambda(S) \leq \lambda(S') \).

**Proof.** It is sufficient to prove that, if \( \lambda > \lambda_c(S') \), then \( \lambda \geq \lambda_c(S) \). Let \( v' \) be a solution of (3.42) with \( n_j = n'_j, j = 1, \cdots, m \) and \( v_0 \) satisfy

\[
\Delta v_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{m} n_j \delta_{p_j},
\]

where \( N = n_1 + \cdots + n_m \). Setting \( v' = v_0 + w_\omega \) we have

\[
\Delta w_\omega = -\lambda e^{G(v_0+w_\omega)}(e^{G(v_0+w_\omega)} - 1)^2 + 4\pi \frac{N}{|\Omega|} + 4\pi \sum_{j=1}^{m} (n'_j - n_j) \delta_{p_j}
\]

\[
\leq -\lambda e^{G(v_0+w_\omega)}(e^{G(v_0+w_\omega)} - 1)^2 + 4\pi \frac{N}{|\Omega|},
\]

in the distributional sense, which implies in particular that \( w_\omega \) is a subsolution of (3.17) in the sense of distribution and (3.21) holds pointwise. It is easy to check that the singularity of \( w_\omega \) is at most of the type \( \ln |x - p_j| \). Hence, the inequality (3.21) still results in the convergence of the sequence of \( \{w_n\} \) to a solution of (3.17) in any \( C^k \)-norm. Indeed, by (3.21), we see that \( \{w_n\} \) converges almost everywhere and is bounded in the \( L^2 \)-norm. Therefore, the sequence converges in \( L^2(\Omega) \). Similarly, we see that the right-hand side of (3.20) also converges in \( L^2(\Omega) \). Then, it follows from the standard \( L^2 \)-estimate that the sequence \( \{w_n\} \) converges in \( W^{2,2}(\Omega) \) to a strong solution of (3.17). Therefore, we can get a classical solution of (3.17). By a bootstrap argument, we can obtain the convergence in any \( C^k \)-norm. Then we have \( \lambda \geq \lambda_c(\Omega) \). Therefore, \( \lambda(S) \leq \lambda(S') \).

Then, Theorem 3.1 follows from Lemmas 3.1–3.8

### 4 Generalized Abelian Higgs vortices

In this section, we construct a multivortex solution for the generalized self-dual Abelian Higgs equation also proposed in [5] over the doubly periodic domain \( \Omega \), using our results of the last section.
Recall that in [5] the Hamiltonian of the generalized Abelian Higgs model can be written as

\[ H = \int_{\Omega} dx \left\{ \kappa^2 \left[ (1 - |\phi|^2) F_{12} - i (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) \right]^2 + 2 (1 - |\phi|^2)^2 (|D_1 \phi|^2 + |D_2 \phi|^2) + V \right\}. \]

With the choice of the Higgs potential

\[ V = \frac{1}{4 \kappa^2} (1 - |\phi|^2)^4, \]

as in section 2, we rewrite \( H \) as

\[ H = \int_{\Omega} dx \left\{ \kappa \left[ (1 - |\phi|^2) F_{12} - i (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) \right] - \frac{1}{2 \kappa} (1 - |\phi|^2)^2 \right\}^2 + 2 (1 - |\phi|^2)^2 |D_1 \phi - i D_2 \phi|^2 + (1 - |\phi|^2)^3 F_{12} - 3i (1 - |\phi|^2)^2 (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) \}

\[ = \int_{\Omega} dx \left\{ \kappa \left[ (1 - |\phi|^2) F_{12} - i (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) \right] - \frac{1}{2 \kappa} (1 - |\phi|^2)^2 \right\}^2 + 2 (1 - |\phi|^2)^2 |D_1 \phi - i D_2 \phi|^2 + F_{12} - 3i \varepsilon_{ij} \partial_i \left[ \left( 1 - |\phi|^2 + \frac{1}{3} |\phi|^4 \right) \phi \partial_j \phi \right]. \]

Then we obtain

\[ H \geq \int_{\Omega} F_{12} dx, \]

and this lower bound is saturated if and only if \((\phi, A)\) satisfies the self-dual equations

\[ D_1 \phi = i D_2 \phi, \quad (4.1) \]

\[ F_{12} = i (D_1 \phi \overline{D_2 \phi} - \overline{D_1 \phi} D_2 \phi) + \frac{1}{2 \kappa^2} (1 - |\phi|^2)^2. \quad (4.2) \]

The structure of (4.1) and (4.2) is similar to that of (2.7) and (2.8). However, the approach dealing with (2.7) and (2.8) cannot be directly used to (4.1) and (4.2). Fortunately, based on the obtained solution of (2.7) and (2.8), we can establish a solution of (4.1) and (4.2).

Following a similar procedure as in section 2, we can reduce the equations (4.1) and (4.2) into the quasilinear elliptic equation

\[ \Delta u - e^u |\nabla u|^2 = -\lambda (e^u - 1)^2 + 4 \pi \sum_{s=1}^{N} \delta_{p_s} \quad \text{in} \quad \Omega, \quad (4.3) \]

where \( \lambda = \frac{1}{\kappa^2} \). Using \( v = F(u) = 1 + u - e^u \) again, we have

\[ \Delta v = -\lambda (e^{G(v)} - 1)^2 + 4 \pi \sum_{s=1}^{N} \delta_{p_s} \quad \text{in} \quad \Omega. \quad (4.4) \]

Let \( v = v_0 + w \), where \( v_0 \) is defined by (3.16). Then the equation (4.4) is modified into

\[ \Delta w = -\lambda (e^{G(v_0 + w)} - 1)^2 + \frac{4 \pi N}{|\Omega|} \quad \text{in} \quad \Omega. \quad (4.5) \]
Let \( w \) be a solution of (3.17). Then we have \( v_0 + w < 0 \) in \( \Omega \). As a result, \( e^{(v_0+w)} < 1 \), which implies

\[
\Delta w = -\lambda e^{G(v_0+w)} (e^{G(v_0+w)} - 1)^2 + \frac{4\pi N}{|\Omega|} \geq -\lambda (e^{G(v_0+w)} - 1)^2 + \frac{4\pi N}{|\Omega|}.
\]

Thus we see that \( w \) is a subsolution of (4.5). It is easy to see that \( -v_0 \) is also a supersolution of (4.5). Therefore we can modify the iteration scheme (3.20) to establish a solution \( w \) of (4.5), satisfying \( w < w < -v_0 \). Indeed, we can get the following theorem.

**Theorem 4.1** For any prescribed points \( p_1, \ldots, p_N \in \Omega \), there is a critical value of \( \lambda \), say \( \lambda_c \), such that, for \( \lambda > \lambda_c \), the equation (4.3) has a solution, while for \( \lambda \leq \lambda_c \), the equation (4.3) has no solution.

**Remark 1** It was shown in [30], for the Abelian Higgs equation,

\[
\Delta u = \lambda (e^u - 1) + 4\pi \sum_{j=1}^N \delta_{p_j} \quad \text{in} \quad \Omega,
\]

the critical value of the coupling parameter is

\[\lambda_c = \frac{4\pi N}{|\Omega|}.
\]

But, at \( \lambda = \lambda_c \), the equation (4.6) has no solution.

Consequently, by Theorem 4.1, we can recover a solution to (4.1) and (4.2) by the transformation (3.5) and (3.6).

**Theorem 4.2** Let \( p_1, p_2, \ldots, p_m \in \Omega \), \( n_1, n_2, \ldots, n_m \) be some positive integers and \( N = \sum_{i=1}^m n_i \). There exists a critical value of the coupling parameter, say \( \kappa_c \), satisfying

\[0 < \kappa_c \leq \frac{1}{2} \sqrt{\frac{|\Omega|}{\pi N}},
\]

such that, for \( 0 < \kappa < \kappa_c \), the self-dual equations (4.1) and (4.2) admit a solution \((\phi, A)\) for which \( p_1, p_2, \ldots, p_m \) are zeros of \( \phi \) with multiplicities \( n_1, n_2, \ldots, n_m \), while for \( \kappa \geq \kappa_c \), the equations (4.1) and (4.2) have no solution.

**References**

[1] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, *Sov. Phys. JETP* 5 (1957) 1174–1182.

[2] T. Aubin, *Nonlinear Analysis on Manifolds: Monge–Ampère Equations*, Springer, Berlin and New York, 1982.
[3] A. Bezryadina, E. Eugenieva, and Z. Chen, Self-trapping and flipping of double-charged vortices in optically induced photonic lattices, *Optics Lett.* 31 (2006) 2456–2458.

[4] E. B. Bogomol’ny, The stability of classical solutions, *Sov. J. Nucl. Phys.* 24 (1976) 449–454.

[5] J. Burzlaff, A. Chakrabarti, and D. H. Tchrakian, Generalized self-dual Chern–Simons vortices, *Phys. Lett.* B 293 (1992) 127–131.

[6] L. Caffarelli and Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, *Comm. Math. Phys.* 168 (1995) 321–336.

[7] D. Chae and O.Y. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, *Comm. Math. Phys.* 215 (2000) 119–142.

[8] H. Chan, C. C. Fu, and C. S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation, *Comm. Math. Phys.* 231 (2002) 189–221.

[9] X. Chen, S. Hastings, J. B. McLeod and Y. Yang, A nonlinear elliptic equation arising from gauge field theory and cosmology, *Proc. R. Soc.* (London) A 446 (1994) 453–478.

[10] K. Choe, Asymptotic behavior of condensate solutions in the Chern–Simons–Higgs theory, *J. Math. Phy.* 48 (2007) 103501.

[11] W. Ding, J. Jost, J. Li, and G. Wang, An analysis of the two-vortex case in the Chern–Simons–Higgs model, *Calc. Var. P.D.E.* 7 (1998) 87-97.

[12] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity. In: *Collected Papers of L. D. Landau* (edited by D. Ter Haar), New York: Pergamon, 1965, 546–568.

[13] M. B. Hindmarsh and T. W. B. Kibble, Cosmic strings, *Rep. Prog. Phys.* 58 (1995) 477–562.

[14] G. ’t Hooft, A property of electric and magnetic flux in nonabelian gauge theories, *Nucl. Phys. B* 153 (1979) 141–160.

[15] J. Hong, Y. Kim and P.Y. Pac, Multivortex solutions of the Abelian Chern–Simons theory, *Phys. Rev. Lett.* 64 (1990) 2230–2233.

[16] R. W. Jackiw, and E. J. Weinberg, Self-dual Chern–Simons vortices, *Phys. Rev. Lett.* 64 (1990), 2234–2237.

[17] A. Jaffe and C. H. Taubes, *Vortices and Monopoles*, Birkhäuser, Boston, 1980.

[18] T. W. B. Kibble, Some implications of a cosmological phase transition, *Phys. Rep.* 67 (1980) 183–199.

[19] C. S. Lin and S. Yan, Bubbling solutions for relativistic abelian Chern–Simons model on a torus, *Comm. Math. Phys.* 297 (2010) 733–758.

[20] H. B. Nielsen and P. Olesen, Vortex-line models for dual-strings, *Nucl. Phys. B* 61 (1973) 45–61.

[21] M. Nolasco and G. Tarantello, Double vortex condensates in the Chern–Simons–Higgs theory, *Calc. Var. P.D.E.* 9 (1999) 31–94.

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[22] J. B. Sokoloff, Charged vortex excitations in quantum Hall systems, *Phys. Rev. B* 31 (1985) 1924–1928.

[23] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern–Simons theory: existence and approximation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 12 (1995) 75–97.

[24] J. Spruck and Y. Yang, The existence of non-topological solitons in the self-dual Chern–Simons theory, *Comm. Math. Phys.* 149 (1992) 361–376.

[25] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, *J. Math. Phys.* 37 (1996) 3769–3796.

[26] G. Tarantello, Uniqueness of self-dual periodic Chern–Simons vortices of topological-type, *Calc. Var. P.D.E.* 28 (2007) 191–217.

[27] D. H. Tchrakian and Y. Yang, The existence of generalized self-dual Chern–Simons vortices, *Lett. Math. Phys.* 36 (1996) 403–413.

[28] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge: Cambridge U. Press, 1994.

[29] R. Wang, The existence of Chern–Simons vortices, *Comm. Math. Phys.* 137 (1991) 587–597.

[30] S. Wang and Y. Yang, Abrikosov’s vortices in the critical coupling, *SIAM J. Math. Anal.* 23 (1992) 1125–1140.

[31] Y. Yang, Chern–Simons soliton and a nonlienar elliptic equation, *Helv. Phys. Acta* 71 (1998) 573–585.

[32] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, New York, 2001.