Non-Gaussian Limit Theorem for Non-Linear Langevin Equations Driven by Lévy Noise

Alexei Kulik†‡ and Ilya Pavlyukevich§

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Abstract

In this paper, we study the small noise behaviour of solutions of a non-linear second order Langevin equation $\ddot{x}_\varepsilon + |\dot{x}_\varepsilon|^{\beta} = Z_\varepsilon$, $\beta \in \mathbb{R}$, driven by symmetric non-Gaussian Lévy processes $Z_\varepsilon$. This equation describes the dynamics of a one-degree-of-freedom mechanical system subject to non-linear friction and noisy vibrations. For a compound Poisson noise, the process $x_\varepsilon$ on the macroscopic time scale $t/\varepsilon$ has a natural interpretation as a non-linear filter which responds to each single jump of the driving process. We prove that a system driven by a general symmetric Lévy noise exhibits essentially the same asymptotic behaviour under the principal condition $\alpha + 2\beta < 4$, where $\alpha \in [0, 2]$ is the “uniform” Blumenthal–Getoor index of the family $\{Z_\varepsilon\}_{\varepsilon > 0}$.

Keywords: Lévy process; Langevin equation; non-linear friction; Hölder-continuous drift; singular drift; stable Lévy process; Blumenthal–Getoor index; ergodic Markov process; Lyapunov function.

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†Institute of Mathematics, Ukrainian National Academy of Sciences, Tereshchenkivska Str. 3, 01601 Kiev, Ukraine
‡Institut für Mathematik, Strasse des 17. Juni 136, D-10623 Berlin, Germany; kulik.alex.m@gmail.com
§Institut für Mathematik, Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany; ilya.pavlyukevich@uni-jena.de
1 Introduction and motivation

In this paper we study a non-linear response of a one-dimensional system to both external stochastic excitation and non-linear friction. In the simplest mathematical setting in the absence of external forcing, one can assume that the friction force is proportional to a power ($\beta \in \mathbb{R}$) of the particle's velocity; that is, the equation of motion has the form

$$\ddot{x}_t = -|\dot{x}_t|^{\beta} \text{sgn} \dot{x}_t.$$ 

This model covers such prominent particular cases as the linear viscous (Stokes) friction $\beta = 1$, the dry (Coulomb) friction $\beta = 0$, and the high-speed limit of the Rayleigh friction $\beta = 2$ (see Persson (2000); Popov (2010); Sergienko and Bukharov (2015)). As usual, the second-order equation (1.1) can be written as a first order system

$$\begin{align*}
\dot{x}_t &= v_t, \\
\dot{v}_t &= -|v_t|^{\beta} \text{sgn} v_t,
\end{align*}$$

which is a particular case of a (non-linear) Langevin equation. The second equation in this system is autonomous, and the corresponding velocity component can be given explicitly, once its initial value $v_0$ is fixed:

$$v_t = \begin{cases} 
v_0 e^{-t}, & \beta = 1; \\
(v_0|1-\beta|-(1-\beta)t)^{1/(1-\beta)} \text{sgn} v_0, & \text{otherwise.}
\end{cases} \quad (1.3)$$

Clearly, for any $\beta \in \mathbb{R}$ and $v_0 \in \mathbb{R}$ such a solution tends to 0 as $t \to \infty$; that is, in any case, the velocity component of the system dissipates. The complete picture which also involves the position component, is more sophisticated. Clearly,

$$x_t = x_0 + \int_0^t v_s \, ds,$$

and one can easily observe that $v = (v_t)_{t \geq 0}$ is integrable on $\mathbb{R}_+$ if $\beta < 2$. In this case the position component $x = (x_t)_{t \geq 0}$ dissipates as well and tends to a limiting value

$$x_t \to x_\infty = x_0 + F(v_0), \quad t \to \infty, \quad F(v) = \frac{1}{2-\beta}|v|^{2-\beta} \text{sgn}(v).$$

The function $F(v)$ has the meaning of a complete response of the system to the instant perturbation of its velocity by $v$. For $\beta \geq 2$, the integral of $v_t$ over $\mathbb{R}_+$ diverges, and $x_t$ tends to $\pm \infty$ depending on the sign of $v_0$. In other words, the friction in the system in the vicinity of zero is too weak to slow down the particle.

In this paper we consider the interplay between the non-linear dissipation and the weak random vibrations of the particle, namely we study perturbations of the velocity by a weak (symmetric) Lévy process $Z$,

$$\begin{align*}
\dot{x}^\varepsilon_t &= v^\varepsilon_t, \\
\dot{v}^\varepsilon_t &= -|v^\varepsilon_t|^{\beta} \text{sgn} v^\varepsilon_t + \dot{Z}_{\varepsilon t},
\end{align*}$$

in the small noise limit $\varepsilon \to 0$.

Heuristically, we consider a system, which consists of two different components acting on different time scales. The microscopic behaviour of the system is primarily determined by the non-linear model under random perturbations of low intensity. It is clear that neither these perturbations themselves nor their impact on the system are visible on the microscopic time scale; that is on any finite time interval $[0,T]$, $Z_{\varepsilon t}$ tends to 0, and $(x^\varepsilon_t, v^\varepsilon_t)$ become close to $(x_t, v_t)$ as $\varepsilon \to 0$.

The influence of random perturbations becomes significant on the macroscopic time scale $\varepsilon^{-1} t$ which suggests to focus our analysis on the limit behaviour of the pair

$$(X^\varepsilon_t, V^\varepsilon_t) := \left(x^\varepsilon_{\varepsilon^{-1}t}, v^\varepsilon_{\varepsilon^{-1}t}\right).$$

\[1\] Proof of \[5.10\]
satisfying the system of SDEs
\[ \begin{align*}
\frac{dX^\varepsilon_t}{\varepsilon} &= \frac{1}{\varepsilon}V^\varepsilon_t \, dt, \\
\frac{dV^\varepsilon_t}{\varepsilon} &= -\frac{1}{\varepsilon}(|V^\varepsilon_t|^{\beta} \sgn V^\varepsilon_t \varepsilon^{-1/\beta} + \varepsilon^{1/\beta} \, dZ_t).
\end{align*} \] (1.6)

We will look for a non-trivial limit for the position process $X^\varepsilon$ as $\varepsilon \to 0$, in dependence on the friction exponent $\beta$ and the properties of the process $Z$.

The case of Stokes friction $\beta = 1$ is probably the simplest one: the system (1.6) is linear, and under zero initial conditions $X^\varepsilon_0 = V^\varepsilon_0 = 0$, its solution $X^\varepsilon$ is found explicitly as a convolution integral
\[ X^\varepsilon_t = \int_0^t (1 - e^{-(t-s)/\varepsilon}) \, dZ_s. \]

Hintze and Pavlyukevich (2014) showed, that for any Lévy forcing $Z$, $X^\varepsilon$ converges to $Z$ in the sense of finite-dimensional distributions. It is worth noticing that although $X^\varepsilon$ is an absolutely continuous process, the limit is in general a jump process. In that case, a functional limit theorem requires the convergence in non-standard Skorokhod topologies such as the $M_1$-Skorokhod topology.

Non-linear ($\beta \neq 1$) stochastic systems of the type (1.6) driven by Brownian motion, $Z = B$, have been studied in recent years both in physical and mathematical literature, see Lindner (2007, 2008, 2010), Lisý et al. (2014) for the analysis for $\beta = 1, 2, 3, 5$, Baule and Solich (2012), Touchette et al. (2010), de Gennes (2003), Hayakawa (2003), Kawarada and Hayakawa (2004), Mauzer (2006) for the important case of dry (Coulomb) friction $\beta = 0$, and Goohpattader and Chaudhury (2010) for experiments and simulations for the dry friction $\beta = 0$ and irregular friction $\beta = 0.4$. The main goal of these papers was to determine on the physical level of rigour how the so-called effective diffusion coefficient, which is roughly speaking the variance of the particle’s position, depends on $\varepsilon$. In mathematical terms, the result from Hintze and Pavlyukevich (2014) gave convergence $X^\varepsilon \Rightarrow B$ for $\beta = 1$, whereas Eon and Gradinaru (2015) proved that for $\beta > -1$, the scaled process $\varepsilon^{2(\beta-1)/(\beta+1)} X^\varepsilon$ weakly converges in the uniform topology to a Brownian motion whose variance is calculated explicitly.

The limiting behaviour of (1.6) with a symmetric $\alpha$-stable Lévy forcing was also the subject of the paper by Eon and Gradinaru (2015). Under the condition $\alpha + 2\beta > 4$ they proved that the scaled process $\varepsilon^{\alpha/(\alpha+2\beta-4)/(\alpha+\beta-1)} X^\varepsilon$ weakly converges to a Brownian motion. The proof is based on the application of the central limit theorem for ergodic processes.

In the present paper, we establish a principally different type of the limit behaviour of the process $X^\varepsilon$. We specify a condition on the Lévy noise $Z$, which ensures that $X^\varepsilon$, without any additional scaling, converges to a non-Gaussian limit. Such a behaviour is easy to understand once $Z$ is a compound Poisson process, which is the simplest model for mechanical or physical shocks. If $\beta < 2$, the position process $X^\varepsilon$ is a composition of individual responses of the deterministic system (1.1) on a series of rare impulse perturbations. Since a general (say, symmetric) non-Gaussian Lévy process $Z$ can be interpreted as limit of compound Poisson processes, one can naively guess that the same effect should be observed for (1.6) in the general case as well. This guess is not completely true for the “large jumps” part of the noise (being, of course, a compound Poisson process) now interferes with the “small jumps” via a non-linear drift $|v|^{\beta}\sgn v$. To guarantee that the “small jump” are indeed negligible, we have to impose a balance condition between the non-linearity index $\beta$ and the proper version of the Blumenthal–Getoor index $\alpha_{BG}(Z)$ of the Lévy noise, namely we require that
\[ \alpha_{BG}(Z) + 2\beta < 4. \] (1.7)

Combined with the aforementioned analysis of the symmetric $\alpha$-stable case by Eon and Gradinaru (2015), this clearly separates two alternatives available for the system (1.6). Once (1.7) holds true, the small jumps are negligible, and $X^\varepsilon$ converges to a non-Gaussian limit; otherwise, the small jumps dominate, and $X^\varepsilon$ is subject to the central limit theorem, i.e. after a proper scaling one gets a Gaussian limit for it. Note that since (1.7) necessitate the bound $\beta < 2$, a non-Gaussian limit for $X^\varepsilon$ can be observed only when both the velocity and the position components of (1.2) are dissipative.

The rest of the paper is organized as follows. In Section 2 we introduce the setting and formulate the main results of the paper. To clarify the presentation, we separate two preparatory results: Theorem 2.1 for the system (1.6) with the compound Poisson noise, and Theorem 2.2 which describes the asymptotic
properties of the velocity component of a general system. The proofs of the preparatory results are contained in Section \ref{sec:prelim}. The proof of the main statement of the paper, Theorem \ref{thm:main}, is given separately in the regular case and in the non-regular/quasi-ergodic case in Section \ref{sec:regular} and Section \ref{sec:nonregular} respectively; see discussion of the terminology therein. Some technical auxiliary results are postponed to Appendix.

2 Main results

2.1 Notation and preliminaries

For $a \in \mathbb{R}$, we denote $a_+ = \max\{a, 0\}$, $a \wedge b = \min\{a, b\}$

$$\text{sgn} x = \begin{cases} 
-1, & x < 0, \\
0, & x = 0, \\
1, & x > 0, 
\end{cases}$$

$X \xrightarrow{f.d.d.} X$ denotes convergence in the sense of finite dimensional distributions.

Throughout the paper, $Z$ is a one-dimensional symmetric non-Gaussian Lévy process with the Lévy measure $\mu$. In Section \ref{sec:notation}, we assume that $Z$ is a compound Poisson process with $\mu(\mathbb{R}) \in (0, \infty)$ which is not necessarily symmetric. In both cases, the Lévy–Hinchin formula for $Z$ reads

$$\mathbb{E}e^{i\lambda Z_t} = \exp \left( t \int (e^{i\lambda z} - 1) \mu(\mathrm{d}z) \right), \quad \lambda \in \mathbb{R}, \ t \geq 0.$$ 

The Blumenthal–Getoor index $\alpha_{\text{BG}}(Z)$ of $Z$ is defined by

$$\alpha_{\text{BG}}(Z) = \inf \left\{ \alpha > 0 : \sup_{r \in (0,1)} r^\alpha \mu(\{z : |z| > r\}) < \infty \right\}.$$ 

Note that for an arbitrary Lévy measure $\mu$ the following estimate holds true:

$$r^2 \mu(\{z : |z| > r\}) = r^2 \int_{|z| > r} \mu(\mathrm{d}z) \leq \int_{\mathbb{R}} \min(z^2, 1) \mu(\mathrm{d}z) < \infty, \quad (2.1)$$ 

that is, $\alpha_{\text{BG}}(Z) \in [0, 2]$.

We always assume $\mu(\{0\}) = 0$. If $\mu(\mathbb{R}) \in (0, \infty)$, then $Z$ is a compound Poisson process, and in that case we write

$$Z_t = \sum_{k=1}^{\infty} J_k \mathbb{1}_{[\tau_k, \infty)}(t),$$

where $\{\tau_k\}_{k \geq 1}$ are jump arrival times of $Z$, and $\{J_k\}_{k \geq 1}$ are jump amplitudes. For $Z$ with infinite Lévy measure, an analogue of this representation is given by the Itô–Lévy decomposition

$$Z_t = \int_0^t \int_{|z| \leq 1} z \tilde{N}(\mathrm{d}z \, \mathrm{d}s) + \int_0^t \int_{|z| > 1} z N(\mathrm{d}z \, \mathrm{d}s),$$

where $N(\mathrm{d}z \, \mathrm{d}t)$ is the Poisson point measure associated with $Z$, $\tilde{N}(\mathrm{d}z \, \mathrm{d}t) = N(\mathrm{d}z \, \mathrm{d}t) - \mu(\mathrm{d}z) \mathrm{d}t$ is corresponding compensated measure.

We do not specifically address the question of the existence and uniqueness of solutions of the system \ref{eq:system}, assuming these solutions to be well defined. Let us briefly mention several facts about that.

1. If $Z$ is a compound Poisson process then the system \ref{eq:system} can be uniquely solved path-by-path for any $\beta \in \mathbb{R}$.

2. For $\beta \in \mathbb{R}$ and general $Z$, it is natural to understand the drift $b(v) = |v|^\beta \text{sgn} v$ in the following set-valued sense:

$$b(v) = |v|^\beta \text{sgn} v, \quad v \neq 0,$$

$$b(v) \in I_\beta, \quad v = 0.$$
where

\[ I_\beta = \begin{cases} 
0, & \beta > 0, \\
[-1, 1], & \beta = 0, \\
\mathbb{R}, & \beta < 0;
\end{cases} \]

see, e.g. Pardoux and Răşcanu (2014).

3. For \( \beta \geq 0 \), the SDE (1.6) has unique strong solution, which follows by monotonicity of the drift \( b \). In Pardoux and Răşcanu (2014), this is proved for an SDE with Brownian noise, for the SDE (1.6) with additive Lévy noise the argument remains literally the same.

4. The existence and uniqueness of solutions for \( \beta < 0 \) and arbitrary (symmetric) Lévy noise seems to be an open question. We mention here results by Tanaka et al. (1974); Portenko (1994); Aryasova and Pilipenko (2012); Priola (2012); Zhang (2013).

2.2 The simplest non-Gaussian case: compound Poisson impulses

Let \( Z \) be a compound Poisson process and denote

\[ N_t = \sum_{k=1}^{\infty} \mathbb{I}_{(\tau_k, \infty)}(t), \quad t \geq 0, \]

its counting process, so that

\[ Z_t = \sum_{k=1}^{N_t} J_k. \]

Let the initial position and velocity \( x_0, v_0 \) be fixed, and let \((X_t^\varepsilon, V_t^\varepsilon)_{t \geq 0}\) be the solution to the system (1.6) with the initial condition \((x_0, v_0)\).

**Theorem 2.1** For any \( t > 0 \), we have the following convergence a.s. as \( \varepsilon \to 0 \):

1. for \( \beta < 2 \),

\[ X_t^\varepsilon \to x_0 + \frac{1}{2 - \beta} |v_0|^{2-\beta} v_0 + \frac{1}{2 - \beta} \sum_{k=1}^{N_t} |J_k|^{2-\beta} \text{sgn} J_k, \tag{2.2} \]

2. for \( \beta = 2 \),

\[ \left( \ln \frac{1}{\varepsilon} \right)^{-1} X_t^\varepsilon \to \text{sgn} v_0 + \sum_{k=1}^{N_t} \text{sgn} J_k, \]

3. for \( \beta \geq 2 \),

\[ \varepsilon^{\frac{\beta-1}{\beta-2}} X_t^\varepsilon \to \left( \frac{\beta-1}{\beta-2} \right)^{\frac{\beta-1}{\beta-2}} \sum_{k=2}^{N_t} (\tau_k - \tau_{k-1})^{\frac{\beta-1}{\beta-2}} \text{sgn} J_{k-1} + (t - \tau_{N_t})^{\frac{\beta-1}{\beta-2}} \text{sgn} J_{N_t}. \]

In the above Theorem, the considerably different limits in the case 1 and the cases 2, 3 are caused by the different dissipativity properties of the system (1.2) discussed in the Introduction. For \( \beta < 2 \), the complete response to the perturbation of the velocity is finite, and is given by the function

\[ F(v) = \frac{1}{2 - \beta} |v|^{2-\beta} \text{sgn}(v). \tag{2.3} \]

Note that the right hand side in (2.2) is just the sum of the initial position \( x_0 \), the response which corresponds to the initial velocity \( v_0 \), and the responses to the random impulses which had arrived into the system up to the time \( t \). Similar additive structure remains true in the cases 2 and 3 as well, however for \( \beta \geq 2 \) the complete response of the system to every single perturbation is infinite, which explains the necessity to introduce a proper scaling. For \( \beta > 2 \), this also leads to necessity to take into account the jump arrival times. Note that in all three regimes, the initial value \( v_0 \) of the velocity has a natural interpretation as a single jump with the amplitude \( J_0 = v_0 \), which occurs at the initial time instant \( \tau_0 = 0 \).
2.3 General setup

In the main part of the paper, we adopt even a more general setup, than the one explained in the Introduction. Namely, we consider a system

\[ \begin{align*}
\frac{dX_t^\varepsilon}{\varepsilon} & = V_t^\varepsilon \, dt, \\
\frac{dV_t^\varepsilon}{\varepsilon} & = -\frac{1}{\varepsilon}|V_t^\varepsilon|^\beta \text{sgn } V_t^\varepsilon \, dt + dZ_t^\varepsilon
\end{align*} \tag{2.4} \]

with a family of Lévy processes \( \{Z_t^\varepsilon\}_{\varepsilon \in (0,1]} \). Such a setting allows for taking into account small uncertainties in the random perturbations. It also allows one to avoid certain technical issues, preserving the model’s physical relevance. For instance, for \( \beta < 0 \) and infinite \( \mu \), it may be difficult to specify the solution to (1.6), but such a solution is well defined for each compound Poisson approximation \( Z^\varepsilon \) to \( Z \), where all the jumps of \( Z \) with amplitudes smaller than some threshold \( \ell(\varepsilon) \) are truncated.

The first statement in this section actually shows that the velocity component of the system (2.4), under very wide assumptions on the Lévy noise, has a dissipative behaviour similar to the one of \( v_t \), discussed in the Introduction.

**Theorem 2.2** Let \( \{Z_t^\varepsilon\}_{\varepsilon \in (0,1]} \) be a family of symmetric pure jump Lévy processes, and let \( Z^\varepsilon \overset{\text{f.d.d.}}{\rightarrow} Z \) as \( \varepsilon \to 0 \). Then for any \( \beta \in \mathbb{R} \) the following hold true:

(i) for any \( T > 0 \) and any initial value \( v_0 \),

\[ \lim_{R \to \infty} \sup_{\varepsilon \in (0,1]} \mathbb{P} \left( \sup_{t \in [0,T]} |V_t^\varepsilon| > R \right) = 0; \tag{2.5} \]

(ii) for any \( t > 0 \), any initial value \( v_0 \), and any \( \delta > 0 \),

\[ \lim_{\varepsilon \to 0} \mathbb{P}(|V_t^\varepsilon| > \delta) = 0. \tag{2.6} \]

The main result of the entire paper is presented in the following Theorem.

**Theorem 2.3** Let conditions of Theorem 2.2 hold true. Assume that, for some \( \alpha \in [0,2] \) such that \( \alpha + 2\beta < 4 \) and some \( C > 0 \), the Lévy measures \( \{\mu^\varepsilon\} \) of the Lévy processes \( \{Z_t^\varepsilon\} \) satisfy

\[ \sup_{\varepsilon \in (0,1]} \mu^\varepsilon(|z| \geq r) \leq Cr^{-\alpha}, \quad r \in (0,1]. \tag{2.7} \]

If \( \alpha = 2 \), then assume additionally that

\[ \lim_{r \to 0} \sup_{\varepsilon \in (0,1]} \int_{|z| \leq r} z^2 \mu^\varepsilon(\text{d}z) = 0. \tag{2.8} \]

Then

\[ X^\varepsilon \overset{\text{f.d.d.}}{\rightarrow} x_0 + \frac{|v_0|^{2-\beta}}{2-\beta} \text{sgn } v_0 + \frac{1}{2-\beta} \int_0^t \int |z|^{2-\beta} \text{sgn } z \bar{N}(\text{d}z \, \text{d}s), \quad \varepsilon \to 0, \tag{2.9} \]

on \( t \in (0,\infty) \), where \( \bar{N} \) is the compensated Poisson random measure, which corresponds to the Lévy process \( Z \).

Inequality (2.7) is a uniform analogue of the one from the definition of the Blumenthal–Getoor index. Namely, if \( \{\mu^\varepsilon\} \) consists of a single Lévy measure \( \mu \), (2.7) holds true for any \( \alpha > \alpha_{BG}(Z) \). Condition (2.8) prevents accumulation of small jumps for the family \( \{\mu^\varepsilon\} \), and also holds true once \( \{\mu^\varepsilon\} \) consists of one measure. This leads to the following

**Corollary 2.1** Let \( Z \) be a symmetric pure jump Lévy process with the Blumenthal–Getoor index satisfying \( \alpha_{BG} + 2\beta < 4 \). Let either \( Z^\varepsilon = Z \), or \( Z^\varepsilon \) be a compound Poisson process, obtained from \( Z \) by truncations of the jumps with amplitudes smaller than \( \ell(\varepsilon) \), and let

\[ \ell(\varepsilon) \to 0, \quad \varepsilon \to 0. \]

Then the position component \( X^\varepsilon \) of the system (2.4) satisfies (2.9).
Note that the right hand side in (2.9) is a Lévy process with the Lévy measure
\[ \mu^X(B) = \mu\left(\left\{z : \frac{|z|^{2-\beta} \text{sgn } z}{2-\beta} \in B\right\}\right), \quad B \in \mathcal{B}(\mathbb{R}). \] (2.10)

Theorem 2.3 actually shows that the Langevin equation (1.4) with small Lévy noise, considered at the macroscopic time scale, performs a non-linear filter of the noise, with the transformation of the jump intensities given by (2.10). Since \(\mu\) is symmetric and the response function \(F(v) = \frac{1}{2-\beta} |v|^{2-\beta} \text{sgn } v\) is odd,
\[
\int_0^t \int F(z) \tilde{N}(dz \, ds) = L^2 - \lim_{\delta \to 0} \sum_{s \leq t} F(|\Delta Z_s|) \cdot \mathbb{I}(|\Delta Z_s| > \delta).
\]

In other words, the right hand side in (2.9) has exactly the same form as (2.2). Note that the assumption \(\alpha + 2\beta < 4\) again requires \(\beta < 2\), since \(\alpha \geq 0\). Hence, the operation of the aforementioned non-linear filter can be shortly described as follows: every jump \(z\) of the input process \(Z\) is transformed to the jump \(F(z)\) of the output process. From this point of view, the assumption \(\alpha + 2\beta < 4\) can be interpreted as a condition for the jumps to arrive “sparsely” enough, for the system to be able to filter them independently. The following example, in particular, shows that this assumption is sharp, and once it fails, the asymptotic regime for (2.4) may change drastically.

Example 2.1 Let \(Z\) be a symmetric \(\alpha\)-stable process with the Lévy measure
\[ \mu(dz) = c \frac{dz}{|z|^\alpha+1}, \quad c > 0. \]

Then the right hand side in (2.9) is also a symmetric stable process with the Lévy measure
\[ \mu^X(dz) = c_X \frac{dz}{|z|^\alpha_X+1}, \]

where
\[ \alpha_X = \frac{\alpha}{2-\beta}, \quad c_X = c(2-\beta)^{-\alpha_X-1}. \]

Note that the new stability index \(\alpha_X\) is smaller than 2 exactly when \(\alpha + 2\beta < 4\). That is, in the symmetric stable setting, Theorem 2.3 obviously fails when the latter condition fails. This is not surprising because we know from Eon and Gradinaru (2014) that, once \(\alpha + 2\beta > 4\), the properly scaled process \(X^\varepsilon\) has a Gaussian limit. The boundary case \(\alpha + 2\beta = 4\) is yet open for a study.

Before proceeding with the proofs, let us give two more remarks. First, it will be seen from the proofs that for any \(t > 0\)
\[
X^\varepsilon_t - x_0 - \frac{|v_0|^{2-\beta} \text{sgn } v_0}{2-\beta} \cdot \frac{1}{2-\beta} \int_0^t \int |z|^{2-\beta} \text{sgn } z \tilde{N}(dz \, ds) \to 0, \quad \varepsilon \to 0,
\] (2.11)
in probability, where \(\tilde{N}\) denotes the compensated Poisson random measure for the process \(Z^\varepsilon\). This is a stronger feature than just the weak convergence stated in Theorem 2.3. Hence the non-linear filter, discussed above, actually operates with the trajectories of the noise rather than with its law.

Second, in order to make exposition considerably simple and compact, we restrict ourselves to the f.d.d. weak convergence (actually, the point-wise convergence in probability), rather than the functional convergence. We believe that (2.3) holds true in the \(M_1\)-topology, similarly to the case \(\beta = 1\) studied in Hintze and Pavlyukevich (2014). This guess can be easily verified in the context of Theorem 2.1: the explicit trajectory-wise calculations from its proof can be slightly modified in order to show that the convergence holds true in \(M_1\)-topology for \(\beta \leq 2\), and in the uniform topology for \(\beta > 2\).
3 Proofs of preparatory results

3.1 Proof of Theorem 2.1

The solution of the system (1.6) can be written explicitly. Namely, denote

\[ V_\varepsilon^t(v) = \begin{cases} \varepsilon e^{-t/\varepsilon}, & \beta = 1; \\ (|v|^{1-\beta} - t(1-\beta)/\varepsilon)^{1/(1-\beta)} \text{sgn} v, & \text{otherwise}, \end{cases} \]

which is just the velocity component of the system (1.2) with \( v_0 = v \), taken at the macroscopic time scale \( \varepsilon^{-1} t \); see (1.3). The integral of the velocity

\[ I_\varepsilon^t(v) = \frac{1}{\varepsilon} \int_0^t V_\varepsilon^s(v) \, ds, \]

can be also easily computed:

\[ I_\varepsilon^t(v) = \begin{cases} \varepsilon \left( 1 - e^{-t/\varepsilon} \right), & \beta = 1; \\ \ln \left( 1 + \frac{|v| t}{\varepsilon} \right) \text{sgn} v, & \beta = 2; \\ - \frac{1}{\beta - 2} \left( |v|^{1-\beta} - (1-\beta) t/\varepsilon \right)^{\beta-2} + |v|^{2-\beta} \text{sgn} v, & \text{otherwise}. \end{cases} \]

Then \((X_\varepsilon^t, V_\varepsilon^t)\), defined by (1.6), can be expressed as follows:

\[ V_\varepsilon^t = \sum_{k=0}^{\infty} V_\varepsilon^{t-\tau_k}(\tau_k) \left( V_\varepsilon^{\tau_k} + J_k \right) I_{\tau_k, \infty}(t) \]  

and

\[ X_\varepsilon^t = x_0 + \sum_{k=0}^{\infty} I_{(t-\tau_k)^{\tau_k}} \left( V_\varepsilon^{\tau_k} + J_k \right) \cdot I_{(t-\tau_k, \infty)}(t), \]

where we adopt the notation

\[ \tau_0 = 0, \quad J_0 = v_0, \quad V_\varepsilon^{\tau_0} = 0. \]

Since \( V_\varepsilon^t(v) \) and \( I_\varepsilon^t(v) \) are given explicitly, we now easily obtain the required statements. First, observe that for each \( t > 0 \) and \( v \in \mathbb{R} \),

\[ V_\varepsilon^t(v) \to 0, \quad \varepsilon \to 0, \]

hence

\[ V_\varepsilon^{\tau_k} \to 0, \quad \varepsilon \to 0, \quad k \geq 0, \]

almost surely. Next, we have for \( \beta < 2 \) for any \( t > 0, v \in \mathbb{R} \)

\[ I_\varepsilon^t(v) \to F(v) = \frac{1}{2-\beta} |v|^{2-\beta} \text{sgn}(v), \quad \varepsilon \to 0. \]

Since any fixed time instant \( t > 0 \) with probability 1 does not belong to the set \( \{\tau_k\}_{k \geq 0} \), the latter relation combined with (3.3) gives

\[ X_\varepsilon^t \to x_0 + \frac{1}{2-\beta} \sum_{k=0}^{N_l} |J_k|^{2-\beta} \text{sgn} J_k, \quad \varepsilon \to 0, \]

almost surely. For \( \beta = 2 \), for any for \( t > 0, v \in \mathbb{R} \) we have

\[ I_\varepsilon^t(v) - \left( \ln \frac{1}{\varepsilon} \right) \text{sgn} v \to \ln(|v| t) \cdot \text{sgn} v, \quad \varepsilon \to 0. \]

Combined with (3.3), this gives

\[ \left( \ln \frac{1}{\varepsilon} \right)^{-1} X_\varepsilon^t \to \sum_{k=0}^{N_l} \text{sgn} J_k, \quad \varepsilon \to 0, \]
almost surely. In the case $\beta > 2$ the argument is completely analogous, and is based on the relation

$$
\varepsilon^{ \frac{\beta-2}{\beta-2}} \mathbf{1}_{\varepsilon}^{(v)} \to \left( \frac{\beta - 1}{\beta - 2} \right) \varepsilon^{ \frac{\beta-2}{\beta-2}} \text{sgn } v, \quad \varepsilon \to 0, \quad t > 0,
$$

and we omit the details.

### 3.2 Proof of Theorem 2.2

1. In what follows, we assume that all the processes $\{Z^\varepsilon\}^{\varepsilon \in (0,1]}$ are defined on the same filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. We will systematically use the following “truncation of large jumps” procedure. For $A > 1$, denote by $Z^{\varepsilon,A}_t$ the truncation of the Lévy process $Z^\varepsilon$ at the level $A$, namely

$$
Z^{\varepsilon,A}_t = \int_0^t \int_{|z| \leq 1} z \tilde{N}^\varepsilon(ds, dz) + \int_0^t \int_{|z| > A} \tilde{N}^\varepsilon(ds, dz).
$$

For a given $T > 0$,

$$
\mathbb{P}(Z^\varepsilon_t = Z^{\varepsilon,A}_t, t \in [0, T]) = \mathbb{P}(N^\varepsilon(\{z: |z| > A\} \times [0, T]) = 0) = 1 - \exp\left(-T \int |z| > A \mu^\varepsilon(dz)\right).
$$

Recall that the convergence $Z^{\varepsilon} \xrightarrow{d} Z$, $\varepsilon \to 0$, of Lévy processes yields

$$
\lim_{\varepsilon \downarrow 0} \int f(z) \mu^\varepsilon(dz) = \int f(z) \mu(dz) \tag{3.4}
$$

for any $f \in C_b(\mathbb{R}, \mathbb{R})$ such that $f(z) = 0$ in a neighbourhood of the origin. This means that the tails of the Lévy measures $\mu^\varepsilon$ uniformly vanish at $\infty$:

$$
\sup_{\varepsilon \in (0,1]} \mu^\varepsilon(z: |z| > A) \to 0, \quad A \to \infty.
$$

That is, for any $T > 0$ and $\theta > 0$ we can fix $A > 0$ large enough such that

$$
\inf_{\varepsilon \in (0,1]} \mathbb{P}(Z^\varepsilon_t = Z^{\varepsilon,A}_t, t \in [0, T]) \geq 1 - \theta.
$$

Assume that for such $A$ we manage to prove statements (i), (ii) of the Theorem for the system (2.4) driven by $Z^{\varepsilon,A}$ instead of $Z^\varepsilon$. Since this system coincides with the original one on a set of probability larger than $1 - \theta$, we immediately get the following weaker versions of (2.5) and (2.6):

$$
\limsup_{N \to \infty} \sup_{\varepsilon \in (0,1]} \mathbb{P}\left( \sup_{t \in [0, T]} |V^\varepsilon_t| > N \right) \leq \theta,
$$

$$
\limsup_{\varepsilon \searrow 0} \mathbb{P}(|V^\varepsilon_t| > \delta) \leq \theta.
$$

Taking $A$ large enough, we can make $\theta$ arbitrarily small. Hence, in order to get the required statements, it is sufficient to prove the same statements under the additional assumption that, for some $A$,

$$
\supp \mu^\varepsilon \subseteq [-A, A], \quad \varepsilon \in (0, 1]. \tag{3.5}
$$

2. Let us proceed with the proof of (2.2). By (3.5) and the symmetry of $\mu^\varepsilon$, we have that

$$
Z^\varepsilon_t = \int_0^t \int_{-A}^A z \tilde{N}^\varepsilon(ds, dz),
$$

which is a square integrable martingale. With the help of Itô’s formula applied to the process $V^\varepsilon$ we get

$$
|V^\varepsilon_t|^2 = v_0^2 - \frac{2}{\varepsilon} \int_0^t |V^\varepsilon_s|^\beta 1_{V^\varepsilon_s \neq 0} ds + t \int_{-A}^A z^2 \mu^\varepsilon(dz) + M^\varepsilon_t, \tag{3.6}
$$

for some square integrable martingale $M^\varepsilon_t$. The proof is complete.
where
\[ M^\varepsilon_t = 2 \int_0^t \int_{-A}^A z V^{\varepsilon}_{s-} \tilde{N} \varepsilon (ds \, dz) \] (3.7)
is a local martingale. The sequence
\[ \tau^\varepsilon_m := \inf \{ t \geq 0 : |V^\varepsilon_t| > m \}, \quad m \geq 1, \]
is a localizing sequence for \( M^\varepsilon \) and thus
\[ |V^\varepsilon_{t \wedge \tau^\varepsilon_m}|^2 \leq v_0^2 + T \int_{-A}^A z^2 \mu^\varepsilon (dz) + M^\varepsilon_{t \wedge \tau^\varepsilon_m}. \]

By the Doob maximal inequality,
\[ E \sup_{t \in [0,T]} |M^\varepsilon_{t \wedge \tau^\varepsilon_m}|^2 \leq 4E |M^\varepsilon_{T \wedge \tau^\varepsilon_m}|^2 = 16 \cdot E \int_0^{T \wedge \tau^\varepsilon_m} |V^\varepsilon_s|^2 ds \cdot \int_{-A}^A z^2 \mu^\varepsilon (dz). \]

This yields
\[ E \sup_{t \in [0,T]} |V^\varepsilon_{t \wedge \tau^\varepsilon_m}|^2 \leq v_0^2 + T \int_{-A}^A z^2 \mu^\varepsilon (dz) + 4 \cdot \left( E \sup_{t \in [0,T]} |V^\varepsilon_{t \wedge \tau^\varepsilon_m}|^2 \right)^{1/2}. \]

Thus these exist a constant \( C > 0 \), independent on \( \varepsilon \), such that
\[ \sup_m E \sup_{t \in [0,T]} |V^\varepsilon_{t \wedge \tau^\varepsilon_m}|^2 \leq C. \] (3.8)

Since \( \tau^\varepsilon_m \to \infty, m \to \infty \), a.s., by the Fatou lemma we get
\[ E \sup_{t \in [0,T]} |V^\varepsilon_t|^2 \leq C. \]

This yields (2.5) by the Chebyshev inequality.

To prove (2.6), we note that \( M^\varepsilon \) defined in (3.7) is a square integrable martingale by (3.8). Then by (3.6), we have
\[ E |V^\varepsilon_T|^2 = v_0^2 - \frac{2}{\varepsilon} E \int_0^T |V^\varepsilon_s|^\beta I_{V^\varepsilon_s \neq 0} ds + T \int_{-A}^A z^2 \mu^\varepsilon (dz). \] (3.9)

Hence
\[ E \int_0^T |V^\varepsilon_s|^\beta I_{V^\varepsilon_s \neq 0} ds \leq \frac{\varepsilon}{2} \left( v_0^2 + T \int_{-A}^A z^2 \mu^\varepsilon (dz) \right) \to 0, \quad \varepsilon \to 0. \]

For \( \beta > -1 \) this yields that, for any \( \delta > 0 \),
\[ \int_0^T I_{|V^\varepsilon_s| > \delta} ds \to 0, \quad \varepsilon \to 0, \] (3.10)
in probability. For \( \beta \leq -1 \), combined with (2.5), this gives even more:
\[ \int_0^T I_{|V^\varepsilon_s| \neq 0} ds \to 0, \quad \varepsilon \to 0, \]
in probability. In each of these cases, we have that, for any given \( \zeta > 0 \), \( t_0 \geq 0 \), the stopping times
\[ \theta^\varepsilon_{t_0} = \inf \{ t \geq t_0 : |V^\varepsilon_t| \leq \zeta \} \]
satisfy
\[ \theta^\varepsilon_{t_0} \to t_0, \quad \varepsilon \to 0 \] (3.11)
Now we can finalize the proof of (2.6). For a given \( t > 0 \), fix \( t_0 \in (0, t) \) and \( \zeta > 0 \), and consider the set

\[
C_{\zeta,t_0,t} = \{ \theta_{\zeta}(t_0) \leq t \} \subseteq \mathcal{F}_{\theta_{\zeta}(t_0)}.
\]

Then by (3.6) and Doob’s optional sampling theorem, we have

\[
E|V^\varepsilon_t|^2 \mathbb{I}_{C_{\zeta,t_0,t}} \leq E|V^\varepsilon_{\theta_{\zeta}(t_0)}|^2 \mathbb{I}_{C_{\zeta,t_0,t}} + E(t - \theta_{\zeta}(t_0)) \mathbb{I}_{C_{\zeta,t_0,t}} \left( \int_{-A}^A z^2 \mu^\varepsilon(dz) \right).
\]

This implies that

\[
P(|V^\varepsilon_t| > \delta) \leq P(\Omega \setminus C_{\zeta,t_0,t}) + \frac{\zeta^2}{\delta^2} + \frac{t - t_0}{\delta^2} \left( \int_{-A}^A z^2 \mu^\varepsilon(dz) \right),
\]

and by (3.11) we have

\[
\limsup_{\varepsilon \to 0} P(|V^\varepsilon_t| > \delta) \leq \frac{\zeta^2}{\delta^2} + \frac{t - t_0}{\delta^2} \left( \int_{-A}^A z^2 \mu^\varepsilon(dz) \right).
\]

Since \( \zeta > 0 \) and \( t_0 < t \) are arbitrary, this proves (2.6). \( \blacksquare \)

4 Proof of Theorem 2.3: regular case

4.1 Outline

In this section, we prove Theorem 2.3 assuming

\[
(\alpha, \beta) \in \Xi_{\text{regular}} = \left\{ \alpha \in [0, 2], \alpha + \beta < 2 \right\} \cup \{(2, 0)\},
\]

see Fig. 1. We call this case regular. Let us explain this name together with the main idea of the proof.

Let us apply, yet just formally, the Itô formula to the function \( F(v) = \frac{1}{1+2-\beta} |v|^{2-\beta} \text{sgn} v \) and the process \( V^\varepsilon \) given by (2.4):

\[
F(V^\varepsilon_t) = F(v_0) - \frac{1}{\varepsilon} \int_0^t V^\varepsilon_s \, ds + M^\varepsilon_t + \int_0^t H^\varepsilon(V^\varepsilon_s) \, ds,
\]

where

\[
M^\varepsilon_t = \int_0^t \int_R \left( F(V^\varepsilon_s + z) - F(V^\varepsilon_s) \right) \tilde{N}^\varepsilon(dz) \, ds,
\]

\[
H^\varepsilon(v) = \int_0^\infty \left( F(v + z) + F(v - z) - 2F(v) \right) \mu^\varepsilon(dz).
\]
Then
\[ X_\varepsilon^t + F(V_\varepsilon^t) = x_0 + \frac{1}{\varepsilon} \int_0^t V_\varepsilon^s \, ds + F(V_\varepsilon^0) = x_0 + F(v_0) + M_\varepsilon^t + \int_0^t H_\varepsilon(V_\varepsilon^s) \, ds, \tag{4.5} \]

By Theorem 2.2 we have
\[ F(V_\varepsilon^t) \to 0, \quad \varepsilon \to 0 \]
in probability, and by \( (3.10) \) one can expect to have
\[ M_\varepsilon^t - M_\varepsilon^0 \to 0, \quad \varepsilon \to 0 \tag{4.6} \]
in probability; here and below we denote
\[ M_\varepsilon^t := \int_0^t \left( F(0 + z) - F(0) \right) \tilde{N}_\varepsilon(dz) \, ds = \int_0^t F(z) \tilde{N}_\varepsilon(dz) \, ds. \]

It is easy to show that
\[ M_\varepsilon^t \overset{\text{f.d.d.}}{\to} \frac{1}{2 - \beta} \int_0^\varepsilon \int_0^\varepsilon |z|^{2-\beta} \text{sgn} z \, d\tilde{N}(dz) \, d\varepsilon, \quad \varepsilon \to 0. \tag{4.7} \]

Hence, to prove the required statement, it will be enough to show that
\[ \int_0^t H_\varepsilon(V_\varepsilon^s) \, ds \to 0, \quad \varepsilon \to 0. \tag{4.8} \]

We note that, up to a certain point, this argument follows the strategy, frequently used in limit theorems, based on the use of a correction term. In one of its standard forms, which dates back to Gordin (1969) (see also Gordin and Lifshits (1978)), the correction term approach assumes that one adds to the process an asymptotically negligible term, which transforms it into a martingale. In our framework, the classical correction term would have the form \( F_\varepsilon(V_\varepsilon^t) \), where \( F_\varepsilon \) is the solution to the Poisson equation
\[ L^\varepsilon F_\varepsilon(v) = -v, \]
where
\[ L^\varepsilon f(v) = -|v|^\beta \text{sgn} v \cdot f'(v) + \varepsilon \int_\mathbb{R} \left( f(v + z) - f(v) - f'(v)z \right) \mu_\varepsilon(dz) \]
is the generator of the velocity process \( v^\varepsilon \) at the “microscopic time scale”. Since we are not able to specify the solution \( F_\varepsilon \) to the Poisson equation, we use instead the function \( F \), which in this context is just the solution to equation
\[ L^0 F(v) = -v, \quad L^0 f(v) = -|v|^\beta \text{sgn} v \cdot f'(v). \]
Hence \( F \) can be understood as an approximate solution to the Poisson equation, and thus we call the entire argument the approximate correction term approach. Note that the non-martingale term
\[ \int_0^t H_\varepsilon(V_\varepsilon^s) \, ds, \]
appears in \( (4.5) \) exactly because the exact solution to the Poisson equation is replaced by an approximate one. In what follows we will show that such an approximation is precise enough, and this integral term is negligible.

Of course, this is just an outline of the argument, and we have to take care about numerous technicalities. For \( \beta < 0 \) or \( \beta = 1 \), the function \( F \) belongs to \( C^2(\mathbb{R}, \mathbb{R}) \) and thus \( (4.2) \) follows by the usual Itô formula. Otherwise, we yet have to justify this relation, e.g. by an approximation procedure. We are actually able to do that when \( (\alpha, \beta) \in \Xi_{\text{regular}} \); see Lemma A.2 in Appendix. Note that this is exactly the case, where the functions \( H_\varepsilon \) can be proved to be equicontinuous at the point \( v = 0 \), see Lemma A.1. Otherwise, the functions \( H_\varepsilon \) are typically discontinuous, or even unbounded near the origin (see Fig. 2) which makes the entire approach hardly applicable.

To summarize: when \( (\alpha, \beta) \in \Xi_{\text{regular}} \), the function \( F \) is regular enough to allow the Itô formula to be applied, and the family \( \{H_\varepsilon\} \) is equicontinuous at \( v = 0 \), which makes it possible to derive \( (4.5) \) from the convergence \( V_\varepsilon^t \to 0, \varepsilon \to 0 \). This is why we call this case regular.
Hence the above estimate provides that for each \( \delta > 0 \),

By (2.5),

\[ A > 0. \]

prove (2.11) under the additional assumption (3.5), then we actually have (2.11) in the general setting.

We will use the same “truncation of large jumps” argument which now has the following form: if we can

\[ \{ \alpha, \beta \} = (1.2, 1.1) \]

\[ (\alpha, \beta) \notin \Xi_{\text{regular}} \]

\[ (\alpha, \beta) = (1.2, 0.8) \]

\[ (\alpha, \beta) \in \Xi_{\text{regular}} \]

\[ (\alpha, \beta) = (1.2, 0.4) \]

\[ (\alpha, \beta) \in \Xi_{\text{regular}} \]

\[ \tau R = \inf \{ t : |V R(\epsilon)| > R \}, \]

\[ M_{t, \delta} = \int_0^t \int_{|z| > R} \left( F(V s_+ - z) - F(V s_-) \right) \bar{N}_\delta(z) dz \]

\[ M_{t, \delta}^\epsilon = \int_0^t \int_{|z| > R} \left( F(V s_+ - z) - F(V s_-) \right) \bar{N}_\delta(z) dz. \]

\[ \sup_{\epsilon \in (0, 1]} P(\tau R^\epsilon < t) \rightarrow 0, \quad R \rightarrow \infty. \]

(4.9)

Hence the above estimate provides that for each \( \delta > 0 \),

\[ M_{t, \delta}^\epsilon - M_{t, \delta} \rightarrow 0, \quad \epsilon \rightarrow 0 \]

in probability.

Next, we have

\[ \sup_{\epsilon \in (0, 1]} E \left( M_{t, \delta}^\epsilon - M_{t, \delta} \right)^2 \leq t \int_{|z| \leq \delta} \left( F(z) \right)^2 \mu'(dz) \rightarrow \epsilon 0. \]

(4.11)
Figure 3: The domain of parameters \((\alpha, \beta)\) corresponding to the non-regular/quasi-ergodic case.

If \(\beta \in [1, 2)\), the function \(F\) is Hölder continuous with the index \(2 - \beta\), and for \(M^\varepsilon\) we have essentially the same estimate:

\[
\sup_{\varepsilon \in (0, 1]} \mathbf{E}\left(M_t^\varepsilon - M_t^{\varepsilon, \delta}\right)^2 = t \int_{|z| \leq \delta} (F(z))^2 \mu^\varepsilon(dz) \leq C\delta^{4 - \alpha - 2\beta} \rightarrow 0, \quad \delta \rightarrow 0.
\]

If \(\beta < 1\), the function \(F\) has a locally bounded derivative, which gives for arbitrary \(R\)

\[
\sup_{\varepsilon \in (0, 1]} \mathbf{E}\left(M_{t \wedge \tau_R}^\varepsilon - M_{t \wedge \tau_R}^{\varepsilon, \delta}\right)^2 \leq tC_R \sup_{\varepsilon \in (0, 1]} \int_{|z| \leq \delta} z^2 \mu^\varepsilon(dz) \rightarrow 0, \quad \delta \rightarrow 0.
\]

In both these cases, we have for arbitrary \(c > 0\)

\[
\sup_{\varepsilon \in (0, 1]} \mathbf{P}\left(|M_t^\varepsilon - M_t^{\varepsilon, \delta}| > c\right) \rightarrow 0, \quad \delta \rightarrow 0. \tag{4.12}
\]

Combining (4.10), (4.11), and (4.12), we complete the proof of (4.6).

Each \(M^\varepsilon\) is a Lévy process. Since (3.4) and (4.11) hold true and \(F\) is continuous, we have for any \(t \geq 0\) and \(\lambda \in \mathbb{R}\)

\[
\mathbf{E}e^{i\lambda M_t^\varepsilon} = \exp\left(t \int (e^{i\lambda F(z)} - 1) \mu^\varepsilon(dz)\right) \rightarrow \exp\left(t \int (e^{i\lambda F(z)} - 1) \mu(dz)\right), \quad \varepsilon \rightarrow 0,
\]

which gives (4.7). This completes the proof of the Theorem.

**Remark 4.1** In the proof of (4.6) and (4.7), we have not used the regularity assumption \((\alpha, \beta) \in \Xi_{\text{regular}}\) and proved these relations under the principal assumption \(\alpha + 2\beta < 4\) combined with the auxiliary truncation assumption (3.5).

5 Proof of Theorem 2.3: non-regular/quasi-ergodic case

5.1 Outline

In this section, we prove Theorem 2.3 assuming

\[
(\alpha, \beta) \notin \Xi_{\text{regular}}.
\]

Combined with the principal assumption \(\alpha + 2\beta < 4\), this yields

\[
\alpha > 0, \quad \beta > 0,
\]

see Fig. 3.
We call this case non-regular and quasi-ergodic. Let us explain the latter name and outline the proof. We make the change of variables

$$Y_t^\varepsilon = \varepsilon^{-\gamma} V_{t\varepsilon^{\alpha}}, \quad \gamma = \frac{1}{\alpha + \beta - 1} > 0,$$

so that the new process $Y^\varepsilon$ satisfies the SDE

$$Y_t^\varepsilon = Y_0^\varepsilon - \int_0^t |Y_s^\varepsilon|^\beta \text{sgn} Y_s^\varepsilon \, ds + U_t^\varepsilon$$

(5.1)

with a Lévy process

$$U_t^\varepsilon = \varepsilon^{-\gamma} Z_{t\varepsilon^{\alpha}},$$

(5.2)

with a symmetric jump measure $\nu$. Such a space-time rescaling transforms the equation for the velocity in the original system (2.4) to a similar one, but without the term $1/\varepsilon$. In terms of $Y^\varepsilon$, the expression for $X^\varepsilon$ takes the form

$$X_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t V_s^\varepsilon \, ds = \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} Y_s^\varepsilon \, ds.$$  

(5.3)

In the particularly important case where $Z^\varepsilon = Z$ and $Z$ is symmetric $\alpha$-stable, each process $U^\varepsilon$ has the same law as $Z$, and thus the law of the solution to (5.1) does not depend on $\varepsilon$. The corresponding Markov processes $Y^\varepsilon$ are also equal in law and ergodic for $\alpha + \beta > 1$, see [Kulik, 2017, Section 3.4]. Hence one can expect the limit behaviour of the re-scaled integral functional (5.3) to be well controllable. We confirm this conjecture in the general (not necessarily $\alpha$-stable) case, which we call quasi-ergodic because, instead of one ergodic process $Y$ we have to consider a family of processes $\{Y^\varepsilon\}$, which, however, possesses a certain uniform stabilization property as $t \to \infty$ thanks to dissipativity of the drift coefficient in (5.1).

To study the limit behaviour of $X^\varepsilon$, we will follow the approximate corrector term approach, similar to the one used in Section 4. On this way, we meet two new difficulties. The first one is minor and technical: since we assume $(\alpha, \beta) \not\in \Xi_{\text{regular}}$, we are not able to apply the Itô formula to the function $F$, see Fig. 2.

Consequently we consider a mollified function

$$\hat{F} = F + \tilde{F},$$

where $\tilde{F}$ is an odd continuous function, vanishing outside of $[-1,1]$, and such that $\hat{F} \in C^3(\mathbb{R}, \mathbb{R})$. Now the Itô formula is applicable:

$$\hat{F}(Y_t^\varepsilon) = \hat{F}(Y_0^\varepsilon) - \int_0^t \hat{F}(Y_s^\varepsilon)|Y_s^\varepsilon|^\beta \text{sgn}(Y_s^\varepsilon) \, ds + m_t^\varepsilon + \int_0^t \tilde{J}^s(Y_s^\varepsilon) \, ds,$$

where

$$m_t^\varepsilon = \int_0^t \int_{\mathbb{R}} (\hat{F}(Y_{s-}^\varepsilon) + u) - \hat{F}(Y_{s-}^\varepsilon) \, \nu^\varepsilon(du) \, ds,$$

$$\tilde{J}^s(y) = \int_0^\infty (\hat{F}(y + u) + \tilde{F}(y - u) - 2\hat{F}(y)) \nu^\varepsilon(du),$$

see the notation in Section 5.2 below. This gives

$$X_t^\varepsilon + \varepsilon^{(2-\beta)\gamma} \hat{F}(Y_{t\varepsilon^{\alpha\gamma}}^\varepsilon) = x_0 + \varepsilon^{(2-\beta)\gamma} \hat{F}(Y_0^\varepsilon) + \varepsilon^{(2-\beta)\gamma} m_{t\varepsilon^{-\alpha\gamma}} + \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\alpha\gamma}} R^\varepsilon(Y_s^\varepsilon) \, ds,$$

(5.4)

where

$$R^\varepsilon(y) = -\hat{F}'(y)|y|^\beta \text{sgn} y + y \, \hat{J}^s(y) = -\hat{F}'(y)|y|^\beta \text{sgn} y + \hat{J}^s(y).$$

(5.5)

This representation is close to (1.3). This relation becomes even more visible, when one observes that

$$\varepsilon^{(2-\beta)\gamma} F(Y_{t\varepsilon^{\alpha\gamma}}^\varepsilon) = F(V_t^\varepsilon).$$
Then (5.4) can be written as

\[
X_t^\varepsilon + F(Y_t^\varepsilon) = x_0 + F(v_0) + M_t^\varepsilon + \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{\alpha-\gamma}} R^\varepsilon(Y_s^\varepsilon) \, ds \\
- \varepsilon^{(2-\beta)\gamma} \bar{F}(Y_{t\varepsilon^{\alpha-\gamma}}^\varepsilon) + \varepsilon^{(2-\beta)\gamma} \bar{F}(Y_0^\varepsilon) + \varepsilon^{(2-\beta)\gamma} \bar{m}_t^\varepsilon_{t\varepsilon^{\alpha-\gamma}}
\]

with

\[
\bar{m}_t^\varepsilon = \int_R \int_\mathbb{R} (\bar{F}(Y_{s+}^\varepsilon + u) - \bar{F}(Y_{s}^\varepsilon)) \bar{n}(du \, ds).
\]

Since \(\bar{F}\) is bounded and \(\beta < 2\), the terms \(\varepsilon^{(2-\beta)\gamma} \bar{F}(Y_{t\varepsilon^{\alpha-\gamma}}^\varepsilon)\) and \(\varepsilon^{(2-\beta)\gamma} \bar{F}(Y_0^\varepsilon)\) are obviously negligible. Also, it will be not difficult to show that the last term in (5.6) is negligible, as well:

\[
\varepsilon^{(2-\beta)\gamma} \bar{m}_t^\varepsilon_{t\varepsilon^{\alpha-\gamma}} \to 0, \quad \varepsilon \to 0,
\]

in probability. Recall that we have (4.6) and (4.7), see Remark 4.1. Eventually, to establish (2.11), it is enough to show that

\[
\varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{\alpha-\gamma}} R^\varepsilon(Y_s^\varepsilon) \, ds \to 0, \quad \varepsilon \to 0,
\]

in probability. The second, more significant, difficulty which we encounter now is that this relation cannot be obtained in the same way we did that in Section 4. Instead, we will prove (5.8) using the stabilization properties in the neighbourhood of the point \(v = 0\), see Fig. 2. We have for each \(\delta > 0\)

\[
\sup_{|v| > \delta} |\tilde{H}^\varepsilon(v) - H^\varepsilon(v)| \to 0, \quad \varepsilon \to 0,
\]

the proof is postponed to Appendix C. Thus the family \(\{\tilde{H}^\varepsilon\}_{\varepsilon \in (0,1]}\) is unbounded, and one can hardly derive (5.8) from (5.10), like we did that in Section 4. Instead, we will prove (5.8) using the stabilization properties of the family \(\{Y^\varepsilon\}\).

### 5.2 Preliminaries to the proof

In what follows we assume (3.5) to hold true, i.e. the jumps of the processes \(Z^\varepsilon\) are bounded by some \(A > 0\). Using the “truncation of large jumps” trick from the previous section, we guarantee that this assumption does not restrict the generality. We denote by \(\nu^\varepsilon\) the Lévy measure of the Lévy process \(U^\varepsilon\) introduced in (5.2), and by \(n^\varepsilon\) and \(\bar{n}^\varepsilon\) the corresponding Poisson and compensated Poisson random measures. More precisely, for \(B \in \mathcal{B}(\mathbb{R})\) and \(s \geq 0\)

\[
\nu^\varepsilon(B) := \varepsilon^{\alpha\gamma} \mu(z : \varepsilon^{-\gamma}z \in B),
\]

\[
n^\varepsilon(B \times [0, s]) := N^\varepsilon \left( (z, t) : (\varepsilon^{-\gamma}z, \varepsilon^{\alpha\gamma}t) \in B \times [0, s] \right),
\]

\[
\bar{n}^\varepsilon(du \, ds) := n^\varepsilon(du \, ds) - \nu^\varepsilon(du) \, ds.
\]

Each of the measures \(\nu^\varepsilon\) is symmetric, and

\[
\nu^\varepsilon(u : |u| > r) = \varepsilon^{\alpha\gamma} \mu(z : |z| > \varepsilon^{\gamma}r), \quad r > 0.
\]

Hence we have the following analogue of (2.7):

\[
\sup_{\varepsilon \in (0,1]} \nu^\varepsilon(u : |u| \geq r) \leq Cr^{-\alpha}, \quad r > 0,
\]

(5.11)
see also (A.1). In addition, we have

\[ \text{supp } \nu^\varepsilon \subseteq [-A \varepsilon^{-\gamma}, A \varepsilon^{-\gamma}] \]  

(5.12)

by the assumption (5.6), and

\[ \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (u^2 \land 1) \nu^\varepsilon(du) < \infty. \]  

(5.13)

The latter inequality follows directly from (5.11) for \( \alpha < 2 \). For \( \alpha = 2 \), one should also use (2.8), which gives

\[ \int_{|u| \leq 1} u^2 \nu^\varepsilon(du) = \int_{|z| \leq \varepsilon^\gamma} u^2 \mu^\varepsilon(dz) \to 0, \quad \varepsilon \to 0. \]

Using these relations, it is easy to derive (5.7). Since \( \hat{F} \in C^2(\mathbb{R}, \mathbb{R}) \) and \( \bar{F} = \hat{F} - F \) is compactly supported, \( \bar{F} \) is \((2 - \beta)\)-Hölder continuous for \( \beta \geq 1 \) and is Lipschitz continuous if \( \beta < 1 \). In addition, \( \bar{F} \) is bounded, which gives

\[ \mathbb{E}(\varepsilon^{(2-\beta)\gamma} \bar{m}_{1-\alpha\gamma}^\varepsilon)^2 \leq C\varepsilon^{4-2\beta - \alpha} \int_{\mathbb{R}} (|u|^{4-2\beta} \land 1) \nu^\varepsilon(du) \]

if \( \beta \geq 1 \), and

\[ \mathbb{E}(\varepsilon^{(2-\beta)\gamma} \bar{m}_{1-\alpha\gamma}^\varepsilon)^2 \leq C\varepsilon^{4-2\beta - \alpha} \int_{\mathbb{R}} (u^2 \land 1) \nu^\varepsilon(du) \]

if \( \beta < 1 \). In the latter case, (5.7) follows by (5.13) and the basic assumption \( \alpha + 2\beta < 4 \). For \( \beta \geq 1 \), we have (5.7) by

\[ \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (|u|^{4-2\beta} \land 1) \nu^\varepsilon(du) < \infty, \]

which follows from (5.11).

Let us explain the strategy of the proof of (5.8). The process \( Y^\varepsilon \) being a solution to (5.1) is a Markov process. Let us denote by \( P^y_{Y^\varepsilon} \) its law of this process with \( Y^\varepsilon_0 = y \), and by \( E^y_{Y^\varepsilon} \) the corresponding expectation. Then

\[ \mathbb{E} \left( \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\gamma}} R^\varepsilon(Y^\varepsilon_s) \, ds \right)^2 = 2\varepsilon^{(4-2\beta)\gamma} \mathbb{E} \int_0^{t\varepsilon^{-\gamma}} \left( R^\varepsilon(Y^\varepsilon_s) \cdot E^y_{Y^\varepsilon} \int_0^{t\varepsilon^{-\gamma}-s} R^\varepsilon(Y^\varepsilon_r) \, dr \right) \, ds. \]

Our aim will be to construct a non-negative function \( Q \) such that, for some \( c, C > 0 \),

- for all \( y \in \mathbb{R}, t > 0, \) and \( \varepsilon > 0 \)

\[ R^\varepsilon(y) E^y_{Y^\varepsilon} \int_0^t R^\varepsilon(Y^\varepsilon_s) \, ds \leq c Q(y); \]  

(5.14)

- for all \( t > 0 \) and \( \varepsilon > 0 \)

\[ \mathbb{E} \int_0^t Q(Y^\varepsilon_s) \, ds \leq c \cdot C \left( 1 + t + |Y^\varepsilon_0|^\alpha \right). \]  

(5.15)

Since \( Y^\varepsilon_0 = \varepsilon^{-\gamma} \nu_0 \), this will provide (5.8) since

\[ \mathbb{E} \left( \varepsilon^{(2-\beta)\gamma} \int_0^{t\varepsilon^{-\gamma}} R^\varepsilon(Y^\varepsilon_s) \, ds \right)^2 \leq C\varepsilon^{(4-2\beta)\gamma} \left( 1 + t \varepsilon^{-\alpha\gamma} + |\nu_0|^{\alpha \varepsilon^{-\alpha\gamma}} \right) \to 0 \]

by the principal assumption \( \alpha + 2\beta < 4 \).

The inequality (5.14) can be obtained in quite a standard way, based on a proper Lyapunov-type condition, see e.g. Section 2.8.2 and Section 3.2 in Kulik (2017). For the reader’s convenience, we explain how this simple, but important argument can be applied in the current setting. Denote for \( g \in C^2(\mathbb{R}, \mathbb{R}) \)

\[ \mathcal{L}^\varepsilon g(y) = -|y|^\beta \text{sgn } v \cdot g'(y) + \int_0^\infty \left( g(y + u) + g(y - u) - 2g(y) \right) \nu^\varepsilon(du). \]  

(5.16)
Lemma 5.1 Let a non-negative $G \in C^2(\mathbb{R}, \mathbb{R})$ be such that for some $c_1, c_2 > 0$

$$\mathcal{A}^\varepsilon G(y) \leq -c_1 Q(y) + c_2, \quad \varepsilon > 0. \quad (5.17)$$

Then for all $t \geq 0$ and $\varepsilon > 0$

$$\mathbb{E} \int_0^t Q(Y_s^\varepsilon) \, ds \leq \frac{1}{c_1} G(Y_0^\varepsilon) + \frac{c_2}{c_1} t. \quad (5.18)$$

Proof: By the Itô formula,

$$G(Y_t^\varepsilon) = \int_0^t \mathcal{A}^\varepsilon G(Y_s^\varepsilon) \, ds + \mathcal{M}_t^\varepsilon,$$

where $\mathcal{M}_t^\varepsilon$ is a local martingale. Let $\tau_n^\varepsilon \nearrow \infty$ be a localizing sequence for $\mathcal{M}_t^\varepsilon$, then

$$\mathbb{E} \int_0^{t \wedge \tau_n^\varepsilon} Q(Y_s^\varepsilon) \, ds \leq \frac{c_2}{c_1} t - \frac{1}{c_1} \mathbb{E} \int_0^{t \wedge \tau_n^\varepsilon} \mathcal{A}^\varepsilon G(Y_s^\varepsilon) \, ds$$

$$= \frac{c_2}{c_1} t + \frac{1}{c_1} \mathbb{E} G(Y_0^\varepsilon) - \frac{1}{c_1} \mathbb{E} G(Y_{t \wedge \tau_n^\varepsilon}) \leq \frac{c_2}{c_1} t + \frac{1}{c_1} G(Y_0^\varepsilon).$$

We complete the proof passing to the limit $n \to \infty$ and applying the Fatou lemma.

Now we specify the functions $G$ and $Q$ which we plug into this general statement. Fix

$$p \in (\beta - 1, \alpha + \beta - 1), \quad (5.18)$$

recall that $\alpha > 0$ and therefore the above interval is non-empty. Let a non-negative $G \in C^2(\mathbb{R}, \mathbb{R})$ be such that

$$G(y) \equiv 0 \text{ in some neighbourhood of 0},$$

$$G(y) \leq |y|^{p+1-\beta}, \quad |y| \leq 1, \quad (5.19)$$

$$G(y) = |y|^{p+1-\beta}, \quad |y| > 1.$$

Then for $|y| \geq 1$

$$\mathcal{A}^\varepsilon G(y) = -(p + 1 - \beta)|y|^p + K^\varepsilon(y), \quad (5.20)$$

where

$$K^\varepsilon(y) = \int_0^\infty \left(G(y + u) + G(y - u) - 2G(y)\right) \nu^\varepsilon(du).$$

The function $G$ satisfies the assumptions of Lemma 3.1 with $\sigma = p + 1 - \beta$; note that assumption 6.18 means that $\sigma \in (0, \alpha)$. Since

$$\sigma - \alpha = p + 1 - \alpha - \beta < p,$$

we have by Lemma 3.1

$$\sup_{\varepsilon \in (0,1)} |y|^{-p} K^\varepsilon(y) \to 0, \quad y \to \infty. \quad (5.21)$$

In addition, by the same Lemma the family $\{K^\varepsilon\}_{\varepsilon \in (0,1]}$ is uniformly bounded on each bounded set, hence the same property holds true for the family $\{\mathcal{A}^\varepsilon G\}_{\varepsilon \in (0,1]}$. This provides (5.17) with $G$ specified above,

$$Q(y) = 1 + |y|^p,$$

and properly chosen $c_1, c_2$. Eventually by construction we have

$$G(y) \leq |y|^{p+1-\beta} \leq C(1 + |y|^\alpha),$$

therefore (5.15) holds true by Lemma 5.1.

By Lemma 3.2 the family $\{R^\varepsilon\}_{\varepsilon \in (0,1]}$ satisfies

$$|R^\varepsilon(y)| \leq C(1 + |y|)^{2-\alpha-\beta} \ln(2 + |y|).$$
Hence, to prove the bound (5.14) with $Q$ specified above, it is enough to show that, for some $p' < p$

$$\left| E_{y}^{\varepsilon} \int_{0}^{t} R^{\varepsilon}(Y_{s}^{\varepsilon}) \, ds \right| \leq C(1 + |y|)^{p' + \alpha + \beta - 2}, \quad t > 0, \quad \varepsilon > 0. \quad (5.22)$$

In the rest of the proof, we verify this relation for properly chosen $p'$. We fix $y$, and (with a slight abuse of notation) denote by $Y^{\varepsilon}$, $Y_{0}^{\varepsilon}$ the strong solutions to (5.11) with the same process $U^{\varepsilon}$ and initial conditions $Y_{0}^{\varepsilon} = y$, $Y_{0}^{\varepsilon} = 0$. Recall that the Lévy process $U^{\varepsilon}$ is symmetric. Since the drift coefficient $-|y|^{\beta} \text{ sgn} \, y$ in (5.11) is odd, the law of $Y_{0}^{\varepsilon}$ is symmetric as well. By Lemma B.2 the family of functions $\{R^{\varepsilon}\}_{\varepsilon \in (0,1]}$ is bounded: if $\alpha + \beta > 2$ this is straightforward, for $\alpha + \beta = 2$ one should recall that in the non-regular case this identity excludes the case $\alpha = 2$, see Fig. 3. It is also easy to verify that functions $R^{\varepsilon}$ are odd, which gives

$$E R^{\varepsilon}(Y_{t}^{\varepsilon}) = 0, \quad t \geq 0, \quad \varepsilon > 0. \quad \text{(5.23)}$$

Then

$$\left| E_{y}^{\varepsilon} \int_{0}^{t} R^{\varepsilon}(Y_{s}^{\varepsilon}) \, ds \right| = \left| E \int_{0}^{t} R^{\varepsilon}(Y_{s}^{\varepsilon}) \, ds - E \int_{0}^{t} R^{\varepsilon}(Y_{s}^{\varepsilon}) \, ds \right| \leq \int_{0}^{t} E|R^{\varepsilon}(Y_{s}^{\varepsilon}) - R^{\varepsilon}(Y_{s}^{\varepsilon})| \, ds.$$ 

This bound will allow us to prove (5.22) using the dissipation, brought to the system by the drift coefficient $-|y|^{\beta} \text{ sgn} \, y$. In what follows, we consider separately two cases: $\beta \in [1, 2)$ ("strong dissipation") and $\beta \in (0, 1)$ ("Hölder dissipation").

### 5.3 Strong dissipation: $\beta \in [1, 2)$

Since the noise in the SDE (5.1) is additive, the difference $t \mapsto \Delta_{t}^{\varepsilon} = Y_{t}^{\varepsilon} - Y_{t}^{\varepsilon,0}$ is an absolutely continuous function and

$$d\Delta_{t}^{\varepsilon} = \left(|Y_{t}^{\varepsilon}|^{\beta} \text{ sgn} \, Y_{t}^{\varepsilon} - |Y_{t}^{\varepsilon,0}|^{\beta} \text{ sgn} \, Y_{t}^{\varepsilon,0}\right) \, dt.$$ 

Since $\Delta \mapsto |\Delta|$ is Lipschitz continuous, $t \mapsto |\Delta_{t}^{\varepsilon}|$ is an absolutely continuous function as well with

$$d|\Delta_{t}^{\varepsilon}| = -\left(|Y_{t}^{\varepsilon}|^{\beta} \text{ sgn} \, Y_{t}^{\varepsilon} - |Y_{t}^{\varepsilon,0}|^{\beta} \text{ sgn} \, Y_{t}^{\varepsilon,0}\right) \text{sgn}(Y_{t}^{\varepsilon} - Y_{t}^{\varepsilon,0}) \, dt.$$ 

For $\beta \in (1, 2)$ we have the inequality

$$-(|y_{1}|^{\beta} \text{ sgn} \, y_{1} - |y_{2}|^{\beta} \text{ sgn} \, y_{2}) \text{ sgn}(y_{1} - y_{2}) \leq -2^{-\beta}|y_{1} - y_{2}|^{\beta}, \quad y_{1}, y_{2} \in \mathbb{R},$$

hence

$$\frac{d}{dt}|\Delta_{t}^{\varepsilon}| \leq -2^{-\beta}|\Delta_{t}^{\varepsilon}|^{\beta}.$$ 

Denote by $\Upsilon$ the solution to the ODE

$$\frac{d}{dt} \Upsilon_{t} = -2^{-\beta} \Upsilon_{t}^{\beta}, \quad \Upsilon_{0} = |\Delta_{0}^{\varepsilon}| = |y|.$$ 

Then by the comparison theorem [Lakshmikantham and Leela, 1969, Theorem 1.4.1] $|\Delta_{t}^{\varepsilon}| \leq \Upsilon_{t}, t \geq 0$. This solution is explicit:

$$\Upsilon_{t} = \begin{cases} |y|e^{-2^{-\beta}t}, & \beta = 1, \\ \left(|y|^{1-\beta} + 2^{-\beta}(\beta - 1)t\right)^{\frac{1}{\beta - 1}}, & \beta > 1, \end{cases}$$ 

and we have

$$\int_{0}^{\infty} \Upsilon_{t} \, dt = \frac{2^{\beta}}{2 - \beta}|y|^{2-\beta}.$$ 

By Lemma B.3 derivatives of the functions $R^{\varepsilon}$ are uniformly bounded, which gives for some $C > 0$

$$\left| E_{y}^{\varepsilon} \int_{0}^{t} R^{\varepsilon}(Y_{s}^{\varepsilon}) \, ds \right| \leq C \int_{0}^{t} E|\Delta_{s}^{\varepsilon}| \, ds \leq C \int_{0}^{t} \Upsilon_{s} \, ds \leq C|y|^{2-\beta}.$$
Eventually we obtain \( (5.22) \) with
\[
p' = 4 - \alpha - 2\beta > 0.
\]

If \( \beta \in (1, 2) \), we have
\[
p' - \alpha = 2(2 - \alpha - \beta) \leq 0 < \beta - 1,
\]
that is,
\[
p' < \alpha + \beta - 1.
\]
Then we can take \( p \in (p', \alpha + \beta - 1) \) and get that, for \( Q(y) = 1 + |y|^p \), both \( (5.14) \) and \( (5.15) \) hold true which provides \( (5.8) \) and completes the entire proof.

For \( \beta = 1 \), the same argument applies with just a minor modification. Namely, since the functions \( \{R^n\}_{\varepsilon \in (0,1]} \) are uniformly bounded and have uniformly bounded derivatives, for each \( \kappa \in (0, 1) \) these functions are uniformly \( \kappa \)-Hölder equicontinuous:
\[
|R^n(y_1) - R^n(y_2)| \leq C|y_1 - y_2|\kappa, \quad y_1, y_2 \in \mathbb{R}, \quad \varepsilon > 0.
\]
Hence
\[
\left|E^{\varepsilon, \kappa}_y \int_0^t R^n(Y^n_s) \, ds \right| \leq C \int_0^t E[|D^n s|] \, ds \leq C \int_0^t \gamma^n s \, ds \leq C|y|^\kappa.
\]
That is, we have \( (5.22) \) with
\[
p' = \kappa + 1 - \alpha.
\]
Note that for \( \beta = 1 \) the principal assumption \( \alpha + 2\beta < 4 \) yields \( \alpha < 2 \), hence \( p' \) can be made positive by taking \( \kappa < 1 \) close enough to 1. On the other hand, we are considering the non-regular case now, hence \( \alpha \geq 2 - \beta = 1 \). That is,
\[
p' \leq \kappa < 1 \leq \alpha + \beta - 1.
\]
Again, we can take \( p \in (p', \alpha + \beta - 1) \) and get that, for \( Q(y) = 1 + |y|^p \), both \( (5.14) \) and \( (5.15) \) hold true, which provides \( (5.8) \) and completes the entire proof.

### 5.4 Hölder dissipation: \( \beta \in (0, 1) \)

Now the situation is more subtle because, instead of \( (5.23) \), which holds true on the entire \( \mathbb{R} \), we have only a family of local inequalities. Namely, one can easily show that, for each \( D > 0 \), there exists \( c_D > 0 \) such that
\[
-([y_1]^{\beta} \text{sgn} y_1 - [y_2]^{\beta} \text{sgn} y_2) \text{sgn}(y_1 - y_2) \leq -c_D |y_1 - y_2|^{\beta} \wedge |y_1 - y_2| \cdot \mathbb{I}_{|y_2| \leq D}, \quad y_1, y_2 \in \mathbb{R}.
\]
We will prove \( (5.22) \) in two steps, considering separately the cases \( |y| \leq D \) and \( |y| > D \) for some fixed \( D \). In both these cases, we will require the following recurrence bound. Denote
\[
\theta_D = \inf\{t \geq 0 : |Y^n_t| \leq D\}.
\]

**Lemma 5.2** Let \( p \in (0, \alpha + \beta - 1) \) be fixed. Then there exist \( D > 1 \) large enough and a constant \( C > 0 \) such that
\[
E^{\varepsilon, \kappa}_y (\theta_D^{p+1-\beta}) \leq C|y|^{p+1-\beta}, \quad y \in \mathbb{R}.
\]

**Proof:** Since \( \beta \in (0, 1) \), we have \( p > 0 > \beta - 1 \). That is, \( p \) satisfies \( (6.11) \), and for the function \( G \) given by \( (6.19) \), the inequality \( (6.20) \) holds true. By \( (6.21) \), we can fix \( D \) large enough and some \( c > 0 \) such that
\[
\mathcal{A}^{\varepsilon} G(y) \leq -c(G(y))^{p/(p+1-\beta)}, \quad |y| > D.
\]
Note that, by the Itô formula,
\[
G(Y^n_t) = G(y) + \int_0^t \mathcal{A}^{\varepsilon} G(Y^n_s) \, ds + M^{G, \varepsilon}_t,
\]
where the local martingale $M^{G,\epsilon}$ is given by

$$M^{G,\epsilon}_t = \int_0^t \int_{\mathbb{R}} \left(G(Y^\epsilon_{s-} + u) - G(Y^\epsilon_{s-})\right) \tilde{n}^\epsilon(du \, ds).$$

The rest of the proof is based on the general argument explained in [Hairer, 2016, Section 4.1.2]; see also [Kuik, 2017, Lemma 3.2.4] and [Eberle et al., 2010, Lemma 2]. Denote $q = (p + 1 - \beta)/(1 - \beta) > 1$ and let

$$H(t, g) = \left(\frac{c}{q} t + g^{1/q}\right)^q.$$

Then

$$H'_t(t, g) = c \left(\frac{c}{q} t + g^{1/q}\right)^{q-1} = c H(t, g)^{p/(p+1-\beta)}$$

and

$$H'_g(t, g) = g^{-(q-1)/q} \left(\frac{c}{q} t + g^{1/q}\right)^{q-1} = g^{-p/(p+1-\beta)} H(t, g)^{p/(p+1-\beta)},$$

and the function $g \mapsto H(t, g)$ is concave for each $t \geq 0$. Then by the Itô formula

$$H(t, G(Y^\epsilon_t)) = G(y) + \int_0^t \left[ c + (G(Y^\epsilon_s))^\beta \mathcal{A}^\epsilon G(Y^\epsilon_s) \right] H(s, G(Y^\epsilon_s))^{p/(p+1-\beta)} \, ds$$

$$+ \int_0^t \Psi^\epsilon_s \, ds + M^{H,\epsilon}_t,$$

where $M^{H,\epsilon}$ is a local martingale and

$$\Psi^\epsilon_t = \int_{\mathbb{R}} \left[ H(s, G(Y^\epsilon_{s-} + u)) - H(s, G(Y^\epsilon_{s-})) - H'_g(s, G(Y^\epsilon_{s-}))(G(Y^\epsilon_{s-} + u) - G(Y^\epsilon_{s-})) \right] U^\epsilon(du) \leq 0$$

since $H(t, \cdot)$ is concave. Combined with (5.27), this provides that

$$H(t \wedge \theta^\epsilon_D, G(Y^\epsilon_{t \wedge \theta^\epsilon_D})) \leq G(y), \quad t \geq 0$$

is a local super-martingale. Then, by the Fatou lemma,

$$E^Y_{y,\epsilon} H(t \wedge \theta^\epsilon_D, G(Y^\epsilon_{t \wedge \theta^\epsilon_D})) \leq G(y), \quad t \geq 0.$$

Note that $G(y) = |y|^p$ for $|y| > D$, and

$$H(t, g) \geq \left(\frac{c}{q}\right)^q.$$

This gives (5.28) for $|y| > D$. For $|y| \leq D$ we have $\theta^\epsilon_D = 0$ $\mathbb{P}^Y_{Y,\epsilon}$-a.s., and (5.28) holds true trivially. By Jensen’s inequality, (5.28) yields

$$E^Y_{y,\epsilon} \theta^\epsilon_D \leq C|y|^{1-\beta}, \quad y \in \mathbb{R}. \quad (5.28)$$

Since functions $R^\epsilon$ are uniformly bounded, by the strong Markov property this leads to the bound

$$|E^Y_{y,\epsilon} \int_0^t R^\epsilon(Y^\epsilon_s) \, ds| \leq \left| E^Y_{y,\epsilon} \int_0^{t \wedge \theta^\epsilon_D} R^\epsilon(Y^\epsilon_s) \, ds \right| + \left| E^Y_{y,\epsilon} \int_{t \wedge \theta^\epsilon_D}^t R^\epsilon(Y^\epsilon_s) \, ds \right|$$

$$\leq CE^Y_{y,\epsilon} \theta^\epsilon_D + \left| E^Y_{y,\epsilon} \left[ E^Y_{y,\epsilon} \int_0^{t \wedge \theta^\epsilon_D} R^\epsilon(Y^\epsilon_s) \, ds \right]_{y = Y^\epsilon_{t \wedge \theta^\epsilon_D}} \right|$$

$$\leq C|y|^{1-\beta} + \sup_{|y'| \leq D, t' \leq t} \left| E^Y_{y,\epsilon} \int_0^{t'} R^\epsilon(Y^\epsilon_s) \, ds \right|.$$

That is, if we manage to show that

$$\sup_{|y| \leq D, t \geq 0, \epsilon \in (0, 1)} \left| E^Y_{y,\epsilon} \int_0^t R^\epsilon(Y^\epsilon_s) \, ds \right| < \infty, \quad (5.29)$$
then we have (5.22) with
\[ p' = (1 - \beta) - (\alpha + \beta - 2) = 3 - \alpha - 2\beta. \]

Since \( \beta > 0 \), we have
\[ 4 - 2\alpha - 3\beta = 2(2 - \alpha - \beta) - \beta < 0 \implies p' < \alpha + \beta - 1. \]

Taking \( p \in (p' \lor 0, \alpha + \beta - 1) \), we will get that, for \( Q(y) = 1 + |y|^p \), both (5.14) and (5.15) hold true, which will provide (5.8) and complete the entire proof.

To prove (5.29), we modify the dissipativity-based argument from the previous section. Namely, denote
\[ \lambda_D(t) = c_D \| Y_t^\epsilon \| \leq D, \]
then by (5.25)
\[ \frac{d}{dt} |\Delta|^\beta \leq -\lambda_D(t) \left( |\Delta|^\beta \land |\Delta|^\beta \right). \]

Denote by \( \Upsilon(t,r) \) the solution to the Cauchy problem
\[ \frac{d}{dt} \Upsilon(t,r) = -\Upsilon(t,r)^\beta \land \Upsilon(t,r), \quad \Upsilon(0,r) = r, \]
then again by the comparison theorem
\[ |\Delta|^\beta \leq \Upsilon(\Lambda_D(t), r), \quad \Lambda_D(t) = \int_0^t \lambda_D(s) \, ds. \]

We have \( \Upsilon(t,r) \leq r \) for \( t \geq 0 \), and
\[ \Upsilon(t,r) = e^{t\epsilon} \cdot e^{-t}, \quad t \geq t_r, \]
where
\[ t_r = \frac{(r^{1-\beta} - 1)_+}{1 - \beta}. \]

Since the derivatives of \( R^\epsilon, \epsilon > 0 \) are uniformly bounded, this provides for \( |y| \leq D \)
\[ \mathbb{E} Y^\epsilon \| \int_0^t R^\epsilon(Y_s^\epsilon) \, ds \| \leq CDN + Ce^{tD} \mathbb{E} \int_0^\infty e^{-\Lambda_D(t)} \, dt. \] (5.30)

The rest of the proof is contained in the following

**Lemma 5.3** For any \( q < \alpha/(2 - 2\beta) \), there exist \( D > 0, a > 0, \) and \( C \) such that
\[ \mathbb{P}(\Lambda_D(t) \leq at) \leq C(1 + t)^{-q}, \quad t \geq 0, \quad \epsilon > 0. \] (5.31)

Once Lemma (5.3) is proved, we easily complete the entire proof. Namely, because \( \alpha + \beta \geq 2 \) and \( \beta > 0 \), we have
\[ \frac{\alpha}{2 - 2\beta} \geq \frac{2 - \beta}{2 - 2\beta} > 1. \]

That is, (5.31) holds true for some \( D > 1, a > 0, \) and \( q > 1 \). Using the estimate
\[ \mathbb{E} e^{-\Lambda_D(t)} \leq \mathbb{P}(\Lambda_D(t) \leq at) + e^{-at} \]
and (5.30), we guarantee (5.29) and complete the proof of Theorem 2.3.

**Proof of Lemma 5.3** Without loss of generality, we can assume that \( q > 1/2 \). Let \( p \) be such that
\[ q = \frac{p + 1 - \beta}{2 - 2\beta}, \]
then \( p \in (0, \alpha + \beta - 1) \), and Lemma 5.2 is applicable. Let \( D_0 > 1 \) be such that (5.26) holds true with \( p \) specified above and \( D = D_0 \).
There exists $D > D_0$ large enough, such that
\[
\sup_{|y| \leq D_0} \mathbf{P}^{Y_{\epsilon}} \left( \sup_{t \in [0,1]} |Y_{\epsilon, t}| \geq D \right) \leq \frac{1}{2};
\]
the calculation here is the same as in Section 3.2 and we omit the details. We fix these two levels $D_0, D$ and define iteratively the sequence of stopping times
\[
\theta^\epsilon_0 = 0, \quad \chi^\epsilon_k = \inf \{ t \geq \theta^\epsilon_{k-1} : |Y_{\epsilon, 0}^\epsilon | \geq D \} \wedge (\theta^\epsilon_{k-1} + 1),
\]
\[
\theta^\epsilon_k = \inf \{ t \geq \chi^\epsilon_k : |Y_{\epsilon, 0}^\epsilon | \leq D \}
\]
k \geq 1.

We denote
\[
S_{\epsilon, \uparrow}^\epsilon_k = \sum_{j=1}^k (\chi^\epsilon_j - \theta^\epsilon_{j-1}), \quad S^\epsilon_{\downarrow} = \sum_{j=1}^k (\theta^\epsilon_j - \chi^\epsilon_j), \quad S^\epsilon_k = S^\epsilon_{\uparrow} + S^\epsilon_{\downarrow} = \theta^\epsilon_k, \quad k \geq 1,
\]
and
\[
N^\epsilon_t = \min \{ k \geq 1 : S^\epsilon_k \geq t \}, \quad t > 0.
\]

Then
\[
\Lambda^\epsilon(t) \geq cD S_{N^\epsilon_t}^\epsilon,
\]
and thus for arbitrary $b > 0$ we have
\[
\mathbf{P}(\Lambda^\epsilon(t) \leq at) \leq \mathbf{P}
\left( S_{[bt]}^\epsilon \leq \frac{a}{cD} t \right) + \mathbf{P}(N^\epsilon_t \leq bt).
\]

On the other hand,
\[
S^\epsilon_k \leq S^\epsilon_{\downarrow} + k,
\]
which gives
\[
\mathbf{P}(N^\epsilon_t \leq bt) = \mathbf{P}
\left( S_{[bt]}^\epsilon \geq t \right) \leq \mathbf{P}
\left( S_{[bt]}^\epsilon \geq t - bt \right).
\]

In what follows, we show that there exists $c > 0$ small enough, such that
\[
\mathbf{P}
\left( S_{[bt]}^\epsilon \geq c^{-1}k \right) \leq Ck^{-q} \quad \text{and} \quad \mathbf{P}
\left( S_{[bt]}^\epsilon \leq ck \right) \leq Ck^{-q}, \quad k \geq 1.
\]

Once we do that, the rest of the proof is easy. Namely, we take
\[
b < (1 + c^{-1})^{-1},
\]
then $(1 - b)/b > c^{-1}$, and by the first inequality in (5.33) we get
\[
\mathbf{P}(N^\epsilon_t \leq bt) \leq \mathbf{P}
\left( S_{[bt]}^\epsilon \geq t - bt \right) \leq C|bt|^{-q}.
\]

Then taking
\[
a < \frac{cDbc}{2}
\]
we will have by the second inequality in (5.33)
\[
\mathbf{P}
\left( S_{[bt]}^\epsilon \leq \frac{a}{cD} t \right) \leq C|bt|^{-q}.
\]

Let us proceed with the proof of the first inequality in (5.33). Denote
\[
s_{k}^\epsilon = S_{k}^\epsilon - S_{k-1}^\epsilon = \theta^\epsilon_k - \chi^\epsilon_k, \quad \mathcal{F}_{k}^\epsilon = \mathcal{F}_{\theta^\epsilon_k}, \quad k \geq 0,
\]
then
\[ M^\varepsilon_{k+1} = \sum_{j=1}^{k} \left( s^\varepsilon_j - E[s^\varepsilon_j \mid \mathcal{F}^\varepsilon_{j-1}] \right), \quad k \geq 0 \]

is an \{\mathcal{F}^\varepsilon_{k+1}\}-martingale. By Lemma 5.2 applied to \( D = D_0 \), and the strong Markov property, we have
\[ E \left[ (s^\varepsilon_k)^2 \mid \mathcal{F}^\varepsilon_{k-1} \right] = E \left[ (\theta^\varepsilon_k - \chi^\varepsilon_k)^{p+1-\beta}/(1-\beta) \right] \leq CE^\varepsilon_{y,\chi^\varepsilon_k}|Y^\varepsilon|^{p+1-\beta} \]

Note that
\[ |Y^\varepsilon_{\theta^\varepsilon_k}| \leq D_0, \quad k \geq 0 \]

by construction. Next, it is easy to show that
\[ \sup_{|y| \leq D_0, \varepsilon \in (0,1]} E^Y_{\varepsilon,\chi^\varepsilon}|Y^\varepsilon|^{p+1-\beta} < \infty. \] (5.34)

Indeed, let \( g \in C^2(\mathbb{R}, \mathbb{R}) \) be such that \( g(y) \geq |y|^{p+1-\beta} \) and \( g(y) = |y|^{p+1-\beta} \) for \( |y| \geq 1 \). Since \( p < \alpha + \beta - 1 \), we have \( p+1-\beta < \alpha \) and thus by (5.10) the family of functions \( \{\mathcal{A}^\varepsilon g\}_{\varepsilon \in (0,1]} \) is well defined and is uniformly bounded on the set \( \{|y| \leq D_0\} \). We have
\[ E^Y_{\varepsilon,\chi^\varepsilon}|Y^\varepsilon|^{p+1-\beta} \leq g(y) + E^Y_{\varepsilon,\chi^\varepsilon} \int_0^{Y^\varepsilon} \mathcal{A}^\varepsilon g(Y_s^\varepsilon) \, ds \leq g(y) + \sup_{|y'| \leq D_0} |\mathcal{A}^\varepsilon g(y')| \]

since \( \chi^\varepsilon_1 \leq 1 \) by construction. This yields (5.34). Summarizing the above calculation, we conclude that
\[ E \left[ (s^\varepsilon_k)^2 \mid \mathcal{F}^\varepsilon_{k-1} \right] \leq C, \quad k \geq 1. \] (5.35)

Consequently, for some \( c^1 > 0 \) we have
\[ E \left[ s^\varepsilon_k \mid \mathcal{F}^\varepsilon_{k-1} \right] \leq c^1, \quad k \geq 1, \] (5.36)

and
\[ E \left[ |M^\varepsilon_k - M^\varepsilon_{k-1}|^2 \mid \mathcal{F}^\varepsilon_{k-1} \right] \leq C, \quad k \geq 1. \]

By the Burkholder–Davis–Gundy inequality \[ \text{[Kallenberg, 2002, Theorem 23.12]} \], and Jensen’s inequality, we have
\[ E \left[ |M^\varepsilon_k|^2 \right] \leq C(p) E \left( \sum_{j=1}^{k} \left( M^\varepsilon_j - M^\varepsilon_{j-1} \right)^2 \right)^q \leq C(q) k^{q-1} \sum_{j=1}^{k} E \left( M^\varepsilon_j - M^\varepsilon_{j-1} \right)^2 \leq Ck^q. \]

Now we obtain the first inequality in (5.33): if \( c > 0 \) is such that \( c^{-1} > c^1 \), then
\[ P \left( S^\varepsilon_k \geq c^{-1} k \right) \leq P \left( M^\varepsilon_k \geq (c^{-1} - c^1) k \right) \leq (c^{-1} - c^1)^{-2} k^{-2} \beta^{2-2^q} E \left[ M^\varepsilon_k \right]^2 k^q \leq C k^{-q}. \]

The proof of the second inequality in (5.33) is similar and simpler. We denote
\[ s^\varepsilon_k = s^\varepsilon_k - \theta^\varepsilon_k, \quad \chi^\varepsilon_k = \mathcal{A}^\varepsilon_k, \quad k \geq 1, \quad \mathcal{F}^\varepsilon_{\theta^\varepsilon_k} = \mathcal{F}_0, \]

and put
\[ M^\varepsilon_k = \sum_{j=1}^{k} \left( s^\varepsilon_j - E[s^\varepsilon_j \mid \mathcal{F}^\varepsilon_{j-1}] \right), \quad k \geq 0. \]

Now \( s^\varepsilon_k \leq 1 \) by construction, hence analogues of (5.33) and (5.36) trivially hold true, which gives
\[ E \left[ |M^\varepsilon_k|^2 \right] \leq Ck^q. \]
On the other hand, by (5.32) and the strong Markov property,

\[ E \left[ s_k \mid \mathcal{F}_{k-1}^{\varepsilon, \uparrow} \right] \geq \frac{1}{2}, \quad k \geq 1. \]

Then for \( c < 1/2 \) we have

\[ P \left( S_k^{\varepsilon, \uparrow} \leq ck \right) \leq P \left( |M_k^{\varepsilon, \uparrow}| \geq \left( \frac{1}{2} - c \right) k \right) \leq \left( \frac{1}{2} - c \right)^{-2q} k^{-2q} E \left| M_k^{\varepsilon, \uparrow} \right|^{2q} \leq C k^{-q}. \]

\[ \blacksquare \]

A Auxiliaries to the proof of Theorem 2.3: regular case

In this section, we assume conditions of Theorem 2.3 to hold true, and (3.5) to hold true for some \( A \). First, we give some basic integral estimates. Denote

\[ T_\varepsilon(r) = \mu_\varepsilon([r, \infty)), \quad r > 0, \]

the tail function for \( \mu_\varepsilon \). By (3.5), \( T_\varepsilon(r) = \), \( r > A \), and by (3.4), for each \( r_0 > 0 \)

\[ \sup_{r > r_0} \sup_{\varepsilon > 0} T_\varepsilon(r) < \infty. \]

Hence by (2.7) we have

\[ \sup_{r > 0} \sup_{\varepsilon \in (0, 1]} r^\alpha T_\varepsilon(r) < \infty. \]  

(A.1)

Next, for each \( c > 0 \)

\[ \limsup_{\delta \downarrow 0} \sup_{\varepsilon \in (0, 1]} \delta^2 T_\varepsilon(\delta) \leq \limsup_{\delta \downarrow 0} \sup_{\varepsilon \in (0, 1]} \delta^2 \mu_\varepsilon([c, \infty)) + \limsup_{\delta \downarrow 0} \sup_{\varepsilon \in (0, 1]} \int_{[\delta, c]} z^2 \mu_\varepsilon(dz) = \sup_{\varepsilon > 0} \int_{(0, c)} z^2 \mu_\varepsilon(dz). \]

Since \( c > 0 \) is arbitrary, the above inequality and (2.8) yield

\[ \lim_{\delta \to 0} \sup_{\varepsilon \in (0, 1]} \delta^2 T_\varepsilon(\delta) = 0 \]  

(A.2)

for \( \alpha = 2 \). The same assertion holds true for \( \alpha < 2 \) by (A.1).

Using (A.2), we can perform integration by parts:

\[ 2 \int_0^r z T_\varepsilon(z) \, dz = z^2 T_\varepsilon(z) \bigg|_{0^+}^r - \int_0^r z^2 dT_\varepsilon(z) = r^2 T_\varepsilon(r) + \int_0^r z^2 \mu_\varepsilon(dz). \]  

(A.3)

For \( \alpha < 2 \), by (A.1) and (3.5), this immediately gives

\[ \sup_{\varepsilon \in (0, 1]} \int_0^\infty z^2 \mu_\varepsilon(dz) \leq 2 \sup_{\varepsilon \in (0, 1]} \int_0^\infty z T_\varepsilon(z) \, dz < \infty \]  

(A.4)

and

\[ \sup_{\varepsilon \in (0, 1]} \int_0^\delta z^2 \mu_\varepsilon(dz) \leq 2 \sup_{\varepsilon \in (0, 1]} \int_0^\delta z T_\varepsilon(z) \, dz \to 0, \quad \delta \to 0. \]  

(A.5)

For \( \alpha = 2 \), the same relations hold true by (2.8).

From now on, we assume that \( (\alpha, \beta) \in \Xi_{\text{regular}} \). The following lemma describes the local \( (v \to 0) \) and the asymptotic \( (v \to \infty) \) behavior of the functions \( H_\varepsilon \) defined in (4.4).
Lemma A.1 For each $\varepsilon \in (0,1]$, the function

$$H^\varepsilon(v) = \int_0^A \left(F(v + z) + F(v - z) - 2F(v)\right) \mu^\varepsilon(dz)$$

is well defined, continuous, and odd. In addition,

$$\lim_{v \to 0} \sup_{\varepsilon \in (0,1]} |H^\varepsilon(v)| = 0, \quad (A.6)$$

and, for every $\delta > 0$,

$$\sup_{|v| \geq \delta} \sup_{\varepsilon \in (0,1]} |v|^\beta |H^\varepsilon(v)| < \infty. \quad (A.7)$$

Proof: First, let us consider the case $\alpha < 2$, note that in this case we have $\alpha + \beta < 2$. By the Fubini theorem,

$$H^\varepsilon(v) = \int_0^A \left(F(v + z) + F(v - z) - 2F(v)\right) \mu^\varepsilon(dz)$$

$$= -\int_0^A \int_0^z \left(F'(v + w) - F'(v - w)\right) dw T^\varepsilon(dw) = \int_0^A \left(F'(v + w) - F'(v - w)\right) T^\varepsilon(w) dw. \quad (A.8)$$

The r.h.s. in (A.8) is well defined because, by (A.1), for $v > 0$

$$|H^\varepsilon(v)| \leq C \int_0^A \left|v + w\right|^{1-\beta} - \left|v - w\right|^{1-\beta} \frac{dw}{w^{\alpha}} = C|v|^{2-\alpha-\beta} \int_0^A \frac{\psi(\rho)}{\rho^{\alpha}} d\rho, \quad (A.9)$$

where we denote

$$\psi(\rho) = \left|(1 + \rho)^{1-\beta} - |1 - \rho|^{1-\beta}\right|. \quad (A.10)$$

The latter integral in (A.9) is finite because

$$\frac{\psi(\rho)}{\rho^{\alpha}} \sim 2(1 - \beta)\rho^{1-\alpha}, \quad \rho \to 0,$$

and the function $\psi$ either is continuous for $\beta \leq 1$, or satisfies

$$\psi(\rho) \sim \frac{1}{|1 - \rho|^{\beta-1}}, \quad \rho \to 1$$

for $\beta \in (1,2)$. Since

$$\psi(\rho) \sim 2(1 - \beta)\rho^{-\beta}, \quad \rho \to +\infty$$

one can easily derive for the function

$$I(\sigma) = \int_0^\sigma \frac{\psi(\rho)}{\rho^{\alpha}} d\rho$$

the following:

$$I(\sigma) \sim \begin{cases} 
2(1 - \beta)\sigma^{1-\alpha-\beta}, & \alpha + \beta < 1, \\
\frac{1 - \alpha - \beta}{1 + \beta}\sigma^{1-\alpha-\beta}, & \alpha + \beta = 1, \\
c_1, & 1 < \alpha + \beta < 2, \\
\sigma \to \infty,
\end{cases} \quad (A.10)$$

where

$$c_1 = \int_0^\infty \frac{\psi(\rho)}{\rho^{\alpha}} d\rho \in (0,\infty), \quad \alpha + \beta > 1.$$

Thus there exist $v_0 > 0$ and $C > 0$ such that

$$|H^\varepsilon(v)| \leq C \cdot \begin{cases} 
|v|, & \alpha + \beta < 1, \\
|v| \ln \frac{1}{|v|}, & \alpha + \beta = 1, \\
|v|^{2-\alpha-\beta}, & 1 < \alpha + \beta < 2, \\
|v| \leq v_0,
\end{cases} \quad (A.11)$$
which gives (A.6). The proof of (A.7) is similar and is based on the relation
\[ I(\sigma) \sim \frac{2|1-\beta|}{2-\alpha} \sigma^{2-\alpha}, \quad \sigma \to 0, \]
we omit the details.

Next, let \( \alpha = 2 \); note that, in this case \( \beta \leq 0 \). Then
\[ \left| F'(v + w) - F'(v - w) \right| \leq C(1 + |v|^{-\beta})w, \quad w \in (0, A), \quad (A.12) \]
and the integral in the right hand side of (A.8) is well defined by (A.4). The same inequality yields (A.7).

To prove (A.6), we restrict ourselves to the case \( 0 < |v| \leq 2A \), and decompose
\[ H^\varepsilon(v) = \left( \int_0^{||v||/2} + \int_{||v||/2}^2 \right) \left( F'(v + w) - F'(v - w) \right) T^\varepsilon(w) \, dw =: H^\varepsilon_1(v) + H^\varepsilon_2(v). \]
The term \( H^\varepsilon_2 \) admits estimates similar to those we had above. Namely, we have
\[ |H^\varepsilon_2(v)| \leq C|v|^{2-\beta} I_2(A\left|\frac{A}{v}\right|), \quad I_2(\sigma) = \int_{1/2}^\sigma \left( (1 + \rho)^{1-\beta} - |1 - \rho|^1 \right) \frac{d\rho}{\rho^2}. \]
For \( I_2 \) analogue of (A.10) holds true, and thus \( H^\varepsilon_2 \) satisfies (A.11). To estimate \( H^\varepsilon_1 \) we use the Lipschitz condition (A.12) and assumption \( v \leq 2A \):
\[ |H^\varepsilon_1(v)| \leq C \sup_{\varepsilon \in (0,1]} \int_0^{||v||/2} u T^\varepsilon(w) \, dw < \infty. \]

Now (A.6) follows by (A.5). \( \square \)

In the following lemma, we justify the formal relation (4.2).

**Lemma A.2** Identity (4.2) holds true with the local martingale \( M^\varepsilon \) defined by (4.3).

**Proof:** For \( \beta < 0 \), \( F \in C^2(\mathbb{R}, \mathbb{R}) \), and the standard Itô formula holds. For \( \beta \in [0, 2) \), we consider an approximative family \( F_\delta \in C^2(\mathbb{R}, \mathbb{R}) \), \( \delta \in (0, 1] \), for \( F \), which satisfies the following:
\[ \sup_{v \in \mathbb{R}} |F(v) - F_\delta(v)| \leq C \delta^{2-\beta}, \quad (A.13) \]
\[ F'(v) \equiv F_\delta'(v) \text{ for } |v| \geq \delta, \]
\[ \sup_{|v| \leq \delta} |F_\delta'(v)|^\beta \text{ sgn}(v) - v \leq c\delta \quad \text{and} \quad F_\delta'(v)|^\beta \text{ sgn}(v) \equiv v \text{ for } |v| \geq \delta \quad (A.14) \]
\[ \lim_{\delta \to 0} F_\delta''(v) = F''(v) \text{ for any } v \neq 0. \]

One particular example of such a family is given by
\[ F_\delta(v) = \begin{cases} \frac{1-\beta^2}{3(2-\beta)} \delta^{2-\beta} + \frac{1}{2(2-\beta)} |v|^{2-\beta} \text{ sgn}(v), & |v| \geq \delta, \\ \frac{1+\beta}{2} \delta^{1-\beta} v + \frac{1-\beta}{6} |v|^\beta, & |v| < \delta. \end{cases} \]

The Itô formula applied to \( F_\delta \) yields
\[ F_\delta(V_\varepsilon^\gamma) = F_\delta(V_0^\gamma) - \frac{1}{\varepsilon} \int_0^t F_\delta'(V_\varepsilon^\gamma)|V_\varepsilon^\gamma|^\beta \text{ sgn}(V_\varepsilon^\gamma) \, ds + M_\delta(t) + \int_0^t H^{\varepsilon,\delta}(V_\varepsilon^\gamma) \, ds, \]
\[ M_\delta(t) = \int_0^t \int_{|z| \leq A} F_\delta(V_\varepsilon^\gamma + z) - F_\delta(V_\varepsilon^\gamma - z) \, \tilde{N}(ds, dz), \quad (A.15) \]
\[ H^{\varepsilon,\delta}(v) = \int_0^\infty \left( F_\delta(v + z) + F_\delta(v - z) - 2F_\delta(v) \right) \mu^\varepsilon(dz). \]
By construction, we have  
\[ F_\delta(V_\epsilon^\delta) \to F(V_\epsilon^\delta), \quad F_\delta(v_0^\delta) \to F(v_0^\delta), \quad \delta \to 0, \]
\[ \int_0^t \left( F_\delta'(V_\epsilon^\delta)|V_\epsilon^\delta|^\beta \text{sgn}(V_\epsilon^\delta) - V_\epsilon^\beta \right) ds \to 0, \quad \delta \to 0, \]
in probability. To analyse the behaviour of the martingale part \( M_\delta^\epsilon \), we repeat, with proper changes, the argument used to prove (4.6). Namely, truncating the small jumps, stopping the processes at the time moments \( \tau_R^\epsilon = \inf \{ t : |V_\epsilon^\delta| > R \}, \quad R > 0, \)
and using Theorem \ref{thm:ito} we can show that  
\[ M_\delta^\epsilon(t) \to M_1^\epsilon, \quad \delta \to 0, \]
in probability. Finally, repeating with minor changes the estimates from the proof of Lemma \ref{lem:a1} we can show that  
\[ H^{\epsilon, \delta} \to H^{\epsilon}, \quad \delta \to 0 \]
uniformly of any bounded set. Taking \( \delta \to 0 \) in (\ref{eq:approx}), we obtain the required Itô formula. \( \blacksquare \)

**B** Auxiliaries to the proof of Theorem \ref{thm:main}: non-regular case

**Lemma B.1** Let \( G \in C^2(\mathbb{R}, bR) \) be such that for some \( \sigma \in (0, \alpha) \) and all \( |y| \geq 1 \)
\[ |G'(y)| \leq C|y|^\sigma - 1 \quad \text{and} \quad |G''(y)| \leq C|y|^{\sigma - 2}. \quad (B.1) \]
Then the family
\[ K^\epsilon(y) = \int_0^\infty \left( G(y + u) + G(y - u) - 2G(y) \right) \nu^\epsilon(du), \quad \epsilon \in (0, 1], \quad (B.2) \]
satisfies
\[ \sup_{\epsilon \in (0, 1]} |K^\epsilon(y)| \leq C(1 + |y|)^{\sigma - \alpha} \]
for \( \alpha \in (0, 2) \) and
\[ \sup_{\epsilon \in (0, 1]} |K^\epsilon(y)| \leq C(1 + |y|)^{\sigma - 2} \ln (2 + |y|) \]
for \( \alpha = 2. \)

**Proof:** To simplify the notation, we assume \( y \geq 0 \); clearly, this does not restrict the generality. For \( y \leq 2 \), we decompose
\[ K^\epsilon(y) = \int_0^3 \left( G(y + u) + G(y - u) - 2G(y) \right) \nu^\epsilon(du) \]
\[ + \int_3^\infty \left( G(y + u) + G(y - u) - 2G(y) \right) \nu^\epsilon(du) =: K^\epsilon_1(y) + K^\epsilon_2(y). \]
We have  
\[ |K^\epsilon_1(y)| \leq \max_{|v| \leq 3} |G''(v)| \sup_{\epsilon \in (0, 1]} \int_{|y| < 3} u^2 \nu^\epsilon(du) < \infty, \]
see (\ref{eq:approx}). Next, we transform \( K^\epsilon_2(y) \) using the Newton–Leibniz formula and the Fubini theorem:
\[ K^\epsilon_2(y) = \int_3^\infty \int_0^u \left( G'(y + v) - G'(y - v) \right) dv \nu^\epsilon(du) \]
\[ = \int_0^\infty \left( G'(y + v) - G'(y - v) \right) \nu^\epsilon([3 \vee v, \infty)) dv. \]
Since $G'$ is locally bounded, by (5.11) this gives for some $C > 0$

$$|K_2(y)| \leq C + C \int_3^{\infty} |G'(y + v) - G'(y - v)| v^{-\alpha} \, dv.$$  

For $y \in [0, 2]$ and $v \geq 3$ we have $|y \pm v| \geq 1$, hence we can continue the above estimate:

$$|K_2(y)| \leq C \left( 1 + \int_3^{\infty} (|y + v|^\sigma - 1 + |y - v|^\sigma - 1) \, v^{-\alpha} \, dv \right)$$

$$\leq C \left( 1 + \int_3^{\infty} (v^{\sigma - 1} + (v - 2)^{\sigma - 1}) \, v^{-\alpha} \, dv \right) < \infty,$$

where the integral is finite because $\sigma < \alpha$. That is,

$$\sup_{y \leq 2} \sup_{v \in (0, 1]} |K^\nu(y)| < \infty.$$  \hspace{1cm} (B.3)

For $y > 2$, we use another decomposition:

$$K^\nu(y) = \int_0^{y/2} \left( G(y + u) + G(y - u) - 2G(y) \right) u^\nu \, (du)$$

$$+ \int_{y/2}^{\infty} \left( G(y + u) + G(y - u) - 2G(y) \right) u^\nu \, (du) =: K^\nu_3(y) + K^\nu_4(y).$$

We have $\sigma < \alpha \leq 2$. Hence, for $u \leq y/2$,

$$\left| G(y + u) + G(y - u) - 2G(y) \right| \leq u^2 \sup_{v \geq y/2} \left| G''(v) \right| \leq Cu^2 \gamma^{\sigma - 2},$$

and

$$|K^\nu_3(y)| \leq Cy^{\sigma - 2} \int_0^{y/2} u^2 v^\nu \, (du).$$

We have by (5.11) and (5.13)

$$\int_0^r u^2 v^\nu \, (du) = \int_0^1 u^2 v^\nu \, (du) + \int_1^r u^2 v^\nu \, (du) \leq C r^{2 - \alpha} + C \int_1^r u^{1 - \alpha} \, du.$$

That is, we have for $y > 2$

$$|K^\nu_3(y)| \leq Cy^{\sigma - \alpha}$$

if $\alpha \in (0, 2)$, and

$$|K^\nu_3(y)| \leq Cy^{\sigma - 2} \ln(2 + y)$$

if $\alpha = 2$.

For $K^\nu_4(y)$, we again use the Fubini theorem:

$$K^\nu_4(y) = \int_{y/2}^{\infty} \int_0^u \left( G'(y + v) - G'(y - v) \right) \, dv \, v^\nu \, (du)$$

$$= \int_0^{\infty} \left( G'(y + v) - G'(y - v) \right) v^\nu ((y/2) v, \infty) \, dv$$

$$= \left[ \int_0^{y/2} + \int_{y/2}^{y-1} + \int_{y-1}^{y+1} + \int_{y+1}^{\infty} \right] \left( G'(y + v) - G'(y - v) \right) v^\nu ((y/2) v, \infty) \, dv$$

$$=: \sum_{j=1}^{4} K^\nu_{4,j}(y).$$

Since $y > 2$ we have $y + v > 1$ for any $v > 0$, and thus

$$|G'(y + v)| \leq C(y + v)^{\sigma - 1}.$$
In addition, for \( v \in [0, y - 1] \) we have

\[
|G'(y + v)| \leq C(y - v)^{\sigma - 1}.
\]

Thus by (5.11)

\[
|K_{4,1}'(y)| \leq Cy^{-\alpha} \int_{\frac{y}{2}}^{\frac{y}{2}} \left( (y + v)^{\sigma - 1} + (y - v)^{\sigma - 1} \right) dv \leq Cy^{\sigma - \alpha},
\]

\[
|K_{4,2}'(y)| \leq C \int_{\frac{y}{2}}^{\frac{y}{2}} \left( (y + v)^{\sigma - 1} + (y - v)^{\sigma - 1} \right) v^{-\alpha} dv
\]

\[
\leq Cy^{\sigma - \alpha} \int_{\frac{1}{2}}^{1} \left( (1 + \rho)^{\sigma - 1} + (1 - \rho)^{\sigma - 1} \right) \rho^{-\alpha} d\rho,
\]

note that the latter integral is finite because \( \sigma > 0 \). Similarly,

\[
|K_{4,4}'(y)| \leq C \int_{\frac{y}{2}}^{\infty} \left( (y + v)^{\sigma - 1} + (v - y)^{\sigma - 1} \right) v^{-\alpha} dv
\]

\[
\leq Cy^{\sigma - \alpha} \int_{1}^{\infty} \left( (1 + \rho)^{\sigma - 1} + (\rho - 1)^{\sigma - 1} \right) \rho^{-\alpha} d\rho,
\]

and the latter integral is finite because \( \sigma > 0 \) and \( \sigma < \alpha \). Finally, since \( G' \) is locally bounded,

\[
|K_{4,3}'(y)| \leq C \int_{y-1}^{y+1} (y + v)^{\sigma - 1} + 1 v^{-\alpha} dv \leq C \left( y^{\sigma - \alpha - 1} + y^{-\alpha} \right).
\]

Combining the estimates for \( K_{4}' \) and for \( K_{4,j}' \), \( j = 1, \ldots, 4 \), we complete the proof. \( \blacksquare \)

**Lemma B.2** The functions \( R^\varepsilon \), \( \varepsilon \in (0, 1) \) satisfy

\[
|R^\varepsilon(y)| \leq C(1 + |y|)^{2-\alpha-\beta}
\]

if \( \alpha \in (0, 2) \), and

\[
|R^\varepsilon(y)| \leq C(1 + |y|)^{2-\alpha-\beta} \ln(2 + |y|)
\]

if \( \alpha = 2 \).

**Proof:** The family \( R^\varepsilon \), \( \varepsilon \in (0, 1) \) has the form [5.2] with \( G = \hat{F} \), and this function satisfies [5.1] with \( \sigma = 2 - \beta > 0 \). Hence, for \( \alpha + \beta > 2 \), the required statement follows directly from Lemma [5.1]. Let us prove this statement in the boundary case \( \alpha + \beta = 2 \). One can see that the estimates for \( K_{4}' \), \( K_{3}' \) and \( K_{4,j}' \), \( j = 1, 2, 3 \), from the previous proof remain true under the assumption \( \sigma = 0 \) as well. Next, for \( G = \hat{F} \) we have

\[
G'(y) = \begin{cases} 
y^{1-\beta}, & y \geq 1, \\
(-y)^{1-\beta}, & y \leq -1.
\end{cases}
\]

Then for \( y \leq 2 \)

\[
K_{2}'(y) = \int_{0}^{\infty} \left( (y + v)^{1-\beta} - (v - y)^{1-\beta} \right) \nu^\varepsilon([3 \lor v, \infty)) dv,
\]

and therefore

\[
|K_{2}'(y)| \leq C \left( 1 + \int_{3}^{\infty} \left| v^{\sigma - 1} - (v - 2)^{\sigma - 1} \right| v^{-\alpha} dv \right).
\]

The latter integral is finite for \( \sigma > \alpha - 1 \) because

\[
\left| v^{\sigma - 1} - (v - 2)^{\sigma - 1} \right| \sim cv^{\sigma - 2}, \quad v \to \infty.
\]

Similarly,

\[
|K_{4,4}'(y)| \leq Cy^{\sigma - \alpha} \int_{1}^{\infty} \left| (1 + \rho)^{\sigma - 1} - (\rho - 1)^{\sigma - 1} \right| \rho^{-\alpha} d\rho,
\]

and the latter integral is finite for \( \sigma > \alpha - 1 \). \( \blacksquare \)
**Lemma B.3** The derivatives of functions $R^\varepsilon$, $\varepsilon \in (0,1]$, are uniformly bounded, namely there exists $C > 0$ such that

$$ |\frac{d}{dy} R^\varepsilon(y) | \leq C, \quad y \in \mathbb{R}, \quad \varepsilon \in (0,1]. $$

**Proof:** We have

$$ \frac{d}{dy} R^\varepsilon(y) = \int_0^\infty \left( \hat{F}'(y + u) + \hat{F}'(y - u) - 2\hat{F}'(y) \right) \nu^\varepsilon(du), $$

and the integral is well defined because $\hat{F}' \in C^2(\mathbb{R}, \mathbb{R})$. We have that the second derivative $(\hat{F}')'' = \hat{F}'''$ of $\hat{F}'$ is bounded, and $\hat{F}'$ is either bounded for $\beta \geq 1$, or $(1-\beta)$-Hölder continuous for $\beta \in (0,1)$. In the first case, we just have

$$ \sup_{y \in \mathbb{R}, \varepsilon \in (0,1]} |\frac{d}{dy} R^\varepsilon(y) | \leq C \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (u^2 \wedge 1) \nu^\varepsilon(du) < \infty, $$

see (5.13). In the second case we have

$$ \sup_{y \in \mathbb{R}, \varepsilon \in (0,1]} |\frac{d}{dy} R^\varepsilon(y) | \leq C \sup_{\varepsilon \in (0,1]} \int_{\mathbb{R}} (u^2 \wedge |u|^{1-\beta}) \nu^\varepsilon(du). $$

By (5.13),

$$ \int_{|u| > 1} |u|^{1-\beta} \nu^\varepsilon(du) = 2 \nu^\varepsilon([1,\infty)) + 2(1-\beta) \int_1^\infty \int_1^u \nu^\varepsilon(du) \leq C + C \int_1^\infty v^{-\alpha-\beta} dv < \infty, $$

where we have used (5.11) and the assumption $\alpha + \beta \geq 2 > 1$. This provides the required statement for $\beta \in (0,1)$. \qed

### C Proof of (5.10)

First, we observe that

$$ \varepsilon^{(2-\alpha-\beta)\gamma} \int_0^\infty (F(\varepsilon^{-\gamma}v + u) + F(\varepsilon^{-\gamma}v - u) - 2F(\varepsilon^{-\gamma}v)) \nu^\varepsilon(du) $$

$$ = \varepsilon^{(2-\beta)\gamma} \int_0^\infty (F(\varepsilon^{-\gamma}v + \varepsilon^{-\gamma}z) + F(\varepsilon^{-\gamma}v - \varepsilon^{-\gamma}z) - 2F(\varepsilon^{-\gamma}v)) \mu^\varepsilon(dz) $$

$$ = \int_0^\infty (F(v + z) + F(v - z) - 2F(v)) \mu^\varepsilon(dz) = H^\varepsilon(v). $$

Hence

$$ \tilde{H}^\varepsilon(v) - H^\varepsilon(v) = \varepsilon^{(2-\alpha-\beta)\gamma} R^{0,\varepsilon}(\varepsilon^{-\gamma}v) + \varepsilon^{(2-\alpha-\beta)\gamma} R^{1,\varepsilon}(\varepsilon^{-\gamma}v), $$

where

$$ R^{0,\varepsilon}(y) = -\hat{F}'(y) |y|^\beta \mathrm{sgn} y, \quad R^{1,\varepsilon}(y) = \int_0^\infty (\hat{F}(y + u) + \hat{F}(y - u) - 2\hat{F}(y)) \nu^\varepsilon(du). $$

Since $\tilde{F}$ vanishes outside of $[-1,1]$, so does $R^{0,\varepsilon}(y)$, and for $y > 1$ we have

$$ |R^{1,\varepsilon}(y)| = \left| \int_{|y-1,y+1]} F(y-u) \nu^\varepsilon(du) \right| \leq C \nu^\varepsilon([y-1,\infty)) \leq C(y-1)^{-\alpha}; $$

here we have used that $\tilde{F}$ is bounded and (5.11). Hence

$$ \sup_{|v| > 2^\gamma} |\tilde{H}^\varepsilon(v) - H^\varepsilon(v)| \leq C \varepsilon^{(2-\beta)\gamma} \to 0, \quad \varepsilon \to 0, $$

which yields (5.10).
References

O. V. Aryasova and A. Yu. Pilipenko. On the strong uniqueness of a solution to singular stochastic differential equations. *Theory of Stochastic Processes*, 17(33)(2):1–15, 2012.

A. Baule and P. Sollich. Singular features in noise-induced transport with dry friction. *Europhysics Letters*, 97(2):20001, 2012.

P.-G. de Gennes. Brownian motion with dry friction. *Journal of Statistical Physics*, 119(5-6):953–962, 2005.

A. Eberle, A. Guillin, and R. Zimmer. Quantitative Harris type theorems for diffusions and McKean–Vlasov processes. *arXiv preprint arXiv:1606.06012*, 2016.

R. Eon and M. Gradinaru. Gaussian asymptotics for a non-linear Langevin type equation driven by a symmetric $\alpha$-stable Lévy noise. *Electronic Journal of Probability*, 20(100):1–19, 2015.

P. S. Goohpattader and M. K. Chaudhury. Diffusive motion with nonlinear friction: apparently Brownian. *The Journal of Chemical Physics*, 133(2):024702, 2010.

M. I. Gordin. The central limit theorem for stationary processes. *Soviet Mathematics. Doklady*, 10:1174–1176, 1969.

M. I. Gordin and B. A. Lifshits. The central limit theorem for stationary Markov processes. *Soviet Mathematics. Doklady*, 19:392–394, 1978.

M. Hairer. Convergence of Markov processes. *Lecture notes, www.hairer.org/notes/Convergence.pdf*, 2016.

H. Hayakawa. Langevin equation with Coulomb friction. *Physica D: Nonlinear Phenomena*, 205(1):48–56, 2005.

R. Hintze and I. Pavlyukevich. Small noise asymptotics and first passage times of integrated Ornstein–Uhlenbeck processes driven by $\alpha$-stable Lévy processes. *Bernoulli*, 20(1):265–281, 2014.

O. Kallenberg. *Foundations of Modern Probability*. Probability and its Applications. Springer, New York, second edition, 2002.

A. Kawarada and H. Hayakawa. Non-Gaussian velocity distribution function in a vibrating granular bed. *Journal of the Physical Society of Japan*, 73(8):2037–2040, 2004.

A. Kulik. *Ergodic Behavior of Markov Processes*. De Gruyter, Berlin, 2017.

V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities: Theory and Applications. Volume I: Ordinary Differential Equations*, volume 55 of *Mathematics in Science and Engineering*. Academic Press, New York, 1969.

B. Lindner. The diffusion coefficient of nonlinear Brownian motion. *New Journal of Physics*, 9(5):136, 2007.

B. Lindner. Diffusion coefficient of a Brownian particle with a friction function given by a power law. *Journal of Statistical Physics*, 130(3):523–533, 2008.

B. Lindner. Diffusion of particles subject to nonlinear friction and a colored noise. *New Journal of Physics*, 12(6):063026, 2010.

V. Lišý, J. Tóthová, and L. Glod. Diffusion in a medium with nonlinear friction. *International Journal of Thermophysics*, 35(11):2001–2010, 2014.

A. Mauger. Anomalous motion generated by the Coulomb friction in the Langevin equation. *Physica A: Statistical Mechanics and its Applications*, 367:129–135, 2006.

E. Pardoux and A. Răşcanu. *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, volume 69 of *Stochastic Modelling and Applied Probability*. Springer, Cham, 2014.
B. J. N. Persson. *Sliding Friction: Physical Principles and Applications*. Springer, Berlin, second edition, 2000.

V. L. Popov. *Contact Mechanics and Friction: Physical Principles and Applications*. Springer, Heidelberg, 2010.

N. I. Portenko. Some perturbations of drift-type for symmetric stable processes. *Random Operators and Stochastic Equations*, 2(3):211–224, 1994.

E. Priola. Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka Journal of Mathematics*, 49(2):421–447, 2012.

V. P. Sergienko and S. N. Bukharov. *Noise and Vibration in Friction Systems*, volume 212 of *Springer Series in Materials Science*. Springer, Cham, 2015.

H. Tanaka, M. Tsuchiya, and S. Watanabe. Perturbation of drift-type for Lévy processes. *Journal of Mathematics of Kyoto University*, 14(1):73–92, 1974.

H. Touchette, E. Van der Straeten, and W. Just. Brownian motion with dry friction: Fokker–Planck approach. *Journal of Physics A: Mathematical and Theoretical*, 43(44):445002, 2010.

X. Zhang. Stochastic differential equations with Sobolev drifts and driven by α-stable processes. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 49(4):1057–1079, 2013.