A connection between supersymmetric quantum mechanics and Painlevé V equation

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Dedicated to Professor Bogdan Mielnik for his 50 years of scientific career.

Abstract. In this article we introduce the relation between supersymmetric quantum mechanics (SUSY QM) and a second-order non-linear differential equation known as Painlevé V (PV) equation. To that end, we will first make a swift examination on the SUSY QM treatment of the radial oscillator and we will revisit its relation with the polynomial Heisenberg algebras (PHA). After that, we will formulate a theorem that connects SUSY QM to a set of solutions of the PV equation through specific PHA.

1. Introduction

In this article we will introduce a link joining supersymmetric quantum mechanics (SUSY QM) for the radial oscillator to analytical solutions of the PV equation. We will apply an approach that has already been proven efficient to solve the Painlevé IV equation [1–6]. This link has already been used both in the context of dressing chains [7–9] and in SUSY QM [1,10–12].

The Painlevé V (PV) equation can be applied in the description of several systems in solid state physics [13], electrodynamics [14] and condense matter [15], thereby leading to studies of its geometric properties [16], numerical solutions [17], Bäcklund transformations [18], $q$-deformations [19], discrete versions [20], and others.

The main idea is that through SUSY QM we can obtain systems ruled by third-order polynomial Heisenberg algebras (PHA), generated by natural fourth-order differential ladder operators. In this work we will prove that these systems are connected with the PV equation. Up to our knowledge, this link was first realized in 1980 by Ablowitz et al. [21] and Flaschka [22]. Then, both subjects were connected through first-order SUSY QM [7–9] and later on using the higher-order case [1,6,10–12,23,24].

In view of this, we will first formulate a reduction theorem for the SUSY partners of the radial oscillator, which contains the necessary conditions on the transformation functions to reduce the order of the natural ladder operators associated with the SUSY partners from higher to fourth order. Then, we will study the properties of these fourth-order ladder operators in order to analyze the consequences of the theorem. In particular, we will study the different types of PV solutions that can be obtained with this method.
2. Supersymmetric quantum mechanics
In the \( k \)-th order SUSY QM one starts from a given solvable Hamiltonian

\[
H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x),
\]

and generates a chain of intertwined Hamiltonians as follows [25,26]

\[
H_j A_j^+ = A_j^+ H_{j-1}, \quad H_{j-1} A_j^- = A_j^- H_j,
\]

\[
H_j = \frac{1}{2} \frac{d^2}{dx^2} + V_j(x),
\]

\[
A_j^\pm = \frac{1}{\sqrt{2}} \left[ \mp \frac{d}{dx} + \alpha_j(x, \epsilon_j) \right], \quad j = 1, \ldots, k.
\]

Therefore, the next equations have to be fulfilled:

\[
\alpha_j'(x, \epsilon_j) + \alpha_j^2(x, \epsilon_j) = 2[V_{j-1}(x) - \epsilon_j],
\]

\[
V_j(x) = V_{j-1}(x) - \alpha_j'(x, \epsilon_j).
\]

The final Riccati solution \( \alpha_k(x, \epsilon_k) \) is determined by \( k \) solutions \( u_j \) of the associated Schrödinger equation:

\[
H_0 u_j = -\frac{1}{2} u_j'' + V_0(x) u_j = \epsilon_j u_j, \quad j = 1, \ldots, k,
\]

which are connected with the initial Riccati solutions through \( \alpha_1(x, \epsilon_j) = u_j'/u_j \).

Thus, there is a pair of \( k \)-th order operators intertwining the initial \( H_0 \) and the final Hamiltonians \( H_k \) in the way

\[
H_k B_k^+ = B_k^+ H_0, \quad H_0 B_k^- = B_k^- H_k,
\]

where \( B_k^+ = A_k^+ \ldots A_1^+ \) and \( B_k^- = A_k^- \ldots A_1^- \).

The normalized eigenfunctions \( \psi_n^{(k)} \) of \( H_k \), associated to the eigenvalues \( E_n \), are proportional to the action of \( B_k^+ \) onto the corresponding ones of \( H_0 \) (\( \psi_n, n = 0,1, \ldots \)). Moreover, there are \( k \) additional eigenstates \( \psi_{\epsilon_j}^{(k)} \) associated to the eigenvalues \( \epsilon_j \) (\( j = 1, \ldots, k \)), which are simultaneously annihilated by \( B_k^- \). Their corresponding explicit expressions are given by [1,27]:

\[
\psi_n^{(k)} = \frac{B_k^+ \psi_n}{[(E_n - \epsilon_1) \ldots (E_n - \epsilon_k)]^{1/2}}, \quad E_n,
\]

\[
\psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, \ldots, u_k)}, \quad \epsilon_j.
\]

Furthermore, the restriction \( \epsilon_j < E_0 \) naturally arises in this treatment since in this way we avoid singularities in \( V_k(x) \). A diagram representing a second-order transformation is shown in Fig. 1.

3. Polynomial Heisenberg algebras
Systems described by \( (m - 1) \)-th order polynomial Heisenberg algebras (PHA) possess three generators, one of them (the Hamiltonian \( H \)) commutes with the other two (\( \mathcal{L}_m^\pm \)) as ladder operators do, but the commutator \( [\mathcal{L}_m^-, \mathcal{L}_m^+] \) is a polynomial in \( H \), i.e.,

\[
[H, \mathcal{L}_m^\pm] = \pm \mathcal{L}_m^\pm, \quad (7a)
\]

\[
[\mathcal{L}_m^-, \mathcal{L}_m^+] \equiv N_m(H + 1) - N_m(H) = P_{m-1}(H), \quad (7b)
\]
where $P_{m-1}(H)$ and $N_m(H) = L_m^+ L_m^-$ are polynomials in $H$ of degrees $m-1$ and $m$, respectively. $N_m(H)$ is the analogous to the number operator for the harmonic oscillator, which can be factorized as

$$N_m(H) = \prod_{i=1}^{m} (H - \epsilon_i), \quad (8)$$

and $\epsilon_i$ are the energies associated with the extremal states.

Note that we can realize these algebras through $m$th-order differential ladder operators; nevertheless, what really defines the order of the PHA is the degree $m-1$ of the polynomial $P_{m-1}(H)$ characterizing the deformation (see commutator in Eq. (7b)).

The spectrum of $H$ can be built up by acting iteratively $L_m^+$ on the extremal states of the system [10, 27, 28], which are annihilated by $L_m^-$ and (in the diagonalizable case) satisfy $H\psi_{\epsilon_i} = \epsilon_i \psi_{\epsilon_i}, i = 1, \ldots, m$.

Now, the general systems ruled by these PHA should be identified: for zeroth- and first-order PHA, the systems are the harmonic and radial oscillators, respectively [28–31]. On the other hand, for second- and third-order PHA, the determination of the potentials reduces to finding solutions to the Painlevé IV and V equations, denoted as PIV and PV, respectively [30, 32]. In this work, such relation will be used in reversed order, i.e., first we will find systems which are described by PHA and then we will obtain solutions to the Painlevé equations. To do that, let us establish the link between third-order PHA and PV equation.

4. Third-order PHA: fourth-order ladder operators.
Let us assume that $L_4^+$ are fourth-order ladder operators, that can be factorized as:

$$L_4^+ = A_4^+ A_3^+ A_2^+ A_1^+ = \frac{1}{2^2} \left( \frac{d}{dx} - f_1 \right) \left( \frac{d}{dx} - f_3 \right) \left( \frac{d}{dx} - f_2 \right) \left( \frac{d}{dx} - f_4 \right), \quad (9a)$$

$$L_4^- = A_4^- A_3^- A_2^- A_1^- = \frac{1}{2^2} \left( \frac{d}{dx} - f_1 \right) \left( \frac{d}{dx} - f_3 \right) \left( \frac{d}{dx} - f_2 \right) \left( \frac{d}{dx} - f_4 \right). \quad (9b)$$

Now, let us build a closed-chain, such that each pair of operators $A_j^\pm$ intertwines two Schrödinger Hamiltonians [9] as:

$$H_{j+1} A_j^+ = A_j^+ H_{j}, \quad H_{j} A_j^- = A_j^- H_{j+1}, \quad (10)$$
where \( j = 1, 2, 3, 4 \). This leads to the following factorizations of the Hamiltonians

\[
H_1 = A_1^+ A_1^- + \epsilon_1, \tag{11a}
\]

\[
H_2 = A_2^+ A_2^- + \epsilon_2 = A_2^+ A_2^- + \epsilon_2, \tag{11b}
\]

\[
H_3 = A_3^+ A_3^- + \epsilon_3 = A_3^+ A_3^- + \epsilon_3, \tag{11c}
\]

\[
H_4 = A_4^+ A_4^- + \epsilon_4 = A_4^+ A_4^- + \epsilon_4, \tag{11d}
\]

\[
H_5 = A_5^+ A_5^- + \epsilon_5, \tag{11e}
\]

where the fifth Hamiltonian is not independent, but is rather given by the closure condition:

\[
H_5 = H_1 - 1 \equiv H - 1. \tag{12}
\]

A diagram representing the transformations and the closure relation is shown in Fig. 2.

![Diagram](image)

**Figure 2.** Diagram representing the two equivalent SUSY transformations. Above: the four-step transformation induced by the first-order operators \( A_1^+, A_2^+, A_3^+ \) and \( A_4^+ \). Below: the direct transformation achieved through the fourth-order operators \( L_4^\pm \).

By making the corresponding operator products of Eqs. (11), after a long calculation it turns out that all the involved functions and the potential \( V(x) \) become expressed in terms of just one, \( w(z) \) with \( z = x^2 \), which satisfies [33]:

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{w - 1} \frac{dw}{dz} + \frac{(w - 1)^2}{z^2} \left( aw + b \right) + c \frac{w}{z} + d \frac{w(w + 1)}{w - 1}, \tag{13}
\]

which is the Painlevé V equation with parameters

\[
a = \frac{\alpha_1^2}{2}, \quad b = -\frac{\alpha_2^2}{2}, \quad c = \frac{\alpha_2 - \alpha_4}{2}, \quad d = -\frac{1}{8}, \tag{14}
\]

where \( \alpha_1 = \varepsilon_1 - \varepsilon_1, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_4 - \varepsilon_1 + 1 \).

The spectrum of \( H \) could contain four independent equidistant energy ladders starting from the extremal states [11], here we need only the expression of two of them:

\[
\psi_{\varepsilon_3} \propto \exp \left[ \int \left( \frac{h'}{2h} + \frac{h}{2h} - \frac{\alpha_3}{h} \right) dx \right], \tag{15a}
\]

\[
\psi_{\varepsilon_4} \propto \exp \left[ \int \left( \frac{h'}{2h} + \frac{h}{2h} + \frac{\alpha_3}{h} \right) dx \right], \tag{15b}
\]

where \( h(x) = -x - g(x) \), with \( g(x) \) being related to \( w \) through \( g(x) = x/[w(x^2) - 1] \). Note that the number operator \( N_4(H) \) for this system is of fourth degree, \( N_4(H) = (H - \varepsilon_1)(H - \varepsilon_2)(H - \varepsilon_3)(H - \varepsilon_4) \). From Eqs. (9–12) it is straightforward to obtain the energies of the extremal states in terms of the factorization energies as \( \varepsilon_j = \epsilon_j + 1 \) for \( j = 1, 2, 3, 4 \).
On the other hand, if a system having fourth-order ladder operators is found, then it is possible to design a mechanism for generating solutions to the PV equation, similar to the one implemented for the PIV equation [3, 4, 6]. The key point is to identify once again the extremal states of our system. Then, from Eqs. (15) it is straightforward to show that

\[ h(x) = \frac{2\alpha_3}{\left[ \ln \left( \frac{\psi_4}{\psi_3} \right) \right]'} = \left\{ \ln [W(\psi_{\varepsilon_3}, \psi_{\varepsilon_4})] \right\}'. \]  

Furthermore, we obtain an expression for \( g(x) \) given by

\[ g(x) = -x - h(x) = -x - \left\{ \ln [W(\psi_{\varepsilon_3}, \psi_{\varepsilon_4})] \right\}'. \]  

Finally, \( g(x) \) is related with the solution \( w(z) \) of the PV equation through

\[ w(z) = 1 + \frac{z^{1/2}}{g(z^{1/2})}, \]  

Thus, we have introduced a simple recipe to generate solutions to the PV equation, based on the identification of the extremal states for systems ruled by third-order PHA, which have differential ladder operators of fourth order.

5. SUSY partners of the radial oscillator

In this section we will apply SUSY QM to the radial oscillator Hamiltonian, which is given by

\[ H_\ell = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{8} + \frac{\ell(\ell + 1)}{2x^2}, \quad \ell \geq 0, \quad x \geq 0, \]  

where we have added the subscript \( \ell \) to denote the dependence of the Hamiltonian on the angular momentum index. First, let us solve the spectral problem associated to \( H_\ell \) [11, 34, 35]. In order to do that, let us take \( b_\ell^\pm \) such that

\[ b_\ell^\pm = \frac{1}{2} \left( \frac{d^2}{dx^2} \mp x \frac{d}{dx} + \frac{x^2}{4} - \frac{\ell(\ell + 1)}{x^2} \pm \frac{1}{2} \right). \]  

Then, it is easily shown that \( H_\ell b_\ell^- = b_\ell^-(H_\ell - 1) \) and \( H_\ell b_\ell^+ = b_\ell^+(H_\ell + 1) \), i.e., the following commutators are obeyed \([H_\ell, b_\ell^\pm] = \pm b_\ell^\pm\). Now, we can obtain the eigenstates of \( H_\ell \) departing from the extremal states \( \psi_{\varepsilon_i}, i = 1, 2 \), two solutions of Schrödinger equation for \( H_\ell \) such that \( b_\ell^- \psi_{\varepsilon_i} = 0 \), which are given by:

\[ \psi_{\varepsilon_1} \propto x^{\ell+1} \exp(-x^2/4), \quad \varepsilon_1 = \frac{\ell}{2} + \frac{3}{4} \equiv E_{0\ell}, \]  

\[ \psi_{\varepsilon_2} \propto x^{-\ell} \exp(-x^2/4), \quad \varepsilon_2 = -\frac{\ell}{2} + \frac{1}{4} = -E_{0\ell} + 1. \]  

As only the first one fulfills the boundary conditions, therefore it leads to a ladder of physical eigenfunctions. The spectrum of the radial oscillator is therefore

\[ \text{Sp}(H_\ell) = \{ E_{n\ell} = n + \frac{\ell}{2} + \frac{3}{4}, n = 0, 1, \ldots \}. \]  

Note that an analogue of the number operator for the radial oscillator can be defined as

\[ b_\ell^+ b_\ell^- = (H_\ell - \varepsilon_1)(H_\ell - \varepsilon_2) = (H_\ell - E_{0\ell})(H_\ell + E_{0\ell} - 1), \]  

thus, we have introduced a simple recipe to generate solutions to the PV equation, based on the identification of the extremal states for systems ruled by third-order PHA, which have differential ladder operators of fourth order.
which is a second degree polynomial in $H_\ell$, i.e., the radial oscillator is ruled by a first-order PHA.

In order to implement the SUSY transformations, we employ the general solution of the stationary Schrödinger equation for any factorization energy $\epsilon$, which is given by \[11,36,37\]

$$u(x) = x^{-\ell} e^{-x^2/4} \left[ 1 F_1 \left( \frac{1}{4} - \frac{2\ell - 4\epsilon}{2} ; \frac{x^2}{2} \right) + \nu \frac{\Gamma \left( \frac{3+2\ell-4\epsilon}{4} \right)}{\Gamma \left( \frac{3+2\ell}{2} \right)} 1 F_1 \left( \frac{3+2\ell-4\epsilon}{4} ; \frac{3+2\ell}{2} ; \frac{x^2}{2} \right) \right].$$  \(24\)

Thus, the first-order SUSY partner potential of the radial oscillator becomes

$$V_1(x) = \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2} - [\ln u(x)]''.$$ \(25\)

The conditions that must be fulfilled to produce a non-singular transformation are

$$x > 0, \quad \epsilon < E_0 \ell, \quad \nu \geq -\frac{\Gamma \left( \frac{1-2\ell}{4} \right)}{\Gamma \left( \frac{1-2\ell-4\epsilon}{4} \right)}.$$ \(26\)

For a $k$-th order SUSY transformation let us take $k$ appropriate solutions \{$u_k, \ldots, u_1\}$ in the form given in Eq. \(24\), for the factorization energies $\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_1 < E_0 \ell$. The SUSY partner potential of the radial oscillator becomes now:

$$V_k(x) = \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2} - [\ln W(u_1, \ldots, u_k)]''.$$ \(27\)

with spectrum given by $\text{Sp}(H_k) = \{\epsilon_k, \ldots, \epsilon_1, E_0 \ell, E_1 \ell, \ldots\}$. In Fig. 3 we show examples of first (left) and second-order (right) SUSY partner potentials of the radial oscillator.

![Figure 3](image)

**Figure 3.** Radial oscillator potential (blue) and its SUSY partners for $k = 1$, $\ell = 2$, $\epsilon = 1/2$, and $\nu = \{-0.59 \text{ (magenta)}, -0.4 \text{ (yellow)}, 1 \text{ (green)}\}$ (left plot). The right plot corresponds to $k = 2$, $\ell = 5$, $\nu_1 = 1$, and $\epsilon_1 = \{0 \text{ (magenta)}, -2 \text{ (yellow)}, -4 \text{ (green)}\}$ with $u_2 = b_\ell^+ u_1$ and $\epsilon_2 = \epsilon_1 - 1$.

Now we define a natural pair of ladder operators $L_k^\pm$ for $H_k$ as

$$L_k^\pm = B_k^+ b_\ell^+ B_k^-,$$ \(28\)

which are of $(2k+2)$th-order and fulfill

$$[H_k, L_k^\pm] = \pm L_k^\pm.$$ \(29\)
From the intertwining relations we can obtain the analogue of the number operator for the \( k \)-th order SUSY partners of the radial oscillator as

\[
N(H_k) = L_k^+ L_k^- = (H_k - E_0) (H_k + E_0 - 1) \prod_{j=1}^{k} (H_k - \epsilon_j)(H_k - \epsilon_j - 1).
\]

(30)

Since both \( \epsilon_j \) and \( \epsilon_j + 1 \) are in the set of roots, there is a finite ladder starting and ending at \( \epsilon_j \). As the index \( j \) runs from 1 to \( k \), this means that \( \text{Sp}(H_k) \) contains \( k \) one-step ladders. Note that \( E_0 \) and \( -E_0 + 1 \) are also roots of \( N(H_k) \), then \( \text{Sp}(H_k) \) could have in principle two infinite ladders, but just the one starting from \( E_0 \) is physical. We conclude that \( \{H_k, L_k^-, L_k^+\} \) generates a \((2k + 1)\)th-order polynomial Heisenberg algebra.

6. Reduction theorem for the SUSY generated Hamiltonians \( H_k \)

In order to implement the prescription pointed out at the end of Section 4 to produce solutions to the PV equation, first of all we need to identify systems ruled by a third-order PHA, generated by fourth-order ladder operators. Note that for \( k - 1 \) the set \( \{H_k, L_k^-, L_k^+\} \) generates a PHA of order greater than three. Would it be possible to identify a subfamily of the \( k \)-th order SUSY partners of the radial oscillator Hamiltonian which, in addition of having the natural \((2k + 2)\)th-order ladder operators \( L_k^\pm \) would have fourth-order ones? If so, we could generate additional solutions to the PV equation. The answer to this question turns out to be positive, and the conditions required to produce such a reduction process are contained in the following theorem [6].

**Theorem.** Let \( H_k \) be the \( k \)th-order SUSY partner of the radial oscillator Hamiltonian \( H_0 \) generated by \( k \) Schrödinger seed solutions \( u_i \) which are connected by the annihilation operator \( b^- \) as

\[
u_1 \geq \frac{\Gamma\left(\frac{1-2\ell}{2}\right)}{\Gamma\left(\frac{1-2\ell-4\epsilon_1}{4}\right)}.
\]

(32)

Therefore, the natural \((2k + 2)\)th-order ladder operator \( L_k^+ = B_k^+ b^+ B_k^- \) of \( H_k \) turn out to be factorized in the form

\[
L_k^+ = P_{k-1}(H_k) l_k^+,
\]

(33)

where \( P_{k-1}(H_k) = (H_k - \epsilon_1) \ldots (H_k - \epsilon_{k-1}) \) is a polynomial of degree \( k - 1 \) in \( H_k \) and \( l_k^+ \) is a fourth-order differential ladder operator,

\[
[H_k, l_k^+] = l_k^+,
\]

(34)

such that

\[
l_k^+ l_k^- = (H_k - E_0) (H_k + E_0 - 1) (H_k - \epsilon_k)(H_k - \epsilon_1 - 1).
\]

(35)

The proof of this theorem will be given in a forthcoming paper [38]. It implies that once the factorization in Eq. (33) is fulfilled, the natural ladder operators \( L_k^\pm \) are expressed as a product of a polynomial of degree \( k - 1 \) in \( H_k \) times the fourth-order ladder operator \( l_k^+ \). This means that the \((2k + 1)\)th-order PHA obtained through the SUSY transformation involving \( k \) connected solutions with \( \epsilon_i = \epsilon_1 - (i - 1) \), \( i = 1, \ldots, k \), can be reduced to a third-order one [39].
Recall that these algebras are closely related to the PV equation. This means that when we reduce the higher-order algebras, we open the possibility of obtaining new solutions of the PV equation, similar to what happens in the case of second-order PHA and PIV equation [1–6]. In the following sections we will discuss the method to obtain solutions of the PV equation.

7. Solutions to the PV equation through SUSY QM

In order to generate solutions to the PV equation we need to find systems ruled by third-order PHA, with fourth-order ladder operators. In fact, the key point of the technique is to identify the extremal states of the system, as well as their associated energies. Then, we use Eqs. (17)-(18) to calculate the PV solution and Eq. (14) to determine the parameters, both of which are symmetric under the exchanges $\varepsilon_1 \leftrightarrow \varepsilon_2$ and $\varepsilon_3 \leftrightarrow \varepsilon_4$. Thus, from the $4! = 24$ possible permutations of the four indexes we will get six different solutions to the PV equation, some of which could have singularities. Let us apply the technique to the radial oscillator potential and to its first-order SUSY partners.

We have seen that the radial oscillator Hamiltonian has second-order differential ladder operators $b^\pm_\ell$ leading to a first-order PHA. We can construct also pairs of fourth-order differential ladder operators which will give place to a third-order PHA, for example, $L^\pm_\ell = a^\pm_\ell + b^\pm_\ell a^\mp_\ell$, where $\sqrt{2} a^\pm_\ell \equiv \mp d/dx - (\ell + 1)/x \times x/2$. The analogue to the number operator becomes $L^\pm_\ell L^\mp_\ell = (H_\ell + E_0\ell - 1)(H_\ell - E_0\ell)(H_\ell - E_{1\ell})^2$. Its roots suggest naturally three extremal states: two of them are those associated to the first-order PHA, and another one is the first excited state $\psi_{1\ell \ell}$ of $H_\ell$ with eigenvalue $E_{1\ell \ell}$. The last one is chosen as the ‘mathematical eigenstate’ of $H_\ell$ to the eigenvalue $E_{1\ell}$, denoted as $\psi_{1\ell}$, which satisfies $W(\psi_{1\ell}, \psi_{1\ell}) = 1$. Now, we number them as follows:

\[
\begin{align*}
\psi_{1\ell} &\propto x^{\ell+1} \exp(-x^2/4), & \varepsilon_1 &= E_0\ell, \\
\psi_{2\ell} &\propto x^{-\ell} \exp(-x^2/4), & \varepsilon_2 &= -E_0\ell + 1, \\
\psi_{3\ell} &= \psi_{1\ell}, & \varepsilon_3 &= E_{1\ell}, \\
\psi_{4\ell} &= \psi_{1\ell}^\dagger, & \varepsilon_4 &= E_{1\ell}.
\end{align*}
\]

Using these expressions and Eqs. (14),(17),(18), it is straightforward to obtain the PV solution with parameters:

\[
a = \frac{(2\ell + 1)^2}{8}, \quad b = 0, \quad c = -\frac{(2\ell + 7)}{4}, \quad d = -\frac{1}{8}.
\]

Now, by exploring all possible permutations of the indexes leading to different PV solutions, we obtain six combinations of parameters, which are shown in Table 1. For each one of them we obtain a solution of the PV equation, of which three non-trivial examples are:

\[
\begin{align*}
w(z) &= 1 + \frac{z}{2\ell + 1}, & (38a) \\
w(z) &= 1 - \frac{2\ell + z + 1}{2}, & (38b) \\
w(z) &= 1 + \frac{z\Gamma(\ell + 1/2, 0, z/2) - 2^{\ell+1/2}c^{z/2}(-z)^{\ell+1/2}}{(2\ell - 1)\Gamma(\ell + 1/2, 0, z/2)}, & (38c)
\end{align*}
\]

where $\Gamma(a, z_0, z_1)$ is the generalized incomplete gamma function. An illustration of the radial oscillator potential for different values of $\ell$ and the corresponding PV solutions $w(z)$ of equation (38c) are shown in Fig. 4.
Figure 4. Radial oscillator potentials (left) and the corresponding PV solutions $w(z)$ of Eq. (38c) (right) for $\ell = 3/2$ (blue), 5/2 (magenta), 7/2 (yellow) and 9/2 (green).

| Order  | $8a$           | $8b$           | $4c$         |
|--------|----------------|----------------|--------------|
| 1234   | $(2\ell + 1)^2$ | 0              | $-2\ell - 7$ |
| 1324   | 4              | $-(2\ell + 3)^2$ | $2\ell - 1$ |
| 1423   | 4              | $-(2\ell + 3)^2$ | $2\ell - 1$ |
| 2314   | $(2\ell + 3)^2$ | $-4$           | $-2\ell - 3$ |
| 2413   | $(2\ell + 3)^2$ | $-4$           | $-2\ell - 3$ |
| 3412   | 0              | $-(2\ell + 1)^2$ | $-2\ell - 3$ |

Table 1. Parameters of PV equation for the six permutations of the radial oscillator potential.

We must remember that, due to the radial oscillator is also described by a first-order PHA, then the third-order case must reduce to the original first-order PHA. Indeed, this is what happens since $L_4^+ = b_\ell^+ (H_\ell - E_0\ell)$ and $L_4^- = (H_\ell - E_0\ell) b_\ell^-.$

On the other hand, the first-order SUSY partners of the radial oscillator Hamiltonian possess also a pair of natural fourth-order differential ladder operators

$$L_4^+ = A_{\ell}^+ b_\ell^+ A_1^-,$$  

(39)

where $A_{\ell}^+$ are the first-order intertwining operators and $b_\ell^+$ are the second-order ladder operators of Section 5. The operators $L_4^+$ give place to a third-order PHA since

$$N(H_1) = L_4^+ L_4^- = (H_\ell + E_0\ell - 1) (H_\ell - E_0\ell) (H_\ell - \epsilon) (H_\ell - \epsilon - 1).$$  

(40)

The roots of $N(H_1)$ and the SUSY procedure suggest now the following extremal states: two of them are the SUSY transformed extremal states of the radial oscillator, another one is the new ground state created by the SUSY transformation. The last one will be an unphysical eigenstate of $H_1$ associated to $\epsilon + 1.$ Let us choose now the following initial ordering:

$$\psi_{\epsilon_1} \propto A_{\ell}^+ b_\ell^+ u,$$  

$\epsilon_1 = \epsilon + 1,$  

(41a)

$$\psi_{\epsilon_2} \propto A_{\ell}^+ [x^{-\ell} \exp \left( -\frac{x^2}{4} \right)],$$  

$\epsilon_2 = -E_0\ell + 1,$  

(41b)

$$\psi_{\epsilon_3} \propto \frac{1}{u},$$  

$\epsilon_3 = \epsilon,$  

(41c)

$$\psi_{\epsilon_4} \propto A_{\ell}^+ [x^{\ell+1} \exp \left( -\frac{x^2}{4} \right)],$$  

$\epsilon_4 = E_0\ell.$  

(41d)
Using these expressions and Eqs. (14), (17), (18), one can obtain the PV solution and its corresponding parameters:

\[
\begin{align*}
a &= \frac{(4\epsilon + 2\ell + 3)^2}{32}, & b &= -\frac{(4\epsilon - 2\ell - 3)^2}{32}, & c &= -\frac{(2\ell + 1)}{4}, & d &= -\frac{1}{8}.
\end{align*}
\]

Now, by permuting the indexes of Eqs. (41) we can obtain different PV solutions. Although they can be quickly calculated with any symbolic software, they are quite large to be written down explicitly (see some plots of them in Figure 5). For that reason, we choose a simple case, using as seed solution \(u(x)\) the ground state of the radial oscillator. In this case, of the six possible solutions obtained by permutations, half of them reduce to zero and the other half to rational functions. The corresponding parameters of the PV equation are given in Table 2 and the nontrivial solutions are the following:

\[
\begin{align*}
w(z) &= 1 + \frac{z}{2\ell - z + 1}, \\
w(z) &= 1 + \frac{z[4\ell^2 + 8\ell^3 - 15z^2 - 2\ell(2 + 5z^2)]}{16\ell(-1 + \ell + 2\ell^2)}, \\
w(z) &= 1 + \frac{z[8\ell^3 - 4\ell^2(z - 1) + 2(5z^2 + 2z + 2) + 5(z - 3)z^2]}{-8\ell^3(z - 4) + 4\ell^2(z^2 - 3z + 4) + 2\ell(5z^3 + 2z^2 - 2z - 8) - 5(z - 1)z^3}.
\end{align*}
\]

| Order   | 8a          | 8b          | 4c          |
|---------|-------------|-------------|-------------|
| 1234    | (2\ell + 3)^2 | 0           | -2\ell - 1 |
| 1324    | 4           | -(2\ell + 1)^2 | 2\ell + 1  |
| 1423    | 4           | -(2\ell + 1)^2 | 2\ell + 1  |
| 2314    | (2\ell + 1)^2 | -4          | -2\ell - 5 |
| 2413    | (2\ell + 1)^2 | -4          | -2\ell - 5 |
| 3412    | 0           | -(2\ell + 3)^2 | 2\ell - 3  |

Table 2. Parameters of the PV equation for the six permutations of the first-order SUSY partners of the radial oscillator.

Figure 5. SUSY partner potential \(V_1(x)\) of the radial oscillator (black) (left) and the solutions \(w_1(z)\) to PV equation (right) for \(\ell = 1, \epsilon_1 = 1,\) and \(\nu_1 = \{0.905\) (blue), 0.913 (magenta), 1 (yellow), 10 (green)\}. 


8. Conclusions
In this paper we introduced an algebraic technique to solve the PV equation and, based on this method, we were able to obtain some of its solutions. In order to do that, we derived the SUSY partners of the radial oscillator and we obtained the natural ladder operators connected with these Hamiltonians. Then we formulated a reduction theorem, in which we point out the requirements for the natural \((2k + 2)\)th-order ladder operators associated to the \(k\)th-order SUSY partners of the radial oscillator to be reduced to fourth-order ones, which are related with third-order PHA and the PV equation.

Using the connection between the SUSY partners of the radial oscillator and third-order PHA, in Section 7 we were able to derive a method to obtain solutions of the PV equation. We presented the general method to get explicit solutions through the radial oscillator, and its first-order SUSY partners.

This paper further expands previous works of the same authors related with the Painlevé IV equation. The difference is that the connection with the PV equation is more elaborated, and working out explicit solutions becomes increasingly complicated, even at low orders of SUSY. We realize also that there are many other solutions that can be obtained by this method. We will continue working in this direction in the near future.

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