ON UNIVERSALLY STABLE ELEMENTS

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Abstract. We show that certain subrings of the cohomology of a finite $p$-group $P$ may be realised as the images of restriction from suitable virtually free groups. We deduce that the cohomology of $P$ is a finite module for any such subring. Examples include the ring of ‘universally stable elements’ defined by Evens and Priddy, and rings of invariants such as the mod-2 Dickson algebras.

Let $P$ be a finite $p$-group, and let $C_u$ be the category whose objects are the subgroups of $P$, with morphisms all injective group homomorphisms. Let $C$ be any subcategory of $C_u$ such that $P$ is an object of $C$, and such that for any object $Q$ of $C$, the inclusion of $Q$ in $P$ is a morphism in $C$. Let $H^*(\cdot)$ stand for mod-$p$ group cohomology, which may be viewed as a contravariant functor from $C_u$ to $\mathbb{F}_p$-algebras. We shall study the limit $I(P, C)$ of this functor:

$$I(P, C) = \lim_{Q \in C} H^*(Q).$$

Given our assumptions on $C$, we may identify $I(P, C)$ with a subring of $H^*(P)$. In the final remarks we discuss generalizations of our results in which most of the conditions that we impose upon $P$, $C$, and $H^*(\cdot)$ are weakened.

The classical case of this construction occurs in Cartan-Eilenberg’s description of the image of the cohomology of a finite group in the cohomology of its Sylow subgroup as the ‘stable elements’ [2]. Let $G$ be a finite group with $P$ as its Sylow subgroup, and let $C_G$ be the subcategory of $C_u$ containing all the objects, but with morphisms only those homomorphisms $Q$ to $Q'$ induced by conjugation by some element of $G$. Then the image, $\text{Im}(\text{Res}^G_P)$, of $H^*(G)$ in $H^*(P)$ is equal to $I(P, C_G)$.

Rings of invariants also arise in this way. If $C$ is a category whose only object is $P$, with morphisms a subgroup $H$ of $\text{Aut}(P)$, then $I(P, C)$ is just the subring $H^*(P)^H$ of invariants under the action of $H$.

Another case considered already, which motivated our work, is the ring of universally stable elements defined by Evens-Priddy in [4]. Let $C_s$ be the subcategory of $C_u$ generated by all subcategories of the form $C_G$ as defined above. Then $I(P, C_s)$ is the subring $I(P)$ of $H^*(P)$ introduced in [4], consisting of those elements of $H^*(P)$ which are in the image of $\text{Res}^G_P$ for every finite group $G$ with Sylow subgroup $P$.
A fourth case of interest is \( I(P, C_u) \), which might be viewed as the elements of \( H^*(P) \) which are ‘even more stable’ than the elements of \( I(P, C_s) \). It is easy to see that in general \( C_s \) is strictly contained in \( C_u \). For example, the endomorphism monoid \( \text{Hom}_{C_s}(P, P) \) of \( P \) is the subgroup of \( \text{Aut}(P) \) generated by elements of order coprime to \( p \), whereas \( \text{Hom}_{C_u}(P, P) \) is the whole of \( \text{Aut}(P) \). Our main result is the following theorem.

**Theorem 1.** Let \( P \) be a finite \( p \)-group, and let \( C \) be any subcategory of \( C_u = C_u(P) \) satisfying the conditions stated in the first paragraph. Then there exists a discrete group \( \Gamma \) containing \( P \) as a subgroup such that:

a) \( \text{Im}(\text{Res}_P^\Gamma) \) is equal to \( I(P, C) \);

b) \( (\text{Ker}(\text{Res}_P^\Gamma))^2 \) is trivial;

c) \( \text{Res}_P^\Gamma \) induces an isomorphism from \( H^*(\Gamma)/\sqrt{0} \) to \( I(P, C)/\sqrt{0} \);

d) \( \Gamma \) is virtually free. More precisely, \( \Gamma \) has a free normal subgroup of index dividing \(|P|! \).

If \( \Gamma' \) is a free normal subgroup of \( \Gamma \) of finite index, then \( P \) maps injectively to the finite group \( \Gamma/\Gamma' \), so by the Evens-Venkov theorem [5], \( H^*(P) \) is a finite module for \( H^*(\Gamma/\Gamma') \) and hence a fortiori for \( H^*(\Gamma) \). Thus one obtains the following corollary.

**Corollary 2.** Let \( P \) and \( C \) be as in the statement of Theorem 1. Then \( H^*(P) \) is a finite module for its subring \( I(P, C) \).

The case \( C = C_s \) is Theorem A of [4]. Our result is stronger, since it applies to categories such as \( C_u \) itself, and our proof is more elementary. There is an even shorter proof of Corollary 2 however, which is to deduce it from the following simpler theorem.

**Theorem 3.** Let \( P \) be a finite \( p \)-group, and let \( G \) be the symmetric group on a set \( X \) bijective with \( P \). Regard \( P \) as a subgroup of \( G \) via a Cayley embedding (or regular permutation representation). Then \( \text{Im}(\text{Res}_P^G) \) is contained in \( I(P, C_u) \).

To deduce Corollary 2 from Theorem 3, note that for any \( C \) as above, one has

\[
\text{Im}(\text{Res}_P^G) \subseteq I(P, C_u) \subseteq I(P, C) \subseteq H^*(P),
\]

and \( H^*(P) \) is a finite module for \( \text{Im}(\text{Res}_P^G) \) by the Evens-Venkov theorem.

**Proof of Theorem 3.** We deduce Theorem 3 from the following group-theoretic lemma.

**Lemma 4.** Let \( Q \leq P \leq G \) be as in the statement of Theorem 3, and let \( \phi \) be any injective homomorphism from \( Q \) to \( P \). Then there exists \( g \in G \) such that for all \( q \in Q \),

\[
\phi(q) = gqq^{-1}.
\]

**Proof.** Fix a bijection between \( P \) and the set \( X \) permuted by \( G \). This fixes an embedding \( i_P \) of \( P \) in \( G \). Let \( i_Q \) be the induced inclusion of \( Q \) in \( G \). Write \( i_P X \) for \( X \) viewed as a
Thus \( i_X \) is a free \( P \)-set of rank one. There are two ways to view \( X \) as a \( Q \)-set, either via \( i_Q \) or \( i_P \circ \phi \). The \( Q \)-sets \( i_Q X \) and \( i_P \circ \phi X \) are both free of rank equal to the index, \( |P : Q| \), of \( Q \) in \( P \). Let \( g \) be an isomorphism of \( Q \)-sets between \( i_Q X \) and \( i_P \circ \phi X \). Then \( g \) is an element of \( G \) having the required property, because for each \( x \in X \) and \( q \in Q \), \( g \cdot q \cdot x = \phi(q) \cdot g \cdot x \).

Returning now to the proof of Theorem 3, any morphism in \( C \) factors as the composite of an isomorphism followed by an inclusion. Thus it suffices to show that for \( \phi \) as in Lemma 4, \( \text{Res}^G_Q \) and \( \phi^* \circ \text{Res}^G_P \) are equal. Writing \( c_g \) for the automorphism of \( G \) given by conjugation by \( g \), we have shown that there exists \( g \) such that \( c_g \circ i_Q = i_P \circ \phi \). But \( c_g^* \) is the identity map on \( H^*(G) \), and hence \( i_Q^* = \phi^* \circ i_P^* \) as required.

This completes the proofs of all of our statements except for Theorem 1. For the proof of Theorem 1 we recall the following theorem (see for example [3], I.7.4 or IV.1.6).

**Theorem 5.** Let \( \Gamma \) be a group that acts simplicially (i.e., without reversing any edges) on a tree with all stabilizer groups of order dividing a fixed integer \( M \). Then there is a homomorphism from \( \Gamma \) to the symmetric group on \( M \) letters whose kernel, \( K \), is torsion-free. Since \( K \) acts freely, simplicially on the tree, it follows that \( K \) is a free group.

In fact, the short direct proof of Theorem 3 is based on some of the ideas in the proof of Theorem 5 given in [3].

**Proof of Theorem 1.** The group \( \Gamma \) will be constructed as the fundamental group of a graph of groups (see [3], I.3, [6], I.5, or [1], VII.9 for the definitions and basic theorems). Let \( Q_1, \ldots, Q_M \) be the objects of \( C \), and let \( \phi_1, \ldots, \phi_N \) be the morphisms of \( C \). Define a function \( m \) so that the domain of \( \phi_i \) is \( Q_{m(i)} \). Now let \( \Gamma \) be the group generated by the elements of \( P \) and new elements \( t_1, \ldots, t_N \) subject to all relations that hold in \( P \), together with the relations

\[
t_i q t_i^{-1} = \phi_i(q),
\]

for all \( i \in \{1, \ldots, N\} \) and all \( q \in Q_{m(i)} \). Thus \( \Gamma \) is the fundamental group of a graph of groups with one vertex and \( N \) edges. The vertex group is of course \( P \) and the \( i \)th edge group is \( Q_{m(i)} \). The two maps from the \( i \)th edge group to the vertex group (corresponding to its initial and terminal ends) are the inclusion and \( \phi_i \).

The group \( \Gamma \) as defined above has the following properties (see any of the references listed above): \( P \) is a subgroup of \( \Gamma \); for each \( i \), the homomorphism \( \phi_i: Q_{m(i)} \rightarrow P \) is inner in \( \Gamma \) (i.e., is induced by conjugation by the element \( t_i \)); \( \Gamma \) acts simplicially on a tree \( T \), with one orbit of vertices and \( N \) orbits of edges, with \( P \) being a vertex stabilizer and \( Q_{m(i)} \) being the stabilizer of some edge in the \( i \)th orbit. The quotient \( T/\Gamma \) is the graph used in defining \( \Gamma \).
Recall from [1], VII.7–VII.9 that for any \( \Gamma \)-CW-complex \( X \), there is a spectral sequence, with \( E_{p,q}^1 = \bigoplus_{\sigma} H^q(\Gamma_{\sigma}) \), where the sum is over a set of orbit representatives of \( p \)-cells in \( X \). For coefficients in a ring with trivial \( \Gamma \)-action (such as the field of \( p \) elements), this is a spectral sequence of rings. When \( X \) is acyclic the spectral sequence converges to a filtration of \( H^{p+q}(\Gamma) \). We apply this spectral sequence in the case when \( X = T \). In this case

\[
E_{0,q}^1 \cong H^q(P), \quad E_{1,q}^1 \cong \bigoplus_{i=1}^N H^q(Q_{m(i)}),
\]

and \( E_{p,q}^1 = 0 \) for \( p > 1 \). Under this isomorphism the differential \( d_1 : E_{0,q}^1 \to E_{1,q}^1 \) has \( i \)-th coordinate \( \text{Res}_P^P \phi_i^* - \phi_i^* \), and so \( E_{2,q}^2 \) is isomorphic to \( I(P,C) \). The fact that \( E_{p,q}^2 = 0 \) for \( p > 1 \) implies that the spectral sequence collapses at the \( E_2 \)-page. The edge homomorphism from \( E_{0,*}^\infty \) to \( H^*(P) \) may be identified with \( \text{Res}_P^\Gamma \) (consider the map of spectral sequences induced by the inclusion of the vertex set of the tree in the whole tree, viewed as a map of \( \Gamma \)-spaces), and so a) is proved. For b), note that since \( E_{2,q}^2 = 0 \) for \( p > 1 \), elements of \( E_{1,*}^2 \) uniquely determine elements of \( H^*(\Gamma) \), and the product of any two such elements is zero in \( H^*(\Gamma) \). Since \( \text{Ker}(\text{Res}_P^\Gamma) \) may be identified with \( E_{2,*}^2 \), b) follows, and c) follows immediately from b). Finally, d) follows from Theorem 5 stated above.

Remarks. 1) There are alternatives to using the equivariant cohomology spectral sequence in the proof of Theorem 1, but following a suggestion of the referee we decided to explain just one method in detail in the proof. Since the spectral sequence has only two non-zero rows it is essentially just a long exact sequence. This long exact sequence may be obtained by applying \( H^*(\Gamma; \cdot) \) to the augmented chain complex for the tree \( T \), modulo an application of the Eckmann-Shapiro lemma. We felt, however, that the ring structure of \( H^*(\Gamma) \) is more easily understood in terms of the spectral sequence.

2) We believe that \( I(P,C_u) \) has some advantages over \( I(P,C_s) \). Both of these rings enjoy the finiteness property stated in Corollary 2. To compute \( I(P,C_s) \) one needs to know something about the \( p \)-local structure of all groups with Sylow subgroup \( P \), whereas \( I(P,C_u) \) requires only knowledge of \( P \).

3) On the other hand, \( I(P,C_u) \) does not retain much information concerning \( P \). Let \( W(P) \) be the variety of all ring homomorphisms from \( I(P,C_u) \) to an algebraically closed field \( k \) of characteristic \( p \). Then \( W(P) \) is determined up to homeomorphism by the \( p \)-rank of \( P \): If \( P \) has \( p \)-rank \( n \), then \( W(P) \) is homeomorphic to \( k^n/\text{GL}_n(F_p) \), and if \( E \) is an elementary abelian subgroup of \( P \) of rank \( n \), then the induced map from \( W(E) \) to \( W(P) \) is an homeomorphism. These assertions concerning \( W(P) \) follow easily from Quillen’s
theorem describing the variety of homomorphisms from $H^*(P)$ to $k$ (see for example [5], chap. 9). Note that this the only place where we use Quillen’s theorem.

4) The definitions and theorems that we state remain valid if $P$ is any finite group. We restrict to the case when $P$ is a $p$-group only because this is the case occurring naturally in the work of Cartan-Eilenberg and Evens-Priddy.

5) The reader may have noticed that Theorems 1 and 3 work perfectly well for cohomology with coefficients in any ring $R$ (viewed as a trivial $P$-module). Corollary 2 is valid for cohomology with coefficients in any ring $R$ for which the Evens-Venkov theorem holds (see [5], 7.4 for a general statement).

6) The easiest way to relax the restrictions on the category $C$ is to consider arbitrary finite categories with objects finite groups and morphisms injective group homomorphisms (it is unhelpful to view the groups as subgroups of a single group if the inclusion maps are not in the category). Define $I(C)$ to be the limit over this category and for any group $\Gamma$, define $D(\Gamma)$ to be the category of finite subgroups of $\Gamma$, with morphisms inclusions and conjugation by elements of $\Gamma$. Then one obtains

**Theorem 1’.** Let $C$ be a connected finite category of finite groups and injective homomorphisms. Then there exists a discrete group $\Gamma$ and a natural transformation from $C$ to $D(\Gamma)$ such that $\Gamma$ and the induced map from $H^*(\Gamma)$ to $I(C)$ satisfy properties a) to d) of Theorem 1.

Recall that a category is said to be connected if the equivalence relation on objects generated by ‘there is a morphism from $Q$ to $Q’’ has exactly one class. Note that there cannot be a direct analogue of Theorem 1 unless the category $C$ is connected, since the degree zero part of $I(C)$ is an $F_p$ vector space of dimension the number of components of $C$, whereas $H^0(\Gamma) \cong F_p$. The proof of Theorem 1’ is very similar to the proof of Theorem 1, except that one creates a graph of groups with one vertex for every object of $C$. The restriction to connected categories is not serious, since given any category $C$ as above, one may make a connected category $C^+$ by adding a trivial group to $C$ as an initial object (i.e., add one new object, a trivial group, and one morphism from this object to every other object). The natural map from $I(C^+)$ to $I(C)$ is an isomorphism, except in degree zero.

The analogue of Corollary 2 in this generality, for which $C$ need not be assumed to be connected, is:

**Corollary 2’.** Let $C$ be a finite category of finite groups and injective homomorphisms. Then $\prod_{Q \in C} H^*(Q)$ is a finite module for $I(C)$.
7) The following instance of Theorem 1 seems worthy of special note. Let $P$ be an elementary abelian 2-group of rank $n$, let $C$ be the category whose only object is $P$ and whose morphisms are the group $GL(n, F_2)$. Then $H^*(BG)$ is a ring whose radical is invariant under the action of the Steenrod algebra, and $H^*(BG)/\sqrt{0}$ is isomorphic to the Dickson algebra $D_n = F_2[x_1, \ldots, x_n]^{GL(n, F_2)}$. On the other hand it is known that for $n \geq 6$, $D_n$ itself cannot be the cohomology of any space [7].

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