Enhancement of the superconducting transition temperature from the competition between electron-electron correlations and electron-phonon interactions

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We uncover that the competition between electron-electron correlations and electron-phonon interactions gives rise to unexpectedly huge enhancement of the superconducting transition temperature, several hundreds percent larger ($\geq 200$ K) than that of the case when only one of the two is taken into account ($\sim 30$ K). Our renormalization group analysis claims that this mechanism for the enhancement of the critical temperature is not limited on superconductivity but applied to various Fermi surface instabilities, proposing an underlying universal structure, which turns out to be essentially identical to that of a recent study [Phys. Rev. Lett. 108, 046601 (2012)] on the enhancement of the Kondo temperature in the presence of Rashba spin-orbit interactions. We also discuss the stability of superconductivity against nonmagnetic randomness.

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It is our aspiration to increase the superconducting transition temperature. The BCS (Bardeen-Cooper-Schriefer) theory [1] has been our paradigm for the mechanism of superconductivity. Unfortunately, this fundamental theory does not allow us to enhance the critical temperature as much as what we want. Most efforts are dedicated to searching “the beyond-BCS theory”, where electron correlations are proposed to cause Cooper pairing, the source of superconductivity. These theories for unconventional superconductivity can be classified into Fermi-liquid based theory [2], spin-liquid based theory [3], quantum-critical-metal based theory [3], and etc [3], based on the mother state for the superconductivity. Although these theoretical frameworks have their own predictions in physical spectra, thermodynamics, and transport for various unconventional superconductors such as high $T_c$ cuprates, organic materials, heavy-fermion systems, pnictide superconductors, and etc, many fundamental questions remain unanswered, in particular, even the simplest question: how can we enhance the critical temperature?

In this letter we revisit this simple but fundamental question, resorting to the Fermi-liquid based theory and the BCS mechanism, which allows us to avoid any artificial complexity and give a definite answer. We introduce both electron-electron correlations and electron-phonon interactions, which favor $s^+$ and $s^+$ pairing symmetries, respectively, in the FeAs-type two-band structure [4]. The naive expectation is that the critical temperature ($T_c$) decreases down to zero around the region where both effective interactions become identical. However, we observe unexpectedly huge enhancement of $T_c$, several hundreds percent larger ($\geq 200$ K) than that of the case when only one of the two is taken into account ($\sim 30$ K). Solving coupled BCS gap equations both numerically and analytically, we prove this $T_c$ enhancement.

An interesting aspect of our study is to claim that there exists an underlying universal structure for the $T_c$ enhancement. The renormalization group analysis clarifies such a structure, which occurs from Fermi surface instabilities. We observe that our coupled renormalization group equations for two competing superconducting correlations are essentially identical to those of a recent study [5] on the enhancement of the Kondo temperature, where the interplay between Rashba spin-orbit interactions and the Kondo effect strengthens the Kondo effect. It is rather unexpected that both systems have basically the same structure in the renormalization group sense.

Our model consists of two types of Fermi surfaces, the hole Fermi surface near the $\Gamma$ point ($c_{k\alpha}$) and the electron one near the $M$ point ($f_{k\alpha}$), sometimes regarded as an effective Hamiltonian for $FeAs$ superconductors [7]. We consider two competing interactions for superconductivity, one of which results from electron-phonon correlations to describe pair hopping from the hole Fermi surface to the electron one and vice versa, $H_{\Delta}^{el-ph} = \frac{1}{2} \sum_{k,p} V_{\alpha\beta\alpha'}^{el-ph}(k,p)(c_{k\alpha}^\dagger c_{p\beta'}^\dagger f_{-p\beta}^\dagger f_{p\alpha'},)$, and the other of which originates from electron-electron interactions to introduce conventional BCS pairing on each Fermi surface, $H_{\Delta}^{el-el} = \frac{1}{2} \sum_{k,p} V_{\alpha\beta\alpha'}^{el-el}(k,p)(c_{k\alpha}^\dagger c_{-p\beta}^\dagger f_{p\beta'}^\dagger f_{-p\alpha'},)$, where $V_{\beta\alpha\alpha'}^{el-el}(k,p) = V_{\beta\alpha\alpha'}^{el-el}(i\sigma\nu)^{\dagger}_{\alpha\beta}(i\sigma\nu)^{\dagger}_{\beta\alpha'}$ is the effective coupling constant for the singlet superconductivity. Here, we neglect their momentum dependencies, i.e., $V_{\beta\alpha\alpha'}^{el-el}(k,p) = V_{\beta e\alpha'}^{el-el} = V_{ee}^{el-el}$.

One may be concerned with spin-density-wave instability, expected to compete with superconducting instability. Recently, one of us discussed that marginal breakdown of the Fermi-surface nesting favors superconductivity instead of spin-density-wave ordering [6]. In this study we assume the regime to favor superconducting in-
stability instead of spin-density-wave ordering. We do not take into account the competition between antiferromagnetism and superconductivity.

We perform the standard mean-field analysis within the BCS framework. Electron-electron correlations give rise to $s^+$ pairing while electron-phonon interactions result in $s^+$ pairing [1]. Introducing this pairing symmetry into the mean-field analysis, we obtain coupled gap equations,

$$
\sum_k \frac{\Delta_c}{E_c(k)} \tanh \left( \frac{\beta E_c(k)}{2} \right) + \frac{2V_{ee}}{V_{ee}^2 - V_{ep}^2} \Delta_f + \frac{2V_{ep}}{V_{ee}^2 - V_{ep}^2} \Delta_c = 0,
$$

$$
\sum_k \frac{\Delta_f}{E_f(k)} \tanh \left( \frac{\beta E_f(k)}{2} \right) + \frac{2V_{ee}}{V_{ee}^2 - V_{ep}^2} \Delta_c + \frac{2V_{ep}}{V_{ee}^2 - V_{ep}^2} \Delta_f = 0, \quad (1)
$$

![Graph](image)

**FIG. 1:** Left (Right) : The critical temperature $T_c(R)$ from Eq. (1) as a function of the interaction ratio $R = V_{ee}/V_{ep}$ with a fixed $V_{ep}$ ($V_{ee}$). The black-circle line is obtained when $V_{ee} = 15\omega_D$ ($V_{ee} = 15\omega_D$) and the red-triangle line, $V_{ee} = 20\omega_D$ ($V_{ee} = 20\omega_D$), where $\omega_D$ is the Debye frequency. The main feature is that the critical temperature becomes enhanced at $V_{ee} \approx V_{ep}$, i.e., $T_c(R \approx 1^+) \geq 200K$ ($T_c(R \approx 1^+) \geq 250K$), several hundreds percent more than the electron-phonon (electron-electron) driven critical temperature $T_c(R = 0) \approx 25K$ ($T_c(R = 0) \approx 30K$). Inset : The critical temperature $T_c(R)$ from the analytic formula of Eq. (2). Although the qualitative consistency confirms our numerical analysis, the quantitative difference results from the choice of cutoff used to determine $T_c(R)$.

where $\Delta_c(f)$ is the total pairing amplitude for the hole (electron) Fermi surface, which results from both effective interactions, and $E_c(f)(k)$ is the conventional BCS-quasiparticle spectrum [1] for each Fermi surface. An interesting point in these equations is the presence of divergence when both interactions become identical, i.e., $V_{ee} = V_{ep}$, implying that the pairing amplitude vanishes. However, we find a non-monotonic behavior for the critical temperature near the point of $V_{ee} = V_{ep}$.

Fig. 1 shows how the critical temperature $T_c(R)$ changes as a function of the interaction ratio $R = V_{ee}/V_{ep}$. First, we start from the regime where the electron-electron interaction is smaller than the electron-phonon interaction, i.e., $R < 1$ (Left in Fig. 1). We note that the critical temperature at $R = 0$, i.e., the electron-phonon driven BCS transition temperature is about the order of 25 K. Increasing electron-electron correlations in $R < 1$, the critical temperature decreases down monotonically and touches the zero point before $R = 1$. This result is physically natural, where the $s^+$ pairing competes with the $s^+$ pairing and the total Cooper-pairing amplitude vanishes near $R = 1$. On the other hand, if $R$ increases further to cross the $R = 1$ point, we observe an abrupt enhancement of $T_c$, i.e., $T_c(R = 1^+) \gg T_c(R = 0)$. Further increase of electron-electron correlations leads $T_c(R \gg 1)$ to follow $V_{ee}$ in the large-$V_{ee}$ region. Second, we start from the regime of $R \to \infty$, i.e., $V_{ee} = 0$, where the electron-correlation driven $s^+$ superconducting transition temperature is about 30 K (Right in Fig. 1). Increasing the electron-phonon interaction, the critical temperature also increases monotonically, touching the highest temperature larger than 200 K near $R \approx 1^+$. $T_c$ becomes enhanced several hundreds percent more! On the other hand, the critical temperature vanishes near $R = 1^-$ when $R$ crosses the $R = 1$ point from the $R > 1$ side. It increases monotonically as $R$ decreases further in $R < 1$, and reaches its saturate value at $R = 0$ ($V_{ph} \to \infty$).

One can obtain an analytic expression for the critical temperature in the case of twin Fermi-surfaces, given by

$$
T_c(R) = D \exp \left( \frac{1}{N_F V_{ee} R^2} - 1 \right),
$$

where $N_F$ is the density of states at the Fermi energy and $D$ is the half bandwidth. As shown in Fig. 1 (Inset), this analytic expression displays qualitatively the same behavior as the numerical result of $T_c(R)$. The quantitative difference results from the choice of cutoff, used to determine $T_c(R)$.

Fig. 2 shows the temperature dependencies of two superconducting order parameters at each value of $R$.
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abruptly. However, we do not understand the reason
cparameters at low temperatures. We interpret that this
observed. Fig. 2 displays bifurcation behaviors for order
the superconducting order parameter in the zero temper-
R < 1 or
R > 1. In
1/2 ≤ R < 1 both ∆c(T) and ∆f(T) follow that of the
electron-phonon mediated s++ superconductivity while the
sign difference of ∆c(T) > 0 and ∆f(T) < 0 reflects the s+-
pairing symmetry due to electron-electron correlations in R > 1. On the other hand, completely unexpected behaviors appear near R ≈ 1+. Both supercondu-
cing order parameters exhibit non-monotonic temperature dependencies, where their amplitudes become suppressed at low temperatures although they start to arise at much higher temperature than the case of either R < 1 or R > 1. In R ≈ 1/2 more exotic behaviors are observed. Fig. 2 displays bifurcation behaviors for order parameters at low temperatures. We interpret that this bifurcation behavior originates from the second critical temperature, attributed to the electron-correlation s+-
superconductivity, where ∆c(f)(T) increases (decreases) abruptly. However, we do not understand the reason why this behavior does appear only in R ≪ 1/2, not in R ≥ 1/2.

FIG. 2: Temperature dependencies of both superconducting order parameters with a fixed Vee, where ∆c/f(T) is represented by the dashed (thick) line in 1 ≤ R while they are expressed by the upper (lower) curve in R ≪ 1/2. The main feature is that both order parameters display non-monotonic behaviors near R ≈ 1+, where they become suppressed at low temperatures, while they exhibit conventional BCS-type behaviors in both 1/2 ≤ R < 1 and R ≫ 1. On the other hand, bifurcation behaviors are observed at low temperatures in R ≪ 1/2. We interpret that this bifurcation behavior originates from the second critical temperature, attributed to the electron-correlation s+- superconductivity, where ∆c(f)(T) increases (decreases) abruptly. Inset: The total Cooper pairing amplitude ∆c(R) in the zero temperature limit, based on the analytic formulae Eq. (3).

One can find an analytic formula for the amplitude of the superconducting order parameter in the zero temper-

1/2 ≤ R < 1 both ∆c(T) and ∆f(T) follow that of the
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although the enhancement of Tc has been proved within the mean-field theory, the physical mechanism for the enhancement of Tc has not been identified yet. Performing the standard renormalization group analysis in the one-loop level, we obtain coupled renormalization group equations for both effective interactions,

\[ \frac{dv_{ep}}{d\ln \mu} = -v^2_{ep} - v^2_{ee}, \quad \frac{dv_{ee}}{d\ln \mu} = -v_{ep} v_{ee}, \] (4)

where both coupling constants are scaled with the density of states, respectively, and \( \mu \) is the typical scale for the renormalization group analysis. Here, a positive numerical constant \( c \) counts the difference of two Fermi surfaces, and \( c = 2 \) describes twin Fermi surfaces.

Let’s focus on the case of twin Fermi surfaces for simplicity. Then, these renormalization group equations can be rewritten as follows, \( \frac{dv_{ep}}{d\ln \mu} = \mp v^2_{ee} \), where \( v_{\pm} = v_{ee} \pm v_{ep} \) are effective coupling constants for superconductivity. It is straightforward to solve these equations and find \( v_{\pm}(T) = \frac{v^2_{ee}}{1 \mp v^2_{ee} \ln(D/T)} \) as a function of temperature. The critical temperature is identified as \( v_{\pm}(T_c) \rightarrow \infty \), given by \( T_c = D \exp\left( -\frac{1}{v_{ee} \mp v_{ep}} \right) \) when \( R > 1 \) and \( T_c = D \exp\left( \frac{1}{v_{ee} \mp v_{ep}} \right) \) when \( R < 1 \). The second expression results in \( T_c \rightarrow 0 \) as \( v_{ee} \rightarrow v^2_{ep} \) in \( R < 1 \). In fact, these expressions for \( T_c \) coincide with the above analytic formula [Eq. (2)] from the coupled BCS gap equations [Eq. (1)], which crosschecks the whole procedure of our analysis.

It is rather remarkable to observe that our renormalization group equations [Eq. (4)] are essentially identical to those in a recent study [8] on the enhancement of the Kondo temperature due to the interplay between Rashba spin-orbit interactions and the Kondo effect. Indeed, one can identify the electron-phonon interaction with the Kondo coupling constant while one may match the electron-electron correlation with the Rashba spin-orbit interaction. In this respect we propose a general scheme on the way how to enhance the critical temperature, which occurs from the Fermi-surface instability.
The presence of competing interactions gives rise to the enhancement of the critical temperature, where “competition” means that the critical temperature vanishes or becomes suppressed when one interaction parameter approaches the other in one direction. More precisely, the system described by the renormalization group equations, Eq. (4) is the rule-model for the enhancement of the critical temperature.

It is important to check the stability of the superconducting state against weak nonmagnetic randomness since we are considering unconventional superconductivity and the Anderson theorem [11] does not work in this situation. There are two types of nonmagnetic impurities. One causes intra-scattering within the same Fermi surface, and the other generates inter-scattering between different Fermi surfaces. It has been both intensively and extensively investigated that the intra-scattering events do not affect the $s^{++}$ superconducting properties such as the critical temperature and gap size much while the inter-scattering reduce both the critical temperature and gap size seriously [12]. Let’s apply these results to our mixed superconductivity. In $R < 1$ the superconducting transition is driven by the electron-phonon interaction, although the critical temperature decreases monotonically down to zero as a function of $R$ due to the renormalization effect from the electron-electron correlation. It is natural to expect that the Anderson theorem will work in $R < 1$. However, we can also observe the second critical temperature in $R \ll 1/2$, given by the electron-electron correlation. As a result, the low temperature superconductivity will be suppressed by the inter-scattering events and thus, the abrupt increase or decrease in superconducting order parameters may appear at much lower temperature. Furthermore, it is difficult to guarantee that the Anderson theorem holds in $1/2 \leq R < 1$ because the amplitude of the superconducting order parameter can be smaller than the disorder strength. In this respect the competition between weak $s^{++}$ superconductivity and weak nonmagnetic randomness will be more complicated. In $R > 1$ the inter-scattering events will reduce the critical temperature pretty much. Short-range nonmagnetic impurities will give negative effects on the enhancement of $T_c$. More quantitative and self-consistent analysis is necessary.

In conclusion, we proposed a general scheme on the way how to enhance the critical temperature for superconductivity. An essential ingredient is to introduce competing interactions. Here, we considered both electron-phonon interactions and electron-electron correlations and demonstrated that the superconducting transition temperature becomes much enhanced more than several hundreds percent when the electron-phonon interaction increases up to the electron-electron correlation. We claim that this phenomenon will not be limited on superconductivity but be generalized to various Fermi-surface instabilities. The renormalization group analysis uncovers the underlying general structure for the enhancement of the critical temperature, given by Fermi-surface instabilities. Indeed, the Kondo effect in the presence of the Rashba spin-orbit interaction [8] shows essentially the same renormalization group structure as the electron-electron vs. electron-phonon competing superconductivity.

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