Matrix at slow roll: nonrelativistic and perturbative

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Abstract

We analyze the slow roll dynamics of the massive version of Yang–Mills-type matrix mechanics. We find that one can reproduce in this limit the one-loop non-Hermitian matrix model, describing the dilatations of the local gauge invariant composite operators in the scalar sector of $\mathcal{N} = 4$ super Yang–Mills theory. This possibility is explained through the fact that the dilatation operator of the last can be identified with the radial-time Hamiltonian. We also explore the finite mass corrections to the slow roll action.

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1. Introduction

Matrix models appear as a convenient tool for encoding complicated combinatorics in various statistical physics problems. Thus, IKKT and BFSS matrix models [1, 2] were proposed as models describing IIB string with quantized worldsheet, and light-cone quantized membrane, respectively. These models belong to the type of so-called Yang–Mills matrix models, because they can be obtained by reduction to one or zero dimensions from the Yang–Mills theory in higher dimensions. They also proved to be relevant for background independent formulation of noncommutative Yang–Mills theory in various dimensions (see e.g. [3]).

Generically, one can obtain matrix mechanical models by the worldsheet quantization of a ‘sliced’ description of extended objects such as membranes [4, 5]. In the flat spacetime, such models are massless, but they may acquire a mass in a non-trivial background such as gravitational plane waves [6, 7]. In this case, the mass parameter corresponds to the strength of the gravity background. The large mass limit means strongly coupled gravity. According to the AdS/CFT conjecture this corresponds to a weakly coupled gauge theory. On the other hand, the large mass parameter admits a slow roll expansion similar to the nonrelativistic limit. This expansion is the subject of this work.

Massive Yang–Mills-type matrix models also naturally arise in the radial-time quantization [8] of Yang–Mills theories [9]. As seen from the point of view of the realtime
formalism, the radial-time Hamiltonian corresponds to the dilatation operator. Therefore, the large mass expansion of these matrix models should correspond to the perturbative expansion of the dilatation operator in the real time (see [10]). Due to the scale transmutation, however, it is not clear whether the radial- and realtime quantizations are always equivalent. As we show below, even in the conformal theories, for which this certainly should be true, this mechanism implies nontrivial relations between the classical and one-loop level quantities.

The plan of the paper is as follows. In the next section, we introduce the massive matrix theory and argue that the perturbation theory expansion should be equivalent to the large mass expansion. Then, in the following section we take the slow roll limit of the above model and show that the resulting theory has exactly the same structure and can be associated with the dilatation operator in the scalar sector of $N = 4$ super Yang–Mills theory. Next, we find the leading finite mass corrections to the slow roll action and compare it to the two-loop dilatation operator of $N = 4$ super Yang–Mills theory. Finally, we conclude by discussion of the results.

2. Massive matrix theory

Consider a purely bosonic massive matrix model described by the action

$$S = \int dt \left( \frac{1}{2} \text{tr}(\nabla_0 X_i)^2 - \frac{1}{2} m^2 X_i^2 + \frac{g^2}{4} \text{tr}[X_i, X_j]^2 \right).$$

(2.1)

The indices $i, j$ run through 1, . . . , $D$, and the covariant time derivative is defined by

$$\nabla_0 X_i = \dot{X}_i + [A, X_i],$$

(2.2)

where $A$ is a non-dynamical SU($N$) gauge field. The dot denotes the time derivative. The system is invariant with respect to SU($N$) time-dependent gauge transformations:

$$X_i \rightarrow U^{-1}X_iU, \quad A \rightarrow U^{-1}AU + U^{-1}\dot{U}.$$  

(2.3)

Let us consider the independent parameters controlling this model. A rescaling of the matrix variables $X_i \rightarrow g^{-\frac{1}{3}}X_i$ associated with the time rescaling $t \rightarrow g^{-\frac{2}{3}}t$ will set the coupling to $g = 1$, changing at the same time the mass parameter $m \rightarrow mg^{-\frac{2}{3}}$. So the zero temperature theory effectively depends only on the combination $m^3/g^2$ of mass and coupling. Therefore, the large mass limit is equivalent to the weak coupling limit and vice versa. In the case of a finite temperature, the above transformation modifies in addition the inverse temperature $\beta \rightarrow \beta g^{\frac{2}{3}}$.

The model described by action (2.1) can be obtained, among others, as the bosonic part of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the radial quantization [8]. In [12] this model was conjectured to provide (for $D = 6$) a nonperturbative description for dilatations in the BMN limit [6]. In contrast, in [13], it was argued that it is a non-Hermitian matrix model, which governs the dynamics associated with the dilatation operator at the one-loop level. In what follows we will consider the limit of slow dynamics for action (2.1), which is justified in the case of large mass, and show that the non-Hermitian matrix model is reproduced in this limit.

3. The Slow roll limit

Consider the slow roll expansion of the massive matrix model (2.1). In order to do this, let us express the matrix $X_i$ as a combination of ‘mostly forward’ and ‘mostly backward’ running matrices $\Phi_i$ and $\Phi_i$, respectively:

$$X_i = \frac{1}{\sqrt{2m}}(e^{-imt} \Phi_i(t) + e^{imt} \Phi_i(t)).$$

(3.1)
Expansion (3.1) of the matrix $X_i$ is similar to the particle–antiparticle expansion in the non-relativistic limit in field theory.

In order to take the slow roll limit, we substitute the decomposition (3.1) into the action (2.1) then take the limit of large mass, keeping the finite fixed value for
\[ g_{YM}^2 = 4\pi^2(g/m)^2, \]  
(3.2)
dropping the terms vanishing in the $m \to \infty$ limit. As we aim the slow rolling limit we have also to discard the high frequency modes $\omega \gtrsim m$. This is equivalent to introducing UV cutoff $\Lambda \sim m$. The resulting action takes the following form:
\[
S_{\text{slow}} = \int dt \left( \frac{i}{2} \left( \dot{\Phi}_i \nabla_0 \Phi_i - \nabla_0 \Phi_i \Phi_i \right) \right. \\
\left. + \frac{g_{YM}^2}{16\pi^2} \left( \text{tr}[\Phi_i, \Phi_j][\Phi_i, \Phi_j] + \frac{1}{2} \text{tr}[\Phi_i, \Phi_j][\Phi_i, \Phi_j] \right) \right). 
\]  
(3.3)

This action coincides with the one corresponding to the dilatation operator for the scalar sector of $\cal{N} = 4$ SYM constructed in [13]. The dilatation operator itself was first constructed in [14]. (See also [15] for the construction of dilatation operators in the general case of renormalizable theories.) This coincidence is not accidental. It has an explanation in the conformal symmetry of $\cal{N} = 4$ SYM. As a conformal theory, it can be equivalently quantized either in the real-time framework or in the radial-time formalism [8], in which the Hamiltonian of theory acquires an additional mass term corresponding to the engineering dimensions of the fields. It is also clear that the identification can be made with some field scale and time parameter redefinitions, which implies a certain relation between the couplings in two theories.

The model described by action (3.3) is quite well understood. Its symmetries are well known and the lowest energy eigenvalues and states can be found exactly in the framework of spin bit model [16]. Its thermodynamical analysis is performed in [13].

Let us note that, since the effective coupling is $g^2/m^3 \sim g_{YM}^2/m$, the original matrix theory remains effectively perturbative for at least as far as $\lambda = g_{YM}^2/4\pi^2 m$ is small, i.e. $g_{YM}$ itself can be very large.

4 ‘Relativistic’ $1/m$ corrections

The slow roll limit provided an easy way to retrieve otherwise more involved one-loop result of [14, 15]. Starting from this, it is interesting to investigate whether we can gain more information on perturbative properties of the theory and obtain $1/m$ corrections to the slow roll action (3.3) in the form of a systematic expansion. In the classical one-particle dynamics the answer to this question is easy: these corrections are casted in the energy expansion formula
\[ E = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m} - \frac{p^4}{8m^4} + \ldots. \]

In the interacting quantum theory, as we will see, the situation is more tricky.

Indeed, to rich the slow roll action (3.3) we discarded the following terms:
\[
\Delta S_{\text{non-slow}} = \int dt \left( \frac{1}{2m} \dot{\Phi}_i \dot{\Phi}_i + \frac{1}{4m} e^{-2imt} \dot{\Phi}_i^2 + \frac{1}{4m} e^{2imt} \dot{\Phi}_i^2 \\
+ \frac{g_{YM}^2}{16\pi^2} \text{tr}[\Phi_i, \Phi_j][\Phi_i, \Phi_j] + \frac{g_{YM}^2}{16\pi^2} e^{2imt} \text{tr}[\Phi_i, \Phi_j][\Phi_i, \Phi_j] \\
+ \frac{g_{YM}^2}{64\pi^2} e^{-4imt} \text{tr}[\Phi_i, \Phi_j]^2 + \frac{g_{YM}^2}{64\pi^2} e^{-4imt} \text{tr}[\Phi_i, \Phi_j]^2 \right), 
\]  
(4.1)
The terms in the first line of (4.1) are suppressed among others, because they are of the order $\sim m^{-1}$. These terms correspond to the correction terms in the low speed expansion of the relativistic energy formula. The additional terms which are appearing in the second and third lines have the order $\sim 1$. However, they contain fast oscillating phase factors; therefore, for slowly varying fields $\Phi$ and $\bar{\Phi}$ these terms are strongly suppressed, at least classically. In the quantum theory, however, they produce a coupling between the slow and fast modes. Because of this, the situation becomes more subtle: the coupling to higher frequencies can result in a correction to the dynamics of slow fields.

From the point of view of the original matrix theory (2.1), the slow roll sector corresponds to the bunch of modes around the free on-shell resonant modes $\omega = \pm m$. These modes, however, are coupled to modes with higher frequencies $|\omega| \gtrsim m$, as well as to the modes with lower frequencies: $|\omega| \lesssim m$. According to the general practice, in order to get the effective action for slow roll modes one should eliminate both higher and lower frequencies by integrating them, after introducing appropriate cutoffs\(^1\).

The terms in the first line of equation (4.1) are coming from the oscillator part of the action. When the high frequency modes are cut off, the remaining term modifies the symplectic structure in such a way that the number of degrees of freedom seems doubled. It is clear that this phenomenon will occur even in the free massive theory, where we expect to get no $1/m$ corrections at all. This can be explained by the fact that the substitution (3.1) is degenerate for finite $m$: transition from $X_i$ to $\Phi_i$ introduces a gauge symmetry mixing mode of different frequencies, which breaks in the $m \to \infty$ limit.

To avoid the above confusion, let us formulate a more accurate expansion. In order to do this let us proceed along the following steps. Let us (i) go to the first-order action description (after fixing the temporal gauge $A = 0$), by introducing the canonical momentum,

$$\Pi_i = \frac{\partial L}{\partial X_i} = \dot{X}_i, \quad (4.2)$$

and performing a Legendre transform, then (ii) go to the complex matrix field

$$\hat{\Phi}_i = \frac{1}{\sqrt{2m}} (\Pi_i - imX_i), \quad (4.3)$$

and finally (iii) do a time-dependent variable substitution

$$\Phi_i = \hat{\Phi}_i e^{imt}. \quad (4.4)$$

Then the action becomes

$$S = \int dt \text{tr} \left\{ \frac{i}{2} (\hat{\Phi}_i \dot{\hat{\Phi}}_i - \dot{\hat{\Phi}}_i \hat{\Phi}_i) + \frac{g^2}{m^2} [ (\Phi_i e^{-imt} - \hat{\Phi}_i e^{imt}) , (\Phi_j e^{-imt} - \bar{\Phi}_j e^{imt}) ]^2 \right\}. \quad (4.5)$$

Applied to the pure matrix oscillator, this procedure completely removes the dependence of the action on the mass\(^2\), pushing it to the boundary conditions, therefore reducing the theory to a topological model with zero Hamiltonian. Since all transformations are non-degenerate, action (4.5) is equivalent to the original action (2.1) at any $m$. In the slow roll limit, the large frequency modes are suppressed and we get the slow roll action (3.3). At the same time, the only finite-$m$ effect is the coupling between the low and high frequency modes due to the presence of the phase factors $e^{\pm imt}$ in the interaction potential. Hence, the $1/m$ corrections can be taken care of by merely integrating out the high frequency modes.

\(^1\) Let us note that since expansion (3.1) is shifting the frequency of fields $\Phi$ with respect to the frequency of $X$, the notions of ‘high frequencies’ and ‘low frequencies’ are different for these two actions. Thus, both ‘low frequency’ and ‘high frequency’ for the original action (2.1) will translate to ‘high frequency’ of the reduced action (3.3).

\(^2\) Apart from the phase factors in the interaction term.
So, to find the contribution from the high frequency modes let us split the original matrix field \( \Phi_i \) into the low and high frequency parts \( \phi_i \) and \( u_i \), respectively:

\[
\Phi_i = \phi_i + u_i ,
\]

where \( \phi_i \) contains only Fourier modes of \( \Phi_i \) with the frequency \( \omega < m \), while \( u_i \) only those with \( \omega \geq m \). This implies the use of a sharp cutoff. In the spirit of the Wilsonian effective action, we have to keep the low frequency mode \( \phi_i \), while integrating out the fast modes \( u_i \). Due to non-trivial coupling between low and high frequency modes the leading correction is produced already at the tree level. For the tree level evaluation we have to keep only linear in \( u_i \) terms in the interaction potential. Up to this order the action (4.5) reads

\[
S_{(1)} = S_0 + \int dt \text{tr} \left[ \frac{1}{2} (\ddot{u}_i u_i - \ddot{\bar{u}}_i u_i) + \frac{4g^2}{m^2} \left\{ \ddot{u}_j (-[[\phi_i, \phi_j], \phi_i]) e^{-3 \text{int}} + \frac{4}{3} \left( [\phi_i, \phi_j], \phi_i \right) + ([\phi_i, \phi_j], \phi_i) e^{3 \text{int}} \right\} \right],
\]

where the part \( S_0 \) is the slow roll action (3.3), depending only on the slow field \( \phi_i \). The fast field \( u_i \) has the frequencies \( m \leq \omega < \infty \) and the Hermitian conjugate \( \bar{u}_i \) carries the frequencies \( \infty < \omega \leq -m \).

Integrating out the \( u_i \)-matrix together with its Hermitian conjugate and expanding the result in \( 1/m \) will produce the following corrections to the action \( S_0 \):

\[
S_{1/m} = \frac{g_{YM}^2}{\pi^4 m} \int dt \text{tr} \left\{ \frac{1}{3} \left( [\phi_i, \phi_j], \phi_i \right) [\phi_k, \phi_j], \phi_k \right\} + (\text{terms related to the ordering prescription})
\]

\[
\times (\text{terms related to the ordering prescription})
\]

(4.8)

where we replaced the original matrix coupling by \( g_{YM} \) according to (3.2). The term in the first line of the rhs of equation (4.8) is the contribution by fast modes with the frequency \( \omega \sim \pm 3m \), while the last two lines are the result of integration of modes with \( \omega \sim \pm m \).

In terms of super Yang–Mills theory, this contribution should represent the two-loop connection to the dilatation operator. Although there is no reason to expect the complete coincidence of the bosonic model (2.1), which is still very different from the \( N = 4 \) SYM, it is still instructive to compare the correction (4.8) with the two-loop contribution to the dilatation operator in \( N = 4 \) SYM theory, which was constructed in [14]. The dilatation operator is formulated in terms of letter insertion and removal operators, \( \Phi_i \) and \( \Phi_i \), respectively, which roughly correspond to quantized versions of the fields \( \phi_i \) and \( \phi_i \). In the case of the scalar SO(6) sector this contribution reads

\[
D_i \propto \text{tr} \Phi_i [\Phi_j, [\Phi_k, [\Phi_l, [\Phi_m, \Phi_i]]] + \ldots,
\]

(4.9)

where \( i, j, k, l = 1, \ldots, 6 \) and the dots stand for the terms related to the ordering prescription for the operators \( \Phi_i \) and \( \Phi_i \).

By a direct inspection of expressions (4.8) and (4.9) it can be verified that the terms in the last two lines of (4.8) can be brought to a form similar to (4.9), while the term in the first line has a different structure. This discrepancy, however, is expected to disappear if the appropriate matrix theory related to the \( N = 4 \) super Yang–Mills theory is used.

Another remark is in order. The time-dependent variable change done in order to obtain action (4.5) modifies the boundary conditions. Therefore, in the case of finite temperature, the periodic in-time boundary conditions should be replaced by a more generic form:

\[
\Phi_i (t + \beta) = e^{i \beta m} \Phi_i (t),
\]

(4.10)
where $\beta$ is the inverse temperature. In the case of a pure matrix oscillator, it is the non-trivial boundary conditions which render the correct partition function, since the dynamics in new coordinates is trivial.

5. Discussion

In this paper, we considered the slow roll limit of a massive matrix theory. In particular, it was shown that the limit applied to the matrix model resulting from the radial quantization of the compact sector of $\mathcal{N} = 4$ super Yang–Mills theory reproduces the model for one-loop anomalous dimensions of local gauge invariant composite operators in this sector. This is explained by the coincidence of the perturbative dilatation operator with the radial-time Hamiltonian in the slow roll limit. Even so, this is remarkable since it implies that the result of one-loop calculations can be easily obtained by a limiting procedure. Inspired by this we performed the calculation of leading $1/m$ correction to the slow roll action. The structure of corrections as it appears in equation (4.8) is reminiscent of that of a two-loop effective action.

It would be interesting to apply the above analysis to the PSU(2, 2$|$4) matrix model or similar models to compare the result to the two-loop contribution to the dilatation operator there. Therefore, a natural generalization of the above analysis is to extend it to the whole spectrum of super Yang–Mills theory. The super Yang–Mills theory in the radial time can itself be represented as a matrix theory with an infinite number of matrices. Then, the possibility of obtaining the complete one-loop dilatation operator by the described procedure is very likely. It may also appear possible that the complete two-loop operator, which was not yet fully constructed can be also obtained. This point requires a further study.

It is also very tempting to apply this analysis to the ordinary Yang–Mills theory or QCD in the radial time. The arguments from section 2 suggest that the $1/m$ expansion has to do with the perturbation expansion rather than with any supersymmetry properties of the model. However, these arguments are based on the scaling properties and in the presence of UV divergences, renormalization and dimensional transmutation they break, so for a conformally non-invariant theory it is hard to predict the behavior of expansion beyond the leading order.

Another point worth being mentioned is the relation to the integrability. As it was shown in [13], the planar limit of the perturbative non-Hermitian matrix model (3.3) is equivalent to an integrable model consisting of collections of independent Heisenberg spin chains. The correspondence just shown in this paper implies that the massive matrix model in the large mass limit is planar integrable too. It is interesting to check if this planar integrability holds for the higher order corrections as well. If what we obtained is the true two-loop correction as discussed above, this should be the case. On the other hand, if the model remains integrable at any order in inverse mass, then there is a high chance that it is planar integrable for a finite mass or even in the massless limit as well. This would be an interesting conclusion for this type of matrix models.

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