Formalization of the optimality criteria based on fuzzy predicates

T Ledeneva and N Kaplieva
Department of Applied Mathematics, Informatics and Mechanics, Voronezh State University, 1, University Square, Voronezh, 394018, Russia
E-mail: ledeneva-tm@yandex.ru

Abstract. The paper considers logical approach to solution of multi-objective optimization problems, which consists in representing a certain optimality principle in the form of a predicate with subsequent generalization in the form of a membership function of a fuzzy set of optimal solutions. The results of the study of the dependence of the optimal solution of the fuzzy multi-objective problem on the principle of optimality, functional representation and parameters of fuzzy logical connectives are presented.

1. Introduction
Multi-objective (vector) optimization is one of the main trends of the decision theory, within the framework of which the following problem is solved [1]. Let $X$ be a set of possible solutions, and here, for selection of the best solution, $n$ of $f_1, \ldots, f_n$ criteria is used, such that $\forall i = 1, n (f_i : X \rightarrow R)$. It is necessary to define the solution $x^* \in C$, which optimizes the set criteria in a sense. $C \subseteq X$ is the set of admissible solution conforming to the limitation of a certain problem. Essentially, in the general case, this problem is a constrained optimization problem. If by optimization we understand maximization of criteria, then formal problem setting may be presented in the following form:

$$
\left\{ \begin{array}{c}
    f_i(x) \rightarrow \max (i = 1, N), \\
    x \in C.
\end{array} \right.
$$

The functions $f_i(x)$ are often referred to as objective ones or objectives, and the problem (1) — as a multipurpose one. Note that the objectives, as a rule, cannot be achieved simultaneously, therefore certain additional information is required for reaching compromise. This information applies to refinement, which should be understood as optimal solution in the presence of many criteria (objectives). For formalization of this notion, various optimality principles are used [1, 2]. The idea of the approach consists in formation of a fuzzy set of optimal solutions in accordance with the set principle. Then the solution of the problem (1) is the solution with maximum value of the membership function. The advantage of this approach is that the objective functions $f_i(x)$ and the set of admissible solutions $C$ may be set approximately, for example, with the help of the notions of the fuzzy sets theory. For Pareto principle, this approach was considered in [3]. The purpose of the paper is its generalisation for other optimality principles and analysis of the implementation peculiarities.
2. Materials and methods

2.1. Justification of the approach

The idea of the approach is to use predicates to formalise the basic definitions and relationships between concepts. This idea is based on the fact that predicates are the basis of the logical model of knowledge representation [4].

By a predicate is meant a function defined on a set of arbitrary nature, which takes on values from the set \{0, 1\} [5]. A predicate can be considered as an indefinite proposition, which, when substituting specific values of predicate variables, takes on one of two values — 1 (true) or 0 (false). A one-place predicate defines a property, and polyadic predicates define relations. All known logical operations apply to predicates. Special operations include quantification (\forall and \exists). Note that the universal quantifier \forall generalizes the conjunction, and the existential quantifier \exists generalizes the disjunction.

By dominance relation we shall understand antireflexive and asymmetrical binary relation \( R \subseteq X^2 \) specified at the set \( X \), which is interpreted as "preference", "superiority", etc. [6].

Let \( x, y \in X \). We shall say that the solution \( x \) dominates the solution \( y \) if \( (x, y) \in R \), or, which is the same, \( xRy \). The solution \( x \) is called optimal with respect to the dominance \( R \) if in \( X \) there is no solution \( y \), which dominates \( x \).

Let us define the predicate \( S_R(x) = (\text{The admissible solution } x \in C \text{ belongs to the set of optimal ones with respect to the dominance of } R \text{ solutions}) \), which may be written in the following form:

\[
S_R(x) \leftrightarrow (x \in C \land \neg(\exists y(y \in C \land yRx)))
\] (2)

or, subject to implication definition \( a \rightarrow b = \pi \lor b \), in the form

\[
S_R(x) \leftrightarrow (x \in C \land \forall y(y \in C \rightarrow \neg(yRx))).
\] (3)

Note that for various values \( x \), this predicate turns into true or false statement. For generalization in a fuzzy case for all elementary predicates, let us introduce the truth function \( t[-] \) with values from \([0, 1]\), which defines the truth degree of a respective statement under certain values of predicate variables. The formula (3) also includes the universal quantifier and logical operations \( \land \) and \( \rightarrow \). Let us indicate logical connectives by \( F_\theta \), where \( \theta \in \{\neg, \land, \lor, \rightarrow, \leftrightarrow, \forall, \exists\} \). Apart from this, it is known, that the quantifier \( \forall \) generalizes conjunction, and \( \exists \) generalizes disjunction for infinite number of predicate variables. Thus, the expression (3) takes the following form:

\[
t(S_R(x)) = F_\land(t[x \in C], F_\forall y(F_\rightarrow(t[y \in C], F_\neg(t[yRx])))\) (4)

and allows, for each value \( x \), calculating the truth degree of the predicate \( S_R(x) \), which may be interpreted as a degree of membership of this solution \( x \) in the set of optimal solution.

In passing to a fuzzy problem option, the truth function is substituted with a membership function of a respective fuzzy set [5, 7], this is why if the set of admissible solutions is fuzzy, then the truth function, for example \( t[x \in C] \), corresponds to the membership function \( \mu_C(x) \), the fuzzy dominance relation \( R \) is defined by the membership function \( \mu_R(x, y) \).

Let us indicate the membership function of the fuzzy set of optimal solutions by \( \mu_{S_R}(x) \). In this case, (4) takes the following form:

\[
\mu_{S_R}(x) = F_\land(\mu_C(x), F_\forall y(F_\rightarrow(\mu_C(y), F_\neg(\mu_R(y, x))))) \) (5)

Representation of fuzzy logical connectives included in this formula allows concretising the membership function \( \mu_{S_R}(x) \). It may be used for finding the fuzzy solution of multi-objective optimization problems, both with fuzzy criteria and with fuzzy admissible set of solutions.

A reasonable approach to selection of a "crisp" solution is to choose \( x^* \) with maximum value of the membership function \( \mu_{S_R}(x^*) = \max_{x \in X} \mu_{S_R}(x) \). Another approach consists in using a suitable defuzzification [5].
2.2. Formation of membership functions for various optimality criteria
Consider the problem of representing various options of dominance relations, which will respectively specify various optimality principles [2, 8].

1) Strong optimality principle
According to this principle, optimal point is considered a point at which each objective function reaches its maximum value. Note that such a point might not exist (and this situation is most probably typical), this is why it is appropriate to build a fuzzy set of solutions for such problem setting.

Consider the predicate \( S(x) = \) (The functions \( f_1, \ldots, f_n \) reach their maximum values at the point \( x \in C \)). Its meaning is reduced to the following: \( x \in C \) and for any \( y \) such that if \( y \) belongs to \( C \), then there will be no such \( i \in \overline{1, n} \), that \( f_i(y) > f_i(x) \). Otherwise formally

\[
S(x) \leftrightarrow (x \in C \land \forall y(y \in C \rightarrow \neg (\exists i \ (f_i(y) > f_i(x)))))) \tag{6}
\]

or in the form

\[
S(x) \leftrightarrow (x \in C \land \forall y(y \in C \rightarrow \forall i \ (f_i(y) \leq f_i(x)))). \tag{7}
\]

Taking into account the above considerations, we obtain that, for the given case, the problem (1) may be written in the following form:

\[
\mu_{str}(x) = \min \left( t[x \in C], \inf_y \left( 1 - t[y \in C] + t[y \in C] \cdot \min_i (t[f_i(y) \leq f_i(x)]) \right) \right). \tag{8}
\]

Various representations of the implication \( I(x, y) \) allow concretizing the formula (8). Some examples are presented below [9]:

a) based on Mizumoto implication (table 1)

\[
\mu_{str}(x) = \min \left( t[x \in C], \inf_y \left( 1 - t[y \in C] + t[y \in C] \cdot \min_i (t[f_i(y) \leq f_i(x)]) \right) \right);
\]

b) based on Lukasiewicz implication (table 1)

\[
\mu_{str}(x) = \min \left( t[x \in C], \inf_y \left( 1 - t[y \in C] - \min_i (t[f_i(y) \leq f_i(x)]) + 2 \cdot t[y \in C] \cdot \min_i (t[f_i(y) \leq f_i(x)]) \right) \right).
\]

2) Pareto principle
The point \( x \in C \) is referred to as Pareto optimal for the problem (1), if for any \( y \) from the fact that \( y \in C \), follows that the following condition will not be met: \( f_i(y) \geq f_i(x) \) for all \( i = 1, n \), and there will be at least one index \( j \), such that \( f_j(y) > f_j(x) \).

The predicate \( S(x) = \) (solution \( x \) is Pareto optimal) is formally represented as follows:

\[
S(x) \leftrightarrow x \in C \land \forall y(y \in C \rightarrow \neg ((\forall i (f_i(x) \leq f_i(y))) \land (\exists i (f_i(x) < f_i(y)))))). \tag{9}
\]

Denoting the set of Pareto optimal solutions by \( P(X) \), we obtain a membership function of fuzzy set of Pareto optimal solutions for the problem (1) in the following form:

\[
\mu_P(x) = F_\land \left( t[x \in C], F_{\lor y} \left( F_{\rightarrow} \left( t[y \in C], F_\land \left( F_{\lor i} \left( F_\land \left( F_{\rightarrow} (t[f_i(y) \geq f_i(x)]) \right) \right) \right) \right) \right) \right). \tag{10}
\]

3) Slater principle The point \( x \in X \) is referred to as weakly efficient solution of the problem (1), or Slater optimal solution, if \( x \in C \) and for any other solution \( y \), such that if \( y \in C \), then \( f_i(y) > f_i(x) \) is not met for all \( i = 1, n \).
The predicate \( S(x) = \text{(solution x is Slater optimal)} \) has the following form:

\[
S(x) \leftrightarrow x \in C \land \forall y (y \in C \rightarrow \neg(\forall i(f_i(y) > f_i(x))).
\]

(11)

Denoting the set of Slater optimal solutions by \( Sl(x) \), we obtain a membership function of a respective fuzzy set as

\[
\mu_{Sl}(x) = F_{\land}(t[x \in C], F_{\lor}y(F_{\rightarrow}(t[y \in C], F_{\rightarrow}(F_{\lor}i(F_{\land}y(F_{\lor}t[f_i(y) > f_i(x)))))),
\]

(12)

and its further use for solution of the problem (1) is associated with concretization of functional representation of logical connectives.

4) Smale principle

The point \( x \in X \) is referred to as strictly efficient solution of the problem (1), or Smale optimal solution, if \( x \in C \) and for all \( y \neq x \) such that if \( y \) belongs to \( C \), then \( f_i(y) \geq f_i(x) \) is not met for all \( i = 1, n \).

The predicate \( S(x) = \text{(solution x is Smale optimal)} \) is formally represented in the following form:

\[
S(x) \leftrightarrow x \in C \land \forall y (y \in C \rightarrow \neg((y \neq x) \land \forall i(f_i(y) \geq f_i(x))),
\]

(13)

and subject to interpretation of logical connectives, it has the following form

\[
\mu_{Sm}(x) = F_{\land}(t[x \in C], F_{\lor}y(F_{\rightarrow}(t[y \in C], F_{\rightarrow}(F_{\lor}i(F_{\land}y(F_{\lor}t[f_i(y) \geq f_i(x))])))),
\]

(14)

and defines the fuzzy set of Smale optimal solutions.

5) Lexicographic optimality

The point \( x \in X \) is referred to as lexicographically optimal solution of the problem (1), if \( x \in C \) and for all \( y \neq x \) such that if \( y \) belongs to \( C \), then the following condition is not met: here we have the index \( i \) such that \( f_i(y) < f_i(x) \) and for all \( j < i \) we have \( f_i(y) = f_i(x) \).

The predicate \( S(x) = \text{(x is a lexicographically optimal solution)} \) is formally represented in the following form:

\[
S(x) \leftrightarrow x \in C \land \forall y (y \in C \rightarrow \neg((\exists i)((f_i(y) > f_i(x)) \land \forall j((j < i) \land (f_i(y) = f_j(x)))))),
\]

(15)

or subject to interpretation of logical connectives

\[
\mu_{Lex}(x) = F_{\land}(F_{\lor}y(F_{\rightarrow}(F_{\rightarrow}(F_{\lor}i(F_{\land}y(F_{\lor}t[f_i(y) = f_i(x)][F_{\land}t[x \in C], F_{\land}t[y \in C], F_{\rightarrow}(F_{\lor}i(F_{\land}y(F_{\lor}t[f_j(y) = f_j(x)]))))))))).
\]

(16)

3. Results and discussion

The computer program [10] was developed for a simulation experiment, the main purpose of which is formation of a membership function of a set of optimal solutions subject to various optimality principles. A series of linear problems was considered at a fuzzy admissible set with crisp and fuzzy objective functions, where coefficients were represented by fuzzy trapezoid and triangular numbers. The value of the sought membership function of a fuzzy set of optimal solutions was calculated at the nodes of a grid specified at the set of admissible solutions.

The purpose of the simulation experiment was to research the solution of a linear problem of multi-objective optimization with fuzzy objective functions in the following aspects: revealing dependence of solution on the selection of optimality principle; comparing the solutions of
the fuzzy problem with its crisp equivalent for each selected optimality principle; establishing relationship for fuzzy sets of optimal solutions; defining dependence of solution on the selection of fuzzy implication and parameters of parametric implications; revealing the influence of various fuzzy numbers comparison methods.

Note that all obtained formulas (8), (10), (12), (14), (16) contain fuzzy implication and suggest comparison of fuzzy numbers. For research of the influence of implication on the fuzzy set of optimal solutions, we selected operations presented in table 1, which refer to various classes.

**Table 1.** Fuzzy implications [5, 9].

| Implication name | Formula |
|------------------|---------|
| Diene            | $I_{Diene}(x, y) = \max(1 - x), y$ |
| Mizumoto         | $I_{Mizumoto}(x, y) = 1 - x + xy$ |
| Goguen           | $I_{Goguen}(x, y) = \begin{cases} \min \left(1, \frac{x}{y}\right), & x \neq 0, \\ 1, & \text{otherwise} \end{cases}$ |
| Mamdani          | $I_{Mamdani}(x, y) = \min(x, y)$ |
| Lukasiewicz      | $I_{Lukasiewicz}(x, y) = \min(1 - x + y, 1)$ |
| Godel            | $I_{Godel}(x, y) = \begin{cases} 1, & x \leq y, \\ y, & \text{otherwise} \end{cases}$ |
| Larsen           | $I_{Larsen}(x, y) = x \cdot y$ |
| Jager            | $I_{Jager}(x, y) = x^y$ |
| QM-max           | $I_{QM-max}(x, y) = \max(1 - x, \min(x, y))$ |
| QM-product       | $I_{QM-product}(x, y) = 1 - x \cdot (1 - x \cdot y)$ |
| QM-min           | $I_{QM-min}(x, y) = \min(1 - x + \max(x + y - 1, 0), 1)$ |
| t-parametric     | $I_t(x, y) = \frac{1 - x - y + (t + 1)xy}{1 + (t - 1)x - y + xy}$ |
| Two-parameter    | $I_{\alpha\beta}(x, y) = \frac{1 - x + \beta y + \left(\frac{1 + \beta}{\alpha} - \beta\right)xy}{1 + \left(\frac{1 + \beta}{\alpha} - 1\right)x + \beta y - \beta xy}$ |
Defuzzification methods and ranking indices were used for comparison of fuzzy numbers, as well as an original method was developed based on a special fuzzy binary relation of order for fuzzy numbers [10].

The analysis of the simulation experiment results allowed making the following basic conclusions.

1. The results of solution of the problems of multi-objective optimization with fuzzy criteria may quite considerably differ from the solutions of its crisp analogue, especially in case of strong fuzziness of the objective functions’ criteria, which manifests itself in a large value of the length of carriers of respective fuzzy sets.

2. For fuzzy sets of optimal solutions corresponding to various dominance relations, in case of fuzzy setting of multi-objective optimization problem, we observe the same inclusion chain as in the crisp case, namely $Sm(X) \subseteq P(X) \subseteq Sl(X)$.

3. Selection of fuzzy implication plays considerable role for formation of membership function of fuzzy sets of optimal solutions. Implications of Mamdani, Jager and Larsen appeared to be unsuitable for the given problem.

4. The results of the problem solution depending on the selection of fuzzy implication correlate with each other (in the value of maximum) regardless of the selection of a certain optimality principle.

5. The result ($x^*$) of solution of the problem is considerably influenced by the selection of fuzzy numbers comparison function. The use of comparison functions based in introduction of fuzzy relations of order allows obtaining more diverse solutions with stronger dependence on the fuzzy implication functions being used, and the solutions may rather strongly differ from those of the crisp case.

Thus, logical approach to solution of multi-objective optimization problems allows applying it to various modifications of the main problem setting, demonstrating conformity with the crisp option. The main issue of its practical implementation consists in formalization of fuzzy logical connectives.

References

[1] Steuer R 1992 Multi-criteria optimisation: Theory, computing, and applications (Moscow: Radio and communications)
[2] Nogin V D 2005 Decision making in a multi-objective environment (Moscow: FIZMATLIT)
[3] Ledeneva T M and Semenov B A 2007 Bulletin of Voronezh State University. Series: System analysis and information technologies 2 50–54
[4] Ledeneva T M, Podvalniy S I and Vasiliev V I 2005 Artificial intelligence and decision-making systems (Ufa: Ufa State Aeronautical Technical University)
[5] Kiiro G and Yuan B 1995 Fuzzy Set and fuzzy logic: theory and applications (New Jersey: Prentice Hall)
[6] Yudin D B 1989 Computational methods of decision theory (Moscow: Chief editorial board of the physico-mathematical literature)
[7] Kolman A 1982 Introduction to the theory of fuzzy sets (Moscow: Radio and communications)
[8] Katulev A N and Severtsev N I 2000 Research of operations and security assurance: principles of decision making and security assurance. Manual for higher education institutions (Moscow: Physico-mathematical literature)
[9] Ledeneva T M and Gribovsky A V 2003 Bulletin of Voronezh State University. Series: Physics, Mathematics 2 189–196
[10] Ledeneva T M and Obolentsev V N 2012 Modern problems of applied mathematics, computer science and mechanics: Proceedings of the International Conference vol 2 (Voronezh) pp 185–191