High-dimensional regression with noisy and missing data: Provable guarantees with non-convexity

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Abstract

Although the standard formulations of prediction problems involve fully-observed and noiseless data drawn in an i.i.d. manner, many applications involve noisy and/or missing data, possibly involving dependence, as well. We study these issues in the context of high-dimensional sparse linear regression, and propose novel estimators for the cases of noisy, missing, and/or dependent data. Many standard approaches to noisy or missing data, such as those using the EM algorithm, lead to optimization problems that are inherently non-convex, and it is difficult to establish theoretical guarantees on practical algorithms. While our approach also involves optimizing non-convex programs, we are able to both analyze the statistical error associated with any global optimum, and more surprisingly, to prove that a simple algorithm based on projected gradient descent will converge in polynomial time to a small neighborhood of the set of all global minimizers. On the statistical side, we provide non-asymptotic bounds that hold with high probability for the cases of noisy, missing, and/or dependent data. On the computational side, we prove that under the same types of conditions required for statistical consistency, the projected gradient descent algorithm is guaranteed to converge at a geometric rate to a near-global minimizer. We illustrate these theoretical predictions with simulations, showing close agreement with the predicted scalings.

Keywords: High-dimensional statistics; missing data; non-convexity; regularization; sparse linear regression; \textit{M}-estimation.

1 Introduction

In standard formulations of prediction problems, it is assumed that the covariates are fully-observed and sampled independently from some underlying distribution. However, these assumptions are not realistic for many applications, in which covariates may be observed only partially, observed subject to corruption, or exhibit some type of dependency. Consider the problem of modeling the voting behavior of politicians: in this setting, votes may be missing due to abstentions, and temporally dependent due to collusion or “tit-for-tat” behavior. Similarly, surveys often suffer from the missing data problem, since users fail to respond to all questions. Sensor network data also tends to be both noisy due to measurement error, and partially missing due to failures or drop-outs of sensors.

There are a variety of methods for dealing with noisy and/or missing data, including various heuristic methods, as well as likelihood-based methods involving the expectation-maximization (EM) algorithm (e.g., see the book [11] and references therein). A challenge in this context is the possible non-convexity of associated optimization problems. For instance,
in applications of EM, problems in which the negative likelihood is a convex function often become non-convex with missing or noisy data. Consequently, although the EM algorithm will converge to a local minimum, it is difficult to guarantee that the local optimum is close to a global minimum.

In this paper, we study these issues in the context of high-dimensional sparse linear regression—in particular, in the case when the predictors or covariates are noisy, missing, and/or dependent. Our main contribution is to develop and study simple methods for handling these issues, and to prove theoretical results about both the associated statistical error and the optimization error. Like EM-based approaches, our estimators are based on solving optimization problems that may be non-convex; however, despite this non-convexity, we are still able to prove that a simple form of projected gradient descent will produce an output that is “sufficiently close”—as small as the statistical error—to any global optimum. As a second result, we bound the statistical error, showing that it has the same scaling as the minimax rates for the classical cases of perfectly observed and independently sampled covariates. In this way, we obtain estimators for noisy, missing, and/or dependent data that have the same scaling behavior as the usual fully-observed and independent case. The resulting estimators allow us to solve the problem of high-dimensional Gaussian graphical model selection with missing data.

There is a large body of work on the problem of corrupted covariates or error-in-variables for regression problems (e.g., see the papers and books [7, 3, 8, 25], as well as references therein). Much of the earlier theoretical work is classical in nature, meaning that it requires that the sample size $n$ diverges with the dimension $p$ fixed. Most relevant to this paper is more recent work that has examined issues of corrupted and/or missing data in the context of high-dimensional sparse linear models, allowing for $n \ll p$. Städler and Bühlmann [21] developed an EM-based method for sparse inverse covariance matrix estimation in the missing data regime, and used this result to derive an algorithm for sparse linear regression with missing data. As mentioned above, however, it is difficult to guarantee that EM will converge to a point close to a global optimum of the likelihood, in contrast to the methods studied here. Rosenbaum and Tsybakov [17] studied the sparse linear model when the covariates are corrupted by noise, and proposed a modified form of the Dantzig selector (see the discussion following our main results for a detailed comparison to this past work, and also to concurrent work [18] by the same authors). For the particular case of multiplicative noise, the type of estimator that we consider here has been studied in past work [25]; however, this theoretical analysis is of the classical type, holding only for $n \gg p$, in contrast to the high-dimensional models that are of interest here.

The remainder of this paper is organized as follows. We begin in Section 2 with background and a precise description of the problem. We then introduce the class of estimators we will consider and the form of the projected gradient descent algorithm. Section 3 is devoted to a description of our main results, including a pair of general theorems on the statistical and optimization error, and then a series of corollaries applying our results to the cases of noisy, missing, and dependent data. In Section 4, we demonstrate simulations to confirm that our methods work in practice, and verify the theoretically-predicted scaling laws. Section 5 contains proofs of some of the main results, with the remaining proofs contained in the supplementary Appendix.

**Notation.** For a matrix $M$, we write $\|M\|_{\text{max}} := \max_{i,j} |m_{ij}|$ to be the elementwise $\ell_\infty$-norm of $M$. Furthermore, $\|M\|_1$ denotes the induced $\ell_1$-operator norm (maximum absolute column sum) of $M$, and $\|M\|_{\text{op}}$ is the spectral norm of $M$. We write $\kappa(M) := \frac{\lambda_{\text{max}}(M)}{\lambda_{\text{min}}(M)}$, the
condition number of \( M \). For matrices \( M_1, M_2 \), we write \( M_1 \odot M_2 \) to denote the componentwise Hadamard product, and write \( M_1 \oplus M_2 \) to denote componentwise division. For functions \( f(n) \) and \( g(n) \), we write \( f(n) \preceq g(n) \) to mean that \( f(n) \leq c g(n) \) for a universal constant \( c \in (0, \infty) \), and similarly, \( f(n) \succeq g(n) \) when \( f(n) \geq c' g(n) \) for some universal constant \( c' \in (0, \infty) \). Finally, we write \( f(n) \asymp g(n) \) when \( f(n) \preceq g(n) \) and \( f(n) \succeq g(n) \) hold simultaneously.

## 2 Background and problem setup

In this section, we provide background and a precise description of the problem, and then motivate the class of estimators analyzed in this paper. We then discuss a simple class of projected gradient descent algorithms that can be used to obtain an estimator.

### 2.1 Observation model and high-dimensional framework

Suppose we observe a response variable \( y_i \in \mathbb{R} \) linked to a covariate vector \( x_i \in \mathbb{R}^p \) via the linear model

\[
y_i = \langle x_i, \beta^* \rangle + \epsilon_i, \quad \text{for } i = 1, 2, \ldots, n. \tag{1}
\]

Here, the regression vector \( \beta^* \in \mathbb{R}^p \) is unknown, and \( \epsilon_i \in \mathbb{R} \) is observation noise, independent of \( x_i \). Rather than directly observing each \( x_i \in \mathbb{R}^p \), we observe a vector \( z_i \in \mathbb{R}^p \) linked to \( x_i \) via some conditional distribution, i.e.,

\[
z_i \sim \mathcal{Q}(\cdot \mid x_i), \quad \text{for } i = 1, 2, \ldots, n. \tag{2}
\]

This setup applies to various disturbances to the covariates, including:

(a) **Covariates with additive noise:** We observe \( z_i = x_i + w_i \), where \( w_i \in \mathbb{R}^p \) is a random vector independent of \( x_i \), say zero-mean with known covariance matrix \( \Sigma_w \).

(b) **Missing data:** For some fraction \( \rho \in [0, 1) \), we observe a random vector \( z_i \in \mathbb{R}^p \) such that for each component \( j \), we independently observe \( z_{ij} = x_{ij} \) with probability \( 1 - \rho \), and \( z_{ij} = * \) with probability \( \rho \). We can also consider the case when the entries in the \( j \)-th column have a different probability \( \rho_j \) of being missing.

(c) **Covariates with multiplicative noise:** Generalizing the missing data problem, suppose we observe \( z_i = x_i \odot u_i \), where \( u_i \in \mathbb{R}^p \) is again a random vector independent of \( x_i \), and \( \odot \) is the Hadamard product. The problem of missing data is a special case of multiplicative noise, where all \( u_{ij} \)'s are independent and \( u_{ij} \sim \text{Bernoulli}(1 - \rho_j) \).

Our first set of results is deterministic, depending on specific instantiations of the observations \( \{(y_i, z_i)\}_{i=1}^n \). However, we are also interested in results that hold with high probability when the \( x_i \)'s and \( z_i \)'s are drawn at random. We consider both the case when the \( x_i \)'s are drawn i.i.d. from a fixed distribution; and the case of dependent covariates, when the \( x_i \)'s are generated according to a stationary vector autoregressive (VAR) process.

We work within a high-dimensional framework that allows the number of predictors \( p \) to grow and possibly exceed the sample size \( n \). Of course, consistent estimation when \( n \ll p \) is impossible unless the model is endowed with additional structure—for instance, sparsity in the parameter vector \( \beta^* \). Consequently, we study the class of models where \( \beta^* \) has at most \( k \) non-zero parameters, where \( k \) is also allowed to increase to infinity with \( p \) and \( n \).
2.2 M-estimators for noisy and missing covariates

In order to motivate the class of estimators we will consider, let us begin by examining a simple deterministic problem. Let \( \Sigma_x \succ 0 \) be the covariance matrix of the covariates, and consider the \( \ell_1 \)-constrained quadratic program

\[
\hat{\beta} \in \arg \min_{\|\beta\|_1 \leq R} \left\{ \frac{1}{2} \beta^T \Sigma_x \beta - \langle \Sigma_x \beta^*, \beta \rangle \right\}.
\]  

As long as the constraint radius \( R \) is at least \( \|\beta^*\|_1 \), the unique solution to this convex program is \( \hat{\beta} = \beta^* \). Of course, this program is an idealization, since in practice we may not know the covariance matrix \( \Sigma_x \), and we certainly do not know \( \Sigma_x \beta^* \)—after all, \( \beta^* \) is the quantity we are trying to estimate!

Nonetheless, this idealization still provides useful intuition, as it suggests various estimators based on the plug-in principle. Given a set of samples, it is natural to form estimates of the quantities \( \Sigma_x \) and \( \Sigma_x \beta^* \), which we denote by \( \hat{\Gamma} \in \mathbb{R}^{p \times p} \) and \( \hat{\gamma} \in \mathbb{R}^p \), respectively, and to consider the modified program

\[
\hat{\beta} \in \arg \min_{\|\beta\|_1 \leq R} \left\{ \frac{1}{2} \beta^T \hat{\Gamma} \beta - \langle \hat{\gamma}, \beta \rangle \right\},
\]

or alternatively, the regularized version

\[
\hat{\beta} \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \beta^T \hat{\Gamma} \beta - \langle \hat{\gamma}, \beta \rangle + \lambda_n \|\beta\|_1 \right\},
\]

where \( \lambda_n > 0 \) is a user-defined regularization parameter. Note that the two programs are equivalent by Lagrangian duality when the objectives are convex, but not in the case of a non-convex objective. The Lasso \([22, 4]\) is a special case of these programs, obtained by setting

\[
\hat{\Gamma}_{\text{Las}} := \frac{1}{n} X^T X \quad \text{and} \quad \hat{\gamma}_{\text{Las}} := \frac{1}{n} X^T y,
\]

where we have introduced the shorthand \( y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \), and \( X \in \mathbb{R}^{n \times p} \), with \( x_i^T \) as its \( i^{th} \) row. A simple calculation shows that \( (\hat{\Gamma}_{\text{Las}}, \hat{\gamma}_{\text{Las}}) \) are unbiased estimators of the pair \( (\Sigma_x, \Sigma_x \beta^*) \). This unbiasedness and additional concentration inequalities (to be described in the sequel) underlie the well-known analysis of the Lasso in the high-dimensional regime.

In this paper, we focus on more general instantiations of the programs (4) and (5), involving different choices of the pair \( (\hat{\Gamma}, \hat{\gamma}) \) that are adapted to the cases of noisy and/or missing data. Note that the matrix \( \hat{\Gamma}_{\text{Las}} \) is positive semidefinite, so the Lasso program is convex. In sharp contrast, for the case of noisy or missing data, the most natural choice of the matrix \( \hat{\Gamma} \) is not positive semidefinite, hence the quadratic losses appearing in the problems (4) and (5) are non-convex. Furthermore, when \( \hat{\Gamma} \) has negative eigenvalues, the objective in equation (5) is unbounded from below. Hence, we make use of the following regularized estimator:

\[
\hat{\beta} \in \arg \min_{\|\beta\|_1 \leq b_0 \sqrt{k}} \left\{ \frac{1}{2} \beta^T \hat{\Gamma} \beta - \langle \hat{\gamma}, \beta \rangle + \lambda_n \|\beta\|_1 \right\},
\]

for a suitable constant \( b_0 \).

In the presence of non-convexity, it is generally impossible to provide a polynomial-time algorithm that converges to a (near) global optimum, due to the presence of local minima.
Remarkably, we are able to prove that this issue is not significant in our setting, and a simple projected gradient descent algorithm applied to the programs (4) or (7) converges with high probability to a vector extremely close to any global optimum.

Let us illustrate these ideas with some examples. Recall that \((\hat{\Gamma}, \hat{\gamma})\) serve as unbiased estimators for \((\Sigma_x, \Sigma_x\beta^*)\).

**Example 1** (Additive noise). Suppose we observe \(Z = X + W\), where \(W\) is a random matrix independent of \(X\), with rows \(w_i\) drawn i.i.d. from a zero-mean distribution with known covariance \(\Sigma_w\). We consider the pair

\[
\hat{\Gamma}_{\text{add}} := \frac{1}{n} Z^T Z - \Sigma_w \quad \text{and} \quad \hat{\gamma}_{\text{add}} := \frac{1}{n} Z^T y. \tag{8}
\]

Note that when \(\Sigma_w = 0\) (corresponding to the noiseless case), the estimators reduce to the standard Lasso. However, when \(\Sigma_w \neq 0\), the matrix \(\hat{\Gamma}_{\text{add}}\) is not positive semidefinite in the high-dimensional regime \((n \ll p)\). Indeed, since the matrix \(\frac{1}{n} Z^T Z\) has rank at most \(n\), the subtracted matrix \(\Sigma_w\) may cause \(\hat{\Gamma}_{\text{add}}\) to have a large number of negative eigenvalues. For instance, if \(\Sigma_w = \sigma_w^2 I\) for \(\sigma_w^2 > 0\), then \(\hat{\Gamma}_{\text{add}}\) has \(p - n\) eigenvalues equal to \(-\sigma_w^2\).

**Example 2** (Missing data). We now consider the case where the entries of \(X\) are missing at random. Let us first describe an estimator for the special case where each entry is missing at random, independently with some constant probability \(\rho \in [0, 1)\). (In Example 3 to follow, we will describe the extension to general missing probabilities.) Consequently, we observe the matrix \(Z \in \mathbb{R}^{n \times p}\) with entries

\[
Z_{ij} = \begin{cases} X_{ij} & \text{with probability } 1 - \rho, \\ 0 & \text{otherwise}. \end{cases}
\]

Given the observed matrix \(Z \in \mathbb{R}^{n \times p}\), we use

\[
\hat{\Gamma}_{\text{mis}} := \frac{Z^T Z}{n} - \rho \text{ diag} \left( \frac{Z^T Z}{n} \right) \quad \text{and} \quad \hat{\gamma}_{\text{mis}} := \frac{1}{n} Z^T y, \tag{9}
\]

where \(Z_{ij} = Z_{ij}/(1 - \rho)\). It is easy to see that the pair \((\hat{\Gamma}_{\text{mis}}, \hat{\gamma}_{\text{mis}})\) reduces to the pair \((\hat{\Gamma}_{\text{Lass}}, \hat{\gamma}_{\text{Lass}})\) for the standard Lasso when \(\rho = 0\), corresponding to no missing data. In the more interesting case when \(\rho \in (0, 1)\), the matrix \(\frac{Z^T Z}{n}\) in equation (9) has rank at most \(n\), so the subtracted diagonal matrix may cause the matrix \(\hat{\Gamma}_{\text{mis}}\) to have a large number of negative eigenvalues when \(n \ll p\). As a consequence, the matrix \(\hat{\Gamma}_{\text{mis}}\) is not (in general) positive semidefinite, so the associated quadratic function is not convex.

**Example 3** (Multiplicative noise). As a generalization of the previous example, we now consider the case of multiplicative noise. In particular, suppose we observe the quantity \(Z = X \odot U\), where \(U\) is a matrix of nonnegative noise variables. In many applications, it is natural to assume that the rows \(u_i\) of \(U\) are drawn in an i.i.d. manner, say from some distribution in which both the vector \(E[u_1]\) and the matrix \(E[u_1 u_1^T]\) have strictly positive entries. This general family of multiplicative noise models arises in various applications; we refer the reader to the papers [7, 3, 8, 25] for more discussion and examples. A natural choice of the pair \((\hat{\Gamma}, \hat{\gamma})\) is given by the quantities

\[
\hat{\Gamma}_{\text{mul}} := \frac{1}{n} Z^T Z \odot E(u_1 u_1^T) \quad \text{and} \quad \hat{\gamma}_{\text{mul}} := \frac{1}{n} Z^T y \odot E(u_1), \tag{10}
\]
where \( \odot \) denotes elementwise division. A small calculation shows that these are unbiased estimators of \( \Sigma_x \) and \( \Sigma_x \beta^\ast \), respectively. The estimators (10) have been studied in past work [25], but only under classical scaling \( (n \gg p) \).

As a special case of the estimators (10), suppose the entries \( u_{ij} \) of \( U \) are independent Bernoulli\((1 - \rho_j)\) random variables. Then the observed matrix \( Z = X \odot U \) corresponds to a missing-data matrix, where each element of the \( j \)th column has probability \( \rho_j \) of being missing. In this case, the estimators (10) become

\[
\hat{\Gamma}_{\text{mis}} = \frac{Z^T Z}{n} \odot M \quad \text{and} \quad \hat{\gamma}_{\text{mis}} = \frac{1}{n} Z^T y \odot (1 - \rho),
\]

where \( M := \mathbb{E}(u_1 u_1^T) \) satisfies

\[
M_{ij} = \begin{cases} 
(1 - \rho_i)(1 - \rho_j) & \text{if } i \neq j \\
1 - \rho_i & \text{if } i = j,
\end{cases}
\]

\( \rho \) is the parameter vector containing the \( \rho_j \)'s, and \( 1 \) is the vector of all 1’s. In this way, we obtain a generalization of the estimator discussed in Example 2.

### 2.3 Restricted eigenvalue conditions

Given an estimate \( \hat{\beta} \), there are various ways to assess its closeness to \( \beta^\ast \). In this paper, we focus on the \( \ell_2 \)-norm \( \| \hat{\beta} - \beta^\ast \|_2 \), as well as the closely related \( \ell_1 \)-norm \( \| \hat{\beta} - \beta^\ast \|_1 \). When the covariate matrix \( X \) is fully observed (so that the Lasso can be applied), it is now well understood that a sufficient condition for \( \ell_2 \)-recovery is that the matrix \( \hat{\Gamma}_{\text{Las}} = \frac{1}{n} X^T X \) satisfy a certain type of restricted eigenvalue (RE) condition (e.g., [2, 23]). In this paper, we make use of the following condition.

**Definition 1** (Lower-RE condition). The matrix \( \hat{\Gamma} \) satisfies a lower restricted eigenvalue condition with curvature \( \alpha_1 > 0 \) and tolerance \( \tau(n, p) > 0 \) if

\[
\theta^T \hat{\Gamma} \theta \geq \alpha_1 \| \theta \|_2^2 - \tau(n, p) \| \theta \|_1^2 \quad \text{for all } \theta \in \mathbb{R}^p.
\]

It can be shown that when the Lasso matrix \( \hat{\Gamma}_{\text{Las}} = \frac{1}{n} X^T X \) satisfies this RE condition (12), the Lasso estimate has low \( \ell_2 \)-error for any vector \( \beta^\ast \) supported on any subset of size at most \( k \lesssim \frac{1}{\tau(n, p)} \). In particular, bound (12) implies a sparse RE condition for all \( k \) of this magnitude, and conversely, Lemma 11 in the Appendix shows that a sparse RE condition implies bound (12). In this paper, we work with condition (12), since it is especially convenient for analyzing optimization algorithms.

In the standard setting (with uncorrupted and fully observed design matrices), it is known that for many choices of the design matrix \( X \) (with rows having covariance \( \Sigma \)), the Lasso matrix \( \hat{\Gamma}_{\text{Las}} \) will satisfy such an RE condition with high probability (e.g., [16, 20]) with \( \alpha_1 = \frac{1}{2} \lambda_{\min}(\Sigma) \) and \( \tau(n, p) \asymp \frac{\log p}{n} \). A significant portion of the analysis in this paper is devoted to proving that different choices of \( \hat{\Gamma} \), such as the matrices \( \hat{\Gamma}_{\text{add}} \) and \( \hat{\Gamma}_{\text{mis}} \) defined earlier, also satisfy condition (12) with high probability. This fact is by no means obvious, since as previously discussed, the matrices \( \hat{\Gamma}_{\text{add}} \) and \( \hat{\Gamma}_{\text{mis}} \) generally have large numbers of negative eigenvalues.

Finally, although such upper bounds are not necessary for statistical consistency, our algorithmic results make use of the analogous upper restricted eigenvalue condition, formalized in the following:
Definition 2 (Upper-RE condition). The matrix $\hat{\Gamma}$ satisfies an upper restricted eigenvalue condition with smoothness $\alpha_2 > 0$ and tolerance $\tau(n, p) > 0$ if
\[
\theta^T\hat{\Gamma}\theta \leq \alpha_2\|\theta\|^2_2 + \tau(n, p)\|\theta\|^2_1 \quad \text{for all } \theta \in \mathbb{R}^p.
\] (13)

In recent work on high-dimensional projected gradient descent, Agarwal et al. [1] make use of a more general form of the lower and upper bounds (12) and (13), applicable to non-quadratic losses as well, which are referred to as the restricted strong convexity (RSC) and restricted smoothness (RSM) conditions, respectively. For various class of random design matrices, it can be shown that the Lasso matrix $\hat{\Gamma}_{Las}$ satisfies the upper bound (13) with $\alpha_2 = 2\lambda_{\max}(\Sigma_x)$ and $\tau(n, p) \approx \frac{\log p}{n}$; see Raskutti et al. [16] for the Gaussian case and Rudelson and Zhou [20] for the sub-Gaussian setting. We will establish similar scaling for our choices of $\hat{\Gamma}$.

2.4 Gradient descent algorithms

In addition to proving results about the global minima of the (possibly non-convex) programs (4) and (5), we are also interested in polynomial-time procedures for approximating such optima. In this paper, we analyze some simple algorithms for solving either the constrained program (4) or the Lagrangian version (7). Note that the gradient of the quadratic loss function takes the form $\nabla L(\beta) = \hat{\Gamma}\beta - \hat{\gamma}$. In application to the constrained version, the method of projected gradient descent generates a sequence of iterates $\{\beta^t, t = 0, 1, 2, \ldots\}$ by the recursion
\[
\beta^{t+1} = \arg \min_{\|\beta\|_1 \leq R} \left\{ L(\beta^t) + \langle \nabla L(\beta^t), \beta - \beta^t \rangle + \frac{\eta}{2}\|\beta - \beta^t\|^2_2 \right\},
\] (14)
where $\eta > 0$ is a stepsize parameter. Equivalently, this update can be written as $\beta^{t+1} = \Pi(\beta^t - \frac{1}{\eta}\nabla L(\beta^t))$, where $\Pi$ denotes the $\ell_2$-projection onto the $\ell_1$-ball of radius $R$. This projection can be computed rapidly in $O(p)$ time using a procedure due to Duchi et al. [5].

For the Lagrangian update, we use a slight variant of the projected gradient update (14), namely
\[
\beta^{t+1} = \arg \min_{\|\beta\|_1 \leq R} \left\{ L(\beta^t) + \langle \nabla L(\beta^t), \beta - \beta^t \rangle + \frac{\eta}{2}\|\beta - \beta^t\|^2_2 + \lambda_n\|\beta\|_1 \right\},
\] (15)
with the only difference being the inclusion of the regularization term. This update can also performed efficiently by performing two projections onto the $\ell_1$-ball (see the paper [1] for details).

When the objective function is convex (equivalently, $\hat{\Gamma}$ is positive semidefinite), the iterates (14) or (15) are guaranteed to converge to a global minimum of the objective functions (4) and (7), respectively. In our setting, the matrix $\hat{\Gamma}$ need not be positive semidefinite, so the best generic guarantee is that the iterates converge to a local optimum. However, our analysis shows that for the family of programs (4) or (7), under a reasonable set of conditions satisfied by various statistical models, the iterates actually converge to a point extremely close to any global optimum in both $\ell_1$-norm and $\ell_2$-norm; see Theorem 2 to follow for a more detailed statement.

3 Main results and consequences

We now state our main results and discuss their consequences for noisy, missing, and dependent data.
3.1 General results

We provide theoretical guarantees for both the constrained estimator (4) and the Lagrangian version (7). Note that we obtain different optimization problems as we vary the choice of the pair \((\hat{\Gamma}, \hat{\gamma}) \in \mathbb{R}^{p \times p} \times \mathbb{R}^p\). We begin by stating a pair of general results, applicable to any pair that satisfies certain conditions. Our first result (Theorem 1) provides bounds on the statistical error, namely the quantity \(\|\hat{\beta} - \beta^*\|_2\), as well as the corresponding \(\ell_1\)-error, where \(\hat{\beta}\) is any global optimum of the programs (4) or (7). Since the problem may be non-convex in general, it is not immediately obvious that one can obtain a provably good approximation to any global optimum without resorting to costly search methods. In order to assuage this concern, our second result (Theorem 2) provides rigorous bounds on the optimization error, namely the differences \(\|\beta_t - \hat{\beta}\|_2\) and \(\|\beta_t - \hat{\beta}\|_1\) incurred by the iterate \(\beta_t\) after running \(t\) rounds of the projected gradient descent updates (14) or (15).

3.1.1 Statistical error

In controlling the statistical error, we assume that the matrix \(\hat{\Gamma}\) satisfies a lower-RE condition with curvature \(\alpha_1\) and tolerance \(\tau(n, p)\), as previously defined (12). Recall that \(\hat{\Gamma}\) and \(\hat{\gamma}\) serve as surrogates to the deterministic quantities \(\Sigma_x \in \mathbb{R}^{p \times p}\) and \(\Sigma_x \beta^* \in \mathbb{R}^p\), respectively. Our results also involve a measure of deviation in these surrogates. In particular, we assume that there is some function \(\varphi(Q, \sigma_\epsilon)\), depending on the two sources of noise in our problem: the standard deviation \(\sigma_\epsilon\) of the observation noise vector \(\epsilon\) from equation (1), and the conditional distribution \(Q\) from equation (2) that links the covariates \(x_i\) to the observed versions \(z_i\). With this notation, we consider the deviation condition

\[
\|\hat{\gamma} - \hat{\Gamma} \beta^*\|_\infty \leq \varphi(Q, \sigma_\epsilon) \sqrt{\frac{\log p}{n}}. \tag{16}
\]

To aid intuition, note that inequality (16) holds whenever the following two deviation conditions are satisfied:

\[
\|\hat{\gamma} - \Sigma_x \beta^*\|_\infty \leq \varphi(Q, \sigma_\epsilon) \sqrt{\frac{\log p}{n}} \quad \text{and} \quad \|\hat{\Gamma} - \Sigma_x\|_\infty \leq \varphi(Q, \sigma_\epsilon) \sqrt{\frac{\log p}{n}}. \tag{17}
\]

The pair of inequalities (17) clearly measures the deviation of the estimators \((\hat{\Gamma}, \hat{\gamma})\) from their population versions, and they are sometimes easier to verify theoretically. However, inequality (16) may be used directly to derive tighter bounds (e.g., in the additive noise case). Indeed, the bounds established via inequalities (17) is not sharp in the limit of low noise on the covariates, due to the second inequality. In the proofs of our corollaries to follow, we will verify the deviation conditions for various forms of noisy, missing, and dependent data, with the quantity \(\varphi(Q, \sigma_\epsilon)\) changing depending on the model. We have the following result, which applies to any global optimum \(\hat{\beta}\) of the regularized version (7) with \(\lambda_n \geq 4 \varphi(Q, \sigma_\epsilon) \sqrt{\frac{\log p}{n}}\):

**Theorem 1** (Statistical error). Suppose the surrogates \((\hat{\Gamma}, \hat{\gamma})\) satisfy the deviation bound (16), and the matrix \(\hat{\Gamma}\) satisfies the lower-RE condition (12) with parameters \((\alpha_1, \tau)\) such that

\[
\sqrt{k} \tau(n, p) \leq \min\left\{ \frac{\alpha_1}{128 \sqrt{k}}, \frac{\varphi(Q, \sigma_\epsilon)}{b_0} \sqrt{\frac{\log p}{n}} \right\}. \tag{18}
\]
Then for any vector $\beta^*$ with sparsity at most $k$, there is a universal positive constant $c_0$ such that any global optimum $\hat{\beta}$ of the Lagrangian program (7) with any $b_0 \geq \|\beta^*\|_2$ satisfies the bounds

$$
\|\hat{\beta} - \beta^*\|_2 \leq \frac{c_0 \sqrt{k}}{\alpha_1} \max \left\{ \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}}, \lambda_n \right\}, \quad \text{and} \quad (19a)
$$

$$
\|\hat{\beta} - \beta^*\|_1 \leq \frac{8 c_0 k}{\alpha_1} \max \left\{ \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}}, \lambda_n \right\}. \quad (19b)
$$

The same bounds (without $\lambda_n$) also apply to the constrained program (4) with radius choice $R = \|\beta^*\|_1$.

**Remarks** To be clear, all the claims of Theorem 1 are deterministic. Probabilistic conditions will enter when we analyze specific statistical models and certify that the RE condition (18) and deviation conditions are satisfied by a random pair $(\hat{\Gamma}, \hat{\gamma})$ with high probability. We note that for the standard Lasso choice $(\hat{\Gamma}_{\text{Las}}, \hat{\gamma}_{\text{Las}})$ of this matrix-vector pair, bounds of the form (19) for sub-Gaussian noise are well known from past work (e.g., [2, 27, 13, 14]). The novelty of Theorem 1 is in allowing for general pairs of such surrogates, which—as shown by the examples discussed earlier—can lead to non-convexity in the underlying $M$-estimator. Moreover, some interesting differences arise due to the term $\varphi(Q, \sigma)$, which changes depending on the nature of the model (missing, noisy, and/or dependent). As will be clarified in the sequel, proving that the conditions of Theorem 1 are satisfied with high probability for noisy/missing data requires some non-trivial analysis, involving both concentration inequalities and random matrix theory.

Note that in the presence of non-convexity, it is possible in principle for the optimization problems (4) and (7) to have *many* global optima that are separated by large distances. Interestingly, Theorem 1 guarantees that this unpleasant feature does not arise under the stated conditions: given any two global optima $\hat{\beta}$ and $\tilde{\beta}$ of the program (4), Theorem 1 combined with the triangle inequality guarantees that

$$
\|\hat{\beta} - \tilde{\beta}\|_2 \leq \|\hat{\beta} - \beta^*\|_2 + \|\tilde{\beta} - \beta^*\|_2 \leq 2c_0 \varphi(Q, \sigma) \sqrt{\frac{k \log p}{n}}
$$

and similarly for the program (7). Consequently, under any scaling such that $\frac{k \log p}{n} = o(1)$, the set of all global optima must lie within an $\ell_2$-ball whose radius shrinks to zero.

In addition, it is worth observing that Theorem 1 makes a specific prediction for the scaling behavior of the $\ell_2$-error $\|\hat{\beta} - \beta^*\|_2$. In order to study this scaling prediction, we performed simulations under the additive noise model described in Example 1, using the parameter setting $\Sigma_x = I$ and $\Sigma_w = \sigma_w^2 I$ with $\sigma_w = 0.2$. Panel (a) of Figure 1 provides plots$^1$ of the error $\|\hat{\beta} - \beta^*\|_2$ versus the sample size $n$, for problem dimensions $p \in \{128, 256, 512\}$. Note that for all three choices of dimensions, the error decreases to zero as the sample size $n$ increases, showing consistency of the method. The curves also shift to the right as the dimension $p$ increases, reflecting the natural intuition that larger problems are harder in a certain sense. Theorem 1 makes a specific prediction about this scaling behavior: in particular, if we plot the $\ell_2$-error versus the rescaled sample size $n/(k \log p)$, the curves should roughly align for

---

$^1$Corollary 1, to be stated shortly, guarantees that the conditions of Theorem 1 are satisfied with high probability for the additive noise model. In addition, Theorem 2 to follow provides an efficient method of obtaining an accurate approximation of the global optimum.
Figure 1. Plots of the error $\|\hat{\beta} - \beta^*\|_2$ after running projected gradient descent on the non-convex objective, with sparsity $k \approx \sqrt{p}$. Plot (a) is an error plot for i.i.d. data with additive noise, and plot (b) shows $\ell_2$-error versus the rescaled sample size $\frac{n}{k \log p}$. As predicted by Theorem 1, the curves align for different values of $p$ in the rescaled plot.

different values of $p$. Panel (b) shows the same data re-plotted on these rescaled axes, thus verifying the predicted “stacking behavior.”

Finally, as noted by a reviewer, the constraint $R = \|\beta^*\|_1$ in the program (4) is rather restrictive, since $\beta^*$ is unknown. Theorem 1 merely establishes a heuristic for the scaling expected for this optimal radius. In this regard, the Lagrangian estimator (7) is more appealing, since it only requires choosing $b_0$ to be larger than $\|\beta^*\|_2$, and the conditions on the regularizer $\lambda_n$ are the standard ones from past work on the Lasso.

3.1.2 Optimization error

Although Theorem 1 provides guarantees that hold uniformly for any global minimizer, it does not provide guidance on how to approximate such a global minimizer using a polynomial-time algorithm. Indeed, for non-convex programs in general, gradient-type methods may become trapped in local minima, and it is impossible to guarantee that all such local minima are close to a global optimum. Nonetheless, we are able to show that for the family of programs (4), under reasonable conditions on $\hat{\Gamma}$ satisfied in various settings, simple gradient methods will converge geometrically fast to a very good approximation of any global optimum. The following theorem supposes that we apply the projected gradient updates (14) to the constrained program (4), or the composite updates (15) to the Lagrangian program (7), with stepsize $\eta = 2 \alpha^2$. In both cases, we assume that $n \gtrsim k \log p$, as is required for statistical consistency in Theorem 1.

Theorem 2 (Optimization error). Under the conditions of Theorem 1:

(a) For any global optimum $\hat{\beta}$ of the constrained program (4), there are universal positive constants $(c_1, c_2)$ and a contraction coefficient $\gamma \in (0, 1)$, independent of $(n, p, k)$, such that the gradient descent iterates (14) satisfy the bounds

$$\|\beta^t - \hat{\beta}\|_2 \leq \gamma^t \|\beta^0 - \hat{\beta}\|_2 + c_1 \log p \frac{\|\hat{\beta} - \beta^*\|_1}{n} + c_2 \|\hat{\beta} - \beta^*\|_2,$$

$$\|\beta^t - \hat{\beta}\|_1 \leq 2\sqrt{k} \|\beta^t - \hat{\beta}\|_2 + 2 \sqrt{k} \|\hat{\beta} - \beta^*\|_2 + 2 \|\hat{\beta} - \beta^*\|_1,$$
for all \( t \geq 0 \).

(b) Letting \( \phi \) denote the objective function of Lagrangian program (7) with global optimum \( \hat{\beta} \), and applying composite gradient updates (15), there are universal positive constants \( (c_1, c_2) \) and a contraction coefficient \( \gamma \in (0, 1) \), independent of \( (n, p, k) \), such that

\[
\| \beta^t - \hat{\beta} \|_2^2 \leq c_1 \| \hat{\beta} - \beta^* \|_2^2 \quad \text{for all iterates } t \geq T, \tag{22}
\]

where \( T := c_2 \log \frac{(\phi(\beta^0) - \phi(\hat{\beta}))}{\delta^2} / \log(1/\gamma) \).

Remarks As with Theorem 1, these claims are deterministic in nature. Probabilistic conditions will enter into the corollaries, which involve proving that the surrogate matrices \( \hat{\Gamma} \) used for noisy, missing, and/or dependent data satisfy the lower- and upper-RE conditions with high probability. The proof of Theorem 2 itself is based on an extension of a result due to Agarwal et al. [1] on the convergence of projected gradient descent and composite gradient descent in high dimensions. Their result as originally stated imposed convexity of the loss function, but the proof can be modified so as to apply to the non-convex loss functions of interest here. As noted following Theorem 1, all global minimizers of the non-convex program (4) lie within a small ball. In addition, Theorem 2 guarantees that the local minimizers also lie within a ball of the same magnitude. Note that in order to show that Theorem 2 can be applied to the specific statistical models of interest in this paper, a considerable amount of technical analysis remains in order to establish that its conditions hold with high probability.

In order to understand the significance of the bounds (20) and (22), note that they provide upper bounds for the \( \ell_2 \)-distance between the iterate \( \beta^t \) at time \( t \), which is easily computed in polynomial-time, and any global optimum \( \hat{\beta} \) of the program (4) or (7), which may be difficult to compute. Focusing on bound (20), since \( \gamma \in (0, 1) \), the first term in the bound vanishes as \( t \) increases. The remaining terms involve the statistical errors \( \| \hat{\beta} - \beta^* \|_q \), for \( q = 1, 2 \), which are controlled in Theorem 1. It can be verified that the two terms involving the statistical error on the right-hand side are bounded as \( O(\frac{k \log p}{n}) \), so Theorem 2 guarantees that projected gradient descent produce an output that is essentially as good—in terms of statistical error—as any global optimum of the program (4). Bound (22) provides a similar guarantee for composite gradient descent applied to the Lagrangian version.

Experimentally, we have found that the predictions of Theorem 2 are borne out in simulations. Figure 2 shows the results of applying the projected gradient descent method to solve the optimization problem (4) in the case of additive noise (panel (a)), and missing data (panel (b)). In each case, we generated a random problem instance, and then applied the projected gradient descent method to compute an estimate \( \hat{\beta} \). We then reapplied the projected gradient method to the same problem instance 10 times, each time with a random starting point, and measured the error \( \| \beta^t - \hat{\beta} \|_2 \) between the iterates and the first estimate (optimization error), and the error \( \| \beta^t - \beta^* \|_2 \) between the iterates and the truth (statistical error). Within each panel, the blue traces show the optimization error over 10 trials, and the red traces show the statistical error. On the logarithmic scale given, a geometric rate of convergence corresponds to a straight line. As predicted by Theorem 2, regardless of the starting point, the iterates \( \{ \beta^t \} \) exhibit geometric convergence to the same fixed point.\(^2\) The statistical error contracts geometrically up to a certain point, then flattens out.

\(^2\)To be precise, Theorem 2 states that the iterates will converge geometrically to a small neighborhood of all the global optima.
3.2 Some consequences

As discussed previously, both Theorems 1 and 2 are deterministic results. Applying them to specific statistical models requires some additional work in order to establish that the stated conditions are met. We now turn to the statements of some consequences of these theorems for different cases of noisy, missing, and dependent data. In all the corollaries below, the claims hold with probability greater than \(1 - c_1 \exp(-c_2 \log p)\), where \((c_1, c_2)\) are universal positive constants, independent of all other problem parameters. Note that in all corollaries, the triplet \((n, p, k)\) is assumed to satisfy scaling of the form \(n \sim k \log p\), as is necessary for \(\ell_2\)-consistent estimation of \(k\)-sparse vectors in \(p\) dimensions.

**Definition 3.** We say that a random matrix \(X \in \mathbb{R}^{n \times p}\) is sub-Gaussian with parameters \((\Sigma, \sigma^2)\) if:

(a) each row \(x_i^T \in \mathbb{R}^p\) is sampled independently from a zero-mean distribution with covariance \(\Sigma\), and

(b) for any unit vector \(u \in \mathbb{R}^p\), the random variable \(u^T x_i\) is sub-Gaussian with parameter at most \(\sigma\).

For instance, if we form a random matrix by drawing each row independently from the distribution \(N(0, \Sigma)\), then the resulting matrix \(X \in \mathbb{R}^{n \times p}\) is a sub-Gaussian matrix with parameters \((\Sigma, \|\Sigma\|_{\text{op}})\).

3.2.1 Bounds for additive noise: i.i.d. case

We begin with the case of i.i.d. samples with additive noise, as described in Example 1.

**Corollary 1.** Suppose that we observe \(Z = X + W\), where the random matrices \(X, W \in \mathbb{R}^{n \times p}\) are sub-Gaussian with parameters \((\Sigma_x, \sigma_x^2)\), and let \(\epsilon\) be an i.i.d. sub-Gaussian vector with...
parameter $\sigma_x^2$. Let $\sigma_w^2 = \sigma_x^2 + \sigma_\epsilon^2$. Then under the scaling $n \gtrsim \max \{ \frac{\sigma_w^2}{\lambda_{\min}(\Sigma_x)}, 1 \} k \log p$, for the $M$-estimator based on the surrogates $(\Gamma_{\text{add}}, \gamma_{\text{add}})$, the results of Theorems 1 and 2 hold with parameters $\alpha_1 = \frac{1}{2}\lambda_{\min}(\Sigma_x)$ and $\varphi(Q, \sigma_\epsilon) = c_0\sigma_x(\sigma_w + \sigma_\epsilon)\|\beta^*\|_2$, with probability at least $1 - c_1 \exp(-c_2 \log p)$.

Remarks
(a) Consequently, the $\ell_2$-error of any optimal solution $\hat{\beta}$ satisfies the bound
\[
\|\hat{\beta} - \beta^*\|_2 \lesssim \frac{\sigma_x(\sigma_w + \sigma_\epsilon)}{\lambda_{\min}(\Sigma_x)} \|\beta^*\|_2 \sqrt{\frac{k \log p}{n}}
\]
with high probability. The prefactor in this bound has a natural interpretation as an inverse signal-to-noise ratio; for instance, when $X$ and $W$ are zero-mean Gaussian matrices with row covariances $\Sigma_x = \sigma_x^2 I$ and $\Sigma_w = \sigma_w^2 I$, respectively, we have $\lambda_{\min}(\Sigma_x) = \sigma_x^2$, so
\[
\frac{(\sigma_w + \sigma_\epsilon)\sqrt{\sigma_x^2 + \sigma_w^2}}{\lambda_{\min}(\Sigma_x)} = \frac{\sigma_w + \sigma_\epsilon}{\sigma_x} \sqrt{1 + \frac{\sigma_w^2}{\sigma_x^2}}
\]
This quantity grows with the ratios $\sigma_w/\sigma_x$ and $\sigma_\epsilon/\sigma_x$, which measure the SNR of the observed covariates and predictors, respectively. Note that when $\sigma_w = 0$, corresponding to the case of uncorrupted covariates, the bound on $\ell_2$-error agrees with known results. See Section 4 for simulations and further discussions of the consequences of Corollary 1.

(b) We may also compare the results in (a) with bounds from past work on high-dimensional sparse regression with noisy covariates [18]. In this work, Rosenbaum and Tsybakov derive similar concentration bounds on sub-Gaussian matrices. The tolerance parameters are all $O(\sqrt{\frac{\log p}{n}})$, with prefactors depending on the sub-Gaussian parameters of the matrices. In particular, in their notation,
\[
\nu \times (\sigma_x \sigma_w + \sigma_w \sigma_\epsilon + \sigma_w^2) \sqrt{\frac{\log p}{n}} \|\beta^*\|_1,
\]
leading to the bound (cf. Theorem 2 of Rosenbaum and Tsybakov [18])
\[
\|\hat{\beta} - \beta^*\|_2 \lesssim \frac{\nu \sqrt{k}}{\lambda_{\min}(\Sigma_x)} \times \frac{\sigma_w^2}{\lambda_{\min}(\Sigma_x)} \sqrt{\frac{k \log p}{n}} \|\beta^*\|_1.
\]

Extensions to unknown noise covariance: Situations may arise where the noise covariance $\Sigma_w$ is unknown, and must be estimated from the data. One simple method is to assume that $\Sigma_w$ is estimated from independent observations of the noise. In this case, suppose we independently observe a matrix $W_0 \in \mathbb{R}^{n \times p}$ with $n$ i.i.d. vectors of noise. Then we use $\hat{\Sigma}_w = \frac{1}{n} W_0^T W_0$ as our estimate of $\Sigma_w$. A more sophisticated variant of this method (cf. Chapter 4 of Carroll et al. [3]) assumes that we observe $k_i$ replicate measurements $Z_{i1}, \ldots, Z_{ik}$ for each $x_i$ and form the estimator
\[
\hat{\Sigma}_w = \frac{\sum_{i=1}^n \sum_{j=1}^{k_i} (Z_{ij} - \bar{Z}_i)(Z_{ij} - \bar{Z}_i)^T}{\sum_{i=1}^n (k_i - 1)}.\tag{23}
\]
Based on the estimator $\hat{\Sigma}_w$, we form the pair $(\tilde{\Gamma}, \tilde{\gamma})$ such that $\tilde{\gamma} = \frac{1}{n} Z^T y$ and $\tilde{\Gamma} = \frac{1}{n} Z^T Z - \hat{\Sigma}_w$. In the proofs of Section 5, we will analyze the case where $\hat{\Sigma}_w = \frac{1}{n} W_0^T W_0$ and show that the result of Corollary 1 still holds when $\Sigma_w$ must be estimated from the data. Note that if $\hat{\Sigma}_w$ is used in equation (23) will also yield the same result, but the analysis is more complicated.
3.2.2 Bounds for missing data: i.i.d. case

Next, we turn to the case of i.i.d. samples with missing data, as discussed in Example 3. For a missing data parameter vector $\rho$, we define $\rho_{\max} := \max_j \rho_j$, and assume $\rho_{\max} < 1$.

**Corollary 2.** Let $X \in \mathbb{R}^{n \times p}$ be sub-Gaussian with parameters $(\Sigma_x, \sigma_x^2)$, and $Z$ the missing data matrix with parameter $\rho$. Let $\epsilon$ be an i.i.d. sub-Gaussian vector with parameter $\sigma_\epsilon^2$. If $n \gtrsim \max \left( \frac{1}{1-\rho_{\max}} \frac{\sigma_x^2}{\lambda_{\min}(\Sigma_x)} \right) k \log p$, then Theorems 1 and 2 hold with probability at least $1 - c_1 \exp \left( -c_2 \log p \right)$ for $\alpha_1 = \frac{1}{2} \frac{\sigma_x}{\lambda_{\min}(\Sigma_x)}$ and $\varphi(Q, \sigma_\epsilon) = c_0 \frac{\sigma_x}{1-\rho_{\max}} \| \beta^* \|_2$.

**Remarks** Suppose $X$ is a Gaussian random matrix and $\rho_j = \rho$ for all $j$. In this case, the ratio $\frac{\sigma_x^2}{\lambda_{\min}(\Sigma_x)} = \frac{\lambda_{\max}(\Sigma_x)}{\lambda_{\min}(\Sigma_x)} = \kappa(\Sigma_x)$ is the condition number of $\Sigma_x$. Then

$$\varphi(Q, \sigma_\epsilon) \propto \left( \frac{1}{\lambda_{\min}(\Sigma_x)} \frac{\sigma_x}{1-\rho} + \frac{\kappa(\Sigma_x)}{(1-\rho)^2} \right) \| \beta^* \|_2,$$

a quantity that depends on both the conditioning of $\Sigma_x$, and the fraction $\rho \in [0, 1)$ of missing data. We will consider the results of Corollary 2 applied to this example in the simulations of Section 4.

**Extensions to unknown $\rho$:** As in the additive noise case, we may wish to consider the case when the missing data parameters $\rho$ are not observed and must be estimated from the data. For each $j = 1, 2, \ldots, p$, we estimate $\rho_j$ using $\hat{\rho}_j$, the empirical average of the number of observed entries per column. Let $\hat{\rho} \in \mathbb{R}^p$ denote the resulting estimator of $\rho$. Naturally, we use the pair of estimators $(\hat{\Gamma}, \hat{\gamma})$ defined by

$$\hat{\Gamma} = \frac{Z^T Z}{n} \bigoplus \hat{M} \quad \text{and} \quad \hat{\gamma} = \frac{1}{n} Z^T y \bigoplus (1 - \hat{\rho}),$$

(24)

where

$$\hat{M}_{ij} = \begin{cases} (1 - \hat{\rho}_i)(1 - \hat{\rho}_j) & \text{if } i \neq j \\ 1 - \hat{\rho}_i & \text{if } i = j. \end{cases}$$

We will show in Section 5 that Corollary 2 holds when $\rho$ is estimated by $\hat{\rho}$.

3.2.3 Bounds for dependent data

Turning to the case of dependent data, we consider the setting where the rows of $X$ are drawn from a stationary vector autoregressive (VAR) process according to

$$x_{i+1} = Ax_i + v_i, \quad \text{for } i = 1, 2, \ldots, n - 1,$$

(25)

where $v_i \in \mathbb{R}^p$ is a zero-mean noise vector with covariance matrix $\Sigma_v$, and $A \in \mathbb{R}^{p \times p}$ is a driving matrix with spectral norm $\|A\|_2 < 1$. We assume the rows of $X$ are drawn from a Gaussian distribution with covariance $\Sigma_x$, such that $\Sigma_x = A \Sigma_x A^T + \Sigma_v$. Hence, the rows of $X$ are identically distributed but not independent, with the choice $A = 0$ giving rise to the i.i.d. scenario. Corollaries 3 and 4 correspond to the case of additive noise and missing data for a Gaussian VAR process.
Corollary 3. Suppose the rows of $X$ are drawn according to a Gaussian VAR process with driving matrix $A$. Suppose the additive noise matrix $W$ is i.i.d. with Gaussian rows, and let $\epsilon$ be an i.i.d. sub-Gaussian vector with parameter $\sigma_\epsilon^2$. If $n \gtrsim \max \left( \frac{c_4}{\lambda_{\min}(\Sigma_x)}, 1 \right) k \log p$, with $\zeta^2 = \|\Sigma_w\|_{op} + \frac{2\|\Sigma_x\|_{op} \rho_{\max}}{1 - \rho_{\max}^2}$, then Theorems 1 and 2 hold with probability at least $1 - c_1 \exp(-c_2 \log p)$ for $\alpha_1 = \frac{1}{2} \lambda_{\min}(\Sigma_x)$ and $\varphi(\sigma, \epsilon_\Sigma) = c_0(\sigma_\epsilon \zeta + \zeta^2)\|\beta^*\|_2$.

Corollary 4. Suppose the rows of $X$ are drawn according to a Gaussian VAR process with driving matrix $A$, and $Z$ is the observed matrix subject to missing data, with parameter $\rho$. Let $\epsilon$ be an i.i.d. sub-Gaussian vector with parameter $\sigma_\epsilon^2$. If $n \gtrsim \max \left( \frac{c_4}{\lambda_{\min}(\Sigma_x)}, 1 \right) k \log p$, with $\zeta^2 = \frac{1}{(1 - \rho_{\max})^2} \frac{2\|\Sigma_x\|_{op} \rho_{\max}}{1 - \rho_{\max}^2}$, then Theorems 1 and 2 hold with probability at least $1 - c_1 \exp(-c_2 \log p)$ for $\alpha_1 = \frac{1}{2} \lambda_{\min}(\Sigma_x)$ and $\varphi(\sigma, \epsilon_\Sigma) = c_0(\sigma_\epsilon \zeta + \zeta^2)\|\beta^*\|_2$.

Remarks. Note that the scaling and the form of $\varphi$ in Corollaries 2-4 are very similar, except with different effective variances $\sigma_\epsilon^2 = \frac{\sigma_\epsilon^4}{(1 - \rho_{\max})^2}$, $\zeta^2$, or $\zeta^2$, depending on the type of corruption in the data. As we will see in Section 5, the proofs involve verifying the deviation conditions (17) using similar techniques. On the other hand, the proof of Corollary 1 proceeds via deviation condition (16), which produces a tighter bound.

We may also extend the cases of dependent data to situations when $\Sigma$ is sparse. Our method bears similarity to the method of Yuan [26], but is valid for $\alpha$-independent data, and does not require an i.i.d. sub-Gaussian assumption. When $\Sigma$ is sparse, our methods enables recovery of the support of the precision matrix $\Theta$ when each column of $\Theta$ is bounded in $\sigma_\epsilon$-cept with different effective variances $\sigma_\epsilon^2 = \frac{\sigma_\epsilon^4}{(1 - \rho_{\max})^2}$, $\zeta^2$, or $\zeta^2$, depending on the type of corruption in the data. As we will see in Section 5, the proofs involve verifying the deviation conditions (17) using similar techniques. On the other hand, the proof of Corollary 1 proceeds via deviation condition (16), which produces a tighter bound.

3.3 Application to graphical model inverse covariance estimation

The problem of inverse covariance estimation for a Gaussian graphical model is also related to the Lasso. Meinshausen and Bühlmann [12] prescribed a way to recover the support of the precision matrix $\Theta$ when each column of $\Theta$ is $k$-sparse, via linear regression and the Lasso. More recently, Yuan [26] proposed a method for estimating $\Theta$ using the Dantzig selector, and obtained error bounds on $\|\hat{\Theta} - \Theta\|_1$ when the columns of $\Theta$ are bounded in $\ell_1$. Both of these results assume that $X$ is fully-observed and has i.i.d. rows.

Suppose we are given a matrix $X \in \mathbb{R}^{n \times p}$ of samples from a multivariate Gaussian distribution, where each row is distributed according to $N(0, \Sigma)$. We assume the rows of $X$ are either i.i.d. or sampled from a Gaussian VAR process. Based on the modified Lasso of the previous section, we devise a method to estimate $\Theta$ based on a corrupted observation matrix $Z$, when $\Theta$ is sparse. Our method bears similarity to the method of Yuan [26], but is valid in the case of corrupted data, and does not require an $\ell_1$ column bound. Let $X^j$ denote the $j$th column of $X$, and let $X^{-j}$ denote the matrix $X$ with $j$th column removed. By standard results on Gaussian graphical models, there exists a vector $\theta^j \in \mathbb{R}^{p-1}$ such that

$$X^j = X^{-j}\theta^j + \epsilon^j,$$

where $\epsilon^j$ is a vector of i.i.d. Gaussians and $\epsilon^j \perp X^{-j}$. Setting $a_j := -(\Sigma_{jj} - \Sigma_{j,-j}\theta^j)^{-1}$, we can verify that $\Theta_{j,-j} = a_j \theta^j$. Our algorithm, described below, forms estimates $\hat{\theta}^j$ and $\hat{a}_j$ for each $j$, then combines the estimates to obtain an estimate $\hat{\Theta}_{j,-j} = \hat{a}_j \hat{\theta}^j$.

In the additive noise case, we observe the matrix $Z = X + W$. From the equations (26), we obtain $Z^j = X^{-j}\theta^j + (\epsilon^j + W^j)$. Note that $\delta^j = \epsilon^j + W^j$ is a vector of i.i.d. Gaussians, and since $X \perp W$, we have $\delta^j \perp X^{-j}$. Hence, our results on covariates with additive noise allow us to recover $\theta^j$ from $Z$. We can verify that this reduces to solving the program (4) or (7) with the pair $(\hat{\Gamma}^{(j)}, \hat{\gamma}^{(j)}) = (\hat{\Sigma}_{j,-j}, \frac{1}{n} Z^{-j}Z^j)$, where $\hat{\Sigma} = \frac{1}{n} Z^T Z - \Sigma_w$. 

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When \( Z \) is a missing-data version of \( X \), we similarly estimate the vectors \( \theta^j \) via equation (26), using our results on the Lasso with missing covariates. Here, both covariates and responses are subject to missing data, but this makes no difference in our theoretical results. For each \( j \), we use the pair

\[
(\hat{\Gamma}^{(j)}, \hat{\gamma}^{(j)}) = (\hat{\Sigma}_{-j,-j}, 1/n Z^{-jT} Z^j \oplus (1 - \rho^{-j})(1 - \rho_j)),
\]

where \( \hat{\Sigma} = 1/n Z^T Z \oplus M \), and \( M \) is defined as in Example 3.

To obtain the estimate \( \hat{\Theta} \), we therefore propose the following procedure, based on the estimators \( \{(\hat{\Gamma}^{(j)}, \hat{\gamma}^{(j)})\}_{j=1}^p \) and \( \bar{\Sigma} \).

**Algorithm 3.1.**

1. Perform \( p \) linear regressions of the variables \( Z^j \) upon the remaining variables \( Z^{-j} \), using the program (4) or (7) with the estimators \( (\hat{\Gamma}^{(j)}, \hat{\gamma}^{(j)}) \), to obtain estimates \( \hat{\theta}^j \) of \( \theta^j \).

2. Estimate the scalars \( a_j \) using the quantity \( \hat{a}_j := -\bar{\Sigma}_{jj} - \hat{\Sigma}_{j,-j} \hat{\theta}^j \), based on the estimator \( \hat{\Sigma} \). Form \( \hat{\Theta} \) with \( \hat{\Theta}_{j,-j} = \hat{a}_j \hat{\theta}^j \) and \( \hat{\Theta}_{jj} = -\hat{a}_j \).

3. Set \( \hat{\Theta} = \arg \min_{\Theta \in S^p} \| \Theta - \hat{\Theta} \|_1 \), where \( S^p \) is the set of symmetric matrices.

Note that the minimization in step (3) is a linear program, so is easily solved with standard methods. We have the following corollary about \( \hat{\Theta} \):

**Corollary 5.** Suppose the columns of the matrix \( \Theta \) are \( k \)-sparse, and suppose the condition number \( \kappa(\Theta) \) is nonzero and finite. Suppose we have

\[
\| \hat{\gamma}^{(j)} - \hat{\Gamma}^{(j)} \hat{\theta}^j \|_\infty \leq \varphi(Q, \sigma_e) \sqrt{\log p \over n}, \quad \forall j,
\]

and suppose we have the following additional deviation condition on \( \bar{\Sigma} \):

\[
\| \bar{\Sigma} - \Sigma \|_{\max} \leq c_\varphi(Q, \sigma_e) \sqrt{\log p \over n}.
\]

Finally, suppose the lower-RE condition holds uniformly over the matrices \( \hat{\Gamma}^{(j)} \) with the scaling (18). Then under the estimation procedure of Algorithm 3.1, there exists a universal constant \( c_0 \) such that

\[
\| \hat{\Theta} - \Theta \|_p \leq c_0 \kappa^2(\Sigma) \left( \varphi(Q, \sigma_e) \over \lambda_{\min}(\Sigma) \right) \left( \varphi(Q, \sigma_e) \over \lambda_{\min}(\Sigma) \alpha_1 \right) k \sqrt{\log p \over n}.
\]

**Remarks.** Note that Corollary 5 is again a deterministic result, with parallel structure to Theorem 1. Furthermore, the deviation bounds (27) and (28) hold for all scenarios considered in Section 3.2 above, using Corollaries 1-4 for the first two inequalities, and a similar bounding technique for \( \| \bar{\Sigma} - \Sigma \|_{\max} \) and the lower-RE condition holds over all matrices \( \hat{\Gamma}^{(j)} \) by the same technique used to establish the lower-RE condition for \( \hat{\Gamma} \). The uniformity of the lower-RE bound over all sub-matrices holds because

\[
0 < \lambda_{\min}(\Sigma) \leq \lambda_{\min}(\Sigma_{-j,-j}) \leq \lambda_{\max}(\Sigma_{-j,-j}) \leq \lambda_{\max}(\Sigma) < \infty.
\]

Hence, the error bound in Corollary 5 holds with probability at least \( 1 - c_1 \exp(-c_2 \log p) \) when \( n \gtrsim k \log p \), for the appropriate values of \( \varphi \) and \( \alpha_1 \).
4 Simulations

In this section, we report some additional simulation results to confirm that the scalings predicted by our theory are sharp. In Figure 1 following Theorem 1, we showed that the error curves align when plotted against a suitably rescaled sample size, in the case of additive noise perturbations. Panel (a) of Figure 3 shows these same types of rescaled curves for the case of missing data, with sparsity $k \approx \sqrt{p}$, covariate matrix $\Sigma_x = I$, and missing fraction $\rho = 0.2$, whereas panel (b) shows the rescaled plots for the vector autoregressive case with additive noise perturbations, using a driving matrix $A$ with $\|A\|_op = 0.2$. Each point corresponds to an average over 100 trials. Once again, we see excellent agreement with the scaling law provided by Theorem 1.

![Figure 3. Plots of the error $\|\hat{\beta} - \beta^*\|_2$ after running projected gradient descent on the non-convex objective, with sparsity $k \approx \sqrt{p}$. In all cases, we plotted the error versus the rescaled sample size $kn/(k \log p)$. As predicted by Theorems 1 and 2, the curves align for different values of $p$ when plotted in this rescaled manner. (a) Missing data case with i.i.d. covariates. (b) Vector autoregressive data with additive noise. Each point represents an average over 100 trials.](image)

We also ran simulations to verify the form of the function $\varphi(Q, \sigma)$ appearing in Corollaries 1 and 2. In the additive noise setting for i.i.d. data, we set $\Sigma_x = I$ and $\epsilon$ equal to i.i.d. Gaussian noise with $\sigma_{\epsilon} = 0.5$. For a fixed value of the parameters $p = 256$ and $k \approx \log p$, we ran the projected gradient descent algorithm for different values of $\sigma_w \in (0.1, 0.3)$, such that $\Sigma_w = \sigma_w^2 I$ and $n \approx 60(1 + \sigma_w^2)^2 k \log p$, with $\|\beta^*\|_2 = 1$. According to the theory, $\varphi(Q, \sigma_{\epsilon}) \approx (\sigma_w + 0.5) \sqrt{1 + \sigma_w^2}$, so that

$$\|\hat{\beta} - \beta^*\|_2 \lesssim (\sigma_w + 0.5) \sqrt{1 + \sigma_w^2} \sqrt{\frac{k \log p}{(1 + \sigma_w^2)^2 k \log p}} \times \frac{\sigma_w + 0.5}{\sqrt{1 + \sigma_w^2}}.$$  

In order to verify this prediction, we plotted $\sigma_w$ versus the rescaled error $\frac{\sqrt{1 + \sigma_w^2}}{\sigma_w + 0.5} \|\hat{\beta} - \beta^*\|_2$. As shown by panel (a) of Figure 4(a), the curve is roughly constant, as predicted by the theory.

Similarly, in the missing data setting for i.i.d. data, we set $\Sigma_x = I$ and $\epsilon$ equal to i.i.d. Gaussian noise with $\sigma_{\epsilon} = 0.5$. For a fixed value of the parameters $p = 128$ and $k \approx \log p$, we ran simulations for different values of the missing data parameter $\rho \in (0, 0.3)$, such that
n \approx \frac{60}{(1-\rho)^2} k \log p. \quad \text{According to the theory, } \frac{\varphi(Q, \tau)}{\alpha} \leq \frac{\sigma_x}{1-\rho} + \frac{1}{(1-\rho)^2}. \quad \text{Consequently, with our specified scalings of } (n, p, k), \text{ we should expect a bound of the form}

\| \hat{\beta} - \beta^* \|_2 \lesssim \frac{\varphi(Q, \tau)}{\alpha} \sqrt{\frac{k \log p}{n}} \approx 1 + 0.5(1 - \rho).

The plot of $\rho$ versus the rescaled error $\| \hat{\beta} - \beta^* \|_2$ is shown in Figure 4(b). The curve is again roughly constant, agreeing with theoretical results.

Finally, we studied the behavior of the inverse covariance matrix estimation algorithm on three types of Gaussian graphical models:

(a) **Chain-structured graphs.** In this case, all nodes of the graph are arranged in a linear chain. Hence, each node (except the two end nodes) has degree $k = 2$. The diagonal entries of $\Theta$ are set equal to 1, and all entries corresponding to links in the chain are set equal to 0.1. Then $\Theta$ is rescaled so $\| \Theta \|_{op} = 1$.

(b) **Star-structured graphs.** In this case, all nodes are connected to a central node, which has degree $k \approx 0.1 p$. All other nodes have degree 1. The diagonal entries of $\Theta$ are set equal to 1, and all entries corresponding to edges in the graph are set equal to 0.1. Then $\Theta$ is rescaled so $\| \Theta \|_{op} = 1$.

(c) **Erdős-Renyi graphs.** This example comes from Rothman et al. [19]. For a sparsity parameter $k \approx \log p$, we randomly generate the matrix $\Theta$ by first generating the matrix $B$ such that the diagonal entries are 0, and all other entries are independently equal to 0.5 with probability $k/p$, and 0 otherwise. Then $\delta$ is chosen so that $\Theta = B + \delta I$ has condition number $p$. Finally, $\Theta$ is rescaled so $\| \Theta \|_{op} = 1$.

After generating the matrix $X$ of $n$ i.i.d. samples from the appropriate graphical model, with covariance matrix $\Sigma_x = \Theta^{-1}$, we generated the corrupted matrix $Z = X + W$ with
\[ \Sigma_w = (0.2)^2 I \] in the additive noise case, or the missing data matrix \( Z \) with \( \rho = 0.2 \) in the missing data case.

Panels (a) and (c) in Figure 5 show the rescaled \( \ell_2 \)-error \( \frac{1}{\sqrt{k}} \| \hat{\Theta} - \Theta \|_{op} \) plotted against the sample size \( n \) for a chain-structured graph. In panels (b) and (d), we have \( \ell_2 \)-error plotted against the rescaled sample size, \( n/(k \log p) \). Once again, we see good agreement with the theoretical predictions. We have obtained qualitatively similar results for the star and Erdős-Rényi graphs.

**Figure 5.** (a) Plots of the error \( \| \hat{\Theta} - \Theta \|_{op} \) after running projected gradient descent on the non-convex objective for a chain-structured Gaussian graphical model with additive noise. As predicted by Theorems 1 and 2, all curves align when the error is rescaled by \( \frac{1}{\sqrt{k}} \) and plotted against the ratio \( \frac{n}{k \log p} \), as shown in (b). Plots (c) and (d) show the results of simulations on missing data sets. Each point represents the average over 50 trials.

5 Proofs

In this section, we turn to the proofs of our two main theorems, as well as their various corollaries. Included in the main text are the primary steps in the proofs, with more technical aspects of the proofs deferred to the appendices.
5.1 Proof of Theorem 1

Let $L(\beta) = \frac{1}{2} \beta^T \hat{\Gamma} \beta - \langle \hat{\gamma}, \beta \rangle + \lambda_n \Vert \beta \Vert_1$ denote the loss function to be minimized. This definition captures both the estimator (4) with $\lambda_n = 0$ and the estimator (7) with the choice of $\lambda_n$ given in the theorem statement. For either estimator, we are guaranteed that $\beta^*$ is feasible and $\hat{\beta}$ is optimal for the program, so $L(\hat{\beta}) \leq L(\beta^*)$. Indeed, in the regularized case, the $k$-sparsity of $\beta^*$ implies that $\Vert \beta^* \Vert_1 \leq \sqrt{k} \Vert \beta^* \Vert_2 \leq b_0 \sqrt{k}$. Defining the error vector $\nu := \hat{\beta} - \beta^*$ and performing some algebra leads to the equivalent inequality

$$\frac{1}{2} \nu^T \hat{\Gamma} \nu \leq \langle \nu, \hat{\gamma} - \hat{\Gamma} \beta^* \rangle + \lambda_n \{ \Vert \beta^* \Vert_1 - \Vert \beta^* + \nu \Vert_1 \}. \quad (29)$$

In the remainder of the proof, we first derive an upper bound for the right-hand side of this inequality. We then use this upper bound and the lower-RE condition to show that the error vector $\nu$ must satisfy the inequality

$$\Vert \nu \Vert_1 \leq 8 \sqrt{k} \Vert \nu \Vert_2. \quad (30)$$

Finally, we combine the inequality (30) with the lower-RE condition to derive a lower bound on the left-hand side of the basic inequality (29). Combined with our earlier upper bound on the right-hand side, some algebra yields the claim.

**Upper bound on right-hand side** We first upper-bound the RHS of inequality (29). Hölder’s inequality gives $\langle \nu, \hat{\gamma} - \hat{\Gamma} \beta^* \rangle \leq \Vert \nu \Vert_1 \Vert \hat{\gamma} - \hat{\Gamma} \beta^* \Vert_\infty$. By the triangle inequality, we have

$$\Vert \hat{\gamma} - \hat{\Gamma} \beta^* \Vert_\infty \leq \Vert \hat{\gamma} - \Sigma_x \beta^* \Vert_\infty + \Vert (\Sigma_x - \hat{\Gamma}) \beta^* \Vert_\infty \overset{(i)}{\leq} 2 \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}},$$

where inequality (i) follows from the deviation conditions (17). Combining the pieces, we conclude that

$$\langle \nu, \hat{\gamma} - \hat{\Gamma} \beta^* \rangle \leq 2 \Vert \nu \Vert_1 \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}} = \left( \Vert \nu_S \Vert_1 + \Vert \nu_{S^c} \Vert_1 \right) 2 \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}}. \quad (31)$$

On the other hand, we have

$$\Vert \beta^* + \nu \Vert_1 - \Vert \beta^* \Vert_1 \geq \{ \Vert \beta_S^* \Vert_1 - \Vert \nu_S \Vert_1 \} + \Vert \nu_{S^c} \Vert_1 - \Vert \beta^* \Vert_1 = \Vert \nu_{S^c} \Vert_1 - \Vert \nu_S \Vert_1, \quad (32)$$

where we have exploited the sparsity of $\beta^*$ and applied the triangle inequality. Combining the pieces, we conclude that the right-hand side of inequality (29) is upper-bounded by

$$2 \varphi(Q, \sigma) \sqrt{\frac{\log p}{n}} (\Vert \nu_S \Vert_1 + \Vert \nu_{S^c} \Vert_1) + \lambda_n \{ \Vert \nu_S \Vert_1 - \Vert \nu_{S^c} \Vert_1 \}, \quad (33)$$

a bound that holds for any nonnegative choice of $\lambda_n$.

**Proof of inequality (30)** In the case of the constrained estimator (4) with $R = \Vert \beta^* \Vert_1$, we have $\Vert \hat{\beta} \Vert_1 = \Vert \beta^* + \nu \Vert_1 \leq \Vert \beta^* \Vert_1$. Combined with inequality (32), we conclude that $\Vert \nu_{S^c} \Vert_1 \leq \Vert \nu_S \Vert_1$. Consequently, we have the inequality $\Vert \nu \Vert_1 \leq 2 \Vert \nu_S \Vert_1 \leq 2 \sqrt{k} \Vert \nu \Vert_2$, which is a slightly stronger form of the bound (30).

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For the regularized estimator (7), we first note that our choice of $\lambda_n$ guarantees that the term (33) is at most $\frac{3\lambda_n}{2}||\hat{\nu}_S||_1 - \frac{\lambda_n}{4}||\hat{\nu}_{Sc}||_1$. Returning to the basic inequality, we apply the lower-RE condition to lower-bound the left-hand side, thereby obtaining the inequality

$$-\frac{\tau}{2}||\hat{\nu}||^2_1 \leq \frac{1}{2}(\alpha_1||\hat{\nu}||^2_2 - \tau||\hat{\nu}||^2_1) \leq \frac{3\lambda_n}{2}||\hat{\nu}_S||_1 - \frac{\lambda_n}{2}||\hat{\nu}_{Sc}||_1.$$  

By the triangle inequality, we have $||\hat{\nu}||_1 \leq ||\hat{\beta}||_1 + ||\beta^*||_1 \leq 2b_0\sqrt{k}$. Since we have assumed $\sqrt{k}\tau(n, p) \leq \frac{\phi(Q, \sigma_\epsilon)}{b_0} \sqrt{\frac{\log p}{n}}$, we are guaranteed that

$$\frac{\tau(n, p)}{2}||\hat{\nu}||^2_1 \leq \phi(Q, \sigma_\epsilon)\sqrt{\frac{\log p}{n}}||\hat{\nu}||_1 \leq \frac{\lambda_n}{4}||\hat{\nu}||_1,$$

by our choice of $\lambda_n$. Combining the pieces, we conclude that

$$0 \leq \frac{3\lambda_n}{2}||\hat{\nu}_S||_1 - \frac{\lambda_n}{2}||\hat{\nu}_{Sc}||_1 + \frac{\lambda_n}{4}(||\hat{\nu}_S||_1 + ||\hat{\nu}_{Sc}||_1) = \frac{7\lambda_n}{4}||\hat{\nu}_S||_1 - \frac{\lambda_n}{4}||\hat{\nu}_{Sc}||_1,$$

and rearranging implies $||\hat{\nu}_{Sc}||_1 \leq 7||\hat{\nu}_S||_1$, from which we conclude that $||\hat{\nu}||_1 \leq 8\sqrt{k}||\hat{\nu}||_2$, as claimed.

**Lower bound on left-hand side** We now derive a lower bound on the left-hand side of inequality (29). Combining inequality (30) with the RE condition (12) gives

$$\hat{\nu}^T\hat{\Gamma}\hat{\nu} \geq \alpha_1||\hat{\nu}||^2_2 - \tau(n, p)||\hat{\nu}||^2_1 \geq \{\alpha_1 - 64\kappa\tau(n, p)\}||\hat{\nu}||^2_2 \geq \frac{\alpha_1}{2}||\hat{\nu}||^2_2,$$  

(34)

where the final step uses our assumption that $k\tau(n, p) \leq \frac{\alpha_1}{128}$.

Finally, combining bounds (33), (30), and (34) gives

$$\frac{\alpha_1}{4}||\hat{\nu}||^2_2 \leq 2\max\{2\phi(Q, \sigma_\epsilon)\sqrt{\frac{\log p}{n}}, \lambda_n\}||\hat{\nu}||_1 \leq 32\sqrt{k}\max\{\phi(Q, \sigma_\epsilon)\sqrt{\frac{\log p}{n}}, \lambda_n\}||\hat{\nu}||_2,$$

yielding inequality (19a). Using inequality (30) again gives inequality (19b).

### 5.2 Proof of Theorem 2

We begin by proving the claims for the constrained problem, and projected gradient descent. For the $\ell_2$-error bound, we make use of Theorem 1 in the pre-print of Agarwal et al. [1].

Their theory, as originally stated, requires that the loss function be convex, but a careful examination of their proof shows that their arguments hinge on restricted strong convexity and smoothness assumptions, corresponding to a more general version of the lower- and upper-RE conditions given here. Apart from these conditions, the proof exploits the fact that the sub-problems defining the gradient updates (14) and (15) are convex. Since the loss function itself appears only in a linear term, their theory still applies.

In order to apply Theorem 1 in their paper, we first need to compute the tolerance parameter $\epsilon^2$ defined there; since $\beta^*$ is supported on the set $S$ with $|S| = k$ and the RE conditions
hold with $\tau \asymp \frac{\log p}{n}$, we find that

$$
e^2 \leq c \frac{\log p}{\alpha_2 n} (\sqrt{k} \parallel \hat{\beta} - \beta^* \parallel_2 + 2 \parallel \hat{\beta} - \beta^* \parallel_1)^2$$

\begin{align*}
&\leq c' \frac{k \log p}{\alpha_2 n} \parallel \hat{\beta} - \beta^* \parallel_2^2 + c_1 \frac{\log p}{\alpha_2 n} \parallel \hat{\beta} - \beta^* \parallel_1^2 \\
&\leq c_2 \parallel \beta - \beta^* \parallel_2^2 + c_1 \frac{\log p}{\alpha_2 n} \parallel \beta - \beta^* \parallel_1^2,
\end{align*}

where the final inequality makes use of the assumption that $n \gtrsim k \log p$. Similarly, we may compute the contraction coefficient to be

$$\gamma = \left(1 - \frac{\alpha_1}{\alpha_2} + \frac{c_1 k \log p}{\alpha_2 n}\right) \left(1 - \frac{c_2 k \log p}{\alpha_2 n}\right)^{-1},$$

so $\gamma \in (0, 1)$ for $n \gtrsim k \log p$.

We now establish the $\ell_1$-error bound. First, let $\Delta^t := \beta^t - \beta^*$. Since $\beta^t$ is feasible and $\hat{\beta}$ is optimal with an active constraint, we have $\parallel \beta^t \parallel_1 \leq \parallel \hat{\beta} \parallel_1$. Applying the triangle inequality gives

$$\parallel \hat{\beta} \parallel_1 \leq \parallel \beta^* \parallel_1 + \parallel \beta - \beta^* \parallel_1 = \parallel \beta^*_s \parallel_1 + \parallel \hat{\beta} - \beta^*_s \parallel_1,$$

$$\parallel \beta^t \parallel_1 \leq \parallel \beta^* + \Delta^t \parallel_1 \leq \parallel \beta^*_s + \Delta^t_S \parallel_1 - \parallel \Delta^t_S \parallel_1 = \parallel \beta^*_s \parallel_1 + \parallel \Delta^t_S \parallel_1 - \parallel \Delta^t_S \parallel_1,$$

combining the bounds yields $\parallel \Delta^t_S \parallel_1 \leq \parallel \beta^*_s \parallel_1 + \parallel \hat{\beta} - \beta^*_s \parallel_1$. Then

$$\parallel \Delta^t \parallel_1 \leq \parallel \Delta^t_S \parallel_1 + \parallel \hat{\beta} - \beta^*_s \parallel_1 \leq 2 \sqrt{k} \parallel \Delta^t \parallel_2 + \parallel \hat{\beta} - \beta^* \parallel_1,$$

so

$$\parallel \beta^t - \hat{\beta} \parallel_1 \leq \parallel \beta^* \parallel_1 + \parallel \Delta^t \parallel_1 \leq 2 \sqrt{k} \parallel \beta^t - \hat{\beta} \parallel_2 + \parallel \hat{\beta} - \beta^* \parallel_2 + 2 \parallel \hat{\beta} - \beta^* \parallel_1.$$

Turning to the Lagrangian version, we exploit Theorem 2 in Agarwal et al., with $\mathcal{M}$ corresponding to the subspace of all vectors with support contained within the support set of $\beta^*$. With this choice, we have $\psi(\mathcal{M}) = \sqrt{k}$, and the contraction coefficient $\gamma$ takes the previous form (35), so that the assumption $n \gtrsim k \log p$ guarantees that $\gamma \in (0, 1)$. It remains to verify that the requirements are satisfied. From the conditions in our Theorem 2 and using the notation of Agarwal et al., we have $\beta(\mathcal{M}) = O\left(\frac{\log p}{n}\right)$ and $\bar{\sigma} = \sqrt{k}$, and the condition $n \gtrsim k \log p$ implies that $\xi(\mathcal{M}) = O(1)$. Putting together the pieces, we find that the compound tolerance parameter $\epsilon^2$ satisfies the bound $\epsilon^2 = O\left(\frac{k \log p}{n} \parallel \beta - \beta^* \parallel_2^2\right) = O\left(\parallel \beta - \beta^* \parallel_2^2\right)$, so the claim follows.

### 5.3 Proof of Corollary 1

The proof of this corollary is based on two technical lemmas, one establishing that the lower- and upper-RE conditions hold with high probability, and the other proving a form of the deviation bounds (17).

**Lemma 1** (RE conditions, i.i.d. with additive noise). Under the conditions of Corollary 1, there are universal positive constants $c_i$ such that the matrix $\tilde{F}_{\text{add}}$ satisfies the lower- and upper-RE conditions with parameters $\alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}$, $\alpha_2 = \frac{3}{2} \lambda_{\max}(\Sigma_x)$, and

$$\tau(n, p) = c_0 \lambda_{\min}(\Sigma_x) \max\left(\frac{(\sigma_x^2 + \sigma_{wi}^2)^2}{\lambda_{\min}^2(\Sigma_x)}, 1\right) \log p \frac{1}{n},$$

with probability at least $1 - c_1 \exp\left(-c_2 n \min\left(\frac{\lambda_{\min}^2(\Sigma_x)}{(\sigma_x^2 + \sigma_{wi}^2)^2}, 1\right)\right)$. 

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Proof. Using Lemma 13 in Appendix A, together with the substitutions

\[
\hat{\gamma} - \Sigma_x = \frac{Z^T Z}{n} - \Sigma_z, \quad \text{and} \quad s := \frac{1}{c \log p} \min \left\{ \frac{\lambda_{\min}^2(\Sigma_x)}{\sigma^4}, 1 \right\}, (36)
\]

where \( \sigma^2 = \sigma^2_x + \sigma^2_w \) and \( c \) is chosen sufficiently small so \( s \geq 1 \), we see that it suffices to show that

\[
\sup_{v \in K(2s)} |v^T \left( \frac{Z^T Z}{n} - \Sigma_z \right) v| \leq \frac{\lambda_{\min}(\Sigma_x)}{54}
\]

with high probability.

Note that the matrix \( Z \) is sub-Gaussian with parameters \((\Sigma_x + \Sigma_w, \sigma^2)\). Consequently, by Lemma 15 in Appendix B, we have

\[
P[D(s) \geq t] \leq 2 \exp \left( -c' n \min \left( \frac{t^2}{\sigma^4}, \frac{t}{\sigma^2} \right) + 2s \log p \right),
\]

for some universal constant \( c' > 0 \). Setting \( t = \frac{\lambda_{\min}(\Sigma_x)}{54} \), we see that as long as the constant \( c \) in the definition (36) is chosen sufficiently small, we are guaranteed that

\[
P[D(s) \geq \frac{\lambda_{\min}(\Sigma_x)}{54}] \leq 2 \exp \left( -c_2 n \min \left( \frac{\lambda_{\min}^2(\Sigma_x)}{(\sigma^2_x + \sigma^2_w)^2}, 1 \right) \right), (37)
\]

which establishes the result.

\[\square\]

**Lemma 2** (Deviation conditions, additive noise). Under the conditions of Corollary 1, there are universal positive constants \( c_i \) such the deviation bound (16) holds with parameter

\[
\varphi(Q, \sigma) = c_0 \sigma_z (\sigma_w + \sigma) \| \beta^* \|_2,
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log p) \).

Proof. Using the fact that \( y = X\beta^* + \epsilon \), we may write

\[
\| \hat{\gamma} - \hat{\Gamma} \beta^* \|_\infty = \| \frac{Z^T y}{n} - \left( \frac{Z^T Z}{n} - \Sigma_w \right) \beta^* \|_\infty
\]

\[
= \| \frac{Z^T (X \beta^* + \epsilon)}{n} - \left( \frac{Z^T Z}{n} - \Sigma_w \right) \beta^* \|_\infty
\]

\[
\leq \| \frac{Z^T \epsilon}{n} \|_\infty + \left\| (\Sigma_w - \frac{Z^T W}{n}) \beta^* \right\|_\infty.
\]

Hence, the conclusion follows easily from Lemma 14 in Appendix B.

\[\square\]
Extension to unknown $\Sigma_w$. In the case when $\Sigma_w$ is unknown, we first verify the deviation bound (16). Note that the form of $\hat{\gamma}$ is the same as in the case when $\Sigma_w$ is known, so it suffices to bound the quantity $\| (\tilde{\Gamma} - \Sigma_x) \beta^* \|_\infty$ w.h.p. Furthermore,

$$\| (\tilde{\Gamma} - \Sigma_x) \beta^* \|_\infty \leq \| (\tilde{\Gamma} - \hat{\Gamma}) \beta^* \|_\infty + \| (\hat{\Gamma} - \Sigma_x) \beta^* \|_\infty = \| (\tilde{\Sigma}_w - \Sigma_w) \beta^* \|_\infty + \| (\tilde{\Gamma} - \Sigma_x) \beta^* \|_\infty,$$

and the second term is bounded by $c\sigma_w^2 \sqrt{\frac{\log p}{n}}$ w.h.p., by Lemma 14 in Appendix B. If we use the estimator $\hat{\Sigma}_w = \frac{1}{n} W_0^T W_0$, then

$$\mathbb{P}(\| (\tilde{\Sigma}_w - \Sigma_w) \beta^* \|_\infty \leq c\sigma_w^2 \sqrt{\frac{\log p}{n}}) \geq 1 - c_1 \exp(-c_2 \log p)$$

by the same sub-Gaussian tail bounds. Since $\sigma_w^2 \leq \sigma_z^2$, we conclude that

$$\| (\tilde{\Gamma} - \Sigma_x) \beta^* \|_\infty \leq c\sigma_z^2 \sqrt{\frac{\log p}{n}}$$

with probability at least $1 - c_1 \exp(-c_2 \log p)$, as wanted.

Turning to the RE conditions, we similarly write

$$|v^T (\tilde{\Gamma} - \Sigma_x) v| \leq |v^T (\tilde{\Gamma} - \hat{\Gamma}) v| + |v^T (\hat{\Gamma} - \Sigma_x) v|$$

$$= |v^T (\tilde{\Sigma}_w - \Sigma_w) v| + |v^T (\hat{\Gamma} - \Sigma_x) v|.$$

Then applying Lemma 15 to both terms, followed by Lemma 13, yields the required bounds.

5.4 Proof of Corollary 2

We now turn to the proof of Corollary 2, which applies to the case of missing data, based on the general $M$-estimator using the pair $(\tilde{\Gamma}_{\min}, \hat{\gamma}_{\min})$ defined in equation (11). We will establish that the RE conditions and deviation conditions (17) hold with high probability.

Lemma 3 (RE conditions, i.i.d. with missing data). Under the conditions of Corollary 2, there are universal positive constants $c_i$ such that $\tilde{\Gamma}_{\min}$ satisfies the lower- and upper-RE conditions with parameters $\alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}$, $\alpha_2 = \frac{3}{2} \lambda_{\max}(\Sigma_x)$, and

$$\tau(n, p) = c_0 \lambda_{\min}(\Sigma_x) \max \left( \frac{1}{(1 - \rho_{\max})^2}, \frac{\sigma_w^4}{\lambda_{\min}(\Sigma_x)}, 1 \right) \frac{\log p}{n},$$

with probability at least $1 - c_1 \exp(\ -c_2 n \min \left( (1 - \rho_{\max})^4, \frac{\lambda_{\min}(\Sigma_x)}{\sigma_z^2} \right))$.

Proof. This proof parallels the proof of Lemma 1 for the additive noise case. We make use of Lemma 13. This time, we have

$$\tilde{\Gamma} - \Sigma_x = \frac{Z^T Z}{n} \oplus M - \Sigma_x = (\frac{Z^T Z}{n} - \Sigma_z) \oplus M,$$
with the parameter $s$ defined as in equation (36), with $\sigma^2 = \frac{\sigma^2}{(1-\rho_{\text{max}})^2}$. Note that for a vector $v \in \mathbb{R}^p$, we have

$$|v^T(\hat{\Sigma} - \Sigma_x)v| = |v^T\left(\frac{Z^T Z}{n} - \Sigma_x\right)\otimes M|$$

$$\leq \frac{1}{\|M\|_{\text{min}}} |v^T\left(\frac{Z^T Z}{n} - \Sigma_x\right)v|$$

$$\leq \frac{1}{(1 - \rho_{\text{max}})^2} |v^T\left(\frac{Z^T Z}{n} - \Sigma_x\right)v|.$$  \hspace{1cm} (38)

Furthermore, $Z$ is a sub-Gaussian matrix with parameters $(\Sigma_z, \sigma_z^2)$, so applying Lemma 15 in Appendix B with $t = (1 - \rho_{\text{max}})^2 \lambda_{\text{min}}(\Sigma_z)$ to the right-hand expression, we obtain the bound

$$\mathbb{P}[D(s) \geq \frac{\lambda_{\text{min}}(\Sigma_z)}{54}] \leq 2 \exp(-c_2 n \min((1 - \rho_{\text{max}})^4, \frac{\lambda_{\text{min}}^2(\Sigma_x)}{\sigma_x^4}, 1)).$$

\[\square\]

**Lemma 4** (Deviation conditions, missing data). Under the conditions of Corollary 2, there are universal positive constants $c_1$ such the deviation bounds (17) hold with parameter

$$\varphi(Q, \sigma_\epsilon) = c_0 \frac{\sigma_x}{1 - \rho_{\text{max}}} (\sigma_\epsilon + \frac{\sigma_x}{1 - \rho_{\text{max}}}),$$

with probability at least $1 - c_1 \exp(-c_2 \log p)$.

**Proof.** The key idea is to note that the observed matrix $Z$ is a sub-Gaussian matrix with parameter $\sigma_x^2$. Indeed, recalling that the hidden matrix $X$ is sub-Gaussian with parameter $\sigma_x^2$, we see that for any unit vector $v \in \mathbb{R}^p$, and any missing value pattern of $X_i$, we have

$$\mathbb{E}[\exp(\lambda Z v) \mid \text{missing values}] = \mathbb{E}(\exp(\lambda X_i u_i)) \leq \exp\left(\frac{\sigma_x^2 \lambda^2}{2}\right),$$

where the vector $u \in \mathbb{R}^p$ has entries $u_i = v_i$ when entry $i$ is observed, and $u_i = 0$ otherwise. By the tower property of conditional expectation, it follows that the moment generating function of $Zv$ is upper-bounded by the same quantity, so $Z$ is also a sub-Gaussian matrix with parameter at most $\sigma_x^2$.

Observe that

$$\|\hat{\gamma} - \Sigma_x \beta^*\|_\infty = \left\|\left(\frac{1}{n} (Z^T y - \text{cov}(Z_i, y)) \otimes (1 - \rho)\right) \beta^*\right\|$$

$$\leq \frac{1}{1 - \rho_{\text{max}}} \left\|\frac{1}{n} (Z^T y - \text{cov}(z_i, y)) \beta^*\right\|_\infty$$

$$\leq \frac{1}{1 - \rho_{\text{max}}} \left(\left\|\frac{1}{n} (Z^T X - \text{cov}(z_i, z_i)) \beta^*\right\|_\infty + \left\|\frac{Z^T \epsilon}{n}\right\|_\infty\right).$$

Using the sub-Gaussianity of the matrices $X$, $Z$, and $\epsilon$, and Lemma 14, the two terms may be bounded as

$$\mathbb{P}\left[T_1 \geq c_0 \frac{\sigma_x^2}{(1 - \rho_{\text{max}})^2} \sqrt{\log p} \frac{\log p}{n}\right] \leq c_1 \exp(-c_2 \log p),$$

$$\mathbb{P}\left[T_2 \geq c_0 \frac{\sigma_x \sigma_\epsilon}{(1 - \rho_{\text{max}})} \sqrt{\log p} \frac{\log p}{n}\right] \leq c_1 \exp(-c_2 \log p).$$

(41a)

(41b)
Now consider the quantity \( \| (\hat{\Gamma} - \Sigma_x)\beta^* \|_\infty \). By a similar manipulation, we have

\[
\| (\hat{\Gamma} - \Sigma_x)\beta^* \|_\infty = \| (\frac{Z^T Z}{n} - \Sigma_x) \otimes M \beta^* \|_\infty \leq \frac{1}{(1 - \rho_{\text{max}})^2} \| (\frac{Z^T Z}{n} - \Sigma_x)\beta^* \|_\infty,
\]

so using Lemma 14 yields

\[
\| (\hat{\Gamma} - \Sigma_x)\beta^* \|_\infty \leq c_0 \frac{\sigma_x^2}{(1 - \rho_{\text{max}})^2} \sqrt{\frac{\log p}{n}},
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log p) \). Combining bounds (41) and (44), we conclude that the deviation conditions (17) both hold with parameter

\[
\varphi(Q, \sigma) = c_0 \frac{\sigma_x}{1 - \rho_{\text{max}}} \left( \frac{\sigma_x}{1 - \rho_{\text{max}}} + \sigma_\epsilon \right),
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log p) \), as claimed.

**Extension to unknown \( \rho_j \)**

We now consider the more challenging case when the missing probabilities \( \rho_j \) are unknown. Note that the estimates \( \hat{\rho}_j \) satisfy the deviation bound

\[
P\left( \max_j |\hat{\rho}_j - \rho_j| \geq t \right) \leq c_1 \exp(-c_2 nt^2 + \log p),
\]

by a Hoeffding bound for Bernoulli random variables, together with a union bound. In particular, taking \( t = c_0 \sqrt{\frac{\log p}{n}} \), we have

\[
\| \hat{\rho} - \rho \|_\infty \leq c_0 \sqrt{\frac{\log p}{n}},
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log p) \).

As long as \( n \gtrsim \frac{\log p}{(1 - \rho_{\text{max}})^2} \), as required by our results, the deviation condition (46) implies that \( |\hat{\rho}_j - \rho_j| \leq \frac{1 - \rho_{\text{max}}}{2} \) for each \( j \), so

\[
(1 - \hat{\rho}_j) \geq (1 - \rho_j) - \frac{1 - \rho_{\text{max}}}{2} \geq \frac{1}{2} (1 - \rho_j).
\]

In particular, we obtain the bound

\[
\max_j \left| \frac{1 - \rho_j}{1 - \hat{\rho}_j} - 1 \right| \leq \max_j \frac{|\rho_j - \hat{\rho}_j|}{(1 - \rho_{\text{max}})(1 - \hat{\rho}_j)} \leq \frac{2}{(1 - \rho_{\text{max}})^2} \max_j |\rho_j - \hat{\rho}_j|,
\]

and since \( \left| \frac{1 - \rho_j}{1 - \hat{\rho}_j} - 1 \right| \leq 1 \) by inequality (47), we also have

\[
\max_{i,j} \left| \frac{(1 - \rho_i)(1 - \rho_j)}{(1 - \hat{\rho}_i)(1 - \hat{\rho}_j)} - 1 \right| = \left| \frac{1 - \rho_i}{1 - \hat{\rho}_i} - 1 \right| \left( \frac{1 - \rho_j}{1 - \hat{\rho}_j} - 1 \right) \leq 3 \max_j \left| \frac{1 - \rho_j}{1 - \hat{\rho}_j} - 1 \right| \leq 6 \frac{\max_j |\rho_j - \hat{\rho}_j|}{(1 - \rho_{\text{max}})^2},
\]

26
using the triangle inequality and inequality (48).

We will use these bounds to verify the results of Lemmas 3 and 4 for the estimators (24).

For the deviation bounds, we begin by writing

$$
\|(\hat{\Gamma} - \Sigma_x)\beta^*\|_\infty \leq \|(\hat{\Gamma} - \hat{\Gamma})\beta^*\|_\infty + \|(\hat{\Gamma} - \Sigma_x)\beta^*\|_\infty.
$$

Note that we have already bounded $$\|(\hat{\Gamma} - \Sigma_x)\beta^*\|_\infty$$ in inequality (44). Furthermore,

$$
\|(\hat{\Gamma} - \Gamma)\beta^*\|_\infty = \|(\frac{Z^T Z}{n} \odot \tilde{M} - \frac{Z^T Z}{n} \odot M)\beta^*\|_\infty
$$

$$
= \|(\frac{Z^T Z}{n} \odot M) \odot (M \odot \tilde{M} - \mathbf{11}^T)\beta^*\|_\infty
$$

$$
\leq \|M \odot \tilde{M} - \mathbf{11}^T\|_{\max}\|(\frac{Z^T Z}{n} \odot M)\beta^*\|_\infty
$$

$$
\leq \frac{2}{(1 - \rho_{\max})^2} \max_j |\rho_j - \hat{\rho}_j| (\|(\hat{\Gamma} - \Sigma_x)\beta^*\|_\infty + \|\Sigma_x\beta^*\|_\infty),
$$

where we have used inequality (48) and the triangle inequality in the last inequality above.

Noting that $$\|\Sigma_x\beta^*\|_\infty \leq \lambda_{\max}(\Sigma_x)\|\beta^*\|_2 \leq c\sigma_x^2$$ and using the bounds (44) and (46), we obtain

$$
\|(\hat{\Gamma} - \Sigma_x)\beta^*\|_\infty \leq \frac{c\sigma_x^2}{(1 - \rho_{\max})^2} \sqrt{\frac{\log p}{n}}.
$$

Combining the pieces, we conclude that the deviation conditions (17) are satisfied with

$$
u(Q, \sigma) = \alpha_1 \frac{\sigma_x}{\rho_{\max}} \left(1 - \frac{\sigma_x}{\rho_{\max}} + \sigma_x\right),$$

as claimed.

For the RE conditions, we use a similar argument. Note that by Lemma 13, we need to show that $$|v^T (\hat{\Gamma} - \Sigma_x)v| \leq \frac{\lambda_{\min}(\Sigma_x)}{24}$$ for all $$v \in \mathbb{K}(2s)$$, with high probability. We write

$$
|v^T (\hat{\Gamma} - \Sigma_x)v| \leq |v^T (\hat{\Gamma} - \Gamma)v| + |v^T (\Gamma - \Sigma_x)v|,
$$

and note that we have already shown how to upper-bound $$|v^T (\hat{\Gamma} - \Gamma)v|$$ by $$c\lambda_{\min}(\Sigma_x)$$ with high probability. Furthermore,

$$
|v^T (\hat{\Gamma} - \Gamma)v| = |v^T (\frac{Z^T Z}{n} \odot \tilde{M} - \frac{Z^T Z}{n} \odot M)v|
$$

$$
= |v^T (\frac{Z^T Z}{n} \odot M) \odot (\tilde{M} \odot M - \mathbf{11}^T)v|
$$

$$
\leq \|\tilde{M} \odot M - \mathbf{11}^T\|_{\max}|v^T (\frac{Z^T Z}{n} \odot M)v|
$$

$$
\leq \|\tilde{M} \odot M - \mathbf{11}^T\|_{\max}(|v^T (\hat{\Gamma} - \Sigma_x)v| + |v^T \Sigma_xv|).
$$

Making use of inequality (49) and the concentration bound (45) with

$$
t = c\frac{\lambda_{\min}(\Sigma_x)}{\lambda_{\max}(\Sigma_x)} (1 - \rho_{\max})^2,
$$

we obtain

$$
\|\tilde{M} \odot M - \mathbf{11}^T\|_{\max}|v^T \Sigma_xv| \leq c\lambda_{\min}(\Sigma_x)
$$

with probability at least

$$
1 - c_1 \exp \left[ - c_2 n (1 - \rho_{\max})^4 \frac{\lambda_{\min}^2(\Sigma_x)}{\lambda_{\max}^2(\Sigma_x)} \right] \geq 1 - c_1 \exp \left[ - c_2 n (1 - \rho_{\max})^4 \frac{\lambda_{\min}^2(\Sigma_x)}{\sigma_x^4} \right].
$$

Note that $$t \leq c'$$, so the earlier upper bound on $$|v^T (\hat{\Gamma} - \Sigma_x)v|$$ is sufficient to ensure that

$$
|v^T (\hat{\Gamma} - \Sigma_x)v| \leq \frac{\lambda_{\min}(\Sigma_x)}{54}
$$

with the required probability.
5.5 Proof of Corollary 3

We now need to establish the RE conditions and deviation bounds (17) for the Gaussian VAR case, which we summarize in the following:

Lemma 5 (RE conditions, dependent case with missing data). Under the conditions of Corollary 3, there are universal positive constants $c_i$ such that $\hat{\Gamma}_{mis}$ satisfies the lower- and upper-RE conditions with

\[ \alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}, \quad \alpha_2 = \frac{3}{2} \lambda_{\max}(\Sigma_x), \]

and

\[ \tau(n, p) = c_0 \lambda_{\min}(\Sigma_x) \max \left( \frac{\zeta^4}{\lambda_{\min}^2(\Sigma_x)}, 1 \right) \frac{\log p}{n}, \]

with probability at least $1 - c_1 \exp \left( -c_2 n \min \left( \frac{\lambda_{\min}^2(\Sigma_x)}{\zeta^4}, 1 \right) \right)$.

Proof. The proof is identical to the proof of Lemma 1, except we use Lemma 18 instead of Lemma 15 in Appendix B. \qed

Lemma 6 (Deviation conditions, VAR with additive noise). Under the conditions of Corollary 3, there are universal positive constants $c_i$ such the deviation bounds (17) hold with parameter

\[ \varphi(Q, \sigma) = c_0 \zeta (\zeta + \sigma), \]

with probability at least $1 - c_1 \exp(-c_2 \log p)$.

Proof. We begin by bounding the term

\[ \| (\hat{\Gamma} - \Sigma_x)\beta^* \|_\infty = \max_{1 \leq j \leq p} \left| e_j^T \left( \frac{1}{n} Z^T Z - \Sigma_x \right) \beta^* \right|. \]

Define the function $\Phi(u, v) := u^T \left( \frac{1}{n} Z^T Z - \Sigma_x \right) v$ and rewrite the term as $\max_{1 \leq j \leq p} |\Phi(e_j, \beta^*)|$. For each fixed $j$, some simple algebra shows that

\[ \Phi(e_j, \beta^*) = \frac{1}{2} \left\{ \Phi(e_j, \beta^*, e_j + \beta^*) - \Phi(e_j, e_j) - \Phi(\beta^*, \beta^*) \right\}, \]

so it suffices to have a high-probability upper bound on the quantity $\Phi(v, v)$ for each fixed unit vector $v$. In particular, combining inequality (76) from Lemma 17 (see Appendix B) with the union bound and the relation (51), we conclude that

\[ \mathbb{P}[\| (\hat{\Gamma} - \Sigma_x)\beta^* \|_\infty \geq c_0 \zeta^2 \sqrt{\log p} \] \leq c_1 \exp(-c_2 \log p). \]

We now turn to the quantity $\| \hat{\gamma} - \Sigma_x\beta^* \|_\infty$, which by the triangle inequality may be upper-bounded as

\[ \| \hat{\gamma} - \Sigma_x\beta^* \|_\infty \leq \left\| \left( \frac{1}{n} Z^T Z - \Sigma_x \right) \beta^* \right\|_\infty + \left\| \left( \frac{1}{n} W^T W - \Sigma_w \right) \beta^* \right\|_\infty + \left\| \frac{1}{n} Z^T \epsilon \right\|_\infty + \left\| \frac{1}{n} X^T W \beta^* \right\|_\infty. \]

We have already bounded the first term in inequality (52) above. As for the second term, the matrix $W$ is sub-Gaussian, so that Lemma 14 can be used to control it (as in previous
arguments). In order to upper-bound the third term on the RHS of equation (53), we first condition on $Z$. Under this conditioning, the third term may be written as $\max_{\ell=1,\ldots,p} v_\ell$, where $v_\ell := \frac{1}{n} \langle Ze_\ell, \epsilon \rangle$ is a zero-mean Gaussian variable with variance at most $\frac{\sigma^2}{n} \left( \frac{\|Ze_\ell\|_2}{\sqrt{n}} \right)^2$. Combining the union bound and the deviation bound (76) with $t=1$, we conclude that as long as $n \gtrsim \log p$, then
\[ P\left[ \max_{\ell=1,\ldots,p} \left( \|Ze_\ell\|_2 \sqrt{\frac{n}{\sqrt{n}}} \right)^2 \geq c\zeta^2 \right] \leq c_1 \exp \left( -c_2 \log p \right). \]
Conditioning on this event and applying standard tail bounds to control $\{v_\ell\}$, we conclude that $\mathbb{P}[|v_\ell| \geq t] \leq \exp \left( -c_2 \sigma \zeta \sqrt{\log p} \right)$ and then taking a union bound over $\ell \in \{1,\ldots,p\}$ yields the desired result. A similar analysis can be used to bound the fourth term, since the matrices $X$ and $W$ are independent. Combining the pieces yields the claim.

5.6 Proof of Corollary 4

Lemma 7 (RE conditions, dependent case with missing data). Under the conditions of Corollary 4, there are universal positive constants $c_i$ such that $\hat{\Gamma}_{mis}$ satisfies the lower- and upper-RE conditions with $\alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}$, $\alpha_2 = \frac{3}{2} \lambda_{\max}(\Sigma_x)$, and
\[ \tau(n,p) = c_0 \lambda_{\min}(\Sigma_x) \max \left( \frac{\zeta^4}{\lambda_{\min}^2(\Sigma_x)}, 1 \right) \frac{\log p}{n}, \]
with probability at least $1 - c_1 \exp \left( -c_2 n \min \left( \frac{\lambda_{\min}(\Sigma_x)}{\zeta^4}, 1 \right) \right)$.

Proof. Again, we simply substitute the bound of Lemma 18 for the bound of Lemma 15 in the proof of Lemma 3.

The final step is verify the deviation bounds (17).

Lemma 8 (Deviation conditions, VAR with missing data). Under the conditions of Corollary 4, there are universal positive constants $c_i$ such the deviation bounds (17) hold with parameter
\[ \varphi(Q, \sigma_\epsilon) = c_0 \zeta' \left( \zeta' + \sigma_\epsilon \right), \]
with probability at least $1 - c_1 \exp(-c_2 \log p)$.

Proof. To control the term $\|\hat{\Gamma} - \Sigma_x\|_{\infty}$, we use the same argument as in Lemma 6 to obtain inequality (52). For the term $\|\hat{\gamma} - \Sigma_x\|_{\infty}$, we use the expansion (40) from the i.i.d. case. We show how to bound the terms $T_1$ and $T_2$ appearing in the expansion.

For a vector $v \in \mathbb{R}^p$, write $\Psi(v) = \frac{\|v\|^2}{n} - \mathbb{E}\left[ \frac{\|v\|^2}{n} \right]$, and note that
\[ T_1 = \max_j \frac{1}{2} \left[ \Psi(Ze_j + X\beta^*) - \Psi(Ze_j) - \Psi(X\beta^*) \right]. \] (54)

By Lemma 17, we may upper-bound the last term in equation (54) by $C \zeta'^2 (1 - \rho_{\max})^2 \sqrt{\log p}$, with probability at least $1 - c_1 \exp(-c_2 \log p)$. In order to bound the other two terms,
we again use Lemma 17. Note that $Z e_j$ is a mixture of Gaussians $N(0, Q_j)$, each with $\|Q_j\|_{\text{op}} \leq (1 - \rho_{\text{max}})^2 \zeta^2$. Then $Z e_j + X^* \beta^*$ is also a mixture of Gaussians $N(0, Q'_j)$, and we have the bound $\|Q'_j\|_{\text{op}} \leq 4 \zeta^2 (1 - \rho_{\text{max}})^2$. Hence, by Lemma 17 and a union bound, we conclude that $T_1 \leq c \zeta^2 (1 - \rho_{\text{max}})^2 \sqrt{\frac{\log p}{n}}$, with probability at least $1 - c_1 \exp(-c_2 \log p)$. Turning to term $T_2$ in the expansion (40), we condition on $Z$. Repeating the argument in Lemma 6, we obtain the bound

$$T_2 \leq c_0 \sigma \epsilon \zeta' (1 - \rho_{\text{max}}) \sqrt{\frac{\log p}{n}}.$$  

Finally, plugging back into inequality (40), we arrive at the bound

$$\|\bar{\gamma} - \Sigma \beta^*\|_\infty \leq c \zeta (1 - \rho_{\text{max}}) + \sigma \epsilon \zeta'.$$

Altogether, we have the form of $\varphi$ given by $\varphi(Q, \sigma) = c_0 \sigma \zeta + \zeta^2$, as claimed.

5.7 Proof of Corollary 5

First note that by Theorem 1, we have the bounds

$$\|\hat{\theta}^j - \theta^j\|_1 \leq \frac{c \varphi(Q, \sigma_\epsilon)}{\alpha_1} k \sqrt{\frac{\log p}{n}},$$

$$\|\hat{\theta}^j - \theta^j\|_2 \leq \frac{c \varphi(Q, \sigma_\epsilon)}{\alpha_1} \sqrt{\frac{k \log p}{n}}.$$  

We now establish the following lemma, which we will use to prove the theorem.

**Lemma 9.** For each $1 \leq j \leq p$, we have

$$\frac{1}{\lambda_{\text{max}}(\Sigma)} \leq |a_j| \leq \frac{1}{\lambda_{\text{min}}(\Sigma)} \quad \text{and} \quad \|\theta^j\|_2 \leq 1.$$  

**Proof.** Observe that $|a_j| \leq \max_j |\Theta_{jj}| \leq \lambda_{\text{max}}(\Theta) = \frac{1}{\lambda_{\text{min}}(\Sigma)}$, and similarly, $|a_j| \geq \frac{1}{\lambda_{\text{max}}(\Sigma)}$, which establishes the first inequality (57). Next, note that the rows (and also columns) of $\Theta$ are bounded in $\ell_2$-norm according to the inequality

$$\|\Theta_j\|_2 = \|\Theta e_j\|_2 \leq \lambda_{\text{max}}(\Theta) = \frac{1}{\lambda_{\text{min}}(\Sigma)},$$

which implies that $\|\theta^j\|_2 = \|\Theta_j / a_j\|_2 = \|\Theta_j/\| / |a_j| \leq 1$, as claimed.

Moving forward, we establish the following deviation inequalities between $a_j$ and $\Theta_j$ and their respective estimators.

**Lemma 10.** For all $j$, we have the following deviation inequalities:

$$|\hat{a}_j - a_j| \leq \frac{c \varphi(Q, \sigma_\epsilon)}{\lambda_{\text{min}}(\Sigma)} \left( \frac{\varphi(Q, \sigma_\epsilon)}{\lambda_{\text{min}}(\Sigma)} + \frac{\varphi(Q, \sigma_\epsilon)}{\alpha_1} \right) \sqrt{\frac{k \log p}{n}},$$

$$\|\hat{\Theta}_j - \Theta_j\|_1 \leq \frac{c \zeta^2(\Sigma)}{\lambda_{\text{min}}(\Sigma)} \left( \frac{\varphi(Q, \sigma_\epsilon)}{\lambda_{\text{min}}(\Sigma)} + \frac{\varphi(Q, \sigma_\epsilon)}{\alpha_1} \right) \sqrt{\frac{\log p}{n}}.$$  

\[30\]
Proof. We first derive inequality (58). Since the columns of $\Theta$ are $k$-sparse, $\|\theta^j\|_1 \leq \sqrt{k} \|\theta^j\|_2$. Then
\[
\|\hat{\theta}^j\|_1 \leq \|\theta^j\|_1 + \|\hat{\theta}^j - \theta^j\|_1 \leq c\sqrt{k} (\|\theta^j\|_2 + \frac{\varphi(Q,\sigma)}{\alpha_1} \sqrt{\frac{k \log p}{n}}) \leq c\sqrt{k} (\kappa(\Sigma) + \frac{\varphi(Q,\sigma)}{\alpha_1} \sqrt{\frac{\log p}{n}}),
\]
where we have used Lemma 9 and inequality (55). Under the assumed scaling $n \gtrsim k \log p$, this simplifies to the inequality
\[
\|\hat{\theta}^j\|_1 \leq c(k) \sqrt{k}.
\]
We now have
\[
|\bar{a}_j^{-1} - a_j^{-1}| = |(\hat{\Sigma}_{jj} - \Sigma_{jj}) - (\Sigma_{jj} - \Sigma_{jj} \theta^j)| 
\leq |\hat{\Sigma}_{jj} - \Sigma_{jj}| + |\Sigma_{jj} - \Sigma_{jj} \theta^j|.
\]
Using inequality (28), we have $T_1 \leq c\varphi(Q,\sigma) \sqrt{\frac{\log p}{n}}$. Furthermore,
\[
T_2 \leq |(\hat{\Sigma}_{jj} - \Sigma_{jj} \theta^j) - (\Sigma_{jj} - \Sigma_{jj} \theta^j)| 
\leq \|\Sigma - \Sigma\|_{\max} \|\hat{\theta}^j\|_1 + \|\Sigma_{jj}\|_2 \|\hat{\theta}^j - \theta^j\|_2
\leq c(\varphi(Q,\sigma_\epsilon) \kappa(\Sigma) + \lambda_{\max}(\Sigma) \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1}) \sqrt{\frac{k \log p}{n}},
\]
using inequality (60) and inequality (56). Substituting back into inequality (61), we obtain
\[
|\bar{a}_j^{-1} - a_j^{-1}| \leq c(\varphi(Q,\sigma_\epsilon) \kappa(\Sigma) + \lambda_{\max}(\Sigma) \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1}) \sqrt{\frac{k \log p}{n}}
\]
for all $j$. Hence,
\[
\left| \frac{a_i}{\bar{a}_i} - 1 \right| = |a_i| |\bar{a}_i^{-1} - a_i^{-1}| \leq c\kappa(\Sigma) \left( \frac{\varphi(Q,\sigma_\epsilon)}{\lambda_{\min}(\Sigma)} + \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1} \right) \sqrt{\frac{k \log p}{n}},
\]
using Lemma 9, so $|\bar{a}_j| \leq 2a_j$ for $n \gtrsim k \log p$, and
\[
|\hat{a}_j - a_j| = |\bar{a}_j| \left| \frac{a_j}{\hat{a}_j} - 1 \right| \leq \frac{c\kappa(\Sigma)}{\lambda_{\min}(\Sigma)} \left( \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1} \right) \sqrt{\frac{k \log p}{n}},
\]
which establishes inequality (58).

Turning to inequality (59), we have
\[
\|\tilde{\Theta}^j - \Theta^j\|_1 = |\bar{a}_j - a_j| + |\hat{a}_j \hat{\theta}^j - a_j \theta^j|_1
\leq |\bar{a}_j - a_j| + |a_j| \|\hat{\theta}^j - \theta^j\|_1 + |\hat{a}_j - a_j| \|\hat{\theta}^j\|_1
\leq c \left( \frac{1}{\lambda_{\min}(\Sigma)} \right) \left( \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1} + \frac{\kappa^2(\Sigma)}{\lambda_{\min}(\Sigma)} \left( \frac{\varphi(Q,\sigma_\epsilon)}{\lambda_{\min}(\Sigma)} + \frac{\varphi(Q,\sigma_\epsilon)}{\alpha_1} \right) \right) k \sqrt{\frac{\log p}{n}},
\]
by a combination of inequalities (55), (58), (60), and (62). Noting that $\kappa(\Sigma) > 1$, we arrive at inequality (59). \qed
Returning to the proof of Corollary 5, observe that since \( \hat{\Theta} \) and \( \Theta \) are symmetric, we have \( \|\hat{\Theta} - \Theta\|_\infty \leq \|\hat{\Theta} - \Theta\|_1 \). Furthermore, by the triangle inequality and the definition of \( \hat{\Theta} \),

\[
\|\hat{\Theta} - \Theta\|_1 \leq \|\hat{\Theta} - \tilde{\Theta}\|_1 + \|\tilde{\Theta} - \Theta\|_1 \leq 2\|\hat{\Theta} - \Theta\|_1 = 2 \max_j \|\hat{\Theta}_{-j} - \Theta_{-j}\|_1,
\]

so that the union bound and inequality (59) yield the claim.

6 Discussion

In this paper, we formulated an \( \ell_1 \)-constrained minimization problem for sparse linear regression on corrupted data. The source of corruption may be additive noise or missing data, and although the resulting objective is not generally convex, we showed that projected gradient descent is guaranteed to converge to a point within statistical precision of the optimum. In addition, we established \( \ell_1 \)- and \( \ell_2 \)-error bounds that hold with high probability when the data are drawn i.i.d. from a sub-Gaussian distribution, or drawn from a Gaussian vector autoregressive process. Finally, we used our results on linear regression to perform sparse inverse covariance estimation for a Gaussian graphical model, where the data are observed subject to corruption. The bounds we obtain for the spectral norm of the error are of the same order as existing bounds for inverse covariance matrix estimation when the data are uncorrupted and i.i.d.

Future directions of research include studying more general types of dependencies or corruption in the covariates of regression, such as more general types of multiplicative noise; and performing sparse linear regression for corrupted data with additive noise when the noise covariance is unknown and replicates of the data may be unavailable. As pointed out by a reviewer, it would also be interesting to study the performance of our algorithms on data that are not sub-Gaussian, or even under model mismatch. In addition, one might consider other loss functions, where it is more difficult to correct the objective for corrupted covariates. Finally, it remains to be seen whether or not our techniques—used here to show that certain non-convex problems can solved to statistical precision—can be applied more broadly.

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A Restricted eigenvalue conditions

In this appendix, we provide the proofs for various lemmas used to establish restricted eigenvalue conditions for different classes of random matrices, depending on the observation model. We begin by establishing two auxiliary lemmas, and then proceed to the main lemma used directly in the proofs of the corollaries. Our first result shows how to bound the intersection of the \( \ell_1 \)-ball with the \( \ell_2 \)-ball in terms of a simpler set.

Lemma 11. For any constant \( s \geq 1 \), we have

\[
B_1(\sqrt{s}) \cap B_2(1) \subseteq 3 \text{ cl}\{\text{conv} \{B_0(s) \cap B_2(1)\}\},
\]

(63)

32
where the balls are taken in $p$-dimensional space, and $\text{cl}\{\cdot\}$ and $\text{conv}\{\cdot\}$ denote the topological closure and convex hull, respectively.

Proof. Note that when $s > p$, the containment is trivial, since the right-hand set equals $\mathbb{B}_2(3)$, and the left-hand set is contained in $\mathbb{B}_2(1)$. Hence, we will assume $1 \leq s \leq p$.

Let $A, B \subseteq \mathbb{R}^p$ be closed convex sets, with support function given by $\phi_A(z) = \sup_{\theta \in A} \langle \theta, z \rangle$ and $\phi_B$ similarly defined. It is a well-known fact that $\phi_A(z) \leq \phi_B(z)$ if and only if $A \subseteq B$ (cf. Theorem 2.3.1 of Hug and Weil [6]). We now check this condition for the pair of sets $A = \mathbb{B}_1(\sqrt{s}) \cap \mathbb{B}_2(1)$ and $B = 3 \text{ cl} \{ \text{ conv } (\mathbb{B}_0(s) \cap \mathbb{B}_2(1)) \}$.

For any $z \in \mathbb{R}^p$, let $S \subseteq \{1, 2, \ldots, p\}$ be the subset that indexes the top $\lfloor s \rfloor$ elements of $z$ in absolute value. Then $\|z_{S^c}\|_{\infty} \leq |z_j|$ for all $j \in S$, whence

$$\|z_{S^c}\|_{\infty} \leq \frac{1}{\lfloor s \rfloor} \|z_S\|_1 \leq \frac{1}{\sqrt{\lfloor s \rfloor}} \|z_S\|_2.$$  \hfill (64)

We now split the supremum over $A$ into two parts, corresponding to the elements indexed by $S$ and its complement $S^c$, thereby obtaining

$$\phi_A(z) = \sup_{\theta \in A} \langle \theta, z \rangle \leq \sup_{\|\theta_S\|_2 \leq 1} \langle \theta_S, z_S \rangle + \sup_{\|\theta_{S^c}\|_1 \leq \sqrt{s}} \langle \theta_{S^c}, z_{S^c} \rangle \leq \|z_S\|_2 + \sqrt{s} \|z_{S^c}\|_{\infty} \leq \left(1 + \sqrt{\frac{s}{\lfloor s \rfloor}}\right) \|z_S\|_2 \leq 3 \|z_S\|_2,$$

where step (i) makes use of inequality (64). Finally, we recognize that

$$\phi_B(z) = \sup_{\theta \in B} \langle \theta, z \rangle = 3 \max_{|U| = \lfloor s \rfloor} \sup_{\|\theta_U\|_2 \leq 1} \langle \theta_U, z_U \rangle = 3 \|z_S\|_2,$$

from which the claim follows. \hfill $\square$

For ease of notation, define the sparse set $K(s) := \mathbb{B}_0(s) \cap \mathbb{B}_2(1)$ and the cone set $C(s) := \{ v : \|v\|_1 \leq \sqrt{s} \|v\|_2 \}$. Our next result builds on Lemma 11, showing how to control deviations uniformly over vectors in $\mathbb{R}^p$.

**Lemma 12.** For a fixed matrix $\Gamma \in \mathbb{R}^{p \times p}$, parameter $s \geq 1$, and tolerance $\delta > 0$, suppose we have the deviation condition

$$|v^T \Gamma v| \leq \delta \quad \forall v \in K(2s).$$  \hfill (65)

Then

$$|v^T \Gamma v| \leq 27 \delta (\|v\|_2^2 + \frac{1}{s} \|v\|_1^2) \quad \forall v \in \mathbb{R}^p.$$  \hfill (66)

Proof. We begin by establishing the inequalities

$$|v^T \Gamma v| \leq 27 \delta \|v\|_2^2 \quad \forall v \in C(s),$$  \hfill (67a)

$$|v^T \Gamma v| \leq \frac{27 \delta}{s} \|v\|_1^2 \quad \forall v \not\in C(s).$$  \hfill (67b)

Inequality (66) then follows immediately.
By rescaling, inequality (67a) follows if we can show that

\[ |v^T \Gamma v| \leq 27 \delta \quad \text{for all } v \text{ such that } \|v\|_2 = 1 \text{ and } \|v\|_1 \leq \sqrt{3}. \]  

(68)

By Lemma 11 and continuity, we further reduce the problem to proving the bound (68) for all vectors \( v \in 3 \text{conv } \{K(s) \} = \text{conv } \{B_0(s) \cap B_2(3) \} \). Consider a weighted linear combination of the form \( v = \sum_i \alpha_i v_i \), with weights \( \alpha_i \geq 0 \) such that \( \sum_i \alpha_i = 1 \), and \( \|v_i\|_0 \leq s \) and \( \|v_i\|_2 \leq 3 \) for each \( i \). Expanding, we can write

\[ v^T \Gamma v = \left( \sum_i \alpha_i v_i \right)^T \Gamma \left( \sum_i \alpha_i v_i \right) = \sum_{i,j} \alpha_i \alpha_j (v_i^T \Gamma v_j). \]

Applying inequality (65) to the vectors \( \frac{1}{3} v_i \), \( \frac{1}{3} v_j \), and \( \frac{1}{6} (v_i + v_j) \), we have

\[ |v_i^T \Gamma v_j| = \frac{1}{2} \| (v_i + v_j)^T \Gamma (v_i + v_j) - v_i^T \Gamma v_i - v_j^T \Gamma v_j | \leq \frac{1}{2} (36 \delta + 9 \delta + 9 \delta) = 27 \delta \]

for all \( i, j \), and hence \( |v^T \Gamma v| \leq \sum_{i,j} \alpha_i \alpha_j (27 \delta) = 27 \delta \|v\|_2^2 = 27 \delta \), establishing inequality (67a).

Turning to inequality (67b), first note that for \( v \notin \mathcal{C}(s) \), we have

\[ \frac{|v^T \Gamma v|}{\|v\|_1^2} \leq \frac{1}{s} \sup_{\|u\| \leq 1, \|u\|_2 \leq 1} |u^T \Gamma u| \leq \frac{27 \delta}{s}, \]  

(69)

where the first inequality follows by the substitution \( u = \sqrt{s \frac{v}{\|v\|_2^2}} \), and the second follows by the same argument used to establish inequality (67a), since \( u \) is in the set appearing in Lemma 11. Rearranging inequality (69) yields inequality (67b).

\[ \square \]

**Lemma 13** (RE conditions). Suppose \( s \geq 1 \) and \( \hat{\Gamma} \) is an estimator of \( \Sigma_x \) satisfying the deviation condition

\[ |v^T (\hat{\Gamma} - \Sigma_x) v| \leq \frac{\lambda_{\min}(\Sigma_x)}{54} \quad \forall v \in K(2s). \]

Then we have the lower-RE condition

\[ v^T \hat{\Gamma} v \geq \frac{\lambda_{\min}(\Sigma_x)}{2} \|v\|_2^2 - \frac{\lambda_{\min}(\Sigma_x)}{2s} \|v\|_1^2 \]  

(70)

and the upper-RE condition

\[ v^T \hat{\Gamma} v \leq \frac{3}{2} \lambda_{\max}(\Sigma_x) \|v\|_2^2 + \frac{\lambda_{\min}(\Sigma_x)}{2s} \|v\|_1^2. \]  

(71)

\[ \square \]

**Proof.** This result follows easily from Lemma 12. Setting \( \Gamma = \hat{\Gamma} - \Sigma_x \) and \( \delta = \frac{\lambda_{\min}(\Sigma_x)}{54} \), we have the bound

\[ |v^T (\hat{\Gamma} - \Sigma_x) v| \leq \frac{\lambda_{\min}(\Sigma_x)}{2} (\|v\|_2^2 + \frac{1}{s} \|v\|_1^2). \]

Then

\[ v^T \hat{\Gamma} v \geq v^T \Sigma_x v - \frac{\lambda_{\min}(\Sigma_x)}{2} (\|v\|_2^2 + \frac{1}{s} \|v\|_1^2) \quad \text{and} \]

\[ v^T \hat{\Gamma} v \leq v^T \Sigma_x v + \frac{\lambda_{\min}(\Sigma_x)}{2} (\|v\|_2^2 + \frac{1}{s} \|v\|_1^2), \]

so the inequalities follow from \( \lambda_{\min}(\Sigma_x) \|v\|_2^2 \leq v^T \Sigma_x v \leq \lambda_{\max}(\Sigma_x) \|v\|_2^2 \).

\[ \square \]
B Deviation bounds

In this appendix, we state and prove some deviation bounds for various types of random matrices.

B.1 Bounds in the i.i.d. setting

Given a zero-mean random variable $Y$, we refer to the quantity $\|Y\|_{\psi_1} := \sup_{\ell \geq 1} \ell^{-1}(\mathbb{E}|Y|^{\ell})^{1/\ell}$ as its sub-exponential parameter. The finiteness of this quantity guarantees existence of all moments, and hence large-deviation bounds of the Bernstein type.

By Lemma 14 of Vershynin [24], if $X$ is a zero-mean sub-Gaussian random variable with parameter $\sigma$, then the random variable $Y = X^2 - \mathbb{E}(X^2)$ is sub-exponential with $\|Y\|_{\psi_1} \leq 2\sigma^2$. It then follows that if $X_1, \ldots, X_n$ are zero-mean i.i.d. sub-Gaussian variables, we have the deviation inequality

$$
\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \mathbb{E}[X_i^2] \geq t \right] \leq 2 \exp \left( -c \min \left( \frac{nt^2}{4\sigma^4}, \frac{nt}{2\sigma^2} \right) \right) \quad \text{for all } t > 0,
$$

where $c > 0$ is a universal constant (see Proposition 16 in Vershynin [24]). This deviation bound may be used to establish the following useful result:

**Lemma 14.** If $X \in \mathbb{R}^{n \times p_1}$ is a zero-mean sub-Gaussian matrix with parameters $(\Sigma_x, \sigma^2_x)$, then for any fixed (unit) vector $v \in \mathbb{R}^{p_1}$, we have

$$
\mathbb{P}\left[ \|Xv\|_2^2 - \mathbb{E}[\|Xv\|_2^2] \geq nt \right] \leq 2 \exp \left( -c n \min \left( \frac{t^2}{\sigma_x^4}, \frac{t}{\sigma_x^2} \right) \right). \quad (72)
$$

Moreover, if $Y \in \mathbb{R}^{n \times p_2}$ is a zero-mean sub-Gaussian matrix with parameters $(\Sigma_y, \sigma^2_y)$, then

$$
\mathbb{P}\left( \left\| \frac{Y^T X}{n} - \text{cov}(y_i, x_i) \right\|_{\max} \geq t \right) \leq 6p_1p_2 \exp \left( -c n \min \left( \frac{t^2}{(\sigma_x\sigma_y)^2}, \frac{t}{\sigma_x\sigma_y} \right) \right), \quad (73)
$$

where $X_i$ and $Y_i$ are the $i^{th}$ rows of $X$ and $Y$, respectively. In particular, if $n \gtrsim \log p$, then

$$
\mathbb{P}\left( \left\| \frac{Y^T X}{n} - \text{cov}(y_i, x_i) \right\|_{\max} \geq c_0 \sigma_x \sigma_y \sqrt{\frac{\log p}{n}} \right) \leq c_1 \exp(-c_2 \log p). \quad (74)
$$

**Proof.** Inequality (72) follows from the above discussion and the fact that $Xv$ is a vector of i.i.d. sub-Gaussians with parameter $\sigma$. In order to prove inequality (73), we first note that if $Z$ is a zero-mean sub-Gaussian variable with parameter $\sigma_z$, then the rescaled variable $Z/\sigma_z$ is sub-Gaussian with parameter 1. Consequently, we may assume that $\sigma_x = \sigma_y = 1$ without loss of generality, rescaling as necessary. We then observe that

$$
e_i^T \left\{ \frac{Y^T X}{n} - \text{cov}(y_i, x_i) \right\} e_j = \frac{1}{2} \left[ \Phi(Xe_j + Ye_i) - \Phi(Xe_j) - \Phi(Ye_i) \right],
$$

where we have defined $\Phi(v) := \frac{\|v\|_2^2}{n} - \mathbb{E}\left( \frac{\|v\|_2^2}{n} \right)$. Since $Xe_j + Ye_i$ is sub-Gaussian with parameter at most 4, we may apply inequality (72) to each of the three terms, to obtain

$$
\left| e_i^T \left( \frac{Y^T X}{n} - \text{cov}(y_i, x_i) \right) e_j \right| \leq \frac{3t}{2}
$$
with probability at least \(1 - 6 \exp(-cn \min\{t^2, t\})\). Taking a union bound over all \(1 \leq i \leq p_1\) and \(1 \leq j \leq p_2\) yields inequality (73). Finally, setting \(t = c_0 \sigma_x \sigma_y \sqrt{\log p / n}\) and using the assumption \(n \gtrsim \log p\) yields inequality (74).

We combine this lemma with a discretization argument and union bound to obtain the next result. For a parameter \(s \geq 1\), recall the notation \(K(s) := \{v \in \mathbb{R}^p : \|v\|_2 \leq 1, \|v\|_0 \leq s\}\).

**Lemma 15.** If \(X \in \mathbb{R}^{n \times p}\) is a zero-mean sub-Gaussian matrix with parameters \((\Sigma, \sigma^2)\), then there is a universal constant \(c > 0\) such that

\[
\mathbb{P}\left( \sup_{v \in K(2s)} \left| \frac{\|Xv\|_2^2}{n} - \mathbb{E}\left[ \frac{\|Xv\|_2^2}{n} \right] \right| \geq t \right) \leq 2 \exp\left( -cn \min\left( \frac{t^2}{\sigma^4}, \frac{t}{\sigma^2} \right) + 2s \log p \right). \tag{75}
\]

**Proof.** Given \(U \subseteq \{1, \ldots, p\}\), define \(S_U = \{v \in \mathbb{R}^p : \|v\|_2 \leq 1, \text{supp}(v) \subseteq U\}\), and note that \(K(2s) = \bigcup_{|U| \leq 2s} S_U\). If \(A = \{u_1, \ldots, u_m\}\) is a \(1/3\)-cover of \(S_U\), then for every \(v \in S_U\), there is some \(u_i \in A\) such that \(\|\Delta v\|_2 \leq \frac{1}{3}\), where \(\Delta v = v - u_i\). It is known [10] that we can construct \(A\) with \(|A| \leq 9^s\). If we define \(\Phi(v, v) = v_1^T (\frac{X^T X}{n} - \Sigma) v_2\), we have

\[
\sup_{v \in S_U} |\Phi(v, v)| \leq \max_i |\Phi(u_i, u_i)| + 2 \sup_{v \in S_U} |\max_i \Phi(\Delta v, u_i)| + \sup_{v \in S_U} |\Phi(\Delta v, \Delta v)|.
\]

Since \(3\Delta v \in S_U\), it follows that

\[
\sup_{v \in S_U} |\Phi(v, v)| \leq \max_i |\Phi(u_i, u_i)| + \sup_{v \in S_U} \left( \frac{2}{3} |\Phi(v, v)| + \frac{1}{9} |\Phi(v, v)| \right),
\]

hence \(\sup_{v \in S_U} |\Phi(v, v)| \leq \frac{9}{2} \max_i |\Phi(u_i, u_i)|\). By Lemma 14 and a union bound, we obtain

\[
\mathbb{P}\left( \sup_{v \in S_U} \left| \frac{\|Xv\|_2^2}{n} - \mathbb{E}\left( \frac{\|Xv\|_2^2}{n} \right) \right| \geq t \right) \leq 9^s \cdot 2 \exp\left( -cn \min\left( \frac{t^2}{\sigma^4}, \frac{t}{\sigma^2} \right) \right).
\]

Finally, taking a union bound over the \(\binom{p}{\lfloor 2s \rfloor}\) \(p^s\) choices of \(U\) yields

\[
\mathbb{P}\left( \sup_{v \in K(2s)} \left| \frac{\|Xv\|_2^2}{n} - \mathbb{E}\left( \frac{\|Xv\|_2^2}{n} \right) \right| \geq t \right) \leq 2 \exp\left( -cn \min\left( \frac{t^2}{\sigma^4}, \frac{t}{\sigma^2} \right) + 2s \log p \right),
\]

as claimed. \( \square \)

**B.2 Bounds for autoregressive processes**

We base our analysis of Gaussian autoregressive matrices on the following lemma:

**Lemma 16.** Suppose \(Y \in \mathbb{R}^m\) is a mixture of multivariate Gaussians \(Y_j \sim N(0, Q_j)\), and let \(\sigma^2 = \sup_j \|Q_j\|_{op}\). Then for all \(t > \frac{2}{\sqrt{m}}\), we have

\[
\mathbb{P}\left[ \frac{1}{n} \|Y\|_2^2 - \mathbb{E}(\|Y\|_2^2) > 4t\sigma^2 \right] \leq 2 \exp\left( -\frac{m(t - \frac{2}{\sqrt{m}})^2}{2} \right) + 2 \exp(-m/2).
\]
Proof. This result is a generalization of Lemma I.2 in the paper [15]. By definition, the random vector \( Y \) is a mixture of random vectors of the form \( \sqrt{Q_j} X_j \), where \( X_j \sim N(0, I_n) \). For each index \( j \), the function \( f_j(x) = \| \sqrt{Q_j} x \|_2 / \sqrt{m} \) is Lipschitz with constant \( \| \sqrt{Q_j} \|_{op} / \sqrt{m} \). Since each \( X_j \) is Gaussian, it follows from the concentration for Lipschitz functions of Gaussians [9] that \( f_j(X_j) \) is a sub-Gaussian random variable with parameter \( \sigma_j^2 = \| Q_j \|_{op}/m \). Therefore, the mixture \( \| Y \|_2 / \sqrt{m} \) is sub-Gaussian with parameter \( \sigma^2 = \frac{1}{m} \sup_j \| Q_j \|_{op} \). The remainder of the proof proceeds as in the paper [15].

We now specialize the preceding lemma to the cases of additive noise and missing data appearing in our paper.

**Lemma 17.** Let \( X \in \mathbb{R}^{n \times p} \) be a Gaussian random matrix, with rows \( x_i \) generated according to a vector autoregression (25) with driving matrix \( A \). Let \( v \in \mathbb{R}^p \) be a fixed vector with unit norm. Then for all \( t > \frac{2}{\sqrt{n}} \),

\[
\mathbb{P}\left[ |v^T (\hat{\Sigma} - \Sigma_x) v| \geq 4t \zeta^2 \right] \leq 2 \exp\left( - \frac{n(t - \frac{2}{\sqrt{n}})^2}{2} \right) + 2 \exp(-n/2),
\]

where

\[
\zeta^2 := \left\{ \begin{array}{ll}
\frac{\| \Sigma_w \|_{op} + 2 \| \Sigma_x \|_{op}}{1 - \max_{1 \leq \ell \leq n} \| A \|_{op}} & \text{(additive noise case),} \\
\frac{\| \Sigma_w \|_{op} + 2 \| \Sigma_x \|_{op}}{1} & \text{(missing data case).}
\end{array} \right.
\]

**Proof.** First consider the additive noise case, where \( \hat{\Sigma} - \Sigma_x = \frac{Z^T Z}{n} - \Sigma_x \). For any fixed vector with \( \| v \|_2 = 1 \), the variable \( Z v \in \mathbb{R}^n \) is a zero-mean Gaussian random variable with covariance matrix, say \( Q \succeq 0 \). In order to apply Lemma 16, we need to upper-bound the spectral norm of \( Q \), which we do using the elementary upper bound \( \| Q \|_{op} \leq \max_{1 \leq i \leq n} \sum_{\ell=1}^n |Q_{i\ell}| \). For each pair \( i, \ell \in \{1, 2, \ldots, n\} \), we have

\[
|Q_{i\ell}| = |\text{cov}(e^T_i Z v, e^T_\ell Z v)| = |v^T \text{cov}(Z_i, Z_\ell) v|,
\]

where \( Z_i \) and \( Z_\ell \) are the \( i \)th and \( \ell \)th rows of \( Z \), and \( \| v \|_2 = 1 \). For \( i \neq \ell \), we have

\[
|v^T \text{cov}(Z_i, Z_\ell) v| = |v^T \text{cov}(X_i, X_\ell) v| = |v^T A^{i-\ell} \Sigma_x v| \leq \| \Sigma_x \|_{op} \| A \|_{op}^{i-\ell},
\]

and for \( i = \ell \), we have \( |v^T \text{cov}(Z^i, Z^i) v| \leq \| \Sigma_x \|_{op} \leq \| \Sigma_w \|_{op} + \| \Sigma_x \|_{op} \). Putting together the pieces, we conclude that \( \| Q \|_{op} \leq \zeta^2 \), with \( \zeta \) as defined in the lemma statement. Consequently, the bound (76) follows from Lemma 16.

In the missing data case, the variable \( Z v \) is a zero-mean mixture of Gaussians, conditioned on the positions of the missing data. Suppose \( Z^c \) is the random matrix \( Z \) corresponding to a given positioning scheme (with 0’s in the missing positions). We claim that

\[
\| Q_j \|_{op} \leq \frac{2 \| \Sigma_x \|_{op}}{1 - \| A \|_{op}},
\]

where \( Q_j = \text{Cov}(Z^c v) \). Indeed, we write \( \| Q_j \|_{op} \leq \max_i \sum_{\ell=1}^n |Q_{i\ell}| \), and for each pair \( (i, \ell) \),

\[
|Q_{i\ell}| = |\text{cov}(e^T_i Z^c v, e^T_\ell Z^c v)| = |\text{cov}(Z^c_i v, Z^c_\ell v)| = |\text{cov}(Z^c i v_1, Z^c_\ell v_2)|,
\]

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where \( v_1 \) and \( v_2 \) are the vector \( v \) with 0’s in the positions corresponding to the 0’s of \( Z^i \) and \( Z^\ell \), respectively. Since \( |\text{cov}(Z^i v_1, Z^\ell v_2)| \leq \|\Sigma_x\| \|A\|^{i-\ell} \) for \( i \neq \ell \) and

\[
|\text{cov}(Z^i v_1, Z^\ell v_2)| \leq \|\Sigma_w\|_{\text{op}} + \|\Sigma_x\|_{\text{op}}
\]

by a similar argument as before, the claim (77) follows. By the bounding technique (38) earlier in the paper, together with Lemma 16, we arrive at inequality (76).

\[\blacksquare\]

**Lemma 18.** Let \( X \) be a Gaussian matrix with rows generated from a vector autoregression with driving matrix \( A \). Let \( s \geq 1 \). Then for all \( t > \frac{2}{\sqrt{n}} \),

\[
P\left[ \sup_{v \in K(2s)} |v^T (\hat{\Gamma} - \Sigma_x) v| \geq 4t\zeta^2 \right] \leq 4 \exp\left( -cn \min\left( \left( t - \frac{2}{\sqrt{n}} \right)^2, 1 \right) + 2s \log p \right),
\]

(78)

with \( \zeta \) as defined in Lemma 17.

*Proof.* We use the single-deviation bounds from Lemma 17, together with a discretization argument identical to that of Lemma 15.

\[\blacksquare\]

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