Temporal evolution of one-dimensional fermion liquid with particle loss

Wei-Zhu Yi, 1, 2 Hao-Jie Lin, 2 Ze-Xun Lin, 3, 4 and Wei-Qiang Chen 1, 2, 5, *

1 Shenzhen Key Laboratory of Advanced Quantum Functional Materials and Devices, Southern University of Science and Technology, Shenzhen 518055, China
2 Department of Physics and Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China
3 Department of Physics, The University of Texas at Austin, Austin, TX 78712, USA
4 Department of Physics, Northeastern University, Boston, MA 02115, USA
5 International Quantum Academy, and Shenzhen Branch, Hefei National Laboratory, Futian District, Shenzhen, P. R. China

Intriguing phenomenon emerge with the sacrifice of Hermiticity in quantum systems, which can be complemented in open system. Open quantum systems behave non-trivially from closed systems especially in temporal evolution. In this work, we study the dynamical properties of a generic one-dimensional fermionic system with particle loss. The short-time behavior of the system is described by a non-Hermitian effective Hamiltonian, while the long-time dynamics is governed by Lindblad master equation. We show that the time-dependent von Neumann entropy with spatial bipartition of the system has universal behaviors hinging on the Liouvillian spectra. On account of thermalization, the entropy increases rapidly in short time when turning on the quantum jumps. Whether the Liouvillian gap closes or not affects the long-time decaying. The left-right asymmetry of quasiparticles in momentum space is observed as a result of non-reciprocal hopping in the effective Hamiltonian induced by correlated quantum-jump operators. This will also induce an interaction-strength-independent momentum-space entanglement in early time. The phenomenon in momentum-space share the same origin with the renowned non-Hermitian skin effect.

Introduction.— Open quantum system and non-Hermitian physics have become an increasingly attractive topic for research. In recent years, there are lots of important findings in this area, both theoretically[1–11] and experimentally[12–21]. Especially, in condensed matter physics, there have been lots of studies focusing on non-Hermitian skin effect[22–30], symmetry and topology of non-Hermitian quantum phases[22, 24, 27, 31–45], the quantum phase transition and quantum criticality of non-Hermitian systems[46–52], and the exceptional points in non-Hermitian system [28, 33, 34, 53–55]. However, most of them are based on the non-Hermitian band theory and short-time dynamics [6, 15, 56–60], which derives from a single-particle non-Hermitian Hamiltonian. Though the effective non-Hermitian Hamiltonian can describe certain effects in many cases, it fails to capture the long-time features of the open system due to its limitations. Therefore, a study of many-body non-Hermitian problem with the long-time dynamics taken into account is in demand. Generally speaking, the non-hermiticity of the effective Hamiltonian of an open system mainly comes from the coupling between the system and its environment. It can be contained in the self-energy of single-particle Green’s function resulting from the interactions, or equivalently it can be induced by coupling to a Markovian environment while neglecting quantum jumps or by postselection[61, 62]. Complementing this short-time effective description, the full dynamics is governed by the so-called Lindblad master equation[63, 64],

\[ \dot{\rho} = L[\rho] := -i[H_0, \rho] + \sum_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \left[ L_i^\dagger L_i, \rho \right] \right) \]  

(1)

where \( \rho \) is the density matrix of the system, \( L_i \)'s are jump operators originating from the coupling between the system and the environment. The effective non-Hermitian Hamiltonian expresses as \( H_{\text{eff}} = H_0 - \frac{i}{2} \sum_i L_i^\dagger L_i \) by ignoring \( L_i \rho L_i^\dagger \) terms.

Recent progress in quantum information provides some novel tools to study quantum many-body systems, among that the von Neumann entropy (the entanglement entropy) is the most powerful in detecting system’s universal properties, which is widely applied in multifarious fields[65–67]. In the case of zero-temperature gapped quantum system, the entanglement entropy follows the area law[66, 68, 69]. For critical systems, entanglement entropy has logarithmic scaling behavior[66, 70, 71]. Under temporal evolution, Von Neumann entropy also tracks the universal dynamical properties of correlations in various systems, including unitary[72–77] or non-unitary quantum quench[78, 79], and generic dissipative system[80–83] and generic stochastic [84–92] dynamics in isolated, time-periodic driven system[93–96], or non-equilibrium systems[90, 97–99].

In this letter, we study a generalized dimerized model whose coupling to the environment is described by dissipative quantum-jump terms. It is observed that the effective Hamiltonian shares particular information with the Liouvillian in terms of eigenspectrum. Via the solving of time-dependent correlation function, we show that long-time behavior is also highly related to the effective non-Hermitian Hamiltonian albeit not governed by it. Through time-dependent correlation matrix method we see the universal features can be reflected by system’s time-dependent entropy. We show that when the quantum jumps are turned on, the system’s entropy surges in a short time signaling it quickly thermalized. The entropy then decreases, which manifests the quantum dissipative effect. The decaying behavior depends not on whether
the effective Hamiltonian is gapped or not, but dominantly on whether the Liouvillian spectra of the system is gapped or gapless. Next, we move to momentum space, and show that its $k$-space left and right movers are asymmetric in dynamics, which serves as a reminiscent of the Non-Hermitian skin effect (NHSE). At the end, we also include a detailed discussion of the interaction-induced momentum-space entanglement between bosonic quasi-particles, which is intertwined with the left-right asymmetry. We believe our work serves as a guideline for future study of many-body problems in open quantum systems and dynamical non-Hermitian systems.

**The dissipative model.**– We consider a generic Hermitian dimerized chain with hopping parameters $iw$ and $iv$, where $w, v$ are real numbers, undergoing correlated decaying quantum jumps described by $L_{n,A} = \sqrt{\gamma_A}(c_{n,A} - c_{n,B})$ and $L_{n,B} = \sqrt{\gamma_B}(c_{n,B} - c_{n+1,A})$, as shown in Fig. 1. For simplicity, we consider only the case with $w = (1 - \eta)\nu, v = \eta, \gamma_A = (1 - 2\lambda)(1 - \eta),$ and $\gamma_B = (1 - 2\lambda)\eta$ with $0 < \lambda, \eta < 1$. The Liouvillian is thus written as

$$\mathcal{L}\rho = -i[H_{\text{eff}}, \rho]_{\text{ahl}} + \gamma_A \sum_n (c_{n,A} - c_{n,B})\rho(c_{n,A}^\dagger - c_{n,B}^\dagger) + \gamma_B \sum_n (c_{n+1,A} - c_{n,B})\rho(c_{n+1,A}^\dagger - c_{n,B}^\dagger),$$  

(2)

where $\rho$ denotes the (many-body) density matrix of the system [100] and $\gamma = \gamma_A + \gamma_B$. The non-Hermitian commutator $-i[H_{\text{eff}}, \rho]_{\text{ahl}} = -i \left( H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger \right)$. The effective Hamiltonian is given by

$$H_{\text{eff}} = H_{\text{hop}} + H_{\text{on-site}},$$

(3)

where $H_{\text{hop}} = \sum_{n=1}^{N-1} (it_1 c_{n+1,A}^\dagger c_{n,B} + it_2 c_{n,B}^\dagger c_{n+1,A}) + \sum_{n=1}^{N} (i\mu c_{n,B}^\dagger c_{n,A} + i\mu c_{n,A}^\dagger c_{n,B})$ is the non-reciprocal hopping part and $H_{\text{on-site}} = i\mu \sum_{n=1}^{N} (c_{n,A}^\dagger c_{n,A} + c_{n,B}^\dagger c_{n,B})$ is the on-site part with imaginary chemical potential. And the parameters are given by $t_1 = (1 - \eta)(1 - \lambda), t_2 = \eta(1 - \lambda); t_1' = \lambda(1 - \eta), t_2' = -\lambda\eta; \mu = (2\lambda - 1)/2$.

The effective Hamiltonian (3) gives the quasi-steady state and information of the short-time evolution of the system, while the Liouvillian (2) determines the long-time dynamics of this system. Due to its quadratic form, the Liouvillian spectrum can be obtained by diagonalizing $\mathcal{L}$ with the so-called third quantization [101] as $\mathcal{L} = \sum_n \lambda_n |n\rangle\langle n|$, where $|n\rangle, \langle n|$ are fermionic operators and $\lambda_n$ denotes the eigenvalues of excitations above the Fock vacuum of $\eta_L$. The Liouvillian super-operator acts on the (many-body) density matrix, therefore its full spectrum is a many-body version of eigenspectrum generated from single-particle eigenvalues $\lambda_L$.

For a non-interacting system, all the information has been encoded in the two-point correlation functions, which are defined as,

$$C_{ij}(t) = \text{Tr}[c_i^\dagger c_j \rho(t)]$$

(4)

where $i, j$ are index of lattice sites. The temporal evolution of the correlation function is determined by $C_{ij}(t) = \text{Tr}[c_i^\dagger c_j \rho(t)]$.
FIG. 2. (a) The “phase diagram” of the non-Hermitian Hamiltonian (3) as \( \lambda \) and \( \eta \) vary. The green area represents the real spectra gapped phase (line gapped in full spectra) and the pink area represents the real spectra gapless phase (point gapped in full spectra), they are separated by the cross lines, on which exceptional points exists. The \( \lambda = 1/2 \) yellow dash line corresponds to the Hermitian case. \( \lambda > 1/2(\lambda < 1/2) \) corresponds to a positive (negative) imaginary part spectrum. The brown dash line represents that the model reduces to single band. Therefore the exceptional point is fake. Representative points as denoted by different shapes on the diagram, their spectra are shown on the right panel. In the context, we mainly focus on (III), (I), (IV). (b) Full spectra of distinct phases: (I), (III) and (VI) are point-gapped phases; (II) is Hermitian gapped phase; (IV) is the line gapped phase; (V) is the exceptional phase. (c) The particle number density \( n_A(t) \) evolve with time in k-space, there is left-right asymmetry. (d) the real space particle number density \( n_A(0) \) elove with time under open boundary condition (OBC). It is a remanent of non-Hermitian skin effect (NHSE), also dubbed as chiral damping effect [29].

tems to a maximally entangled initial state as a ground state of Eq. (3), \( |\psi(0)\rangle_{AB} = \sqrt{\frac{1}{2}}|0\rangle_{AB} - \sqrt{\frac{1}{2}}|1\rangle_{AB} \), the time-dependent density matrix can be analytically obtained from the Lindblad master equation \( \rho_t = \exp(\mathcal{L}_{AB}t)\rho_0 \), such that

\[
\rho_t = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & e^{-4\gamma t} & 0 & 0 \\
0 & 0 & e^{-4\gamma t} & 0 \\
0 & 0 & 0 & 2 - 2e^{-4\gamma t}
\end{pmatrix}
\]

The reduced density matrix of A is \( \rho_{A,t} = \text{Tr}_B\rho_t = \text{Diag}\{\exp(-4\gamma t)/2, 1 - \exp(-4\gamma t)/2\} \). For a pure state, the quantum entanglement between bipartite parts A and B can be measured by the von Neumann entropy (entanglement entropy) \( S_{A(B)} = \text{Tr}\rho_{A(B)}\ln\rho_{A(B)} \). For a thermalized mixed state, classical information is included in von Neumann entropy. Thus, one can adopt the entanglement of formation [106, 107], \( \mathcal{E}_F(\rho_{\text{mixed}}) = \min \sum_i p_i S(\rho_{p_{\text{pure}}}) \) to measure the quantum entanglement, which counts the minimum number of singlets in demand to create an ensemble of pure states since singlets are basic unit of entanglement. The minimum is over all possible pure-state decompositions of \( \rho_{\text{mixed}} \) with probabilities \( p_i \). Unfortunately, it is difficult to calculate in general cases. In our case, the entanglement of formation (EoF) can be exactly obtained from the two-qubit concurrence \( \mathcal{C}(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \} \), where \( \lambda_i \) are square roots of the eigenvalues of the spin-flipped state matrix in decreasing order. The EoF is hence given by [106, 107] \( \mathcal{E}_F(\rho) = h(\frac{1+\sqrt{1-4\gamma}}{2}) \) [108]. We obtain \( \mathcal{C}(t) \sim e^{-4\gamma t} \), which coincides with the non-trivial eigenvalues of \( \rho_t \). The time-dependent EoF decreases monotonically as \( \mathcal{E}_F(t) \propto \gamma \exp(-8\gamma t) \).

Now we come back to the von Neumann entropy of the two-qubit mixed state \( S_{AB}(t) = \text{Tr}\rho_t\ln\rho_t = e^{-4\gamma t}[4\gamma t + (1 - e^{4\gamma t})\ln(1 - e^{-4\gamma t})] \). It grows from 0 to a maximum entanglement at \( t = \ln2/(4\gamma) \) as a result of thermalization decreases to 0 as \( S_{AB}(t) \propto \gamma \exp(-4\gamma t) \) after long-time evolution for particle loss effect (evaporization) predominates. It behaves slightly different from the entropy of subsystem A(B), which is monotonically decreasing and scaling as \( S_{A(B)}(t) \propto \gamma \exp(-4\gamma t) \) from \( \ln2 \) to 0. Another popular entanglement measurement is mutual information (MI). In such case, it is defined as \( I(A : B) = S_A + S_B - S_{AB} \). The calculated mutual information decreases monotonically \( I(A : B)_t \propto \gamma \exp(-4\gamma t) \), which is quite similar to the \( S_{A(B)}(t) \). As we can see from the results above, the information contained in EoF and MI data do not go beyond vNE in our case of study. For short-time dynamics, the classical information proliferates as a manifestation of rapid thermalization, in which the entropy can capture the physics while EoF ignores it. For long-time dynamics, the quantum effect manifests in both vNE, MI and EoF in similar form. Therefore, we only care about the vNE data with various time scales.

Then we consider an unentangled initial state \( |\psi(0)\rangle_{AB} = |1\rangle_1 \), the entanglement entropy at small t increases as \( S_A(t) \propto -\gamma \ln(\gamma t) \) since the quantum-jump operator \( L = \sqrt{\gamma}(c_A - c_B) \) correlates A and B and brings about entanglement (A is from a pure state to a mixed state), after an intermediate pe-
riod of decaying oscillation, the entanglement entropy finally decays to 0 as $S_A(t) \sim \gamma t \exp(-2\gamma t)$. The full process displays both thermalization and evaporation of the subsystem A.

Now we come to the real many-body problem. To calculate the entanglement entropy with spatial bipartition, we use the correlation matrix method developed in Ref.[109–111], since the wick theorem still holds for non-hermitian and open systems[112–115]. However, calculating it in Hilbert space is exponentially complexity in terms of computation. Considering the non-interacting nature and quadratic form of the Liouvillian, we can actually still take the single-particle picture to describe the many body problem and note that the single-particle density matrix is not trace normalizing to 1. The entanglement Hamiltonian is still written as quadratic form and hence can be simultaneously diagonalized. Thus, the reduced (biorthogonal[116]) density matrix is

$$\tilde{\rho}_t = \exp\left(\sum_\sigma c_\sigma(t) a_\sigma^\dagger a_\sigma\right)/Z_0,$$

where $Z_0$ is the partition function as well as the normalization factor with $Tr\tilde{\rho}_t = 1$.

We prepare the $t = 0$ ground state as the half-filling of real energy band. The von Neumann entropy is expressed by

$$S(t) = -\sum_\sigma [C_\sigma(t) \ln C_\sigma(t) - C_\sigma(t) \ln (Z_t \mathcal{E}_0 - C_\sigma(t))] + Z_t \mathcal{E}_0 \ln (1 - C_\sigma(0))$$

(7)

where $C_\sigma(t)$ are the eigenvalues of the reduced correlation matrix $\tilde{\rho}_t$, and $Z_t = Z_0 Tr\tilde{\rho}_0$.

We calculate numerically the entanglement entropy with a spatial partition of the observed system with total size $L$ to part A with length $l$ and part B with length $L - l$. At $t = 0$, for the gapped phases, the entropy $S = a_0$, as shown in Fig. 3(d), where $a_0$ is size-independent constant, and hence respects the area law. For the gapless phase, the entropy shown in Fig. 3(b)(c) fits with a subsystem size $l$ as $S = \alpha \ln (\sin(\pi l/L)) + c_0$ with $\alpha = 0.333$, which corresponds to a unitary conformal field theory (CFT) with central charge $c = 1$[117, 118]. The results above can be obtained via Toeplitz determinants[119], the vanishing $l$-linear dependence of the entanglement entropy originates from two point-like Fermi surface in 1d. And the logarithmic scaling is ascribed to the discontinuities of occupied states on the Fermi points. However, when we turn on the quantum jumps, there is no longer a well-defined Fermi surface as time evolves. Therefore we expect a $l$-linear-dependent entropy to appear as temporal evolution, which images the thermalization of the system.

We then consider the temporal evolution of $S$. We first consider an equal bipartition of the system. The time-dependent entropy in Fig. 3(a) shows a similar growth in short time, which fits as $S(t) \sim -tlnt$. This agrees with the expression of the short-time entropy increase of one-dimensional dissipative interacting fermions studied in Ref.[83]. It signals the rapid thermalization. However, they behave different in long time. In gapped Liouvillian spectrum cases, the entropy decays as $S(t) \sim t \exp(-\Lambda t)$ where $\Lambda$ being the Liouvillian gap. For the Liouvillian spectrum gapless cases, the long-time behavior of entropy is $S(t) \sim t^{-1/2}\ln t$. The long-time behaviors are originating from the particle vaporization. Then we calculated the $l$-dependent entropy and the results are shown in Fig. 3(b)-(d). It can be seen that the scaling behaviors are quite similar as $l$-linear form at long time, though the short-time ones are determined by the Hamiltonian.

To sum up, we can use $S(t) \sim a(t)\ln l + b(t)l + c(t)$ to cover the full-time scale behaviors of all cases, a careful fitting demonstrates that $a(t)$ and $c(t)$ decrease monotonically as $t^{-1/2}$ for gapless Liouvillian spectrum and $t^{-1/2}\ln t$ for gapped ones. The time dependence of $b(t)$ is similar as the one of $S(t)$. It increases as $-t\ln t$ at small time $t$ and decays as $b(t) \sim t^{-1/2}\ln t$ for gapless Liouvillian spectrum and $b(t) \sim t \exp(-\Lambda t)$ for gapped Liouvillian spectrum[120]. The decaying is a quantum effect due to particle losing, which is also reflected in the concurrence decaying for large $t$.

**Correlation in Momentum Space.**—By Fourier transforming the correlator (5) to momentum-space, one can immediately get the dynamics of quasi-particles in k-space, which is depicted in Fig. 2(d). It shows a left-right asymmetry, where the left-moving (right-moving) particle is identified with the sign of the group velocity $v = dE/dk$, $\text{sign}[v] = -1(+1)$ corresponds left-movers (right-movers). Such an asymmetry originates from the non-reciprocal hoppings and indicate the NHSE in OBC since particles will pile up at one side if there
are endpoints.

Now we consider the time evolution of a system in presence of both a density-density interaction and the particle loss. Assume the interaction and the particle loss is turned on simultaneously at $t = 0$, at short time $t$, the Fermi surface is still well-defined. Thus with the bosonization, the short time dynamics can be studied with the Luttinger liquid Hamiltonian

$$H = \sum_{q \neq 0} v(q)|b_q^\dagger b_q + i\frac{g_2}{2}|b_q b_{-q} + b_q^\dagger b_{-q}\rangle \langle b_q b_{-q} + b_q^\dagger b_{-q}|$$  \hspace{1cm} (8)$$

where $v(q)$ is the spectrum with velocity $v$, $b_q^\dagger = i\sqrt{1/|q|} \sum c_{k+q} c_k$ : create electron-hole pairs around the Fermi surface, $g_2$ is the interaction strength. After the bosonization, the quantum jump operator composes of single fermionic annihilation operator becomes an exponential of a linear combination of boson operators $\psi_R(x) \propto \exp \left( \sum_{n>0} (\alpha_n(x)b_b^R + \beta_n(x)b_b^R) \right)$. Since the Hamiltonian is quadratic, the nonunitary dynamics of $\rho$ is encoded in the momentum-space correlation function $C_{k+q,k}(t) = \text{Tr} \left[ c_{k+q} c_k \rho(t) \right]$. With Eq. (2), the equation of motion of $b_q(t)$ is given by $\partial_t b_q(t) = -4\gamma b_q + i(v(q)b_q - ig_2)\langle b_{-q}b_q^{\dagger}\rangle = -i\gamma v(q)b_{-q} - ig_2 b_{q}b_{-q}$. Though the full solution is very tedious and lengthy, if we focus on the short-time dynamics, we can express $b_q(t)$ and $b_{-q}(t)$ as $b_q(t) = \exp(i\gamma v(q)t - 4\gamma t)b_{q}^{\dagger} + ig_2 tb_{-q} + \cdots$ and $b_{-q}(t) = \exp(-i\gamma v(q)t)(1-ig_2)q^2b_{-q} - \frac{1}{2}g_2^2|q|^2r^2b_{-q} + \cdots$.

In general, the system cannot be equally bipartitioned into $q > 0$ and $q < 0$ in momentum space, because the $q > 0$ part decays exponentially, while the particle numbers at $q < 0$ are nearly unchanged. And the momentum-space entanglement entropy (MSEE) should have a overall decay factor as $\exp(-8\gamma t)$ for the summation of $q$. However, if the time $t$ is short enough such that the loss event does not happen, the density matrix should be renormalized. And the canonical commutation relation of $[b_q(t), b_{-q}^{\dagger}(t)] = 1$ can be satisfied by setting $b_q^\dagger(t) = u_q(t)b_q^\dagger + v_q(t)b_{-q}$ [121]. It is reasonable to assume $|u_q(t)|^2 + |v_q(t)|^2 = 1$ as the $\mathcal{PT}$-symmetric non-Hermitian case in Ref. [79]. Thereout, the short-time pseudo-Bogoliubov coefficients can be calculated as $u_q(t) \sim 1 + iv_q(t) - 4\gamma t$ and $v_q(t) \sim -2\sqrt{2r^2}t^{1/2}$. The $g_2$ dependence only appears in higher order terms. This will give an increase of MSEE with time as $S \sim -t \ln t (t \ll 1)$. This result suggests an interaction-independent system, which originates from the coherent nature of $L_q$. It also serves as a reflection of the correlation effect of the asymmetric hopping, which has the same origin with NHSE.

**Discussion.**– We explore the entanglement entropy dynamics of a one-dimensional quantum liquid in several cases. Though the von Neumann entropy contains information beyond quantum entanglement, it provides a better description on the universal behavior of temporal evolution of the system. The pure quantum entanglement spectrum can be analyzed via the logarithmic entanglement negativity[122, 123], however its computational complexity is very large. The many-body open quantum systems can be experimentally engineered in cold atomic optical lattice systems, which is fine tuned by atomic couplings and parallel inciding lasers. Alternatively, one can simulate the many-body Liouvillian dynamics on large-scale dissipative three-level superconducting qubit circuits via the control of driven frequencies and couplings between qudits[124], where the non-Hermitian ground state is prepared by quantum tomography[125] and entanglement can be measured in several protocols[126–129]. Other numeric or qudits quantum simulation methods for open systems are also applicable[130, 131].

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* chenwq@sustech.edu.cn

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