COMPLEXITY OF SPARSE POLYNOMIAL SOLVING 2: RENORMALIZATION

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Abstract. Renormalized homotopy continuation on toric varieties is introduced as a tool for solving sparse systems of polynomial equations, or sparse systems of exponential sums. The cost of continuation depends on a renormalized condition length, that is on a line integral of the renormalized condition number along all the lifted paths.

The theory developed in this paper leads to a continuation algorithm tracking all the solutions between two arbitrary systems of the same structure. The algorithm is randomized, in the sense that it follows a random path between the two systems. It can be modified to succeed with probability one. In order to produce an expected cost bound, several invariants depending solely of the supports of the equations are introduced here. For instance, the mixed surface is a quermassintegral that generalizes surface in the same way that mixed volume generalizes ordinary volume. The face gap measures how close is the set of supporting hyperplanes for a direction in the 0-fan from the nearest vertex. Once the supports are fixed, the expected cost depends on the input coefficients solely through two invariants: the renormalized toric condition number and the imbalance of the absolute values of the coefficients. This leads to a non-uniform complexity bound for polynomial solving in terms of those two invariants. Up to logarithms, it is quadratic in the first invariant and linear in the last one.

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List of Notations

The following convention applies: vectorial quantities are typeset in boldface (e.g. $f$) while scalar quantities are not (e.g. $f$). Multi-indices and vectors of coefficients of polynomials and exponential sums are treated as covectors (row vectors), as well as the momentum map.

$\mathcal{P}_A$ Space of Laurent polynomials with support $A$ 5

$A_i$ Support for the $i$-th equation 8

$V_A$ Veronese embedding for exponential sums with support $A$ 8

$V_{ia}$ $a$-th coordinate of the Veronese embedding for support $A_i \ni a$ 8

$\rho_{ia}$ Constant coefficient for $V_{ia}$ 8

$\mathcal{F}_A$ Space of exponential sums with support $A$ 8

$\mathcal{M}$ Domain of the main chart for the toric variety 9

$\Lambda$ Integer lattice spanned by the union of the sets $A_i - A_i$ 9

[·] Natural quotient or projection, e.g. in multi-projective space 11
\begin{align*}
V &\quad \text{The toric variety associated to } (A_1, \ldots, A_n) \\
A_i &\quad \text{Convex hull of } A_i \\
\mathbf{m}_i &\quad \text{The momentum map for the } i\text{-th support} \\
\delta_i &\quad \text{Radius of } A_i \\
\| \cdot \|_{i,\mathbf{x}} &\quad i\text{-th toric metric at the point } \mathbf{x} \in \mathcal{M} \\
\| \cdot \|_{\mathbf{x}} &\quad \text{Toric norm, } \| \cdot \|_{\mathbf{x}} = \sum_i \| \cdot \|_{i,\mathbf{x}} \\
\mu(f, \mathbf{x}) &\quad \text{Toric condition number of } f \text{ at } \mathbf{x} \in \mathcal{M} \\
M(f, \mathbf{x}) &\quad \text{Unscaled condition matrix for } f \text{ at } \mathbf{x} \\
\nu_i &\quad \text{Distortion invariant for the } i\text{-th support} \\
\nu &\quad \text{Distortion invariant} \\
S &\quad \text{Solution variety} \\
\alpha(f, \mathbf{z}) &\quad \text{Smale's alpha invariant for } f \text{ at } \mathbf{z} \\
\gamma(f, \mathbf{z}) &\quad \text{Smale's gamma invariant for } f \text{ at } \mathbf{z} \\
\beta(f, \mathbf{z}) &\quad \text{Smale's beta invariant for } f \text{ at } \mathbf{z} \\
N &\quad \text{Newton iteration} \\
\mathcal{L}(f_i, \mathbf{z}_i; a, b) &\quad \text{Condition length (non-renormalized) of the path } \\
&\quad \quad (f_i, \mathbf{z}_i)_{t \in [a, b]} \\
\mathcal{L}^*(f_i, \mathbf{z}_i; a, b) &\quad \text{Renormalized condition length for the same path} \\
R_i, R &\quad \text{Renormalization operator} \\
V = V(A_1, \ldots, A_n) &\quad \text{Mixed volume of the tuple } (A_1, \ldots, A_n) \\
V' = V'(A_1, \ldots, A_n) &\quad \text{Mixed surface of the tuple } (A_1, \ldots, A_n) \\
\kappa_{\rho_i} &\quad \text{Imbalance invariant for the coefficients } \rho_{i\mathbf{a}} \\
Z(\mathbf{q}) &\quad \text{Zero set of } \mathbf{q} \\
Z_H(\mathbf{q}) &\quad \text{Set of zeros of } \mathbf{q} \text{ with } \max(|z_i|) \leq H
\end{align*}
1. Introduction

Classical foundational results on polynomial system solving refer to the possibility of solving them by an algorithm such as elimination or homotopy. A theory capable to explain and predict the computational cost of solving polynomial systems over \( \mathbb{C} \) using homotopy algorithms was developed over the last thirty years (Smale, 1987; Kostlan, 1993; Shub, 1993; Shub and Smale, 1993a; 1993b; 1993c; 1996; 1994; Dedieu and Shub, 2000; Beltrán and Pardo, 2008; 2009; 2011; Shub, 2009; Beltrán and Shub, 2009; 2010; Beltrán et al., 2010; 2012; Beltrán, 2011; Bürgisser and Cucker, 2011; Dedieu et al., 2013; Armentano et al., 2016; Lairez, 2017; 2020). As explained in the books by Blum et al. (1998) and Bürgisser and Cucker (2013), most results in this theory were obtained through the use of unitary symmetry. This setting limited its reach to the realm of dense polynomial systems, or to multi-homogeneous ones.

This paper extends the theory of homotopy algorithms to more general sparse systems. A common misconception is to consider sparse systems as a particular case of dense systems, with some vanishing coefficients. This is not true from the algorithmic viewpoint. The vanishing coefficients introduce exponentially many artifact solutions. To see that, compare the classical Bézout bound to the mixed volume bound in Theorems 1.5.2 and 1.5.6 below.

A theory of homotopy algorithms featuring toric varieties as a replacement for the classical projective space was proposed by Malajovich (2019) in a previous attempt. Unfortunately, no clear complexity bound could be obtained independently of integrals along the homotopy path. Much stronger results are derived...
here through the introduction of another symmetry group, that I call renormalization. Essentially, renormalization lifts the algorithm domain from the toric variety to its tangent space. Before going further, it is necessary to explain the basic idea of renormalization and how it replaces unitary invariance.

1.1. Symmetry and renormalization. Solutions for systems of $n$ homogeneous polynomial equations in $n + 1$ variables are complex rays through the origin, so the natural solution locus is projective space $\mathbb{P}^n$. The unitary group $U(n + 1)$ acts transitively and isometrically on projective space, and this induces an action on the space $H_d$ of degree $d$ homogeneous polynomials.

A rotation $Q \in U(n + 1)$ acts on a polynomial $f$ by composition $f \circ Q^*$, so that every pair $(f, [X]) \in H_d \times \mathbb{P}^n$ with $f(X) = 0$ is mapped to the pair $(f \circ Q^*, [QX])$, and $(f \circ Q^*)(QX) = f(X) = 0$. For the correct choice of a Hermitian inner product in $H_d$, the group $U(n + 1)$ acts by isometries. As a consequence, all of the invariants used in the theory are $U(n + 1)$-invariants.

The canonical argument of dense polynomial solving goes as follows: suppose that one wants to prove a Lemma for some system $f = (f_1, \ldots, f_n)$ of polynomials $f_i \in H_{d_i}$, $d_i \in \mathbb{N}$, at some point $[X] \in \mathbb{P}^n$. Then one assumes without loss of generality that $X = [1 \ 0 \ \ldots \ 0]^T$. Intrincate lemmas become simple calculations.

Early tentatives to develop a complexity theory for solving sparse polynomial systems were hindered by the lack of a similar action (Malajovich and Rojas, 2002; 2004). For instance, the complexity bounds obtained by Malajovich (2019) depend on a condition length, which is the line integral along a path of solutions $(f_t, X_t)$ of the condition number, times a geometric distortion invariant $\nu(X_t)$. No bound on the expectation of this integral is known.

It is customary in the sparse case to look at roots $X \in \mathbb{C}^n$ with $X_i \neq 0$, that is on the multiplicative group $\mathbb{C}_\times^n$. The toric variety from equation (3) below is a convenient closure of $\mathbb{C}_\times^n$. The contributions in this paper stem from the action of the multiplicative group $\mathbb{C}_\times^n$ onto itself, and onto spaces of sparse polynomials. Each element $U \in \mathbb{C}_\times^n$ acts on $X \in \mathbb{C}_\times^n$ by componentwise multiplication. Let $A \subseteq \mathbb{Z}^n$ be finite, and let $[\mathcal{P}_A]$ be the set of Laurent polynomials of the form

$$F(Z) = \sum_{a \in A} f_a Z_1^{a_1} Z_2^{a_2} \cdots Z_n^{a_n}.$$ 

The element $U \in \mathbb{C}_\times^n$ acts by sending $F(Z)$ into $F(U^{-1}Z)$. This allows to send a pair $(F(\cdot), X)$ into the pair $(F(X^{-1} \cdot), 1)$ where $1 = (1 \ 1 \ \ldots \ 1)^T$ is the unit of $\mathbb{C}_\times^n$. One can also replace the unit of $\mathbb{C}_\times^n$ by an arbitrary point. This is the renormalization used in this paper.

The main results in this paper can now be stated informally. They will be formalized later, using logarithmic coordinates that make polynomials into exponential sums. While this last formulation is sharper and more elegant, we start with the primary results.
1.2. Main results.

1.2.1. Renormalization. The Renormalized Newton Iteration is constructed in Section 1.4 and Renormalized Homotopy in Definition 1.4.6. The computational cost of this homotopy, that is the number of renormalized Newton iteration steps, is bounded in Theorem [A] as proportional to a certain variational invariant, the renormalized condition length.

1.2.2. Expected condition. Let \( q \) be a random, Gaussian sparse polynomial system and let \( Z(q) \) denote the set of roots of \( q \) in the toric variety. In order to investigate the renormalized condition length, one would like to bound the average of the sum over \( Z(q) \) of the squared renormalized condition number. The bound obtained in this paper is more technical: Theorem B provides a conditional bound, only the roots away from ‘toric infinity’ are counted. The most troubling issue is that the upper bound does not depend solely on the mixed volume, but also on the mixed surface. This is another quermassintegral generalizing the surface of a convex body.

1.2.3. Toric infinity. Will the roots close to ‘toric infinity’ make the bound from Theorem B worthless? Theorem 1.6.5 establishes a perturbation bound in terms of the distance to the locus of sparse systems with solution at ‘toric infinity’. The degree of this locus is bounded in Theorem [C] In the particular case where the supports are general enough (strongly mixed supports), this degree is no larger than the number of 1-cones in the fan of the tuple of supports. Those two results can be used in Theorem 1.6.14 to bound the probability that a linear homotopy path fails fails the condition in Theorem B, that is the probability that it crosses the set of systems with at least one root close to ‘toric infinity’.

1.2.4. Expected condition length, conditional. Since this is an exploratory paper, we choose for simplicity a homotopy path of the form \( g + tf \), where \( g \) has iid Gaussian coefficients and \( f \) is fixed and outside a certain variety. If the supports are strongly mixed, the only requirement is that the coefficients of \( f \) are non-zero.

With probability one, this homotopy path lifts to \( n!V \) solution paths, where \( V \) is Minkowski’s mixed volume. The global cost of Renormalized Homotopy along this homotopy path, \( 0 \leq t \leq \infty \), is given by the sum over all the solution paths of the condition length from Theorem [A] Theorem [D] states that with probability at least 3/4, this sum of condition lengths is no more than a constant times

\[
Q n^{3/2} \max_i (S_i^{3/2}) K \left( K + \sqrt{1 + K^4/4} + \kappa \eta \sqrt[3]{8} \right) \kappa \mu_2 \nu_0 \\
\times \left( \log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_\kappa) + \log(\eta) \right)
\]

where

- The invariant \( Q \) depends solely of the tuple of supports: Let \( n \) be the dimension of the system, \( V \) be Minkowski’s mixed volume, \( V' \) the mixed surface, \( \delta_i \) the radius of each support, and \( \eta \) the surface gap, which measures the ‘quality’ of the tuple of supports. The lattice determinant \( \det \Lambda \geq 1 \) will be explained later. Then,

\[
Q \overset{\text{def}}{=} \eta^{-2} \left( \sum_{i=1}^n \delta_i^2 \right) \max(n!V, n - 1!V' \eta) / \det \Lambda.
\]
The precise definition of $\eta$ is postponed to equation [10] but the reader can think of $\eta^{-1} \sqrt{\sum_{i=1}^{n} \frac{1}{\delta_i^2}}$ as an adjusted, intrinsic ‘radius’ for the tuple. When all the supports are scaled by the same positive integer factor, neither of the product above nor $Q$ does change.

- $S_i$ is the size (number of points) of the support for the $i$-th equation, and $S = \sum_{i=1}^{n} S_i$ is the input size.
- $K = \left(1 + \sqrt{\frac{\log(n) + \log(10)}{\min(S_i)}}\right)$.
- The number $d_r$ is the degree of the locus of systems with a root at toric infinity.
- $\nu_0$ is the geometric invariant from [Malajovich, 2019], evaluated at one point that we take to be the origin.
- The bound depends on the coefficients of the target system $f$ solely through the renormalized condition number $\mu_f$ of the target system $f$, and through the invariant $\kappa_f$ that measures the imbalance between the absolute value of the coefficients.

1.2.5. The cost of homotopy. The result in Theorem [E] allows to solve a random system $g$, given the set of solutions of a suitable system $h$ with same support. It also allows to solve a suitable arbitrary system $f$ of same support, given the solutions of the random system $g$. In order to obtain a more decisive complexity bound, we consider the problem of finding the set of solutions of $f$ in terms of the set of solutions of $h$. The procedure goes through a random system $g$, in a manner akin to the Cheater’s Homotopy suggested by Li et al. (1989). The randomized algorithm in Theorem [F] will perform this task with probability one, and expected cost linear on

$$QnS^{3/2} \max_i(S_i^{3/2})K \left( K + \sqrt{1 + K/4 + \kappa/8} \right) \kappa(\mu_f^2 + \mu_h^2)\nu_0 \times \left( \log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_f) + \log(\mu_h) + \log(\kappa_f) \right)$$

where $\kappa = \max(\kappa_f, \kappa_h)$. In particular, once one convenient start system $h$ with small $\kappa_h$ and small condition number is known, we obtain a non-uniform complexity bound: the cost of solving a polynomial system $f$ with the same support as $h$ is is

$$O(\mu_f^2 \log(\mu_f) \kappa_f \log(\kappa_f)).$$

1.2.6. Organization of the paper. In the next subsection we revisit the notations, basic definitions and facts needed in the sequel. The main results are formally stated in the remaining subsections. The proof of the main statements is postponed to Sections [2] to [5]. The final section lists some open problems and other remaining issues.

1.3. Background: exponential sums, toric varieties and condition. This paper is built on top of the theory of Newton iteration and homotopy on toric varieties proposed by [Malajovich, 2019]. We review in this section the notations and results that are necessary to formally state the main theorems of this paper.

As in the previous work, logarithmic coordinates are used to represent polynomial roots, exact or approximate. Polynomials get replaced by exponential sums. For instance if $Z$ is a root of $F(X) = 0$,

$$F(X) = \sum_{\alpha \in A} f_\alpha X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$$
for a finite set \( A \subset \mathbb{Z}^n \), then we write
\[
f(x) = \sum_{a \in A} f_a e^{ax}, \quad x_i = \log X_i \quad \text{and} \quad z_i = \log Z_i
\]
so that \( F = f \circ \exp \) and \( f(z) = 0 \).

Malajovich (2019) considered the action of the additive group \( \mathbb{R}^n \) by shifting supports. This leads us to consider more general exponential sums, where we cannot assume any more that \( A \subset \mathbb{Z}^n \). We assume instead that \( A - A = \{ a - a', a, a' \in A \} \) is a finite subset of \( \mathbb{Z}^n \). It obviously contains the origin.

We will actually deal with \textit{systems} of equations with possibly different supports. Suppose that we are given finite sets \( A_1, \ldots, A_n \subset \mathbb{R}^n \) such that \( A_i - A_i \subset \mathbb{Z}^n \). To each \( a \in A_i \) we associate a function
\[
V_{ia} : \mathbb{C}^n \longrightarrow \mathbb{C} \quad x \longrightarrow V_{ia}(x) = \rho_{ia} e^{ax}
\]
where \( \rho_{ia} > 0 \) is a fixed real number. We denote by \( \mathcal{F}_{A_i} \) the complex vector space of exponential sums of the form
\[
f : \mathbb{C}^n \longrightarrow \mathbb{C} \quad x \longrightarrow \sum_{a \in A_i} f_a V_{ia}(x)
\]
Solving systems of sparse polynomial systems with support \((A_1, \ldots, A_n)\) is equivalent to solving systems of exponential sums in \( \mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n} \). The following conventions apply: the inner product on \( \mathcal{F}_{A_i} \) is the inner product that makes the basis \((V_{ia})_{a \in A_i}\) orthonormal. Objects in \( \mathcal{F}_{A_i} \) will be represented in coordinates as ‘row vectors’ \( f = (\ldots f_{ia} \ldots)_{a \in A_i} \) and objects in \( \mathcal{F}_{A_i}^\dagger \) will be represented as column vectors. We denote by \( V_{A_i} \) the vector valued \textit{Veronese} map
\[
V_{A_i} : \mathbb{C}^n \longrightarrow \mathcal{F}_{A_i}^\dagger = \mathbb{C}^{\# A_i} \quad x \longrightarrow V_{A_i}(x) = \left( \begin{array}{c} \vdots \\ V_{ia}(x) \\ \vdots \\ \vdots \end{array} \right)_{a \in A_i}.
\]
Then evaluation of \( f_i \) at \( x \) is given by the pairing
\[
f_i(x) = f_i \cdot V_{A_i}(x) = (\ldots f_{ia} \ldots)_{a \in A_i} \left( \begin{array}{c} \vdots \\ V_{ia}(x) \\ \vdots \\ \vdots \end{array} \right)_{a \in A_i}.
\]

\textbf{Example 1.3.1.} Let \( A = A_1 = \{ 0, 1, \ldots, d \} \subset \mathbb{Z} \). The roots of the polynomial \( F(X) = \sum_{a=0}^d F_a X^a \) are of the form \( X = e^x \), where \( x \) is a solution of the exponential sum equation \( f(x) \overset{\text{def}}{=} \sum_{a=0}^d F_a e^{ax} = 0 \). Notice that if \( f(x) = 0 \) then \( f(x + 2k\pi \sqrt{-1}) = 0 \) for all \( k \in \mathbb{Z} \), so roots of \( F(X) = 0 \) in \( \mathbb{C} \) are in bijection with roots of \( f(x) = 0 \) in \( \mathbb{C} \mod 2\pi \sqrt{-1} \mathbb{Z} \).

\textbf{Example 1.3.2 (Weyl metric).} The unitary invariant inner product introduced by Weyl (1931) and also known as the Bombieri inner product plays a prominent role in the theory of dense homotopy algorithms (Blum et al., 1998). Let \( A = \{ a \in
\( \mathbb{N}_0^n \) s.t. \( \sum_{j=1}^n a_j \leq d \). If \( F, G \) are degree \( d \) polynomials in \( n \) variables, \( F(X) = \sum_{a \in A} F_a X^a \) and \( G(X) = \sum_{a \in A} G_a X^a \), Weyl’s inner product is by definition

\[
\langle F, G \rangle_{d,n} = \sum_{a_1 + \cdots + a_n \leq d} \frac{F_a G_a}{(d/a)!}
\]

where the multinomial coefficient

\[
\binom{d}{a} = \frac{d!}{a_1! a_2! \cdots a_n!}
\]

is the coefficient of \( W_1^{a_1} W_2^{a_2} \cdots W_n^{a_n} \) in \( (1 + W_1 + \cdots + W_n)^d \). We set

\[
\rho_a = \sqrt{\binom{d}{a}} \\
V_a(x) = \rho_a e^{ax}, \quad \text{and} \quad f_a = \frac{F_a}{\rho_a}
\]

As before, \( f \cdot V_A(x) = F(e^x) \). The exponential sum \( f \) is represented in orthonormal coordinates \( f_a \) with respect to Weyl’s metric.

Once we fixed the supports (finite sets) \( A_1, \ldots, A_n \), \( A_i - A_i \in \mathbb{Z}^n \), and picked the coefficients \( \rho_a \), we would like to be able to solve the system of equations

\[
f(x) = \left( f_1(x) \right) = 0,
\]

with \( f \in \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n} \). If \( x \in \mathbb{C}^n \) is a solution of \( f(x) = 0 \), then \( f(x + 2\pi \sqrt{-1} k) = 0 \) for all \( k \in \mathbb{Z}^n \). It makes sense therefore to consider solutions in \( \mathbb{C}^n \mod 2\pi \sqrt{-1} \mathbb{Z}^n \) instead. It turns out that in many situations we can do better.

**Example 1.3.3** (Generalized biquadratic trick). Let \( A = \{0, d, 2d, \ldots cd\} \) for \( c, d \in \mathbb{N} \). The degree \( cd \) polynomial

\[
F(X) = \sum_{a \in A} F_a X^a
\]

can be solved by finding the roots of the degree \( c \) polynomial equation \( G(W) = \sum_{i=0}^c F_i d^i W^i = 0 \) and then taking \( d \)-th roots. This is the same as solving the exponential sum \( g(w) = \sum_{i=0}^c F_i d^i e^{iw} \) in \( \mathbb{C} \mod 2\pi \sqrt{-1} \mathbb{Z} \) and dividing by \( d \). Or solving \( f(x) = \sum_{a \in A} F_a e^{ax} = 0 \) in \( \mathbb{C} \mod \frac{2\pi}{d} \sqrt{-1} \).

There is a multi-dimensional analogous to the situation in example 1.3.3. A lot of work can be saved by exploiting this fact. After we fixed the \( A_i - A_i \)'s, we want to declare \( x \) and \( w \in \mathbb{C}^n \mod 2\pi \sqrt{-1} \mathbb{Z}^n \) equivalent if for all \( f = (f_1, \ldots, f_n) \in \mathcal{F} \)

\[
f(x) = 0 \iff f(w) = 0.
\]

To do this formally, let

\[
[V_{A_i}] : \mathbb{C}^n \to \mathbb{P}(\mathcal{F}_{A_i}^*) \quad x \mapsto [V_{A_i}(x)]
\]

be the differentiable map induced by \( V_{A_i} \). The equivalence relation below has the properties of [1]

\[
x \sim w \iff \forall i, [V_{A_i}(x)] = [V_{A_i}(w)].
\]

Then we quotient \( \mathbb{M} = \mathbb{C}^n / \sim \). If the mixed volume \( V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)) \) is non-zero, then \( \mathbb{M} \) turns out to be \( n \)-dimensional [Malajovich, 2019] Lemma 3.3.1
and Remark 3.3.2). In general, the natural projection \( \mathbb{C}^n \mod 2\pi \sqrt{-1} \mathbb{Z}^n \to \mathcal{M} \) is many-to-one, and its degree is given by the determinant of a certain lattice. More precisely, let \( \mathcal{A} \subset \mathbb{Z}^n \) be the \( \mathbb{Z} \)-module spanned by the union of all the \( A_i - A_i \). Assuming again non-zero mixed volume, \( \Lambda \) has rank \( n \). This means that the linear span of \( \Lambda \) is an \( n \)-dimensional vector space. In example 1.3.3 we had \( \Lambda = d\mathbb{Z} \).

Before going further, let us recall some basic definitions about lattices. For further details, the reader is referred to the textbook by Lovász (1986).

**Definition 1.3.4.**

(a) A full rank lattice \( \Lambda \subset \mathbb{R}^n \) is a \( \mathbb{Z} \)-module so that there are \( u_1, \ldots, u_n \in \Lambda \) linearly independent over \( \mathbb{R} \), and such that every \( u \in \Lambda \) is an integral linear combination of the \( u_i \). A list \( (u_1, \ldots, u_n) \) with that property is called a basis of \( \Lambda \).

(b) If \( \Lambda \subset \mathbb{R}^n \) is a full rank lattice, then we define its determinant as \( \det \Lambda = |\det U| \) where \( U \) is a matrix with columns \( u_1, \ldots, u_n \) of a basis of \( \Lambda \). The determinant does not depend on the choice of the basis.

(c) The dual of a full rank lattice \( \Lambda \subset \mathbb{R}^n \) is the set

\[
\Lambda^\ast = \{v \in (\mathbb{R}^n)^\ast : \forall u \in \Lambda, v(u) \in \mathbb{Z}\}.
\]

It turns out that \( \Lambda^\ast \) is also a full rank lattice. If \( \Lambda \) is full rank and a basis of \( \Lambda \) is given by the columns of a matrix \( U \), then \( U \) is invertible and a basis for \( \Lambda^\ast \) is given by the rows of \( U^{-1} \). In general, if the columns of \( U \) are a basis for a general lattice \( \Lambda \), then the rows of its Moore-Penrose pseudo-inverse \( U^\dagger \) are a basis for \( \Lambda^\ast \).

We can now give a more precise description of \( \mathcal{M} \):

**Lemma 1.3.5.**

\[
\mathcal{M} = \mathbb{C}^n \mod 2\pi \sqrt{-1} \Lambda^\ast
\]

**Proof.** The relation \( x \sim w \) in equation (2) is equivalent to:

\[
\forall i, \exists s_i \in \mathbb{C} \setminus \{0\} \text{ such that } \forall a \in A_i, a(x - w) \equiv s_i \mod 2\pi \sqrt{-1} \mathbb{Z}^n.
\]

We can eliminate the \( s_i \) to obtain an equivalent statement,

\[
\forall i, \forall a, a' \in A_i, (a - a')(x - w) \equiv 0 \mod 2\pi \sqrt{-1}.
\]

This is the same as

\[
\forall \lambda \in \Lambda, \lambda(x - w) \equiv 0 \mod 2\pi \sqrt{-1}.
\]

Thus, \( x \sim w \) is equivalent to

\[
x \equiv w \mod 2\pi \sqrt{-1} \Lambda^\ast.
\]

\( \square \)

There is a natural metric structure on \( \mathcal{M} \). Recall that each \( V_{A_i} \) induces a differentiable map

\[
[V_{A_i}] : \mathcal{M} \to \mathbb{P}(\mathcal{F}^\ast_{A_i}) \quad x \mapsto [V_{A_i}(x)].
\]

Let \( \omega_{A_i} \) denote the pull-back of the Fubini-Study metric in \( \mathbb{P}(\mathcal{F}^\ast_{A_i}) \) to \( \mathcal{M} \). The Hermitian inner product associated to this Kähler form is denoted by \( \langle \cdot, \cdot \rangle_i \).

**Example 1.3.6.** If \( A = \{0, e_1, \ldots, e_n\} \) and \( \rho_A = 1 \), then \( \mathbb{P}(\mathcal{F}_A) = \mathbb{P}^n \) and \( \langle \cdot, \cdot \rangle_1 \) is just the pull-back of the Fubini-Study metric. More generally, in the setting of example 1.3.2 we notice that

\[
\|V_A(x)\| = \|(1, X_1, \ldots, X_n)\|_d.
\]
As a consequence, the inner product is $d^2$ times the Fubini-Study metric.

Let $\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n}$. Let $V = (V_{A_1}, \ldots, V_{A_n})$. The coordinatewise coupling is denoted by

$$f \cdot V(x) = \begin{pmatrix} f_1 \cdot V_{A_1}(x) \\ \vdots \\ f_n \cdot V_{A_n}(x) \end{pmatrix}.$$  

The zero-set of $f$ is

$$Z(f) = \{ x \in M : f \cdot V(x) = 0 \}$$

Assuming again that $\Lambda$ has full rank, the immersion

$$\begin{pmatrix} V \end{pmatrix} : M \rightarrow \mathbb{P}(\mathcal{F}_{A_1}^*) \times \cdots \times \mathbb{P}(\mathcal{F}_{A_n}^*)$$

$$x \mapsto \begin{pmatrix} [V_1(x)] \\ \vdots \\ [V_n(x)] \end{pmatrix}$$

turns out to be an embedding. The $n$-dimensional toric variety

$$(3) \quad \mathcal{V} = \{ [V(x)] : x \in M \}$$

is the natural locus for roots of sparse polynomial systems (aka exponential sums). Points in $\mathcal{V}$ that are not of the form $r_i V_1(x)$ are said to be at toric infinity. The main chart for $\mathcal{V}$ is the map $\begin{pmatrix} V \end{pmatrix} : M \rightarrow \mathcal{V}$. Its range contains the 'finite' points of $\mathcal{V}$, that is the points not at toric infinity.

The momentum map

$$\begin{pmatrix} m_i \end{pmatrix} : M \rightarrow A_i = \text{Conv}(A_i)$$

$$x \mapsto m_i(x) = \sum_{a \in A_i} [V_{\alpha}(x)]^2 a$$

is a surjective volume preserving map (up to a constant) from $(M, \langle \cdot, \cdot \rangle)$ into the interior of $A_i$. The constant is precisely $\pi^n = n! \text{Vol}(\mathbb{P}^n)$, so that a generic $f \in \mathcal{F}_{A_i}^n$ has $n! \text{Vol} A_i$ roots in $M$ (see Malajovich (2019) and references). The following result is a coarse, although handy bound of the toric norm in terms of the Hermitian norm:

**Lemma 1.3.7.** Let $x \in M$ and $u \in T_x M \cong \mathbb{C}^n$. Let $\| \cdot \|$ be the canonical Hermitian norm. Then,

$$\| u \|_{1,x} - \| D[V_{A_i}](x) u \| \leq \delta_i \| u \|.$$

where

$$\frac{1}{2} \text{diam(Conv}(A_i)) \leq \delta_i \overset{\text{def}}{=} \max_{a \in A_i} \| a - m_i(x) \| \leq \text{diam(Conv}(A_i)).$$

**Proof.** The upper bound $\delta_i \leq \text{diam(Conv}(A_i))$ is trivial. For the lower bound, let $a, a' \in A_i$ maximize $\| a - a' \| = \text{diam(Conv}(A_i))$. Let $c = \frac{1}{2}(a + a')$. Assume without loss of generality that $\| m_i(x) - a \| \geq \| m_i(x) - a' \|$. Then, $\| m_i(x) - a \| \geq \| c - a \| = \frac{1}{2} \text{diam(Conv}(A_i))$. 


In order to bound $D[V_A](x)$, we choose $\frac{V_A(x)}{V_A(x)}$ as representative of the projective point $[V_A(x)]$. We differentiate:

\[
D[V_A](x)u = P_{V_A(x)} \frac{1}{\|V_A(x)\|} DV_A(x)u \\
= \left(I - \frac{1}{\|V_A(x)\|^2} V_A(x)V_A(x)^*\right) \frac{1}{\|V_A(x)\|} DV_A(x)u \\
= \frac{1}{\|V_A(x)\|} (DV_A(x) - V_A(x)m_i(x)) u.
\]

The $a$-th coordinate of the expression above is bounded by

\[
|(D[V_A](x)u)_a| = \frac{|V_a(x)|}{\|V_A(x)\|} |(a - m_i(x))u| \leq \frac{|V_a(x)|}{\|V_A(x)\|} \max_{a \in A_i} \|a - m_i(x)\| \|u\|.
\]

Normwise,

\[
\|D[V_A](x)u\| \leq \max_{a \in A_i} \|a - m_i(x)\| \|u\| = \delta_i \|u\|.
\]

We define an inner product on $M$ as the pull-back of the Fubini-Study volume in $V$, namely

\[
\langle , \rangle_x = \langle , \rangle_1,1 + \cdots + \langle , \rangle_n,n.
\]

Its associated norm is denoted by $\| \cdot \|_x$. The previous complexity analysis by Malajovich (2019) relied on two main invariants. The toric condition number is defined for $f \in \mathcal{F}$ and $x \in M$ by

\[
\mu(f, x) = \|M(f, x)^{-1} \text{diag}(\|f_i\|)\|_x
\]

where

\[
M(f, x) = \begin{pmatrix}
\frac{1}{\|V_A(x)\|} f_1 \cdot P_{V_A(x)} DV_A(x) \\
\frac{1}{\|V_A(x)\|} f_2 \cdot P_{V_A(x)} DV_A(x) \\
\vdots \\
\frac{1}{\|V_A(x)\|} f_n \cdot P_{V_A(x)} DV_A(x)
\end{pmatrix}
\]

and $\| \cdot \|_x$ is the operator norm for linear maps from $\mathbb{C}^n$ (canonical norm assumed) into $(M, \| \cdot \|_x)$. In terms of the momentum map, we can also write

\[
M(f, x) = \begin{pmatrix}
\frac{1}{\|V_A(x)\|} f_1 \cdot (DV_A(x) - V_A(x)m_1(x)) \\
\vdots \\
\frac{1}{\|V_A(x)\|} f_n \cdot (DV_A(x) - V_A(x)m_n(x))
\end{pmatrix}.
\]

If $m_i(x) = 0$ for all $i$, or if $x$ is a zero for $f$, we have just

\[
M(f, x) = \begin{pmatrix}
\frac{1}{\|V_A(x)\|} f_1 DV_A(x) \\
\vdots \\
\frac{1}{\|V_A(x)\|} f_n DV_A(x)
\end{pmatrix}.
\]

The second invariant bounds the distortion when passing from $\| \cdot \|_{1,x}$ to $\| \cdot \|_{1,y}$ (Malajovich, 2019) Lemma 3.4.5). It can also be understood as the ‘radius’ of the support with respect to the momentum $m_i(x)$, in the dual metric to $\| \cdot \|_{1,x}$. More precisely, it is defined as $R(x) = \max_i u_i(x)$ with

\[
\psi(x) = \sup_{\|u\|_{1,x} \leq 1} \sup_{a \in A_i} \|a - m_i(x)\| u
\]
Figure 1. The solution variety. Each path in the space $\mathcal{F}$ of equations lifts onto possibly several paths. Each of those corresponds to a different renormalized path in $\mathcal{F}_0 = \{ f \in \mathcal{F} : f \cdot V(0) = 0 \}$.

Remark 1.3.8. Lemma 1.3.7 provides a lower bound

$$1 \leq \sup_{\delta, |u| \leq 1} |(a - m_i(x))u| \leq \nu_i(x)$$

Expressions for both invariants can be simplified by shifting each support $A_i$, so that $m_i(x) = 0$. By shifting supports, we still obtain finite sets $A_i \subseteq \mathbb{R}^n$ with the property that $A_i - A_i \in \mathbb{Z}^n$. The following estimates will be needed:

**Proposition 1.3.9.** Let $f, g \in \mathcal{F}$ and $x \in \mathcal{M}$.

(a) $1 \leq \mu(f, x)$.

(b) If $\|f - g\|\mu(f, x) < 1$, then

$$\frac{\mu(f, x)}{1 + d_P(f, g)\mu(f, x)} \leq \mu(g, x) \leq \frac{\mu(f, x)}{1 - d_P(f, g)\mu(f, x)}$$

(c) Let $\gamma$ be defined by

$$\gamma(f, x) = \sup_{k \geq 2} \left( \frac{1}{k!} \| (f \cdot P_{V(x)} D V(x))^{-1} (f \cdot P_{V(x)} D^k V(x)) \|_x \right)^{1/(k-1)},$$

then $\gamma(f, x) \leq \frac{1}{2} \mu(f, x) \nu(x)$.

This Proposition aggregates miscellaneous results from Malajovich (2019). Item (a) is Equation (5), item (b) is a particular case of Theorem 4.3.1 with $s = 0$ and item (c) is Theorem 3.6.1.

1.4. The renormalized Newton operator, and homotopy. The main result in my previous paper (Malajovich, 2019) was a step count for path-following in terms of a certain condition length. Let $\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n}$ and let

$$S = \{ (f, z) \in \mathcal{F} \times \mathcal{M} : f \cdot V(z) = 0 \}$$
be the solution variety (Figure 1). The condition length of the path \((f_t, z_t)_{t \in [a, b]} \in \mathcal{S}\) is

\[
\mathcal{L}(f_t, z_t; a, b) = \int_a^b \sqrt{\| \dot{f}_t \|_F^2 + \| \dot{z}_t \|_Z^2} \mu(f_t, z_t) \nu(z_t) \, dt,
\]

where \(\| \dot{f} \|_F\) is the norm in \(T_f \mathbb{P}(\mathcal{F})\), namely \(\| \dot{f} \|_F^2 = \sum |P_{f_t} \cdot \dot{f}_t |^2 / \| f_t \|_F^2\). It is assumed that the path is smooth enough for the integral to exist, namely of class \(W^{2, \infty}\). At this time, no average estimate of \(\mathcal{L}\) is known in the sparse setting. The main obstructions to obtaining such bound seem to be the invariant \(\nu(z)\) and the dependence of the condition number on the toric norm at the point \(z\). When \(z\) approaches ‘toric infinity’, that is \(m_1(z)\) approaches \(e \text{Conv}(A_i)\) for some \(i\), the invariant \(\nu(z)\) can become arbitrarily large. This occurs in particular during polyhedral or ‘cheater’s’ homotopy, where the starting system has roots at ‘toric infinity’.

The renormalization approach in this paper is intended to overcome those difficulties. First, we fix once and for all a privileged point in \(M\). In this paper, we take this point to be the origin \(0\). By shifting the supports \(A_i\), we can further assume that \(m_1(0) = 0\).

**Definition 1.4.1.** The renormalization operator \(R = R(u)\) is the operator

\[
R(u) : \mathcal{F} \rightarrow \mathcal{F}
\]

\[
f \mapsto f \cdot R(u) = \begin{pmatrix} f_1 \cdot R_1(u) \\ \vdots \\ f_n \cdot R_n(u) \end{pmatrix}
\]

with

\[
R_i(u) : \mathcal{F}_{A_i} \rightarrow \mathcal{F}_{A_i}, \quad f_i = [\ldots, f_{i_n}, \ldots] \mapsto f_i \cdot R_i(u) = [\ldots, f_{i_n}e^{au - \ell_i(u)}, \ldots]
\]

and \(\ell_i(u) = \max_{a \in A_i} a \text{Re}(u)\).

In particular, \(f_i \cdot V_{A_i}(x) = e^{\ell_i(x)}(f_i \cdot R(x)) \cdot V_{A_i}(0)\). Let \(\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{F}_{A_1}) \times \cdots \times \mathbb{P}(\mathcal{F}_{A_n}) \) and \(\mathbb{P}([f] = ([f_1], \ldots, [f_n]) \in \mathbb{P}(\mathcal{F})\). The renormalization operator acts on \(\mathbb{P}(\mathcal{F})\):

**Theorem 1.4.2.** Under the notations above:

(a) The renormalization operator induces an action also denoted \(R(u)\) of the additive group \(\mathbb{C}^n\) into the projectivized solution variety \(\mathbb{P}(\mathcal{S}) = \{( [f], z) \in \mathbb{P}(\mathcal{F}) \times \mathcal{M} : f \cdot V(z) = 0 \}\), namely \(([f], z) \mapsto [f \cdot R(u)], (z - u)\).

(b) If \(u\) is pure imaginary, then the map \(R(u) : \mathcal{S} \rightarrow \mathcal{S}\) is an isometry, as well as the coordinate maps \(f \mapsto f \cdot R(u)\) and \(z \mapsto z - u\).

(c) In general, \(\| f \cdot R(u) \| \leq \| f \| \) and \(\| f_i \cdot R_i(u) \| \leq \| f_i \|\).

The action of \(\mathbb{R}^n\) by imaginary renormalization \(R(u\sqrt{-1})\) is also known as toric action. The word toric comes from the fact that this action is actually an action of \(\mathbb{R}^n \mod 2\pi \mathbb{Z}^n \simeq (S^1)^n\). A more elaborate version of the real action was used by Verschelde (2000), with additional variables.

The toric Newton operator was defined in (Malajovich, 2019) by

\[
\mathcal{N} : \mathcal{F} \times \mathcal{M} \rightarrow \mathcal{M}
\]

\[
(f, z) \mapsto z - (f \cdot P_{V(z)} \cdot DV(z))^{-1}(f \cdot V(z))
\]
and its analysis was done in terms of the toric version of Smale’s invariants

\[ \gamma(f, z) = \beta(f, z) \gamma(f, z) \]

\[ \beta(f, z) = \left\| (f \cdot P_{V(z)}^{-1} DV(z))^{-1} (f \cdot V(z)) \right\|_z \]

\[ \gamma(f, z) = \sup_{k \geq 2} \left( \frac{1}{k!} \left\| (f \cdot P_{V(z)}^{-1} DV(z))^{-1} (f \cdot P_{V(z)}^{-1} D^k V(z)) \right\|_z \right)^{1/(k-1)}. \]

All those definitions scale with respect to each \( f_i \). By renormalizing \( g = f R(x) \) at the origin and assuming that \( m_i(0) = 0 \) for all \( i \), we obtain simpler expressions:

\[ N(g, 0) = - (g \cdot DV(0))^{-1} (g \cdot V(0)), \]

\[ \beta(g, 0) = \left\| (g \cdot DV(0))^{-1} (g \cdot V(0)) \right\|_0, \]

\[ \gamma(g, 0) = \sup_{k \geq 2} \left( \frac{1}{k!} \left\| (g \cdot DV(0))^{-1} (g \cdot D^k V(0)) \right\|_0 \right)^{1/(k-1)}. \]

Let \( x_0 \) be a point of \( \mathcal{M} \). The renormalized iterates of \( x_0 \) are defined inductively by

\[ x_{i+1} = N(f \cdot R(x_i), 0) + x_i. \]

They can be compared to the actual Newton iterates of \( y_0 \equiv x_0 \) in \( T_0 M = \mathbb{C}^n \) for a suitable function, namely:

**Lemma 1.4.3.** Assume \( m_i(0) = 0 \) for all \( i \). Let

\[ F : T_0 M \to \mathbb{C}^n \]

\[ y \mapsto F(y) = f \cdot V(y). \]

If \( y = x \), then

(a) \( N(f \cdot R(x), 0) = N(F, y) - y \)

(b) \( \beta(f \cdot R(x), 0) = \beta(F, y) \)

(c) \( \gamma(f \cdot R(x), 0) = \gamma(F, y) \)

where the left-hand-sides use the notations in (Malajovich, 2019) and the right-hand-sides are the classical Smale’s invariants in \( (T_0 M, \| \cdot \|_0) \).

**Proof of Lemma 1.4.3.** We establish first item (a):

\[ N(f \cdot R(x), 0) = -(f \cdot R(x) \cdot DV(0))^{-1} (f \cdot R(x) \cdot V(0)) \]

\[ = -(f \cdot DV(0))^{-1} (f \cdot V(0)) \]

\[ = -(DF(y))^{-1} F(y) \]

\[ = -y + N(F, y) \]

Items (b) and (c) are similar, so we just prove item (c). For each \( k \geq 2 \),

\[ \frac{1}{k!} \left\| (f \cdot R(x) \cdot DV(0))^{-1} (f \cdot R(x) \cdot D^k V(0)) \right\|_0 = \]

\[ \frac{1}{k!} \left\| DF(y)^{-1} D^k F(y) \right\|_0. \]

Taking \( k - 1 \)-th roots and taking the sup, we obtain

\[ \gamma(f \cdot R(x), 0) = \gamma(F, y) \]

as stated. \( \square \)
Lemma 1.4.3 immediately implies a renormalized version of the classical Smale’s theorems on quadratic convergence of Newton iteration without the necessity of dealing with different metrics at different points like (Malajovich, 2019). See also Theorem 2.1.1, 2.1.2 and references ibidem. Smale’s quadratic convergence theorems become, in this context:

**Theorem 1.4.4** ($\gamma$-theorem). Let $\zeta \in \mathcal{M}$ be a non-degenerate zero of $f \cdot V(\zeta) = 0$. If $x_0 \in \mathcal{M}$ satisfies

$$|\zeta - x_0|_0 \gamma(f \cdot R(\zeta), 0) \leq \frac{3 - \sqrt{7}}{2},$$

then the sequence $x_{i+1} = N(f \cdot R(x_i), 0)$ is well-defined and

$$\|\zeta - x_i\|_0 \leq 2^{-2^i+1}\|\zeta - x_0\|_0.$$

**Theorem 1.4.5** ($\alpha$-theorem). Let

$$\alpha \leq \alpha_0 = \frac{13 - 3\sqrt{17}}{4},$$

$$r_0 = \frac{1 + \alpha - \sqrt{1 - 6\alpha + \alpha^2}}{4\alpha} \quad \text{and} \quad r_1 = \frac{1 - 3\alpha - \sqrt{1 - 6\alpha + \alpha^2}}{4\alpha}.$$  

If $x_0 \in \mathcal{M}$ satisfies $\alpha(f \cdot R(x_0), 0) \defeq \beta(f \cdot R(x_0), 0) \gamma(f \cdot R(x_0), 0) \leq \alpha$, then the sequence defined recursively by $x_{i+1} = N(f \cdot R(x_i), 0)$ is well-defined and converges to a limit $\zeta$ so that $f \cdot V(\zeta) = 0$. Furthermore,

(a) $|x_i - \zeta|_0 \leq 2^{-2^i+1}|x_1 - x_0|_0$

(b) $|x_0 - \zeta|_0 \leq r_0 \beta(f \cdot R(x_0), 0)$

(c) $|x_1 - \zeta|_0 \leq r_1 \beta(f \cdot R(x_0), 0)$.

Loosely speaking, approximate roots of $f$ are points $x_0$ satisfying the conclusions of Theorem 1.4.4 (approximate roots of the second kind) or of Theorem 1.4.5(a) (approximate roots of the first kind). Sometimes it is useful to have an effective certification of the hypotheses of either theorem, and this can be achieved by replacing the invariant $\gamma(f \cdot R(x_0), 0)$ by its upper bound $\frac{1}{2}\mu(f \cdot R(x_0), 0)\nu(0)$ from Proposition 1.3.9.

**Definition 1.4.6.** Let $(q_\tau)_{\tau \in \mathcal{T}}$ be a path in $\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n}$ and $x_0 \in \mathcal{M}$ be a suitable starting solution. The renormalized homotopy at the origin is given by the recurrence:

$$\begin{aligned}
q_{j+1} &= N(q_j, R(x_j), 0) + x_j \\
\tau_{j+1} &= \min\left\{ \tau, \inf \left\{ t > \tau_j : \frac{1}{2}\beta(q_t, R(x_j+1), 0) \times \mu(q_t, R(x_j+1), 0) \nu_0 > \alpha_0 \right\} \right\}
\end{aligned}$$

**Definition 1.4.7.** Let $(q_\tau, z_\tau)_{\tau \in [t, t']}$ be a path in the solution variety $\mathcal{S}$. Its renormalized condition length is defined by:

$$\mathcal{L}((q_\tau, z_\tau); t, t') = \int_t^{t'} \left( \left| \frac{\partial}{\partial \tau} (q_\tau \cdot R(z_\tau)) \right|_{q_\tau \cdot R(z_\tau)} + \nu_0 \|z_\tau\|_0 \right) \mu(q_\tau \cdot R(z_\tau), 0) \, d\tau$$

where

$$\left| \frac{\partial}{\partial \tau} (q_\tau \cdot R(z_\tau)) \right|_{q_\tau \cdot R(z_\tau)} = \sqrt{\sum_{i=1}^{n} \left| P_{q_\tau \cdot R(z_\tau)} (q_{i\tau} \cdot R(z_\tau)) \right|^2 \left| q_{i\tau} R(z_\tau) \right|^2}.$$
and $\nu_0 = \nu(0)$.

Definition 1.4.7 makes sense for $-\infty \leq t \leq t' \leq \infty$. For instance, we can consider a linear homotopy $q_t = f + t g$ for $0 \leq t \leq \infty$. This line projects onto a finite path in $\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{F}_{A_1}) \times \cdots \times \mathbb{P}(\mathcal{F}_{A_n})$. If the condition number $\mu(q_t \cdot R(z_t), 0)$ is bounded for all $t$ and for all solutions paths $z_t$, then the renormalized condition length will be finite (see Section 5). Of course, it may happen to the condition length to be infinite. The number of homotopy steps required by Definition 1.4.6 can be bounded in terms of the condition length:

**Main Theorem A.** There are constants $0 < \alpha_* \simeq 0.074 \cdots < \alpha_0$, $0 < u_* = u_*(\alpha_*) \simeq 0.129 \cdots$, $0 < \delta_* = \delta_*(\alpha_*) \simeq 0.085 \cdots$ with the following properties. Let $-\infty \leq t_0 < T \leq \infty$. For any path $(q_t)_{t \in [t_0,T]}$ of class $C^{1 + Lip}$ in $\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n}$ and for any $x_0 \in \mathbb{C}^n$, if the pair $(q_{t_0}, x_0)$ satisfies

$$
\frac{1}{2} \beta(q_{t_0} R(x_0), 0) \mu(q_{t_0} R(x_0), 0) \nu_0 \leq \alpha_*,
$$

the recurrence (4) is well-defined and there is a $C^{1 + Lip}$ path $(q_t, z_t) \in \mathcal{S}$ satisfying, for all $t_j \leq t < t_{j+1}$,

$$
u_j(t) \overset{\text{def}}{=} \frac{1}{2} \| z_t - z_{j+1} \|_0 \mu(q_t R(z_t), 0) \nu_0 \leq u_*.
$$

Moreover, $\mathcal{L}(t_j, t_{j+1}) \geq \delta_*$ whenever $t_{j+1} < T$. In particular, whenever $\mathcal{L}(t_0, T)$ is finite, there is $N \in \mathbb{N}$ with $t_N = T$ and

$$N \leq 1 + \frac{1}{\delta_*} \mathcal{L}(t_0, T).
$$

If we set

$$x_{N+1} = N(q_T R(x_N), 0) + x_N
$$

then

$$
\frac{1}{2} \beta(q_T R(x_{N+1}), 0) \mu(q_T R(x_{N+1}), 0) \nu_0 \leq \alpha_*,
$$

In conclusion, the recurrence terminates after at most

$$1 + \frac{1}{\delta_*} \mathcal{L}(t_0, T)
$$

iterations. With an extra iteration more, we recover an approximate root of $q_T$, in the sense of the definition below:

**Definition 1.4.8.** An approximate root of $f \in \mathcal{F}$ is some $y \in \mathcal{M}$ satisfying

$$
\frac{1}{2} \beta(f \cdot R(y), 0) \mu(f \cdot R(y), 0) \nu_0 < \alpha_0.
$$

Let $(x_r)$ be the sequence of renormalized Newton iterates of $x$, viz. $x_0 = x$ and $x_{r+1} = N(f \cdot R(x_r), 0) + x_r$. We say that $\zeta = \lim_{r \to \infty} x_r$ is the associated root to $x$.

Theorem A provides a way to produce approximate roots for $q_T$ from the knowledge of approximate roots of $q_0$. Approximate roots with different associated roots of $q_0$ give rise to approximate roots for $q_T$ with different associated roots. In case the renormalized condition length associated to all homotopy paths is finite, this allows to ‘approximately solve’ $q_T$ in terms of a maximal set of ‘approximate solutions’ to $q_0$. The renormalized condition length will be infinite if for instance one
of the roots is degenerate or at ‘toric infinity’. Theorem A is proved in Section 2 below.

Remark 1.4.9. The word renormalization is taken here in the sense of dynamical systems or cellular automata. Strictly speaking, the renormalization operator should be allowed to be time-dependent as in the renormalized Graeffe iteration by Malajovich and Zubelli (2001a, 2001b).

1.5. Expectation of the renormalized condition number. Theorem A reduces the problem of obtaining a global complexity estimate to the evaluation of the renormalized condition length as in Definition 1.4.7. We will take a random homotopy path, with one of the endpoints fixed. Before computing its renormalized condition length, we will need to compute the expected squared condition number for a given time \( \tau \). We will actually obtain a conditional expectation. This will be enough to produce an algorithm with a bounded absolute expectation, as it will be explained in section 1.7. This bound on the conditional expectation of the squared condition number depends on a generalization of Minkowski’s mixed volume, that we call the mixed surface. We recall the definition of mixed volume first.

**Definition 1.5.1.** The mixed volume of an \( n \)-tuple of compact convex sets \( (A_1, \ldots, A_n) \) in \( \mathbb{R}^n \) is

\[
V(A_1, \ldots, A_n) \overset{\text{def}}{=} \frac{1}{n!} \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} \text{Vol}(t_1 A_1 + \cdots + t_n A_n)
\]

where \( t_1, \ldots, t_n \geq 0 \) and the derivative is taken at \( t_1 = \cdots = t_n = 0 \).

The normalization factor \( 1/n! \) ensures that

\[
\text{Vol}(A) = V(A, \ldots, A).
\]

The mixed volume is also known to be monotonic, symmetric, translation invariant and linear in each \( A_i \) with respect to Minkowski linear combinations. Those five properties also define the mixed volume. The Bernstein-Kushirenko-Khovanskii bound can be stated in terms of polynomials or exponential sums, we state it here in terms of exponential sums.

**Theorem 1.5.2** (Bernstein, 1975; Kushnirenko and Khovanskii, 1976). Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z}^n \), \( A_i = \text{Conv}(A_i) \), \( i = 1, \ldots, n \), and let \( \mathbf{f}_i \in \mathcal{F}_{A_i} \). Then, the system

\[
f_i V_{A_i}(z) = 0
\]

has at most \( n! V(A_1, \ldots, A_n) \) isolated roots in \( \mathbb{C}^n \mod 2\pi \sqrt{-1} \mathbb{Z}^n \). This bound is attained for generic \( \mathbf{f} \).

This statement is equivalent to the polynomial version, because

\[
\exp : \mathbb{C}^n \mod 2\pi \sqrt{-1} \mathbb{Z} \rightarrow \mathbb{C}^n_{\mathbb{R}} \quad z \mapsto \exp(z) = (e^{z_1} \cdots e^{z_n})
\]

is a bijection.
Figure 2. The mixed volume and the mixed surface. Top, the support for the system of polynomials $F_1(X_1, X_2, X_3) = 1 + X_2 + X_3 + X_2X_3$, $F_2(X_1, X_2, X_3) = 1 + X_1 + X_3 + X_1X_3$, $F_3(X_1, X_2, X_3) = 1 + X_1 + X_2 + X_1X_2$. Each support is represented in a different color. Bottom left, the Minkowski linear combination of the supports. Only the two cubes with edges of all colors, aka ‘mixed cells’, have a volume term in $t_1t_2t_3$. The mixed volume $V = \frac{1}{4}2 = 1/3$, and this means that if the coefficients are replaced by generic coefficients the system has 2 roots in $\mathbb{C}_x$. Bottom right, the cells with surface multiple of $t_1t_2$ are one of the parcels in the mixed surface. In this example there are 6 of them, therefore $V(\text{Conv}(A_1), \text{Conv}(A_2), B^3) = 1$ and by permuting supports, the mixed surface $V'$ is equal to 3.

The mixed volume was originally defined by Minkowski (1901) in connection with surface and curvature of convex bodies. Assume that $A \subset \mathbb{R}^n$ is a compact convex body with smooth boundary. Its surface or $n - 1$ dimensional volume of its boundary $\partial A$ is given by

$$V' = V'(A) = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} \text{Vol}(A + \epsilon B^n) = nV(A, \ldots, A, B^n).$$
Figure 3. The Steiner polynomial of a convex body $C$ is the volume of $C + \epsilon B^n$. In two dimensions, $\text{Vol}(C + \epsilon B^2) = \text{Vol}(C) + 2\epsilon \text{Length}(\partial C) + \pi \epsilon^2$. With the proper normalization, the coefficients of this polynomial are known as the intrinsic volumes of a convex polytope.

A generalization of Minkowski’s surface quermassintegral turns out to be an important invariant for homotopy continuation in the sparse case, namely

$$(9) \quad V'(A_1, \ldots, A_n) = \frac{\partial}{\partial \epsilon} |_{\epsilon = 0} V(A_1 + \epsilon B^n, \ldots, A_n + \epsilon B^n) = \sum_i V(A_1, \ldots, B^n, \ldots, A_n)_{i\text{-th}}$$

This quermassintegral will be called Mixed Surface in analogy with the ordinary surface of the boundary of a convex set. Definitions (8) and (9) coincide when $A_1 = \cdots = A_n = A$. Another manifestation of this invariant arises when one of the supports in replaced by a unit simplex $\Delta_n = \{0, e_1, \ldots, e_n\}$ as a result of eliminating one variable, see for instance [Herrero et al., 2019] and references. Recall that $\Delta_n$ has circumscribed radius $\sqrt{\frac{1}{n}}$ so that

$$\sum_i V(A_1, \ldots, \text{Conv}(\Delta_n), \ldots, A_n) \leq \sqrt{1 - \frac{1}{n}} V' \leq V'.$$

Example 1.5.3. Suppose that the convex sets have the same shape, say $A_i = d_i A$ for $d_i > 0$. Then,

$$V = V(A_1, \ldots, A_n) = \prod_{i=1}^n d_i \text{Vol} A$$

and

$$\left( \min_{1 \leq j \leq n} \prod_{i \neq j} d_i \right) V'(A) \leq V'(A_1, \ldots, A_n) \leq \left( \max_{1 \leq j \leq n} \prod_{i \neq j} d_i \right) V'(A).$$

In this example,

$$\frac{V(A_1, \ldots, A_n)}{V(A) \max_j d_j} \leq \frac{V'(A_1, \ldots, A_n)}{V'(A)} \leq \frac{V(A_1, \ldots, A_n)}{V(A) \min_j d_j}$$

and the isoperimetric inequality [Khovanskii, 1989] $V'(A) \geq nV(A)^{\frac{n-1}{2}} \text{Vol}(B^n)^{\frac{1}{2}}$ provides the bound

$$V(A_1, \ldots, A_n) \leq \frac{(\max_j d_j) \sqrt{\pi V(A)^{1/n}}}{n \Gamma(n/2)^{1/n}} V'(A_1, \ldots, A_n)$$

where $\Gamma(n/2)^{1/n} \approx \sqrt{\frac{2n}{\pi}}$.

Example 1.5.4. In the specific case when $A = \text{Conv}(\Delta_n)$, $n!V = \prod d_i$ is the Bézout number and $n!V' = (n^2 + n\sqrt{n+1}) \sum_i \prod_{j \neq i} d_j$. 
Example 1.5.5. Let \( p, q \in \mathbb{N} \) be relatively prime. By Euclid’s algorithm there are \( r, s \in \mathbb{Z} \) so that \( pr + qs = 1 \). Let \( A_1 = \{0, 0, [p, q]\}, A_2 = \{0, 0, [-s, r]\} \) and \( A_i = \text{Conv}(A_i) \). Then \( V(A_1, A_2) = t \) and \( V’(A_1, A_2) = \sqrt{t^2 + q^2 + r^2 + s^2} \). We see from this example that \( V’/V \) can be arbitrarily large.

We saw in the previous section that the natural map \( \mathbb{C}^n \mod 2\pi\sqrt{-1} \mathbb{Z}^n \rightarrow \mathcal{M} \) is \( (\det \Lambda) \) to 1, so we may restate Theorem 1.5.2 as follows:

**Theorem 1.5.6.** Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{C}^n \), \( A_i = \text{Conv}(A_i), i = 1, \ldots, n \), and let \( f_i \in \mathcal{F}_{A_i} \). Then, the system

\[
\begin{align*}
  f_1 V_{A_1}(z) &= 0 \\
  &\vdots \\
  f_n V_{A_n}(z) &= 0
\end{align*}
\]

has at most \( n!V(A_1, \ldots, A_n)/(\det \Lambda) \) isolated roots in \( \mathcal{M} \). This bound is attained for \( f \) generic.

**Remark 1.5.7.** The bound in Theorem 1.5.6 is basis invariant in the following sense: if one replaces the \( A_i \) by \( A_i \Lambda \) for an arbitrary matrix \( \Lambda \) with integer coefficients, invertible over \( \mathbb{Q} \), then the number \( \frac{n!V(A_1, \ldots, A_n)}{\det \Lambda} \) does not change.

We assume from now on that the mixed volume \( V(A_1, \ldots, A_n) \) is non-zero, and in particular \( \Lambda \) has rank \( n \). Then we consider a random Gaussian complex polynomial system \( q, q_i \in \mathcal{F}_{A_i} \). For simplicity let \( f_i = \mathbb{E}(q_i) \) be the average, \( g_i = q_i - f_i \), and \( \Sigma_i^2 = \mathbb{E}(g_i^* g_i) \) be the covariance matrix. Recall that \( g_i \) is a covector, so \( g_i^* \) is a vector so \( \Sigma_i^2 \) is indeed an Hermitian matrix. In this sense, \( q_i \sim N(f_i, \Sigma_i^2; \mathcal{F}_{A_i}) \). For short,

\[
\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n},
\]

\[
q \sim N(f, \Sigma^2; \mathcal{F}) \quad \text{and} \quad g = q - f \sim N(0, \Sigma^2; \mathcal{F})
\]

We will need a distortion bound to state the next Theorem. The condition number depends on the metric we choose on the \( \mathcal{F}_{A_i} \)’s. In this paper, the metric is specified by the choice of the coefficients \( \rho_{ia} \). Therefore we introduce a distortion bound for these coefficients. Let \( A_i’ \) denote the set of vertices of \( \text{Conv}(A_i) \). We set

\[
\kappa_{\rho_{ia}} \overset{\text{def}}{=} \sqrt{\frac{\sum_{a \in A_i} \rho_{ia}^2}{\min_{a \in A_i’} \rho_{ia}}}
\]

**Remark 1.5.8.** In the particular case \( \rho_{ia} = 1 \) for all \( a \), we have \( \kappa_{\rho_{ia}} = \sqrt{\Sigma_i} \).

**Remark 1.5.9.** In the dense case of degree \( d_i \) with the coefficients of Example 1.3.2 \( \kappa_{\rho_{ia}} = (n + 1)^{\frac{d_i}{2}} \) so this distortion can be much larger than \( \sqrt{\Sigma_i} = \left( \frac{d_i + n}{n} \right)^{\frac{1}{2}} \).

**Remark 1.5.10.** In general, \( \nu_i(0) \leq \kappa_{\rho_{ia}} \). Indeed, assume without loss of generality that \( m_i(0) = 0 \). Then,

\[
\|u\|_{\mathcal{F}_{A_i}} \overset{\text{def}}{=} \sqrt{\sum_{a \in A_i} (\rho_{ia} au)^2} \geq \max_{a \in A_i} |au| = \max_{a \in A_i} |au|.
\]

Therefore, if \( \|u\|_{\mathcal{F}_{A_i}} \leq 1 \) implies that \( \max_{a \in A_i} |au| \leq \kappa_{\rho_{ia}} \).
We also define zero-sets

\[ Z(\mathbf{q}) = \{ \mathbf{z} \in \mathcal{M} : \mathbf{q} \cdot \mathbf{V}(\mathbf{z}) = 0 \} \quad \text{and} \quad Z_H(\mathbf{q}) = \{ \mathbf{z} \in Z(\mathbf{q}) : \|\text{Re}(\mathbf{z})\|_\infty \leq H \}. \]

For generic \( \mathbf{q} \), Theorem 1.5.6 implies that

\[ \#Z_H(\mathbf{q}) \leq \#Z(\mathbf{q}) = \frac{n!V(A_1, \ldots, A_n)}{\det \Lambda}. \]

**Main Theorem B.** Let \( A_1, \ldots, A_n \in \mathbb{R}^n \) be such that the \( \mathbb{Z} \)-module \( \Lambda \) generated by \( \bigcup_{i=1}^n A_i - A_i \) is a subset of \( \mathbb{Z}^n \), and suppose that the mixed volume \( V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)) \) is non-zero. Let \( \mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n} \). Let \( S \) = \#\( A_i = \dim \mathcal{C}(\mathcal{F}_{A_i}) \geq 2 \). For each \( i \), let \( \Sigma_i \) be diagonal and positive definite in \( \mathcal{F}_{A_i} \). Let \( L = \max_i |f_i|^{-1} \sqrt{S_i} \). Then, for any fixed real number \( H > 0 \),

\[
\mathbb{E}_{\mathbf{q} \sim \mathcal{N}(f, \Sigma^2, \mathcal{F})} \left( \sum_{i \in Z_H(\mathbf{q})} \mu^2(\mathbf{q} \cdot R(z), 0) \right) \leq \frac{2.5eH \sqrt{n}}{\det(\Lambda)} \left( 1 + 3L + \sqrt{\log(n)} \frac{\log(n)}{\min(S_i)} + 2 \log(3/2) \right)^2 \times \prod_{i=1}^{n} \left( \frac{S_i \mu_n \max_{\mathbf{a} \in A_i} (\sigma_{i,a}^2)}{\min_{\mathbf{a}} (\sigma_{i,a}^2)} \right) (\sum_{i=1}^{n} \delta_i^2) (n-1)!V'
\]

where

\[ V' = \sum_{i=1}^{n} V(A_1, \ldots, B^{i-1}, B^i, \ldots, A_n) \]

is the mixed surface.

**Remark 1.5.11.** The choice of the coefficients \( \rho_{i,a} \) allows for more flexibility to the theory. Besides the trivial choice \( \rho_{i,a} = 1 \) and the Weyl metric \( \rho_{i,a} = \sqrt{\frac{|f_i|}{a}} \), another interesting possibility is \( \rho_{i,a} = |f_i| \) for a fixed system \( f \). In this case, we recover a condition number \( \mu(f, z) \) (resp. a renormalized condition number \( \mu(f \cdot R(z)) \) with respect to coefficientwise relative error. Numerical evidence supporting this choice was presented by [Malajovich and Rojas (2002)] in the context of the non-renormalized condition number.

**Remark 1.5.12.** The interest of varying the \( \sigma_{i,a} \) while keeping the \( \rho_{i,a} \) fixed arises from the theoretical analysis of non-linear homotopy paths. For instance, one can consider a Gaussian system \( g \), and produce a non-linear homotopy by setting \( g_{i,a} = e^{-th_{\mathbf{a}}} b_{i,a} \) for random real coefficients \( b_{i,a} \). This is equivalent to polyhedral homotopy as described by [Verschelde et al. (1994)] or [Huber and Sturmfels (1995)]. No a priori step count bound is known at this time. This avenue of research will be pursued in a future paper.

**Remark 1.5.13.** Let \( d \in \mathbb{N} \). If we replace each \( A_i \) in Theorem B by \( d A_i \), we will replace the mixed surface \( V' \) by \( d^{n-1} V' \) and each \( \delta_i \) by \( d \delta_i \). Moreover, \( \det d \Lambda = d^n \det \Lambda \). To keep the same solution set, we must replace \( H \) by \( H/d \). The right hand bound is therefore invariant. It is not, unfortunately, a lattice basis invariant.

Below is a simplified statement of Theorem B. We assume a centered, uniform Gaussian distribution with coefficients \( \rho_{i,a} = 1 \). The expectancy grows mildly in...
tems of $n$ and $H$. However, it grows as the square of the generalized degrees. In the unmixed case $A_1 = \cdots = A_n$, the number $V'$ is precisely the area of $\text{Conv}(A_i)$.

**Corollary 1.5.14.** Under the hypotheses of Theorem [B] with $\rho_{\mathbf{a}} = \rho_i$, 
\[
\mathbb{E}_{\mathbf{g} \sim N(0,f,\mathcal{F})} \left( \sum_{z \in Z_{\mathcal{F}}(\mathbf{g})} \mu^2(\mathbf{g} \cdot R(z), 0) \right) \leq \frac{25 \sqrt{n}}{\det(\Lambda)} \left( \frac{\log(n)}{\min(S_i)} + 2 \log(3/2) \right)^2 \times \max_i \left( \sum \delta_i^2 \right) (n-1)!V'
\]

**Example 1.5.15 (dense case).** Assume that $n, d \in \mathbb{N}$ and $A_i = \{a \in \mathbb{N}_0^n : \sum a_i \leq d\}$ in that case $V = \frac{n}{\pi^2}, V' = \frac{d}{n+1}((n+\sqrt{n})^2) = \left( n + \frac{d}{n} \right)$. Choosing the center of gravity as the origin, $\delta_i = d \left( 1 - \frac{n+1}{(n+1)^2} \right) \leq d$. We obtain a bound of
\[
O \left( Hn^{3/2}d^{n+1} \left( \frac{d+n}{n} \right)^2 \right).
\]

The only known results that are vaguely similar to Theorem [B] are Theorems 1 and 5 by Malajovich and Rojas (2004). The mixed case (Theorem 5) depends on a quantity called the \textit{mixed dilation}. This is equal to 1 in the unmixed case, but the mixed dilation is hard to bound in general. The definition of the condition number is different but coincides up to scaling, in the unmixed case, with the definition here: what appears as $\mu$ is actually $\sqrt{n}\tilde{\mu}$ in this paper. Volumes are also differently scaled. There is no renormalization, so there is no need for $H$. We obtain an imperfect comparison to Theorem 1 of Malajovich and Rojas (2004), after rescalings:

**Theorem 1.5.16.** Let $A_1 = \cdots = A_n$ and $\rho_{\mathbf{a}} = 1$. Then,
\[
\text{Prob}_{\mathbf{g} \sim N(0,f,\mathcal{F})} \left[ \max_{z \in Z(\mathbf{g})} \mu(\mathbf{g}, z) > \epsilon^{-1} \right] \leq n(n+1)(S_i - 1)(S_i - 2) n!V \epsilon^4.
\]

Recall that $\mu(\mathbf{g}, z) \geq 1$. Integrating with respect to $t = \epsilon^{-2}$, we recover
\[
\mathbb{E}_{\mathbf{g} \sim N(0,f,\mathcal{F})} \left( \max_{z \in Z(\mathbf{g})} \mu^2(\mathbf{g} \cdot R(z), 0) \right) \leq n(n+1)(S_i - 1)(S_i - 2) n!V \int_1^{\infty} t^{-2} \, dt = n(n+1)(S_i - 1)(S_i - 2) n!V
\]

and none of the bounds implies the other. Finally, we state below another simplified version of Theorem [B]. It is the cornerstone of this paper, as it is part of the proof of Theorems [B] and [5.1.1]. It can be directly compared to Theorem 18.4 by Bürgisser and Cucker (2013).

**Theorem 1.5.17.** Let $A_1, \ldots, A_n \in \mathbb{R}^n$ be such that the $\mathbb{Z}$-module $\Lambda$ generated by $\bigcup_{i=1}^n A_i - A_i$ is a subset of $\mathbb{Z}^n$, and suppose that the mixed volume $V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n))$ is non-zero. Let $\mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n}$. Let $S_i = \#A_i\dim_{\mathbb{C}}(\mathcal{F}_{A_i}) \geq 2$. Let $\Sigma$ be diagonal and positive definite in $\mathcal{F}$. Fix some real number $H > 0$, and denote by $Z_H(\mathbf{q})$ be the set of isolated roots of $\mathbf{q} \in \mathcal{F}$ in $\mathcal{M}$ with $\|\text{Re}(\mathbf{z})\|_\infty \leq H$. Then, regardless of $\tilde{f} \in \mathcal{F}$,
\[
\mathbb{E}_{\mathbf{q} \sim N(\tilde{f},\Sigma^2)} \left( \sum_{z \in Z_H(\mathbf{q})} \|M(\mathbf{q}, z)^{-1}\|^2_2 \right) \leq \frac{2H\sqrt{n}}{\det(\Lambda) \min_{\mathbf{a}} \sigma_{\mathbf{a}}^n} (n-1)!V'.
\]
In this paper we pursue the latter choice. Other remedy is to show that roots with a large infinity norm have low probability. It is to change coordinates ‘near infinity’ and use another sort of renormalization. The hindrance to renormalized homotopy. There are two obvious remedies. One of them

To pay for greater generality is of modest

\[ \frac{n(n+1)\sqrt{\sigma^2}}{\sigma^2} \]

while the particular case of Theorem \ref{thm:1.5.17} would be

\[ \frac{2H\sqrt{n(n+1)(n-1)!V'}}{n!V}. \]

The isoperimetric ratio in this example is \( \frac{(n-1)!V'}{n!V} = \frac{n+\sqrt{n}}{d}. \) As we see, the price to pay for greater generality is of modest \( O(H/n/d). \)

1.6. On infinity. Theorem \ref{thm:1.5.17} suggests that roots with large infinity norm are a hindrance to renormalized homotopy. There are two obvious remedies. One of them is to change coordinates ‘near infinity’ and use another sort of renormalization. The other remedy is to show that roots with a large infinity norm have low probability. In this paper we pursue the latter choice.

Recall that the toric variety was defined as the Zariski closure

\[ V = \{(V(x)) : x \in \mathcal{M} = \{(V_1(x)),\ldots,(V_n(x)) : x \in \mathcal{M}\} \subset \mathbb{P}(\mathcal{F}_1) \times \cdots \times \mathbb{P}(\mathcal{F}_n). \]

Points of the form \([V(x)], x \in \mathcal{M}\) are deemed finite, all the other points are said to be at toric infinity.

Example 1.6.1 (Linear case). Assume that \( A_i = \Delta_n = \{0,e_1,\ldots,e_n\} \) for \( i = 1,\ldots,n \). Then for any nonempty proper subset \( B \) of \( A_i \) we set \( y_j = 0 \) for \( j \in B \), \( y_j = -1 \) for \( j \notin B \). For any \( 1 \leq j \leq n \), choose \(-\pi < \omega_j < \pi\). Define

\[ x(t) = \left( \begin{array}{c} (y_1 - y_0)t + \omega_1 \sqrt{t} \\ (y_2 - y_0)t + \omega_2 \sqrt{t} \\ \vdots \\ (y_n - y_0)t + \omega_n \sqrt{t} \end{array} \right) \]

so the point \([w] = \lim_{t \to +\infty} [V(x(t))]\) is a point at toric infinity. The reader can easily check that each choice of \( B \) defines a different set of points at toric infinity in \( Y \), with \( w_\alpha \neq 0 \) if and only if \( \alpha \in B \).

In order to clarify what does ‘toric infinity’ look like in general, we may introduce the Legendre transform associated to each polytope,

\[ \lambda_i : \mathbb{R}^n \to \mathbb{R} \quad \xi \to \lambda_i(\xi) \overset{\text{def}}{=} \max_{\alpha \in A_i} \langle a, \xi \rangle. \]

Always, \( \alpha \xi - \lambda_i(\xi) \leq 0 \), \( \forall \alpha \in A_i \). The convex closure \( \text{Conv}(A_i) \) is the intersection of all half-spaces \( x\xi - \lambda_i(\xi) \leq 0 \) for \( \xi \in S^{n-1} \). Its ‘supporting’ facet in the direction \( \xi \) is \( \text{Conv}(A_i^\xi) \) for the subset

\[ A_i^\xi = \{ \alpha \in A_i : \lambda_i(\xi) - \langle a, \xi \rangle = 0 \}. \]
We also define
\[ \eta_i : \mathbb{R}^n \rightarrow \mathbb{R} \]
\[ \xi \rightarrow \eta_i(\xi) = \min_{a \in A_i \setminus A_i^\xi} \lambda_i(\xi) - a \xi \]
Each function \( \eta_i \) is lower semi-continuous, with \( \inf_{\xi \in S^{n-1}} \eta_i(\xi) = 0 \). Yet, the \( \eta_i \) will provide us with an important invariant to assess the ‘quality’ of a tuple of supports \((A_1, \ldots, A_n)\). Before, we need to introduce the associated \emph{fan}, which is the dual structure to this tuple.

**Definition 1.6.2.** For any tuple of non-empty subsets \( B_1 \subset A_1, \ldots, B_n \subset A_n \), the \emph{open cone above} \((B_1, \ldots, B_n)\) is
\[ C(B_1, \ldots, B_n) = \{ 0 \neq \xi \in \mathbb{R}^n : B_i = A_i^\xi \} \]
The \emph{closed cone} \( \bar{C}(B_1, \ldots, B_n) \) is the topological closure of \( C(B_1, \ldots, B_n) \) in \( \mathbb{R}^n \).

The open cone above \((B_1, \ldots, B_n)\) may possibly be the empty set. It is always a polyhedral cone, and its closure is either empty or a pointed closed cone. For \( j = 0, \ldots, n-1 \), let \( \mathfrak{F}_j \) be the set of non-empty oriented \( j+1 \)-dimensional closed cones of the form \( C(B_1, \ldots, B_n) \) for \( \emptyset \neq B_k \subset A_k \).

**Definition 1.6.3.** The \emph{fan} associated to the tuple \((A_1, \ldots, A_n)\) is the tuple \((\mathfrak{F}_{n-1}, \ldots, \mathfrak{F}_0)\).

**Remark 1.6.4.** If \( \mathbb{Z}^\delta_i \) denotes the \( \mathbb{Z} \) algebra of \( \mathfrak{F}_j \), then we obtain an exact sequence
\[ \mathbb{Z}^\delta_{n-1} \overset{\partial}{\rightarrow} \cdots \overset{\partial}{\rightarrow} \mathbb{Z}^\delta_0 \overset{\partial}{\rightarrow} \mathbb{Z}^{(0)} \rightarrow 0 \]
where \( \partial \) is the border operator. Moreover, the union of the cones in \( \bigcup_{0 \leq i \leq n-1} \mathfrak{F}_i \) is \( \mathbb{R}^n \setminus \{0\} \).

The quality or \emph{condition} of the tuple of supports can now be measured in terms of the face gaps
\[ \begin{align*}
\eta_i &= \min_{\xi \in \mathfrak{F}_0 \cap S^{n-1}} \eta_i(\xi) \quad \text{and} \quad \eta = \min_{i} \eta_i.
\end{align*} \tag{10} \]
(See figure 4). A related invariant is the gap associated to the lattice \( \Lambda \) generated by \( \cup A_i - A_i \):
\[ \eta_\Lambda = \min_{\xi \in \mathfrak{F}_0 \cap S^{n-1}} \min_{b \in \Lambda : b \xi \neq 0} |b \xi|. \]

We will prove that if a system of sparse exponential sums has a root with a large infinity norm, then it is close to the variety of systems with a root at toric infinity:

**Theorem 1.6.5.** Let \( H > 0 \). Let \( q \in \mathcal{F} \) and suppose that there is \( z \in Z(q) \) with \( \|\text{Re}(z)\|_2 \geq H \). Then, there are \( \xi \in \mathfrak{F}_0 \cap S^{n-1} \) and \( r \in \mathcal{F} \) such that
\[ \frac{\|r_i\|}{\|q_i\|} \leq \kappa_p e^{-\eta_i H/n}, \quad i = 1, \ldots, n \]
with the property that \( \lim_{t \rightarrow \infty} |V(z + t \xi)| \) is a root at infinity for \( q + r \). This means that for all \( i \),
\[ (q_i + r_i) \cdot \left( \lim_{t \rightarrow \infty} \frac{1}{\|V_{A_i}(z + t \xi)\|} V_{A_i}(z + t \xi) \right) = 0. \]
Let \( \Sigma^\infty \) be the set of all systems \( \mathbf{p} \in \mathcal{F} \) that admit a root at toric infinity. The condition number theorem above says that the ‘reciprocal condition’ of \( \mathbf{q} \) with respect to \( \Sigma^\infty \) is bounded above by \( S_i \kappa_{p_i} e^{-\eta H/n} \). The role of \( \Sigma^\infty \) can be clarified by translating Bernstein’s second theorem into the language of this paper.

**Theorem 1.6.6** (Bernstein [1975], Theorem B). Assume that the mixed volume \( V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)) \) does not vanish. The mixed volume bound in Theorem 1.5.2 (resp. 1.5.6) for the number of isolated roots is attained if and only if \( \mathbf{f} \notin \Sigma^\infty \).

**Remark 1.6.7.** The Theorem above says nothing about the condition of the isolated roots for \( \mathbf{f} \notin \Sigma^\infty \). However, it clarifies that isolated roots have multiplicity one.

**Remark 1.6.8.** From Lemma 1.3.7, we immediately recover the estimate

\[
(11) \quad \eta_{\Lambda} \leq \eta \leq \eta_i \leq \text{diam}(\text{Conv}(A_i)) \leq 2\delta_i
\]
Now we need to consider homotopy paths, and we pick the easiest example: set $q(t) = g + tf$, where $f$ is fixed and $g$ is a Gaussian random variable. But we will need paths avoiding $\Sigma^\infty$. This is possible if we take $t$ real.

At this point we need to mention a particular class of tuples $(A_1, \ldots, A_n)$ that guarantee that $\Sigma^\infty$ is empty, for $q_i \neq 0$. Recall that the tropical semi-ring is $\mathbb{R} \cup \{-\infty\}$, with sum $x \boxplus y = \max(x, y)$ and product $x \boxtimes y = x + y$. We will keep the notation $x^a = \boxtimes a$ times. To a tropical polynomial $\boxplus a_i G_i a_i X^a$ we associate the set of $X$ where the maximum of $f_a \boxtimes X^a$ is attained twice. This is called the tropical surface associated to the polynomial. To a system of tropical polynomial equations, one associates the tropical prevariety, that is the intersection of the tropical surfaces from each equation.

**Definition 1.6.9.** A system of $n$-variate polynomials $G_1, \ldots, G_n$, with support $(A_1, \ldots, A_n)$,

$$G_i(X) = \sum_{a \in A_i} G_i a_i X^a \quad i = 1, \ldots, n,$$

$G_i a_i \neq 0$, is **strongly mixed** if and only if, the tropical polynomial system

$$H_i(\xi) = \boxplus_{a \in A_i} \xi^a$$

has tropical prevariety $\{0\}$. The same definition holds for exponential sums, and we will loosely say that $(A_1, \ldots, A_n)$ is a strongly mixed support.

**Remark 1.6.10.** The tropicalization of the polynomial $G_i(X)$ is usually defined as

$$\text{trop}(G_i) = \boxplus_{a \in A_i} (-v(G_i a_i)) \boxtimes \xi^a$$

where $v$ is a non-trivial valuation. This is different from the polynomials $H_i(\xi)$ above. For more details on tropical geometry, the reader is refered to the book by Maclagan and Sturmfels (2015).

**Remark 1.6.11.** Let $n \geq 2$. A system that is strongly mixed cannot be unmixed. It cannot have a repeated support. It cannot have a support that is a scaled translation of another support.

We expect generic systems of $m$ tropical polynomials in $n$ variables to have $n - m$ dimensional tropical prevarieties, but the system above is not generic. All the tropical monomials have identical coefficient, so this particular tropical prevariety is a union of polyhedral cones, plus the origin. More precisely, the tropical prevariety is the union of the origin and the cones of a (possibly empty) subset of $\mathfrak{f}_{n-2} \cup \cdots \cup \mathfrak{f}_0$.

**Example 1.6.12 (Dense linear case).** If $A_1 = \cdots = A_m = A = \Delta_n = \{0, e_1, \ldots, e_n\}$, then the tropical prevariety is the union of all the cones in $\mathfrak{f}_{n-2}$.

**Lemma 1.6.13.** Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be finite, with $A_i - A_i \subset \mathbb{Z}^n$. The following are equivalent:

(a) The exponential sum

$$\sum_{a \in A_i} q_i a_i z^a, \quad i = 1, \ldots, n$$

is strongly mixed.

(b) For each $\xi \in \mathfrak{f}_0 \cap S^{n-1}$, there is $i$ such that $|A_i^\xi| = 1$. 
\( \Sigma^\infty \) is contained in an union of \( \#\mathcal{S}_0 \) hyperplanes of the form \( q_{ia} = 0 \).

Before stating the next result, we introduce the polynomial
\[
v(t) \overset{\text{def}}{=} \frac{\partial}{\partial \epsilon} \Bigg|_{\epsilon=0} \text{Vol}(\text{Conv}(A_1) + \epsilon B^n + t A, \ldots, \text{Conv}(A_n) + \epsilon B^n + t A)
\]
\[
= \sum_{k=0}^{n} v_k t^k.
\]

where \( B^n \) denotes the unit \( n \)-ball, \( A = \text{Conv}(A_1 + \cdots + A_n) \) and \( V \) is the mixed volume. Notice that \( v_0 = V' \) is the mixed surface. When \( A_1 = \cdots = A_n \), \( v(t) = (1 + tn)^{n-1}V' \). The coefficients \( v_k \) can be seen up to scaling as a mixed, non-smooth analogue of the curvature integrals in Weyl's tube formula [Bürgisser and Cucker, 2013, Theorem 21.9].

**Main Theorem C.** Let \( A_1, \ldots, A_n \subset \mathbb{Z}^n \) be finite, with non-zero mixed volume. Under the notations above, the following hold:

(a) The set \( \Sigma^\infty \) is contained in the zero set of a polynomial of degree
\[
d_r \leq \frac{e^2 \eta \Lambda}{\sqrt{4\pi} \det \Lambda} \max_{k \geq 0} (n - k! k! v_k) \#\mathcal{S}_0
\]

(b) If the system is strongly mixed, then \( d_r = \#\mathcal{S}_0 \) and \( \Sigma^\infty = Z(r) \).

(c) Assume that \( r(f) \neq 0 \). Then, the set \( \Sigma_f^\infty = \{ g \in \mathcal{F} : \exists t \in \mathbb{R}, g + tf \in \Sigma^\infty \} \) is contained, as a subset of \( \mathbb{R}^{2S} \), in the zero set of a real polynomial \( s = s(g) \) of degree at most \( d_r^2 \).

When \( f \) is fixed with \( r(f) \neq 0 \), the set of paths with some root of large norm is a neighborhood of \( \Sigma_f^\infty \) in the usual topology. More specifically, define
\[
\Omega_H = \Omega_{f,T,H} = \{ g \in \mathcal{F} : \exists t \in [0,T], \exists z \in Z(\mathcal{M},\|\text{Re}(z)\|_\infty \geq H) \}.
\]

At this point and in the next section, we choose always coefficients \( \rho_{ia} = 1 \) and variance \( \Sigma^2 = I \), in order to keep statements short.

**Theorem 1.6.14.** Assume that \( \rho_{ia} = 1 \) for all \( i, a \). Assume that \( r(f) \neq 0 \). Let
\[
0 < \delta < \frac{1}{2(2d_r^2 + 1)S} \text{ and assume that }
\]
\[
H \geq \frac{n}{\eta} \log \left( 16e d_r^{-1} d_r^2 S \max_i (S_i) \left( 1 + \frac{T|f|_{\mathcal{F}}}{\sqrt{\delta \sqrt{S(1-S)}}} \right) \right).
\]

Then,
\[
\text{Prob}_{g \sim N(0,t;\mathcal{F})} [g \in \Omega_H] \leq \delta.
\]

**Remark 1.6.15.** The coefficients \( v_k \) can be bounded in terms of a more classical-looking Quermassintegral. Indeed, let
\[
w(\tau) \overset{\text{def}}{=} \text{Vol}(\text{Conv}(A_1) + \tau B^n, \ldots, \text{Conv}(A_n) + \tau B^n)
\]
\[
= \sum_{k=0}^{n} w_k \tau^k.
\]
The first terms are \( w_0 = V \) and \( w_1 = V' \). Since \( \mathcal{A} \subset (\sum_{i=0}^{n} \delta_i)B^n \), we bound

\[
v_k \leq n\left( \sum_{i=0}^{n} \delta_i \right)^k w_{k+1} \leq n\left( \sum_{i=0}^{n} \delta_i \right)^{n-1} \left( \frac{n}{k+1} \right) \text{Vol}(B^n).
\]

In particular,

\[
\log(d_r) \leq O\left( \log(\#\mathcal{S}_0) + n \log(n) + \log(\eta_\Lambda/\det(\Lambda)) + n \log\left( \sum_{i=0}^{n} \delta_i \right) \right).
\]

**Remark 1.6.16.** The coefficients \( w_k \) from the preceding remark are closely related to well-known invariants in convex geometry, the intrinsic volumes \( V_k(C) \) of a convex body \( C \). In the unmixed case \( A_1 = A_2 = \cdots = A_n \), one has

\[
\left( \begin{array}{c} n \\ k \end{array} \right) w_k = \text{Vol}(B^{n-k})V_k(\text{Conv}(A_1)).
\]

1.7. **Analysis of the homotopy algorithm.** The ‘renormalization’ process comes at a cost. We will define below three sets that must be excluded from the choice of \( \mathbf{g} \) for the algorithm to behave well. Choices of \( \mathbf{g} \) in one of those sets may lead to a ‘failed’ computation, and we have to start over.

The first of those sets is easy to describe and easy to avoid. It corresponds to paths \( q_t = \mathbf{q} + t\mathbf{f} \) with no known decent upper bound for the renormalized speed vector

\[
\left\| \frac{\partial}{\partial t} (\mathbf{q}_t \cdot R(\mathbf{z}_t)) \right\|_{\mathbf{q}_t \cdot R(\mathbf{z}_t)}
\]

for some continuous \( \mathbf{z}_t \in \mathcal{Z}(\mathbf{q}_t) \). We will see that this set is confined to a product of slices in the complex plane, one slice from each coordinate:

\[
\Lambda_i \overset{\text{def}}{=} \{ \mathbf{g} \in \mathcal{F} : \exists 1 \leq i \leq n, \exists \mathbf{a} \in A_i, \arg\left( \frac{g_{ia}}{\bar{f}_{ia}} \right) \geq \pi - \epsilon \}.
\]

Notice that \( \text{Prob}_{\mathbf{g} \sim \mathcal{N}(0,1,\mathcal{F})} [\mathbf{g} \in \Lambda_i] \leq S \frac{S}{\sqrt{2}} \), \( S = \dim_c(\mathcal{F}) \). For simplicity we will take \( \epsilon = \frac{\pi}{7} \) so that \( \text{Prob}_{\mathbf{g} \sim \mathcal{N}(0,1,\mathcal{F})} [\mathbf{g} \in \Lambda_i] \leq 1/72 \).

The second exclusion set is more subtle. We want to remove the set

\[ \Omega_H = \Omega_{L, T, H} \]

from Theorem 1.6.14. While we do not know a priori if \( \mathbf{g} \in \Omega_H \), we can execute the path-following algorithm and stop in case of failure, that is in case \( \| \text{Re}(\mathbf{z}(t)) \|_{\infty} \geq H \). Failure is also likely if \( \mathbf{f} \) has a root at infinity. We pick \( H \) so that \( \delta = \text{Prob}_{\mathbf{g} \sim \mathcal{N}(0,1,\mathcal{F})} [\mathbf{g} \in \Omega_H] \leq \frac{1}{2^{1/72 + 1/72}} \) in order to apply Theorem 1.6.14. Since \( S \geq 4 \) and \( d_r \geq \#\mathcal{S}_0 \geq 2 \), we have always \( \delta \leq \frac{1}{72} \). We deduce that with probability \( \geq 71/72 \), \( \mathbf{g} \notin \Omega_H \) for

\[ H = \frac{n}{\eta} O\left( \log(d_r) + \log(S) + \log(T) \right) \]

The third set to avoid is

\[ Y_K = \{ \mathbf{g} \in \mathcal{F} : \exists i, \| g_i \| \geq K \sqrt{S_i} \} \]

Using the large deviations estimate, we will show that for \( K = 1 + \sqrt{\frac{\log(n) + \log(10)}{\min(S_i)}} \) we have \( \text{Prob}_{\mathbf{g} \sim \mathcal{N}(0,1,\mathcal{F})} [\mathbf{g} \in Y_K] \leq 1/10 \).

**Main Theorem D.** There is a constant \( C \) with the following property:
(a) Assume that \( A_1, \ldots, A_n \in \mathbb{R}^n \) are such that the \( \mathbb{Z} \)-module \( \Lambda \) generated by \( \bigcup_{i=1}^{n} A_i - A_i \) is a subset of \( \mathbb{Z}^n \), and that the mixed volume \( V = V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)) \) is non-zero. Also, let \( V' \) be the mixed surface as in \( \text{[1]} \).

(b) Let \( \mathcal{F} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_n} \) where it is assumed that for all \( i, a \in A_i, \rho_{i,a} = 1 \). Denote by \( S_i \) the complex dimension of \( \mathcal{F}_{A_i} \), \( S_i = \#A_i \), and assume that \( S_i \geq 2 \).

(c) Let \( f \in \mathcal{F} \) with \( r(f) \neq 0 \) where \( r \) is the polynomial from Theorem [E]. Suppose also that \( f \) is scaled in such a way that \( \|f_i\| = \sqrt{S_i} \).

(d) Let \( K = \left( 1 + \sqrt{\frac{\log(n) + \log(10)}{\min(S_i)}} \right) \).

(e) Define also
\[
\kappa_f = \max_{i,a} \frac{\|f_i\|}{\|f_{i,a}\|},
\]
and
\[
\mu_f = \max_{a \in \mathbb{Z}(f)} \mu(f \cdot R(z), 0).
\]

(f) Take \( g \sim N(0, I; \mathcal{F}) \), and and consider the random path \( q_t = g + tf \in \mathcal{F} \). To this path associate the set \( \mathcal{Z}(q_t) \) be the set of continuous solutions of \( q_t \cdot V(z_t) = 0 \).

(\( g \)
\[
Q \overset{\text{def}}{=} \eta^{-2} \left( \sum_{i=1}^{n} \delta_i^2 \right) \max(n!V, n - 1!V'\eta) / \det \Lambda
\]
where \( \eta \) was defined in \( \text{[10]} \) and \( V' \) is the mixed surface as in Theorem [E].

(h) \( \text{LOGS} = \log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_f) + \log(\kappa_f) \)
where \( d_r \) is the degree of \( r \). Then with probability 1, all \( z_t \in \mathcal{Z}(q_t) \) are continuous for \( t \in [0, \infty] \). With probability \( \geq 3/4 \),
\[
\sum_{z_t \in \mathcal{Z}(q_t)} \mathcal{L}(q_t, z_t; 0, \infty) \leq CQn^{3/2} \max_i (S_i^{3/2}) K(1 + \sqrt{1 + K/4 + \kappa_f/8})
\times \kappa_f \mu_f^2 \nu_0 \text{LOGS}
\]

Remark 1.7.1. Remark [16.8] implies that \( \frac{1}{2} \leq \frac{n}{4} \leq \eta^{-2} \sum \delta_i^2 \). The number of paths satisfies with probability one:
\[
\# \mathcal{Z}(q_t) = \frac{n!V}{\Lambda} \leq 2Q.
\]

Remark 1.7.2. The quantity \( Q \) is invariant by uniform integer scaling \( A_i \rightarrow dA_i, \) \( d \in \mathbb{N} \).

Remark 1.7.3. In the particular case \( f \) is strongly mixed, \( d_r \leq \#S_0 \). Otherwise a bound for \( d_r \) is provided by Theorem [C].

The proof of Theorem [E] is postponed to Section [5].

Definition 1.7.4. Assume that \( f \in \mathcal{F} \) is non-degenerate with no root at infinity. A certified solution set for \( f \) is a set \( X \) of certified approximate roots for \( f \), with different associated roots, with
\[
\#X = n!V(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)).
\]
Theorems [A] and [D] above give complexity estimates for producing certified solution sets for some \( f \in \mathcal{F} \) from a certified solution set for a random \( g \in \mathcal{F} \) and vice-versa. First of all, assume that there is a procedure to generate a random solution sets for some \( N \) and the other is the final refining step. Hence, the total number of extra steps is at most two extra steps, one comes from the bound from Theorem A with probability one. The expected number of renormalized Newton iterations is needed to know \( Q \) again. However, the results in this paper would be completely useless if one really needed to know \( Q \). Here is what we can do:

Let \( h \in \mathcal{F} \) is given, \( r(h) \neq 0 \), together with a certified solution set \( X_h \) for \( h \). Let \( \mu_h = \max_{\zeta \in Z(h)} \mu(h \cdot R(\zeta), 0) \) be finite. We can apply the algorithm of Theorem [A] to the homotopy path \( p_t = t^{-1}g + f \), \( t \in [0, \infty] \). With probability at least 3/4, this will produce a certified solution set for \( g \). The condition length is given in Theorem [D]. We can compose the two procedures: given \( h \) and a certified solution set, produce a certified solution set for \( g \) and finally produce a certified solution set for \( f \).

**Proposition 1.7.5.** Let \( f, h \in \mathcal{F} \) be given \( r(f) \neq 0 \), \( r(h) \neq 0 \), together with a certified solution set \( X_h \) for \( h \). Let \( \kappa = \max(\kappa_f, \kappa_g) \). Let \( g \in N(0, I; \mathcal{F}) \). Then with probability \( \geq 1/2 \), the procedure above will produce a certified solution set \( X_f \) for \( f \) in at most

\[
Q(4 + \delta_f^{-1}CnS^{3/2} \max_i(S_i^{3/2})K(1 + \sqrt{1 + K/4 + \kappa/8})\kappa(\mu_f^2 + \mu_h^2)\nu_0 \ LOGS)
\]

renormalized Newton iterations.

\[
LOGS' = \log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_f) + \log(\mu_h) \log(\kappa)
\]

This algorithm can be modified to ‘give up’ for an unlucky choice of \( g \) and start again. However, the results in this paper would be completely useless if one really needed to know \( Q \), \( d_r \) and \( \mu_f \) in order to produce a probability one algorithm for homotopy. Here is what we can do:

Let

\[
N^\ast = Q(4 + \delta_f^{-1}CnS^{3/2} \max_i(S_i^{3/2})K(1 + \sqrt{1 + K/4 + \kappa/8})\kappa(\mu_f^2 + \mu_h^2)\nu_0 \ LOGS')
\]

be the exact bound. From remark [1.7.1] there are at most \( 2Q \) paths to follow. Each path requires at most two extra steps, one comes from the bound from Theorem A and the other is the final refining step. Hence, the total number of extra steps is at most \( 4Q \).

We do not assume \( N^\ast \) to be known. Since the probability \( 1/2 \) procedure of Proposition [1.7.5] requires at most \( N^\ast \) renormalized Newton iterations, we will proceed as follows:

1. Stipulate an arbitrary value \( N_0 \) and set \( k = 0 \).
2. Repeat
   2.1 Execute the algorithm of Proposition [1.7.5] up to \( N_k \) renormalized Newton iterations.
   2.2 If the algorithm terminated with a set \( X_f \) of approximate solutions for \( f \) and \( \#X_f = \#X_h \), then output \( X_f \) and terminate.
   2.3 Set \( N_{k+1} = \sqrt{2}N_k \) and increase \( k \) by one.

Eventually for some value of \( k \), \( N_k \leq N^\ast < N_{k+1} \) so this algorithm succeeds with probability one. The expected number of renormalized Newton iterations is

\[
\bar{N} = N_0 + \cdots + N_k + \frac{1}{2}N_{k+1} + \frac{1}{4}N_{k+2} + \cdots
\]
Trivially,
\[ N_0 + \cdots + N_k \leq N_\ast \left( 1 + \frac{1}{\sqrt{2}} + \left( \frac{1}{\sqrt{2}} \right)^2 + \cdots \right) = (2 + \sqrt{2})N_\ast \]

while
\[ \frac{1}{2} N_{k+1} + \frac{1}{4} N_{k+2} + \cdots \leq N_\ast \left( \frac{1}{\sqrt{2}} + \left( \frac{1}{\sqrt{2}} \right)^2 + \cdots \right) = (1 + \sqrt{2})N_\ast \]

It follows that
\[ \hat{N} \leq (3 + 2\sqrt{2})N_\ast. \]

We proved:

**Main Theorem E.** There is a probability 1 algorithm with input \( n, A_1, \ldots, A_n, f \in \mathcal{F}, h \in \mathcal{F}, X_h \) and output \( X_f \) with the following properties. If \( r(f) \neq 0, r(h) \neq 0, N_\ast \) is finite and \( X_h \) is an approximate solution set for \( h \), then \( X_f \) is an approximate solution set for \( f \). This algorithm will perform at most
\[ (3 + 2\sqrt{2})N_\ast. \]

renormalized Newton iterations on average.

If a system \( h \in \mathcal{F} \) is given together with all its solutions, \( \mu_h, \kappa_h \) are finite and \( r(h) \neq 0 \), then we can solve arbitrary systems in \( \mathcal{F} \) at a cost that depends polynomially on \( \mu_f \) and \( \kappa_f \). This leads to a non-uniform algorithm, with a non-uniform bound depending also on the supports \( A_1, \ldots, A_n \):

**Corollary 1.7.6.** Let \( \mathcal{F} \) be fixed. There is a non-uniform randomized algorithm that finds all the roots of \( f \in \mathcal{F}, r(f) \neq 0 \), with probability 1 and expected cost
\[ O(\mu_f^2 \log(\mu_f)\kappa_f \log(\kappa_f)). \]

1.8. **Related work.** [Shub (2009)] introduced the condition length in the solution variety and related it to the number of Newton steps in a homotopy continuation method. The step selection problem was dealt independently by [Beltrán (2011)] and [Dedieu et al. (2013)]. The integral bounds obtained in those papers would apply to any subspace of the space of dense polynomials. As explained before, lack of unitary invariance prevented obtaining global complexity bounds in this setting. Recently, [Ergür et al. (2019) TA] introduced new techniques in the context of real polynomial solving that may overcome this difficulty.

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2. Renormalized homotopy

| Constants in Theorem A and its proof |
|---|
| $\alpha_k = 0.074, 609, 958 \cdots$ & $\alpha_k = 0.096, 917, 682 \cdots$ & $\delta_k = 0.085, 180, 825 \cdots$ |
| $u_k = 0.129, 283, 177 \cdots$ & $u_k = 0.007, 556, 641 \cdots$ & $u_k = 0.059, 668, 617 \cdots$ |

The objective of this section is to prove Theorem A. We first prove a technical result for later use.

2.1. Technical Lemma.

**Lemma 2.1.1.** Let $f_i \in \mathcal{F}_{A_i}$ and $x \in \mathbb{C}^n$. Assume that $m_i(0) = 0$ for $i = 0, \ldots, n$. Write $\nu_0 = \nu(0)$. Then for all $i = 1, \ldots, n$,

$$d_P(f_i, f_i \cdot R_i(x)) \leq \sqrt{5} \|x\|_{i,0} \nu_i(0).$$

Moreover,

$$d_P(f, f \cdot R(x)) \leq \sqrt{5} \|x\|_0 \nu_0.$$

**Proof.** Without loss of generality, let $\|f_i\| = 1$. First, assume that $x$ is a real vector. Then,

$$d_P(f_i, f_i \cdot R_i(x)) = \inf_{t \in \mathbb{C}} \frac{\|f_i - tf_i \cdot R_i(x)\|}{\|f_i\|} \leq \sqrt{\sum_{a \in A_i} |f_i, a|^2 |1 - e^{ax - \ell(x)}|^2}$$

where we set $t = 1$ and $\ell(x) = \max_{a \in A_i} aRe(x)$, $aRe(x)$ is the Legendre transform of the trivial map $A_i \to \mathbb{R}$, $\alpha \mapsto 0$. For all $a \in A_i$, $ax - \ell(x) \leq 0$. The mean value theorem applied to $t \mapsto e^{t(ax - \ell(x))}$ implies that

$$d_P(f_i, f_i \cdot R_i(x)) \leq \sqrt{\sum_{a \in A_i} |f_i, a|^2 |ax - \ell(x)|^2} \leq \max_{a \in A_i} |ax - \ell(x)|.$$

Now, assume that $x$ is pure imaginary.

$$d_P(f_i, f_i \cdot R(x)) = \inf_{t \in \mathbb{C}} \frac{\|f_i - tf_i \cdot R_i(x)\|}{\|f_i\|} \leq \inf_{t \in \mathbb{C}} \|f_i - f_i \cdot R_i(x)\| \leq \sqrt{\sum_{a \in A_i} |f_i, a|^2 |1 - e^{ax}|^2} \leq \max_{a \in A_i} |ax|.$$

For a general $x \in \mathbb{C}^n$, triangular inequality and Theorem 1.4.2(c) imply that

$$d_P(f_i, f_i \cdot R_i(x)) \leq d_P(f_i, f_i \cdot R_i(Re(x))) + d_P(f_i, R_i(Re(x))),$$

$$\leq \max_{a \in A_i} |a(Re(x)) - \ell(x)| + \|f_i \cdot R_i(Re(x))\| \max_{a \in A_i} |a(Im(x))|$$

$$\leq 2 \max_{a \in A_i} |a(Re(x))| + \max_{a \in A_i} |a(Im(x))|$$

$$\leq \sqrt{5} \|x\|_{i,0} \nu_i(0)$$
where the last inequality comes from:
\[ \max_{c_i^2, x_i^2 \leq 1} 2c + s = \max_{0 \leq t \leq 2\pi} 2\cos(t) + \sin(t) = \sqrt{5}. \]

Finally,
\[ d_P(f, f \cdot R(x))^2 = \sum_i d_P(f_i, f_i \cdot R_i(x))^2 \leq 5\nu_i(0) \sum_i \|x\|_i^2 \leq 5\nu_0\|x\|_0^2. \]

2.2. Proof of Theorem \[A\]. We claim first that for \(\alpha_a\) small enough, the recurrence \[4\] of Definition \[1.4.6\] is well-defined in the sense that given previously produced \(t_j < T\) and \(x_j\), there is \(t_{j+1} > t_j\) satisfying the condition in \[4\]. This will follow from the intermediate value theorem after replacing \(f\) by \(q_{t_j}\) in the Lemma below.

**Lemma 2.2.1.** Assume that
\[ \frac{1}{2} \beta(f \cdot R(x_j), 0) \mu(f \cdot R(x_j), 0) \nu_0 \leq \alpha \leq \alpha_0. \]
Moreover, set \(x_{j+1} = N(fR(x_j), 0) + x_j\) as in \[4\]. Then,
\[ \frac{1}{2} \beta(f \cdot R(x_{j+1}), 0) \mu(f \cdot R(x_{j+1}), 0) \nu_0 \leq \alpha^2 \frac{(1 - \alpha) \psi(\alpha)(1 - 2\sqrt{5} \alpha)}{\psi(\alpha) (1 - 2\sqrt{5} \alpha)}. \]

Numerically, the bound in the right-hand side of \[12\] is smaller than \(\alpha\) for all \(0 < \alpha \leq 0.155, 0.98\ldots\) and \(\alpha_a < 1.555\).

**Proof.** Proposition \[1.3.9\] applied to \((f \cdot R(x_j), 0)\) yields
\( \beta(f \cdot R(x_j), 0) \gamma(f \cdot R(x_j), 0) \leq \frac{1}{2} \beta(f \cdot R(x_j), 0) \mu(f \cdot R(x_j), 0) \nu_0 \leq \alpha. \)
Let \(y_0 = x_j\) and \(F(y) = f \cdot V(y)\). Lemma \[1.4.3\] implies that
\( \beta(F, y_0) \gamma(F, y_0) \leq \alpha \leq \alpha_0 \)
so we are in the conditions of Theorem \[1.4.5\]. Moreover, \(x_{j+1} = y_1 = N(F, y_0)\) so that according to Shub and Smale (1993a, Prop. 3 p.478),
\[ \beta(F, y_1) \leq \frac{1 - \alpha}{\psi(\alpha)} \alpha \beta(F, y_0) \]
where \(\psi(u) = 1 - 4u + 2u^2\). Let \(g_i = f_i \cdot R(x_j) = f_i \cdot R(y_0)\). Lemma \[2.1.1\] yields
\[ d_P(f_i \cdot R(x_j), f_i \cdot R(x_{j+1})) = d_P(g_i, g_i \cdot R(x_{j+1} - x_j)) \leq \sqrt{5} \|x_{j+1} - x_j\|_0 \nu_0. \]
Hence,
\[ d_P(f \cdot R(x_j), f \cdot R(x_{j+1})) \leq \sqrt{5} \|x_{j+1} - x_j\|_0 \nu_0 = \sqrt{5} \beta(f \cdot R(x_j), 0) \nu_0 \]
From Proposition \[1.3.9\],
\[ \mu(f \cdot R(x_{j+1}), 0) \leq \frac{\mu(f \cdot R(x_j), 0)}{1 - \mu(f \cdot R(x_j), 0)} d_P(f \cdot R(x_j), f \cdot R(x_{j+1})) \leq \mu(f \cdot R(x_j), 0) \frac{1 - 2\sqrt{5} \alpha}{1 - 2\sqrt{5} \alpha} \]
Putting all together,
\[ \frac{1}{2} \beta(f \cdot R(x_{j+1}), 0) \mu(f \cdot R(x_{j+1}), 0) \nu_0 \leq \alpha^2 \frac{(1 - \alpha)}{\psi(\alpha)(1 - 2\sqrt{5} \alpha)}. \]
\[ \square \]
Equation (6) follows from item (2) of the Lemma 2.2.2 below. The other two items will be instrumental to the complexity bound.

Lemma 2.2.2. With the notations of Theorem 1.4.5, let $u_0 = \frac{5}{4}$. For $0 < \alpha < \alpha_0$, define

\[
\begin{align*}
&u_\alpha = \frac{\alpha r_0(\alpha)}{1 - 2\sqrt{r_0(\alpha)\alpha}} \\
u_{\alpha} = \frac{\alpha r_1(\alpha)}{1 - 2\sqrt{r_1(\alpha)\alpha}}
\end{align*}
\]

and $u_{\alpha\alpha} = \frac{\psi(u_\alpha)}{u_\alpha + \psi(u_\alpha)} \alpha$. Then,

(a) for $t_j \leq t \leq t_{j+1}$, $u_j(t) < u_\alpha < u_0$,  
(b) $u_j(t_j) \leq u_{\alpha\alpha}$, and  
(c) $u_j(t_{j+1}) \geq u_{\alpha\alpha}$.

The graphs of $u_\alpha$, $u_{\alpha\alpha}$ and $u_{\alpha\alpha\alpha}$ as functions of $\alpha$ are plotted in Figure 5. In view of Lemma 2.2.2, we define $\alpha_{\alpha\alpha} \approx 0.096,917\ldots$ as the solution of $u_\alpha(\alpha_{\alpha\alpha}) = u_0$. The bound in the right-hand side of (12) is strictly smaller than $\alpha$ for all $0 < \alpha \leq \alpha_{\alpha\alpha} \leq 0.155,098\ldots$. Also notice that $\alpha_{\alpha\alpha}$ is smaller than the constant $\alpha_0$ from Theorem 1.4.3.

Proof. Apply Smale’s alpha-theorem (Theorem 1.4.5) to the point $y_1 = x_{j+1}$, for $F(y) = q_j \cdot V(y)$, $t_j \leq t \leq t_{j+1}$. From the construction of $t_{j+1}$ in (4) we know that $\alpha(F, y) = \alpha(F, x_{j+1}) \leq \alpha \leq \alpha_0$. Theorem 1.4.5(b) asserts that the Newton iterates of $y_1$ converge to a zero $z_\alpha$ of $F$ with

\[
\|y_1 - z_\alpha\|_0 \leq r_0(\alpha) \beta(q_j R(x_{j+1}), 0)
\]
From Proposition 1.3.9 [b] followed by Lemma 2.1.1 by the bound above, then by the hypothesis \( \frac{1}{2} \beta(q_t \cdot R(y_1), 0) \mu(q_t \cdot R(y_1), 0) \nu_0 \leq \alpha \):

\[
\mu(q_t \cdot R(z_t), 0) \leq \frac{\mu(q_t \cdot R(y_1), 0)}{1 - \mu(q_t \cdot R(y_1), 0) d_p(q_t \cdot R(y_1), q_t \cdot R(z_t))} \\
\leq \frac{\mu(q_t \cdot R(y_1), 0)}{1 - \sqrt{5} \mu(q_t \cdot R(y_1), 0) \|y_1 - z_t\|_0 \nu_0} \\
\leq \frac{\mu(q_t \cdot R(y_1), 0)}{1 - 2 \sqrt{5} r_0(\alpha) \alpha}
\]

Therefore,

\[
u_j(t) = \frac{1}{2} \|z_t - x_{j+1}\| \mu(q_t \cdot R(z_t), 0) \nu_0 \leq \frac{\alpha r_0(\alpha)}{1 - 2 \sqrt{5} r_0(\alpha) \alpha} = u_*.
\]

By construction \( u_* < u_0 \), and equation (6) follows.

We may obtain a sharper estimate for \( u_j(t_j) \), since \( y_1 = x_{j+1} \) is the iterate of \( y_0 = x_j \). In that case,

\[\|y_1 - z_t\|_0 \leq r_1(\alpha) \beta(q_t \cdot R(x_j), 0)\]

and by the very same reasoning,

\[
u_j(t_j) \leq \frac{\alpha r_1(\alpha)}{1 - 2 \sqrt{5} r_1(\alpha) \alpha} = u_{**}
\]

If \( t_j+1 \neq T \), then by construction

\[
\frac{1}{2} \beta(q_t \cdot R(x_{j+1}), 0) \mu(q_t, R(x_{j+1}), 0) \nu_0 = \alpha
\]

Thus,

\[
\frac{1}{2} \|x_{j+1} - z_{t_{j+1}}\|_0 \mu(q_t \cdot R(x_{j+1}), 0) \nu_0 \leq u_* < u_0
\]

Let \( y_0 = x_{j+1} \). From [Blum et al., 1998 Proposition 1 p. 157] the Newton iterate \( y_1 = N(F, y_0) \) satisfies

\[\|y_1 - z_{t_{j+1}}\|_0 \leq \frac{u_*}{\psi(u_*)} \|y_0 - z_{t_{j+1}}\|_0.\]

Therefore,

\[
\beta(q_t \cdot R(x_{j+1}), 0) = \|y_0 - y_1\|_0 \\
\leq \|y_0 - z_{t_{j+1}}\|_0 + \|z_{t_{j+1}} - y_1\|_0 \\
\leq \left(1 + \frac{u_*}{\psi(u_*)}\right) \|x_{j+1} - z_{t_{j+1}}\|_0.
\]

It follows that

\[
\alpha \leq \frac{1}{2} \left(1 + \frac{u_*}{\psi(u_*)}\right) \|x_{j+1} - z_{t_{j+1}}\|_0 \mu(q_t \cdot R(x_{j+1}), 0) \nu_0 \\
\leq \left(1 + \frac{u_*}{\psi(u_*)}\right) u_j(t_{j+1})
\]

Since \( u_{**} = \frac{\psi(u_*)}{\psi(u_*) + u_*} \alpha \),

\[u_{**} \leq u_j(t_{j+1}).\]
Towards the proof of Theorem A let \( \mu_j = \mu(q_{t_j} \cdot R(z_{t_j}), 0) \) and let

\[
d_{\text{max}}(t) = \max_{t_j \leq t \leq t_{j+1}} d_P(q_{t} \cdot R(z_{t}), q_{t_j} \cdot R(z_{t_j})).
\]

Clearly \( d_{\text{max}}(0) = 0 \) and \( d_{\text{max}}(t) \) is a continuous function.

**Lemma 2.2.3.**

\[
d_{\text{max}}(t_{j+1}) \mu_j \leq (1 + d_{\text{max}}(t_{j+1}) \mu_j) \mathcal{L}(t_j, t_{j+1}).
\]

Furthermore if \( \mathcal{L}(t_j, t_{j+1}) < 1 \),

\[
d_{\text{max}}(t_{j+1}) \mu_j \leq \frac{\mathcal{L}(t_j, t_{j+1})}{1 - \mathcal{L}(t_j, t_{j+1})}.
\]

**Proof.** The projective distance is always less than the Riemannian metric, since they share the arc length element. The Riemannian distance between two points is smaller or equal than the Riemannian length of an arbitrary path between those two points. We obtain the upper bound

\[
d_{\text{max}}(t_{j+1}) \leq \max_{t_j \leq \tau \leq t_{j+1}} \int_{t_j}^{\tau} \left\| \frac{\partial}{\partial \tau} (q_{\tau} \cdot R(z_{\tau})) \right\|_{q_{\tau} \cdot R(z_{\tau})} d\tau
\]

and \( 0 \leq d_{\text{max}}(t) \leq d_{\text{max}}(t_{j+1}) \) for \( t_j \leq t \leq t_{j+1} \). From the definition of \( d_{\text{max}}(t) \), we have a trivial lower bound

\[
d_P(q_{t_j} \cdot R(z_{t_j}), q_{t} \cdot R(z_{t})) \leq d_{\text{max}}(t_{j+1}).
\]

Proposition 1.3.9(b) combined with equation (13) yields the estimate

\[
\frac{\mu_j}{1 + d_{\text{max}}(t_{j+1}) \mu_j} \leq \mu(q_{t} \cdot R(z_{t}), 0) \leq \frac{\mu_j}{1 - d_{\text{max}}(t_{j+1}) \mu_j}.
\]

We can combine the upper and lower bounds:

\[
d_{\text{max}}(t_{j+1}) \mu_j \leq \int_{t_j}^{t_{j+1}} \mu_j \left\| \frac{\partial}{\partial \tau} (q_{\tau} \cdot R(z_{\tau})) \right\|_{q_{\tau} \cdot R(z_{\tau})} d\tau
\]

\[
= (1 + d_{\text{max}}(t_{j+1}) \mu_j) \int_{t_j}^{t_{j+1}} \mu(q_{\tau} \cdot R(z_{\tau}))
\]

\[
\times \left\| \frac{\partial}{\partial \tau} (q_{\tau} \cdot R(z_{\tau})) \right\|_{q_{\tau} \cdot R(z_{\tau})} d\tau
\]

\[
= (1 + d_{\text{max}}(t_{j+1}) \mu_j) \mathcal{L}(t_j, t_{j+1}).
\]

Rearranging terms under the assumption \( \mathcal{L}(t_j, t_{j+1}) < 1 \),

\[
d_{\text{max}}(t_{j+1}) \mu_j \leq \frac{\mathcal{L}(t_j, t_{j+1})}{1 - \mathcal{L}(t_j, t_{j+1})}.
\]

**Proof of Theorem A.** A path \((z_t)_{t \in [t_j, t_{j+1}]}\) can be produced as in Lemma 2.2.2 for each value of \( j \) by extending the previous definition to \( t_{j+1} \): For each \( t \in [t_j, t_{j+1}] \), define \( y_1(t) = x_{j+1} \) and inductively, \( y_{k+1}(t) \) as the Newton iterate of \( y_k(t) \) for the system \( F_i(y) = q_{t} \cdot V(y) \). Equation \( \text{(6)} \) guarantees quadratic convergence to a zero \( z_t \) because \( u_* < u_0 = \frac{3 - \sqrt{5}}{2} \), so we can apply Theorem 1.4.4 combined with
Proposition 1.3.9(c). Moreover, for \( t_j \leq t \leq t_{j+1} \), \( \|y_k(t) - z_t\|_0 \leq 2^{-2^k+1} u_\alpha \) so the convergence is uniform.

We claim that each \((z_t)_{t \in [t_j, t_{j+1}]}\) is continuous. Indeed, let \( \epsilon > 0 \). There is \( k \) such that for all \( \tau \in [t_j, t_{j+1}] \), \( \|y_k(\tau) - z_\tau\|_0 \leq \epsilon/3 \). Moreover, \( y_k(\tau) \) is continuous in \( t \) so there is \( \delta > 0 \) with the property that for all \( t' \in [t_j, t_{j+1}] \), \( |t - t'| < \delta \) implies that \( \|y_k(t) - y_k(t')\|_0 < \epsilon/3 \). Whence,

\[
\|z_t - z_{t'}\|_0 \leq \|z_t - y_k(t)\|_0 + \|y_k(t) - y_k(t')\|_0 + \|y_k(t') - z_{t'}\|_0 < \epsilon.
\]

To check that the constructed paths \((z_t)_{t \in [t_j, t_{j+1}]}\) and \((z_t)_{t \in [t_{j+1}, t_{j+2}]}\) patch together, we need to compare the end-points at \( t_{j+1} \). Recall that \( x_{j+2} = N(q_{t_{j+1}}, R(x_{j+1}, 0)) + x_{j+1} = N(F_{t_{j+1}}, x_{j+1}) \) using Lemma 1.4.3(a). Let \( \hat{y}_k(t) \) denote the \( k - 1 \)-th Newton iterate of \( x_{j+2} \) for the system \( F_t = q_t \cdot V(\hat{y}) \). By construction, \( \hat{y}_k(t_{j+1}) = y_k(t_{j+1}) \). Therefore,

\[
\lim_{k \to \infty} \hat{y}_k(t_{j+1}) = \lim_{k \to \infty} y_{k+1}(t_{j+1})
\]

and the endpoints at \( t_{j+1} \) are the same. Because of Lemma 2.2.2(b) \( \mu(q_t, R(z_t), 0) \) is finite and hence the implicit function theorem guarantees that \( z_t \) has the same differentiability class than \( q_t \).

We proceed now to the lower bound \( \mathcal{L}(t_j, t_{j+1}) \geq \delta_\alpha \). Assume without loss of generality that \( \delta_\alpha < 1/2 \), and that \( \delta = \mathcal{L}(t_j, t_{j+1}) < \delta_\alpha \). Lemma 2.2.3 provides an upper bound for \( d_{\max}(t_{j+1}) \mu_j \). The rightmost inequality of (14) with \( t = t_{j+1} \) implies that

\[
u_j(t_{j+1}) \leq \frac{1}{2} \frac{\mu_j \mu_0 \|x_{t_{j+1}} - x_{j+1}\|_0}{1 - d_{\max}(t_{j+1}) \mu_j}.
\]
From Lemma 2.2.3 and rearranging terms,
\[
(1 - d_{\max}(t_{j+1})\mu_j) u_{**} \leq \frac{1}{2} \mu_j \nu_0 \| z_{t_{j+1}} - z_t \|_0.
\]

Triangular inequality yields
\[
\frac{1}{2} \mu_j \nu_0 \| z_{t_{j+1}} - z_t \|_0 \geq \frac{1}{2} \mu_j \nu_0 \| z_{t_{j+1}} - x_{j+1} \|_0 - \frac{1}{2} \mu_j \nu_0 \| z_t - x_{j+1} \|_0
\geq u_{**} (1 - d_{\max}(t_{j+1})\mu_j) - u_{**}
\]

On the other hand,
\[
\frac{1}{2} \mu_j \nu_0 \| z_{t_{j+1}} - z_t \|_0 \leq \frac{1}{2} \mu_j \nu_0 \int_{t_j}^{t_{j+1}} \| \dot{z}_t \|_0 dt
\leq \frac{1}{2} \left( 1 + d_{\max}(t_{j+1})\mu_j \right) \int_{t_j}^{t_{j+1}} \| \dot{z}_t \|_0 \mu(q_t \cdot R(z_t), 0) \nu_0 dt
\]

Thus,
\[
u_{**} (1 - d_{\max}(t_{j+1})\mu_j) - u_{**} \leq \frac{1}{2} \mathcal{L}(t_j, t_{j+1}) (1 + d_{\max}(t_{j+1})\mu_j)
\]

Assume that \( \delta = \mathcal{L}(t_j, t_{j+1}) < 1/2 \). Lemma 2.2.3 above implies that \( 1 - d_{\max}(t_{j+1})\mu_j \geq 1 - \frac{\delta}{1 - \delta} = \frac{1 - 2\delta}{1 - \delta} \). Similarly, \( 1 + d_{\max}(t_{j+1}) \leq 1 + \frac{\delta}{1 - \delta} \). We have
\[
\left( u_{**} - \frac{1 - 2\delta}{1 - \delta} - u_{**} \right) \leq \frac{\delta}{2(1 - \delta)}
\]

Rearranging terms,
\[
\delta \geq \frac{u_{**} - u_{**}}{2u_{**} - u_{**} + \frac{1}{2}}
\]

By construction \( \alpha_{**} \) satisfies \( u_{**}(\alpha_{**}) = u_0 \). The right hand side is a smooth function of \( \alpha \in (0, \alpha_{**}) \) (See figure 6). Its maximum is attained for \( \alpha_{**} \sim 0.074, 609 \ldots \) with value \( \delta_{**} = 0.085, 180 \ldots \)

Lemma 2.2.1 implies that \( x_{N+1} = N(q_T R(x_N), 0) + x_N \) is an approximate root for \( q_T \) as in equation (5).

3. The expectation of the squared condition number

3.1. On conditional and unconditional Gaussians. The condition number used in this paper and the previous one [Malajovich, 2019] is invariant under independent scalings of each coordinate polynomial. This multi-homogeneous invariance introduced by [Malajovich and Rojas (2004)] breaks with the tradition in dense polynomial systems [Blum et al., 1998] [Bürgisser and Cucker, 2013]. This richer invariance will be strongly exploited in this section.

Recall that \( Z(q) \) denotes the set of isolated roots for a system of exponential sums \( q \in \mathcal{F} \). It would be desirable here to bound the expectancy of \( \sum_{z \in Z(q)} \mu(q \cdot R(z), 0)^2 \) where \( q \sim N(f, \Sigma^2) \). This author was unable to compute the expectancy above. There is no reason to believe at this point that this expectancy is finite. Instead, let \( H > 0 \) be arbitrary. Recall that \( Z_H(q) = \{ z \in Z(q) : \| \text{Re}(z) \|_\infty \leq H \} \). We
will bound the expectancy of \( \sum_{z \in \mathcal{M}(q)} \mu(q \cdot R(z), 0)^2 \) where \( q \sim N(f, \Sigma^2) \). More precisely, let

\[
E_{f, \Sigma^2} := \mathbb{E}_{q \sim N(f, \Sigma^2)} \left( \sum_{z \in \mathcal{M}(q)} \mu(qR(z), 0)^2 \right).
\]

The conditional expectancy below will turn out to be easier to bound:

\[
E_{f, \Sigma^2, K} := \mathbb{E}_{q \sim N(f, \Sigma^2)} \left( \sum_{z \in \mathcal{M}(q)} \mu(qR(z), 0)^2 \mid \|q_i - f_i\Sigma^{-1}\| \leq K \sqrt{S_i}, \quad i = 1, \ldots, n \right)
\]

with \( S_i = \#A_i = \dim_{\mathbb{C}} \mathcal{F}_{A_i} \). When \( f = 0 \), conditional and unconditional expectancies coincide, due to scaling invariance:

\[
E_{0, \Sigma^2} = E_{0, \Sigma^2, K}
\]

for any \( K > 0 \). For a general \( f \),

\[
E_{f, \Sigma^2, K} \leq \frac{E_{f, \Sigma^2}}{\operatorname{Prob}_{q \sim N(f, \Sigma^2)} \left( \|q_i - f_i\Sigma^{-1}\| \leq K \sqrt{S_i}, \quad i = 1, \ldots, n \right)}.
\]

The reciprocal inequality for a non-centered Gaussian probability distribution is more elusive. In the next section we prove:

**Proposition 3.1.1.** Let \( S_i = \#A_i \) and \( L > 0 \). Suppose that \( K \geq 1 + 2L + \sqrt{\log(n)} + 2 \log(3/2) \). If \( f \in \mathcal{F} \) satisfies \( \|f_i\Sigma^{-1}\| \leq L \sqrt{S_i} \), \( i = 1, \ldots, n \), then

\[
E_{f, \Sigma^2} \leq e \sup_{\|f_i\Sigma^{-1}\| \leq L \sqrt{S_i}} E_{f, \Sigma^2, K}.
\]

### 3.2. The truncated non-centered Gaussian

Bürgisser and Cucker (2011) developed a truncated Gaussian technique to bound the expected value of the squared condition number for the dense case. This technique will be generalized to the toric setting in order to prove Proposition 3.1.1. It should be stressed that we do not have unitary invariance and that the condition number in this paper is multi-homogeneous invariant. The following result will be used:

**Lemma 3.2.1.**

(a) Let \( \varphi : \mathbb{R}^{S_1} \times \cdots \times \mathbb{R}^{S_n} \to \mathbb{R} \cup \{\infty\} \) be measurable, positive and scaling invariant: for all \( 0 \neq \lambda \in \mathbb{R}^n \),

\[
\varphi(\lambda w_1, \ldots, \lambda w_n) = \varphi(w_1, \ldots, w_n).
\]

Let \( u \in \mathbb{C}^{S_1} \times \cdots \times \mathbb{C}^{S_n} \). Let \( L = \max(\|u_i\|/\sqrt{S_i}) \) and \( K \geq 1 + 2L + \sqrt{\log(n)}/\min(S_i) \) + 2 log(3/2). Write \( S = \sum S_i \). Then,

\[
\mathbb{E}_{w \sim N(u, I, \mathbb{R}^n)} (\varphi(w)) \leq \sqrt{e} \sup_{|u_i| \leq \sqrt{S_i} L} G_{u, K}
\]

with

\[
G_{u, K} := \mathbb{E}_{w \sim N(u, I, \mathbb{R}^n)} (\varphi(w) \mid \|w - u\| \leq \sqrt{S_i} K).
\]
(b) Let $\varphi : \mathbb{C}^S_1 \times \cdots \times \mathbb{C}^S_n \to \mathbb{R} \cup \{\infty\}$ be measurable, positive and scaling invariant: for all $0 \neq \lambda \in \mathbb{C}^n$,

$$\varphi(\lambda_1 w_1, \ldots, \lambda_n w_n) = \varphi(w_1, \ldots, w_n).$$

Let $u \in \mathbb{C}^S_1 \times \cdots \times \mathbb{C}^S_n$. Let $L = \max(\|u_i\|/\sqrt{S_i})$ and $K \geq 1 + 2L + \sqrt{\log(n)/\min(S_i)} + 2 \log(3/2)$. Write $S = \sum S_i$. Then,

$$\mathbb{E}_{w \sim N(u, I; \mathbb{C}^S)} (\varphi(w)) \leq e \sup_{|u_i| \leq \sqrt{S_i}L} G_{\bar{u}, K}$$

with

$$G_{\bar{u}, K} \overset{\text{def}}{=} \mathbb{E}_{w \sim N(u, I; \mathbb{C}^S)} (\varphi(w) \mid \|w\| \leq \sqrt{S_i}K).$$

This Lemma will follow from the large deviations estimate:

**Lemma 3.2.2.** Let $s, t > 0$. Then,

(a) $$\operatorname{Prob}_{u \sim N(0, I; \mathbb{R}^N)} \left[ |u| \geq \sqrt{N} + t \right] \leq e^{-\frac{t^2}{2}},$$

(b) and $$\operatorname{Prob}_{w \sim N(0, I; \mathbb{C}^N)} \left[ \|w\| \geq \sqrt{N} + s \right] \leq e^{-s^2}.$$

**Proof.** Item (a) is borrowed from Bürgisser and Cucker (2013 Corollary 4.6). For item (b), set $u = \sqrt{2} \operatorname{Re}(w)$ and $v = \sqrt{2} \operatorname{Im}(w)$. Then, $[u, v] \sim N(0, I; \mathbb{R}^{2n})$ and

$$\operatorname{Prob}_{w \sim N(0, I; \mathbb{C}^N)} \left[ \|w\| \geq \sqrt{N} + s \right] = \operatorname{Prob}_{[u, v] \sim N(0, I; \mathbb{R}^{2n})} \left[ \|[u, v]\| \geq \sqrt{2N} + s\sqrt{2} \right] \leq e^{-s^2}$$

using item (a). \qed

We will also use the following elementary bound.

**Lemma 3.2.3.** Let $u, v \in \mathbb{C}^N$ (resp. $\mathbb{R}^N$) with $\|u\| \leq L < K \leq \|v\|$. Then,

$$-\|v\|^2 < -\frac{\|u + v\|^2}{\sigma^2}$$

with $\sigma = 1 + \frac{L}{K}$.

**Proof.** By hypothesis $\frac{\|u\|}{\|v\|} < \frac{L}{K}$. Therefore,

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$< \|v\|^2 \left( \frac{L^2}{K^2} + 2 \frac{L}{K} + 1 \right)$$

$$= \|v\|^2 \left( 1 + \frac{L}{K} \right)^2$$

$$= \|v\|^2 \sigma^2.$$

Thus,

$$-\|v\|^2 < -\frac{\|u + v\|^2}{\sigma^2}.$$ 

\qed
Proof of Lemma 3.2.1. The proof for the real and complex case is essentially the same, so we only write down the proof for the complex case (b). Let \( w = u + v \), \( v \sim N(0, I; \mathbb{C}^S) \), so the vectors \( v \) are independent. We subdivide the domain \( \mathbb{C}^S \) in cells indexed by each \( J \subseteq [n] = \{1, \ldots, n\} \) as follows:

\[
V_J = \{ v = (v_1, \ldots, v_n) \in \mathbb{C}^{S_1} \times \cdots \times \mathbb{C}^{S_n} : |v_i| > \sqrt{S_i}K \Leftrightarrow i \in J \}.
\]

Under this notation \( \mathbb{C}^S \) is the disjoint union of all the \( V_J, J \subseteq [n] \). We also define sets \( \tilde{V}_J \subseteq V_J \) by

\[
\tilde{V}_J = \{ v = (v_1, \ldots, v_n) \in \mathbb{C}^{S_1} \times \cdots \times \mathbb{C}^{S_n} : |v_i| > \sqrt{S_i}(K - L) \text{ for } i \in J \}
\]

and

\[
|v_i| \leq \sqrt{S_i}K \text{ for } i \notin J.
\]

with \( \sigma = 1 + \frac{L}{K} > 1 \). The expectation satisfies:

\[
\mathbb{E}_{w \sim N(u, I; \mathbb{C}^S)}(\varphi(w)) = \sum_{J \subseteq [n]} \int_{v \in V_J} \varphi(u + v) \frac{e^{-|v|^2}}{\pi^S} \text{d} \mathbb{C}^S(v).
\]

We will change variables inside each integral. If \( i \in J \), we set \( \hat{u}_i = 0 \) and \( \hat{v}_i = u_i + v_i \). If \( i \notin J \), \( \hat{u}_i = u_i \) and \( \hat{v}_i = v_i \). In all cases, \( \hat{u} + \hat{v} = u + v \). If \( i \in J \), we have \( |v_i| \leq L\sqrt{S_i} < K\sqrt{S_i} \leq \sqrt{S_i} |v_i| \) so that Lemma 3.2.3 yields \( -|v_i|^2 < -|\hat{v}_i|^2/\sigma^2 \) with \( \sigma = 1 + L/K \). Each integral can be bounded as follows:

\[
\int_{v \in V_J} \varphi(u + v) \frac{e^{-|v|^2}}{\pi^S} \text{d} \mathbb{C}^S(v) \leq \int_{v \in \tilde{V}_J} \varphi(\hat{u} + \hat{v}) \prod_{i \in J} \frac{e^{-|\hat{v}_i|^2/\sigma^2}}{\pi^{S_i}} \prod_{i \notin J} \frac{e^{-|\hat{v}_i|^2}}{\pi^{S_i}} \text{d} \mathbb{C}^S(\hat{v}).
\]

For each \( j \in J \), \( \hat{u}_j = 0 \) and the function \( \varphi(\hat{u} + \hat{v}) \) is invariant by scaling \( \hat{v}_j \). We will replace \( \hat{v}_j = \sigma \hat{v}_j \) for \( j \in J \), and \( \hat{v}_j = \hat{v}_j \) otherwise. Now,

\[
\int_{v \in V_J} \varphi(u + v) \frac{e^{-|v|^2}}{\pi^S} \text{d} \mathbb{C}^S(v) \leq \sigma^2 \sum_{i \in J} \int_{v \in \tilde{V}_J} \varphi(\hat{u} + \hat{v}) \frac{e^{-|\hat{S}_i|^2}}{\pi^S} \text{d} \mathbb{C}^S(\hat{v}).
\]

Again, we take advantage of the scaling invariance of the function \( \varphi \) with respect to \( \hat{v}_i \) for \( i \in J \). For those indices, we can replace the domain of integration \( [\hat{v}_i] > \sqrt{S_i}K - L \) by \( [\hat{v}_i] \leq \sqrt{S_i}K \) as long as we take into account the full probability of each domain. Namely,

\[
\int_{v \in V_J} \varphi(u + v) \frac{e^{-|v|^2}}{\pi^S} \text{d} \mathbb{C}^S(v) \leq \prod_{i \in J} \sigma^2 S_i \int_{\hat{v}_i \leq \sqrt{S_i}K} \varphi(\hat{u} + \hat{v}) \frac{e^{-|\hat{S}_i|^2}}{\pi^S} \text{d} \mathbb{C}^S(\hat{v}).
\]
The product of integrals is precisely
\[
\prod_{i=1}^{n} \int_{\|\tilde{v}_i\| \leq \sqrt{S_i}K} \varphi(\tilde{u} + \tilde{v}) \frac{e^{-\|\tilde{v}\|^2}}{\pi^{S_i}} d\mathbb{C}^S(\tilde{v}) = G_{\tilde{u},K} \prod_{i=1}^{n} \text{Prob}_{\tilde{v}_i \in N(0,I;\mathbb{C}^{S_i})} \left[ \|\tilde{v}_i\| \leq \sqrt{S_i}K \right]
\]
where
\[
G_{\tilde{u},K} \overset{\text{def}}{=} \mathbb{E}_{w \sim N(\tilde{u},I;\mathbb{C}^N)} \left( \varphi(w) \mid \|w - \tilde{u}\| \leq \sqrt{S_i}K \right).
\]
In the equation above, \(\tilde{u}\) depends on the choice of \(J\). This is why we take the supremum \(\sup_{\|\tilde{u}\| \leq K\sqrt{S_i}} G_{\tilde{u},K}\) in the main statement. Lemma 3.2.2(b) provides the bound
\[
\text{Prob}_{\tilde{v}_i \in N(0,I;\mathbb{C}^{S_i})} \left[ \|\tilde{v}_i\| > \frac{\sqrt{S_i}(K - L)}{\sigma} \right] \leq e^{-S_i(\frac{K - L}{\sigma} - 1)^2}.
\]
For \(j \in J\), the probability that \(\|\tilde{v}_i\| \leq \sqrt{S_i}K\) appears in the numerator and in the denominator, so it cancels. For \(j \notin J\), we use the trivial bound
\[
\text{Prob}_{\tilde{v}_i \in N(0,I;\mathbb{C}^{S_i})} \left[ \|\tilde{v}_i\| < \sqrt{S_i}K \right] \leq 1.
\]
Adding for all subsets \(J\),
\[
\mathbb{E}_{w \sim N(\tilde{u},I;\mathbb{C}^N)} (\varphi(w)) \leq \sum_{J \subseteq [n]} \prod_{i \in J} e^{S_i \left(2\log(\sigma) - \left(\frac{K - L}{\sigma} - 1\right)^2 \right)} \sup_{\|\tilde{u}\| \leq K\sqrt{S_i}} G_{\tilde{u},K} \leq \prod_{i=1}^{n} \left( 1 + e^{S_i \left(2\log(\sigma) - \left(\frac{K - L}{\sigma} - 1\right)^2 \right)} \right) \sup_{\|\tilde{u}\| \leq K\sqrt{S_i}} G_{\tilde{u},K}.
\]
We choose \(K \geq 1 + 2L + t\), with \(t\) to be determined. In this case \(\sigma = 1 + L/K < 3/2\) and
\[
\frac{K - L}{\sigma} - 1 = \frac{K^2 - KL - K - L}{K + L} = K - 1 - 2\frac{KL}{K + L} \geq K - 1 - 2L \geq t.
\]
By setting \(t = \sqrt{\frac{\log(n)}{S_i}} + 2\log(3/2)\), we obtain
\[
\prod_{i=1}^{n} \left( 1 + e^{S_i \left(2\log(\sigma) - \left(\frac{K - L}{\sigma} - 1\right)^2 \right)} \right) \leq \left( 1 + \frac{1}{n} \right)^n \leq e
\]
and thus
\[
\mathbb{E}_{w \sim N(\tilde{u},I;\mathbb{C}^N)} (\varphi(w)) \leq e \sup_{\|\tilde{u}\| \leq K\sqrt{S_i}} G_{\tilde{u},K}.
\]

The proof for the real case is the same, with the following changes. In equation (15), the Jacobian is \(\sigma^{\sum_{i \in J} S_i}\) instead of \(\sigma^{2\sum_{i \in J} S_i}\). Lemma 3.2.2(a) yields
\[
\text{Prob}_{\tilde{v}_i \in N(0,I;\mathbb{R}^{S_i})} \left[ \|\tilde{v}_i\| > \frac{\sqrt{S_i}(K - L)}{\sigma} \right] \leq e^{-S_i(\frac{K - L}{\sigma} - 1)^2/2}.
\]
Therefore, with the same choice of \(t\),
\[
\prod_{i=1}^{n} \left( 1 + e^{S_i \left(\log(\sigma) - \left(\frac{K - L}{\sigma} - 1\right)^2 \right)/2 \right) \leq \left( 1 + \frac{1}{2n} \right)^n \leq e^{t/4}.
\]
Proof of Proposition \([3.1.1]\) Recall that the \(f_i \in \mathcal{F}_{A_i}\) are always written as covectors, so \(f_i \Sigma_i^{-1}\) is the product of \(f_i\) by the matrix \(\Sigma_i^{-1}\). Under that notation, write \(u_i = f_i \Sigma_i^{-1}\) and \(w_i = q_i \Sigma_i^{-1}\). By hypothesis, \(\|u_i\| \leq L \sqrt{\Sigma_i}\). Let
\[
\varphi(w) \overset{\text{def}}{=} \sum_{z \in \mathcal{Z}(w) \Sigma} \mu(w \cdot \Sigma \cdot R(z), 0)^2.
\]
In the notations of Lemma \([3.2.1]\) we have
\[
E_{\hat{f},\Sigma^2} = \mathbb{E}_{q \sim N(f, \Sigma^2)} \left( \varphi(q \Sigma_i^{-1}) \right) = \mathbb{E}_{w \sim N(u, I)} (\varphi(w)) = G_{u, I}
\]
and similarly
\[
E_{\hat{f},\Sigma^2, K} = G_{u, I, K}
\]
for \(\hat{f} = \hat{u}_i \Sigma_i\). It follows from Lemma \([3.2.1]\) that
\[
E_{\hat{f},\Sigma^2} \leq e \sup_{\|\hat{f},\Sigma_i^{-1}\| \leq L \sqrt{\Sigma_i}} E_{\hat{f},\Sigma^2, K}.
\]
\[
\square
\]

3.3. Renormalization and the condition number. Towards the proof of Main Theorem \([\mathbb{B}]\) we will estimate the expectation \(E_{\hat{f},\Sigma^2, K}\) from Proposition \([3.1.1]\) in terms of the following integral:
\[
I_{\hat{f},\Sigma^2} \overset{\text{def}}{=} \mathbb{E}_{q \sim N(f, \Sigma^2)} \left( \sum_{z \in \mathcal{Z}(q) \Sigma} \|M(q, z)^{-1}\|^2_F \right)
\]
The bound for \(I_{\hat{f},\Sigma^2}\) is given in Theorem \([1.5.17]\). We will prove:

**Proposition 3.3.1.** For \(i = 1, \ldots, n\), let \(S_i = \#A_i \geq 2\) and \(\delta_i = \max_{a \in A_i} \|a - m_i(0)\|.\) Let \(\|f_i \Sigma_i^{-1}\| \leq \sqrt{\Sigma_i} L\). Let \(K \geq 1 + 2L + \sqrt{\frac{\log(n)}{\min(S_i)}} + 2 \log(3/2)\). Then,
\[
E_{\hat{f},\Sigma^2} < 1.25 e (K + L)^2 \left( \sum_i \delta_i^2 \right) \max_i \left( S_i K \right) \max_{a \in A_i} \left( \sigma_i^2 \right) \sup_{|\hat{f},\Sigma_i^{-1}| \leq L \sqrt{\Sigma_i}} I_{\hat{f},\Sigma^2}.
\]

**Proof of Theorem \([\mathbb{B}]\)** Plugging Theorem \([1.5.17]\) into Proposition \([3.3.1]\)
\[
E_{\hat{f},\Sigma^2} \leq \frac{2.5 e H \sqrt{n}}{\det(A)} \left( 1 + 3L \sqrt{\frac{\log(n)}{\min(S_i)}} + 2 \log(3/2) \right)^2 \times \max_i \left( S_i K \right) \max_{a \in A_i} \left( \sigma_i^2 \right) \times \left( \sum_i \delta_i^2 \right)^{(n - 1)!V^l}
\]
with \(L = \max \|f_i \Sigma_i^{-1}\| / \sqrt{\Sigma_i}\).
\[
\square
\]

It remains to prove Proposition \([3.3.1]\) and Theorem \([1.5.17]\). In order to prove Proposition \([3.3.1]\) we need two preliminary results.

**Lemma 3.3.2.** Let \(q \in \mathcal{F}\) and \(z \in \mathcal{Z}(q) \subset \mathcal{M}\). Then,

1. \(\|M(q, z)^{-1}\|_0 \leq \sqrt{\sum_i \delta_i^2 \|M(q, z)^{-1}\|_F} \),

Comparing coordinate by coordinate, in particular, if the coefficients \( \rho \) then,

\[
\| V_A(z) \| \leq e^{\ell_i(z)} \| V_A(0) \| \leq \kappa_{p_i} \| V_A(z) \|
\]

In particular, if the coefficients \( \rho_{i,a} = \rho_i \) are the same, \( \kappa_{p_i} = \sqrt{S_i} \) and

\[
e^{\ell_i(z)} \| V_A(0) \| \leq \sqrt{S_i} \| V_A(z) \|.
\]

Remark 3.3.4. In the context of example 1.3.2 if \( \rho_{i,a} = \sqrt{d_i} \) and \( A_i = \{a \in \mathbb{Z}^n : 0 \leq a_i, \sum a_i \leq d_i \} \), \( \kappa_{p_i} = (n+1)d_i \) and all we have is

\[
e^{d_i, \max(\max_i(\text{Re}(z)))} \| V_A(0) \| \leq (n+1)^d \| V_A(z) \|.
\]

Proof. In order to prove the first inequality, we claim that 0 is a global minimum of \( \| V_A(z) \| \). Indeed, \( m_i(z) \) is precisely the derivative of the convex potential

\[
\psi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \rightarrow \frac{1}{2} \log(\| V_A(x) \|^2).
\]

Since \( V_A \) and \( m_i \) depend only on the real part of their argument and \( m_i(0) = 0 \), the point 0 is a global minimum of the convex potential \( \psi \).

For the next two inequalities, we compare the norms of

\[
e^{\ell_i(z)} V_A(0) = \begin{pmatrix} \rho_{i,a} e^{\ell_i(z)} \\ \vdots \\ \rho_{i,a} e^{\ell_i(z)} \end{pmatrix}_{a \in A_i} \quad \text{and} \quad V_A(z) = \begin{pmatrix} e^{\rho_{i,a} z} \\ \vdots \\ e^{\rho_{i,a} z} \end{pmatrix}_{a \in A_i}.
\]

Comparing coordinate by coordinate,

\[
\| V_A(z) \| \leq e^{\ell_i(z)} \| V_A(0) \|
\]

The maximum of \( a \text{ Re}(z) \) is attained for some \( a^* \in A_i' \). Hence,

\[
e^{\ell_i(z)} \| V_A(0) \| = e^{\ell_i(z)} \sqrt{\sum_{a \in A_i} \rho_{i,a}^2} = e^{a^* \text{ Re}(z)} \sqrt{\sum_{a \in A_i} \rho_{i,a}^2} \leq \sqrt{\sum_{a \in A_i} \rho_{i,a}^2} \| V_A(z) \|.
\]

Proof of Lemma 3.3.3

Item (a): We have to prove that that

\[
\| M(q, z)^{-1} \|_0 \leq \sqrt{\sum_i \delta_i^2} \| M(q, z)^{-1} \|_F
\]
where $\|X\|_F = \sqrt{\sum_{ij} |X_{ij}|^2}$ is the Frobenius norm.

$$\|M(q, z)^{-1}\|_0 = \|D[V](0) M(q, z)^{-1}\|_0 \leq \|D[V](0)\| \|M(q, z)^{-1}\|_0 \leq \|D[V](0)\| \|M(q, z)^{-1}\|_F.$$ 

Lemma [1.3.7] yields, for each $i = 1, \ldots n$,

$$\|D[V_{A_i}](0)\| \leq \delta_i$$

and hence

$$\|D[V](0)\|_2 \leq \sqrt{\sum \delta_i^2}.$$ 

**Item (b):** We can assume without loss of generality that $m_i(0) = 0$ for each $i$. Indeed, subtracting $m_i(0)$ from each $a \in A_i$ will multiply $V_{A_i}(z)$, $DV_{A_i}(z)$ by the same constant $e^{-m_i(0)z}$. In particular, $V_{A_i}(0)$, $DV_{A_i}(0)$ do not change and the metric of $T_0M$ is the same. Also, the quantities $M(q, z)$ and $q_i \cdot R(z)$ do not change. Under the hypothesis $m_i(0) = 0$,

$$M(q \cdot R(z), 0) = \left(\begin{array}{c}
\frac{1}{|V_{A_1}(0)|} q_1 R_1(z) D V_{A_1}(0) \\
\vdots \\
\frac{1}{|V_{A_n}(0)|} q_n R_n(z) D V_{A_n}(0)
\end{array}\right)$$

$$= \left(\begin{array}{c}
\frac{e^{-\ell_1(z)}}{|V_{A_1}(0)|} q_1 D V_{A_1}(z) \\
\vdots \\
\frac{e^{-\ell_n(z)}}{|V_{A_n}(0)|} q_n D V_{A_n}(z)
\end{array}\right).$$

Therefore,

$$\mu(q \cdot R(z), 0) = \|M(q \cdot R(z), 0)^{-1} \text{diag} (\|q_i \circ R(z)\|)\|_0$$

$$= \left(\text{diag} \left(\frac{|V_{A_i}(z)| e^{-\ell_i(z)}}{|V_{A_i}(0)|} M(q, z)\right)^{-1} \text{diag} (\|q_i \circ R(z)\|)\right)_0$$

$$= \left\|M(q, z)^{-1} \text{diag} \left(\frac{\|q_i \circ R(z)\| |V_{A_i}(0)| e^{\ell_i(z)}}{|V_{A_i}(z)|}\right)\right\|_0$$

$$\leq \|M(q, z)^{-1}\|_0 \max_i \frac{\|q_i \circ R(z)\| |V_{A_i}(0)| e^{\ell_i(z)}}{|V_{A_i}(z)|}$$

by definition, by the previous item, and then trivially. From Lemma [3.3.3]

$$e^{\ell_i(z)} |V_{A_i}(0)| \leq |V_{A_i}(z)| \kappa_{p_i}.$$ 

Theorem [1.4.2] (c) states that $\|q_i R_i(z)\| \leq \|q_i\|$. Combining those bounds with item (a),

$$\mu(q \cdot R(z), 0) \leq \sqrt{\sum_i \delta_i^2 \left\|M(q, z)^{-1}\right\|_F \max_i (\kappa_{p_i}, \|q_i\|)}.$$ 

\[\square\]
Proof of Proposition 3.3.1 From Proposition 3.1.1

\[ E_{f, \Sigma^2} \leq e \sup_{\|\hat{f}, \Sigma^{-1}\| \leq L \sqrt{S_i}} E_{I_f, \Sigma^2, K} \]
\[ \leq e \sup_{\|\hat{f}, \Sigma^{-1}\| \leq L \sqrt{S_i}} \mathbb{E} \left( \sum_{z \in Z_H(q)} \mu(qR(z), 0)^2 \right) \cdot \|(q_i - \hat{f}_i)\Sigma^{-1}\| \leq K \sqrt{S_i}, \quad i = 1, \ldots, n \).

The condition \(\|(q_i - \hat{f}_i)\Sigma^{-1}\| \leq K \sqrt{S_i}\) implies that \(|q_i| \leq (K + L) \sqrt{S_i} \max_{a \in A_i} \sigma_{ia}\).

From Lemma 3.3.2
\[ \mu(qR(z), 0)^2 \leq (K + L)^2 \left( \sum_i \delta_i^2 \right) \max_i \left( S_i \kappa_i^2 \max_{a \in A_i} \sigma_{ia}^2 \right) \|M(q, z)^{-1}\|_F^2. \]

It follows that
\[ E_{f, \Sigma^2} \leq e(K + L)^2 \left( \sum_i \delta_i^2 \right) \max_i \left( S_i \kappa_i^2 \max_{a \in A_i} \sigma_{ia}^2 \right) \cdot \sup_{\|\hat{f}, \Sigma^{-1}\| \leq L \sqrt{S_i}} \mathbb{E} \left( \sum_{z \in Z_H(q)} \|M(q, z)^{-1}\|_F^2 \right) \cdot \|(q_i - \hat{f}_i)\Sigma^{-1}\| \leq K \sqrt{S_i}, \quad i = 1, \ldots, n \).

The conditional expectancy times the probability that \(\|(q_i - \hat{f}_i)\Sigma^{-1}\| \leq K \sqrt{S_i}\) is bounded above by \(I_{f, \Sigma^2}\). Thus,
\[ E_{f, \Sigma^2} \leq \frac{e(K + L)^2 \left( \sum_i \delta_i^2 \right) \max_i \left( S_i \kappa_i^2 \max_{a \in A_i} \sigma_{ia}^2 \right)}{\operatorname{Prob}_{q \sim N(f, \Sigma^2)} \left( \|(q_i - \hat{f}_i)\Sigma^{-1}\| > K \sqrt{S_i}, \quad i = 1, \ldots, n \right) \sup_{\|\hat{f}, \Sigma^{-1}\| \leq L \sqrt{S_i}} (I_{f, \Sigma^2})} \]
From Lemma 3.2.2(b),
\[ \operatorname{Prob}_{q \sim N(f, \Sigma^2)} \left( \|(q_i - \hat{f}_i)\Sigma^{-1}\| > K \sqrt{S_i} \right) \leq e^{-S_i(K-1)^2} < \frac{1}{n} \left( \frac{4}{9} \right)^{S_i}. \]
Thus, the probability that \(\|(q_i - \hat{f}_i)\Sigma^{-1}\| > K \sqrt{S_i}\) for some \(i\) is at most \((4/9)^{\min(S_i)}\) \(\leq \frac{16}{81}\). The probability of the opposite event is therefore at least 65/81. In the final expression, we replaced the factor 81/65 by its approximation 1.25. \(\square\)

3.4. Proof of Theorem 1.5.17 Theorem 1.5.17 claims a bound for the integral
\[ I_{f, \Sigma^2} = \mathbb{E}_{q \sim N(f, \Sigma^2)} \left( \sum_{z \in Z_H(q)} \|M(q, z)^{-1}\|_F^2 \right) \]
which is independent of \(\hat{f}\). This bound will be derived from the coarea formula, see for instance [Blum et al., 1998 Th.4 p.241]. This statement is also known in other communities as the Rice formula [Azaïs and Wschebor, 2009] or the Crofton
formula. However, it is convenient to restate this result in terms of complex fiber bundles (See figure 7).

Recall that the solution variety is \( S = \{(q, z) \in \mathcal{F} \times \mathcal{M} : q \cdot V(z) = 0\} \). Let \( \pi_1 : S \to \mathcal{F} \) and \( \pi_2 : S \to \mathcal{M} \) be the canonical projections. Then \( (S, \mathcal{M}, \pi, \mathcal{F}) \) is a complex smooth fiber bundle, where \( F\Omega_1 = \pi_1 \circ \pi_2^{-1}(\Omega) \subset \mathcal{F} \). The solution variety \( S \) will be endowed here with the pull-back metric \( dS = \pi_1^* d\mathcal{F} \). We write

\[
I_{f,\Sigma} = \int_{(f + g, z) \in S} \|M(f + g, z)^{-1}\|_F^2 \exp\left(-\|g\Sigma^{-1}\|^2\right) \frac{\exp\left(-\|g\Sigma^{-1}\|^2\right)}{\pi^S \prod_i |\det \Sigma_i|^2} dS
\]

with \( S = \dim_{\mathbb{C}} \mathcal{F} = \sum \#A_i \). As explained by Malajovich (2011) Th. 4.9), the coarea formula may be restated in terms of fiber bundles:

**Theorem 3.4.1.** Let \( (\mathcal{S}, \mathcal{M}, \pi, F) \) be a complex smooth fiber bundle. Assume that \( \mathcal{M} \) is finite dimensional. Let \( \phi : \mathcal{S} \to \mathbb{R}_{\geq 0} \) be measurable. Then whenever the last integral exists,

\[
\int_{\mathcal{S}} \phi(p) dS(p) = \int_{\mathcal{M}} d\mathcal{M}(z) \int_{\pi_2^{-1}(z)} \det(D\pi_2(p)D\pi_2(p)^*)^{-1} \phi(p) \, d\pi_2^{-1}(z)(p).
\]
Because the metric in $S$ is the pull-back of the metric in $F$, this is the same as
\[
\int_S \phi(p) \, dS(p) = \int_M dM(z) \int_{\pi_1^{-1}(z)} \det(DG(q)DG(q)^*)^{-1} \phi(q, z) \, dF(q)
\]
with $G : f \in U \subset F \to M$ the local implicit function, that is the local branch of $\pi_2 \circ \pi_1^{-1}$. While the integral on the left is independent of the volume form $dM(z)$ on $M$, the value of the determinants of $(D\pi(p)D\pi(p)*)$ and of $DG(q)DG(q)^*$ depend on this volume form. In order to simplify computations, we choose to endow $M$ with the canonical Hermitian metric of $C^n$. We can now compute the normal Jacobian.

**Lemma 3.4.2.** Under the assumptions above,
\[
\det(DG(q)DG(q)^*) = |\det M(q, z)|^2.
\]

**Proof.** Assume that $M(q, z)$ is invertible, otherwise both sides are zero. Then we can parameterize a neighborhood of $(q, z) \in S$ by a map $h \mapsto (h, G(h))$ where the implicit function $G(h)$ is defined in a neighborhood of $q$, satisfies $G(q) = z$ and $h \cdot V(G(h)) = 0$. The derivative of the implicit function can be obtained by differentiation at $(q, z)$, viz.
\[
q \cdot DV(z)\dot{z} + \dot{q} \cdot V(z) = 0.
\]
The reproducing Kernel property allows to write
\[
\dot{q} \cdot V(z) = \begin{pmatrix}
\langle q_1, K_1(\cdot, z) \rangle_{F_{\lambda_1}} \\
\vdots \\
\langle q_n, K_n(\cdot, z) \rangle_{F_{\lambda_n}}
\end{pmatrix} = \begin{pmatrix}
K_1(\cdot, z)^* \\
\vdots \\
K_n(\cdot, z)^*
\end{pmatrix} \dot{q}.
\]
It follows that
\[
DG(q, z) = -M(q, z)^{-1} \begin{pmatrix}
\frac{1}{|V_1(z)|} K_1(\cdot, z)^* \\
\vdots \\
\frac{1}{|V_n(z)|} K_n(\cdot, z)^*
\end{pmatrix}
\]
Because of the reproducing kernel property, $|V_{\lambda_i}(z)|^2 = K_i(z, z)$ and hence
\[
\det(DG(q, z)^* DG(q, z)) = \det(M(q, z)^{-*} M(q, z)^{-1}) = |\det M(q, z)|^{-2}.
\]

Let $M_H = \{z \in M : |\Re(z)| \leq H\}$ and let $\chi_H(z)$ be its indicator function. We can now compute $I_{F, |\Sigma|^2}$ by replacing, in the statement of Theorem 3.4.1
\[
\phi(q, z) = \|M(\hat{f} + g, z)^{-1}\|^2_F \exp \left( -\|g\Sigma^{-1}\|_F^2 \right) \frac{\pi^s \prod_i |\det \Sigma_i|^2}{|\det(z)|^2} \chi_H(z).
\]
We introduce the notation $F_z = (\pi_1 \circ \pi_2^{-1})(z)$. The Lemma above yields
\[
I_{F, |\Sigma|^2} = \int_{M_H} dM(z) \int_{F_z} |\det M(\hat{f} + g, z)|^2 \|M(\hat{f} + g, z)^{-1}\|^2_F \times \exp \left( -\|g\Sigma^{-1}\|_F^2 \right) \frac{\pi^s \prod_i |\det \Sigma_i|^2}{|\det(z)|^2} \, dF(g).
\]
introduced the following technique to integrate $|\det(M(\hat{f} + g, z))|^2 \cdot |M(\hat{f} + g, z)^{-1}f|^2$. Let $S(M, k, \mathbf{w})$ be the determinant of the matrix obtained by replacing the $k$-th row of $M$ with $\mathbf{w}$. Cramer’s rule yields

$$|\det M|^2 \cdot |M^{-1}|^2 = \sum_{k} |S(M, k, e_k)|^2.$$ 

Therefore,

$$I_{f, \Sigma^2} = \int_{M_H} dM(z) \sum_{k,l} |S(M(\hat{f} + g, z), k, e_l)|^2 \exp \left( - \frac{\|g \Sigma^{-1}\|_F^2}{\pi^S \prod_{i} |\det \Sigma_i|^2} \right) dF(g).$$

Armentano et al. (2016) computed that integral in the dense case. We will proceed in a different manner. First we notice that for each $k$, the first term inside the sum is independent of $g_k$, while the second part is a Gaussian. So we may integrate out $g_k$:

$$I_{f, \Sigma^2} = \sum_{k,l} \int_{M_H} \frac{|\det(\Sigma_k)_{(\mathcal{F}_A)_x}|^2}{\pi |\det(\Sigma_k)|^2} dM(z) \sum_{\mathcal{F} \neq k} |S(M(\hat{f} + g, z), k, e_l)|^2 \exp \left( - \frac{\sum_{i \neq k} \|g_i \Sigma_i^{-1}\|_F^2}{\pi^S \prod_{i \neq k} |\det \Sigma_i|^2} \right) \sum_{\mathcal{F} \neq k} d(\mathcal{F}_A)_x(g).$$

Above, $(\mathcal{F}_A)_x = K_A(\cdot, z)^\perp \in \mathcal{F}_A$ is the $i$-th component space of $\mathcal{F}_z$. At this point we need the following Lemma

**Lemma 3.4.3.** Let $H$ be a Hermitian positive matrix, with eigenvalues $\sigma_1 \geq \cdots \geq \sigma_n$. Let $\mathbf{w} = \mathbf{u} + i \mathbf{v}$ be a non-zero vector in $\mathbb{C}^N$, with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then,

$$\frac{\det H}{\sigma_1} \geq \det H_{|\mathbf{w}|} \leq \frac{\det H}{\sigma_N}.$$ 

It follows from the Courant-Fischer minimax theorem (Demmel, 1997, Theorem 5.2). Only the first part of the statement is quoted below. Recall that the Grassmanian $\text{Gr}(k, V)$ is the set of $k$-dimensional linear subspaces in a real space $V$:

**Theorem 3.4.4.** Let $\alpha_1 \geq \cdots \geq \alpha_m$ be the eigenvalues of a symmetric matrix $A$, and let $\rho(r, A) = \frac{r^T A r}{\|r\|^2}$ be the Rayleigh quotient. Then,

$$\max_{\text{ReGr}(j, \mathbb{R}^m) \neq A} \min_{r \in \mathbb{R}^m} \rho(r, A) = \alpha_j = \min_{\text{ReGr}(m-j+1, \mathbb{R}^m) \neq A} \min_{r \in \mathbb{R}^m} \rho(r, A).$$

**Proof of Lemma 3.4.3.** We assume without loss of generality that $W$ is diagonal and real. A vector $z = x + iy \in \mathbb{C}^N$ is complex-orthogonal to $w = u + iv$ with $x, y, u$ and $v$ real, if and only if $\begin{pmatrix} x \\ y \end{pmatrix}$ is real-orthogonal to $\begin{pmatrix} u \\ v \end{pmatrix}$ and $\begin{pmatrix} -v \\ u \end{pmatrix}$. We consider
also the diagonal matrix

\[
A = \begin{pmatrix}
\sigma_1 & & \\
& \ddots & \\
& & \sigma_N \\
& & \\
& & & \sigma_1 \\
& & & & \ddots \\
& & & & & \sigma_N
\end{pmatrix}.
\]

It satisfies \( \det(A) = \det(W)^2 \). Let \( T \) be the space orthogonal to \( \begin{pmatrix} u \\ v \end{pmatrix} \) and \( \begin{pmatrix} -v \\ u \end{pmatrix} \).

We denote by \( A|_T \) the restriction of \( A \) to \( T \), as a bilinear form. The restriction \( A|_T \) is still symmetric and positive. Let \( \alpha_1 = \sigma_1 \geq \alpha_2 = \sigma_1 \geq \alpha_3 = \sigma_2 \geq \cdots \geq \alpha_{2N} = \sigma_n \) be the eigenvalues of \( A \), and \( \lambda_1 \geq \cdots \geq \lambda_{2N-2} \) be the eigenvalues of \( A|_T \). Courant-Fischer theorem yields:

\[
\lambda_j = \max_{\text{ReGr}(j,T)} \min_{0 \neq r \in R} \rho(r,A|_T) \leq \max_{\text{ReGr}(j,\mathbb{R}^{2N})} \min_{0 \neq r \in R} \rho(r,A) = \alpha_j
\]

and

\[
\alpha_j = \min_{\text{Segr}(2N-j+1,\mathbb{R}^{2N})} \min_{0 \neq s \in S} \rho(s,A) \leq \min_{\text{Segr}(2N-2-(j-2)+1,1)} \min_{0 \neq s \in S} \rho(s,A) = \lambda_{j-2}
\]

It follows that \( \lambda_j \leq \alpha_j \leq \lambda_{j-2} \). Since this holds for all \( j \),

\[
\frac{\det(A)}{\sigma_1^2} = \prod_{j=1}^{2N} \lambda_j \leq \prod_{j=1}^{2N} \alpha_j \leq \prod_{j=1}^{2N} \frac{\det(A)}{\sigma_N^2}
\]

and hence

\[
\frac{\det(H)}{\sigma_1} \leq \det(H_{w'}) \leq \frac{\det(H)}{\sigma_N}.
\]

From Lemma 3.4.3,

\[
\max_a \sigma_{ka}^2 \leq \det(\Sigma_k)^2 \leq \det(\Sigma_k)^2 |(\mathcal{F}_{\lambda_k})_k|^2 \leq \min_a \sigma_{ka}^2.
\]

We have proved:

**Proposition 3.4.5.**

\[
I_{f,\Sigma^2} \leq \pi^{-1} \sum_{k,l=1}^{n} \frac{1}{\min_a \sigma_{ka}^2} I_{kl}
\]

with

\[
I_{kl} \overset{\text{def}}{=} \int_{\mathcal{M}_H} dM(z) \int_{\Theta_{i\neq k}(\mathcal{F}_{\lambda_k})_k} |\det S(M(f + g, z), k, e_l)|^2 \exp \left( -\sum_{i \neq k} \|g \Sigma_i^{-1}\| \right) \prod_{i \neq k} \frac{1}{\min_a \sigma_{ia}^2} \mathcal{A}(\mathcal{F}_{\lambda_k})_k(g).
\]

The integrals \( I_{kl} \) of Proposition 3.4.5 do not need to be computed. Instead, one can interpret them as the expected number of roots of certain random mixed systems of polynomials and exponential sums. Such objects were studied by ?Malajovich-expected.
Proposition 3.4.6. Let \( p_l : \mathbb{R}^n \rightarrow e_l^1 \simeq \mathbb{R}^{n-1} \) be the orthogonal projection. Then,
\[
I_{kl} = \frac{4\pi H}{\det \Lambda} (n-1)! V(p_l(A_1), \ldots, p_l(A_j), p_l(A_{j+1}), \ldots, p_l(A_n))
\]
where \( V \) is the \( n-1 \)-dimensional mixed volume operator.

Proof of Proposition 3.4.6. We only need to prove Proposition 3.4.6 for \( k = l = n \). If \( q \in \mathcal{F}_{A_i} \), then \( z \in Z(q) \). Recall that in that case,
\[
M(q, z) = \begin{pmatrix}
\frac{1}{\|V_{A_1}(z)\|} q_1 \cdot DV_{A_1}(z) \\
\vdots \\
\frac{1}{\|V_{A_n}(z)\|} q_n \cdot DV_{A_n}(z)
\end{pmatrix}
\]
and
\[
DV_{A_i}(z) = \text{diag} (V_{A_i}(z)) A_i
\]
where on the right, \( A_i \) stands for the matrix with rows \( a \in A_i \). The rows of matrix \( S(M(q, z), n, e_n) \) are:
\[
S(M(q, z), n, e_n) = \begin{pmatrix}
\frac{1}{\|V_{A_1}(z)\|} \text{diag}_{a \in A_1} (q_1 a V_{1a}(z)) A_1 \\
\vdots \\
\frac{1}{\|V_{A_{n-1}}(z)\|} \text{diag}_{a \in A_{n-1}} (q_{n-1} a V_{n-1a}(z)) A_{n-1}
\end{pmatrix}
\]
We claim that \(| \det S(M(q, z), n, e_n) |^{-2} \) is the normal Jacobian for a certain system of polynomial equations. We previously defined \( \mathcal{F}_{A_1}, \ldots, \mathcal{F}_{A_{n-1}} \) as spaces of polynomials over the complex manifold \( \mathcal{M} = \mathbb{C}^n \) mod \( 2\pi \sqrt{-1} \Lambda^* \), where \( \Lambda \) is the lattice generated by the \( A_i - A_i, 1 \leq i \leq n \) and \( \Lambda^* \) is its dual.

Let \( \mathcal{N} \subset \mathbb{C} \) be the strip \( -\pi < \text{Im}(z) \leq \pi \). Each point of \( \mathbb{C}^n \) mod \( 2\pi \sqrt{-1} \mathbb{Z}^n \) is represented by a unique point in \( \mathcal{N}^n \). Moreover, the natural projection
\[
\mathcal{N}^n \rightarrow \mathcal{M}
\]
is a \( n \)-to-1 local isometry. In the same spirit, we define \( \mathcal{N}_{H} \subset \mathcal{N} \) as the domain \( \mathcal{N}_{H} = \{ z \in \mathcal{N} : H \leq \text{Re}(z) \leq H, -\pi < \text{Im}(z) \leq \pi \}. \) Now each point of \( \mathbb{C}^n \) mod \( 2\pi \sqrt{-1} \mathbb{Z}^n \) with \( \| \text{Re}(z) \|_{\infty} \leq H \) is represented by a unique point in \( \mathcal{N}_{H}^n \). The natural projection
\[
\mathcal{N}_{H}^n \rightarrow \mathcal{M}_{H}
\]
is again a \( n \)-to-1 local isometry.

We extend all our spaces \( \mathcal{F}_{A_i} \) to spaces of functions on \( \mathcal{N}_{H}^n \), and write
\[
I_{nn} = \frac{1}{\det \Lambda} \int_{\mathcal{N}_{H}^n} \text{d}\mathbb{C}^n(z) \int_{\mathcal{M}_{H}(\mathcal{F}_{A_i})_{*}} \left| \det S(M(q, z), k, e_n) \right|^2 \\
\exp \left( -\sum_{i \neq n} \| (q_i - \hat{f}_i) \Sigma_i^{-1} \|^2 \right) \\
\prod_{i \neq n} \left| \det \Sigma_i \right|^2 \\
\text{d} (\mathcal{F}_{A_i})(q).
\]
We will recognize in the formula above the average number of zeros of a certain system of polynomial equations. This will be done through direct application of Theorem 3.4.1 (coarea formula). We need first a fiber bundle.
Define $\mathcal{H} = \mathcal{F}_{A_1} \times \cdots \times \mathcal{F}_{A_{n-1}} \times \mathcal{N}_H$, and endow this space with the product metric (the space $\mathcal{C}$ is endowed with the canonical metric). The solution variety $S_H \subset \mathcal{H} \times \mathcal{N}_H^n$ will be

$$S_H = \{(q_1, \ldots, q_{n-1}, w; z) \in \mathcal{H} \times \mathcal{N}_H^n : q_1 \cdot V_{A_1}(z) = \ldots = q_{n-1} \cdot V_{A_{n-1}}(z) = z_n - w = 0\}$$

with canonical projections $\pi_1 : S_H \to \mathcal{H}$ and $\pi_2 : S_H \to \mathcal{N}_H^n$. The inner product in $S_H$ is the pull-back of the inner product of $\mathcal{H}$ by $\pi_1$. Then the bundle $(S_H, \mathcal{N}_H^n, \pi, F)$ is a fiber bundle with fiber $F = (\mathcal{F}_{A_1})_0 \times \cdots \times (\mathcal{F}_{A_{n-1}})_0 \times \mathcal{C}$.

In order to compute the normal Jacobian, we differentiate the implicit function $\hat{G}$ for

$$\Phi(q_1, \ldots, q_{n-1}, w; z) = \begin{pmatrix} q_1 \cdot V_{A_1}(z) \\ \vdots \\ q_{n-1} \cdot V_{A_{n-1}}(z) \\ z_n - w \end{pmatrix} = 0,$$

Let $\hat{q} = (q_1, \ldots, q_{n-1}, w)$. We obtain:

$$D\hat{G}(\hat{q}; z) = -Dz\Phi(\hat{q}; z)^{-1}Dq\Phi(\hat{q}; z) = -S(M(q, z), e_n, n) \left( \prod_{1 \leq i \leq n} \frac{1}{\|V_{A_i}(z)\|} K_{A_i}(\cdot, z)^* \right) \left( \prod_{1 \leq i \leq n-1} \frac{1}{\|V_{A_{i-1}}(z)\|} K_{A_{i-1}}(\cdot, z)^* \right)^{-1}.$$

Recall that $\|V_{A_i}(z)\|^2 = K_{A_i}(z, z)$. The Normal Jacobian is therefore

$$NJ^2 = |\det D\hat{G}(\hat{q}; z)D\hat{G}(\hat{q}; z)^*| = |S(M(q, z), n, e_n)|^{-2}.$$

The coarea formula (Theorem 3.4.1) yields

$$I_{nn} = \frac{1}{\det \Lambda} \int_{S_H} \int_{\mathcal{H}} d\mathbb{C}^n(z) J_{nn}^{\mathcal{H}} 2N^{-1} \exp \left( -\sum_{i \neq n} \frac{\| (q_i - \hat{f}_i) \Sigma^{-1}_i \|^2}{\prod_{i \neq n} \| \Sigma_i \|^2} \right) \left( \prod_{i \neq n} \frac{1}{\| \Sigma_i \|^2} \right) dS_H(\hat{q}, z) = \frac{1}{\det \Lambda} \int_{\mathcal{H}} \#^{\pi_1^{-1}(\hat{q})} \exp \left( -\sum_{i \neq n} \frac{\| (q_i - \hat{f}_i) \Sigma^{-1}_i \|^2}{\prod_{i \neq n} \| \Sigma_i \|^2} \right) \left( \prod_{i \neq n} \frac{1}{\| \Sigma_i \|^2} \right) d\mathcal{H}(\hat{q}).$$

The integral is the average number of roots of the fenvomial system $\Phi(\hat{q}, z) = 0$. The value of the variable $z_n$ at a solution is precisely $w$. If we eliminate this variable from the other equations, we obtain a system of exponential sums with support $p_n(A_i)$, $1 \leq i \leq n - 1$. Theorem 1.3.2 implies that those systems have generically (and at most) $(n-1)!V(p_1(A_1), \ldots, p_{n-1}(A_{n-1}))$ isolated roots in $\mathcal{N}^n$,.
whence at most $(n - 1)!V(p_1(A_1), \ldots, p_{n-1}(A_{n-1}))$ isolated roots in $\mathcal{A}_H$. Thus, we integrate for $-H \leq \text{Re}(w) \leq H$ and $-\pi < \text{Im}(w) \leq \pi$ to obtain:

$$I_{nn} \leq \frac{4\pi H}{\det \Lambda} (n - 1)! V(p_1(A_1), \ldots, p_{n-1}(A_{n-1})).$$

\[\square\]

**Proof of Theorem 1.5.1** We make $K = 1 + 2L + \sqrt{\frac{\log(n)}{\min_i \sigma^2_{ik}} + 2 \log(3/2)}$, $L = \max \|f_i \Sigma_i^{-1}\|/\sqrt{S_i}$. Recall from Proposition 3.3.1 that

$$E_{f, S^2} \leq 1.25e(K + L)^2 \left( \sum_{k=1}^{n} \delta_{ii}^2 \max_{\rho: \lambda_i \in A_{k,i}, a \in A_i} S_i \right) \sup_{|f_i \Sigma_i^{-1}| \leq L} I_{f, S^2}.$$

Combining propositions 3.4.5 and 3.4.6

$$I_{f, S^2} \leq 4H \det(\Lambda) \frac{\sum_{k,i=1}^{n} \frac{1}{\min_{n} \sigma^2_{ik}} (n - 1)!V(p_1(A_1), \ldots, p_{n}(A_{n})))}{\det(\Lambda) \min_{n} \sigma^2_{ik} (n - 1)!V'}.$$

Since

$$V(p_1(A_1), \ldots, p_{n}(A_{n})) = V(A_1, \ldots, [0e_1], \ldots, A_n),$$

the additive properties of mixed volume imply that

$$\sum_{i} V(p_1(A_1), \ldots, p_{n}(A_{n})) = V(A_1, \ldots, [0, 1]^n, \ldots, A_n) \leq \frac{\sqrt{n}}{2} V(A_1, \ldots, B^n, \ldots, A_n)$$

and hence

$$I_{f, S^2} \leq 2H \frac{\sqrt{n}}{2} \frac{1}{\det(\Lambda) \min_{n} \sigma^2_{ik}} (n - 1)!V'.$$

\[\square\]

4. **Toric infinity**

4.1. **Proof of Theorem 1.6.5** Let $0 \neq \xi_1, \ldots, \xi_m \in \mathbb{R}^n$. The closed polyhedral cone spanned by the $\xi_i$ is

$$\text{Cone}(\xi_1, \ldots, \xi_m) = \{ s_1 \xi_1 + \cdots + s_m \xi_m : s_1, \ldots, s_m \geq 0 \}.$$

It turns out that any $k$-dimensional polyhedral cone is actually a union of cones of the form $\text{Cone}(\xi_i)$ where $#I = k$:

**Theorem 4.1.1** (Carathéodory). Let $0 \neq \xi_1, \ldots, \xi_m \in \mathbb{R}^n$ and let $x \in \text{Cone}(\xi_1, \ldots, \xi_n)$. Then there is $I \subset [m], \#I \leq n$, such that $x \in \text{Cone}(\xi_I)$.

A proof of Carathéodory’s Theorem can be found in the book by Blum et al. (1998, Cor. 2 p.168). We will apply this theorem here to the cones of the fan of a tuple of supports $A_1, \ldots, A_n$. Recall from section 1.6 that given subsets $B_1 \subset A_1, \ldots, B_n \subset A_n$, the (open) cone above $(B_1, \ldots, B_n)$ is

$$C(B_1, \ldots, B_n) = \{ 0 \neq \xi \in \mathbb{R}^n : B_i = A_i^\xi \}$$

where $A_i^\xi$ is the set of $a \in A_i$ maximizing $a \xi$; This cone belongs to some stratum $\mathcal{F}_{k-1}$ of the fan. The closure of $C(B_1, \ldots, B_n)$ is a polyhedral cone, and its vertices are all elements of the 0-stratum $\mathcal{F}_0$. Carathéodory’s theorem directly implies:
Corollary 4.1.2. Let $0 \neq x \in C(B_1, \ldots, B_n)$. Then there are $k \leq n$, and $\xi_1, \ldots, \xi_k \in S^n \cap \mathbb{F}_0$ so that

$$x = s_1 \xi_1 + \cdots + s_k \xi_k$$

for some $s_1, \ldots, s_k > 0$.

Proof of Theorem 1.6.5. Let $z = x + \sqrt{-1}y \in Z(q)$ be such that $\| \text{Re}(z) \| \geq H$. By Corollary 4.1.2, there are $\xi_1, \ldots, \xi_k \in S^n \cap \mathbb{F}_0$, $k \leq n$, so that

$$x = s_1 \xi_1 + \cdots + s_k \xi_k$$

with $s_1, \ldots, s_k > 0$. By permuting the $\xi_i$'s one can assume that $s_1 \geq s_2 \geq \cdots \geq s_k > 0$. In particular,

$$H \leq |x| \leq s_1 \| \xi_1 \| + \cdots + s_k \| \xi_k \| \leq ks_1 \leq ns_1.$$

This provides a lower bound $s_1 \geq H/n$. Suppose now that $ax$ is maximal for $a \in A_i$. Since $a \in A_i = A_i^{\xi_i} \cap A_i^{\xi_2} \cap \cdots \cap A_i^{\xi_k}$,

$$ax = s_1 \lambda_i(\xi_1) + s_2 \lambda_i(\xi_2) + \cdots + s_k \lambda_i(\xi_k).$$

Now suppose that $a' \in A_i \backslash A_i^{\xi_i}$. In that case

$$a'x = s_1 a' \xi_1 + s_2 a' \xi_2 + \cdots + s_k a' \xi_k.$$

Subtracting the two expressions,

$$(a' - a)x = s_1 (a' \xi_1 - \lambda_i(\xi_1)) + \cdots + s_k (a' \xi_k - \lambda_i(\xi_k)).$$

For $j > 1$, we estimate $a' \xi_j - \lambda_i(\xi_j) \leq 0$. But for $j = 1$, we can bound

$$a' \xi_1 - \lambda_i(\xi_1) \leq -\eta_i(\xi_1) \leq -\eta_i.$$

Therefore, $(a' - a)x \leq -\eta_i H/n$. In order to produce the perturbation $r_i$, define first

$$g_i = \sum_{a' \in A_i \backslash A_i^{\xi_i}} q_{ia'} V_{ia'}(z) = -\sum_{a \in A_i^{\xi_i}} q_{ia} V_{ia}(z).$$

For $a \in A_i^{\xi_i}$, set

$$r_{ia} = g_i \frac{V_{ia}(z)}{\sum_{a \in A_i^{\xi_i}} |V_{ia}(z)|^2}$$

and set $r_{ia'} = 0$ for $a' \in A_i \backslash A_i^{\xi_i}$. Let

$$W_i = \lim_{t \to \infty} \frac{1}{\| V_{Ai}(z + t\xi_1) \|} V_{Ai}(z + t\xi_1)$$

so that $[W] = \lim_{t \to \infty} [V(z + t\xi_1)]$. By construction,

$$(q_i + r_i)W_i = \sum_{a \in A_i} (q_{ia} + r_{ia}) W_{ia} = \sum_{a \in A_i^{\xi_i}} (q_{ia} + r_{ia}) W_{ia} = \frac{\sum_{a \in A_i^{\xi_i}} q_{ia} V_{ia}(z) - g_i}{\sqrt{\sum_{a \in A_i^{\xi_i}} |V_{ia}(z)|^2}} = 0.$$
The norm of the perturbation can be estimated as follows. For each \( a' \in A_i \setminus A_i^\xi \),
\[ |e^{a'z}| \leq e^{-\eta_i H/n + \lambda_i(x)} \]
and hence
\[ |g_i| = \left| \sum_{a' \in A_i \setminus A_i^\xi} q_i a^a V_{in^a}(z) \right| \leq \|q_i\| \sqrt{\sum_{a' \in A_i \setminus A_i^\xi} \rho_{in^a}^2 e^{-\eta_i H/n + \lambda_i(x)}}. \]

By construction, \( \lambda_i(x) = a''x \) for some vertex \( a'' \) of \( A_i \). Then,
\[ |g_i| \leq \|q_i\| \sqrt{\sum_{a'' \in A_i \setminus A_i^\xi} \rho_{in^a}^2 e^{-\eta_i H/n}} |V_{in^a}(z)| \leq \|q_i\| \kappa_{\rho_i} e^{-\eta_i H/n} \max_{a \in A_i^\xi} |V_{in^a}(z)|. \]

It follows that
\[ \frac{\|r_i\|}{\|q_i\|} \leq \kappa_{\rho_i} e^{-\eta_i H/n}. \]

\[ \square \]

4.2. Proof of Lemma 1.6.13

Proof of Lemma 1.6.13 We prove first the equivalence between (a) and (b). Assume that
\[ \sum_{a \in A_i} q_i a^a z^a \quad i = 1, \ldots, n \]
is strongly mixed. This means that for all \( \xi \neq 0 \), there is some \( i = i(\xi) \) with \( 1 \leq i \leq n \) such that there is a unique \( a^* = a^*(\xi) \in A_i \) satisfying
\[ H_i(\xi) = \bigoplus_{a \in A_i} \xi^a = \xi^{a^*}. \]

This condition holds in particular for all \( \xi \in \mathbb{R}_{0} \cap S^{n-1} \), so (a) implies (b). For the contrapositive, suppose that there is \( \xi \in S^{n-1} \) so that for all \( i \),
\[ H_i(\xi) = \bigoplus_{a \in A_i} \xi^a = \xi^{a^*} = \xi^{a^{**}} \]
with \( a^* \neq a^{**} \). If \( \xi \) belongs to a cone in \( \mathbb{R}_{0} \), we are done. Otherwise, it belongs to some closed cone \( \bar{C}_k \in \mathbb{R}_{0} \) with \( k \geq 1 \) minimal. But then the property above holds for any \( \xi \in \bar{C}_k \), hence it holds in \( \partial C_k \) and \( k \) was not minimal. Therefore, \(-a \Rightarrow -b\).

To show that (b) implies (c), we choose for each \( \xi \in \mathbb{R}_{0} \) a pair \((i, a^*)\) with \( A_i^\xi = \{a^*\} \). Then we suppose that \( q_{in^a} \neq 0 \) for all those pairs. In particular, for all \( 0 \neq \xi \in \mathbb{R}_{0} \), \( q \in \Sigma^x \) are unique by contradiction that \( q \in \Sigma^x \). We claim that \( q \in \Sigma^x \) for some \( x \). Indeed, there is some \( [V] \in \mathcal{Y} \setminus [V(M)] \) with \( q_i \cdot \frac{1}{|V_i|} V_{in^a} = 0 \) for all \( i \). Then there is a path \( z(t) \in \mathcal{M} \) with \( \lim [V(z(t))] = [V] \). By compactness of the sphere \( S^{n-1} \), there is an accumulation point \( x \in S^{n-1} \) of \( z(t) \) and \( [V] \in \Sigma^x \).

As before, let \( k \) be minimal so that \( x \) belong to a cone \( \bar{C}_k \) in \( \mathbb{R}_{0} \). We showed that \( k \neq 0 \). But if \( [V] \in \Sigma^x \), \( [V] \in \Sigma^x \) for any \( \xi \in \bar{C}_k \). In particular, this holds for \( \xi \in \partial C_k \), contradiction.
Now suppose that (b) does not hold for a certain $\xi$. Let $[V] = \lim_{t \to \infty} [V(t\xi)]$. We choose coefficients $q_{ia}$ so that

$$\sum_{a \in A^\xi} \rho_{ia} q_{ia} = 0, \ i = 1, \ldots, n.$$  

The value of the other $q_{ia} \neq 0$ is irrelevant. Then we have

$$q_i \frac{1}{\|V_i\|} V_{Ai} = 0.$$  

Thus, $\neg(b)$ implies $\neg(c)$.

4.3. The variety of systems with solutions at toric infinity. We assume in this section that $0 \neq \xi \in \mathbb{Z}^n$ and that $\gcd(\xi_i) = 1$. We define $\Sigma^\xi$ as the Zariski closure of the set of all $q \in \mathcal{F}$ with a root at infinity in the direction $\xi$. This means that the overdetermined system

$$\sum_{a \in A^\xi} q_{ia} V_{i,a}(z)$$

has a common root in $\xi^\perp \subset \mathbb{C}^n$, possibly at infinity. More formally, for each $i$ we can write $\mathcal{F}_{Ai} = \mathcal{F}_{A_i^\xi} \times \mathcal{F}_{A_i \setminus A_i^\xi}$. Let $\Sigma^\xi_0$ be the subvariety of systems in $\mathcal{F}_{A_1^\xi} \times \cdots \times \mathcal{F}_{A_n^\xi}$ that are solvable in $\xi^\perp$. Then

$$\Sigma^\xi = \Sigma^\xi_0 \times \mathcal{F}_{A_1 \setminus A_1^\xi} \times \cdots \times \mathcal{F}_{A_n \setminus A_n^\xi}.$$  

It follows that

$$\text{codim}(\Sigma^\xi) = \text{codim}(\Sigma^\xi_0)$$

both as subvarieties of a linear space or as subvarieties of a projective space. Pedersen and Sturmfels (1993) and Sturmfels (1994, Lemma 1.1) proved that the closure of the locus of sparse overdetermined systems with a common root is an irreducible variety in the product of the projectivizations of the coefficient spaces. This variety is defined over the rationals. In the setting of this paper, this implies that whenever $\xi \in \mathfrak{S}_0$, $\Sigma^\xi_0$ is an irreducible variety in

$$\mathbb{P}(\mathbb{C}^{\#A_1^\xi}) \times \cdots \times \mathbb{P}(\mathbb{C}^{\#A_n^\xi})$$

with rational coefficients. The same is trivially true for $\Sigma^\xi$.

Sturmfels (1994) defined the sparse mixed resultant in the codimension one case as the generating polynomial of the ideal of the variety of systems with a common root. If the codimension is more than one, he defined the sparse resultant as 1.

In this paper we proceed differently. We do not actually need to find the sparse resultant ideal $I(\Sigma^\xi)$ but just a non-zero polynomial in it. If there is $i$ with $\#A_i^\xi = 1$ for instance, the variety $\Sigma^\xi$ is contained into the hyperplane $q_{ia} = 0$. This will be enough to prove Theorem C and to derive probabilistic complexity bounds. Item (b) of the Theorem follows trivially from Lemma 1.6.13(c). Therefore we assume from now on the hypothesis $\#A_i^\xi \geq 2$.

If $I \subset \{i\}$, then we denote by $A_I$ the lattice spanned by $\bigcup_{i \in I} (A_i - A_i)$, and by $A_I^\xi$ the lattice spanned by $\bigcup_{i \in I} (A_i^\xi - A_i^\xi)$. The variety $\Sigma^\xi$ is the variety of solvable systems with support $(A_i^\xi)_{i=1,\ldots,n}$. 


The codimension of $\Sigma_0^\xi$ is known. Sturmfels (1994) computed the codimension of the variety of solvable systems. In the particular case of $\Sigma_0^\xi$, his bound reads:

**Theorem 4.3.1.** (Sturmfels, 1994)

$$\text{codim}(\Sigma_0^\xi) = \max_{I \subseteq [n]} \left( \#I - \text{rank}(\Lambda_0^\xi) \right).$$

**Corollary 4.3.2.**

$$\text{codim}(\Sigma^\xi) = \max_{I \subseteq [n]} \left( \#I - \text{rank}(\Lambda_0^\xi) \right).$$

From this result it is easy to construct an example of supports with non-zero mixed volume and a variety $\Sigma^\xi$ of large codimension.

**Example 4.3.3.** Choose $\xi = -e_n$. Let $A_n$ be the hypercube $A_n = \{a : a_i \in \{0,1\}, i = 1, \ldots, n\}$. Let $0 \leq k < n$ be arbitrary, and let $\Delta_k$ be the $k$-dimensional simplex,

$$\Delta_k = \{0, e_1, \ldots, e_k\}$$

and set

$$A_1 = \cdots = A_{n-1} = \Delta_k \cup (e_n + A_n)$$

For all non-empty $I \subseteq [n]$, rank$\Lambda^\xi_I = k$ and therefore the maximum is attained for $I[n]$. The codimension of $\Sigma^\xi$ is precisely $n - k$, which ranges between 1 and $n - 1$.

**Theorem 4.3.4.** Let $\Lambda$ be the lattice spanned by $\bigcup_i A_i - A_i$, let $B^n$ be the unit ball, and let $A = \text{Conv}(A_1) + \cdots + \text{Conv}(A_n)$. Let $\xi \in \mathfrak{S}_0$. Then the variety $\Sigma^\xi$ is contained in some surface of the form $Z(p)$, where $p$ is an irreducible polynomial of degree at most

$$d \leq \frac{e^2 \eta \lambda}{4\pi \det \Lambda} \max_{0 \leq k \leq n-1} (n-k!k!v_k).$$

In the strongly mixed case, $p$ is linear. In the unmixed case $A_1 = A_2 = \cdots = A_n$, or if there is some $A$ with $A_1 = d_1 A$, $\ldots$, $A_n = d_n A$, then $p$ is an irreducible polynomial of degree at most

$$d \leq \frac{e^2 \eta \lambda}{\sqrt{4\pi} \det \Lambda} n! v_0$$

**Proof.** Let $I$ be minimal such that

$$\#I - \text{rank}(\Lambda_0^\xi) = 1 < \text{codim}\Sigma^\xi.$$ 

In the unmixed case, we suppose that rank$(\Lambda_I^\xi) = n - 1$ for all $I$, so the only possible choice is $I = [n]$. The strongly mixed case has minimal $I$ with $\#I = 1$ and $p$ is therefore linear. We consider the general case now. In order to simplify notations, we reorder the supports so that $I = [r+1]$ where $r = \text{rank}(\Lambda_I^\xi)$. Suppose that $q \in \Sigma^\xi$. In that case, there is $z$ such that

(16) \[ \sum_{a \in A^\xi_i} q_{ia} V_{ia}(z) = 0, \quad 1 \leq i \leq n \]

and in particular the subsystem

(17) \[ \sum_{a \in A^\xi_i} q_{ia} V_{ia}(z) = 0, \quad 1 \leq i \leq r + 1 \]
admits a solution \( z \in \mathbb{C}^n \). Let \( Z \) be the complex linear span of \( \Lambda_\xi \). Let \( Z' \) be the smallest complex space containing \( Z \) and \( \xi \). For later usage, let \( Z_R \) be the real linear span of \( \Lambda_\xi \) and let \( Z'_R \) be the smallest real space containing \( Z_R \) and \( \xi \).

Without loss of generality, we can assume that \( z \in Z \).

Since \( I \) is minimal, the set of systems \( q \) for which such a solution of (17) exists is a hypersurface. Its degree \( d \) is the number of values \( t \in \mathbb{C} \) for which the system below admits a solution \( z \in Z 
\begin{equation}
  t \sum_{a \in A_\xi} f_{i(a)} \rho_{i, a} e^{az} + \sum_{a \in A_\xi} g_{i(a)} \rho_{i, a} e^{az} = 0, \quad 1 \leq i \leq r + 1.
\end{equation}

We assume \( f \) and \( g \) generic so there are no solutions for \( t = 0 \) or for \( t = \infty \). If we set \( t = e^{\|\xi\|^2} \), we recover a solution \( w = z + \|\xi\|^{-2} s \xi \in Z' \) for the system
\begin{equation}
  \sum_{a \in A_\xi} f_{i(a)} \rho_{i, a} e^{(n+\xi^T)w} + \sum_{a \in A_\xi} g_{i(a)} \rho_{i, a} e^{aw} = 0, \quad 1 \leq i \leq r + 1.
\end{equation}

Let \( \Lambda' \) be the lattice spanned by \( A_\xi - A_\xi, \ldots, A_{r+1}^\xi - A_{r+1}^\xi, \) and \( \{\xi^T\} \). The dual lattice of \( \Lambda' \) is \( (\Lambda_\xi^*) \) or \( \{\xi^T : k \in \mathbb{Z}\} \). For each solution of the system (19) in \( \mathbb{C}^n \), we recover a solution \( w = z + \|\xi\|^{-2} s \xi \in Z' \) for the system (18) has a solution in \( \mathbb{C}^n \mod 2\pi \sqrt{-1}(\Lambda_\xi)^* \).

We scaled \( \xi \) so that \( \xi \in \mathbb{Z}^n \) and the system (19) is an exponential sum with integer coefficients. According to Theorem 3.5.6, its generic number of solutions in \( Z' \) is precisely
\begin{equation}
  d_{n-r}^{\xi} = r + 1!
\end{equation}

using the identity \( \det \Lambda' = \|\xi\| \det \Lambda_\xi \). In the formula above, \( V_{Z_R} \) denotes the mixed volume restricted to the \( r + 1 \)-dimensional space \( Z_R \).

We assumed that \( \Lambda \) was an \( n \)-dimensional lattice, and that \( \xi \in \mathfrak{g}_0 \). These hypotheses imply that \( \xi \) is orthogonal to a sublattice \( \Lambda_\xi \subset \Lambda \) of rank \( n \), and we have the inclusion
\[
  \Lambda_\xi \subset \Lambda \cap Z_R \subset \Lambda \cap \xi^\perp \subset \Lambda.
\]

The first inclusion is equidimensional. There is a lattice basis \( (u_1, \ldots, u_n) \) of \( \Lambda \) with the following properties: \( u_1, \ldots, u_r \in \Lambda \cap Z_R \) and \( u_{r+1}, \ldots, u_{n-1} \in \Lambda \cap \xi^\perp \), with \( u_n = \xi \).

Let \( C_\xi = [0u_1] + \cdots + [0u_n] \). In particular, \( C_n \) is a fundamental domain of \( \Lambda \) and \( C_{n-1} \) is a fundamental domain of \( \Lambda \cap \xi^\perp \). We denote by \( \pi \) the orthogonal projection onto \( (Z_R^\perp)^\perp \). In particular, \( C' = \pi(C_{n-1}) = [0, \pi(u_{r+1})] + \cdots + [0, \pi(u_{n-1})] \) is a fundamental domain of \( \pi(\Lambda) \subset (Z_R^\perp)^\perp \).

If \( \eta_\Lambda(\xi) \) is the minimum of \( \|b\xi\| \) for \( b \in \Lambda \\setminus (\Lambda \cap \xi^\perp) \), we have
\[
  \det \Lambda = \eta_\Lambda(\xi) \det(\Lambda \cap Z_R) \vol(C') \leq \eta_\Lambda(\xi) \det(\Lambda_\xi^*) \vol(C').
\]
Equation (20) implies the bound:
\begin{equation}
  d_{n-r}^{\xi} \leq r + 1! \eta_\Lambda(\xi) \vol(C') \frac{V_{Z_R}(\conv(A_\xi^\perp) + [0, \xi^T], \ldots, \conv(A_{r+1}^\xi) + [0, \xi^T])}{\|\xi\| \det \Lambda}
\end{equation}
The domain $C'$ is orthogonal to $Z_{\mathbb{R}} \supset \text{Conv}(A^\xi), [0\xi]$. Therefore,
\[
d_{n-r}^\xi \leq r + 1! \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, \text{Conv}(A^\xi_{n+1}) + [0\xi], \ldots, [0\pi(\mathbf{u}_{r+1}), \ldots, [0\pi(\mathbf{u}_{n-1})])
\]
\[
= \frac{r + 1!}{n - r - 1!} \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, \text{Conv}(A^\xi_{r+1}) + [0\xi],
\]
\[
C', C'').
\]

To simplify notations we assume now that $0 \in A_i$ for each $i$. The sum $A = \text{Conv}(A_1) + \cdots + \text{Conv}(A_n)$ is $n$-dimensional, so it admits a linearly independent set of vectors $\mathbf{w}_j = \sum_i a_{ij}\mathbf{a}_j$ with $a_{ij} \in A_i, 1 \leq j \leq n$. At least $n - r - 1$ of the $\pi(\mathbf{w}_j)$ are linearly independent, say those are $\pi(\mathbf{w}_{r+1}), \ldots, \pi(\mathbf{w}_{n-1})$. Since that $C'$ is a fundamental domain of $\pi(A), \text{Vol}(C') \leq \text{Vol}(\pi(C''))$, for
\[
C'' = [0\mathbf{w}_{r+1}] + \cdots + [0\mathbf{w}_{n-1}]
\]
and hence
\[
d_{n-r}^\xi \leq \frac{r + 1!}{n - r - 1!} \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, \text{Conv}(A^\xi_{n+1}) + [0\xi], \ldots, \pi(C''), \ldots, \pi(C''))
\]
\[
= \frac{r + 1!}{n - r - 1!} \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, \text{Conv}(A^\xi_{r+1}) + [0\xi],
\]
\[
C'', \ldots, C'').
\]

By convexity, the simplex with vertices $0, \mathbf{w}_{r+1}, \ldots, \mathbf{w}_{n-1}$ is contained in $A$. It follows that $C'' \subset (n - r - 1)A$ and thus
\[
d_{n-r}^\xi \leq \frac{r + 1!}{n - r - 1!} \frac{(n - r - 1)^{n-r-1}}{n - r - 1!} \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, 
\]
\[
\text{Conv}(A^\xi_{r+1}) + [0\xi], A, \ldots, A)
\]

In the particular case $r = n - 1$, what we get is
\[
d_1^\xi \leq n! \frac{\eta A(\xi)}{||\xi|| \det A} V(\text{Conv}(A^\xi) + [0\xi], \ldots, \text{Conv}(A^\xi_{n+1}) + [0\xi])
\]
\[
\leq n! \eta \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon = 0} V(\text{Conv}(A^\xi) + \epsilon B^n, \ldots, \text{Conv}(A^\xi_n) + \epsilon B^n)
\]
\[
\leq \frac{n! \eta}{2 \det A} V'.
\]

For the general case $r < n - 1$ we will need Stirling’s approximation for $l = n - r - 1$:
\[
1^{l/2} \leq \frac{l^{l/2} e^l}{\sqrt{2\pi l^{l+1/2}}} \leq \frac{1}{\sqrt{2\pi}} e^{(l+1/2)\log(l)} \leq \frac{e}{\sqrt{2\pi}}
\]
Using this trick,
\[
\phi_{n-r}^{\xi} \leq \frac{r + 1!}{4\pi \det \Lambda} \frac{e^{2\eta}}{\epsilon} \frac{c_{n-r-1}}{c_{n-r-1}} V(\text{Conv}(\bar{A}_\xi) + \epsilon B^n + \delta A, \ldots)
\]
\[
\leq \frac{r + 1!}{4\pi \det \Lambda} \frac{e^{2\eta}}{\epsilon} \frac{c_{n-r-1}}{c_{n-r-1}} V(\text{Conv}(A_1) + \epsilon B^n + \delta A, \ldots)
\]
\[
= \frac{r + 1!}{4\pi \det \Lambda} n - r - 1 - \frac{e^{2\eta}}{4\pi \det \Lambda} \rho_{n-r-1}
\]
Finally, we set \( k = n - r - 1 \). When \( r + 1 = n, k = 0 \).

**Proof of Main Theorem**

We claim that
\[
\Sigma^\infty = \bigcup_{\xi \in \delta_0} \Sigma^\xi.
\]
Indeed, let \( q \in \Sigma^\infty \). Let \( [W] \in V \) be the root at infinity, so \( q \cdot W = 0 \). Let \( B_i = \{ a \in A_i : W_ia \neq 0 \} \). There must exist \( x \in S^{n-1} \) with \( B_i \subset A^x_i \), for otherwise there would exist some \( z \in M \) with \( [W] = [V(z)] \). The closed cone \( \bar{C} = \bar{C}(A^x_1, \ldots, A^x_n) \) is a polyhedral cone with vertices in \( \delta_0 \). For each vertex \( \xi \in \delta_0 \), we can write \( q \) as the limit of a family \( q_0 \in \Sigma^\xi \) (just multiply the coefficients \( q_{au} \) by \( t \) for \( a \in \bar{A}_i \setminus A^x_i \). Since \( \Sigma^\xi \) is Zariski closed, \( q \in \Sigma^\xi \) for \( \xi \in \delta_0 \cap S^{n-1} \). Thus, \( \Sigma^\infty \) is a finite union of \( \Sigma^\xi \) for \( \xi \in \delta_0 \) in a subset of \( \delta_0 \cap S^{n-1} \).

Item (a) follows directly from Proposition 4.3.4. The particular case in Item (b) was already proved in Lemma 1.6.13(c). Let’s prove item (c) now.

Let \( r : \mathcal{F} \to \mathbb{C} \) be the polynomial of item (a). We produce now two real polynomials \( p \) and \( q \) depending on the real and on the imaginary parts of \( g \in \mathcal{F} \), and on a real parameter \( t \):

\[
p(\text{Re}(g), \text{Im}(g), t) = \text{Re}(r(g + tf))
\]
\[
q(\text{Re}(g), \text{Im}(g), t) = \text{Im}(r(g + tf)).
\]

We also write \( p(t) = p(\text{Re}(g), \text{Im}(g), t) = \sum_{i=0}^{d_r} p_i t^i \) and similarly for \( q \). The coefficient \( p_i \) (resp. \( q_i \)) is a polynomial of degree at most \( d_r - i \) on the real and imaginary parts of \( g \). We assumed that \( r(f) \neq 0 \), therefore the leading term \( t^{d_r} r(f) \) of \( r(g + tf) \) does not vanish, and \( r(g + tf) \) has degree \( d_r \) in \( t \). We do not know the degree of \( p(t) \) and \( q(t) \). We can nevertheless assume that \( \deg p(t) = a \) and \( \deg q(t) = b \) with \( \max(a, b) = d_r \). The Sylvester resultant is

\[
s(g) \overset{\text{def}}{=} S(\text{Re}(g), \text{Im}(g)) = \det
\[
\begin{pmatrix}
  b \text{ columns} & a \text{ columns} \\
  p_0 & q_0 \\
  p_1 & q_1 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  p_a & q_a \\
  \vdots & \vdots \\
  \end{pmatrix}
\]
which has clearly degree $ab \leq d^2$. If $r(g + tf)$ for a real, possibly infinite $t$, then $p(t)$ and $q(t)$ have a common real root. In particular, $s(g) = 0$. The situation $s \equiv 0$ cannot arise because $r(f) \neq 0$.

Notice also that the Sylvester resultant can vanish when $p(t)$ and $q(t)$ have a common non-real root, which does not correspond to a $g + tf \in \Sigma^\infty$, $t \in \mathbb{R}$.

### 4.4 Probabilistic estimates

We start with a trivial Lemma:

**Lemma 4.4.1.** Let $c > 0$ and $S \in \mathbb{N}$ be arbitrary.

$$\Pr_{x \sim N(0, I; S)} \left[ \|x\| \leq c\sqrt{S} \right] \leq \frac{e^{S(1 + 2 \log(c))}}{\sqrt{2\pi S}}.$$

**Proof.**

$$\Pr_{x \sim N(0, I; S)} \left[ \|x\| \leq c\sqrt{S} \right] = \int_{B(0, c\sqrt{S}; S^2)} e^{-\|x\|^2} dC^S(x) \leq \frac{e^{S(1 + 2 \log(c))}}{\sqrt{2\pi S}}.$$

using Stirling’s approximation $S! \geq \sqrt{2\pi S} S^{1/2} e^{-S}$.

In order to prove Theorem 1.6.14 we introduce now the conic condition number

$$\mathcal{C}(g) \defeq \inf_{h \in \mathcal{F} : x(g + h) = 0} \frac{\|g\|}{\|h\|} = \frac{\sqrt{\sum_{i=1}^N \|g_i\|^2}}{\inf_{h \in \mathcal{F} : x(g + h) = 0} \sqrt{\sum_{i=1}^N \|h_i\|^2}}.

We stress that this is the only part of this paper where we forsake the complex multi-projective structure (and invariance) in $\mathcal{F}$. At this point, we look at the zero-set of $s$ as a real algebraic variety in $\mathbb{R}^{2S}$ or in $\mathbb{S}^{2S-1}$. According to Bürgisser and Cucker (2013, Theorem 21.1),

$$\Pr_{g \sim N(0, I; \mathcal{F})} \left[ \mathcal{C}(g) \geq \epsilon^{-1} \right] \leq 8eS \epsilon$$

for $\epsilon^{-1} > 2(2d^2 + 1)S$.

**Proof of Theorem 1.6.14** Let $c$ be a number to be determined. We consider first $g$ such that $\|g\| > c\sqrt{S}$.

Assume furthermore that $g \in \Omega_H$. In particular there are $t \in [0, T]$, $z \in Z(g + tf)$ with $\|\text{Re}(z)\|_\infty \geq H$. From Theorem 1.6.5 there is $h$ such that $g + h + tf \in \Sigma^\infty \subset Z(r)$, with

$$\frac{\|h_i\|}{\|g_i + tf_i\|} \leq \kappa_{\rho_i} e^{-\eta_i H/n} \quad i = 1, \ldots, n.$$

In particular,

$$\|h\| \leq \|g + tf\| \max_{i} (\kappa_{\rho_i}) e^{-\eta H/n}.$$
We can bound \( \|g + tf\| \leq \|g\| + T|f| \leq \|g\| \left(1 + \frac{T|f|}{c\sqrt{S}}\right) \). Therefore,

\[
\mathcal{E}(g)^{-1} \leq \frac{\|h\|}{\|g\|} \leq \epsilon \overset{\text{def}}{=} \left(1 + \frac{T|f|}{c\sqrt{S}}\right)_{i=1}^{\max} \kappa_{\rho_i} e^{-\eta H/n}.
\]

We can now pass to probabilities:

\[
\begin{align*}
\text{Prob}_{g \sim N(0, I; F)} \left[ g \in \Omega_H \right] & \leq \text{Prob}_{g \sim N(0, I; F)} \left[ \|g\| \leq c\sqrt{S} \right] + \text{Prob}_{g \sim N(0, I; F)} \left[ \mathcal{E}(g)^{-1} > \epsilon^{-1} \right] \\
& \leq \frac{e^{S(1+2\log(c))}}{\sqrt{2\pi S}} + 8ed_n^2 \epsilon \\
& \leq \frac{e^{S(1+2\log(c))}}{\sqrt{2\pi S}} + 8ed_n^2 \max_{i=1}^{\max} |\sigma_{ia}| \left(1 + \frac{T|f|}{c\sqrt{S}}\right)_{i=1}^{\max} \kappa_{\rho_i} e^{-\eta H/n}.
\end{align*}
\]

We want this expression smaller than an arbitrary \( \delta > 0 \). A non-optimal solution is to set \( c = e^{\frac{\log(\delta)}{2\eta}} - \frac{1}{2} \). This guarantees that

\[
\frac{e^{S(1+2\log(c))}}{\sqrt{2\pi S}} \leq \frac{\delta}{2}.
\]

Then we can set

\[
H \geq \frac{n}{\eta} \log \left(16e\delta^{-1}d_n^2 \max_{i=1}^{\max} (\kappa_{\rho_i}) \left(1 + \frac{T|f|}{c\sqrt{S}}\right)_{i=1}^{\max}\right).
\]

Because the \( \rho_{ia} = 1 \) are constant, we can replace \( \max_i (\kappa_{\rho_i}) \) by \( \max_i (S_i) \).

\[
\square
\]

5. Analysis of Linear Homotopy

5.1. Overview. The proof of Theorem \ref{thm:main} is long. Recall that \( \mathbf{q}_t = g + tf \), where \( f \) is assumed ‘fixed’ and \( g \) is assumed ‘Gaussian’, conditional to the event \( g \notin \Lambda_I \cup \Omega_H \cup Y_K \), where

\[
\Lambda_I = \left\{ g \in F : \exists 1 \leq i \leq n, \exists \mathbf{a} \in A_i, \arg \left(\frac{g_{ia}}{f_{ia}}\right) \geq \pi - \epsilon, \right\}
\]

\[
\Omega_H = \Omega_{t,r,H} = \{ g \in F : \exists t \in [0, T], \exists \mathbf{z} \in \mathbb{Z}(g + tf) \subset \mathcal{M}, \| \text{Re}(\mathbf{z}) \|_\infty \geq H \}
\]

\[
Y_K = \{ g \in F : \exists i, \|g_i\| \geq K \sqrt{S_i} \}.
\]

The first two sets were assumed with probability \( \leq 1/72 \). The event \( g \in \Lambda_I \cup \Omega_H \) has therefore probability \( \leq 1/36 \). By choosing \( K = 1 + \sqrt{\frac{\log(n) + \log(10)}{\min_i S_i}} \), Lemma \ref{lem:bounds} implies that the event \( \|g_i\| \geq K \sqrt{S_i} \) has probability \( \leq \frac{1}{10} \), hence the event \( g \in Y_K \) has probability \( \leq 1/10 \). Moreover, it is independent from the two other events. Thus the event \( g \notin \Lambda_I \cup \Omega_H \cup Y_K \) has probability \( \geq 7/8 \). Under this condition we need to estimate the expectation of the condition length

\[
\mathcal{L}(\mathbf{q}_t, \mathbf{z}_r; 0, T) = \int_0^T \left\| \frac{\partial}{\partial \tau} \left( \mathbf{q}_t \cdot R(\mathbf{z}_r) \right) \right\|_{\mathbf{q}_t, R(\mathbf{z}_r)} + \nu_0 \| \dot{\mathbf{z}}_r \|_0 \right\| \mu(\mathbf{q}_t \cdot R(\mathbf{z}_r), 0) \right\| d\tau
\]

with

\[
\left\| \frac{\partial}{\partial \tau} \left( \mathbf{q}_t \cdot R(\mathbf{z}_r) \right) \right\|_{\mathbf{q}_t, R(\mathbf{z}_r)} = \sum_{i=1}^{n} \left\| \frac{P_{q_is} R(z_r)}{\|q_is R(z_r)\|} \right\|^2.
\]
Theorem 5.1.1. There is a constant $C_0$ with the following properties. Assume that the hypotheses $[\text{a}]$ to $[\text{d}]$ of Theorem 5.1 hold. Let $T > 2K$, and set

$$\text{LOGS}_0 \overset{\text{def}}{=} \log(d_s) + \log(S) + \log(T).$$

Let $f \in \mathcal{F}$ with $r(f) \neq 0$, and suppose that $f$ is scaled in such a way that $\|f\| = \sqrt{S}$. Let $K = \left(1 + \frac{\log(n) + \log(10)}{\min(S_i)}\right)$. Define also

$$\kappa_f = \max_{i,a} \frac{\|f_i\|}{\|f_{\mathbf{a}}\|}.$$

Take $g \sim N(0,I;\mathcal{F})$ conditional to $g \notin \Lambda_{\varepsilon} \cup \Omega_{\varepsilon} \cup Y_{K}$, and and consider the random path $\hat{g}_f = g + tf \in \mathcal{F}$. To this path associate the set $\mathcal{Z}(\hat{g}_f)$ be the set of continuous solutions of $q_t \cdot V(z_t) = 0$. Suppose that $T > 2K$. Then with probability $\geq 6/7$,

$$\sum_{\mathcal{Z}_t \in \mathcal{Z}(\hat{g}_f)} \mathcal{L}(q_t, z_t; 0, T) \leq C_0 q_n S \max_i (S_i) (K + \sqrt{1 + K/4 + \kappa_f/8}) T \text{ LOGS}_0.$$

Corollary 5.1.2. If the conditional probability for $g$ is replaced by unconditional $g \sim N(0,I;\mathcal{F})$, then the inequality above holds with probability $\geq \frac{6}{7} \frac{9}{8} = \frac{3}{4}$.

The first step toward the proof of Theorem 5.1.1 is to break the condition length into two integrals,

$$\mathcal{L}_1((q_t, z_t); 0, T) = \int_0^T \|q_{\mathbf{r}} \cdot R(z_t)\|_{q_{\mathbf{r}} \cdot R(z_t)} \mu(q_{\mathbf{r}} \cdot R(z_t), 0) \, d\tau,$$

$$\mathcal{L}_2((q_t, z_t); 0, T) = 2\nu_0 \int_0^T \|\hat{z}_{\mathbf{r}}\|_{\mu(q_{\mathbf{r}} \cdot R(z_t), 0)} \, d\tau.$$

Lemma 5.1.3. Assume that $q_{\mathbf{r}} \cdot V(z_t) = 0$ and $m_i(0) = 0$ for all $i$. Then

(a) $$\left\| \frac{\partial}{\partial \tau} q_{\mathbf{r}} \cdot R(z_t) \right\|_{q_{\mathbf{r}} \cdot R(z_t)} \leq \left\| (q_{\mathbf{r}} \cdot R(z_t)) \right\|_{q_{\mathbf{r}} \cdot R(z_t)} + \nu_0 \|\hat{z}_{\mathbf{r}}\|.$$

(b) $$\mathcal{L}(q_t, z_t; 0, T) \leq \mathcal{L}_1((q_t, z_t); 0, T) + \mathcal{L}_2((q_t, z_t); 0, T).$$

Proof. Item (a) follows from the product rule and some easy estimates: in each $\mathcal{P}(\hat{F}_A)$,

$$[q_{i\tau} R_i(z_t)] = [q_{i\tau} R_i(z_t) e^{\mathbf{r} \cdot \mathbf{z}_t}] = [q_{i\tau} \hat{R}_i(z_t)]$$

with $\hat{R}_i(z_t) = \left(\begin{array}{c} e^{n_{z_{2i}}} \\
_{z_t} \end{array}\right)_{a \in A_i}$. By replacing each $q_i(\tau) R_i(z_t)$ by the more suitable representative $q_i(\tau) \hat{R}_i(z_t)$, one obtains:

$$\left\| \frac{\partial}{\partial \tau} (q_{\mathbf{r}} \cdot R(z_t)) \right\|_{q_{\mathbf{r}} \cdot R(z_t)} \leq \left\| \frac{\partial}{\partial \tau} (q_{\mathbf{r}} \cdot \hat{R}(z_t)) \right\|_{q_{\mathbf{r}} \cdot \hat{R}(z_t)} \leq \|q_{\mathbf{r}} \cdot \hat{R}(z_t)\|_{q_{\mathbf{r}} \cdot \hat{R}(z_t)} + \max_{i,a} |a z_{2i}|$$

$$\leq \|q_{\mathbf{r}} \cdot R(z_t)\|_{q_{\mathbf{r}} \cdot R(z_t)} + \max_{i,a} |a z_{2i}|$$

$$\leq \|q_{\mathbf{r}} \cdot R(z_t)\|_{q_{\mathbf{r}} \cdot R(z_t)} + \nu_0 \|\hat{z}_{\mathbf{r}}\|.$$
Item (b) is now trivial.

5.2. Expectation of the condition length (part 1). The objective of this section is to bound the expectancy of the integral

\[ \mathcal{L}_1((q_t(z_t);0,T) = \int_0^T \|q_t \cdot R(z_t)\|_{q_t \cdot R(z_t)} \mu(q_t \cdot R(z_t),0) \, dt. \]

**Proposition 5.2.1.** Under the conditions of Theorem 5.1.1, there is a constant \( C_1 \) such that

\[
\mathbb{E}_{g \sim N(0,1)} \left( \sum_{z \in \mathcal{Z}(q_t)} \mathcal{L}_1((q_t(z_t);0,T) \left| g \notin \Lambda_t \cup \Omega_H \cup Y_K \right. \right) \leq \leq C_1 Q^{n/2} \sqrt{1 + K/4 + \kappa T/8} \max(S_i) T \log S_0
\]

We need first a preliminary result:

**Lemma 5.2.2.** Let \( q_t \in \mathcal{X} \) be smooth at \( t_0 \), with \( \|q_t\|_{t_0} \neq 0 \) for \( i = 1, \ldots, n \). Let \( z \in \mathbb{C}^n \). Then for all \( i \),

\[ \|q_{it} \cdot R_i(z)\|_{q_{it} \cdot R_i(z)} \leq \max_{a \in A_i} \frac{|(q_{ia})|_{t_0}}{|(q_{ia})|_{t_0}} \]

In particular, if \( q_t = tf + g \in \mathcal{X} \), then

\[ \|q_{it} \cdot R_i(z)\|_{q_{it} \cdot R_i(z)} \leq \max_{a \in A_i} \frac{1}{|g_{ia}| + t} \]

and

\[ \|q_t \cdot R(z)\|_{q_t \cdot R(z)} \leq \sqrt{\sum_{i} \left( \max_{a \in A_i} \frac{1}{|g_{ia}| + t} \right)^2} \]

**Proof.** Let \( u = (q_t)_{t_0} \cdot R_i(z) \) and \( v = (q_t)_{t_0} \cdot R_i(z) \). We compute

\[ \|v\|_u^2 = \frac{1}{|u|_6} \|u\|_6 v - \langle v, u \rangle u \|^2 = \frac{1}{|u|_6^2} (\|u\|_6^2 - \langle v, u \rangle^2) \]

Let \( w_{ia} = v_{ia}/u_{ia} \). Then,

\[ \|v\|_u^2 = \sum_{a,b} |u_{ia}|^2 |u_{ib}|^2 w_{ia} (\bar{w}_{ia} - \bar{w}_{ib}) = \frac{1}{2} \sum_{a,b} |u_{ia}|^2 |u_{ib}|^2 |w_{ia} - w_{ib}|^2 \]

and so \( |v|_u \leq \max |w_{ia}| \). This proves the first part of the Lemma. The second part comes from taking absolute values of the expression

\[ w_{ia} = \frac{f_{ia}}{g_{ia} + tf_{ia}} = \frac{1}{|g_{ia}| + t} \]

**Lemma 5.2.2** yields a convenient bound for the integral \( \mathcal{L}_1 \), viz.

\[ \mathcal{L}_1((q_t(z_t);0,T) \leq \int_0^T \sqrt{\sum_{i} \max_{a \in A_i} \left( \frac{1}{|g_{ia}| + t} \right)^2} \mu(q_t \cdot R(z_t),0) \, dt, \]
and we also apply Lemma 3.3.2(b) with \(\|(\mathbf{q}_i)_t\| \leq (t + K)\sqrt{S_t}\). Adding over all paths,

\[
\sum_{\mathbf{z}_r \in Z(q_r)} \mathcal{L}_1((\mathbf{q}_r, \mathbf{z}_r); 0, T) \leq c_1 \int_0^T \left( \sum_{i \in A_1} \max_{\mathbf{a}_i} \left( \frac{t + K}{q_{r,i} + t} \right) \right)^2 \sum_{\mathbf{z} \in Z(q_t)} \|M(\mathbf{q}_t, \mathbf{z})^{-1}\|_F^2 \, dt.
\]

with \(c_1 = \sqrt{\sum_i \delta_i^2 \max_i(S_i)}\). Cauchy-Schwartz inequality applied first to the rightmost sum and then to the integral yields

\[
\sum_{\mathbf{z}_r \in Z(q_r)} \mathcal{L}_1((\mathbf{q}_r, \mathbf{z}_r); 0, T) \leq c_2 \sqrt{\int_0^T \sum_{i \in A_1} \max_{\mathbf{a}_i} \left( \frac{(t + K)^2}{q_{r,i} + t} \right)^2 \, dt} \sum_{\mathbf{z} \in Z(q_t)} \|M(\mathbf{q}_t, \mathbf{z}_t)^{-1}\|_F^2 \, dt,
\]

with \(c_2 = \sqrt{\sum_i \delta_i^2 \max_i(S_i)} \sqrt{\frac{\text{det} \Lambda}{\text{det} \Sigma}}\). We would like at this point to pass to the conditional expectancies. Using Cauchy-Schwartz again,

\[
\mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \sum_{\mathbf{z}_r \in Z(q_r)} \mathcal{L}_1((\mathbf{q}_r, \mathbf{z}_r); 0, T) \mid \mathbf{g} \notin \Lambda_e \cup \Omega_H \cup Y_K \right) \leq \]

\[
\leq c_2 \sqrt{\mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \sum_{i \in A_1} \max_{\mathbf{a}_i} \left( \frac{(t + K)^2}{q_{r,i} + t} \right)^2 \mid \mathbf{g} \notin \Lambda_e \cup \Omega_H \cup Y_K \right)} \sum_{\mathbf{z} \in Z(q_t)} \|M(\mathbf{q}_t, \mathbf{z}_t)^{-1}\|_F^2 \, dt \sum_{\mathbf{g} \notin \Lambda_e \cup \Omega_H \cup Y_K}
\]

Above, we used the fact that \(\mathbf{g} \notin \Omega_H\). The sets \(Z(q_t)\) and \(Z_H(q_t)\) are therefore the same. For any positive measurable function \(\phi : \mathcal{F} \to \mathbb{R}\), we have

\[
\mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \phi(g) \mid g \notin \Lambda_e \cup \Omega_H \cup Y_K \right) \leq \]

\[
\leq \mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \phi(g) \mid g \notin \Lambda_e \right) \frac{\text{Prob}_{\mathbf{g} \sim N(0, I; \mathcal{F})} [\mathbf{g} \notin \Lambda_e]}{\text{Prob}_{\mathbf{g} \sim N(0, I; \mathcal{F})} [\mathbf{g} \notin \Lambda_e \cup \Omega_H \cup Y_K]}
\]

\[
\leq \frac{71}{63} \mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \phi(g) \mid g \notin \Lambda_e \right)
\]

and similarly:

\[
\mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \phi(g) \mid g \notin \Lambda_e \cup \Omega_H \cup Y_K \right) \leq \frac{8}{7} \mathbb{E}_{\mathbf{g} \sim N(0, I; \mathcal{F})} \left( \phi(g) \right)
\]

We have proved that
Lemma 5.2.3. On the hypotheses above,

\[
E_{g \sim N(0, I; F)} \left( \sum_{z_t \in Z(q_t)} L_1((q_t, z_t); 0, T) \mid g \notin \Lambda_T \cup \Omega_H \cup Y_K \right) \leq \]

\[
\leq c_3 \sqrt{E_{g \sim N(0, I; F)} \left( \int_0^T \sum_{a \in A} \frac{(t + K)^2}{q_{i/n} + t} dt \mid g \notin \Lambda_T \right)} \times \sqrt{E_{g \sim N(0, I; F)} \left( \int_0^T \sum_{z \in Z_H(q_t)} \|M(q_t, z_t)^{-1}\|_F^2 dt \right)}. \]

with \(c_3 = \frac{2\sqrt{12}}{\pi} \sum_i \delta_i^2 \max_i(S_i) \sqrt{\frac{nV}{\det(\Lambda)}}.\)

Now we need to bound the two expectations in the bound above. The second one is easy: from Theorem 1.5.17.

Lemma 5.2.4.

\[
E_{g \sim N(0, I; F)} \left( \int_0^T \sum_{z \in Z_H(q_t)} \|M(q_t, z_t)^{-1}\|_F^2 dt \right) \leq \frac{2HT \sqrt{n}}{\det(\Lambda)} (n - 1)!V'. \]

Proof.

\[
\leq \int_0^T \sum_{z \in Z_H(q_t)} \|M(q_t, z_t)^{-1}\|_F^2 dt \leq \frac{2HT \sqrt{n}}{\det(\Lambda)} (n - 1)!V'. \]

Lemma 5.2.5.

\[
E_{g \sim N(0, I; F)} \left( \sum_{a \in A} \frac{(t + K)^2}{q_{i/n} + t} dt \mid g \notin \Lambda_T \right) \leq \left( \pi \log(2/e) + \frac{\epsilon^2}{\sin(\epsilon)} \right) \times \left( \frac{1}{2} K^2 S \sqrt{\pi} + KS + \frac{1}{4} K_I S \sqrt{\pi} \right) + 2ST. \]

Proof. We start by bounding the maximum by the sum,

\[
\int_0^T \sum_{a \in A} \frac{(t + K)^2}{q_{i/n} + t} dt \leq \int_0^T \sum_{a \in A} \frac{(t + K)^2}{q_{i/n} + t} dt = \sum_{a \in A} \frac{(t + K)^2}{q_{i/n} + t} dt. \]
Let \( \Delta = \{ z \in \mathbb{C} : -\pi + \epsilon \leq \arg(z) \leq \pi - \epsilon \} \), and notice that the variables \( g_{ia}/f_{ia} \) are independently distributed in \( \Delta \), with probability density function

\[
\frac{|f_{ia}|^2}{\pi - \epsilon} e^{-\frac{|g_{ia}/f_{ia}|^2}{|f_{ia}|^2}}.
\]

An elementary change of variables yields

\[
\int_0^T \frac{(t + K)^2}{|f_{ia}|^2} dt = \int_0^T \frac{\left( \left| f_{ia} \right| t + K \right)^2}{\left| f_{ia} \right|^2 + \left| f_{ia} \right| t} dt = \left| f_{ia} \right|^{-1} \int_0^T \frac{(s + K |f_{ia}|)^2}{|z + s|^2} ds
\]

where the random variable \( z = g_{ia}/f_{ia} \in \Delta \) has probability density function

\[
\frac{1}{\pi - \epsilon} e^{-|z|^2}.
\]

We need another change of variables: write \( z = x + \sqrt{-1}y \) and replace \( s = y\tau - x \).

Assume first that \( y > 0 \):

\[
\int_0^T \frac{(t + K)^2}{|f_{ia}|^2} dt = \left| f_{ia} \right|^{-1} \frac{1}{y} \int_{x/y}^{(t + K)^2} \frac{r^2((t + x)^2)/y}{1 + \tau^2} \frac{(y\tau - x + K |f_{ia}|)^2}{1 + \tau^2} \, d\tau
\]

\[
\leq 2 \left| f_{ia} \right|^{-1} \frac{1}{y} \int_{x/y}^{(t + K)^2} \frac{r^2\tau^2 + (x - K |f_{ia}|)^2}{1 + \tau^2} \, d\tau
\]

The case \( y < 0 \) is the same with the sign and the integration limits reversed. The expression above can be expanded as follows:

\[
\int_0^T \frac{(t + K)^2}{|f_{ia}|^2} dt \leq 2 \left( \left| f_{ia} \right| K^2 y^{-1} - 2Kxy^{-1} + \left| f_{ia} \right|^{-1} x^2 y^{-1} \right) A_0(z) + 2 \left| f_{ia} \right|^{-1} y A_2(z)
\]

with

\[
A_1(z) = \int_{x/y}^{(t + K)^2} \frac{\tau^2}{1 + \tau^2} \, d\tau.
\]

We can integrate, assuming again \( y > 0 \):

\[
A_0(z) = \left[ \arctan(\tau) \right]_{\tau=xy^{-1}}^{\infty} = \left[ \arctan(\tau) \right]_{\tau=xy^{-1}}^{xy^{-1}} = \arg(z)
\]

\[
A_2(z) = |\tau - \arctan(\tau)|_{\tau=xy^{-1}}^{xy^{-1}} = |f_{ia}|T y^{-1} - A_0(z)
\]

Replacing in equation (23),

\[
\int_0^T \frac{(t + K)^2}{|f_{ia}|^2} dt \leq 2 \left( \left| f_{ia} \right| K^2 - 2Kx + \left| f_{ia} \right|^{-1} (x^2 - y^2) \right) \frac{A_0(z)}{y} + 2T
\]

Passing to polar coordinates \( x = r \cos(\theta), y = r \sin(\theta) \) we can bound

\[
\int_0^T \frac{(t + K)^2}{|f_{ia}|^2} dt \leq 2 \left| f_{ia} \right| K^2 + 2Kr + \left( \left| f_{ia} \right|^{-1} r^2 \right) \frac{\theta}{r \sin(\theta)} + 2T
\]
Above, we used trivial bounds \(-1 \leq \cos(\theta) \leq 1, \cos^2(\theta) - \sin^2(\theta) \leq 1\). Notice also that the left hand side of (23) is symmetric with respect to \(y = \text{Im} \left( \frac{q_n}{f_{in}} \right) |f_{in}| \). We need to bound

\[
\mathbb{E}_{z \in \Delta_n} \left( \sum_{i,a} \int_0^T \frac{(t + K)^2}{|q_n|/f_{in} + t} \, dt \right) \leq \frac{2}{\pi - \epsilon} \sum_{i,a} \int_{-\pi + \epsilon}^{\pi - \epsilon} |f_{in}|K^2 + 2Kr + (|f_{in}|^{-1}r^2) e^{-r^2} \frac{\theta}{\sin(\theta)} \, dr \, d\theta + 2ST
\]

The integral above clearly splits. The integral in \(r\) is trivial:

\[
\sum_{i,a} \int_0^{\pi - \epsilon} (|f_{in}|K^2 + 2Kr + |f_{in}|^{-1}r^2) e^{-r^2} \, dr =
\]

\[
= \frac{1}{2} \sum_{i,a} \left( |f_{in}|K^2/\sqrt{\pi} + 2K + \frac{1}{2}|f_{in}|^{-1}/\sqrt{\pi} \right)
\]

\[
\leq \frac{1}{2} |f|K^2/\sqrt{\pi} + KS + \frac{1}{4}\sqrt{\pi} \sum_{i,a} |f_{in}|^{-1}
\]

\[
\leq \frac{1}{2} K^2S/\sqrt{\pi} + KS + \frac{1}{4}Kr\sqrt{\pi}S/\sqrt{\pi}
\]

A primitive \(F(\theta)\) for \(\frac{\theta}{\sin(\theta)}\) is known:

\[
F(\theta) = \theta \left( \log \left( 1 - e^{-\sqrt{\pi} \theta} \right) - \log \left( 1 + e^{-\sqrt{\pi} \theta} \right) \right) + \sqrt{-1} \left( \text{Li}_2 \left( -e^{-\sqrt{\pi} \theta} \right) - \text{Li}_2 \left( e^{-\sqrt{\pi} \theta} \right) \right)
\]

where \(\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}\) is the polylogarithm or Jonquières’s function, and the identity \(F'(\theta) = -\frac{\theta}{\sin(\theta)}\) can be deduced from the property \(\text{Li}_2(z)' = -\frac{\log(1-z)}{z}\). Since \(\lim_{x \to 0} \text{Re}(F(\theta)) = 0\), we have

\[
\int_0^{\pi - \epsilon} \frac{\theta}{\sin(\theta)} \, d\theta = \text{Re}(F(\pi - \epsilon))
\]

\[
= \pi \text{Re} \left( \log \left( 1 + e^{-\sqrt{\pi} \epsilon} \right) - \log \left( 1 - e^{-\sqrt{\pi} \epsilon} \right) \right) - \text{Re}(F(-\epsilon))
\]

\[
\leq \pi \log(2/\epsilon) + \epsilon^2/\sin(\epsilon)
\]

Putting all together and bounding \(\sqrt{n} \leq \sqrt{S}\),

\[
\mathbb{E}_{z \in \Delta_n} \left( \sum_{i,a} \int_0^T \frac{(t + K)^2}{|q_n|/f_{in} + t} \, dt \right) \leq \left( \pi \log(2/\epsilon) + \frac{\epsilon^2}{\sin(\epsilon)} \right) \left( \frac{1}{2} K^2S/\sqrt{\pi} + KS + \frac{KrS/\sqrt{\pi}}{4} \right) + 2ST
\]

\[
\square
\]

**Proof of Proposition 5.2.1**: In Lemma 5.2.5 we replace \(\epsilon\) by \(\pi/(32S)\), in order that \(\log(2/\epsilon) \leq \log S + \log(64/\pi)\). Also \(\epsilon^2/\sin(\epsilon)\) is bounded by \(\epsilon < \log(2/\epsilon)\) and

\[
\pi \log(2/\epsilon) + \frac{\epsilon^2}{\sin(\epsilon)} \leq c_4 \log(S)
\]
for some constant \( c_4 \). An elementary bound is

\[
\frac{1}{2}K^2S\sqrt{\pi} + KS + \frac{1}{4}\kappa_fS\sqrt{\pi} \leq c_5(K^2 + \kappa_f/2)S
\]

for \( c_5 \) constant, whence

\[
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \int_{0}^{T} \max_{l, a} \left( \frac{(t + K)^2}{2} \right) dt \mid g \notin \Lambda_e \right) \leq \sqrt{c_4 c_5 (K^2 + \kappa_f/2) S \log(S) + 2ST}
\]

This can be simplified if \( T \geq 2K \geq 2 \),

\[
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \int_{0}^{T} \max_{l, a} \left( \frac{(t + K)^2}{2} \right) dt \mid g \notin \Lambda_e \right) \leq \sqrt{2c_4 c_5 S \log(S)} \times \sqrt{1 + K/4 + \kappa_f/8 \sqrt{T}}
\]

Combining with Lemma 5.2.4 and multiplying by

\[
c_3 = \frac{2\sqrt{142}}{21} \sqrt{\sum \delta_i^2 \max(S_i)} \sqrt{n! V \det \Lambda},
\]

we obtain

\[
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{\mathcal{Z}_1((q_r, z_r); 0, T) \mid g \notin \Lambda_e \cup \Omega_H \cup Y_K} \right) \leq \frac{8\sqrt{T}I}{21} c_4 c_5 \sqrt{H} \sqrt{\sum \delta_i^2 \max(S_i)} \sqrt{n! V \det(\Lambda)} \times \sqrt{1 + K/4 + \kappa_f/8 \sqrt{T}}
\]

We picked \( H \leq \frac{n}{\eta} \) LOGS. Recall from Remark 1.6.8 that \( \eta_i \leq 2\delta_i \), whence \( \eta = \min \eta_i \leq 2\delta_i \) for all \( i \). Thus,

\[
\eta \leq \frac{2}{\sqrt{n}} \sqrt{\sum \delta_i^2}
\]

This allows us to bound

\[
\sqrt{H} \leq \frac{\sqrt{n}}{\eta^{3/2}} \eta \sqrt{\text{LOGS}} \leq \frac{2}{\eta^{3/2}} \sqrt{\sum \delta_i^2 \sqrt{\text{LOGS}}}
\]

Using also that \( \log(S) \leq \text{LOGS} \), we conclude that

\[
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{\mathcal{Z}_1((q_r, z_r); 0, T) \mid g \notin \Lambda_e \cup \Omega_H \cup Y_K} \right) \leq \quad C_1 Qn^4 S \max(S_i) \sqrt{1 + K/4 + \kappa_f/8 \sqrt{T} \text{LOGS}}
\]
5.3. Expectation of the condition length (part 2).

**Proposition 5.3.1.** Under the conditions of Theorem 5.1.1, there is a constant $C_2$ such that

$$
\mathbb{E}_{g \sim \mathcal{N}(0, I \cdot \mathcal{F})} \left( \sum_{z_r \in \mathcal{F}(q_r)} \mathbb{L}_2((q_r, z_r); 0, T) : g \not\in \Lambda_r \cup \Omega_H \cup Y_k \right) \leq C_2 Q n^{3/2} \sqrt{S} \max(S_i) K T \text{LOGS}_0
$$

We need first an auxiliary Lemma.

**Lemma 5.3.2.**

(a) Assume that $q_r \cdot V(z_r) \equiv 0$. Then for any $y \in \mathcal{M}$,

$$
\| \hat{z}_r \|_y \leq \| P_{q_r}(\hat{q}_r) \| M(q_r, z_r)^{-1} \|_y.
$$

(b) Assume that $\tau \neq 0$ and $q_r = \tau f + g$. Then,

$$
\| P_{f+g}(\hat{f}) \| \leq \min(\| f \|, 2 \| g \|/\tau).
$$

(c) Assume $\tau \neq 0$, $q_r = \tau f + g$ and $q_r \cdot V(z_r) \equiv 0$. Then,

$$
\| \hat{z}_r \|_{0\mu(q_r \cdot R(z_r), 0)} \leq \left( \sum_i \delta_i^2 \right) \min(\| f \|, 2 \| g \|/\tau) \| M(q_r, z_r)^{-1} \|_p^2 \max(\sqrt{S_i} \| q_i \|) .
$$

**Proof of Lemma 5.3.2.** Item (a) is obtained by differentiating $q_r \cdot V(z_r) = 0$:

$$
\begin{pmatrix}
\vdots \\
\| V_{A_i}(z_r) \| \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\hat{q}_r \\
\hat{z}_r
\end{pmatrix} = -q_r \cdot V(z_r).
$$

But because $q_r \cdot V(z_r) = 0$, this is equivalent to

$$
\begin{pmatrix}
\vdots \\
\| V_{A_i}(z_r) \| \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\hat{q}_r \\
\hat{z}_r
\end{pmatrix} = -P_{q_r}(\hat{q}_r) \cdot V(z_r).
$$

Thus,

$$
\hat{z}_r = -M(q_r, z_r)^{-1} \left( P_{q_r}(\hat{q}_r) \cdot \begin{pmatrix}
\frac{1}{\| V_{A_i}(z_r) \|} V_{A_i}(z_r) \\
\vdots
\end{pmatrix} \right)
$$

Now we prove item (b). Because $P$ is a projection operator, $\| P_{f+g}(\hat{f}) \| \leq \| f \|$. We prove the remaining inequality below: For each $i$, let $q_i = \tau f_i + g_i$.

$$
P_{q_r}(f_i) = \frac{1}{\tau} P_{q_r}(q_i - g_i)
$$

$$
= \frac{1}{\tau} (I - \frac{1}{\| q_i \|^2}) q_i q_i^*(q_i - g_i)
$$

$$
= \frac{1}{\tau} \left( q_i - g_i - q_i + q_i \frac{\langle g_i, q_i \rangle}{\| q_i \|^2} \right)
$$

$$
= \frac{1}{\tau} \left( -g_i + q_i \frac{\langle g_i, q_i \rangle}{\| q_i \|^2} \right)
$$
so passing to norms, \(\|P_{r_i}^\dagger(f_i)\| \leq 2\|\theta_i\|\) and \(\|P_{r+g}^\dagger(f)\| \leq 2\|\theta\|/\tau\).

To prove item (c), recall from Lemma 3.3.2(b) that

\[
\mu(q \cdot R(z), 0) \leq \sqrt{\sum_i \delta_i^2 \|M(q, z)^{-1}\|_F \max_\tau \kappa_{\rho_i} \|q_i\|}
\]

with \(\kappa_{\rho_i} = \sqrt{S_i}\) because \(\rho_i, \alpha = 1\). \(\Box\)

Proof of Proposition 5.3.1 Since \(\|f\| = \sqrt{S}\), \(\|f\| = \sqrt{S}\) where \(S = \sum S_i\). For \(g \notin Y_K\), we also have \(\|g\| \leq K\sqrt{S}\). As in the proof of Proposition 5.2.1 we bound the conditional expectation by

\[
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{z \in Z(q_{r_i})} \mathcal{L}_2((q_{r_i}, z_{r_i}); 0, T) \mid g \notin \Lambda_r \cup \Omega_H \cup Y_K \right) \leq \frac{71}{70} \lim_{v_0} \int_{\mathcal{F} \setminus \Omega_H \cup Y_K} e^{-\|z\|^2} \sum_{z \in Z(q_{r_i})} \mathcal{L}_2((q_{r_i}, z_{r_i}); 0, T) \ d\mathcal{F}(g)
\]

\[
\leq \frac{71}{35} \int_{\mathcal{F} \setminus \Omega_H \cup Y_K} e^{-\|z\|^2} \sum_{z \in Z(q_{r_i})} \mathcal{L}_2((q_{r_i}, z_{r_i}); 0, T) \ d\mathcal{F}(g)
\]

\[
\leq \frac{71}{35} \int_{\mathcal{F} \setminus \Omega_H \cup Y_K} e^{-\|z\|^2} \sum_{z \in Z(q_{r_i})} \mathcal{L}_2((q_{r_i}, z_{r_i}); 0, T) \ d\mathcal{F}(g) \ d\tau
\]

where the last step is Lemma 5.3.2(c) above. We can bound

\[
\min(\|f\|, 2\|g\|/\tau) \leq \sqrt{S} \min(1, 2K/\tau)
\]

and

\[
\max_\tau \|q_i\| \leq \max_\tau \sqrt{S_i}(\tau + K).
\]

As in sections 3.3 and 3.4 but with \(\Sigma^2 = I\), define

\[
I_{f,l} = \mathbb{E}_{q \sim N(f, I)} \left( \sum_{z \in Z_H(q)} \|M(q, z)^{-1}\|_F^2 \right)
\]

\[
= \int_{\mathcal{F}} e^{-\|z\|^2} \sum_{z \in Z_H(f+g)} \|M(f + g, z)^{-1}\|_F^2 \ d\mathcal{F}(g).
\]
However, this time $\hat{f} = \tau f$. We have

$$
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{z_{\tau} \in \mathcal{F}((q_{\tau}))} \mathcal{L}_2((q_{\tau}, z_{\tau}); 0, T) \mid g \notin \Lambda_c \cup \Omega_H \cup Y_K \right) \leq \frac{71}{35} v_0 \sqrt{n} \max_i (S_i) \left( \sum_i \delta_i^2 \right) \int_0^T (\tau + K) \min(1, 2K/\tau) \, I_{\tau, I} \, d\tau
$$

From Theorem 1.5.17 we recover:

$$
I_{\tau, I} = I_{\hat{f}, I} \leq \frac{2H \sqrt{n}}{\det(\Lambda)} (n - 1)!V'.
$$

Finally, we integrate for $T \geq 2K$

$$
\int_0^T (\tau + K) \min(1, 2K/\tau) \, d\tau = \int_0^{2K} \tau + K \, d\tau + \int_{2K}^T 2K + 2K^2/\tau \, d\tau = 2KT + 2K^2 \log \left( \frac{T}{2K} \right) \leq 3KT.
$$

Putting all together,

$$
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{z_{\tau} \in \mathcal{F}((q_{\tau}))} \mathcal{L}_2((q_{\tau}, z_{\tau}); 0, T) \mid g \notin \Lambda_c \cup \Omega_H \cup Y_K \right) \leq \frac{426}{35} H v_0 \sqrt{n} \sqrt{S} \max_i (S_i) KT \left( \sum_i \delta_i^2 \right) (n - 1)!V' \frac{\log(\frac{T}{2K})}{\det(\Lambda)}
$$

Recall that

$$
Q = \eta^{-2} \left( \sum_{i=1}^n \delta_i^2 \right) \frac{\max(n!V, n - 1)!V'\eta}{\det(\Lambda)}
$$

and that $H \leq \frac{n}{\eta} \LOGS_n$. This allows to simplify expression 24 to

$$
\mathbb{E}_{g \sim N(0, I; \mathcal{F})} \left( \sum_{z_{\tau} \in \mathcal{F}((q_{\tau}))} \mathcal{L}_2((q_{\tau}, z_{\tau}); 0, T) \mid g \notin \Lambda_c \cup \Omega_H \cup Y_K \right) \leq C_2 Q n^{3/2} \sqrt{S} \max_i (S_i) KT \LOGS_n
$$

for some constant $C_2$.

5.4. Proof of Theorem 5.1.1.
Proof of Theorem 5.1.1. We need to put together Propositions 5.2.1 and 5.3.1.

We have

\[
\mathbb{E} \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L}_1((q_r, z_r); 0, T) \quad \mathbb{P} \{ g \notin \Lambda_c \cup \Omega_H \cup Y_K \} \leq C_1 Qn^{-\frac{3}{2}} S \max_i (S_i) \sqrt{1 + K/4 + \kappa_f/8} T \log S_0
\]

\[
\mathbb{E} \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L}_2((q_r, z_r); 0, T) \quad \mathbb{P} \{ g \notin \Lambda_c \cup \Omega_H \cup Y_K \} \leq C_2 Qn^{-\frac{3}{2}} S \max_i (S_i) K T \log S_0
\]

Adding together, we conclude that

\[
\mathbb{E} \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L}((q_r, z_r); 0, T) \quad \mathbb{P} \{ g \notin \Lambda_c \cup \Omega_H \cup Y_K \} \leq (C_1 + C_2) Qn S \max_i (S_i)(K + \sqrt{1 + K/4 + \kappa_f/8}) T \log S_0
\]

To simplify notations, let \( \mathcal{L} \) be the random variable \( \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L}((q_r, z_r); 0, T) \) and let

\[ E = Qn S \max_i (S_i)(K + \sqrt{1 + K/4 + \kappa_f/8}) T \log S_0. \]

The expectation above is:

\[
\mathbb{E} \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L} \quad \mathbb{P} \{ g \notin \Lambda_c \cup \Omega_H \cup Y_K \} \leq (C_1 + C_2) E.
\]

Let \( C_0 = t(C_1 + C_2) \) for \( t > 1 \). Markov’s inequality says that

\[
\mathbb{P} \frac{\mathbb{E} \sum_{z_r \in \mathcal{Z}(q_r)} \mathcal{L} \quad \mathbb{P} \{ g \notin \Lambda_c \cup \Omega_H \cup Y_K \}}{C_0 E} \leq 1/t
\]

The main statement follows from setting \( t = 7 \).

\[
\triangleq
\]

5.5. Finite, non-degenerate roots. Assume that \( q_r = g + tf \in \mathcal{F} \) is given, and that there is a continuous path \( z_t \in M \) with \( q_r \cdot V(z_t) = 0 \). Assume furthermore that \( \zeta = \lim_{t \to \infty} z_t \) exists and is a point in \( M \). The Lemma below gives precise values for \( T \) so that Theorem 5.1.1 can be used for tracking the homotopy path: we may want to find approximations of \( z_0 \) out of approximations of \( \zeta \), or the opposite. Define

\[
\Delta_0(T) = \max_{\tau > T} d_\mathcal{F}(q_r \cdot R(\zeta), f \cdot R(\zeta)),
\]

\[
\Delta_1(T) = \max_{\tau > T} d_\mathcal{F}(q_r \cdot R(z_r), f \cdot R(\zeta))
\]

and

\[
\Delta_2(T) = \nu_0 \max_{\tau > T} \|z_r - \zeta\| \leq \Delta_2(T) = \nu_0 \int_T^\infty \|z_r\| \, dt.
\]

Those functions are decreasing and, by continuity,

\[
\lim_{T \to \infty} \Delta_0(T) = \lim_{T \to \infty} \Delta_1(T) = \lim_{T \to \infty} \Delta_2(T) = 0.
\]

Also, write \( \mu = \mu(f \cdot R(\zeta), 0) \). Recall that \( \kappa_f = \max_{i,a} |\frac{f_i}{f_{ia}}| \).
Lemma 5.5.1. Assume that \( n \geq 2 \), \( S_i \geq 2 \) for all \( i \), and \( \rho_{in} = 1 \) always. Suppose that \( f \) is scaled so that \( ||f_i|| = \sqrt{S_i} \) exactly, for each \( i \). Assume that \( ||g_i|| \leq K \sqrt{S_i} \) for some constant \( K \). with \( K = 1 + \sqrt{\frac{\log(n)+\log(10)}{\min(S_i)}} \). Assume that

\[
T \geq \theta \kappa \sqrt{S} \max_i(\sqrt{S_i}) \mu^2 \nu_0
\]

for \( \theta \geq \theta_0 \approx 14.113,684 \ldots \), and that there is a smooth path \( \mathbf{z}_t \in \mathcal{M}, t \geq T \), with \( \mathbf{q}_t \cdot \mathbf{V}(\mathbf{z}_t) = 0 \) for \( \mathbf{q}_t = \mathbf{g} + tf \) with \( \zeta = \lim_{t \to \infty} \mathbf{z}_t \). Then,

\[
\Delta_0(T) \mu \leq \frac{1}{2} \theta^{-1},
\]

\[
\Delta_1(T) \mu < k_1 \theta^{-1} \quad \text{with} \quad k_1 \approx 7.056, 842 \ldots, \text{and}
\]

\[
\Delta_2(T) \mu \leq \tilde{\Delta}_2(T) \mu < k_2 \theta^{-1} \quad \text{with} \quad k_2 \approx 2.932, 308 \ldots
\]

Proof. We start with \( \Delta_0(T) \):

\[
\Delta_0(T) = d_\mathcal{P}(\mathbf{q}_T \cdot R(\zeta), \mathbf{f} \cdot R(\zeta))
\]

\[
\leq \left\| \sum_{i=1}^{n} \frac{||\frac{d}{dt}(\mathbf{q}_i) - \mathbf{f}_i \cdot R_i(\zeta)||^2}{||\mathbf{f}_i \cdot R_i(\zeta)||^2} \right\|
\]

\[
= \frac{1}{T} \left\| \sum_{i=1}^{n} \frac{||g_i \cdot R_i(\zeta)||^2}{||\mathbf{f}_i \cdot R_i(\zeta)||^2} \right\|
\]

\[
\leq \frac{\kappa \mu}{T} \sum_{i=1}^{n} \frac{||g_i||^2}{||\mathbf{f}_i||^2}
\]

\[
\leq \frac{\kappa \mu K \sqrt{n}}{T}.
\]

The estimate (25) follows now from bounds \( \nu_0 \geq 1, \mu \geq 1, \sqrt{S_i} \geq 2 \) and \( \sqrt{S} \geq \sqrt{2n} \). We bound now \( \Delta_1(T) \). Suppose that the maximum in its definition is attained for \( \tau = t \geq T \).

\[
\Delta_1(T) = \Delta_1(t) = d_\mathcal{P}(\mathbf{q}_t \cdot R(\mathbf{z}_t), \mathbf{f} \cdot R(\zeta))
\]

\[
\leq d_\mathcal{P}(\mathbf{q}_t \cdot R(\mathbf{z}_t), \mathbf{q}_t \cdot R(\zeta)) + d_\mathcal{P}(\mathbf{q}_t \cdot R(\mathbf{z}_t), \mathbf{q}_t \cdot R(\zeta), \mathbf{f} \cdot R(\zeta))
\]

\[
= d_\mathcal{P}(\mathbf{q}_t \cdot R(\mathbf{z}_t), \mathbf{q}_t \cdot R(\zeta)) + \Delta_0(t)
\]

Setting \( \mathbf{h} = \mathbf{q}_t \cdot R(\mathbf{z}_t) \), Lemma 2.1.1 applied to the first term yields

\[
d_\mathcal{P}(\mathbf{h} \cdot R(\mathbf{z}_t - \zeta), \mathbf{h}) \leq \sqrt{5}||\mathbf{z}_t - \zeta||_0 \nu_0 \leq \sqrt{5} \Delta_2(t).
\]

Thus,

\[
\Delta_1(T) \leq \sqrt{5} \Delta_2(T) + \frac{1}{20 \mu}.
\]
The bound on $\Delta_2$ is obtained by integration:

$$\Delta_2(T) \leq \nu_0 \Delta_2(T)$$

$$= \nu_0 \int_T^\infty \|\tilde{z}_\tau\|_0 \, d\tau$$

$$\leq \nu_0 \int_T^\infty \|M(q_\tau, z_\tau)^{-1}\|_0 \min(\|f\|, 2\|g\|)^{1-\nu} \, d\tau$$

$$\leq 2K\nu_0 \sqrt{S} \int_T^\infty \|M(q_\tau, z_\tau)^{-1}\|_0 \tau^{1-\nu} \, d\tau$$

Using Lemma 5.3.2(a) and (b). The last step follows when $\tau \geq T \geq 2K$. Notice that

$$M(q_\tau, z_\tau) = \text{diag} \left( \frac{\|V_{A_\tau}(0)\|}{\|V_{A_\tau}(z_\tau)\|}, \tau^{\nu} \right) \cdot M(q_\tau, R(z_\tau), 0)$$

so Lemma 3.3.3 implies:

$$\|M(q_\tau, z_\tau)^{-1}\|_0 \leq \max_{i} (\sqrt{S_i}) \|M(q_\tau, R(z_\tau), 0)^{-1}\|_0.$$ 

Triangular inequality yields $|q_\tau \cdot R_i(z_\tau)| \geq \tau |f_i \cdot R_i(z_\tau)| - |g_i \cdot R_i(z_\tau)| \geq \tau \min |f_{ia}| - K\sqrt{S_i}$. In particular, if $\tau \geq T$, $K\sqrt{S_i} \leq \tau \kappa_\tau^{1-\frac{1}{2}} \nu_o \sqrt{S_{\mu}} \Theta$ we obtain

$$\min(|q_\tau \cdot R_i(z_\tau)|) \geq (\tau \kappa_\tau^{1-\frac{1}{2}} - K) \sqrt{\frac{S_i}{\tau}} \geq \tau \kappa_\tau^{1-\frac{1}{2}} \left( 1 - \frac{1}{2\theta} \right).$$

This means that

$$\|M(q_\tau, z_\tau)^{-1}\|_0 \leq \mu(q_\tau, R(z_\tau), 0)^{\kappa_\tau \max_i (\sqrt{S_i}) \sqrt{\frac{2}{2-\theta^{-1}}}}$$

Proposition 1.3.9(b) yields:

$$\mu(q_\tau, R(z_\tau), 0) \leq \frac{\mu}{1 - \mu \Delta_1(\tau)}$$

so we can bound

$$\Delta_2(T) \leq \frac{2\sqrt{2} \kappa_\tau K \sqrt{S}}{2 - \theta^{-1}} \frac{\mu_0}{1 - \mu \Delta_1(T)} \int_T^\infty \tau^{-2} \, d\tau.$$ 

Integrating,

$$\Delta_2(T) \leq \frac{2\sqrt{2} \kappa_\tau K \sqrt{S}}{2 - \theta^{-1}} \frac{\mu_0}{1 - \mu \Delta_1(T)} T^{1-\frac{1}{2}}.$$ 

We claim that $\mu \Delta_1(T) \leq \frac{1}{2}$. Suppose by contradiction that $\mu \Delta_1(T) > \frac{1}{2}$, then we can increase $T$ such that $\mu \Delta_1(T) = \frac{1}{2}$. From equation (28),

$$\mu \Delta_1(T) \leq \sqrt{5} \mu \Delta_2(T) + \frac{1}{2\theta}.$$ 

Equation (29) implies

$$\mu \Delta_2(T) \leq \mu \Delta_2(T) \leq \frac{4\sqrt{2}}{2\theta - 1}.$$ 

Combining the two bounds,

$$\frac{1}{2} \leq \frac{4\sqrt{10}}{2\theta - 1} + \frac{1}{2\theta}.$$
When \( \theta \to \infty \), we clearly get a contradiction. To find the smaller \( \theta_0 \) that guarantees equality, we compute the largest solution of

\[
\theta_0^2 - \left( \frac{3}{2} + 4\sqrt{10} \right) \theta_0 + \frac{1}{2} = 0
\]

that is \( \theta_0 \approx 14.113,684 \ldots < 15 \). This contradiction establishes that \( \mu \Delta_1(T) \leq \frac{1}{2} \).

The very same calculations imply, for \( \theta \geq \theta_0 \):

\[
\mu \Delta_1(T) \leq \frac{4\sqrt{10}}{2\theta - 1} + \frac{1}{2\theta}
\]

but this bound is inconvenient. Notice that for \( \theta \geq \theta_0 \), \( \frac{1}{2\theta - 1} - \frac{1}{2\theta} = \frac{1}{(2\theta_0 - 1)2\theta} \). Using this bound, we obtain numerically

\[
\mu \Delta_1(T) \leq \frac{1}{\theta} \left( 2\sqrt{10} \left( 1 + \frac{1}{2\theta_0 - 1} \right) + \frac{1}{2} \right) \leq k_1 \theta^{-1}
\]

with \( k_1 \approx 7.056, 842 \ldots \) and

\[
\mu \Delta_2(T) \leq \mu \Delta_2(T) \leq \frac{1}{\theta} \left( 2\sqrt{2} \left( 1 + \frac{1}{2\theta_0 - 1} \right) \right) \leq k_2 \theta^{-1}
\]

with \( k_2 \approx 2.932, 308 \ldots \)

\( \square \)

5.6. **Expectation of the condition length (part 3).**

**Proposition 5.6.1.** Assume that the hypotheses [a] to [e] and [f] of Theorem 5.2 hold. Let \( q_t = g + tf \), where \( g \in \mathcal{F} \) satisfies \( \|g\| \leq K \sqrt{S_t} \). To this path associate the set \( \mathcal{Z}(q_t) \) be the set of continuous solutions of \( q_t \cdot V(z_t) = 0 \). Suppose that \( T \geq \theta \kappa f K \sqrt{S} \max_i(\sqrt{S_i}) \mu_i^2 \nu_0 \) with \( \theta \geq \theta_0 \approx 14.113,684 \ldots \) and \( \mu_f = \max_{z \in \mathcal{Z}(T)} \mu(f \cdot R(z), 0) \). Then unconditionally,

\[
\sum_{z_t \in \mathcal{Z}(q_t)} \mathcal{Z}(q_t, z_t; T, \infty) \leq 2Q k_3.
\]

for \( k_3 \leq 0.867781 \ldots \).

**Lemma 5.6.2.** Assume that \( t \geq T \). Under the hypotheses of Proposition 5.6.1

\[
\| \dot{q}_t \cdot R(z_t) \|_{q_t \cdot R(z_t)} \leq 1 \sum_{i=1}^{N} \frac{K}{1 - T^{-1} k_f \sqrt{S_t}}.
\]

**Proof.**

\[
\| \dot{q}_t \cdot R(z_t) \|_{q_t \cdot R(z_t)} = \left\| \dot{q}_t \cdot R(z_t) \right\|_{q_t \cdot R(z_t)} = \frac{1}{t} \left\| q_t \cdot R(z_t) - g_t \cdot R(z_t) \right\|_{q_t \cdot R(z_t)} \leq \frac{1}{t} \left\| q_t \cdot R(z_t) \right\|_{q_t \cdot R(z_t)} + \frac{1}{t} \left\| g \cdot R(z_t) \right\|_{q_t \cdot R(z_t)}
\]

The first term vanishes. The second term admits a trivial bound:

\[
\frac{1}{t} \left\| g_t \cdot R(z_t) \right\|_{(q_t) \cdot R(z_t)} = \frac{1}{t} \left\| (q_t)_{1t} \cdot R(z_t) \right\| \leq \frac{1}{t} \frac{K \sqrt{S_t}}{\kappa_f - K \sqrt{S_t}}
\]

It follows that

\[
\| \dot{q}_t \cdot R(z_t) \|_{q_t \cdot R(z_t)} \leq \frac{1}{t^2} \sum_{i=1}^{N} \frac{K}{1 - T^{-1} K \sqrt{S_t}}
\]
Proof of Proposition 5.6.1. Since we supposed that $T \geq \theta \kappa T K \sqrt{S} \max_i (\sqrt{S_i}) \mu_i^2 \nu_0$, we recover from Lemma 5.6.2 that

$$\|q_t \cdot R(z_t)\|_{q_t, R(z_t)} \leq \frac{1}{L^2} \sum_{i=1}^{N} \frac{K}{1 - \frac{1}{\theta - 1} \sqrt{S_i}} \leq \frac{1}{L^2} \frac{K \sqrt{n}}{1 - \frac{1}{\theta - 1}}.$$

We can now bound

$$\mathcal{L}_1((q_t, z_t); T, \infty) = \int_T^{\infty} \|q_t \cdot R(z_t)\|_{q_t, R(z_t)} \mu(q_t \cdot R(z_t), 0) \nu_0 \, dt$$

$$\leq \frac{\mu \nu \mu \Delta_1(T)}{1 - \mu \nu \Delta_1(T)} \int_T^{\infty} \|q_t \cdot R(z_t)\|_{q_t, R(z_t)} \, dt$$

$$\leq T^{-1} \frac{\mu \nu \mu \Delta_1(T)}{1 - k_1 \nu \nu \mu \Delta_1(T)} \frac{K \sqrt{n}}{1 - \frac{1}{\theta - 1}}$$

$$\leq \frac{1}{2\theta - 1}$$

Similarly,

$$\mathcal{L}_2((q_t, z_t); T, \infty) = 2\nu_0 \int_T^{\infty} \|z_t\|_{z_t} \mu(q_t \cdot R(z_t), 0) \nu_0 \, dt$$

$$\leq 2 \frac{\mu \nu \Delta_2(T)}{1 - \mu \nu \Delta_1(T)}$$

$$\leq \frac{2k_2}{\theta - k_1}$$

It follows from Lemma 5.1.3 that for every solution path $z_t \in Z(q_t)$,

$$\mathcal{L}((q_t, z_t); T, \infty) \leq \mathcal{L}_1((q_t, z_t); T, \infty) + \mathcal{L}_2((q_t, z_t); T, \infty) \leq \frac{1}{2\theta - 1} + \frac{2k_2}{\theta - k_1}$$

We set $k_3 = \frac{1}{2\theta - 1} + \frac{2k_2}{\theta - k_1} \approx 0.867781...$ From Remark 1.7.1, the total number of paths is at most $2Q$. The Proposition follows.

Proof of Main Theorem 5.1.1 We will combine Theorem 5.1.1 with Proposition 5.6.1

Fix $T = \theta_0 \kappa T K \sqrt{S} \max_i (\sqrt{S_i}) \mu_i^2 \nu_0$. With probability at least $7/8$, the random system $g$ does not belong to the exclusion set $\Lambda_0 \cup \Omega_H \cup Y_K$. In that case,

$$\sum_{z_t \in X'(q_t)} \mathcal{L}(q_t, z_t; 0, \infty) = \sum_{z_t \in X'(q_t)} \mathcal{L}(q_t, z_t; 0, T) + \sum_{z_t \in X'(q_t)} \mathcal{L}(q_t, z_t; T, \infty)$$

and with probability at least $6/7$,

$$\sum_{z_t \in X'(q_t)} \mathcal{L}(q_t, z_t; 0, T) \leq C_0 QnS \max_i (S_i)(K + \sqrt{1 + K/4 + \kappa T/8}) T \log S_0$$

Also, we know from Proposition 5.6.1 that

$$\sum_{z_t \in X'(q_t)} \mathcal{L}(q_t, z_t; T, \infty) \leq Qk_3/2.$$
Adding and replacing $T$ by its value, we obtain that
\[
\sum_{\mathbf{z}_1 \in \mathcal{X}(\mathbf{q}_1)} \mathcal{L}(\mathbf{q}_1, \mathbf{z}_1; 0, \infty) \leq (2k_3 + C_0 \theta_0) Q n S^{3/2} \max_i (S_i^{3/2}) \times K(K + \sqrt{1 + K/4 + \kappa_\ell/8}) \kappa_\ell \mu_\ell \theta_0 \ \text{LOGS}_0
\]
with
\[
\text{LOGS}_0 \in O(\log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_\ell) + \log(\kappa_\ell)).
\]
The constant $C$ is the product of $2k_3 + C_0 \theta_0$ times the constant in LOGS$_0$.

6. Conclusions and further research

A theory of homotopy algorithms over toric varieties is now within reach. In this paper, the renormalization technique allowed to obtain complexity bounds for homotopy between two fixed systems, as long as they satisfy some conditions: they should be well-posed, and have no root at infinity. New invariants that play an important role in the theory were identified: the mixed surface, and the face gap $\eta$. The cost of a ‘cheater’s homotopy’ between two fixed, non-degenerate systems with same support was bounded here.

Theorem 1.6.5 paves the way for rigorously detecting roots at infinity, and furthermore finding out which toric infinity the root may be converging to. Then one can think of replacing the original system with the appropriate overdetermined system at infinity, and attempt to solve it. There are some technical difficulties to certify the global solution set with roots at toric infinity, that also deserve some investigation.

Degenerate roots are more challenging. The hypothesis $d_r(f) \neq 0$ in Theorems D and E already imply that $f \notin \Sigma^\infty$ and hence, by Bernstein’s second theorem (Th. 1.6.6 here) the number of finite roots is $n! V / \det(\Lambda)$ and the roots are isolated. If we do not assume $d_r(f) \neq 0$, then more general singular solutions may arise. There are numerical methods to deal with this situation, see for instance Sommese and Verschelde (2000), Dayton et al. (2011), Giusti and Yakoubsohn (2013), Li and Sang (2015), and Hauenstein et al. (2017) and references.

Finding a convenient starting system is usually one of the big challenges for homotopy algorithms. In the sparse setting there are several viable options. One of them is the use of polyhedral homotopy, also known as nonlinear homotopy (Verschelde et al., 1994; Huber and Sturmfels, 1995; Li, 1999; Verschelde, 1999). It ‘reduces’ a generic system to a tropical polynomial system. Several approaches are available for solving tropical polynomial systems. A complexity bound in terms of mixed volumes and quermassintegrals for solving generic tropical systems was given by (Malajovich, 2017). A procedure to solve arbitrary tropical systems with roughly the same complexity bound was given independently by Jensen (2016a) and Jensen (2016b). The results of those papers disprove the belief by practitioners that

In general, finding the exact maximal root count for given sparse structure and setting up a compatible homotopy is a combinatorial process with a high computational complexity (Bates et al., 2013, p.71)
and provide usable implementations for finding the starting systems.

The situation is different for polyhedral homotopy continuation itself. While the same numerical evidence, as together as this author’s experience show that this is a highly effective numerical method, theoretical justifications are missing. It is important here to point out our findings in Theorem B: the variance of the coefficients appears in the average bound for the condition, and this precludes obtaining a decent complexity estimate with the tools in this paper. No complexity bound for polyhedral homotopy is known at this time.

Polyhedral homotopy is not the only possible algorithm for solving sparse systems. One can also experiment with monodromy as in (Krone and Leykin, 2017; Leykin et al., 2018; Duff et al., 2019; Brysiewicz et al., 2020). Finding a point in the solution variety is easy, just project a random system into the subspace vanishing at a fixed point. Then the other roots can be found by homotopy continuation through several random loops. No complexity analysis for this procedure is known either.

Finally, there is the situation where many systems inside a space \( F \) need to be found, and in this case one just needs to solve one generic system in \( F \). The cost of obtaining this ‘cheater’ system is then irrelevant, and it can be obtained by total degree homotopy as in (Breiding et al., 2020).

Experimental validation of the results in this paper is still to be done. Theorem D uses a conditional probability estimate. By performing experiments with this conditional probability or with adversarial probability distributions, one can determine if the domains in the proof of the Theorem are really necessary or if they are a side-product of the proof technique.

The complexity bound in this paper,

\[
Q n S^{3/2} \max \{ S_i^{3/2} \} K \left( K + \sqrt{1 + K/4 + \kappa/8} \right) \kappa (\mu_f^2 + \mu_h^2) \nu_0 
\times \left( \log(d_r) + \log(S) + \log(\nu_0) + \log(\mu_f) + \log(\mu_h) + \log(\kappa_f) \right)
\]

should be compared to the problem size. Since we are considering the problem of finding all the roots, a reasonable definition for the problem size should be \( n!V S \) times some function depending solely on the coefficients. This function can be thought as the logarithm of some abstract condition number, see for instance (Cucker, 2015; Malajovich and Shub, 2019) for the rationale of introducing such an object. While there is no hope for the algorithm in this paper to be uniform polynomial time, one can still define the problem size as \( n!V S \kappa (\mu_f^2 + \mu_h^2) \) and ask whether this algorithm behaves experimentally as if it was asymptotically polynomial time for natural, easy to define families of examples.

Last but not least, a large number of implementation issues remain unsettled. Several choices in this paper were done to simplify the theory, but do not seem reasonable in practice. For instance, it would be reasonable to replace the trial and error procedure of Theorem E by early detection that the Gaussian system is outside of the domain of the conditional probability. Also, there is nothing special about straight lines and using great circles instead looks more natural.

Complexity analysis in this paper is done in terms of total cost. But each path can be followed independently of the others, so the algorithm is massively parallelizable. In those situations, the computational bottleneck is the communication between processes. If it is possible to detect failure early from data at the path,
one can avoid communication almost completely. It would be desirable in this case to estimate the expected parallel running time.

A more fundamental question is the following: most implementation of homotopy algorithms use a predictor-corrector scheme, as explained for instance by Allgower and Georg (1990). Up to now, the tightest rigorous complexity bounds for homotopy algorithms refer to a corrector-only homotopy, which no one actually uses in practice. Is it possible to improve the complexity bound of Theorem A by more than a constant by using a higher order method? What about the bound in Theorem D?

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