Symplectic nilmanifolds with a symplectic non-free $\mathbb{Z}_3$-action

Marisa Fernández and Vicente Muñoz

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Abstract

In [15] the authors have introduced a new technique to produce symplectic manifolds. It consists on taking a symplectic non-free action of a finite group on a symplectic manifold and resolving symplectically the singularities of the quotient. This has allowed to produce the first example of a non-formal simply connected compact symplectic manifold of dimension 8. Here we present another description of such a manifold and we expand on some of the details concerning its properties.

1 Introduction

In [15], the authors have produced the first example of a simply connected compact symplectic manifold of dimension 8 which is non-formal.

In general, simply connected compact manifolds of dimension less than or equal to 6 are formal [29, 13], and there are simply connected compact manifolds of dimension greater than or equal to 7 which are non-formal [31, 12, 10, 6, 14]. This is a problem that can be tackled by using minimal models [9] and suitable constructions of differentiable manifolds.

However, if we consider symplectic manifolds, the story is not so straightforward, basically due to the fact that there are not so many constructions of symplectic manifolds. In [1, 2] Babenko and Taimanov give examples of non-formal simply connected compact symplectic manifolds of any dimension bigger than or equal to 10, by using the symplectic blow-up [26]. They raise the question of the existence of non-formal simply connected compact symplectic manifolds of dimension 8. Examples of these cannot be constructed by means of symplectic blow-ups. Other methods of construction of symplectic manifolds, like the connected sum along codimension two submanifolds [17], or symplectic fibrations [27, 33, 35] have not produced such examples so far.

The solution to this question presented in [15] uses a new and simple method of construction of symplectic manifolds. This method consists on taking quotients of symplectic manifolds by finite groups and resolving symplectically the singularities. Starting with a suitable compact non-formal nilmanifold of dimension 8, on which the finite group $\mathbb{Z}_3$ acts with simply connected quotient, one gets a simply connected compact symplectic non-formal 8–manifold.

In this note, we expand on some of the issues touched in [15]. First we present an alternative description of the manifold constructed in [15], by using real Lie groups instead of complex Lie groups (see Section 3). Actually this is the way in which we first obtained the example; introducing complex Lie groups was an ulterior simplification. The reason for our choice of symplectic 8–dimensional nilmanifold $\mathcal{M}$ becomes transparent with the description that we give in Section 4: it is the simplest case in which the group $\mathbb{Z}_3$ acts not having any invariant part in the cohomology of degree 1. In this way we have a chance to get a simply connected symplectic
orbifold $\tilde{M} = M/\mathbb{Z}_3$, as we prove later it is the case with our particular choice of $M$ and $\mathbb{Z}_3$-action. To get a smooth 8–dimensional symplectic manifold, we have to resolve symplectically the singularities. For this, in Section 6, we take suitable Kähler models around each singular point. It is clear that this method can be used in much greater generality.

The last issue concerns with the non-formality of the constructed manifold. Our example of symplectic 8–manifold has vanishing odd Betti numbers, therefore its triple Massey products are zero. Thus the natural way to prove non-formality is to produce the minimal model of $\tilde{M}$, but this can be a lengthy task for large Betti numbers.

In [13] the concept of formality is extended to a weaker notion named as $s$–formality. We shall not review this notion here, but we want to mention that it has always been a guidance for us when trying to write down an obstruction to detect non-formality. Actually, when we spelt out the condition for 3–formality, we realised that there is an easily described new type of obstruction to formality constructed with differential forms, in spirit similar to Massey products, and which we christen here as $G$-Massey product (see Definition 2.2 in Section 2). Then, in Section 5, the non-formality of $\tilde{M}$ is easily checked via a non-trivial $G$-Massey product. There, we also see that a suitable quadruple Massey product of $\tilde{M}$ is non-trivial, although the proof is definitely more obscure. So we have decided to include both proofs of the non-formality of $\tilde{M}$. It would be interesting to find a space with non-trivial $G$-Massey products but with all multiple Massey products trivial.

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2 Formality and $G$-Massey products

Before we start with the construction of symplectic manifolds, we shall briefly review the notion of formality [13, 9], and we shall introduce the $G$-Massey products as an obstruction to this property. Let $(A, d)$ be a differential graded commutative algebra over the field $\mathbb{R}$ of real numbers. Then $(A, d)$ is said to be minimal if:

1. $A$ is free as an algebra, that is, $A$ is the free algebra $\wedge V$ over a graded vector space $V = \oplus V^i$, and

2. there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set $I$, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each $da_\tau$ is expressed in terms of preceding $a_\mu$ ($\mu < \tau$).

This implies that $da_\tau$ does not have a linear part, i.e., it lives in $\wedge V^0 \cdot \wedge V^0 \subset \wedge V$.

Given a differential algebra $(A, d)$, we denote by $H^*(A)$ its cohomology. $(A, d)$ is connected if $H^0(A) = \mathbb{R}$. We shall say that $(M, d)$ is a minimal model of the differential algebra $(A, d)$ if $(M, d)$ is minimal and there exists a morphism of differential graded algebras $\rho: (M, d) \rightarrow (A, d)$ inducing an isomorphism $\rho^*: H^*(M) \rightarrow H^*(A)$ on cohomology. In [20] Halperin proved that any connected differential algebra $(A, d)$ has a minimal model unique up to isomorphism.

A minimal model of a connected differentiable manifold $X$ is a minimal model $(\wedge V, d)$ for the de Rham complex $(\Omega X, d)$ of differential forms on $X$. If $X$ is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(X) \otimes \mathbb{R}$ is isomorphic to $V^i$ for any $i$. This relation also happens when $i > 1$ and $X$ is nilpotent, that is, the fundamental group $\pi_1(X)$ is nilpotent and its action on $\pi_j(X)$ is nilpotent for $j > 1$ (see [9]).
A minimal model $(\mathcal{M}, d)$ is said to be \textit{formal} if there is a morphism of differential algebras $\psi: (\mathcal{M}, d) \to (H^*(\mathcal{M}), d = 0)$ that induces the identity on cohomology. We shall say that $X$ is \textit{formal} if its minimal model is formal or, equivalently, the differential algebras $(\Omega^*X, d)$ and $(H^*(X), d = 0)$ have the same minimal model. Therefore, if $X$ is formal and simply connected, then the real homotopy groups $\pi_i(X) \otimes \mathbb{R}$ are obtained from the minimal model of $(H^*(X), d = 0)$.

Many examples of formal manifolds are known: spheres, projective spaces, compact Kähler manifolds. The importance of formality in symplectic geometry stems from the fact that it allows to distinguish between symplectic manifolds which admit Kähler structures and some which do not [34].

In order to detect non-formality, instead of computing the minimal model, which usually is a lengthy process, we can use Massey products, which are obstructions to formality. Let us recall its definition. The simplest type of Massey product is the triple (also known as ordinary) Massey product. Let $X$ be a (not necessarily simply connected) manifold and let $a_i \in H^{p_i}(X)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. The (triple) Massey product of the classes $a_i$ is defined as the set

$$\langle a_1, a_2, a_3 \rangle = \{ [\alpha_1 \wedge (-1)^{p_1+1} \xi \wedge \alpha_3] \mid a_i = [\alpha_i], \alpha_1 \wedge \alpha_2 = d\xi, \alpha_2 \wedge \alpha_3 = d\eta \} \subset H^{p_1+p_2+p_3-1}(X).$$

(The same set is obtained if we fix the forms $\alpha_i$ such that $a_i = [\alpha_i]$ and we only let $\eta$ and $\xi$ vary.) It is easily seen that $\langle a_1, a_2, a_3 \rangle$ is a set of the form $b + (a_1 \cup H^{p_2+p_3-1}(X) + H^{p_1+p_2-1}(X) \cup a_3)$, so it gives a well-defined element in

$$\frac{H^{p_1+p_2+p_3-1}(X)}{a_1 \cup H^{p_2+p_3-1}(X) + H^{p_1+p_2-1}(X) \cup a_3}.$$ We say that $\langle a_1, a_2, a_3 \rangle$ is trivial if $0 \in \langle a_1, a_2, a_3 \rangle$.

The definition of higher Massey products is as follows (see [22, 24, 32]). The Massey product $\langle a_1, a_2, \ldots, a_t \rangle$, $a_i \in H^{p_i}(X)$, $1 \leq i \leq t$, $t \geq 3$, is defined if there are differential forms $\alpha_{i,j}$ on $X$, with $1 \leq i \leq j \leq t$, except for the case $(i, j) = (1, t)$, such that

$$a_i = [\alpha_{i,i}], \quad d \alpha_{i,j} = \sum_{k=i}^{j-1} \tilde{\alpha}_{i,k} \wedge \alpha_{k+1,j},$$

where $\tilde{\alpha} = (-1)^{\deg(\alpha)} \alpha$. Then the Massey product is

$$\langle a_1, a_2, \ldots, a_t \rangle = \left\{ \sum_{k=1}^{t-1} \tilde{\alpha}_{1,k} \wedge \alpha_{k+1,t} \mid \alpha_{i,j} \text{ as in } (1) \right\} \subset H^{p_1+\cdots+p_t-(t-2)}(X).$$

We say that the Massey product is trivial if $0 \in \langle a_1, a_2, \ldots, a_t \rangle$. Note that for $\langle a_1, a_2, \ldots, a_t \rangle$ to be defined it is necessary that $\langle a_1, \ldots, a_{t-1} \rangle$ and $\langle a_2, \ldots, a_t \rangle$ are defined and trivial.

The existence of a non-trivial Massey product is an obstruction to formality. Concretely, we have the following result, for whose proof we refer to [9, 32].

\textbf{Lemma 2.1} If $X$ has a non-trivial Massey product then $X$ is non-formal. \hfill \Box

Next, we introduce another obstruction to formality, which we call G-Massey product, since it is a generalization, in spirit, of the Massey products. This product has the advantage of being simpler for computations than the multiple Massey products.
Definition 2.2 Let $X$ be a manifold of any dimension. Let $a, x_1, x_2, x_3 \in H^2(X)$ be degree 2 cohomology classes satisfying that $a \cup x_i = 0$, $i = 1, 2, 3$. We define the G-Massey product $\langle a; x_1, x_2, x_3 \rangle$ as the subset

$$\langle a; x_1, x_2, x_3 \rangle = \{ [\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] | a = [\alpha], x_i = [\beta_i], \alpha \wedge \beta_i = d\xi_i, i = 1, 2, 3 \} \subset H^8(X).$$

We say that the G-Massey product is trivial if $0 \in \langle a; x_1, x_2, x_3 \rangle$.

We must notice that

\begin{equation}
[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] \in H^8(X)
\end{equation}

is a well defined cohomology class, for forms $\alpha, \beta_i \in \Omega^2(X)$ and $\xi_i \in \Omega^3(X)$, with $a = [\alpha], x_i = [\beta_i]$ and $\alpha \wedge \beta_i = d\xi_i, i = 1, 2, 3$. In fact, we have

$$d(\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2) = \alpha \wedge \beta_1 \wedge \xi_2 \wedge \beta_3 - \xi_1 \wedge \alpha \wedge \beta_2 \wedge \beta_3 +$$

$$+ \alpha \wedge \beta_2 \wedge \xi_3 \wedge \beta_1 - \xi_2 \wedge \alpha \wedge \beta_3 \wedge \beta_1 + \alpha \wedge \beta_3 \wedge \xi_1 \wedge \beta_2 - \xi_3 \wedge \alpha \wedge \beta_1 \wedge \beta_2 = 0.$$

Lemma 2.3 Let $a, x_1, x_2, x_3 \in H^2(X)$ be degree 2 cohomology classes satisfying that $a \cup x_i = 0$, $i = 1, 2, 3$. Then

$$b_1, b_2 \in \langle a; x_1, x_2, x_3 \rangle \implies b_1 - b_2 \in W,$$

where $W = \langle x_1, a, x_2 \rangle \cup H^3(X) + \langle x_1, a, x_3 \rangle \cup H^3(X) + \langle x_2, a, x_3 \rangle \cup H^3(X) \subset H^8(X)$.

Proof Choose forms $\alpha, \beta_i \in \Omega^2(X)$ and $\xi_i \in \Omega^3(X)$, with $a = [\alpha], x_i = [\beta_i]$ and $\alpha \wedge \beta_i = d\xi_i, i = 1, 2, 3$. First of all, note that the conditions $a \cup x_i = 0$, $i = 1, 2, 3$, ensure that the triple Massey products $\langle x_1, a, x_2 \rangle, \langle x_1, a, x_3 \rangle, \langle x_2, a, x_3 \rangle$ are well defined.

Now suppose that we write $a = [\alpha + df], f \in \Omega^1(X)$. Then $(\alpha + df) \wedge \beta_i = d(\xi_i + f \wedge \beta_i)$ and

$$(\xi_1 + f \wedge \beta_1) \wedge (\xi_2 + f \wedge \beta_2) \wedge (\xi_3 + f \wedge \beta_3) \wedge \beta_1 + (\xi_3 + f \wedge \beta_3) \wedge (\xi_1 + f \wedge \beta_1) \wedge \beta_2 =$$

$$= \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2,$$

so the cohomology class (2) does not change by changing the representative of $a$. If we change the representatives of $x_i$, say for instance $x_1 = [\beta_1 + df], f \in \Omega^1(X)$, then $\alpha \wedge (\beta_1 + df) = d(\xi_1 + \alpha \wedge f)$ and

$$(\xi_1 + \alpha \wedge f) \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge (\beta_1 + df) + \xi_3 \wedge (\xi_1 + \alpha \wedge f) \wedge \beta_2 =$$

$$= \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 + d(f \wedge \xi_2 \wedge \xi_3),$$

thus the cohomology class (2) does not change again. Finally, if we change the form $\xi_1$ to $\xi_1 + g, g \in \Omega^3(X)$ closed, then

$$(\xi_1 + g) \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge (\beta_1 + df) + \xi_3 \wedge (\xi_1 + g) \wedge \beta_2 =$$

$$= \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 + g \wedge (\xi_2 \wedge \beta_3 - \xi_3 \wedge \beta_2),$$

and $\xi_2 \wedge \beta_3 - \xi_3 \wedge \beta_2$ is a representative of $\langle x_2, a, x_3 \rangle$. \hfill \Box

The indeterminacy of a subset $S$ of a vector space $V$ is the subspace of $W \subset V$ generated by the differences $s_1 - s_2 \in S$ for all $s_1, s_2 \in S$. In this situation, $S$ defines an element in $V/W$. Lemma 2.3 says that the indeterminacy of $\langle a; x_1, x_2, x_3 \rangle$ is contained in $W = \langle x_1, a, x_2 \rangle \cup$
$H^3(X) + \langle x_1, a, x_3 \rangle \cup H^3(X) + \langle x_2, a, x_3 \rangle \cup H^3(X)$, hence the G-Massey product $\langle a; x_1, x_2, x_3 \rangle$ gives a well-defined element in

$$H^8(X) \cap (\langle x_1, a, x_2 \rangle \cup H^3(X) + \langle x_1, a, x_3 \rangle \cup H^3(X) + \langle x_2, a, x_3 \rangle \cup H^3(X)).$$

The relevance of the G-Massey product for formality is given in the following result.

**Proposition 2.4** Let $a, x_1, x_2, x_3 \in H^2(X)$ be cohomology classes satisfying that $a \cup x_i = 0$, $i = 1, 2, 3$. Suppose that $\langle a; x_1, x_2, x_3 \rangle$ is a non-trivial G-Massey product. Then $X$ is not formal.

**Proof** Let $\psi : (\bigwedge V, d) \to (\Omega^*(X), d)$ be the minimal model for $X$. Then there are closed elements $\hat{a}, \hat{x}_i \in \bigwedge V^2$ whose images are 2-forms $\alpha, \beta_i$ representing $a, x_i$. Since $[\hat{a} \cdot \hat{x}_i] = 0$, there are elements $\xi_i \in (\bigwedge V)^3$ such that $d\xi_i = \hat{a} \cdot \hat{x}_i$. Let $\xi_i = \psi(\xi_i) \in \Omega^3(X)$.

If $X$ is formal, then there exists a quasi-isomorphism $\psi' : (\bigwedge V, d) \to (H^*(X), 0)$. Note that by adding a closed element to $\hat{\xi}$, we can suppose that $\psi'(\hat{\xi}_i) = 0$. Then

$$[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] = \psi'(\xi_1 \wedge \xi_2 \wedge \hat{x}_3 + \hat{\xi}_2 \wedge \hat{\xi}_3 \wedge \hat{x}_1 + \hat{\xi}_3 \wedge \hat{\xi}_1 \wedge \hat{x}_2) = 0$$

belongs to $\langle a; x_1, x_2, x_3 \rangle$. □

Actually, the G-Massey product is the first obstruction to formality that appears as an obstruction to 3-formality [13] for a simply connected manifold, and which is different from a Massey product.

The G-Massey product can be related, in some situations, with the multiple Massey products. However, it cannot be written in terms of the higher Massey products [22] (or even the matric Massey products [25, 3]) because the indeterminacy of Massey products is usually much bigger. (A similar phenomenon happens to the product $\langle a \rangle^k$ discussed in [22 Section 3].)

**Lemma 2.5** Let $X$ be a manifold of any dimension. Let $a, x_1, x_2, x_3 \in H^2(X)$ be cohomology classes satisfying that $a \cup x_i = 0$, $i = 1, 2, 3$, and $a \cup a = 0$. Suppose that $H^3(X) = 0$. Then

$$\langle a; x_1, x_2, x_3 \rangle \cap (x_3 \langle x_1, a, a, x_2 \rangle + x_2 \langle x_3, a, a, x_1 \rangle + x_1 \langle x_2, a, a, x_3 \rangle) \neq \emptyset.$$

**Proof** Write $a = [a]$, $x_i = [\beta_i]$, $\alpha \wedge \beta_i = d\xi_i$, $i = 1, 2, 3$ and $\alpha \wedge \alpha = d\chi$. The triple Massey products $\langle x_i, a, a \rangle$ and $\langle a, a, x_i \rangle$ are defined and zero, since $H^3(X) = 0$. Then we can write $\xi_i \wedge \alpha - \beta_i \wedge \chi = d\eta_i$. By definition,

$$\eta_1 \wedge \beta_2 - \eta_2 \wedge \beta_1 + \xi_1 \wedge \xi_2 \in \langle x_1, a, a, x_2 \rangle,$$

and analogously for the others. Thus the element

$$\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 =$$

$$= \beta_3 \wedge (\eta_1 \wedge \beta_2 - \eta_2 \wedge \beta_1 + \xi_1 \wedge \xi_2) + \beta_2 \wedge (\eta_3 \wedge \beta_1 - \eta_1 \wedge \beta_3 + \xi_3 \wedge \xi_1) +$$

$$+ \beta_1 \wedge (\eta_2 \wedge \beta_3 - \eta_3 \wedge \beta_2 + \xi_2 \wedge \xi_3)$$

is in the intersection [3]. □

**Remark 2.6** Note that

$$S = x_3 \langle x_1, a, a, x_2 \rangle + x_2 \langle x_3, a, a, x_1 \rangle + x_1 \langle x_2, a, a, x_3 \rangle \subset H^8(X)$$

is only defined if $a \cup a = 0$. Therefore the G-Massey product can be understood as a refinement of the subset $S$. Moreover, the indeterminacy of $S$ is different (and usually bigger) than that of the G-Massey product $\langle a; x_1, x_2, x_3 \rangle$. 


The concept of formality is also defined for nilpotent CW-complexes, and all the discussion above can be extended to them by using piecewise polynomial differential forms instead of differential forms. Also in this case G-Massey products can be defined. We shall not need this in full generality, but we shall use the case when $X$ is an orbifold.

An orbifold is a topological space $X$ with an atlas with charts modelled on $U/\Pi_p$, where $U$ is an open set of $\mathbb{R}^n$ and $\Pi_p$ is a finite group acting linearly on $U$ with only one fixed point $p \in U$. For an orbifold $X$, we define $\Omega^k(X)$ as the space of orbifold differential forms, i.e., forms such that in each chart are $\Pi_p$-invariant elements of $\Omega^k(U)$. The orbifold minimal model of $X$ is defined as the minimal model $(\wedge V, d)$ of $(\Omega^k(X), d)$.

**Lemma 2.7** Suppose that $X$ is a smooth manifold with minimal model $(\wedge V, d)$. Let $\Pi$ be a finite group acting on $X$ with only isolated points with non-trivial isotropy, and consider the orbifold $\tilde{X} = X/\Pi$. Let $(\wedge W, d)$ be the minimal model of the differential algebra $((\wedge V)^{\Pi}, d)$. Then

- $(\wedge W, d)$ is the orbifold minimal model of $\tilde{X}$.
- Consider $\tilde{X}$ as a topological space (actually it is naturally a CW-complex). If $\tilde{X}$ is nilpotent, then $(\wedge W, d)$ is its minimal model.

**Proof** In this situation, $\Omega^k(\tilde{X}) = \Omega^k(X)^{\Pi}$. The action of $\Pi$ on $(\Omega(X), d)$ lifts to an action on the minimal model $(\wedge V, d)$ (see [11] for example). As $(\wedge V, d) \to (\Omega(X), d)$ is a quasi-isomorphism, $(\wedge V^{\Pi}, d) \to (\Omega(X)^{\Pi}, d)$ is also. Thus $(\wedge W, d)$ is the orbifold minimal model of $X$.

For the second item, triangulate $\tilde{X}$ in such a way that the orbifolds points are vertices of the triangulation. The algebra of piecewise polynomial differential forms is quasi-isomorphic to the algebra $(\Omega^k(\tilde{X}), d)$ of piecewise smooth differential forms [19]. Now the natural map $(\Omega^k(\tilde{X}), d) \to (\Omega^k(\tilde{X}), d)$ is a quasi-isomorphism since $H^*(\Omega^k(\tilde{X}), d) \cong H^*(X)^{\Pi} \cong H^*(\tilde{X})$. So the minimal model of $\tilde{X}$ is also $(\wedge W, d)$. \hfill $\blacksquare$

### 3 A nilmanifold of dimension 6

Let $G$ be the simply connected nilpotent Lie group of dimension 6 defined by the structure equations

\begin{align*}
d\beta_i &= 0, \quad i = 1, 2 \\
d\gamma_i &= 0, \quad i = 1, 2 \\
d\eta_1 &= -\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 + 2\beta_2 \wedge \gamma_2, \\
d\eta_2 &= 2\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_2,
\end{align*}

where $\{\beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$ is a basis of the left invariant 1–forms on $G$. Because the structure constants are rational numbers, Mal’cev theorem [23] implies the existence of a discrete subgroup $\Gamma$ of $G$ such that the quotient space $N = \Gamma \backslash G$ is compact.

Using Nomizu theorem [30], we can compute the real cohomology of $N$. We get

\begin{align*}
H^0(N) &= \langle 1 \rangle, \\
H^1(N) &= \langle [\beta_1], [\beta_2], [\gamma_1], [\gamma_2] \rangle, \\
H^2(N) &= \langle [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_1], [\beta_1 \wedge \gamma_2], [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2] \rangle, \\
H^3(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \eta_1], [\beta_1 \wedge \beta_2 \wedge \eta_2], [\gamma_1 \wedge \gamma_2 \wedge \eta_1], [\gamma_1 \wedge \gamma_2 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge (\eta_1 + 2\eta_2)] \rangle,
\end{align*}
convenience, we shall take the subgroup

\[ \Gamma = \{(y_1, y_2, z_1, z_2, v_1, v_2) \in \mathbb{Z}^6 \mid v_1 \equiv v_2 \pmod{3}\} \subset G, \]

and define the nilmanifold

\[ N = \Gamma \backslash G. \]

In terms of a (global) system of coordinates \((y_1, y_2, z_1, z_2, v_1, v_2)\) for \(G\), the 1–forms \(\beta_i, \gamma_i\) and \(\eta_i, 1 \leq i \leq 2\), are given by

\[ \beta_i = dy_i, \quad 1 \leq i \leq 2, \]
\[ \gamma_i = dz_i, \quad 1 \leq i \leq 2, \]
\[ \eta_1 = dv_1 - y_1dz_1 + y_2dz_1 + y_1dz_2 + 2y_2dz_2, \]
\[ \eta_2 = dv_2 + 2y_1dz_1 + y_2dz_1 + y_1dz_2 - y_2dz_2. \]

Note that \(N\) is a principal torus bundle

\[ T^2 = \mathbb{Z} \langle (1,1), (3,0) \rangle \backslash \mathbb{R}^2 \hookrightarrow N \longrightarrow T^4 = \mathbb{Z}^4 \backslash \mathbb{R}^4, \]

with the projection \((y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (y_1, y_2, z_1, z_2)\).

The Lie group \(G\) can be also described as follows. Consider the basis \(\{\mu_i, \nu_i, \theta_i; 1 \leq i \leq 2\}\) of the left invariant 1–forms on \(G\) given by

\[ \mu_1 = \beta_1 + \frac{1 + \sqrt{3}}{2} \beta_2, \quad \mu_2 = \beta_1 + \frac{1 - \sqrt{3}}{2} \beta_2, \]
\[ \nu_1 = \gamma_1 + \frac{1 + \sqrt{3}}{2} \gamma_2, \quad \nu_2 = \gamma_1 + \frac{1 - \sqrt{3}}{2} \gamma_2, \]
\[ \theta_1 = \frac{2}{\sqrt{3}} \eta_1 + \frac{1}{\sqrt{3}} \eta_2, \quad \theta_2 = \eta_2. \]
Hence, the structure equations can be rewritten as
\[
\begin{align*}
    d\mu_i &= 0, \quad 1 \leq i \leq 2, \\
    d\nu_i &= 0, \quad 1 \leq i \leq 2, \\
    d\theta_1 &= \mu_1 \wedge \nu_1 - \mu_2 \wedge \nu_2, \\
    d\theta_2 &= \mu_1 \wedge \nu_2 + \mu_2 \wedge \nu_1.
\end{align*}
\]  
(6)

This means that $G$ is the complex Heisenberg group $H_C$, that is, the complex nilpotent Lie group of complex matrices of the form
\[
\begin{pmatrix}
1 & u_2 & u_3 \\
0 & 1 & u_1 \\
0 & 0 & 1
\end{pmatrix}.
\]

In fact, in terms of the natural (complex) coordinate functions $(u_1, u_2, u_3)$ on $H_C$, we have that the complex 1–forms
\[
\mu = du_1, \quad \nu = du_2, \quad \theta = du_3 - u_2 du_1
\]
are left invariant and $d\mu = d\nu = 0$, $d\theta = \mu \wedge \nu$. Now, it is enough to take $\mu_1 = \text{Re}(\mu)$, $\mu_2 = \text{Im}(\mu)$, $\nu_1 = \text{Re}(\nu)$, $\nu_2 = \text{Im}(\nu)$, $\theta_1 = \text{Re}(\theta)$, $\theta_2 = \text{Im}(\theta)$ to recover equations (6).

**Lemma 3.1** Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, and consider the discrete subgroup $\Gamma_H \subset H_C$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda$. Then there is a natural identification of $N = \Gamma \backslash G$ with the quotient $\Gamma_H \backslash H_C$.

**Proof** We have constructed above an isomorphism of Lie groups $G \to H_C$, whose explicit equations are
\[
(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (u_1, u_2, u_3),
\]
where
\[
\begin{align*}
    u_1 &= \left( y_1 + \frac{1 + \sqrt{3}}{2} y_2 \right) + i \left( y_1 + \frac{1 - \sqrt{3}}{2} y_2 \right), \\
    u_2 &= \left( z_1 + \frac{1 + \sqrt{3}}{2} z_2 \right) + i \left( z_1 + \frac{1 - \sqrt{3}}{2} z_2 \right), \\
    u_3 &= \frac{1}{\sqrt{3}} \left( 2v_1 + v_2 + 3z_1 y_2 + 3z_2 y_1 + 3z_2 y_2 \right) + i \left( v_2 + 2z_1 y_1 + z_2 y_1 + z_1 y_2 - z_2 y_2 \right).
\end{align*}
\]

Note that the formula for $u_3$ can be deduced from
\[
du_3 - u_2 du_1 = \theta = \left( \frac{2}{\sqrt{3}} \eta_1 + \frac{1}{\sqrt{3}} \eta_2 \right) + i \eta_2.
\]

Now the group $\Gamma \subset G$ corresponds under this isomorphism to
\[
\left\{ (u_1, u_2, u_3) | u_1, u_2 \in \mathbb{Z} \left( 1 + i, \frac{1 + \sqrt{3}}{2} \right) + \mathbb{Z} \left( \frac{1}{2}, \frac{1 - \sqrt{3}}{2} \right), u_3 \in \mathbb{Z} \left( 2\sqrt{3}, \sqrt{3} \right) + i \right\}.
\]

Using the isomorphism of Lie groups $H_C \to H_C$ given by
\[
(u_1, u_2, u_3) \mapsto (u_1', u_2', u_3') = \left( \frac{u_1}{1 + i}, \frac{u_2}{1 + i}, \frac{u_3}{(1 + i)^2} \right),
\]
we get that $u_1', u_2', u_3' \in \Lambda = \mathbb{Z}(1, \zeta)$, which completes the proof. \qed
Remark 3.2 If we had considered the discrete subgroup $\mathbb{Z}^6 \subset G$ instead of $\Gamma \subset G$, then we would not have obtained the fact $u'_2 \in \Lambda$ in the proof of Lemma 3.1. Actually the manifold $\mathbb{Z}^6 \setminus G$ is not diffeomorphic to $N = \Gamma \setminus G$, as can be proved as in Proposition 3.3 below. Note that $N = \Gamma \setminus G \to \mathbb{Z}^6 \setminus G$ is a $3 : 1$ covering. (However the nilmanifold $\mathbb{Z}^6 \setminus G$ could also have been used as a starting point to construct a simply connected compact symplectic non-formal 8–manifold with the arguments of [15].)

Under the identification $N = \Gamma \setminus G \cong \Gamma_H \setminus H_C$, $N$ is a principal torus bundle

$$T^2 = \Lambda \setminus \mathbb{C} \hookrightarrow N \longrightarrow T^4 = \Lambda^2 \setminus \mathbb{C}^2,$$

via the projection $(u_1, u_2, u_3) \mapsto (u_1, u_2)$.

The manifold $(\Gamma_H \setminus H_C) \times (\Lambda \setminus \mathbb{C})$ is the 8–dimensional compact nilmanifold $M$ defined in [15, Section 2]. It is interesting here to compare $N$ with the Iwasawa manifold. Let us recall its definition. Let $\Lambda' \subset \mathbb{C}$ be the Gaussian integers, i.e., the lattice generated by 1 and $i$, and consider the discrete subgroup $\Gamma_0 \subset H_C$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda'$. Then the Iwasawa manifold is defined as the quotient $\mathbb{S} \setminus \mathbb{I} \setminus \mathbb{H} \setminus \mathbb{S} \setminus \mathbb{H} \setminus \mathbb{S} \setminus \mathbb{H}$

$$N' = \Gamma_0 \setminus H_C.$$

Note that $N'$ is also a principal torus bundle

$$T^2 = \Lambda' \setminus \mathbb{C} \hookrightarrow N' \longrightarrow T^4 = (\Lambda')^2 \setminus \mathbb{C}^2,$$

Proposition 3.3 The fundamental groups $\pi_1(N)$ and $\pi_1(N')$ are not isomorphic. In particular, $N$ and $N'$ are not diffeomorphic.

Proof Let us first consider the manifold $N$. It is a principal torus bundle over $T^4 = \Lambda^2 \setminus \mathbb{C}^2$, the action of $T^2 = \Lambda \setminus \mathbb{C}$ being by translations in the $u_3$ coordinate. The 1–form $\theta = du_3 - u_2 du_1 \in \Omega^1(N, \mathbb{C})$ is a connection 1–form with values in $\mathbb{C}$, the Lie algebra of $T^2$. The curvature form $F = d\theta$ is the lift of the 2–form $\mu \wedge \nu = du_1 \wedge du_2 \in \Omega^2(T^4, \mathbb{C}^2)$. The cohomology class defined by the curvature is

$$[F] \in H^2(T^4, \mathbb{C}) = \text{Hom}(H_2(T^4, \mathbb{Z}), \mathbb{C}).$$

The image of this map lies in $\Lambda \subset \mathbb{C}$. Actually, the $T^2 = \Lambda \setminus \mathbb{C}$-principal bundles over a space $X$ are classified by

$$[X, B(\Lambda \setminus \mathbb{C})] = [X, K(\Lambda, 2)] = H^2(X, \Lambda),$$

and $[F]$ gives the required element classifying $N$. Since $H_k(T^4, \mathbb{Z}) = \bigwedge^k(\Lambda^2)$, we can view intrinsically

$$[F] \in \text{Hom}(\bigwedge^2(\Lambda^2), \Lambda).$$

To compute $[F]$, consider the basis for $H_1(T^4, \mathbb{Z})$ given as $\{e_1 = (1, 0), e_2 = (\zeta, 0), e_3 = (0, 1), e_4 = (0, \zeta)\}$. This basis gives us an isomorphism $H_1(T^4, \mathbb{Z}) = \Lambda^2 \cong \mathbb{Z}^4$. Then a basis for the 2–homology $H_2(T^4, \mathbb{Z}) \cong \bigwedge^2 \mathbb{Z}^4$ is $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$. Also use the basis $\{1, \zeta\}$ for $\Lambda$. We compute $[F] \in H^2(T^4, \Lambda) = \text{Hom}(\bigwedge^2(\Lambda^2), \Lambda) \cong \text{Hom}(\bigwedge^2(\mathbb{Z}^4), \mathbb{Z})$ in terms of these bases:

$$[F](e_1 \wedge e_2) = \int_{e_1 \wedge e_2} F = \int_{u_1 \in (\Lambda \setminus \mathbb{C}), u_2 = 0} du_1 \wedge du_2 = 0,$$

$$[F](e_1 \wedge e_3) = \int_{e_1 \wedge e_3} F = \int_{u_1 = t_1, u_2 = t_2, 0 \leq t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 dt_1 dt_2 = 1,$$
\[ [F](e_1 \wedge e_4) = \int_{e_1 \wedge e_4} F = \int_{t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 \zeta dt_1 dt_2 = \zeta, \]

\[ [F](e_2 \wedge e_3) = \int_{e_2 \wedge e_3} F = \int_{t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 \zeta dt_1 dt_2 = \zeta, \]

\[ [F](e_2 \wedge e_4) = \int_{e_2 \wedge e_4} F = \int_{t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 \zeta^2 dt_1 dt_2 = \zeta^2 = -1 - \zeta, \]

\[ [F](e_3 \wedge e_4) = \int_{e_3 \wedge e_4} F = \int_{u_1 = 0, \ u_2 \in (\Lambda \setminus \mathbb{C})} du_1 \wedge du_2 = 0. \]

In terms of the given bases, \([F]\) is the matrix

\[
(F) = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & -1 & 0
\end{pmatrix}.
\]

We can similarly work out the case of the Iwasawa manifold \(N'\). Again it is a principal \(T^2\)-torus bundle over \(T^4\), where \(T^2 = \Lambda \setminus \mathbb{C}\) and \(T^4 = (\Lambda')^2 \setminus \mathbb{C}^2\). Working analogously as before, the curvature \(F'\) of this principal bundle is \(F' = du_1 \wedge du_2\) and the cohomology class \([F']\) \(\in\) Hom \((\Lambda^2, (\Lambda')^2)\) is computed as follows: consider the basis \(\{e_1 = (1, 0), e_2 = (i, 0), e_3 = (0, 1), e_4 = (0, i)\}\) for \(H_1(T^4, \mathbb{Z})\) and the basis \(\{1, i\}\) for \(\Lambda'.\) Then

\[ [F'](e_1 \wedge e_2) = \int_{e_1 \wedge e_2} F = \int_{u_1 \in (\Lambda' \setminus \mathbb{C}), \ u_2 = 0} du_1 \wedge du_2 = 0, \]

\[ [F'](e_1 \wedge e_3) = \int_{e_1 \wedge e_3} F = \int_{u_1 = t_1, u_2 = t_2, 0 \leq t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 dt_1 dt_2 = 1, \]

\[ [F'](e_1 \wedge e_4) = \int_{e_1 \wedge e_4} F = \int_{u_1 = t_1, u_2 = t_2, 0 \leq t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 idt_1 dt_2 = i, \]

\[ [F'](e_2 \wedge e_3) = \int_{e_2 \wedge e_3} F = \int_{u_1 = t_1, u_2 = t_2, 0 \leq t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 idt_1 dt_2 = i, \]

\[ [F'](e_2 \wedge e_4) = \int_{e_2 \wedge e_4} F = \int_{u_1 = t_1, u_2 = t_2, 0 \leq t_1, t_2 \leq 1} du_1 \wedge du_2 = \int_0^1 \int_0^1 i^2 dt_1 dt_2 = -1, \]

\[ [F'](e_3 \wedge e_4) = \int_{e_3 \wedge e_4} F = \int_{u_1 = 0, \ u_2 \in (\Lambda' \setminus \mathbb{C})} du_1 \wedge du_2 = 0. \]

So the corresponding matrix is

\[
(F') = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & -1 & 0
\end{pmatrix}.
\]

Since \(F\) and \(F'\) are different, the two torus bundles are not isomorphic (as principal bundles over \(T^4\)). Moreover they are not even isomorphic even after an automorphism of the basis, since \([F]\) and \([F']\) are inequivalent under the natural action of \(GL(4, \mathbb{Z}) \times GL(2, \mathbb{Z})\) (by changes of basis in the lattices corresponding to base and fiber, respectively). This can be seen by considering the intersection pairing \(Q\) on \(\wedge^2 \mathbb{Z}^4 \times \wedge^2 \mathbb{Z}^4 \to \wedge^4 \mathbb{Z}^4 \cong \mathbb{Z}\) (well-defined modulo sign) and looking at the determinant of the image lattice of \([F]\) and \([F']\), respectively. As \(\text{Im}[F] = \{(0, 1, 0, 0, -1, 0), (0, 0, 1, 1, -1, 0)\}\) and \(\text{Im}[F'] = \{(0, 1, 0, 0, -1, 0), (0, 0, 1, 1, 0, 0)\}\), then

\[
\det(Q|\text{Im}[F]) = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3
\]
and
\[ \det(Q|_{\text{Im}[F]}) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4. \]

A relevant point here is that the fundamental group can be read off from the classifying cohomology class. For instance, the fundamental group of \( N \) and this is determined by the commutator bracket \( F \) which in turn coincides with the linear map \( \pi \). Hence the orbit of \([F] \) under \( \text{GL}(4,\mathbb{Z}) \times \text{GL}(2,\mathbb{Z}) \) determines the fundamental group \( \pi_1(N) \). A similar fact happens for \( \pi_1(N') = \Gamma_0 \). Therefore, \( \pi_1(N) \not\cong \pi_1(N') \), and \( N \) and \( N' \) are not diffeomorphic.

\section{Quotient of a nilmanifold by a \( \mathbb{Z}_3 \)-action}

We define the 8-dimensional compact nilmanifold \( M \) as the product

\[ M = T^2 \times N. \]

By Lemma \ref{lem:iso} there is an isomorphism between \( M \) and the manifold \( (\Gamma_H \setminus H_C) \times (\Lambda \setminus \mathbb{C}) \) studied in \cite{15} Section 2 \( ( \) we have to send the factor \( T^2 \) of \( M \) to the factor \( \Lambda \setminus \mathbb{C} \). Clearly, \( M \) is a principal torus bundle

\[ T^2 \hookrightarrow M \xrightarrow{\pi} T^6. \]

Let \( (x_1, x_2) \) be the Lie algebra coordinates for \( T^2 \), so that \( (x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) \) are coordinates for the Lie algebra \( \mathbb{R}^2 \times G \) of \( M \). Then \( \pi(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (x_1, x_2, y_1, y_2, z_1, z_2) \).

A basis for the left invariant (closed) 1–forms on \( T^2 \) is given by \( \{\alpha_1, \alpha_2\} \), where \( \alpha_1 = dx_1 \) and \( \alpha_2 = dx_2 \). Then \( \{\alpha_i, \beta_i, \gamma_i; 1 \leq i \leq 2\} \) constitutes a (global) basis for the left invariant 1–forms on \( M \). Note that \( \{\alpha_i, \beta_i, \gamma_i; 1 \leq i \leq 2\} \) is a basis for the left invariant closed 1–forms on the base \( T^6 \). (We use the same notation for the differential forms on \( T^6 \) and their pullbacks to \( M \))

Using the computation of the cohomology of \( N \), we get that the Betti numbers of \( M \) are: \( b_0(M) = b_5(M) = 1, b_1(M) = b_7(M) = 6, b_2(M) = b_6(M) = 17, b_3(M) = b_5(M) = 30, b_4(M) = 36. \) In particular, \( \chi(M) = 0 \), as for any nilmanifold.

Let us now write the minimal model of the nilmanifold \( M \). Nomizu’s theorem \cite{30} gives that the minimal model of \( M \) is the differential graded commutative algebra

\[ (\bigwedge W, d) = (\bigwedge (a_1, a_2, b_1, b_2, c_1, c_2, e_1, e_2), d), \]

whose generators \( a_i, b_i, c_i \) and \( e_i \), \( 1 \leq i \leq 2 \), have degree 1, the differential \( d \) is given by

\[ da_i = db_i = dc_i = 0, \quad 1 \leq i \leq 2, \]
\[ de_1 = -b_1 \cdot c_1 + b_2 \cdot c_1 + b_1 \cdot c_2 + 2b_2 \cdot c_2, \]
\[ de_2 = 2b_1 \cdot c_1 + b_2 \cdot c_1 + b_1 \cdot c_2 - b_2 \cdot c_2, \]
and the morphism $\phi: (\wedge (a_i, b_i, c_i, e_i), d) \to (\Omega(M), d)$, inducing an isomorphism on cohomology, is defined by $\phi(a_i) = \alpha_i$, $\phi(b_i) = \beta_i$, $\phi(c_i) = \gamma_i$, $\phi(e_i) = \eta_i$, for $1 \leq i \leq 2$, where $(\Omega(M), d)$ denotes the de Rham complex of differential forms on $M$.

Consider the action of the finite group $\mathbb{Z}_3$ on $\mathbb{R}^2$ given by

$$\rho(x_1, x_2) = (-x_1 - x_2, x_1),$$

for $(x_1, x_2) \in \mathbb{R}^2$, $\rho$ being the generator of $\mathbb{Z}_3$. Clearly $\rho(\mathbb{Z}^2) = \mathbb{Z}^2$, and so $\rho$ defines an action of $\mathbb{Z}_3$ on the 2-torus $T^2 = \mathbb{Z}^2 \setminus \mathbb{R}^2$ with 3 fixed points: $(0, 0)$, $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$. The quotient space $T^2 / \mathbb{Z}_3$ is the orbifold 2-sphere $S^2$ with 3 points of multiplicity 3. Let $x_1, x_2$ denote the natural coordinates functions on $\mathbb{R}^2$. Then the 1-forms $dx_1$, $dx_2$ satisfy $\rho^*(dx_1) = -dx_1 - dx_2$ and $\rho^*(dx_2) = dx_1$, hence $\rho^*(-dx_1 - dx_2) = dx_2$. Thus, we can take the 1-forms $\alpha_1$ and $\alpha_2$ on $T^2$ such that

$$\rho^*(\alpha_1) = -\alpha_1 - \alpha_2, \quad \rho^*(\alpha_2) = \alpha_1.$$

We denote by $A$ the 2-dimensional representation of $\mathbb{Z}_3$ given by

$$\begin{align*}
\mathbb{Z}_3 & \to \text{GL}(2, \mathbb{R}) \\
\rho & \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}
\end{align*}$$

Then the cohomology group $H^1(T^2) \cong A$, as $\mathbb{Z}_3$-representations.

It is easy to see the following isomorphisms of representations [16]:

$$A \wedge A \cong \mathbb{R}, \quad A \otimes A \cong \mathbb{R} \oplus \mathbb{R} \oplus A,$$

where $\mathbb{R}$ denotes the trivial 1-dimensional representation.

Define the following action of $\mathbb{Z}_3$ on $M$, given, at the level of Lie groups, by $\rho: \mathbb{R}^2 \times \mathbb{R}^6 \to \mathbb{R}^2 \times \mathbb{R}^6$,

$$\rho(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (-x_1 - x_2, x_1, -y_1 - y_2, y_1, -z_1 - z_2, z_1, -v_1 - v_2, v_1).$$

Note that $m(\rho(p'), \rho(p)) = \rho(m(p', p))$, for all $p, p' \in G$, where $m$ is the multiplication map for $G$. Also $\Gamma \subset G$ is stable by $\rho$ since

$$v_1 \equiv v_2 \pmod{3} \implies -v_1 - v_2 \equiv v_1 \pmod{3}.$$ 

Therefore there is a induced map $\rho: M \to M$, and this covers the action $\rho: T^6 \to T^6$ on the 6-torus $T^6 = T^2 \times T^2 \times T^2$ (defined as the action $\rho$ on each of the three factors simultaneously). The action of $\rho$ on the fiber $T^2 = \mathbb{Z} \langle (1, 1), (3, 0) \rangle$ has also 3 fixed points: $(0, 0)$, $(1, 0)$ and $(2, 0)$. Hence there are $3^4 = 81$ fixed points on $M$.

**Remark 4.1** Under the isomorphism $M \cong (\Gamma_H \setminus H_\mathbb{C}) \times (\Lambda \setminus \mathbb{C})$, we have that the action of $\rho$ becomes $\rho(u_1, u_2, u_3) = (\zeta u_1, \zeta u_2, \zeta u_3)$, where $\zeta = e^{2\pi i / 3}$. Composing the isomorphism of Lemma 3.4 with the conjugation $(u_1, u_2, u_3) \mapsto (v_1, v_2, v_3) = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ (which is an isomorphism of Lie groups $H_\mathbb{C} \to H_\mathbb{C}$ leaving $\Gamma_H$ invariant), we have that the action of $\rho$ becomes $\rho(v_1, v_2, v_3) = (\zeta v_1, \zeta v_2, \zeta^2 v_3)$. This is the action used in [17].

We take the basis $\{\alpha_i, \beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$ of the 1-forms on $M$ considered above. The 1-forms $dy_i$, $dz_i$, $dv_i$, $1 \leq i \leq 2$, on $G$ satisfy the following conditions similar to (3): $\rho^*(dy_1) =$
\[-dy_1 - dy_2, \rho^*(dy_2) = dy_1, \rho^*(dz_1) = -dz_1 - dz_2, \rho^*(dz_2) = dz_1, \rho^*(dv_1) = -dv_1 - dv_2, \rho^*(dv_2) = dv_1.\] So

\[
\rho^*(\alpha_1) = -\alpha_1 - \alpha_2, \quad \rho^*(\alpha_2) = \alpha_1, \\
\rho^*(\beta_1) = -\beta_1 - \beta_2, \quad \rho^*(\beta_2) = \beta_1, \\
\rho^*(\gamma_1) = -\gamma_1 - \gamma_2, \quad \rho^*(\gamma_2) = \gamma_1, \\
\rho^*(\eta_1) = -\eta_1 - \eta_2, \quad \rho^*(\eta_2) = \eta_1.
\]

(12)

**Remark 4.2** If we define the 1-forms \(\alpha_3 = -\alpha_1 - \alpha_2, \beta_3 = -\beta_1 - \beta_2, \gamma_3 = -\gamma_1 - \gamma_2\) and \(\eta_3 = -\eta_1 - \eta_2\), then we have \(\rho^*(\alpha_1) = \alpha_3, \rho^*(\alpha_2) = \alpha_1, \rho^*(\alpha_3) = \alpha_2\), and analogously for the others.

Note that there is also a \(\mathbb{Z}_3\)-action on the minimal model \((\bigwedge W, d)_M\) of \(M\) defined analogously to \((12)\). As \(\mathbb{Z}_3\)-representations, we have an isomorphism \(W \cong A^4\). This gives, using \((11)\), the following decomposition of the minimal model as \(\mathbb{Z}_3\)-representation:

\[
\begin{align*}
(\bigwedge W)^1 & \cong \mathbb{R}^4, \\
(\bigwedge W)^2 & \cong \mathbb{R}^{16} \oplus A^6, \\
(\bigwedge W)^3 & \cong \mathbb{R}^8 \oplus A^{24}, \\
(\bigwedge W)^4 & \cong \mathbb{R}^{36} \oplus A^{17}, \\
(\bigwedge W)^5 & \cong \mathbb{R}^8 \oplus A^{24}, \\
(\bigwedge W)^6 & \cong \mathbb{R}^{16} \oplus A^6, \\
(\bigwedge W)^7 & \cong A^4, \\
(\bigwedge W)^8 & \cong \mathbb{R}.
\end{align*}
\]

(13)

Define the quotient space

\(\widehat{M} = M/\mathbb{Z}_3\),

and denote by \(\varphi : M \to \widehat{M}\) the projection. It is an orbifold, but we can compute the rational homotopy type of the underlying topological manifold using Lemma 2.7. A model for \(\widehat{M}\) is given by the \(\mathbb{Z}_3\)-invariant part \((\bigwedge W)^{\mathbb{Z}_3}, d)\) of the minimal model of \(M\). This corresponds to the \(\mathbb{R}\)-factors of \((13)\). Since \((\bigwedge W)^1 = W \cong A^4\), the invariant part \(W^{\mathbb{Z}_3}\) is zero. This means that the first stage of the minimal model of \(\widehat{M}\) is zero and hence \(b_1(\widehat{M}) = 0\). This was the starting point that led us to consider the equations \((1)\) to define \(M^\circ\).

One can compute explicitly the differential \(d : (\bigwedge W)^{i} \to (\bigwedge W)^{i+1}\mathbb{Z}_3\) to get the cohomology of \(\widehat{M}\). For instance,

\[
\begin{align*}
H^1(\widehat{M}) &= 0, \\
H^2(\widehat{M}) &= \langle [\alpha_1 \cap \alpha_2], [\alpha_1 \cap \beta_2 - \alpha_2 \cap \beta_1], [\alpha_1 \cap \beta_1 + \alpha_1 \cap \beta_2 + \alpha_2 \cap \beta_2], \\
&\quad [\alpha_1 \cap \gamma_2 - \alpha_2 \cap \gamma_1], [\alpha_1 \cap \gamma_1 + \alpha_1 \cap \gamma_2 + \alpha_2 \cap \gamma_2], [\beta_1 \cap \beta_2], [\beta_1 \cap \gamma_2 - \beta_2 \cap \gamma_1], \\
&\quad [\beta_1 \cap \gamma_1 + \beta_1 \cap \gamma_2 + \beta_2 \cap \gamma_2], [\beta_1 \cap \eta_2 - \beta_2 \cap \eta_1], [\beta_1 \cap \eta_1 + \beta_1 \cap \eta_2 + \beta_2 \cap \eta_2], \\
&\quad [\gamma_1 \cap \gamma_2], [\gamma_1 \cap \eta_2 - \gamma_2 \cap \eta_1], [\gamma_1 \cap \eta_1 + \gamma_1 \cap \eta_2 + \gamma_2 \cap \eta_2], \rangle, \\
H^3(\widehat{M}) &= 0.
\end{align*}
\]

**Remark 4.3** The Euler characteristic of \(\widehat{M}\) can be computed via the formula for finite group action quotients: let \(\Pi\) be the cyclic group of order \(n\), acting on a space \(X\) almost freely. Then

\[
\chi(X/\Pi) = \frac{1}{n} \chi(X) + \sum_p \left(1 - \frac{1}{\#\Pi_p}\right),
\]

where \(\Pi_p \subset \Pi\) is the isotropy group of \(p \in X\). In our case \(\chi(\widehat{M}) = \frac{1}{3} \chi(M) + 81(1 - \frac{1}{3}) = 54\).
Using this remark and the previous calculation, we get that \( b_1(\hat{M}) = b_7(\hat{M}) = 0, \) \( b_2(\hat{M}) = b_6(\hat{M}) = 13, \) \( b_3(\hat{M}) = b_5(\hat{M}) = 0 \) and \( b_4(\hat{M}) = 26. \) Note that \( \hat{M} \) satisfies Poincaré duality since \( H^*(\hat{M}) = H^*(\hat{M})_{\mathbb{Z}_3} \) and \( H^*(M) \) satisfies Poincaré duality.

**Proposition 4.4** \( \hat{M} \) is simply connected.

**Proof** Let \( p_0 \in M \) be a fixed point of the \( \mathbb{Z}_3 \)-action and let \( \hat{p}_0 = \varphi(p_0) \). There is (see [5]) an epimorphism of fundamental groups

\[
\Gamma = \pi_1(M, p_0) \twoheadrightarrow \pi_1(\hat{M}, \hat{p}_0).
\]

This holds since every path in \( \hat{M} \) can be lifted to \( M \), in an unique way as long as it does not touch a singular point, an in three different ways when it does.

Since the nilmanifold \( M \) is a principal torus bundle over the 6–torus \( T^6 \), we have

\[
\mathbb{Z}^2 \hookrightarrow \Gamma \to \mathbb{Z}^6.
\]

Consider \( p_0 \in M \) a fixed point of the \( \mathbb{Z}_3 \)-action and \( \hat{p}_0 = \pi(p_0) \), where \( \pi: M \to T^6 \) is the projection of the torus bundle. Then \( \mathbb{Z}_3 \) acts on \( \pi^{-1}(\hat{p}_0) \cong T^2 \), and the restriction to \( \mathbb{Z}^2 \) of the map \( \Gamma \to \pi_1(\hat{M}) \) factors through \( \pi_1(T^2/\mathbb{Z}_3) = \{1\} \). So, the map \( \Gamma \to \pi_1(\hat{M}) \) factors also through the quotient, \( \mathbb{Z}^6 \to \pi_1(\hat{M}). \) But \( M \) contains three 2–tori, \( T_1, T_2 \) and \( T_3 \), which are the images of \( \{(x_1, x_2, 0, 0, 0, 0, 0, 0, 0)\}, \{(0, 0, y_1, y_2, 0, 0, 0, 0, 0)\} \) and \( \{(0, 0, 0, 0, z_1, z_2, 0, 0, 0)\} \), and \( \pi_1(\hat{M}) \) is generated by the images of \( \pi_1(T_1), \pi_1(T_2) \) and \( \pi_1(T_3). \) Clearly, \( \mathbb{Z}_3 \) acts in the standard way on each \( T_i \). Therefore \( \pi_1(\hat{M}) \) is generated by \( \pi_1(T_i/\mathbb{Z}_3) = \{1\}, \) which proves that \( \pi_1(\hat{M}) = \{1\}. \)

\( \square \)

## 5 Non-formality of the quotient orbifold

Now we want to prove the non-formality of the orbifold \( \hat{M} \) constructed in the previous section. By the results of [21, 34], \( M \) is non-formal since it is a nilmanifold which is not a torus. We shall see that this property is inherited by the quotient space \( \hat{M} = M/\mathbb{Z}_3. \) For this, we study the Massey products on \( \hat{M}. \)

**Lemma 5.1** \( \hat{M} \) has a non-trivial Massey product if and only if \( M \) has a non-trivial Massey product with all cohomology classes \( a_i \in H^*(M) \) being \( \mathbb{Z}_3 \)-invariant cohomology classes.

**Proof** We shall do the case of triple Massey products, since the general case is similar. Suppose that \( \langle a_1, a_2, a_3 \rangle, a_i \in H^{p_i}(\hat{M}), \) \( 1 \leq i \leq 3 \) is a non-trivial Massey product on \( \hat{M}. \) Let \( a_i = [\alpha_i], \) where \( \alpha_i \in \Omega^*(\hat{M}). \) We pull-back the cohomology classes \( \alpha_i \) via \( \varphi^*: \Omega^*(\hat{M}) \to \Omega^*(M) \) to get a Massey product \( \langle [\varphi^*\alpha_1], [\varphi^*\alpha_2], [\varphi^*\alpha_3] \rangle. \) Suppose that this is trivial on \( M, \) then \( \varphi^*\alpha_1 \wedge \varphi^*\alpha_2 = d\xi, \varphi^*\alpha_2 \wedge \varphi^*\alpha_3 = d\eta, \) with \( \xi, \eta \in \Omega^*(M), \) and \( \varphi^*\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \varphi^*\alpha_3 = df. \) Then \( \bar{\eta} = (\eta + \rho^*\eta + (\rho^*)^2\eta)/3, \bar{\xi} = (\xi + \rho^*\xi + (\rho^*)^2\xi)/3 \) and \( \bar{f} = (f + \rho^*\eta + (\rho^*)^2\eta)/3 \) are \( \mathbb{Z}_3 \)-invariant and \( \varphi^*\alpha_1 \wedge \bar{\eta} + (-1)^{p_1+1} \bar{\xi} \wedge \varphi^*\alpha_3 = d\bar{f}. \)

Conversely, suppose that \( \langle a_1, a_2, a_3 \rangle, a_i \in H^{p_i}(M)_{\mathbb{Z}_3}, 1 \leq i \leq 3, \) is a non-trivial Massey product on \( M. \) Then we can represent \( a_i = [\alpha_i] \) by \( \mathbb{Z}_3 \)-invariant differential forms \( \alpha_i \in \Omega^{p_i}(M). \) Let \( \hat{\alpha}_i \) be the induced form on \( \hat{M}. \) Then \( \langle [\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3] \rangle \) is a non-trivial Massey product on \( \hat{M}. \) For if it were trivial then pulling-back by \( \varphi, \) we would get \( 0 \in \langle \varphi^*[\hat{\alpha}_1], \varphi^*[\hat{\alpha}_2], \varphi^*[\hat{\alpha}_3] \rangle = \langle a_1, a_2, a_3 \rangle. \)

\( \square \)
Remark 5.2 As $M$ is a nilmanifold which is not a torus, by [13, Lemma 2.6], it is not 1–formal. On the other hand, $\widehat{M}$ is simply connected by Proposition 4.4 and hence it is 2–formal. By the results of [12], since $\widehat{M}$ is of dimension 8, the only possibility that it be non-formal is not to be 3–formal. This means that we have to compute the minimal model up to degree 3, which is a lengthy task, given that $b_2(\widehat{M}) = 13$ is quite large. Therefore it is more convenient to find a suitable non-trivial Massey product.

In our case, all the triple and quintuple Massey products on $\widehat{M}$ are trivial. For instance, for a Massey product of the form $\langle a_1, a_2, a_3 \rangle$, all $a_i$ should have even degree, since $H^1(\widehat{M}) = H^3(\widehat{M}) = H^5(\widehat{M}) = H^7(\widehat{M}) = 0$. Therefore the degree of the cohomology classes in $\langle a_1, a_2, a_3 \rangle$ is odd, hence they are zero.

Since the dimension of $\widehat{M}$ is 8, there is no room for sextuple Massey products or higher, since the degree of $\langle a_1, a_2, \ldots, a_s \rangle$ is at least $s + 2$, as deg $a_i \geq 2$. For $s = 6$, a sextuple Massey product of cohomology classes of degree 2 would live in the top degree cohomology. For computing an element of $\langle a_1, \ldots, a_6 \rangle$, we have to choose $\alpha_{i,j}$ in (I). But then adding a closed form $\phi$ with $a_1 \cup [\phi] = \lambda[\widehat{M}] \in H^8(\widehat{M})$ to $a_{2,6}$ we can get another element of $\langle a_1, \ldots, a_6 \rangle$ which is the previous one plus $\lambda[\widehat{M}]$. For suitable $\lambda$ the we get $0 \in \langle a_1, \ldots, a_6 \rangle$.

The only possibility for checking the non-formality of $\widehat{M}$ via Massey products is to get a non-trivial quadruple Massey product.

Because from now on, we will denote by the same symbol a $\mathbb{Z}_3$-invariant form on $M$ and that induced on $\widehat{M}$. Notice that the 2 forms $\gamma_1 \wedge \gamma_2$, $\beta_1 \wedge \beta_2$ and $\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$ are $\mathbb{Z}_3$-invariant forms on $M$, hence they descend to the quotient $\widehat{M} = M/\mathbb{Z}_3$. We have the following:

**Proposition 5.3** The quadruple Massey product

$$\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\gamma_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \rangle$$

is non-trivial on $\widehat{M}$. Therefore, the space $\widehat{M}$ is non-formal.

**Proof** First we see that

$$\begin{align*}
(\gamma_1 \wedge \gamma_2) \wedge (\beta_1 \wedge \beta_2) &= d\xi, \\
(\beta_1 \wedge \beta_2) \wedge (\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2) &= d\zeta,
\end{align*}$$

where $\xi$ and $\zeta$ are the differential 3–forms on $\widehat{M}$ given by

$$\begin{align*}
\xi &= -\frac{1}{6} (\beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2 + \beta_2 \wedge \eta_1) + \gamma_2 \wedge (\beta_1 \wedge \eta_2 + \beta_1 \wedge \eta_1 + \beta_2 \wedge \eta_1), \\
\zeta &= \frac{1}{3} (-\alpha_1 \wedge (\eta_2 \wedge \beta_1 + \eta_1 \wedge \beta_1 + \eta_1 \wedge \beta_2) + \alpha_2 \wedge (\eta_2 \wedge \beta_2 - \eta_1 \wedge \beta_1)).
\end{align*}$$

Therefore, the triple Massey products $\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2] \rangle$ and $\langle [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 \wedge \gamma_2] \rangle$ are defined, and they are trivial because all the (triple) Massey products on $\widehat{M}$ are trivial. (Notice that the forms $\xi$ and $\zeta$ are $\mathbb{Z}_3$-invariant on $M$ and so descend to $\widehat{M}$.) Therefore, the quadruple Massey product $\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 \wedge \gamma_2] \rangle$ is defined on $\widehat{M}$. Moreover, it is trivial on $\widehat{M}$ if and only if there are differential forms $f_i \in \Omega^4(\widehat{M})$, $1 \leq i \leq 3$, and $g_j \in \Omega^4(\widehat{M})$, $1 \leq j \leq 2$, such that

$$(\gamma_1 \wedge \gamma_2) \wedge (\beta_1 \wedge \beta_2) = d(\xi + f_1),$$

where...
\[(\beta_1 \land \beta_2) \land (\beta_1 \land \beta_2) = df_2,\]
\[(\beta_1 \land \beta_2) \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) = d(\varsigma + f_3),\]
\[(\gamma_1 \land \gamma_2) \land f_2 - (\xi + f_1) \land (\beta_1 \land \beta_2) = dg_1,\]
\[(\beta_1 \land \beta_2) \land (\varsigma + f_3) - f_2 \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) = dg_2,\]

and the 6–form given by

\[\Psi = - (\gamma_1 \land \gamma_2) \land g_2 - g_1 \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) + (\xi + f_1) \land (\varsigma + f_3)\]

defines the zero class in \(H^6(\hat{M})\). Clearly \(f_1, f_2\) and \(f_3\) are closed 3–forms. Since \(H^3(\hat{M}) = 0\), we can write \(f_1 = df_1', \ f_2 = df_2'\) and \(f_3 = df_3'\) for some differential 2–forms \(f_1', f_2'\) and \(f_3'\) \(\in \Omega^2(\hat{M})\). Now, multiplying \([\Psi]\) by the cohomology class \([\sigma] \in H^2(\hat{M})\), where \(\sigma = 2\alpha_1 \land \gamma_2 - \alpha_2 \land \gamma_1 + \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_2\) we get

\[\sigma \land \Psi = \frac{1}{3}(\alpha_1 \land \alpha_2 \land \beta_1 \land \beta_2 \land \gamma_1 \land \gamma_2 \land \eta_1 \land \eta_2) + d(\sigma \land \xi \land f_3' + \sigma \land \varsigma \land f_1' + \sigma \land f_1' \land df_3').\]

Hence, \([2\alpha_1 \land \gamma_2 - \alpha_2 \land \gamma_1 + \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_2] \cup [\Psi] \neq 0\), which implies that \([\Psi]\) is non-zero in \(H^6(\hat{M})\). This proves that the Massey product \(\langle [\gamma_1 \land \gamma_2], [\beta_1 \land \beta_2], [\beta_1 \land \beta_2], [\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2]\rangle\) is non-trivial, and so \(\hat{M}\) is non-formal. \(\square\)

Let us see that \(\hat{M}\) is non-formal by proving that it has a non-zero G-Massey product.

**Proposition 5.4** Consider the following closed 2–forms on \(\hat{M}\)

\[\vartheta = \beta_1 \land \beta_2, \ \tau_1 = 2\alpha_1 \land \gamma_2 - \alpha_2 \land \gamma_1 + \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_2, \ \tau_2 = \gamma_1 \land \gamma_2, \ \tau_3 = \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2.\]

Then the G-Massey product \(\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle\) is non-trivial on \(\hat{M}\).

**Proof** A direct calculation shows that

\[\vartheta \land \tau_1 = d\kappa, \ \vartheta \land \tau_2 = d\xi, \ \vartheta \land \tau_3 = d\varsigma,\]

where \(\xi\) and \(\varsigma\) are the 3–forms given in the proof of Proposition 5.3 and \(\kappa\) is the 3–form

\[\kappa = \frac{1}{3}(\alpha_1 \land \beta_1 \land \eta_1 - \alpha_1 \land \beta_1 \land \eta_2 - \alpha_1 \land \beta_2 \land \eta_1 - 2\alpha_1 \land \beta_2 \land \eta_2 - 2\alpha_2 \land \beta_2 \land \eta_1 - 2\alpha_2 \land \beta_2 \land \eta_2).
\]

We know that the forms \(\xi\) and \(\varsigma\) are \(\mathbb{Z}_2\)-invariant on \(M\), and one can check that the form \(\kappa\) is also. The G-Massey product \(\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle\) is defined. As \(H^3(\hat{M}) = 0\), we have that \(W = 0\) in Lemma 2.3 so \(\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle\) consists of one element. This is

\[\left[ -\frac{4}{3}(\alpha_1 \land \alpha_2 \land \beta_1 \land \beta_2 \land \gamma_1 \land \gamma_2 \land \eta_1 \land \eta_2) \right] \in H^8(\hat{M}),\]

which is non-zero. So \(\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle\) is non-trivial. \(\square\)
6 Symplectic resolution of singularities

In this section we resolve symplectically the singularities of $\tilde{M}$ to produce a smooth symplectic 8-manifold $\tilde{M}$ which is simply connected and non-formal. For this, we need the following results:

**Proposition 6.1** The 2-form $\omega$ on $M$ defined by

$$\omega = \alpha_1 \wedge \alpha_2 + \eta_2 \wedge \beta_1 - \eta_1 \wedge \beta_2 + \gamma_1 \wedge \gamma_2$$

is a $\mathbb{Z}_3$-invariant symplectic form on $M$. Therefore it induces $\tilde{\omega} \in \Omega^2(\tilde{M})$, such that $(\tilde{M}, \tilde{\omega})$ is a symplectic orbifold.

**Proof** Clearly $\omega^4 \neq 0$. Using (12) we have that $\rho^*(\omega) = (-\alpha_1 - \alpha_2) \wedge \alpha_1 + \eta_1 \wedge (-\beta_1 - \beta_2) + (\eta_1 + \eta_2) \wedge \beta_1 + (-\gamma_1 - \gamma_2) \wedge \gamma_1 = \omega$, so $\omega$ is $\mathbb{Z}_3$-invariant. Finally,

$$d\omega = d\eta_2 \wedge \beta_1 - d\eta_1 \wedge \beta_2 = (\beta_2 \wedge \gamma_1 - \beta_2 \wedge \gamma_2) \wedge \beta_1 - (-\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2) \wedge \beta_2 = 0.$$

$\square$

**Lemma 6.2** Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$-action. Then there exists a system of complex coordinates $(w_1, w_2, w_3, w_4)$ around $p$ such that the symplectic form $\omega$ defined in Proposition 6.1 can be expressed as

$$\omega = i(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3 + dw_4 \wedge d\bar{w}_4).$$

Moreover, with respect to these coordinates, the $\mathbb{Z}_3$-action $\rho$ on $M$ is given as

$$\rho(w_1, w_2, w_3, w_4) = (\zeta^2 w_1, \zeta^2 w_2, \zeta w_3, \zeta w_4),$$

where $\zeta = e^{2\pi i/3}$.

**Proof** Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$-action. Let $g \in G$ be a group element taking $p$ to the point $p_0 = (0, \ldots, 0) \in M$. Writing $m_g = m(g, \cdot)$, we have that $\rho \circ m_g = m_{\rho(g)} \circ \rho$, so that $m_{\rho(g)}(p) = \rho(m_g(p)) = \rho(p_0) = p_0 = m_g(p)$, therefore $\rho(g)$ coincides with $g$ modulo $\Gamma$, and hence $\rho \circ m_g = m_g \circ \rho$ on $M$. So we may suppose that the fixed point is $p = p_0$.

The coordinates for $G$ yield coordinates $(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2)$ for $M$ in a ball $B$ around $p_0$ in which $p_0$ is mapped to the origin. The symplectic form $\omega$ at the point $p_0$ is

$$\omega_0 = dx_1 \wedge dx_2 + dv_2 \wedge dy_1 - dv_1 \wedge dy_2 + dz_1 \wedge dz_2.$$

Take now $\mathbb{Z}_3$-equivariant Darboux coordinates $\Phi: (B, \omega) \rightarrow (B^\ast_{\mathbb{C}^4}(0, \epsilon), \omega_0)$, for some $\epsilon > 0$. This means that $\Phi \circ d\rho_{p_0} = \rho \circ \Phi$ and $\Phi^* \omega_0 = \omega$. The proof of the existence of usual Darboux coordinates in [27] pp. 91–93] carry over to this case, only being careful that all the objects constructed should be $\mathbb{Z}_3$-equivariant.

In terms of the complex coordinates $x = x_1 + ix_2$, $y = y_1 + iy_2$, $z = z_1 + iz_2$, $v = v_1 + iv_2$ of $\mathbb{C}^4$, the form $\omega$ is written as

$$\omega = i(dx \wedge d\bar{x} + dv \wedge d\bar{y} + dy \wedge d\bar{v} + dz \wedge d\bar{z}).$$
Now, we define the functions \( u = \frac{1}{\sqrt{2}}(v + y) \) and \( w = \frac{1}{\sqrt{2}}(v - y) \). Since \( dv \wedge dy + d\bar{y} \wedge d\bar{v} = du \wedge d\bar{u} - dw \wedge d\bar{w} \), the symplectic form \( \omega \) is expressed as

\[
\omega = i(dx \wedge d\bar{x} + du \wedge d\bar{u} - dw \wedge d\bar{w} + dz \wedge d\bar{z}).
\]

Consider new complex functions \( x' = x_1 + ix_2', \ u' = u_1 + iu_2', \ w' = w_1 + iw_2' \) and \( z' = z_1 + iz_2' \), where

\[
\begin{align*}
x'_1 &= x_1 - \frac{1}{2}x_2, & x'_2 &= -\frac{\sqrt{3}}{2}x_2, \\
u'_1 &= u_1 - \frac{1}{2}u_2, & u'_2 &= -\frac{\sqrt{3}}{2}u_2,
\end{align*}
\]

So,

\[
\rho(x', u', w', z') = (\zeta x', \zeta u', \zeta w', \zeta z'),
\]

with \( \zeta = e^{\frac{2\pi i}{3}} \) (by using that \( \rho \) corresponds to the matrix in \([\Omega]\) for the coordinates \( x, u, w, z \)).

Since \( dx' \wedge dx' = -\frac{\sqrt{3}}{2}dx \wedge d\bar{x} \), we have

\[
\omega = -\frac{2i}{\sqrt{3}}(dx' \wedge d\bar{x}' + du' \wedge d\bar{u}' - dw' \wedge d\bar{w}' + dz' \wedge d\bar{z}').
\]

Finally, the set of coordinates \((w_1, w_2, w_3, w_4) = \sqrt{\frac{2}{\sqrt{3}}}(x', \bar{u}', \bar{w}', \bar{z}')\) gives the desired result. \( \square \)

Next, we see how it is possible to desingularize the space \( \widetilde{M} \). We use the following result which is \([\Omega] \) Lemma 2.2].

**Lemma 6.3** Let \((B, \omega_0)\) be the standard Kähler ball in \( \mathbb{C}^n \), \( n > 1 \), and let \( \Pi \) be a finite group acting linearly (by complex isometries) on \( B \) whose only fixed point is the origin. Let \( \phi : (\widetilde{B}, \omega_1) \to (B/\Pi, \omega_0) \) be a Kähler resolution of the singularity of the quotient. Then there is a Kähler form on \( \widetilde{B} \) such that it coincides with \( \omega_0 \) near the boundary, and with a positive multiple of \( \omega_1 \) near the exceptional divisor \( \Pi^{-1}(0) \). \( \square \)

**Theorem 6.4** There is a smooth compact symplectic manifold \( (\widetilde{M}, \tilde{\omega}) \) which is isomorphic to \( (\tilde{M}, \tilde{\omega}) \) outside the singular points.

**Proof** Let \( p \) be a fixed point of the \( \mathbb{Z}_3 \)-action. By Lemma 6.2 we have a Kähler model for a neighbourhood \( B \) of \( p \), where the action is of the form \((w_1, w_2, w_3, w_4) \mapsto (\zeta^2 w_1, \zeta^2 w_2, \zeta w_3, \zeta^2 w_4) \). We may resolve the singularity of \( B/\mathbb{Z}_3 \) with a Kähler model. We do the resolution of singularities via iterated blow-ups as it is a standard procedure for algebraic manifolds.

Blow up \( B \) at \( p \) to get \( \widetilde{B} \). This replaces the point \( p \) by a complex projective space \( F = \mathbb{P}^3 \) in which \( \mathbb{Z}_3 \) acts as

\[
[w_1, w_2, w_3, w_4] \mapsto [\zeta^2 w_1, \zeta^2 w_2, \zeta w_3, \zeta^2 w_4] = [w_1, w_2, \zeta^2 w_3, w_4].
\]

Therefore there are two components of the fix-point locus of the \( \mathbb{Z}_3 \)-action on \( \widetilde{B} \), namely the point \( q = [0, 0, 1, 0] \) and the complex projective plane \( H = \{[w_1, w_2, 0, w_4] \} \subset F = \mathbb{P}^3 \). Next blow up \( \widetilde{B} \) at \( q \) and at \( H \) to get \( \widetilde{\tilde{B}} \). The point \( q \) is substituted by a projective space \( H_1 = \mathbb{P}^3 \). The normal bundle of \( H \subset \widetilde{B} \) is the sum of the normal bundle of \( H \subset F \), which is \( O_{\mathbb{P}^2}(1) \), and the restriction of the normal bundle of \( F \subset \widetilde{B} \) to \( H \), which is \( O_{\mathbb{P}^3}(-1)|_{\mathbb{P}^2} = O_{\mathbb{P}^2}(-1) \). Therefore the second blow-up replaces the plane \( H \) by the \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \) defined as \( H_2 = \mathbb{P}(O_{\mathbb{P}^2}(-1) \oplus O_{\mathbb{P}^2}(1)) \).
The strict transform of \( F \subset \tilde{B} \) under the second blow-up is the blow up \( \tilde{F} \) of \( F = \mathbb{P}^3 \) at \( q \), which is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \), actually \( \tilde{F} = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(1)) \).

The fix-point locus of the \( \mathbb{Z}_3 \)-action on \( \tilde{B} \) consists of the two disjoint divisors \( H_1 \) and \( H_2 \). Therefore the quotient \( \tilde{B}/\mathbb{Z}_3 \) is a smooth Kähler manifold [4, page 82]. This provides a symplectic resolution of the singularity \( B/\mathbb{Z}_3 \). To glue this Kähler model to the symplectic form in the complement of the singular point using Lemma 6.3. We do this at every fixed point to get a smooth symplectic resolution of \( \tilde{M} \).

\[ \square \]

**Theorem 6.5** The manifold \( \tilde{M} \) is non-formal.

**Proof** All the forms of the proof of either Proposition 5.3 or Proposition 5.4 can be defined on the resolution \( \tilde{M} \) as follows: take a \( \mathbb{Z}_3 \)-equivariant map \( \psi : \tilde{M} \to M \) which is the identity outside small balls around the fixed points, and contracts smaller balls onto the fixed points. Substitute the forms \( \vartheta, \tau_1, \kappa, \xi, \ldots \) by \( \psi^* \vartheta, \psi^* \tau_1, \psi^* \kappa, \psi^* \xi, \ldots \). Then the corresponding elements in the quadruple Massey product or the G-Massey product are non-zero, but these forms are zero in a neighbourhood of the fixed points. Therefore they define forms on \( \tilde{M} \), by extending them by zero.

\[ \square \]

**Theorem 6.6** The manifold \( \tilde{M} \) is simply connected.

**Proof** We have already seen in Proposition 5.4 that \( \tilde{M} \) is simply connected. The resolution \( \tilde{M} \to \hat{M} \) consists of substituting, for each singular point \( p \), a neighbourhood \( B/\mathbb{Z}_3 \) of it by the non-singular model \( \tilde{B}/\mathbb{Z}_3 \). The fiber over the origin of \( \tilde{B}/\mathbb{Z}_3 \to B/\mathbb{Z}_3 \) is simply connected: it consists of the union of the three divisors \( H_1 = \mathbb{P}^3, H_2 = \mathbb{P}(O_{\mathbb{P}^2}(-1) \oplus O_{\mathbb{P}^2}(1)) \) and \( \tilde{F}/\mathbb{Z}_3 = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(3)) \), all of them are simply connected spaces, and their intersection pattern forms no cycles. A Seifert-Van Kampen argument proves that \( \tilde{M} \) is simply connected.

\[ \square \]

**Remark 6.7** The second Betti number of \( \tilde{M} \) increases in 3 by a desingularisation of a fixed point. As there are 81 fixed points, we have \( b_2(\tilde{M}) = b_2(\hat{M}) + 81 \cdot 3 = 256 \). This makes very difficult to write down the minimal model of \( \tilde{M} \) to degree 3 to check non-formality.

**Remark 6.8** The symplectic orbifold \( (\hat{M}, \omega) \) is not hard Lefschetz: consider the \( \mathbb{Z}_3 \)-invariant forms \( \beta_1 \wedge \beta_2 \) and \( \alpha_1 \wedge \alpha_2 \wedge \xi \) on \( \hat{M} \), where \( \xi \) is the 3-form given in the proof of Proposition 5.3. As they are \( \mathbb{Z}_3 \)-invariant forms, they descend to \( \tilde{M} \). But then

\[
\omega^2 \wedge (\beta_1 \wedge \beta_2) = \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 = d(\alpha_1 \wedge \alpha_2 \wedge \xi),
\]

which means that the map \([\omega]^2 : H^2(\hat{M}) \to H^6(\hat{M})\) is not a monomorphism. These forms can be extended to \( \tilde{M} \) via the process carried out in the proof of Theorem 6.3. Therefore, the map \([\omega]^2 : H^2(\tilde{M}) \to H^6(\tilde{M})\) is not injective.

This raises the question of the existence of a non-formal simply connected compact symplectic 8–manifold satisfying the hard Lefschetz property (Cavalanti [7] has given examples for dimensions \( \geq 10 \)).
References

[1] I.K. Babenko, I.A. Taimanov, On existence of non-formal simply connected symplectic manifolds, *Russian Math. Surveys* **53** (1998), 1082–1083.

[2] I.K. Babenko, I.A. Taimanov, On non-formal simply connected symplectic manifolds, *Siberian Math. J.* **41** (2000), 204–217.

[3] I.K. Babenko, I.A. Taimanov, Massey products in symplectic manifolds, *Sb. Math.* **191** (2000), 3–44.

[4] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, 1984.

[5] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[6] G.R. Cavalcanti, Formality of \(k\)-connected spaces in \(4k+3\) and \(4k+4\) dimensions, *Math. Proc. Cam. Phil. Soc.* **141** (2006), 101–112.

[7] G.R. Cavalcanti, The Lefschetz property, formality and blowing up in symplectic geometry, *Trans. Amer. Math. Soc.* **359** (2007), 333–348.

[8] S.S. Chern, *Complex manifolds without potential theory*, Springer-Verlag, Berlin, 1979.

[9] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* **29** (1975), 245–274.

[10] A. Dranishnikov, Y. Rudyak, Examples of non-formal closed \((k-1)\)-connected manifolds of dimensions \(4k-1\) and more, *Proc. Amer. Math. Soc.* **133** (2005), 1557–1561.

[11] Y. Félix, D. Tanré, The cohomology algebra of unordered configuration spaces, *J. Lond. Math. Soc.* **72** (2005), 525–544.

[12] M. Fernández, V. Muñoz, On non-formal simply connected manifolds, *Topology and its Appl.* **135** (2004), 111–117.

[13] M. Fernández, V. Muñoz, Formality of Donaldson submanifolds, *Math. Zeit.* **250** (2005), 149–175.

[14] M. Fernández, V. Muñoz, Non-formal compact manifolds with small Betti numbers, Proceedings of the Conference Contemporary Geometry and Related Topics (Belgrade 2005), to appear.

[15] M. Fernández, V. Muñoz, An \(8\)-dimensional non-formal simply connected symplectic manifold, Preprint [math.SG/0506449](http://arxiv.org/abs/math.SG/0506449).

[16] W. Fulton, J. Harris, *Representation Theory. A First Course*, Graduate Text in Math. **129**, Springer Verlag, Berlin, 1999.

[17] R. Gompf, A new construction of symplectic manifolds, *Ann. of Math.* **142** (1995), 527–597.

[18] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, John Wiley, New York, 1978.

[19] P. Griffiths, J.W. Morgan, *Rational homotopy theory and differential forms*, Progress in Math. **16**, Birkhäuser, 1981.

[20] S. Halperin, *Lectures on minimal models*, Mém. Soc. Math. France **230**, 1983.

[21] K. Hasegawa, Minimal models of nilmanifolds, *Proc. Amer. Math. Soc.* **106** (1989), 65–71.

[22] D. Kraines, Massey higher products, *Trans. Amer. Math. Soc.* **124** (1966), 431–449.

[23] A.I. Mal’cev, A class of homogeneous spaces, *Izvestia Akademii Nauk S.S.S.R. Seriya Matematicheskaya* **13** (1949), 9–32. English translation: *Amer. Math. Soc. Transl.* **39** (1951).

[24] W.S. Massey, Some higher order cohomology operations, *Int. Symp. Alg. Top. Mexico* (1958), 145–154.

[25] J.P. May, Matric Massey products, *J. Algebra* **12** (1969) 533–568.

[26] D. McDuff, Examples of symplectic simply connected manifolds with no Kähler structure, *J. Diff. Geom.* **20** (1984), 267–277.
[27] D. McDuff, D. Salamon, *Introduction to symplectic geometry*, second edition, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1998.

[28] J. Morrow, K. Kodaira, *Complex manifolds*, Holt Rinehart Winston, New York, 1971.

[29] J. Neisendorfer, T.J. Miller, Formal and coformal spaces. *Illinois. J. Math.* 22 (1978), 565–580.

[30] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* 59 (1954), 531–538.

[31] J. Oprea, The Samelson space of a fibration, *Michigan Math. J.* 34 (1987), 127–141.

[32] D. Tanr´e, *Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Notes in Math. 1025, Springer–Verlag, 1983.

[33] W.P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* 55 (1976), 467–468.

[34] A. Tralle, J. Oprea, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Math. 1661, Springer–Verlag, 1997.

[35] A. Weinstein, Fat bundles and symplectic manifolds, *Advances Math.* 37 (1980), 239–250.

M. Fernández: Departamento de Matemáticas, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain.

E-mail: marisa.fernandez@ehu.es

V. Muñoz: Departamento de Matemáticas, Consejo Superior de Investigaciones Científicas, C/ Serrano 113bis, 28006 Madrid, Spain.

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain.

E-mail: vicente.munoz@imaff.cfmac.csic.es