MULTIPLICATIVELY SPECTRUM-PRESERVING AND
NORM-PRESERVING MAPS BETWEEN INVERTIBLE
GROUPS OF COMMUTATIVE BANACH ALGEBRAS

OSAMU HATORI, TAKESHI MIURA, AND HIROYUKI TAKAGI

Abstract. Let $A$ and $B$ be unital semisimple commutative Banach algebras and $T$ a map from the invertible group $A^{-1}$ onto $B^{-1}$. Linearity and multiplicativity of the map are not assumed. We consider the hypotheses on $T$: (1) $\sigma(TfTg) = \sigma(fg)$; (2) $\sigma_\pi(TfTg - \alpha) \cap \sigma_\pi(fg - \alpha) \neq \emptyset$; (3) $r(TfTg - \alpha) = r(fg - \alpha)$ hold for some non-zero complex number $\alpha$ and for every $f, g \in A^{-1}$, where $\sigma(\cdot)$ (resp. $\sigma_\pi(\cdot)$) denotes the (resp. peripheral) spectrum and $r(\cdot)$ denotes the spectral radius. Under each of the hypotheses we show representations for $T$ and under additional assumptions we show that $T$ is extended to an algebra isomorphism. In particular, if $T$ is a surjective group homomorphism such that $T$ preserves the spectrum or $T$ is a surjective isometry with respect to the spectral radius, then $T$ is extended to an algebra isomorphism. Similar results holds for maps from $A$ onto $B$.

1. Introduction

Recently spectrum-preserving maps on Banach algebras which are not assumed to be linear are studied by several authors including [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In this paper we mainly consider maps which are defined on the invertible groups of commutative Banach algebras.

For unital Banach algebras the groups of all invertible elements can be isomorphic as groups to each other while these Banach algebras are not algebraically isomorphic to each other. Let $C([0, 1])$ be the Banach algebra of all continuous complex-valued functions on the closed unit interval. Then the group $C([0, 1])^{-1}$ of all invertible elements of $C([0, 1])$ is isomorphic as a group to the group of all non-zero complex numbers, the invertible group of the one dimensional Banach algebra $\mathbb{C}$; the complex number field. We show a proof for a convenience.
\{z_\alpha\}$ be a Hamel basis for the linear space $\mathbb{C}$ over the rational number field. We may assume that $\pi \in \{z_\alpha\}$. Since the cardinal number of $C([0, 1])$ is the continuum $\mathfrak{c}$, the cardinal number of a Hamel basis for $C([0, 1])$ over the rational number field is also the continuum. It follows that there is a linear isomorphism $S$ (over the rational number field) from $\mathbb{C}$ onto $C([0, 1])$ such that $S(\pi) = \pi$. Put the map $T$ from $\exp \mathbb{C}$ onto $\exp(C([0, 1]))$ as $T(\exp(z)) = \exp(S(z))$: $T$ is well-defined since $S(\pi) = \pi$. By the definition $T$ is multiplicative. Since the equality $\exp(C([0, 1])) = C([0, 1])^{-1}$ holds (by a theorem of Arens and Royden [2, Corollary III. 7. 4], for example), we see that $T$ is a group isomorphism from $\{z \in \mathbb{C} : z \neq 0\}$ onto $C([0, 1])^{-1}$. In the same way there are various pairs of unital commutative Banach algebras which are not isomorphic as algebras while the invertible groups are isomorphic as groups. An interesting example of a group isomorphism between invertible groups of two non-isomorphic commutative $C^*$-algebras which is also a homeomorphism with respect to the relative topology induced by the norms on the algebras is presented in the monograph of Želazko [14, Remark 1. 7. 8].

In spite of the above Hochwald [6] proved that if a group homomorphism from the group of all invertible matrices in $M_n$ of all $n \times n$ matrices into itself preserves the spectrum, then it is extended to an algebra automorphism on $M_n$. We show a type of a theorem of Hochwald for the case of unital semisimple commutative Banach algebras (cf. Corollary 4.3). We also show that a group homomorphism between the invertible groups of unital semisimple Banach algebra is extended to a real-algebra isomorphisms between underlying algebras if $T$ is isometric with respect to the spectral radius (cf. Corollary 5.15). In this paper we consider not only group homomorphisms with additional topological properties between invertible groups but also multiplicatively (resp. peripheral) spectrum-preserving maps and norm-preserving maps. We say that a map $T$ between Banach algebras is multiplicatively spectrum-preserving if

$$\sigma(TfTg) = \sigma(fg)$$

holds for every pair $f$ and $g$ in the domain of $T$, where $\sigma(\cdot)$ denotes the spectrum. The peripheral spectrum $\{z \in \sigma(f) : |z| = r(f)\}$ is denoted by $\sigma_\pi(f)$, where $r(f)$ is the spectral radius for $f$. We say that $T$ is multiplicatively peripheral spectrum-preserving if

$$\sigma_\pi(TfTg) = \sigma_\pi(fg)$$
holds for every pair $f$ and $g$ in the domain of $T$, and $T$ is multiplicatively norm-preserving if

$$\|TfTg\| = \|fg\|$$

holds for every pair $f$ and $g$ in the domain of $T$ for a certain norm $\| \cdot \|$ including the spectral radius. The study of multiplicatively spectrum-preserving maps between Banach algebras was initiated by Molnár [9] and he characterized algebra isomorphisms in terms of multiplicative spectrum-preservingness. Rao and Roy [12] and Hatori, Miura and Takagi [4] generalized the theorem of Molnár for uniform algebras. Hatori, Miura and Takagi [5] also generalizes for unital semisimple commutative Banach algebras. Luttman and Tonev [8] introduced multiplicatively peripheral spectrum-preserving maps and generalizes results of Rao and Roy [12] and Hatori, Miura and Takagi [4] in the case of uniform algebras. Lambert, Luttman and Tonev [7] considered the maps with much weaker conditions such as multiplicatively norm-preservingsess or weakly periodically-multiplicativity.

After some preliminaries in the next section, we study multiplicatively norm-preserving maps between the invertible groups of commutative Banach algebras in section three. We show that commutative $C^*$-algebras are algebraically isomorphic if there exists a multiplicatively norm-preserving surjection, in particular, a norm-preserving group isomorphism between the invertible groups of the algebras.

In section four we consider multiplicatively spectrum-preserving maps, in particular, peripheral spectrum-preserving group isomorphisms between invertible groups of uniform algebras and show that they are extended to algebra isomorphisms between underling algebras.

In section five we consider non-symmetric multiplicatively norm-preserving maps and we consider non-symmetric multiplicatively (resp. peripheral) spectrum-preserving maps in section six. We say that a map $T$ is non-symmetric multiplicatively norm-preserving if

$$\|TfTg - \alpha\| = \|fg - \alpha\|$$

holds for every pair $f$ and $g$ in the domain of $T$ and some non-zero complex number $\alpha$. We say that $T$ is non-symmetric multiplicatively spectrum-preserving if

$$\sigma(TfTg - \alpha) = \sigma(fg - \alpha)$$

holds for every pair $f$ and $g$ in the domain of $T$ and some non-zero complex number $\alpha$. We say that $T$ is non-symmetric multiplicatively peripheral spectrum-preserving if

$$\sigma_\pi(TfTg - \alpha) = \sigma_\pi(fg - \alpha)$$
holds for every pair \( f \) and \( g \) in the domain of \( T \) and some non-zero complex number \( \alpha \).

In the last section we consider maps between commutative Banach algebras which are multiplicatively norm-preserving, and show a generalization of a theorem of Luttman and Tonev [8]. We also study non-symmetric multiplicatively norm-preserving maps between commutative Banach algebras.

2. Preliminaries

Suppose that \( A \) is a unital commutative Banach algebra. We call the group of all invertible elements in \( A \) the invertible group of \( A \). The invertible group of \( A \) is denoted by \( A^{-1} \). The spectrum of \( f \in A \) is denoted by \( \sigma(f) \). The peripheral spectrum \( \sigma_\pi(f) \) is the set \( \{ z \in \sigma(f) : |z| = r(f) \} \), where \( r(f) \) denotes the spectral radius of \( f \in A \). The maximal ideal space of \( A \) is a compact Hausdorff space and is denoted by \( M_A \). Then by Gelfand theory, the equality \( \sigma(f) = \hat{f}(M_A) \) holds, where \( \hat{f} \) is the Gelfand transform of \( f \). For a compact Hausdorff space \( X \), the algebra of all complex-valued continuous functions on \( X \) is denoted by \( C(X) \). We denote \( \text{cl} A \) the uniform closure of the Gelfand transform \( \hat{A} \) of \( A \) in \( C(M_A) \). Then \( \text{cl} A \) is a uniform algebra on \( M_A \).

In this paper we denote the Gelfand transform of \( f \) in \( A \) also by \( f \); omitting \( \hat{\cdot} \), if \( A \) is semisimple.

Let \( \mathcal{A} \) be a uniform algebra on a compact Hausdorff space \( X \), that is, \( \mathcal{A} \) is a uniformly closed subalgebra of \( C(X) \) which contains constants and separates the points of \( X \). For a subset \( K \) of \( X \) the supremum norm on \( K \) is denoted by \( \|g\|_{\infty(K)} \) for \( g \in \mathcal{A} \). A function \( f \in \mathcal{A} \) is said to be a peaking function if \( \sigma_\pi(f) = \{1\} \). Since the spectral radius and the supremum norm on \( X \) coincide for uniform algebras, we see by a simple calculation that \( \sigma_\pi(g) = \{z \in g(X) : |z| = \|g\|_{\infty(X)} \} \) for every \( g \in \mathcal{A} \). We denote the set of all peaking function by \( P_A \) and \( P_A(x) = \{f \in P_A : f(x) = 1\} \) for \( x \in X \). We also denote

\[
P_A^0 = \{f \in P_A : 0 \notin \sigma(f)\}
\]

and

\[
P_A^0(x) = \{f \in P_A(x) : 0 \notin \sigma(f)\}
\]

for \( x \in X \). For a closed subset \( K \) of \( X \), we say that \( K \) is a peak set if there is a peaking function \( f \in \mathcal{A} \) such that \( K = f^{-1}(1) \). If a peak set is a singleton, then the unique element of the set is said to be a peak point. An intersection of peak sets is said to be a \( p \)-set. If a \( p \)-set is a singleton, then the unique element of the set is said to be a \( p \)-point. The Choquet boundary for \( \mathcal{A} \) is denoted by \( \text{Ch}(\mathcal{A}) \). Note that \( \text{Ch}(\mathcal{A}) \)
For a positive integer \( n \) positive real number \( \varepsilon \) \{complex number\} on \( \bar{\Omega} \) and \( \bar{\Omega} = D \) is analytic on the open unit disk is approximated by analytic polynomials on \( \bar{\Omega} \) and analytic on \( D \) since it is continuous \( D \) onto \( \bar{\Omega} \) such that \( \bar{\Omega} = D \) is analytic on the open unit disk \( D \) onto \( \bar{\Omega} \). For every positive real number \( \delta \) there exists a Möbius transformation \( \phi_\delta \) from \( D \) onto \( D \) such that \( \phi_\delta(1) = 1 \) and \( |\phi_\delta(z) + 1| < \delta \) holds for every complex number \( z \) with \( |z| < 1/2 \). Put \( u_\delta = \pi \circ \phi_\delta \circ f_{\bar{\Omega}} \circ v_n \). Note that \( \pi \circ \phi_\delta \) is approximated by analytic polynomials on \( D \) since \( \pi \circ \phi_\delta \) is analytic on \( D \). For every positive integer \( n \), put \( u_n = u_0 \sum_{k=1}^{n} \frac{u_k}{2^k} \).
The above series is majorized by the convergent series $\sum \frac{1}{2^k}$, so $u$ is in $A$. Since $Re u > 0$ on $X$, we see that $u \in \exp A \subset A^{-1}$. Moreover, $u$ is easily seen to be a function in $P^0_A(x_0)$.

Put $g = (1/\lambda)f u$. Then this $g$ is a desired function. Choose an arbitrary $x \in X$. If $x \in F_0$, then we have

$$|g(x)| = \frac{1}{|\lambda|} |f(x)| |u_0(x)| \sum_{k=1}^{\infty} \frac{|u_k(x)|}{2^k} < \frac{1}{|\lambda|} \|f\|_{\infty(X)} \frac{|\lambda|}{\|f\|_{\infty(X)}} \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$ 

If $x \in F_n$ for some positive integer $n$, then

$$|g(x)| = \frac{1}{|\lambda|} |f(x)| |u_0(x)| \left( \frac{|u_n(x)|}{2^n} + \sum_{k \neq n} \frac{|u_k(x)|}{2^k} \right) < \frac{1}{|\lambda|} \left( \frac{|\lambda|}{2^n} + |\lambda| \right) \left( \frac{1}{2^n} \frac{1}{2^n + 1} + 1 - \frac{1}{2^n} \right) = 1.$$ 

If $x \in X \setminus \bigcup_{n=0}^{\infty} F_n$, then $f(x) = \lambda$ and so $g(x) = u(x)$. Thus we have that $g(X) \subset D \cup \{1\}$ and $g(x_0) = u(x_0) = 1$, so the proof is completed. \qed

3. Multiplicatively norm-preserving maps between invertible groups

Multiplicatively norm-preserving maps on uniform algebras are recently studied by Lambert, Luttmann and Tonev\(^{[7]}\). We show the following, which is a generalization of Theorem 1 in\(^{[7]}\).

**Theorem 3.1.** Let $A$ and $B$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Let $T$ be a map from $A^{-1}$ onto $B^{-1}$. Suppose that

$$\|TfTg\|_{\infty(Y)} = \|fg\|_{\infty(X)}$$

holds for every $f, g \in A^{-1}$. Then there exists a homeomorphism $\phi$ from $Ch(B)$ onto $Ch(A)$ such that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in A^{-1}$ and $y \in Ch(B)$.

Note that $T$ need not be injective.
Proof. Let \( y \in \text{Ch}(\mathcal{B}) \) and put
\[
L_y = \{ x \in X : |f(x)| = 1 \text{ for every } f \in \mathcal{A}^{-1} \text{ with } |Tf(y)| = 1 = \|Tf\|_{\infty(Y)} \}.
\]
We show that \( L_y \neq \emptyset \). Let \( f_1, \ldots, f_n \in \mathcal{A}^{-1} \) such that \( |Tf_j(y)| = 1 = \|Tf_j\|_{\infty(Y)} \) for \( j = 1, 2, \ldots, n \). We show that
\[
\bigcap_{j=1}^n |f_j|^{-1}(1) \neq \emptyset.
\]
By \( \|T1T1\|_{\infty(Y)} = \|1^2\|_{\infty(X)} = 1 \) we have that \( |T1| \leq 1 \) on \( \text{Ch}(\mathcal{B}) \). We show that \( |T1| = 1 \) on \( \text{Ch}(\mathcal{B}) \). Suppose that \( |T1(y)| < 1 \) for some \( y \in \text{Ch}(\mathcal{B}) \). Then by Lemma 2.1 there exists an \( H \in P^2_B(y) \) such that \( \|T1H\|_{\infty(Y)} = |T1(y)| \). Choose an \( h \in \mathcal{A}^{-1} \) with \( Th = H \). Then we have that \( \|h\|_{\infty(X)} = 1 \) since \( \|h^2\|_{\infty(X)} = \|H^2\|_{\infty(Y)} = 1 \) and \( \| \cdot \|_{\infty(Y)} \) is a uniform norm. It follows that \( \|T1H\|_{\infty(Y)} = \|h\|_{\infty(X)} = 1 \), which is a contradiction. Thus we see that \( |T1| = 1 \) on \( \text{Ch}(\mathcal{B}) \). Thus we see that
\[
\|f\|_{\infty(X)} = \|1f\|_{\infty(X)} = \|T1Tf\|_{\infty(Y)} = \|Tf\|_{\infty(Y)}
\]
holds for every \( f \in \mathcal{A}^{-1} \). Put \( F = \prod_{j=1}^n Tf_j \). Then \( |F(y)| = 1 = \|F\|_{\infty(Y)} \). Since \( T \) is a surjection we can choose \( f \in \mathcal{A}^{-1} \) such that \( Tf = F \). Then there exists \( x_0 \in X \) with
\[
|f(x_0)| = \|f\|_{\infty(X)} = \|F\|_{\infty(Y)} = 1
\]
since \( \|f\|_{\infty(X)} = \|Tf\|_{\infty(Y)} = \|F\|_{\infty(Y)} = 1 \). Since \( Tf = \prod_{j=1}^n Tf_j \) and \( \|Tf_j\|_{\infty(Y)} = 1 = \|Tf\|_{\infty(Y)} \), we have \( |f_j|^{-1}(1) \subseteq |Tf_j|^{-1}(1) \), so
\[
|f|^{-1}(1) \cap \text{Ch}(\mathcal{A}) \subseteq |f_j|^{-1}(1).
\]
(Suppose that there exists \( x_1 \in \text{Ch}(\mathcal{A}) \) such that \( |f(x_1)| = 1 \) and \( |f_j(x_1)| \neq 1 \). Then \( |f_j(x_1)| < 1 \) for \( |f_j|_{\infty(X)} = 1 \). Then there exists a \( u \in P^2_A(x_1) \) such that \( |uf_j| < 1 \) on \( X \), so that \( 1 > \|uf_j\|_{\infty(X)} = \|TuTf_j\|_{\infty(Y)} \). On the other hand, since \( 1 = \|uf\|_{\infty(X)} = \|TuTf\|_{\infty(Y)} \), there exists a \( y_1 \in Y \) such that \( |Tu(y_1)Tf(y_1)| = 1 \), so that
\[
|Tu(y_1)| = 1 = |Tf(y_1)|
\]
holds since \( \|Tu\|_{\infty(Y)} = \|u\|_{\infty(X)} \) and \( \|Tf\|_{\infty(Y)} = 1 \). Since \( |Tf|^{-1}(1) \subseteq |Tf_j|^{-1}(1) \) we see that \( |Tu(y_1)Tf(y_1)| = 1 \), so \( \|TuTf_j\|_{\infty(Y)} = 1 \), which is a contradiction.) Thus we see that
\[
\bigcap_{j=1}^n |f_j|^{-1}(1) \cap \text{Ch}(\mathcal{A}).
\]
Note that \( (|f|^{-1}(1)) \cap \text{Ch}(\mathcal{A}) \neq \emptyset \). (Since \( \|f\|_{\infty(X)} = 1 \), there is an \( x \in \text{Ch}(\mathcal{A}) \) with \( |f(x)| = 1 \). Then \( f^{-1}(f(x)) \) is a peak set since \( (1+f(x)/f)/2 \) peaks on \( f^{-1}(f(x)) \). Then
\[
\emptyset \neq \text{Ch}(\mathcal{A}) \cap f^{-1}(f(x)) \subseteq \text{Ch}(\mathcal{A}) \cap |f|^{-1}(1).\)
Then \( \cap_{y=1}^{n} |f_y|^{-1}(1) \neq \emptyset. \) By the finite intersection property we see that \( L_y \neq \emptyset. \)

Next we show that \( L_y \) is a singleton and the unique element in \( L_y \) is an element in \( \text{Ch}(\mathcal{A}). \) Let \( x_2 \in L_y \) and \( f \in \mathcal{A}^{-1} \) such that \( |Tf(y)| = 1 = \|Tf\|_{\infty(Y)}. \) Then we have \( |f(x_2)| = 1 \) by the definition of \( L_y. \) Put \( \bar{f} = (f(x_2)f + 1)/2. \) Then \( \bar{f} \) is a peak function such that \( \bar{f}(x_2) = 1. \)

We see that \( \bar{f}^{-1}(1) \subset |f|^{-1}(1), \) so

\[ x_2 \in \cap_{f} \bar{f}^{-1}(1) \subset L_y. \]

Since \( \cap_{f} \bar{f}^{-1}(1) \) is a \( \rho \)-set, there is an

\[ x_0 \in \left( \cap_{f} \bar{f}^{-1}(1) \right) \subset \text{Ch}(\mathcal{A}). \]

We show that \( L_y = \{x_0\}. \) Suppose that \( x_3 \in L_y \setminus \{x_0\}. \) Then there exists a \( u \in P_{\mathcal{A}}^0(x_0) \) such that \( |u(x_3)| < 1. \) Since \( \|Tu\|_{\infty(Y)} = 1 \) we see that \( |Tu(y)| < 1 \) by the definition of \( L_y. \) (Suppose that \( |Tu(y)| = 1. \) Then \( |u| = 1 \) on \( L_y, \) which is a contradiction. So \( |Tu(y)| \neq 1, \) and since \( \|Tu\|_{\infty(Y)} = 1, \) we have that \( |Tu(y)| < 1 \).

Then there exists a \( F' \in P_{\mathcal{B}}^0(y) \) such that \( \|F'Tu\|_{\infty(Y)} < 1 \). Since \( T \) is a surjection, there is an \( f' \in \mathcal{A}^{-1} \) such that \( Tf' = F'. \) Then we have that

\[ 1 > \|Tf'Tu\|_{\infty(Y)} = \|f'u\|_{\infty(X)}. \]

Since

\[ Tf'(y) = F'(y) = 1 = \|F'\|_{\infty(Y)} = \|Tf'\|_{\infty(Y)}, \]

we have \( |f'| = 1 \) on \( L_y, \) so we see that \( |f'(x_0)u(x_0)| = 1, \) so \( \|f'u\|_{\infty(X)} = 1, \) which is a contradiction. We see that \( L_y = \{x_0\} \) and \( x_0 \in \text{Ch}(\mathcal{A}). \)

Put a function \( \phi \) from \( \text{Ch}(\mathcal{B}) \) into \( \text{Ch}(\mathcal{A}) \) by \( \phi(y) = x_0, \) the unique element in \( L_y. \) We show that \( |Tf(y)| = |f(\phi(y))| \) holds for every \( f \in \mathcal{A}^{-1} \) and \( y \in \text{Ch}(\mathcal{B}). \) For the case where \( f \in \mathcal{A}^{-1} \) satisfies that \( |Tf(y)| = 1 = \|Tf\|_{\infty(Y)}, \) we see that \( |Tf(y)| = |f(\phi(y))| \) holds for every \( y \in \text{Ch}(\mathcal{B}) \) by the definition of \( \phi. \) Let \( f \) be an arbitrary function in \( \mathcal{A}^{-1}. \) Since \( f(\phi(y)) \neq 0, \) there exists an \( h \in P_{\mathcal{A}}^0(\phi(y)) \) such that \( \sigma_\pi(fh) = \{f(\phi(y))\} \) by Lemma 2.1, so we see that

\[ |f(\phi(y))| = \|fh\|_{\infty(X)} = \|TfTh\|_{\infty(Y)} \geq |Tf(y)Th(y)|. \]

We see that \( |Th(y)| = 1. \) (Suppose not. Then \( |Th(y)| < 1 \) since \( \|h\|_{\infty(X)} = \|Th\|_{\infty(Y)} = 1. \) Then there is an \( H' \in P_{\mathcal{B}}^0(y) \) such that \( \|ThH'\|_{\infty(Y)} < 1. \) Choose an \( h' \in \mathcal{A}^{-1} \) with \( Th' = H'. \) Then \( |h'(\phi(y))| = 1 \) by the definition of \( L_y. \) We see that

\[ \|ThH'\|_{\infty(Y)} = \|hh'\|_{\infty(X)} \geq |h(\phi(y))h'(\phi(y))| = 1, \]

which is a contradiction.) It follows that \( |f(\phi(y))| \geq |Tf(y)|. \) On the other hand there exists an \( H'' \in P_{\mathcal{B}}^0(y) \) such that \( \sigma_\phi(TfH'') = \{Tf(y)\}. \)
Choose an $h'' \in A^{-1}$ with $Th'' = H''$. Since $Th''(y) = 1 = ||Th''||$, we have $|h''(\phi(y))| = 1$ by the definition of $L_y$. Thus

\begin{equation}
|Tf(y)| = ||TfH''||_{\infty(Y)} = ||fh''||_{\infty(X)} \\
\geq |f(\phi(y))h''(\phi(y))| = |f(\phi(y))|.
\end{equation}

It follows that $|Tf(y)| = |f(\phi(y))|$. 

Next we show that $\phi$ is continuous. Let $y \in Ch(B)$. Suppose that 

\{ $y_\alpha$ \} is a net in Ch(B) which converges to $y$. Let $f \in A^{-1}$. Since $Tf(y_\alpha) \to Tf(y)$, and $|Tf(y_\alpha)| = |f(\phi(y_\alpha))|$ and $|Tf(y)| = |f(\phi(y))|$, we see that $|f(\phi(y_\alpha))| \to |f(\phi(y))|$. By the Alexandroff theorem the original topology on $X$ coincides with the weak topology on $X$ which is induced by the family \{ $|f| : f \in A^{-1}$ \}. It follows that $\phi(y_\alpha) \to \phi(y)$. We see that $\phi$ is a continuous map from Ch(B) into Ch(A).

We show that $\phi$ is a homeomorphism. For that purpose we show that there exists a continuous function $\psi$ from Ch(A) into Ch(B) such that $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions on Ch(A) and Ch(B) respectively. Although we need some consideration since $T$ needs not be injective, a proof of the existence of $\psi$ is similar to that of $\phi$. Let $x \in Ch(A)$. Put

$$K_x = \{ y \in Y : |Tf(y)| = 1 \text{ for every } f \in A^{-1} \text{ with } |f(x)| = 1 = ||f||_{\infty(X)} \}.$$

Suppose that $f_1, \ldots, f_n \in A^{-1}$ with $|f_j(x)| = 1 = ||f_j||_{\infty(X)}$ for every $j = 1, \ldots, n$. We show that $\cap_{j=1}^n |Tf_j|^{-1}(1) \neq \emptyset$. It will follow that $K_x \neq \emptyset$ by the finite intersection property. Put $f = \prod_{j=1}^n f_j$. Since $|f_j(x)| = 1 = ||f_j||_{\infty(X)}$ we have that $|f(x)| = 1 = ||f||_{\infty(X)}$. Put $Tf = F$. Since $|T1| = 1$ on Ch(B), we see that

$$||F||_{\infty(Y)} = ||F||_{\infty(Ch(B))} = ||FT1||_{\infty(Ch(B))} = ||FT1||_{\infty(Y)} = ||f||_{\infty(X)} = 1.$$

Thus there exists a $y_0 \in Ch(B)$ with $|F(y_0)| = 1$. Since $|f|^{-1}(1) \subset |f_j|^{-1}(1)$, we have that

$Ch(B) \cap |F|^{-1}(1) \subset |Tf_j|^{-1}(1)$. 

(Suppose not. There exists $y_1 \in Ch(B)$ with

$$|F(y_1)| = 1 > |Tf_j(y_1)|$$

since $||Tf_j||_{\infty(Y)} = 1$. Then there exists an

$$H \in P^0_B(y_1)$$
with $\|Tf_H\|_{\infty(Y)} < 1$. Since $|FH(y_1)| = 1$, we have $\|FH\|_{\infty(Y)} = 1$. Choose an $h \in A^{-1}$ with $Th = H$. Then

$$\|f_j h\|_{\infty(X)} = \|Tf_j Th\|_{\infty(Y)} = \|Tf_j H\|_{\infty(Y)} < 1.$$  

On the other hand, we have $1 = \|FH\|_{\infty(Y)} = \|fh\|_{\infty(X)}$. Since $|f|^{-1}(1) \subset |f_j|^{-1}(1)$, we have that $\|f_j h\|_{\infty(X)} = \|fh\|_{\infty(X)} = 1$, which is a contradiction. Thus we have that

$$\text{Ch}(B) \cap |F|^{-1} \subset \cap_{j=1}^{n} |Tf_j|^{-1}(1).$$

It follows that $\cap_{j=1}^{n} |Tf_j|^{-1}(1) \neq \emptyset$ since $\text{Ch}(B) \cap |F|^{-1}(1) \neq \emptyset$. (There exists a $y_2 \in Y$ with $|F(y_2)| = 1$ since $\|F\| = 1$. Then $(F(y_2)F + 1)/2$ is a peak function which peaks on $F^{-1}(F(y_2))$. Thus we have that

$$y_2 \in \text{Ch}(B) \cap F^{-1}(F(y_2)) \subset \text{Ch}(B) \cap |F|^{-1}(1).$$

Next we show that $K_x$ is a singleton. Suppose that $y_1 \in K_x$. Then $|Tf(y_1)| = 1$ for every $f \in A^{-1}$ with $|f(x)| = 1 = \|f\|_{\infty(X)}$. For an $f \in A^{-1}$ with

$$|f(x)| = 1 = \|f\|_{\infty(X)},$$

put

$$\tilde{F} = (Tf(y_1)Tf + 1)/2.$$  

Then $\tilde{F}$ is a peak function such that $\tilde{F}(y_1) = 1$ since $\|Tf\|_{\infty(Y)} = 1$. Thus $y_1 \in \text{Ch}(B) \cap |F|^{-1}(1) \subset K_x$, where $\cap$ takes for all $\tilde{F} = (Tf(y_1)Tf + 1)/2$ for $f \in A^{-1}$ with $|f(x)| = 1 = \|f\|_{\infty(X)}$. Since $\cap \tilde{F}^{-1}(1)$ is a $p$-set, there exists an $y_0 \in \bigcap_{j=1}^{n} \tilde{F}^{-1}(1) \text{Ch}(B)$. We show that $\{y_0\} = K_x$. Suppose that $y_3 \in K_x \setminus \{y_0\}$. Then there exists an $H \in P_{B}(y_0)$ with $|H(y_3)| < 1$. Choose an $h \in A^{-1}$ with $Th = H$. Then $\|h\|_{\infty(X)} = \|H\|_{\infty(Y)} = 1$. Suppose that $|h(x)| = 1$. Then $|H| = 1$ on $K_x$ by the definition of $K_x$, which contradicts to $|H(y_3)| < 1$. Thus we have that $|h(x)| \neq 1$, so $|h(x)| < 1$ since $\|h\|_{\infty(X)} = \|H\|_{\infty(Y)} = 1$. Thus there exists an $f \in P_{A}(x)$ with $\|f_h\|_{\infty(X)} < 1$, so $\|Tf H\|_{\infty(Y)} = \|fh\|_{\infty(X)} < 1$. On the other hand, we have that $|Tf(y_0)| = 1$ since $|Tf| = 1$ on $K_x$ by the definition of $K_x$. Thus we have that $\|Tf H\|_{\infty(Y)} = 1$, which is a contradiction proving that $K_x = \{y_0\}$.

Put a function $\psi$ from $\text{Ch}(A)$ into $\text{Ch}(B)$ by $\psi(x) = y_0$, the unique element of $K_x$. Then by the definition of $K_x$, we have that $|Tf(\psi(x))| = |f(x)|$ for every $f \in A^{-1}$ with $|f(x)| = 1 = \|f\|_{\infty(X)}$. We show that $|Tf(\psi(x))| = |f(x)|$ holds for every $f \in A^{-1}$. Let $f \in A^{-1}$. Then there exists an $h \in P_{A}(x)$ with $\sigma_{A}(fh) = \{f(x)\}$. Thus we see that

$$|f(x)| = \|fh\|_{\infty(X)} = \|TfTh\|_{\infty(Y)} \geq |Tf(\psi(x))Th(\psi(x))|. $$
Choose a function \( h' \in A^{-1} \) with \( Th' = H' \). Then
\[
\|h'\|_{\infty(X)} = \|H'\|_{\infty(Y)} = 1.
\]
We also see that \( |h'(x)| = 1 \). (Suppose not. Then \( |h'(x)| < 1 \). Then there exists an \( h'' \in P_{A}\{x\} \) with \( \|h'h''\| < 1 \). We also have that
\[
\|h''\|_{\infty(X)} = \|H'Th''\|_{\infty(Y)} \geq |H'(\psi(x))|T\|h''(\psi(x))|.
\]
By the definition of \( K_x \), we have that \( |T\|\psi(x))|_1 = 1 \) since \( h''(x) = 1 = \|h''\|_{\infty(X)} \). Since \( H' \in P_{B}\{x\} \), we have that \( |H'(\psi(x))| = 1 \), so \( \|h'h''\|_{\infty(X)} \geq 1 \), which is a contradiction.) Thus we have that
\[
|Tf(\psi(x))| = |TfH'|_{\infty(Y)} = \|fh'\|_{\infty(X)} \geq |f(x)h'(x)| = |f(x)|,
\]
so that \( |Tf(\psi(x))| = |f(x)| \).

We show that \( \psi \) is continuous. Suppose that \( x \in Ch(A) \) and \( \{x_\alpha\} \) is a net which converges to \( x \). Then for every \( f \in A^{-1} \) we have that
\[
|Tf(\psi(x_\alpha))| = |f(x_\alpha)| \to |f(x)| = |Tf(x_\alpha)|.
\]
Since \( T \) is a surjection, we see that \( |F(\psi(x_\alpha))| \to |F(x)| \) holds for every \( F \in B^{-1} \). By the Alexandroff theorem the original topology on \( Y \) and the weak topology on \( Y \) induced by the family \( \{|F| : F \in B^{-1}\} \) coincides. So we see that \( \psi(x_\alpha) \to \psi(x) \). Thus we see that \( \psi \) is continuous on \( Ch(A) \).

By the first part of the proof we have that \( |Tf(y)| = |f(\psi(y))| \) holds for every \( f \in A^{-1} \) and \( y \in Ch(B) \), so we see that
\[
|Tf(y)| = |f(\psi(y))| = |Tf(\psi(\phi(y)))|
\]
hold for every \( f \in A^{-1} \). Since \( T(A^{-1}) = B^{-1} \) and \( \{|F| : F \in B^{-1}\} \) separates the points of \( Y \), we see that \( \psi \circ \phi(y) = y \) holds for every \( y \in Ch(B) \). Since
\[
|f(x)| = |Tf(\psi(x))| = |f(\phi(\psi(x)))|
\]
hold for every \( x \in Ch(A) \), in a way similar, we also see that \( \phi \circ \psi(x) = x \) holds for every \( x \in Ch(A) \). It follows that \( \phi \) and \( \psi \) are bijections and \( \phi^{-1} = \psi \). Since \( \phi \) and \( \psi \) are continuous, we see that \( \phi \) is a homeomorphism from \( Ch(B) \) onto \( Ch(A) \). \( \square \)

Note that under the hypotheses of Theorem 3.1 the map \( T \) need not be extended to a linear map from \( A \) into \( B \). On the other hand we show that two unital commutative \( C^* \)-algebras are algebraically isomorphic to each other if there exists a surjective group homomorphisms which
preserves the norm (cf. Corollary 3.4). The results compares with the following example which is presented in [14, Remark 1.7.8].

**Example 3.2.** [14] Let \( X_1 = [0, 1] \cup \{2\} \) and \( X_2 = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \). Suppose that \( T \) is a group isomorphism from \( C(X_1)^{-1} \) onto \( C(X_2)^{-1} \) such that

\[
Tf(y) = \begin{cases} 
f(y + 1), & y \in [-1, -\frac{1}{2}] \\
f(2y), & y \in [\frac{1}{2}, 1]. 
\end{cases}
\]

Then \( T \) is a homeomorphism with respect to the relative topologies on \( C(X_1)^{-1} \) and \( C(X_2)^{-1} \) which are induced by the supremum norms on \( C(X_1) \) and \( C(X_2) \) respectively. On the other hand \( C(X_1) \) is not algebraically isomorphic to \( C(X_2) \) since \( X_1 \) and \( X_2 \) is not homeomorphic.

In this example \( \sup \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} = \infty \) and \( \inf \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} = 0 \). In Corollary 3.4 we show that if a group homomorphism from \( C(X)^{-1} \) onto \( C(Y)^{-1} \) for compact Hausdorff spaces \( X \) and \( Y \) satisfies that \( \sup \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} < \infty \) and \( \inf \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} > 0 \), then \( C(X) \) is algebraically isomorphic to \( C(Y) \). Note that such a \( T \) may not be extended an algebra isomorphism (cf. Example 3.6).

**Corollary 3.3.** Let \( A \) and \( B \) be uniform algebras on compact Hausdorff spaces \( X \) and \( Y \) respectively. Suppose that \( T \) is a group homomorphism from \( A^{-1} \) onto \( B^{-1} \) which satisfies that \( \inf \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} > 0 \) and \( \sup \frac{\|Tf\|_{\infty(X_2)}}{\|f\|_{\infty(X_1)}} < \infty \). Then there is a homeomorphism \( \phi \) from \( \text{Ch}(B) \) onto \( \text{Ch}(A) \) such that an equality \( |Tf(y)| = |f(\phi(y))| \) holds for every \( f \in A^{-1} \) and \( y \in \text{Ch}(B) \).

**Proof.** First we show that \( \|Tf\|_{\infty(Y)} = \|f\|_{\infty(X)} \) holds for every \( f \in A \). Suppose not. If \( \|Tf_1\|_{\infty(Y)} > \|f_1\|_{\infty(X)} \) for some \( f_1 \in A^{-1} \), then we have that \( \lim_{n \to \infty} \frac{\|Tf_1^n\|_{\infty(Y)}}{\|f_1^n\|_{\infty(X)}} = \infty \) for \( T \) is multiplicative. If \( \|Tf_2\|_{\infty(Y)} < \|f_2\|_{\infty(X)} \) for some \( f_2 \in A^{-1} \), then we have that \( \lim_{n \to \infty} \frac{\|Tf_2^n\|_{\infty(Y)}}{\|f_2^n\|_{\infty(X)}} = 0 \). In any case we have a contradiction. Thus we see that \( \|Tf\|_{\infty(Y)} = \|f\|_{\infty(X)} \) for every \( f \in A^{-1} \). Since \( T \) preserves multiplicity we see that

\[
\|Tfg\|_{\infty(Y)} = \|fg\|_{\infty(X)}
\]

holds for every pair \( f \) and \( g \) in \( A \). Then by Theorem 3.1 we see that the conclusion holds. \( \square \)
Corollary 3.4. Suppose that $X$ and $Y$ are compact Hausdorff spaces and $T : C(X)^{-1} \to C(Y)^{-1}$ is a surjective group homomorphism. If \( \sup_{f \in C(X)} \frac{\|Tf\|_{\infty(Y)}}{\|f\|_{\infty(X)}} < \infty \) and \( \inf_{f \in C(X)} \frac{\|Tf\|_{\infty(Y)}}{\|f\|_{\infty(X)}} > 0 \), then $C(X)$ is isometrically and algebraically isomorphic to $C(Y)$.

Proof. By Proposition 3.3 there is a homeomorphism from $X$ onto $Y$, so that $C(X)$ is isometrically and algebraically isomorphic to $C(Y)$.

Note that a norm preserving group homomorphism from $C(X)^{-1}$ onto $C(Y)^{-1}$ need not be injective.

Example 3.5. Let $X$ be the closed unit interval of the real numbers. Put $T : C(X)^{-1} \to C(X)^{-1}$ be defined as $T(f) = f^2/|f|$ for each $f \in C(X)^{-1}$. Then $T$ is a norm-preserving group homomorphism. For any $F \in C(X)^{-1}$, choose $f \in C(X)^{-1}$ with $f^2 = F|F|$. Such an $f$ exists since $C(X)^{-1}$ is closed under the square-root-operation (cf. [3]), that is, for every $h \in C(X)^{-1}$ there exists $g \in C(X)^{-1}$ with $g^2 = h$. Then $Tf = F$, so we see that $T$ is a surjection onto $C(X)^{-1}$. Since $T(f) = T(-f)$, $T$ is not injective.

Note also that the map $T$ in Corollary 3.4 need not be extended to an algebra isomorphism from $C(X)$ onto $C(Y)$ even if $T$ is injective. An example is as follows.

Example 3.6. Let $C(X)$ be the usual Banach algebra of complex-valued continuous functions on a compact Hausdorff space $X$ and $A$ the direct sum $C(X) \oplus C(X)$. Note that $A$ is isometrically isomorphic to $C(X_1 \cup X_2)$, where $X_1$ and $X_2$ are two copies of $X$. Let $T$ be a map from $A^{-1}$ into itself defined by

\[
T(f \oplus g) = \frac{f^2 g}{|fg|} \oplus \frac{f^3 g^2}{|f^3 g|}, \quad f \oplus g \in A.
\]

Then $T$ is a norm preserving group automorphism on $A^{-1}$ while $T$ is not extended to a linear map on $A$.

Proof. Clearly $T$ is a norm preserving group endomorphism on $A^{-1}$. We only show that $T$ is a bijection. Let $h_1 \oplus h_2$ be an arbitrary function in $A^{-1}$. Put $f = (h_1^2|h_2|)/(h_2|h_1|)$ and $g = (h_2^2|h_1|^3)/(h_2|h_2|)$. Then by a simple calculation that $T(f \oplus g) = h_1 \oplus h_2$ and this $f \oplus g$ is the only a function with $T(f \oplus g) = h_1 \oplus h_2$.

Even if a group isomorphism between the invertible groups of uniform algebras preserve the norm, it can be discontinuous.

Example 3.7. Let $C_b([0, 1])$ denote the real Banach space of all real-valued continuous functions on the closed unit interval $[0, 1]$ and $\{u_\lambda\}_{\lambda \in \Lambda}$
a basis for $C_\mathbb{R}([0, 1])$ as a real linear space such that $1 \in \{u_\lambda\}_{\lambda \in \Lambda}$ and $\|u_\lambda\|_{\infty([0, 1])} = 1$ for every $\lambda \in \Lambda$. Suppose that $\{u_n\}$ and $\{v_n\}$ are disjoint countable subsets of $\{u_\lambda\}_{\lambda \in \Lambda}$ and $u_1 = 1$. Without loss of generality we may assume that $u_n([0, 1]) = [0, 1]$ and $v_n([0, 1]) = [0, 1]$ for every positive integer $n$. Let $R$ be the linear isomorphism from $C_\mathbb{R}([0, 1])$ onto itself such that

$$R(u_\lambda) = \begin{cases} u_\lambda, & \text{if } u_\lambda \in \{u_\lambda\}_{\lambda \in \Lambda} \setminus (\{u_n\} \cup \{v_n\}) \\ \left(\frac{1}{3n} + \frac{1}{2}\right)v_n, & \text{if } u_\lambda = v_n \text{ for some } n, \\ nu_n, & \text{if } u_\lambda = u_n \text{ for some } n. \end{cases}$$

By a simple calculation we have that $R$ is not a bounded as linear transformation on the Banach space $C_\mathbb{R}([0, 1])$ and $2R - I$ is a linear isomorphism from $C_\mathbb{R}([0, 1])$ onto itself, where $I$ is the identity operator. Put $T : \exp C([0, 1]) \to \exp C([0, 1])$ defined by

$$T(\exp f) = \exp(f - i2R(\text{Im} f)), \quad \exp f \in \exp C([0, 1]),$$

where $\text{Im} f$ denotes the imaginary part of $f$. Since $R(u_1) = u_1$ and $\exp C([0, 1]) = (C([0, 1]))^{-1}$ by [2] Corollary III.7.4, it is easy to see that $T$ is well-defined and is a group isomorphism form $(C([0, 1]))^{-1}$ onto itself such that $\|T \|_{\infty([0, 1])} = \|\|_{\infty([0, 1])}$ for every $g \in (C([0, 1]))^{-1}$.

Put $f_n = \frac{i}{\sqrt{n}}u_n$ for every positive integer $n$. Then $\|\exp f_n - 1\|_{\infty([0, 1])} \to 0$ as $n \to \infty$. On the other hand we have that $T(\exp f_n) = \exp(\frac{1}{\sqrt{n}} - 2\sqrt{n})i\text{Im} u_n)$, so that $T(\exp f_n) \not\to 1 = T1$ since $u_n([0, 1]) = [0, 1]$ for every $n$; $T$ is not continuous. Put $g_n = -\frac{i}{3\sqrt{n}}u_n$. We see in a way similar to the above that $\|\exp g_n - 1\|_{\infty([0, 1])} \to 0$ and $T^{-1}(\exp g_n) \not\to 1 = T1$; $T^{-1}$ is not continuous.

4. Multiplicatively spectrum-preserving maps between invertible groups

The following corollary is a version of a theorem of Luttman and Tonev [3]. Another slight generalization of the theorem of Luttman and Tonev is given in the last section (cf. [7, 3]).

**Corollary 4.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively and $T$ a map from $\mathcal{A}^{-1}$ onto $\mathcal{B}^{-1}$ such that the inclusion

$$\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$$

holds for every pair $f, g \in \mathcal{A}^{-1}$. Then $(T1)^2 = 1$ and there exists a homeomorphism $\phi$ from $\text{Ch}(\mathcal{B})$ onto $\text{Ch}(\mathcal{A})$ such that the equality

$$Tf(y) = T1(y)f(\phi(y)), \quad y \in \text{Ch}(\mathcal{B})$$
holds for every $f \in \mathcal{A}^{-1}$. Thus $T/T1$ is extended to an isometrical algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. In particular, $T$ is extended to an isometrical algebra isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ if $T1 = 1$.

Proof. First we consider the case where $T1 = 1$. For every pair $f$ and $g$ in $\mathcal{A}^{-1}$ the equality $\|Tfg\|_{\infty(Y)} = \|fg\|_{\infty(X)}$ holds since $\sigma_{\pi}(Tfg) \subseteq \sigma_{\pi}(fg)$. Then by Theorem 3.1 there exists a homeomorphism $\phi$ from Ch($\mathcal{B}$) onto Ch($\mathcal{A}$) such that the equality $|Tf(y)| = |f(\phi(y))|$ holds for every $y \in$ Ch($\mathcal{A}$) and $f \in \mathcal{A}^{-1}$.

Let $y \in$ Ch($\mathcal{B}$) and $u \in P_B^0(\phi(y))$. Since we have assumed that $T1 = 1$, we see that

\[
\sigma_{\pi}(Tu) = \sigma_{\pi}(Tu1) \subseteq \sigma_{\pi}(u \cdot 1) = \{1\},
\]

that is, $Tu \in P_B^0$ and $\sigma_{\pi}(Tu) = \{1\}$. On the other hand since $|Tu(y)| = |u(\phi(y))| = 1$, we see that $Tu(y) = 1$ and so $Tu \in P_B^0(y)$. Thus we conclude that

\[
T(P_B^0(\phi(y))) \subseteq P_B^0(y).
\]

Let $f \in \mathcal{A}^{-1}$ and $y \in$ Ch($\mathcal{B}$). Then by Lemma 2.1 there exists a $u \in P_B^0(\phi(y))$ such that $\sigma_{\pi}(fu) = \{f(\phi(y))\}$. Then we have that $\sigma_{\pi}(Tfu) = \{f(\phi(y))\}$. Thus we see that

\[
|f(\phi(y))| = \|Tfu\|_{\infty(Y)} \geq |Tf(y)Tu(y)| = |Tu(y)| = |f(\phi(y))|
\]

hold since $Tu \in P_B^0(y)$, so $\|Tfu\|_{\infty(Y)} = |Tf(y)|$. It follows that $Tf(y) = f(\phi(y))$ since $\sigma_{\pi}(Tfu) = \{f(\phi(y))\}$.

We consider the general case and prove that $(T1)^2 = 1$. First we have that $\|T1\|_{\infty(Y)} = 1$ since $\sigma_{\pi}(T11) \subseteq \sigma_{\pi}(1) = \{1\}$ and $\|(T1)^2\|_{\infty(Y)} = \|T1\|_{\infty(Y)}$. Suppose that there exists a $y \in$ Ch($\mathcal{B}$) with $(T1(y))^2 \neq 1$. Then we see that $|T1(y)| < 1$ since $\sigma_{\pi}((T1)^2) \subseteq \{1\}$. Then by Lemma 2.1 there exists a $U \in P_B^0(y)$ such that $\|T1U\|_{\infty(Y)} < 1$. Since $TA^{-1} = B^{-1}$, there exists a $u \in \mathcal{A}^{-1}$ with $Tu = U$. Then $\sigma_{\pi}(T1U) = \{1\}$ since $\sigma_{\pi}(T1U) \subseteq \sigma_{\pi}(u1) = \{1\}$ and $\sigma_{\pi}(T1U) \neq \emptyset$. This contradicts to $\|T1U\|_{\infty(Y)} < 1$. We conclude that $(T1(y))^2 = 1$ for every $y \in$ Ch($\mathcal{B}$), and so $(T1)^2 = 1$ for Ch($\mathcal{B}$) is a boundary for $\mathcal{B}$. Put $\tilde{T} = T/T1$. Then $\tilde{T}$ is a well-defined map from $\mathcal{A}^{-1}$ into $\mathcal{B}^{-1}$. By a simple calculation we see that $\tilde{T}(\mathcal{A}^{-1}) = \mathcal{B}^{-1}$. We see that $\tilde{T}1 = 1$ by the definition of $\tilde{T}$ and that

\[
\sigma_{\pi}(\tilde{T}fg) \subseteq \sigma_{\pi}(fg)
\]

holds for every pair $f$ and $g$ in $\mathcal{A}^{-1}$ since $(T1)^2 = 1$. Then by the first part of the proof there exists a homeomorphism $\phi$ from Ch($\mathcal{B}$) onto Ch($\mathcal{A}$) such that

\[
\tilde{T}f(y) = f(\phi(y)), \quad y \in$ Ch($\mathcal{B}$)
holds for every $f \in \mathcal{A}^{-1}$. Thus we see that
\[ Tf(y) = T1(y)f(\phi(y)), \quad y \in \text{Ch}(\mathcal{B}) \]
holds for every $f \in \mathcal{A}^{-1}$. Since Ch(\mathcal{B}) is a boundary for \mathcal{B}, the restriction map $R : \mathcal{B} \to \mathcal{B}|\text{Ch}(\mathcal{B})$ is a bijective isometrical algebra isomorphism. Put $T_{e} : \mathcal{A} \to \mathcal{B}|\text{Ch}(\mathcal{B})$ by $T_{e}f(y) = f(\phi(y))$ for $f \in \mathcal{A}$. Then $T_{e}$ is well-defined and it is easy to see that $R^{-1} \circ T_{e}$ is an isometrical algebra isomorphism from \mathcal{A} onto \mathcal{B} which is an extension of $T$. \qed

**Corollary 4.2.** Let \mathcal{A} and \mathcal{B} be uniform algebras and $T$ a group homomorphism from \mathcal{A}^{-1} onto \mathcal{B}^{-1} which satisfies that
\[ \sigma_{\pi}(Tf) \subset \sigma_{\pi}(f) \]
holds for every $f \in \mathcal{A}^{-1}$. Then $T$ is extended to an isometrical algebra isomorphism from \mathcal{A} onto \mathcal{B}.

**Proof.** We see that $T1 = 1$ since $T$ is a surjective group homomorphism. We also see that $\sigma_{\pi}(TfTg) \subset \sigma_{\pi}(fg)$ holds for every pair $f$ and $g$ in \mathcal{A}^{-1} since $TfTg = Tfg$. The conclusion follows from Corollary 4.1. \qed

**Corollary 4.3.** Let \mathcal{A} be a unital semisimple commutative Banach algebra and \mathcal{B} a unital commutative Banach algebra. Suppose that $T$ is a surjective group homomorphism from \mathcal{A}^{-1} onto \mathcal{B}^{-1} such that
\[ \sigma(Tf) = \sigma(f) \]
holds for every $f \in \mathcal{A}^{-1}$. Then $T$ is extended to an algebra isomorphism from \mathcal{A} onto \mathcal{B}. In particular, \mathcal{B} is semisimple.

**Proof.** Since
\[ \sigma(TfTg) = \sigma(T(\frac{f}{g})) = \sigma(\frac{f}{g}) \]
hold for every pair $f$ and $g$ in \mathcal{A}^{-1}, we see that $T$ is injective.

First we consider the case where \mathcal{B} is semisimple. Recall that we denote the uniform closure of the Gelfand transform of \mathcal{A} in $C(M_{A})$ (resp. \mathcal{B} in $C(M_{B})$) by cl\mathcal{A} (resp. cl\mathcal{B}). We may consider that \mathcal{A} \subset cl\mathcal{A} (resp. \mathcal{B} \subset cl\mathcal{B}) since \mathcal{A} (resp. \mathcal{B}) is semisimple. Note that the maximal ideal space $M_{cl\mathcal{A}}$ (resp. $M_{cl\mathcal{B}}$) is homeomorphic to $M_{A}$ (resp. $M_{B}$). In fact, the correspondence between complex homomorphisms on cl\mathcal{A} (resp. cl\mathcal{B}) and its restrictions on \mathcal{A} (resp. \mathcal{B}) gives a homeomorphism between $M_{cl\mathcal{A}}$ and $M_{A}$ (resp. $M_{cl\mathcal{B}}$ and $M_{B}$). Thus the spectrum of $f \in cl\mathcal{A}$ (resp. $f \in cl\mathcal{B}$) is coincide with $f(M_{A})$ (resp. $f(M_{B})$); we denote the spectrum with respect to cl\mathcal{A} (resp. cl\mathcal{B}) also by $\sigma(f)$.

We extend $T$ to a group homomorphism from $(cl\mathcal{A})^{-1}$ onto $(cl\mathcal{B})^{-1}$ such that $\sigma(Tf) = \sigma(f)$ holds for every $f \in (cl\mathcal{A})^{-1}$. Suppose that $f \in$
(clA)^{-1}. Then there is a sequence \{f_n\} in A such that \|f_n - f\|_{\infty(M_A)} \to 0 as \(n \to \infty\). We may assume that \{f_n\} \subset A^{-1}. Since \(|f| > 0\) on \(M_A\), we may assume without loss of generality that there is a positive number \(M\) such that \(\frac{1}{M} < |f_n| < M\) on \(M_A\). So \(\frac{1}{M} - 1 \leq M|f_m - f_n|\) on \(M_A\). Since \(T\) is a group homomorphism with the assumption concerning the spectrum we see that

\[
\sigma(Tf_n) = \sigma(T(f_n/f_m)) = \sigma(f_n/f_m),
\]

so that \(\|Tf_n - 1\|_{\infty(M_B)} = \|f_n - 1\|_{\infty(M_A)}\). It follows that

\[
\|Tf_n - Tf_m\|_{\infty(M_B)} \leq \|Tf_n - 1\|_{\infty(M_B)} \|Tf_m\|_{\infty(M_B)} \leq M^2 \|f_n - f_m\|_{\infty(M_A)}
\]

since \(\|Tf_m\|_{\infty(M_B)} = \|f_m\|_{\infty(M_A)}\). Therefore \(\{Tf_n\}\) is a Cauchy sequence with respect to the supremum norm and the uniform limit \(F\) is in clB. Since \(\sigma(Tf_n) = \sigma(f_n)\) and \(\frac{1}{M} < |f_n| < M\) holds on \(M_A\) for every \(n\), we see that \(\frac{1}{M} < |Tf_n| < M\) holds on \(M_B\) for every \(n\), so \(|F| \geq \frac{1}{M}\) on \(M_B\). Thus we see that \(F \in (clB)^{-1}\) since \(M_{clB} = M_B\). By a routine calculation the function \(F\) is independent of the choice of the sequence \(\{f_n\}\). Put \(\widetilde{T}f = F\). Then \(\widetilde{T}\) is a function from (clA)^{-1} into (clB)^{-1} and \(\widetilde{T} = T\) on \(A^{-1}\).

We show that \(\widetilde{T}\) is a surjection. Let \(F \in (clB)^{-1}\). Then there is a sequence \(\{F_n\}\) in \(B^{-1}\) which uniformly converges to \(F\). In the way similar to the above, we see that \(\{T^{-1}F_n\}\) is a Cauchy sequence in \(A^{-1}\) and converges uniformly to a function \(f \in (clA)^{-1}\). Then by the definition of \(\widetilde{T}\) we see that \(\widetilde{T}f = F\).

We show that \(\sigma(\widetilde{T}f) = \sigma(f)\) holds for every \(f \in (clA)^{-1}\). Suppose that \(f \in (clA)^{-1}\). Then there exists a sequence \(\{f_n\}\) in \(A^{-1}\) such that \(\|f_n - f\|_{\infty(M_A)} \to 0\) as \(n \to \infty\). Then by the definition of \(\widetilde{T}\) we see that \(\|Tf_n - \widetilde{T}f\|_{\infty(M_B)} \to 0\) as \(n \to \infty\). Suppose that \(\lambda \in \sigma(f)\). Then there is an \(x \in M_A\) with \(\lambda = f(x)\). Put \(\lambda_n = f_n(x)\), so \(\lambda_n \to \lambda\) as \(n \to \infty\). On the other hand, for each positive integer \(n\), there is \(y_n \in M_B\) such that \(\lambda_n = Tf_n(y_n)\) since \(\sigma(f_n) = \sigma(Tf_n)\) for every \(n\). Thus we have

\[
|\lambda - \widetilde{T}f(y_n)| \leq |\lambda - \lambda_n| + \|Tf_n - \widetilde{T}f\|_{\infty(M_B)} \to 0
\]
as \(n \to \infty\). Thus we have that \(\lambda \in \sigma(\widetilde{T}f)\), so that \(\sigma(f) \subset \sigma(\widetilde{T}f)\). The reverse inclusion is proven in the same way; we see that \(\sigma(f) = \sigma(\widetilde{T}f)\).

We show that \(\widetilde{T}\) is a group homomorphism. Let \(f, g \in (clA)^{-1}\). Then there are sequences \(\{f_n\}\) and \(\{g_n\}\) in \(A^{-1}\) such that \(\|f_n - f\|_{\infty(M_A)} \to 0\) and
and \( \|g_n - g\|_{\infty(M_B)} \to 0 \) as \( n \to \infty \). So \( \|f_n g_n - fg\|_{\infty(M_A)} \to 0 \) as \( n \to \infty \). Then we see that
\[
\|Tf_n - \tilde{T}f\|_{\infty(M_B)} \to 0, \quad \|Tg_n - \tilde{T}g\|_{\infty(M_B)} \to 0
\]
and so
\[
\|Tf_n Tg_n - \tilde{T}f \tilde{T}g\|_{\infty(M_B)} \to 0
\]
as \( n \to \infty \). We also have that
\[
\|T(f_n g_n) - \tilde{T}(fg)\|_{\infty(M_B)} \to 0
\]
as \( n \to \infty \) since \( \|f_n g_n - fg\|_{\infty(M_A)} \to 0 \) as \( n \to \infty \). It follows that
\[
\tilde{T}f \tilde{T}g = \tilde{T}(fg) \text{ since } T(f_n g_n) = Tf_n Tg_n.
\]
Since \( \sigma(\tilde{T}f) = \sigma(f) \) holds for every \( f \in (\text{cl}A)^{-1} \), \( \sigma_{\pi}(\tilde{T}f) = \sigma_{\pi}(f) \) holds for every \( f \in (\text{cl}A)^{-1} \). Thus applying Corollary 4.2 we see that \( \tilde{T} \) is extended to an algebra isomorphism from \( \text{cl}A \) onto \( \text{cl}B \). By a simple calculation we see that the restriction of the extended isomorphism to \( A \) is an algebra isomorphism from \( A \) onto \( B \).

Finally we consider the general case. Let \( \Gamma_B \) denote the Gelfand transform of \( B \). Then by a simple calculation we see that \( \Gamma_B \circ T \) is a group homomorphism from \( A^{-1} \) onto \( (\Gamma_B(B))^{-1} \), since \( \Gamma_B(B^{-1}) = (\Gamma_B(B))^{-1} \). Then by the first part of the proof, we see that \( \Gamma_B \circ T \) is extended to an algebra isomorphism from \( A \) onto \( \Gamma_B(B) \). Then we see that \( \Gamma_B \) is injection from \( B^{-1} \) onto \( (\Gamma_B(B))^{-1} \). (Suppose that \( g_1, g_2 \in B^{-1} \) with \( \Gamma_B(g_1) = \Gamma_B(g_2) \). Since \( T \) is surjection from \( A^{-1} \) onto \( B^{-1} \), there are \( f_1, f_2 \in A^{-1} \) with \( Tf_1 = g_1 \) and \( Tf_2 = g_2 \). Then we have
\[
\Gamma_B \circ T(f_1) = \Gamma_B(g_1) = \Gamma_B(g_2) = \Gamma_B \circ T(f_2).
\]
Since \( \Gamma_B \circ T \) is an injection we see that \( f_1 = f_2 \), thus \( g_1 = g_2 \), that is, \( \Gamma_B \) is injective on \( B^{-1} \). It follows by a simple calculation that \( \Gamma_B \) is an injection on \( B \). Thus \( B \) is semisimple, and \( T \) is extended to an algebra isomorphism from \( A \) onto \( B \) applying the first part of the proof. \( \square \)

5. Non-symmetric multiplicatively norm-preserving maps between invertible groups

Under the hypotheses in Theorem 3.1, the map \( T \) appearing in Theorem 3.1 need not be linear nor multiplicative. For example, let \( A \) be a uniform algebra on \( X \) and put a map \( \varepsilon \) from \( A \) into \( \{-1, 1\} \). Then the map \( T : A \to A \) defined by \( Tf = \varepsilon(f)f, \ f \in A \) satisfies that the equality \( \|TfTg\|_{\infty(X)} = \|fg\|_{\infty(X)} \) holds for every pair \( f \) and \( g \) in \( A \) and \( T \) can be surjective which is not linear nor multiplicative according to the choice of \( \varepsilon \).
In this section we consider non-symmetric multiplicatively norm-preserving maps between invertible groups. Let $A$ and $B$ be unital commutative Banach algebras and $T$ a map from $A^{-1}$ into $B^{-1}$. We say that $T$ is non-symmetric multiplicatively (spectral) norm-preserving if there exists a nonzero complex number $\alpha$ such that

$$\|\hat{T}f\hat{T}g - \alpha\|_{\infty(M_B)} = \|\hat{f}\hat{g} - \alpha\|_{\infty(M_A)}$$

holds for every pair $f$ and $g$ in $A$, where $M_A$ (resp. $M_B$) denotes the maximal ideal space of $A$ (resp. $B$). Multiplicatively norm-preserving map corresponds to the case where $\alpha = 0$. Although multiplicatively norm-preserving maps need not be extended to linear nor multiplicative maps, we show that non-symmetric ones are extended to real-linear and multiplicative maps if $T1 = 1$ and $T$ is surjective.

In the following, from Lemma 5.1 to Lemma 5.11, $A$ and $B$ are uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively and $T$ is a map from $A^{-1}$ onto $B^{-1}$ such that

$$\|TfTg - 1\|_{\infty(Y)} = \|fg - 1\|_{\infty(X)}$$

holds for every $f, g \in A^{-1}$.

**Lemma 5.1.** $T$ is an injection. The equality $\|TfTg\|_{\infty(Y)} = \|fg\|_{\infty(X)}$ holds for every $f, g \in A^{-1}$.

**Proof.** First we show that $T$ is an injection. Suppose that $Tf = Tg$. Then we have that

$$0 = \|gg^{-1}\|_{\infty(X)} = \|TgTg^{-1} - 1\|_{\infty(Y)} = \|TfTg^{-1} - 1\|_{\infty(Y)} = \|fg^{-1} - 1\|_{\infty(X)}.$$

Thus we see that $fg^{-1} = 1$, so $f = g$.

Next we show that $\|TfTg\|_{\infty(Y)} = \|fg\|_{\infty(X)}$ holds for every $f, g \in A^{-1}$. Since $(T1)^2 = 1$, we have for every $19$
positive integer \( n \) that
\[
\| \tilde{T}n - 1 \|_{\infty(Y)} = \| \tilde{T}n - 1 \|_{\infty(X)} = \| n \cdot 1 - 1 \|_{\infty(X)} = n - 1.
\]
So we have that
\[
n - 2 \leq \| \tilde{T}n \|_{\infty(Y)} \leq n.
\]
Let \( f \in A^{-1} \). Then we have that \((\tilde{T}f)^{-1} = \tilde{T}(f^{-1})\) since
\[
\| \tilde{T}f \tilde{T}(f^{-1}) - 1 \|_{\infty(Y)} = \| ff^{-1} - 1 \|_{\infty(X)} = 0.
\]
Put \( K_n = \tilde{T}(nf)\tilde{T}(f^{-1}) \). We see that
\[
\| K_n - 1 \|_{\infty(Y)} = \| \tilde{T}(nf)\tilde{T}(f^{-1}) - 1 \|_{\infty(Y)} = \| nff^{-1} - 1 \|_{\infty(X)} = n - 1,
\]
so we have that \( \| K_n \|_{\infty(Y)} \leq n \). For every \( g \in A^{-1} \), we have that
\[
\| nfg \|_{\infty(X)} - 1 \leq \| nfg - 1 \|_{\infty(X)} = \| \tilde{T}(nf)\tilde{T}g - 1 \|_{\infty(Y)} \leq \| K_n \|_{\infty(Y)} \| \tilde{T}f\tilde{T}g \|_{\infty(Y)} + 1.
\]
It follows that
\[
\| fg \|_{\infty(X)} - \frac{1}{n} \leq \frac{\| K_n \|_{\infty(Y)} \| \tilde{T}f\tilde{T}g \|_{\infty(Y)}}{n} + \frac{1}{n} \leq \| \tilde{T}f\tilde{T}g \|_{\infty(Y)} + \frac{1}{n},
\]
and letting \( n \to \infty \), we have that the inequality \( \| fg \|_{\infty(X)} \leq \| \tilde{T}f\tilde{T}g \|_{\infty(Y)} \) holds for every \( f, g \in A^{-1} \). \( \tilde{T} \) is injective since \( T \) is. Applying the similar argument to \((\tilde{T})^{-1}\) instead of \( \tilde{T} \), we have that
\[
\| FG \|_{\infty(Y)} \leq \| (\tilde{T})^{-1}F(\tilde{T})^{-1}G \|_{\infty(X)}
\]
holds for every \( F, G \in B^{-1} \). It follows that the equality \( \| fg \|_{\infty(X)} = \| \tilde{T}f\tilde{T}g \|_{\infty(Y)} \) holds for every \( f, g \in A^{-1} \). Since \((T1)^2 = 1\) we conclude that the equality \( \| TfTg \|_{\infty(Y)} = \| fg \|_{\infty(X)} \) holds for every \( f, g \in A^{-1} \).

By Theorem 3.1 we see that there exists a homeomorphism \( \phi \) from \( \text{Ch}(B) \) onto \( \text{Ch}(A) \) such that
\[
|Tf(y)| = |f(\phi(y))|, \quad y \in \text{Ch}(B)
\]
holds for every \( f \in A \). In the following up to Lemma 5.11 \( \phi \) denotes this homeomorphism. Moreover we see that the following.

**Lemma 5.2.** \( |T\lambda| = |\lambda| \) on \( \text{Ch}(B) \) and \( |T^{-1}\lambda| = |\lambda| \) on \( \text{Ch}(A) \) for every complex number \( \lambda \).
Lemma 5.3. Suppose that $T1 = 1$. For every complex number $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1, -1$, we have that

$$(T^{-1}(\lambda))(\text{Ch}(A)) \subset \Lambda_1 \cup \Lambda_2,$$

where $\Lambda_1$ (resp. $\Lambda_2$) is the closed arc on the closed unit circle with the end points $\lambda$ and $-\lambda$ (resp. $\overline{\lambda}$ and $-\lambda$) which does not contain the real number.

Proof. Note that $T^{-1}$ is well-defined since $T$ is an injection by Lemma 5.1. Then we have

$$\|T^{-1}\lambda - 1\|_{\infty(X)} = \|\lambda - 1\|_{\infty(Y)} = |\lambda - 1|.$$  

Since

$$\|T(-1)T(-1) - 1\|_{\infty(Y)} = \|(-1)^2 - 1\|_{\infty(X)} = 0,$$

we have that $(T(-1))^2 = 1$. Since $T$ is injective by Lemma 5.1, $T(-1) \neq 1$, so there exists $y \in \text{Ch}(B)$ such that $(T(-1))(y) = -1$. We have that

$$\|T^{-1}\lambda + 1\|_{\infty(X)} = \|T^{-1}\lambda - 1\|_{\infty(X)} = \|T(-1)\lambda - 1\|_{\infty(Y)}.$$  

Suppose that $\text{Re}\lambda \geq 0$. We see that

$$\|T(-1)\lambda - 1\|_{\infty(Y)} = |\lambda + 1|$$

since $(T(-1))^2 = 1$ and $T(-1)$ takes the value $-1$. Thus we have by the equation (5.2) that $\|T^{-1}\lambda + 1\|_{\infty(X)} = |\lambda + 1|$ if $\text{Re}\lambda \geq 0$. Recall that $\|T^{-1}\lambda - 1\|_{\infty(X)} = |\lambda - 1|$ by the equation (5.1) and $|T^{-1}\lambda| = |\lambda|$ on $\text{Ch}(A)$ by Lemma 5.2. It follows that $(T^{-1}\lambda)(\text{Ch}(A)) \subset \{|\lambda, \overline{\lambda} \}$ if $\text{Re}\lambda \geq 0$. Suppose that $\text{Re}\lambda < 0$. Then $\|T(-1)\lambda - 1\|_{\infty(Y)} \leq |\lambda - 1|$ since $(T(-1))^2 = 1$. Thus by the equations (5.1) and (5.2) we see that

$$(T^{-1}(\lambda))(\text{Ch}(A)) \subset \Lambda_1 \cup \Lambda_2.$$  

In any case we have the conclusion.  

Lemma 5.4. Suppose that $T1 = 1$. Then $T(P_A^0(\phi(y))) = P_B^0(y)$ holds for every $y \in \text{Ch}(B)$.

Proof. Suppose that $u \in P_A^0(\phi(y))$. Then by Lemma 5.1 we see that

$$\|Tu\|_{\infty(Y)} = \|u\|_{\infty(X)} = 1$$

since $T1 = 1$. First we show that $\sigma_\pi(Tu) = \{1\}$. Suppose that $\alpha \in \sigma_\pi(Tu) \setminus \{1\}$. We have that

$$\|Tu - 1\|_{\infty(Y)} = \|u - 1\|_{\infty(X)} < 2$$

since $T1 = 1$ and $\sigma_\pi(u) = \{1\}$, so $\alpha \neq -1$. Thus

$$\|T^{-1}(\alpha\overline{\alpha})u - 1\|_{\infty(X)} = \|T^{-1}(\alpha\overline{\alpha})u - 1\|_{\infty(\text{Ch}(A))} < 2.$$
by Lemma [5.3]. On the other hand, \(\| - \alpha Tu - 1\|_{\infty(Y)} = 2\) since \(\alpha \in \sigma_\pi(Tu)\), which is a contradiction since

\[
\| - \alpha Tu - 1\|_{\infty(Y)} = \|T^{-1}(-\alpha)u - 1\|_{\infty(X)}.
\]

We have that \(\sigma_\pi(Tu) = \{1\}\) since \(\sigma_\pi(Tu)\) is not empty. By Theorem 3.1 and Lemma 5.1 we have that

\[
|Tu(y)| = |u(\phi(y))| = 1,
\]

so that \(Tu(y) = 1\) since \(\sigma_\pi(Tu) = \{1\}\). We have proved that \(Tu \in P_B^0(y)\) for every \(u \in P_A^0(\phi(y))\). Since \(T\) is injection by Lemma 5.1 we have in a way similar to the above that \(T^{-1}(P_B^0(y)) \subset P_A^0(\phi(y))\). Then the conclusion holds.

**Lemma 5.5.** Suppose that \(T1 = 1\). Then \(T(-1) = -1\).

**Proof.** Since

\[
\|T(-1)T(-1) - 1\|_{\infty(Y)} = \|(-1)^2 - 1\|_{\infty(X)} = 0,
\]

we see that \((T(-1))^2 = 1\). Suppose that there is a \(y \in \text{Ch}(B)\) such that \((T(-1))(y) = 1\). Then there exists a \(U \in P_B^0(y)\) with \(\sigma_\pi(T(-1)U) = \{1\}\) by Lemma 2.1. Thus

\[
\| - T^{-1}U - 1\|_{\infty(X)} = \|T(-1)U - 1\|_{\infty(Y)} < 2.
\]

On the other hand \(T^{-1}U \in P_A^0(\phi(y))\) by Lemma 5.4 so

\[
\|T(-1)U - 1\|_{\infty(Y)} = \| - T^{-1}U - 1\|_{\infty(X)} = 2,
\]

which is a contradiction.

**Lemma 5.6.** Suppose that \(T1 = 1\). Then \((T\lambda)(\text{Ch}(B)) \subset \{\lambda, \bar{\lambda}\}\) for every complex number \(\lambda\) with the unit absolute value.

**Proof.** By Lemma 5.2 we see that \(|T\lambda| = |\lambda|\) holds on \(\text{Ch}(B)\). Since \(T1 = 1\), we have \(\|T\lambda - 1\|_{\infty(Y)} = |\lambda - 1|\). Since \(T(-1) = -1\) by Lemma 5.5 we also see that

\[
\|T\lambda + 1\|_{\infty(Y)} = \| - T\lambda - 1\|_{\infty(Y)} = \|T(-1)T\lambda - 1\|_{\infty(Y)} = | - \lambda - 1| = |\lambda + 1|.
\]

It follows by a simple calculation that the conclusion holds.

**Lemma 5.7.** Suppose that \(T1 = 1\). Then \(T(-i) = -Ti\).

**Proof.** Since \(\|T(-i)Ti - 1\|_{\infty(Y)} = \| - i \cdot i - 1\|_{\infty(X)} = 0\), we have that \(T(-i)Ti = 1\). By Lemma 5.6 we see that \((Ti)^2(\text{Ch}(A)) = \{-1\}\), so \((Ti)^2 = -1\). Thus we see that the conclusion holds.
Definition 5.8. Suppose that $T1 = 1$. Put
\[ K = \{ y \in \text{Ch}(B) : Ti(y) = i \} . \]

Lemma 5.9. Suppose that $T1 = 1$. Then $K$ is a clopen subset of \text{Ch}(B) and $Ti = -i$ on $\text{Ch}(B) \setminus K$.

Proof. Since $Ti$ is continuous on $\text{Ch}(B)$ and $Ti(\text{Ch}(B)) \subset \{ i, -i \}$ by Lemma 5.6, $K$ is a clopen subset of $\text{Ch}(B)$ and $T(i) = -i$ on $\text{Ch}(B) \setminus K$. □

Lemma 5.10. Suppose that $T1 = 1$. For every complex number $\alpha$ with the absolute value 1, $T\alpha(y) = \alpha$ if $y \in K$ and $T\alpha(y) = \bar{\alpha}$ if $y \in \text{Ch}(B) \setminus K$. Thus $T(\alpha \beta) = T\alpha T\beta$ holds for every pair of complex numbers $\alpha$ and $\beta$ with unit absolute values.

Proof. If $\alpha = -1$, then $T(\alpha) = \alpha$ by Lemma 5.5. We consider the case where $\alpha$ is a imaginary number. By Lemma 5.1, we have that
\[ |TiT\alpha + 1|_{\infty(Y)} = |T(-i)T\alpha - 1|_{\infty(Y)} \]
\[ = |i\alpha - 1|_{\infty(X)} = ||i\alpha + 1||_{\infty(X)}. \]

We also see that $|TiT\alpha - 1|_{\infty(Y)} = |i\alpha - 1|_{\infty(X)}$. By Lemma 5.2, we see that $|TiT\alpha| = 1$ on $\text{Ch}(B)$. It follows by a simple calculation that
\[ (TiT\alpha)(\text{Ch}(B)) \subset \{ i\alpha, i\bar{\alpha} \}. \]

Suppose that $y \in K$. Then $Ti(y) = i$. By Lemma 5.6, $T\alpha(y) = \alpha$ or $\bar{\alpha}$. Suppose that $T\alpha(y) = \bar{\alpha}$. Then $(TiT\alpha)(y) = i\bar{\alpha}$, which contradicts to $(TiT\alpha)(\text{Ch}(B)) \subset \{ i\alpha, i\bar{\alpha} \}$. Thus we see that $T\alpha(y) = \alpha$ if $y \in K$. In a way similar, we see that $T\alpha(y) = \bar{\alpha}$ if $y \in \text{Ch}(B) \setminus K$. Thus $T(\alpha \beta) = T(\alpha)T(\beta)$ holds on $\text{Ch}(B)$ and so we have that $T(\alpha \beta) = T(\alpha)T(\beta)$. □

Lemma 5.11. Suppose that $T1 = 1$. Then $T(\alpha P^0_A(\phi(y))) = (T\alpha)P^0_B(y)$ holds for every $y \in \text{Ch}(B)$ and every complex number $\alpha$ with the unit absolute value.

Proof. First we show that $T(\alpha P^0_A) \subset (T\alpha)P^0_B$. Suppose that $u \in P^0_A$. Since
\[ 2 = | -\bar{\alpha}au - 1|_{\infty(X)} = ||T(-\bar{\alpha})T(\alpha u) - 1||_{\infty(Y)}, \]
we have that $-1 \in \sigma_p(T(-\bar{\alpha})T(\alpha u))$. Suppose that
\[ \beta \in \sigma_p(T(-\bar{\alpha})T(\alpha u)) \setminus \{-1\} . \]
Note that $|\beta| = 1$ since
\[ ||T(-\bar{\alpha})T(\alpha u)||_{\infty(Y)} = 1. \]
Since \( \sigma_\pi(F) \subset F(\text{Ch}(B)) \) holds for every \( F \in B \), there exists a \( y \in \text{Ch}(B) \) with \( (T(-\bar{\alpha})T(\alpha u))(y) = \beta \). If \( y \in K \), then
\[
(T((-\bar{\beta})(-\bar{\alpha}))T(\alpha u))(y) = (T(-\bar{\beta})T(-\bar{\alpha})T(\alpha u))(y) = -1
\]
since \( T(-\bar{\beta}) = -\bar{\beta} \) on \( K \) by Lemma \( 5.10 \) so that
\[
2 = \|T((-\bar{\beta})(-\bar{\alpha}))T(\alpha u) - 1\|_{\infty(Y)} = \|\bar{\beta}u - 1\|_{\infty(X)}.
\]
Since \( u \in P^0_A \) we see that \( \bar{\beta} = -1 \), so \( \beta = -1 \), which is a contradiction.

If \( y \in \text{Ch}(B) \setminus K \), then we have, in a way similar to the above, a contradiction. It follows that
\[
\sigma_\pi(T(-\bar{\alpha})T(\alpha u)) = \{-1\}.
\]
Thus we see that
\[
T(-\bar{\alpha})T(\alpha P^0_A) \subset -P^0_B.
\]
and
\[
-T(-\alpha) = T(-1)T(-\alpha) = T(\alpha)
\]
bv Lemma \( 5.5 \) and \( 5.10 \) we see that
\[
T(\alpha P^0_A) \subset T(\alpha)P^0_B.
\]
Next we show that
\[
T(\alpha P^0_A(\phi(y))) \subset T(\alpha)P^0_B(y).
\]
Suppose that \( u \in P^0_A(\phi(y)) \). Then by the above we have that \( T(\alpha u) \in T(\alpha)P^0_B \). Since \( T(\bar{\alpha})T(\alpha) = T1 = 1 \) by Lemma \( 5.10 \) we see that \( T(\bar{\alpha})T(\alpha u) \in P^0_B \). On the other hand, by Theorem \( 3.1 \) and Lemma \( 5.1 \)
\[
|(T(\bar{\alpha})T(\alpha u))(y)| = |\bar{\alpha}(\phi(y))||\alpha u(\phi(y))| = 1,
\]
so that \( (T(\bar{\alpha})T(\alpha u))(y) = 1 \) since \( T(\bar{\alpha})T(\alpha u) \in P^0_B \). It follows that \( T(\bar{\alpha})T(\alpha u) \in P^0_B(y) \) and so \( T(\alpha u) \in (T\alpha)P^0_B(y) \). We see that
\[
T(\alpha P^0_A(\phi(y))) \subset T(\alpha)P^0_B(y).
\]
We show that
\[
T(\alpha P^0_A(\phi(y))) \supset T(\alpha)P^0_B(y).
\]
Suppose that \( U \in P^0_B(y) \). Then
\[
\| -\bar{\alpha}T^{-1}((T\alpha)U) - 1\|_{\infty(X)} = \|T(-\bar{\alpha})T(\alpha U) - 1\|_{\infty(Y)} = \| - U - 1\|_{\infty(Y)} = 2,
\]
\[24\]
so $-1 \in \sigma_x(-\bar{\alpha}T^{-1}((T\alpha)U))$. Suppose that $\beta \in \sigma_x(-\bar{\alpha}T^{-1}((T\alpha)U))$. Then we have that $|\beta| = 1$. We also have
\[2 = \|(-\bar{\beta})(-\bar{\alpha}T^{-1}((T\alpha)U)) - 1\|_{\infty(X)}\]
\[= \|T((-\bar{\beta})(-\bar{\alpha}))(T\alpha)U - 1\|_{\infty(Y)} = \|(T((-\bar{\beta})(-\bar{\alpha}))U - 1\|_{\infty(Y)}\]
\[= \|(T(\bar{\beta}))U - 1\|_{\infty(Y)}.
\]
Since $|T(\bar{\beta})| = 1$ on Ch(B), we see that $-1 \in (T(\bar{\beta}))(\text{Ch}(B))$, so by Lemma 5.6 we have that $\beta = -1$. We see that $\sigma_x(-\bar{\alpha}T^{-1}((T\alpha)U)) = \{-1\}$, so $-\bar{\alpha}T^{-1}((T\alpha)U) \in -P^0_A$. On the other hand,
\[|(-\bar{\alpha}T^{-1}((T\alpha)U)) (\phi(y))| = |T^{-1}((T\alpha)U)(\phi(y))|
\[= |(T(\alpha)U)(y)| = |U(y)| = 1,
\]
so
\[(-\bar{\alpha}T^{-1}((T\alpha)U))(\phi(y)) = -1
\]
since $-\bar{\alpha}T^{-1}((T\alpha)U) \in -P^0_A$. Thus
\[-\bar{\alpha}T^{-1}((T\alpha)U) \in -P^0_A(\phi(y)),
\]
so
\[T^{-1}((T\alpha)U) \in \alpha P^0_A(\phi(y)).
\]
Thus $(T\alpha)(U) \in T(\alpha P^0_A(\phi(y)))$. Since $U \in P^0_B(y)$ is arbitrary, we see that $(T\alpha)(P^0_B(y)) \subset T(\alpha P^0_A(\phi(y)))$. It follows that
\[(T\alpha)(P^0_B(y)) = T(\alpha P^0_A(\phi(y))).
\]

\[\square\]

**Theorem 5.12.** Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Suppose that $T$ is a map from $\mathcal{A}^{-1}$ onto $\mathcal{B}^{-1}$. Suppose that the equality
\[\|TfTg - 1\|_{\infty(Y)} = \|fg - 1\|_{\infty(X)}
\]
holds for every $f, g \in \mathcal{A}^{-1}$. Then $(T1)^2 = 1$ and the map $T$ is extended to a map $T_E$ from $\mathcal{A}$ onto $\mathcal{B}$, and there exists a homeomorphism $\phi$ from Ch(B) onto Ch(A) and a clopen subset $K$ of Ch(B) such that the equality
\[T_E f(y) = T1(y) \times \begin{cases} f(\phi(y)), & y \in K \\ \tilde{f}(\phi(y)), & y \in \text{Ch}(B) \setminus K \end{cases}
\]
holds for every $f \in \mathcal{A}$. In particular, $T_E$ is a real-algebra isomorphism. Thus we see that the equality
\[\|T_E fT_E g - 1\|_{\infty(Y)} = \|fg - 1\|_{\infty(X)}
\]
holds for every pair $f$ and $g$ in $\mathcal{A}$.
Proof. Since
\[ \|T1T1 - 1\|_{\infty(Y)} = \|1^2 - 1\|_{\infty(X)} = 0, \]
we see that \((T1)^2 = 1\). First we consider the case where \(T1 = 1\). By Lemma 5.11 we see that \(T\) is injective and \(\|TfTg\|_{\infty(Y)} = \|fg\|_{\infty(X)}\) for every \(f, g \in A^{-1}\). Then by Theorem 5.1 there exists a homeomorphism \(\phi\) from \(Ch(B)\) onto \(Ch(A)\) such that \(|Tf(y)| = |f(\phi(y))|\) holds for every \(f \in A^{-1}\) and every \(y \in Ch(B)\). We show that \(Tf(y) = f(\phi(y))\) if \(y \in K\) and \(Tf(y) = f(\phi(y))\) if \(y \in Ch(B) \setminus K\), where \(K\) is defined as in Definition 5.8.

Suppose that \(y \in K\) and \(Tf(y) \neq f(\phi(y))\). By Lemma 2.1 there exists an \(H \in P_R(\phi(y))\) such that \(\sigma_x(TfH) = \{Tf(y)\}\). Since \(Ti = i\) on \(K\) and \(Ti = -i\) on \(Ch(B) \setminus K\) we see that the closure \(\bar{K}\) of \(K\) in \(Y\) and the closure \(Ch(B) \setminus \bar{K}\) of \(Ch(B) \setminus K\) are disjoint. Thus we may assume that \(|TfH| < |Tf(y)|\) on \(Ch(B) \setminus K\). Put \(h = T^{-1}H\). Then by Lemma 5.11 we see that \(h \in P_A(\phi(y))\). Put \(\alpha = \frac{-f(\phi(y))}{f(\phi(y))}\). Then we have that
\[ \|T\alpha TfTh - 1\|_{\infty(Y)} < |Tf(y)| + 1. \]
(Suppose not; \(\|T\alpha TfTh - 1\|_{\infty(Y)} \geq |Tf(y)| + 1\). Since
\[ \|T\alpha TfTh - 1\|_{\infty(Y)} \leq \|T\alpha\|_{\infty(Y)} \|TfTTh\|_{\infty(Y)} + 1 = |Tf(y)| + 1, \]
we have that
\[ \|T\alpha TfTh - 1\|_{\infty(Y)} = |Tf(y)| + 1, \]
so there exists a \(z \in Ch(B)\) such that
\[ |(T\alpha TfTh)(z) - 1| = |Tf(y)| + 1. \]
Since \(|T\alpha| = 1\) on \(Ch(B)\), we have that
\[ \|T\alpha TfTh\|_{\infty(Y)} = \|T\alpha TfTh\|_{\infty(Ch(B))} = \|TfTTh\|_{\infty(Ch(B))} = \|TfTTh\|_{\infty(Y)} = |Tf(y)|, \]
thus
\[ (T\alpha TfTh)(z) = -|Tf(y)|. \]
Then we see that \(z\) is a point in \(K\). Suppose that \(z \in Ch(B) \setminus K\). Then by the definition of \(H\), \(|(T\alpha TfTh)(z)| < |Tf(y)|\) holds. Since \(|T\alpha| = 1\) we have that
\[ |(T\alpha TfTh)(z) - 1| \leq |(T\alpha TfTh)(z)| + 1 < |Tf(y)| + 1, \]
which is a contradiction proving \(z \in K\). So \(T\alpha(z) = \alpha\) by Lemma 5.10 and thus we have by the equation (5.3) that
\[ \frac{-f(\phi(y))}{|f(\phi(y))|} Tf(z) Th(z) = -|Tf(y)|. \]
So $|Tf(z)Th(z)| = |Tf(y)|$. Since $\sigma_{\pi}(TfTh) = \{Tf(y)\}$, we see that $Tf(z)Th(z) = Tf(y)$, so that $\frac{f(\phi(y))}{f(\phi(y))}Tf(y) = |Tf(y)|$. Then $\frac{f(\phi(y))}{f(\phi(y))}Tf(y) = |Tf(y)|^2$ holds since $|Tf(y)| = |f(\phi(y))|$, so that $f(\phi(y)) = Tf(y)$, which is a contradiction.) Then by Lemma 5.11, there exists an $h' \in \mathcal{P}_A(\phi(y))$ such that $T(\alpha h') = (T\alpha)(Th)$. Thus

$$|Tf(y)| + 1 > \|T\alpha TfTh - 1\|_{\infty(Y)}$$

$$= \|T(\alpha h')Tf - 1\|_{\infty(Y)} = \|\alpha h'f - 1\|_{\infty(X)}$$

$$\geq |\alpha h'(\phi(y))f(\phi(y)) - 1| = |f(\phi(y))| + 1 = |Tf(y)| + 1,$$

which is a contradiction proving that $Tf(y) = f(\phi(y))$.

Suppose that $y \in Ch(B) \setminus K$. We show that $Tf(y) = \frac{f(\phi(y))}{f(\phi(y))}$. Suppose not. By Lemma 2.1 and Lemma 5.11 there exists an $h \in \mathcal{P}_A(\phi(y))$ such that $\sigma_{\pi}(TfTh) = \{Tf(y)\}$. We may assume that $|TfTh| < |Tf(y)|$ on $K$ as the same way as in the case that $y \in K$. Put $\alpha = \frac{-T\phi(y)}{f(\phi(y))}$. Then as in the same way as in the case where $y \in K$, we see that

$$|T\alpha TfTh - 1|_{\infty(Y)} < |Tf(y)| + 1.$$

By Lemma 5.11, there exists an $h' \in \mathcal{P}_A(\phi(y))$ such that $T(\alpha h') = T(\alpha)Th$. Then we have that

$$\|T\alpha TfTh - 1\|_{\infty(Y)} = \|T(\alpha h')Tf - 1\|_{\infty(Y)} = \|\alpha h'f - 1\|_{\infty(X)}$$

$$\geq |\alpha h'(\phi(y))f(\phi(y)) - 1| = |f(\phi(y))| + 1 = |Tf(y)| + 1,$$

which is a contradiction proving that $Tf(y) = \frac{f(\phi(y))}{f(\phi(y))}$. So we see that the equality

$$Tf(y) = \begin{cases} f(\phi(y)), & y \in K, \\ \frac{f(\phi(y))}{f(\phi(y))}, & y \in Ch(B) \setminus K \end{cases}$$

holds for every $f \in \mathcal{A}^{-1}$ if $T1 = 1$.

We consider the general case. Put $\tilde{T} = \frac{T1}{T1}$. By a simple calculation we see that $\tilde{T}$ is a map from $\mathcal{A}^{-1}$ onto $\mathcal{B}^{-1}$ such that

$$\|\tilde{T}f\tilde{T}g - 1\|_{\infty(Y)} = \|fg - 1\|_{\infty(X)}$$

holds for every $f, g \in \mathcal{A}^{-1}$ and $\tilde{T}1 = 1$. Then by the first part of the proof we see that there is a homeomorphism $\phi$ from $Ch(B)$ onto $Ch(A)$ and a clopen subset $K$ of $Ch(B)$ such that the equality

$$\frac{Tf(y)}{T1(y)} = \tilde{T}(f)(y) = \begin{cases} f(\phi(y)), & y \in K, \\ \frac{f(\phi(y))}{f(\phi(y))}, & y \in Ch(B) \setminus K. \end{cases}$$
holds for every \( f \in \mathcal{A}^{-1} \). Let \( I_A \) (resp. \( I_B \)) denote the map \( I_A f = f|_{\text{Ch}(\mathcal{A})} \) (resp. \( I_B f = f|_{\text{Ch}(\mathcal{B})} \)) from \( \mathcal{A} \) (resp. \( \mathcal{B} \)) onto \( \mathcal{A}|_{\text{Ch}(\mathcal{A})} \) (resp. \( \mathcal{B}|_{\text{Ch}(\mathcal{B})} \)). Then \( I_A \) (resp. \( I_B \)) is an algebra isomorphism from \( \mathcal{A} \) (resp. \( \mathcal{B} \)) onto \( \mathcal{A}|_{\text{Ch}(\mathcal{A})} \) (resp. \( \mathcal{B}|_{\text{Ch}(\mathcal{B})} \)). Put a map \( T_0 \) from \( \mathcal{A}|_{\text{Ch}(\mathcal{A})} \) into \( \mathcal{B}|_{\text{Ch}(\mathcal{B})} \) by

\[
T_0(f|_{\text{Ch}(\mathcal{A})})(y) = T_1(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(\mathcal{B}) \setminus K \end{cases}
\]

for \( f|_{\text{Ch}(\mathcal{A})} \in \mathcal{A}|_{\text{Ch}(\mathcal{A})} \). Put

\[
T_E = I_B^{-1} \circ T_0 \circ I_A.
\]

Then it is easy to see that \( T_E \) is a bijection from \( \mathcal{A} \) onto \( \mathcal{B} \) which is an extension of \( T \), and \( \frac{T_E}{T_1} \) is a real-algebra isomorphism from \( \mathcal{A} \) onto \( \mathcal{B} \) which is an extension of \( \frac{T}{T_1} \). By the definition we see that the equality

\[
T_E f(y) = T_1(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(\mathcal{B}) \setminus K \end{cases}
\]

holds for every \( f \in \mathcal{A} \).

Let \( \mathcal{A} \) be a unital commutative Banach algebra. Recall that the spectral radius of \( f \in \mathcal{A} \) is denoted by \( r(f) \). Then by the definition

\[
r(f) = \| \hat{f} \|_{\infty(M_A)}
\]

holds for every \( f \in \mathcal{A} \), where \( \hat{f} \) denotes the Gelfand transformation in \( \mathcal{A} \).

Recall that \( \text{cl} \mathcal{A} \) is the uniform closure of the Gelfand transform of \( \mathcal{A} \) in \( C(M_A) \). If \( \mathcal{A} \) is semisimple, then we may suppose that \( \mathcal{A} \subset \text{cl} \mathcal{A} \). Recall also that the Gelfand transform of \( f \in \mathcal{A} \) is denoted also by \( f \) omitting \( \hat{\cdot} \) for simplicity.

**Corollary 5.13.** Let \( \mathcal{A} \) be a unital semisimple commutative Banach algebra and \( \mathcal{B} \) a unital commutative Banach algebra. Suppose that \( T \) is a map from \( \mathcal{A}^{-1} \) onto \( \mathcal{B}^{-1} \) which satisfies that the equation

\[
r(T f T g - 1) = r(f g - 1)
\]

holds for every pair \( f \) and \( g \) in \( \mathcal{A}^{-1} \). Then \( \mathcal{B} \) is semisimple and \( (T I)^2 = 1 \), and \( T \) is extended to a map \( T_E \) from \( \text{cl} \mathcal{A} \) onto \( \text{cl} \mathcal{B} \) such that \( T_E \mathcal{A} = \mathcal{B} \), and there exist a homeomorphism \( \phi \) from \( \text{Ch}(\text{cl} \mathcal{B}) \) onto \( \text{Ch}(\text{cl} \mathcal{A}) \) and a clopen subset \( K \) of \( \text{Ch}(\text{cl} \mathcal{B}) \) which satisfies that the equation

\[
T_E f(y) = T_1(y) \times \begin{cases} f(\phi(y)), & y \in K \\ \frac{f(\phi(y))}{f(\phi(y))}, & y \in \text{Ch}(\text{cl} \mathcal{B}) \setminus K \end{cases}
\]
holds for every \( f \in \text{cl}A \). In particular, \( T_E \) is a real-algebra isomorphism from \( \text{cl}A \) onto \( \text{cl}B \) and \( T_E(A) = B \); \( A \) is real-algebraically isomorphic to \( B \). Thus the equality

\[
\|T_E f T_E g - 1\|_{\infty(M_B)} = \|fg - 1\|_{\infty(M_A)}
\]

holds for every pair \( f \) and \( g \) in \( \text{cl}A \) and \( T_E((\text{cl}A)^{-1}) = (\text{cl}B)^{-1} \).

**Proof.** First we consider the case where \( B \) is semisimple. Let \( f \in \text{cl}A^{-1} \). Since

\[
r(TfTf^{-1} - 1) = r(ff - 1) = 0
\]

and \( B \) is assumed to be semisimple, we have that \( Tf^{-1} = (Tf)^{-1} \). So

\[
r(TfTg - 1) = r(fg - 1)
\]

holds for every pair \( f \) and \( g \) in \( A^{-1} \). (Recall that we denote the Gelfand transform of \( f \in A \) also by \( f \); omitting \( \hat{\cdot} \), and we suppose that \( A \subset \text{cl}A \subset C(M_A) \), \( B \subset \text{cl}B \subset C(M_B) \), so that such a formula like \( f g \) is well-defined and so on.) We show that \( T \) is extended to a map from \( (\text{cl}A)^{-1} \) onto \( (\text{cl}B)^{-1} \). Note that \( M_A = M_{\text{cl}A} \) and so \( \text{cl}A \cap C(M_A)^{-1} = (\text{cl}A)^{-1} \). Let \( f \in (\text{cl}A)^{-1} \). Then there exists a sequence \( \{f_n\} \) in \( A^{-1} \) such that \( \|f - f_n\|_{\infty(M_A)} \to 0 \) as \( n \to \infty \). Since \( f \) is invertible there exists a positive number \( M \) such that \( \frac{1}{M} < |f_n| < M \) holds on \( M_A \) for every \( f_n \). Thus

\[
\frac{1}{M}|f_n(x) - f_m(x)| \leq |\frac{f_n(x)}{f_m(x)} - 1| \leq M|f_n(x) - f_m(x)|
\]

hold for every \( x \in M_A \), so we have that

\[
\frac{1}{M}r(f_n - f_m) \leq r(\frac{Tf_n}{Tf_m} - 1) \leq Mr(f_n - f_m)
\]

since \( r(\frac{Tf_n}{Tf_m} - 1) = r(\frac{f_n}{f_m} - 1) \). Since \( r(T1T1 - 1) = r(1^2 - 1) = 0 \) we have that \( (T1)^2 = 1 \) and so we have that

\[
r(Tf_m) = r(Tf_mT1) \leq r(Tf_mT1 - 1) + 1 = r(f_m - 1) + 1 \leq M + 2.
\]

It follows that

\[
r(Tf_n - Tf_m) \leq r(Tf_m)r(\frac{Tf_n}{Tf_m} - 1) \leq M(M + 2)r(f_n - f_m),
\]

so that \( \{Tf_n\} \) is a Cauchy sequence in \( B^{-1} \) with respect to the supremum norm on \( M_B \). Thus there is an \( F \in C(M_B) \) with

\[
\|Tf_n - F\|_{\infty(M_B)} \to 0
\]
as $n \to \infty$. In a way similar to the above we see that $r(Tf_m) \le M + 2$ and so that $\frac{1}{M+2} \le |Tf_m|$ holds on $M_B$. Thus we see that $\frac{1}{M+2} \le |F|$ on $M_B$. It follows that $F$ is invertible in $C(M_B)$ and so $F \in (clB)^{-1}$ since $M_{clB} = M_B$. In a routine argument we see that for each $f \in (clA)^{-1}$ $F$ is uniquely determined; it does not depend on the choice of the sequence $\{f_n\}$ which converges to $f$. We define a map $T_e$ from $(clA)^{-1}$ into $(clB)^{-1}$ by $T_eg = F$. We show that $T_e$ is a surjection. Suppose that $F \in (clB)^{-1}$. Then there is a sequence $\{F_n\}$ in $B$ such that $\|F_n - F\|_{\infty(M_B)} \to 0$ as $n \to \infty$. Since $F \in (clB)^{-1}$ and $M_{clB} = M_B$, we may assume that $F_n \in B^{-1}$ for every $n$. Since $TA^{-1} = B^{-1}$, there exists a sequence $\{f_n\}$ in $A^{-1}$ with $Tf_n = F_n$ for every positive integer $n$. As in a way similar to the above we see $\{f_n\}$ is a Cauchy sequence which uniformly converges to some $f \in (clA)^{-1}$. By the definition of $T_e$ we have that $T_ef = F$; $T_e$ is a surjection. Suppose that $f, g \in (clA)^{-1}$. Then there are some $\{f_n\}$ and $\{g_n\}$ in $A^{-1}$ such that $f_n \to f$ and $g_n \to g$ as $n \to \infty$. Then we see that

$$r(Tf_ng_n - 1) = r(f_ng_n - 1)$$

and letting $n \to \infty$ we have that

$$\|T_eT_eg - 1\|_{\infty(M_B)} = \|fg - 1\|_{\infty(M_A)}.$$ 

Applying Theorem 5.12 to the map $T_e$ from $(clA)^{-1}$ onto $(clB)^{-1}$, we see that there exists a clopen subset $K$ of $Ch(clB)$ and $T_e$ is extended to a map $T_E$ from $clA$ onto $clB$ such that the equality

$$T_Ef(y) = T1(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in Ch(clB) \setminus K \end{cases}$$

holds for every $f \in clA$. It follows that $\frac{T_E}{T1}$ is a real-algebra isomorphism from $clA$ onto $clB$. We also see that $\frac{T_E}{T1}(A) = B$. (Since $T_E = T$ on $A^{-1}$ and $T1 \in B^{-1}$ we see that $\frac{T_E}{T1}(A^{-1}) = B^{-1}$). Suppose that $F \in B$. Then there exist an $F_0 \in B^{-1}$ and a complex number $\lambda$ with $F = F_0 + \lambda$. If $\lambda = 0$, then $F = F_0$ and there exists an $f_0 \in A^{-1}$ with $Tf_0 = T1F_0$, so that $\frac{T_E}{T1}(f_0) = F_0 = F$. If $\lambda \neq 0$, then there exist an $f_0 \in A^{-1}$ and $f_\lambda \in A^{-1}$ with $Tf_0 = T1F_0$ and $Tf_\lambda = \lambda T1$. Since $T_E$ is real-linear, we have that $T_E(f_0 + f_\lambda) = T1F$. We see that $\frac{T_E}{T1}(A) = B$.

Finally we consider the general case; $B$ is not assumed to be semisimple. Let $\Gamma$ denote the Gelfand transform on $B$. Then $\Gamma \circ T$ is a function from $A^{-1}$ onto $(\hat{B})^{-1}$, where $\hat{B}$ is the Gelfand transform of $B$. Then by the first part of the proof we see that there exist a homeomorphism $\phi$ from $Ch(cl\hat{B})$ onto $Ch(clA)$ and a clopen subset $K$ of $Ch(cl\hat{B})$ such
that the equality
\[ \Gamma \circ Tf(y) = \Gamma \circ T_1(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(\text{cl} \hat{B}) \setminus K. \end{cases} \]
holds for every \( f \in A^{-1} \). In particular, we see that \( \Gamma \circ T \) is an injection from \( A^{-1} \) onto \( (\hat{B})^{-1} \), so \( \Gamma \) is injective on \( B^{-1} \) and so \( \Gamma \) is injective on \( B \) by a simple calculation. Thus we see that \( B \) is semisimple. Then applying the first case we have the conclusion.

**Corollary 5.14.** Let \( A \) be a unital semisimple commutative Banach algebra and \( B \) a unital commutative Banach algebra. Suppose that \( T \) is a map from \( A^{-1} \) onto \( B^{-1} \) which satisfies that the equality
\[ r(TfTg - 1) = r(fg - 1) \]
holds for every pair \( f \) and \( g \) in \( A^{-1} \). Suppose that there exists a \( \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \) such that \( T\lambda = \lambda \). Then \( B \) is semisimple, \( T \) is extended to a complex-algebra isomorphism \( T_E \) from \( \text{cl} A \) onto \( \text{cl} B \) such that \( T_E A = B \) and \( T_E((\text{cl} A)^{-1}) = ((\text{cl} B)^{-1}, \) and there exists a homeomorphism \( \phi \) from \( \text{Ch}(\text{cl} B) \) onto \( \text{Ch}(\text{cl} A) \) such that the equality
\[ T_E f(y) = f(\phi(y)), \quad y \in \text{Ch}(\text{cl} B) \]
holds for every \( f \in \text{cl} A \).

**Proof.** By Corollary 5.13 and its proof, we see that \( B \) is semisimple and \( (T1)^2 = 1 \), and \( T \) is extended to a map \( T_E \) from \( \text{cl} A \) onto \( \text{cl} B \) such that there exists a homeomorphism \( \phi \) from \( \text{Ch}(\text{cl} B) \) onto \( \text{Ch}(\text{cl} A) \) and a clopen subset \( K \) of \( \text{Ch}(\text{cl} B) \) which satisfies that the equation
\[ T_E f(y) = T_1(y) \times \begin{cases} f(\phi(y)), & y \in K \\ f(\phi(y)), & y \in \text{Ch}(\text{cl} B) \setminus K \end{cases} \]
holds for every \( f \in \text{cl} A \). In particular, \( \frac{T_E}{T_1} \) is a real-algebra isomorphism from \( \text{cl} A \) onto \( \text{cl} B \) such that \( \frac{T_E}{T_1}(A) = B \). We show that \( K = \text{Ch}(\text{cl} B) \). Suppose not. Then there is a \( y \in \text{Ch}(\text{cl} B) \setminus K \). So by the hypothesis,
\[ \lambda = T\lambda(y) = T_1(y)\overline{\lambda(\phi(y))} = \lambda T_1(y), \]
which is a contradiction since \( T_1(y) = 1 \) or \(-1\) and \( \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \). Then we also see that \( T1 = 1 \) since
\[ \lambda = T\lambda = T_E\lambda = T_1\lambda \]
on \( \text{Ch}(\text{cl} B) \).
Corollary 5.15. Let $A$ and $B$ be unital semisimple commutative Banach algebras. Suppose that $T$ is a group homomorphism from $A^{-1}$ onto $B^{-1}$. Then the following are equivalent.

(i) $T$ is isometry with respect to the spectral radius, that is, the equality
$$r(Tf - Tg) = r(f - g)$$
holds for every pair $f$ and $g$ in $A^{-1}$.

(ii) The equality
$$r(Tf - 1) = r(f - 1)$$
holds for every $f \in A^{-1}$.

(iii) There exist a homeomorphism from $\text{Ch(cl}\ B\ \text{cl})$ onto $\text{Ch(cl}\ A\ \text{cl})$ and a clopen subset $K$ of $\text{Ch(cl}\ B\ \text{cl})$ such that the equality
$$Tf(y) = \begin{cases} f(\phi(y)), & y \in K \\ f(\bar{\phi}(y)), & y \in \text{Ch(cl} B \text{cl}) \setminus K \end{cases}$$
holds for every $f \in A^{-1}$.

Thus if one of the above holds, then $T$ is extended to a real-algebra isomorphism from $A$ onto $B$; $A$ is real-algebraically isomorphic to $B$.

Proof. The inclusion (iii) $\rightarrow$ (i) is trivial. Since $T1 = 1$, (i) implies (ii) is also trivial. By Corollary 5.13 we see that (ii) implies (iii). Thus $T$ is extended to a real-algebra isomorphism from $A$ onto $B$ if at least one of (i), (ii) and (iii) holds. \qed

The group isomorphism in Corollary 5.15 is not extended to the complex algebra isomorphism in general, as the following example shows.

Example 5.16. Let $A(\bar{D})$ be the disk algebra on the closed unit disk; the algebra of all complex-valued continuous functions on the closed unit disk $\bar{D}$ which are analytic on the interior of the disk, and $\overline{A(\bar{D})} = \{ f \in C(\bar{D}) | f \in A(\bar{D}) \}$. Put $A = A(\bar{D}) \oplus A(\bar{D})$, the direct sum of two copies of $A(\bar{D})$ and $B = A(\bar{D}) \oplus A(\bar{D})$, the direct sum of $A(\bar{D})$ and $A(\bar{D})$. Then $A$ and $B$ are uniform algebras on $X = \bar{D} \times \{1, 2\}$. Then $T(f \oplus g) = f \oplus \bar{g}$ defined on $A^{-1}$ is a group isomorphisms onto $B^{-1}$ such that
$$\|T(f \oplus g) - 1\|_{\infty(X)} = \|f \oplus g - 1\|_{\infty(X)}.$$ 
On the other hand $A$ is not algebraically isomorphic to $B$ as a complex algebra.

Corollary 5.17. Let $A$ and $B$ be unital semisimple commutative Banach algebras. Suppose that $T$ is a group homomorphism from $A^{-1}$
onto $B^{-1}$ which satisfies that the equality
\[ r(Tf - 1) = r(f - 1) \]
holds for every $f \in A^{-1}$. Suppose that there exists a $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $T\lambda = \lambda$. Then $T$ is extended to a complex-algebra isomorphism $T_E$ from $\text{cl} A$ onto $\text{cl} B$ with $T_E A = B$, and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl} B)$ onto $\text{Ch}(\text{cl} A)$ such that the equality
\[ T_E f(y) = f(\phi(y)), \quad y \in \text{Ch}(\text{cl} B) \]
holds for every $f \in cl A$.

**Proof.** By Corollary 5.15 we see that there exist a homeomorphism from $\text{Ch}(\text{cl} B)$ onto $\text{Ch}(\text{cl} A)$ and a clopen subset $K$ of $\text{Ch}(\text{cl} B)$ such that the equality
\[ Tf(y) = \begin{cases} f(\phi(y)), & y \in K \\ f(\phi(y)), & y \in \text{Ch}(\text{cl} B) \setminus K \end{cases} \]
holds for every $f \in A^{-1}$. We also see that $K = \text{Ch}(\text{cl} B)$. Suppose that $y \in \text{Ch}(\text{cl} B) \setminus K$, then we have that
\[ \lambda = T\lambda(y) = \overline{\lambda(\phi(y))} = \bar{\lambda}, \]
which is a contradiction since $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Thus $K = \text{Ch}(\text{cl} B)$. As in the same way as the proof of Theorem 5.12 $T$ is extended to the desired $T_E$. \qed

**Corollary 5.18.** Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $T$ is a map from $A^{-1}$ onto $B^{-1}$ and $\alpha$ is a non-zero complex number which satisfies that the equality
\[ r(TfTg - \alpha) = r(fg - \alpha) \]
holds for every pair $f$ and $g$ in $A^{-1}$. Then $B$ is semisimple, and $T$ is extended to a map $T_E$ from $\text{cl} A$ onto $\text{cl} B$ such that $T_E A = B$, and there exist an element $\eta \in B^{-1}$ such that $\eta^2 = 1$, a homeomorphism $\phi$ from $\text{Ch}(\text{cl} B)$ onto $\text{Ch}(\text{cl} A)$ and a clopen subset $K$ of $\text{Ch}(\text{cl} B)$ which satisfies that the equality
\[ T_E f(y) = \eta(y) \times \begin{cases} f(\phi(y)), & y \in K \\ \frac{\alpha}{|\alpha|} f(\phi(y)), & y \in \text{Ch}(\text{cl} B) \setminus K \end{cases} \]
holds for every $f \in \text{cl} A$. Furthermore, if $T1 = 1$, then the equality
\[ T_E f(y) = \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(\text{cl} B) \setminus K \end{cases} \]
holds for every \( f \in \text{cl}A \). Thus the equality
\[
\|T_{E}fT_{E}g - \alpha\|_{\infty(M_{B})} = \|fg - \alpha\|_{\infty(M_{A})}
\]
holds for every pair \( f \) and \( g \) in \( \text{cl}A \), and \( T_{E}((\text{cl}A)^{-1}) = (\text{cl}B)^{-1} \) holds. Furthermore, if \( T_{1} = 1 \) and \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) such that \( T\lambda = \lambda \), then the equality
\[
T_{E}(f) = f(\phi(y)), \quad y \in \text{Ch(clB)}
\]
holds for every \( f \in \text{cl}A \). In this case, \( T_{E} \) is extended to a complex-algebra isomorphism from \( \text{cl}A \) onto \( \text{cl}B \) with \( T_{E}A = B \).

**Proof.** Let \( \beta \) be a complex number with \( \beta^{2} = \alpha \). Put a function \( T_{\beta} \) from \( A^{-1} \) into \( B^{-1} \) by
\[
T_{\beta}f = \frac{1}{\beta}T(\beta f) \quad f \in A^{-1}.
\]
Then by a simple calculation \( T_{\beta}A^{-1} = B^{-1} \). Since \( T_{\beta}fT_{\beta}g = \frac{1}{\alpha}T(\beta f)T(\beta g) \), we have that
\[
r(T_{\beta}fT_{\beta}g - 1) = \frac{1}{|\alpha|}r(T(\beta f)T(\beta g) - \alpha)
\]
\[
= \frac{1}{|\alpha|}r(\beta f + \alpha) = r(fg - 1).
\]
Then by Corollary 5.13, \( B \) is semisimple and \( (T_{\beta}1)^{2} = 1 \). We also see by Corollary 5.13 that \( T_{\beta} \) is extended to a function \( (T_{\beta})_{E} \) from \( \text{cl}A \) onto \( \text{cl}B \) and that there exists a homeomorphism \( \phi \) from \( \text{Ch(clB)} \) onto \( \text{Ch(clA)} \) and a clopen subset of \( \text{Ch(clB)} \) such that the equation
\[
(T_{\beta})_{E}f(y) = T_{\beta}1(y) \times \begin{cases} 
    f(\phi(y)), & y \in K, \\
    \overline{f(\phi(y))}, & y \in \text{Ch(clB)} \setminus K
\end{cases}
\]
holds for every \( f \in \text{cl}A \). Put \( \eta = T_{\beta}1 \) and put
\[
T_{E}f = \beta(T_{\beta})_{E}(\frac{f}{\beta})
\]
for \( f \in \text{cl}A \). By a simple calculation we see that \( T_{E} \) is an extension of \( T \) which maps from \( \text{cl}A \) onto \( \text{cl}B \) and we have that
\[
T_{E}f(y) = \eta(y) \times \begin{cases} 
    f(\phi(y)), & y \in K, \\
    \frac{\alpha}{|\alpha|}f(\phi(y)), & y \in \text{Ch(clB)} \setminus K
\end{cases}
\]
since \( \frac{\beta}{\beta} = \frac{\alpha}{|\alpha|} \).
Suppose that $T1 = 1$. Then by the above equation we have that $1 = \eta$ on $K$ and $1 = \frac{\alpha}{|\alpha|} \eta$ on $\text{Ch}(\text{cl}B) \setminus K$. Thus we see that the equality

\[(5.4) \quad T_E f(y) = \begin{cases} f(\phi(y)), & y \in K, \\ \frac{f(\phi(y))}{|\alpha|}, & y \in \text{Ch}(\text{cl}B) \setminus K. \end{cases}\]

holds for every $f \in \text{cl}A$.

Furthermore suppose that $T1 = 1$, and $\alpha \in \mathbb{C} \setminus \mathbb{R}$ or $T\lambda = \lambda$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. First we consider the case where $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Since $B$ is semisimple and $T1 = 1$ we have that

\[r(T\alpha - \alpha) = r(T\alpha T1 - \alpha) = r(\alpha1 - \alpha) = 0,\]

so $T\alpha = \alpha$ on $\text{Ch}(\text{cl}B)$. On the other hand by putting $f = \alpha$ in the above equation $[5.4]$ we have that

\[T_E \alpha(y) = \begin{cases} \alpha, & y \in K, \\ |\alpha|, & y \in \text{Ch}(\text{cl}B) \setminus K. \end{cases}\]

Since $\alpha \in \mathbb{C} \setminus \mathbb{R}$ we see that $K = \text{Ch}(\text{cl}B)$ and so the equality

\[T_E f(y) = f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)\]

holds for every $f \in \text{cl}A$. Next we consider the case where $T\lambda = \lambda$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then by the above equation $[5.4]$ we have that

\[T\lambda(y) = T_E \lambda(y) = \begin{cases} \lambda, & y \in K, \\ \lambda, & y \in \text{Ch}(\text{cl}B) \setminus K. \end{cases}\]

It follows that $K = \text{Ch}(\text{cl}B)$ since $T\lambda = \lambda$. So we have that

\[T_E f(y) = f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)\]

holds for every $f \in \text{cl}A$. \qed

**Corollary 5.19.** Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $T$ is a group homomorphism from $A^{-1}$ onto $B^{-1}$ which satisfies that there exists a nonzero complex number $\alpha$ such that the equality

\[r(Tf - \alpha) = r(f - \alpha)\]

holds for every $f \in A^{-1}$. Then $B$ is semisimple, $T$ is extended to a real-algebra isomorphism $T_E$ from $\text{cl}A$ onto $\text{cl}B$ with $T_E A = B$ and $T_E((\text{cl}A)^{-1}) = (\text{cl}B)^{-1}$, and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ and a clopen subset $K$ of $\text{Ch}(\text{cl}B)$ such that

\[T_E f(y) = \begin{cases} f(\phi(y)), & y \in K, \\ \frac{f(\phi(y))}{f(\phi(y))}, & y \in \text{Ch}(\text{cl}B) \setminus K. \end{cases}\]
holds for every \( f \in \text{cl}A \). Furthermore if \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists a \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) such that \( T\lambda = \lambda \), then

\[
T_E f(y) = f(\phi(y)), \quad y \in \text{Ch(cl}B)
\]

holds for every \( f \in \text{cl}A \); In this case \( T_E \) is a complex-algebra isomorphism.

**Proof.** Since \( T \) is a group homomorphism we see that

\[
r(TfTg - \alpha) = r(fg - \alpha)
\]

holds for every \( f \) and \( g \) in \( A^{-1} \). Then \( B \) is semisimple by Corollary \ref{cor:5.18}. We also see by Corollary \ref{cor:5.18} that \( T \) is extended to a real-algebra isomorphism \( T_E \) from \( \text{cl}A \) onto \( \text{cl}B \) with \( T_E A = B \), and there exist a homeomorphism \( \phi \) from \( \text{Ch(cl}B) \) onto \( \text{Ch(cl}A) \) and a clopen subset \( K \) of \( \text{Ch(cl}B) \) which satisfies that the equality

\[
T_E f(y) = \begin{cases} 
  f(\phi(y)), & y \in K, \\
  f(\phi(y)), & y \in \text{Ch(cl}B) \setminus K
\end{cases}
\]

holds for every \( f \in \text{cl}A \). Note that \( T1 = 1 \) since \( T \) is a group homomorphism from \( A^{-1} \) onto \( B^{-1} \).

Suppose that \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists a \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) such that \( T\lambda = \lambda \). Then by Corollary \ref{cor:5.18} we see that the equality

\[
T_E f(y) = f(\phi(y)), \quad y \in \text{Ch(cl}B)
\]

holds for every \( f \in \text{cl}A \). \( \square \)

6. **Non-symmetric multiplicatively spectrum and peripheral spectrum-preserving maps between invertible groups**

In this section we consider non-symmetric multiplicatively (peripheral) spectrum-preserving maps. We say that a map \( T \) from the invertible group \( A^{-1} \) of a unital commutative Banach algebra \( A \) into the invertible group \( B^{-1} \) of a unital commutative Banach algebra \( B \) is non-symmetric multiplicatively (resp. peripheral) spectrum-preserving if there exists a nonzero complex number \( \alpha \) such that

\[
\sigma(TfTg - \alpha) = \sigma(fg - \alpha) \\
(\text{resp.} \sigma(\pi(TfTg - \alpha) = \sigma(\pi(fg - \alpha))
\]

holds for every pair \( f \) and \( g \) in \( A^{-1} \). In this section we show, under some additional assumption, that non-symmetric multiplicatively peripheral spectrum-preserving maps from \( A^{-1} \) onto \( B^{-1} \) are extended to algebra isomorphisms from \( A \) onto \( B \).
Corollary 6.1. Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $T$ is a map from $A^{-1}$ onto $B^{-1}$ which satisfies that there exists a nonzero complex number $\alpha$ such that

$$\sigma_\pi(TfTg - \alpha) \cap \sigma_\pi(fg - \alpha) \neq \emptyset$$

holds for every pair $f$ and $g$ in $A^{-1}$. Then $B$ is semisimple, and $T$ is extended to a map $T_E$ from $\text{cl}A$ onto $\text{cl}B$ with $T_E A = B$, and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ such that the equality

$$T_E f(y) = T1(y) f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$

holds for every $f \in \text{cl}A$. In particular, $T_E$ is extended to a complex-algebra isomorphism $\frac{T_E}{T}$ from $\text{cl}A$ onto $\text{cl}B$ with $\frac{T_E}{T}(A) = B$.

Proof. First we consider the case where $\alpha = 1$. We have that the equality

$$r(TfTg - 1) = r(fg - 1), \quad f, g \in A^{-1}$$

holds since

$$\sigma_\pi(TfTg - 1) \cap \sigma_\pi(fg - 1) \neq \emptyset$$

holds for every pair $f$ and $g$ in $A^{-1}$. Thus by Corollary 5.13 we see that $B$ is semisimple, and $T$ is extended to the map $T_E$ from $\text{cl}A$ onto $\text{cl}B$, and there exist a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ and a clopen subset $K$ of $\text{Ch}(\text{cl}B)$ such that the equality

$$T_E f(y) = T1(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(\text{cl}B) \setminus K \end{cases}$$

holds for every $f \in \text{cl}A$, where $(T1)^2 = 1$. Let $\lambda \in \sigma_\pi(f)$ for an $f \in A^{-1}$ (resp. $B^{-1}$). Then there exists an $x \in M_A$ (resp. $M_B$) with $f(x) = \lambda$. Since $|\lambda| = \|f\|_{\infty(Y)}$ (resp. $|\lambda| = \|f\|_{\infty(X)}$), we have that $f^{-1}(\lambda)$ is a peak set for $\text{cl}A$ (resp. $\text{cl}B$). Thus there exists an $x_0 \in \text{Ch}(\text{cl}A) \cap f^{-1}(\lambda)$ (resp. $x_0 \in \text{Ch}(\text{cl}B) \cap f^{-1}(\lambda)$) by Corollary 2.4.6 in [11]. It follows that $\lambda = f(x_0) \in f(\text{Ch}(\text{cl}A))$ (resp. $\lambda = f(x_0) \in f(\text{Ch}(\text{cl}B))$), so that $\sigma_\pi(f) \subset f(\text{Ch}(\text{cl}A))$ (resp. $\sigma_\pi(f) \subset f(\text{Ch}(\text{cl}B))$) holds for every $f \in A^{-1}$ (resp. $B^{-1}$).

We show that $K = \text{Ch}(\text{cl}B)$. Suppose not; There exists a $y_0 \in \text{Ch}(\text{cl}B) \setminus K$. By the definition of $K$ we have that $K \cap \text{Ch}(\text{cl}B) \setminus K = \emptyset$, where $\cdot$ denotes the closure in $M_B$. Thus there exists a $U \in B^{-1}$ such that $U(y_0) = i$, $|U| < \frac{1}{3}$ on $K$, and $\text{Im}U > 0$ on $\text{Ch}(\text{cl}B)$. Then there
exists a $u \in A^{-1}$ with $Tu = T_E u = \frac{u}{T_1}$ since $TA^{-1} = B^{-1}$, so

$$U(y) = Tu(y) T_1(y) = \begin{cases} u \circ \phi(y), & y \in K \\ \overline{u \circ \phi(y)}, & y \in \text{Ch}(\text{cl}B) \setminus K \end{cases}$$

since $(T_1)^2 = 1$. Thus

$$\sigma(\pi(TuT_1 - 1)) = \sigma(\pi(U - 1))$$

$$\subset U(\text{Ch}(\text{cl}B)) - 1 \subset \{z \in \mathbb{C} : \text{Im}z > 0\}.$$ 

Since

$$i = U(y_0) = \overline{u \circ \phi(y_0)},$$

we have $u(\phi(y_0)) = -i$. Since $U = \overline{u \circ \phi}$ on $\text{Ch}(\text{cl}B) \setminus K$, we have $\text{Im}u < 0$ on $\phi(\text{Ch}(\text{cl}B) \setminus K)$. Since $U = u \circ \phi$ on $K$, we have that $|u| < \frac{1}{\sqrt{\beta}}$ on $\phi(K)$. Since $(u - 1)(\phi(y_0)) = -i - 1$, we have

$$\sigma(u - 1) \subset \{z \in \mathbb{C} : |z| \geq \sqrt{2}\}.$$ 

Thus $(u - 1)(\phi(K)) \cap \sigma(u - 1) = \emptyset$. It follows that

$$\sigma(u - 1) \subset \{z \in \mathbb{C} : \text{Im}z < 0\}.$$ 

So we see that

$$\sigma(TuT_1 - 1) \cap \sigma(u - 1) = \emptyset,$$

which is a contradiction proving that $K = \text{Ch}(\text{cl}B)$. It follows that the equality

$$T_E f(y) = T_1(y) f(\phi(y))$$

holds for every $f \in \text{cl}A$ and $y \in \text{Ch}(\text{cl}B)$.

Finally we consider the general case for $\alpha$. Let $\beta$ be a complex number with $\beta^2 = \alpha$. Put a function $T_\beta$ from $A^{-1}$ onto $B^{-1}$ by $T_\beta f = \frac{1}{\beta} T(\beta f)$ for $f \in A^{-1}$. Then $T_\beta$ is well-defined and we see by a simple calculation that $T_\beta A^{-1} = B^{-1}$. Then we see that the equalities

$$\sigma(T_\beta fT_\beta g - 1) = \sigma(\frac{1}{\alpha} T(\beta f)T(\beta g) - 1)$$

$$= \frac{1}{\alpha} \sigma(T(\beta f)T(\beta g) - \alpha) = \sigma(fg - 1)$$

hold for every pair $f$ and $g$ in $A^{-1}$. Thus by the first part of the proof we see that $B$ is semisimple, and $T_\beta$ is extended to a map $(T_\beta)_E$ from $\text{cl}A$ onto $\text{cl}B$, and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ such that the equality

$$(T_\beta)_Ef(y) = T_\beta 1(y) f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$
holds for every $f \in \text{cl}A$. Put $T_E : \text{cl}A \to \text{cl}B$ by $T_E f = \beta(T_\beta E(f))$ for $f \in \text{cl}A$. Then since we see that the equality

$$T_E f(y) = \beta(T_\beta E(f))(y) = T_\beta 1(y) f(\phi(y)), \quad y \in \text{Ch(cl}B)$$

holds for every $f \in \text{cl}A$. Then we have

$$T 1(y) = T_E 1(y) = T_\beta 1(y), \quad y \in \text{Ch(cl}B)$$

holds. Thus we see that $T_\beta 1 = T 1$ and the conclusion holds. \qed

**Corollary 6.2.** Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $T$ is a group homomorphism from $A^{-1}$ onto $B^{-1}$ which satisfies that there exists a non-zero complex number $\alpha$ such that

$$\sigma_\pi(T f - \alpha) \cap \sigma_\pi(f - \alpha) \neq \emptyset$$

holds for every $f$ in $A^{-1}$. Then $B$ is semisimple, and $T$ is extended to a complex-algebra isomorphism $T_E$ from $\text{cl}A$ onto $\text{cl}B$ with $T_E A = B$, and there exists a homeomorphism $\phi$ from $\text{Ch(cl}B)$ onto $\text{Ch(cl}A)$ such that the equality

$$T_E f(y) = f(\phi(y)), \quad y \in \text{Ch(cl}B)$$

holds for every $f \in \text{cl}A$.

**Proof.** Since $T$ is a group homomorphism we see that

$$\sigma_\pi(T fT g - \alpha) \cap \sigma_\pi(fg - \alpha) = \sigma_\pi(T(fg) - \alpha) \cap \sigma_\pi(fg - \alpha) \neq \emptyset$$

holds for every pair $f$ and $g$ in $A^{-1}$. Thus by Corollary 6.1 $T$ is extended to a map $T_E$ from $\text{cl}A$ onto $\text{cl}B$ with $T_E A = B$, and there exists a homeomorphism $\phi$ from $\text{Ch(cl}B)$ onto $\text{Ch(cl}A)$ such that the equality

$$T_E f(y) = T 1(y) f(\phi(y)), \quad y \in \text{Ch(cl}B)$$

holds for every $f \in \text{cl}A$. Since $T$ is a group homomorphism from $A^{-1}$ onto $B^{-1}$ we have that $T 1 = 1$, so we conclude that the equation

$$T_E f(y) = f(\phi(y)), \quad y \in \text{Ch(cl}B)$$

holds for every $f \in \text{cl}A$. \qed

**7. Surjections between commutative Banach algebras**

Let $A$ and $B$ be unital semisimple commutative Banach algebras. In this section we consider the maps from $A$ onto $B$, which satisfy the similar conditions for maps from $A^{-1}$ onto $B^{-1}$ in the previous sections. Multiplicatively spectrum-preserving maps are initiated by Molnár [9], Rao and Roy [12] and Hatori, Miura and Takagi [4] extended the results of Molnár for uniform algebras. Luttman and Tonev [8]
extended the results of Rao and Roy and Hatori, Miura and Takagi (for uniform algebras) in the case where the maps between uniform algebras are multiplicatively peripheral spectrum-preserving. Inspired by the theorem of Luttman and Tonev we have considered the following question.

**Question 7.1.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras and $T$ is a map from $\mathcal{A}$ onto $\mathcal{B}$. Suppose that

$$\|TfTg + 1\|_\infty(Y) = \|fg + 1\|_\infty(X)$$

holds for every pair $f$ and $g$ in $\mathcal{A}$ and $T\lambda = \lambda$ for every complex number $\lambda$. Does it follow that $T$ is an algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$?

In this section we give a complete solution to the above question in more general form (cf. Theorem 7.4 and Corollary 7.5). We also give a generalization of a theorem of Luttman and Tonev (cf. Corollary 7.3.)

**Theorem 7.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively and $S$ a map from $\mathcal{A}$ onto $\mathcal{B}$. Suppose that the equality $\|SfSg\|_\infty(Y) = \|fg\|_\infty(X)$ holds for every pair $f$ and $g$ in $\mathcal{A}$. Then there exists a homeomorphism $\phi$ from $\text{Ch}(\mathcal{B})$ onto $\text{Ch}(\mathcal{A})$ such that the equality

$$|Sf(y)| = |f(\phi(y))|,$$

holds for every $f \in \mathcal{A}$.

Note that $S$ need not be injective.

**Proof.** We can prove Theorem 7.2 in a way similar to the proof of Theorem 3.1 and we only show a sketch of the proof.

In a way similar in the proof of Theorem 3.1 we see that $|S1(y)| = 1$ for every $y \in \text{Ch}(\mathcal{B})$. Thus we see that the equality $\|Sf\|_\infty(Y) = \|f\|_\infty(X)$ holds for every $f \in \mathcal{A}$. Let $y \in \text{Ch}(\mathcal{B})$ and put

$$L_y = \{x \in X : |f(x)| = 1 \text{ for every } f \in \mathcal{A} \} \cap \{y \in \text{Ch}(\mathcal{B}) : \|Sf\|_\infty(Y) = 1 = \|Sf\|_\infty(Y)\}.$$

Then $L_y$ is a singleton whose element is a point in $\text{Ch}(\mathcal{A})$. Put a function $\phi$ from $\text{Ch}(\mathcal{B})$ into $\text{Ch}(\mathcal{A})$ by $\phi(y) = \phi(y)$ is the unique element of $L_y$. Then in the same way as in the proof of Theorem 3.1 we see that $|Sf(y)| = |f(\phi(y))|$ holds for every $f \in \mathcal{A}$ and $y \in \text{Ch}(\mathcal{B})$ if $Sf(y) \neq 0$ and $f(\phi(y)) \neq 0$. We show that $|Sf(y)| = |f(\phi(y))|$ holds even if $Sf(y) = 0$ or $f(\phi(y)) = 0$. Suppose that $Sf(y) = 0$. Then for every positive $\varepsilon$ there exists an $H_\varepsilon \in P_0^\mathcal{B}(y)$ such that $\|SfH_\varepsilon\|_\infty(Y) < \varepsilon$. 

Since $S$ is a surjection there is an $h_ε ∈ A$ with $Sh_ε = H_ε$. Then by the definition of $φ(y)$, $|h_ε(φ(y))| = 1$ holds since

$$Sh_ε(y) = H_ε(y) = 1 = \|H_ε\|_∞(Y) = \|Sh_ε\|_∞(Y).$$

Thus we have that

$$ε > \|SfH_ε\|_∞(Y) = \|fh_ε\|_∞(X) ≥ |f(φ(y))h_ε(φ(y))| = |f(φ(y))|,$$

so the equalities $f(φ(y)) = 0 = Sf(y)$ holds since $ε$ is arbitrary. Suppose that $f(φ(y)) = 0$. Then for every positive $ε$, there exists a $u_ε ∈ P^0_A(φ(y))$ such that $∥fu_ε∥_∞(X) < ε$. We see that $|Su_ε(y)| = 1$.

Suppose not. Then $|Su_ε(y)| < 1$ since $∥Su_ε∥_∞(Y) = ∥u_ε∥_∞(X) = 1$. So there exists an $H ∈ P^0_B(y)$ with $∥Su_εH∥_∞(Y) < 1$. Since $S$ is a surjection there is an $h ∈ A$ with $Sh = H$. Then by the definition of $φ(y)$ we see that $|h(φ(y))| = 1$ since $Sh(y) = 1 = \|Sh\|_∞(Y)$. It follows that

$$1 > \|Su_εH\|_∞(Y) = \|u_εh\|_∞(X) ≥ |u_ε(φ(y))h(φ(y))| = |u_ε(φ(y))|,$$

which is a contradiction proving that $|Su_ε(y)| = 1$. Then we have that

$$ε > ∥fu_ε∥_∞(X) = ∥SfSu_ε∥_∞(Y) ≥ |Sf(y)Su_ε(y)| = |Sf(y)|,$$

and so we have that $Sf(y) = 0 = f(φ(y))$ since $ε$ is arbitrary. We conclude that the equality

$$|Sf(y)| = |f(φ(y))|, \quad y ∈ Ch(ℬ)$$

holds for every $f ∈ A$. In the same way as in the proof of Theorem 3.1 we see that $φ$ is continuous.

Next let $x ∈ Ch(A)$ and put

$$K_x = \{y ∈ Y : |Sf(y)| = 1 \text{ for every } f ∈ A \text{ with } |f(x)| = 1 = ∥f∥_∞(X)\}.$$

In a way similar in the proof of Theorem 3.1 we see that $K_x$ is a singleton which consists of a point in $Ch(ℬ)$. Put a function $ψ$ from $Ch(A)$ into $Ch(ℬ)$ such that $ψ(x)$ is the unique element in $K_x$. In a way similar in the proof of Theorem 3.1 and the first part of the proof we see that the equality

$$|Sf(ψ(x))| = |f(x)|, \quad x ∈ Ch(A)$$

holds for every $f ∈ A$ and $ψ$ is continuous on $Ch(A)$. Again in a way similar in the proof of Theorem 3.1 we see that $φ ∘ ψ$ and $ψ ∘ φ$ are identity functions on $Ch(A)$ and $Ch(ℬ)$ respectively, so that $φ$ is a homeomorphism from $Ch(ℬ)$ onto $Ch(A)$.

The following is a generalization of a theorem of Luttman and Tonev 8 and it is related to Corollary 3 in [7].
Corollary 7.3. Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Suppose that $S$ is a map from $\mathcal{A}$ onto $\mathcal{B}$ such that the inclusion
\[
\sigma_\pi(SfSg) \subset \sigma_\pi(fg)
\]
holds for every pair $f$ and $g$ in $\mathcal{A}$. Then $(S1)^2 = 1$ and there exists a homeomorphism $\phi$ from $\text{Ch}(\mathcal{B})$ onto $\text{Ch}(\mathcal{A})$ such that the equality
\[
Sf(y) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\mathcal{B})
\]
holds for every $f \in \mathcal{A}$. In particular, $\frac{S}{S1}$ is an isometrical algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Proof. A proof is similar to that of Corollary 4.1 and we sketch a proof. First we consider the case where $S1 = 1$. By the inclusion $\sigma_\pi(SfSg) \subset \sigma_\pi(fg)$, we have that $\|SfSg\|_{\infty(Y)} = \|fg\|_{\infty(X)}$ holds for every pair $f$ and $g$ in $\mathcal{A}$. Thus by Theorem 7.2 there exists a homeomorphism $\phi$ from $\text{Ch}(\mathcal{B})$ onto $\text{Ch}(\mathcal{A})$ such that the equality
\[
(7.1) \quad |Sf(y)| = |f(\phi(y))|, \quad y \in \text{Ch}(\mathcal{B})
\]
holds for every $f \in \mathcal{A}$. We show that the equality
\[
Sf(y) = f(\phi(y)), \quad y \in \text{Ch}(\mathcal{B})
\]
holds for every $f \in \mathcal{A}$. If $f(\phi(y)) \neq 0$, then the proof is similar to that in the proof of Corollary 4.1. If $f(\phi(y)) = 0$, then $Sf(y) = f(\phi(y))$ holds by the above equation (7.1).

Finally we consider the general case; We do not assume $S1 = 1$. In a way similar to the proof of Corollary 4.1 we see that $(S1)^2 = 1$. So $S1 \in \mathcal{B}^{-1}$. Put a map $\tilde{S}$ from $\mathcal{A}$ into $\mathcal{B}$ by $\tilde{S}f = \frac{Sf}{S1}$. Then $\tilde{S}$ is a surjection and the inclusion
\[
\sigma_\pi(\tilde{S}f\tilde{S}g) \subset \sigma_\pi(fg)
\]
holds for every pair $f$ and $g$ in $\mathcal{A}$ since $(S1)^2 = 1$. By the first part of the proof we see that the conclusion holds.

Theorem 7.4. Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $S$ is a map from $A$ onto $B$ which satisfies that there exists a non-zero complex number $\alpha$ such that the equality
\[
r(SfSg - \alpha) = r(fg - \alpha)
\]
holds for every pair $f$ and $g$ in $A$. Then $B$ is semisimple and there exist an $\eta \in B$ with $\eta^2 = 1$, a homeomorphism $\phi$ from $\text{Ch}(\text{cl} B)$ onto
Ch(clA) and a clopen subset K of Ch(clB) such that the equality

\[ Sf(y) = \eta(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ \frac{\alpha}{|\alpha|}f(\phi(y)), & y \in \text{Ch(clB) \setminus K} \end{cases} \]

holds for every \( f \in A \). Furthermore if \( S1 = 1 \), then \( \eta = 1 \). Furthermore if \( S1 = 1 \), and \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists a \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) such that \( S\lambda = \lambda \), then \( K = \text{Ch(clB)} \) and the equality

\[ Sf(y) = f(\phi(y)), \quad y \in \text{Ch(clB)} \]

holds for every \( f \in A \).

Proof. First we consider the case where \( B \) is semi-simple. We show that \( SA^{-1} = B^{-1} \). Let \( f \in A^{-1} \). Put \( g = \alpha f^{-1} \). Then we have that

\[ 0 = r(fg - \alpha) = r(SfSg - \alpha), \]

so \( SfSg = \alpha \) for \( B \) is semisimple. Since \( \alpha \) is a non-zero complex number we see that \( Sf \in B^{-1} \); we have proved that \( SA^{-1} \subset B^{-1} \). Suppose that \( F \in B^{-1} \). Since \( SA = B \), there exist an \( f \) and a \( g \) in \( A \) with \( Sf = F \) and \( Sg = \alpha F^{-1} \). Then we have that

\[ 0 = r(SfSg - \alpha) = r(fg - \alpha), \]

so \( fg = \alpha \) for \( A \) is semisimple. Thus we see that \( f \in A^{-1} \). It follows that \( SA^{-1} = B^{-1} \). Applying Corollary 5.18 to \( S|A^{-1} \) we see that there corresponds the extended map \( (S|A^{-1})E \) from clA onto clB such that the equality

\[ \|(S|A^{-1})E(f)(S|A^{-1})_Eg - \alpha\|_{\infty(M_B)} = \|fg - \alpha\|_{\infty(M_A)} \]  

(7.2)

holds for every pair \( f \) and \( g \) in clA. We also see by Corollary 5.18 that there exist an \( \eta \in B^{-1} \) with \( \eta^2 = 1 \), a homeomorphism \( \phi \) from \( \text{Ch(clB)} \) onto \( \text{Ch(clA)} \) and a clopen subset \( K \) of \( \text{clB} \) which satisfy that

\[ ((S|A^{-1})E(f))(y) = \eta(y) \times \begin{cases} f(\phi(y)), & y \in K, \\ \frac{\alpha}{|\alpha|}f(\phi(y)), & y \in \text{Ch(clB) \setminus K} \end{cases} \]

If \( S1 = 1 \), and \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists a \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) with \( S\lambda = \lambda \), then

\[ ((S|A^{-1})E(f))(y) = f(\phi(y)), \quad y \in \text{Ch(clB)} \]

holds for every \( f \in A \) by Corollary 5.18.

We show that \((S|A^{-1})_E = S\) on \( A \). By the definition of \((S|A^{-1})_E\) we have that for every \( g \in (\text{clA})^{-1} \) there exists a sequence \( \{g_n\} \) in \( A^{-1} \) with \( \|g - g_n\|_{\infty(M_A)} \to 0 \) as \( n \to \infty \), and the equality

\[ \|(S|A^{-1})_Eg - Sg_n\|_{\infty(M_B)} \to 0 \]
holds as \( n \to \infty \). Thus for every \( f \in A \) and \( g \in (\text{cl}A)^{-1} \) the equality
\[
\|SfSg_n - \alpha\|_{\infty(M_B)} = \|fg_n - \alpha\|_{\infty(M_A)}
\]
holds, where \( \{g_n\} \subset A^{-1} \) and \(\|g - g_n\|_{\infty(M_A)} \to 0 \) as \( n \to \infty \). Letting \( n \to \infty \) we see that the equality
\[
(7.3) \quad \|Sf(SA^{-1})_Eg - \alpha\|_{\infty(M_B)} = \|fg - \alpha\|_{\infty(M_A)}
\]
holds for every \( f \in A \) and \( g \in (\text{cl}A)^{-1} \). Since \( (SA^{-1})_E((\text{cl}A)^{-1}) = (\text{cl}B)^{-1} \) holds we see that the equality
\[
\|SfG - \alpha\|_{\infty(M_B)} = \|(SA^{-1})_EfG - \alpha\|_{\infty(M_B)}
\]
holds for every \( f \in A \) and \( G \in (\text{cl}B)^{-1} \) by the equations (7.2) and (7.3). Applying peaking function argument as before it follows that
\[
Sf(y) = (SA^{-1})_Ef(y), \quad y \in \text{Ch}(\text{cl}B)
\]
holds for every \( f \in A \). We show a proof for a convenience. Substituting \( G \) by \( nG \) for positive integer \( n \), we have that
\[
\|Sf(nG) - \alpha\|_{\infty(M_B)} = \|(SA^{-1})_Ef(nG) - \alpha\|_{\infty(M_B)},
\]
so
\[
\|SfG - \frac{\alpha}{n}\|_{\infty(M_B)} = \|(SA^{-1})_EfG - \frac{\alpha}{n}\|_{\infty(M_B)},
\]
and letting \( n \to \infty \) we have
\[
\|SfG\|_{\infty(M_B)} = \|(SA^{-1})_EfG\|_{\infty(M_B)}
\]
holds for every pair \( f \in A \) and \( G \in (\text{cl}B)^{-1} \). Suppose that \( f \in A \) and \( y \in \text{Ch}(\text{cl}B) \). If \( Sf(y) = 0 \), then there exists a sequence \( \{G_n\} \) in \( P_{\text{cl}B}^0(y) \) with
\[
\|SfG_n\|_{\infty(M_B)} \to 0,
\]
as \( n \to \infty \) so that
\[
\|(SA^{-1})_EfG_n\|_{\infty(M_B)} \to 0.
\]
as \( n \to \infty \). It follows that \( (SA^{-1})_Ef(y) = 0 \). In the same way we see that \( Sf(y) = 0 \) if \( (SA^{-1})_Ef(y) = 0 \). Suppose that \( Sf(y) \neq 0 \) and \( (SA^{-1})_Ef(y) \neq 0 \). Applying Lemma 2.1 we see that there exists a \( G \in P_{\text{cl}B}^0(y) \) such that
\[
(7.4) \quad \sigma_\pi(SfG) = \{Sf(y)\}, \quad \sigma_\pi((SA^{-1})_EfG) = \{(SA^{-1})_Ef(y)\}.
\]
Thus we see that
\[
(7.5) \quad |Sf(y)| = \|SfG\|_{\infty(M_B)} = \|(SA^{-1})_EfG\|_{\infty(M_B)} = |(SA^{-1})_Ef(y)|.
\]
Put \( \mu = -\frac{\alpha Sf(y)}{|Sf(y)|} \). Then we have
\[
\|\mu SfG - \alpha\|_{\infty(M_B)} = |\alpha| \|\frac{Sf(y)}{|Sf(y)|} SfG + 1\|_{\infty(M_B)} = |\alpha| (|Sf(y)| + 1).
\]

On the other hand
\[
\|\mu SfG - \alpha\|_{\infty(M_B)} = \|\mu (S|A^{-1})_E fG - \alpha\|_{\infty(M_B)}
= |\alpha| \frac{Sf(y)}{|Sf(y)|} (S|A^{-1})_E fG + 1\|_{\infty(M_B)}.
\]

Applying the equations (7.4) and (7.5) we see that \( Sf = (S|A^{-1})_E fG \). Thus we see that \( B \) is semisimple.

Finally we consider the general case, where we do not assume that \( B \) is semisimple. Let \( \Gamma \) be the Gelfand transform on \( B \). By applying the conclusion of the first part of the proof, we see that the map \( \Gamma \circ S \) from \( A \) onto \( \hat{B} \), the Gelfand transform of \( B \), is injective. It follows that \( \Gamma \) is injective. Thus we see that \( B \) is semisimple. Applying the first part of the proof we see that the conclusion holds.

Since uniform algebras are unital semisimple commutative Banach algebras, we see that the following holds.

**Corollary 7.5.** Let \( A \) and \( B \) be uniform algebras on compact Hausdorff spaces \( X \) and \( Y \) respectively. Suppose that \( S \) is a map from \( A \) onto \( B \) which satisfies that there exists a nonzero complex number \( \alpha \) such that the equality
\[
\|SfSg - \alpha\|_{\infty(Y)} = \|fg - \alpha\|_{\infty(X)}
\]
holds for every pair \( f \) and \( g \) in \( A \). Then there exist an \( \eta \in B \) with \( \eta^2 = 1 \), a homeomorphism \( \phi \) from \( \text{Ch}(B) \) onto \( \text{Ch}(A) \), and a clopen subset \( K \) of \( \text{Ch}(B) \) such that the equality
\[
Sf(y) = \eta(y) \times \begin{cases} \frac{f(\phi(y))}{|f(\phi(y))|}, & y \in K, \\ \frac{\alpha}{|\alpha|} f(\phi(y)), & y \in \text{Ch}(B) \setminus K \end{cases}
\]
holds for every \( f \in A \). Furthermore if \( T1 = 1 \), then the equality
\[
Sf(y) = \begin{cases} f(\phi(y)), & y \in K, \\ f(\phi(y)), & y \in \text{Ch}(B) \setminus K \end{cases}
\]
holds for every \( f \in A \). Furthermore if \( T1 = 1 \), and \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) or there exists a \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) with \( T\lambda = \lambda \), then the equality
\[
Tf(y) = f(\phi(y)), \quad y \in \text{Ch}(B)
\]
holds for every \( f \in A \).
Corollary 7.6. Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital commutative Banach algebra. Suppose that $S$ is a map from $A$ onto $B$ which satisfies that there exists a non-zero complex number $\alpha$ such that

$$\sigma(\pi(SfSg - \alpha)) \cap \sigma(\pi(fg - \alpha)) \neq \emptyset$$

holds for every pair $f$ and $g$ in $A$. Then $B$ is semisimple and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ such that the equality

$$Sf(y) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$

holds for every $f \in A$.

Proof. Since

$$\sigma(\pi(SfSg - \alpha)) \cap \sigma(\pi(fg - \alpha)) \neq \emptyset,$$

holds for every $f, g \in A$, then

$$r(SfSg - \alpha) = r(fg - \alpha)$$

holds for every $f, g \in A$. Then by Theorem 7.4 we see that $B$ is semisimple and $S$ is real-linear. On the other hand we see that $SA^{-1} = B^{-1}$. Suppose that $f \in A^{-1}$. Then

$$\sigma(f(\alpha^{-1}f) - \alpha) = \{0\},$$

so

$$\sigma(SfS(\alpha^{-1}f) - \alpha) \supset \{0\}.$$ 

It follows that

$$\sigma(SfS(\alpha^{-1}f) - \alpha) = \{0\}.$$ 

We see that $Sf \in B^{-1}$. Thus we see that $SA^{-1} \subset B^{-1}$. Suppose conversely that $F \in B^{-1}$, then there exist an $f$ and a $g$ in $A$ with

$Sf = F$ and $Sg = \alpha^{-1}F$. Then

$$\sigma(SfSg - \alpha) = \{0\},$$

so

$$\sigma(fg - \alpha) = \{0\}.$$ 

We see that $f \in A^{-1}$. Thus we see that $B^{-1} \subset A^{-1}$, and thus $SA^{-1} = B^{-1}$. Then by Corollary 6.1 there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ such that the equality

(7.6) 

$$Sf(y) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$

holds for every $f \in A^{-1}$. We show that the equation (7.6) holds for every $f \in A$. Let $f \in A$. Then there exist $f_0 \in A^{-1}$ and a complex number $\mu$ with $f = f_0 + \mu$. If $\mu = 0$, then

$$Sf(y) = Sf_0(y) = S1(y)f(\phi(y)) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$

for every $y \in \text{Ch}(\text{cl}B)$. If $\mu \neq 0$, then

$$Sf(y) = S(f_0 + \mu)(y) = S1(y)f_0(\phi(y)) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B).$$

Thus the equality (7.6) holds for every $f \in A$. Therefore $B$ is semisimple and there exists a homeomorphism $\phi$ from $\text{Ch}(\text{cl}B)$ onto $\text{Ch}(\text{cl}A)$ such that the equality

(7.6) 

$$Sf(y) = S1(y)f(\phi(y)), \quad y \in \text{Ch}(\text{cl}B)$$

holds for every $f \in A$. Thus the proof is completed.
holds. If \( \mu \neq 0 \), then, by real-linearity of \( S \) and the equation (7.6) we have

\[
S f(y) = S f_0(y) + S\lambda(y) = S1(y)f_0(\phi(y)) + S1(y)\lambda = S1(y)f(\phi(y))
\]

holds for every \( y \in \text{Ch(cl}B) \).

\[ \square \]

The authors do not know if a corresponding result for \( \alpha = 0 \) holds.

References

[1] A. Browder, “Introduction to Function Algebras”, W.A. Benjamin, 1969.
[2] T. W. Gamelin, “Uniform Algebras 2nd ed.”, Chelsea Publishing Company, 1984.
[3] O. Hatori and T. Miura, On a characterization of the maximal ideal spaces of commutative \( C^* \)-algebras in which every elements is the square of another, Proc. Amer. Math. Soc., 128(2000), 1185–1189
[4] O. Hatori, T. Miura and H. Takagi, Characterizations of isometric isomorphisms between uniform algebras via non-linear range-preserving properties, Proc. Amer. Math. Soc., to appear
[5] O. Hatori, T. Miura and H. Takagi, Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative, Jour. Math. Anal. Appl., to appear
[6] S. H. Hochwald, Multiplicative maps on matrices that preserves the spectrum, Linear Algebra and Its Appl., 212/213(1994), 339–351
[7] S. Lambert, A. Luttman and T. Tonev, Weakly peripherally-multiplicative operators between uniform algebras, Contemp. Math., 435 (2007), 265–281.
[8] A. Luttman and T. Tonev, Uniform algebra isomorphisms and peripheral multiplicity, Proc. Amer. Math. Soc., 135 (2007), 3589–3598.
[9] L. Molnár, Some characterizations of the automorphisms of \( B(H) \) and \( C(X) \), Proc. Amer. Math. Soc., 130(2002), 111–120
[10] T. Petek and P. Šemrl, Characterization of Jordan Homomorphisms on \( M_n \) using preserver properties, Linear Algebra and Its Appl., 269(1998), 33–46
[11] T. Ransford, A Cartan theorem for Banach algebras, Proc. Amer. Math. Soc., 124(1996), 243–247
[12] N. V. Rao and A. K. Roy, Multiplicatively spectrum-preserving maps of function algebras, Proc. Amer. Math. Soc., 133(2005), 1135–1142
[13] N. V. Rao and A. K. Roy Multiplicatively spectrum-preserving maps of function algebras.II, Proc. Edin. Math. Soc., 48(2005), 219–229
[14] W. Želazko, “Banach Algebras”, Elsevier, 1973
Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181 Japan
E-mail address: hatori@math.sc.niigata-u.ac.jp

Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510 Japan
E-mail address: miura@yz.yamagata-u.ac.jp

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621 Japan
E-mail address: takagi@math.shinshu-u.ac.jp