On submaximal plane curves

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Abstract

We prove that a submaximal curve in $\mathbb{P}^2$ has sequence of multiplicities $(\mu, \nu, \ldots, \nu)$, with $\mu < s\nu$ for every integer $s$ with $(s - 1)^2(s + 2)^2 \geq 6.76(r - 1)$.

This note is a sequel to [10], where a specialization method was developed in order to bound the degree of singular plane curves. The problem under consideration is, given a system of multiplicities $(m) = (m_1, m_2, \ldots, m_r) \in \mathbb{Z}^r$ and points $p_1, \ldots, p_r \in \mathbb{P}^2$, which we shall always assume to be in general position, to determine the minimal degree $\alpha(m)$ of a curve with multiplicity $m_i$ at each point $p_i$. In [10], the focus was on homogeneous $(m)$, (i.e., $m_1 = m_2 = \cdots = m_r$), but the method applies in general; here it is used to show that if one of the multiplicities is much bigger than the others, in a sense we make precise below (see theorem 1), then

$$\alpha(m) > \frac{\sum_{i=1}^{r} m_i}{\sqrt{r}}.$$  \hspace{1cm} (1)

In connection with his solution to the fourteenth problem of Hilbert, M. Nagata conjectured in 1959 that the inequality (1) holds for all $(m)$ provided $r > 9$, and proved it in the case when $r$ is a perfect square (see [7] or [8]). Since then, many partial results have been proved by several authors (see for instance [13], [3], [4], [5], [11], [12]), but as far as we know the conjecture remains open in general. One of the research lines in this area is the study of submaximal curves that arise in the context of Seshadri constants. A submaximal curve is an irreducible curve which causes the $(r$-point) Seshadri constant of a surface to be non-maximal; in the case of $\mathbb{P}^2$ it is just an irreducible curve which causes (1) to fail, and Nagata’s conjecture can be equivalently stated by saying that there exist no submaximal curves for $r > 9$. T. Szemberg proved in [11, 4.6] that every submaximal curve on a surface with Picard number $\rho = 1$ whose multiple points are in general position is quasi-homogeneous, i.e., has $m_2 = \cdots = m_r$ for a suitable ordering of the points. In the case of the projective plane, our result shows that moreover $m_1$ can not be much bigger than $m_2$, constraining further the range of possible counterexamples to Nagata’s conjecture. It is worth mentioning that quasi-homogeneous curves are relevant also for the method of C. Ciliberto and R. Miranda [4] to compute the dimension of (homogeneous) linear systems.

The approach is based on the specialization introduced in [10]. Roughly speaking, one proves that if there exists a curve with given multiplicities at $r$ general points, then
by semicontinuity there must also exist curves with the same degree and (virtual) multiplicities at \( r \) points which satisfy some well chosen proximity relations. The proximity inequalities impose then that the effective multiplicity of the specialized curve must grow, and one uses this bigger multiplicity as a bound for the degree of the curve.

In order to give a brief explicit description of the specialization, let us recall the notations of [10] (see [11] for generalities on clusters and unloading, and [9] for a general approach to specializations parameterized by varieties of clusters). We work on \( \mathbb{P}^2 \), and consider both proper and infinitely near points (which are those lying on a smooth surface that dominates \( \mathbb{P}^2 \) birationally. A cluster is a set \( K \) of points of \( \mathbb{P}^2 \) such that if \( p \in K \) and \( p \) is infinitely near to \( q \) (i.e., \( p \) lies on the exceptional divisor of \( q \) after blowing up a sequence of points) then \( q \in K \). Write \( \pi_K : S_K \to \mathbb{P}^2 \) for the blowing up of all points in \( K \). For each \( i = 2, \ldots, r \), we denote \( U_i \) the set of clusters \( K = \{ p_1, \ldots, p_r \} \) such that

- \( p_2, \ldots, p_i \) are proximate to \( p_1 \),
- \( p_j \) is proximate to \( p_{j-1} \) for all \( j = 2, \ldots, r \), and
- there are no other proximity relations.

In other words, denoting by \( E_j \) the (total) exceptional divisor of blowing up \( p_j \) this can be expressed by saying that the divisors \( E_1 = E_1 \) and \( E_j = E_j - E_{j+1} \), \( j = 2, \ldots, r - 1 \) on the surface \( S_K \) are effective and irreducible.

The sets \( U_i \) are nonempty and have a natural structure of smooth irreducible locally closed subvarieties in a projective variety (the iterated blowing-up \( X_{r-1} \) of Kleiman [9]), and they satisfy

\[
U_2 \supset U_3 \supset \cdots \supset U_r.
\]

The specialization we use works stepwise. Begin with a general cluster of \( r \) distinct points, \( K = \{ p_1, p_2, \ldots, p_r \} \), and a curve \( C \) with multiplicities \( (m) = (m_1, m_2, \ldots, m_r) \) at these points, assuming \( m_1 \geq m_2 \geq \cdots \geq m_r \). Then specialize \( K \) to a cluster \( K_3 \) general in \( U_3 \). If \( m_1 < m_2 + m_3 \), then the specialized curve \( C_3 \) cuts negatively the irreducible divisor \( E_1 = E_1 - E_2 - E_3 \), so \( E_1 \) is a component of \( \pi_K(C_3) = m_1 E_1 - \cdots - m_r E_r \), and the effective multiplicity of \( C_3 \) at \( p_1 \) is bigger than \( m_1 \). Call \( (m^{(3)}) \) the system of multiplicities obtained after unloading multiplicities (i.e., substracting the \( \hat{E}_j \) which are cut negatively): \( C_3 \) goes through the cluster \( K_3 \) with multiplicities \( (m^{(3)}) \). Then specialize \( K_3 \) to a general \( K_4 \in U_4 \), and successively to a \( K_5 \in U_5, \ldots \), to a \( K_r \in U_r \), performing unloadings whenever necessary. The first multiplicity of the last system \( (m^{(r)}) \) is a lower bound for the degree of \( C \), and therefore \( \alpha(m) \geq m_1^{(r)} \).

This multiplicity is not hard to compute in each particular case; in [10] a bound was given that holds in general and is asymptotically sharp but that in many particular cases can be improved, especially when the multiplicities are relatively small. Now we are interested in the case that \( m_1 \) is much bigger than the other multiplicities, in which one can show that the inequality \( \Box \) holds:

**Theorem 1.** Let \( s \) be such that \( (s - 1)^2(s + 2)^2 \geq 6.76(r - 1) \), \( r \geq 9 \), and assume \( m_1 \geq m_2 \geq \cdots \geq m_r \). If moreover

\[
m_1 \geq \sum_{i=2}^{s+1} m_i \]

then \( \alpha(m) > \sum_{i=1}^r m_i / \sqrt{r} \).
Proof. Using notations as above, the preceding discussion shows that it suffices to prove
\[ m_1^{(r)} > \sum_{i=1}^{r} m_i / \sqrt{r}. \]
The hypothesis implies that the system of multiplicities \((m)\) is consistent for all clusters in \(U_3, \ldots, U_{s+1}\) (no unloading is needed for these) so 
\( (m) = (m^{(3)}) = \cdots = (m^{(s+1)}) \). Then, apply [10, lemma 3.5] as in the proof of [10, theorem 4.1] to obtain
\[ m_1^{(r)} \geq \sum_{i=1}^{r} m_i \left( 1 - \frac{1}{r} \right) \prod_{k=s+1}^{r-1} \left( 1 - \frac{k}{k^2 + r - 1} \right). \]

We have to see that this is bigger than \( \sum m_i / \sqrt{r} \). Because of [10, proposition 5.1], it will be enough to prove
\[ \prod_{k=2}^{s} \left( 1 - \frac{k}{k^2 + r - 1} \right)^{-1} > \frac{\sqrt{r}}{\sqrt{r - 1} - \frac{2}{8}}. \] (2)

Write \( x^2 = r - 1 \). As \( r > 9 \), we have \( x \geq 3 \). The term on the left in (2) is
\[ \prod_{k=2}^{s} \left( 1 + \frac{k}{k(k+1) + x^2} \right) > 1 + \sum_{k=2}^{s} \frac{k}{k(k+1) + x^2}. \]
Let \( s_0 \) be the minimum integer such that \( (s_0 - 1)^2(s_0 + 2)^2 \geq 6.76x^2 \); the hypothesis on \( s \) says that \( s \geq s_0 \), and it is clear that \( s_0(s_0 + 1) \leq x^2 \). Using this we get
\[ 1 + \sum_{k=2}^{s} \frac{k}{k(k+1) + x^2} > 1 + \sum_{k=2}^{s_0} \frac{k}{2x^2} = 1 + \frac{(s_0 - 1)(s_0 + 2)}{4x^2} \geq 1 + \frac{\sqrt{6.76x}}{4x^2} = 1 + \frac{65}{x}. \]

On the other hand, the term on the right in (2) can be written as
\[ \frac{x}{x - \pi/8} \sqrt{1 + \frac{1}{x^2}} \leq \left( 1 + \frac{\pi}{8x - \pi} \right) \left( 1 + \frac{1}{2x^2} \right) \]
which for \( x \geq 3 \) is less or equal to \( 1 + .65/x \), and the proof is complete. \( \square \)

Corollary 2. Let \( C \) be a submaximal curve with respect to general points \( p_1, p_2, \ldots, p_r \in \mathbb{P}^2 \). Then for some reordering of the points, the system \((m)\) of multiplicities of \( C \) at \( p_1, p_2, \ldots, p_r \) is \( (m) = (\mu, \nu, \ldots, \nu) \) with \( \mu < s\nu \) for every integer \( s \) with \( (s-1)^2(s+2)^2 \geq 6.76(r-1) \).

Proof. The system \((m)\) of multiplicities of \( C \) at \( p_1, p_2, \ldots, p_r \) is \( (m) = (\mu, \nu, \ldots, \nu) \) because of [11, corollary 4.6], so it is enough to prove that \( \mu \geq s\nu \) for some integer \( s \) with \( (s-1)^2(s+2)^2 \geq 6.76(r-1) \) leads to contradiction. But theorem [11] shows that there are no submaximal curves when \( \mu \geq s\nu \) for some integer \( s \) with \( (s-1)^2(s+2)^2 \geq 6.76(r-1) \), so we are done. \( \square \)

We finish by an example, considering the smallest values \( r \) for which Nagata’s conjecture is unknown. Corollary 2 says that a submaximal curve with respect to \( r \) general points, \( 10 \leq r \leq 15 \) has system of multiplicities \((m) = (\mu, \nu, \ldots, \nu) \) with \( \mu < 3\nu \).
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