Quantum Harmonic Oscillator Algebra and Link Invariants

César Gómez and Germán Sierra
Instituto de Física Fundamental, Serrano 123,
Madrid, Spain

October, 1991

The $q$–deformation $U_q(h_4)$ of the harmonic oscillator algebra is defined and proved to be a Ribbon Hopf algebra. Associated with this Hopf algebra we define an infinite dimensional braid group representation on the Hilbert space of the harmonic oscillator, and an extended Yang–Baxter system in the sense of Turaev. The corresponding link invariant is computed in some particular cases and coincides with the inverse of the Alexander–Conway polynomial. The $R$ matrix of $U_q(h_4)$ can be interpreted as defining a baxterization of the intertwiners for semicyclic representations of $SU(2)_q$ at $q = e^{2\pi i/N}$ in the $N \to \infty$ limit. Finally we define new multicolored braid group representations and study their relation to the multivariable Alexander–Conway polynomial.
1 Introduction

The connection between quantum groups [1] and link invariants was first established in [2]. The simplest way to describe this connection is by associating with a given quantum group an extended Yang–Baxter system in the sense of Turaev [3]. Using this procedure for the case of $U_q(SU(2))$ one reobtains Jones’s invariant [4]. Given the finite dimensional irrep of spin $j$ of $U_q(SU(2))$, $V^{(j)}$, a realization of the braid group $B_n$ in $\text{End}(\bigotimes^n V^{(j)})$ is defined in terms of the quantum $R$–matrix of $U_q(SU(2))$ in the representation $j$, as follows:

$$\rho : \sigma_i \rightarrow \otimes \cdots \otimes R^{ij}_{(i,i+1)} \otimes \cdots \otimes 1 \quad (1)$$

For any word $\alpha$ in $B_n$ the corresponding link invariant is defined by:

$$T(\alpha) = a^{-\omega(\alpha)} b^{-n \text{tr}(\rho(\alpha)\mu^n)} \quad (2)$$

where $\mu$ is a diagonal homeomorphism: $\mu : V^{(j)} \rightarrow V^{(j)}$ satisfying:

$$(\mu_i\mu_j - \mu_k\mu_\ell)R^{k\ell}_{ij} = 0 \quad (3)$$

$$\sum_j R^{kj}_{ij} \mu_j = ab\delta^k_i$$

$$\sum_j (R^{-1})^{kj}_{ij} \mu_j = a^{-1}b\delta^k_i$$

For the $U_q(SU(2))$ $-R$ matrix in the representation $j$ the corresponding extended Yang–Baxter system defined by (3) is given by

$$\mu = q^{H/2}$$

$$a = q^c (-1)^{2j}$$

$$b = [2j + 1] \quad (4)$$

In the particular case $j = 1/2$ the invariant (2) coincides with Jones’s polynomial. The state model, in the spirit of Kauffman [5], associated with the invariant (2) can be defined using the quantum $6j$–symbols of $U_q(SU(2))$. Moreover the relationship with vertex integrable models [6] follows
by defining the representation (1) of $B_n$ as the very anisotropic limit of the trigonometric solution for the Yang–Baxter operators.

In the last few months some new connections relating the Alexander–Conway polynomial with quantum groups have been discovered [7–9]. In particular in reference [7] the quantum group realization of the Alexander polynomial was obtained starting with the supergroup GL $(1,1)$ which turns out to define a free fermion model of the invariant. In a completely different context Date et al make contact with the Alexander polynomial in their study of the braid group representations arising from the chiral Potts model [9]. This result should indicate a new interplay between the Alexander polynomial and quantum groups, based on the characterization of the solution to the start triangle equation for the chiral Potts model in terms of intertwiners for the cyclic representations of $U_q(SU(2))$ at roots of unit [10].

In a previous work we have found the intertwiners $R$ matrices for semicyclic representations of $U_q(SU(2))$ with $q = e^{2\pi i/N}$ (for definitions and more details see references [11, 12]). In the limit of $N$ going to infinity these $R$ matrices give rise to a new braid group representation realized on tensor product of an infinite dimensional space which in fact is isomorphic to the Hilbert space of a harmonic oscillator [12]. This braid group representation admits a baxterization [13] which allows the introduction of new parameters and, interesting enough, the baxterized $R$–matrices satisfies the Turaev conditions (3) for having a link and knot invariant which was already defined in [12]. The aim of this paper is twofold: i) clarify the quantum group origin of this infinite dimensional braid group representation and ii) give a proper definition and identification of the link invariant obtained with these new $R$ matrices.

The main results that we obtain are:

1. The relation of the baxterized $R$–matrices, coming from the semicyclic irreps of $U_q(SU(2))$, with the $R$–matrices of a quantum group deformation of the harmonic oscillator algebra $h_4$.

2. The identification of the invariant of reference [12] with the inverse of the Alexander–Conway polynomial.

3. The construction of new colored braid group representations associated with the quantum group $U_q(h_4)$ and their relation with the multivariable Alexander–Conway polynomial.

These results show that the inverse of the Alexander polynomial admits a bosonic model in close analogy with the free fermion model of the Alexander
invariant presented in references [7]. The fermionic version of this invariant is based on a quantum deformation of the supergroup GL (1,1), which in our bosonic model is replaced by the harmonic oscillator algebra $h_4$.

The plan of the paper is the following: in section 2 we review the infinite dimensional representations of the braid group founded in [12]. In section 3 we define the quantum deformation of the harmonic oscillator algebra $h_4$ and establish the connection between the universal $R$ matrix of this Hopf algebra and the infinite dimensional braid group representation introduced in section 2. In section 4 we define the link invariants and we extend the result to the case of the multivariable Alexander–Conway polynomial. In Appendix A we perform some explicit computation of link and knots invariants, in Appendix B we study the Clebsch–Gordan decomposition of the irreps of $U_q(h_4)$, and finally in appendix C we discuss the ribbon structure of $U_q(h_4)$.

2 The Harmonic Oscillator Braid group representation

In reference [12] we have considered the $N \to \infty$ limit ($q = e^{2\pi i/N}$) of the $U_q(SU(2))$ Hopf algebra:

$$
\begin{align*}
EF &= q^2 FE = 1 - K^2 \\
KE &= q^{-2} EK \\
KF &= q^2 FK
\end{align*}
$$

with the comultiplication:

$$
\begin{align*}
\Delta E &= E \otimes 1 + K \otimes E \\
\Delta F &= F \otimes 1 + K \otimes F \\
\Delta K &= K \otimes K
\end{align*}
$$

In this limit the Hopf algebra (5) becomes:

$$
[E, F] = 1 - K^2 \quad [K, E] = [K, F] = 0
$$

which is isomorphic to the Heisenberg algebra for the harmonic oscillator:

$$
a \equiv \frac{1}{1 + k} E \quad a^+ \equiv \frac{1}{1 - k} F$$
\[ [a, a^+] = 1 \quad [k, a] = [k, a^+] = 0 \] (8)

The semicyclic representations in this limit are now \( \infty \)-dimensional and they are labelled by the eigenvalues of \( K \):

\[
F e_r = (1 - \lambda) e_{r+1} \\
E e_r = r(1 + \lambda) e_{r-1} \\
K e_r = \lambda e_r
\] (9)

Denoting \( H_\lambda \) these irreps, it was proved in reference [12] that there exist an intertwiner \( R^{\lambda_1, \lambda_2} : H^{\lambda_1} \otimes H^{\lambda_2} \rightarrow H^{\lambda_2} \otimes H^{\lambda_1} \) satisfying the Yang Baxter equation. The explicit form of the intertwiner is given by

\[
R_{r_1, r_2}^{r_1 + \ell, r_2 - \ell}(u) = (1 + u)^{r_1 - \ell} u^{r_2 - \ell} P^{(r_2 - \ell, r_1 - \ell)}_{\ell}(1 - 2u^2)
\] (10)

with \( u = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \) and \( P^{(\alpha, \beta)}_n(x) \) the Jacobi polynomials:

\[
P^{(\alpha, \beta)}_n(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n + \alpha}{m} \binom{n + \beta}{n - m} (x - 1)^{n-m} (x + 1)^m
\] (11)

The Yang–Baxter equation for the \( R \)-matrix (10) reads:

\[
R_i(u) R_{i+1} \left( \frac{u + v}{1 + u v} \right) R_i(v) = R_{i+1}(v) R_i \left( \frac{u + v}{1 + u v} \right) R_{i+1}(u)
\] (12)

The non trivial braid limit of (12) is obtained for \( u = v = \pm 1 \). In this limit the braid group generators are given by:

\[
R_{r_1, r_2}^{r_1', r_2'}(+) = \delta_{r_1 + r_2, r_1' + r_2'} (-1)^{r_2'} 2^{r_1 - r_2'} \begin{pmatrix} r_1 \\ r_2' \end{pmatrix}
\] (13)

\[
R_{r_1, r_2}^{r_1', r_2'}(-) = \delta_{r_1 + r_2, r_1' + r_2'} (-1)^{r_1'} 2^{r_2 - r_1'} \begin{pmatrix} r_2 \\ r_1' \end{pmatrix}
\] (14)

which satisfy:

\[
R(+) R(-1) = 1
\] (15)

The representantion of \( B_n \) as \( \text{End}(\otimes^n \mathcal{H}) \) is defined by:
\[ \sigma_i^\pm \rightarrow 1 \otimes \cdots R_i(\pm) \otimes \cdots 1 \]  
(16)

with \( \mathcal{H} \) isomorphic to the Hilbert space of the harmonic oscillator. We will refer to this representation of the braid group as the harmonic oscillator representation.

A compact way to rewrite eqs (13) and (14) is in terms of the “universal” \( \mathcal{R} \) matrix defined as \( \mathcal{R} = PR \) with \( P \) the permutation operator \( P(\varphi_1 \otimes \varphi_2) = \varphi_1 \otimes \varphi_1(\varphi_1, \varphi_2 \in \mathcal{H}) \). Using the creation and annihilation operators \( a, a^+ \) we find that \( R(\pm) \) admit the following nice representation:

\[ R(+)= (e^{i\pi N} \otimes 1)e^{2a^+a} \]

\[ R(-)=(1 \otimes e^{i\pi N})e^{2a^+a} \]

(17)

with \( N \) the number operator: \( N = a^+a \).

The harmonic oscillator braid group representation (17) admit a baxter-ization [13] in the following sense:

**Proposition 2.1** For arbitrary complex numbers \( x, y \) the \( R \) matrices:

\[ R(x,y,+) = (x^N \otimes y^{-N})e^{(y-x)a^+a} \]

\[ R(x,y,-) = e^{(x-y)a^+a}(y^N \otimes x^{-N}) \]

(18)

satisfy the Yang Baxter relation:

\[ \mathcal{R}_{12}(\pm)\mathcal{R}_{13}(\pm)\mathcal{R}_{23}(\pm) = \mathcal{R}_{23}(\pm)\mathcal{R}_{13}(\pm)\mathcal{R}_{12}(\pm) \]

(19)

where \( \mathcal{R}_{12}(+) = (x^N \otimes y^{-N} \otimes 1)e^{(y-x)a^+a} \otimes 1, \) etc.

From this proposition if follows that the matrices \( R(x, y, \pm) = PR(x, y, \pm) \):

\[ R_{r_1 r_2}(x,y,+) = \delta_{r_1+r_2, r'_1 + r'_2} \begin{pmatrix} r_1 \\ r'_2 \end{pmatrix} (y-x)^{r_1-r'_2}x^{r'_2}y^{-r'_1} \]

\[ R_{r_1 r_2}(x,y,-) = \delta_{r_1+r_2, r'_1 + r'_2} \begin{pmatrix} r_2 \\ r'_1 \end{pmatrix} (x-y)^{r_2-r'_1}x^{r_1}y^{r'_1} \]

(20)
define a \( \infty \) dimensional representation of the braid group \( \pi_{x,y} : B_n \to \text{End}(H^\otimes n) \). Notice that we recover the \( R \)-matrices (13), (14) from eq (20) in the case where \( x = -1, y = 1 \).

At a first stage it may appear that we are dealing in (20) with a two parameter family of inequivalent braid group representations, but this is not the case since we can prove the equivalence between the representations \( \pi_{x,y} \) and \( \pi_{\alpha x, \alpha y} \) for any \( \alpha \neq 0 \). This follows automatically from the relation:

\[
R(\alpha x, \alpha y; \pm) = (\alpha^{-N} \otimes 1)R(x, y)(\alpha^N \otimes 1)
\]

(21)

Therefore the representation is really characterized by the ratio \( \frac{x}{y} \).

In order to use the braid group representations \( \pi_{x,y} \) to define link invariants we need the following result.

**Proposition 2.2** The braid group representation \( \pi_{x,y} \) admits, for arbitrary \( x, y \), an extension a la Turaev.

In fact it is easy to check that for \( \mu \) the identity operator and \( a = b^{-1} = \sqrt{(y/x)} \) the set of equations (3) are satisfied for \( R = R(x, y) \) with arbitrary values of \( x \) and \( y \). In these conditions the link invariant (2) becomes:

\[
T_{x,y}(\alpha) = (x/y)^{1/2(\omega(\alpha)-n)}\text{tr}[\pi_{x,y}(\alpha)]
\]

(22)

Notice that the invariant only depends on the ratio \( x/y \). This fact follows from the scaling transformations law (21). A proper way to regularize (22) preserving its invariance under Markov moves will be defined in section 4.

3 A quantum deformation of the Harmonic oscillator algebra

The harmonic oscillator algebra \( h_4 \) contains four generators \( N, a, a^+, E \) subjected to the relations:

\[
[Na^+] = a^+
\]

\[
[N, a] = -a
\]

(23)

\[
[a, a^+] = E
\]
and with $E$ commuting with all the generators.

This algebra can be obtained as a particular contraction of $U(2)$. The two casimirs are:

\[
\begin{align*}
c_1 &= EN - a^+a \\
c_2 &= E^2
\end{align*}
\]

Next we proceed to define a quantum deformation\footnote{This quantum deformation should not be confused with the $q$–oscillators defined in reference [14]} of $h_4$.

**Definition 3.1** The quantum deformed harmonic oscillator algebra $U_q(h_4)$ is defined by:

\[
\begin{align*}
[a, a^+] &= \frac{qE - q^{-E}}{q - q^{-1}} \\
[N, a] &= -a \\
[N, a^+] &= a^+
\end{align*}
\]

with the comultiplication:

\[
\begin{align*}
\Delta a &= a \otimes q^{E/2} + q^{-E/2} \otimes a \\
\Delta a^+ &= a^+ \otimes q^{E/2} + q^{-E/2} \otimes a^+ \\
\Delta E &= E \otimes 1 + 1 \otimes E \\
\Delta N &= N \otimes 1 + 1 \otimes N
\end{align*}
\]

The antipode:

\[\gamma(x) = -x \quad x = a, a^+, E, N\]

and counit:

\[\varepsilon(x) = 0\]
Proposition 3.1 The algebra $U_q(\mathfrak{h}_4)$ is a quasitriangular Ribbon Hopf algebra.

The universal $R$ matrix satisfying:

$$(\Delta \otimes \text{id}) R = R_{13} R_{23}$$
$$(\text{id} \otimes \Delta) R = R_{13} R_{12}$$
$$(\gamma \otimes \text{id}) R = R^{-1}$$
$$\Delta' = R \Delta R^{-1}$$

is given by:

$$R = q^{-\left(E \otimes N + N \otimes E\right)} e(q^{-1})(q^{E/2} \otimes q^{-E/2}) a \otimes a^+$$

The ribbon structure of $U_q(\mathfrak{h}_4)$ will be studied in section C.

Each irrep of $U_q(\mathfrak{h}_4)$ is labelled by the values of the two Casimirs $c_1, c_2 (= qE-q^{-E}/2 - a^- a^+)$ which is equivalent to give the eigenvalues $(e, n)$ of $E$ and $N$. A generic irrep $(e, n)$ is defined as follows:

$$a | r > = [e]^{1/2} \sqrt{r} | r - 1 >$$
$$a^+ | r > = [e]^{1/2} \sqrt{r + 1} | r + 1 >$$
$$E | r > = e | r >$$
$$N | r > = (r + n) | r >$$

where $\{| r >\}_{r=0}^{\infty}$ is an orthonormal basis and the $q$–number $[e]$ is defined as $[x] = \frac{q^x-q^{-x}}{q-q^{-1}}$.

We shall always consider the cases with $[e] \neq 0$. Evaluating the universal $R$ matrix (30) in the tensor product $(e_1 n_1) \otimes (e_2 n_2)$ we define the matrix:

$$R^{e_1 e_2} \equiv R(e_1 e_2) = q^{e_1 n_2 + e_2 n_1} P \, R^{(e_1 n_1),(e_2 n_2)}$$

where $P$ is the usual permutation operator. In the basis (31) we get

$$R_{r_1 r_2}^{r'_1 r'_2}(e_1, e_2) = \delta_{r_1+r_2, r'_1+r'_2} \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right)^{1/2} \left( \begin{array}{c} r'_1 \\ r'_2 \end{array} \right)^{1/2}$$

$$[(q^{e_1} - q^{-e_1})(q^{e_2} - q^{-e_2})]^{1/2} q^{\frac{s_1-s_2}{2}(r_1-r'_2)} q^{-e_1 r'_1 - e_2 r'_2}$$
\[(R^{-1})_{r_1 r_2}^{r_1' r_2'}(e_1, e_2) = \delta_{r_1+r_2, r_1'+r_2'} \left( \begin{array}{c} r_2' \\ r_1' \end{array} \right) \frac{1}{2} \left( \begin{array}{c} r_2 \\ r_1 \end{array} \right) \frac{1}{2}
\]

\[\left[(q^{e_1} - q^{-e_1})(q^{e_2} - q^{-e_2})\right]^{r_2-r_1'} \left(-1\right)^{r_2-r_1'} q^{-\frac{e_1-e_2}{2}(r_1-r_2')} q^{e_1 r_2 + e_2 r_1} (33)\]

These \(R\)–matrices satisfy the colored–braid relation:

\[R_{i}^{e_2 e_3} R_{i+1}^{e_1 e_3} R_{i}^{e_1 e_2} = R_{i+1}^{e_1 e_3} R_{i}^{e_1 e_2} R_{i}^{e_2 e_3} (34)\]

which reduces to the ordinary braid group relation if \(e_1 = e_2 = e_3\).

In addition to (34) one has the following properties

**P1: Hermiticity**

\[R_{r_1 r_2}^{r_1' r_2'}(e_1, e_2) = q^{(e_1-e_2)(r_1'-r_1)} R_{r_1 r_2}^{r_1' r_2'}(e_1, e_2) (35)\]

**P2: Reflection Symmetry:**

\[R_{r_2 r_1}^{r_2' r_1'}(e_2 e_1) = (-1)^{r_1'-r_2} q^{-e_1(r_2+r_1')-e_2(r_1+r_2')} (R^{-1})_{r_1 r_2}^{r_1' r_2'}(e_1, e_2) (36)\]

Next we would like to compare the \(R\) matrices of the quantum group \(U_q(h_4)\) with the matrix \(R(x, y, \pm)\) introduced in the previous section (eqs (20)).

**Proposition 3.2** The \(R\) matrices \(R(x, y, \pm)\) are equivalent to the \(R\) matrix of \(U_q(h_4)\) \(R^{\pm1}(e_1, e_2)\) for \(y = x^{-1} = q^{e_1} = q^{e_2}\).

To prove this proposition we to compare eqs. (20) with the \(R\) matrices given in eqs. (A2, A3), which are related to eqs (33) by a similarity transformation.

Now from propositions (2.2) and (3.2) it follows that the matrices \(R(e, e)\) and \(R^{-1}(e, e)\) satisfy the Turaev conditions (3) with \(\mu = \text{id}, a = b^{-1} = q^e\).

Consequently the link invariant (2) defined by the extended Yang Baxter system associated with \(U_q(h_4)\) is the same as the one defined in section two using the braid group representation \(\pi_{x,y}\) with \(y/x = q^{2e}\).
4 The harmonic oscillator link invariants and the Alexander polynomial

In this section we will proceed to define a regularized version of (22) which preserves the invariance under Markov moves.

Given an irrep \((e, n)\) of \(U_q(h_4)\) the link invariant defined by the associated extended Yang–Baxter system is:

\[
T(\alpha) = q^e(-\omega(\alpha)+m)\text{Tr}[\pi_e(\alpha)]
\]

with \(\alpha \in B_m\) and \(\pi_e(\alpha)\) the infinite dimensional braid group representation \(\pi_{x,y}\) with \(y = x^{-1} = q^e\). By construction \(T(\alpha)\) is invariant under the two Markov moves \([3]\):

\[
(M \cdot I) \quad T(\alpha\beta) = T(\beta\alpha) \\
(M \cdot II) \quad T(\alpha\sigma_m) = T(\alpha) \quad \alpha \in B_{m-1}
\]

To make sense of the infinite dimensional trace in (37) we will proceed to define a regularization similar to the one used in references \([7]\) \([9]\).

**Definition 4.1** Given \(\alpha \in B_m\) such that the corresponding link \(\hat{\alpha}\) is connected, and the irrep \((e, n)\) of \(U_q(h_4)\), we define a regularized trace \(T^r(\pi_e(\alpha))\) as follows:

\[
T^r(\pi_e(\alpha)) = \text{Tr}_{2...m}(\pi_e(\alpha))
\]

Properly speaking the regularized trace defined by (39) is associated with the tangle obtained by cutting one strand of the link \(\hat{\alpha}\). If \(\hat{\alpha}\) is disconnected we must, in order to define the regularized trace, to cut one strand of each component.

**Proposition 4.1** The regularized trace \(T^r\) satisfies Markov I.

The proof of this proposition goes as follows. First of all we must notice that given the tangle obtained by cutting one strand, let say the \(i^{th}\) strand, of the link \(\hat{\alpha}\) we can always find another strand \(i'\) such that the tangle obtained by cutting the \(i'^{th}\) strand of \(\hat{\beta}\) is equivalent to the original one. Using this result, the invariance under Markov I of the regularized trace (39) reduces to prove that it is independent of which strand we chose to cut, namely:
\[ T r_{2,-m}(\pi_e(\alpha)) = T r_{1,-i,i+2,-m}(\pi_e(\alpha)) \]  

(40)

The identity (40) is a formal consequence of the following lemma.

**Lemma 4.1**

\[ T' r(\pi_e(\alpha)) \propto \mathbb{1} \]  

(41)

Proof: Using the quasitriangularity of \( U_q(h_4) \): \( \Delta' = R \Delta R^{-1} \), we obtain:

\[ \Delta^{(m)}(a^+)^{\pi_e(\alpha)} = \pi_e(\alpha) \Delta^{(m)}(a^+) \]  

(42)

for any \( \alpha \in B_m \). Exponentiating equation (42) we get:

\[ \pi_e(\alpha)(e^{2a^+} \otimes \mathbb{1} \otimes ...) = (e^{2a^+} \otimes \mathbb{1} \otimes ...)(\mathbb{1} \otimes \Omega)\pi_e(\alpha)(\mathbb{1} \otimes \Omega^{-1}) \]  

(43)

where \( \Omega \) is a similarity transformation. Now defining the trace in the basis of coherent states: \( |z> = e^{za^+}|0> \) and using (43) we obtain the desired result (41).

Invariance under Markov II is obtained by including the same prefactor as in (37). At this point we can define a normalized link invariant by:

\[ Z_{\hat{\alpha}}(q^e) = q^{e(-\omega(\alpha)+m-1)}T' r(\pi_e(\alpha)) \]  

(44)

where the normalization is given by:

\[ Z(unknot) = 1 \]  

(45)

The invariant \( Z_{\hat{\alpha}} \) is a polynomial in \( q^e \). In all the examples we have considered (see Appendix A for some non trivial cases) we have found the result:

\[ Z_{\hat{\alpha}}(q^e) = \frac{1}{\Delta_{\hat{\alpha}}(t)} \]  

(46)

with \( \Delta_{\hat{\alpha}}(t = q^e) \) the Alexander polynomial. The result (46) is very natural in our approach where we have considered as the starting point the Hopf algebra \( U_q(h_4) \) which can be thought as a bosonic version of GL (1,1). We do not yet have a general proof of (46). Notice that although \( \Delta_{\hat{\alpha}} \) satisfies a skein rule its inverse does not. This is of course related to the fact that the braid group representation underlying \( Z_{\hat{\alpha}} \) is infinite dimensional and that \( R(e,e) \) has an infinite number of different eigenvalues for generic values of
This means that the proof of (46) cannot probably proceed through the skein rule as is usually done for other invariants coming from quantum groups.

The colored version of the previous invariant is easily obtained once we have a colored braid group representation as in (34). Proceeding as the authors of ref [9] we first renormalized the matrix $R(e_1, e_2)$ according to:

$$\tilde{R}(e_1, e_2) \equiv q^{-\frac{e_1+e_2}{2}} R(e_1, e_2)$$

(47)

such that $\tilde{R}(e_1, e_2)$ do satisfies the braid group relation while the Turaev conditions (3) holds now with

$$\sum_j \tilde{R}(e, e)_{ij}^{kj} = \sum_j \tilde{R}^{-1}(e, e)_{ij}^{kj} = q^{-e/2} \delta_i^k$$

(48)

with this redefinition the colored version of the invariant (37) reads:

$$T(\hat{\alpha}) = q^{\sum_{i=1}^{n} e_i} Tr \tilde{\pi}(\alpha) = \prod_{i=1}^{n} t_i Tr \tilde{\pi}(\alpha)$$

(49)

where $\tilde{\pi}(\alpha)$ is the representation of $\alpha$ in terms of the matrix $\tilde{R}$. Notice that (49) reduces to (37) if $e_1 = e_2 = .. = e_n = e$.

Finally if we normalized the invariant (49) to be equal to $(t_1 - t_1^{-1})$ for the unknot with color 1, then it becomes:

$$Z_{\hat{\alpha}} = (t_1 - t_1^{-1}) \prod_{i=2}^{n} t_i T' r \tilde{\pi}(\alpha)$$

(50)

as an exercise one can check that

$$Z_{\hat{\sigma}_{1}} = (t_1 - t_1^{-1}) t_2 \ T' r \tilde{R}^{e_1e_2} \tilde{R}^{e_1e_2} = 1$$

(51)

which is the correct value for the two component link in the normalization adopted above.

Here again we conjecture that the invariant (50) is the inverse of the multivariable Alexander Conway polynomial.
Appendix A

Calculation of some links and knots invariants

In this appendix we shall compute some invariants of knots and links which shall support the conjecture (46), i.e. that the invariant $\hat{Z}_\alpha$ is the inverse of the Alexander–Conway polynomial.

Instead of using the normalized basis $|r>$ defined in eqs. (31) it will be more convenient to use the following basis for the irrep $(e, n)$ of $U_q(h_4)$:

\[
\begin{align*}
\alpha v_r &= [e] r v_{r-1} \\
\alpha^+ v_r &= v_{r+1} \\
N v_r &= (r + n) v_r \\
E v_r &= e v_r
\end{align*}
\]

The braiding matrices (33) now read:

\[
\begin{align*}
R_{r_1 r_2}^{r'_1 r'_2}(e_1, e_2) &= \delta_{r_1 + r_2, r'_1 + r'_2} \\
(R^{-1})_{r_1 r_2}^{r'_1 r'_2}(e_1, e_2) &= \delta_{r_1 + r_2, r'_1 + r'_2}
\end{align*}
\]

where $t_1 = q^{e_1}$ and $t_2 = q^{e_2}$.

It is clear from (41) that the invariant $Z_\alpha$ given in eq. (44) can also be compute as:

\[
Z_\alpha = (1 - \mu) q^{(m-\omega(\alpha)-1)} Tr_{\pi_e(\alpha)}(\mu^{N-n} \otimes \mathbb{1} - \mathbb{1})
\]

since the trace of $\mu^{N-n}$ in the first space is equal to $(1 - \mu)^{-1}$ for any $\mu \in C(|\mu| < 1)$.

Let us consider the case of the link obtained as $\hat{\sigma}_1^3$ with $\sigma_1 \in B_2$. Here we have to evaluate
$$\text{Tr}(R^{ee} R^{ee}(\mu^{N-n} \otimes \mathbb{1})) = \sum_{r_1r_2 = 0}^{\ell_1\ell_2} R^\ell_1\ell_2(e, e) R^{r_1r_2}_e(e, e) \mu^{\ell_1}$$

$$= \sum_{r_1r_2\ell_1\ell_2} \delta_{r_1+r_2,\ell_1+\ell_2} \binom{r_1}{\ell_1} \binom{r_2}{\ell_2} (t - t^{-1})^r_1 - r_2 2^{\ell_1+\ell_2} \mu^{\ell_1} \times t^{-(r_1+r_2+\ell_1+\ell_2)} \mu^\ell_1$$

(A5)

First we sum $r_1$ in order to eliminate the delta “function” obtaining:

$$\sum_{r_2\ell_2} \binom{-1 - \ell_1 + r_2}{\ell_2} \binom{\ell_1}{r_2} (-1)^{\ell_2} (t - t^{-1})^2(\ell_1-r_2) t^{-2(\ell_1+\ell_2)} \mu^{\ell_1} \quad (A6)$$

where we have used the formula:

$$\binom{r}{n} = (-1)^n \binom{-1 + n - r}{n} \quad (A7)$$

to rewrite the combinatorial number $\binom{r_1}{\ell_2}$. Now (A6) can be computed performing first the sum over $\ell_2$, next over $r_2$ and finally over $\ell_1$, the final net result is:

$$\text{Tr}(R^{ee} R^{ee}(\mu^{N-n} \otimes \mathbb{1})) = \frac{1}{(1 - t^{-2})(1 - \mu)}$$

(A8)

so that

$$Z_{\sigma_1} = \frac{1}{t - t^{-1}}$$

(A9)

which is precisely the inverse of the Alexander–Conway polynomial of the link.

Similarly we obtain for the reverse link

$$Z_{\sigma_1^{-2}} = \frac{-1}{t - t^{-1}}$$

(A10)
For more complex links or knots the calculation of the invariant requires
the knowledge of complicated combinatorial formulae. There is however a
way to overcome this difficulty, which consist in making an integral repre-
sentation of the δ “functions” in $R$ and $R^{-1}$; namely:

$$\delta_{r_1+r_2,r'_1+r'_2} = \oint \frac{dz}{2\pi i} z^{-1+r_1+r_2-r'_1-r'_2}$$

(A11)

where the contour integral is performed around the origin. Thus we associate
to each braiding matrix $R$ or $R^{-1}$ a complex variable $z$; having done this
we can perform all the sums over the entries so that $\text{tr}(\pi(\alpha)\mu^N \otimes \cdots \otimes \mathbb{I})$ gets converted into a multicontour integral.

Let us exemplify this method with the computation of the invariant of
the trefoil.

$$Z(\sigma_1^3 \in B_2) = (1 - \mu) t^{-2} \text{Tr} R^3(e,e)(\mu^N \otimes \mathbb{I})$$

$$\text{Tr}_\mu(R^3(e,e)) = \sum_{r,\ell,m} R_{m_1 m_2}^{r_1 r_2} R_{\ell_1 \ell_2}^{r_1 r_2} \mu^{m_1}$$

$$= \sum_{r,\ell,m} \oint \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} z_1^{-1+r_1+r_2-m_1-m_2} z_2^{-1+\ell_1+\ell_2-r_1-r_2} \mu^{m_1}$$

$$\begin{pmatrix} r_1 \\ m_2 \end{pmatrix} \begin{pmatrix} \ell_1 \\ r_2 \end{pmatrix} \begin{pmatrix} m_1 \\ \ell_2 \end{pmatrix} (t^{-1})^{r_1+\ell_1+m_1-r_2-\ell_2-m_2} t^{-(r_1+\ell_1+m_1+r_2+\ell_2+m_2)}$$

(A12)

Notice that we have not introduced an integral representation for the
delta $\delta_{\ell_1+\ell_2,m_1+m_2}$, since this is implied by the first two deltas in (A12).
The sum over $r_2, \ell_2$ and $m_2$ can be done straightforwardly yielding:

$$\oint \frac{dz_1 dz_2}{(2\pi i)^2} z_1^{-1} z_2^{-1}$$

$$\sum_{r_1} \left( \frac{1 + t(t^{-1}) z_1}{t^2 z_2} \right)^{r_1} \sum_{\ell_1} \left( \frac{z_1 + z_2 t(t^{-1})}{t^2} \right)^{\ell_1} \sum_{m_1} \left( \frac{\mu z_2 + (t^{-1})}{t^2 z_1} \right)^{m_1}$$

$$= \oint \frac{dz_1 dz_2}{(2\pi i)^2} \frac{t^6}{(t^2 z_2 - t(t^{-1}) z_1 - 1)(t^2 - z_1 - z_2 t(t^{-1}))(t^2 z_1 - \mu z_2 - \mu t(t^{-1}))}$$

(A13)
The convergence of the geometric sums in $r_1, \ell_1$ and $m_1$ impose the following restrictions on $z_1, z_2$ and $\mu$:

$$|1 + t(t-t^{-1})z_1| < |t^2z_2|$$

$$|z_1 + z_2t(t-t^{-1})| < |t^2|$$  \hspace{1cm} (A14)

$$|\mu(z_2 + t(t-t^{-1})| < |t^2z_1|$$

If $\mu = 0$ we see from (A13) that the integrand has a pole at $z_1 = 0$, integrating around this pole we get

$$\oint \frac{dz_2}{2\pi i} \frac{t}{(t^{-1} - t)} \frac{1}{(z_2 - \frac{1}{t})(z_2 - \frac{t}{t-t^{-1}})}$$  \hspace{1cm} (A15)

and from (A14) we see that

$$\frac{1}{|t^2|} < |z_2| < \left| \frac{t}{t-t^{-1}} \right|$$  \hspace{1cm} (A16)

So that finally

$$TrR^3 = \frac{t^2}{t^2 + t^{-2} - 1}$$  \hspace{1cm} (A17)

which yields:

$$Z(\text{Trefoil}) = \frac{1}{1 + (t-t^{-1})^2}$$  \hspace{1cm} (A18)

according to the conjecture (46).

We have also checked $\sigma_1^4$ in $B_2$ and $\sigma_1^2\sigma_2^2$ and $(\sigma_1\sigma_2^{-1})^3$ in $B_3$, obtaining the result (44).
Appendix B

Clebsch–Gordan coefficients of $U_q(h_4)$

If $(e_1, n_1)$ and $(e_2, n_2)$ are two irreps of $U_q(h_4)$ with $e_1, e_2$ and $e_1 + e_2$ different from zero then the tensor product $(e_1, n_1) \otimes (e_2, n_2)$ decomposes into irreps as follows:

$$(e_1, n_1) \otimes (e_2, n_2) = \bigoplus_{n \geq 0} (e_1 + e_2, n_1 + n_2 + n) \quad (B1)$$

The normalized highest weight vector of the irrep $(e_1 + e_2, n_1 + n_2 + n)$ is given by:

$$|0; e_1 + e_2, n_1 + n_2 + n> = \frac{1}{[e_1 + e_2]^{n/2}}$$

$$\times \sum_{r \geq 0} (-1)^r [e_1] \frac{a^{e_1}}{2} [e_2] \frac{a^{e_2}}{2} q^{-\frac{e_1 e_2}{2}} q^{-\frac{e_1 + e_2}{2} n} \left( \frac{n}{r} \right)^{1/2}$$

$$\times |r; e_1, n_1 > \otimes |n - r; e_2, n_2> \quad (B2)$$

The whole CG–coefficients for the tensor product decomposition (B1) can be found from (B2) acting with the creation operators $a^+$. An interesting application of (B2) is to find the braiding factors $\phi$ associated with the $R$–matrix (33) which we define as:

$$R^{(e_1 n_1), (e_2 n_2)}_{(e_1 + e_2, n_1 + n_2 + n)} K^{(e_1 n_1), (e_2 n_2)}_{(e_1 + e_2, n_1 + n_2 + n)}$$

$$= \phi^{(e_1 n_1), (e_2 n_2)}_{(e_1 + e_2, n_1 + n_2 + n)} K^{(e_2 n_2), (e_1 n_1)}_{(e_1 + e_2, n_1 + n_2 + n)} \quad (B3)$$

where $K^{12}$ is the CG operator: $V_3 \to V_1 \otimes V_2$, which maps the irrep “3” into the tensor product “1 $\otimes$ 2”.

Using (33) and (B2) one finds

$$\phi^{(e_1 n_1), (e_2 n_2)}_{(e_1 + e_2, n_1 + n_2 + n)} = (-1)^n q^{-e_1 n_2 - e_2 n_1} \quad (B4)$$

Putting back the factor $q^{-e_1 n_2 - e_2 n_1}$, that we discard in the definition of $R^{e_1 e_2}$ (eq. 32), we see that the whole braiding factor induced by the $R$ matrix is given by:
\[ (-1)^n q^{e_1n_1+e_2n_2-(e_1+e_2)(n_1+n_2+n_1)} \quad (B5) \]

We recognize in this expression the classical casimir $C_1$ in the exponential $q^n$, which is the analog of the braiding factor $q^{j(j+1)}$ of the quantum group $U_q(SU(4))$. Similarly $(-1)^n$ is a parity factor analogous to $(-1)^j$ in SU(2).
Appendix C

$U_q(h_4)$ and Ribbon Hopf Algebras

Any quasitriangular $A$ has an invertible element, usually called $u$, with the property that

$$\gamma^2(a) = uau^{-1} \quad \forall a \in A \quad (C1)$$

The element $u$ and its inverse $u^{-1}$ can be obtained from the universal $R$ matrix as follows:

$$u = m[\gamma \otimes id(\sigma(R))] \quad (C2)$$

$$u^{-1} = m[id \otimes \gamma^2(\sigma(R))]$$

In the particular case of the universal $R$–matrix (30) of the quantum group $U_q(h_4)$ we obtain:

$$u = \sum_{\ell \geq 0} \frac{(q^{-1}-q)^\ell}{\ell!} q^{-E}a^+a^{\ell}q^{-2E} \quad (C3)$$

$$u^{-1} = \sum_{\ell \geq 0} \frac{(q-q^{-1})^\ell}{\ell!} q^E(a^+)a^{\ell}q^{-2E}$$

The element $u$ of the quantum deformation $U_q(G)$ of a complex simple Lie algebra $G$ does not in general commute with the others elements of the algebra, however for $U_q(h_4)$ we have from eqs (27) that $\gamma^2 = id$ so that $u$ is central. Indeed using (C3) and (31) we obtain:

$$u|r; (e, n) >= q^{2E}n|r; (e, n) > \quad (C4)$$

Similarly $\gamma(u)$ is also central and one can prove that

$$\gamma(u) = q^{-2E}u \quad (C5)$$

We are now in conditions to study whether $U_q(h_4)$ is a Ribbon Hopf algebra. Following Reshetikhin and Turaev [2] we define a Ribbon Hopf Algebra $(A, R, v)$ as a quasitriangular Hopf algebra with a universal $R$ matrix and the choice of an element $v$ such that:
\[
v \quad \text{is central}
\]
\[
v^2 = u\gamma(u)
\]
\[
\gamma(v) = v
\]
\[
\varepsilon(v) = 1
\]
\[
\Delta(vu^{-1}) = vu^{-1} \otimes vu^{-1}
\]

In the case of \( U_q(h_4) \) we can easily see using eq. (C5) that the element \( v \) is given by:

\[
v = q^{-E} u
\]

which in the irrep \((e, n)\) takes the value

\[
v|r; (e, n) >= q^{(2n-1)e}|r; (e, n) >
\]

The fact that \( v^{-1}u = q^E \) explains the presence of the term \( q^m \) in the invariant (37) or the corresponding term in the multicolored version (49), and it is the \( U_q(h_4) \) analogue of the operator \( q^{H/2} \) of the quantum group \( U_q(SU(2)) \).

The value of \( v \) in a given representation of a quantum group \( A \) contains some interesting information of the corresponding conformal field theory associated to \( A \). In the cases where \( A = U_q(\mathcal{G}) \) with \( \mathcal{G} \) a simple Lie algebra it was shown in reference [14] that the conformal weight \( \Delta_\alpha \) of a primary field of the WZW model \( \hat{\mathcal{G}}_k \) is related to the value \( v_\alpha \) by:

\[
v_\alpha = e^{2\pi i \Delta_\alpha}
\]

where \( v_\alpha \) is the value of \( v \) on the irrep \( \alpha \) of \( U_q(\mathcal{G}) \) associated to the primary field \( \alpha \). If eq. (C9) would holds true for \( h_4 \) it would imply that

\[
q^{(2n-1)e} = e^{2\pi i \Delta(\epsilon,e,n)}
\]

In ref [7], where it is consider a WZW theory based on the supergroup \( GL(1,1) \), eq. (C10) is violated by higher order corrections of order \( 1/k^2 \) with \( k \) the level of the Kac–Moody supergroup GL(1,1).

In our case we do not know yet whether there is an underlying conformal field theory associated to the quantum group \( U_q(h_4) \).
References

[1] V.G. Drinfeld, “Quantum Groups”, Proceedings of the 1986 International Congress of Mathematics at Berkeley ed. A.M. Gleason (1987) Am. Math. Soc., 1, p. 798.
   M. Jimbo, Comm. Math. Phys., 102 (1986) 537.

[2] A.N. Kirillov and N.Yu. Reshetikhin, ‘Representations of the algebra $U_q(SU(2))$, $q$–orthogonal polynomials and invariants of links’. Leningrad preprint LOMI-E-9-88.
   N.Yu. Reshetikhin and V.G. Turaev. Comm. Math. Phys., 127 (1990) 1.

[3] V.G. Turaev. Invent. Math., 92 (1988) 527.

[4] V.F.R. Jones. Ann. Math., 126 (1987) 335.

[5] L.H. Kauffman. “On Knots” Ann. of Math. Studies No 115. Princeton University Press 1987.

[6] Y. Akutsu, A. Kuniba and M. Wadati. Phys. Rep., 180 (1989) 427.

[7] L.H. Kauffman and H. Saleur. “Free Fermions and the Conway Alexander polynomial”. Comm. Math. Phys. (1991) in press.
   L. Rozansky and H. Saleur. YCTP-P20-91.

[8] E. Date, M. Jimbo, K. Miki and T. Miwa. RIMS-729 preprint (1990).

[9] Y. Akutsu and T.K. Deguchi. Phys. Rev. Lett., 67 (1991) 777.

[10] E. Date, M. Jimbo, K. Miki and T. Miwa. RIMS-715 (1990), RIMS-706 (1990), RIMS-703 (1990).

[11] C. Gómez, M. Ruiz Altaba and G. Sierra. Phys. Lett., B265 (1991) 95.

[12] C. Gómez and G. Sierra. “A New Solution to the Star–Triangle Equation based on $U_q(SU(2))$ at roots of unit”. CERN preprint TH 6200/91.

[13] V.F.R. Jones. Int. Journ. of Modern Phys. A Vol. 6 (1991) 2035.

[14] A.J. MacFarlane. J. Phys. A: Math. Gen., 22 (1989) 4581.
   L.C. Biedenharn. J. Phys. A: Math. Gen., 22 (1989) L.873.

[15] L. Alvarez–Gaumé, C. Gómez and G. Sierra. Nucl. Phys., B330 (1990) 347.