Permutation symmetry and entanglement in quantum states of heterogeneous systems

Gururaj Kadiri and S. Sivakumar

Materials Science Group, Indira Gandhi Centre for Atomic Research, Kalpakkam, Tamilnadu, India. Pin: 603102

Permutation symmetries of multipartite quantum states are defined only when the constituent subsystems are of equal dimensions. In this work we extend this notion of permutation symmetry to heterogeneous systems, that is, systems composed of subsystems having unequal dimensions. Given a tensor product space of \( k \) subsystems (of arbitrary dimensions) and a permutation operation \( \sigma \) over \( k \) symbols, these states are such that they have identical decompositions (up to an overall phase) in the given tensor product space and the tensor product space obtained by the permuting the subsystems by \( \sigma \). Towards this, we construct a matrix whose action is to simultaneously permute the subsystem label and subsystem dimension of a given state according to permutation \( \sigma \). Eigenvectors of this matrix have the required symmetry. We then examine entanglement of states in the eigenspaces of these matrices. It is found that all nonsymmetric eigenspaces of such matrices are completely entangled subspaces, with states being equally entangled in both the given tensor product space and the permuted tensor product space.

I. INTRODUCTION

Quantum theory is usually formulated in terms of states vectors which are considered as elements of a suitable Hilbert space. For each classical degree of freedom, the quantum formulation requires a corresponding Hilbert space. Thus, a system of two 1D oscillators requires two Hilbert spaces. If there are quantum degrees of freedom such as the spin of a particle, they too will have their respective Hilbert spaces. The right way of describing the system with more than one degree of freedom turns out to be the tensor product of the Hilbert spaces corresponding to the various degrees of freedom relevant to the system, so such tensor product spaces are central for the description of multipartite quantum systems. While pure states of multiparticle quantum systems could also be represented as a ray in \( \mathbb{C}^N \) for an appropriate \( N \), the counterintuitive features of such states, like the nonlocality, entanglement etc, do not manifest in this unfactored space \( \mathbb{C}^N \) but will manifest only in the tensor product of the Hilbert spaces of the constituent systems.

A tensor product space (TPS) is homogeneous if the constituent subsystems are of equal dimension. Otherwise, it is said to be heterogeneous [1]. In the homogeneous \( k \)–partite TPS having \( d \) dimensional subsystems, the “symmetric subspace” of \( \mathbb{C}^N \) (where \( N = d^k \)) consists of states that remain invariant under arbitrary permutation of their subsystem labels. The symmetric subspace is interesting because its dimension scales with \( k \) like the binomial coefficient \( (d+k-1)C_k \), while the dimension of the composite system increases exponentially like \( d^k \). In the case of qubits (\( d = 2 \)), the symmetric subspace is spanned by the Dicke basis [2]. Symmetric states of homogeneous systems, particularly of the multiparticle qubits, have been extensively studied both experimentally [3–7] and theoretically [8–12], with respect to their tomography [13–15], entanglement [16–19] etc. Though heterogeneous systems also have been studied theoretically [20–25], and experimentally [26, 27], the notion of permutation symmetry is not readily extendable to them. In this work, we demonstrate that there is a natural way to extend the conventional notion of permutation symmetry to heterogeneous systems.

To motivate such a construction, consider a quantum system \( S \), whose Hilbert space is \( H_S \) of dimension \( d_S \). Assume that \( S \) is allowed to interact with the environment \( E \), whose Hilbert space is \( H_E \) of dimension \( d_E \). The state of the composite system \( (S + E) \) can be represented in the tensor product space \( H_S \otimes H_E \) or the tensor product space \( H_E \otimes H_S \). Consider an arbitrary state \( \psi_{SE} \) in the TPS \( H_S \otimes H_E \):

\[
|\psi_{SE}\rangle = \sum_{i,j} \alpha_{ij} |i \rangle \otimes |j \rangle
\]

(1)

where \( \{|i\rangle\}_{i=0}^{d_S-1} \) and \( \{|j\rangle\}_{j=0}^{d_E-1} \) are orthornomal bases for the system and reservoir respectively. The state “physically equivalent” to \( |\psi_{SE}\rangle \) in the TPS \( H_E \otimes H_S \), is

\[
|\psi_{ES}\rangle = \sum_{i,j} \alpha_{ij} |j \rangle \otimes |i \rangle
\]

(2)

State \( |\psi_{ES}\rangle \) is physically equivalent to \( |\psi_{SE}\rangle \) in the sense that the expectation value of any operator \( \hat{M} \) of the system \( S \) is identical in both the states:

\[
\langle \psi_{SE}| \hat{M} \otimes \hat{I}_E |\psi_{SE}\rangle = \langle \psi_{ES}| \hat{I}_E \otimes \hat{M} |\psi_{ES}\rangle
\]

where \( \hat{I}_E \) is the identity \( H_E \). Similarly, reduced density matrices corresponding to \( S \), obtained by tracing out the second subsystem from \( |\psi_{SE}\rangle \langle \psi_{SE}| \) or the first subsystem from \( |\psi_{ES}\rangle \langle \psi_{ES}| \) are identical. Further, the numerical measure of entanglement of the state \( |\psi_{SE}\rangle \) in the tensor product space \( H_S \otimes H_E \) is identical to that of the state \( |\psi_{ES}\rangle \) in the tensor product space \( H_E \otimes H_S \).

However, as tensor product operation is not commutative, \( |j \rangle \otimes |i \rangle \) is not necessarily equal to \( |i \rangle \otimes |j \rangle \), and hence states \( |\psi_{SE}\rangle \) and \( |\psi_{ES}\rangle \) can be distinct when seen
as states in $\mathbb{C}^N$. A state $|\psi_{SE}\rangle$ is called exchange invariant if it is identical to $|\psi_{ES}\rangle$, upto an overall phase factor. In other words, a state $|\psi\rangle \in \mathbb{C}^N$ is exchange invariant if it remains invariant under the transformation

$$|i\rangle \otimes |j\rangle \rightarrow |j\rangle \otimes |i\rangle$$

for all $i = 0, \ldots, d_S - 1$ and $j = 0, \ldots, d_E - 1$, where $\{|i\rangle\}_{i=0}^{d_S-1}$ and $\{|j\rangle\}_{j=0}^{d_E-1}$ are two arbitrary orthonormal basis of the two subsystems.

For example, consider $d_S = 2$ and $d_E = 3$. Consider the computational basis state $|3\rangle$ in $\mathbb{C}^5$. This state in the $\mathbb{C}^2 \otimes \mathbb{C}^3$ tensor product space is $|1\rangle \otimes |0\rangle$. The physical equivalent state of this in $\mathbb{C}^3 \otimes \mathbb{C}^2$ is $|0\rangle \otimes |1\rangle$. But $|0\rangle \otimes |1\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^2$ corresponds to the state $|1\rangle$ in $\mathbb{C}^6$, rather than $|3\rangle$ we began with. So state $|3\rangle$ is not symmetric in the qubit-qutrit decomposition. Consider, on the other hand, the computational basis state $|5\rangle$. This state in $\mathbb{C}^2 \otimes \mathbb{C}^3$ is $|1\rangle \otimes |2\rangle$. The physical equivalent state of this in the $\mathbb{C}^3 \otimes \mathbb{C}^2$ is $|2\rangle \otimes |1\rangle$. Since $|2\rangle \otimes |1\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^2$ corresponds to the same state $|5\rangle$ in $\mathbb{C}^6$, state $|5\rangle$ is a symmetric state.

Similarly, consider the state $\frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |4\rangle)$ in $\mathbb{C}^6$. This state in the $\mathbb{C}^2 \otimes \mathbb{C}^4$ is $\frac{1}{\sqrt{3}} (|01\rangle + |02\rangle + |10\rangle)$. The physical equivalent state to this in the $\mathbb{C}^4 \otimes \mathbb{C}^2$ is $\frac{1}{\sqrt{3}} (|10\rangle + |20\rangle + |01\rangle)$. This state also corresponds to the same state $\frac{1}{\sqrt{3}} (|2\rangle + |4\rangle + |1\rangle)$ in $\mathbb{C}^8$, so $\frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |4\rangle)$ is a symmetric state in the qubit-qutrit bipartite system.

This notion of exchange symmetry in heterogenous bipartite systems can be extended to permutation symmetry of multipartite heterogenous systems as well. First, notations to be used subsequently are explained. A multiplicative partition of $N$ is represented by the $k$-tuple $d = [d_1, d_2, \ldots, d_k]$, where $d_i$s are positive integers greater than 1 such that $\prod d_i = N$. The number of elements in $d$ is denoted by $n(d)$. Corresponding to this $d$, the $k$-partite TPS $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$ is represented by $\mathbb{C}^d$.

Let $\sigma$ be one of the elements of $S_{n(d)}$, the group of permutations over $n(d)$ symbols. Given $d$ and a $\sigma$, another multiplicative partition $\sigma(d)$ of $N$ is obtained by permuting the entries in $d$ by $\sigma$, that is, $\sigma(d) = [d_{\sigma^{-1}(1)}, d_{\sigma^{-1}(2)}, \ldots, d_{\sigma^{-1}(k)}]$. The TPS corresponding to this partition is $\mathbb{C}^{\sigma(d)}$. As in the bipartite case, the state of a multipartite composite system can be represented equally well in any of the TPS, $\mathbb{C}^{\sigma(d)}$ for any $\sigma \in S_{n(d)}$, although the number of subsystems $n(d)$ and the dimension $d_i$ of each subsystem are decided by the experiment.

Given the $k$-partite TPS $\mathbb{C}^d$, a basis for $\mathbb{C}^N$ is constructed from the tensor product of the $k$ bases $\mathbb{B}_{d_1}, \mathbb{B}_{d_2}, \ldots, \mathbb{B}_{d_k}$ of the individual subsystems where $\mathbb{B}_{d_i} = \{|i\rangle\}_{i=0}^{d_i-1}$ is an orthonormal basis for $\mathbb{C}^{d_i}$. This tensor product basis is denoted by $\mathbb{B}_d$. An element in $\mathbb{B}_d$ is of the form $|i_1 \rangle \otimes |i_2 \rangle \otimes \cdots \otimes |i_k \rangle$, where $|i_r \rangle \in \mathbb{B}_{d_r}$. This state is expressed in short notation as $|i_1 i_2 \cdots i_k\rangle_d$.

Similarly, another basis for $\mathbb{C}^N$ could be the tensor product of the bases in the permuted order: $\mathbb{B}_{d_{\sigma^{-1}(1)}} \otimes \mathbb{B}_{d_{\sigma^{-1}(2)}} \otimes \cdots \otimes \mathbb{B}_{d_{\sigma^{-1}(k)}}$. This basis is denoted as $\mathbb{B}_{\sigma(d)}$, suffix indicating that it has been obtained by a permutation of another basis. An element in $\mathbb{B}_{\sigma(d)}$ is of the form $|i_{\sigma^{-1}(1)}\rangle \otimes |i_{\sigma^{-1}(2)}\rangle \otimes \cdots \otimes |i_{\sigma^{-1}(k)}\rangle$ where $|i_r\rangle \in \mathbb{B}_{d_r}$. A short notation for this state is as $|i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(k)}\rangle_{d_{\sigma(d)}}$.

Given a TPS $\mathbb{C}^d$ and a permutation $\sigma \in S_{n(d)}$, a state $|\psi\rangle$ is invariant under permutation $\sigma$ if it remains invariant under the mapping

$$|i_1 i_2 \cdots i_k\rangle_d \rightarrow |i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(k)}\rangle_{d_{\sigma(d)}}$$

for all $0 \leq i_r \leq d_r - 1$ and $1 \leq r \leq k$ where $k = n(d)$. In the bipartite case, $\sigma$ is the permutation $(1, 2)$.

Towards achieving this mapping we construct an operator $\hat{T}_{d,\sigma} : \mathbb{B}_d \rightarrow \mathbb{B}_{\sigma(d)}$, such that

$$\hat{T}_{d,\sigma} |i_1 i_2 \cdots i_k\rangle_d = |i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(k)}\rangle_{\sigma(d)}.$$ 

Eigenvectors of $\hat{T}_{d,\sigma}$ are the states satisfying the desired mapping defined in Eqn. 4. Being a unitary transformation in $\mathbb{C}^N$, its eigenvalues are complex numbers of unit moduli. Given a TPS $\mathbb{C}^d$ and a permutation $\sigma$, the Hilbert space of the composite systems $\mathbb{C}^N$ thus splits into disjoint eigenspaces of $\hat{T}_{d,\sigma}$:

$$\mathbb{C}^N \simeq \bigoplus_{\eta} S_{n(d)}^{\eta}_{d,\sigma}.$$ 

Here $S_{n(d)}^{\eta}_{d,\sigma}$ is a subspace of $\mathbb{C}^N$, composed of eigenstates of $\hat{T}_{d,\sigma}$ with eigenvalue $\eta$. States in the subspaces $S_{n(d)}^{\eta}_{d,\sigma}$ are such that the reduced density matrix of the $r$th subsystem in TPS $\mathbb{C}^d$ is identical to the reduced density matrix of the $\sigma(r)$th subsystem in TPS $\mathbb{C}^\sigma(d)$. This work provides a prescription for obtaining the dimensions and bases of these subspaces.

The paper is organized as follows. A procedure for constructing bipartite exchange invariant states is detailed in Section II. A multipartite extension of this construction to obtain states that are invariant under an arbitrary permutation of subsystems is provided in Section III. In Section IV, we examine the entanglement of states in the subspaces $S_{n(d)}^{\eta}_{d,\sigma}$, with respect to both the TPSSs, $\mathbb{C}^d$ and $\mathbb{C}^\sigma(d)$. In recent years, it has been argued that entanglement needs to be defined with respect to a distinguished set of observables rather than with respect to a distinguished tensor product space [28–32]. However, in this paper we stick to the conventional notion of entanglement, but examine it in different tensor product spaces. Results are summarized in Section V.

We provide a list of symbols appearing in this paper along with their brief description in Tables VII and VIII.
II. BIPARTITE EXCHANGE SYMMETRY

In the bipartite case, the action of the matrix \( \hat{T}_{[d_1,d_2]} \) on the product state \( |i⟩ \otimes |j⟩ \in \mathbb{B}_{d_1} \otimes \mathbb{B}_{d_2} \) is given by

\[
\hat{T}_{[d_1,d_2]} (|i⟩ \otimes |j⟩) = |j⟩ \otimes |i⟩.
\]

(7)

The matrix representation of \( \hat{T}_{[d_1,d_2]} \) is the tensor commutator matrix (TCM) \([33]\). The subscript \([d_1,d_2]\) indicates that \( \hat{T}_{[d_1,d_2]} \) maps product states in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \) to the corresponding product states in \( \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1} \). The eigenvectors of \( \hat{T}_{[d_1,d_2]} \) are the states that are exchange invariant.

The explicit form of \( \hat{T}_{[d_1,d_2]} \) defined as a mapping on the span of \( \mathbb{B}_{d_1} \otimes \mathbb{B}_{d_2} \) is

\[
\hat{T}_{[d_1,d_2]} = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} (|j⟩ \otimes |i⟩) (⟨i| \otimes ⟨j|),
\]

(8)

where \( \{|i⟩\}_{i=0}^{d_1-1} \) and \( \{|j⟩\}_{j=0}^{d_2-1} \) are arbitrary bases for \( \mathbb{C}^{d_1} \) and \( \mathbb{C}^{d_2} \) respectively. In the computational basis of \( \mathbb{C}^N \) the matrix elements of \( \hat{T}_{[d_1,d_2]} \) are:

\[
\hat{T}_{[d_1,d_2]} = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{m-1}{d_1} & \cdots & \frac{m-1}{d_1} \\
\vdots & \ddots & \vdots \\
\frac{m-1}{d_1} & \cdots & \frac{m-1}{d_1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{n-1}{d_2} & \cdots & \frac{n-1}{d_2} \\
\vdots & \ddots & \vdots \\
\frac{n-1}{d_2} & \cdots & \frac{n-1}{d_2}
\end{bmatrix}
\]

is three-dimensional, \( S_{[2,3]} \) is denoted by \( S_{[2,3]} \). It is easy to see that every vector in \( S_{[2,3]} \) is indeed exchange invariant. The subspace associated with eigenvalue \(-1\) is one-dimensional,

\[
S_{[2,3]} = \text{span}\left\{\frac{1}{2}(|1⟩ - |2⟩ - |3⟩ + |4⟩)\right\}
\]

(13)

The eigenvectors of \( \hat{T}_{[2,3]} \) have been expressed in the basis for \( \mathbb{C}^6 \). To see their exchange symmetry, the states are expressed in the \( \mathbb{B}_{[2,3]} \) and \( \mathbb{B}_{[3,2]} \) bases. This requires to establish a correspondence between the states in \( \mathbb{B}_N \) and those in \( \mathbb{B}_{[d_1,d_2]} \). Given one of the computational basis states \(|m⟩\) in \( \mathbb{B}_N \), its representation in the tensor product basis \( \mathbb{B}_{[d_1,d_2]} \) is

\[
|m⟩ = |i⟩_{d_1} \otimes |j⟩_{d_2} = |i,j⟩_{[d_1,d_2]},
\]

(14)

where \( i = \left\lfloor \frac{m}{d_1} \right\rfloor \), \( j = \text{mod}(m,d_2) \). Conversely, given a state \(|i,j⟩_{[d_1,d_2]} \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \), its representation in \( \mathbb{C}^N \) is

\[
|i,j⟩_{[d_1,d_2]} = |i \times d_2 + j⟩
\]

(15)

\[
S_{[2,3]}^{-1} \text{ expressed in } \mathbb{C}^2 \otimes \mathbb{C}^3 \text{ is}
\]

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{bmatrix}
\]

whereas in \( \mathbb{C}^3 \otimes \mathbb{C}^2 \) this is

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

which acquires an overall negative sign under simultaneous exchange of subsystem states and dimensions. The respective reduced density matrices are also identical,

\[
[2,3]ρ_1 = [3,2]ρ_2 = \frac{1}{4} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

and
where $|d_i,d_j\rangle_{\rho_i}$, $i=1,2$, refers to the reduced density matrix of the $i^{th}$ subsystem after tracing out the other subsystem for a state $|\psi\rangle$ in the $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ decomposition. The corresponding reduced density matrices are identical for exchange symmetric states. However, an arbitrary state $|\psi\rangle \in \mathbb{C}^6$ need not yield identical reduced density matrices as in this example.

For instance, consider the state

$$|\psi(p)\rangle = \sqrt{p}|0\rangle + \sqrt{1-p} \left( \frac{1}{2} (|1\rangle - |2\rangle - |3\rangle + |4\rangle) \right),$$

which is a linear combination of one of the symmetric states $|0\rangle$ and anti-symmetric state of Eqn. 13. This state is not exchange symmetric unless $p = 0, 1$. Other values of $p$ correspond to the state being asymmetric. The relevant reduced density matrices of suitable dimensions are compared using trace distance. Denoting the trace distance between the $2 \times 2$ density matrices $[2,3]_{\rho_1}$ and $[3,2]_{\rho_2}$ by $d_2(p)$ and that between the $3 \times 3$ density matrices $[2,3]_{\rho_2}$ and $[3,2]_{\rho_1}$ by $d_3(p)$, we have

$$d_2(p) = \frac{1}{2} \sum_{i} |\lambda_{2,i}|, \quad d_3(p) = \frac{1}{2} \sum_{i} |\lambda_{3,i}|$$

(17)

where $\lambda_{2,i}$ are eigenvalues of $(|2,3\rangle_{\rho_1} - |3,2\rangle_{\rho_2})$ and $\lambda_{3,i}$ are eigenvalues of $(|2,3\rangle_{\rho_2} - |3,2\rangle_{\rho_1})$. Figure 1 shows the variation of $d_2(p)$ (blue plot) and $d_3(p)$ (green plot) as a function of $p$.

![Figure 1. Trace distances $d_2(p)$ (blue plot) and $d_3(p)$ (green plot) as function of $p$, for $|\psi(p)\rangle$ given in Eqn. 16.](image)

The trace distances are symmetric about $p = 1/2$, which corresponds to the most asymmetric state. Any deviation from $p = 1/2$ takes the state $|\psi(p)\rangle$ closer to either symmetric ($p < 1/2$) or antisymmetric ($p > 1/2$) state. The trace distance peaks at $p = 1/2$, for which the $3 \times 3$ reduced density matrices $[2,3]_{\rho_2}$ and $[3,2]_{\rho_1}$ orthogonal to each other:

$$[2,3]_{\rho_2} = \frac{1}{4} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$  

(20)

It is to be noted that $[2,3]_{\rho_2}$ is a mixed state whereas $[3,2]_{\rho_1}$ is a pure state. This implies that the state $|\psi(p = 0.5)\rangle$ is entangled in $[2,3]$ partition but separable in $[3,2]$ partition.

The other two eigenvectors of $\hat{T}_{[2,3]}$ also give identical reduced density matrices in both the decompositions. One marked difference between the case $d = [2,2]$ discussed earlier and $d = [2,3]$ case is the emergence of eigenstates which acquire a phase $\neq 0, \pi$ under subsystem exchange operation. It will be demonstrated, for every heterogeneous bipartite decomposition ($d_1 \neq d_2$), there are subspaces spanned by those states that acquire a phase $e^{i\phi}$, $\phi \neq 0, \pi$ under exchange of subsystems.

### A. $\hat{T}_{[d_1,d_2]}$ as a permutation matrix

Rules for relating the vectors in $\mathbb{B}$, and the TPS $\mathbb{B}_d$ are already given in Eqs. 14 and 15. Vectors in the basis $\mathbb{B}_{[d_1,d_2]}$ and $\mathbb{B}_{[d_2,d_1]}$ are related by the mapping $\hat{T}_{[d_1,d_2]}$, whose matrix representation in the computational basis is a permutation matrix. Here, the permutation effected by this matrix on the basis states is identified.

![Figure 2. Schematic of the procedure employed to obtain the permutation corresponding to the permutation matrix $\hat{T}_{[d_1,d_2]}$.](image)

Towards this, begin with a state $|L_n\rangle \in \mathbb{B}$ where $0 \leq L_n \leq d_1 d_2 - 1$. Let the representation of this state in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ partition be $|i,j\rangle_{[d_1,d_2]}$ (refer Eqn. 14). The action of $\hat{T}_{[d_1,d_2]}$ is to map this state into the state $|j,i\rangle_{[d_2,d_1]} \in \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$. This corresponds to a state $|L_{n+1}\rangle$ in the unpartitioned space, where $L_{n+1} = j \times d_1 + i$ (refer Eqn 15). See Fig. 2 for the sequence of operations. This $|L_{n+1}\rangle$ can once again be expressed in the $[d_1,d_2]$ partition from which $L_{n+2}$ can be obtained by swapping the indices along with the dimensions as was done for $L_{n+1}$. This process is repeated until $L_{n+m+1}$ becomes $L_n$ for some $m$, which is guaranteed since $\hat{T}_{[d_1,d_2]}$ is one-to-one and, therefore, invertible. States $(|L_n\rangle, |L_{n+1}\rangle, \ldots, |L_{n+m}\rangle)$ form one cycle. This cycle is called an $m$-cycle as it has $m$ states.
It is seen that index numbers \( L_{n+1} \) can be obtained from \( L_n \) using the relation:

\[
L_{n+1} = d_1 L_n - \frac{L_n}{d_2} (N - 1).
\]

To generate another cycle, pick up another state from \( \mathfrak{B} \) not already present in any cycle as \( |L_n| \) and generate another cycle in the same manner as above (or using Eqn. 18). Repeat the process until every state in \( \mathfrak{B} \) is accommodated in some cycle. It may be noted that the sum of the lengths of the cycles equals the dimension of \( \mathfrak{B} \).

The action of \( \hat{T}_{[d_1,d_2]} \) is to group the basis vectors of \( \mathfrak{B} \) into disjoint sets corresponding to each cycle. The vectors in a given disjoint set are in the orbit of the mapping \( \hat{T}_{[d_1,d_2]} \). Hence, this cycle decomposition represents the permutation \( \pi \) corresponding to \( \hat{T}_{[d_1,d_2]} \).

As an illustration, \( \pi (2,3) \) is explicitly constructed. Here, \( \mathfrak{B} \) is \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}. Consider state \{0\} of \( \mathfrak{B} \). Its representation in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is \( |0\rangle \otimes |0\rangle \) which under the action of \( \hat{T}_{[2,3]} \) goes over to \( |0\rangle \otimes |0\rangle \) in \( \mathbb{C}^3 \otimes \mathbb{C}^2 \) which again corresponds to \{0\} in \( \mathbb{C}^6 \). Similarly, \{1\} in \( \mathbb{C}^6 \) corresponds to \{0\} \( \otimes \{1\} \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) which under the action of \( \hat{T}_{[2,3]} \) goes to \{1\} \( \otimes \{1\} \) in \( \mathbb{C}^3 \otimes \mathbb{C}^2 \) which corresponds to \{2\} in \( \mathbb{C}^6 \). This is illustrated in Table I. One this is done, the cycles can be obtained easily: \{0\} in the left-most column is getting mapped to \{0\} in the right-most column, so \{0\} is a 1-cycle. Similarly we have a sequence of states \{1\} \( \rightarrow \{2\} \rightarrow \{4\} \rightarrow \{3\} \rightarrow \{1\} \) so \{1, 2, 4, 3\} is another cycle. And \{5\} is another 1-cycle. So the cycle decomposition corresponding to \( \hat{T}_{[2,3]} \) is \( \pi (2,3) = (0, 1, 2, 4, 3, 5) \).

For illustration, consider \( d_1 = 2 \) and \( d_2 = 4 \). Since \( \pi (2,4) = (0, 1, 2, 4, 3, 6, 5, 7) \), eigenvalues of \( \hat{T}_{[2,4]} \) are \( \{1, \omega, \omega^2 \} \) where \( \omega \) is the primitive cube root of unity, \( e^{2\pi i/3} \). The symmetric subspace \( S^1_{[2,4]} \) is four-dimensional, given by:

\[
\text{span} \left\{ |0\rangle, \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |4\rangle), \frac{1}{\sqrt{3}} (|3\rangle + |6\rangle + |5\rangle), |7\rangle \right\}
\]

Similarly, eigenspaces corresponding to \( \omega \) and \( \omega^2 \), \( S^\omega_{[2,3]} \) and \( S^{\omega^2}_{[2,4]} \) are given by:

\[
\text{span} \left\{ \frac{1}{\sqrt{3}} (\omega^2 |1\rangle + \omega |2\rangle + |4\rangle), \frac{1}{\sqrt{3}} (\omega^2 |3\rangle + \omega |5\rangle + |6\rangle) \right\}
\]
and

$$\text{span} \left\{ \frac{1}{\sqrt{3}} \left( \omega \ket{1} + \omega^2 \ket{2} + \ket{4} \right), \frac{1}{\sqrt{3}} \left( \omega \ket{3} + \omega^2 \ket{5} + \ket{6} \right) \right\}$$

respectively. As there are no cycles of even length in $\pi(2,4)$, there is no anti-symmetric subspace here.

When $d_1 = d_2 = d$, the matrix $\hat{T}_{[d,d]}$ is involutory and its eigenvalues are $\{\pm 1\}$. Consequently, there are only symmetric and anti-symmetric states. This feature is brought out in $\pi(d,d)$ as well; it has $d$ fixed points (cycles of unit length) corresponding to the states $\ket{(d+1)i}$, $i = 0 \cdots d-1$ and the rest are 2-cycles. For example, $\pi(3,3)$ Table II is seen to have these features.

The dimensions of symmetric and anti-symmetric subspaces do not necessarily increase with increasing $N$ when $d_1 \neq d_2$. This is also evident from Table II where the number of cycles in $\pi(d_1,d_2)$ does not necessarily increase with $d_1$ or $d_2$. Figure 3 shows the dimensions of the symmetric and anti-symmetric subspaces for $\hat{T}_{[2,d]}$ for $d = 2$ to 29.

![Figure 3. Dimension of the symmetric (diamond) and anti-symmetric (star) subspaces of $\hat{T}_{[2,d]}$, for different values of $d$.](image)

The symmetric subspace of any $\hat{T}_{[d_1,d_2]}$ is at least three-dimensional, as any $\pi(d_1,d_2)$ has at least three cycles: two cycles corresponding to states $\ket{0}$ and $\ket{N-1}$ and another cycle comprising of the rest of the states. Note that from Eqn. 18 it follows that $\{0\}$ and $\{N-1\}$ are two fixed points in $\pi(d_1,d_2)$ for all $d_1$ and $d_2$.

Given $d_1$ and $d_2$, the eigenstates of $\hat{T}_{[d_1,d_2]}$ furnish a special basis for $C^N$. These basis vectors have identical decompositions in $C^{d_1} \otimes C^{d_2}$ and $C^{d_2} \otimes C^{d_2}$ partitions. This special basis is denoted by $B_{T_{[d_1,d_2]}}$.

From Eqn. 7, it is seen that $\hat{T}_{[d_2,d_1]}$ is the inverse of $\hat{T}_{[d_1,d_2]}$. The eigenvalues of $\hat{T}_{[d_1,d_2]}$ come in complex conjugate pairs. Hence $\hat{T}_{[d_1,d_2]}$ and $\hat{T}_{[d_2,d_1]}$ share the same set of eigenvalues. Further, the eigenspace corresponding to an eigenvalue $\eta$ of $\hat{T}_{[d_1,d_2]}$ will correspond to that of $\eta^{-1}$ for $\hat{T}_{[d_2,d_1]}$ matrix and vice-versa:

$$S_{T_{[d_1,d_2]}}^\eta = S_{T_{[d_2,d_1]}}^{-\eta}$$

This is reflected in the cycle decomposition also. By interchanging $d_1$ and $d_2$ in Eqn. 18, the cycles will be generated in the reverse order so that $\pi(d_2,d_1) = \pi^{-1}(d_1,d_2)$. For example, $\pi(2,4) = (\{0\}, \{1,2,4\}, \{3,6,5\}, \{7\})$ implies $\pi(4,2) = (\{0\}, \{1,4,2\}, \{3,5,6\}, \{7\})$.

If $m$ is the order of the cycle $\pi(d_1,d_2)$, then $\hat{T}_{[d_1,d_2]}^m = I$. More insights about cycle structure of $\pi(d_1,d_2)$ can be obtained by examining the characteristic equation of $\hat{T}_{[d_1,d_2]}$ derived in reference [36]. A cycle of length $l$ exists in $\pi(d_1,d_2)$ if $l$ divides $l_*$, where

$$l_* = \min \{p|d_2^p \equiv 1 \pmod{N-1}\}$$

The number of cycles of length $l$ in $\pi(d_1,d_2)$, denoted by $\sigma(l)$, is given by

$$\sigma(l) = \gcd(d_2-1, N-1) + 1$$

if $l = 1$

$$= \frac{1}{l} \sum_{d_1|l} \mu \left( \frac{l}{d} \right) \gcd \left( d_2^l-1, N-1 \right)$$

if $l > 1$ (22)

Here $\mu$ is the Mobius function defined on integers,

$$\mu(n) = \begin{cases} 1 & \text{if } n=1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

and $a|b$ stands for $a$ is divisor of $b$.

There is no anti-symmetric subspace when $l_*$ is odd, as there are no cycles of even length, a consequence of the fact that the factors of odd number are odd. For example, $l_*$ in the $[2,12]$ decomposition is 11, so there is no anti-symmetric subspace for $\hat{T}_{[2,12]}$ (see figure 3). Further, when $d_1 = d_2 = d$, $l_*$ is always 2, giving $\sigma(1) = d$ and $\sigma(2) = \frac{d(d-1)}{2}$, so that symmetric subspace is $\frac{d(d+1)}{2}$ dimensional and anti-symmetric subspace is $\frac{d(d-1)}{2}$ dimensional.

III. EXTENSION TO MULTIPARTITE QUANTUM STATES

In the previous section, the notion of exchange symmetry for heterogeneous bipartite systems has been discussed. In this section, the question of generalizing this notion to multi-partite heterogeneous systems is addressed.
A. Multipartite subsystem permutation

The first requirement is to identify a map between the computation basis vectors of $\mathcal{B}$ of $\mathbb{C}^N$ to those in the $\mathcal{B}_d$ partition, akin to Eqn. 14 for the bipartite case. Given any vector $|L\rangle$ in $\mathcal{B}$, its representation in $\mathcal{B}_d$ partition is $|i_1i_2\cdots i_k\rangle_d$ where each $i_r$ can be obtained successively as

$$i_r = \left\lfloor \frac{L - \sum_{j=1}^{r-1} j \prod_{l=j+1}^{k} d_l}{\prod_{m=r+1}^{k} d_m} \right\rfloor + 1,$$

(24)

where $\lfloor x \rfloor$ represents the integer part of $x$.

Conversely, given a basis state in $|i_1i_2\cdots i_k\rangle_d \in \mathcal{B}_d$, the corresponding state in $\mathcal{B}$ is $|L\rangle$ where

$$L = \sum_{r=1}^{k} i_r \times \left( -\prod_{j=r+1}^{k} d_j \right) .$$

(25)

Note that in Eqns. 24 and 25, the product of empty set is taken to be 1.

As in the bipartite case, the matrix representation of the mapping $T_{d,\sigma}$ in the computational basis of $\mathbb{C}^N$ yields a permutation matrix.

The schematic for this construction given in figure 4, is explained here. Start with one of the states $|L_n\rangle$ of $\mathcal{B}$, $0 \leq L_n \leq N - 1$. Its representation of in $\mathcal{d}$ partition is $|i_1i_2\cdots i_k\rangle_d$ which is obtained from Eqn. 24. The action of $T_{d,\sigma}$ is to map this state to the state $|i_{\sigma^{-1}(1)}i_{\sigma^{-1}(2)}\cdots i_{\sigma^{-1}(k)}\rangle_{\sigma(d)}$. Let $|L_{n+1}\rangle$ be the representation of this state in the unpartitioned space, obtained through Eqn. 25 but using permuted $d_j$s. Now $|L_{n+1}\rangle$ is expressed in $\mathcal{d}$ partition (using Eqn. 24), its labels and subsystems permuted and the new state is represented again in the unpartitioned space using Eqn. 25. Let that state be $|L_{n+2}\rangle$. This process is repeated until $L_{n+m+1}$ becomes equal to $L_n$ for some $m$, which is guaranteed since $T_{d,\sigma}$ is one-to-one and invertible. This way we have a mapping of states: $|L_n\rangle \rightarrow |L_{n+1}\rangle \rightarrow \cdots \rightarrow |L_{n+m+1}\rangle \rightarrow |L_n\rangle$, which we shall indicate as an $m-$cycle $(L_n, L_{n+1}, \cdots, L_{n+m})$. To proceed, pick-up another vector from $\mathcal{B}$ not already appearing in cycles as $L_n$ and generate another cycle of states as above. The process is repeated until every vector in $\mathcal{B}$ is accommodated in some cycle. The permutation $\pi \in S_N$ corresponding to the permutation matrix $T_{d,\sigma}$ is denoted by $\pi(d, \sigma)$.

To illustrate the scheme, $\pi([2, 2, 3], (1, 2, 3))$ is constructed. Towards this, begin with the computational basis $\mathcal{B}$ of $\mathbb{C}^12$. Consider one of the states of $\mathcal{B}$, say $|0\rangle$. This state in $[2, 2, 3]$ is $|001\rangle_{[2, 2, 3]}$. Under $(1, 2, 3)$ permutation, this state goes over to the state $|100\rangle_{[3, 2, 2]}$ in $[3, 2, 2]$ decomposition, which corresponds to state $|4\rangle$ in $\mathcal{B}$. Similarly, the mapping for all the elements of $\mathcal{B}$ can be found as given in Table III.

| $\mathcal{B}$ | $\mathcal{B}_{[2, 2, 3]}$ | $\mathcal{B}_{[3, 2, 2]}$ | $\mathcal{B}$ |
|----------|-----------------|-----------------|----------|
| $|0\rangle$ | $|000\rangle$ | $|000\rangle$ | $|0\rangle$ |
| $|1\rangle$ | $|001\rangle$ | $|100\rangle$ | $|4\rangle$ |
| $|2\rangle$ | $|002\rangle$ | $|200\rangle$ | $|8\rangle$ |
| $|3\rangle$ | $|010\rangle$ | $|001\rangle$ | $|1\rangle$ |
| $|4\rangle$ | $|011\rangle$ | $|101\rangle$ | $|5\rangle$ |
| $|5\rangle$ | $|012\rangle$ | $|201\rangle$ | $|9\rangle$ |
| $|6\rangle$ | $|100\rangle$ | $|010\rangle$ | $|2\rangle$ |
| $|7\rangle$ | $|101\rangle$ | $|110\rangle$ | $|6\rangle$ |
| $|8\rangle$ | $|102\rangle$ | $|210\rangle$ | $|10\rangle$ |
| $|9\rangle$ | $|110\rangle$ | $|111\rangle$ | $|5\rangle$ |
| $|10\rangle$ | $|111\rangle$ | $|111\rangle$ | $|7\rangle$ |
| $|11\rangle$ | $|112\rangle$ | $|211\rangle$ | $|11\rangle$ |

Table III. Procedure for obtaining the permutation $\pi([2, 2, 3], (1, 2, 3))$

To generate the cycle, start with any element, say $|1\rangle$, in the left-most column $\mathcal{B}$. This state is mapped to state $|4\rangle$ of the right most column $\mathcal{B}$. State $|4\rangle$ on the left most column is mapped to $|5\rangle$ on the right-most column and so on. This generates an orbit $|1\rangle \rightarrow |4\rangle \rightarrow |5\rangle \rightarrow |9\rangle \rightarrow |3\rangle \rightarrow |1\rangle$ so one of the cycles is $(1, 4, 5, 9, 3)$. Similarly starting with $|2\rangle$ one generates another orbit $|2\rangle \rightarrow |8\rangle \rightarrow |10\rangle \rightarrow |4\rangle \rightarrow |7\rangle \rightarrow |6\rangle \rightarrow |2\rangle$, yielding a cycle $(2, 8, 10, 4, 7, 6)$. Further, there are two $1-$cycles, $(0)$ and $(11)$, so that $\pi([2, 2, 3], (1, 2, 3)) = ((0), (1, 4, 5, 9, 3), (2, 8, 10, 4, 7, 6), (11))$.

Using the same symbol $\pi$ for bipartite and multipartite cases should not lead to any confusion, as the arguments of $\pi$ are different in the two cases. In fact, $\pi(d_1, d_2)$ is a shorthand notation for $\pi((d_1, d_2), (1, 2))$. Also, note that $\sigma$ represents one of the permutations of the subsystems, whereas $\pi$ refers to one of the permutations on the computational basis vector labels: $\sigma \in S_n(d)$ while $\pi \in S_N$. When $\sigma$ is the identity permutation over $k-$symbols, $\sigma = ((1), (2), \cdots, (k))$ where $k = n(d)$, $\pi(d, \sigma)$ is the identity permutation of $N$ symbols: $\pi(d, \sigma) = ((1), (2), \cdots, (N))$.

For illustration, possible cycle decompositions $\pi([2, 2, 3], \sigma)$ for all non-trivial permutations $\sigma \in S_3$ are
given in Table IV.

| $\sigma$   | $\pi(d, \sigma)$ |
|------------|-------------------|
| $((1, 2), (3))$ | $((0), (1), (2), (3, 6), (4, 7), (5, 8), (9), (10), (11))$ |
| $((1, 3), (2))$ | $((0), (1, 4, 6), (2, 8, 9, 3), (5, 10, 7), (11))$ |
| $((1), (2, 3))$ | $((0), (1, 2, 4, 3), (5), (6), (7, 8, 10, 9), (11))$ |
| $(1, 2, 3)$ | $((0), (1, 4, 5, 9, 3), (2, 8, 10, 7, 6), (11))$ |
| $(1, 3, 2)$ | $((0), (1, 2, 4, 5, 10, 9, 7, 3, 6), (11))$ |

Table IV. Permutation symmetries of $[2, 2, 3]$ decomposition of $N = 12$.

Once the cycle decomposition is available, obtaining eigenstates and eigenvalues proceeds as in the bipartite case. For example, consider the second entry of Table IV corresponding to the exchange of first and third subsystems in the $[2, 2, 3]$ decomposition. One of the cycles in $\pi([2, 2, 3], (1, 3), (2))$ is $(1, 4, 6)$. It contributes three eigenstates to $\hat{T}_{[2, 2, 3], (1, 3), (2)}$, one of which is the symmetric state

$$
\frac{1}{\sqrt{3}} (|1| + |4| + |6|) = \frac{1}{\sqrt{3}} (|001| + |011| + |100|)_{[2, 2, 3]}
$$

$$
= \frac{1}{\sqrt{3}} (|001| + |100| + |110|)_{[3, 2, 2]}.
$$

Many of the observations made in bipartite case hold in the multipartite case as well. When $\sigma$ is transposition of two subsystems of same dimensions, there are no cycles beyond two-cycles in $\pi(d, \sigma)$, as exemplified by the entry corresponding to $\pi([2, 2, 3], (1, 2), (3))$ in Table IV. Next, the inverse of $\hat{T}_{d, \sigma}$ is not $\hat{T}_{d, \sigma}^{-1}$ but $\hat{T}_{\sigma(d), \sigma^{-1}}^{-1}$:

$$
\hat{T}_{d, \sigma} = \hat{T}_{\sigma(d), \sigma^{-1}},
$$

from which it follows that $\pi(\sigma(d), \sigma^{-1})$ is the inverse of $\pi(d, \sigma)$ rather than $\pi(d, \sigma^{-1})$. Finally, given two permutations $\sigma_1, \sigma_2 \in S_n(d)$ the following relation holds:

$$
\pi(d, \sigma_1 \circ \sigma_2) = \pi(\sigma_2(d), \sigma_1) \circ \pi(d, \sigma_2)
$$

where $\circ$ denotes the composition of permutations.

**B. Projection to the completely symmetric and antisymmetric subspaces**

For a homogenous $k$-partite partition $d$, the completely symmetric projector $\hat{S}_d$ and completely antisymmetric projector $\hat{A}_d$ are defined as

$$
\hat{S}_d = \frac{1}{k!} \sum_{\sigma} \hat{T}_{d, \sigma},
$$

and

$$
\hat{A}_d = \frac{1}{k!} \sum_{\sigma} (-1)^{sgn(\sigma)} \hat{T}_{d, \sigma},
$$

where summation is over all $\sigma \in S_k$ (including the identity element, for which $\hat{T}_{d, \sigma}$ is the identity matrix) and $sgn(\sigma)$ is the parity of the permutation $\sigma$. It is evident that in the homogenous case both $\hat{S}_d$ and $\hat{A}_d$ are projection operators, i.e., their eigenvalues are +1 and 0. Indeed, $(d+k-1)! C_k$ eigenvalues of $\hat{S}_d$ are +1 and rest of them are 0. Similarly, $\hat{A}_d$ has $\frac{k}{d} C_k$ eigenvalues as +1 and the other eigenvalues are 0. If the eigenspaces of these operators corresponding to eigenvalue +1 and −1 are $S_d$ and $A_d$ respectively, then

$$
|\psi\rangle \in S_d \implies \hat{T}_{d, \sigma} |\psi\rangle = |\psi\rangle, \forall \sigma \in S_{n(d)},
$$

and

$$
|\psi\rangle \in A_d \implies \hat{T}_{d, \sigma} |\psi\rangle = (-1)^{sgn(\sigma)} |\psi\rangle, \forall \sigma \in S_{n(d)}.
$$

When $k > d$, $\hat{A}_d$ is a zero matrix and there is no completely antisymmetric subspace in that case [19].

From the projectors $\hat{S}_d$ and $\hat{A}_d$, two (mixed) states $\rho_S$ and $\rho_A$ are defined to be

$$
\rho_S = \frac{1}{(d+k-1)! C_k} \hat{S}_d,
$$

$$
\rho_A = \frac{1}{d C_k} \hat{A}_d.
$$

The density matrix $\rho_A$ is called the “antisymmetric state” and its entanglement is studied in [37]. $\rho_A$ is found to be maximally steerable for all dimensions [38]. A one parameter family of states is constructed using these states as

$$
\rho(p) = pp_S + (1-p) p_A
$$

where $p \in [0,1]$. The states $\rho(p)$ are such that they remain invariant under any local unitary transformation acting identically on all the subsystems:

$$
\rho(p) = \left( \sum_{k} U_d \otimes U_d \otimes \cdots \otimes U_d \right)^{k \text{ times}}
\rho(p) \left( \sum_{k} U_d^\dagger \otimes U_d^\dagger \otimes \cdots \otimes U_d^\dagger \right)^{k \text{ times}}
$$

where $U_d$ is a $d \times d$ unitary matrix. In the bipartite setting, $k = 2$, $\rho(p)$ are the well-known Werner states [39]. Separability of the these states in the tripartite case is discussed in [40].

For heterogeneous $d$, $\hat{S}_d$ is no longer a projector since some of its eigenvalues are different from 0 and 1. Eigenspace of $\hat{S}_d$ corresponding to an eigenvalue +1 is three-dimensional, with three eigenvectors being $|0\rangle$, $|N - 1\rangle$ and $|\Gamma_N\rangle$, for all $d \in S_N$, where $|\Gamma_N\rangle$ is defined as

$$
|\Gamma_N\rangle = \frac{1}{\sqrt{N-2}} \left( \sum_{n=1}^{N-2} |n\rangle \right).
$$
where \(|n|\) is the \((n + 1)^{th}\) computational basis for \(C^N\). We refer to the subspace spanned by the three vectors \(\{0\}, \{N - 1\} \) and \(|\Gamma_N\rangle\) as the “generalized symmetric subspace”, as it belongs to the symmetric subspace corresponding to any permutation in any TPS:

\[
\text{span} \{\{0\}, \{N - 1\}, |\Gamma_N\rangle\} \subseteq \mathbb{S}_d, \forall d \in \mathbb{P}(N), \sigma \in S_{n(d)}
\]

Similarly, \(\hat{A}_d\) is not a projection operator in the heterogeneous case and its eigenvalues are of magnitude strictly less than one.

C. Equivalent decompositions and Coarse-graining

Given a state in \(C^N\), there could be different tensor product spaces \(\mathbb{C}^{d_1}\) and \(\mathbb{C}^{d_2}\) consistent with \(N\), but \(d_1\) and \(d_2\) not related by any permutation symmetry. For example, \(N = 8\) may be realized in two ways: \(d_1 = [2, 2, 2]\) and \(d_2 = [2, 4]\). How are the cycle decompositions \(\pi(d_1, \sigma)\) and \(\pi(d_2, \sigma)\) related?

We represent all the multiplicative partitions of \(N\) (including those that differ in the order of subsystems) as \(\mathbb{P}(N)\):

\[
\mathbb{P}(N) = \left\{ d : \prod d_i = N \right\} \tag{37}
\]

For example, \(N = 12\) allows for the following seven multiplicative partitions:

\[
\mathbb{P}(12) = \{[2, 2, 3], [2, 3, 2], [2, 6], [3, 2, 2], [3, 4], [4, 3], [6, 2]\}.
\]

Among these, let \(\mathbb{P}_k(N)\) denote the set of all partitions \(d \in \mathbb{P}(N)\) having \(n(d) = k\). For example,

\[
\mathbb{P}_2(12) = \{[2, 6], [3, 4], [4, 3], [6, 2]\}
\]

and

\[
\mathbb{P}_3(12) = \{[2, 2, 3], [2, 3, 2], [3, 2, 2]\}.
\]

The largest value of \(n(d)\) is equal to \(N\), the number of prime factors of \(N\) (allowing for repetitions). Given a partition \(d\) and a permutation \(\sigma \in S_{n(d)}\), define \(\sigma(d)\) as the \(k\)-tuple \([d_{\sigma^{-1}(1)}, d_{\sigma^{-1}(2)}, \ldots, d_{\sigma^{-1}(k)}]\). Further, we denote the equivalence class (under permutation) of set of all decompositions connected to a partition \(d_c\) by \(E\)

\[
E(d_c) = \{d' \in \mathbb{P}_{n(d)}(N) | \exists \sigma \in S_{n(d)}, d' = \sigma(d_c)\} \tag{38}
\]

For example, in case of \(N = 12\), we have three distinct classes

\[
E([2, 6]) = \{[2, 6], [6, 2]\}
\]

\[
E([3, 4]) = \{[3, 4], [4, 3]\}
\]

\[
E([2, 2, 3]) = \{[2, 2, 3], [2, 3, 2], [3, 2, 2]\}
\]

Here an equivalence class is labeled by one of its members \(d_c\), whose entries are arranged in increasing order: \(d_i \leq d_j\), if \(i < j\). We call \(d_c\) a representative partition of the class to which it belongs.

Among the representative partitions of \(N\), we identify one which contains only prime \(d_i\)s. We call this, the “primitive decomposition” and represent it by \(d_p\). For example, for \(N = 24\), the primitive partition is \(d_p = [2, 2, 2, 3]\). Further, we call partitions \(d \in E(d_p)\), the prime partitions. By the uniqueness of prime factorization we have \(E(d_p) = \mathbb{P}_{n(d_p)}(N)\).

If the cycle-decomposition \(\pi(d_c, \sigma)\) of the representative partitions \(d_c\) is obtained for all \(\sigma \in S_{n(d_c)}\), the decompositions \(\pi(d_c, \sigma)\), corresponding to any other partition \(d\) belonging to the same class \(E(d_c)\) can be obtained. Permutations \(\pi(d_c, \sigma)\), for \(d \in E(d_c)\) and \(\sigma_2 \in S_{n(d_p)}\), can be obtained from the permutations corresponding to the representative partition \(d_c\) through the relation

\[
\pi(d_c, \sigma_2) = \pi(\sigma_1(d_c), \sigma_2) = \pi(d_c, \sigma_2 \circ \sigma_1) \circ \pi^{-1}(d_c, \sigma_1) \tag{39}
\]

This relation is obtained by just rearranging the Eqn. 27. As \(\sigma_2 \circ \sigma_1\) is another permutation belonging to \(S_{n(d_c)}\), it follows that permutation symmetries of every tensor product space can be obtained using the permutation symmetries of representative decomposition \(d_c\) alone.

Now, consider a TPS \(\mathbb{C}^{d_p}\), where \(d_p \in \mathbb{P}_{n(d_p)}(N)\) for \(k' < \Omega(N)\). Such partitions with fewer number of subsystems than the prime partition are called coarse-grained partitions. It is important to know whether the permutations \(\pi(d_c', \sigma')\) of the coarse-grained partitions are related to those of the primitive decomposition, \(\pi(d_c, \sigma)\).

As a coarse-grained partition \(d_c'\) involves a fewer number of tensor products to generate \(C^N\) than the maximal number of tensor products \(\Omega_N\) in \(d_p\). Therefore the coarse-grained partition can be expressed by combining (via tensoring) some of the prime dimensional Hilbert spaces. Each of the dimensions \(d_i'\) in \(d_c'\) is a product of one or more \(d_i\)'s of \(d_p\). Hence, the cycle decomposition \(\pi(d_c', \sigma')\) is identical to that of \(\pi(d_c, \sigma)\) for some \(d \in E(d_p)\) and \(\sigma_2 \in S_{n(d_p)}\). In essence, given \(\pi(d_c', \sigma')\), it is always possible to find two permutations \(\sigma_1, \sigma_2 \in S_{n(d_p)}\) such that

\[
\pi(d_c', \sigma') = \pi(\sigma_1(d_c), \sigma_2) \tag{40}
\]

For example, consider \(N = 24\). Its primitive decomposition is \(d_p = [2, 2, 2, 3]\). Consider a coarse-grained decomposition of \(N\), say, \(d_p' = [4, 3, 2]\) and the permutation operation to be the anti-cyclic rotation \(\sigma' = (1, 3, 2)\). In this case, \(\pi(d_c', \sigma')\) is:

\[
\pi(d_c', \sigma') = \begin{pmatrix}
(0), \\
(1, 4, 16, 18, 3, 12, 2, 8, 9, 13, 6), \\
(5, 20, 11, 21, 15, 14, 10, 17, 22, 19, 7), \\
(23)
\end{pmatrix}
\]
Permutations \( \sigma_1, \sigma_2 \in S_4 \) such that \( \pi(\sigma_1(d_1), \sigma_2) = \pi\left([4,3,2], (1,3,2)\right) \) is \( \sigma_1 = ((1), (2), (3), 4) \) and \( \sigma_2 = (1,3), (2,4) \).

If attention is restricted to bipartite partitioning \( d' = [d_1', d_2'] \), where the only non-trivial permutation is the subsystem exchange \( \sigma' = (2,1) \), it is possible to find suitable \( \sigma_1, \sigma_2 \in S_{\Omega(N)} \) such that \( \pi(\sigma_1(d_1), \sigma_2) = \pi(d_1', d_2') \) where \( \sigma(d_1', d_2') \) is the permutation corresponding to the bipartite exchange. This is illustrated with an example. If \( N = 24 \), the allowed bipartite partitions are

\[
\mathbb{P}_2(24) = \{[2,12],[3,8],[4,6],[6,4],[8,3],[12,2]\}
\]

The primitive decomposition \( d_\pi \) for \( N = 24 \) is \( d_\pi = [2,2,2,3] \). For every \( d' \in \mathbb{P}_2(24) \), Table V shows possible \( \sigma_1, \sigma_2 \in S_4 \) satisfying Eqn. 40, that is \( \pi(\sigma_1(2,2,2,3), \sigma_2) = \pi(d', (1,2)) \).

| \( d \) | \( \sigma_1 \) | \( \sigma_1(d_\pi) \) | \( \sigma_2 \) |
|---|---|---|---|
| [2,12] | ((1), (2), (3), (4)) | [2,2,2,3] | (1,4,3,2) |
| [3,8] | (1,2,3,4) | [3,2,2,2] | (1,4,3,2) |
| [4,6] | ((1), (2), (3), (4)) | [2,2,3,2] | (1,3,2,4) |
| [6,4] | ([1), (2), (3), (4)] | [2,3,2,2] | (1,3,2,4) |
| [8,3] | ([1), (2), (3), (4)] | [2,2,2,3] | (1,2,3,4) |
| [12,2] | ([1), (2), (3), (4)] | [2,2,3,2] | (1,2,3,4) |

Table V. \( \sigma_1 \) and \( \sigma_2 \) values satisfying Eqn. 40 for exchange symmetry in all bipartite decompositions of \( N = 24 \).

The cycle decomposition corresponding to cyclic shift of subsystems, \( \sigma_c = (1,2, \cdots, k) \) is related to that of bipartite exchange symmetry by

\[
\pi([d_1, d_2, \cdots d_k], \sigma_c) = \pi(d', d_k) \quad \text{where} \quad d' = \prod_{i=1}^{k-1} d_i.
\]

Similarly, \( \pi([d_1, d_2, \cdots d_k], \sigma_c^{-1}) = \pi(d_1, d) \) where \( d = \prod_{i=2}^k d_i \).

D. Cyclic invariance in equi-dimensional multipartitioning

It may appear that the eigenvalues of \( \hat{T}_{d, \sigma_c} \) do not equal to \( \pm 1 \) exist only when \( \sigma(d) \neq d \), that is, only when subsystems of distinct dimensions are permuted. However, this is not the case. Consider an \( k \)-partite decomposition \( d \) where all the subsystems are of equal dimensions \( d \), such that \( N = d^k \). Given a TPS \( C^d \), consider the permutation \( \sigma_c = (1,2, \cdots, k) \) which is the cyclic permutation of \( k \)-subsystems where \( n = (d) \):

\[
\sigma_c(i) = (i + 1) \mod k, \quad \text{for} \quad i = 1, \cdots, k. \quad (41)
\]

Given \( k \) qudits, and \( k \) parties \( A_1, A_2, \cdots, A_k \), the eigenstates of \( \hat{T}_{d, \sigma_c} \) are such that their interpretation remains identical irrespective of which qudit each party makes the measurement on, as long as the measurements are done in the order \( A_1, A_2, \cdots, A_k \). Now, since \( \hat{T}_{d, \sigma_c} \) and \( \hat{T}_{d, \sigma_{c}^{-1}} \) share same eigenvectors, these states have identical interpretation when the measurements are carried out even in the anticyclic order \( A_1, A_k, A_{k-1}, \cdots, A_2 \).

For example, consider \( d = [2,2,2,2] \) and \( \sigma_c = (1,2,3,4) \). The cycle decomposition \( \pi([2,2,2,2], (1,2,3,4)) \) is

\[
((0), (1,8,4,2), (3,9,12,6), (5,10), (7,11,13,14), (15)),
\]

from which the cyclic shift invariant states can be obtained. For example, the \( 4 \)-cycle \( (1,8,4,2) \) contributes 4 eigenstates: a symmetric state

\[
\frac{1}{2}(|0001\rangle + |1000\rangle + |0100\rangle + |0010\rangle),
\]

an anti-symmetric state

\[
\frac{1}{2}(-|0001\rangle + |1000\rangle - |0100\rangle + |0010\rangle),
\]

an eigenstate with eigenvalue \( i \):

\[
\frac{1}{2}(-i|0001\rangle - |1000\rangle + i|0100\rangle + |0010\rangle),
\]

and an eigenstate with eigenvalue \(-i\):

\[
\frac{1}{2}(i|0001\rangle - |1000\rangle - i|0100\rangle + |0010\rangle).
\]

Symmetric subspace \( \mathbb{S}_{d, \sigma}^1 \) is six-dimensional and the anti-symmetric subspace \( \mathbb{S}_{d, \sigma}^{-1} \) is four-dimensional. The other two eigenspaces \( \mathbb{S}_{d, \sigma}^2 \) and \( \mathbb{S}_{d, \sigma}^{-2} \) are both three-dimensional.

The eigenvalues of the cyclic shift operator and dimensions for the corresponding eigenspaces for few \( d \) are shown in Table VI for illustration.

| \( d \) | \( k \) | \( \sigma_c \) | \( \epsilon \) | \text{Dimension of } \mathbb{S}_{d, \sigma_c} | \nmid 2 | 2 | 3 | 3 | 4 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| [2,2] | [1,2,3] | \( \epsilon^{\pm} \) | \( \epsilon^{\mp} \) | 4 | 2 | 2 | 2 | 2 | 2 | 2 |
| [2,2,2] | [1,2,3,4] | \( \epsilon^{\pm} \) | \( \epsilon^{\pm} \) | 6 | 3 | 4 | 3 | 4 | 4 | 4 |
| [3,3,3] | [1,2,3] | \( \epsilon^{\pm} \) | \( \epsilon^{\pm} \) | 11 | 8 | 8 | 8 | 8 | 8 | 8 |
| [3,3,3,3] | [1,2,3,4] | \( \epsilon^{\pm} \) | \( \epsilon^{\pm} \) | 24 | 18 | 18 | 18 | 18 | 18 | 18 |
| [4,4,4] | [1,2,3] | \( \epsilon^{\pm} \) | \( \epsilon^{\pm} \) | 24 | 20 | 20 | 20 | 20 | 20 | 20 |
| [4,4,4,4] | [1,2,3,4] | \( \epsilon^{\pm} \) | \( \epsilon^{\pm} \) | 70 | 60 | 60 | 60 | 60 | 60 | 60 |

Table VI. Eigenvalues and dimensions of eigenspaces of circular permutation invariant states of different \( k \) and \( d \).

The eigenvalues of these permutations remain independent of \( d \) and depend only on \( k \). Further, the cycle lengths in the cycle decomposition \( \pi(d, \sigma_c) \) are factors of \( k \), so there is no anti-symmetric subspace when \( k \) is odd. It also follows that if \( k \) is prime then \( \pi(d, \sigma_c) \) contains \( \text{mod}(d^k -2, k) + 2 \) number of 1-cycles and \( \text{mod}(d^k -2, k) \) number of \( k \)-cycles and no other cycles. Hence, the dimension of the symmetric subspace in this case is \( \text{mod}(d^k -2, k) + 2 \).
IV. PERMUTATION SYMMETRY AND ENTANGLEMENT

Entanglement of multipartite heterogeneous states has been extensively studied in the recent years. The standard notion of entanglement presupposes an underlying TPS $\mathbb{C}^d$. Given a TPS $\mathbb{C}^d$, a pure state $|\psi\rangle$ is separable if it is of the form $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_k\rangle$, where $|\psi_i\rangle \in \mathbb{C}^{d_i}$. Otherwise, the state is entangled. It is easy to see that entangled states in a TPS need not be entangled in another. For instance, consider $\sqrt{2} (|1\rangle + |2\rangle) \in \mathbb{C}^6$. Using the rule of association given in Section (?), this is identified as $|0\rangle_2 \otimes \frac{1}{\sqrt{2}} (|1\rangle_3 + |2\rangle_3) \in \mathbb{C}^2 \otimes \mathbb{C}^3$, which is a product state. The corresponding state is $\frac{1}{\sqrt{2}} (|0\rangle_1 \otimes |1\rangle_2 + |1\rangle_1 \otimes |0\rangle_2) \in \mathbb{C}^3 \otimes \mathbb{C}^2$, which is entangled.

As the focus of this work is on extending the notion of permutation symmetry to heterogeneous systems, a suitable measure of entanglement is required. Most of the multipartite entanglement measures exist only in case of $d_1 = d_2 = \cdots = d_k = 2$, that is, they are defined only for $k$-partite qubit states. A recently proposed measure [41], based on the degree of the mixedness of the reduced density matrices, is

$$E_t (|\psi\rangle) = \min_{|A|=t} \sqrt{\frac{d}{d-1} \left(1 - \text{tr} (\rho_A^2)\right)}, \quad d = \prod_{i \in A} d_i \quad (42)$$

where $|\psi\rangle$ is an arbitrary $k$-qubit pure state belonging to $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$ and $t = 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$ where $\left\lfloor \frac{k}{2} \right\rfloor$ is the integral part of $\frac{k}{2}$ and $A$ is an arbitrary set of $t$ qudits among the $k$ of them. Here $\rho_A = Tr_{\overline{A}} (|\psi\rangle \langle \psi|)$ is the reduced density matrix of the subsystem $A$. The quantity $\sqrt{\frac{d}{d-1} \left(1 - \text{tr} (\rho_A^2)\right)}$ measures the degree of mixedness associated with a specific bipartition $\{A, \overline{A}\}$ where $\overline{A}$ is the complement of $A$. $E_t (|\psi\rangle)$ refers to the minimum of this quantity among all possible bipartitions $\{A, \overline{A}\}$ where $|A| = t$. For example, $E_2 (|\psi\rangle)$ refers to the minimum of the entanglement existing every pair of systems considered as a unit and the rest.

The maximally entangled state for an equi-dimensional $k$-partite system is the generalized GHZ state,

$$|\text{GHZ}_{k,d}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \left( \underbrace{i \cdots i}_{k} \right) \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |\alpha_i\rangle,$$  

(43)

where $\alpha = \frac{d^k - 1}{d - 1}$. The prefactors of Eqn. This state has an entanglement equal to 1, with respect to the measure defined in Eqn 42. In the case of heterogeneous $\mathbb{C}^d$, a state of the form of Eqn. 43, with $d = \min (d)$ is considered as a possible generalization. Entanglement of this state is

$$E_1 (|\text{GHZ}_{k,d_{min}}\rangle) = \sqrt{\frac{d_{max} (d_{min} - 1)}{d_{min} (d_{max} - 1)}}, \quad (44)$$

where $d_{min} = \min (d)$ and $d_{max} = \max (d)$. This state is maximally entangled state when $k = 2$, though the numerical value of $E_1 (|\text{GHZ}_{k,d_{min}}\rangle)$ measure is less than 1. Further, the entanglement of this state is identical in all decompositions $\sigma (d)$, for $\sigma \in S_n (d)$.

A. Bipartite exchange symmetry and entanglement

1. A measure of entanglement

As $t = 1$ for bipartite ($k = 2$) decompositions, the entanglement measure is denoted as $E$, without the subscript $t$. However, $E (|\psi\rangle)$ depends on the decomposition $[d_1, d_2]$, which is indicated with a suitable subscript as in $E (|\psi\rangle)$. For example for the $[d_1, d_2]$ bipartition,

$$[d_1, d_2] E (|\psi\rangle) = \sqrt{\frac{d_{max} (1 - \text{tr} (\rho_i^2))}{d_{max} - 1}}, \quad (45)$$

where $d_{min} = \max (d_1, d_2)$ and $\rho_i$ could be either of the reduced density matrices with $|\psi\rangle$ expressed in $[d_1, d_2]$ partition. Similarly, $[d_1, d_2] E (|\psi\rangle)$ can be calculated. The entanglements differ in the way the reduced density matrices are computed. The reduced density matrix of the first subsystem after tracing over the second subsystem from $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is:

$$[d_1, d_2] \rho_1 = \sum_{j=0}^{d_2-1} (|I_{d_1} \otimes \langle j|\rangle) |\psi\rangle \langle \psi| (|I_{d_1} \otimes |j\rangle) \quad (46)$$

where $\{\langle j|\rangle_{j=0}^{d_2-1}$ is a basis for $\mathbb{C}^{d_2}$ and $I_{d_1}$ is the identity matrix in $\mathbb{C}^{d_1}$. In $[d_1, d_2] \rho_1$ notation, the prefix indicates the tensor product space and the suffix indicates the subsystem in the factorization. The three other relevant reduced density matrices are

$$[d_2, d_1] \rho_2 = \sum_{j=0}^{d_2-1} (|j\rangle \otimes |I_{d_1}\rangle) |\psi\rangle \langle \psi| (|j\rangle \otimes |I_{d_1}\rangle), \quad (47)$$

$$[d_1, d_2] \rho_2 = \sum_{i=0}^{d_1-1} (|i\rangle \otimes |I_{d_2}\rangle) |\psi\rangle \langle \psi| (|i\rangle \otimes |I_{d_2}\rangle), \quad (48)$$

$$[d_2, d_1] \rho_1 = \sum_{i=0}^{d_1-1} (|I_{d_2} \otimes \langle i|\rangle) |\psi\rangle \langle \psi| (|I_{d_2} \otimes |i\rangle). \quad (49)$$

Of these four reduced density matrices, $[d_1, d_2] \rho_1$ and $[d_2, d_1] \rho_2$ are $d_1$-dimensional whereas $[d_2, d_1] \rho_1$ and
\[ \text{and } [d_1, d_2] \rho_2 \text{ are } d_2 \text{-dimensional. For a generic } |\psi\rangle, [d_1, d_2] \rho_1 \text{ need not be equal to } [d_1, d_2] \rho_2 \text{ and } [d_1, d_2] \rho_2 \text{ need not be equal to } [d_2, d_1] \rho_1. \text{ Therefore, entanglement of these states, namely, } [d_1, d_2] E(|\psi\rangle) \text{ and } [d_2, d_1] E(|\psi\rangle) \text{ are different. Nevertheless, if the state is exchange invariant, it follows that}
\]
\[ [d_1, d_2] E(|\psi\rangle) = [d_2, d_1] E\left(\tilde{T}_{[d_1, d_2]} |\psi\rangle\right). \tag{50} \]

One consequence of Eqn. 50 when \(d_1 = d_2 = d\) is that the states \(|\psi\rangle\) and \(\tilde{T}_{[d_1]} |\psi\rangle\) are equally entangled, for arbitrary \(|\psi\rangle\). Further, when \(d_1 \neq d_2\), the eigenstates of \(\tilde{T}_{[d_1, d_2]}\) are equally entangled in both the partitions. This is a special case of more general result. If \(|\psi\rangle\) and \(\tilde{T}_{[d_1, d_2]} |\psi\rangle\) are related as
\[
\tilde{T}_{[d_1, d_2]} |\psi\rangle = \tilde{U}_{d_2} \otimes \tilde{U}_{d_1} |\psi\rangle, \tag{51}
\]
where \(\tilde{U}_{d_i}\) is a local unitary operator of dimension \(d_i\), Eqn. 50 yields
\[
[d_1, d_2] E(|\psi\rangle) = [d_2, d_1] E\left(\tilde{U}_{d_2} \otimes \tilde{U}_{d_1} |\psi\rangle\right) = [d_2, d_1] E(|\psi\rangle). \tag{52}
\]

The entanglement of the state \(|\Gamma_N\rangle\), defined in Eqn. 35, in \([d_1, d_2]\) partition is
\[
[d_1, d_2] E(|\Gamma_N\rangle) = \sqrt{\frac{d}{d - 1}} \frac{4 \left(d_1 - 1\right) \left(d_2 - 1\right) - 2}{\left(d_1 d_2 - 2\right)^2} \neq 0 \tag{53}
\]
where \(d = \text{max}(d_1, d_2)\). Therefore, \(|\Gamma_N\rangle\) is entangled in every bipartition. For example, \(|\Gamma_4\rangle\) is one of the Bell states, \(\frac{1}{\sqrt{2}} ((01) + (10))_{[2, 2]}\), which is maximally entangled in \(C^2 \otimes C^2\).

2. Entanglement in the symmetric subspace

Product states in the symmetric subspace:

Product states completely residing in the symmetric subspace of multipartite qubit states are extensively studied in various contexts such as the geometric measure of entanglement [43], qubit spin coherent states in Majorana representation [44], etc. Here conditions on product state in \(C^{d_1} \otimes C^{d_2}\) to belong to the symmetric subspace of \(\tilde{T}_{[d_1, d_2]}\) are derived.

It is easy to see that product states \(|0\rangle\) and \(|N - 1\rangle\) belong to the symmetric subspace for every bipartition of \(N\). Consider the uniform state \(|\Sigma_N\rangle\), defined as
\[
|\Sigma_N\rangle = \frac{1}{\sqrt{N}} \left(\sum_{n=0}^{N-1} |n\rangle\right), \tag{54}
\]
where \(\{|n\rangle\}_{n=0}^{N-1}\) is the computational basis for \(C^N\) [45]. This state differs from \(|\Gamma_N\rangle\) defined in Eqn. 35, in that the summation in \(|\Sigma_N\rangle\) includes \(|0\rangle\) and \(|N - 1\rangle\). This state also belongs to the symmetric subspace (as it is a superposition of symmetric states \(|0\rangle\), \(|\Gamma_N\rangle\) and \(|N - 1\rangle\)), and is a product state in any bipartition \([d_1, d_2]\) as
\[
|\Sigma_N\rangle = \left(\frac{1}{\sqrt{d_1}} \sum_{i=0}^{d_1-1} |i\rangle\right) \otimes \left(\frac{1}{\sqrt{d_2}} \sum_{j=0}^{d_2-1} |j\rangle\right), \tag{55}
\]
where \(\{|i\rangle\}_{i=0}^{d_1-1}\) and \(\{|j\rangle\}_{j=0}^{d_2-1}\) are the computational bases of dimensions \(d_1\) and \(d_2\), respectively. Hence states \(|\Sigma_N\rangle\), \(|0\rangle\) and \(|N - 1\rangle\) are symmetric product states in every partition. These product states in the symmetric subspace are referred as trivial product states. It would be interesting to see whether there are other product states in the symmetric subspace apart from these trivial ones. That is, states \(|\phi\rangle \in C^{d_1}\) and \(|\psi\rangle \in C^{d_2}\) satisfying:
\[
|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle. \tag{56}
\]
In case of \(d_1 = d_2 = d\), symmetric product states are of the form
\[
|\psi^{\text{sym}}\rangle \triangleq |\epsilon\rangle \otimes |\epsilon\rangle, \tag{57}
\]
where \(|\epsilon\rangle \in C^d\). When \(d_1 \neq d_2\), finding states satisfying Eqn. 56 is more involved [46]. Cycle decomposition will aid in identifying the symmetric product states.

An arbitrary product state in the \([d_1, d_2]\) bipartition can be written in the computation basis as:
\[
|\phi\rangle = \left(\sum_{i=0}^{d_1-1} \alpha_i |i\rangle\right) \otimes \left(\sum_{j=0}^{d_2-1} \beta_j |j\rangle\right) = \sum_{i,j} \alpha_i \beta_j |ij\rangle_{[d_1, d_2]} \tag{58}
\]
where $\alpha_i$ and $\beta_j$ are complex numbers, such that
\[
\sum_{i=0}^{d_1-1} |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_{i=0}^{d_2-1} |\beta_i|^2 = 1.
\]
For $|\phi\rangle$ to be an eigenstate of $T_{[d_1,d_2]}$, $\alpha_i$ and $\beta_j$ need to satisfy constraints arising due to each cycle in $\pi(d_1, d_2)$. Consider one of the cycles $(L_1, L_2, \ldots, L_l)$ in $\pi(d_1, d_2)$. Recall that $L_1, L_2, \ldots, L_l$ are all integers between 0 and $d_1 d_2 - 1$. For notational convenience, we use the following symbols $\vec{x} = \left[ \begin{array}{c} x \end{array} \right]$ and $\pi \equiv \text{mod}(x, d_2)$ so that the state $|L_r\rangle$ in $[d_1, d_2]$ decomposition is $| \vec{L_r}, \vec{L_r} \rangle_{[d_1, d_2]}$, and from Eqn. 58 it can be seen that in the expansion of $|\phi\rangle$, the coefficient of $| \vec{L_r}, \vec{L_r} \rangle_{[d_1, d_2]}$ is $\alpha_{L_r} \beta_{L_r}$. For the state $|\phi\rangle$ to remain invariant under $\hat{T}_{[d_1, d_2]}$, the complex coefficients $\alpha_i$ and $\beta_j$ of Eqn. 58 have to satisfy the following constraints:
\[
\alpha_{L_i} \beta_{L_i} = \alpha_{L_2} \beta_{L_2} = \cdots = \alpha_{L_l} \beta_{L_l}. \quad (59)
\]
For every cycle of length $l$ greater than 1, there are $\binom{2}{l}$ similar such equalities on the coefficients $\alpha_i$ and $\beta_j$. For example, consider the [2,3] partition which has $\pi(2, 3) = ((0), (1, 2, 4, 3), (5))$. Consider one of the cycles of $\pi(2, 3)$, say $(1, 2, 4, 3)$. For product state $|\phi\rangle$ to be a symmetric state, the coefficients $\alpha_i$ and $\beta_j$ are required to satisfy (see Eqn. 59) the following three independent constraints:
\[
\alpha_0 \beta_1 = \alpha_0 \beta_2 = \alpha_1 \beta_1 = \alpha_1 \beta_0. \quad (60)
\]
The other two cycles $(0)$ and $(5)$ correspond to symmetric eigenstates by themselves and do not yield any additional constraints. The only state satisfying these three constraints is $|\Sigma_6\rangle$. There are no other symmetric product states in $\mathbb{C}^2 \otimes \mathbb{C}^3$ apart from $|0\rangle$, $|5\rangle$ and $|\Sigma_6\rangle$. In fact, for situations where $\pi(d_1, d_2)$ has only three cycles (that is two 1-cycles and one $d_1 d_2 - 2$ cycle; see for example $\pi(2, 6)$ in Table II), it is easy to see that there are no other symmetric product states apart from the trivial ones.

On the other hand, $\mathbb{C}^2 \otimes \mathbb{C}^4$ has $\pi(2, 4) = ((0), (1, 2, 4), (3, 6, 5), (7))$. The cycle $(1, 2, 4)$ offers three constraints $\alpha_0 \beta_1 = \alpha_0 \beta_2 = \alpha_1 \beta_0$ and the cycle $(3, 6, 5)$ contributes three more constraints, $\alpha_0 \beta_3 = \alpha_1 \beta_1 = \alpha_1 \beta_0$. These six constraints are satisfied provided $\beta_1 = \beta_2 = \alpha_1 \beta_0 = \alpha_0 \beta_3 = \alpha_1 \beta_1$.

States appearing as fixed-points are symmetric product states. For example, $\pi(3, 5)$ (see Table II) has state $|\uparrow\rangle$ appearing as a 1-cycle, which in $\mathbb{C}^3 \otimes \mathbb{C}^5$ decomposition is $|1\rangle_3 \otimes |2\rangle_5$ and in $\mathbb{C}^3 \otimes \mathbb{C}^3$ decomposition is $|2\rangle_3 \otimes |1\rangle_5$. Incidentally, $\mathbb{C}^3 \otimes \mathbb{C}^5$ is the smallest (in terms of $N$) heterogenous bipartite TPS where $|\hat{d}_1, \hat{d}_2\rangle$ has $1$-cycles other than $|0\rangle$ and $|d_1 d_2 - 1\rangle$: in other words smallest $d_1$ and $d_2 \neq d_1$ for which the matrix $\hat{T}_{[d_1, d_2]}$ has trace greater than two. It follows from Eqn. 22 that when $d_1$ or $d_2$ is 2, $\sigma(1)$ is 2. In that case, there are only two fixed points in $\pi(2, d)$ and $\pi(d, 2)$. Similarly, cycle decomposition $\pi(3, 4)$ also has no cycle of length one apart from $(0)$ and $(11)$, see Table II.

When $N$ is of the form $d^k$, where $d$ is a prime number, then recall that the symmetric product states in the homogenous $k$-partite decomposition are of the form $|\epsilon\rangle \otimes |\epsilon\rangle \otimes \cdots \otimes |\epsilon\rangle$, where $|\epsilon\rangle \in \mathbb{C}^d$ is a normalized pure state. Now, it is easy to see these states would remain symmetric product states in any coarse grained decomposition $\mathbb{C}^d$, where $d \in \mathbb{P}(N)$.

3. Entanglement in the non-symmetric eigenspaces of $\hat{T}_{[d_1, d_2]}$

The central result of this paper is the observation that the non-symmetric eigenspaces of $\hat{T}_{[d_1, d_2]}, \Sigma_{\mathbb{C}, \mathbb{P}, \eta}^\eta$, $\eta \neq 1$ are completely entangled. There are no product states in either partitioning in these subspaces. To see this, assume on the contrary that a $[d_1, d_2]$ product state $|\psi\rangle \otimes |\phi\rangle$ belongs to the non-symmetric eigenspace of $\hat{T}_{[d_1, d_2]}$. Then $\hat{T}_{[d_1, d_2]}(|\psi\rangle \otimes |\phi\rangle) = e^{\frac{2\pi i}{\Delta d}} |\psi\rangle \otimes |\phi\rangle$, for some integers $n$ and $k$ such that $0 < k < n$. But this is impossible as the real matrix $\hat{T}_{[d_1, d_2]}$ only permutes the entries of $|\psi\rangle \otimes |\phi\rangle$ and cannot introduce a complex phase. It is known that non-symmetric eigenspaces of $\hat{T}_{[d,d]}$ are completely entangled [17]. Our result generalization to heterogenous systems.

As eigenstates of $\hat{T}_{[d_1, d_2]}$ have equal entanglement in both $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$, the non-symmetric eigenspaces of $\hat{T}_{[d_1, d_2]}$ are completely entangled subspaces in both of them. This way, given $d_1$ and $d_2$, one obtains as many completely entangled subspaces as there are distinct non-unit eigenvalues of $\hat{T}_{[d_1, d_2]}$, given by Eqn. 19.

The largest subspace of a TPS where every vector is entangled is discussed in [47] and an explicit construction of a basis for such a subspace is provided in [48]. Given $d_1$ and $d_2$, the largest completely entangled subspaces (CES) in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$ are $(d_1 - 1)(d_2 - 1)$ dimensional [47].

Given $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the largest CES is the one orthogonal to $R^\perp$ given as [48]:
\[
R^\perp = \text{span} \left\{ \sum_{i_1 + i_2 = n} |i_1\rangle \otimes |i_2\rangle, n = 0, \ldots, n_{\text{max}} \right\},
\]
(61) where $\{ |i_1\rangle \}_{i_1=0}^{d_1-1}$ is an orthonormal basis in $\mathbb{C}^{d_1}$, $\{ |i_2\rangle \}_{i_2=0}^{d_2-1}$ is an orthonormal basis in $\mathbb{C}^{d_2}$, $n_{\text{max}} = d_1 + d_2 - 2$ and $P$ is the normalization constant.

The largest CES $R_{[d_1,d_2]}$ and $R_{[d_2,d_1]}$ are related as
\[
R_{[d_1,d_2]} = \hat{T}_{[d_1,d_2]} R_{[d_1,d_2]},
\]
(62) where $\hat{T}_{[d_1,d_2]} R_{[d_1,d_2]}$ stands for the subspace spanned by the vectors of the form $\hat{T}_{[d_1,d_2]} |\psi\rangle$ where $|\psi\rangle$ span $R_{[d_1,d_2]}$. A subscript is used to $R$ to denote the TPS in
which it is completely entangled. Note that vectors in $R_{[d_1, d_2]}$ need not be entangled when viewed as states in $[d_2, \ d_1]$ partition and vice-versa.

Given two CES $R_{[d_1, d_2]}$ and $R_{[d_2, d_1]}$, their intersection $R_{[d_1, d_2]} \cap R_{[d_2, d_1]}$ is also a CES in which every vector is entangled in both $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$. The states in the intersection, however, generally have different entanglement in the two TPSs. The non-symmetric eigenspaces of $T_{[d_1, d_2]}$, on the other hand, are CES in which every vector is equally entangled in both the partitions.

Again, to make progress we study qubit-qudit bipartite TPS. The largest CES subspaces in $[2, d]$ and $[d, 2]$ partitions are both $d - 1$ dimensional, given by

$$R_{[2,d]} = \text{span}\left\{ \frac{1}{\sqrt{2}} ((0,i) - (1,i-1))_{[2,d]} \right\}, \quad (63)$$

$$R_{[d,2]} = \text{span}\left\{ \frac{1}{\sqrt{2}} ((i-1,1) - (i,0))_{[d,2]} \right\}, \quad (64)$$

where $i$ runs from 1 to $d - 1$.

The basis vectors of $R_{[2,d]}$ (resp $R_{[d,2]}$) are all equally entangled in the $[2,d]$ (resp $[d,2]$) partition with $E = \sqrt{\frac{d}{2d-2}}$, which is the maximum entanglement in the $[2,d]$ (resp $[d,2]$) partition (see Eqn. 44). The dimension of the intersection of $R_{[2,d]}$ and $R_{[d,2]}$ subspaces depends on whether $d$ is odd or even. If $d$ is odd, the intersection is $\frac{d-1}{2}$ dimensional, and it is the span of $\left\{ \frac{1}{\sqrt{2}} ((2i-1) - (2i) - (d + 2i - 2)) \right\}$, for $i = 1, 2, ..., \frac{d-1}{2}$. If $d$ is even, it is one-dimensional, spanned by

$$R_{[2,d]} \cap R_{[d,2]} = \frac{1}{\sqrt{2(d-1)}} \left( \sum_{i=1}^{2(d-1)} (-1)^{i+1} |i\rangle \right), \quad (65)$$

When $d_1 = d_2 = d$, there is only one non-symmetric eigenspace, the $\frac{d}{2}d(d-1)$ dimensional anti-symmetric subspace $A_{[d,d]}$ given by:

$$A_{[d,d]} = \text{span}\left\{ \frac{1}{\sqrt{2}} ((i \otimes j) - (j \otimes i)) \right\}, \quad (66)$$

for $i, j \in (0, \ldots, d-1)$ and $i > j$. In this case, $A_{[d,d]} \subseteq R_{[d,d]}$ with the equality holding only when $d = 2$.

All the basis vectors of $A_{[d,d]}$ listed above have $\text{Tr}(\rho_A^2) = \frac{1}{2}$, for all $d$. Hence, entanglement of any of the basis vectors is $\sqrt{\frac{d}{2d-1}}$. Further, it has been numerically verified (for over $10^4$ states, sampled randomly with respect to Haar measure [49]) that the lowest entanglement in the anti-symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ is $\sqrt{\frac{d}{2d-1}}$.

### B. Multipartite permutation symmetry and Entanglement

For a general state $|\psi\rangle$, a decomposition $d$ and a permutation $\sigma$, analogous to Eqn. 50, the following relation holds:

$$dE (|\psi\rangle) = \sigma (d) E (\tilde{T}_{d,\sigma} |\psi\rangle), \quad (67)$$

for all $t = 1, 2, \ldots, \left[ \frac{d}{2} \right]$ in Eqn. 42. As in the bipartite case (Eqn. 51), if $|\psi\rangle$ and $\tilde{T}_{d,\sigma} |\psi\rangle$ are related as:

$$\tilde{T}_{d,\sigma} |\psi\rangle = \otimes \tilde{U}_{\sigma^{-1}(i)} |\psi\rangle, \quad (68)$$

where $\tilde{U}_i$ is the local unitary transformation of dimension $d_i$, then eqn. 67 is satisfied.

For a given $N$, the states $|0\rangle$, $|N-1\rangle$, $|\Sigma_N\rangle$ and $|\sigma N\rangle$ belong to the symmetric subspace in $\mathbb{C}^d$, for any $d \in \mathbb{P} (N)$ and any $\sigma \in S_n(d)$. Of these, states $|0\rangle$, $|N-1\rangle$ and $|\Sigma_N\rangle$ are product states in every partition $d$, whereas $|\sigma N\rangle$ is entangled. The entanglement in the later is given by

$$dE (|\sigma N\rangle) = \sqrt{\frac{d-1}{2} \left( \frac{d(d-1)}{N-2} - 1 \right)}, \quad (69)$$

where $d = \max (d)$ and $d' = \frac{N}{d}$.

It will be instructive to examine entanglement of states in the generalized symmetric subspace, defined in Eqn 36, in all representative partitions $d_i$. Consider two families of states:

$$|\chi_1 (p)\rangle = \sqrt{p} \frac{1}{\sqrt{2}} (|0\rangle + |N-1\rangle) + \sqrt{1-p} |\Sigma_N\rangle \quad (70)$$

where relative phase $\phi (p)$ is a random variable between 0 to $2\pi$. Figure 5 shows the variation of entanglement of these two families of states for $0 \leq p \leq 1$ and $N = 24$. These states belong to the symmetric subspace for all permutations $\sigma$, therefore it enough to study their entanglement in the representative decompositions of $N$.

There are six representative factorizations of $\mathbb{C}^{24}$: three bipartite, two tripartite, and the four-partite primitive decomposition. Variation of entanglement with $p$ of these families of states is given in Fig 5. Entanglement is largest in the primitive decomposition $d_p = [2, 2, 2, 3]$ and least in the $[2, 12]$ decomposition for both $|\chi_1 (p)\rangle$ and $|\chi_2 (p)\rangle$, for all values of $p$. For $p = 0$, $|\chi_1 (p)\rangle$ is $|\Sigma_N\rangle$, which is entangled in every partition of $N$ (see Eqn. 69). At $p = 1$, it corresponds to the GHZ-like state having equal superposition of two product states $|0\rangle$ and $|N-1\rangle$. From Fig. 5(a) it is seen that this state is more entangled than $|\Sigma_N\rangle$. At $p = \frac{4}{24}$,
The state $|\chi_1(p)\rangle$ is $|\sum_N\rangle$ of Eqn. 54, which is a product state in every decomposition $d$, which explains the dip at $p = \frac{1}{N}$ for $N = 24$ in all the plots of Fig. 5(a).

Fig. 5(b) is plot of entanglement in the states $|\chi_2(p)\rangle$, which are superpositions of the product state $|0\rangle$ and $|\Gamma_N\rangle$ with a random relative phase. It is evident that entanglement of $|\chi_2(p)\rangle$ shows identical variation with $p$ in all TPSs. These observations are independent of $N$. Here $N = 24$ was chosen only because it has a number of distinct partitions.

So far, entanglement in the symmetric subspace has been discussed. Now, entanglement in the nonsymmetric eigenspaces $S_{d,\sigma}^{\eta}$, $\eta \neq 1$ of $T_{d,\sigma}$ will be examined. As an illustration, consider $d = [2, 2, 3]$ and $\sigma = ((1, 2), (3))$. There are three cycles of even lengths in the cycle decomposition $\pi([2, 2, 3], ((1, 2), (3)))$ (see last row of Table IV). This implies that the anti-symmetric subspace is three dimensional:

$$S_{d,\sigma}^{-1} = \text{span} \left\{ \begin{array}{c} \frac{1}{\sqrt{2}} (|101\rangle - |100\rangle)_{[2, 2, 3]} \\ \frac{1}{\sqrt{2}} (|101\rangle - |101\rangle)_{[2, 2, 3]} \\ \frac{1}{\sqrt{2}} (|012\rangle - |102\rangle)_{[2, 2, 3]} \end{array} \right\}$$

Subspace $S_{d,\sigma}^{-1}$ is a CES in the sense that there are no product state of the form $|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes |\beta\rangle$ in this subspace where $|\alpha_i\rangle \in \mathbb{C}^2$ and $|\beta\rangle \in \mathbb{C}^3$. But states in $S_{d,\sigma}^{-1}$ are entangled only with respect to the first and second subsystems. Therefore, there is no genuine tripartite entanglement in this subspace. Indeed, the entanglement of states in this subspace with respect to the measure Eqn. 42 is zero:

$$E_1(|\psi\rangle) = 0 \forall |\psi\rangle \in S_{d,\sigma}^{-1}$$

Similarly, there is no genuine tripartite entanglement in subspaces $S_{d,\sigma}^{-1}$ for $\sigma = ((1, 3), (2))$ and $\sigma = ((1), (2, 3))$. It can be inferred from this example that if $\sigma$ involves permutation of only a subset of subsystems, the corresponding non-symmetric eigenspaces will be genuinely entangled only with respect to those subsystems. The states in the subspace will be separable with respect to the rest of the subsystems.

Now, consider a permutation $\sigma$ such that $\sigma(i) \neq i$ for $i = 1, \cdots, k$. In this case, the non-symmetric eigenspaces of $T_{d,\sigma}$ are all completely entangled in both $d$ and $\sigma (d)$ partitions. For example, $\pi([2, 2, 3], (1, 3, 2))$ (see last row of Table IV) has one even length cycle. The corresponding anti-symmetric state is genuinely entangled. For this state, the quantum of entanglement with respect to the measure defined in Eqn. 42 is 0.9. Permutation $\sigma_e$ discussed in section III D is another example where the non-symmetric eigenspaces are genuinely multipartite entangled. To the best of our knowledge, there is no other prescription for generating genuinely completely entangled subspaces. For example, the construction discussed in [48], in case of $k$ partite qubit system, generates the subspace orthogonal to the conventional symmetric subspace (the space spanned by the Dicke basis). This CES is $2^k - (k + 1)$ dimensional, but it has states which do not have genuine entanglement.

V. SUMMARY

Symmetry is one of the fundamental notions in physics, and its role in quantum mechanics cannot be overstated. In multipartite quantum systems, a natural symmetry operation is permutation symmetry. For homogenous $k$-partite systems, one identifies the “symmetric subspace” as the span of the states that remain invariant under any permutation of the subsystem labels.

Permutation symmetry of multipartite quantum states is generally considered only in the homogenous setting. A way of extending this symmetry to the case when subsystems are of unequal dimensions has been established here. This extension has been achieved via the natural isomorphism existing between the unfactored Hilbert space and the tensor product of the heterogeneous subsystems taken in different ordering. This extension recovers the conventional definition of permutation symmetry in the homogenous case. This has been accomplished.
by extending the idea of permutation matrix in the bipartite homogeneous case to multipartite heterogenous case. In the computational basis of $\mathbb{C}^N$, these matrices are permutation matrices. An algorithm for obtaining the permutations $\pi \in S_N$, corresponding to these matrices has been provided. The eigenvectors of $\hat{T}_{d,\sigma}$ are such that they have identical representation in both the tensor product spaces $\mathbb{C}^d$ and $\mathbb{C}^{\sigma(d)}$. The eigenspaces of $\hat{T}_{d,\sigma}$ corresponding to eigenvalue $+1$ are symmetric subspaces and eigenvalue $-1$ are anti-symmetric subspaces. This definition is meaningful as it gives rise to the conventional notions of symmetric and anti-symmetric states when $d = \sigma(d)$, which is possible if the system is homogeneous or the permutation is among the subsystems of equal dimensions. Moreover, this extension gives rise to classes of states other than the symmetric and antisymmetric ones. These are states which acquire a global complex phase ($\neq \pm 1$) under action of $\hat{T}_{d,\sigma}$. A procedure to obtain the dimension of each of these eigenspaces of $\hat{T}_{d,\sigma}$ by examining the corresponding permutation $\pi(d, \sigma)$ has been discussed. Further, it has been shown that all the nonsymmetric eigenspaces (i.e., eigenspaces corresponding to eigenvalues $\neq 1$) of $\hat{T}_{d,\sigma}$ are completely entangled subspaces. There are no product states in these subspaces. Further, these states have equal entanglement in both $\mathbb{C}^d$ and $\mathbb{C}^{\sigma(d)}$. These completely entangled subspaces are distinct from those discussed by Bhat [48]. If $\sigma$ is such that it has no cycles of length one, the states in these completely entangled subspaces are also genuinely entangled in the sense they remain entangled under arbitrary bipartitions.

For a given unfactored space of dimension $N$, we have identified a unique tensor product space composed of subspaces whose dimensions are the prime factors of $N$, tensored in the order of increasing subsystem dimensions. This unique tensor product space has the maximum number of subsystems and every other coarse-grained tensor product space consistent with $N$ can be obtained by permutation (if needed) and merging of the subsystems of this unique factorization. It has been established that the permutation symmetries of such coarse-grained tensor product spaces are expressible in terms of the permutation symmetries of this unique tensor product space.

ACKNOWLEDGMENTS

We thank Ludovic Arnaud for his insightful feedback on the manuscript. We also thank A.K. Rajgopal, Ajit Iqbal Singh and D. Goyeneche for their useful comments.

[1] D. Goyeneche, J. Bielawski, and K. Życzkowski, Physical Review A 94, 012346 (2016).
[2] R. H. Dicke, Physical Review 93, 99 (1954).
[3] G. Tóth, J. Opt. Soc. Am. B 24, 275 (2007).
[4] M. Gärttner, Phys. Rev. A 92, 013629 (2015).
[5] X. Wei and M.-F. Chen, International Journal of Theoretical Physics 54, 812 (2015).
[6] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, M. Harlander, W. Hänsel, M. Henrich, and R. Blatt, Phys. Rev. Lett. 106, 130506 (2011).
[7] A. Chiuri, C. Greganti, M. Paternostro, G. Vallone, and P. Mataloni, Phys. Rev. Lett. 109, 173604 (2012).
[8] D. J. Markham, Physical Review A 83, 042332 (2011).
[9] M. Aulbach, International Journal of Quantum Information 10, 1230004 (2012).
[10] L. Novo, T. Moroder, and O. Gühne, Phys. Rev. A 88, 012305 (2013).
[11] A. K. Rajagopal and R. W. Rendell, Phys. Rev. A 65, 032328 (2002).
[12] A. W. Harrow, ArXiv e-prints (2013), arXiv:1308.6595 [quant-ph].
[13] A. B. Klömv, G. Björk, and L. L. Sánchez-Soto, Phys. Rev. A 87, 012109 (2013).
[14] G. Tóth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Phys. Rev. Lett. 105, 250403 (2010).
[15] T. Moroder, P. Hyllus, G. Torh, C. Schwemmer, A. Niggebaum, S. Gaile, O. Gühne, and H. Weinfurter, New Journal of Physics 14, 105001 (2012).
[16] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A 67, 022112 (2003).
[17] T. Ichikawa, T. Sasaki, I. Tsutsui, and N. Yonezawa, Phys. Rev. A 78, 052105 (2008).
[18] T.-C. Wei, Phys. Rev. A 81, 054102 (2010).
[19] L. Arnaud, Physical Review A 93, 012320 (2016).
[20] C.-s. Yu, L. Zhou, and H.-s. Song, Physical Review A 77, 022313 (2008).
[21] A. Miyake and F. Verstraete, Phys. Rev. A 69, 012101 (2004).
[22] L. Chen, Y.-X. Chen, and Y.-X. Mei, Phys. Rev. A 74, 052331 (2006).
[23] S. Wang, Y. Lu, and G.-L. Long, Phys. Rev. A 87, 062305 (2013).
[24] N. Johnston, Phys. Rev. A 88, 062330 (2013).
[25] L. Chen and D. Z. Dokovic, Journal of Mathematical Physics 54, 022201 (2013).
[26] M. Malik, M. Erhard, M. Huber, M. Krenn, R. Fickler, and A. Zeilinger, Nature Photonics 10, 248 (2016).
[27] X. Xiao, Physica Scripta 89, 065102 (2014).
[28] P. Zanardi, Physical Review Letters 87, 077901 (2001).
[29] P. Zanardi, D. A. Lidar, and S. Lloyd, Phys. Rev. Lett. 92, 060402 (2004).
[30] L. Viola and H. Barnum, Philosophy of Quantum Information and Entanglement; Bokulich, A., Jaeger, G., Eds, 16 (2010).
[31] A. De la Torre, D. Goyeneche, and L. Leitao, European Journal of Physics 31, 325 (2010).
[32] W. Thirring, R. A. Bertlmann, P. Köhler, and H. Narnhofer, The European Physical Journal D 4, 181 (2011).
[33] J. R. Magnus and H. Neudecker, The Annals of Statistics, 381 (1979).
[34] J. L. Stuart and J. R. Weaver, Linear Algebra and Its Applications 150, 255 (1991).
[35] M. I. García Planas et al., Advances in Pure Mathematics
5, 390 (2015).
[36] F. H. Don and A. P. van der Plas, Linear Algebra and its Applications 37, 135 (1981).
[37] M. Christandl, N. Schuch, and A. Winter, Communications in Mathematical Physics 311, 397 (2012).
[38] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Phys. Rev. Lett. 112, 180404 (2014).
[39] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[40] T. Eggeling and R. F. Werner, Phys. Rev. A 63, 042111 (2001).
[41] C. Zhao, G.-w. Yang, and X.-y. Li, International Journal of Theoretical Physics 55, 1668 (2016).
[42] J. Eakins and G. Jaroszkiewicz, Journal of Physics A: Mathematical and General 36, 517 (2002).
[43] T.-C. Wei and P. M. Goldbart, Phys. Rev. A 68, 042307 (2003).
[44] M. Aulbach, D. Markham, and M. Murao, in Conference on Quantum Computation, Communication, and Cryptography (Springer, 2010) pp. 141–158.
[45] N. R. Wallach, “Quantum computing and entanglement for mathematicians,” in Representation Theory and Complex Analysis: Lectures given at the C.I.M.E. Summer School held in Venice, Italy June 10–17, 2004, edited by E. C. Tarabusi, A. D’Agnolo, and M. Picardello (Springer Berlin Heidelberg, Berlin, Heidelberg, 2008) pp. 345–376.
[46] R. Horn and C. Johnson, Topics in Matrix Analysis (Cambridge University Press, 1994).
[47] K. Parthasarathy, Proceedings Mathematical Sciences 114, 365 (2004).
[48] B. R. Bhat, International Journal of Quantum Information 4, 325 (2006).
[49] M. Ozols, “How to generate a random unitary matrix,” (2009).
| Symbol | Description |
|--------|-------------|
| $\mathbb{C}^N$ | Complex vector space of dimension $N$ |
| $[d_1, d_2]$ | A bipartite decomposition of $N = d_1d_2$. |
| $|i\rangle_{d_1}$ | $(i + 1)^{th}$ computational basis vector in $\mathbb{C}^{d_1}$. A $d_1$-dimensional column vector having 1 in $(i + 1)^{th}$ position and 0 everywhere else. |
| $\mathbb{B}_{d_1}$ | Computational basis of $\mathbb{C}^{d_1}$. |
| $\mathbb{B}$ | Computational basis of $\mathbb{C}^N$. |
| $T_{[d_1,d_2]}$ | Subsystem permutation operator mapping product state $|a\rangle \otimes |b\rangle$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ to $|b\rangle \otimes |a\rangle$ in $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$. |
| $\mathbb{B}_{[d_1,d_2]}$ | $\mathbb{B}_{d_1} \otimes \mathbb{B}_{d_2}$, tensor product of the computational bases of $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$. |
| $|ij\rangle_{[d_1,d_2]}$ | An element of $\mathbb{B}_{[d_1,d_2]}$, stands for the state $|i\rangle_{d_1} \otimes |j\rangle_{d_2}$. |
| $[d_1,d_2] \rho_j (\chi)$ | $d_j$-dimensional reduced density matrix corresponding to the second subsystem, after tracing out $d_i$-dimensional first subsystem from a state $|\chi\rangle$ in $\mathbb{C}^{d_i} \otimes \mathbb{C}^{d_j}$ tensor product space. |
| $[d_1,d_2] \rho_k (\chi)$ | $d_i$-dimensional reduced density matrix corresponding to the first subsystem, after tracing out $d_j$-dimensional second subsystem from a state $|\chi\rangle$ in $\mathbb{C}^{d_i} \otimes \mathbb{C}^{d_j}$ tensor product space. |
| $S_N$ | Permutation group of $N$-symbols. |
| $\pi (d_1,d_2)$ | Permutation corresponding to the permutation matrix $T_{[d_1,d_2]}$. Element of the permutation $S_{N=d_1d_2}$. |
| $\mathbb{B}_{[d_1,d_2]}^T$ | Set of eigenvectors of $T_{[d_1,d_2]}$, seen as a basis for $\mathbb{C}^N$. Not related to $\mathbb{B}_{[d_1,d_2]}$ (except through a unitary transformation). |
| $S^0_{[d_1,d_2]}$ | Eigenspace of $T_{[d_1,d_2]}$ corresponding to eigenvalue $\eta$. $S^0_{[d_1,d_2]}$ is the symmetric subspace and $S^{-1}_{[d_1,d_2]}$ is the anti-symmetric subspace. |
| $R_{[d_1,d_2]}$ | Completely entangled subspace in the $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ tensor product space according to Bhat. |

Table VII. List of symbols relevant to the bipartite decomposition.

| Symbol | Description |
|--------|-------------|
| $d = [d_1,d_2,\ldots,d_k]$ | A multiplication decomposition of $N$. Positive integers $> 1$ such that $\prod d_i = N$. |
| $\mathbb{P}(N)$ | All multiplicative partitions of $N$. $[1,N]$ and $[N,1]$ are not included in the definition. |
| $\mathbb{P}_r(N)$ | All multiplicative partitions of $N$ having $k$ terms. |
| $\mathbb{E}(d)$ | Appears along with $d$. Refers to any permutation of $k$ symbols, where $k = n(d)$. |
| $\sigma(d)$ | Shorthand notation for $[d_{\sigma^{-1}(1)},d_{\sigma^{-1}(2)},\ldots,d_{\sigma^{-1}(k)}]$. |
| $\mathbb{C}^d$ | Tensor product space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$. |
| $\mathbb{C}^{\sigma(d)}$ | Tensor product space $\mathbb{C}^{d_{\sigma^{-1}(1)}} \otimes \mathbb{C}^{d_{\sigma^{-1}(2)}} \otimes \cdots \otimes \mathbb{C}^{d_{\sigma^{-1}(k)}}$. |
| $\mathbb{B}_d$ | Tensor product of the $k$ computational bases $\mathbb{B}_{d_1}, \mathbb{B}_{d_2}, \ldots, \mathbb{B}_{d_k}$ in that order. |
| $|i_1i_2\cdots i_k\rangle_d$ | An element of $\mathbb{B}_d$. Shorthand notation for $|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_k\rangle$ where each $|i_j\rangle \in \mathbb{B}_{d_j}$. |
| $T_{d,\sigma}$ | A mapping between states $|i_1i_2\cdots i_k\rangle_d$ and $|\sigma(i_1)\sigma(i_2)\cdots \sigma(i_k)\rangle_{d(\sigma)}$. |
| $S^\eta_{d,\sigma}$ | Eigenspace of $T_{d,\sigma}$ corresponding to an eigenvalue $\eta$. $S^\eta_{d,\sigma}$ is the symmetric subspace and $S^{-1}_{d,\sigma}$ represents the anti-symmetric subspace. |
| $\pi(d,\sigma)$ | Permutation matrix corresponding to the permutation $T_{d,\sigma}$. |
| $\Omega(N)$ | Number of prime factors of $N$, allowing for repetition. |
| $d_\sigma$ | A prime partition $[d_1,d_2,\ldots,d_{\Omega(N)}]$, such that all $d_s$ are prime and $d_i \leq d_j$ if $i < j$. |
| $d$ | A coarse-grained partition. $d$ with $n(d) < \Omega(N)$. |
| $\sigma'$ | Appears along with $d$. Permutation $\in S_n(d')$. |
| $\sigma_c$ | Given along with a $d$, refers to the cyclic shift of subsystems, $(1,2,\ldots,k)$ where $k = n(d)$. |

Table VIII. List of symbols relevant to multipartite decomposition.