Vehicular traffic flow at an intersection with the possibility of turning

M Ebrahim Foulaadvand\textsuperscript{1,2} and Somayyeh Belbasi\textsuperscript{1}

\textsuperscript{1} Department of Physics, Zanjan University, PO Box 19839-313, Zanjan, Iran
\textsuperscript{2} Computational Physical Sciences Laboratory, Department of Nano-Science, Institute for Research in Fundamental Sciences (IPM), PO Box 19395-5531, Tehran, Iran

E-mail: foolad@iasbs.ac.ir

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Abstract
We have developed a Nagel–Schreckenberg cellular automata model for describing a vehicular traffic flow at a single intersection. A set of traffic lights operating in a fixed-time scheme controls the traffic flow. An open boundary condition is applied to the streets each of which conducts a unidirectional flow. Streets are single lane and cars can turn upon reaching to the intersection with prescribed probabilities. Extensive Monte Carlo simulations are carried out to find the model flow characteristics. In particular, we investigate the flow dependence on signalization parameters, turning probabilities and input rates. It is shown that for each set of parameters, there exists a plateau region inside which the total outflow from the intersection remains almost constant. We also compute total waiting time of vehicles per cycle behind red lights for various control parameters.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Simulation of the urban traffic flow has shown to be of prime importance for optimization and control purposes as the number of vehicles increases continuously and traffic conditions deteriorate. Modelling traffic flow dynamics by cellular automata has constituted the subject of intensive research by statistical physics during the past few years [1–3]. Understanding the characteristics of the city traffic is one of the most essential parts of the traffic research and was an early simulation target for statistical physicists [4–12]. Despite elementary attempts for simulation of perpendicular flows [13–17] the first serious cellular automata describing the flow at a signalized intersection was proposed by authors in [18]. Recently, physicists have notably attempted to simulate the traffic flow at intersections and other traffic designations such as roundabouts [19–32]. Further recent progresses include various features such as
self-organized controlling of traffic lights and flows in networks [33, 34], the modelling of decision making at intersections [35], mixture of motorized and non-motorized vehicles [36], simulation of the traffic flow at T-shaped intersections [37, 38], data collection in the city traffic [39], application of fuzzy logic in signalization of intersections [40], visual simulation of the vehicular traffic [41] and the dynamic route finding strategy [42]. Vehicular flow at the intersection of two roads can be controlled via two distinctive schemes. In the first scheme, the traffic is controlled without traffic lights [28, 43]. In the second scheme, signalized traffic lights control the flow. In [18], we have modelled the traffic flow at a single intersection with open boundary conditions applied to the streets. In a recent study, a single intersection operating under fixed time and traffic responsive schemes has been explored with a closed boundary condition [44]. In the above-mentioned papers cars were restricted to move straight and not allowed to change their directions when they reach the intersection. In this work we incorporate the possibility of turning. This incorporation constitutes a crucial step for a more realistic description of the traffic flow and would certainly help us in a proper coordination of traffic lights.

2. Description of the problem

Consider the traffic flow at the intersection of two streets. Each street conducts a unidirectional flow and has a single lane. Here for simplicity we exclude the possibility of overtaking which requires having more than one lane and some laws for lane-changing [45]. The flow directions are taken to be south–north in street 1 and east–west in street 2. Vehicles can turn when reaching to the intersection. A northward moving car can turn left and a westward moving car can turn right (towards north) when reaching to the intersection. We model each street by a chain of \( L \) sites. These perpendicular chains intersect each other at the middle sites \( i_1 = i_2 = \frac{L}{2} \) (see figure 1 for illustration). The discretization of space is such that each car occupies an integer number of cells denoted by \( N_{\text{cell}} \). Cell length is denoted by \( \Delta x \). The car position is denoted by the location of its head cell. Time elapses in discrete steps of \( \Delta t \).
and velocities take discrete values 0, 1, 2, . . . , \(v_{\text{max}}\) in which \(v_{\text{max}}\) is the maximum velocity measured in unit of \(\Delta x / \Delta t\). The probability of turning a north-moving car to west is denoted by \(p_{\text{sw}}\). Correspondingly, the probability of turning a west-moving car to north is denoted by \(p_{\text{en}}\).

At each step of time, the system is characterized by the position and velocity configurations of cars. The system evolves under the Nagel–Schreckenberg (NS) dynamics [46]. Let us now specify the physical values of our time and space units. The length of each car, \(L_{\text{car}}\), is taken to be 4.5 m. Therefore, the spatial grid \(\Delta x\) (cell length) is equal to \(\frac{4.5}{N_{\text{cell}}}\) m. We take the time step \(\Delta t = 1\) s. Furthermore, we adopt a speed limit of 74 km h\(^{-1}\). In addition, each discrete increment of velocity signifies a value of \(\Delta v = \frac{4.5}{N_{\text{cell}}}\) m s\(^{-1}\) which is also equivalent to the acceleration value. For \(N_{\text{cell}} = 5\) we have \(v_{\text{max}} = 23\) cells per time step. Moreover, the acceleration takes the comfort value \(a = 0.9\) m s\(^{-2}\). The value of random braking parameter is taken to be \(p = 0.1\) throughout this paper. A set of traffic lights controls the traffic flow in a fixed-time scheme as follows.

2.1. Fixed time signalization of lights

In this scheme traffic lights periodically turn into red and green. The period \(T\), hereafter referred to as the cycle time, is divided into two phases. In the first phase with duration \(T_{g}\), the lights are green for the westward street and red for the northward one. In the second phase which lasts for \(T - T_{g}\) time steps the lights change their colour and become red for the westward and green for the northward street. The gaps of all cars are updated with their leader vehicle except the ones (in each street) which are the nearest approaching cars to the intersection. These two cars need special attention. For these cars gap should be adjusted with the signal in its red phase. In this case, the gap is defined as the number of cells immediately after the car’s head to the intersection point \(L_{2}\). If the head position of the approaching car lies in the crossing point the gap is 0. Note that for \(N_{\text{cell}} > 1\) at a time step when a light goes red for a direction, portion of a passing car from that direction can occupy the crossing point. In this case the leading car of the queue in the other direction should wait until the passing car from the crossing point completely passes the intersection, i.e. its tail cell position becomes larger than \(L_{2}\).

2.2. Modelling the car turning

To model car turning we proceed as follows: once a car approaches to the intersection we draw a random number \(r\) (uniformly chosen from the unit interval). If \(r\) is less than the corresponding turning probability, the car turning label becomes 1; otherwise it remains 0. In the subsequent time steps the approaching car’s gap is adjusted appropriately: if its turning label is 0, its gap is adjusted via its forward leading car; otherwise if its turning label is 1, the gap is adjusted via the first car in the perpendicular direction which has already passed the intersection (see figure 2).

2.3. Entrance/removal of cars to/from the intersection

The type of boundary condition we implement in this paper is open. The reason is that if one tries to imply a close-type boundary condition, after some time steps most of the cars will appear in that street which has a higher turning probability in it. This effect arises because of accumulation of cars in the street with the higher inward turning probability. To avoid such a spurious effect which is surely contradicting to what occurs in reality we impose an open boundary condition. This type of boundary condition is more compatible to the realistic traffic
flow. For this purpose, we define two integer-valued parameters $d_1 \geq 1$ and $d_2 \geq 1$ for streets 1 and 2, respectively. $d_i$ is inversely proportional to the traffic density in street $i$. Once the distance of the nearest car to the first chain cell $j = 1$ (farthest car to the intersection) in street $i (i = 1, 2)$ exceeds $d_i$ a new car is inserted at site $j = 1$ of street $i$ with maximum velocity $v_{\text{max}}$. The removal of cars from a street is modelled as follows: when the position of a car which has passed the intersection becomes larger than $L - N_{\text{cell}}$, this car is removed from the system and the number of exited cars is increased by 1 correspondingly. The output current from street $i$ is defined as the number of cars exited from street $i$ during a time interval $[T_1, T_2]$ divided by $T_2 - T_1$. We denote the output currents by $J_1$ and $J_2$ for streets 1 and 2, respectively.

3. Simulation results

Now all the computational ingredients for simulation are at our disposal. We have taken the street sizes equal to 1350m. For $N_{\text{cell}} = 1$ this corresponds to $L_1 = L_2 = 300$ cells. The system is updated for $2 \times 10^5$ time steps. After transients, two streets maintain steady-state currents $J_1$ and $J_2$ which are defined as the number of vehicles passing the intersection per time step. They are functions of input rates $\frac{1}{d_1}$ and $\frac{1}{d_2}$, turning probabilities $p_{\text{sw}}, p_{\text{en}}$ and signal times $T$ and $T_g$.

3.1. $N_{\text{cell}} = 1$

In this case, each cell accommodates a car. Figure 3 shows the variation of $J_1$ versus green period of street 2 (east–west) $T_g$ for some values of cycle time $T$. The maximum of $J_1$ is less than $T_g = \frac{L}{2}$ because street 2 is less loaded than street 1. The inequality of input rates makes the slope $J_1$ different at the vicinity of $\frac{L}{2}$. For a given set of parameters, one can maximize the output current of street 1 by setting $T_g$ to the maximising value. A sharp fall in $J_1$ is noticeable and is related to enhancing of the jammed region in street 1.

Figure 4 depicts the behaviour of $J_1$ versus $T_g$ but this time for various values of the input parameter $d_1$. Other parameters are specified in the figure. By decreasing the green time of street 1, its output current slightly increases until it reaches the plateau region on which $J_1$ is maintained constant. The reason is that the red cycle of light optimally organizes the output
Figure 3. $J_1$ versus $T_g$ at $v_{\text{max}} = 4$ for cycle time $T = 30, 40, 50, 60$ and 70 s. The values of other parameters are specified in the figure. Street 1 has a higher input rate than street 2. Turning probabilities are equal to each other.

Figure 4. $J_1$ versus $T_g$ for various values of $d_1$ at $T = 50$ and $v_{\text{max}} = 4$. Other parameter values are specified in the figure.

flow from the intersection. This might seem counterintuitive to the expectation that the less the red period the larger the outflow. However, we should note that turning cars from the east–west street contribute to $J_1$ too.

Figure 5 exhibits $J_2$ versus green time devoted to street 2 for various values of input rates of street 1. Two distinctive behaviours can be identified. For a given $d_1$ the current $J_2$ increases
by increasing street 2 green time $T_g$ up to a certain value. Afterwards, $J_2$ remains constant on a small plateau region and starts to decrease in a slight manner. The reason is that by increasing $T_g$, the green time of street 1 decreases. This reduces the current of the southwest turning car into street 2. In fact, $J_2$ has two sources: straight moving east–west cars in street 2 and southwest turning cars in street 1. Beyond a certain $T_g$, the contribution of the second source dramatically falls which makes $J_2$ show a maximum.

Let us now look at the behaviour of the total flow $J_{tot} = J_1 + J_2$ versus $T_g$ and $d_1$. Figure 6 shows this variation versus street 2 green time $T_g$ for various values of cycle time $T$. 

Figure 5. $J_2$ versus $T_g$ of street 2 for various values of $d_1$ at $v_{max} = 4$. Other parameters values are specified in the figure.

Figure 6. $J_{tot}$ versus $T_g$ of street 2 for various values of $T$ at $v_{max} = 4$. Other parameters values are specified in the figure.
The notable point is that the maximum total flow only weakly depends on the cycle time $T$. For each $T$ there is a plateau region maintaining a maximum flow. This region is centred at a $T_g = \alpha T$ and increases with cycle time $T$. $\alpha$ depends on the street input rates and is less than 0.5 when $d_2$ is less than $d_1$. Roughly speaking $\alpha = \frac{d_1}{d_1 + d_2}$. The slope is symmetrical on both sides of the plateau region.

In figure 7, $J_{tot}$ is sketched versus $T_g$, this time for various values of street 1 input rate $d_1$. Typically the total flow shows a three-regime behaviour: an increasing portion, a plateau and eventually a decreasing portion. The plateau length decreases with the increase in $d_1$. These three portions are associated with the competition of two competing features. The direct current and the turning current. The plateau region corresponds to the situation where two features are balanced and equally contribute to the total current.

To gain a deeper insight into the problem, we have obtained the mutual dependence of the total flow $J_{tot}$ on the input rate parameters $d_1$ and $d_2$ for equal turning probability $P_{sw} = P_{en} = 0.1$ in figure 8. Other parameters are specified in the figure. We see the existence of a two-dimensional plateau on which the total flow is maximized. The maximum throughput flow is slightly less than 0.8. Moreover, the total current $J_{tot}$ is symmetric with respect to the plane $d_1 = d_2$ as it should be. The staircase-like topography of the current surface and the appearance of semi-plateau regions are noticeable. We see the dramatic fall of the total current in the vicinity of the plane $d_1 = d_2$. This marks the inefficiency of the fixed-time scheme in organizing the flow via equal distribution of cycle times to equally loaded streets. In figures 9 and 10, we have sketched $J_{tot}$ versus $T$ and $T_g$ for given input rates. As you can see there is a relatively large plateau on which the total current is maximal. The shape, size and orientation of this plateau show strong dependence on input rates $d_1$ and $d_2$. The maximal current value poorly depends on input rates.

The dependence of $J_{tot}$ on turning probabilities could serve us to achieve a better understanding from the overall flow characteristics of our single intersection. Figure 11 exhibits such behaviour. The current surface has interesting properties. There is a wide
Figure 8. Three-dimensional plot of $J_{\text{tot}}$ versus $d_1$ and $d_2$. $T = 30$ s, $T_g = 15$ s and $v_{\text{max}} = 4$. Turning probabilities are equally set at 0.1.

Figure 9. Three-dimensional plot of $J_{\text{tot}}$ versus $T$ and $T_g$. $d_1 = d_2 = 8$, $v_{\text{max}} = 4$ and turning probabilities are equally set at 0.1. $J_{\text{tot}}$ retains a constant value on a wide plateau region and sharply falls outside it.
plateau region maintaining a large current between 0.8 and 0.85. A sharp decrease is followed when $|p_{sw} - p_{en}|$ becomes large. To see the reason consider a situation where $p_{sw}$ is large whereas $p_{en}$ is small. Therefore, most of the entrant cars to street 2 move straight to the west. On the other hand most of the entrant cars to street 1 prefer to turn left (towards west) when reaching the intersection. This causes a dramatic jam in the vicinity of the intersection which leads to a sharp decrease in the total current.

In figure 12 we have depicted a similar 3D plot, this times for unequal input rates $d_1 \neq d_2$. Other parameters are identical to those in figure 11. The prominent effect of having unequal input rates is the reduction of the maximal current value in the plateau region from roughly 0.82 to 0.71. Other characteristics are analogous to the case $d_1 = d_2$. A comparison of figures 10 and 11 suggests that fixed-time signalization can poorly adjust itself to the variation of turning probabilities and the implementation of more efficient adaptive strategies is inevitable.

3.1.1. Waiting times. Despite maximizing the output currents is highly desirable for us it may not be the ultimate task in optimization of the traffic flow. Another important parameter for the optimization of the traffic flow at intersections is aggregate waiting times of cars stopping at queues formed in the red cycle of the traffic lights. In our model this total waiting time, denoted by $WT_{tot}$ onward, can be simply be evaluated. Once a car reaches to the end of a queue the $WT_{tot}$ counter is increased by 1 at each time step. This counting is paused once the corresponding light goes green. In figure 13 we show the $WT_{tot}$ per cycle versus green time of street 2 for various values of the cycle time $T$. The system has evolved for $10^5$ timesteps. The first 20000 timesteps have been discarded for equilibration. Contrary to the total current
dependence on $T^{\nu}$, here we do not have a minimal plateau region but rather there exists a global minimum for $W T_{\text{tot}}$. This is helpful to us in achieving the optimization of the traffic light by
simultaneously maximizing the total current and minimizing of the total waiting time. The results of figures 13 and 6 suggest to take $T_g$ at the end of the current maximal plateau as the optimal choice for the green time distribution.

To gain a deeper insight, we have drawn a 3D plot of the $WT_{tot}$ versus various control parameters in figures 14 and 16. In figure 14 the mutual dependence of $WT_{tot}$ on the input rates $d_1$ and $d_2$ for $T = 30$ and $T_g = 15$ is plotted. The qualitative topography of the $WT_{tot}$ surface resembles much with figure 8. For both $d_1$ and $d_2$ larger than 13 (light traffic in both streets) we have a relatively small $WT_{tot}$. Once either of $d_1$ or $d_2$ becomes less than 11 a dramatic increase in $WT_{tot}$ is observed. Figure 15 sheds more light on the problem. This figure is a 2D view of figure 14 and clearly shows the staircase-like structure of the $WT_{tot}$ surface. The sharp fall of $WT_{tot}$ is noticeable.

In figure 16 the total waiting time per cycle is mutually sketched versus turning probabilities for fixed equal input rates. We see that there is a wide minimal plateau on which the total waiting time retains its constant minimal value. On the boundaries where $|p_{sw} - p_{en}|$ becomes large the waiting time dramatically begins to increase. The behaviours of total current and total waiting times are complementary to each other, i.e. the maximum of $J_{tot}$ coincides with the minimum of $WT_{tot}$. It is our expectation that such complementarity between $WT_{tot}$ and $J_{tot}$ remains valid for the cases in which the input rates are not equal.

We end this section by showing the 3D plots of $WT_{tot}$ mutually versus $T$ and $T_g$. In figure 17, $WT_{tot}$ is given for $d_1 = d_2 = 8$. There is no minimal plateau sharply separated from high values. The curve is very smooth and the low value region gently crosses over to high values of waiting times. When considering nonequal input rates, the situation becomes worse. In figure 18, $WT_{tot}$ is sketched for $d_1 = 7 \neq d_2 = 15$. Once again we see the region of the maximal total current does not entirely overlap with the region of the minimal waiting time. This demonstrates that the optimization of the traffic flow is not a simple task at least within the framework of the NS model. Obviously empirical data are needed and can illuminate the problem.
Figure 14. Total waiting time per cycle versus $d_1$ and $d_2$. Other parameters: $v_{\text{max}} = 4$, $T = 30$, $T_g = 15$ and turning probabilities are equally taken to be 0.1. Three plateau regions are identified.

Figure 15. Dependence of $WT_{\text{tot}}$ per cycle on $d_1$ for fixed values of $d_2$. Other parameters: $v_{\text{max}} = 4$, $T = 30$, $T_g = 15$ and turning probabilities are equally taken to be 0.1.

3.2. $N_{\text{cell}} = 5$

In this section we show the results for a multi-cell occupation of cars. Here we take $N_{\text{cell}} = 5$ as discussed in section 2. In this case we have the comfort acceleration $a = 0.9 \text{ m s}^{-2}$. 
Figure 16. Total waiting time per cycle versus turning probabilities. Input rates are $d_1 = 8$ and $d_2 = 8$. $v_{\text{max}} = 4$ and turning probabilities are equally taken to be 0.1.

Figure 17. Total waiting time per cycle versus $T$ and $T_g$. Input rates are equal $d_1 = d_2 = 8$. $v_{\text{max}} = 4$ and turning probabilities are equally taken to be 0.1.
Figure 18. Total waiting time per cycle versus $T$ and $T_g$. Input rates are not equal: $d_1 = 7$, $d_2 = 15$. $v_{\text{max}} = 4$ and turning probabilities are equally taken to be 0.1.

Figure 19. $J_1$ versus $T_g$ at $v_{\text{max}} = 23$ for $N_{\text{cell}} = 5$. The values of other parameters are identical to figure 3.

We have reproduced figures 3–7 to see the similarities/differences imposed by varying $N_{\text{cell}}$. Figures 19 and 20 show exactly the same graphs as those shown in figures 3 and 4. The only difference is due to $N_{\text{cell}}$ which is now equal to 5.
The most prominent difference is that in $N_{\text{cell}} = 5$ the current $J_1$ is decreasing whereas in $N_{\text{cell}} = 1$ it slightly increases with $T_g$ and then starts to decrease. Currents $J_1$ in the uni-cell, i.e. $N_{\text{cell}} = 1$, are in general larger than their counterparts in the multi-cell case. This is due to unrealistic large acceleration in the NS model for uni-cell occupation. Figure 21 sketches the same diagram as shown in figure 5 for $N_{\text{cell}} = 5$. As shown, $J_2$ is an increasing function of $T_g$ while in the uni-call case it reaches to a maximum, and then after a short plateau it declines.

Figures 22 and 23 exhibit the dependence of the total current $J_{\text{tot}}$ on $T_g$ for various values of the cycle time $T$ and the input rate $d_1$.

Here also we encounter substantial differences. The most distinguishable feature is the convex nature of the total waiting time curve. In sharp contrast to the uni-cell case, in multi-cell occupation, we do not have a plateau region. Instead a smooth decrease is observed up to a minimum and then it symmetrically begins to increase. Analogously, the total current value is less than the uni-cell due to the same reason explained for $J_1$ and $J_2$. In figure 23, for $d_1 = 5$, the behaviour for small $T_g$ is similar to the uni-cell case, i.e. $J_{\text{tot}}$ increases. The difference is that in the uni-cell it reaches a maximum and then increases sharply, whereas in multi-cell occupation $J_{\text{tot}}$ does not decrease after the linear increasing regime is ended. For $d_1 > 5$ we do not have the increasing behaviour for small $T_g$.

4. Summary and concluding remarks

In summary, we have simulated the vehicular traffic flow at a single intersection in which the possibility of turning when cars reach to the intersection is augmented to the problem. A set of traffic lights operating in a fixed time scheme controls the flow. We have tried to shed some light on the problem of traffic optimization by extensive simulations. The total current is shown to remain constant on a plateau of the green time period given to one of the directions. The three-dimensional plot of the total current for given input rates of vehicles in terms of signalization...
parameters shows the existence of a 2D plateau region encompassed by almost flat planes of sharp decreasing currents. By extensive simulations we have examined the effect of turning on the output current. The dependence of $J_{\text{tot}}$ on the whole range of turning probabilities for fixed values of other parameters has been computed including the equal and nonequal input rates. The main finding is the appearance of a plateau current associated with the central region of the $P_{\text{sw}}-P_{\text{en}}$ plane. The current sharply decreases when $|P_{\text{sw}}-P_{\text{en}}|$ becomes large. Needless to say,
the plateau characteristics, i.e. its size and height, depend on the input rate and signalization parameters $T$ and $T_g$. Similar properties are observed if instead of $J_{tot}$ one looks at the total waiting time per cycle. Note here that the waiting time sharply increases when $|P_{sw} - P_{en}|$ becomes large. Besides the total current, the total waiting time per cycle has been computed. Our investigations reveal that in the parameter space, the minimization of the total waiting time per cycle does not fully coincide with the maximization of the total current. This arises the natural question: what quantity should be optimized in order to acquire the most efficiency for the intersection? In our Nagel–Schreckenberg cellular automata, not only the case of unip cell occupation, where each cell accommodates a car, has been considered but also the more realistic case of multi-cell occupation in which a car occupying more than one cell has been investigated. The corresponding results differ quantitatively and in some case qualitatively. Seeking for advanced models properly designated for modelling the behaviours of drivers at intersections is inevitable and unavoidable. We would like to end by emphasizing on the role of empirical data for adjusting the parameters of any intersection traffic model. Despite utilizing multi-cell occupation can render the deceleration/acceleration value of moving cars to a realistic one, the behaviour of halted cars in the queue when the light goes green is still suffering from a satisfactory modelling.

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