LARGE-TIME BEHAVIORS OF THE SOLUTION TO 3D COMPRESSIONABLE NAVIER-STOKES EQUATIONS IN HALF SPACE WITH NAVIER BOUNDARY CONDITIONS

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Dedicated to Professor Shuxing Chen for his 80th birthday

Abstract. We are concerned with the large-time asymptotic behaviors towards the planar rarefaction wave to the three-dimensional (3D) compressible and isentropic Navier-Stokes equations in half space with Navier boundary conditions. It is proved that the planar rarefaction wave is time-asymptotically stable for the 3D initial-boundary value problem of the compressible Navier-Stokes equations in $\mathbb{R}^+ \times T^2$ with arbitrarily large wave strength. Compared with the previous work [17, 16] for the whole space problem, Navier boundary conditions, which state that the impermeable wall condition holds for the normal velocity and the fluid tangential velocity is proportional to the tangential component of the viscous stress tensor on the boundary, are crucially used for the stability analysis of the 3D initial-boundary value problem.

1. Introduction. We are concerned with the large-time behavior of the solutions to the half space problem of the 3D compressible and isentropic Navier-Stokes equations

$$
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u,
\end{align*}
$$

where $t \geq 0$ is the time variable and $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$ is the spatial variables and the half space domain $\Omega := \mathbb{R}^+ \times T^2 = \{ x = (x_1, x') : x_1 > 0, x' := (x_2, x_3) \in T^2 \}$ with $\mathbb{R}^+$ being a real half line and $T^2 := (\mathbb{R}/\mathbb{Z})^2$ being a two-dimensional unit flat torus. The functions $\rho, u = (u_1, u_2, u_3)^t$ and $p(\rho) = a \rho^\gamma (a > 0, \gamma > 1)$

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represent respectively the fluid density, velocity and pressure. Both the shear and bulk viscosity coefficients $\mu$ and $\lambda$ are constants and satisfy the physical constraints

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$  

We consider the system (1.1) under the initial conditions

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x) \to (\rho_+, u_+), \quad \text{as} \quad x_1 \to +\infty,$$  

where $\rho_+ > 0$ and $u_+ = (u_1+, 0, 0)^t$ with $u_{1+} > 0$ (corresponding to the case of the planar rarefaction wave) are prescribed constant states, and Navier boundary conditions on the boundary $x_1 = 0$,

$$(u_1(0, x'), u_2(0, x'), u_3(0, x')) = k(x')(0, \partial_{x_1} u_2(0, x'), \partial_{x_1} u_3(0, x')),$$  

where the function $k(x')$ is periodic and smooth in $x' = (x_2, x_3) \in \mathbb{T}^2$ and in particular, $k \leq k(x') \leq \bar{k}$ for some positive constants $k, \bar{k}$, and the periodic boundary conditions are imposed on $(x_2, x_3) \in \mathbb{T}^2$ for the solution $(\rho, u)$. Navier boundary conditions (1.3) state that the impermeable wall condition holds for the normal velocity and the fluid tangential velocity is proportional to the tangential component of the viscous stress tensor on the boundary.

The large-time behavior of the solution to the 3D initial-boundary value problem (1.1)-(1.3) is expected to be determined by the planar Riemann problem of the corresponding 3D compressible Euler system:

$$\begin{cases} 
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) = 0,
\end{cases}$$  

with the planar Riemann initial data

$$(\rho, u)(t = 0, x) = (\rho_0, u_0)(x_1) = \begin{cases} 
(\rho_-, u_-), & x_1 < 0, \\
(\rho_+, u_+), & x_1 > 0,
\end{cases}$$  

where $(\rho_+, u_+)$ is given by the far-fields condition in (1.2), $u_- = 0$ is determined by Navier boundary condition (1.3), and $\rho_-$ can be uniquely determined by the corresponding wave curve. In the present paper, we are concerned with the time-asymptotic stability of planar rarefaction wave to (1.1)-(1.3), and therefore, $\rho_-$ can be uniquely determined by $u_\pm, \rho_+$ through the rarefaction wave curve (1.8) in the following.

Generally speaking, the solution of the compressible Euler equations (1.4) may form the discontinuities, such as shock wave, no matter how smooth or small the initial values are. To describe the development of the discontinuity to Euler system (1.4), the Riemann problem was first formulated and studied by Riemann [28] in 1860s for the one-dimensional (1D) isentropic model and then was extended to the more general 1D system of the hyperbolic conservation laws by Lax [14] in 1957. There are three basic wave patterns, that is, two nonlinear waves, shock and rarefaction waves, and one linearly degenerate wave, contact discontinuity, for the solution to the Riemann problem. These three basic waves and their linear superpositions, called Riemann solutions, are scaling and shift invariant, and Riemann solution is fundamental in the theory of the system of conservation laws since it not only captures the local and global behavior of the general solution to Euler equations, but also determines the large-time behavior of the solution to the corresponding Navier-Stokes equations. In particular, for the large-time asymptotic behaviors of the solution to compressible Navier-Stokes equations (1.1), which is a typical example of the system of the viscous conservation laws, the above three basic waves
become viscous shock wave, smooth rarefaction wave and viscous contact wave, respectively.

There have been many literatures concerned with the time-asymptotic behaviors towards the viscous basic wave patterns for the solution to the Cauchy problem of the viscous conservation laws. In the spatial 1D case, we refer to [8, 22, 18, 26, 20] for the stability of viscous shock wave, [23, 24] for the rarefaction wave, and [19, 12] for the viscous contact wave. In the spatial multi-dimensional (multi-D) case, there are only partial results on the time-asymptotic stability of planar rarefaction wave and planar viscous shock wave to the multi-D viscous conservation laws due to the essential difficulties caused by high-dimensionality. Xin [30] first proved the asymptotic stability of planar rarefaction wave to the scalar viscous conservation laws in multi-dimensions by the elementary $L^2$-energy method. For an $2 \times 2$ artificial system with positively definite viscosity matrix, Hokari and Matsumura [10] proved the stability of the planar rarefaction wave in two-dimensional case, which crucially depends on the strict positivity of the viscosity matrix and can not be applied to the compressible Navier-Stokes system (1.1) with physical viscosities. Recently, Li and Wang [17], Li, Wang and Wang [16] proved the time-asymptotic stability of the planar rarefaction wave to the multi-dimensional compressible Navier-Stokes equations for the periodic transverse directions. For the planar viscous shock wave, Goodman [9] first proved its stability to the scalar viscous conservation laws by introducing suitable shift function. Then Humpherys, Lynh and Zumbrun [13] presented a first numerical computation of an Evans function associated with the multidimensional stability of a planar viscous shock wave. Note that it is still open for the time-asymptotic stability of planar viscous contact wave in spatial multi-D case. Besides the planar wave patterns, there are more complicated generic wave structure to the multi-D Riemann problem (1.4)-(1.5), as described in Chang-Hsiao [1] and Shu-Xing Chen’s series of works [3, 4, 5], while the large-time behaviors towards these generic wave patterns to the multi-D viscous conservation laws are completely open as far as we know.

On the other hand, for the large-time behavior towards the viscous basic wave patterns to the initial boundary value problem to the viscous conservation laws, in particular, the compressible Navier-Stokes equations, most existing results are concerned with spatial 1D case, and we refer to the survey paper by Matsumura [21] and the references therein and thereafter. However, for the stability of the planar wave patterns to the multi-dimensional compressible Navier-Stokes equations (1.1) with boundaries, there are very few results due to the difficulties caused by the high dimensionality and physical boundaries. In the present paper, as a first step, we are concerned with the half space problem to 3D compressible Navier-Stokes equations (1.1) with Navier boundary condition (1.3) and we prove the time-asymptotic stability of planar rarefaction wave to 3D initial and boundary value problem (1.1)-(1.3).

To state our main result, we first recall the planar rarefaction wave to (1.4)-(1.5). On one hand, the planar rarefaction wave is the self-similar solution to the corresponding 1D Riemann problem

\begin{equation}
\begin{cases}
\partial_t \rho + \partial_{x_1} (\rho u_1) = 0, \\
\partial_t (\rho u_1) + \partial_{x_1} (\rho u_1^2 + p) = 0,
\end{cases}
\end{equation}
with the rarefaction Riemann initial data

\[
(\rho, u_1)(0, x_1) = (\rho_0, u_{10})(x_1) = \begin{cases} 
(\rho_-, u_{1-}), & x_1 < 0, \\
(\rho_+, u_{1+}), & x_1 > 0,
\end{cases}
\]

(1.7)

and at the same time, the planar rarefaction wave is the unique solution to the multi-D planar Riemann problem (1.4)-(1.5) with the rarefaction Riemann initial data in the class of bounded weak solutions (cf. [2, 6, 7]).

It is straight to calculate that the Euler system (1.6) for \((\rho, u_1)\) has two distinct eigenvalues

\[
\lambda_i(\rho, u_1) = u_1 + (-1)^i \sqrt{\rho'(\rho)}, \quad i = 1, 2,
\]

with corresponding right eigenvectors

\[
r_i(\rho) = (1, (-1)^i \sqrt{\rho'(\rho)})^t, \quad i = 1, 2,
\]

such that

\[
r_i(\rho) \cdot \nabla_{(\rho, u_1)} \lambda_i(\rho, u_1) = (-1)^i \rho \rho''(\rho) + 2 \rho'(\rho) \neq 0, \quad i = 1, 2,
\]

in the non-vacuum region. Thus the two \(i\)-Riemann invariants \(\Sigma_i(\rho, u_1)\) can be defined by (cf. [25])

\[
\Sigma_i(\rho, u_1) = u_1 + (-1)^{i+1} \int_0^\rho \frac{\sqrt{\rho'(s)}}{s} ds, \quad i = 1, 2,
\]

(1.8)

such that

\[
\nabla_{(\rho, u_1)} \Sigma_i(\rho, u_1) \cdot r_i(\rho) \equiv 0, \quad i = 1, 2.
\]

Here we only consider the time-asymptotic stability of planar 2–rarefaction wave to the Euler system (1.6)-(1.7) in the present paper and the stability of planar 1–rarefaction wave can be proved similarly. For 2–rarefaction wave, 2–Riemann invariant \(\Sigma_2(\rho, u_1)\) is constant and therefore we can derive \(\rho_-\) uniquely by the relation

\[
u_{1+} - \int_{\rho_-}^{\rho_+} \frac{\sqrt{\rho'(s)}}{s} ds = u_{1-} - \int_{\rho_-}^{\rho_+} \frac{\sqrt{\rho'(s)}}{s} ds.
\]

(1.9)

Then the 2-rarefaction wave solution \((\rho^r, u_1^r(x_1))\) to the compressible Euler system (1.6)-(1.7) can be defined explicitly by

\[
\begin{cases}
B_\pm = \lambda_2(\rho_\pm, u_{1\pm}) = u_{1\pm} + \sqrt{\rho'(\rho_\pm)}, \\
B^r(\frac{x_1}{t}) = \lambda_2(\rho^r(\frac{x_1}{t}), u_1^r(\frac{x_1}{t})) = u_1^r(t, x_1) + \sqrt{\rho'(\rho^r(\frac{x_1}{t}, x_1))}, \\
\Sigma_2(\rho^r, u_1^r) = \Sigma_2(\rho_\pm, u_{1\pm}), \quad u_i^r \equiv 0, \quad i = 2, 3.
\end{cases}
\]

(1.10)

where \(B_- < B_+\) and \(B^r(\frac{x_1}{t})\) is the solution to the Riemann problem of the following inviscid Burgers equation:

\[
\begin{cases}
\partial_t B + B \partial_x B = 0, \\
B(0, x_1) = \begin{cases} B_-, & x_1 < 0, \\
B_+, & x_1 > 0,
\end{cases}
\end{cases}
\]

(1.11)

with

\[
B^r(\frac{x_1}{t}) = \begin{cases} B_-, & \frac{x_1}{t} \leq B_-, \\
\frac{x_1}{t}, & B_- < \frac{x_1}{t} \leq B_+, \\
B_+, & \frac{x_1}{t} \geq B_+.
\end{cases}
\]

(1.12)
Now we can state the main result in this paper as follows.

**Theorem 1.1.** Let \((\rho^r, u^r)(\frac{t}{T})\) be the planar 2—rarefaction wave defined in (1.10) with \(u_{1+} > 0\). Then there exists a positive constant \(\varepsilon_0\), such that if

\[
\| (\rho_0 - \rho_+, u_0 - u_+) \|_{H^2(\Omega)} \leq \varepsilon_0,
\]

then 3D initial and boundary value problem (1.1)–(1.3) admits a unique global solution \((\rho, u)\) satisfying

\[
\begin{cases}
(\rho - \rho^r, u - u^r) \in C(0, +\infty; L^2(\Omega)), \\
\nabla(\rho, u) \in C(0, +\infty; H^1(\Omega)), \\
\nabla^2 \rho \in L^2(0, +\infty; L^2(\Omega)), \\
\nabla^2 u \in L^2(0, +\infty; H^1(\Omega)),
\end{cases}
\]

and the time-asymptotic stability toward the planar rarefaction wave \((\rho^r, u^r)(\frac{t}{T})\) holds true:

\[
\lim_{t \to \infty} \sup_{x \in \Omega} |(\rho, u)(t, x) - (\rho^r, u^r)(\frac{x_1}{x})| = 0.
\]

**Remark 1.1.** This result generalize the previous works on the whole space problem in [17, 16] to the half space problem with Navier boundary condition (1.3) and is the first result about the nonlinear stability of planar basic wave patterns for 3D initial and boundary value problem (1.1)–(1.3) as far as we know.

The rest part of the paper is arranged as follows. In Section 2 we first construct an approximate rarefaction wave and present its properties. Then, the energy estimates to the perturbation will be carried out in Section 3. Finally, in Section 4 we prove our main Theorem 1.1 based on the uniform a priori estimates.

**Notations.** Throughout this paper, several positive generic constants are denoted by \(C\) if without confusions. For a nonnegative integer \(m\) we denote by \(H^m\) the usual \(m\)–th order \(L^2\) Sobolev space on \(\Omega\) with norm \(\| \cdot \|_{H^m}\) and \(\| \cdot \|_{L^2(\Omega)}\). We denote the integral on \(\Omega\), \(\int_\Omega dx\), simply by \(\int dx\).

### 2. Approximate Rarefaction wave
Since the rarefaction wave \((\rho^r, u^r)(\frac{t}{T})\) in (1.10) is only Lipschitz continuous, we first construct a smooth approximation for the rarefaction wave. For the Burgers equation (1.11)-(1.12), the smooth rarefaction wave profile can be constructed by

\[
\begin{cases}
\partial_t \tilde{B} + \tilde{B} \partial_{x_1} \tilde{B} = 0, \\
\tilde{B}(0, x_1) = \tilde{B}_0(x_1) = \begin{cases} 
B_-, \\
B_- + (B_+ - B_-)k_q \int_0^{\varepsilon x_1} y^q e^{-y} dy, & x_1 \geq 0,
\end{cases}
\end{cases}
\]

where both \(\varepsilon > 0\) being suitably small and \(q \geq 14\) are to be determined, and \(k_q\) is a constant such that \(k_q \int_0^{+\infty} y^q e^{-y} dy = 1\). Note that for \(B_- < B_+\), the problem (2.1) admits a unique classical solution \(\tilde{B}(t, x_1)\) given by

\[
\tilde{B}(t, x_1) = \tilde{B}_0(x_0(t, x_1)), \quad x_1 = x_0(t, x_1) + \tilde{B}_0(x_0(t, x_1))t.
\]

Correspondingly, the smooth 2—rarefaction wave profile \(\tilde{\rho}(\tilde{u})(t, x_1)\) to compressible Euler system (1.6)-(1.7) can be defined by

\[
\begin{cases}
B_\pm = \lambda_2(\rho_\pm, u_{1\pm}) = u_{1\pm} + \sqrt{p'(\rho_\pm)}, \\
\tilde{B}(1 + t, x_1) = \lambda_2(\tilde{\rho}, \tilde{u}_1)(t, x_1) = \tilde{u}_1(t, x_1) + \sqrt{p'(\tilde{\rho}(t, x_1))}, \\
\Sigma_2(\tilde{\rho}, \tilde{u}_i) = \Sigma_2(\rho_\pm, u_{1\pm}), \quad \tilde{u}_i \equiv 0, \quad i = 2, 3.
\end{cases}
\]

where \(\tilde{B}(t, x_1)\) is the solution of Burgers equation (2.1) given by (2.2).
The solution \((\bar{\rho}, \bar{u}) (t, x_1)\) defined in (2.3) satisfies the Euler system
\[
\begin{align*}
\partial_t \bar{\rho} + \partial_{x_1} (\bar{\rho} \bar{u}_1) &= 0, \\
\partial_t (\bar{\rho} \bar{u}_1) + \partial_{x_1} (\bar{\rho} \bar{u}_2^2 + p(\bar{\rho})) &= 0.
\end{align*}
\] (2.4)

Now we list the following properties of the smooth 2—rarefaction wave profile \((\bar{\rho}, \bar{u}_1)\).

**Lemma 2.1** ([11]). The following properties hold:

1. For \(t \geq 0, x_1 \in \mathbb{R}\), there exists a constant \(C\) such that
   \[
   \frac{\sqrt{p'(\bar{\rho})}}{\bar{\rho}} \partial_{x_1} \bar{\rho} = \partial_{x_1} \bar{u}_1 > 0, \quad |\partial_{x_1} \bar{u}_1| \leq C \varepsilon |\partial_{x_1} \bar{u}_1|, \quad |\partial_{x_1}^2 \bar{u}_1| \leq C \varepsilon |\partial_{x_1} \bar{u}_1|.
   \]
2. For any \(t > 0\) and \(p \in [1, +\infty]\), there exists a constant \(C_{pq}\) such that
   \[
   \|\partial_{x_1} (\bar{\rho}, \bar{u}_1)(t, \cdot)\|_{L^p} \leq C_{pq} \min\{\varepsilon^{1-\frac{1}{p}}, (1+t)^{-1+\frac{1}{p}}\},
   \]
   \[
   \|\partial_{x_1}^2 (\bar{\rho}, \bar{u}_1)(t, \cdot)\|_{L^p} \leq C_{pq} \min\{\varepsilon^{2-\frac{1}{p}}, (1+t)^{-1+\frac{1}{p}}\}.
   \]
3. Smooth and inviscid 2—rarefaction waves are equivalent time-asymptotically, i.e.,
   \[
   \lim_{t \to +\infty} \sup_{x_1 \in \mathbb{R}^+} \|(\bar{\rho}, \bar{u}_1)(t, x_1) - (\rho^*, u_1^*)(\frac{t}{t})\|_{L^2} = 0.
   \]

Finally, we present the Gagliardo-Nirenberg inequality in domain \(\Omega := \mathbb{R}_+ \times \mathbb{T}^2\), whose proof can be found in [15, 29].

**Lemma 2.2.** There exists some generic constant \(C\) such that any for \(g(x) \in H^2(\Omega)\), we have
\[
\|g\|_{L^\infty(\Omega)} \leq \sqrt{2} \|g\|_{L^2(\Omega)} \|\partial_{x_1} g\|_{L^2(\Omega)} + C \|\nabla g\|_{L^2(\Omega)} \|\nabla^2 g\|_{L^2(\Omega)}.
\] (2.5)

3. A priori estimates. Before we present the energy estimates, we first set the perturbation \((\phi, \psi)(t, x) = (\rho - \bar{\rho}, u - \bar{u})(t, x)\), then problem (1.1)-(1.3) is transformed into
\[
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi + \rho \text{div} \psi &= f, \\
\partial_t \psi + u \cdot \nabla \psi + p'(\rho) \nabla \phi - \mu \Delta \psi - (\mu + \lambda) \nabla \text{div} \psi &= g,
\end{align*}
\] (3.1)

with the initial and boundary conditions
\[
\begin{align*}
(\phi, \psi)|_{t=0} &= (\phi_0, \psi_0), \\
(\psi_1, \psi_2, \psi_3)|_{x_1=0} &= k(x')(0, \partial_{x_1} \psi_2, \partial_{x_1} \psi_3)|_{x_1=0}, \\
(\phi, \psi) &\to (0, 0, 0, 0), \quad \text{as} \quad x_1 \to +\infty,
\end{align*}
\] (3.2)

where
\[
\begin{align*}
f &= -\psi_1 \partial_{x_1} \bar{\rho} - \phi \partial_{x_1} \bar{u}_1, \\
g &= (g_1, 0, 0)^t, \quad g_1 = -(p'(\rho) - \frac{\rho}{\rho} p'(\bar{\rho})) \partial_{x_1} \bar{\rho} - \rho \psi_1 \partial_{x_1} \bar{u}_1 + (2\mu + \lambda) \partial_{x_1}^2 \bar{u}_1.
\end{align*}
\] (3.3)

The solution \((\phi, \psi)(t, x)\) is sought in the set of functional space \(X(0, +\infty)\) defined by
\[
X(0, T) = \left\{ (\phi, \psi) | (\phi, \psi) \in C(0, T; H^2(\Omega)), \nabla \phi \in L^2(0, T; H^1(\Omega)), \nabla \psi \in L^2(0, T; H^2(\Omega)) \quad \sup_{0 \leq t \leq T} \|\psi(t)\|_{H^2} \leq \chi \right\},
\]
with \(0 \leq T \leq +\infty\).
Note that if \( \chi \) is suitably small, then the condition \( \sup_{0 \leq t \leq T} \| (\phi, \psi)(t) \|_{L^2} \leq \chi \) and Sobolev embedding theorem imply that \( ||\phi, \psi|| \leq \frac{1}{2} \rho_- \) and \( |u| = |(u_1, u_2, u_3)| \leq C \) with \( C \) being a positive constant which only depends on \( \rho_+, \ u_\pm \). Therefore, the density function \( \rho(t, x) := \rho(t, x_1) + \phi(t, x) \) satisfies that
\[
0 < \frac{1}{2} \rho_- \leq \rho(t, x) \leq \frac{1}{2} \rho_- + \rho_+. \tag{3.4}
\]

Since the proof for the local-in-time existence and uniqueness of the classical solution to (1.1)-(1.3) is standard for the suitably small perturbation of the solution around the planar rarefaction wave satisfying the property (3.4) (for instance, one can refer to [27]), the details will be omitted. To prove Theorem 1.1, it suffices to show the following a priori estimates.

**Proposition 3.1.** (A priori estimates) Suppose that the reformulated problem (3.1) admits a solution \( (\phi, \psi) \in X(0, T) \) for some \( T > 0 \). Then there exist positive constants \( \chi \) and \( C \) independent of \( T \), such that if
\[
E(t) \triangleq \sup_{0 \leq t \leq T} \| (\phi, \psi)(\cdot, t) \|_{L^2} \leq \chi, \tag{3.5}
\]
then the following uniform estimates hold:
\[
\begin{align*}
\sup_{0 \leq t \leq T} \left( \| (\phi, \psi)(\cdot, t) \|_{L^2}^2 + \| \psi' \r|_{x_1=0}^2 + \| \partial_x \psi' \r|_{x_1=0}^2 & + \| \partial_x \phi \r|_{H^1}^2 + \| \partial_t \phi \r|_{H^1}^2 \right) \\
\phantom{=} & + \int_0^T \left( \| \sqrt{\partial_x \bar{u}_1}(\phi, \psi_1) \r|^2 + \| \partial_t \phi(\psi) \r|_{H^1}^2 + \| \nabla \phi \r|_{H^1}^2 + \| \nabla \psi \r|_{H^2}^2 \right) d\tau \\
\phantom{=} & + \int_0^T \left( \| \partial_x \psi' \r|_{x_1=0}^2 + \| \partial_x ^2 \psi' \r|_{x_1=0}^2 + \| \nabla \phi \r|_{H^1}^2 + \| \nabla \psi \r|_{H^2}^2 \right) d\tau \leq C \| (\phi_0, \psi_0) \r|_{L^2}^2 + \varepsilon^\frac{1}{2},
\end{align*}
\]
where and in the sequel \( \psi' = (\psi_2, \psi_3), \ \partial_{x^i} = \partial_{x_2} \) or \( \partial_{x_3} \) and \( |\alpha| = 0, 1, 2 \).

From now on, we always assume that \( \chi + \varepsilon \leq 1 \). For convenience, we define \( M(t) \geq 0 \) by
\[
M^2(t) = (\chi + \varepsilon) \left( E^2(t) + \int_0^t \left( \| \sqrt{\partial_x \bar{u}_1}(\phi, \psi_1) \r|^2 + \| \partial_t \phi(\psi) \r|_{H^1}^2 + \| \nabla \phi \r|_{H^1}^2 + \| \nabla \psi \r|_{H^2}^2 \right) d\tau \right).
\]
First, we have the following \( L^2 \) estimate.

**Lemma 3.1.** For \( T > 0 \) and \( (\phi, \psi) \in X(0, T) \) satisfying a priori assumption (3.5) with suitably small \( \chi + \varepsilon \), we have for \( t \in [0, T] \),
\[
\begin{align*}
\| (\phi, \psi)(t) \|^2 & + \int_0^t \left( \| \sqrt{\partial_x \bar{u}_1}(\phi, \psi_1) \r|^2 + \| \nabla \psi \r|^2 + \| \psi' \r|_{x_1=0}^2 + \| \nabla \phi \r|_{H^1}^2 + \| \nabla \psi \r|_{H^2}^2 \right) d\tau \\
& \leq C \| (\phi_0, \psi_0) \r|_{L^2}^2 + C \varepsilon^\frac{1}{2}. \tag{3.7}
\end{align*}
\]

**Proof.** A direct calculation shows
\[
\partial_t (\rho \mathcal{E}) + \text{div}(\rho \mathcal{E} + (p(\rho) - p(\bar{\rho}))(\phi + \mu + \lambda)(\partial_t \phi + \mu + \lambda)(\partial_t \psi)) = \mu \nabla \psi|^2 + (\mu + \lambda)(\partial_t \phi + \mu + \lambda)(\partial_t \psi) + (\mu + \lambda)(\partial_t \psi)(\partial_t \psi)
\]
\[
\text{where}
\rho \mathcal{E} = \rho (\Phi(\rho, \bar{\rho}) + \frac{|\psi|^2}{2}),
\]
and
\[
\Phi(\rho, \bar{\rho}) = \int_{\bar{\rho}}^\rho \frac{p(s) - p(\bar{\rho})}{s^2} ds = \frac{1}{\gamma - 1} \rho (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi).
\]
Integrating the above equality with respect to \( t, x \) over \((0, t) \times \Omega\) and using the boundary conditions (3.13) and (3.2), one has

\[
\begin{align*}
\int_0^t \int_\Omega \rho \mathcal{E} \, dx \, d\tau \bigg|_{\tau = 0} + \int_0^t \int_\Omega \left( (\mu |\nabla \psi|^2 + (\mu + \lambda) \text{div} \psi)^2 \right) \, dx \, d\tau + \\
\int_0^t \int_{\partial x_1, \bar{u}_1 (\rho \psi_1^2 + p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi)} dx \, d\tau + \\
\mu \int_0^t \int_{\mathbb{T}^2} \frac{1}{k(x')} |\psi'|^2 \bigg|_{x_1 = 0} \, dx_2 dx_3 d\tau = (2\mu + \lambda) \int_0^t \int_{\partial x_1, \bar{u}_1 \psi_1} dx \, d\tau, \quad (3.9)
\end{align*}
\]

where the boundary terms can be obtained under the conditions (1.3) or (3.2) as follows,

\[
\begin{align*}
\int_0^t \int_\Omega \text{div}(\rho u \mathcal{E} + (p(\rho) - p(\bar{\rho})) \psi) \, dx \, d\tau \\
= \int_0^t \int_{\partial x_1} (\rho u_1 \mathcal{E} + (p(\rho) - p(\bar{\rho})) \psi_1) \, dx_2 dx_3 d\tau \\
= \int_0^t \int_{\mathbb{T}^2} (\rho u_1 \mathcal{E} + (p(\rho) - p(\bar{\rho})) \psi_1) \bigg|_{x_1 = 0} dx_2 dx_3 d\tau = 0. \quad (3.10)
\end{align*}
\]

Similarly,

\[
-(\mu + \lambda) \int_0^t \int_{\mathbb{T}^2} \text{div}(\psi \mathcal{E}) \, dx \, d\tau = -(\mu + \lambda) \int_0^t \int_{\mathbb{T}^2} \psi_1 \text{div}\psi \bigg|_{x_1 = 0} dx_2 dx_3 d\tau = 0. \quad (3.11)
\]

It follows from (3.2) that

\[
(2\mu + \lambda) \int_0^t \int_{\mathbb{T}^2} \partial_{x_1}^2 \bar{u}_1 \psi_1 \, dx \, d\tau \leq C \int_0^t \int_{\mathbb{T}^2} \|\psi_1\|_{L_{x_1}^{2}} \left\| \partial_{x_1, \bar{u}_1} \right\|_{L_{x_1}^{1}} \, dx_2 dx_3 d\tau \leq C \varepsilon \int_0^t \int_{\mathbb{T}^2} \left( (1 + \tau)^{\frac{3}{2}(-1 + \frac{\lambda}{\varepsilon}) \left( \int_{\mathbb{T}^2} \|\psi_1\|_{L_{x_1}^{2}} \|\partial_{x_1, \psi_1}\|_{L_{x_1}^{2}} \, dx_2 dx_3 \right) \right) \, d\tau \]

Substituting the (3.13) into (3.9), and noting that \( p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi \) is equivalent to \( \phi^2 \) for suitably small \( \phi \), we can obtain (3.7) for suitably small \( \varepsilon \) and finish the proof of Lemma 3.1.

Next, different from [17, 16] for the whole space problem, the derivative estimates are obtained by the tangential direction and normal direction respectively due to

\[
\begin{align*}
\int_0^t \int_{\mathbb{T}^2} \left( \partial_{x_1}^2 \bar{u}_1 \psi_1 \right) \, dx \, d\tau \\
\leq C \varepsilon \int_0^t \int_{\mathbb{T}^2} \left( (1 + \tau)^{\frac{3}{2}(-1 + \frac{\lambda}{\varepsilon}) \left( \int_{\mathbb{T}^2} \|\psi_1\|_{L_{x_1}^{2}} \|\partial_{x_1, \psi_1}\|_{L_{x_1}^{2}} \, dx_2 dx_3 \right) \right) \, d\tau \]
\]
the boundary effect. We begin with the first-order tangential derivative estimates of $(\phi, \psi)$, based on the symmetric hyperbolic-parabolic structure of the system.

**Lemma 3.2.** For $T > 0$ and $(\phi, \psi) \in X(0, T)$ satisfying a priori assumption (3.5) with suitably small $\chi + \varepsilon$, it holds that for $t \in [0, T]$,

$$
\|\partial_{x'}(\phi, \psi)(t)\|^2 + \int_0^t \|\nabla \partial_{x'} \psi\|^2 d\tau + \int_0^t \|\partial_{x'} \psi'\|_{x_1=0}^2 d\tau \\
\leq C(E^2(0) + M^2(t) + \varepsilon^\frac{4}{3}),
$$

(3.14)

and

$$
\|\partial_t(\phi, \psi)(t)\|^2 + \int_0^t \|\nabla \partial_t \psi\|^2 d\tau + \int_0^t \|\partial_t \psi'\|_{x_1=0}^2 d\tau \\
\leq C(E^2(0) + M^2(t) + \varepsilon).
$$

(3.15)

**Proof.** First, we derive the first-order tangential derivative in space. Applying $\partial_{x'}$ to (3.1) yields

$$
\begin{aligned}
\begin{cases}
\partial_t \phi + u \cdot \nabla \phi + \rho \partial_{x'} \phi = -\partial_x u \cdot \nabla \phi - \partial_{x'} \rho \partial_{x'} \phi + \partial_{x'} f, \\
\rho (\partial_t \psi + u \cdot \nabla \psi) + \nabla (p'(\rho) \partial_{x'} \phi) - \mu \Delta \partial_{x'} \psi - (\mu + \lambda) \nabla \partial_{x'} \psi = p''(\rho) \nabla \partial_{x'} \phi - \partial_{x'} \rho \partial_t \psi - \partial_{x'} (\rho u) \cdot \nabla \psi + \partial_{x'} g.
\end{cases}
\end{aligned}
$$

(3.16)

Multiplying the first equation by $\frac{p'(\rho)}{\rho} \partial_{x'} \phi$, the second equation by $\partial_{x'} \psi$, adding the two resulted equations together and then integrating it with respect to $t, x$ over $(0, t) \times \Omega$, we can obtain

$$
\begin{aligned}
&\int \left( \frac{p'(\rho)}{\rho} \left( \frac{\partial_{x'} \phi}{2} \right)^2 + \rho \left| \partial_{x'} \psi \right|^2 \right) dx \bigg|_{\tau=0}^{t} + \int_0^t \int \left( \mu \left| \nabla \partial_{x'} \psi \right|^2 + (\mu + \lambda)(\nabla \partial_{x'} \psi)^2 \right) dx d\tau \\
&\quad + \mu \int_0^t \int_{\mathbb{T}^2} \left( (\partial_{x'} \psi_2)^2 + (\partial_{x'} \psi_3)^2 \right) \left|_{x_1=0} \right. dx_2 dx_3 d\tau \\
&= -\mu \sum_{i=2}^3 \int_0^t \int_{\mathbb{T}^2} \partial_{x'} \left( \frac{1}{k(x')} \right) \psi_i \partial_{x'} \psi_i \left|_{x_1=0} \right. dx_2 dx_3 d\tau \\
&\quad + \int_0^t \int \left( (3 - \gamma) \frac{p'(\rho)}{\rho} \partial_{x'} \phi \right) dx d\tau + p''(\rho) \int_0^t \int \left| \partial_{x'} \phi \right|^2 dx d\tau \\
&\quad - \int_0^t \int \left( \rho \left( \partial_t \psi + \nabla \phi + \partial_{x'} \rho \partial_{x'} \psi \right) \right) \frac{p'(\rho)}{\rho} \partial_{x'} \phi dx d\tau \\
&\quad + \int_0^t \int \left( \frac{p'(\rho)}{\rho} \partial_{x'} f \partial_{x'} \phi + \partial_{x'} g \cdot \partial_{x'} \psi \right) dx d\tau := 4 \sum_{i=1}^4 I_i.
\end{aligned}
$$

(3.17)

First, for the boundary term $I_1$, we have

$$
I_1 \leq \frac{\mu}{2} \int_0^t \int_{\mathbb{T}^2} \left( (\partial_{x'} \psi_2)^2 + (\partial_{x'} \psi_3)^2 \right) \left|_{x_1=0} \right. dx_2 dx_3 d\tau + C \int_0^t \int_{\mathbb{T}^2} \left| \psi' \right|^2 \left|_{x_1=0} \right. dx_2 dx_3 d\tau.
$$

(3.18)

Using Cauchy’s inequality, Sobolev’s inequality, and the assumption (3.5), we have

$$
I_2 \leq C \int_0^t \left( \|\nabla \psi\|_{L^\infty} \|\partial_{x'} \phi\|^2 + \|\partial_{x'} \mu \|_{L^\infty} \|\partial_{x'} \phi\|^2 + \|\partial_{x'} \rho \|_{L^\infty} \|\partial_{x'} \psi_1\| \|\partial_{x'} \phi\| \right) dx.
$$
noting that Substituting (3.18)-(3.21) into (3.17) leads to (3.14). Next, applying where we have used the boundary conditions (3.2) to compute that to \( t, x \)

\[ \leq C(\chi + \varepsilon) \int_0^t (||\partial_x \phi||^2 + ||\partial_x \psi_1||^2 + ||\nabla^2 \psi||^2) \, dt. \]  

(3.19)

It follows from the Cauchy’s inequality, Sobolev’s inequality and the assumption (3.5) that

\[ I_3 \leq C \int_0^t (||\partial_x \phi||_{L^2} ||\nabla \phi||_{L^6} + ||\partial_x \phi||_{L^1} ||\nabla \psi||_{L^6}) ||\partial_{xx} \phi|| \, dt \]

\[ + C \int_0^t (||\partial_x \phi||_{H^{1/2}} ||\partial_t \phi||_{L^6} + ||\partial_x (\phi, \psi)||_{H^1} ||\nabla \psi||_{L^6}) ||\partial_{xx} \phi|| \, dt \]

\[ \leq C \int_0^t (||\partial_x \psi||_{H^{1/2}} ||\nabla \phi||_{H^1} + ||\partial_x \phi||_{H^{1/2}} ||\nabla \psi||_{H^1}) ||\partial_{xx} \phi|| \, dt \]

\[ + C \int_0^t (||\partial_x \phi||_{H^{1/2}} ||\partial_x \phi||_{H^{1/2}} ||\partial_t \psi||_{H^1} + ||\partial_x (\phi, \psi)||_{H^1} ||\nabla \psi||_{H^1}) ||\partial_{xx} \phi|| \, dt \]

\[ \leq C \chi \int_0^t (||\nabla \phi||_{H^1}^2 + ||\nabla \psi||_{H^1}^2 + ||\partial_t \psi||_{H^1}^2). \]  

(3.20)

By Cauchy’s inequality and (3.3), it holds

\[ I_4 \leq C \varepsilon \int_0^t (||\partial_x \phi||^2 + ||\partial_x \psi||^2) \, dt. \]  

(3.21)

Substituting (3.18)-(3.21) into (3.17) leads to (3.14). Next, applying \( \partial_t \) to (3.1) and noting that

\[ ||\partial_t (\phi, \psi)(0)||^2 \leq C||\nabla^2 \psi_0||^2 + C||\nabla (\phi_0, \psi_0)||^2 + C\varepsilon||\nabla (\phi_0, \psi_0)||^2 \]

(3.15) can be obtained similarly, and the proof of Lemma 3.2 is completed. \( \Box \)

We next derive the \( H^1 \)–parabolic estimates for \( \psi \).

**Lemma 3.3.** For \( T > 0 \) and \( (\phi, \psi) \in X(0, T) \) satisfying a priori assumption (3.5) with suitably small \( \chi + \varepsilon \), we have for \( t \in [0, T] \),

\[ ||\nabla \psi(t)||^2 + ||\psi'(t)||_{L^2(\gamma \Omega)}^2 + \int_0^t ||\partial_t \psi||^2 \, dt \]  

\[ \leq C||\phi(t)||^2 + C\eta \int_0^t ||\nabla \phi||^2 \, dt + C\eta \int_0^t ||\nabla \psi||^2 \, dt + C(E^2(0) + M^2(t)) + C_\eta \varepsilon^2. \]  

(3.22)

**Proof.** Multiplying (3.1) by \( \partial_t \psi \) and integrating the resulting equality with respect to \( t, x \) over \([0, t] \times \Omega\), we have

\[ \int_{\Omega} \left( \frac{\mu}{2} ||\nabla \psi||^2 + \frac{\mu + \lambda}{2} (\div \psi)^2 \right) \, dx \bigg|_{\tau = 0}^{\tau = t} + \frac{\mu}{2} \int \left| \frac{\psi'^2}{2} \right| \, dx \bigg|_{x_1 = 0}^{x_1 = \tau} + \int_0^t \rho ||\partial_t \psi||^2 \, dx \, dt \]

\[ = - \int_0^t \int \rho' \nabla \phi \cdot \partial_t \psi \, dx \, dt - \int_0^t \int \rho u \cdot \nabla \psi \cdot \partial_t \psi \, dx \, dt + \int_0^t \int g_1 \cdot \partial_t \psi \, dx \, dt, \]  

(3.23)

where we have used the boundary conditions (3.2) to compute that

\[ - \mu \int_0^t \int \div (\nabla \psi \cdot \partial_t \psi) \, dx \, dt \]  

(3.24)
and

 integration by parts implies that

\[ \int_0^t \frac{1}{k(x')} (\psi_2 \partial_t \psi_2 + \psi_3 \partial_t \psi_3) \bigg|_{x_1=0} \, dx_2 dx_3 d\tau = \frac{\mu}{2} \int_0^t \frac{|\psi'|^2}{k(x')} \bigg|_{x_1=0} \, dx_2 dx_3 \bigg|_{\tau=0}^t, \]

and

\[ -(\mu + \lambda) \int_0^t \int \text{div}(\partial_t \psi \text{div} \psi) \, dx d\tau \]

\[ = (\mu + \lambda) \int_0^t \int \partial_t \psi_1 \text{div} \psi \bigg|_{x_1=0} \, dx_2 dx_3 d\tau = 0. \quad (3.25) \]

For the first term on the right hand side of (3.23), integration by parts implies that

\[ - \int_0^t \int p'(\rho) \nabla \cdot \partial_t \rho \psi \psi d\tau d\tau \]

\[ = - \int_0^t \int p'(\rho) \nabla \phi \cdot \partial_t \rho \psi \psi d\tau d\tau + \int_0^t \int \partial_t (p(\rho) \nabla \phi) \cdot \psi \psi d\tau d\tau \]

\[ = \int_0^t \int p'(\rho) \text{div} \psi d\tau d\tau + \int_0^t \int \phi \psi \cdot \nabla p'(\rho) d\tau d\tau + \int_0^t \int \partial_t (p(\rho) \nabla \phi) \cdot \psi \psi d\tau d\tau. \]

Now we handle the last term above. Direct calculation gives

\[ \partial_t (p'(\rho) \nabla \phi) + u \cdot \nabla (p'(\rho) \nabla \phi) + p'(\rho) \rho \nabla \text{div} \psi \]

\[ = p''(\rho)(\partial_t \rho + u \cdot \nabla \rho) \nabla \phi + p'(\rho)(\partial_t \nabla \phi + u \cdot \nabla (\nabla \phi) + \rho \nabla \text{div} \psi) \]

\[ = - p''(\rho) \rho \text{div} \psi \nabla \phi + p'(\rho)(-\nabla u \cdot \nabla \phi - \rho \text{div} \psi + \nabla f). \]

Therefore, similar to (3.19) and (3.20), we have

\[ \int_0^t \int \partial_t (p'(\rho) \nabla \phi) \cdot \psi \psi d\tau d\tau \]

\[ = - \int_0^t \int u \cdot \nabla (p'(\rho) \nabla \phi) \cdot \psi \psi d\tau + \int_0^t \int p'(\rho) \rho \psi \cdot \nabla \psi \psi d\tau d\tau \]

\[ - \int_0^t \int p''(\rho) \rho \text{div} u \nabla \phi \psi d\tau d\tau + \int_0^t \int p'(\rho)(-\nabla u \cdot \nabla \phi - \rho \text{div} \psi + \nabla f) \cdot \psi 
\]

\[ = \int_0^t \int p'(\rho)(u \cdot \nabla \psi \cdot \nabla \phi + \rho (\text{div} \psi)^2 + \psi \cdot \nabla \phi \psi d\tau + \int_0^t \int p'(\rho)(-\nabla u \cdot \nabla \phi - \rho \text{div} \psi + \nabla f) \cdot \psi d\tau d\tau \]

\[ + \int_0^t \int p'(\rho) \psi \cdot \nabla f d\tau d\tau + \int_0^t \int p'(\rho) \partial_x \bar{u}_1(\psi \cdot \nabla \phi - \psi \partial_x \phi) d\tau d\tau \]

\[ + \int_0^t \int p''(\rho)(\psi_1 \partial_{x_1} \bar{u}_1 \psi \cdot \nabla \phi d\tau d\tau := \sum_{i=5}^8 I_i. \quad (3.27) \]

By Cauchy’s inequality, we have

\[ I_5 \leq \eta \int_0^t \|
\]
Similar to (3.13), one has

\[
I_6 \leq \eta \int_0^t \|\nabla \phi\|^2 dt + C_0 \int_0^t \|\partial_x u_1 \| \phi \|^2 dt
+ C \int_0^t \|\nabla \psi\|^2 dt + C \int_0^t \sqrt{\partial_x u_1 (\phi, \psi_1)} \|^2 dt
\]

\[
\leq \eta \int_0^t \|\nabla \phi\|^2 dt + C_0 \int_0^t \|\partial_x \psi\|^2 dt + \varepsilon \frac{1}{\tau}
+ C \int_0^t \|\nabla \psi\|^2 dt + C_0 \int_0^t \sqrt{\partial_x u_1 (\phi, \psi_1)} \|^2 dt.
\]

Substituting (3.28)-(3.29) into (3.27), we can get

\[
\int_0^t \int \partial_t (p' (\rho) \nabla \phi) \cdot \psi dxd\tau
\]

\[
\leq \eta \int_0^t \|\nabla \phi\|^2 dt + C_0 \int_0^t \|\nabla \psi\|^2 dt + CM^2(t) + C_0 \varepsilon \frac{1}{\tau}.
\]

Therefore, by (3.26), (3.30) and Cauchy’s inequality, it holds

\[
- \int_0^t \int p' (\rho) \nabla \phi \cdot \partial_x \psi dxd\tau
\]

\[
\leq \frac{\mu + \lambda}{4} \|\text{div} (\psi (t))\|^2 + C \|\phi (t)\|^2 + C (\text{E}^2 (0) + M^2 (t))
+ C (\chi + \varepsilon) \|\phi (t)\|^2_{L^1} + \|\psi (t)\|^2 + \eta \int_0^t \|\nabla \phi\|^2 dt + C_0 \int_0^t \|\nabla \psi\|^2 dt + C_0 \varepsilon \frac{1}{\tau}.
\]

For the second term on the right hand side of (3.23), by Cauchy’s inequality, it holds

\[
- \int_0^t \int \rho u \cdot \nabla \psi \cdot \partial_x \psi dxd\tau \leq \frac{1}{4} \int_0^t \|\sqrt{\rho} \partial_x \psi\|^2 + C \int_0^t \|\nabla \psi\|^2 dt,
\]

and

\[
\int_0^t \int g_1 \cdot \partial_t \psi_1 dxd\tau \leq \frac{1}{4} \int_0^t \|\sqrt{\rho} \partial_x \psi_1\|^2 + C \int_0^t \|\sqrt{\partial_x u_1 (\phi, \psi_1)}\|^2 dt + C \varepsilon.
\]

Finally, noting that

\[
\frac{\mu}{2} \int_{\gamma} \frac{|\psi'|^2}{k (x')} dx' \bigg|_{x_1 = 0} \leq C \int_{\gamma} \|\psi_0\|_{L^2} dx_2 dx_3
\]

\[
\leq C \int \|\psi_0\|_{L^2} \|\partial_x \psi_0\|_{L^2} dx_2 dx_3 \leq C \int \|\psi_0\|_{L^2} \|\partial_x \psi_0\|_{L^2} dx_2 dx_3
\]

\[
\leq C \|\psi_0\|^2 + \|\partial_x \psi_0\|^2,
\]

and substituting (3.31)-(3.34) into (3.23), we can obtain (3.22). The proof of Lemma 3.3 is completed. \qed
Then, we derive the dissipative estimates for normal derivative of $\phi$, i.e., $\|\partial_{x_1}\phi\|^2$, which follows from a hyperbolic-parabolic structure of the perturbation system (3.1).

**Lemma 3.4.** For $T > 0$ and $(\phi, \psi) \in X(0, T)$ satisfying a priori assumption (3.5) with suitably small $\chi + \varepsilon$, it holds that for $t \in [0, T]$,

$$
\|\partial_{x_1}\phi(t)\|^2 + \int_0^t (\|\partial_{x_1}\phi\|^2 + \|\partial_{x_1}^2\psi_1\|^2) dt \\
\leq C \int_0^t (\|\partial_t\psi_1\|^2 + \|\nabla\partial_{x_1}\psi\|^2) dt + C(E^2(0) + M^2(t) + \varepsilon). 
$$

(3.35)

**Proof.** Applying $\partial_{x_1}$ to (3.1)$_1$ and rewriting (3.1)$_2$, we have

$$\left\{
\begin{array}{l}
\partial_t \partial_{x_1} \phi + u \cdot \nabla \partial_{x_1} \phi + \rho \partial_{x_1}^2 \psi_1 + \rho \partial_{x_1} \nabla \phi \cdot \partial_{x_1} \psi + \partial_{x_1} u \cdot \nabla \phi + \partial_{x_1} \rho \partial_{x_1} \phi = \partial_{x_1} f, \\
\rho (\partial_t \psi_1 + u \cdot \nabla \psi_1) + \partial_t \phi (\rho \partial_{x_1} \phi - (2\mu + \lambda) \partial_{x_1}^2 \psi_1 - \mu \Delta' \psi_1 - (\mu + \lambda) \partial_{x_1} \nabla \phi \cdot \partial_{x_1} \psi') = g_1.
\end{array}
\right.
$$

where $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, $\nabla' = (\partial_{x_2} \partial_{x_1})$ and $\nabla' \cdot \psi' = \partial_{x_2} \psi_2 + \partial_{x_3} \psi_3$. Multiplying (3.36)$_1$ by $\frac{1}{\rho} \partial_{x_1} \phi$ and (3.36)$_2$ by $\frac{1}{\rho} (\partial_{x_1} \phi)$ and adding the resulted equations together and then integrating the resulted equality with respect to $t$, $x$ over $[0, t] \times \Omega$, it holds

$$
\int_0^t \left( \frac{1}{\rho} \frac{\partial_t \phi}{2} \right)^2 dx |_{t=0} + \int_0^t \frac{\partial_t \phi}{2 \mu + \lambda} (\partial_{x_1} \phi)^2 dx dt \\
= \int_0^t \left( \mu (\Delta' \psi_1 - \partial_{x_1} \nabla \phi \cdot \psi') - \rho (\partial_t \psi_1 + u \cdot \nabla \psi_1) \right) \frac{\partial_{x_1} \phi}{2 \mu + \lambda} dx dt \\
+ \int_0^t \left( \frac{\mu}{\rho} (\partial_{x_1} \phi)^2 - \frac{1}{\rho} \partial_{x_1} u \cdot \nabla \phi \partial_{x_1} \phi \right) dx dt \\
+ \int_0^t \left( \frac{1}{\rho} \partial_{x_1} f \partial_{x_1} \phi + \frac{1}{2 \mu + \lambda} g_1 \partial_{x_1} \phi \right) dx dt := \sum_{i=9}^{11} I_i. 
$$

(3.37)

Here the boundary term vanishes under the condition (3.2). By Cauchy’s inequality, one has

$$I_9 \leq \frac{1}{8 (2 \mu + \lambda)} \int_0^t \left( \frac{\rho' \phi}{\rho} \right)^2 dx + C \int_0^t (\|\nabla \partial_{x_1} \psi\|^2 + \|\partial_t \psi_1\|^2 + \|\nabla \psi_1\|^2) dt. 
$$

(3.38)

It follows from Cauchy’s inequality and the assumption (3.5) that

$$I_{10} = \int_0^t \int \frac{1}{\rho} \left( - \partial_{x_1} \psi \cdot \nabla \phi \partial_{x_1} \phi - \partial_{x_1} \rho \partial_{x_1} \nabla \psi \phi \right) dx dt \\
\leq C \int_0^t (\|\partial_{x_1} \psi\|_{L^4} \|\nabla \phi\|_{L^8} \|\partial_{x_1} \phi\| + \varepsilon \|\nabla \psi\| \|\partial_{x_1} \phi\|) dt \\
\leq C(\chi + \varepsilon) \int_0^t (\|\nabla \psi\|^2_{L^2} + \|\partial_{x_1} \phi\|^2). 
$$

(3.39)

By Cauchy’s inequality and Lemma 2.1, we have

$$I_{11} \leq \frac{1}{8 (2 \mu + \lambda)} \int_0^t \left( \frac{\rho' \phi}{\rho} \right)^2 dx dt \\
+ C \varepsilon \int_0^t (\|\partial_{x_1} \phi\|^2 + \|\partial_{x_1} \psi_1\|^2 + \|\sqrt{\partial_{x_1} \phi_1} (\phi, \psi_1)\|^2) dt + C \varepsilon. 
$$

(3.40)
Substituting (3.38)-(3.40) into (3.37) gives

\[
\|\partial_x \phi(t)\|^2 + \int_0^t \|\partial_x \phi\|^2 \, d\tau \\
\leq C \|\partial_x \phi_0\|^2 + C \int_0^t (\|\partial_t \psi_1\|^2 + \|\nabla \psi_1\|^2 + \|\nabla \partial_x \psi\|^2) \, d\tau \\
+ C (\chi + \varepsilon) \int_0^t \|\nabla \psi\|_{H^1}^2 \, d\tau + C \varepsilon \int_0^t \|\sqrt{\partial_x u_1(\phi, \psi)}\|^2 \, d\tau + C \varepsilon. \tag{3.41}
\]

It follows from (3.36) that

\[
\int_0^t \|\partial^2_x \psi_1\|^2 \, d\tau \leq C \int_0^t (\|\partial_x \phi\|^2 + \|\partial_t \psi_1\|^2 + \|\nabla \psi_1\|^2 + \|\nabla \partial_x \psi\|^2) \, d\tau \\
+ C \varepsilon \int_0^t \|\sqrt{\partial_x u_1(\phi, \psi)}\|^2 \, d\tau + C \varepsilon. \tag{3.42}
\]

Then, multiplying (3.41) by a large constant \(C\) and combining (3.42), we can get (3.35). The proof of Lemma 3.4 is completed.

We next want to estimate the tangential derivatives of \(\phi\) by the momentum equations.

**Lemma 3.5.** For \(T > 0\) and \((\phi, \psi) \in X(0, T)\) satisfying a priori assumption (3.5) with suitably small \(\chi + \varepsilon\), it holds that for \(t \in [0, T]\),

\[
\int_0^t (\|\partial_x \phi\|^2 + \|\partial^2_x \psi\|^2) \, d\tau \\
\leq C \eta \int_0^t (\|\partial_x \phi\|^2 \, d\tau + C \eta \int_0^t \|\nabla \partial_x \psi\|^2 \, d\tau \\
+ C \int_0^t (\|\partial_t \psi\|^2 + \|\nabla \psi\|^2 + \|\partial_x \partial_x \phi\|^2 + ||\psi||_{L^2(T^2)}^2) \, d\tau. \tag{3.43}
\]

**Proof.** We rewrite the second perturbed momentum equations in (3.1) as

\[
\rho(\partial_t \psi_2 + u \cdot \nabla \psi_2) + p'(\rho)(\partial_x \phi) - \mu \partial^2_x \psi_2 - \mu \Delta \psi_2 - (\mu + \lambda) \partial_x \div \psi = 0. \tag{3.44}
\]

Multiplying the above equation by \(\partial_x \phi\) and integrating the resulting equation with respect to \(t, x\) over \((0, t) \times \Omega\), we have

\[
\int_0^t \int \rho' (\rho)(\partial_x \phi)^2 \, dx \, d\tau \tag{3.45}
\]

\[
= \int_0^t \mu \partial_x \psi_2 \partial_x \phi \, dx \, d\tau + \int_0^t \int (\mu \Delta \psi_2 + (\mu + \lambda) \div \psi - \rho(\partial_t \psi_2 + u \cdot \nabla \psi_2)) \partial_x \phi \, dx \, d\tau.
\]

Integration by parts under the boundary conditions (3.2) leads to

\[
\int \mu \int \partial^2_x \psi_2 \partial_x \phi \, dx \, d\tau = - \mu \int \int \partial^2_x \phi \, dx \, d\tau
\]

\[
= \mu \int \int \partial_x \partial_x \psi_2 \phi \, dx \, d\tau + \mu \int \int \partial_x \partial_x \partial_x \psi_2 \phi \, dx \, d\tau
\]

\[
= - \mu \int \int \partial_x \partial_x \psi_2 \phi \, dx \, d\tau + \mu \int \int \partial_x \partial_x \partial_x \psi_2 \phi \, dx \, d\tau
\]
By Cauchy’s inequality, it holds

\[
\leq C \int_0^t \left( \int_{\mathbb{T}^3} |\psi|_{x_1=0} \| \partial_x \phi \|_{L^\infty} \, dx \right) \, dt
\]

Substituting the above inequality into (3.46) and using Cauchy’s inequality, we can obtain

\[
\mu \int_0^t \int_{\mathbb{T}^3} \partial_{x_1} \psi \partial_{x_2} \phi \, dx \, dt
\]

By Cauchy’s inequality, it holds

\[
\int_0^t \left( \int_{\mathbb{T}^3} (\mu \Delta \psi + (\mu + \lambda) \partial_x \phi) \partial_x \phi \, dx \right) \, dt
\]

Substitution (3.48) and (3.49) into (3.45) yields

\[
\int_0^t \left( \int_{\mathbb{T}^3} (\mu \Delta \psi + (\mu + \lambda) \partial_x \phi) \partial_x \phi \, dx \right) \, dt
\]

It follows from (3.44) that

\[
\int_0^t \left( \int_{\mathbb{T}^3} (\mu \Delta \psi + (\mu + \lambda) \partial_x \phi) \partial_x \phi \, dx \right) \, dt
\]

Then, multiplying (3.50) by a large constant \( C \) and combining (3.51) together, we have

\[
\int_0^t \left( \int_{\mathbb{T}^3} (\mu \Delta \psi + (\mu + \lambda) \partial_x \phi) \partial_x \phi \, dx \right) \, dt
\]

Similar to (3.52), one can obtain the estimate of \( \int_0^t (\| \partial_x \phi \|^2 + \| \partial_x^2 \psi \|^2) \, dt \). Then we have (3.43), and the proof of Lemma 3.5 is completed.
In the following, the higher order derivative estimates can be obtained by the same idea as the lower order derivative estimates. We start with the tangential derivative estimates.

**Lemma 3.6.** For $T > 0$ and $(\phi, \psi) \in X(0, T)$ satisfying a priori assumption (3.5) with suitably small $\chi + \varepsilon$, it holds that for $t \in [0, T]$,

$$
\|\partial_x^3 (\phi, \psi)(t)\|^2 + \int_0^t \|\nabla \partial_x^2 \psi\|^2 d\tau + \int_0^t \|\partial_x^2 \psi'\|_{x_1=0}^2 d\tau \\
\leq C(E^2(0) + M^2(t) + \varepsilon^\frac{1}{2}).
$$

(3.53)

**Proof.** Applying $\partial_x^3$ to (3.1), we have

\[
\begin{cases}
\partial_t \partial_x^2 \phi + u \cdot \nabla \partial_x^2 \phi + p \text{div} \partial_x^2 \psi = -[\partial_x^2, u \cdot \nabla] \phi - [\partial_x^2, \rho] \text{div} \psi + \partial_x^2 f, \\
\rho(\partial_t \partial_x^2 \psi + u \cdot \nabla \partial_x^2 \psi) + \nabla(p'\rho) \partial_x^2 \phi - \mu \Delta \partial_x^2 \psi - (\mu + \lambda) \nabla \text{div} \partial_x^2 \psi \\
= p''(\rho) \nabla \rho \partial_x^2 \phi - [\partial_x^2, \rho] \partial_t \psi - [\partial_x^2, \rho u \cdot \nabla] \psi - [\partial_x^2, p'\rho] \nabla \phi + \partial_x^2 g.
\end{cases}
\]

(3.54)

Here $[A, B] = AB - BA$ is the commutator of $A$ and $B$. Multiplying the equation (3.54)$_1$ by $\frac{p''(\rho)}{\rho} \partial_x^2 \phi$, (3.54)$_2$ by $\partial_x^2 \psi$ and adding the resulted equations together, then integrating it with respect to $t, x$ over $[0, t] \times \Omega$, similar to (3.17), one has

\[
\begin{align*}
&\int \left( \frac{p''(\rho)}{\rho} \frac{(\partial_x^2 \phi)^2}{2} + \frac{\partial_x^2 \psi^2}{2} \right) dx \bigg|_{t=0}^{t} + \int \mu \int_0^t \int_{\mathbb{T}^2} \left( (\partial_x^2 \phi)^2 + (\partial_x^2 \psi)^2 \right) dx d\tau \\
&\quad + \mu \sum_{i=1}^6 \int_0^t \int_{\mathbb{T}^2} \left( \frac{1}{k(x')} \psi_i \partial_x^2 \psi_i + 2 \partial_x^2 \left( \frac{1}{k(x')} \partial_x^2 \psi \partial_x^2 \psi_i \right) \right) dx_3 d\tau \\
&\quad + \int_0^t \int_0^t \int_{\mathbb{T}^2} (3 - \gamma) \frac{p''(\rho)}{\rho} \text{div} \left( \frac{\partial_x^2 \phi}{2} \right) dx_3 d\tau + \int_0^t \int_0^t \int_{\mathbb{T}^2} p''(\rho) \partial_x^2 \phi \cdot \nabla \rho \partial_x^2 \phi dx_3 d\tau \\
&\quad - \int_0^t \int_0^t \int_{\mathbb{T}^2} \left( [\partial_x^2, u \cdot \nabla] \phi + [\partial_x^2, \rho] \text{div} \psi \right) p''(\rho) \partial_x^2 \phi dx_3 d\tau \\
&\quad - \int_0^t \int_0^t \int_{\mathbb{T}^2} \left( [\partial_x^2, \rho] \partial_t \psi + [\partial_x^2, \rho u \cdot \nabla] \psi + [\partial_x^2, p'\rho] \nabla \phi \right) \cdot \partial_x^2 \psi dx_3 d\tau \\
&\quad + \int_0^t \int_0^t \int_{\mathbb{T}^2} \left( \frac{p''(\rho)}{\rho} \partial_x^2 f \partial_x^2 \phi + \partial_x^2 g \cdot \partial_x^2 \psi \right) dx_3 d\tau := \sum_{i=1}^6 J_i.
\end{align*}
\]

(3.55)

For $J_1$, it holds

\[
J_1 \leq \frac{\mu}{2} \int_0^t \int_{\mathbb{T}^2} \frac{1}{k(x')} \left( (\partial_x^2 \phi)^2 + (\partial_x^2 \psi)^2 \right) dx_3 d\tau + C \int_0^t \int_{\mathbb{T}^2} \left( \psi^2 + |\partial_x \psi|^2 \right) dx_3 d\tau.
\]

(3.56)

Similar to (3.19), we have

\[
J_2 \leq C(\chi + \varepsilon) \int_0^t \|\partial_x^2 \phi\|^2 + \|\nabla^2 \text{div} \psi\|^2 d\tau.
\]

(3.57)
It follows from the Cauchy’s inequality, Sobolev’s inequality and assumption (3.5) that

\[
J_3 \leq C \int_0^t (\|\partial_x^2 \psi\|_{L^2} \|\nabla \phi\|_{L^2} \|\partial_x^2 \phi\| + \|\partial_x \tilde{\mu}\|_{L^\infty} \|\partial_x^2 \psi_1\| \|\partial_x^2 \phi\|) d\tau
\]

\[
\leq C(\chi + \varepsilon) \int_0^t (\|\partial_x^2 \psi\|_{H^1}^2 + \|\partial_x^2 \phi\|^2) d\tau.
\] (3.58)

Note that

\[
\begin{align*}
[\partial_x^2, u \cdot \nabla] \phi &= \partial_x^2 (u \cdot \nabla \phi) - u \cdot \nabla \partial_x^2 \phi = \partial_x^2 u \cdot \nabla \phi + \partial_x \partial_x u \cdot \nabla \phi, \\
[\partial_x^2, \rho] \text{div} \psi &= \partial_x^2 (\rho \text{div} \psi) - \rho \text{div} \partial_x^2 \psi = \partial_x^2 \rho \text{div} \psi + \partial_x \rho \text{div} \psi,
\end{align*}
\]

by Cauchy’s inequality and Sobolev’s inequality, we have

\[
J_4 \leq C \int_0^t (\|\nabla \partial_x \psi\|_{L^2} \|\nabla \phi\|_{L^2} + \|\nabla \psi\|_{L^\infty} \|\nabla \phi\|) \|\partial_x^2 \phi\| d\tau dx d\tau
\]

\[
\leq C\chi \int_0^t (\|\nabla^2 \psi\|_{H^1}^2 + \|\nabla \phi\|_{H^1}^2) d\tau.
\] (3.59)

Similar to (3.59), it holds

\[
J_5 \leq C\chi \int_0^t (\|\partial_x^2 \psi\|_{H^1}^2 + \|\partial_x^2 \phi\|_{H^1}^2) d\tau.
\] (3.60)

By Cauchy’s inequality and (3.3), it holds

\[
J_6 \leq C\varepsilon \int_0^t (\|\partial_x \phi\|_{H^1}^2 + \|\partial_x \psi\|_{H^1}^2) d\tau.
\] (3.61)

Substitution (3.60)-(3.61) into (3.55) leads to (3.53), therefore, the proof of Lemma 3.6 is completed.

**Lemma 3.7.** For $T > 0$ and $(\phi, \psi) \in X(0, T)$ satisfying a priori assumption (3.5) with suitably small $\chi + \varepsilon$, we have for $t \in [0, T]$,

\[
\|\nabla^2 \psi(t)\|^2 + \|\partial_x \psi'\|_{x_1 = 0}(t)^2 \leq C(\|\nabla^2 \phi(t)\|^2 + \|\nabla^2 \psi(t)\|^2) + C\|\phi\|_{L^2(T^2)}^2 + C\varepsilon \|\psi\|_{L^2(T^2)} + C\varepsilon^3.
\] (3.62)

**Proof.** Since

\[
\mu \int |\Delta \psi|^2 dx
\]

\[
= \mu \int (|\partial_{x_1}^2 \psi|^2 + |\partial_{x_2}^2 \psi|^2 + |\partial_{x_2}^2 \psi|^2) dx
\]

\[
+ 2\mu \int (\partial_{x_2}^2 \psi \cdot \partial_{x_3}^2 \psi + \partial_{x_1}^2 \psi \cdot \partial_{x_2}^2 \psi + \partial_{x_1}^2 \psi \cdot \partial_{x_3}^2 \psi) dx
\]

\[
= \mu \int (|\partial_{x_1}^2 \psi|^2 + |\partial_{x_2}^2 \psi|^2 + |\partial_{x_3}^2 \psi|^2 + 2|\partial_{x_2} \partial_{x_3} \psi|^2) dx
\]

\[
- 2\mu \int (\partial_{x_1}^2 \partial_{x_2} \psi \cdot \partial_{x_3} \psi + \partial_{x_1}^2 \partial_{x_3} \psi \cdot \partial_{x_2} \psi) dx
\]

\[
= \mu \|\nabla^2 \psi\|^2 + 2\mu \int_{\Gamma^2} (\partial_{x_1} \partial_{x_2} \psi \cdot \partial_{x_3} \psi + \partial_{x_1} \partial_{x_3} \psi \cdot \partial_{x_2} \psi)^3 |_{x_1 = 0} dx_2 dx_3
\]

\[
= \mu \|\nabla^2 \psi\|^2 + 2\mu \int_{\Gamma^2} \frac{1}{k(x')} |\partial_{x'} \psi'|^2 |_{x_1 = 0} dx_2 dx_3
\]
\begin{align*}
+ 2\mu \sum_{i=2}^{3} \int_{\Omega} \left[ \partial_{x_2} \left( \frac{1}{k(x')} \right) \partial_{x_2} \psi_i \psi_i + \partial_{x_3} \left( \frac{1}{k(x')} \right) \partial_{x_3} \psi_i \psi_i \right]_{x_1 = 0} \, dx_2 dx_3,
\end{align*} 
and
\begin{align*}
(\mu + \lambda) \int \Delta \psi \cdot \nabla \text{div} \psi \, dx \\
= (\mu + \lambda) \int (\text{div}(\Delta \psi \text{div} \psi) - \Delta \text{div} \psi \text{div} \psi) \, dx \\
= (\mu + \lambda) \int \text{div}(\Delta \psi \text{div} \psi - \nabla \text{div} \psi \text{div} \psi) \, dx + (\mu + \lambda) \| \nabla \text{div} \psi \|^2 \\
= (\mu + \lambda) \| \nabla \text{div} \psi \|^2 - (\mu + \lambda) \int_{\Omega} (\Delta \psi_1 - \partial_{x_1} \text{div} \psi)_{x_1 = 0} \, dx_2 dx_3 \\
= (\mu + \lambda) \| \nabla \text{div} \psi \|^2 + (\mu + \lambda) \int_{\Omega} (\partial_{x_2} \partial_{x_2} \psi_2 + \partial_{x_3} \partial_{x_3} \psi_3)_{x_1 = 0} \, dx_2 dx_3 \\
= (\mu + \lambda) \| \nabla \text{div} \psi \|^2 + (\mu + \lambda) \int_{\Omega} \left( \frac{\psi_2}{k(x')} + \frac{\psi_3}{k(x')} \right)_{x_1 = 0} \, dx_2 dx_3 \\
= (\mu + \lambda) \| \nabla \text{div} \psi \|^2 + (\mu + \lambda) \int_{\Omega} \frac{1}{k(x')} | \nabla' \cdot \psi' |^2_{x_1 = 0} \, dx_2 dx_3 \\
+ (\mu + \lambda) \sum_{i=2}^{3} \int_{\Omega} \psi_i \partial_{x_i} \left( \frac{1}{k(x')} \right)_{x_1 = 0} \, dx_2 dx_3,
\end{align*} 

multiplying (3.1)_2 by \(-\Delta \psi\) and then integrating it with respect to \(x\) over \(\Omega\), it holds
\begin{align*}
\mu \| \nabla^2 \psi \|^2 + (\mu + \lambda) \| \nabla \text{div} \psi \|^2 + 2\mu \int_{\Omega} \frac{1}{k(x')} | \partial_{x'} \psi' |^2_{x_1 = 0} \, dx_2 dx_3 \\
+ (\mu + \lambda) \int_{\Omega} \frac{1}{k(x')} | \nabla' \cdot \psi' |^2_{x_1 = 0} \, dx_2 dx_3 \\
= -(\mu + \lambda) \int_{\Omega} \frac{1}{k(x')} \nabla' \cdot \psi' \partial_{x_1} \psi_1_{x_1 = 0} \, dx_2 dx_3 \\
- 2\mu \sum_{i=2}^{3} \int_{\Omega} \left[ \partial_{x_2} \left( \frac{1}{k(x')} \right) \partial_{x_2} \psi_i \psi_i + \partial_{x_3} \left( \frac{1}{k(x')} \right) \partial_{x_3} \psi_i \psi_i \right]_{x_1 = 0} \, dx_2 dx_3 \\
- (\mu + \lambda) \sum_{i=2}^{3} \int_{\Omega} \psi_i \partial_{x_i} \left( \frac{1}{k(x')} \right)_{x_1 = 0} \, dx_2 dx_3 \\
+ \int (p'(\rho) \nabla \phi + \rho \partial \psi + \rho u \cdot \nabla \psi - g) \cdot \Delta \psi \, dx.
\end{align*} 

By Cauchy’s inequality and Sobolev’s inequality, we have
\begin{align*}
- (\mu + \lambda) \int_{\Omega} \frac{1}{k(x')} \nabla' \cdot \psi' \partial_{x_1} \psi_1_{x_1 = 0} \, dx_2 dx_3 \\
\leq C \int_{\Omega} \| \nabla' \cdot \psi' \|_1 \| \partial_{x_1} \psi_1 \|_2 \, dx_2 dx_3 \\
\leq C \int_{\Omega} \| \nabla' \cdot \psi' \|_1 \| \partial_{x_1} \nabla' \cdot \psi' \|_1 \| \partial_{x_1} \psi_1 \|_2 \| \partial_{x_1} \psi_1 \|_2 \, dx_2 dx_3
\end{align*}
\[ C \| \nabla' \cdot \psi' \|_2^2 \| \partial_{x_1} \nabla' \cdot \psi' \|_2^2 \| \partial_{x_2} \psi_1 \|_2^2 \| \partial_{x_3} \psi_1 \|_2^2 \leq \frac{\mu}{4} \| \nabla^2 \psi \|^2 + C \| \nabla \psi \|^2. \]  
(3.66)

Then we have
\[
-2\mu \sum_{i=2}^{3} \int_{T_2} \left[ \partial_{x_i} \left( \frac{1}{k(x')} \right) \partial_{x_i} \psi_i + \partial_{x_3} \left( \frac{1}{k(x')} \right) \partial_{x_3} \psi_i \right] \bigg|_{x_1=0} dx_2 dx_3 
\leq \frac{\mu}{2} \int_{T_2} \frac{1}{k(x')} |\partial_{x_3} \psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 + C \int_{T_2} |\psi'|^2 \bigg|_{x_1=0} dx_2 dx_3, \tag{3.67}
\]
and
\[
- (\mu + \lambda) \sum_{i=2}^{3} \int_{T_2} \psi_i \partial_{x_i} \left( \frac{1}{k(x')} \right) \text{div} \psi \bigg|_{x_1=0} dx_2 dx_3 \leq \frac{\mu}{2} \int_{T_2} \frac{1}{k(x')} |\partial_{x_3} \psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 + C \int_{T_2} |\psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 
+ C \int_{T_2} |\psi'|_{x_1=0} \| \partial_{x_1} \psi_1 \|_{L^\infty_{x_1}} dx_2 dx_3 \leq \frac{\mu}{2} \int_{T_2} \frac{1}{k(x')} |\partial_{x_3} \psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 + C \int_{T_2} |\psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 
+ C \int_{T_2} |\psi'|_{x_1=0} \| \partial_{x_1} \psi_1 \|_{L^2_{x_1}} \| \partial_{x_3} \psi_1 \|_{L^2_{x_1}} dx_2 dx_3 \leq \frac{\mu}{2} \int_{T_2} \frac{1}{k(x')} |\partial_{x_3} \psi'|^2 \bigg|_{x_1=0} dx_2 dx_3 + \frac{\mu}{4} \| \nabla^2 \psi \|^2 + C \| \psi' \|_{x_1=0} \| \nabla^2 \psi \|_{x_1=0} + \| \partial_{x_1} \psi_1 \|^2. \]

Then
\[
\int (p'(\rho) \nabla \phi + \rho \partial_t \psi + \rho u \cdot \nabla \psi - g) \cdot \Delta \psi dx 
\leq \frac{\mu}{4} \| \nabla^2 \psi \|^2 + C(\| \nabla \phi \|^2 + \| \nabla \psi \|^2 + \| \partial_t \psi \|^2) + C\varepsilon(\| \phi \|^2 + \| \psi_1 \|^2) + C\varepsilon^3. \tag{3.69}
\]
Substitution (3.66)-(3.69) into (3.65) gives (3.62), and the proof of Lemma 3.7 is completed. \qed

Now, we derive the second order normal derivative estimates of \( \phi \).

**Lemma 3.8.** For \( T > 0 \) and \( (\phi, \psi) \in X(0, T) \) satisfying a priori assumption (3.5) with suitably small \( \chi + \varepsilon \), it holds that for \( t \in [0, T] \),
\[
\left\| \partial_{x_1}^2 \phi(t) \right\|^2 + \int_0^t \left( \| \partial_{x_1}^2 \phi \|^2 + \| \partial_{x_1} \psi_1 \|^2 \right) d\tau 
\leq C \int_0^t \left( \| \partial_t \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1} \partial_{x'} \psi \|^2 \right) d\tau + C(E^2(0) + M^2(t) + \varepsilon). \tag{3.70}
\]
\[
\left\| \partial_{x_1} \partial_{x'} \phi(t) \right\|^2 + \int_0^t \left( \| \partial_{x_1} \partial_{x'} \phi \|^2 + \| \partial_{x_1} \partial_{x'} \psi_1 \|^2 \right) d\tau \leq C(E^2(0) + M^2(t) + \varepsilon). \tag{3.71}
\]

**Proof.** Applying \( \partial_{x_1}^2 \) to (3.1)_1 and \( \partial_{x_1} \) to (3.1)_2, we have
\[
\begin{cases}
\partial_t \partial_{x_1}^2 \phi + u \cdot \nabla \partial_{x_1}^2 \phi + \rho \partial_{x_1}^2 \psi_1 + \rho \partial_{x_1} \psi_1 \nabla' \cdot \psi' \\
\rho(\partial_t \partial_{x_1} \psi_1 + u \cdot \nabla \partial_{x_1} \psi_1) + p'(\rho) \partial_{x_1}^2 \phi + \partial_{x_1} \rho \partial_t \psi_1 + \partial_{x_1}(\rho u) \cdot \nabla \psi_1 \\
+ p''(\rho) \partial_{x_1} \rho \partial_{x_1} \phi - (2\mu + \lambda) \partial_{x_1}^3 \psi_1 - \mu \partial_{x_1} \Delta \psi_1 - (\mu + \lambda) \partial_{x_1}^2 \nabla' \cdot \psi' = \partial_{x_1} g_1.
\end{cases}
\tag{3.72}
\]
Similar to Lemma 3.4, multiplying (3.72)\textsubscript{1} by \(\frac{1}{\rho}\partial_{x_1}^2\phi\) and (3.72)\textsubscript{2} by \(\frac{1}{2\mu + \lambda}\partial_{x_1}^2\phi\), then adding the resulted equations together and then integrating it with respect to \(t\), \(x\) over \((0, t) \times \Omega\), it holds

\[
\int_0^t \int \left( \frac{1}{\rho} \left( \partial_{x_1}^2 \phi \right)^2 \right) dx \, d\tau = \frac{1}{4\mu + \lambda} \int_0^t \int \left( \rho \left( \partial_{x_1} \phi \right)^2 \right) dx \, d\tau.
\]

\[
= \int_0^t \int \left( \mu \left( \partial_{x_1} \Delta \psi_1 - \partial_{x_1}^2 \nabla' \cdot \psi' \right) - \rho \left( \partial_{x_1} \psi_1 + u \cdot \nabla \psi_1 \right) \right) \partial_{x_1} \phi \, dx \, d\tau
+ \int_0^t \int \left( \frac{\text{div} u}{\rho} \left( \partial_{x_1}^2 \phi \right)^2 - \frac{1}{\rho} \partial_{x_1} \phi \partial_{x_1} \psi_1, u \cdot \nabla \phi \right) \, dx \, d\tau
+ \int_0^t \int \left( \partial_{x_1} \rho \partial_{x_1} \psi_1 + \partial_{x_1} (\rho u) \cdot \nabla \psi_1 + p''(\rho) \partial_{x_1} \rho \partial_{x_1} \phi \right) \partial_{x_1} \phi \, dx \, d\tau.
\]

By Cauchy’s inequality, one has

\[
J_7 \leq \frac{1}{8(2\mu + \lambda)} \int_0^t \left( \int \left\| \sqrt{p'(\rho)} \partial_{x_1} \phi \right\|^2 \, d\tau \right)
+ C \int_0^t \left( \left\| \nabla \partial_{x_1} \partial_{x_1} \psi_1 \right\|^2 + \left\| \partial_{x_1} \partial_{x_1} \psi_1 \right\|^2 + \left\| \nabla \partial_{x_1} \psi_1 \right\|^2 \right) \, d\tau.
\]

Note that

\[
[\partial_{x_1}^2, u \cdot \nabla] \phi = \partial_{x_1}^2 (u \cdot \nabla) - u \cdot \nabla \partial_{x_1}^2 \phi = \partial_{x_1}^2 u \cdot \nabla \phi + \partial_{x_1} u \cdot \nabla \partial_{x_1} \phi,
\]

\[
[\partial_{x_1}^2, \rho] \text{div} \psi = \partial_{x_1}^2 (\rho \text{div} \psi) - \rho \text{div} \partial_{x_1}^2 \psi = \partial_{x_1}^2 \rho \text{div} \psi + \partial_{x_1} \rho \text{div} \partial_{x_1} \psi,
\]

therefore, we can obtain

\[
\int \left( \frac{\text{div} u}{\rho} \left( \partial_{x_1}^2 \phi \right)^2 - \frac{1}{\rho} \partial_{x_1} \phi \partial_{x_1} \psi_1, u \cdot \nabla \phi \right) \, dx \, d\tau
+ \int \left( \partial_{x_1} \rho \partial_{x_1} \psi_1 + \partial_{x_1} (\rho u) \cdot \nabla \psi_1 \right) \partial_{x_1} \phi \, dx \, d\tau.
\]

By Cauchy’s inequality and assumption (3.5), similar to (3.19), it holds

\[
J_8 \leq \int_0^t \left( \left\| \partial_{x_1} \psi_1 \right\|_{L^\infty} \left\| \nabla \partial_{x_1} \phi \right\|^2 + \left\| \partial_{x_1} \phi \right\| \left\| \nabla^2 \phi \right\|_{L^\infty} \left\| \nabla \phi \right\| \right) \, d\tau
+ \epsilon \left\| \partial_{x_1} \phi \right\| \left( \left\| \nabla^2 \phi \right\| + \left\| \nabla (\phi, \psi) \right\| \right) \right) \, d\tau
\]

\[
\leq C(\chi + \epsilon) \int_0^t \left( \left\| \nabla \partial_{x_1} \phi \right\|^2 + \left\| \nabla^2 \phi \right\|^2_{H^1} + \left\| \nabla (\phi, \psi) \right\|^2 \right) \, d\tau \leq CM^2(t).
\]

Similar to (3.76), it follows from Cauchy’s inequality and assumption (3.5) that

\[
J_9 \leq C(\chi + \epsilon) \int_0^t \left( \left\| \partial_{x_1} \phi \right\|^2_{H^1} + \left\| \partial_t \psi_1 \right\|^2_{H^1} + \left\| \nabla \psi \right\|^2_{H^1} \right) \, d\tau \leq CM^2(t).
\]
By Cauchy’s inequality and Lemma 2.1, we have

\[ J_{10} \leq C \varepsilon \int_0^t (\| \partial^2_{x_1} (\phi, \psi_1) \|^2 + \| \partial_{x_1} (\phi, \psi_1) \|^2) d\tau + C \varepsilon \leq C(M^2(t) + \varepsilon). \]  

(3.78)

Hence, substituting (3.74), (3.76)-(3.78) into (3.73) and choosing \( \chi + \varepsilon \) suitably small, we can obtain

\[ \| \partial_{x_1}^2 \phi(t) \|^2 + \int_0^t \| \partial_{x_1}^2 \phi \|^2 d\tau \leq C \int_0^t (\| \partial_{x_1} \partial_{x_2} \phi \|^2 + \| \nabla \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1} \partial_{x_2} \psi \|^2) d\tau + C(E^2(0) + M^2(t) + \varepsilon). \] 

(3.79)

It follows from (3.72) that

\[ \int_0^t \| \partial_{x_1}^2 \psi_1 \|^2 \leq C \int_0^t (\| \partial_{x_1}^2 \| + \| \partial_{x_1} \partial_{x_2} \| + \| \nabla \partial_{x_1} \| + \| \nabla \partial_{x_1} \partial_{x_2} \|) d\tau + C(M^2(t) + \varepsilon). \] 

(3.80)

Multiplying (3.79) by a large constant \( C \) and combining with (3.80), we have (3.70).

Next, applying \( \partial_{x'} \) to (3.36), it holds

\[ \partial_t \partial_{x_1} \partial_{x'} \phi + u \cdot \nabla \partial_{x_1} \partial_{x'} \phi + \rho \partial_{x_1}^2 \partial_{x'} \psi_1 + \rho \partial_{x_1} \partial_{x'} \nabla' \cdot \psi' \]
\[ = -[\partial_{x_1} \partial_{x'}, u \cdot \nabla] \phi - [\partial_{x_1} \partial_{x'}, \rho] \text{div} \psi + \partial_{x_1} \partial_{x'} f, \]
\[ \rho(\partial_t \partial_{x_1} \partial_{x'} \psi_1 + u \cdot \nabla \partial_{x_1} \partial_{x'} \psi_1) + p'(\rho) \partial_{x_1} \partial_{x'} \phi + \rho \partial_{x_1} \partial_{x'} \psi_1 + \partial_{x_1} \partial_{x'} (\rho u) \cdot \nabla \psi_1 \]
\[ + p''(\rho) \partial_{x_1} \partial_{x'} \phi - (2\mu + \lambda) \partial_{x_1}^2 \partial_{x'} \psi_1 - \mu \partial_{x'} \Delta' \psi_1 - (\mu + \lambda) \partial_{x_1} \partial_{x'} \nabla' \cdot \psi' \]
\[ = \partial_{x_1} g_1. \]

(3.81)

Multiplying (3.81)_1 by \( \frac{1}{\varepsilon} \partial_{x_1} \partial_{x'} \phi \) and (3.81)_2 by \( \frac{1}{2\mu + \lambda} \partial_{x_1} \partial_{x'} \phi \), then adding the resulted equations together and integrating it with respect to \( t, x \) over \((0, t) \times \Omega\), similar to (3.79), it holds

\[ \| \partial_{x_1} \partial_{x'} \phi(t) \|^2 + \int_0^t \| \partial_{x_1} \partial_{x'} \phi \|^2 d\tau \]
\[ \leq C \int_0^t (\| \partial_{x_1} \partial_{x'} \psi_1 \|^2 + \| \nabla \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1}^2 \psi \|^2) d\tau + C(E^2(0) + M^2(t)). \] 

(3.82)

It follows from (3.81) that

\[ \int_0^t \| \partial_{x_1}^2 \partial_{x'} \psi_1 \|^2 d\tau \]
\[ \leq C \int_0^t (\| \partial_{x_1} \partial_{x'} \psi \|^2 + \| \partial_t \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1} \psi_1 \|^2 + \| \nabla \partial_{x_1}^2 \psi \|^2) d\tau + C M^2(t). \] 

(3.83)

Multiplying (3.82) by a large constant \( C \), combining with (3.83), and together with Lemma 3.2 and Lemma 3.6, we have (3.71). Therefore the proof of Lemma 3.8 is completed.

In order to close the a priori assumption (3.5), we need to derive the higher order tangential derivative estimates of \( \phi \).
Lemma 3.9. For $T > 0$ and $(\phi, \psi) \in X(0, T)$ satisfying a priori assumption (3.5) with suitably small $\chi + \varepsilon$, it holds that for $t \in [0, T]$,

\[
\int_0^t \left( \|\partial^2_{x'} \phi\|^2 + \|\partial^2_{x'} \partial_{x'} \psi\|^2 + \|\partial^2_{x'} \partial_{x'} \phi\|^2 \right) d\tau \\
\leq C \int_0^t \left( \|\partial_{x'} \phi\|^2 + \|\nabla \partial_{x'} \psi\|^2 + \|\nabla^2 \psi\|^2 \right) d\tau + C(E^2(0) + M^2(t) + \varepsilon^2). \tag{3.84}
\]

Proof. Applying $\partial_{x'}$ to (3.44) yields

\[
\rho(\partial_t \partial_{x'} \psi_2 + u \cdot \nabla \partial_{x'} \psi_2) + p'(\rho) \partial_{x'} \phi + \partial_{x'} (\rho \partial_t \psi_2) + \partial_{x'} (\rho u) \cdot \nabla \psi_2 \\
+ p''(\rho) \partial_{x'} \phi - \mu \partial^2_{x_1} \partial_{x'} \psi_2 - \mu \Delta' \partial_{x'} \psi_2 - (\mu + \lambda) \partial_{x_2} \partial_{x'} \psi_2 - \partial_{x'} \partial_{x'} \phi = 0. \tag{3.85}
\]

Multiplying the above equation by $\partial_{x_2} \partial_{x'} \phi$, then integrating it with respect to $t$, $x$ over $(0, t) \times \Omega$, we have

\[
\int_0^t \int \rho'(\rho) (\partial_{x_2} \partial_{x'} \phi)^2 dxd\tau \\
= \int_0^t \int \mu \partial^2_{x_2} \partial_{x'} \phi \partial_{x_2} \partial_{x'} \phi dxd\tau \\
+ \int_0^t \int (\mu \Delta' \partial_{x'} \psi_2 + (\mu + \lambda) \partial_{x_2} \partial_{x'} \psi_2 - \rho(\partial_t \partial_{x'} \psi_2 + u \cdot \nabla \partial_{x'} \psi_2) - \rho(\partial_t \partial_{x'} \phi - \mu \partial^2_{x_1} \partial_{x'} \psi_2 - \mu \Delta' \partial_{x'} \psi_2 - (\mu + \lambda) \partial_{x_2} \partial_{x'} \psi_2 - \partial_{x'} \partial_{x'} \phi) dxd\tau \\
- \int_0^t \int (\partial_{x'} \partial_{x_2} \psi_2 + \partial_{x'} (\rho u) \cdot \nabla \psi_2 + p''(\rho) \partial_{x'} \phi \partial_{x_2} \partial_{x'} \phi dxd\tau. \tag{3.86}
\]

Integration by parts under the boundary conditions (3.2) leads to

\[
\mu \int_0^t \int \partial^2_{x_2} \partial_{x'} \phi \partial_{x_2} \partial_{x'} \phi dxd\tau \\
= -\mu \int_0^t \int \partial^2_{x_2} \partial_{x_2} \partial_{x'} \phi \partial_{x'} \phi dxd\tau \\
= \mu \int_0^t \int \partial_{x_2} \partial_{x_2} \partial_{x'} \phi \partial_{x'} \phi \bigg|_{x_2=0} \ dxd\tau + \mu \int_0^t \int \partial_{x_1} \partial_{x_1} \partial_{x'} \phi \partial_{x'} \phi \ dxd\tau \\
= \mu \int_0^t \int \partial_{x_2} \partial_{x'} \left( \frac{\psi_2}{k(x')} \right) \partial_{x'} \phi \bigg|_{x_2=0} \ dxd\tau + \mu \int_0^t \int \partial_{x_1} \partial_{x_1} \partial_{x'} \phi \partial_{x'} \phi \ dxd\tau,
\]

while

\[
\mu \int_0^t \int \partial_{x_2} \partial_{x'} \left( \frac{\psi_2}{k(x')} \right) \partial_{x'} \phi \bigg|_{x_2=0} \ dxd\tau \\
\leq C \int_0^t \int \left[ \left| \partial_{x_2} \partial_{x'} \psi_2 \right|_{x_1=0} + \left| \partial_{x'} \psi_2 \right|_{x_1=0} + \left| \partial_{x'} \phi \right|_{x_1=0} \right] \left\| \partial_{x'} \phi \right\|_{L^\infty} \ dxd\tau \\
\leq C \int_0^t \int \left[ \left| \partial_{x_2} \partial_{x'} \psi_2 \right|_{x_1=0} + \left| \partial_{x'} \psi_2 \right|_{x_1=0} + \left| \partial_{x'} \phi \right|_{x_1=0} \right] \left\| \partial_{x'} \phi \right\|_{L^\infty} \ dxd\tau \\
\leq C \int_0^t \left[ \left\| \partial_{x_2} \partial_{x'} \psi_2 \right\|_{L^2(\Omega)} + \left\| \partial_{x'} \psi_2 \right\|_{L^2(\Omega)} + \left\| \partial_{x'} \phi \right\|_{L^2(\Omega)} \right] \left\| \partial_{x'} \phi \right\|_{L^2(\Omega)} \ d\tau.
\]
\[ C \int_0^t \left[ \| \partial_{x'} \phi \|^2 + \| \partial_{x_1} \partial_{x'} \phi \|^2 + \| \partial_{x_2} \partial_{x'} \psi_2 \|_{x_1=0}^2 \|_{L^2(T_2)}^2 + \| \partial_{x'} \psi_2 \|_{x_1=0}^2 \|_{L^2(T_2)}^2 \right] d\tau, \tag{3.88} \]

Substituting the above inequality into (3.87) together with Cauchy’s inequality, we can obtain

\[ \mu \int_0^t \int \partial_{x_1}^2 \partial_{x'} \psi_2 \partial_{x_2} \partial_{x'} \phi dxd\tau \]

\[ \leq C \int_0^t \left( \| \partial_{x'} \phi \|^2 + \| \partial_{x_1} \partial_{x'} \phi \|^2 + \| \partial_{x_2} \partial_{x'} \psi_2 \|^2 + \| \partial_{x_1} \partial_{x'} \psi_2 \|^2 \|_{x_1=0}^2 \|_{L^2(T_2)}^2 \right) + \| \partial_{x'} \psi_2 \|^2 \|_{x_1=0}^2 \|_{L^2(T_2)}^2 + \| \psi_2 \|^2 \|_{x_1=0}^2 \|_{L^2(T_2)}^2 \] d\tau. \tag{3.89} \]

By Cauchy’s inequality, it holds

\[ \int_0^t \left( \mu \Delta' \partial_{x'} \psi_2 + (\mu + \lambda) \partial_{x_1} \partial_{x'} \psi_2 \right) \partial_{x_2} \partial_{x'} \phi dxd\tau \leq \frac{1}{4} \int_0^t \| \sqrt{P(\rho)} \partial_{x_2} \partial_{x'} \phi \|^2 d\tau + C \int_0^t (\| \partial_t \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \phi \|^2 \| \right) d\tau. \tag{3.90} \]

Similar to (3.74) and (3.76), it follows from Sobolev’s inequality and the assumption (3.5) that

\[ \int_0^t \| \partial_{x_2} \partial_{x'} \phi \|^2 d\tau \leq C \int_0^t (\| \partial_t \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \phi \|^2 \| \right) d\tau + C M^2(t). \tag{3.91} \]

Substituting (3.89)-(3.91) into (3.86) yields

\[ \int_0^t \| \partial_{x_2} \partial_{x'} \phi \|^2 d\tau \leq C \int_0^t (\| \partial_t \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \phi \|^2 \| \right) d\tau + C M^2(t). \tag{3.92} \]

It follows from (3.85) that

\[ \int_0^t \| \partial_{x_1}^2 \partial_{x'} \psi_2 \|^2 d\tau \leq C \int_0^t (\| \partial_{x_2} \partial_{x'} \phi \|^2 + \| \partial_t \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \psi_2 \|^2 + \| \nabla \partial_{x'} \phi \|^2 \| \right) d\tau + C M^2(t). \tag{3.93} \]

Then, multiplying (3.92) by a large constant \( C \), together with (3.93) implies

\[ \int_0^t (\| \partial_{x_2} \partial_{x'} \phi \|^2 + \| \partial_{x_1}^2 \partial_{x'} \psi_2 \|^2) d\tau \]
\[
\begin{aligned}
&\leq C\int_0^t (||\partial_x, \partial_t\phi||^2 + ||\partial_x\phi||^2) \, dt + CM^2(t) + C \int_0^t \left[ ||\partial_t \partial_x \psi_2||^2 + ||\nabla \partial_x \psi_2||^2 \\
&+ ||\nabla \partial_x^2 \psi||^2 + ||\partial_x \partial_t \psi_2||_{x=0}^2 \right]_{L^2(T^2)} \, dt.
\end{aligned}
\]

Applying \(\partial_{x_1}\) to \((3.44)\) yields
\[
\begin{aligned}
&\rho(\partial_t \partial_x \psi_2 + u \cdot \nabla \partial_x \psi_2 + p'(\rho) \partial_x \partial_t \psi_2 + \partial_x \rho \partial_t \psi_2 + \partial_x (\rho u) \cdot \nabla \psi_2 + p''(\rho) \partial_x \rho \partial_x \psi - \\
&- \mu \partial_x^2 \psi_2 - \mu \Delta \partial_x \psi_2 - (\mu + \lambda) \partial_x^2 \psi_1 - (\mu + \lambda) \partial_x \partial_x \nabla' \cdot \psi' = 0.
\end{aligned}
\]

By Cauchy's inequality, we have
\[
\int_0^t ||\partial_{x_1} \psi_2||^2 \, dt \leq C \int_0^t (||\partial_{x_1} \partial_{x_2} \phi||^2 + ||\partial_{x_1} \partial_x \psi_1||^2 + ||\partial_{x_1} \partial_x^2 \psi||^2) \, dt
\]
\[
+ C \int_0^t (||\partial_t \partial_x \psi_2||^2 + ||\nabla \partial_x \psi_2||^2) \, dt + CM^2(t),
\]
which together with \((3.94)\) leads to
\[
\begin{aligned}
&\int_0^t (||\partial_x \partial_{x'} \phi||^2 + ||\partial_x^2 \partial_{x'} \psi_2||^2 + ||\partial_x^2 \psi_2||^2) \, dt \\
&\leq C \int_0^t (||\partial_{x_1} \partial_{x'} \phi||^2 + ||\partial_{x_1} \partial_x \psi_1||^2 + ||\partial_x \phi||^2) \, dt + CM^2(t) \\
&+ C \int_0^t (||\nabla \partial_x \psi_2||^2 + ||\nabla \partial_x \psi||^2 + ||\partial_x \partial_{x'} \psi_2||^2 + ||\partial_x \partial_{x'} \psi_1||^2 + ||\partial_x \partial_x \nabla' \cdot \psi' ||_{x=0}^2) \, dt \\
&\leq C \int_0^t (||\partial_{x'} \phi||^2 + ||\nabla \partial_x \psi||^2 + ||\nabla \partial_x \psi||^2) \, dt + C(E^2(0) + M^2(t) + \varepsilon^{\frac{1}{2}}),
\end{aligned}
\]
where in the last inequality we have used \((3.7), (3.14), (3.53)\) and \((3.71)\). Note that the estimates of \(\int_0^t (||\partial_{x_1} \partial_{x'} \phi||^2 + ||\partial_{x_1} \partial_x \psi_3||^2 + ||\partial_{x_1} \partial_x \psi_3||^2) \, dt\) can be obtained similarly as \((3.97)\). Hence, we can derive \((3.84)\), and complete the proof of Lemma 3.9. \(\square\)

Now we prove Proposition 3.1 by combining the above lemmas.

**Proof of Proposition 3.1:** First, combining \((3.35)\) and \((3.43)\) together, and choosing \(\eta\) suitably small, we can obtain
\[
\begin{aligned}
&||\partial_x, \phi(t)||^2 + \int_0^t (||\nabla \phi||^2 + ||\partial_x^2 \psi||^2) \, dt \\
&\leq C(M^2(t) + E^2(0) + \varepsilon)
\end{aligned}
\]
\[
+ C \int_0^t (||\partial_x \psi||^2 + ||\nabla \partial_x \psi||^2 + ||\partial_x \partial_{x'} \phi||^2 + ||\partial_t \partial_x^2 \psi||_{x=0}^2) \, dt,
\]
which together with \((3.14)\) and \((3.15)\) yields
\[
\begin{aligned}
&||\partial_t(\phi, \psi)(t)||^2 + ||\nabla \phi(t)||^2 + \int_0^t (||\nabla \partial_t \psi||^2 + ||\nabla \phi||^2 + ||\nabla^2 \psi||^2) \, dt \\
&\leq C(E^2(0) + M^2(t) + \varepsilon) + C \int_0^t (||\partial_t \psi||^2 + ||\nabla \psi||^2 + ||\partial_t \partial_x \phi||^2 + ||\partial_t \partial_x \psi||_{x=0}^2) \, dt.
\end{aligned}
\]
Then, multiplying (3.22) by a large constant $C$, then combining it with the above
inequality together and choosing $\eta$ suitably small, we have
\[
\| \partial_t (\phi, \psi)(t) \|^2 + \| \nabla (\phi, \psi)(t) \|^2 + \int_0^t \left( \| \partial_t \psi \|^2_{H^1} + \| \nabla \phi \|^2 + \| \nabla^2 \psi \|^2 \right) \, d\tau \quad \text{(3.100)}
\]
\[
+ \| \psi' \|_{x_1=0}^2 \| L^2(T_2) + \int_0^t \left( \| \partial_t \psi' \|_{x_1=0}^2 \| L^2(T_2) + \| \partial_{x'} \psi' \|_{x_1=0}^2 \right) \, d\tau \\
\leq C \| \phi(t) \|^2 + C \int_0^t \| \nabla \psi \| + \| \psi' \|_{x_1=0}^2 \| L^2(T_2) \) \, d\tau + C(E^2(0) + M^2(t) + \varepsilon^\frac{1}{2}).
\]
By (3.1), one has
\[
\int_0^t \| \partial_t \phi \| \, d\tau \leq C \int_0^t \| \nabla (\phi, \psi) \| \, d\tau + C \varepsilon \int_0^t \| \partial_{x_1} \nabla u_1 (\phi, \psi) \| \, d\tau.
\]
(3.101)
Therefore, multiplying (3.100) by a large constant $C$ and adding (3.101) together, we have
\[
\| \partial_t (\phi, \psi)(t) \|^2 + \| \nabla (\phi, \psi)(t) \|^2 + \int_0^t \left( \| \partial_t \psi \|^2_{H^1} + \| \partial_t \phi \|^2 + \| \nabla \phi \|^2 + \| \nabla^2 \psi \|^2 \right) \, d\tau \\
+ \| \psi' \|_{x_1=0}^2 \| L^2(T_2) + \int_0^t \left( \| \partial_t \psi' \|_{x_1=0}^2 \| L^2(T_2) + \| \partial_{x'} \psi' \|_{x_1=0}^2 \right) \, d\tau \\
\leq C \| \phi(t) \|^2 + C \int_0^t \| \partial_{x_1} \partial_{x'} \psi \|^2 + \| \nabla \psi \|^2 + \| \psi' \|_{x_1=0}^2 \| L^2(T_2) \) \, d\tau \\
+ C(E^2(0) + M^2(t) + \varepsilon^\frac{1}{2}) \\
\leq C \| \phi(t) \|^2 + C \int_0^t \| \nabla \psi \|^2 + \| \psi' \|_{x_1=0}^2 \| L^2(T_2) \) \, d\tau + C(E^2(0) + M^2(t) + \varepsilon^\frac{1}{2}),
\]
(3.102)
where in the last inequality we have used (3.71).

Next, we collect the higher order derivative estimates. The combination of (3.53),
(3.70) and (3.71) leads to
\[
\| \nabla^2 \phi(t) \|^2 + \int_0^t \left( \| \nabla \partial_{x_1} \phi \|^2 + \| \partial_{x_1}^3 \psi_1 \|^2 + \| \partial_{x_1}^2 \partial_{x'} \psi_1 \|^2 \right) \, d\tau \\
\leq C \int_0^t \left( \| \partial_{x_1} \partial_{x'} \psi_1 \|^2 + \| \nabla \partial_{x_1} \psi_1 \|^2 + \| \partial_{x_1} \partial_{x'} \psi' \|^2 \right) \, d\tau + C(E^2(0) + M^2(t) + \varepsilon).
\]
(3.103)

Multiplying (3.84) by a large constant $C$ and then adding it with the above
inequality, we have
\[
\| \nabla^2 \phi(t) \|^2 + \int_0^t \left( \| \nabla^2 \phi \|^2 + \| \nabla^3 \psi \|^2 + \| \partial_{x'}^2 \psi' \|_{x_1=0}^2 \right) \, d\tau \\
\leq C \int_0^t \left( \| \nabla \partial_t \psi \|^2 + \| \nabla^2 \psi \|^2 + \| \partial_{x'} \phi \|^2 \right) \, d\tau + C(E^2(0) + M^2(t) + \varepsilon).
\]
(3.104)

The combination (3.62) and (3.104) yields
\[
\| \nabla^2 (\phi, \psi)(t) \|^2 + \| \partial_{x'} \psi' \|_{x_1=0}^2 \| L^2(T_2) + \int_0^t \left( \| \nabla^2 \phi \|^2 + \| \nabla^3 \psi \|^2 + \| \partial_{x'}^2 \psi' \|_{x_1=0}^2 \right) \, d\tau \\
\leq C(\| \partial_t \phi(t) \|^2 + \| \nabla (\phi, \psi)(t) \|^2 + \| \psi' \|_{x_1=0}^2 (t) \| L^2(T_2) \).
\]
Proof of Theorem 1.1.

By (3.16) and (3.36), we have
\[
\int_0^t \|\nabla \partial_t \phi \|^2 \, d\tau \leq C \int_0^t \|\nabla^2 (\phi, \psi) \|^2 \, d\tau + C(\chi + \varepsilon) \int_0^t \|\nabla (\phi, \psi) \|^2 \, d\tau + C\varepsilon. \tag{3.106}
\]

Multiplying (3.105) by a large constant \(C\), then using the above inequality, we have
\[
\|\nabla^2 (\phi, \psi)(t) \|^2 + \|\nabla \psi' \|_{\ell^2_x}^2 \tag{4.2}
\]
\[
+ \int_0^t \|\nabla \partial_t \phi \|^2 + \|\nabla^2 \phi \|^2 + \|\nabla^3 \psi \|^2 \, d\tau \leq C(\|\partial_t (\phi, \psi)(t) \|^2 + \|\nabla (\phi, \psi)(t) \|^2 + \|\psi' \|_{\ell^2_x}^2) \tag{4.3}
\]
\[
+ C \int_0^t \|\nabla^2 \psi \|^2 \, d\tau + C(E^2(0) + M^2(t) + \varepsilon). \tag{4.4}
\]

Finally, multiplying (3.7) by a large constant \(C\), then adding it with (3.108), and choosing \(\chi + \varepsilon\) suitably small, we can obtain (3.6) and prove Proposition 3.1.

4. Proof of Theorem 1.1.

Proof of Theorem 1.1: We now finish the proof of the main result in Theorem 1.1. The global existence result follows immediately from Proposition 3.1 (A priori estimates) and local existence which can be obtained similarly as in [27]. To complete the proof of Theorem 1.1, we only need to justify the time-asymptotic behavior (1.15). In fact, from the estimates (3.6), it holds
\[
\int_0^\infty \left( \|\nabla (\phi, \psi) \|^2 + \left| \frac{d}{dt} \|\nabla (\phi, \psi) \|^2 \right| \right) \, d\tau < \infty, \tag{4.1}
\]
which implies
\[
\lim_{t \to \infty} \|\nabla (\phi, \psi)(t) \|^2 = 0. \tag{4.2}
\]

By Lemma 2.2, one has
\[
\| (\phi, \psi)(t) \|_{\ell^\infty} \leq C(\| (\phi, \psi)(t) \| + \|\nabla (\phi, \psi)(t) \| + C\|\nabla^2 (\phi, \psi)(t) \| \|\nabla^2 (\phi, \psi)(t) \|), \tag{4.3}
\]
which together with (3.6) and (4.2) yields
\[
\lim_{t \to \infty} \| (\phi, \psi)(t) \|_{\ell^\infty} = 0. \tag{4.4}
\]

Hence (4.4) and Lemma 2.1 (3) imply (1.15) and proof of Theorem 1.1 is complete.
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