THE IGUSA QUARTIC AND THE PRYM MAP.
(I) SOME RATIONALITY RESULTS

ALESSANDRO VERRA

Abstract. Let $b = 0$ be the equation of the Igusa quartic $B \subset \mathbb{P}^4$. The notorious ubiquity of $B$ shows up again in this paper: now in its relation to the Prym map $P_6 : \mathcal{R}_6 \rightarrow \mathcal{A}_5$. We study several families related to $B$ and $P_6$, studying their geometry and proving the rationality of their moduli. One of these is the family $X$ of quartic threefolds of equation $a^2 + b = 0$, a quadratic form. A general $X \in X$ is a 30-nodal quartic threefold. The natural desingularization $X'$ of $X$ is endowed with six pairs of conic bundle structures $X' \rightarrow S$, where $S$ is the unnodal quintic Del Pezzo surface. The discriminants of these conic bundles are genus six Prym curves $(C, \eta)$ whose Prym is the intermediate Jacobian $JX'$. This is related to the fibre of the Prym map $P_6$ and the 27 lines of a cubic surface. More families geometrically related to $P_6$, and to $B$ via the Double fourfold, are described. In particular we prove the rationality of the Prym moduli space $\mathcal{R}_6^{ps}$ of 4-nodal plane sextics.

1. Introduction

The Igusa quartic is a well known quartic threefold of the complex projective space $\mathbb{P}^4$. It originates from classical Algebraic Geometry and Invariant Theory, see [B] chapter V. In more recent times it has been reconsidered frequently, starting from some work of Igusa and of van der Geer, [I, VdG]. We will denote it as $B$. Actually $B$ belongs to the unique pencil of quartic threefolds which are invariant by the standard action of the symmetric group $S_6$ on $\mathbb{P}^4$, see [CKS, VdG]. Identifying $\mathbb{P}^4$ to the hyperplane $\{z_1 + \ldots + z_6 = 0\}$ of $\mathbb{P}^5$, the equation of the Igusa quartic is

$$z_1^4 + \ldots + z_6^4 - \frac{1}{4}(z_1^2 + \ldots + z_6^2)^2 = 0.$$ 

Sing $B$ is a well known configuration of 15 lines and the dual of $B$ is the classical Segre’s cubic primal. In section 3 more informations are recollected about this matter. However it is due to remind since now of the notorious ubiquity of $B$ in Algebraic Geometry, which implies that $B$ is marking beautifully the landscape at several spots.

In this paper the ubiquity of $B$ shows up at another place again. Here we address the links of $B$ to the moduli space $\mathcal{R}_6$ of genus 6 Prym curves and to the Prym map

$$P : \mathcal{R}_6 \rightarrow \mathcal{A}_5.$$ 

The map $P$ is an equally pleasant topic and its connections to $B$ naturally fit in a very nice groundfield, certainly deserving further investigations. We are going to present a view of these connections and to deduce, along the way, the rationality of most of the moduli spaces involved. Our work frequently relies on the papers [CKS] and [FV].

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We begin from the unnodal quintic Del Pezzo surface $S$ and its rank 3 Mukai bundle $\mathcal{M}$. This defines an embedding of $S$ in the Grassmannian $G$ of planes of $\mathbb{P}^4$. More precisely $S$ is a linear section of the Plücker embedding of $G$. Hence we have:

$$S = \mathbb{P}^5 \cap G \subset \mathbb{P}^9.$$  

We have $c_1(\mathcal{M}) \cong \omega_S^{-1}$ and $\deg c_2(\mathcal{M}) = 2$. Let us consider the diagram

$$\mathbb{P}^4 \leftarrow t \mathbb{P} \dot{\longrightarrow} u S,$$

where $u$ is the universal plane over $S$. Then $\mathbb{P}$ is the projective bundle associated to $\mathcal{M}$ and $t$ is the tautological morphism of $\mathbb{P}$. In particular we have $\mathcal{O}_\mathbb{P}(1) \cong t^* \mathcal{O}_{\mathbb{P}^4}(1)$. Since $\deg c_2(\mathcal{M}) = 2$ then $t$ is a morphism of degree 2. $B$ appears at this step and we can start our description. Consider indeed the Stein factorization of $t$, namely

$$\mathbb{P} \xrightarrow{c} \mathbb{P} \xrightarrow{t} \mathbb{P}^4.$$

It is known that the branch divisor of $t$ is $B$ and that $c$ is a small resolution of $\mathbb{P}$, see [CKS] 2. Often $\mathbb{P}$ is called the Couble fourfold. It turns out that a very important element of our picture is also represented by the linear system of conic bundles $|\mathcal{O}_\mathbb{P}(2)|$.

This can be studied as in [FV] 2. Let $Q \in |\mathcal{O}_\mathbb{P}(2)|$ be general then $Q$ is smooth and $u|Q : Q \to S$ is a conic bundle. Its discriminant is a smooth Prym curve $(C, \eta)$ of genus 6. $C$ is canonically embedded in $S \subset \mathbb{P}^5$ and has general moduli. So we meet here the moduli space $\mathcal{R}_6$. On the other hand we also meet the moduli space

$$\mathcal{R}_6^{cb}$$

of conic bundles $u|Q : Q \to S$. Both the moduli $\mathcal{R}_6$ and $\mathcal{R}_6^{cb}$ are relevant in this work. In particular we show in theorem 8.3 that $\mathcal{R}_6^{cb}$ is rational. However, profiting of some construction and methods in [FV], it is convenient to introduce a new actor of our story: the moduli of 4-nodal Prym plane sextics

$$\mathcal{R}_6^{ps}.$$ 

These are just singular Prym curves $(C', \eta')$ such that: $C'$ is a nodal plane sextic, $\eta' \in \text{Pic} C'$ is a non zero 2-torsion element, $\text{Sing} C'$ is a set of four points in general position. Actually one rationality result to be proven here is:

**Theorem A** The moduli space of 4-nodal Prym plane sextics is rational.

See theorem (8.5). This follows from the rationality of $\mathcal{R}_6^{cb}$, see proof of (8.5) and theorem (6.3), where the birationality of $\mathcal{R}_6^{cb}$ and $\mathcal{R}_6^{ps}$ is proven. That follows from a construction and methods in [FV] 2, see also section 4. In short one has a commutative diagram

$$\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{f} \mathbb{P} \xrightarrow{t} \mathbb{P}^4 \xrightarrow{t'} \mathbb{P}^2.$$
where $\epsilon$ is birational and $t'$ is as follows. Let $e \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a set of 4 points in general position and $\mathcal{I}_e$ its ideal sheaf. Then $t'$ is the rational map of degree 2 defined by $|\mathcal{I}_e(1,1)|$. In particular the strict transform by $\epsilon$ defines a linear isomorphism

$$\epsilon' : |\mathcal{O}_\mathbb{P}(2)| \to |\mathcal{I}_e^2(2,2)|.$$  

Let $Q' = \epsilon'(Q)$ be general and $u' : Q' \to \mathbb{P}^2$ its first projection, then $u'$ is a conic bundle with discriminant a 4-nodal Prym plane sextic. Moreover $u'$ is obtained via a suitable base change from $u : Q \to S$. This defines a birational map between $\mathcal{R}_6^b$ and $\mathcal{R}_6^{ps}$. See theorem (6.3). On the other hand, concentrating our view towards $B$ and $\mathbb{P}^4$, very interesting families of threefolds, and of geometric configurations appear, relating $B$ and the Prym map. Let us describe all that and our further results, though very briefly.

Starting from $Q$ we list some geometric objects related to it and fix the notation for their moduli spaces. Let $\iota$ be the birational involution induced on $\mathbb{P}$ by $t : \mathbb{P} \to \mathbb{P}^4$ and $Q$ the strict transform of $Q$ by $\iota$. Notice that $\overline{Q} \in |\mathcal{O}_\mathbb{P}(2)|$ and that $\mathcal{R} := t^{-1}(B) \in |\mathcal{O}_\mathbb{P}(2)|$. Let $\tau$ be the involution of $|\mathcal{O}_\mathbb{P}(2)|$ such that $\tau(Q) = \overline{Q}$, then its set of fixed points is $\tau^*|\mathcal{O}_\mathbb{P}(4)| \cup \{R\}$.

A first object associated to $Q$ is $X := t_sQ = t_s\overline{Q}$. $X$ is a quartic threefold with 30 nodes, since $\text{Sing} \ X = X \cap \text{Sing} \ B$. Let $\tilde{X}$ be $X$ blown up at $\text{Sing} \ X$. Then its intermediate Jacobian is the Prym variety $P(C, \eta)$ of the discriminant of the conic bundle $u : Q \to S$. Since $t^*X = Q + \overline{Q}$ is split and $B$ is the branch locus of $t$, we have $B \cdot X = 2A$ where $A$ is a quadric. Hence $X$ defines a pencil of quartic threefolds with 30 nodes

$$\lambda a^2 + \mu b = 0,$$

where $B = \text{div}(b)$ and $A = \text{div}(a)$. We say that a pencil $P$, generated by $b$ and some $a^2$, is an Igusa pencil. Let $X \in P$, for some reason we also say that $X$ is an $E_6$-quartic threefold. Finally we denote the moduli spaces of $P$ and of $X$ respectively as

$$\mathcal{P}^I, \ Q^{E_6}.$$  

Clearly one has $t^*X = Q + \overline{Q}$ as above and $Q, \overline{Q}$ generate the pencil $t^*P$. Now consider the plane $\mathcal{P}_y = t_su^*(y), y \in S$. Restricting $X$ to it we have a union of two conics:

$$X \cdot \mathcal{P}_y = t_s(Q \cdot \mathcal{P}_y) \cup t_s(\overline{Q} \cdot \mathcal{P}_y).$$

So $X$ has two family of conics, parametrized by $S$ and birationally defining on it the conic bundle structures corresponding to $Q, \overline{Q}$. Notice that $\mathcal{P}_y \cdot B$ is a double conic. Indeed $B$ is the focal locus of the congruence of planes $S$, see section 3. Now the intriguing feature of $B$ shows up: including $S, B$ is the focal hypersurface of exactly six congruences of planes which are smooth linear sections of $G$, cfr. [H] 3.2, 3.3, [D] 2. Hence $X$ is endowed with six pairs of conic bundle structures as above, a double six we can say. We say that $(X, Q, \overline{Q})$ is a marked $E_6$-quartic and denote its moduli space as $\tilde{Q}^{E_6}$.

The relations proven between the moduli spaces considered are summed up in the next theorem and diagram. The links to the Prym map and the 27 lines of the cubic surface are evident and partially discussed in section 8.

Let $d : \tilde{Q}^{E_6} \to Q^{E_6}$ and $p : \mathcal{R}_6^{ps} \to \mathcal{R}_6$ be the natural forgetful maps. Moreover let $J : Q^{E_6} \to A_5$ be defined by the assignement of $\tilde{X}$ to its intermediate Jacobian. Finally,
under the birational identification $\mathcal{R}_{6}^{ps} = \mathcal{R}_{6}^{cb}$ considered, let $q : \mathcal{R}_{6}^{ps} \to \tilde{Q}_{6}^{E_{6}}$ be the quotient map of the involution induced by the exchange of $Q$ and $\overline{Q}$. We have:

**Theorem B** The next diagram of dominant rational maps is commutative:

\[ \begin{array}{ccc}
\mathcal{R}_{6}^{ps} & \xrightarrow{\text{deg}} & \mathcal{Q}_{E_{6}}^{6} \\
p & \downarrow{q} & \downarrow{d} \\
\mathcal{R}_{6} & \xrightarrow{p} & A_{5}.
\end{array} \]

One has: $\deg q = 2$, $\deg d = 6$ and $\deg p = 80$. Moreover $\deg J = 180$ and $\deg P = 27$.

See (58) and the concluding remarks of section 9, where the diagram is reconsidered. More geometric informations on these maps are given along the paper. Moreover we use all these constructions to prove the rationality of the following moduli spaces:

**Theorem C** The moduli spaces of $E_{6}$-quartic threefolds and of Igusa pencils are rational. The same is true for the moduli of marked $E_{6}$-quartic threefolds.

See sections 7 and 8, in particular theorems (7.3), (8.1), (8.3). However the most interesting moduli spaces $\mathcal{R}_{6}$ and $A_{5}$ are missed in these rationality results. We hope that this paper could be useful in this open direction.

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2. **The Del Pezzo quintic and its Mukai bundle**

In the Introduction we have already fixed the notation $S$ for the unnodal quintic Del Pezzo surface and mentioned some preliminary properties. As the reader knows an enormous literature exists on $S$, we refer to [DI] for a fundamental account. We have also mentioned the vector bundle $\mathcal{M}$, characterized by the next theorem.

**Theorem 2.1.** On $S$ a unique stable rank 3 vector bundle $\mathcal{M}$ exists so that

\[ \det \mathcal{M} \cong \mathcal{O}_{S}(-K_{S}) \text{ and } \deg c_{2}(\mathcal{M}) = 2. \]

We say that $\mathcal{M}$ is the *Mukai bundle of S*. We recall that $h^{0}(\mathcal{M}) = 5$ and $h^{i}(\mathcal{M}) = 0$ for $i \geq 1$. $\mathcal{M}$ defines an embedding of $S$ as a linear section of the Plücker embedding of the Grassmannian $G$ of planes of $\mathbb{P}^{4}$. Therefore we assume as in (1)

\[ S = \mathbb{P}^{5} \cap G \subset \mathbb{P}^{9}. \]

Let $u : \mathbb{P} \to S$ be the universal plane over $S$. As in (2) we have the diagram

\[ \begin{array}{ccc}
\mathbb{P}^{4} & \xrightarrow{t} & \mathbb{P} & \xrightarrow{u} & S,
\end{array} \]

where $t$ is the tautological morphism of $\mathbb{P}$ and $\mathcal{O}_{\mathbb{P}}(1) \cong t^{*}\mathcal{O}_{\mathbb{P}^{4}}(1)$. Since the Chern class $c_{2}(\mathcal{M})$ has degree two it follows that $t : \mathbb{P} \to \mathbb{P}^{4}$ is a morphism of degree two. We want
to add more geometry about \( t \), therefore we consider the incidence correspondence

\[
\mathbb{I} := \{ (x, y) \in \mathbb{P}^4 \times \mathbb{G} \mid x \in P_y \}
\]

and its projections

\[
\mathbb{P}^4 \xleftarrow{t} \mathbb{I} \xrightarrow{u} \mathbb{G},
\]

where the plane with parameter point \( y \in \mathbb{G} \) is denoted by

\[
P_y \subset \mathbb{P}^4.
\]

We have \( u = u|P \) and \( t = t|P \). Let \( t^*(x) := \mathbb{G}_x \), then \( \mathbb{G}_x = \{ (x, y) | x \in P_y \} \) is a codimension 2 Schubert cycle embedded as a smooth quadric. Notice also that \( t \) is a Grassmann bundle with fibre the Grassmannian of lines of \( \mathbb{P}^3 \). Now, since \( S \) is a codimension 4 linear section, it is immediate that \( S \cdot \mathbb{G}_x \) consists of two points for a general \( x \). Notice also that its class in the Chow ring of \( \mathbb{G} \) is \( c_2(\mathcal{M}) \). More preliminary results on the classical map \( t \) are reported in the next section. Here let us sketch in advance the picture of its fundamental locus. Consider the curve

\[
C_\ell \subset S \subset \mathbb{P}^5,
\]

union of the ten lines of \( S \). As is well known \( C_\ell \) is a quadratic section of \( S \) and a stable canonical curve of arithmetic genus 6. In particular \( \text{Sing} C_\ell \) is a set of 15 nodes: 3 for each line of \( C_\ell \). Let \( \Gamma_\ell \) be the graph curve associated to \( C_\ell \), then \( \Gamma_\ell \) is defined by the Petersen graph \( P \). \( \Gamma_\ell \) is the union of 15 copies of \( \mathbb{P}^1 \), corresponding to the edges of \( P \), and \( \text{Sing} \Gamma_\ell \) consists of 10 triple points, corresponding to the 10 vertices. Finally let \( F \subset \mathbb{P}^4 \) be the fundamental locus of \( t \). \( F \) can be studied discussing when \( t^*(x) = \mathbb{G}_x \cdot S \) is a curve, that is a line or a conic. The description of \( F \) is however well known. We can summarize it as follows, see for instance [D3] 2, [Hu] 3.3, 3.4.

**Theorem 2.2.** \( F \) is biregular to \( \Gamma_\ell \) and union of 15 lines \( L_1 \ldots L_{15} \), moreover

\[
t^*(F) = E_1 \cup \ldots \cup E_{15}
\]

where \( E_i \) is an unnodal quintic Del Pezzo, \( t(E_i) = L_i \) and \( t|E_i \) is a conic bundle.

3. The tautological map and the Igusa quartic

Sometimes, adopting the classical language, we will say that a surface \( Y \subset \mathbb{G} \) is a congruence of planes of \( \mathbb{P}^4 \). In the same language the order \( m \) of \( Y \) is the number of planes of the family passing through a general point and its class \( n \) is the number of planes cutting a general plane along a line. For \( S \) we have \((m,n) = (2,3)\). Moreover

\[
[S] = 2\sigma_l + 3\sigma_p,
\]

where \( \sigma_l, \sigma_p \) are the classes of Schubert cycles generating \( \mathcal{C} \mathcal{H}^4(\mathbb{G}) \). Then \( \sigma_l \) is the class of a family of planes containing a fixed line and \( \sigma_p \) the class of a star of planes in a fixed hyperplane. In the same vein we can say that the branch divisor of \( t \) is the focal locus of the congruence \( S \). We keep the notation \( B \) for it. The ramification of \( t \) will be

\[
R \subset \mathbb{P}.
\]

**Proposition 3.1.** \( B \) is a quartic threefold and \( R \) is an element of \(|\mathcal{O}_{\mathbb{P}}(2)|\).

**Proof.** Let \( K \) be a canonical divisor of \( \mathbb{P} \) and \( H \in |\mathcal{O}_{\mathbb{P}}(1)| \), from the morphism \( t \) we have \( K \sim -5H + R \). Since \( \text{det} \mathcal{M} \otimes \mathcal{O}_S(K_S) \cong \mathcal{O}_S \), the formula for the canonical class of the projective bundle \( \mathbb{P} \) implies \( K \sim -3H \). Then \( R \sim 2H \) and \( B = t_*R \) is a quartic. \( \square \)
B is crucial for our work and a very peculiar quartic threefold, one has:

**Theorem 3.2.** The focal locus B is the Igusa quartic threefold and \( F = \text{Sing } B \).

See [CKS]. The Igusa quartic B is a very classical object, see [RI]. Its geometry is exposed in chapter V of Baker’s treatise [B]. Its ubiquity in Algebraic Geometry has been mentioned already, see [CKS] D1 D2 [VG] and [Hu] in particular. This implies that B bears more names, like Castelnuovo - Richmond quartic, and appears in relation to several moduli spaces. For instance consider the Stein factorization

\[
\mathbb{P}^4 \xrightarrow{t} S_{\mathbb{P}^4} \xrightarrow{e} \mathbb{P}^2.
\]

Then \( \mathbb{P}^4 \) is the moduli space of 6 ordered points of \( \mathbb{P}^2 \), D1.9.4.17. We will see new incarnations of B relating it to Prym curves of genus 6 and their Prym varieties. Now let us introduce some useful properties of B. We recall that B is invariant under the action of the symmetric group \( S_6 \) and that \( \text{Aut } B \cong S_6 \). Let \( (z_1 : \ldots : z_6) \) be coordinates on \( \mathbb{P}^5 \) and \( s_k := z_1^k + \ldots + z_6^k \), putting \( \mathbb{P}^4 := \{ s_1 = 0 \} \) the equation of B in \( \mathbb{P}^4 \) is \( s_4 - \frac{1}{4}s_2^2 = 0 \).

To continue let us consider a point \( y \in S \) and the fibre of \( u|R : R \to S \) at \( y \), say

\[
R_y := R \cdot \mathbb{P}_y.
\]

Since the ramification divisor R is an element of \( |\mathcal{O}_{\mathbb{P}^2}(2)| \), and no plane is in B, then \( R_y \) is always a conic. Consider the plane \( \mathbb{P}_y = t(\mathbb{P}_y) \) and \( B_y := t_*R_y \). Then it follows

\[
P_y \cdot B = 2B_y.
\]

Indeed the branch divisor of \( t : t^*(\mathbb{P}_y) \to \mathbb{P}_y \) is \( \mathbb{P}_y \cdot B \) and \( t^*(\mathbb{P}_y) \) is reducible since it contains \( \mathbb{P}_y \). This implies \( \mathbb{P}_y \cdot B = 2B_y \). In other words B cuts on \( \mathbb{P}_y \) twice the conic \( B_y \).

According to plane geometry in \( \mathbb{P}^4 \), \( B_y \) is the focal conic of \( \mathbb{P}_y \). Considering the induced action of \( S_6 \) on the Grassmannian \( G \), it is well known that the orbit of S is the union

\[
S_{\mathbb{P}^4} = S_1 \cup \ldots \cup S_6,
\]

of six Del Pezzo surfaces and that the stabilizer of S is \( \text{Aut } S \cong S_5 \). Each surface \( S_i \) is endowed as in [2] with its universal plane \( \mathbb{P}_i \), \( i = 1 \ldots 6 \), and the usual diagram

\[
\mathbb{P}^4 \xleftarrow{t_i} \mathbb{P}_i \xrightarrow{u_i} S_i.
\]

We fix the convention: \( S = S_1 \) so that \( t = t_1 \) and \( u = u_1 \). The following well known characterization of B will be also coming into the play.

**Theorem 3.3.** B is the strict dual hypersurface of the Segre cubic primal.

Up to projective automorphisms, the Segre primal is the unique cubic threefold \( B^* \) in \( \mathbb{P}^{4*} \) whose singular locus consists of exactly 10 double points. 10 is also the maximal number of isolated singular points carried by a cubic threefold. Notice that the dual map \( B^* \to B \) contracts the 15 planes of \( B^* \) to the 15 singular lines of B. The set of 15 planes of \( B^* \) and Sing \( B^* \) define well known (154) and (106) incidence configurations: 4 nodes in each plane and 6 planes passing through each node, see e.g. D3 2.2. Finally we recall that the Fano variety of lines of \( B^* \) is the union of 15 planes and 6 Del Pezzo surfaces \( S_1^* \ldots S_6^* \) in the Grassmannian \( G^* \) of lines of \( \mathbb{P}^{4*} \). Under the natural duality between \( G^* \) and \( G \), these surfaces are birational to the previous surfaces \( S_1 \ldots S_6 \).
4. Conic bundles associated to a genus 6 Prym curve

Now we study the linear system $|\mathcal{O}_\mathbb{P}(2)|$ in order to use it.

**Proposition 4.1.** For $m \geq 1$ one has

$$(14) \quad h^0(\mathcal{O}_\mathbb{P}(m)) = \binom{m+4}{4} + \binom{m+2}{4}, \quad h^i(\mathcal{O}_\mathbb{P}(m)) = 0 \text{ for } i \geq 1.$$ 

We omit the standard proof and only compute $h^0(\mathcal{O}_\mathbb{P}(m))$ for future use. Let

$$\begin{array}{ccc}
\mathbb{P} & \xrightarrow{c} & \tilde{\mathbb{P}} \\
\downarrow t & & \downarrow \tilde{t} \\
\mathbb{P}^4 & & \\
\end{array}$$

be the Stein factorization of $t$, we consider the birational involution induced by $t$:

$$(15) \quad \iota : \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}.$$ 

By (2.2) $c$ is a small contraction, then the map $c_* : H^0(\mathcal{O}_\mathbb{P}(m)) \to H^0(\mathcal{T}^*\mathcal{O}_{\mathbb{P}^4}(m))$ is an isomorphism. Let $\tilde{t} : \overline{\mathbb{P}} \to \overline{\mathbb{P}}$ be the biregular involution induced by $\tilde{t}$. In particular

$$(16) \quad t^* := c_*^{-1} \circ \tau^* \circ c_* : H^0(\mathcal{O}_\mathbb{P}(m)) \to H^0(\mathcal{O}_\mathbb{P}(m))$$

is an involution with eigenspaces

$$(17) \quad H^+_m := t^* H^0(\mathcal{O}_{\mathbb{P}^4}(m)) \text{ and } H^-_m := t^* H^0(\mathcal{O}_{\mathbb{P}^4}(m-2)) \otimes <q_>-,$$

where $\text{div}(q_-) = R = t^{-1}(B)$. This implies the previous equality:

$$h^0(\mathcal{O}_\mathbb{P}(m)) = \dim H^+_m + \dim H^-_m = \binom{m+4}{4} + \binom{m+2}{4}.$$ 

We will denote by $Q$ the elements of $|\mathcal{O}_\mathbb{P}(2)|$, let us introduce some of their properties. Clearly the general fibre of $u|Q : Q \to S$ is a conic. With some abuse of language we will say that $u|Q$ is a conic bundle structure on $Q$. From [FV] 2 we have:

**Proposition 4.2.** A general $Q$ is smooth and each fibre of $u|Q$ is a conic of rank $\geq 2$.

Assume $Q$ is general as in the above statement and let $Q_y := (u|Q)^*(y)$. Then

$$(18) \quad C := \{ y \in S \mid \text{rank } Q_y = 2 \}$$

is a smooth curve, see [BL] I. $C$ is endowed with the map $s : C \to \mathbb{P}$, sending $y \in C$ to $\text{Sing} Q_y$. Under our assumptions $s$ is an embedding.

**Definition 4.1.** $s : C \to \mathbb{P}$ is the Steiner embedding of $C$.

Let $Q_C = u^*(C)$ then $\text{Sing} Q_C = s(C)$. Moreover it is well known and easy to see that the normalization map $n : \tilde{Q}_C \to Q_C$ defines the Cartesian square

$$\begin{array}{ccc}
\tilde{Q}_C & \xrightarrow{n} & Q_C \\
\downarrow \tilde{u}_C & & \downarrow u_C \\
\tilde{C} & \xrightarrow{\pi} & C.
\end{array}$$

Here $u_C$ is $u|Q_C$, $\tilde{u}_C$ is the Stein factorization of $n \circ u_C$, $\pi$ is an étale double covering. In particular $\pi$ is defined by a 2-torsion element of $\text{Pic} C$: we will denote it as $\eta$. The next theorem summarizes what happens in our situation.
Theorem 4.3. Let $Q$ be general as above:

1. $C \in |-2K_S|$, 
2. $\eta$ is non trivial, 
3. $C$ is a general genus 6 curve, 
4. $u : C \rightarrow S \subset \mathbb{P}^3$ is the canonical embedding, 
5. $t \circ s : C \rightarrow \mathbb{P}^4$ is defined by $\omega_C \otimes \eta$.

Proof. Some basic results on conic bundles and genus 6 curves imply the statement. (1) is given by the formula for the discriminant of a conic bundle, see [S]. (2), (3), (4) are known properties one retrieves in [FV]. (5) We summarize a well known proof for our case. Consider the biregular section $\tilde{C} := n^*(C)$ of $\tilde{u}_C : \tilde{Q}_C \rightarrow \tilde{C}$ and $\tilde{H} = n^*H$, where $H \in |\mathcal{O}_P(1)|$. Let $h = s(C) \cdot \tilde{H}$ and $\tilde{h} = \tilde{C} \cdot \tilde{H}$ then $\tilde{h} = \pi^*h$. Applying adjunction formula in $\tilde{Q}_C$, it turns out that $\tilde{h}$ is a canonical divisor of $\tilde{C}$. Moreover $\tilde{h}$ is defined by an antiinvariant section of the involution induced by $\pi$. This implies $h \in |\omega_C \otimes \eta|$. □

Remark 4.1. As is well known the Prym canonical map of a Prym curve $(C, \eta)$ is the map defined by $\omega_C \otimes \eta$. It follows from [CD] 0.6 that $\omega_C \otimes \eta$ is very ample for a general $C$ of genus 6. From now we assume this general property.

We say that $(C, \eta)$ is the discriminant of $u : Q \rightarrow S$. $(C, \eta)$ defines a point in the moduli space $\mathcal{R}_6$ of genus 6 Prym curves. Starting from that we will be also interested to the possible realizations of $(C, \eta)$ as the discriminant of some $Q \in |\mathcal{O}_P(2)|$.

To study $|\mathcal{O}_P(2)|$ we are going to use the main diagram of rational maps in [FV] p. 529. This induces a linear isomorphism between $|\mathcal{O}_P(2)|$ and $|\mathcal{I}_e(2, 2)|$, where $\mathcal{I}_e$ is the ideal sheaf of a set $e$ of four general points in $\mathbb{P}^2 \times \mathbb{P}^2$. Let us explain this precisely.

The transformation depends on the choice of a contraction $\sigma : S \rightarrow \mathbb{P}^2$ of four disjoint exceptional lines $E_1 \ldots E_4$ of $S$. Let $E = E_1 + \ldots + E_4$ and $e_i = \sigma(E_i)$, $i = 1 \ldots 4$. Then $\{e_1 \ldots e_4\}$ is a set of points in general position. The same is true for its diagonal embedding $e \subset \mathbb{P}^2 \times \mathbb{P}^2$. Notice that the contractions $\sigma$ are five and that $\dim |\mathcal{I}_e(1, 1)| = 4$.

Definition 4.2. $t_e : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ is the rational map associated to $|\mathcal{I}_e(1, 1)|$.

Now the Segre embedding, defined by $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1)$, has degree 6, hence $\deg t_e = 2$. This map is studied in detail in [GKS], where it is shown that the Igusa quartic is the branch divisor of $t_e$. The same is true replacing the rational map $t_e$ by

$$t_E := t_e \circ (\sigma \times id_{\mathbb{P}^2}) : S \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^4.$$

The relation between the tautological morphism $t$ and the rational map $t_e$ is described by the mentioned commutative diagram in [FV]. It can be written as follows:

$$\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\epsilon_1} & \mathbb{P} \\
\downarrow{\epsilon_2} & & \downarrow{t} \\
\mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\sigma \times id_{\mathbb{P}^2}} & S \times \mathbb{P}^2 \xrightarrow{t_E} \mathbb{P}^4
\end{array}$$

where $\epsilon_1, \epsilon_2$ are birational morphisms and the indeterminacy of the birational map

$$\epsilon := \epsilon_1 \circ \epsilon_2^{-1} : S \times \mathbb{P}^2 \dashrightarrow \mathbb{P}$$

is solved by the diagram

$$\begin{array}{ccc}
S \times \mathbb{P}^2 & \xleftarrow{\epsilon_2} & \mathbb{P} \\
\xrightarrow{\epsilon_1} \mathbb{P}
\end{array}$$
From [FV] 2 we recall for further use how this diagram of birational maps works. Let
\[ u|P_E : P_E \to E \]
be the restriction of \( u \) over the exceptional divisor \( E \) of \( \sigma : S \to P^2 \). Then \( P_E \) is the union of the four disjoint divisors \( P_{E_i} := u^{-1}(E_i), i = 1...4 \). It turns out that each \( P_{E_i} \) is \( P^3 \) blown up along a line. Moreover let \( F_i \subset P_{E_i} \) be the corresponding exceptional divisor, then \( u|F_i : F_i \to E_i \) is the trivial \( \mathbb{P}^1 \)-bundle \( E_i \times \mathbb{P}^1 \to E_i \). We have:

**Theorem 4.4.**

1. \( \epsilon_1 : \tilde{P} \to P \) is the blowing up of the disjoint union \( F_1 \cup ... \cup F_4 \).
2. Let \( \tilde{F}_i := \epsilon_1^{-1}(F_i) \): each fibre of \( u \circ \epsilon_1 : \tilde{F}_i \to E_i \) is \( P^2 \) blown up in a point.
3. \( \epsilon_2(\tilde{F}_i) \) is \( E_i \times P^2 \) and \( \epsilon_2 : \tilde{F}_i \to E_i \times P^2 \) is the blowing up of \( E_i \times \{e_i\} \).
4. Let \( P'_{E_i} \) be the strict transform of \( P_{E_i} \) by \( \epsilon_1 \), then \( \epsilon_1(P'_{E_i}) = E_i \times \{e_i\} \).

Now let \( \phi := (\sigma \times id_{P^2}) \circ \epsilon \), we consider the birational map
\[ \phi : P \dashrightarrow P^2 \times P^2. \]

Let \( Q \in |O_P(2)| \) be general and let \( Q' \subset P^2 \times P^2 \) be its strict transform by \( \phi \). To understand \( Q' \) consider \( Q_{E_i} := Q \cdot P_{E_i} \). It turns out that \( Q_{E_i} \), \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up in two distinct points \( o_1, o_2 \) and that \( u : Q_{E_i} \to E_i \) is the conic bundle defined by the pencil \( |z_{o_1,o_2}(1,1)| \), where \( z_{o_1,o_2} \) is the ideal sheaf of \( \{o_1, o_2\} \) in \( Q_{E_i} \). These define sections of \( u : Q_{E_i} \to E_i \). Then \( \phi \) blows up each of these sections and flops down the resulting surfaces to two lines. These intersect at a double point of \( Q' \) which is the contraction of \( Q_{E_i} \). Moreover \( \phi : Q \to Q' \) is biregular on \( U = Q \setminus (\bigcup_{i=1,4} Q_{E_i}) \).

The resulting \( Q' \) is singular at \( e \) and has bidegree \( (2,2) \). Let \( \mathcal{I}_e \) be the ideal sheaf of \( e \) in \( P^2 \times P^2 \), then \( Q' \in |\mathcal{I}_e(2,2)| \). \( \phi \) is studied in all details in [FV] 2. In particular let
\[ h := (\sigma \times id_{P^2}) \circ \epsilon_1 \circ \epsilon_1^* \]
we conclude pointing out the following property:

**Theorem 4.5.** \( h : H^0(O_P(2)) \to H^0(\mathcal{I}_e^2(2,2)) \) is an isomorphism.

5. **IGUSA PENCILS AND \( E_6 \)-QUARTIC THREEFOLDS**

The property is a useful tool in the sequel. A second useful tool, we are going to introduce, is the notion of *Igusa pencil of quartic threefolds*. As in [15] let us consider the involution \( \iota^* : H^0(O_P(2)) \to H^0(O_P(2)) \) and its eigenspaces \( H_+^+, H^- \). Dropping the suffix 2 out, we denote these as \( H^+, H^- \). So far we have:
\[ H^0(O_P(2)) = H^+ \oplus H^- = \iota^* H^0(O_P(2)) \oplus q_+ < q_+ >, \]
where \( \text{div}(q_-) = R \). Let \( 0 \neq q \in H^0(O_P(2)) \) and \( Q = \text{div}(q) \), we fix the notation
\[ \overline{q} = \iota^* q \text{ and } \overline{Q} := \iota^* Q. \]

Let \( U \subset H^+ \oplus H^- \) be any 2-dimensional subspace containing \( H^- \). Then we have
\[ U \cap H^+ = < q_+ > \text{, } U \cap H^- = < q_- > \]
for some non zero \( q^+_+ \in H^+ \). Equivalently \( \iota^* \) restricts to a non identical involution on \( U \) which is generated by some \( q_+, \overline{q} \) as above. In other words \( P(U) \) is the pull-back by \( t \) of a pencil of quartic threefolds containing the Igusa quartic and a double quadric, say
\[ P \subset |O_{P^4}(4)|. \]
Definition 5.1. We say that $P$ is an Igusa pencil of quartic threefolds.

Definition 5.2. We say that a threefold $X \in P$ is an $E_6$-quartic threefold.

Notice that $t^*\lambda$ acts on $\mathbb{P}(\mathcal{U})$ exchanging $Q$ with $\overline{Q}$ and fixing $Q_+ := \text{div}(q_+)$ and $R$. Clearly $q_+ = t^*(a)$, where $a$ is a non zero quadratic form. Let $\text{div}(b) = B$ then

\begin{equation}
\lambda a^2 + \mu b = 0
\end{equation}

is the equation of the Igusa pencil $P$. The family of these pencils is parametrized by $|\mathcal{O}_{\mathbb{P}^1}(2)|$. Assuming $P$ general and $X$ general in $P$, let us see some interesting properties of $X$ and of $P$. At first we stress that the threefold $t^*X$ is split, namely

\[ t^*X = Q + \overline{Q}, \]

where $Q, \overline{Q}$ are smooth and $Q \neq \overline{Q}$. The base scheme of $P$ is the contact surface

\[ \{a^2 = b = 0\} = 2A \cdot X \subset \mathbb{P}^4, \]

where $A = \text{div}(a)$ is a smooth quadric transversal to $B$. Let $y \in S$ and $P_y \subset \mathbb{P}^4$ the corresponding plane. We know that $P_y \cdot B = 2B_y$, where $B_y$ is a conic. Let $b_y$ be the equation of $B_y$ in $P_y$. Restricting $\lambda a^2 + \mu b$ to $P_y$ we obtain the sum of squares

\[ \lambda a^2 + \mu b^2. \]

This defines the union of two conics of $P_y$, namely $t_*Q_y$ and $t_*\overline{Q}_y$. Furthermore the covering $t : t^{-1}(P_y) \rightarrow P_y$ splits, since it is branched on $2B_y$. Let $P_y := t^*(P_y)$, then

\[ t^*(P_y) = P_y + \overline{P_y}. \]

In particular $[P_y + \overline{P_y}] = [H^2]$ in $CH^2(\mathbb{P})$ and $\overline{P_y}$ is a birational section of $u : \mathbb{P} \rightarrow S$.

Theorem 5.1. Assume $X$ and $P$ are general as above then:

1. $\text{Sing} \ X$ is $A \cap \text{Sing} B$ and consists of 30 ordinary double points.
2. The discriminants of $u|Q$, $u|\overline{Q}$ are smooth genus 6 Prym curves.
3. $t|Q : Q \rightarrow X$ and $t|\overline{Q} : \rightarrow X$ are small contractions to $X$.

Proof. By Bertini theorem $X \in P$ is smooth out of the surface $B_a := \{a = b = 0\}$. Moreover $\text{Sing} B_a = \text{Sing} B \cap A$ and this set consists of 30 ordinary double points of $X$. Let $x \in B_a - \text{Sing} B_a$ then $B$ is smooth at $x$ and hence $\{a^2 = 0\}$ is the unique element of $P = \{\lambda a^2 + \mu b = 0\}$ which is singular at $x$. This implies (1). (2) follows from theorem [1]. Since $Q, \overline{Q}$ are general in $|\mathcal{O}_{\mathbb{P}^2}(2)|$, then they are smooth with no fibre of rank $\leq 1$. For (3) it suffices to recall that the fibre of $t$ at a general $x \in \text{Sing} B$ is $\mathbb{P}^1$. \hfill \Box

Let $Q, \overline{Q}$ be as above, consider the Cartesian square of birational morphisms

\begin{equation}
\begin{array}{ccc}
\hat{X} & \longrightarrow & Q \\
\downarrow & & \downarrow t|Q \\
\overline{Q} & \longrightarrow & X.
\end{array}
\end{equation}

Since $X$ is general then $\text{Sing} X = X \cap \text{Sing} B$. By theorem (2.2) the fibre of $t|Q$ and $t|\overline{Q}$ at $o \in \text{Sing} X$ is $\mathbb{P}^1$. Let $\beta : \hat{X} \rightarrow X$ be the birational morphism induced by the
Notice also that \(\beta^*(\alpha) = \mathbb{P}^1 \times \mathbb{P}^1\). It is easy to see that \(\beta\) is the blowing up of \(\text{Sing} \, X\). Then \(\hat{X}\) is smooth and, denoting the intermediate Jacobian of \(T\) by \(JT\), we have:

\[
JQ \cong J\hat{X} \cong J\bar{Q}.
\]

See [CG], lemma 3.11. Finally let \((C, \eta)\) and \((\bar{C}, \bar{\eta})\) be the discriminants of \(u|Q\) and \(u|\bar{Q}\). Consider the corresponding Prym varieties \(P(C, \eta)\) and \(P(\bar{C}, \bar{\eta})\) as in [B1], then

\[
JQ \cong P(C, \eta), \quad J\bar{Q} \cong P(\bar{C}, \bar{\eta})\tag{31}
\]

Now let us recall that \(t : \mathbb{P} \to \mathbb{P}^4\) is just one of the six double coverings

\[
t_i : \mathbb{P}_i \to \mathbb{P}^4, \quad i = 1\ldots 6,
\]

considered in [12]. Hence \(t^*X = Q + \bar{Q}\) is just one of the splittings

\[
t_i^*X = Q_i + \bar{Q}_i \subset \mathbb{P}_i, \quad i = 1\ldots 6.
\]

Hence, birationally, \(X\) is naturally endowed with six pairs of conic bundle structures:

\[
\{u_i|Q_i : Q_i \to S_i, \ u_i|\bar{Q}_i : \bar{Q}_i \to S_i\}. \tag{32}
\]

so that \(Q_i \cong X \cong \bar{Q}_i\) and \(JQ_i \cong J\hat{X} \cong J\bar{Q}_i\). To simplify our notation we set:

\[
u_i := u_i|Q_i, \quad \bar{\nu}_i := u_i|\bar{Q}_i. \tag{33}
\]

**Definition 5.3.** \(\{u_i, \ \bar{\nu}_i \mid i = 1\ldots 6\}\) is the double six of conic bundle structures of \(X\).

The set of corresponding discriminants is the double six of Prym curves of \(X\):

\[
\{(C_i, \eta_i), (\bar{C}_i, \bar{\eta}_i), \mid i = 1\ldots 6\}. \tag{34}
\]

The expert reader certainly recognizes here the shadow of the set of 27 lines on the cubic surface, via its 36 Schlafli double sixers. Consider the Prym map in genus 6

\[
P : \mathcal{R}_6 \to \mathcal{A}_5, \tag{35}
\]

assigning to \((C, \eta)\) its Prym \(P(C, \eta)\). \(P\) is described in [DS] [D1]. It has degree 27 and the group defined by its field extension is the Weyl group of \(E_6\). This is the group preserving the incidence of the 27 lines. Our discussion implies that: the double six of Prym curves of \(X\) defines a subset of the fibre of \(P\) at the moduli point of \(J\hat{X}\).

### 6. Moduli of Prym Sextics of Genus 6

According to the program of our Introduction, it is time to address the rationality results for several moduli spaces related to \(\mathcal{R}_6\) and the relations between these moduli. This involves at first the moduli of 4-nodal Prym plane sextics of genus six.

With this in mind we start from a general genus 6 Prym curve \((C, \eta)\). Then we know from theorem [13] and [14] that \((C, \eta)\) is birational to the discriminant of a smooth \(\mathcal{Q} \in |\mathcal{O}_\mathbb{P}(2)|\). According to the same theorem we know more: consider

\[
u|\mathcal{Q} : \mathcal{Q} \to S \subset \mathbb{P}^5
\]

then \(C\) is canonical in \(\mathbb{P}^5\). Moreover let \(s : C \to \mathbb{P}\) be the Steiner map sending \(y \in C\) to \(\text{Sing} \, \mathcal{Q}_y\). Then \(s \circ t : C \to \mathbb{P}^4\) is the Prym canonical embedding of \((C, \eta)\). Now consider \(C_s := s(C)\) and its ideal sheaf \(\mathcal{I}_{C_s}(2)\) in \(\mathbb{P}\), we have the standard exact sequence

\[
0 \to \mathcal{I}_{C_s}(2) \to \mathcal{O}_\mathbb{P}(2) \to \mathcal{O}_{C_s}(2) \to 0.
\]

Notice also that \(\mathcal{O}_{C_s}(1) \cong \omega_C \otimes \eta\) and \(\omega_{C_s}(2) \cong \omega_C^{\otimes 2}\). Let \((C, \eta)\) be general then:
Proposition 6.1. One has $h^0(\mathcal{I}_{C_s}(2)) = 1$ and $h^i(\mathcal{I}_{C_s}(2)) = 0$, $i \geq 1$.

Proof. We have $h^0(\mathcal{O}_{C_s}(2)) = 15$ and $h^0(\mathcal{O}_P(2)) = 16$ from [1,1]. Hence the previous exact sequence implies $h^0(\mathcal{I}_{C_s}(2)) \geq 1$. Assume the latter inequality is strict, then it follows $\dim H^+ \cap H^0(\mathcal{I}_{C_s}(2)) \geq 1$ in $H^0(\mathcal{O}_P(2))$. Equivalently a non zero quadratic form is zero on the Prym canonical model $t(C_s)$. But then the moduli point of $(C, \eta)$ is in the ramification of the Prym map $P$, [B1] 7.5, and $(C, \eta)$ is not general: a contradiction. \qed

Let $\mathbb{P}_C \subset \mathbb{P}$ be the universal plane over $C$. Then $\mathbb{P}_C$ is defined by the embedding of $C$ in $\mathbb{P}$ defined by the Mukai bundle $\mathcal{M}_C := \mathcal{M} \otimes \mathcal{O}_C$. This is unique on $C$, see [M1]. Moreover the Steiner map $s : C \to \mathbb{P}_C \subset \mathbb{P}$ is uniquely defined by an exact sequence

$$0 \to \mathcal{N}_C \to \mathcal{M}_C \xrightarrow{s} \omega_C \otimes \eta \to 0$$

for a given 1-dimensional space $<v_s> \subset \text{Hom}(\mathcal{M}_C, \omega_C \otimes \eta)$.

Remark 6.1. Conversely, starting from a general $v \in \text{Hom}(\mathcal{M}_C, \omega_C \otimes \eta)$, one obtains an exact sequence as above and a section $s_v : C \to \mathbb{P}_C \subset \mathbb{P}$. For a general $v$, $s_v(C)$ is certainly contained in a unique $Q_v \in |\mathcal{O}_P(2)|$, as prescribed by proposition (6.1). However $s_v(C)$ is not necessarily the relative singular locus of $u|Q_v : Q_v \to S$ and $s_v$ is not the Steiner map of the discriminant of $u|Q_v$. This brings us to the next definition.

Let $(C, \eta, s)$ be a triple such that:

1. $C \in |\mathcal{O}_S(2)|$ is smooth and $(C, \eta)$ is a Prym curve,
2. the map $s : C \to \mathbb{P}_C \subset \mathbb{P}$ is a regular section,
3. $s$ is defined by $v \in \text{Hom}(\mathcal{M}_C, \omega_C \otimes \eta)$,
4. a unique $Q \in |\mathcal{O}_P(2)|$ contains $s(C)$.

Definition 6.1. $(C, \eta, s)$ is a Steiner map of $(C, \eta)$ if the discriminant of $u|Q$ is $(C, \eta)$ and $s : C \to Q$ is the Steiner map of $u|Q$.

The generality conditions we assume for $(C, \eta, s)$ imply that $Q$ is smooth, $C$ is smooth and $Q$ is the only 2-tautological divisor containing $s(C)$. Conversely every such a $Q$ defines a Steiner map. Notice that $(C, \eta, s)$ does not depend on $\mathcal{M}_C$ up to isomorphisms. Indeed $\mathcal{M}_C$ is the unique rank 3 Mukai bundle on $C$ and satisfies $\mathcal{M}_C \cong \mathcal{M} \otimes \mathcal{O}_C$.

Definition 6.2. $\mathcal{R}_6^{cb}$ is the moduli space of the Steiner maps $(C, \eta, s)$.

The assignement of $Q$ to $(C, \eta, s)$ defines a surjective moduli map

$$m : |\mathcal{O}_P(2)| \twoheadrightarrow \mathcal{R}_6^{cb}.$$ 

Since $m$ is surjective $\mathcal{R}_6^{cb}$ is irreducible. Let $f : \mathcal{R}_6^{cb} \to \mathcal{R}_6$ be the forgetful map, we know that $f \circ m$ is dominant and that $\dim |\mathcal{O}_P(2)| = 15$. This implies the next statement.

Proposition 6.2. $\mathcal{R}_6^{cb}$ is integral of dimension 15.

Now we introduce and study some other moduli spaces, related to $\mathcal{R}_6^{cb}$ and relevant for this paper. The first one is the moduli space of Prym plane sextics of genus 6.

Definition 6.3. A Prym plane sextic of genus 6 is a pair $(C', \eta')$ such that:

1. $C' \subset \mathbb{P}^2$ is a 4-nodal plane sextic,
2. Sing$C'$ is a set of 4 points in general position,
3. $\eta' \in \text{Pic} C'$ is a non zero 2-torsion element,
4. $\eta'$ is endowed with an isomorphism $\eta'^{\otimes 2} \cong \mathcal{O}_{C'}$. 
For obvious reasons we often adopt the name genus 6 Prym sextic for a pair \((C', \eta')\) as above, implicitly assuming an inclusion \(C'' \subset \mathbb{P}^2\).

**Definition 6.4.** \(\mathcal{R}_{6}^{ps}\) is the moduli space of genus 6 Prym sextics \((C', \eta')\).

In particular it follows that \(C'\) is integral and stable. \(C'\) has arithmetic genus 10. Let \(\overline{\mathcal{R}}_g\) be the well known compactification of \(\mathcal{R}_g\) constructed in [FL]. Then \((C', \eta')\) defines a point in the boundary \(\overline{\mathcal{R}}_{10} \setminus \mathcal{R}_{10}\). Notice that the assignment of \((C', \eta')\) to such a point defines a rational map \(\mathcal{R}_{6}^{ps} \rightarrow \overline{\mathcal{R}}_{10}\) which is birational onto its image.

The same moduli space of genus 6 Prym sextics is already studied in detail in [FV], where it is proven that \(\mathcal{R}_{6}^{ps}\) is irreducible of dimension 15. In the sequel we prove that \(\mathcal{R}_{6}^{ps}\) and \(\mathcal{R}_{6}^{sb}\) are birational. Then we will work on \(\mathcal{R}_{6}^{sb}\) to finally obtain its rationality and hence the rationality of the moduli of genus 6 Prym sextics. Let

\[ \nu : C \rightarrow C' \]

be the normalization map, then \(\nu^*\) defines the exact sequence of 2-torsion groups

\[ 0 \rightarrow \mathbb{Z}_2^4 \rightarrow \text{Pic} C' \xrightarrow{\nu^*} \text{Pic} C \rightarrow 0. \]  

(37)

A more general standard property implies indeed that the Kernel of \(\nu^* : \text{Pic} C' \rightarrow \text{Pic} C\) is determined by the 4-nodes of \(C'\) as \(C^{**}\). In what follows we will also set

\[ \eta := \nu^* \eta'. \]

As previously we will denote by \(e \subset \mathbb{P}^2 \times \mathbb{P}^2\) the diagonal embedding of \(\text{Sing} C'\) and by \(\mathcal{I}_e\) its ideal sheaf. Our goal is now to prove the following theorem.

**Theorem 6.3.** \(\mathcal{R}_{6}^{sb}\) and \(\mathcal{R}_{6}^{ps}\) are birational.

The proof relies on [FV] and [B2]. We use diagram (21) and the isomorphism

\[ h : H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \rightarrow H^0(\mathcal{I}_e^2(2, 2)), \]

defined in (26). We assume that \(Q' \in |\mathcal{I}_e^2(2, 2)|\) is general so that \(\text{Sing} Q'\) consists of 4 nodes at \(e\). Let \((z_1 : z_2 : z_3)\) be coordinates on \(\mathbb{P}^2\) then the equation of \(Q'\) is

\[ (z_1 \ z_2 \ z_3) \ A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0, \]

(39)

where \(A\) is a symmetric matrix of quadratic forms. Let \(u' : \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2\) be the first projection then \(u'|Q' : Q' \rightarrow \mathbb{P}^2\) is a conic bundle. Since \(Q'\) is general we can also assume that every fibre of \(u'|Q'\) is a conic of rank \(\geq 2\). Moreover the discriminant curve of \(u'|Q'\) is a 4-nodal sextic, singular at \(u'(e)\) and it is defined by \(\det A = 0\). Finally, for such a general \(Q'\), the matrix \(A\) defines an exact sequence of vector bundles

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \rightarrow \eta' \rightarrow 0 \]

(40)

where \(\eta'\) is a non zero 2-torsion element of \(\text{Pic} C'\). \((C', \eta')\) is a genus 6 Prym plane sextic. Conversely a general Prym plane sextic \((C', \eta')\) uniquely defines a resolution of \(\eta'\) as above up to isomorphisms. See [B2] thm. B and [FV] 1. The resolution of \(\eta'\) defines a conic bundle \(Q' \in |\mathcal{I}_e^2(2, 2)|\) as in (39), up to the action of \(\text{PGL}(3)\) on \(A\).
Remark 6.2. We will say that \((C', \eta')\) is the discriminant of \(u'| Q'\). Let

\[ s' : C' \to Q' \subset P^2 \times P^2 \]

be the Steiner map, defined as in (4.1). It is easy to see and well known that \(s'\) embeds \(C'\) in \(P^2 \times P^2\), moreover the exact sequence (40) implies

\[ O_{C'}(1, 0) \cong O_{C'}(1), \quad O_{C'}(0, 1) \cong \eta'(2). \]

Proof of theorem (6.3). From diagram (21) we have the commutative diagram

\[ \begin{array}{ccc}
P^2 \times P^2 & \xrightarrow{\phi} & P \\
\downarrow u' & & \downarrow u \\
P^2 & \xrightarrow{\sigma^{-1}} & S
\end{array} \]

where \(\phi := \epsilon \circ (\sigma^{-1} \times id_{P^2})\) and \(\epsilon = \epsilon_2 \circ \epsilon_1^{-1}\) as in (21). In particular the strict transform by \(\phi\) of a general \(Q \in |O_P(2)|\) is defined by the projective isomorphism

\[ \phi' : |O_P(2)| \to |I_6^2(2, 2)| \]

associated to the isomorphism \(h^{-1} : H^0(O_P(2)) \to H^0(I_6^2(2, 2))\) of theorem 4.5. Let \(Q' = \phi'(Q)\) be the strict transform of \(Q\), then \(\phi'(Q') : Q' \to Q\) is birational. Moreover the conic bundle \(u'| Q'\) is the birational pull-back of \(u| Q\) defined by the diagram

\[ \begin{array}{ccc}
Q' & \xrightarrow{\phi'} & Q \\
\downarrow u' & & \downarrow u \\
P^2 & \xrightarrow{\sigma^{-1}} & S
\end{array} \]

Now, following a typical method in these circumstances, we use the above discussion to define rational maps \(\alpha : R^6_{cb} \dashrightarrow R^6_{ps}\) and \(\beta : R^6_{ps} \dashrightarrow R^6_{cb}\) which are one inverse to the other. Since \(R^6_{cb}\) and \(R^6_{ps}\) are irreducible, it follows that these spaces are birational.

Let \((C, \eta, s)\) be general and \(Q \in |O_P(2)|\) such that the discriminant of \(u| Q\) is \((C, \eta)\) and \(s\) is the Steiner map. Consider \(Q' = \phi'(Q)\) then the discriminant of \(u'| Q'\) is a genus 6 Prym sextic \((C', \eta')\) such that the following diagram is commutative:

\[ s'(C') \xleftarrow{\phi^{-1}} s(C) \]

\[ s' \uparrow \quad \uparrow s \quad \]

\[ C' \xleftarrow{\sigma} C \]

So we have \(C' = \sigma_* C\) and \(\sigma^* \eta' \cong \eta\). The construction of \((C', \eta')\) from \((C, \eta, s)\) is clearly modular. Hence it defines a rational map \(\alpha : R^6_{cb} \dashrightarrow R^6_{ps}\) and \(\beta : R^6_{ps} \dashrightarrow R^6_{cb}\). The inverse construction is also clear. From a general \((C', \eta')\) one retrieves the exact sequence (40) and hence \(Q'\).

7. Moduli of Igusa pencils and rationality

In this and the next section we study two new moduli spaces related to the previous ones, namely the *moduli of Igusa pencils* and the *moduli of \(E_6\)-quartic threefolds*. Both are defined as GIT quotients and we prove that both are rational. Then we will discuss how the four spaces considered are connected geometrically by some rational maps.
As we know an Igusa pencil is generated by a quartic which is projective isomorphic to the Igusa quartic \( B \) and by a double quadric. Up to Aut \( \mathbb{P}^4 \) its equation is \( \lambda a^2 + \mu b = 0 \), where \( b \) is the equation of \( B \) and \( a \) is a non zero quadratic form. Let
\[
V \subset \mathbb{P}^{64} := |\mathcal{O}_{\mathbb{P}^4}(4)|
\]
be the union of the lines which are Igusa pencils as above. Then \( V \) is a cone, of vertex the element \( B \), over the 2-Veronese embedding of \( \mathbb{P}^{14} := |\mathcal{O}_{\mathbb{P}^4}(2)| \). The latter is just
\[
V_2 := \{ 2A, \ A \in \mathbb{P}^{14} \} \subset V.
\]
Notice also that each Igusa pencil \( P \) contains a unique double quadric \( 2A \), otherwise each element of \( P \) would be union of two quadrics. Let \( P = \{ \lambda a^2 + \mu b = 0 \} \) be general and \( X \) general in \( P \). Then Sing \( X \) is the transversal intersection \( A \cap B \) and consists of 30 ordinary double points, distributed in pairs on the 15 lines whose union is Sing \( B \).

**Proposition 7.1.** No element of \( P \setminus \{ B \} \) is projective equivalent to \( B \).

**Proof.** Let \( t : \mathbb{P} \rightarrow \mathbb{P}^4 \) be our usual tautological morphism and \( \overline{t} : \overline{\mathbb{P}} \rightarrow \mathbb{P}^4 \) its Stein factorization as in (9). Then the pencil \( \overline{t} P \) is generated by \( \overline{B} := \overline{t}^{-1}(B) \) and \( \overline{A} = \overline{t}^{-1}(A) \).

For a general \( P \) the finite double cover \( \overline{t} : \overline{A} \rightarrow A \) is branched on a surface \( A \cap B \) which is singular at the finite set Sing \( B \cap A \). Hence \( \overline{A} \) has isolated singularities. Let \( u : \overline{\mathbb{P}} \rightarrow \mathbb{P}^1 \) be the rational map defined by \( \overline{t} P \), we consider on \( P \) the unique open condition:
1. every \( \overline{X} \in \overline{t} P \setminus \{ \overline{B} \} \) has isolated singularities,
2. the ramification scheme of \( u \) is reduced at Sing \( \overline{B} \).

We claim that this condition is not empty. This easily implies the statement. To prove the claim a well known pencil is available. This is the unique pencil of \( G_6 \)-invariant quartic forms: \( \{ s_1 = \lambda a^2 + \mu s_4 = 0 \} \). This satisfies (1), see [VdG] theorem 4.1 and [CKS] theorem 3.3. The proof of (2) is a straightforward computation. \( \square \)

**Proposition 7.2.** Two general Igusa pencils \( P_1, P_2 \) in \( \mathbb{V} \) are projectively equivalent iff
\[
\exists \ \psi \in \text{Aut} \ B \ | \ P_2 = \psi(P_1).
\]

In the same way two general \( X_1, X_2 \in \mathbb{V} \) are projectively equivalent iff
\[
\exists \ \psi \in \text{Aut} \ B \ | \ P_2 = \psi(P_1).
\]

**Proof.** Let \( P_1, P_2 \) be projectively equivalent then \( P_2 = \psi(P_1) \) for some \( \psi \in \text{Aut} \mathbb{P}^4 \). By the previous proposition no element of \( P_2 \setminus \{ B \} \) is projective equivalent to \( B \). Hence it follows \( \psi(B) = B \) and \( \psi \in \text{Aut} \ B \). The converse is obvious. In the same way let \( X_1, X_2 \in \mathbb{V} \) be general. Then \( X_i, (i = 1, 2) \), belongs to a unique Igusa pencil contained in the cone \( \mathbb{V} \), say \( P_i = \{ \lambda a_i^2 + \mu b = 0 \} \). Moreover Sing \( X_i \) is the transversal intersection Sing \( B \cap A_i \), where \( A_i = \{ a_i = 0 \} \). We claim that \( h^0(I_{\text{Sing} X_i}(2)) = 1 \). Under this claim assume that \( \psi \) is a projective isomorphism such that \( \psi(X_1) = X_2 \). Then it follows \( \psi(\text{Sing} X_1) = \text{Sing} X_2 \) and \( \psi(A_1) = A_2 \). In particular this also implies that \( \psi(P_1) = P_2 \). Hence, by the former proof, \( \psi \in \text{Aut} \ B \). This implies the statement up to proving the claim. For this consider the standard exact sequence of ideal sheaves
\[
0 \rightarrow I_{\text{Sing} B}(2) \rightarrow I_{\text{Sing} X_1}(2) \rightarrow I_{\text{Sing} X | \text{Sing} B}(2) \rightarrow 0.
\]
Then observe that \( I_{\text{Sing} X | \text{Sing} B}(2) \cong \mathcal{O}_{\text{Sing} B} \) and notice that no quadric contains the singular locus of \( B \). Then the claim follows from the associated long exact sequence. \( \square \)
Now recall that Aut $B = \mathcal{S}_6$ and consider its action on $\mathbb{V}_2$ induced by the standard action of $\mathcal{S}_6$ on $\mathbb{P}^4$. The latter proposition motivates the next definitions.

**Definition 7.1.** The moduli space of Igusa pencils is the GIT quotient
\begin{equation}
\mathcal{P}^I := \mathbb{V}_2//\mathcal{S}_6.
\end{equation}

**Definition 7.2.** The moduli space of $E_6$-quartic treefolds is the GIT quotient
\begin{equation}
\mathcal{Q}^{E_6} := \mathbb{V}/\mathcal{S}_6.
\end{equation}

One can deduce the rationality of $\mathcal{P}^I$ from the geometry of the Segre cubic primal $B^* \subset \mathbb{P}^{4*}$, the dual of $B$ appearing in theorem (3.3). The proof goes as follows.

**Theorem 7.3.** The moduli space $\mathcal{P}^I$ of Igusa pencils is rational.

**Proof.** Let $\mathcal{I}_{\text{Sing}B^*}$ be the ideal sheaf of Sing $B^*$. Consider the standard exact sequence
\begin{equation}
0 \to \mathcal{I}_{\text{Sing}B^*}(2) \to \mathcal{O}_{\mathbb{P}^{4*}}(2) \to \mathcal{O}_{\text{Sing}B^*}(2) \to 0,
\end{equation}
as is well known its associated long exact sequence is
\begin{equation}
0 \to H^0(\mathcal{I}_{\text{Sing}B^*}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{4*}}(2)) \to H^0(\mathcal{O}_{\text{Sing}B^*}(2)) \to 0.
\end{equation}
After dualizing and projectivizing one obtains from it a linear projection
$$p : [\mathcal{O}_{\mathbb{P}^{4*}}(2)]^* \to \mathbb{P}^4 := \mathbb{P}(I^*).$$
where $I = H^0(\mathcal{I}_{\text{Sing}B^*}(2))$ is the space of quadratic forms vanishing on the ten singular points of the Segre primal $B^*$. Consider the action of $\mathcal{S}_6$ on $[\mathcal{O}_{\mathbb{P}^{4*}}(2)]$ induced by its action on $\mathbb{P}^{4*}$. Clearly this is linear and equivariant on the center and the target space of $p$. In particular, over a non empty open set $U$, $p$ descends to a projective bundle
$$\mathcal{P} : [\mathcal{O}_{\mathbb{P}^{4*}}(2)]^* // \mathcal{S}_6 \to U \subset \mathbb{P}^4//\mathcal{S}_6,$$
cfr. [MP] 7.1. This is birational to $\mathbb{V}_2//\mathcal{S}_6$. Finally the quotient $\mathbb{P}^4//\mathcal{S}_6$ is rational: it is the weighted projective space $\mathbb{P}(2,3,4,5,6)$ and, by the way, the moduli space of Schl"afli double sixers of cubic surfaces, [D2] p.281 and 9.1. Hence $\mathbb{V}_2//\mathcal{S}_6$ is rational. \[\square\]

8. Rationality of moduli of genus 6 Prym sextics and more

To begin this section let us define the dominant rational map
\begin{equation}
f : \mathcal{Q}^{E_6} \dashrightarrow \mathcal{P}^I.
\end{equation}
Let $X \in \mathbb{V}$ be general and $P$ the unique Igusa pencil containing it. Denoting by $[X]$ and $[P]$ their moduli, we set by definition $f([X]) = [P]$. Then the fibre of $f$ at $[P]$ is the image of $P$ in $\mathcal{Q}^{E_6}$ via the moduli map. Therefore $f$ is a fibration in rational curves.

**Theorem 8.1.** The moduli space $\mathcal{Q}^{E_6}$ is rational.

**Proof.** Let $f' := f \circ \sigma$, where $\sigma : \mathcal{Q}' \to \mathcal{Q}^{E_6}$ is a birational morphism and $\mathcal{Q}'$ is smooth. Then the general fibre of $f'$ is $\mathbb{P}^1$. Therefore it suffices to show that $f'$ admits a rational section $s : \mathcal{P}^I \dashrightarrow \mathcal{Q}'$. This implies that $\mathcal{Q}^{E_6}$ is birational to $\mathcal{P}^I \times \mathbb{P}^1$ and the statement follows because $\mathcal{P}^I$ is rational. To construct $s$ we use the exceptional divisor of the cone $\mathbb{V}$ blown up in its vertex $v$. Let $\tilde{\sigma} : \tilde{\mathbb{V}} \to \mathbb{V}$ be such a blow up and $\tilde{E}$ the exceptional divisor. Then the projection of $\tilde{\mathbb{V}}$, from $v$ onto $\mathbb{V}_2$, lifts to a $\mathbb{P}^1$-bundle $\tilde{p} : \tilde{\mathbb{V}} \to \mathbb{V}_2$ such that $\tilde{E}$ is the image of the obvious biregular section $\tilde{s} : \mathbb{V}_2 \to \tilde{\mathbb{V}}$. The action of $\mathcal{S}_6$ on $\mathbb{V}$ lifts to $\tilde{\mathbb{V}}$ and coincides on $\tilde{E}$ with the standard action of $\mathcal{S}_6$ on $\mathbb{V}_2$. Passing to the corresponding GIT quotients, it is clear that $\tilde{s}$ is the pull-back of a section $s$ of $f'$. \[\square\]
Let \( \mathcal{H} \subseteq \mathfrak{S}_6 \) be a subgroup, we can consider more in general the GIT quotients
\[
\mathcal{P}^I_\mathcal{H} := \mathbb{V}/\mathcal{H}, \quad \mathcal{Q}^{E_6}_\mathcal{H} := \mathbb{V}/\mathcal{H}
\]
and the rational map
\[
f_\mathcal{H} : \mathcal{Q}^{E_6}_\mathcal{H} \rightarrow \mathcal{P}^I_\mathcal{H},
\]
defined by the assignment of \( X \) to the unique Igusa pencil \( P \subset \mathbb{V} \) containing \( X \). Then, applying to \( f_\mathcal{H} \) the same method and proof as above, the next result is immediate.

**Proposition 8.2.** Assume \( \mathcal{P}^I_\mathcal{H} \) is rational, then \( \mathcal{Q}^{E_6}_\mathcal{H} \) is rational.

A case with geometric meaning, where this result applies, is when \( \mathcal{H} \subseteq \mathfrak{S}_6 \) is the stabilizer of our usual Del Pezzo quintic \( S \). Recall that \( S \in \{ S_1 \ldots S_6 \} \), where \( \{ S_1, \ldots, S_6 \} \) is the configuration of six Del Pezzo surfaces with focal locus \( B \) as in \((12)\). Then \( \mathfrak{S}_6 = \text{Aut } B \) acts on this set and, moreover, the stabilizer of \( S \) is
\[
\mathfrak{S}_5 = \text{Aut } S \subset \text{Aut } B.
\]
To explain the geometric meaning of \( \mathbb{V}/\mathfrak{S}_5 \) we recall that a general \( X \in \mathbb{V} \) is endowed, as in definition \( [5.3] \), with its double six of conic bundles structures
\[
\{ u_i|Q_i : Q_i \rightarrow S_i, \ u_i|\overline{Q}_i : \overline{Q}_i \rightarrow S_i, \ i = 1 \ldots 6 \}.
\]
Let us keep the same notation used there, so we have \( Q_i + \overline{Q}_i = t_i^* X \).

**Definition 8.1.** A marked \( E_6 \)-quartic threefold is a pair \( (X, Q_i + \overline{Q}_i) \) such that:
\[
X \in \mathbb{V} \quad \text{and} \quad Q_i + \overline{Q}_i = t_i^* X \quad \text{for some } i \in \{1 \ldots 6\}.
\]

Note that \( Q_i \) and \( \overline{Q}_i \) are conic bundles over \( S_i \in \{ S_1 \ldots S_6 \} \) via the restriction of the corresponding map \( u_i : \mathbb{P}_i \rightarrow S_i \). Let \( (X, Q_i + \overline{Q}_i) \) and \( (X', Q'_i + \overline{Q}'_i) \) be pairs as above, we say that they are isomorphic if there exists \( \psi \in \text{Aut } \mathbb{P}_i \) such that
\[
\psi(X) = X' \quad \text{and} \quad \psi_S(S_i) = S'_i,
\]
where \( \psi_S \in \text{Aut } \mathfrak{S}_6 \) denotes the induced automorphism. For such a \( \psi \) proposition (7.2) implies that \( \psi \in \text{Aut } B = \mathfrak{S}_6 \), moreover \( \mathfrak{S}_6 \) acts transitively on \( \{ S_1 \ldots S_6 \} \). Therefore two pairs are isomorphic iff \( i = i' \) and there exists \( \psi \) as above. For this reason it is not restrictive to assume \( i = i' = 1 \) when marking \( X \) modulo isomorphisms. We have:

\[
(50) \quad \text{Two pairs are isomorphic iff } \exists \ \psi \in \text{Aut } \mathfrak{S}_5 \ | \ \psi(X) = X',
\]
where \( S = S_1 \). This legitimates the next definiton and implies the next theorem.

**Definition 8.2.** The moduli space of marked \( E_6 \)-quartic threefolds is the GIT quotient
\[
\overline{\mathcal{Q}}^{E_6} := \mathbb{V}/\mathfrak{S}_5.
\]

**Theorem 8.3.** The moduli space of marked \( E_6 \)-quartic threefold is rational.

**Proof.** By \([7.2]\) it suffices to show that \( \mathbb{V}/\mathfrak{S}_5 \) is rational. We have \( \mathbb{V} = |\mathcal{O}_{\mathbb{P}^4}(2)| \) and the action of \( \mathfrak{S}_5 \) is induced by the linearized standard action on \( |\mathcal{O}_{\mathbb{P}^4}(1)| \), which implies the rationality of \( \mathbb{V}/\mathfrak{S}_5 \). This also follows from the same proof of theorem \((7.3)\); replacing \( \mathfrak{S}_6 \) by \( \mathfrak{S}_5 \) and observing at the end that \( |\mathcal{O}_{\mathbb{P}^4}(1)|/\mathfrak{S}_5 \) is of course rational. \( \square \)
Coming to moduli $\mathcal{R}_{6}^{ps}$ of genus 6 Prym sextics, we first consider the involution
\begin{equation}
\iota_{cb} : \mathcal{R}_{6}^{cb} \to \mathcal{R}_{6}^{cb}
\end{equation}
induced by our usual involution $\iota : \mathbb{P} \to \mathbb{P}$. By definition $\iota_{cb}$ works as follows. Let $(C, \eta, s)$ be a Steiner map with moduli point a general $q \in \mathcal{R}_{6}^{cb}$. As we know $s(C)$ is contained in a unique $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ so that $(C, \eta)$ is the discriminant of the conic bundle $Q$ and $s : C \to Q$ is its Steiner map. Let $\overline{Q} := t^*Q$ and $(\overline{C}, \overline{\eta}, \overline{s})$ its associated triple. The construction of the triple is modular and uniquely defines a point
\[
\overline{q} := \iota_{cb}(q) \in \mathcal{R}_{6}^{cb}.
\]
It is also clear that $Q + \overline{Q}$ uniquely defines $X \in \mathbb{V}$ so that $t^*X = Q + \overline{Q}$.

**Theorem 8.4.** $\mathcal{R}_{6}^{ps} / \langle \iota_{ps} \rangle$ and $\mathcal{R}_{6}^{cb} / \langle \iota_{cb} \rangle$ are rational varieties.

**Proof.** By theorem (6.3) the isomorphism $h : H^0(\mathcal{O}_{\mathbb{P}}(2)) \to H^0(T_6^2(2, 2))$ induces a birational map $\phi_h : \mathcal{R}_{6}^{cb} \to \mathcal{R}_{6}^{ps}$. Setting $t_{ps} := \phi_h \circ \iota_{cb} \circ \phi_h^{-1}$, we then have a birational involution $\mathcal{R}_{6}^{cb} \to \mathcal{R}_{6}^{ps}$. Clearly the quotients $\mathcal{R}_{6}^{cb} / \langle \iota_{cb} \rangle$ and $\mathcal{R}_{6}^{ps} / \langle \iota_{ps} \rangle$ are birational. Moreover the definition of marked $E_6$-quartic threefold implies that $\mathcal{R}_{6}^{cb} / \langle \iota_{cb} \rangle$ is just birational to the moduli space $\tilde{Q}_{E_6}^6$. Finally the latter is rational by theorem (8.3). □

**Definition 8.3.** The quotient map of $\iota_{cb}$ is $q : \mathcal{R}_{6}^{cb} \to \tilde{Q}_{E_6}^6$.

Finally, for the moduli space $\mathcal{R}_{6}^{ps}$, we conclude as follows.

**Theorem 8.5.** The moduli space of genus 6 Prym plane sextics is rational.

**Proof.** By (6.3) it suffices to show that $\mathcal{R}_{6}^{cb}$ is rational. We have the diagram
\begin{equation}
\begin{array}{ccc}
\mathcal{R}_{6}^{cb} & \xrightarrow{q} & \tilde{Q}_{E_6}^6 \\
& \xrightarrow{f \circ q} & \mathcal{P}^I,
\end{array}
\end{equation}
where $f$ is the fibration in rational curves in (39). Let $[P] \in \mathcal{P}^I$ be a general point and $\tilde{P} := \{Q, \overline{Q} \mid Q + \overline{Q} = t^*X, X \in P\}$.

We consider the $2 : 1$ cover of rational curves $f_P : \tilde{P} \to P$ sending $Q, \overline{Q}$ to $X$. Let $F = f^{-1}([P])$ and $\tilde{F} = (f \circ q)^{-1}([P])$, then $f_P$ induces the morphism $q|\tilde{F} : \tilde{F} \to F$. In particular it follows that $f \circ q : \mathcal{R}_{6}^{cb} \to \mathcal{P}^I$ is a fibration in rational curves, with fibre $\tilde{F}$ at $[P]$. Moreover $f \circ q$ admits a rational section $\tilde{s} : \mathcal{P}^I \to \tilde{Q}_{E_6}$. To prove this one applies to $f \circ q$ the same argument used for $f$ in the proof of theorem 8.1. In the same way a section $\tilde{s}$ is defined by a distinguished element of $\tilde{P}$, namely $R = t^{-1}(B)$. This is a distinguished ramification point of $f_P : \tilde{P} \to P$, rationally defined over $\mathcal{P}^I$. The existence of $\tilde{s}$ implies that $\mathcal{P}^I \times \mathbb{P}^1$ is birational to $\mathcal{R}_{6}^{cb}$. Hence the latter is rational. □

9. **Revisiting the Prym map: a few concluding remarks**

At the end of this work we want to discuss a bit more on its relation to the Prym map in genus six and related topics. This map is in fact behind most results and geometric constructions we have seen so far. We are going to outline a possibly wider picture, including Igusa pencils, $E_6$-quartic threefolds and related conic bundles. This
mainly represents a way of revisiting the beautiful and well known picture of the Prym map \( P \) from [Do] and [DS]. Nevertheless more investigations, on all the geometric matter retrieved here, appear to be of interest for future developments, in particular for addressing the rationality problems of \( \mathcal{R}_6 \) and \( \mathcal{A}_5 \).

We can think of a general fibre \( \mathbb{F} \) of \( P \) as of the configuration of 27 lines of a cubic surface. From Donagi's tetragonal construction it follows that two distinct elements of \( \mathbb{F} \) are incident lines iff they are directly associated by such a construction as follows, see [Do] 2.5 and 4.1. Let \((C, \eta)\) be a general genus 6 Prym curve with moduli point \( l \in \mathbb{F} \). Notice that the Brill-Noether locus

\[
W^1_4(C) := \{ L \in \text{Pic}^4(C) \mid h^0(L) \geq 2 \}
\]

consists of 5 elements and that \( h^0(L) = 2 \). From the triple \((C, \eta, L)\) one constructs as in [Do] two new triples \((C', \eta', L')\) and \((C'', \eta'', L'')\). These define two points \( l', l'' \in \mathbb{F} \) so that \( l, l', l'' \) are coplanar lines of \( \mathbb{F} \). One says that \( l', l'', l \) form a triality. In what follows we will also say that \( l', l'' \) are directly associated to \( l \). After labeling by the suffix \( i = 1...5 \), we obtain from the five triples \((C, \eta, L_i)\) exactly eleven elements of \( \mathbb{F} \): \( l, l_i', l_i'' \). These are the lines of \( \mathbb{F} \) incident to \( l \). Let us fix the notation

\[
\mathbb{F}^+ = \{ l_i', l_i'' \mid i = 1...5 \}, \quad \mathbb{F}^- = \{ n_j, j = 1...16 \}
\]

for the set of lines respectively intersecting or non intersecting \( l \). Then \( \mathbb{F} \) decomposes as

\[
\mathbb{F} = \{ l \} \cup \mathbb{F}^+ \cup \mathbb{F}^-.
\]

Now let us point out the bijection induced by Serre duality, namely

\[
W^1_4(C) \leftrightarrow W^2_6(C).
\]

This is sending \( L \in W^1_4(C) \) to \( M := \omega_C \otimes L^{-1} \), where \( M \) belongs to \( W^2_6(C) \) and

\[
W^2_6(C) := \{ M \in \text{Pic}^6 C \mid h^0(M) \geq 3 \}.
\]

The latter set consists of five line bundles, defining five sextic models \( C' \subset \mathbb{P}^2 \) of \( C \) modulo \( \text{Aut} \mathbb{P}^2 \). This brings us to revisit the fibre \( \mathbb{F} \) of \( P \) in terms of Prym plane sextics. Therefore let \((C', \eta')\) be a Prym plane sextic of genus 6 so that \( \nu : C \to C' \) is the normalization and \( \eta \cong \nu^* \eta' \). As already remarked we have the exact sequence

\[
0 \to (\mathbb{C}^*)^4 \to \text{Pic}^0 C' \xrightarrow{\nu^*} \text{Pic}^0 C \to 0
\]

of 2-torsion subgroups and \( \eta' \in \nu^* \eta \). Moreover the embedding \( C' \subset \mathbb{P}^2 \) is determined by the line bundle \( M \equiv \nu^*\mathcal{O}_{\mathbb{P}^2}(1) \in W^2_6(C) \) such that \( M \cong \omega_C \otimes L^{-1} \). Finally let

\[
f : \mathcal{R}^{ps} \to \mathcal{R}_6,
\]

be the rational map induced by the assignements \((C', \eta') \to (C, \eta, M) \to (C, \eta)\). It is not difficult to compute the degree of \( f \), as remarked in [PV]. Since \( |W^2_6(C)| = 5 \) and the elements \( \eta' \) in the above exact sequence are 16, the next property follows.

**Proposition 9.1.** The natural map \( f : \mathcal{R}^{ps} \to \mathcal{R}_6 \) has degree 80.

To go further fix the moduli point \( l \in \mathbb{F} \) of \((C, \eta)\) and recall from [Do] that the set \( \mathbb{F}^+ \), of ten incident lines to \( l \) distinct from \( l \), is recovered applying just once the tetragonal constructions to the five line bundles of \( W^1_4(C) \). In particular, fixing an element \([C, \eta]\) in the fibre \( \mathbb{F} \) of the Prym map \( P \), one has the decomposition

\[
\mathbb{F} = \{ [C, \eta] \} \cup \{ [C_i^+, \eta_i^+] \mid i = 1...5 \} \cup \{ [C_j^-, \eta_j^-] \mid j = 1...16 \} = \{ l \} \cup \mathbb{F}^+ \cup \mathbb{F}^-.
\]
Here the elements labeled by + are obtained from \((C, \eta)\) applying once the tetragonal construction to each \(L \in W_4^4(C)\). The elements labeled by - correspond to the elements of the set \(\mathbb{F}^-\) of lines disjoint from \(l\). Starting from \((C, \eta)\) these are obtained after a sequence of two tetragonal constructions applied to the elements of \(\mathbb{F}\).

Notice that the number 16 does not appear by chance when counting the number of Schl"afli double sixers containing as an element the line \(l\). Indeed there exist 36 double sixers and \(36 \times 12 = 27 \times 16\). Actually a natural bijection exists between \(\mathbb{F}^-\) and the set of double sixers containing \(l\) and \(n\) as elements not in the same sixer. The bijection is constructed as follows. Let \(n \in \mathbb{F}^-\) then a standard exercise on the configuration \(\mathbb{F}\) of 27 lines shows that exactly five lines of \(n_{1...n_5} \in \mathbb{F}^-\) satisfy the following conditions:

\[
|n_i \cap n_j| = \delta_{ij}, |l \cap n_k| = 0, |n \cap n_k| = 1,
\]

for \(1 \leq i, j \leq 5\) and \(1 \leq k \leq 5\). Since \(|l \cap n| = 0\), then \(n\) defines the sixer of disjoint lines \(\{l \ n_{1...n_5}\}\) and hence a unique double sixer

\[
(57) \quad \{l \ n_{1...n_5}\}, \ {n \ m_{1...m_5}}
\]

where \(\{m_{1...m_5}\} \subset \mathbb{F}^+\). We do not enter in further details, nor expand this matter, but for introducing, and partially discuss, the next and conclusive commutative diagram:

\[
\begin{array}{ccc}
\tilde{Q}^{E_6} & \xrightarrow{d} & Q^{E_6} \\
\mathcal{R}_6^{ps} & \xrightarrow{\text{deg}} & \mathcal{Q}^{E_6} \\
\mathcal{R}_6 & \xrightarrow{p} & A_5.
\end{array}
\]

(58)

Let us explain it: \(q\) is the quotient map of the involution \(\iota_{ps}: \mathcal{R}_6^{ps} \to \mathcal{R}_6^{ps}\) considered in (51). The map \(d: \tilde{Q}^{E_6} \to Q^{E_6}\) is induced by the forgetful map, sending \((X, Q, \tilde{Q})\) to \(X\). Consider the inclusion \(\mathcal{G}_5 = \text{Aut } S \subset \text{Aut } B = \mathcal{G}_6\). Then \(d\) is the natural map

\[
d: \mathbb{V}/\mathcal{G}_5 \to \mathbb{V}/\mathcal{G}_6,
\]

as follows from the definitions of \(\tilde{Q}^{E_6}\) and \(Q^{E_6}\). Hence we have \(\text{deg } d = 6\) and we know that \(\text{deg } p = 80\). Finally the fatal number appears: \(27 = \text{deg } P\). So we have

\[
\text{deg } P \times \text{deg } p = 27 \times (16 \times 5) = 27 \times (36 \times 5) = \text{deg}(d \circ q) \times \text{deg } J.
\]

\(J\) is the period map, sending the moduli point of an \(E_6\)-quartic threefold \(X\) to that of the intermediate Jacobian \(J\tilde{X}\), see (29). A transparent suggestion of the diagram is that the general fibre of \(J\) could reflect the configuration of the set of double sixers of a cubic surface, up to some base change of degree 5 determined by the plane sextic models.

So \(J\) seems to outline some intrinsic properties of \(A_5\). For instance the double six

\[
\{(C_i, \eta_i), (\tilde{C}_i, \tilde{\eta}_i), \ i = 1...6\},
\]

of discriminants of the conic bundle structures of \(X\), defines six surfaces in \(J(\tilde{X})\). These are difference surfaces \(\tilde{C}_i - \tilde{C}_i\), where \(\tilde{C}_i, \tilde{\eta}_i\) are the Abel-Prym embeddings of the covers defined by \(\eta_i, \tilde{\eta}_i\). This possibly relates to the very recent and interesting paper [CDJ]. In it 27 surfaces are constructed, via the Prym map, in the theta divisor of a general
p.p. abelian variety \((A, \Theta)\) of dimension 5. It is shown that these generate the algebraic part of \(H^4(\Theta, \mathbb{Q})\) and that \(H^4(\Theta, \mathbb{Q})\) contains a natural copy of the \(E_6\)-lattice.

Summing up, some interesting geometry still waits to be enlightened in the matter reported here. Likely there will be further investigations by interested readers. There are few doubts, indeed, about the fatal attraction that 27 lines on a cubic surface could exert on an algebraic geometer. As magistrally the authors warn in [DS] at page 1.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI ROMA TRE, LARGO SAN LEONARDO MURIALDO, 00146 ROMA, ITALY
E-mail address: verra@mat.uniroma3.it