Convoluted solutions in supergravity

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Abstract. Inspired by the convoluted solutions for two intersecting M2 branes in eleven-dimensional supergravity, in which one brane in the system is completely localized along the overall and relative transverse coordinates while the other brane in the system is localized only along the overall transverse coordinates, we construct two classes of exact solutions to Einstein-Maxwell theory in six and higher dimensions. We show that the membrane configuration preserves four supersymmetries and upon dimensional reduction, the solutions provide intersecting configurations of three D-branes in type IIA supergravity. Moreover, we show that the metric functions in six and higher dimensions can be written as convolution-like integrals of two special functions. The solutions are regular everywhere and show a bolt structure on a single point in any dimensionality. Also, we find the exact nonstationary solutions to the Einstein-Maxwell theory with positive cosmological constant. We show that the cosmological solutions are expanding patches in asymptotically de Sitter spacetime.

1. Introduction
Finding the supergravity solutions for the intersecting D-branes is relatively easy as long as the D-branes are sufficiently smeared [1]-[4]. However, it is far harder to find exact solutions for the fully localized D-branes that are valid everywhere; near, as well as far from the core of D-branes. To overcome this problem, in an interesting paper [5] the authors constructed the exact solutions for the fully localized type IIA D2 branes intersecting D6 branes. By lifting a D6 brane to a four-dimensional Taub-NUT geometry embedded in M-theory and then placing M2 branes in the Taub-NUT background geometry, the authors found convolution-like solutions for the M2 branes. The solutions for the system of intersecting D-branes can be obtained by compactifying the corresponding convolution-like M2 brane solutions over a circle of transverse self-dual Taub-NUT geometry. The solutions are not restricted to near core of D6 branes and moreover the solutions are supersymmetric. Inspired by other related works in [6]-[41] to find exact solutions for M-branes and intersecting D-branes as well as solutions to Einstein-Maxwell theory with different matter fields, we can find some exact convoluted solutions to six and higher dimensional Einstein-Maxwell theory [42]. The purpose of this talk (article) is to summarize the progress in finding the convoluted solutions in supergravity and Einstein-Maxwell theory presented in references [14],[21],[42]. In section 2, we present the different M2 brane solutions that upon compactification lead to different intersecting brane systems. In section 3, we present the solutions for two M2 branes and then in section 4 we explicitly calculate the number of preserved supersymmetries. In section 5, we present the exact convoluted cosmological solutions to Einstein-Maxwell theory in D-dimensions and discuss the properties of the solutions. The concluding remarks are in section 6.
2. Convoluted M2 brane solutions in $D = 11$ supergravity

We consider the bosonic ground state of $D = 11$ supergravity in which the vacuum expectation value of any fermionic field vanishes. The theory contains two fields, $g_{MN}$ and $A_{MNP}$ where the capital indices $M, N, P = 0 \cdots 10$ show the eleven-dimensional world coordinates. The solutions in the theory upon reduction to ten dimensions, successfully generate the intersecting D-branes in type IIA string theory. The equations of motion are given by

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3} \left[ F_{MPQR} F_{N}^{PQR} - \frac{1}{8} g_{MN} F_{PQRS} F^{PQRS} \right],$$

(1)

$$\nabla_M F^{MNPQ} = -\frac{1}{576} \varepsilon^{M_1 \cdots M_8 NPQ} F_{M_1 \cdots M_8} F_{M_9 \cdots M_8},$$

(2)

where the four-form field strength $F_{MNPQ}$ is

$$F_{MNPQ} = 4 \partial [M A_{NPQ}] = \frac{1}{2} \left[ A_{MNPQ} - A_{NPQ,M} + A_{PQM,N} - A_{QMN,P} \right].$$

(3)

We consider the general ansatz for an M2 brane solution [43]

$$ds^2 = H^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + H^{1/3} \left( ds_{mtrc1}^2 + ds_{mtrc2}^2 \right),$$

(4)

where in general the metric function $H$ can depend on the coordinates transverse to the brane $H = H(x_3, \ldots, x_{10})$, and we also consider the only non-zero component of $A_{MNP}$ as $A_{tx_1 x_2} = 1/H$. The two transverse metrics $ds_{mtrc1}^2$ and $ds_{mtrc2}^2$ to M2 brane can be any combination of flat space, $k$-dimensional Taub-NUT/Bolt, Eguchi-Hanson (type I or II), Atiyah-Hitchin and Bianchi type IX spaces. As an example, we consider embedding the flat space and Taub-NUT space as $ds_{mtrc1}^2$ and $ds_{mtrc2}^2$ in (4). The Taub-NUT metric is given by

$$ds_{TN}^2 = V(r) \left( dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right) + \left( \frac{(4n)^2}{V(r)} \right) \left( d\psi + \frac{1}{2} \cos(\theta) d\phi \right)^2,$$

(5)

where $V(r) = \left( 1 + \frac{2a}{r} \right)$. The M2 brane metric function $H(y, r)$ depends on $r$ and $y$ which is the radius of sphere in $ds_{mtrc1}^2$. The equations of motion (1) and (2) lead to the partial differential equation

$$\frac{3(r + 2n)}{y} \frac{\partial H(y, r)}{\partial y} + (r + 2n) \frac{\partial^2 H(y, r)}{\partial y^2} + 2 \frac{\partial H(y, r)}{\partial r} + r \frac{\partial^2 H(y, r)}{\partial r^2} = 0.$$  

(6)

Separating the coordinates according to $H(y, r) = 1 + QY(y)R(r)$ leads to two ordinary differential equations for $R(r)$ and $Y(y)$ where $Q$ is the charge of M-brane. The solutions for $R(r)$ are given by product of trigonometric functions and hypergeometric functions and the solutions for $Y(y)$ are given by modified Bessel function. So, if we superimpose all the solutions corresponding to the different separation constants, we get,

$$H(y, r) = 1 + Q \int_0^\infty dc R_c(r) Y_c(y) = 1 + Q \int_0^\infty dc f(c)e^{-icr} G(1 + icn, 2, 2icr) \frac{K_1(cy)}{y},$$

(7)

where $f(c)$ depends on separation constant $c$. The measure function $f(c)$ can be determined by matching the general solution (7) to the metric function of an M2 brane in the near horizon limit. In the near horizon limit, the metric (5) reduces to $R^4$ and the metric function of an M2 brane in this limit is $1 + \frac{Q}{(y^2 + 8nr)^3}$. Matching the solutions leads to an integral equation for $f(c)$
with the solution as $\psi = \frac{d}{\partial r}$. Dimensional reduction of M2 brane solution along the coordinate $\psi$ yields the type IIA supergravity solution

$$ds_{10}^2 = H^{-1/2}V^{-1/2}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/2}V^{1/2}(dy_2^2 + d\Omega_3^2) + H^{1/2}V^{1/2}(dr^2 + r^2d\Omega_5^2),$$

which describes a D2 brane localized along the world-volume of D6 brane. The NSNS fields are $\Phi = \frac{3}{4} \ln \left( \frac{H^{1/2}}{f_4} \right)$ and $B_{\mu\nu} = 0$ and the Ramond-Ramond (RR) fields are $C_\phi = 2n\cos(\theta)$ and $A_{lx_1x_2} = \frac{1}{\pi}$. It is straightforward to show that the M2 brane solution (4) with (7) preserve 1/4 of supersymmetry (section 4). Embedding the other four-dimensional metrics including Bianchi type IX, Eguchi-Hanson (type I and II) and Atiyah-Hitchin as $ds_{mtrc2}^2$ with a flat space as $ds_{mtrc2}^1$ leads to supersymmetric solutions with eight superchargers and the dimensional reduction of the M2 brane leads to a D2 brane localized along the world-volume of D6 brane [8],[11]. Moreover embedding any combination of Taub-NUT, Eguchi Hanson (type I and II) and Atiyah-Hitchin as $ds_{mtrc2}^1$ and $ds_{mtrc2}^2$ leads to supersymmetric solutions that preserve 1/4 of supersymmetry. We also note that embedding Taub-Bolt and six and eight dimensional Taub-NUT and Taub-Bolt in the transverse space to the M2 brane leads to non-supersymmetric solutions [14]. We found that in all of these solutions, the M2 brane metric function is convolution-like of an exponentially decaying radial function with a damped oscillating function. The radial functions vanish far from the M2 brane and diverge near the core of the brane. Moreover, one can embed Taub-NUT, Eguchi-Hanson (type I and II) and Atiyah-Hitchin as a part of transverse space to an M5 brane [13]. The solutions are supersymmetric and upon dimensional reduction lead to a system of type IIA NS5 branes completely localized in the worldvolume of D6 branes. However embedding the four dimensional Taub-Bolt leads to a non-supersymmetric solution.

3. Convolved two M2 branes solutions in $D = 11$ supergravity

In this section, we consider a system of two membranes in eleven-dimensional supergravity and find an exact analytic class of solutions for the metric functions. In the system, the first brane is completely localized along the entire overall and relative transverse coordinates. However, the second brane is localized only along the overall transverse coordinates. We show that the system preserves four supersymmetries. Let’s consider the following ansatz for the system of two M2 branes [44]

$$ds_{11}^2 = H_1^2H_2^2\left( -\frac{dt^2}{H_1H_2} + (dx_1^2 + dx_2^2) + \frac{1}{H_1}(dy_1^2 + dy_2^2) + d\rho^2 + \rho^2d\eta^2 \right),$$

where the branes are located in $\rho = 0$ and $r = 0$ respectively. The coordinates $\rho$ and $\eta$ belong to $0 < \rho < +\infty$ and $0 \leq \eta < 2\pi$. The metric functions $H_1$ and $H_2$ depend on transverse coordinates as $H_1 = H_1(x_1, x_2, \rho, r)$ and $H_2 = H_2(\rho, r)$. Moreover, we consider the four-form field strength, given by

$$F = \frac{1}{2} \left( d\left( \frac{1}{H_2} \right) \wedge dt \wedge dx_1 \wedge dx_2 + d\left( \frac{1}{H_1} \right) \wedge dt \wedge dy_1 \wedge dy_2 \right),$$

in terms of two metric functions. The equations of motion (1) and (2) imply that the metric functions satisfy the two partial differential equations [21]

$$\frac{1}{\rho} \frac{\partial H_2}{\partial \rho} + \frac{\partial^2 H_2}{\partial \rho^2} + \frac{1}{V(r)} \left( \frac{2\partial H_2}{r \partial r} + \frac{\partial^2 H_2}{\partial r^2} \right) = 0,$$

$$\frac{1}{\rho} \frac{\partial H_1}{\partial \rho} + \frac{\partial^2 H_1}{\partial \rho^2} + \frac{1}{V(r)} \left( \frac{2\partial H_1}{r \partial r} + \frac{\partial^2 H_1}{\partial r^2} \right) = -H_2 \left( \frac{\partial^2 H_1}{\partial x_1^2} + \frac{\partial^2 H_1}{\partial x_2^2} \right).$$

\[3\]
We note that the solutions for $H_1$ depend explicitly on the solutions for $H_2$. We can solve equation (11) by separating the coordinates according to

$$H_2 = Q_2 P(\rho)R(r),$$

(13)

where $Q_2$ is the charge of M2 brane. We find two ordinary differential equations for $P(\rho)$ and $R(r)$ as

$$\frac{d^2 P(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{dP(\rho)}{d\rho} - c^2 P(\rho) = 0,$$

(14)

and

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + c^2 V(r)R(r) = 0.$$

(15)

The equation (14) is the modified Bessel equation and the solutions to the equation are given by

$$P(\rho) = C_1 I_0(c\rho) + C_2 K_0(c\rho).$$

(16)

Moreover, the solution to equation (15) is given by

$$R(r) = B e^{-icr} K_M(1 + \frac{1}{2}icn, 2, 2icr),$$

(17)

where $K_M$ stands for the Kummer function of type $M$. We note that we do not consider the second independent solution to (15) because the solution does not have proper behaviour in the near horizon limit. We choose $C_1 = 0$ as the corresponding term in (16) diverges far from the branes and so does not lead to proper behaviour for the metric function at infinity. So, the most general solution for the metric function is a superposition of all possible solutions (16) and (17) with different values of $c$

$$H_2(\rho, r) = Q_2 \int_0^{+\infty} f(c) K_0(c\rho)e^{-icr} K_M(1 + \frac{1}{2}icn, 2, 2icr) dc,$$

(18)

where $f(c)$ is an arbitrary function of separation constant. To find the proper $f(c)$, we look at the near horizon of the system where $r \to 0$ and moreover we consider $n \to +\infty$ in such a way that $nr$ remains fixed. In these limits, the Taub-NUT space approaches to

$$ds_{TN}^2 \to dz^2 + z^2 d\Omega_3^2,$$

(19)

where $z = 2\sqrt{n}r$ [21]. So, the transverse space to M2 branes, in these limits, is simply equivalent to the product of a ball of radius $z$ and a disc of radius $\rho$. The metric function for an M2 brane with such a transverse space is $\frac{Q_2}{(z^2 + \rho^2)^{\frac{3}{2}}}$. Demanding the most general solution (18) for the metric function $H_2$ approaches to $\frac{Q_2}{(z^2 + \rho^2)^{\frac{3}{2}}}$, leads to an integral equation for the measure function $f(c)$ after we take the proper limits of the Kummer function in (18). The integral equation for $f(c)$ is given by

$$\int_0^{+\infty} f(c) K_0(c\rho) J_1(2c\sqrt{n}r) \frac{dc}{c} = \frac{\sqrt{n}r}{(4nr + \rho^2)^2}.$$ 

(20)

The solution to (20) is given by $f(c) = \frac{c^3}{4}$ and so we find the metric function $H_2$ as

$$H_2(\rho, r) = \frac{Q_2}{4} \int_0^{+\infty} e^{-icr} K_0(c\rho) K_M(1 + \frac{1}{2}icn, 2, 2icr) dc.$$ 

(21)
The exact form of the general solution for \( H_2(\rho, r) \) enables us find the general solution to (12) for the metric function \( H_1(x_1, x_2, \rho, r) \). Separating the coordinates as

\[
H_1(x_1, x_2, \rho, r) = 1 + Q_1(x_1^2 + x_2^2) + P(\rho)S(r),
\]

where \( Q_1 \) is the charge of M2 brane, leads to separability of the differential equation (12). The equation for \( S(r) \) is

\[
\frac{d^2 S(r)}{dr^2} + \frac{2}{r} \frac{dS(r)}{dr} + c^2 V(r)S(r) - G(r) = 0,
\]

where \( G(r) = -4Q_1Q_2R(r)V(r) \). The solution to (23) is

\[
S(r) = -R(r) \int \frac{G(r)\tilde{R}(r)dr}{\mathcal{W}(R(r), \tilde{R}(r))} + \tilde{R}(r) \int \frac{G(r)R(r)dr}{\mathcal{W}(R(r), \tilde{R}(r))},
\]

where

\[
\tilde{R}(r) = R(r) \int \frac{dr}{r^2 R(r)^2}.
\]

and \( \mathcal{W}(R(r), \tilde{R}(r)) \) is the Wronskian of \( R(r) \) and \( \tilde{R}(r) \). Again, we can write the most general solution for \( H_1(x_1, x_2, \rho, r) \) as the superposition of all possible \( P(\rho) \) and \( S(r) \) with different values of separation constant \( c \)

\[
H_1(x_1, x_2, \rho, r) = 1 + Q_1(x_1^2 + x_2^2) + \int_0^\infty \tilde{f}(c)P(\rho)S(r)dc.
\]

We use the same near horizon limits to find the measure function \( \tilde{f}(c) \). After taking the limit of \( G(r) \) and finding the solution to (23) in the near horizon limit, we find an integral equation for the measure function \( \tilde{f}(c) \) as

\[
\int_0^\infty \frac{\tilde{f}(c)}{c^2}K_0(\tilde{c}p)J_0(2c\sqrt{nr})dc = \frac{1}{4(4nr + \rho^2)}.
\]

The solution to equation (27) is \( \tilde{f}(c) = \frac{\tilde{c}^3}{4} \) and so the most general solution for the metric function \( H_1 \) is

\[
H_1(x_1, x_2, \rho, r) = 1 + Q_1(x_1^2 + x_2^2) + \frac{1}{4} \int_0^\infty cK_0(\tilde{c}p)S(r)dc,
\]

where \( S(r) \) is given by (24). Moreover, we can find another independent set of solutions for the metric functions \( H_1 \) and \( H_2 \) by analytically continuing the separation constant \( c \) to \( i\tilde{c} \). We skip the explicit derivation and just mention the final results for the membrane metric functions. The metric functions are [21]

\[
\tilde{H}_2(\rho, r) = Q_2 \int_0^{+\infty} -\frac{\tilde{c}^3}{8} \Gamma(\frac{\tilde{c}n}{2}) e^{-\tilde{c}r} K_U(1 + \frac{\tilde{c}n}{2}, 2, 2\tilde{c}r)J_0(\tilde{c}p)dc,
\]

and

\[
\tilde{H}_1(x_1, x_2, \rho, r) = 1 + Q_1(x_1^2 + x_2^2) + \int_0^\infty \frac{\tilde{c}^2}{4} J_0(\tilde{c}p)\tilde{h}(r)dc,
\]

where \( \tilde{h}(r) \) is given by

\[
\tilde{h}(r) = -\tilde{\mathcal{F}}_1(r) \int \frac{\tilde{G}(r)\tilde{F}_2(r)dr}{\mathcal{W}(\tilde{F}_1(r), \tilde{F}_2(r))} + \tilde{\mathcal{F}}_2(r) \int \frac{\tilde{G}(r)\tilde{F}_1(r)dr}{\mathcal{W}(\tilde{F}_1(r), \tilde{F}_2(r))}.
\]
The functions \( \tilde{G}(r) \), \( \tilde{F}_1(r) \) and \( \tilde{F}_2(r) \) are

\[
\tilde{G}(r) = -4Q_1 Q_2 (1 + \frac{n}{r}) e^{-\epsilon r} K_U (1 + \frac{1}{2} \tilde{c} n, 2, 2 \tilde{c} r),
\]

(32)

\[
\tilde{F}_1(r) = e^{-\epsilon r} K_M (1 + \frac{1}{2} \tilde{c} n, 2, 2 \tilde{c} r),
\]

(33)

and

\[
\tilde{F}_2(r) = e^{-\epsilon r} K_U (1 + \frac{1}{2} \tilde{c} n, 2, 2 \tilde{c} r),
\]

(34)

where \( K_U \) and \( K_M \) are the Kummer functions of type \( U \) and \( M \) respectively and \( \mathcal{W}(\tilde{F}_1(r), \tilde{F}_2(r)) \) is the Wronskian of \( \tilde{F}_1(r) \) and \( \tilde{F}_2(r) \). Dimensional reduction of all two-membrane solutions in this section yields the field content and metric of type IIA string theory. The dimensional reduction along the coordinate \( \psi \) of (5) gives the NSNS dilaton and the RR one-form by

\[
\Phi = \frac{3}{4} \ln (H_1^3 H_2^3 V_1^3) \quad \text{and} \quad C_\phi = n \cos(\theta). \]

The antisymmetric NSNS two-form is zero and the only non-zero components of RR three-form are \( A_{1x_1 x_2} = \frac{1}{2H_2} \) and \( A_{y_1 y_2} = \frac{1}{2H_1} \). The ten-dimensional metric read as

\[
ds^2 = -H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} V^{-\frac{1}{2}} dt^2 + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} V^{-\frac{1}{2}} (dx_1^2 + dx_2^2) + H_1^{-\frac{1}{2}} H_2^{\frac{1}{2}} V^{-\frac{1}{2}} (dy_1^2 + dy_2^2) + \]

(35)

\[
+ H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} V^{-\frac{1}{2}} (d\rho^2 + \rho^2 d\alpha^2) + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} V^{\frac{1}{2}} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right],
\]

that describes a system of three D branes. The convoluted supergravity solutions constructed in this section preserve 1/8 of supersymmetry [21]. Moreover all the supergravity solutions constructed in section 2 preserves 1/4 of supersymmetry. In next section, we find in detail the number of preserved supersymmetries for some of the solutions.

4. The Killing spinor equation and the number of preserved supersymmetries

To find out the number of preserved supersymmetries for any solution of supergravity in eleven dimensions, one must determine the number of solutions to the Killing spinor equation that is given by [43]

\[
\partial_M \xi + \frac{1}{4} \epsilon_{abM} \Gamma^{ab} \xi + \frac{1}{144} \Gamma_M^{NPQR} F_{NPQR} \xi - \frac{1}{18} \Gamma^{NPQR} F_{MPQR} \xi = 0.
\]

(36)

In (36), the \( \Gamma^M \)'s are the Dirac matrices in eleven dimensions while \( \epsilon_{abM} \)'s are the spin connection coefficients. We use the capital indices \( M, N, ... \) to label the eleven dimensional world coordinates and the lowercase indices \( a, b, ... \) to label the tangent space coordinates in eleven dimensions. The Dirac matrices in tangent space \( \Gamma^a \) satisfy the following Clifford algebra

\[
\left\{ \Gamma^a, \Gamma^b \right\} = -2 \eta^{ab},
\]

(37)

where the metric for the tangent space is \([-1, +1, \ldots, +1]\). A very useful representation of the algebra (37) is given by

\[
\Gamma^i = \gamma^i \otimes 1_8,
\]

(38)

and

\[
\Gamma_{\tilde{\xi} + 4} = \gamma_5 \otimes \tilde{\Gamma}_{\xi},
\]

(39)

where \( \tilde{i} = 0, 1, 2, 3 \) and \( \xi = 0, 1, \ldots, 6 \) show respectively the labels of the tangent space groups \( SO(1,3) \) and \( SO(7) \). Moreover, we define the \( \tilde{\Gamma}_{\xi} \)'s that are given by

\[
\tilde{\Gamma}_0 = \gamma_0 \otimes 1_2,
\]

\[
\tilde{\Gamma}_1 = \gamma_1 \otimes 1_2,
\]

\[
\tilde{\Gamma}_{i+3} = \gamma_5 \otimes \sigma_i,
\]

(40)
in terms of the Pauli matrices $\sigma_i$ ($i = 1, 2, 3$), $\gamma_0 = \begin{pmatrix} 0 & 116 \\ 116 & 0 \end{pmatrix}$, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. We note that the $\Gamma_{\xi+4}$ and also $\tilde{\Gamma}_{\xi}$ satisfy the usual Dirac algebras as

$$\{\Gamma_{\xi+4}, \Gamma_{\xi+4}\} = \{\tilde{\Gamma}_{\xi}, \tilde{\Gamma}_{\xi}\} = -2\delta_{\xi\xi}. \tag{41}$$

Moreover

$$\begin{align*}
\omega^{a}_{bc} &= g^{a}_{bc}e^{b} \wedge e^{c}, \tag{42} \\
\omega^{a}_{bc} &= \frac{1}{2} \left( g^{a}_{bc} + g^{b}_{ca} - g^{c}_{ab} \right), \tag{43} \\
\omega_{dbM} &= \omega^{a}_{bc}\eta_{ad}e^{e}_{M}, \tag{44}
\end{align*}$$

where the usual definitions and properties

$$e^{a} = e^{a}_{M}dx^{M}g_{MN} = \eta^{a}_{bc}e^{b}, \tag{45}$$

(etc.) hold. The $\Gamma$'s in (36) are given by

$$\Gamma^{A_1\ldots A_p} = \Gamma^{[A_1 \ldots A_p]}, \tag{46}$$

The longitudinal components of the Killing spinor equation (36) for the M2 brane solutions with transverse Taub-NUT lead to a projection equation $(1 + \Gamma^{x_1x_2})\epsilon = 0$ for the Killing spinor $\epsilon$ where all superscripts are tangent space indices. The projection equation eliminates half of the supersymmetry. The only other non-zero terms in the remaining transverse components of the Killing spinor equation are given by

$$\begin{align*}
\partial_{\alpha_1} \epsilon &= -\frac{1}{2} \Gamma^{\gamma_{\alpha_1}} \epsilon = 0, \tag{47} \\
\partial_{\alpha_2} \epsilon &= -\frac{\sin \alpha_1 \Gamma^{\gamma_{\alpha_2}} \epsilon - \cos \alpha_1}{2} \Gamma^{\alpha_1\alpha_2} \epsilon = 0, \tag{48} \\
\partial_{\alpha_3} \epsilon &= -\frac{\sin \alpha_1 \sin \alpha_2 \Gamma^{\gamma_{\alpha_3}} \epsilon - \sin \alpha_2 \cos \alpha_1}{2} \Gamma^{\alpha_1\alpha_2} \epsilon - \frac{\cos \alpha_2}{2} \Gamma^{\alpha_2\alpha_3} \epsilon = 0, \tag{49} \\
\partial_{\psi} \epsilon + \frac{n}{2(r + n)^2} \left[ \Gamma^{\psi} + \Gamma^{\theta} \right] \epsilon &= 0, \tag{50} \\
\partial_{\theta} \epsilon + \frac{n}{2(r + n)^2} \left[ \Gamma^{\psi} + \Gamma^{\theta} \right] \epsilon &= 0, \tag{51} \\
\partial_{\phi} \epsilon + \frac{n^2 \cos \theta}{(r + n)^2} \left[ \Gamma^{\psi} + \Gamma^{\phi} \right] \epsilon &= \frac{n \sin \theta}{2(r + n)} \Gamma^{\psi} \epsilon - \frac{r \sin \theta}{2(r + n)} \Gamma^{\phi} \epsilon - \frac{\cos \theta}{2} \Gamma^{\phi} \epsilon = 0, \tag{52}
\end{align*}$$

where $\alpha_i$, $i = 1, 2, 3$ and $y$ are three coordinates of $ds^2_{mtrc1}$. All the superscripts in equations (47)-(52) are tangent space indices while the subscripts of derivative operators are world indices. If we consider $\epsilon$ independent of coordinate $\psi$, then the only equation that leads to a second projection equation is equation (50). The equation (50) restricts $\epsilon$ obeys $\left(1 - \Gamma^{\psi}\Gamma^{\phi}\right)\epsilon = 0$. This projection equation eliminates another half of the supersymmetry. We can solve the other equations (47)-(49), (51) and (52) to write down the structure of the Killing spinor equation as the Lorentz rotated of an arbitrary spinor. We then conclude M2 brane solution (4) with Taub-NUT as a part of transverse space, preserves 1/4 of supersymmetry. Similar lengthy calculations show that embedding other four-dimensional self-dual metrics including Bianchi type IX, Eguchi-Hanson (type I and II) and Atiyah-Hitchin as $ds^2_{mtrc2}$ with a flat space as $ds^4_{mtrc2}$ leads to supersymmetric solutions. All the solutions preserve 1/4 of supersymmetry. If we embed any combination of Taub-NUT, Eguchi-Hanson (type I and II) and Atiyah-Hitchin as $ds^4_{mtrc2}$ and $ds^2_{mtrc2}$, we find supersymmetric solutions that preserve 1/4 of supersymmetry. On the other hand, the supergravity solutions presented in section 3 lead to three projection equations and so they preserve 1/8 of supersymmetry [21].
5. Convoluted solutions in $D$-dimensional Einstein-Maxwell theory

Inspired by the convoluted solutions in $D = 11$ supergravity in section 2 and 3, in this section we review the convoluted solutions in $D \geq 6$ dimensional Einstein-Maxwell theory. We consider the $D$-dimensional metric as

$$ds^2_D = -H(r,x)^{-2}dt^2 + H(r,x)^{(2/(D-3))}(dx^2 + x^2d\Omega_{D-6} + ds^2_{TN}),$$  \hspace{1cm} (53)

where $d\Omega_{D-6}$ is the metric on unit sphere $S^{D-6}$. We consider the only non-zero component of the gauge field as

$$A_t = \sqrt{\frac{D-2}{D-3}}H^{-1}(r,x).$$  \hspace{1cm} (54)

The metric ansatz (53) and the gauge field (54) satisfy the gravitational and electromagnetic field equations

$$R_{\mu\nu} = F^\lambda_{\mu\nu}F_{\nu\lambda} - \frac{1}{8}F^2,$$  \hspace{1cm} (55)

$$F_{\mu\nu} = 0,$$  \hspace{1cm} (56)

if the metric function $H(r,x)$ in (53) satisfies

$$(rxV(r)\left(\frac{\partial^2}{\partial x^2}H(r,x)\right) + (D-6)rV(r)\frac{\partial}{\partial x}H(r,x) + rx\left(\frac{\partial^2}{\partial r^2}H(r,x)\right) + 2x\frac{\partial}{\partial r}H(r,x) = 0. \hspace{1cm} (57)$$

The equation (57) is separable if we consider $H(r,x) = 1 + hH_1(r)H_2(x)$. We get two ordinary differential equations for $H_1(r)$ and $H_2(x)$ as

$$r\frac{d^2}{dr^2}H_1(r) + 2\frac{d}{dr}H_1(r) - \epsilon C^2 rV(r)H_1(r) = 0,$$  \hspace{1cm} (58)

and

$$x\frac{d^2}{dx^2}H_2(x) + (D-6)\frac{d}{dx}H_2(x) + \epsilon C^2 xH_2(x) = 0,$$  \hspace{1cm} (59)

where $C$ is the separation constant and $\epsilon = \pm 1$. We find that the solutions to equation (58) are given by

$$H_1(r) = h_{1,M}e^{-\sqrt{r}Cr}{\mathcal{M}}(1 + 1/2\sqrt{r}Cn, 2, 2\sqrt{r}Cr) + h_{1,U}e^{-\sqrt{r}Cr}{\mathcal{U}}(1 + 1/2\sqrt{r}Cn, 2, 2\sqrt{r}Cr),$$  \hspace{1cm} (60)

where $\mathcal{M}$ and $\mathcal{U}$ are the Kummer $M$ and $U$ functions. We consider first the case where $\epsilon = +1$. In this case, the term which is proportional to the Kummer $M$ function doesn’t lead to a consistent finite metric function for large values of the radial coordinate $r$ [42]. Hence we set $h_{1,M} = 0$. The solutions to the second differential equation (59) are

$$H_2(x) = h_J x^{\frac{7-D}{2}} J_{\frac{7-D}{2}}(Cx) + h_Y x^{\frac{7-D}{2}} Y_{\frac{7-D}{2}}(Cx),$$  \hspace{1cm} (61)

in terms of Bessel functions. Superimposing all the solutions (60) and (61) corresponding to different values of the separation constant $C$, we find that the most general solution to the equations of motion is given by

$$H(r,x) = 1 + \int_0^\infty dC e^{-Cr}{\mathcal{U}}(1 + 1/2 Cn, 2, 2Cr)\{f_{J,D}(C) x^{\frac{7-D}{2}} J_{\frac{7-D}{2}}(Cx) + f_{Y,D}(C) x^{\frac{7-D}{2}} Y_{\frac{7-D}{2}}(Cx)\},$$  \hspace{1cm} (62)
where we denote the two measure functions by \( f_{I,D}(C) \) and \( f_{Y,D}(C) \). To find these two measure functions, we consider the limits where \( r \to 0 \) and \( n \to \infty \). We can find then an exact \( D \)-dimensional solutions to the Einstein-Maxwell theory as

\[
\text{d} s^2 = - \frac{1}{H_{0,D}(r,x)} \, \text{d} t^2 + H_{0,D}(r,x)^{(2/(D-3))} (x^2 + x^2 \text{d} \Omega_{D-6} + w^2 \text{d} \Omega_3^2),
\]

where \( H_{0,D}(r,x) \) is given by

\[
H_{0,D}(r,x) = 1 + \frac{B}{(r^2 + x^2)^{D-3}},
\]

and \( B \) is a constant. The gauge field is still given by (54). Demanding the most general solution (62) reduces to (64) in the above mentioned limits, we find an integral equation for the measure functions \( f_{I,D}(C) \) and \( f_{Y,D}(C) \) that its solutions are

\[
f_{Y,D}(C) = 0,
\]

and

\[
f_{I,D}(C) = B \frac{C^{D-1}}{\phi_D} \Gamma(C/2).
\]

The \( \phi_D \) for even dimensions \( D = 2k, k = 3, 4, \cdots \) is equal to

\[
\phi_{2k} = 2\sqrt{2\pi} \prod_{n=0}^{k-3} (2n + 1),
\]

while for odd dimensions \( D = 2k + 1, k = 3, 4, \cdots \) is

\[
\phi_{2k+1} = 4 \prod_{n=1}^{k-2} (2n).
\]

As the final result, the most general \( D \)-dimensional metric function is

\[
H(r,x) = 1 + \frac{B}{\phi_D} x^{\frac{7-D}{2}} \int_0^\infty \text{d} C C^{\frac{D-1}{2}} \Gamma(Cn/2) e^{-CrU} (1 + 1/2 Cn, 2, 2Cr) J_{D-2}(Cx).
\]

We also would like to mention that if we consider the \( D \)-dimensional Einstein-Maxwell theory in presence of positive cosmological constant, we can find exact convoluted cosmological solutions to the equations of motion. We consider the metric ansatz that is very similar to (53), however we let the metric function depends on time, too

\[
\text{d} s_D^2 = -H(t,r,x)^{-2} \text{d} t^2 + H(t,r,x)^{2/(D-3)} (x^2 + x^2 \text{d} \Omega_{D-6} + \text{d} s_N^2).
\]

The equations of motion lead to a partial differential equation for the metric function that can be separated by taking \( H(t,r,x) = H_1(t)H_2(x) + H_3(t) \). After separation the coordinates, we find two ordinary differential equations for \( H_1(t) \) and \( H_2(x) \) that are exactly equations (58) and (59). The solutions to the third ordinary differential equation for \( H_3(t) \) are simply.

\[
H_3(t) = \sqrt{2} t \Gamma(3/2).
\]

After writing the most general \( D \)-dimensional superposition of the solutions and fixing the measure functions, we find that the most general cosmological \( D \)-dimensional solutions read as

\[
H(t,r,x) = 1 + \sqrt{\xi_D} t + \frac{B}{\phi_D} x^{\frac{7-D}{2}} \int_0^\infty \text{d} C C^{\frac{D-1}{2}} \Gamma(Cn/2) e^{-CrU} (1 + 1/2 Cn, 2, 2Cr) J_{D-2}(Cx).
\]
We note that the general solutions (69) and (71) correspond to choosing $\epsilon = 1$ in equations (58) and (59). However, we can get another independent set of solutions (with and without cosmological constant) in $D$-dimensions if we choose $\epsilon = -1$. We only report the solutions here. We find the cosmological solution as

$$H(t, r, x) = 1 + \sqrt{\xi D \Lambda t + \frac{2B}{\nu D} x^{-\frac{2-D}{2}}} \int_0^\infty dC C^{D-1} e^{-iCr} \mathcal{M} (1 + 1/2iCn, 2, 2iCr) K_{D-7} (Cx),$$

(72)

and the solution without the cosmological constant can be obtained from (72) by setting $\Lambda = 0$. As we notice, the general cosmological solutions (71) and (72) describe asymptotically dS spacetime. We can show that the cosmological $c$-function for $D$-dimensional solutions (71) and (72) monotonically increases as a function of time [42]. The monotonically increasing behaviour of $c$-function as a function of time shows that the flow of renormalization group is toward UV for the $D$-dimensional asymptotically dS spacetimes with the metric functions (71) and (72).

6. Conclusions

In this article, we review the explicit construction of the supergravity solutions for fully localized intersections of D2 branes in worldvolume of D6 brane without restricting to the near core region of the D6 branes by dimensional reduction of convoluted M2 brane solutions. We construct the exact convoluted solutions for M2 brane as well as two M2 branes. The common feature of these solutions is that the brane metric function is a convolution-like of a decaying function with a damped oscillating function. The metric functions vanish far from the branes and diverge near the brane cores. The solutions preserve 1/4 of supersymmetry for a single M2 brane and 1/8 of supersymmetry for two M2 branes. Inspired by these convoluted solutions in $D = 11$ supergravity, we also construct exact convoluted solutions to $D \geq 6$ dimensional Einstein-Maxwell theory.

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