Nonexistence of Wandering Domains for Infinitely Renormalizable Hénon Maps

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Abstract This article extends the theorem of the absence of wandering domains from unimodal maps to infinitely period-doubling renormalizable Hénon-like maps in the strongly dissipative (area contracting) regime. The theorem solves an open problem proposed by several authors [64,44], and covers a class of maps in the nonhyperbolic higher dimensional setting. The classical proof for unimodal maps breaks down in the Hénon settings, and two techniques, “the area argument” and “the good region and the bad region”, are introduced to resolve the main difficulty.

The theorem also helps to understand the topological structure of the heteroclinic web for such kind of maps: the union of the stable manifolds for all periodic points is dense.

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1 Introduction

This article studies the question of the existence of wandering domains for Hénon-like maps. A Hénon-like map is a real two-dimensional continuous map that has the form

\[ F(x,y) = (f(x) - \varepsilon(x,y), x) \]  \hspace{1cm} (1.1)

where \( f \) is a unimodal map (will be defined later) and \( \varepsilon \) is a small perturbation. For renormalization purposes, the Hénon-like maps in consideration are all real analytic and strongly dissipative (the Jacobian \( \left| \frac{\partial \varepsilon}{\partial y} \right| \) is small\(^{1}\)). One can see from the definition,
Hénon-like maps are a generalization of classical Hénon maps [30] (two-parameters polynomial maps) to the analytic settings and extension of unimodal maps to higher dimensions.

Strongly dissipative Hénon-like maps are the maps that are close to unimodal maps. They share some dynamical properties with unimodal maps. For example, the tool of unimodal renormalization can be adopted to Hénon-like maps [12,44,28], the renormalization operator is hyperbolic [12, Theorem 4.1], and an infinitely renormalizable Hénon-like map has an attracting Cantor set [12, Section 5.2]. However, there are also some properties that make Hénon-like maps distinct from unimodal maps. For example, the Cantor set for infinitely renormalizable Hénon-like maps is not rigid [12, Theorem 10.1] and a universal model can not be presented by a finite dimensional family of Hénon-like maps [29]. In the degenerate case, unimodal maps do not have wandering intervals [24,50,51,45]. It is natural to ask whether this property can be promoted to Hénon-like maps.

The study of wandering domains has a broad interest in the field of dynamics. In one-dimension, the problem has been widely studied and there are many important consequences due to the absence of wandering intervals/domains. However, only a few systems in higher dimensions were known not having wandering domains.

In real one-dimension, showing the absence of wandering intervals in a system is important to solve the classification problem. For circle homeomorphisms, a sequence of works [25,67,32,57] follows after Denjoy [14] showed that a circle homeomorphism with irrational rotation number (the average rotation angle is irrational) does not have a wandering domain if the map is smooth enough. Those maps are conjugated to the rigid rotation with the same rotation number by a classical theorem from Poincare [62]. For multimodal maps, a full family is a family of multimodal maps that exhibits all relevant dynamical behavior. A multimodal map that does not have a wandering interval is conjugated to an element in a full family [24,50,51,43,8,45].

In complex dimension one, Sullivan’s no-wandering-domain theorem [65] fully solves the problem for rational maps. The theorem says that a rational map on the Riemann sphere does not have wandering Fatou components. As a consequence, this theorem completes the last puzzle for the classification of Fatou components [19,35]. Thus, the main interest turns to transcendental maps. In general, there are transcendental maps that have wandering domains [3,4,31,65,16,7,18]. There are also some types of transcendental maps that do not have wandering domains [23,17,6,53].

In real higher dimensions, the problem for wandering domains is still wide open. There is no reason to expect the absence of wandering domains [66], especially when the regularity is not enough as in one-dimension [48,49,9]. The classification problem fails between any two different levels of differentiability for diffeomorphisms on $d$-manifold with $d \neq 1, 4$ [26,27]. Examples are found in polynomial skew-product maps having wandering domains [2]. Non-hyperbolic phenomena also play a role in building counterexamples [13,39,37]. A relevant work by Kiriki and Soma [39] found Hénon-like maps having wandering domains by using a homoclinic tangency of some saddle fixed point [36,38]. On the other hand, there are studies [58,41,40,55] suggests that some types of systems may not have wandering domains. However, only a few [56,9] were discovered not having wandering domains.
In complex higher dimension, counterexamples in transcendental maps can be constructed from one-dimensional examples [20] by taking direct products. For polynomial maps, very little was known about the existence of wandering Fatou components until recent developments on polynomial skew-products [42, 60, 59, 61], which are the maps of the form

\[ F(z, w) = (f(z, w), g(w)) \].

The first example was given by Astorg, Buff, Dujardin, Peters, and Raissy [2], who found a polynomial skew-product possessing a wandering Fatou component as the quasi-conformal methods break down. The reader can refer to the survey [63] for more details about other relevant work on polynomial skew-product [42, 60, 59, 61].

For complex Hénon maps [1], a recent paper by Leandro Arosio, Anna Miriam Benini, John Erik Fornaess, and Han Peters [1] found a transcendental Hénon map exhibiting a wandering domain. Nevertheless, the problem is still unsolved [5] for complex polynomial Hénon maps [33, 34].

In this paper, a wandering domain is a nonempty open set that does not intersect the stable manifold of any saddle periodic points. This definition is weaker than the classical notion because it excludes the condition having a disjoint orbit. The reason to drop this condition is to allow the usage study the topological structure of attractors which will be discussed later. Dropping the condition also makes the conclusion of the theorem stronger compared to the classical definition. In fact, this condition is redundant in the unimodal setting (See Remark 6.2).

The main result of this article, Theorem 10.16, is stated as follow.

**Theorem** A strongly dissipative infinitely period-doubling renormalizable Hénon-like map does not have wandering domains.

The theorem covers a class of maps in the higher dimensional nonhyperbolic setting [44, Corollary 6.2] and solves an open problem proposed by van Strien [44], Lyubich, and Martens [44]. The result does not overlap with the previous work by Kiriki and Soma [39]. The Hénon-like maps in this article are real analytic and the fixed points are far away from having a homoclinic tangency, while the examples they found having wandering domains only have finite differentiability and their construction relies on the existence of a homoclinic tangency of a fixed point [36, 38].

The condition of being infinitely renormalizable is imposed to the theorem to gain a self-similarity between different scale. A map is renormalizable means that a higher iterate of the map has a similar topological structure on a smaller scale. Several papers [9, 46, 47] in different contexts show that this condition will ensure the absence of wandering domains. In this paper, we center on infinitely renormalizable maps of period-doubling combinatorics type which is one of the most fundamental types of maps. The reader will see later in the proof that the condition infinitely renormalizable is essential because that the area where bad things (nonhyperbolic phenomena) happens, called the bad region, becomes smaller when a map gets renormalized more times.

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2 The study of complex Hénon maps covers a broader class of functions motivated by the classification of polynomial automorphisms [21]. It allows the map \( f \) in [44] to be any polynomial or analytic map.
The theorem is important because it helps us to understand the structure of attractors. An attracting set is a closed set such that many points evolve toward the set. Hénon maps are famous for its chaotic limiting behavior since Hénon first discovered the strange attractor in the classical Hénon family \[30\]. The $\omega$-limit set of a point is an attractor which characterizes the long-time behavior of a single orbit. For an infinitely period-doubling renormalizable Hénon-like map, the map has only two types of $\omega$-limit set \[22,44\]: a saddle periodic orbit and the renormalization Cantor attractor. From this dichotomy, a wandering domain is equivalently a non-empty open subset of the basin of the Cantor attractor. The theorem of the absence of wandering domains implies that the union of the stable manifolds is dense. In other words, the basin of the Cantor set has no interior even though it has full Lebesgue measure.

Two tools are introduced to prove the theorem: the bad region and the area argument (thickness). The bad region is a set in the domain where the length expansion argument from unimodal maps breaks down. This is where the main difficulty of extending the theorem occurs. The solution to this is the area argument which is also a dimension two feature. These two concepts make the Hénon-like maps different from the unimodal maps.

The tools may be used to prove the nonexistence of wandering domains in other contexts. One is infinitely renormalizable Hénon-like maps with arbitrary combinatorics \[28\]. The definition of the bad region carries over to the arbitrary combinatorics case directly. It is also possible to generalize the area argument because the tip of those maps also has a universal shape \[28\, Theorem 6.1\]. However, the expansion argument breaks down for other combinatorics. This may be solved by studying the hyperbolic length instead of the Euclidean length.

Outline of the article

In this article, chapters, sections, or statements marked with a star sign “\*” means that the main theorem, Theorem \[10.16\] does not depend on them. Terminologies in the outline will be defined precisely in later chapters.

Chapters \[2,3,4,5\] are the preliminaries of the theorem. The chapters include basic knowledge and conventions that will be used in the proof. Most of the theorems in Chapter \[4\] and Section \[5.1\] can be found in \[12,44\].

The proof for the nonexistence of wandering domains is motivated by the proof of the degenerate case. A Hénon-like map is degenerate means that $\varepsilon = 0$ in \(1.1\). In this case, the dynamics of the map degenerates to the unimodal dynamics. In Chapter \[7\] a short proof for the nonexistence of wandering intervals for infinitely renormalizable unimodal maps is presented by identifying a unimodal map as a degenerate Hénon-like map. The proof assumes the contrapositive, there exists a wandering interval $J$. Then we apply the Hénon renormalization instead of the standard unimodal renormalization to study the dynamics of the rescaled orbit of $J$ that closest approaches the critical value. The rescaled orbit is called the $J$-closest approach \(Definition 6.1\). The proof argues that the length of the elements in the rescaled orbit approaches infinity by a length expansion argument which leads to a contradiction. The expansion argument motivates the proof for the Hénon case.
The proof of the main theorem is covered by Chapters 6, 8, 9, and 10. The structure is explained as follows.

Assume the contrapositive, a Hénon-like map has a wandering domain \( J \). In Chapter 6, we study the rescaled orbit \( \{J_n\}_{n \geq 0} \) of \( J \) that closest approaches to the tip, called the \( J \)-closest approach. Each element \( J_n \) belongs to some appropriate renormalization scale (the domain of the \( r(n) \)-th renormalization \( R^{r(n)}F \) for some nonnegative integer \( r(n) \)). The transition between two constitutive sequence elements \( J_n \to J_{n+1} \) is called one step. Motivated by the expansion argument from the degenerate case, we estimate the change rate of the horizontal size \( l_n \) in each step. The horizontal size of a set is the size of its projection to the first coordinate (Definition 6.8). Our final goal is to show that the horizontal size of the sequence elements approaches infinity to obtain a contradiction.

In the degenerate case, the expansion argument says that the horizontal size expands at a uniform rate and hence the horizontal size of the sequence elements approaches infinity. Unfortunately, the argument breaks down in the non-degenerate case. There are two features that make the non-degenerate case special:

1. The good region and the bad region.
2. Thickness.

The good region and the bad region, introduced in Chapter 8, divide the phase space of a Hénon-like map into two regions by how similar the Hénon-like map behaves like unimodal maps. Each renormalization scale (domain of the \( n \)-th renormalization \( R^nF \) for some \( n \)) has its own good region and bad region, and the size of the bad regions contract super-exponentially as the renormalization applies to the map more times ([12, Theorem 4.1] and Definition 8.1). When the elements in a closest approach stay in the good regions of some appropriate scale, we show that the expansion argument can be generalized to the Hénon-like maps. Thus, the horizontal size expands at a uniform rate (Proposition 9.2). However, when an element \( J_n \) enters the bad region of the renormalization scale of the set, the expansion argument breaks down. At this moment, another quantity, called the thickness, offers a way to estimate the horizontal size of the next element \( J_{n+1} \) (Definition 10.2). The reader should imagine the thickness of a set is the same as its area. We will show that the thickness has a uniform contraction rate proportion to the Jacobian of the map (Proposition 10.6). For a strongly dissipative Hénon-like map, the Jacobian is small and hence the contraction is strong. This strong contraction yields the main obstruction toward our final goal.

The breakthrough is the discovery that the elements in a closest approach can at most enter the bad regions finitely many times (Proposition 10.15). When an element \( J_n \) enters the bad region, the horizontal size contracts and the following element \( J_{n+1} \) belongs to a deeper renormalization scale. But the size of the bad region in the deeper scale (scale of \( J_{n+1} \)) is much smaller than the bad region of the original scale (scale of \( J_n \)). Roughly speaking, we found that the contraction of the size of the bad region is faster than the contraction of the horizontal size so that the elements cannot enter the bad region infinitely times. The actual proof is more delicate because another quantity, the time span in the good regions (Definition 10.9), also involves in the competition. The two-row lemma (Lemma 10.13) is the key lemma that gives an estimate for the competitions between the contraction of the thickness, the expansion
of the horizontal size in the good region, the time span in the good region, and the size of the bad region when the closest approach enters the bad region twice. The conclusion follows after applying the two-row lemma inductively (Lemma 10.14).

In summary, the horizontal size of the elements in a closest approach expands in the good regions, while contracts in the bad regions. However, the contraction happens only finitely many times. This shows that the horizontal size approaches infinity which is a contradiction. Therefore, wandering domains cannot exist.

2 Notations

Common terminologies from Dynamical Systems will be adopted in this article. The reader can refer to standard textbook (e.g. [10]) for more information.

Let $I$ be an interval on the real line. The complex $\varepsilon$-neighborhood $I(\varepsilon)$ of $I$ is defined to be the open set $I(\varepsilon) = \{z \in \mathbb{C}; |z - z'| < \varepsilon$ for some $z' \in I\}$. The length of $I$ is defined to be $|I| = \sup \{|b - a|; a, b \in I\}$.

Assume that $X \subset \mathbb{R}^2$ is open and $F : X \to \mathbb{R}^2$ is differentiable. The Jacobian of $F$ is the function $\det DF$.

The projections $\pi_x$ and $\pi_y$ are the maps $\pi_x(x, y) = x$ and $\pi_y(x, y) = y$.

2.1 Functions

Assume that $S$ is a set and $f$ is a real- or complex-valued function on $S$. The sup norm on $S' \subset S$ is denoted by

$$\|f\|_{S'} = \sup \{|f(x)|; x \in S'\}.$$  

The subscript is neglected whenever the context is clear.

Assume that $V$ and $W$ are Banach spaces. A function $f : V \to W$ is called Lipschitz continuous with constant $L$, or $L$-Lipschitz, if

$$|f(y) - f(x)| \leq L|y - x|$$

for all $x, y \in V$. The space of $L$-Lipschitz functions equipped with the sup norm is complete. The space of $C^n(I)$ functions on a closed interval $I$ is the collection of functions $f : I \to \mathbb{R}$ that are $n$-times differentiable with continuous derivatives.

For a holomorphic function, the size of its derivatives can be estimate by the sup norm of the map from the Cauchy integration formula.

**Lemma 2.1** Assume that $U$ is open in $\mathbb{C}$ and $K \Subset U$. For each integer $n \geq 1$, there exists a constant $c > 0$ such that

$$\left\|f^{(n)}\right\|_K \leq c \|f\|_U.$$  

for all holomorphic maps $f$ on $U$ where $f^{(n)}$ means the $n$-th derivative of $f$. A similar estimate holds for multi-variable holomorphic maps and partial derivatives.

This estimate will be frequently used in the setting where $K$ is a real interval $I$ and $U$ is the complex $\delta$-neighborhood $I(\delta)$. 
2.2 Schwarzian derivative

In this section, we recall the definition and the properties of Schwarzian derivative. The proof for the properties stated in this section can be found in [52]. These properties will be used only in Chapter 3.

**Definition 2.2 (Schwarzian Derivative)** Assume that $f$ is a $C^3$ real valued function on an interval. The Schwarzian derivative of $f$ is defined by

$$(Sf)(x) = \left( \frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

whenever $f'(x) \neq 0$. The map $f$ is said to have negative Schwarzian derivative if $Sf(x) < 0$ for all $x \in I$ with $f'(x) \neq 0$.

Negative Schwarzian derivative is preserved under iteration.

**Proposition 2.3** If $f$ has negative Schwarzian derivative, then $f^n$ also has negative Schwarzian derivative for all $n > 0$.

**Proposition 2.4 (Minimal Principle)** Assume that $J$ is a bounded closed interval and $f : J \to \mathbb{R}$ is a $C^3$ map with negative Schwarzian derivative. If $f'(x) \neq 0$ for all $x \in J$, then $|f'(x)|$ does not attain a local minimum in the interior of $J$.

3 Unimodal Maps

In this chapter, we give a short review over the procedure for unimodal renormalization. The goal is to introduce the hyperbolic fixed point for the renormalization operator (Proposition 3.8) and establish the estimations for its derivative (Subsection 3.2.2).

**Definition 3.1 (Unimodal Map)** Let $I = [-1, 1]$. A unimodal map in this paper is a smooth map $f : I \to I$ such that

1. the point $-1$ is the unique fixed point with a positive multiplier,
2. $f(1) = -1$, and
3. the map $f$ has a unique maximum at $c \in \text{int}(I)$ and the point $c$ is a non-degenerate critical point, i.e. $f'(c) = 0$ and $f''(c) \neq 0$.

The class of analytic unimodal maps $f : I \to I$ is denoted as $\mathcal{U}$.

**Definition 3.2 (Critical Orbit)** For a unimodal map $f \in \mathcal{U}$, let $c^{(0)} = c^{(0)}(f) \in I$ be the critical point of $f$. The critical orbit is denoted as $c^{(n)} = f^n(c^{(0)})$ for all $n > 0$.

**Definition 3.3 (Reflection)** Assume that $f \in \mathcal{U}$ and $x \in I$. If $x \neq c^{(0)}$, define the reflection of $x$ to be the point $\hat{x} \in I$ such that $f(\hat{x}) = f(x)$ and $\hat{x} \neq x$. If $x = c^{(0)}$, define $\hat{x} = c^{(0)}$. 
3.1 The renormalization of a unimodal map

To define the period-doubling renormalization operator for unimodal maps, we introduce a partition on $I$ that allows us to define the first return map for a renormalizable unimodal map.

**Definition 3.4** Assume that $f \in \mathcal{U}$ has a unique fixed point $p(0) \in I$ with a negative multiplier. Let $p^{(1)} = \hat{p}(0)$ and $p^{(2)}$ be the point such that $f(p^{(2)}) = p^{(1)}$ and $p^{(2)} > c^{(0)}$. Define $A = (-1, p^{(1)}) \cup (p^{(2)}, 1)$, $B = (p^{(1)}, p(0))$, and $C = (p(0), p^{(2)})$. The sets $A = A(f)$, $B = B(f)$, and $C = C(f)$ form a partition of the domain $D \equiv I$. See Figure 3.1 for an illustration.

The property “renormalizable” is defined by using the partition elements.

**Definition 3.5 (Renormalizable)** A unimodal map $f \in \mathcal{U}$ is (period-doubling) renormalizable if it has a fixed point $p(0)$ with a negative multiplier and $f(B) \subset C$. The class of renormalizable unimodal maps is denoted as $\mathcal{U}'$.

**Remark 3.6** Most of the articles define the unimodal renormalization by using the critical orbit. However, here we choose to use an orbit that maps to the fixed point with a negative multiplier instead. The purpose of doing this is to make the partition consistent with the partition defined for Hénon-like maps (Definition 4.14) because Hénon-like maps do not have a critical point.

For a renormalizable unimodal map, an orbit that is not eventually periodic follows the paths in the following diagram.

![Diagram](image)

This allows us to define the first return map on $B$ and the period-doubling renormalization.
Definition 3.7 (Renormalization) Assume that $f \in \mathcal{U}$. The renormalization of $f$ is the map $Rf = s \circ f^2 \circ s^{-1}$ where $s$ is the orientation-reversing affine rescaling such that $s(p(0)) = -1$ and $s(p'(1)) = 1$.

The renormalization operator is a map $R : \mathcal{U} \rightarrow \mathcal{U}$. If the procedure of renormalization can be done recurrently infinitely many times, then the map is called infinitely (period-doubling) renormalizable. The class of infinitely renormalizable unimodal maps is denoted as $\mathcal{I}$.

3.2 The fixed point of the renormalization operator

In this section, we study the fixed point $g$ of the renormalization operator. The map $g$ is also important for the Hénon case because it also defines the hyperbolic fixed point of the Hénon renormalization operator [12, Theorem 4.1].

The existence and uniqueness of the fixed point was proved in [13]. Here, the theorem is restated in the coordinate system used in this paper.

Proposition 3.8 There exists a unique constant $\lambda = 2.5029...$ and a unique solution $g \in \mathcal{I}$ of the Cvitanović-Feigenbaum-Coullet-Tresser functional equation

$$g(x) = -\lambda g^2 \left( -\frac{x}{\lambda} \right)$$

for $-1 \leq x \leq 1$ with the following properties:

1. $g$ is analytic in a complex neighborhood of $[-1, 1]$.
2. $g$ is even.
3. $g$ is concave on $[-c(1), c(1)]$.
4. $g(c(1)) = -\frac{1}{\lambda} c(1)$ and $g'(c(1)) = -\lambda$.
5. $g$ has negative Schwarzian derivative.

Corollary 3.9 The map $g$ satisfies the following property

$$g^{2n} \left( \frac{1}{(-\lambda)^n x} \right) = \frac{1}{(-\lambda)^n} g(x)$$

for all $n \geq 0$ and all $x \in I$.

Proof The proof follows from the functional equation (3.1). $\square$

In the remaining part of the section, the notations for the unimodal maps will be applied to the map $g$. For example, $\{ c^{(j)} = c^{(j)}(g) \}_{j \geq 0}$ is the critical orbit and the sets $A = A(g), B = B(g),$ and $C = C(g)$ form a partition of the domain $D = I$. 
3.2.1 A backward orbit of the critical point

In this section, we establish a backward orbit \( b^{(2)} \to b^{(1)} \to c^{(0)} \) of the critical point \( c^{(0)} \). Let \( b^{(1)} \in [0, c^{(1)}] \) be the point such that \( g(b^{(1)}) = 0 \). Set \( b^{(2)} = \frac{1}{\lambda} b^{(1)} \).

**Lemma 3.10** We have
\[
g (b^{(2)}) = b^{(1)}.
\]

*Proof* Since \( g \) is even, the only two roots of \( g \) are \(-b^{(1)}\) and \( b^{(1)}\). By the functional equation (3.2), we have
\[
g^2 (b^{(2)}) = -\frac{1}{\lambda} g (-b^{(1)}) = 0.
\]
Thus, \( g (b^{(2)}) = -b^{(1)} \) or \( b^{(1)} \). Also, \( g (b^{(2)}) \neq -b^{(1)} \) because \( b^{(2)} \in (0, b^{(1)}) \) and \( g (x) > 0 \) on \((0, b^{(1)})\). Therefore, \( g (b^{(2)}) = b^{(1)} \). \( \square \)

3.2.2 Estimations for the derivative

Apply the chain rule to the functional equation (3.1), we have
\[
g' (x) = g' (-\frac{x}{\lambda}) g' \circ g (-\frac{x}{\lambda}) \quad (3.3)
\]
for \( x \in I \). We will use this formula to derive the values for the derivative of \( g \) at some particular values.

**Lemma 3.11** The slope at \( b^{(2)} \) is
\[
g' (b^{(2)}) = -1. \quad (3.4)
\]

*Proof* From (3.3) and \( g \) is even, we have
\[
g' (b^{(1)}) = g' (-b^{(2)}) g' \circ g (-b^{(2)}) = -g' (b^{(2)}) g' (b^{(1)}).
\]
We solve \( g' (b^{(2)}) = -1 \). \( \square \)

Let \( q(-1) = -1 \) (the fixed point with a positive multiplier) and \( q(0) \) be the fixed point with a negative multiplier. From the functional equation (3.1), we get \( q(0) = \frac{1}{\lambda} \).

**Lemma 3.12** The slopes at the fixed points satisfy the relation
\[
g' (q(-1)) = [g' (q(0))]^2. \quad (3.5)
\]

*Proof* From (3.3), compute
\[
g' (q(-1)) = g' (-1) = g' \left( \frac{1}{\lambda} \right) g' \circ g \left( \frac{1}{\lambda} \right) = g' (q(0)) g' \circ g (q(0)) = [g' (q(0))]^2.
\]
\( \square \)
Finally, we prove that the map $g$ is expanding on $A$ and $C$.

**Proposition 3.13** The slope of $g$ is bounded below by

$$|g'(x)| \geq |g'(q(0))| > 1$$

for all $x \in [q(-1), \hat{q}(0)] \cup [q(0), \hat{q}(-1)]$.

**Proof** It is enough to prove the case when $x \in [q(0), \hat{q}(-1)]$ since $g$ is even.

First, we consider the interval $[b(2), c(1)]$. We have $b(2) < q(0) < c(1)$. By (3.4) and Proposition 3.8, the derivatives of the boundaries are $g'(b(2)) = -1$ and $g'(c(1)) = -\lambda$. We get $|g'(q(0))| > 1$ by the minimal principle (Proposition 2.4).

Next, we consider the interval $[q(0), \hat{q}(-1)]$. From (3.5), we also get $|g'({\hat{q}}(-1))| > 1$. Therefore, the proposition follows from the minimal principle (Proposition 2.4). \qed 

4 Hénon-like Maps

In this chapter, we give an introduction to the theory of Hénon renormalization in the strongly dissipative regime developed by [12,44]. Their theorems are adopted to fit the notations and the coordinate system used in this article.

4.1 The class of unimodal maps

**Definition 4.1 (Class of unimodal maps)** Assume that $\delta > 0$, $\kappa > 0$, and $I^h \equiv I \equiv [-1, 1]$. Let $\mathscr{U}_{\delta, \kappa}(I^h) \subset \mathscr{U}$ be the class of analytic unimodal maps $f : I^h \to I^h$ such that

1. $f$ has a unique critical point $c$ such that $c \leq f(c) - \kappa$ and $f(c) \leq 1 - \kappa$,
2. $f$ has two fixed points $-1$ and $p$ such that $-1$ has an expanding positive multiplier and $p$ has a negative multiplier,
3. $f$ has holomorphic extension to $I^h(\delta)$.
4. $f$ can be factorized as $f = Q \circ \phi$ where $Q(x) = c^{(1)} - (c^{(1)} + 1)x^2$, $c^{(1)}$ is the critical value, and $\phi$ is an $\mathbb{R}$-symmetric univalent map on $I^h(\delta)$, and
5. $f$ has negative Schwarzian derivative.

In the remaining article, we fix a small $\kappa > 0$ such that the class contains the renormalization fixed point $g$, and we suppress the subscript from the notation $\mathscr{U}_{\delta}(I^h) = \mathscr{U}_{\delta, \kappa}(I^h)$.

**Remark 4.2** From the conditions $f(-1) = -1$ and $f(1) = -1$, this forces $\phi(-1) = -1$ and $\phi(1) = 1$. Thus, $\mathscr{U}_{\delta}$ forms a normal family by [54 Theorem 3.2].
4.2 The class of Hénon-like maps

**Definition 4.3 (Hénon-like map)** Assume that \( I^v \supset I^h \supset I \) are closed intervals. A Hénon-like map is a smooth map \( F : I^h \times I^v \to \mathbb{R}^2 \) of the form

\[
F(x,y) = (f(x) - \epsilon(x,y), x)
\]

where \( f \) is a unimodal map and \( \epsilon \) is a small perturbation. The function \( h \) will also be used to express the \( x \)-component, \( h_y(x) = h(x,y) = \pi_cF(x,y) \). A representation of \( F \) will be expressed in the form \( F = (f - \epsilon, x) \).

The function spaces of the Hénon-like map is defined as follows.

**Definition 4.4 (Class of Hénon-like maps)** Assume that \( I^v \supset I^h \supset I \) and \( \delta > 0 \).

1. Denote \( \mathcal{H}_\delta(I^h \times I^v) \) to be the class of real analytic Hénon-like maps \( F : I^h \times I^v \to \mathbb{R}^2 \) that have the following properties:
   - (a) It has a representation \( F = (f - \epsilon, x) \) such that \( f \in \mathcal{U}_\delta(I^h) \).
   - (b) It has a saddle fixed point \( p(-1) \) near the point \((-1, -1)\). The fixed point has an expanding positive multiplier.
   - (c) The \( x \)-component \( h(x,y) \) has a holomorphic extension to \( I^h(\delta) \times I^v(\delta) \to \mathbb{C} \).
2. Given \( \overline{\epsilon} > 0 \) and \( f \in \mathcal{U}_\delta(I^h) \). Denote \( \mathcal{H}_\delta(I^h \times I^v, f, \overline{\epsilon}) \) to be the class of Hénon-like maps \( F \in \mathcal{H}_\delta(I^h \times I^v) \) with the form \( F = (f - \epsilon, x) \) such that \( \|\epsilon\| < \overline{\epsilon} \).
3. Denote \( \mathcal{H}_\delta(I^h \times I^v, \overline{\epsilon}) = \cup \mathcal{H}_\delta(I^h \times I^v, f, \overline{\epsilon}) \) where the union is taken over all \( f \in \mathcal{U}_\delta(I^h) \).

**Remark 4.5** The domain \( I^h \times I^v \) used in this article is slightly larger than the domain studied in the two original papers [12,44]. Their domain is equivalent to the dynamical interval \([f^2(c), f(c)]\) for unimodal maps which does not include the fixed point with positive multiplier. The larger domain is necessary in this article to study the rescaled orbit of a point. See Proposition 4.11, Proposition 4.16, and Proposition 5.3. Their work also holds on the larger domain \( I^h \times I^v \). See for examples [12, Footnote 7, Section 3.4] and [44, Lemma 3.3, Proposition 3.5, Theorem 4.1]. However, reproving their theorem on the larger domain is not the aim here. This article will assume the results from [12,44] also hold in the larger domain and rephrase them in the notations used in this article without reproving. See also Remarks 4.24, and 10.17.

From the definition, it follows immediately that

**Lemma 4.6** Given \( I^v \supset I^h \supset I \), \( \delta > 0 \), \( \overline{\epsilon} > 0 \), and \( f \in \mathcal{U}_\delta(I^h) \).

1. If \( \overline{\epsilon}_1 < \overline{\epsilon}_2 \) then \( \mathcal{H}_\delta(f, \overline{\epsilon}_1) \subset \mathcal{H}_\delta(f, \overline{\epsilon}_2) \).
2. If \( I \subset I_1^h \subset I_2^h \subset I^h \) and \( f \in \mathcal{U}_\delta(I_2^h) \), then \( \mathcal{H}_\delta(I_1^h) \supset \mathcal{H}_\delta(I_2^h) \) and \( \mathcal{H}_\delta(I_1^h \times I^v, f, \overline{\epsilon}) \supset \mathcal{H}_\delta(I_2^h \times I^v, f, \overline{\epsilon}) \).

An important property of a Hénon-like map is that it maps vertical lines to horizontal lines; it maps horizontal lines to parabola-like arcs.
Example 4.7 (Degenerate case) Assume that $F(x, y) = (f(x) - \varepsilon(x, y), x)$ is a Hénon-like map. The map is called a degenerate Hénon-like map if $\frac{\partial \pi_x F}{\partial y} = \frac{\partial \varepsilon}{\partial y} = 0$; a non-degenerate Hénon-like map if $\frac{\partial \pi_x F}{\partial y} = \frac{\partial \varepsilon}{\partial y} \neq 0$.

If $F$ is degenerate, then $\varepsilon$ only depends on $x$. In this case, without lose of generality, we will assume the Hénon-like map has the representation $F(x, y) = (f(x), x)$ where $f = \pi_x F$ and $\varepsilon = 0$.

For the degenerate case, the dynamics of the Hénon-like map is completely determined by its unimodal component. So it will also be called as the unimodal case in this article.

The degenerate case is an important example in this article. A proof for the nonexistence of wandering intervals for unimodal maps will be presented in Chapter 7 by identifying a unimodal map as a degenerate Hénon-like map. The expansion argument in the proof motivates the proof for the non-degenerate case. The difference between the degenerate case and the non-degenerate case produces the main difficulty (explained in Chapter 8 and Chapter 10) of extending the proof to the non-degenerate case.

Example 4.8 (Classical Hénon maps) The classical Hénon family is a two-parameter family of the form $F_{a,b}(x, y) = (-1 + a(1 - x^2) - by, x)$ where $a, b > 0$. These are Hénon-like maps $F_{a,b} \in H(\mathbb{R}^2, -1 + a(1 - x^2), b[|P'| + 2\delta])$ for all $\delta > 0$ and $P' \supset I^h$.

4.3 Local stable manifolds and partition of a Hénon-like map

To study the dynamics of a Hénon-like map, we need to find a domain $D \subset I^h \times I^v$ that turns the Hénon-like map into a self-map. Also, to renormalize a Hénon-like map, we need to find a subdomain $C \subset D$ that defines a first return map. Motivated from unimodal maps, one can construct a partition of the domain $I^h \times I^v$ to find the domains. In the unimodal case, an orbit that maps to the fixed point $p(0)$ with an expanding multiplier splits the domain $D$ into a partition $\{A, B, C\}$ (Definition 4.4). For a strongly dissipative Hénon-like map, the orbit becomes components of the stable manifold of the saddle fixed point $p(0)$. These components are vertical graphs that split the domain into multiple vertical strips.

Definition 4.9 A set $\Gamma$ is a vertical graph if there exists a continuous function $\gamma : I^v \rightarrow I^h$ such that $\Gamma = \{(\gamma(t), t); t \in I^v\}$. The vertical graph $\Gamma$ is said to have Lipschitz constant $L$ if the function $\gamma$ is Lipschitz with constant $L$.

In this paper, a local stable manifold is a connected component of a stable manifold. Inspired by [12], the partition will be the vertical strips separated by the associated local stable manifolds.

First, we study the local stable manifolds of the saddle fixed point $p(-1)$ which contains an expanding positive multiplier.

Definition 4.10 (The local stable manifolds of $p(-1)$ and the iteration domain $D$) Given $I^v \supset I^h \supset I$, $\delta > 0$, and $F \in H(\mathbb{R}^2)$. Consider the stable manifold of the saddle fixed point $p(-1)$. 
Fig. 4.1: Local stable manifolds and partition $A$, $B$, $C$ for a Hénon-like map $F$. The shaded area is the image of the Hénon-like map. The vertical graphs are the local stable manifolds $W^0(-1), W^1(0), W^0(0), W^2(0),$ and $W^2(-1)$ from left to right. The arrows illustrate the construction of each local stable manifold.

1. If the connected component that contains the fixed point $p(-1)$ is a vertical graph, let $W^0(-1)$ be the component.
2. Assume that $W^0(-1)$ exists. If $F^{-1}(W^0(-1))$ has two components, one is $W^0(-1)$ and the other is a vertical graph. Let $W^2(-1)$ be the one that is disjoint from $W^0(-1)$.

If the the local stable manifolds $W^0(-1)$ and $W^2(-1)$ exists, define $D = D(F) \subset I^h \times I^v$ to be the open set bounded between the two local stable manifolds. See Figure 4.1 for an illustration.

The domain $D$ turns the Hénon-like map into a self-map.

**Proposition 4.11** Given $\delta > 0$ and intervals $I^h$ and $I^v$ with $I^v \supset I^h \equiv I$. There exists $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathcal{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following properties hold:

1. The sets $W^0(-1), W^2(-1),$ and $D$ exist. The two local stable manifolds are vertical graphs with Lipschitz constant $c \| \varepsilon \|$.
2. $F(D) \subset D$.

**Proof** The first property follows from the graph transformation. The techniques were developed in [44, Chapter 3]. See [44, Lemma 3.1, 3.2].
The second property follows from the definition of the local stable manifolds and $\varepsilon > 0$ is sufficiently small. □

Next, we study the local stable manifolds of the other saddle fixed point $p(0)$ with an expanding negative multiplier to define a partition of $D$.

**Definition 4.12** (The local stable manifolds of $p(0)$) Given $I^v \supset I^h \ni I$, $\delta > 0$, and $F \in \mathcal{H}_\delta(I^h \times I^v)$. Assume that $F$ has a saddle fixed point $p(0)$ with an expanding negative multiplier. Consider the stable manifold of $p(0)$.

1. If the connected component that contains $p(0)$ is a vertical graph, let $W^0(0)$ be the component.
2. Assume that $W^0(0)$ exists. If $F^{-1}(W^0(0))$ has two components, one is $W^0(0)$ and the other is a vertical graph. Let $W^1(0)$ be the one that is disjoint from $W^0(0)$.
3. Assume that $W^0(0)$ and $W^1(0)$ exist. If $F^{-1}(W^1(0))$ has two components and one component is a vertical graph located to the right of $W^0(0)$. Let $W^2(0)$ be the component.

See Figure 4.1 for an illustration.

**Remark 4.13** At this moment, the numbers 0 and $-1$ in the notation of the fixed points $p(0)$ and $p(-1)$ (and also the local stable manifolds) do not have a special meaning. After introducing infinitely renormalizable Hénon-like maps, the notation $p(k)$ will be used to define a periodic point with period $2^k$. See Definition 5.2. The numbers are introduced here for consistency.

The local stable manifolds split the domain $D$ into vertical strips. These strips define a partition of the domain.

**Definition 4.14** ($A$, $B$, and $C$) Given $I^v \supset I^h \ni I$, $\delta > 0$, and $F \in \mathcal{H}_\delta(I^h \times I^v)$. Assume that $F$ has a saddle fixed point $p(0)$ with an expanding negative multiplier, the local stable manifolds in Definition 4.12 exist, and $D$ exists.

1. Define $A = A(F) \subset I^h \times I^v$ to be the union of two sets. One is the open set bounded between $W^0(-1)$ and $W^1(0)$; the other is the open set bounded between $W^2(0)$ and $W^2(-1)$.
2. Define $B = B(F) \subset I^h \times I^v$ to be the open set bounded between $W^0(0)$ and $W^1(0)$.
3. Define $C = C(F) \subset I^h \times I^v$ to be the open set bounded between $W^0(0)$ and $W^2(0)$.

**Remark 4.15** The local stable manifolds $W^0(-1), W^1(0), W^0(0), W^2(0)$, and $W^2(-1)$ are associated to the points $p(-1) = -1, p^{(1)}, p(0), p^{(2)}$, and 1 respectively (Definition 3.4).

For a strongly dissipative Hénon-like map, the local stable manifolds are vertical graphs and the dynamics on the partition is similar to the unimodal case.

**Proposition 4.16** Given $\delta > 0$ and intervals $I^h$ and $I^v$ with $I^v \supset I^h \ni I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \mathcal{H}_\delta(I^h \times I^v, \varepsilon)$ the following properties hold:

1. The sets $W^0(0), W^1(0), W^2(0), A, B, C$ exist. The local stable manifolds are vertical graphs with Lipschitz constant $c\|\varepsilon\|$. 
2. \( F(A) \subset A \cup W^1(0) \cup B \).
3. \( F(C) \subset B \).
4. If \( z \in A \) then its orbit eventually escapes \( A \), i.e. there exists \( n > 0 \) such that \( F^n(z) \notin A \).

**Proof** The first property is proved by graph transformation. See \[44\] Chapter 3.

The second and third properties follows from the definition of the local stable manifolds. See also \[44\] Lemma 4.2.

The last property holds because the only fixed points are \( p(-1) \) and \( p(0) \) so the local unstable manifold of \( p(-1) \) must extends across the whole set \( A \). See also \[44\] Lemma 4.2.

By the definition of \( B \), its iterate \( F(B) \) is contained in the right component of \( D \setminus W^0(0) \). With the third property of Proposition 4.16, we can define the condition “renormalizable” as follows.

**Definition 4.17 (Renormalizable)** Assume that \( \varepsilon > 0 \) is sufficiently small. A Hénon-like map \( F \in \mathcal{H}_\delta(I^h \times I^v, \varepsilon) \) is (period-doubling) renormalizable if it has a saddle fixed point \( p(0) \) with an expanding negative multiplier and \( F(B) \subset C \). The class of renormalizable Hénon-like maps is denoted by \( \mathcal{H}_\delta(I^h \times I^v, \varepsilon) \subset \mathcal{H}_\delta(I^h \times I^v, \varepsilon) \).

**Remark 4.18** The notion of “renormalizable” here is similar to \[12\] Section 3.4] (which they called pre-renormalization) but not exactly the same. The “renormalizable” in their paper is called CLM-renormalizable here to compare the difference. In their article, the set “\( C \)” (they named the set \( D \)) where they define the first return map is a region bounded between \( W^0(0) \) and a section of the unstable manifold of \( p(-1) \). In this article, the set \( C \) is defined to be the largest candidate (around the critical value) that is invariant under \( F^2 \) which only uses the local stable manifolds of \( p(0) \). Thus, the sets \( B \) and \( C \) in this article is slightly larger than theirs.

As a result, the property “renormalizable” in this article is stronger than theirs. If a Hénon-like map is renormalizable then it is also CLM-renormalizable. Although the converse is not true in general, the hyperbolicity of the renormalization operator \[12\] Theorem 4.1] allows us to apply the notion of renormalizable to an infinitely CLM-renormalizable map. This makes the final result, Theorem 10.16, also works for CLM-renormalizable maps. See Remarks 4.24 and 10.17 for more details.

Their definition has some advantages and disadvantages. Their notion of renormalizable does not depend on the size of the vertical domain \( I^v \). However, their sets \( B \) and \( C \) are too small. It may requires more iterations for an orbit to enter their \( B \) and \( C \). See the proof of \[44\] Lemma 4.2]. This is the reason for adjusting their definition.

For a renormalizable Hénon-like map, an orbit that is disjoint from the stable manifold of the fixed points follows the paths in the following diagram.

\[
\begin{array}{c}
A \quad \text{finite iterations} \quad \rightarrow \quad B \quad \leftarrow \quad \rightarrow \quad C
\end{array}
\]

Therefore, a renormalizable map has a first return map on \( C \).
4.4 Renormalization operator

When a Hénon-like map is renormalizable, the map has a first return map on \( C \). However, the first return map is no longer a Hénon-like map by a direct computation

\[
F^2(x, y) = (h_x(h_y(x)), h_y(x)).
\]

The paper \[12\] introduced a nonlinear coordinate change \( H(x, y) \equiv (h_x(x), y) \) that turns the first return map into a Hénon-like map. The next proposition defines the renormalization operator.

**Proposition 4.19 (Renormalization operator)** Given \( \delta > 0 \) and intervals \( I^h, I^v \) with \( I^v \supset I^h \supseteq I \). There exists \( \epsilon > 0 \) and \( c > 0 \) so that for all \( F \in \mathcal{H}^h_{\delta}(I^h \times I^v, \epsilon) \) there exists an \( \mathbb{R} \)-symmetric orientation reversing affine map \( s = s(F) \) which depends continuously on \( F \) such that the following properties hold:

Let \( \Lambda(x, y) = (s(x), s(y)) \) and \( \phi = \Lambda \circ H \).

1. The map \( x \to h_y(x) \) is injective on a neighborhood of \( C(F) \) and hence \( \phi \) is a diffeomorphism from a neighborhood of \( C(F) \) to its image.

2. The renormalization \( R_F = \phi \circ F^2 \circ \phi^{-1} \) is an Hénon-like map defined on \( I^h_R(\delta_R) \times I^v_R(\delta_R) \) for some \( \delta_R > 0 \) and intervals \( I^h_R \) and \( I^v_R \). The intervals satisfy \( I^h_R \supset [-1, 1] \) and \( I^v_R = s(I^v) \).

3. The domain \( I^h_R \times I^v_R \) contains \( D(R_F) \), and the rescaling \( \phi \) maps \( \phi(C(F)) = D(R_F) \).

4. The fixed points satisfy the relation \( \phi(p(0)) = p_{RF}(-1) \) where \( p_{RF}(-1) \) is the saddle fixed point of \( RF \) with an expanding positive multiplier.

5. The renormalization has a representation \( RF = (f_R - \epsilon, x) \) where \( f_R \in \mathcal{U} \). The representation satisfies the relations

\[
\|f_R - R_c f\|_{I^h_R(\delta_R)} < c \|\epsilon\|
\]

and

\[
\|\epsilon_R\|_{I^h_R(\delta_R) \times I^v_R(\delta_R)} < c \|\epsilon\|^2.
\]

**Proof** See \[12\] Section 3.5. \( \square \)

**Remark 4.20** The rescaling \( \phi \) preserves the orientation along the \( x \)-coordinate and reverses the orientation along the \( y \)-coordinate.

A map is called infinitely renormalizable if the procedure of renormalization can be done infinitely many times. The class of infinitely renormalizable Hénon-like map is denoted as \( \mathcal{S}_{\delta}(I^h \times I^v, \epsilon) \subset \mathcal{H}_{\delta}(I^h \times I^v, \epsilon) \).

Assume that \( F \in \mathcal{S}_{\delta}(I^h \times I^v, \epsilon) \), we define \( F_n = R^n F \). The subscript \( n \) is called the renormalization level. The subscript is also used to indicate the associated renormalization level of an object. For example, \( H_n, s_n, \) and \( \Lambda_n \) are the functions in Proposition 4.19 that corresponds to \( F_n \). The vertical domain \( I^v_n \) satisfies \( I^v_0 = I^v \) and \( I^v_{n+1} = s_n(I^v_n) \) for all \( n \geq 0 \). The vertical graphs \( W^s_n(j) \) are the local stable manifolds of \( F_n \). The sets \( A_n, B_n, \) and \( C_n \) form a partition of the dynamical domain \( D_n \) that associates to \( F_n \). The points \( p_n(-1) \) and \( p_n(0) \) are the two saddle fixed points of \( F_n \).

Also, define \( \Phi_n^j = \phi_{n+j-1} \circ \cdots \circ \phi_n \) and \( \lambda_n = s'_n(x) \).
Recall $g \in \mathcal{U}$ is the fixed point of the renormalization operator $R$, and $\lambda$ is the rescaling constant defined in \[3.8\]. Let $G(x, y) = (g(x), x)$ be the induced degenerate Hénon-like map.

The renormalization operator is hyperbolic. The next proposition lists the properties of infinitely renormalizable Hénon-like maps.

**Proposition 4.21 (Hyperbolicity of the Renormalization operator)** Given $\delta > 0$ and intervals $I^h, I^v$ with $I^v \supset I^h \supset I$. There exists $\rho < 1$ (universal), $\varepsilon > 0$, $c > 0$ such that for all $F \in \mathcal{A}_\mathcal{R}(I^h \times I^v, \varepsilon)$ there exists $0 < \delta_R < \delta$, an interval $I_R^h$ with $I^h \supset I_R^h \supset I$, and $b \in \mathbb{R}$ such that the following properties hold:

Let $F_n = R^n F$ be the sequence of renormalizations of $F$. Then $F_n \in \mathcal{A}_{\delta_R}(I_R^h \times I_R^v)$ for all $n \geq 0$. Also, the sequence has a representation $F_n = (f_n - \varepsilon_n, x)$ with $f_n \in \mathcal{Y}_{\delta_R}(I_R^h)$ that satisfies

1. $\|f_n - g\|_{I_R^h(\delta_R)} < c \rho^n \|F - G\|_{I_R^h(\delta_R)}$
2. $\|\varepsilon_{n+1}\|_{I_R^h(\delta_R) \times I_R^{v_1}(\delta_R)} < c \|\varepsilon_n\|_{I_R^h(\delta_R) \times I_R^{v_0}(\delta_R)}$
3. $\|f_{n+1} - s_n \circ f_n^2 \circ s_n^{-1}\|_{I_R^h(\delta_R)} < c \|\varepsilon_n\|_{I_R^h(\delta_R) \times I_R^{v_0}(\delta_R)}$
4. $\lambda_n - \lambda | < c \rho^n \|F - G\|_{I_R^h(\delta_R) \times I_R^{v_0}(\delta_R)}$ and
5. $\varepsilon_n(x, y) = b^n a(x)(1 + O(\rho^n))$ (universality)

for all $n \geq 0$ where $a(x)$ is a universal analytic positive function. The value $\delta_R$ in the estimates can be replaced by any positive number that is smaller than $\delta_R$. \hfill \square

**Proof** See [12, Theorem 3.5, 4.1, 7.9, and Lemma 7.4].

**Remark 4.22** The constant $b$ is called the average Jacobian of $F$. See [12, Section 6].

**Remark 4.23** The Hénon-renormalization is an operation that renormalizes around the critical value. However, the renormalization $F_n$ converges to the fixed point $G$ of the unimodal-renormalization that renormalizes around the critical point. This is because of the nonlinear rescaling $H$ maps the domain from $C$ to $B$ in the degenerate case. See Chapter 7 for a more detail explanation.

**Remark 4.24** Although infinitely CLM-renormalizable in general does not imply infinitely renormalizable, the hyperbolicity provides a connection between the two notions of infinitely renormalizable. Assume that $F$ is infinitely CLM-renormalizable. The hyperbolicity of the renormalizable operator [12, Theorem 4.1] says that $R^n F$ converges to the fixed point $G$. This means that $R^n F$ is also infinitely renormalizable for all $n$ sufficiently large. This makes Theorem [10.16] also applies to infinitely CLM-renormalizable Hénon-like maps. See Remark 10.17 for more details.

From now on, for any infinitely renormalizable map $F$, we fix a representation $F_n = (f_n - \varepsilon_n, x)$ such that the maps $f_n$ and $\varepsilon_n$ satisfy the properties given in Proposition 4.21. Also, we neglect the subscript of the supnorms $\|f_n - g\| = \|f_n - g\|_{I_R^h(\delta_R)}$ and $\|\varepsilon_n\| = \|\varepsilon_n\|_{I_R^h(\delta_R) \times I_R^{v_0}(\delta_R)}$ whenever the context is clear.

**Corollary 4.25** There exists a constant $c > 1$ such that

$$\|F_n - G\| < c \rho^n \|F - G\|$$
and
\[ \|\epsilon_{n+t}\| < (c \|\epsilon_n\|)^2^t \]
for all \( t \geq 1 \).

**Lemma 4.26** Assume that \( \epsilon > 0 \) small enough such that Proposition 4.21 holds. There exists a constant \( c_1 > 0 \) such that the inequalities hold
\[ \left| \frac{\partial \epsilon_n}{\partial x}(x,y) \right|, \left| \frac{\partial \epsilon_n}{\partial y}(x,y) \right| \leq c_1 \|\epsilon_n\| \quad (4.1) \]
for all \( F \in \mathcal{A}_\delta(I^h \times I^v, \epsilon) \) and \( (x,y) \in I^h \times I^v_n \). In addition, if \( F \) is non-degenerate, there exists \( N = N(F) \geq 0, \delta_R > 0, \) and \( c_2 > 0 \) such that
\[ \frac{\partial \epsilon_n}{\partial y}(x,y) \geq c_1 \frac{I_v}{|F_n|} \|\epsilon_n\| \quad (4.2) \]
for all \( (x,y) \in I^h \times I^v_n \) and \( n \geq N \).

**Proof** The first inequality (4.1) follows from Lemma 2.1.

By the universality (and the proof of [12, Theorem 7.9]) of the infinitely renormalizable Hénon-like maps, the perturbation \( \epsilon \) and its derivative has the asymptotic form
\[ \epsilon_n(x,y) = b^{2^n} a(x)(1 + O(\rho^n)) \]
and
\[ \frac{\partial \epsilon_n}{\partial y}(x,y) = b^{2^n} a(x)(1 + O(\rho^n)). \]
Since \( a \) is a positive map on a compact set that covers the whole domain, the second inequality follows. \( \Box \)

To study the wandering domains, it is enough to consider Hénon-like maps that are close to the hyperbolic fixed point \( G \). By Corollary 6.4 later, for any integer \( n \geq 0 \), we show an infinitely renormalizable Hénon-like map \( F \) has a wandering domain in \( D(F) \) if and only if \( F_n \) has a wandering domain in \( D(F_n) \). Also, the maps \( F_n \) converge to the hyperbolic fixed point \( G \) as \( n \) approaches to infinity by Proposition 4.21. Thus, we focus on a small neighborhood of the fixed point \( G \).

**Definition 4.27** Given \( \delta > 0 \) and \( I \subseteq I^h \subset I^v \). If \( \epsilon \) is small enough such that Proposition 4.21 holds, define \( \mathcal{A}_\delta(I^h \times I^v, \epsilon) \) to be the class of non-degenerate Hénon-like maps \( F \in \mathcal{A}_\delta(I^h \times I^v, \epsilon) \) such that \( F_n \in \mathcal{A}_\delta(I^h \times I^v_n, \epsilon), \|F_n - G\| < \epsilon, |\lambda_n - \lambda| < \epsilon, \|s_n(x) - (-\lambda)x\|_{\rho^h} < \epsilon, \) and (4.2) holds for all \( n \geq 0 \).

In the remaining part of the article, we will study the dynamics and the topology of Hénon-like maps in this smaller class of maps.

### 5 Structure and Dynamics of Infinitely Renormalizable Hénon-Like Maps

In this chapter, we study the topology of the local stable manifolds and the dynamics on the partition for a infinitely renormalizable Hénon-like map.
5.1 Rescaling levels

This section introduces a finer partition of $C$, called the rescaling levels, based on the maximum possible rescalings of a point in $C$.

For each two consecutive levels of renormalization $n$ and $n+1$, the maps $F_n$ and $F_{n+1}$ are conjugated by the nonlinear rescaling $\phi_n$. The rescaling $\phi_n$ relates the two renormalization levels as follow.

**Lemma 5.1** Given $\delta > 0$ and $I' \supset I^h \supset I$. There exists $\varepsilon > 0$ such that for all $F \in \mathcal{F}_{\delta}(I^h \times I', \varepsilon)$ the following properties hold for all $n \geq 0$:

1. $\phi_n(p_n(0)) = p_{n+1}(-1)$.
2. $\phi_n(W^k_n(0)) = W^k_{n+1}(-1)$ for $k = 0, 2$, and
3. $\phi_n : C_n \rightarrow D_n$ is a diffeomorphism.

The itinerary of a point follows the arrows in the diagram.

The diagram says, if $z_0 \in C_n$, then we can rescale the point. The rescaled point $z_1 = \phi_n(z_0)$ enters the domain $D_{n+1}$ of the next renormalization level $n+1$ by Lemma 5.1. On the renormalization level $n+1$, the rescaled point $z_1$ belongs to one of the sets $A_{n+1}$, $B_{n+1}$, or $C_{n+1}$ if it is disjoint from the stable manifolds. The process of rescaling stops if $z_1$ belongs to $A_{n+1}$ or $B_{n+1}$ and $z_0$ can be rescaled at most one time. If $z_1$ belongs to $C_{n+1}$, we can continue to rescale the point. The rescaled point $z_2 = \phi_{n+1}(z_1)$ enters the domain $D_{n+2}$ of the next renormalization level $n+2$. Similarly, the process of rescaling stops if $z_2$ belongs to $A_{n+2}$ or $B_{n+2}$ and $z_0$ can be rescaled at most two times. If $z_2$ belongs to $C_{n+2}$, we can rescale again and repeat the procedure until the rescaled point enters the sets $A$ or $B$ of some deeper renormalization level.

Motivated from the diagram, we define the finer partition $C_n(j)$ on $C_n$ by the maximal possible rescalings as follows.

**Definition 5.2** For consistency, set $C_n(0) = A_n \cup W^1_n(0) \cup B_n$. Given a positive integer $j$. The $j$-th rescaling level in $C$ is defined as $C_n(j) = (\Phi_n)^{-1}(C_{n+j}(0))$ and the $j$-th rescaling level in $B$ is defined as $B_n(j) = F_{n}^{-1}(C_n(j))$. Also, set $p_n(j) = (\Phi_n)^{-1}(p_{n+j}(0))$ and $W^t_n(j) = (\Phi_n)^{-1}(W^t_{n+j}(0))$ for $t = 0, 2$. 


The diagram explains the definition of a rescaling level.

\[
\begin{array}{c}
C_n(j) \xrightarrow[F^{2j}_n]{\Phi_n^j} C_n(j) \\
\downarrow \Phi_n^j \downarrow \\
D_{n+j} \xrightarrow[F^{2j}_{n+j}]{\Phi_n^j} D_{n+j}
\end{array}
\]

From the definition, the relations of the rescaling levels between two different renormalization level are listed as follow.

**Proposition 5.3** Given \( \delta > 0 \) and \( I' \supset I^h \ni I \). There exists \( \overline{\varepsilon} > 0 \) such that for all \( F \in \mathcal{H}_\delta(I^h \times I', \overline{\varepsilon}) \) the following properties hold for all \( n \geq 0 \):

1. \( p_n(j) \) is a periodic point of \( F_n \) with period \( 2^j \) for \( j \geq 0 \).
2. \( W^s_n(j) \) is a local stable manifold of \( p_n(j) \) for \( j \geq 0 \) and \( t = 0, 2 \).
3. \( \Phi_0^j(W^s_n(j)) = W^s_{n+k}(j-k) \) and \( \Phi_0^j(p_n(j)) = p_{n+k}(j-k) \) for \( j \geq k-1 \) and \( t = 0, 2 \).
4. The map \( \Phi_0^j : C_n(j) \to C_{n+k}(j-k) \) is a diffeomorphism for \( j \geq k \), and
5. For each \( j \geq 0 \), the set \( C_n(j) \) contains two components. The left component \( C_n^1(j) \) is the set bounded between \( W^s_n(j-1) \) and \( W^u_n(j) \) and the right component \( C_n^2(j) \) is the set bounded between \( W^u_n(j) \) and \( W^s_n(j-1) \).

The partition and the local stable manifolds \( W^s_n(j) \) are illustrated in Figure 5.1. The sets \( \{C_n(j)\}_{j \geq 1} \) form a partition of \( C_n \) and the sets \( \{B_n(j)\}_{j \geq 1} \) form a partition of \( B_n \).

Next, we introduce the tip to study the geometric structure of the rescaling levels in \( C \). Recall from [12] Section 7.2] that

**Definition 5.4 (Tip)** Assume that \( \overline{\varepsilon} > 0 \) is sufficiently small. The tip \( \tau \) of an infinitely renormalizable Hénon-like map \( F \in \mathcal{H}_\delta(I^h \times I', \overline{\varepsilon}) \) is the unique point such that

\[
\{\tau\} = \cap_{j=N}^{\infty} \left( \Phi_0^j \right)^{-1}(D_j \cap I^h \times I^h)
\]

for all \( N \geq 0 \).

The tip is an analog of the critical value in the non-degenerate case. Roughly speaking, the tip generates the attracting Cantor set of a Hénon-like map. See [12] Chapter 5 for more information.

From Proposition 5.3, a rescaling level \( C_n(j) \) contains two components which are both bounded by two local stable manifolds. The following proposition lists the geometric properties of the local stable manifolds.

**Proposition 5.5** Given \( \delta > 0 \) and \( I' \supset I^h \ni I \). There exists \( \varepsilon > 0 \), \( c > 0 \) and \( c' > 1 \) such that for all \( F \in \mathcal{H}_\delta(I^h \times I', \varepsilon) \) the following properties hold for all \( n \geq 0 \):

1. \( W^s_n(j) \) is a vertical graph with Lipschitz constant \( c \|\varepsilon_n\| \) for all \( j \geq -1 \) and \( t = 0, 2 \).
2. \( \frac{1}{c'} \left( \frac{1}{2} \right)^{2^j} < |\tau_n| < c' \left( \frac{1}{2} \right)^{2^j} \) for all \( j \geq -1 \) and \( t = 0, 2 \) where \( \tau_n(j) \) is the intersection point of \( W^s_n(j) \) with the horizontal line through \( \tau_n \). See Figure 5.2.

**Proof** See [44] Lemma 3.4 and Proposition 3.5].
Fig. 5.1: The partition and the local stable manifolds of two renormalization levels $F_0$ and $F_1$ from the left to the right. The rescaling levels 1, 2, 3, and below 4 are shaded from light to dark as shown in the legend.

Fig. 5.2: The structure of the partition of the domain. The figure shows the partition and the local stable manifolds on the horizontal cross section that intersects the tip.

Finally, we study the geometric structure of the rescaling levels in $B$. In the degenerate case, each rescaling level $B_n(j)$ contains two components which are bounded by local stable manifolds. In the non-degenerate case, fix an integer $j$, the rescaling level $B_n(j)$ also contains two components when $\varepsilon$ is small enough. The geometric properties of the boundary local stable manifolds are listed as follow.
Proposition 5.6. Given $\delta > 0$ and $I' \supset I^h \ni I$. For all $j \geq 0$ and $d > 0$, there exists $\varepsilon = \varepsilon(j,d) > 0$ and $c = c(j) > 0$ such that for all $F \in \mathcal{F}_\delta(I^h \times I', \varepsilon)$ the following properties hold for all $n \geq 0$:

1. $F_n^{-1}(W^0_n(j))$ has exactly two components $W^l_n(j) \subset [q^l(j) - d, q^l(j) + d] \times I^l_n$ and $W^r_n(j) \subset [q^r(j) - d, q^r(j) + d] \times I^r_n$.
2. Both components $W^l_n(j)$ and $W^r_n(j)$ are vertical graphs with Lipschitz constant $c\|\varepsilon_0\|$.

Proof. The proof is similar to Proposition 5.5 \hfill \Box

Remark 5.7. Unlike Proposition 5.5, here the constant $\varepsilon$ is not uniform on $j \geq 0$. For a non-degenerate Hénon-like map, the structure of the local stable manifolds is similar to degenerate case when $j$ is large. The local stable manifold $W^0_n(j)$ is far away from the tip and hence the pullback $F_n^{-1}(W^0_n(j))$ is the union of two vertical graphs in $B_n$. However, the structure turns to be different when $j$ is large. The local stable manifold is close to the tip and the vertical line argument in Chapter 8 shows that the pullback $F_n^{-1}(W^0_n(j))$ is a concave curve in $B_n$.

5.2 Asymptotic behavior near $G$

In this section, we estimate the derivatives of a Hénon-like map that is close to the hyperbolic fixed point $G$. Define $v_n \in I^h$ to be the critical point of $f_n$ and $w_n = f_n(v_n)$ be the critical value.

The first lemma proves that a Hénon-like map acts like a quadratic map on $B$.

Lemma 5.8. Given $\delta > 0$ and $I' \supset I^h \ni I$. There exists $a > 0$ (universal), $\varepsilon > 0$, and an interval $I^B \subset I^h$ (universal) such that for all $F \in \mathcal{F}_\delta(I^h \times I', \varepsilon)$ the following properties hold for all $n \geq 0$:

The interior of $I^B$ contains $\hat{q}(0)$ and $q(0)$, $I^B \times I^l_n \ni B_n$,

$$\frac{1}{a} |x - v_n| \leq |f_n'(x)| \leq a |x - v_n|,$$

and

$$\frac{1}{2a} (x - v_n)^2 \leq |f_n(x) - f_n(v_n)| \leq \frac{a}{2} (x - v_n)^2$$

for all $x \in I^B$.

Proof. The lemma is true because the map $F$ is close to the hyperbolic fixed point $G$ and the map $g$ is concave on the compact set $[-c^{(1)}, c^{(1)}]$ by Proposition 5.8 \hfill \Box

The next lemma shows that a Hénon-like map is expanding on $A$ and $C$ in the $x$-coordinate when it is close enough to the fixed point $G$.

Lemma 5.9. Given $\delta > 0$ and $I' \supset I^h \ni I$. There exists $E > 1$ (universal), $\varepsilon > 0$, and a union of two intervals $I^{AC} \subset I^h$ such that for all $F \in \mathcal{F}_\delta(I^h \times I', \varepsilon)$ the following properties hold for all $n \geq 0$:
The interior of $I^{AC}$ contains $q(-1), \hat{q}(0), q(0),$ and $\hat{q}(-1)$, $I^{AC} \times I'_n \supset A_n \cup W_n^2(0) \cup C_n$, and
\[
\left| \frac{\partial h_n}{\partial x} (x,y) \right| \geq E
\]
for all $(x,y) \in I^{AC} \times I'_n$.

Proof The lemma is true because the map $g$ is expanding on $A$ and $C$ by Proposition 3.13 and the Hénon-like map $F$ is close to the hyperbolic fixed point $G$ of the renormalization operator. \hfill \square

5.3 Relation between the tip and the critical value

In Lemma 5.8 we proved that a Hénon-like map behaves like a quadratic map when a point is close to the critical point $v_n$ of $f_n$ for the representation $F_n = \left( f_n - \varepsilon_n, x \right)$. However, the critical point $v_n$ and the critical value $w_n$ in the estimates depend on the representation.

In this section, we show that the critical value $w_n$ (for any representation) is $\| \varepsilon_n \|$-close to the tip $\tau_n$ in Proposition 5.13. This allows us to replace $v_n$ and $w_n$ by the representation independent quantity $\tau_n$. This makes the quadratic estimations in Lemma 5.8 useful when a point is $\| \varepsilon_n \|$-away from the tip.

To estimate the distance from the tip to the critical value, we write $\tau_n = (a_n, b_n)$. Since the rescaling $h_n$ maps a horizontal line to a horizontal line, we focus on the horizontal slice that intersects the tip in each renormalization level. Define the restriction of the rescaling map $\phi$ to the slice as
\[
\eta_n(x) = \pi_x \circ h_n(x, b_n) = s_n \circ h_n(x, b_n).
\]

By the definition of the tip, the quantities satisfy the recurrence relations $\phi_n(\tau_n) = \tau_{n+1}$, $\eta_n(a_n) = a_{n+1}$, and $s_n(b_n) = b_{n+1}$.

First, we prove a lemma that allows us to compare the critical value between two renormalization levels.

Lemma 5.10 Given $\delta > 0$ and $I' \supset I^b \equiv I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \mathcal{I}_\delta(I^b \times I', \varepsilon)$ we have
\[
|w_{n+1} - \eta_n(w_n)| < c \| \varepsilon_n \|
\]
for all $n \geq 0$.

Proof First, we compare the critical points $v_n$ and $v_{n+1}$. By Proposition 4.21, we have
\[
\| f'_{n+1} - (s_n \circ f_n^2 \circ s_n^{-1})' \|_{p^b} < c \| f_{n+1} - s_n \circ f_n^2 \circ s_n^{-1} \|_{p^b} < c \| \varepsilon_n \|
\]
for some constant $c > 0$ when $\varepsilon > 0$ is sufficiently small. Since the critical point of the map $f_{n+1}$ is nondegenerate and the map $(s_n \circ f_n^2 \circ s_n^{-1})'$ is a small perturbation of $f'_{n+1}$, the root $s_n(v_n)$ of $(s_n \circ f_n^2 \circ s_n^{-1})'$ is also a small perturbation of the root $v_{n+1}$ of $f'_{n+1}$. That is, there exists $c' > 0$ such that
\[
|v_{n+1} - s_n(v_n)| \leq c' \| \varepsilon_n \|.
\]
The constant $c'$ can be chosen to be independent of $F$ because $F$ is close to $G$.

Moreover, by the quadratic estimates in Lemma 5.8, we get

$$\left| f_{n+1}(v_{n+1}) - s_n \circ f_n^2(v_n) \right| \leq \left| f_{n+1}(v_{n+1}) - f_{n+1}(s_n(v_n)) \right| + \left| f_{n+1}(s_n(v_n)) - s_n \circ f_n^2(v_n) \right|$$

$$\leq \frac{a}{2} |v_{n+1} - s_n(v_n)|^2 + \left| f_{n+1}(s_n(v_n)) - s_n \circ f_n^2 \circ s_n^{-1}(s_n(v_n)) \right|$$

$$\leq \frac{ac'2}{2} \|\varepsilon_n\|^2 + c \|\varepsilon_n\|$$

$$\leq c'' \|\varepsilon_n\|$$

for some constant $c'' > 0$.

Finally, we compute the critical values $w_n$ and $w_{n+1}$. Compute

$$|w_{n+1} - \eta_n(w_n)| = \left| f_{n+1}(v_{n+1}) - s_n\big(f_n^2(v_n) - \varepsilon_n(f_n(v_n), b_n)\big) \right|$$

$$\leq \left| f_{n+1}(v_{n+1}) - f_{n+1} \circ f_n^2(v_n) \right| + \lambda_n \left| \varepsilon_n(f_n(v_n), b_n) \right|$$

$$\leq c'' \|\varepsilon_n\| + 2\lambda \|\varepsilon_n\|$$

$$= (c'' + 2\lambda) \|\varepsilon_n\|$$

for all $n \geq 0$ whenever $\overline{v}$ is small enough such that $\lambda_n \leq 2\lambda$.

The rescaling maps $\{\eta_n\}_{n \geq 0}$ can be viewed as a non-autonomous dynamical system (system that depends on time). An orbit is defined as follows.

**Definition 5.11 (Orbit of Non-Autonomous Systems)** Let $Y_n$ be a complete metric space, $X_n \subset Y_n$ be a closed subset, and $f_n : X_n \to Y_{n+1}$ be a continuous map for all $n \geq 1$. A sequence $\{x_n\}_{n=1}^\infty$ is an orbit of the non-autonomous system $\{f_n\}_{n=1}^\infty$ if $x_n \in X_n$ and $x_{n+1} = f_n(x_n)$ for all $n \geq 1$. A sequence $\{x_n\}_{n=1}^\infty$ is an $\varepsilon$-orbit of the non-autonomous system $\{f_n\}_{n=1}^\infty$ if $x_n \in X_n$ and $|x_{n+1} - f_n(x_n)| < \varepsilon$ for all $n \geq 1$.

Next, we state an analog of the shadowing theorem for non-autonomous systems.

**Lemma 5.12 (Shadowing Theorem for Non-Autonomous Systems)** For each $n \geq 1$, let $Y_n$ be a complete metric space equipped with a metric $d$ (the metric depends on $n$), $X_n \subset Y_n$ be a closed subset, and $f_n : X_n \to Y_{n+1}$ be a homeomorphism. Also assume that the non-autonomous system $\{f_n\}_{n=1}^\infty$ has a uniform expansion. That is, there exists a constant $L > 1$ such that $|f_n(a) - f_n(b)| \geq L|a - b|$ for all $a, b \in X_n$ and $n \geq 1$.

If $\{x_n\}_{n=1}^\infty$ is an $\varepsilon$-orbit of $\{f_n\}_{n=1}^\infty$, there exists a unique orbit $\{u_n\}_{n=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $d(x_n, u_n) \leq \frac{\varepsilon}{L - 1}$

for all $n \geq 1$. In addition, if $\{X_n\}_{n=1}^\infty$ is uniformly bounded, then the non-autonomous system $\{f_n\}_{n=1}^\infty$ has exactly one orbit $\{u_n\}_{n=1}^\infty$.

This Lemma is an analog of the Anosov’s Shadowing Theorem. See [10, Exercise 5.1.3, Corollary 5.3.2] for the version of autonomous systems. The proof is left to the reader.

The result from Lemma 5.10 shows that the sequence of critical values $w_n$ is an $\varepsilon$-orbit of the expanding non-autonomous system $\eta_n$. With the help from the Shadowing Theorem, we are able to obtain the goal for this section.
Proposition 5.13 Given $\delta > 0$ and $I^v \supset I^h \supset I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \mathcal{J}_\delta(I^h \times I^v, \varepsilon)$ we have

$$|f_n(v_n) - \pi_x(\tau_n)| < c\|\varepsilon_n\|$$

for all $n \geq 0$.

Proof Fix $n \geq 0$. The critical values $\{w_j\}_{j \geq n}$ form an $\|\varepsilon_n\|$-orbit and the tips $\{\tau_j\}_{j \geq n}$ form an orbit of the perturbed maps $\{\eta_j\}_{j \geq n}$. Also, the perturbed maps are uniform expanding by Lemma 5.9 and $\lambda_n > 1$. Therefore, the proposition follows by Lemma 5.12.

In addition, we can also estimate the distance from the critical point to the preimage of the tip.

Corollary 5.14 Given $\delta > 0$ and $I^v \supset I^h \supset I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \mathcal{J}_\delta(I^h \times I^v, \varepsilon)$ we have

$$|v_n - \pi_x(\tau_n)| < c\sqrt{\|\varepsilon_n\|}$$

for all $n \geq 0$.

Proof The corollary is true because the map $f_n$ behaves like a quadratic map near the critical point by Lemma 5.8.

6 Closest Approach

The proof for the nonexistence of wandering domains begins from this chapter. We assume the contrapositive: there exists a wandering domain $J$.

In this chapter, we construct a rescaled orbit $\{J_n\}_{n=0}^\infty$ of an wandering domain $J$ which is called the $J$-closest approach. Then we define the horizontal size $l_n$, the vertical size $h_n$, and the rescaling level $k_n$ of an element $J_n$.

Recall the definition of wandering domains.

Definition 6.1 (Wandering Domain) Assume that $F \in \mathcal{H}_\delta(I^h \times I^v)$, $D(F)$ exists, and $F$ is an open map (diffeomorphism from $D(F)$ to the image). A nonempty connected open set $J \subset D(F)$ is a wandering domain of $F$ if the orbit $\{F^n(J)\}_{n \geq 0}$ does not intersect the stable manifold of a periodic point.

Remark 6.2 The classical definition of wandering intervals contains one additional condition: the elements of the orbit do not intersect. This condition is redundant for case of the unimodal maps. Assume that $J$ is an nonempty open interval that does not contain points from the basin of a periodic orbit. If the elements in the orbit of $J$ intersect, then take a connected component $A$ of the union of the orbit that contains at least two elements from the orbit. Then, there exists a positive integer $n$ such that $F^n(U) \subset U$. It is easy to show that $f^n$ has a fixed point in the interior of $U$ by applying the Brouwer fixed-point theorem several times which leads to a contradiction. Therefore, the orbit elements of $J$ are disjoint.
The following proposition allow us to generate wandering domains by iteration and rescaling.

**Proposition 6.3** Given \( \delta > 0 \) and \( I' \supset I^h \ni I \). There exists \( \bar{\varepsilon} > 0 \) such that for all open maps \( F \in \mathcal{H}^r_\delta(I^h \times I') \), the following properties hold:

1. A set \( J \subset D(F) \) is a wandering domain of \( F \) if and only if \( F(J) \) is a wandering domain of \( F \).
2. A set \( J \subset C(F) \) is a wandering domain of \( F \) if and only if \( \phi(J) \subset D(RF) \) is a wandering domain of \( RF \).

**Proof** The proposition is true because the stable manifold of a periodic orbit is invariant under iteration and the rescaling of a stable manifold is also a stable manifold. \( \square \)

**Corollary 6.4** Given \( \delta > 0 \) and \( I' \supset I^h \ni I \). There exists \( \bar{\varepsilon} > 0 \) such that for all open maps \( F \in \mathcal{H}^r_\delta(I^h \times I', \bar{\varepsilon}) \), \( F \) has a wandering domain in \( D(F) \) if and only if \( RF \) has a wandering domain in \( D(RF) \).

**Proof** Assume that \( J \subset D(F) \) is a wandering domain. If \( J \subset C \), then \( RF \) has a wandering domain by Proposition 6.3. If \( J \subset A \), there exists \( n \geq 1 \) such that \( F^n(J) \subset B \) by Proposition 4.16. If \( J \subset B \), then \( F(J) \subset C \) by Proposition 4.16. Thus, \( RF \) has a wandering domain by Proposition 6.3.

The converse follows from the second property of Proposition 6.3. \( \square \)

Also, we define the rescaling level of a wandering domain in \( B \).

**Definition 6.5 (Rescaling level)** Assume that \( U \subset A_n \cup B_n \) is a connected set that does not intersect any of the stable manifolds. Define the rescaling level \( k(U) \) as follows. If \( U \subset B_n \), set \( k(U) \) to be the integer such that \( U \subset B_n(k(U)) \); otherwise if \( U \subset A_n \), set \( k(U) = 0 \).

To study the dynamics of a wandering domain, we apply the procedure of renormalization. If a wandering domain is contained in \( A_0 \) or \( B_0 \), then its orbit will eventually leave \( A_0 \) and \( B_0 \) and enter \( C_0 \). If the orbit of the wandering domain enters \( C_0 \), we rescale the orbit element by \( \phi_0, \phi_1, \ldots \) as many times as possible until it lands on one of the sets \( A_n \) or \( B_n \) of some renormalization level \( n \), then study the dynamics of the rescaled orbit by the renormalized map \( F_n \). If the rescaled orbit enters \( C_n \) again, then we rescale it and repeat the same procedure again. By this process, we construct a rescaled orbit as follows.

**Definition 6.6 (Closest approach)** Assume that \( \bar{\varepsilon} > 0 \) is sufficiently small and \( F \in \mathcal{H}^r_\delta(I^h \times I', \bar{\varepsilon}) \).

Given a set \( J \subset A \cup B \) such that it does not intersect any of the stable manifolds. Define a sequence of sets \( \{J_n\}_{n=0}^\infty \) and the associate renormalization levels \( \{r(n)\}_{n=0}^\infty \) by induction such that \( J_n \subset A_{r(n)} \cup B_{r(n)} \) for all \( n \geq 0 \).

1. Set \( J_0 = J \) and \( r(0) = 0 \).
2. Write the rescaling level of \( J_n \) as \( k_n = k(J_n) \) whenever \( J_n \) is defined.
3. If \( J_n \subset A_{r(n)} \), set \( J_{n+1} = F_{r(n)}(J_n) \) and \( r(n+1) = r(n) \).
Fig. 6.1: The construction of a closest approach $J_n$. The graphs are the domains and the partitions of $F_0$ and $F_1$ from the left to the right.

4. If $J_n \subset B_{r(n)}$, set $J_{n+1} = \Phi_{r(n)}^{k_n} \circ F_{r(n)}(J_n)$ and $r(n+1) = r(n) + k_n$.

The transition between two constitutive sequence elements, one iteration together with rescaling (if possible), is called one step. The sequence $\{J_n\}_{n=0}^{\infty}$ is called the rescaled iterations of $J$ that closest approaches the tip, or $J$-closest approach for short.

The itinerary of a closest approach is summarized by the following diagram.

Example 6.7 In this example, we explain the construction of a closest approach and demonstrate the idea of proving the nonexistence of wandering domains. Let $F = (f - \varepsilon, x)$ be a Hénon-like map such that $f(x) = 1.7996565(1 + x)(1 - x) - 1$ and $\varepsilon(x, y) = 0.025y$. The map $F$ is numerically checked to be seven times renormalizable. Given a set $J = (-0.950, -0.947) \times (0.042, 0.045) \subset A$. We show that the set is not a wandering domain by contradiction.

If $J$ is a wandering domain, we construct a $J$-closest approach as shown in Figure 6.1. Set $J_0 = J$ and $r(0) = 0$. The set $J_0$ is contained in $A_{r(0)}$. The next element is defined to be $J_1 = F_{r(0)}(J_0)$ and $r(1) = r(0) = 0$. The set $J_1$ is also contained in $A_{r(1)}$. Set $J_2 = F_{r(1)}(J_1)$ and $r(2) = r(1) = 0$. The set $J_2$ is contained in $B_{r(2)}(1)$. Set $k_2 = 1$, $r(3) = r(2) + k_2 = 1$, and $J_3 = \Phi_{r(2)}^{k_2} \circ F_{r(2)}(J_2) = \phi_0 \circ F_0(J_2)$. The set $J_3$ is contained in $A_{r(3)}$. Set $J_4 = F_{r(3)}(J_3)$ and $r(4) = r(3) = 1$.

From the graph, we see that the sizes of the elements $\{J_n\}$ grow as the procedure continues and the element $J_4 \subset B_1$ becomes so large that it intersects some local stable manifolds. This leads to a contradiction. Therefore, $J$ is not a wandering domain.
Motivated from the example, we study the growth of the horizontal size and prove the sizes of the elements approach to infinity to obtain a contradiction. The size is defined as follow.

**Definition 6.8 (Horizontal and Vertical size)** Assume that $J \subset \mathbb{R}^2$. Define the horizontal size as

$$l(J) = \sup \{|x_1 - x_2|; (x_1, y_1), (x_2, y_2) \in J\} = |\pi_x J|$$

and the vertical size as

$$h(J) = \sup \{|y_1 - y_2|; (x_1, y_1), (x_2, y_2) \in U\} = |\pi_y J|.$$ 

If $J$ is compact, the horizontal endpoints of $J$ are two points in the set that determines $l(J)$.

Figure 10.1 illustrates the horizontal size and the vertical size of a set $J$. For a Hénon-like map $F \in \mathcal{H}_\delta(I^h \times I^v)$, it follows from the definition that

$$h(F(J)) = l(J)$$

for all $J \subset I^h \times I^v$.

For simplicity, we start from a closed subset $J$ of a wandering domain such that $\text{int}(J) = J$. Then consider the $J$-closest approach $\{J_n\}_{n \geq 0}$ instead to ensure the horizontal endpoints exist. Note that the sequence element $J_n$ is also a subset of a wandering domain of $F_{r(n)}$. For elements in a closest approach, set $l_n = l(J_n)$ and $h_n = h(J_n)$.

Our final goal is to show that the horizontal size $l_n$ approaches to infinity and hence wandering domains cannot exist.

### 7 *The Degenerate Case*

In this chapter, we study the relationship between the unimodal renormalization and the Hénon renormalization by identifying a unimodal map as a degenerate Hénon-like map. The main goal is to present a short proof for the nonexistence of wandering intervals for an infinitely renormalizable unimodal map at the end of this chapter. It is well known that a unimodal map (under some regularity condition) does not have wandering interval \([50][51][43][8][45]\). Here, we give a different proof by using the Hénon renormalization instead of the unimodal renormalization. The expansion argument introduced in the proof motivates the proof for the non-degenerate case.

#### 7.1 Local stable manifolds and partition

First, we adopt the notations from unimodal maps and Hénon-like maps. Let $F$ be a degenerate Hénon-like map

$$F(x, y) = (f(x), x).$$

We use the super-scripts “$u$” and “$h$” to distinguish the difference between the notations for unimodal maps and Hénon-like maps to avoid confusion. For example,
$p^μ(-1) = -1$ and $p^μ(0)$ are the fixed points of $f$; $p^h(-1) = (-1, -1)$ and $p^h(0)$ are the saddle fixed points of $F$. $A^u, B^u, C^u \subset I$ is the partition defined for $f$; $A^h, B^h, C^h \subset I^h \times I^v$ is the partition defined for $F$.

The next lemma gives the relations between the local stable manifolds for the degenerate Hénon-like map with the fixed points and their preimages for the unimodal maps. Recall that $p^{(1)}$ and $p^{(2)}$ are the points such that $f(p^{(2)}) = p^{(1)}$, $f(p^{(1)}) = p^μ(0)$, and $p^{(1)} < p^μ(0) < p^{(2)}$ (Definition 3.4); $W^0(-1)$ and $W^2(-1)$ are the local stable manifolds of $p^h(-1)$ (Definition 4.10); $W^0(0), W^1(0), W^2(0)$ are the local stable manifolds of $p^h(0)$ (Definition 4.12).

**Lemma 7.1 (Fixed points and local stable manifolds)** Assume that $F \in \mathcal{H}_D(I^h \times I^v)$ is a degenerate Hénon-like map. Then

1. $p^h(j) = (p^u(j), p^u(j))$ for $j = -1, 0$,
2. the local stable manifold $W^0(j)$ is the vertical line $x = p^u(j)$ for $j = -1, 0$,
3. the local stable manifold $W^2(-1)$ is the vertical line $x = p^μ(-1)$,
4. the local stable manifold $W^1(0)$ is the vertical line $x = p^{(1)}$, and
5. the local stable manifold $W^2(0)$ is the vertical line $x = p^{(2)}$.

It follows from the definition that the partition for unimodal maps and degenerate Hénon-like maps coincide.

**Corollary 7.2 (Partition)** Assume that $F \in \mathcal{H}_D(I^h \times I^v)$ is a degenerate Hénon-like map. Then $A^h = A^u \times I^v$, $B^h = B^u \times I^v$, $C^h = C^u \times I^v$, and $D^h = I \times I^v$.

7.2 Renormalization operator

Next we compare the renormalization operator for Hénon-like maps with the renormalization operator for unimodal maps. Recall the definitions of the rescaling maps. For a degenerate renormalizable Hénon-like map $F$, the rescaling map has the form $\phi = \Lambda \circ H$ where $\Lambda(x, y) = (s^h(x), s^h(y))$, $s^h$ is the affine rescaling map, and $H(x, y) = (f(x), y)$ is the nonlinear rescaling term. The renormalized map is $RF = \phi \circ F^2 \circ \phi^{-1}$. For a renormalizable unimodal map $f$, $s^u$ is the affine rescaling and $Rf = s^u \circ f^2 \circ (s^u)^{-1}$ is the renormalization about the critical point.

Although the Hénon renormalization rescales the first return map around the “critical value”, the operation acts like the unimodal renormalization which rescales the first return map around the “critical point”. This is because of the nonlinear rescaling term $H$ for the Hénon-renormalization. Let $A^h_0, B^h_0, C^h_0$ be the partition for $F$ and $D^h_1$ be the domain for $RF$. The rescaling map $\phi(x, y) = (s^h \circ f(x), s^h(y))$ maps $C^h_0$ to $D^h_1$. This means the operation $f$ in the $x$-component maps $C^h_0$ to $B^h_0$ and the affine map $s^h$ maps $C^h_0$ back to the unit size $I$. Thus, the two affine maps $s^u$ and $s^h$ are the same and

$$H \circ F^2 \circ H^{-1}(x, y) = (f^2 \big|_{B^h_0}(x), x)$$

is the first return map on $B^h_0$. Therefore, the two renormalizations coincide

$$RF(x, y) = (s^u \circ f^2 \circ (s^u)^{-1}(x), x) = (Rf(x), x).$$
This also explains why \( R^nF \) converges to the fixed point \( g \) of the unimodal renormalization operator.

The observation is summarized as follows.

**Lemma 7.3 (Renormalization operator)** Assume that \( F \in H_\delta(I^h \times I^v) \) is a degenerate Hénon-like map. Then \( F \) is Hénon renormalizable if and only if \( f \) is unimodal renormalizable. When the map is renormalizable, we have

1. \( s^h = s^u \) and
2. \( RF(x,y) = (Rf(x),x) \).

In fact, if \( F \) is infinitely renormalizable, then the affine term \( \Lambda_n : B_n(j) \to B_{n+1}(j-1) \) is a bijection for all \( n \geq 0 \) and \( j \geq 1 \) where \( B_n(0) \equiv A_n \cup W^2_n(0) \cup C_n \).

From now on, we remove the super-script from \( s \) because the maps are the same.

For an infinitely renormalizable Hénon-like map, we also adopt the subscript used for the renormalization levels to the degenerate case. Assume that a degenerate Hénon-like map \( F(x,y) = (f(x),x) \) is infinitely renormalizable. Let \( F_n = R^nF \) and \( f_n = R^nf \). Then \( F_n(x,y) = (f_n(x),x) \) by the second property of Lemma 7.3.

Next proposition proves an important equality which will be used to prove the nonexistence of wandering intervals for infinitely renormalizable unimodal maps. The expansion argument comes from this proposition.

**Proposition 7.4 (Rescaling trick)** Assume that \( f \in \mathcal{S} \). Then

\[
(s_{n+j-1} \circ f_{n+j-1}) \circ \cdots \circ (s_n \circ f_n) \circ f_n = f_{n+j} \circ s_{n+j-1} \circ \cdots \circ s_n
\]

for all integers \( n \geq 0 \) and \( j \geq 0 \).

**Proof** Prove by induction on \( j \). It is clear that the equality holds when \( j = 0 \).

Assume that the equality holds for some \( j \). Then

\[
(s_{n+j} \circ f_{n+j}) \circ (s_{n+j-1} \circ f_{n+j-1}) \circ \cdots \circ (s_n \circ f_n) \circ f_n
\]

\[
= (s_{n+j} \circ f_{n+j}) \circ f_{n+j} \circ s_{n+j-1} \circ \cdots \circ s_n
\]

\[
= (s_{n+j} \circ f_{n+j} \circ f_{n+j} \circ s_{n+j-1}^{-1}) \circ s_{n+j} \circ s_{n+j-1} \circ \cdots \circ s_n
\]

\[
= f_{n+j+1} \circ s_{n+j} \circ s_{n+j-1} \circ \cdots \circ s_n.
\]

Therefore, the lemma is proved by induction.

By Lemma 7.3 and Proposition 7.3, we get

**Corollary 7.5** Assume that \( F \in \mathcal{S}_\delta(I^h \times I^v) \) is a degenerate Hénon-like map. Then

\[
\Phi_{n+j}^j \circ F_n = F_{n+j} \circ \Lambda_{n+j-1} \circ \cdots \circ \Lambda_n
\]

for all integers \( n \geq 0 \) and \( j \geq 0 \).
7.3 Nonexistence of wandering intervals

In this section, we present a proof for the nonexistence of wandering intervals for infinitely renormalizable unimodal maps by identifying a unimodal map as a degenerate Hénon-like map and using the Hénon renormalization. A wandering interval is a nonempty interval such that its orbit does not intersect itself and the omega limit set does not contain a periodic point.

**Proposition 7.6** A infinitely renormalizable unimodal map does not have a wandering interval.

**Proof** (Sketch of the proof) Prove by contradiction. Assume that \( f \) is an infinitely renormalizable unimodal map that has a wandering interval \( J^u \). Without loss of generality, we may assume that the map is close to the fixed point \( g \) of the renormalization operator because the sequence of renormalizations \( R^nf \) converges to \( g \) as \( n \) approaches to infinity. Let \( F = (f,x) \). Then \( F \) is a degenerate infinitely renormalizable Hénon-like map. Assume that \( J^u \subset I \) is a wandering interval of \( f_0 \). Let \( J^h = J^u \times \{0\} \) and \( J_n \subset A_{r(n)} \cup B_{r(n)} \) be the \( J^h \)-closest approach. The projection \( \pi_n J_n \) is a wandering interval of \( f_{r(n)} \) and the horizontal size \( l_n \) is the length of the projection.

If \( J_n \subset A_{r(n)} \), then

\[
l_{n+1} > E l_n
\]

for some constant \( E > 1 \) because \( g \) is expanding on \( A(g) \) by Proposition 3.13 and the map \( f_{r(n)} \) is close to \( g \).

If \( J_n \subset B_{r(n)}(k_n) \), then \( J_{n+1} = F_{r(n+1)} \circ A_{r(n)+k_n-1} \circ \cdots \circ A_{r(n)}(J_n) \) by Corollary 7.5. The rescaling maps \( A_{r(n)} \cdots A_{r(n)+k_n-1} \) expands the horizontal size. The map \( F_{r(n+1)} \) also expands the horizontal size because \( A_{r(n)+k_n-1} \circ \cdots \circ A_{r(n)}(J_n) \subset A_{r(n+1)} \cup C_{r(n+1)} \), \( g \) is expanding on \( A(g) \cup C(g) \) by Proposition 3.13 and the map \( f_{r(n+1)} \) is close to \( g \). Thus,

\[
l_{n+1} > E' l_n
\]

for some constant \( E' > 1 \).

This shows that the horizontal size \( l_n \) approaches to infinity which yields a contradiction. Therefore, wandering intervals cannot exist. \( \Box \)

In the proof, we showed that the horizontal size expands at a definite size in each size. This motivates the proof for the non-degenerate case. In the remain part of the article, we will study the growth rate or contraction rate of the horizontal size. In Chapter 9, we will show this is also true for the non-degenerate case under some conditions.

8 The Good Region and the Bad Region

In this chapter, we group the sub-partitions of \( \{B_n(j)\}_{j=1}^\infty \) and \( \{C_n(j)\}_{j=1}^\infty \) into two regions by the following phenomena. Assume that \( \{J_n\}_{n=0}^\infty \) is the \( J \)-closest approach and \( J_n \in B_{r(n)}(k_n) \) for some \( n \).

When \( J_n \) is far from the center, i.e. \( k_n \) is small, the topology of \( B_{r(n)}(k_n) \) and the dynamics of \( F_{r(n)} \) behave like the unimodal case. It can be proved that the boundaries
of $B_n(k_n)$ are vertical graphs of small Lipschitz constant. Also, studying the iteration of horizontal endpoints of a wandering domain provides a good approximation to the expansion rate of the horizontal size. Chapter 9 will show the expansion argument works in this case. This group is called “the good region”.

However, when $J_n$ is close to the center, i.e. $k_n$ is large, the topology of $B_r(n)(k_n)$ and dynamics of $F_r(n)$ is different from the unimodal case. In this group, the two boundary local stable manifolds of $C_r(n)(k_n)$ are so close to the tip $\tau_{r(n)}$ that they only intersect the image $F_r(n)(D_r(n))$ once. Thus, the preimage of a local stable manifold becomes concave and hence $B_r(n)(j)$ becomes an arch-like domain. See the left graph of Figure 6.1 and the next paragraph. Also, the expansion argument fails. The iteration of horizontal endpoints of a wandering domain fails to provide an approximation for the change rate of the horizontal size. In fact, we show that the $x$-coordinate of the two iterated horizontal endpoints can be as close as possible in the next paragraph. This group is called “the bad region”.

The vertical line argument in Figure 8.1 explains why the expansion argument fails in the bad region. The construction is as follows. Draw a vertical line (dashed vertical line in the figure) so close to the tip that its intersection with the image of $F_r(n)$ only has one component. Take the preimage of the intersection. Unlike the case in the good region, the preimage is not a vertical graph. Instead, it is a concave curve that has a $y$-extremal point close to the center of the domain. When a sequence element $J_n$ is in the bad region, it is close to the center. The size of $J_n$ is small because the size of bad region is small and hence $F_r(n)$ acts like a linear map. If the line $\overrightarrow{UV}$ connecting horizontal endpoints $U$ and $V$ of $J_n$ is also parallel to the concave curve as in Figure 8.1a, the image of the horizontal endpoints will also be parallel to the vertical line as Figure 8.1b shows. In this case, the iterated horizontal endpoints forms a vertical line that has no $x$-displacement. Therefore, the horizontal size shrinks and the horizontal endpoints fail to estimate the change of horizontal size when a sequence element enters the bad region.

From the vertical line argument, it becomes crucial to group the sets $\{C_n(j)\}_{j \geq 1}$ by how close the set to the tip is. The size of the image is $\|e_n\|$. To avoid a vertical line intersecting the image only once, the line has to be $\|e_n\|$ away from the tip. This motivates the definition of the boundary sequence $\{K_n\}_{n \geq 0}$, the good region, and the bad region.

**Definition 8.1 (Good and Bad Regions)** Fixed $b > 0$. Assume that $\mathcal{E} > 0$ is sufficiently small so that Proposition 5.5 holds and $F \in \mathcal{F}_0(I^h \times I^h, \mathcal{E})$. For each $n \geq 0$, define $K_n = K_n(b)$ to be the largest positive integer such that

$$|\pi_x z - \pi_x \tau_n| > b \|e_n\|$$

for all $z \in W^0_n(K_n) \cap (I^h \times I^h)$.

The set $C_n(j)$ (resp. $B_n(j)$) is in the good region if $j \leq K_n$; in the bad region if $j > K_n$. The sequence $K_n$ is called the boundary for the good region and the bad region. See Figure 8.2.

**Remark 8.2** It is enough to consider the subdomain $I^h \times I^h \subset I^h \times I^h_n$ in the definition because $F_n(D_n) \subset I^h \times I^h$.
Fig. 8.1: Vertical line argument. The scales in (a) and (b) are chosen to be the same for the reader to compare the change of horizontal size.

Remark 8.3 Here the boundary sequence \( \{K_n\}_{n \geq 0} \) depends on the constant \( b \) and we make \( b \) flexible. In the theorems of this chapter, we will prove that each property holds for all \( b \) that satisfies certain constraints. At the end, we will fix a constant \( b \) sufficiently large that makes all theorems work. So the sequence \( \{K_n\}_{n \geq 0} \) will be fixed in the remaining article.

Remark 8.4 One can see that the bad region is a special feature for the Hénon case. For the degenerate case, \( \varepsilon_n = 0 \) and hence \( K_n = \infty \). This means that there are no bad region for the degenerate case.
Our goal in this chapter is to study the geometric properties for the good region and the bad region. The main theorem is stated as follows.

**Proposition 8.5 (Geometric properties for the good region and the bad region)**

Given \( \delta > 0 \) and \( I' \supset I^h \supset I \). There exists \( \varepsilon > 0, \bar{b} > 0 \), and \( c > 1 \) such that for all \( F \in \mathcal{F}_{\delta}(I^h \times I', \varpi) \) and \( b > \bar{b} \) the following properties hold for all \( n \geq 0 \):

The boundary \( K_n \) is bounded by

\[
\frac{1}{c} \frac{1}{b \| \varepsilon_n \|} \leq \lambda^{K_n} \leq c \frac{1}{\sqrt{b \| \varepsilon_n \|}}. \tag{8.1}
\]

For the good region \( 1 \leq j \leq K_n \), we have

1. \( C_n^j(j) \cap F_n(D_n) = \phi \),
2. \( |\pi_z - \pi_{\tau_n}| > b \| \varepsilon_n \| \) for all \( z \in C_n(j) \cap F_n(D_n) \),
3. \( |\pi_z - v_n| > \frac{1}{c} \sqrt{b \| \varepsilon_n \|} \) for all \( z \in B_n(j) \), and
4. \( \frac{1}{\varepsilon} \left( \frac{\varepsilon}{\lambda} \right)^2 \leq \frac{1}{c} \left( \frac{\varepsilon}{\lambda} \right)^2 \) for all \( z \in C_n(j) \cap F_n(D_n) \).

For the bad region \( j > K_n \), we have
1. \( |\pi_x z - \pi_x \tau_n| < cb \| \varepsilon_n \| \) for all \( z \in C_n(j) \cap F_n(D_n) \) and 
2. \( |\pi_x z - v_n| < c \sqrt{b} \| \varepsilon_n \| \) for all \( z \in B_n(j) \).

This proposition will be proved by the lemmas in this chapter.

First, we estimate the bounds for the boundary \( K_n \).

**Lemma 8.6** Given \( \delta > 0 \) and \( I' \supset I^h \). There exists \( \overline{e} > 0, \overline{b} > 0 \), and \( c > 1 \) such that for all \( F \in \mathcal{F}_\delta(I^h \times I^r, \overline{e}) \) and \( b > \overline{b} \) we have

\[
\frac{1}{c} \frac{1}{\sqrt{b \| \varepsilon_n \|}} \leq \lambda^{K_n} \leq c \frac{1}{\sqrt{b \| \varepsilon_n \|}}
\]

for all \( n \geq 0 \).

**Proof** In the proof, we apply Proposition 5.5 to relate the rescaling level \( K_n \) with the \( x \)-coordinate of the local stable manifold. Assume that \( \overline{e} > 0 \) is small enough.

For the upper bound, by the definition of \( K_n \) and Proposition 5.5, we have

\[
c' \left( \frac{1}{\lambda} \right)^{2K_n} \geq \left| z^{(0)}_n(K_n) - \tau_n \right| \geq b \| \varepsilon_n \|.\]

for some constant \( c' > 1 \). Thus,

\[
\lambda^{K_n} \leq \sqrt{\frac{c'}{b \| \varepsilon_n \|}}.
\]

For the lower bound, by the definition of \( K_n \), there exists \( z \in W^0_n(K_n + 1) \cap I^h \times I^h \) such that \( |\pi_x z - \pi_x \tau_n| \leq b \| \varepsilon_n \| \). Apply Proposition 5.5, we get

\[
\frac{1}{c'} \left( \frac{1}{\lambda} \right)^{2(K_n + 1)} \leq \left| z^{(0)}_n(K_n + 1) - \tau_n \right| \leq |\pi_x z - \pi_x \tau_n| + \left| \pi_x z - \pi_x z^{(0)}_n(K_n + 1) \right| \leq b \| \varepsilon_n \| + c \| I^h \| \| \varepsilon_n \| .
\]

for some constant \( c > 0 \). We solved

\[
\lambda^{K_n} \geq \frac{1}{\lambda} \sqrt{\frac{1}{c'(b + c \| I^h \|) \| \varepsilon_n \|}} \geq \frac{1}{2} \sqrt{b \| \varepsilon_n \|}
\]

when \( b \geq c \| I^h \| \).

\( \square \)
8.1 Properties for the good region

To prove the properties, the strategy is to first estimate the \( x \)-location of the local stable manifolds \( W^s_n(j) \). Since the local stable manifolds bounds \( C_n(j) \), the properties for \( C_n(j) \) follows. Properties for \( B_n(j) \) follows directly by the quadratic estimations from Lemma 5.8 and Proposition 5.13.

**Lemma 8.7** Given \( \delta > 0 \) and \( I' \supset I^h \supset I \). There exists \( \bar{v} > 0, \bar{b} > 0 \), and \( c > 1 \) such that for all \( F \in \hat{F}_\delta(I^h \times I', \bar{v}) \) and \( b > \bar{b} \) we have

\[
\frac{1}{c} \left( \frac{1}{\lambda} \right)^{2j} \leq |\pi_x z - \pi_x \tau_n| \leq c \left( \frac{1}{\lambda} \right)^{2j}
\]

for all \( z \in W^s_n(j) \cap (I^h \times I^h) \) with \( t \in \{0, 2\} \), \( 0 \leq j \leq K_n \), and \( n \geq 0 \).

**Proof** In the proof, we apply Proposition 5.5 to relate the rescaling level \( j \) with the \( x \)-coordinate of the local stable manifold \( W^s_n(j) \). Assume that \( \bar{v} > 0 \) is sufficiently small. We prove the case for \( t = 0 \) and the other case is similar.

Let \( z \in W^0_n(j) \cap (I^h \times I^h) \) and \( b > \bar{b} \) where \( \bar{b} \) is given by Lemma 8.6.

To prove the lower bound, apply Proposition 5.5, we get

\[
|\pi_x z - \pi_x \tau_n| \geq \left| z_n^{(0)}(j) - \tau_n \right| - |\pi_x z - \pi_x z_n^{(0)}(j)| \\
\geq \frac{1}{c} \left( \frac{1}{\lambda} \right)^{2j} - c \| \epsilon_n \| |I^h| \\
\geq \left( \frac{1}{c} - c \| \epsilon_n \| \lambda^{2K_n} \right) \left( \frac{1}{\lambda} \right)^{2j}
\]

for some constant \( c > 1 \). By Lemma 8.6 there exists \( c' > 1 \) such that

\[
|\pi_x z - \pi_x \tau_n| \geq \left( \frac{1}{c} - \frac{c'2 |I^h|}{b} \right) \left( \frac{1}{\lambda} \right)^{2j} \\
\geq \frac{1}{2c} \left( \frac{1}{\lambda} \right)^{2j}.
\]

Here we assume that \( b \geq 2c^2 c'^2 |I^h| \).

Similarly, to prove the upper bound, apply Proposition 5.5 we get

\[
|\pi_x z - \pi_x \tau_n| \leq \left| z_n^{(0)}(j) - \tau_n \right| + |\pi_x z - \pi_x z_n^{(0)}(j)| \\
\leq c \left( \frac{1}{\lambda} \right)^{2j} + c \| \epsilon_n \| |I^h| \\
\leq \left( c + c |I^h| \| \epsilon_n \| \lambda^{2K_n} \right) \left( \frac{1}{\lambda} \right)^{2j}.
\]
By Lemma 8.6 we get
\[ |\pi x z - \pi x \tau| \leq \left( \frac{1}{c} + \frac{cc^2 |f|}{b} \right)^2 \left( \frac{1}{\lambda} \right)^{2j} \leq \frac{3}{2c} \left( \frac{1}{\lambda} \right)^{2j}. \]
\[ \square \]

We prove the first property for the good region.

Lemma 8.8 Given \( \delta > 0 \) and \( I' \supset I^h \supseteq I \). There exists \( \bar{c} > 0 \) and \( b > 0 \) such that for all \( F \in \mathcal{F}_\delta(I^h \times I', \bar{c}) \) and \( b > b \) we have
\[ C^c_r(n) \cap F_n(D_n) = \phi \]
for all \( 1 \leq j \leq K_n \) and \( n \geq 0 \).

Proof Since \( \cup_{j=1}^{K_n} C^c_r(j) \) is bounded by the local manifolds \( W^2_n(K_n) \) and \( W^2_n(0) \), it suffices to prove the local stable manifold \( W^2_n(K_n) \) is far away from the image.

We have
\[ f_n(v_n) - \|\varepsilon_n\| \leq \sup_{z' \in D_n} h_n(z') = \sup_{z' \in D_n} \left( f_n(\pi x z') + \varepsilon_n(z') \right) \leq f_n(v_n) + \|\varepsilon_n\|. \]

Apply Proposition 5.13, Lemma 8.6, and Lemma 8.7 there exists constants \( c > 0 \) and \( a > 1 \) such that
\[ \pi x z - \sup_{z' \in D_n} h_n(z') \geq (\pi x z - \pi x \tau) - |\pi x \tau - f_n(v_n)| - \left| f_n(v_n) - \sup_{z' \in D_n} h_n(z') \right| \]
\[ \geq \frac{1}{a} \left( \frac{1}{\lambda} \right)^{2K_n} - c \|\varepsilon_n\| - \|\varepsilon_n\| \]
\[ \geq \left( \frac{b}{a} - c - 1 \right) \|\varepsilon_n\| \]
for all \( z \in W^2_n(K_n) \cap (I^h \times I^h) \). The coefficient is positive when \( b > 0 \) is large enough. Consequently, \( C^c_r(j) \cap F_n(D_n) = \phi \) for all \( 1 \leq j \leq K_n \). \[ \square \]

By the previous lemma, it is enough to prove the rest of the properties for the left component \( C^c_l(j) \). The second property follows directly from the definition of \( K_n \).

Lemma 8.9 Given \( \delta > 0 \) and \( I' \supset I^h \supseteq I \). There exists \( \bar{c} > 0 \) and \( b > 0 \) such that for all \( F \in \mathcal{F}_\delta(I^h \times I', \bar{c}) \) and \( b > \bar{b} \) we have
\[ |\pi x z - \pi x \tau| > b \|\varepsilon_n\| \]
for all \( z \in C^c_n(j) \cap F_n(D_n) \) with \( 1 \leq j \leq K_n \) and \( n \geq 0 \).
Proof By Lemma 8.8, only the left component $C_{n}(j)$ intersects the image. Also the set $C_{n}(j)$ is bounded by the local stable manifolds $W_{0}^{1}(j-1)$ and $W_{0}^{0}(j)$. Thus, the lemma follows from the definition of $K_{n}$. □

The third property of the good regions follows by the previous proposition and the quadratic estimates from Lemma 5.8.

**Corollary 8.10** Given $\delta > 0$ and $I^{v} \supset I^{h} \ni I$. There exists $\overline{\epsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \mathcal{F}_{\delta}(I^{h} \times I^{v}, \overline{\epsilon})$ and $b > \overline{b}$ so that the following property hold for all $n \geq 0$:

If $z \in I^{B} \times I^{v}$ satisfies $|h_{n}(z) - \pi_{x} \tau_{n}| \geq b \|\epsilon_{n}\|$, then

$$|\pi_{x}z - v_{n}| \geq c \sqrt{b \|\epsilon_{n}\|}.$$ (8.2)

In particular, (8.2) holds for all $z \in B_{n}(j)$ with $1 \leq j \leq K_{n}$.

Proof Assume that $b > 0$ is large enough such that Lemma 8.9 holds. Let $z \in I^{B} \times I^{v}$ be such that $|h_{n}(z) - \pi_{x} \tau_{n}| > b \|\epsilon_{n}\|$. Apply Proposition 5.13, we get

$$|f_{n}(\pi_{x}z) - f_{n}(v_{n})| \geq |h_{n}(z) - \pi_{x} \tau_{n}| - |f_{n}(\pi_{x}z) - h_{n}(z)| - |\pi_{x} \tau_{n} - f_{n}(v_{n})|$$

$$\geq (b - 1 - c) \|\epsilon_{n}\|$$

$$> \frac{b}{2} \|\epsilon_{n}\|$$

for some $c > 0$ when $b > 2(1 + c)$.

Moreover, by the quadratic estimates from Lemma 5.8, there exists $a > 1$ such that

$$|f_{n}(\pi_{x}z) - f_{n}(v_{n})| \leq \frac{a}{2} (\pi_{x}z - v_{n})^{2}$$

for all $n \geq 0$ when $\overline{\epsilon} > 0$ is small enough. Therefore,

$$|\pi_{x}z - v_{n}| \geq \sqrt{\frac{1}{a} \sqrt{b \|\epsilon_{n}\|}}.$$ □

The fourth property for the good region gives an estimate for the $x$-location of $C_{n}(j)$ in terms of the rescaling level $j$. To prove the property, we use the boundary stable manifolds $W_{0}^{1}(j-1)$ and $W_{0}^{0}(j)$ to estimate the $x$-location of $C_{n}(j)$.

**Corollary 8.11** Given $\delta > 0$ and $I^{v} \supset I^{h} \ni I$. There exists $\overline{\epsilon} > 0$, $\overline{b} > 0$, and $c > 1$ such that for all $F \in \mathcal{F}_{\delta}(I^{h} \times I^{v}, \overline{\epsilon})$ and $b > \overline{b}$ we have

$$\frac{1}{c} \left(\frac{1}{\lambda}\right)^{2j} < |\pi_{x}z - \pi_{x} \tau_{n}| < c \left(\frac{1}{\lambda}\right)^{2j}$$

for all $z \in C_{n}(j) \cap F_{n}(D_{n})$ with $1 \leq j \leq K_{n}$ and $n \geq 0$. 
Proof. We use the property that $C_n(j)$ is bounded by $W_n^0(j-1)$ and $W_n^2(j)$ then apply the estimations from the local stable manifolds Lemma 8.7. Assume that $b > 0$ is large enough such that Lemma 8.7 and Lemma 8.8 hold.

For all $z \in C_n(j) \cap F_n(D_n)$ with $1 \leq j \leq K_n$, there exists $z_1 \in W_n^0(j-1) \cap (I^h \times I^h)$ and $z_2 \in W_n^0(j) \cap (I^h \times I^h)$ such that $\pi \tau z = \pi \tau z_1 = \pi \tau z_2$ since the local stable manifolds are vertical graphs. From Lemma 8.7, we obtain

$$\frac{1}{c} \left( \frac{1}{\lambda} \right)^{2j} \leq |\pi \tau z \tau - \pi \tau \tau n| \leq |\pi \tau z - \pi \tau \tau n| \leq |\pi \tau z \tau_1 - \pi \tau \tau n| \leq c \lambda^2 \left( \frac{1}{\lambda} \right)^{2j}.$$

8.2 Properties for the bad region

We prove the first property for the bad region by applying Lemma 8.7 to the boundary local stable manifolds $W_n^0(K_n)$ and $W_n^2(K_n)$.

Lemma 8.12. Given $\delta > 0$ and $I^v \supset I^h \ni I$. There exists $\varepsilon > 0$, $b > 0$, and $c > 0$ such that for all $F \in \hat{J}_\delta(I^h \times I^v, \varepsilon)$ and $b > b$ we have

$$|\pi \tau z \tau - \pi \tau \tau n| < cb \|\varepsilon\|$$

for all $z \in C_n(j) \cap F_n(D_n)$ with $j > K_n$ and $n \geq 0$.

Proof. Since the bad region $\bigcup_{j > K_n} C_n(j)$ is bounded by the local stable manifolds $W_n^0(K_n)$ and $W_n^2(K_n)$, it is sufficient to estimate the location of $W_n^0(K_n)$ and $W_n^2(K_n)$.

Assume that $z \in W_n^t(K_n) \cap (I^h \times I^h)$ with $t \in \{0, 2\}$. By Lemma 8.6 and Lemma 8.7, there exists $c > 1$ such that

$$|\pi \tau z \tau - \pi \tau \tau n| \leq c \left( \frac{1}{\lambda} \right)^{2K_n} \leq c^3 b \|\varepsilon\|$$

for all $b > 0$ sufficiently large.

The second property for the bad region follows from the quadratic estimates Lemma 5.8.

Corollary 8.13. Given $\delta > 0$ and $I^v \supset I^h \ni I$. There exists $\varepsilon > 0$, $b > 0$, and $c > 0$ such that for all $F \in \hat{J}_\delta(I^h \times I^v, \varepsilon)$ and $b > \varepsilon$ we have

$$|\pi \tau z - v_n| < c \sqrt{b} \|\varepsilon\|$$

for all $z \in B_n(j)$ with $j > K_n$ and $n \geq 0$. 
Proof Assume that $z \in B_n(j)$. Then $F_n(z) \in C_n(j) \cap F_n(D_n)$. By Proposition 5.13 and Lemma 8.12 there exists $c > 0$ such that
\[
|f_n(\pi_x z) - f_n(v_n)| \leq |h_n(z) - \pi_x \tau_n| + |f_n(\pi_x z) - h_n(z)| + |\pi_x \tau_n - f_n(v_n)| \\
\leq (cb + 1 + c) \|\varepsilon_n\| \\
< 2cb \|\varepsilon_n\|
\]
for all $b > 0$ sufficiently large. Also, by Lemma 5.8 we have
\[
|f_n(\pi_x z) - f_n(v_n)| \geq \frac{1}{2a} (\pi_x z - v_n)^2
\]
for some constant $a > 0$. Combine the two inequalities, we obtain
\[
|\pi_x z - v_n| \leq \sqrt{4ac} \sqrt{b \|\varepsilon_n\|}.
\]

9 The Good Region and the Expansion Argument

Our goal in this chapter is to prove the expansion argument in the good region. Proposition 9.2, the horizontal size of the closest approach expands when the wandering domains stay in the good region. This shows that Hénon-like maps behaves like unimodal maps in the good region.

From now on, fix $b > 0$ sufficiently large so that Proposition 8.5 holds and the sequence $\{K_n\}_{n \geq 0}$ depends only on $F$.

To prove the horizontal size expands, we iterate the horizontal endpoints to estimate the expansion of the horizontal size. The vertical line argument in Chapter 8 showed that the iteration of the horizontal endpoints fails to approximate the expansion rate when the line connecting the two horizontal endpoints is parallel to the preimage of a vertical line. The following condition, $R$-regular, provides a criteria to ensure “parallel” does not happen in the good region.

Definition 9.1 (Regular) Let $R > 0$. A set $U \subset D(F)$ is $R$-regular if
\[
\frac{h(U)}{l(U)} \leq R \frac{1}{\|\varepsilon\|^{1/4}}. \tag{9.1}
\]

To see $R$-regular implies not parallel, we estimate the slope of the preimage of a vertical line. Assume that $\gamma: I' \to I^n$ is the vertical graph of the preimage of some vertical line $x = x_0$ by the Hénon-like map $F_n$ and the vertical graph is in the good region. Then
\[
h_n(\gamma(y), y) = x_0.
\]

Apply the derivative in terms of $y$ to the both sides, we solved
\[
\gamma'(y) = \frac{\frac{\partial h_n}{\partial y}(\gamma(y), y)}{f'_n(\gamma(y)) - \frac{\partial h_n}{\partial x}(\gamma(y), y)}.
\]
By Lemma 5.8 and Proposition 8.5, we get
\[
\left| f_n'(y) - \frac{\partial \varepsilon_n}{\partial x}(\gamma(y), y) \right| \geq \frac{1}{a} |\gamma(y) - v_n| - \frac{1}{\delta} \|\varepsilon_n\|
\geq \frac{c}{d} \sqrt{\|\varepsilon_n\|} - \frac{1}{\delta} \|\varepsilon_n\|
\geq \frac{c}{2d} \sqrt{\|\varepsilon_n\|}
\]
when $\varepsilon$ is small enough. This yields
\[
|\gamma'(y)| \leq c' \sqrt{\|\varepsilon_n\|} \quad (9.2)
\]
for some constant $c' > 0$.

The condition $R$-regular says that the vertical slope of the line determined by the horizontal endpoints $(x_1, y_1)$ and $(x_2, y_2)$ of $J$ is bounded by
\[
\frac{|x_2 - x_1|}{|y_2 - y_1|} \geq \frac{l(J)}{h(J)} \geq \frac{1}{R} \|\varepsilon_n\|^{1/4}. \quad (9.3)
\]
From (9.2) and (9.3), we get
\[
\frac{|x_2 - x_1|}{|y_2 - y_1|} \gg |\gamma'(y)|.
\]
This concludes that the line connecting the horizontal endpoints is not parallel to the preimage of a vertical line if the wandering domain is $R$-regular.

Now we state the main proposition of this chapter.

**Proposition 9.2 (Expansion argument)** Given $\delta > 0$ and $I' \supset I^h \supseteq I$. There exists $\varepsilon > 0$, $E > 1$, and $R > 0$ such that for all $F \in \mathcal{F}_\delta(I^h \times I', \varepsilon)$ the following property hold:

Assume that $J \subset A \cup B$ is a $R$-regular closed subset of a wandering domain for $F$ and $\{J_n\}_{n=0}^\infty$ is the $J$-closest approach. If $k_n \leq K_{r(n)}$ for all $n \leq m$, then $J_n$ is $R$-regular for all $n \leq m + 1$ and
\[
l_{n+1} \geq El_n \quad (9.4)
\]
for all $n \leq m$.

**Proof** The proof is based on the estimations for the expansion rate developed later in this chapter. The expansion rate will be computed in three different cases:

1. $J_n \subset A_{r(n)}$, proved in Lemma 9.3
2. $J_n \subset B_{r(n)}(k_n)$ for the intermediate region $1 \leq k_n < \overline{K}$ where $\overline{K}$ is some constant, proved in Lemma 9.14
3. $J_n \subset B_{r(n)}(k_n)$ for the region close to the center $\overline{K} \leq k_n \leq K_{r(n)}$, proved in Lemma 9.4
Here we assume the three lemmas to prove this proposition.

Fixed $R > 0$ to be the constant given by Lemma 9.4. Also, fixed $K \geq 1$ to be an integer large enough such that

$$cE_K^2 > 1$$

from (9.7) where $c > 0$ and $E_2 > 1$ are the constants in Lemma 9.4. Let $E_1 > 1$ be the expansion constant in Lemma 9.3 and $E_3 > 1$ be the expansion constant in Lemma 9.14. Set $E = \min(E_1, cE_K^2, E_3) > 1$. Let $\varepsilon > 0$ be small enough such that Proposition 8.5, Lemma 9.3, Lemma 9.4, and Lemma 9.14 hold.

Assume that $F \in \hat{I}^l(I^h \times I^v, \varepsilon)$ and $J \subset A \cup B$ is a closed $R$-regular subset that is a wandering domain of $F$. We prove that $J_n$ is $R$-regular by induction then (9.4) follows by the three lemmas.

For the base case, $J_0$ is $R$-regular by assumption.

If $J_n$ is $R$-regular and $k_n \leq K_{r(n)}$ for some $n \geq 0$. If $J_n \subset A_{r(n)}$, then $J_{n+1}$ is $R$-regular and

$$l_{n+1} \geq E_1 l_n \geq El_n$$

by Lemma 9.3 If $J_n \subset B_{r(n)}(k_n)$ with $1 \leq k_n \leq K$, then $J_{n+1}$ is $R$-regular and

$$l_{n+1} \geq E_3 \lambda^{k_n} l_n \geq El_n$$

by Lemma 9.14 If $J_n \subset B_{r(n)}(k_n)$ with $K \leq k_n \leq K_n$, then $J_{n+1}$ is $R$-regular and

$$l_{n+1} \geq cE_2^{k_n} l_n \geq cE_2^K l_n \geq El_n$$

by Lemma 9.4.

Therefore, the theorem is proved by induction. \hfill \Box

9.1 Case $J_n \subset A_{r(n)}$

In this section, we compute the expansion rate when a wandering domain $J$ lies in $A$. We use the property that $F_n$ is close to the fixed point $G$ then apply the properties for $g$ in Section 3.2 to estimate the expansion.

**Lemma 9.3** Given $\delta > 0$ and $I' \supset I^h \equiv I$. For all $R > 0$, there exists $\varepsilon = \varepsilon(R) > 0$ and $E > 1$ such that for all $F \in \hat{I}^l(I^h \times I^v, \varepsilon)$ the following property hold for all $n \geq 0$:

Assume that $J \subset A_n$ is an $R$-regular closed set. Then $J' \subset C_n(0) = A_n \cup W_n^1(0) \cup B_n$ is $R$-regular and

$$l(J') \geq El(J)$$

where $J' = F_n(J)$.

**Proof** Let $E > 1$ be the constant defined in Lemma 5.9 and $\Delta E > 0$ be small enough such that $E' \equiv E - \Delta E > 1$. Assume that $\varepsilon > 0$ is small enough such that Lemma 5.9 holds and

$$\frac{R}{\delta} \|e_n\|^{3/4} < \Delta E$$

(9.5)
for all \( n \geq 0 \).

To prove the inequality, let \((x_1, y_1), (x_2, y_2) \in J\) such that \(x_2 - x_1 = l(J)\). Then \(h(J) \geq |y_2 - y_1|\).

Compute

\[
l'(J) \geq |\pi_x[F_n(x_2, y_2) - F_n(x_1, y_1)]|
\]

\[
\geq |\pi_x[F_n(x_2, y_2) - F_n(x_1, y_2)]| - |\pi_x[F_n(x_1, y_2) - F_n(x_1, y_1)]|.
\]

By the mean value theorem, there exists \( \xi \in (x_1, x_2) \) and \( \eta \in (y_1, y_2) \) such that

\[
\pi_x[F_n(x_2, y_2) - F_n(x_1, y_2)] = \frac{\partial h_n}{\partial x} (\xi, y_2) (x_2 - x_1)
\]

and

\[
\pi_x[F_n(x_1, y_2) - F_n(x_1, y_1)] = \frac{\partial \epsilon_n}{\partial y} (x_1, \eta) (y_2 - y_1).
\]

Since \((x_1, y_1), (x_2, y_2) \in A_n \subset I^{AC} \times I_0^v\), we have \((\xi, y_2) \in I^{AC} \times I_0^v\). By Lemma 4.26 and Lemma 5.9 we get

\[
l'(J) \geq E l(J) - \frac{1}{\delta} \|\epsilon_n\| h(J)
\]

\[
= \left( E - \frac{1}{\delta} \|\epsilon_n\| \frac{h(J)}{l(J)} \right) l(J).
\]

Also, by \( J \) is \( R \)-regular and (9.5), this yields

\[
l'(J) \geq \left( E - \frac{R}{\delta} \|\epsilon_n\|^{3/4} \right) l(J)
\]

\[
\geq E' l(J). \tag{9.6}
\]

To prove that \( J' \) is \( R \)-regular, we apply (9.6) and \( h(J') = l(J) \). We get

\[
\frac{h(J')}{l(J')} \leq \frac{1}{E'}.
\]

Also assume that \( \overline{\epsilon} \) is small enough such that \( \frac{1}{E'} \leq R \|\epsilon_n\|^{-1/4} \) for all \( n \geq 0 \). This proves that \( J' \) is \( R \)-regular. \( \square \)

### 9.2 Case \( J_n \subset B_{r(n)}(k_n) \), \( \overline{K} \leq k_n \leq K_{r(n)} \)

In this section, we prove the horizontal size expands when a wandering domains is in the center of good region.

Unfortunately, the rescaling trick, Proposition 7.4 does not work in the non-degenerate case. In the degenerate case, the affine rescaling coincide with the rescaling for the unimodal renormalization about the critical point. Thus, the affine rescaling \( \Lambda_n \) maps rescaling levels in \( B_n \) to renormalization levels in \( B_{n+1} \) (Proposition 7.3).
However, in the non-degenerate case, the affine rescaling has no geometrical and dynamical meaning. So the proofs for Proposition 7.4 and Corollary 7.5 do not apply to the non-degenerate case. We have to find another strategy to estimate the expansion rate for the horizontal size.

The idea of the proof is as follows. When the sequence element \( J_n \) enters \( B_{r(n)} \), the step from \( J_n \) to \( J_{n+1} \) contains an iteration \( F_{r(n)} \) and a composition of rescalings \( \Phi_{r(n)}^{k_n} \). On the one hand, the horizontal size contracts by the iteration \( F_{r(n)} \) because the Hénon-like map acts like a quadratic map. Lemma 5.8 says that the contraction becomes strong as \( J_n \) approach to the center. On the other hand, the horizontal size expands by the rescaling \( \Phi_{r(n)}^{k_n} \) (Lemma 5.9). Proposition 8.5 says the number of rescaling \( k_n \) becomes large as \( J_n \) approach to the center. We will show the expansion compensates with the contraction. This yields the following lemma.

**Lemma 9.4** Given \( \delta > 0 \) and \( I' \supset I^0 \supseteq I \). There exists \( \varepsilon > 0 \), \( E > 1 \), \( R > 0 \), and \( c > 0 \) such that for all \( F \in \mathcal{J}_\delta(I^0 \times I', \varepsilon) \) the following property holds for all \( n \geq 0 \):

Assume that \( J \subset B_n(k) \) is an \( R \)-regular closed set and \( 1 \leq k \leq K_n \). Then \( J' \subset C_{n+k}(0) = A_{n+k} \cup W_{n+k}^1(0) \cup B_{n+k} \) is \( R \)-regular and

\[
|I(J')| \geq cE^k|I(J)| \tag{9.7}
\]

where \( J' = \Phi_n^k \circ F_n(J) \).

In the remaining part of this chapter, we will fix \( \bar{K} > 0 \) to be sufficiently large so that (9.7) provides a strict expansion to the horizontal size for all \( k \geq \bar{K} \).

To prove this lemma, we set up the notations. Given a closed set \( J \subset B_n(k) \). Let \( (x_1, y_1), (x_2, y_2) \in J \) be such that \( |I(J)| = |x_2 - x_1| \). Then \( h(J) \geq |y_2 - y_1| \). We define

\[
(x_1^{(j)}, y_1^{(j)}) = \Phi_n^j \circ F_n(x_1, y_1) \quad \text{and} \quad (x_2^{(j)}, y_2^{(j)}) = \Phi_n^j \circ F_n(x_2, y_2)
\]

for \( j = 0, \cdots, k \). Also, let \( x \in \{x_1, x_2\} \) be such that \( |x - v_n| = \min_{i=1,2} |x_i - v_n| \).

We first prove the following estimate.

**Lemma 9.5** Given \( \delta > 0 \) and \( I' \supset I^0 \supseteq I \). For all \( R > 0 \), there exists \( \varepsilon = \varepsilon(R) > 0 \), \( E > 1 \), \( a > 1 \), and \( R' > 0 \) such that for all \( F \in \mathcal{J}_\delta(I^0 \times I', \varepsilon) \) the following property holds for all \( n \geq 0 \):

Assume that \( J \subset B_n(k) \) is an \( R \)-regular closed set and \( k \leq K_n \) then

\[
\left| x_2^{(j)} - x_1^{(j)} \right| \geq \frac{1}{2a} |x - v_n| (\lambda E)^j |I(J)|
\]

and

\[
\left| \frac{y_2^{(j)} - y_1^{(j)}}{x_2^{(j)} - x_1^{(j)}} \right| \leq R' \frac{1}{\sqrt{\|\varepsilon_n\|}}
\]

for \( j = 0, \cdots, k \).

The constants \( E, a, \) and \( R' \) does not depend on \( R \).

**Proof** Let \( \varepsilon > 0 \) be small enough so that Lemma 4.26, Lemma 5.8, and Proposition 8.5 hold. Let \( E' > 1 \) be the constant defined in Lemma 5.9 and \( E \) be a constant such that \( E' > E > 1 \). We prove the lemma by induction on \( j \).
For the case $j = 0$, we have
\[
\left| x_2^{(0)} - x_1^{(0)} \right| = |\pi_x (F_n(x_2,y_2) - F_n(x_1,y_1))|
\geq |\pi_x (F_n(x_2,y_2) - F_n(x_1,y_1))| - |\pi_x (F_n(x_1,y_2) - F_n(x_1,y_1))| \quad (9.8)
\]

Apply the mean value theorem, there exists $\xi \in (x_1,x_2)$ and $\eta \in (y_1,y_2)$ such that
\[
\pi_x (F_n(x_2,y_2) - F_n(x_1,y_2)) = \left[ f'_n(\xi) - \frac{\partial e_n}{\partial x}(\xi,y_2) \right] (x_2-x_1) \quad (9.9)
\]

and
\[
\pi_x (F_n(x_1,y_2) - F_n(x_1,y_1)) = -\frac{\partial e_n}{\partial y}(x_1,\eta)(y_2-y_1). \quad (9.10)
\]

Then $\xi \in \mathcal{I}^B$ since $(x_1,y_1), (x_2,y_2) \in \mathcal{B}_n \subset \mathcal{I}^B \times \mathcal{I}^B$. By Lemma 5.8 (9.9) yields
\[
|\pi_x (F_n(x_2,y_2) - F_n(x_1,y_2))| \geq \left( |f'_n(\xi)| - |\frac{\partial e_n}{\partial x}(\xi,y_2)| \right) l(J)
\geq \left( \frac{1}{a} |x - v_n| - \frac{1}{\delta} \|e_n\| \right) l(J). \quad (9.11)
\]

Also, since $J$ is $R$-regular, (9.10) yields
\[
\left| \pi_x (F_n(x_1,y_2) - F_n(x_1,y_1)) \right| \leq \frac{1}{\delta} \|e_n\| h(J) \leq \frac{R}{\delta} \|e_n\|^{3/4} l(J). \quad (9.12)
\]

Combine (9.8), (9.11), and (9.12), we get
\[
\left| x_2^{(0)} - x_1^{(0)} \right| \geq \left( \frac{1}{a} |x - v_n| - \frac{1}{\delta} \left( \|e_n\|^{1/2} + R \|e_n\|^{1/4} \right) \sqrt{\|e_n\|} \right) l(J)
\]

By Proposition 8.5 $c |x - v_n| > \sqrt{\|e_n\|}$ for some constant $c > 1$. Also, assume that $\bar{e} = \bar{e}(R)$ is small enough such that $\frac{R}{\delta} \left( \|e_n\|^{1/2} + R \|e_n\|^{1/4} \right) < \frac{1}{2a}$ for all $n \geq 0$. We obtain
\[
\left| x_2^{(0)} - x_1^{(0)} \right| \geq \frac{1}{2a} |x - v_n| l(J).
\]

Moreover, by applying $y_2^{(0)} - y_1^{(0)} = l(J)$ and the previous inequality, we have
\[
\left| \frac{y_2^{(0)} - y_1^{(0)}}{x_2^{(0)} - x_1^{(0)}} \right| \leq \frac{2a}{|x - v_n|}.
\]

By Proposition 8.5 again, we get
\[
\left| \frac{y_2^{(0)} - y_1^{(0)}}{x_2^{(0)} - x_1^{(0)}} \right| \leq R' \frac{1}{\sqrt{\|e_n\|}}
\]

where $R' = 2ac$. 

Assume that the two inequalities are true for $j \leq k$. We prove the inequalities for $j + 1 \leq k$. We have $(x_1^{(j)}, y_1^{(j)}) \in C_{n+j}(k-j)$. By the mean value theorem, there exists $\xi_j \in (x_1^{(j)}, x_2^{(j)}) \subset I^C$ and $\eta_j \in (y_1^{(j)}, y_2^{(j)})$ such that

$$\pi_x \left( \phi_{n+j}(x_2^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) \right) = -\lambda_{n+j} \frac{\partial h_n}{\partial x}(\xi_j, y_2^{(j)}) \left( x_2^{(j)} - x_1^{(j)} \right)$$

(9.13)

and

$$\pi_x \left( \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_1^{(j)}) \right) = \lambda_{n+j} \frac{\partial \epsilon_{n+j}}{\partial y}(x_1^{(j)}, \eta_j) \left( y_2^{(j)} - y_1^{(j)} \right)$$

(9.14)

Apply Lemma 5.9 to (9.13), we have

$$\left| \pi_x \left( \phi_{n+j}(x_2^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) \right) \right| \geq \lambda_{n+j} E' \left| x_2^{(j)} - x_1^{(j)} \right|$$

(9.15)

for some constant $E' > 1$. Also apply the induction hypothesis to (9.14), we have

$$\left| \pi_x \left( \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_1^{(j)}) \right) \right| \leq \frac{\lambda_{n+j}}{\delta} \left\| \epsilon_{n+j} \right\| \left| y_2^{(j)} - y_1^{(j)} \right|$$

$$\leq \frac{\lambda_{n+j} R'}{\delta} \sqrt{\left\| \epsilon_n \right\|} \left| x_2^{(j)} - x_1^{(j)} \right|$$

(9.16)

By the triangular inequality, (9.15), and (9.16), we get

$$\left| x_2^{(j+1)} - x_1^{(j+1)} \right| \leq \pi_x \left( \phi_{n+j}(x_2^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) \right) - \pi_x \left( \phi_{n+j}(x_1^{(j)}, y_2^{(j)}) - \phi_{n+j}(x_1^{(j)}, y_1^{(j)}) \right)$$

$$\leq \lambda_{n+j} \left( E' - \frac{R'}{\delta} \sqrt{\left\| \epsilon_n \right\|} \right) \left| x_2^{(j)} - x_1^{(j)} \right|$$

(9.17)

Assume that $\bar{e}$ is sufficiently small such that $\lambda_{n+j} \left( E' - \frac{R'}{\delta} \sqrt{\left\| \epsilon_n \right\|} \right) > \lambda E$ for all $n \geq 0$ and $j \geq 0$ since $E' > E > 1$ and $|\lambda_n - \lambda| < \bar{e}$. Apply the induction hypothesis to (9.17), we get

$$\left| x_2^{(j+1)} - x_1^{(j+1)} \right| \geq \lambda E \left| x_2^{(j)} - x_1^{(j)} \right| \geq \frac{1}{2a} |x - v_n| (\lambda E)^{j+1} l(J).$$

Moreover, by applying $y_2^{(j+1)} - y_1^{(j+1)} = \lambda_{n+j} \left( y_2^{(j)} - y_1^{(j)} \right)$, (9.17), and the induction hypothesis, we get

$$\left| y_2^{(j+1)} - y_1^{(j+1)} \right| \leq \frac{1}{E' - \frac{R'}{\delta} \sqrt{\left\| \epsilon_n \right\|}} \left| y_2^{(j)} - y_1^{(j)} \right| \leq \frac{R'}{\delta} \sqrt{\left\| \epsilon_n \right\|} \left| x_2^{(j)} - x_1^{(j)} \right|$$

since $E' - \frac{R'}{\delta} \sqrt{\left\| \epsilon_n \right\|} > 1$ when $\bar{e}$ is small enough.

Therefore, the two inequalities are proved by induction. \square
We also need a lemma to relate the rescaling level \( k_n \) with the distance from the wandering domain \( J_n \) to the center.

**Lemma 9.6** Given \( \delta > 0 \) and \( I' \supset I^h \supset I \). There exists \( \bar{c} > 0 \) and \( c > 0 \) such that for all \( F \in \mathcal{F}_\delta(I^h \times I', \bar{c}) \) the following property holds for all \( n \geq 0 \):

If \((x, y) \in B_n(k)\) with \( k \leq K_n \) then \( k \) is bounded below by

\[
\lambda^k > \frac{c}{|x - v_n|}.
\]

**Proof** Let \( \bar{c} \) be sufficiently small. Apply Proposition 8.5 and triangular inequality, we have

\[
\frac{1}{c} \left( \frac{1}{\lambda} \right)^{2k} < |h_n(x, y) - \pi_x \tau_n| \leq |f_n(x) - f_n(v_n)| + |f_n(v_n) - \pi_x \tau_n| + |h_n(x, y) - f_n(x)|
\]

for some constant \( c > 0 \) since \( F_n(x, y) \in C_n(k) \). By Lemma 5.8, Lemma 5.13, and Proposition 8.5, we get

\[
\frac{1}{c} \left( \frac{1}{\lambda} \right)^{2k} \leq \frac{a}{2} (x - v_n)^2 + (c' + 1) \|\varepsilon_n\|
\]

\[
\leq \left[ \frac{a}{2} + (c' + 1)c' \right] (x - v_n)^2
\]

for some constants \( a > 0 \) and \( c' > 0 \). Therefore, the lemma is proved. \( \square \)

Now we are ready to prove the main lemma of this section.

**Proof** (Proof of Lemma 9.4) By Lemma 9.5, we have

\[
|J'| \geq \left| x_2^{(k)} - x_1^{(k)} \right| \geq \frac{1}{2a} |x - v_n| (\lambda E)^k l(J).
\]

for some constant \( a > 0 \). Apply Lemma 9.6, we get

\[
l(J') \geq \frac{c}{2a} E^k l(J)
\]

for some constant \( c > 0 \). This proves (9.7).

It remains to show that \( J' \) is \( R \)-regular. By Proposition 4.21, we have

\[
\|\varepsilon_{n+k}\| \leq \|\varepsilon_{n+1}\| \leq c \|\varepsilon_n\|^2
\]

for some constant \( c > 0 \) since \( k \geq 1 \). Also, the Hénon-like map \( F_n \) maps the \( x \)-th coordinate to \( y \)-th coordinate and \( \Phi_n^k \) rescales the \( y \)-th coordinate affinely, we have \( h(J') = |y_2^{(k)} - y_1^{(k)}| \). By Lemma 9.5, we get

\[
\frac{h(J')}{l(J')} \leq R' \frac{1}{\sqrt{\|\varepsilon_n\|}} \leq R' c \frac{1}{\|\varepsilon_{n+k}\|^{1/4}}.
\]

Set \( R = R'c \). Then \( J' \) is \( R \)-regular. Here, we fixed \( R \) so \( \bar{c} \) is a constant. \( \square \)
9.3 Case $J_n \subset B_{r(n)}(k_n)$, $1 \leq k_n < K$

In Lemma 9.4, the expansion of horizontal size only works for levels $k \geq K$ which are close to the center. In this section, we prove the expansion argument also holds in the intermediate region $1 \leq k < K$.

Although the rescaling trick does not apply to the non-degenerate case, we still can apply it to the limiting degenerate Hénon-like map $G$. In the limiting case, the horizontal size expands by the rescaling trick. Because $K$ is a fixed number, we will show the horizontal size also expands in the intermediate region of a non-degenerate Hénon-like map when it is close enough to the limiting function $G$.

Observe in the limiting case, we have

$$\lim_{n \to \infty} F_n(x, y) = (g(x), x),$$

and

$$\lim_{n \to \infty} \phi_n(x, y) = (-\lambda)(g(x), y).$$

Then

$$\lim_{n \to \infty} \Phi^j_n(x, y) = \left(\left(-\lambda\right)g\right)^j(x) = \left(\left(-\lambda\right)g\right)^j((x, y))$$

and

$$\lim_{n \to \infty} \Phi^j_n \circ F_n(x, y) = \left(\left(-\lambda\right)g\right)^j \circ g(x) = \left(\left(-\lambda\right)g\right)^j((x, y))$$

where $\left[\left(-\lambda\right)g\right]^j$ means the function $x \to (-\lambda)g(x)$ is composed $j$ times.

The following is the rescaling trick, Lemma 7.4, for the limiting case.

**Lemma 9.7 (Rescaling trick)** Assume that $j \geq 0$ is an integer. Then

$$\left(\left(-\lambda\right)g\right)^j \circ g(x) = g((\left(-\lambda\right)g)^j(x))$$

(9.19)

for all $-(\frac{1}{\lambda})^j \leq x \leq (\frac{1}{\lambda})^j$.

**Proof** The lemma follows either from the functional equation 3.1 or Proposition 7.4.

By the rescaling trick, we are able to estimate the derivative for the limiting case as follows.

**Lemma 9.8** There exists universal constants $E, E' > 1$ such that for all integer $j \geq 0$ we have

$$E\lambda^j \leq \left| \frac{d}{dx} \left[\left(-\lambda\right)g\right]^j \circ g(x) \right| \leq E'\lambda^j$$

(9.20)

for all and $\left(\frac{1}{\lambda}\right)^{j+1} \leq |x| \leq \left(\frac{1}{\lambda}\right)^j$. 

Lemma 9.9

Given $\Phi_{9.11}$, allow us to apply the function $\Phi_{9.10}$ on Hénon-like maps that are close enough to the fixed point $G$. By the rescaling trick and chain rule, we get

$$\frac{d}{dx}((-\lambda)g)^j \circ g(x) = (-\lambda)^j g'((-\lambda)^j x)$$

for all $|x| \leq (\frac{1}{\lambda})^j$. By Proposition 3.13, there exists $E > 1$ such that

$$|g'(x)| \geq E$$

for all $\frac{1}{\lambda} \leq |x| \leq 1$. Also, by compactness, there exists $E' > 0$ such that

$$|g'(x)| \leq E'$$

for all $x \in I$. This yields (9.20) since $\frac{1}{\lambda} \leq |(-\lambda)^j x| \leq 1$ for all $(\frac{1}{\lambda})^{j+1} \leq |x| \leq (\frac{1}{\lambda})^j$. □

Next, we need to do some hard work to make these expansion estimates also work on Hénon-like maps that are close enough to the fixed point $G$.

One of the difficulty is the subpartitions on $B$ and $C$ are not rectangular in the non-degenerate case. The following lemmas, Lemma 9.9, Lemma 9.10, and Corollary 9.11 allow us to apply the function $\Phi_{9.10}$ on a rectangular neighborhood of $B_n(j)$.

Lemma 9.9

Given $\delta > 0$ and $I' \supseteq I^h \supseteq I$. For all $d > 0$, there exists $\bar{e} = \bar{e}(d) > 0$ and $d' = d'(d)$ such that for all $F \in \mathcal{F}_{\delta}^r(I^h \times I, \bar{e})$ we have

$$s_n \circ h_n([a - d', b + d'], y) \subset [(\lambda)g(a) - d, (\lambda)g(b) + d]$$

for all $[a - d', b + d'] \subset I'^{AC,r}$, $y \in I'_n$, and $n \geq 0$ where $I'^{AC,r}$ is the right component of $I'^{AC}$ defined in Lemma 5.9.

Proof

By the compactness of $I^h$, there exists $E \geq 1$ such that $|g'(x)| \leq E$ for all $x \in I^h$. Then

$$s_n \circ h_n(a - d') \geq (-\lambda)g(a - d') - |(-\lambda)g(a - d') - (-\lambda)h_n(a - d')| - |(-\lambda)h_n(a - d') - s_n \circ h_n(a - d')|$$

$$\geq (-\lambda)g(a) - \lambda Ed' - \lambda\|F_n - G\| - \|s_n(x) - (-\lambda)x\|_{I^h}$$

$$\geq (-\lambda)g(a) - d'$$

when $d'$ is small enough such that $\lambda Ed' < \frac{d}{2}$ and $\bar{e}$ is small enough such that $\lambda\|F_n - G\| + \|s_n(x) - (-\lambda)x\|_{I^h} < \frac{d}{2}$ for all $n \geq 0$.

Similarly,

$$s_n \circ h_n(b + d') \leq (-\lambda)g(b + d') + |(-\lambda)g(b + d') - (-\lambda)h_n(b + d')| + |(-\lambda)h_n(b + d') - s_n \circ h_n(b + d')|$$

$$\leq (-\lambda)g(b) + \lambda Ed' + \lambda\|F_n - G\| + \|s_n(x) - (-\lambda)x\|_{I^h}$$

$$\leq (-\lambda)g(b) + d.$$

Therefore, the lemma is proved. □
Given $\delta > 0$ and $I^r \supset I^h \equiv I$. For all integer $j \geq 1$, there exists $\varepsilon(j) > 0$ and $d_C(j) > 0$ such that for all $F \in J_\delta^b(I^r \times I^r, \varepsilon)$ the rescaling $\Phi^j_n$ is defined on $[q(j-1) - d_C(j), q(j) + d_C(j)] \times I^r_n$ for all $n \geq 0$ and $[q(0) - d_C(1), q(1) + d_C(1)] \subset I^{AC}$.

In addition,
\[
\phi_n([q(k-1) - d_C(k), q(k) + d_C(k)] \times I^r_n) \subset [q(k-2) - d_C(k-1), q(k-1) + d_C(k-1)] \times I^r_{n+1}
\]
for all $F \in J_\delta^b(I^r \times I^r, \varepsilon(j))$, $n \geq 0$, and $2 \leq k \leq j$.

Proof: We prove by induction on $j \geq 1$.

For the base case $j = 1$, $\Phi^1_0 = \phi_0$ is defined on $I^{AC,F} \times I^r_0$ by Lemma 9.10 where $I^{AC,F}$ is the right component of $I^{AC}$.

Assume that there exists $d > 0$ such that $\Phi^j_t$ is defined on $[q(j-1) - d, q(j) + d]$ for all $n \geq 0$.

For the case $j + 1$, we know that $\lambda g(q(j)) = q(j-1)$ and $\lambda g(q(j+1)) = q(j)$. By Lemma 9.9, there exists $d' > 0$ and $\varepsilon(j+1) \geq \varepsilon(j)$ such that
\[
s_n \circ h_n([q(j) - d', q(j) + d'), y] \subset [q(j-1) - d, q(j) + d]
\]
for all $F \in J_\delta^b(I^r \times I^r, \varepsilon(j+1))$, $y \in I^r_n$, and $n \geq 0$. Then $\pi_t \circ \phi_n([q(j) - d', q(j) + d'] \times I^r_n) \subset [q(j-1) - d, q(j) + d] \times I^r_{n+1}$. Therefore, $\Phi^j_{n+1} = \Phi^j_{n+1} \circ \phi_n$ is defined on $[q(j) - d', q(j+1) + d'] \times I^r_n$ by the induction hypothesis.

The relation (9.21) follows from the definition of $d(j)$.

Recall $q^r(j) = -|q^f(j)|$ and $q^l(j) = |q^f(j)|$ from Definition 9.8.

Given $\delta > 0$ and $I^r \supset I^h \equiv I$. For all integer $j \geq 1$, there exists $\varepsilon(j) > 0$, $d_C(j) > 0$, $E > 1$, and $E' > 1$ such that for all $F \in J_\delta^b(I^r \times I^r, \varepsilon)$ the following properties hold for all $n \geq 0$:

1. $F_n([q^l(j-1) - d_B^l(j), q^l(j) + d_B^l(j)] \times I^r_n) \subset [q(j-1) - d_C(j), q(j) + d_C(j)] \times I^r_n$ and $F_n([q^l(j-1) - d_B^l(j), q^l(j) + d_B^l(j)] \times I^r_n) \subset [q(j-1) - d_C(j), q(j) + d_C(j)] \times I^r_n$.

That is, $\Phi^j_n \circ F_n$ is defined on $[q^l(j-1) - d_B^l(j), q^l(j) + d_B^l(j)] \cup [q^r(j-1) - d_B^r(j), q^r(j)] \times I^r_n$.

2. $B^l_n(j) \subset [q^l(j-1) - d_B^l(j), q^l(j) + d_B^l(j)] \times I^r_n$ and $B^r_n(j) \subset [q^r(j-1) - d_B^r(j), q^r(j)] \times I^r_n$.

Here $B^l_n(j)$ and $B^r_n(j)$ are the left and right components of $B_n(j)$ respectively.

Proof: Fixed $j \geq 1$.

There exists $d' > 0$ and $\varepsilon > 0$ small enough such that $\Phi^j_n$ is defined on $[q(j-1) - d', q(j) + d'] \times I^r_n$ for all $n \geq 0$. By the continuity of $g$, there exists $d > 0$ such that $g([q(j-1) - d, q(j) + d]) \subset [q(j-1) - d', q(j) + d']$. Then
\[
h_n(q^l(j-1) - d, y) \geq g(q^l(j-1) - d) - ||h_n - g||_{\pi_0(\delta)} \times I^r_0(\delta)
\]
\[
\geq q(j-1) - d'
\]
for all \( n \geq 0 \). Here, we assume that \( \overline{\varepsilon} \) is small enough such that \( \|h_n - g\|_{p(h) \times I_n(\delta)} < \frac{d'}{2} \). Similarly,

\[
h_n(q^j(j) + d, y) \geq g(q^j(j) + d) - \|h_n - g\|_{p(h) \times I_n(\delta)} \geq q(j) - d'.
\]

Thus, \( F_n([q^j(j - 1) - d, q^j(j) + d] \times I_n) \subset [q(j - 1) - d', q(j) + d'] \times I_n \). This proves that \( \Phi^j_n \circ F_n \) is defined on \( (q^j(j - 1) - d, q^j(j) + d) \times I_n \).

Similarly, we can choose \( d > 0 \) to be small enough such that \( F_n([q^j(j - 1) - d, q^j(j) + d] \times I_n) \subset [q(j - 1) - d', q(j) + d'] \times I_n \). Consequently, the first property is proved.

By Proposition 5.6, we may also assume that \( \overline{\varepsilon} = \overline{\varepsilon}(j) \) is small enough such that \( W^j_n(j - 1) \subset [q^j(j - 1) - d, q^j(j - 1) + d] \times I_n \), \( W^j_n(j) \subset [q^j(j) - d, q^j(j) + d] \times I_n \), and \( W^j_n(j - 1) \subset [q^j(j - 1) - d, q^j(j - 1) + d] \times I_n \). This proves the second property.

\( \square \)

The next lemma will show that the expansion of a Hénon-like map is close to the limiting case when \( \overline{\varepsilon} \) is small.

**Lemma 9.12** Given \( \delta > 0 \) and \( I^v \supset I^h \supset I \). For all \( \hat{\varepsilon} > 0 \) and integer \( j \geq 1 \), there exists \( \overline{\varepsilon} = \overline{\varepsilon}(\hat{\varepsilon}, j) > 0 \) such that

\[
\left| \frac{\partial \pi_x \circ \Phi^j_n \circ F_n(x, y)}{\partial x} \right| \frac{d [(-\lambda)g]^{j} \circ g(x)}{dx} < \hat{\varepsilon}
\]

(9.22)

for all \( F \in \mathcal{S}_\delta(I^h \times I^v, \overline{\varepsilon}) \), \( n \geq 0 \), and \( (x, y) \in ([q^j(j - 1) - d^B(j), q^j(j) + d^B(j)] \cup [q^j(j) - d^B(j), q^j(j - 1) + d^B(j)]) \times I_n^j \).

**Proof** The Lemma is true because \( j \) is fixed and the Hénon-like maps \( F_n \) are close to the fixed point \( G \) when \( \overline{\varepsilon} \) is small. The proof is left as an exercise to the reader. \( \square \)

**Corollary 9.13** Given \( \delta > 0 \) and \( I^v \supset I^h \supset I \). For all integer \( j \geq 1 \), there exists \( \overline{\varepsilon}(j) > 0 \), \( d^B(j) \), \( E > 1 \), and \( E' > 1 \) such that for all \( F \in \mathcal{S}_\delta(I^h \times I^v, \overline{\varepsilon}) \) the following properties hold for all \( n \geq 0 \):

\[
E \lambda^j \leq \left| \frac{\partial \pi_x \circ \Phi^j_n \circ F_n(x, y)}{\partial x} \right| \leq E' \lambda^j
\]

for all \( x \in [q^j(j - 1) - d^B(j), q^j(j) + d^B(j)] \cup [q^j(j) - d^B(j), q^j(j - 1) + d^B(j)] \) and \( y \in I_n^j \).

**Proof** By the continuity of \( g \) and (9.20), we may assume that \( \hat{d}^B(j) < d^B(j) \) is small enough and \( E' > E > 1 \) such that

\[
E \lambda^j \leq \left| \frac{d [(-\lambda)g]^{j} \circ g(x)}{dx} \right| \leq E' \lambda^j
\]

for all \( \left( \frac{1}{\lambda} \right)^{j+1} - \hat{d}^B(j) \leq |x| \leq \left( \frac{1}{\lambda} \right)^j + \hat{d}^B(j) \).
By Lemma 9.12 we get
\[ \sqrt{E} \lambda^j \leq \left| \frac{\partial \pi_\varepsilon \circ \Phi^j_n \circ F_n}{\partial x}(x,y) \right| \leq \sqrt{E} \lambda^j \]
for all \((\frac{1}{\lambda})^{j+1} - \delta^B(j) \leq |x| \leq (\frac{1}{\lambda})^j + \delta^B(j), y \in I^*_n\), and \(n \geq 0\) when \(\varepsilon\) is small enough. \(\square\)

By applying the estimates from the limiting case, we are able to estimate the expanding rate for the intermediate case as follows.

**Lemma 9.14** Given \(\delta > 0\) and \(I^\nu \supset I^h \supset I\). For all \(\overline{K} > 0\) and \(R > 0\), there exists \(\varepsilon = \varepsilon(\overline{K}, R) > 0\) and \(E > 1\) such that for all \(F \in \mathcal{J}_\delta(I^h \times I^\nu, \varepsilon)\) the following properties hold for all \(n \geq 0\):

Assume that \(J \subset B_n(k)\) is a connected closed \(R\)-regular set and \(k \leq \min(\overline{K}, K_n)\). Then \(J' \subset C_{n+k}(0) = A_{n+k} \cup W_{n+k}^1(0) \cup B_{n+k}\) is \(R\)-regular and
\[ l(J') \geq E \lambda^k l(J) \] (9.23)

where \(J' = \Phi^k_n \circ F_n(J)\).

**Proof** Since \(\overline{K}\) is fixed, we may assume that \(\varepsilon = \varepsilon(\overline{K}) > 0\) is sufficiently small such that the properties in Corollary 9.13 hold for all \(j \leq \overline{K}\).

Given a connected closed \(R\)-regular set \(J \subset B_n(k)\) with \(k \leq \min(\overline{K}, K_n)\) and \(n \geq 0\). For convenience, let \(G = \Phi^k_n \circ F_n\) and denote \(G_\varepsilon = \pi_\varepsilon \circ G\). We have \(J' = G(J)\).

To prove (9.23), assume the case that \(J \subset B_n^I(k)\). The other case \(J \subset B_n^O(k)\) is similar. Let \((x_1, y_1), (x_2, y_2) \in J\) such that \(l(J) = x_2 - x_1\). From Corollary 9.13, \(G\) is defined on \([q^j(k-1) - d(k), q^j(k) + d(k)] \times I^\nu_n\) and \(x_1, x_2 \in [q^j(k-1) - d(k), q^j(k) + d(k)]\). We can apply the mean value theorem. There exists \(\xi \in (x_1, x_2)\) and \(\eta \in (y_1, y_2)\) such that
\[ G_\varepsilon(x_2, y_2) - G_\varepsilon(x_1, y_2) = \frac{\partial G_\varepsilon}{\partial x}(\xi, y_2)(x_2 - x_1) \]
and
\[ G_\varepsilon(x_1, y_2) - G_\varepsilon(x_1, y_1) = \frac{\partial G_\varepsilon}{\partial y}(x_1, \eta)(y_2 - y_1). \]

By triangular inequality and \(J\) is \(R\)-regular, we get
\[ l(J') \geq |G_\varepsilon(x_2, y_2) - G_\varepsilon(x_1, y_2)| - |G_\varepsilon(x_1, y_2) - G_\varepsilon(x_1, y_1)| \]
(9.24)
\[ \geq \left| \frac{\partial G_\varepsilon}{\partial x}(\xi, y_2) \right| l(J) - \left| \frac{\partial G_\varepsilon}{\partial y}(x_1, \eta) \right| h(J) \]
\[ \geq \left| \frac{\partial G_\varepsilon}{\partial x}(\xi, y_2) \right| - \left| \frac{\partial G_\varepsilon}{\partial y}(x_1, \eta) \right| R \|e_n\|^{-1/4} \right) l(J). \]

The first term \(\frac{\partial G_\varepsilon}{\partial x}(\xi, y_2)\) can be bounded by Corollary 9.13. That is
\[ E \lambda^k < \left| \frac{\partial G_\varepsilon}{\partial x}(\xi, y_2) \right| < E' \lambda^k \] (9.25)
for some constants $E' > E > 1$.

To bound the second term $\frac{\partial G_x}{\partial y}(x_1, \eta)$, compute by the chain rule and Corollary 9.13 we get

$$E' \lambda^k > \left| \frac{\partial G_x}{\partial x}(x_1, \eta) \right| \geq \left| \frac{\partial \pi \circ \Phi_n^k \circ F_n(x_1, \eta)}{\partial x} \frac{\partial h_n}{\partial x}(x_1, \eta) \right|$$

$$\geq \left| \frac{\partial \pi \circ \Phi_n^k \circ F_n(x_1, \eta)}{\partial y} \frac{\partial h_n}{\partial y}(x_1, \eta) \right|$$

and

$$\left| \frac{\partial G_x}{\partial y}(x_1, \eta) \right| = \left| \frac{\partial \pi \circ \Phi_n^k \circ F_n(x_1, \eta)}{\partial x} \frac{\partial h_n}{\partial x}(x_1, \eta) \right|$$

$$\leq (E' \lambda^k + E' \lambda^k) \left( \left| \frac{\partial \pi \circ \Phi_n^k \circ F_n(x_1, \eta)}{\partial y} \frac{\partial h_n}{\partial y}(x_1, \eta) \right| \right) \left( \left| f_n'(x_1) \right| - \frac{1}{\delta} \| e_n \| \right)^{-1} \frac{1}{\delta} \| e_n \|.$$

Assume that $\bar{e} = \bar{e}(K)$ is small enough such that $\left| \frac{\partial \pi \circ \Phi_n^k \circ F_n(x, y)}{\partial y} \right| < E'$ for all $k \leq K$ and $n \geq 0$. This is possible because of Proposition 9.18, the definition of $J'$, and $K$ is a fixed bounded number. Apply Lemma 5.8, we get

$$\left| \frac{\partial G_x}{\partial y}(x_1, \eta) \right| \leq 2E' \lambda^k \left( \frac{1}{a} |x_1 - v_n| - \frac{1}{\delta} \| e_n \| \right)^{-1} \frac{1}{\delta} \| e_n \|.$$

for some constant $a > 1$. Also, by Proposition 8.5, we obtain

$$\left| \frac{\partial G_x}{\partial y}(x_1, \eta) \right| \leq 2E' \lambda^k \left( \frac{c}{a} \| e_n \|^{1/2} - \frac{1}{\delta} \| e_n \| \right)^{-1} \frac{1}{\delta} \| e_n \| \leq \frac{4E' a}{\delta c} \lambda^k \| e_n \|^{1/2}$$

(9.26)

for some constant $c > 0$ when $\bar{e}$ is small enough.

Combine (9.24), (9.25), and (9.26), we obtain

$$l(J') \geq \left( E \lambda^k - \frac{4E' ac}{\delta} \lambda^k \| e_n \|^{1/2} R \| e_n \|^{-1/4} \right) l(J)$$

(9.27)

when $\bar{e}$ is small enough.

To prove that $J'$ is $R$-regular, we apply (9.27) and $h(J') = \left( \prod_{j=0}^{k(J)-1} \lambda_{j+n} \right) l(J)$. Assume that $\bar{e} = \bar{e}(K)$ is small enough such that $\prod_{j=0}^{k(J)} \lambda_{j+n} \leq 2\lambda^j$ for all $1 \leq i \leq K$ and $n \geq 0$. Thus,

$$\frac{h(J')}{l(J')} \leq \frac{2\lambda^k l(J)}{\sqrt{E} \lambda^k l(J)} = \frac{2}{\sqrt{E}} \leq R \| e_{n+k} \|^{-1/4}$$

when $\bar{e} = \bar{e}(R)$ is small enough. □
10 The Bad Region and the Thickness

In the good region, we showed the expansion argument holds by studying the iteration of the horizontal endpoints. However, in the bad region, the iteration of horizontal endpoints fails to estimate the change rate of the horizontal size. In fact, the \( x \)-displacement of the endpoints can shrink as small as possible by the vertical line argument in Chapter 8. Even worse, the case of entering the bad region is unavoidable. The next lemma shows that an infinitely renormalizable Hénon-like map must have a wandering domain in the bad region if it has any wandering domain.

**Lemma 10.1** *Given \( \delta > 0 \) and \( I' \supset I^h \supset I \). There exists \( \varepsilon > 0 \) such that for all non-degenerate Hénon-like maps \( F \in \hat{\mathcal{H}}_\delta(I^h \times I', \varepsilon) \) the following property holds. If \( F \) has a wandering domain in \( D \) then \( F \) has a wandering domain in the bad region of \( B \) and a wandering domain in the bad region of \( C \).*

**Proof** Recall \( K_0 \) is the boundary of the good and bad region for \( F_0 = F \). See Definition 8.1. Let \( j > K_0 \).

If \( F \) has a wandering domain in \( D \), then \( F_j \) also has a wandering domain \( J' \) in \( D_j \) by Corollary 6.4. By iterating the wandering domain, we can assume without lose of generality that \( J' \subset F_j(D_j) \). Set \( J_C = \left( \Phi_0^j \right)^{-1}(J') \). Then \( J_C \subset C \) is a wandering domain of \( F \).

Moreover, since \( J' \subset F_j(D_j) \), we have \( J_C \subset \left( \Phi_0^j \right)^{-1}(F_j(D_j)) \subset F(D) \). Let \( J_B = F^{-1}(J_C) \). Then \( J_B \subset B \) is a wandering domain of \( F \) in the bad region. \( \square \)

The case of entering the bad region becomes the main difficulty for proving the nonexistence of wandering domain.

In this chapter, we will first introduce a new quantity “thickness” that approximates the horizontal or vertical cross-section of a wandering domain. When a sequence element \( J_n \) enters the bad region, we showed by the vertical line argument in Chapter 8 that the next sequence element \( J_{n+1} \) can turn so vertical that the iteration of horizontal endpoints fails to estimate the horizontal size of \( J_{n+1} \). However, the horizontal size \( l_{n+1} \) is not zero because \( J_{n+1} \) has area as shown in Figure 8.1c. This is because the Hénon-like map is non-degenerated. The Jacobian is not zero. Thus, thickness (horizontal cross-section) provides an approximation for the horizontal size \( l_{n+1} \).

Before giving a precise definition for the thickness and rigorous computation for its change rate, here we present a lax estimation on the thickness by using an area argument to explain the relations between the thickness, horizontal size, and vertical size in a closest approach.

Assume that we start from a square subset \( J_0 \) of a wandering domain. Let \( \{J_n\}_{n=0}^\infty \) be the \( J \)-closest approach, \( a_n \) be the area of \( J_n \), and \( w_n \) be the thickness of \( J_n \). Assume that \( J_0, J_1, \ldots, J_{n-1} \) stays in the good region and \( J_n \) enters the bad region. Since \( J_0 \) is a square, we have \( l_0 = h_0 \).

Some assumptions are made here to simplify the argument. The contribution from the rescaling is neglected. When the wandering domain is \( R \)-regular, assume that the horizontal size is comparable to the vertical size, i.e. \( l_n \sim h_n \). Also, assume that the...
thickness $w_n$ is determined by the horizontal cross-section which can be approximated by $w_n \sim \frac{a_n}{n}w_n$.

In the good region, the estimations in Chapter 9 determines the relation of the horizontal size. Proposition 9.2 says that $l_{m+1} \sim El_m$ for all $m \leq n - 1$ where $E > 1$ is a constant.

However, for the wandering domain $J_{n+1}$, the horizontal size $l_n$ fails to estimate $l_{n+1}$ because $J_n$ enters the bad region. We need to use the thickness to approximate the horizontal size. That is, $l_{n+1} \sim w_{n+1}$. The only known relation between the horizontal size and the thickness prior entering the bad region is $w_0 = l_0$. This is because $J_0$ is a square. To relate $l_{n+1}$ with $l_0$, we need to go back to study the change rate of the thickness in each step.

We use the area to study the change rate of thickness. In each step, the change rate of the area is determined by the Jacobian $\text{Jac} \phi_0 \sim \|\varepsilon_0\|$ of the Hénon-like map, i.e. $a_{m+1} \sim \|\varepsilon_{r(m)}\| a_n$. We get

$$w_{m+1} \sim \frac{a_{m+1}}{h_{m+1}} \sim \|\varepsilon_{r(m)}\| \frac{a_m}{l_m} \sim \|\varepsilon_{r(m)}\| \frac{a_m}{h_m} \sim \|\varepsilon_{r(m)}\| w_m.$$ 

This allows us to relate $l_{n+1}$ with $l_0$ by

$$l_{n+1} \sim w_{n+1} \sim \left( \prod_{m=0}^{n} \|\varepsilon_{r(m)}\| \right) w_0 \sim \left( \prod_{m=0}^{n} \|\varepsilon_{r(m)}\| \right) l_0.$$ 

Consequently, the horizontal size becomes extremely small when the $J$-closest approach leaves the bad region.

One can see two problems from the estimations.

One problem is the wandering domain $J_{n+1}$ fail to be $R$-regular after the $J$-closest approach leaves the bad region. This is because

$$\frac{h_{n+1}}{l_{n+1}} \sim \frac{l_n}{l_{n+1}} \sim \frac{E^n l_0}{\left( \prod_{m=0}^{n} \|\varepsilon_{r(m)}\| \right) l_0} = E^n \left( \prod_{m=0}^{n} \|\varepsilon_{r(m)}\| \right)^{-1}. $$

Thus, the expansion argument does not work for the later sequence element even if the $J$-closest approach does not enter the bad region again. This problem will be resolved by introducing the largest square subset.

Another problem is the strong contraction of horizontal size when the wandering domain enters the bad region. We will show that this strong contraction happens every time when the $J$-closest approach enters the bad region. If the $J$-closest approach enters the bad region infinitely many time, then the horizontal size may fails to tend to infinity because this strong contraction happens infinity many times. This problem will be resolved in Section 10.3 by proving the closest approach $J_n$ can only enter the bad region at most finitely many times.

Finally, after combining all of the ingredients in this article together, we will show wandering domains do not exist in Section 10.4.
10.1 Thickness and largest square subset

When the wandering domain \( J_n \) enters the bad region, there are two issues that stop us from proceeding. First, the horizontal size of \( J_{n+1} \) cannot be estimated by the expansion argument in Chapter 9. Instead, it is determined by its horizontal cross-section that is not comparable to the horizontal size of \( J_n \). Second, \( J_{n+1} \) fail to be \( R \)-regular. The estimations for the expansion rate of the horizontal size in Proposition 9.2 does not apply to the later steps \( J_{n+1} \to J_{n+2} \to \cdots \) in the sequence.

To resolve the two issues, we need the following:

1. A quantity to approximate the horizontal cross-section of a wandering domain, called the thickness.
2. Keep track of the thickness in each step of the closest approach. This will provide the information for the horizontal size when the wandering domain enters the bad region.
3. A method to select a subset from the wandering domain \( J_{n+1} \) that makes the subset to be \( R \)-regular and has approximately the same horizontal size as \( J_{n+1} \). The subset will be defined to be a largest square subset of \( J_{n+1} \).

In this section, we define thickness and largest square subset then study the properties of these two objects in a closest approach.

First, define

**Definition 10.2 (Square, Largest square subset, and Thickness)** A set \( I \subset \mathbb{R}^2 \) is a square if \( I = [x_1, x_2] \times [y_1, y_2] \) with \( x_2 - x_1 = y_2 - y_1 \). This means that \( I \) is a closed square with horizontal and vertical sides.

Assume that \( J \subset \mathbb{R}^2 \). Define the thickness of \( J \) to be the quantity \( w(J) = \sup \{ l(I) \} \) where the supremum is taken over all square subsets \( I \subset J \).

A subset \( I \subset J \) is a largest square subset of \( J \) if \( I \) is a square such that \( l(I) = w(J) \).

The definition is illustrated as in Figure 10.1.

**Lemma 10.3** A largest square subset of a compact set exists.

**Proof** The lemma follows from compactness. \( \square \)

To keep track of the thickness in each step, the following two lemmas estimate the change rate of a square under iteration and rescaling.

**Lemma 10.4** Given \( \delta > 0 \) and \( I' \supset I \), there exists \( \varepsilon > 0 \) and \( c > 0 \) such that for all \( F \in \hat{\mathcal{H}}_\delta(I' \times I', \varepsilon) \) the following property holds for all \( n \geq 0 \):

If \( I \subset D_n \) is a square, there exists a square \( I' \subset F_n(I) \) such that

\[
|I'| \geq \frac{c\|\varepsilon_n\|}{|I_n|} l(I).
\]

**Proof** The lemma is trivial when \( F \) is degenerate. We assume that \( F \) is non-degenerate.

By the definition of \( \hat{\mathcal{H}} \) and (4.2) we have \( \frac{\partial F}{\partial y} > 0 \) for all \( n \geq 0 \). Write \( I = [x, a] \times [y_1, y_2] \). Fixed \( b > 0 \) to be sufficiently small. Let \( (x_1, x) = F_n(x, y_2), (x_2, x) = \)
Fig. 10.1: Comparison of the horizontal size $l$, the vertical size $h$, and the thickness $w$ for $J$. In this picture, $I$ is a largest square subset of $J$.

Fig. 10.2: Four points on the cross section $y = t$.

$F_n(x, y_1)$, and $W = b(x'_2 - x'_1) = b [\varepsilon_n(x, y_2) - \varepsilon_n(x, y_1)] > 0$. Define $x' = \frac{x'_1 + x'_2}{2}$ and $I' = \left[ x' - \frac{1}{2}W, x' + \frac{1}{2}W \right] \times [x, x + W]$.

To prove that $I' \subset F_n(I)$ for some $b > 0$ sufficiently small, it suffice to prove the inequality

$$h_n(t, y_2) < x' - \frac{1}{2}W < x' + \frac{1}{2}W < h_n(t, y_1) \quad (10.1)$$

that corresponds to the four points on a horizontal cross section at $y = t$ for $x \leq t \leq x + W$. See Figure 10.2. If this is true, then by the mean value theorem, there exists $\eta \in (y_1, y_2)$ such that

$$l(I') = W = b \frac{\partial \varepsilon_n}{\partial y}(x, \eta) l(I).$$

Also, by (4.2), we obtain

$$l(I') \geq \frac{bc}{\|\varepsilon_n\|} l(I)$$

since $F \in \hat{\mathcal{G}}(I^h \times I^v, \varepsilon)$ which proves the lemma.
First, we prove the left inequality of (10.1)
\[ h_n(t, y_2) < x' - \frac{1}{2} W. \]

By the mean value theorem and the compactness of the domain, there exists \( \xi \in (x, t) \) and \( E > 1 \) such that
\[
|h_n(t, y_2) - x'_1| = |h_n(t, y_2) - h_n(x, y_2)| = \left| \frac{\partial h_n}{\partial x}(\xi, y_2) \right| |t - x| \leq EW.
\]

We get
\[
\left( x' - \frac{1}{2} W \right) - h_n(t, y_2) = \left[ \left( x' - \frac{1}{2} W \right) - x'_1 \right] - [h_n(t, y_2) - x'_1] \\
\geq \left( \frac{x'_2 - x'_1}{2} - \frac{1}{2} W \right) - EW \\
= \left[ \frac{1}{2} - \left( \frac{1}{2} + E \right) b \right] (x'_2 - x'_1) \\
> 0
\]
when \( b < \frac{1}{1+2E} \). Note that \( b \) can chosen to be universal. Thus, the left inequality is proved.

Similarly, we prove the right inequality of (10.1)
\[ x' + \frac{1}{2} W < h_n(t, y_1). \]

By the mean value theorem, there exists \( \xi \in (x, t) \) such that
\[
|h_n(t, y_1) - x'_2| = |h_n(t, y_1) - h_n(x, y_1)| = \left| \frac{\partial h_n}{\partial x}(\xi, y_1) \right| |t - x| \leq EW.
\]

Similarly, we get
\[
h_n(t, y_1) - \left( x' + \frac{1}{2} W \right) = \left[ x'_2 - \left( x' + \frac{1}{2} W \right) \right] - [x'_2 - h_n(t, y_1)] \\
\geq \left( \frac{x'_2 - x'_1}{2} - \frac{1}{2} W \right) - EW \\
= \left[ \frac{1}{2} - \left( \frac{1}{2} + E \right) b \right] (x'_2 - x'_1) \\
> 0.
\]

Thus, the right inequality is proved.

\[\square\]

**Lemma 10.5** Given \( \delta > 0 \) and \( I' \supset I^h \ni I \). There exists \( \epsilon > 0 \) such that for all \( F \in \mathcal{S}_\delta(I^h \times I', \epsilon) \) the following property holds for all \( n \geq 0 \):

If \( I \subset C_n \) is a square, there exists a square \( I' \subset \phi_n(I) \) such that
\[ l(I') = \lambda_n l(I). \]
Proof Let $I = [x_1, x_2] \times [y_1, y_2]$, $W = l(I)$, $x = \frac{1}{2} [h_n(x_2, y_1) + h_n(x_1, y_1)]$, and $I'' = [x - \frac{1}{2} W, x + \frac{1}{2} W] \times [y_1, y_2]$. Then $I''$ is a square with $l(I'') = l(I)$.

First we prove that $I'' \subset H_n(I)$. It suffice to prove the inequality

$$h_n(x_2, t) < x - \frac{1}{2} W < x + \frac{1}{2} W < h_n(x_1, t)$$

that corresponds to the four points on a horizontal cross section at $y = t$ for $y_1 \leq t \leq y_2$. See Figure 10.3.

To prove the left inequality, by the mean value theorem, there exists $\xi \in (x_1, x_2)$ and $\eta \in (y_1, t)$ such that

$$h_n(x_1, y_1) - h_n(x_2, y_1) = \left| \frac{\partial h_n}{\partial x}(\xi, y_1) \right| (x_2 - x_1)$$

and

$$\varepsilon_n(x_2, t) - \varepsilon_n(x_2, y_1) = \frac{\partial \varepsilon_n}{\partial y}(x_2, \eta) (t - y_1).$$

By Lemma 5.9 there exists $E > 1$ such that

$$\left( x - \frac{1}{2} W \right) - h_n(x_2, t) = [x - h_n(x_2, y_1)] - [\varepsilon_n(x_2, y_1) - \varepsilon_n(x_2, t)] - \frac{1}{2} W$$

$$\geq \frac{1}{2} \left| \frac{\partial h_n}{\partial x}(\xi, y_1) \right| (x_2 - x_1) - \left| \frac{\partial \varepsilon_n}{\partial y}(x_2, \eta) \right| (t - y_1) - \frac{1}{2} W$$

$$\geq \left( \frac{E}{2} - \frac{1}{\delta} \| \varepsilon_n \| - \frac{1}{2} \right) W$$

$$> 0$$
when $\varepsilon > 0$ is sufficiently small. Thus, the left inequality is proved.

Similarly, to prove the right inequality, by the mean value theorem, there exists $\eta' \in (y_1, t)$ such that

$$\varepsilon_n(x_1, t) - \varepsilon_n(x_1, y_1) = \frac{\partial \varepsilon_n}{\partial y}(x_1, \eta')(t - y_1).$$

Compute

$$h_n(x_1, t) - \left(x + \frac{1}{2}W\right) = [h_n(x_1, y_1) - x] - [\varepsilon_n(x_1, t) - \varepsilon_n(x_1, y_1)] - \frac{1}{2}W$$

$$\geq \frac{1}{2} \left|\frac{\partial h_n}{\partial x}(\xi, y_1)\right| (x_2 - x_1) - \left|\frac{\partial \varepsilon_n}{\partial y}(x_1, \eta')(t - y_1) - \frac{1}{2}W\right|$$

$$\geq \left(\frac{E}{2} - \frac{1}{\delta} \|\varepsilon_n\| - \frac{1}{2}\right)W$$

$$> 0.$$  

Thus, the right inequality is proved.

Finally, let $I' = A_n(I'')$. Then $I' \subset \phi_h(I)$ and

$$l(I') = \lambda_n l(I'') = \lambda_n l(I).$$

$\square$

As before we abbreviate $w_n = w(J_n)$ for a closest approach $\{J_n\}_{n=0}^{\infty}$. The next proposition allows us to estimate the contraction rate of the thickness for a closest approach.

**Proposition 10.6** Given $\delta > 0$ and $I' \supset I$ and $I'' \supset I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \hat{\mathcal{F}}_\delta(I'' \times I', \varepsilon)$ the following property holds:

Assume that $J \subset A \cup B$ is a compact subset of a wandering domain of $F$ and $\{J_n\}_{n=0}^{\infty}$ is the $J$-closest approach. Then

$$w_{n+1} \geq c \left\|\frac{\varepsilon_r(n)}{I'_r(n)}\right\| w_n$$

for all $n \geq 0$.

**Proof** Let $\varepsilon > 0$ be small enough such that Lemma 10.4 and Lemma 10.5 holds. The sets $\{J_n\}_{n=0}^{\infty}$ are compact by the continuity of Hénon-like maps and rescaling.

For the case that $J_n \subset \hat{A}_{r(n)}$, let $I$ be a largest square of $J_n$. By Proposition 10.4, there exists a square $I' \subset F_{r(n)}(I) \subset J_{n+1}$ such that

$$l(I') \geq c \left\|\frac{\varepsilon_r(n)}{I'_r(n)}\right\| l(I).$$

We get

$$w_{n+1} \geq l(I') \geq c \left\|\frac{\varepsilon_r(n)}{I'_r(n)}\right\| l(I) \geq c \left\|\frac{\varepsilon_r(n)}{I'_r(n)}\right\| w_n.$$
For the case that $J_n \subset B_r(n)$, let $I$ be a largest square of $J_n$. By Proposition 10.4, there exists a square $I_0 \subset F_r(n)(I) \subset F_r(n)(J_n)$ such that

$$l(I_0) \geq c \frac{\|e_r(n)\|}{l'(r(n))} l(I).$$

Also by Proposition 10.5, there exists a square $I_{j+1} \subset \Phi_{r(n)+j}(I_j) \subset \Phi_{r(n)}(J_n)$ such that

$$l(I_{j+1}) = \lambda_{r(n)+j} l(I_j)$$

for all $0 \leq j < k_n$. We get

$$w_{n+1} \geq l(I_{k_n}) = \left( \prod_{j=0}^{k_n-1} \lambda_{r(n)+j} \right) l(I_0) \geq c \frac{\|e_r(n)\|}{l'(r(n))} l(I) = c \frac{\|e_r(n)\|}{l'(r(n))} w_n.$$

\[\square\]

**Remark 10.7** The original proof was based on the area and horizontal cross-section estimates briefly mentioned in the beginning of this chapter instead of tracking the size of largest square subset. However, the area argument is discarded by two reasons. First, to estimate the horizontal cross-section of a set, we need to find the lower bound of $a/l$. This means that we need to repeat the arguments in Chapter 9 to find the upper bound for $l$ and the lower bound for $a$. This makes the argument several times longer than the current one. Second, to select a subset from the wandering domain after it enters the bad region, the area approach makes it hard to find the upper bound of $l$ for the subset.

Since $\|e_n\|$ decreases super-exponentially and $|I_n^w|$ increases exponentially, we can simplify

**Corollary 10.8** Given $\delta > 0$ and $I^v \supset I^h \supset I$. There exists $\varepsilon > 0$ and $c > 0$ such that for all $F \in \mathcal{S}_{\delta}(I^h \times I^v, \varepsilon)$ the following property holds:

Assume that $J \subset A \cup B$ is a compact subset of a wandering domain of $F$ and $\{J_n\}_{n=0}^\infty$ is the $J$-closest approach. Then

$$w_{n+1} \geq c \left\| e_r(n) \right\|^3 2 w_n$$

for all $n \geq 0$.

10.2 Double sequence

Next, we study the number of times that a closest approach enters the bad region by defining a double sequence (two-dimensional sequence/sequence of two indices) of sets. The double sequence consists of rows. Each row is a closest approach in the sense of Definition 6.6. When the sequence first enters the bad region in a row, the horizontal size of the next step is dominated by its thickness. Add a new row by selecting a largest square subset then generate the closest approach starting from the
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**Definition 10.9 (Double sequence, Row, and Time span in the good regions)** Given \( \delta > 0 \) and \( I' \supset I' \supset I \). Assume that \( \bar{v} > 0 \) be sufficiently small so that Proposition 8.5 holds and \( F \in \hat{F}_\delta(I' \times I', v) \) is a non-degenerate open map.

Given a square subset \( J \subset A \cup B \) of a wandering domain for \( F \). Define \( \{J_n^{(j)}\}_{n \geq 0} \), \( \{J_n^{(0)}\}_{n \geq 0} \), \( \{J_n^{(3)}\}_{n \geq 0} \), \( \{J_n^{(1)}\}_{n \geq 0} \), and \( \{J_n^{(2)}\}_{n \geq 0} \) for some \( \bar{v} \in \mathbb{N} \cup \{0, \infty\} \). by induction on \( j \) such that the following properties hold.

1. For \( j = 0 \), set \( J_0^{(0)} = J \) and \( F_0^{(0)} = F \).
2. The super-script \( j \) is called row. The initial set \( J_0^{(j)} \) for each row \( j \) is a square in \( A(F_0^{(j)}) \cup B(F_0^{(j)}) \).
3. Each row \( j \) is a \( J_0^{(j)} \)-closest approach. Precisely, if \( J_0^{(j)} \) and \( F_0^{(j)} \) are defined, set \( F_0^{(j)} = R^0 F_0^{(j)} \) and \( K_0^{(j)} \) be the boundary of good and bad region for \( F_0^{(j)} \). Let \( \{J_n^{(j)}\}_{n \geq 0} \) and \( \{J_n^{(3)}(n)\}_{n \geq 0} \) be the \( J_0^{(j)} \)-closest approach. See Definition 6.6 and Definition 8.1.
4. For a row \( j \), if there exists some \( n \geq 0 \) such that \( k_n^{(j)} > K_n^{(j)} \), set \( n^{(j)} \) to be the smallest integer with this property. The set \( J_n^{(j)}(n) \) is the first set in row \( j \) that enters the bad regions. The nonnegative integer \( n^{(j)} \) is called the time span in the good regions for row \( j \). Otherwise, if the row never enters the bad region, set \( n^{(j)} = \infty \) and \( \bar{v} = j = j \) and the construction stops.

---

\[\begin{array}{c|c|c}
\text{Square} & \text{Good} & \text{Bad} \\
\hline
J_0^{(3)} & \ldots & J_0^{(1)} \\
\vdots & \\
J_0^{(j-1)} & \ldots & J_0^{(j-1)} \\
\hline
J_0^{(0)} & \ldots & J_0^{(3)} \\
\hline
\end{array}\]

Fig. 10.4: Construction of a double sequence.

subset. Thus, each row in the double sequence corresponds to enter the bad region once.

The precise definition of the double sequence is as follows. Figure 10.4 illustrates the construction.
5. If \( n^{(j)} < \infty \), construct a new row \( j + 1 \) by defining \( J^{(j+1)}_0 \) to be a largest square subset of \( J^{(j)}_{n^{(j)+1}} \) and set \( F^{(j+1)}_0 = F^{(j)}_{r^{(j)}(n^{(j)+1})} \).

6. If the procedure never stop, i.e. enters the bad region infinitely many times, set \( \bar{j} = \infty \).

The two dimensional sequence \( \{ J^{(j)}_n \}_{n \geq 0} \) is called a double sequence generated by \( J \) or a \( J \)-double sequence. The integer \( \bar{j} \) is the number of rows for the double sequence (enters the bad region \( \bar{j} \) times).

To be consistent and avoid confusion, the superscript is assigned for the row and the subscript is assigned for the renormalization level or the index of sequence element in the closest approach. For example, abbreviate \( A^{(j)}_n = A(F^{(j)}_n) \), \( B^{(j)}_n = B(F^{(j)}_n) \), \( C^{(j)}_n = C(F^{(j)}_n) \), \( D^{(j)}_n = D(F^{(j)}_n) \), \( l^{(j)}_n = l(J^{(j)}_n) \), \( h^{(j)}_n = h(J^{(j)}_n) \), \( w^{(j)}_n = w(J^{(j)}_n) \), and \( k^{(j)}_n = k(J^{(j)}_n) \) as before.

In the following, we abbreviate \( r^{(j)}(n) = r(n) \) when the context is clear, for example \( F^{(j)}_{r^{(j)}(n^{(j)+1})} = F^{(j)}_{r^{(j)}(n^{(j)+1})} \). Also, write \( \varepsilon^{(j)} = \varepsilon^{(j)}_{r^{(j)}(n^{(j)})} \), \( K^{(j)} = K^{(j)}_{r^{(j)}(n^{(j)})} \), and \( k^{(j)} = k^{(j)}_{n^{(j)}} \).

For convenience, let \( m^{(j)} = n^{(j)} + 1 \).

**Example 10.10** Figure 10.5 gives an example of constructing a double sequence.

In this example, we choose the same Hénon-like map as in Example 6.7. Select an initial square set \( J^{(0)}_0 = [-0.6642, -0.6632] \times [0.3200, 0.321] \subset A^{(0)}_0 \) and set \( r^{(0)}(0) = 0 \).

By the construction of the closest approach, \( J^{(0)}_1 = F^{(0)}_{r^{(0)}(J^{(0)}_0)} \) and \( r^{(0)}(1) = r^{(0)}(0) = 0 \). From the figure, we see that \( J^{(0)}_1 \subset B_{r^{(0)}(1)} \). In this example, \( \varepsilon \) is chosen to be so large that \( C^{(0)}_1 \) intersects the image \( F_0(D_0) \). Thus \( K^{(0)}_{r^{(1)}} = K^{(0)}_0 = 0 \) and \( J^{(0)}_1 \) lies in the bad region. Set \( n^{(0)} = 1 \).

By the construction, \( J^{(0)}_{n^{(0)}+1} = \Phi^{(0)}_{r^{(n^{(0)})}} \circ F^{(0)}_{r^{(n^{(0)})}}(J^{(0)}_{n^{(0)})}) \). The double sequence in this example is chosen in purpose to demonstrate the set \( J^{(0)}_{n^{(0)}+1} \) turns so vertical that the thickness dominates the horizontal size as in Figure 10.5d. Select a largest square subset \( J^{(1)}_0 \) from \( J^{(0)}_{n^{(0)}+1} \) as in Figure 10.5e. Set \( F^{(1)}_0 = F^{(0)}_{n^{(0)}+1} \).

The procedure is repeated until the sequence does not enter the bad region again.

Next, we study the relation between horizontal size and thickness in a double sequence. For each row \( j \), the first set \( J^{(j)}_0 \) is a square so \( l^{(j)}_0 = w^{(j)}_0 \). When the row stays in the good region \( (n < n^{(j)}_n) \), the next horizontal size can be estimated by expansion argument \( l^{(j)}_{n^{(j)+1}} \geq E l^{(j)}_0 \) (Proposition 10.2). When the row first enters the bad region \( n = n^{(j)}_n \), the expansion argument fails. The vertical line argument in Chapter 8 shows that the only way to estimate the horizontal size \( l^{(j)}_{n^{(j)+1}} \) is to use the thickness \( w^{(j)}_{n^{(j)+1}} \). That is, \( l^{(j)}_{n^{(j)+1}} \geq w^{(j)}_{n^{(j)+1}} \). Proposition 10.6 provides the relation between \( l^{(j)}_{n^{(j)+1}} \) and \( l^{(j)}_0 \) by using the thickness. Finally, the horizontal size \( l^{(j+1)}_0 \) and thickness \( w^{(j+1)}_0 \) of the first set \( J^{(j+1)}_0 \) in the next row \( j + 1 \) is obtained by the thickness \( w^{(j)}_{n^{(j)+1}} \) by definition.
Fig. 10.5: Construction of a double sequence. The left and right are the graphs for $F^{(0)}_0$ and $F^{(0)}_1 = F^{(1)}_0$ respectively. The arrows indicate the iteration and rescaling in the construction of the double sequence. The sub-figures (a), (b), (c), and (d) are the zoomed double sequence elements. The scale of (a), (b), (c), and (d) are chosen to be the same for the reader to compare the change of the horizontal size.
From the discussion, the horizontal size of any set in the double sequence can be estimated as follows.

**Proposition 10.11** Given $\delta > 0$ and $I^v \supset I^h \equiv I$, There exists $\bar{\epsilon} > 0$ and $E > 1$ such that for all non-degenerate open maps $F \in \hat{\mathcal{H}}_\delta(I^h \times I^v, \bar{\epsilon})$ the following property holds:

Let $J \subset A \cup B$ be a square subset of a wandering domain of $F$ and $\{ J_n^{(j)} \}_{n \geq 0, 0 \leq j \leq J}$ be a $J$-double sequence. Then

1. $\ln l_0^{(j+1)} \geq 2m^{(j)} \ln \| e^{(j)} \| + \ln l_0^{(j)}$ for all $0 \leq j \leq J - 1$ and
2. $l_n^{(j)} \geq El_n^{(j)}$ for all $n < n^{(j)}$ and all $0 \leq j \leq J$.

**Proof** Let $\bar{\epsilon} > 0$ be small enough such that Proposition 9.2 and Corollary 10.8 hold.

With the help Corollary 10.8 we are able to compare $l_0^{(j+1)}$ with $l_0^{(j)}$ by using the thickness. That is

$$ l_0^{(j+1)} = w_{n^{(j)}+1}^{(j)} $$

$$ \geq \left( \prod_{n=0}^{n^{(j)}} c^{(n^{(j)})} \right)^{\frac{3}{2}} w_0^{(j)} \geq \left( c^{\frac{3}{2}} \| e^{(j)} \|^{(n^{(j)})} \right)^{\frac{2}{3}} w_0^{(j)} $$

$$ = \left( c^{\frac{3}{2}} \| e^{(j)} \| \right)^{\frac{2}{3}} l_0^{(j)} $$

where $c > 0$ is a constant. Apply nature logarithm to both sides, we get

$$ \ln l_0^{(j+1)} \geq \frac{3}{2} m^{(j)} \left( \ln \| e^{(j)} \| + \frac{2}{3} \ln c \right) + \ln l_0^{(j)} $$

$$ \geq 2m^{(j)} \ln \| e^{(j)} \| + \ln l_0^{(j)} $$

Here we assume that $\bar{\epsilon}$ is small enough so that $\frac{2}{3} \ln c \geq \frac{1}{2} \ln \| e^{(j)} \|$ for all $0 \leq j \leq J - 1$ to assimilate the constants.

The second inequality follows directly from Proposition 9.2, the definition of $n^{(j)}$, and a square is $R$-regular when $\bar{\epsilon}$ is small enough.

The next proposition provides the relation of the perturbation $\epsilon$ between two rows.

**Proposition 10.12** Given $\delta > 0$ and $I^v \supset I^h \equiv I$. There exists $\bar{\epsilon} > 0$ and $\alpha > 0$ (universal) such that for all non-degenerate open maps $F \in \hat{\mathcal{H}}_\delta(I^h \times I^v, \bar{\epsilon})$ we have

$$ \| e^{(j+1)} \| \leq \| e^{(j)} \| \| e^{(j)} \|^{-2\alpha} \tag{10.2} $$

for all $0 \leq j \leq J - 1$. 
Proof By definition and Proposition 4.21 we have
\[ \|e^{(j+1)}\| = \|e^{(j+1)}_{r(n^{(j+1)})}\| \leq \|e^{(j)}_{r(n^{(j)})}\| \leq \left( c \|e^{(j)}_{r(n^{(j)})}\| \right)^{2^{k(j)}} = \left( c \|e^{(j)}\| \right)^{2^{k(j)}} \]
for some constant \( c > 0 \). Apply logarithm to both sides, we get
\[ \ln \|e^{(j+1)}\| \leq 2^{k(j)} \left( \ln \|e^{(j)}\| + \ln c \right) \leq 2^{k(j)-1} \ln \|e^{(j)}\| \tag{10.3} \]
Here we assume that \( \bar{e} > 0 \) is small enough such that
\[ -\frac{1}{2} \ln \|e^{(j)}\| > \ln c \]
for all \( j \geq 0 \).

Since \( J^{(j)}_{n^{(j)}} \) enters the bad region, we have \( k^{(j)} > K^{(j)} \). By Proposition 8.5 and the change base formula, we get
\[ 2^{k^{(j)}} > 2^{K^{(j)}} = \left( \lambda K^{(j)} \right) \ln 2^{\ln \lambda} \geq c' \left( \frac{1}{\|e^{(j)}\|} \right)^{\ln 2^{\ln \lambda}} \tag{10.4} \]
for some constant \( c' > 0 \). Let \( \alpha = \frac{\ln 2}{\ln \lambda} > 0 \). Combine (10.3) and (10.4), we obtain
\[ \ln \|e^{(j+1)}\| \leq \frac{c'}{2} \left( \frac{\|e^{(j)}\|^{3\alpha}}{\|e^{(j)}\|} \right) \ln \|e^{(j)}\| < \left( \frac{\|e^{(j)}\|^2}{\|e^{(j)}\|} \right) 2^{\alpha} \ln \|e^{(j)}\| . \]
Note that \( \ln \|e^{(j)}\| < 0 \). Here we also assume that \( \bar{e} \) is small enough such that
\[ \frac{c'}{2} \left( \frac{\|e^{(j)}\|^\alpha}{\|e^{(j)}\|} \right) \geq \frac{c'}{2} \left( \frac{\|e^{(j)}\}^{\alpha}}{\|e^{(j)}\|} \right) > 1 \]
for all \( j \geq 0 \). This proves the proposition. \( \square \)

10.3 Closest approach cannot enter the bad region infinitely many times

A strong contraction on the horizontal size occurs each time when the double sequence (or closest approach) enters the bad region as proved in Proposition 10.11. The contraction produces an obstruction to the expansion argument. This section will resolve the problem by proving the double sequence can have at most finitely many rows.

Although entering the bad region produces an obstruction to the expansion argument, it also provides a restriction to the sequence element \( J^{(j)}_{n^{(j)}} \), its horizontal size \( l^{(j)}_{n^{(j)}} \) cannot exceed the size of bad region (Proposition 8.5). The Two Row Lemma, which is the final key toward the proof, studies the interaction between the obstruction and restriction between two consecutive rows as illustrated in Figure 10.6.
Given Lemma 10.13 (Two Row Lemma) contraction of thickness produces the contraction of horizontal size from sequence element \( l \) of expansion on row \( j \) prior to \( J \). Let \( z \) cannot exceed the size of bad region. Let \( \alpha \leq 1 \) for all \( \alpha > 0 \) (universal) such that for all non-degenerate open maps \( F \in \mathcal{J}_\delta(I^h \times I') \) the following property holds:

Fig. 10.6: Relations of horizontal size and thickness in two rows \( j \) and \( j + 1 \).

Assume the two rows \( j \) and \( j + 1 \) both enter the bad region.

On row \( j + 1 \), the sequence element \( J_{n(j+1)} \) enters the bad region. The size of bad region provides the restriction to the horizontal size \( l \). For the sequence elements prior to \( J_{n(j+1)} \) on the same row, the horizontal size expand. This means that the initial sequence element \( l \) is restricted by both the size of bad region and the amount of expansion on row \( j + 1 \).

On row \( j \), the thickness determines the horizontal size of the next row \( j + 1 \). The contraction of thickness produces the contraction of horizontal size from \( l \) to \( l \). This cause the obstruction toward the expansion argument.

The following lemma summarize the discussion.

\textbf{Lemma 10.13 (Two Row Lemma)} Given \( \delta > 0 \) and \( I' \supset I^h \supset I \). There exists \( \mathcal{E} > 0 \), \( E > 1 \), \( \alpha > 0 \) such that for all non-degenerate open maps \( F \in \mathcal{J}_\delta(I^h \times I', \mathcal{E}) \) the following property holds:

Let \( J \subset A \cup B \) be a square subset of a wandering domain of \( F \) and \( \{ J_n \} \) be a \( J \)-double sequence. Then the time span in the good regions \( n(j) = m(j) - 1 \) for row \( j \) is bounded below by

\[ m(j) > \frac{\ln E}{-2\ln \| e(j) \|} m(j+1) + \left( \frac{1}{\| e(j) \|} \right)^\alpha + \frac{1}{-2\ln \| e(j) \|} \ln l_0^{(j)} \quad (10.5) \]

for all \( 0 \leq j \leq \overline{j} - 2 \).

\textbf{Proof} The idea of the proof comes from Figure 10.6.

On row \( j + 1 \), \( J_{n(j+1)} \) is in the bad region since \( j + 1 \leq \overline{j} - 1 \). The size of \( J_{n(j+1)} \) cannot exceed the size of bad region. Let \( z_1, z_2 \in J_{n(j+1)} \) be such that \( |\pi z_2 - \pi z_1| = l_{n(j+1)} \). Apply Proposition 8.5 to bound the horizontal size. We get

\[ l_{n(j+1)}^{(j+1)} \leq |\pi z_2 - v_{n(j+1)}^{(j+1)}| + |\pi z_1 - v_{n(j+1)}^{(j+1)}| \leq 2c \sqrt{\| e_{n(j+1)}^{(j+1)} \|} = 2c \sqrt{\| e_{n(j+1)}^{(j)} \|} \]
for some constant $c > 0$.

Also, the horizontal size expands on row $j + 1$. Proposition 10.11 yields

$$E^{n(j+1)} l_0^{(j+1)} \leq l_{n(j+1)}^{(j+1)} \leq 2c \sqrt{\|\epsilon(j+1)\|}.$$ 

Apply natural logarithm to both sides, we get

$$\ln l_0^{(j+1)} < -n^{(j+1)} \ln E + \frac{1}{2} \ln \|\epsilon(j+1)\| + \ln 2c$$

$$= -m^{(j+1)} \ln E + \frac{1}{2} \ln \|\epsilon(j+1)\| + (\ln E + \ln 2c).$$

On row $j$, the thickness contracts. Proposition 10.11 provides the contraction as

$$2m^{(j)} \ln \|\epsilon(j)\| \leq \ln l_0^{(j+1)} - \ln l_0^{(j)}$$

$$< -m^{(j+1)} \ln E + \frac{1}{2} \ln \|\epsilon(j+1)\| + (\ln E + \ln 2c) - \ln l_0^{(j)}.$$

Since $\ln \|\epsilon(j)\| < 0$, we solved

$$m^{(j)} > \frac{\ln E}{-2 \ln \|\epsilon(j)\|} m^{(j+1)} + \frac{1}{4} \ln \|\epsilon(j+1)\| + \frac{\ln E + \ln 2c}{2 \ln \|\epsilon(j)\|} + \frac{\ln l_0^{(j)}}{-2 \ln \|\epsilon(j)\|}.$$ 

To simplify the second term, apply Proposition 10.12. We obtain

$$m^{(j)} > \frac{\ln E}{-2 \ln \|\epsilon(j)\|} m^{(j+1)} + \frac{1}{4} \left( \frac{1}{\|\epsilon(j)\|} \right)^{2\alpha} \ln E + \ln 2c + \frac{1}{2 \ln \|\epsilon(j)\|} \ln l_0^{(j)}$$

$$= \frac{\ln E}{-2 \ln \|\epsilon(j)\|} m^{(j+1)} + \left( \frac{1}{\|\epsilon(j)\|} \right)^{\alpha} \left[ \frac{1}{4} \left( \frac{1}{\|\epsilon(j)\|} \right)^{\alpha} + \frac{\ln E + \ln 2c}{2 \ln \|\epsilon(j)\|} \|\epsilon(j)\|^{\alpha} \right]$$

$$+ \frac{1}{-2 \ln \|\epsilon(j)\|} \ln l_0^{(j)}$$

$$> \frac{\ln E}{-2 \ln \|\epsilon(j)\|} m^{(j+1)} + \left( \frac{1}{\|\epsilon(j)\|} \right)^{\alpha} + \frac{1}{-2 \ln \|\epsilon(j)\|} \ln l_0^{(j)}.$$ 

Here we assume that $\overline{\epsilon}$ is sufficiently small such that

$$\frac{1}{4} \left( \frac{1}{\|\epsilon(j)\|} \right)^{\alpha} + \frac{\ln E + \ln 2c}{2 \ln \|\epsilon(j)\|} \|\epsilon(j)\|^{\alpha} > 1$$

for all $j \geq 0$ to assimilate the constants. \qed
If the double sequence has infinite rows, then the obstruction and restriction both happen infinitely many times. For this to happen, the obstruction must beats (or balance with) the restriction. However, it is not possible to compare the contraction of the horizontal size with the size of bad region directly because the time span in the good regions also interacts with the obstruction and restriction as \(\text{[10.5]}\) shows. So we turn to analyze the relation between the time span in the good regions versus the the number of rows in a double sequence.

If the double sequence enters the bad region twice, we apply the Two Row Lemma to row 0 and 1. The restriction from the bad region says the the horizontal size is bounded by the size of bad region \(\left\| e^{(1)} \right\| \). At this moment, there are no information for the expansion on row 1. So the restriction comes only from the size of bad region. To balance the obstruction with the restriction, the total contraction \(\left\| e^{(0)} \right\|^m(0)\) on row 0 must have at least the same order as the size of bad region. Thus, the time span in the good regions \(m(0)\) must be large \((\approx \left\| e^{(0)} \right\|^{-1})\) by Proposition \[10.12\] because the contraction and the size of the bad region come from the perturbation on two different rows.

If a double sequence enters the bad region three times, we apply the Two Row Lemma twice. First, we apply the lemma to row 1 and 2. Unlike the previous paragraph, now the expansion on row 1 is determined when the Two Row Lemma is applied to row 1 and 2. So the restriction comes from both the size of bad region \(\left\| e^{(1)} \right\| \) and the expansion of horizontal size \(E^{m(1)} \sim E\left\| e^{(1)} \right\|^{-1}\).

To balance the obstruction with the restriction, the total contraction \(\left\| e^{(0)} \right\|^m(0)\) on row 0 must have at least the same order as \(E^{-\left\| e^{(1)} \right\|^{-1}} \left\| e^{(1)} \right\| \). This yields a larger estimate (compare to the previous paragraph) for the time span in the good regions \(m(0)\) because of the expansion.

If the double sequence enters the bad region infinite times, we start from any arbitrary row \(j + k + 1\) then apply the Two Row Lemma recurrently to the rows \(j, j + 1, \cdots, j + k + 1\) in reverse order. It is important that the contribution of obstruction and restriction comes from the perturbation in two different rows as illustrated in Figure \[10.6\]. With the help from Proposition \[10.12\] the contribution from different rows makes the time span in the good regions increases each time when the Two Row Lemma is applied. This gives the following lemma

**Lemma 10.14** Given \(\delta > 0\) and \(I^\prime \supset I^h \supset I\). There exists \(\epsilon > 0\) such that for all non-degenerate open maps \(F \in \mathcal{F}_\delta(I^h \times I^\prime, \epsilon)\) the following property holds:

Let \(J \subset A \cup B\) be a square subset of a wandering domain of \(F\) and \(\left\{J_n^{(j)}\right\}_{n \geq 0, 0 \leq j \leq J}\) be a \(J\)-double sequence. Then the time span in the good regions \(n^{(j)}\) for row \(j\) is bounded below by

\[
m^{(j)} = n^{(j)} + 1 > \frac{2^k}{\left\| e^{(j)} \right\|^\alpha} + \frac{1}{-2ln\left\| e^{(j)} \right\|} \ln l_0^{(j)} \quad (10.6)
\]
for all $j$ and $k$ with $0 \leq j \leq \overline{j} - 2$ and $0 \leq k \leq (\overline{j} - 2) - j$ where $\alpha > 0$ is a universal constant.

In particular for the case $j = 0$

$$m^{(0)} = n^{(0)} + 1 > \frac{2^k}{\|e^{(0)}\|^{\alpha}} + \frac{1}{-2\ln\|e^{(0)}\|} \ln l_0^{(0)}$$

(10.7)

for all $0 \leq k \leq \overline{j} - 2$.

Proof We prove (10.6) holds for all $0 \leq j \leq \overline{j} - k - 2$ by induction on $k \leq \overline{j} - 2$. Let $\varepsilon$ be small enough such that Proposition 10.11, Proposition 10.12, and Lemma 10.13 hold.

For the base case $k = 0$. Apply (10.5), we have

$$m^{(j)} > \frac{\ln E}{-2\ln\|e^{(j)}\|} m^{(j+1)} + \left(\frac{1}{\|e^{(j)}\|}\right)^\alpha + \frac{1}{-2\ln\|e^{(j)}\|} \ln l_0^{(j)}$$

$$> \frac{1}{\|e^{(j)}\|^{\alpha}} + \frac{1}{-2\ln\|e^{(j)}\|} \ln l_0^{(j)}$$

for all $j$ with $0 \leq j \leq \overline{j} - 2$.

Assume that there exists $k$ with $1 \leq k \leq \overline{j} - 2$ such that (10.6) holds for all $j$ with $0 \leq j \leq \overline{j} - k - 2$. If $k + 1 \leq \overline{j} - 2$ and $0 \leq j < \overline{j} - (k + 1) - 2$, then $k \leq \overline{j} - 2$ and $1 \leq j + 1 \leq \overline{j} - k - 2$. The induction hypothesis yields

$$m^{(j+1)} > \frac{2^k}{\|e^{(j+1)}\|^{\alpha}} + \frac{1}{-2\ln\|e^{(j+1)}\|} \ln l_0^{(j+1)}.$$ (10.8)

Substitute (10.8) into (10.5), we get

$$m^{(j)} > \frac{\ln E}{-2\ln\|e^{(j)}\|} \left(\frac{2^k}{\|e^{(j+1)}\|^{\alpha}} + \frac{\ln E}{-2\ln\|e^{(j+1)}\|} \frac{1}{-2\ln\|e^{(j+1)}\|} \ln l_0^{(j+1)}\right)$$

$$+ \frac{1}{-2\ln\|e^{(j)}\|} \ln l_0^{(j)}.$$ (10.9)

For the first term of (10.9), we have

$$\ln \frac{1}{\|e^{(j)}\|} < \frac{1}{\|e^{(j)}\|}.$$

Together with (10.2), we get

$$\ln E \cdot \frac{\|e^{(j)}\|}{-2\ln\|e^{(j)}\|} \left(\frac{2^k}{\|e^{(j+1)}\|^{\alpha}} \right) > 2^k \left[\frac{\ln E}{2} \left(\frac{1}{\|e^{(j)}\|}\right)^{\alpha\|e^{(j)}\|^{-2\alpha - 1}}\right] > 2^{k+2} \left(\frac{1}{\|e^{(j)}\|}\right)^\alpha.$$

Here, we assume that $\varepsilon$ is small enough such that

$$\frac{\ln E}{8} > \|e^{(j)}\|.$$
\[ \alpha \left\| \varepsilon^{(j)} \right\|^{-2\alpha} - 2 > \alpha \]

for all \( j \geq 0 \).

For the second term of (10.9), apply Proposition 10.11 We get

\[
\frac{\ln E}{-2 \ln \left\| \varepsilon^{(j)} \right\|} - \frac{1}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \ln l_0^{(j+1)} + \frac{\ln E}{2 \ln \left\| \varepsilon^{(j+1)} \right\|} \ln l_0^{(j+1)} > \frac{\ln E}{-2 \ln \left\| \varepsilon^{(j)} \right\|} \ln l_0^{(j+1)} + \frac{1}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \ln l_0^{(j+1)}.
\]

Combine the results to (10.9), we obtain

\[
m^{(j)} > 2^{k+2} \left( \frac{1}{\left\| \varepsilon^{(j)} \right\|} \right)^{\alpha} + \frac{\ln E}{2 \ln \left\| \varepsilon^{(j+1)} \right\|} m^{(j)} + \frac{1}{-2 \ln \left\| \varepsilon^{(j)} \right\|} \left( 1 + \frac{\ln E}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \right) \ln l_0^{(j+1)}.
\]

Then

\[
\left( 1 + \frac{\ln E}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \right) m^{(j)} > 2^{k+2} \left( \frac{1}{\left\| \varepsilon^{(j)} \right\|} \right)^{\alpha} + \frac{1}{-2 \ln \left\| \varepsilon^{(j)} \right\|} \left( 1 + \frac{\ln E}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \right) \ln l_0^{(j+1)}.
\]

Solve for \( m^{(j)} \), we get

\[
m^{(j)} > 2^{k+2} \left( 1 + \frac{\ln E}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \right)^{-1} \left( \frac{1}{\left\| \varepsilon^{(j)} \right\|} \right)^{\alpha} + \frac{1}{-2 \ln \left\| \varepsilon^{(j)} \right\|} \ln l_0^{(j+1)}.
\]

To simplify the inequality, we assume that \( \varepsilon \) is small enough such that

\[
\frac{\ln E}{-2 \ln \left\| \varepsilon^{(j+1)} \right\|} \leq \frac{\ln E}{-2 \ln \varepsilon} < 1
\]

for all \( j \geq 0 \). Therefore,

\[
m^{(j)} > \frac{2^{k+1}}{\left\| \varepsilon^{(j)} \right\|^{\alpha}} + \frac{1}{-2 \ln \left\| \varepsilon^{(j)} \right\|} \ln l_0^{(j+1)}
\]

and the lemma is proved by induction.

The lemma shows that the restriction beats the obstruction because (10.7) approaches infinity as the total number of rows in a double sequence increases. This proves

**Proposition 10.15** Given \( \delta > 0 \) and \( I' \supset I \). There exists \( \varepsilon > 0 \) such that for all non-degenerate open maps \( F \in \hat{I'}_{\delta}(I', I) \) the following property holds:

Let \( J \subset A \cup B \) be a square subset of a wandering domain of \( F \) and \( \{ J_n^{(j)} \}_{n \geq 0, 0 \leq j \leq \overline{j}} \) be a \( J \)-double sequence. Then the number of rows \( \overline{j} \) for the double sequence is finite.
10.4 Nonexistence of wandering domain

Finally, the main theorem is concluded as follows.

**Theorem 10.16** Given $\delta > 0$ and $I' \supset I^h \ni I$. There exists $\epsilon > 0$ such that every non-degenerate open Hénon-like map $F \in \mathcal{F}_\delta(I^h \times I', \epsilon)$ does not have wandering domains.

**Proof** Assume that $\epsilon > 0$ is small enough such that Proposition 4.21 holds and $F \in \mathcal{F}_\delta(I^h \times I', \epsilon)$. There exists $0 < \delta_R < \delta$ and $I \subset I_R^h \subset I^h$ such that $F_n \in \mathcal{H}_{\delta_R}(I_R^h \times I_n^h, \epsilon)$ for all $n \geq 0$.

Prove by contradiction. Assume that $F$ has a wandering domain. Let $\epsilon' > 0$ be small enough such that Proposition 10.15 holds for $\epsilon'$. By Proposition 4.21, there exists $N \geq 0$ such that $F_N \in \mathcal{F}_{\delta_R}(I_R^h \times I_N^h, \epsilon')$. Set $\tilde{F} = F_N|_{I_R^h \times I_R^h}$.

By Corollary 6.4, $F_N$ has a wandering domain $J$ in $D(F_N) \subset I^h(F_N) \times I_N^h$. If $J \subset B(F_N)$, then $J \subset I_R^h \times I_N^h$ and so $F^2(J) \subset B(F_N) \cap (I_R^h \times I_R^h)$. If $J \subset A(F_N)$, there exists $n > 0$ such that $F^n(J) \subset B(F_N)$ by Proposition 4.16. If $J \subset C(F_N)$, then $J \subset B(\tilde{F})$. Without lose of generality, we may assume that $J \subset B(F_N) \cap (I_R^h \times I_R^h)$. Hence, $J \subset B(\tilde{F})$ is a wandering domain of the restriction $\tilde{F}$.

Let $\hat{J}$ be a nonempty square subset of $J$ and $\{J_n^{(j)}\}_{n \geq 0, 0 \leq j \leq \hat{J}}$ be a $\hat{J}$-double sequence. By Proposition 10.15, $\hat{J}$ is finite. Then the second property of Proposition 10.15 implies that

$$\lim_{n \to \infty} l_n^{(\hat{J})} = \infty$$

which is a contraction. Therefore, $F$ does not have wandering domains. \(\square\)

**Remark 10.17** The result for Theorem 10.16 also applies to infinitely CLM-renormalizable maps if all levels of renormalization are defined on a sufficiently large domain. This is because of the hyperbolicity of the Hénon renormalization operator.

Assume that $F$ is a strongly dissipative infinitely CLM-renormalizable Hénon-like map. By the hyperbolicity of the renormalization operator [12, Theorem 4.1], there exists $N \geq 0$ such that $F_n$ is sufficiently close to the fixed point $G$ for all $n \geq N$. This means that $F_n$ is renormalizable for all $n \geq N$ and hence $F_N$ is infinitely renormalizable in the sense of this article. Thus, we can apply the theorem to $F_N$ to conclude $F$ does not have wandering domains.

As a consequence, the absence of wandering domains provides the information of the topology as follows.

**Corollary 10.18** Given $\delta > 0$ and $I' \supset I^h \ni I$. There exists $\epsilon > 0$ such that for any non-degenerate open map $F \in \mathcal{F}_\delta(I^h \times I', \epsilon)$, the union of the stable manifolds for the period doubling periodic points is dense in the domain.
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Nomenclature

A, B, C Partition of the domain $D$ for Hénon-like map, page 15
A, B, C Partition of the domain $D$ for unimodal map, page 8
$c^{(n)}$ Critical point and its orbit, page 7
$C_n(j)$ Subpartition for $C_n$ with rescaling level $j$, page 20
$C_n^L(j)$ Left component of $C_n(j)$, page 21
$C_n^R(j)$ Right component of $C_n(j)$, page 21
$D$ The Hénon-like map is defined to be a self-map on $D \subset I^h \times I^v$, page 14
$\epsilon$ Perturbation component for Hénon-like map, page 12
$F$ Hénon-like map, page 12
$f$ Unimodal component for Hénon-like map, page 12
$G$ Fixed point for $R$, page 18
$g$ Fixed point for $R_c$, page 9
$\mathcal{H}$ Class of Hénon-like maps, page 12
$H$ Nonlinear part of the Hénon rescaling, page 17
$h$ $x$-component for Hénon-like map, page 12
$h$ Vertical size, page 29
$\hat{\cdot}$ Reflection point, page 7
$\mathcal{H}_{\delta}$ Class of infinite renormalizable Hénon-like maps., page 17
$I(\epsilon)$ Complex $\epsilon$-neighborhood of the interval $I$, page 6
$I^h$ Horizontal domain for a Hénon-like map, page 12
$I_v$ Vertical domain for a Hénon-like map, page 12
$J_n$ $J$-closest approach, page 27
$k$ Level of rescaling, page 27
$K$ Boundary for good and bad regions, page 33
$l$ Horizontal size, page 29
$\Lambda$ Affine part of the Hénon rescaling, page 17
$\lambda_n$ $s_{n+1}$, page 17
$n^{(j)}$ Time span in the good region for row $j$ in a double sequence of wandering domain, page 63
$p_n(j)$ Periodic point with period $2^j$ for the Hénon-Like map $F_n$, page 20
$\phi$ Hénon rescaling, page 17
$\Phi_n^j$ Nonlinear rescaling from renormalization level $n$ to $n + j$, page 17
$r(n)$ Level of renormalization of the sequence of wandering domain $J_n$, page 27
$R_c$ Renormalization operator about the critical point, page 9
$S$ Schwarzian derivative, page 7
$s$ Affine part of the Hénon rescaling, page 17
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