How unfair is private learning?

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Abstract

As machine learning algorithms are deployed on sensitive data in critical decision making processes, it is becoming increasingly important that they are also private and fair. In this paper, we show that, when the data has a long-tailed structure, it is not possible to build accurate learning algorithms that are both private and results in higher accuracy on minority subpopulations. We further show that relaxing overall accuracy can lead to good fairness even with strict privacy requirements. To corroborate our theoretical results in practice, we provide an extensive set of experimental results using a variety of synthetic, vision (CIFAR-10 and CelebA), and tabular (Law School) datasets and learning algorithms.

1 Introduction

In recent years, reliability of machine learning algorithms have become ever more important due to their widespread use in daily life. Fairness and privacy are two instances of such reliability traits that are desirable but often absent in modern machine learning algorithms [11, 15, 42]. As a result, there has been a flurry of recent works that aim to improve these properties in commonly used learning algorithms. However, most of these works discuss these two properties individually with relatively less attention paid to how they affect each other.

There is a multitude of definitions for privacy and fairness in their respective literatures. Perhaps the most widespread statistical notion of privacy is that of Differential Privacy [17] and its slightly relaxed variant, Approximate Differential Privacy [18]. Despite its marginally weaker privacy guarantees, Approximate Differential Privacy enjoys better theoretical guarantees in terms of statistical complexity for learning [8, 23]. It is also more widely used in practice [1] [34]. Thus, we always use approximate differential privacy in this paper and for the sake of brevity, refer to it as differential privacy (DP). Intuitively, DP limits the amount of influence any single data point has on the output of the DP algorithm. This ensures that DP algorithms do not leak information about whether any particular data point was given as input to the algorithm. While DP was initially popular as a theoretical construct, it has recently been put to practical use by large companies [21, 44] and governments [35] alike. Its popularity is largely due to its strong privacy guarantees, ease of implementation, and the quantitative nature of differential privacy.

There are many notions of fairness in machine learning [19, 29, 30]. Minority or worst group accuracy [16, 32] and its difference from the overall accuracy is a common notion of fairness used in recent works. We define this difference as accuracy discrepancy and use it to measure the degree of unfairness in this paper. However, we expect our results to translate to other related fairness metrics as well. Sagawa* et al. [40] observed that robust optimisation methods (under appropriate regularisations) obtain higher minority group accuracy but at the cost of a lower overall accuracy compared to vanilla training. Several other works have also observed this behavior in practice thereby suggesting a possible trade-off between overall accuracy and fairness metrics. Subsequent works including Goel et al. [28] and Menon et al. [37] have tried to minimise this trade-off.

While there have been a large body of works that aim to minimising the trade-off between accuracy and fairness [16, 28, 40] and between accuracy and privacy [18, 27], there is relatively few works [25] that...
investigate the intersection of privacy, fairness, and accuracy. In this paper, we provide theoretical and experimental results to show that private and accurate algorithms are necessarily unfair. We further show that achieving privacy and fairness simultaneously leads to inaccurate algorithms.

Contributions Our main contributions can be stated as —

- In Theorem 1 and 2, we provide asymptotic lower bounds for unfairness (accuracy discrepancy) of DP algorithms, that are accurate, showing that privacy and accuracy comes at the cost of fairness.
- In Theorem 3, we show that in a very strict privacy regime, fairness can be achieved at the cost of accuracy.
- In Section 3, we conduct experiments using multiple architectures on synthetic and real world data-sets (CelebA, CIFAR-10, and Law School) to validate our theory.

Related works

It is now well understood that by imposing these additional conditions of DP more data are required to achieve high accuracy. A string of theoretical works [6, 10, 23] have shown that the sample complexity of learning certain concept classes privately and accurately can be arbitrarily larger than learning the same classes non-privately (i.e. with high accuracy but without privacy). On the other hand, it is easy to guarantee any arbitrary level of differential privacy if high accuracy is not desired. This can be achieved by simply composing the output of an accurate classifier with a properly calibrated randomised response mechanism [49]. This allows for a tradeoff between differential privacy and accuracy.

One of the most popular notions of fairness is group fairness that compares the performance of the algorithm on a minority group with other groups in the data. A popular instantiation of this, especially for deep learning algorithms, is comparing the accuracy on the minority group against the entire population [16, 28, 32, 40]. In the fairness literature, Buolamwini and Gebru [11], Raz et al. [39] shows extensively that this discrepancy is large between different groups of people for popular facial recognition systems.

DP-SGD [1] is a widely used algorithm for implementing differentially private deep learning models. Bagdasaryan et al. [5] provides some experimental evidence that DP-SGD can have disparate impact on accuracy. Tran et al. [45] provides theoretical explanation for why DP-SGD and output perturbation based DP methods suffer from fairness issues. Conversely, Chang and Shokri [12] shows, experimentally, that fairness aware machine learning algorithms suffer from less privacy. However, unlike these works, we provide theoretical results that are model agnostic and that discuss the dependance of the trade-off on the subpopulation sizes and frequencies.

Cummings et al. [14] and Agarwal [2] were one of the first to consider the impact of privacy on fairness theoretically. They construct a distribution where any algorithm that is always fair and private will necessarily output a trivial constant classifier, thereby suggesting a tradeoff between fairness and privacy. However, there are multiple drawbacks with their work. First, their work only discusses pure differential privacy which is not only theoretically more restrictive than approximate differential privacy [7, 8, 23] but also rarely used in practice. Second, their proof heavily relies on it being pure differential privacy and the algorithm being always fair; and their proofs are not amenable to relaxations of these assumptions. Third, their result does not show how unfairness increases with stricter privacy or different properties of the distribution. Further, they do not provide experiments to corroborate their theory perhaps due to the unrealistic requirements of pure DP. On the other hand, we look at approximate DP (which is a stronger result than pure DP), construct bounds for both fairness and error, and provide experimental results to support our theory. Perhaps, most closely related to our work is that of Feldman [22], who studies, mainly, the impact of memorisation on test accuracy for long-tailed distributions. However, neither does their work focus on differential privacy nor fairness.

2 Theoretical results

The main contribution of our work is to provide a qualitative explanation for why and when differentially private algorithms cannot be simultaneously accurate and fair. Real world data distributions often contain a
large number of subpopulations with very few examples in each of them and a few subpopulations with a large number of examples.

For example, Figure 1 (left) depicts the distribution of subpopulations in CelebA. Using the 40 attributes of the CelebA dataset [34], we partition the training set (of size $m = 160k$) into $2^{40}$ subpopulation bins. The blue shaded area shows the group of the subpopulations with large number of examples (probability mass greater than $\frac{1}{m}$) and the red shaded area corresponds to subpopulations which contain just one example in them. We refer to the subpopulations in the red area as \textit{minority subpopulations} and the subpopulations in the blue area as \textit{majority subpopulations}. This long-tailed structure over subpopulations have also been observed in Zhu et al. [55] in other vision datasets like the SUN [52] and PASCAL [20] datasets. Babbar and Schölkopf [4] observes this structure in extreme multilabel classification datasets like Amazon-670K [36] and Wikipedia-31k [9] datasets. Various other works [13, 33, 46, 48] have shown this structure in a range of datasets including eBird [33], Visual Genome [33], Pasadena trees [50], and iNaturalist [17].

### 2.1 Problem setup

Mathematically, the large number of small subpopulations discussed above constitute the \textit{long tail} of the distribution. We use this structure of data distributions to illustrate the tension between accuracy and fairness of private algorithms. For our theoretical results, we view each subpopulation as an element of a discrete set $X$ without any intrinsic structure such as distance. This distribution is inspired by the use of a similar distribution in Feldman [22]. Next, we define a distribution over the subpopulations in $X$ that reflect the \textit{long-tailed} structure.

\textbf{Definition 1} ($\left(p, N, k \right)$-long-tailed distribution on $X$). Given $p \in (0, 1)$, $N \in \mathbb{N}$, and $1 < k \ll N$, define two groups (i) the group of \textit{majority subpopulations} $X_1 \subset X$ where $|X_1| = (1 - p)k$ and (ii) the group of \textit{minority subpopulations} $X_2 \subset X \setminus X_1$, where $|X_2| = N$. Now, define the distribution $\Pi_{p,N,k}$ as

$$
\Pi_{p,N,k}(x) = \begin{cases} 
\frac{1}{p} & x \in X_1 \\
\frac{1}{N} & x \in X_2.
\end{cases}
$$

We use the terms \textit{group of majority subpopulations} and \textit{majority group} interchangeably to denote $X_1$ and the \textit{terms group of minority subpopulations and minority group to refer to $X_2$ respectively.}

We provide an illustration of the distribution in Figure 1 (right). Here, $p$ denotes the total probability mass of the group of minority subpopulations under $\Pi_{p,N,k}$ and $N$ denotes the number of minority subpopulations. We let $N$ go to $\infty$ and treat $k$ as a constant. Thus, for the sake of simplicity, we remove $k$ from the notation of the distribution. Note that each minority subpopulation i.e. each element in $X_2$ has a probability mass of the order of $O\left(\frac{1}{\sqrt{N}}\right)$ which is much smaller than $\frac{1}{k} = \Omega(1)$, i.e. the probability mass of each element in $X_1$.

Note that in the distribution for CelebA (Figure 1 (left)) the probability masses of the majority subpopulations not exactly equal to $\frac{1}{k}$ as in the distribution of Definition 1. However, all our results hold true even if the majority subpopulations have different probability masses, as long as $\Pi_{p,N}(x) = \Omega\left(\frac{1}{k}\right)$ for some $k = O(1)$ for all $x \in X_1$. We set them to $\frac{1}{k}$ only for simplicity of the theoretical results.

As we deal with a multiclass classification setup, we also define a label space $\mathcal{Y}$ and a function space $\mathcal{F}$ of labelling functions. We use $\mathcal{F}$ to represent a distribution on the function space $\mathcal{F}$ and refer to this distribution as the label prior. Our results do not restrict the size of $\mathcal{Y}$ and hence, cover both binary and multi-classification settings.

For any $\zeta \subset X$ and labelling $Q \in \mathcal{Y}^{\left|\zeta\right|}$, let $\mathcal{F}_{\zeta,Q}$ denote the marginal distribution over the subset of labelling functions that satisfy the labelling $Q$ on $\zeta$. For each $x \in X \setminus \zeta$, this induces a probability mass function over $\mathcal{Y}$ as $\mathcal{F}_{\zeta,Q}(x) = \mathbb{P}_{f \sim \mathcal{F}_{\zeta,Q}}[f(x) = y]$. We say a label prior $\mathcal{F}$ is \textit{subpopulation-wise independent} if the distribution $\mathcal{F}_{\zeta,Q}(x)$ is independent of the choice of $Q$ and $\zeta$ for all $x \in X$ i.e. $\mathcal{F}(x) = \mathcal{F}_{\zeta,Q}(x)$ for all $x$.\footnote{\text{WLOG we will assume that $k$ is such that $(1 - p)k$ is an integer and if not, replace $k$ with the closest number such that $(1 - p)k$ is an integer.}}
\[ \|F\|_\infty = \max_{x \in X, y \in Y} F(x)[y]. \] (2)

To generate a dataset of size \( m \) from a \( (p,N) \)-long-tailed distribution \( \Pi_{p,N} \) on \( X \), first sample an unlabelled dataset \( S = \{x_1, \ldots, x_m\} \) of size \( m \) from \( \Pi_{p,N} \). Then, generate the labelled dataset \( S_f = \{(x_1, f(x_1)), \ldots, (x_m, f(x_m))\} \) using a labelling function \( f \sim F \). In all our theoretical results, we consider an asymptotic regime where \( \frac{N}{m} \rightarrow c \) as \( N, m \rightarrow \infty \). This is common in high-dimensional statistics where the number of dimensions often grows to \( \infty \) along with the sample size. Intuitively, \( c \) quantifies the hardness of the learning problem as it is inversely proportional to the number of data points observed per minority subpopulation.

Next, we define the error and fairness measure of an algorithm on the distribution defined above. Consider a domain \( X \), a label space \( \mathcal{Y} \), a set of labelling functions \( \mathcal{F} \), a label prior \( \mathcal{F} \), and a distribution \( \Pi_{p,N} \) on \( X \).

### 2.2 Privacy, error, and fairness

In the context of this paper, a differentially private (randomised) learning algorithm generates similar distributions over classifiers when trained on neighbouring datasets. Two datasets are neighbouring when they differ in one entry. Formally,

**Definition 2** (Approximate Differential Privacy [17, 18]). Given any two neighbouring datasets \( S, S' \), \( \epsilon > 0 \), and \( \delta \in (0,1) \) an algorithm \( A \) is called \((\epsilon, \delta)\)-differentially private if for all sets of outputs \( \mathcal{Z} \), the following holds

\[ P[A(S) \in \mathcal{Z}] \leq e^\epsilon P[A(S') \in \mathcal{Z}] + \delta. \]

Next, we define the error of an algorithm in our problem setup. For a randomised learning algorithm \( A \), a distribution \( \Pi_{p,N} \), a label prior \( \mathcal{F} \), we can define the error of the algorithm as follows
Table 1: A table of notations frequently used in the text

| Notation | Description |
|----------|-------------|
| m        | Size of dataset |
| N        | Number of minority subpopulations |
| p        | Probability of minority group |
| k        | Reciprocal of the probability of individual subpopulations in $X_2$ |
| c        | Ratio of $N$ and $m$ |
| $\epsilon, \delta$ | Privacy parameters of Approximate Differential Privacy |
| $p_1$   | Parameter in Assumption A1 |
| $\mathcal{A}$ | Randomised learning algorithm |
| $X, X_1, X_2$ | Entire data domain, majority, and minority group respectively |
| $\Pi_{p,N}, \Pi_{p,N}^2$ | Data distribution on $X$ and marginal distributions on $X_2$ respectively |
| S, $S_f$ | m-sized unlabelled and labelled (with $f \in F$) dataset respectively |
| $S^{\ell}$ | All points that appear $\ell$ times in $S$ |
| $F, \mathcal{F}$ | Set of labelling functions and distribution over the set respectively |

Definition 3 (Error measure on $\Pi_{p,N}$). The error of the algorithm $\mathcal{A}$ trained on a dataset of size $m$ from the distribution $\Pi_{p,N}$ with respect to a label prior $\mathcal{F}$ is

$$\text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = \mathbb{E} \left[ I \{ h(x) \neq f(x) \} \right]$$

where $I \{ \cdot \}$ is the indicator function and the expectation is over $S \sim \Pi_{p,N}^m, f \sim \mathcal{F}, h \sim \mathcal{A}(S_f)$, and $x \sim \Pi_{p,N}$.

Note that the error metric with an expectation over $\mathcal{F}$, was previously used in Feldman [22]. In fact, for the purpose of lower bounds on unfairness, this is a stronger notion than the worst case $f \in \mathcal{F}$ as, here, the lower bound is on the expectation which is stronger than a lower bound on the worst case. Next, we define the accuracy discrepancy of an algorithm, represented by $\Gamma$ over the distribution $\Pi_{p,N}$. For this purpose, for any $\Pi_{p,N}$, define the marginal distribution on the group of minority subpopulations $X_2$ as

$$\Pi_{p,N}^2(x) = \begin{cases} \frac{\Pi_{p,N}(x)}{\sum_{x \in X_2} \Pi_{p,N}(x)} & x \in X_2 \\ 0 & x \notin X_2 \end{cases}$$

Definition 4 (Accuracy discrepancy on $\Pi_{p,N}$). For $X, \mathcal{F}, \Pi_{p,N}$, and $\Pi_{p,N}^2$ as defined above, the accuracy discrepancy of the algorithm $\mathcal{A}$ trained on a dataset of size $m$ on the distribution $\Pi_{p,N}$, with respect to the label prior $\mathcal{F}$, is

$$\Gamma_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = \text{err}_m (\mathcal{A}, \Pi_{p,N}^2, \mathcal{F}) - \text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F})$$

where $\text{err}_m (\cdot)$ is as defined in Definition 3.

This notion of group fairness is similar to the notion of subgroup performance gap used in Goel et al. [28] and has also been implicitly used in multiple works Du et al. [16], Koh et al. [32], Sagawa et al. [40] as discussed before. It has also been used in works related to the privacy [5,12] and fairness literature [11,39].

2.3 Privacy and accuracy at the cost of fairness

The theoretical results below use the definitions and notations described above and summarised in Table 1. First, in Theorem 1, we show that there are distributions (within the family of distributions defined in Definition 1) where any accurate and approximately differentially private algorithm (with additional assumptions) is
necessarily unfair. In Theorem 2, we relax some of the stronger assumptions and present a more general result. As discussed above, the dataset size \( m \) and the number of minority subpopulations \( N \) both simultaneously go to \( \infty \) and the ratio \( \frac{N}{m} \) asymptotically approaches \( \frac{\alpha}{\epsilon} \). Throughout this section, we also use the notation \( \operatorname{err}(A, \Pi_{p,N}, F) = \lim_{N,m \to \infty} \operatorname{err}(A, \Pi_{p,N}, F) \) and \( \Gamma(A, \Pi_{p,N}, F) = \lim_{N,m \to \infty} \Gamma(A, \Pi_{p,N}, F) \) to denote the asymptotic limit for the error and the accuracy discrepancy metrics as \( m, N \to \infty \).

**Theorem 1.** For \( \epsilon \in (0, 1.11) \) and \( \delta \in (0, 0.01) \), consider any \((\epsilon, \delta)\)-DP algorithm \( A \) that does not make mistakes on subpopulations occurring more than once in the dataset. Then, there exists a family of label priors \( F \) where for any \( \alpha \in (0, 0.025) \), there exists \( p \in (0, 1/2), c > 0 \) such that,

\[
\operatorname{err}(A, \Pi_{p,N}, F) \leq \alpha \quad \text{and} \quad \Gamma(A, \Pi_{p,N}, F) \geq 0.5.
\]

where \( \frac{N}{m} \to c \) as \( N, m \to \infty \).

An immediate consequence of unfairness \( \Gamma_m \) being greater than 0.5, coupled with the very small error \( \alpha \), is that the algorithm essentially behaves worse than random chance on the minority subpopulations thereby rendering the algorithm useless for these subpopulations. Below, we present a proof sketch and discuss the results. A detailed version of the theorem along with its full proof is relegated to Appendix A.4

**Proof sketch** By Definition 1, the probability mass of each majority subpopulation is \( \Omega(1) \) whereas the probability mass of each minority subpopulation is \( O\left(\frac{1}{m}\right)\). Thus, for a large enough dataset (i.e. large \( m \)), we show that almost all majority subpopulations appear more than once and consequently, the algorithm in Theorem 1 makes fewer mistakes on the majority subpopulations. As a result, the error and the accuracy discrepancy are both caused by mistakes, primarily, on the minority subpopulations. We, then, count the number of subpopulations that do not appear or appear just once among the minority subpopulations and use that to provide the upper bounds for error and lower bound for unfairness (accuracy discrepancy). As these bounds are expressed in terms of \( p \) and \( c \), the proof then follows by showing the existence of \( p, c \) that satisfy the inequalities in the theorem.

While Theorem 1 shows the existence of distributions under which private and accurate algorithms are necessarily unfair, in Theorem 2, we provide a quantitative lower bound for unfairness of private algorithms. In addition, we also generalise Theorem 1 to include a much broader set of algorithms. For this, we state an assumption below that we refer to as the accuracy assumption. For any \( \ell \in \mathbb{N} \), define \( S^\ell \) to denote the set of examples that appear exactly \( \ell \) times in \( S \).

**Accuracy Assumption** Given \( s_0 \in \mathbb{N} \) and \( p_1 \in (0, 1) \), we state that the algorithm \( A \) satisfies Assumption A1 with parameters \( p_1 \) and \( s_0 \) if for all datasets \( S \), for all \( \ell > s_0 \), and for all \( x \in S^\ell \),

\[
\mathbb{P}_{f \sim \mathcal{F}, h \sim \mathcal{A}(S)}[h(x) \neq f(x)] \leq p_1.
\]

(A1)

Assumption A1 essentially requires the algorithm to have small overall train error. Note that since our domain is discrete, high training accuracy translates to high test accuracy in particular as the sample size approaches infinity. When \( p_1 \) is small, algorithm \( A \) obtains low training (and hence test) error on frequently occurring or typical data (i.e. \( \ell \geq s_0 \)) where \( s_0 \) can be interpreted as the minimum frequency with which typical data appears.

We note that the accuracy assumption in Theorem 1 that the algorithm does not make mistakes on subpopulations that appear more than once is an instantiation of Assumption A1 with the parameters \( s_0 = 1 \) and \( p_1 = 0 \). In the next theorem, we present a detailed result showing how the unfairness of a DP algorithm varies with a wide range of \( s_0, p_1 \) in Assumption A1 privacy parameters \( \epsilon, \delta \) and distributional parameters \( p, c \). For easier interpretation, we show a simplified version in Theorem 2 and highlight the key takeaways, and provide a detailed version in Appendix A.2.

**Theorem 2.** For any \( \epsilon \in (0, \frac{1}{2}) \), \( c > 0 \) such that \( \frac{c}{\epsilon} \leq 1 \), consider the distribution \( \Pi_{p,k,N} \) where \( \frac{N}{m} \to c \) as \( N, m \to \infty \) and \( k = O(1) \). For any \( \epsilon > 0, \delta \in (0, \frac{1}{2}) \), such that \( \frac{\epsilon}{\delta}, \log \left(\frac{1}{\delta}\right) = o(m) \), consider an \((\epsilon, \delta)\)-DP algorithm \( A \) that satisfies Assumption A1 with \( s_0 = \left\lfloor \min \left(\frac{1}{2} \log 2, \log \left(\frac{1}{\delta} \right)\right) \right\rfloor \) and some \( p_1 \in (0, 1) \). Also
assume that the label prior $F$ is subpopulation wise independent. Then, the accuracy discrepancy is lower bounded as
\[
\Gamma (A, \Pi_{p,N}, F) \geq \frac{(1 - \|F\|_\infty)(1 - p)}{3} \gamma_0
\]
where $\gamma_0$ is some constant depending on $c, p, \epsilon$, and $\delta$. Further, the error of the algorithm is upper-bounded by $\text{err}(A, \Pi_{p,N}, F) \leq (1 - p_1)p_{00} + p_1$ and $a_0$ depends on $c, \epsilon, \delta,$ and $p$. In the asymptotic limit $c \to \infty$ and $\epsilon \to 0$, $\gamma_0$ increases as $1 - O(\frac{e^{-\epsilon^2c}}{\sqrt{\epsilon}})$. As $p \to 0$, $a_0$ increases as $1 - O(\sqrt{p}\text{e}^{-\epsilon^2p})$.

The detailed expressions of $a_0$ and $\gamma_0$ (including its dependance on $c, \delta, c,$ and $p$) can be found in Theorem 5 in Appendix A.2. We now briefly discuss how the theorem characterizes the effect of privacy and accuracy (via $\epsilon$ and $s_0$ respectively) of the algorithm and the distribution and number of of subpopulations (via $p$, $c$ respectively) on the accuracy discrepancy. Theorem 2 indicates that as $c$ increases and $\epsilon$ decreases, $\gamma_0$ and hence unfairness increases. Intuitively, this is because minority subpopulations appear infrequently and the algorithm is more likely to be incorrect on infrequent subpopulations. Therefore, the lower bound on unfairness increases with $c$ and decreases with $\epsilon$. Further, as $\|F\|_\infty$ decreases, which can be interpreted as an increase in the inherent uncertainty of the labelling function, the unfairness increases.

**Proof Sketch** Next, we provide a proof sketch of Theorem 2 and relegate the full proof to Appendix A.2.

First, note that when the ratio $c$ of the number of minority subpopulations with respect to the sample size is large, most minority subpopulations appear infrequently in the observed dataset. Second, a key component of the proof is showing that when $\epsilon$ and $\|F\|_\infty$ is relatively small, most infrequently observed subpopulations are misclassified with a non-trivial probability. This is formalised by Lemma 1 below which shows that subpopulations with frequency less than $\frac{\log 2}{\epsilon / \sqrt{2\epsilon}}$ (for sufficiently small $\delta$) is misclassified with probability greater than $\frac{1 - \|F\|_\infty}{3}$.

**Lemma 1.** Let $F$ be any subpopulation-wise independent label prior over all labelling functions and $S$ be any dataset. For any $(\epsilon, \delta)$-differentially private algorithm $A$, and for all subpopulations $x \in X$ that appear fewer than $s_0 = \left\lceil \frac{1}{2} \log 2, \log \left(\frac{1}{2\epsilon} \right) \right\rceil$ times in the dataset $S$, we have that
\[
P_{f \sim F, h \sim A(S_f)} [h(x) \neq f(x)] > \frac{(1 - \|F\|_\infty)}{3}.
\]

A detailed version of this lemma along with its proof is presented in Lemma 3 in the Appendix. Combining this with Assumption A1 and using standard inequalities in probability theory gives us theorem 2.

**An illustrative example** We now provide an example of a simple learning problem and a differentially private learning algorithm for the problem that satisfies Assumption A1. The sole purpose of this is to help explain the concepts in the preceding sections with a simple constructive example. In particular, consider a binary classification problem on the domain $X$. Let $A_\eta$ be an algorithm that accepts an $m$-sized dataset $S_f \in (X \times \{0, 1\})^m$ and a noise rate $\eta \in (0, \frac{1}{2})$ as input and outputs a dictionary matching every subpopulation in $X$ to a label in $\{0, 1\}$. The algorithm first creates a dictionary where the set $X$ is the set of keys. In order to assign values to every key, it first randomly flips the label of every element in $S_f$ with probability $\eta$, then for every unique key in $S_f$, the algorithm computes the majority label of that key in the flipped dataset and assigns that majority label to the corresponding key. For elements in $X$ not present in $S_f$, it assigns a random label. Lemma 2 provides privacy and accuracy guarantees for this algorithm.

**Lemma 2.** The algorithm $A_\eta$ is $\left(\log \left(\frac{1 - \eta}{\eta}\right), 0\right)$-DP. Further, for any dataset $S_f$ and $s_0 \in \mathbb{N}$,

- if a subpopulation $x$ appears more than $s_0$ times in $S$, $P_{h \sim A_\eta(S_f)} [h(x) \neq f(x)] \leq e^{-\gamma_0(1 - 2\eta)^2 / s(1 - \eta)}$ and

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2While the discussion here assumes the asymptotic limit for $c$, and $p$ as $m \to \infty$, our results in Appendix A.2 shows non-asymptotic dependence on these terms.
Figure 2: Each figure plots the accuracy discrepancy ($\Gamma$; higher is less fair) in green dashed line, the accuracy of the minority group with red, and the overall accuracy with blue on the y-axis and the parameter $c$ in the X-axis. The left most ($\epsilon = 1$) achieves the strictest level of privacy and the right most ($\epsilon = \infty$) is vanilla training without any privacy constraints. The two figures in between achieve intermediate levels of privacy. Here $p = 0.2$. Experiment for $p = 0.5$ is in Appendix B.1.

- if a subpopulation $x$ appears less than $s_0$ times in $S$, $P_{h \sim A_\eta(S_f)}[h(x) \neq f(x)] \geq \frac{1}{\sqrt{2s_0}}(4\eta(1 - \eta))^{s_0/2}$.

Equivalently, algorithm $A_\eta$ satisfies Assumption A1 with $p_1 = e^{-\frac{\alpha(1-2\alpha)}{2(1-\eta)}}$.

Lemma 2 shows that for all $\epsilon > 0$, we can find an $\eta = \frac{1}{1+\epsilon^2}$ such that $A_\eta$ is $(\epsilon, 0)$-differentially private. For example, $\eta \approx 0.475$ provides $(0.1, 0)$ differential privacy. Further, this algorithm is more accurate on frequently occurring subpopulations and inaccurate on rare subpopulations. For example, with $\eta = 0.475$, $P_{h \sim A_\eta(S_f)}[h(x) \neq f(x)] \geq 0.34$ for subpopulations occurring less than 5 times. Similarly, for subpopulations occurring more than 6000 times, $P_{h \sim A_\eta(S_f)}[h(x) \neq f(x)] \leq 0.05$. If we assume that the data is distributed according to the long-tailed distribution defined in with $N = 10^5$, $k = 10$, $p = 0.2$ and a dataset of size $m = 10^5$ is drawn then almost all subpopulations either occur more than 6000 times or less than 5 times.

2.4 Privacy and fairness at the cost of accuracy

So far we have shown that under strict privacy and high average accuracy requirements on the algorithm, fairness necessarily suffers. A natural question to ask is whether it is possible to sacrifice accuracy for fairness. We present a simplified theorem statement here for easier interpretation and prove a more precise version in Appendix A.3 along with a discussion. In words, the theorem states that for very strict privacy parameters, fairness can be achieved at the cost of accuracy.

**Theorem 3.** For any $p \in (0, 1/2), c > 0$ such that $p/c \leq 1$, consider the distribution $\Pi_{p,N}$ where $N$ is the number of minority subpopulations. For any $\alpha > 0$, $\epsilon = \frac{\log_2 \frac{1}{\alpha \sqrt{m + \frac{1}{2\alpha^2}}}}{\alpha \sqrt{m + \frac{1}{2\alpha^2}}} + \frac{1}{\alpha \sqrt{m + \frac{1}{2\alpha^2}}} k$ and $\delta < 2^{-\frac{1}{2} - 1}$ consider any $(\epsilon, \delta)$-DP algorithm $A$. Further, let $N/m \rightarrow c$ as $m, N \rightarrow \infty$. Then,

$$
\text{err}(A, \Pi_{p,N}, \mathcal{F}) \geq \left(1 - \frac{\|\mathcal{F}\|_{\infty}}{3}ight) \left(1 - (1 - p) \left(e^{-c_1 \sigma^2} \right) \right)
$$

and

$$
\Gamma(A, \Pi_{p,N}, \mathcal{F}) \leq (1 - p) \left[1 - \frac{\|\mathcal{F}\|_{\infty}}{3} \left(1 - e^{-c_1 \sigma^2} \right) \right]
$$

This is possibly a loose lower bound. Running a simulation shows that the lower bound is at least 0.17 which matches the lower bound from Lemma 1 with $\|\mathcal{F}\|_{\infty} = 0.5$ (equal probability for both classes).
where \( c_1 = \frac{4(2-p)(2-3p)}{(2-3p)^2} \).

A detailed version of the theorem with the full proof can be found in Theorem 6 in Appendix A.3. We now briefly discuss how the theorem characterises the effect of privacy and accuracy (via \( \alpha \)) on fairness. When \( \alpha \) is large, \( \epsilon \) is small, therefore making the algorithm more private. From Lemma 3, we know that a small \( \epsilon \) renders the algorithm inaccurate on frequent subpopulations (i.e. majority subpopulations). Thus, \( \alpha \), essentially, characterises what fraction of majority subpopulations the algorithm is incorrect on. First, Theorem 3 shows that when \( \alpha \) is large, the overall error increases. Intuitively, this is because, with increasing \( \alpha \), the algorithm is incorrect, not only on minority subpopulations, but also on majority subpopulations. Second, Theorem 3 shows that as \( \alpha \) increases, i.e. privacy increases, the unfairness decreases. This shows that by making the algorithm more private (by increasing \( \alpha \)), fairness can be achieved but at the cost of overall accuracy.

3 Experimental results

In this section, we conduct experiments to support our theoretical results from Section 2. We note that our theoretical results are model-agnostic and to demonstrate the universality of our result, we conduct a broad set of experiments on both synthetic (in Section 3.1), and real world datasets (in Section 3.2 and 3.3), using multiple machine learning models including deep neural networks and random forests.

3.1 Synthetic experiments

First, we look at a synthetic data distribution that closely emulates the data distribution we use in our theoretical results in Theorem 1 and 2. Given \( N, k \in \mathbb{N}, c \in \mathbb{R}^+, \) and \( p \in (0, 0.5) \), we construct a continuous version of the long-tailed distribution \( \Pi_{p,N,k} \) (Definition 1) on a domain \( X \). First of all, since the domain \( X \) is discrete, we can place each element on a vertex of a \( O(\log(N)) \)-dimensional hypercube. The continuous distribution we use in our experiments is a mixture of Gaussians where each Gaussian is centered around the vertices of the hypercube. In the experiments, we choose \( k = 64, m = 10^4 \), vary the ratio \( c \) from 0.01 to 0.2, set the number of minority subpopulations to \( N = mc \), and choose \( p \in \{0.2, 0.5\} \). We train a five-layer fully connected neural network with ReLU activations using DP-SGD \[1\] for varying levels of \( \epsilon \) while setting \( \delta = 10^{-3} \). We refer to Appendix B for a more detailed description of the data distribution and the training algorithm.

Unfairness aggravates with increasing number of minority subpopulations As discussed in Section 2.1, increasing the number of subpopulations compared to the number of samples via \( c \) decreases accuracy...
on the minority subpopulations while the majority subpopulations remain unaffected. Figure 2 shows how increasing $c$ hurts fairness since the accuracy discrepancy (green dashed line) increases, most pronounced for small values $\epsilon$ (i.e. more private algorithms). This corroborates our theoretical results from Theorem 2 regarding the dependence of accuracy discrepancy on $\epsilon$. We further observe that the increase in unfairness is almost entirely due to the drop in the minority accuracy (red solid) whereas the overall accuracy (blue) stays relatively constant. This highlights our claim that, in the presence of strong privacy, fairness can be poor even when overall error is low.

Privacy constraints hurt fairness for accurate models In this section, we analyse the dependence of fairness on the privacy parameter $\epsilon$ for a fixed $c$. In Figure 3 (left), we plot the disparate accuracies $\Gamma$ for varying privacy parameter $\epsilon$ and Figure 3 (right) depicts the minority and overall accuracy as a function of $\epsilon$.

There are two distinct phases in the development of the accuracy discrepancy with increasing $\epsilon$ separated by the gray dashed line: For a very small $\epsilon$, the learned classifier is essentially a trivial classifier as evidenced by the very low overall accuracy ($\approx 60\%$). This is a trivial way of achieving fairness without learning an accurate classifier and is explained by Theorem 3 in our theoretical section. As the privacy restrictions are relaxed, the classifier becomes more accurate and less fair in the first phase.

The interesting regime is when classifier obtains decent overall accuracy ($\approx 80\%$) and is marked by the vertical gray dashed line. In the region to the right of the vertical dashed line, assumption $\frac{1}{\epsilon} = o(m)$ is fulfilled and Figure 3 (left) reflects the behavior as predicted in Theorem 2: loosening privacy increases fairness or smaller $\epsilon$ implies larger accuracy discrepancy.

3.2 Experiments on vision datasets

In this section, we show that our claims resulting from Theorem 2 and 3 do not only hold in synthetic settings but can also be observed in real-world computer vision datasets. In particular, we conduct experiments on two popular computer vision datasets — CelebA and CIFAR-10. CelebA is a dataset of approximately 160k training images of dimension $178 \times 218$ and another 20k of the same dimension for testing. CIFAR-10 is a 10-class classification dataset where there are 50k training images and 10k test images of dimension $3 \times 32 \times 32$. For CIFAR-10, we use a ResNet-18 and for CelebA, we use a ResNet-50 architecture.

3.2.1 Minority and majority subpopulations

In practice, datasets like CelebA and CIFAR-10 often do not come with a label of what constitutes a subpopulation. In this section, we describe how we define the minority and majority subpopulations for CIFAR-10 and CelebA.
CelebA The CelebA dataset provides 40 attributes for each image including characteristics like gender, hair color, facial hair etc. We create a binary classification problem by using the gender attribute as the target label. In addition, we use 11 of the remaining 39 binary attributes to create $2^{11}$ subpopulations and categorise each example into one of these $2^{11}$ subpopulations. Then, we create various groups of minority subpopulations by aggregating the samples of all the unique subpopulations that appear less than $s \in \{5, 10, 20, 40, 60, 80, 100\}$ times in the test set. The remaining examples constitute the majority group. In this section, we run experiments using $s = 40$. We report results for the other values of $s$ in Appendix B.2.

CIFAR-10 Unlike the synthetic distribution and CelebA as described above, CIFAR-10 cannot be readily grouped into subpopulations using explicit attributes. However, recent works [41, 54] have shown the presence of subpopulations in CIFAR-10 in the context of influence functions and adversarial training respectively. We use the influence score estimates from Zhang and Feldman [54] to create the minority and majority subpopulations. Intuitively, we treat examples that are atypical i.e. unlike any other examples in the dataset as minority examples belonging to minority subpopulations; and examples that are typical i.e. similar to a significant number of other examples in the dataset as examples belonging to majority subpopulations.

To define these subpopulations, first, we sort the examples in the training set according to their self-influence [54]. We define all of those that surpass a threshold $\rho$ as minority populations. In order to find the samples belonging to each subpopulation $x$ in the test set, we search for images that are heavily influenced (influence score is greater than the threshold) by at least one of the samples in $x$ in the training set. In this section, we report results with $\rho = 0.1$. Other values of $\rho$ show a similar trend and we plot results using $\rho = 0.01$ in Appendix B.3.

3.2.2 Privacy leads to worse fairness for accurate models

In this section, we use the above definitions of minority and majority groups to measure the impact of privacy on fairness (using Definition 4). Like Section 3.1, we measure both the accuracy discrepancy and the individual minority and overall accuracies.

CelebA Figure 1 plots the change in accuracy discrepancy, with respect to the $\epsilon$ parameter of differential privacy (smaller $\epsilon$ indicates stricter privacy). Figure 4 (left) shows that smaller $\epsilon$, with high accuracy (see Figure 4 (right)) implies a larger accuracy discrepancy. This aligns with our theoretical results from Section 2. Note that unlike Figure 3 (left), the accuracy discrepancy here monotonically decreases with increasing $\epsilon$ without exhibiting a two-phase behavior. Figure 4 (right) shows that, the reason why we do not observe the two-phase behavior is that throughout the range of observed $\epsilon$, we are in the regime of high accuracy.
3.3 Experiments on tabular data

To show that our observations hold across a wider range of publicly used datasets, we next conduct similar experiments using tabular data. We run our experiments on the the Law school dataset \cite{dwork2012fairness} that has previously been used in fairness-awareness studies like Quy et al. \cite{quy2018fairness}. It is a binary classification dataset with 21k data points and 12 dimensional features. Out of the 12 attributes, two binary attributes are used to obtain the minority group as defined in Quy et al. \cite{quy2018fairness}.

In contrast to previous experiments, we use random forest model from Fletcher and Islam \cite{fletcher2020private} instead of neural networks as in Section 3.1 and 3.2. For our implementation, we use the publicly available code in Holohan et al. \cite{holohan2020differentially} with 10 trees and a maximum depth 50. The results are plotted in Figure 6 and they show a similar two phase behavior as in our previous experiments with CIFAR-10.

Thus, all our experiments provide empirical evidence in support of the theoretical arguments in Section 2. The behavior is consistent across multiple kinds of datasets, machine learning models, and learning algorithms.

4 Future work

The experimental results on CelebA in Section 3.2 shows that when the minority group is composed of small sized subpopulations, differential privacy requirements hurt the fairness of the algorithm. Here, we highlight that not all small-sized subpopulations are hurt equally in this process. Figure 7 shows that a different partition of CelebA composed of similar sized populations do not show similar behaviours in terms of how accuracy discrepancy changes with sizes of subpopulations. We refer to the 11 attributes we chose to
partition the testset for our experiments so far as Set A and, here, we choose another set of 11 attributes and refer to them as Set B. Figure 7 (left) shows that both Set A and Set B induce a very similar distribution over sizes of subpopulations on the test set. However, Figure 7 (right) shows that while the group of minority subpopulations induced by Set A suffers very high accuracy discrepancy from private training compared to vanilla training, Set B does not (see Appendix B.2 for more details on Set A and Set B). This indicates that, irrespective of sizes, private training hurts fairness disproportionately more for certain subpopulations compared to others. In particular, an interesting direction of further research is to investigate where these minority subpopulations that are worse-affected by private training intersects with the subpopulations that are relevant for the specific domain. While most past works [5, 12] have also used sizes of subpopulations to differentiate between disparately impacted subpopulations, this suggests that that is not always the case.

In this paper, we have shown theoretically that when the minority group in the data is composed of multiple subpopulations, a DP algorithm can achieve very low error but necessarily incurs worse fairness. Further, we corroborated our theoretical results with experimental evidence on synthetic and real world computer vision datasets. However, our model-agnostic results, that shed a rather pessimistic light on algorithmic fairness and differential privacy, only apply under certain distributional assumptions. It is possible that in some real-world datasets there are fair and private algorithms that achieve a more optimistic trade-off. This begs further research to develop fair and private algorithms that are closer to the pareto optimal frontier.

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A Proofs

A.1 Proof for Theorem 1

**Theorem 4** (Detailed version of Theorem 1). For any $\alpha \in (0, 0.025)$, $\epsilon \in (0, 1.1)$, $\delta \in (0, 0.01)$, consider any $(\epsilon, \delta)$-DP algorithm $A$ that does not make mistakes on points occurring more than once in the dataset. Then, there exists $p \in (0, 1/2), c > 0$ such that

\[
err_m (A, \Pi_{p,N}, \mathcal{F}) \leq \alpha
\]

and

\[
\Gamma_m (A, \Pi_{p,N}, \mathcal{F}) \geq 0.5
\]

where $N/m \to c$ as $N,m \to \infty$ and $\mathcal{F}$ belongs to a family of label priors such that $\max_{x \in X} ||\mathcal{F}(x)||_{\infty} \leq 0.1$ for $||\mathcal{F}(x)||_{\infty} = \max_{y \in Y} P_{f \sim \mathcal{F}}[f(x) = y]$.

**Proof.** Recall the definition of the majority and minority subpopulations $X_1$ and $X_2$ from Definition 4. Given a dataset $S$, define $S_1$ to be the partition of $S$ that belongs to $X_1$ and $m_1 = |S_1|$ to be the number of examples in $S$ that belong to the majority subpopulation. Similarly, define $S_2$ and $m_2 = m - m_1$ as the set of minority examples and the size of the set of minority examples respectively. We also use $S^i_1$ to denote the set of $x \in X$ that appears $\ell$ times in $S_i$.

First, we expand the expression for error, defined in Definition 3 as follows.

\[
err_m (A, \Pi_{p,N}, \mathcal{F}) = \mathbb{E}_{S,h,f} \left[ \sum_{x \in X} \Pi_{p,N}(x) \mathbb{I}\{h(x) \neq f(x)\} \right]
\]

\[
= \mathbb{E}_{S,f} \left[ \sum_{\ell,m,i=1} \sum_{x \in S'_{i+1}} \Pi_{p,N}(x) \mathbb{P}_{f \sim \mathcal{F}}[h(x) \neq f(x)] \right]
\]

\[
= \mathbb{E}_{S,f} \left[ \sum_{\ell,i=0} \sum_{x \in S'^i_{i+1}} \Pi_{p,N}(x) \mathbb{P}_{f \sim \mathcal{F}}[h(x) \neq f(x)] \right]
\]

\[
\leq \frac{1}{k} \mathbb{E}_S \left[ |S^0_1| + |S^1_1| \right] + \frac{c_1 p}{N} \mathbb{E}_S \left[ |S^0_2| + |S^1_2| \right]
\]

where $\mathbb{P}_{f \sim \mathcal{F}, h \sim \mathcal{A}(S_f)}[h(x) \neq f(x)] \leq \max_{x \in X_2} \max_{y \in Y} \mathbb{P}_{f \sim \mathcal{F}}[f(x) \neq y] := c_1 \leq 0.1$. To see why, note that this upper bound can be achieved by an algorithm that ignores the labels in the dataset $S_f$ and returns a deterministic classifier that predicts a fixed label for an example, possibly different for different examples. Similarly, we decompose the expression of accuracy discrepancy. For this purpose, for any $\Pi_{p,N}$, define the marginal distribution on the group of *majority* subpopulations $X_2$ as

\[
\Pi^1_{p,N}(x) = \begin{cases} \frac{\Pi_{p,N}(x)}{\sum_{x \in X_1} \Pi_{p,N}(x)} & x \in X_1 \\ 0 & x \notin X_1 \end{cases}
\]

Now, we decompose the definition of accuracy discrepancy from Definition 3 as follows

\[
\Gamma_m (A, \Pi_{p,N}, \mathcal{F}) = \text{err}_m (A, \Pi^2_{p,N}, \mathcal{F}) - \text{err}_m (A, \Pi_{p,N}, \mathcal{F})
\]

\[
= (1-p) \left[ \text{err}_m (A, \Pi^2_{p,N}, \mathcal{F}) - \text{err}_m (A, \Pi^1_{p,N}, \mathcal{F}) \right]
\]

\[
= (1-p) \sum_{\ell,i=0} \mathbb{E}_{S,f} \left[ (-1)^{i+1} \sum_{x \in S'_{i+1}} \Pi^{i+1}_{p,N}(x) \mathbb{P}_{f \sim \mathcal{F}}[h(x) \neq f(x)] \right]
\]

\[
\geq \frac{c_2 (1-p)}{k} \mathbb{E}_S \left[ |S^1_1| + |S^0_1| \right] - \frac{(1-p)}{k} \mathbb{E}_S \left[ |S^1_2| + |S^0_2| \right]
\]
where we define $c_2$ as follows. Let $\tilde{f}$ be sampled from a class of functions that only differ from $f$ at $x$, then $S_{\tilde{f}}$ and $S_f$ are neighboring datasets differing only at $x$. As $\mathcal{A}$ is $(\epsilon, \delta)$-differentially private, for all $S \sim (\Pi_{p,N})^n$ and for all $x \in S_2$, we have the following inequality by the definition of differential privacy

$$
P_{h \sim \mathcal{A}(S_{\tilde{f}})}[h(x) \neq f(x)] = 1 - P_{h \sim \mathcal{A}(S_f)}[h(x) = f(x)] \geq 1 - \epsilon \| \mathcal{F} \|_\infty - \delta \geq \min_{x \in X_2} 1 - \epsilon \max_{y \in Y} P_{f \sim \mathcal{F}}[f(x) = y] - \delta = \min_{x \in X_2} 1 - \epsilon \| \mathcal{F}(x) \|_\infty - \delta := c_2$$

where $\| \mathcal{F}(x) \|_\infty = \max_{y \in Y} P_{f \sim \mathcal{F}}[f(x) = y]$. Now, we will bound the expectation of the terms $|S_1^0|, |S_1^1|, |S_2^0|, |S_2^1|$ individually to obtain the relevant upper and lower bounds. First, we define the following random event over the sampling of the $m$-sized dataset from $\Pi_{p,N}$.

$$\mathcal{E} = \left\{ \frac{p}{2} \leq \frac{m_2}{m} \leq \frac{3p}{2} \right\}$$

As $m_2$ is a binomial distribution with parameter $(m, p)$, as $m \to \infty$ it could be estimated with a Gaussian distribution with mean $mp$ and variance $mp(1-p)$ by Central Limit Theorem (CLT). Using Hoeffding's inequality and the assumption $p < \frac{1}{2}$, we lower bound the probability of $\mathcal{E}$ as follows:

$$P[\mathcal{E}] = 1 - P\left[ m_2 \leq \frac{mp}{2} \right] - P\left[ m_2 \geq \frac{3mp}{2} \right] = 1 - P\left[ |m_2 - mp| \geq \frac{mp}{2} \right] \geq 1 - 2 \exp\left( -\frac{1}{2} mp^2 \right) \to 0$$

Then, by law of total expectation, we have that

$$\lim_{m \to \infty} \mathbb{E}_S[|S_1^0|] = \lim_{m \to \infty} \mathbb{E}_S[|S_1^0| | \mathcal{E}] P[\mathcal{E}] + \lim_{m \to \infty} \mathbb{E}_S[|S_1^0| | \mathcal{E}^c] P[\mathcal{E}^c] = \lim_{m \to \infty} \sum_{x \in X_1} P_{S_1 \sim \Pi_{p,N}^m}\{ x \text{ occurs 0 times in } S_1 | \mathcal{E} \} P[\mathcal{E}] \leq \lim_{m \to \infty} k(1-p) \left( 1 - \frac{1}{k(1-p)} \right)^{(1-\frac{3p}{2})m} P[\mathcal{E}] = 0$$

where the last step follows because on the event $\mathcal{E}$, $m_1 = m - m_2 \geq (1 - \frac{3p}{2}) m$ and $\frac{1}{k(1-p)} \in (0, 1)$. It follows from similar arguments that $\lim_{m \to \infty} \mathbb{E}_S[|S_1^1|] = 0$.

Next, we compute $\mathbb{E}_S[|S_2^0|]$ and $\mathbb{E}_S[|S_2^1|]$. By simple counting argument, we have that

$$\mathbb{E}_S[|S_2^\ell|] = \sum_{x \in X_2} P_{S_2 \sim \Pi_{p,N}^m}\{ x \text{ occurs } \ell \text{ times in } S_2 \} = N\left( \frac{m_2}{\ell} \right) \left( \frac{1}{N} \right)^\ell \left( 1 - \frac{1}{N} \right)^{m_2 - \ell}$$

Plugging in $\ell = 1$ and $\ell = 0$, we get

$$\mathbb{E}_S[|S_2^1|] = m_2 \left( 1 - \frac{1}{N} \right)^{m_2 - 1} \quad \text{and} \quad \mathbb{E}_S[|S_2^0|] = N \left( 1 - \frac{1}{N} \right)^{m_2}$$
respectively. Plugging these expressions back into the expression of error in Equation (6), we obtain the following upper bound

\[
\text{err} (\mathcal{A}, \Pi_{p,N,F}) = \lim_{m,N \to \infty} \text{err}_m (\mathcal{A}, \Pi_{p,N,F}) \\
\leq \frac{1}{k} \lim_{m \to \infty} \mathbb{E}_S [ |S_0^0| + |S_1^0| ] + \lim_{m,N \to \infty} \frac{c_1 p}{N} \mathbb{E}_S [ |S_0^0| + |S_1^0| ] \\
= \lim_{m,N \to \infty} c_1 p \left( \frac{3pm}{2N} \left( 1 - \frac{1}{N} \right)^{p/n-1} + \left( 1 - \frac{1}{N} \right)^{p/n} \right) \\
= c_1 p \left( \frac{3p}{2c} e^{-\eta/2c} + e^{-\eta/2c} \right)
\]

where the last step follows from limit rules because \( \frac{N}{m} \to c \) as \( m, N \to \infty \).

Similarly, we simplify the expression of accuracy discrepancy from Equation (8).

\[
\Gamma (\mathcal{A}, \Pi_{p,N,F}) = \lim_{m,N \to \infty} \Gamma_m (\mathcal{A}, \Pi_{p,N,F}) \\
\geq \lim_{m,N \to \infty} \frac{c_2 (1-p)}{N} \mathbb{E}_S [ |S_2^0| + |S_1^0| ] - \lim_{m,N \to \infty} \frac{(1-p)}{k} \mathbb{E}_S [ |S_1^0| + |S_0^0| ] = 0 \\
\geq c_2 (1-p) \left( \frac{p}{2c} e^{-3p/2c} + e^{-3p/2c} \right)
\]

Next, we show that for any \( \alpha \in [0, 1] \), there is some \( p, c \) such that \( \text{err} (\mathcal{A}, \Pi_{p,N,F}) \leq \alpha \) and \( \Gamma (\mathcal{A}, \Pi_{p,N,F}) \geq 0.5 \).

Setting \( c = 10, p = 0.26 \), the minority group accuracy discrepancy evaluates to \( \Gamma (\mathcal{A}, \Pi_{p,N,F}) \geq 0.5 \). For \( p \in (0, 0.26) \), the lower bound of accuracy discrepancy increases with decreasing \( p \) as its derivative is negative. Therefore, \( \Gamma (\mathcal{A}, \Pi_{p,N,F}) \geq 0.5 \) is always satisfied for all \( p \leq 0.26 \) and \( c = 10 \).

Now, we show that for \( \alpha \in [0, 1] \), there exists a distribution \( \Pi_{p,N} \) with \( p \leq 0.26, c = 10 \) such that \( \text{err} (\mathcal{A}, \Pi_{p,N,F}) \leq \alpha \). Specifically, we first argue that the upper bound of error monotonically increases with \( p \) for \( p \) in \( [0, 0.26] \). For \( c = 10 \), then we show that the upper bound achieves its maximum and minimum values at 0 and 0.26 respectively and invoke the intermediate value theorem to complete the proof.

Taking the derivative of the upper bound of the error, for \( p \in [0, 0.26] \) and \( c = 10 \), we observe that the upper bound on error monotonically increases from \( p = 0 \) to \( p = 0.26 \) for \( c = 10 \). Plugging in \( p = 0.26 \) and \( p = 0 \), we obtain the maximum and minimum values to be 0.027 and 0 respectively. Invoking the intermediate value theorem on this completes the proof.

\[ \square \]

### A.2 Proof for Theorem 2

In this section, we prove the following more detailed version of Theorem 2.

**Theorem 5** (Detailed version of Theorem 2). For any \( p \in (0, \frac{1}{2}) \), \( c > 0 \) such that \( \frac{c}{p} \leq 1 \), consider the distribution \( \Pi_{p,k,N} \) where \( N/m \to c \) as \( N,m \to \infty \), \( k = o(m) \), and \( k(1-p) \geq 2 \). For any \( \epsilon > 0, \frac{c}{p} \geq \delta > 0 \), such that \( \frac{1}{c} \log \left( \frac{1}{\epsilon} \right) = o(m) \), consider an \((\epsilon, \delta)\)-DP algorithm \( \mathcal{A} \) that satisfies Assumption A with \( s_0 = \lfloor \min \left( \frac{1}{c} \log 2, \log \left( \frac{1}{\epsilon^2} \right) \right) \rfloor \) and some \( p_1 \in (0, 1) \). Also, let \( F \) be subpopulation-wise independent. Then, for the limits \( \text{err} (\mathcal{A}, \Pi_{p,N,F}) = \lim_{m,N \to \infty} \text{err}_m (\mathcal{A}, \Pi_{p,N,F}) \) and \( \Gamma (\mathcal{A}, \Pi_{p,N,F}) = \lim_{m,N \to \infty} \Gamma_m (\mathcal{A}, \Pi_{p,N,F}) \), we have

\[
\Gamma (\mathcal{A}, \Pi_{p,N,F}) \geq \frac{(1-\|F\|_\infty)(1-p)}{3} \left( 1 - \sqrt{\frac{3p e^{-\pi c(p_0 - e^{-\pi c p_0})^2}}{2 \left( s_0 - \frac{3p}{2c} \right)}} \right) - (1-p)p_1
\]
while the error is upper bounded by

$$\text{err} (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) \leq (1 - p_1)p \left( 1 - \frac{(s_0 - p)^2 \frac{2e}{p} - 1}{\sqrt{2\pi}(s_0 - p)^2 \left( \frac{2e}{p} \right)^2} \right) + p_1$$

where $\|\mathcal{F}\|_\infty = \max_{x \in X, y \in Y} \mathbb{P}_{f \sim \mathcal{F}} [f(x) = y]$.

**Proof.** We use the same extra notation for $\{m_i, S_i, S_i^t\}$ for $i \in \{1, 2\}$ and $\ell \in \mathbb{N}$, $\ell \leq m$ as in the proof of Theorem 4. First, we decompose the overall error.

\[
\text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = \mathbb{E}_{S,f} \left[ \sum_{\ell=0}^{s_0} \sum_{x \in S^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim \mathcal{A}(S_i)} [h(x) \neq f(x)] + \sum_{\ell=s_0+1}^{m} \sum_{x \in S^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim \mathcal{A}(S_j)} [h(x) \neq f(x)] \right] \\
\leq \sum_{\ell=0}^{s_0} \mathbb{E}_{S} \left[ \sum_{x \in S^t} \Pi_{p,N}(x) \right] + p_1 \mathbb{E}_{S} \left[ \sum_{\ell=0}^{m} \sum_{x \in S^t} \Pi_{p,N}(x) \right] - p_1 \sum_{\ell=0}^{s_0} \mathbb{E}_{S} \left[ \sum_{x \in S^t} \Pi_{p,N}(x) \right] \\
= (1 - p_1) \sum_{\ell=0}^{s_0} \mathbb{E}_{S} \left[ \sum_{x \in S^t} \Pi_{p,N}(x) \right] + p_1 \\
\leq (1 - p_1) \sum_{\ell=0}^{s_0} (\mathbb{E}_{S} [|S_i^t|] + |S_i^t|) + p_1
\]

where $S, f$ are distributed as $S \sim \Pi_{p,N}^m$ and $f \sim \mathcal{F}$. In step (a), the second expectation vanishes by noting that $\sum_{\ell=0}^{m} \sum_{x \in S^t} \Pi_{p,N}(x) = 1$.

Further, by definition of accuracy discrepancy in Equation (5) we can write,

$$\Gamma_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = (1 - p) \left( \text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) - \text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) \right).$$

We now expand the above two terms in the decomposition of $\Gamma_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F})$.

\[
\text{err}_m (\mathcal{A}, \Pi_{p,N}^2, \mathcal{F}) = \mathbb{E}_{S} \left[ \sum_{\ell=0}^{s_0} \sum_{x \in S_i^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim \mathcal{A}(S_i)} [h(x) \neq f(x)] \right] \\
\geq \sum_{\ell=0}^{s_0} \mathbb{E}_{S} \left[ \sum_{x \in S_i^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim \mathcal{A}(S_i), f \sim \mathcal{F}} [h(x) \neq f(x)] \right] \\
\geq \frac{(1 - \|\mathcal{F}\|_\infty)}{3N} \sum_{\ell=0}^{s_0} \mathbb{E}_{S} \left[ |S_i^t| \right]
\]

where the last step follows by applying Lemma 3 (which is the detailed version of Lemma 1 in the main text) with the subpopulation-wise independent $\mathcal{F}$ and with $s_0 = \left\lceil \frac{1}{e} \log 2, \log \left( \frac{1}{2\delta} \right) \right\rceil$. Next, we upper
bound the error of the majority group.

\[
\text{err}_m (A, \Pi_{p,N}^1, F) = \mathbb{E}_{S,F} \left[ \sum_{\ell=0}^{s_0} \sum_{x \in S_{\ell}^1} \Pi_{p,N}^1(x) \mathbb{P}_{h \sim A(S_f)}[h(x) \neq f(x)] \right] + \mathbb{E}_{S,F} \left[ \sum_{\ell=s_0+1}^{m_1} \sum_{x \in S_{\ell}^1} \Pi_{p,N}^1(x) \mathbb{P}_{h \sim A(S_f)}[h(x) \neq f(x)] \right] \leq \mathbb{E}_{S} \left[ \sum_{\ell=0}^{s_0} \sum_{x \in S_{\ell}^1} \Pi_{p,N}^1(x) \right] + p_1 \mathbb{E}_{S} \left[ \sum_{\ell=s_0+1}^{m_1} \sum_{x \in S_{\ell}^1} \Pi_{p,N}^1(x) \right] \leq \frac{1}{k(1-p)} \mathbb{E}_{S} \left[ \sum_{\ell=0}^{s_0} |S_{\ell}| \right] + p_1
\]

where step \(a\) follows from Assumption \(A1\) and step \(b\) follows because \(\sum_{\ell=s_0+1}^{m_1} \sum_{x \in S_{\ell}^1} \frac{1}{k(1-p)} \Pi_{p,N}^1(x) \leq 1\) and \(\Pi_{p,N}^1(x) = \frac{1}{k(1-p)}\).

In order to complete the proof, next we evaluate the terms \(S_{\ell}^1\) and \(S_{\ell}^2\) respectively. We recall the definition of the random event \(\mathcal{E}\) from the proof of Theorem 4

\[\mathcal{E} = \left\{ \frac{p}{2} \leq \frac{m_2}{m} \leq \frac{3p}{2} \right\} \quad \text{and} \quad \mathbb{P}[\mathcal{E}] = 1 - o\left(e^{-\frac{m-p}{2}}\right) \quad \text{as} \quad m \to \infty\]

Then, by law of total expectation, for all \(\ell\), we have

\[
\lim_{m \to \infty} \mathbb{E}[|S_{\ell}^1|] = \lim_{m \to \infty} \mathbb{E}[|S_{\ell}^1| | \mathcal{E}] \mathbb{P}[\mathcal{E}] + \lim_{m \to \infty} \mathbb{E}[|S_{\ell}^1| | \mathcal{E}^c] \mathbb{P}[\mathcal{E}^c]
\]

\[
= \lim_{m \to \infty} \sum_{x \in X_1} \mathbb{P}_{S_1 \sim \Pi_{p,N}^1(m_1)}[x \text{ occurs } \ell \text{ times in } S_1] \mathbb{P}[\mathcal{E}]
\]

\[
= \lim_{m \to \infty} k(1-p) \binom{m_1}{\ell} \left( \frac{1}{k(1-p)} \right)^\ell \left( 1 - \frac{1}{k(1-p)} \right)^{m_1-\ell} \mathbb{P}[\mathcal{E}]
\]

\[
\leq \lim_{m \to \infty} k(1-p) \binom{em_2}{\ell} \left( \frac{1}{k(1-p)} \right)^\ell \left( 1 - \frac{1}{k(1-p)} \right)^{m_2-\ell} \mathbb{P}[\mathcal{E}]
\]

where step \(a\) follows from a simple counting argument, step \(b\) follows from the well known inequality \(\binom{m_1}{\ell} \leq \binom{em_2}{\ell}\). Using standard limit rules we have

\[
\lim_{m \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{\ell}^1|] = 0
\]

under the assumption that \(s_0, k = o(m)\) and conditioned on the event \(\mathcal{E}\) which requires \(m_1 \geq \frac{(1-p)m}{2}\).

Using a similar technique, we next lower bound \(\lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{\ell}^2|]\)

\[
\lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{\ell}^2|] = \lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} N \binom{m_2}{\ell} \left( \frac{1}{N} \right)^\ell \left( 1 - \frac{1}{N} \right)^{m_2-\ell} \mathbb{P}[\mathcal{E}]
\]

\[
= \lim_{m,N \to \infty} N \Phi_{\ell \sim \text{binom}(m_2, \frac{1}{N})} \left[ \ell \leq s_0 \right] \mathbb{P}[\mathcal{E}]
\]

\[
\leq N \Phi \left( \frac{s_0 - 3p}{2c} \sqrt{\frac{2c}{3p}} \right),
\]

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Then, we have the following lower and upper tail bounds on $W$ for all $x > 0$:

$$
\left(1 - \frac{1}{x^3}\right) \frac{e^{-\frac{x^2}{2} \pi}}{\sqrt{2\pi}} \leq \Pr[W \geq \mu + \sigma x] \leq \frac{e^{-\frac{x^2}{2} \pi}}{x \sqrt{2\pi}}.
$$

This gives the following upper bound on $\lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{2,\ell}^m|]$:

$$
\lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{2,\ell}^m|] \leq N \left(1 - \frac{(s_0 - \frac{p}{2c})^2 2x}{p} - 1\right) \frac{\sqrt{2\pi}}{\sqrt{2\pi}} e^{-(s_0-p/2c)^2/p},
$$

and the following lower bound

$$
\lim_{m,N \to \infty} \sum_{\ell=0}^{s_0} \mathbb{E}[|S_{2,\ell}^m|] \geq N \left(1 - \sqrt{\frac{3p}{\pi c}} \frac{e^{-\frac{\pi}{\pi c}}(s_0 - \frac{3x}{2c})^2}{2(s_0 - \frac{3x}{2c})}\right).
$$

Plugging Equations (19) and (22) in Equation (15), we obtain the upper bound on the (overall) error in the asymptotic limit

$$
\text{err} (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = \lim_{m,N \to \infty} \text{err}_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) 
\leq (1 - p_1) p \left(1 - \frac{(s_0 - \frac{p}{2c})^2 2x}{p} - 1\right) \frac{\sqrt{2\pi}}{\sqrt{2\pi}} e^{-(s_0-p/2c)^2/p} + p_1.
$$

Plugging Equation (23) into Equation (17) and Equation (19) into Equation (18), we obtain the required expressions for accuracy discrepancy in Equation (16) and the desired result follows.

$$
\Gamma (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) = \lim_{m,N \to \infty} \Gamma_m (\mathcal{A}, \Pi_{p,N}, \mathcal{F}) 
\geq \frac{(1 - \|\mathcal{F}\|_\infty) (1 - p)}{3} \left(1 - \sqrt{\frac{3p}{\pi c}} \frac{e^{-\frac{\pi}{\pi c}}(s_0 - \frac{3x}{2c})^2}{2(s_0 - \frac{3x}{2c})}\right) - (1 - p)p_1.
$$

\[\square\]

A.3 Proof for Theorem 3

In this section, we prove the following more detailed version of Theorem 3.
Theorem 6 (Detailed version of Theorem 3). For any \( p \in (0, 1/2), c > 0 \) such that \( p/c \leq 1 \), consider the distribution \( \Pi_{p,N} \) where \( N \) is the number of minority subpopulations. The number of majority subpopulations is \( k(1 - p) \) which satisfies \( k(1 - p) \geq 2 \) and \( k = o(m) \). For any \( \alpha > 0 \), \( \epsilon = \frac{\log^2 2}{\alpha \sqrt{m} + \log m} \) and \( \delta < 2^{-\frac{1}{2}} \), consider any \((\epsilon, \delta)\)-DP algorithm \( A \). Then, for the limits \( N/m \to c \) as \( m, N \to \infty \) we have

\[
\text{err}(A, \Pi_{p,N}, \mathcal{F}) \geq \left(1 - \frac{\| \mathcal{F} \|_\infty}{3} \right) \left(1 - (1 - p) \left( e^{-(2-p)\epsilon^2} \right) \right)
\]

where \( \text{err}(A, \Pi_{p,N}, \mathcal{F}) = \lim_{N \to \infty} \text{err}_m(A, \Pi_{p,N}, \mathcal{F}) \), and \( \Gamma(A, \Pi_{p,N}, \mathcal{F}) = \lim_{N \to \infty} \Gamma_m(A, \Pi_{p,N}, \mathcal{F}) \).

Proof. We use the same extra notations as in the proof of Theorem 4 and 5. In addition, we define \( s_0 = \left\lfloor \frac{1}{2} \log 2, \log (\frac{1}{1 - p}) \right\rfloor \). As in previous proofs, we first decompose the expression for overall error as

\[
\text{err}_m(A, \Pi_{p,N}, \mathcal{F}) = E_S \left[ \sum_{\ell = 0}^{s_0} \sum_{x \in S_\ell^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim A(S_f), f \sim \mathcal{F}}[h(x) \neq f(x)] \right] + E_S \left[ \sum_{\ell = s_0 + 1}^{m} \sum_{x \in S_\ell^t} \Pi_{p,N}(x) \mathbb{P}_{h \sim A(S_f), f \sim \mathcal{F}}[h(x) \neq f(x)] \right] \geq \frac{(1 - \| \mathcal{F} \|_\infty)}{3} \sum_{\ell = 0}^{s_0} \left( \frac{1}{k} E_S [||S_\ell^t||] + \frac{p}{N} E_S [||S_\ell^t||] \right),
\]

where the last step follows from the definition of \( s_0 \) and Lemma 3. Further, by definition of accuracy discrepancy in Equation (5)

\[
\Gamma_m(A, \Pi_{p,N}, \mathcal{F}) = (1 - p) \left( \text{err}_m(A, \Pi_{p,N}^2, \mathcal{F}) - \text{err}_m(A, \Pi_{p,N}^1, \mathcal{F}) \right) \leq (1 - p) \left( 1 - \text{err}_m(A, \Pi_{p,N}^1, \mathcal{F}) \right),
\]

where we use the simple upper bound \( \text{err}_m(A, \Pi_{p,N}^2, \mathcal{F}) \leq 1 \) for error in the minority subpopulations. We next lower bound the error of the majority subpopulations.

\[
\text{err}_m(A, \Pi_{p,N}^1, \mathcal{F}) = E_S \left[ \sum_{x \in X_1} \Pi_{p,N}^1(x) \mathbb{P}_{f \sim \mathcal{F}, h \sim A(S_f)}[h(x) \neq f(x)] \right] \geq E_S \left[ \sum_{\ell = 0}^{s_0} \sum_{x \in S_\ell^t} \frac{1}{k(1 - p)} \left( 1 - \| \mathcal{F} \|_\infty \right) \right] \geq \frac{(1 - \| \mathcal{F} \|_\infty)}{3k(1 - p)} \sum_{\ell = 0}^{s_0} E_S [||S_\ell^t||].
\]

In step (a), we drop the terms with \( \ell > s_0 \) and then apply Lemma 3 with the definition of \( s_0 \). We recall the definition of the random event \( \mathcal{E} \) from the proof of Theorem 4

\[
\mathcal{E} = \left\{ \frac{p}{2} \leq \frac{m_2}{m} \leq \frac{3p}{2} \right\} \quad \text{and} \quad \mathbb{P}[\mathcal{E}] = 1 - o \left( e^{-\frac{m_2}{2}} \right) \quad \text{as} \quad m \to \infty
\]

Next, we show that all points in the minority group occur less than \( s_0 \) times in \( S \).
\[
\lim_{m,N \to \infty} \sum_{\ell = s_0 + 1}^{m_2} \mathbb{E}_S [ |S'_2| ] = \lim_{m,N \to \infty} \sum_{\ell = s_0 + 1}^{m_2} \mathbb{E}_S [ |S'_2| ] \mathbb{P}_\epsilon(\mathcal{E}) + \mathbb{E}_S [ |S'_2| ] \mathbb{P}(\mathcal{E}'^c) \\
= \lim_{m,N \to \infty} \sum_{\ell = s_0 + 1}^{m_2} N \left( \frac{m_2}{\ell} \right) \left( \frac{1}{N} \right)^{\ell} \left( 1 - \frac{1}{N} \right)^{m_2 - \ell} \mathbb{P}_\epsilon(\mathcal{E}) \\
\leq \lim_{m,N \to \infty} N e^{-2m_2 \left(1 - \frac{1}{N} - \frac{m_2 - s_0}{m_2} \right)^2}
\]
where the last step is an instantiation of the Hoeffding’s inequality on the upper tail of a Bernoulli random variable. Noting that \( \delta < 2^{-\frac{1}{2}} \), we have \( s_0 = \log \frac{2}{\delta} = \alpha \sqrt{m} + \frac{(1 - \frac{1}{2}p)}{k(1-p)} m \). Using the limit \( \frac{N}{m} \to c \) as \( m, N \to \infty \) and \( m_2 \geq \frac{em}{2} \) due to the conditioning on the event \( \mathcal{E} \),

\[
\lim_{m,N \to \infty} \sum_{\ell = s_0 + 1}^{m_2} \mathbb{E}_S [ |S'_2| ] \leq \lim_{N,m \to \infty} Ne^{-m \left( \frac{2\sqrt{\alpha \sqrt{m}}}{p} + \frac{2 - \alpha}{2\sqrt{m} (1-p)} - \frac{\epsilon}{2} \right)} = 0. \tag{29}
\]

Therefore,

\[
\lim_{m,N \to \infty} \sum_{\ell = 0}^{s_0} \mathbb{E}_S [ |S'_2| ] = N - \lim_{m,N \to \infty} \sum_{\ell = s_0 + 1}^{m_2} \mathbb{E}_S [ |S'_2| ] = N \tag{30}
\]

Similarly, using the law of total expectation and a counting argument, we lower bound \( \sum_{\ell = 0}^{s_0} \mathbb{E}_S [ |S'_1| ] \)

\[
\sum_{\ell = 0}^{s_0} \mathbb{E}_S [ |S'_1| ] \geq \sum_{\ell = 0}^{s_0} k(1-p) \left( \frac{m_1}{\ell} \right) \left( \frac{1}{k(1-p)} \right)^{\ell} \left( 1 - \frac{1}{k(1-p)} \right)^{m_1 - \ell} \\
= k(1-p) \mathbb{P}_{\ell \sim \text{binom}(m_1, \frac{1}{k(1-p)})} [\ell \leq s_0]. \tag{31}
\]

To calculate \( \mathbb{P}_{\ell \sim \text{binom}(m_1, \frac{1}{k(1-p)})} [\ell \leq s_0] \), we apply Hoeffding’s inequality on Bernoulli distribution.

\[
\mathbb{P}_{\ell \sim \text{binom}(m_1, \frac{1}{k(1-p)})} [\ell \leq s_0] \geq 1 - e^{-2m_1 \left(1 - \frac{1}{k(1-p)} - \frac{m_1 - s_0}{m_1} \right)^2} \\
\overset{(a)}{=} 1 - e^{-\frac{(2-p)m}{k(1-p)}} \left( \frac{m_1 - s_0}{m_1} \right)^2 \\
= 1 - e^{-\frac{(2-p)m^2}{(1-p)^2k}} \tag{32}
\]

where step \( (a) \) follows from conditioning on \( \mathcal{E} \) and due to \( s_0 = \frac{1 - 3p/2}{k(1-p)} m + \alpha \sqrt{m} \). Plugging Equation \( 32 \) into Equation \( 31 \), we have

\[
\lim_{m,N \to \infty} \sum_{\ell = 0}^{s_0} \mathbb{E}_S [ |S'_1| ] \geq k(1-p) \left( 1 - e^{-\frac{(2-p)m^2}{(1-p)^2k}} \right). \tag{33}
\]

Plugging Equations \( 30 \) and \( 33 \) into Equation \( 26 \), we lower bound overall error

\[
\text{err}(A, \Pi_p, \mathcal{F}) = \lim_{m,N \to \infty} \text{err}_m(A, \Pi_p, \mathcal{F}) \\
\geq \left( \frac{1 - \| \mathcal{F} \|_{\infty}}{3} \right) \left( 1 - (1-p) \left( e^{-\frac{(2-p)m^2}{(1-p)^2k}} \right) \right). \tag{34}
\]

Further, plugging Equations \( 28 \) and \( 33 \) in Equation \( 27 \) we upper bound unfairness

\[
\Gamma(A, \Pi_p, \mathcal{F}) = \lim_{m,N \to \infty} \Gamma_m(A, \Pi_p, \mathcal{F}) \\
\leq (1-p) \left[ 1 - \left( \frac{1 - \| \mathcal{F} \|_{\infty}}{3} \right) \left( 1 - e^{-\frac{(2-p)m^2}{(1-p)^2k}} \right) \right], \tag{35}
\]

which completes the proof of Theorem \( \emptyset \).
A.4 Proof of Lemma 1

Next, we restate and prove Lemma 1. The proof makes use of the concept of vertex cover and independent set that we define next.

**Definition 5** (Vertex Cover and Independent Set). Consider any undirected graph $G = (V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. Also, let $P$ define a probability distribution over the vertices $V$.

- A set $C \subseteq V$ is called the vertex cover of $G$ if for all edges $(u, v) \in E$ at least one of $u$ and $v$ belongs to $C$.

The smallest such set is known as the minimum vertex cover. The vertex cover with the smallest probability mass under $P$ is known as the probabilistically minimum vertex cover ($P$MVC ($G$)). If a vertex cover does not remain a vertex cover upon the removal of any of its vertices, then it is known as a minimal vertex cover.

- A set $I \subseteq V$ is called an independent set of $G$ if for any two vertices $u, v \in I$, the edge $(u, v) \notin E$. The largest such set is known as the maximum independent set and the independent set with the largest probability mass is known as the probabilistically maximum independent set. If an independent set does not remain an independent set upon including a vertex from outside the set, then it is known as a maximal independent set.

- All minimum vertex covers and maximum independent sets are minimal vertex covers and maximal independent sets respectively but not vice versa.

The complement of any vertex cover $C$ is an independent set $I = V \setminus C$. Therefore, the complement of a minimum vertex cover is a maximum independent set.

Further, for some $x \in S$, we let $Q \in \mathcal{Y}^{m-1}$ represent an arbitrary labeling of the points in $S \setminus \{x\}$ and use $\mathcal{F}|_{S \setminus \{x\}, Q}$ to denote the marginal distribution of $\mathcal{F}$ over the set of functions that satisfies the labeling $Q$ on $S \setminus \{x\}$ and $\mathcal{F}|_{S \setminus \{x\}, Q}$ denote the support of $\mathcal{F}|_{S \setminus \{x\}, Q}$. We also define conditions $\text{C1}$ and $\text{C2}$ on labeling functions as follows.

Let $S$ be a dataset of size $m$. Then, for any $x \in S$, we say that two labelling functions $f_1, f_2$ satisfy Condition $\text{C1}$ with respect to $x$ if they differ only in $x$ i.e.

\[
f_1(z) = f_2(z) \text{ for all } z \neq x, \quad z \in S \quad \text{ (C1)}
\]

\[
f_1(z) \neq f_2(z) \text{ for } z = x.
\]

Further, a labelling function $f$ satisfies condition $\text{C2}$ with respect to $x$ and an algorithm $\mathcal{A}$ with parameter $q$ if

\[
\mathbb{P}_{h \sim \mathcal{A}(S_f)}[h(x) \neq f(x)] > 1 - q. \quad \text{(C2)}
\]

**Lemma 3.** [Detailed version of Lemma 1] Consider a label prior $\mathcal{F}$, an $m$-sized unlabeled $S \sim \Pi_{n,N}^m$, and an $(\epsilon, \delta)$-DP algorithm $\mathcal{A}$. For any $q \in (0, 1)$, let $s_0$ be the largest integer that satisfies $q \geq \frac{1 + \log m - \frac{\delta}{\epsilon}}{1 + \log m - \frac{\delta}{\epsilon}}$. Then for any $x \in S'$, where $\ell \in \mathbb{N}$, $\ell \leq s_0$, there exists a set of labeling functions $\hat{F}(x) \subset F$ such that for all functions $f \in \hat{F}(x)$,

\[
\mathbb{P}_{h \sim \mathcal{A}(S_f)}[h(x) \neq f(x)] > 1 - q
\]

and the size of $\hat{F}(x)$ is large

\[
|\hat{F}(x)| \geq |F| - \sum_{Q \in \mathcal{Y}^{m-1}} \max_{y \in \mathcal{Y}} \left| \{f \in F|_{S \setminus \{x\}, Q} : f(x) = y \} \right|.
\]

In particular, if $\mathcal{F}$ is subpopulation-wise independent, for $s_0 = \lceil \min \left( 0.5 \log 2, \log \left( \frac{4}{\epsilon} \right) \right) \rceil$, we have

\[
\mathbb{P}_{h \sim \mathcal{A}(S_f), f \sim \mathcal{F}}[h(x) \neq f(x)] > 1 - \frac{\|\mathcal{F}\|_{\infty}}{3}.
\]

**Proof of Lemma 3.** Let $S$ be a dataset of size $m$. By Lemma 4, if $q \geq \frac{1 + \log m - \frac{\delta}{\epsilon}}{1 + \log m - \frac{\delta}{\epsilon}}$ and if $f_1, f_2$ satisfy Condition $\text{C1}$ then at least one of them satisfies condition $\text{C2}$ with respect to all $x \in S'$ for $\ell \leq s_0$ and all $(\epsilon, \delta)$-DP algorithm $\mathcal{A}$ with parameter $q$ i.e.
\[ P_{h \sim A(S_j)}[h(x) \neq f(x)] > 1 - q. \]

Next, we first measure the minimum size of the set of such labelling functions \( f \) and then lower bound the probability with which such a labelling function will be drawn when sampling from \( \mathcal{F} \). Let \( \zeta(Q, x) \) denote the total number of functions in \( F_{S \setminus \{x\}, Q} \) that satisfy condition \( \mathcal{C}2 \). We show that \( \zeta(Q, x) \) is bigger than the size of the minimum vertex cover of the following graph.

Construct a graph \( G_{S, x, Q} \) where every function \( f_i \in F_{S \setminus \{x\}, Q} \) represents a vertex. Construct an edge between two vertices \( f_i, f_j \) if they satisfy Condition \( \mathcal{C}1 \), i.e., \( f_i(x) \neq f_j(x) \). It is easy to see that this is a complete \( |Y| \)-partite graph where each partition consists of functions that predict a different label \( y \) on \( x \). They are defined as \( \{ f \in F_{S \setminus \{x\}, Q} : f(x) = y \} \). A vertex cover of a graph (defined in Definition 5) is a set of vertices such that for every edge at least one of its endpoints are included in the set. As for every edge in \( G_{S, x, Q} \), both its endpoints satisfy condition \( \mathcal{C}1 \) at least one of them should satisfy condition \( \mathcal{C}2 \). Therefore, the total number of functions that satisfy condition \( \mathcal{C}2 \) is at least the size of the minimum vertex cover of \( G_{S, x, Q} \). Using Lemma 5, we can lower bound the probability mass of \( \hat{Q} \).

The total number of labeling functions that satisfy condition \( \mathcal{C}2 \) is

\[
\zeta(Q, x) \geq |F_{S \setminus \{x\}, Q}| - \max_{y \in Y} \left| \{ f \in F_{S \setminus \{x\}, Q} : f(x) = y \} \right|.
\]

Summing this up for all labellings, the total number of labeling functions that satisfy condition \( \mathcal{C}2 \) is

\[
\left| \hat{F}(x) \right| = \sum_{Q \in \mathcal{Y}^{-1}} \zeta(Q, x) \geq \sum_{Q \in \mathcal{Y}^{-1}} |F_{S \setminus \{x\}, Q}| - \max_{y \in Y} \left| \{ f \in F_{S \setminus \{x\}, Q} : f(x) = y \} \right| = |F| - \sum_{Q \in \mathcal{Y}^{-1}} \max_{y \in Y} \left| \{ f \in F_{S \setminus \{x\}, Q} : f(x) = y \} \right|.
\]

This completes the proof of Equation (37). To prove Equation (38), consider \( \mathcal{F} \) to be a subpopulation-wise independent label prior and recall that for all \( Q \in \mathcal{Y}^{-1} \), \( G_{S, x, Q} \) is a complete multi-partite graph. Using the same argument as above, with Lemma 4, we can lower bound the probability mass of \( \hat{F}(x) \)

\[
P_{f \sim \mathcal{F}} \left[ f \in \hat{F}(x) \right] \geq \sum_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in \text{PMVC}(G_{S, x, Q}) \right]
\]

\[
= \sum_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in \text{PMVC}(G_{S, x, Q}) \right] P_{f} \left[ f \in F_{S \setminus \{x\}, Q} \right]
\]

\[
\geq \min_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in \text{PMVC}(G_{S, x, Q}) \right] \sum_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in F_{S \setminus \{x\}, Q} \right] = (1 - \|F\|_{\infty})
\]

where \( \sum_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in F_{S \setminus \{x\}, Q} \right] = 1 \) by the law of total probability. Using the assumption that \( \mathcal{F} \) is subpopulation-wise independent and the definition of \( \|F\|_{\infty} \), we obtain

\[
\min_{Q \in \mathcal{Y}^{-1}} P_{f} \left[ f \in \text{PMVC}(G_{S, x, Q}) \right] \geq (1 - \|F\|_{\infty})
\]

and thus

\[
P_{f \sim \mathcal{F}} \left[ f \in \hat{F}(x) \right] \geq (1 - \|F\|_{\infty}).
\]

Using Lemma 4 and Equation (40), we can lower bound

\[
P_{f \sim \mathcal{F}, h \sim A(S_j)} \left[ h(x) \neq f(x) \right] \geq P_{f \sim \mathcal{F}, h \sim A(S_j)} \left[ h(x) \neq f(x) \mid f \in \hat{F}(x) \right] P_{f} \left[ f \in \hat{F}(x) \right]
\]

\[
\geq (1 - q)(1 - \|F\|_{\infty})
\]

Substituting the value of \( s_0 = \min \left( \frac{1}{2} \log 2, \log \left( \frac{1}{2} \right) \right) \), we obtain \( q = \frac{2}{3} \geq \frac{1 + \log e^{-\delta}}{1 + e^{-\delta}} \). Using \( q = \frac{2}{3} \) in Equation (41) completes the proof of Equation (38).
Lemma 4. Let $S$ be a dataset of size $m$ and $q \in (0,1)$. Further, let $s_0$ be the largest integer that fulfills $q \geq \frac{1 + s_0 e^{-\epsilon \delta}}{1 + e^{-\epsilon \delta}}$. Then, for an $(\epsilon, \delta)$-differentially private algorithm $A$, for any $x \in S^\ell$ where $\ell \in \mathbb{N}, \ell \leq s_0$, if two functions $f_1, f_2$ agree on all points in $S \setminus \{x\}$ but disagree on $x$, i.e. satisfies condition (C1), then at least one of the two functions $f \in \{f_1, f_2\}$ satisfies condition (C2)

$$\mathbb{P}_{h \sim A(S)} [h(x) \neq f(x)] > 1 - q.$$  

Proof of Lemma 4. We prove this via contradiction. Let’s assume that two labeling functions $f_1, f_2$ satisfying condition (C1) also satisfies

$$\mathbb{P}_{h \sim A(S)} [h(x) \neq f(x)] \leq 1 - q$$  

for some $x \in S^\ell$. Then, we lower bound the probability that an $(\epsilon, \delta)$-DP algorithm trained on a dataset, labeled with $\tilde{f}$, disagrees with $f$ on $x \in S^\ell$.

$$\mathbb{P}_{h \sim A(S,j)} [h(x) = f(x)] \geq \left(\mathbb{P}_{h \sim A(S,j)} [h(x) = \tilde{f}(x)] - \ell e^{(\ell-1)\epsilon \delta}\epsilon^{-\ell\epsilon}\right) e^{-\ell\epsilon}$$

The first inequality follows from the notion of group differential privacy and the second inequality follows from Assumption (42). Further, by definition of $f_1, f_2$ and Assumption (42), we have

$$\mathbb{P}_{h \sim A(S)} [h(x) = f(x)] \leq \mathbb{P}_{h \sim A(S)} [h(x) \neq \tilde{f}(x)] \leq 1 - q.$$  

Combining the two inequalities (43,44), it has to hold that $q \leq \frac{1 + s_0 e^{-\epsilon \delta}}{1 + e^{-\epsilon \delta}}$. However, this leads to a contradiction as $q \geq \frac{1 + s_0 e^{-\epsilon \delta}}{1 + e^{-\epsilon \delta}}$. This implies that for any $\ell \leq s_0$ such that $q \geq \frac{1 + s_0 e^{-\epsilon \delta}}{1 + e^{-\epsilon \delta}}$, Assumption (42) is incorrect which concludes our proof.

Lemma 5. Given $k$ sets of vertices $\{V_i\}_{i=1}^k$, consider a complete $k$-partite graph $G = (V, E)$ where the $k^{th}$ partition consists of $V_i$. Then the size of the minimum vertex cover is $|V| - \max_i |V_i|$. Further, if there exists a probability distribution $\mu$ over the vertices, the probabilistically minimum vertex cover (or PMVC($G$)) has a probability mass of $1 - \max_i \mu |V_i|$ where $\mu |V_i| = \mathbb{P}_{V \sim \mu} [V \in V_i]$.

Proof of Lemma 5. A set of vertices is known as an independent set if and only if its compliment is a vertex cover. Therefore, we use the identity that the size of the minimum vertex cover and the maximum independent set of a graph equals to the total number of vertices of the graph. Therefore, the problem of finding the minimum vertex cover is equivalent to finding the maximum independent set. 

By the definition of independent set, no two vertices in the independent set are adjacent. Without loss of generality, let $v \in V_i$ for some $i$ be in the independent set $\mathcal{I}$. Then, by definition of complete $k$-partite graph, all vertices belonging to $\bigcup_{j \neq i} V_j$ are adjacent to $v$ and therefore cannot be in $\mathcal{I}$. However, all other vertices in $V_i$ can be included in $\mathcal{I}$. This creates a maximal independent set. Thus, there are only $k$ maximal independent sets and the set corresponding to $i^* = \arg\max_i |V_i|$ is the maximum independent set. Therefore, the size of the minimum vertex cover is $k - \max |V_i|$.

By a similar reasoning, the set corresponding to $i^* = \arg\max_i \mu |V_i|$ is the probabilistically maximum independent set. Therefore, the probability mass of PMVC($G$) is $1 - \max_i \mu |V_i|$.

A.5 Proof of Lemma 2

We restate a detailed version of Lemma 2 and provide the proof below.

Lemma 6. The algorithm $A^\eta : S \rightarrow Y^X$ is \((\log \left(\frac{1 - \eta (|Y| - 1)}{\eta}\right), 0)\)-differentially private. Further for any dataset $S_f$ and $s_0 \in \mathbb{N}$,
• if a subpopulation $x$ appears more than $s_0$ times in $S$, $P_{h \sim A_\eta(S_f)}[h(x) \neq f(x)] \leq e^{-\alpha(1-2\eta^2)s/(1-\eta)}$ and 

• if a subpopulation $x$ appears less than $s_0$ times in $S$, $P_{h \sim A_\eta(S_f)}[h(x) \neq f(x)] \geq \frac{1}{\sqrt{2\pi} \eta(1-\eta)^{\eta/2}}$.

Proof. We restate the algorithm for the sake of completeness. The algorithm $A_\eta$ accepts a dataset $S_f \in (X \times Y)^\eta$ and a noise rate $\eta \in (0, \frac{1}{2})$ as input. Then it creates a dictionary where the set $X$ is the set of keys. In order to assign values to every key, it first randomly flips the label of every element in $S_f$ with probability $\eta$, then for every unique key in $S_f$, the algorithm computes the majority label of that key in the flipped dataset and assigns that majority label to the corresponding key. For elements in $X$ not present in $S_f$, it assigns a random element from $\mathcal{Y}$.

Privacy: We first prove that $A_\eta$ is $(\epsilon, 0)$-differentially private. For any two neighboring datasets $S_f$ and $S_{f'}$ that differ in one element, the classifiers obtained from the algorithm $A_\eta$ on the two datasets differs the most if the datasets differ on a singleton point $x'$ i.e. a point which appears just once in the dataset.

$$\begin{align*}
\text{For all } x' \in S, f \in F, \text{ and } \tilde{f} \in F_{S \setminus \{x'\}, f} \\
\frac{P_{h \sim A(S_f)}[h(x) = f(x), \forall x \in S]}{P_{h \sim A(S_f)}[h(x) = f(x), \forall x \in S]} = \frac{\prod_{x \in S} P_{h \sim A(S_f)}[h(x) = f(x)]}{\prod_{x \in S} P_{h \sim A(S_f)}[h(x) = f(x)]} \\
= \frac{P_{h \sim A(S_f)}[h(x') = f(x')]}{P_{h \sim A(S_f)}[h(x') = f(x')]} \\
= \frac{P_{h \sim A(S_f)}[x' \text{ is not flipped}]}{P_{h \sim A(S_f)}[x' \text{ is flipped and flipped to } f(x')]} \\
= \frac{1 - \eta}{\eta |\mathcal{Y}| - 1} = \frac{1 - \eta}{\eta (|\mathcal{Y}| - 1)}
\end{align*}$$

where $F_{S \setminus \{x'\}, f}$ is the subset of $F$ consisting of all labeling functions that agree with $f$ at all points in $S$ except $x$. This shows that the algorithm $A_\eta$ is $\left(O \left( \log \frac{1}{\eta} \right), 0 \right)$-DP.

Utility: Next, we bound the probability that $A_\eta$ misclassifies a point occurring $\ell$ times in $S$ in order to establish the utility of the algorithm. For a point $x$ occurring $\ell$ times in $S$, we denote all replicates of this point in the dataset as $\{X_2, ..., x_\ell\}$. As each point is flipped with probability $\eta$, let $k_i^x = \mathbb{I}_{\{x_i \text{ is not flipped}\}}$, then $\sum_{i=1}^\ell k_i^x$ is a binomial random variable with $\ell$ trials and success probability $1 - \eta$. The algorithm $A_\eta$ would correctly classify $x$ if $\sum_{i=1}^\ell k_i^x > \ell/2$.

The upper bound of the error probability is shown in Sanyal et al. [11], but we reproduce it here for completeness.

$$\begin{align*}
P \left[ \sum_{i=1}^\ell k_i^x \leq \frac{\ell}{2} \right] &= P \left[ \sum_{i=1}^\ell k_i^x \leq \frac{\ell}{2} \mu + \mu - \mu \right] \\
&= P \left[ \sum_{i=1}^\ell k_i^x \leq \left(1 - \left(1 - \frac{\ell}{2\mu}\right)\right) \mu \right] \\
&\leq e^{-\frac{(1-2\eta)^2}{8(1-\eta)^2} \mu} \quad \text{[a]} \\
&= e^{-\frac{(1-2\eta)^2}{8(1-\eta)^2} \ell} \quad \text{[b]}
\end{align*}$$

where step [a] follows from Chernoff’s concentration bound on the lower tail and step [b] due to $\mu = (1 - \eta)\ell$. 28
Then, we show a lower bound for the error probability using an anti-concentration bound on the lower tail. Define $D(a||p)$ to be the Kullback-Leibler divergence between two Bernoulli random variables with parameter $a$ and $p$, i.e. $D(a||p) = a \log \left( \frac{a}{p} \right) + (1-a) \log \left( \frac{1-a}{1-p} \right)$.

\[
\mathbb{P} \left[ \sum_{i=1}^{\ell} k_i^x \leq \frac{\ell}{2} \right] \geq \frac{1}{\sqrt{2\ell}} e^{-\ell D(\frac{1}{2}\|1-\eta)} \geq \frac{1}{\sqrt{2\ell}} \frac{e^{-\frac{1}{4}(\log(1/2(1-\eta))+\log(1/2\eta))}}{\sqrt{2\ell}} \left(4\eta (1-\eta)\right)^{\ell/2}
\]

(47)

where step [a] follows from the binomial anti-concentration bound as in Inequality 2. This completes the proof.

**Inequality 2** (From Ash [3]). Let $X$ be a binomial random variable with parameters $(m,p)$, then for any $k < mp$, we can lower bound the left tail of $X$ as

\[
\mathbb{P}[X \leq k] \geq \frac{1}{\sqrt{2m}} e^{-m D(\frac{1}{2}\|p)}.
\]

\[\Box\]

**B Experimental Details and Additional Experiments**

**B.1 Synthetic data**

**Constructing the synthetic data distribution** Given $m \in \mathbb{N}, p \in (0,0.5), C > 0$, and $k \in \mathbb{N}$, we first create two boolean hypercubes of dimension $d_{\text{min}}$ and $d_{\text{maj}}$ respectively, where $d_{\text{min}} = O \left( \lceil \log Cm \rceil \right)$ and $d_{\text{maj}} = O \left( \lceil \log (k(1-p)) \rceil \right)$ respectively. For the minority subpopulation, the data is distributed uniformly across a mixture of $N$ Gaussian clusters positioned on a randomly chosen set of $d_{\text{min}}$ vertices of this hyper-cube. Similarly, for the majority class the data is distributed across a uniform mixture of $d_{\text{maj}}$ Gaussian clusters situated on randomly chosen vertices of the second hypercube. Each of the clusters is randomly assigned a binary label. Finally, $m$ samples are sampled from the minority distribution, along with their labels, and each data covariate is appended with a $d_{\text{maj}}$-dimensional vector of all $10^{-4}$ to create a $d_{\text{min}} + d_{\text{maj}}$ dimensional dataset of size $m$. Similarly, $m$ samples are sampled from the majority distribution and the covariates are appended with a $d_{\text{min}}$ vector of all $10^{-4}$ to create a $d_{\text{min}} + d_{\text{maj}}$ dimensional dataset of size $m$. Finally, $m_1$ points are chosen randomly from the minority dataset and $m - m_1$ points are chosen randomly from the majority dataset where $m_1$ is sampled from a binomial distribution with parameters $m$ and $p$.

**DP Algorithm for synthetic data** We use PyTorch Opacus [53] for all our experiments. We use multiple values of $\epsilon$ in the range $(0,1,10)$. We clip the maximum gradient norm to 1, and then train the algorithm for $10^3$ epochs. Note that the clipping is necessary to control the lipschitzness of the gradient descent algorithm update step. The DP-SGD algorithm uses the value of $\delta$, the maximum gradient norm, and the number of epochs mentioned above to compute the noise-multiplier of the algorithm that is necessary to attain the required privacy guarantee.

**Additional weight of minority group** In this section, we plot additional experiments with the same setup as Section 3.1. In particular, we show results from the same experiment as Figure 2 but with $p = 0.5$ instead of $p = 0.2$. Our experiments show that the same trend is held in particular for how the accuracy discrepancy changes with $c$ and with $\epsilon$. 

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Figure 8: Here \( p = 0.5 \). Each figure plots the disparate accuracy (higher is less fair) in green dashed line, the accuracy of the minority group with red, and the overall accuracy with blue on the y-axis and the parameter \( c \) in the X-axis. The left most (\( \epsilon = 1 \)) achieves the strictest level of privacy and the right most (\( \epsilon = \infty \)) is vanilla training without any privacy constraints.

Figure 9: Accuracy Discrepancy against Privacy

B.2 Additional sizes of subpopulations in CelebA

To construct the subpopulations in CelebA, we use the following attributes 1. Pointy Nose, 2. Wearing Earrings, 3. Wavy Hair, 4. Wearing Lipstick, 5. Heavy Makeup, 6. Attractive, 7. Receding Hairline, 8. Blurry, 9. Bangs, 10. Wearing Hat, 11. Eyeglasses. This also corresponds to Set A used in Figure 7.

We use the following features for Set B in our experiments in Figure 7 1. Arched Eyebrows 2. Bags Under Eyes 3. Blond Hair 4. Double Chin 5. High Cheekbones 6. Pale Skin 7. Rosy Cheeks 8. Straight Hair 9. Wearing Necklace 10. Wearing Necktie 11. Young
In Figure 4, we reported results with the subpopulation size set to 40. In this section, we report results where the size belongs to \{5, 10, 20, 60, 80, 100\}. In Figure 9, we show how disparate accuracy changes with $\epsilon$ for various subpopulation sizes. CelebA-x is the minority majority division where the minority subpopulations' sizes are less than $x$. In Figure 10, we show how the minority group accuracy and the overall accuracy changes with $\epsilon$ for various subpopulation sizes. Overall the results, reflect the same trend as Figure 4 but the worst case accuracy discrepancy decreases for increasing subpopulation sizes.

![Graphs showing accuracy changes](image)

Figure 10: Minority group and overall accuracy against privacy.

### B.3 Additional details and experiments on CIFAR-10

In Figure 11, we plot the same results as Figure 5 but with the threshold $\rho$ set to 0.01.
Figure 11: CIFAR-10—**Left** Disparate accuracy decreases (low is fairer) with increasing $\epsilon$ (high is less privacy) i.e. fairness is worse for stricter privacy. **Right** figure shows that both minority and overall accuracy increases with $\epsilon$. Dashed lines show the corresponding metric of a vanilla model (no privacy criterion).