Multiple teleportation via the partially entangled states

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We investigate the multiple teleportation with some nonmaximally entangled channels. The efficiencies of two multiple teleportation protocols, the separate multiple teleportation protocol (SMTP) and the global multiple teleportation protocol (GMTP), are calculated. We show that GMTP is more efficient than SMTP.

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Quantum teleportation is one of the most significant components in quantum information processing, which allows indirect transmission of quantum information between distant parties by using previously shared entanglement and classical communication between them. Indeed, it is considered as a basic building block of quantum communication nowadays. Not only is it one of the most intriguing phenomena in the quantum world, but also a very useful tool to perform various tasks in quantum information processing and quantum computing. For example, controlled quantum gates are implemented by means of quantum teleportation, which is very important in linear optical quantum computation. Recently, the original scheme for teleporting a qubit has been widely generalized in many different ways. In the previous teleportation protocols and in many other applications of teleportation, we want to construct an unknown input state with unity fidelity at another location while destroying the original copy, which is always achieved if two parties share a maximally entangled state. However, it might happen that our parties do not share a maximally entangled state. This limitation can be overcome by distilling out of an ensemble of partially entangled states a maximally entangled one. But this approach requires a large amount of copies of partially entangled states to succeed. Another way to achieve unity fidelity teleportation with limited resources is based on the probabilistic quantum teleportation protocols of Refs. [6, 7, 8].

Recently, in an interesting work, Modlawska and Grudka showed that if the qubit is teleported several times via some nonmaximally entangled states, then the “errors” introduced in the previous teleportations can be corrected by the “errors” introduced in the following teleportations. Their strategy was developed in the framework of the scheme proposed in Ref. [3] for linear optical teleportation. In this paper, we show that this feature of the multiple teleportation of Ref. [16] is not restricted to the teleportation scheme stated in Ref. [3].

Based on the general teleportation language of the original proposal shown in Ref. [1], we compare the efficiencies of two multiple teleportation protocols, the separate multiple teleportation and the global multiple teleportation. In the former protocol, a complete teleportation including error correction is strictly executed by neighboring parties. On the other hand, in the latter protocol, all errors introduced in the teleportation are corrected by the final receiver. We find the global multiple teleportation is more efficient than the separate multiple teleportation.

To illustrate two protocols clearly, let us first begin with the multiple teleportation in the case of three parties.

Alice wants to teleport an unknown quantum state

\[ |\psi\rangle = a|0\rangle + b|1\rangle \]  \( (1) \)

to Bob, where \( a, b \in \mathbb{C} \) and \( |a|^2 + |b|^2 = 1 \). There is no direct entanglement resource between Alice and Bob, fortunately, Alice and the third party Charlie have a partially entangled state

\[ |\Psi\rangle = \alpha|00\rangle + \beta|11\rangle, \]  \( (2) \)

while Charlie and Bob share the same entanglement resource, where \( \alpha \) and \( \beta \) are real numbers and satisfy \( \alpha^2 + \beta^2 = 1 \). Without loss of generality, we suppose \( |\alpha| \leq |\beta| \).

The simplest and directest strategy is to perform two separate teleportations, i.e., Alice teleports the quantum state \( |\psi\rangle \) to Charlie via the first teleportation. Then Charlie teleports it to Bob via the second teleportation. Because this protocol consists of two separate teleportations, we call it the separate multiple teleportation protocol (SMTP).

According to the standard probabilistic teleportation protocol, in the first separate teleportation, Alice performs the Bell-basis measurement (BM) on the teleported qubit and the entangled qubit in her side. Charlie can apply the corresponding Pauli transformation conditioned on the result of BM, i.e., \( I \) if the BM yields \( |\Phi^+\rangle \), \( \sigma_z \) for \( |\Phi^-\rangle \), \( \sigma_x \) for \( |\Psi^+\rangle \), and \( i\sigma_y \) for \( |\Psi^-\rangle \), where \( I \) is the
identity, $\sigma_x, \sigma_y, \sigma_z$ are standard Pauli matrices and

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle),$$

(3)

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle).$$

(4)

Finally, the state Charlie received becomes

$$|\psi_1\rangle = \frac{1}{\sqrt{p_1}} (a|0\rangle + b|1\rangle)$$

(5)

with the probability $p_1 = |a|^2 + |b|^2$ or

$$|\psi_2\rangle = \frac{1}{\sqrt{p_2}} (\beta|0\rangle + \alpha|1\rangle)$$

(6)

with the probability $p_2 = |a|^2 + |b|^2$. These states are in accordance with the original state $|\psi\rangle$ only if the quantum channel is a maximally entangled state, i.e. $\alpha = \beta$. For the case of non-maximally entangled channel, there exists the “error” in $|\psi_1\rangle$ and $|\psi_2\rangle$. These states can be returned to the original state with certain probability by performing the generalized measurement given by Kraus operators:

$$E_{S1} = |0\rangle\langle 0| + \frac{\alpha}{\beta} |1\rangle\langle 1|,$$

(7a)

$$E_{F1} = \sqrt{1 - \frac{\alpha^2}{\beta^2}} |1\rangle\langle 1|$$

(7b)

for $|\psi_1\rangle$ and

$$E_{S2} = \frac{\alpha}{\beta} |0\rangle\langle 0| + |1\rangle\langle 1|,$$

(8a)

$$E_{F2} = \sqrt{1 - \frac{\alpha^2}{\beta^2}} |0\rangle\langle 0|$$

(8b)

for $|\psi_2\rangle$. When $E_S$ is obtained, the qubit ends in its original state $|\psi\rangle = a|0\rangle + b|1\rangle$. The success probability in the first teleportation is

$$p = \sum_{i=1}^2 p_i \langle \psi_i | E_{S1} E_{S2} | \psi_i \rangle = 2\alpha^2.$$ 

(9)

Next, Charlie teleports the recovered quantum state to Bob by the similar process. Combining these two teleportations, the total probability that Bob receives the quantum state $|\psi\rangle$ is

$$P_S = p^2 = 4\alpha^4.$$ 

(10)

However, the above teleportation protocol is not the optimal strategy. In fact, the third party Charlie does not need to recover the quantum state to be teleported, but teleports the “error state” to Bob directly. Lastly, Bob corrects all “errors” of the quantum state in the teleportation process. Formally, either Alice and Charlie or Charlie and Bob do not complete an intact separate teleportation, so we call it the global multiple teleportation protocol (GMTP).

Let us, thus, assume that Charlie does not correct the “error” introduced in the first teleportation, he only makes a Pauli transformation according to Alice’s measurement outcome, then he also performs BM on his two qubits and broadcasts the measurement outcome to Bob. After making the corresponding Pauli transformation conditioned on Charlie’s measurement outcome, Bob’s qubit will collapse into one of the following states

$$|\phi_1\rangle = \frac{1}{\sqrt{p_1}} (\alpha^2 a|0\rangle + \beta^2 b|1\rangle),$$

(11a)

$$|\phi_2\rangle = \frac{1}{\sqrt{p_2}} (\beta^2 a|0\rangle + \alpha^2 b|1\rangle),$$

(11b)

$$|\phi_3\rangle = a|0\rangle + b|1\rangle$$

(11c)

with the probabilities $p'_1 = \alpha^4 |a|^2 + \beta^4 |b|^2$, $p'_2 = \beta^4 |a|^2 + \alpha^4 |b|^2$, $p'_3 = 2\alpha^2 \beta^2$ respectively. When the state is in $|\phi_3\rangle$, we do not have to perform the error correction. It is very joyful to see that the second teleportation corrects the “error” introduced by the first teleportation. This effect is called error self-correction. For $|\phi_1\rangle$ and $|\phi_2\rangle$, one can recover the original state by performing generalized measurement given by Kraus operators:

$$E'_{S1} = |0\rangle\langle 0| + \frac{\alpha^2}{\beta^2} |1\rangle\langle 1|,$$

(12a)

$$E'_{F1} = \sqrt{1 - \frac{\alpha^4}{\beta^2}} |1\rangle\langle 1|$$

(12b)

and

$$E'_{S2} = \frac{\alpha^2}{\beta^2} |0\rangle\langle 0| + |1\rangle\langle 1|,$$

(13a)

$$E'_{F2} = \sqrt{1 - \frac{\alpha^4}{\beta^2}} |0\rangle\langle 0|$$

(13b)

respectively. The total probability of successfully recovering the original state is

$$P_G(3) = 2\alpha^2 \beta^2 + \sum_{i=1}^2 p'_i |\phi_i\rangle E_{S1} E_{S2} |\phi_i\rangle = 2\alpha^2.$$ 

(14)

The ratio of efficiency of GMTP to that of SMTP

$$P_G(3)/P_S = \frac{1}{2\alpha^2}.$$ 

(15)

We can easily see $P_G/P_S \geq 1$ because of $\alpha \leq \frac{1}{\sqrt{2}}$. It is obvious that for the maximally entangled channel, the two protocols are equivalent, but for the partially entangled channel, GMTP is more efficient than SMTP. Moreover, the less $\alpha$ is, the more efficient the GMTP is.

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It is straightforward to generalize the above two protocols to arbitrary parties. Let us first discuss the GMTP. Since error self-correction only appears in the even times Bell-basis measurements, so here we discuss the \((2N+1)\) -party teleportation. Suppose that Alice 1 wants to teleport a quantum state \(|\psi\rangle = a|0\rangle + b|1\rangle\) to Alice \(2N + 1\). There is no direct entanglement resource between them, but they can link through \(2N - 1\) intermediaries called Alice 2, Alice 3, \cdots, Alice \(2N\), respectively. Two neighboring parties share the partially entangled state described by Eq.(2). They can complete the task through cooperative teleportation. After \(2N\) Bell-basis measurements and corresponding Pauli transformations conditioned on previous parties, the final receiver’s qubit will be in one of the states

\[
|\phi_i\rangle = \frac{1}{\sqrt{p_i}}(\alpha^{2N-i}e^{i\theta}a|0\rangle + \alpha^{i}\beta^{2N-i}|b\rangle), \quad (16)
\]

with the probability \(C_{2N}^{i}p_{i}^{G} \equiv C_{2N}^{i}(\alpha^{2(2N-i)}\beta^{2i}|a|^2 + \alpha^{2i}\beta^{2(2N-i)}|b|^2), \quad i = 0, 1, 2, \cdots, 2N\). By correcting the error, the total success probability is

\[
P_{G}(2N + 1) = C_{2N}^{G}\alpha^{2N}\beta^{2N} + 2 \sum_{i=0}^{N-1} C_{2N}^{i}\alpha^{2(2N-i)}\beta^{2i}.
\]

On the other hand, in the case of SMTP, we must perform \(2N\) separate teleportations, then the total success probability equals

\[
P_{S}(2N + 1) = p^{2N} = 2^{2N}\alpha^{4N}.
\]

It is easy to verify that \(P_{G}(2N + 1) \geq P_{S}(2N + 1)\).

In order to show how the total success probabilities of two protocols depend on the entanglement of channels for different \(N\), we will choose concurrence \(C\) defined by Wootters as a convenient measure of entanglement \[17\]. The concurrence varies from \(C = 0\) of a separable state to \(C = 1\) of a maximally entangled state. For a pure partially entangled state described by Eq. (2), the concurrence may be expressed explicitly by \(C = 2|\alpha\beta|\).

In Fig.1, we plot \(P_{S}\) and \(P_{G}\) as the function of concurrence \(C\) for different \(N\). We can see that both the total success probabilities of two protocols declines with the decrease of the entanglement of channels. Moreover, the greater \(N\) is, the more sharper the success probabilities declines. It shows that the quantum channel with small entanglement will become unpractical with the increase of \(N\). Fig.1 also indicates explicitly that the GMTP is more efficient than SMTP. For example, for the case of \(N = 10\), the total success probability of GMTP \(P_{G}\) \(\approx 21\%)\) while the total success probability of SMTP \(P_{S}\) only attains \(0.14\%)\) when the concurrence of channels is \(C = 0.96\).

The ratio of \(P_{G}\) to \(P_{S}\) as a function of \(C\) for different \(N\) is illustrated in Fig.2. Here we only take the concurrence from \(0.9\) to \(1\) because the small entanglement channels are unpractical for large \(N\). From Fig.2, we can see that the greater \(N\) is, the larger \(P_{G}/P_{S}\) is. In other words, the efficiency of GMTP is far higher than that of SMTP when the steps of teleportation increase.

When the entanglement of the quantum channel is different between neighboring parties, the circumstance becomes complicated. Here we only consider the case of three parties.

Alice wants to teleport an unknown quantum state \(|\psi\rangle = a|0\rangle + b|1\rangle\) to Bob. There is no direct entanglement resource between Alice and Bob, fortunately, Alice and the third party Charlie share a partially entangled state

\[
|\Psi_1\rangle = \alpha_1|00\rangle + \beta_1|11\rangle,
\]

while Charlie and Bob share another entanglement resource

\[
|\Psi_2\rangle = \alpha_2|00\rangle + \beta_2|11\rangle,
\]

where \(\alpha_i\) and \(\beta_i\) are real numbers and satisfy \(|\alpha_i| \leq |\beta_i|\) and \(\alpha_i^2 + \beta_i^2 = 1\). After two Bell-basis measurements and
Pauli operations, the qubit of Bob will be in one of the following states

\[
|\psi_{ij}\rangle = \frac{1}{\sqrt{p_{ij}}}(\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j a|0\rangle + \alpha_1^{2-i}\beta_i^{1-j}\beta_2 b|1\rangle)
\]

with the probabilities

\[
p_{ij} = |\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j a|^2 + |\alpha_1^{2-i}\beta_i^{1-j}\beta_2 b|^2 (i, j = 0, 1)
\]

respectively. The qubit can be returned to its original state by performing the generalized measurement given by Kraus operators:

\[
E_S = |0\rangle\langle 0| + \frac{\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j}{\alpha_1^{2-i}\beta_i^{1-j}\beta_2 b}|1\rangle\langle 1|,
\]

\[
E_F = \sqrt{1 - |\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j|^2} |1\rangle\langle 1|
\]

for \(|\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j|^2 \leq |\alpha_1^{2-i}\beta_i^{1-j}\beta_2|^2\). A similar measurement exists if \(|\alpha_1^{1-i}\alpha_2^{1-j}\beta_i\beta_j|^2 \geq |\alpha_1^{2-i}\beta_i^{1-j}\beta_2|^2\).

By tedious but standard calculation we can obtain the success probability of teleportation

\[
P = \min\{2\alpha_1^2, 2\alpha_2^2\},
\]

It is an interesting result, the success probability of teleportation is completely determined by the channel of less entanglement. For another channel of more entanglement, its entanglement does not affect the success probability at all. In other words, the channel of more entanglement is equivalent to the maximally entangled channel in the total teleportation process.

In summary, we have presented two multiple teleportation protocols via some partially entangled state, the separate multiple teleportation and the globe multiple teleportation. In the former protocol, a complete teleportation including error correction is strictly executed by neighboring parties. However, in the latter protocol, all errors introduced in the teleportation are corrected by the final receiver. It has been shown that the property of self error-correction is a general feature of multiple teleportations, not being restricted to the scheme proposed in Ref. [3]. We also have compared the efficiencies of the two multiple teleportation protocols and found the globe multiple teleportation is more efficient than the separate multiple teleportation due to the property of self error-correction.

[1] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[2] X.B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, Phys. Rep. 448, 1 (2007).
[3] G.L. Long, F.G. Deng, C. Wang, X.H. Li, K. Wen, and W.Y. Wang, Front. Phys. China 2, 251 (2007).
[4] D. Gottesman and I.L. Chuang, Nature (London) 402, 390 (1999).
[5] E. Knill, R. Laflamme, and G.J. Milburn, Nature (London) 409, 46 (2001).
[6] P. Agrawal and A.K. Pati, Phys. Lett. A 305, 12 (2002).
[7] G. Gordon and G. Rigolin, Phys. Rev. A 73, 042309 (2006).
[8] W.L. Li, C.F. Li, and G.C. Guo, Phys. Rev. A 61, 034301 (2000).
[9] F.G. Deng, C.Y. Li, Y.S. Li, H.Y. Zhou, and Y. Wang, Phys. Rev. A 72, 022338 (2005).
[10] F.L. Yan and D. Wang, Phys. Lett. A 316, 297 (2003).
[11] T. Gao, F.L. Yan, and Y.C. Li, Europhys. Lett. 84, 50001 (2008).
[12] M.Y. Wang and F.L. Yan, Phys. Lett. A 355, 94 (2006).
[13] M.Y. Wang, F.L. Yan, T. Gao, and Y.C. Li, International Journal of Quantum Information 6, 201 (2008).
[14] T. Gao, F.L. Yan, and Z.X. Wang, Quantum Information and Computation 4, 186 (2004).
[15] C.H. Bennett1, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, and W.K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
[16] J. Modlawska and A. Grudka, Phys. Rev. Lett. 100, 110503 (2008).
[17] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).