Input-to-State Stability of Networked and Quantized Control Systems

Wei Ren*, Junlin Xiong**

* Division of Decision and Control Systems, EECS, KTH Royal Institute of Technology, SE-10044, Stockholm, Sweden. (emails: weire@kth.se).
** Department of Automation, University of Science and Technology of China, Hefei, 230026, China. (emails: junlin.xiong@gmail.com).

Abstract: This paper studies input-to-state stability of networked and quantized control systems with external disturbances. Since the external disturbance is considered, the state may escape from the quantization regions, which thus leads to the introduction of the quantization mechanism with zooming-out and zooming-in stages. We first propose a novel hybrid system to incorporate both the open-loop and closed-loop systems. Based on the developed hybrid model, we establish the boundedness of the state in the zooming-out stage, and the convergence of the state in the zooming-in stage. Furthermore, sufficient conditions are established for input-to-state stability in the case of switching between the zooming-in and zooming-out stages. Finally, a numerical example is presented to illustrate the obtained results.

Keywords: Hybrid systems, networked control systems, quantized control, system modelling, input-to-state stability, average dwell-time.

1. INTRODUCTION

With the increasing control applications emerging in many fields like vehicle industry and teleoperation, control of systems via communication channels has been given great attentions in the last few decades; see Zhang et al. (2013); Lunze (2014). In these control applications, system components, such as sensors, actuators, controllers and quantizers, are connected through a communication network. The presence of the network provides many benefits in terms of installation, maintenance, flexible architecture, high reliability and wireless. However, because of the limited capacity of the network, some unavoidable issues are induced, such as time-varying transmission intervals, packet dropouts, quantization effects and communication constraints. See survey papers Zhang et al. (2013); Baillieul and Antsaklis (2001). Observing from these two modelling strategies, we can conclude that the system is always closed-loop in NCSs (Nešić and Liberzon (2009)), whereas the system is always open-loop in QCSs (Liberzon (2003a); Liberzon and Nešić (2007)); the quantizer and network can be seen as information processing devices and only provide part of information (Liberzon (2003b); Nešić and Teel (2004)). Based on these similarities, both QCSs and NCSs can be combined and such control systems are called networked and quantized control systems (NQCSs); see van Loon et al. (2013); Ren and Xiong (2018b); Tian et al. (2007); Nešić and Liberzon (2009). However, the differences between QCSs and NCSs lead to additional assumptions. For instance, the encoding part is open-loop in QCSs (Liberzon (2003a); Liberzon and Nešić (2007)), whereas the system is always closed-loop in NCSs (Nešić and Liberzon (2009); Heemels et al. (2009)). Hence, the absence of the external disturbance (Ren and Xiong (2018b); van Loon et al. (2013)) or the assumption to constrain the system in the quantization regions (Heemels et al. (2009); Nešić and Liberzon (2009)) are required, which lead to conservatism of the obtained results. In this paper, we follows this direction and study this topic further.

In this paper, we focus on input-to-state stability (ISS) of networked and quantized control systems with external disturbance. Since the external disturbance is addressed here, which may result in escaping of the state from the quantization regions, it is necessary to address the quantization mechanism with two stages: zooming-out and zooming-in stages, which thus leads to reconsideration of the system model. As a result, our first contribution is to propose a novel hybrid model for NQCSs with external disturbance based on the formalism of Goebel et al. (2012). Since the general assumption, which requires the system state to be always in the quantization regions (Nešić and Liberzon (2009)), is not needed here, the main challenge is to model the data transmissions on the one hand and the switches between zooming-out and the zooming-

* This work was supported by National Natural Science Foundation of China, under Grant 61773357.
in stages on the other hand. To this end, we describe in detail the quantization mechanism with two stages, and then introduce two clock variables to represent respectively the time elapsed since the last data transmission and the time elapsed since the last stage switch. Both the open-loop and closed-loop cases are included, and thus a high fidelity model is developed to be amenable to controller design and stability analysis. With the developed hybrid model, our second contribution is to derive sufficient conditions to guarantee ISS of the whole system. To this end, we derive the boundedness of the state in the zooming-out stage, the convergence of the state in zooming-in stage, and further study ISS of the whole system in terms of average dwell-time. Finally, a numerical example is presented to illustrate the obtained results.

2. PRELIMINARIES

Basic definitions and notations are introduced in this section. \( \mathbb{R} := (-\infty, +\infty) ; \mathbb{R}_+: = [0, +\infty) ; \mathbb{N} := \{0, 1, 2, \ldots\}; \mathbb{N}_+: = \{1, 2, \ldots\} . \) \( | \cdot | \) stands for the Euclidean norm. Given two vectors \( x, y \in \mathbb{R}^n \), \( (x, y) := (x^\top, y^\top) \) for simplicity of notation, and \( (x, y) \) denotes the usual inner product. \( I \) represents the identity matrix with the appropriate dimension; \( \text{diag}\{ \cdot \} \) to denote the block diagonal matrix. The symbols \( \land \) and \( \lor \) denote separately ‘and’ and ‘or’ in logic. \( f \) denotes the identity function. Given a piecewise continuous function \( f : \mathbb{R}^+ \to \mathbb{R} \), \( f(t^+) := \lim_{t \to t^+} f(t) \) if it is continuous, \( f(t^+) := \text{ess. sup}_{t \to t^+} f(t) \) with given interval \( [a, b] \subseteq \mathbb{R} \); and \( \| \cdot \| := \| \cdot \|_{(a, b)} \) if \( b \to \infty \). A function \( \alpha : \mathbb{R}_+^n \to \mathbb{R}_+^n \) is of class \( K \) if it is continuous, \( \alpha(0) = 0 \), and strictly increasing; it is of class \( K \) and unbounded. A function \( \beta : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n \) is of class \( K \) if \( \beta(s, t) \) is of class \( \mathcal{K} \) for each fixed \( s \geq 0 \) and \( \beta(s, t) \) decreases to zero as \( t \to 0 \) for each fixed \( s \). A function \( \beta : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) is of class \( \mathcal{K} \) if \( \beta(s, t) \) is of class \( \mathcal{K} \) for each fixed \( s \geq 0 \) and of class \( \mathcal{K} \) if \( \beta(s, t) \) for each fixed \( t \geq 0 \).

Consider the nonlinear hybrid system (Cai and Teel (2009)):

\[
\begin{align*}
\dot{x} &= F(x, w), \quad (x, w) \in C, \\
x^+ &= G(x, w), \quad (x, w) \in D,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the system state, \( w \in \mathbb{R}^m \) is the external input, \( y \in \mathbb{R}^p \) is the system output, \( F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is the flow map, \( G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) is the jump map, \( C \subset \mathbb{R}^n \times \mathbb{R}^m \) is the flow set, and \( D \subset \mathbb{R}^n \times \mathbb{R}^m \) is the jump set. For the hybrid system (1), the following basic assumptions are given in Goebel et al. (2012): the sets \( C, D \subset \mathbb{R}^n \times \mathbb{R}^m \) are closed; \( F \) is continuous on \( C \); \( G \) is continuous on \( D \).

Definition 1. (Cai and Teel (2009)). The system (1) is input-to-state stable (ISS), if there exist \( \beta, \gamma \in \mathcal{K} \) such that \( \beta(|x(t, j)|) \leq \beta(|x(0, j)|) \gamma(\|w\|) \) for all \( (t, j) \in \text{dom} x \) and all the solution to (1).

3. SYSTEM MODEL AND PROBLEM FORMULATION

Consider the plant of the following form

\[
\begin{align*}
\dot{x}_p &= A_p x_p + B_p u + E_p w, \quad y = C_p x_p,
\end{align*}
\]

where \( x_p \in \mathbb{R}^n_p \) is the plant state, \( u \in \mathbb{R}^m \) is the control output, \( w \in \mathbb{R}^m \) is the unknown disturbance, and \( y \in \mathbb{R}^p \) is the plant output. The controller designed for the plant (3) is given by

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y + E_c w, \quad u = C_c x_c + D_c y,
\end{align*}
\]

where \( x_c \in \mathbb{R}^n_c \) is the controller state. In (3)-(4), \( w \in \mathbb{R}^m \) is assumed to be Lebesgue measurable and locally bounded. Our objective is to implement the controller (3) over the quantizer and network, and to show that under reasonable assumptions, the assumed stability property of the system (2)-(3) will be preserved for NQCSs.

At the transmission times, denoted by \( t_i \in \mathbb{R}^+_n, i \in \mathbb{N} \), the outputs of the plant (2) and the controller (3) are sampled, quantized and transmitted via the network. According to the band-limited network and the spatial locations of the sensors and actuators, the sensors and actuators are partitioned into \( \ell \in \mathbb{N}_+ \) nodes connecting the network. Correspondingly, the transmitted information is also divided into \( \ell \) parts. At \( t_i \in \mathbb{R}^+_n \), one and only one node is allowed to access to the network to transmit its data packet, and which node is granted is determined by the time-scheduling protocols; see Donkers et al. (2011); Nešić and Liberzon (2009); Nešić and Teel (2004). The transmission time sequence \( \{t_i \in \mathbb{R}^+_n : i \in \mathbb{N} \} \) is strictly increasing, and thus the transmission intervals are defined as \( h_i := t_{i+1} - t_i, i \in \mathbb{N} \). The following assumption is made for the transmission intervals; see also Nešić and Liberzon (2009); Nešić and Teel (2004).

Assumption 1. There exist \( h_{\text{max}} > \varepsilon > 0 \) such that \( \varepsilon \leq h_i \leq h_{\text{max}} \) holds for all \( i \in \mathbb{N} \).

3.1 Quantization

After sampling via the sensors, the sampled measurements \( y(t_i) \) and \( u(t_i) \) are quantized to match the bandwidth limitation of the network. A quantizer is a piecewise constant function \( q : \mathbb{R}^n \to \mathcal{Q} \). To well define the quantizer, the following assumption is made; see also Ren and Xiong (2018b,a); Liberzon (2003a); Liberzon and Nešić (2007).

Assumption 2. Each node has a quantizer \( q_j \) with \( j \in \mathcal{J} := \{1, \ldots, \ell\} \). For each \( j \in \mathcal{J} \), there exist \( M_j > \Delta_j > 0 \) and \( \Delta_0 > 0 \) such that for all \( z_j \in \mathbb{R}^n \),

\[
|z_j| \leq M_j \Rightarrow |q_j(z_j) - z_j| \leq \Delta_j,
\]

\[
|z_j| > M_j \Rightarrow |q_j(z_j)| > M_j - \Delta_j,
\]

\[
|z_j| \leq \Delta_0 \Rightarrow q_j(z_j) = 0.
\]

The dynamic quantizer applied in this paper is given by

\[
q(\mu, z) := \mu q(\mu, z), \quad \mu > 0,
\]

where \( \mu \in \mathbb{R}^+ \) is the quantization parameter. All the quantization parameters are combined as \( \mu := (\mu_1, \ldots, \mu_\ell) \) in \( \mathbb{R}^\ell \), and the overall quantizer is given by

\[
q(\mu, z) := (q_1(\mu_1, z), \ldots, q_\ell(\mu_\ell, z)).
\]

The quantization parameter \( \mu \in \mathbb{R}^\ell \) evolves according to a hybrid dynamics, which will be discussed later. The quantized measurements are defined as \( \tilde{y} := q(\mu, y) \in \mathbb{R}^\ell \) and \( \tilde{u} := q(\mu, u) \in \mathbb{R}^\ell \); the quantization errors are defined as \( e_i := y - \tilde{y} \in \mathbb{R}^\ell \) and \( e_u := u - \tilde{u} \in \mathbb{R}^\ell \).

Since the external disturbance are studied and the dynamic quantizer (7) is applied, the quantization mechanism is introduced with two stages: zooming-out and zooming-in stages. Since the initial state is unknown \( a \text{ priori} \) and the state may escape from the quantization regions due to the effects of the external disturbance, the system is in the zooming-out stage, and \( \mu \) increases such that the quantization regions are expanded to cover the state. Meanwhile, the quantization error is getting large with the increase of \( \mu \). Once the state is captured by the
quantization regions, the system switches into the zooming-in stage, and $\mu$ decreases such that the quantization regions are contracted to drive the state to be bounded and even convergent. Because of these two stages in the quantization mechanism, the applied control strategy also has two stages: the controller is not activated in the zooming-out stage until the state is captured in the quantization regions, whereas activated in the zooming-in stage such that the control input is transmitted via the network.

In the following, we introduce a logical variable $c \in \{0, 1\}$ to distinguish the zooming-out and zooming-in stages, and a timer $\tau_2 \in \mathbb{R}^+$ to keep track of these two stages. In particular, $c = 0$ means that the system is in the zooming-out stage, whereas $c = 1$ means that the system is in the zooming-in stage. The dynamics of the variable $(\tau_2, c)$ is given by

\[
(\dot{\tau}_2, c) = (1, 0), \quad (\tau_2, c) \in C^{\text{avg}},
\]

\[
(\tau_2^+, c^+) = (0, 1 - c), \quad (\tau_2, c) \in D^{\text{avg}},
\]

(8)

with

\[
C^{\text{avg}} := ([0, T_1(x)] \times \{0\}) \cup ([0, T_2(x)] \times \{1\}),
\]

\[
D^{\text{avg}} := ([T_1(x)] \times \{0\}) \cup ([T_2(x)] \times \{1\}),
\]

(9)

where $x := (x_0, x_L) \in \mathbb{R}^{n_x}$ is the augmented state with $n_x = n_p + n_c$. In (9), $T_1(x) := \inf\{t \in \mathbb{R}^+ : \exists j \in \mathcal{L}, (|y_j(t)| \land |u_j(t)|) \leq M_j |\mu_j(t)| \land (\exists j \in \mathcal{L}, (|y_j(0)| \lor |u_j(0)|) > M_j |\mu_j(0)|)\}$ is the time instant from which the quantizer switches into the zooming-out stage from the zooming-in stage, and $T_2(x) := \inf\{t \in \mathbb{R}^+ : \exists j \in \mathcal{L}, (|y_j(t)| \land |u_j(t)|) \leq M_j |\mu_j(t)| \land (\forall j \in \mathcal{L}, (|y_j(0)| \lor |u_j(0)|) \leq M_j |\mu_j(0)|)\}$ is the time instant from which the quantizer switches into the zooming-in stage from the zooming-out stage. Both $T_1(x)$ and $T_2(x)$ are time-varying and state-dependent. Hence, the dynamics (8) with (9) is state-dependent and event-triggered, and the switching between the zooming-out and zooming-in stages is event-triggered. The event-triggered conditions are presented in $T_1(x)$ and $T_2(x)$ to verify whether the outputs of the plant and controller are in the quantization regions. Because of the strictly increasing transmission time sequence and Assumption 1, the Zeno phenomena are ruled out, which implies the positiveness of $T_1(x)$ and $T_2(x)$.

### 3.2 Zooming-out Stage

In the zooming-out stage, the system is open-loop, the quantization parameter increases and the controller is not activated. The following assumption is made first for the increase of the quantization parameter.

**Assumption 3.** In the zooming-out stage, the quantization parameter $\mu$ increases periodically, and the period is $\delta \in \mathbb{R}^+$ satisfying $\delta \in (0, h_{\text{max}}]$.

Since the system is open-loop in the zooming-out stage, the transmission times do not affect the increase of the quantization parameter $\mu$, which thus implies that Assumption 3 is reasonable. It has to be noted that the increase of $\mu$ is also periodic in the previous works; see Liberzon (2014, 2003a); Liberzon and Hespanha (2005). The period $\delta \in (0, h_{\text{max}}]$ can be set arbitrarily, and the smaller $\delta$ is, the larger the increase frequency of $\mu$ is. We can set $\delta$ as the inter-sampling period as in Liberzon and Hespanha (2005) for the sake of convenience.

Next, we introduce a timer $\tau_1 \in \mathbb{R}$ to keep track of the transmission interval (which refers to the update interval of $\mu$ in the zooming-out stage). Hence, the open-loop system in the zooming-out stage is denoted by $\mathcal{F}_1$ and modelled below.

\[
\begin{align*}
\dot{x} &= \delta_1 x + \delta_2 w, \\
\mu &= 0, \\
\tau_1 &= 0, \\
x^+ &= x, \\
\mu^+ &= \Omega_{\text{out}}^\mu, \\
\tau_1^+ &= 0.
\end{align*}
\]

(10a)

(10b)

with

\[
\delta_1 := \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad \delta_2 := \begin{bmatrix} E_p \\ E_c \end{bmatrix}, \\
C_1 := \left[ [0, \delta] \times [0, T_1(x)] \times \{0\}, \right. \\
D_1^\text{jump} := \left\{ (\delta) \times [0, T_1(x)] \times \{0\}, \right. \\
\text{where } \Omega_{\text{out}} = \text{diag}\{\Omega_{\text{out}, 1}, \ldots, \Omega_{\text{out}, \ell}\} \in \mathbb{R}^\ell \text{ with the constants } \Omega_{\text{out}, j} > 1 \text{ given for all } j \in \mathcal{L}.
\]

The dynamics in (10a)-(10b) implies that the timer $\tau_1$ is reset to 0 when it reaches $\delta$ during the zooming-out stage, i.e., when $c = 0$ and $\tau_2 \in [0, T_1(x)]$. It is obvious that the switching times for the solutions of the hybrid system (10) are consistent with Assumption 3. In Subsection 3.1, we will establish the existence of the time instants switching from the zooming-out stage to the zooming-in stage, which shows the boundedness of $T_1(x)$. After the zooming-out stage, the state is captured by the quantization regions and thus the system switches into the zooming-in stage.

### 3.3 Zooming-in Stage

In the zooming-in stage, the controller is activated and the system is closed-loop such that the quantized measurements $\hat{y}$ and $\hat{u}$ are transmitted via the network. Here, the quantization measurements are assumed to be time-invariant in the transmission intervals and updated at the transmission times. The received measurements, denoted by $\hat{y}$ and $\hat{u}$, are supposed to be time-invariant in the transmission intervals $[t_i, t_{i+1}]$, $i \in \mathbb{N}$; see for instance van Loon et al. (2013); Heemels et al. (2010). At the transmission times $t_i$, $i \in \mathbb{N}$, the updates of $\hat{y}$ and $\hat{u}$ are given as follows.

\[
\begin{align*}
\hat{y}(t_i^+) &= \hat{y}(t_i) + h_i(i, e_i(t_i), e_j(t_i)), \\
\hat{u}(t_i^+) &= \hat{u}(t_i) + h_u(i, e_u(t_i), e_j(t_i)),
\end{align*}
\]

(12)

where $h_i$ and $h_u$ are the update functions, and depend on the time-scheduling protocols. As a result, the update of the network-induced errors are given by

\[
\begin{align*}
e_i(t_i^+) &= \hat{y}(t_i^+) - y(t_i^+), \\
e_j(t_i^+) &= \hat{u}(t_i^+) - \hat{u}(t_i).
\end{align*}
\]

(13)

where the second "\text{--}" holds due to (12). Similarly,

\[
e_u(t_i^+) := \hat{u}(t_i(i, x_2(t_i), e_u(t_i), \mu(t_i))).
\]

(14)

Since the functions $h_i$ and $h_u$ model the time-scheduling mechanism, which determines the transmissions between the controller and the plant. Following the terminology of Nešić and Teel (2004), the following is referred to as the transmission protocol

\[
e^+ = h(x, e, \mu),
\]

(15)

where $e := (e_y, e_u) \in \mathbb{R}^{n_e}$ with $n_e := n_p + n_u$, and $h(i, x, e, \mu) := (h_i(i, x_p, e_y, \mu), h_u(i, x_2, e_u, \mu))$. According to the number of the nodes, $e$ is partitioned as $e = (e_1, \ldots, e_\ell)$. The transmission protocol (15) is to determine the node to transmit its data packet at the transmission times. That is, at each transmission instant $t_i$, if the node $j \in \mathcal{L}$ gets access to the network, then
the corresponding component of \(e_j\) is updated whereas other components of \(e\) is kept.

The dynamics of \(\mu\) in the zooming-in stage is given by

\[
\dot{\mu} = 0, \quad t \in [t_1, t_2 + 1],
\]

\[
\mu(t_2) = \Psi \Omega_{in} \mu(t) + \psi(1 - \psi) \mu(t),
\]

(16)

where \(\Omega_{in} = \text{diag}\{\Omega_{in,1}, \ldots, \Omega_{in,\ell}\} \in \mathbb{R}^{\ell} \times \mathbb{R}^\ell \) with the constants \(\Omega_{in,j} \in (0, 1)\) given for all \(j \in \Sigma\). Note that the dynamic of \(\mu\) in the zooming-in stage is similar to those in Heemels et al. (2009); Nešić and Liberzon (2009).

Combining all the variables and their relationships in (2)-(3), (12)-(16), the closed-loop system in the zooming-in stage is denoted by \(\mathcal{F}_2\) and modelled as follows.

\[
\dot{x} = A_0 + B_0 + B_1 C_0 x + B_1 C_0 + B_2 C_0 x, \\
\dot{e} = A_2 e + B_2 C_0 x + B_2 C_0 + B_3 C_0 e,
\]

(17a)

\[
\dot{\mu} = 0, \quad \tau_1 = 0
\]

\[
x^+ = x
\]

\[
\tau_2 = \tau_1 \quad (\tau_1, \tau_2, c) \in C_2,
\]

(17b)

with

\[
C_2 := [0, h_{\text{mat}}] \times [0, T_2(x)] \times \{1\},
\]

\[
D_2^{\text{jump}} := [\epsilon, h_{\text{mat}}] \times [0, T_2(x)] \times \{1\},
\]

(18)

where

\[
A_0 := A, \quad B_0 := B_1 C_0, \quad B_1 := B_1 C_0 + B_2 C_0,
\]

\[
B_2 := B_2 C_0, \quad B_3 := -C_2 B_3, \quad B_3 := -C_2 B_3
\]

Different from the periodic increase of the quantization parameter in the zooming-out stage in Subsection 3.2, the transmissions do not occur periodically \textit{a priori} in the zooming-in stage. Hence, both the flow and jump are allowed to occur when \(\tau_1 \in [0, \epsilon]\). If there is a solution to (17a)-(17b) such that \(\tau_1 \in [0, \epsilon]\) and \(\tau_2 = T_2(x)\), then it stops to exist, which is not an issue since this case does not occur in practice and the stability in Section 5 still holds.

4. HYBRID MODEL

We combine the system models in both the zooming-out and zooming-in stages, and construct a unified hybrid model as the formalism in Goebel et al. (2012). To begin with, we introduce a counter \(\kappa_1 \in \mathbb{N}\) to keep track of the number of the transmissions needed to implement some time-scheduling protocols like the RT protocol, a counter \(\kappa_2 \in \mathbb{N}\) to keep track of the number of the switching between the zooming-in and the zooming-out stage, and a timer \(\tau \in \mathbb{R}\) to keep track of the length of all the zooming-out stages. Denote \(\mathcal{X} := (x, e, \mu, \kappa_1, \kappa_2, \tau, \tau_1, \tau_2, c) \in \mathcal{H} := \mathbb{R}^n_0 \times \mathbb{R}^{\ell} \times \mathbb{R}^{\ell} \times \mathbb{N} \times \mathbb{R} \times \{0, 1\}\), and the NCQS is modelled as

\[
\mathcal{H} : \begin{cases}
\dot{x} = F(x, w), & (x, w) \in C \times \mathbb{R}^{n_0}, \\
\dot{X} = G(x), & (x, w) \in D \times \mathbb{R}^{n_0},
\end{cases}
\]

(19)

where

\[
C := \mathbb{R}^{n_0} \times \mathbb{R}^{n_0} \times \mathbb{R}^\ell \times \mathbb{N} \times \mathbb{R} \times (C_1 \cup C_2),
\]

\[
D := \mathbb{R}^{n_0} \times \mathbb{R}^{n_0} \times \mathbb{R}^\ell \times \mathbb{N} \times \mathbb{R} \times (D_1 \cup D_2),
\]

(20)

with \(C_1\) and \(C_2\) defined respectively from (11) and (18), \(D_1 := D_1^{\text{out}} \cup D_1^{\text{jump}}\), \(D_2 := D_2^{\text{out}} \cup D_2^{\text{jump}}\) with \(D_1^{\text{jump}}\) and \(D_2^{\text{jump}}\) defined in (11) and (18) respectively, and

\[
\begin{align*}
D_1^{\text{out}} & := \{0\} \times \{T_1(x)\} \times \{0\}, \\
D_2^{\text{out}} & := \{0\} \times \{T_2(x)\} \times \{1\}.
\end{align*}
\]

(21)

The mapping \(F\) in (19) is defined as

\[
F(x, w) := \begin{cases}
F_1(x), & (\tau_1, \tau_2, c) \in C_1, \\
F_2(x), & (\tau_1, \tau_2, c) \in C_2,
\end{cases}
\]

(22)

where \(F_1(x) := (\sigma x + \delta_1 x, -c_1 \sigma x - c_1 \delta_1 w, 0, 0, 1, 1, 0)\) corresponds to the flow (10a) for the zooming-out stage, and \(F_2(x) := (\sigma x + \delta_2 x, \sigma x + \delta_2 x, \delta_2 w, 0, 0, 0, 1, 1, 0)\) corresponds to the flow (17a) for the zooming-in stage. The mapping \(G\) in (19) is defined as

\[
\begin{align*}
G_1(x) & := (G_1(x), (\tau_1, \tau_2, c) \in D_1^{\text{jump}}), \\
G_2(x) & := (G_2(x), (\tau_1, \tau_2, c) \in D_2^{\text{jump}}), \\
G_3(x) & := (G_3(x), (\tau_1, \tau_2, c) \in D_1^{\text{out}} \cup D_2^{\text{out}}).
\end{align*}
\]

(23)

where \(G_1(x) := (\tau_1, \tau_2, c) \in D_1^{\text{jump}}\) corresponds to a transmission jump (10b) in the zooming-out stage, \(G_2(x) \in \{x, \epsilon, \Omega_{out} \mu, \kappa_1, \kappa_2, \tau, 0, t_2, c\} \) corresponds to the transmission jump (17b) in the zooming-in stage, and \(G_3(x) := (x, \epsilon, \Omega_{out} \mu, 0, 0, 0, 1, 0, c) \) corresponds to the jump caused by the switching between the zooming-out and zooming-in stages.

5. MAIN RESULTS

In this section, we first study the evolution of the system state in the zooming-out stage, and then the convergence of the system state in the zooming-in stage.

5.1 Boundedness in Zooming-Out Stage

The boundedness of the system state in the zooming-out stage is investigated in this subsection.

Theorem 1. Consider the system \(\mathcal{H}\) in the zooming-out stage, and let \(\delta \in (0, \ln \Omega_{\mu_{\text{out}}}^{\text{min}} / \|\mathcal{A}\|)\). There exists a finite hybrid time instant \((t_1, j_1)\) such that for all \((t_0, j_0) \geq (t_1, j_1)\),

\[
|x(t, j)| \leq \beta_\delta (x(t_0, j_0) - t - t_0, j - j_0) + \gamma_\delta (\|w\|),
\]

(24)

\[
|\mu(t, j)| \leq \beta_\delta (x(t_0, j_0) - t - t_0, j - j_0) + \gamma_\delta (\|w\|),
\]

(25)

where \(\beta_\delta, \beta_\delta \in \mathcal{K} \cap \mathcal{L}\) and \(\gamma_\delta, \gamma_\delta \in \mathcal{K}\). In addition, \(x(t_1, j_1)\) is in the quantization regions.

Theorem 1 ensures that the state and the quantization parameter are bounded in each zooming-out stage, and the existence of \((t_1, j_1)\) ensures the finite length of each zooming-out stage, which further implies that \(T_1(x)\) is bounded. In addition, the state is in the quantization regions at the end of the zooming-out stages. As a result, there always exists a hybrid time instant such that the system switches from the zooming-out stage into the zooming-in stage.

In quantized control systems (Ren and Xiong (2018a); Liberzon and Nešić (2007)), the existence of the time instant \((t_1, j_1)\) implies that the state will be in the quantization regions from such a time instant despite the effects of the external disturbance. If this is also the case for the NCQS studied in this paper, then we obtain that \(T_1(x)\) is finite, \(t(t) \equiv 1\) for all \(t \in [T_1(x), \infty)\), and the input-to-state stability of the hybrid system \(\mathcal{H}\) depends on
the convergence of the state in the zooming-in stage, which will be studied in Subsection 5.2. However, due to the effects of the external disturbance and the network-induced errors, the state may escape from the quantization regions such that the system $\mathcal{H}$ switches back to the zooming-out stage.

5.2 Convergence in Zooming-In Stage

We next study the input-to-state stability of the system $\mathcal{H}$ in the zooming-in stage. In order to ensure ISS, some assumptions are provided for the $x$ and $(e, \mu)$ subsystems.

**Assumption 4.** There exist a function $W : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}_0^+$ which is locally Lipschitz in $(e, \mu)$ for all $k \in \mathbb{N}$, $\alpha_{1W}, \alpha_{2W} \in \mathcal{K}_{\infty}$ and $\lambda \in [0,1]$ such that for all $(k, e, \mu) \in \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^r$, $\alpha_{1W}(||e, \mu||) \leq W(k, e, \mu) \leq \alpha_{2W}(||e, \mu||)$, $W(k+1, h(k,x,e,\mu), \Omega_{in}\mu) \leq \lambda W(k, e, \mu)$, for all $x \in \mathbb{R}^n$ and almost all $e \in \mathbb{R}^n$, $\mu \in \mathbb{R}^r$, $\frac{\partial W(k,e,\mu)}{\partial e} + \alpha_{3W} x + \beta_{3W} e + \gamma_{3W} \mu \leq \lambda W(k, e, \mu) + |\tilde{y}|$, (26)

where $\tilde{y} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on $(x,e,w)$.

In Assumption 4, the function $W$ is used to analyze the stability of $(e, \mu)$-subsystem. Assumption 4 is to estimate the jumps of $W$ at the discrete-time instants and the derivative of $W$ in the continuous-time intervals.

**Assumption 5.** There exist a locally Lipschitz function $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\alpha_{1V}, \alpha_{2V} \in \mathcal{K}_{\infty}$, and constants $\rho, \theta, \gamma > 0$, such that such that $\alpha_{1V}(|x|) \leq V(x) \leq \alpha_{2V}(|x|)$ holds for all $x \in \mathbb{R}^n$; and $\left\langle \nabla V(x), \alpha_{3W} x + \beta_{3W} e + \gamma_{3W} \mu \right\rangle \leq (\gamma^2 - \theta) W(k, e, \mu) + \phi(|w|) - |\tilde{y}|^2 - \rho V(x)$ holds for all $(e, \mu) \in \mathbb{R}^n \times \mathbb{R}^r$ and almost all $x \in \mathbb{R}^n$, where $\tilde{y}$ is given in Assumption 4.

The last assumption is on the transmission intervals, and is to bound the MATI in Assumption 1. Consider the following differential equation

$$\dot{\phi} = -2L\phi - \gamma \phi^2 + 1, \quad \phi(0) = \lambda^{-1}, \quad (27)$$

where $L \geq 0$ and $\gamma > 0$ are given respectively in Assumptions 4-5. It is noted from Heemels et al. (2010) that the solution to (27) is strictly decreasing as long as $\phi(\tau_1) \geq 0$. With the equation (27), the upper bound $h_{mati}$ satisfies $\phi(h_{mati}) = \lambda$; see Jentzen et al. (2010).

**Theorem 2.** Consider the system $\mathcal{H}$ in the zooming-in stage, and let Assumptions 4-5 hold if $\text{MATI h_{mati}}$ satisfies $\phi(h_{mati}) = \lambda$, then the state of the system $\mathcal{H}$ is convergent.

6. SWITCHING BETWEEN TWO STAGES

According to Section 5, we can construct the following Lyapunov function for the system $\mathcal{H}$ in (19):

$$U(X) := (1-c)U_1(X) + cU_2(X),$$

where $U_1(X)$ is the Lyapunov function for the zooming-out stage and $U_2(X)$ is the Lyapunov function applied in the zooming-in stage. For the sake of simplicity, we can take the Lyapunov function below

$$\tilde{U}(X) := V(x) + c\phi(\tau_1) W^2(k,e,\mu), \quad (28)$$

where $V(x)$ and $W(k,e,\mu)$ are given in the assumptions in Subsection 5.1. Hence, $U_1(X) = V(x)$ and $U_1(X) = V(x) + c\phi(\tau_1) W^2(k,e,\mu)$.

For the zooming-out stage, we have from (10) and Theorem 1 that there exist $r_1 \in \mathbb{R}^+$ related to the function $f$ and $r_2 \in \mathcal{H}$ such that

$$\langle \nabla U(X), F_1(X,w) \rangle \leq r_1 U(X) + r_2 |w|^2, \quad (29)$$

where $r_1 := \lambda_{\text{min}}(\mathcal{P}) \lambda_{\text{max}}(\phi' + \mathcal{P}(\phi' + \mathcal{P}\phi'))$ and $r_2 := \mathcal{B}$. For the zooming-in stage, we have from Subsection 5.2 that there exist $\sigma > 0$ and $r_3 \in \mathcal{H}$ such that

$$\langle \nabla U_2(X), F_2(X,w) \rangle \leq -\sigma U_2(X) + r_3 |w|^2. \quad (30)$$

**Proposition 3.** Consider the system (19), and let Assumptions 1-5 hold. If there exists $\xi > 1$ such that

$$\gamma(x) \leq \xi - 1 \nu(x),$$

$$\tau \leq (r_1 + \sigma)^{-1} \omega, \quad \omega \in (\sigma/2, \sigma),$$

$$\kappa_2 \leq N_0 + \frac{2(\omega - 0\omega)}{\ln \xi}, \quad 0 < \omega \in (2\omega - \sigma, \omega), \quad (33)$$

then the system (19) is ISS.

In Proposition 3, Condition (31) is to ensure that $U_2(X) \leq \xi U_1(X)$ at the stage switches, which holds by choosing an appropriate constant $\xi > 1$. Condition (32) is to constrain the time length of the zooming-out stage, and Condition (33) is to constrain the number of the switching from the zooming-in stage to the zooming-out stage. In addition, Conditions (32)-(33) can be treated as the average dwell-time (ADT) conditions, and $0.5(\omega - 0\omega)^{-1} \ln \xi$ is the ADT. Conditions (32)-(33) can be satisfied via increasing the quantization parameter as fast as possible to ensure the time length of the zooming-out stage as small as possible. The smaller the constant $\delta > 0$ in $\mathcal{H}$ is, the more frequent the increase of $\mu$ is, and thus the shorter of the time length of the zooming-out stage is.

7. NUMERICAL EXAMPLE

In this section, we apply our results to stabilize the origin of a batch reactor in Heemels et al. (2010). Consider the plant below

$$\dot{x}_p = A_p x_p + B_p u + C_p w, \quad y = D_p x_p, \quad (34)$$

and the designed controller given by

$$\dot{x}_c = A_c x_c + B_c u, \quad u = C_c x_c + D_c y, \quad (35)$$

where $x_p \in \mathbb{R}^4, y \in \mathbb{R}^2, x_c \in \mathbb{R}^3, y \in \mathbb{R}^2$ and $u \in \mathbb{R}^2$; the matrices in (34)-(35) can be found in Nešić and Teel (2004); Heemels et al. (2010).

As a result, we have that the open-loop model is

$$\dot{x}_p = A_p x_p + C_p w, \quad (36)$$

which means that the system is in the zooming-out stage. Let $\mu$ be initialized at $E$, and $\Omega_{\text{out}} := \exp(2\delta ||A_p||)/\delta$, where $\delta > 0$ is given in Subsection 3.2. Thus, there exists a time instance such that $|x| \leq M||\mu||$, and such time instance can be set as the initial time for the zooming-in stage.

In the zooming-in stage, the closed-loop model is

$$\dot{x} = A_{11} x + A_{12} e + A_{13} w, \quad \dot{e} = A_{21} x + A_{22} e + A_{23} w, \quad (37)$$

where the matrices in (37) can be found in Heemels et al. (2010). Here, we choose $\max_j A_{1j} = 0.8$ and $\max_j \Omega_{mj} = 0.6$, $j \in \{1, 2\}$, which implies that $\sqrt{(l-1)}/l \geq \max_j \Omega_{mj} + \epsilon \max_j A_{1j}/\sqrt{l}$ holds for the TOD protocol case and $\sqrt{(l-1)}/l \geq \max_j \Omega_{mj} + \epsilon \sqrt{l} \max_j A_{1j}$ holds for the RR protocol case. As a result, Assumption 4-5 hold with $\lambda = \sqrt{2}/2, \rho = M + 1, L = 15.730, \gamma = 21.5275$, and $\tilde{y} = \sqrt{2}A_{21} x$ for the RR protocol case.
Fig. 1. The state response in the TOD protocol case.

Table 1. Bounds on MATI in Section 5

|        | RR | TOD |
|--------|----|-----|
| Theorem 1 | 0.0010 | 0.0029 |
| Simulations | 0.0663 | 0.0961 |

and with $\lambda = \sqrt{2}/2$, $\rho = M + 1$, $L = 15.730$, $\gamma = 15.9222$, and $\tilde{y} = A_{21}x$ for the TOD protocol case. The MATI bounds are summarized and compared to the bounds estimated via simulations in Table 1.

Given a switching time instant $T_k \in \mathbb{R}^+$, the next switching time from the zooming-out stage to the zooming-in stage is obtained by $T_1(x) = T_{2k+1} = \inf\{t \geq T_2 : (|y(t)| \lor |u(t)|) \leq M\mu(t), l = 1, 2\}$, and the next switching time from the zooming-in stage to the zooming-out stage is obtained by $T_2(x) = T_{2k+2} = \inf\{t \geq T_{2k+1} : (|y(t)| \lor |u(t)|) > M\mu(t), l = 1, 2\}$. Hence, we verify whether the state is in the quantization regions, and then determine the switching time instants and further which stage that the system is in. Under the external disturbance $w(t) = (10\sin(50t), 5\cos(20t))$ and $M = 5$, the state response is shown in Fig. 1, which implies that the system state converges to a region around the origin, which is caused due to the external disturbance, and implies input-to-state stability of the whole system.

8. CONCLUSIONS

In this paper, system modelling and input-to-state stability were studied for networked and quantized control systems with external disturbances. A unified hybrid system framework was developed for networked and quantized control systems. In particular, the quantization mechanism was considered. According to the developed system model, a Lyapunov ISS protocol was proposed and sufficient conditions for input-to-state stability were established. A numerical example was used to illustrate the developed theory.

REFERENCES

Bailioul, J. and Antsaklis, P.J. (2007). Control and communication challenges in networked real-time systems. in Proceedings of the IEEE, 95(1), 9–28.

Brockett, R.W. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems. IEEE Transactions on Automatic Control, 45(7), 1279–1289.

Cai, C. and Teel, A.R. (2009). Characterizations of input-to-state stability for hybrid systems. Systems & Control Letters, 58(1), 47–53.

Donkers, M., Heemels, W., Van De Wouw, N., and Hetel, L. (2011). Stability analysis of networked control systems using a switched linear systems approach. IEEE Transactions on Automatic Control, 56(9), 2101–2115.

Goebel, R., Teel, A.R., and Sanfelice, R.G. (2012). Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press.

Heemels, W.H., Teel, A.R., van de Wouw, N., and Nesic, D. (2010). Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance. IEEE Transactions on Automatic Control, 55(8), 1781–1796.

Heemels, W., Nesic, D., Teel, A.R., and van de Wouw, N. (2009). Networked and quantized control systems with communication delays. in Proceedings of IEEE Conference on Decision and Control, 7929–7935.

Jentzen, A., Leber, F., Schneisgen, D., Berger, A., and Siegmund, S. (2010). An improved maximum allowable transfer interval for $\mathcal{L}_p$ stability of networked control systems. IEEE Transactions on Automatic Control, 55(1), 179–184.

Liberzon, D. (2003a). Hybrid feedback stabilization of systems with quantized signals. Automatica, 39(9), 1543–1554.

Liberzon, D. (2003b). On stabilization of linear systems with limited information. IEEE Transactions on Automatic Control, 48(2), 304–307.

Liberzon, D. (2014). Finite data-rate feedback stabilization of switched and hybrid linear systems. Automatica, 50(2), 409–420.

Liberzon, D. and Hespanha, J.P. (2005). Stabilization of nonlinear systems with limited information feedback. IEEE Transactions on Automatic Control, 50(6), 910–915.

Liberzon, D. and Nešić, D. (2007). Input-to-state stabilization of linear systems with quantized state measurements. IEEE Transactions on Automatic Control, 52(5), 767–781.

Lunze, J. (2014). Control Theory of Digitally Networked Dynamic Systems. Springer.

Nešić, D. and Liberzon, D. (2009). A unified framework for design and analysis of networked and quantized control systems. IEEE Transactions on Automatic Control, 54(4), 732–747.

Nešić, D. and Teel, A.R. (2004). Input-output stability properties of networked control systems. IEEE Transactions on Automatic Control, 49(10), 1650–1667.

Ren, W. and Xiong, J. (2018a). Quantized feedback stabilization of nonlinear systems with external disturbance. IEEE Transactions on Automatic Control, 63(9), 3167–3172.

Ren, W. and Xiong, J. (2018b). Tracking control of networked and quantized control systems. In IEEE Conference on Decision and Control, 5844–5849. IEEE.

Tian, E., Yue, D., and Zhao, X. (2007). Quantised control design for networked control systems. IET Control Theory & Applications, 1(6), 1693–1699.

van Loon, S., Donkers, M., van de Wouw, N., and Heemels, W. (2013). Stability analysis of networked and quantized linear control systems. Nonlinear Analysis: Hybrid Systems, 10, 111–125.

Walsh, G.C., Beldiman, O., and Bushnell, L.G. (2001). Asymptotic behavior of nonlinear networked control systems. IEEE Transactions on Automatic Control, 46(7), 1093–1097.

Walsh, G.C., Ye, H., and Bushnell, L.G. (2002). Stability analysis of networked control systems. IEEE Transactions on Control Systems Technology, 10(3), 438–446.

Zhang, L., Gao, H., and Kaynak, O. (2013). Network-induced constraints in networked control systems—A survey. IEEE Transactions on Industrial Informatics, 9(1), 403–416.