Stabilized by quantum fluctuations, dipolar Bose-Einstein condensates can form self-bound liquid-like droplets. However in the Bogoliubov theory, there are imaginary phonon energies in the long-wavelength limit, implying dynamical instability of this system. A similar instability appears in the Bogoliubov theory of a binary quantum droplet, and is removed due to higher-order quantum fluctuations as shown recently [1]. In this work, we study the excitation energy of a dipolar quantum droplet in the Beliaev formalism, and find that quantum fluctuations significantly enhance the phonon stability. We adopt a self-consistent approach without the problem of complex excitation energy in the Bogoliubov theory, and obtain a stable anisotropic sound velocity which is consistent with the superfluid hydrodynamic theory, but slightly different from the result of the extended Gross-Pitaevski (GP) equation due to quantum depletion. A modified GP-equation in agreement with the Beliaev theory is proposed, which takes the effect of quantum fluctuations into account more completely.

Quantum droplets have been realized in different systems, including single-component dipolar Bose gases of $^{164}$Dy [2–6], $^{166}$Er [7] atoms, nonmagnetic binary $^{39}$K mixture [8,10], and heteronuclear $^{39}$K–$^{85}$Rb mixtures [11]. There are also proposals to create quantum droplets in binary dipolar mixtures [12,13]. Usually an ultracold bosonic gas is a dilute weakly-interacting system in which the contribution from quantum fluctuations are typically small compared with the mean-field contribution, but quantum droplets are just exceptions. Due to the repulsive Lee-Huang-Yang (LHY) energy [14] which overcomes the net attractive mean-field energy near the mean-field unstable point, a nonmagnetic binary bosonic mixture displays a self-bound liquid-like state [15], as observed in experiments [8,11]. The LHY energy [14] also stabilizes the single-component dipolar quantum droplet in the dipole-dominated regime where the dipolar interaction dominates over the repulsive s-wave interaction. Phase-coherent droplet arrays have also been observed in experiments [17,20], where broken translational symmetry and superfluid order appear at the same time, consistent with a supersolid [21,22].

In a nonmagnetic boson mixture, quantum droplets are formed in the mean-field unstable region where the total mean-field energy is attractive. As a result, in the Bogoliubov theory, there are imaginary excitation energies in the long wavelength limit, implying the dynamical instability. However these excitations have little contribution to LHY energy [15], and it is postulated that these excitations are stable after integrating out high-energy excitations. The LHY energy is thus obtained in the Bogoliubov theory with the imaginary part neglected, and used in the construction of the extended Gross-Pitaevskii equation (EGPE). The EGPE approach has been quite successful in interpreting experimental results [23,26], with support to some extent by quantum Monte Carlo calculations [27,30]. Despite the success of the EGPE, how the dynamical instability is avoided in quantum droplets, or even, whether the Bose-Einstein-condensation (BEC) state is a stable ground state has become a pressing issue. The pairing [31] and Gaussian states [32] have been proposed as ground states. Recently in a Beliaev approach, it is found that the phonon energy is always positive after considering the interaction between spin and density fluctuations [11], showing that the BEC state is a stable ground state for a nonmagnetic binary quantum droplet.

In the Bogoliubov theory of a uniform dipolar Bose gas [16], in the quantum droplet region the excitation energy in the direction perpendicular to the polarization is imaginary in the long-wavelength limit, suggesting dynamic instability. The Gaussian state was also proposed to be the ground state of dipolar quantum droplets [32]. In this work, we study the excitation stability of dipolar quantum droplets by going beyond the Bogoliubov theory and calculating the excitation energy in the Beliaev formalism [33]. In the quantum droplet region, as there are imaginary excitation energies in the Bogoliubov theory, we propose a self-consistent method to take account of the fluctuation effect from the start and avoid the complex excitation energy. In the long-wavelength limit the excitation energy is stable with relative dipole-dipole interaction (DDI) strength $\epsilon_{dd} < 1.35$ for the $^{166}$Er system. We obtain the anisotropic phonon velocity consistent with the superfluid hydrodynamic theory, but slightly different from the EGPE result due to the quantum depletion. We propose a modified EGPE to produce the correct phonon velocity. Finally we discuss implication of our results in experiment.

Beliaev theory of a dipolar Bose gas. We consider a homogeneous gas of bosonic particles with mass $m$ and a finite magnetic dipole moment $d$ at zero temperature, with condensate density $n_0$. The magnetic dipoles align along the $z$ axis with the interaction potential is given by

$$V_{int}(r) = g|\delta(r)| + \frac{3\epsilon_{dd}}{4\pi} \frac{z^2}{r^2} (1 - 3 \frac{z^2}{r^2}),$$

(1)
where $g$ is the s-wave coupling constant, $g = 4\pi\hbar^2a_s/m$, $a_s$ is the s-wave scattering length, $\epsilon_{dd} = a_{dd}/a_s$ is the relative interaction strength, $a_{dd} = \mu_0d^2m/12\pi\hbar^2$ is the characteristic length of DDI, and $\mu_0$ is the magnetic permeability in vacuum. The Hamiltonian of a dipolar Bose gas is given by
\[
\hat{H} = \int d^3r \hat{\psi}^\dagger(r)(-\frac{\hbar^2\nabla^2}{2m})\hat{\psi}(r) + \frac{1}{2}\int d^3rd^3r'\hat{\psi}^\dagger(r)\hat{\psi}^\dagger(r')V_{\text{int}}(r-r')\hat{\psi}(r')\hat{\psi}(r),
\]
where $\hat{\psi}(r)$ is the boson field operator. The Fourier transform of the interaction potential is given by
\[
U(p) = g[1 + \epsilon_{dd}(3\cos^2\theta - 1)],
\]
where $\theta$ is the angle between $p$ and the z axis [34].

In the BEC phase, there are macroscopic number of atoms in the zero-momentum state, and the boson field operators of the zero-momentum state $\hat{a}_0$ and $\hat{a}_0^\dagger$ can be replaced by a c-number $N_0^{1/2}$, where $N_0 = n_0V$ is the number of bosons in the condensate and $V$ is the volume [35]. The Boson field operator can be separated into two parts,
\[
\hat{\psi}(r) \equiv \frac{\hat{a}_0}{\sqrt{V}} + \sqrt{N_0}\sum_k U(k)\hat{\psi}_k\hat{a}_k + \hat{\psi}_k^\dagger\hat{a}_k,
\]
where $\hat{\psi}_k^\dagger(r)$ is the field operator of bosons outside the condensate. In this work, we consider the dilute case where the interaction term in the Hamiltonian can be expanded in terms of $n_0^{1/2}$. The Bogoliubov Hamiltonian describes the quadratic fluctuation, given by
\[
\hat{H}_B = \frac{1}{2}n_0^2VU(0) + \frac{1}{2}n_0\sum_k U(k)(\hat{\psi}_k^\dagger\hat{\psi}_k - \hat{\psi}_k\hat{\psi}_k^\dagger) + \sum_k [\epsilon_k + n_0U(0) + n_0U(k)]\hat{\psi}_k^\dagger\hat{\psi}_k,
\]
where $\hat{\psi}_k$ is the boson annihilation operator in the momentum space. The quasi-particle excitation energy is obtained by diagonalizing the Bogoliubov Hamiltonian,
\[
\epsilon_p = \sqrt{\epsilon_p^0(n_0U(p) + \epsilon_p^0)},
\]
where $\epsilon_p^0 = \hbar^2p^2/2m$ is the kinetic energy of the atom. In the BEC phase, the boson Green’s function is a matrix, defined as [33]
\[
G(p, t_1 - t_2) = -i\langle T\{\Psi_p(t_1)\Psi_p^\dagger(t_2)\}\rangle,
\]
where $\Psi_p^\dagger(t) = [\hat{a}_p^\dagger(t), \hat{a}^\dagger_{-p}(t)]$, and $T\{\ldots\}$ is the time-ordering operator. In the momentum and energy space, the Dyson’s equation is given by
\[
G(p) = G^{(0)}(p) + G^{(0)}(p)\Sigma(p)G(p),
\]
where $p = (p, p^0)$, and $\Sigma(p)$ is the self-energy. In the matrices of the Green’s function and self-energy,
\[
G(p) = \begin{bmatrix} G_{11}(p) & G_{12}(p) \\ G_{21}(p) & G_{11}(-p) \end{bmatrix},
\]
\[
G^{(0)}(p) = \begin{bmatrix} G_0(p) & 0 \\ 0 & G_0(-p) \end{bmatrix},
\]
\[
\Sigma(p) = \begin{bmatrix} \Sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{11}(-p) \end{bmatrix},
\]
the subscript 11 refers to an ingoing and an outgoing external line in the Feynman diagrams, the subscript 12 refers to two outgoing lines, and the superscript 21 refers to two ingoing lines. The free-boson Green’s function is given by
\[
G_0(p) = 1/(p^0 - \epsilon_p^0 - \mu + i0^+),
\]
where $\mu$ is the chemical potential. From the pole-equation of the Green’s function $\det[G^{-1}(p)] = 0$, the excitation spectrum can be solved.

For a dilute Bose gas, the self-energy matrix $\Sigma(p)$ can be expanded in terms of the condensate density $n_0$,
\[
\Sigma(p) = \Sigma^{(1)}(p) + \Sigma^{(2)}(p) + \ldots,
\]
where the first-order self-energy matrix is given by
\[
\Sigma^{(1)}(p) = \begin{bmatrix} n_0U(0) & n_0U(p) \\ n_0U(p) & n_0U(0) + n_0U(p) \end{bmatrix}.
\]
The chemical potential can be expanded as well, and the first-order term is given by $\mu^{(1)} = n_0U(0)$. The first-order Green’s function is given by
\[
G_{11}(p) = \frac{A_p}{p^0 - \epsilon_p + i0^+} - \frac{B_p}{p^0 + \epsilon_p - i0^+},
\]
\[
G_{12}(p) = -C_p(\frac{1}{p^0 - \epsilon_p + i0^+} - \frac{1}{p^0 + \epsilon_p - i0^+}) = G_{21}(p),
\]
where
\[
A_p = [\epsilon_p + \epsilon_p^0 + n_0U(p)]/2\epsilon_p,
\]
\[
B_p = [-\epsilon_p + \epsilon_p^0 + n_0U(p)]/2\epsilon_p,
\]
\[
C_p = n_0U(p)/2\epsilon_p.
\]
From the pole of the Green’s function, the quasiparticle excitation energy can be obtained, same as that in Eq. [7] from the Bogoliubov theory. In the droplet regime with $\epsilon_{dd} > 1$, the interaction $U(p)$ is attractive in the direction perpendicular to the polarization with $\theta = \pi/2$, and the excitation energy becomes imaginary in the long wavelength limit, indicating the dynamical instability.

In Ref. [33], Beliaev calculated the second-order self-energy from one-loop diagrams and obtained the correction to excitation spectrum of a Bose gas with a s-wave interaction. Here for a dipolar Bose gas, we adopt Beliaev’s approach to consider the fluctuation effect beyond the Bogoliubov theory, and obtain the second-order self-energy shown in Fig. [1] given by
where $\Sigma$ avoid double-counting. interaction and the other for an exchange interaction. In the diagrams with four external lines, two lines are with zero

![Feynman diagrams of second-order self-energies. A filled rectangle denotes a sum of two rectangles, one for a direct interaction and the other for an exchange interaction. In the diagrams with four external lines, two lines are with zero momentum. The internal lines are first-order Green's functions. The line with one arrow is the normal part of the Green's function, and that with two arrows is the anomalous part. A dashed line represents subtracting the contribution of $G^{(0)}$ to avoid double-counting.

\[
\Sigma_{11}^{(2)}(p) = \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} \frac{1}{p^0 - \varepsilon_k - \varepsilon_q + i0^+} \times \left\{ [U(p) + U(k)](A_k; B_q) + 2[U(p) + U(k)]U(p) + U(q)\right\}C_k C_q \\
- \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} \frac{1}{p^0 + \varepsilon_k + \varepsilon_q - i0^+} \times \left\{ [U(p) + U(k)](A_k; B_q) + 2[U(p) + U(k)]U(p) + U(q)\right\}C_k C_q \\
+ \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} U^2(q) \left( \frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_q} \right) \int \frac{dq}{(2\pi)^3} \left( \frac{1}{\varepsilon_p^0} - \frac{4}{\varepsilon_q^0 + \varepsilon_q^0 + i0^+} + \frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_q}, \right) \\
\Sigma_{12}^{(2)}(p) = \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} \left( \frac{1}{p^0 - \varepsilon_k - \varepsilon_q + i0^+} - \frac{1}{p^0 + \varepsilon_k + \varepsilon_q - i0^+} \right) \times \left\{ [U(p) + U(k)](A_k; B_q) + 2[U(p) + U(q)]U(p) + U(q)\right\}(A_k; B_q) \\
+ \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} U^2(q) \left( \frac{1}{\varepsilon_q^0} - \frac{1}{\varepsilon_q} \right), \tag{19}
\]

where $k = p - q$, the symbol $(;)$ denotes a symmetrized product, $(A_k; B_q) = A_k B_q + B_k A_q$. The second-order chemical potential satisfies the Hugenholtz-Pines theorem [36],

\[
\mu^{(2)} = \Sigma_{11}^{(2)}(0) - \Sigma_{12}^{(2)}(0) = \int \frac{dq}{(2\pi)^3} [U(0) + U(q)]B_q + \frac{1}{2} n_0 \int \frac{dq}{(2\pi)^3} U^2(q) \left( \frac{1}{\varepsilon_q^0} - \frac{1}{\varepsilon_q} \right). \tag{20}
\]

The pole equation of the Green’s function up to the second order is given by

\[
[p^0 - \frac{1}{2}(\Sigma_{11}^{(2)} - \Sigma_{12}^{(2)})]^2 = [\varepsilon_p^0 + 2n_0 U(p) + \frac{1}{2}(\Sigma_{11}^{(2)} + \Sigma_{12}^{(2)}) - \mu^{(2)} + \Sigma_{12}^{(2)}] \times [\varepsilon_p^0 + \frac{1}{2}(\Sigma_{11}^{(2)} - \Sigma_{12}^{(2)}) - \mu^{(2)} - \Sigma_{12}^{(2)}], \tag{21}
\]

where $\Sigma_{11}^{(2)}(p) = \Sigma_{11}(\pm p)$.

The energy is given by

\[
\Sigma_{11}^{(2)} - \Sigma_{11}^{(2)} = 2\alpha p^0, \\
\frac{1}{2}(\Sigma_{11}^{(2)} + \Sigma_{12}^{(2)}) - \mu^{(2)} + \Sigma_{12}^{(2)} = \beta n_0 U(p), \\
\frac{1}{2}(\Sigma_{11}^{(2)} + \Sigma_{12}^{(2)}) - \mu^{(2)} - \Sigma_{12}^{(2)} = \gamma \varepsilon_p^0.
\]

In the long-wavelength limit, the second-order self-
where \( \alpha, \beta, \) and \( \gamma \) are coefficients. Since the second-order self-energies have one more power of \( \sqrt{n_0} \) than the first-order self-energies, these coefficients can be treated as small quantities. To determine the leading correction to the phonon spectrum, we linearize the pole equation Eq. (21), and obtain in the long-wavelength limit

\[
\varepsilon_p^B = \sqrt{n_0 U(p)} \varepsilon_p^{(0)} [2 + (4\alpha + 2\gamma + \beta)]. \tag{22}
\]

It is worth noting that each matrix element of the second-order self-energy is divergent in the long-wavelength limit, but the divergencies cancel each other in the final expression for the phonon energy in the combination \( 4\alpha + 2\gamma + \beta \) as in the original Beliaev theory [33]. We separate the divergent parts, \( \alpha_d, \beta_d, \) and \( \gamma_d, \) from coefficients \( \alpha, \beta, \) and \( \gamma, \)

\[
\alpha_d = \frac{n_0^2}{2} \int dq |U(p)U^2(q)| \left( \frac{1}{\varepsilon_q^3} \right),
\]

\[
\beta_d = -n_0^2 \int dq |U(p)U^2(q)| \left( \frac{1}{\varepsilon_q^3} \right),
\]

\[
\gamma_d = -\frac{n_0^3}{2} \int dq U(p)U^2(q) \left( \frac{1}{\varepsilon_q^3} \right). \tag{23}
\]

which satisfy \( 4\alpha_d + 2\gamma_d + \beta_d = 0. \) The convergent parts of the coefficients are denoted by \( \alpha_c, \beta_c, \) and \( \gamma_c, \)

\[
\alpha_c = -4\sqrt{n_0 a_0^3/\pi Q_3(\epsilon_{dd})},
\]

\[
\beta_c = 16\sqrt{n_0 a_0^3/\pi Q_3(\epsilon_{dd})} + \frac{32g}{U(p)} \sqrt{n_0 a_0^3/\pi Q_5(\epsilon_{dd})},
\]

\[
\gamma_c = \frac{8}{3} \sqrt{n_0 a_0^3/\pi Q_3(\epsilon_{dd})}. \tag{24}
\]

The functions \( Q_3(\epsilon_{dd}) \) and \( Q_5(\epsilon_{dd}) \) describe the dipolar enhancement [34, 38], given by

\[
Q_3(\epsilon_{dd}) = \frac{(3\epsilon_{dd})^{3/2}}{8} \left[ (2 + 5y)\sqrt{1 + y} + 3y^2 \ln \frac{1 + \sqrt{1 + y}}{\sqrt{y}} \right],
\]

\[
= 1 + \frac{3}{10} \frac{\epsilon_{dd}^2}{\epsilon_{dd}} + O(\epsilon_{dd}^3), \tag{25}
\]

\[
Q_5(\epsilon_{dd}) = \frac{(3\epsilon_{dd})^{5/2}}{48} \left[ (8 + 26y + 33y^2)\sqrt{1 + y} \right.
\]

\[
+ 15y^3 \ln \frac{1 + \sqrt{1 + y}}{\sqrt{y}}],
\]

\[
= 1 + \frac{3}{2} \frac{\epsilon_{dd}^4}{\epsilon_{dd}} + O(\epsilon_{dd}^5), \tag{26}
\]

where \( y = (1 - \epsilon_{dd})/3\epsilon_{dd}. \) The second-order chemical potential can be written explicitly in terms of these functions,

\[
\mu^{(2)} = \frac{40n_0 a_0^3}{3} \sqrt{n_0 a_0^3/\pi Q_5(\epsilon_{dd})}. \tag{27}
\]

Thus in the Beliaev theory of a dipolar Bose gas, the phonon energy in the long-wavelength limit is given by

\[
\varepsilon_p^B = \sqrt{n_0 U(p)} \varepsilon_p^{(0)} [2 + (4\alpha_c + 2\gamma_c + \beta_c)] = v|p|, \tag{28}
\]

where the phonon velocity \( v \) is given by

\[
v = \left( \frac{n_0}{m} \right)^{1/2} \left[ U(p) \right]^{1/2} \left[ 1 + \frac{8}{3} \sqrt{n_0 a_0^3/\pi Q_3(\epsilon_{dd})} \right]
\]

\[
+ 16g \sqrt{n_0 a_0^3/\pi Q_5(\epsilon_{dd})} \right]^{1/2}. \tag{29}
\]

In comparison, in the Bogoliubov theory, the phonon velocity is given by \( v_B = \sqrt{n_0 U(p)/m}. \) The main difference comes from the last term in the square root in Eq. (29) which effectively increases the \( s \)-wave interaction and changes the anisotropy of the total interaction.

The modified EGPE. The beyond-mean-field effect of a dipolar Bose gas has also been considered in the superfluid hydrodynamic and EGPE by taking account of the LHY energy. In the superfluid hydrodynamic theory, with the LHY energy considered, the sound velocity is given by [38]

\[
v_s = \sqrt{\frac{n}{m} \{ U(p) + 16g \sqrt{n_0 a_0^3/\pi Q_5(\epsilon_{dd})} \}}, \tag{30}
\]

where \( n \) is the total boson density. Up to the first-two orders, the total density is given by \( n = n_0 + n_d, \) where the quantum-depletion density is given by [38]

\[
n_d = \frac{8}{3} n_0 \sqrt{n_0 a_0^3/\pi Q_3(\epsilon_{dd})}. \tag{31}
\]

The last term inside the square root on r.h.s of Eq. (30) is from LHY energy, where the density \( n \) can be replaced by the condensation density \( n_0 \) with negligible difference of higher orders. Thus the sound velocity given by Eq. (29) in the Beliaev theory is in agreement with the result in the superfluid hydrodynamic theory.

The sound velocity of a uniform dipolar Bose gas can be also obtained from the EGPE [23, 24, 39],

\[
\hbar \psi = L_{GP} \psi,
\]

where \( \psi \) is the wavefunction of the condensate, and

\[
L_{GP} = -\frac{\hbar^2 \nabla^2}{2m} + \int dx V_{int}(\mathbf{x} - \mathbf{x}')(\psi(\mathbf{x}'))^2 \tag{32}
\]

\[
+ \frac{32}{3} \frac{g}{\sqrt{a_0^3/\pi}} \sqrt{Q_5(\epsilon_{dd})}|\psi|^3.
\]

By linearizing both sides of the EGPE around the uniform solution \( \psi_0 = \sqrt{n_0}, \) one can obtain the sound velocity given by

\[
v_{GP} = \sqrt{\frac{n_0}{m} \{ U(p) + 16g \sqrt{n_0 a_0^3/\pi Q_5(\epsilon_{dd})} \}}. \tag{33}
\]

Compared with Eq. (30), in the first term inside the square root of \( v_{GP} \) is the condensation density \( n_0, \) not the total density \( n. \) This is due to the fact that in the EGPE the mean-field energy is computed only in the condensate, whereas in superfluid hydrodynamics the mean-field energy includes also the contribution from the quantum depletion. As this difference is of the same order of the
LHY term, in principle the EGPE should be modified to take it into account. It is negligible in the dilute system with \( n \approx n_0 \), but inessential in the system with significant quantum depletion. Therefore to be consistent with the Beliaev and superfluid hydrodynamic theories, the EGPE should be modified as

\[
i \hbar \psi = L'_{GP} \psi,
\]

where the mean-field term is modified,
probably due to the finite-size effect which effectively increases the kinetic energy and overcomes the interaction energy in the direction perpendicular to the polarization. In the presence of a trap, the system size increases and the linear phonon dispersion is found in an EGPE study [41]. We note that recently the sound velocity of the two-component dipolar Bose gas was derived in a hydrodynamic framework [42].

Concluding remarks. In conclusion, we go beyond Bogoliubov approximation to obtain the phonon energy of a dipolar Bose gas in the Beliaev formalism and found that higher-order quantum fluctuations significantly increase the stability region of the quantum droplet. The quantum fluctuations effectively enhance the s-wave interaction and change the anisotropy of the total interaction. A modified EGPE in agreement with the Beliaev theory and superfluid hydrodynamics is proposed, which takes into account quantum fluctuation effects more completely. We would like to thank Z.-Q. Yu and Q. Gu for helpful discussions. This work is supported by the National Basic Research Program of China under Grant No. 2016YFA0301501.

[1] Qi Gu and Lan Yin. Phonon stability and sound velocity of quantum droplets in a boson mixture. Physical Review B, 102(22):220503, 2020. (document)
[2] Holger Kadau, Matthias Schmitt, Matthias Wenzel, Clarissa Wink, Thomas Maier, Igor Ferrier-Barbut, and Tilman Pfau. Observing the rosenweig instability of a quantum ferrofluid. Nature, 530(7589):194–197, 2016. (document)
[3] Igor Ferrier-Barbut, Holger Kadau, Matthias Schmitt, Matthias Wenzel, and Tilman Pfau. Observation of quantum droplets in a strongly dipolar bose gas. Physical Review Letters, 116(21):215301, 2016. (document)
[4] Igor Ferrier-Barbut, Matthias Schmitt, Matthias Wenzel, Holger Kadau, and Tilman Pfau. Liquid quantum droplets of ultracold magnetic atoms. Journal of Physics B: Atomic, Molecular and Optical Physics, 49(21):214004, 2016.
[5] Matthias Schmitt, Matthias Wenzel, Fabian Böttcher, Igor Ferrier-Barbut, and Tilman Pfau. Self-bound droplets of a dilute magnetic quantum liquid. Nature, 539(7628):259–262, 2016.
[6] Matthias Wenzel, Fabian Böttcher, Tim Langen, Igor Ferrier-Barbut, and Tilman Pfau. Striped states in a many-body system of tilted dipoles. Physical Review A, 96(5):053630, 2017. (document)
[7] L Chomaz, S Baier, D Petter, MJ Mark, F Wächter, Luis Santos, and F Ferlaino. Quantum-fluctuation-driven crossover from a dilute bose-einstein condensate to a macrodroplet in a dipolar quantum fluid. Physical Review X, 6(4):041039, 2016. (document)
[8] CR Cabrera, L Tanzi, J Sanz, B Naylor, P Thomas, P Cheiney, and L Tarruell. Quantum liquid droplets in a mixture of bose-einstein condensates. Science, 359(6373):301–304, 2018. (document)
[9] P Cheiney, CR Cabrera, J Sanz, B Naylor, L Tanzi, and L Tarruell. Bright solitons to quantum droplet transition in a mixture of bose-einstein condensates. Physical review letters, 120(13):135301, 2018.
[10] G Semeghini, G Fierioli, L Masi, C Mazzinghi, L Wolswijk, F Minardi, M Modugno, G Modugno, M Inguscio, and M Fattori. Self-bound quantum droplets of atomic mixtures in free space. Physical review letters, 120(23):235301, 2018. (document)
[11] C D’Errico, A Burchianti, M Prevedelli, L Salasnich, F Ancilotto, M Modugno, F Minardi, and C Fort. Observation of quantum droplets in a heteronuclear bosonic mixture. Physical Review Research, 1(3):033155, 2019. (document)
[12] RN Bisset, LA Peña Ardila, and Luis Santos. Quantum droplets of dipolar mixtures. Physical Review Letters, 126(2):025301, 2021. (document)
[13] Joseph C Smith, D Baillie, and PB Blakie. Quantum droplet states of a binary magnetic gas. Physical Review Letters, 126(2):025302, 2021. (document)
[14] Tsin D Lee, Kerson Huang, and Chen N Yang. Eigenvalues and eigenfunctions of a bose system of hard spheres and its low-temperature properties. Physical Review, 106(6):1135, 1957. (document)
[15] DS Petrov. Quantum mechanical stabilization of a collapsing bose-bose mixture. Physical review letters, 115(15):155302, 2015. (document)
[16] Aristen RP Lima and Axel Polster. Beyond mean-field low-lying excitations of dipolar bose gases. Physical Review A, 86(6):063609, 2012. (document)
[17] R Bombin, J Boronat, and F Mazzanti. Dipolar bose supersolid stripes. Physical review letters, 119(25):255302, 2017. (document)
[18] Luca Tanzi, Eleonora Lucioni, Francesca Famà, Jacopo Catani, Andrea Fioretti, Carlo Gabbanini, Russell N Bisset, Luis Santos, and Giovanni Modugno. Observation of a dipolar quantum gas with metastable supersolid properties. Physical review letters, 122(13):130405, 2019.
[19] Santo Maria Roccuzzo and Francesco Ancilotto. Supersolid behavior of a dipolar bose-einstein condensate confined in a tube. Physical Review A, 99(4):041601, 2019.
[20] Fabian Böttcher, Jan-Niklas Schmidt, Matthias Wenzel, Jens Hertkorn, Mingyang Guo, Tim Langen, and Tilman Pfau. Transient supersolid properties in an array of dipolar quantum droplets. Physical Review X, 9(1):011051, 2019. (document)
[21] Jun-Ru Li, Jeongwon Lee, Wujie Huang, Sean Burchesky, Boris Shteynas, Furkan Çağrı Top, Alan O Jamison, and Wolfgang Ketterle. A stripe phase with supersolid properties in spin–orbit-coupled bose–einstein condensates. Nature, 543(7643):91–94, 2017. (document)
[22] Julian Léonard, Andrea Morales, Philip Zupancic, Tilman Esslinger, and Tobias Donner. Supersolid formation in a quantum gas breaking a continuous translational symmetry. Nature, 543(7643):87–90, 2017. (document)
[23] F Wächter and L Santos. Ground-state properties and elementary excitations of quantum droplets in dipolar bose-einstein condensates. Physical Review A, 94(4):043618, 2016. (document)
[24] D Baillie, RM Wilson, RN Bisset, and PB Blakie. Self-
bound dipolar droplet: A localized matter wave in free space. *Physical Review A*, 94(2):021602, 2016. [document]

[25] DS Petrov and GE Astrakharchik. Ultradilute low-dimensional liquids. *Physical Review letters*, 117(10):100401, 2016.

[26] Yaroslav V Kartashov, Boris A Malomed, and Lluis Torner. Metastability of quantum droplet clusters. *Physical review letters*, 122(19):193902, 2019. [document]

[27] Hiroki Saito. Path-integral monte carlo study on a droplet of a dipolar bose–einstein condensate stabilized by quantum fluctuation. *Journal of the Physical Society of Japan*, 85(5):053001, 2016. [document]

[28] A Macia, Juan Sánchez-Baena, J Boronat, and F Mazzanti. Droplets of trapped quantum dipolar bosons. *Physical review letters*, 117(20):205301, 2016.

[29] Fabio Cinti, Alberto Cappellaro, Luca Salasnich, and Tommaso Macrì. Superfluid filaments of dipolar bosons in free space. *Physical review letters*, 119(21):215302, 2017.

[30] Fabian Böttcher, Matthias Wenzel, Jan-Niklas Schmidt, Mingyang Guo, Tim Langen, Igor Ferrier-Barbut, Tilman Pfau, Raúl Bombín, Joan Sánchez-Baena, Jordi Boronat, et al. Dilute dipolar quantum droplets beyond the extended gross-pitaevskii equation. *Physical Review Research*, 1(3):033088, 2019. [document]

[31] Hui Hu and Xia-Ji Liu. Consistent theory of self-bound quantum droplets with bosonic pairing. *Physical Review Letters*, 125(19):195302, 2020. [document]

[32] Yuqi Wang, Longfei Guo, Su Yi, and Tao Shi. Theory for self-bound states of dipolar bose-einstein condensates. *Physical Review Research*, 2(4):043074, 2020. [document]

[33] ST Beliaev. Energy spectrum of a non-ideal bose gas. *Sov. Phys. JETP*, 7(2):299–307, 1958. [document]

[34] Thierry Lahaye, C Menotti, L Santos, M Lewenstein, and T Pfau. The physics of dipolar bosonic quantum gases. *Reports on Progress in Physics*, 72(12):126401, 2009. [document]

[35] Ralf Schützhold, Michael Uhlmann, Yan Xu, and Uwe R Fischer. Mean-field expansion in bose-einstein condensates with finite-range interactions. *International Journal of Modern Physics B*, 20(24):3555–3565, 2006. [document]

[36] NM Hugenholtz and David Pines. Ground-state energy and excitation spectrum of a system of interacting bosons. *Physical Review*, 116(3):489, 1959. [document]

[37] RN Bisset, RM Wilson, D Baillie, and PB Blakie. Ground-state phase diagram of a dipolar condensate with quantum fluctuations. *Physical Review A*, 94(3):033619, 2016. [document]

[38] Aristeu RP Lima and Axel Pelster. Beyond mean-field low-lying excitations of dipolar bose gases. *Physical Review A*, 86(6):063609, 2012. [document]

[39] D Baillie, RM Wilson, and PB Blakie. Collective excitations of self-bound droplets of a dipolar quantum fluid. *Physical review letters*, 119(25):255302, 2017. [document]

[40] Enes Aybar and MÖ Oktel. Temperature-dependent density profiles of dipolar droplets. *Physical Review A*, 99(1):013620, 2019. [document]

[41] G Natale, RMW van Bijnen, A Patscheider, D Petter, MJ Mark, L Chomaz, and F Ferlaino. Excitation spectrum of a trapped dipolar supersolid and its experimental evidence. *Physical review letters*, 123(5):050402, 2019. [document]

[42] Volodymyr Pastukhov. Beyond mean-field properties of binary dipolar bose mixtures at low temperatures. *Physical Review A*, 95(2):023614, 2017. [document]