Towards a global structure theory for
semilattices of infinite breadth

Yemon Choi, Mahya Ghandehari, Hung Le Pham

19th November 2018

Abstract

We obtain two global structure theorems for semilattices of infinite breadth, exploiting their representations as union-closed set systems. The first of these theorems implies that such a semilattice admits a subquotient isomorphic to one of three natural examples. The second has a Ramsey-theoretic flavour: we study how a union-closed set system interacts with a given decomposition of the underlying set, and show that there is either extreme fragmentation or some controlled structure. In subsequent work, we shall apply these structure theorems to the study of character stability for certain Banach algebras.

Keywords: breadth, semilattice, set system, subquotient.

MSC 2010: Primary 06A12, 06A07. Secondary 05D10.

1 Introduction

Semilattices are basic objects in both algebra and combinatorics, occurring as: particular kinds of semigroups (namely, those which are commutative and generated by their idempotents); particular kinds of posets (namely, those in which any two elements have a greatest lower bound); and particular kinds of set systems (namely, those subsets of the powerset $\mathcal{P}(\Omega)$ that are closed under taking finite unions).

To illustrate the importance of semilattices, we present some of their uses, both old and new. In semigroup theory, they arise whenever we study inverse semigroups, since the set of idempotents in an inverse semigroup $S$ forms a semilattice $L$. Moreover, when $L$ is also a central subset of $S$, we have a decomposition $S = \bigsqcup_{e \in L} G_e$ where each $G_e$ is a group and $G_e G_f \subseteq G_{ef}$: this represents $S$ as a family of groups glued together along a skeleton formed by $L$ (see [How76, Section IV.2]). There are also important examples of semigroups that are built out of the initial datum of a given semilattice $L$, such as the Munn semigroup associated to $L$ (see e.g. [How76, Section V.4]), and more recently the notion of an Ehresmann monoid arising from order-preserving maps of $L$ (see [BGG15, BGGW18]).

On the combinatorial or order-theoretic side: the notion of semilattice turns out to be a fundamental concept for dataflow analysis in computer science [KU76, KSS09]; and quite recently, semilattices have also been proposed as models for distributed data structures [ASB, SPB+11], which form an active research topic in modern software engineering.

We also note that an interesting source of semilattices with infinite breadth (to be defined below in Definition 1.1) is provided by studying binary relations that are stable in the sense of model theory (see [ADH+16, Proposition 2.20]), and in this setting, the semilattices occur naturally as union-closed set systems.

Given that semilattices occur naturally, it is natural to seek some kind of structure theory for them. For general semilattices little seems to be known, although more can be
said under assumptions of “low complexity” in various senses. One measure of complexity is breadth (which we shall define in Section 2) and indeed there has been work on semilattices with small breadth [Dit84, Gie94, Law71]. However, the present paper appears to be the first attempt to obtain structural results for semilattices of infinite breadth: Theorem 1.3 below, while not the sharpest form of our results, is already a significant advance on what was previously known. While the proofs presented in this article are rather technical and complicated, we obtain powerful structure theorems that could be useful for further study of such semilattices. (See Remark 1.5 for some more specific motivation.)

To motivate and explain our main results, it is necessary to quickly introduce certain key examples. First we describe three infinite semilattices that in some sense have minimal complexity:

- $N_{\text{max}}$ is the set $\mathbb{N}$ equipped with binary operation $m \cdot n := \max(m, n)$;
- $N_{\text{min}}$ is the set $\mathbb{N}$ equipped with binary operation $m \cdot n := \min(m, n)$;
- $N_{\text{ort}}$ is the set $\mathbb{N}_0$ equipped with the following binary operation

\[
m \cdot n := \begin{cases} n & \text{if } m = n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}
\]

On the other hand, one can have finite semilattices with high complexity: take $F_n$ to be the set of all non-empty subsets of $\{1, \ldots, n\}$, equipped with the binary operation $a \cdot b := a \cup b$. We call $F_n$, and any example isomorphic to it, the free semilattice of rank $n$. The element $\{1, \ldots, n\} \in F_n$ is called the zero element of $F_n$, for reasons that will be explained in Section 2.

**Definition 1.1.** A semilattice $S$ has infinite breadth if for each $n$ it contains a subsemilattice isomorphic to $F_n$. That is: there are arbitrarily large finite subsets of $S$, each of which generates a free sub-semilattice.

For any infinite set $\Omega$, the semilattice $\mathcal{P}(\Omega)$ has infinite breadth, but there are many other possibilities. To see why a wide range of behaviour is possible, note that in Definition 1.1, we know $S$ contains copies of $F_n$ for each $n$, but we have no further information on how these copies sit inside $S$ and how they relate to each other. Such missing information is what we refer to in our title when we speak of a global structure theory, rather than just knowledge of local behaviour.

With this in mind, consider the following sets

\[
T_{\text{max}} = \prod_{n \in \mathbb{N}} F_n, \quad T_{\text{min}} = \prod_{n \in \mathbb{N}} F_n, \quad T_{\text{ort}} = \{0\} \cup \prod_{n \in \mathbb{N}} F_n
\]

where $F_n$ is as defined above. In $T_{\text{max}}$, the product of $a \in F_m$ and $b \in F_n$ is defined to be their usual product if $m = n$, but if $m \neq n$ we define it to be the element $a$ or $b$ that belongs to $F_{\max(m,n)}$. Similar definitions apply for $T_{\text{min}}$ and $T_{\text{ort}}$. Informally, in each of these three cases, we are gluing together finite free semilattices of increasing rank, along a skeleton that is isomorphic to $\mathbb{N}_{\text{max}}, \mathbb{N}_{\text{min}}$ or $\mathbb{N}_{\text{ort}}$: so locally we have high complexity, but the global arrangement of the relevant pieces has low complexity.

**Remark 1.2.** $T_{\text{max}}, T_{\text{min}}$ and $T_{\text{ort}}$ are known examples in the lattice-theoretic literature: for instance, $T_{\text{min}}$ corresponds to [Mis86, Example 1]. None of these three examples contain copies of $\{0, 1\}_{\mathbb{N}} \cong \mathcal{P}(\mathbb{N})$, a property which is relevant to harmonic analysis on these semilattices: see [Mis86] for some further details and references.
We can now state the following striking consequence of our main theorems. It shows that $T_{\text{max}}$, $T_{\text{min}}$ and $T_{ort}$ are not just natural and tractable examples; they are unavoidable when dealing with semilattices of infinite breadth.

**Theorem 1.3 (Subquotient theorem, abstract version).** Let $S$ be a semilattice with infinite breadth. Then there is a homomorphic image of $S$ which contains a copy of either $T_{\text{max}}$, $T_{\text{min}}$ or $T_{ort}$.

It is easy to check that subsemigroups of homomorphic images are examples of subquotient semigroups, so Theorem 1.3 implies that every semilattice of infinite breadth has either $T_{\text{max}}$, $T_{\text{min}}$ or $T_{ort}$ among its subquotients. Since none of $T_{\text{max}}$, $T_{\text{min}}$ or $T_{ort}$ can occur as subquotients of each other, our result is sharp.

Theorem 1.3 will follow from a more precise but more technical version, Theorem 3.1, which is the first main theorem of our paper. In order to state and prove this theorem, we represent semilattices of infinite breadth as union-closed set systems. We believe that this perspective may be useful for further work on semilattices, in much the same way that an “abstract” group can be studied through a suitable representation as a “concrete” group of permutations. The proof of Theorem 3.1 takes up all of Section 3. Our approach is self-contained and combinatorial in flavour, but is rather long and requires a detailed case-by-case analysis, as well as various auxiliary results to control the relative positions of embedded copies of $F_n$.

We now discuss the second main theorem of our paper, Theorem 4.8, which has a Ramsey-theoretic flavour. Given a union-closed set system $S \subseteq \mathcal{P}(\Omega)$, the statement and proof of Theorem 3.1 uses the notion of a spread in $\Omega$. In Section 4 we pursue a deeper study of how the set system can interact with such a spread, building up to Theorem 4.8. The result requires too many technical definitions to be stated here, but loosely speaking it says that when $S$ interacts with a spread, we can control the complexity in a certain technical sense via a colouring of the spread, unless we are in a special situation with very high complexity (“shattering”).

Combining these two main theorems with some auxiliary definitions yields a refinement of Theorem 1.3. To give some idea of what is involved, and to indicate what we mean by a Ramsey-theoretic flavour, we shall state this refinement here without defining what various terminology means: the necessary definitions will be given in later sections.

**Theorem 1.4 (Statement of Corollary 4.12).** Let $S$ be a semilattice with infinite breadth. Then at least one of the following statements holds.

(i) There is a concrete representation $S$ of $S$ on some set $\Omega$ with the property that, for every countable concrete semilattice $Q$ on $\mathbb{N}$, there exists a countably infinite subset $\Gamma \subseteq \Omega$ with a type $\omega_0$ well-ordering such that $S \not\preceq \Gamma$ contains $Q(\Gamma)$.

(ii) For any concrete representation $S$ of $S$ on any set $\Omega$, there exist a spread $\mathcal{E}$ in $\Omega$ and an $S$-decisive colouring $C$ of $\mathcal{E}$, such that $S \not\preceq \text{join}(\mathcal{E})$ contains either one of $T_{\text{max}}(\mathcal{E}), T_{\text{min}}(\mathcal{E})$, or $T_{ort}(\mathcal{E})$.

The results of Section 4 illustrate that particular concrete representations of an abstract semilattice $S$ can display rich behaviour; it would be interesting to investigate this diverse behaviour in future work. In Section 5 we present some examples to show that a given semilattice $S$ can have two different concrete representations that display different behaviour from the point of view of Theorem 3.1, suggesting that more work is needed to understand how “concrete subquotients” may behave differently from “abstract subquotients”.
Remark 1.5. Our original motivation for seeking some kind of global structural theory for semilattices of infinite breadth came from a rather different area: namely, studying (in)stability of characters in Banach algebra theory, as set out in papers such as [Jar97, Joh86], and answering questions arising from previous work of the first author [Cho13]. These applications of our structure theory will be addressed in a companion paper [CGP], which makes essential use of both Theorems 3.1 and 4.8.

2 Preliminaries

In this section we collect some basic definitions and terminology which will be needed for the proofs of our main results. Since we hope that these results will be of interest to both algebraists and combinatorists, we have repeated several standard definitions that may be less familiar to one or other of these audiences.

2.1 Abstract semilattices

A semilattice is a commutative semigroup $S$ satisfying $x^2 = x$ for all $x \in S$. For two elements $x, y \in S$, we say that $y$ is a multiple of $x$ or $x$ is a factor of $y$ or $x$ divides $y$ and write $x \mid y$ if there exists $z \in S$ such that $y = xz$; which for the semilattice $S$ is simply equivalent to $xy = y$.

The divisibility relation provides $S$ with a standard and canonical partial order, and to signify this aspect of the relation, we sometimes write $y \preceq x$ instead of $x \mid y$. With respect to this particular partial order, $xy$ is the meet (or greatest lower bound) of $x$ and $y$; this gives an alternative, order-theoretic definition of a semilattice. One could work with the opposite partial order (replacing meets by joins), but our convention is motivated by the following example.

Example 2.1 (The 2-element semilattice). Consider $\{0, 1\}$ equipped with usual multiplication. This is a semilattice, and we have $1 \mid 0$ or, writing differently, $0 \preceq 1$. More generally, in a semilattice $S$ with a unit element 1 (satisfying $1x = x$ for all $x \in S$) and a zero element $\theta$ (satisfying $\theta x = \theta$ for all $x \in S$) we have $\theta \preceq x \preceq 1$ i.e. $1 \mid x$ and $x \mid \theta$ for all $x \in S$.

Definition 2.2 (Compressible/incompressible subsets of semilattices). Let $S$ be a semilattice. Given a finite, non-empty subset $E \subseteq S$, we say $E$ is compressible if there exists a proper subset $E' \subset E$ such that $\prod_{x \in E} x = \prod_{x \in E'} x$; otherwise, we say $E$ is incompressible.

Remark 2.3. The same property is referred to in [LLM77, Mis86] as “meet irredundant”. However, “incompressible” seemed to be better terminology when we come to work with union-closed set systems (see also Remark 2.9), so we use the same terminology here to be consistent.

To aid intuition, we observe that for any finite subset $E \subseteq S$, the subsemigroup of $S$ generated by $E$ has cardinality at most $2^{|E|} - 1$; equality occurs if and only if $E$ is incompressible. The proof is straightforward.

Definition 2.4 (Breadth of a semilattice). The breadth of a semilattice $S$ is defined to be

$$b(S) = \inf \{n \in \mathbb{N} : \text{every } E \subseteq S \text{ with } n + 1 \text{ members is compressible}\}$$

$$= \sup \{n \in \mathbb{N} : S \text{ has an incompressible subset with } n \text{ elements}\}.$$
The breadth of a semilattice sheds some light on its structure, and is related to more familiar order-theoretic concepts such as height and width. (Some basic links, with references, are surveyed in [ADH+13, Section 4.1].) For instance, by examining incompressible subsets, one sees that if \( b(S) \geq n \) then \( S \) contains a chain (totally ordered subset) and an antichain (subset in which no two elements are comparable) both of cardinality \( n \). In particular, a semilattice \( S \) has breadth 1 exactly when the poset \((S, \preceq)\) is totally ordered.

However, diverse behaviour occurs even among semilattices of breadth 2. For instance, the following example shows that every infinite \( k \)-ary rooted tree \((k \geq 2)\) is a semilattice with breadth 2 that contains infinite chains and infinite antichains.

**Example 2.5 (Infinite \( k \)-ary rooted tree).** Let \( k \geq 2 \). An infinite \( k \)-ary rooted tree is an infinite rooted tree in which every vertex has \( k \) children. If \( x \) and \( y \) are vertices in the tree then they have a “youngest” common ancestor, which we denote by \( x \land y \). Clearly \( \land \) is a commutative, associative and idempotent binary operation; so the set of vertices becomes a semilattice \((T, \land)\), and the partial order \( \preceq \) becomes “is an ancestor of”.

There is an infinite path \( P \subset T \) obtained by starting at the root and successively taking the 1st child; this gives us an infinite chain in \((T, \preceq)\). If instead we take the 2nd child of each element of \( P \), this gives us an infinite antichain in \((T, \preceq)\).

On the other hand, let \( x, y, z \in T \), and let \( p = x \land y \land z \). Then either \( x \land y \) or \( y \land z \) is equal to \( p \); for if not, then the set \( \{p, x \land y, y \land z, y\} \) would form a cycle of length \( \geq 3 \) in the tree \( T \), which is impossible. Thus every 3-element subset of \( S \) is compressible, and so \( b(S) \leq 2 \). On the other hand, \( b(S) \geq 2 \), since \( S \) is not totally ordered.

### 2.2 Concrete semilattices

Let \( \Omega \) be a non-empty set. We write \( \mathcal{P}(\Omega) \) for its power set, \( \mathcal{P}^\text{fin}(\Omega) \) for the set of all finite subsets, and \( \mathcal{P}^\text{fin}_s(\Omega) \) for the set of all non-empty, finite subsets of \( \Omega \). We also write \( \mathcal{P}^\text{cofin}(\Omega) \) for the set of all subsets with finite complement in \( \Omega \). Elements of \( \Omega \) will usually be denoted by lower-case Greek letters.

Set systems on \( \Omega \) (i.e. subsets of \( \mathcal{P}(\Omega) \)) will usually be denoted by letters such as \( \mathcal{B}, \mathcal{S}, \) etc. If \( \mathcal{B} \) is such a set system, we refer to *members of \( \mathcal{B} \)* rather than elements. If \( \mathcal{B} \) and \( \mathcal{S} \) are set systems on \( \Omega \) we denote their union and intersection by \( \mathcal{B} \cup \mathcal{S} \) and \( \mathcal{B} \cap \mathcal{S} \).

Members of a set system \( \mathcal{S} \) will be denoted by letters such as \( a, b, p, \) etc., and we write \( a \cup b \) and \( a \cap b \) for their union and intersection respectively. If it happens that \( a \) and \( b \) are disjoint subsets of \( \Omega \) we shall sometimes emphasise this by writing their union as \( a \cup b \).

**Definition 2.6.** A *union-closed set system* or *concrete semilattice* on \( \Omega \) is a subset \( \mathcal{S} \subseteq \mathcal{P}(\Omega) \) which is closed under taking finite unions; this is clearly a semilattice, where set-union serves as the binary operation. If \( \mathcal{T} \) is a subsemilattice of a concrete semilattice \( \mathcal{S} \), we also say that \( \mathcal{T} \) is a (union-closed) *subsystem* of \( \mathcal{S} \).

**Remark 2.7.** A concrete semilattice on \( \Omega \) may be identified with a subsemilattice of the semilattice \( \{0,1\}^\Omega \) in which multiplication is defined coordinatewise. Particular examples of concrete semilattices are \( \mathcal{P}^\text{fin}(\Omega) \), \( \mathcal{P}^\text{fin}_s(\Omega) \), and \( \mathcal{P}^\text{cofin}(\Omega) \).

Every semilattice can be viewed as a concrete semilattice, using the following construction.

**Example 2.8.** Let \( S \) be a semilattice. For \( x \in S \) let \( E_x := S \setminus \{y \in S : x \not\preceq y\} \). It is easily checked that \( E_x \cup E_y = E_{xy} \) for all \( x, y \in S \). Therefore, the function \( E_\cdot : S \to \mathcal{P}(S), \ x \mapsto E_x \), defines an injective semilattice homomorphism from \( S \) into \((\mathcal{P}(S), \cup)\). This is sometimes known as the *Cayley embedding* of a semilattice.
Remark 2.9. If $S$ is a union-closed set system, then $b \mid a$ if and only if $b \subseteq a$. One should beware that the canonical partial order of $S$ as a semilattice is not given by inclusion but by containment: $a \leq b \iff a \supseteq b$. So for $a, b \in S$, the $\leq$-meet of $a$ and $b$ is not $a \cap b$, but rather $a \cup b$. Thus, we are really working with “join-semilattices” inside $P(\Omega)$. This is one reason for our terminology “incompressible” instead of “meet-irredundant”.

To avoid cluttered formulas, we introduce the following notation.

Definition 2.10. Given $F \subseteq P(\Omega)$, the join of $F$ is the set $\text{join}(F) := \bigcup_{x \in F} x$. If $F$ is a finite subset of a union-closed set system $S$, then $\text{join}(F) \in S$.

Thus a finite, non-empty subset $E$ of a union-closed set system $S$ is compressible if and only if there exists $E' \subseteq E$ such that $\text{join}(E') = \text{join}(E)$.

Note that when $\Omega$ is an infinite set, $P^{\text{fin}}(\Omega)$ is an easy example of a concrete semilattice with infinite breadth, since for any $\gamma_1, \ldots, \gamma_n \in \Omega$ the set $\{\{\gamma_j\} : 1 \leq j \leq n\}$ is an incompressible subset of $P^{\text{fin}}(\Omega)$. Similarly, $P^{\text{cofin}}(\Omega)$ is a concrete semilattice with infinite breadth.

More generally, if $b(S) = \infty$, then there are arbitrarily large finite subsets of $S$ that are incompressible. However, unlike the example of $P^{\text{fin}}(\Omega)$, we cannot always arrange for these to be nested in an infinite sequence $E_1 \subseteq E_2 \subseteq \ldots$. This can be seen very clearly with the three key examples that will be introduced in Definition 2.12.

2.3 Concrete realisations of our three key examples

Definition 2.11. A spread is a sequence $E = (E_n)_{n \geq 1}$ of finite non-empty subsets of some set $\Omega$ which are pairwise disjoint and satisfy $|E_n| \to \infty$. Note that a spread $E = (E_n)_{n \geq 1}$ does not need to cover $\Omega$. A refinement of $E$ is a spread $F = (F_j)_{j \geq 1}$ with the property that each $F_j$ is contained in some $E_n(j)$ and $n(j) \neq n(k)$ whenever $j \neq k$.

Definition 2.12 (Three special set systems). Let $E = (E_n)_{n \geq 1}$ be a spread. For $n \in \mathbb{N}$, let $E_{< n} := E_1 \cup \cdots \cup E_{n-1}$ (with the convention that $E_{< 1} = \emptyset$) and let $E_{\geq n} := \bigcup_{j \geq n+1} E_j$.

We now define the following set systems on $\text{join}(E)$:

$$T_{\text{max}}(E) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{E_{< n} \cup a\},$$

$$T_{\text{min}}(E) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{a \cup E_{> n}\},$$

$$T_{\text{ort}}(E) := \bigvee_{n \geq 1} \bigvee_{\emptyset \neq a \subseteq E_n} \{E_{< n} \cup a \cup E_{> n}\}.$$

By construction, each of these set systems has a natural partition into “levels” indexed by $n \in \mathbb{N}$. Each level of this partition is itself union-closed, and is isomorphic (as a concrete semilattice) to $P_*(E_n)$; in particular, in each level there is an incompressible subset with cardinality $|E_n|$. (See Figure 1.)

Remark 2.13. If $|E_n| = n$ for all $n$, then the set systems in Definition 2.12 provide concrete realisations of the semilattices $T_{\text{max}}$, $T_{\text{min}}$ and $T_{\text{ort}}$ that appeared in the Introduction. More generally: for any spread $E$, the set systems $T_{\text{max}}(E)$, $T_{\text{min}}(E)$ and $T_{\text{ort}}(E)$ contain subsemilattices isomorphic to $T_{\text{max}}$, $T_{\text{min}}$ and $T_{\text{ort}}$ respectively (since for any two spreads $E$ and $F$, $T_{\text{max}}(E)$ is isomorphic to a subsemilattice of $T_{\text{max}}(F)$, etc.).
Figure 1: Typical members of $T_{\text{max}}$, $T_{\text{min}}$, and $T_{\text{ort}}$ at level $n$

Remark 2.14. Given a spread $\mathcal{E}$ on $\Omega$, we always have

$$T_{\text{max}}(\mathcal{E}) \subseteq \mathcal{P}^{\text{fin}}(\Omega) \quad \text{and} \quad T_{\text{min}}(\mathcal{E}) \cup T_{\text{ort}}(\mathcal{E}) \subseteq \mathcal{P}^{\text{cofin}}(\Omega).$$

Moreover, these examples have the following fundamental feature: if $T$ is one of them and $F$ is an incompressible subset of $T$, with $|F| \geq 3$, then all members of $F$ must belong to the same “level”. We thus have extremely precise control over the incompressible subsets in these examples.

In what follows, we will frequently need to take a union-closed set system $S \subseteq \mathcal{P}(\Omega)$, and a prescribed subset of $\Omega$, and do one of two things: either we “restrict” to those members of $S$ that are contained in this subset, obtaining a subsystem of $S$; or we “project” by intersecting every member of $S$ with the given subset, obtaining a homomorphic image of $S$. We therefore introduce some more notation. Given $S \subseteq \mathcal{P}(\Omega)$, $a, \Gamma \in \mathcal{P}(\Omega)$, we define

$$S^{-\Gamma} := \{x \in S: x \cap \Gamma = \emptyset\} \subseteq S, \quad S \sqcap a := \{x \setminus a: x \in S\} \subseteq \mathcal{P}(\Omega),$$

$$S^{-\Gamma} \sqcap \Gamma := \{x \in S: x \supseteq \Gamma\} \subseteq S, \quad S \sqcup a := \{x \cap a: x \in S\} \subseteq \mathcal{P}(\Omega). \quad (2)$$

Clearly, each of these is union-closed if $S$ is. Note that the obvious maps $S \to S \sqcap a$ and $S \to S \sqcup a$ are lattice homomorphisms.

Remark 2.15. One can use these operations to illustrate the very different behaviour of the set systems $T_{\text{max}}(\mathcal{E})$, $T_{\text{min}}(\mathcal{E})$ and $T_{\text{ort}}(\mathcal{E})$. For example, it is easy to see that $T_{\text{max}}(\mathcal{E})^{-\Gamma}$ and $T_{\text{ort}}(\mathcal{E})^{-\Gamma}$ are finite set systems, for any $\emptyset \neq \Gamma \subseteq \text{join}(\mathcal{E})$. However, for every finite subset $\Gamma$ of $\Omega$, $b(T_{\text{min}}^{-\Gamma}) = \infty$. On the other hand, $T_{\text{min}} \sqcap a$ and $T_{\text{ort}} \sqcap a$ are finite but $b(T_{\text{max}} \sqcap a) = \infty$, whenever $a$ belongs to the corresponding set system.

3 The first structure theorem

We now state the first main theorem of this paper, from which Theorem 1.3 immediately follows.

Theorem 3.1 (The subquotient theorem). Let $S$ be a semilattice with infinite breadth, and let $S$ be a concrete representation of $S$ as a union-closed set system on some set $\Omega$. Then at least one of the following statements holds.

(i) There is a spread $\mathcal{E}_1$ in $\Omega$ such that $S \sqcap \text{join}(\mathcal{E}_1) \supseteq T_{\text{max}}(\mathcal{E}_1)$.

(ii) There is a spread $\mathcal{E}_2$ in $\Omega$ such that $S \sqcap \text{join}(\mathcal{E}_2) \supseteq T_{\text{min}}(\mathcal{E}_2)$.
(iii) There is a spread $E_3$ in $\Omega$ such that $S \boxast \text{join}(E_3) \supseteq T_{\text{ort}}(E_3)$.

**Remark 3.2.** As stated in the Introduction, we already know at the “abstract” level that none of $T_{\text{max}}$, $T_{\text{min}}$ and $T_{\text{ort}}$ can arise as subquotients of the other two. However, because Theorem 3.1 involves concrete projection homomorphisms, there is a subtlety here: it is possible that a semilattice $S$ has two different concrete representations $\mathcal{S}$ and $\mathcal{S}'$ such that $S$ satisfies case (ii) but not case (iii) of the theorem, while $\mathcal{S}'$ satisfies case (iii) but not case (ii). Details will be given in Section 5.

The rest of this section is devoted to proving Theorem 3.1. A key idea, which we will use repeatedly, is to understand a given incompressible $F \subset \mathcal{S}$ by finding a simpler model or ‘skeleton’ for $F$. More precisely: consider $\{x_1, \ldots, x_m\} \subseteq \mathcal{P}(\Omega)$. It is straightforward to check that the following statements are equivalent:

(i) there exists $j$ such that $x_j \cap \bigcap_{i \neq j} x_i^c = \emptyset$;

(ii) there exists $j$ such that $x_j \subseteq \bigcup_{i \neq j} x_i$;

(iii) $\{x_1, \ldots, x_m\}$ is compressible.

Thus, a finite subset $\{x_1, \ldots, x_m\}$ of $\mathcal{P}(\Omega)$ is incompressible if and only if, for every $j$, there exists $\gamma_j \in \Omega$ such that

$$\gamma_j \in x_j \quad \text{and} \quad \gamma_j \notin \text{join}(\{x_i : i \neq j\}). \quad (3)$$

**Definition 3.3.** Let $F = \{x_1, \ldots, x_m\} \subset \mathcal{P}(\Omega)$. A subset $E = \{\gamma_1, \ldots, \gamma_m\}$ of $\Omega$ is called a witness of incompressibility for $F$, or just a witness for $F$, if it satisfies (3).

Witnesses of incompressibility will be useful in creating the spreads $E_1$, $E_2$ and $E_3$ in Theorem 3.1. The following terminology will also be useful.

**Definition 3.4.** Let $S \subseteq \mathcal{P}(\Omega)$ be union-closed, and let $a \in \mathcal{P}(\Omega)$. We say $S$ is thick in $a$ if $S \boxast a$ has infinite breadth, and thin in $a$ if $S \boxast a$ has finite breadth.

**Lemma 3.5.** Let $a_1, \ldots, a_n \subseteq \Omega$ and set $a := \bigcup_{i=1}^n a_i$. If $S$ is thin in all the $a_i$, it is thin in $a$.

**Proof.** Let $m_i = b(S \boxast a_i)$ and let $n = \sum_{i=1}^n m_i$. We will show $b(S \boxast a) \leq n$. Let $F \subseteq S$ with $|F| > n$. For each $i$, since $S \boxast a_i$ has breadth $m_i$, there exists $F_i \subseteq F$ with $|F_i| \leq m_i$ and

$$\bigcup_{x \in F_i} x \cap a_i = \bigcup_{x \in F} x \cap a_i. \quad (**)$$

Let $F' = \bigvee_{i=1}^n F_i$; then $|F'| \leq n$ and

$$\text{join}(F \boxast a) = \bigcup_{x \in F} x \cap \left( \bigcup_{i=1}^n a_i \right) = \bigcup_{x \in F \cup F} x \cap a_i = \bigcup_{x \in F} \left( \bigcup_{i=1}^n x \cap a_i \right)$$

A similar calculation shows that $\text{join}(F' \boxast a) = \bigcup_{i=1}^n \left( \bigcup_{x \in F'} x \cap a_i \right)$. Hence, by the identity $(**)$, $\text{join}(F \boxast a) = \text{join}(F' \boxast a)$. \hfill \Box

Finite union-closed set systems have finite breadth, so taking $n = 2$ in Lemma 3.5 yields:
Corollary 3.6. Let $S \subset \mathcal{P}(\Omega)$ be union-closed, with infinite breadth. If $p$ is a finite subset of $\Omega$ then $S$ is thick in $p^c$.

To go further, we need to introduce two auxiliary conditions.

Definition 3.7 (Auxiliary conditions).

(A) for every $k \in \mathbb{N}$ and every $p \in S \cup \{\emptyset\}$ such that $S$ is thick in $p^c$, there is an incompressible size $k$-subset $F \subset S \sqcup p$ such that $S \sqcap p$ is thick in $\text{join}(F)^c$.

(B) for every $k \in \mathbb{N}$ and every $c, \Gamma \subseteq \Omega$ such that $S^{-\Gamma}$ is thick in $c$, there is an incompressible size $k$-subset $F \subset S^{-\Gamma} \sqcap c$ with a witness $E$ such that $S^{-\Gamma \cup E}$ is thick in $c$.

Note that Condition (A) implies that $(S \sqcap p \sqcap \text{join}(F)) = S \sqcap (p \cup \text{join}(F))$ has infinite breadth, and $p \cup \text{join}(F) = \bigcup_{x \in F}(p \cup x)$ is again a member of $S$.

Amongst our three set systems, $T_{\text{max}}$ is the only one that satisfies Condition (A), and $T_{\text{min}}$ is the only one that satisfies Condition (B), while $T_{\text{opt}}$ does not satisfy either condition. The first stage in proving the subquotient theorem is to show that conditions (A) and (B) may be used to induce at least one set system of the form $T_{\text{max}}$ or $T_{\text{min}}$.

Lemma 3.8. Suppose that $S$ has infinite breadth and satisfies (A). Then there is a spread $E = (E_n)_{n \geq 1}$ such that $S \sqcap \text{join}(E)$ contains the set system $T_{\text{max}}(E)$.

Proof. First, by induction we shall construct sequences $(d_n)_{n=1}^\infty \subseteq \mathcal{P}(\Omega)$ and $(F_n)_{n=1}^\infty \subseteq \mathcal{P}(\mathcal{P}(\Omega))$ with the following properties:

(i) $S_n := S \sqcup \bigcup_{j=1}^{n-1} d_j$ has infinite breadth for each $n$;

(ii) $F'_n$ is an incompressible finite subset of $S_n$ of size $n$, and $\text{join}(F'_n) = d_n$;

(iii) $\bigcup_{j=1}^{n-1} d_j \in S \cup \{\emptyset\}$.

Suppose we have found $S_n$, with infinite breadth, which is of the form $S \sqcup p_n$ for some $p_n = \bigcup_{j=1}^{n-1} d_j \in S \cup \{\emptyset\}$. (When $n = 1$, $S_1 = S$ has infinite breadth by assumption.) By Condition (A), there is an incompressible size-$n$ subset $F'_n \subset S_n$, such that $S_n$ is thick in $\text{join}(F'_n)^c$. Set $d_n := \text{join}(F'_n)$ and set $S_{n+1} := S_n \sqcup d_n = S \sqcup \bigcup_{j=1}^{n} d_j$. Since $p_n \in S \cup \{\emptyset\}$, we have $x \cup p_n \in S$ for every $x \in S_n$. Let $p_{n+1} := \bigcup_{j=1}^{n} d_j$: then $p_{n+1} = p_n \cup d_n \in S$. Thus the inductive construction can be continued. By induction, the sets $d_n$ ($n \in \mathbb{N}$) are disjoint.

For each $n$, choose a witness for the incompressibility of $F'_n$, say $E_n$. Then $E_n \subseteq d_n$. Set

$$F_n := \{x \cup p_n = x \cup \bigcup_{j=1}^{n-1} d_j : x \in F'_n\},$$

and take $E := (E_n)_{n \geq 1}$. Then $E$ is a spread in $\Omega$ and $(\bigcup_{n=1}^\infty F_n) \sqcap E$ generates the set system $T_{\text{max}}(E)$. Since $F_n \subseteq S$, this completes the proof.

Lemma 3.9. Suppose that $S$ has infinite breadth and satisfies (B). Suppose also that for every spread $E$ in $\Omega$, the set system $S \sqcap \text{join}(E)$ does not contain $T_{\text{max}}(E)$. Then there exists a spread $E$ in $\Omega$ such that $S \sqcap \text{join}(E)$ contains $T_{\text{min}}(E)$.

Proof. Our hypothesis, together with Lemma 3.8 and the fact that $S^{-\Gamma} \sqcap c$ is a subsystem of $S \sqcap c$, leads to the following observation.
\((\ast)\) If \(c, \Gamma \subseteq \Omega\) such that \(b(S^{-\Gamma} \sqcup c) = \infty\) then \(S^{-\Gamma} \sqcup c\) doesn’t satisfy Condition (A).

We use this to set up an inductive argument. Suppose that \((E_j)_{j=1}^{n-1}\) and \((c_j)_{j=1}^{n-1}\) have been constructed such that, when we take \(G_{n-1} = \bigcup_{j=1}^{n-1} E_j\) and \(d_{n-1} = \bigcap_{j=1}^{n-1} c_j\), then \(S^{-G_{n-1}}\) is thick in \(d_{n-1}\). Put \(S_n := S^{-G_{n-1}} \sqcup d_{n-1}\), which has infinite breadth by the inductive hypothesis. (When \(n = 1\) the usual conventions give \(S_1 = S\), which has infinite breadth by assumption.)

By \((\ast)\), \(S_n\) cannot satisfy Condition (A). Hence, there exist \(k \in \mathbb{N}\) and \(p_n \subseteq d_{n-1} \setminus G_{n-1}\), with \(S_n\) thick in \(p_n^c\), such that

for every incompressible size-\(k\)-subset \(F \subseteq S_n \sqcup p_n\), \(b \left( (S_n \sqcup p_n) \sqcup \text{join}(F) \right) < \infty\). \((4)\)

Since

\[ S_n \sqcup p_n = S^{-G_{n-1}} \sqcup (d_{n-1} \setminus p_n), \]

by Condition (B) there exists an incompressible subset \(F_n\) of \(S_n \sqcup p_n\) of size \(kn\) with a witness \(E'\) such that:

\[ S^{-G_{n-1} \cup E'} \text{ is thick in } d_{n-1} \setminus p_n. \] \((6)\)

Partition \(F_n\) into \(n\) parts each of size \(k\), and let \(F_n'\) be the collection of the \(n\) unions of the parts. Then \(F_n'\) is an incompressible subset of \(S_n \sqcup p_n\) of size \(n\), and we can also choose a witness for \(F_n'\), call it \(E_n\), such that \(E_n \subseteq E'\).

Set \(G_n = G_{n-1} \cup E_n\). From \((6)\), \(S^{-G_n}\) is thick in \(d_{n-1} \setminus p_n\). By \((4)\), for each \(x'' \in F_n''\) the set system \(S_n \sqcup p_n) \sqcup x''\) has finite breadth, and hence so does its subsystem

\[ (S^{-G_{n-1} \cup E_n} \sqcup (d_{n-1} \setminus p_n)) \sqcup x''. \]

Therefore, by Lemma 3.5, \(S^{-G_n}\) is thick in

\[ (d_{n-1} \setminus p_n) \cap \bigcap_{x'' \in F_n''} x'' = d_{n-1} \cap \bigcap_{x'' \in F_n''} x''. \]

Thus, setting \(c_n := \bigcap_{x'' \in F_n''} x''\) will allow us to continue the induction.

For each \(n \in \mathbb{N}\), by \((5)\), we form a subset \(F_n \subseteq S\) of size \(n\) by choosing for each \(x'' \in F_n''\) some \(x \in S\) with

\[ x \cap G_{n-1} = \emptyset \quad \text{and} \quad x'' = x \cap d_{n-1} \setminus p_n. \] \((7)\)

By construction, \(E_n \subseteq d_{n-1} = \bigcap_{j=1}^{n-1} c_j\). So for each \(x \in F_n\), \(x \supseteq c_n \supseteq E_j \) \((j > n)\). Moreover, since \(E_n\) is a witness for \(F_n''\), for each \(x \in F_n\) the set \(x \cap E_n\) is a singleton. Finally, from \((7)\) we see that for \(j < n\)

\[ E_n \cap E_j \subseteq \text{join}(F_n) \cap G_j \subseteq \text{join}(F_n) \cap G_{n-1} = \emptyset. \]

So if we set \(\mathcal{E} := (E_n)_{n \geq 1}\), which is a spread in \(\Omega\), then \(S \sqcup \text{join}(\mathcal{E})\) contains the subsystem generated by \(\bigvee_{n=1}^{\infty} F_n\) \(\sqcup \text{join}(\mathcal{E})\), which is the set system \(T_{\text{min}}(\mathcal{E})\).

We can now complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Assume that Case (i) does not hold. Then, by Lemma 3.8, \(S\) cannot satisfy Condition (A). Hence, there exist \(k \in \mathbb{N}\) and \(p \in S \cup \{\emptyset\}\), such that \(S\) is thick in \(p^c\), yet for every incompressible size-\(k\)-subset \(F \subseteq S \sqcup p\), \(S \sqcup p\) is thin in \(\text{join}(F)^c\).

Replacing \(S\) by \(S \sqcup p\) if necessary, we suppose from now on that \(S\) has infinite breadth, and satisfies
(C1) \( S \) is thin in \( \text{join}(\mathcal{F}) \), for every incompressible size-\( k \)-subset \( \mathcal{F} \subset S \).

Replacing \( S \) by \( S \sqsupset p \) does not change the property that Case (i) fails to hold. So if \( S \) satisfies Condition (B), we are in case (ii) by Lemma 3.9.

From here on, we assume \( S \) does not satisfy Condition (A) and does not satisfy Condition (B). By the previous observations, there exist \( d, \Gamma \subset \Omega \) such that \( S^{-\Gamma} \) is thick in \( d \). By enlarging \( k \) if necessary, we may also assume that for each incompressible size-\( k \)-subset \( \mathcal{F} \subset S^{-\Gamma} \sqsupset d \) and every \( E \) which is a witness for \( \mathcal{F} \), the set system \( S^{-\Gamma \cup E} \) is thin in \( d \).

In such a situation \( E \subset d \). Hence,

\[
S^{-\Gamma \cup E} \sqsupset d = (S^{-\Gamma} \sqsupset d)^{-E}.
\]

Thus, replacing \( S \) by \( S^{-\Gamma} \sqsupset d \) if necessary, we shall suppose from now on that \( S \) satisfies the following condition:

(C2) for every incompressible subset \( \mathcal{F} \) of size \( k \) of \( S \) and every \( E \) that is a witness for \( \mathcal{F} \), the set system \( S^{-E} \) has finite breadth.

Clearly, given \( y_1, \ldots, y_n \subset S^{-\Gamma} \), if \( \{y_1 \cap d, \ldots, y_n \cap d\} \) is incompressible in \( S^{-\Gamma} \sqsupset d \) then \( \{y_1, \ldots, y_n\} \) is incompressible in \( S^{-\Gamma} \). Also, for any \( y \in S^{-\Gamma} \)

\[
[S^{-\Gamma} \sqsupset d] \sqsupset (y \cap d) = [S^{-\Gamma} \sqsupset y] \sqsupset d = [S \sqsupset y]^{-\Gamma \sqsupset d}.
\]

Therefore, after replacing \( S \) by \( S^{-\Gamma} \sqsupset d \), we can assume that \( S \) satisfies both Condition (C1) and Condition (C2).

In fact, more is true: for every \( D, q \subset \Omega \), the set system \( S_D \sqsupset q \) satisfies conditions (C1) and (C2). To see this, let \( \mathcal{F} \) be an incompressible size-\( k \)-subset of \( S_D \sqsupset q \), with a witness \( E \). Let \( G \) be a \( k \)-element subset of \( S_D \) such that \( \mathcal{F} = G \sqsupset q \). Then \( G \) is incompressible, since \( E \) is a witness for \( G \) as well. Thus,

\[
(S_D \sqsupset q) \sqsupset \text{join}(\mathcal{F}) = (S_D \sqsupset \text{join}(G)) \sqsupset q \subset (S \sqsupset \text{join}(G)) \sqsupset q
\]

has finite breadth as \( S \) satisfies Condition (C1). Moreover, since \( E \subset q \),

\[
(S_D \sqsupset q)^{-E} = (S_D)^{-E} \sqsupset q \subset S^{-E} \sqsupset q,
\]

and this has finite breadth since \( S \) satisfies (C2).

We are now ready for the inductive construction of our spread \( E \). Suppose that we already have \( S_n = (S_D) \sqsupset q \) that has infinite breadth where \( D = \bigcup_{j=1}^{n-1} E_j \) and \( q = \bigcap_{j=1}^{n-1} \epsilon_j \) are subsets of \( \Omega \). (When \( n = 1 \), the usual convention gives \( S_1 = S \) which has infinite breadth by assumption.) Take an incompressible subset of \( S_n \) of size \( k(n+k) \), partition that subset into \( n+k \) parts each of size \( k \), and let \( G'_n \) be the collection of the \( n+k \) unions of the parts. Then \( G'_n \) is an incompressible subset of \( S_n \) of size \( n+k \). Moreover, each \( S_n \sqsupset x \) has finite breadth for every \( x \in G'_n \), since \( S_n \) satisfies (C1).

Choose a witness for \( G'_n \), denoted by \( E'_n \). From Condition (C2) for \( S_n \), we observe that \( S_n^{-E} \) has finite breadth for every \( k \)-element subset \( E \) of \( E'_n \). Since

\[
S_n = \left( \bigvee_{E \subset E'_n, |E|=k} S_n^{-E} \right) \lor \left( \bigvee_{E \subset E'_n, |E|=n} (S_n)_E \right),
\]

there exists a subset \( E_n \) of \( E'_n \) of size \( n \) such that \((S_n)_{E_n}\) has infinite breadth. Let \( G''_n \) be the subset of \( G'_n \) corresponding to \( E_n \). Then, form \( G_n \subset S_D \subseteq S \) by choosing, for each \( x'' \in G''_n \), some \( x \in S_D \) such that \( x'' = x \cap q \).
Set \( c_n := \bigcap_{x \in G_n} x \) and \( S_{n+1} := (S_n)_{E_n} \setminus c_n \). Then \( S_{n+1} \) has infinite breadth by Lemma 3.5; noting that \((S_n)_{E_n}\) has infinite breadth while
\[
(S_n)_{E_n} \cup x \subseteq S_n \cup x = S_n \cup x''
\]
has finite breadth for every \( x \in G_n \), where \( x'' = x \cap q \in G''_n \subseteq G'_n \). Also,
\[
S_{n+1} = (S_{D \cup E_n}) \setminus (q \cap c_n),
\]
and so the induction can be continued.

Note that:

(i) each \( G_n \) is an incompressible subset of \( S \), with witness \( E_n \) of size \( n \),

(ii) \( x \supseteq \bigcup_{j \neq n} E_j \) for every \( x \in G_n \), and

(iii) \( E_j \cap E_n = \emptyset \) whenever \( j < n \).

To see (ii), note that \( G_n \subseteq S_D \), so \( \bigcup_{j=1}^{n-1} E_j \subseteq x \) for every \( x \in G_n \). On the other hand, by the inductive definition of \( c_n \) we have
\[
E_n \subseteq q = \bigcap_{j=1}^{n-1} c_j \subseteq \bigcap_{j=1}^{n-1} \bigcap_{x \in G_j} x,
\]
which proves \( E_n \subseteq x \) whenever \( x \in G_m \) with \( m < n \). To see (iii), note that \( E_n \) is a witness for the subset \( G_n \) of \( S_D \), and so \( E_n \cap D = \emptyset \) when \( n > 1 \). Thus if we set \( \mathcal{E} = (E_n)_{n \geq 1} \), this is a spread in \( \Omega \); and \( S \setminus \text{join}(\mathcal{E}) \) contains the subsystem generated by \( (\bigvee_{n=1}^{\infty} G_n) \setminus \text{join}(\mathcal{E}) \), which is the set system \( T_{\text{ort}}(\mathcal{E}) \). This completes the proof. \( \square \)

4 The second structure theorem

When dealing with semilattices of infinite breadth, it is natural to inquire about the nature of their incompressible subsets. In particular, one may ask whether a semilattice \( S \) of infinite breadth contains a nested chain of incompressible sets of growing sizes. It is easy to see that the answer to this question is positive for \( \mathcal{P}^{\text{fin}}(\Omega) \) when \( \Omega \) is infinite, but negative for any semilattice of the form \( T_{\text{max}}, T_{\text{min}} \) or \( T_{\text{ort}} \). In this section, we study the structure of semilattices which possibly fail this condition.

We first investigate what occurs when we do have such a nested chain.

**Lemma 4.1.** Let \( m, k \in \mathbb{N} \). Suppose \( x_1, \ldots, x_{m+k} \) are distinct non-empty subsets of \( \Omega \), such that \( \{x_1, \ldots, x_{m+k}\} \) is incompressible. Let \( d = x_1 \cup \cdots \cup x_m \). Then \( \{x_{m+1} \setminus d, \ldots, x_{m+k} \setminus d\} \) is an incompressible subset of \( \mathcal{P}(\Omega) \setminus d \).

**Proof.** Consider a witness of incompressibility for \( \{x_i : 1 \leq i \leq m+k\} \), with \( \alpha_i \in x_i \). Then the elements of this witness that correspond to \( x_{m+j} \) (1 \( \leq j \leq k \)) form a witness for the incompressibility of \( \{x_{m+j} \setminus d : 1 \leq j \leq k\} \). \( \square \)

**Proposition 4.2.** Let \( S \subseteq \mathcal{P}(\Omega) \) be a concrete semilattice. Suppose there is a sequence \( (a_n)_{n \geq 1} \) of distinct members of \( S \) with the following property: for each \( m \in \mathbb{N} \), the set \( \{a_1, \ldots, a_m\} \) is incompressible. Then there is a spread \( \mathcal{E} \) in \( \Omega \) such that \( S \setminus \text{join}(\mathcal{E}) \) contains \( T_{\text{max}}(\mathcal{E}) \).
This follows immediately from Lemma 3.8, as the subsystem of $S$ generated by the sequence $(a_n)$ satisfies the auxiliary Condition (A) from the previous section, even in a rather strong sense (see Lemma 4.1). However, we can also give a direct argument, as follows.

**Proof.** Fix positive integers $n_1 < n_2 < \ldots$ such that $n_{k+1} - n_k \to \infty$ (for instance we could take $n_k = k^2$). Let $d_k = \bigcup_{i=1}^{n_k} a_i \in S$, and for convenience set $d_0 = 0$, $n_0 = 0$.

By Lemma 4.1, for each $k \in \mathbb{N}$ the set $F_k := \{a_j \setminus d_{k-1} : n_k - 1 \leq j \leq n_k \}$ is an incompressible subset of $S \sqcup d_{k-1}$. Let $E_k$ be a witness for $F_k$. Since $E_k \subseteq d_{k+1} \setminus d_k$ and $|E_k| = n_{k+1} - n_k$, the sequence $E = (E_k)_{k \in \mathbb{N}}$ is a spread in $\Omega$.

To finish, it suffices to show that given $k \in \mathbb{N}$ and some $\omega \in E_k$, there exists some $z \in S$ such that $z \cap \text{join}(E) = E \cup \{\omega\}$. Since $E_k$ is a witness for $F_k$, there exists $z' \in F_k$ such that $z' \cap E_k = \{\omega\}$. Our construction also ensures that $E_1 \cup \cdots \cup E_{k-1} = d_{k-1} \cap \text{join}(E)$. Put $z = z' \cup d_{k-1}$: this satisfies $z \cap \text{join}(E) = E \cup \{\omega\}$, and we must show $z \in S$. But since $z'$ has the form $a_i \setminus d_{k-1}$ for some $i \in \mathbb{N}$, we have $z = a_i \cup d_{k-1} \in S$, as required.

The following definition describes a much stronger property than having a nested sequence of incompressible sets.

**Definition 4.3 (Shattering a spread).** Let $E = (E_n)_{n \geq 1}$ be a spread in $\Omega$ and let $(a_j)_{j \geq 1}$ be a sequence of subsets of $\Omega$. We say that this sequence **shatters** $E$ if, for every $m \in \mathbb{N}$ and every $m$-tuple $(y_1, \ldots, y_m)$ such that $y_j \in \{a_j, a_j^c\}$ for $j = 1, \ldots, m$,

$$\lim_{n \to \infty} \left| E_n \cap \bigcap_{j=1}^{m} y_j \right| = \infty.$$

Recall from the remarks before Equation (3) that the condition

$$y_1 \cap \cdots \cap y_m \neq \emptyset$$

implies that $a_1, \ldots, a_m$ are mutually distinct and form an incompressible subset of $\mathcal{P}(\Omega)$. So if we start with a shattering sequence $(a_j)$ and allow ourselves to take complements and binary unions (i.e. we generate a ring of sets) then the resulting collection has high complexity; moreover, this complexity is seen inside each $E_n$ once we take $n$ sufficiently large. In fact, if $(a_j)_{j \geq 1}$ shatters some spread, then the concrete semilattice generated by $(a_j)_{j \geq 1}$ is **universal** in a certain sense, which will be made precise in Lemma 4.6.

**Definition 4.4.** Let $Q$ be a concrete semilattice on $\mathbb{N}$, and let $\Gamma$ be a well-ordered set of order type $\omega_0$ (put it another way, $\Gamma := (\gamma_j)_{j \geq 1}$ is a sequence of distinct elements). Define $Q(\Gamma)$ to be the concrete semilattice on $\Gamma$ that is “pulled-back” from $Q$ through the unique order-isomorphism $\Gamma \to \mathbb{N}$.

**Remark 4.5.** We are interested in semilattices of the form $Q(\Gamma)$ where $Q$ is also countable. Examples of such includes $\mathcal{P}^\text{fin}(\Gamma)$, $\mathcal{P}^\text{cofin}(\Gamma)$, and $\mathcal{P}^\text{fin}(\Gamma) \setminus \mathcal{P}^\text{cofin}(\Gamma)$ for any countably infinite set $\Gamma$. Moreover, every countable semilattice is isomorphic to some $Q(\Gamma)$, thanks to its Cayley embedding.

**Lemma 4.6.** Let $E = (E_n)_{n \geq 1}$ be a spread in $\Omega$ and let $(a_j)_{j \geq 1}$ be a sequence of subsets of $\Omega$ that shatters $E$. Denote by $\overline{a}$ the closure of $a$ in the Stone–Cech compactification $\beta \Omega$ of the discrete space $\Omega$ for each subset $a$ of $\Omega$. Then the concrete semilattice $S$ generated by $(\overline{a_j})_{j \geq 1}$, which is isomorphic to the concrete semilattice generated by $(a_j)_{j \geq 1}$, has the property that, for every countable concrete semilattice $Q$ on $\mathbb{N}$, there exists a well-ordered set $\Gamma \subseteq \beta \Omega$ of order type $\omega_0$ such that $S \sqsubseteq \Gamma$ contains $Q(\Gamma)$. 

13
Proof. Replacing $\mathcal{Q}$ by the semilattice generated by $\mathcal{Q} \cup \mathcal{P}^{\text{fin}}(\mathbb{N})$, we may and shall assume that $\mathcal{Q}$ “separates” points of $\mathbb{N}$. Let $\mathcal{Q} = \{c_n : n \geq 1\}$ be a listing of $\mathcal{Q}$ without repetition. For each $k \in \mathbb{N}$, define

$$A_k := \{n \in \mathbb{N} : k \in c_n\} \quad \text{and} \quad B_k := \{n \in \mathbb{N} : k \notin c_n\}.$$ 

Since

$$y_1 \cap \cdots \cap y_m \neq \emptyset \quad \text{whenever} \quad y_j \in \{\overline{a}_j, \beta \Omega \setminus \overline{a}_j\} \quad \text{for} \quad 1 \leq j \leq m \quad \text{and} \quad m \in \mathbb{N},$$

the compactness allows us to find $\gamma_k \in \beta \Omega$ such that

$$\gamma_k \in \bigcap_{n \in A_k} \overline{a}_n \setminus \bigcup_{m \in B_k} \overline{a}_m.$$ 

Then $\gamma_k \in \overline{a}_n$ if and only if $k \in c_n$, for every $k, n \in \mathbb{N}$. Since $\mathcal{Q}$ separates points of $\mathbb{N}$, we see that $\gamma_k \neq \gamma_l$ whenever $k \neq l$. Set $\Gamma := \{\gamma_k : k \in \mathbb{N}\}$ with the well-order being defined on $\Gamma$ in an obvious manner. Then obviously $\mathcal{S} \uplus \Gamma \supseteq Q(\Gamma)$. \hfill $\square$

**Definition 4.7 (Decisive colourings).** Given a spread $\mathcal{E} = (E_n)_{n \geq 1}$, a partition of $\Omega$ into finitely many subsets $\Omega = C_1 \cup \cdots \cup C_d$ is said to colour $\mathcal{E}$ if $\lim_n |C_j \cap E_n| = \infty$ for each $j$. We call $\mathcal{C} = \{C_1, \ldots, C_d\}$ a colouring of $\mathcal{E}$. If we are also given a set system $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, we say that the colouring $\mathcal{C}$ decides $\mathcal{S}$, or is $\mathcal{S}$-decisive, if there exists a colour class $C_0 \in \mathcal{C}$ with the following property: every $x \in \mathcal{S}$ satisfies

$$\sup_{n \geq 1} \min\{|x \cap C_0 \cap E_n| : |x^c \cap C \cap E_n|, C \in \mathcal{C}\} < \infty.$$  \hfill (8)

In this context we say $C_0$ is a **decisive colour class**. A colouring (of $\mathcal{E}$) that has no decisive colour class is said to be **indecisive** (for $\mathcal{S}$).

Informally speaking, when we have a decisive colour class $C_0$, each $x \in \mathcal{S}$ must either have small intersection with $C_0 \cap E_n$, or else have large intersection with $C \cap E_n$ for some $C \in \mathcal{C}$, once $n$ is sufficiently large.

We can now state the second main theorem of this paper. It tells us, loosely speaking, that unless we are in the highly fragmented case, we can find a decisive colouring.

**Theorem 4.8 (Controlled or fragmented).** Let $\mathcal{S}$ be a semilattice, and let $\mathcal{S}$ be a concrete representation of $\mathcal{S}$ on some set $\Omega$. Let $\mathcal{E}$ be a spread in $\Omega$. At least one of the following conclusions holds:

1. **(D1)** there is a spread $\mathcal{G}$ that refines $\mathcal{E}$, and a sequence $(a_i)_{i \geq 1}$ in $\mathcal{S}$ which shatters $\mathcal{G}$;
2. **(D2)** there is a spread $\mathcal{F}$ that refines $\mathcal{E}$, and an $\mathcal{S}$-decisive colouring $\mathcal{C}$ of $\mathcal{F}$.

Unlike the results of the previous section, this theorem applies to semilattices $\mathcal{S}$ with finite breadth as well as those with infinite breadth. For example, we have the following corollary, whose proof is immediate.

**Corollary 4.9.** Let $\mathcal{S}$ be a semilattice with finite breadth. Then for any concrete representation $\mathcal{S}$ of $\mathcal{S}$ on some infinite set $\Omega$ and any spread $\mathcal{E}$ in $\Omega$, there is a spread $\mathcal{F}$ that refines $\mathcal{E}$, and an $\mathcal{S}$-decisive colouring $\mathcal{C}$ of $\mathcal{F}$.
The following example shows that in the conclusion of the previous theorem, condition (8) of a decisive colouring cannot be strengthened to

$$\min\{\sup_{n \geq 1} |x \cap C_0 \cap E_n| ; \sup_{n \geq 1} |x^c \cap C \cap E_n|, C \in C\} < \infty.$$  

(9)

Example 4.10. Let $\mathcal{E} = (E_n)_{n \geq 1}$ be any spread on any infinite set $\Omega$, and let $\mathcal{S}$ be a concrete semilattice on $\Omega$ that consists of those $a \subseteq \Omega$ such that, for each $n \in \mathbb{N}$, either $E_n \subseteq a$ or $E_n \cap a = \emptyset$. Then we see that there is no sequence $(a_i)_{i \geq 1}$ in $\mathcal{S}$ which shatters $\mathcal{E}$, and that there is no colouring $C$ of $\mathcal{F}$ for which (9) holds for all $x \in \mathcal{S}$; for the latter, one simply chooses an $x$ such that $E_n \subseteq x$ if and only if $n$ is even. These conditions pass down to any spread $\mathcal{F} = (F_n)_{n=1}^\infty$ that refines $\mathcal{E}$, since such an $\mathcal{F}$ also satisfies that, for each $a \in \mathcal{S}$ and each $n \in \mathbb{N}$, either $F_n \subseteq a$ or $F_n \cap a = \emptyset$.

The proof of Theorem 4.8 requires a double induction, and we isolate part of it as a separate lemma. The following terminology is introduced to streamline the presentation. Let $\mathcal{E} = (E_n)_{n \geq 1}$ be a spread, and let $D$ and $F$ be subsets of $\Omega$. We say that $F$ halves $D$ with respect to $\mathcal{E}$ if

$$\lim_{n \to \infty} |D \cap F \cap E_n| = \lim_{n \to \infty} |(D \setminus F) \cap E_n| = \infty.$$  

Also, if $N \subseteq \mathbb{N}$ is an infinite subset, and $(t_n)_{n \geq 1}$ is a sequence in $[0, \infty)$, we say that $t_n \to \infty$ along $n \in N$ if $\lim_{k \to \infty} t_{n_k} = \infty$, where $N = \{n_k : k \in \mathbb{N}\}$.

Lemma 4.11. Let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ be a concrete semilattice. Let $\mathcal{E} = (E_n)_{n \geq 1}$ be a spread in $\Omega$, and let $\mathcal{C}$ be a colouring of $\mathcal{E}$. Then at least one of the following conclusions holds:

(a) there is an infinite set $N \subseteq \mathbb{N}$ and some $y \in \mathcal{S}$ which halves $C$ with respect to $(E_n)_{n \in N}$ for every $C \in \mathcal{C}$;

(b) there is a spread $\mathcal{F}$ refining $\mathcal{E}$, such that $\mathcal{C}$ is an $\mathcal{S}$-decisive colouring of $\mathcal{F}$.

Proof. The idea is as follows: we attempt to construct, by iteration, members of $\mathcal{S}$ that are closer and closer to satisfying the property in Case (a). At each stage of the iteration we will be able to continue, unless we find ourselves in Case (b). Therefore, if Case (b) does not hold, our iteration will run successfully until Case (a) is satisfied.

Assume from now on that Case (b) does not hold, and enumerate the members of $\mathcal{C}$ as $C_1, \ldots, C_d$. Since $\mathcal{C}$ is not an $\mathcal{S}$-decisive colouring of $\mathcal{E}$, there exists $y_1 \in \mathcal{S}$ such that

$$\sup_n \min\{|y_1 \cap C_1 \cap E_n| ; |y_1^c \cap C_j \cap E_n|, 1 \leq j \leq d\} = \infty.$$  

Passing to an appropriate subsequence, there exists an infinite $N_1 \subseteq \mathbb{N}$ such that:

- $|y_1 \cap C_1 \cap E_n| \to \infty$ along $n \in N_1$; and
- $|y_1^c \cap C_j \cap E_n| \to \infty$ along $n \in N_1$ for every $1 \leq j \leq d$.

Now let $2 \leq k \leq d$. Suppose there are $y_{k-1} \in \mathcal{S}$ and an infinite $N_{k-1} \subseteq \mathbb{N}$ such that:

(i) $|y_{k-1} \cap C_i \cap E_n| \to \infty$ along $n \in N_{k-1}$ for all $1 \leq i \leq k - 1$; and

(ii) $|y_{k-1}^c \cap C_j \cap E_n| \to \infty$ along $n \in N_{k-1}$ for all $1 \leq j \leq d$. 

15
Case (a).

Suppose that Case (D2) of the theorem does not hold. The following notation will be useful: given $a_1, \ldots, a_m \in P(\Omega)$, consider

$$\Gamma(a_1, \ldots, a_m) := \left\{ \bigcap_{j=1}^m y_j : y_j \in \{x_j, x_j^c\} \text{ for each } j = 1, \ldots, m \right\}.$$ 

This is a partition of $\Omega$, although some members of the partition might be empty.

Now let $E_0 = \emptyset$ and let $C_0$ denote the trivial colouring $\emptyset$. Apply Lemma 4.11 to the pair $(E_0, C_0)$. Case (b) of the lemma does not hold, since otherwise we would be in Case (D2) of the theorem. Hence we are in Case (a) of the lemma, so there exists an infinite $N_k \subseteq N_{k-1}$ such that:

(iii) $|x \cap C_k \cap (y_{k-1} \cap E_n)| \to \infty$ along $n \in N_k$; and

(iv) $|x \cap C_j \cap (y_{k-1} \cap E_n)| \to \infty$ along $n \in N_k$, for every $1 \leq j \leq d$.

Let $y_k := y_{k-1} \cup x$, which belongs to $S$ since $S$ is union-closed. Since $N_k \subseteq N_{k-1}$ and $y_k \supseteq y_{k-1}$, condition (i) implies

$$|y_k \cap C_i \cap E_n| \to \infty \text{ along } n \in N_k, \text{ for all } 1 \leq i \leq k-1;$$

and since $y_k \supseteq x$, condition (iii) implies

$$|y_k \cap C_k \cap E_n| \to \infty \text{ along } n \in N_k.$$ 

But by the definition of $y_k$, condition (iv) can be rephrased as

$$|y_k \cap C_j \cap E_n| \to \infty \text{ along } n \in N_k, \text{ for every } 1 \leq j \leq d.$$ 

Thus the induction can continue. We end up with $y_d \in S$ and an infinite subset $N_d \subseteq N$ such that $y_d$ halves $C_j$ with respect to $(E_n)_{n \in N_d}$ for every $1 \leq j \leq d$, i.e. we are in Case (a).

**Proof of Theorem 4.48.** Suppose that Case (D2) of the theorem does not hold. The following notation will be useful: given $x_1, \ldots, x_m \in P(\Omega)$, consider

$$\Gamma(x_1, \ldots, x_m) := \left\{ \bigcap_{j=1}^m y_j : y_j \in \{x_j, x_j^c\} \text{ for each } j = 1, \ldots, m \right\}.$$ 

This is a partition of $\Omega$, although some members of the partition might be empty.

Now let $E_0 = \emptyset$ and let $C_0$ denote the trivial colouring $\emptyset$. Apply Lemma 4.11 to the pair $(E_0, C_0)$. Case (b) of the lemma does not hold, since otherwise we would be in Case (D2) of the theorem. Hence we are in Case (a) of the lemma, so there exists an infinite set $N_1 \subseteq N$ and some $a_1 \in S$ which halves $\Omega$ with respect to $(E_n)_{n \in N_1}$.

Suppose that for some $k \geq 1$, we have found $a_1, \ldots, a_k \in S$ and an infinite subset $N_k \subseteq N$, such that $\Gamma(a_1, \ldots, a_k)$ colours the spread $(E_n)_{n \in N_k}$. This colouring cannot be $S$-decisive (otherwise we would be in Case (D2) of the theorem, contrary to assumption). Hence, by Lemma 4.11 there exist some infinite $N_{k+1} \subseteq N_k$ and some $a_{k+1} \in S$, such that $a_{k+1}$ halves $C$ with respect to $(E_n)_{n \in N_{k+1}}$ for each $C \in \Gamma(a_1, \ldots, a_k)$. Now $\Gamma(a_1, \ldots, a_k)$ is a colouring of the spread $(E_n)_{n \in N_{k+1}}$.

Continuing in this way, we inductively construct a sequence $(a_n)_{n \geq 1}$ in $S$, and a descending chain of infinite subsets of $N$, $N_1 \supseteq N_2 \supseteq \ldots$, such that:

for each $m \geq 1$ and each $C \in \Gamma(a_1, \ldots, a_m)$, $|C \cap E_n| \to \infty$ along $n \in N_m$. 

16
Since \( N_1 \supseteq N_2 \supseteq \ldots \) is a decreasing sequence of infinite subsets of \( \mathbb{N} \), we can extract a diagonal subsequence \( n(1) < n(2) < n(3) < \ldots \) satisfying \( n(k) \in N_j \) for every \( j \leq k \). For each \( m \), \((n(i))_{i \geq m}\) is a subsequence of \( N_m \), and so for each \( C \in \Gamma(a_1, \ldots, a_m) \) we have
\[
\lim_{i \to \infty} |C \cap E_{n(i)}| = \lim_{N_m \ni n \to \infty} |C \cap E_n| = \infty.
\]
Therefore, if we define \( G_i = E_{n(i)} \), the sequence \((G_i)_{i \geq 1}\) is a spread which is shattered by the sequence \((a_j)_{j \geq 1}\), and we are in Case (D1) as required.

Combining Lemma 4.6 with Theorem 4.8 and Theorem 3.1 yields the following refinement of the structure of infinite breadth semilattices. (It was already stated in the introduction as Theorem 1.4.)

**Corollary 4.12.** Let \( S \) be a semilattice with infinite breadth. Then at least one of the following statements holds.

(i) There is a concrete representation \( S \) of \( S \) on some set \( \Omega \) with the property that, for every countable concrete semilattice \( Q \) on \( \mathbb{N} \), there exists a countably infinite subset \( \Gamma \subseteq \Omega \) with a type \( \omega_0 \) well-ordering such that \( S \sqsupseteq \Gamma \) contains \( Q(\Gamma) \).

(ii) For any concrete representation \( S \) of \( S \) on any set \( \Omega \), there exist a spread \( E \) in \( \Omega \) and an \( S \)-decisive colouring \( C \) of \( E \), such that \( S \sqsupseteq \text{join}(E) \) contains either one of \( T_{	ext{max}}(E), T_{	ext{min}}(E) \), or \( T_{\text{ort}}(E) \).

**Remark 4.13.** The extra level of control provided by Corollary 4.12 is crucial for the application of this structure theory to the study of (weighted) instability of filters and characters, which will appear in [CGP]. In the following section, we will see that one can strengthen the statement of Case (ii) of the corollary a little, as indicated in Proposition 5.2, but not much as shown by Example 5.1.

## 5 Further examples

In this closing section, we give the details of the claims made in Remark 3.2. That is, we provide an explicit example of a semilattice \( S \) and two different concrete representations of \( S \) where one has to settle for different types of either \( T_{\text{min}} \) or \( T_{\text{ort}} \).

**Example 5.1.** Set \( E_n := \{(n,k) : 1 \leq k \leq n\} \), and set \( E := (E_n)_{n \geq 1}, \quad \Omega_0 := \text{join}(E), \quad \text{and} \quad \Omega := \Omega_0 \cup \mathbb{N}. \)

For each \( a \in T_{\text{min}}(E) \), let us define \( \text{level}(a) \) to be the level of \( a \) as indicated in Figure 1. Define
\[
S := \{a \sqcup \{1, \ldots, m\} : a \in T_{\text{min}}(E), \ m \in \mathbb{N}, \ m \geq \text{level}(a) \geq 2\}.
\]
(The requirement of \( \text{level}(a) \geq 2 \) here is just to guarantee that \( S \) is isomorphic to \( S' \) constructed in (10).) Then \( S \) is an infinite breadth concrete semilattice on \( \Omega \) that satisfies neither (i) nor (iii) of Theorem 3.1 (but certainly, it satisfies (ii)).

Now let us find a different concrete representation of \( S \). Set \( \Omega' := \Omega_0 \times \mathbb{N} \), and define \( S' := \{(a \times \mathbb{N}) \cup (\Omega_0 \times \{1, \ldots, m\}) : a \in T_{\text{min}}(E), \ m \in \mathbb{N}, \ m \geq \text{level}(a) \geq 2\} \)
\[
:= \{(a \times \mathbb{N}) \cup (E_{\leq \text{level}(a)} \times \{1, \ldots, m\}) : a \in T_{\text{min}}(E), \ m \in \mathbb{N}, \ m \geq \text{level}(a) \geq 2\}. \tag{10}
\]
Then a little thought will convince us that $S'$ is a concrete semilattice on $\Omega'$ that is isomorphic to $S$, and that the pair $(S', \Omega')$ satisfies neither (i) nor (ii) of Theorem 3.1 (but certainly, it satisfies (iii)).

**Proof.** First of all, it can be seen that for any sequence $(x_j)_{j \geq 1}$ of distinct members of $S'$ one has $\bigcup_{j=1}^{\infty} x_j = \Omega'$. Assume now that there were a spread $\mathcal{F} = (F_k)_{k \geq 1}$ such that $S' \sqcup \text{join}(\mathcal{F})$ contains $T_{\min}(\mathcal{F})$. Let us then choose $x_j \in S'$ such that $x_j \land \text{join}(\mathcal{F})$ is of level $j + 2$ in $T_{\min}(\mathcal{F})$. Then

$$\text{join}(\mathcal{F}) \neq \bigcap_{j=1}^{\infty} (x_j \land \text{join}(\mathcal{F})) = \left( \bigcup_{j=1}^{\infty} x_j \right) \land \text{join}(\mathcal{F}) = \Omega' \land \text{join}(\mathcal{F}) = \text{join}(\mathcal{F})$$

a contradiction. \qed

In contrast, the $T_{\max}(\mathcal{E})$ case of Theorem 3.1 is better behaved.

**Proposition 5.2.** If an infinite breadth semilattice $S$ possesses a concrete representation $S$ on some $\Omega$ with a spread $\mathcal{E}$ in $\Omega$ such that $S \sqcup \text{join}(\mathcal{E}) \supseteq T_{\max}(\mathcal{E})$, then on every other concrete representation $(S', \Omega')$ of $S$, there is a spread $\mathcal{E}'$ in $\Omega'$ such that $S' \sqcup \text{join}(\mathcal{E}') \supseteq T_{\max}(\mathcal{E}')$.

**Proof.** For simplicity of notation, we shall write $a$ or $b_n$ for the corresponding members of $S$ and $a'$ or $b'_n$ for the corresponding members of $S'$. Also, without loss of generality, we suppose that $\mathcal{E} = (E_n)_{n \geq 1}$ with $|E_n| = n$ for all $n$; say $E_n = \{ \gamma_{nj} : 1 \leq j \leq n \}$. For $1 \leq j \leq n$, let $a_{nj}$ be an element of $S$ such that $a_{nj} \land \text{join}(\mathcal{E})$ is a member of $T_{\max}(\mathcal{E})$ that meets $E_n$ at the singleton $\{ \gamma_{nj} \}$. Then, for each $n \in \mathbb{N}$, $a_{nj}$ does not divide the product of $a_{ml}$ and $a_{nk}$ where $m < n$ and $k \neq j$, and so we can find an element $\gamma''_{nj}$ of $\Omega'$ that belongs to $a'_{nj}$ but not to any of $a'_{ml}$ and $a'_{nk}$ where $m < n$ and $k \neq j$. Define $E'_n := \{ \gamma''_{nj} : 1 \leq j \leq n \}$, and then $\mathcal{E}' := (E'_n)_{n \geq 1}$. Then $\mathcal{E}'$ is a spread in $\Omega'$, and $S' \sqcup \text{join}(\mathcal{E}') \supseteq T_{\max}(\mathcal{E}')$: for each $1 \leq j \leq n$, we have

$$\left( \bigcup_{1 \leq l \leq m < n} a'_{ml} \cup a'_{nj} \right) \land \text{join}(\mathcal{E}') = E'_{\leq n} \cup \{ \gamma''_{nj} \}.$$

\qed

In light of these results, the different behaviour of the same abstract semilattice in different (faithful) concrete representations seems to deserve further study. We leave this as a possible line of enquiry for future work.

**Acknowledgements**

This work grew out of conversations between the authors while attending the conference “Banach Algebras and Applications”, held in Gothenburg, Sweden, July–August 2013, and was further developed while the authors were attending the thematic program “Abstract Harmonic Analysis, Banach and Operator Algebras” at the Fields Institute, Canada, during March–April 2014. The authors thank the organizers of these meetings for invitations to attend and for pleasant environments to discuss research.
The first author acknowledges the financial support of the Faculty of Science and Technology at Lancaster University, in the form of a travel grant to attend the latter meeting. The second author acknowledges financial support from University of Delaware Research Foundation. The third author acknowledges the financial supports of a Fast Start Marsden Grant and of Victoria University of Wellington to attend these meetings.

Further work was done during the second author’s visit to Lancaster University in November 2014, which was supported by a Scheme 2 grant from the London Mathematical Society. She thanks the Department of Mathematics and Statistics at Lancaster for their hospitality.

References

[ADH+13] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson, and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, II, Notre Dame J. Form. Log., 54 (2013), pp. 311–363.

[ADH+16] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson, and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, I, Trans. Amer. Math. Soc., 368 (2016), pp. 5889–5949.

[ASB] P. S. Almeida, A. Shoker, and C. Baquero, Delta state replicated data types, Journal of Parallel and Distributed Computing, 111 (2018), pp. 162–173.

[BGG15] M. J. J. Branco, G. M. S. Gomes, and V. Gould, Ehresmann monoids, J. Algebra, 443 (2015), pp. 349–382.

[BGGW18] M. J. J. Branco, G. M. S. Gomes, V. Gould, and Y. Wang, Ehresmann monoids: adequacy and expansions, J. Algebra, 513 (2018), pp. 344–367.

[Cho13] Y. Choi, Approximately multiplicative maps from weighted semilattice algebras, J. Aust. Math. Soc., 95 (2013), pp. 36–67.

[CGP] Y. Choi, M. Ghandehari, and H. L. Pham, The stability problem for characters on weighted semilattice algebras. Preprint, see arXiv.

[Dit84] S. Z. Ditor, Cardinality questions concerning semilattices of finite breadth, Discrete Math., 48 (1984), pp. 47–59.

[Gie94] G. Gierz, Level sets in finite distributive lattices of breadth 3, Discrete Math., 132 (1994), pp. 51–63.

[How76] J. M. Howie, An introduction to semigroup theory, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. L.M.S. Monographs, No. 7.

[Jar97] K. Jarosz, Almost multiplicative functionals, Studia Math., 124 (1997), pp. 37–58.

[Joh86] B. E. Johnson, Approximately multiplicative functionals, J. London Math. Soc. (2), 34 (1986), pp. 489–510.

[KSS09] U. Khedkher, A. Sanyai, and B. Sathe, Data Flow Analysis: Theory and Practice, CRC Press, 2009.
[KU76] J. B. Kam and J. D. Ullman, *Global data flow analysis and iterative algorithms*, J. Assoc. Comput. Mach., 23 (1976), pp. 158–171.

[Law71] J. D. Lawson, *The relation of breadth and codimension in topological semilattices. II*, Duke Math. J., 38 (1971), pp. 555–559.

[LLM77] J. D. Lawson, J. R. Liukkonen, and M. W. Mislove, *Measure algebras of semilattices with finite breadth*, Pacific J. Math., 69 (1977), pp. 125–139.

[Mis86] M. Mislove, *Detecting local finite breadth in continuous lattices and semilattices*, in Mathematical foundations of programming semantics (Manhattan, Kan., 1985), vol. 239 of Lecture Notes in Comput. Sci., Springer, Berlin, 1986, pp. 205–214.

[SPB+11] M. Shapiro, N. Preguiça, C. Baquero, and M. Zawirski, *Conflict-free replicated data types*, in Proceedings of the 13th International Conference on Stabilization, Safety, and Security of Distributed Systems, SSS’11, Berlin, Heidelberg, 2011, Springer-Verlag, pp. 386–400.

Yemon Choi, Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom.

y.choi1@lancaster.ac.uk

Mahya Ghandehari, Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, United States of America.

mahya@udel.edu

Hung Le Pham, School of Mathematics and Statistics, Victoria University of Wellington, Wellington 6140, New Zealand.

hung.pham@vuw.ac.nz