Abstraction. Let $G$ be an almost simple simply-connected affine algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. If $G$ has type $B_n$, $C_n$ or $F_4$, we assume that $p > 2$, and if $G$ has type $G_2$, we assume that $p > 3$. Let $P \subset G$ be a parabolic subgroup. We prove that the tangent bundle of $G/P$ is Frobenius stable with respect to the anticanonical polarization on $G/P$.

1. Introduction

Let us recall the notion of slope stability of a sheaf over a polarized projective scheme. The slope of a sheaf $\mathcal{F}$ is defined as the quotient of its degree by its rank: it will be denoted by $\mu(\mathcal{F})$. A sheaf $\mathcal{F}$ is called stable (respectively, semi-stable) if for any strict subsheaf $\mathcal{G}$ we have $\mu(\mathcal{G}) < \mu(\mathcal{F})$ (respectively, $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$). Throughout, (semi)stability will mean slope (semi)stability.

Let $G$ be an almost simple simply-connected affine algebraic group over an algebraically closed field $k$, and let $P \subset G$ be a parabolic subgroup. If the characteristic $\text{char}(k)$ is zero, then it is known that the tangent bundle of $G/P$ is stable with respect to the anticanonical polarization on $G/P$.

In fact, if $G/P$ is a Hermitian symmetric space, the result goes back to the sixties [Ram66]. In the complex case, it was proved long ago that this bundle admits a Kähler-Einstein metric (see [Ko55] or [Be87, Chapter 8]), which implies polystability. Simplicity of this bundle was proved in [AB10], proving the stability; A. Boralevi proved stability of $T(G/P)$ when $G$ is of type ADE [Bor12, Theorem C].

Our aim here is to address stability of $T(G/P)$ in the case where $\text{char}(k)$ is positive. If $G$ is of type $B_n$, $C_n$ or $F_4$, we assume that $\text{char}(k) > 2$; if $G$ is of type $G_2$, we assume that $\text{char}(k) > 3$. The main Theorem of this note says that under the above assumption, the tangent bundle of $G/P$ and all its iterated Frobenius pull-backs are stable with respect to the anticanonical polarization on $G/P$.

The method of proof of the main Theorem is as follows. We prove that the stability of $T(G/P)$ is equivalent to a certain statement on the quotient $\text{Lie}(G)/\text{Lie}(P)$ considered as a $P$–module. The statement in question is shown to be independent of the characteristic of $k$ (as long as the above assumptions hold). Finally, the main Theorem follows from the fact that $T(G/P)$ is stable if $\text{char}(k) = 0$.

A natural question to ask is whether $T(G/P)$ remains stable with respect to polarizations on $G/P$ other than the anticanonical one. A. Boralevi gave a negative answer to this question. She constructed examples of $G/P$ and polarizations on them with respect to which $T(G/P)$ is not even semi-stable [Bor12, Theorem D].

Another natural question is to understand what happens when one relaxes the hypothesis on the characteristic. We are quite far from having a complete answer to this question. However, in the last section, we give an example of a homogeneous space in type $C_n$ and in characteristic 2 whose tangent bundle is not stable, but semi-stable. We have not been able to find any example of a tangent bundle which is not semi-stable.

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Let $G$ be an almost simple simply-connected affine algebraic group defined over an algebraically closed field $k$. Let $P \subseteq G$ be a parabolic subgroup. The Lie algebra of $G, P$ will be denoted by $g, p$. The nilpotent ideal of $p$ will be denoted by $n$.

Fix a maximal torus $T \subseteq G$ and a Borel subgroup $B$. Assume $T < B < P$. Let $R$ denote the set of roots of $g$. The set of positive (respectively, negative) roots of $g$ will be denoted by $R^+$ (respectively, $R^-$). The eigenspace corresponding to any $\alpha \in R$ will be denoted by $g^\alpha$.

A subsheaf $E \subset T(G/P)$ is called $G$-stable if it is preserved by the left action of $G$ on $T(G/P)$. Since the left translation action of $G$ on $G/P$ is transitiv, any $G$-stable subsheaf of $T(G/P)$ is a subbundle. Sheaves with a $G$-action are called linearized sheaves. For a coherent sheaf on $G/P$, there is an open subset where it is free as a $\mathcal{O}_{G/P}$-module. Therefore, linearized sheaves are locally free. Moreover, we recall the well-known correspondence between $G$-linearized sheaves and $P$-modules (details can be found in [Bri09, 2.1] and [Jan03, 5.9]):

**Proposition 2.1.** There is an equivalence of categories between $G$-linearized vector bundles on $G/P$ and $P$-modules. On the one hand, to a $G$-linearized vector bundle $E$, one can associate the fiber $E_o$ at the $P$-stable point. On the other hand, to a $P$-module $M$, one can associate the bundle $(G \times M)/P$ over $G/P$.

If $E$ is the $G$-linearized vector bundle corresponding to a $P$-module $M$, since this correspondence is functorial, it induces a correspondence between $G$-linearized subbundles of $E$ and $P$-submodules of $M$.

We now impose the following assumptions on the characteristic of the field $k$ ($\alpha^\vee$ is the coroot corresponding to $\alpha$):

**Working assumption.**

- The characteristic $\text{char}(k)$ of $k$ is positive, and
- $\text{char}(k)$ is bigger than all the coefficients $|\langle \alpha^\vee, \beta \rangle|$ for all roots $\alpha, \beta$ of $G$ with $\alpha \neq \pm \beta$.

In other words, if the root system of $G$ is simply-laced, then $\text{char}(k)$ is only assumed to be positive; if $G$ is any of $B_n, C_n$ and $F_4$, we assume that $\text{char}(k) > 2$; if $G = G_2$, we assume that $\text{char}(k) > 3$. Recall that a bundle is said to be Frobenius stable with respect to a given polarization if it is stable and all its iterated Frobenius pull-backs are again stable.

**Main Theorem.** Under the previous assumption, the tangent bundle $T(G/P)$ is Frobenius stable with respect to the anticanonical polarization on $G/P$.

The rest of the article is devoted to the proof of this theorem. This is essentially given by reduction to characteristic zero. This reduction is achieved using the following construction: let $G_\mathbb{Z}$ be the split simply-connected Chevalley group scheme over $\mathbb{Z}$ having the same root system as $G$. By the theory of reductive algebraic group schemes, as the root system characterizes simply-connected groups up to isomorphism, we have $G \simeq G_\mathbb{Z} \otimes \text{Spec} \mathbb{Z}$. On the other hand, we denote $G_\mathbb{Z} \otimes \text{Spec} \mathbb{C}$ by $G_\mathbb{C}$, and we denote by $g_\mathbb{C}$ its Lie algebra. There exists a parabolic group $P_\mathbb{Z} \subset G_\mathbb{Z}$ such that $P_\mathbb{Z} \otimes \text{Spec} \mathbb{K}$ is conjugate to $P$. The parabolic subgroup $P_\mathbb{Z} \otimes \text{Spec} \mathbb{C}$ of $G_\mathbb{C}$ will be denoted by $P_\mathbb{C}$.

**3. Proof of the main result**

The set of roots $\alpha$ such that $g^\alpha \subset p$ will be denoted by $I(P)$. Let $x_0 \in G/P$ denote the base point. We have

$$T_{x_0}(G/P) \cong g/p \cong \bigoplus_{\alpha \in R \setminus I(P)} g^\alpha.$$

Thus, the vector space $\bigoplus_{\alpha \in R \setminus I(P)} g^\alpha$, has a natural $P$-module structure, which is the one we consider in the following lemma.
Lemma 3.1. Let \( I \subset R \setminus I(\mathcal{P}) \) be a set of negative roots. Then the sum \( M(I) := \bigoplus_{\alpha \in I} \mathfrak{g}^\alpha \) is a \( \mathcal{P} \)-stable submodule of \( \bigoplus_{\alpha \in R \setminus I(\mathcal{P})} \mathfrak{g}^\alpha \) if, and only if,

\[
\forall \beta \in I(\mathcal{P}), \, \forall \alpha \in I, \, \alpha + \beta \in R \setminus I(\mathcal{P}) \implies \alpha + \beta \in I.
\]

Proof. Take \( \alpha \in I \) and \( \beta \in I(\mathcal{P}) \) such that \( \alpha + \beta \in R \setminus I(\mathcal{P}) \). In particular, we have \( \beta \neq \pm \alpha \). Since \( \mathcal{G} \) is simply-connected, \( \mathfrak{g} \) is the Lie algebra defined by Serre’s relations (this is explained for example in [CR10, Remark 2.2.3]), so we can choose a basis of \( \mathfrak{g} \) such that the coefficients of the Lie bracket are those of the Chevalley basis [Ca72]. Consider the biggest integer \( p \) such that \( \alpha - p\beta \in R \). This \( p \) is smaller than the length of the \( \beta \)-string of roots through \( \alpha \) minus 1 (since \( \alpha + \beta \in R \)), and thus, by the working Assumption, we have \( p \leq \text{char}(k) - 2 \). This implies that \( p + 1 < \text{char}(k) \). It now follows from [Ca72, Theorem 4.2.1] that \([\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha + \beta}\). Assuring that \( M(I) \) is \( \mathcal{P} \)-stable, we have it to be \( p \)-stable, and therefore \( \alpha + \beta \in I \).

On the other hand, let \( U_\beta \subset \mathcal{G} \) be the one-parameter additive subgroup corresponding to the root \( \beta \). Since \( \mathcal{G}_0 \cdot \mathfrak{g}^\alpha \subset \bigoplus_{k \geq 0} \mathfrak{g}^{\alpha + k\beta} \), from (1) it follows that \( M(I) \) is \( U_\beta \)-stable for any root \( \beta \) in \( I(\mathcal{P}) \), and thus \( M(I) \) is \( \mathcal{P} \)-stable. \( \square \)

The anticanonical line bundles of \( \mathcal{G}/\mathcal{P} \) and \( \mathcal{G}_C/P_C \) are ample. Fix the anticanonical polarization on \( \mathcal{G}/\mathcal{P} \) and also on \( \mathcal{G}_C/P_C \).

Proposition 3.2. Let \( E \subset T(\mathcal{G}/\mathcal{P}) \) be a \( \mathcal{G} \)-stable subbundle of \( T(\mathcal{G}/\mathcal{P}) \). There exists a subbundle \( E_C \subset T(\mathcal{G}_C/P_C) \) such that \( \text{rk}(E_C) = \text{rk}(E) \) and \( \deg(E_C) = \deg(E) \).

Proof. Under the correspondence of Proposition 2.1, let \( M \) be the \( \mathcal{P} \)-submodule of \( \bigoplus_{\alpha \notin I(\mathcal{P})} \mathfrak{g}^\alpha \) corresponding to \( E \). Since \( M \) is a \( T \)-stable subspace of \( \bigoplus_{\alpha \notin I(\mathcal{P})} \mathfrak{g}^\alpha \), there is a subset \( I(M) \subset R \setminus I(\mathcal{P}) \) such that \( M = \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha \). By Lemma 3.1, we have

\[
\forall \beta \in I(\mathcal{P}), \, \forall \alpha \in I(M), \, \alpha + \beta \in R \setminus I(\mathcal{P}) \implies \alpha + \beta \in I(M).
\]

Thus, \( M_C := \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha \) is a \( \mathcal{P}_C \)-submodule of \( \bigoplus_{\alpha \notin I(\mathcal{P})} \mathfrak{g}^\alpha \) and the subbundle \( E_C \subset T(\mathcal{G}_C/P_C) \) corresponding to \( M_C \) has the same rank as \( E \).

Note that there is a corresponding vector bundle \( E_Z \) over \( \mathcal{G}_Z/P_Z \), since the \( \mathcal{P} \)-module \( \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha \) is defined over \( \mathbb{Z} \). Since this is a flat bundle, we get that \( \deg(E_C) = \deg(E) \). \( \square \)

Lemma 3.3. The tangent bundle \( T(\mathcal{G}/\mathcal{P}) \) is polystable.

Proof. Let \( E \) be the first term of the Harder-Narasimhan filtration of \( T(\mathcal{G}/\mathcal{P}) \). First assume \( E \neq T(\mathcal{G}/\mathcal{P}) \), so

\[
\mu(E) > \mu(T(\mathcal{G}/\mathcal{P})).
\]

Since the anticanonical polarization of \( \mathcal{G}/\mathcal{P} \) is fixed by \( \mathcal{G} \), from the uniqueness of the Harder-Narasimhan filtration it follows that \( E \) is \( \mathcal{G} \)-stable. By Proposition 3.2 and stability of \( T(\mathcal{G}_C/P_C) \) in characteristic 0 [AB10, Theorem 2.1], we thus have \( \mu(E) < \mu(T(\mathcal{G}/\mathcal{P})) \) which contradicts (2). So \( T(\mathcal{G}/\mathcal{P}) \) is semi-stable.

We can then similarly argue with the polystable socle (cf. [HL97, page 23, Lemma 1.5.5]) of \( T(\mathcal{G}/\mathcal{P}) \) to deduce that \( T(\mathcal{G}/\mathcal{P}) \) is polystable. \( \square \)

Since \( T(\mathcal{G}/\mathcal{P}) \) is polystable, it is isomorphic to

\[
\bigoplus_{i=1}^{r} E_i^{\oplus m_i},
\]

such that

- each \( E_i \) is stable with \( \mu(E_i) = \mu(T(\mathcal{G}/\mathcal{P})) \),
- \( m_i \geq 1 \), and
- \( E_i \neq E_j \) if \( i \neq j \).
We note that the isomorphism classes of \( E_1, \ldots, E_r \) are unique up to permutations of \( \{1, \ldots, r\} \). Let
\[
\text{Hom}(E_i, T(G/P)) = H^0(G/P, T(G/P) \otimes E_i^*)
\]
be the space of homomorphisms. Now consider the natural homomorphism
\[
\bigoplus_{i=1}^{r} \text{Hom}(E_i, T(G/P)) \otimes E_i \longrightarrow T(G/P)
\]
that sends any \( s \otimes v \), where \( s \in \text{Hom}(E_i, T(G/P)) \) and \( v \in (E_i)_x \) to \( s(x)(v) \in T(G/P)_x \). Since \( \text{Hom}(E_i, E_j) = 0 \) if \( i \neq j \), and \( \text{Hom}(E_i, E_i) = k \), it follows that the homomorphism in (3) is an isomorphism.

**Lemma 3.4.** Take any \( g \in G \) and integer \( 1 \leq j \leq r \). Then \( g^* E_j \cong E_j \) as vector bundles on \( G/P \).

**Proof.** Let \( \phi : G \times (G/P) \longrightarrow G/P \) be the left-translation action. Let \( p_2 : G \times (G/P) \longrightarrow G/P \) be the projection to the second factor. The action \( \phi \) produces an isomorphism of vector bundles
\[
\phi : \bigoplus_{i=1}^{r} \text{Hom}(E_i, T(G/P)) \otimes \phi^* E_i = \phi^* T(G/P) \longrightarrow p_2^* T(G/P) = \bigoplus_{i=1}^{r} \text{Hom}(E_i, T(G/P)) \otimes p_2^* E_i.
\]
For \( i \neq \ell \), as \( E_i \) and \( E_\ell \) are stable of the same slope, we have
\[
\text{Hom}((\phi^* E_i)|_{[g]} \times G/P, (p_2^* E_\ell)|_{[g]} \times G/P) = \text{Hom}(E_i, E_\ell) = 0.
\]
Hence, using the semi-continuity of the function \( (g_1, g_2) \rightarrow \dim \text{Hom}((\phi^* E_i)|_{[g_i]} \times G/P, (p_2^* E_\ell)|_{[g_\ell]} \times G/P) \), we get
\[
\text{Hom}(\phi^* E_i, p_2^* E_\ell) = 0.
\]
From (5) it follows immediately that \( \Phi \) in (4) takes \( \text{Hom}(E_i, T(G/P)) \otimes \phi^* E_i \) to itself for every \( 1 \leq i \leq r \). In particular, we have \( \text{Hom}(E_j, T(G/P)) \otimes \phi^* E_j \cong \text{Hom}(E_j, T(G/P)) \otimes p_2^* E_j \). Fix \( g \in G \): restricting to \( [g] \times G/P \), we get
\[
\text{Hom}(E_j, T(G/P)) \otimes g^* E_j \cong \text{Hom}(E_j, T(G/P)) \otimes E_j.
\]
Since \( E_j \) is stable, we know that \( g^* E_j \) is indecomposable. Now in view of the uniqueness of the decomposition into a direct sum of indecomposable vector bundles (see [At56, p. 315, Theorem 2]), from (6) we conclude that \( g^* E_j \cong E_j \).

**Lemma 3.5.** For all \( j \in \{1, \ldots, r\} \), the vector bundle \( E_j \) is \( G \)-linearized.

**Proof.** Fix an integer \( 1 \leq j \leq r \). We now introduce the group of symmetries of the vector bundle \( E_j \). Let \( \tilde{G} \) denote the set of pairs \( (g, h) \), where \( g \in G \) and \( h \in \text{Aut}(E_j) \), such that the diagram
\[
\begin{array}{ccc}
E_j & \xrightarrow{h} & E_j \\
\downarrow & & \downarrow \\
G/P & \xrightarrow{g} & G/P
\end{array}
\]
commutes. Since \( E_j \) is simple, \( \text{Aut}_{G/P}(E_j) \cong G_m \), and therefore we get a central extension
\[
1 \longrightarrow G_m \longrightarrow \tilde{G} \xrightarrow{pr_1} G \longrightarrow 1.
\]
By Lemma 3.4, the above homomorphism \( pr_1 \) is surjective. This \( \tilde{G} \) is an algebraic group. To see this, consider the direct image \( p_2^*: \mathcal{F}so(\phi^* E_j, p_2^* E_j) \), where \( \phi \) and \( p_2 \) are the projections in the proof of Lemma 3.4, and \( \mathcal{F}so(\phi^* E_j, p_2^* E_j) \) is the sheaf of isomorphisms between the two vector bundles \( \phi^* E_j \) and \( p_2^* E_j \). This direct image is a principal \( G_m \)-bundle over \( G/P \). The total space of this principal \( G_m \)-bundle is identified with \( \tilde{G} \).

We consider the derived subgroup \( [\tilde{G}, \tilde{G}] \). Since \( G \) is simple and not abelian, we have \( [G, G] = G \), so \( \pi([\tilde{G}, \tilde{G}]) = G \). The unipotent radical of \( \tilde{G} \) is trivial. Indeed, the unipotent radical is mapped to the trivial subgroup of \( G \) since \( G \) is simple. Therefore it is included in \( G_m \) and so the unipotent radical is trivial.
Since $\tilde{G}$ is reductive, $[\tilde{G}, \tilde{G}]$ is semi-simple, hence a proper subgroup of $\tilde{G}$ (the radical of $\tilde{G}$ contains $\mathbb{G}_m$ hence $\tilde{G}$ is not semi-simple). Thus the restriction of $p r_1$ to $[\tilde{G}, \tilde{G}]$ is an isogeny. Since $G$ is simply-connected, the restriction of $p r_1$ to $[\tilde{G}, \tilde{G}]$ is an isomorphism. Consequently, the tautological action of $[\tilde{G}, \tilde{G}]$ on $E_j$ makes it a $G$-linearized bundle.

**Lemma 3.6. The integer $r$ in (3) is 1.**

**Proof.** Since $\text{Hom}(E_1, T(G/P)) \otimes E_1$ is a direct summand of $T(G/P)$ (see (3)), from Lemma 3.3 we know that the slope of $\text{Hom}(E_1, T(G/P)) \otimes E_1$ coincides with the slope of $T(G/P)$. In the proof of Lemma 3.5 we saw that $\text{Hom}(E_1, T(G/P)) \otimes E_1$ is a $G$-equivariant direct summand of $T(G/P)$. As $T(G/P)$ is stable, $[\text{AB}10, \text{Theorem 2.1}]$, from Proposition 3.2 it now follows that $\text{Hom}(E_1, T(G/P)) \otimes E_1 = T(G/P)$.

The following proposition holds without any restriction on the characteristic.

**Proposition 3.7.** Let $M_1, M_2$ be two $G$-modules such that $H^0(G/P, T(G/P)) = M_1 \otimes M_2$ as $G$-modules. Then either $M_1 = k$ or $M_2 = k$.

**Proof.** Let $\theta$ be the highest root of $\mathfrak{g}$. We claim that $\theta$ is a maximal weight of $H^0(G/P, T(G/P))$ in the sense that $\theta + \alpha$ is not a weight of $H^0(G/P, T(G/P))$ for any positive root $\alpha$. To prove this, first note that if $H^0(G/P, T(G/P)) = \mathfrak{g}$, then this is in fact the definition of the highest root. By [De77, Théorème 1], there are only three cases where $H^0(G/P, T(G/P)) \neq \mathfrak{g}$:

1. $G = \text{Sp}(2n)$ of type $C_n$ with $G/P$ a projective space and $H^0(G/P, T(G/P)) = \mathfrak{sl}(2n)$,
2. $G = \text{SO}(n+2)$ of type $B_n$ with $G/P$ a spinor variety and $H^0(G/P, T(G/P)) = \mathfrak{so}(2n+2)$, and
3. $G = G_2$ with $G/P$ a quadric and $H^0(G/P, T(G/P)) = \mathfrak{so}(7)$.

In these three cases, we have exceptional automorphisms that account for additional vector fields and we have $H^0(G/P, T(G/P)) = \mathfrak{g} \oplus V$, where $V$ has a unique highest weight which is not higher than $\theta$. For example, if $G = \text{Sp}(2n)$, then $G/P = \text{SL}(2n)/\mathbb{P}\mathfrak{sl}(2n)$ is a projective space of dimension $2n-1$, so that $H^0(G/P, T(G/P))$ is $\mathfrak{sl}(2n)$. Then $V$ is a module with unique highest weight $e_1 + e_2$, whereas $\theta = 2e_1$ (in the notation of [Bou05, Chap VI, Planches]). So the claim is proved.

As $\theta$ is a maximal weight of $H^0(G/P, T(G/P)) = M_1 \otimes M_2$, there are maximal weights $\omega_1$ and $\omega_2$ of $M_1$ and $M_2$ respectively, such that

$$\theta = \omega_1 + \omega_2. \tag{7}$$

Since $\omega_1$ and $\omega_2$ are maximal, they are dominant. In all types except $A_n$ and $C_n$, we have $\theta$ to be a fundamental weight. Therefore, from the equality in (7) it follows that either $\omega_1 = 0$ or $\omega_2 = 0$, hence the proposition is proved in these cases.

For the remaining cases of $A_n$ and $C_n$, assume that $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Let $\omega_i$ denote the $i$-th fundamental weight. In case of $A_n$, we have $\theta = \omega_1 + \omega_n$, so up to a permutation, $\omega_1 = \omega_1$ and $\omega_2 = \omega_n$. Since the Weyl group orbits of both $\omega_1$ and $\omega_n$ have $n+1$ elements, it follows that $\dim M_1 \geq n + 1$ and $\dim M_2 \geq n + 1$. This implies that $\dim H^0(G/P, T(G/P)) \geq (n+1)^2$ which is a contradiction. In case of $C_n$, we have $\theta = 2\omega_1$, so similarly we get $\omega_1 = \omega_2 = \omega_1$, and $\dim H^0(G/P, T(G/P)) \geq (2n)^2$. This is again a contradiction.

**Lemma 3.8.** $\dim \text{Hom}(E_1, T(G/P)) = 1$.

**Proof.** From Lemma 3.6 we have $H^0(G/P, T(G/P)) = \text{Hom}(E_1, T(G/P)) \otimes H^0(G/P, E_1)$. Since $T(G/P)$ is globally generated, so is $E_1$ and thus $\dim H^0(G/P, E_1) > 1$. Thus, as $E_1$ is $G$-linearized, the lemma follows from Proposition 3.7.

From equation (3) and Lemma 3.6, we get that $T(G/P) = \text{Hom}(E_1, T(G/P)) \otimes E_1$. By Lemma 3.8, $\text{Hom}(E_1, T(G/P)) \approx k$, thus $T(G/P) \approx E_1$ and it is stable.

The following lemma completes the proof of the main Theorem.
Lemma 3.9. Let \( E \) be a semi-stable (respectively, stable) \( G \)-linearized vector bundle on \( G/P \). Then \( E \) is Frobenius semi-stable (respectively, Frobenius stable).

Proof. The absolute Frobenius morphism on \( G/P \) will be denoted by \( F \). First assume that \( E \) is semi-stable. Let \( W \) be the first term of the Harder-Narasimhan filtration of \( F^*E \). We use the correspondence between vector bundles on \( G/P \) and \( P \)-modules given in Proposition 2.1. Thus \( W \) corresponds to a \( P \)-stable subspace of \((F^*E)_x \), the fiber of \( F^*E \) at the base point in \( G/P \). This is the same as an \( F^*P \)-stable subspace \( S \) of \( E_x \). Since \( F : P \to P \) is bijective, this \( S \) is also a \( P \)-submodule of \( E_x \). Thus, there exists a subbundle \( W' \subset E \) of slope \( \frac{\mu(W)}{\text{char}(k)} \geq \frac{\mu(F^*E)}{\text{char}(k)} = \mu(E) \) such that \( W = F^*W' \). By semi-stability of \( E \), we have \( W' = E \). Thus we get that \( W = F^*E \).

Assume now that \( E \) is stable. So, as above, \( F^*E \) is semi-stable. Let \( W \subset F^*E \) be a subbundle with \( \mu(W) = \mu(F^*E) \). We consider the Cartier connection \( F^*E \to F^*E \otimes \Omega^1_{G/P} \). By [Ka76, Theorem 5.1], the subbundle \( W \) is a Frobenius pull-back if and only if its image under the composition

\[
W \to F^*E \to F^*E \otimes \Omega^1_{G/P}
\]

is contained in \( W \otimes \Omega^1_{G/P} \). Let \( f : W \to F^*E \otimes \Omega^1_{G/P} \) be the above composition.

To prove that \( f(W) \subset W \otimes \Omega^1_{G/P} \), consider the composition

\[
W \xrightarrow{f} F^*E \otimes \Omega^1_{G/P} \to ((F^*E)/W) \otimes \Omega^1_{G/P}.
\]

Since \( E \) is Frobenius semi-stable, the pullback \( F^*E \) is Frobenius semi-stable. As \( \mu(W) = \mu(F^*E) \), and \( F^*E \) is Frobenius semi-stable, it follows that both \( W \) and \((F^*E)/W \) are Frobenius semi-stable with

\[
\mu(W) = \mu((F^*E)/W).
\]

Now, since \( \Omega^1_{G/P} \) is also Frobenius semi-stable, it follows that \((F^*E)/W \otimes \Omega^1_{G/P} \) is semi-stable [RR84, p. 285, Theorem 3.18]. From (9) and the fact that \( \mu(\Omega^1_{G/P}) < 0 \) it follows that

\[
\mu(W) > \mu((F^*E)/W) \otimes \Omega^1_{G/P}.
\]

So the composition in (8) being \( \Theta_{G/P} \)-linear has to vanish, meaning we have \( f(W) \subset W \otimes \Omega^1_{G/P} \). Therefore, let \( W' \subset E \) be such that \( W = F^*W' \). We have \( \mu(W') = \mu(E) \). By stability of \( E \), we get that \( W' = E \) and hence \( W = F^*E \). \( \square \)

Remark 3.10. It is not true that for any semistable vector bundle \( V \) on a smooth projective variety, the pullback of \( V \) by the Frobenius map of the variety is semistable. In fact, there are stable vector bundles on curves whose Frobenius pullback is semistable; see [LP08].

4. An example in small characteristic

We give an example of a tangent bundle which is semi-stable but not stable. We do not know if there are some tangent bundles to homogeneous spaces which are not semi-stable in positive characteristic.

The example is that of \( G/P = G_{so}(n, 2n) \), the Grassmaennian of Lagrangian spaces in a symplectic space of dimension \( 2n \), and we assume that \( k \) has characteristic 2. Namely, \( G \) is \( Sp(2n) \) and \( P \) is the maximal parabolic subgroup corresponding to the long simple root. Let \( U \) denote the universal bundle on \( G/P \), of rank \( n \) and degree \(-1\). Then \( T(G/P) \) is a submodule of \( U^* \otimes U^* \); in fact if \( S^2 U \) denotes the symmetric quotient of \( U \otimes U \), then \( T(G/P) = (S^2 U)^* \).

We will implicitly use the correspondence between \( P \)-modules and \( G \)-linearized homogeneous bundles on \( G/P \) (Proposition 2.1). Note that the reductive quotient of \( P \) is \( GL(U) \). Since \( \text{char} k = 2 \), we have an additive map \( U \to S^2 U, x \to x^2 \), which can be seen as a linear application \( F^* U \to S^2 U \) (recall that \( F \) denotes the Frobenius morphism). It is \( GL(U) \)-equivariant, so this defines an exact sequence of bundles on \( G/P \):

\[
0 \to F^* U \to S^2 U \to K \to 0
\]

(10)
It follows that there is a subbundle \( K^* \subset T(G/P) \). Since \( \mu(F^*U) = \mu(S^2U) = 2\mu(U) \), we get \( \mu(K^*) = \mu(T(G/P)) \) and \( T(G/P) \) is not stable. However since \( F^*U \) is the only \( GL(U) \)-invariant subspace in \( S^2U \), \( K^* \) is the only linearized subbundle in \( T(G/P) \). Thus the semi-stability inequality holds for this subbundle. Arguing as in the proof of Lemma 3.3, we deduce that \( T(G/P) \) is semi-stable.

For general homogeneous spaces \( G/P \), we face two difficulties:

- There are linearized subbundles in \( T(G/P) \) which do not lift to characteristic 0, and contrary to the above example, they are numerous in general.
- The stability of \( T(G/P) \) for characteristic 0 says nothing about \( \mu(E) \) of such a subbundle \( E \subset T(G/P) \). It is difficult to compute the \((\dim(G/P) - 1)\)-th power of the anticanonical polarization to be able to show the semi-stability inequality for \( E \).

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